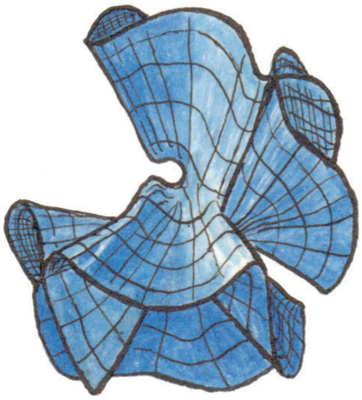
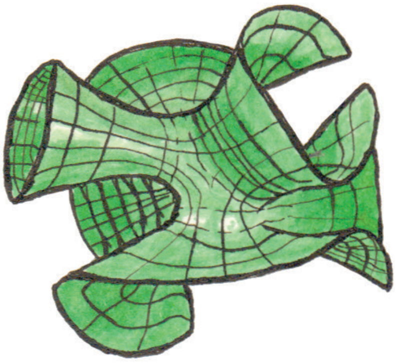
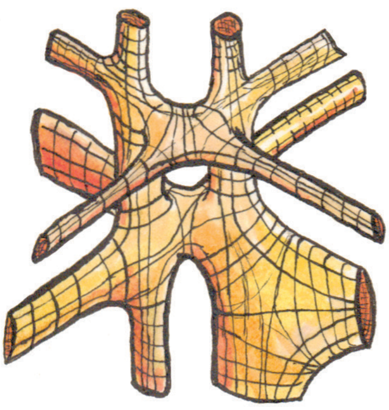
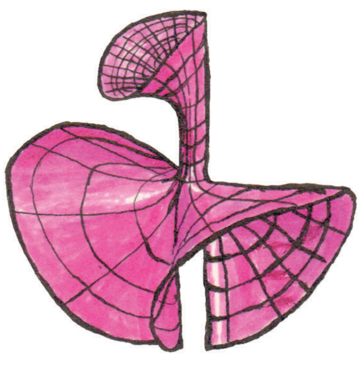




Quantum reconstruction for Fano bundles

Andrew Strangeway





Quantum Reconstruction for Fano Bundles

A thesis presented for the degree of

Doctor of Philosophy of Imperial College London

by

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Declaration of Originality

I certify that the contents of this dissertation are the product of my own work. Ideas or quotations from the work of others are appropriately acknowledged.

Andrew Strangeway

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Abstract

This dissertation discusses Fano vector bundles on projective space and the quantum cohomology of the associated projective bundle. A bundle is said to be Fano if the associated projective bundle is a Fano variety.

We prove a theorem that reconstructs the small quantum cohomology ring for the projective bundle from a subset of the quantum multiplication-table. Equivalently, in the spirit of Kontsevich's first reconstruction theorem, the theorem reconstructs all genus-0 Gromov-Witten invariants from a small set of 2-point invariants. The theorem improves on Kontsevich's reconstruction theorem, where they both apply, as the required subset of invariants is smaller.

Qin and Ruan note that, unlike in the classical case, the quantum cohomology of a projectivised bundle need not be a module over the quantum cohomology of the base. The reconstruction theorem described in this thesis demonstrates that this 'non-modularity' contains all the essential quantum information.

As an example we discuss the Fano bundle $\Omega^2(2)_{\mathbb{P}^4}$, the second wedge of the cotangent bundle on $\mathbb{P}^4$, and calculate the quantum cohomology of the Fano 9-fold given by its projectivisation using the reconstruction theorem. We apply Givental formalism to this result to obtain the quantum periods for some Fano subvarieties embedded in $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$. These novel calculations are of interest to the Fanosearch programme. In particular, we describe two 4-dimensional Fano subvarieties that have quantum periods suggestive of 4-folds which are not of toric complete-intersection type; they are not contained in a conjecturally complete list of such objects.

We observe the presence of the Apéry numbers in J-function of the Fano 9-fold $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ and comment, after Golyshev, that this may indicate hidden modularity in the J-function.

To Mum and Dad
To Merlin and Quinn
with love.

Acknowledgements

During my studies at Imperial I have learned much from many people; my time at Imperial has been both productive and enjoyable. For their friendship and enlightening conversations I am particularly grateful to Nick Addington, John Calabrese, Lorenzo Foscolo, Paul Johnson, Andrew Macpherson, and Thomas Walpuski.

I would like to thank Martin Guest for hosting me during a recent trip to Tokyo and for sharing his time and ideas generously. Conversations with Hiroshi Iritani carried out during my visit have led to improvements in my understanding and exposition. I am indebted to an anonymous referee of my recent paper, who offered extremely valuable comments, suggestions and, in particular, simplifications to the proof of Lemma 3.2.1.

I would like to thank my supervisors: Alessio Corti for introducing me to the subject, for sharing with me his viewpoint on many topics, and for his enthusiasm for bikes; and Tom Coates, for his guidance, generosity, and patience.

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Chapter -1
Conventions

We use throughout the following conventions which apply unless otherwise stated.

X is a smooth complex variety

E is a Fano bundle of rank r on $\mathbb{P}^n$

$\mathbb{P}(E)$ denotes the associated projective bundle of lines in E

N is the dimension of $H^\bullet(X, \mathbb{Z})$

R is the dimension of $H^2(X; \mathbb{Z})$

$\phi_0, .., \phi_{N-1}$ is a basis for $H^\bullet(X, \mathbb{Z})$ with $\phi^0, .., \phi^{N-1}$ a dual basis under intersection pairing. So $\int \phi_i \cup \phi^j = \delta_i^j$. Further $\phi_1, .., \phi_R$ form a basis for $H^2(X; \mathbb{Z})$.

In the case where $X = \mathbb{P}(E)$ with E a Fano bundle $\phi_1, .., \phi_N$ will denote the basis generated by p and ξ with lexicographical ordering.

$c_i(E)$ denotes the ith Chern class of a bundle E.

c_i denotes the ith Chern number of a normalised bundle.

Chapter 0
Introduction

0.0.1 Some Philosophy

Quantum cohomology is supposed to fall under the banner of algebraic geometry. As such, it ought to be understood how quantum cohomology is affected by those geometric operations which march under the same colours. These include: the intersecting of hyperplanes; the blowing up of subvarieties, and other birational operations; and the projectivisation of vector bundles.

The behaviour of quantum cohomology under taking hyperplanes is well understood by the quantum Lefschetz hyperplane theorem of Coates–Givental [11], which we will make use of later in this dissertation. The behaviour under birational operations is typically not well understood, and though results exist [15, 29, 26] they are difficult to use. In this dissertation we describe the case of projectivising vector bundles, restricting our attention to bundles on projective space with the extra property that the associated projective bundle is a Fano variety: so called Fano bundles. Even in this restricted case getting to grips with the quantum cohomology turns out to be surprisingly hard.

The relation between the ordinary cohomology of a base space and the cohomology of a projectivised bundle over it is well understood: the latter is a module over the former, and relations in the module are determined by Chern classes of the bundle. This relation does not persist to the quantum case. Indeed the conclusion of the presented thesis is that it is precisely in the 'non-modularity' of the quantum cohomology of bundle that the entirety of the quantum information lies.

0.0.2 Quantum Cohomology of Fano Bundles

An early paper from '95 by Qin and Ruan [33] studies quantum cohomology of projective bundles on $\mathbb{P}^n$. They restrict their focus to Fano bundles, primarily those that are totally split. They demonstrate that the quantum cohomology of a Fano bundle is quantum generated by two classes p and ξ (the hyperplane on $\mathbb{P}^n$ and relative hyperplane on $\mathbb{P}(E)$ respectively) and has two relations. Qin and Ruan give examples of Fano bundles for which the quantum cohomology of $\mathbb{P}(E)$ is not a module over the quantum cohomology of $\mathbb{P}^n$, unlike what we might expect from the classical case. They calculate the relations for some split bundle examples and the tangent bundle to projective space (note that they use the opposite convention for the associate projective bundle, so for us this would be the cotangent bundle).

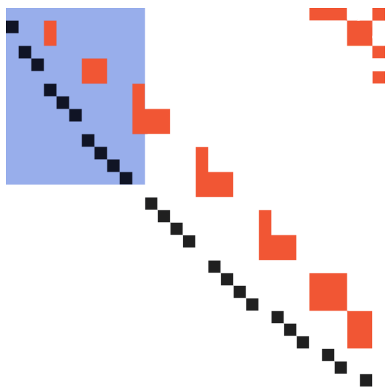
In their '04 paper Ancona and Maggesi [1] calculate the quantum cohomology for 3 non-splitting rank-2 Fano bundles on $\mathbb{P}^2$ using ad-hoc techniques based on those used in the Qin-Ruan paper. These calculations conclude all rank-2 Fano bundles on the projective plane. The techniques used in this paper were very useful when starting out calculating Gromov–Witten invariants for other bundles.

Szurek and Wiśniewski have shown that Fano vector bundles exist only on Fano manifolds [35]. In this dissertation we will consider Fano vector bundles on projective space.

0.0.3 Quantum Reconstruction

The quantum reconstruction theorem for Fano bundles is in the same spirit as Kontsevich's more general reconstruction theorem; it says that we can recover the full small quantum cohomology from a small number of low degree Gromov-Witten invariants. The number required depends on the specific Fano bundle in question. The reconstruction theorem presented in this dissertation improves upon the Kontsevich's reconstruction theorem when they both apply, as it requires significantly fewer input invariants. Indeed, reconstruction for Fano bundles essentially produces the required input for Kontsevich reconstruction from a small subset of the data.

The following figure gives a diagrammatic explanation of the result. It depicts a typical matrix for quantum multiplication by an H^2 class with respect to an obvious (lexicographically ordered) basis to be discussed later (Subsection 1.2). The black squares correspond to the classical multiplication data which is easily determined from the the presentation of the (classical) cohomology ring given in Section 1.2. The red squares correspond to quantum corrections and are determined by certain Gromov-Witten invariants.



The quantum reconstruction theorem for Fano bundles says that if we have the data in the blue rectangle then we can reconstruct the full matrix of quantum multiplication using only linear algebra.

The higher above the classical data the red squares appear, the higher the degree of the associated Gromov–Witten invariants. Typically, higher degree invariants are harder to calculate. That is to say, the red squares in the top right hand corner are difficult to calculate as they often involve excess intersection theory and other complications. It is therefore pleasing to not have to calculate them.

As the quantum corrections for the quantum multiplication of any two classes may be determined from those contained in the multiplication of $p \star p^i$, $0 \leq i \leq n$, we see that the non-modularity information determines the quantum cohomology.

Quantum reconstruction is more effective for bundles where the rank is high compared with the dimension of the base as the required invariants are a relatively smaller subset of the whole. On the other hand it is known that rank-2 Fano bundles on $\mathbb{P}^n$ split for $n \geq 3$ [2]. It is not inconceivable that there are few low-rank non-split Fano bundles on projective space, suggesting that quantum reconstruction is a useful tool.

Chapter 1
Fano Bundles on Projective Space

This chapter introduces the star of the show; enter the Fano bundle. A bundle is said to be Fano if the associated projective bundle is a Fano variety. For a bundle the property of being Fano is a normalisation-stable version of a bundle being nef/ample [27]. We discuss the modular nature of the ordinary cohomology and describe the structure of the Mori cone of curves of the projectivisation of a Fano bundle.

1.1 Definition and Normalisation

In this section we define Fano bundles and fix a normalisation, that is to say a fixed tensor with a certain invertible sheaf, which we will use throughout. Recall that an n-dimensional variety M is said to be Fano if the anticanonical class $-K = c_1(\wedge^n T_M)$ is ample.

Definition 1.1.1. *Let E be a vector bundle on a variety Y. E is said to be Fano if the associated projective bundle of lines in the fibre of E, $\mathbb{P}(E)$, is a Fano variety.*

It is known by the work of Szurek and Wiśniewski [35] that Fano vector bundles exist only on Fano manifolds. In this dissertation we consider Fano vector bundles on projective space. Henceforth E will denote a Fano bundle of rank r on $\mathbb{P}^n$.

Consider the variety $\mathbb{P}(E)$, with $\pi : \mathbb{P}(E) \to \mathbb{P}^n$ the induced projective bundle structure.

We denote by

$$p = \pi^*(c_1(\mathcal{O}(1)_{\mathbb{P}^n})) \quad \text{and} \quad \xi_E = c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$$

the pullback of the hyperplane class on $\mathbb{P}^n$ and the relative hyperplane class on $\mathbb{P}(E)$. By abuse of notation we will make no distinction between $c_1(\mathcal{O}(1)_{\mathbb{P}^n})$ and $\pi^*(c_1(\mathcal{O}(1)_{\mathbb{P}^n}))$ and so denote both by p.

The anticanonical class on $\mathbb{P}(E)$ is given by

$$-K_{\mathbb{P}(E)} = (n+1)p + c_1(E) + r\xi.$$

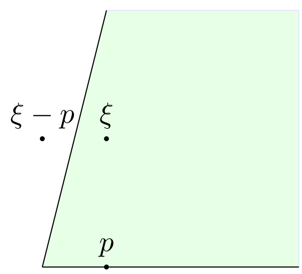
A vector bundle E is Fano if this class is ample. While it appears that this class will vary under tensoring E with $\mathcal{O}(1)$ it in fact does not. We shall fix a normalisation of the bundle to remove any apparent ambiguity.

1.1.1 Normalisation of Fano Bundles

For any invertible sheaf $\mathcal{O}(d)$ on $\mathbb{P}^n$, $\mathbb{P}(E)$ and $\mathbb{P}(E \otimes \mathcal{O}(d))$ are isomorphic. However the relative hyperplane class, ξ_E, depends on E [20, Lemma 7.9, pg.161]. Let $f : \mathbb{P}(E) \to \mathbb{P}(E \otimes \mathcal{O}(d))$ denote the isomorphism between $\mathbb{P}(E)$ and $\mathbb{P}(E \otimes \mathcal{O}(d))$, then the relative hyperplane classes are related by the following formula*

$$\xi_E = f^* \xi_{E \otimes \mathcal{O}(d)} + dp.$$

We now fix a normalisation of E, a fixed tensor with $\mathcal{O}(d)_{\mathbb{P}^n}$, such that ξ has an unambiguous meaning. Given any Fano bundle E on projective space we can always find a normalisation of E such that ξ_E is nef and $\xi_{E(1)} = f^* \xi_E - p$ is not. We shall use this normalisation throughout and shall simply write E for such a normalised bundle, with the normalisation assumed. We shall denote by ξ the relative hyperplane class, ξ_E, for this normalisation. We illustrate this discussion in the following diagram of the nef cone of $\mathbb{P}(E)$, indicating ξ for the normalised E.



Recall that the bundle $\mathcal{O}(1)_{\mathbb{P}(E)}$ is defined as a quotient of $\pi^*(E^\vee)$ via the relative Euler sequence

$$0 \to \Omega_{\mathbb{P}(E)}(1) \to \pi^*(E^\vee) \to \mathcal{O}(1)_{\mathbb{P}(E)} \to 0.$$

In particular if $E^\vee$ is generated by global sections then $\mathcal{O}(1)_{\mathbb{P}(E)}$ is also generated by global sections and hence ξ is nef. Indeed, as the $\mathcal{O}(1)_{\mathbb{P}(E)}$ is the quotient of a bundle that is generated by global sections ξ is base point free. we make use of this fact in Chapter5.

1.2 Ordinary Cohomology Ring of $\mathbb{P}(E)$

We describe the ordinary integer cohomology of $\mathbb{P}(E)$ and fix a basis that we will use in later sections.

*This formula may seem counter intuitive, this is due to our convention of taking one dimensional subspaces rather than one dimensional quotients

1.2.1 Modularity of $H^\bullet(\mathbb{P}(E);\mathbb{Z})$

The ordinary cohomology of $\mathbb{P}(E)$, $H^\bullet(\mathbb{P}(E);\mathbb{Z})$, is a module over the cohomology ring of the base, $H^\bullet(\mathbb{P}^n;\mathbb{Z})$. The cohomology ring $H^\bullet(\mathbb{P}(E);\mathbb{Z})$ is generated by the classes p and ξ, with Leray-Hirsch relations determined by the Chern classes of the bundle E.

We write the total Chern class of E as $c(E) = \sum_{i=0}^{r} c_i(E)$, where $c_i(E)$ denotes the ith Chern class of E.

The integer cohomology of $\mathbb{P}(E)$ is given by

$$H^\bullet(\mathbb{P}(E);\mathbb{Z}) = \frac{\mathbb{Z}[p,\xi]}{(p^{n+1}, \xi^r + c_1(E)\xi^{r-1} + \cdots + c_r(E))}. \tag{1.1}$$

1.2.2 Fixing a Basis for Cohomology

We fix the basis $\phi_0, .., \phi_{(n+1)r-1}$ for the ordinary cohomology of $H^\bullet(\mathbb{P}(E);\mathbb{Z})$ viewed as a vector space. The basis is given by the monomials $p^i\xi^j$ for $0 \le i \le n$, $0 \le j \le r-1$. We give this basis lexicographical ordering, taking p before ξ. In particular $\phi_0 = 1, \phi_1 = p, \phi_2 = \xi$. Let ϕ^i be the dual basis defined by the intersections product: $\int_X \phi_i \cup \phi^j = \delta_i^j$.

1.3 Splitting of Vector Bundles

We describe the splitting of vector bundles over rational curves. For further details see [32]. The splitting type of a bundle and the nature of the jumping loci will be key in our discussion of the extremal rays of the Mori cone, and curves whose class lie on these rays typically lie over the 'highest' jumping locus. Phrased differently curves which do not move much lie in the locus where the vector bundle E is 'least nef'.

A theorem of Grothendieck states that any holomorphic vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles (see e.g. [32, page 22]). Given a line L in $\mathbb{P}^n$ then the restriction of E to L splits as a direct sum of line bundles:

$$E|_L = \bigoplus_{i=1}^{r} \mathcal{O}(a_i(L))_L$$

where we order the $a_i(L)$ such that $a_1(L) \ge a_2(L) \ge \cdots \ge a_r(L)$.

We write $\underline{a(L)} := (a_1(L), .., a_r(L))$ for the vector of the $a_i(L)$s. Given two lines $L, L' \subset \mathbb{P}^n$, we say that $\underline{a}(L) > \underline{a}(L')$ if the first non-zero difference $a_i(L) - a_i(L')$ is positive.

Over lines, one has the so called generic splitting type of a bundle given by $\underline{a} := \min_{L\subset\mathbb{P}^n}\{\underline{a}(L)\}$. It is known that the subset of lines in $\mathbb{P}^n$ on which the splitting type of E is given by $\underline{a}$ is dense in the Grassmann of lines in $\mathbb{P}^n$. A line L on which the splitting type $\underline{a}(L) > \underline{a}$ is called a jumping line.

More generally for a rational curve C of degree d given by $\mathbb{P}^1 \to \mathbb{P}^n$ we have $f^*E|_C = \bigoplus_{i=1}^r \mathcal{O}(a_i(C))_{\mathbb{P}^1}$. Likewise we can order such $\underline{a}(C)$ in the same manner as for lines. Note that c_1 is not preserved under pullback by f if f is of degree greater than one. We cannot therefore directly compare splitting over curves of different degrees, however the notion of a jumping curve does make sense within curves of the same degree.

Remark 1.3.1. *Our choice of normalisation gives* $0 \geq a_i(C)$ *for all* C *- if this were this not the case* ξ *would fail to be nef over* C. *This guarantees that* $0 \geq c_1$, *since* $c_1 = \sum a_i(C)$.

Finally, if $c_1 = 0$ *then* $E = \bigoplus_{i=1}^r \mathcal{O}$. *This holds as the generic splitting would necessarily be trivial on all lines, and in particular trivial on all lines through some fixed point. It is then known that* E *is of the form claimed by application of Theorem 3.2.1 from [32, page 51].*

1.4 Extremal Rays in the Mori Cone of $\mathbb{P}(E)$

Using the splitting of the vector bundle over rational curves, we will argue that we have a restriction on the form that the extremal curves of the Mori cone can take.

As the projectivisation of a vector bundle on $\mathbb{P}^n$, $\mathbb{P}(E)$ has Picard-rank 2. Since $\mathbb{P}(E)$ is Fano, we know, by Mori's cone theorem [6], that the cone of curves has exactly two extremal rays, R_1 and R_2, that form the boundary of the Mori cone of curves. We denote by A_1, A_2 the primitive generators of the extremal rays of the Mori cone, so that $R_i = \mathbb{R}_+ A_i$.

Up to a choice of ordering, which we fix now once and for all, it is well known that $A_2 = \mathrm{PD}(p^n \xi^{r-2})$. The class A_2 is the class of a line in a fibre of π. The class A_1 is more difficult to pin down, and is determined by the splitting behaviour of the vector bundle.

Over a given curve C the splitting of $f^*(E|C)$ describes an embedding of a rational curve into $\mathbb{P}(E)$ in the following manner. Consider the following diagram of bundles. Note that if C is a line then f is the identity.

$$\begin{tikzcd}
\mathcal{O}(a_1(C)) \arrow[r, hook] \arrow[d] & \bigoplus \mathcal{O}(a_i(C)) \arrow[r] \arrow[d] & E|_C \arrow[r, hook] \arrow[d] & E \arrow[d] \\
\mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \arrow[r, "f"] & C \arrow[r, hook] & \mathbb{P}^n
\end{tikzcd}$$

The associated projectivised version of this diagram yields an embedding ι : $\mathbb{P}(\mathcal{O}(a_1)) \hookrightarrow \mathbb{P}(E)^\dagger$. This embedding is given by a section of $\mathcal{O}(-a_1(C))$: recall that $a_1(C)$ is non-positive, hence $\mathcal{O}(-a_1(C))$ has such a section. Further, $\mathbb{P}(\mathcal{O}(a_1(C)))$ is isomorphic to $\mathbb{P}^1$ and this embedding gives a rational curve A in $\mathbb{P}(E)$. The

†The fact that arrows are not reversed in this diagram is again a result of our convention of taking the associated bundle of lines curve A satisfies the following: $p \cdot A = \deg(f)$ the degree of the embedding f, and $\xi \cdot A = -a_1(C)$. This can be seen by pulling back the bundle ξ along ι: it is clear that it is given by $\mathcal{O}(-a_1(C))$.

Lemma 1.4.1. *Let C be a rational curve of degree d in $\mathbb{P}^n$ over which E splits as*

$$\oplus_{i=1}^{r}\mathcal{O}(a_i),$$

then the class $A_C = \mathrm{PD}(dp^{n-1}\xi^{r-1} + (dc_1 - a_1)p^n\xi^{r-2})$, given by $\iota_*[C]$, lies on the extremal ray generated by A_1 if and only if $\xi + \frac{a_1}{d}p$ is nef.

In particular if $\exists C$ such that $a_1(C) = 0$, then $A_1 = \mathrm{PD}(p^{n-1}\xi^{r-1} + c_1p^n\xi^{r-2})$.

Proof. If ι_*C lies on the extremal ray generated by A_1, then A_1 is given by $\frac{1}{e}\iota_*C$, for some $e \in \mathbb{Z}$. The class of any effective curve may be written as $\lambda_1A_1 + \lambda_2A_2$ for $\lambda_i \geq 0$. We have $(\xi + \frac{a_1}{d}p) \cdot (\lambda_1A_1 + \lambda_2A_2) = \lambda_2 \geq 0$ hence $(\xi + \frac{a_1}{d}p)$ is nef.

Conversely, if $(\xi + \frac{a_1}{d}p)$ is nef then consider an effective curve C with class

$$[C] = \mathrm{PD}(\lambda_1p^{n-1}\xi^{r-1} + \lambda_2p^n\xi^{r-2}).$$

We have $0 \leq (\xi + \frac{a_1}{d}p) \cdot [C] = \lambda_1(\frac{a_1}{d} - c_1) + \lambda_2$ and $0 \leq p \cdot [C] = \lambda_1$, so we can write

$$[C'] = \frac{\lambda_1}{d}\,\mathrm{PD}(dp^{n-1}\xi^{r-1} + (dc_1 - a_1)p^n\xi^{r-2}) + (\lambda_1(\frac{a_1}{d} - c_1) + \lambda_2)\,\mathrm{PD}(p^n\xi^{r-2}).$$

That is, we have shown that any effective curve class may be written in the form

$$[C'] = \lambda_1'A_C + \lambda_2'A_2$$

where $\lambda_1' = \frac{\lambda_1}{d}$ and $\lambda_2' = (\lambda_1(\frac{a_1}{d} - c_1) + \lambda_2)$ (with A_C and A_2 as above). Hence we see that A_C does indeed lie on the extremal ray and must be (some multiple of) the primitive generator A_1. The special case follows from normalisation: ξ is chosen to be nef. □

Remark 1.4.2. *In practice, checking the nefness of $\xi - sp$, for $0 < s < 1$ is challenging. Indeed, finding the appropriate normalisation for E is non-trivial.*

However, let F be a Fano bundle that is not necessarily normalised, but presented as a subbundle of a direct sum of line bundles. That is, we have an exact sequence beginning

$$0 \to F \to \bigoplus \mathcal{O}(b_i) \to \ldots$$

We can freely assume that $b_i \leq 0$, simply by tensoring by an appropriate power of $\mathcal{O}(1)$. This implies that $F^\vee$ is generated by global sections, as it is the quotient of a bundle with this property, and hence ξ_F is nef. One sees this by considering the dual sequence

$$\cdots \to \bigoplus \mathcal{O} \to \bigoplus \mathcal{O}(-b_i) \to F^\vee \to 0.$$

If one then finds a splitting of $f^*F|_C := \bigoplus \mathcal{O}(a_i)$ over some curve C such that $a_1(C) = 0$ then Lemma 1.4.1 shows: that the bundle F is indeed normalised (the existence of the class A_C demonstrates that ξ_F is strictly nef); and that the extremal ray A_1 is given by $\mathrm{PD}(p^{n-1}\xi^{r-1} + c_1 p^n \xi^{r-2})$.

Remark 1.4.3. *Kollár has remarked that though A_1 may be the primitive generator for the extremal ray, it may not be representable by a rational curve. For example it is possible that $2A_1$ is representable by a curve which maps one-to-one to a conic under π, but A_1 is not representable by a curve. We note that we have seen no examples of such in the realm of Fano bundles.*

1.4.1 Conjectural Form of A_1 for Fano Bundles

We wrap up this section by demonstrating a number of equivalent conditions for the class A_1 to be representable by a curve mapping under π to a line $L \subset \mathbb{P}^n$. We conjecture that for Fano bundles A_1 may always be represented in this manner.

Lemma 1.4.4. *Let E be a Fano bundle, which is not necessarily normalised. The following statements are equivalent*

1. *A_1 is the class of a curve in $\mathbb{P}(E)$ which maps one-to-one to a line in $\mathbb{P}^n$ under π;*
2. *E is nef if and only if $E|L$ is nef for all lines $L \subset \mathbb{P}^n$;*
3. *E is normalised if and only if there exists a line L such that $E|_L = \oplus_{i=1}^r \mathcal{O}(a_i)$ with $a_1 = 0$.*

Proof. 1. $\Longrightarrow$ 2. If E is nef then $E|L$ is necessarily nef for all $L \subset \mathbb{P}^n$. It remains to show that nefness over lines and the form of the extremal ray implies nefness of the bundle.

In order to check the ξ_E is nef we need only evaluate against A_1 and A_2, that is check that $\xi_E \cdot A_1 \geq 0$ and $\xi_E \cdot A_2 \geq 0$. As A_2 may be represented by a line in the fibre of π, it is clear that $\xi_E \cdot A_2 = 0$.

We must then check $\xi_E \cdot A_1$. Since A_1 maps down to a line L, we need only check that the restriction of ξ_E to $\pi^{-1}(L)$ is nef. By functoriality of Chern classes this is equivalent to checking that $\xi_{E|_L}$ is nef, which holds by assumption. Hence E is nef.

2. $\Longrightarrow$ 3.

Assuming that E is nef then, using the notation of Section 1.3, for any L we have $a_i(L) \leq 0$. Let $a_1^{\max} = \max_{L \subset \mathbb{P}^n}(a_1(L))$. It is clear that $E \otimes \mathcal{O}(-a_1^{\max})$ is also nef, since it is nef for every line in $\mathbb{P}^n$. On the other hand it is also clear that $E \otimes \mathcal{O}(-a_1^{\max} + 1)$ is not nef, as in particular, on the line L the restricted bundle $E \otimes \mathcal{O}(-a_1^{\max} + 1)|_L$ is not nef ($a_1 = 1$). Hence $E \otimes \mathcal{O}(-a_1^{\max})$ is normalised.

On the other hand if E is normalised and $a_1^{\max} < 0$ we arrive at a contradiction; for $E(1)$ the restriction to any line is still nef.

3. $\implies$ 1

This follows trivially from Lemma 1.4.1, since the existence of such L gives an embedding of a line into $\mathbb{P}(E)$ realising the claimed extremal ray. $\square$

We conjecture that for E a Fano bundle A_1 may be represented by a curve that maps one-to-one to a line in $\mathbb{P}^n$ under π. The author knows of no Fano bundles which contradict this conjecture. For vector bundles with a equivariant torus action it is known that nefness holds if and only if the restriction to torus invariant lines is nef [21]. This is at least consistent with the above conjecture, though we accept that such a torus action is a strong condition.

Chapter 2
Quantum Cohomology

Quantum cohomology is a deformation of the ordinary classical cohomology of an algebraic variety.

2.1 Motivation

The motivating ideas for Gromov–Witten invariants and quantum cohomology arise in string theory, in particular in the study of non-linear sigma models [22]. In this picture the quantum cohomology describes interactions occurring in a physical model built from the data of a Calabi-Yau 3-fold. Both Gromov–Witten invariants and quantum cohomology have been put on a sound mathematical footing, via Kontsevich's notion of stable maps to X [25, 12]. Leaving aside the physical interpretation, we can consider Gromov–Witten invariants and quantum cohomology on any algebraic variety, not just those of Calabi-Yau type. The moduli space of stable maps gives a compactification of the space of holomorphic maps from marked smooth Riemannian surfaces into X. Gromov-Witten invariants are intersection numbers in the moduli space of stable maps; they carry information about the enumerative geometry of the target space X.

2.2 Gromov–Witten Invariants

Roughly speaking, genus-0 Gromov–Witten invariants count rational curves of a fixed homological class ('degree') that intersect a number of fixed homology classes. This enumerative interpretation is only valid under very favourable circumstances, but gives a reasonable first impression. In order to give the correct definition we briefly recall some details regarding stable maps. An exceptionally readable first introduction to this material is given by [23], while more detailed references are [13, 12]. Since we only require genus-0 stable maps to define quantum cohomology we restrict attention to this case. The discussions here are not exhaustive; we aim to fix notation and recall some properties which will be important for later discussions.

Gromov–Witten invariants are defined by producing a moduli space of stable maps to a space X, equipping this space with a virtual fundamental class, and then integrating pullbacks of cohomology classes from X against this virtual fundamental class.

2.2.1 Stable Maps

Here we define stable maps to target space X, families of stable maps and the target class of a stable map.

Definition 2.2.1. *A genus-*0 *prestable curve with n marked points $(C; p_1, .., p_n)$ is a curve C of arithmetic genus* 0 *with at worst nodal singularities, together with n distinct, smooth, marked points $p_1, .., p_n$ on C.*

Definition 2.2.2. *A genus-*0*, n-pointed stable map to X is a genus-*0 *prestable curve with n marked points $(C; p_1, .., p_n)$ together with a morphism $\mu : C \to X$ such that if C_i is a component of C which is mapped to a point by μ, then C_i contains at least 3 special points (either marked points or nodes).*

Definition 2.2.3. *Let B be a scheme. A family of stable maps over B is a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & X \\ B & \end{tikzcd}$$

*together with sections $p_1, .., p_n$ of π such that the fibre over each geometric point $b \in B$ is a genus-*0*, n-pointed stable map to X.*

Definition 2.2.4. *We say that a stable map has target class β if the push-forward under μ of the fundamental curve class $[C]$ equals β. That is if $\mu_* [C] = \beta$. A family of stable maps is said to have target class β if the same condition holds for each member of the family.*

2.2.2 $X_{0,n,\beta}$ - Moduli Space of Stable Maps

Let $X_{0,n,\beta}$ denote the moduli space of genus-0 n-pointed stable maps into X with class $\beta \in H_2(X; \mathbb{Z})$ [24, 13]. These spaces are compact by construction, though in other ways they may be very badly behaved.

The expected dimension of $X_{0,n,\beta}$ is obtained via a Riemann-Roch calculation and is referred to as the virtual dimension. It is given by:

$$\dim_{\text{vir}}(X_{0,n,\beta}) = \dim_{\mathbb{C}}(X) - 3 + n - K_X \cdot \beta. \tag{2.1}$$

In general the moduli space may be singular and not purely of expected dimension; it may have components of higher dimension. One can construct a virtual fundamental class which is of the expected dimension [28, 4, 3]. This is denoted by $[X_{0,n,\beta}]^{\text{vir}}$. The details of the construction are not relevant for the present discussion. Equipping the moduli space with the virtual fundamental class allows us to construct a 'virtual intersection theory' which is well-behaved under deformations.

2.2.3 Gromov–Witten Invariants

We now use the preceding discussion to define Gromow–Witten invariants. There are evaluation maps at the marked points

$$\begin{aligned} \mathrm{ev}_i : X_{0,n,\beta} &\to X & 1 \leq i \leq n \\ (C; p_1, .., p_n; \mu) &\mapsto \mu(p_i). \end{aligned}$$

These allow us to pull back cohomology classes from the target space.

Definition 2.2.5. *Given cohomology classes $\alpha_1, .., \alpha_n \in H^\bullet(X)$, we define genus zero Gromov–Witten invariants as*

$$\langle \alpha_1, .., \alpha_n \rangle^X_{0,n,\beta} := \int_{[X_{0,n,\beta}]^{\mathrm{vir}}} \mathrm{ev}_1^* \alpha_1 \wedge .. \wedge \mathrm{ev}_n^* \alpha_n$$

where $[X_{0,n,\beta}]^{\mathrm{vir}}$ is the virtual fundamental class of $X_{0,n,\beta}$ (op. cit.).

2.2.4 Some Properties of Gromov–Witten Invariants

We state here some of the key properties of Gromov–Witten invariants. This list is not exhaustive, we merely emphasise those which form essential parts of later arguments. A full list may be found in [12, pp 191-194].

The divisor axiom is an important property of Gromov–Witten invariants that we will make use of in the proof of the main theorem so we recall it here.

Lemma 2.2.6 (Divisor Axiom)**.** *Suppose that $\alpha_n \in H^2(X)$. If $\beta \neq 0$ or if $n \geq 3$ then*

$$\langle \alpha_1, .., \alpha_n \rangle^X_{0,n,\beta} = \left(\int_\beta \alpha_n \right) \langle \alpha_1, .., \alpha_{n-1} \rangle^X_{0,n-1,\beta}$$

This result is clear from the enumerative point of view; a generic curve with class β and a hypersurface dual to α_n intersect $\int_\beta \alpha_n$ times, giving this number of choices of where to place the nth marked point on the curve. We therefore expect to see this multiplicity in the invariant.

Lemma 2.2.7 (Degree Axiom)**.** *The Gromov–Witten invariant $\langle \alpha_1, .., \alpha_n \rangle^X_{0,n,\beta}$ vanishes unless the degrees of the insertion classes α_i sum to the virtual dimension of the corresponding moduli space $X_{0,n,\beta}$.*

The reason for this is clear, if the combined codimension of the α_i is not equal to the dimension of the virtual fundamental class then the integral

$$\int_{[X_{0,n,\beta}]^{\mathrm{vir}}} \mathrm{ev}_1^* \alpha_1 \wedge .. \wedge \mathrm{ev}_n^* \alpha_n$$

will be zero.

2.3 Quantum Cohomology

We use Gromow–Witten invariants as structure constants to define a deformation of the usual cup product on cohomology. This deformation is called the quantum product and the cohomology ring with this product is called quantum cohomology.

In the smooth case we can think of the ordinary cup product as the intersection of cycles that represent the classes we are multiplying. Classes only have non-zero cup product if the cycles intersect. In quantum cohomology we have so called quantum corrections, which make the intersection 'fuzzy'. These corrections, roughly speaking, represent the possibility of joining the two cycles by some rational curve. This is where Gromov Witten invariants come in as they determine these quantum corrections.

Quantum cup product ($\star$) versus classical cup product ($\cup$):

$$\alpha_1 \star \alpha_2 = \alpha_1 \cup \alpha_2 + \text{quantum corrections}$$

The task in calculating quantum cohomology is to calculate the Gromov–Witten invariants which determine these quantum corrections.

2.3.1 A Basis for Cohomology

The examples that we wish to consider all have the property that cohomology is generated as a ring by degree-2 classes, and that $H^2(X;\mathbb{Z})$ is torsion-free. For notational simplicity we will assume this in what follows. In other words: let X be a smooth Fano manifold with $H^\bullet(X;\mathbb{Z})$ generated as a ring by $H^2(X;\mathbb{Z})$ and with $H^2(X;\mathbb{Z})$ torsion-free.

We shall fix a basis for $H^\bullet$ in a similar and compatible fashion to the case of projectivisation of a Fano bundle. Let $\{\phi_1,..,\phi_R\}$ be a nef integer basis for $H^2(X;\mathbb{Z})$, $\phi_0 = 1$ and ϕ_i, $i = 0,..,N-1$ a basis for $H^\bullet(X;\mathbb{Z})$ consisting of monomials in $\phi_1,..,\phi_R$ given lexicographical ordering. Finally let ϕ^i be the dual basis defined by the intersection product: $\int_X \phi_i \cup \phi^j = \delta_i^j$. Note that this is wholly compatible with the basis we defined for the cohomology of $\mathbb{P}(E)$

2.3.2 Formal Quantum Variables

Let $\Lambda := \mathbb{C}[H_2(X;\mathbb{Z})]$ be the group ring, an element of which is a finite sum $\sum_{\beta\in H^2(X;\mathbb{Z})} \lambda_\beta q^\beta$, with symbols multiplying as follows, $q^{\beta_1}q^{\beta_2} = q^{\beta_1+\beta_2}$. We let $q_i = q^{\mathrm{PD}\,\phi^i}$ be the quantum corrections associated to the basis $\mathrm{PD}\,\phi^i$, $i = 1,..,r$ of $H_2(X;\mathbb{Z})$.

Note that since the basis ϕ_i was chosen nef we know that the poincare dual basis above is given curve classes which are either on the boundary of the effective cone, or lie outside of it. As such any effective curve may be written as a non-negative linear combination of $\mathrm{PD}\,\phi^i$, $i = 1,..,R$. In particular for any β an effective curve class q^β maybe be written as

$$q^\beta = \prod_i q_i^{\lambda_i}$$

with $\lambda_i \geq 0$.

2.3.3 Quantum Multiplication

Definition 2.3.1. *Let X be a smooth algebraic variety. We consider the small quantum product as the following operation:*

$$\star : H^\bullet(X;\mathbb{C}) \times H^\bullet(X;\mathbb{C}) \to H^\bullet(X;\mathbb{C}) \otimes \Lambda$$

with

$$\alpha_1 \star \alpha_2 = \sum_{i=0}^{\dim(H^\bullet(X))-1} \sum_{\beta \in H_2(X;\mathbb{Z})} q^\beta \langle \alpha_1, \alpha_2, \phi_i \rangle_{0,3,\beta}^X \phi^i.$$

Note that this is independent of our choice of dual bases $\{\phi_i\}, \{\phi^i\}$.

We define $\deg(q_i) = -K \cdot A_i$; since X is Fano, $\deg q_i$ is strictly positive. With this definition, quantum product makes $H^\bullet(X, \Lambda)$ into a graded ring. This fact, which is a simple consequence of the virtual dimension of $X_{0,n,\beta}$ and the degree axiom (Lemma 2.2.7), constrains the degree of image class β for the Gromov–Witten invariants associated to quantum corrections based on the degree of the classes being multiplied.

The upshot is that in the multiplication of classes of low cohomological degree only low-degree Gromov-Witten invariants play a role, whereas considering multiplication of classes of high degree involves high-degree invariants. Typically the high degree invariants are significantly harder to calculate. To be precise, when multiplying two classes with degree k_1 and k_2 we need only consider maps to curves β such that $-K \cdot \beta \leq k_1 + k_2$.

Remark 2.3.2. *One could work with a somewhat different definition of quantum multiplication which depends on a class $t \in H_2(X)$. This multiplication, denoted $\underset{t}{\star}$, is defined by:*

$$\alpha_1 \underset{t}{\star} \alpha_2 = \sum_{i=0}^{\dim(H^\bullet(X))-1} \sum_{n \geq 0} \sum_{\beta \in H_2(X;\mathbb{Z})} \frac{q^\beta}{n!} \langle \alpha_1, \alpha_2, \phi_i, \overbrace{t, .., t}^{n} \rangle_{0,n+3,\beta}^X \phi^i. \tag{2.2}$$

By applying the divisor equation to this we obtain

$$\begin{aligned}\alpha_1 \underset{t}{\star} \alpha_2 &= \sum_{i=0}^{\dim(H^\bullet(X))-1} \sum_{n\geq 0} \sum_{\beta\in H_2(X;\mathbb{Z})} \frac{q^\beta}{n!}(\beta.t)^n \langle \alpha_1, \alpha_2, \phi_i\rangle^X_{0,n+3,\beta}\phi^i \\ &= \sum_{i=0}^{\dim(H^\bullet(X))-1} \sum_{\beta\in H_2(X;\mathbb{Z})} q^\beta e^{\beta.t}\langle \alpha_1, \alpha_2, \phi_i\rangle^X_{0,3,\beta}\phi^i\end{aligned}$$

So, we can obtain Definition 2.3.1, with which we work, simply by setting $t = 0$ *in* (2.2). *On the other hand we can recover* $\star_t$ *from* $\star$ *by writing* $q^\beta e^{\beta.t}$ *instead of* q^β*:*

$$\underset{t}{\star} = \underset{0}{\star}\big|_{q^\beta\mapsto q^\beta e^{\beta.t}}.$$

2.3.4 Reduction to Divisor Multiplication

As we assume that $H^2(X;\mathbb{Z})$ generates $H^\bullet(X;\mathbb{Z})$ as a ring we can reduce all quantum multiplication calculations to multiplication by divisor classes. This is easily seen with basis elements. Given a class α of degree d, we can write it as $\alpha_1 \star \alpha' - f$ where α_1 is a divisor class, α' is of degree $d-1$ and f the quantum correction for $\alpha_1 \star \alpha'$. Note that q_i has positive degree since X is Fano. Hence, as quantum corrections, the cohomological degree of f is less than d. This procedure can be iterated to produce an expression for α in terms of quantum multiplication of divisor classes and formal quantum variables.

We see, therefore, that the quantum multiplication of any class is entirely determined by quantum multiplication by divisor classes. We can express this information in terms of matrices, M_i, $i = 1,..,R$, where M_i corresponds to multiplication by the basis class ϕ_i with respect to the fixed basis $\phi_0,..,\phi_{N-1}$.

One consequence of this fact is that we only need quantum multiplication matrices for the H^2 basis $\phi_1,..,\phi_R$. A corollary to this is that we only ever need consider 2-point stable maps, as the Gromov–Witten invariants we consider will always have divisor insertions and we can therefore use the divisor axiom (Lemma 2.2.6) to reduce to the 2-point case.

2.4 Givental Formalism

Givental has developed a powerful heuristic, based on loop-spaces, for computing with Gromov–Witten invariants, and also a rigorous framework, motivated by this heuristic, in terms of infinite-dimensional symplectic geometry and D-modules. Givental's quantum D-module arises from the Dubrovin connection, which is a pencil of flat connections on the tangent bundle $TH^\bullet(X)$ defined in terms of the quantum product. In this section we describe the parts of this theory that we need. We emulate the style, if not the accomplishment, of the excellent expository papers of Guest

[18, 19].

Our aim is to define and calculate the J-function, a generating function for certain genus-zero Gromov–Witten invariants . The J-function is part of the fundamental solution to the Dubrovin connection.

The J-function neatly packages the information of quantum cohomology and allows comparisons between the quantum cohomology of an ambient space and the quantum cohomology of a complete intersection contained therein via the quantum Lefschetz theorem [11]. We aim here to introduce an approach to those techniques which makes use of computer algebra packages. As the output of the quantum reconstruction theorem will be quantum multiplication matrices our target is to describe a process of using these matrices to obtain the J-function. Using the J-function we can obtain information about the quantum cohomology of complete intersection contained within the Fano bundle. This is of particular interest in light of the Fanosearch project as it allows us to obtain the quantum period for a number of Fano varieties of dimension greater than three, which conjecturally fits them into a classification.

2.4.1 J-function

Recall, from Subsection 2.3.1, $\phi_0, .., \phi_{N-1}$ is a basis of $H^\bullet(X;\mathbb{Z})$ with $\phi_0 = 1$ and $\phi_1, .., \phi_R$ a basis for $H^2(X;\mathbb{Z})$. Additionally $\phi^0, .., \phi^{N-1}$ is the dual basis given by the intersection pairing. The J-function of X is the function $H^2(X;\mathbb{C}) \to H^\bullet(X;\mathbb{C}) \otimes \mathbb{C}[[1/z]]$ defined by

$$J_X(t) = e^{t/z}\left(1 + \sum_{\epsilon=0}^{N-1} \sum_{\beta \in \mathrm{NE}(X)} q^\beta e^{\beta \cdot t} \left\langle \frac{\phi^\epsilon}{z(z-\psi)} \right\rangle_{0,1,\beta}^X \phi_\epsilon\right)$$

where we expand $\left\langle \frac{\phi^\epsilon}{z(z-\psi)} \right\rangle_{0,1,\beta}$ as $\sum_{k\geq 0} \langle \phi^\epsilon \psi^k \rangle_{0,1,\beta} \frac{1}{z^{k+2}}$ and $t \in H^2(X;\mathbb{C})$.

Since t is nilpotent as an element of $H^\bullet(X)$ the expression $e^{t/z}$ makes sense in $H^\bullet(X) \otimes \mathbb{C}[[1/z]]$. In a similar fashion to Remark 2.3.2 we can recover the J-function at general $t \in H_2(X)$ from its value at zero $J_X(0)$. This is clear, since

$$J_X(0) = \left(1 + \sum_{\epsilon=0}^{N-1} \sum_{\beta \in \mathrm{NE}(X)} q^\beta \left\langle \frac{\phi^\epsilon}{z(z-\psi)} \right\rangle_{0,1,\beta}^X \phi_\epsilon\right)$$

Hence:

$$J_X(t) = e^{t/z} J_X(0)|_{q^d \mapsto q^d e^{t.d}}$$

We can then consider the J-function as the function $J_X(0)$ of $q_1, .., q_R$ alone without losing any information.

Much of the time we will consider the identity component of $J_X(q)$ which we denote by $J_X^0(q)$. This is given by $\langle J_X(q), \phi^0\rangle$. This can be written as a power series

$$J_X^0 = \sum_{i_j \geq 0} c_{i_1,..,i_R} q_1^{i_1} \dots q_R^{i_R}$$

2.4.2 Quantum Differential Equations

In this section we describe the D-module viewpoint of the J-function, which uses the quantum multiplication matrices defined in Subsection 2.3.4. Here we follow closely the excellent papers of Guest [18, 19].

The J-function of X, J_X, satisfies a system of differential operators, called quantum differential operators [19]. Let $M_i(q)$ denote the matrices of quantum multiplication by ϕ_i $i = 1, .., R$ with respect to the basis $\phi_0, .., \phi_{N-1}$.

Consider the system of differential equations:

$$z\, q_i \frac{\partial}{\partial q_i} s = M_i(q) s \quad i = 1, .., R$$

where s is a vector valued function of $t \in H^2(X;\mathbb{C})$, or equivalently, a multivalued vector function of $q_1, .., q_R$. This defines a flat connection on the bundle $TH^\bullet(X)$. This is called the Dubrovin connection.

The system admits a fundamental solution matrix, equivalently a flat section, the rows of which are given by vectors J_i

$$S = \begin{pmatrix} \text{-} & \text{-} & J_0 & \text{-} & \text{-} \\ \text{-} & \text{-} & J_2 & \text{-} & \text{-} \\ & & \vdots & & \\ \text{-} & \text{-} & J_{N-1} & \text{-} & \text{-} \end{pmatrix}$$

The row-vector J_{N-1} is the expansion of the J-function, $J_X(q)$, as a vector valued function in $H^\bullet(X;\mathbb{C}) \otimes \mathbb{C}[[1/z]]$ in terms of the basis $\phi_0, .., \phi_{N-1}$. The differential system is equivalent to RN differential equations in J_0 through J_{N-1}. By solving for the rows J_0 through J_{N-2} in terms of J_{N-1} we are left with $(R-1)N+1$ differential equations in J_{N-1}. Applying Groebner basis techniques we find a generating set for the ideal formed by these equations in the Weyl algebra. These (non-unique) differential equations are *quantum differential equations* and define J_X up to scalar. We fix this scalar by demanding that the constant term coefficient is equal to one.

Note that up until this point we have produced precise answers. In examples it is not difficult to calculate the differential equations we have described. In the following we shall describe a method for producing a *partial* solution to them. That is, a polynomial solution of the power series, that is correct up to some degree. In the examples section we shall aim to make it clear when we are working up to some finite degree of precision, or when answers are fully precise.

2.4.3 Solving the QDEs to Finite Degree

In this section we describe a method to obtain an arbitrary (finite) number of terms in the power series expansion of the J-function of X, by solving the quantum differential equations for X. These differential equations represent an overdetermined system, and, in general, are difficult to solve analytically. The method that we use is not especially pretty or elegant, but it is practical and allows calculation of the J-function to sufficiently high degree for our purposes.

The identity component of the J-function is necessarily of the form

$$J^0 = \sum_{\beta \geq 0} c_\beta q^\beta = \sum_{i_j \geq 0} c_{i_1,..,i_R} q_1^{i_1} \dots q_R^{i_R}.$$

We use the computer package Maple to perform calculations. Define a finite order polynomial approximation to J^0,

$$J^0_{\text{poly}} = \sum_{i=0}^{K} c_i q^i$$

where for brevity $i = (i_1, .., i_R)$ and $q^i = q_1^{i_1} \dots q_R^{i_R}$.

By applying the quantum differential operators to this function we obtain a number of polynomial expressions. Should J^0_{poly} solve the differential equations, these would vanish as polynomials. Extract the coefficients for each monomial in q_i from each expression, set these expressions to zero and form a large system of linear equations. A number of these linear equations are incomplete as J^0_{poly} is only polynomial. We then throw out those equations that are not complete.

We solve the system of linear equations setting the initial condition $c_{0,..,0} = 1$. As we have mentioned above, this technique allows us to solve the equations to some finite degree; we do not obtain the closed form solution. These systems are overdetermined and we often find that they are hard to solve analytically. On the other hand, it is sometimes possible to produce a solution to some finite degree and then 'eyeball' the correction closed form solution.

2.4.4 Regularised Quantum Period

The quantum period is a useful deformation invariant for a Fano manifold derived from the J-function. It carries sufficient information about the manifold to identify it (up to deformation, of course) uniquely, at least for Fano 3-folds. It is used in the paper *Quantum Periods for 3-Dimensional Fano Manifolds* [8] to identify Fano manifolds with their mirror polytopes. We describe here the process of obtaining the quantum period from the J-function, and what it means to normalise it.

The quantum period G_X is the identity component of the J-function, restricted to lie over $-K_X$ and evaluated at $t = 0, z = 1$. It is obtained from J_X via the following set of operations.

1. take the identity component of J_X;
2. set $t = 0, z = 1$;
3. replace q^β with $Q^{\langle \beta, -K_X \rangle}$.

The quantum period is a power series in Q, and may be written

$$G_X(Q) = 1 + \sum_{i \geq 2} c_i Q^i$$

The normalised quantum period is given by

$$\widehat{G}_X(Q) = 1 + \sum_{i \geq 2} i! c_i Q^i$$

2.5 Quantum Lefschetz Theorem

The quantum Lefschetz theorem, like its classical namesake, allows us to gain information on the quantum cohomology for a complete intersection M from the quantum cohomology of the ambient space X. The input needed for the quantum Lefschetz theorem is the J-function of X and the direct sum of line bundles which describe M as a complete intersection.

The quantum Lefschetz theorem allows us to recover part of the J-function for M from that of X. It does not recover the full J_M, in particular it can only give information about the part of cohomology of M which descends from X. In the examples we discuss we are only interested in the identity component of the J-function, the quantum Lefschetz theorem provides the entirety of J^0_M from J^0_X. We do not aim to explain how the quantum Lefschetz theorem works, but how one uses it in practical examples.

With this in mind we briefly outline the process of applying Quantum Lefschetz. There are four formal functions which play a role in the theorem: The J-function for X, J_X; the I function, $I_{X,M}$, which is produced from J_X; the twisted J-function $J_{X,M}$; and the J-function for M, J_M. The relations between these functions may seem somewhat mystical, but we take it as a black box.

Let $\mathcal{E} = \bigoplus_{i=1}^{\mathrm{codim}(M)} L_i$ be the direct sum of line bundles corresponding to the complete intersection M and $\rho_i = c_1(L_i)$ the first Chern class of the line bundle summands. Given the J-function of X, $J_X(t,z) = \sum_{d \in H_2(X)} J_d(t,z) q^d$, one forms the hypergeometric modification

$$I_{X,M}(t,z) = \sum_{d \in H_2(X;\mathbb{Z})} J_d(t,z) q^d \prod_{i}^{\mathrm{codim}(M)} \prod_{k=1}^{\rho_i \cdot d} (\rho_i + kz)$$

We consider the formal function $J_{X,M}(t,z)$, defined in [11], which has the same domain and target as $I_{X,M}$. It satisfies the following property

$$e(\mathcal{E})J_{X,M}(j^*u,z) = j_*J_M(u,z)$$

where $e(\mathcal{E}) = \prod_i \rho_i$ is the Euler class of the bundle $\mathcal{E}$. The relation between J_M and J_X is indirectly realised by the mirror map, which relates $I_{X,M}$ to $J_{X,M}$. The mirror map is determined by comparing the asymptotics of the expressions. Considered as a power series in z^{-1}, $J_{X,M}$ is the unique function with the form $J_{X,M} = z+t+O(z^{-1})$. We may write $I_{X,M}$ in the form $f(t)z + g^0(q,t)\phi_0 + \sum_{i=1}^{r} g^i(t)\phi_i + O(z^{-1})$. By homogeneity considerations writing $I_{X,M}$ in this form is practicable, even in the case that J_X is only known up to finite order in q, as in the case in point. The mirror map is given by the following;

$$\frac{I_{M,X}}{f(t)} = J_{X,M}\left(\frac{g^0(q,t)}{f(t)}\phi_0 + \sum_{i=1}^{r}\frac{g^i(t)}{f(t)}\phi_i, z\right) = e^{\frac{g^0(q,t)}{f(t)}\phi_0} J_{X,M}(\tau, q) \tag{2.3}$$

where $\tau = \sum_{i=1}^{r} \frac{g^i(t)}{f(t)}\phi_i$. The second equality follows from the string equation and the definition of J.

The procedure may summarised as follows:

1. Calculate J_X;
2. Produce the hypergeometric modification $I_{X,M}$ using the Chern classes of the bundle $\mathcal{E}$;
3. Calculate the mirror map from the asymptotics of $I_{X,M}$;
4. Produce $J_{X,M}$ from $I_{X,M}$ using the mirror map.

Remark 2.5.1. *Note that we do not obtain the entirety of J_M by comparison with $J_{X,M}$: some information is lost in the pushforward. However, we can recover the full identity component J^0_M by following the described method with J^0_X, since pushforward of the identity* is *cup with the Euler class.*

Remark 2.5.2. *It is possible, indeed often practical, to calculate the mirror map given only finite terms from the power-series J^0. So we can calculate the mirror map precisely using the first few terms J^0_{poly} by considering the asymptotics. This fact is particularly useful for our application, as we calculate J^0_{poly} using the technique described in Subsection 2.4.3*

Chapter 3
Basic Results in Quantum Cohomology

In this chapter we present some basic results concerning Gromov–Witten invariants and bundle maps. The ideas in these results are certainly used elsewhere (e.g. [1, 33] but it appears they are not stated in this particular form and generality. These will be used in the reconstruction statements in Chapter 4.

3.1 Gromov–Witten Invariants and Bundle Maps

Recall that, for E a Fano bundle, one of the extremal rays A_2 corresponds to a line in the fibre of $\mathbb{P}(E)$. We present here a lemma which determines the Gromov–Witten invariants corresponding to maps to multiples of this class.

Lemma 3.1.1. *Let $E \to Y$ be a Fano bundle of rank $r \geq 2$, with Y smooth of dimension n, not necessarily isomorphic to $\mathbb{P}^n$. Let $X = \mathbb{P}(E)$ and let ξ be the relative hyperplane class on X, $\pi : X \to Y$ the induced bundle map, and A_2 the extremal curve corresponding to lines in the fibre of π. Then, for any $\alpha_1, \alpha_2 \in H^\bullet(X; \mathbb{Z})$, with* $\deg(\alpha_1) + \deg(\alpha_2) = \dim_{\mathrm{vir}}(X_{0,2,kA_2})$

$$\langle \alpha_1, \alpha_2 \rangle^X_{0,2,kA_2} = \begin{cases} \pi_*\alpha_1 \cdot \pi_*\alpha_2 & k = 1 \\ 0 & k \geq 2 \end{cases}$$

Proof. This result generalises Lemmas 3.6 and 3.7 of [33]. We use a dimension argument to demonstrate that for $k \geq 2$ the invariants vanish as stated.

Consider the following diagram of morphisms

$$\begin{tikzcd} X_{0,2,kA_2} \arrow[r, "\mathrm{ev}_1 \times \mathrm{ev}_2"] \arrow[dr, bend right, "P"'] & X \times X \arrow[d, "\overline{P}"] & X \times_Y X \arrow[l] \\ & \mathrm{pt} & \end{tikzcd}$$

The Gromov–Witten invariant $\langle \alpha_1, \alpha_2 \rangle_{0,2,kA_2}$ is given by

$$P_* \left([X_{0,2,kA_2}]^{\mathrm{vir}} \cap (\mathrm{ev}_1 \times \mathrm{ev}_2)^* (\alpha_1 \boxtimes \alpha_2) \right)$$

We use a push-pull argument (i.e. apply the projection formula [14]) to demonstrate that this vanishes. Since P is the map to a point, the above is the integral as expected. We factor P through the above diagram as $\overline{P} \circ (\mathrm{ev}_1 \times \mathrm{ev}_2)$, since the map to a point is unique.

$$
\begin{aligned}
&P_*\left([X_{0,2,kA_2}]^{\mathrm{vir}} \cap (\mathrm{ev}_1 \times \mathrm{ev}_2)^* (\alpha_1 \boxtimes \alpha_2)\right)\\
=&\overline{P}_*\left((\mathrm{ev}_1 \times \mathrm{ev}_2)_* [X_{0,2,kA_2}]^{\mathrm{vir}} \cap (\alpha_1 \boxtimes \alpha_2)\right)
\end{aligned}
$$

Connected curves of the class kA_2 are restricted to lie in a single fibre of π, so both points of a stable map in $X_{0,2,kA_2}$ map to the same fibre of π. It is clear that the image $(\mathrm{ev}_1 \times \mathrm{ev}_2)(X_{0,2,kA_2})$ lies in $X \times_Y X$, which has dimension $n+2r-2$. The pushforward of the virtual fundamental class, $(\mathrm{ev}_1 \times \mathrm{ev}_2)_* [X_{0,2,kA_2}]^{\mathrm{vir}}$, is supported on this set. On the other hand the virtual dimension of $X_{0,2,kA_2}$ is $n+r-1+kr$, in particular for $k \geq 2$ the virtual dimension is at least $n+3r-1$. Since the virtual dimension is greater than the dimension of the set on which the pushforward is supported, $(\mathrm{ev}_1 \times \mathrm{ev}_2)_* [X_{0,2,kA_2}]^{\mathrm{vir}}$ vanishes, and so too does the Gromov–Witten invariant.

To prove that $\langle \alpha_1, \alpha_2 \rangle^X_{0,2,A_2} = \pi_*\alpha_1 \cdot \pi_*\alpha_2$ we show that the Gromov–Witten invariants are enumerative in this case. In order to demonstrate that the invariants are enumerative, we must prove that the corresponding moduli space of stable maps $X_{0,2,A_2}$ is smooth and of expected dimension.

Note that A_2 can only be written as the class of a line in the fibre of π, and in particular cannot be written as the class of an irreducible or non-reduced curve. Hence the moduli space of maps from $\mathbb{P}^1$, which forms the open, dense component of the moduli space of stable maps, is compact and has expected dimension. In particular $X_{0,0,A_2}$ is the Grassmann bundle of 2-dimensional vector subspaces in E, or equivalently the bundle of lines in $\mathbb{P}(E)$. It is smooth and of the expected dimension

$$n + 2(r-2) = n + 2r - 4 = (n+r-1) - 3 + r.$$

The Gromov–Witten invariants are therefore enumerative and can be calculated by counting curves of class A_2 on which the insertion classes are supported.

This is identical to the problem of counting fibres of π common to the support of both α_1 and α_2. This counting is carried out by pushdown via π followed by intersection in the cohomology of the base. This yields the claimed formula.

□

3.2 Gromov–Witten Invariants and Smooth Locus Blow-ups

In this section we develop the ideas of Lemma 3.1.1 to calculate the Gromov–Witten invariants corresponding to maps to lines in the extremal fibres of a smooth blow-up. In his paper *Gromov-Witten invariants of blow-ups* Gathmann [15] presents a theory on the Gromov–Witten invariants of the blow-up of points in projective space. We generalise the result of Lemma 2.4 regarding maps to extremal classes of blow-ups. Maggesi presents a generalisation in the form of blow-ups of linear subvarieties of $\mathbb{P}^n$ [29]. Our work here is easier as we are interested only in stable maps with target class given by a line in exceptional fibres of the blow-up.

Lemma 3.2.1. *Let Y be a smooth algebraic variety, $Z \subset Y$ a smooth, connected subvariety of codimension $r \geq 2$, and $X := Bl_Z Y$ the blow-up of Y along Z. Let A_1 be the extremal ray contracted by the blow-up map $\Xi : X \to Y$. A_1 is the class of a line in the fibre of the exceptional divisor D of the blow-up. Then, for any $\alpha_1, \alpha_2 \in H^\bullet(X;\mathbb{Z})$, with $\deg(\alpha_1) + \deg(\alpha_2) = \dim_{\mathrm{vir}}(X_{0,2,kA_1})$*

$$\langle \alpha_1, \alpha_2 \rangle^X_{0,2,kA_1} = \begin{cases} \Xi|_* \iota^* \alpha_1 \cdot \Xi|_* \iota^* \alpha_2 & k = 1, \\ 0 & k \geq 2 \end{cases}$$

where $\Xi|$ is the restriction of Ξ to $D \subset X$, the exceptional divisor of the blow-up, and $\iota : D \hookrightarrow X$ is the embedding of D in X.

Proof. We summarise the geometry of the situation in the following diagram

$$\begin{tikzcd} D \arrow[r, hook, "\iota"] \arrow[d, "\Xi|"] & X \arrow[d, "\Xi"] \\ Z \arrow[r, hook] & Y \end{tikzcd}$$

Note that $D = \mathbb{P}(N_{Z/Y})$ is the projectivisation of the normal bundle to Z in Y so we are in a similar situation to that discussed to Lemma 3.1.1. Many of the arguments in this proof differ little from that of the preceding lemma.

We first prove vanishing for $k \geq 2$ using a push-pull argument.

Consider the following diagram of morphisms.

$$\begin{tikzcd} X_{0,2,kA_1} \arrow[r, "\mathrm{ev}_1 \times \mathrm{ev}_2"] \arrow[dr, bend right, "P"'] & X \times X \arrow[d, "\overline{P}"] & D \times_Z D \arrow[l] \\ & \mathrm{pt} & \end{tikzcd}$$

The Gromov–Witten invariant $\langle \alpha_1, \alpha_2 \rangle_{0,2,kA_1}$ is given by

$$P_* \left([X_{0,2,kA_1}]^{\mathrm{vir}} \cap (\mathrm{ev}_1 \times \mathrm{ev}_2)^*(\alpha_1 \boxtimes \alpha_2) \right)$$

Since P is the map to a point, the above is an integral as expected. If we can show that the integrand vanishes we have shown vanishing for the Gromov–Witten invariant. We factor P through the above diagram as $\overline{P} \circ (\mathrm{ev}_1 \times \mathrm{ev}_2)$, since the map to a point is unique. We manipulate this expression using the projection formula [14].

$$\begin{aligned}&P_*\left([X_{0,2,kA_1}]^{\mathrm{vir}}\cap(\mathrm{ev}_1\times\mathrm{ev}_2)^*(\alpha_1\boxtimes\alpha_2)\right)\\=&\overline{P}_*\left((\mathrm{ev}_1\times\mathrm{ev}_2)_*\,[X_{0,2,kA_1}]^{\mathrm{vir}}\cap(\alpha_1\boxtimes\alpha_2)\right)\end{aligned}$$

Since connected curves of class kA_1 are restricted to a single fibre of $\Xi|$, it is clear that the image $(\mathrm{ev}_1\times\mathrm{ev}_2)(X_{0,2,kA_1})$ lies in $D\times_Z D$, which has dimension $n+r-2$. The pushforward of the virtual fundamental class, $(\mathrm{ev}_1\times\mathrm{ev}_2)_*\,[X_{0,2,kA_1}]^{\mathrm{vir}}$, is supported on this set. On the other hand the virtual dimension of $X_{0,2,kA_1}$ is $n-1+k(r-1)$, in particular for $k\geq 2$ the virtual dimension is at least $n+2r-3$. Since the virtual dimension is greater than the dimension of the set on which the pushforward is supported, $(\mathrm{ev}_1\times\mathrm{ev}_2)_*\,[X_{0,2,kA_1}]^{\mathrm{vir}}$ vanishes, and so too does the Gromov–Witten invariant.

We now prove the invariants $\langle\alpha_1,\alpha_2\rangle^X_{0,2,A_1}$ can be calculated as claimed. The class A_1 is irreducible and reduced and can only be represented as a line in the fibre of the exceptional locus, hence the moduli space of holomorphic maps $\mathbb{P}^1\to X$ with target class A_1 is compact and of expected dimension. In particular the moduli space $X_{0,0,A_1}$ is given by the Grassmann bundle of lines in the fibre of $\mathbb{P}(N_{Z/Y})$. It is smooth and of the expected dimension

$$(n-r)+2(r-2)=n+r-4=n-3+r-1.$$

The Gromov–Witten invariants are therefore enumerative and can be calculated by counting curves of class A_1 on which the insertion classes are supported.

This is identical to the problem of counting lines in fibres over the exceptional locus on which $\iota^*\alpha_1$ and $\iota^*\alpha_2$ are supported. As the exceptional divisor is the blow-up of the normal bundle, we are in the cases of Lemma 3.1.1 and are counting common fibres of Ξ in the support of $\iota^*\alpha_1$ and $\iota^*\alpha_2$. This counting is carried out by pushdown via Ξ followed by intersection in the cohomology of the base. This yields the claimed formula.

$\square$

Chapter 4
Quantum Reconstruction for Fano Bundles

The quantum reconstruction theorem for Fano bundles on projective space states that we can recover the small quantum product from a small part of low-degree quantum multiplication. The usefulness of this result increases with the relative rank of the bundle as compared with the base projective space, as the required input becomes relatively smaller when compared with the full multiplication table.

4.1 Reconstruction Lemmas

Before we state the reconstruction theorem we prove two Lemmas that recover the quantum multiplication of certain basis elements from previously known data. With care one could make similar statements basis free, however since we have fixed a basis and quantum cohomology is distributive this basis dependent form will suffice.

4.1.1 ξ-Lemma

We prove a lemma stating that the quantum multiplication of a basis class by ξ can be recovered from the quantum multiplication of the same class by p.

Lemma 4.1.1 (ξ-Lemma)**.** *Let i, j be non negative integers. The Gromov–Witten invariants which determine the quantum corrections in $\xi \star p^i\xi^j$ are determined by Lemma 3.1.1 and the Gromov–Witten invariants corresponding to the corrections in $p \star p^i\xi^j$.*

Proof. As a consequence of the definition of quantum multiplication and the degree axiom of Gromov–Witten invariants (Lemma 2.2.7) the quantum corrections in $p \star p^i\xi^j$ are determined by, and hence determine, all Gromov–Witten invariants of the form

$$\langle p^i\xi^j, \alpha\rangle^X_{0,2,\beta}$$

for all $\beta \in H_2(X;\mathbb{Z})$ such that $-K \cdot \beta \leq i + j + 1$ and $p \cdot \beta > 0$ and $\alpha \in H^\bullet(X;\mathbb{Z})$.

Likewise, the quantum corrections for $\xi \star p^i\xi^j$ are determined by invariants of the form

$$\langle p^i\xi^j, \alpha\rangle^X_{0,2,\beta}$$

for all $\beta \in H_2(X;\mathbb{Z})$ such that $-K \cdot \beta \leq i + j + 1$ and $\xi \cdot \beta > 0$ and $\alpha \in H^\bullet(X;\mathbb{Z})$.

It is clear that the only Gromov–Witten invariants required to determine the quantum corrections in $\xi \star p^i\xi^j$ that are not already determined by $p \star p^i\xi^j$ are those where the target class β satisfies $p \cdot \beta = 0$. Following the discussion of Section 1.4 it is clear that the class β therefore lies on the extremal ray generated by A_2 and is therefore some multiple of A_2. All the invariants with target class kA_2 are determined by Lemma 3.1.1.

□

4.1.2 p-Lemma

We prove a lemma that recovers p-multiplication from a related ξ-multiplication and lower degree data. When multiplying classes which contain $p^n\xi$ as a factor only the lower degree data is required.

Lemma 4.1.2 (p-Lemma)**.** *Let i, j be non-negative integers, with $j \geq 1$. The quantum multiplication $p \star p^i\xi^j$ is determined by $\xi \star p^{i+1}\xi^{j-1}$ and quantum multiplication by p and ξ of classes of degree $\leq i+j-1$.*

Note that for $i = n$, $\xi \star p^{n+1}\xi^{j-1} = 0$, since $p^{n+1} = 0$. This special case of the lemma states that $p \star p^n\xi^j$, ($j \geq 1$) is determined purely by lower degree multiplication.

Proof. Since quantum cohomology is both associative and commutative, we can use the knowledge of multiplication in lower degree to make the following manipulations:

$$\begin{aligned} p \star p^i\xi^j =& p \star (\xi \star p^i\xi^{j-1} - h) \\ &= \xi \star \left(p \star p^i\xi^{j-1}\right) - p \star h \\ &= \xi \star \left(p^{i+1}\xi^{j-1} + g\right) - p \star h \\ &= \xi \star p^{i+1}\xi^{j-1} + \xi \star g - p \star h \end{aligned}$$

where g is the quantum correction from $p \star p^i\xi^{j-1}$ and h is the correction from $\xi \star p^i\xi^{j-1}$.

Note that as g and h are corrections they necessarily include a quantum variable factor. As such the cohomological factor is of strictly lower degree. The multiplication of the cohomological content by the divisor classes p and ξ is, by assumption, known.

□

4.2 Quantum Reconstruction for Fano Bundles

4.2.1 Main Theorem

We combine the above lemmas to produce the following theorem.

Theorem 4.2.1. *Let $E \to \mathbb{P}^n$ be a Fano bundle on projective space. The quantum cohomology for $\mathbb{P}(E)$ is determined by the ordinary cohomology and the quantum multiplication $p \star p^k$ for $0 \le k \le n$.*

Proof. The proof proceeds by induction. This is essentially the construction of an algorithm. We produce an algorithm in pseudocode in Appendix B.

Recall that for Fano bundles $H^\bullet(\mathbb{P}(E))$ is generated as a ring by the classes in $H^2(\mathbb{P}(E))$ and as such quantum multiplication is determined by quantum multiplication by divisor classes (see Subsection 2.3.4). It is therefore sufficient to calculate multiplication by p and ξ only.

Recall that we fix a basis $\phi_0, .., \phi_{r(n+1)-1}$. Since the basis is given lex ordering, it is clear that given i, the set $\{\phi_j | j < i\}$ contains all basis classes of lower degree than that of ϕ_i.

The induction proceeds as follows. We assume that quantum multiplication, by p and ξ, for all classes ϕ_j with $j < i$ is known. Note that by hypothesis the multiplication $p \star p^k$ for $0 \le k \le n$ is also known. We then prove that we can obtain the quantum multiplication of ϕ_i via Lemma 4.1.2 and Lemma 4.1.1.

The base case is trivial: simply take $i = 1$. Since $\phi_0 = 1$, $p \star \phi_0$ is known by hypothesis. We obtain $\xi \star \phi_0$ by application of the ξ-Lemma (Lemma 4.1.1).

We now prove the inductive step. ϕ_i is necessarily of the form

$$\phi_i = \begin{cases} p^k & 0 \le k \le n, \\ p^n \xi^l & 1 \le l \le r-1, \\ p^k \xi^l & 0 \le k \le n-1, 1 \le l \le r-1. \end{cases}$$

In the first case the multiplication $p \star \phi_i = p^k$ is known by hypothesis. In the second case we are in the special case of the p-Lemma 4.1.2. In the final case it is clear by the above discussion that we have quantum multiplication for lower degree classes and that $\phi_{i-1} = p^{k+1}\xi^{l-1}$, so we are in the general case of the p-Lemma.

In any case we obtain the quantum multiplication $p \star \phi_i$, whether trivially or by application of the p-Lemma. We now apply the ξ-Lemma to obtain $\xi \star \phi_i$. This completes the inductive step of the proof.

□

4.2.2 Corollary à la Kontsevich

We describe a theorem which combines quantum reconstruction for Fano bundles with Kontsevich's first reconstruction theorem, providing a list of invariants from which all other invariants may be reconstructed.

The following corollary is essentially trivial. It states that the quantum cohomology determines all three-point invariants with one divisor insertion class and restriction on the degree of the target class β. The purpose of this corollary is to demonstrate that the quantum cohomology produces the required input for Kontsevich's first reconstruction theorem.

Corollary 4.2.2. *The invariants of the form $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta}$, for $\alpha_1 \in H^2(X)$ and for all β with $-K \cdot \beta \leq 2 \dim_{\mathbb{C}} X + 1$, are determined by the following set of invariants*

$$\{\langle \alpha, p^i \rangle_{0,2,\beta} \quad | 0 \leq i \leq n; \quad -K \cdot \beta \leq n+1; \quad deg(\alpha) + i = -K \cdot \beta + n + r - 2\}$$

Proof. The set of invariants which we claim are sufficient are precisely those which determine the quantum multiplication $p \star p^i$, for i from 0 to n. By Theorem 4.2.1, these yield the full small quantum product. For any invariant $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,3,\beta}$ we can assume that the α_i are basis elements, since Gromov–Witten invariants are multilinear. It is clear from the definition of quantum multiplication that such an invariant appears in the multiplication $\alpha_1 \star \alpha_2$ and hence may be determined from the quantum product. ☐

We recall here the statement of Kontsevich's first reconstruction theorem [24].

Theorem 4.2.3. *[Kontsevich] Let X be a smooth projective variety with $H^\bullet(X; \mathbb{Q})$ generated by $H^2(X; \mathbb{Q})$. All genus zero invariants can be uniquely reconstructed from the following set of invariants:*

$$\{\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,2,\beta} \quad | -K \cdot \beta \leq 2 \dim_{\mathbb{C}} X + 1; \deg \alpha_1 = 1; \sum_{i=1}^{3} \deg \alpha_i = -K \cdot \beta + \dim_{\mathbb{C}} X\}$$

The following theorem combines Corollary 4.2.2 with Kontsevich's first reconstruction theorem. The proof follows trivially, as the statement of Corollary 4.2.2 gives the required input for Theorem 4.2.3. We arrive at a strengthening of Kontsevich's reconstruction in the case of Fano bundles.

Theorem 4.2.4. *Let $E \to \mathbb{P}^n$ be a Fano bundle. All genus-zero Gromov–Witten invariants of $\mathbb{P}(E)$ may be determined from the following set of invariants*

$$\{\langle \alpha, p^i \rangle_{0,2,\beta} \quad | 0 \leq i \leq n, \quad -K \cdot \beta \leq n+1, \quad deg(\alpha) + i = -K \cdot \beta + n + r - 2\}$$

4.2.3 $QH\mathbb{P}(E)$ is not a module over $QH\mathbb{P}^n$ (and that's ok)

In their paper, Qin and Ruan note that, counter to the intuition given by classical cohomology, the quantum cohomology ring of a projectivised bundle $QH(\mathbb{P}(E))$ need not be a module over the quantum cohomology ring of the base $QH\mathbb{P}^n$. With the above theorem we see that all the essential information of the quantum cohomology ring is contained in this 'non-modularity'; the information is contained in the difference between $p \star p^i$ in the cohomology of the projectivised bundle and $p \star p^i$ in the cohomology of the base $\mathbb{P}^n$. This seems to be a good indiction that the quantum cohomology of a Fano bundle is not determined by the Chern classes of the bundle alone.

4.3 Special Cases

There are two special cases which are of particular interest to us. First, in the case that the bundles rank is high with respect to the first Chern number we can reduce the invariants required to those which map to multiples of the class A_1. The second case is a subset of the first, where additionally the extremal map which collapses A_1 is given by the blow-up of a smooth subvariety in smooth ambient. In this case all invariants are determined by the lemmas of Chapter 3.

4.3.1 High Rank - $r + c_1 > 0$

Let E be a Fano bundle normalised in the fashion of Section 1.1. The first Chern number c_1 is then well defined as an invariant of the projectivisation $\mathbb{P}(E)$ and we can speak of it without ambiguity. In the case that $r + c_1 > 0$ the statement of the reconstruction theorem is strengthened.

First note that given $r + c_1 > 0$ the primitive generator of the first extremal ray R_1 is given by A_1 with the following intersection properties. $p \cdot A_1 = 1$, $\xi \cdot A_1 = 0$. This follows from the fact that on any line $a_1(l) = 0$. So Lemma 1.4.1 demonstrates that the ray is as described.

The degree of the curve A_1+A_2 is given by $-K\cdot(A_1+A_2) = ((n+1+c_1)p + r\xi)\cdot(A_1 + A_2) = n+1+c_1+r > n+1$ hence it is clear that invariants with target class $A_1 + A_2$ do not appear in the quantum corrections of $p \star p^i$ for $0 \leq i \leq n$. Indeed curves of class $kA_1 + lA_2$ with $k, l > 0$ do not appear as corrections here.

Corollary 4.3.1. *Let $E \to \mathbb{P}^n$ be a rank-r, normalised Fano bundle on projective space with $c_1 + r > 0$. Let A_1 be the class $\mathrm{PD}(p^{n-1}\xi^{r-1} + c_1 p^n \xi^{r-2})$. The quantum cohomology for $\mathbb{P}(E)$ is determined by the multiplication the following set of invariants.*

$$\{\langle \alpha, p^i\rangle_{0,2,kA_1} \quad | k(n+1+c_1) \leq n+1, \quad deg(\alpha) + i = k(n+1+c_1) + n + r - 2\}$$

Proof. The proof of the statement is simply an application of the degree axiom Lemma 2.2.7. A curve of class A_1+A_2 has degree $n+1+c_1+r$. Under the assumption $r + c_1 > 0$ such a curve need not be considered as target of stable maps involved in the multiplication $p \star p^i$, $0 \leq i \leq n$ and hence, by Theorem 4.2.1, are not involved in determining the quantum cohomology. It is then clear that the only invariants required to determine the quantum cohomology are those listed above.

□

Remark 4.3.2. *If E is a normalised Fano bundle and the first extremal class A_1 is representable by a curve which maps down to a line under π, then ξ is necessarily strictly nef by Lemma 1.4.4. It is then clear that, since $-K_{\mathbb{P}(E)}$ is ample, $n+1+c_1 > 0$. Then $r > n$ is sufficient to guarantee that $r + c_1 > 0$. Should the conjecture made at the end of Chapter 1 hold, then the above corollary holds automatically for bundles with $r > n$. This is of particular interest as one might expect that typically low rank Fano bundles split. This is certainly true in the rank-2 case where all rank-2 Fano bundles on $\mathbb{P}^n$ split for $n \geq 3$ [2].*

4.3.2 Smooth Blow-Up

When the extremal map that collapses A_1 realises the Fano bundle as a blow-up along a smooth subvariety in smooth ambient space we can reduce all calculation to classical intersection. This combines the previous corollary to reconstruction with the results of Chapter 3.

Corollary 4.3.3. *Let $E \to \mathbb{P}^n$ be a rank-r, normalised Fano bundle on projective space with $c_1 + r > 0$. Further let $\mathbb{P}(E)$ be realised as the blow-up of a smooth subvariety. Then the quantum cohomology of $\mathbb{P}(E)$ is determined by Lemma 3.2.1.*

Proof. As in Corollary 4.3.1 the only invariants required are those which map to multiples of A_1. In this case these are maps to lines in the fibres of the exceptional locus of the blow-up. These are determined by Lemma 3.2.1 as claimed. $\square$

Chapter 5
Worked Example - $\Omega^2(2)_{\mathbb{P}^n}$

In this chapter we discuss the example of the Fano bundle $(\wedge^2\Omega)\otimes\mathcal{O}(2)$, the second wedge of the cotangent bundle on $\mathbb{P}^n$ twisted by $\mathcal{O}(2)^*$. We denote this bundle $\Omega^2(2)$. We first describe the bundle on general $\mathbb{P}^n$ and demonstrate that it is Fano and normalised in the sense of Subsection 1.1.

We describe in detail the geometry of $X := \Omega^2(2)_{\mathbb{P}^4}$ over $\mathbb{P}^4$ and that of a particular rank-2 Fano 3-fold number 17 in Mori-Mukai [30], which we denote by M. We also note that two Fano 4-folds exist as complete intersections in X.

We apply techniques developed above, in particular Corollary 4.3.3, to recover the quantum cohomology for X. Using the method described in Subsection 2.4.3 we obtain a finite degree part of the J-series for X. Proceeding using the quantum Lefschetz theorem (see Section 2.5) we calculate a finite degree (i.e. polynomial) part of the identity component of J-function of M. This finite part is of sufficiently high degree to precisely determine the Picard-Fuchs operator. We describe how this calculation fits into the Fanosearch classification of Fano 3-folds [7].

The Fanosearch team, in particular Alessio Corti, used an alternative calculation technique, the abelian/non-abelian correspondence, to obtain the J-function for M [8]. This calculation bypasses the Fano bundle calculation and provides a closed form expression for the J-function of M. In Subsection 5.4.1 we use this closed form and an 'inverse quantum Lefschetz' technique to recover a closed form expression for the J-series of X. Finally, using this closed form expression we obtain the J-series for the two Fano 4-folds embedded in X. We conjecture that these 4-folds are not of toric complete-intersection type as their J-series do not appear in a conjecturally complete list of such Fano 4-folds [10].

5.1 The Geometry of $\Omega^2(2)_{\mathbb{P}^n}$

We denote by $\Omega^2_{\mathbb{P}^n} := \wedge^2\Omega_{\mathbb{P}^n}$ the second wedge of the cotangent bundle on $\mathbb{P}^n$. We tensor by $\mathcal{O}(2)$ since this gives us the desired normalisation, as we will see shortly. For brevity let $E := \Omega^2(2)$ and $X := \mathbb{P}(E)$.

On projective space we have the following sequence of vector bundles, known as the Euler sequence.

$$0 \to \Omega(1) \to V^\vee \otimes \mathcal{O} \to \mathcal{O}(1) \to 0$$

*note that if we used the opposite convention for the associated projective bundle the cotangent bundle is simply replaced by the tangent bundle

We can take the second wedge power of the Euler sequence, a procedure described in Hartshorne [20], in order to obtain the following sequence.

$$0 \to E \to \wedge^2 V^\vee \otimes \mathcal{O} \xrightarrow{v \lrcorner} V^\vee \otimes \mathcal{O}(1) \to \mathcal{O}(2) \to 0 \tag{5.1}$$

As discussed in Section 1.1, this implies that ξ_E is nef, since $E^\vee$ is generated by global sections. Indeed, as noted in that section ξ_E is base point free. From (5.1) one calculates:

$$c_i(E) = (-1)^i \frac{(n+i)! - 2(n+i-1)!}{n!} p^i. \tag{5.2}$$

In particular the first Chern number $c_1 = -(n-1)$. This implies that E is Fano for any n since ξ is nef and $n+1+c_1 > 0$. The bundle is rank $\frac{n(n-1)}{2}$. In particular this shows that the bundle is normalised as $r + c_1 > 0$, and (from Lemma 1.4.1) that the extremal ray A_1 is given as follows.

$$A_1 = PD(p^{n-1}\xi^{r-1} + c_1 p^n \xi^{r-2}).$$

5.1.1 Classical Geometry of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$

We focus our attention on the example of this bundle over $\mathbb{P}^4$, and here we describe the geometry of the projectivisation, in particular the nature of the extremal contractions. The following discussion shows that X is given by the blow-up of the Plücker embedding of $G(2, V^\vee)$ inside $\mathbb{P}(\wedge^2 V^\vee)$.

We see, from the sequence (5.1), that we have an embedding of $X = \mathbb{P}(E)$ into $\mathbb{P}^4 \times \mathbb{P}^9$, with projections to $\mathbb{P}^4$ and $\mathbb{P}^9$ given by extremal contractions corresponding to p and ξ respectively.

$$\begin{tikzcd} X \arrow[r, "\Xi"] \arrow[d, "P"'] & \mathbb{P}(\wedge^2 V^\vee) \\ \mathbb{P}(V) & \end{tikzcd}$$

The equations for the embedding of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ inside $\mathbb{P}^4 \times \mathbb{P}^9$ are given by the map $v\lrcorner$:

$$(5.3)\quad \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ -a_0 & 0 & 0 & 0 & a_2 & a_3 & a_4 & 0 & 0 & 0 \\ 0 & -a_0 & 0 & 0 & -a_1 & 0 & 0 & a_3 & a_4 & 0 \\ 0 & 0 & -a_0 & 0 & 0 & -a_1 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & 0 & -a_0 & 0 & 0 & -a_1 & 0 & -a_2 & -a_3 \end{pmatrix} \cdot \begin{pmatrix} y_{01} \\ y_{02} \\ y_{03} \\ y_{04} \\ y_{12} \\ y_{13} \\ y_{14} \\ y_{23} \\ y_{24} \\ y_{34} \end{pmatrix} = 0$$

where the a_i are homogenous coordinates on $\mathbb{P}^4 = \mathbb{P}(V)$ and the y_{ij} are the obvious coordinates on $\mathbb{P}^9 = \mathbb{P}(\wedge^2 V^\vee)$. From the form above we can see that the extremal contraction P down to $\mathbb{P}^4$ is a bundle map (as expected) since the matrix on the left of (5.3) is of constant rank.

We can rewrite the defining equation as:

$$(5.4)\quad \begin{pmatrix} 0 & y_{01} & y_{02} & y_{03} & y_{04} \\ -y_{01} & 0 & y_{12} & y_{13} & y_{14} \\ -y_{02} & -y_{12} & 0 & y_{23} & y_{24} \\ -y_{03} & -y_{13} & -y_{23} & 0 & y_{34} \\ -y_{04} & -y_{14} & -y_{24} & -y_{34} & 0 \end{pmatrix} \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0.$$

When written in the form of (5.4) we can see that the map Ξ is an isomorphism outside of a codimension 1 subset. This subset is given by the locus where the antisymmetric matrix in (5.4) drops rank. This occurs when the Pfaffians of the 4×4 minors vanish – matrices drop rank when the determinant of their minors drop rank and the determinant of an antisymmetric matrix is the square of the Pfaffian.

The vanishing of these Pfaffians inside $\mathbb{P}(\wedge^2 V^\vee)$ precisely describes the Plücker embedding of the Grassmannian $Gr(2, V^\vee)$ in $\mathbb{P}(\wedge^2 V^\vee)$ (see e.g. [17, pg. 211]). Since X, $\mathbb{P}^9$ and $Gr(2, V^\vee)$ are all smooth and the map Ξ is an extremal contraction the map is necessarily the blow up of the Grassmannian inside $\mathbb{P}(\wedge^2 V^\vee)$.

We can alternatively arrive at the same conclusion by considering the rational map $\mathbb{P}(\wedge^2 V^\vee) \dashrightarrow \mathbb{P}(V)$ that the bundle canonically induces. The map is given on the dense set of rank-4 matrices by sending a matrix to its kernel.

Lemma 5.1.1. *The rational map $\mathbb{P}(\wedge^2 V^\vee) \dashrightarrow \mathbb{P}(V)$ is given by the linear system of quadrics containing $G(2, V^\vee)$.*

Proof. The map sends a 2-form w, thought of as an antisymmetric 5×5 matrix $A\colon V \to V^\vee$, to its kernel. By a version of the Cramer rule, we can describe the map explicitly by sending A to the vector of 4×4 Pfaffians:

$$\mathrm{pf}(A) = (\mathrm{pf}_0(A), \ldots, \mathrm{pf}_4(A)).$$

The statement then reduces to the fact that these Pfaffians generate the ideal of $G(2,V) \subset \mathbb{P}(\wedge^2 V^\vee)$. $\square$

5.2 Fano Submanifolds of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$

We now consider Fano submanifolds of X, and describe them as complete intersections. In particular we consider the rank-2 Fano 3-fold, number 17 in the Mori-Mukai list [30] and two Fano 4-folds.

5.2.1 Fano 3-fold 2–17

We denote by M the Fano 3-fold No. 17 in the Mori–Mukai list of rank 2 Fano 3-folds [30]. According to Mori–Mukai, M is the blow-up of a 3-dimensional quadric $Q \subset \mathbb{P}^4$ with centre $\Gamma \subset Q$, a nonsingular curve of genus 1 and degree 5.

Lemma 5.2.1. *M is a complete intersection of type $p \cap \xi^5$ in $\mathbb{P}(E)$*

Proof. It is well–known that the Plucker embedding $G(2,V^\vee) \hookrightarrow \mathbb{P}(\wedge^2 V^\vee)$ is of degree five. Using adjunction, one can easily check that the curve given by the complete intersection of 5 general hyperplanes with the Grassmannian has trivial canonical bundle and hence is genus 1:

$$\Gamma = h_1 \cap \cdots \cap h_5 \cap G(2,V^\vee) \subset h_1 \cap \cdots \cap h_5 \cong \mathbb{P}^4,$$

all of this taking place in the natural ambient $\mathbb{P}(\wedge^2 V^\vee)$. Since $\Xi\colon X \to \mathbb{P}(\wedge^2 V^\vee)$ is the blow-up of $\mathbb{P}(\wedge^2 V^\vee)$ along the Plücker embedding of $G(2,V^\vee)$, the discussion makes it clear that M is the complete intersection

$$M = \Xi^*(h_1) \cap \cdots \Xi^*(h_5) \cap \widetilde{Q} \subset X$$

where $\widetilde{Q}$ is the proper transform of a quadric containing $G(2,V^\vee)$, i.e., by Lemma 5.1.1, a section of p. $\square$

Remark 5.2.2. *Since p and ξ are basepoint free (see Subsection 1.1.1) such a complete intersection is a smooth variety.*

Corollary 5.2.3. *Since M is smooth we can apply adjunction to see that $-K_M = (p+\xi)|_M$.*

5.2.2 Fano 4-folds in $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$

There are two distinct Fano 4-folds which exist as complete intersections in X. These are given by the intersection of type ξ^5 and $p \cap \xi^5$. As there is not a complete classification for Fano 4-folds we do not currently have a description of these 4-folds which is analogous to that of the 3-fold M. Nevertheless, as these 4-folds are complete intersections within X we can obtain their regularised quantum period sequences using quantum Lefschetz, and do so in Subsection 5.4.2.

5.3 The Quantum Cohomology of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$

In this section we use the reconstruction theorem (Theorem 4.2.1) to calculate the quantum cohomology of $X := \mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$. In particular we will make use of Corollary 4.3.3 since the geometry $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ is of the required form: $c_1 + r > 0$ and X is the blow-up of a smooth subvariety in smooth ambient. We produce a polynomial approximation to the J-function for X from the quantum differential operators as described in Subsection 2.4.3. We then use Quantum Lefschetz [11] to obtain information about the quantum cohomology of M, the Fano 3-fold embedded in X.

5.3.1 Presentation of the Quantum Cohomology Ring

Since $X = \mathbb{P}(E)$ is the projectivisation of a Fano bundle and the extremal contraction Ξ is the blow-up of $G(2,5) \subset \mathbb{P}^9$ the quantum cohomology follows from Corollary 4.3.3.

We make use of the Schubert calculus for $G(2,5)$, following notational conventions from [17]. In particular, let σ_i be the ith Chern class of the tautological quotient bundle Q on $G(2,V^\vee)$. For example σ_1 is the class of the Schubert cell of lines which intersect a given plane. We describe the Schubert cells and produce the multiplication table in Appendix A.

Let N be the normal bundle to the embedding of $G(2,V^\vee)$ in $\mathbb{P}(\wedge^2 V^\vee)$. The exceptional divisor D is given by the projectivisation of N. The normal bundle to the embedding into $\mathbb{P}(\wedge^2 V^\vee)$ is given by $Q^\vee(2\sigma_1)$, though for the sake of convenient relations in cohomology we will work instead with $Q^\vee$ (of course $\mathbb{P}(N)$ and $\mathbb{P}(Q^\vee)$ are isomorphic). The cohomology of this bundle is given by standard calculation as:

$$H^\bullet(\mathbb{P}(Q^\vee)) = \frac{H^\bullet(G(2,V^\vee))[\eta]}{(\eta^3 - \sigma_1\eta^2 + \sigma_2\eta - \sigma_3)}$$

where η is the relative hyperplane class of $\mathbb{P}(Q^\vee)$.

The following diagram describes the geometry of the situation showing the exceptional locus $\mathbb{P}(Q^\vee)$ embedding inside X.

$$\begin{tikzcd}
\mathbb{P}(Q^\vee) \arrow[r, hook, "\iota"] \arrow[d, "\Xi|"] & X \arrow[d, "\Xi"] \\
G(2,V^\vee) \arrow[r, hook] & \mathbb{P}(\wedge^2 V^\vee)
\end{tikzcd}$$

For the purpose of calculation, note that $\iota^*\xi = \sigma_1$ and $\iota^* p = \eta$. This is relatively easy to check. It is clear the $\iota^*\xi = \sigma_1$ since ξ corresponds to $\mathcal{O}(1)_{\mathbb{P}(\wedge^2 V^\vee)}$ under pullback by the Plücker embedding. Then, since p and η are both nef, primitive and contracted by the respective extremal contractions, it is clear that $\iota^* p = \eta + c\sigma_1$ for some $c \in \mathbb{Z}$. One can determine $c = 0$ by checking that $(\iota^* p)^5 = \iota^*(p^5) = 0$.

Theorem 5.3.1.

$$QH^\bullet(X) = \frac{\mathbb{C}[p, \xi, q_1, q_2]}{(R_1, R_2)}$$

where

$$R_1 = p^{\star 5} - q_1^2 p + 2q_1^2 \xi + 2q_1 p^{\star 3} - 2q_1 p^{\star 2} \star \xi - q_1 p \star \xi^{\star 2} - q_1 \xi^{\star 3}$$

and

$$R_2 = \xi^{\star 6} - 3p \star \xi^{\star 5} + 5p^{\star 2} \star \xi^{\star 4} - 5p^{\star 3} \star \xi^{\star 3} - q_2 - 5q_1 p \star \xi^{\star 3} + 10q_1 \xi^{\star 4}$$

Proof. Quantum multiplication is determined by Corollary 4.3.3. The classical cohomology of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ is calculated using (1.1) and the Chern classes determined in (5.2).

$$H^\bullet(X) = \frac{\mathbb{C}[p, \xi]}{(p^5, \xi^6 - 3p\xi^5 + 5p^2\xi^4 - 5p^3\xi^3)} \tag{5.5}$$

From Theorem 2.2 in [34], all that then remains is to evaluate the relations of the classical cohomology (5.5), replacing classical multiplication with quantum multiplication, from which the statement follows.

□

5.3.2 Producing Partial J-function

We produce the system of quantum differential operators as described in Subsection 2.4.2. As the quantum multiplication matrices for X are large, we relegate them to Appendix C, where they are reproduced in full.

Lemma 5.3.2. *The ideal of quantum differential operators for X is generated by*

$$\begin{aligned} \Delta_1 = D_2^{10} &- q_2 D_1^4 - 2q_2 D_1^3 D_2 - 4q_2 D_1^2 D_2^2 - 3q_2 D_1 D_2^3 - q_2 D_2^4 \\ &- 2q_1 q_2 D_1^2 - 2zq_2 D_1^3 - 2zq_1 q_2 D_1 D_2 - 8zq_2 D_1^2 D_2 - 3q_1 q_2 D_2^2 \\ &- 9zq_2 D_1 D_2^2 - 4zq_2 D_2^3 - q_1^2 q_2 - 4zq_1 q_2 D_1 - 4z^2 q_2 D_1^2 - 7zq_1 q_2 D_2 \\ &- 9z^2 q_2 D_1 D_2 - 6z^2 q_2 D_2^2 - 5z^2 q_1 q_2 - 3z^3 q_2 D_1 - 4z^3 q_2 D_2 - z^4 q_2, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \Delta_2 = D_1 D_2^7 - 2D_2^8 + 5q_2 D_1^2 + 5q_2 D_1 D_2 + 2q_2 D_2^2 + 5q_1 q_2 + 5zq_2 D_1 \\ + 4zq_2 D_2 + 2z^2 q_2, \end{aligned} \tag{5.7}$$

$$\Delta_3 = 5D_1^3 D_2^3 - 5D_1^2 D_2^4 + 3D_1 D_2^5 - D_2^6 + 5q_1 D_1 D_2^3 - 10q_1 D_2^4 + q_2, \tag{5.8}$$

$$(5.9)\quad \Delta_4 = D_1^5 + 2q_1D_1^3 - 2q_1D_1^2D_2 - q_1D_1D_2^2 - q_1D_2^3 + q_1^2D_1 + 2zq_1D_1^2 \\ - 2q_1^2D_2 - 3zq_1D_1D_2 - 2zq_1D_2^2 + z^2q_1D_1 - 2z^2q_1D_2,$$

where $D_i = zq_i\frac{\partial}{\partial q_i}$.

Note that these quantum differential equations are determined by the quantum cohomology and are known to full precision.

The identity component of J_X, denoted J_X^0, is a power series in q_1 and q_2, i.e. $J_X^0 = \langle J_X, \phi^0\rangle = \sum_{i,j\geq 0} c_{i,j}q_1^iq_2^j$. The differential system gives recursion relations for the coefficients in this power series. The coefficients are fixed by demanding that $c_{0,0} = 1$.

We can find J^0_{poly} up to arbitrary order as described in Subsection 2.4.3. One might hope that by observation of finite terms one could find a closed form solution for the coefficients $c_{i,j}$ that solves the differential system. In this case such a closed form was not clear by observation, though we obtain it later, in Subsection 5.4.1, by alternative means.

We present the coefficients $c_{i,j}$ for $i \leq 7, j \leq 7$ in the the following matrix $A = (a_{i,j})$. We have cleared the denominators by setting $a_{i+1,j+1} = i!^2j!^6c_{i,j}$. We have also set $z = 1$, so the coefficients are integers. The index of z associated to each coefficient can be recovered from the fact that the J-function is homogenous.

$$A := \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 5 & 20 & 51 & 104 & 185 & 300 & 455 \\ 0 & 4 & 73 & 447 & 1756 & 5320 & 13539 & 30373 \\ 0 & 0 & 90 & 1445 & 10904 & 55220 & 216110 & 703955 \\ 0 & 0 & 36 & 2148 & 33001 & 282085 & 1690515 & 7926751 \\ 0 & 0 & 0 & 1500 & 54500 & 819005 & 7606080 & 51405305 \\ 0 & 0 & 0 & 400 & 50350 & 1447150 & 21460825 & 211463875 \\ 0 & 0 & 0 & 0 & 24500 & 1590050 & 39750270 & 584307365 \end{bmatrix}$$

We remark that the coefficients are all zero below the leading 'slant diagonal'. Additionally, in the form presented where we have cleared denominators, the leading diagonal $a_{i,i}$ is given by the Apéry numbers [31]. The occurrence of the Apery numbers here may indicate hidden modular symmetries of J_0 (cf [5, 36]). Golyshev has observed a striking connection between the quantum differential equations for Fano 3-folds of Picard rank one and modular forms [16], and it is possible that this connection persists to the case of higher Picard rank.

5.4 J-function for M

In this section we calculate the identity component of the J-function for M, $J^0(M)$ using the J-function for X and the quantum Lefschetz theorem described in Section 2.5. As described in Section 5.2.1, M is a complete intersection in X given by the intersection of one hyperplane of class $\mathrm{PD}(p)$ by 5 of $\mathrm{PD}(\xi)$. The hypergeometric modification of $J_X = \sum_{a,b} J_{a,b}(t,z)q_1^a q_2^b$ is given by

$$I_{X,M}(t,z) = \sum_{a,b} J_{a,b}(t,z)q_1^a q_2^b \prod_{k=1}^{a}(p+kz)\prod_{k=1}^{b}(\xi+kz)^5$$

Restricting our attention to the identity component $J_X^0 = \sum_{i,j} c_{i,j}q_1^i q_2^j$, we have $I_{X,M}^0 = (I_{X,M},\phi^0) = \sum_{i,j} d_{i,j}q_1^i q_2^j$, with $d_{i,j} = c_{i,j}i!(j!)^5$

If we set the degree of $\deg(z) = 1$ and $\deg(t^i) = 1 - \deg(\phi_i)$, J_X is known to be homogeneous of degree 1. One can see that in our example the only possible contributions to the mirror map come from the identity component, J_X^0, and in particular we need only consider $c_{0,0}, c_{1,0}, c_{0,1}$. We find that $f(t) = 1$, $g^0(q,t) = 1+q_2$, $g^1(t) = g^2(t) = 1$.

Lemma 5.4.1. *The mirror map is given by;*

$$J_{M,X}(\tau_0,t_1,t_2,z) = I_{M,X}(t_0,t_1,t_2,z)$$

where $\tau_0 = t_0 + q_2$. By applying the string equation we can more conveniently write this as

$$J_{M,X}(t_0,t_1,t_2,z) = e^{-q_2}I_{M,X}(t_0,t_1,t_2,z)$$

As noted in Remark 2.5.2 this mirror map is known precisely, even though J_X is only known in polynomial terms.

To produce the quantum period sequence from J_M^0 we restrict t to the anti-canonical direction in $H^2(M;\mathbb{C})$. As previously stated $-K_M = p + \xi$, and so restricting to the anti-canonical direction has the effect of setting $q_1 = q_2 = q = e^t$. The effect on $J_M^0 = \sum_{i,j} c_{i,j}$ is to collapse the sum to a power series in one variable with coefficients $d_i = \sum_{j+k=i} c_{j,k}$. The first ten terms in the regularised period sequence are:

$$1, 0, 10, 42, 414, 3300, 29890, 275940, 2608270, 25305000.$$

Since it is known that J_M^0 satisfies quantum differential equations, the period sequence also does. There are *a priori* bounds on the order of the Picard–Fuchs operator, which annihilates the sequence, and so given a finite but sufficient number of entries in the power series of the period sequence we may find the Picard–Fuchs operator precisely.

The Picard–Fuchs operator for the regularised period sequence of M is given by

$$
\begin{aligned}
(5.10)\quad & -17727940t^9D^4 - 47452732t^8D^4 - 177279400t^9D^3 - 51239477t^7D^4 \\
& - 400876912t^8D^3 - 620477900t^9D^2 - 28719434t^6D^4 - 363088702t^7D^3 \\
& - 1218943172t^8D^2 - 886397000t^9D - 8782543t^5D^4 - 169273876t^6D^3 \\
& - 958664473t^7D^2 - 1562482112t^8D - 425470560t^9 - 1322684t^4D^4 \\
& - 42555106t^5D^3 - 384463114t^6D^2 - 1102964660t^7D - 696963120t^8 \\
& - 37187t^3D^4 - 5281118t^4D^3 - 80112855t^5D^2 - 392394560t^6D - 456149412t^7 \\
& + 13026t^2D^4 - 238966t^3D^3 - 7132816t^4D^2 - 69331328t^5D - 148485888t^6 \\
& + 995tD^4 - 11442t^2D^3 - 3879t^3D^2 - 4318688t^4D - 22881836t^5 - 24D^4 \\
& - 2278tD^3 + 16030t^2D^2 + 146332t^3D - 928456t^4 + 24D^3 + 35tD^2 + 9600t^2D \\
& + 76072t^3 + 1920t^2
\end{aligned}
$$

where $D = t\frac{d}{dt}$ [9].

This matches the Picard–Fuchs operator predicted by the mirror polytope [7, 8].

5.4.1 Recovering the J-function of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$

The Fanosearch group later discovered that a different method, the abelian/non-abelian correspondence, could be used to access the J-function of the Fano 3-fold directly, without passing through $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ [8]. Furthermore this technique allows one to obtain the entirety of the J-function in closed form, rather than a finite number of terms in the power series expansion of the identity component. All calculations carried out in this section are done to full precision and we work with power series rather than polynomials.

We can use this result to obtain a closed form solution for the Fano 9-fold $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ via 'inverse quantum Lefschetz'. Since the entirety of the identity component of the J-function for the ambient space passes down to the identity component for the J-function of the complete intersection, we can recover the identity component of the ambient space J-function from that of the complete intersection.

From the paper *Quantum Periods For 3-Dimensional Fano Manifolds* [8], we have the following expression for a particular twisted J-function of an non-abelian quotient (which we do not aim to explain), closely associated with the J-function for Fano 3-fold 2–17, J_M †.

† Note that due to convention differences the roles of q_1 and q_2 are the opposite of their appearance in the original paper.

(5.11) $$\tilde{J}_{V_G}(0) \cup \Omega = e^{-(q_1+q_2)/z} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_3=0}^{\infty} \frac{(-1)^{l_1+l_2} q_1^{l_3} q_2^{l_1+l_2} \Big(\prod_{1 \le i < j \le 3} \prod_{k=1}^{l_i+l_j} (\lambda + p_i + p_j + kz) \Big)}{\prod_{j=1}^{j=3} \prod_{k=1}^{k=l_j} (p_j + kz)^4} \times \Big(\prod_{k=1}^{l_1+l_2+l_3} (\lambda + p_1 + p_2 + p_3 + kz) \Big) \big(p_2 - p_1 + (l_2 - l_1)z\big)$$

To obtain the identity component of the J-function for Fano 3-fold 2-17 J_M^0 we wish to extract from (5.11) the p_2 component. We then set $\lambda = 0$ and $p_1 = p_2 = p_3 = 0$.

We obtain the following expression for the J-function of M:

$$J_M^0 = e^{-(q_1+q_2)} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_3=0}^{\infty} (-1)^{l_1+l_2} q_1^{l_3} q_2^{l_1+l_2} \frac{(l_1+l_2)!(l_1+l_3)!(l_2+l_3)!(l_1+l_2+l_3)!}{(l_1!)^4 (l_2!)^4 (l_3!)^4} \times \Big(1 + (l_2 - l_1)(H_{l_2+l_3} - 4H_{l_2}) \Big)$$

where H_i denotes the ith harmonic number:

$$H_i := \sum_{j=1}^{i} \frac{1}{j}.$$

Next, we perform inverse quantum Lefschetz. Recall from Subsection 5.3.2 that we considered the identity component of J_X to be a power series $J_X^0 = \sum_{i,j \ge 0} c_{i,j} q_1^i q_2^j$. We performed quantum Lefschetz by multiplying $c_{i,j}$ by $i!(j!)^5$. This formed the I-function that we identified with J_M^0 via mirror map: $J_M^0 = e^{-q_2} I$. Our aim is to invert this by performing the opposite operations.

Since our expression for J_M^0 already includes a factor of $e^{-(q_1+q_2)}$, the easiest method is to set $l_1 + l_2 = j$ and $l_3 = i$, find the term for fixed (i,j) (ignoring the factor of e^{-q_1}), then modify this term to account for the remaining multiplication by e^{-q_1}. Finally we multiply this term by $\frac{1}{i!(j!)^5}$. We obtain an expression for the degree (i,j) coefficient of J_X^0.

$$c_{i,j} = \sum_{m=0}^{i} \sum_{l=0}^{j} (-1)^{m+d_2} \frac{(i+j-l-m)!(l+i-m)!(i+j-m)!}{((j-l)!)^4 (l!)^4 ((i-m)!)^4 m! i! (j!)^4} \times \Big(1 + (2l - j)(H_{l+i-m} - 4H_l) \Big)$$

The complexity of this J-function perhaps partly accounts for the difficulty in spotting the closed form from the polynomial solution J_{poly}^0 noted in Subsection 5.3.2.

Note that by taking the diagonal (i.e. setting $i = j$) and multiplying by $(i!)^8$ we obtain the following expression for the Apéry numbers:

$$A(i) = \sum_{m=0}^{i} \sum_{l=0}^{i} (-1)^{m+i} \frac{(i!)^3 (2i-l-m)!(l+i-m)!(2i-m)!}{((i-l)!)^4 (l!)^4 ((i-m)!)^4 m!} \times \Big(1 + (2l-i)(H_{l+i-m} - 4H_l)\Big)$$

5.4.2 Fano 4-folds Embedded in $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$

Using either the polynomial J^0_{poly} or the precise expression for J^0_X obtained above we can also calculate the regularised quantum period sequence for the two Fano 4-folds which are given by hyperplane sections of X. These 4-folds are obtained by taking the hyperplane sections $p\xi^4$ and ξ^5 as described in Subsection 5.2.2. The quantum period sequences obtained via the quantum Lefschetz theorem begin as

ξ^5:

$$1, 0, 0, 30, 120, 240, 5850, 50400, 214200$$

$p\xi^4$:

$$1, 0, 2, 30, 54, 600, 6590, 26040, 265510$$

The first 9 terms in the quantum period sequence are sufficient to compare the quantum periods with those contained in a conjecturally complete list of toric complete intersection Fano 4-folds due to Coates and Kasprzyk [10]. We find that the above periods are not contained in the list and hence conjecturally represent some new non toric-complete-intersection Fano 4-folds.

Appendix A
Schubert Calculus on $G(2,5)$

We recall the Schubert calculus on $G(2,5)$, as it features heavily in the calculations behind the quantum cohomology of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$ discussed in Subsection 5.3.1. We first describe the Schubert cells as subsets of the set of all lines in $\mathbb{P}^4$, and then give the multiplication table for the cohomology.

$\sigma_{1,0}$ Lines intersecting a fixed 2-plane.

$\sigma_{2,0}$ Lines intersecting a fixed line.

$\sigma_{1,1}$ Lines contained in a fixed 3-plane.

$\sigma_{3,0}$ Lines through a fixed point.

$\sigma_{2,1}$ Lines contained in a fixed 3-plane that intersect a given line.

$\sigma_{3,1}$ Lines contained in a fixed 3-plane that pass through a given point.

$\sigma_{2,2}$ Lines contained in a fixed 2-plane.

$\sigma_{3,2}$ Lines contained in a fixed 2-plane that pass through a given point.

$\sigma_{3,3}$ A fixed line i.e. a point in $G(2,5)$.

	$\sigma_{1,0}$	$\sigma_{2,0}$	$\sigma_{1,1}$	$\sigma_{3,0}$	$\sigma_{2,1}$	$\sigma_{3,1}$	$\sigma_{2,2}$	$\sigma_{3,2}$
$\sigma_{1,0}$	$\sigma_{2,0}+\sigma_{1,1}$	$\sigma_{3,0}+\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{3,1}$	$\sigma_{3,1}+\sigma_{2,2}$	$\sigma_{3,2}$	$\sigma_{3,2}$	$\sigma_{3,3}$
$\sigma_{2,0}$	$\sigma_{3,0}+\sigma_{2,1}$	$\sigma_{3,1}+\sigma_{2,2}$	$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,2}$	$\sigma_{3,3}$	0	0
$\sigma_{1,1}$	$\sigma_{2,1}$	$\sigma_{3,1}$	$\sigma_{2,2}$	0	$\sigma_{3,2}$	0	$\sigma_{3,3}$	0
$\sigma_{3,0}$	$\sigma_{3,1}$	$\sigma_{3,2}$	0	$\sigma_{3,3}$	0	0	0	0
$\sigma_{2,1}$	$\sigma_{3,1}+\sigma_{2,2}$	$\sigma_{3,2}$	$\sigma_{3,2}$	0	$\sigma_{3,3}$	0	0	0
$\sigma_{3,1}$	$\sigma_{3,2}$	$\sigma_{3,3}$	0	0	0	0	0	0
$\sigma_{2,2}$	$\sigma_{3,2}$	0	$\sigma_{3,3}$	0	0	0	0	0
$\sigma_{3,2}$	$\sigma_{3,3}$	0	0	0	0	0	0	0

Appendix B
Reconstruction Algorithm

In the process of producing an algorithm from Lemmas 4.1.1 and 4.1.2 to carry out the reconstruction process, we consider cohomology classes as vectors in the lexicographical basis ϕ_i. Quantum multiplication by the basis classes p and ξ can be considered as left multiplication of cohomology vectors by $r(n+1) \times r(n+1)$ matrices, M_p and M_ξ respectively.

With this viewpoint in mind, we see that the lemmas can be reinterpreted.

Lemma B.0.2 (p-lemma as linear algebra)**.** *Given the i^{th} column of M_p we can determine the i^{th} column of M_ξ.*

Lemma B.0.3 (ξ-lemma as linear algebra)**.** *Assume the first i columns of M_p and M_q have been determined. We can calculate the $i+1^{st}$ column of M_p using linear algebra.*

The input for the reconstruction process is a pair of $r(n+1) \times r(n+1)$ matrices M_p and M_ξ. In M_p the columns corresponding to multiplication $p \star p^i$, for $0 \leq i \leq n$, are known, and we initialise the unknown entries as zero. We can initialise M_ξ as the zero matrix. By convention we label arrays with the first entry given index 1. We may think of the reconstruction process as giving an algorithm to fill in the rest of the matrices M_p and M_ξ

The result of Lemma 4.1.2 can usefully be written as:

$$p \star p^a \xi^b = \xi \star p \star p^a \xi^{b-1} - p \star (\xi \star p^a \xi^{b-1} - p^a \xi^b)$$

We use this form in the algorithm as it lends itself easily to calculation by matrix multiplication.

We define the following functions for use in our pseudo-code: Let numelts (d) be the number of basis elements ϕ_i in degree up to degree d.

$$\operatorname{numelts}(d) = \begin{cases} \frac{1}{2}(d+1)(d+2) & \text{for } d \leq n \text{ and } d \leq r-1 \\ d+1+n(d+\frac{n}{2}+\frac{1}{2}), & \text{for } d > n \text{ and } d \leq r-1 \\ d+1+(r-1)(d-\frac{r}{2}+1), & \text{for } d \leq n \text{ and } d > r-1 \\ (d+1)(1-\frac{d}{2})+n(d-\frac{n}{2}+\frac{1}{2})+ & \\ \quad +(r-1)(d-\frac{r}{2}+1), & \text{for } d > n \text{ and } d > r-1 \end{cases}$$

Let Ind (a,b) be the position that $p^a\xi^b$ appears in the basis of $H^\bullet(X)$, when given lexicographical ordering. This is given by

$$\operatorname{Ind}(a,b) = \begin{cases} \operatorname{numelts}(a+b-1)+(b+1), & \text{for } a+b \le n \\ \operatorname{numelts}(a+b-1)+(n-1+a), & \text{for } a+b > n \end{cases}$$

Next, $V_{(a,b)}$ is the vector corresponding to the monomial $p^a\xi^b$

$$V_{(a,b)}[j] = \begin{cases} 1 & \text{for } j = \operatorname{Ind}(a,b) \\ 0 & \text{otherwise} \end{cases}$$

Finally we define the Term function, which extracts from a polynomial the term specified along with the corresponding coefficient. It is given by the expression $\operatorname{Term}(\text{polynomial}, \text{monomial})$. e.g. $\operatorname{Term}\left(5x^2+3xy+2x+1, x\right) = 2x$

Let C be an r-vector with $C[i] := -c_i(E)$, note that we set the entries to $-c_i(E)$ to determine the form ξ^r takes.

We produce the following algorithm. Note that we aim for clarity of exposition and as such the algorithm is not minimised in terms of calculational expense.

d loops over all the degrees to calculate
for $d = 0 \to r+n$ **do**
  ## We loop over all classes in degree d
  ## Note that the expression below in I(.,.) determines the number of
  ## basis elements of the given degree
  ## together a and b determine which column of the matrix
  ## we're filling in
  for $k = 0 \to (\operatorname{numelts}(d) - \operatorname{numelts}(d-1) - 1)$ **do**
    ## amax:= $\overline{a}$ and bmin:= $\underline{b}$ determine the exponent of elements in the degree.
    $\overline{a} := min(d,n)$
    $\underline{b} := max(d-n, 0)$
    ## test to see if p-lemma should be applied
    ## to rule out columns corresponding to multiplication of p^d

    if $d > n$ **or** $k > 0$ **then**
      $HOLD := M_\xi M_p V_{(\overline{a}-k,\underline{b}+k-1)} - M_p(M_\xi V_{(\overline{a}-k,\underline{b}+k-1)} - V_{(\overline{a}-k,\underline{b}+k)})$
      ## the following loop copies HOLD into the appropriate column of M_p
      for $j = 1 \to (n+1)*r$ **do**
        $P\left[j, \operatorname{Ind}(\overline{a}-k, \underline{b}+k)\right] := HOLD[j]$
      end for
    end if
    ## ξ-lemma
    ## can improve on the bound s

```
for s = 1 → (n+1)r do
    ## Extracting the quantum corrections from the relevant
    ## p multiplication. We just pull out the data
    ## from the corresponding column in M_p
    ## Note the important factor 1/s
    ## from the divisor axiom
    for j = 1 → (n+1)r do
```

$$M_\xi\left[j, \mathrm{Ind}\left(\overline{a}-k, \underline{b}+k\right)\right] := M_\xi\left[j, \mathrm{Ind}\left(\overline{a}-k, \underline{b}+k\right)\right] + \frac{1}{s}\mathrm{Term}\left(M_p\left[j, \mathrm{Ind}\left(\overline{a}-k, \underline{b}+k\right)\right], s\right)$$

```
    end for
end for
## Testing for special case where class = p^α ξ^{r−1}

if b + k = r − 1 then
    ## Insert additional quantum correction in special case
```

$$M_\xi\left[\mathrm{Ind}\left(\overline{a}-k, 0\right), \mathrm{Ind}\left(\overline{a}-k, \underline{b}+k\right)\right] := q_2$$

```
    ## Insert classical multiplication in special case
    ## i.e. copy accross some Chern classes
    ## Note that we shift the entries as we copy them to account
    ## for the powers of p that our class includes
    ## s is restricted so that we account for the vanishing of p^{n+1}

    for s = 1 → n + r − 1 − d do
        M_ξ[Ind(ā − k + s, r − s), Ind(ā − k, b + k)] := C[s]
    end for
else
    ## Insert classical multiplication in generic case
    M_ξ[Ind(ā − k, b + k + 1), Ind(ā − k, b + k)] := 1
end if
    end for
end for
```

Appendix C
Quantum multiplication matrices

We reproduce below the matrices for multiplication, by p and ξ respectively, for quantum cohomology of $\mathbb{P}(\Omega^2(2)_{\mathbb{P}^4})$. These matrices given the quantum multiplication with respect to the lexicographically ordered basis defined in Section 1.2.

M_p

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2 & 3\,q_1q_2 & 5\,q_1q_2 & 0 & 0 & 2\,{q_1}^2q_2 \\
1 & 0 & 0 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2 & 3\,q_1q_2 & 0 \\
0 & 0 & 0 & 2\,q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2 & 4\,q_1q_2 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2 \\
0 & 0 & 1 & 0 & 0 & 0 & -q_1 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\,q_1 & 2\,q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1q_2 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -q_1 & -q_1 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 2\,q_1 & 2\,q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -q_1 & -q_1 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 2\,q_1 & 2\,q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -q_1 & -q_1 & -q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 2\,q_1 & 2\,q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4\,q_1 & 10\,q_1 & 10\,q_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4\,q_1 & -10\,q_1 & -10\,q_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2\,q_1 & 5\,q_1 & 5\,q_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 10\,q_1 & 25\,q_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -6\,q_1 & -15\,q_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2\,q_1 & 5\,q_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

M_ξ

$$
\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_2&0&0&0&0&q_1q_2&3\,q_1q_2&5\,q_1q_2&0&0&{q_1}^2q_2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_2&0&0&0&q_1q_2&3\,q_1q_2&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_1q_2&4\,q_1q_2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_2&0&0&q_1q_2\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_1q_2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_2&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&q_2\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&5&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&-5&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&3&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&5&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&-5&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&3&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&-5&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&3&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&3&0
\end{pmatrix}
$$

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