

Abelian Profinite Groups and the Discontinuous Isomorphism Problem

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Jonathan A. Kiehlmann

Department of Mathematics
Imperial College
180 Queen's Gate, London SW7 2BZ

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Abstract

We investigate the question: “Can there be a non-continuous isomorphism between two profinite groups which are not topologically isomorphic?” On one end of the spectrum, we show that branch and semisimple profinite groups have no non-continuous automorphisms. On the other, many abelian pro- p groups are abstractly but not topologically isomorphic.

In [13], we totally answered the question for countably-based profinite groups. There are many examples of such groups which are abstractly but not topologically isomorphic: we give explicit constructions of such non-topological isomorphisms

We used Pontryagin duality to reduce the question of classifying countably based abelian pro- p groups to that of countable abelian p -groups. In the 1930s Ulm and Zipin classified countable abelian p -groups. This work was expanded in the 1970s, to give the theory of totally projective abelian p -groups. We survey the structural theory of these groups and construct their duals, the totally injective groups. These provide more positive answers to our question: every dual-reduced totally injective pro- p group is abstractly isomorphic to the closure of its torsion subgroup, although in most cases these groups are not topologically isomorphic.

We proceed to give a detailed discussion on the features of the abstract and of the topological subgroup structures of such groups.

We introduce a new invariant, unbounded multiplicity, of Cartesian products of finite p -groups, in the above proof. This allows us to use infinite combinatorial arguments which give more results. Two of these Cartesian groups are isomorphic modulo their torsion subgroups if and only if they have the same unbounded multiplicity. A totally injective pro- p group will be abstractly isomorphic to its closed torsion subgroup whenever the unbounded multiplicity of this subgroup bounds the dimension of continuous torsion-free quotients.

Additionally, we construct a new class of commutative, unital pro- p rings. For each totally injective abelian pro- p group G , we construct a pro- p ring R with $(R, +) = G$.

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El original dice catorce, pero sobran motivos para inferir que en boca de Asterión, ese adjetivo numeral vale por infinitos. – La casa de Asterión, Jorge Luis Borges.

Introduction

The profinite groups are a class of infinite groups whose behaviour is determined by their finite quotients. They are in some sense the limits of their finite discrete quotients. Profinite groups have been a key object of study for group theorists for the last thirty years. We can form a profinite group from the discrete quotients of a residually finite group. This profinite completion encodes all the information of the group, but allows us to use topological methods and results. For instance, the classification of p -adic Lie groups leads to the structure of analytic pro- p groups, which has given rise to rich group theory. Lubotzky's linearity condition, [3, B.6], uses this classification, with implications for subgroup growth and more. Zelmanov's celebrated solution to the restricted Burnside problem also uses the theory of analytic pro- p groups. Furthermore, profinite groups are linked to many groups with interesting properties. The class of branch groups, including Grigorchuk's famous examples of groups (see [6] for an account), with intermediate growth, are strongly linked to profinite groups. In 1980, Leedham-Green and Newman, [16], introduced finite coclass groups to outline an ambitious asymptotic classification of all finite p -groups.

One recent development has been work on the algebraic structure of profinite groups. In [19], Nikolov and Segal showed that finitely generated profinite groups are strongly complete, i.e. all of their finite index subgroups are closed. This is equivalent to the statement "every group homomorphism from a topologically finitely-generated profinite group to a profinite group is continuous" and is a generalisation from a result due to Serre on pro- p groups. In [20], Nikolov and Segal explore non-continuous homomorphic images of profinite groups and build from this to show that topologically finitely generated compact groups have countably infinite image if and only if they are FAb. (A group G is said to be FAb if every subgroup H of finite index in G has $H/[H, H]$ finite.)

In this thesis, we explore non-continuous automorphisms of profinite groups. By Nikolov and Segal's strong completeness result, we need only consider the infinitely generated case.

In particular, we look at the question “When can topologically non-isomorphic profinite groups be abstractly isomorphic?” and give extensive answers in the abelian case, with some smaller non-abelian results. We call this question “the discontinuous isomorphism problem”.

Chapter 2 outlines results on some non-abelian groups which are in some sense very far from being abelian along with some entirely cardinality-based abelian results. Profinite groups arise as subgroups, closed in the product topology, of Cartesian products of finite groups each with discrete topology. Tychonoff’s Theorem says that a Cartesian product of compact sets is compact. Without the equivalent axiom of choice, we cannot say that infinite products of non-empty sets are non-empty. Many of the results in Chapter 2, as indeed the thesis as a whole, come from considering profinite groups as Cartesian products or subgroups of Cartesian products. We shall explore the structure arising from infinite products and the way that Cartesian (full) products differ from restricted products (which we shall refer to as direct sums).

In the non-abelian case, we show that some groups with a specific, restrictive structure have no non-continuous automorphisms:

Theorem.²

Let G be a profinite branch group or a Cartesian product of finite groups, each with trivial centre. Then every automorphism of G is continuous.

The abelian case is the opposite: in general, we find many discontinuous automorphisms. The strong condition of “commuting” forces out a lot of structure. The theory of abelian p -groups boils down to largely set-theoretic methods: by restricting to a prime p , we even lose the possibility using of number theory.

Statement. The *Continuum Hypothesis* is the statement that the first uncountable cardinal $\aleph_1$ is equal to $2^{\aleph_0}$, the cardinality of the continuum. The *Generalised Continuum Hypothesis* (*GCH*) is a claim about the nature of the cardinals. It states that for any infinite cardinal X , there is no cardinal between X and 2^X .

The continuum hypothesis is a deep statement known to be independent of the standard Zermelo-Fraenkel axioms of set theory. In fact, the continuum hypothesis is independent of ZFC, ZF with the axiom of choice. GCH is known to be independent of ZFC and

²This combines Theorem 2.3 and the corollary to Theorem 2.4.

is highly controversial as it implies the continuum hypothesis. Many of the elementary abelian examples end up depending on how much the Generalised Continuum Hypothesis fails. For instance:

Theorem.³

Suppose there exist distinct cardinals X and Y with $2^X = 2^Y$. Then for any pro-cyclic group G , the Cartesian product of X copies of G will be abstractly isomorphic to the Cartesian product of Y copies of G , though not topologically. We write G^X to denote the Cartesian product of X copies of G .

Definition. *For convenience, we shall say an abelian pro- p group is Cartesian if it is a Cartesian product of a collection of cyclic p -groups.*

We consider groups that are as far from FAb as possible: those groups which have infinite abelianisation. This thesis mostly deals with groups which are abelian, infinitely generated and have elements of arbitrary large finite order. We shall make statements about profinite abelian groups or torsion abelian groups in general: the finite exponent case is generally a trivial example.

Lemma.⁴

A torsion-free profinite abelian group is a Cartesian product of copies of the q -adic integers, for various primes q .

When dealing with profinite abelian groups, we have the very powerful tool of Pontryagin duality. As outlined in Chapter 1, this gives a duality of categories between profinite abelian groups and discrete abelian torsion groups.

Chapter 3 outlines the theory classifying abelian p -groups and thus abelian torsion groups. Work of Ulm [27] and Zippin in the 1930s classified all countable abelian torsion groups. Section 3.3.1 gives an overview of the proof of the classification of countable abelian p -groups. In brief: Ulm theory gives invariants for an abelian p -group by looking at transfinite iterations of the functor on the class of all such groups given by $G \mapsto pG$. Ulm's Theorem (Theorem 3.7) states that two groups with the same Ulm–Kaplansky invariant function are isomorphic. It generalises the notion of height (which means roughly “being a multiple/power of p^n ”) and Zippin gives a means of constructing groups with any possible

³Theorem 2.7 shows this is true for finite groups, Theorem 7.2 for torsion-free pro- p groups and Proposition 4.2 passes from prime to composite cases.

⁴This is implied by Corollary 4.4 and Proposition 4.2.

invariants. We present a brief surveying account of the work P. Hill, Nunke and others in the late 1960s and 1970s, outlining the theory of totally projective abelian p -groups. The totally projective groups are the largest class of groups generalising the class of countable abelian p -groups, that is, the largest class of groups for which an extension of Ulm's Theorem (Theorem 3.31) holds.

Chapter 4 presents the dual notion of totally injective groups. These profinite groups have been studied (without using the word profinite) in [12] and [17]. We note some results dual to 1960s discrete results that have not been previously recorded: Chapter 4 is essentially dual to Chapter 3.

Definition. ⁵

For a Cartesian group $A = \prod_{i \in \mathbb{N}} (C_{p^i})^{\alpha_i}$, we define a combinatorial infinite cardinal $\text{um}(A) = \aleph_0 \cdot \limsup_n (\alpha_n)$, the unbounded multiplicity of A . This is the greatest cardinal α such that A is the Cartesian product of α Cartesian groups without finite exponent.

The notion of unbounded multiplicity is important when discussing Cartesian groups and those constructed from them.

In [13] I solved the discontinuous isomorphism problem in the countably-based case, independently of the results on totally projective groups. Chapter 5 gives a structure theorem for arbitrary totally injective abelian pro- p groups, Theorem 5.5. This class includes all countably based abelian pro- p groups. By the work of Chapter 3, this completes the classification of these groups up to topological isomorphism.

Chapter 6 continues generalising the results of [13] to the much larger class of totally injective groups. It happens that almost every one of these groups is abstractly isomorphic to some Cartesian group: the only exceptions are those groups which are not Cartesian and which do not contain elements of arbitrarily large finite order.

Theorem. ⁶

Let G be a totally injective abelian pro- p group. If $t(G)$, the torsion subgroup of G , is of finite exponent then it is a closed subgroup of G and G is (topologically) of the form

$$\prod_{i=1}^e (C_{p^i})^{\alpha_i} \times \mathbb{Z}_p^{r(F(G))}$$

⁵We show this definition gives a well-defined maximum in Theorem 5.3. We give the initial definition of unbounded multiplicity in Chapter 5.

⁶This is Theorem 6.1.

for some $e \in \mathbb{N}$, (α_i) a sequence of cardinals and $r(F(G))$ the maximum minimal cardinality of a topological generating set of a continuous torsion-free image of G . If $t(G)$ contains elements of unbounded order, the dual-reduced part⁷ of G is isomorphic to $\overline{t(G)}$ as an abstract group and $\overline{t(G)}$ is of the form

$$\prod_{i \in \mathbb{N}} (C_{p^i})^{\alpha_i},$$

for (α_i) a sequence of cardinals not tending to 0. Furthermore, if the rank of $F(G)$ (a maximal torsion-free closed-continuous direct summand of G) is no greater than $\text{um}(\overline{t(G)}) = \aleph_0 \cdot \limsup_n \{\alpha_n\}$, then G is abstractly isomorphic to the closure of the torsion subgroup of G .

Theorem.⁸

Let A be a Cartesian group. There are precisely $2^{\text{um}(A)}$ pairwise topologically non-isomorphic totally injective pro- p groups abstractly isomorphic to A .

These will be the totally injective groups which have closure of the set of torsion elements topologically isomorphic to A .

In particular, in the proof of Lemma 6.2, we give explicit constructions for non-continuous isomorphisms between totally projective groups.

Example. The Cartesian group $\prod_{n \in \mathbb{N}} C_{p^n}$, $\mathbb{Z}_p \times \prod_{n \in \mathbb{N}} C_{p^n}$ and the group $H_{\omega+1}$, first mentioned as Γ in Example 1 in Chapter 1, which has $\overline{t(H_{\omega+1})} \cong \prod_{n \in \mathbb{N}} C_{p^n}$, are all discontinuously isomorphic.

Lemma 6.2 gives in its proof an explicit isomorphism between $\prod_{n \in \mathbb{N}} C_{p^n}$ and $\mathbb{Z}_p \times \prod_{n \in \mathbb{N}} C_{p^n}$. Then Corollary 6.3 (to Lemma 6.2) and Lemma 6.4 give a way to extend it to an explicit isomorphism between $H_{\omega+1}$ and $\overline{t(H_{\omega+1})}$.

The totally injective groups are of further significance due to the following result:

Lemma.⁹

Let G be an abelian pro- p group of generalised exponent τ . Then G is topologically isomorphic to a cobalanced (closed) subgroup of a totally injective group.

⁷See Definition 30

⁸This is Theorem 6.5.

⁹This is Theorem 4.19.

(We define cobalanced subgroups in the eponymous subsection of Chapter 4. We define generalised exponent at the start of Subsection 4.2.1.)

Chapter 7 is concerned with the abstract group structure of abelian profinite groups. The prior chapters show that the totally injective groups are almost all abstractly isomorphic to Cartesian products of cyclic p -groups. Certainly, they are all abstractly isomorphic to a Cartesian product of procyclic pro- p groups, and almost all are not topologically isomorphic to such groups. This is itself entirely determined by the abstract isomorphism type of the non-closed torsion subgroup. Here we consider the abstract structure of such groups.

Theorem. ¹⁰

Let A be a Cartesian pro- p group with $\text{um}(A) > 0$, such that B is a basic subgroup of $t(A)$.

Then, $B \cong \bigoplus_n C_{p^n}^{f_A(n)}$, where $f_A(n)$ is the pro-Ulm function¹¹ of A and

$$A/B \cong \bigoplus_{2^{\text{um}(A)}} C_{p^\infty} \times \bigoplus_{2^{\text{um}(A)}} \mathbb{Q} \times \mathbb{Z}_p^{\text{um}(A)}.$$

This gives the following:

Corollary. ¹²

If G, H are dual-reduced Cartesian pro- p groups of the same unbounded multiplicity, then $G/t(G) \cong H/t(H)$.

In fact, we can clarify what these will look like:

Theorem. ¹³

Let G be a profinite abelian group. Suppose that, for each prime q , the q -Sylow subgroup G_q is totally injective.

Then $G/t(G)$ is a direct sum of a rational vector space and a Cartesian product of copies of the q -adic integers, for various primes q .

It is interesting to ask if this is true in greater generality.

One might ask why we study profinite abelian groups. After all, these are much larger

¹⁰This follows from Theorem 7.6.

¹¹See section 4.2.1

¹²This is a weaker form of Corollary 7.11.

¹³Theorem 7.12 and Corollary 7.11.

than their dual groups and, by Pontryagin duality, all profinite information can be found in the dual, which is easier to work with. But profinite groups have two structures: we can look at the abstract structure of these profinite groups and see what can be said about the profinite structure, and vice versa. In particular, Chapter 6 shows many cases where abelian profinite groups which are not isomorphic as topological groups are abstractly isomorphic. Furthermore, Chapter 8 demonstrates a way in which profinite abelian groups give rise to profinite rings.

Theorem. ¹⁴

Let G be a totally injective non-trivial abelian pro- p group. Then there exists some commutative ring with identity, R , such that $(R, +)$ is topologically isomorphic to G .

While there will be many such ring structures on each pro- p group, many infinite discrete p -groups do not admit the structure of a ring. These are useful and open the possibility of constructing new profinite groups as matrix or polynomial groups over these rings.

¹⁴This is Theorem 8.1.

Chapter 1

Background Basics and Notation

In this section we run through some reminders of basic concepts.

1.1 Topological Groups

Recall the definition of a topological group.

Definition 1. A topological group is a pair $G = (\Gamma, T)$ where Γ is a group, T is a topology on Γ such that the inversion map $x \mapsto x^{-1}$ is a homeomorphism on G and the multiplication map

$$M : \Gamma \times \Gamma \rightarrow \Gamma$$

given by $(x, y) \mapsto x.y$ is continuous when considering $\Gamma \times \Gamma$ under the product topology.

We sometimes refer to groups without topologies as *abstract groups* to indicate that we are discussing the abstract group structure without the information added by the topology. Note that algebraic groups, though commonly encountered groups with topologies are not topological groups. Their multiplication is not necessarily continuous with the map given above.

1.2 Infinite Products and Sums

One of the most common constructions in mathematics is the direct product. Given a set (or group, etc.) X , we construct $X \times X$, the set of pairs of elements in X . We can take a product of any finite number of copies: how do we take a product of infinitely many sets?

Definition 2. Let $(X_i)_{i \in I}$ be an infinite collection of non-empty sets, indexed by some index set I .

We define the Cartesian Product of these sets to be the set

$$\prod_{i \in I} X_i = \{(x_i)_{i \in I} \mid x_i \in X_i\}.$$

If the X_i are groups, this is a group with multiplication given by component-wise multiplication.

As long as infinitely many of the X_i contain at least two elements, this gives rise to an infinite set.

If each $|X_i|$ is at most $|I|$, then $\prod_I X_i$ is of cardinality at most $2^{|I|}$. This will be of cardinality exactly $2^{|I|}$ as long as at least $|I|$ of the X_i contain at least two elements.

This is one of two ways to extend taking direct products to combine infinitely many groups.

Definition 3. Let $(X_i)_{i \in I}$ be an infinite collection of groups, rings, or vector spaces (over the same field) indexed by some index set I .

We define the restricted direct product (or direct sum) of the X_i to be the subobject

$$\bigoplus_{i \in I} X_i = \left\{ (x_i)_{i \in I} \in \prod_{i \in I} X_i \mid x_i \text{ is non-trivial for finitely many } i \right\}.$$

of the Cartesian product.

As long as infinitely many of the X_i contain two elements, the direct sum is also infinite, but if each X_i is of cardinality at most $|I|$, $\bigoplus_I X_i$ is of cardinality $|I|$.

The major distinction is how these object behave with respect to morphisms. The direct sum is the minimal object which the X_i embed in, whereas the product is the minimal group which has maps surjectively to X_i independently and simultaneously. (These surjections are the canonical projections by taking the i -th entry.)

In terms of category theory, the Cartesian product is the (categorical) product of the X_i , while in the category of abelian groups, the direct sum is the coproduct.

The Cartesian product of topological spaces has the natural product topology. This allows us to think of issues of convergence in our new space. It preserves a lot of properties of the X_i .

Theorem 1.1. *Tychonoff's Theorem*

Let $(X_i)_{i \in I}$ be an infinite collection of non-empty topological spaces, indexed by some index set I .

Then, if each X_i is compact, $\prod_I X_i$ is compact.

This well-known result motivated the definition of the product topology. It is equivalent to the axiom of choice. Indeed, the axiom of choice is also equivalent to the statement “The product of a collection of non-empty sets is non-empty”; as we work extensively with infinite products, we do not hesitate in using this.

A central part of this thesis is taken up by examining the difference between the direct sum and Cartesian product of a family: see especially Chapter 7.

Throughout, we write G^X to denote the Cartesian product of $|X|$ copies of G .

1.3 Profinite Groups

The profinite groups are a class of topological groups which are formed from finite groups: they are limits of families of finite groups.

Definition 4. A directed set is a non-empty partially ordered set $(\Lambda, \leq)$, such that for every $\mu, \lambda \in \Lambda$ there is some $\nu \in \Lambda$ with $\nu \geq \lambda$ and $\nu \geq \mu$.

An inverse system of objects over the directed set Λ is a collection $(X_\lambda)_{\lambda \in \Lambda}$ of objects with morphisms $\phi_{\lambda\mu} : X_\lambda \rightarrow X_\mu$ whenever $\lambda \geq \mu$ satisfying the compatibility condition

$$\phi_{\lambda\lambda} = \text{Id}_{X_\lambda}, \quad \phi_{\mu\nu}\phi_{\lambda\mu} = \phi_{\lambda\nu},$$

for every $\lambda \geq \mu \geq \nu$.

Let $(X_\lambda)_{\lambda \in \Lambda}$ be an inverse system. Suppose there is an object X with canonical morphisms

$$(\pi_\mu : X \rightarrow X_\mu)_{\mu \in \Lambda},$$

such that $\phi_{\lambda\mu}\pi_\lambda = \pi_\mu$ whenever $\lambda \geq \mu$. Assume that the following universal condition is satisfied: for each object Y with morphisms $\psi_\lambda : Y \rightarrow X_\lambda$ such that $\phi_{\lambda\mu}\psi_\lambda = \psi_\mu$ whenever $\lambda \geq \mu$, we have a unique morphism $\psi : Y \rightarrow X$ such that $\pi_\lambda\psi = \psi_\lambda$ for each $\lambda \in \Lambda$. That

is, such that the following diagram commutes.

$$\begin{array}{ccc}
 & Y & \\
 \psi_\lambda \swarrow & \downarrow \exists \psi & \searrow \psi_\mu \\
 & X & \\
 \pi_\lambda \swarrow & & \searrow \pi_\mu \\
 X_\lambda & \xrightarrow{\phi_{\lambda\mu}} & X_\mu
 \end{array}$$

Then we say that X is the inverse limit of $(X_\lambda)_{\lambda \in \Lambda}$. We write $X = \varprojlim_{\lambda \in \Lambda} X_\lambda$ to denote this.

It follows from this definition that any two inverse limits of such an inverse system are unique up to unique isomorphism. As a result, we speak of “the inverse limit” of an inverse system, rather than “an inverse limit”.

This is a general category-theoretical definition. We shall mainly care about inverse limits of topological groups, abstract groups and to a lesser extent rings and sets. In these situations we have a concrete realisation of the inverse limit.

Lemma 1.2. *Let $(X_\lambda)_{\lambda \in \Lambda}$ be an inverse system of objects in category $\mathcal{C}$ with inverse limit $\varprojlim_{\lambda \in \Lambda} X_\lambda$.*

If each object in $\mathcal{C}$ is a set, all morphisms in $\mathcal{C}$ are set maps and $\prod_{\lambda \in \Lambda} X_\lambda$ is in $\mathcal{C}$, then $\varprojlim_{\lambda \in \Lambda} X_\lambda$ is isomorphic to

$$\{(x_\lambda)_{\lambda \in \Lambda} \in \prod_{\lambda \in \Lambda} X_\lambda \mid \phi_{\lambda\mu} x_\lambda = x_\mu, \forall \lambda \geq \mu\},$$

if this is in $\mathcal{C}$. Furthermore, if objects in $\mathcal{C}$ are topological spaces and morphisms are continuous, then this subspace of the Cartesian product is closed.

There are several equivalent definitions of profinite groups.

Theorem 1.3. *Let G be a topological group. The following are equivalent:*

1. G is compact, Hausdorff and has a neighbourhood base for 1 given by open subgroups;
2. G is an inverse limit of an inverse system of finite groups with the discrete topology;
3. G is a subgroup of a Cartesian product, $\prod_{i \in I} K_i$, of finite groups K_i , each with discrete topology, which is closed in the product topology;

4. G is compact and totally disconnected.

To see the equivalence of the first three, note that the first is closed under taking closed subgroups. By Tychonoff's Theorem, this holds for any closed subgroup of such a Cartesian product and hence (1) and (3) are equivalent. By Lemma 1.2, (2) implies (3). Suppose G satisfies (1). As G is compact, open subgroups must be finite index. With little work, one can see that every open subset of G contains some open normal subgroup N of G . Also, as the open normal subgroups of G intersect trivially, $G \cong \varprojlim_{N \triangleleft_o G} G/N$ and so we are done.

Full details of the proof of this theorem can be found in the introduction to any textbook on profinite groups such as [3], [23] or [29].

Definition 5. A group G satisfying one and hence all of the above conditions is called a profinite group.

A profinite group G is called *pro- p* if and only if every open subgroup of G is of index some power of p .

The standard definition of a pro- p group is an inverse limit of finite p -groups. This is equivalent to every open subgroup of a profinite group being of p -power index. This is shown in, for instance [3, 1.18].

In using profinite groups to construct other profinite groups, we use the following construction:

Definition 6. For X, Y abelian profinite groups $x \in X, y \in Y$, we write the profinite presentation

$$\langle X, Y \mid x = y \rangle$$

to denote the quotient $X \times Y / \overline{\langle x - y \rangle}$.

1.4 Pontryagin Duality

We write $\mathbb{T}$ for the group $\mathbb{R}/\mathbb{Z}$. Note that this is isomorphic to the circle group, $\{z \in \mathbb{C} \mid |z| = 1\}$, of modulus 1 complex numbers.

Definition 7. The dual group G^* of a locally compact abelian topological group G is defined by

$$G^* = \text{Hom}(G, \mathbb{T})$$

the group of continuous homomorphisms from G to $\mathbb{T}$, equipped with the compact-open

topology. This is the topology with basis given by the sets

$$\{f \in \text{Hom}(G, \mathbb{T}) \mid f(K) \subseteq U\}$$

for each compact $K \subseteq G$ and open $U \subseteq \mathbb{T}$.

We shall be most interested in abelian groups that are profinite or discrete torsion groups. These groups cannot map to most of $\mathbb{R}/\mathbb{Z}$: in fact any continuous image of such a group in $\mathbb{R}/\mathbb{Z}$ must be contained in $\mathbb{Q}/\mathbb{Z}$. We have the following results about duality of these groups.

Theorem 1.4. *Pontryagin Duality*

1. If G is an abelian torsion group equipped with the discrete topology or an abelian profinite group, then $G^* = \{f : G \rightarrow \mathbb{Q}/\mathbb{Z} \mid f \text{ is a continuous homomorphism}\}$.
2. If G is an abelian profinite group, G^* is a discrete abelian torsion group. If G is a discrete abelian torsion group, G^* is an abelian profinite group.
3. If G is an abelian profinite group or a discrete abelian torsion group the (canonical) homomorphism

$$\alpha_G : G \rightarrow G^{**}$$

which sends $g \mapsto (\alpha_g : f \mapsto f(g))$ is an isomorphism of topological groups.

Proofs of these results can be found in [23, 2.9].

This is crucial to our work with abelian groups as Pontryagin duality provides a contravariant functor between discrete abelian torsion groups and abelian profinite groups.

Theorem 1.5. *Basic Dual Results*

Let G be a discrete torsion abelian group or a profinite abelian group and G_i be a collection of profinite abelian groups. Then

1. If G is finite, $G^* \cong G$.
2. $(\varprojlim G_i)^* \cong \varinjlim (G_i^*)$.
3. $(\mathbb{Z}_p)^* \cong C_{p^\infty}$.
4. $(\prod G_i)^* \cong \bigoplus (G_i^*)$.

Note that each of these, by duality, implies their duals, e.g. $(C_{p^\infty})^* \cong \mathbb{Z}_p$. For proofs of these results see [23, 2.9.3-5].

Definition 8. Let G be an abelian profinite group or discrete abelian torsion group with subset X . We write $\text{Ann}_{G^*}(X)$ for the annihilator of X in G^* . This is defined to be the set of elements of G^* which send every $x \in X$ to the identity.

Theorem 1.6. Let G be a discrete torsion abelian group or profinite abelian group and H be a closed subgroup of G . Then

$$H^* \cong G^* / \text{Ann}_{G^*}(H).$$

A simple proof of this can be found in [18], where it is Theorem 28.

Corollary 1.7. Let G be a discrete torsion abelian group or profinite abelian group, with H a closed subgroup of G .

Then

$$(G/H)^* \cong \text{Ann}_{G^*}(H).$$

Proof. Recall that $K = \text{Ann}_{G^*}(H)$ is closed in G^* .

Consider $\alpha_G(H)$, the image of H under the canonical isomorphism α_G . As G^{**} is the set of all homomorphisms from G^* to $\mathbb{Q}/\mathbb{Z}$, we have

$$\alpha_G(H) = \{\alpha_h \mid h \in H\}$$

where α_h is defined by $f \mapsto f(h)$. This annihilates $f \in G^*$ if and only if $f(h) = 0$ for every $h \in H$. Hence $\alpha_G(H) = \text{Ann}_{G^{**}}(\text{Ann}_{G^*}(H))$. But $\text{Ann}_{G^{**}}(K) = \text{Ann}_{G^{**}}(\text{Ann}_{G^*}(H))$. Hence $G^{**}/\text{Ann}_{G^{**}}(K)$ is isomorphic to G/H .

By Theorem 1.6 K is isomorphic to the dual of $G^{**}/\text{Ann}_{G^{**}}(K)$ and the result follows. $\square$

This demonstrates that Pontryagin duality is a duality (in the categorical sense) between discrete torsion abelian groups and profinite abelian groups. That is, it is a contravariant functor which is its own ‘inverse’. (It is its own inverse in the sense that it is defined for any locally compact abelian topological group and applying it twice to a discrete torsion or profinite group G produces an isomorphic copy of G . Additionally, every finite discrete group is isomorphic to its dual, Theorem 1.5 above.)

Theorem 1.8. *Pontryagin duality is a contravariant categorical duality between discrete abelian torsion groups and abelian profinite groups.*

See, for instance, [29, 6.4.1], which is a strictly stronger statement.

This means we can translate many classical results from the theory of abelian torsion groups to the setting of profinite groups.

First, we note how duality affects the set of n -th powers.

We write $G[n]$ to denote the kernel of the n -th power map, that is, the set of elements of order dividing n in G and nG to denote the set of multiples of n in G .

Theorem 1.9. *Let G be a discrete torsion abelian group or profinite abelian group. Then*

$$\text{Ann}_{G^*}(nG) = G^*[n] \text{ and } \text{Ann}_{G^*}(G[n]) = n(G)^*.$$

See [RZ], where it is 2.9.11.

This duality acts on the Boolean lattice of closed subgroups by swapping meet and join, as one would expect.

Theorem 1.10. *Let G be an abelian profinite group or discrete abelian torsion group with closed subgroups H_1, H_2 .*

Then

$$\text{Ann}_{G^*}(H_1 H_2) = \text{Ann}_{G^*}(H_1) \cap \text{Ann}_{G^*}(H_2)$$

and

$$\text{Ann}_{G^*}(H_1 \cap H_2) = \text{Ann}_{G^*}(H_1) \text{Ann}_{G^*}(H_2).$$

See [RZ], where it is 2.9.10.

For the remainder of the paper, we do not calculate duals. We use only the above results on annihilators, the category theoretical notion of a dual and that finite groups are self-dual.

Example 1. Consider the abelian p -group

$$\Gamma = \langle z, \{x_n \mid n \in \mathbb{N}\} \mid pz = 0, p^n x_n = z \rangle.$$

From the above, it is clear that $\bigcap_n p^n \Gamma = \langle z \rangle$ and $\Gamma / \langle z \rangle$ is isomorphic to $\bigoplus_n C_{p^n}$.

Define, for each natural number n , the subgroup $\Gamma_n = \langle z, x_1, x_2, \dots, x_n \rangle$ of Γ . We can

obtain Γ as the direct limit of the following direct system

$$\langle z, x_1 \rangle \rightarrow \langle z, x_1, x_2 \rangle \rightarrow \langle z, x_1, x_2, x_3 \rangle \rightarrow \langle z, x_1, \dots, x_4 \rangle \rightarrow \dots$$

with the obvious inclusion maps. These groups are of the following isomorphism types:

$$C_{p^2} \rightarrow C_{p^1} \times C_{p^3} \rightarrow C_{p^1} \times C_{p^2} \times C_{p^4} \rightarrow C_{p^1} \times C_{p^2} \times C_{p^3} \times C_{p^5} \rightarrow \dots$$

Consider the profinite group

$$X = \left\langle \prod_i C_{p^i} = \prod_{i \in \mathbb{N}} \langle d_i \mid p^i d_i = 0 \rangle, \mathbb{Z}_p = \overline{\langle a \rangle} \mid pa = \prod_{i \in \mathbb{N}} d_i \right\rangle.$$

We claim $X \cong \Gamma^*$.

For each natural number n , write $D_n = \overline{\langle d_i \mid i > n \rangle}$. It is clear to see that X is the inverse limit of the quotients X/D_n . That is,

$$X \cong \varprojlim_n \langle a, d_1, \dots, d_n \mid p^i d_i = 0, pa = d_1 + d_2 + \dots + d_n \rangle.$$

Hence X is the inverse limit of the following inverse system of finite groups. (We have in each presentation the relations $p^i d_i = 0$ and $pa = \prod_{i \in \mathbb{N}} d_i$.)

$$\langle a, d_1 \rangle \leftarrow \langle a, d_1, d_2 \rangle \leftarrow \langle a, d_1, d_2, d_3 \rangle \leftarrow \dots \leftarrow \langle a, d_1, \dots, d_4 \rangle \leftarrow \dots$$

These groups are of the following isomorphism types.

$$C_{p^2} \leftarrow C_{p^1} \times C_{p^3} \leftarrow C_{p^1} \times C_{p^2} \times C_{p^4} \leftarrow C_{p^1} \times C_{p^2} \times C_{p^3} \times C_{p^5} \leftarrow \dots$$

Where the maps come from canonical maps between the following system of quotients of X :

$$\frac{X}{\langle d_2, \dots \rangle} \leftarrow \frac{X}{\langle d_3, \dots \rangle} \leftarrow \frac{X}{\langle d_4, \dots \rangle} \leftarrow \frac{X}{\langle d_5, \dots \rangle} \leftarrow \dots$$

It is easy to see that each Γ_n^* is isomorphic to

$$X_n = \langle a, d_1, \dots, d_n \mid p^i d_i = 0, pa = d_1 + d_2 + \dots + d_n \rangle.$$

But, by Theorem 1.6, above, Γ^* surjects onto each Γ_n^* , as $\Gamma_n^* \cong \Gamma^* / \text{Ann}_{\Gamma^*}(\Gamma_n)$.

As $X_{n+1} \cong \Gamma_{n+1}^*$, we can consider X_{n+1} as a set of functions on Γ_{n+1} . By considering the restriction of these functions to Γ_n , we get a map to $\Gamma_n^* \cong X_n$. It is clear, on consideration, that this is the same map as the map $X_{n+1} \rightarrow X_n$ in the inverse system above. Hence the Γ_n^* form an inverse system isomorphic to the inverse system of quotients of X above. Hence,

by the universal property of the inverse limit, Γ^* continuously surjects onto X , with kernel contained in $\bigcap_n \text{Ann}_{\Gamma^*}(\Gamma_n)$.

But $\bigcap_n \text{Ann}_{\Gamma^*}(\Gamma_n)$ is equal to $\text{Ann}_{\Gamma^*}(\langle \Gamma_n \mid n \in \mathbb{N} \rangle)$, by Theorem 1.10 above. But $\langle \Gamma_n \mid n \in \mathbb{N} \rangle = \Gamma$ and so this annihilator is trivial, hence $X \cong \Gamma^*$.

This is a somewhat drawn-out example: in general we shall use somewhat less detail. This is a crucial example for our purposes: it is the prototypical example of a non-trivial abelian pro- p group which has no continuous torsion-free images and which is not the Cartesian product of cyclic groups. To see this, consider the closed subgroup $D = \overline{\langle d_i \mid i \in \mathbb{N} \rangle}$. As each d_i is contained in D , it emerges that every torsion element of X is contained in D and so $D = \overline{t(X)}$ and so X is not Cartesian.

We shall frequently return to this group. In Chapter 4, we call this $H_{\omega+1}$, one of the generalised Kiefer groups.

We also make the following technical definition.

Definition 9. *Let G be a profinite group.*

We say that a normal subgroup H of G is a closed-continuous direct summand if and only if H is closed and there exists a closed normal subgroup K of G such that

$$HK = G \text{ and } H \cap K = \{1\}.$$

We call K a closed complement of H .

This definition is needed: a subgroup is a closed-continuous direct summand if and only if it is a topological splitting. Merely having a closed direct summand is not sufficient: Lemma 6.2, gives rise to an example of a closed direct summand with no closed complement.

It is elementary to see that a closed subgroup H is a closed-continuous direct summand of a profinite abelian group G if and only if $\text{Ann}_{G^*}(H)$ is a direct summand of G^* .

Chapter 2

The Discontinuous Isomorphism Problem in Profinite Groups of Extremal Commutativity

Suppose f is an isomorphism of abstract groups from a profinite group G to a profinite group H . If f is not a homeomorphism (i.e. not continuous), we shall say that f is discontinuous.

We shall call the question of when we can find a discontinuous isomorphism $f : G \rightarrow H$ between two profinite groups which are not isomorphic in the category of topological groups “the Discontinuous Isomorphism Problem”. (Many profinite groups have automorphisms that are not continuous. If every automorphism of a profinite group G is continuous then the profinite topology of G is determined by the group structure. Theorem 2.6 gives an example of a discontinuous isomorphism of $\prod_X C_p$, for infinite X .)

In this chapter a profinite group will be denoted by $G = (\Gamma, T)$, where T is a topology on the group Γ . In subsequent chapters, when we will not need to explicitly refer to T , we will use G to denote the underlying abstract group. We note some useful immediate results.

Lemma 2.1. *Let (Γ, T_0) be a profinite group. If T_1 is a refinement of T such that (Γ, T_1) is a profinite group, then $T = T_1$.*

Proof. We shall in fact prove that a Hausdorff topological space (X, T_2) cannot have a proper compact refinement T_1 .

If T_1 refines T_2 , $Id_X : (X, T_1) \rightarrow (X, T_2)$ is continuous. But if (X, T_1) is compact and (X, T_2) is Hausdorff, Id_X sends closed sets to closed sets and hence every closed set in T_1

is also closed in T_2 and so $T_1 = T_2$.

□

This means that, given a profinite group (Γ, T) , any group topology on Γ which is strictly finer or coarser than T cannot be profinite. So, if we can establish enough sets which must be open based only on algebraic information about some profinite G , then we can determine when the topology of G is determined by the algebraic structure.

Proposition 2.2. *Suppose (Γ, T) is a profinite group G . Then*

1. *every finite $X \subseteq \Gamma$ is closed in G ;*
2. *for each discrete group word w , Γ_w , the set of values w takes over Γ , is closed in G ;*
3. *for each $Y \subseteq \Gamma$, $C_\Gamma(Y)$, the centraliser of Y in Γ is closed in G .*

Proof. 1 holds as profinite groups are Hausdorff.

A group word is a continuous map from G^n to G . As multiplication and inversion are continuous maps, 2 follows.

Now, for any $y \in \Gamma$, $C_\Gamma(y)$ is the pre-image in Γ of the (closed) trivial subgroup under the continuous map $g \mapsto [x, g]$. By the argument above, this is necessarily continuous and hence $C_\Gamma(y)$ is closed in G . But

$$C_\Gamma(Y) = \bigcap_{y \in Y} C_\Gamma(y),$$

and so we have 3. □

The third of these allows us to get some basic results about some classes of very non-abelian groups which have large centralisers.

Recall the definition of a branch group.

Definition 10. (*Branch Groups*)

Let G be a profinite group.

If there exist two decreasing sequences of closed subgroups $(L_i)_{i \in \mathbb{N}}$ and $(H_i)_{i \in \mathbb{N}}$ and a sequence $(k_i)_{i \in \mathbb{N}}$ of positive integers such that: $L_0 = K_0 = G$ and $k_0 = 1$ and for each i

1. *H_i is normal and of finite index in G ,*

2. there exist closed subgroups $L_i^{(1)}, L_i^{(2)}, \dots, L_i^{(k_1 k_2 \dots k_i)}$ of G such that

$$H_i = L_i^{(1)} \times L_i^{(2)} \times \dots \times L_i^{(k_i)},$$

with each $L_i^{(r)}$ isomorphic (as a topological group) to L_i ,

3. at each level, the product decomposition refines that of the previous level, i.e. $L_{i+1}^{(r)}$ is contained in $L_i^{(\lceil r/k_{i+1} \rceil)}$,

4. conjugation by elements of G transitively permutes the $L_i^{(r)}$,

we say that G is a branch group. Such a collection of subgroups is called a branch system for G .

Branch groups are significant as they have faithful continuous actions on spherically homogenous rooted trees which are transitive on ends. Under this action the subgroup H_i is the subgroup which fixes all vertices of distance at most i from the root, and $L_i^{(r)}$ fixes all vertices except the descendants of $v_r^{(i)}$, the r th vertex of distance i from the root. (We call a vertex v a descendent of $v_r^{(i)}$ if the path from v to the root passes through $v_r^{(i)}$.) These subgroups are known as *level stabilisers* and *restricted vertex stabilisers*, respectively. For more detail about branch groups, see [6].

Theorem 2.3. *Let Γ be a group with topology T_0 such that $G = (\Gamma, T_0)$ is a profinite branch group.*

Then, if T_1 is a topology on Γ such that (Γ, T_1) is a profinite group, $T_1 = T_0$.

Proof. We shall refer to the subgroups in the branch system of G as $H_i, L_i^{(r)}$, both as subgroups of Γ and closed subgroups of G .

A rooted tree automorphism will centralise a restricted vertex stabiliser if and only if it fixes every vertex in the subtree of its descendants and so

$$L_i^{(r)} = \bigcap_{1 \leq s \leq k_1 k_2 \dots k_i, s \neq r} C_\Gamma(L_i^{(s)}).$$

Hence each $L_i^{(r)}$ is an intersection of stabilisers in Γ and hence must be closed in each profinite topology on Γ . Moreover, each H_i , as a finite product of closed subsets, must also be closed in each profinite topology on Γ .

As the H_i are closed in (G, T_1) and of finite index, they are each open in T_1 . But the H_i form a basis for the usual branch topology for T_0 and hence T_1 is a refinement of T_0 . Hence, by Lemma 2.1, $T_1 = T_0$. $\square$

It is immediate that we can use a similar argument to get the following.

Theorem 2.4. *Let Γ be the abstract group structure of the profinite group*

$$G = \prod_{i \in I} S_i,$$

for some collection $\{S_i \mid i \in I\}$ of finite groups with trivial centres.

Then, if T_1 is a topology on Γ such that (Γ, T_1) is a profinite group, $(\Gamma, T_1) = G$.

Proof. Identifying S_k , $k \in I$ with the subgroup of G consisting of strings with all co-ordinates except the k th entry trivial, we have

$$C_\Gamma(S_k) = \{(x_i) \in G \mid x_i = 1 \forall i \neq k\},$$

the kernel of the projection from the k th co-ordinate. Hence these and so their intersections are closed in every topological group structure on Γ . But these subgroups form a neighbourhood basis for 1 in G . Hence, if T_1 is a topological group topology on Γ , every open set in G must be open in (Γ, T_1) . By Lemma 2.1, it follows that $(\Gamma, T_1) = G$. $\square$

In fact, this proof shows more:

Corollary 2.5. *Let (Γ, T) be the profinite group*

$$G = \prod_{i \in I} S_i,$$

for some collection $\{S_i : i \in I\}$ of finite groups. Write K_i for the kernel of the projection onto the i th co-ordinate.

Then, for each i and each subset T_i of S_i containing $Z(S_i)$, $T_i K_i$ is contained in every topology T_1 on Γ such that (Γ, T_1) is a profinite group, $(\Gamma, T_1) = G$.

But what of groups equal to their centres, what of abelian groups?

A product of simple abelian groups can have many different profinite topologies.

Theorem 2.6. *The elementary pro- p group C_p^X has unique profinite topology if and only if X is finite.*

Proof. It suffices to show that for any infinite cardinal X we can find a non-continuous automorphism of C_p^X .

Any non-continuous automorphism α of $C_p^X = (\Gamma, T)$ will give rise to a different topology: $\alpha(T) \neq T$ as α is not continuous.

But $C_p^X = \prod_X C_p$ is an $\mathbb{F}_p$ -vector space of dimension 2^X . Hence $\text{Aut}(C_p^X)$ is transitive on subspaces of codimension 1 (i.e. subgroups of index p) and so it suffices to find a non-open subspace of codimension 1. As C_p^X has precisely X open subgroups, only X out of 2^{2^X} subspaces of codimension 1 can possibly be open. So, using the axiom of choice, take some non-open H of codimension 1 and find some group automorphism α taking some distinct arbitrary open codimension 1 subgroup to H . This α will be non-continuous: a discontinuous isomorphism. $\square$

Theorem 2.7. *Let X and Y be cardinals such that $X \neq Y$ but $2^X = 2^Y$, then for each natural number n $C_{p^n}^X$ is abstractly but not topologically isomorphic to $C_{p^n}^Y$.*

Proof. Firstly, if $X \neq Y$, $C_{p^n}^X$ and $C_{p^n}^Y$ cannot be topologically isomorphic, or even homeomorphic. The topology of $C_{p^n}^X$ is X -based: it has precisely X open subgroups, as X is infinite. As $C_{p^n}^Y$ has exactly Y open subgroups and thus Y open sets, these spaces cannot be homeomorphic.

To show these are abstractly isomorphic, we shall prove that

$$\prod_A C_{p^n} \cong \bigoplus_{2^A} C_{p^n},$$

for each infinite ordinal A , from which the theorem is immediately apparent.

We proceed by induction on n .

Our base case is $n = 1$. We know that $\prod_A C_p$ is an $\mathbb{F}_p$ -space and that it is of cardinality, and hence dimension, 2^A . The axiom of choice gives us a basis X of order 2^A . By the definition of a basis, $\prod_A C_p = \bigoplus_{x \in X} \langle x \rangle \cong \bigoplus_X C_p$.

Now, suppose the statement is true for $n = k$. We have that $\prod_A C_{p^k} \cong \bigoplus_{2^A} C_{p^k}$ and consider $G = \prod_A C_{p^{k+1}}$.

Consider pG , the image of G under the map $x \mapsto px$. As $pG \cong \prod_A C_{p^k}$, by the inductive hypothesis this is a direct sum of 2^A copies of C_{p^k} and we can take some basis X for pG .

For each $x \in X$, there is some $y_x \in G$ with $py = x$.

Write $Y = \{y_x \mid x \in X\}$. Each y_x in Y is of order p^{k+1} . This set is independent modulo pG as X is a basis for pG . As $Y + pG$ is linearly independent, Y is linearly independent. Now, as $\prod_A C_{p^{k+1}}$ has no direct summands isomorphic to C_p , Y is a basis for G , which completes our induction. $\square$

We make a similar argument for torsion-free profinite abelian groups in Theorem 7.2.

Chapter 3

Background Abelian Group Theory

Much of the work in this thesis builds on classical results in abelian group theory. This chapter has two aims.

Firstly, to detail the results which we want to translate through Pontryagin duality in the next chapter. Secondly, to provide an introductory survey of some results on abelian groups, including an account of Ulm theory, which should be both understandable and helpful for a graduate student with knowledge of group theory. The material on divisible groups and some of 3.3.1 come from [10]. Much of the rest of the chapter follows the material of [4].

Theorem 3.1. *Let G be a torsion abelian group. Then*

$$G = \bigoplus_{p \text{ prime}} \{x \in G \mid o(x) = p^n, n \in \mathbb{N}\}$$

A key concept in the structure of abelian groups is that of height, dual to order. This is a codification and extension of the notion of “being a multiple of p ”.

Definition 11. *Let G be an abelian p -group. The height of x in G is the greatest n such that x is in $p^n G$. We say that x is of infinite height in G if x is in $\bigcap_{n \in \mathbb{N}} p^n G$.*

As we care about height of group elements, it is important to note that height of a (non-trivial) element x in a group G depends on G . When we pass to subgroups, the height may be reduced. For instance, the height of any non-trivial x in $\langle x \rangle$ is 0, but most groups contain elements of non-zero height.

We care about subgroups which preserve height.

Definition 12. Let G be an abelian group and H a subgroup of G .

We say that H is pure in G if

$$nG \cap H = nH$$

for every natural number n .

The simplest example of a p -group with elements of infinite height is the *Prüfer quasi-cyclic group*, C_{p^∞} .

$$C_{p^\infty} = \langle x_1, x_2, \dots \mid px_1 = 0, px_{i+1} = x_i, i \geq 1 \rangle$$

This group is the direct limit of the system $(\mathbb{Z}/p^i\mathbb{Z})_{i \in \mathbb{N}}$, with the natural homomorphisms from $\mathbb{Z}/p^{i+j}\mathbb{Z}$ to $\mathbb{Z}/p^i\mathbb{Z}$ given by reducing modulo p^i . It is the maximal p -subgroup of $\mathbb{Q}/\mathbb{Z}$: in fact $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the direct sum of each C_{q^∞} as q ranges across all primes. The Prüfer quasicyclic group is a member of an important class of groups.

3.1 Divisible Groups

Definition 13. An abelian group D is said to be divisible if for every s in D and natural n there is some t in D with $nt = s$.

The archetypal example of a divisible group is the additive group of the rationals, $\mathbb{Q}$. Indeed, the additive group of any characteristic 0 field is divisible.

In fact, it is easy to classify these groups.

Theorem 3.2. Let G be a divisible abelian group. Then G is isomorphic to a direct sum of copies of $\mathbb{Q}$ and C_{p^∞} , for various primes p .

Proof of this result follows from a series of properties of divisible groups and is presented after two other results on divisible groups.

Corollary 3.3. For every infinite cardinal X

$$\prod_X C_{p^\infty} \cong \left(\bigoplus_{2^X} C_{p^\infty} \right) \oplus \left(\bigoplus_{2^X} \mathbb{Q} \right).$$

Proof. As X is infinite, we have $X = X \cdot X$. Thus we can rewrite $\prod_X C_{p^\infty} = G$ as $G = \prod_X H$, where H is isomorphic to G .

As the product of X non-empty, non-singleton sets, each of order less than 2^X , G is of order 2^X . Now $t(G) \cong t(\prod_X t(G))$, which contains a subgroup isomorphic to $\prod_X C_p$ of order 2^X and so $|t(G)| = 2^X$. Similarly, as $G/t(G)$ maps onto $\prod_X G/t(G)$, this quotient must be of order 2^X .

From Theorem 3.2, we can see that $t(G)$ as divisible p -group of order 2^X , must be isomorphic to $\bigoplus_{2^X} C_{p^\infty}$. From a similar argument on the torsion-free part, the result follows. $\square$

This is a basic case in our theme of examining a Cartesian product using the direct sum.

Lemma 3.4. *Let G be an abelian group. If D is a divisible subgroup of G , then D is a direct summand of G .*

Proof. By Zorn's Lemma, there is a $K \leq G$, intersecting trivially with D , maximal with respect to the latter property. We claim that $G = D + K$. Suppose otherwise. Then there is some $x \in G \setminus (D + K)$. We consider $K' = \langle K, x \rangle$. By the maximality of K , there is some non-trivial $d \in D \cap K'$, and then $d = k + nx$ for some $k \in K, n \in \mathbb{N}$ and hence $nx \in D + K$. We can assume, without loss of generality, that n is minimal such that $nx \in D + K$. Take some prime p dividing n . Then $(n/p)x$ is not in $D + K$, but $p(n/p)x = nx = d - k$. By the divisibility of D , there is some $d_1 \in D$ with $d = pd_1$ and so

$$(n/p)x - d_1$$

is not in $D + K$, but $p((n/p)x - d_1) = -k$ is in K . As K is a maximal subgroup of G which intersects trivially with D and $(n/p)x - d_1$ is not in K , it follows that $\langle K, (n/p)x - d_1 \rangle \cap D \neq 0$. So, we can find some non-zero $d_2 \in D, k_2 \in K, m \in \mathbb{Z}$, with $d_2 = k_2 + m((n/p)x - d_1)$. But d_2 is a non-zero element of D , so $m((n/p)x - d_1)$ is not in K and so p does not divide m . So m and p are coprime and we can find integral r, t with $rm + pt = 1$. Now,

$$((n/p)x - d_1) = rm((n/p)x - d_1) + tp((n/p)x - d_1) \in D + K$$

which is a contradiction. $\square$

Proof. (of Theorem 3.2).

It is easy to see that $t(G)$, the subgroup of torsion elements of G , must also be divisible. By the previous lemma, it follows that $G = t(G) \oplus G/t(G)$. $G/t(G)$ is the torsion-free part

of G and, as a quotient, is also divisible. As it is torsion free, for every $x \in G/t(G)$ and integer n there is a *unique* y such that $x = ny$. This means that $(1/n)x$ is well-defined and hence $G/t(G)$ is a $\mathbb{Q}$ -space. As every vector space has a basis, $G/t(G)$ is a direct sum of copies of $\mathbb{Q}$.

We can now restrict, by 3.1, to the case when G is a p -group. Consider P , the set of elements of order p . P is an $\mathbb{F}_p$ -space and hence we can pick a basis X of P . As G is divisible, for each $x \in X$ we can find $x_1, x_2, \dots, x_n, \dots$ with $x = px_1, x_i = px_{i+1}$, for each i . Hence we have a copy of C_{p^∞} for each element of X . The set X_1 of elements of order p^2 collected this way must similarly be independent, by the independence of X and generates the subgroup of all elements of order p^2 , as each of these elements multiplied by p is of order p . Hence we can see that these X_n will be independent generating sets for the subgroup of elements of order p^n and so we get, as required,

$$G = \bigoplus_X C_{p^\infty}.$$

□

Theorem 3.5. *Every abelian group has a unique maximal divisible subgroup.*

Proof. The sum of all the divisible subgroups of an abelian group G is divisible and hence is the unique maximal divisible subgroup of G . □

Definition 14. *A group is said to be reduced if it has no non-trivial divisible subgroups.*

With these results on divisible groups, it is clear that all abelian groups are the direct sum of an divisible and a reduced group. From the structure theorem for divisible group, it is now only necessary to consider reduced abelian p -groups.

It is possible for these to have elements of infinite height.

Example 2. Consider the abelian p -group we considered in Example 1 given by

$$\Gamma = \langle x_0, x_1, x_2, \dots, x_n, \dots \mid px_0 = 0, p^i x_i = x_0 \rangle.$$

Recall that $\Gamma/\langle x_0 \rangle$ is isomorphic to $\bigoplus_{i \in \mathbb{N}} C_{p^i}$.

This group is reduced, as quotients of divisible groups are divisible but x_0 is of infinite height in Γ . (This follows from observing that for every $a \in \Gamma$ with $p^n a = x_0$ for some positive n , there is some natural number m such that there is no $b \in \Gamma$ with $p^m b = a$.)

Knowledge of divisible groups makes it possible to discuss basic subgroups.

Definition 15. *Let A be an abelian torsion group. We say that a subgroup B is a basic subgroup of A if B is a pure subgroup of A with A/B divisible and B is a direct sum of cyclic groups.*

These are structurally significant, as the following theorem shows.

Theorem 3.6. *Let A be an abelian torsion group. Then A has a basic subgroup and all basic subgroups of A are isomorphic.*

This combines [25, 4.3.4] and [25, 4.3.6]: it is due to Kulikov and Fuchs.

Proof. The proof of existence comes to a Zorn's Lemma argument providing a pure subgroup maximal with respect to appropriate conditions and arguing that it must have divisible quotient: see [25, 4.3.4] for details.

We can assume A is a p -group. Then, by the above, there is a basic subgroup B . As a direct sum of cyclic p -groups, the isomorphism type of B is entirely determined by the inverse system $(B/p^n B)_n$: the dimensions of the kernels will tell us how many direct summands B has of isomorphism type C_{p^r} , for each r . (The basic idea behind this way of classifying groups is at the heart of Ulm theory. The remainder of this chapter will independently show that counting kernels of iterations of the map $x \mapsto px$, under the name "Ulm–Kaplansky invariants" classifies all countable torsion groups.)

As the quotient A/B is divisible, $A = B + p^n A$, for all natural numbers n . But by the definition of purity, $B \cap p^n A = p^n B$ and so $A/p^n A \cong B/p^n B$. Hence the inverse system $(B/p^n B)_n$ is isomorphic to $(A/p^n A)_n$ and so all basic subgroups of A are isomorphic. $\square$

The notion of basic subgroups and their structural significance is particularly relevant to the work in Chapter 7.

3.2 Ulm Theory for Countable p -groups

For Γ any abelian p -group, we have a chain of characteristic subgroups

$$1\Gamma > p\Gamma > p^2\Gamma > \cdots > p^i\Gamma > \cdots$$

which intersects in $\text{ih}(\Gamma)$, the elements of infinite height in Γ . This contains a series of subgroups characteristic in Γ :

$$1 \text{ ih}(\Gamma) > p \text{ ih}(\Gamma) > p^2 \text{ ih}(\Gamma) > \cdots > p^i \text{ ih}(\Gamma) > \cdots$$

which in turn intersect in $\text{ih}(\text{ih}(\Gamma))$. This gives us a transfinite series of subgroups, defined by

$$\Gamma_0 = \Gamma$$

$$\Gamma_{\alpha+n} = p^n \Gamma_\alpha$$

$$\Gamma_\varepsilon = \bigcap_{\alpha < \varepsilon} \Gamma_\alpha$$

for any ordinal α , natural number n and limit ordinal ε . (Recall that by *limit ordinal* we mean an ordinal that is not given by the successor function on ordinals $\alpha \mapsto \alpha + 1$.) Eventually this series becomes constant: there is some least λ such that $\Gamma_\beta = \Gamma_\lambda$ for all $\beta \geq \lambda$. Γ_λ is then the reduced part of Γ and we call λ the *length* of Γ . This series is obviously an invariant of Γ . It gives rise to the Ulm–Kaplansky invariant function.

Definition 16. *The Ulm–Kaplansky invariant function of Γ is the function f_Γ from ordinals less than λ to cardinals less than $|\Gamma|$ given by*

$$\alpha \mapsto \dim_{\mathbb{F}_p}(\Gamma_\alpha \cap \Gamma[p]) / (\Gamma_{\alpha+1} \cap \Gamma[p])$$

The values taken by this function are the Ulm–Kaplansky invariants of Γ .

These invariants were not in fact used by Ulm: his proofs involved transfinite matrices. Rather, they were introduced by Irving Kaplansky and George Mackey in [11] and are substantially more concrete than Ulm’s original paper. The n -th Ulm–Kaplansky invariant counts the number of cyclic summands of order p^n of Γ . That is, Γ has a direct summand isomorphic to $\bigoplus_\alpha C_{p^n}$ if and only if $\alpha \leq f_\Gamma(n)$.

We can generalise the definition of height.

Definition 17. *Let Γ be an abelian p -group.*

Recall we defined the chain of subgroups via

$$\Gamma_0 = \Gamma$$

$$\Gamma_{\alpha+n} = p^n \Gamma_\alpha$$

$$\Gamma_\varepsilon = \bigcap_{\alpha < \varepsilon} \Gamma_\alpha$$

for any ordinal α , natural number n and limit ordinal ε .

For any x in Γ , the (generalised) height of x in Γ is the maximal ordinal α such that $x \in \Gamma_\alpha$.

As a result, we also define $p^\alpha \Gamma$ to be the subgroup we call Γ_α above: the elements of height at least α .

Note that this agrees with the earlier definition of height for any element of finite height.

Note that, for any limit ordinal ε , $p^{\varepsilon+\omega} \Gamma$ is the subgroup of elements of infinite height in $p^\varepsilon \Gamma$. We can thus look at the characteristic transfinite series

$$\Gamma = p^0 \Gamma > p^\omega \Gamma > p^{2\omega} \Gamma > \dots > p^\varepsilon \Gamma > p^{\varepsilon+\omega} \Gamma > \dots$$

where $0, \omega, 2\omega, \dots, \varepsilon, \varepsilon + \omega, \dots$ are the limit ordinals. This has a sequence of consecutive quotients $\Gamma/p^\omega \Gamma, p^\omega \Gamma/p^{2\omega} \Gamma, \dots, p^\varepsilon \Gamma/p^{\varepsilon+\omega} \Gamma, \dots$, each of which contains no elements of infinite height. This is known as the *Ulm sequence* of Γ . We would like this to determine the isomorphism class of Γ and indeed, for countable Γ , it does.

Theorem 3.7. *If Γ and Δ are countable reduced abelian p -groups with isomorphic Ulm sequences (or identical Ulm invariant functions), then Γ and Δ are isomorphic.*

Note that Prüfer's theorem [25, 4.3.15], which says that any countable abelian group with no elements of infinite height is a direct sum of cyclic groups, follows immediately from considering groups of length ω .

The proof of this theorem can be found as Theorem 14 in [10], or 37.1 of [5]. I do not believe there is an enlightening translation of this through duality. We outline a proof of Theorem 3.7 at the start of the next section. After that proof, the remainder of this chapter is devoted to exploring the most general class of groups where the basic idea of this proof works.

Ulm's Theorem provides us with a way of distinguishing groups by invariants. We now turn to look at Zippin's Theorem, which determines for which potential invariants there exist groups.

Any function from countable ordinals to $\mathbb{N} \cup \{0, \aleph_0\}$ which is constantly 0 for all ordinals

greater than λ can only be a possible Ulm–Kaplansky invariant function for a countable p -group if it takes non-zero values on infinitely many values between every consecutive pair of limit ordinals less than λ . This is necessary to ensure that each term of the Ulm sequence except the last is of unbounded exponent, as required.

The following definition is helpful:

Definition 18. *A function f from ordinals less than (a fixed ordinal) τ to cardinals is said to be τ -admissible (or just admissible) if the following conditions hold*

1.

$$\tau = \sup\{\sigma + 1 \mid f(\sigma) \neq 0\},$$

2.

$$\text{for each } \sigma \text{ with } \sigma + \omega < \tau \text{ we have } \sum_{\rho \geq \sigma + \omega} f(\rho) \leq \sum_{n < \omega} f(\sigma + n).$$

We say that τ is the length of f .

In the countable case, this is equivalent to a function from ordinals less than or equal to (some countable) λ to countable cardinals, taking infinitely many non-zero values between any two limit ordinals less than λ . We shall later see that the definition of admissibility is helpful for talking about Ulm–Kaplansky invariant functions of groups of a larger class.

In fact, for each such function, there exists a group.

Theorem 3.8. *Zippin’s Theorem (for countable p -groups) ([30])*

Let (Γ_α) be a transfinite sequence, of countable length, of countable abelian p -groups with no elements of infinite height, such that all except possibly the last are of unbounded exponent. Then there exists a reduced abelian p -group Γ with Ulm sequence (Γ_α) .

Equivalently, for f_Γ a function from ordinals less than or equal to λ to countable cardinals, which takes infinitely many non-zero values between any two limit ordinals less than λ , there is a p -group Γ with Ulm invariant function f_Γ .

[Note that the conditions on this second statement do not involve p at all.] In his original paper [30], Zippin claims to have worked by proving this existence theorem in the dual case, then translating these results into the discrete case. As far as I am able to tell, a proof of this working only in the dual has not been previously published.

We omit the proof and construction of such groups: instead, [13]’s construction of pro- p groups is clearer and through the duality subsequently described, utterly equivalent. This

construction is that given in Theorem 5.5. The proof is the same, with the minor addition of discussions of unbounded multiplicity for when we must concern ourself with uncountably infinite cases. In the profinite case, we inductively build groups with previous groups as subgroups; in the discrete case we have previous groups as quotients. The profinite version of the proof is arguably clearer and easier to understand, as adding elements is simpler than adding relations. For an alternative proof, of Theorem 3.8, see [5, 36].

The following definition occurs naturally when one attempts to construct arbitrary totally projective groups via Zippin's Theorem.

Definition 19. *Let Γ be an abelian p -group. We say that Γ is simply presented if it can be given as a presentation (of abelian groups)*

$$\Gamma = \langle X = \{x_i \mid i \in I\} \mid R \rangle$$

where the set of relations R consists of relations of the form

$$p^n x_i = 0 \text{ or } p^m x_i = x_j$$

for $i, j \in I$ and n, m natural numbers.

To understand this: suppose an (abelian) presentation $\langle X \mid R \rangle$ satisfies the above conditions. This is true if and only if there is some presentation of Γ

$$\langle X' \subseteq \{p^n x \mid x \in X, n \in \mathbb{N}\} \mid R \rangle,$$

with the same relation set R such that the graph with vertices $X \cup \{0\}$ which has an edge from a to b whenever $p^n a = b \in R$, for some $n \in \mathbb{N}$ forms a tree.

This is an intuitively clear notion, which is easy to understand and says much about the structure of discrete abelian p -groups. However the dual notion is less useful when it comes to the pro- p sense, so we shall merely mention it here for completeness.

The definition is due to Crawley and Hales in [2]. The class of totally projective groups, studied in the next section, turns out to co-incide precisely with that of simply presented groups.

Ulm's Theorem and Zippin's Theorem are true in larger generality than just countable groups. Fifteen years after Ulm's original proof, in 1960, Kolettis (a student of Irving Kaplansky) showed the analogous result holds for direct sums of countable p -groups, [15].

(A countable p -group has length strictly less than ω_1 , the first uncountable ordinal. A direct sum of countable p -groups has length less than or equal to ω_1 . It transpires that these groups are precisely the totally projective groups of length at most ω_1 .)

It was shown in [24] that this result can be reduced to a two-page solution which involves no group theory whatsoever.

P. Hill in [7], building on work of Nunke [21], showed that the largest class of groups which Ulm's Theorem applies to are the so-called Totally Projective Groups.

3.3 Totally Projective Groups

To understand these groups, we first look at subgroup structure in abelian groups.

The material in this section is heavily based on [4, 73–83]. It was first studied in [7]. Most sources heavily reference this paper, which first demonstrated the full extension of Ulm's Theorem. Unfortunately, P. Hill in a 2002 talk [8] cites this as “P.Hill, On the classification of abelian groups, photocopied manuscript, University of Houston, Texas, 1967”. The apparent lack of availability of this manuscript is slightly unhelpful to the modern reader.

We care about preserving height in quotients. This is codified in the following definition, which is fundamental for the proof outlined in 3.3.1 and thus the structural features we exploit to classify abelian p -groups.

Definition 20. *Let Γ be an abelian group with subgroup Δ . We say that an element x is proper with respect to Δ if it is of maximum height in Γ among elements of $\Delta + x$.*

(It follows that x is proper with respect to Δ if and only if the height of $x + \Delta$ in Γ/Δ is equal to the height of x in Γ .)

A crucial fact used at each step of the countable proof is that we can always find elements in a coset proper with respect to a coset. We codify this with the following definition.

Definition 21. *Let G be an abelian p -group and K a subgroup of G .*

We say that K is nice in G if every non-zero coset of G modulo K contains an element which is proper with respect to K . That is, for every $x + K$ (with $x \in G \setminus K$) we have some $k(x) \in K$ such that

$$h_{G/K}(x + K) = h_G(x + k(x))$$

where indices refer to the group in which height is being calculated. (These indices are not

strictly needed, as there is no danger of confusion, and we shall generally omit them from now on.)

In order to best understand why we make these definitions and to understand the structure of the remainder of this chapter, we consider the following sketch of the proof of Theorem 3.7:

3.3.1 Kaplansky's Proof of Ulm's Theorem

Irving Kaplansky's version of the proof of Ulm's Theorem, Theorem 3.7, first published in [11], introduced the notion of Ulm-Kaplansky invariants. This modern proof of the theorem is based on the following induction, which inspired subsequent research and the discovery of the class of totally projective groups.

Let $\Gamma = \{g_1, g_2, \dots, g_n, \dots\}$ and $\Delta = \{d_1, d_2, \dots, d_n, \dots\}$ be enumerations of countable reduced abelian p -groups. We construct isomorphisms between finite subgroups of Γ and Δ . We shall at each step make sure that all of these isomorphisms are height-preserving. (By this we refer to height relative to Γ, Δ .) At the n -th step of our induction, we have the following situation.

We have some $\{\Gamma, \Delta\}$ -height-preserving isomorphism μ_n from Φ_n , a finite subgroup of Γ , to Ξ_n , finite subgroup of Δ . We extend μ_n to a height-preserving isomorphism

$$\mu'_n : \langle \Phi_n, g_n \rangle \rightarrow \Xi'_n,$$

and then extend $(\mu'_n)^{-1}$ to a height-preserving isomorphism

$$\mu_{n+1}^{-1} : \langle \Xi'_n, d_n \rangle = \Xi_{n+1} \rightarrow \Phi_{n+1}.$$

We thus ensure that all elements will ultimately be hit, by 'doubling back' with our isomorphisms in this way.

The problem is reduced to showing that this map $\mu : \Phi \rightarrow \Xi$ extends to a height-preserving isomorphism from the subgroup $\langle \Phi, x \rangle$ to some Ξ' by finding an appropriate element x of Δ , where we have $px \in \Phi$.

As we work only with finite subgroups, we can assume without loss of generality that x is proper with respect to Φ . Furthermore, we also use the assumption that, among all elements of $x + S$ of maximal height, the height of px is maximal.

These two assumptions do not lose generality as these cosets are finite. However these assumptions essentially give us the proof. In order to prove Ulm's theorem in an uncountable case, we must find a context where we can regain these conditions.

The remainder of the proof of the countable case, given Lemma 3.9, is elementary but not enlightening. We re-present that result below, but first make some technical book-keeping notational definitions. (Fuchs in [4] codifies it in as its Lemma 77.1. That result extends a result due to Kaplansky and Mackey [11].) We present it below in a form that will be useful in the rest of this chapter, but with less than full generality.

First, some technical definitions.

Definition 22. Let Γ be a reduced abelian p -group with subgroup Φ . For each ordinal α , we define

$$\Phi(\alpha) = (p^{\alpha+1}\Gamma + \Phi) \cap (p^\alpha\Gamma)[p].$$

(This subgroup consists of the union of the trivial subgroup with the set of all elements of Γ of height α and order p which are not proper with respect to Φ .)

Now, the cardinal

$$f_{\Gamma, \Phi}(\alpha) = \dim_{\mathbb{F}_p}((p^\alpha\Gamma)[p]/\Phi(\alpha)),$$

is well-defined. This is called the α -th Ulm–Kaplansky invariant of Γ relative to Φ .

This definition of relative Ulm–Kaplansky invariant functions was first given by P. Hill in [7].

Each relative Ulm–Kaplansky invariant function is less than the Ulm invariant function of Γ , defined in Definition 16. That is to say, $f_{\Gamma, \Phi}(\alpha) \leq f_\Gamma(\alpha)$ for each ordinal α and subgroup Φ ; and $f_{\Gamma, 0} = f_\Gamma$.

The crucial result is this:

Lemma 3.9. Let Γ, Δ be reduced abelian p -groups with $\Phi \leq \Gamma$ and $\Xi \leq \Delta$ nice subgroups with $\mu : \Phi \rightarrow \Xi$ an isomorphism with

$$h_\Gamma(\mu f) = h_\Delta(\phi) \text{ for each } f \in \Phi.$$

(We shall say that such an isomorphism is height-preserving.)

Further, let

$$\nu_\alpha : (p^\alpha\Gamma)[p]/\Phi(\alpha) \rightarrow (p^\alpha\Delta)[p]/\Xi(\alpha)$$

be arbitrary monomorphisms for each α .

Suppose that

$$f_{\Gamma, \Phi}(\alpha) \leq f_{\Delta, \Xi}(\alpha)$$

for each α .

If $g \in \Gamma$ is proper with respect to Φ and $pg \in \Phi$, then μ can be extended to a height-preserving isomorphism

$$\mu' : \langle \Phi, g \rangle \rightarrow \langle \Xi, d \rangle,$$

(for some appropriate/suitable $d \in \Delta$) such that, for each α , ν_α maps $\langle \Phi, g \rangle / \Phi(\alpha)$ onto $\langle \Xi, d \rangle / \Xi(\alpha)$.

Furthermore, if we have equality of relative Ulm–Kaplansky invariants in the hypothesis, we have

$$f_{\Gamma, \langle \Phi, g \rangle}(\alpha) = f_{\Delta, \langle \Xi, d \rangle}(\alpha),$$

for each α .

(We do not need to have the ν_α specified, but it may be useful later to have notation to refer to them.)

As noted, this is a restatement of Lemma 77.1 and Corollary 77.2 of [4], where a proof can be found.

3.3.2 Nice Subgroups

Studying the details of Kaplansky’s proof of Ulm’s Theorem led P. Hill (in [7]) to discover a new class of subgroups, important to the structure of abelian groups.

Over this and the next sections, we introduce a series of definitions and results which build towards the main results of this chapter: Theorems 3.32 and 3.31 which characterise the maximal class of groups where Ulm’s Theorem holds.

Recall Definition 21.

Definition. Let Γ be an abelian p -group and Δ a subgroup of Γ .

We say that Δ is nice in Γ if every non-zero coset of Γ modulo Δ contains an element which is proper with respect to Δ . That is, for every $x + \Delta$ (with $x \in \Gamma \setminus \Delta$) we have some $k(x) \in \Delta$ such that

$$h_{\Gamma/\Delta}(x + \Delta) = h_{\Gamma}(x + k(x))$$

where indices refer to the group in which height is being calculated. (These indices are not strictly needed, as there is no danger of confusion, and we shall omit them from now on.)

It is clear that $\sigma\Gamma$ is nice in Γ , for any group Γ , ordinal σ

The following Lemma is almost a corollary to this definition:

Lemma 3.10. *A subgroup Δ of a p -group Γ is nice if and only if*

$$(p^\alpha\Gamma + \Delta)/\Delta = p^\alpha(\Gamma/\Delta)$$

for each ordinal α .

Proof. We always have $(p^\alpha\Gamma + \Delta)/\Delta \supseteq p^\alpha(\Gamma/\Delta)$.

Now, Δ is nice if and only if every coset of Γ/Δ -height α contains an element of Γ -height α , which is true if and only if $p^\alpha(\Gamma/\Delta)$ contains $(p^\alpha\Gamma + \Delta)/\Delta$. $\square$

We note some structural signifiers of nice subgroups

Lemma 3.11. *Let Γ be an abelian p -group.*

Then the following hold.

1. *Direct summands of Γ are nice subgroups.*
2. *Let $\Delta_i, i \in I$ be subgroups of abelian p -groups Γ_i . Now $\bigoplus_{i \in I} \Delta_i$ is nice in $\bigoplus_{i \in I} \Gamma_i$ if and only if each Δ_i is nice in Γ_i .*
3. *For each ordinal α , $p^\alpha\Gamma$ is nice in Γ .*
4. *If Δ, Λ are subgroups of Γ such that Δ is of finite index in Λ and nice in Γ , then Λ is nice in Γ .*
5. *Let $\Delta \leq \Lambda \leq \Gamma$. Then*

- (a) *if Λ is nice in Γ then Λ/Δ is nice in Γ/Δ ;*
- (b) *if Λ/Δ is nice in Γ/Δ and Δ is nice in Γ , then Λ is nice in Γ .*

Hence if Δ is a nice subgroup of Γ , then in the natural correspondence of subgroups of Γ/Δ and subgroups of Γ containing Δ , nice subgroups correspond to nice subgroups.

6. *If Δ is a nice subgroup of $p^\alpha\Gamma$, for any ordinal α , Δ is nice in Γ . If Λ is nice in Γ , then $\Lambda \cap p^\alpha\Gamma$ is nice in $p^\alpha\Gamma$.*

Each of these results is specifically needed for later proofs included in this thesis.

Proof. The first three statements are clear. The fourth follows as we have only a finite difference in heights and nice-ness is always present unless we have infinite differences.

The fifth is 79.3 of [4]: a proof can be found there.

For any abelian p -group Γ and ordinal α , we have, for each $x \in p^\alpha\Gamma$, $h_\Gamma(x) = \alpha + h_{\Gamma/p^\alpha\Gamma}(x)$. Hence nice subgroups of $p^\alpha\Gamma$ are nice in Γ . It follows from the identity in 5 that for Λ nice in Γ , $\Lambda \cap p^\alpha\Gamma$ is nice in $p^\alpha\Gamma$.

□

Example 3. Consider the group of Example 1, given by

$$\Gamma = \langle x_0, x_1, x_2, \dots, x_n, \dots \mid px_0 = 0, p^i x_i = x_0 \rangle.$$

The subgroup

$$\Xi = \langle x_1 - px_2, x_2 - px_3, \dots, x_n - px_{n+1}, \dots \rangle$$

is the kernel of the homomorphism $\phi : \Gamma \rightarrow \mathbb{Q}/\mathbb{Z}$ given by $x_n \mapsto p^{-1-n}$. The image of ϕ is isomorphic to the Prüfer quasicyclic group, which is divisible. Consequently,

$$p^\omega(\Gamma/\Xi) = \Gamma/\Xi > (\langle x_0 \rangle + \Xi)/\Xi = (p^\omega\Gamma + \Xi)/\Xi$$

and so, by the above Lemma 3.10, Ξ is not nice.

Niceness of a subgroup depends on how height behaves within the cosets of that subgroup: it is a quotient property. We shall see how to construct “nice systems” and “nice composition series” to ensure that this segment of the proof can be translated beyond the countable case.

The other important detail of Kaplansky’s proof that we constantly work with is “height-preserving isomorphisms” (i.e. isomorphisms between subgroups of overgroups which preserve the heights of the elements in their relative overgroups).

3.3.3 Isotype Subgroups

As we have seen, height is very important in the structure of abelian p -groups.

Recall we defined the concept of a pure subgroups, which are “(finite-)height-preserving subgroups”. We generalise the notion of a pure subgroup.

The results in this subsection are from [4, 80].

Definition 23. Let Γ be an abelian p -group and Δ a subgroup of Γ .

We say that Δ is an isotype subgroup of Γ if

$$p^\sigma \Gamma \cap \Delta = p^\sigma \Delta$$

for each ordinal σ .

This definition is due to Kulikov [14].

We look at some examples.

Example 4. Recall the group

$$\Gamma = \langle x_0, x_1, x_2, \dots, x_n, \dots \mid px_0 = 0, p^i x_i = x_0 \rangle,$$

introduced in Chapter 1, where it is Example 1. (We shall see in a few pages that this is isomorphic to the generalised Prüfer group $\Xi_{\omega+1}$.)

Clearly $p^\omega \Gamma = \langle x_0 \rangle$. However, any finite subgroup F which contains x_0 will have $p^\omega F = 0$ as finite p -groups have finite exponent. Hence any finite subgroup F which contains x_0 (and hence any x_i) cannot be isotype.

On the other hand $\langle x_1 - px_2 \rangle$, as a subgroup Δ isomorphic to C_p , with $\Delta \cap p\Gamma = \{0\}$ is isotype.

We note several conditions equivalent to being isotype.

Lemma 3.12. Let Γ be an abelian p -group with subgroup Δ . The following are equivalent:

1. Δ is an isotype subgroup of Γ ;
2. the short exact sequence

$$0 \longrightarrow \Delta \longrightarrow \Gamma \longrightarrow \Gamma/\Delta \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow p^\alpha \Delta \longrightarrow p^\alpha \Gamma \longrightarrow p^\alpha (\Gamma/\Delta)$$

for each ordinal α ;

3. the (generalised) height of each element of Δ in Δ is equal to its height in the overgroup Γ ;
4. $p^\alpha \Delta[p] = \Delta \cap p^\alpha \Gamma[p]$, for every ordinal α ;
5. $p^\alpha \Delta$ is pure in $p^\alpha \Gamma$, for every ordinal α .

This follows straightforwardly from observation: the third, fourth and fifth are clearly restatements of the definition of being isotype. The first and second are equivalent as any short exact sequence

$$0 \rightarrow \Delta \rightarrow \Gamma \rightarrow \Gamma/\Delta \rightarrow 0$$

induces an exact sequence

$$0 \rightarrow \Delta \cap p^\alpha \Gamma \rightarrow p^\alpha \Gamma \rightarrow p^\alpha(\Gamma/\Delta).$$

Being isotype is in many senses a transitive property.

Lemma 3.13. *Let Γ be an abelian p -group. Then*

1. an isotype subgroup Δ of Γ is isotype in every intermediate subgroup;
2. if $\Delta \leq \Lambda \leq \Gamma$ with Δ isotype in Λ and Λ isotype in Γ , then Δ is isotype in Γ ;
3. if Δ is an isotype subgroup of Γ , then, for each ordinal α , (the canonical image of) $\Delta/p^\alpha \Delta$ is isotype in $\Gamma/p^\alpha \Gamma$;
4. any union of an ascending chain of isotype subgroups of Γ is itself isotype;
5. if $\Delta \leq \Gamma$ is isotype, then $f_\Delta(\alpha) \leq f_\Gamma(\alpha)$ for each ordinal α .

Proof. Let $\Delta \leq \Lambda \leq \Gamma$ with Δ isotype in Γ . For each ordinal α ,

$$p^\alpha \Delta \leq p^\alpha \Lambda \cap \Delta \leq (p^\alpha \Gamma \cap \Lambda) \cap \Delta \leq p^\alpha \Gamma \cap \Delta = p^\alpha \Delta$$

and so Δ is isotype in Λ .

Suppose $\Delta \leq \Lambda \leq \Gamma$ with Δ isotype in Λ and Λ isotype in Γ . This gives $p^\alpha \Delta = p^\alpha \Lambda \cap \Delta = (p^\alpha \Gamma \cap \Lambda) \cap \Delta = p^\alpha \Gamma \cap \Delta$, proving the second statement.

Suppose Δ isotype in Γ . Then, by a clear argument as above, $\Delta/p^\alpha \Delta$ is isotype in $\Gamma/p^\alpha \Gamma$, for each ordinal α . But by the second isomorphism theorem we have a canonical

isomorphism taking $\Delta/p^\alpha\Delta = \Delta/(p^\alpha\Gamma \cap \Delta)$ to $(\Delta + p^\alpha\Gamma)/p^\alpha\Gamma$. Identifying this way, we see that this subgroup is still isotype and thus the third statement holds.

Let $(\Delta_\beta)_\beta$ be an ascending chain of isotype subgroups of Γ . Write Δ for the union of the Δ_β . Then for each ordinal α , if $x \in p^\alpha\Gamma \cap \Delta$, then $x \in \Delta_\gamma$, for some γ . But then $x \in p^\alpha\Delta_\gamma \leq p^\alpha\Delta$ and so Δ is isotype and the fourth statement holds.

The final statement follows at once from Lemma 3.12 (4) and the definition of Ulm–Kaplansky invariants. $\square$

3.3.4 Balanced Subgroups

We have just defined two subgroup properties. What can we say about those subgroups which have both?

The results in this subsection are from [4, 80]; we follow their proofs.

Definition 24. *A subgroup Δ of an abelian p -group Γ is said to be balanced if it is both nice and isotype.*

We note some immediate consequences.

Lemma 3.14. *Let Γ be an abelian p -group.*

1. *Direct summands of Γ are balanced.*
2. *If $\Delta \leq \Lambda \leq \Gamma$ with Δ balanced in Γ , then Δ is balanced in Λ .*
3. *If $\Delta \leq \Lambda \leq \Gamma$ with Λ balanced in Γ , then Λ/Δ is balanced in Γ/Δ .*
4. *If $\Delta \leq \Lambda \leq \Gamma$ with Δ balanced in Γ and Λ/Δ balanced in Γ/Δ , then Λ is balanced in Γ .*
5. *If $\Delta \leq \Lambda \leq \Gamma$ with Δ balanced in Λ and Λ balanced in Γ , then Δ is balanced in Γ .*

Proof. 1. This is very clear.

2. We showed in Lemma 3.13 1 that Δ is isotype in Λ . It remains to show Δ is nice in Λ .

Suppose there exists $a \in \Lambda$ such that $h_{\Lambda/\Delta}(a) = \rho$, for some limit ordinal ρ , and that, as an inductive hypothesis, all cosets in Λ/Δ of height less than ρ contain elements proper with respect to Δ . Then, for each $\sigma < \rho$ there is some $b_\sigma \in \Delta$ with

$a + b_\sigma \in p^\sigma \Delta$. If $b \in \Delta$ satisfies $a + b \in p^\rho \Delta$, then $b - b_\sigma \in \Delta \cap p^\sigma \Lambda = p^\sigma \Delta$. But then $a + b = (a + b_\sigma) + (b - b_\sigma) \in p^\sigma \Lambda$ and now $h_\Lambda(a + b) = \rho$, giving niceness by induction.

3. By Lemma 3.11 5 (a), Λ/Δ is nice. Suppose $x \in \Lambda$ such that $x + \Delta$ is in $p^\alpha(\Gamma/\Delta)$. There is some $y \in p^\alpha \Gamma$ and $z \in \Delta$ with $y + z = x \in \Lambda \cap p^\alpha \Gamma = p^\alpha \Lambda$. So $x + \Delta$ is in $p^\alpha \Lambda + \Delta \leq p^\alpha(\Lambda/\Delta)$ and so Λ/Δ is isotype in Γ/Δ .

4. Again by Lemma 3.11 5 (b), Λ/Δ is nice.

Whenever $x \in \Lambda \cap p^\alpha \Gamma$, its coset $x + \Delta \in \Lambda/\Delta \cap p^\alpha(\Gamma/\Delta) = p^\alpha(\Lambda/\Delta)$. This gives $x - y \in \Delta$, for some $y \in p^\alpha \Lambda$, and hence $x - y \in \Delta \cap p^\alpha \Gamma = p^\alpha \Delta$. Now, as y and $x - y \in p^\alpha \Lambda$, it follows that $x \in p^\alpha \Lambda$ and so Λ/Δ is isotype.

5. Lemma 3.13 2 assures us that Δ is isotype in Γ .

Suppose there exists $a \in \Gamma$ such that $h_{\Gamma/\Delta}(a + \Delta) = \rho$. If $a \in \Gamma \setminus \Delta$, then $h_{\Gamma/\Lambda}(a + \Lambda) \geq \rho$, with $h_\Gamma(a + b) = h_{\Gamma/\Lambda}(a + \Lambda)$, for some $b \in \Lambda$. It follows that $h_{\Gamma/\Delta}(a - b - a + \Delta) \geq \rho$, giving $h_\Gamma(-b + c) \geq \rho$ for some $c \in \Delta$. But then $h_\Gamma(a + b - b + c) = h_\Gamma(a + c) \geq \rho$. But this forces $h_\Gamma(a + c) = \rho$.

On the other hand if $a \in \Lambda$, then $h_{\Lambda/\Delta}(a + \Delta) = \rho$ and so, as Δ is nice in Λ , there is some $c \in \Delta$ with $h_{\Gamma/\Delta}(a + \Delta) = h_{\Lambda/\Delta}(a + \Delta) = h_\Lambda(a) = h_\Gamma(a) = \rho$. Hence Δ is balanced in Γ .

□

From this, it is clear that being balanced is a very strong property. Meeting these strong conditions determines much of the relationship between the subgroup and the rest of the structure of the group. For instance, being balanced is preserved in quotients and transitive in almost every sense possible.

We note some equivalent statements of the definition of balanced.

Lemma 3.15. *Let*

$$0 \longrightarrow \Delta \longrightarrow \Gamma \xrightarrow{\phi} \Lambda \longrightarrow 0$$

be a short exact sequence of abelian p -groups.

Then the following are equivalent:

1. Δ is balanced in Γ ;

2. for each ordinal α , the induced sequence

$$0 \longrightarrow p^\alpha \Delta \longrightarrow p^\alpha \Gamma \longrightarrow p^\alpha \Lambda \longrightarrow 0$$

is exact;

3. for each ordinal α , the induced sequence

$$0 \longrightarrow \Delta/p^\alpha \Delta \longrightarrow \Gamma/p^\alpha \Gamma \longrightarrow \Lambda/p^\alpha \Lambda \longrightarrow 0$$

is exact;

4. for each ordinal α , $\phi(p^\alpha \Gamma[p]) = p^\alpha \Lambda[p]$.

We omit the proof of this result and note that it is entirely a consequence of Lemma 4.12, which is proved independently later. Alternatively, it is proved as Proposition 80.2 in [4].

The following lemma allows us to make a series of strong statements about balanced subgroups. In [4], this is Lemma 80.3: it is crucial in [4]’s proof of Ulm’s Theorem. We will not use these results to prove Ulm’s Theorem in the larger case, but present these to show just how strong a condition “being balanced” is.

Lemma 3.16. *Let $\Delta, \Gamma, \Lambda, \Xi, \Phi$ be abelian p -groups such that*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & \Gamma & & \\ & & \downarrow \phi & & \downarrow \psi & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \Xi & \xrightarrow{\eta} & \Phi \longrightarrow 0 \end{array}$$

is commutative with exact rows, where Λ is a balanced subgroup of Ξ , and ϕ does not decrease height in Γ . If $g \in \Gamma$ is proper with respect to Δ and $pg \in \Delta$, we can extend ϕ to a map

$$\phi' : \langle \Delta, g \rangle \rightarrow \Xi$$

which also is not height-decreasing, such that $\eta\phi'g = \psi g$.

Proof. Write $\rho = h_\Gamma(g)$.

As η is surjective, there is some $x \in p^\rho \Xi$ with $\eta(x) = \psi(g)$. As the diagram commutes and ϕ is not height-decreasing, $\eta(px - \phi(pg)) = 0$ and so $px - \phi(pg) \in \Lambda \cap p^{\rho+1}\Xi$. As Λ is

balanced and so isotype, this is $p^{\rho+1}\Lambda$ and so there is some $y \in p^\rho\Lambda$ such that $py = px - \phi(pg)$.

We define ϕ' as $\phi'(g) = x - y$. This gives us a homomorphism $\langle \Delta, g \rangle \rightarrow \Xi$ such that $\eta(\phi'(g)) = \eta(x) = \psi(g)$. It follows that $h(x - y) \geq \rho = h(g)$.

To show ϕ' is not height-decreasing, we need only check that $h(g + a) \leq h(x - y + \phi a)$ for every $a \in \Delta$.

But Λ is balanced, so is nice and thus $h(x - y + \phi a) \geq \min\{h(x - y), h(\phi a)\} \geq \min\{h(g), h(a)\}$. Then g is proper with respect to Δ , so $\min\{h(g), h(a)\} = h(g + a)$, and we are done. $\square$

In particular, a repeated application of this result allows us to assemble powerful inductive proofs.

Corollary 3.17. 1. Let Γ be an abelian p -group with nice subgroup Θ of countable index. If

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Theta & \longrightarrow & \Gamma & & \\ & & \downarrow \phi & & \downarrow \psi & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \Xi & \xrightarrow{\eta} & \Phi \longrightarrow 0 \end{array}$$

is commutative with exact rows, Λ is balanced in Ξ and ϕ does not decrease height in Γ , we can extend ϕ to a homomorphism $\phi' : \Gamma \rightarrow \Xi$ with $\eta\phi' = \psi$.

2. A balanced subgroup of countable index in an abelian p -group is a direct summand. In particular, any balanced subgroup of a countable abelian p -group is a direct summand.

Proof. The first statement follows from Lemma 3.16: we apply this countably many times.

Specifically: we can enumerate elements of Γ/Θ as $\{x_1 + \Theta, x_2 + \Theta, \dots\}$. Writing $\Theta_{n+1} = \langle \Theta_n, x_{n+1} \rangle$, we have a series $\Theta = \Theta_0 \leq \Theta_1 \leq \Theta_2 \leq \dots$. Without loss of generality, we can choose the x_i such that each Θ_i is of index at most p in Θ_{i+1} . Each Θ_i is nice in Γ , by Lemma 3.11 (4). Inductively, we have for each n , the situation of Lemma 3.16, where Θ_n replaces Δ . Now, $\Gamma = \bigcup_n \Theta_n$, and so we get a map ϕ' from $\Gamma \rightarrow \Xi$ and the first statement follows.

The second statement follows from the first by taking Θ to be trivial, Γ to be Φ and ψ to be the identity on Γ . This gives a splitting of Φ as the product of Λ with its countable quotient Φ . $\square$

The attentive reader will have noticed that when talking of balanced subgroups, we generally cared about short exact sequences where the image of the embedding is a balanced subgroup.

This continues through what follows. We mainly care less about being balanced (which is a very strong property: in a countable universe, there is no example of a balanced subgroup which is not a direct summand) but about what it means for its quotient. As a result, many statements involving balanced subgroups are much more natural in the dual – we go from caring about the quotient Γ/Δ modulo a balanced subgroup to having $(\Gamma/\Delta)^*$ isomorphic to $\text{Ann}_{\Gamma^*}(\Delta)$, a (cobalanced) closed subgroup of Γ .

3.3.5 Nice Composition Series

Aiming to reconstruct the inductive frameworks used in Kaplansky's proof (see Section 3.3.1), we make the following definition.

Definition 25. *Let Γ be a p -group and*

$$0 = \Lambda_0 < \Lambda_1 < \dots < \Lambda_\alpha < \dots < \Lambda_\mu = \Gamma$$

be a well-ordered strictly ascending chain of subgroups of Γ such that:

1. $\Lambda_0 = 0$ and $\Lambda_\mu = \Gamma$;
2. for each ordinal α , Λ_α is a nice subgroup of Γ ;
3. for each α with $\alpha + 1 \leq \mu$, the quotient $\Lambda_{\alpha+1}/\Lambda_\alpha$ is cyclic of order p ;
4. for each limit ordinal δ , $\Lambda_\delta = \bigcup_{\beta < \delta} \Lambda_\beta$.

We call such a chain a nice composition series for Γ .

By the axiom of choice, any countable p -group will have such a series: it can be realised as an ascending chain of finite subgroups, which must be nice. This replicates the conditions outlined in subsection 3.3.1 before. This sets up a stronger version of Lemma 3.3.1.

Lemma 3.18. *Let Δ, Γ be reduced abelian p -groups and let μ be a height-preserving isomorphism between Φ , a nice subgroup of Γ , and Ξ , an arbitrary subgroup of Δ . Now, if Γ/Φ has a nice composition series and*

$$f_{\Gamma, \Phi}(\alpha) \leq f_{\Delta, \Xi}(\alpha), \text{ for each ordinal } \alpha,$$

then there is an extension of μ to some height-preserving monomorphism μ' of Γ into Δ .

(This is Theorem 81.2 of [4]: we follow their proof.)

Proof. This proof heavily depends on the axiom of choice.

First, we choose arbitrary monomorphisms $\nu_\alpha : (p^\alpha \Gamma)[p]/\Phi(\alpha) \rightarrow (p^\alpha \Delta)[p]/\Xi(\alpha)$, for every ordinal α .

We take a nice composition series between Φ and Γ :

$$\Phi = \Lambda_0 < \Lambda_1 < \cdots < \Lambda_\alpha < \cdots < \Lambda_\varepsilon = \Gamma.$$

Considering the set of pairs $(\Lambda_\alpha, \mu_\alpha)$ such that

- μ_α takes Λ_α isomorphically to some $\Theta_\alpha \leq \Delta$ and is height-preserving;
- μ_α restricted to Φ is equal to μ ;
- ν_β induces an isomorphism $\Lambda_\alpha(\beta)/\Phi(\beta) \rightarrow \Theta_\alpha(\beta)/\Xi(\beta)$, for every ordinal β .

There is a natural partial order induced by ordinals. By Zorn's Lemma, we can pick some $(\Lambda_\kappa, \nu_\kappa)$ maximal in this set.

The third condition ensures that $f_{\Gamma, \Lambda_\alpha}(\beta) \leq f_{\Delta, \Theta_\alpha}(\beta)$, for each ordinal β . Hence the first condition means that we are in the situation of Lemma 3.9 and hence μ_κ can be extended into a height-preserving isomorphism $\mu_{\kappa+1}$ of $\Lambda_{\kappa+1}$ to some subgroup $\Theta_{\kappa+1}$ of Δ , still satisfying the third condition above. This contradicts the maximality of $(\Lambda_\kappa, \nu_\kappa)$ unless $\kappa = \varepsilon$ and so $\Lambda_\kappa = \Gamma$. $\square$

This gives us an embedding result (81.3 of [4]).

Corollary 3.19. *A reduced abelian p -group Γ with nice composition series embeds in a p -group Δ as an isotype subgroup if and only if $f_\Gamma(\alpha) \leq f_\Delta(\alpha)$ for each ordinal α .*

Proof. By Lemma 3.13 (v), one direction is clear. Now apply Lemma 3.18 with $\Xi = \Phi = 0$ to get an embedding preserving generalised height. $\square$

This shows why Lemma 3.18 alone is not enough to complete the proof of Ulm's Theorem in groups with nice composition series. Although, by Lemma 3.18, two such groups with equal Ulm–Kaplansky invariant functions will embed in each other, these groups are not

co-Hopfian: they can have proper subgroups to which they are isomorphic. (For instance, $\bigoplus_n C_{p^n}$ is isomorphic to $p(\bigoplus_n C_{p^n})$, which is certainly proper.)

We present Lemma 3.18 above to show how a nice composition series allows us to extend homomorphisms inductively.

The following result (81.4 of [4]) is used to establish Lemma 3.27, Lemma 3.28 and Theorem 3.29.

Lemma 3.20. *Let Γ, Δ be reduced abelian p -groups and ϕ a homomorphism of a nice subgroup Λ of Γ into Δ which does not decrease heights. If Γ/Λ has a nice composition series, then ϕ can be extended to a homomorphism $\phi' : \Gamma \rightarrow \Delta$ which is also non-decreasing on heights.*

Note the intuitive similarity to Lemma 3.16 and its corollary: the proofs of these results are independent.

Proof. Consider the ϕ -induced height-preserving automorphism μ of $\Lambda \oplus \Delta$ given by $(l, d) \mapsto (l, d + \phi l)$. Now $\Lambda \oplus \Delta$ is nice in $\Gamma \oplus \Delta$ and this gives rise to an induced nice composition series of $(\Gamma \oplus \Delta)/(\Lambda \oplus \Delta)$.

Hence we are in the situation of Lemma 3.18 and so have a height-preserving isomorphism μ' of $\Gamma \oplus \Delta$ to itself. Now, the composite $\rho \circ \mu' \circ \pi$, where ρ is the natural injection of Γ into $\Gamma \oplus \Delta$ and π is the projection $\Gamma \oplus \Delta \rightarrow \Delta$, is the required map ϕ' . $\square$

We are working to present a characterisation of the largest class of abelian p -groups where Ulm's theorem holds. To this end, the following closure property is useful.

Theorem 3.21. *Let Γ be an abelian p -group.*

Then, for each ordinal α , Γ has a nice composition series if and only if $p^\alpha \Gamma$ and $\Gamma/p^\alpha \Gamma$ both have nice composition series.

Proof. Suppose Γ has a nice composition series

$$1 = \Lambda_0 < \dots < \Lambda_\sigma < \Lambda_{\sigma+1} < \dots < \Lambda_\mu = \Gamma.$$

For each α , there is some ordinal β maximal such that $p^\alpha \Gamma \leq \Lambda_\beta$.

The set

$$\{\Lambda_\delta/p^\alpha \Gamma \mid \delta \geq \beta\} \cup \{p^\alpha \Gamma\},$$

ordered by containment will be a nice composition series for $\Gamma/p^\alpha\Gamma$. This follows immediately from Lemma 3.11, 5 (a).

We claim that the set

$$\{p^\alpha\Gamma \cap \Lambda_\delta \mid \delta \leq \beta\},$$

ordered by containment, will be a nice composition series for $p^\alpha\Gamma$.

For each δ , $p^\alpha\Gamma \cap \Lambda_\delta$ is of index at most p in Λ_δ . Hence, by Lemma 3.11 6, this follows.

Conversely, suppose we have some ordinal α such that $p^\alpha\Gamma$ and $\Gamma/p^\alpha\Gamma$ have nice composition series. Then, we can write

$$1 = \Lambda_0 < \dots < \Lambda_\sigma < \Lambda_{\sigma+1} < \dots < \Lambda_\mu = p^\alpha\Gamma = \Delta_0 < \dots < \Delta_\sigma < \dots < \Delta_\kappa = \Gamma,$$

where the Λ_γ are a nice composition series for $p^\alpha\Gamma$ and the Δ_θ/Δ_0 form a nice composition series for $\Gamma/p^\alpha\Gamma$. By Lemma 3.11, parts 3, 5 (b) and 6, each of these subgroups is nice in Γ and hence this is a nice composition series for Γ . $\square$

3.3.6 Nice Systems

Thinking about the idea of a nice composition series leads to the following generalisation, first studied by P. Hill in [9].

Definition 26. *Let Γ be a p -group. If Γ has a system $\mathbf{\Lambda}$ of nice subgroups such that*

1. $\{0\} \in \mathbf{\Lambda}$;
2. for any subset $\{\Lambda_i\}_{i \in I}$ of $\mathbf{\Lambda}$, $\Sigma_i \Lambda_i \in \mathbf{\Lambda}$;
3. for any $\Lambda \in \mathbf{\Lambda}$ and countable $X \subseteq \Gamma$, there is some $\Delta \in \mathbf{\Lambda}$ with

$$\langle \Lambda, X \rangle \leq \Delta \text{ and } |\Delta : \Lambda| \leq \aleph_0,$$

we say that $\mathbf{\Lambda}$ is a nice system for Γ .

This is referred to by P. Hill as “the third axiom of countability”. (The first two axioms are countability and decomposition into a direct sum of countable groups, representing consecutive classes of groups for which Ulm’s theorem was shown to hold. This attempts to model the system of subgroups of a countable group.)

We have no real examples of this beyond those we have already seen: we shall introduce new examples, but first prove the major result of this chapter.

This class of groups has some well-behaved properties.

Lemma 3.22. (*P. Hill, [7]*)

The class of groups with nice systems is closed under direct sums and direct summands.

This is [4, 81.5]: a proof can be found there.

Theorem 3.23. *Ulm's Theorem*

Let Γ and Δ be reduced abelian p -groups with nice subgroups Φ and Ξ , respectively, such that Γ/Φ and Δ/Ξ have nice systems, and $f_{\Gamma,\Phi} = f_{\Delta,\Xi}$.

Then any height-preserving isomorphism $\mu : \Phi \rightarrow \Xi$ extends to an isomorphism $\Gamma \rightarrow \Delta$.

This proof, due to Walker [28], is similar to Hill's in the mythical [7], but uses Zorn's Lemma instead of transfinite induction.

Proof. First, for each ordinal α , we choose arbitrary monomorphisms $\nu_\alpha : (p^\alpha \Gamma)[p]/\Phi(\alpha) \rightarrow (p^\alpha \Delta)[p]/\Xi(\alpha)$.

Write $\mathbf{\Lambda}$ and $\mathbf{\Theta}$ for nice systems of Γ/Φ and Δ/Ξ , respectively. Write X for the set of all pairs (Λ, Θ) of subgroups $\Phi \leq \Lambda \leq \Gamma$ and $\Xi \leq \Theta \leq \Delta$, such that $\Lambda/\Phi \in \mathbf{\Lambda}$, $\Theta/\Xi \in \mathbf{\Theta}$. Define the following family of isomorphic extensions of μ :

$$\Psi = \{ \psi : \Lambda \rightarrow \Theta \text{ isomorphism} \mid (\Lambda, \Theta) \in X, \text{ each } \nu_\alpha \text{ induces iso'm } \Lambda(\alpha)/\Phi(\alpha) \rightarrow \Theta(\alpha)/\Xi(\alpha) \}.$$

There is an order on this set induced by the inclusion order on $\mathbf{\Lambda}$ or $\mathbf{\Theta}$. We can, by Zorn's lemma, pick a maximal $\psi_0 \in \Psi$, with $\psi_0 : \Lambda_0 \rightarrow \Theta_0$.

By definition, ν_α induces an isomorphism, and so we have $f_{\Lambda_0,\Phi} = f_{\Theta_0,\Xi}$. We can also deduce that each ν_α induces an isomorphism $(p^\alpha \Gamma)[p]/\Lambda(\alpha) \rightarrow (p^\alpha \Delta)[p]/\Theta(\alpha)$. By Lemma 3.11, 5(b), Λ, Θ are nice in Γ, Δ respectively.

Suppose $\Gamma \neq \Lambda$: take $g \in \Gamma \setminus \Lambda$. By Lemma 3.9, there is a height-preserving isomorphism $\psi'_0 : \Lambda_1 = \langle \Lambda, g \rangle \rightarrow \Theta_1$ extending ψ_0 , with the ν_α s inducing isomorphisms $\Lambda_1(\alpha)/\Lambda(\alpha) \rightarrow \Theta_1(\alpha)/\Theta(\alpha)$. By Lemma 3.11 4, Λ_1 and Θ_1 are nice subgroups of Γ and Δ respectively.

By the definition of nice systems, we can find some $\Lambda^{(1)}$ with $\Lambda^{(1)}/\Phi \in \mathbf{\Phi}$ and

$$\Lambda^{(1)} = \langle \Lambda_1, x_{1,1}, x_{1,2}, \dots, x_{1,n}, \dots \rangle.$$

Now, again using Lemma 3.9, we have a height-preserving isomorphism $\langle \Lambda_1, x_{1,1} \rangle \rightarrow \Theta_2$, satisfying the condition on ν_α , extending our previous isomorphism. Similarly, we can find $\Theta^{(2)}$ with $\Theta^{(2)}/\Xi \in \Phi$ and

$$\Theta^{(2)} = \langle \Theta_2, y_{2,1}, y_{2,2}, \dots, y_{2,n}, \dots \rangle.$$

From this, again using Lemma 3.9, we get a height-preserving isomorphism $\Lambda_2 \rightarrow \langle \Theta_2, y_{2,1} \rangle$ satisfying our condition.

We can repeat this: Λ_2 is of countable index in some

$$\Lambda^{(2)} = \langle \Lambda_2, x_{2,1}, x_{2,2}, \dots, x_{2,n}, \dots \rangle$$

coming from the nice system. We find, using Lemma 3.9, some height-preserving isomorphism $\langle \Lambda_2, x_{1,2}, x_{2,1} \rangle \rightarrow \Theta_3$ extending our previous one. Once again, we extend: Θ_3 is of countable index in some

$$\Theta^{(3)} = \langle \Theta_3, y_{3,1}, y_{3,2}, \dots, y_{3,n}, \dots \rangle$$

from the nice system. Again, using Lemma 3.9, extend to a height-preserving isomorphism $\Lambda_3 \rightarrow \langle \Theta_3, y_{2,2}, y_{3,1} \rangle$, satisfying the ν_α conditions.

None of these isomorphisms in this sequence will necessarily be isomorphisms between elements of the nice systems: however, by construction, we will get $\bigcup \Lambda^{(i)} = \bigcup \Lambda_i$. This has inductively constructed a height-preserving homomorphism $\bigcup \Lambda^{(i)} = \bigcup \Lambda_i \rightarrow \bigcup \Theta^{(i)} = \bigcup \Theta_i$, extending ψ_0 , with ν_α s inducing appropriate isomorphisms.

But, from our definition of a nice system, $(\bigcup \Lambda^{(i)})/\Phi \in \mathbf{\Lambda}$ and $(\bigcup \Theta^{(i)})/\Xi \in \mathbf{\Theta}$. This contradicts the maximality of ψ_0 and hence $\Lambda = \Gamma$.

An analogous argument shows that $\Theta = \Delta$ and hence our proof is complete. $\square$

This immediately gives us Ulm's Theorem for groups with nice systems.

Corollary 3.24. *Two reduced abelian p -groups with nice systems are isomorphic if and only if they have the same Ulm invariant function.*

Putting this to one side, we have a discursion on the structure of these groups and some proofs of alternative characterisations of this class to follow.

3.3.7 Generalised Prüfer Groups

We define an important family of groups with nice systems. These were first noted by [21].

Definition 27. *Generalised Prüfer Groups*

Set $\Xi_0 = \{0\}$. We recursively define a family of reduced p -groups Ξ_α , for arbitrary ordinal α , such that:

1. $p^\alpha \Xi_{\alpha+1}$ is cyclic of order p and $\Xi_{\alpha+1}/p^\alpha \Xi_{\alpha+1} \cong \Xi_\alpha$; and
2. $\Xi_\alpha = \bigoplus_{\beta < \alpha} \Xi_\beta$ whenever α is a limit ordinal.

We call these the generalised Prüfer groups.

(Note that the Prüfer (quasicyclic) group is not a generalised Prüfer group.)

It follows that each Ξ_α is of length α and that each Ulm–Kaplansky invariant of Ξ_α is at most $|\alpha|$.

Immediately: we can see that for each natural number n , Ξ_n is cyclic of order p^n . From this, we get $\Xi_\omega \cong \bigoplus_{n \in \mathbb{N}} C_{p^n}$.

A version of Definition 27 without the first clause of the second item would define families of groups. The definition as given, we shall go on to see defines a specific family: the family $\Upsilon_i = \Xi_i \times \Xi_i$ would also satisfy the abbreviated definition.

Lemma 3.25. *The generalised Prüfer groups exist.*

Proof. We proceed, unsurprisingly, by transfinite induction. For a limit ordinal δ , if Ξ_β is known for every $\beta < \delta$, by the definition we can construct

$$\Xi_\delta = \bigoplus_{\beta < \delta} \Xi_\beta.$$

Hence $\Xi_\omega = \bigoplus_{n \in \mathbb{N}} C_{p^n}$. (Note that this immediately implies $\Xi_{\omega+1}$ is the group of Example 2 and other previous Examples, the smallest infinite non-separable reduced p -group.)

Now, let $\alpha + 1$ be a successor ordinal and assume we have constructed Ξ_β for each $\beta < \alpha$. We have two cases.

Firstly, let α be a successor ordinal: we write $\alpha + 1 = \tau + 2$.

Recall Ξ_2, Ξ_1 are cyclic groups of order p^2 and p , respectively. Write ϕ for some arbitrary epimorphism $\Xi_2 \rightarrow \Xi_1$. The induced map $\hat{\phi} : \text{Ext}(\Xi_\tau, \Xi_2) \rightarrow \text{Ext}(\Xi_\tau, \Xi_1)$ is surjective and

thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Xi_2 & \dashrightarrow & \Delta & \dashrightarrow & \Xi_\tau \longrightarrow 0 \\
 & & \downarrow \phi & & \downarrow \psi & & \downarrow = \\
 0 & \longrightarrow & \Xi_1 & \xrightarrow{\psi} & \Xi_{\tau+1} & \longrightarrow & \Xi_\tau \longrightarrow 0
 \end{array}$$

where $\psi\Xi_1 = p^\tau\Xi_{\tau+1}$. Now, $|p^\tau\Delta|$ is at most p^2 , as $p^\tau\Xi_\tau = 0$. But if $|p^\tau\Delta| < p^2$, we would have $p^\tau\Xi_{\tau+1} = 0$, a contradiction. Hence $p^\tau\Delta$ is cyclic of order p^2 and so we define $\Xi_{\tau+2} = \Delta$ to complete the induction.

Finally, suppose α is a limit ordinal and that we know Ξ_β for each $\beta < \alpha$.

From above,

$$\Xi_\alpha = \bigoplus_{\beta < \alpha} (\Xi_\beta) \cong \bigoplus_{\beta < \alpha} (\Xi_{\beta+1}/p^\beta\Xi_{\beta+1}),$$

and each $p^\beta\Xi_{\beta+1}$ is cyclic of order p . Consider the so-called codiagonal map

$$\psi : \bigoplus_{\beta} p^\beta\Xi_{\beta+1} \rightarrow C_p$$

given by taking the sum over all co-ordinates. This produces the following commutative diagram, with pushout Λ :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \bigoplus p^\beta\Xi_{\beta+1} & \longrightarrow & \bigoplus \Xi_{\beta+1} & \longrightarrow & \Xi_\alpha \longrightarrow 0 \\
 & & \downarrow \psi & & \downarrow \psi & & \nearrow \\
 0 & \longrightarrow & \Xi_1 & \dashrightarrow & \Lambda & &
 \end{array}$$

where the direct sums are over $\beta < \alpha$. As ψ is an epimorphism, $\Xi_1 \leq p^\alpha\Lambda$ and so we have a short exact sequence

$$0 \longrightarrow \Xi_1 \longrightarrow \Lambda \longrightarrow \Xi_\alpha \longrightarrow 0$$

and so can define $\Xi_{\alpha+1}$ to be Λ . □

The groups constructed do not depend on the arbitrary epimorphisms used: the generalised Prüfer groups are unique to isomorphism. This follows from the following lemma and Theorem 3.23, upon considering the Ulm–Kaplansky invariants.

Lemma 3.26. *The generalised Prüfer groups have nice systems.*

Proof. This follows inductively. For α a limit ordinal, we noted that $\Xi_\alpha = \bigoplus_{\beta < \alpha} \Xi_\beta$. Hence,

by Lemma 3.22 it inherits a nice system from those of its summands.

Now, let Ξ_α have a nice system. We have an epimorphism $\Xi_{\alpha+1} \rightarrow \Xi_\alpha$, with nice kernel $p^\alpha \Xi_{\alpha+1}$. The pre-image of a nice system of Ξ_α in this map will form a nice system for $\Xi_{\alpha+1}$. $\square$

The concept behind the family of generalised Prüfer groups is an important one. Yet for any specific group, the property of “being a generalised Prüfer group” is not hugely important. As with the cyclic groups, we care more that we have a family of groups with similar behaviour than of the structure of any one specific member of the family. Most statements true of the family is true of any direct summands of a direct sum of Prüfer groups.

The Prüfer groups are in some sense an extension of the family of cyclic p -groups. The generalised Prüfer groups share an important injectivity property with the cyclic p -groups.

Lemma 3.27. *For any reduced abelian p -group Γ and any $\gamma \in \Gamma$ of height at least α and order at most p^n , there is a homomorphism*

$$\phi : \Xi_{\alpha+n} \rightarrow \Gamma \text{ such that } \xi\phi = \gamma$$

for ξ a generator of $p^\alpha \Xi_{\alpha+n}$.

This is due to Nunke [21].

Proof. Setting up the map $\xi \mapsto \gamma$ gives us a homomorphism $\psi : p^\alpha \Xi_{\alpha+n} \rightarrow \langle \gamma \rangle$, which does not decrease heights (relative to $\Xi_{\alpha+n}$ and to Γ). But $p^\alpha \Xi_{\alpha+n}$ is nice in $\Xi_{\alpha+n}$ and its quotient has a nice composition series. Hence, by Lemma 3.20, ψ will extend to a homomorphism ϕ , as required. $\square$

Lemma 3.28. *Let Γ be a reduced p -group of length τ . Then, there exists a short exact sequence*

$$0 \longrightarrow \Delta \longrightarrow \Xi \xrightarrow{\phi} \Gamma \longrightarrow 0$$

where Ξ is a direct sum of generalised Prüfer groups of length no more than τ with Δ balanced in Ξ .

Proof. Every non-zero element of Γ is contained in some $p^\alpha \Gamma[p^n]$. For each non-zero $\gamma_i \in$

$p^\alpha \Gamma[p^n]$, (where $\alpha + n < \tau$), pick a generalised Prüfer group $\Xi_i \cong \Xi_{\alpha+n}$ and a homomorphism $\phi_i : \Xi_i \rightarrow \Gamma$ satisfying the conditions of Lemma 3.27. Define Ξ to be $\bigoplus_i \Xi_i$ and an epimorphism $\phi : \Xi \rightarrow \Gamma$ given by $\phi(\sum_i \gamma_i) = \sum_i \phi_i(\gamma_i)$.

Now $\phi(p^\alpha \Xi[p]) = p^\alpha \Gamma[p]$ for each α and so Lemma 3.15 shows that Δ is balanced. $\square$

We introduce this since the dual of this result, Theorem 4.19, has a straightforward statement and had not previously appeared.

This also allows us our first example of a balanced subgroup that is not a direct summand. The torsion subgroup T of $\prod_n C_{p^n}$ has no elements of infinite height and is hence reduced. Lemma 3.28 thus gives us an embedding of T as a balanced subgroup of a direct sum of a generalised Prüfer groups of length ω : a direct sum of cyclic p -groups. But, we show in Chapter 7 that T has a proper basic subgroup and thus is not a direct sum of cyclic groups. Our next result implies that T cannot be a direct summand.

We need only one more definition before proving our major classification result.

Definition 28. *We say that an abelian p -group Γ has the projective property with respect to balanced-exact sequences if, whenever we have a diagram of abelian p -groups*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Delta & \longrightarrow & \Phi & \longrightarrow & \Lambda \longrightarrow 0 \\ & & & & & & \uparrow \psi \\ & & & & & & \Gamma \end{array}$$

with first row a balanced-exact sequence, then there is some map $\phi : \Gamma \rightarrow \Phi$ which makes the diagram commute.

3.3.8 Characterisations of Totally Projective Groups

Lemma 3.28 allows us to prove the following important intermediate result, following [4, 81.9].

Theorem 3.29. *Let Γ be a reduced abelian p -group. The following are equivalent:*

1. Γ has a nice system;
2. Γ has a nice composition series;
3. Γ has the projective property with respect to all balanced-exact sequences

$$0 \longrightarrow \Delta \longrightarrow \Phi \longrightarrow \Lambda \longrightarrow 0$$

of abelian p -groups;

4. Γ is a summand of a direct sum of generalised Prüfer groups.

Proof. From the definitions, (1) trivially implies (2).

Now, the Corollary to Lemma 3.16 shows that (2) implies (3). Suppose Γ has a nice composition series, (Γ_α) . Then, whenever we have a map $f : \Gamma \rightarrow \Lambda$ where $0 \rightarrow \Delta \rightarrow \Phi \xrightarrow{g} \Lambda \rightarrow 0$ is a balanced-exact sequence, we can define a map $h : \Gamma \rightarrow \Phi$ via transfinite recursion. We define a map h_α from Γ_α to Φ such that

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Gamma_\alpha & \hookrightarrow & \Gamma & & \\ & & \downarrow h_\alpha & & \downarrow f & & \\ 0 & \longrightarrow & \Delta & \longrightarrow & \Phi & \xrightarrow{g} & \Lambda \longrightarrow 0 \end{array}$$

commutes. As Γ_0 is trivial, h_0 is straightforward to define. Whenever we have h_α , Lemma 3.16 will give us $h_{\alpha+1}$, as Γ_α is of countable index in $\Gamma_{\alpha+1}$, and we can consider $f|_{\Gamma_{\alpha+1}}$. Whenever α is a limit ordinal, we define h_α on $\Gamma_\alpha = \bigcup_{\beta < \alpha} \Gamma_\beta$ as the $h_\beta(x)$ when $x \in \Gamma_\beta$.

Suppose (3). Then, Lemma 3.28 gives us a balanced-exact sequence

$$0 \longrightarrow \Delta \longrightarrow \Xi \xrightarrow{\varphi} \Gamma \longrightarrow 0$$

where Ξ is a direct sum of Generalised Prüfer groups. By the projective property, we have some map $\psi : \Gamma \rightarrow \Xi$ with $\phi\psi = 1_\Gamma$ and so the sequence splits. Hence (3) implies (4).

In Lemma 3.26 and Lemma 3.22, we have already shown that (4) implies (1). $\square$

For historical reasons, we present the definition of totally projective groups

Definition 29. Let α be an ordinal. An abelian p -group Γ is said to be p^α -projective if, for every abelian group Δ ,

$$p^\alpha \text{Ext}(\Gamma, \Delta) = 0.$$

We say that a reduced abelian p -group Γ is totally projective if

$$p^\alpha \text{Ext}(\Gamma/p^\alpha \Gamma, \Delta) = 0 \text{ for each ordinal } \alpha \text{ and abelian group } \Delta.$$

This class of groups was first studied by Nunke, in [21], from a homological viewpoint. He introduced the class of totally projective abelian p -groups and showed that the abelian p -groups of length at most ω_1 , the first uncountable ordinal, are precisely the direct sums of

countable p -groups. Due to Kolettis' result [15], work proceeded on attempting to establish a version of Ulm's Theorem for this class.

It was shown in [7] that this class of groups is precisely the class of groups with nice systems. In [2], Crawley and Hales introduced the class of simply presented groups (under the name “ T -groups”) and showed that this class coincides with the totally projective groups.

We shall never use this original definition of totally projective groups and instead use, without proof, the fact that the class of groups in Theorem 3.29 is the class of totally projective groups. We outline two key characterisations of this class and then collect these results.

Theorem 3.30. *The class of totally projective abelian p -groups is the smallest class of groups $\mathcal{C}$ such that*

- $\mathcal{C}$ contains C_p ,
- $\mathcal{C}$ is closed under taking direct sums and summands,
- for any abelian p -group Γ , $\Gamma \in \mathcal{C}$ if and only if $p^\alpha \Gamma, \Gamma/p^\alpha \Gamma \in \mathcal{C}$, for each ordinal α .

This is due to Parker and Walker, [22].

Proof. Our class of groups is that of direct summands of direct sums of generalised Prüfer groups. By the definition of the generalised Prüfer groups, each Ξ_α is in $\mathcal{C}$ by induction.

Hence the totally-projective groups are contained in $\mathcal{C}$. By Theorem 3.21, any group in $\mathcal{C}$ must have a nice composition series and so this is precisely our class. $\square$

As noted, the class of totally projective groups is maximal with respect to the property “its members are distinguished by Ulm invariants” [4, 83.7].

Theorem 3.31. *Let $\mathcal{C}$ be a group theoretic class of abelian p -groups such that*

1. $\mathcal{C}$ contains all totally projective p -groups;
2. $\mathcal{C}$ is closed under taking direct summands;
3. if Γ, Δ in $\mathcal{C}$ are non-isomorphic, then they have different Ulm–Kaplansky invariants.

Then $\mathcal{C}$ is the class of totally projective p -groups.

Proof. Firstly, Theorem 3.23 shows that the class of totally projective groups satisfies the third condition and so that $\mathcal{C}$ exists.

Now, suppose $\Gamma \in \mathcal{C}$ is of length τ . Write $\lambda = |\Gamma| \cdot \aleph_0$. Consider the group $\Xi = \bigoplus_{\lambda} (\bigoplus_{\alpha \leq \tau} \Xi_{\alpha})$. From item 4 of Definition 27, the Ulm–Kaplansky invariant $f_{\Xi}(\beta) = \lambda$ for all $\beta < \tau$. Hence $\Gamma \oplus \Xi \cong \Xi$. Hence Γ is a direct summand of the totally projective group Ξ and so is totally projective. $\square$

We collect the results on characterisation:

Theorem 3.32. *Let Γ be an abelian p -group.*

The following are equivalent:

1. Γ has a nice composition series;
2. Γ has a nice system;
3. Γ is a direct summand of a direct sum of generalised Prüfer groups;
4. Γ has the projective property relative to balanced-exact sequences of p -groups;
5. Γ is simply presented;
6. Γ is totally projective;
7. Γ is a member of the smallest class of groups which contains C_p , is closed under taking direct sums, and which contains a group Δ if, for some ordinal α , it contains both $p^{\alpha}\Delta$ and $\Delta/p^{\alpha}\Delta$;
8. Γ is in the class of groups $\mathcal{C}$ of Theorem 3.31: the class of groups which contains all totally projective groups, is closed under summands and in which Ulm’s Theorem holds.

Proof. Theorem 3.29 shows the equivalence of the first four conditions. The equivalence of the third and sixth conditions was first shown by Hill in [7]: a more accessible version is [4, 82.3]. That being simply presented is equivalent was first shown by Crawley and Hales in [2]. Theorems 3.30 and 3.31, prove that the eighth and seventh conditions, respectively, are equivalent to the sixth. $\square$

This is why the class of totally projective p -groups is of interest.

Recall the definition of an admissible function, Definition 18.

Definition. A function f from ordinals less than (a fixed ordinal) τ to cardinals is said to be τ -admissible (or just admissible) if and only if the following conditions hold

1.

$$\tau = \sup\{\sigma + 1 \mid f(\sigma) \neq 0\},$$

2.

$$\text{for each } \sigma \text{ with } \sigma + \omega < \tau \text{ we have } \sum_{\rho \geq \sigma + \omega} f(\rho) \leq \sum_{n < \omega} f(\sigma + n).$$

We say that τ is the length of f .

In the countable case, Zippin's Theorem showed a bijection between Ulm–Kaplansky invariant functions of reduced abelian p -groups and admissible functions. To finish this chapter, we give the equivalent result for all totally projective groups.

Theorem 3.33. (Crawley and Hales [2], Hill [7])

Let τ be a given ordinal. If Γ is a totally projective p -group of length τ then the Ulm function of Γ is an admissible function of length τ .

Theorem 3.33 shows that admissible functions of totally projective groups are admissible. The original result actually is an equivalence: see [4, 83.6]¹.

Proof. Let Γ be an arbitrary totally projective p -group of length τ .

It is clear to see that $\tau = \sup\{\sigma + 1 \mid f_\Gamma(\sigma) \neq 0\}$. To see that the second condition holds, we use induction on τ .

The base case, τ a finite ordinal, holds trivially. The case where τ is a limit ordinal is also straightforward: to move from one term in the Ulm sequence to another, we must have elements of unbounded finite height.

Let $\tau = \rho + n$ for some infinite limit ordinal ρ and natural number n . For all $\sigma < \rho$, $p^\sigma \Gamma / p^\rho \Gamma$ is infinite and, from the definition of the Ulm–Kaplansky invariant function,

$$|p^\rho \Gamma| \geq \sum_{\rho \leq \sigma < \tau} f_\Gamma(\sigma).$$

¹In fact, results in subsequent chapters (Lemma 4.8, Lemma 5.4 and Theorem 5.5) comprise a remarkably roundabout proof of the converse. Given an arbitrary sequence of groups, which is subject to the conditions codified in Definition 47, Theorem 5.5 constructs pro- p groups: Lemma 4.8 shows that a totally projective group has such an Ulm sequence if and only if has admissible Ulm–Kaplansky invariant function.

Hence, Pontryagin duality applied to Theorem 5.5 will, given an arbitrary sequence of reduced p -groups, subject to conditions which are shown to be necessary, construct an arbitrary totally projective p -group with this Ulm sequence.

For every $x \in p^\rho \Gamma$, the set $N_x = \{y \in p^\sigma \Gamma \mid \exists n \in \mathbb{N}, p^n y = x\}$ is infinite, as $p^\rho \Gamma \subseteq \text{ih}(p^\sigma \Gamma)$ and these groups are torsion. Now, $\bigcup_{x \in p^\rho \Gamma} N_x = p^\sigma \Gamma$ and each $y \in p^\sigma \Gamma$ is in only finitely many N_x . As $p^\sigma \Gamma / p^\rho \Gamma$ is infinite, this implies $|p^\sigma \Gamma / p^\rho \Gamma| \geq |p^\rho \Gamma|$ and so

$$|p^\sigma \Gamma / p^\rho \Gamma| \geq |p^\rho \Gamma| \geq \sum_{\rho \leq \sigma < \tau} f_\Gamma(\sigma).$$

Inductively, $f_{\Gamma/p^\rho \Gamma}$ is admissible. Now, the result holds as, for Γ an infinite reduced totally projective group, $|\Gamma| = \sum_\alpha f_\Gamma(\alpha)$, see [4, 83 (g)].

□

Chapter 4

Profinite Abelian Groups

The central idea of this chapter is to pull the material of the previous chapter through Pontryagin duality. This has previously been studied in [12], [17] and [13].

4.1 Basics

We look at effects of duality on the classification of countable abelian torsion groups. This recaps some material from [13] and uses proofs based heavily on that paper.

Firstly, from Theorem 3.1 and Pontryagin duality, we deduce the following

Corollary 4.1. *Let G be a profinite abelian group. Then we can find, for each prime p , a closed pro- p subgroup $G_{[p]}$ of G such that $G = \prod_p G_{[p]}$.*

More than this, it emerges that these are not dependant on the topological structure of G .

Proposition 4.2. *Let G be a profinite abelian group.*

Then, for each prime p , the unique p -Sylow subgroup $G_{[p]}$ of G is given by

$$\bigcap_{q \text{ prime, } q \neq p} q^\omega G.$$

Hence this is closed in any profinite topology and in each profinite topology we have $G = \prod_p G_{[p]}$.

In fact, the same result holds for pro-nilpotent groups, as they are the Cartesian products of their p -Sylow subgroups.

Proof. In any profinite group structure on G , $G_{[p]}$ is the image of the necessarily continuous map (see Proposition 2.2) from G to G given by $x \mapsto px$ and is hence closed. The result follows immediately from the fact that, for q prime, any pro- q group Q satisfies $q^\omega Q = 1$. $\square$

Recall that Pontryagin duality sends powers to heights (Theorem 1.9): with this, we can work towards constructing the dual of an Ulm sequence.

Theorem 4.3. *Let G be an abelian profinite group or a discrete abelian torsion group. Then,*

$$(\overline{t(G)})^* \cong G^* / \overline{\text{ih}(G^*)}$$

as topological groups, where $\text{ih}(G^*)$ denotes the set of elements of infinite height in G^* .

Recall we introduced the concept of infinite height as Definition 11, at the start of Chapter 3.

This is a strengthening of [23, 2.9.12].

Proof. By definition, $\text{Ann}_{G^*}(\overline{t(G)})$ is the minimal closed set containing

$$\bigcap_{n \in \mathbb{N}} \text{Ann}_{G^*}(G[n])$$

which, by Theorem 1.9, is equal to

$$\bigcap_{n \in \mathbb{N}} n(G^*)$$

which is of course the collection of all elements of infinite height in G^* . $\square$

From this we can see that torsion-free pro- p groups are dual to divisible p -groups (those where all elements are of infinite height). As we have shown in Theorem 3.2, divisible abelian groups are classified; a divisible abelian group is a direct sum of copies of the additive group of $\mathbb{Q}$ and of quasicyclic groups. From this, we can immediately derive the following.

Corollary 4.4. *Let $\phi : G \rightarrow H$ be a continuous homomorphism from an abelian pro- p group to a torsion-free Hausdorff group. Then there is some closed subgroup K of G isomorphic to $\prod_I \mathbb{Z}_p$ (isomorphic to the image of ϕ), for some index set I , with*

$$G = K \times \ker \phi.$$

This follows immediately from Pontryagin duality and the fact that a divisible subgroup of an abelian group is a direct summand. Hence all torsion-free quotients of abelian pro- p groups split and all torsion-free abelian pro- p groups are direct sums of copies of $\mathbb{Z}_p$.

This is stronger than [26, I.1, Exercise 1]. This corollary says that any torsion-free quotient of an abelian pro- p group has a complement and so splits as a direct factor. A divisible subgroup of any group must be contained in the finite residual. But in all profinite groups, this subgroup is trivial, by residual finiteness. Hence all profinite groups are reduced in the traditional sense of having trivial maximal divisible subgroup. We introduce a concept dual to reduction.

Definition 30. *We shall say a profinite group is dual-reduced if it has no non-trivial continuous torsion-free quotients: Theorem 4.3, shows this is equivalent to having a reduced dual.*

Note that we define “dual-reduced” to mean “having no torsion-free quotients”. This is important to remember.

Theorem 4.5. *Let G be a pro- p abelian group. Then*

$$G = G_0 \times F$$

with the closed-continuous direct summand $F \cong (\mathbb{Z}_p)^X$, for some cardinal X , where G_0 is the maximal dual-reduced closed subgroup.

As $\mathbb{Z}_p^*$ is isomorphic to the quasicyclic group C_{p^∞} , this follows from considering the dual of a discrete p -group, which must decompose as a sum of a reduced and (maximal) divisible part.

Definition 31. *Let G be an abelian pro- p group. We write $F(G)$ to denote a closed complement to the maximal dual-reduced subgroup of G . We write $r(F(G))$ to denote the (topological) rank of this group.*

It follows that $F(G) \cong \mathbb{Z}_p^{r(F(G))}$. The invariant $r(F(G))$ is an important one to be able to call upon.

We introduce a topologically characteristic series of subgroups of a profinite group.

Definition 32. *Let G be an arbitrary abelian pro- p group.*

Set $T_0(G)$ to be the trivial subgroup. Now, we recursively define

$$T_{\alpha+1}(G)/T_\alpha(G) = \overline{t(G/T_\alpha(G))}$$

for any ordinal α and

$$T_\delta(G) = \overline{\langle T_\alpha(G) \mid \alpha < \delta \rangle}$$

for δ any limit ordinal.

We call this chain the *torsion series* of G . As this chain is increasing there will be some ordinal τ such that $T_\tau(G) = T_{\tau+1}(G)$. We call the least such τ the *torsion type* of G .

We set G_{T_α} to be $T_{\alpha+1}(G)/T_\alpha(G)$. Now, we call the well-ordered transfinite sequence of quotients

$$G_{T_0}, G_{T_1}, \dots, G_{T_\alpha}, \dots$$

the *torsion sequence* of G .

Remark 4.1. The torsion type of a group is not always the supremum of the torsion types of its subgroups. Indeed, Section 6.1 shows that any countably based dual-reduced pro- p group is isomorphic to a closed subgroup of any countably based dual-reduced pro- p group which contains torsion of unbounded order. On the other hand, all continuous quotients of countably based dual-reduced pro- p groups will have torsion types bounded by that of the original group.

Remark 4.2. This sequence, by Theorem 4.3, is dual to the Ulm sequence of a discrete torsion group. That is, the α -th term of the Ulm sequence of G^* is isomorphic to the dual of G_{T_α} . As Ulm's Theorem says that the isomorphism type of a countable group is determined by its Ulm sequence, this gives a classification result.

The torsion sequence completely determines the topological-group structure of the dual-reduced part of a countably-based abelian pro- p group, as outlined in [13].

We shall say “the same torsion sequence” to mean that G and H are of the same torsion type and that $G_{T_\alpha} \cong H_{T_\alpha}$ for each α .

Theorem 4.6. *Let G and H be countably based abelian dual-reduced pro- p groups with the same torsion sequence. Then G and H are isomorphic (as topological groups).*

Proof. The dual groups G^* and H^* are countable abelian p -groups. Repeated applications of Theorem 4.3, shows that the dual of the torsion sequence of a pro- p group is isomorphic

to the Ulm sequence of its dual. Hence, by Ulm's Theorem, Theorem 3.7, $G^* \cong H^*$. Now, by Pontryagin duality G and H are topologically isomorphic. $\square$

This uses only the fact that two countable reduced abelian p -groups with isomorphic Ulm sequences are isomorphic, i.e. that Ulm's Theorem holds for countable abelian p -groups. As Ulm's Theorem also holds for the larger class of totally projective groups, the same results of [13] hold in a larger class of pro- p groups. We shall investigate this in 4.2.

4.2 Subgroup Structure of Profinite Abelian Groups

4.2.1 Pro-Ulm Theory

Following the notion of the torsion sequence given in Definition 32, dual to Ulm sequence, we define the dual notion to generalised height.

Definition 33. *Let G be a pro- p group.*

We define a chain of subgroups, $G[p^\alpha]$. We set $G[1]$ to be the trivial subgroup and define

- 1. for arbitrary ordinal α , $G[p^{\alpha+1}] = \{x \in G \mid px \in G[p^\alpha]\};$*
- 2. for limit ordinal β , $G[p^\beta]$ to be the closure of $\bigcup_{\delta < \beta} G[p^\delta]$.*

For $\alpha < \omega$, $G[p^\alpha]$ is, as usual, the elements of G of order at most p^α .

For a non-trivial g in G , we shall define the (generalised) order of g to be the ordinal γ such that

$$g \in G[p^\gamma] \setminus G[p^{\gamma+1}],$$

if such an ordinal exists, ∞ otherwise. As usual, the order of the identity is set to be $p^0 = 1$.

There is some τ at which this chain stabilises: we call such τ the generalised exponent of G .

(Note that the sequence $(0, G[p^\beta])$, where β runs through all limit ordinals exactly coincides with that of the torsion series. From this, we can see that for $x \in G$, $o_G(x) = \infty$ if and only if x is not contained in any term of the torsion series of G , i.e. $f(x) \neq 0$ for some continuous $f : G \rightarrow \mathbb{Z}_p$.)

This is exactly dual to generalised height.

We have a strengthening of Theorem 1.10.

Corollary 4.7. *Let G be an abelian profinite group.*

Then, for each α ,

$$\text{Ann}_{G^*}(G[p^\alpha]) = p^\alpha(G^*).$$

We now dualize the material on subgroup structure from Section 3.3. This was first studied in [12].

Note that, in the same way that the height of an element depends on the group we are in, the generalised order of an element is calculated relative to our group G . If infinite, the element can have different order relative to other subgroups.

For example, $\prod_{i \in \mathbb{N}} C_{p^i}$ is the closure of its torsion subgroup. Hence the order of any infinite order element x will be p^ω . On the other hand, $\overline{\langle x \rangle}[p^n]$ is trivial for each $n \in \mathbb{N}$ and so $o_{\overline{\langle x \rangle}}(x) = \infty$.

When we have G a pro- p group with closed subgroup H , for each $x \in H$, $o_G(x) \leq o_H(x)$. Hence, we have

$$o_G(x) = \min \{ o_K(x) \mid K \leq_C G, K \ni x \}.$$

This leads us to the function dual to the Ulm function.

Definition 34. *Let G be an abelian pro- p group.*

The pro- p Ulm invariants of G are the values of the function

$$f_G : \delta \mapsto d \left(\frac{G[p^\delta]G^p}{G[p^{\delta+1}]G^p} \right)$$

where $d(K)$ denotes the minimal cardinality of a topological generating set for a topological group K .

Lemma 4.8. *Let G be an abelian pro- p group.*

Then the pro-Ulm function f_G is equal to f_{G^*} , the abstract Ulm–Kaplansky invariant function of its dual.

Proof. Let G be an abelian pro- p group.

Then, for every ordinal δ we have

$$f_G(\delta) = d \left(\frac{G[p^\delta]G^p}{G[p^{\delta+1}]G^p} \right) = \dim_{\mathbb{F}_p} \left(\left(\frac{G[p^\delta]G^p}{G[p^{\delta+1}]G^p} \right)^* \right) = \dim_{\mathbb{F}_p} \left(\frac{\text{Ann}_{G^*}(G[p^\delta]G^p)}{\text{Ann}_{G^*}(G[p^{\delta+1}]G^p)} \right)$$

$$= \dim_{\mathbb{F}_p} \left(\frac{\text{Ann}_{G^*}(G[p^\delta]) \cap \text{Ann}_{G^*}(G^p)}{\text{Ann}_{G^*}(G[p^{\delta+1}]) \cap \text{Ann}_{G^*}(G^p)} \right) = \dim_{\mathbb{F}_p} \left(\frac{p^\delta G^* \cap G^*[p]}{p^{\delta+1} G^* \cap G^*[p]} \right) = f_{G^*}(\delta).$$

□

Attempting to dualise the proof of Ulm's Theorem outlined in Subsection 3.3.1 is not straightforward. Instead of enumerating the countable elements of countable group, we would have to enumerate the countable collection of continuous homomorphisms to finite groups. This is the equivalent of enumerating open subgroups.

The body of the proof, through duality, comes down to the following. We have a list of open subgroups $N_1, N_2, \dots$ of G and a list of open subgroups $M_1, M_2, \dots$ of H . We then have to construct a succession of quotients of G and H which are isomorphic to each other in an “order-preserving” way.

As in the discrete case, we alternate: we have an open subgroup N of G and an open subgroup M of H , with a (G, H) -structure-preserving isomorphism from $G/N \rightarrow H/M$.

At the k -th step, we find a suitable pull-back of this map from G/N to $G/(N \cap N_k)$ to some isomorphic quotient of H which factors through M . Write H/L for this quotient. We reverse the new isomorphism and extend it from $H/(L \cap M_k)$ to some quotient of G .

This is harder to deal with for a few reasons. Tracking quotients is harder than tracking elements. Elements also have convenient operations on them: the group operation, multiplication by p , etc. It is hard to construct similar operation on quotients without getting to a point where one is working on the dual in all but name.

I do not give full details of such a proof here as I have none which structurally adds anything. Instead, we keep this in mind as we continue dualising Chapter 3.

4.2.2 Smart Subgroups

The material in the next three sections is based on dualising the previous chapter in a straightforward way. This was first done, to my knowledge, by Kiefer in [12]. Subsequently, Loth slightly expands on Kiefer's results in [17].

We make the same definitions, but dualise some results that had not previously been dualised.

Definition 35. *Let G be an abelian pro- p group, with closed subgroup H .*

We say that H is smart in G if and only if, for each ordinal α ,

$$H[p^\alpha] = G[p^\alpha] \cap H.$$

Smart subgroups are dual to nice subgroups. These were first studied and named in [12].

Lemma 4.9. *Let G be an abelian pro- p group, with closed subgroup H .*

Then H is smart in G if and only if $\text{Ann}_{G^*}(H)$ is nice in G^* .

Proof. By definition H is smart in G if and only if $\text{Ann}_{G^*}(H[p^\alpha]) = \text{Ann}_{G^*}(G[p^\alpha] \cap H)$ for each ordinal α . But by Lemma 1.9, $\text{Ann}_{G^*}(H[p^\alpha])$ must be the subgroup of G^* naturally corresponding to $p^\alpha(G^*/\text{Ann}_{G^*}(H))$ in $G^*/\text{Ann}_{G^*}(H)$.

Now, by Theorem 1.10, $\text{Ann}_{G^*}(G[p^\alpha] \cap H) = p^\alpha G^* + \text{Ann}_{G^*}(H)$. Consequently,

$$G[p^\alpha] \cap H = H[p^\alpha],$$

if and only if

$$(p^\alpha G^* + \text{Ann}_{G^*}(H))/\text{Ann}_{G^*}(H) = p^\alpha(G^*/\text{Ann}_{G^*}(H)).$$

This holds for each ordinal α and so Lemma 3.10 concludes the proof. $\square$

Lemma 4.10. *Let G be an abelian pro- p group.*

Then the following hold.

1. *Closed-continuous direct summands of G are smart subgroups.*
2. *For each ordinal α , $G[p^\alpha]$ is smart in G .*
3. *Let $K \leq H \leq G$. Then*

- (a) *if K is smart in G then K is smart in H ;*
- (b) *if K is smart in H and H is smart in G , then K is smart in G .*

This is new.

Proof. The first result is clear.

The second follows from the fact that the $G[p^\alpha]$ are a chain.

If K is smart in G , then $K[p^\alpha] = K \cap G[p^\alpha] = K \cap H[p^\alpha]$ as $H[p^\alpha]$ contains $K[p^\alpha]$ and so K is smart in H .

Let K be smart in H and H be smart in G . Then, $K[p^\alpha] = K \cap H[p^\alpha] = K \cap (H \cap G[p^\alpha]) = K \cap G[p^\alpha]$ and so the result holds. $\square$

In the discrete torsion case, isotype subgroups are somehow more simply defined than nice subgroups. But as we work in the mirror universe of the dual, the definition of isotype gets more complicated.

4.2.3 Duotype Subgroups

Definition 36. Let G be an abelian pro- p group with closed subgroup H .

We say that H is duotype in G if, for each ordinal α ,

$$(G/H)[p^\alpha] = (G[p^\alpha] + H)/H.$$

These were first studied and named in [12].

Proposition 4.11. Let G be an abelian pro- p group with closed subgroup H . The following are equivalent.

1. H is duotype in G ;
2. $\text{Ann}_{G^*}(H)$ is isotype in G^* ;
3. each non-trivial coset $x + H$ contains an element $x + h$ such that $o(x + h)$ in G is equal to $o(x + H)$ in G/H ;
4. for each non-trivial coset $x + H$, $\min\{o(x + h) \mid h \in H\} = o(x + H)$.

This is based on [17, 5.8], following [12].

Proof. We shall prove that (1) is equivalent to (2).

Write $\Gamma = G^*$ and $\Delta = \text{Ann}_{G^*}(H) \leq \Gamma$. We have, by Theorem 1.4, $\Gamma^* \cong G$ and $(\Gamma/\Delta)^* \cong H$. Under this identification $\text{Ann}_G(\Delta)$ corresponds to H : in a mild abuse of notation they are identified for the purpose of this proof.

We now essentially follow the proof of Lemma 4.9, only working on the opposite side of the duality.

From the definition of the annihilator, it is easy to see that, in our canonical identification of Δ^* and G/H , a subgroup K of G containing H is sent to $\text{Ann}_{G^*}(K)/\text{Ann}_{G^*}(H)$.

By definition Δ is isotype in Γ if and only if $p^\alpha \Delta = \Delta \cap p^\alpha \Gamma$ for each ordinal α . But by Lemma 1.9, $p^\alpha \Delta$ must be the subgroup of Δ naturally corresponding to $(G/H)[p^\alpha]$ in the canonical identification of Δ and $(G/H)^*$.

Now, by Theorem 1.10, $\Delta \cap p^\alpha \Gamma = \text{Ann}_\Gamma(G[p^\alpha] + H)$. Consequently,

$$(G/H)[p^\alpha] = (G[p^\alpha] + H)/H,$$

if and only if

$$p^\alpha \Delta = \Delta \cap p^\alpha \Gamma,$$

for an ordinal α and hence (1) is equivalent to (2).

The third and fourth statements are also equivalent to the dual of Lemma 3.12 3. The specific statement of these is from [17].

Firstly, recall that $\text{Ann}_{G^*}(H)$ is isotype in G^* if and only if $p^\alpha \text{Ann}_{G^*}(H) = p^\alpha G^* \cap \text{Ann}_{G^*}(H)$. We use the fact that the duality functor turns the $p^\alpha X$ functor to the $X[p^\alpha]$ functor. The left hand side of this expression is dual to $(G/H)[p^\alpha]$.

Recall that $(G/H)[p^\alpha]$ is the set of elements in G/H which are of order dividing p^α . Meanwhile, $(G[p^\alpha] + H)/H$ is the image of $G[p^\alpha]$ in the canonical homomorphism $G \mapsto G/H$.

Any continuous homomorphism f is non-increasing on generalised orders, as $p^\alpha f(x)$ is trivial whenever $p^\alpha x$ is trivial. Hence, for each α , $(G/H)[p^\alpha] \supseteq (G[p^\alpha] + H)/H$. We have equality for each α if and only if (3) holds. Hence (3) is equivalent to (1).

Also, as any homomorphism is non-increasing on orders, (3) is equivalent to (4).

□

Example 5. 1. Let $G = \prod_i C_{p^i}$. For any $x \in G \setminus (t(G) + pG)$, the subgroup $\overline{\langle x \rangle}$ is duotype, but not smart.

2. If G is a dual-reduced countably-based pro- p group, with $G[p^\omega]$ a proper subgroup of G , $G[p^\omega]$ is smart but not duotype.

The first example can be seen to be duotype from inspection: it is not smart as $o_G(x) = p^\omega$, because x is contained in $G = G[p^\omega]$, but $o_{\overline{\langle x \rangle}}(x) = \infty$. The second is smart from Lemma 4.10 2 and cannot be duotype, by the definition of the torsion sequence. That is, for dual-reduced G , $(G/G[p^\omega])[p^\omega]$ is equal to $(G[p^\omega] + G[p^\omega])/G[p^\omega]$ only if $G = G[p^\omega]$.

4.2.4 Cobalanced Subgroups

Definition 37. We say that a closed subgroup H of an abelian pro- p group G is cobalanced if it is both smart and duotype.

Being cobalanced is, unsurprisingly, dual to being balanced.

As in the discrete case, this is a strong condition which is equivalent to powerful statements.

Lemma 4.12. *Let*

$$0 \longrightarrow H \longrightarrow G \xrightarrow{\phi} K \longrightarrow 0$$

be a short exact sequence of abelian pro- p groups. (i.e. the maps are continuous.)

Then the following are equivalent:

1. H is cobalanced in G ;
2. $\text{Ann}_{G^*}(H)$ is balanced in G^* ;
3. for each ordinal α , the induced sequence

$$0 \longrightarrow H/H[p^\alpha] \longrightarrow G/G[p^\alpha] \longrightarrow K/K[p^\alpha] \longrightarrow 0$$

is exact;

4. for each ordinal α , the induced sequence

$$0 \longrightarrow H[p^\alpha] \longrightarrow G[p^\alpha] \longrightarrow K[p^\alpha] \longrightarrow 0$$

is exact;

5. for each ordinal α , $pH + H[p^\alpha] = H \cap (pG + G[p^\alpha])$;
6. for each ordinal α , $H[p^\alpha] = H \cap G[p^\alpha]$ and $\text{Ann}_{G^*}(H) \cap p^\alpha G^* = p^\alpha \text{Ann}_{G^*}(H)$.

The equivalence of the first two were noted by Loth and Kiefer. The sixth statement, particularly, is a nice display of the symmetry of the definitions of nice/smart and iso-type/duotype.

Proof. “Being a balanced subgroup” is dual to “being a cobalanced subgroup” as we have shown that “smart” and “duotype” are dual to “nice” and “isotype”, respectively. Hence (1) if and only if (2), by Proposition 4.11 and Lemma 4.9.

Condition (3) is equivalent to the statement: for each ordinal α , $H[p^\alpha] = G[p^\alpha] \cap H$, and $(G/H)[p^\alpha] = (G[p^\alpha] + H)/H$. That is to say, H is smart and duotype in G .

As to (3) and (4), being equivalent, this is elementary group theory which has arisen before, in the dual context, in Lemma 3.15, conditions 2 and 3.

We shall prove (2) and (5) are equivalent. Write, as in the proof of Lemma 4.9, Γ for G^* and Δ for $\text{Ann}_{G^*}(H)$. In a minor abuse of notation, we consider Γ^* to be G and $\text{Ann}_G(\Gamma)$ to be H . Similarly, we can consider H^* to be Γ/Δ and Δ^* to be G/H .

Now,

$$H \cap (pG + G[p^\alpha]) = \text{Ann}_G(\Delta) \cap \text{Ann}_G(p^\alpha \Gamma \cap \Gamma[p]) = \text{Ann}_G(\Delta + (p^\alpha \Gamma \cap \Gamma[p])),$$

by Theorem 1.10. Similarly $pH + H[p^\alpha] = \text{Ann}_H(p^\alpha(\Gamma/\Delta) \cap (\Gamma/\Delta)[p])$. Hence (5) is equivalent to

$$p^\alpha(\Gamma/\Delta) \cap (\Gamma/\Delta)[p] = (\Delta + (p^\alpha \Gamma \cap \Gamma[p]))/\Delta.$$

This is shown to be equivalent in [4, 80.2].

The equivalence of (1) and (6) follows from Lemma 4.11. □

Note that the proof here does not use Lemma 3.15, but implies it. The proof here is substantially simpler and more elementary than in the discrete case: as noted earlier, it does seem that “being balanced/cobalanced” is more natural to consider in the compact than discrete case.

Again, this strong property respects a lot of transitivity properties.

Lemma 4.13. *Let G be an abelian pro- p group, with closed subgroups H, K such that $K \leq H$. Then*

1. *closed direct summands of G are cobalanced;*
2. *if H is cobalanced in G , then H/K is cobalanced in G/K ;*
3. *if K is cobalanced in G , then K is cobalanced in H ;*
4. *if K is cobalanced in H and H is cobalanced in G , then K is cobalanced in G ;*

5. if K is cobalanced in G and H/K is cobalanced in G/K , then H is cobalanced in G ;

This has not to my knowledge appeared before.

Proof. If we have an abelian pro- p group G with closed subgroups H, K with $H \geq K$, we have the following situation.

The chain $1 \leq K \leq H \leq G$ dualises to $G^* = \Gamma \geq \Lambda \geq \Delta \geq 0$, with

$$\text{Ann}_{G^*}(G) = 0, \text{Ann}_{G^*}(H) = \Delta, \text{Ann}_{G^*}(K) = \Lambda, \text{Ann}_{G^*}(1) = \Gamma.$$

We prove this lemma by noting the following equivalences

- A) H is cobalanced in G if and only if Δ is balanced in Γ ;
- B) K is cobalanced in G if and only if Λ is balanced in Γ ;
- C) K is cobalanced in H if and only if Λ/Δ is balanced in Γ/Δ ;
- D) H/K is cobalanced in G/K if and only if Δ is balanced in Λ .

The above statements are true if we replace (cobalanced, balanced) with (smart, nice) or (duotype, isotype).

The first two we already know. We have K cobalanced in H if and only if $\text{Ann}_{H^*}(K)$ is balanced in $\text{Ann}_{H^*}(0) \cong H^* \cong \Gamma/\Delta$, by canonical isomorphisms, thanks to Theorem 1.6.

But $\text{Ann}_{H^*}(K)$ is the set of continuous homomorphisms from H to the circle group, which annihilates K : this is identified canonically with $\text{Ann}_{(H/K)^*}(0) \cong (H/K)^*$. But, again thanks to Theorem 1.6, we have $\Gamma/\Lambda \cong K^* \cong H^*/\text{Ann}_{H^*}(K)$. Now, by the isomorphism theorems, it follows that we have the canonical identification $(H/K)^* \cong \Lambda/\Delta$. Hence K is cobalanced in H if and only if Λ/Δ is balanced in Γ/Δ .

The last statement follows by the same argument, under the action of the permutation $(G\Gamma)(H\Lambda)(K\Delta)$.

With these equivalences, the lemma follows immediately from Lemma 3.14. $\square$

Being cobalanced is a very strong condition: every cobalanced subgroup of a countably-based group is a closed-continuous direct summand. This allows us to extend homomorphisms in a way that facilitates induction.

Theorem 4.14. *If H is a cobalanced subgroup of an abelian pro- p group G with G/H countably-based, H is a closed-continuous direct summand. In particular, any cobalanced subgroup of a countably-based abelian pro- p group is a closed-continuous direct summand.*

Proof. This follows immediately from the previous result and Corollary 3.17, recalling that a profinite abelian group is countably based if and only if it is an inverse limit of a countable collection of finite groups. This is equivalent to its dual being a direct limit of a countable collection of finite groups, i.e. countable. $\square$

4.3 Classifying Totally Injective Groups

The central aim of this section is Theorem 4.22, which is dual to Theorem 3.32.

We proceed by giving the definitions that will be used in it. All of these definitions originated in [12].

As we generalise a composition series to mimic Kaplansky's proof and get an (ascending) nice composition series, the dual structure has a descending smart composition series.

Definition 38. *Let G be an abelian pro- p group with a well-ordered strictly descending chain of normal subgroups of G*

$$G = K_0 > K_1 > K_2 > \cdots K_\alpha > \cdots > K_\mu$$

be a such that:

1. $K_0 = G$ and $K_\mu = 0$;
2. for each ordinal α , K_α is a smart subgroup of G ;
3. for each α with $\alpha + 1 \leq \mu$, the quotient $K_\alpha/K_{\alpha+1}$ is cyclic of order p ;
4. for each limit ordinal δ , $K_\delta = \bigcap_{\beta < \delta} K_\beta$.

We call such a chain a smart composition series for G .

(As with the Generalised Prüfer Groups, we could give an abbreviated definition, but such a definition would lose the uniqueness of the family.)

Lemma 4.15. *Let G be an abelian pro- p group. Then G has a smart composition series if and only if G^* has a nice composition series*

Proof. By Lemma 4.9, it follows that if G has a smart composition series, (K_α) , then $(\text{Ann}_{G^*}(K_\alpha))$ is a nice composition series for G^* and vice versa. $\square$

Definition 39. Let G be an abelian pro- p group. If G has a system $\mathbf{K}$ of closed smart subgroups such that

1. $G \in \mathbf{K}$;
2. for any subset $\{K_i\}_{i \in I}$ of $\mathbf{K}$, $\bigcap_{i \in I} K_i \in \mathbf{K}$;
3. for any $K \in \mathbf{K}$ and closed $M \leq G$ with G/M countably-based, there is some $N \in \mathbf{K}$ with $N \leq M$ and G/N countably based,

we say that G has a smart system.

We could consider this “the third axiom of being countably-based”.

Lemma 4.16. An abelian pro- p group G has a smart system if and only if G^* has a nice system.

Proof. It is clear to see that G has a smart system $\mathbf{K}$ if and only if $\Theta = \{\text{Ann}_{G^*}(K) \mid K \in \mathbf{K}\}$ is a nice system for G^* . $\square$

Being simply presented is probably the most concrete and understandable of the equivalent conditions for a group to be totally projective. Unfortunately, this does not translate well to the profinite case.

Definition 40. Let G be an abelian group.

We say that G is simply given if there is some prime p , some index set I , which is the disjoint union of J and K , such that G is topologically isomorphic to some group C , of form

$$C = \{(x_i)_I \in \prod_I \mathbb{R}/\mathbb{Z} \mid px_i = 0 \text{ if } i \in J \text{ and } px_i = x_{f(i)} \text{ if } i \in K\},$$

for some map $f : K \rightarrow I$. (The group $\prod_I \mathbb{R}/\mathbb{Z}$ is equipped with the product topology, where each copy of $\mathbb{R}/\mathbb{Z}$ has its natural topology.)

Note that being simply given does not guarantee being pro- p . A group of the above form will be pro- p if and only if iterating f annihilates K . That is, if $\bigcap_{n \in \mathbb{N}} f^n(K) = \emptyset$, where $f^n(K)$ denotes iterating f , n times. (i.e. $f^1(K) = f(K)$, $f^{i+1}(K) = f|_{K \cap f^i(K)}(K \cap f^i(K))$.)

This definition seems less aesthetically pleasing than the discrete case. In fact, I know of no use of this definition outside of the following result.

Lemma 4.17. *An abelian pro- p group G is simply given if and only if its dual is simply presented.*

The definition is a somewhat literal-minded translation of that of simply-presented. Hence proof of this lemma is obvious and we do not include it.

Definition 41. *Generalised Kiefer Groups*

Set $H_0 = \{0\}$. We recursively define a family of dual-reduced abelian pro- p groups H_α , for arbitrary ordinal α , such that:

1. $H_{\alpha+1}/H_{\alpha+1}[p^\alpha]$ is cyclic of order p and $H_{\alpha+1}[p^\alpha] \cong H_\alpha$; and
2. $H_\alpha = \prod_{\beta < \alpha} H_\beta$ whenever α is a limit ordinal.

We call these the generalised Kiefer groups.

It is worth noting H_α is of length α and each pro-Ulm–Kaplansky invariant of H_α is at most $|\alpha|$.

Lemma 4.18. *The generalised Kiefer groups exist and $H_\alpha^* \cong \Xi_\alpha$ for every ordinal α .*

This follows immediately from Pontryagin duality. We have $H_n \cong C_{p^n}$, $H_\omega = \prod_n C_{p^n}$ and $H_{\omega+1}$ is the group X in Example 1.

Theorem 4.19. *Let G be a dual-reduced abelian pro- p group of length α .*

Then G is (topologically) isomorphic to a cobalanced subgroup of a Cartesian product of generalised Kiefer groups of length at most α .

This is a new result. It had not been stated previously and is another demonstration of the significance of the totally injective pro- p groups.

Proof. This follows at once from Lemma 3.28 and the work we have considered. □

Definition 42. *Let G be an abelian pro- p group.*

We say that G has the injective property relative to balanced embeddings of pro- p groups if, whenever we have an abelian pro- p group H with cobalanced subgroup K , every continuous homomorphism $\phi : K \rightarrow G$ extends to a continuous homomorphism $\psi : H \rightarrow G$.

Definition 43. *Let G be an abelian pro- p group. We say that G is totally injective if G^* is totally projective.*

More generally, an abelian profinite group is totally injective whenever all its p -Sylow subgroups are totally injective.

We can immediately generalise Theorem 4.6:

Theorem 4.20. *Let G, H be totally injective abelian dual-reduced pro- p groups with the same torsion sequence. Then G and H are isomorphic (as topological groups).*

The proof is exactly the same as before.

Proof. The dual groups G^*, H^* are totally projective p -groups. Repeated applications of Theorem 4.3, shows that the dual of the torsion sequence of a pro- p group is isomorphic to the Ulm sequence of its dual. Hence, by Ulm's theorem, ([5], Theorem 37.1 or 3.31), $G^* \cong H^*$. Now, by Pontryagin duality G and H are topologically isomorphic. $\square$

From this, it follows that all countably-based dual-reduced pro- p groups are totally injective.

Theorem 4.21. *The class $\mathcal{TI}_p$ of totally injective pro- p groups is the maximal class containing $\mathcal{TI}_p$ which*

1. *contains all totally injective pro- p groups;*
2. *is closed under taking closed-continuous direct summands;*
3. *contains no pair of non-isomorphic groups with the same pro-Ulm invariants.*

Again, this follows from the same result in the discrete case: it is a precise translation of Theorem 3.31

We can collect a number of results.

Theorem 4.22. *Let G be a dual-reduced abelian pro- p group. The following are equivalent.*

1. *G is totally injective;*
2. *G has a smart system;*
3. *G has a smart composition series;*
4. *G is a closed-continuous direct summand of a Cartesian product of generalised Kiefer groups;*
5. *G has the injective property with respect to balanced embeddings;*

6. G is simply given;
7. G is a member of the smallest class $\mathcal{TI}_p$ of pro- p groups which:
 - (a) contains C_p ;
 - (b) is closed under taking Cartesian products and closed-continuous direct summands;
 - (c) contains a pro- p group H if, for some ordinal α , $\mathcal{TI}_p$ contains $H[p^\alpha]$ and $H/H[p^\alpha]$.

This was first noted by [12] and is clearly immediate from Theorem 3.32 and the results we have noted. We now have outlined a great body of theory which we can draw upon: we now proceed to consider how to construct these groups, before looking at their abstract structure.

An obvious question is “Given an arbitrary sequence of groups which could occur as the torsion sequence of a totally injective pro- p group, can we construct such a group with this torsion sequence?”. We answer this positively in the following section.

Chapter 5

How to Build Abelian Pro- p Groups

In this section we describe how to construct an arbitrary totally injective abelian pro- p group. This generalises [13]: many of the proofs are repeated from there with more details to take care of uncountability.

Recall the definition of a profinite presentation:

Definition 44. For X, Y abelian profinite groups with $x \in X, y \in Y$, we write the profinite presentation

$$\langle X, Y \mid x = y \rangle$$

to denote the quotient $X \times Y / \overline{\langle x - y \rangle}$.

We mainly use this in the following case.

Lemma 5.1. Let X be an abelian pro- p group, $Y = \overline{\langle y \rangle} \cong \mathbb{Z}_p$ and a be a natural number.

Then, if $x \in X \setminus (t(X) + pX)$,

$$G = \langle X, Y \mid p^a y = x \rangle$$

is a pro- p extension of X by C_{p^a} and $t(G) \subseteq X$.

Proof. Firstly, X is a subgroup of G due to the conditions on x : these prevent $x - y$ from being a element of X and we have $G/X = \{X, y + X, \dots, (p^a - 1)y + X\}$. Clearly, $y + X$ generates this cyclic quotient.

Secondly, $t(X)$ is contained in X . Suppose there is some $z \in G \setminus X$ of finite order. Without loss of generality, we can choose z such that $pz \in X$. In fact, without losing generality, we can write $z = w + p^{a-1}y$, for some $w \in X$, as the choice of coset is unimportant.

But then $pz = x + pw$ and so $x = pz - pw$. But $pz \in t(X)$ and $pw \in pX$ and hence $x \in t(X) + pX$, which is a contradiction. $\square$

It is helpful to consider only those profinite groups with closure of torsion elements a Cartesian product of cyclic p -groups, which we call Cartesian. It transpires that this class includes all totally injective pro- p groups.

Proposition 5.2. *Let G be a totally injective pro- p group. Then $\overline{t(G)}$ and each term of the torsion sequence of G is Cartesian.*

Proof. Inductively, it is sufficient only to consider $\overline{t(G)}$.

There is a result [4, 81.10], which says that the exact dual of this holds. That is, every term in the Ulm series of any group which satisfies one of the conditions listed in Theorem 3.29 is a direct sum of cyclic groups. This follows from noticing that this must be true of any direct summand of a direct sum of Prüfer groups, which follows elementarily from the definition of the generalised Prüfer groups. Hence $\overline{t(G)}$ will be isomorphic to a Cartesian product of cyclic p -groups. $\square$

5.1 Unbounded Multiplicity

Theorem 2.4 of [13] outlines a construction of abelian pro- p groups. For a sequence, of countable length τ , $(N_\alpha)_{\alpha < \tau}$ of Cartesian abelian pro- p groups, with every non-final term of the sequence (N_α) of unbounded exponent, it gives a construction of a dual-reduced abelian pro- p group with torsion sequence (N_α) .

As totally injective groups are not restricted to the relatively small case of countably-based groups, some delicate cardinality conditions must be considered.

To this end, we introduce the notion of unbounded multiplicity.

Recall the definition of the supremum limit of a sequence of cardinals $(\gamma_n)_{n \in \mathbb{N}}$.

Definition 45. *Let $(\gamma_n)_{n \in \mathbb{N}}$ be a sequence of cardinals. The supremum limit of this sequence, $\limsup_{n \in \mathbb{N}}(\gamma_n)$, is a generalisation of the notion of limit, given by*

$$\limsup_{n \in \mathbb{N}}(\gamma_n) = \inf_{n \in \mathbb{N}}(\sup_{m \geq n}(\gamma_m)).$$

Recall also that, if $(\kappa_n)_{n \in \mathbb{N}}$ is a sequence of cardinals with at least one infinite term, then $\sup_n(\kappa_n) \leq \sum_n \kappa_n$, as larger infinite terms dominate.

Definition 46. Let G be a Cartesian group.

For a cardinal α , we say that G is α -unbounded if there exist Cartesian groups H_x of unbounded exponent, for x in index set X , with $|X| = \alpha$ such that

$$K = \prod_{x \in X} H_x,$$

for some subgroup K closed in G . If there is some maximum α such that G is α -unbounded, we call this invariant the unbounded multiplicity of G and write $\text{um}(G)$ to denote this.

Similarly, we define the multiplicity of G , $m(G)$ to be the greatest cardinal α such that G is the Cartesian product of α non-trivial groups.

NB: It is immediately apparent that, for any abelian pro- p group G , G is 0-unbounded, $\text{um}(G)$ is defined and at least 0 and $\text{um}(G) \leq m(G)$. (From the definition, such a group G with finite exponent has $\text{um}(G) = 0$.)

It is clear that $\{\gamma \mid G \text{ is } \gamma\text{-unbounded}\}$ has a supremum. Theorem 5.3, below, shows that this is in fact an attained maximum and thus that every Cartesian group has well-defined unbounded multiplicity.

A Cartesian pro- p group of unbounded exponent can always be written as a product of $\aleph_0$ such groups, by partitioning the natural numbers. From this, it is immediately apparent that $\text{um}(G)$ is always zero or an infinite cardinal.

In fact, we can determine what value the unbounded multiplicity is.

Theorem 5.3. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of cardinals. Then the pro- p abelian group

$$G = \prod_n (C_{p^n}^{\alpha_n})$$

is $\limsup_{n \in \mathbb{N}}(\alpha_n)$ -unbounded and $\aleph_0 \cdot \limsup_{n \in \mathbb{N}}(\alpha_n)$ -unbounded.

The cardinal $\aleph_0 \cdot \limsup_{n \in \mathbb{N}}(\alpha_n)$ is the greatest α such that G is α -unbounded.

This theorem shows that unbounded multiplicity of Cartesian groups is well-defined and that it is equal to the supremum limit above. In particular, there is a maximum that is always attained.

Proof. Let $\alpha_\omega = \limsup_{n \in \mathbb{N}}(\alpha_n)$.

Note that $\alpha_\omega = 0$ if and only if (α_n) has only finitely many non-zero terms: that is, if and only if G is of finite exponent. Hence we need only consider the case where G is 1-unbounded, that is, $\alpha_\omega \geq 1$: the definition shows that for every $\beta > \alpha$ every β -unbounded group is α -unbounded.

Firstly, suppose α_ω is finite.

The definition of α_ω ensures that (α_n) can be decomposed as the sum of α_ω sequences $(\beta_n^{(i)})$, as i ranges over some index set I with $|I| = \alpha_\omega$, with

$$\alpha_n = \sum_{i \in I} \beta_n^{(i)}, \text{ for every } n$$

and with $\limsup_{n \in \mathbb{N}} (\beta_n^{(i)}) = 1$ for each $i \in I$. This allows us to write G as a product of Cartesian groups $G = \prod_{i \in I} H_i$, where $H_i = \prod_n (C_{p^n}^{\beta_n^{(i)}})$. Hence, to prove the statement holds when α_ω is finite, it suffices to prove that it holds when $\alpha_\omega = 1$.

Suppose $\alpha_\omega = 1$. Rewrite G as a product of Cartesian subgroups $G = \prod_{x \in X} K_x$, with each K_x of unbounded multiplicity. Each K_x must contain cyclic direct summands of G of infinitely many different orders. Hence, simple counting shows that the unbounded multiplicity of G is no more than $\aleph_0 \cdot \limsup_{n \in \mathbb{N}} (\alpha_n) = \aleph_0 \cdot \alpha_\omega = \aleph_0$. Hence, for any cardinal $\alpha > \aleph_0$, G cannot be α -unbounded, as G is a product of some torsion Cartesian group with $\aleph_0$ many finite cyclic groups. It suffices to show that G is α_ω -unbounded and then $\aleph_0 \cdot \limsup_{n \in \mathbb{N}} (\alpha_n)$ -unbounded to prove the theorem. As $\alpha_\omega = 1$, G must be of unbounded exponent and hence G is 1-unbounded. To see that G is $\aleph_0$ -unbounded, partition the infinite set $\{n | \alpha_n > 0\}$ as infinitely many disjoint sets J_i as i ranges over the natural numbers. Then the infinite decomposition

$$G = T \prod_{i \in \mathbb{N}} (C_{p^i}^{\alpha_i})$$

shows that G is $\aleph_0$ -unbounded.

Conversely, suppose α_ω is an infinite cardinal. Here $\alpha_\omega = \aleph_0 \cdot \alpha_\omega$.

In this case, we can assume without loss of generality that

$$\alpha_\omega = \limsup_{n \in \mathbb{N}} (\alpha_n) = \sup_{n \in \mathbb{N}} (\alpha_n).$$

Otherwise, $\sup_{n \in \mathbb{N}} (\alpha_n) = \beta > \limsup_{n \in \mathbb{N}} (\alpha_n)$ and there is some natural number t such that these values would coincide for $G/p^t G$. This implies that for any cardinal $\kappa > \alpha_\omega$, G cannot

be κ -unbounded, as G is a product of a torsion group with α_ω finite cyclic groups.

If the sequence α_n reaches α_ω infinitely many times, we can write G as the product of α_ω groups of unbounded exponent, in exactly the same way as the finite case.

It remains only to consider the case where this upper bound is not attained by any α_n , where the supremum limit α_ω is equal to the supremum $\sup_{n \in \mathbb{N}}(\alpha_n)$.

We have $\alpha_\omega = \sum_n \alpha_n$.

Now, from the definition, $G = \prod_{i \in I} L_i$, for some index set I of cardinality α_ω , for finite cyclic subgroups L_i . But I can be partitioned into α_ω infinite sets as there is a bijection from $I \times \mathbb{N}$ to I . Take any such partition.

This gives rise to G as a product of α_ω Cartesian groups M_j . For each natural number n there are at most α_n groups of exponent p^n . Hence α_ω of the M_j are of unbounded exponent and so, by definition, G is α_ω -unbounded. $\square$

Note that, in the class of countably based Cartesian pro- p groups the condition “every non-final term of the sequence $(N_\alpha)_{\alpha < \tau}$ is of unbounded exponent” (used in Theorem 2.4 of [13]) means exactly the same as “ $\text{um}(N_\alpha) \geq \text{um}(N_\beta)$ whenever $\alpha \leq \beta < \tau$ ”. In a more general setting, the second statement implies the first.

We can now state the conditions which must be met to extend Theorem 2.4 of [13] to uncountably based groups.

5.2 Admissible Sequences

Definition 47. Let $(N_\alpha)_{\alpha < \tau}$ be a sequence of Cartesian pro- p groups of length τ , some ordinal.

Then we say that (N_α) is an admissible sequence if and only if:

1.

$$\forall 1 < \alpha \leq \tau, \forall \alpha < \beta < \tau \text{ we have } \text{um}(N_\alpha) \geq \text{um}(N_\beta)$$

and,

2. for each ordinal α there are no more than $\text{um}(N_\alpha)$ ordinals β such that $\alpha < \beta$ with N_β non-trivial.

Note that the first statement is equivalent to “whenever $\alpha < \beta < \tau$, for each cardinal X such that N_β is the Cartesian product of X non-trivial groups, N_α is X -unbounded,” and

the second to “for each ordinal α , N_α is $|\{\beta > \alpha \mid N_\beta \text{ non-trivial}\}|$ -unbounded.” We use these equivalent statements in the following proof.

The concept of an admissible sequence is related to the concept of an admissible function by the following lemma.

Lemma 5.4. *Let G be a totally injective abelian pro- p group. Then the torsion sequence of G is admissible.*

Proof. Recall that the pro-Ulm function of a totally injective abelian pro- p groups, which is a function from ordinals to cardinals, is the Ulm–Kaplansky function of its totally projective dual. Recall that, by Theorem 3.33, a function from ordinals to cardinals can be the Ulm–Kaplansky function of a totally projective p -group if and only if it is admissible. (We only proved the “only if” part which suffices for this proof.) We shall write f_G for the (admissible) pro-Ulm function of G .

Let α be such that $T_{\alpha+1}(G) < G$, i.e. G_{T_α} is not the final term of the torsion sequence. (The conditions hold trivially for the final term.) As the closure of $G/T_\alpha(G)$ is not of finite exponent, G_{T_α} is of unbounded exponent. Hence G_{T_α} is $\aleph_0$ -unbounded and so G_{T_α} is X -bounded for every finite X .

We write γ such that $G[p^\gamma] = T_\alpha(G)$.

Now, suppose there is some $\beta > \alpha$ with $m(G_{T_\beta})$ an uncountably infinite cardinal. As G_{T_β} is a Cartesian group, there must be some positive integer k such that

$$m(G_{T_\beta}) = f_G(\delta + k),$$

for δ with $G[p^\delta] = T_\beta(G)$. We have $\delta \geq \gamma + \omega$.

For any infinite cardinal X , $\text{um}(G_{T_\alpha}) \geq X$ if and only if, for each natural number m ,

$$\sum_{n < \omega} f_G(\gamma + m + n) \geq X.$$

To see this, note that the proof of Theorem 5.3 shows that Cartesian group $\prod_n (C_{p^n})^{a_n}$ is X -unbounded if and only if $\sum_n a_n \geq X$.

Now,

$$m(G_{T_\beta}) = f_G(\delta + k) \leq \sum_{\rho \geq (\gamma+m)+\omega} f_G(\rho) \leq \sum_{n < \omega} f_G(\gamma + m + n),$$

for each $m \in \mathbb{N}$ and so $\text{um}(G_{T_\alpha}) \geq m(G_{T_\beta})$. Hence the torsion sequence of a totally injective

group must satisfy the first condition of admissibility.

The sum, $\sum_{\rho \geq (\gamma+m)+\omega} f_G(\rho)$ counts cyclic (closed-continuous) direct summands of each G_{T_β} occurring after G_{T_α} in the torsion sequence and so is a very large upper bound for $|\{\beta > \alpha \mid |N_\beta| > 1\}|$. Hence, for every natural number m ,

$$\sum_{n < \omega} f_G(\gamma + m + n) \geq \sum_{\rho \geq (\gamma+m)+\omega} f_G(\rho) \geq |\{\beta > \alpha : |N_\beta| > 1\}|$$

and so G_{T_α} is $|\{\beta > \alpha \mid |N_\beta| > 1\}|$ -unbounded. Hence the torsion sequence of a totally injective abelian pro- p group is admissible. □

This enables us to state the following central theorem for this thesis.

5.3 Construction of Arbitrary Totally Injective Abelian Pro- p Groups

Theorem 5.5. *Let $(N_\alpha)_{\alpha < \tau}$ be a sequence of Cartesian pro- p groups of length τ , some ordinal.*

Then there exists a totally injective dual-reduced abelian pro- p group G with torsion sequence (N_α) if and only if (N_α) is an admissible sequence.

Our proof comes largely from applying Pontryagin duality to the proof of Zippin's Theorem given in ([5], §36). We give this dual construction (which seems to have first appeared in [13], in the countably-based case) as analysis of this construction is useful for proving Theorem 6.1.

We follow the proof by [13], reproducing it with the new extensions using my new definition of unbounded multiplicity.

Proof. By Lemma 5.4, every totally injective abelian pro- p group has admissible torsion sequence. Hence we need only prove that, given an admissible sequence (N_α) , we can construct a totally injective G with torsion sequence (N_α) .

We first give a construction for groups, then will verify that these groups are in fact totally injective.

We proceed by induction on τ .

The base case is $\tau = 2$, where we have one group which is the closure of its torsion elements and $G = N_0$ is a Cartesian group. As the Cartesian product of totally injective

(cyclic) groups, this is totally injective.

We now split into five cases.

Case I: $\tau - 2$ exists and $N_{\tau-1}$ is a cyclic group, of order p^r .

By induction, there exists a totally injective group H with torsion sequence $(N_0, \dots, N_{\tau-2})$. Now, $H/T_{\tau-2}(H) \cong \prod_{i \in I} X_i$, where each X_i is a cyclic group. So we have canonical epimorphisms $\theta_i : H \rightarrow X_i$, for $i \in I$. Choose $\delta \in H$ which is sent by each θ_i to an element in X_i which generates that group. Then, take the abelian group with profinite presentation (as defined above)

$$G = \langle H, X = \overline{\langle x \rangle} \mid p^r x = \delta \rangle$$

where X is an infinite procyclic pro- p group topologically generated by x .

We claim that $T_\alpha(G) = T_\alpha(H)$ for each $\alpha < \tau - 1$. Suppose otherwise. Take $\beta \leq \tau - 2$ to be minimal such that $T_\beta(G) \neq T_\beta(H)$. If β is a limit ordinal

$$T_\beta(H) = \overline{\bigcup_{\mu < \beta} T_\mu(H)} = \overline{\bigcup_{\mu < \beta} T_\mu(G)} = T_\beta(G)$$

and we have a contradiction.

So β is a successor ordinal: write $\beta = \gamma + 1$.

Note that $T_{\gamma+1}(G)$ is not contained in H . Otherwise,

$$T_{\gamma+1}(G)/T_\gamma(G) = T_{\gamma+1}(G)/T_\gamma(H) \leq_C H/T_\gamma(H)$$

and so $T_{\gamma+1}(G) \subseteq T_{\gamma+1}(H)$. Then

$$T_{\gamma+1}(G) = T_{\gamma+1}(H),$$

contradicting our assumption on β .

Take some $a \in T_{\gamma+1}(G) \setminus H$ which is of finite order modulo $T_\gamma(G) = T_\gamma(H)$. So, we have $a \in nx + H$ for some $0 < n \leq p^r - 1$. As $\gamma = \beta - 1 \leq \tau - 2$, $a + T_{\tau-2}(H)$ is of finite order. But we chose δ so that $\delta + T_{\tau-2}(H)$ and so $x + T_{\tau-2}(H)$ must be of infinite order in $H/T_{\tau-2}(H)$, so $nx + H$ cannot contain elements of finite order in $H/T_{\tau-2}(H)$. This is a contradiction. Hence $T_{\tau-2}(G) = H$ and $T_\alpha(G) = T_\alpha(H)$, for each $\alpha \leq \tau - 2$.

Thus, as G/H is cyclic of order p^r , it follows that G has the required torsion sequence. As the Cartesian group $G/T_{\tau-2}(G)$ and $T_{\tau-2}(G) = H$ are both totally injective, G is totally

injective.

Case II: $\tau - 2$ exists and $N_{\tau-1}$ is not cyclic.

By admissibility, for each $\alpha < \tau - 1$, $\text{um}(N_\alpha)$ is not less than $m(N_{\tau-1})$. There exist cyclic groups, $\{N_{\tau-1,i}\}_{i \in I}$ such that $N_{\tau-1} = \prod_{i \in I} N_{\tau-1,i}$. For each $\varepsilon < \tau - 1$, we can find some decomposition $N_\varepsilon = \prod_{i \in I} N_{\varepsilon,i}$, such that each $(N_{\varepsilon,i})_{\varepsilon < \tau-1}$ is an admissible sequence. For example, let $N_\varepsilon = \prod_{i \in I} C_{p^{m_i}}$, for some set $M = \{m_i\}_{i \in I}$ of natural numbers. Taking any partition of I into infinitely many infinite pairwise disjoint subsets $(K_t)_{t \in T}$, with $\bigcup_{t \in T} K_t = I$, there is a decomposition $N_\varepsilon = \prod_{t \in T} \prod_{i \in K_t} C_{p^{m_i}}$.

Hence, by Case I, we can construct totally injective groups H_i with torsion sequence $(N_{\alpha,i})_{\alpha < \tau-1,i}$, for each i . Now

$$G = \prod_{i \in I} H_i$$

has the required torsion sequence, as the product $\prod_{i \in I} T_\alpha(H_i)$ will be isomorphic to $T_\alpha(\prod_{i \in I} H_i)$. As G is a Cartesian product of totally injective groups, it is totally injective.

Case III: τ is a limit ordinal.

Take a set $\{\sigma_i \mid i \in I\}$ of ordinals less than τ and with supremum σ .

Claim: The methods in Case II above can be used to prove the existence of Cartesian groups $N_{\varepsilon,i}$ for all $\varepsilon < \sigma_i$ such that, for all $i \in I$, $(N_{\varepsilon,i})_{\varepsilon < \sigma_i}$ is an admissible sequence and $\prod_i N_{\varepsilon,i} = N_\varepsilon$ for each $\varepsilon < \tau$.

This is the point at which we need the second part of admissibility. The first condition is needed, as we have seen before, to construct the successor-ordinal-indexed terms of the torsion sequence. By the second condition of admissibility, we can topologically split each N_α enough times to provide splittings of the sequence as provided. This is because this condition is equivalent to saying that, for each α , we can write

$$N_\alpha = \prod_{x_\alpha \in X_\alpha} M_{x_\alpha}, \text{ where } X_\alpha = \{\beta > \alpha \mid |N_\beta| > 1\},$$

where each M_{x_α} is Cartesian of unbounded exponent.

By inductive hypothesis we can construct a totally injective group H_i with (admissible) torsion sequence $(N_{\varepsilon,i})$, for each i . The group

$$G = \prod_{i \in I} H_i$$

is totally injective and has the required torsion sequence, as the torsion sequence of a product is the product of torsion sequences, as in Case II.

Case IV: $\tau - 1$ exists and is a limit ordinal, σ , and N_σ is cyclic of order p^r .

Take a set $\{\sigma_i : i \in I\}$ of ordinals less than and with supremum σ .

Construct, as in Case III, new Cartesian groups $N_{\varepsilon,i}$ with $\prod_i N_{\varepsilon,i} = N_\varepsilon$ such that $N_{\varepsilon,i}$ is trivial if $\sigma_i < \varepsilon$ and such that each $(N_{\varepsilon,i})_{\varepsilon < \sigma_i}$ is an admissible sequence. Furthermore, without loss of generality, we can choose each $N_{\sigma_i,i}$ to be cyclic. By the inductive hypothesis we can construct a totally injective group H_i with torsion sequence $(N_{\varepsilon,i})$, for each i . For each i , we can choose $\gamma_i \in H_i$ such that $\gamma_i + T_{\sigma_i-1}(H_i)$ is a generator for $H_i/T_{\sigma_i-1}(H_i)$. Choose $\gamma \in H = \prod_i H_i$ such that, for each i , the canonical epimorphism $\phi_i : H \rightarrow H_i$ sends γ to γ_i . Note that H has torsion type $\tau - 1$ and torsion sequence $(N_\varepsilon)_{\varepsilon < \tau}$.

In the same way as Case I, we construct an extension of a Cartesian product of groups by C_{p^r} . Consider

$$G = \langle \prod_i H_i, X = \overline{\langle x \rangle} \mid p^r x = \gamma \rangle$$

where X is an infinite procyclic pro- p group topologically generated by x . Now $G/H \cong C_{p^r}$ and so to show G has the required torsion sequence, it remains only to show $T_\alpha(G) = T_\alpha(H)$ for each $\alpha \leq \tau$.

Suppose otherwise: we can pick least ordinal β with $T_\beta(G) \neq T_\beta(H)$. If β is a limit ordinal

$$T_\beta(H) = \bigcup_{\mu < \beta} \overline{T_\mu(H)} = \bigcup_{\mu < \beta} \overline{T_\mu(G)} = T_\beta(G)$$

and we have a contradiction. Hence β must be a successor ordinal and so $\beta < \tau - 1$. It is straightforward to see that $T_\alpha(G) \geq T_\alpha(H)$ for each α . As in Case I, $T_\beta(G)$ cannot be contained in H .

Take some $a \in T_\beta(G) \setminus T_\beta(H)$, such that a must be of finite order modulo $T_{\beta-1}(H)$. If $a \in H$, then $a + T_{\beta-1}(H) \in \overline{t(H/T_{\beta-1}(H))}$ and so $a \in T_\beta(H)$. Hence $a \in T_\beta(G) \setminus H$.

We have $a \in nx + H$, for some $1 \leq n \leq p^r - 1$. But we chose γ so that for each σ_i , $\gamma + T_{\sigma_i}(H)$, and so $x + T_{\sigma_i}(H)$ must be of infinite order in $H/T_{\sigma_i}(H)$. Thus, taking $\sigma_i > \beta - 1$, we see that a cannot be of finite order in $H/T_{\beta-1}(H)$. This is a contradiction.

Hence $T_\alpha(G) = T_\alpha(H)$ for each $\alpha \leq \tau - 1$ and $T_{\tau-1}(G) = H$. As G/H is cyclic of order p^r , it follows that G has the required torsion sequence and is totally injective.

Case V: $\tau - 1$ exists and is a limit ordinal, σ , and N_σ is not cyclic.

This follows from Case IV in exactly the same way as Case II follows from Case I. By admissibility, for each $\alpha < \tau - 1$, $\text{um}(N_\alpha)$ is greater than $m(N_{\tau-1})$. Hence, as in Case II, we find cyclic groups, $\{N_{\sigma,i}\}_{i \in I}$ such that $N_\sigma = \prod_{i \in I} N_{\sigma,i}$. For each $\varepsilon < \sigma$, we can find some decomposition $N_\varepsilon = \prod_{i \in I} N_{\varepsilon,i}$ with each $(N_{\varepsilon,i})_{\varepsilon < \sigma_i}$ an admissible sequence. .

By the inductive hypothesis we can construct a totally injective group H_i with (admissible) torsion sequence $(N_{\alpha,i})_{\alpha \leq \sigma}$, for each i . But then

$$G = \prod_{i \in I} H_i$$

has the required torsion sequence, as before.

To verify that G is totally injective, recall that the class $\mathcal{TI}_p$ of totally injective groups contains C_p and is closed under taking Cartesian products, or closed-continuous direct summands and that G is in $\mathcal{TI}_p$ whenever $G[\sigma], G/G[\sigma] \in \mathcal{TI}_p$.

So, if G is of torsion length $\tau = \sigma + 1$, $T_{\tau-1}(G)$ is totally injective and $G_{T_{\tau-1}} = G/T_{\tau-1}(G)$ is Cartesian, G must be totally injective. On the other hand, if τ is a limit ordinal and every (proper) term of the torsion sequence of G is totally injective, we constructed a G which decomposes as a Cartesian product of closed-continuous direct summands of lower terms of its torsion sequence. But each of these summands must be totally injective and hence as the Cartesian product of these, G must be totally injective. Hence, as every proper subgroup in the torsion sequence of G is totally injective, it follows that G is totally injective.

□

In the proof, we use the axiom of choice extensively and have many choices of ways to decompose Cartesian groups of unbounded exponent as products of infinitely many Cartesian groups of unbounded exponent. The choices made at each stage do not matter, as Theorem 4.20 shows.

By Theorem 4.5, Theorem 5.5 and Theorem 4.20 we have thus classified all totally injective abelian pro- p groups. Hence we can construct any totally injective abelian profinite group as a product of its Sylow pro- p subgroups.

This completely classifies totally injective profinite groups up to continuous isomorphism.

Proposition 5.6. *Let G be an infinite totally injective dual-reduced abelian pro- p group of*

torsion type τ . Then there exist non-trivial closed subgroups K_i of G for an infinite index set I such that

1. we have $G = \prod_{i \in I} K_i$;
2. (a) if $G_{T_{\tau-1}}$ is finite, each K_i is of torsion type at least $\tau - 1$;
(b) if $G_{T_{\tau-1}}$ is infinite or τ is a limit ordinal, each K_i is of torsion type τ , and;
3. each K_i of torsion type some successor ordinal α_i has $(K_i)_{T_{\alpha_i-1}}$ cyclic.

This is stronger than the result dual to ([4], Proposition 77.5) and, aside from the use of Ulm's theorem in our classification, proves it independently.

Proof. We prove the first two parts together. As G is totally injective, we can see that each G_{T_α} must be Cartesian. Now, we can write each $G_{T_\alpha} = \prod_{n \in \mathbb{N}} H_{\alpha,n}$, subject to the following conditions: for each α, n

1. $H_{\alpha,n}$ is Cartesian,
2. For each cardinal X , $H_{\alpha,n}$ is X -unbounded if and only if G_{T_α} is X -unbounded,
3. $H_{\alpha,n}$ is finite if and only if G_{T_α} is finite.

Note that each $(H_{\alpha,n})_\alpha$ is an admissible sequence. By Theorem 5.5, we can find groups H_n with torsion sequence $(H_{\alpha,n})_\alpha$. Now, by Theorem 4.20, we have $G \cong \prod_{n \in \mathbb{N}} H_n$.

We claim that the H_n are all non-trivial and satisfy the conditions on the K_i given in the second part of the statement. We proceed by induction on τ . As G is infinite, $T_{G_0} = \overline{t(G)}$ must also be infinite. As an infinite Cartesian group, it is by definition a product of infinitely many non-trivial groups and so in the base case, $\tau = 2$, the proposition holds.

If $\tau > 1$, $H_{0,n}$ is of unbounded exponent and so non-trivial, for each n , and so each H_n must also be of torsion type at least 1. If τ is a limit ordinal, then, for each $\alpha < \tau$, G_{T_α} must be of unbounded exponent and so each $H_{\alpha,n}$ must be of unbounded exponent. Hence each H_n must be of torsion type τ . If τ is a successor ordinal and $G_{T_{\tau-1}}$ is finite then all but finitely many H_n will be of torsion type $\tau - 1$: the remainder will be of type τ . Otherwise, if $G_{T_{\tau-1}}$ is infinitely generated, each $H_{\tau-1,n}$ must also be infinitely generated and so each H_n must be of torsion type τ . Thus $G \cong \prod_{n \in \mathbb{N}} H_n$ and so we can find closed subgroups K_i satisfying the first two parts of the statement.

We can decompose the G_{T_α} in different ways and thus get decompositions with different properties. In particular, we can always pick $H_{\alpha,n}$ such that each H_n of torsion type $\tau_n = \varepsilon_n + 1$ has cyclic $(H_n)_{T_{\varepsilon_n}}$. By the same argument as above, this proves the third part, completing the proof. $\square$

Corollary 5.7. *If G is an infinitely generated, totally injective abelian profinite group, G can be decomposed as a product of infinitely many non-trivial groups.*

Note the converse of this is not true: the group $\prod_{p \text{ prime}} C_p$ is topologically 1-generated.

Proof. An abelian profinite group G is equal to $\prod_{p \text{ prime}} G_p$, where G_p is a p -Sylow subgroup of G . If G is infinitely generated, either infinitely many of these are non-trivial and we are done, or there is some infinitely generated G_p . Consider G_p , an infinitely generated pro- p group. By Theorem 4.5, $G_p = H \times F$, where H is dual-reduced and F is torsion-free and so one of these must be infinitely generated. The group F , dual to a sum of Prüfer quasicyclic groups must be isomorphic to $\prod_{i \in I} \mathbb{Z}_p$, a product of infinite procyclic groups. If F is infinitely generated, it is therefore the product of infinitely many infinite procyclic groups and our result holds. If H is infinitely generated it must be totally injective, as G is totally injective.

And now Proposition 5.6 gives our result. $\square$

We now look at which of these groups are isomorphic as abstract groups.

Chapter 6

Discontinuous Isomorphisms

This chapter solves the discontinuous isomorphism problem for abelian totally injective pro- p groups. This essentially reproduces section 3 of [13], with some extra twists to allow things to work in greater generality. Since submitting [13], it has been drawn to my attention that Z. Chatzidakis in [1], proved that $\prod_n C_{p^n}$ is abstractly isomorphic to $\mathbb{Z}_p \times \prod_n C_{p^n}$. This chapter (and [13]) independently present a proof of this result, as well as a very strong inductive generalisation. I am very grateful to have this result brought to my attention: reading [1] has been helpful and clarified ideas regarding Chapter 7. The proof presented in this section proceeds by a totally different method to that of [1] and provides explicit construction (modulo the use of an ultrafilter and the axiom of choice) of non-continuous isomorphisms between abelian pro- p groups.

Theorem 6.1. *Discontinuous Abelian Isomorphisms*

Let G be a totally injective abelian pro- p group. If $t(G)$, the torsion subgroup of G , is of finite exponent then it is a closed subgroup of G and G is of the form

$$\prod_{i=1}^e (C_{p^i})^{\alpha_i} \times \mathbb{Z}_p^X$$

for some $e \in \mathbb{N}$, (α_i) a sequence of cardinals and X some cardinal. If $t(G)$ contains elements of unbounded order, the dual-reduced part of G is isomorphic to $\overline{t(G)}$ as an abstract group and $\overline{t(G)}$ is of the form

$$\prod_{i \in \mathbb{N}} (C_{p^i})^{\alpha_i},$$

for (α_i) a sequence of cardinals not tending to 0. Furthermore, if the rank of a maximal

torsion-free closed-continuous direct summand, $F(G)$, of G is no greater than $\text{um}(T_0(G)) = \limsup_n \{\alpha_n\}$, then G is abstractly isomorphic to the closure of the torsion subgroup of G .

In fact, it is not hard to see that $\alpha_i = \dim_{\mathbb{F}_p}(G[p^i]/(pG \cap G[p^i]))$. (This function is the pro-Ulm invariant function.) Specifically, the α_i are invariants of the torsion part of G and so do not depend on the topological structure of G .

To prove Theorem 6.1, we first prove the following crucial lemma:

Lemma 6.2. *Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of cardinals with infinitely many non-zero terms and I be some arbitrary index set such that*

$$|I| \leq \text{um} \left(\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \right).$$

Then

$$\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \cong \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \times \prod_I \mathbb{Z}_p$$

as abstract groups.

We do this by using a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ to define a series of homomorphisms in a similar way to an ultralimit.

Definition 48. *An ultrafilter $\mathcal{U}$ on a set X is a subset of the power set of X such that the function defined by $m(A) = 1$ if $A \in \mathcal{U}$, $m(A) = 0$ otherwise, is a finitely additive measure. It is said to be principal if it has some least element, $\{x\} \subseteq X$ and non-principal otherwise.*

The existence of non-principal ultrafilters on infinite sets is equivalent to the Boolean Prime Ideal Theorem.

Proof. (of Lemma 6.2.) We can write $\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i}$ as

$$\prod_{i \in I} \prod_{j \in \mathbb{N}} (\mathbb{Z}/p^j \mathbb{Z})^{\alpha_j^{(i)}},$$

for index set I of cardinality at most $\text{um}(T_0(G))$, such that each $(\alpha_j^{(i)})$ is a sequence in $\mathbb{N} \cup \{\aleph_0\}$ with infinitely many non-zero entries.

In this way, we need only show that for any sequence (α_i) in $\mathbb{N} \cup \{0, \aleph_0\}$ with infinitely many non-zero terms

$$\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \cong \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \times \mathbb{Z}_p$$

as abstract groups.

Without loss of generality, we can assume each $\alpha_i \in \{0, 1\}$, by considering a direct factor of a partial product. Define a set of homomorphisms $\phi_j^i : (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i} \rightarrow \mathbb{Z}/p^j\mathbb{Z}$ by

$$x\phi_j^i \equiv x \pmod{p^j} \text{ for } j \leq i, \text{ and } x\phi_j^i = 0 \text{ for } j > i.$$

For $(x_1, x_2, \dots) \in \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i}$ and each $n \in \mathbb{Z}/p^j\mathbb{Z}$, set

$$I_j^n((x_1, x_2, \dots)) = \{m \in \mathbb{N} \mid x_m \phi_j^m = n\}$$

giving, for each $j \in \mathbb{N}$, a function from our Cartesian group to the power set of $\mathbb{N}$.

We define a new map by setting $(x_1, x_2, \dots)\psi_j^{(\mathcal{U})}$ to be the unique $n \in \mathbb{Z}/p^j\mathbb{Z}$ such that $I_j^n((x_1, x_2, \dots))$ is in $\mathcal{U}$, a non-principal ultrafilter on $\mathbb{N}$.

Now, for $x, y \in \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i}$, we have

$$I_j^{x\psi_j^{(\mathcal{U})}}(x) \cap I_j^{y\psi_j^{(\mathcal{U})}}(y) \subseteq I_j^{x\psi_j^{(\mathcal{U})} + y\psi_j^{(\mathcal{U})}}(x + y).$$

The left hand side is the intersection of two elements of $\mathcal{U}$ and so is in $\mathcal{U}$, as it is an ultrafilter. Thus the right hand side contains an element of $\mathcal{U}$ and so is also in $\mathcal{U}$ and so we have

$$(x + y)\psi_j^{(\mathcal{U})} = x\psi_j^{(\mathcal{U})} + y\psi_j^{(\mathcal{U})}.$$

So $\psi_j^{(\mathcal{U})} : \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i} \rightarrow \mathbb{Z}/p^j\mathbb{Z}$ is a homomorphism. Now, the $\mathbb{Z}/p^j\mathbb{Z}$ with maps $\mu_{mn} : \mathbb{Z}/p^m\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ given by

$$(a + p^m\mathbb{Z})\mu_{mn} = a + p^n\mathbb{Z}, \text{ for each } n \leq m$$

form a surjective inverse system. As ϕ_j^i and μ_{mn} commute (for each i, j, m, n) so do $\psi_j^{(\mathcal{U})}$ and μ_{mn} . Now, we have surjective maps from $\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i}$ to each term of the surjective inverse system and all terms commute. By the universal property, these pull back to a unique surjective homomorphism, call it $\psi^{(\mathcal{U})}$, to the inverse limit. This map,

$$\psi^{(\mathcal{U})} : \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i\mathbb{Z})^{\alpha_i} \rightarrow \mathbb{Z}_p$$

is given by

$$a \mapsto \lim_{j \in \mathbb{N}} ((a)\psi_j^{(\mathcal{U})})$$

(where we refer to the limit under the p -adic norm). Now $\mathbb{Z}_p$ is a torsion-free group, but $\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i}$ is dual-reduced, so $\psi^{(\mathcal{U})}$ cannot be continuous and so has non-closed kernel. Consider the diagonal element $\eta = (x_i)$, $x_i = \alpha_i \in \{0, 1\}$. It is straightforward to see that

$$\overline{\langle \eta \rangle} = \{(x_i) \mid \forall m, n \in \mathbb{N}, \alpha_m = \alpha_n = 1 \implies x_m \mu_{m,n} = x_n\}$$

and so $\overline{\langle \eta \rangle} \psi^{(\mathcal{U})} = \mathbb{Z}_p$. Now, $\ker \psi$ is a complement to $\overline{\langle \eta \rangle}$, so we have

$$\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} = \ker \psi^{(\mathcal{U})} \times \overline{\langle \eta \rangle} = \ker \psi^{(\mathcal{U})} \times \overline{\langle \eta \rangle}$$

as we are working with abelian groups. It follows that

$$\ker \psi^{(\mathcal{U})} \cong \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} / \overline{\langle \eta \rangle}$$

as abstract groups. But consider the endomorphism θ of $\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i}$ defined by

$$(x_i)_{i \in \mathbb{N}} \mapsto (x_i - x_{i+t_i} \mu_{i+t_i, i})_{i \in \mathbb{N}}$$

where t_i is chosen to be the minimal positive number such that $\alpha_{i+t_i} = 1$ if $\alpha_i = 1$ and to be 0 when $\alpha_i = 0$. It is surjective and has kernel precisely $\overline{\langle \eta \rangle}$ and so our quotient is topologically isomorphic to $\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i}$.

Now, the map $\theta \times \psi^{(\mathcal{U})} : \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \rightarrow \prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i} \times \mathbb{Z}_p$ given by

$$(x_i)_{i \in \mathbb{N}} \mapsto ((x_i)\theta, (x_i)\psi^{(\mathcal{U})}) = ((x_i - x_{i+t_i} \mu_{i+t_i, i})_{i \in \mathbb{N}}, \lim_{j \in \mathbb{N}} (((x_i)_{i \in \mathbb{N}})\psi_j^{(\mathcal{U})})),$$

is our desired isomorphism, which completes the proof. $\square$

Note that $\psi^{(\mathcal{U})}$ above is in fact a ring homomorphism, as each homomorphism described above can be shown with little work to be a ring homomorphism. The direct product decomposition described above is not a decomposition as rings, as $\overline{\langle \eta \rangle}$ is not an ideal. The ideas of Chapter 8 are partially affected by this.

This gives us a specific non-continuous automorphism and allows us to construct others.

We, in fact, get numerous homomorphisms by considering different ultrafilters. Also, decompositions of $\prod_{i \in \mathbb{N}} (\mathbb{Z}/p^i \mathbb{Z})^{\alpha_i}$ via partitions of $\mathbb{N}$ may give different non-continuous isomorphisms. This is an area that is well worth further study.

Note that a group can be written as $\prod_{i \in I} C_{p^{\alpha_i}}$ for some $\{\alpha_i \mid i \in I\} \subseteq \mathbb{N}$ with no finite upper bound if and only if it can be written as $\prod_{i \in \mathbb{N}} (C_{p^i})^{\beta_i}$ for (β_i) some sequence of cardinals not tending to 0.

Lemma 6.2 is not hard to strengthen.

Corollary 6.3. *Let $\{\alpha_i \mid i \in I\} \subseteq \mathbb{N}$ with no finite upper bound.*

Write $G = \prod_{i \in I} C_{p^{\alpha_i}}$.

If $y \in G \setminus (pG + t(G))$ then

$$G = K \times \overline{\langle y \rangle}$$

where $K \cong G$ (as abstract groups).

Proof. Write $[x]_i$ for the sequence with i th entry x and every other entry trivial.

Write $y = (y_i)_{i \in I}$. Each y_i is either a generator for $C_{p^{\alpha_i}}$ or a non-generator. As $y \in G \setminus (pG + t(G))$ the set $N_y = \{i \in I \mid \langle y_i \rangle = C_{p^{\alpha_i}}\}$ has no upper bound. This is because $z = (z_i)$ is in $t(G)$ if and only if the set $\{o(z_i) \mid i \in I\}$ is bounded and because p th powers in cyclic groups are not generators.

Suppose every y_i is a generator. Then we can construct a continuous isomorphism $\theta : G \rightarrow \prod_{i \in I} \mathbb{Z}/p^{\alpha_i} \mathbb{Z}$ by sending each $[y_i]_i$ to $[1]_i$ and extending. Then Lemma 6.2 applied to $G\theta$ shows that $G = K \times \overline{\langle y \rangle}$, where K is the pre-image in G of the non-closed subgroup $\ker \psi^{(\mathcal{U})}$. (We use the notation given previously: $\psi^{(\mathcal{U})}$ is the non-continuous homomorphism defined in the proof of Lemma 6.2.)

Conversely, if y has any non-generator entries, we consider the canonical projection $\pi_{N_y} : G \rightarrow \prod_{i \in N_y} C_{p^{\alpha_i}}$. Every entry of $y\pi_{N_y}$ is a generator and we are in the situation described above.

Hence $\prod_{i \in N_y} C_{p^{\alpha_i}} = K_y \times \overline{\langle y\pi_{N_y} \rangle}$, with $\prod_{i \in N_y} C_{p^{\alpha_i}} \cong K_y$, as abstract groups. We have

$$K_y \pi_{N_y}^{-1} \cong \prod_{i \in N_y} C_{p^{\alpha_i}} \times \prod_{i \in I \setminus N_y} C_{p^{\alpha_i}}.$$

Now, to show $K_y \pi_{N_y}^{-1} \times \overline{\langle y \rangle} = G$, it is sufficient to show that

$$K_y \pi_{N_y}^{-1} \cap \overline{\langle y \rangle} = \{0\}.$$

Suppose $\beta y \in K_y \pi_{N_y}^{-1} \cap \overline{\langle y \rangle}$. Then, for each $n \in N_y$, the n th entry of βy must be trivial and hence $\beta \equiv 0$ modulo p^{α_n} . But N_y is infinite and so $\beta = 0$. $\square$

We need another strengthening of this result to prove Theorem 6.1.

Lemma 6.4. *Let $\{\alpha_i \mid i \in I\} \subseteq \mathbb{N}$ with no finite upper bound and $H \cong \prod_{i \in I} C_{p^{\alpha_i}}$. Let A be an abstract abelian group containing H with $t(A) \subseteq H$ and $A/H \cong C_{p^k}$, for some $k \in \mathbb{N}$. Then A is isomorphic to H as an abstract group.*

Proof. Choose any $x \in A$ such that $\langle x \rangle + H = A$: we have $p^k x = y \in H$. Note that $y \in H \setminus (t(A) + pH)$. Otherwise, $y = pz + t$ for some torsion t and $z \in H$. But then $p^{k-1}x - z$ is torsion and so in H , as $p(p^{k-1}x - z) = t$ is torsion. Hence x has order at most p^{k-1} in A/H and so H is of index at most p^{k-1} in A , which is a contradiction.

We can consider H to be a pro- p group under the product topology. Write C to denote the subgroup of H topologically generated by y and D to be the subgroup topologically generated by x . From the previous result, Corollary 6.3, we have that $H = K \times C$ where K is isomorphic to H as an abstract group. But, $D = \langle x, C \mid p^k x = y \rangle \cong C \cong \mathbb{Z}_p$ trivially. Now,

$$A = \langle x \rangle + H = \langle H, x \rangle = \langle K \times C, x \rangle = K \times D.$$

It follows, by Corollary 6.3, that A is isomorphic to H as an abstract group. $\square$

Proof. (of Theorem 6.1)

Suppose G is dual-reduced. By Theorem 4.20, G is topologically isomorphic to a group constructed as in the proof of Theorem 5.5. We now proceed by induction on τ , the torsion type of G . The base case, $\tau = 2$ holds as the closure of the torsion subgroup of an abelian totally injective pro- p group must be Cartesian.

If τ is a limit ordinal, then the proof of Theorem 5.5 shows that G is the product of groups of lower torsion types. By the inductive hypothesis, each of these is isomorphic to the closure of its torsion subgroup and so G must be isomorphic to $\overline{t(G)}$.

If τ is a successor ordinal, by the argument of Proposition 5.6, we can write

$$G = \prod_{i \in I} K_i,$$

where I is not necessarily infinite, such that each K_i has torsion type τ and each $(K_i)_{T_{\tau-1}}$ is cyclic. Hence, without loss of generality, we can assume that $G_{T_{\tau-1}}$ is a cyclic group, of order

p^k for some $k \in \mathbb{N}$. But then $T_{\tau-1}(G)$ is a subgroup of G , which, by definition contains $t(G)$ and $G/T_{\tau-1}(G) \cong C_{p^k}$. But by the inductive hypothesis $T_{\tau-1}(G)$ is abstractly isomorphic to $\overline{t(G)}$, a Cartesian group of unbounded exponent. Now Lemma 6.4 gives the result.

This classifies all dual-reduced pro- p groups and so, with Theorem 4.5 and Lemma 6.2, we are done. $\square$

Combining Theorem 6.1 and Theorem 5.5 gives the following result.

Theorem 6.5. *Let A be a Cartesian group of unbounded exponent. There are $2^{\text{um}(A)}$ topologically non-isomorphic totally injective pro- p groups abstractly isomorphic to A .*

Proof. By Theorem 5.5 and Theorem 4.20 (Ulm's Theorem for totally injective pro- p groups), Theorem 6.1 implies that the number of topologically distinct totally injective pro- p group structures on A is at least the number of admissible sequences beginning with A . (In fact, it offers $\text{um}(A)$ times this many, by considering dual-reduced groups and those with non-trivial torsion-free quotients.)

By the second part of the definition of an admissible sequence, such an admissible sequence can have at most $\text{um}(A)$ non-trivial entries. The first condition in that definition shows that each term after A must have multiplicity at most $\text{um}(A)$. Write X for the cardinality of the set of cardinals less than or equal to $\text{um}(A)$. There are at most $2^X \leq 2^{\text{um}(A)}$ Cartesian groups of multiplicity at most $\text{um}(A)$.

Theorem 5.3 shows that $J = \{j \in \mathbb{N} \mid f_A(j) \geq \text{um}(A)\}$ is an infinite subset of $\mathbb{N}$. Take M any one of the $2^{\aleph_0}$ infinite subsets of $\mathbb{N}$ with $(\mathbb{N} \setminus M) \cap J$ infinite. Then taking $A^{(M)} = \prod_{m \in M} C_{p^m}^{\text{um}(A)}$ gives us $2^{\aleph_0}$ Cartesian groups $A^{(M)}$ with $m(A^{(M)}) = \text{um}(A^{(M)}) = \text{um}(A)$. For each $A^{(M)}$, there are 2^X Cartesian groups $\prod_{j \in J \setminus M} C_{p^j}^{x_j}$ of multiplicity $\text{um}(A)$. Thus we have 2^X Cartesian groups with multiplicity and unbounded multiplicity $\text{um}(A)$.

So, to construct an admissible sequence beginning with A , we choose $\text{um}(A)$ many times from 2^X choices. Hence there are $(2^X)^{\text{um}(A)} = 2^{X \text{um}(A)} = 2^{\text{um}(A)}$ non-isomorphic totally injective pro- p groups abstractly isomorphic to A . $\square$

This gives us one useful negative result.

Theorem 6.6. *Let C be an infinite Cartesian pro- p group.*

1. $|C| = |t(C)| = 2^{m(C)}$ and;

2. C has a torsion-free subgroup of order 2^X if and only if C is X -unbounded.

In the second case, C has such a subgroup as a closed direct summand which is not closed-continuous.

Proof. In Chapter 7 below, we shall show that the torsion subgroup of an infinite Cartesian group C with $m(C) = X$ is of order 2^X and that $|C/t(C)| \leq 2^X$. This implies $|C| = |t(C)| = 2^{m(C)}$. This also implies that a Cartesian group with unbounded multiplicity X cannot contain a torsion-free subgroup of order greater than 2^X .

Then Lemma 6.3 completes the proof. $\square$

Theorem 6.1 implies the following description of the abstract group structure of totally injective abelian pro- p groups.

Corollary 6.7. *Let G, H be dual-reduced totally injective abelian pro- p groups with unbounded torsion. Then G and H are abstractly isomorphic if and only if $t(G)$ is isomorphic to $t(H)$.*

Furthermore, if G, H are totally injective abelian pro- p groups with isomorphic torsion and $F(G), F(H)$ not (abstractly) isomorphic, G and H are abstractly isomorphic if and only if $2^{\max\{r(F(G)), r(F(H))\}} \leq 2^{\text{um}(T_0(G))}$.

Recall $F(G)$ is a maximal continuous torsion-free direct summand of G .

Proof. As G has unbounded torsion, by Theorem 6.1, it is abstractly isomorphic to $\prod_i (C_{p^i})^{\alpha_i}$, for some sequence (α_i) . The abstract isomorphism class of G is totally determined by the abstract isomorphism class of $t(G)$. This is because

$$\prod_i (C_{p^i})^{\alpha_i} \cong \prod_i (C_{p^i})^{\beta_i}$$

(as abstract groups) if and only if $2^{\alpha_i} = 2^{\beta_i}$ for each $i \in \mathbb{N}$. (This condition implies isomorphism by Lemma 2.7. If $2^{\alpha_i} > 2^{\beta_i}$, only one of the two groups has a direct summand isomorphic to $\bigoplus (C_{p^n})^{2^{\alpha_i}}$ and so they cannot be isomorphic.)

Now, suppose G, H are totally injective abelian pro- p groups with isomorphic torsion and $F(G), F(H)$ not abstractly isomorphic. To see the second part, Lemma 6.2 will show that G is abstractly isomorphic to

$$T_0(G) \bigoplus (\mathbb{Z}_p)^{\max\{\text{um}(T_0(G)), r(F(G))\}},$$

with H abstractly isomorphic to

$$T_0(G) \bigoplus (\mathbb{Z}_p)^{\max\{\text{um}(T_0(G)), r(F(H))\}},$$

as $T_0(G)$ and $T_0(H)$ are abstractly isomorphic.

But now, if $\text{um}(T_0(G)) \geq r(F(G))$, then G is abstractly isomorphic to $T_0(G)$. Similarly, if $\text{um}(T_0(G)) \geq r(F(H))$, then H is abstractly isomorphic to $T_0(G)$ and we are done.

If $r(F(H)) > \text{um}(T_0(G))$ then H can be written as a direct sum of a group of finite exponent, a Cartesian group of order $2^{\text{um}(T_0(G))}$ and a torsion-free group of order $2^{r(F(H))}$. By Theorem 6.6, this cannot be isomorphic to a Cartesian group.

The same argument holds for $r(F(G))$ and the result follows as $F(G), F(H)$ are not abstractly isomorphic. $\square$

We have shown that the abstract isomorphism class of a totally injective pro- p abelian group is determined by its abstract torsion subgroup, and is a Cartesian pro- p abelian group.

But how much does the structure of a Cartesian pro- p abelian group depend on its torsion subgroup?

Let $(a_i)_{i \in I}, (b_i)_{i \in I}$ be sequences of natural numbers. Is there much difference between $\prod_{i \in I} C_{p^{a_i}}$ and $\prod_{i \in I} C_{p^{b_i}}$ modulo their abstract torsion subgroups? We answer shall this question in the next chapter.

6.1 Embeddings

The statement of Theorem 6.1 leads one to ask if every totally injective abelian pro- p group with unbounded torsion is isomorphic (as a profinite group) to a closed subgroup of the closure of its torsion subgroup. In fact, a much stronger result is true.

Theorem 6.8. *Let A, B be totally injective abelian pro- p groups. If $\text{um}(\overline{t(B)})$ is greater than 0 and an upper bound for the number of open subgroups of A , then A is (topologically) isomorphic to a closed subgroup of $\overline{t(B)}$.*

Proof. Every profinite group is isomorphic to a closed subgroup of the Cartesian product of its discrete images: so every totally injective abelian pro- p group is a closed subgroup of a product of cyclic p -groups. Specifically, A is a closed subgroup of a Cartesian product of $\text{um}(\overline{t(G)})$ cyclic groups.

But, by Proposition 5.2, $\overline{t(B)}$ is Cartesian group. Hence, $\overline{t(B)}$ is a Cartesian product of $\text{um}(\overline{t(G)})$ unbounded Cartesian groups and so contains isomorphic copies of every Cartesian product of $\text{um}(\overline{t(B)})$ cyclic groups as closed subgroups. As A is isomorphic to a closed subgroup of one of these, it follows that $\overline{t(B)}$ contains a closed subgroup isomorphic to A . $\square$

Chapter 7

The abstract algebraic structure of pro- p abelian groups

In [13], I showed that every abelian countably based pro- p group which contains torsion elements of unbounded order is abstractly isomorphic to a Cartesian product of a collection of cyclic groups of unbounded order. This means that all countably based abelian pro- p groups are either abstractly isomorphic to a Cartesian product of cyclic groups or topologically isomorphic to a Cartesian product of cyclic groups of bounded order and copies of the p -adic integers.

Two abelian countably based pro- p groups with elements of unbounded order are abstractly isomorphic to each other if and only if they have isomorphic torsion subgroups. This leads us to ask “What do all these pro- p groups look like modulo their abstract torsion subgroups?”

In this chapter, we show that Cartesian pro- p groups of unbounded exponent have the same structure modulo their torsion subgroups. By Corollary 6.7, the same is true for all totally injective groups with appropriate unbounded multiplicity and torsion-free rank.

To do this we follow the ideas of [1], Appendix 1, and generalise the result there.

We also make some general observations about the subgroup structure of $\mathbb{Z}_p$.

First, however, an aside on p -adic topology.

7.1 p -adic Topology

Recall the notion of the p -adic topology.

Definition 49. Let G be an abelian group with $\bigcap_n p^n G = 1$. The p -adic metric on G is the metric on G given by

$$d_p(x, y) = p^{-k}, \text{ where } k = \max\{i \in \mathbb{N} \mid x - y \in p^i G\}.$$

The topology induced by this metric is called the p -adic topology on G .

The p -adic topology on a $\mathbb{Z}_p$ -module M has basis $\{p^i M\}$.

For a given $\mathbb{Z}_p$ -module M , we shall write $\widetilde{M}$ to denote the completion of M with respect to its p -adic topology.

Lemma 7.1. Let L be a cardinal. The p -adic completion $\widetilde{\bigoplus_L \mathbb{Z}_p}$ is abstractly isomorphic to the inverse limit $\varprojlim_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$.

Note that this is the inverse limit in the category of abstract groups; no topology is involved as the p -adic topology is completely determined by the group-theoretic structure.

Proof. The inverse limit $\varprojlim_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$ can be considered as the subgroup

$$\{(y_n)_{n \in \mathbb{N}}, y_n \in \bigoplus_L \mathbb{Z}/p^n \mathbb{Z} \mid \forall i \geq j, y_i \equiv y_j \pmod{p^j}\}$$

of “coherent sequences”, in the Cartesian group $\prod_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$.

Consider X , the set of sequences in $\bigoplus_L \mathbb{Z}_p$ which are Cauchy under the p -adic topology. These are just strings $(x_i)_{i \in \mathbb{N}}$, with entries $x_i \in \bigoplus_L \mathbb{Z}_p$ such that for each $j \in \mathbb{N}$ there is some $N \in \mathbb{N}$ such that for all $k \geq j$, $x_j - x_k \in p^N (\bigoplus_L \mathbb{Z}_p)$. It forms a subgroup of the (abstract) group $\prod_{i \in \mathbb{N}} (\bigoplus_L \mathbb{Z}_p)$.

Take a Cauchy sequence $x = (x_i)_{i \in \mathbb{N}} \in X$. For each $n \in \mathbb{N}$ there is some $k(n)$ such that, for each $M \geq k(n)$, we have

$$x_M \equiv x_{k(n)} \equiv \theta_n(x) \pmod{p^n \bigoplus_L \mathbb{Z}_p},$$

for some $\theta_n(x) \in \bigoplus_L (\{0, 1, \dots, p^n - 1\})$.

Now, consider the map $\theta : X \rightarrow \varprojlim_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$ given by $x \mapsto (\theta_n(x))_{n \in \mathbb{N}}$. From the definition, above, it is apparent that

$$\theta_n(x) \equiv \theta_{n+1}(x) \pmod{p^n \bigoplus_L \mathbb{Z}_p},$$

as if $a_n \equiv a \pmod{p^n}$ and $a_{n+1} \equiv a \pmod{p^{n+1}}$, then $a_{n+1} \equiv a_n \pmod{p^n}$. Hence θ is a well-defined map. As all operations involved are addition of integers, sometimes p -adic, sometimes modulo p^n , sometimes componentwise, this is a homomorphism of groups.

Suppose $(x_i)_{i \in \mathbb{N}} = x \in X$ satisfies $\theta(x) = 0$. Then for each $n \in \mathbb{N}$, there is some $k(n) \in \mathbb{N}$ such that for every $j \geq k(n)$, x_j is a p^n th power. Hence x tends to 0 in the p -adic topology. So $\ker \theta$ is the set of all p -adically Cauchy sequences tending to 0, consequently the image of θ is isomorphic to the set of their limits, $\widehat{\bigoplus_L \mathbb{Z}_p}$. Hence it remains only to show that θ is surjective.

Given any $(y_i)_{i \in \mathbb{N}} \in \varprojlim_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$, we want to find a Cauchy sequence x , with entries from $\bigoplus_L \mathbb{Z}_p$ with $\theta(x) = ((y_i)_{i \in \mathbb{N}})$. But $\bigoplus_L \mathbb{Z}_p$ maps to each $\bigoplus_L \mathbb{Z}/p^n \mathbb{Z}$ and hence, by the universal property, to $\varprojlim_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$ in a way that commutes with all the appropriate maps. So we can consider each y_i to be a coset represented by an element $y'_i \in \bigoplus_L \mathbb{Z}_p$. Now $(y'_i)_{i \in \mathbb{N}}$ is p -adically Cauchy, by construction and so θ is surjective. $\square$

This allows us to give a torsion-free version of Theorem 2.7.

Theorem 7.2. *Let G be a torsion-free abelian pro- p group. Then G is abstractly isomorphic to the completion of a direct sum of copies of $\mathbb{Z}_p$ with respect to its p -adic topology.*

In particular, $\mathbb{Z}_p^X$ and $\mathbb{Z}_p^Y$ are abstractly isomorphic if and only if $2^X = 2^Y$.

Proof. Certainly, each torsion-free abelian pro- p group is a module for the discrete valuation ring $\mathbb{Z}_p$, which is complete with respect to its p -adic topology. Pro- p groups are residually finite, so contain no elements of infinite height. Hence, by Theorem 7.7, G must be a completion of a direct sum of copies of $\mathbb{Z}_p$.

By Lemma 7.1, G is abstractly isomorphic to a group of form $\varprojlim_{n \in \mathbb{N}} (\bigoplus_L \mathbb{Z}/p^n \mathbb{Z})$, for some cardinal L . But for G to be (abstractly isomorphic to) this group, L must be the $\mathbb{F}_p$ -dimension of the abstract quotient G/pG .

But $\mathbb{Z}_p^X/p\mathbb{Z}_p^X$ is abstractly isomorphic to $(C_p)^X$ and $\mathbb{Z}_p^Y/p\mathbb{Z}_p^Y$ is abstractly isomorphic to $(C_p)^Y$. The second statement follows, by Theorem 2.7. $\square$

7.2 The Abstract Algebraic Structure of Cartesian Abelian Groups

For this section, we fix the following notation. Let $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence of cardinals.

We set $A = \prod_{n \in \mathbb{N}} (\mathbb{Z}/p^n \mathbb{Z})^{\alpha_n} \cong \prod_{n \in \mathbb{N}} (C_{p^n})^{\alpha_n}$. Write α for the unbounded multiplicity of A . As the torsion of A is of unbounded exponent, $\alpha \geq \aleph_0$. In fact,

$$\alpha = \max\{\beta \mid \beta = \alpha_n \text{ for infinitely many } n \in \mathbb{N}\}.$$

Suppose $m(A) > \alpha = \limsup_n \{\alpha_n\}$. Then A is infinite and we have $A = F \times \prod_{n \in \mathbb{N}} (C_{p^n})^{\gamma_n}$, where F is of finite exponent and of order $2^{m(A)}$, and $\max_n \{\gamma_n\} \leq \alpha$. At this point we can switch to considering the group $\prod_{n \in \mathbb{N}} (C_{p^n})^{\gamma_n}$. Modulo basic subgroups of torsion subgroups, this group and A will be isomorphic.

Define T to be the torsion subgroup of A .

Note

$$T = \{(x_n) \mid \exists i \text{ such that } \forall n \text{ such that } n \geq i, x_n \in p^{n-i}(C_{p^n})^{\alpha_n}\},$$

and if A is infinite, $|A| = |T| = 2^{m(A)}$.

We consider A as an abstract $\mathbb{Z}_p$ -module.

Proposition 7.3.

$$B = \{(x_n) \mid \exists i \text{ such that, } \forall n \geq i, x_n = 0\} \leq A$$

is a basic submodule of T .

By “basic submodule”, we mean “a basic subgroup of a module which is also a submodule”. As we are working with torsion groups, a subset is a subgroup if and only if it is a submodule.

Proof. First,

$$B \cong \bigoplus_{n \in \mathbb{N}} \left(\prod_{\alpha_i = n} \mathbb{Z}/p^n \mathbb{Z} \right)^{\alpha_n},$$

and so, as vector spaces have bases, is a direct sum of cyclic groups. If $(x_n) = p(y_n) \in B$ with $(y_n) \in T$, then each x_n is in $p(C_{p^n})^{\alpha_n}$. Hence $(x_n) \in pB$ and so B is a pure subgroup of T . Hence $T/B = p(T/B)$ and so T/B is divisible, so B is a basic subgroup of A . $\square$

Proposition 7.4. *A is complete in its p-adic topology.*

This follows straightforwardly from the fact that the product of a family of complete groups is complete.

Lemma 7.5. *The subset*

$$C = \{(x_n) \mid \forall i \exists j \text{ such that, } \forall k \text{ such that } k \geq j, x_k \in p^i(C_{p^k})^{\alpha_k}\}$$

is the closure of B in the p -adic topology of A . It contains all torsion elements of A .

The above equation considers A as an abstract $\mathbb{Z}_p$ -module here. Consequently, this subgroup (and submodule) C is abstractly characteristic, in addition to being the closure of the torsion part of A in its p -adic topology. This subgroup plays an important role and we shall call it *the p -adic torsion subgroup* of A .

Proof. Let $((x_n^{(i)})_{n \in \mathbb{N}})_i$ be a Cauchy sequence in B tending to $x = (x_n)$ in A . Fix $j \in \mathbb{N}$. There is some N such that for all $k \geq N$, $x_n - x_n^{(k)} \in p^j A$. Hence, for all n , we have $x_n - x_n^{(k)} \in p^j(C_{p^n})^{\alpha_n}$.

Consider $(x_n^{(N)})$. As it is an element of B , there is some $M \geq N$ such that for all $k \geq M$, $x_k^{(N)} = 0$. Hence for all $k \geq M$, $x_k \in p^j(C_{p^k})^{\alpha_k}$ and so the closure of B is contained in C . Now, let $y = (y_n) \in C$ be arbitrary. Set $z_n^{(i)} = y_n$ if $i \geq n$ and $z_n^{(i)} = 0$ otherwise. Now, for each i , $(z_n^{(i)})$ is in B . The sequence $((z_n^{(i)})_n)_i$ is Cauchy and tends to y . Hence, C is the closure of B in A .

Now, all torsion elements of A lie in C .

For if $t = (t_n)$ is a torsion element of A of order p^i then, for every $j \geq i$, t_j must lie in $p^{j-i}(C_{p^j})^{\alpha_j}$. Hence $B \subseteq T \subseteq C$ and C is the closure of B in the p -adic topology. $\square$

Consider the series

$$0 \leq B \leq T \leq C \leq A.$$

What can we say about the quotients of this series?

Theorem 7.6. *With the notation above, $A = \prod_{n \in \mathbb{N}} (\mathbb{Z}/p^n \mathbb{Z})^{\alpha_n}$, B a basic submodule of T , the torsion part of A , we have*

$$A/B \cong \bigoplus_X C_{p^\infty} \times \bigoplus_X \mathbb{Q} \times \widetilde{\bigoplus_X \mathbb{Z}_p}$$

$$A/T \cong \bigoplus_X \mathbb{Q} \times \widetilde{\bigoplus_X \mathbb{Z}_p}$$

$$A/C \cong \widetilde{\bigoplus_X \mathbb{Z}_p}$$

where α is the unbounded multiplicity of A and $|X| = 2^\alpha$.

We shall prove this via a series of smaller results.

We call upon some results on general discrete valuation rings which we shall need in the proof of Theorem 7.6.

Theorem 7.7. *Let R be a discrete valuation ring with residue field finite of characteristic p , M an R -module with no elements of infinite height, such that M is complete with respect to its p -adic topology. Then M is the (p -adic) completion of a direct sum of cyclic modules.*

For proof, see [10], where it is Theorem 22.

(This gives us a torsion-free version of Theorem 2.7: see Theorem 7.2 below.)

Theorem 7.8. *Let R be a discrete valuation ring with residue field finite of characteristic p , M an R -module, S a pure submodule of M . If S has no elements of infinite height and is complete with respect to the p -adic topology, then S is a direct summand of M .*

For proof, see [10], where it is Theorem 23.

This allows us to prove that modulo its p -adic torsion subgroup, a Cartesian pro- p group is the p -adic completion of a direct sum of copies of $\mathbb{Z}_p$ following the argument of [1].

Recall that we have a Cartesian group A with torsion subgroup T which has basic subgroup B , with C the closure of B in the p -adic topology of A .

Proposition 7.9. *The quotient C/B is divisible and isomorphic to $\bigoplus_{2^\alpha} C_{p^\infty} \times \bigoplus_{2^\alpha} \mathbb{Q}$.*

Note, as a consequence, that $A/B \cong A/C \times C/B$. But Lemma 7.10 tells us that $A/B \cong \widetilde{\bigoplus_\alpha \mathbb{Z}_p}$ and so with the following proof we have proved Theorem 7.6.

Proof. First, we show that C/B is divisible. As we showed above, B is a basic subgroup of T and the factor T/B is divisible. As divisible submodules are direct summands, $C/B \cong T/B \times C/T$.

Now C is the closure of T in A and so $C = \bigcap_{n \in \mathbb{N}} (p^n A + T)$.

Now, as C/T is divisible and torsion-free, it must be the direct sum of some number of copies of $\mathbb{Q}$.

However, as A is α -unbounded, we can write $A = \prod_{i \in I} A_i$, where $|I| = \alpha$, each A_i is a Cartesian product of cyclic p -groups, with torsion elements of unbounded order.

Hence, by the arguments of this chapter, for each $i \in I$,

$$A_i \geq C_i \geq T_i \geq B_i$$

with

$$A = \prod_{i \in I} A_i, C = \prod_{i \in I} C_i, T = \prod_{i \in I} T_i,$$

and versions of every statement we have proved about A, C, T and B holding for each A_i and its submodules. While we do not necessarily have $B = \prod_{i \in I} B_i$, it is certainly true that, for $X \in \{A, C, T\}$, X/B has surjective image $\prod_i X_i/B_i$.

However each T_i/B_i and each C_i/T_i , is non-empty. Write $A_i = \prod_{j \in J} \mathbb{Z}/p^{b_j} \mathbb{Z}$, for each i . Now (p^{b_j-1}) gives a non-zero element of T_i/B_i , as it is of order p , but has non-zero entries in co-ordinates corresponding to factors of arbitrarily high order. Also, $(p^{\lfloor b_j/2 \rfloor})$ is not in T_i , as there is no bound on the order of the entries of its co-ordinates, but is in C_i , giving a non-zero element of C_i/T_i . (For any $k \in \mathbb{N}$, we have $k/2$ such that $\forall l$ such that $b_l \geq k/2$, $p^{\lfloor b_j/2 \rfloor} \in p^k \mathbb{Z}/p^{b_l} \mathbb{Z}$, so this is indeed in C_i .)

Hence C/B must contain a subgroup isomorphic to $\prod_{\alpha} C_{p^\infty} \times \mathbb{Q}$. (This product gives us a subgroup of an image: that there is a subgroup follows as C/B is divisible, so splitting occurs.) But $C/B \cong C/T \times T/B$, $|C/B| \leq 2^\alpha$, as it is a submodule of A , and from the above, $|C/T|, |T/B| \geq 2^\alpha$. Hence,

$$C/B \cong \prod_{\alpha} (C_{p^\infty} \times \mathbb{Q}) \cong \bigoplus_{2^\alpha} (C_{p^\infty} \times \mathbb{Q}).$$

□

This completes the proof of Theorem 7.6.

Lemma 7.10. *There is a $\mathbb{Z}_p$ -submodule V of A such that $A = C \times V$. Furthermore, $V \cong \widetilde{\bigoplus_{2^\alpha} \mathbb{Z}_p}$.*

Proof. Now, C is the closure of T in the p -adic topology.

As C is the closure in the p -adic topology of a pure submodule B , C is pure in A . (This is [10, exercise 55].) As A is residually a finite p -group, $\bigcap_{n \in \mathbb{N}} p^n A$ is trivial, so A and C contain no non-trivial elements of infinite height.

Now Theorem 7.8 shows that $A = C \times V$, for V some closed submodule of A . We can

see that V must be complete and, as $T \subseteq C$, torsion-free.

By Theorem 7.7, V , as a complete pure submodule of A which has no non-trivial elements of infinite height, is the completion of a direct sum of copies of $\mathbb{Z}_p$ under the p -adic topology.

Now, consider $\dim(V/pV)$. By the argument of the proof of Proposition 7.9, we can see that V is isomorphic to the Cartesian product of α unbounded Cartesian groups, modulo their p -adic torsion subgroups. Any unbounded Cartesian group is non-trivial modulo this subgroup: any sequences all of whose entries are of height 0 will be outside this group. This means that V/pV has a subgroup isomorphic to $C_p^\alpha \cong \bigoplus_{2^\alpha} C_p$. Hence $\dim(V/pV)$ is at least 2^α . As this dimension is bounded above by this, due to the size of the group (see the discussion following Theorem 5.3), it follows that $\dim(V/pV) = 2^\alpha$.

Lemma 7.1 shows that if W is the completion of a direct sum of L copies of $\mathbb{Z}_p$ under the p -adic topology, then $\dim(W/pW) = L$. Hence if V is the completion of a direct sum of copies of $\mathbb{Z}_p$ in the p -adic topology and $\dim(V/pV) = 2^\alpha$, then $V \cong \bigoplus_{2^\alpha} \mathbb{Z}_p$. $\square$

This gives us a powerful statement.

Corollary 7.11. *Let G be a totally injective pro- p group.*

Then

$$G/t(G) \cong \prod_{\max\{\text{um}(\overline{t(G)}), r(F(G))\}} \mathbb{Z}_p \oplus \bigoplus_{2^{\text{um}(\overline{t(G)})}} \mathbb{Q}.$$

Proof. By Theorem 6.1, G is abstractly isomorphic to $t(G) \times \mathbb{Z}_p^{r(F(G))}$.

Now, Theorem 7.6, with Theorem 7.2, implies the result. $\square$

This result as written cannot be true for all abelian pro- p groups: unbounded multiplicity is only defined for Cartesian groups. If we try to consider, for instance, the dual T^* of the abstract torsion subgroup of the Cartesian product $\prod_n C_{p^n}$, it would not be clear what cardinal to use. However, it does lead to an interesting question.

Is $G/t(G)$ a direct sum of a $\mathbb{Q}$ -space and a product of copies of $\mathbb{Z}_p$ for any abelian pro- p group G ? This would be an interesting question to look into: a possible counterexample is the pro- p completion of a direct sum of countably many copies of $\mathbb{Z}_p$: it is not clear to me whether or not this works. (Similarly, could it be isomorphic to a direct sum of a rational space with a completion of a direct sum of copies of $\mathbb{Z}_p$ under its p -adic topology? This is strictly weaker than the previous question.)

In fact, we can generalise this.

Theorem 7.12. *Let G be a profinite abelian group. Write G_q for the q -Sylow subgroup of G .*

Then

$$\frac{\prod_{q \text{ prime}} t(G_q)}{t(G)} \cong \bigoplus_X \mathbb{Q}$$

for some cardinal X .

If each G_q is abstractly isomorphic to a Cartesian product of procyclic groups, then $X = 2^Y$, where $Y = \limsup_q \{\dim_{\mathbb{F}_q}(t(G_q)/qt(G_q))\}$.

(As previously noted, each totally injective pro- p group is abstractly isomorphic to a Cartesian product of procyclic groups.)

Proof. Write $K = \frac{\prod_{q \text{ prime}} t(G_q)}{t(G)}$.

Let $x \in K$. From Proposition 4.2, for $q \neq p$ every element of G_q is a multiple of p . Hence, modulo $t(G_p)$, x is a multiple of p . So K is divisible. Clearly K is also torsion-free, hence a rational space.

Now, suppose each G_q is totally injective. If $Y = 0$, the result holds. Otherwise, we can write $t(G)$ as the product of Y infinite countably-based groups of torsion type 1, $(N_y)_{y \in Y}$, each involving infinitely many primes. Then each $N_y + t(G)/t(G)$ contains a copy of $\mathbb{Q}$ and hence we have $X \leq 2^Y$. But we can remove all primes q such that $\dim_{\mathbb{F}_q}(t(G_q)/qt(G_q)) \geq Y$ as direct summands without affecting K . This leaves a group whose order cannot be more than 2^Y , which gives the result. $\square$

7.3 The Abstract Subgroup Structure of $\mathbb{Z}_p$

Similarly to the above, we can make some observations about the abstract subgroup structure of $\mathbb{Z}_p$.

Firstly, we note:

Theorem 7.13. *Let $x \in \mathbb{Z}_p$ be non-trivial.*

Then,

$$\mathbb{Z}_p/\langle x \rangle \cong C_{p^n} \oplus \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_{q \text{ prime}, q \neq p} C_{q^\infty},$$

where n is the p -height of x in $\mathbb{Z}_p$.

(Recall that, as a residually finite group, the abstract group $\mathbb{Z}_p$ can contain no elements of infinite height.)

Proof. First, consider $\mathbb{Z}_p/\mathbb{Z}$. This is a subgroup of $\mathbb{Q}_p/\mathbb{Z}$. There is an embedding of C_{p^∞} into $\mathbb{Q}/\mathbb{Z} < \mathbb{Q}_p/\mathbb{Z}$ given by $\langle p^{-n} \mid n \in \mathbb{N} \rangle$. But this group is divisible and so a direct summand of $\mathbb{Q}_p/\mathbb{Z}$. It is plain to see that $\mathbb{Z}_p/\mathbb{Z}$ provides a complement to this group. Hence $\mathbb{Q}_p/\mathbb{Z}_p \cong C_{p^\infty}$ and $\mathbb{Q}_p/\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Z}_p/\mathbb{Z}$.

As $\mathbb{Q}/\mathbb{Z}$ is divisible, we have

$$\mathbb{Q}_p/\mathbb{Z} \cong \mathbb{Q}_p/\mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z},$$

and $\mathbb{Q}_p/\mathbb{Z}$ is divisible as the $\mathbb{Q}$ -space $\mathbb{Q}_p/\mathbb{Q}$ is divisible.

Recall that $\mathbb{Q}/\mathbb{Z} \cong \bigoplus_{q \text{ prime}} C_{q^\infty}$. From our above description we can see $\mathbb{Q}_p/\mathbb{Z}$ has a unique subgroup isomorphic to C_{p^∞} . Hence

$$\mathbb{Z}_p/\mathbb{Z} \cong \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_{q \text{ prime}, q \neq p} C_{q^\infty}.$$

For every $x \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$, we have an automorphism $y \mapsto x.y$ of the p -adic numbers and so x is automorphism-conjugate to 1 in $\mathbb{Z}_p$. This gives rise to an isomorphism between $p^n\mathbb{Z}_p/\langle p^n x \rangle$ and the divisible $\mathbb{Z}_p/\mathbb{Z}$.

By the definition of height, we have $\langle x \rangle \leq p^n\mathbb{Z}_p$, with $x \in p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$. For every $a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$, we have an automorphism $b \mapsto b.a$ of the p -adic numbers and so a is automorphism-conjugate to 1 in $\mathbb{Z}_p$. But there exists some $a \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ with $x = p^n a$ and we thus have an isomorphism between $\mathbb{Z}_p/\mathbb{Z}$ and $p^n\mathbb{Z}_p/\langle p^n a \rangle = p^n\mathbb{Z}_p/\langle x \rangle$.

But above we showed that $p^n\mathbb{Z}_p/\langle x \rangle$ is divisible and so $\mathbb{Z}_p/\langle x \rangle \cong p^n\mathbb{Z}_p/\langle p^n x \rangle \oplus \mathbb{Z}_p/p^n\mathbb{Z}_p$ and so the result holds. $\square$

Theorem 7.14. *Let Γ be a non-trivial, d -generated abstract subgroup of $\mathbb{Z}_p$.*

Then there is a unique $n \in \mathbb{N}$ such that $p^{n+1}\mathbb{Z}_p \not\geq \Gamma \leq p^n\mathbb{Z}_p$. It follows that $\mathbb{Z}_p/\Gamma$ is isomorphic to $C_{p^n} \oplus \Delta$, for some divisible group Δ .

Moreover, for every prime q (including p), the q -primary component of $\mathbb{Z}_p/\Gamma$ is the direct sum of at most d non-trivial groups.

Proof. As $\bigcap_n p^n\mathbb{Z}_p = \{0\}$, there is a unique $n \in \mathbb{N}$ such that $p^{n+1}\mathbb{Z}_p \not\geq \Gamma \leq p^n\mathbb{Z}_p$. Consequently, there is some $x \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$, with $p^n x \in \Gamma$.

By the previous result, $\mathbb{Z}_p/\Gamma$ is isomorphic to a quotient of

$$\mathbb{Z}_p/p^n\mathbb{Z} \cong C_{p^n} \oplus \bigoplus_{2^{\aleph_0}} \mathbb{Q} \oplus \bigoplus_{q \text{ prime, } q \neq p} C_{q^\infty},$$

and so the first statement holds.

We can see that $\Gamma/\langle p^n x \rangle$ will be $(d-1)$ -generated. As all non-trivial elements of $\mathbb{Q}$ are conjugate under an automorphism, for every $r \in \mathbb{Q} \setminus \{0\}$, the isomorphism $\mathbb{Q}/\langle r \rangle \cong \mathbb{Q}/\mathbb{Z}$. By Theorem 3.2, any quotient of a $\mathbb{Q}$ -space is isomorphic to a direct sum of copies of $\mathbb{Q}$ and direct sums of C_{p^∞} . Hence all torsion subgroups of a quotient of an infinite $\mathbb{Q}$ -space by a $(d-1)$ -generated group are subgroups of $(\mathbb{Q}/\mathbb{Z})^{d-1}$. From this, the second statement follows. $\square$

Chapter 8

Rings

One use of abelian groups in group theory is to construct rings. This gives us a large source of potential examples, via groups of matrices over these rings, or over rings of polynomials over these rings. In this section we show how to construct a commutative unital pro- p ring R such that $(R, +)$ is an arbitrary non-trivial totally projective pro- p group G .

Definition 50. Let $G \cong \prod_{i \in \mathbb{N}} (C_{p^i})^{a_i}$ be a Cartesian group.

For each natural number n , write $\sigma_n(G)$ to denote the group $\prod_{i \in \mathbb{N}} (C_{p^{i+n}})^{a_i}$.

More generally, for G a totally injective abelian pro- p group, we write $\sigma_n(G)$ to denote the group $D \oplus F$, where F is the torsion-free part of G and D is the dual-reduced group with torsion sequence $(\sigma_n(G_{T_\alpha}))$.

Note that by definition, $p^n \sigma_n(G) \cong G$.

Theorem 8.1. Let G be a non-trivial totally injective pro- p abelian (dual-reduced) group. Write τ for the torsion type of G .

We can construct a commutative, unital pro- p ring R with $(R, +)$ topologically isomorphic to G . Furthermore, we can choose R such that for each ordinal α , $T_\alpha((R, +))$ is an ideal of R .

This gives us a commutative, unital ring for each non-trivial totally projective profinite group. We do not really need dual-reduced – we get around this by considering the Cartesian product with free $\mathbb{Z}_p$ modules.

Proof. We proceed by induction on τ , the torsion type of the group. We will find the desired ring R_τ as a closed subring of a ring R_α with additive group of torsion type α less than τ . (In fact, the relation between $(R_\alpha, +)$ and G is alluded to by Section 6.1.)

Our base case is $\tau = 2$. By Proposition 5.2, G is topologically isomorphic to $\prod_{i \in \mathbb{N}} C_{p^{n_i}}$ for some sequence (n_i) . Then $R = \prod_{i \in \mathbb{N}} \mathbb{Z}/p^{n_i}\mathbb{Z}$ is a commutative unital ring satisfying the above conditions.

Now, assume statement proved for all groups of torsion type less than τ . Furthermore, by Theorem 5.5 and Proposition 5.6, we can assume without loss of generality that $G_{T_{\tau-1}}$ is cyclic, whenever τ is a successor ordinal.

Case 1: $\tau - 1$ exists and is greater than 1. We have that $G_{T_{\tau-1}}$ is cyclic of order p^n . Now, by the inductive hypothesis, we can construct a ring S , with $(S, +) \cong \sigma_n(T_{\tau-1}(G))$ satisfying the conditions of the theorem.

Define R to be the subring of S generated by 1_S and $p^n S$. Now 1_S is an identity for R and so we can drop the subscript without risk of confusion. It follows that $(p^n S, +) \cong p^n \sigma_n(T_{\tau-1}(G)) \cong T_{\tau-1}(G)$, by definition of σ_n .

All the (additive) torsion elements of $(R, +)$ are contained in $p^n S$, which is an ideal of R . As $p^n S$ is closed, it follows that the torsion series of $(R, +)$ is given by $T_{\alpha}((R, +)) = T_{\alpha}(p^n S, +)$, for each $\alpha \leq \tau - 1$. In particular, $p^n S = T_{\tau-1}((R, +))$. It follows that $R/T_{\tau-1}((R, +))$ is isomorphic to $\mathbb{Z}/p^n\mathbb{Z}$, with isomorphism given by $1 + T_{\tau-1}((R, +)) \mapsto 1 + p^n\mathbb{Z}$. Hence $(R, +)_{T_{\tau-1}}$ is cyclic of order p^n and so $(R, +)$ has the same torsion sequence as G . By the pro- p version of Ulm's Theorem it follows that $(R, +) \cong G$.

Case 2: τ is a limit ordinal.

By Proposition 5.6 or examining the proof of Theorem 5.5, $G = \prod_{i \in I} G_i$ where each G_i is of torsion type τ . By the inductive hypothesis, we can find rings R_i with $(R_i, +) \cong G_i$, which satisfy the appropriate condition. We claim that $R = \prod_{i \in I} R_i$ satisfies the conditions. Firstly, $(R, +) = (\prod R_i, +) = \prod (R_i, +) \cong \prod G_i = G$. Hence $(R, +)_{T_{\alpha}} \cong \prod_i (G_i)_{T_{\alpha}}$ for each α and so condition 1 holds. As $R/T_{\alpha}((R, +)) \cong \prod_i R_i/T_{\alpha}((R_i, +))$, 2 also holds, which completes our induction. □

While constructing these rings, we make a lot of choices: it is not clear whether or not these choices affect the (ring) isomorphism class of ring we construct.

Example 6. For instance, recall, $H_{\omega+1}$ has $t_0(H_{\omega+1}) \cong \prod_n C_{p^n}$. We can write $H_{\omega+1} = K_e \times \prod_n C_{p^{2n-1}} = \prod_n C_{p^{2n}} \times K_o$, for some K_o, K_e closed subgroups of H_{ω} .

Would any of $R_{H_{\omega+1}}$, as constructed above, $R_{K_e} \times \prod_n \mathbb{Z}/p^{2n-1}\mathbb{Z}$, or $\prod_n \mathbb{Z}/p^{2n}\mathbb{Z} \times R_{K_o}$, be isomorphic to each other? This is Question 2 in the next chapter.

This is another way that pro- p groups are more structurally interesting than their duals.

Theorem 8.2. *Let Γ be an abstract abelian p -group without finite exponent.*

Then, there is no commutative unital ring R such that $(R, +) \cong \Gamma$.

Proof. If R is a commutative unital ring, then for each $x \in R$, $(n1_R).x = nx$. Hence the additive order of 1_R is an upper bound for additive orders of elements in R . In fact, as $1_R \in R$, its order is the maximum of these orders. But if Γ is a discrete p -group without finite exponent, there is no maximum. $\square$

This shows that the only abstract abelian p -groups which can be given non-profinite commutative unital ring structure are those of finite exponent. In that case, we can take (finite) direct sums of finite rings with rings of formal power series of the form $(\mathbb{Z}/p^n\mathbb{Z})[[t_n]]$.

Chapter 9

Further Questions

In this, we collect some further questions for study. We have three major specific questions, then a smaller one:

Question 1. Can there be two profinite groups G, H which are in some “genuine” way non-abelian which are not isomorphic topologically but are isomorphic as abstract groups?

(We can easily take, for any finite non-abelian group N , say,

$$G = N \times \prod_n C_{p^n} \text{ and } H = N \times \mathbb{Z}_p \times \prod_n C_{p^n},$$

to get groups which are non-topologically isomorphic. Is it possible to do so if G cannot be written as $G = K \times L$ such that our non-continuous isomorphism $\phi : G \rightarrow H$ restricts to a homeomorphism on K with L abelian?)

One possible way to do this while building on the abelian work would be to attempt to build discontinuously isomorphic commutative unitary pro- p rings R, S and consider matrix groups over these rings. The work in Chapter 8 could help towards this, but we are unable to answer the following question about their structure:

Question 2. Let R, S be commutative unital pro- p rings constructed via Theorem 8.1 with $(R, +)$ and $(S, +)$ topologically isomorphic. When must R and S be topologically isomorphic? When must they be abstractly isomorphic?

If we reduce our requirements to $(R, +)$ and $(S, +)$ abstractly isomorphic, what can we say about abstract isomorphism classes of R, S ?

Example 6 provides a more concrete version of this question.

What can we say about abelian pro- p groups which are not totally injective? The easiest example to handle is the dual of the torsion subgroup of a Cartesian group.

Theorem 9.1. *Let (a_n) be a sequence of cardinals which does not tend to 0. Suppose Γ is the torsion subgroup of $\prod_n (C_{p^n})^{a_n} = A$.*

Then $G = \Gamma^*$ is not abstractly isomorphic to a Cartesian group.

Proof. As always, we can assume without loss of generality that $\text{um}(A) = m(A)$: by the first condition this is infinite.

By the work of Chapter 7, we know that Γ has a basic subgroup Δ isomorphic to $\bigoplus_n (C_{p^n})^{a_n}$. By Theorem 7.6, we know that the divisible group Γ/Δ is isomorphic to $(C_{p^\infty})^{2^{\text{um}(A)}}$. By Theorem 3.2, we know that this is isomorphic to a direct sum of $2^{2^{\text{um}(A)}}$ copies of C_{p^∞} .

By Pontryagin duality, we have

$$G/H \cong \prod_n (C_{p^n})^{a_n},$$

where $H = \text{Ann}_G(\Delta)$. From the equation presented above, we have $\text{um}(G/H) = \text{um}(A)$.

Moreover, from our knowledge of $\Gamma/\Delta \cong H^*$, the torsion-free group H is isomorphic to $\mathbb{Z}_p^{2^{2^{\text{um}(A)}}}$. If x is a non-trivial torsion element of G , x cannot be in H and hence $t(G)$ is isomorphic to a closed subgroup of G/H .

We know that A contains no elements of infinite height: hence Γ has no elements of infinite height and so $G = \overline{t(G)}$. As $G = \overline{t(G)}$, we have

$$|t(G)| \leq |G/H| = 2^{\text{um}(A)}.$$

But for a Cartesian group, Theorem 7.6 shows that $|t(G)| = |H|$, which is greater than $|G/H|$ and so G is not Cartesian. $\square$

This argument shows that for any unbounded Cartesian group A , $t(A)^*$ is not abstractly isomorphic to a Cartesian group.

Question 3. What can we say about the discontinuous isomorphism problem for abelian pro- p groups which are not totally injective?

Can we have a not-totally-injective group G abstractly isomorphic to a Cartesian group?

Corollary 7.11 raised an interesting question:

Question 4. Let G be an abelian pro- p group. Is $G/t(G)$ a direct sum of a $\mathbb{Q}$ -space and a Cartesian product of copies of $\mathbb{Z}_p$?

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¹I have not read this, or known any way to access a copy.

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