

"HYPERSONIC LAMINAR BOUNDARY LAYER THEORY"

M.P. Georgeff, B.Sc., B.E.

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TEXT BOUND CLOSE TO THE SPINE IN THE ORIGINAL THESIS

The stern heart softens, all its pride unlearning,
A shudder passes through me, and I weep.
All that I have stands off from me afar,
And all I lost is real, my guiding-star.

Goethe

ABSTRACT

Methods of predicting the major characteristics of viscous-inviscid interactions at supersonic and hypersonic speeds are examined, particular emphasis being placed on the integral method of Klineberg.

The method of Klineberg is extended to allow the study of viscous interaction phenomena over a continuous range of wall cooling ratio. By suitably normalizing the required profile quantities the dependence on wall cooling ratio is removed, thus considerably simplifying the method and allowing flows with variable wall cooling ratio to be studied.

An extensive theoretical evaluation of the method is made, thus establishing the validity of the integral approach. It is further shown that the properties associated with the stability of the governing differential equations are mathematical properties of the analytic model and should not be associated with any physical characteristics of the boundary layer.

Comparisons show that the method of Klineberg provides a good description of the major features of viscous interaction phenomena and is superior to other existing methods. The general agreement with experiment is good for a wide class of body shapes in both supersonic and hypersonic flow.

A parametric study is carried out, thus isolating the significant parameters of viscous-inviscid interactions.

A further extension of the method to include three dimensional boundary layers with zero transverse pressure gradient is also developed, and applications to general three dimensional flows are indicated. Normal pressure gradient effects are discussed, and methods of allowing for nose bluntness are described.

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NOTATIONSymbols

a	speed of sound; also Klineberg velocity profile parameter
A	quantity defined in equation (112); also typical profile quantity
$\bar{A}$	matrix defined in equation (A15)
$\tilde{A}$	quantity defined in equation (113)
A_1-A_8	functions defined in equation (26)
$\bar{A}_1, \bar{A}_5$	functions defined in equation (97)
A^*	similarity parameter $\frac{\gamma(\gamma - 1)}{2} F$
b	enthalpy profile parameter $-\frac{\partial}{\partial \eta} \left(\frac{S}{S_w} \right)_{\eta=0}$
$b_0, b_1, b_{11}, b_\alpha, b_{\alpha 1}$	coefficients of asymptotic expansions [equations (39) & (41)]
$\bar{b}$	similarity variable b
b	$b - b_B$
b_0, b_1	functions defined in equations (A9) & (A11)
B	$\frac{a_{ep_e}}{a_{\infty p_{\infty}}}$
B_1-B_8	functions defined in equation (25)
C	constant in viscosity relationship $\left(\frac{\mu}{\mu_{\infty}} \right) / \left(\frac{T}{T_{\infty}} \right)$
C_0-C_6	coefficients of polynomial functions; also functions defined in equation (A3)

- C_F skin friction coefficient $\left(\mu \frac{\partial u}{\partial y} \right)_w / (\frac{1}{2} \rho_\infty u_\infty^2)$
- C_{F_z} skin friction coefficient (transverse plane) $\left(\mu \frac{\partial w}{\partial y} \right)_w / (\frac{1}{2} \rho_\infty u_\infty^2)$
- C_H heat transfer coefficient $\left(k \frac{\partial T}{\partial y} \right)_w / \rho_\infty u_\infty (h_{o_e} - h_{o_w})$
- D determinant of governing set of differential equations [equation (24)]; also function defined in equation (A8)
- $D_1 - D_4$ functions defined in equation (A3); also as in equations (A16) & (A17).
- E $\frac{1}{\delta_i^*} \int_0^{\delta_i} \frac{S}{S_w} dy$
- $\bar{E}$ $E - k_1 G$
- f function defined in equation (19); also function related to stream function [equation (100)]
- $\bar{f}$ function defined in equation (96)
- F $H + \frac{1 + m_e}{m_e} (1 + S_w E)$
- $\bar{F}$ $H + \frac{1 + m_e}{m_e} (1 + S_w \bar{E})$
- g transverse velocity ratio: plane flow $\left(1 - \frac{w}{w_e} \right)$,
axisymmetric flow $\frac{w}{\Omega r_w}$
- G $\frac{1}{\delta_i^*} \int_0^{\delta_i} \left(\frac{S}{S_w} \right)^2 dy$
- h static enthalpy; also function defined in equation (19)
- h_o total enthalpy
- $\bar{h}$ function defined in equation (111)
- H $\frac{\theta_i}{\delta_i^*}$

$H_0, H_1, H_{11},$
 $H_\alpha, H_{\alpha 1}$ coefficients of asymptotic expansions [equations (39) & (41)]

$\bar{H}$ similarity variable H

H $H - H_B$

H_0, H_1 functions defined in equations (A9) & (A11)

I
$$- \frac{\delta_i^{*2}}{(1 + S_w)} \frac{\partial^2}{\partial Y^2} \left(\frac{U}{U_e} \right)$$

J
$$\frac{\theta_i^*}{\delta_i^*}$$

k thermal conductivity; also constant defined in equation (31).

k_c leading edge drag coefficient

k_1
$$\frac{\omega}{S_w}$$

k_2 quantity defined in equation (114)

K $M_\infty \alpha_i$; also constant in Stollery-Hankey relationship $J=KH$

K_1
$$\frac{3\gamma - 1}{\gamma - 1} \frac{m_\infty}{1 + m_\infty}$$

K constant

L characteristic length

m
$$\frac{\gamma - 1}{2} M^2$$

$m_1, m_2, m_{11}, m_{12}, m_{21}$ coefficients of asymptotic expansions [equation (41)]

$\bar{m}$ ratio of mass entrained by boundary layer to that displaced

M Mach number

$\bar{M}$ similarity variable
$$\frac{M_e}{M_\infty}$$

n	exponent [equation (98)]
N_1-N_4	functions defined in equation (24)
$\overline{N}_1-\overline{N}_4$	functions defined in equation (A14)
P	static pressure
$\overline{P}$	P_e/P_∞
$P_0, P_1, P_\alpha, P_{\alpha 1}$	coefficients of asymptotic expansions [equation (39)]
P	$\delta_i^* \left\{ \frac{\partial}{\partial Y} \left(\frac{U}{U_e} \right) \right\}_{Y=0}$
P_1	$\frac{(P - P_B)}{(1 + 0.25S_w)}$
$\overline{P}$	function defined in equation (121)
Pr	Prandtl number
Q	$-\delta_i^* \left\{ \frac{\partial}{\partial Y} \left(\frac{S}{S_w} \right) \right\}_{Y=0}$
$\overline{Q}$	function defined in equation (19); also function defined in equation (111)
r_w	local body radius
R	$2\delta_i^* \int_0^1 \left\{ \frac{\partial}{\partial Y} \left(\frac{U}{U_e} \right) \right\}^2 dY$
$\overline{R}$	function defined in equation (111)
Re_x	$\frac{a_\infty}{v_\infty} M_\infty x$
Re_{δ^*}	$\frac{a_\infty}{v_\infty} M_\infty \delta^*$
$Re_{\delta_i^*}$	$\frac{a_\infty}{v_\infty} M_\infty \delta_i^*$
Reynolds numbers	

$\overline{R}_{e_{\delta_1^*}}$	similarity variable $\frac{R_{e_{\delta_1^*}}}{M_\infty^3 C}$ or $\frac{R_{e_{\delta_1^*}}}{FM_\infty^3 C}$
s	S/S_w
S	total enthalpy function $\left[\frac{h_o}{h_{oe}} - 1 \right], \left[\frac{h_{otr}}{h_{oe_{tr}}} - 1 \right]$
t	leading edge thickness
T	$\int_0^{(\eta) U/U_e=0.99} \frac{U}{U_e} \frac{S}{S_w} d\eta$; also static temperature
T^*	$\frac{1}{\delta_1^*} \int_0^{\delta_1} \frac{U}{U_e} \frac{S}{S_w} dY$
$\overline{T}^*$	$T^* - k_1 W^*$
u, v, w	longitudinal, normal and transverse velocity components
U, V	Stewartson-illingworth transformed velocity components
W	$\int_0^{(\eta) U/U_e=0.99} \frac{U}{U_e} \left(\frac{S}{S_w} \right)^2 d\eta$
W^*	$\frac{1}{\delta_1^*} \int_0^{\delta_1} \frac{U}{U_e} \left(\frac{S}{S_w} \right)^2 dY$
x, y, z	curvilinear co-ordinates, longitudinal, normal and transverse (see Figure 1a)
$\overline{x}$	x/x_o
X, Y	Stewartson-illingworth transformed co-ordinates; also cartesian co-ordinates (see Figure 1a)
Z	$\frac{1}{\delta_1^*} \int_0^{\delta_1} \frac{U}{U_e} dY$
α	$\left(\int_0^{(\eta) U/U_e=0.99} (1 - U/U_e) d\eta \right)^{-1}$; also local surface slope (see Figure 1a)

- $\alpha_1 - \alpha_3$ functions defined in equation (A8)
- β Cohen-Reshotko pressure gradient parameter, $\frac{2n}{n+1}$
- $\beta_1 - \beta_3$ functions defined in equation (A8)
- γ ratio of specific heats
- $\gamma_1 - \gamma_3$ functions defined in equation (A8)
- δ boundary layer thickness $\int_0^{(\delta) u/u_e=0.99} dy$
- δ_i transformed boundary layer thickness $\int_0^{(\delta_i) U/U_e=0.99} dY$
- δ^* boundary layer displacement thickness $\int_0^{\delta} \left(1 - \frac{\rho u}{\rho_e u_e} \right) dy$
- δ_i^* transformed boundary layer displacement thickness $\int_0^i \left(1 - \frac{U}{U_e} \right) dY$
- δ_o $\frac{x_o}{(R_e x_o / C)^{1/2}}$
- $\delta_o, \delta_1, \delta_{11}, \delta_\alpha, \delta_{\alpha 1}$ coefficients of asymptotic expansions [equations (39), (40) & (41)]
- $\frac{\delta}{\delta_o}$ $\frac{4}{(\gamma - 1)^2} \frac{R_{e_{\delta_i^*}}}{M_\infty^3 C}$; also $\frac{\delta_i^*}{\delta_o}$ or $\frac{\delta^*}{M_\infty^2 \delta_o}$
- ε perturbation parameter
- $\frac{1}{\varepsilon}$ small positive number
- ζ function defined in equation (A6)
- ζ_o function defined in equation (A9)
- η Cohen-Reshotko similarity variable
- θ_e local streamline inclination at edge of boundary layer

$$\theta_i \quad \int_0^{\delta_i} \frac{U}{U_e} \left(1 - \frac{U}{U_e} \right) dy$$

$$\theta_i^* \quad \int_0^{\delta_i} \frac{U}{U_e} \left(1 - \frac{U^2}{U_e^2} \right) dy$$

$$\kappa \quad \int_0^{(\eta)_{U/U_e=0.99}} \left(\frac{S}{S_w} \right)^2 d\eta$$

$\lambda, \lambda_1 - \lambda_4$ characteristic exponents (equation A15)

μ viscosity coefficient

ν kinematic viscosity $\frac{\mu}{\rho}$; also Prandtl Meyer function

ρ gas density

$\sigma \quad \int_0^{(\eta)_{U/U_e=0.99}} \frac{S}{S_w} d\eta$; also function defined in equation (A11)

$\tau_0 - \tau_5$ coefficients of polynomial functions

ϕ shock wave angle

χ hypersonic viscous interaction parameter $\frac{M_\infty^3 \sqrt{C}}{\sqrt{R_{ex}}}$

ψ stream function

$$\omega \quad \frac{w_e^2}{2 \left(h_{oe} \right)_{tr}}$$

Ω angular velocity of body

Subscripts

B Blasius values

e local flow outside boundary layer (external)

HL corner of flat plate-wedge (hinge line)

i	transformed, incompressible; also incipient separation condition
k,j	summation coefficients
le	leading edge
r	ramp, flap, wedge
R	reattachment point
s	shock wave, outer edge of entropy layer
S	separation point
SI	strong interaction
TE	trailing edge
tr	values determined with respect to co-ordinate frame moving in the z direction with velocity w_e
w	wall (surface)
WI	weak interaction
o	beginning of interaction, reference
1,2	upstream, downstream of 'jump', shock wave
$\infty, \infty-$	freestream conditions upstream of interaction
$\infty+$	conditions far downstream of interaction

A parameter (or co-ordinate) used as subscript represents partial differentiation with respect to the parameter (or co-ordinate).

Other Notation

'	partial differentiation with respect to η, H
.	partial differentiation with respect to b
~	perturbation quantities

INTRODUCTION

With the advent of high altitude supersonic and hypersonic flight the analysis of laminar viscous interaction phenomena has become of considerable practical importance. Thus it is essential not only to seek a greater understanding of the structure of viscous-inviscid interactions, but also to develop workable methods capable of accurately predicting the overall characteristics of the interaction.

At high Mach numbers, the development of the viscous layer can significantly modify the external flow field. The usual methods of boundary layer analysis, where viscous effects are considered as a perturbation on an already existing flow, are therefore inapplicable. However, the viscous interaction can be described by employing the boundary layer equations together with a "coupling" equation relating the development of the inner viscous flow to the outer inviscid flow. The governing partial differential equations can then be solved using finite difference techniques or can be expressed in integral form and solved as ordinary differential equations.

Successful finite difference methods include those of Reyhner & Flügge-Lotz (1966), Baum (1967) and Smith (1972), the latter two being restricted to attached flow. These methods provide the most accurate and general solution to the governing differential equations as no assumptions are involved in describing the boundary layer profiles. However, integral methods have usually been preferred because of their relative simplicity.

Two of the most successful integral methods are those of Lees and Reeves (1964) and Nielsen, Lynes and Goodwin (1965). Nielsen employs a family of velocity and enthalpy profiles to provide the required relations

between profile quantities and solves the first four moments of the momentum equation and the first moment of the energy equation. The method has been extended to include two dimensional plane and axisymmetric flows with heat transfer (Nielsen 1967, 1969, 1970).

Lees and Reeves use the Cohen-Reshotko similarity solutions (1956a) to provide the required relations between boundary layer quantities and solve the integral equations of momentum and moment of momentum together with a coupling equation based on the integral form of the continuity equation. By incorporating the energy equation Holden (1965, 1967, 1968) and Klineberg (1968) extended the method to include non-adiabatic isothermal wall conditions. Later, Holden (1969) attempted to include the effects of normal pressure gradient, but the underlying theory can be shown to be invalid. The axisymmetric case was also considered (Horton 1970), and by including the effects of spin, Leblanc, Horton and Ginoux (1971) further extended the method to three dimensional adiabatic flows.

Some interesting problems arose during the development of these methods. It was found that the governing differential equations could exhibit either stable (often termed "supercritical") or unstable ("subcritical") behaviour depending on the condition of the boundary layer. This aspect of the problem was investigated by a number of authors (Crocco 1955, Garvine 1968, Weinbaum and Garvine 1969, Shamroth 1969, Stollery and Hankey 1970a, Georgeff 1971) although the physical significance of such behaviour was not adequately resolved. However, by comparing a number of methods based on the Lees-Reeves and Holden-Klineberg formulations, the author (1972a) was able to show that the stability of the governing differential equations was related to the formulation of the problem and not to any physical characteristics of the boundary layer. In the same report the necessity of incorporating the energy equation to des-

cribe non-adiabatic flows was demonstrated and the integral formulation put on firmer ground by showing that the choice of governing differential equations or profile quantity relationships was not critical.

The comparisons of Murphy (1969) showed that both the Klineberg and Nielsen methods provide a good description of the major characteristics of viscous interaction phenomena at low Mach numbers. Both methods are computationally quite efficient, but a great disadvantage of the Klineberg method is that the relations between the boundary layer profile quantities are dependent on the wall cooling ratio. Additional similarity solutions must therefore be calculated at each value of wall cooling ratio in order to provide the required polynomial functions used to describe these relations.

It was shown (Georgeff 1972b), however, that this dependence on wall cooling ratio can be removed by suitably normalizing the profile quantities. The advantages thus derived are considerable. The onerous task of recalculating the polynomial functions at each required value of wall cooling ratio is thereby avoided and non-isothermal surfaces can be included without difficulty.

In the same paper it was also shown that the normalized governing differential equations become similar in form to those of a three dimensional boundary layer provided that the flow gradients in the transverse plane are negligible. This enabled a further extension of the Klineberg method to include three dimensional flows with heat transfer.

This thesis describes the author's investigations and subsequent extension of the Lees-Reeves and Klineberg methods to three dimensional flows with arbitrary wall cooling ratio. The theory has been generalized since the earlier reports appeared and some further theoretical considerations are advanced. A number of previously unpublished experimental comparisons are also included.

2. ANALYSIS

2.1 APPROACH

The basic approach is as follows. In order to make the viscous interaction problem amenable to analysis, the flow field is divided into two regions: a viscous inner layer and an inviscid external flow (Figure 1b). The development of the external flow is related to the growth of the viscous layer by demanding compatibility at the boundary between the two regions. This provides the so-called 'coupling equation'.

The physical model of the external flow can be quite complex despite the assumption that viscous effects are negligible. Incident and reflected shock and expansion waves, together with other non-isentropic effects, considerably complicate the flow pattern. However, in general it is possible to provide a relatively simple model to enable the main effects of the interaction to be accommodated.

The behaviour of the inner viscous layer is determined by the Navier-Stokes equations. However, in general these prove difficult to solve. It is therefore assumed that the behaviour of the inner viscous layer is primarily governed by the first order terms in the Navier-Stokes equations and consequently the boundary layer equations can be used.

The boundary conditions must also be specified. However, unlike boundary layer problems with prescribed pressure gradient, viscous interaction problems are not well posed initial value problems. Through the mechanism of viscous interaction contained in the coupling equation, the viscous layer is sensitive to downstream conditions. Hence both upstream and downstream boundary conditions must be specified.

In order to solve the governing differential equations, some form of "marching" procedure is adopted. Consequently it is not possible to determine the correct solution trajectory a priori and the upstream boundary

conditions must be arbitrarily perturbed until the appropriate downstream boundary conditions are satisfied. It is assumed that the solution thus obtained is unique.

2.2 GOVERNING EQUATIONS

The governing equations for a steady two-dimensional compressible laminar boundary layer are (Schlichting 1955)

Continuity:
$$\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0 \quad (1)$$

Momentum:
$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) \quad (2)$$

$$\frac{\partial p}{\partial y} = 0$$

Energy:
$$\rho u \frac{\partial h_o}{\partial x} + \rho v \frac{\partial h_o}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\mu}{Pr} \frac{\partial h_o}{\partial y} \right) - \frac{\partial}{\partial y} \left[\mu \left(\frac{1}{Pr} - 1 \right) u \frac{\partial u}{\partial y} \right] \quad (3)$$

(All symbols are defined under 'Notation'.)

Assuming

(i) The gas is calorically perfect

(ii) $Pr = 1$

(iii) $h_{oe} = \text{constant}$

(iv) $\frac{\mu}{\mu_\infty} = C \frac{T}{T_\infty}$ where C is a constant of the Chapman-Rubesin type,

and applying the Stewartson-illingworth transformation

$$dX = C \frac{a_e p_e}{a_\infty p_\infty} dx \quad (4)$$

$$dY = \frac{a_e \rho}{a_\infty \rho_\infty} dy$$

the following integral forms of the boundary layer equations can be derived (Klineberg 1968):

Momentum:

$$H \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + (2H + 1 + S_w E) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{P}{\delta_i^* R_e \delta_i^*} \quad (5)$$

Moment of Momentum:

$$J \frac{d \ln \delta_i^*}{dx} + \frac{dJ}{dx} + (3J + 2S_w T^*) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{R}{\delta_i^* R_e \delta_i^*} \quad (6)$$

Energy:

$$T^* \frac{d \ln \delta_i^*}{dx} + \frac{dT^*}{dx} + T^* \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{Q}{\delta_i^* R_e \delta_i^*} - T^* \frac{d \ln S_w}{dx} \quad (7)$$

where

$$B = \frac{a_e p_e}{a_\infty p_\infty}$$

$$R_{e \delta_i^*} = \frac{a_\infty}{v_\infty} M_\infty \delta_i^*$$

and the transformed profile quantities are:

$$\begin{aligned} U &= \frac{a_\infty}{a_e} u & \delta_i &= \int_0^{\delta_i} dy & \delta_i^* &= \int_0^{\delta_i} \left(1 - \frac{U}{U_e}\right) dy \\ \theta_i &= \int_0^{\delta_i} \frac{U}{U_e} \left(1 - \frac{U}{U_e}\right) dy & \theta_i^* &= \int_0^{\delta_i} \frac{U}{U_e} \left(1 - \frac{U^2}{U_e^2}\right) dy \\ H &= \frac{\theta_i}{\delta_i^*} & J &= \frac{\theta_i^*}{\delta_i^*} & Z &= \frac{1}{\delta_i^*} \int_0^{\delta_i} \frac{U}{U_e} dy \\ R &= 2\delta_i^* \int_0^{\delta_i} \frac{\partial}{\partial Y} \left(\frac{U}{U_e}\right)^2 dy & P &= \delta_i^* \frac{\partial}{\partial Y} \left(\frac{U}{U_e}\right)_{Y=0} \\ Q &= -\delta_i^* \frac{\partial}{\partial Y} \left(\frac{S}{S_w}\right)_{Y=0} & E &= \frac{1}{\delta_i^*} \int_0^{\delta_i} \frac{S}{S_w} dy \\ T^* &= \frac{1}{\delta_i^*} \int_0^{\delta_i} \frac{U}{U_e} \frac{S}{S_w} dy \end{aligned} \quad (8)$$

It is important to note that the enthalpy profile quantities differ from those of Klineberg (1968) in that they are normalized with respect to the wall cooling ratio S_w . Interestingly enough, this is the original form as used by Lees and Reeves (1964) and also by Holden (1965).

The development of the outer inviscid flow can be related to the boundary layer growth by integrating the continuity equation across the boundary layer. This gives as the Coupling equation:

$$\frac{d\delta^*}{dx} - (\delta - \delta^*) \frac{d \ln(\rho_e u_e)}{dx} = \tan(\theta_e - \alpha_w) \quad (9)$$

$$\text{where } \delta = \int_0^\delta dy$$

$$\delta^* = \int_0^\delta \left(1 - \frac{\rho u}{\rho_e u_e} \right) dy$$

α_w is the local surface slope

and θ_e is the streamline inclination at the edge of the boundary layer.

Transforming to Stewartson-illingworth co-ordinates the coupling equation becomes

$$\begin{aligned} F \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + \frac{1+m_e}{m_e} S_w \frac{dE}{dx} + \frac{1}{1+m_e} \left(m_e F + 2H - \frac{1+2m_e}{m_e} Z \right) \frac{d \ln M_e}{dx} \\ - \left(F + \frac{Z}{m_e} \right) \frac{d \ln p_e}{dx} = \frac{B(1+m_e)}{m_e(1+m_w)} \frac{\tan(\theta_e - \alpha_w)}{\delta_i^*} - \frac{1+m_e}{m_e} S_w \frac{d \ln S_w}{dx} \end{aligned} \quad (10)$$

$$\text{where } F = H + \frac{1+m_e}{m_e} (1 + S_w E)$$

$$m = \frac{\gamma - 1}{2} M^2$$

and the local static pressure, p_e , and streamline inclination at the edge of the boundary layer, θ_e , are related to the local Mach number, M_e , by means of the model used to describe the outer inviscid flow (see Section 3).

The present coupling equation is more general than that of Klineberg, where the relationship between local pressure and Mach number gradients applies only for isentropic external flow.

In order to solve equations (5), (6), (7) and (10) the form of dependence of the profile quantities H , J , etc., on each other must be known. It is therefore assumed that the relations established for certain exact solutions are universal and hence valid for all flows. One then obtains a determinate set of ordinary differential equations in M_e , δ_1^* , and the parameters describing these universal relations.

2.3 CHOICE OF UNIVERSAL RELATIONS

As with the method of Klineberg, the Cohen-Reshotko similarity solutions (1956a) are employed to provide the required universal relations between profile quantities. Two parameters are used to describe these relations, one representing the velocity profile and the other the total enthalpy profile. The velocity profile parameter is taken to be the profile quantity 'H' [see equation (8)] and the total enthalpy profile parameter, normalized with respect to S_w , is taken to be

$$b = - \frac{\partial}{\partial \eta} \left(\frac{S}{S_w} \right) \quad (11)$$

where η is the similarity variable of Cohen and Reshotko.

Arbitrarily taking the edge of the boundary layer (η_e) at $\frac{u}{u_e} = 0.99$, all the profile quantities appearing in the governing differential equations can be expressed as functions of these two parameters at any particular value

of S_w . The determination of the functions that depend on both the velocity and enthalpy profiles can be simplified by using the integrals

$$\left. \begin{aligned} \alpha(H) &= \left[\int_0^{\eta_{.99}} \left(1 - \frac{U}{U_e} \right) d\eta \right]^{-1} \\ \sigma(b) &= \int_0^{\eta_{.99}} \frac{S}{S_w} d\eta \\ T(H,b) &= \int_0^{\eta_{.99}} \frac{U}{U_e} \frac{S}{S_w} d\eta \end{aligned} \right\} \quad (12)$$

such that

$$\left. \begin{aligned} Q &= \frac{b}{\alpha(H)} \\ E &= \alpha(H)\sigma(b) \\ T^* &= \alpha(H)T(H,b) \end{aligned} \right\} \quad (13)$$

It is important in determining the function $T(H,b)$ that each velocity profile is multiplied point by point by all the total enthalpy profiles. In this way a complete uncoupling of the enthalpy and velocity profiles is achieved.

Now as shown by Gautier and Ginoux (1971) the enthalpy profile functions of Klineberg are strongly dependent on the wall cooling ratio S_w . However, as shown in Figures (2) and (3) the normalized forms adopted herein are quite independent of S_w .^{*} Similarly, the velocity profile functions are,

*Note that the function $\sigma(b)$ is defined differently by Klineberg and by Gautier and Ginoux. Klineberg employs two functions to represent $\sigma(b)$, one for $b \geq 0.25$ and one for $b \leq 0.25$. Gautier and Ginoux also employ two functions for $\sigma(b)$, one for separated flow and one for attached flow. As separation usually occurs near $b = 0.25$, the difference in definitions is not significant. In the present formulation a single function for $\sigma(b)$ is used.

with the exception of $P(H)$, also independent of S_w (Figures 4-8). The function $P(H)$ shows a significant dependence on S_w , particularly in the separation region. However, the effect of P on the solution is small in this region as those terms containing P are small in comparison with the other terms appearing in the governing differential equations. Hence the solution in terms of the variables M_e , δ_i^* , H and b should not be greatly affected by assuming that P , as a function of H , is also independent of S_w . The skin friction coefficient, which is directly proportional to P , might be considerably in error in the separation region, but in this region accurate prediction of skin friction would not be expected in any case. On the other hand, it is a simple matter to allow for the dependence of $P(H)$ on S_w . For example, the similarity solutions are more accurately represented if, in the governing differential equations, P is expressed as

$$P(H, S_w) = P_1(1 + 0.25S_w) + P_B \quad (14)$$

where $P_1(H)$ and P_B are taken to be independent of S_w , the subscript B denoting Blasius values. P_1 is plotted as a function of H in Figure 8.

It is important to note that the distributions of U/U_e and S/S_w are independent of S_w for the zero pressure gradient similarity solutions. Hence for the Blasius solution, the relations between the normalized profile quantities must be exactly independent of S_w .

The advantage of using the integral quantity ' H ' as a parameter instead of the quantity ' a ' as in Klineberg's report is two-fold. Firstly, the S_w independence of the profile functions is more accurate when these are expressed in terms of H . Secondly, it is not necessary to use different polynomial functions in the attached and separated flow regions as, unlike ' a ', the quantity H remains single valued. This implies a signifi-

cant improvement in computational efficiency.

The profile parameters H and b as determined from the similarity solutions lie in the range

$$0 < H \leq 0.45 \quad \left(\pm .05 \right)$$

$$0 < b \leq 0.55 \quad \left(\pm .05 \right)$$

the upper limits showing a small dependence on S_w (as indicated)*. This range is adequate for solving most viscous interaction problems, but in strong favourable pressure gradients (expansions) the upper limits can be exceeded. In this region a new set of universal relations must be chosen. These could be obtained from any known family of solutions to the boundary layer equations, but the simplest procedure is to retain the similarity representation and extrapolate the similarity profile functions beyond the upper limits. Of course, the extrapolated profile functions do not correspond to any similarity solutions.

All the profile quantities appearing in this report are calculated using the data of Klineberg at $S_w = -0.8$. These quantities are then expressed as polynomial functions of the parameters H and b using a least-squares curve fit. The coefficients of these curve fits are given in Appendix 1. In general it would be preferable to determine the profile functions at some intermediate value of S_w , or alternatively, to take some mean over a number of wall cooling ratios. Furthermore, more accurate representations could probably be obtained by re-evaluating the profile quantities directly from the Cohen-Reshotko similarity solutions rather than via the (approximate) polynomial expressions of Klineberg.

* Klineberg (1968) allows the parameters to exceed these values, and calculates profile quantities for which the pressure gradient parameter, β , is greater than two. Such solutions, however, do not correspond to real values of the velocity component U (viz. $U = C_1 X^n$, $\beta = \frac{2n}{n+1}$). Klineberg also suggests that no upper limit is encountered in non-adiabatic flow. The expansion corner comparisons of Section 9 do not support this view.

2.4 FINAL FORM OF THE GOVERNING DIFFERENTIAL EQUATIONS

The governing equations (5), (6), (7) and (10) can be written in terms of the two profile parameters H and b as:

$$H \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + (2H + 1 + S_w E) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{P}{\delta_i^{*R} e_{\delta_i^*}} \quad (15)$$

$$J \frac{d \ln \delta_i^*}{dx} + \frac{dJ}{dH} \frac{dH}{dx} + (3J + 2S_w T^*) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{R}{\delta_i^{*R} e_{\delta_i^*}} \quad (16)$$

$$T^* \frac{d \ln \delta_i^*}{dx} + \frac{\partial T^*}{\partial H} \frac{dH}{dx} + \frac{\partial T^*}{\partial b} \frac{db}{dx} + T^* \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{\bar{Q}}{\delta_i^{*R} e_{\delta_i^*}} \quad (17)$$

$$F \frac{d \ln \delta_i^*}{dx} + \frac{\partial F}{\partial H} \frac{dH}{dx} + \frac{\partial F}{\partial b} \frac{db}{dx} + f \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{h}{\delta_i^{*R} e_{\delta_i^*}} \quad (18)$$

where

$$\frac{\partial F}{\partial H} = 1 + \left(\frac{1 + m_e}{m_e} \right) S_w \frac{\partial E}{\partial H}$$

$$\frac{\partial F}{\partial b} = \left(\frac{1 + m_e}{m_e} \right) S_w \frac{\partial E}{\partial b}$$

$$\bar{Q} = Q - \frac{\delta_i^{*R} e_{\delta_i^*} M_e}{BC M_\infty} T^* \frac{d \ln S_w}{dx}$$

$$f = \frac{1}{1 + m_e} \left(m_e F + 2H - \frac{1 + 2m_e}{m_e} Z \right) - \left(F + \frac{Z}{m_e} \right) \frac{M_e}{p_e} \frac{dp_e}{dM_e}$$

$$\text{and } h = \frac{M_e}{M_\infty} \frac{1 + m_e}{m_e} \frac{R_{\delta_i^*}}{C} \left[\frac{\tan(\theta_e - \alpha_w)}{1 + m_\infty} - \frac{\delta_i^*}{B} S_w E \frac{d \ln S_w}{dx} \right]$$

(19)

The dependent variables are M_e , δ_i^* , H and b.

This leads to the final form of the governing differential equations suitable for numerical computation:

$$\frac{1}{M_e} \frac{dM_e}{dx} = \frac{BC}{\delta_i^* R_e \delta_i^*} \frac{M_\infty}{M_e} \left[\frac{N_1(M_e, \delta_i^*, H, b)}{D(M_e, H, b)} \right] \quad (20)$$

$$\frac{1}{\delta_i^*} \frac{d\delta_i^*}{dx} = \frac{BC}{\delta_i^* R_e \delta_i^*} \frac{M_\infty}{M_e} \left[\frac{N_2(M_e, \delta_i^*, H, b)}{D(M_e, H, b)} \right] \quad (21)$$

$$\frac{dH}{dx} = \frac{BC}{\delta_i^* R_e \delta_i^*} \frac{M_\infty}{M_e} \left[\frac{N_3(M_e, \delta_i^*, H, b)}{D(M_e, H, b)} \right] \quad (22)$$

$$\frac{db}{dx} = \frac{BC}{\delta_i^* R_e \delta_i^*} \frac{M_\infty}{M_e} \left[\frac{N_4(M_e, \delta_i^*, H, b)}{D(M_e, H, b)} \right] \quad (23)$$

where

$$\begin{aligned} D(M_e, H, b) &= B_1 \frac{\partial T^*}{\partial b} - B_2 \frac{\partial F}{\partial b} \\ N_1(M_e, \delta_i^*, H, b) &= B_3 \frac{\partial T^*}{\partial b} - B_4 \frac{\partial F}{\partial b} \\ N_2(M_e, \delta_i^*, H, b) &= B_5 \frac{\partial T^*}{\partial b} - B_6 \frac{\partial F}{\partial b} \\ N_3(M_e, \delta_i^*, H, b) &= B_7 \frac{\partial T^*}{\partial b} - B_8 \frac{\partial F}{\partial b} \\ N_4(M_e, \delta_i^*, H, b) &= B_4^f + B_6^F + B_8 \frac{\partial F}{\partial H} - B_2^h \end{aligned} \quad (24)$$

and

$$\begin{aligned}
 B_1 &= A_6 \frac{\partial F}{\partial H} + A_3 f - A_8 F \\
 B_2 &= A_6 \frac{\partial T^*}{\partial H} + (A_3 - A_8) T^* \\
 B_3 &= A_2 \frac{\partial F}{\partial H} + A_3 h - A_4 F \\
 B_4 &= A_2 \frac{\partial T^*}{\partial H} + A_3 \bar{Q} - A_4 T^* \\
 B_5 &= A_7 \frac{\partial F}{\partial H} + A_4 f - A_8 h \\
 B_6 &= A_7 \frac{\partial T^*}{\partial H} + A_4 T^* - A_8 \bar{Q} \\
 B_7 &= A_6 h - A_2 f - A_7 F \\
 B_8 &= A_6 \bar{Q} - (A_2 + A_7) T^*
 \end{aligned} \tag{25}$$

where

$$\begin{aligned}
 A_1 &= 2H + 1 + S_w E \\
 A_2 &= PJ - HR \\
 A_3 &= H \frac{dJ}{dH} - J \\
 A_4 &= P \frac{dJ}{dH} - R \\
 A_5 &= 3J + 2S_w T^* \\
 A_6 &= A_1 J - A_5 H \\
 A_7 &= A_1 R - A_5 P \\
 A_8 &= A_1 \frac{dJ}{dH} - A_5
 \end{aligned} \tag{26}$$

3 MODEL FOR THE OUTER INVISCID FLOW

It is necessary to choose a model for the outer inviscid flow in order to fully specify the coupling relationship (equation (10)). Accurate modelling is important because the development of the inner viscous layer is strongly dependent on the behaviour of the outer inviscid flow.

3.1 SUPERSONIC FLOWS

The typical flow pattern produced by the interaction of a viscous layer with a supersonic stream is shown schematically in Figure 9a.* The flow field can be conveniently divided into four regions:

- (1) the region close to the nose, or strong interaction region
($\bar{\chi} \gg 1$)
- (2) the region over the remainder of the flat plate, or weak interaction region ($\bar{\chi} < 1$)
- (3) the region in the neighbourhood of the disturbance (e.g. corner or impinging shock)
- (4) the region far downstream of the disturbance.

3.1.1 STREAMLINES OF THE FLOW

For hypersonic flow it is not difficult to show that very little mass entrainment occurs and that consequently the edge of the boundary layer approximates to a streamline of the inviscid flow field.

The ratio of the mass entrained by the boundary layer to that displaced is given by

$$\bar{m} = \frac{\delta - \delta^*}{\delta^*} \quad (27)$$

*For a blunt leading edge, many of the details of the following description are inapplicable (Lees 1957, Cheng 1961). However, the general considerations are still relevant (Lees 1956b).

In Stewartson-illingworth co-ordinates

$$\delta = \frac{a_{\infty} \rho_{\infty}}{a_e \rho_e} \delta_i^* (m_e F + Z) \quad (28)$$

$$\delta^* = \frac{a_{\infty} \rho_{\infty}}{a_e \rho_e} \delta_i^* (m_e F)$$

so that

$$\bar{m} = \frac{Z}{m_e F} \quad (29)$$

The profile quantities Z and F are of the same order and hence for

$$\frac{\gamma - 1}{2} M_e^2 \gg 1$$

$$\bar{m} \rightarrow 0$$

The integral form of the continuity equation (equation (9)) then gives

$$\frac{d\delta^*}{dx} \rightarrow \tan(\theta_e) \quad (30)$$

which implies that the displacement surface (or outer edge of the boundary layer) is a streamline.

Note that in the strong interaction region the entrainment term is significant as $\frac{d \ln \rho_e u_e}{dx}$ is large and M_e is small.

The typical streamline pattern resulting from the displacement effect of the boundary layer is represented schematically in Figure 10. Lees (1956b) has shown that the streamline crossing the boundary layer edge at $R_{e_{x_1}}$ passes through the shock at $R_{e_{x_0}} \approx (R_{e_{x_1}})^{1/3}$. Hence even in the weak interaction region the streamline crossing the boundary layer has passed through the leading edge shock very close to the nose.

3.1.2 INVISCID FLOW CONSIDERATIONS

It is instructive to examine the inviscid flow shown schematically in Figure 9b. The nose of the body is a circular arc of initial slope α_{1e} and is followed by a parallel section and ramp. The shock wave pattern is similar to that developed in the shock wave boundary layer interaction (Figure 9a), and the surface of the body can be considered to represent a typical streamline of the viscous interaction where the nose semi angle α_{1e} corresponds to the shock deflection angle θ_2 (see Figure 10). As for the viscous interaction problem, the flow is divided into four regions.

The surface pressure and Mach number distributions for freestream Mach number $M_\infty = 10$ and various nose semi-angles α_{1e} as determined by the method of characteristics are shown in Figure 11. The distributions are plotted against surface slope over the circular arc and against stream-wise distance, x/L , over the remainder of the body. Solutions were not obtained beyond the corner (ramp hinge-line).

The results obtained using the tangent wedge rule, the Prandtl-Meyer rule based on freestream conditions and shock expansion theory are included for comparison. Shock expansion theory neglects the effect of reflected waves and thus assumes isentropic (Prandtl-Meyer) flow behind the leading edge shock.

In region 1 it is seen that shock expansion theory provides a very good description of the flow. For moderate α_{1e} , the tangent wedge rule provides a good description of the pressure distribution but considerably overpredicts the Mach number downstream of the leading edge. The flow is poorly described by the Prandtl Meyer rule based on freestream reference conditions.

Over the parallel section of the body (region 2) both the tangent

wedge rule and the Prandtl Meyer rule based on freestream reference conditions overestimate the Mach number and underestimate the pressure ratio. As the waves reflected from the shock and entropy gradient are weak, the flow is quite well described by shock expansion theory for some distance downstream of the nose.

In region 3 the upstream entropy gradient is such that the reflected waves have little effect on the pressure distribution for a considerable distance along the ramp, and shock expansion theory provides a good description of the flow (Bird 1963). Whether the static pressure in this region is above or below that given by the tangent wedge rule depends on the magnitude of the entropy gradient and the distance of the corner from the nose. If the pressure immediately upstream of the corner has returned to its free stream value then the pressure downstream of the corner will be lower than that predicted by the tangent wedge rule. Otherwise the pressure downstream of the corner will tend to be above the tangent wedge value. The effect of the entropy gradient in any case is to cause the pressure to rise along the ramp.

The reflected waves produced by the interaction of the leading edge shock wave with that generated at the corner impinge on the body far downstream (region 4). These reflected waves take the form of expansions (Bird, 1963) and produce a rapid decrease in pressure back towards the value which would occur on a surface of the same slope exposed directly to the freestream. Hence shock expansion theory is no longer adequate and the tangent wedge rule provides a better description of the flow.

It can be concluded that shock expansion theory provides the best description of the flow along the body until the reflected waves from major shock wave interactions reach the surface. Thereafter the tangent wedge rule is to be preferred. The Prandtl Meyer rule based on freestream

reference conditions is not good, except for very weak (almost isentropic) shocks.

In hypersonic flow, the same considerations apply to the Newtonian and Newton-Busemann pressure laws as to the tangent wedge rule. However they are less accurate than the tangent wedge rule and in any computational method are of no advantage.

3.1.3 APPLICATION TO VISCOUS-INVISCID INTERACTIONS

The results of the inviscid investigation can now be employed to determine the most suitable model for the outer flow in the viscous interaction problem.

In the strong interaction region (region 1), shock expansion theory provides the most accurate description of the outer inviscid flow. However, the entropy of each streamline crossing the boundary layer must be known and it is therefore necessary to determine the shape of the leading edge shock wave.

For a flat plate Stewartson (1955a,b) found that in the vicinity of the leading edge the shock shape $y_s(x)$ could be expressed as

$$y_s(x) = k\delta^*(x) = k\delta_0 x^{3/4} \quad (31)$$

where $k = 1.69$ for $\gamma = 1.4$

and δ_0 is determined by series expansion in powers of $(1/\bar{X})$

(see Section 4.2.1).

The corresponding shock wave angle is given by

$$\begin{aligned} \phi(x) &= \frac{d}{dx} y_s \\ &= \frac{3}{4} (k\delta_0)^{4/3} y_s^{-1/3} \end{aligned} \quad (32)$$

A mass flow balance enables the streamline crossing the boundary layer to be located at the shock. Referring to fig. 10, it is readily

shown that

$$\rho_{\infty} u_{\infty} y_S(x_0) = [\rho_e u_e (\delta - \delta^*)]_{x_1} \quad (33)$$

which gives

$$y_S(x_0) = \left[\delta_i^* \frac{M_e}{M_{\infty}} Z \right]_{x_1} \quad (34)$$

and

$$\phi(x_0) = \frac{3}{4} (k \delta_0)^{4/3} \left[\delta_i^* \frac{M_e}{M_{\infty}} Z \right]_{x_1}^{-1/3} \quad (35)$$

Provided no further interactions occur, equation (35) together with the oblique shock relations, determines the entropy of the streamline crossing the boundary layer at the station x_1 .

The flow properties required to completely specify the coupling relationship (equation (10)) are then readily determined. Assuming Prandtl Meyer expansion between the shock and the boundary layer

$$\left. \begin{aligned} p_e &= \left(\frac{1 + m_2}{1 + m_e} \right)^{\gamma/\gamma-1} p_2 \\ \theta_e &= v(M_2) - v(M_e) - \theta_2 \end{aligned} \right\} \quad (36)$$

where the conditions immediately downstream of the shock (subscript 2) are given by equation (35) and the oblique shock relations.

The pressure gradient $\frac{dp_e}{dx}$ is given by

$$\frac{1}{p_e} \frac{dp_e}{dx} = \frac{\gamma M_2}{(1 + m_2)} \frac{dM_2}{dx} - \frac{\gamma M_e}{(1 + m_e)} \frac{dM_e}{dx} + \frac{1}{p_2} \frac{dp_2}{dx} \quad (37)$$

where

$$\left. \begin{aligned} \frac{dM_2}{dx} &= \frac{dM_2}{d\phi} \left(\frac{d\phi}{dx} \right)_{x_1} \\ \frac{dp_2}{dx} &= \frac{dp_2}{d\phi} \left(\frac{d\phi}{dx} \right)_{x_1} \end{aligned} \right\} \quad (38)$$

and

$$\left[\frac{d\Phi}{dx} \right]_{x_1} = - \frac{\Phi}{3} \left[\frac{d \ln \delta_i^*}{dx} + \frac{d \ln M_e}{dx} + \frac{1}{Z} \frac{dZ}{dH} \frac{dH}{dx} \right]$$

Equations (15) - (18) and (35) - (38) thus provide a determinate set of equations in the dependent variables M_e , δ_i^* , H and b^* .

In the moderately weak interaction region ($\bar{\chi} = O(1)$) shock expansion theory again provides the most accurate description of the flow. As the streamlines crossing the boundary layer pass through the leading edge shock wave very close to the nose the approach adopted for the strong interaction region is still applicable.

Far downstream of the leading edge ($\bar{\chi} \ll 1$) a more simple description of the flow is provided by the Prandtl Meyer rule. In this region the boundary layer edge closely approximates a streamline and the Prandtl Meyer rule accurately describes the flow. However, it is difficult to decide on suitable reference conditions. If freestream reference conditions are used then the local static pressure will be lower and the Mach number higher than that actually realized. An examination of a number of viscous interaction problems may lead to a consistent procedure for determining the (average) entropy along the boundary layer edge and hence to more suitable reference conditions.

Finding a suitable model for the outer inviscid flow presents a difficult problem when shock waves interact with the boundary layer (region 3). The complex flow field produced by the reflected pressure waves and by the entropy variation across streamlines passing through different portions of the shock system cannot be readily included in the

* Klineberg (1968) has suggested a similar approach to allow for entropy gradients resulting from the curved leading edge shock wave. However, Klineberg employs an iterative procedure that is unnecessary in the present formulation.

analysis. However, the results of the inviscid investigation again suggest that shock expansion theory is the best method to employ. Some expression for the entropy change due to the compression (shock) waves produced at the surface must be given. For supersonic flows without strong shock waves the following simple flow model of Lees and Reeves (1964) appears to be adequate.

Lees and Reeves postulate that the flow over a flat plate-wedge (Figure 12a) is equivalent to the flow over a flat plate subjected to an impinging shock wave provided that

- (i) Freestream conditions upstream of the disturbance are identical.
- (ii) $R_{e_{x_{\text{corner}}}} = R_{e_{x_{\text{shock}}}}$.
- (iii) Total pressure ratio $\left(\frac{P_{\infty+}}{P_{\infty-}} \right)$ is identical.

They then model only the latter case as shown in Figure 12b. The incident shock wave is considered to impinge on the boundary layer at a point. Because the laminar boundary layer cannot support a sudden rise in pressure, the incident shock wave is reflected as an expansion at this point and the static pressure ratio remains continuous. The velocity and total enthalpy profiles and the transformed displacement thickness are also assumed to remain continuous at this point. The local Mach number downstream of the interaction can then be calculated from the static pressure at impingement and from the entropy change across the incident shock wave, neglecting boundary layer displacement thickness effects (cf. Gautier and Ginoux 1971). All the dependent variables immediately downstream of the shock are therefore known in terms of those immediately upstream.

However, at higher Mach numbers and for strong shock waves the equivalence between the two flows is no longer a valid assumption, and a better model needs to be found.

Far downstream, where the reflected waves from the interacting shock wave system have reached the boundary layer (region 4), the tangent wedge rule better describes the flow than the other methods.

For hypersonic flows, the Newtonian or Newton-Busemann pressure laws can also be employed to describe the outer inviscid flow (e.g. Cheng 1961, Stollery 1970b). However they are less accurate than the tangent wedge rule and offer no computational advantage.

The behaviour of the outer flow can be determined more accurately using the method of characteristics (e.g. McConnell 1971). Although this is not a difficult problem in itself, it involves a considerable increase in computational effort and is not recommended except when other approaches prove unsatisfactory.

3.2 SUBSONIC FLOWS

In subsonic flow the streamline inclination at the edge of the boundary layer depends on the solution to the entire flow field. Hence an iterative procedure, whereby successive approximations to the entire flow field were made, would have to be employed. In many problems small disturbance theory could be used to describe the outer inviscid flow, thus avoiding the complexity of solving for the entire flow field at each iteration (e.g. Green 1966).

3.3 FLOW MODEL ADOPTED

All the comparisons made in this thesis employ a model based on shock expansion theory. Except where shock waves interact with the boundary layer, the outer inviscid flow is assumed to be isentropic and hence the local static pressure and streamline inclination at the edge of the boundary layer can be related to the local Mach number through the Prandtl Meyer relationship. Reference conditions are taken to be those behind the leading edge shock wave neglecting the displacement effects of the boundary layer. Hence for the flat plate at zero incidence freestream conditions are used. It is clear from the preceding discussion that such a model is valid only in the very weak interaction region, but for the sake of convenience and to avoid solving the more complex equations allowing for the curvature of the leading edge shock (i.e. equations 35-38) this simple model is used throughout. An exception is made in the strong interaction region, where the tangent wedge rule is employed. Entropy changes in other regions of shock wave interaction are allowed for using the model of Lees and Reeves.

4 METHOD OF SOLUTION

4.1 STABILITY OF THE GOVERNING DIFFERENTIAL EQUATIONS

Although the stability of the governing differential equations is discussed in detail in Section 7.1, it is introduced here in order to enable the methods of solution described in this section to be fully understood.

In viscous interaction problems the solutions to the governing differential equations usually exhibit a singular dependence on the initial conditions. If some parameter in the initial conditions is varied by an arbitrarily small amount, each resulting solution ultimately diverges from the other and in general only one particular value of this parameter will provide a satisfactory solution. This unstable behaviour of the governing differential equations (often termed 'subcritical' behaviour) implies that the boundary layer can generate its own pressure gradient and a finite pressure change can be produced by means of a small perturbation to the solution for undisturbed flow. Conversely, it is possible for the governing differential equations to be stable ('supercritical') and not to exhibit a singular dependence on the initial conditions. In this case the boundary layer cannot generate its own pressure gradient and in order to initiate compressive interactions a discontinuous 'jump' in profile quantities must be allowed.

Another consequence of this singular behaviour is the necessity to impose some downstream constraint (boundary condition) on the solution. This is supplied by demanding that the solution approaches a physically realistic solution downstream of the corner. Satisfaction of this downstream constraint may require that the solution pass through the Crocco-Lees critical point (Crocco and Lees 1952), a singularity associated

with the vanishing of the determinant, D [equation (24)], of the governing set of equations. At this point the numerators $N_1, \dots, N_4$ must also vanish. At the critical point a change in stability of the governing differential equations occurs (see Appendix 3). This implies that the locus of critical points (the 'critical boundary') will separate regions exhibiting unstable behaviour from those exhibiting stable behaviour, thus determining whether or not a 'jump' is required to initiate interaction.

At hypersonic speeds, the critical boundary is independent of Mach number, and is plotted in $H - b$ space in Figure 13 for a range of wall cooling ratios. It is seen that the critical boundary is strongly dependent on the wall cooling ratio: the flat plate weak interaction solution is stable for $S_w = -1.0$ to $S_w = -0.4$, but at smaller values of S_w is unstable. This is in agreement with the results obtained by Gautier and Ginoux (1971) using the exact Cohen-Reshotko similarity solutions evaluated at discrete values of S_w .

4.2 UPSTREAM BOUNDARY CONDITIONS

The upstream boundary conditions are approximated by the undisturbed flat plate flow solution. For all sharp nosed bodies the flow field can be divided into two regions, representing the limiting cases of strong ($\bar{\chi} \gg 1$) and weak ($\bar{\chi} < 1$) interaction. Asymptotic expansions for the flow variables can be obtained in both these limits, and the solution to the entire flat plate flow found, if necessary, by integration of the governing differential equations in the intermediate region using these asymptotic solutions as boundary conditions.

4.2.1 STRONG INTERACTION REGIME

In this region the flow variables can be expressed as asymptotic expansions in large $\bar{\chi}$.

It is assumed that

$$\frac{\gamma - 1}{2} M_\infty^2 \gg 1 \qquad \frac{\gamma - 1}{2} M_e^2 \gg 1$$

and
$$\frac{dS_w}{dx} = 0.$$

A simple model describing the outer inviscid flow in the strong interaction region must be adopted. It is clear from Section 3 that an accurate description of the flow in this region would have to be based on shock expansion theory (see Stewartson 1955a,b). However, as only a fair approximation to the upstream boundary conditions is required in most viscous interaction problems, it is far simpler to adopt the less accurate tangent wedge or Prandtl Meyer rules based on freestream reference conditions.

Klineberg uses the tangent wedge rule to describe pressure and the Prandtl Meyer rule to describe Mach number. The asymptotic expansions then take the form:

$$\left. \begin{aligned} \left(\frac{p_e}{p_\infty} \right)_{SI} &= p_o \bar{\chi} \left[1 + \frac{p_\alpha K}{\bar{\chi}^{1/2}} + \frac{p_1 + p_{\alpha 1} K^2}{\bar{\chi}} + \dots \right] \\ \left(\frac{R_{e\delta i}^*}{M_\infty^3 C} \right)_{SI} &= \delta_o \bar{\chi}^{-1/2\gamma} \left[1 + \frac{\delta_\alpha K}{\bar{\chi}^{1/2}} + \frac{\delta_1 + \delta_{\alpha 1} K^2}{\bar{\chi}} + \dots \right] \\ H_{SI} &= H_o + \frac{H_\alpha K}{\bar{\chi}^{1/2}} + \frac{H_1 + H_{\alpha 1} K^2}{\bar{\chi}} + \dots \\ b_{SI} &= b_o + \frac{b_\alpha K}{\bar{\chi}^{1/2}} + \frac{b_1 + b_{\alpha 1} K^2}{\bar{\chi}} + \dots \end{aligned} \right\} \quad (39)$$

where $K = M_\infty \alpha_i$

and α_i is the angle of incidence of the flat plate.

It is to be noted that the effect of the entropy gradient and vorticity at the edge of the boundary layer result in correction terms of order $\left(\frac{\bar{\chi}}{M_\infty^2}\right)^n$ where $n = O\left(\frac{1}{2}\right)$ (see Lees 1956b). However, this effect will be largely offset by rarefied gas effects which become important for $\frac{\bar{\chi}}{M_\infty^2} > 0.15$.

Substituting into the governing differential equations (15) - (18) it is possible to obtain the coefficients of the expansions. These coefficients are plotted in Figure 14 as functions of wall cooling ratio S_w . The second order incidence terms (i.e. those of order $\frac{K^2}{\bar{\chi}}$) have not been included. As the analytic relationship between the coefficients and S_w is quite complex, the coefficients are determined for a range of S_w and then expressed as polynomial functions in S_w using a least squares curve fit. The coefficients for $\gamma = 1.4$ are tabulated in Appendix 1. All required boundary layer functions can readily be obtained from these expressions.

Alternatively, the tangent wedge rule can be used to describe both the pressure and Mach number. The tangent wedge rule provides a better estimate of Mach number than the Prandtl Meyer rule but the relation between Mach number gradient and pressure gradient is not so well described.

The resulting expansions are similar in form to those given above except that

$$\left(\frac{R_{e_{\delta_i^*}}}{M_\infty^3 C} \right)_{SI} = \delta_o \left[1 + \frac{\delta_o K}{\frac{-1}{2}\bar{\chi}} + \frac{\delta_1 + \delta_{\alpha 1} K^2}{\bar{\chi}} + \dots \right] \quad (40)$$

The coefficients of the expansions can be found as before.

4.2.2 WEAK INTERACTION REGIME

In this regime the flow variables can be expressed as asymptotic expansions in small $\bar{\chi}$.

It is assumed that

$$\frac{dS_w}{dx} = 0$$

and the Prandtl Meyer rule is used to describe the outer inviscid flow.

Letting

$$\left. \begin{aligned} \left(\frac{M_e}{M_\infty} \right)_{WI} &= 1 + m_1 \bar{\chi} + m_2 \bar{\chi}^2 + \dots \\ \left(\frac{R_{e_{\delta_i^*}}}{M_\infty^3 C} \right)_{WI} &= \frac{\delta_0}{\bar{\chi}} (1 + \delta_1 \bar{\chi} + \dots) \\ H_{WI} &= H_0 + H_1 \bar{\chi} + \dots \\ b_{WI} &= b_0 + b_1 \bar{\chi} + \dots \end{aligned} \right\} \quad (41)$$

and substituting into equations (15) - (18) it is possible to obtain the coefficients of the expansions. Note that as the solution must tend to the Blasius solution in the limit $\bar{\chi} \rightarrow 0$, the leading terms δ_0 , H_0 and b_0 must be independent of wall cooling ratio S_w (see Section 2.3).^{*} The dependence of the remaining terms on S_w is relatively simple and can be obtained directly from the governing equations. Hence it is not necessary to curve fit the coefficients as in the strong interaction regime. The values of the coefficients for $\gamma = 1.4$ are given in Appendix 1.

* The leading terms obtained by Gautier and Ginoux and also by Klineberg show some dependence on wall cooling ratio. This can only derive from numerical errors during computation.

Note that for an inclined flat plate the reference conditions in the strong interaction regime are the freestream conditions, whereas those in the weak interaction regime are the inviscid flow conditions immediately behind the leading edge shock.

4.2.3 ENTIRE FLAT PLATE SOLUTION

Having obtained the asymptotic solutions for flat plate flow it is possible to determine the entire flat plate solution by integrating the governing differential equations in the intermediate region using these asymptotic solutions as boundary conditions.

If the flat plate solution is unstable, integration in the downstream direction proves difficult, as each integral path diverges rapidly from the required solution. It is therefore more convenient to integrate upstream, using the weak interaction asymptotic expansion to provide the required initial conditions.* However, an infinite number of integral paths corresponding to different upstream boundary conditions approach the weak interaction solution in the limit $\bar{\chi} \rightarrow 0$. It is therefore necessary to iterate through the range of neighbouring solutions until the integral path which joins the strong interaction solution is found. The neighbouring solutions in the vicinity of the weak interaction solution can be determined using an analysis similar to that of Ko and Kubota (1968). Solutions can then be determined by selecting initial conditions for integration in the upstream direction such that

* This procedure may also be adopted when investigating accelerating flows. In many such cases the integral paths depart too rapidly from the required solution for numerical suppression to be successful.

$$\left. \begin{aligned}
 M_e &= M_{e_{WI}} + \Delta M_e \\
 R_{e_{\delta_i^*}} &= \left(R_{e_{\delta_i^*}} \right)_{WI} \\
 H &= H_{WI} + \Delta H \\
 b &= b_{WI} + \Delta b
 \end{aligned} \right\} \quad (42)$$

where, as shown in Appendix 2,

$$\left. \begin{aligned}
 \Delta H &= - \frac{4}{(\gamma - 1)^2} \left(\frac{R_{e_{\delta_i^*}}}{M_{\infty}^3 C} \right)_{WI} H_1 \left(\frac{\Delta M_e}{M_{\infty}} \right) \\
 \Delta b &= - \frac{4}{(\gamma - 1)^2} \left(\frac{R_{e_{\delta_i^*}}}{M_{\infty}^3 C} \right)_{WI} b_1 \left(\frac{\Delta M_e}{M_{\infty}} \right)
 \end{aligned} \right\} \quad (43)$$

and where, for $-0.3 \leq S_w \leq 0$, $H_1 \approx b_1 \approx 10^{-2}$.

Note that as the weak interaction solution is an asymptotic expansion in vanishingly small $\bar{\chi}$, it is not possible to integrate in the physical plane, and the governing differential equations must be transformed and integrated in phase space. The solution can be located in physical space after matching to the strong interaction solution (see Klineberg 1968).

If the flat plate solution is stable, the strong interaction asymptotic expansions provide suitable initial conditions (at say $\bar{\chi} = 10$) and the governing differential equations can be integrated downstream without difficulty.

The hypersonic flat plate flow solutions and the corresponding asymptotic expansions are shown in Figures 15-17 for various values of wall cooling ratio, S_w .

4.3 DEPARTURE CONDITIONS

The flat plate solution described above represents the solution upstream of the disturbance. Consideration must now be given to the manner in which the required solution departs from the solution for undisturbed flow.

As the departure paths form a family of solutions an iterative procedure must be used to determine the solution applicable to the given downstream boundary conditions.

4.3.1 UNSTABLE GOVERNING DIFFERENTIAL EQUATIONS

In regions where the governing differential equations are unstable, a small perturbation to the solution representing undisturbed flow will produce departure paths tending towards either separated or expanding flow. These departure paths form a single parameter family of solutions, and hence the perturbations to the undisturbed solution must also be expressed in terms of a single parameter. This can be accomplished by local linearization of the governing differential equations in the manner of Ko (1968). The following form for the perturbations is obtained (Appendix 3):

$$\left. \begin{aligned} M_e &= M_{e_0} (1 + A_3 \epsilon) \\ \delta_i^* &= \delta_{i_0}^* (1 - A_8 \epsilon) \\ H &= H_0 + A_6 \epsilon \\ b &= b_0 - B_2 \epsilon \end{aligned} \right\} \quad (44)$$

where A_3 , A_6 , A_8 and B_2 are as defined in equations (25) and (26).*

* Cf. the expressions obtained in the reports by Klineberg and by Gautier and Ginoux where the perturbations to the parameters 'a' and 'b' are incorrect.

As the perturbation parameter ϵ must be small for the local linearization to remain accurate, the beginning of the interaction x_0 must be iterated at a fixed value of $\epsilon (\sim 10^{-3})$ until approximately the correct departure path is determined. The parameter ϵ can then be varied until the exact solution is found.

4.3.2 STABLE GOVERNING DIFFERENTIAL EQUATIONS

In regimes where the governing differential equations are stable, it is not possible to produce departure paths by means of small perturbations to the solution for undisturbed flow. It is therefore assumed that the velocity and enthalpy profiles undergo very large changes in a very short distance, thus invalidating the boundary layer assumptions. Klineberg, following Crocco (1955), allows a discontinuous change in flow quantities in this region. The relationships between the flow variables upstream and downstream of this discontinuity, or 'jump', are obtained by assuming that the external flow experiences an oblique shock wave at the jump location and by satisfying the conservation laws of mass, momentum, moment of momentum, and energy in the limit as the control volume approaches zero. Although the moment of momentum is not a conservative quantity, in the integral formulation it provides a suitable approximation for the jump conditions. The resulting set of equations is

$$\begin{aligned}
 m_{e2} F_2 \left[\frac{H_1}{m_{e1} F_1} - \frac{H_2}{m_{e2} F_2} \right] + \frac{1}{\gamma M_{e1}^2} \left[\frac{p_2}{p_1} - 1 \right] [m_{e2} F_2 + Z_2] - \\
 - \left[1 - \frac{\rho_{e2} u_{e2}^2}{\rho_{e1} u_{e1}^2} \right] [Z_2 - H_2] + \left[1 - \frac{\rho_{e2} u_{e2}}{\rho_{e1} u_{e1}} \right] \left[1 + \frac{H_1}{m_{e1} F_1} \right] Z_2 = 0 \quad (45)
 \end{aligned}$$

$$m_{e2} F_2 \left[\frac{J_1}{m_{e1} F_1} - \frac{J_2}{m_{e2} F_2} \right] + \frac{2}{\gamma M_{e1}^2} \frac{u_{e2}}{u_{e1}} \left(\frac{p_2}{p_1} - 1 \right) [m_{e2} J_2 + Z_2 + (1 + m_{e2}) S_w T_2^*] -$$

$$- \left[1 - \frac{\rho_{e2} u_{e2}^3}{\rho_{e1} u_{e1}^3} \right] [Z_2 - J_2] + \left[1 - \frac{\rho_{e2} u_{e2}}{\rho_{e1} u_{e1}} \right] \left[1 + \frac{J_1}{m_{e1} F_1} \right] Z_2 = 0 \quad (46)$$

$$m_{e2} F_2 \left[\frac{T_1^*}{m_{e1} F_1} - \frac{T_2^*}{m_{e2} F_2} \right] + \left[1 - \frac{\rho_{e2} u_{e2}}{\rho_{e1} u_{e1}} \right] \left[T_2^* + \frac{T_1^*}{m_{e1} F_1} Z_2 \right] = 0 \quad (47)$$

where the subscripts 1 and 2 denote quantities upstream and downstream of the jump, respectively.

The pressure and density ratios can be determined from the change in Mach number M_e by means of the oblique shock relations, and hence the equations can be iteratively solved for M_{e2} , H_2 and b_2 given upstream conditions.

Behind the jump the governing differential equations are unstable and the integral paths depart towards separation. These departure paths, which include the jump, form a one parameter family of solutions and hence the beginning of interaction (jump location) x_0 can be iterated upon until the correct solution is found.

4.4 DOWNSTREAM BOUNDARY CONDITIONS

In the problems under consideration, the downstream boundary condition can be of two types. If the wedge, or ramp, is very (infinitely) long, the required downstream boundary condition is that the flat plate (Blasius) solution be approached. This requires that N_1 , N_3 and N_4 vanish simultaneously, but D , $N_2 \neq 0$. Conversely, if the wedge is short, the flow will accelerate around the trailing edge. For a sharp trailing edge a good approximation is to require that at this point $M_\infty \rightarrow \infty$ and H , representative of the 'fullness' of the velocity profile, becomes large. This requires

that at the trailing edge $D = 0$ while $N_1, N_2, N_3, N_4 \neq 0$ (Ko 1969).

4.5 INTEGRATION PROCEDURE

Initial conditions are determined as described in sections 4.2 and 4.3 and integration commenced in the downstream direction. Because the parameter H is used in preference to the parameter ' a ' it is not necessary to reduce the step size or to transform from the independent variable ' x ' to ' a ' in the separation and reattachment regions (cf. Gautier and Ginoux 1971). At any discontinuity (e.g. shock impingement point or sharp corner) the change in flow conditions is determined according to the particular flow model adopted and integration then continued in a straightforward manner until the end of the interaction. The beginning of interaction x_0 and/or the parameter ϵ are iterated upon until the downstream boundary conditions (Section 4.4) are satisfied.

In order to limit the number of iterations on x_0 and ϵ new initial conditions are determined further downstream by linearly interpolating between neighbouring divergent trajectories at a point where the difference in values is relatively small. This procedure is repeated at successive locations until the complete downstream solution is obtained.

If the Crocco Lees critical point is encountered during integration, an extrapolation procedure is adopted to allow the solution to pass through this singularity.

Using a fourth order Runge Kutta method to integrate the governing equations 20 - 40 seconds computing time on a CDC 6600 computer is required to accurately solve most viscous interaction problems (of the order 3 seconds per iteration). *

* For a full description of the computer program, prospective users are referred to the report by Georgeff (1972c).

5. THEORETICAL EVALUATION OF THE S_w NORMALIZATION

5.1 CHOICE OF PROFILE QUANTITY 'P'

The profile function $P(H)$ was found to be the only function that showed a significant dependence on wall cooling ratio, S_w . As discussed in section 2.3, this dependence can be taken into account by expressing P as a linear function of the modified profile quantity P_1 and S_w . It is useful to consider the importance of this modification.

In Figures 18 and 19 the results obtained using the modified function $P(H, S_w)$ as given by equation (14) and the unmodified function $P(H)$ evaluated at $S_w = -0.8$ are compared with the theoretical results of Klineberg and the experimental data of Lewis (1968) at $S_w = 0$. As far as comparison with experiment is concerned, the results suggest that it is not necessary to express P as a function of S_w . However, in order to retain good agreement with the 'exact' theoretical results it is preferable to use the modified function $P(H, S_w)$, and this is done in all the comparisons of this report.

5.2 COMPARISONS WITH THE UNMODIFIED KLINEBERG METHOD

Gautier and Ginoux (1971) have studied the effects of wall cooling upon the characteristics of a laminar boundary layer-shock wave interaction using the method of Klineberg. They calculated additional 'similar solutions' of the boundary layer equations at discrete values of wall cooling ratio, S_w , in order to provide the required set of polynomial functions describing the relations between profile quantities. No approximations were made regarding the dependence of these functions on S_w , and the method of Klineberg was faithfully followed. An excellent test of the modification proposed herein is therefore provided by comparison with the theoretical results of Gautier and Ginoux.

The experimental conditions used are those of Lewis (1968) (see Table 2), except that other values of wall cooling ratio are also considered. In Figures 20-22 the distributions of pressure ratio, skin friction coefficient and heat transfer coefficient obtained from the proposed modification are compared with the results of Gautier and Ginoux at values of wall cooling ratio in the range $-0.8 \leq S_w \leq 0$.

The results of Klineberg at wall cooling ratio $S_w = -0.8$ are also included. As the profile quantities used in the proposed modification were all evaluated from Klineberg's data, both theories agree exactly at this value of wall cooling ratio. However, the solution obtained by Gautier and Ginoux does not coincide with that of Klineberg. This probably derives from different approximations to the polynomial functions relating profile quantities, slightly different shock conditions (Gautier and Ginoux take the stream-line inclination into account in determining the stagnation pressure drop across the shock), and numerical errors. The difference is of course not large, and from a practical point of view is negligible, but it serves to indicate the degree of acceptable error. The results of Klineberg for the adiabatic case ($S_w = 0$) lie midway between those obtained using the proposed extension and those of Gautier and Ginoux. For the sake of clarity they are not included in the figures.

Comparing the results of the proposed modification with those of Gautier and Ginoux, it is seen that they are in very good agreement, although a small discrepancy is observed in the distributions of heat transfer coefficient, C_H . However, the distributions of heat transfer coefficient obtained by Gautier and Ginoux and by Klineberg also disagree, which suggests that the discrepancy does not derive from the assumption of S_w independence. Furthermore, if these distributions are normalized

with respect to the heat transfer coefficient at the beginning of interaction (C_{H_o}), the discrepancy is considerably reduced (see Figure 26).

The effects of surface cooling upon the characteristic features of the interaction are indicated in Figures 23-26. The agreement with the results of Gautier and Ginoux is excellent.

6. COMPARISON OF METHODS BASED ON THE KLINEBERG FORMULATION

The main purpose behind the following comparison of methods based on the Klineberg formulation is to provide an understanding of the underlying theory. The comparison also allows an evaluation of the relative merits of the various approaches.

The methods used for this comparison include the earlier formulation of the method by Lees and Reeves (1964) and the proposed modifications of Stollery and Hankey (1970a) and of the author* (Georgeff 1972a).

6.1 DESCRIPTION OF THE METHODS

The methods are briefly described below. All use the same model of the flow and essentially the same approach, differing only in the set of governing equations employed, the assumed relations between profile quantities, and the number of parameters left free.

6.1.1 LEES-REEVES METHOD

The method of Lees and Reeves is similar to the method of Klineberg except that the enthalpy profile is considered to be a function of the velocity profile. The energy integral equation is therefore not required. Using the Cohen-Reshotko similarity solutions to provide the required universal relations, all the profile quantities are expressed as functions of the profile parameter H . The governing set of equations becomes

* The modification referred to here is not the author's extension of the Klineberg method to flows with arbitrary wall cooling ratio.

Momentum:

$$H \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + (2H + 1 + S_w E) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{P}{\delta_i^{*R} e_{\delta_i^*}} \quad (48)$$

Moment of Momentum:

$$J \frac{d \ln \delta_i^*}{dx} + \frac{dJ}{dH} \frac{dH}{dx} + (3J + 2S_w T^*) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{R}{\delta_i^{*R} e_{\delta_i^*}} \quad (49)$$

Coupling:

$$F \frac{d \ln \delta_i^*}{dx} + \frac{dF}{dH} \frac{dH}{dx} + f \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{h}{\delta_i^{*R} e_{\delta_i^*}} \quad (50)$$

Thus equations (48) - (50) provide a determinate set of equations for the three unknowns δ_i^* , H and M_e .

6.1.2 STOLLERY-HANKEY MODIFICATION "a"

Stollery and Hankey (1970a) modified the method of Lees and Reeves by replacing the exact relation between the profile quantities J and H as determined from the Cohen-Reshotko similarity solutions by a linear approximation, viz.

$$J = KH. \quad (51)$$

The governing equations (48) - (50) remain unchanged, although they can be simplified.

6.1.3 STOLLERY-HANKEY MODIFICATION "b"

The same modification can be made to the method of Klineberg. The governing equations (15) - (18) remain unchanged.

6.1.4 GEORGEFF MODIFICATION "a"

The author's proposed modification (Georgeff 1972a) to the method of Lees and Reeves involves replacing the moment of momentum integral equation

(49) by the momentum equation at the wall, which in Stewartson Illingworth co-ordinates is

$$\frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{I}{\delta_1^{*R} \delta_1^{*e} \delta_1^{*}} \quad (52)$$

where

$$I = - \frac{\delta_1^{*2}}{(1 + S_w)} \frac{\partial^2}{\partial y^2} \left(\frac{U}{U_e} \right)$$

As before, the required universal relations are provided by the Cohen-Reshotko similarity solutions, and all the profile quantities (including I) are expressed as functions of the profile parameter H . The governing equations (48), (52) and (50) again provide a determinate set in the three unknowns δ_1^* , H and M_e .

This method is closely related to the methods of Cohen and Reshotko (1956b), Bray, Gadd and Woodger (1960) and Curle (1961) although many of the simplifying assumptions employed in these earlier methods are dropped (see Table 1).

6.1.5 GEORGEFF MODIFICATION "b"

The same modification is made to the method of Klineberg. The moment of momentum integral equation (16) is replaced by the momentum equation at the wall (52), thus providing a determinate set of equations (i.e. equations (15), (52), (17), (18)) in the four unknowns δ_1^* , H , b , and M_e .

The methods in which the velocity and enthalpy profiles are coupled will be referred to as "coupled-profile" methods, and the other methods as "uncoupled-profile" methods. A summary of the methods described is given in Table 1.

6.2 EQUIVALENCE OF THE MODIFICATIONS

It is constructive to consider more closely the relationship between the Stollery-Hankey modification and the modification proposed by the author.

In deriving the Cohen-Reshotko similarity solutions we have

$$\frac{dH}{dx} = \frac{dJ}{dx} = 0$$

and under these conditions the momentum integral equation (5) together with the moment of momentum integral equation (6) yields

$$\frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{1}{\delta_i^* R_e \delta_i^*} \frac{(PJ - HR)}{\{J(1 - H + S_w E) - 2S_w T^* H\}} \quad (53)$$

The momentum equation at the wall (52) is unaltered. As the similarity solutions are exact, both equations (52) and (53) must be satisfied, and therefore

$$I = \frac{(PJ - HR)}{\{J(1 - H + S_w E) - 2S_w T^* H\}} \quad (54)$$

Now consider the general case, where

$$\frac{dH}{dx} \neq 0 \quad \frac{dJ}{dx} \neq 0.$$

Under the modification proposed by Stollery and Hankey, viz. $J = KH$, equations (15) and (16) (or equations (48) and (49)) yield

$$\frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{1}{\delta_i^* R_e \delta_i^*} \frac{(PJ - HR)}{\{J(1 - H + S_w E) - 2S_w T^* H\}} \quad (55)$$

Furthermore, as it is assumed that the Cohen-Reshotko similarity solutions provide relations that are universal, the equality (54) must still hold provided the profile parameters H and b remain coupled. Hence equation (55)

is equivalent to equation (52), the right hand side of the equality (54) being negligibly affected by the linear approximation to J employed in deriving equation (55). The governing equations of the Stollery-Hankey modification "a" and the author's modification "a" are therefore equivalent. When the profile parameters H and b are uncoupled, the equality (54) does not hold. However, it remains a good approximation, and the governing equations of both modifications "b" are therefore essentially equivalent. It follows that there will be very little difference between the solutions obtained with the Stollery-Hankey modification and those obtained with the author's modification.

Note that mathematically it is very difficult to justify the use of a linear approximation to J . The approximation is not as good as was supposed by Stollery and Hankey, as the terms neglected in the linearization, although small, are of the same order as the other terms appearing in the governing differential equations. Furthermore, it is extremely difficult to determine the effect that these terms have on the solution. Of course, it is possible to arbitrarily choose the linear relationship between J and H to be universal, but such an arbitrary choice is not very satisfactory.

6.3 THE CRITICAL BOUNDARY

At hypersonic speeds the critical boundary associated with each of the methods described is independent of Mach number, and is plotted in phase space in Figure 27 for $S_w = -0.8$. For the Lees-Reeves method, in which the enthalpy and velocity profiles are coupled, the critical point occurs at $H = 0.439$ and is represented as a point in phase space. No critical point appears in the modifications "a" of Stollery and Hankey and

of the author.* The separation point and the flat plate viscous interaction solution for hypersonic flow and $S_w = -0.8$ are also indicated in Figure 27.

The sensitivity of the position of the critical boundary to changes in the profile quantities is also shown by comparison of the essentially identical methods of Klineberg (1968) and Holden (1965). Holden's use of slightly different profile quantity relations (due to a different order of curve fit) shifts the critical boundary to the right, so that it lies between the weak interaction solution and the strong interaction solution. Hence using Holden's formulation the flat plate weak interaction solution is unstable whereas using Klineberg's it is stable.

It is important to point out that the appearance of a critical boundary involves no mathematical inconsistency, and computationally is no disadvantage (see Section 7). However, by removing the critical boundary all of solution space is made unstable and thus the necessity to initiate interaction by means of a "jump" is avoided. As no discontinuous change in flow quantities is observed in the real flow, the removal of the critical boundary may provide better agreement with experiment.

6.4 COMPUTATIONAL CONSIDERATIONS

In terms of user convenience the methods described are virtually indistinguishable. The method of Klineberg has a small advantage in that the governing differential equations are stable for cold wall flat plate flow so that the viscous interaction solutions used for providing the initial

* Contrary to the opinion held by Stollery and Hankey, the critical boundary is not completely removed by the linear approximation to J in the methods where the profile parameters b and H are uncoupled.

conditions are more easily obtained. Furthermore, the solution far downstream of the corner is more easily determined. Although the modification of Stollery and Hankey and of the author do not encounter a critical point in a compressive interaction, the governing differential equations can become quite unstable as the downstream flat plate solution is approached due to the proximity of the critical boundary (Figure 27). Neighbouring trajectories rapidly diverge from each other and integration in this region is therefore very difficult.* However, using the method of Klineberg, a simple extrapolation procedure readily enables one to pass through the critical point; thereafter the governing differential equations are stable and integration proceeds smoothly.

The relations between the profile quantities H , J , etc. must be generated externally and curve fitted, which requires a considerable amount of computation, especially when the profile parameters H and b are uncoupled. However, once these relations have been established all the methods are computationally quite efficient.

6.5 COMPARISON OF THE METHODS

The methods are compared on the basis of their ability to predict the behaviour of the boundary layer over a flat plate wedge (compression corner). As discussed in section 10, the significant parameters of the

* In order to continue the integration in the downstream direction, new initial conditions are determined by linearly interpolating between neighbouring divergent trajectories at a point where the difference in values is relatively small (Section 4.5). If this difference in values is not kept small (in some cases very small) it is possible to converge onto a realistic but erroneous solution. However, in all the cases considered by the author, it proved impossible to reach the downstream flat plate solution from this erroneous solution. This may explain the difficulty experienced by Holden (1967) in integrating completely to the end of the interaction.

interaction are $\bar{X}_L$, S_w and $M_\infty \alpha_w$. The effect of S_w is mainly one of scaling, and hence an adequate assessment of the methods is provided by comparing them at two values of $\bar{X}_L$, one moderate and the other large, under cold wall conditions. The dependence of the solutions on the body geometry parameter $M_\infty \alpha_w$ is accounted for by taking three different wedge angles at each value of $\bar{X}_L$.

The data of Bloy (1972) at $\bar{X}_L = 2.45$ and the data of Holden (1968) at $\bar{X}_L = 19.8$ provide the most suitable experimental data for the comparisons. As only a relative measure of the methods is required, a detailed discussion of the experimental comparisons is left to Section 9.

The methods have been used as purely predictive methods and all solutions were obtained for $S_w = -0.8$. As Holden's data was obtained at $S_w = -0.95$ an error in scaling is to be expected.*

In Figures (28-33) the described methods are compared with the experimental data. The method of Cheng (1961) is also included as it can be considered a simplification of the Lees-Reeves method (see Section 8.1). The method is modified to incorporate the same outer flow model as used in the other methods. In general, the "uncoupled profile" (Klineberg type) methods better describe the interaction than the "coupled profile" (Lees-Reeves type) methods. The pressure ratio upstream of the corner is well described by all methods, but downstream of the corner the uncoupled profile methods are more accurate. The distributions of skin friction coefficient and heat transfer coefficient are far better predicted by the uncoupled profile methods. All the methods provide a good

* At the time these comparisons were made the extension of the Klineberg method to flows with wall cooling ratio other than $S_w = -0.8$ had not been developed. Later comparisons using the correct value of S_w are given in Section 9.

description of the displacement thickness. The method of Cheng is clearly inadequate for describing flows with significant upstream influence.

It is clear that the modifications to the two basic methods make little difference to the solutions obtained. As discussed in section 6.2 it is found that the modifications of Stollery and Hankey and of the author to the method of Lees and Reeves give virtually identical solutions. Note that the Crocco-Lees critical point is only encountered with the method of Klineberg, as is indicated in the Figures.

7. ANALYTICAL CONSIDERATIONS

Using the results of the preceding comparisons, it is possible to examine in detail the underlying theory of the Klineberg method.

7.1 STABILITY OF THE GOVERNING DIFFERENTIAL EQUATIONS

7.1.1 BEHAVIOUR OF THE GOVERNING DIFFERENTIAL EQUATIONS

As viscous interaction problems often show some upstream influence, it is to be expected that the solution will depend on both upstream and downstream boundary conditions. At first sight it is difficult to see how the solution could depend on downstream boundary conditions as the governing equations of the inner and outer flow are respectively parabolic and hyperbolic and do not admit the upstream propagation of disturbances. However, Garvine (1968) has shown that through the mechanism of viscous interaction contained in the coupling equation, the governing set of equations are neither parabolic nor hyperbolic but "quasi-elliptic", i.e. they reduce to an integropartial differential equation having an "elliptic" character. The upstream propagation of disturbances is hence possible and the viscous layer is sensitive to the downstream boundary conditions. Neither the neglected second order terms in the Navier Stokes equations (cf. Baum 1968) nor the reversed flow in the separation region are responsible for this upstream influence, but it does depend on the subsonic nature of the inner part of the boundary layer. (cf. Stewartson 1969). Physically, a cycle is set up whereby the growth of the boundary layer produces a main stream pressure gradient which promotes further growth of the boundary layer (Lighthill 1953).

If the equations are to be solved using a marching procedure, then the sensitivity of the boundary layer to upstream influence manifests itself in the instability of the governing differential equations to downstream

integration, i.e. if some parameter in the initial conditions is varied by an arbitrarily small amount each resulting solution ultimately diverges from the other. Hence any given upstream boundary conditions will allow an infinite number of possible solutions, but in general only one will satisfy the downstream boundary conditions. The requirement that the solution satisfies both upstream and downstream boundary conditions therefore determines the correct solution trajectory.

This behaviour is exhibited by the governing differential equations whether expressed in partial differential form or in integral form. However, under certain conditions the governing differential equations may become stable. In this case the solution is insensitive to downstream boundary conditions and there can be no upstream influence. The necessity to impose a jump to initiate compressive interaction and the appearance of the Crocco-Lees critical point result directly from this change in stability.

7.1.2 PHYSICAL SIGNIFICANCE OF CHANGES IN STABILITY

Weinbaum and Garvine (1969) demonstrated that the stability of the governing differential equations was related to the Mach number profile within the viscous layer. The "throat" station, or critical point, was found to occur where the viscous layer changed, in some mean sense, from subsonic to supersonic with a corresponding change in the stability of the governing differential equations. However, the analysis was limited to free shear layers and normal pressure gradients, which could well influence the behaviour of the solution, were neglected.

The most rigorous analysis of boundary layer flows that satisfy a zero-slip boundary condition has been given by Stewartson (1969). The boundary layer was divided into a viscous inner layer and an outer inviscid layer in which normal pressure gradients were included. It was

found that an eigensolution departing towards separation existed under all flow conditions, and no behaviour analogous to a change in stability was observed. Hence, although it may be possible to attribute some physical significance to changes in stability in free shear layers (see also Baum and Denison 1967), this does not appear to be the case in boundary layer flows.

The results of the preceding comparisons enforce this view. It was shown that the stability is strongly dependent on the choice of the governing differential equations and on the assumed relations between profile quantities, whereas the solutions obtained are relatively insensitive to the formulation. Similarly, the position of the critical boundary in the $H - b$ plane was found to depend strongly on the formulation of the problem, yet the solutions that encountered the Crocco-Lees critical point were essentially the same as those that did not encounter this singularity. Furthermore, it was shown that the imposition of a jump does not significantly affect the solution except in the vicinity of the jump. These results imply that the distinction between stable and unstable states is not a physical characteristic of the boundary layer but, on the contrary, is a result of the formulation of the problem.

It thus appears that, in boundary layer flows, efforts to ascribe physical significance to changes in stability are ill-founded (see also Stollery and Hankey 1970, Georgeff 1971). Hence the analogy between stable - unstable (or "supercritical-subcritical") states of the boundary layer and supersonic-subsonic flows (Crocco 1955) must be considered weak, at least from a physical standpoint. Procedures adopted for passing from the stable to the unstable state (e.g. "jump" to separation (Crocco 1955, Holden 1965) "jump" based on the conservation

laws (Klineberg 1968) or subjection to a strong negative pressure gradient (Reyhner and Flügge-Lotz 1966)) are, in the light of the above, purely expedients and also lack physical significance. Furthermore, the constraints imposed at the Crocco-Lees critical point and the velocity profile critical point (see Georgeff 1971) are mathematical in nature and cannot be considered physically unrealistic (cf Murphy 1969 and Shamroth 1969). Even in viscous wake flows, where the analysis of Weinbaum and Garvine is perhaps valid, care must be taken in ascribing physical significance to the stability of the governing equations of any approximate method.

7.1.3 METHODS OF SOLUTION

The choice between approaches exhibiting differences in type (stability) depends on the problem and its application. Computational efficiency is enhanced by using stable governing differential equations, while accurate modelling of upstream influence requires unstable governing differential equations.

Considering methods based on the Klineberg formulation, the preceding comparisons indicate that the solutions obtained are relatively insensitive to the initial stability of the governing differential equations. For these methods differences in stability have only a local effect, and all provide a good description of the overall properties of viscous interaction phenomena.

7.2 "UNHOOKING" OF THE VELOCITY PROFILE PARAMETER

The momentum equation at the wall (equation (52)) can be expressed in terms of the Cohen Reshotko pressure gradient parameter, β , as

$$\frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{1}{\delta_i^* R_{\delta_i^*}} \frac{\beta}{\alpha^2} \quad (56)$$

As α is bound, the local pressure gradient cannot vanish unless β vanishes.

Now Lees and Reeves (1964) pointed out that the velocity profiles measured in the pressure plateau region do not correspond with either of those given by the zero pressure gradient ($\beta = 0$) similarity solutions (i.e. the Blasius solution and Stewartson's limiting reversed flow solution). Hence in this region satisfaction of the momentum equation at the wall denies the possibility of the local pressure gradient reaching zero.

Lees and Reeves, and similarly Klineberg, overcame this difficulty by satisfying the moment of momentum integral equation instead of the momentum equation at the wall, so that the local pressure gradient could equal zero without requiring that β equals zero. Using this approach it is possible to attain a pressure plateau with realistic velocity profiles. The velocity profile parameter is no longer explicitly related to the local pressure gradient i.e. it is "unhooked" from the local pressure gradient.*

* An "explicit" relationship is taken to mean that, given all the local parameters, i.e. M_e , δ_i^* , H (and b), the relationship is defined. Hence although the local pressure gradient is not in general defined by the velocity profile parameter alone, it can be calculated knowing the other local parameters i.e. M_e and δ_i^* (and b). Note also that it is inaccurate to say that the pressure gradient parameter (β) is "unhooked" from the velocity profile parameter (H), as strictly speaking β is not defined for non-similar flows.

Conversely, the modifications of Stollery and Hankey and of the author cannot exhibit a pressure plateau, as the velocity profile parameter is explicitly related to the local pressure gradient through equation (56). However, provided that $Re_{\delta_i^*}$ is large, it is possible to almost reach a pressure plateau. Furthermore, due to the approximate nature of these methods, it is unlikely that any significant features of the flow will be lost by leaving the velocity profile parameter "hooked" to the local pressure gradient.

This is clearly shown in the preceding comparisons, where the modifications of Stollery and Hankey and of the author provide as good agreement with experiment in the pressure plateau region as do the methods of Lees and Reeves and of Klineberg. In fact, where the beginning of interaction occurs at a moderate or small value of $\bar{\chi}$, the modifications give better agreement upstream of the corner because no "jump" is required to initiate interaction.

7.3 VALIDITY OF THE INTEGRAL APPROACH

The comparisons of section 6 show that it is necessary to uncouple the enthalpy and velocity profile parameters and thus to employ the energy integral equation in order to accurately describe non-adiabatic viscous interaction phenomena. Otherwise the choice of governing differential equations or set of relations between profile quantities is not critical. This puts the Klineberg method on firmer ground, as conclusive support for the choice of integral equations and profile functions is not necessary.

The results also suggest that the partial differential equations are well described using integral methods that incorporate the energy equation. This conclusion is supported by the results of Murphy (1969) where the solutions obtained from methods based on the full partial differential

equations and integral methods, utilizing either a family of velocity and enthalpy profiles or a set of universal profile relationships, were shown to be in good agreement with one another. However, none of the methods compared allowed for normal pressure gradients, and in some cases these may have a significant effect on the solution (see section 11.1).

It is to be noted that many of the early objections raised by Brown (1969) and Stewartson (1969) to the Klineberg method do not arise in the present formulation. The 'anomalous' behaviour observed at separation using the profile parameter 'a' is not observed when the profile parameter 'H' is employed. Computation, far from being complex, is relatively simple and is not substantially more difficult for non-adiabatic flows. Furthermore, any doubts about the dropping of the momentum equation at the wall are allayed by the excellent agreement between the solutions obtained using the method of Klineberg and those obtained using the author's modification. The method is "irrational" in Van Dyke's sense (1964) in that it is not exact in any known limit. However, this is not to deny the utility of such an approach.

8. OTHER METHODS

In this section a number of other successful methods for treating viscous interaction problems will briefly be described.

All employ basically the same approach (see Section 2.1), and the model used to describe the outer inviscid flow is not restricted to any particular method. However, the boundary layer equations are approximated or solved differently in each case.

8.1 METHOD OF CHENG

The method of Cheng (1961) and the later modifications of Sullivan (1969) and Stollery (1970b) are in essence simplifications of the Klineberg method. In order to show this clearly, the final form of the governing equations will be derived from the set of integral equations (15) - (18), rather than directly from the boundary layer equations. The latter approach is described by Lees (1956a).

The method of Cheng makes use of the following additional assumptions:

- (i) $\frac{\gamma - 1}{\gamma + 1} \ll 1$
- (ii) $\frac{\gamma - 1}{2} M_{\infty}^2 \gg 1$
- (iii) Local flat plate similarity i.e. changes in the boundary layer velocity and enthalpy profiles can be neglected, these profiles being adequately described by the classical Blasius solution.

In general, pressure changes are taken into account only insofar as local pressure affects the transformation between physical co-ordinates and incompressible (Stewartson-illingworth) co-ordinates, but this is not an additional restriction - rather, provided pressure gradients are not

large, it derives from the other assumptions.

The above set of assumptions give

$$\left. \begin{aligned} \left| \frac{d \ln M_e}{dx} \right| &<< \left| \frac{d}{dx} \left(\ln \frac{p_e}{p_\infty} \right) \right| \\ f \frac{d \ln M_e}{dx} &\approx -F \frac{d}{dx} \left(\ln \frac{p_e}{p_\infty} \right) \\ \frac{dH}{dx} &= 0 \quad H = H_B \end{aligned} \right\} \quad (57)$$

where the subscript B denotes Blasius values, and hence the integral equations (15) and (18) reduce to

$$\frac{d \ln \delta_i^*}{dx} - \frac{d}{dx} \left(\ln \frac{p_e}{p_\infty} \right) = \frac{B(1 + m_e)}{m_e(1 + m_\infty)} \frac{\tan(\theta_e - \alpha_w)}{\delta_{iF_B}^*} \quad (58)$$

$$\frac{d \ln \delta_i^*}{dx} = BC \frac{M_\infty}{M_e} \frac{1}{\delta_{iR_e}^* \delta_i^*} \left(\frac{P}{H} \right)_B \quad (59)$$

Integrating equation (59) yields

$$\frac{\delta_i^*}{L} = \left[2 \left(\frac{P}{H} \right)_B \frac{C}{R_{eL}} \int_0^{x/L} B \frac{M}{M_e} d \left(\frac{x}{L} \right) \right]^{1/2} \quad (60)$$

On transforming to physical co-ordinates

$$\delta^* = \frac{\gamma - 1}{2} M_\infty^2 \frac{F}{B} \delta_i^* \quad (61)$$

so that equation (58) reduces to

$$\frac{d \delta^*}{dx} = \tan \theta_e - \alpha_w \quad (62)$$

and equation (60) becomes

$$M_\infty \frac{\delta^*}{L} = \left(\frac{\gamma - 1}{\gamma + 1} \right) \frac{M_\infty^3}{(R_{eL}/C)^{1/2}} \left[\frac{(\gamma + 1)^2 \left(\frac{F^2 P}{H} \right)_B}{2} \right]^{1/2} \left[\int_0^{x/L} \frac{p_e}{p_\infty} d \left(\frac{x}{L} \right) \right]^{1/2} \left(\frac{p_e}{p_\infty} \right)^{-1} \quad (63)$$

Using the Cohen-Reshotko zero pressure gradient similarity solutions to provide the required Blasius values, equation (63) may be written as

$$M_{\infty} \frac{\delta^*}{L} = \frac{\gamma - 1}{\gamma + 1} \left(0.664 + 1.73 \frac{T_w}{T_o} \right) \bar{\chi}_L \left(\int_0^{x/L} \frac{p_e}{p_{\infty}} d \frac{x}{L} \right)^{1/2} \left(\frac{p_e}{p_{\infty}} \right)^{-1} \quad (64)$$

where

$$\bar{\chi}_L = \frac{M_{\infty}^3}{(R_{e_L}/C)^{1/2}}$$

Equations (62) and (64) represent the final form of the governing equations.

The assumption of a Blasius velocity profile prevents application of the method to separated flows, and the stability of the governing differential equations implies that problems exhibiting upstream influence cannot be treated. However, as a large range of viscous interaction phenomena show negligible upstream influence, the method is still very useful.

As a result of the stability of the equations, the method of Cheng is considerably more efficient in computation than the Klineberg method.

8.2 INTEGRAL APPROACHES USING PROFILE FAMILIES

The most successful use of such an approach to viscous interaction phenomena has been made by Nielsen, Kuhn and Goodwin (1970). As in the Klineberg method normal pressure gradients are neglected. The integral equations employed are the first four moments of the momentum equation and the first moment of the energy equation. These equations are transformed into "Dorodnitsyn co-ordinates" and a quartic representation of the shear and enthalpy profiles is used. The coupling equation is provided by relating the pressure gradient to the growth of the boundary

layer displacement thickness. At high Mach numbers the displacement thickness approximates a streamline of the flow and hence the coupling equation becomes virtually identical to that employed in the Klineberg formulation.

Because of the greater number of governing equations, the method is less efficient computationally than the method of Klineberg.

8.3 FINITE DIFFERENCE METHODS

A number of successful finite difference methods exist, including those of Reyhner and Flügge-Lotz (1966), Baum (1967) and Smith (1972). These methods approximate the partial differential equations by difference equations and employ some implicit scheme for marching downstream. The coupling equations used are as in the Klineberg or Nielsen methods.

Of these methods, the only one capable of treating separated flows is that of Reyhner and Flügge-Lotz. However, in order to obtain a solution it was found necessary to make certain non-physically realistic assumptions in regions of reversed flow. Normal pressure gradient effects are neglected except in the method of Baum.

Another finite difference method has been proposed by Wornom and Werle (1972), who allow for some of the effects of normal pressure gradient. However, the use of the tangent wedge rule to describe the outer inviscid flow is not good (see Section 3) and some aspects of their numerical work seem inaccurate.

Computationally, these methods are far less efficient than the integral approaches.

9. COMPARISON WITH EXPERIMENT

9.1 LIMITATIONS OF THE METHOD

The main limitations on the Klineberg method are as follows:

- (i) The flow must be laminar.* Transition can be detected using the criteria of Ginoux (1968), where it was shown that the extent of upstream influence increases with increasing unit Reynolds number for laminar flow but decreases for transitional flow.
- (ii) The flow must be either two dimensional or such that transverse flow gradients are negligible. Two dimensionality is difficult to verify, but in general if the aspect ratio is large and little spanwise variation in the boundary layer properties is observed, two dimensional flow can be assumed.
- (iii) Normal pressure gradients within the boundary layer should not be excessively large. As discussed in Section 11.1 the assumption $\frac{\partial p}{\partial y} = 0$ is not valid in most viscous interaction problems, but this does not appear to prevent good agreement with experiment from being obtained. Hence unless normal pressure gradients are excessively large, the Klineberg method probably remains viable. Large normal pressure gradients occur when
 - (a) the height of the separation bubble is large with respect to separation length
 - (b) the separation region itself is large
 - (c) large wall curvatures are experienced
 - (d) the flow is hypersonic.

* The method has been extended to turbulent flows using a turbulent eddy viscosity model (Alber and Lees 1968). A better yet considerably different approach is developed by Reeves (1971) using a two layer model. The external pressure gradient is prescribed and a coupling equation is used to relate the growth of the inner (wall) layer to the development of the outer wake layer.

Some additional limitations result from the particular model used in the following comparisons to describe the outer inviscid flow. They are as follows

(i) The shocks produced by or incident upon the boundary layer must not be too strong. Otherwise the resulting large entropy gradients and the complex interacting shock wave pattern invalidate the outer flow model.

(ii) The reflected waves from major shock wave interactions must not impinge on the model, at least in the vicinity of reattachment.

9.2 CHOICE OF EXPERIMENTAL DATA

The choice of experimental data was limited only by the requirements of two-dimensionality and laminar flow. As a check on the method, some earlier comparisons (see Lees 1964 and Klineberg 1968) were repeated. These comparisons are not included herein as the results agree with those previously published.

9.3 METHOD OF COMPARISON

For the cases considered sufficient data is available to provide a unique solution to the analytic problem. Hence the method was used in a purely predictive capacity and no matching procedure was involved.*

The experimental conditions for the cases considered are given in Table 2.

* Note that a number of authors use some form of matching procedure. Murphy (1969) considered that the data available was not known accurately enough to provide a fair assessment of the methods, and hence adopted the procedure of fitting the predicted pressure distribution to the data. Gautier and Ginoux (1971), in comparisons with the data of Needham (1965), match the separation points. These procedures, however, do not provide an adequate test of the methods as predictive theories.

9.4 RESULTS AND DISCUSSION

9.4.1 SHARP FLAT PLATE AT ZERO INCIDENCE

The method of Klineberg is compared with various experimental data in Figures 34 and 35. In order to correlate the data obtained at different freestream conditions and wall cooling ratios the results are expressed in the similarity form of Cheng (1961) (see Section 10).

The theoretical solution was obtained in the hypersonic limit at $S_w = -0.8$. It is to be noted that the similarity scalings do not completely remove the dependence of the solution on the reference conditions. For fixed $\bar{A}\bar{X}$, a decrease in Mach number causes a slight increase in pressure ratio, although the strong interaction asymptotic limit is unchanged. The effect of wall cooling ratio is small.

In the strong interaction region ($\bar{A}\bar{X} > 1$) the theory tends to overpredict the pressure ratio. The discrepancy partly derives from inaccurate modelling of the outer inviscid flow, both in deriving the strong interaction asymptotic expansions and in obtaining the entire flat plate solution (see Sections 3 and 4.2.1). Low density effects, which become important in the experimental regime ($\frac{\bar{X}}{M_\infty^2} \sim 0.1$), also contribute to the disagreement between theory and experiment. The tendency for the pressure data at fixed freestream conditions to "flatten off" as $\bar{X}$ is increased is indicative of low density effects.

In the moderately weak interaction region ($\bar{A}\bar{X} < 1$) the pressure ratio is quite well predicted by theory. This is to be expected as the outer inviscid flow model is quite accurate in this region and non-continuum effects are absent. The scatter of the experimental data partly derives from the difficulty in measuring the very low pressures experienced in this region.

In all regions the heat transfer coefficient is quite accurately predicted.

Also included for comparison are the results obtained using the flat plate similarity methods of Cheng (1961) and Sullivan (1969) and the finite difference method of Smith (1972). The similarity scaling completely removes the dependence on reference conditions in the Cheng and Sullivan methods, but some dependence is to be expected for the method of Smith. The results of Smith were obtained at $M_\infty = 12.2$ and $S_w = -0.78$.

All theories are in general agreement with each other. The lower Mach number considered by Smith is partly responsible for the small disagreement between his results and those obtained using the method of Klineberg. The method of Cheng provides a better estimate of the pressure ratio in the strong interaction region, but farther downstream is significantly inferior to the other methods.

9.4.2 FLAT PLATE - WEDGE (COMPRESSION CORNER)

The data of Bloy (1972) at $M_\infty = 12.2$ and $S_w = -0.78$ provides distributions of heat transfer coefficient and pressure ratio at three different wedge angles. An estimate of the boundary layer displacement thickness was obtained from the schlieren photographs by measuring the distance from the wall to the point of maximum density gradient within the boundary layer. In Figures 36-44 the theoretical results are compared with the experimental data.

The general agreement between theory and experiment is good. Although the boundary layer characteristics exhibit a slight 'lag' in response to the disturbance generated at the corner, the maximum values of heat transfer coefficient and pressure ratio are accurately predicted. The incipient separation angle, observed by Bloy to be 9° , is also well predicted.

The data of Holden (1968) was obtained at $M_\infty = 19.8$ and $S_w = -0.95$. At the larger wedge angles the leading edge shock interacts with the shock wave generated at the corner, and the reflected expansion waves impinge on the wedge downstream of the reattachment point. The expansion waves result in a rapid decrease in pressure ratio towards the inviscid value (see Section 3), and hence in this region the flow model adopted would not be expected to provide good agreement with experiment. As at the largest wedge angle the point of impingement of the reflected waves is almost coincident with the reattachment point, the disturbance could feed into the separation region and affect the flow properties well upstream of this point.

In Figures 45-53 the theoretical results are compared with the experimental data. The approximate beginning of the region of impingement of the reflected expansion waves as determined from the schlieren photographs appearing in Holden's published report is also indicated in the Figures. The general agreement with experiment is quite good, especially when one considers that the entire interaction takes place in the strong interaction region and that the freestream Mach number is very high. The pressure ratio is well predicted up to the region of impingement of the reflected expansion waves. The heat transfer coefficient is also well described, although the maximum value tends to be underpredicted. At the largest wedge angle the agreement appears to be better, but the relatively low experimental values probably result from the interacting expansion waves. The skin friction coefficient is in poor agreement at the larger wedge angles. However, the maximum value is well predicted and the incipient separation angle of 22° is in good agreement with Holden's estimate of 21.6° . At the largest wedge angle the extent of upstream influence and the length of the separation region are considerably underpredicted.

Note that in obtaining the theoretical solution at the smallest wedge angle, it was not possible to introduce a "jump" to initiate interaction. Consequently no upstream influence is predicted.

The disagreement between theory and experiment, especially at the largest wedge angle, results primarily from an inadequate model of the outer inviscid flow. At high Mach numbers and with strong shock waves it is not valid to replace the corner shock system with an equivalent impinging shock system. The Lees-Reeves model should still provide a reasonable description of the overall interaction, which explains the relatively good agreement with experiment downstream of the corner. However, the model does not take account of the shock produced at separation, which provides a mechanism for promoting further separation and increasing the pressure ratio upstream of the corner. Furthermore, the effect of the large entropy gradients experienced in the strong interaction region could well be an important factor.

The data of Johnson (1968, 1970) at $M_\infty = 7.6$ and $S_w = -0.52$ is compared with theory in Figures 54 and 55. The lowest unit Reynolds number and wedge angle were chosen for comparison in order that the effects of transition be absent. As the trailing edge flap is very short, the downstream conditions relative to finite length wedges have been employed in determining the theoretical solution (i.e. $M_\infty \rightarrow \infty$ at the trailing edge).

The pressure ratio distribution is well described, but the heat transfer coefficient is considerably underpredicted downstream of the corner. Also the separation length is not well predicted. However, Johnson's estimate of separation length based on oil flow measurements does not seem particularly accurate. As unit Reynolds number increases the separation length measured by Johnson decreases, which is not in agree-

ment with other theoretical and experimental results for laminar flow, e.g. Klineberg (1968) and Needham (1965).

The data of Ball and Korkegi (1968) at $M_\infty = 12.3$ and $S_w = -0.44$ represents the only data available for an inclined flat plate followed by a wedge, and is compared with theory in Figure 56. As with Johnson's model, the flap is small in length, and the downstream boundary conditions relative to finite length wedges have been employed in determining the theoretical solution.

The large overprediction of pressure ratio throughout the interaction is difficult to explain. At similar values of $\bar{\chi}_L$ and $M_\infty \alpha_w$ the data of Lewis at $S_w = 0$ and $S_w = -0.8$ (e.g. Figure 18) and the data of Johnson at $S_w = -0.52$ (Figure 54) show relatively good agreement with theory. The poor agreement with the data of Ball and Korkegi therefore appears attributable to the different configurations. However, the interaction takes place well into the weak interaction regime where the effect of configuration is not expected to be important. Perhaps some of the discrepancy results from the reference conditions taken over the wedge, which were based on inviscid flow considerations. The effect of the viscous layer near the leading edge is to increase the shock wave angle, which would result in a reduction of pressure ratio relative to the wedge value (see Section 3).

Other comparisons have been made at wall cooling ratios $S_w = 0$ and $S_w = -0.8$ (e.g. Lees 1964, Holden 1965, 1967, 1968, Klineberg 1968, Ginoux 1968, Murphy 1969, Gautier 1971) and in most cases the agreement with experiment is good. The only notable exceptions are the comparisons with the data of Needham (1965). Although acceptable results can be obtained using some form of matching of theoretical and experimental results (e.g. Murphy 1969, Gautier 1971), when used purely as a predictive

method the method of Klineberg is in poor agreement with experiment. Separation length is considerably overpredicted and the pressure and heat transfer distributions, although qualitatively similar, bear no quantitative resemblance to the experimental data. The disagreement arises because the experiments of Needham were conducted in an expanding conical nozzle so that the flow was not two-dimensional.

9.4.3 CURVED COMPRESSION SURFACES

In Figures 57-59 comparisons are made with the data of Mohammadian (1970) for flow over a concave cubic surface. The general agreement between the theory of Klineberg and experiment is adequate over the forward part of the surface. Far downstream the compressive disturbances coalesce to form an imbedded shock, and the resulting reflected waves cause the pressure to drop rapidly back towards the inviscid value. As the flow model adopted does not allow for these reflected waves, the pressure ratio in this region is considerably overpredicted. However, the underprediction of heat transfer coefficient cannot be readily explained by deficiencies in the outer flow model. The physical displacement thickness agrees well with the estimate determined from the schlieren photographs (as in Section 9.4.2), and the flow remains attached throughout the interaction.

The considerations advanced in Section 3 indicate that in this case neither the Prandtl Meyer rule nor the tangent wedge rule can adequately describe the flow property relationships at the edge of the boundary layer. A considerable improvement in the theoretical predictions should be obtainable using a characteristics solution in the outer inviscid region, although the effect on the distribution of heat transfer coefficient is difficult to assess.

The finite difference method of Smith and the flat-plate similarity method of Sullivan are also included for comparison. The results of Smith were obtained at a lower unit Reynolds number ($Re_1 = 0.083/\text{in}$) than in experiment, and mass entrainment at the edge of the boundary layer was not considered in deriving the coupling equation. The effect of both these 'errors' is to improve the agreement with experiment by between one and four percent. Allowing for this small correction, the solutions obtained by the method of Smith and by the method of Klineberg are almost identical, which suggests that the discrepancy between theory and experiment does not result from the integral approximation to the boundary layer equations.

The method of Sullivan gives rather poor agreement with experiment, especially in the distribution of heat transfer coefficient which is not even qualitatively correct.

As the developing boundary layer considerably modifies the effective body shape the inviscid pressure distribution as determined by the Prandtl Meyer rule is not in good agreement with experiment.

Other comparisons have been made for adiabatic flow over curved compression corners (Ko 1969). Excellent agreement between theory and experiment is observed. In these cases no embedded shocks occur and hence the Prandtl Meyer rule provides a good description of the outer inviscid flow throughout the interaction.

9.4.4 INCLINED FLAT PLATE - EXPANSION CORNER

The data of Bloy (1972) obtained on a wedge followed by a sharp expansion corner is compared with theory in Figures 60-66. In obtaining the theoretical solution it was found that the sudden expansion at the corner resulted in the profile parameters H and b exceeding the range of validity of the polynomial functions given in Appendix 1. Following

the procedure described in Section 2.3, very simple and approximate linear extrapolations of the profile functions were made and the profile parameters allowed to exceed the original polynomial limits. The solution was found to be quite insensitive to the order of approximation made.

The general agreement with experiment is good. Upstream of the corner the pressure ratio tends to be underpredicted by a constant amount. This could result from a slightly larger inclination angle on the flat plate than that measured or from the displacement effect of the boundary layer at the leading edge (see Section 3). The heat transfer coefficient is well predicted upstream of the corner, except at the largest inclination angle. This discrepancy may also result from an inaccurate outer flow model, but see Section 9.4.5. Downstream of the corner the agreement with experiment is very good. No upstream influence is predicted as the governing differential equations are stable. This agrees well with experiment, where the extent of upstream influence is seen to be small.

Note that the physical displacement thickness as determined by the method of Klineberg drops at the corner and then begins to rise again. This anomalous behaviour arises because in this case the physical displacement thickness, δ^* , is determined by differencing two relatively large expressions, $(H + 1)$ and $(0.8E)$. Hence any small errors in H or b lead to a relatively large error in δ^* . The inaccuracies of the extrapolation procedure serve to further increase the error. The solution is still valid, however, as δ^* is not a dependent variable of the governing differential equations.

Also included for comparison are the results obtained using the method of Sullivan. At the largest wedge angle the oblique shock relations together with the Prandtl Meyer rule are employed in order to provide a more realistic model of the outer flow than that given by the tangent wedge

rule. Although the distribution of heat transfer coefficient is well predicted, the pressure ratio distribution is in poor agreement with experiment. This is to be expected as the boundary layer profiles are considerably different from the flat plate values and very large pressure gradients are experienced.

For the 10° expansion corner, the heat transfer distribution obtained with the finite difference method of Sells (1966) using the experimentally determined pressure distribution is also shown. The good agreement with the method of Klineberg indicates that the integral method accurately models the full partial differential equations.

9.4.5 CURVED EXPANSION SURFACES

In Figures 67 and 68 comparisons are made with the data of Mohammadian (1970) for flow over a convex surface. The agreement between theory and experiment is reasonable. Over the forward section of the model (18° wedge) the heat transfer data is considerably above that predicted by theory, yet under identical freestream conditions, the data for the 20° wedge (see Figure 65) is considerably below theory. This would suggest that in one case at least, errors in the experimental determination of the heat transfer coefficient are responsible for the disagreement with theory.

The method of Sullivan is also included for comparison. In this case the theory is unmodified and the tangent wedge rule is used to describe the outer inviscid flow. This explains the high pressure ratio over the forward section of the model compared with that predicted by the modified Sullivan method over the 20° wedge (Figure 64). Considering that the pressure and heat transfer data are artificially high upstream of the expansion corner, the agreement with experiment is only fair.

It is noteworthy that the inviscid pressure distribution given by shock expansion theory (i.e. oblique shock with Prandtl Meyer expansion) agrees better with the experimental data than either the Klineberg or Sullivan theories. Hence in this case, the usual methods of boundary layer analysis, where the pressure gradient is prescribed, could successfully be employed.

A number of other comparisons have been made (Victoria and Kubota 1970) for adiabatic flow over sharp and curved expansion corners. The agreement with experiment is very good. For adiabatic flow the governing differential equations are unstable and hence some upstream influence is predicted. The extent of upstream influence also agrees well with experiment.

9.4.6 OTHER COMPARISONS

The Klineberg method has been compared with experiment for axisymmetric adiabatic flows (Horton 1970, Leblanc 1971). Good agreement with experiment was obtained for both stationary and spinning bodies.

9.5 OTHER METHODS

The integral method of Nielsen has been compared with experiment for a large number of plane and axisymmetric adiabatic and non-adiabatic flows (Nielsen 1965, 1967, 1969, 1970, Murphy 1969). In general the pressure distributions are well predicted. However, the heat transfer rate is underpredicted and shows unrealistic minima at separation and downstream of the corner. The Nielsen method has also been compared with the methods of Lees-Reeves and Klineberg (Murphy 1969). The predicted skin friction coefficient and pressure distributions are virtually identical. Unfortunately no comparisons of the heat transfer coefficient were made, but in view of the anomalies encountered with the Nielsen method, the method of

Klineberg can be assumed superior.

The finite difference of Reyhner and Flügge-Lotz shows good agreement with experiment (Reyhner and Flügge-Lotz 1966, Murphy 1969). The comparisons of Murphy show that the predicted skin friction coefficient and pressure distributions are almost identical to those obtained using the Lees-Reeves or Klineberg methods.

The finite difference method of Baum (1967) has been compared with experiment for both sharp and rounded backward facing steps (Baum 1968). The pressure distributions agree well with experiment up to the point of separation (or sharp corner).

The finite difference method of Wornom and Werle (1972) has been compared with the experimental data of Mohammadian (1970) for the concave cubic model. Although the pressure distribution is quite well predicted, the physical displacement thickness is not in good agreement with experiment. Furthermore, the solution indicates that separation occurs which was not observed experimentally. Heat transfer distributions were not given. It is difficult to evaluate the method on this single comparison, but the vague numerical procedures adopted and the lack of experimental agreement cast serious doubts on the validity of the method.

An integral method developed by Rose (1969) has been compared with the data of Needham (1965). The method requires that the extent of upstream influence be known. Having thus matched the data, good agreement with experiment is obtained. Although Rose chooses to call the theory a two layer model, it is very different from either the Lighthill (1953) or Stewartson (1969) two-layer models. In effect it simply provides an ostensibly more accurate model for the outer inviscid flow field for use with either the Klineberg method or any of the methods described in Section 8.

9.6 GENERAL COMMENTS

Although the general agreement with experiment is surprisingly good, it is worthwhile considering whether any significant improvements to the method of Klineberg can be made. The integral approach appears adequate for describing the major characteristics of viscous interaction phenomena. The excellent agreement between all methods, using both finite difference and integral approaches, suggests that in the viscous flow region the partial differential equations are well described using integral methods.

Much of the disagreement between theory and experiment is attributable to the model of the outer inviscid flow. The considerations advanced in Section 3 suggest that the main cause of error is in the expression for the entropy at the edge of the boundary layer. For example, some allowance must be made for the entropy gradient produced by the leading edge shock wave. The effect of reflected waves is in general small, so that given the entropy at the outer edge of the boundary layer, the Prandtl Meyer rule should provide a good description of the outer inviscid flow. The tangent wedge rule is in general less accurate. In some interactions, the reflected waves resulting from strong shock interactions must be taken into account. In these situations it may be necessary to resort to the more complex characteristics method to describe the outer inviscid flow. Considering shock wave-boundary layer interactions, it is quite clear from schlieren photographs and theoretical considerations that the Lees-Reeves model is inadequate for strong shock waves and high freestream Mach numbers. The shock formation at separation and the resulting entropy changes must be more accurately modelled.

Although in most hypersonic viscous layers large normal pressure gradients exist (see Section 11.1), the good agreement of the predicted pressure distributions with experiment suggests that their effect is small.

Similarly, the related supersonic nature of the outer part of the viscous layer appears to have little effect on the solution obtained. However, the application of a comprehensive finite difference method to, for example, the concave cubic surface of Mohammadian (1970) would be of great benefit in establishing the importance of normal pressure gradients.

It is unlikely that the assumed linear viscosity law or unit Prandtl number would significantly affect the solution. This view is supported by the good agreement between the results of Smith at $Pr = 0.7$ and those obtained using the Klineberg method at $Pr = 1.0$.

10 PARAMETRIC STUDIES

10.1 PARAMETRIC FORM OF THE GOVERNING EQUATIONS

The results of parametric studies describe the dependence of the boundary layer characteristics on free stream flow conditions and body geometry. A theoretical study allows for an accurate parametric description as, in contrast to the experimental situation, it is very simple to vary one parameter while keeping all others fixed.

It is first necessary to determine theoretically the important parameters of the interaction. This allows the similarity variables to be extracted and avoids the needless repetition of results.

Assuming hypersonic flow,

$$\frac{\gamma - 1}{2} M_\infty^2 \gg 1$$

the governing differential equations can be expressed in terms of the parameters $\bar{\chi}$, S_w , and $M_\infty \tan(\theta_e - \alpha_w)$ provided the dependent variables are defined as

$$\left. \begin{aligned} \bar{M} &= \frac{M_e}{M_\infty} \\ \bar{R}_{e_{\delta_i^*}} &= \frac{R_{e_{\delta_i^*}}}{M_\infty^3 C} \\ \bar{H} &= H \\ \bar{b} &= b \end{aligned} \right\} \quad (65)$$

As the upstream boundary conditions (initial values) depend only on $\bar{\chi}$ and S_w , then integration of the governing differential equations must yield

$$\left. \begin{aligned}
 \bar{M} &= \bar{M}(\bar{\chi}, S_w, M_\infty \tan(\theta_e - \alpha_w)) \\
 \bar{R}_{e_{\delta_i^*}} &= \bar{R}_{e_{\delta_i^*}}(\bar{\chi}, S_w, M_\infty \tan(\theta_e - \alpha_w)) \\
 \bar{H} &= \bar{H}(\bar{\chi}, S_w, M_\infty \tan(\theta_e - \alpha_w)) \\
 \bar{b} &= \bar{b}(\bar{\chi}, S_w, M_\infty \tan(\theta_e - \alpha_w))
 \end{aligned} \right\} \quad (66)$$

Now if the body slope is not large, i.e.

$$\alpha_w \ll 1$$

then

$$\left. \begin{aligned}
 M_\infty \tan(\theta_e - \alpha_w) &\sim M_\infty \theta_e - M_\infty \alpha_w \\
 &= f(\bar{M}) - M_\infty \alpha_w
 \end{aligned} \right\} \quad (67)$$

where the function f depends on the model used to describe the outer inviscid flow.

Hence the similarity variables can be expressed as functions of the freestream parameters $\bar{\chi}$ and S_w and the body geometry parameter $M_\infty \alpha_w$.

In the physical plane the similarity variables become

$$\left. \begin{aligned}
 \frac{M_e}{M_\infty} \quad \text{or} \quad \frac{P_e}{P_\infty} \\
 M_\infty^3 C_F \quad \text{and} \quad M_\infty^3 C_H \\
 \frac{R_{e_{\delta^*}}}{M_\infty^5 C} \quad \text{or} \quad \frac{M_\infty \delta^*}{x}
 \end{aligned} \right\} \quad (68)$$

The functional dependence of the similarity variables on $\bar{\chi}$, S_w and $M_\infty \alpha_w$ can be determined for geometrically similar bodies by using the Klineberg method and varying each parameter in turn.

10.1.1 DEPENDENCE ON S_w AND γ

Although the dependence of the solution on wall cooling ratio S_w cannot be completely removed, the following approach provides an approximate scaling.

In most viscous interactions, the profile quantity F remains approximately constant. Then assuming that the pressure gradient is not large, the governing differential equations can be expressed in terms of the parameters $F\bar{\chi}$ and $M_\infty \alpha_w$ provided the dependent variable $\bar{R}_{e\delta_i^*}$ is redefined as

$$\bar{R}_{e\delta_i^*} = \frac{R_{e\delta_i^*}}{FM_\infty^3 C} \tag{69}$$

The dependence on the specific heat ratio γ cannot be eliminated from the governing differential equations. However, making certain simplifying assumptions the dependence on γ for either the Prandtl Meyer or tangent wedge inviscid pressure laws gives the similarity parameters

$$\left. \begin{aligned} &A^* \bar{\chi} \text{ and } M_\infty \alpha_w \\ \text{where } &A^* \approx \gamma \left(\frac{\gamma - 1}{2} \right) F \end{aligned} \right\} \tag{70}$$

The similarity variables in the physical plane are then

$$\left. \begin{aligned} &\frac{p_e}{p_\infty} \\ &M_\infty^3 C_F A^* \quad \text{and} \quad M_\infty^3 C_H A^* \\ &\frac{R_{e\delta_i^*}}{A^{*2} M_\infty^5 C} \quad \text{or} \quad \frac{M_\infty \delta_i^*}{x} \end{aligned} \right\} \tag{71}$$

Note that for interactions in the vicinity of the Blasius solution

$$F \sim \left(1 + 2.6 \frac{T_w}{T_o} \right) \tag{72}$$

so that

$$A^* \sim \gamma \left[\frac{\gamma - 1}{2} \right] \left[1 + 2.6 \frac{T_w}{T_o} \right] \quad (73)$$

This scaling agrees in form with those of Chapman (1957), Curle (1961), Cheng (1961), Stewartson (1969) and Stollery (1970b). However, in most viscous interaction problems the pressure gradients experienced are large enough to invalidate the above derivation of S_w dependence. Without a more rigorous investigation of the effects of wall cooling it is therefore inadvisable to combine, in the form of a single similarity parameter, the parameters $\bar{\chi}$ and S_w .

10.1.3 INCIPIENT SEPARATION

It follows from the above that the incipient separation condition (subscript i) for geometrically similar bodies must be of the form

$$(M_\infty \alpha_w)_i = f(\bar{\chi}, S_w) \quad (74)$$

10.2 THEORETICAL AND EXPERIMENTAL COMPARISONS

10.2.1 EVALUATION OF SIMILARITY REPRESENTATION

The similarity form of the governing differential equations derived in the preceding sections can be examined by comparing solutions at fixed values of $\bar{\chi}_L$, S_w and $M_\infty \alpha_w$ while allowing M_∞ and R_{e_L} to vary. Such a comparison for the flat plate-wedge at $S_w = -0.8$ is shown in Figures 69-71. At the hinge line $\bar{\chi} = 1.0$ and the ramp angle (α_r) is such that $M_\infty \alpha_r = 1.745$. For hypersonic flow the solutions in terms of the similarity variables show little dependence on M_∞ (or R_{e_L}), thus confirming the similarity representation. At lower Mach numbers (e.g. $M_\infty = 5$) the similarity scaling is not so accurate, overestimating the effect of a reduction in Mach number.

In Figure 72, the incipient separation condition for $S_w = -0.8$ at three

different values of $\bar{\chi}_L$ is shown as a function of M_∞ (or R_{e_L}). Except at low Mach numbers, it can be seen that incipient separation is a function only of the similarity parameters $\bar{\chi}$ and $M_\infty \alpha_w$ (and S_w).

10.2.2 FUNCTIONAL DEPENDENCE ON $\bar{\chi}_L$, $M_\infty \alpha_w$

The determination of the functional dependence of the similarity variables on the similarity parameters $\bar{\chi}_L$ and $M_\infty \alpha_w$ has not been attempted. However for the flat plate-wedge model consideration of unit Reynolds number and wedge angle effects indicate that the non-dimensionalized separation length $\frac{\Delta x}{x_L}$ increases as $\bar{\chi}_L$ is reduced or as $M_\infty \alpha_w$ is increased.

As the freestream Mach number constitutes part of the freestream similarity parameter $\bar{\chi}_L$ and the geometrical similarity parameter $M_\infty \alpha_w$, it is difficult to determine the effect of varying M_∞ from similarity rules. Riethmuller (1970) has made a study of Mach number effects and finds for the flat plate wedge configuration that reducing the Mach number increases the separation length.

10.2.3 S_w DEPENDENCE

A detailed investigation of the effect of wall cooling ratio in viscous interactions has not been attempted. However, some of the preceding comparisons provide an indication of the dependence of the major characteristics of viscous interactions on S_w .

The flat plate flow solutions are given in Figures 34 and 35. As discussed in Section 9.4.1, the results are virtually independent of S_w when expressed in the similarity form of Cheng (1961).

In Figures 23-26 the effect of wall cooling on the characteristics of a shock wave-boundary layer interaction is presented. (See also Gautier and Ginoux 1971). The results were obtained at $\bar{\chi}_L = 0.57$ and $M_\infty \alpha_r = 1.08$ corresponding to the experimental conditions of Lewis (1968). The variation

with S_w of the characteristic lengths of the interaction and the maximum values of heat transfer and skin friction coefficient, normalized with respect to their values at the beginning of interaction, agrees favourably with the similarity scaling derived in section 10.2.1. The effect of S_w on the characteristics of the pressure distribution is negligible if pressures are referred to the pressure at the beginning of the interaction.

10.2.4 INCIPIENT SEPARATION

Incipient separation conditions for the flat plate-wedge at $S_w = -0.8$ are shown in Figure 73, where $(M_\infty \alpha_r)_i$ is plotted against $\chi_L^{-1/2}$. The results obtained using the method of Klineberg are seen to be in good agreement with the experimental data. Over much of the range considered $(M_\infty \alpha_r)_i$ varies almost linearly with $\chi_L^{-1/2}$, in agreement with the approximate theory of Needham (1965). At large wedge angle or low to moderate Mach number the similarity principle begins to break down, separation occurring at a lower $M_\infty \alpha_r$ than expected (see Figure 72).

11 EXTENSIONS OF THE KLINEBERG METHOD

11.1 NORMAL PRESSURE GRADIENT EFFECTS

In the derivation of the governing set of equations it was assumed that pressure gradients normal to the surface were negligible. However, this is not a valid assumption in many viscous interaction problems.

Significant normal pressure gradients are produced when the separation region becomes large or where large wall curvatures are experienced. The form of the boundary layer equations that includes these "second order" effects has been derived by Maslen (1963). Large normal pressure gradients also occur as a result of the supersonic nature of the flow. At hypersonic speeds a considerable part of the boundary layer is supersonic and in this region pressure waves tend to be propagated along Mach lines (Myring and Young 1967, Baum 1967, Stewartson 1969). Hence not only is the pressure gradient normal to the surface very large, but the domain of dependence does not admit of the immediate propagation of disturbances across the boundary layer.

An attempt to modify the Klineberg method to include the effects of normal pressure gradient has been made by Holden (1969). However, the underlying theory can be shown to be invalid. A number of important second order terms, necessary for a full description of wall curvature effects, are neglected in the governing differential equations and the method allows immediate propagation of disturbances across the boundary layer. Furthermore, dubious numerical procedures are employed and a number of profile quantities (e.g. P_2 , P_{3A} , P_4 , P_5 , P_6 and P_{13}) and functions (most $P_i(a)$) are inconsistently defined.

Although the effects of surface curvature could readily be incorporated into an integral method, it is difficult to see how the supersonic nature of the outer part of the boundary layer could be correctly represented.

The very good agreement of the basic Klineberg method with experiment suggests, however, that this latter effect may not be too important. This may derive from the fact that the major change in the profile parameters occurs near the wall so that the outer supersonic region has little effect on the solution. Alternatively, the boundary layer equations and functional relationships could approximate those based on a non-orthogonal co-ordinate system where the y axis is taken along isobars. Myring and Young (1967) derived a "Mach line" form of the momentum integral equation which would seem to support this view, but the derivation lacks rigour.

In order to accurately model the boundary layer it may be necessary to resort to finite difference methods. Unfortunately no finite difference method has been developed which adequately takes into account all the major effects of normal pressure gradient. Baum (1967) includes the supersonic effect in the normal component of the momentum equation, but it is not clear whether the choice of differencing quotients and step sizes preserve the domain of dependence as determined by the governing differential equations. Also, wall curvature effects are not correctly represented as important second order terms in the governing equations have been neglected. Wornom and Werle (1972) correctly allow for curvature but neglect to include the terms that would account for the supersonic nature of the outer part of the boundary layer.

Another approach is to divide the boundary layer into an inner (mainly subsonic) viscous region where normal pressure gradients are negligible and an outer region where viscous effects are negligible (see Stewartson 1969). Integral methods could still be employed to describe the behaviour of the inner layer and the outer portion could perhaps be described by a modified characteristics method. The propagation of disturbances within the boundary layer would thus be accurately modelled.

11.2 BLUNTNESS EFFECTS

The basic Klineberg method is capable of describing the effects of nose bluntness. The outer inviscid flow could be described using the method of characteristics on a suitably refined grid with strong blast wave theory providing the required initial conditions. However, the complexity of such an approach can be avoided by employing either shock expansion theory or the simple model of Cheng (1961) to provide an estimate of bluntness effects.

The shock expansion approach described in Section 3.3 can be simplified for the blunt leading edge. In this case most of the streamlines crossing the boundary layer pass through a near normal leading edge shock. Hence the shock angle $\Phi \approx 90^\circ$, and the equations describing the outer inviscid flow (i.e. 35-38) can be considerably simplified. However, such an approach is unlikely to be valid far downstream of the strong interaction region, as the large entropy gradients generated near the nose will significantly effect the solution.

The model of Cheng (1961) effectively defines the shape of the leading edge shock and, after making certain assumptions, provides a very simple formulation. The flow field is divided into three regions: an inner viscous layer, an entropy layer, and an outer shock layer (see Figure 74). Assuming that the pressure, p_e , and streamwise velocity, u_e , are approximately constant across the entropy layer, then it can be shown that

$$(y_s - y_e)p_e \approx \frac{\gamma - 1}{2} k_c t \left[\frac{1}{2} \rho_\infty u_\infty^2 \right] \quad (75)$$

where y_s and y_e denote the outer edges of the entropy and boundary layers respectively,

k_c is the leading edge (nose) drag coefficient
and t is the leading edge thickness.

Differentiating, equation (75) yields

$$\theta_s - \theta_e = - \frac{\gamma - 1}{2} k_c t \frac{d \ln p_e}{dx} \frac{\rho_\infty u_\infty^2}{2 p_e} \quad (76)$$

so that the coupling equation (equation (10)) becomes

$$\begin{aligned} F \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + \frac{1 + m_e}{m_e} S_w \frac{dE}{dx} + \frac{1}{1 + m_e} \left(m_e F + 2H - \frac{1 + 2m_e}{m_e} Z \right) \frac{d \ln M_e}{dx} \\ - \left(F + \frac{Z}{m_e} + \frac{\gamma}{2} \frac{M_\infty}{M_e} \frac{k_c t}{\delta_i^*} \right) \frac{d \ln p_e}{dx} = \frac{B(1 + m_e)}{m_e (1 + m_\infty)} \frac{\tan(\theta_s - \alpha_w)}{\delta_i^*} \\ - \frac{1 + m_e}{m_e} S_w \frac{d \ln S_w}{dx} \end{aligned} \quad (77)$$

where θ_s , the streamline inclination at the outer edge of the entropy layer, is related to the pressure, p_e (or Mach number, M_e) by some suitable pressure law (e.g. tangent wedge rule). Otherwise, the governing equations remain unchanged.

The strong interaction asymptotic expansion coefficients for the blunt flat plate can be determined by expressing the flow variables as expansions in large $\bar{\chi}$ and substituting into the governing differential equations. Using these strong interaction asymptotic expansions as upstream boundary conditions, the governing equations (15)-(17), (77) can be solved as before. The increase in computational effort is negligible.

Note that the entropy distribution over the entire flow field can significantly affect the flow properties in other interaction regions (see section 3).

11.3 THREE DIMENSIONAL BOUNDARY LAYERS

The method described in this report can readily be extended to the study of three dimensional adiabatic or non-adiabatic boundary layers. Although the three dimensional flow is assumed to be such that all derivatives with respect to the transverse co-ordinate z vanish (i.e. flow over an infinite yawed wing or axisymmetric flow), the method allows prediction of the major effects in general three dimensional viscous interaction problems.

11.3.1 GOVERNING EQUATIONS FOR THREE DIMENSIONAL PLANE FLOW

The governing equations for a steady, plane three dimensional compressible laminar boundary layer with transverse velocity component $w = w(x,y)$ are the same as those for a two dimensional boundary layer [i.e. equations (1)-(3)] with an additional equation expressing conservation of transverse momentum (Schlichting 1955), viz.

$$\rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} = \frac{\partial}{\partial y} \left(\mu \frac{\partial w}{\partial y} \right) \quad (78)$$

Derivatives with respect to z are taken to be zero.

At the edge of the boundary layer ($y \rightarrow \infty$) equation (78) requires that

$$w_e = w_\infty = \text{constant}$$

Applying the Stewartson-illingworth transformation these equations become (Crabtree 1954, Chu 1954)

$$\text{Continuity:} \quad U_X + V_Y = 0 \quad (79)$$

Longitudinal Momentum:

$$UU_X + VU_Y = U_e (U_e)_X \left[1 + S - \omega \left(1 - \frac{w}{w_e} \right)^2 \right] + v_\infty U_{YY} \quad (80)$$

Transverse Momentum:

$$Uw_X + Vw_Y = v_\infty w_{YY} \quad (81)$$

Energy:

$$US_X + VS_Y = v_\infty S_{YY} \quad (82)$$

where

$$\left. \begin{aligned} U &\equiv \psi_Y = \frac{a_\infty}{a_e} u \\ V &\equiv -\psi_X \\ S &= \frac{(h_o)_{tr}}{(h_o)_{e\ tr}} - 1 \\ \omega &= \frac{w_e}{2(h_o)_{e\ tr}} = \text{constant} \end{aligned} \right\} \quad (83)$$

and where the subscript 'tr' denotes values determined with respect to a co-ordinate frame moving in the z direction with velocity w_e

$$\text{i.e.} \quad (h_o)_{tr} = h + \frac{u^2 + (w - w_e)^2}{2}$$

The boundary conditions are

$$\left. \begin{aligned} U(X,0) &= V(X,0) = 0 \\ w(X,0) &= 0 \quad S(X,0) = S_w \\ \lim_{Y \rightarrow \infty} (U) &= U_e(X) \\ \lim_{Y \rightarrow \infty} (w) &= w_e \quad \lim_{Y \rightarrow \infty} (S) = 0 \end{aligned} \right\} \quad (84)$$

For S_w constant, equations (79)-(82) can be put in the form

$$U_X + V_Y = 0 \quad (79)$$

$$UU_X + VU_Y = U_e(U_e)_X \{1 + S_w s - \omega g^2\} + v_\infty U_{YY} \quad (85)$$

$$Ug_X + Vg_Y = v_\infty g_{YY} \quad (86)$$

$$Us_X + Vs_Y = v_\infty s_{YY} \quad (87)$$

where

$$\left. \begin{aligned} g &= \left(1 - \frac{w}{w_e} \right) \\ s &= \frac{S}{S_w} \end{aligned} \right\} \quad (88)$$

The boundary conditions for 'g' and 's' are

$$\left. \begin{aligned} g(X,0) &= s(X,0) = 1 \\ \lim_{Y \rightarrow \infty} (g) &= \lim_{Y \rightarrow \infty} (s) = 0 \end{aligned} \right\} \quad (89)$$

As the same equations (86 and 87) and boundary conditions (89) apply to both 'g' and 's', a particular solution for 'g' is

$$g = s \quad (90)$$

Then equation (85) reduces to

$$UU_X + VU_Y = U_e(U_e)_X \{1 + S_w(s - k_1 s^2)\} + v_\infty U_{YY} \quad (91)$$

where

$$k_1 = \frac{\omega}{S_w}$$

By integrating across the boundary layer the following integral forms can be obtained:

Longitudinal Momentum:

$$H \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + (2H + 1 + S_w \bar{E}) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{P}{\delta_i^* R_{e \delta_i^*}} \quad (92)$$

Longitudinal Moment of Momentum:

$$J \frac{d \ln \delta_i^*}{dx} + \frac{dJ}{dx} + (3J + 2S_w \bar{T}^*) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{R}{\delta_i^* R_{e \delta_i^*}} \quad (93)$$

$$\text{Energy:} \quad T^* \frac{d \ln \delta_i^*}{dx} + \frac{dT^*}{dx} + T^* \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{Q}{\delta_i^* R_{e \delta_i^*}} \quad (94)$$

Coupling:

$$\bar{F} \frac{d \ln \delta_i^*}{dx} + \left(\frac{dH}{dx} + \frac{1 + m_e}{m_e} S_w \frac{d\bar{E}}{dx} \right) + \bar{f} \frac{d \ln M_e}{dx} = \frac{B(1 + m_e) \tan(\theta_e - \alpha_w)}{m_e (1 + m_\infty) \delta_i^*} \quad (95)$$

where the profile quantities H, J , etc. are as defined by equations (8)

and

$$\left. \begin{aligned} \bar{E} &= E - k_1 G \\ \bar{F} &= H + \frac{1 + m_e}{m_e} (1 + S_w \bar{E}) \\ \bar{f} &= \frac{1}{1 + m_e} \left(m_e \bar{F} + 2H - \frac{1 + 2m_e}{m_e} Z \right) - \left(\bar{F} + \frac{Z}{m_e} \right) \frac{M_e}{p_e} \frac{dp_e}{dM_e} \\ \bar{T}^* &= T^* - k_1 W^* \\ G &= \frac{1}{\delta_i^*} \int_0^\delta \left(\frac{S}{S_w} \right)^2 dy \\ W^* &= \frac{1}{\delta_i^*} \int_0^\delta \frac{U}{U_e} \left(\frac{S}{S_w} \right)^2 dy \end{aligned} \right\} \quad (96)$$

As $g = s$ the transverse velocity profile quantities are equivalent to the corresponding total enthalpy profile quantities.

Following the procedure used for two dimensional flows (Section 2.3), it is assumed that the relations between the profile quantities are the

same as those established for the corresponding similarity solutions. All the profile quantities appearing in the governing differential equations can then be expressed as functions of the velocity profile parameter H and the enthalpy profile parameter b at any particular value of S_w and ω .

The final form of the governing differential equations is as for two dimensional flows [i.e. equations (20)-(23)] with

$$\left. \begin{array}{l} A_1 \text{ replaced by } \bar{A}_1 = 2H + 1 + S_w \bar{E} \\ A_5 \text{ by } \bar{A}_5 = 3J + 2S_w \bar{T}^* \\ F, \frac{\partial F}{\partial H}, \frac{\partial F}{\partial b} \text{ by } \bar{F}, \frac{\partial \bar{F}}{\partial H}, \frac{\partial \bar{F}}{\partial b} \end{array} \right\} \quad (97)$$

and f by $\bar{f}$

It is important to remember that S_w is defined with respect to a co-ordinate frame moving in the z direction with velocity w_e [see equation (83)].

11.3.2 RELATIONS BETWEEN PROFILE QUANTITIES

Now the equations for self-similar flow can be obtained (Cohen and Reshotko 1956a) by letting

$$U_e = C_1 x^n \quad (98)$$

and taking as a similarity variable

$$\eta = y \left[\frac{n+1}{2} \frac{U_e}{v_\infty x} \right]^{1/2} \quad (99)$$

Equations (79), (85)-(87) then reduce to

$$f''' + ff'' + \beta \{1 - f'^2 + S_w (s - k_1 s^2)\} = 0 \quad (100)$$

$$s'' + fs' = 0 \quad (101)$$

$$g = s$$

where

$$f' = \frac{df}{d\eta} = \frac{U}{U_e}$$

and the pressure gradient parameter $\beta = \frac{2n}{n+1}$

The boundary conditions are

$$\left. \begin{array}{ll} f(0) = f'(0) = 0 & f'(\infty) = 1 \\ s(0) = 1 & s(\infty) = 0 \end{array} \right\} \quad (102)$$

These equations can be solved for a range of β , S_w and k_1 (or ω), and the profile quantities H , J , etc. determined as for the two dimensional case (see Klineberg, 1968). The profile quantities G and W^* are determined in the same manner as the related quantities E and T^* , i.e. the quantities are expressed in the form

$$\left. \begin{array}{l} G = \alpha(H) \kappa(b) \\ W^* = \alpha(H) W(H, b) \end{array} \right\} \quad (103)$$

where

$$\left. \begin{array}{l} \kappa(b) = \int_0^{\eta_{.99}} \left(\frac{S}{S_w} \right)^2 d\eta \\ W(H, b) = \int_0^{\eta_{.99}} \frac{U}{U_e} \left(\frac{S}{S_w} \right)^2 d\eta \end{array} \right\} \quad (104)$$

The profile quantity $W(H, b)$ is determined by multiplying each similar velocity profile by the square of each total enthalpy profile point by point.

Now for $k_1 = 0$ ($\omega = 0$), it has been shown that the profile functions $J(H)$, $T(H, b)$, etc. are virtually independent of S_w^* . Similarly, Leblanc,

* As discussed in Section 2.3, the profile quantity P does show some dependence on S_w , but the effect on the solution is not found to be significant (Section 5.1).

Horton and Ginoux (1971) have indicated that for $k_1 = \frac{1}{2}$ ($S_w = 2$), these functions are independent of S_w (or ω). The reason for this independence is that a change in S_w or ω has a similar effect as a related change in the pressure gradient parameter, β , which, although affecting all the profiles, does not affect the relations between profile quantities. It is therefore probable that the profile functions are independent of S_w (and ω) for all values of k_1 , provided S_w (and ω) are of order unity or less. Making this assumption, the functions established for two dimensional boundary layers with heat transfer are applicable to three dimensional boundary layers with transverse velocity component.

In summary: For constant wall cooling ratio S_w , a small modification to the governing differential equations for two dimensional boundary layers with heat transfer allows three dimensional boundary layers with heat transfer and transverse velocity component to be considered. Furthermore, the relations between profile quantities established from the Cohen-Reshotko similarity solutions for two dimensional flow with heat transfer provide all those required for investigating three dimensional flows.

11.3.3 GOVERNING EQUATIONS FOR AXISYMMETRIC FLOW WITH SPIN

The results of the preceding section can be readily generalized to include axisymmetric flow (Chu 1954). It is assumed that

$$\delta \ll r_w$$

and the following quantities are redefined:

$$\left. \begin{aligned} g &= \frac{w}{w_w} = \frac{w}{\Omega r_w} \\ \omega &= \frac{w_w^2}{2h_{oe}} \end{aligned} \right\} \quad (105)$$

where r_w is the local radius of the body and Ω is the angular velocity

of the body.

Note that for axisymmetric flow

$$(h_o)_{tr} = h_o$$

$$\text{and in the special case } \Omega = 0 \quad g = \omega = 0$$

For k_1 constant, i.e.

$$\frac{1}{S_w} \frac{dS_w}{dx} = \frac{1}{\omega} \frac{d\omega}{dx} = \frac{2}{r_w} \frac{dr_w}{dx}$$

then

$$g = s \quad (106)$$

and the integral form of the governing differential equations in Stewartson-Illingworth co-ordinates becomes

Axial Momentum:

$$\begin{aligned} H \frac{d \ln \delta_i^*}{dx} + \frac{dH}{dx} + (2H + 1 + S_w \bar{E}) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{P}{\delta_i^* R_{e \delta_i^*}} - \\ - \left[H + \frac{S_w (1 + m_e)}{m_e} k_1 G \right] \frac{d \ln r_w}{dx} \end{aligned} \quad (107)$$

Axial Moment of Momentum:

$$\begin{aligned} J \frac{d \ln \delta_i^*}{dx} + \frac{dJ}{dx} + (3J + 2S_w \bar{T}^*) \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{R}{\delta_i^* R_{e \delta_i^*}} - \\ - \left[J + \frac{2S_w (1 + m_e)}{m_e} k_1 W^* \right] \frac{d \ln r_w}{dx} \end{aligned} \quad (108)$$

Energy:

$$T^* \frac{d \ln \delta_i^*}{dx} + \frac{dT^*}{dx} + T^* \frac{d \ln M_e}{dx} = BC \frac{M_\infty}{M_e} \frac{Q}{\delta_i^* R_{e \delta_i^*}} - 3T^* \frac{d \ln r_w}{dx} \quad (109)$$

Coupling:

$$\begin{aligned} \bar{F} \frac{d \ln \delta_i^*}{dx} + \left(\frac{dH}{dx} + \frac{1+m_e}{m_e} S_w \frac{d\bar{E}}{dx} \right) + \bar{f} \frac{d \ln M_e}{dx} = \frac{B(1+m_e) \tan(\theta_e - \alpha_w)}{m_e (1+m_\infty)} \frac{\delta_i^*}{\delta_i^*} - \\ - \left(\frac{2S_w(1+m_e)}{m_e} \bar{E} - \frac{Z}{m_e} \right) \frac{d \ln r_w}{dx} \end{aligned} \quad (110)$$

Equations (107)-(110) can be solved as in the case of three dimensional plane flow. The profile functions are unchanged as they are obtained from the similarity solutions with $dr_w/dx = 0$. As before, the transverse velocity profile quantities are equivalent to the corresponding total enthalpy profile quantities. The final form of the governing differential equations is as for two dimensional flows [i.e. equations (20-23)] with

$$\left. \begin{aligned} A_1 \text{ replaced by } \bar{A}_1 &= 2H + 1 + S_w \bar{E} \\ A_5 \text{ by } \bar{A}_5 &= 3J + 2S_w \bar{T}^* \\ F, \frac{\partial F}{\partial H}, \frac{\partial F}{\partial b} \text{ by } \bar{F}, \frac{\partial \bar{F}}{\partial H}, \frac{\partial \bar{F}}{\partial b} \\ f \text{ by } \bar{f} \\ P \text{ by } \bar{P} &= P - \frac{\delta_i^{*R} e \delta_i^*}{BC} \frac{M_e}{M_\infty} \left(H + S_w \left(\frac{1+m_e}{m_e} \right) k_1 G \right) \frac{d \ln r_w}{dx} \\ R \text{ by } \bar{R} &= R - \frac{\delta_i^{*R} e \delta_i^*}{BC} \frac{M_e}{M_\infty} \left(J + 2S_w \left(\frac{1+m_e}{m_e} \right) k_1 W^* \right) \frac{d \ln r_w}{dx} \\ Q \text{ by } \bar{Q} &= Q - \frac{\delta_i^{*R} e \delta_i^*}{BC} \frac{M_e}{M_\infty} (T^*) \frac{d \ln r_w}{dx} \\ h \text{ by } \bar{h} &= h + \frac{\delta_i^{*R} e \delta_i^*}{BC} \frac{M_e}{M_\infty} \left(\frac{Z}{m_e} \right) \frac{d \ln r_w}{dx} \end{aligned} \right\} \quad (111)$$

A different relation between the streamline inclination at the edge of the boundary layer, θ_e , and pressure, p_e , is required as the Prandtl

Meyer rule is not valid in the axisymmetric case. A suitable pressure law is derived by Horton (1970), using the second order shock expansion method of Syvertson and Dennis (1957).

For the adiabatic case $k_1 = \frac{1}{2}$ and these equations reduce to those of Leblanc, Horton and Ginoux (1971) although in a simpler form.* Although Leblanc et al. have noted the similarity between the equations of transverse velocity and total enthalpy, they do not seem to have developed this nor to have used the enthalpy profiles to provide the required transverse velocity profile quantities. In fact, they contrast the two cases, and state that for the case of heat transfer it is necessary to calculate the profile functions for each value of wall cooling ratio S_w .

11.3.4 PROCEDURE FOR THE CASE OF VARIABLE k_1

When k_1 is not constant the governing differential equations for the transverse velocity component and the total enthalpy can no longer be expressed in identical form and

$$g \neq s$$

The transverse moment of momentum equation must therefore be employed and an additional term must be included in the energy and coupling equations (see Section 2.2). The profile functions remain unchanged as they are obtained from the similarity solutions for $dS_w/dx = 0$. However, an additional parameter (analogous to that chosen to represent the total enthalpy profile) is required to represent the transverse velocity profile.

* Equation (22) in Leblanc (1971) is incorrect. The last term should read

$$(H + K_3 H_Y \kappa) \frac{\delta^*_{si}}{r_w} \frac{dr_w}{dx}$$

Note that in all cases (two dimensional flow with heat transfer, three dimensional plane and axisymmetric flow) the leading terms in the weak interaction expansion coefficients (see Section 4.2.2) are the same. This is because the normalized velocity and enthalpy profiles for the limiting solution (i.e. Blasius solution with $\beta = 0$) are (exactly) independent of S_w and ω . Furthermore, in all cases the analysis of the neighbouring solutions at infinity (Appendix 2) is applicable.

11.3.5 AN APPROXIMATE ANALOGY BETWEEN HEAT TRANSFER AND TRANSVERSE VELOCITY

It would be useful to obtain a simple relationship expressing the characteristics of a three dimensional boundary layer with transverse velocity in terms of those of an analogous two dimensional boundary layer with heat transfer.

Consider the transformed governing differential equations for a three dimensional boundary layer assuming S_w and ω are constant (and for axisymmetric flow, r_w large), i.e. equations (79), (86), (87) and (91). The only term containing a measure of the transverse velocity component is

$$A = s - k_1 s^2 \quad (112)$$

If ω is small this term can be approximated by

$$\tilde{A} = k_2 s \quad (113)$$

where

$$k_2 = \frac{\int_0^{.99} (s - k_1 s^2) d\eta}{\int_0^{.99} s d\eta} \quad (114)$$

or, assuming a linear relationship between s and η

$$k_2 = 1 - \frac{2}{3} k_1$$

In replacing the expression A by the approximation $\tilde{A}$ in equation (91) the governing differential equations become identical to those for a two dimensional boundary layer with wall cooling ratio $S_w k_2$. Hence a three dimensional boundary layer with wall cooling ratio S_w and transverse velocity parameter ω is equivalent, in terms of the dependent variables M_e , δ_1^* , H and b , to a two dimensional boundary layer with wall cooling ratio $(S_w)_{eq} = S_w [1 - \frac{2}{3} \frac{\omega}{S_w}]$. The properties of the transverse flow can be readily determined from the corresponding properties of the total enthalpy distribution.

The effect of such gross approximations on the solution is difficult to determine, but it is probable that the analogy is at least qualitatively correct. Theoretical 'experiments' could be performed to check the validity of the analogy and to provide a more accurate expression for k_2 .

Two applications of the analogy will be considered.

(i) Axisymmetric adiabatic flow.

For adiabatic flow

$$S_w = 2\omega$$

and hence for r_w large the flow should be equivalent to a two-dimensional flow with wall cooling ratio

$$(S_w)_{eq} = \frac{4}{3} \omega$$

Rotation is seen to have the same effect as heating the wall (except, of course, that there is no heat transfer to the flow). A plausible physical explanation is that the transverse shear stresses generate the additional heat that in the two dimensional case would result from heating

the wall.

The properties of the flow over the spinning axi-symmetric body can then be readily determined from studies of two dimensional flows. For example, rotation would be expected to increase the length of separation (see Figures 20-26). In the transverse plane, skin friction is given by

$$C_{F_z} = 2 \left(\frac{m_\infty}{1 + m_\infty} \omega \right)^{1/2} \frac{p_e a_e}{p_\infty a_\infty} \frac{Q}{(R_e^*/C) \delta_i^*}$$

which, as expected, increases with spin.

The results of Leblanc (1971) do not agree with the above predictions, indicating that separation length decreases as ω is increased. The disagreement may result from centrifugal effects which, at the value of r_w considered, are significant.

(ii) Infinite swept wing

It is more convenient in this case to relate the flow properties to freestream conditions in a co-ordinate frame fixed with respect to the body. However, the solution to the governing equations must still be obtained in the plane normal to the leading edge. The Mach number in this plane is defined as M_n and the angle of sweep is taken to be Λ . Then for given freestream conditions and wall temperature

$$\omega = \frac{m_\infty}{1 + m_\infty} \tan^2 \Lambda$$

$$\text{and } S_w = \left(\frac{T_w}{T_o} - 1 \right) (1 + \omega) + 2\omega$$

Making use of the two dimensional analogy and considering hypersonic flow

$$(S_w)_{eq} = \left(\frac{T_w}{T_o} - 1 \right) \sec^2 \Lambda + \frac{4}{3} \tan^2 \Lambda$$

The effect of sweep is therefore to increase S_w in the equivalent two dimensional flow. This effect becomes less pronounced as the wall is cooled, and is negligible for the cold wall with moderate sweep.

The other similarity parameters (see Section 10) also depend on sweep. For fixed freestream conditions and body shape

$$\bar{\chi} \sim M_n^{2\frac{1}{2}} \sim (\cos \Lambda)^{2\frac{1}{2}}$$

$$\text{and } M_n \alpha_w \sim \cos \Lambda$$

The major effects of sweep, at least for the cold wall, are seen to result from the change in velocity component normal to the leading edge rather than from the cross flow.

Consider, for example, the cold wall case. Incipient separation is given by (Section 10.2.4)

$$(M_n \alpha_w)_i \sim \bar{\chi}^{1/2}$$

from which it follows that, for given freestream conditions,

$$(\alpha_w)_i \sim (\cos \Lambda)^{1/4}$$

Similarly it is readily shown that the length of separation increases with sweep. The effect of the cross flow in increasing $(S_w)_{eq}$ should further increase the characteristic lengths of the interaction.

The effect of sweep on other flow properties is more difficult to determine without a greater knowledge of the dependence of the similarity variables on Mach number. However, the limited results available suggest that heat transfer coefficient decreases with sweep approximately as $(\cos \Lambda)$ while the total skin friction coefficient remains relatively constant.

The analysis also highlights the inadequacies of strip theory. In inviscid flow Ma is the only similarity parameter and is the same whether

evaluated normal to the leading edge or parallel to the freestream.

However, in viscous flow the parameter $\bar{\chi}$ also appears, and this cannot be identical in both planes. For example, using simple strip theory in planes parallel to the freestream the incipient separation condition is given by

$$(\alpha_w)_i \sim (\cos \Lambda)^{-3/4}$$

In this case such an approach is clearly unsuitable.

11.3.4 GENERAL THREE DIMENSIONAL BOUNDARY LAYERS

The analysis of flows over infinite swept wings or axisymmetric bodies can be extended to more general three dimensional problems using approximate methods.

The simplest approach is provided by "strip theory", where the flow in particular planes is considered independent of the flow in adjacent planes and transverse flow gradients are neglected. However, the above analysis of flow over an infinite swept wing indicates that the results obtained are strongly dependent on the plane in which the "strips" are taken. It appears that the most appropriate choice is that plane normal to the plane in which pressure gradients are a minimum. Hence for flows that approximate the infinite swept wing condition the strips should be taken normal to the leading edge.* On the other hand,

* The equivalence between the flow over a swept wing and axisymmetric flow given by Leblanc (1971) is erroneous. The plane of equivalence for the swept wing is normal to the leading edge and not, as shown in Figure 1 of their report, parallel to the flow.

for highly swept deltas in hypersonic flow the root area flow is dominant and it is more appropriate to take strips parallel to the centre line. Such an approach has been applied to general three dimensional flows by Mirels and Lewellen (1967) and Boger and Aiello (1970), using modified Cheng theory to describe the boundary layer.

The so-called "axisymmetric analogy" may also be used for general three dimensional flows. (Eichelbrenner and Oudart 1955, Bradshaw 1971). Any chordwise section of the body is treated as an infinite swept wing and the spanwise convergence of the stream is allowed for by the "equivalent" axisymmetric terms involving $\frac{dr_w}{dx}$.

The possibility of significant changes in the flow pattern in three dimensional configurations should not be overlooked. Hence for the delta wing the windward side is well described by two dimensional strip theory, but on the leeward side vortex sheets form above the surface and invalidate a two dimensional approach (e.g. Whitehead 1970).

11.4 FUTURE INVESTIGATIONS

In its present form the method of Klineberg provides a good description of the major features of viscous interaction phenomena, and is superior to other existing methods. Further development of the method is therefore recommended, with emphasis on the following areas:

(i) Improvement of the model used to describe the outer inviscid flow. The results of such an investigation should be generally applicable to other methods of viscous layer analysis.

(ii) Analytical study of normal pressure gradient effects and the related supersonic nature of the outer part of the viscous layer.

Accurately formulated finite difference methods offer a most suitable means of determining the importance of these effects. Two layer models

could also be investigated.

(iii) Application of the method as formulated herein to three dimensional flows. Such an investigation should prove valuable in relating the two dimensional results to more general flows. It would also be interesting to compare the predictions of the two dimensional analogy with experiment.

12. CONCLUSIONS

The integral method of Klineberg has been extended to allow the study of viscous interaction phenomena over a continuous range of wall cooling ratio. Suitable normalization of the boundary layer profile quantities obviates the need to recalculate complex polynomial functions at each value of wall cooling ratio. Furthermore, flows with variable wall cooling ratio can be considered.

An extensive theoretical evaluation of the method has been made, thus establishing the validity of the integral approach. It has been shown that it is necessary to "uncouple" the velocity and enthalpy profile parameters and thus to employ the energy integral equation in order to accurately describe non-adiabatic viscous interactions. Otherwise the choice of governing differential equations or profile quantities is not critical. It has also been shown that the properties associated with the stability of the governing differential equations (i.e. "subcritical" and "supercritical" behaviour, the necessity to initiate compressive interaction by means of a "jump", and the appearance of the Crocco-Lees critical point during integration) are mathematical properties of the analytic model and should not be associated with any physical characteristics of the boundary layer.

Comparisons made over a wide class of body shapes show that in general the agreement with experiment is good. Contrary to the conclusions reached by earlier investigators, the comparisons indicate that flows up to quite high Mach numbers are well described by the present Klineberg method. In some regions, however, the agreement with experiment is found to be poor. Consideration of the outer inviscid flow suggests that these discrepancies result primarily from inadequacies in the simple model used to describe the flow properties at the edge of the boundary layer. It has been indi-

cated how improvements to the existing models can be made.

The comparisons also suggest that the method of Klineberg is superior to other methods in predicting the major characteristics of viscous interaction phenomena. Agreement with experiment is significantly better than obtained with the more simple integral methods of Cheng, Sullivan and Stollery, and the method does not show the anomalies in heat transfer rate obtained using the integral method of Nielsen. Existing finite difference methods do not provide significantly better agreement with experiment than the method of Klineberg, and computationally are far less efficient.

A rigorous parametric study has been carried out, thus isolating the significant parameters of viscous interactions. The validity of the analysis is shown by both theoretical and experimental comparisons for incipient and fully separated flows.

The similarity between the governing differential equations of a two dimensional boundary layer with heat transfer and those of a three dimensional boundary layer with zero transverse pressure gradient has also been investigated. Consequently, an extension of the Klineberg method to three dimensional plane and axisymmetric flows has been developed. The three dimensional flows are not limited, as in previous reports, to the adiabatic wall case, but the wall cooling ratio must either be constant or, in the case of axially symmetric flow, vary as the square of local body radius. A very simple analogy between two dimensional flows with heat transfer and three dimensional flows has also been derived. Applications to general three dimensional flows have been indicated.

Normal pressure gradient effects have been discussed, and it is suggested that the large normal pressure gradients experienced in hypersonic flow may not be of importance in determining the major characteristics of the interaction. Methods of allowing for nose bluntness have also been described.

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APPENDIX 1: TABLES OF COEFFICIENTS

1. FUNCTIONAL RELATIONS BETWEEN INTEGRAL PROFILE QUANTITIES

1.1 Velocity Profile Dependent Quantities

The velocity profile quantities can be expressed as polynomial functions of the parameter H , i.e.

$$A(H) = \sum_{k=0}^{k=6} C_k H^k$$

where A is the profile quantity considered. The coefficients C_k are given in Table A1.

1.2 Total Enthalpy Profile Dependent Quantities

The total enthalpy profile quantities can be expressed as polynomial functions of the parameter b , i.e.

$$A(b) = \sum_{k=0}^{k=5} C_k b^k$$

where A is the profile quantity considered. The coefficients C_k are given in Table A1.

1.3 Quantities Depending on Both Velocity and Total Enthalpy Profiles

The profile quantity T can be expressed as a polynomial function of the two parameters H and b , i.e.

$$T(H, b) = \sum_{k=0}^{k=5} \tau_k(b) H^k$$

where $\tau_k(b) = \sum_{j=0}^{j=5} C_j b^j$. The coefficients C_j are given in Table A1.

$\frac{\partial T}{\partial H}$ and $\frac{\partial T}{\partial b}$ can be found by direct differentiation of the polynomial expression for $T(H, b)$.

1.4 Range of Validity

The polynomial coefficients given in Table A1 are valid in the range

$$0.10 \leq H \leq 0.45$$

$$0.05 \leq b \leq 0.55$$

TABLE A1 Polynomial Coefficients for Profile Quantities

A	C ₀	C ₁	C ₂	C ₃	C ₄	C ₅	C ₆
Functions of Parameter H							
J	-2.69029E-02	3.28382E+00	-2.44383E+01	1.45692E+02	-4.46132E+02	6.96279E+02	-4.33666E+02
Z	-4.62537E-01	2.37071E+01	-2.56365E+02	1.58045E+03	-5.13908E+03	8.47436E+03	-5.44929E+03
R	6.27139E+00	-6.32363E+01	4.06480E+02	-1.65901E+03	4.00820E+03	-5.13905E+03	2.70259E+03
P	-5.99772E-01	8.03797E-01	1.54370E+01	-2.97149E+01	-2.22571E+00	3.78547E+01	0.E+00
α	-8.69375E-01	3.11317E+01	-3.66793E+02	2.25117E+03	-7.34057E+03	1.21178E+04	-7.86098E+03
dJ/dH	1.51751E+00	-7.94635E+00	7.51109E+01	-2.68400E+02	4.68822E+02	-3.20584E+02	0.E+00
dα/dH	-1.21455E+01	4.36250E+02	-5.68470E+03	3.76888E+04	-1.33998E+05	2.42718E+05	-1.74007E+05
Functions of Parameter b							
σ	8.22871E+00	-6.59303E+01	3.10672E+02	-8.19937E+02	1.11094E+03	-5.98933E+02	-
dσ/db	-7.10897E+01	7.69209E+02	-3.82623E+03	9.95261E+03	-1.30537E+04	6.80598E+03	-
τ ₀	8.65704E-01	-2.39401E+01	1.98505E+02	-6.81611E+02	1.04910E+03	-5.99044E+02	-
τ ₁	1.84021E+01	-1.66055E+02	-4.51759E+02	4.85173E+03	-1.05336E+04	7.14417E+03	-
τ ₂	7.04240E+01	-2.63002E+02	1.04797E+04	-5.87911E+04	1.16040E+05	-7.68410E+04	-
τ ₃	-5.88181E+02	4.06865E+03	-5.85306E+04	2.90091E+05	-5.55746E+05	3.65010E+05	-
τ ₄	1.47254E+03	-1.08777E+04	1.32368E+05	-6.34553E+05	1.20808E+06	-7.93550E+05	-
τ ₅	-1.25964E+03	9.31064E+03	-1.07102E+05	5.11719E+05	-9.78492E+05	6.45847E+05	-

2. ASYMPTOTIC EXPANSIONS FOR UNDISTURBED FLAT PLATE FLOW

2.1 Strong Interaction Regime

The coefficients of the asymptotic expansions [equation (39)] are functions of the wall cooling ratio S_w , and can be represented by the polynomial function

$$A(S_w) = \sum_{k=0}^{k=3} C_k S_w^k$$

where A is the coefficient considered and the summation coefficients C_k are

A	C_0	C_1	C_2	C_3
P_0	0.50798	0.35000	-0.01797	-
P_1	1.48735	-1.81771	-3.12003	-5.00773
P_α	1.53165	-0.01548	1.19978	-
δ_0	1.18196	0.25693	-0.28668	-
δ_1	0.93945	-0.73468	-1.16796	-1.94352
δ_α	0.94776	0.03070	0.53276	-
H_0	0.42436	0.01941	-0.01193	-
H_1	-0.05441	0.03756	0.03545	- *
H_α	-0.02983	0.01064	0.02345	-
b_0	0.54051	0.00686	-0.05675	-
b_1	-0.09425	0.23683	0.20710	- *
b_α	-0.03955	0.09946	0.11161	-

The polynomial coefficients given above are valid in the range $-1 \leq S_w \leq 0$.

* The coefficients for H_1 , b_1 are more accurate than those given in Georgeff (1972b).

2.2 Weak Interaction Regime

The coefficients of the asymptotic expansions [equation (41)] are given by

$$\begin{aligned} m_1 &= \frac{-(\gamma-1)(1+m_\infty)}{4M_\infty\sqrt{M_\infty^2-1}} \left(m_{11} + \frac{1+m_\infty}{m_\infty} m_{12} \right) \\ m_2 &= \left(\frac{\sqrt{M_\infty^2-1}}{M_\infty^3} m_{21} \right) m_1 + \left(\frac{m_\infty}{1+m_\infty} - \frac{1}{2(M_\infty^2-1)} \right) m_1^2 \\ \delta_1 &= (\delta_{11} - K_1)m_1 \\ H_1 &= H_{11}m_1 \\ b_1 &= b_{11}m_1 \end{aligned}$$

where
$$K_1 = \frac{3\gamma-1}{\gamma-1} \frac{m_\infty}{1+m_\infty} \quad \& \quad m_\infty = \frac{\gamma-1}{2} M_\infty^2$$

The coefficients are all of the form

$$A(S_w) = C_o + C_1 S_w$$

where A is the coefficient considered and the summation coefficients C_o, C_1 are

A	C_o	C_1
m_{11}	0.66208	0.0
m_{12}	1.72403	1.72675
m_{21}	1.61275	0.0
δ_o	1.72403	0.0
δ_{11}	2.12586	1.37934
H_o	0.38404	0.0
H_{11}	-0.27132	-0.22554
b_o	0.47441	0.0
b_{11}	0.03587	-0.15416

The coefficients C_o, C_1 are valid for all S_w .

APPENDIX 2: SOLUTION AT DOWNSTREAM INFINITY

In order to integrate in the upstream direction away from the Blasius point it is necessary to examine the behaviour of the neighbouring solutions. The following analysis follows closely the work of Ko and Kubota (1968). Although the hypersonic approximation ($M_\infty \rightarrow \infty$) has been used wherever applicable, the analysis can be performed without this approximation, the behaviour of the solution remaining qualitatively unchanged.

The following form of the governing differential equations can be obtained:

$$\left. \begin{aligned} \frac{dM}{d\bar{\delta}} &= -\frac{1-M}{\bar{\delta}} \frac{C_1 + D_1 M \bar{\delta}}{C_4 + D_4 M \bar{\delta}} \\ \frac{dH}{d\bar{\delta}} &= \frac{1}{\bar{\delta}} \frac{C_2 + D_2 M \bar{\delta}}{C_4 + D_4 M \bar{\delta}} \\ \frac{db}{d\bar{\delta}} &= \frac{1}{\bar{\delta}} \frac{C_3 + D_3 M \bar{\delta}}{C_4 + D_4 M \bar{\delta}} \end{aligned} \right\} \quad (A1)$$

where

$$\left. \begin{aligned} \bar{\delta} &= \frac{4}{(\gamma-1)^2} \frac{Re \delta_i^*}{M_\infty^3 C} \\ M &= 1 - \frac{M_e}{M_\infty} \\ H &= H - H_B \\ b &= b - b_B \end{aligned} \right\} \quad (A2)$$

and

$$\left. \begin{aligned} C_1 &= A_2 F' - A_4 F - \frac{S_w \dot{E}}{\dot{T}^*} (A_2 T^{*'} + A_3 Q - A_4 T^*) \\ C_2 &= -A_2 f - A_7 F - \frac{S_w \dot{E}}{\dot{T}^*} (A_6 Q - A_2 T^* - A_7 T^*) \\ C_3 &= \frac{1}{\dot{T}^*} (B_4 f + B_6 F + B_8 F') \\ C_4 &= A_7 F' + A_4 f - \frac{S_w \dot{E}}{\dot{T}^*} (A_7 T^{*'} + A_4 T^* - A_8 Q) \\ D_1 &= A_3, \quad D_2 = A_6 \\ D_3 &= -B_2, \quad D_4 = -A_8 \end{aligned} \right\} \quad (A3)$$

and where the superscripts " ' " and " • " represent partial differentiation with respect to H and b , respectively.

For M , H , and b to be bounded as $\bar{\delta}$ approaches infinity (Blasius point) the following relations must hold

$$\left. \begin{aligned} H_B R_B &= P_B J_B \\ Q_B J_B &= T_B R_B \end{aligned} \right\} \quad (A4)$$

and

$$M \approx \frac{F_B P_B}{H_B} \frac{1}{\bar{\delta}} \quad (A5)$$

In order to investigate the neighbouring solutions let

$$M \bar{\delta} = - \frac{(C_2)_B}{(D_2)_B} + \zeta(\bar{\delta}) \quad (A6)$$

where $\zeta \ll 1$.

Thus a linearized approximation for equations (A1) is obtained:

$$\left. \begin{aligned} \frac{d\zeta}{d\bar{\delta}} + \alpha_1 \zeta + \beta_1 H + \gamma_1 b &= - \frac{(C_2)_B}{(D_2)_B} \frac{1}{\bar{\delta}} \\ \frac{dH}{d\bar{\delta}} - \frac{\alpha_2 \zeta}{\bar{\delta}} - \frac{\beta_2 H}{\bar{\delta}} - \frac{\gamma_2 b}{\bar{\delta}} &= 0 \\ \frac{db}{d\bar{\delta}} - \frac{\alpha_3 \zeta}{\bar{\delta}} - \frac{\beta_3 H}{\bar{\delta}} - \frac{\gamma_3 b}{\bar{\delta}} &= 0 \end{aligned} \right\} \quad (A7)$$

where

$$\left. \begin{aligned} \alpha_i &= \frac{(D_i)_B (D_2)_B}{D} \\ \beta_i &= \frac{(C_i')_B (D_2)_B - (D_i')_B (C_2)_B}{D} \\ \gamma_i &= \frac{(\dot{C}_i)_B (D_2)_B - (\dot{D}_i)_B (C_2)_B}{D} \end{aligned} \right\} \quad (A8)$$

$i = 1, 3$

and

$$D = (C_4)_B (D_2)_B - (D_4)_B (C_2)_B$$

The leading terms in the solution for large $\bar{\delta}$ are

$$\left. \begin{aligned} \zeta &= \zeta_0 \frac{(C_2)_B}{(D_2)_B} \frac{1}{\bar{\delta}} + K \bar{\delta}^\sigma \\ H &= H_0 \frac{(C_2)_B}{(D_2)_B} \frac{1}{\bar{\delta}} + H_1 K \bar{\delta}^\sigma \\ b &= b_0 \frac{(C_2)_B}{(D_2)_B} \frac{1}{\bar{\delta}} + b_1 K \bar{\delta}^\sigma \end{aligned} \right\} \quad (A9)$$

where ζ_0 , H_0 , b_0 , H_1 , b_1 and σ are functions of α_i , β_i and γ_i ($i = 1, 3$) and K is a constant.

The first terms of these expressions correspond to the higher order terms in the weak interaction expansion.

The following relationships between neighbouring solutions must therefore hold

$$\left. \begin{aligned} \frac{H - H_{WI}}{M - M_{WI}} &\approx -H_1 \\ \frac{b - b_{WI}}{M - M_{WI}} &\approx -b_1 \end{aligned} \right\} \quad (A10)$$

where the subscript WI denotes weak interaction values and

$$\left. \begin{aligned} H_1 &= - \frac{\alpha_1 \gamma_2 - \alpha_2 \gamma_1}{\beta_1 \gamma_2 - \beta_2 \gamma_1 + \sigma \gamma_1} \\ b_1 &= \frac{\alpha_1 \beta_2 - \alpha_2 \beta_1 - \alpha_1 \sigma}{\beta_1 \gamma_2 - \beta_2 \gamma_1 + \sigma \gamma_1} \\ \sigma &= \frac{1}{2\alpha_1} \left[\alpha_1 \gamma_3 - \alpha_3 \gamma_1 + \alpha_1 \beta_2 - \alpha_2 \beta_1 + \{ (\alpha_1 \gamma_3 - \alpha_3 \gamma_1 + \alpha_1 \beta_2 - \alpha_2 \beta_1)^2 - \right. \\ &\quad \left. - 4\alpha_1 [\alpha_1 (\beta_2 \gamma_3 - \beta_3 \gamma_2) - \alpha_2 (\beta_1 \gamma_3 - \beta_3 \gamma_1) + \right. \\ &\quad \left. + \alpha_3 (\beta_1 \gamma_2 - \beta_2 \gamma_1)] \}^{1/2} \right] \end{aligned} \right\} \quad (A11)$$

Note that for adiabatic flow the expression for σ reduces to

$$\sigma_{ad} = (\alpha_1 \beta_2 - \alpha_2 \beta_1) / \alpha_1$$

This does not agree with the result of Ko and Kubota viz.:

$$\sigma_{ad} = (\beta_2 + \alpha_1 \beta_2 - \alpha_2 \beta_1) / \alpha_1$$

Solutions in the relaxation surface can therefore be determined by selecting initial values for integration in the upstream direction such that

$$\left. \begin{aligned} M_e &= M_{eWI} + \Delta M_e \\ R_{e\delta_i^*} &= \left\{ R_{e\delta_i^*} \right\}_{WI} \\ H &= H_{WI} + \Delta H \\ b &= b_{WI} + \Delta b \end{aligned} \right\} \quad (A12)$$

where

$$\left. \begin{aligned} \Delta H &= - \frac{4}{(\gamma-1)^2} \left(\frac{R_{e\delta_i^*}}{M_\infty^3 C} \right)_{WI} H_1 \left(\frac{\Delta M_e}{M_\infty} \right) \\ \Delta b &= - \frac{4}{(\gamma-1)^2} \left(\frac{R_{e\delta_i^*}}{M_\infty^3 C} \right)_{WI} b_1 \left(\frac{\Delta M_e}{M_\infty} \right) \end{aligned} \right\} \quad (A13)$$

Note that in the range $-0.3 \leq S_w \leq 0.0$

$$H_1 \approx b_1 \approx 10^{-2}$$

These approximate values are probably adequate for the purpose of numerical upstream integration.

APPENDIX 3

STABILITY OF THE GOVERNING EQUATIONS

The stability of the governing differential equations can be determined by examining the stability in the linear first approximation (local linearization).

Letting

$$\bar{x} = \frac{x}{x_0}$$
$$\bar{\delta} = \frac{\delta_i^*}{\delta_0}$$
$$\bar{M} = \frac{M_e}{M_\infty}$$

$$\delta_0 = \frac{x_0}{(Re_{x_0}/C)^{\frac{1}{2}}}$$

the governing differential equations can be transposed to give

$$\frac{d\bar{M}}{d\bar{x}} = \frac{\bar{N}_1}{D}$$
$$\frac{d\bar{\delta}}{d\bar{x}} = \frac{\bar{N}_2}{D}$$
$$\frac{dH}{d\bar{x}} = \frac{\bar{N}_3}{D}$$
$$\frac{db}{d\bar{x}} = \frac{\bar{N}_4}{D}$$

$$\begin{matrix}) \\) \\) \\) \\) \\) \\) \\) \\) \\) \\) \end{matrix}$$

(A.14)

where $\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{N}_4$ and D are functions of the parameters $\bar{M}, \bar{\delta}, H$ and b .

Following the approach given in Sansone (1964), the stability of the system can be determined by examining the characteristic exponents, λ , of the linear reduced system corresponding to the set of equations (A.14).

These are given by the solution of the characteristic equation

$$\det / \bar{A} - \lambda I / = 0$$

(A.15)

$$\text{where } \bar{A} = \begin{vmatrix} \left(\frac{\bar{N}_1}{D}\right)_{\bar{M}} & \left(\frac{\bar{N}_1}{D}\right)_{\bar{\delta}} & \left(\frac{\bar{N}_1}{D}\right)_H & \left(\frac{\bar{N}_1}{D}\right)_b \\ \left(\frac{\bar{N}_2}{D}\right)_{\bar{M}} & \left(\frac{\bar{N}_2}{D}\right)_{\bar{\delta}} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \left(\frac{\bar{N}_4}{D}\right)_b \end{vmatrix}$$

is evaluated at the point x_0 and where the subscripts represent partial differentiation.

The vectors associated with the roots λ of the characteristic equation (A.15) provide the relationships between the perturbations $\hat{M}$, $\hat{\delta}$, $\hat{H}$, and $\hat{b}$ to the quantities $\bar{M}$, $\bar{\delta}$, H and b respectively.

In general the solution of equation (A.15) is quite complex. However, by considering the order of magnitude of the elements of the determinant $\bar{A}$ in the various flow regimes equation (A.15) may be simplified.

1. THE WEAK INTERACTION REGIME

(i) Method of Lees and Reeves

An examination of the orders of magnitude of the terms involved gives

$$\lambda_1 \approx \left(\frac{\bar{N}_1}{D}\right)_{\bar{M}}$$

$$\lambda_2 \approx \lambda_3 \approx 0$$

where

$$\left(\frac{\bar{N}_1}{D}\right)_{\bar{M}} = \frac{4B\bar{M}}{(\gamma-1)^2\bar{\delta}\bar{\chi}_0} \frac{A_3}{D_1} \quad (\text{A.16})$$

and

$$D_1 = A_6 \frac{dF}{dH} + A_3 f - A_8 F$$

$$A_1 = 2H + 1 + S_w E$$

$$A_3 = H \frac{dJ}{dH} - J$$

$$A_5 = 3J + 2S_w T^*$$

$$A_6 = A_1 J - A_5 H$$

$$A_8 = A_1 \frac{dJ}{dH} - A_5$$

In this region λ_1 is positive and hence the governing differential equations are unstable. The vector associated with the characteristic root λ_1 gives

$$\frac{\hat{M}}{\bar{M}A_3} = -\frac{\hat{\delta}}{\bar{\delta} A_8} = \frac{\hat{H}}{A_6} \quad (\text{cf. Ko and Kubota 1968}).$$

(ii) Method of Klineberg

$$\lambda_1 \approx \left(\frac{\bar{N}_1}{D}\right)_M \quad (\text{A.17})$$

$$\lambda_2 \approx \lambda_3 \approx \lambda_4 \approx 0$$

where

$$\left(\frac{\bar{N}_1}{D}\right)_{\bar{M}} = \frac{4 B \bar{M}}{(\gamma-1)^2 \bar{\delta} \bar{\chi}_0} \frac{A_3}{D} \frac{\partial T^*}{\partial b}$$

and

$$D = D_1 \frac{\partial T^*}{\partial b} - D_2 \frac{\partial F}{\partial b}$$

$$D_2 = A_6 \frac{\partial T^*}{\partial H} + (A_3 - A_8) T^* .$$

In this region λ_1 is negative and hence the governing differential equations are stable. A "jump" is therefore required and it is not necessary to determine the relationship between the perturbations $\hat{M}$, $\hat{\delta}$, $\hat{H}$ and $\hat{b}$.

(iii) Stollery - Hankey Modification 'a' and Georgeff Modification 'a'

$$\lambda_1, \lambda_2 \approx \pm \left\{ \left(\frac{\bar{N}_3}{\bar{D}} \right)_{\bar{M}} \left(\frac{\bar{N}_1}{\bar{D}} \right)_H \right\}^{\frac{1}{2}} \quad (\text{A.18})$$

$$\lambda_3 \approx 0$$

where

$$\left(\frac{\bar{N}_3}{\bar{D}} \right)_{\bar{M}} = \frac{4B}{(\gamma-1)^2 \bar{\delta} \bar{\chi}_0} \frac{A_6}{D_1}$$

$$\left(\frac{\bar{N}_1}{\bar{D}} \right)_H = \frac{B}{(\bar{\delta})^2} \left(\frac{A_2}{A_6} \right)_H$$

$\left(\frac{\bar{N}_3}{\bar{D}} \right)_{\bar{M}}$ and $\left(\frac{\bar{N}_1}{\bar{D}} \right)_H$ are both of the same sign and hence the roots

λ_1 and λ_2 are real. The positive root indicates that the equations are unstable provided the perturbations are consistent with the development of the boundary layer in a pressure field, i.e. $\hat{M}$ and $\hat{H}$ must be of the same sign. The relationship between the perturbations is

$$\frac{\hat{M}}{\bar{\epsilon}} = - \frac{\hat{\delta}}{\bar{\delta} A_8} = \frac{\hat{H}}{A_6}$$

where $\bar{\epsilon}$ is a small positive number*.

(iv) Stollery-Hankey Modification 'b' and Georgeff Modification 'b'

$$\lambda_1, \lambda_2 \approx \pm \left\{ \left(\frac{\bar{N}_3}{D} \right)_{\bar{M}} \left(\frac{\bar{N}_1}{D} \right)_H \right\}^{\frac{1}{2}} \quad (\text{A.19})$$

$$\lambda_3 \approx \lambda_4 \approx 0$$

where

$$\left(\frac{\bar{N}_3}{D} \right)_{\bar{M}} = \frac{4B}{(\gamma-1)^2 \bar{\delta} \bar{\chi}_0} \frac{A_6}{D} \frac{\partial T^*}{\partial b}$$

$$\left(\frac{\bar{N}_1}{D} \right)_H = \frac{B}{(\bar{\delta})^2} \left(\frac{A_2}{A_6} \right)_H.$$

As before, $\left(\frac{\bar{N}_3}{D} \right)_{\bar{M}}$ and $\left(\frac{\bar{N}_1}{D} \right)_H$ are both of the same sign and hence the roots λ_1 and λ_2 are real. The equations are unstable provided the relationship between the perturbations is

$$\frac{\hat{M}}{\bar{\epsilon}} = -\frac{\hat{\delta}}{A_8} = \frac{\hat{H}}{A_6} = \frac{\hat{b} \frac{\partial T^*}{\partial b}}{\left\{ A_6 \frac{\partial T^*}{\partial H} + (A_3 - A_8) T^* \right\}}$$

where $\bar{\epsilon}$ is a small positive number.

The largest characteristic root and the resulting stability in various flow regimes is indicated in the following Table. If the largest root is negative, and there exists another root of the same order, this is also indicated.

* It is not necessary to determine $\bar{\epsilon}$ exactly. As a numerical procedure is used to integrate the governing differential equations the positive root will dominate.

$\bar{X}_0$	0.01	0.1	3.0	10.0
Method				
Lees & Reeves	$\left\{ \frac{\bar{N}_1}{D} \right\}_{\bar{M}}$ $\left[+ \frac{10}{\bar{X}_0} \right]$	$\left\{ \frac{\bar{N}_1}{D} \right\}_{\bar{M}}$ $\left[+ \frac{10}{\bar{X}_0} \right]$	$\frac{\left\{ \frac{\bar{N}_3}{D} \right\}_H^2 + 4 \left\{ \frac{\bar{N}_1}{D} \right\}_H \left\{ \frac{\bar{N}_3}{D} \right\}_{\bar{M}}}{2}$ (6 ± 8)	$\left\{ \frac{\bar{N}_3}{D} \right\}_H$ (+8)
Klineberg	$\left\{ \frac{\bar{N}_1}{D} \right\}_{\bar{M}}$ $\left[- \frac{5}{\bar{X}_0} \right]$	$\left\{ \frac{\bar{N}_1}{D} \right\}_{\bar{M}}$ $\left[- \frac{5}{\bar{X}_0} \right]$	$\left\{ \frac{\bar{N}_3}{D} \right\}_H$ (-6)	$\left\{ \frac{\bar{N}_3}{D} \right\}_H$ (-5)
Stollery & Hankey 'a'	$\pm \left\{ \frac{\bar{N}_1}{D} \right\}_H \left\{ \frac{\bar{N}_3}{D} \right\}_{\bar{M}}^{\frac{1}{2}}$ $\left[\pm \frac{5}{\bar{X}_0^{\frac{1}{2}}} \right]$	$\pm \left\{ \frac{\bar{N}_1}{D} \right\}_H \left\{ \frac{\bar{N}_3}{D} \right\}_{\bar{M}}^{\frac{1}{2}}$ $\left[\pm \frac{5}{\bar{X}_0^{\frac{1}{2}}} \right]$	$\frac{\left\{ \frac{\bar{N}_3}{D} \right\}_H^2 + 4 \left\{ \frac{\bar{N}_1}{D} \right\}_H \left\{ \frac{\bar{N}_3}{D} \right\}_{\bar{M}}}{2}$ (1 ± 3)	$\left\{ \frac{\bar{N}_3}{D} \right\}_H$ (+2)
Stollery & Hankey 'b'	$\pm \left\{ \frac{\bar{N}_1}{D} \right\}_H \left\{ \frac{\bar{N}_3}{D} \right\}_{\bar{M}}^{\frac{1}{2}}$ $\left[\pm \frac{30}{\bar{X}_0^{\frac{1}{2}}} \right]$	$\pm \left\{ \frac{\bar{N}_1}{D} \right\}_H \left\{ \frac{\bar{N}_3}{D} \right\}_{\bar{M}}^{\frac{1}{2}}$ $\left[\pm \frac{30}{\bar{X}_0^{\frac{1}{2}}} \right]$	$\left\{ \frac{\bar{N}_3}{D} \right\}_H$ (+1000)	$\left\{ \frac{\bar{N}_3}{D} \right\}_H$ (-100)

2. THE CRITICAL BOUNDARY

The Crocco-Lees critical point is defined as that point where the denominator D vanishes.

In order to obtain a bounded solution it is necessary that the numerators, $\bar{N}_1$, $\bar{N}_2$, $\bar{N}_3$ and $\bar{N}_4$ vanish simultaneously at this point.

Consider the case where the characteristic roots are given by

$$\lambda_1 = \left(\frac{\bar{N}_1}{D} \right)_{\bar{M}} \quad (\text{A.20})$$

$$\lambda_2 \approx \lambda_3 \approx \lambda_4 \approx 0 .$$

Differentiating equation (A.20) by parts gives

$$\lambda_1 = \frac{1}{D} \left(\frac{d\bar{N}_1}{d\bar{M}} - \frac{\bar{N}_1}{D} \frac{dD}{d\bar{M}} \right)_0 .$$

As all terms are evaluated for the zero solution the value of the bracketed expression will remain approximately the same as we pass through the Crocco-Lees critical point, and provided

$$\frac{d\bar{N}_1}{d\bar{M}} - \frac{\bar{N}_1}{D} \frac{dD}{d\bar{M}} \neq 0 \quad \text{at } D = 0$$

will not undergo a change of sign. Hence the stability of the governing differential equations depends on the sign of D in the vicinity of the Crocco-Lees critical point, and on passing through the critical point, the governing differential equations will change from being stable to being unstable, or vice versa. As expected, λ_1 becomes very large in this region.

The same considerations apply to the other important case

$$\lambda_1 = \left(\frac{\bar{N}_3}{D} \right)_H$$

$$\lambda_2 \approx \lambda_3 \approx \lambda_4 \approx 0 .$$

3. METHOD OF CHENG

In the physical plane the governing differential equations can be transposed to give

$$\begin{aligned} \frac{d\bar{\delta}}{d\bar{x}} &= \frac{M_{\infty}^{\theta_e}}{\bar{\chi}_0} \\ \frac{d\bar{p}}{d\bar{x}} &= \left(\frac{\gamma-1}{2} \right)^2 \left(\frac{F^2 P}{H} \right)_B \frac{1}{\bar{\delta}^2} - \frac{\bar{p}}{\bar{\delta}} \frac{M_{\infty}^{\theta_e}}{\bar{\chi}_0} \end{aligned} \quad \begin{matrix}) \\) \\) \\) \\) \\) \\) \\) \end{matrix} \quad (A.21)$$

where

$$\bar{\delta} = \frac{\delta^*}{M_{\infty}^2 \delta_0}$$

and

$$\bar{p} = \frac{p_e}{p_{\infty}}$$

The characteristic roots are given by

$$\lambda^2 + \lambda \left\{ \bar{p} \frac{d\theta_e}{d\bar{p}} + \theta_e \right\} \frac{M_{\infty}}{\bar{\delta} \bar{\chi}_0} + \left\{ \left(\frac{\gamma-1}{2} \right)^2 \left(\frac{F^2 P}{H} \right)_B \frac{\bar{\chi}_0}{M_{\infty} \bar{\delta} \bar{p}} - \theta_e \right\} \bar{p} \frac{d\theta_e}{d\bar{p}} \frac{M_{\infty}^2}{\bar{\delta}^2 \bar{\chi}_0^2} = 0 \quad (A.22)$$

For all physically realistic solutions the characteristic roots are negative. Hence the governing differential equations are stable in all flow regimes.

Author	Governing Equations	Equivalent Dependent Variables	Relations between Profile Quantities	Additional Assumptions
Lees & Reeves	C MI MMI	M_e^*, δ_i^*, H	Cohen-Reshotko similarity Enthalpy & Velocity profiles coupled	-
Klineberg	C MI MMI E	M_e^*, δ_i^*, H, b	Cohen-Reshotko similarity Enthalpy & Velocity profiles uncoupled	-
Stollery & Hankey "a"	C MI MMI	M_e^*, δ_i^*, H	Cohen-Reshotko similarity except $J = KH$ Enthalpy & Velocity profiles coupled	-
Stollery & Hankey "b"	C MI MMI E	M_e^*, δ_i^*, H, b	Cohen-Reshotko similarity except $J = KH$ Enthalpy & Velocity profiles uncoupled	-
Georgeff "a"	C MI MW	M_e^*, δ_i^*, H	Cohen-Reshotko similarity Enthalpy & Velocity profiles coupled	-
Georgeff "b"	C MI MW E	M_e^*, δ_i^*, H, b	Cohen-Reshotko similarity Enthalpy & Velocity profiles uncoupled	*
Cohen & Reshotko	MI MW *	δ_i^*, H	Cohen-Reshotko similarity Enthalpy & Velocity profiles coupled	-

* Note: No coupling equation is used and hence the method can only treat problems with prescribed pressure gradient.

TABLE 1 SUMMARY OF METHODS (continued)

Author	Governing Equations	Equivalent Dependent Variables	Relations between Profile Quantities	Additional Assumptions
Bray, Gadd & Woodger	C MW	M_e^*, δ_i^*	Cohen-Reshotko similarity	$\theta_i = \text{constant}$
Curle	C MI MW	M_e^*, δ_i^*, H	Thwaites, Falkner-Skan & others Enthalpy profile quadratic function of Velocity profile	Complex profile dependencies replaced by linear approximation
Cheng Sullivan Stollery	C MI	M_e^*, δ_i^*	Blasius values	(a) $\frac{\gamma-1}{\gamma+1} \ll 1$ (b) $\frac{\gamma-1}{2} M_e^2 \gg 1$ (c) Local flat-plate similarity $H=H_B, \frac{dH}{dx} = 0$

KEY	Equation	C	-	coupling equation
		MI	-	momentum integral equation
		MMI	-	moment of momentum integral equation
		MW	-	momentum equation at the wall
		E	-	energy equation

TABLE 2 EXPERIMENTAL REFERENCE CONDITIONS

Author	M_∞	S_w	C	$R_e / \text{in} \times 10^{-6}$	$R_e \times 10^{-6}$	$\bar{X}_L^*$	Configuration
Ball & Korkegi (1968)	12.3	-0.44	0.8	0.079	0.276	0.45	Inclined flat plate-wedge 12° + 10°
Bloy (1972)	12.2	-0.78	0.95	0.087	0.522	2.45	Flat plate-wedge 8°, 10°, 12°
Bloy (1972)	"	"	"	"	0.696	0.86 0.40 0.107	Inclined flat plate-expansion corner 5°, 10°, 20° to 0°
Holden (1968)	19.8	-0.95	0.7	0.0087	0.107	19.8	Flat plate-wedge 17.8°, 21.6°, 25.7°
Johnson (1968, 1970)	7.6	-0.52	0.9	0.0183	0.183	0.97	Flat plate-wedge 10°
Lewis (1968)	6.06	0.0 -0.8	1.0 (assumed)	0.0333	0.152	0.57	Flat plate-wedge 10.25°
Mohammadian (1970)	12.2	-0.78	0.95	0.087	0.522	2.45	Concave cubic $y = x^3/150$ (ins) to 28°
Mohammadian (1970)	"	"	"	"	0.870	0.132	Inclined flat plate-convex corner 18°, then $y = .00269x^4 - .0211x^3$ (ins) to 0°

* On inclined surfaces $\bar{X}_L$ is calculated with respect to inviscid conditions behind the leading edge shock wave.

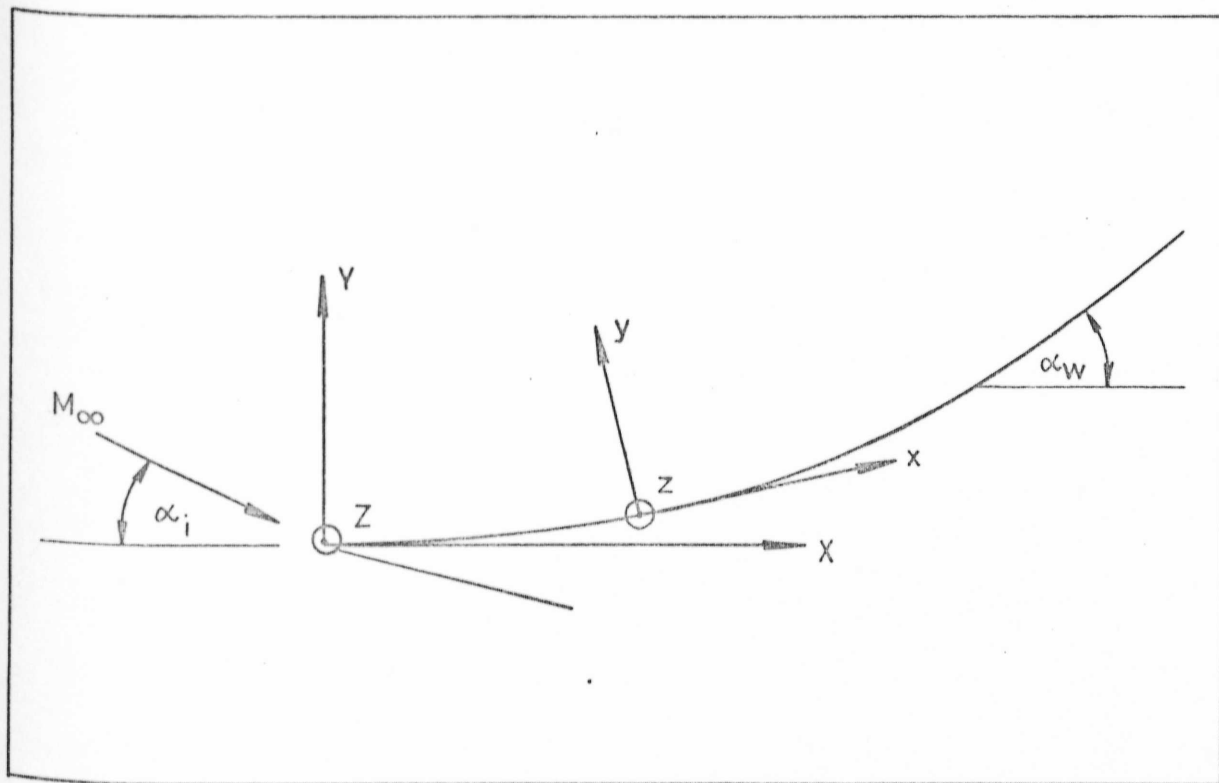


FIG. 1 (a) CO-ORDINATE SYSTEM.

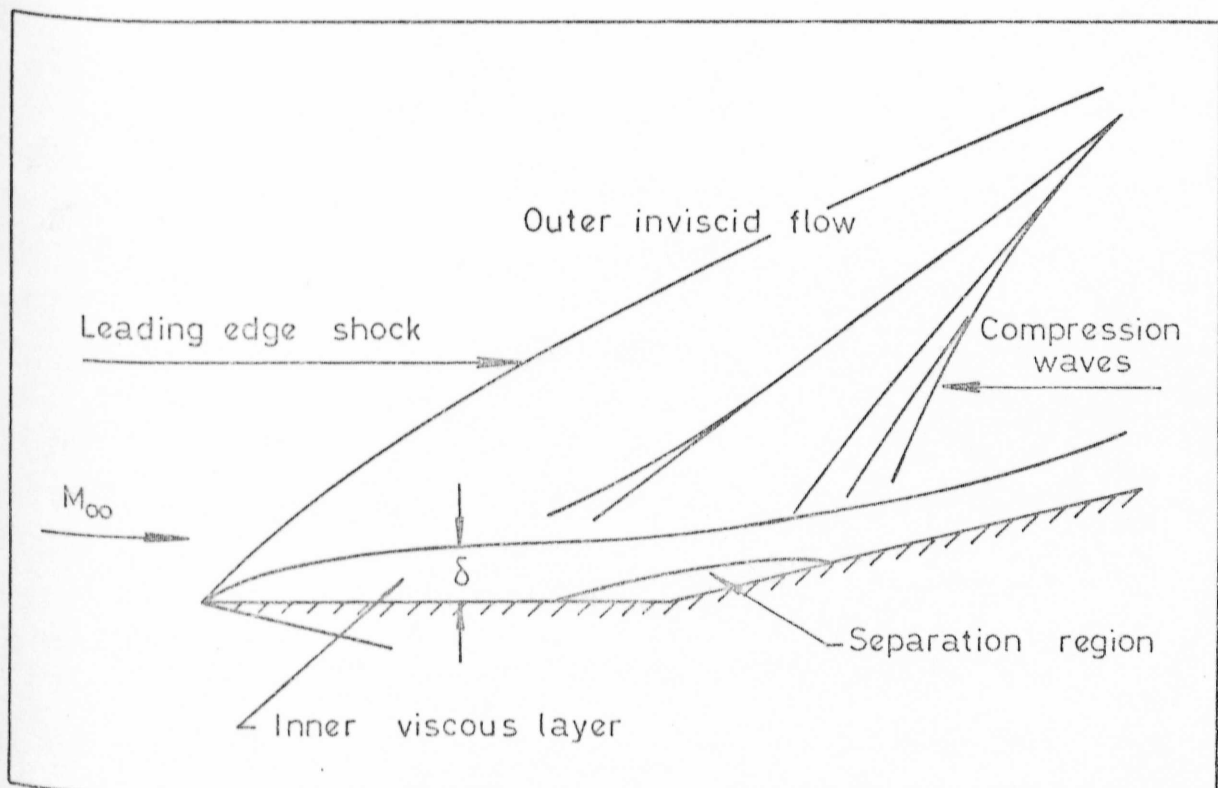


FIG. 1 (b) SCHEMATIC REPRESENTATION OF FLOW.

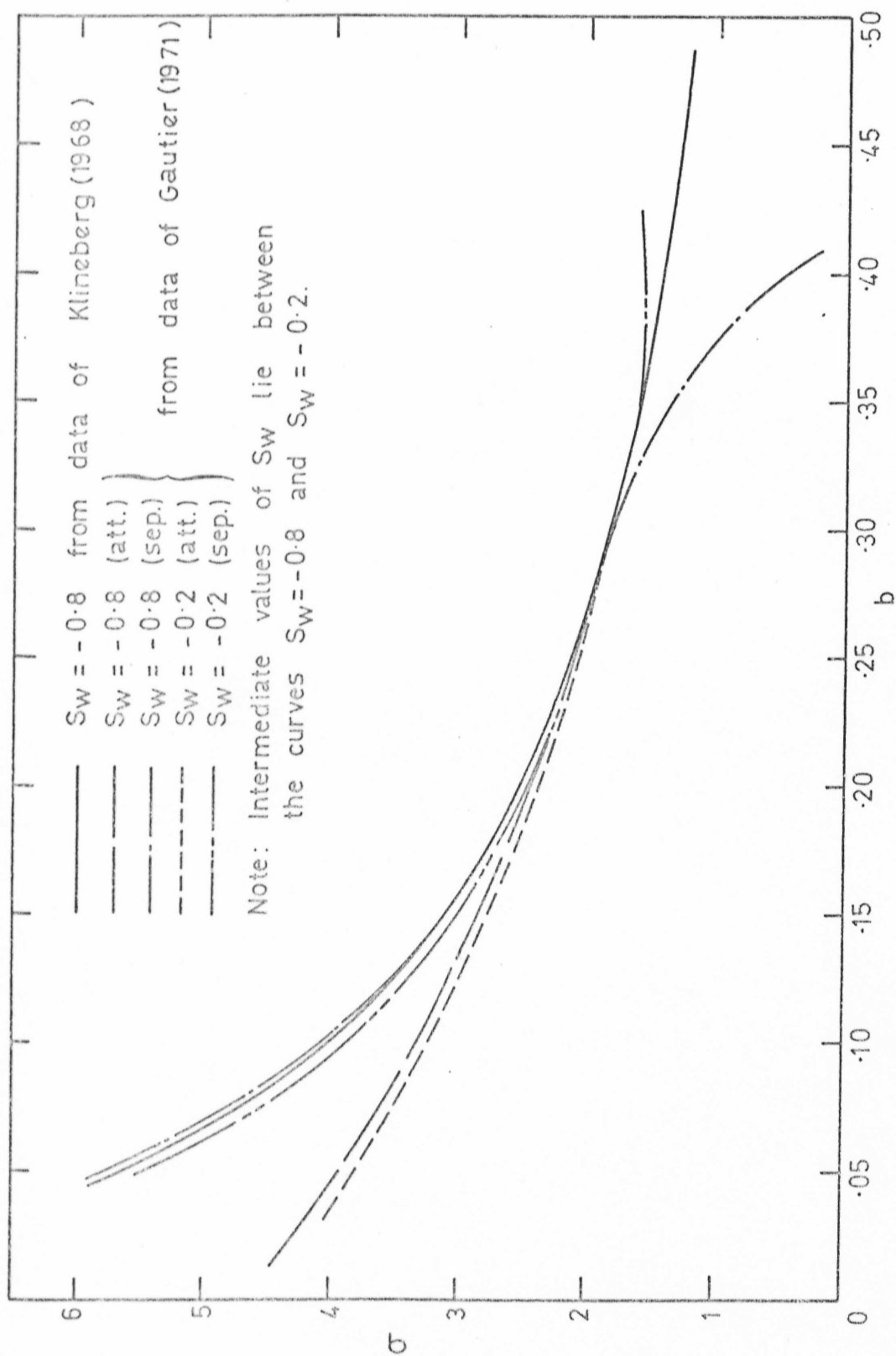


FIG. 2 NORMALISED PROFILE FUNCTION $\sigma(b)$.

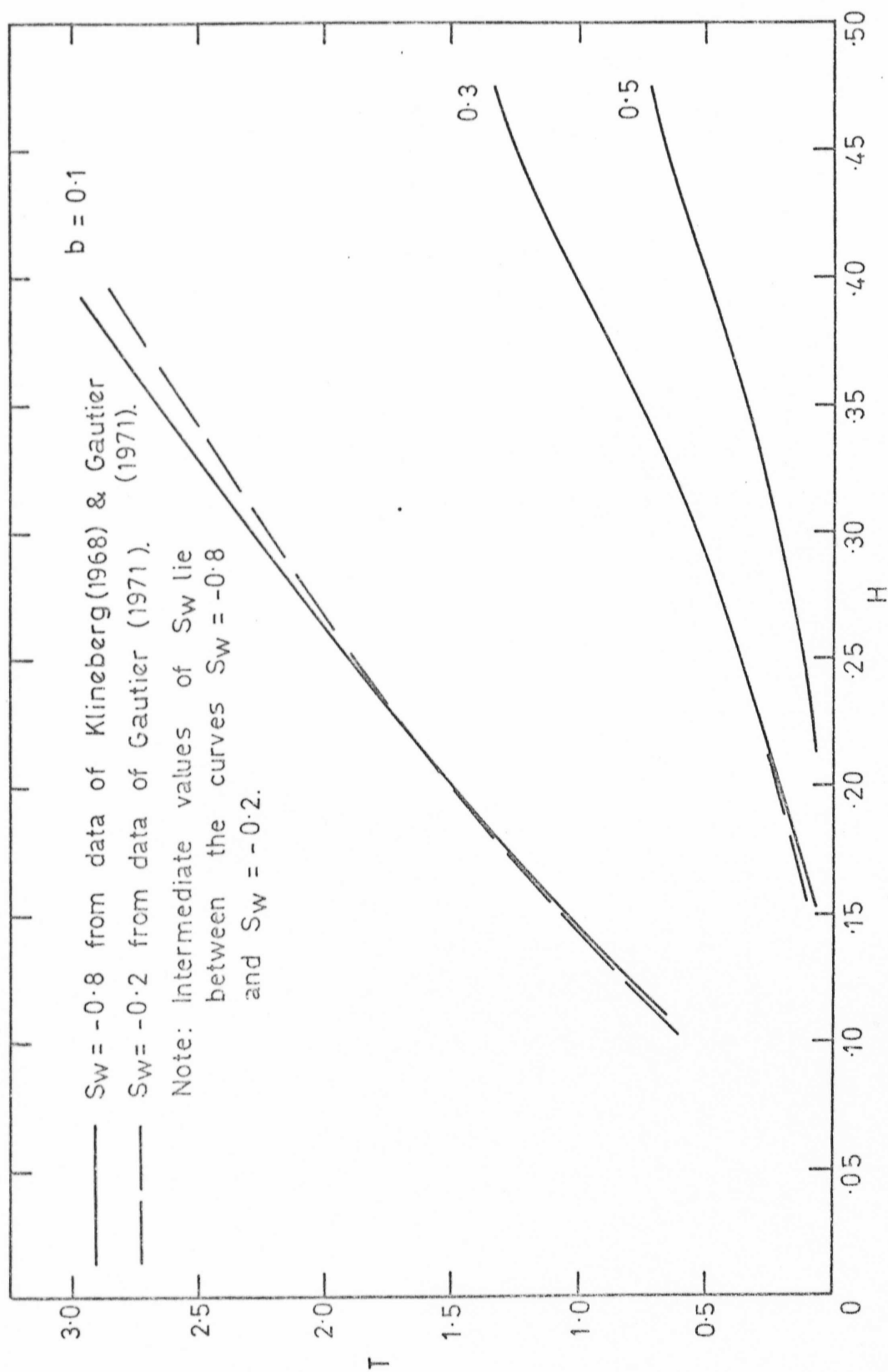


FIG. 3 NORMALISED PROFILE FUNCTION $T(H, b)$

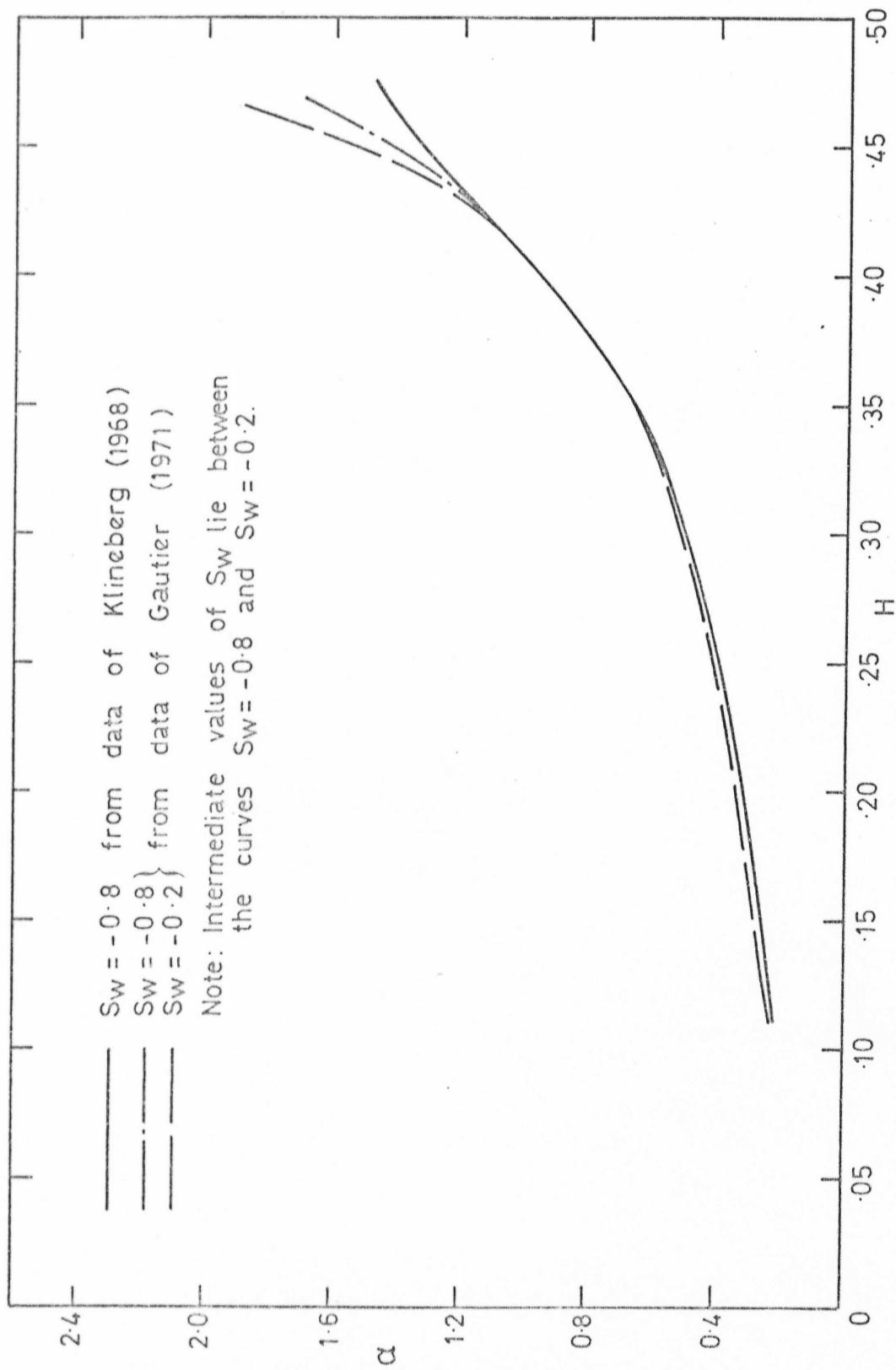


FIG. 4 PROFILE FUNCTION α (H)

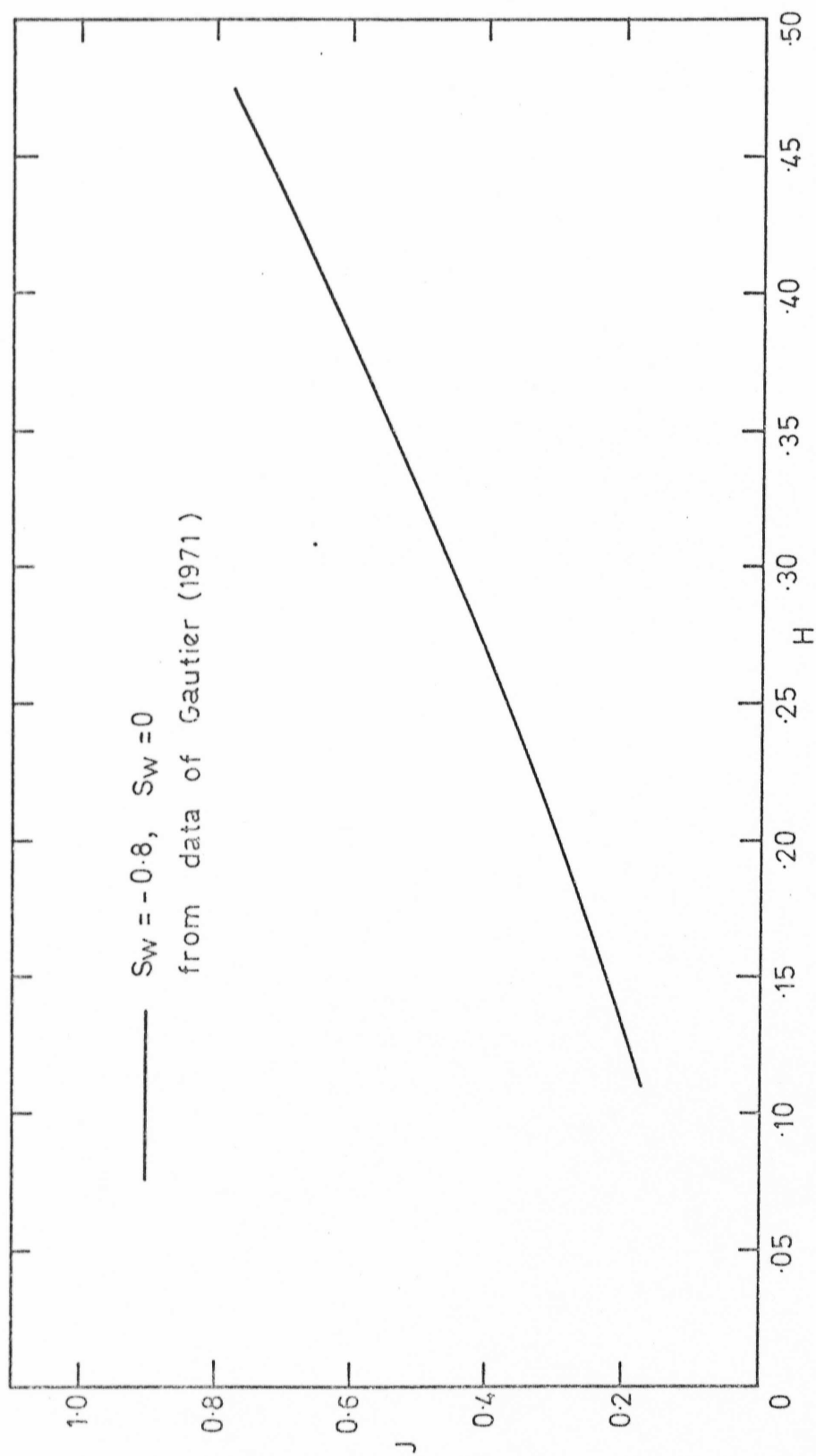


FIG. 5 PROFILE FUNCTION $J(H)$

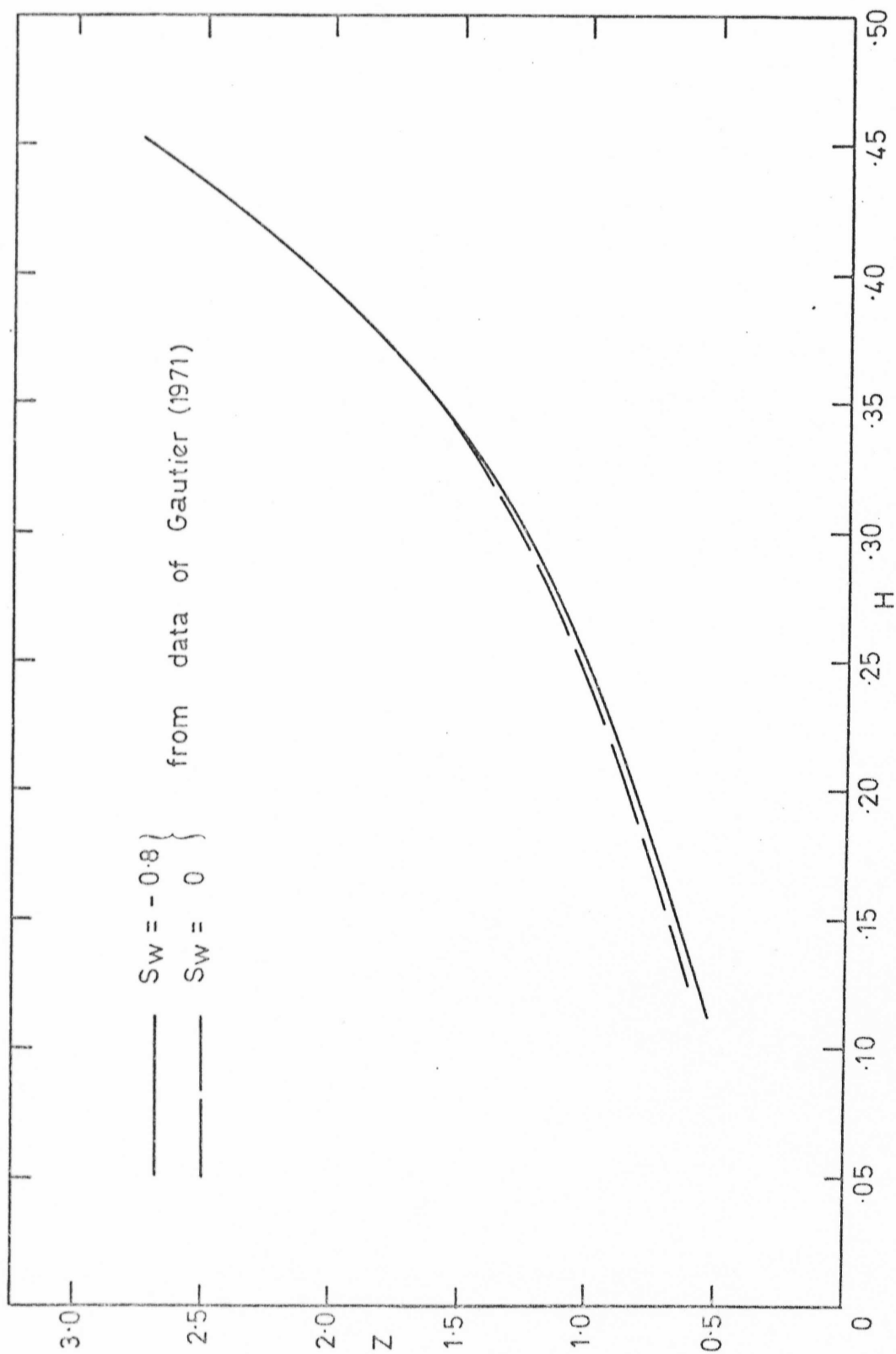


FIG. 6 PROFILE FUNCTION $Z(H)$

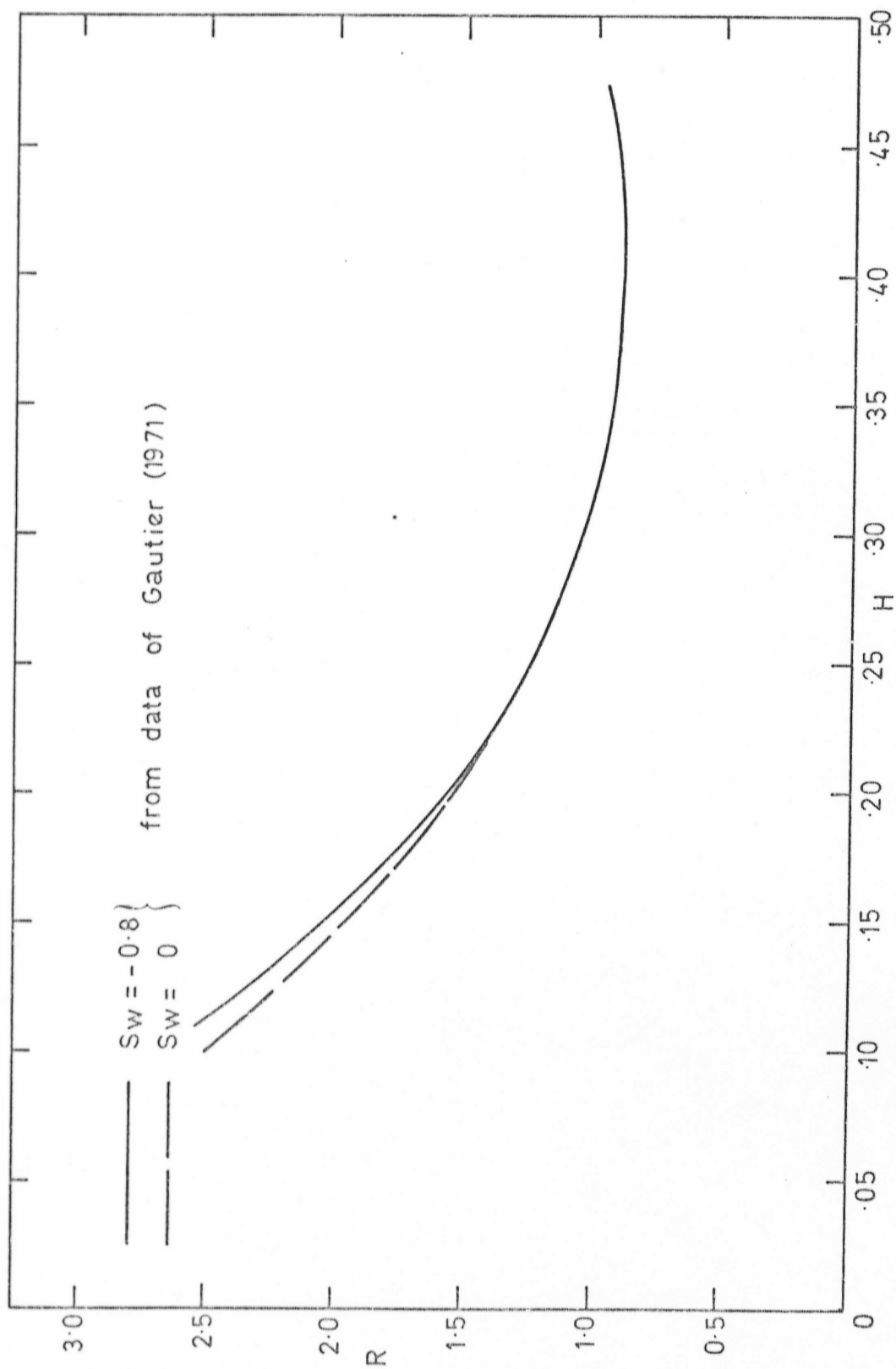


FIG. 7 PROFILE FUNCTION $R(H)$

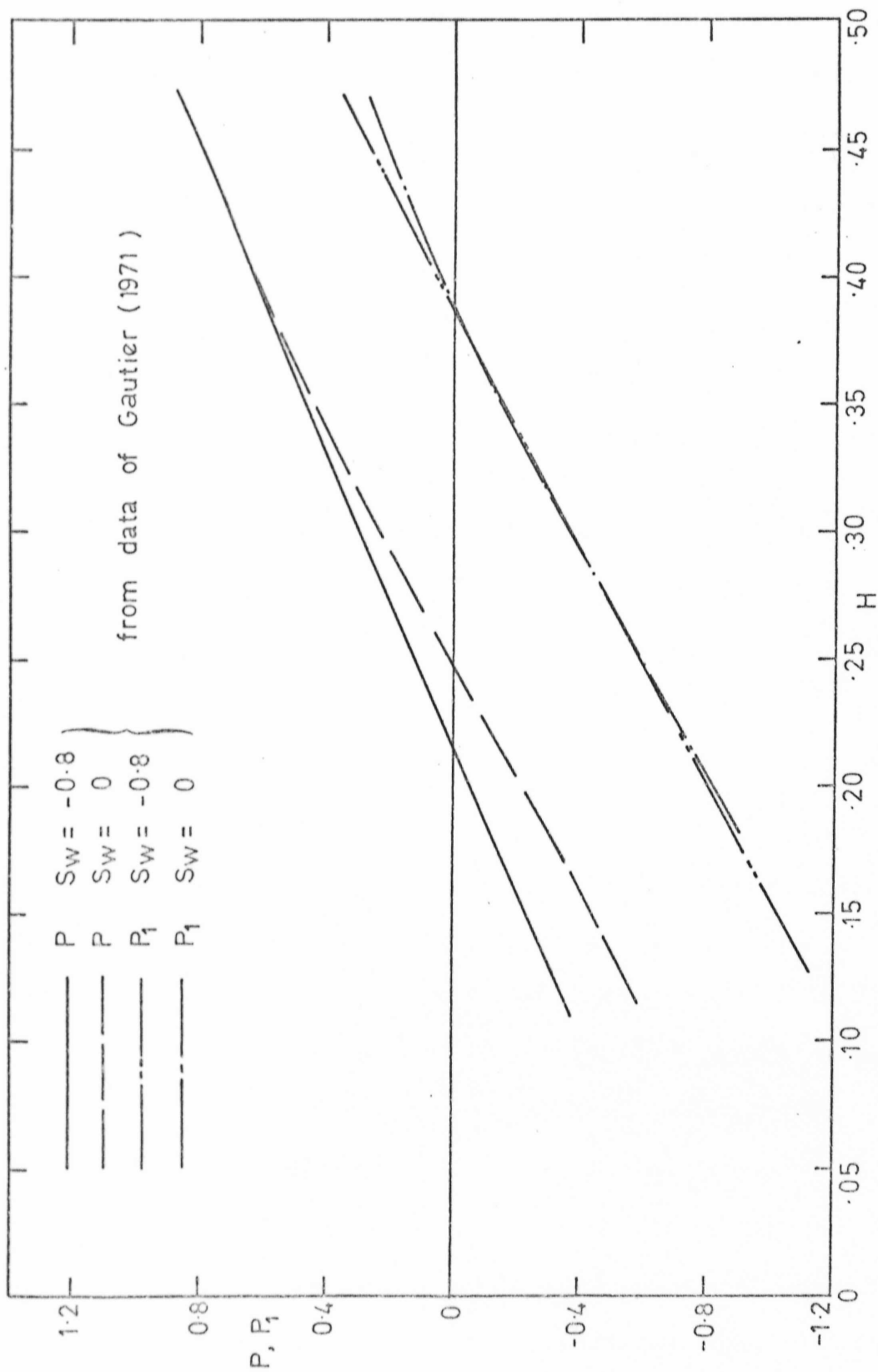
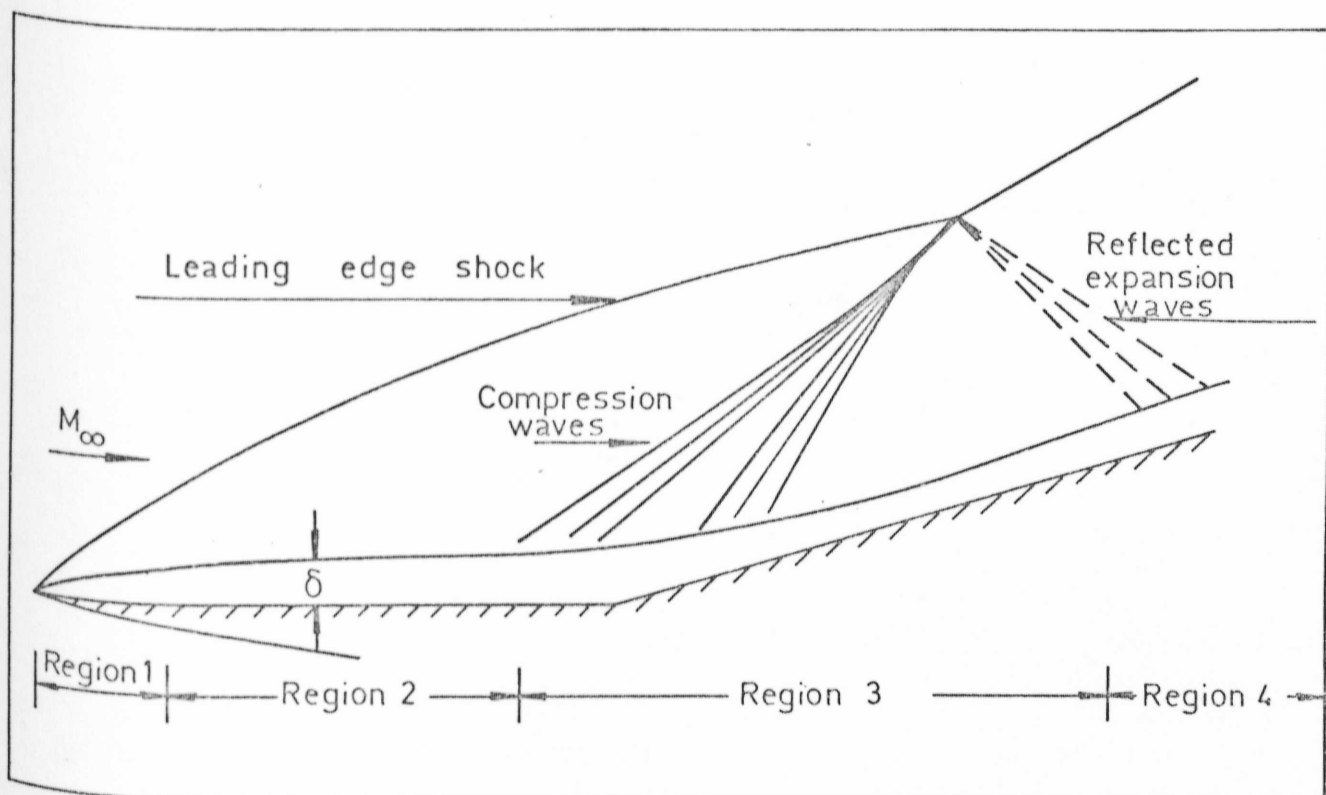
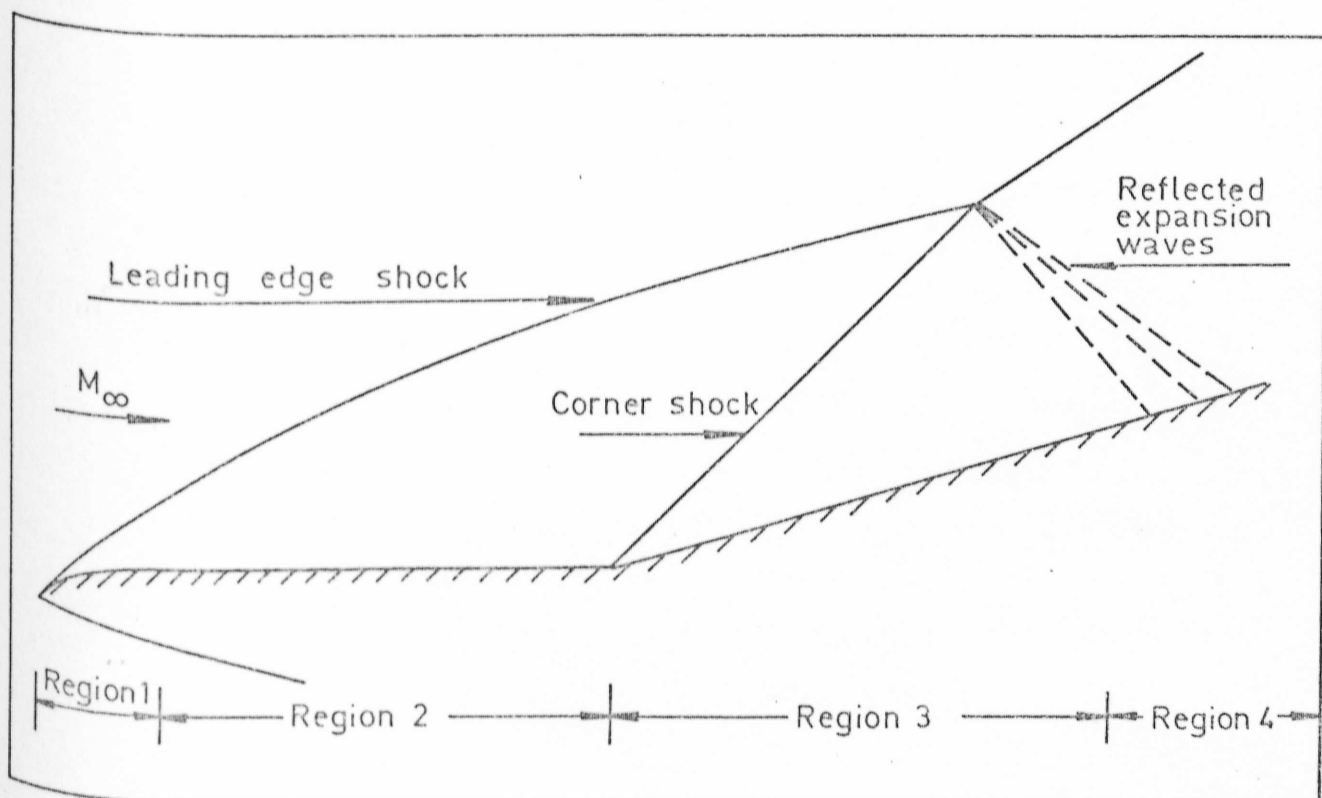


FIG. 8 PROFILE FUNCTIONS $P(H)$ AND $P_1(H)$



9(a) Shock wave boundary layer interaction.



9(b) Inviscid shock wave interaction.

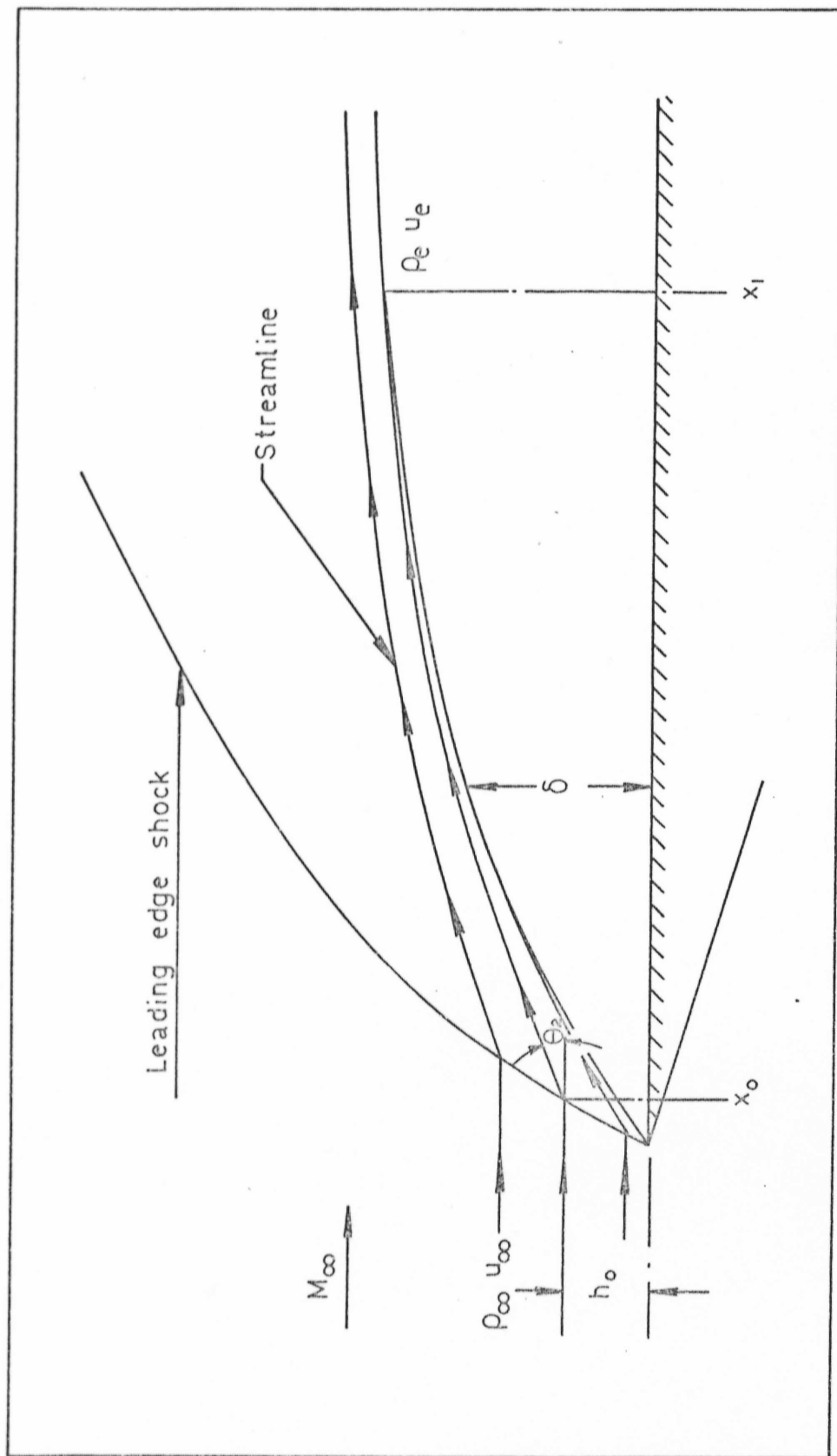


FIG. 10 BOUNDARY LAYER ON FLAT PLATE.

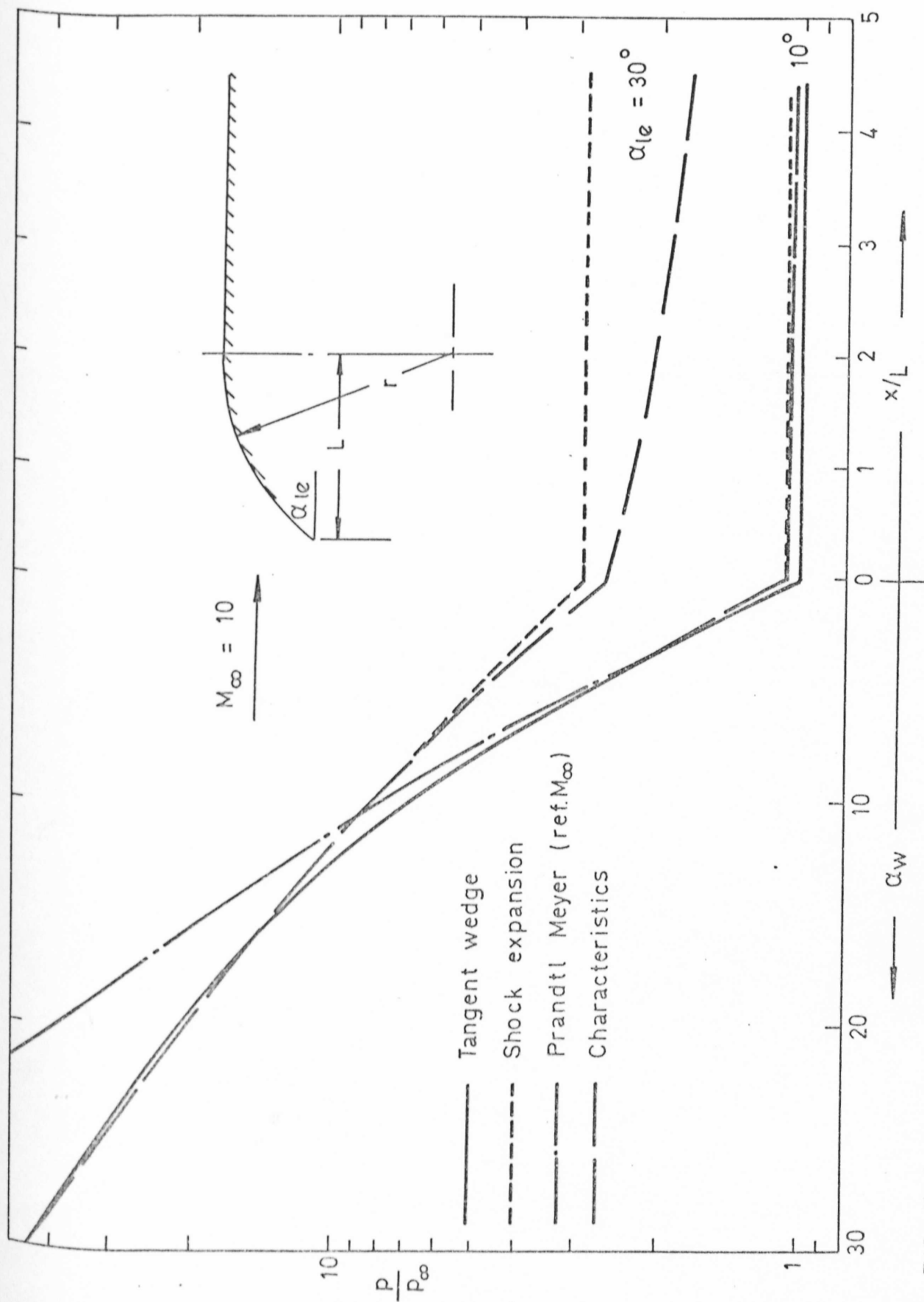


FIG. 11(a) INVISCID PRESSURE DISTRIBUTION ON SURFACE SIMULATING STREAMLINES IN FIG. 10.

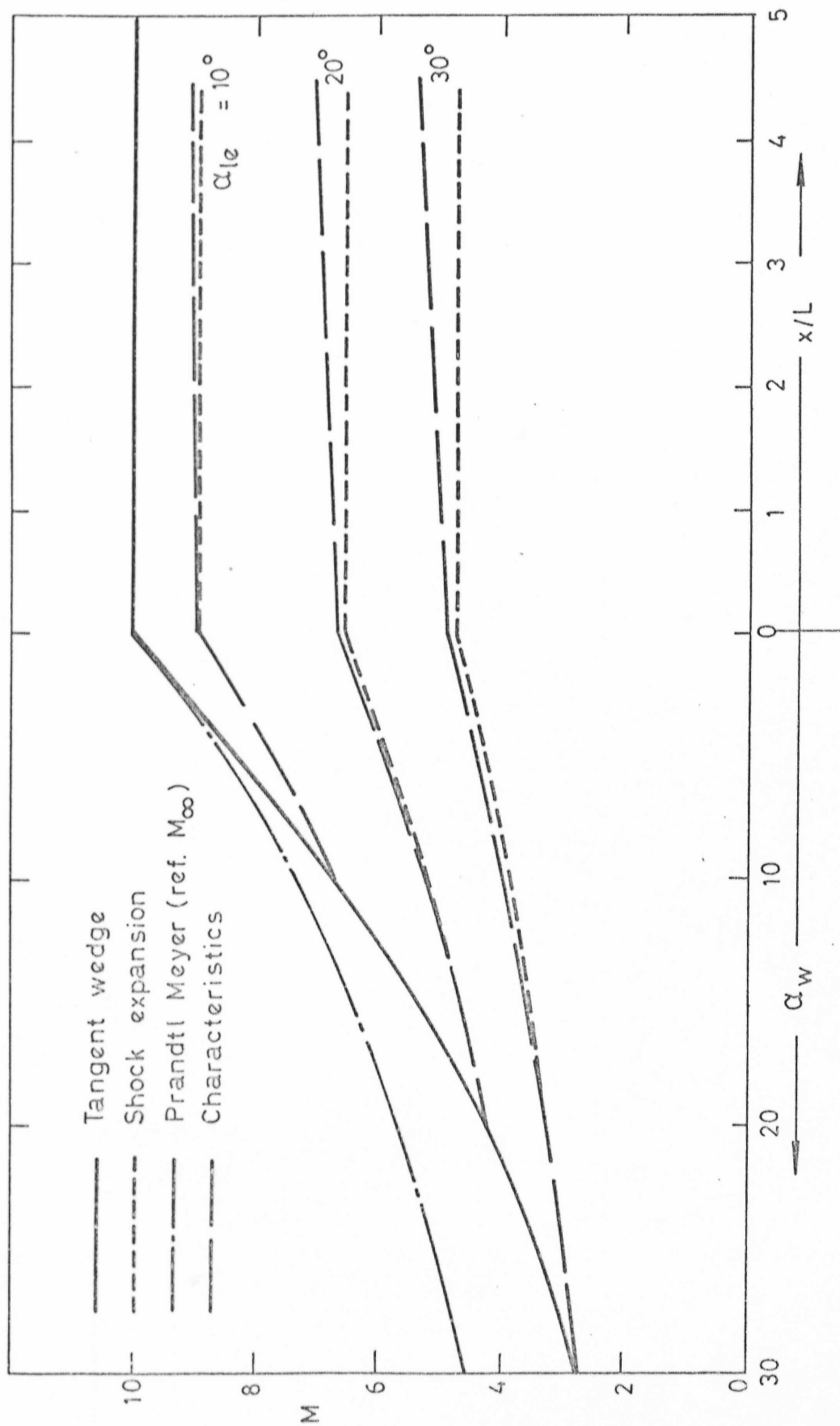
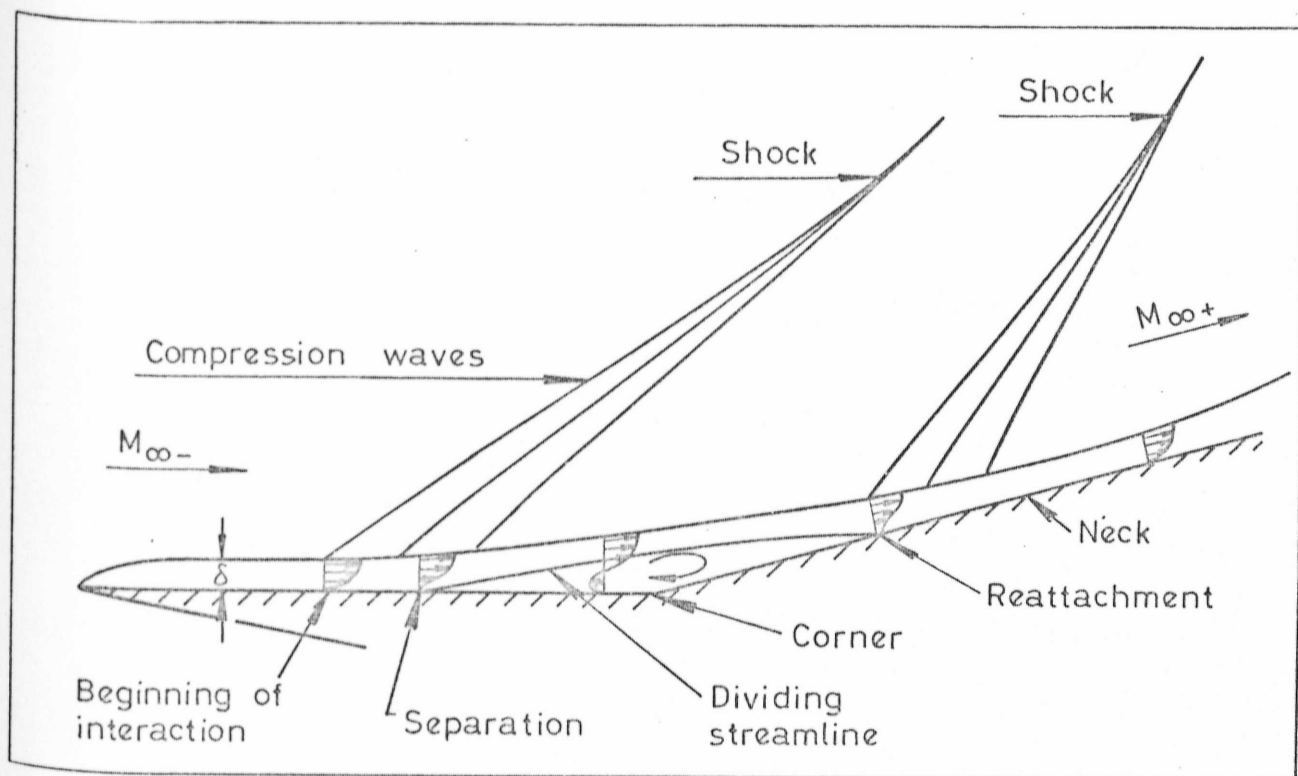
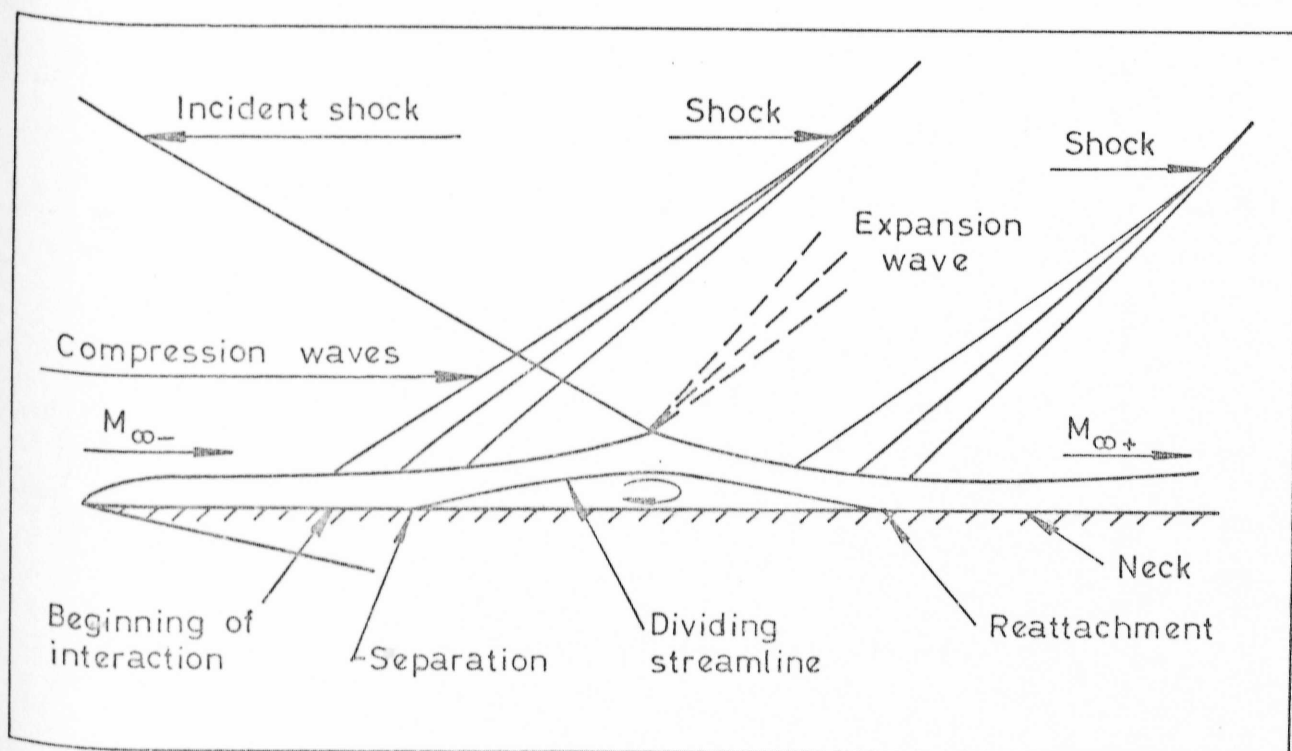


FIG. 11 (b) INVISCID MACH NUMBER DISTRIBUTION ON SURFACE SIMULATING STREAMLINES IN FIG. 10.



12 a FLAT PLATE - WEDGE (COMPRESSION CORNER).



12 b INCIDENT SHOCK ON FLAT PLATE.

FIG. 12 SCHEMATIC REPRESENTATION OF LAMINAR BOUNDARY LAYER-SHOCK WAVE INTERACTION.

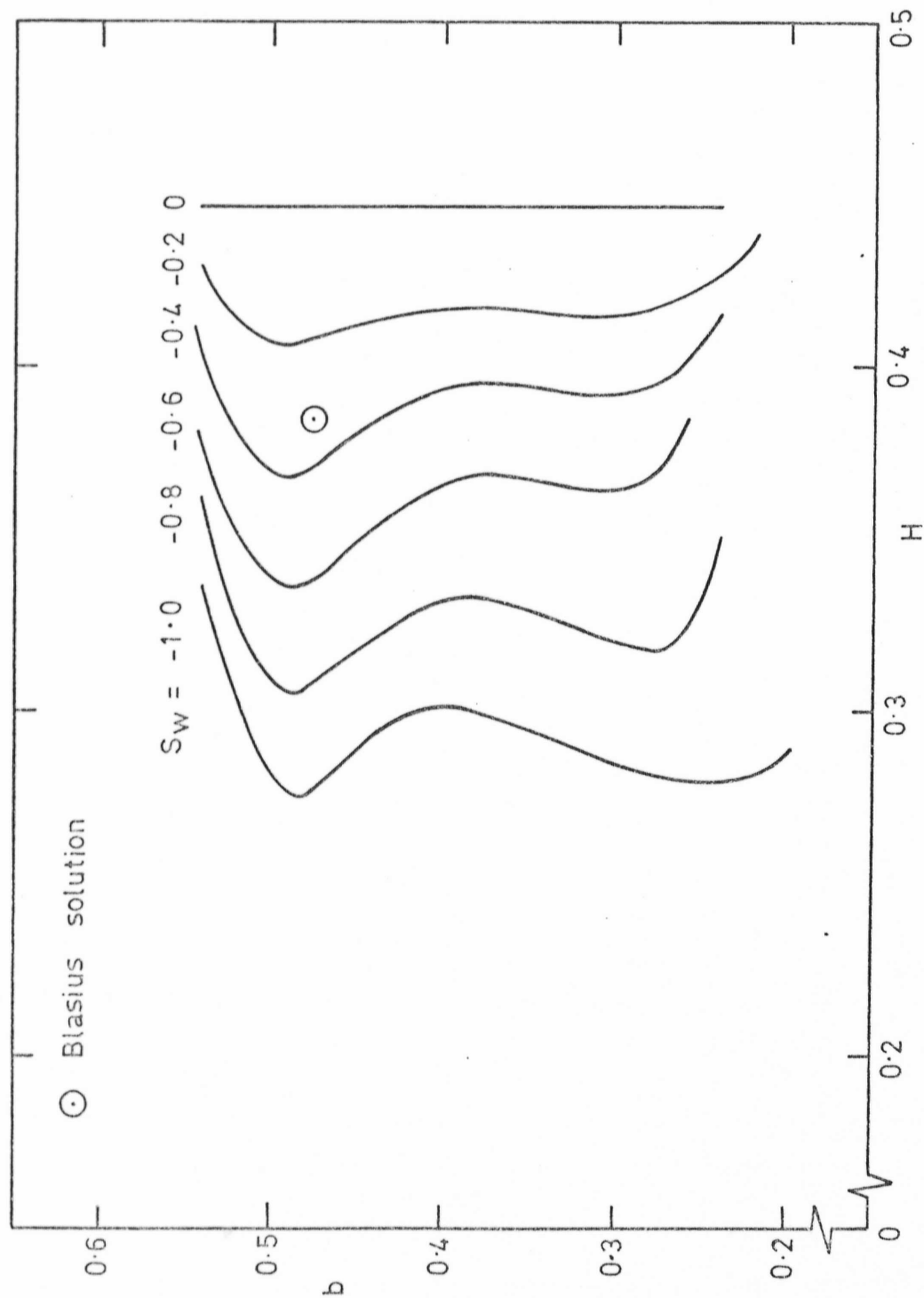


FIG. 13 CRITICAL BOUNDARY FOR HYPERSONIC FLOW.

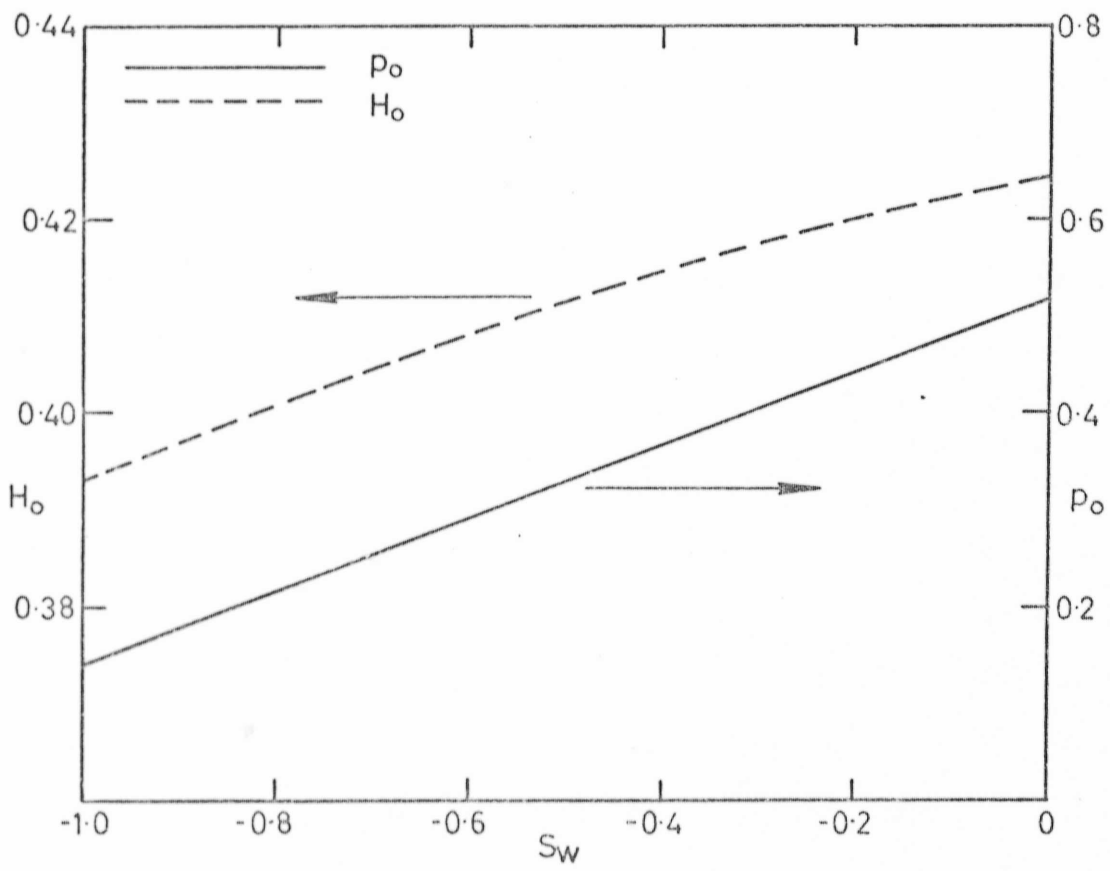
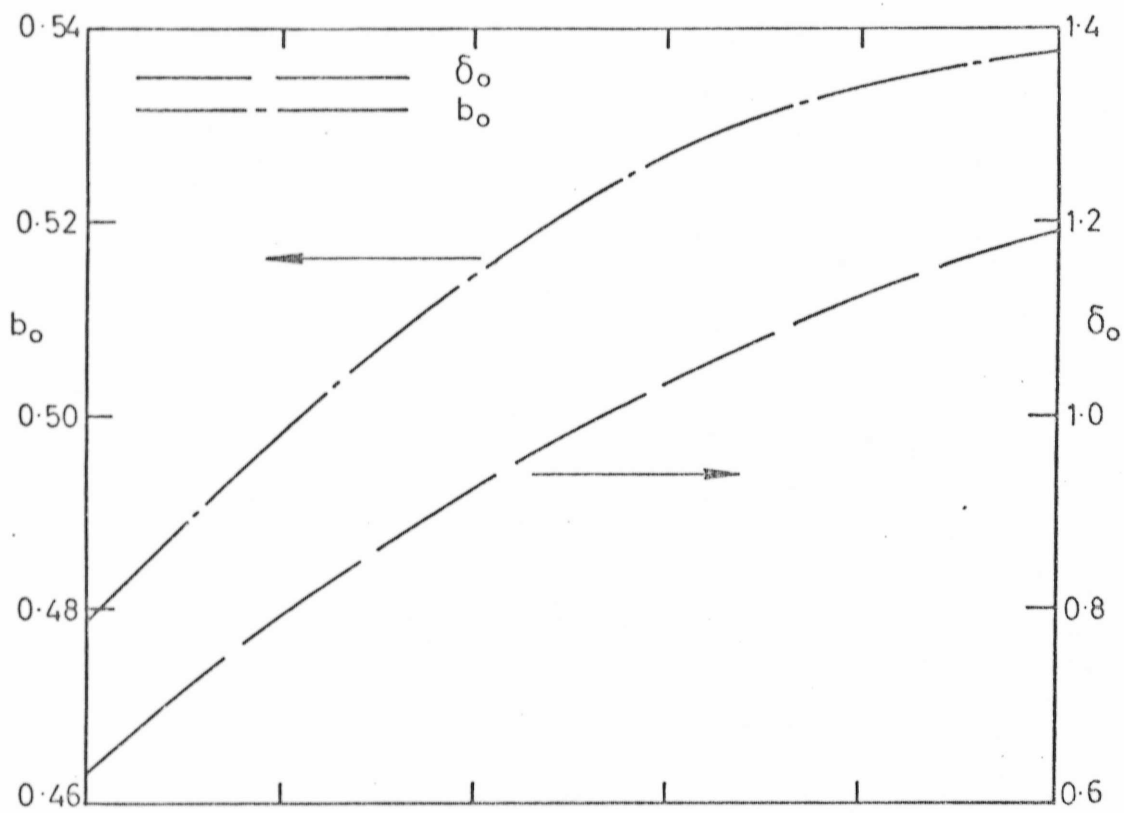


FIG. 14a STRONG INTERACTION EXPANSION COEFFICIENTS
 P_o , δ_o , H_o & b_o .

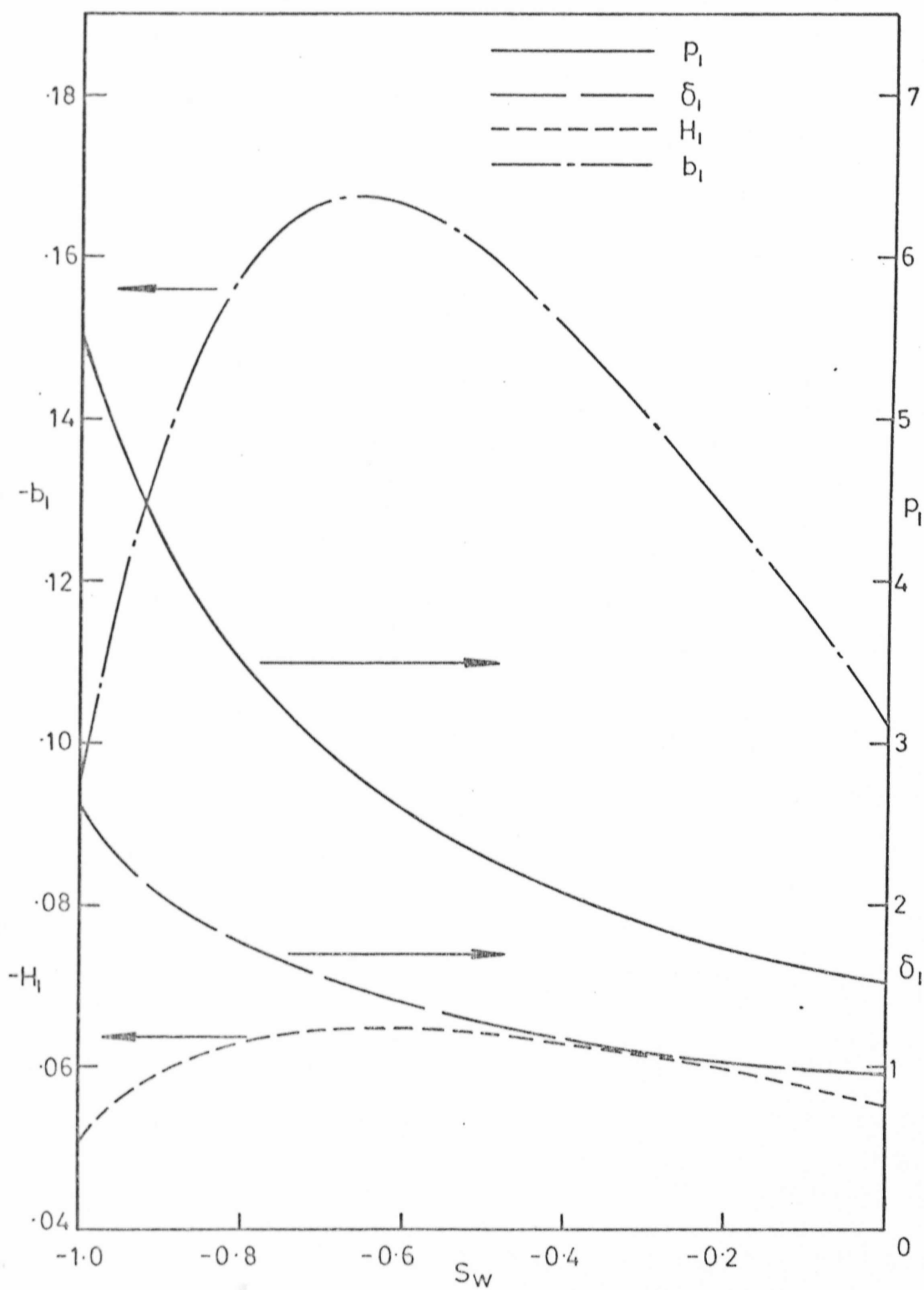


FIG. 14b STRONG INTERACTION EXPANSION COEFFICIENTS
 P_I , δ_I , H_I & b_I .

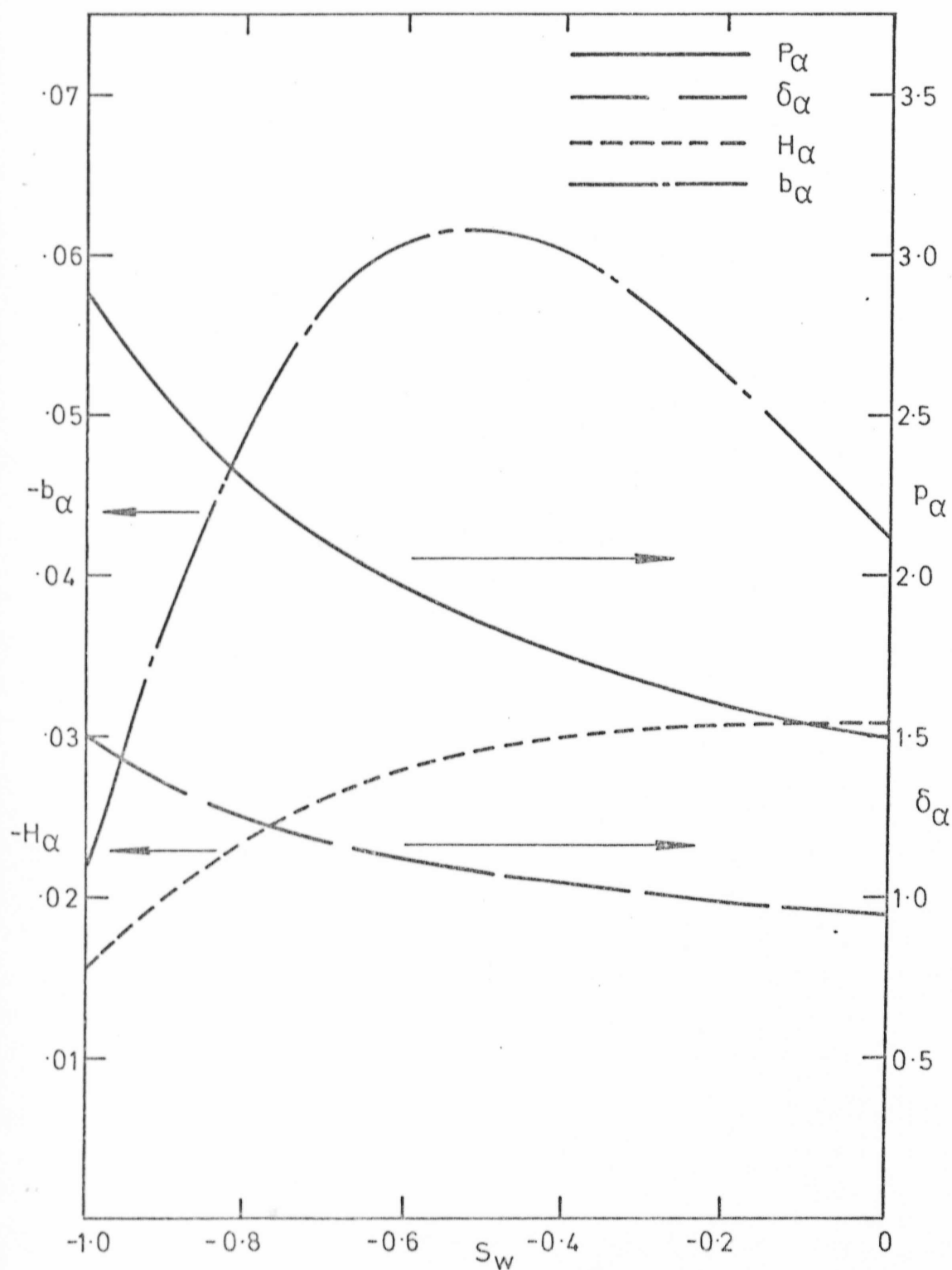


FIG. 14 c STRONG INTERACTION EXPANSION COEFFICIENTS

P_α , δ_α , H_α & b_α .

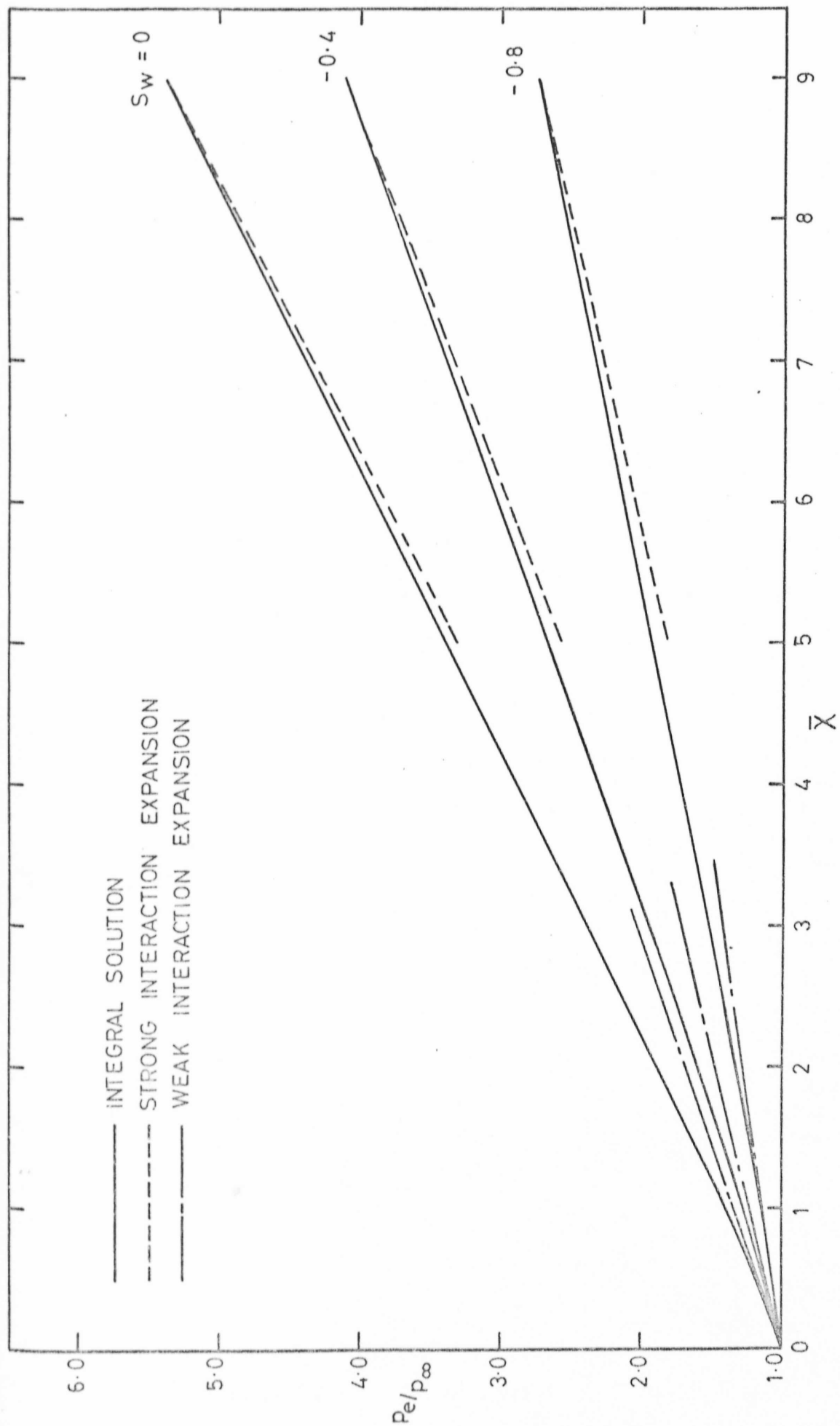


FIG. 15 HYPERSONIC FLAT PLATE PRESSURE DISTRIBUTION.

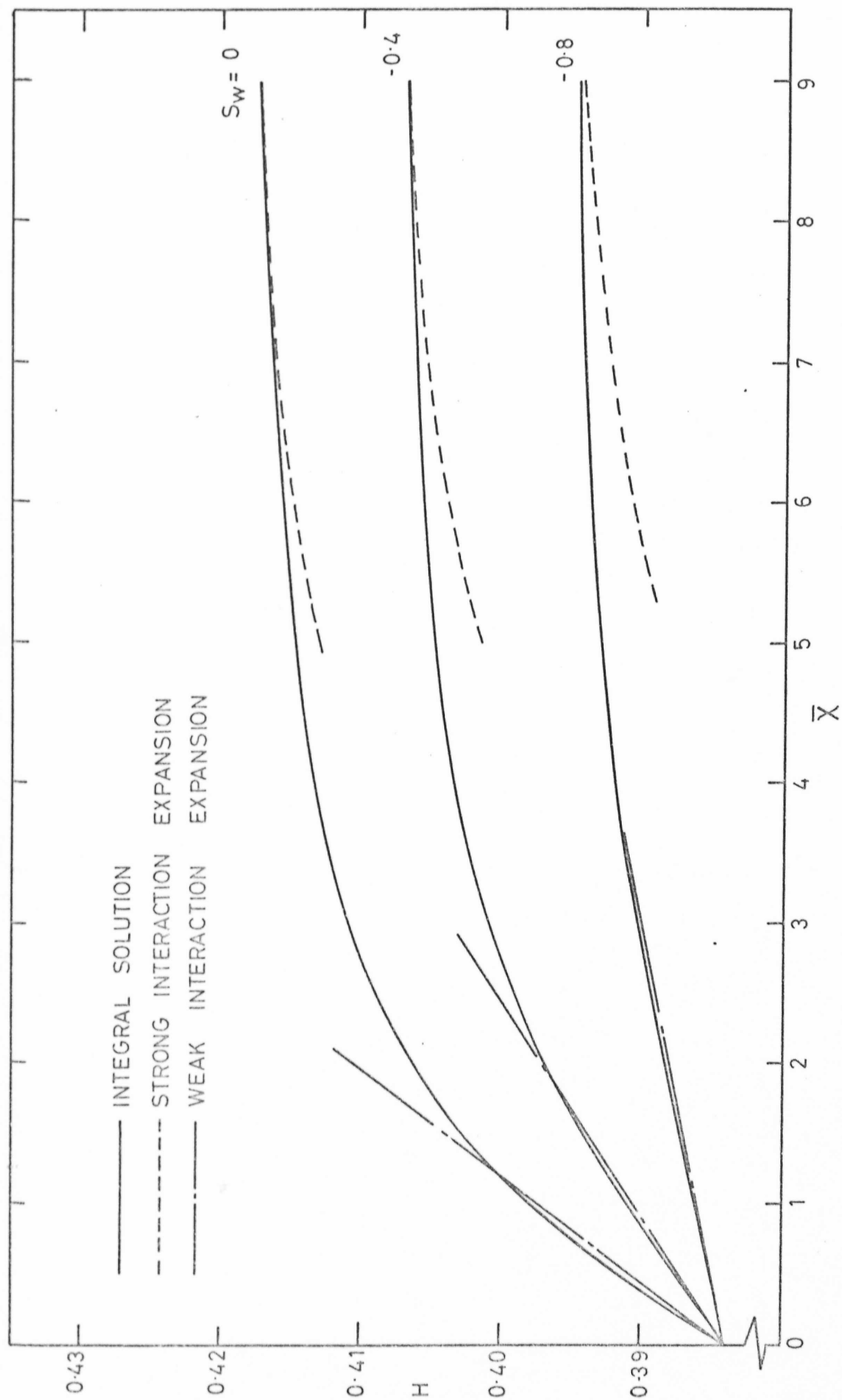


FIG. 16 HYPersonic FLAT PLATE DISTRIBUTION OF H .

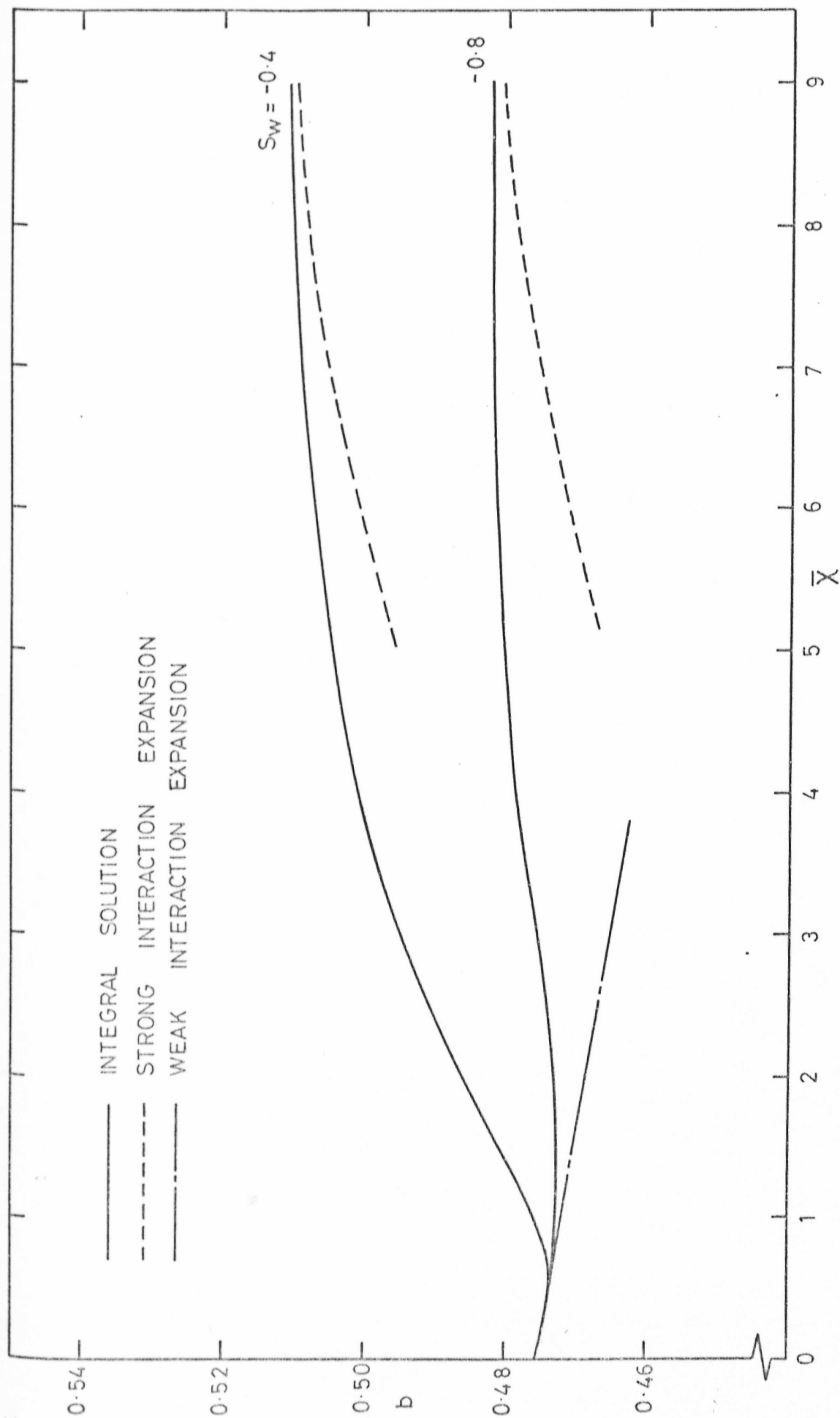


FIG. 17 HYPERSONIC FLAT PLATE DISTRIBUTION OF b .

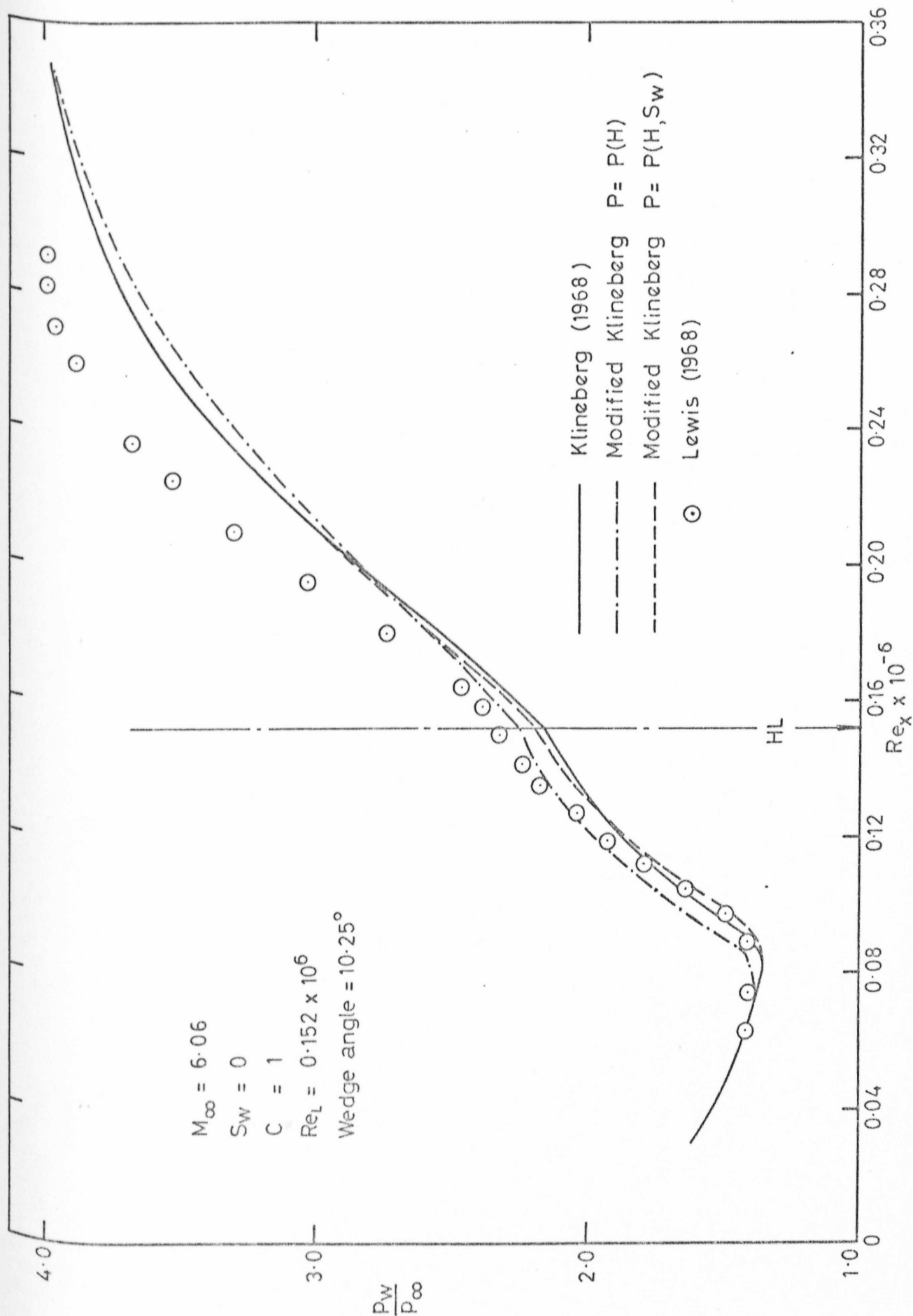


FIG. 18 EFFECT OF PROFILE FUNCTIONS $P(H)$, $P(H, S_w)$ ON PRESSURE DISTRIBUTION.

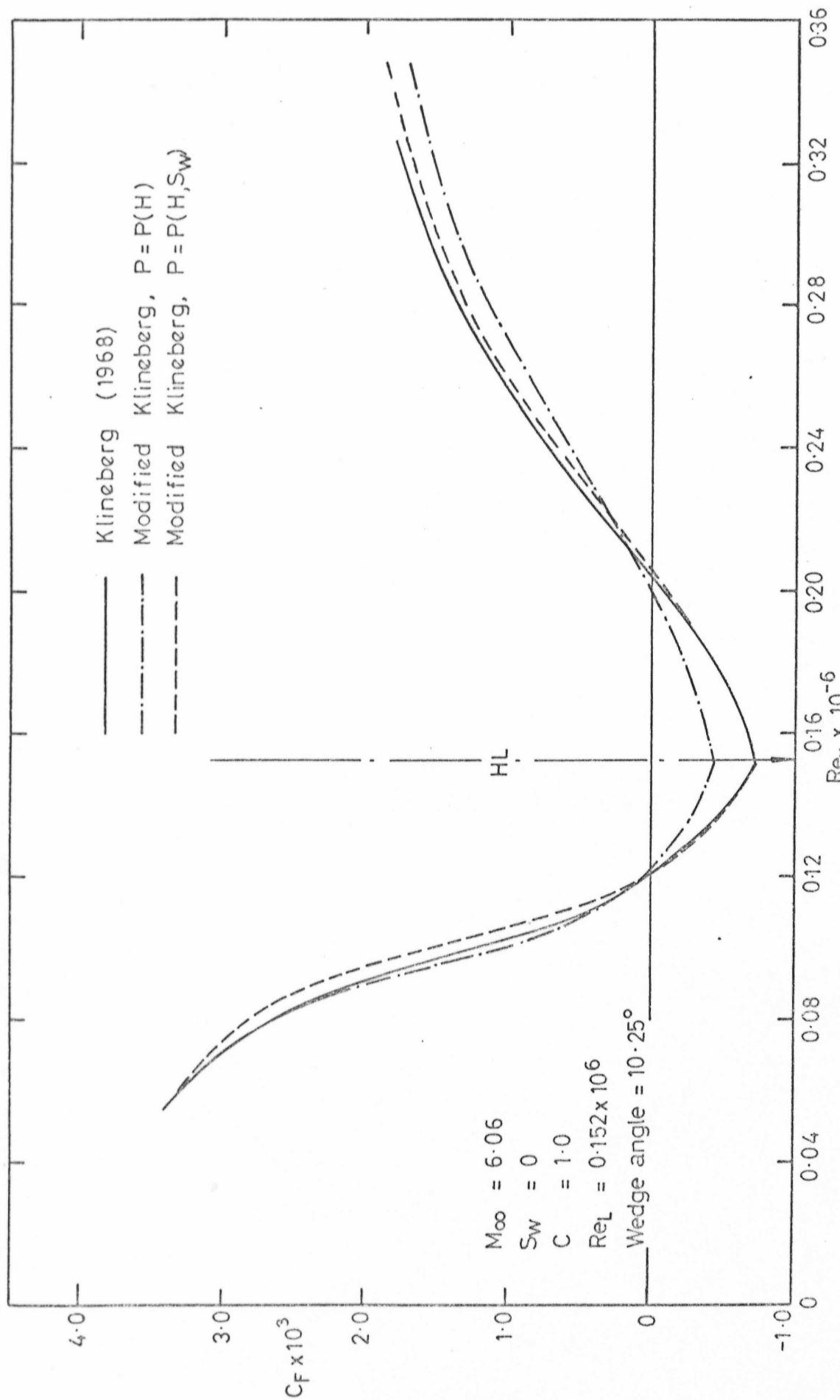


FIG. 19 EFFECT OF PROFILE FUNCTIONS $P(H)$, $P(H, S_w)$ ON DISTRIBUTION OF SKIN FRICTION COEFFICIENT.

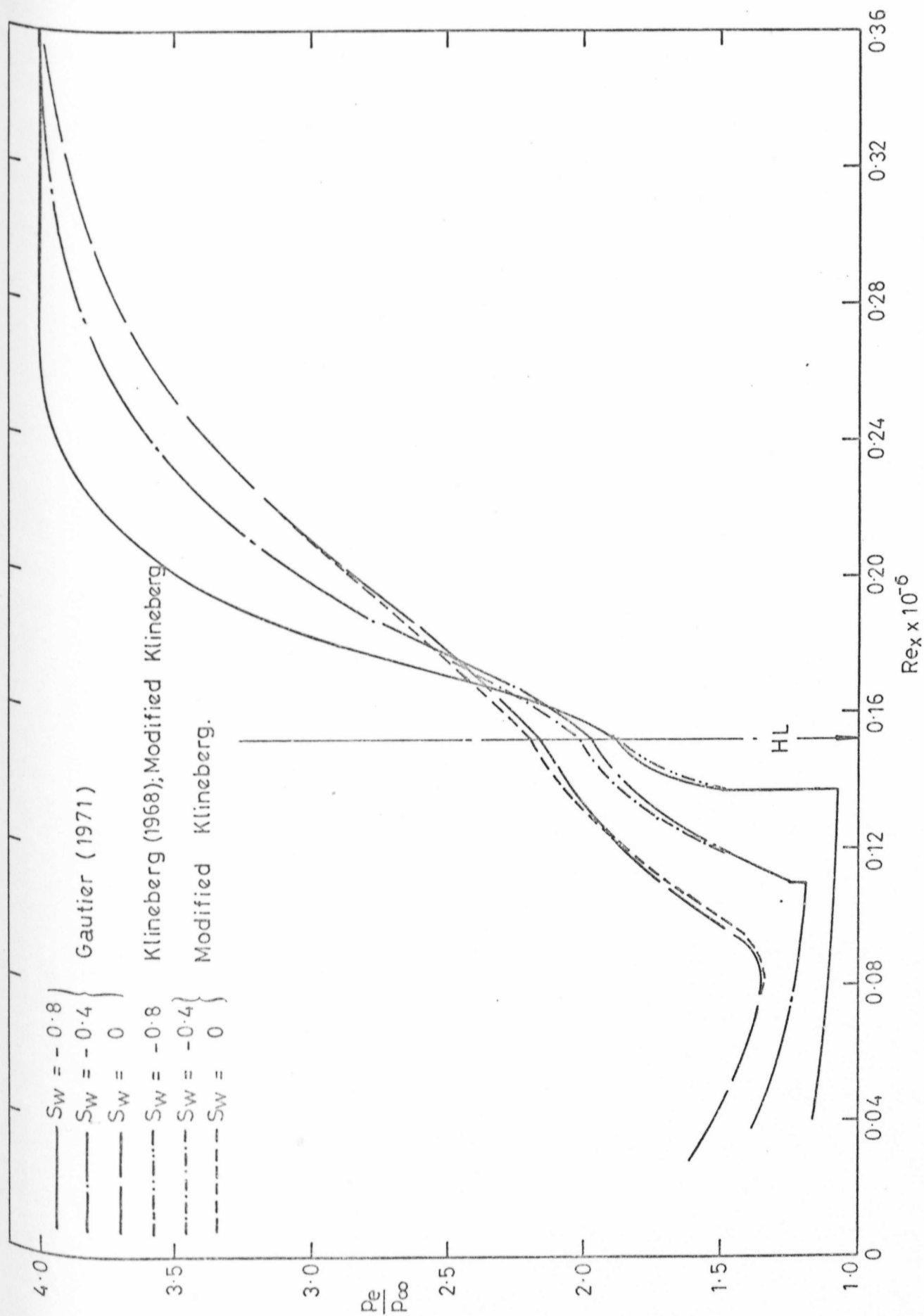


FIG. 20 COMPARISON OF THEORETICAL PRESSURE DISTRIBUTIONS.

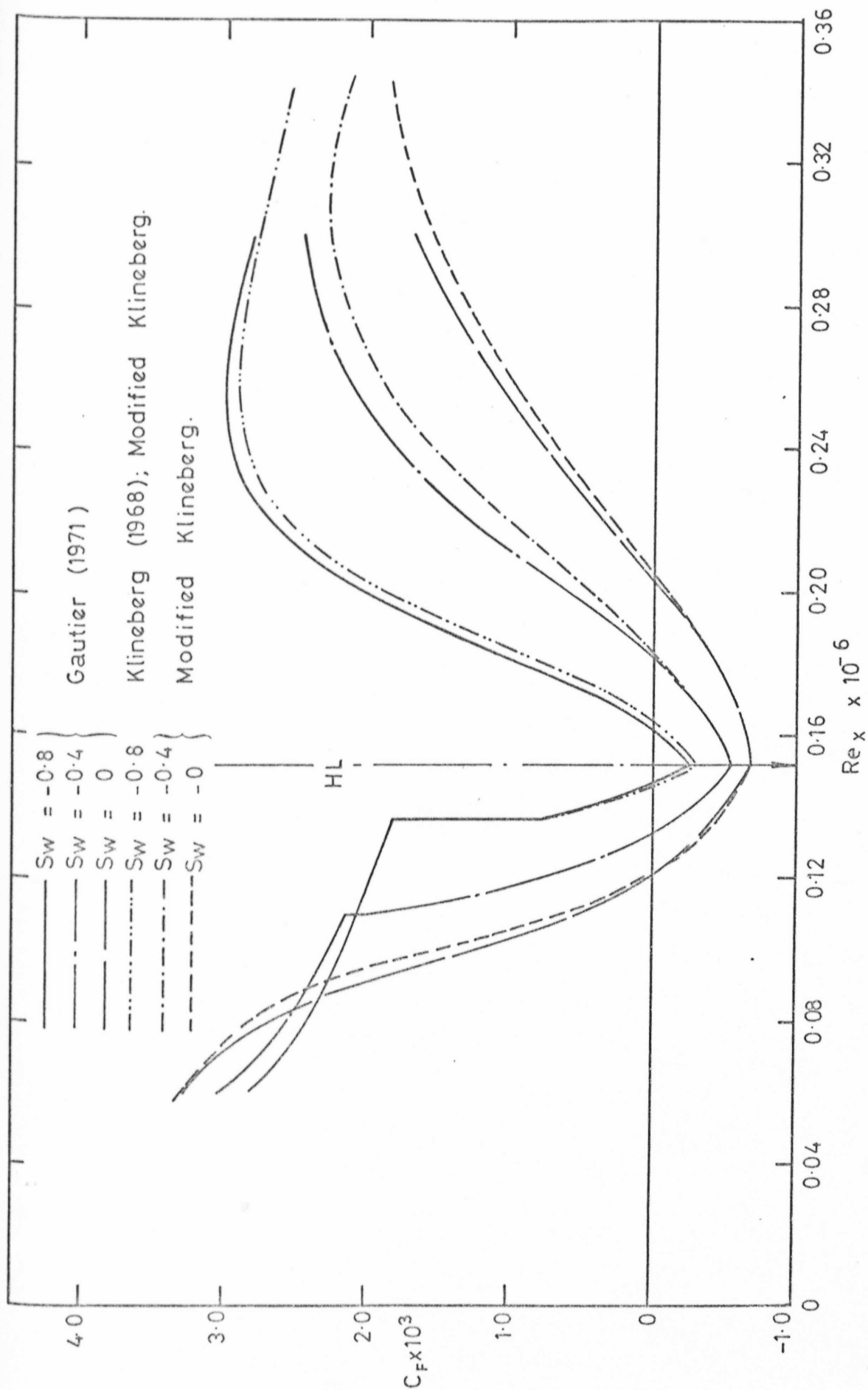


FIG. 21 COMPARISON OF THEORETICAL DISTRIBUTIONS OF SKIN FRICTION COEFFICIENT.

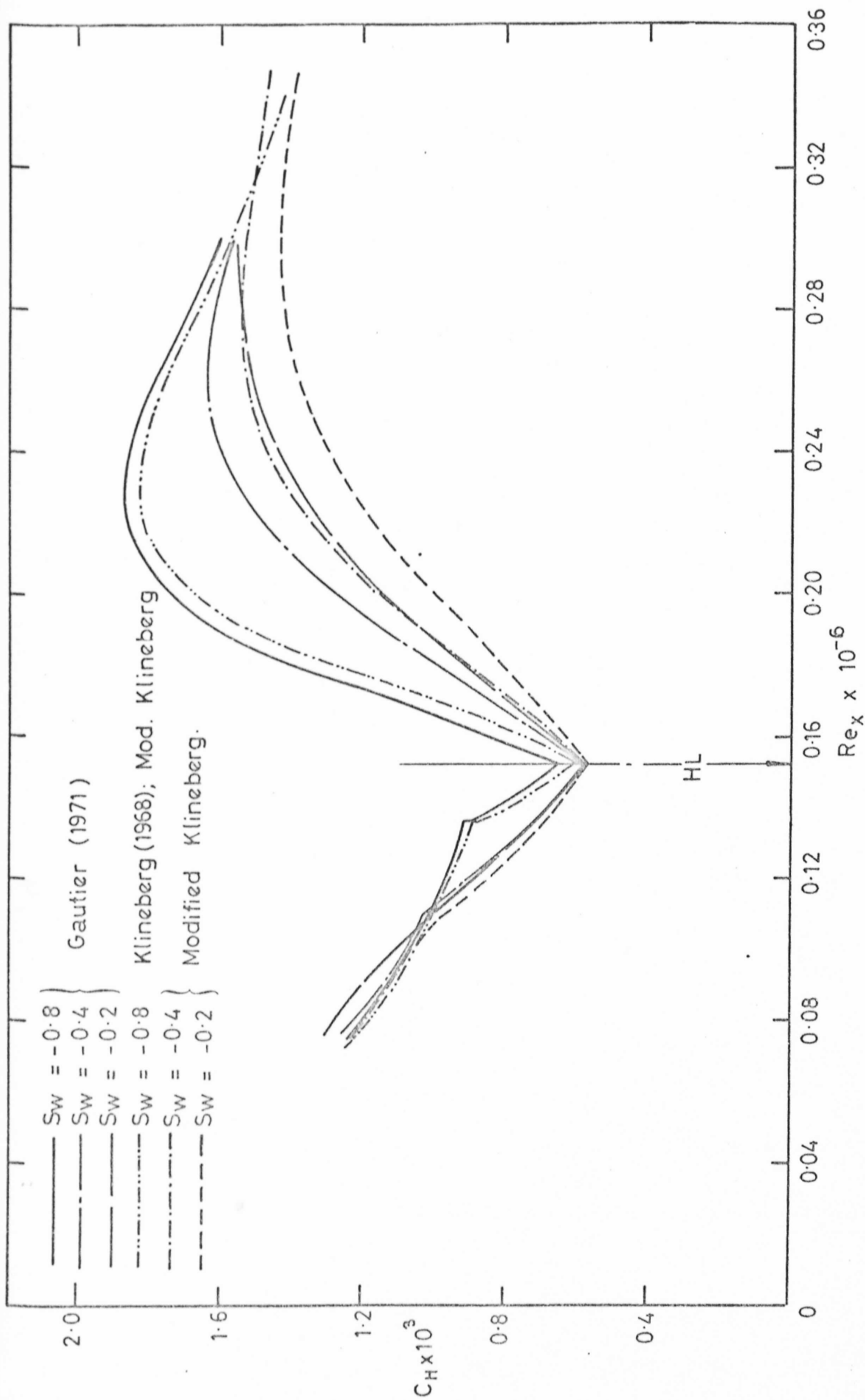


FIG. 22 COMPARISON OF THEORETICAL DISTRIBUTIONS OF HEAT TRANSFER COEFFICIENT

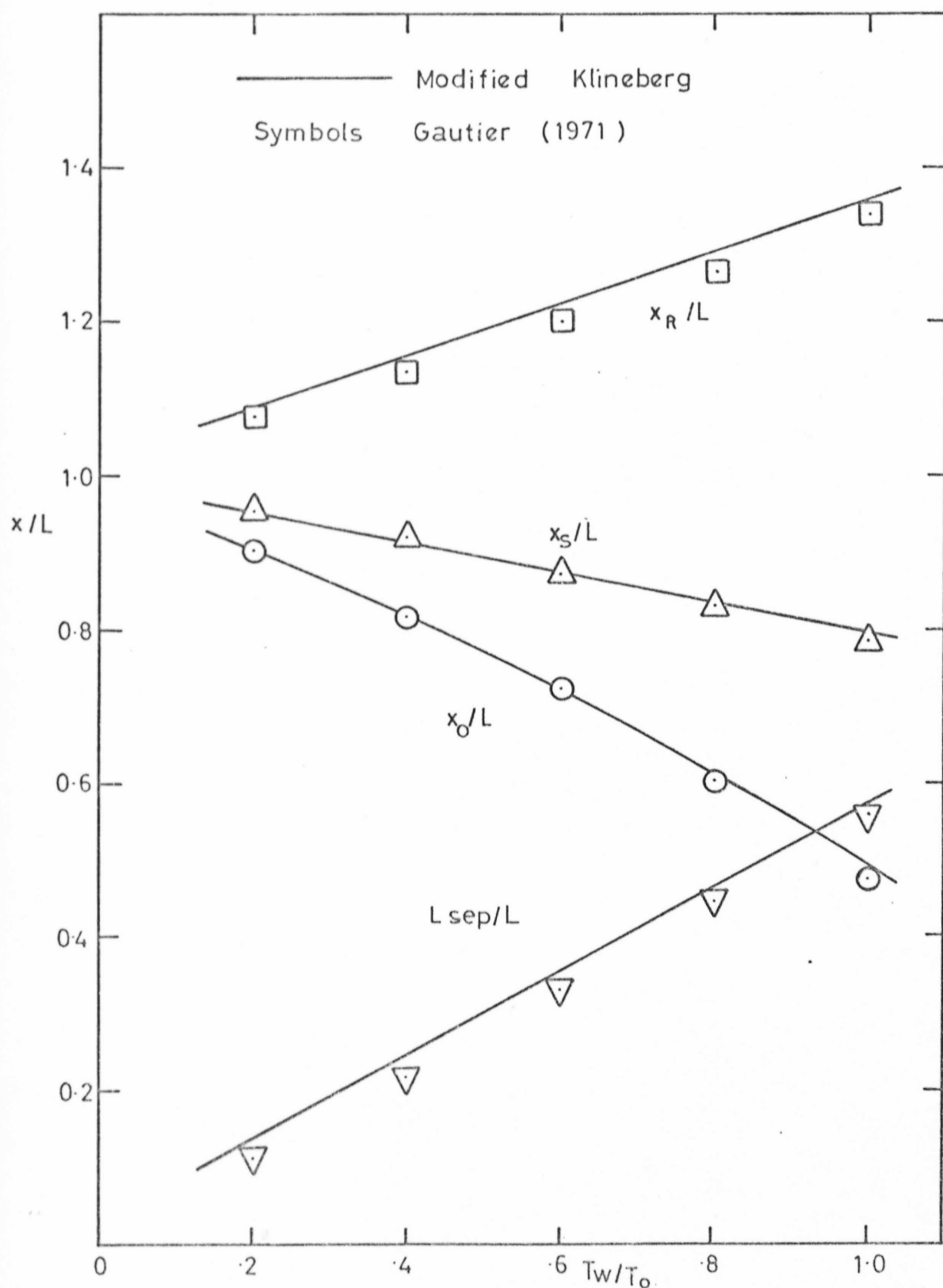


FIG. 23 CHARACTERISTIC LENGTHS — THEORETICAL COMPARISONS.

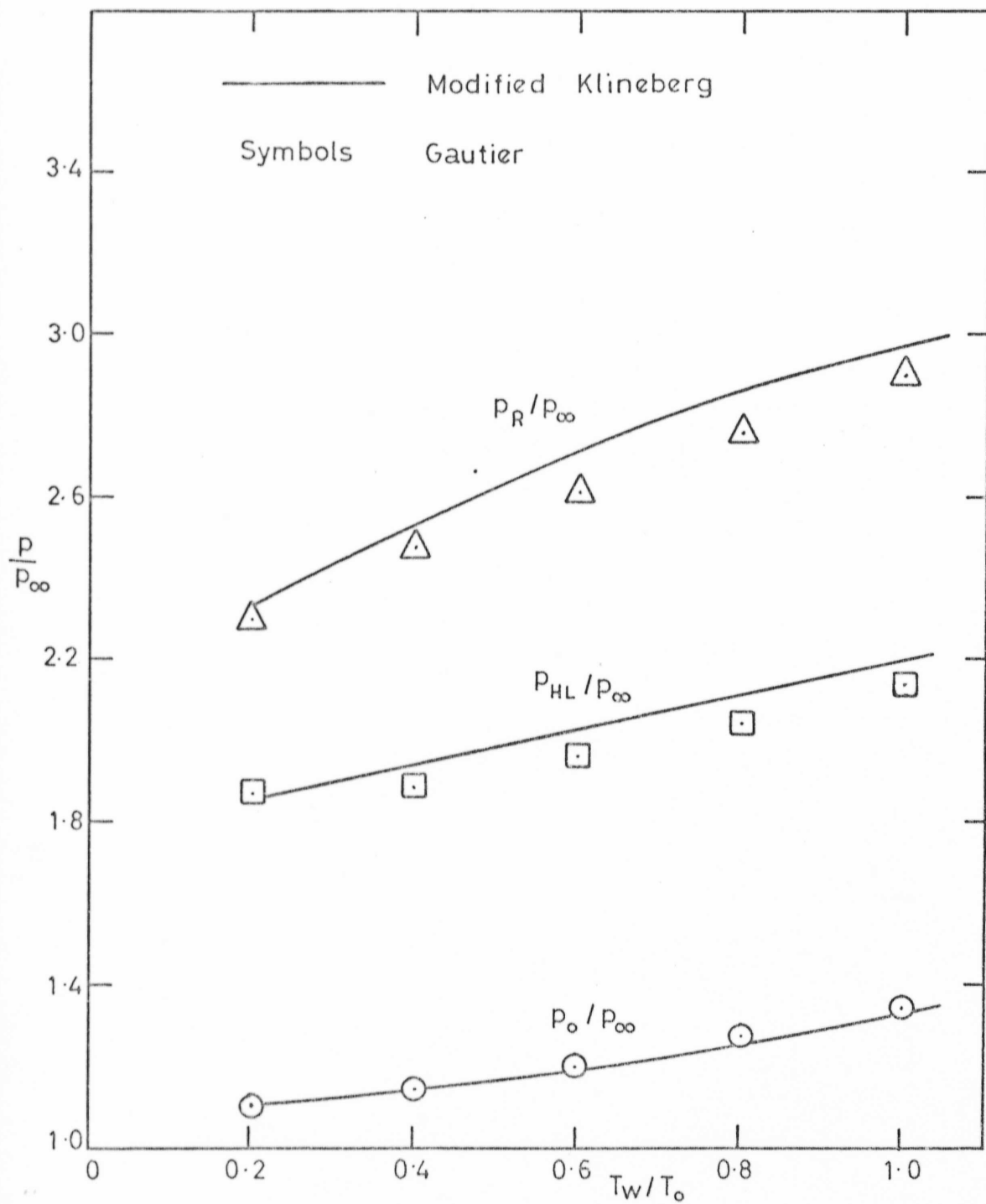


FIG. 24 CHARACTERISTIC PRESSURES — THEORETICAL COMPARISONS

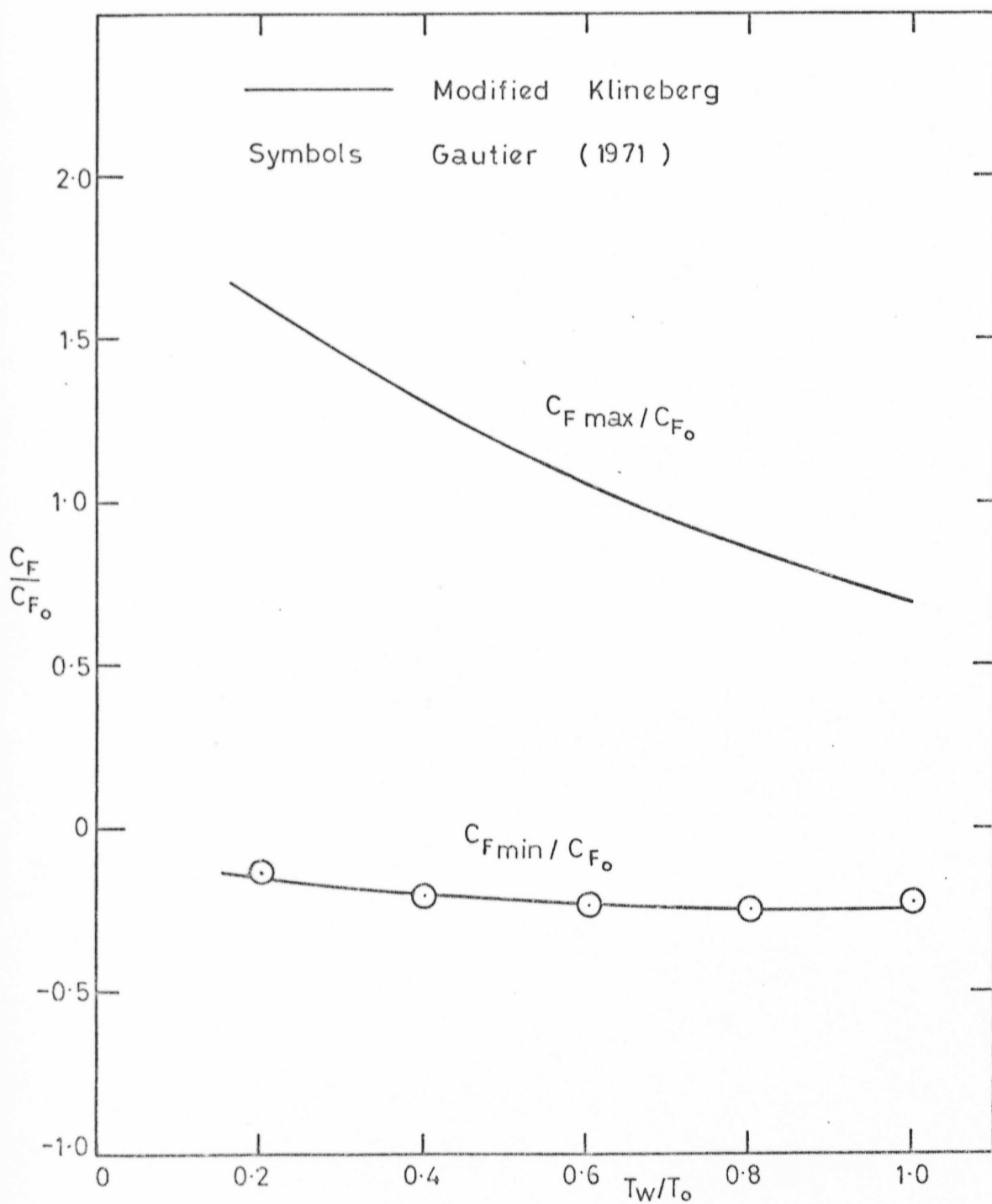


FIG. 25 MAXIMUM AND MINIMUM SKIN FRICTION COEFFICIENTS - THEORETICAL COMPARISONS.

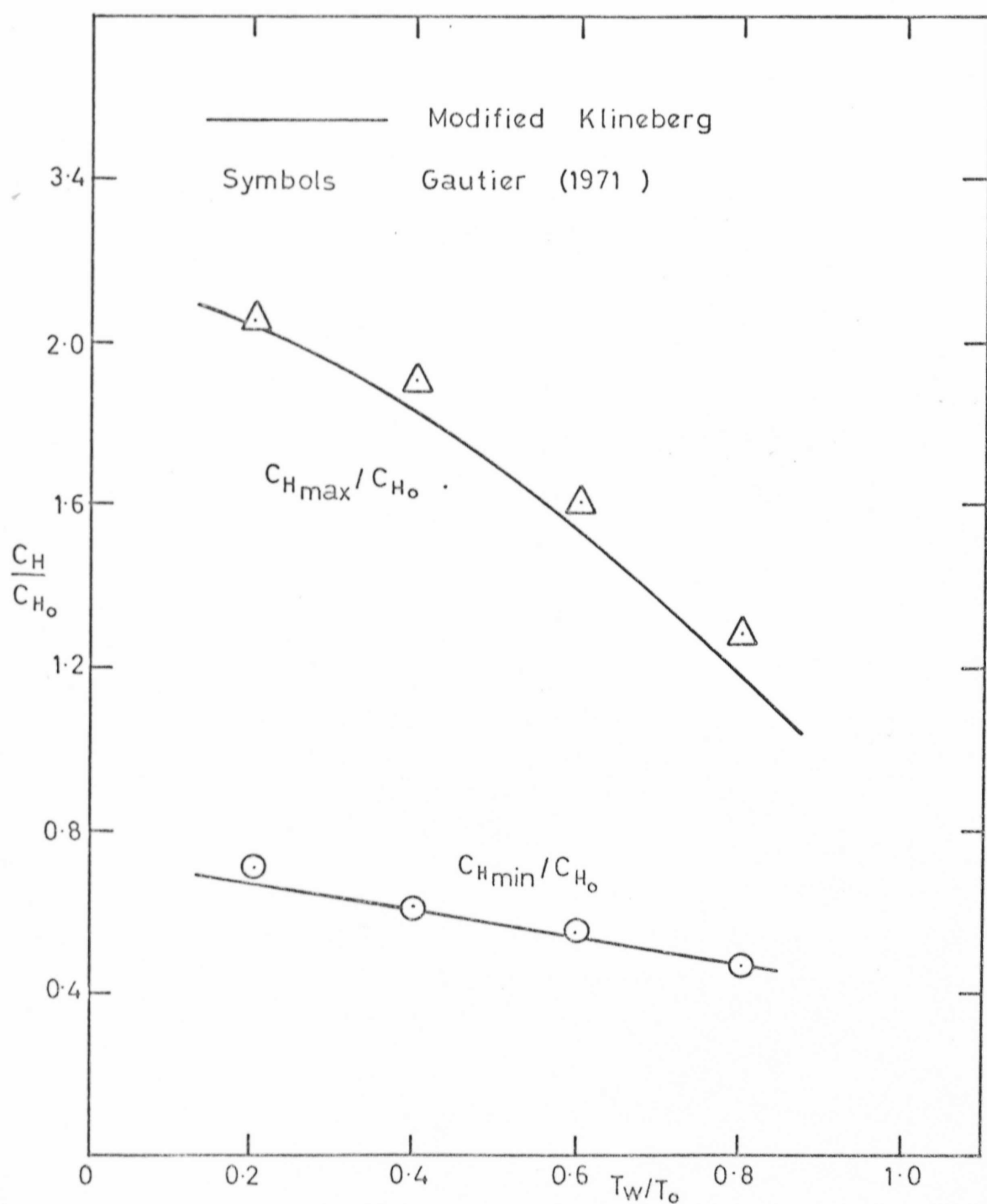


FIG. 26 MAXIMUM AND MINIMUM HEAT TRANSFER COEFFICIENTS - THEORETICAL COMPARISONS.

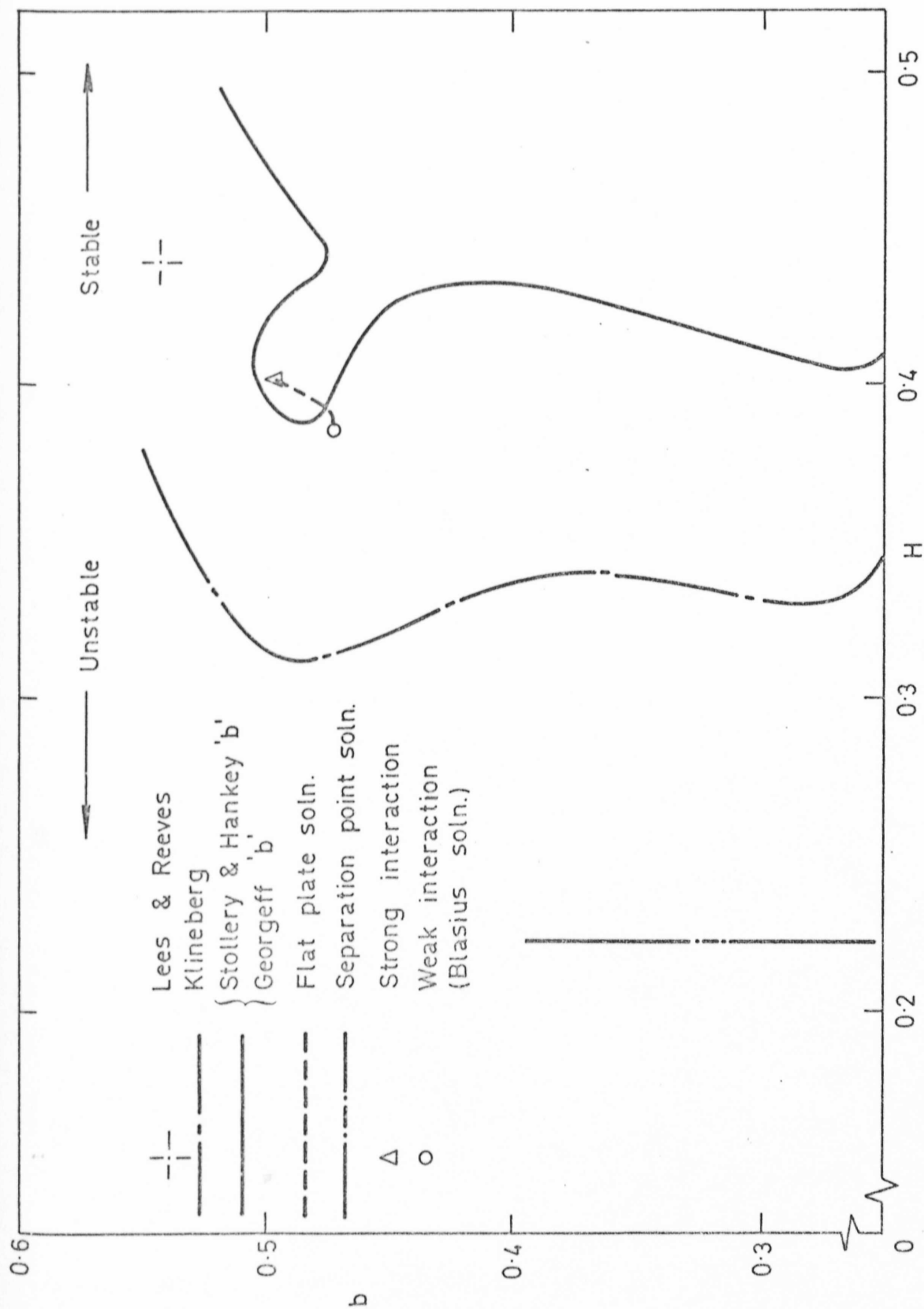
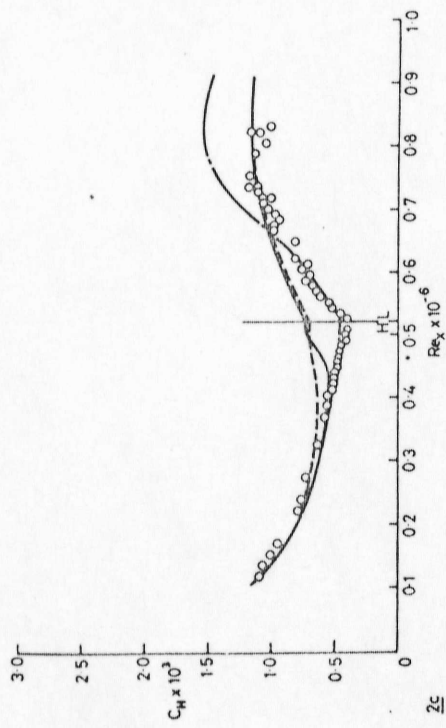
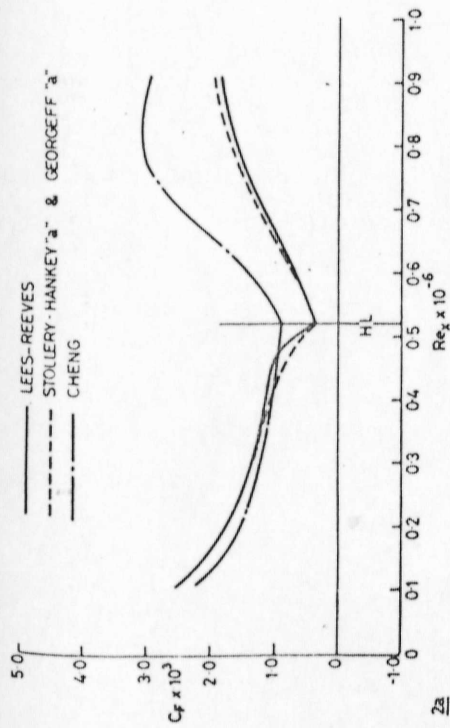
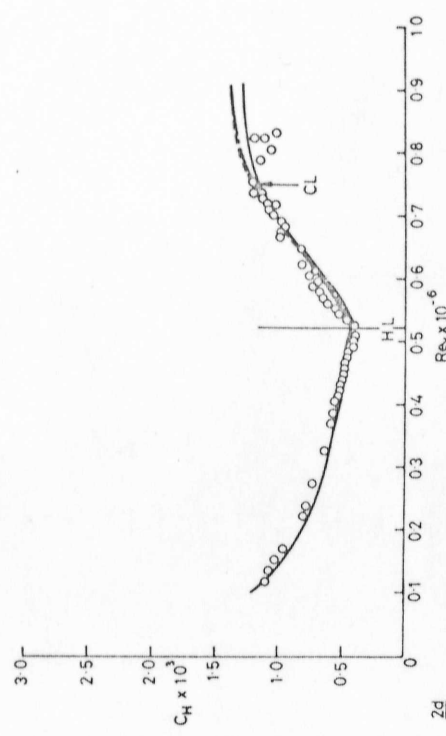
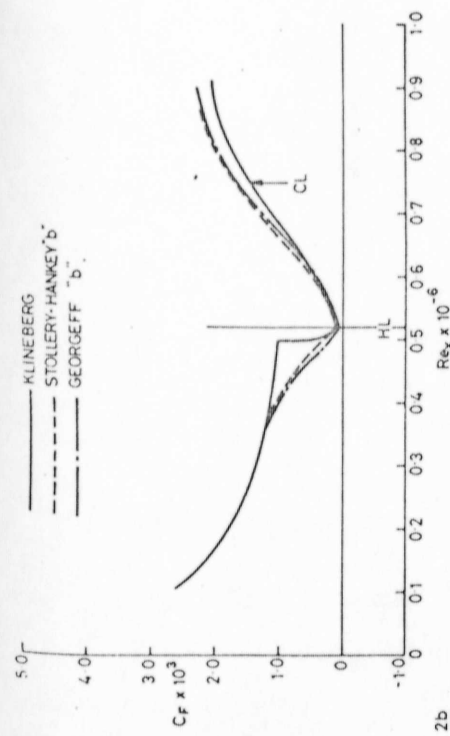


FIG. 27 CRITICAL BOUNDARY FOR HYPERSONIC FLOW AT $S_w = -0.8$.

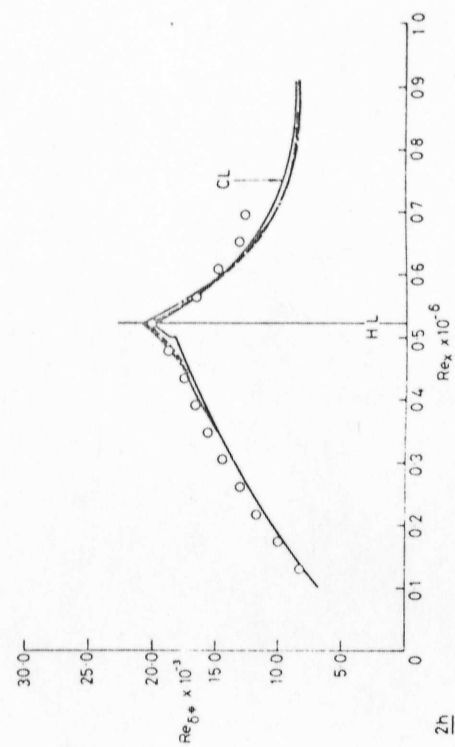
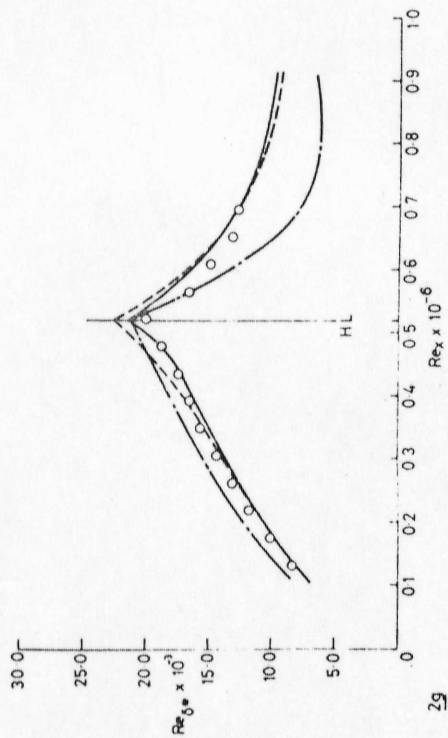
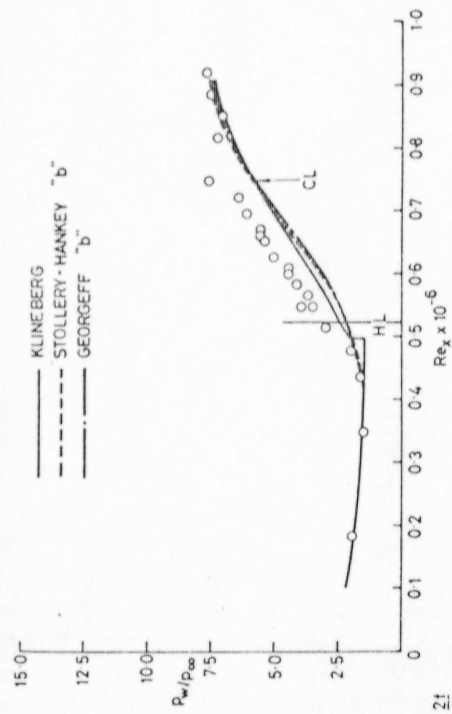
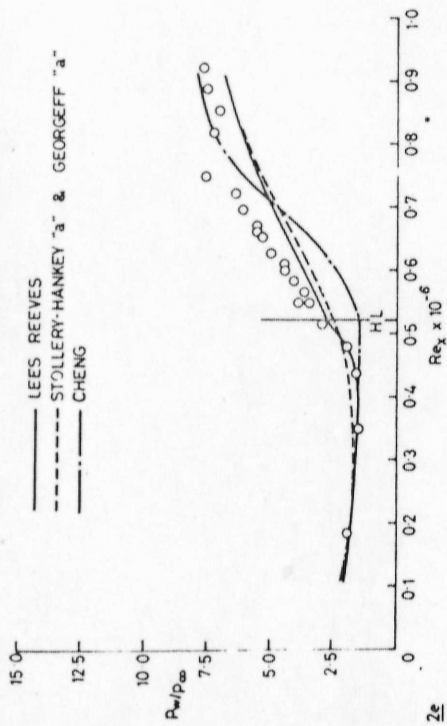


(a & c) "COUPLED PROFILE" & CHENG METHODS



(b & d) "UNCOUPLED PROFILE" METHODS

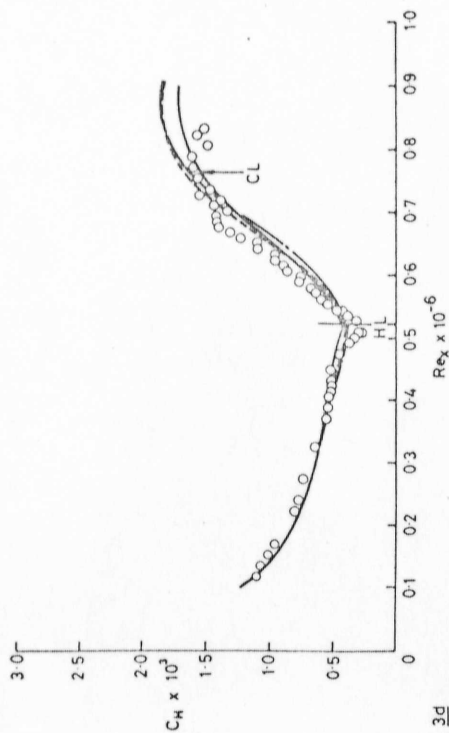
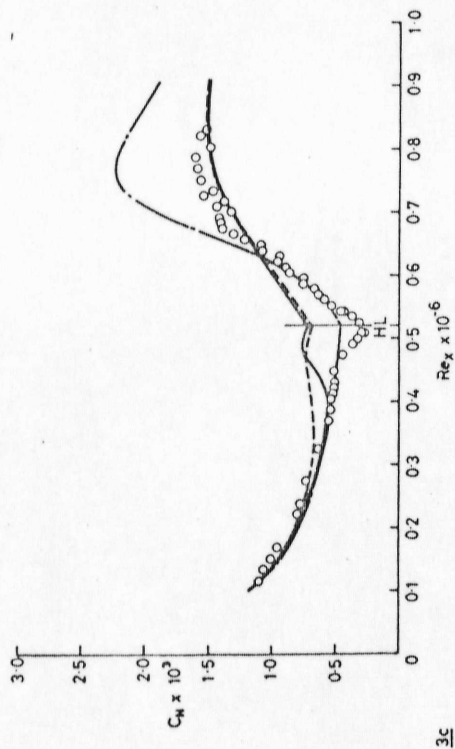
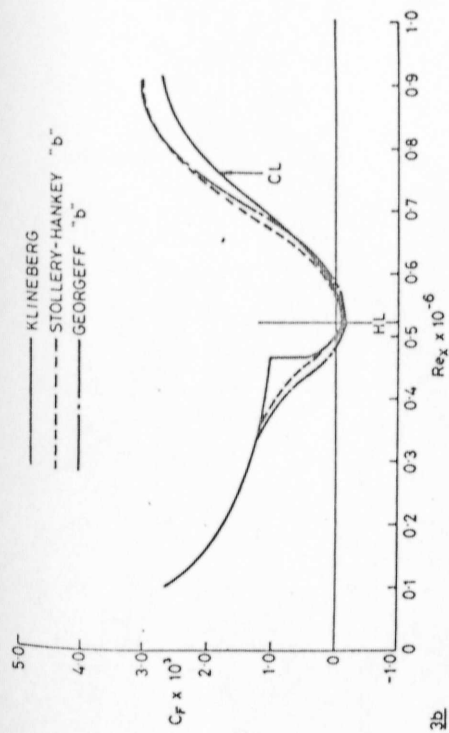
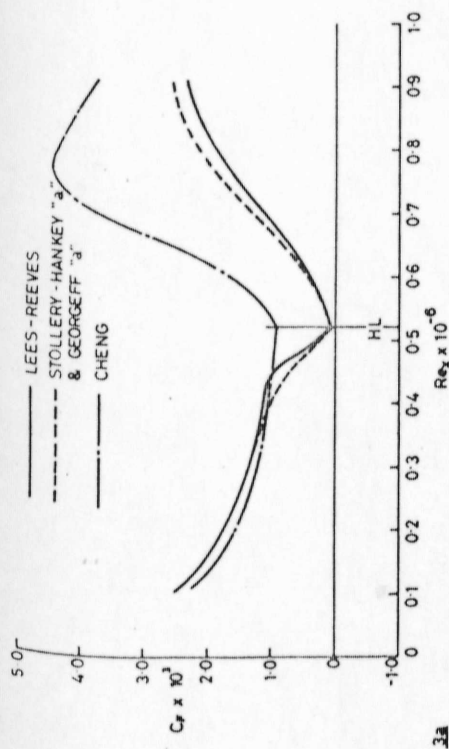
FIG. 28 COMPARISON OF THEORETICAL RESULTS WITH DATA OF BLOY AT WEDGE ANGLE 8°



(e & g) "COUPLED PROFILE" & CHENG METHODS

(f & h) "UNCOUPLED PROFILE" METHODS

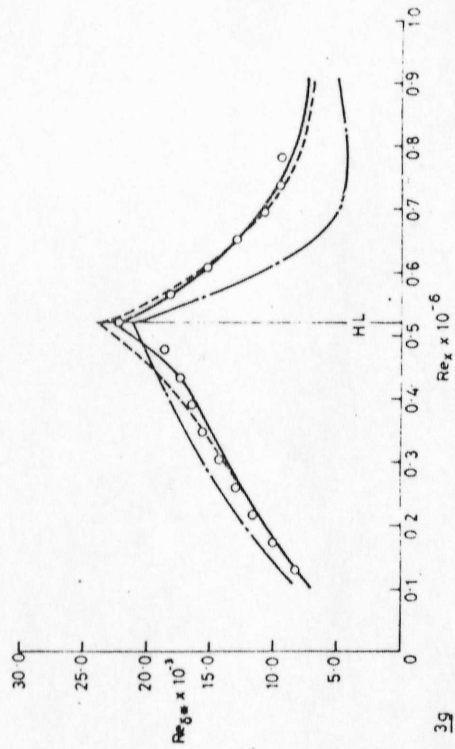
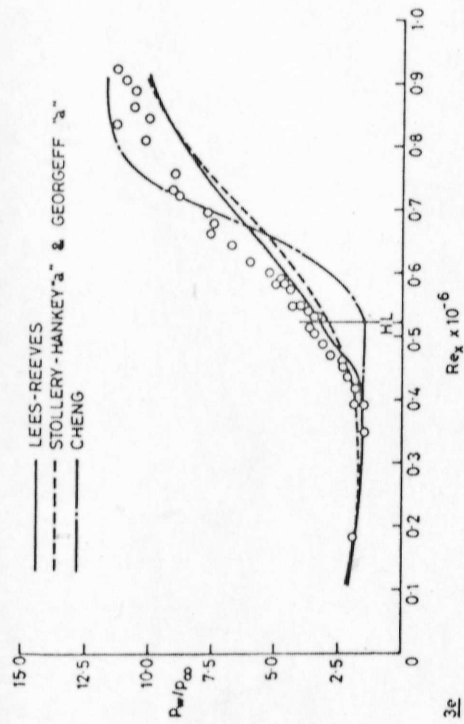
FIG 28 CONCLUDED



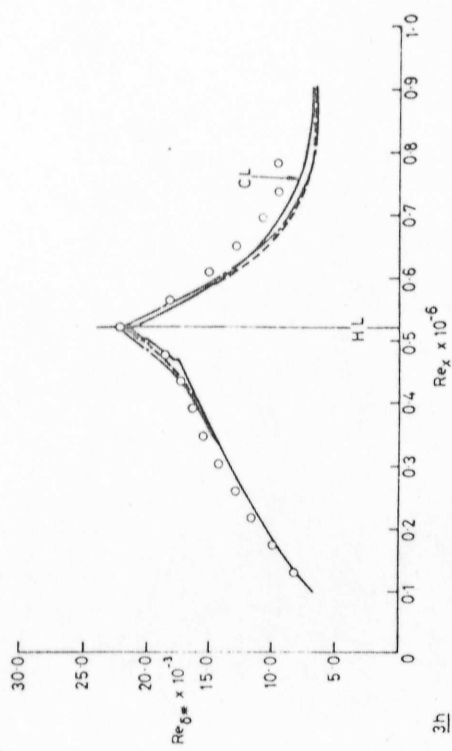
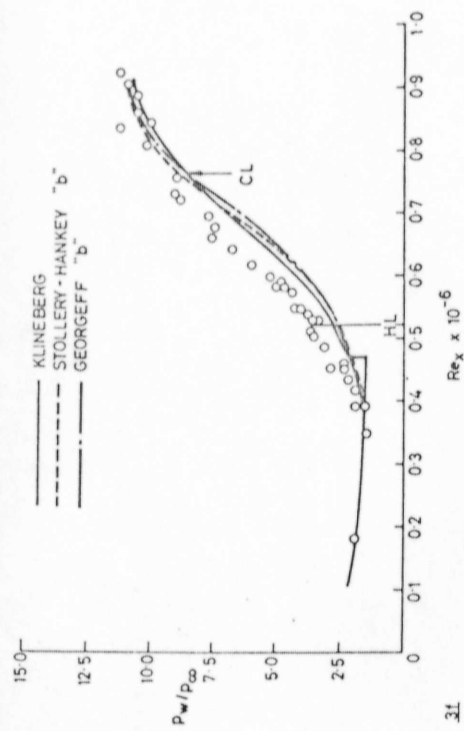
(a & c) "COUPLED PROFILE" & CHENG METHODS.

(b & d) "UNCOUPLD PROFILE" METHODS

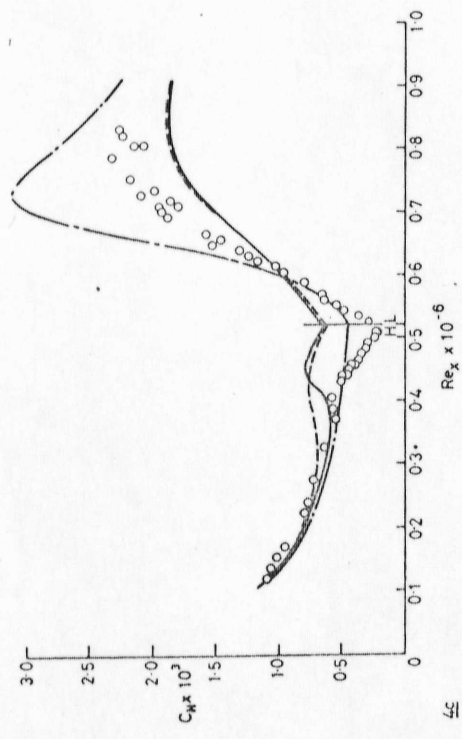
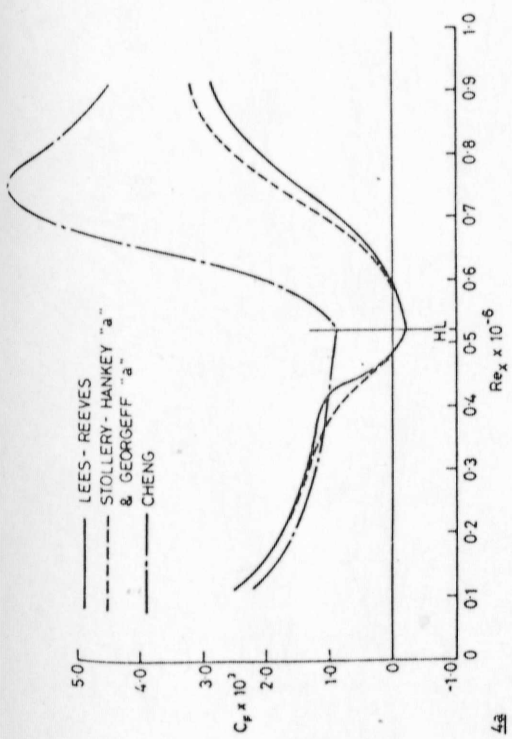
FIG. 29 COMPARISON OF THEORETICAL RESULTS WITH DATA OF BLOY AT WEDGE ANGLE 10°



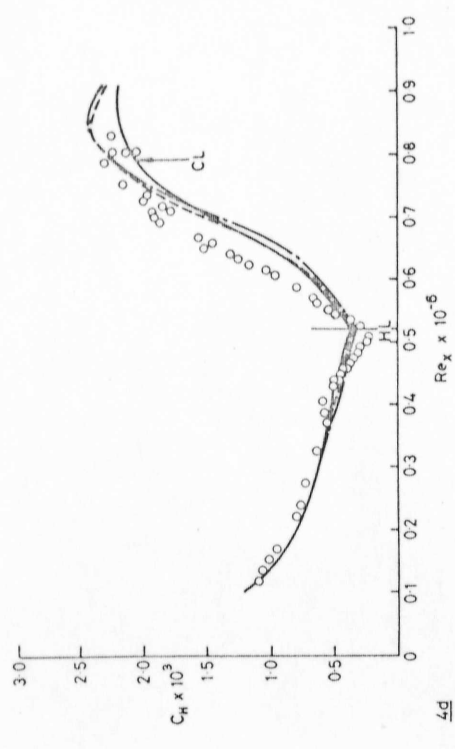
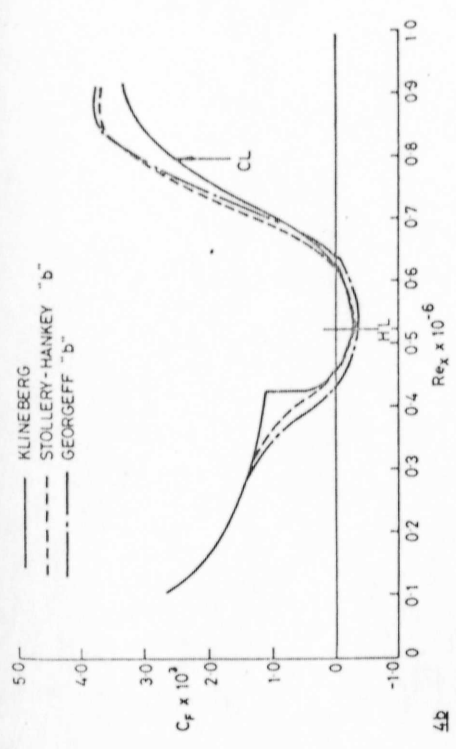
(e & g) "COUPLED PROFILE" & CHENG METHODS



(f & h) "UNCOUPLED PROFILE" METHODS

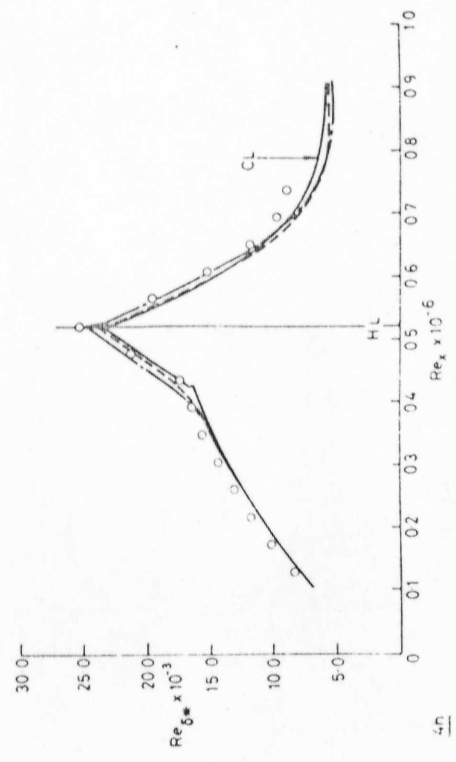
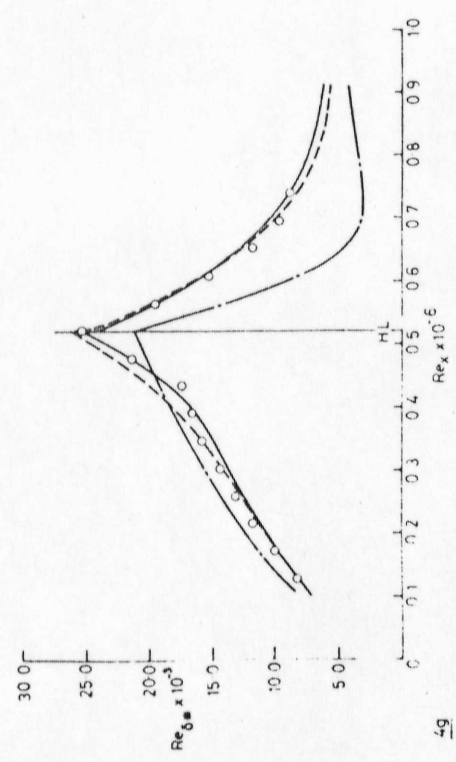
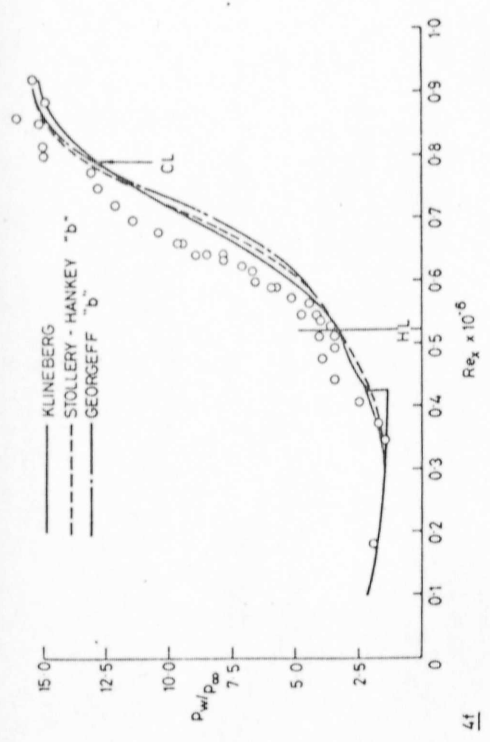
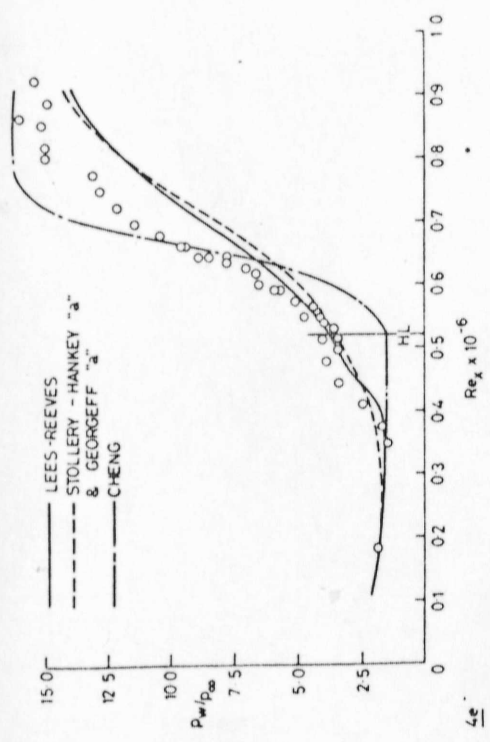


(a & c) "COUPLED PROFILE" & CHENG METHODS.



(b & d) "UNCOUPLED PROFILE" METHODS

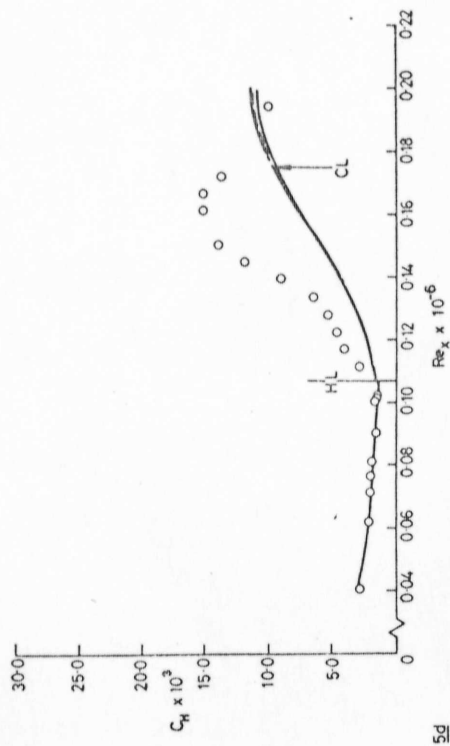
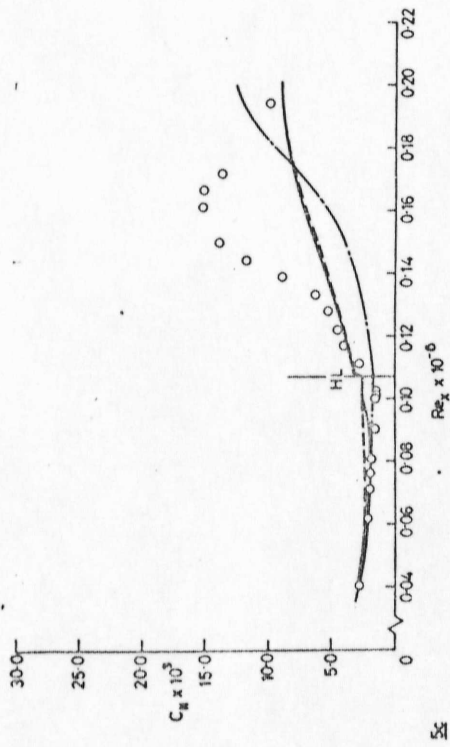
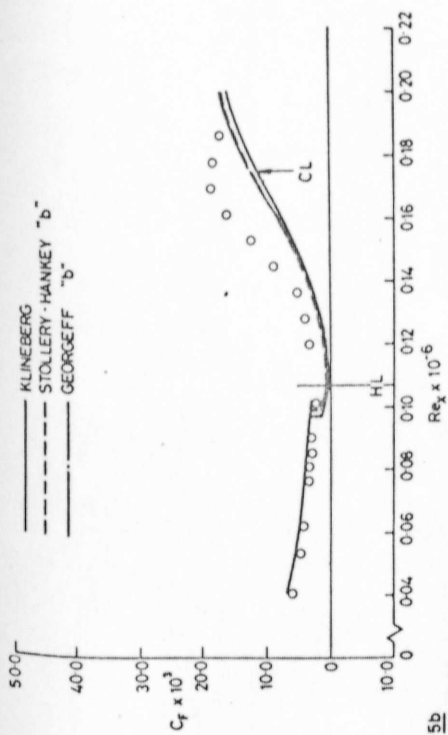
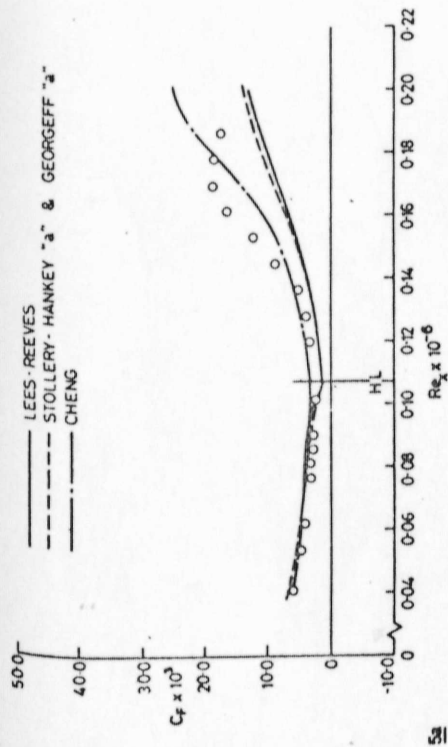
FIG.30 COMPARISON OF THEORETICAL RESULTS WITH DATA OF BLOY AT WEDGE ANGLE 12°



(e & g) "COUPLED PROFILE" CHENG METHODS

(f & h) "UNCOUPLED PROFILE" METHODS

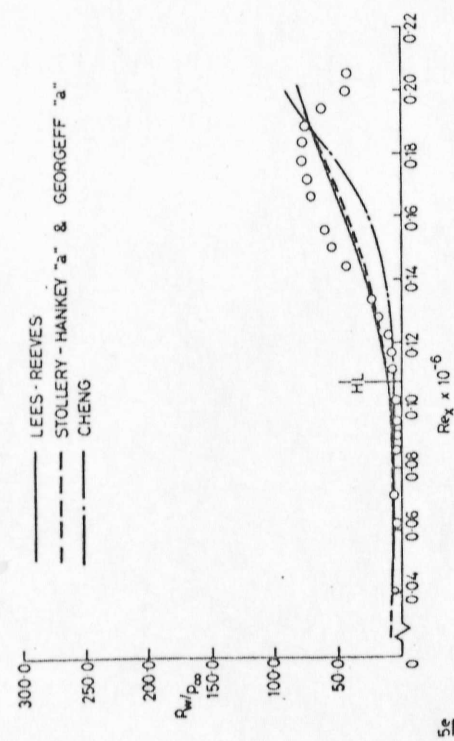
FIG. 30 CONCLUDED



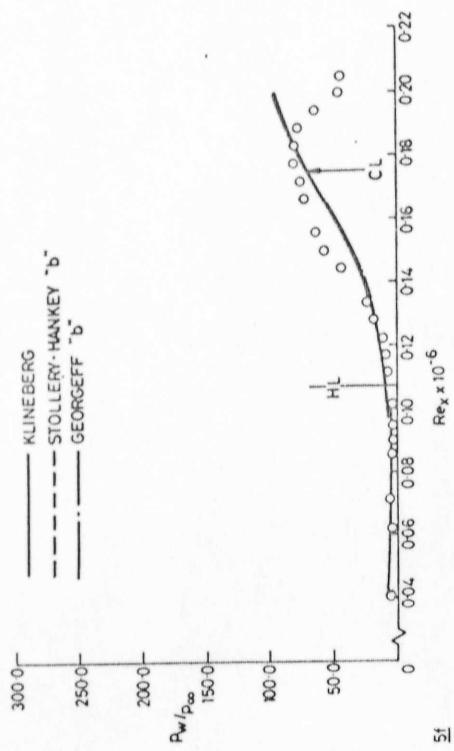
(a&c) "COUPLED PROFILE" & CHENG METHODS

(b&d) "UNCOUPLED PROFILE" METHODS

FIG. 31 COMPARISON OF THEORETICAL RESULTS WITH DATA OF HOLDEN AT WEDGE ANGLE 17.8°

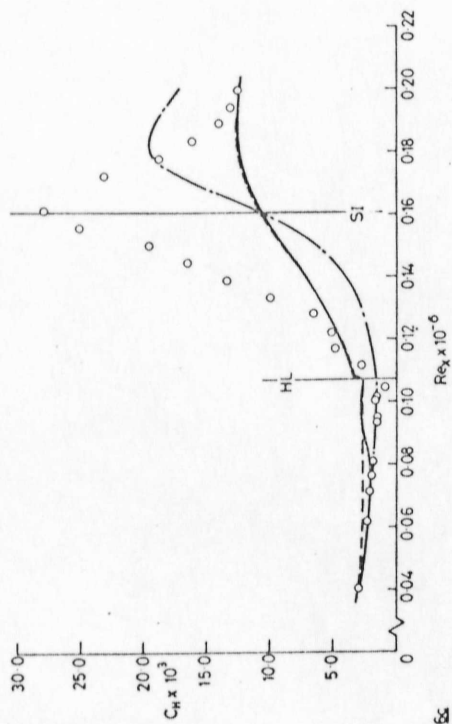
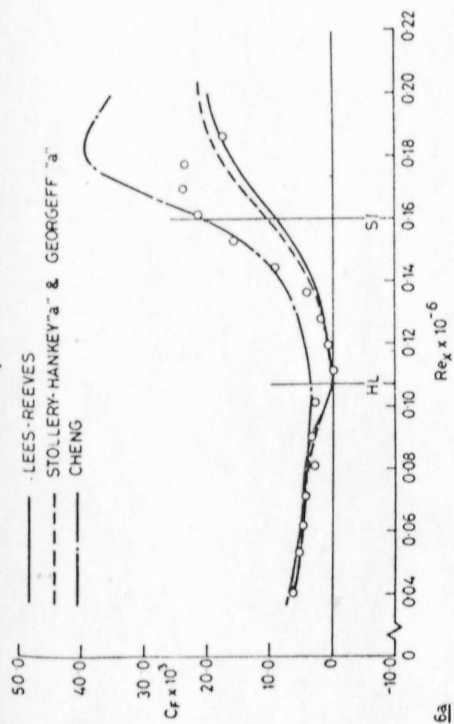


"COUPLED PROFILE" & CHENG METHODS

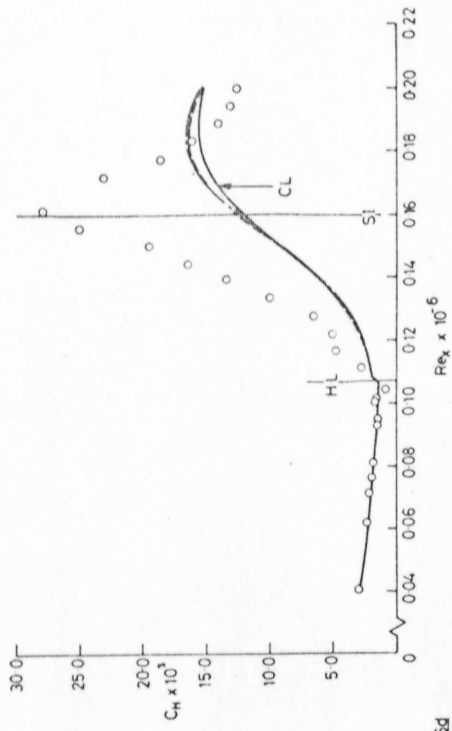
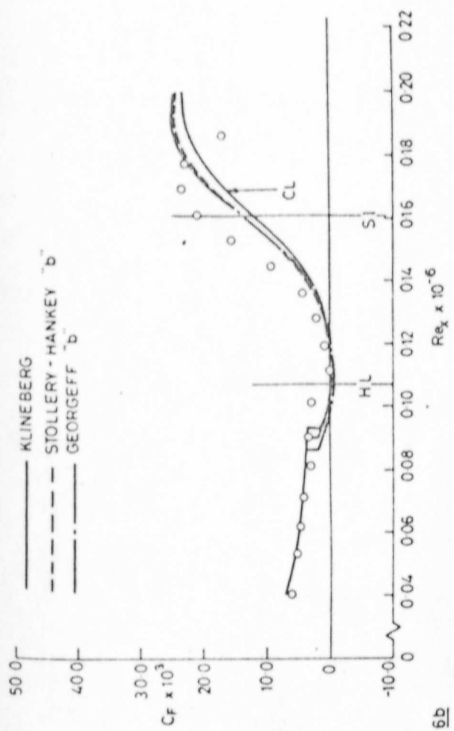


"UNCOUPLED PROFILE" METHODS

FIG. 31 CONCLUDED

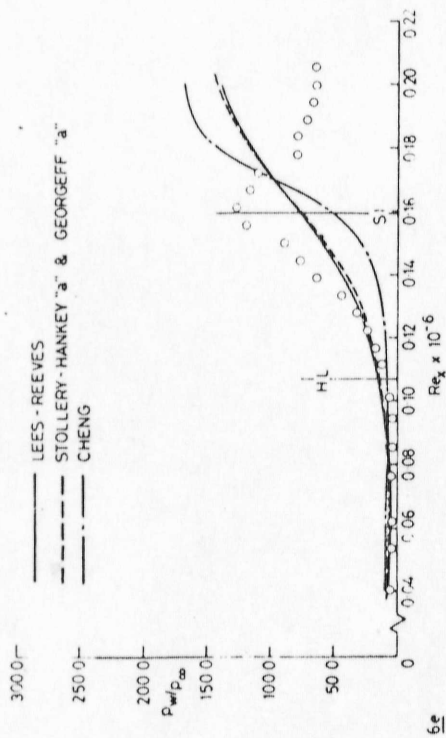


(a & c) "COUPLED PROFILE" & CHENG METHODS

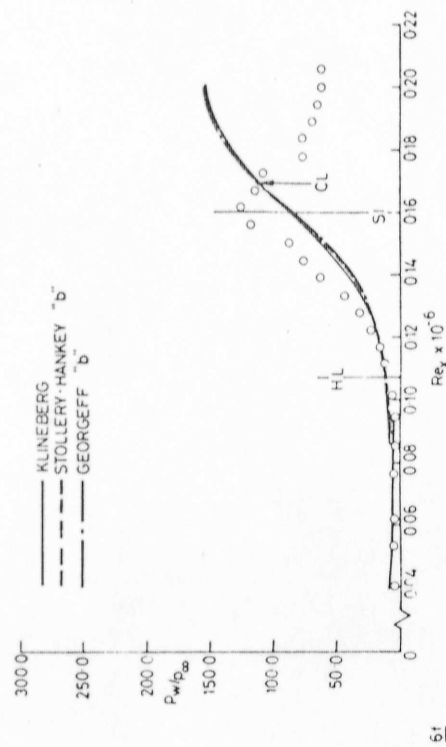


(b & d) "UNCOUPLED PROFILE" METHODS

FIG. 32 COMPARISON OF THEORETICAL RESULTS WITH DATA OF HOLDEN AT WEDGE ANGLE 21.6°

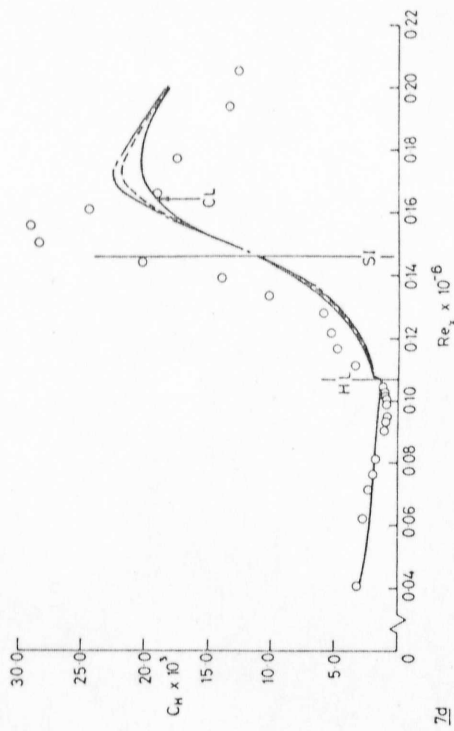
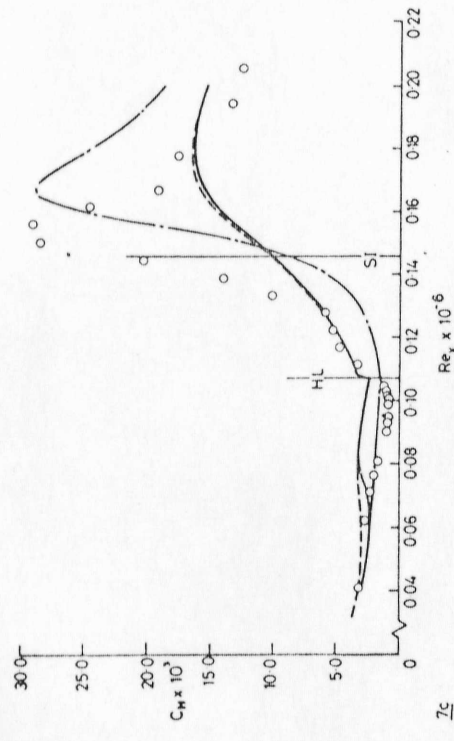
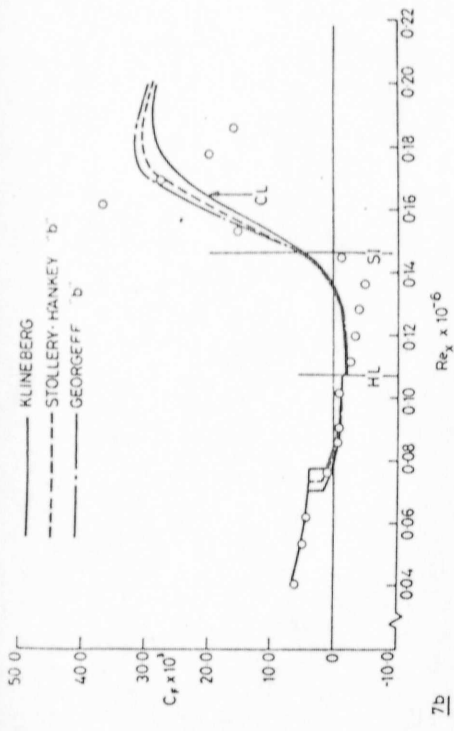
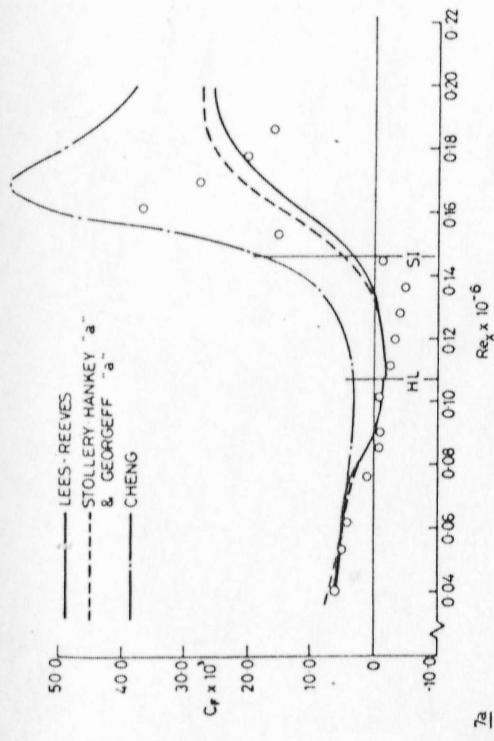


"COUPLED PROFILE" & CHENG METHODS



"UNCOUPLED PROFILE" METHODS

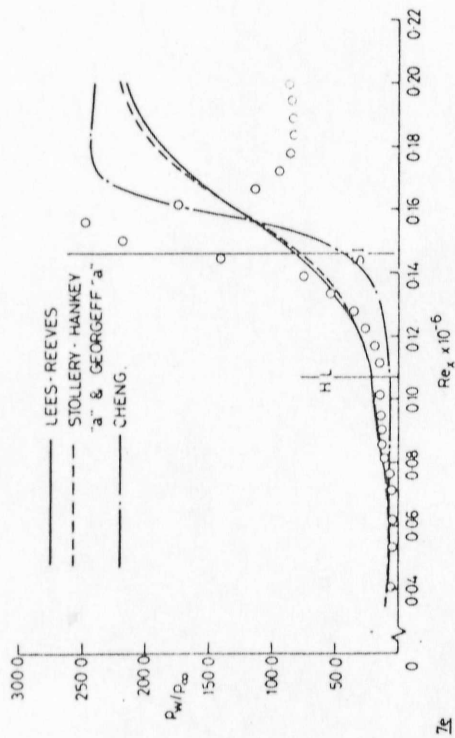
FIG. 32 CONCLUDED



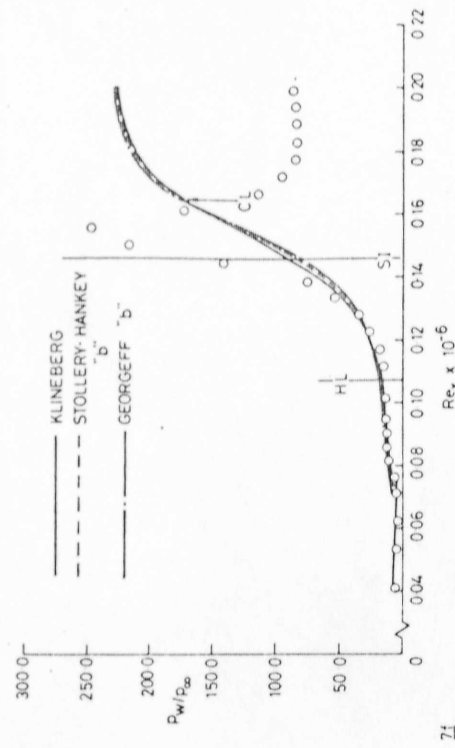
(a & c) "COUPLED PROFILE" & CHENG METHODS

(b & d) "UNCOUPLED PROFILE" METHODS

FIG. 33 COMPARISON OF THEORETICAL RESULTS WITH DATA OF HOLDEN AT WEDGE ANGLE 25.7°



"COUPLED PROFILE" & CHENG METHODS



"UNCOUPLED PROFILE" METHODS

FIG. 33 CONCLUDED

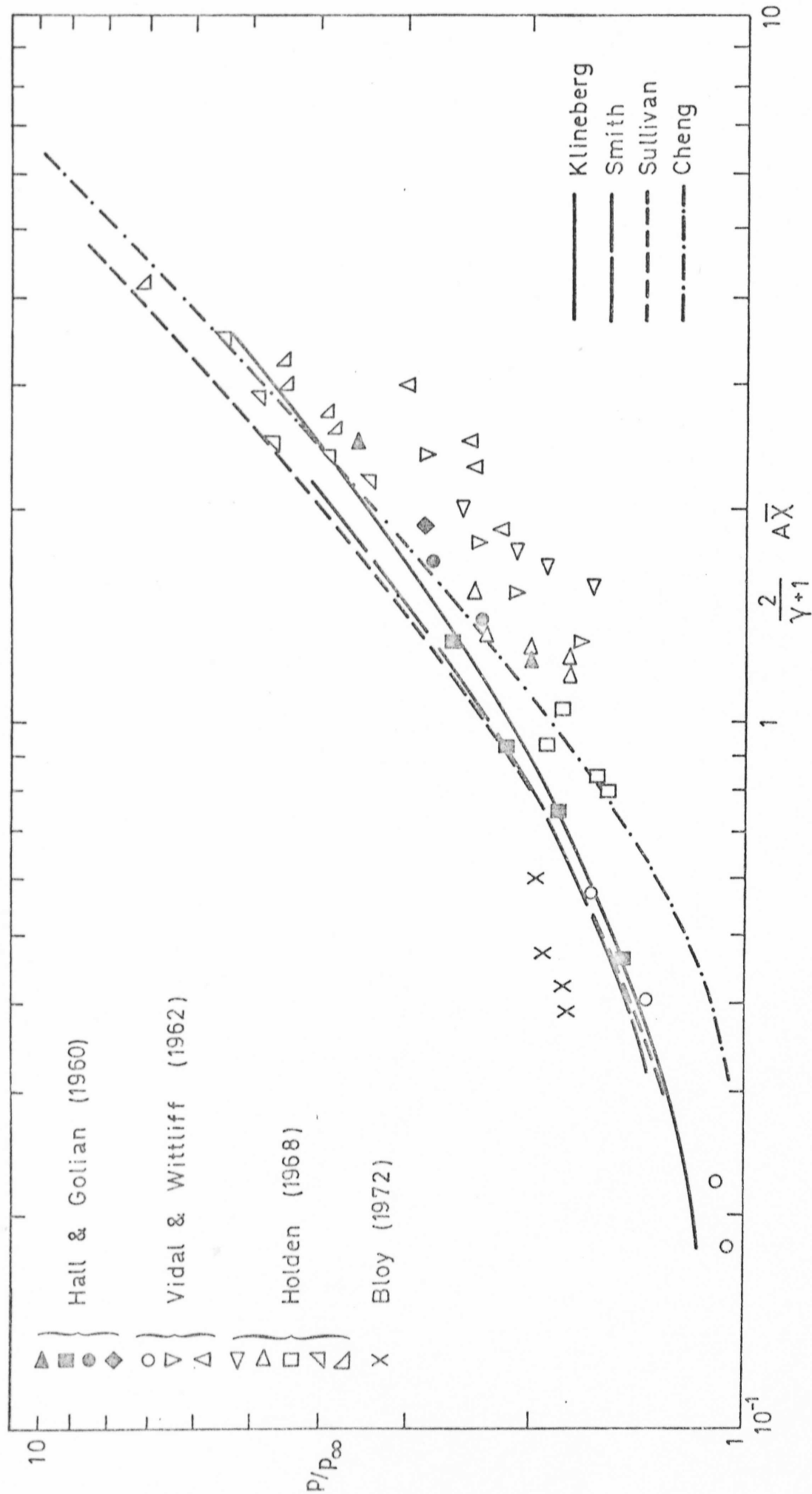


FIG. 34 FLAT PLATE PRESSURE DISTRIBUTION.

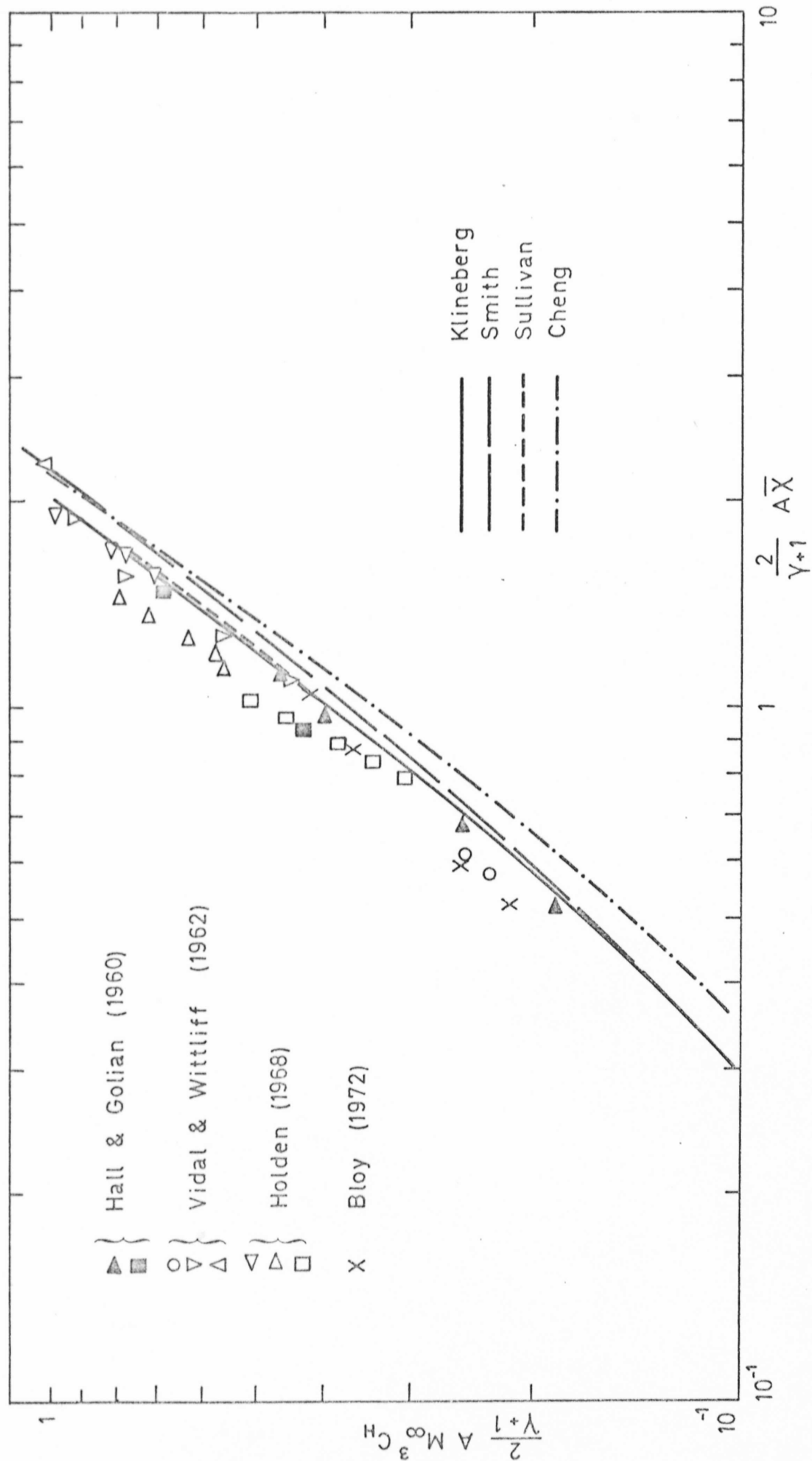


FIG. 35 FLAT PLATE HEAT TRANSFER RATE DISTRIBUTIONS.

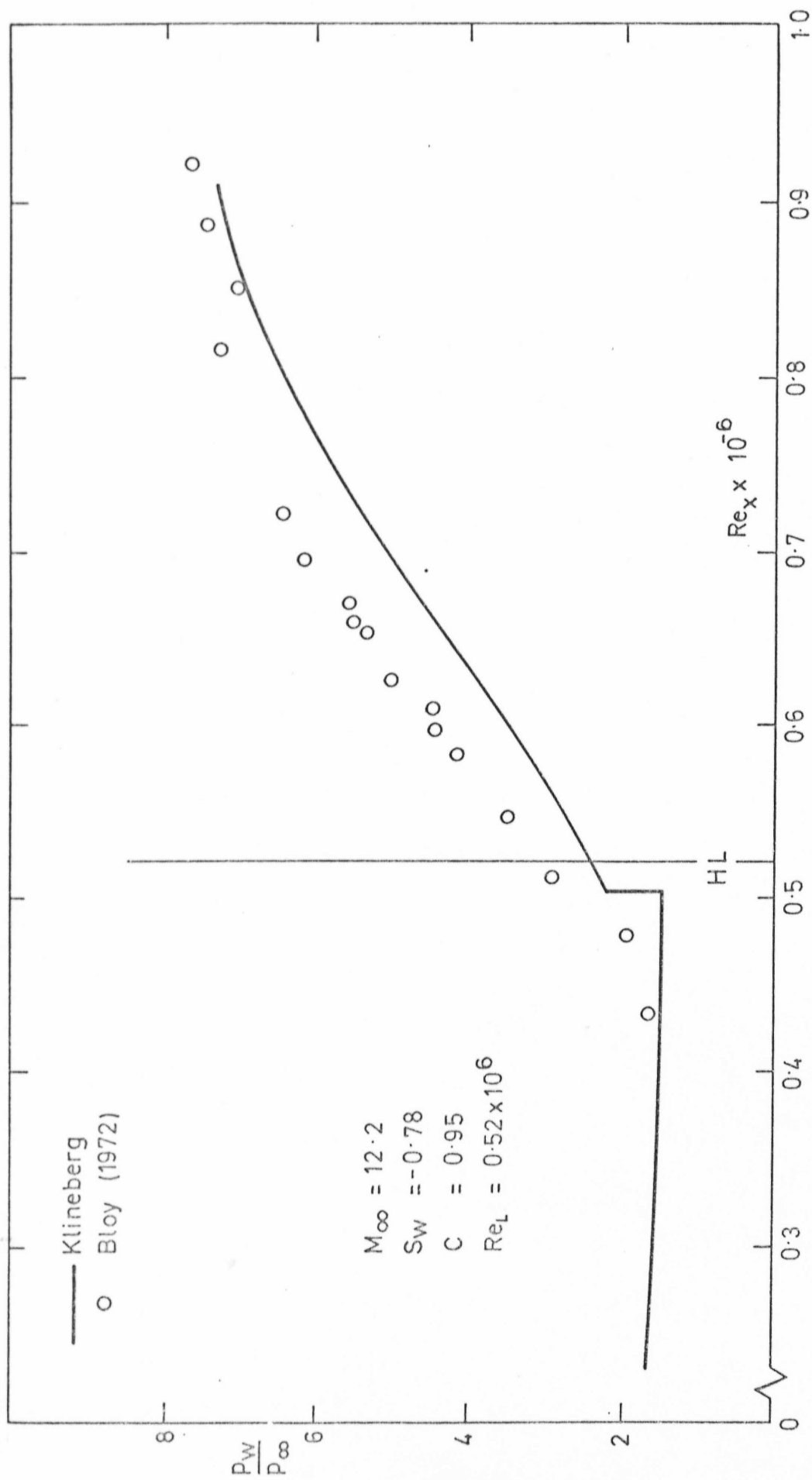


FIG. 36 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 8° .

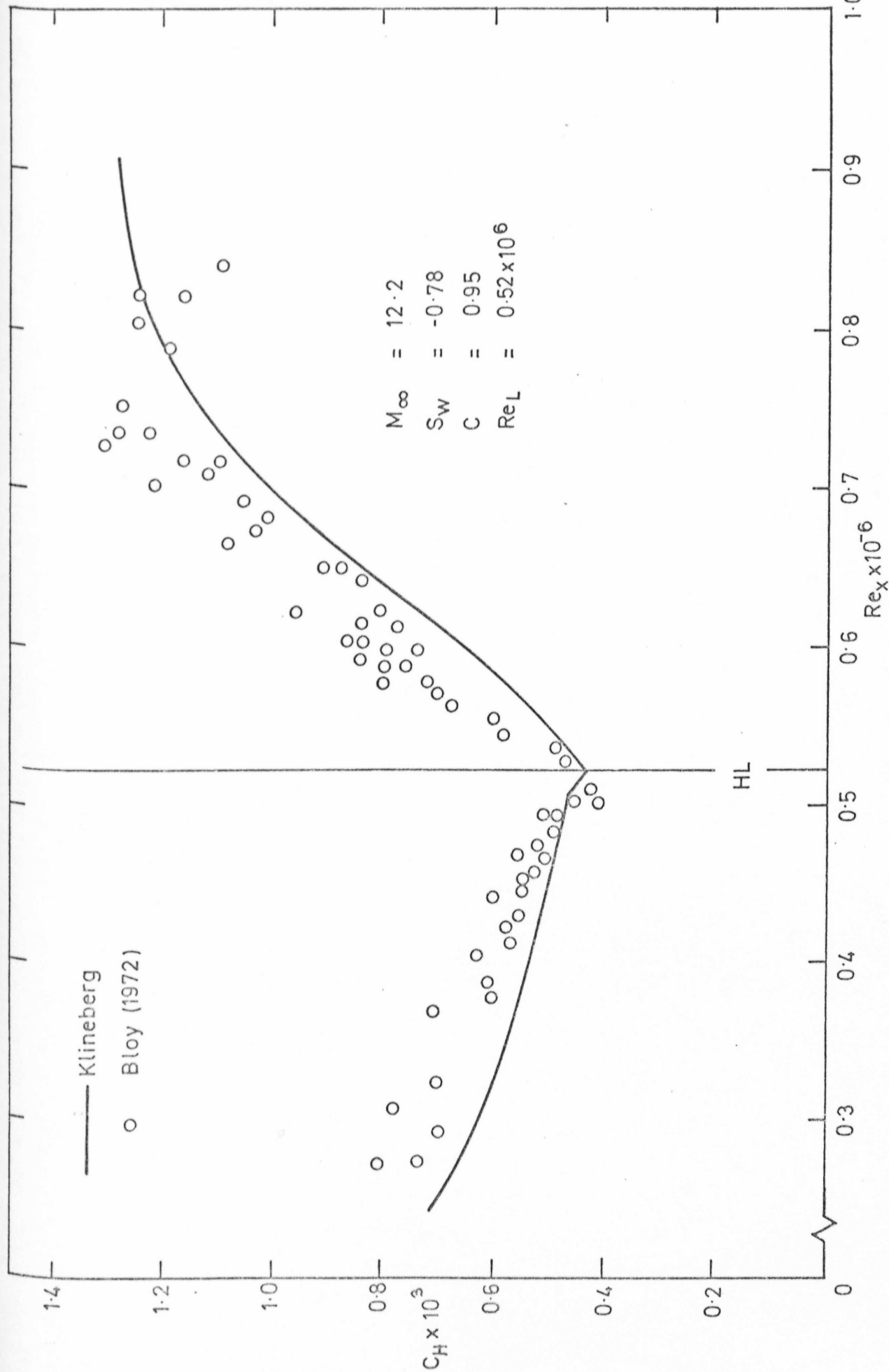


FIG. 37 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT COMPRESSION CORNER AT WEDGE ANGLE 8°.

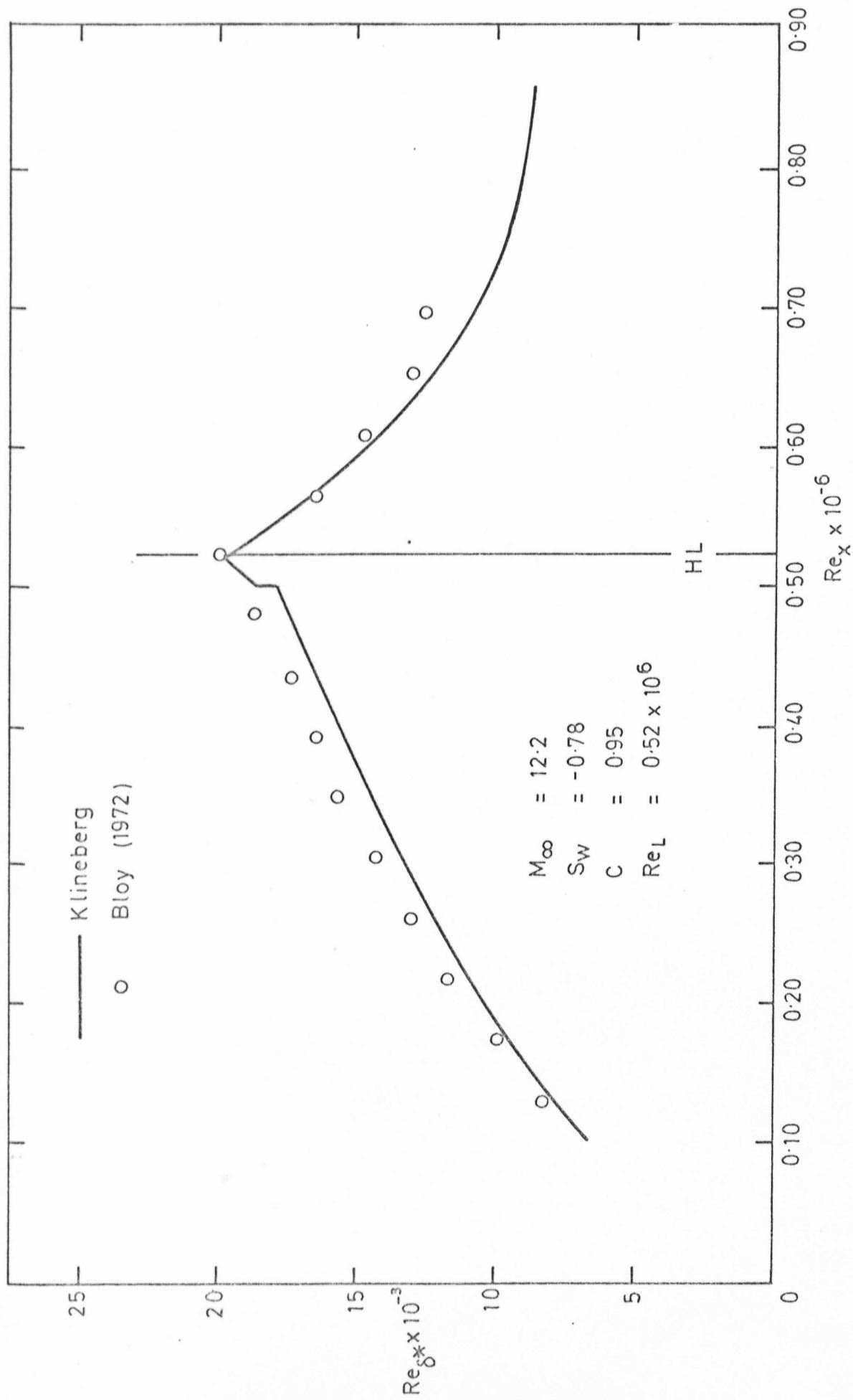


FIG. 38 DISPLACEMENT THICKNESS DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 8° .

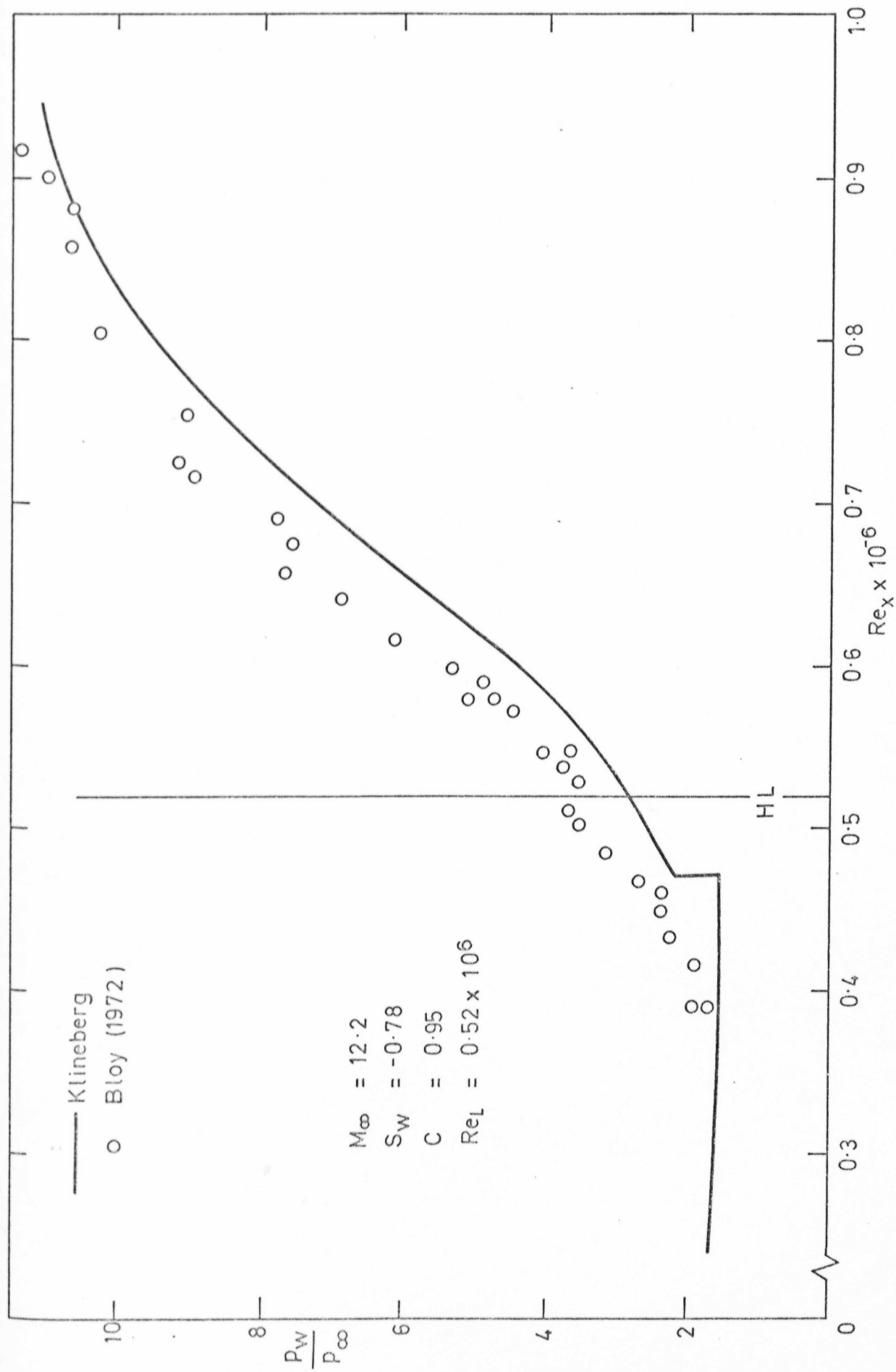


FIG. 39 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 10° .

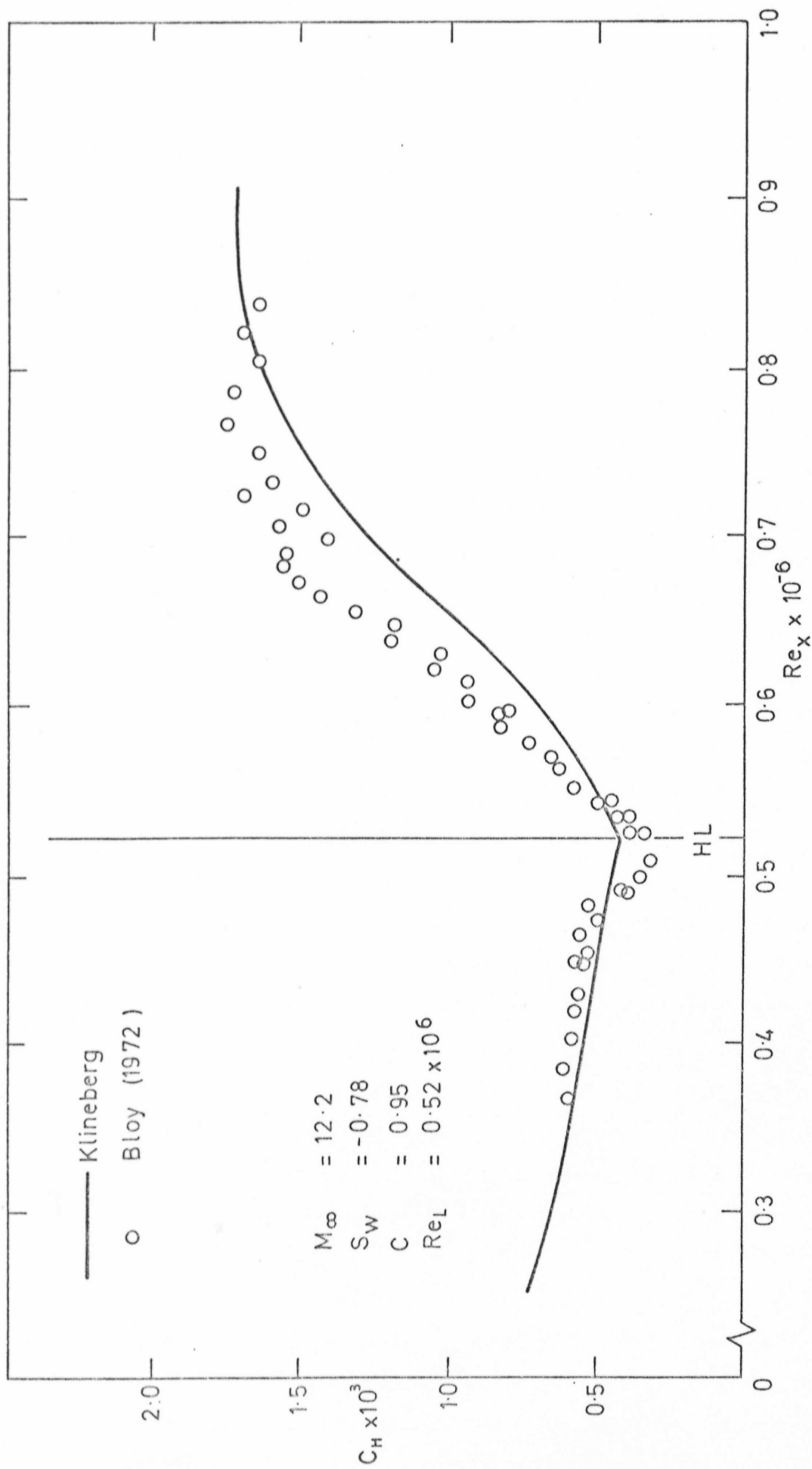


FIG. 40 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 10° .

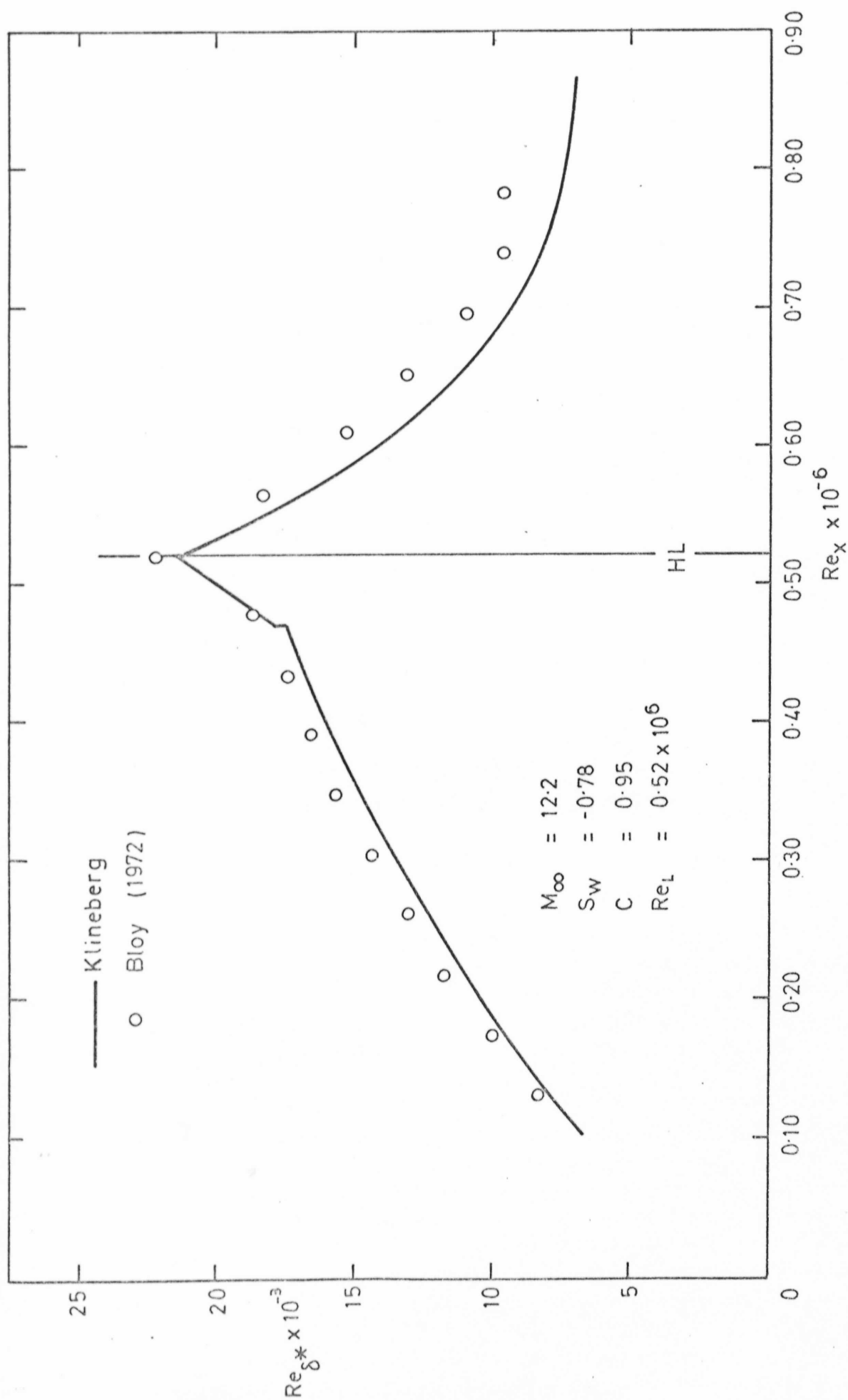


FIG. 41 DISPLACEMENT THICKNESS DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 10° .

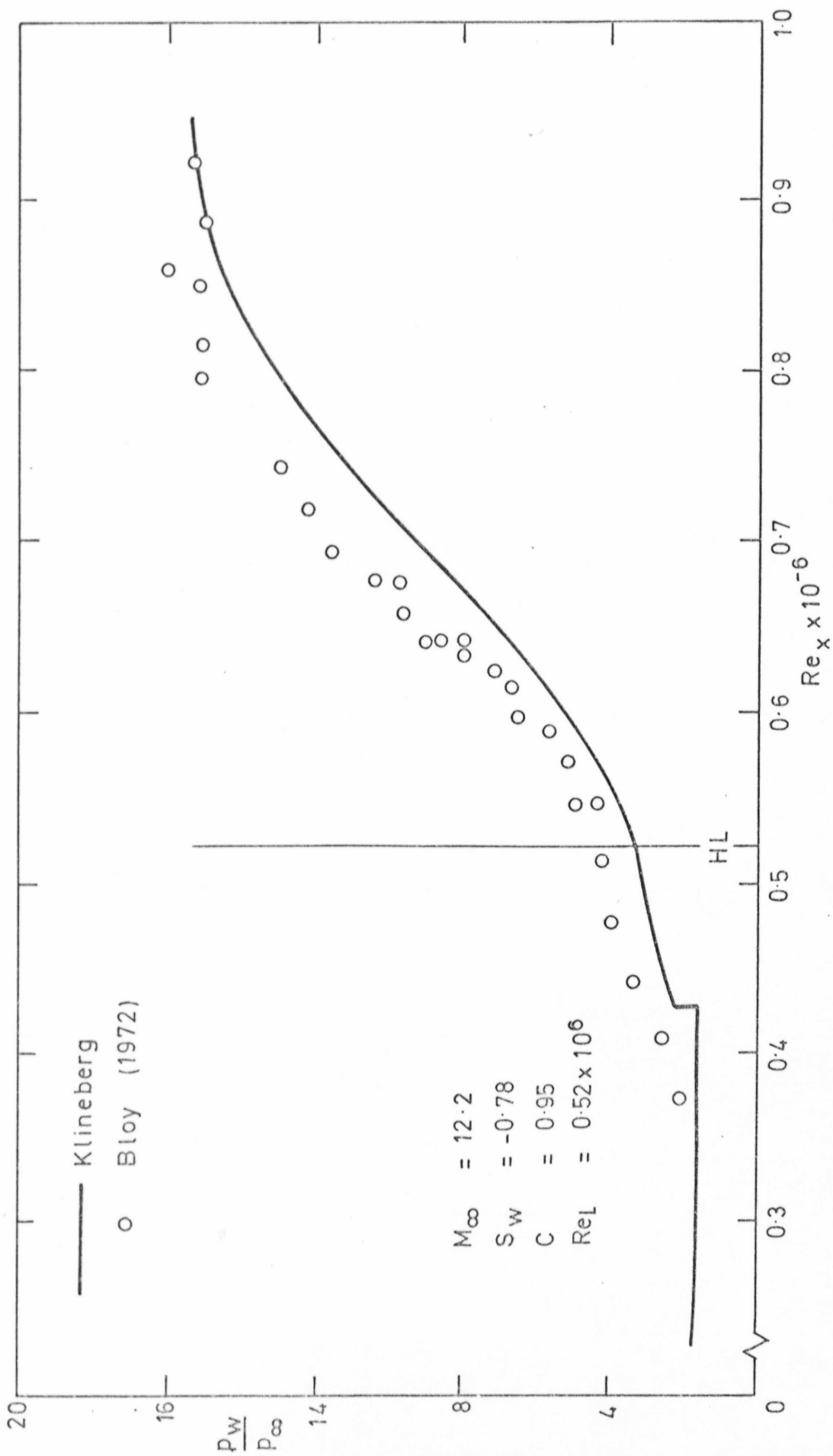


FIG. 42 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 12° .

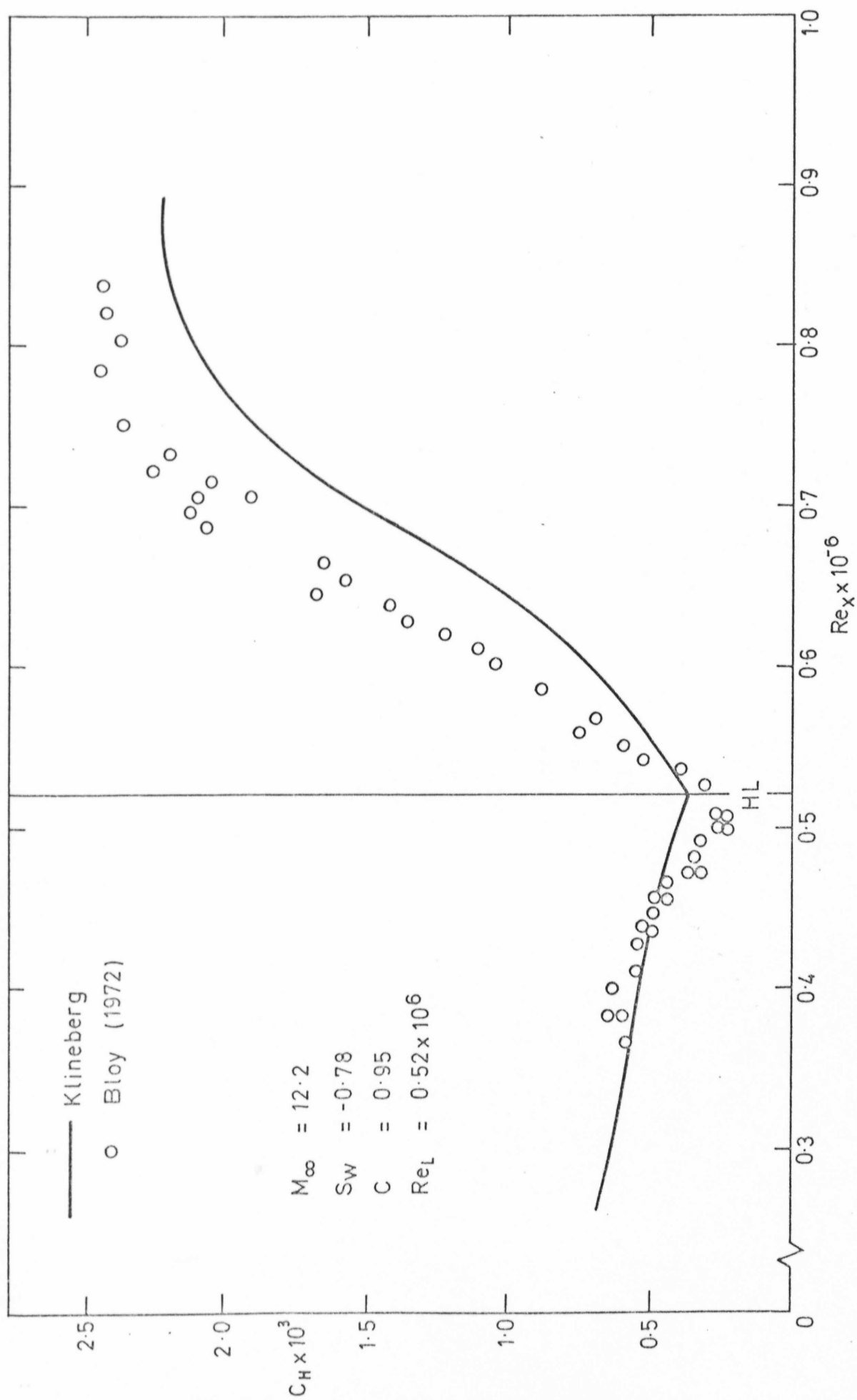


FIG. 43 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 12° .

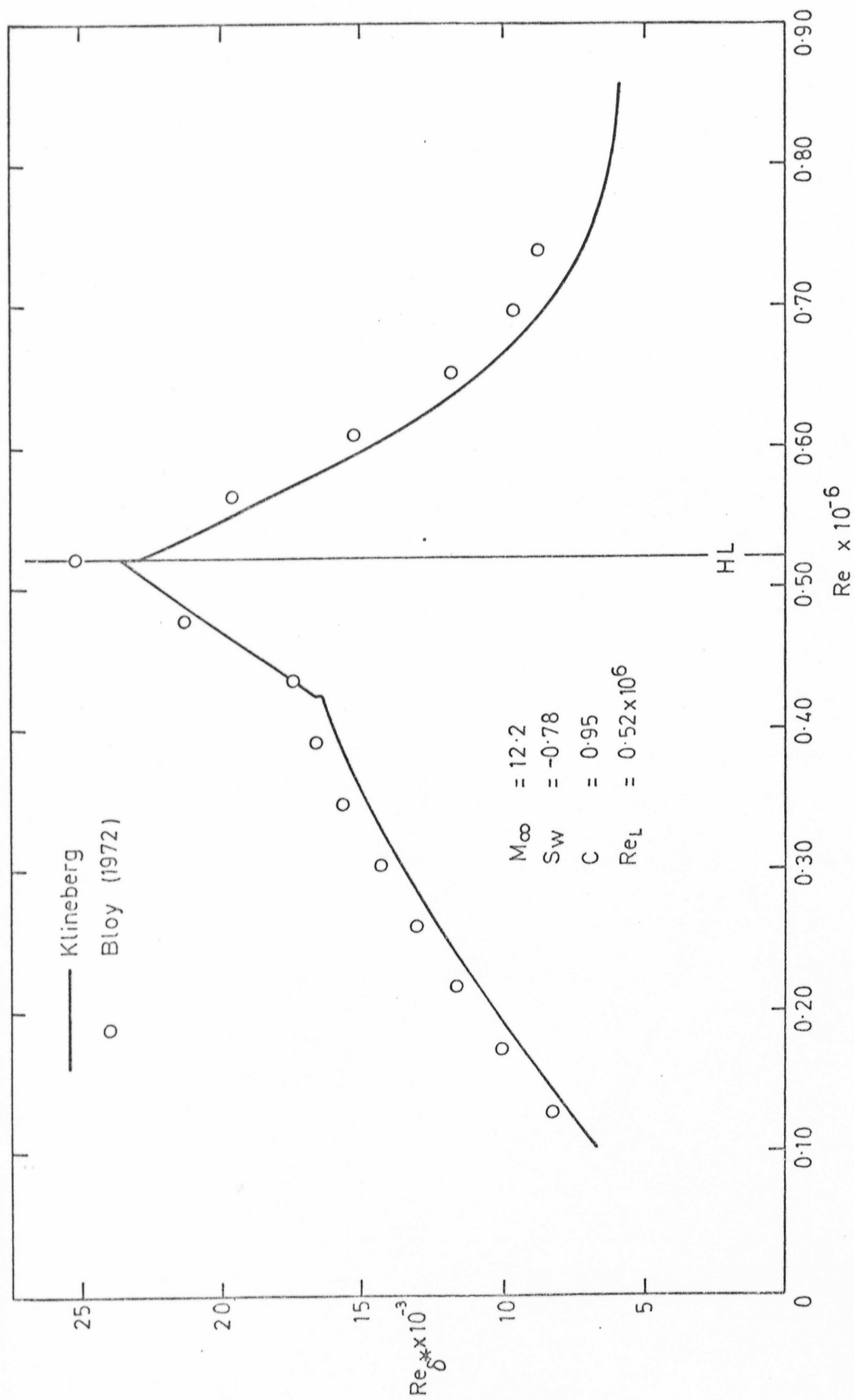


FIG. 44 DISPLACEMENT THICKNESS DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 12°

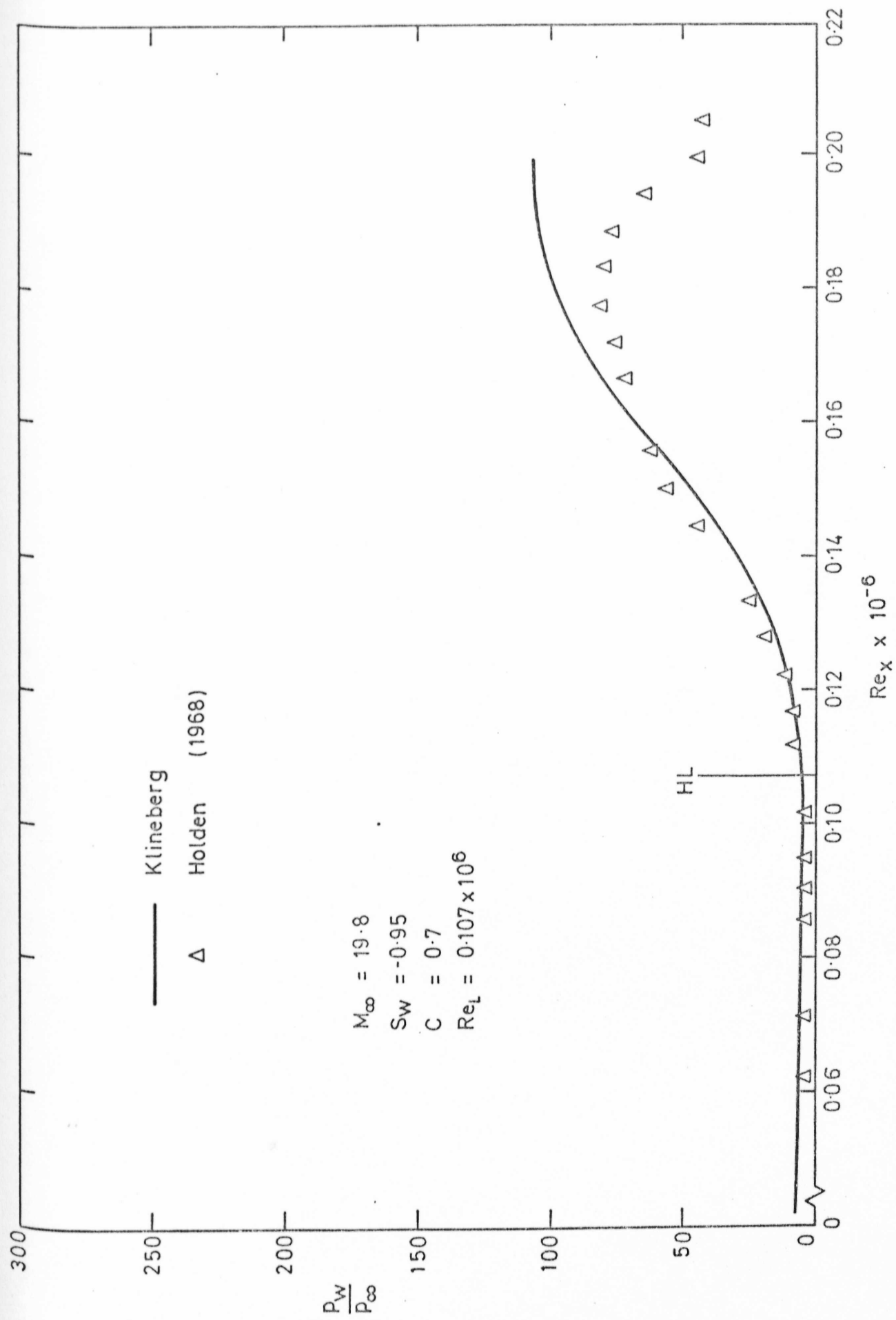


FIG. 45 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 17.8° .

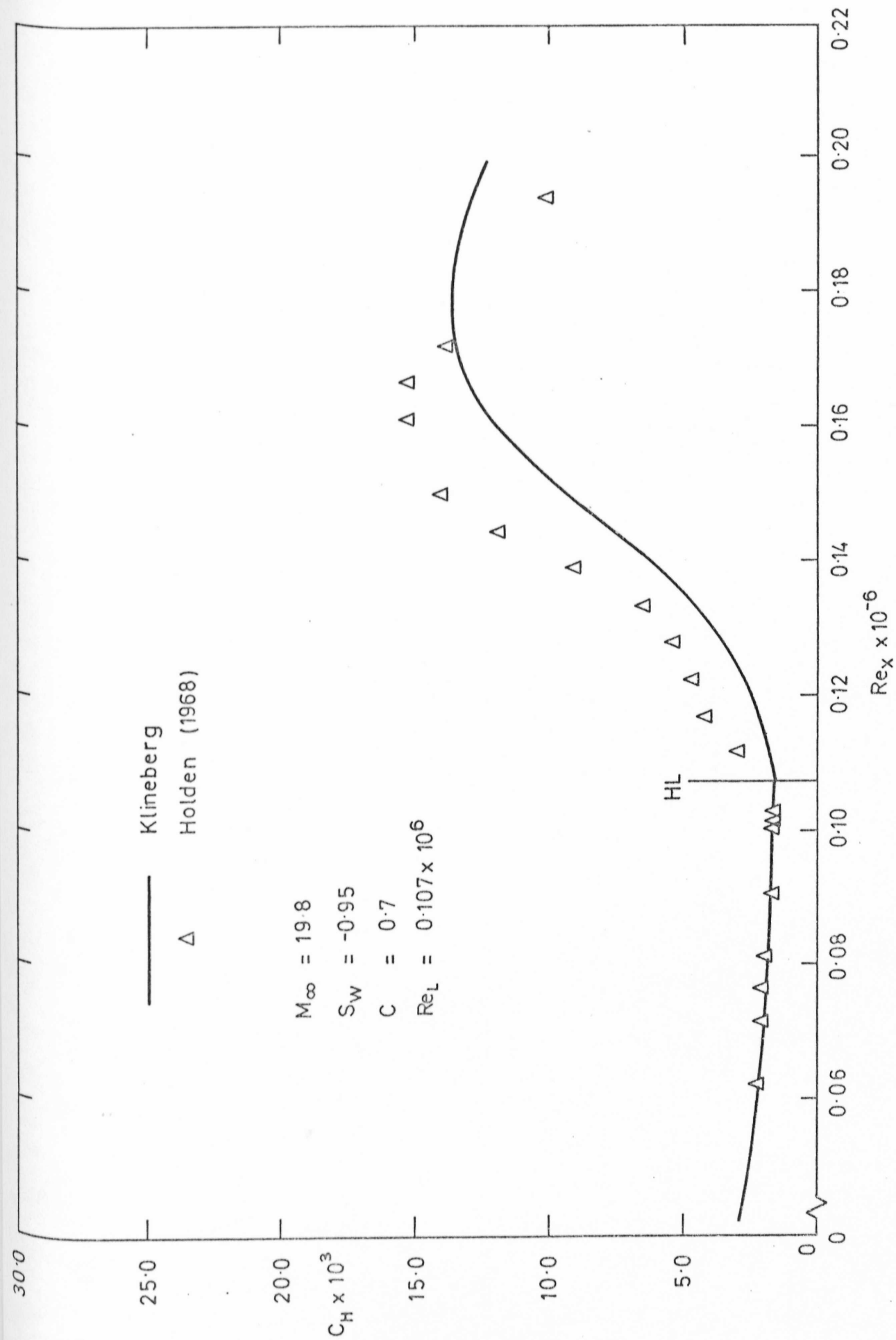


FIG. 46 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 17.8° .

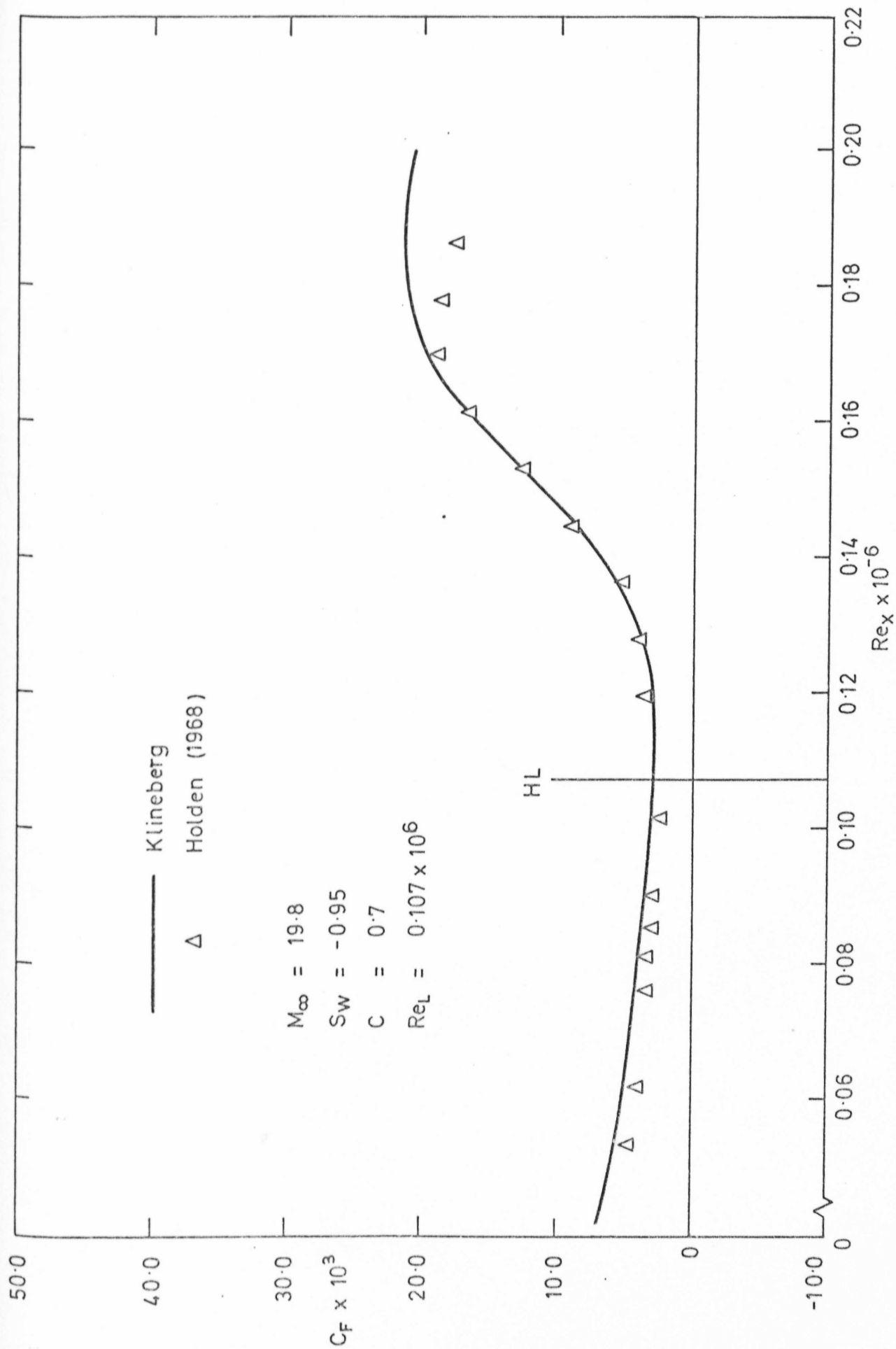


FIG. 4.7 DISTRIBUTION OF SKIN FRICTION COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 17.8° .

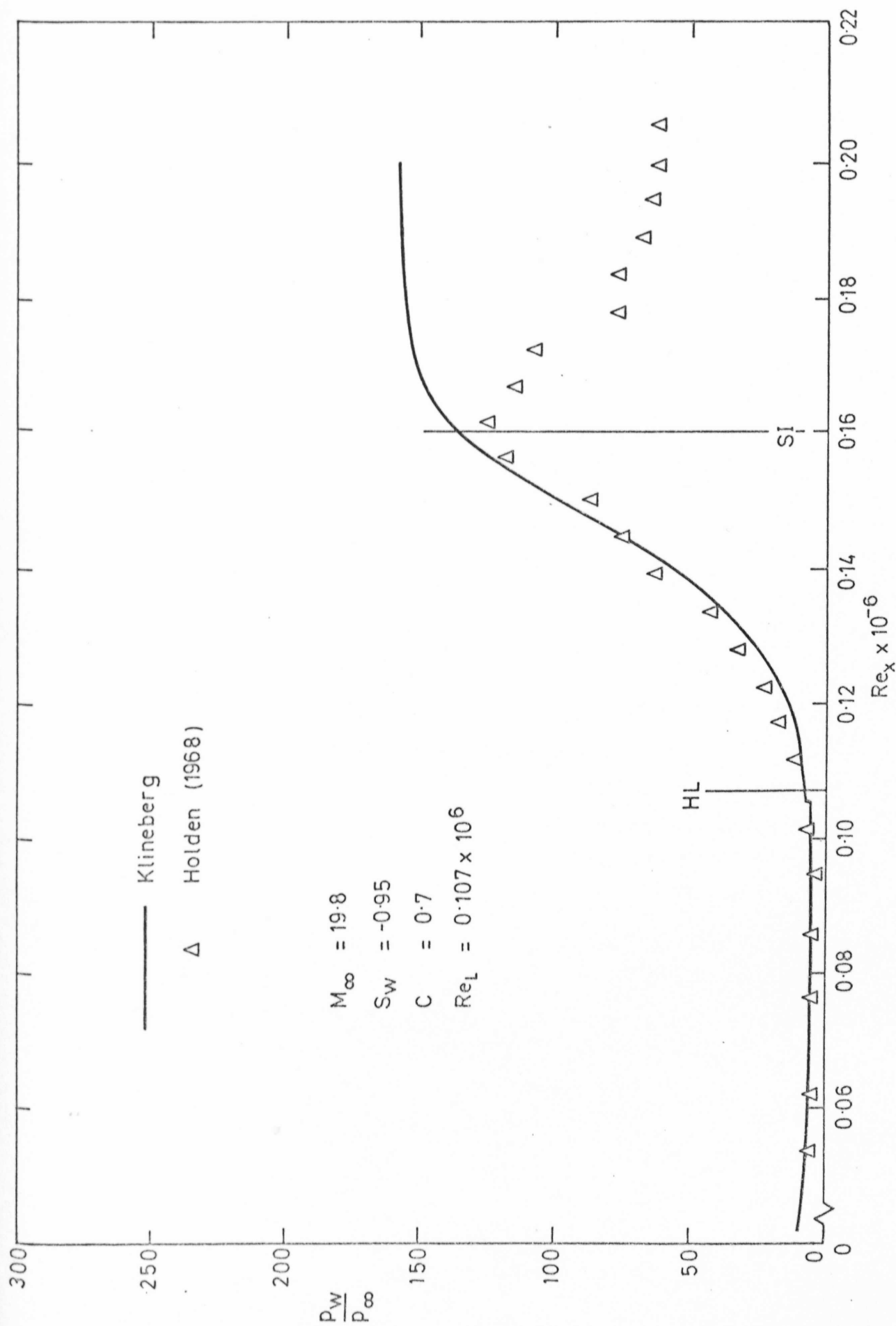


FIG. 48 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 21.6° .

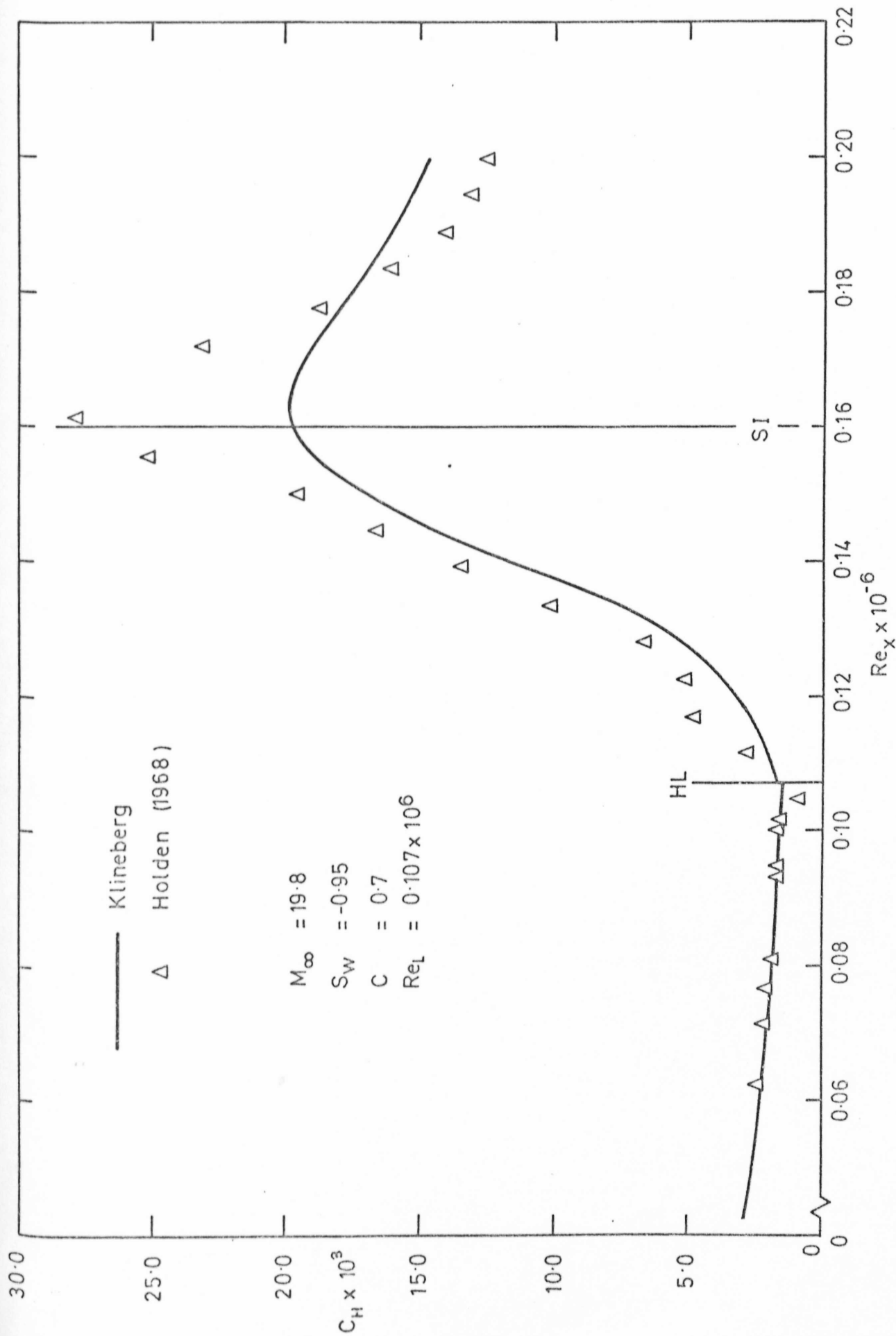


FIG. 49 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 21.6°

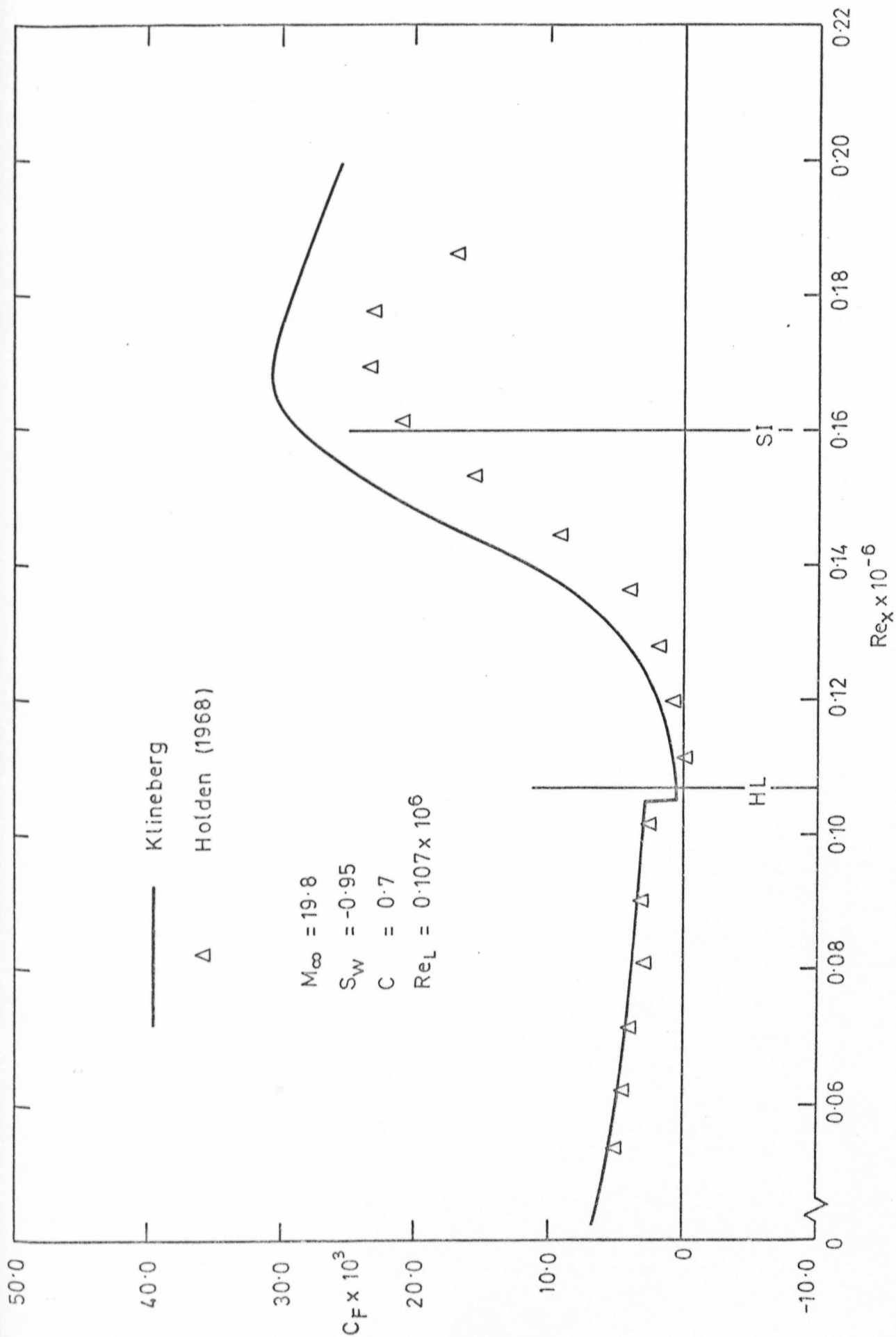


FIG. 50 DISTRIBUTION OF SKIN FRICTION COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 21.6° .

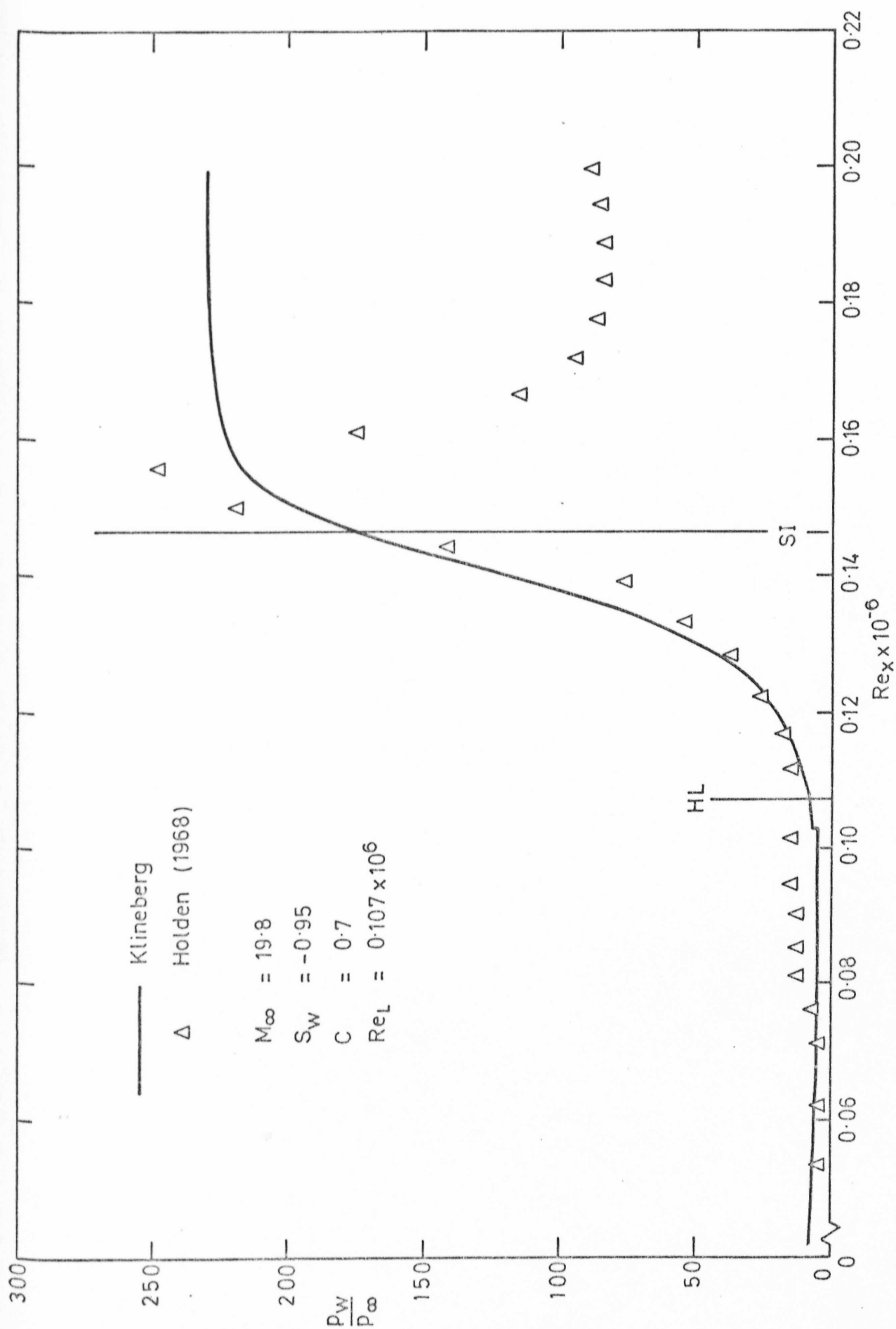


FIG. 51 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 25.7°

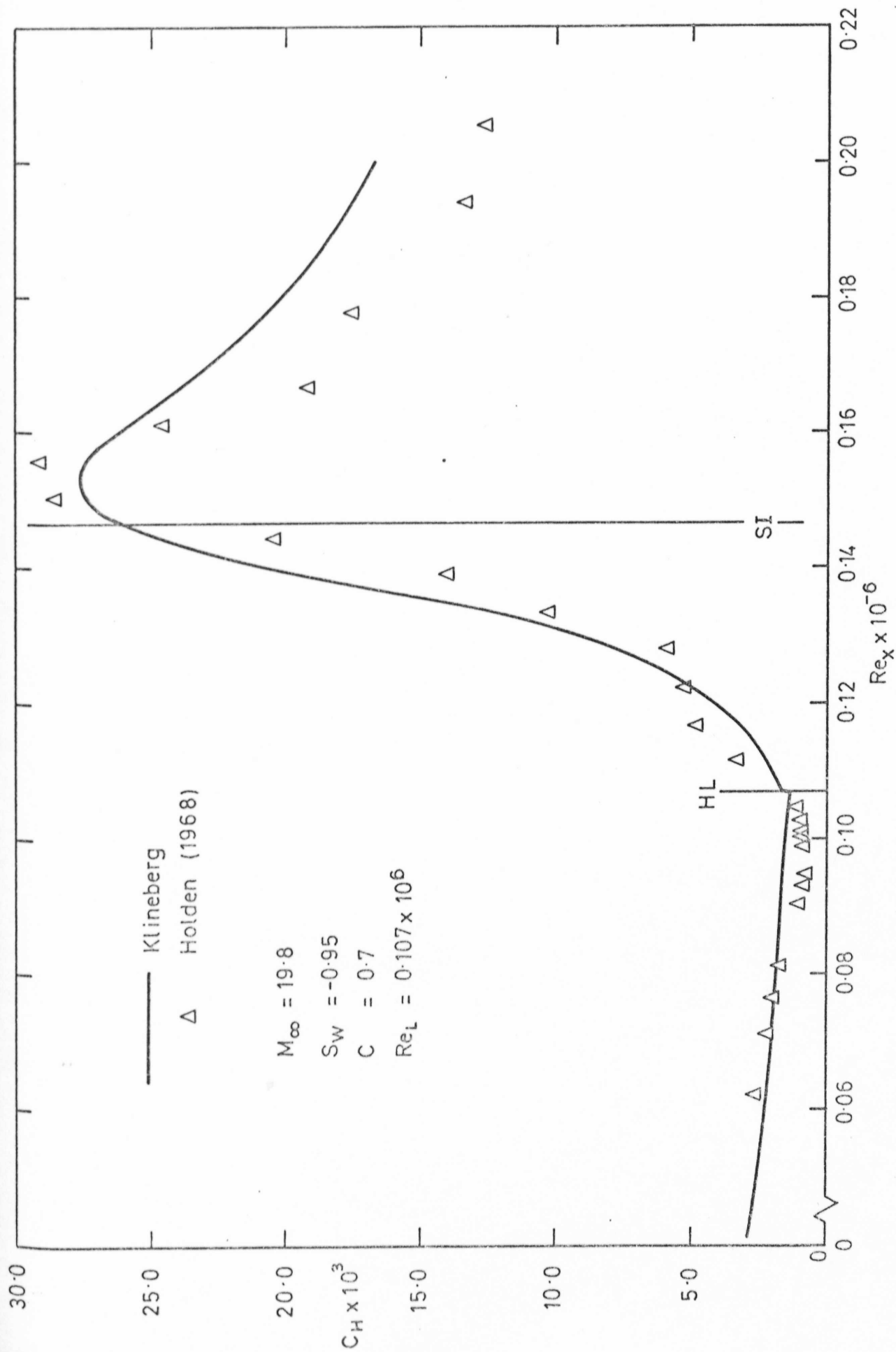


FIG. 52 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 25.7°

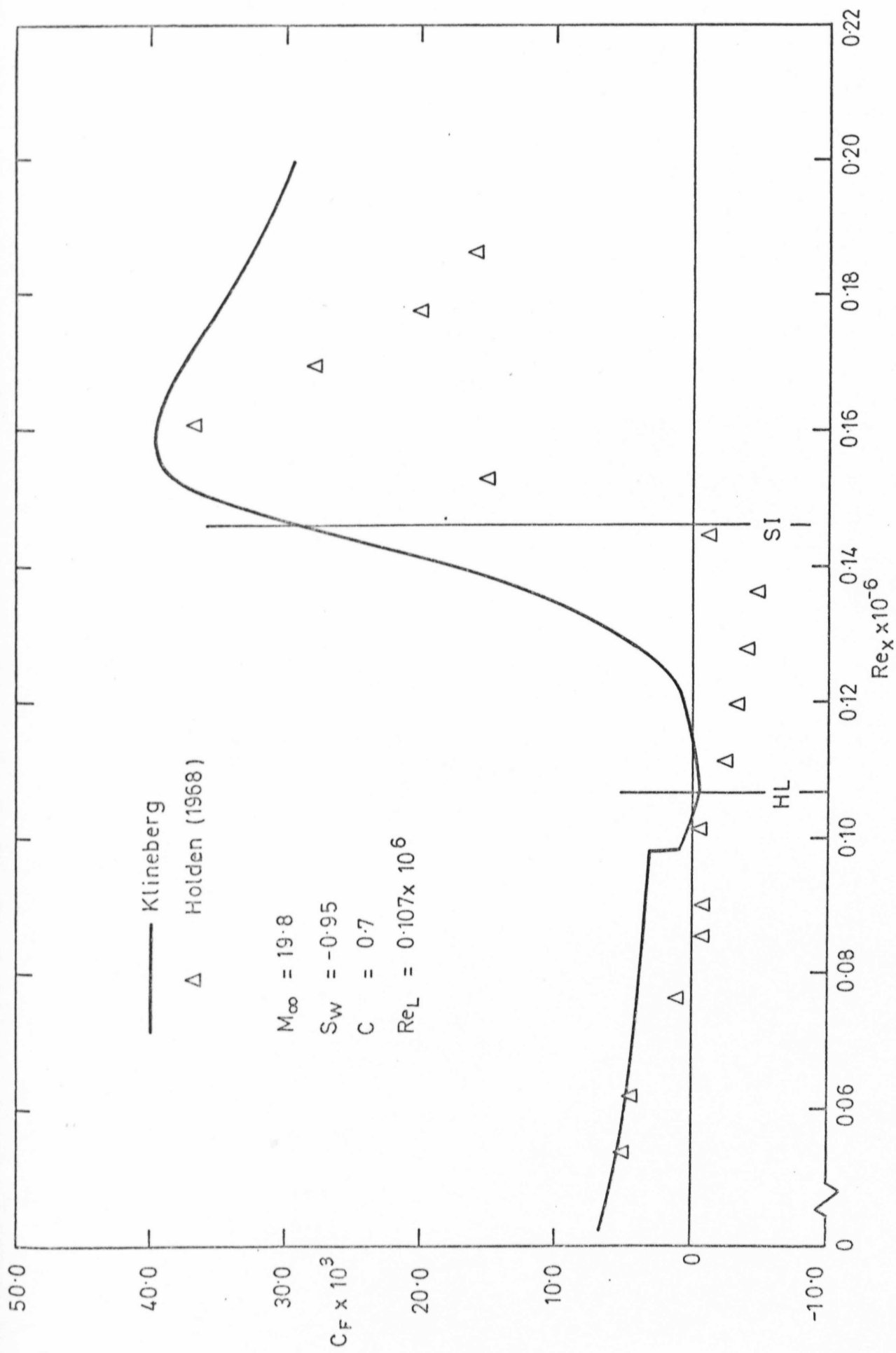


FIG. 53 DISTRIBUTION OF SKIN FRICTION COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 25.7°

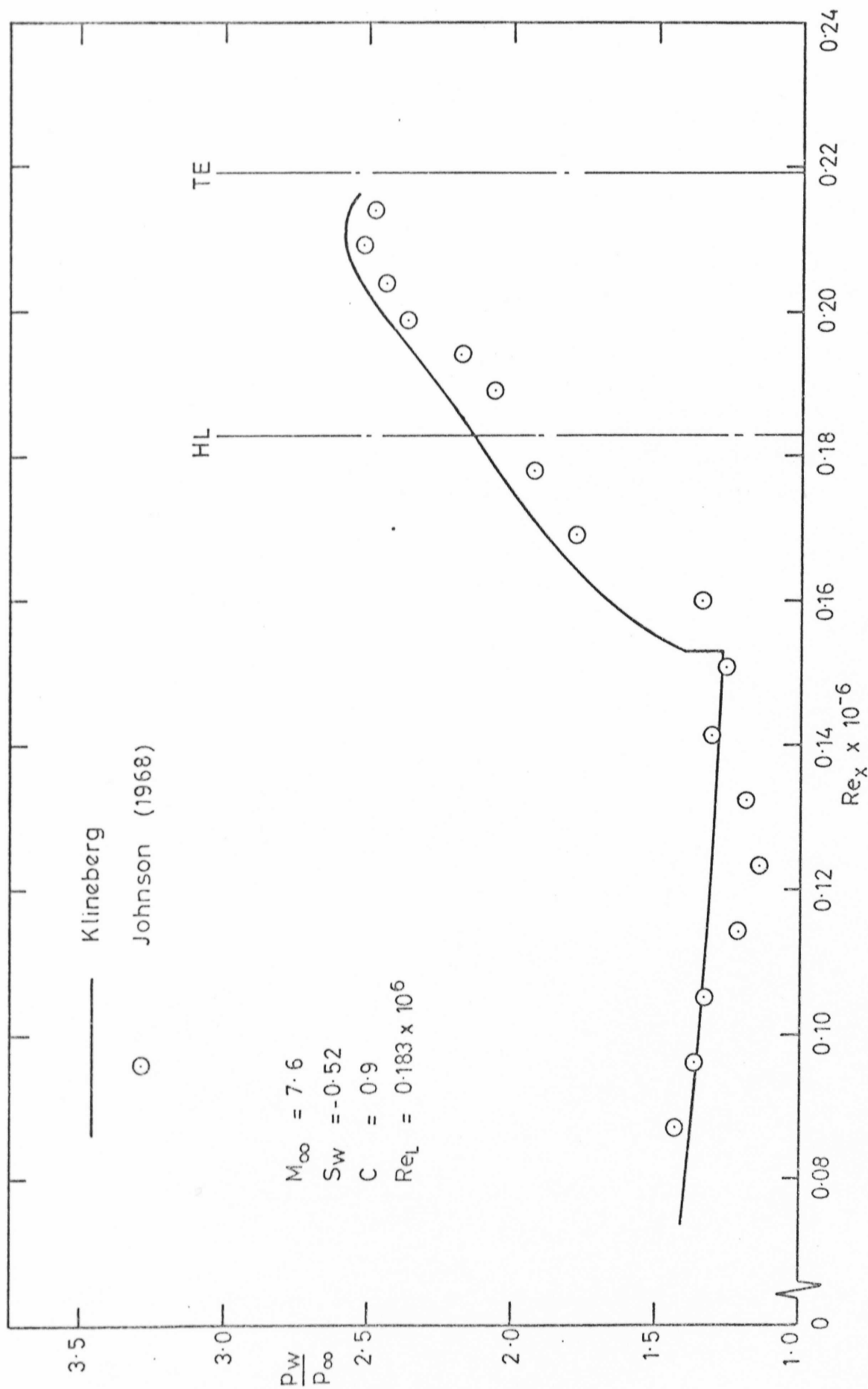


FIG. 54 PRESSURE DISTRIBUTION ON COMPRESSION CORNER AT WEDGE ANGLE 10°

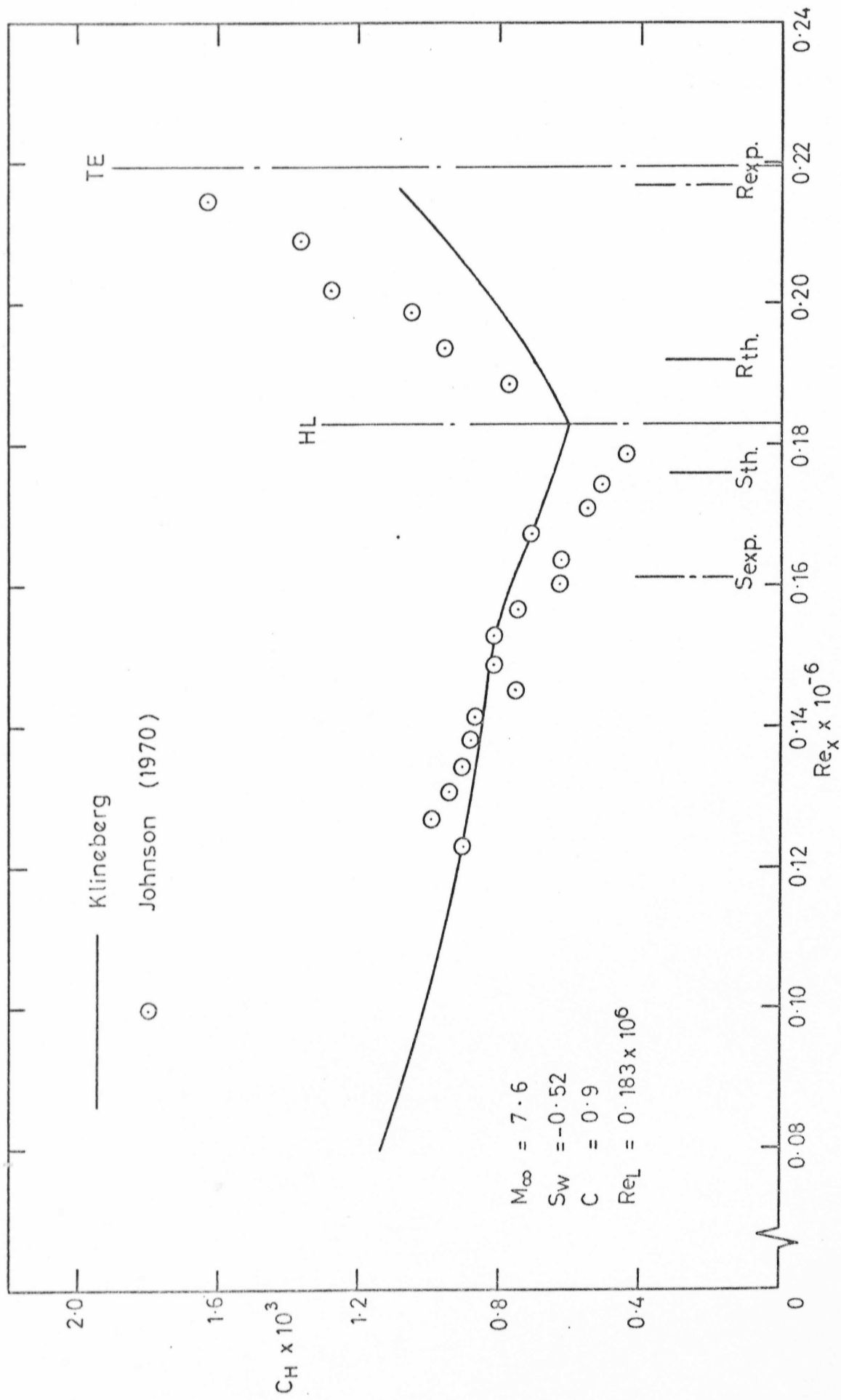


FIG. 55 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON COMPRESSION CORNER AT WEDGE ANGLE 10°

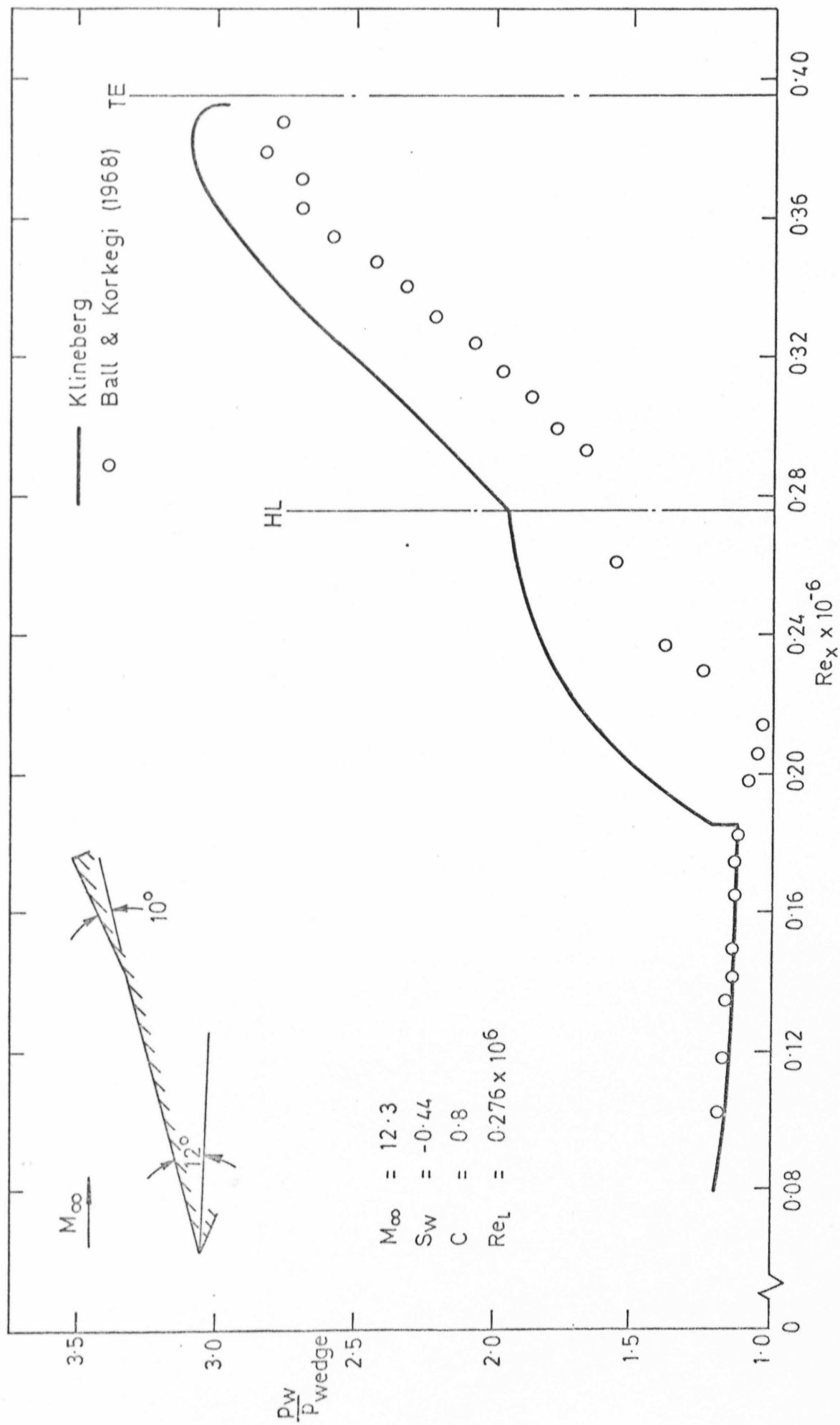


FIG. 56 PRESSURE DISTRIBUTION ON WEDGE - RAMP CONFIGURATION.

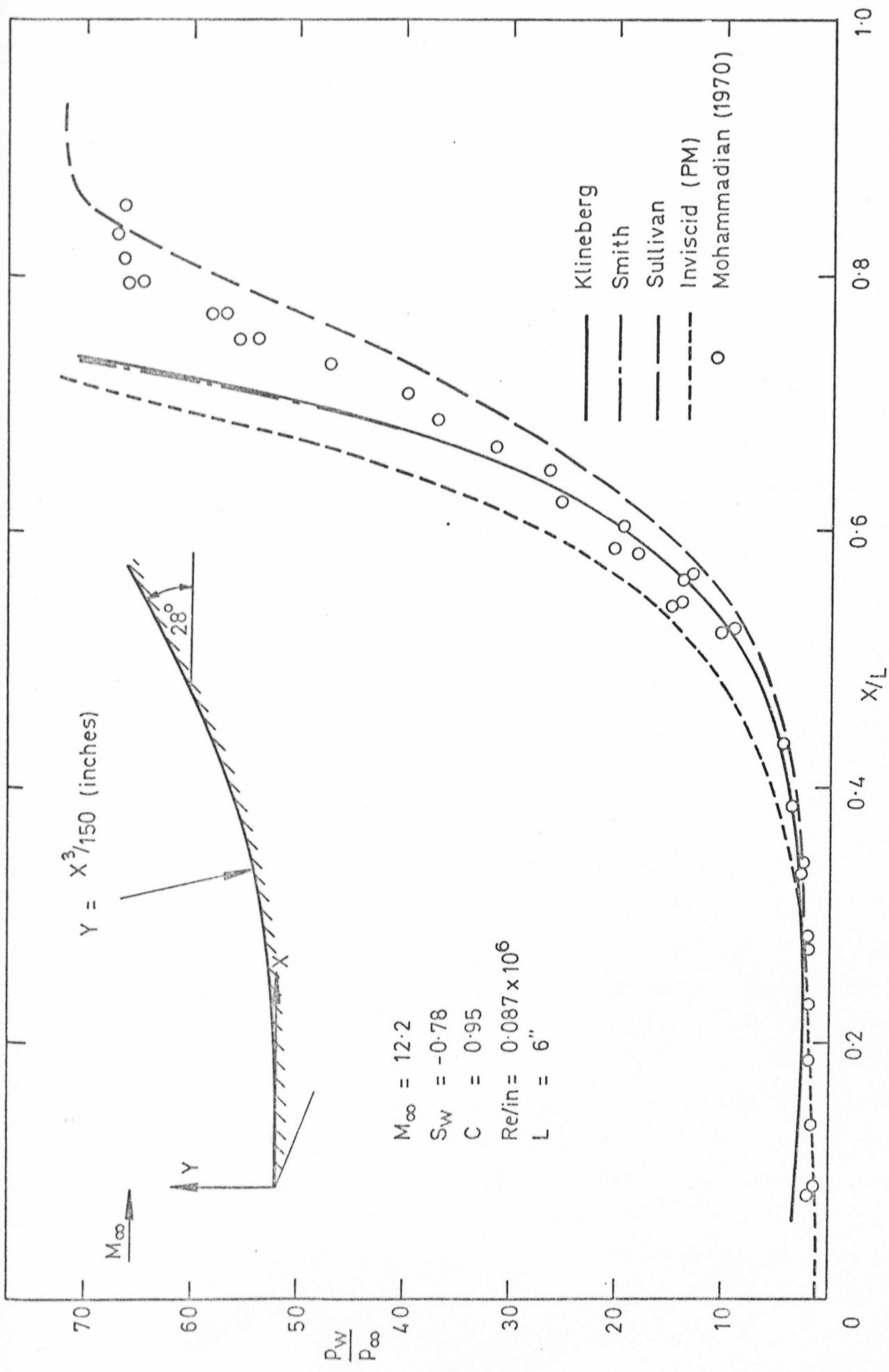


FIG. 57 PRESSURE DISTRIBUTION ON CONCAVE BODY.

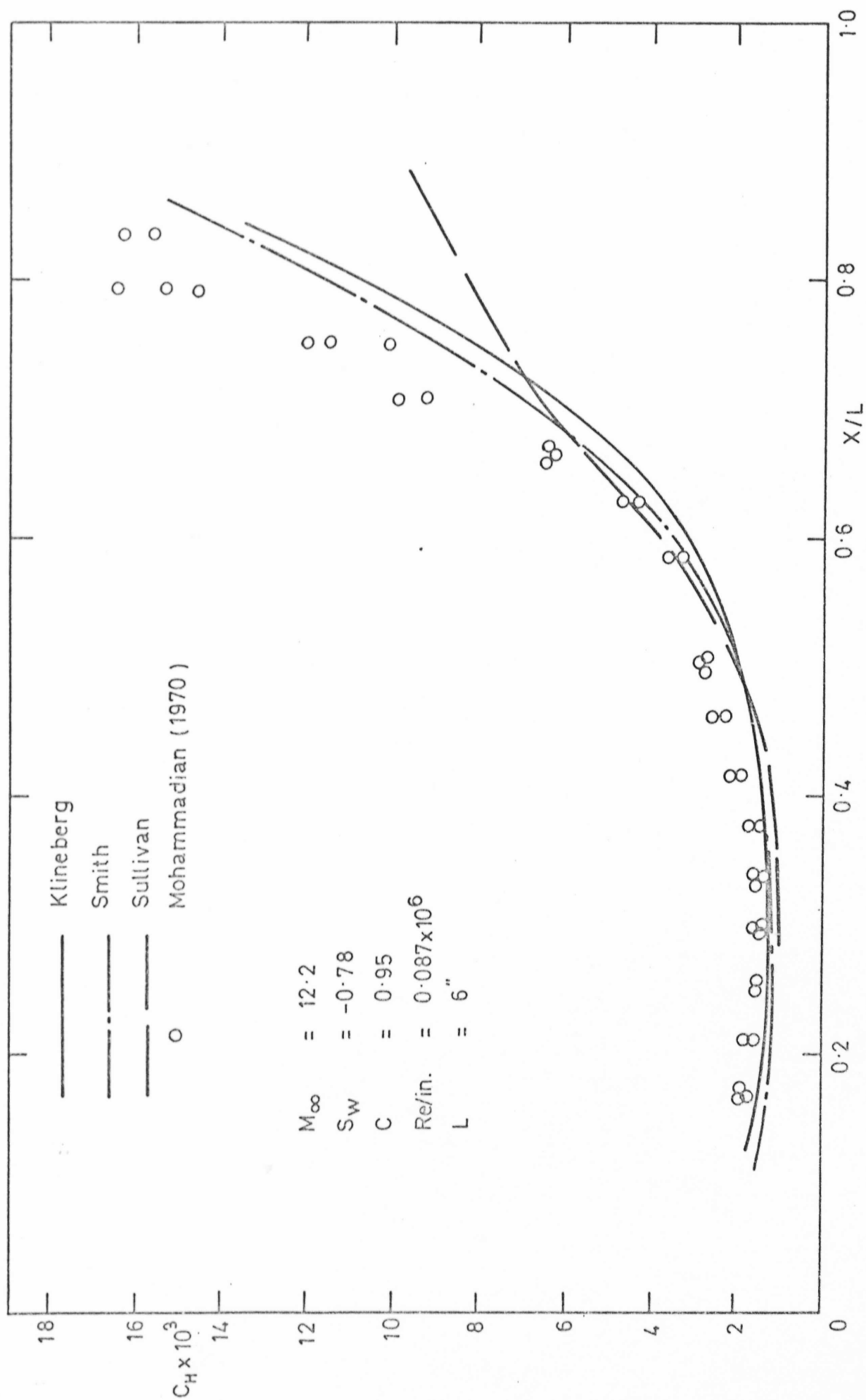


FIG. 58 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON CONCAVE BODY.

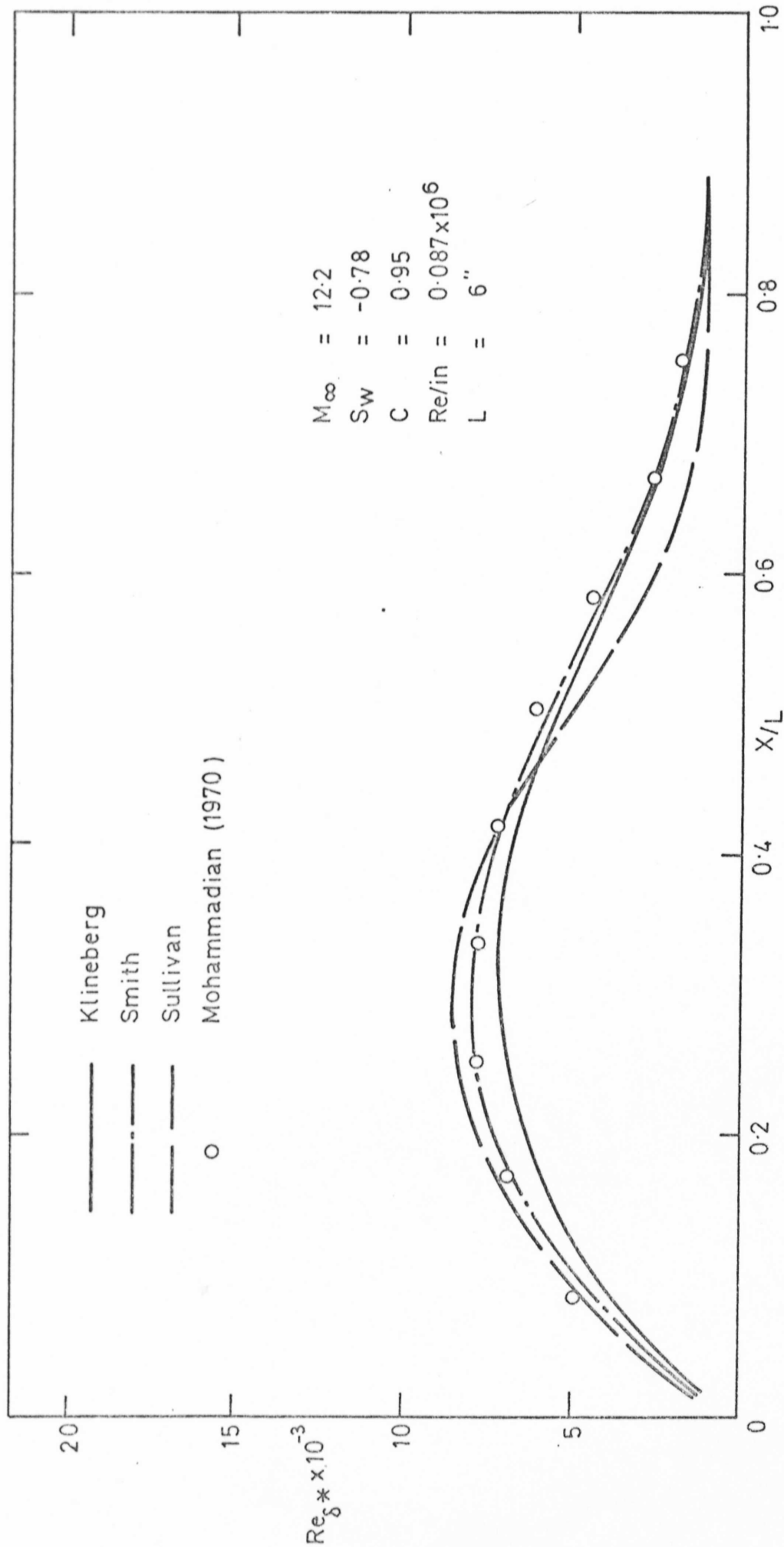


FIG. 59 DISPLACEMENT THICKNESS DISTRIBUTION ON CONCAVE BODY.

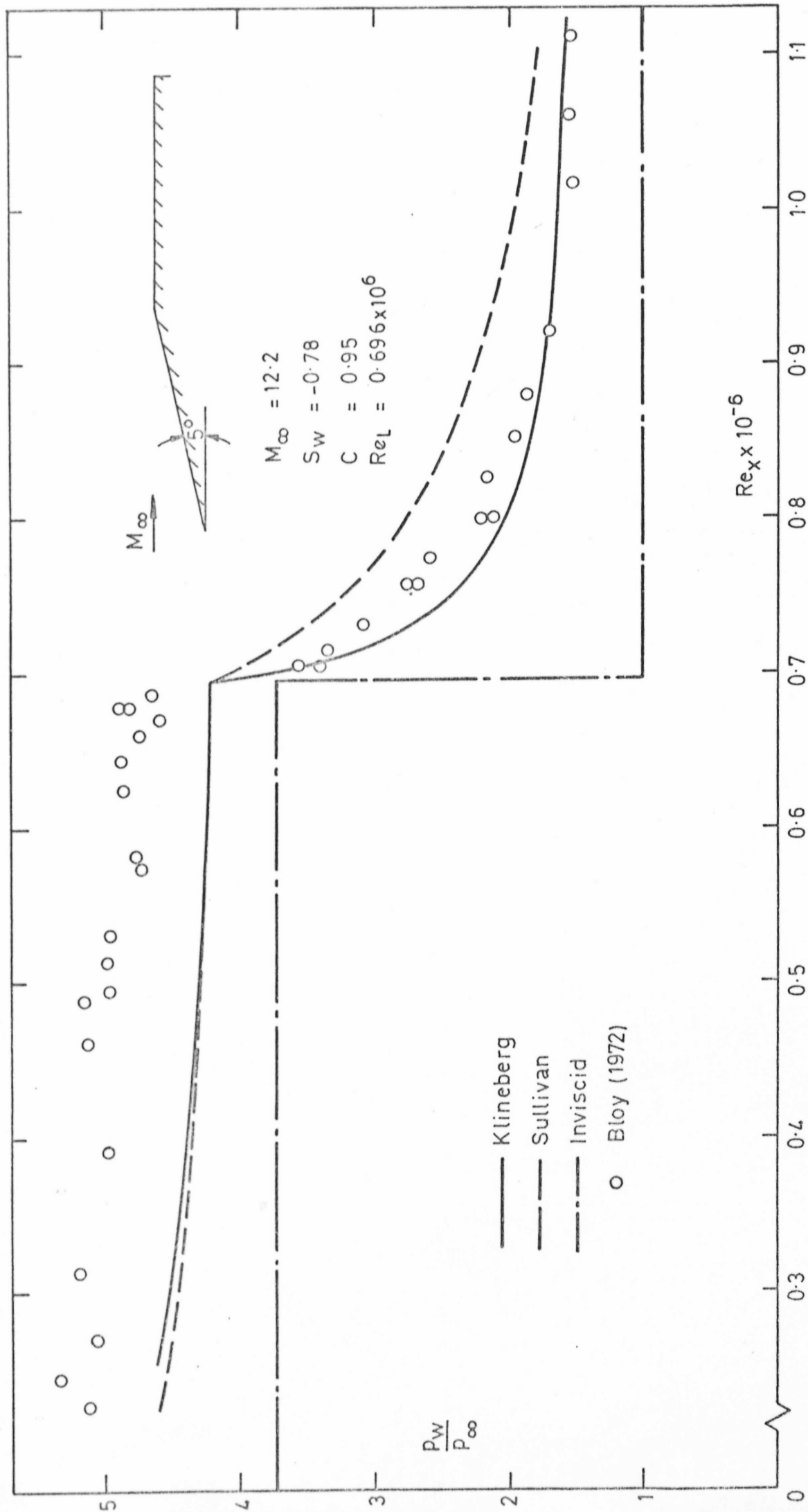


FIG. 60 PRESSURE DISTRIBUTION ON 5° EXPANSION CORNER.

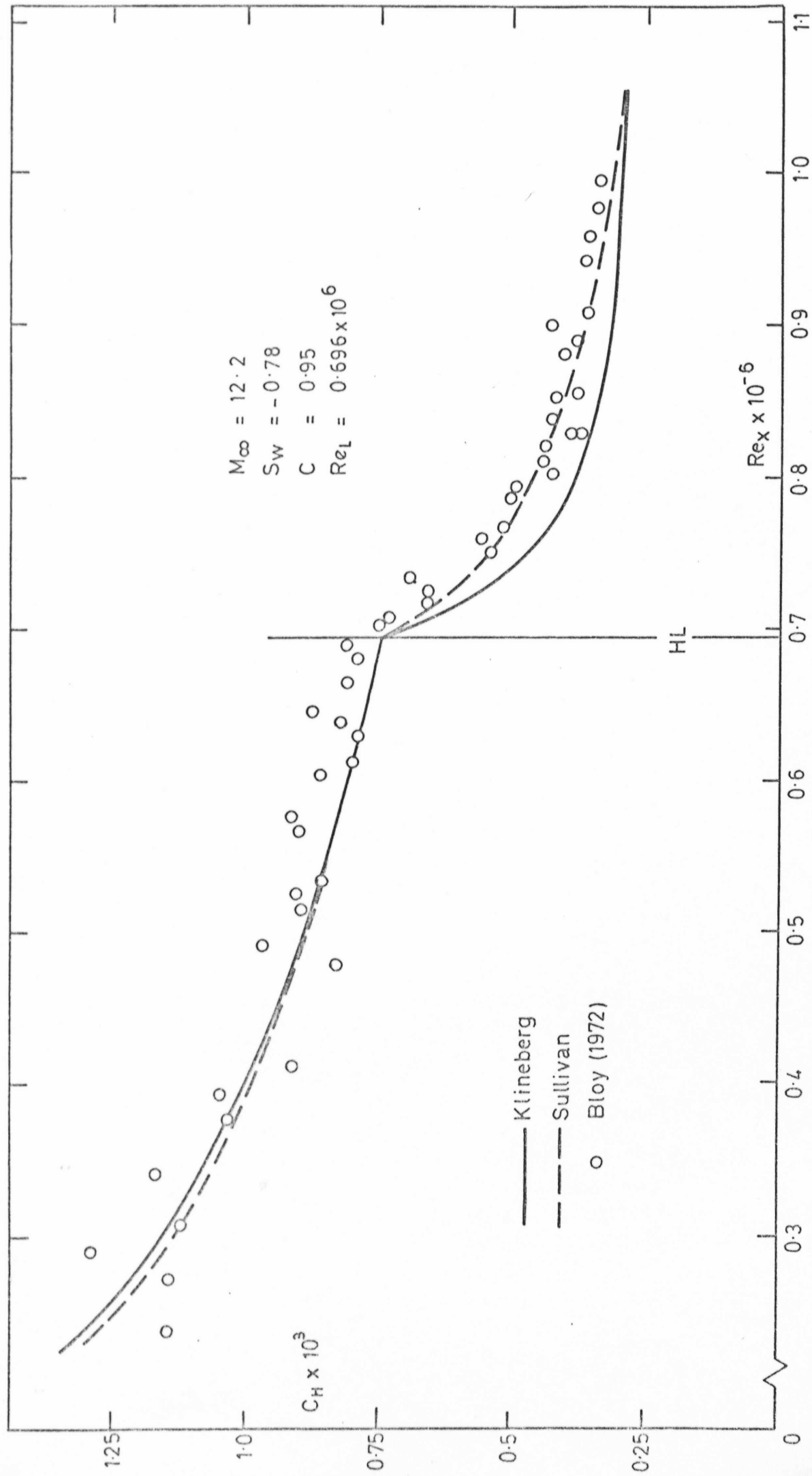


FIG. 61 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON 5° EXPANSION CORNER.

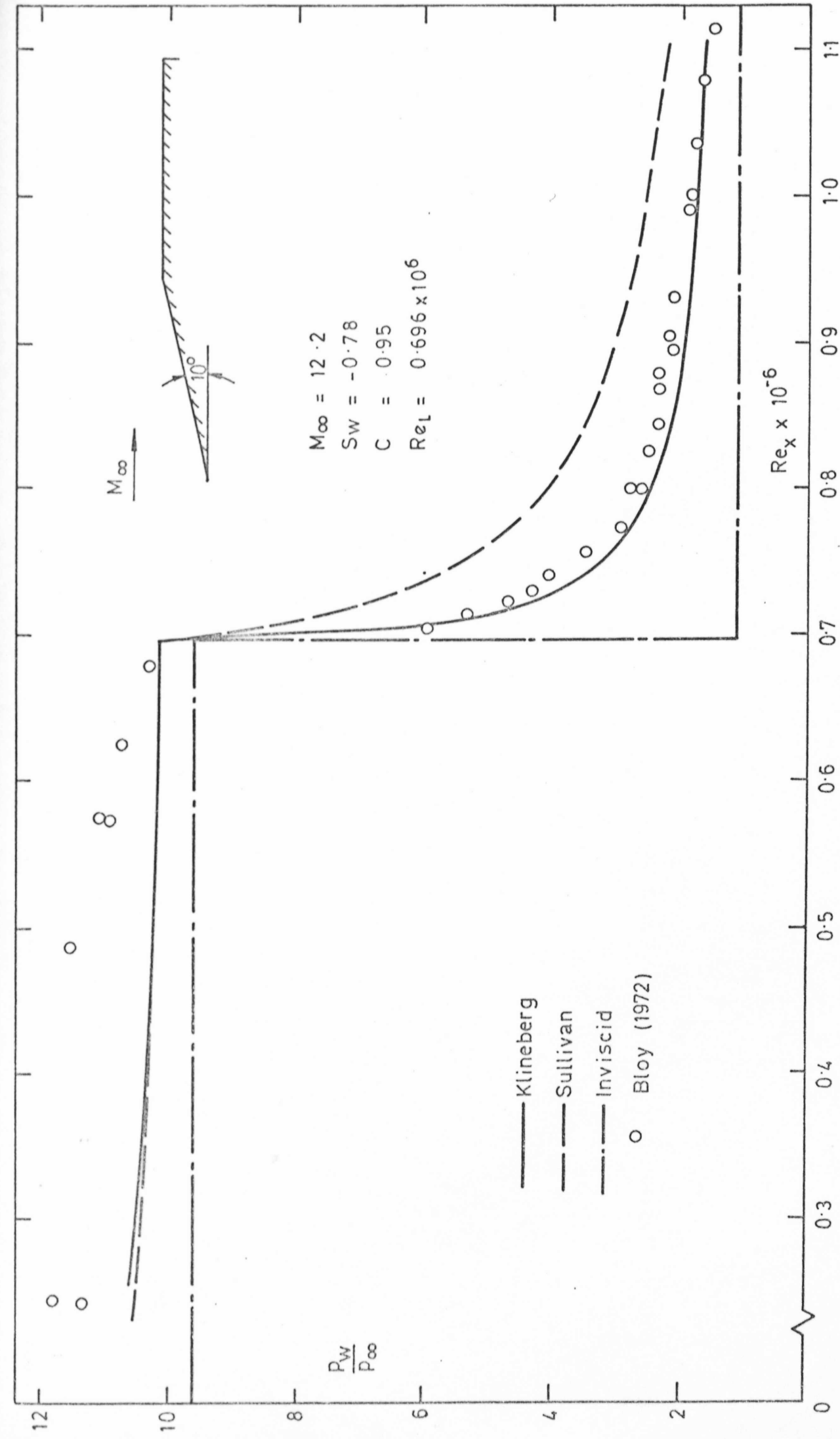


FIG. 62 PRESSURE DISTRIBUTION ON 10° EXPANSION CORNER.

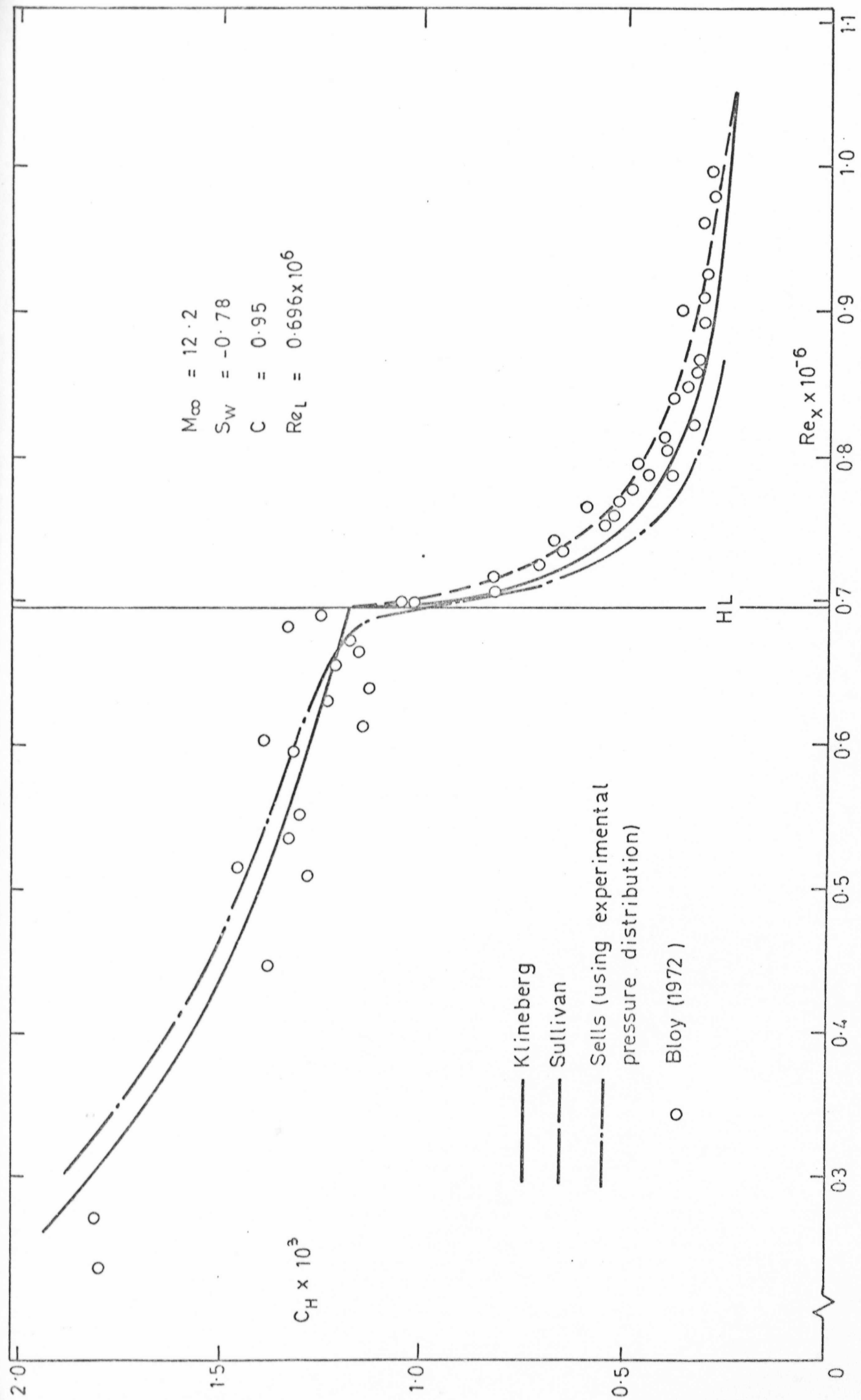


FIG. 63 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON 10° EXPANSION CORNER.

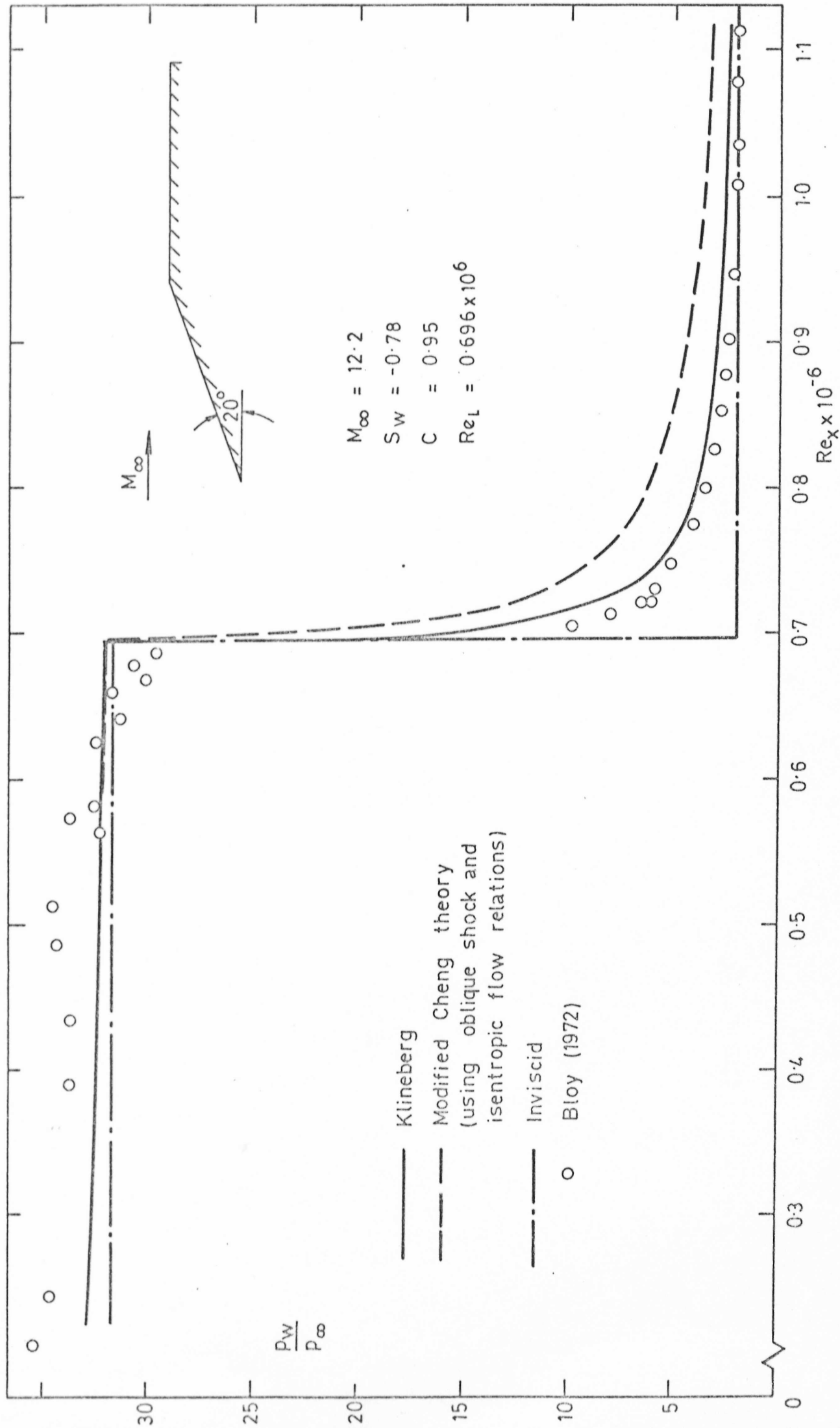


FIG. 64 PRESSURE DISTRIBUTION ON 20° EXPANSION CORNER.

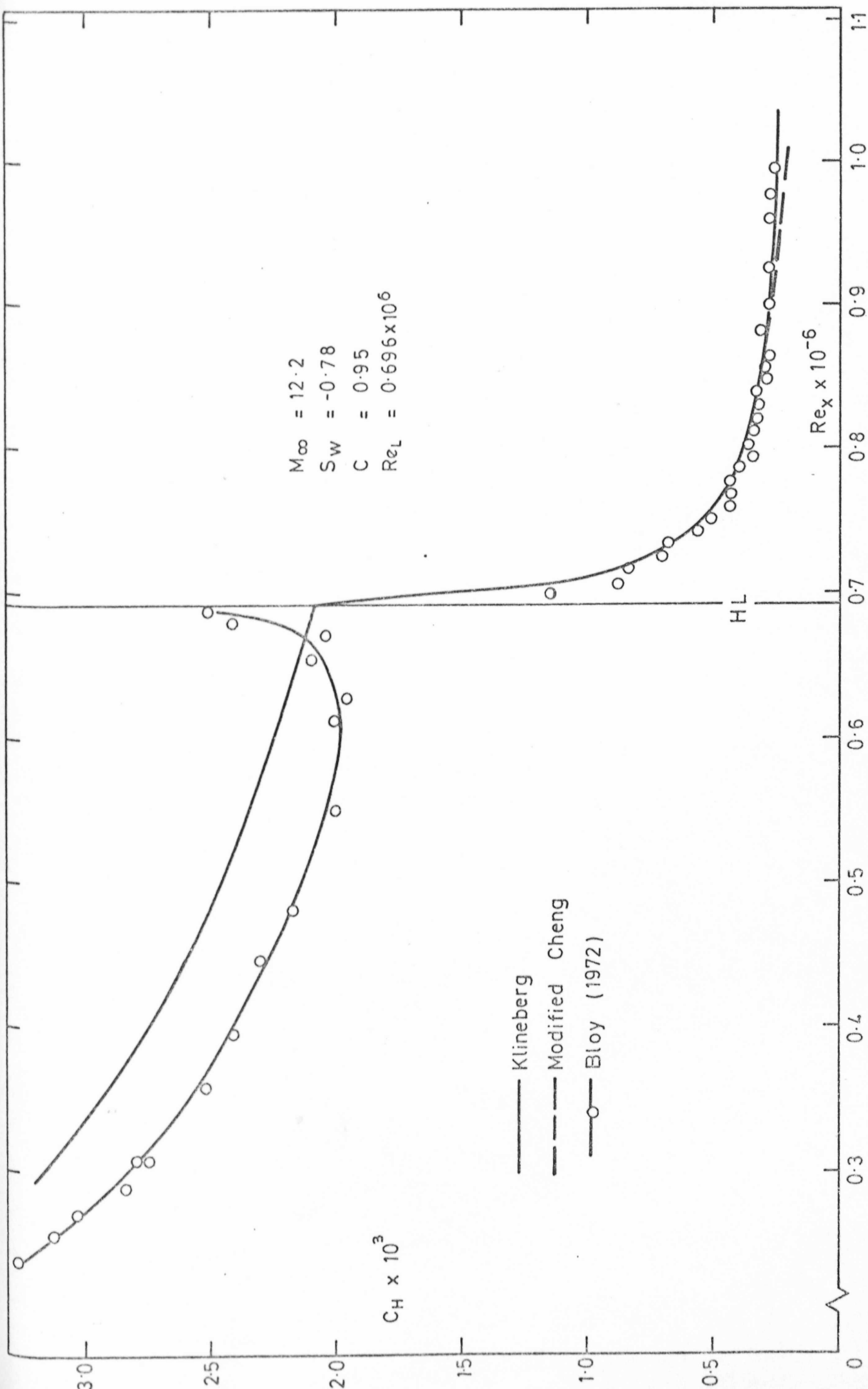


FIG. 65 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON 20° EXPANSION CORNER.

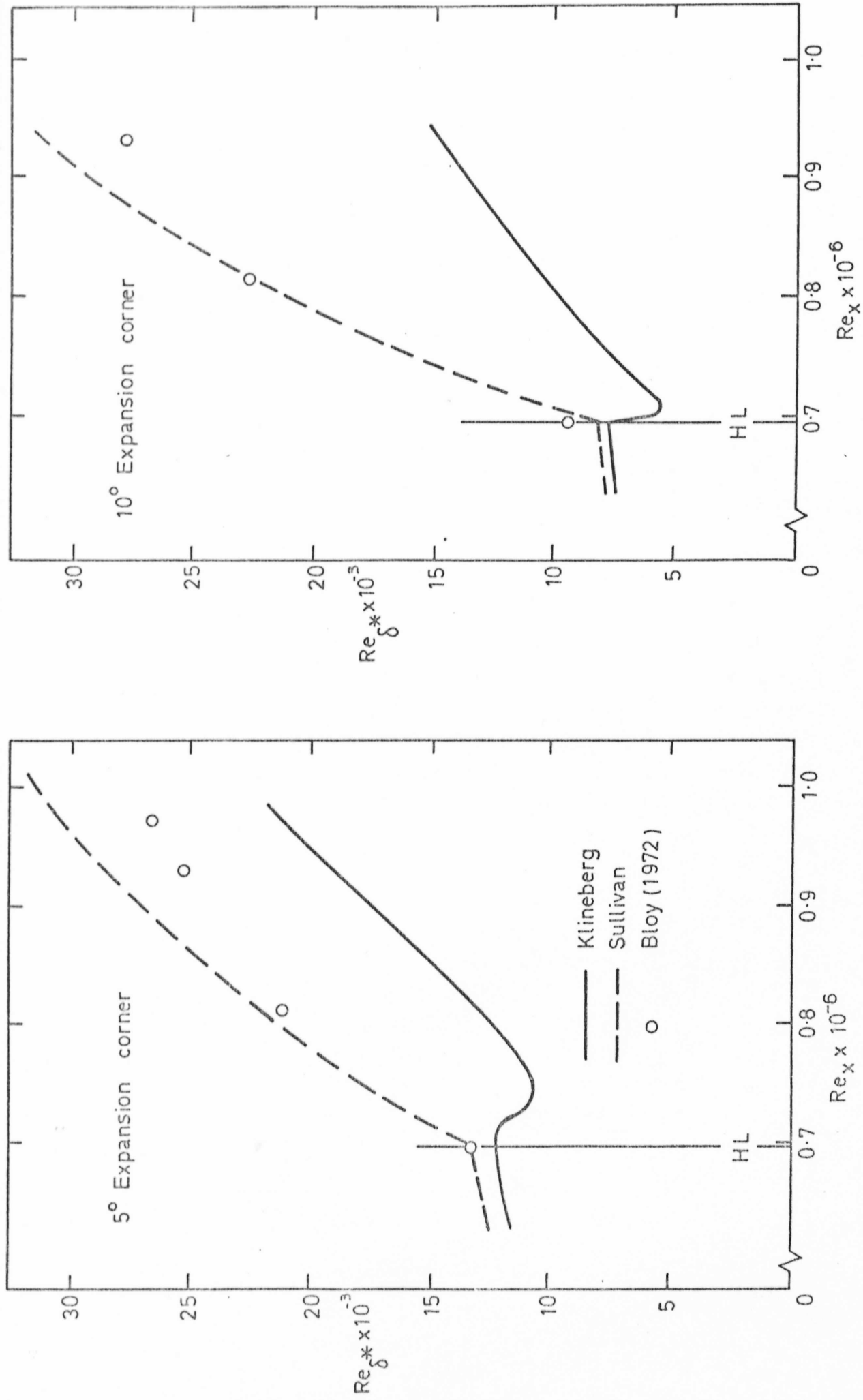


FIG. 66 DISPLACEMENT THICKNESS DISTRIBUTION ON EXPANSION CORNER.

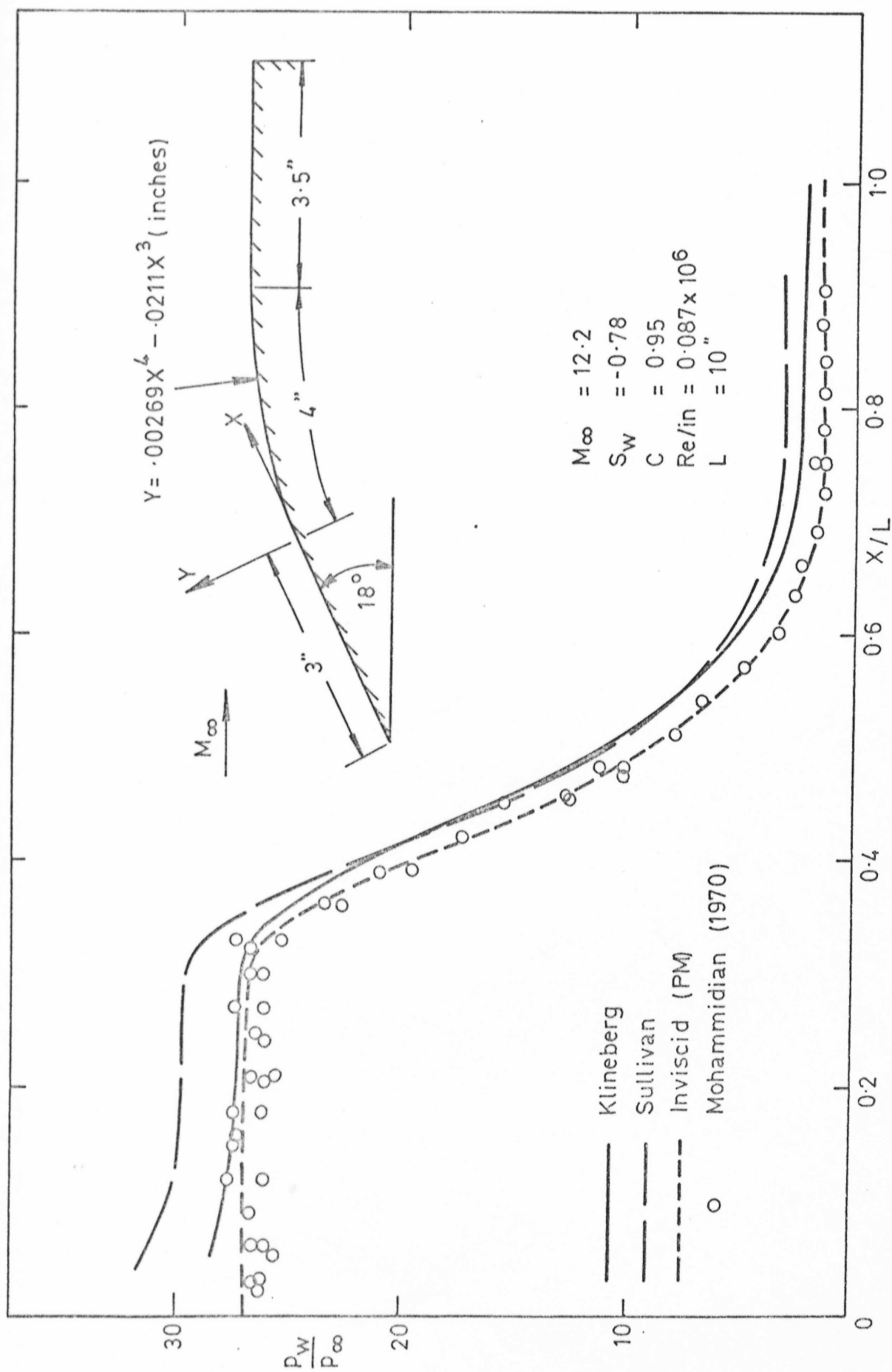


FIG. 67 PRESSURE DISTRIBUTION ON CONVEX BODY.

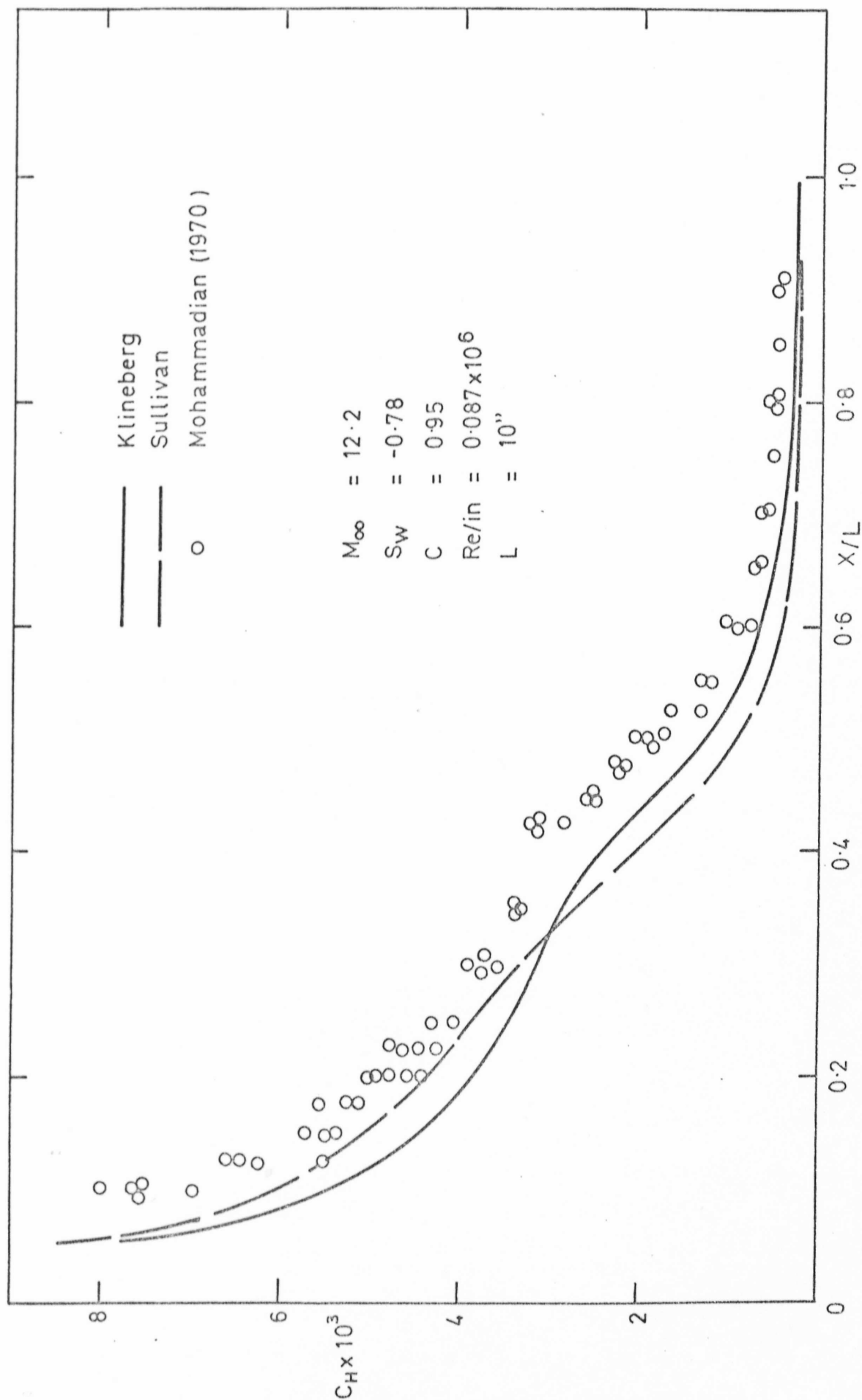


FIG. 68 DISTRIBUTION OF HEAT TRANSFER COEFFICIENT ON CONVEX BODY.

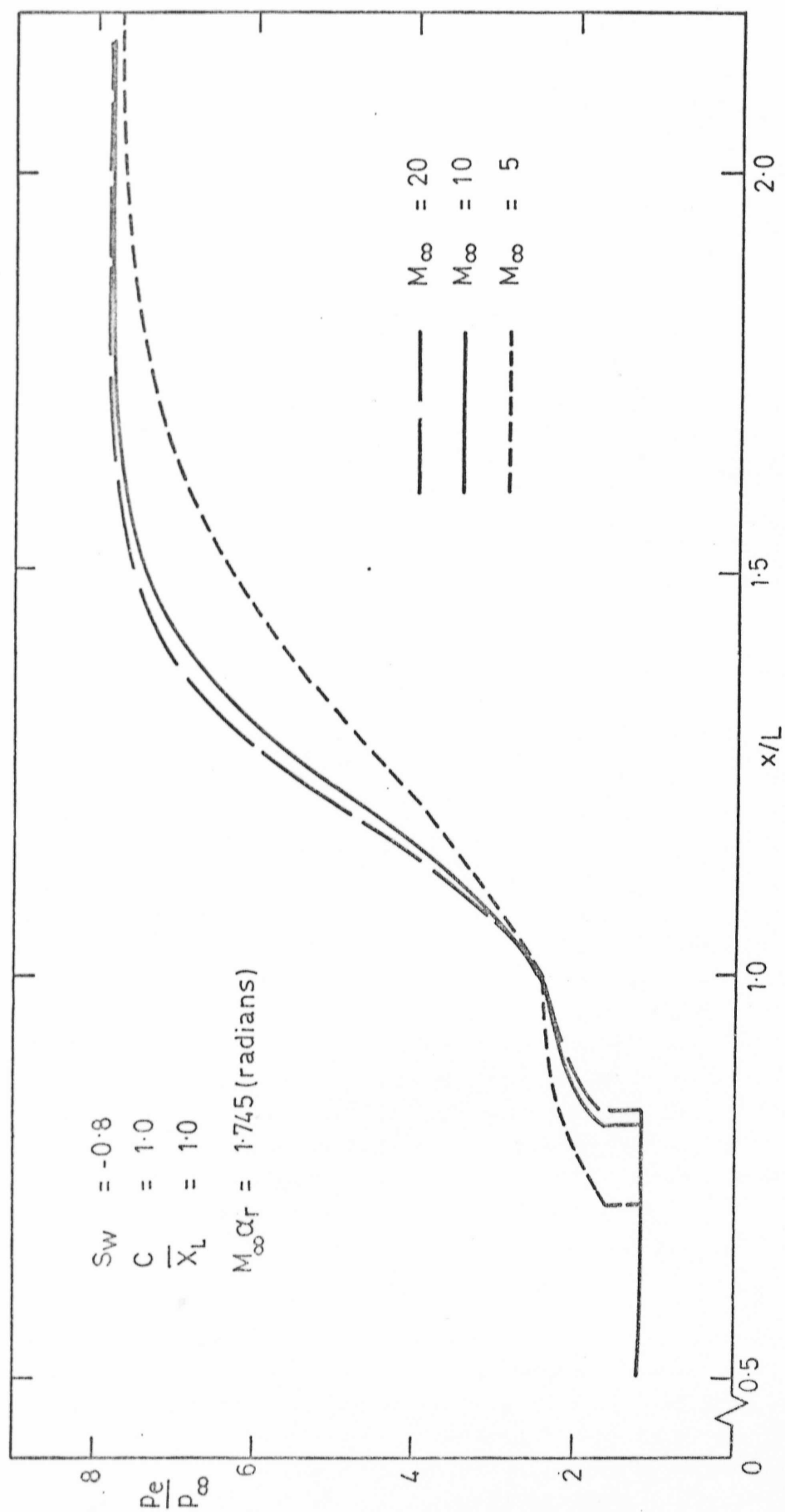


FIG. 69 THEORETICAL COMPARISON OF 'SIMILAR' PRESSURE DISTRIBUTIONS.

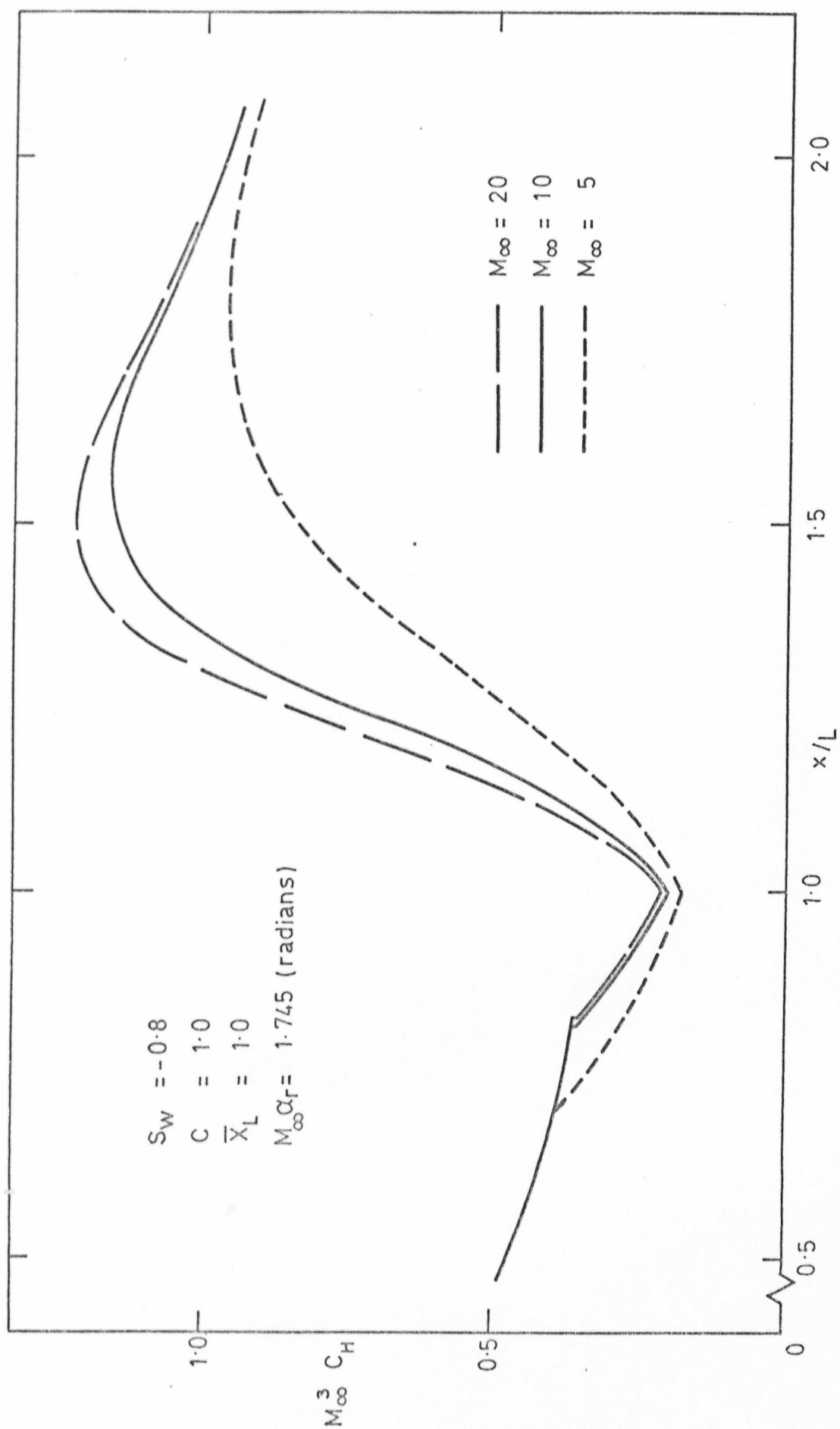


FIG. 70 THEORETICAL COMPARISON OF 'SIMILAR' HEAT TRANSFER COEFFICIENT DISTRIBUTIONS.

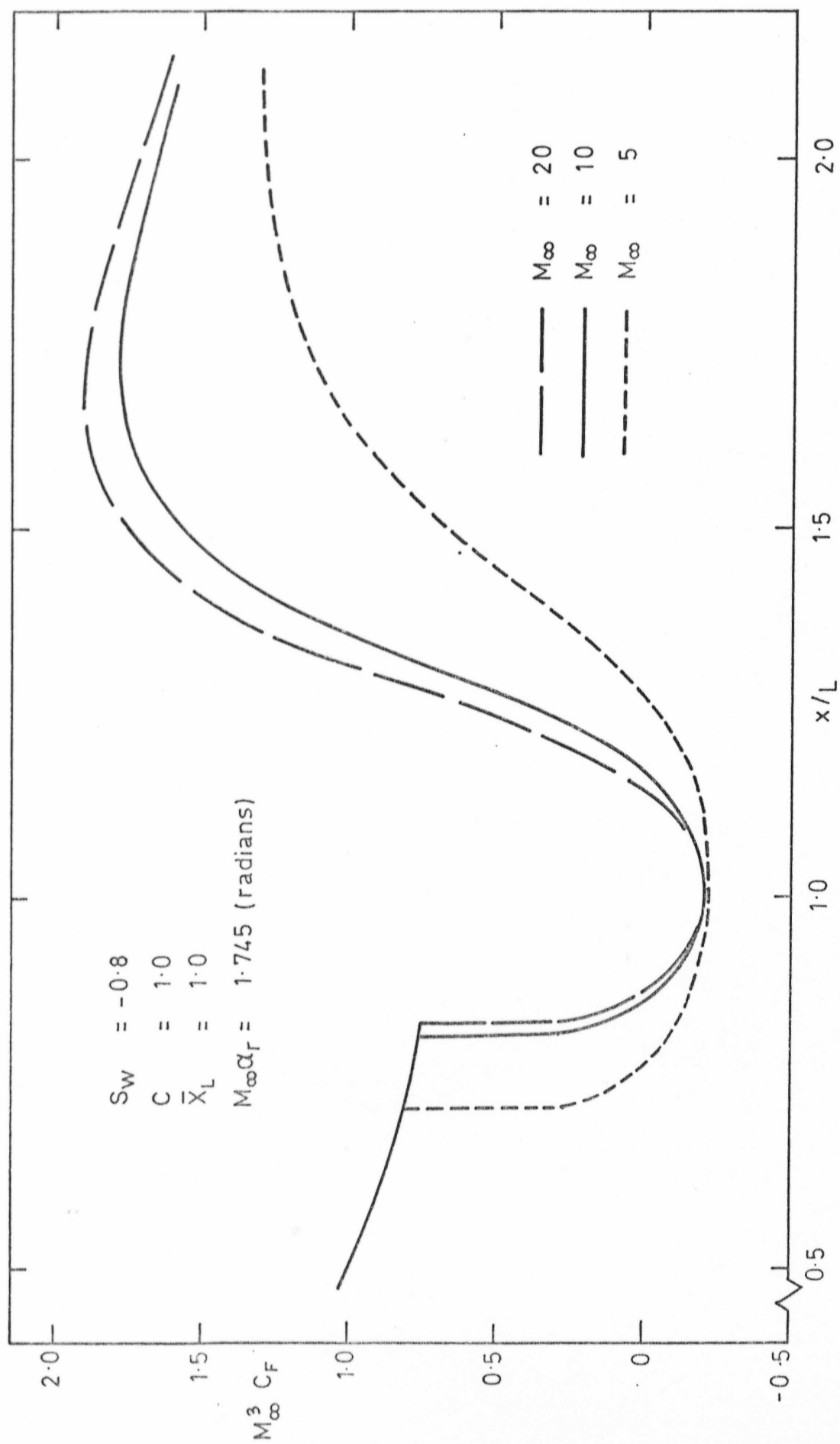


FIG. 71 THEORETICAL COMPARISON OF 'SIMILAR' SKIN FRICTION COEFFICIENT DISTRIBUTIONS.

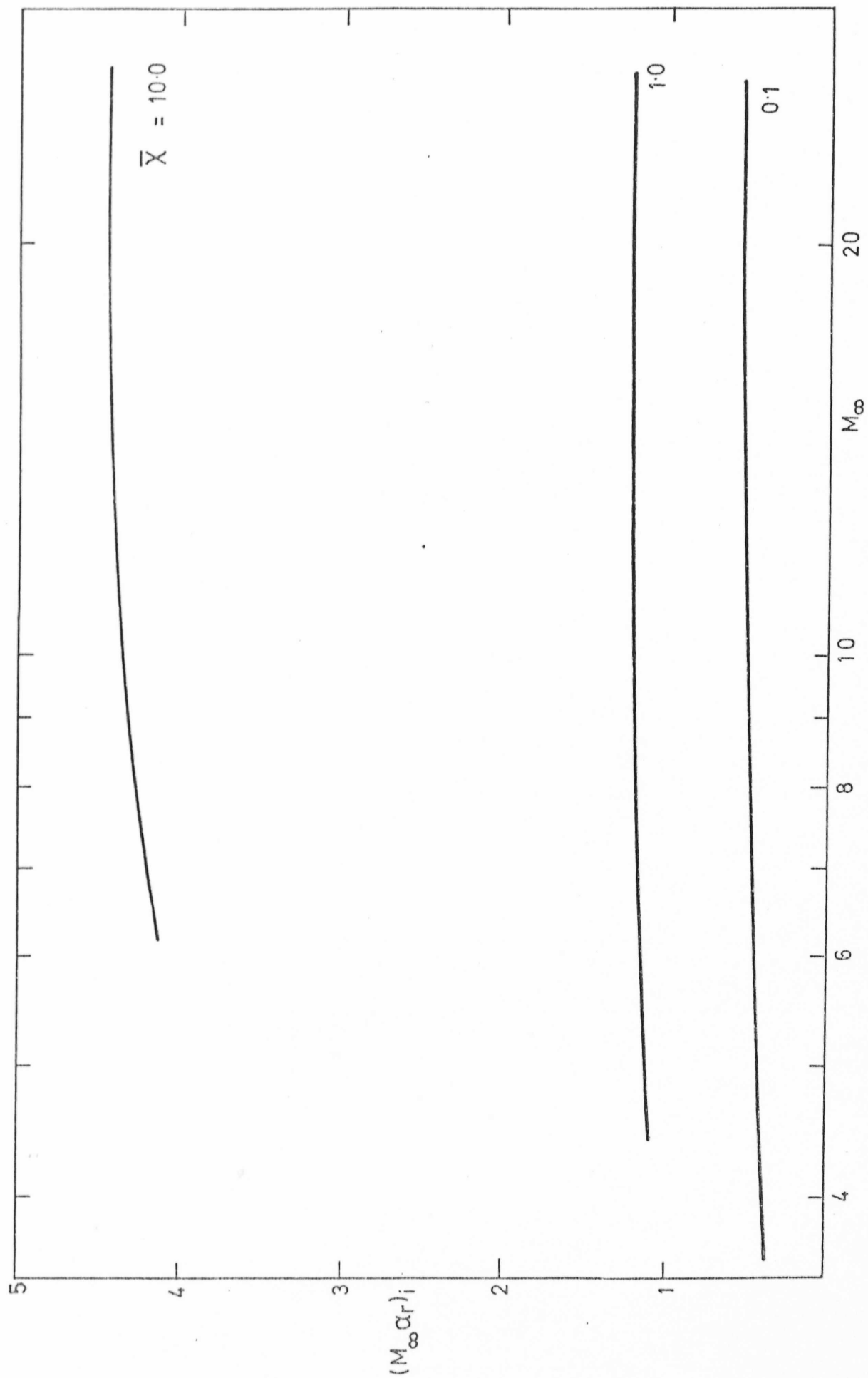


FIG. 72 INCIPIENT SEPARATION CONDITIONS.

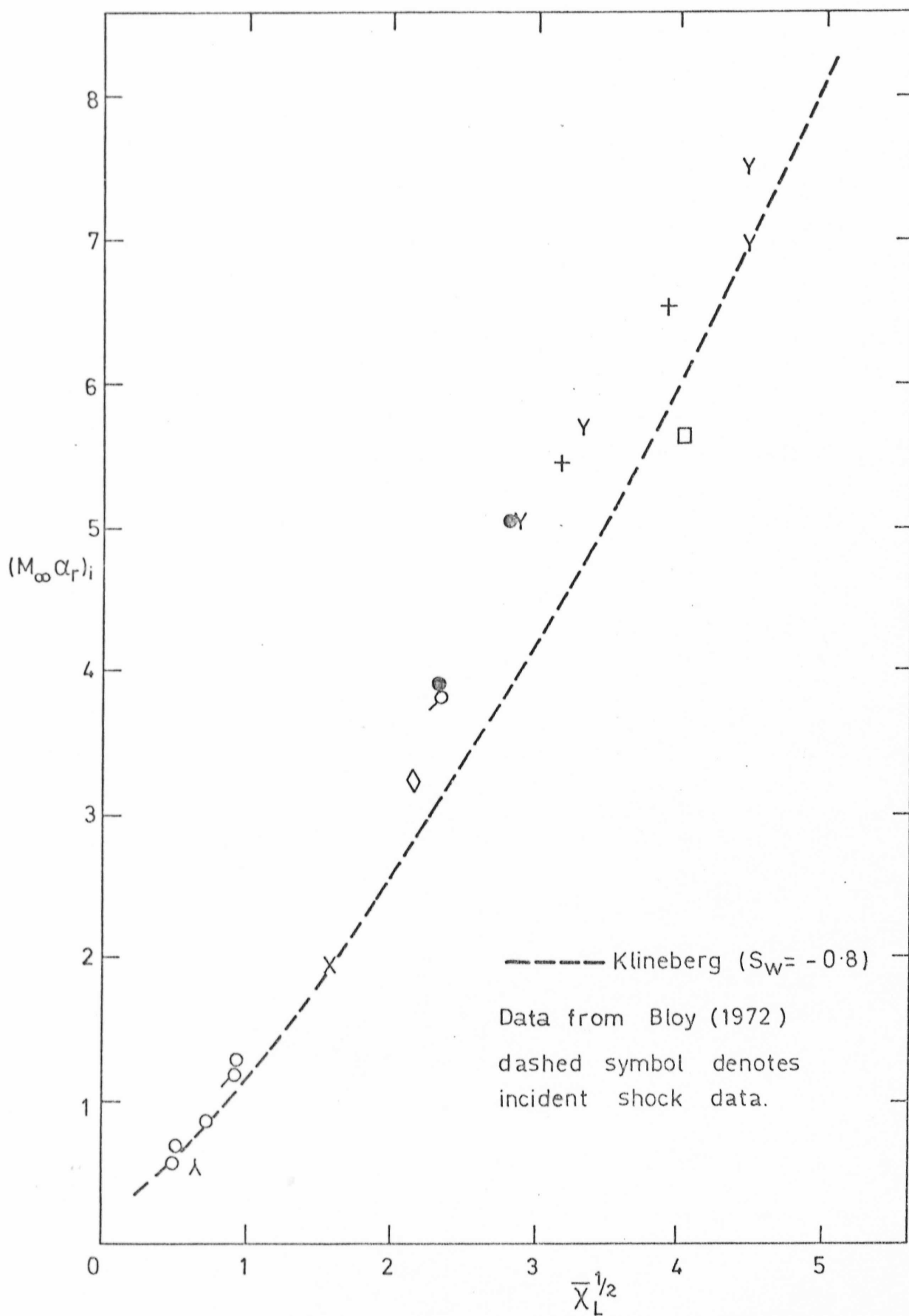


FIG. 73 CORRELATION OF COLD-WALL INCIPIENT SEPARATION DATA.

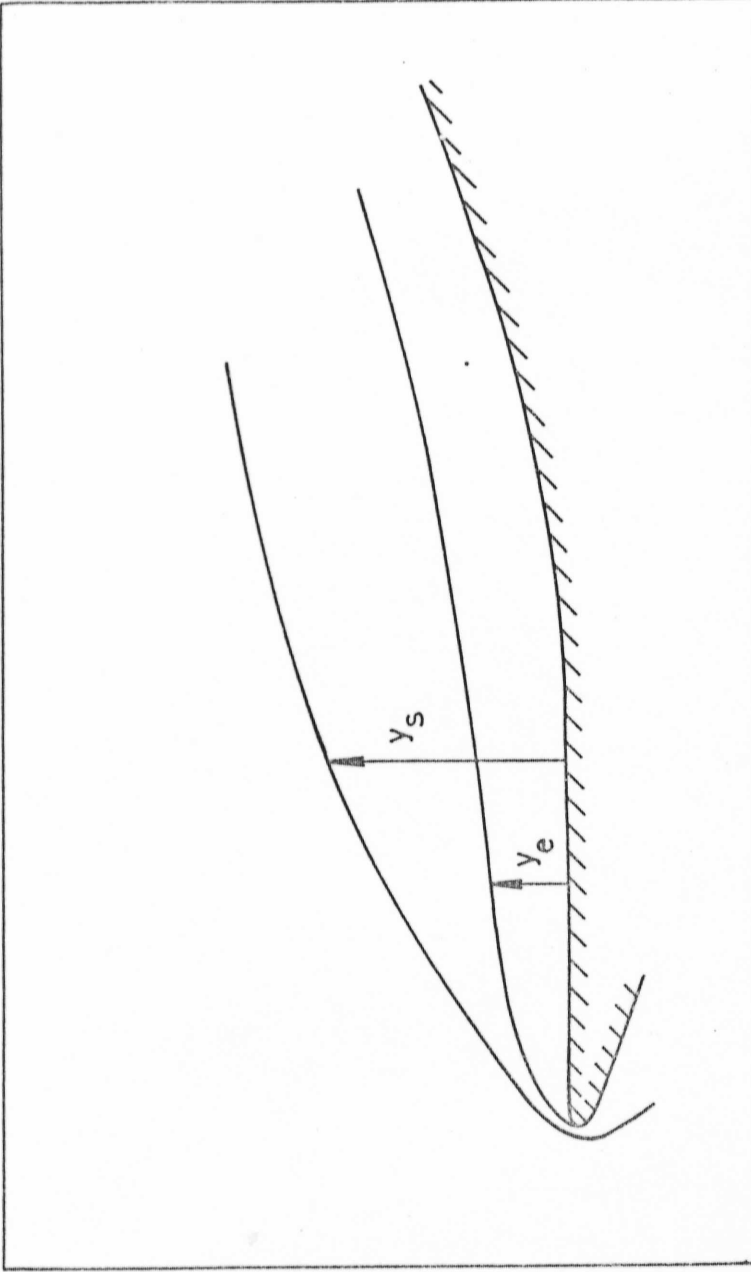


FIG. 74 CHENG MODEL OF VISCOUS INTERACTION FOR
A BLUNT LEADING EDGE.