

RÉNYI ENTROPY AND IMPROVED EQUILIBRATION RATES TO SELF-SIMILARITY FOR NONLINEAR DIFFUSION EQUATIONS.

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ABSTRACT. We investigate the large-time asymptotics of nonlinear diffusion equations $u_t = \Delta u^p$ in dimension $n \geq 1$, in the exponent interval $p > n/(n+2)$, when the initial datum u_0 is of bounded second moment. Precise rates of convergence to the Barenblatt profile in terms of the relative Rényi entropy are demonstrated for finite-mass solutions defined in the whole space when they are re-normalized at each time $t > 0$ with respect to their own second moment, as proposed in [15, 44]. The analysis shows that, in the range $p > \max((n-1)n, n/(n+2))$, the relative Rényi entropy exhibits a better decay, for intermediate times, with respect to the standard Ralston-Newman entropy. The result follows by a suitable use of sharp Gagliardo-Nirenberg-Sobolev inequalities considered in [30], and their information-theoretical proof [42], known as concavity of Rényi entropy power.

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1. INTRODUCTION

In this work, we study the large-time behavior of the positive solutions $v(x, t)$ of the nonlinear diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^p, \quad (x \in \mathbb{R}^n, t > 0), \quad (1) \quad \boxed{\text{poro}}$$

with initial data

$$v(x, t = 0) = v_0(x) \geq 0, \quad (x \in \mathbb{R}^n).$$

Our analysis will be restricted to initial data with finite kinetic energy (second moment), and it will include both the case $p > 1$, (Porous medium equation (PME)), and the case $p < 1$ (Fast diffusion (FD)). We work in dimensions $n \geq 1$, and the range of exponents in the fast diffusion case is $\bar{p} < p < 1$ with $\bar{p} = n/(n+2)$, which is a part of the so-called fast diffusion range $p < 1$. The particular subinterval of p is motivated by the fact that, in this range of the parameter, the so-called Barenblatt solution [5, 6, 51], which serves as the model for the asymptotic behavior of a wide class of solutions, has finite second moment. In the fast diffusion case, the

Barenblatt solution is given by

$$\mathcal{B}_M(x, t) = t^{-n/\lambda} B(|x|t^{-1/\lambda}) = \left(\frac{Ct}{|x|^2 + At^{2/\lambda}} \right)^{1/(1-p)}, \quad (2) \quad \boxed{\text{bar}}$$

where $\lambda = 2 - n(1 - p)$ and $C = 2p\lambda/(1 - p)$ are fixed positive constants. The polynomial decay of $B(r)$ as $r \rightarrow \infty$ establishes a relation between the parameter p and the number of its moments which remain bounded. In case $p > 1 - 2/n$, the Barenblatt solution is integrable, and in this case the constant $A > 0$ is fixed by mass conservation

$$\int_{\mathbb{R}^n} B(|x|) dx = M, \quad t \geq 0.$$

Then the second moment of $\mathcal{B}_M(x, t)$ remains bounded, so that

$$\int_{\mathbb{R}^n} |x|^2 \mathcal{B}_M(x, t) dx = t^{-2/\lambda} \int_{\mathbb{R}^n} |x|^2 B(|x|) dx, \quad t > 0$$

whenever $p > \bar{p}$. The solution to equation (1) satisfies mass and momentum conservations, so that

$$\int_{\mathbb{R}^n} v(x, t) dx = 1; \quad \int_{\mathbb{R}^n} x v(x, t) dx = 0; \quad t \geq 0. \quad (3) \quad \boxed{\text{cons}}$$

Hence, without loss of generality, one can always assume that $v_0(x)$ is a probability density of first moment equal to zero.

The conservation properties of the solution play an important role in connection with the rate of decay towards the self-similar Barenblatt solution. This question, motivated by the search of improved convergence rates when moment conditions are imposed in the framework of Wasserstein, has been addressed in [15, 26, 27]. As far as the conservation of first moment (position of center of mass) is concerned, the question of improved rates has been precisely formulated in [15], and solved in [11, 28] in the weighted L^2 -framework. Indeed, for initial data satisfying (3), it has been found in [15] that the rate of convergence of solutions to the one-dimensional porous medium equation towards a self-similar profile improves by considering Barenblatt solutions satisfying (3).

These results suggested that an improved rate of decay towards Barenblatt solutions could be obtained only by choosing moments in a suitable way. However, since mass and momentum are the only conserved quantities, further improvements which involve conservation of higher order moments could be achieved only asymptotically [11, 28] due to linearizations. Motivated by this remark, a precise analysis of convergence towards the *best-matching* Barenblatt solution has been fully developed in [29, 30]. In contrast with the standard scaling used to treat nonlinear diffusion equations [17, 24], which leads to a Fokker-Planck like equation with constant coefficients, a new scaling was considered in [29] resulting in a nonlinear Fokker-Planck like equation with a non-constant diffusion coefficient. Then, the new equation revealed to be more adapted to study the rate of decay in terms of decay of the Ralston-Newman entropy.

Here, we explore how to get improvements at an initial stage based on a natural scaling argument, as well as on the consideration of Rényi entropy [25], as suggested by the recent papers [42, 46], in place of the Ralston-Newman one.

Let us define the energy $E(v(t))$ of a solution $v(x, t)$ as its second moment

$$E(v(t)) = \int_{\mathbb{R}^n} |x|^2 v(x, t) dx.$$

Then, $E(v(t))$ increases in time from $E_0 = E(v_0)$, and its evolution is given by the nonlinear law

$$\frac{dE(v(t))}{dt} = 2n \int_{\mathbb{R}^n} v^p(x, t) dx \geq 0, \quad (4) \quad \boxed{\text{ene}}$$

which is not explicitly integrable unless $p = 1$. The monotonicity in time of the energy suggests to change variables with the ansatz

$$v(x, t) = \left(\frac{E(v(t))}{E_0} \right)^{-n/2} u \left[x \left(\frac{E(v(t))}{E_0} \right)^{-1/2}, \tau(t) \right], \quad (5) \quad \boxed{\text{cv}}$$

where

$$\tau(t) = \frac{E_0}{2n} \log \left(\frac{E(v(t))}{E_0} \right), \quad \tau(0) = 0. \quad (6) \quad \boxed{\text{new-t}}$$

With this scaling,

$$E(v(t)) = \int_{\mathbb{R}^n} |x|^2 v(x, t) dx = \frac{E(v(t))}{E_0} \int_{\mathbb{R}^n} |y|^2 u(y, \tau) dy.$$

Therefore, for any time $\tau > 0$

$$\int_{\mathbb{R}^n} |y|^2 u(y, \tau) dy = E_0 = \int_{\mathbb{R}^n} |x|^2 v_0(x) dx,$$

and the scaled function $u(y, \tau)$, where $y = (E(v(t))/E_0)^{1/2}x$, maintains its second moment constant in time.

The nonlinear scaling (5) has a robust meaning. As outlined in [\[20\]](#)^{CVQ7}, the second moment corresponds to the kinetic energy of the probability distribution in kinetic theory, where x represents speed, v is the particle density and $E(v)$ is the kinetic energy. In fact, solutions where the only dependence on time is through their temperature are of asymptotic importance in several kinetic models for which related scalings have been used, see for instance [\[9, 10, 18\]](#)^{BCT05, BCT06, CTFE}. On the other hand, this scaling corresponds to the usual normalization for the distribution function of the sum of independent random variables with common distribution function of fixed variance, to obtain the central limit theorem (see [\[44\]](#)^{Los05} for more comments about this analogy and relations to the heat equation). Moreover, since in the original variables the second moment of the solution to equation (1) diverges in time, then scaling back with the second moment will hopefully drive the solutions to a limit from a geometric perspective.

In the field of nonlinear diffusion equations this idea has been at the basis of the study of the large-time behavior of a nonlinear diffusion equation with a general nonlinearity, of type $u_t = \Delta \Phi(u)$ [\[15\]](#)^{CDT05}. More recently, the same scaling has been successfully applied to prove oscillatory behavior for finite-mass solutions defined in the whole space, for particular choices of the function Φ in the diffusion [\[20\]](#)^{CVQ7}. Also, in the fast diffusion regime, improvements based on the second moment were also achieved in [\[29\]](#) (cf. also [\[30\]](#))^{PT}. The strategy in [\[29, 30\]](#)^{PT, PT2} was to introduce an implicit scaling different from (5), which allows to handle in a tricky way, in the

fast diffusion range $p > \max((n-1)n, n/(n+2))$, the time decay of the relative entropy functional by Newman and Ralston [38, 41]

$$\mathcal{F}_p(u|B) = \frac{1}{p-1} \int_{\mathbb{R}^n} [u^p - B^p - pB^{p-1}(u-B)] dx, \quad (7) \quad \boxed{\text{rel-e}}$$

commonly named *relative entropy*.

In the case under study, the nonlinear scaling (5) implies that the new function $u(y, \tau)$ satisfies the nonlinear Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} = \frac{\Delta_y u^p}{\int_{\mathbb{R}^n} u^p dy} + \frac{n}{E_0} \nabla_y \cdot (yu). \quad (8) \quad \boxed{\text{FPR}}$$

Note that (6) implies that $\tau(0) = 0$, while $y(0) = x$. Hence, the initial data of the Fokker-Planck equation (8) coincide with the initial data of the nonlinear diffusion in original variables (1).

At a first look, equation (8) appears more difficult to treat than the original nonlinear diffusion, due to the presence of the unknown denominator of the second order term. This apparent difficulty is balanced by the advantage to have a solution which maintains constant its moments up to the order two. Indeed, by construction, for all times $\tau > 0$ the solution of equation (8) satisfies

$$\int_{\mathbb{R}^n} u(y, \tau) dy = 1, \quad \int_{\mathbb{R}^n} y u(y, \tau) dy = 0, \quad \int_{\mathbb{R}^n} |y|^2 u(y, \tau) dy = E_0. \quad (9) \quad \boxed{\text{norm1}}$$

It is easy to show that the stationary solutions of (8) are Barenblatt solutions of type (2). For example, if $p < 1$, these solutions can be written as

$$B(|y|) = \left(\frac{C_p}{A_p + |y|^2} \right)^{1/(1-p)},$$

where the positive constants A_p and C_p are chosen to satisfy the normalization conditions (9).

Generalized Fokker-Planck equations of type (8) has been introduced into the physical literature in [40], and subsequently studied in a series of papers (cf. the references in [31]), by studying generalized entropies and the consequent statistics. In particular, it has been highlighted in [31] the link between the Fokker-Planck type equation (8) and the Rényi entropy of order p , with $p > 0$, of a probability density f [25], defined by

$$\mathcal{R}_p(f) = \frac{1}{1-p} \log \left(\int_{\mathbb{R}^n} f^p(y) dy \right). \quad (10) \quad \boxed{\text{re}}$$

The Rényi entropy of order 1 is defined as the limit as $p \rightarrow 1$ of $\mathcal{R}_p(f)$. It follows directly from definition (10) that

$$\lim_{p \rightarrow 1} \mathcal{R}_p(f) = \mathcal{R}(u) = - \int_{\mathbb{R}^n} f(y) \log f(y) dy. \quad (11) \quad \boxed{\text{shan}}$$

Then, the (Shannon) entropy is identified with the Rényi entropy of order 1.

Further connections between Rényi entropies and nonlinear diffusion equations have been outlined in [42, 46]. The main contribution of these papers is to establish a strong relationship between Rényi entropies and Barenblatt functions (2), thus clarifying the natural role of these entropies both in information theory and nonlinear partial differential equations.

Coupled to the definition of Rényi entropy of order p , is the definition of relative Rényi entropy of order p , valid for any pair of probability density functions f and g . In contrast to the universal concept of relative Shannon entropy, this definition is not unique in the case of Rényi entropies, even if all definitions converge to the relative Shannon one as $p \rightarrow 1$. In this paper, we consider as relative Rényi entropy of order p the quantity

$$\mathcal{H}_p(f|g) = \frac{1}{(p-1)} \log \int f^p dy + \log \int g^p dy - \frac{p}{(p-1)} \log \int g^{p-1} f dy. \quad (12) \quad \boxed{\text{r-re}}$$

For example, in [36]^{LYZ} the relative Rényi entropy is defined as $\mathcal{H}_p(f|g)/p$.^{LYZ} The relative Rényi entropy of order p is a nonnegative quantity (cf. the proof in [36]), and the relative Shannon entropy is obtained as the limit of $\mathcal{H}_p(f|g)$, when $p \rightarrow 1$

$$\lim_{p \rightarrow 1} \mathcal{H}_p(f|g) = \mathcal{H}(f|g) = \int_{\mathbb{R}^n} f(y) \log \frac{f(y)}{g(y)} dy. \quad (13) \quad \boxed{\text{rsh}}$$

By considering the Rényi entropy of the solution to equation (8) relative to its Barenblatt stationary solution, named B , since both functions satisfy the normalization conditions (9), the relative Rényi entropy is bounded from above by the (positive) difference between the Rényi entropies of the Barenblatt and that of the solution

$$\mathcal{H}_p(u(\tau)|B) \leq \mathcal{RH}_p(u(\tau)) := \mathcal{R}_p(B) - \mathcal{R}_p(u(\tau)),$$

where equality holds if $p \leq 1$. Moreover, we will show that $\mathcal{RH}_p(u)$ satisfies a Csiszar-Kullback type inequality^[23, 35]^{csi, kull}

$$\mathcal{RH}_p(u) \geq C(p, u) \left(\int_{\mathbb{R}^n} |u - B| dx \right)^2.$$

Our main result is exactly concerned with the time decay of $\mathcal{RH}_p(u(\tau))$. We will prove that

$$\mathcal{RH}_p(u(\tau)) \leq -\frac{n}{2} \log \left[1 - \left(1 - \exp \left\{ -\frac{2}{n} \mathcal{RH}_p(v_0) \right\} \right) \exp \left\{ -\frac{2n}{E_0} \tau \right\} \right]. \quad (14) \quad \boxed{\text{dec1}}$$

The decay (14) indicates that in the first part of the time axis, the convergence towards the Barenblatt solution with the same moments up to order two, is faster than the one predicted in terms of the **Newman-Ralston** relative entropy (7). Then, for larger times, the rate of decay stabilizes on the same rate predicted by the **Newman-Ralston** relative entropy. Reverting to the old variables, (14) shows that the polynomial rate of convergence of the solution to (1) towards the self-similar Barenblatt solution is faster by choosing the Barenblatt with the same second moment of the solution $v(t)$. Then, since the second moment of the solution to (1) converges towards the second moment of the corresponding Barenblatt [44]^{pos05}, this improvement is lost for large times. A similar behavior was already observed in [3]^{ack} for the heat equation using the Shannon entropy by optimizing over Gaussians with moments equal up to order two.

The study of the large-time behavior of the solution to (1), and its convergence towards the self-similar solution (2) is a classical problem which has been intensively studied from many years now. When $p > 1$, this behavior has been described in dimension 1 in [33, 34, 47]^{kam1, kam2, va1}, in a L^∞ -setting. These results have been subsequently generalized to the case $n > 1$ [32, 48]^{frka, va}. Among other results, it is remarked in [48]^{va}

that the assumption of finite moments is crucial to obtain convergence with rate. In fact, it is proved that for general data in L^1 no rate can exist.

In [17, 24, 39], a different approach led to new results in the whole space. The entropy dissipation method requires that the initial data possess a sufficiently high number of moments (typically $2 + \delta$ for $\delta > 0$), and it is based on a suitable scaling of the solution, which allows to rephrase the problem of convergence of the solution towards self-similarity for (1) in terms of the convergence of the solution to a Fokker-Planck type equation towards its steady state. The argument usually used in this context is the so-called *entropy-production* method [2, 4], which relies in the study of the relative entropy functional (7).

F. Otto made the link with gradient flows with respect to the Wasserstein distance in [39] allowing to work with initial data assuming only finite kinetic energy. D. Cordero-Erausquin, B. Nazaret and C. Villani gave a proof of the corresponding Gagliardo-Nirenberg inequalities using mass transportation techniques in [21]. In the fast diffusion case, these results are restricted to the range $p > \bar{p}$, due to the fact that, to work with Wasserstein's distance, it is crucial to have second moments bounded (see for instance [26, 27, 28]). Further results relative to the case $p < \bar{p}$ have been presented in a series of recent papers (see [7, 8, 11, 12]) together with a clarification of the strategy of linearization of the relative entropies, at least from the point of view of functional inequalities.

The paper is organized as follows. In Section 2 we use the time-dependent scaling (5) to obtain the Fokker-Planck equation (8), and we describe in details its steady solutions, together with their main properties. The improved rate of decay is derived in Section 3, where the strategy is presented in the linear case $p = 1$, which allows for explicit computations. In this case, the decay is given in terms of the classical relative Shannon entropy defined in (13). The analysis of the linear case outlines in a clear way the connection between the rate of decay of the relative Shannon relative entropy and the concavity property of Shannon entropy power with respect to the solution to the heat equation [22, 50]. The consequences of the concavity of Shannon entropy power have been recently discussed in [45], where the connections between this property and the logarithmic Sobolev inequality have been clarified. The main properties of Rényi entropies are listed in Section 4. In particular, it is shown that Rényi entropies constitute the natural entropies for the study of the Fokker-Planck equation (8). In addition, Csiszar-Kullback type inequalities are obtained both in the fast diffusion and in the porous medium case. Last, Section 5 contains the proof of the main result.

To avoid inessential details, in many places (Sections 3 and 4) computations will not be completely justified. Most details, however, are concerned with analogous ones which can be found in many papers and books on nonlinear diffusion equations [13, 14, 19, 20, 29, 30, 49].

2. SCALED FOKKER-PLANCK EQUATION AND STEADY STATES

sec:FP

The first argument we will deal on is the derivation of the Fokker-Planck equation (8), which is obtained from the nonlinear diffusion (1) under the scaling (5-6). Elementary computations allow to obtain from equation (1) the equation satisfied

by $u(y, \tau)$. It holds

$$\begin{aligned} \frac{\partial v}{\partial t} &= -\frac{n}{2} \left(\frac{E}{E_0} \right)^{-n/2-1} \frac{dE}{dt} \frac{1}{E_0} u + \left(\frac{E}{E_0} \right)^{-n/2} \frac{\partial u}{\partial \tau} \tau'(t) \\ &\quad - \frac{1}{2} \left(\frac{E}{E_0} \right)^{-(n+3)/2} \frac{dE}{dt} \frac{1}{E_0} x \cdot \nabla_y u \\ &= -\frac{1}{2} \left(\frac{E}{E_0} \right)^{-n/2-1} \frac{1}{E_0} \frac{dE}{dt} \nabla_y \cdot (yu) + \left(\frac{E}{E_0} \right)^{-n/2} \frac{\partial u}{\partial \tau} \tau'(t). \end{aligned}$$

Moreover, since

$$\Delta_x v^p = \left(\frac{E}{E_0} \right)^{-np/2-1} \Delta_y u^p,$$

together with (4), we have

$$\frac{dE}{dt} = 2n \int v^p dx = 2n \left(\frac{E}{E_0} \right)^{-n(p-1)/2} \int u^p dy.$$

Then, we obtain that the nonlinear diffusion (1) is equivalent to

$$\left(\frac{E}{E_0} \right)^{n(p-1)/2+1} \frac{\partial u}{\partial \tau} \tau'(t) = \Delta_y u^p + \frac{n}{E_0} \nabla_y \cdot (yu) \int u^p dy.$$

Last, we choose

$$\tau'(t) = \left(\frac{E}{E_0} \right)^{-n(p-1)/2-1} \int u^p dy. \quad (15) \quad \boxed{\text{ok}}$$

Since $\tau'(t) > 0$, (15) gives a good (implicit) time-change of variables. We point out that

$$\tau'(t) = \frac{1}{2n} \frac{E_0}{E} \frac{dE}{dt} = \frac{E_0}{2n} \frac{d}{dt} \log E(v(t)),$$

which implies (6). Hence, $u(y, \tau)$ satisfies the Fokker-Planck equation (8).

Let us now look for the steady states $\bar{B}(y)$ of (8) that solve

$$\nabla_y \cdot \left(\frac{\nabla_y \bar{B}^p}{\int \bar{B}^p dy} + \frac{n}{E_0} y \bar{B} \right) = 0. \quad (16) \quad \boxed{\text{ss}}$$

In the one-dimensional case, an exhaustive description of the properties of the steady states of various generalized Fokker-Planck equations, including (8) can be found in [31]. In particular, from the detailed discussion of the behaviour of the stationary solutions as the exponent p tends to the lower bound $\bar{p}$ ($\bar{p} = 1/3$ in dimension 1), one can argue that this lower bound is of paramount importance in connection with the eventual convergence to equilibrium.

As one can check using classical entropy arguments, to satisfy equation (16) it is sufficient to choose the argument of the divergence equal to zero

$$\frac{\nabla_y \bar{B}^p}{\int \bar{B}^p dy} + \frac{n}{E_0} y \bar{B} = 0. \quad (17) \quad \boxed{\text{ss0}}$$

Multiplying (17) by y and integrating, we obtain

$$\frac{\int y \cdot \nabla_y \bar{B}^p}{\int \bar{B}^p dy} + \frac{n}{E_0} \int |y|^2 \bar{B}(y) dy = 0,$$

or, what is the same

$$\int |y|^2 \bar{B}(y) dy = E_0 = \int |y|^2 u(y, \tau) dy. \quad (18) \quad \boxed{\text{en1}}$$

This fixes the second moment of the steady state to E_0 . Suppose we are given a steady state $\bar{B}(y)$, which has energy E_0 due to (18). Then, by defining for $\sigma > 0$

$$\bar{B}_\sigma(y) = \left(\frac{1}{\sqrt{\sigma}} \right)^{n+2} \bar{B} \left(\frac{y}{\sqrt{\sigma}} \right),$$

we obtain a class of functions $\bar{B}_\sigma$ with the same energy of $\bar{B}$. Moreover, a direct inspection shows that $\bar{B}_\sigma$ still satisfies (17) for all values of σ . Hence, by choosing σ such that

$$\sigma \int \bar{B}_\sigma^p dy = \frac{E_0}{n}, \quad (19) \quad \boxed{\text{gi}}$$

the corresponding steady state satisfies the relationship

$$\sigma \int \bar{B}_\sigma^p dy = \frac{1}{n} \int |y|^2 \bar{B}_\sigma(y) dy. \quad (20) \quad \boxed{\text{ok1}}$$

We can easily argue that it is always possible to choose σ to satisfy (19), and that this choice is unique. In fact, owing to the definition of $\bar{B}_\sigma$, we obtain

$$\phi_p(\sigma) = \sigma \int \bar{B}_\sigma(y)^p dy = \sigma \left(\frac{1}{\sqrt{\sigma}} \right)^{(n+2)p-n} \int B(y)^p dy.$$

Then, when $p < 1$, the function $\phi_p(\sigma)$ is (strictly) increasing starting from $\phi_p(0) = 0$. Analogous conclusion holds for $p > 1$, which generates a strictly decreasing function ϕ_p .

Let now $B_\sigma(y)$ be the unique probability density such that

$$\frac{p}{p-1} \nabla_y B_\sigma^{p-1}(y) + \frac{y}{\sigma} = 0. \quad (21) \quad \boxed{\text{pd}}$$

This probability density satisfies (20), and consequently (17). Therefore, equation (16) has always a unique solution with the right moments up to the order two. Solving equation (21) we obtain for $0 < p < 1$

$$B_\sigma(y) = \left(C_B + \frac{1-p}{p} \frac{|y|^2}{2\sigma} \right)^{1/(p-1)}, \quad (22) \quad \boxed{\text{p1}}$$

on $\mathbb{R}^n$, while, if $p > 1$

$$B_\sigma(y) = \left(C_B - \frac{p-1}{p} \frac{|y|^2}{2\sigma} \right)_+^{1/(p-1)}, \quad (23) \quad \boxed{\text{p2}}$$

where, as usual, $f_+(y)$ denotes the positive part of the function $f(y)$. In (22) and (23) the constants C_B and σ can be chosen in a unique way to fix the mass and the energy.

The previous reasoning shows that, for any given initial datum $u_0(y)$ satisfying conditions (3) and of energy equal to E_0 , the steady state of equation (8) with the same mass, momentum and energy of u_0 can be always represented in the form (22) when $\bar{p} < p < 1$ (respectively (23) when $p > 1$) for a unique choice of the parameters C_B and σ . For this reason, in what follows, for any given centered probability density function f of bounded second moment, we will always denote by $B_{\sigma(f)}$ the Barenblatt solution of type (22) or (23) with the same second moment

of f . This explicit dependence on f will be dropped when there is no possibility of confusion.

sec:linear

3. IMPROVED DECAY IN THE LINEAR CASE

We will start our analysis by considering the linear case $p = 1$. Thanks to mass conservation (3) the Fokker-Planck equation (8) reduces to the classical linear Fokker-Planck equation

$$\frac{\partial u}{\partial \tau} = \Delta_y u + \frac{n}{E_0} \nabla_y(y u). \quad (24) \quad \text{FPL}$$

Equation (24) is obtained from the heat equation

$$\frac{\partial v}{\partial t} = \Delta v, \quad (x \in \mathbb{R}^n, t > 0) \quad (25) \quad \text{heat}$$

by means of the scaling

$$v(x, t) = \left(1 + \frac{2nt}{E_0}\right)^{-n/2} u \left[x \left(1 + \frac{2nt}{E_0}\right)^{-1/2}, \tau(t) \right], \quad (26) \quad \text{cvl}$$

where

$$\tau(t) = \frac{E_0}{2n} \log \left(1 + \frac{2nt}{E_0}\right), \quad \tau(0) = 0. \quad (27) \quad \text{new-tt}$$

If $p = 1$, in fact, the law of evolution of the second moment is explicit, and it is given by the linear growth $E(v(t)) = E_0 + 2nt$. We remark that the presence of the coefficient in front of the divergence term in (24) implies that, for a given initial value that satisfies the normalization conditions (9), the solution to (24) still satisfies (9). As we will see, this property has important consequences.

The steady states of the linear Fokker-Planck equation are given by Gaussian densities. Since any solution to (24) which departs from an initial value that satisfies conditions (9) still satisfies the same conditions, the steady state is given by the Gaussian density

$$M_\sigma(y) = \left(\frac{1}{2\pi\sigma}\right)^{n/2} \exp \left\{ -\frac{|y|^2}{2\sigma} \right\}, \quad (28) \quad \text{max}$$

where $\sigma = E_0/n$. For the sake of simplicity, we will simply write $M_{E_0/n} = M$.

The goal is to study the rate of convergence of the solution of the Fokker-Planck equation (24) towards $M(y)$. Starting from the analysis in [43], the large-time behavior of linear Fokker-Planck type equations has been studied in details by means of entropy-production methods [4]. This method relies in the study of the evolution of the relative entropy, typically the relative Shannon entropy (13). In the case under examination, in reason of the fact that both the solution and M satisfy conditions (9), and in particular are probability densities with the same second moment, the relative entropy coincides with the difference between the Shannon entropies of the Gaussian density and that of the solution

$$\mathcal{H}(u(\tau)|M) = \mathcal{R}(M) - \mathcal{R}(u(\tau)). \quad (29) \quad \text{uB}$$

In (29), $\mathcal{R}(u)$ is given as in (11). Standard computations (cf. [43]) show that, if $u(\tau)$ solves (8)

$$\frac{d}{d\tau} (\mathcal{R}(M) - \mathcal{R}(u)) = -\frac{d}{d\tau} \mathcal{R}(u) = -\left(\mathcal{I}(u) - \frac{n^2}{E_0} \right),$$

where we defined by $\mathcal{I}(f)$ the Fisher information of the probability density f as

$$\mathcal{I}(f) = \int_{\{f>0\}} \frac{|\nabla f|^2}{f} dy. \quad (30) \quad \boxed{\text{fis}}$$

Since $\mathcal{I}(M) = \frac{n^2}{E_0}$, it holds

$$\frac{d}{d\tau} (\mathcal{R}(M) - \mathcal{R}(u)) = -(\mathcal{I}(u) - \mathcal{I}(M)). \quad (31) \quad \boxed{\text{dec11}}$$

The well-known entropy production method consists in bounding from below the difference between the Fisher information of u and that of M , in terms of the relative entropy $\mathcal{R}(M) - \mathcal{R}(u)$. Indeed, for any given probability density f , and for any given $\sigma > 0$, the logarithmic Sobolev inequality [2, 43] takes the form

$$\mathcal{I}(f) - \mathcal{I}(M_\sigma) \geq \frac{2}{n} \mathcal{I}(M_\sigma) (\mathcal{R}(M_\sigma) - \mathcal{R}(f)). \quad (32) \quad \boxed{\text{LS}}$$

Choosing $\sigma = E_0/n$ in (32) one obtains

$$\mathcal{I}(f) - \mathcal{I}(M) \geq \frac{2n}{E_0} (\mathcal{R}(M) - \mathcal{R}(f)). \quad (33) \quad \boxed{\text{lsi}}$$

Using inequality (33) into (31), one shows that the relative entropy decay exponentially at the rate $2n/E_0$

$$\mathcal{H}(u(\tau)|M) \leq \mathcal{H}(v_0|M) \exp \left\{ -\frac{2n}{E_0} \tau \right\}.$$

A different insight into the rate of decay in relative entropy follows by an elementary remark, which simply relies in a different way to write the right-hand side of equation (31). Indeed, equation (31) can be equivalently written as

$$\frac{d}{d\tau} (\mathcal{R}(M) - \mathcal{R}(u)) = -\frac{n^2}{E_0} \left(\frac{\mathcal{I}(u)}{\mathcal{I}(M)} - 1 \right). \quad (34) \quad \boxed{\text{dec2}}$$

The quotient of the two Fisher entropies in (34) can be bounded from below by resorting to the so-called *Isoperimetric Inequality for Entropies* [25, 45]. This inequality relates Fisher information of a probability density with its Shannon entropy power, and asserts that, for any probability density f , and for any $\sigma > 0$ it holds

$$\mathcal{I}(f) \mathcal{N}(f) \geq \mathcal{I}(M_\sigma) \mathcal{N}(M_\sigma). \quad (35) \quad \boxed{\text{b5}}$$

The entropy power of a probability density $f(x)$, where $x \in \mathbb{R}^n$, is defined by

$$\mathcal{N}(f) = \exp \left\{ \frac{2}{n} \mathcal{R}(f) \right\}. \quad (36) \quad \boxed{\text{pow}}$$

Since $\mathcal{R}(M_\sigma) = \frac{n}{2} \log 2\pi\sigma e$, the entropy power of the Gaussian density M_σ defined in (28), in any dimension of the space variable, is proportional to σ , with $\mathcal{N}(M_\sigma) = 2\pi\sigma e$. It is immediate to verify that inequality (35) implies the logarithmic Sobolev inequality (33) [45]. Indeed, inequality (36) can be rewritten in the form

$$\frac{\mathcal{I}(f)}{\mathcal{I}(M_\sigma)} \geq \exp \left\{ -\frac{2}{n} (\mathcal{R}(f) - \mathcal{R}(M_\sigma)) \right\}. \quad (37) \quad \boxed{\text{b55}}$$

Therefore

$$\frac{\mathcal{I}(f)}{\mathcal{I}(M_\sigma)} - 1 \geq \exp \left\{ -\frac{2}{n} (\mathcal{R}(f) - \mathcal{R}(M_\sigma)) \right\} - 1 \geq \frac{2}{n} (\mathcal{R}(M_\sigma) - \mathcal{R}(f)),$$

which gives (32). Inequality (35) has been derived as a consequence of the concavity of Shannon entropy power [25], a property first obtained in a paper by Costa [22] in the framework of information theory. Due to its importance in the forthcoming analysis, let us briefly discuss this result and its physical meaning. By definition, the entropy power of a Gaussian random variable is proportional to its density. Consequently, if M_{2t} denotes the Gaussian self-similar solution to the heat equation (25), its entropy power is a linear function of time. The essence of the concavity of Shannon entropy power is that, among all possible solutions to the heat equation, the Gaussian solution is the only one for which the entropy power is a linear function of time. For all the other solutions, the entropy power is a concave function of time. The original proof of the concavity property of entropy power was simplified by Villani [50], who made an interesting connection between information and kinetic theories. It was shown in [50] that the concavity proof could be based on an old argument introduced by McKean [37] in connection with Kac caricature of the Boltzmann equation. Using inequality (37) into (34), and choosing $\sigma = E_0/n$, one obtains

$$\frac{d}{d\tau} (\mathcal{R}(M) - \mathcal{R}(u)) \leq -\frac{n^2}{E_0} \left(\exp \left\{ -\frac{2}{n} (\mathcal{R}(u) - \mathcal{R}(M)) \right\} - 1 \right). \quad (38) \quad \boxed{\text{dec21}}$$

Equation (38) implies a better decay with respect to the decay obtained from the classical methods based on logarithmic Sobolev inequality. In fact, the relative Shannon entropy $h(\tau) = \mathcal{H}(u(\tau)|M)$ satisfies the differential inequality

$$\frac{dh(\tau)}{dt} \leq -\frac{n^2}{E_0} \left(\exp \left\{ \frac{2}{n} h(\tau) \right\} - 1 \right). \quad (39) \quad \boxed{\text{opt}}$$

Equation (39) can be solved by separation of variables to give

$$h(\tau) \leq -\frac{n}{2} \log \left[1 - \left(1 - \exp \left\{ -\frac{2}{n} h(0) \right\} \right) \exp \left\{ -\frac{2n}{E_0} \tau \right\} \right].$$

Hence, the relative entropy has been shown to satisfy the decay

$$\mathcal{H}(u(\tau)|M) \leq \frac{n}{2} \log \left[1 - \left(1 - \exp \left\{ -\frac{2}{n} \mathcal{H}(v_0|M) \right\} \right) \exp \left\{ -\frac{2n}{E_0} \tau \right\} \right]^{-1}. \quad (40) \quad \boxed{\text{de43}}$$

By comparison, one easily concludes that the decay predicted by (40) is better than the decay found in (34). Indeed, for any given positive constants α and β , the function

$$r_1(\tau) = \frac{n}{2} \log \left[1 - \left(1 - e^{-\frac{2}{n}\alpha} \right) \exp \{-\beta\tau\} \right]^{-1}$$

is always smaller than the function $r_2(\tau) = \alpha e^{-\beta\tau}$. By means of the scaling (26)-(27) the time decay of the relative Shannon entropy of the solution to the Fokker-Planck equation (24) can be translated into the time decay of the relative entropy between the solution $v(x, t)$ of the heat equation (25) and the self-similar Gaussian solution which at time $t = 0$ has the same second moment of $v_0(x)$. However, it is remarkable that inequality (35) can be applied directly to the solution to the heat equation (25), to obtain the time decay of the relative entropy. To this extent, for a given initial datum $v_0(x)$ that satisfies conditions (9), we denote by $M(x, t)$ the Gaussian density, solution to the same heat equation (25), which corresponds to the initial value $M_{E_0/n}$, and still satisfies conditions (9). Then, for all subsequent times $t > 0$, the moments of both $v(x, t)$ and $M(x, t)$ coincide up to order two. In the linear case, in fact, the evolution in time of the second moment of the solution

depends only on the second moment of the initial value. Therefore, for all times $t > 0$

$$\mathcal{H}(v(t)|M(t)) = \mathcal{R}(M(t)) - \mathcal{R}(v(t)).$$

Let us look for the time evolution of the Shannon relative entropy power

$$\mathcal{N}(v(t)|M(t)) = \exp \left\{ \frac{2}{n} (\mathcal{R}(v(t)) - \mathcal{R}(M(t))) \right\}.$$

Since both $u(x, t)$ and $M(x, t)$ satisfy equation (25)

$$\begin{aligned} \frac{d}{dt} \mathcal{N}(v(t)|M(t)) &= \frac{2}{n} (\mathcal{I}(v(t)) - \mathcal{I}(M(t))) \mathcal{N}(v(t)|M(t)) = \\ &= \frac{2}{n} \mathcal{I}(M(t)) \left(\frac{\mathcal{I}(v(t))}{\mathcal{I}(M(t))} - 1 \right) \mathcal{N}(v(t)|M(t)). \end{aligned} \quad (41) \quad \boxed{\text{dec41}}$$

Considering that $\mathcal{I}(M(t)) = n^2/(E_0 + 2nt)$, and making use of the isoperimetric inequality (35) in (41) one obtains

$$\frac{d}{dt} \mathcal{N}(v(t)|M(t)) \geq \frac{2n}{E_0 + 2nt} (1 - \mathcal{N}(v(t)|M(t))). \quad (42) \quad \boxed{\text{dec42}}$$

Inequality (42) can be easily solved to give

$$\mathcal{N}(v(t)|M(t)) \geq 1 - (1 - \mathcal{N}(v_0|M)) \frac{E_0}{E_0 + 2nt}, \quad (43) \quad \boxed{\text{dec43}}$$

or, what is the same

$$\mathcal{H}(v(t)|M(t)) \leq -\frac{n}{2} \log \left[1 - (1 - \mathcal{N}(v_0|M)) \frac{E_0}{E_0 + 2nt} \right].$$

Clearly, since $u(y, \tau)$ and $v(x, t)$ are related by the scaling relations (26) and (27), inequalities (43) and (40) coincide. The previous argument shows that in the linear case there is no necessity to scale the problem, and consequently to study the relaxation to equilibrium for the solution to the Fokker-Planck equation. The results can be obtained directly from the study of the solution to the heat equation. This direct proof is however limited to the linear case. We proved

th:li

Theorem 3.1. *Let $v(x, t)$ be the solution to the initial value problem for equation (25), where the initial value $u_0(x)$ is of bounded Shannon entropy, and satisfies conditions (9). Then, if $M(t)$ denotes the Gaussian density satisfying conditions (9) at time $t = 0$, the relative Shannon entropy $\mathcal{H}(v(t)|M(t))$ decays to zero with time, and satisfies the bound*

$$\mathcal{H}(v(t)|M(t)) \leq -\frac{n}{2} \log \left[1 - \left(1 - \exp \left\{ -\frac{2}{n} \mathcal{H}(v_0|M) \right\} \right) \frac{E_0}{E_0 + 2nt} \right].$$

Theorem 3.1 asserts that the rate of decay to equilibrium in relative entropy, as soon as we choose the self-similar solution with the same energy of the initial value, is bounded in time by a quantity which proportional to the inverse of the rate of growth of the second moment. This decay improves the known results in the first part of the time axis, while maintaining the same rate of decay for large time. This shows that, among all self-similar solutions that represents the intermediate asymptotics of the solution to the linear diffusion equation (25) for large times, in the first part of the time axis, the best approximation to the solution is furnished by the self-similar solution with the same energy.

As we shall prove in the forthcoming section, the same property holds as soon as we are considering the time decay of the relative Rényi entropy.

sec:Renyi

4. RÉNYI ENTROPIES AND NONLINEAR DIFFUSIONS

As briefly explained in the introduction, the second moment of the solution to equation (1) has an important role in connection with the knowledge of the large time behavior of the solution. This importance can be made explicit by the following argument. Let us consider the evolution in time of the Rényi entropy of order p along the solution of the nonlinear diffusion equation (1), with the same exponent p . Integration by parts immediately leads to

$$\frac{d}{dt} \mathcal{R}_p(v(\cdot, t)) = \mathcal{I}_p(v(\cdot, t)), \quad t > 0, \quad (44) \quad \text{e-r}$$

where, for a given probability density f

$$\mathcal{I}_p(f) := \frac{1}{\int_{\mathbb{R}^n} f^p dx} \int_{\{f>0\}} \frac{|\nabla f^p(x)|^2}{f(x)} dx. \quad (45) \quad \text{fis-r}$$

When $p \rightarrow 1$, identity (44) reduces to DeBruijn's identity, which connects Shannon's entropy functional with the Fisher information (30) via the heat equation. Since $\mathcal{I}_p(f) > 0$, identity (44) shows that the Rényi entropy of the solution to equation (1) is increasing in time. However, no information about the possible behavior of the solution follows.

For a given probability density $f(x)$, $x \in \mathbb{R}^n$, we define the dilation of f by

$$f(x) \rightarrow f_a(x) = a^n f(ax), \quad a > 0.$$

Since dilation preserves the total mass, for any given $a > 0$ the function f_a will remain a probability density. Note that Rényi entropies are such that, for all $p > 0$

$$\mathcal{R}_p(f_a) = \mathcal{R}_p(f) - n \log a.$$

Analogously, we have

$$E(f_a) = \int_{\mathbb{R}^n} |x|^2 f_a(x) dx = \frac{1}{a^2} E(f).$$

Therefore, if we define

$$\Lambda_p(f) = \mathcal{R}_p(f) - \frac{n}{2} \log E(f).$$

the functional Λ is invariant under dilation.

Let $v(x, t)$ be a solution to equation (1). If we compute the time derivative of $\Lambda_p(v(t))$, we obtain

$$\frac{d}{dt} \Lambda_p(v(t)) = \mathcal{I}_p(v(t)) - n^2 \frac{\int v^p(t)}{E(v(t))}, \quad (46) \quad \text{der1}$$

which is a direct consequence of both identities (4) and (44).

The right-hand side of (46) is nonnegative. This can be easily shown by an argument which is often used in this type of proofs [46], and goes back at least to

McKean ^{McK}[37]. One obtains

$$\begin{aligned}
0 &\leq \int_{\{v>0\}} \left(\frac{\nabla v^p(x)}{v(x)} + nx \frac{\int v^p}{E(v)} \right)^2 \frac{v(x)}{\int v^p} dx \\
&= \mathcal{I}_p(v) + n^2 \frac{\int v^p}{E(v)^2} \int_{\mathbb{R}^n} |x|^2 v(x) dx + \frac{2n}{E(v)} \int_{\{v>0\}} x \cdot \nabla v^p(x) dx \\
&= \mathcal{I}_p(v) + n^2 \frac{\int v^p}{E(v)} - 2n^2 \frac{\int v^p}{E(v)} = \mathcal{I}_p(v) - n^2 \frac{\int v^p}{E(v)}. \tag{47}
\end{aligned}$$

pos

Note that equality to zero in (47) holds if and only if, when $v(x) > 0$

$$\frac{\nabla v^p(x)}{v(x)} + nx \frac{\int v^p}{E(v)} = 0.$$

This condition can be rewritten as

$$\nabla \left(v^{p-1} + \frac{p-1}{2p} |x|^2 \frac{n \int v^p}{E(v)} \right) = 0$$

which identifies the probability density $v(x)$ as a Barenblatt density in $\mathbb{R}^n$. The simple addition of the second moment to Rényi entropy, coupled with the constraint of invariance of the new functional Λ_p with respect to dilation, allows to identify in a precise way the large-time behavior of the solution to (1).

The argument we presented is twofold. From one side, it represents a powerful tool to study the large-time behavior of solutions to nonlinear diffusion equations. From the other side, it allows to find inequalities by means of solutions to these nonlinear diffusions. Indeed, we proved that, for any probability density function f with bounded second moment

$$\mathcal{R}_p(f) - \frac{n}{2} \log E(f) \leq \mathcal{R}_p(B_\sigma) - \frac{n}{2} \log E(B_\sigma), \quad \text{for all } \sigma > 0,$$

where B_σ is the unique Barenblatt probability density satisfying (21). We remark that for $p > \bar{p} = n/(n+2)$ the quantity $\mathcal{R}_p(B_\sigma) - \frac{n}{2} \log E(B_\sigma)$ is bounded. This is obvious if $p > 1$. On the other hand, whenever $\bar{p} < p < 1$ the second moment of the Barenblatt (2) is bounded. Moreover, thanks to the identity

$$B_\sigma^p(x) = B_\sigma^{p-1}(x) B_\sigma(x) \leq D(1 + |x|^2) B_\sigma(x),$$

where D is a suitable constant, the L^p -norm of the Barenblatt function can be bounded in terms of the mass and the second moment. The previous inequality has an interesting consequence, we will use extensively in the following. Since the functional $\Lambda(f)$ is invariant with respect to dilation, the value of the right-hand side will be invariant with respect to the change of the second moment of the Barenblatt solution. Indeed, the change of the second moment of B_σ can be obtained by dilation. By fixing the second moment of the Barenblatt equal to the second moment of f , one shows that $\mathcal{R}_p(B_{\sigma(f)}) - \mathcal{R}_p(f) \geq 0$. These results are collected next.

Lemma 4.1. *Let $p > \bar{p}$. Then any probability density f with bounded second moment satisfies the inequality*

$$\mathcal{R}_p(f) - \frac{n}{2} \log E(f) \leq \mathcal{R}_p(B) - \frac{n}{2} \log E(B_\sigma), \quad \text{for all } \sigma > 0,$$

where B_σ is the unique Barenblatt probability density given by (21). Moreover, denoting by $B_{\sigma(f)}$ the Barenblatt probability density with the same second moment of f , then

$$\mathcal{R}_p(B_{\sigma(f)}) - \mathcal{R}_p(f) \geq 0. \quad (48) \quad \boxed{\text{posi}}$$

Let us define the nonnegative quantity in (48) as the relative Rényi entropy

$$\mathcal{RH}_p(f) = \mathcal{R}_p(B_{\sigma(f)}) - \mathcal{R}_p(f) = \frac{1}{p-1} \log \left(\frac{\int_{\mathbb{R}^n} f(x)^p dx}{\int_{\mathbb{R}^n} B_{\sigma(f)}(x)^p dx} \right). \quad (49) \quad \boxed{\text{re-r}}$$

Since the second moments of f and $B_{\sigma(f)}$ are equal, we can write the relative Rényi entropy as

$$\begin{aligned} \mathcal{RH}_p(f) &= \frac{1}{p-1} \log \left(\frac{\int_{\mathbb{R}^n} f(x)^p dx}{\left(\int_{\mathbb{R}^n} B_{\sigma(f)}(x)^p dx \right)^{1-p} \left(\int_{\mathbb{R}^n} B_{\sigma(f)}(x)^{p-1} f(x) dx \right)^p} \right) \\ &= \mathcal{H}_p(f|B_{\sigma(f)}), \end{aligned} \quad (50) \quad \boxed{\text{re-r2}}$$

for $\bar{p} < p < 1$, due to (12).

The relative Rényi entropy can be easily related to the relative entropy functional (7) of Newman and Ralston.

Lemma 4.2. *Let $p > \bar{p}$. Then, provided the probability density f and the Barenblatt solution satisfy conditions (9), the relative Ralston entropy (7) can be bounded from above in terms of the relative Rényi entropy (49), and the following bounds hold. If $p < 1$*

$$\mathcal{F}_p(f|B_{\sigma(f)}) \leq \left(\int_{\mathbb{R}^n} B_{\sigma(f)}^p dx \right) \mathcal{RH}_p(f), \quad (51) \quad \boxed{\text{pl1}}$$

while, if $p > 1$

$$\mathcal{F}_p(f|B_{\sigma(f)}) \leq \left(\int_{\mathbb{R}^n} f^p dx \right) \mathcal{RH}_p(f). \quad (52) \quad \boxed{\text{pg1}}$$

Proof. To simplify the notation, we will write B instead of $B_{\sigma(f)}$ in this proof. Consider first the case $p < 1$. Since the second moments of f and B coincide, (48) implies that $\int B^p \geq \int f^p$. Moreover, since B^{p-1} has the form $A + D|x|^2$, with A and D constants,

$$\mathcal{F}_p(f|B) = \frac{1}{p-1} \int_{\mathbb{R}^n} (f^p - B^p) dx$$

Therefore it holds

$$\begin{aligned} \mathcal{RH}_p(f) &= \frac{1}{1-p} \log \frac{\int B^p}{\int f^p} = -\frac{1}{1-p} \log \left(1 - \frac{\int B^p - \int f^p}{\int B^p} \right) \\ &\geq \frac{1}{1-p} \frac{\int B^p - \int f^p}{\int B^p} = \frac{1}{\int B^p} \mathcal{F}_p(f|B). \end{aligned} \quad (53) \quad \boxed{\text{b-re}}$$

In (53) we used the inequality $-\log(1-r) \geq r$, with

$$0 \leq r = \frac{\int B^p - \int f^p}{\int B^p} \leq 1.$$

Let now $p > 1$. In this case, (48) implies $\int B^p \leq \int f^p$. Proceeding as before, we obtain

$$\mathcal{RH}_p(f) = \frac{1}{1-p} \log \frac{\int B^p}{\int f^p} \geq \frac{1}{p-1} \frac{\int f^p - \int B^p}{\int f^p}. \quad (54) \quad \boxed{\text{b1-re}}$$

On the other hand, when $p > 1$, and f and B have the same second moment

$$\begin{aligned}\mathcal{F}_p(f|B) &= \frac{1}{p-1} \int_{\mathbb{R}^n} [f^p - B^p - pB^{p-1}(f-B)] dx \\ &= \frac{1}{p-1} \int_{\mathbb{R}^n} [f^p - B^p] dx - \frac{p}{p-1} \int_{|x|^2 > A} C^{p-1}(A - |x|^2) f dx \\ &\leq \left(\int_{\mathbb{R}^n} f^p dx \right) \mathcal{RH}_p(f),\end{aligned}$$

due to (23) and (54). $\square$

Remark 4.3. *In consequence of inequalities (51) and (52), all inequalities of Csiszar-Kullback type involving the relative Ralston entropy as in [16] remain valid for the relative Rényi entropy.*

The deep connection between Rényi entropies and the nonlinear diffusion equations (1), has been recently outlined in [42, 46], where nonlinear diffusion equations have been introduced as useful instruments to get inequalities in sharp form. The analysis of [42–46] extends to the nonlinear setting analogous results proven in the linear case [45].

The main idea in [42] was to consider a generalization of Shannon entropy power (36) for Rényi entropies. The Rényi entropy power of order p of a probability density $f(x)$, where $x \in \mathbb{R}^n$, has been defined in [42] by

$$\mathcal{N}_p(f) = \exp \left\{ \left(\frac{2}{n} + p - 1 \right) \mathcal{R}_p(f) \right\}. \quad (55) \quad \boxed{\text{powr}}$$

With this definition, one verifies that the Rényi entropy power of the self-similar Barenblatt solution of the nonlinear diffusion equation (1) grows linearly with respect to time. As for Shannon entropy power, it was proven in [42] that this linear growth is restricted to Barenblatt solutions, while for any other solution of (1) the Rényi entropy power (55) is a concave function of time. This property implies, in the range $p > \max((n-1)n, n/(n+2))$, the analogous of the isoperimetric inequality for (Shannon) entropy (35), and rephrases in a new way the Gagliardo-Nirenberg-Sobolev inequalities obtained in [30].

Theorem 4.4. *Let $p > \max((n-1)n, n/(n+2))$. Then, for any probability density f , and for any $\sigma > 0$ it holds*

$$\frac{\mathcal{I}_p(f)}{\mathcal{I}_p(B_\sigma)} \geq \exp \left\{ - \left(\frac{2}{n} + p - 1 \right) (\mathcal{R}_p(f) - \mathcal{R}_p(B_\sigma)) \right\}, \quad (56) \quad \boxed{\text{iso-Rényi}}$$

where B_σ denotes the unique Barenblatt probability density of type (22) if $p < 1$ or (23) if $p > 1$ with the same second moment as f .

Inequalities (56) are in fact equivalent to the sub-family of the Gagliardo-Nirenberg-Sobolev inequalities obtained in [30]. In this sense, the result of [42] does not introduce essential novelties from the point of view of functional analysis. However, the inequalities in [30] do not emphasize neither their information-theoretic meaning nor the fact that they can be obtained directly from the definition of Rényi entropy. Written as in (56), these inequalities allow to better clarify both the role and the importance of the concept of Rényi measure of information in connection with Barenblatt-type functions.

The result of Theorem 4.4 allows to extend the result of Section 3 to the non-linear case.

sec:nl

5. IMPROVED DECAY FOR NONLINEAR DIFFUSIONS

We consider now the evolution of the relative Rényi entropy (49) of the solution of the nonlinear Fokker-Planck equation (8). Proceeding as in Section 3, and using formula (44) that connects Rényi entropy with the generalized Fisher information (45), we obtain

$$\frac{d}{d\tau} \mathcal{RH}_p(u) = -\frac{d}{d\tau} \mathcal{R}_p(u) = -\frac{\mathcal{I}_p(u)}{\int u^p} + \frac{n^2}{E_0}, \quad (57) \quad \text{de15}$$

since B_σ does not depend on τ . Now, consider that, owing to definition (23), the Barenblatt solution satisfies

$$\nabla B_\sigma^p = \frac{p}{p-1} B_\sigma \nabla B_\sigma^{p-1} = -\frac{y}{\sigma} B_\sigma.$$

Therefore, by identity (20)

$$\mathcal{I}_p(B_\sigma) = \frac{1}{\int B_\sigma^p} \int |y|^2 B_\sigma(y) dy = \frac{n}{\sigma}.$$

Last, using the expression of σ , as given by formula (19), one concludes with the identity

$$\frac{n^2}{E_0} = \frac{\mathcal{I}_p(B_\sigma)}{\int B_\sigma^p}. \quad (58) \quad \text{fin22}$$

Making use of (58) into (57) we finally obtain that the time decay of the relative Rényi entropy is given in the form

$$\frac{d}{d\tau} \mathcal{RH}_p(u) = -\left(\frac{\mathcal{I}_p(u)}{\int u^p} - \frac{\mathcal{I}_p(B_\sigma)}{\int B_\sigma^p} \right) = -\frac{n^2}{E_0} \left(\frac{\mathcal{I}_p(u)}{\mathcal{I}_p(B_\sigma)} \frac{\int B_\sigma^p}{\int u^p} - 1 \right).$$

By definition of Rényi entropy, one has

$$\frac{\int B_\sigma^p}{\int u^p} = \exp \{ (p-1) (\mathcal{R}_p(f) - \mathcal{R}_p(B_\sigma)) \}.$$

Finally, formula (56) shows that

$$\frac{d}{d\tau} \mathcal{RH}_p(u) = -\frac{n^2}{E_0} \left(\exp \left\{ \frac{2}{n} \mathcal{RH}_p(u) \right\} - 1 \right).$$

Except for the meaning of the unknown quantity, the differential inequality is identical to (38), relative to the linear case. Consequently, its solution satisfies the bound

$$\mathcal{RH}_p(u(\tau)) \leq -\frac{n}{2} \log \left[1 - \left(1 - \exp \left\{ -\frac{2}{n} \mathcal{RH}_p(v_0) \right\} \right) \exp \left\{ -\frac{2n}{E_0} \tau \right\} \right].$$

Considering that the relative rényi entropy is invariant with respect to dilation, reverting to the original variables, (6) implies the following result:

th:nl

Theorem 5.1. *Let $p > \max((n-1)n, n/(n+2))$, and let $v(x, t)$ be the solution to the initial value problem for the nonlinear diffusion equation (1), where the initial value $v_0(x)$ is of bounded Rényi entropy, and satisfies conditions (9). Then, if*

$B_{\sigma(v_0)}$ denotes the Barenblatt density satisfying conditions (9), the relative Rényi entropy $\mathcal{RH}_p(v(t))$ defined in (49) decays to zero with time, and satisfies the bound

$$\mathcal{RH}_p(v(t)) \leq -\frac{n}{2} \log \left[1 - \left(1 - \exp \left\{ -\frac{2}{n} \mathcal{RH}_p(v_0) \right\} \right) \frac{E_0}{E(v(t))} \right].$$

6. FINAL REMARKS

To finish this work, let us make some remarks about the connections of the scaled equation (8), the relative Rényi entropy functional $\mathcal{RH}_p(u(\tau))$, and gradient flow structures. We know from [39] that the original equation in self-similar variables (1) has the structure of gradient flow of the Ralston-Newman entropy functional $\mathcal{F}_p(f|B_\sigma)$ with respect to the Euclidean Wasserstein distance. This structure is also kept for the scaled equation (8). Indeed, in the parameter interval $\bar{p} < p < 1$ we can check that equation (8) is equivalent to

$$\frac{\partial u}{\partial \tau} = \nabla_y \cdot \left(u \nabla_y \frac{\delta \mathcal{RH}_p(u(\tau)|B_\sigma)}{\delta u} \right).$$

This is immediate. By (50), we get

$$\frac{\delta \mathcal{RH}_p(f|B_\sigma)}{\delta f} = \frac{p}{p-1} \frac{f^{p-1}}{\int_{\mathbb{R}^n} f^p dx} - \frac{p}{p-1} \frac{B_\sigma^{p-1}}{\int_{\mathbb{R}^n} f B_\sigma^{p-1} dx}.$$

Since B_σ has the same mass and second moment of f , and $B_\sigma^{p-1} = A + D|x|^2$, with A and D constants, then for $p < 1$ we get

$$\frac{\delta \mathcal{RH}_p(f|B_\sigma)}{\delta f} = \frac{p}{p-1} \frac{f^{p-1}}{\int_{\mathbb{R}^n} f^p dx} - \frac{p}{p-1} \frac{B_\sigma^{p-1}}{\int_{\mathbb{R}^n} B_\sigma^p dx},$$

that together with (20) and (21) leads to

$$\frac{\delta \mathcal{RH}_p(f|B_\sigma)}{\delta f} = \frac{p}{p-1} \frac{f^{p-1}}{\int_{\mathbb{R}^n} f^p dx} + \frac{n}{2E_0} |y|^2.$$

Therefore, the scaled equation (8) is the formal gradient flow of the relative Rényi entropy functional $\mathcal{RH}_p(f|B_\sigma)$ with respect to the Euclidean Wasserstein distance for $\bar{p} < p < 1$. Moreover, due to the formula (49), we can write the relative Rényi entropy functional $\mathcal{RH}_p(f)$ as

$$\mathcal{RH}_p(f|B_\sigma) = \frac{1}{1-p} \log \left(\frac{\int_{\mathbb{R}^n} B_\sigma(x)^p dx}{(p-1)\mathcal{E}_p(f)} \right), \text{ with } \mathcal{E}_p(f) = \frac{1}{p-1} \int_{\mathbb{R}^n} f(x)^p dx.$$

We know from the by now classical McCann's condition, that the functional $\mathcal{E}_p(f)$ is displacement convex for $(n-1)/n \leq p < 1$. Therefore, it is trivial to check that the relative Rényi entropy functional $\mathcal{RH}_p(f|B_\sigma)$ is displacement convex in the same range, provided $p > \bar{p}$ for being well-defined, since it is the composition of convex functions. Therefore, all the standard theory of gradient flows in probability measures of geodesically convex functionals, as in [1], applies directly to the scaled equation (8) in the range $\max(\frac{n-1}{n}, \frac{n}{n+2}) < p < 1$. This breaks down for $p > 1$ since (50) is no longer true.

The previous discussion, coupled with the results of this work, highlight a strong and new connection between nonlinear Fokker-Planck type equations and Rényi entropies. In particular, they furnish a mathematical structure to the statistical framework introduced in [31] to introduce and justify this type of equations.

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