

AXIOMATIC FORMULATION OF QUANTUM FIELD THEORY

by

ANSARUDDIN SYED

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Department of Theoretical Physics,
Imperial College of Science and Technology
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P R E F A C E

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Except where stated in the text, the work described is original and has not been submitted in this or any other University for any other degree. The work contained in the last chapter was done in collaboration with Mr. M.A. Rashid and the author is grateful to him for allowing to include it in the Thesis. He also wishes to express his acknowledgement to the Pakistan Atomic Energy Commission and the British Department of Technical Cooperation for a financial grant, under the Colombo Plan, during his stay at Imperial College.

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A B S T R A C T

In the first chapter, we give the details of the LSZ formulation of field theory, including the reduction of s-matrix elements to time ordered functions and, for some restricted region in momentum space, to ordinary retarded function. The second chapter starts with (Burgoyne's) definition of generalised retarded operators R_α , and their properties which have been proved by Araki, Burgoyne and Ruelle. Reduction formulae are then proved for them i.e. expressions are obtained for the commutators of an arbitrary generalised retarded operator with in- and out-fields. These formulae lead, in a natural way, to a special subclass of generalised retarded functions whose analyticity and support properties are studied in some detail. We also prove that they are identical with the set of Steinmann retarded operators introduced in reference (8) as well as with Burgoyne's special subclass defined in the appendix of his thesis. In the end, we obtain a generating functional for this particular subclass of retarded operators. In the third chapter, we prove that the matrix element

$$\langle p_1, p_2, \dots, p_m \text{ out } / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle$$

can be reduced, for arbitrary p , to a linear combination of r_β^s , the retarded functions belonging to the above subclass. An explicit expression for this linear combination is obtained for $p \in C_\alpha$ and

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is found to be identical with the expansion of r_α in terms of r_β . This expansion is itself obtained in the fourth chapter, which also contains the proof of linear independence of r_β .

CHAPTER I

LSZ FORMULATION OF FIELD THEORY

1. Introduction

During the past few years, considerable amount of work has been done on the axiomatic approach to quantum field theories. The motivation of this approach has been the fact that although a large number of interesting results, which agree fairly well with experiments, can be obtained by conventional methods, i.e. by using the Lagrangian formulation, the perturbation theory with renormalization, and dispersion techniques, it has not been possible to give a satisfactory answer to the question; whether mathematically rigorous solutions of renormalized field equations exist in any theory. In other words, one is not sure if the very basis of the above techniques is correct. One therefore tries to analyse the structure of field theories by setting up a self-consistent scheme which starts from certain fundamental axioms based on well established physical principles. Fortunately it is found that a fairly small number of axioms lead to a remarkably large amount of interesting and important results.

At present, there exists two important formulations of the axiomatic approach. The first one, essentially due to Wightman¹, starts with certain general principles, the so-called Wightman axioms, such as the postulates of quantum physics, relativistic invariance etc. and studies the properties of the vacuum expectation values (v.e.v.) of products of

(Heisenberg) field operators which follow from these axioms. These v.e.v's are called the Wightman Functions. The second formulation, which is due to Lehmann, Symanzik and Zimmermann^{2,3} and is called LSZ formulation, starts with the same axioms, but studies the properties of v.e.v. of time ordered and retarded products of (Heisenberg) field operator instead of those of Wightman Functions. It then goes on to make a further assumption, the Asymptotic Condition³ and this enables one to give a particle interpretation of the theory, to define the S-matrix and to connect the S-matrix elements to the time ordered and for some restricted regions in momentum space, to the retarded functions.

When the theory is analysed in terms of these functions, it is found that some of the axioms, such as relativistic invariance and locality etc. lead to linear relations on them, while some others such as positive definiteness for Wightman functions and unitarity for retarded functions lead to non-linear relations. This breaking up of the theory into linear and non-linear parts is an important feature of axiomatic approach and is in distinct contrast with the conventional approach where any theory of interaction necessarily has non-linear field equations and is therefore wholly non-linear.

The object of this disseration is to extend the LSZ formulation by reducing the S-matrix elements in terms of retarded functions for every point in momentum space. In other words, we obtain an expression for

$$\langle p_1, p_2, \dots, p_m \text{ out} / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle$$

which is essentially the Fourier transform of a retarded function even when both m and $n-m$ are >2 . As we do not have more than two particles in physical processes, (a case which has already been treated by LSZ), it may seem pointless to consider matrix elements having more than two particles in both the initial and final states. The answer to this objection lies in the way one approaches the whole problem. Here our aim is not to explain some particular reaction between particles, but to formulate a theory which, we hope, contains a complete description of all the reactions. In order to give a full account of this theory, to have a clear understanding of its structure and to get maximum amount of information out of it, we must evidently study all its aspects, i.e. we must study even those reactions and processes which, though not physically possible, are however, contained in it. Further, even though the collision processes occurring in nature have at most two particles in the initial state, their behaviour depends on the properties of processes in which this restriction is not present. This follows from the fact that unitarity of S-matrix leads to many particles intermediate states, so that we get equations relating the matrix element of the original (physical) process with those of processes which have many particles in the (intermediate) initial and final states. Clearly, for a satisfactory analysis of these equations, it is essential to know as much as possible about the general matrix element. The importance of this fact has been recognized for a long time, and some work has already been done in this

direction. In perturbation theory, for example, Nambu²⁴ and Nakanishi²⁵ have obtained expressions for the matrix elements corresponding to general Feynman diagrams having any number of incoming and outgoing particles, and the latter author has also discussed some of their analytic properties. Again, in axiomatic theory, Polkinghorne⁶ has given an expression for the amplitude of the process: one nucleon + n_1 mesons $\rightarrow$ one nucleon + n_2 mesons, and has derived dispersion relations for it.

For the actual process of reduction, we use the generalized retarded functions, which have been defined by Araki⁴, Burgoyne⁵, Polkinghorne⁶ and Ruelle⁷, and are natural generalizations of the ordinary retarded function of LSZ. In spite of the fact that LSZ have already reduced the general matrix element in terms of time ordered functions, our reduction is desirable, because the analytic behaviour of retarded functions is much better than that of time ordered functions. The reason is simple. The Fourier transform

$$\tilde{r}_\alpha(p_1, p_2, \dots, p_n)$$

of each of the generalized retarded functions is the boundary value of an analytic function

$$H(s_1, s_2, \dots, s_n), \quad s'_i = p_i + i q_i$$

as the imaginary part of $s \rightarrow 0$ i.e. $q \equiv (q_1, q_2, \dots, q_n) \rightarrow 0$

in a certain fixed manner. On the other hand, the Fourier transform

$$\tilde{V}(p_1, p_2, \dots, p_n)$$

of the time ordered function is the boundary value of the same analytic function H as the imaginary part of $s \rightarrow 0$ in a way which depends on p , the real part of s i.e. $\tilde{V}(p)$ is a different boundary value of H for different points p . Clearly, it is much easier, mathematically, to deal with a function which is the same boundary value for different values of its argument than one which is not. Another point in this connection is that the restrictions imposed by the causality condition take a very nice form for retarded functions, namely, we get some retardedness (i.e. support) properties of these functions, whereas this is not so in the case of the time ordered functions. Finally, we remark that an analysis of retarded functions is, in fact, nothing but an indirect way of analysing the time ordered functions.

In this chapter, we give the details of LSZ formulation, including the reduction formulae for time ordered and retarded functions and the expressions of the S-matrix elements in terms of these functions. The next chapter starts with the (Burgoyne's) definition⁵ of generalized retarded operators R_α , and their properties, which have been proved by Araki⁴, Burgoyne⁵ and Ruelle⁷. Reduction formulae are then proved for them, i.e. expressions are obtained for the commutators of an arbitrary generalized retarded operator with in- and out-fields. These formulae lead,

in a natural way, to a special subclass of retarded functions, whose analyticity and support properties are studied in some detail. A generating functional is also obtained for them. In the end, we prove that they are identical with the set of Steinmann retarded operators introduced in the ref.8, as well as with Burgoyne's special subclass defined in the appendix of his Thesis⁵. In the third chapter, we prove that the matrix element

$$\langle p_1, p_2, \dots, p_m \text{ out} \mid -p_{m+1}, -p_{m+1}, \dots, -p_n \text{ in} \rangle$$

can be reduced, for arbitrary p , to a linear combination of r_β 's, the retarded functions belonging to the above subclass. An explicit expression for this linear combination is obtained for $p \in c_\alpha$ and is found to be identical with the expansion of r_α in terms of r_β . This expansion is itself obtained in the fourth chapter, which also contains the proof of linear independence of r_β .

2. Axioms of the LSZ formulation of field theory

For simplicity, we consider only a self interacting neutral scalar field $\phi(x)$ of mass m , without any bound states, but the whole formulation can be easily carried over to the cases when we have more than one field, or when we have a general tensor field instead of a neutral scalar one. Let us start by listing the axiom of the theory:

- (1) The usual postulates of quantum mechanics are valid, i.e. states of any system are represented by rays in a (separable) Hilbert space $\mathcal{H}$,

with positive definite metric, and observable by self-adjoint operators in $\mathcal{H}$. A ray is a set of vectors of $\mathcal{H}$ which differ only by a multiplicative constant. We represent it by a normalized vector of the set. We adopt the Heisenberg picture, so that a ray represents the same state for all times.

(2) The correspondence of the rays of $\mathcal{H}$ to physical states leads to a unitary representation of the inhomogeneous Lorentz Group (without reflections) by operators in $\mathcal{H}$, i.e. to every inhomogeneous proper Lorentz transformation (a, Λ) corresponds a unitary operator $U(a, \Lambda)$ such that

$$U(a_1, \Lambda_1) U(a_2, \Lambda_2) = U(a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2) .$$

According to Wigner⁹, this representation is a direct sum of a number of irreducible ones, each of which is characterized by two numbers, the mass m and the spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$. The restriction to scalar states means that we consider only the case in which the mass spectrum consists of the vacuum, a discrete point m which corresponds to $s = 0$, and a continuum starting from $2m$. Clearly, for a physically meaningful theory, m must be real and non-negative.

(3) The (Heisenberg) field is defined as follows. To every point x of space time, corresponds a tempered distribution valued linear operator $\phi(x)$ in $\mathcal{H}$. This means that the operators

$$\begin{aligned} \int dx f(x) \phi(x) + \int dx_1 dx_2 f(x_1, x_2) \phi(x_1) \phi(x_2) + \dots \\ + \int dx_1 dx_2 \dots dx_n f(x_1, x_2, \dots, x_n) \phi(x_1) \phi(x_2) \dots \phi(x_n) \end{aligned}$$

are (unbounded) self-adjoint operators in $\mathcal{H}$. Here $f(x_1, x_2, \dots, x_n)$ are real test functions of 4-vectors x_i belonging to the class $\mathcal{S}_n$, i.e. functions which are infinitely many times differentiable in all variables and vanish at infinity stronger than any power of $|x_i^\mu|$.

(4) The field $\varphi(x)$ is covariant under inhomogeneous proper Lorentz transformations

$$U(a, \Lambda) \varphi(x) U^{-1}(a, \Lambda) = \varphi(\Lambda x + a) .$$

(5) It follows from (3) that the translation operator $U(a, 1)$ is unitary so that it is of the form*

$$U(a, 1) = \exp i P \cdot a$$

where the generators P are self-adjoint operators, with the properties that

$$\varphi(x + a) = e^{iPa} \varphi(x) e^{-iPa} .$$

$$\frac{\partial}{\partial x_\mu} \varphi(x) = i [P_\mu, \varphi(x)] .$$

We interpret them as the energy-momentum operators, and require the energy-momentum spectrum of the states to be reasonable. This latter statement means that there is a unique state ψ_0 , the vacuum, in $\mathcal{H}$, which is simultaneous eigenvector of P corresponding to zero eigenvalues (and is invariant under all the proper Lorentz transformations) i.e. its energy-momentum is zero, and that the energy-momentum of any other state of $\mathcal{H}$ is either timelike or lightlike, with its energy greater than zero.

* We use the metric $x \cdot y = x_0 y_0 - \underline{x} \cdot \underline{y}$

(6) The field $\phi(x)$ is local, i.e.

$$[\phi(x), \phi(y)] = 0 \quad \text{for} \quad (x - y)^2 < 0.$$

This expresses the principle that physical effects cannot travel faster than light, so that a measurement at x cannot have any effect on one at x' if $x - x'$ is spacelike.

(7) Asymptotic condition. Let $f(x)$ be a normalizable solution of Klein Gordon equation of mass m :

$$K_x f(x) \equiv (\square_x + m^2) f(x) = 0$$

$$\square_x = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2},$$

and let us define

$$\begin{aligned} O^f(x_0) &= i \int d\underline{x} O(x) \overleftrightarrow{\frac{\partial}{\partial x_0}} f^x(x) \\ &= i \int d\underline{x} \left[O(x) \frac{\partial}{\partial x_0} f^x(x) - \left\{ \frac{\partial}{\partial x_0} O(x) \right\} f^x(x) \right], \end{aligned}$$

for any operator $O(x)$. Then the limits³

$$\lim_{x_0 \rightarrow -\infty} (\underline{\phi}, \phi^f(x_0) \underline{\psi}) = (\underline{\phi}, \phi_{in}^f \underline{\psi}),$$

$$\lim_{x_0 \rightarrow +\infty} (\underline{\phi}, \phi^f(x_0) \underline{\psi}) = (\underline{\phi}, \phi_{out}^f \underline{\psi})$$

exist, $\underline{\phi}$ and $\underline{\psi}$ being any states of $\mathcal{H}$ in the domain of operators.

Further, $\phi_{in}(x)$ and $\phi_{out}(x)$ are free fields of mass m , i.e.

$$[\varphi_{in}(x), \varphi_{in}(x')] = [\varphi_{out}(x), \varphi_{out}(x')] = i \Delta(x - x'),$$

$$(\square_x + m^2) \varphi_{in}(x) = (\square_x + m^2) \varphi_{out}(x) = 0,$$

where

$$\Delta(x) = \frac{-i}{(2\pi)^3} \int dk e^{ikx} \delta(k^2 - m^2) \varepsilon(k^0),$$

and are irreducible, i.e. the states

$$\psi_0, \varphi_{in \atop out}(f) \psi_0, \varphi_{in \atop out}(f_1) \varphi_{in \atop out}(f_2) \psi_0, \dots,$$

$$\varphi_{in \atop out}(f_1) \varphi_{in \atop out}(f_2) \dots \varphi_{in \atop out}(f_n) \psi_0$$

span the whole of Hilbert space $\mathcal{H}$.

The first six of these are the usual Wightman axioms. The seventh one, i.e. the asymptotic condition, essentially says that in infinite past and infinite future, the field $\varphi(x)$ behaves like a free field. These free fields represent the incoming and outgoing particles, so that we have the particle interpretation of the theory as mentioned in the introduction. It must be noted in this connection that Haag¹⁰ and Ruelle¹¹ have shown that a part of the asymptotic condition, viz. the existence of in- and out-fields follows from the previous axioms provided only that we modify (5) by assuming the existence of a discrete mass state of mass m , which is isolated in the mass spectrum. However, the irreducibility of these fields is an independent axiom and does not

follow from others. As the in- and out-fields are both irreducible, they are related by a unitary operator

$$\varphi_{\text{out}}(x) = S^{-1} \varphi_{\text{in}}(x) S, \quad SS^{\dagger} = 1.$$

This defines the S-matrix of the theory.

The current is defined by

$$K_x \varphi(x) \equiv (\square_x + m^2) \varphi(x) = j(x),$$

and this implies

$$\begin{aligned} \varphi(x) &= \varphi_{\text{in}}(x) + \int dx' \Delta_R(x - x') j(x') \\ &= \varphi_{\text{out}}(x) + \int dx' \Delta_A(x - x') j(x'). \end{aligned}$$

The coefficient of $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$ on RHS is in general a constant which may or may not be equal to one, but we can always arrange it to be one by a suitable redefinition of the field $\varphi(x)$ (by taking the new field to be a constant multiple of the old one).

From $\varphi_{\text{in}}(x)$ and $\varphi_{\text{out}}(x)$ we can construct complete sets of incoming and outgoing states in the usual manner. Let $\{f_a(x)\}$ be a complete set of c-number solutions of Klein Gordon equation of mass m , so that

$$i \int d\underline{x} f_a(x) \overleftrightarrow{\partial}_{x^0} f_b^*(x) = \delta_{ab}$$

$$\sum_a f_a(x) f_a^*(x') = i \Delta^+(x - x')$$

$$\text{where } \Delta_+(x) = \frac{-i}{(2\pi)^{3/2}} \int dK e^{-iKx} \delta(k^2 - m^2) \theta(k^0).$$

Then any self-adjoint operator $A(x)$ can be expanded in terms of $f_a(x)$:

$$A(x) = \sum_a \left[f_a(x) A^a(x_0) + f_a^*(x) A^{a*}(x_0) \right]$$

with the coefficients $A^a(x_0)$ given by

$$A^{a*}(x_0) = i \int d\underline{x} A(x) \overleftrightarrow{\partial}_{x_0} f_a(x)$$

When $A(x) = \phi_{in}^{out}(x)$, the operators $\phi_{in}^{a*}(x_0)$ are independent of time, so that we write them just as ϕ_{in}^{a*out} . They satisfy the commutation relations

$$\left[\begin{matrix} a \\ \phi_{in} \\ out \end{matrix}, \begin{matrix} b^* \\ \phi_{in} \\ out \end{matrix} \right] = \delta_{ab},$$

$$\left[\begin{matrix} a \\ \phi_{in} \\ out \end{matrix}, \begin{matrix} b \\ \phi_{in} \\ out \end{matrix} \right] = \left[\begin{matrix} a^* \\ \phi_{in} \\ out \end{matrix}, \begin{matrix} b^* \\ \phi_{in} \\ out \end{matrix} \right] = 0,$$

and create incoming and outgoing particles in state f_a , so that the complete set of states consists of

$$|a_1, a_2, \dots, a_n \text{ in out}\rangle = \phi_{in \text{ out}}^{a_1*} \phi_{in \text{ out}}^{a_2*} \dots \phi_{in \text{ out}}^{a_n*} / 0\rangle$$

$$n = 0, 1, 2, 3 \dots$$

where $|0\rangle \equiv \psi_0$.

We assume that

$$|a \text{ in}\rangle = |a \text{ out}\rangle.$$

This expresses the fact that one particle states are stable. The in- and out-states are connected by the S-matrix

$$|a_1, a_2, \dots, a_n \text{ in}\rangle = S |a_1, a_2, \dots, a_n \text{ out}\rangle$$

so that

$$\langle b \text{ out} / a \text{ in} \rangle = \langle b \text{ out} / S / a \text{ out} \rangle$$

$$\text{i.e. } S_{ba} = \langle b \text{ out} / a \text{ in} \rangle.$$

Here a and b represent set of indices $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_m$.

If $\{f_a(x)\}$ is taken to be the plane waves, we shall take the normalization

$$f_p(x) = \frac{1}{(2\pi)^{3/2}} e^{ipx}, \quad p^2 = m^2.$$

It follows from the asymptotic condition that the operator which transforms the in- and out-fields under Lorentz transformations is the same as that for $\phi(x)$ i.e.

$$U(a, \Lambda) \begin{matrix} \phi_{\text{in}}(x) \\ \text{out} \end{matrix} U^{-1}(a, \Lambda) = \begin{matrix} \phi_{\text{in}}(\Lambda x + a) \\ \text{out} \end{matrix}.$$

In particular, the energy-momentum operator is also the same so that we have

$$\begin{matrix} \phi_{\text{in}}(x+a) \\ \text{out} \end{matrix} = e^{iPa} \begin{matrix} \phi_{\text{in}}(x) \\ \text{out} \end{matrix} e^{-iPa},$$

$$\partial_\mu \begin{matrix} \phi_{\text{in}}(x) \\ \text{out} \end{matrix} = i \left[P_\mu, \begin{matrix} \phi_{\text{in}}(x) \\ \text{out} \end{matrix} \right].$$

3. Reduction of S-matrix elements

As stated in the introduction, the asymptotic condition may be used to express the S-matrix elements in terms of v.e.v. of certain products of field operators. A useful quantity in this connection is the

time ordered product (the T-product) which is defined as

$$\begin{aligned}
 T(x_1, x_2, \dots, x_n) &= \sum_{\pi} \theta(x_{i_1}^0 - x_{i_2}^0) \theta(x_{i_2}^0 - x_{i_3}^0) \dots \theta(x_{i_{n-1}}^0 - x_{i_n}^0) \\
 &\quad \phi(x_{i_1}) \phi(x_{i_2}) \dots \phi(x_{i_n}) \\
 &\quad \text{for } n = 1, 2, 3, \dots
 \end{aligned} \tag{1}$$

$$= 1 \text{ for } n = 0.$$

Here π is the permutation $k \rightarrow i_k$ and the summation extends over the set of all the permutation of

$$(1, 2, 3, \dots, n).$$

Its vacuum expectation value

$$\mathcal{T}(x_1, x_2, \dots, x_n) = \langle 0 / T(x_1, x_2, \dots, x_n) / 0 \rangle$$

is called the time ordered function ($\mathcal{T}$ function). It is easily seen that

$$T(x_1, x_2, \dots, x_n) = \phi(x_{i_1}) \phi(x_{i_2}) \dots \phi(x_{i_n})$$

where $(i_1, i_2, \dots, i_n)$ is such a permutation of $(1, 2, \dots, n)$ that

$$x_{i_1}^0 > x_{i_2}^0 > x_{i_3}^0 > \dots > x_{i_{n-1}}^0 > x_{i_n}^0.$$

As in (1) we have multiplied a distribution (the product of field operators) with a discontinuous function (the product of θ -functions) the definition is not well defined in the sense that ambiguities appear at the points

$$x_{j_1} = x_{j_2}.$$

We now prove that the time ordered products satisfy the following

reduction formula²

$$\begin{aligned} T(x_1, x_2, \dots, x_n) \varphi_{in}^{a*} &= \varphi_{out}^{a*} T(x_1, x_2, \dots, x_n) \\ &= -i \int dz f_a(z) K_z T(x_1, x_2, \dots, x_n, z) \end{aligned} \quad (2)$$

Let $\underline{\psi}$ and $\underline{\varphi}$ be arbitrary normalizable states of H . Then using the asymptotic condition and the definition of $\varphi^{a*}(z_0)$, we get

$$\begin{aligned} &(\underline{\psi}, [ST(x_1, x_2, \dots, x_n), \varphi_{in}^{a*}] \underline{\varphi}) \\ &= \lim_{z^0 \rightarrow -\infty} (\underline{\psi}, [ST(x_1, x_2, \dots, x_n), \varphi^{a*}(z_0)] \underline{\varphi}) \\ &= -i \lim_{z^0 \rightarrow -\infty} \int dz (\underline{\psi}, [ST(x_1, x_2, \dots, x_n) \varphi(z) - \varphi(z) ST(x_1, x_2, \dots, x_n)] \underline{\varphi}) \overset{\longleftrightarrow}{\delta_{z^0}} f_a(z) \\ &= -i \lim_{z^0 \rightarrow -\infty} \int dz (\underline{\psi}, [ST(x_1, x_2, \dots, x_n) \varphi(z) - \varphi_{in}(z) S T(x_1, x_2, \dots, x_n)] \underline{\varphi}) \overset{\longleftrightarrow}{\delta_{z^0}} f_a(z) \\ &= -i \lim_{z^0 \rightarrow -\infty} \int dz (\underline{\psi}, [ST(x_1, x_2, \dots, x_n, z) - S \varphi_{out}(z) T(x_1, x_2, \dots, x_n)] \underline{\varphi}) \overset{\longleftrightarrow}{\delta_{z^0}} f_a(z) \end{aligned}$$

In the last step, we have used

$$\varphi_{in}(z) S = S \varphi_{out}(z)$$

and the fact that as $z^0 \rightarrow -\infty$, $z^0 < x_i^0$ for all i and so $\varphi(z)$ can

be taken inside the T-product. Now writing

$$- \lim_{z^0 \rightarrow -\infty} \int dz \left(\right) = \int dz \partial_{z^0} \left(\right) - \lim_{z^0 \rightarrow +\infty} \int dz \left(\right)$$

we get

$$\begin{aligned} & (\underline{\psi}, [ST(x_1, x_2, \dots, x_n), \varphi_{in}^{a*}] \underline{\varphi}) \\ &= i \int dz \partial_{z^0} [(\underline{\psi}, \{ST(x_1, x_2, \dots, x_n, z) - S \varphi_{out}(z) T(x_1, x_2, \dots, x_n)\} \underline{\varphi}) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z)] - i \lim_{z^0 \rightarrow +\infty} \int dz (\underline{\psi}, \\ & \{ST(x_1, x_2, \dots, x_n, z) - S \varphi_{out}(z) T(x_1, x_2, \dots, x_n)\} \underline{\varphi}) \\ & \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \end{aligned} \quad (3)$$

The two terms in the second integral vanish, as we have

$$\begin{aligned} & \lim_{z^0 \rightarrow +\infty} S \varphi_{out}(z) T(x_1, x_2, \dots, x_n) \\ &= \lim_{z^0 \rightarrow +\infty} S \varphi(z) T(x_1, x_2, \dots, x_n) \\ &= \lim_{z^0 \rightarrow +\infty} ST(x_1, x_2, \dots, x_n, z) \quad \text{as } z^0 > x_i^0. \end{aligned}$$

Carrying out the indicated differentiation, we see that the first term of the first integral is

$$i \int dz (\underline{\psi}, \{ST(x_1, x_2, \dots, x_n, z) \partial_{z^0}^2 f_a(z) - \partial_{z^0}^2 (ST(x_1, x_2, \dots, x_n, z)) f_a(z)\} \underline{\varphi})$$

$$\begin{aligned}
 &= i \int dz (\underline{\psi}, \{ ST(x_1, x_2, \dots, x_n, z) (\nabla_{\underline{z}}^2 - m^2) f_a(z) - \partial_{z_0}^2 \\
 & (ST(x_1, x_2, \dots, x_n, z)) f_a(z) \} \underline{\varphi}) \quad \text{as } K_z f_a(z) = 0 \\
 &= - i \int dz (\underline{\psi}, \{ f_a(z) (\partial_{z_0}^2 - \nabla_{\underline{z}}^2 + m^2) ST(x_1, x_2, \dots, x_n, z) \} \underline{\varphi}) \\
 &= - i \int dz (\underline{\psi}, S f_a(z) K_z T(x_1, x_2, \dots, x_n, z) \underline{\varphi})
 \end{aligned}$$

where integration by parts over spatial variables has been carried out.

Similarly, the second term of the first integral in (3) is

$$i \int dz (\underline{\psi}, S f_a(z) K_z \varphi_{\text{out}}(z) T(x_1, x_2, \dots, x_n) \underline{\varphi})$$

and this vanishes as $\varphi_{\text{out}}(z)$ satisfies the Klein Gordon equation.

Hence (3) takes the form

$$\begin{aligned}
 & (\underline{\psi}, [ST(x_1, x_2, \dots, x_n) \varphi_{\text{in}}^{a*} - \varphi_{\text{in}}^{a*} ST(x_1, x_2, \dots, x_n)] \underline{\varphi}) \\
 &= - i \int dz (\underline{\psi}, S f_a(z) K_z T(x_1, x_2, \dots, x_n, z) \underline{\varphi})
 \end{aligned}$$

$$\text{or } (\underline{\psi}, \{ T(x_1, x_2, \dots, x_n) \varphi_{\text{in}}^{a*} - \varphi_{\text{out}}^{a*} T(x_1, x_2, \dots, x_n) \} \underline{\varphi})$$

$$= - i \int dz (\underline{\psi}, f_a(z) K_z T(x_1, x_2, \dots, x_n, z) \underline{\varphi})$$

as $S^{-1} \varphi_{\text{in}}^{a*} S = \varphi_{\text{out}}^{a*}$. As $\underline{\psi}$ and $\underline{\varphi}$ are arbitrary states of $\mathcal{H}$, this equation shows that (2) is true.

An alternative form of (2) is obtained as follows. Write it as

$$\left[ST(x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right] = -i \int dz' (z') K_z, ST(x_1, x_2, \dots, x_n, z')$$

The corresponding commutator with ϕ_{in}^a can be evaluated in the same way and is given by

$$\left[ST(x_1, x_2, \dots, x_n), \phi_{in}^a \right] = +i \int dz' f_a^x(z) K_z, ST(x_1, x_2, \dots, x_n, z')$$

Multiply now the first equation by $f_a^x(z)$, the second one by $f_a(z)$, add them and sum over a so as to get

$$\left[ST(x_1, x_2, \dots, x_n), \sum_a \{ \phi_{in}^{a*} f_a^x(z) + \phi_{in}^a f_a(z) \} \right] = -i \int dz'$$

$$\left\{ \sum_a f_a(z') f_a^x(z) - \sum_a f_a^x(z') f_a(z) \right\} K_z, ST(x_1, x_2, \dots, x_n, z')$$

or $\left[ST(x_1, x_2, \dots, x_n), \phi_{in}(z) \right]$

$$= \int dz' \left[\Delta^+(z' - z) - \Delta^+(-z' + z) \right] K_z, ST(x_1, x_2, \dots, x_n, z')$$

As

$$\Delta^+(z' - z) - \Delta^+(-z' + z) = \Delta^+(z' - z) + \Delta^-(-z' - z) = \Delta(z' - z),$$

this gives

$$\left[ST(x_1, x_2, \dots, x_n), \phi_{in}(z) \right] = \int dz' \Delta(z' - z) K_z, ST(x_1, x_2, \dots, x_n, z') \quad (4)$$

Next we come to the reduction² of the matrix element

$$\langle p_1, p_2, \dots, p_m \text{ out } / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle \quad (5)$$

in terms of $\tilde{\omega}$ -functions. The energy-momentum conservation requires

$$p_1 + p_2 + \dots + p_{n-1} + p_n = 0.$$

We assume that none of the partial sums

$$p_{i_1} + p_{i_2} + \dots + p_{i_j}, \quad j < n$$

vanish. Here $(i_1, i_2, \dots, i_j)$ are any j of the indices $(1, 2, \dots, n)$. In conventional theory, this assumption corresponds to considering only the connected Feynman diagrams.

The reduction requires the result

$$\begin{aligned} \varphi_{\nu}^p / k_1, k_2, \dots, k_n \nu \rangle &= 0 \quad \text{if } p \neq k_i, i = 1, 2, \dots, n \\ \nu &\equiv \text{in or out,} \end{aligned} \quad (6)$$

which can be seen to be true as follows. We have

$$\begin{aligned} \varphi_{\nu}^p / k_1, k_2, \dots, k_n \nu \rangle &= \varphi_{\nu}^p \varphi_{\nu}^{k_1*} / k_2, k_3, \dots, k_n \nu \rangle \\ &= \varphi_{\nu}^{k_1*} \varphi_{\nu}^p / k_2, k_3, \dots, k_n \nu \rangle + \delta(p - k_1) / k_2, k_3, \dots, k_n \nu \rangle \\ &= \varphi_{\nu}^{k_1*} \varphi_{\nu}^p / k_2, k_3, \dots, k_n \nu \rangle \end{aligned}$$

by using the commutation relation for φ_{ν}^p and $\varphi_{\nu}^{k_1*}$ and the fact that $p \neq k_1$ so that $\delta(p - k_1) = 0$. Repeating the process $(n-1)$ times, we get

$$\varphi_{\nu}^p / k_1, k_2, \dots, k_n \nu \rangle = \varphi_{\nu}^{k_1*} \varphi_{\nu}^{k_2*} \dots \varphi_{\nu}^{k_n*} \varphi_{\nu}^p / 0 \rangle = 0$$

as $\varphi_{\nu}^p / 0 \rangle = 0$.

Calling the matrix element (5) as X , we see that it can be written as

$$X = \langle p_1, p_2, \dots, p_m \text{ out} / \varphi_{in}^{-p_n*} / -p_{m+1}, -p_{m+2}, \dots, -p_{n-1} \text{ in} \rangle$$

Now (2) with $n = 0$ gives

$$\varphi_{in}^{-p_n^*} = \varphi_{out}^{-p_n^*} - i \int dx_n e^{-i \cdot -p_n x_n} K_{x_n} \varphi(x_n)$$

so that

$$X = -i \int dx_n e^{ip_n x_n} K_{x_n} \langle p_1, p_2, \dots, p_m \text{ out} / \varphi(x_n) / -p_{m+1}, -p_{m+2}, \dots, -p_{n-1} \text{ in} \rangle + \langle p_1, p_2, \dots, p_m \text{ out} / \varphi_{out}^{-p_n^*} / -p_{m+1}, -p_{m+2}, \dots, -p_{n-1} \text{ in} \rangle$$

According to our assumption

$$p_i + p_n \neq 0 \quad i = 1, 2, \dots, m,$$

so that (6) implies that the second term vanishes, i.e.

$$\begin{aligned} X &= -i \int dx_n e^{ip_n x_n} K_{x_n} \langle p_1, p_2, \dots, p_m \text{ out} / \varphi(x_n) \varphi_{out}^{-p_{n-1}^*} / -p_{m+1}, -p_{m+2}, \dots, -p_{n-2} \text{ in} \rangle \\ &= (-i)^2 \int dx_{n-1} dx_n e^{i(p_n x_n + p_{n-1} x_{n-1})} K_{x_n} K_{x_{n-1}} \langle p_1, p_2, \dots, p_m \text{ out} / T(x_{n-1}, x_n) / -p_{m+1}, -p_{m+2}, \dots, -p_{n-2} \text{ in} \rangle - i \int dx_n e^{ip_n x_n} K_{x_n} \langle p_1, p_2, \dots, p_m \text{ out} / \varphi_{out}^{-p_{n-1}^*} \varphi(x_n) / -p_{m+1}, -p_{m+2}, \dots, -p_{n-2} \text{ in} \rangle \\ &= (-i)^2 \int dx_{n-1} dx_n e^{i(p_{n-1} x_{n-1} + p_n x_n)} K_{x_{n-1}} K_{x_n} \langle p_1, p_2, \dots, p_m \text{ out} / T(x_{n-1}, x_n) / -p_{m+1}, -p_{m+2}, \dots, -p_{n-2} \text{ in} \rangle \end{aligned}$$

as the second term again vanishes. Proceeding in this way, we get

$$X = (-i)^{n-m} \int dx_{m+1} dx_{m+2} \dots dx_n \exp(i \sum_{i=1}^n p_i x_i) K_{x_{m+1}} K_{x_{m+2}} \dots K_{x_n}$$

$$\langle p_1, p_2, \dots, p_m \text{ out} / T(x_{m+1}, x_{m+2}, \dots, x_n) / 0 \rangle.$$

Next, using

$$\begin{aligned} \phi_{\text{out}}^a T(x_1, x_2, \dots, x_n) &= T(x_1, x_2, \dots, x_n) \phi_{\text{in}}^a \\ &= -i \int dz f_a^x(z) K_z T(x_1, x_2, \dots, x_n, z) \end{aligned} \quad (2a)$$

which is obtained in the same way as (2), m times, X can be completely reduced to

$$\begin{aligned} X &= \langle p_1, p_2, \dots, p_m \text{ out} / -p_{m+1}, -p_{m+2}, \dots, -p_n \rangle \\ &= (-i)^n \int dx_1 dx_2 \dots dx_n \exp(i \sum_{i=1}^n p_i x_i) K_{x_1} K_{x_2} \dots K_{x_n} \langle 0 / T(x_1, x_2, \dots, x_n) / 0 \rangle \end{aligned} \quad (7)$$

This is the required expression of the matrix element (5) in terms of γ -functions.

In addition to the reduction formula (2) for the T-product, LSZ³ have derived a similar one for retarded product of field operators (R-product) also. This product is defined by

$$\begin{aligned} R(x; x_1, x_2, \dots, x_n) &= \sum_{\pi} \theta(x^0 - x_{i_1}^0) \theta(x_{i_1}^0 - x_{i_2}^0) \dots \theta(x_{i_{n-1}}^0 - x_{i_n}^0) [\dots [\phi(x), \phi(x_{i_1})] , \\ &\quad \dots] , \phi(x_{i_n})] \quad \text{for } n = 1, 2, 3, \dots \\ &= \phi(x) \quad \text{for } n = 0. \end{aligned} \quad (8)$$

where π is the permutation $k \rightarrow i_k$ and the summation extends over the set of all the permutations of

$$(1, 2, 3, \dots, n) .$$

Its v.e.v.

$$r(x; x_1, x_2, \dots, x_n) = \langle 0 / R(x; x_1, x_2, \dots, x_n) / 0 \rangle$$

is called the retarded function (r-function). It is evident from the definition (8) that R (and hence also r) vanishes if any of the x_i^0 is greater than x^0 . The advanced produce is similarly defined by

$$\begin{aligned} & A(x; x_1, \dots, x_n) \\ &= \sum_{\pi} \theta(x_{i_n}^0 - x_{i_{n-1}}) \theta(x_{i_{n-1}}^0 - x_{i_{n-2}}) \dots \theta(x_{i_1}^0 - x^0) [\dots [\varphi(x), \varphi(x_{i_1})] , \\ & \quad \varphi(x_{i_2})] \dots] , \quad \varphi(x_{i_n})] , \quad \text{for } n = 1, 2, 3, \dots \\ &= \varphi(x) \quad \text{for } n = 0 , \end{aligned} \tag{9}$$

$$a(x; x_1, x_2, \dots, x_n) = \langle 0 / A(x; x_1, x_2, \dots, x_n) / 0 \rangle .$$

and vanishes if any of x_i^0 is less than x^0 . The reduction formulae for these products³ are

$$\begin{aligned} & [R(x; x_1, x_2, \dots, x_n) , \varphi_{in}^{a*}] \\ &= -i \int dz f_a(z) K_z R(x; x_1, x_2, \dots, x_n, z) , \end{aligned} \tag{10}$$

$$\begin{aligned} & [A(x; x_1, x_2, \dots, x_n) , \varphi_{out}^{a*}] \\ &= -i \int dz f_a(z) K_x A(x; x_1, x_2, \dots, x_n, z) . \end{aligned} \tag{11}$$

They can be proved as follows. Using the definition of ϕ_{in}^{a*} and asymptotic condition, we get

$$\begin{aligned} & \left[R(x; x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right] \\ = & -i \lim_{z^0 \rightarrow -\infty} \int dz \left[R(x; x_1, x_2, \dots, x_n), \phi(z) \right] \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \end{aligned}$$

in the sense of weak convergence. Now it is easily seen that

$$z^0 \xrightarrow{\text{Lim}} -\infty R(x; x_1, x_2, \dots, x_n, z) = z^0 \xrightarrow{\text{Lim}} -\infty \left[R(x; x_1, x_2, \dots, x_n), \phi(z) \right] .$$

For, as $z^0 < x_i^0$, $i = 1, 2, \dots, n$, a factor $\theta(x_i - z)$ can always be supplied on the RHS, while all the terms appearing in the expansion of the LHS, other than that written on the RHS, have a factor $\theta(z - x_i)$ for some i and so vanish. Thus

$$\begin{aligned} & \left[R(x; x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right] \\ = & -i \lim_{z^0 \rightarrow -\infty} \int dz R(x; x_1, x_2, \dots, x_n, z) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \\ = & -i \int dz \overset{\leftrightarrow}{\partial}_{z^0} \left[R(x; x_1, x_2, \dots, x_n) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \right] - i \lim_{z^0 \rightarrow +\infty} \\ & \int dz R(x; x_1, x_2, \dots, x_n, z) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \\ = & i \int dz \overset{\leftrightarrow}{\partial}_{z^0} \left[R(x; x_1, x_2, \dots, x_n, z) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \right] \end{aligned}$$

the second term being zero due to the fact that as $z^0 \rightarrow +\infty$, $x^0 < z^0$ and so the R-product vanishes. Now carrying out the indicated differentiation,

substituting

$$(\nabla_{\underline{z}}^2 - m^2) f_a(z) \text{ for } \partial_{z^0}^2 f_a(z)$$

and integrating by parts over the spatial variables, we get

$$\begin{aligned} & [R(x; x_1, x_2, \dots, x_n), \varphi_{in}^{a*}] \\ = & i \int dz f_a(z) (-\partial_{z^0}^2 + \nabla_{\underline{z}}^2 - m^2) R(x; x_1, x_2, \dots, x_n, z) \end{aligned}$$

which is the required formula (10). (11) can be proved in an exactly similar manner. The Hermitian conjugates of (10) and (11) are

$$\begin{aligned} & [\varphi_{in}^a, R(x; x_1, x_2, \dots, x_n)] \\ = & -i \int dz f_a^*(z) K_z R(x; x_1, x_2, \dots, x_n, z), \end{aligned} \quad (10a)$$

$$\begin{aligned} & [\varphi_{out}^a, A(x; x_1, x_2, \dots, x_n)] \\ = & -i \int dz f_a^*(z) K_z A(x; x_1, x_2, \dots, x_n, z) \end{aligned} \quad (11a)$$

These are obtained by using the Hermiticity property of R- and A- products

$$\begin{aligned} [R(x; x_1, x_2, \dots, x_n)]^* &= (-1)^n R(x; x_1, x_2, \dots, x_n), \\ [A(x; x_1, x_2, \dots, x_n)]^* &= (-1)^n A(x; x_1, x_2, \dots, x_n), \end{aligned}$$

which can be easily verified.

If (10) is multiplied by $f_a^*(z')$ and added to (10a) multiplied by $-f_a(z')$ and the equation so obtained summed over a , the result is

$$\begin{aligned} & [R(x; x_1, x_2, \dots, x_n), \varphi_{in}(z)] \\ = & \int dz' \Delta(z' - z) K_z R(x; x_1, x_2, \dots, x_n, z') \end{aligned} \quad (12)$$

similarly, (11) leads to

$$\begin{aligned} & \left[A(x; x_1, x_2, \dots, x_n) , \varphi_{\text{out}}(z) \right] \\ = & \int dz' \Delta(z' - z) K_z : A(x; x_1, x_2, \dots, x_n, z') : \end{aligned} \quad (13)$$

LSZ³ have shown that the reduction formulae (10) and (10a) and the assumption of uniqueness of one particle states can be used to express the matrix element (5) in terms of r-functions, in the particular case when $m = 2$. For in this case, we have

$$\begin{aligned} X &= \langle p_1, p_2 \text{ out} / -p_3, -p_4, \dots, -p_n \text{ in} \rangle \\ &= \langle p_2 \text{ in} / \varphi_{\text{out}}^{p_1} / -p_3, -p_4, \dots, -p_n \text{ in} \rangle \quad \text{as } /p_2 \text{ out} \rangle = /p_2 \text{ in} \rangle \\ &= -i \int dx_1 e^{ip_1 x_1} K_{x_1} \langle p_2 \text{ in} / \varphi(x_1) / -p_3, -p_4, \dots, -p_n \text{ in} \rangle \\ &\quad + \langle p_2 / \varphi_{\text{in}}^{p_1} / -p_3, -p_4, \dots, -p_n \text{ in} \rangle . \end{aligned}$$

The second term vanishes as

$$p_1 \neq p_i, \quad i = 3, 4, \dots, n.$$

Hence, using (10) we get

$$\begin{aligned} X &= -i \int dx_1 e^{ip_1 x_1} K_{x_1} \langle p_2 \text{ in} / \varphi(x_1) \text{ in}^{-p_3*} / -p_4, -p_5, \dots, -p_n \text{ in} \rangle \\ &= (-i)^2 \int dx_1 dx_3 e^{i(p_1 x_1 + p_3 x_3)} K_{x_1} K_{x_3} \langle p_2 \text{ in} / R(x_1; x_3) / -p_4, -p_5, \dots, -p_n \text{ in} \rangle \end{aligned}$$

the additional term being again zero, as $-p_3 \neq p_2$. Repeated application of this process leads to

$$\begin{aligned} X &= (-i)^{n-1} \int dx_1 dx_3 \dots dx_n \exp(p_1 x_1 + \sum_3^n p_i x_i) K_{x_1} K_{x_3} \dots K_{x_n} \\ &\quad \langle p_2 \text{ in} / R(x_1; x_3, x_4, \dots, x_n) / 0 \rangle . \end{aligned}$$

Writing now $/p_2$ in $\rangle$ in terms of in-field and using (10a) we finally get

$$X = (-i)^n \int dx_1 dx_2 \dots dx_n e^{i \sum_{i=1}^n p_i x_i} K_{x_1} K_{x_2} \dots K_{x_n} \langle 0 / R(x_1, x_2, x_3, \dots, x_n) / 0 \rangle$$

Clearly, using (11) and (11a), we can similarly express (5) in terms of a-functions when $m = n - 2$. However, it has been pointed out by LSZ³ and Polkinghorne¹² that the above formulae are not sufficient to reduce (5) when $2 \leq m \leq n - 2$. This is essentially due to the fact that we need expressions for the commutators

$$\left[R(x; x_1, x_2, \dots, x_n), \phi_{out}^{a*} \right] \text{ and } \left[A(x; x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right]$$

similar to those for

$$\left[R(x; x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right] \text{ and } \left[A(x; x_1, x_2, \dots, x_n), \phi_{out}^{a*} \right]$$

given by (10) and (11). We shall see later that these expressions involve generalized retarded operators and so, we devote the next chapter to the study of these operators.

CHAPTER II

GENERALIZED RETARDED OPERATORS

1. Definition

The definition of these operators is closely connected with the geometrical properties of certain spaces, and so we start by describing them. First of all, consider the $(n-1)$ dimensional space Q^0 of

$$q^0 = (q_1^0, q_2^0, \dots, q_n^0)$$

with

$$\sum_{i=1}^n q_i^0 = 0.$$

The $2^{n-1} - 1$ distinct hyperplanes

$$q_J^0 = 0, \text{ where}$$

$$J = \text{a subset of } \{1, 2, 3, \dots, n\}$$

$$\text{and } q_J^0 = \sum_{i \in J} q_i^0$$

divide it into a set of convex polyhedral cones, say C_α^0 , each radiating from origin. It is clear that each cone C_α^0 determines uniquely whether $q_J^0 > 0$ or $-q_J^0 > 0$ when $q^0 \in C_\alpha^0$, and that there is at least one J , say J' , such that $q_{J'}^0$ has different signs in C_α^0 and C_β^0 for $\alpha \neq \beta$. Thus there is a one to one correspondence between the cones C_α^0 and a consistent choice of sign of q_J^0 for all J .

Consider next the space Q of 4-vectors

$$q \equiv (q_1, q_2, \dots, q_n)$$

with

$$\sum_{i=1}^n q_i = 0$$

We define the convex cones C_α in this space by the condition that $q \in C_\alpha$ if

- (i) $q_J \in V_+$ for all J such that $q_J^0 > 0$ for $q^0 \in C_\alpha^0$,
- (ii) $q_J \in V_-$ for all J such that $q_J^0 < 0$ for $q^0 \in C_\alpha^0$.

In other words, $q \in C_\alpha$ if $q_J \in V_\pm$ for all J , and the subscript of V , for each J , is such that $q^0 \in C_\alpha^0$.

Corresponding to each of these cones C_α , Burgoyne⁵ now defines a generalized retarded operator by

$$R_\alpha(x_1, x_2, \dots, x_n) = \sum_{\pi} \pm \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \theta_{\pm}(i_{n-1}, i_n) W(i_1, i_2, \dots, i_n) \quad (1)$$

where

- (i) π is the permutation $k \rightarrow i_k$ and

$$W(i_1, i_2, \dots, i_n) = \varphi(x_{i_1}) \varphi(x_{i_2}) \dots \varphi(x_{i_n}).$$

- (ii) The summation extends over the set of all the permutations of

$$(1, 2, 3, \dots, n)$$

- (iii) The choice of sign in

$$\theta_{\pm}(i_k, i_{k+1}) \equiv \theta \left[\pm (x_{i_k}^0 - x_{i_{k+1}}^0) \right]$$

is plus of $q_K \in V_+$ when $q \in C_\alpha$ and minus if $q_K \in V_-$ when $q \in C_\alpha$, K being the set

$$K : (i_1, i_2, \dots, i_k) .$$

(iv) The overall sign of each term in the summation is plus or minus according as the term contains an even or an odd number of θ -minus functions.

These rules define each R_α uniquely. We denote its vacuum expectation value by

$$\langle 0 / R_\alpha(x_1, x_2, \dots, x_n) / 0 \rangle = r_\alpha(x_1, x_2, \dots, x_n) .$$

Araki⁴ and Burgoyne⁵ have proved the following properties of r_α :

(i) It is Lorentz invariant i.e.

$$r_\alpha(\Lambda x_1 + a, \Lambda x_2 + a, \dots, \Lambda x_n + a) = r_\alpha(x_1, x_2, \dots, x_n)$$

That it is a distribution has not yet been proved in general. The difficulty arises due to the fact that a product of Wightman distribution and θ -functions is, in general, not unique. Steinmann¹³ has proved that in 2 dimensional space time it is a distribution. For 4 dimensional space time, the proof is probably complicated, but may well go through. By r_α , we mean any one of the possible definitions.

(ii) It vanishes outside the cone $\overline{C}_\alpha^0$ (in x^0 -space) which is conjugate of the cone C_α^0 (in q^0 -space) i.e. is the set of all x such that

$$x_1^0 q_1^0 + x_2^0 q_2^0 + \dots + x_n^0 q_n^0 > 0$$

for every $q^0 \in C_a^0$. According to the Schwartz Theorem¹⁴ the Fourier transform

$$\tilde{r}_a(p_1, p_2, \dots, p_n), \quad \sum_1^n p_i = 0$$

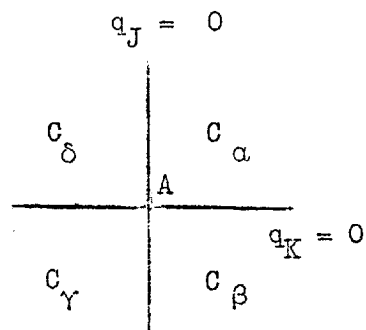
is therefore, the boundary value, as $q \rightarrow 0$, of a function

$$\tilde{r}_a(s_1, s_2, \dots, s_n), \quad \sum_1^n s_i = 0, \quad s_i \equiv p_i + i q_i$$

which is analytic for p arbitrary and q such that

$$q = (q^0, \underline{0}), \quad q^0 \in C_a^0.$$

- (iii) If C_a, C_β, C_γ and C_δ are four cones around some corner of type A (see figure) then



$$r_a - r_\beta + r_\gamma - r_\delta = 0$$

identically, these are called Steinmann relations.

- (iv) If the cone C_a and C_β have a face $q_J = 0$ between them, but are otherwise identical then

$$\tilde{r}_a(p_1, p_2, \dots, p_n) = \tilde{r}_\beta(p_1, p_2, \dots, p_n)$$

for all $(p_1, p_2, \dots, p_n)$ satisfying $p_J^2 < m_J^2$.

These four properties characterize the r -functions completely in the sense that starting from them, we can go backwards and deduce the original axioms of Lorentz covariance, locality and spectrum in terms of Wightman functions. This follows from an inversion theorem proved by

Araki⁴, Burgoyne⁵ and Steinmann¹⁵.

It can be shown¹⁶ that the ordinary retarded and advanced products of LSZ are the generalized retarded operators corresponding to the cones

$$R(x_0; x_1, x_2, \dots, x_n) \rightarrow C_0^-, \quad A(x_0; x_1, x_2, \dots, x_n) \rightarrow C_0^+$$

where $C_0^\pm$ are determined by

$$\begin{aligned} C_0^- &: q_1, q_2, \dots, q_n \in V_- \\ C_0^+ &: q_1, q_2, \dots, q_n \in V_+ \end{aligned}$$

For, we have

$$R(x_0; x_1, x_2, \dots, x_n) = \sum \theta(0, j_1) \theta(j_1, j_2) \dots \theta(j_{n-1}, j_n) \{ [\dots [0, j_1], j_2], \dots], j_n \}$$

in an obvious notation. By expanding the multiple commutator, this can be written as

$$R(x_0; x_1, x_2, \dots, x_n) = \sum_{\pi \in \mathfrak{S}_{n+1}} \mathfrak{S}_\pi W(i_0, i_1, \dots, i_n)$$

where $\mathfrak{S}_\pi$ are C-number functions. To evaluate the coefficient of

$$X = W(i_0, i_1, \dots, i_{k-1}, 0, i_{k+1}, \dots, i_n)$$

where $i_k = 0$, we find all the commutators

$$Y = [[\dots [0, j_1], j_1], \dots], j_n$$

which leads to X in the sense that in their expansion, there is a term

+ X or - X . Let a and b be two positive integers such that

$0 \leq b < a \leq k-1$ and let

$$i_a = j_{1_a}, \quad i_b = j_{1_b}.$$

If $l_a > l_b$, we can write Y as

$$Y = [\dots [0, j_1], j_2], \dots, j_{l_b}], \dots, j_{l_a}], \dots, j_n] .$$

Now, when we write the commutator

$$[[\dots [0, j_1], \dots], j_{l_b-1}], j_{l_b}]$$

as

$$[[\dots [0, j_1], \dots], j_{l_b-1}] j_{l_b} - j_{l_b} [[\dots [0, j_1], \dots], j_{l_b-1}]$$

it is the negative term (i.e. the second term) which can lead to X , as in X , $j_{l_b} \equiv i_b$ appears on the left of zero. Similarly in the expression

$$\begin{aligned} & [[\dots [(j_{l_b} [[\dots [0, j_1], \dots], j_{l_b-1}]), j_{l_b+1}], \dots], j_{l_a-1}], j_{l_a}] \\ &= [[\dots [(j_{l_b} [[\dots [0, j_1], \dots], j_{l_b-1}]), j_{l_b+1}], \dots], j_{l_a-1}] j_{l_a} \\ &- j_{l_a} [[\dots [(j_{l_b} [[\dots [0, j_1], \dots], j_{l_b-1}]), j_{l_b+1}], \dots], j_{l_a-1}] \end{aligned}$$

it is again the negative term which can lead to X . However, it does not do so, as $j_{l_b} = i_b$ appears on the right of $j_{l_a} = i_a$ in it, and on the left of i_a in X . This shows that if Y is to lead to X , we must have

$$l_0 > l_1 > l_2 > \dots > l_{k-1} \quad (2a)$$

where $i_t = j_{l_t}$, $t = 0, 1, 2, \dots, k-1$.

Similarly, considering the indices on the right of zero in X , we can show that if Y is to lead to X , we must also have

$$l_{k+1} < l_{k+2} < \dots < l_n \quad (2b)$$

where $i_t = j_{1_t}$, $t = k+1, k+2, \dots, n$. Further, if (2a,b) are satisfied, Y does lead to a term X. For if we expand Y, starting from the innermost commutator and going outwards and take the positive term or the negative term in j_t commutator according as j_t is on the right or the left side of zero in X, we get X with a coefficient $(-1)^k$ corresponding to the k negative terms. Thus X appears in the expansion of those and only those commutators Y which satisfy (2a,b). Its coefficient in the expansion of $R(x; x_1, x_2, \dots, x_n)$ is therefore

$$B_{\pi} = (-1)^k \sum \theta(0, j_1) \theta(j_1, j_2) \dots \theta(j_{n-1}, j_n)$$

where the sum is over all the sets

$$(j_1, j_2, \dots, j_n)$$

which satisfy (2a,b). We now show that it is equal to

$$B_{\pi} = (-1)^k \theta_{-}(i_0, i_1) \theta_{-}(i_1, i_2) \dots \theta_{-}(i_{k-1}, 0) \theta_{+}(0, i_{k+1}) \dots \theta_{+}(i_{n-1}, i_n).$$

Let

$$x_{i_a}^0 > x_{i_b}^0 \text{ for some } a \text{ and } b \text{ such that}$$

$$0 \leq a < b \leq k. \quad (2c)$$

$a < b$ means that $l_a < l_b$ for all the sets

$$(j_1, j_2, \dots, j_n)$$

satisfy (2a) so that every term in the expression for B_{π} has j_{1_a} on the right of j_{1_b} and hence vanishes, as we have $x_{j_{1_a}}^0 > x_{j_{1_b}}^0$. Thus B_{π} vanishes when (2c) is satisfied. Similarly, using (2b) we can show that

it again vanishes if

$$x_{i_a}^0 < x_{i_b}^0 \text{ for some } a \text{ and } b \text{ such that} \quad (2d)$$

$$k \leq a \leq b \leq n.$$

As $(2c, d)$ are the only points where B_π is zero, we see that β_π is zero whenever B_π is, and so, it is sufficient to prove that $\beta_\pi = B_\pi$ only when B_π is different from zero. This last result follows from the fact that at any point when $B_\pi \neq 0$ (and hence $= (-1)^k$), we have

$$x_0^0 > x_{i_{k+1}}^0 > \dots > x_{i_{k+\delta_1^+}} > x_{i_{k-1}} > \dots > x_{i_{k-\delta_1^-}}^0 > x_{i_{k+\delta_1^++1}} > \\ \dots > x_{i_{k+\delta_2^+}} > x_{i_{k-\delta_1^- - 1}} > \dots > x_{i_{k-\delta_2^-}} > \dots$$

and at such points, all the terms of β_π are zero except the one which corresponds to

$$j_1 = i_{k+1}, \dots, j_{\delta_1^+} = i_{k+\delta_1^+}, j_{\delta_1^++1} = i_{k-1}, \dots, j_{\delta_1^++\delta_1^-} = i_{k-\delta_1^-}, \\ j_{\delta_1^++1+\delta_1^-} = i_{k+\delta_1^++1}, \dots, j_{\delta_2^++\delta_1^-} = i_{k+\delta_2^+}, j_{\delta_2^++\delta_1^-+\delta_1^-} = i_{k-\delta_1^- - 1}, \\ \dots, j_{\delta_2^++\delta_2^-} = i_{k-\delta_2^-}, \dots$$

$$\delta_i^\pm = \text{non negative integers, with } \sum \delta_i^+ = n-k, \sum \delta_i^- = k.$$

and this one is equal to $(-1)^k \cdot 1 = (-1)^k$.

With $\beta_\pi = B_\pi$, the expression for R becomes

$$R(x_0; x_1, x_2, \dots, x_n) = \sum_k \sum_{\pi_n} (-1)^k \theta_-(i_0, i_1) \theta_-(i_1, i_2) \dots \theta_-(i_{n-1}, 0) \\ \theta_+(0, i_{k+1}) \dots \theta_+(i_{n-1}, i_n) W(i_0, i_1, \dots, i_{k-1}, 0, i_{k+1}, \dots, i_n).$$

Also, as the definition of C_o^- gives

$$q_J \in V_- , \quad q_J + q_o \in V_+ \quad \text{when } q \in C_o^-$$

J being any subset of $(1, 2, 3, \dots, n)$,

we see that the series on the RHS is identical with the definition (1) of the generalized retarded operator corresponding to the cone C_o^- . The identity of the advanced product and the operator corresponding to C_o^+ can be similarly proved.

We note that these operators have the following Hermiticity properties

$$R_{\alpha}^* (x_1, x_2, \dots, x_n) = (-1)^{n-1} R_{\alpha} (x_1, x_2, \dots, x_n) . \quad (3)$$

For, we can write

$$\begin{aligned} & R_{\alpha}^* (x_1, x_2, \dots, x_n) \\ = & \left(\sum_{\pi} \sigma_{\pi} \theta_{\sigma_{\pi}^1} (i_2, i_3) \dots \theta_{\sigma_{\pi}^2} (i_2, i_3) \dots \theta_{\sigma_{\pi}^{n-1}} (i_{n-1}, i_n) W(i_1, i_2, \dots, i_n) \right)^* \end{aligned}$$

where σ_j^{π} is (+) or (-) according as q_J is in V_+ or V_- for $q \in C_{\alpha}$, J being the subset

$$J_{\pi} = (i_1, i_2, \dots, i_j)$$

and

$$\sigma_{\pi} = \prod_{j=1}^{n-1} \sigma_j^{\pi} .$$

As $\phi(x)$ is Hermitian (i.e. $\phi(f)$ is Hermitian if $f(x)$ is real) this gives

$$\begin{aligned} & R^* (x_o; x_1, x_2, \dots, x_n) \\ = & \sum_{\pi} \sigma_{\pi} \theta_{\sigma_1^{\pi}} (i_1, i_2) \theta_{\sigma_2^{\pi}} (i_2, i_2) \dots \theta_{\sigma_{n-1}^{\pi}} (i_{n-1}, i_n) W(i_n, i_{n-1}, \dots, i_2, i_1) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{\pi} \sigma_{\pi} \theta_{\overline{\sigma}_{n-1}} \pi(i_n, i_{n-1}) \theta_{\overline{\sigma}_{n-2}} \pi(i_{n-1}, i_{n-2}) \dots \theta_{\overline{\sigma}_1} \pi(i_1, i_2) \\
 &\quad W(i_n, i_{n-1}, \dots, i_2, i_1) \quad [\overline{\sigma} = -\sigma] \\
 &= (-1)^{n-1} \sum_{\pi} (-1)^{n-1} \sigma_{\pi} \theta_{\overline{\sigma}_{n-1}} \pi(i_n, i_{n-1}) \dots \theta_{\overline{\sigma}_1} \pi(i_2, i_1) W(i_n, i_{n-1}, \dots, i_1) .
 \end{aligned}$$

Clearly, $q_{J_{\pi}} \in V_{\overline{\sigma}_j} \pi$ implies $q_{\overline{J}_{\pi}} \in V_{\overline{\sigma}_j} \pi$, $\overline{J}_{\pi}$ being the complement of J_{π} in

$$(1, 2, 3, \dots, n)$$

i.e. the set

$$(i_n, i_{n-1}, \dots, i_{j+2}, i_{j+1}).$$

Also

$$\prod_{j=1}^{n-1} \overline{\sigma}_j \pi = (-1)^{n-1} \prod_{j=1}^{n-1} \sigma_j \pi = (-1)^{n-1} \sigma_{\pi} .$$

Hence the above expression is just equal to

$$(-1)^{n-1} R(x_1, x_2, \dots, x_n)$$

and this proves equation (3).

Further, we give the following result of Burgoyne⁵ for future reference. If $\tilde{f}$ and $\tilde{r}$ denote the Fourier Transforms of the time ordered and generalized retarded functions, then

$$\tilde{f}(p_1, p_2, \dots, p_n) = \tilde{r}_{\alpha}(p_1, p_2, \dots, p_n)$$

for $p \in C_{\alpha}$.

2. Reduction Formulae

We now prove reduction formulae for generalized retarded operators similar to the formulae (10) and (11) for ordinary ones, consider first the commutation

$$\left[R(x_1, x_2, \dots, x_n), \varphi_{in}^{a*} \right]$$

Using the definition of φ_{in}^{a*} and the asymptotic condition, we see that it is equal to

$$z^0 \xrightarrow{\text{Lim}} -\infty -i \int d\underline{z} \left[R(x_1, x_2, \dots, x_n), \varphi(z) \right] \overset{\leftrightarrow}{\partial}_z f_a(z)$$

in the sense of weak convergence. Let us assume that there exists an (n+1) point generalized retarded operator

$$R_{\alpha-}(x_1, x_2, \dots, x_n, z)$$

such that

$$z^0 \xrightarrow{\text{Lim}} -\infty \left[R_{\alpha-}(x_1, x_2, \dots, x_n), \varphi(z) \right] = z^0 \xrightarrow{\text{Lim}} -\infty R_{\alpha-}(x_1, x_2, \dots, x_n, z) \quad (4)$$

In the following section (3) we give a method which explicitly determines $R_{\alpha-}$ when R_{α} is given. In the meantime, using (4), we get

$$\begin{aligned} & \left[R_{\alpha}(x_1, x_2, \dots, x_n), \varphi_{in}^{a*} \right] \\ &= z^0 \xrightarrow{\text{Lim}} -\infty -i \int d\underline{z} R_{\alpha-}(x_1, x_2, \dots, x_n, z) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \\ &= i \int d\underline{z} \partial_{z^0} \left\{ R_{\alpha-}(x_1, x_2, \dots, x_n, z) \overset{\leftrightarrow}{\partial}_{z^0} f_a(z) \right\}, \end{aligned}$$

as it is shown later that

$$z^0 \xrightarrow{\text{Lim}} +\infty R_{\alpha^-}(x_1, x_2, \dots, x_n, z) = 0 \quad (5)$$

Using now the fact that $f_a(z)$ satisfies the Klein Gordon equation, and performing integration by parts over the spatial variables, we get the required formula for the commutator as

$$\begin{aligned} & [R_{\alpha}(x_1, x_2, \dots, x_n), \varphi_{\text{in}}^{a*}] \\ &= -i \int dz f_a(z) K_z R_{\alpha^-}(x_1, x_2, \dots, x_n, z) \end{aligned} \quad (6)$$

where R_{α^-} is obtained from R_{α} according to (4).

Using a similar method, we can show that

$$\begin{aligned} & [R_{\alpha}(x_1, x_2, \dots, x_n), \varphi_{\text{out}}^{a*}] \\ &= -i \int dz f_a(z) K_z R_{\alpha^+}(x_1, x_2, \dots, x_n, z) \end{aligned} \quad (7)$$

where R_{α^+} is the generalized retarded operator such that

$$z^0 \xrightarrow{\text{Lim}} +\infty [R_{\alpha}(x_1, x_2, \dots, x_n), \varphi(z)] = -z^0 \xrightarrow{\text{Lim}} -\infty R_{\alpha^+}(x_1, x_2, \dots, x_n, z) \quad (8)$$

3. Determination of R_{α^+} .

Let us suppose that

$$R_{\alpha}(x_1, x_2, \dots, x_n)$$

is a given generalized retarded operator, i.e. we know whether q_J is in V_+ or V_- for all subsets J of the set

$$(1, 2, 3, \dots, n).$$

We define a cone C_{α^-} of $(n+1)$ points, by the conditions

- (i) $q_J \in V_+$ in C_{α^-} for all J such that $q_J \in V_+$ in C_{α}
 - (ii) $q_J \in V_-$ in C_{α^-} for all J such that $q_J \in V_-$ in C_{α}
 - (iii) $q_{n+1} \in V_-$ and hence $q_1 + q_2 + \dots + q_n \in V_+$ in C_{α^-} .
- (9)

If J is such that $q_J \in V_-$ in C_{α} , then according to (ii), $q_J \in V_-$ in C_{α^-} also so that $q_{n+1} + q_J \in V_-$ in C_{α^-} . If, on the other hand, it is such that $q_J \in V_+$ in C_{α} , then $q_J \in V_-$ in C_{α^-} and hence, by (ii) again, $q_J \in V_-$ in C_{α^-} also, so that $q_J + q_{n+1} \in V_+$ in C_{α^-} . Thus the above conditions determine whether $q_J + q_{n+1}$ is in V_+ or in V_- for each J , i.e. (9) defines the cone C_{α^-} uniquely and completely.

We now show that the operator

$$R_{\alpha^-}(x_1, x_2, \dots, x_n)$$

corresponding to the cone C_{α^-} , satisfies equation (4). From the definition of generalized retarded operators, we have

$$\begin{aligned} x_{n+1}^o \lim_{\rightarrow -\infty} R_{\alpha^-}(x_1, x_2, \dots, x_n, x_{n+1}) \\ = x_{n+1}^o \lim_{\rightarrow -\infty} \sum_{\pi_{n+1}} \pm \theta \pm (i_1, i_2) \theta \pm (i_1, i_3) \dots \theta \pm (i_n, i_{n+1}) \\ W(i_1, i_2, \dots, i_n, i_{n+1}) \\ = x_{n+1}^o \lim_{\rightarrow -\infty} \sum_{l=0}^n \sum_{\pi} \pm \theta (i_1, i_2) \theta \pm (i_2, i_3) \dots \theta \pm (i_1, n+1) \\ \theta \pm (n+1, i_{l+1}) \dots \theta \pm (i_{n-1}, i_n) W(i_1, i_2, \dots, i_{l+1}, i_{l+1}, \dots, i_n) \end{aligned} \quad (10)$$

Consider now the product

$$\theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \left[\theta_{\pm}(i_1, n+1) \theta_{\pm}(n+1, i_{l+1}) \right] \dots \theta_{\pm}(i_{n+1}, i_n) . \quad (11)$$

The subscript of each of the θ -functions is given by the rule (iii) in the definition (1) of retarded operators. As $x_{n+1}^0 \rightarrow -\infty$, the product of the two θ -functions in the square brackets, and hence the product (11) itself, is zero unless the subscript of the first θ -function is (+) and that of the second is (-). This will be true only when

$$q_L \in V_+, \quad q_L + q_{n+1} \in V_- \quad \text{in } C_{\alpha-}$$

$$\text{where } L = (i_1, i_2, \dots, i_l)$$

On the other hand, the conditions (9) defining $C_{\alpha-}$ show that this is not true for $1 \leq l \leq n-1$. Hence in the summation over l in (10), the only terms which can be different from zero are the ones corresponding to $l = 0$ and $l = n$. In the $l = 0$ case, (11) becomes

$$\begin{aligned} & \theta_{-}(n+1, i_1) \theta_{+}(i_1, i_2) \dots \theta_{\pm}(i_{n-1}, i_n) \\ &= - \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \theta_{\pm}(i_{n-1}, i_n) \end{aligned}$$

as $q_{n+1} \in V_-$ in $C_{\alpha-}$ and $\theta_{-}(n+1, i_1) = 1$ in the limit $x_{n+1}^0 \rightarrow -\infty$. The minus sign appears because the left-hand side has one θ -function more than the right-hand side. Thus the term in (10) corresponding to $l = 0$ is

$$\begin{aligned} x^0 \xrightarrow{\text{Lim}} -\infty & - \sum_{\pm} \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \theta_{\pm}(i_{n-1}, i_n) \mathcal{P}(x_{n+1}) \\ & W(i_1, i_2, \dots, i_n) \end{aligned}$$

$$= x_{n+1}^o \xrightarrow{-\infty} \text{Lim} \varphi(x_{n+1}) R_{\alpha}(x_1, x_2, \dots, x_n), \quad (12a)$$

as for any subset J of

$$(1, 2, 3, \dots, n),$$

q_J is in V_+ or in V_- in C_{α} according as it is in V_+ or in V_- in C .

Similarly, when $l = n_1$ (11) becomes

$$\begin{aligned} & \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \theta_{\pm}(i_n, n+1) \\ &= \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \theta_{\pm}(i_{n-1}, i_n) \end{aligned}$$

as $q_1 + q_2 + \dots + q_n \in V_+$ in C_{α} and $\theta_{\pm}(i_n, n+1) = 1$ in the limit

$x_{n+1}^o \rightarrow -\infty$. This time no minus sign appears as the left-hand side and the right-hand side have the same number of θ -minus functions. The term of (10) resulting from $l = n$ is therefore equal to

$$\begin{aligned} & x_{n+1}^o \xrightarrow{-\infty} \text{Lim} \sum \pm \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots \theta_{\pm}(i_{n-1}, i_n) W(i_1, i_2, \dots, i_n) \varphi(x_{n+1}) \\ &= x_{n+1}^o \xrightarrow{-\infty} \text{Lim} R_{\alpha}(x_1, x_2, \dots, x_n) \varphi(x_{n+1}) \end{aligned} \quad (12b)$$

Combining (12a, b), we get

$$\begin{aligned} & x_{n+1}^o \xrightarrow{-\infty} \text{Lim} R_{\alpha}(x_1, x_2, \dots, x_n, x_{n+1}) \\ &= x_{n+1}^o \xrightarrow{-\infty} \text{Lim} \left[-\varphi(x_{n+1}) R_{\alpha}(x_1, x_2, \dots, x_n) + R_{\alpha}(x_1, x_2, \dots, x_n) \varphi(x_{n+1}) \right] \end{aligned}$$

which shows that R_{α} defined by (9) satisfies (4).

In an exactly similar manner, we can show that the operator R_{α}^+

satisfying (8) is the generalized retarded operator corresponding to the cone C_α^+ defined by the conditions

$$\left. \begin{aligned} \text{(i)} \quad q_J \in V_+ \text{ in } C_\alpha^+ \text{ for all } J \text{ such that } q_J \in V_+ \text{ in } C_\alpha \\ \text{(ii)} \quad q_J \in V_- \text{ in } C_\alpha^+ \text{ for all } J \text{ such that } q_J \in V_- \text{ in } C_\alpha \\ \text{(iii)} \quad q_1 + q_2 + \dots + q_n \in V_- \text{ and hence } q_{n+1} \in V_+ \text{ in } C_\alpha^+ \end{aligned} \right\} \quad (13)$$

Let us now verify equation (5). We have

$$\begin{aligned} x_{n+1}^0 \xrightarrow{\text{Lim}} + \infty \quad R_\alpha^-(x_1, x_2, \dots, x_n) \\ = x_{n+1}^0 \xrightarrow{\text{Lir}} + \infty \sum_{l=0} \sum_{\pi_n} \pm \theta_\pm(i_1, i_2) \theta_\pm(i_2, i_3) \dots \theta_\pm(i_1, n+1) \theta_\pm(n+1, \\ i_{l+1}) \dots \\ \theta_\pm(i_{n-1}, i_n) W(i_1, i_2, \dots, i_{l+1}, i_{l+1}, \dots, i_n) \end{aligned} \quad (14)$$

Consider the product (11). As $x_{n+1}^0 \rightarrow +\infty$, the product of the two θ -functions in the square brackets is different from zero only when

$$q_L \in V_- \text{ and } q_L + q_{n+1} \in V_+ \\ \text{where } L = (i_1, i_2, \dots, i_n).$$

This is impossible for $1 \leq l \leq n-1$, as q_L and $q_L + q_{n+1}$ are either both in V_+ or both in V_- in C_α^- . When $l=0$ and $l=n$, (11) is non-zero only if $q_{n+1} \in V_+$ and $q_1 + q_2 + \dots + q_n \in V_-$ respectively, and this is again not the case, as $q_{n+1} \in V_-$ in C_α^- . Hence all the terms in summation over l vanish in (14) so that (5) is automatically satisfied.

To illustrate the results already obtained, let us evaluate the

commutators

$$\left[R_0(x_0, x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right] \text{ and } \left[R_0(x_0, x_1, x_2, \dots, x_n), \phi_{out}^{a*} \right]$$

where R_0 is the ordinary retarded product of LSZ. As proved earlier, the cone C_0 is defined by

$$q_J \in V_- , \quad J \neq \text{null set}$$

$C_0 :$

$$q_0 + q_J \in V_+ , \quad J \neq (1, 2, 3, \dots, n)$$

where J is any subset of $(1, 2, 3, \dots, n)$. Hence if

$$\begin{aligned} & \left[R_0(x_0, x_1, x_2, \dots, x_n), \phi_{in}^{a*} \right] \\ &= -i M_{n+1}^a R_0^-(x_0, x_1, x_2, \dots, x_n, x_{n+1}) \end{aligned} \quad (15)$$

where M is defined as the integral operator

$$M_i^a = \int dx_i f_a(x_i) K_{x_i} , \quad (16)$$

the cone C_{0-} will be defined, according to (9), by

$$q_{n+1}, q_J \in V_- , \quad J \neq \text{null set}$$

$$q_0 + q_J, q_0 + q_J + q_{n+1} \in V_+$$

i.e. by

$$q_I \in V_- , \quad I \neq \text{null set}$$

$$q_0 + q_I \in V_+ , \quad I \neq (1, 2, 3, \dots, n, n+1)$$

where I is any subset of $(1, 2, 3, \dots, n, n+1)$. This shows that R_{0-} is the ordinary retarded product, so that (15) is identical with (1.10) as it should be.

Similarly, if

$$\left[R_0(x_0, x_1, \dots, x_n), \varphi_{out}^{a*} \right] \\ = -i M_{n+1}^2 R_{0+}(x_0, x_1, x_2, \dots, x_n, x_{n+1}),$$

the cone C_{0+} will be defined by

$$q_J, q_J + q_{n+1} \in V_- \quad J \neq \text{null set}$$

$$q_{n+1}, q_0 + q_J \in V_+, \quad J \neq (1, 2, 3, \dots, n)$$

i.e. by

$$q_I \in V_-, \quad I \neq \text{null set or } (n+1)$$

$$q_{n+1}, q_0 + q_I \in V_+, \quad I \neq (1, 2, \dots, n) \text{ and } (1, 2, 3, \dots, n, n+1).$$

This shows that for $n \geq 2$, C_{0+} is the cone C_{0n+1}^- ¹⁷ which is defined by

$$C_{0n+1}^- : -q_{n+1}, q_1 + q_{n+1}, q_2 + q_{n+1}, \dots, q_n + q_{n+1} \in V_-$$

and to which corresponds the generalized retarded operator

$$R_{0n+1}(x_0, x_1, x_2, \dots, x_n, x_{n+1}).$$

For $n = 1$, C_{0+} is identical with the cone

$$C_1^+ : q_0, q_2 \in V_+$$

to which corresponds the advanced function

$$A_1(x_0, x_1, x_2).$$

Thus we have

$$\left[R_0(x_0, x_1), \varphi_{out}^{a*} \right] = -i M_2^2 A_1(x_0, x_1, x_2). \\ \left[R_0(x_0, x_1, x_2, \dots, x_n), \varphi_{out}^{a*} \right] \\ = -i M_{n+1}^2 R_{0n+1}(x_0, x_1, x_2, \dots, x_n, x_{n+1}) \quad \text{for } n \geq 2. \quad (17)$$

4. Analyticity and Support Properties

In this section, we study some of the properties of the retarded operators of the type $R_{\alpha}^{\pm}$ defined by (9) and (13). Let us start by introducing some new notations. If C_{α_n} is an arbitrary n -point cone, we can obtain from it two $(n+1)$ -point cones $C_{\alpha_{n+1}}$ and $C_{\alpha'_{n+1}}$ which are defined by

$$C_{\alpha_{n+1}} : q_J \in V_{\pm} \text{ in } C_{\alpha_{n+1}} \text{ according as it is in } V_{\pm} \text{ in } C_{\alpha_n}, q_{n+1} \in V_{+}$$

$$C_{\alpha'_{n+1}} : q_J \in V_{\pm} \text{ in } C_{\alpha'_{n+1}} \text{ according as it is in } V_{\pm} \text{ in } C_{\alpha_n}, q_{n+1} \in V_{-}$$

$J \equiv$ any proper subset of $(1, 2, \dots, n)$.

We shall denote these cones by

$$C\{\alpha_n, + (n+1)\} \quad \text{and} \quad C\{\alpha_n, - (n+1)\}$$

respectively. If C_{α_n} is itself obtained from an $(n-1)$ -point cone i.e. we have

$$C_{\alpha_n} = C\{\alpha_{n-1}, \pm i_n\}$$

where the prefix $(\pm)$ means either $(+)$ or $(-)$ and $C_{\alpha_{n-1}}$ is an $(n-1)$ -point cone, we shall write

$$C_{\alpha_{n+1}} = C\{\alpha_{n-1}, \pm i_n, + (n+1)\}$$

If we continue this process of expressing $C_{\alpha_{n+1}}$ in terms of C_{α_i} then either (i) it will terminate at a stage $j > 1$ i.e. we will have

$$C_{\alpha_{n+1}} = C\{\alpha_j, \pm i_{j+1}, \dots, \pm i_n, + (n+1)\}$$

where each of the prefixes is either $(+)$ or $(-)$ and C_{α_j} is a

j-point cone such that it can not be obtained from any of the (j-1) point cones. As an example, C_{α_j} may be the 5-point cone C'_{i_j} of reference (17) which can not be obtained from any of the 4-point cones;

or (ii) It will go down to $j = 1$. In this case, we will have

$$C_{\alpha_{n+1}} = C \left\{ \pm i_1, \pm i_2, \pm i_3, \dots, \pm i_n, (n+1) \right\} .$$

Clearly, the prefix of the first and second index must be different. We also note that all the cones up to the 4-point case can be expressed in this form .

Thus we see that an arbitrary n-point cone C_{α_n} either has the form

$$C_{\alpha_n} = C \left\{ \pm i_1, \pm i_2, \pm i_3, \dots, \pm i_n \right\} \quad (18)$$

where

$$(i_1, i_2, \dots, i_n)$$

is a permutation of

$$(1, 2, 3, \dots, n) ,$$

or the form

$$C_{\alpha_n} = C \left\{ \alpha_j, \pm i_{j+1}, \dots, \pm i_n \right\} \quad (18')$$

where the j-point cone C_{α_j} can not be obtained from any of the (j-1)-point cones.

We shall write

$$R_{\alpha_n}(x_1, x_2, \dots, x_n) = R \left\{ \alpha_j, \pm i_{j+1}, \dots, \pm i_n \right\}$$

$$r_{\alpha_n}(x_1, x_2, \dots, x_n) = r \left\{ \alpha_j, \pm i_{j+1}, \dots, \pm i_n \right\}$$

$$\tilde{r}_{\alpha_n}(p_1, p_2, \dots, p_n) = \tilde{r} \left\{ \alpha_j, \pm i_{j+1}, \dots, \pm i_n \right\}$$

where C_{α} is the cone (19').

Clearly, the cones $C_{\alpha}^{\pm}$ defined by (9) and (13) are given by

$$C_{\alpha}^{-} = C\{\alpha, -(n+1)\}, \quad C_{\alpha}^{+} = C\{\alpha, +(n+1)\}$$

so that the reduction formulae (6) and (7) can be written as

$$[R_{\alpha}(x_1, x_2, \dots, x_n), \varphi_{in}^{a*}] = -i M_{n+1}^a R\{\alpha, -(n+1)\} \quad (19a)$$

$$[R_{\alpha}(x_1, x_2, \dots, x_n), \varphi_{out}^{a*}] = -i M_{n+1}^a R\{\alpha, +(n+1)\} \quad (19b)$$

Using the relation (3), we see that the Hermitian conjugate of these formulae are

$$[\varphi_{in}^a, R_{\alpha}(x_1, x_2, \dots, x_n)] = -i M_{n+1}^{a*} R\{\alpha, -(n+1)\}, \quad (19c)$$

$$[\varphi_{out}^a, R_{\alpha}(x_1, x_2, \dots, x_n)] = -i M_{n+1}^{a*} R\{\alpha, +(n+1)\}, \quad (19d)$$

where
$$M_i^{a*} = \int dx_i f_a^{a*}(x_i) K_{x_i}$$

In the next chapter, (19a-d) are used for reduction of S-matrix elements.

In general, an n-point cone has a basis of more than (n-1) elements i.e. it can be completely defined only by more than (n-1) conditions of the form

$$q_I \in V_{\pm} \quad \text{for } q \in C_{\alpha}$$

$$I \equiv \text{a subset of } (1, 2, 3, \dots, n).$$

Thus, for example, the 5-point cone C_{ij}^{+-} , referred to above, has a basis of 6 elements, namely

$$\begin{aligned} q_i + q_1, \quad q_i + q_m, \quad q_i + q_n \\ q_j + q_1, \quad q_j + q_m, \quad q_j + q_n \end{aligned} \in V_{+} \quad (20)$$

(i, j, 1, m, n) $\equiv$ a permutation of (1, 2, 3, 4, 5).

However, cones of the type (18) have a basis of just $(n-1)$ elements.

This follows from the result that a basis of (18) is

$$\left. \begin{aligned} q_{i_1} + q_L &\in V_+ \text{ if the prefix of } i_1 \text{ is } (+) \\ &\in V_- \text{ if the prefix of } i_1 \text{ is } (-) \\ l &= 2, 3, 4, \dots, n \end{aligned} \right\} \quad (21)$$

L = set of all the indices i_t , $t > 1$, having their prefixes different from that of i_1 .

which we now prove. Let

$$J = (k_1, k_2, \dots, k_j)$$

be any subset of

$$(1, 2, 3, \dots, n) .$$

Clearly, we may assume that i_1 is not contained in J , for if it were, we could take $\bar{J}$ and rename it as J . Let k_a be the index of J which is on the left of the rest of k 's in (18), so that (21) implies

$$q_{k_a} + q_A \in \begin{cases} V_+ & \text{if the prefix of } k_a \text{ is } (+) \\ V_- & \text{if the prefix of } k_a \text{ is } (-) \end{cases}$$

A being the set of all the indices of J whose prefix is different from that of k_a . Now if B is the complement of the subset (k_a, A) in J , the prefix of any index of B is the same as that of k_a , and so we get

$$q_{k_a} + q_A + q_B \in \begin{cases} V_+ & \text{if the prefix of } k_a \text{ is } (+) \\ V_- & \text{if the prefix of } k_a \text{ is } (-) . \end{cases}$$

As

$$q_{k_a} + q_A + q_B = q_J ,$$

it follows that the conditions (21) determine uniquely whether q_J is in V_+ or V_- for every subset J of

$$(1, 2, 3, \dots, n)$$

i.e. (21) form a basis of (18).

As an example, we see that the basis of

$$R\{+1, -2, -3, +4, -5\} = R_{14,5}(x_1, x_2, x_3, x_4, x_5)$$

is

$$\begin{array}{ll} q_3 + q_4 \in V_- & q_4 + q_5 \in V_+ \\ q_3 + q_4 \in V_- & q_5 \in V_- \end{array}$$

which is identical with the one given by Burgoyne⁵.

The question now arises: Is the above basis of (18) unique?

The answer is in affirmative and follows from the fact that if any n -point cone C_α has a basis, it is unique. To prove this result, we proceed as follows. Let

$$B : q_{J_1}, q_{J_2}, \dots, q_{J_m} \in V_+$$

be a basis of C_α . This implies that any other basis has just m elements and that any partial sum q_2 can be expanded in the form

$$\begin{aligned} q_L &= \alpha_1 q_{J_1} + \alpha_2 q_{J_2} + \dots + \alpha_m q_{J_m} & \text{if } q_L \in V_+ \text{ in } C_\alpha \\ &= -(\alpha_1 q_{J_1} + \alpha_2 q_{J_2} + \dots + \alpha_m q_{J_m}) & \text{if } q_L \in V_- \text{ in } C_\alpha \end{aligned}$$

where all the α 's are +ve numbers. Now suppose that

$$q_{J_1} = q_{J_1'} + q_{J_1''} \text{ with } q_{J_1'} \in V_+, q_{J_1''} \in V_+ .$$

Then

$$q_{J_1}' = \alpha_1 q_{J_1} + \alpha_2 q_{J_2} + \dots + \alpha_m q_{J_m}$$

where at least one $\alpha_i \neq 0$, $i = 2, 3, \dots, m$, as otherwise q_{J_1}' would become a multiple of q_{J_1} , which is not the case.

$$\therefore q_{J_1}^2 = q_{J_1} - q_{J_1}' = (1 - \alpha_1) q_{J_1} - (\alpha_2 q_{J_2} + \alpha_3 q_{J_3} + \dots + \alpha_m q_{J_m})$$

As q_{J_i} are linearly independent, this is the unique expansion of $q_{J_1}^2$ in terms of q_{J_i} . Hence we get a contradiction, as $q_{J_1}^2$ is to be of the form

$$q_{J_1}^2 = \beta_1 q_{J_1} + \beta_2 q_{J_2} + \dots + \beta_m q_{J_m}$$

where β 's are all to be +ve and at least one $\beta_i \neq 0$, $2 \leq i \leq m$.

This shows that q_{J_1} can not be equal to the sum of two partial sums each of which is in V_+ when $q \in C_\alpha$. Clearly the same will be true for all the elements

$$q_{J_i}, i = 1, 2, 3, \dots, m$$

of the basis B.

Now let

$$B' : q_{J_1}', q_{J_2}', \dots, q_{J_m}' \in V_+$$

be any basis of C_α other than B, and let q_{J_1}' be expanded as

$$q_{J_1}' = \alpha_1' q_{J_1} + \alpha_2' q_{J_2} + \dots + \alpha_m' q_{J_m}, \alpha_i' \geq 0.$$

Then all the α_i' 's must be zero except ^{one} say α_1' , as otherwise q_{J_1}' will be equal to the sum of two partial sums each of which is in V_+ . This will lead to

$$q_{J_1}' = q_{J_1}.$$

Similarly, each of the other q_{j_i} will be just equal to one of the q_{j_i} i.e. B' is identical with B . This proves that there is only one distinct basis of C .

We now explicitly determine the support of

$$r\{\pm i_1, \pm i_2, \pm i_3, \dots, \pm i_n\}$$

In fact, we show that this function vanishes, if any of the following $(n-1)$ set of conditions (corresponding to the $(n-1)$ values $2, 3, 4, \dots, n$ of 1) is satisfied

$$x_{i_1}^0 - x_{i_m}^0 < 0 \text{ if } q_{i_1} \in V_+ \quad (22a)$$

for all $m < 1$ and such that $q_{i_m} \in V_-$,

and

$$x_{i_1}^0 - x_{i_m}^0 > 0 \text{ if } q_{i_1} \in V_- \quad (22b)$$

for all $m < 1$ and such that $q_{i_m} \in V_+$.

The proof depends on the following simple lemma:

LEMMA:- If $(x_1, x_2, \dots, x_n)$ is such that every term in the expansion of

$$R_\alpha(x_1, x_2, \dots, x_n)$$

vanishes identically, then the same is true for every term of

$$R\{\alpha, \pm(n+1), \pm(n+2), \dots, \pm n'\} \quad (23)$$

no matter what values $x_{n+1}, x_{n+2}, \dots, x_{n'}$ have.

For a general term of

$$R\{\alpha, \pm(n+1)\}$$

is

$$\pm \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots [\theta_{\pm}(i_j, i_{j+1}) \theta_{\pm}(i_{j+1}, i_{j+2})] \dots \theta_{\pm}(i_{n-1}, i_n) \\ W(i_1, i_2, \dots, i_j, i_{j+1}, \dots, i_n) \quad (24a)$$

Consider first the case $j \neq 0, n$. As $q_L, q_L + q_{n+1}, L \equiv$ any proper subset of

$$(1, 2, 3, \dots, n)$$

are both in V_+ or both in V_- in

$$C \{a, + (n+1)\}$$

according as $q_L \in V_+$ or V_- in C_a , the product of θ -functions in (24a) is identical with that in the term

$$\pm \theta_{\pm}(i_1, i_2) \theta_{\pm}(i_2, i_3) \dots, \theta_{\pm}(i_j, i_{j+1}) \dots \theta_{\pm}(i_{n-1}, i_n) W(i_1, i_2, \dots, i_n) \quad (24b)$$

in the expansion of R_a except for the fact that the two θ -functions in the square bracket in (24a) are replaced by $\theta_{\pm}(i_j, i_{j+1})$ in (24b). Also, as

$q_J, q_J + q_{n+1}$ where

$$J \equiv (i_1, i_2, \dots, i_j),$$

are again both in V_+ or in V_- in

$$C \{a, + (n+1)\}$$

according as q_J is in V_+ or in V_- in C_a , the subscript of all the three

θ -functions is the same and so, in (24a) we may insert a $\theta_{\pm}(i_j, i_{j+1})$

with the same subscript as that in (24b). The product of θ -functions in

(24a) will now include that in (24b), so that, as the latter vanishes by

hypothesis, so also will do the former. This results in the vanishing of the term (24a) itself. For $j = 0, n$, it is easily verified that the product of Θ -functions in (24a) already includes that in (24b) so that (24a) again vanishes. This proves the lemma for

$$R\{\alpha, +(n+1)\}.$$

The case when the prefix of $(n+1)$ is $(-)$ can be dealt with in an exactly the same way, and a repeated application of these two results shows that the lemma is true for the operators (23) as well.

Clearly, due to this lemma, it is sufficient to prove that every term of

$$r\{\pm i_1, \pm i_2, \dots, \pm i_n\}$$

vanishes when (22a,b) are satisfied for $l = n$. Consider first the function

$$r\{\pm i_1, \pm i_2, \dots, \pm i_{n-1}, + i_n\} \quad (25)$$

a general term in its expansion is

$$\pm \Theta_{\pm}(k_1, k_2) \Theta_{\pm}(k_2, k_3) \dots \Theta_{\pm}(k_j, i_n) \Theta_{\pm}(i_n, k_{j+1}) \dots \Theta_{\pm}(k_{n-2}, k_{n-1}) \\ \langle 0 / W(k_1, k_2, \dots, k_j, i_n, k_{j+1}, \dots, k_{n-2}, k_{n-1}) / 0 \rangle \quad (26)$$

where

$$(k_1, k_2, \dots, k_{n-1}) \equiv \text{a permutation of } (i_1, i_2, \dots, i_n).$$

Now if

$$q_{k_1} + q_{k_2} + \dots + q_{k_j} + q_{i_n} \in V_+$$

there will exist a smallest integer $t > j$ such that

$$q_{k_1} + q_{k_2} + \dots + q_{k_j} + q_{i_n} + q_{k_{j+1}} + \dots + q_{k_t} \in V_-.$$

Clearly, this implies $q_{k_t} \in V_-$. The term (26) will contain the factor

$$\theta_+(i_n, k_{j+1}) \theta_+(k_{j+1}, k_{j+2}) \dots \theta_+(k_{t-1}, k_t)$$

and hence will be zero if

$$x_{i_n}^0 - x_{k_t}^0 < 0.$$

On the other hand, if

$$q_{k_1} + q_{k_2} + \dots + q_{k_j} + q_{i_n} \in V_- ,$$

there will exist a largest integer $t < j$, such that

$$q_{k_1} + q_{k_2} + \dots + q_{k_{t-1}} \in V_+$$

and hence $q_{k_t} \in V_-$. The term (26) will now contain the factor

$$\theta_-(k_t, k_{t+1}) \theta_-(k_{t+1}, k_{t+2}) \dots \theta_-(k_{j-1}, k_j) \theta_-(k_j, i_n)$$

and will therefore again be zero if

$$x_{i_n}^0 - x_{k_t}^0 < 0.$$

Thus every term in the expansion of (25) vanishes if (22a) with $l = n$ is satisfied. To see that all $m < l$ are to be included in (22a), we note that the term

$$\pm \theta_-(k_1, i_n) \theta_{\pm}(i_n, k_2) \theta_{\pm}(k_2, k_3) \dots \theta_{\pm}(k_{n-2}, k_{n-1}) \times \\ \langle 0 / W(k_1, i_n, k_2, k_3, \dots, k_{n-2}, k_{n-1}) / 0 \rangle ,$$

for any k_1 such that $q_{k_1} \in V_-$, vanishes only when

$$x_{i_n}^0 - x_{k_1}^0 < 0.$$

Similarly, we can show that

$$r \left\{ \pm i_1, \pm i_2, \dots, \pm i_{n-1}, -i_n \right\}$$

vanishes if

$$x_{i_n}^0 - x_{i_m}^0 > 0$$

for all $m < n$ and such that $q_{i_m} \in V_+$, as required according to (22b).

Let us make the following remark. It follows from (22b) with $l = 2$ that

$$r \{ + i_1, - i_2, \pm i_3, \pm i_4, \dots, \pm i_n \} = 0 \quad (27a)$$

$$\text{for } x_{i_1}^0 - x_{i_2}^0 < 0$$

so that, as this function is Lorentz invariant, it also vanishes for

$$x_{i_1} - x_{i_2} \notin V_+$$

For if there were a point where $x_{i_1}^0 - x_{i_2}^0 > 0$ but $x_{i_1} - x_{i_2} \notin V_+$, we could find a Lorentz transformation which takes this point to one with $x_{i_1}^0 - x_{i_2}^0 < 0$ and so the function vanishes at such points.

Similarly we have

$$r \{ -i_1, + i_2, \pm i_3, \pm i_4, \dots, \pm i_n \} = 0 \quad (27b)$$

whenever $x_{i_1} - x_{i_2} \notin V_-$.

Now it is shown in Chapter IV that the function

$$r_\alpha = r \{ \pm i_1, \pm i_2, \dots, \pm i_{l-1}, + i_l \pm i_{l+1}, \dots, \pm i_n \}$$

can be expanded in the form

$$r_\alpha = \sum_{\beta} a_{\beta} r_{\beta}$$

where a_{β} are constants and C_{β} are cones of the type

$$C_{\beta} : C \{ + i_1, - k_1, \pm k_2, \pm k_3, \dots, \pm k_{n-1} \},$$

which have the same prefix of any index as in C_α ,

$$(k_1, k_2, k_3, \dots, k_{n-1})$$

being a permutation of

$$(i_1, i_2, \dots, i_{l-1}, i_{l+1}, \dots, i_n).$$

It is easily seen that each of these cones C_β must be such that k_1 is on the left of i_1 in C_α . For if one of them had a k_1 which is on the right of i_1 , the corresponding r -function, and hence also $\sum a_\beta \gamma_\beta$, will not be zero when (22a) are satisfied, while r_α is, so that we get a contradiction. Hence it follows from the above remark that each r_β vanishes when

$$x_{i_1} - x_{i_m} \notin V_+$$

for some $m \leq l$ and such that $q_{i_m} \in V_-$ in C_α

and so

$$r_\alpha = \sum a_\beta \gamma_\beta$$

vanishes when

$$x_{i_1} - x_{i_m} \notin V_+$$

for all $m \leq l$ such that $q_{i_m} \in V_-$ in C_α .

Similar result can be easily obtained for the case when the prefix of i_1 is $(-)$, and so we conclude that the function

$$r\{\pm i_1, \pm i_2, \dots, \pm i_n\}$$

vanishes when any one of the following $(n-1)$ sets of conditions is satisfied

$$\begin{aligned} x_{i_1} - x_{i_m} \notin V_+ & \text{ if } q_{i_1} \in V_+ \\ \text{for all } m \leq l & \text{ and such that } q_{i_m} \in V_- \end{aligned} \quad (28a)$$

and

$$\begin{aligned} x_{i_1} - x_{i_m} &\notin V_- \text{ if } q_{i_1} \in V_- \\ \text{for all } m < l \text{ and such that } q_{i_m} &\in V_+ . \end{aligned} \quad (28b)$$

Clearly, the support implied by (28) is much smaller than that implied by (22).

Next, let C_α be any cone with a basis of $(n-1)$ elements, and let this basis be

$$\begin{aligned} q_{I_1}, q_{I_2}, \dots, q_{I_k} &\in V_+, \quad q_{I_{k+1}}, q_{I_{k+2}}, \dots, q_{I_{n-1}} \in V_- \\ I_1, I_2, \dots, I_{n-1} &\equiv \text{subsets of } (1, 2, 3, \dots, n). \end{aligned}$$

Then, as we can write

$$\begin{aligned} &q_1 x_1 + q_2 x_2 + q_3 x_3 + \dots + q_n x_n \\ = &q_{I_1} x_1 + q_{I_2} x_2 + \dots + q_{I_k} x_k - q_{I_{k+1}} x_{I_{k+1}} - q_{I_{k+2}} x_{I_{k+2}} - \dots - q_{I_{n-1}} x_{I_{n-1}} \end{aligned}$$

where x_i are linear combination of terms of the form

$$x_u - x_v, \quad u, v = 1, 2, \dots, n,$$

the conjugate cone $\bar{C}_\alpha$ is defined by

$$X_1 \in V_+, X_2 \in V_+, \dots, X_{n-1} \in V_+.$$

Thus r_α , whose support is contained in $\bar{C}_\alpha^0$, vanishes if

$$X_1^0 < 0 \text{ or } X_2^0 < 0 \text{ or } \dots \text{ or } X_{n-1}^0 < 0$$

and therefore, due to Lorentz invariance also if

$$X_1 \notin V_+ \text{ or } X_2 \notin V_+ \text{ or } \dots \text{ or } X_{n-1} \notin V_+$$

i.e. if $x \notin \bar{C}_\alpha$. If we now consider the Fourier Transform

$$\tilde{r}_\alpha(s_1, s_2, s_3, \dots, s_n), \quad s = p + iq, \quad \sum_{i=1}^n s_i = 0$$

we see that it is analytic (due to Schwarz Theorem¹⁴) in the tube

$$T_\alpha : p \text{ arbitrary, } q \in C_\alpha,$$

and hence also in the extended tube

$$T'_\alpha = \bigcup_{\Lambda \in L(C)} \Lambda \quad T_\alpha, \quad L(C) \equiv \text{Complex Lorentz Group}$$

by the Theorem of Hall and Wightman¹⁸.

It may be noted that although the analyticity in the tube T_α , and hence also in the extended tube T'_α , can not, in general, be proved when the cone C_α has a basis of more than $(n-1)$ elements, yet this can be explicitly done in the case of the 5-point function r'_{ij} corresponding to the cone C_{ij}^{1-} defined by (20). For using the Steinmann relations¹⁷

$$r_{ij} - r_{ij,k} + r'_{ij,l} - r_{ij,m} = 0$$

$$r'_{ij} - r'_{ij,k} + r_{ij,l} - r'_{ij,m} = 0$$

$$(i, j, k, l, m) \equiv \text{a permutation of } (1, 2, 3, 4, 5),$$

we get

$$r'_{ij} = r_{ij,k} + r_{ij,l} + r_{ij,m} - 2r_{ij} \quad (29)$$

Also, we have (using (28))

$$r_{ij,k} = r \left\{ +i, -l, -m, +j, -k \right\} = 0$$

if

$$(x_l - x_i) \notin V_- \quad \text{or} \quad (x_m - x_i) \notin V_-$$

$$\text{or} \quad \left[(x_j - x_m) \notin V_+ \text{ and } (x_j - x_l) \notin V_+ \right]$$

$$\text{or} \quad \left[(x_k - x_j) \notin V_- \text{ and } (x_k - x_i) \notin V_+ \right]$$

and

$$r_{ij} = r \{ + i, - k, - l, - m, + j \} = 0$$

if

$$(x_k - x_i) \notin V_- \quad \text{or} \quad (x_l - x_i) \notin V_- \quad \text{or} \quad (x_m - x_i) \notin V_-$$

$$\text{or} \quad [(x_j - x_m) \notin V_+ \text{ and } (x_j - x_l) \notin V_+ \text{ and } (x_j - x_k) \notin V_+],$$

so that

$$r'_{ij} = 0$$

if

$$(x_p - x_r) \text{ and any one of } (x_p - x_s), (x_p - x_t) \\ \text{and } (x_q - x_r) \text{ are not in } V_+.$$

Here $(p,q) \equiv$ a permutation of (i,j)

$$(r,s,t) \equiv \text{a permutation of } (k,l,m).$$

This shows that r'_{ij} is different from zero only if

$$\begin{aligned} (x_p - x_r) \in V_+, \quad (x_p - x_s) \in V_+ \\ (x_q - x_r) \in V_+ \quad \text{and} \quad (x_q - x_t) \in V_+ \end{aligned} \tag{30}$$

for every permutation (p,q) of (i,j) and (r,s,t) of (k,l,m) .

Now we can write

$$x_1 q_1 + x_2 q_2 + x_3 q_3 + x_4 q_4 + x_5 q_5 = (x_p - x_r) (q_p + q_s) - (x_p + x_q - x_t - x_s)$$

$$(q_q + q_s + q_t) + (x_q - x_t) (q_q + q_s) + (x_p + x_q - x_s - x_r) (q_q + q_t)$$

and

$$(x_p - x_r), (x_p + x_q - x_t - x_s) = (x_p - x_s) + (x_q - x_t), (x_q - x_t),$$

$$(x_s + x_q - x_s - x_r) = (x_p - x_s) + (x_q - x_r),$$

are all in V_+ when conditions (30) are satisfied, while

$$(q_p + q_s), - (q_q + q_s + q_t) = (q_p + q_r), (q_s + q_q), (q_q + q_t)$$

are all in V_+ when $q \in C_{ij}^{'-}$. We therefore have

$$x_1 q_1 + x_2 q_2 + x_3 q_3 + x_4 q_6 + x_5 q_5 > 0$$

whenever x is such that r_{ij}^i is different from zero and q is in $C_{ij}^{'-}$, and so the Fourier Transform $\tilde{r}_{ij}$ is analytic in the tube

$$T : p, \text{ arbitrary, } q \in C_{ij}^{'-}.$$

The analyticity in the extended tube now follows from the theorem of Hall and Wightman which can be applied in this case also, even though the tube T differs from the tube R_4 of the theorem due to the fact that the cone $C_{12}^{'-}$ has a basis of 6 elements. This can be verified by looking at the Burgmann's proof of the theorem which remains unchanged, even if the tube R_{n-1} is replaced by a tube of the form

$$p, \text{ arbitrary, } q \in C_\alpha$$

and C_α has a basis of more than $(n-1)$ elements.

5. Connection with Steinmann's Retarded Operators

Steinmann¹⁵ has introduced certain retarded operators

$$\varphi(x_{i_1} \updownarrow x_{i_2} \updownarrow x_{i_3} \updownarrow \dots \updownarrow x_{i_n})$$

and has studied some of their properties. Here $\updownarrow$ means either $\uparrow$ or $\downarrow$, and these latter arrows are defined as follows¹⁵:

Let $H(x_1, x_2, \dots, x_n)$ be an operator of the form

$$H(x_1, x_2, \dots, x_n) = \sum_{\pi} f_{\pi}(x_1, x_2, \dots, x_n) \phi(x_{i_1}) \phi(x_{i_2}) \dots \phi(x_{i_n})$$

where the summation extends over the set of all the permutations

$$\pi : j \rightarrow i_j,$$

and f_{π} are arbitrary $\mathbb{C}$ -number functions. Steinmann defines

$$\begin{aligned} H(x_1, x_2, \dots, x_n \uparrow x_{n+1}) &\equiv H(x_1, x_2, \dots, x_n) \uparrow \phi(x_{n+1}) \\ &= \sum_{j=1}^n \theta(x_j^0 - x_{n+1}^0) H(x_1, x_2, \dots, \overline{x_j x_{n+1}}, \dots, x_n) \end{aligned} \quad (31a)$$

$$\begin{aligned} H(x_1, x_2, \dots, x_n \downarrow x_{n+1}) &= H(x_1, x_2, \dots, x_n) \downarrow \phi(x_{n+1}) \\ &= \sum_{j=1}^n \theta(x_{n+1}^0 - x_j^0) H(x_1, x_2, \dots, \overline{x_j x_{n+1}}, \dots, x_n) \end{aligned} \quad (31b)$$

where $\overline{x_j x_{n+1}}$ means that in the definition of H , $\phi(x_j)$ is replaced by the commutator

$$[\phi(x_j), \phi(x_{n+1})]$$

We now prove that the set of all the Steinmann's operators

$$\phi(x_i, \uparrow x_{i_2} \uparrow \dots \uparrow x_{i_n})$$

is identical with the set of retarded operators of the form

$$R \{ \pm i_1, \pm i_2, \dots, \pm i_n \}.$$

Clearly, the identity will be established, if we can show that

$$R_{\alpha_n}(x_1, x_2, \dots, x_n) \uparrow \phi(x_{n+1}) = R \{ \alpha_n, -(n+1) \} \quad (32a)$$

$$R_{\alpha_n}(x_1, x_2, \dots, x_n) \downarrow \phi(x_{n+1}) = -R \{ \alpha_n, +(n+1) \} \quad (32b)$$

where C_{α_n} is an arbitrary n -point cone. Now, we have

$$\begin{aligned} & R_{\alpha_n}(x_1, x_2, \dots, x_n) \uparrow \varphi(x_{n+1}) \\ &= \sum_{j=1}^n \theta(j, n+1) R_{\alpha_n}(x_1, x_2, \dots, \overline{x_j x_{n+1}}, \dots, x_n) \\ &= \sum_{\pi} \sum_{j=1}^n \theta(i_j, n+1) \sigma_{\pi} \theta_{\sigma_1}^{\pi}(i_1, i_2) \theta_{\sigma_2}^{\pi}(i_2, i_3) \dots \theta_{\sigma_{n-1}}^{\pi}(i_{n-1}, i_n) \varphi(x_{i_1}) \\ & \quad \varphi(x_{i_2}) \dots \varphi(x_{i_{j-1}}) [\varphi(x_{i_j}), \varphi(x_{n+1})] \varphi(x_{i_{j+1}}) \dots \varphi(x_{i_n}) \end{aligned}$$

where σ_j^{π} is (+) or (-) according as $q_{j_{\pi}}$ is in V_+ or in V_- for $q \in C_{\alpha_n}$, J_{π} being the subset

$$(i_1, i_2, \dots, i_j)$$

and

$$\sigma_{\pi} = \prod_{j=1}^n \sigma_j^{\pi}.$$

Expanding the commutation, we see that it is equal to

$$\begin{aligned} & \sum_{\pi} \sigma_{\pi} \theta(i_1, n+1) \theta_{\sigma_1}^{\pi}(i_1, i_2) \theta_{\sigma_2}^{\pi}(i_2, i_3) \dots \theta_{\sigma_{n-1}}^{\pi}(i_{n-1}, i_n) W(n+1, \\ & i_1, i_2, \dots, i_n) + \sum_{\pi} \sum_{j=1}^{n-1} \sigma_{\pi} \theta_{\sigma_1}^{\pi}(i_1, i_2) \theta_{\sigma_2}^{\pi}(i_2, i_3) \dots \theta_{\sigma_j}^{\pi}(i_j, i_{j+1}) \\ & [\theta(i_j, n+1) - \theta(i_{j+1}, n+1)] \theta_{\sigma_{j+1}}^{\pi}(i_{j+1}, i_{j+2}) \dots \theta_{\sigma_{n-1}}^{\pi}(i_{n-1}, i_n) W(i_1, i_2, \dots, \\ & i_j, n+1, i_{j+1}, \dots, i_n) + \sum_{\pi} \sigma_{\pi} \theta_{\sigma_1}^{\pi}(i_1, i_2) \theta_{\sigma_2}^{\pi}(i_2, i_3) \dots \theta_{\sigma_{n-1}}^{\pi}(i_{n-1}, i_n) \\ & \theta(i_n, n+1) W(i_1, i_2, \dots, i_n, n+1). \end{aligned}$$

Using now the identities

$$\vartheta_{\sigma_j^\pi}(i_j, i_{j+1}) [\vartheta(i_j, n+1) - \vartheta(i_{j+1}, n+1)] = \sigma_j^\pi \vartheta_{\sigma_j^\pi}(i_j, n+1) \vartheta_{\sigma_j^\pi}(n+1, i_{j+1})$$

which can be verified by explicitly considering the two cases $\sigma_j^\pi = (+)$ and $\sigma_j^\pi = (-)$; we get

$$\begin{aligned} & R_{\alpha_n}(x_1, x_2, \dots, x_n) \uparrow \varphi(x_{n+1}) \\ &= (-\sigma_\pi) \vartheta_{\sigma_\pi}(n+1, i_1) \vartheta_{\sigma_1^\pi}(i_1, i_2) \vartheta_{\sigma_2^\pi}(i_2, i_3) \dots \vartheta_{\sigma_{n-1}^\pi}(i_{n-1}, i_n) W(n+1, i_1, \\ & i_2, \dots, i_n) + \sum_{\pi_j=1}^{n-1} (\sigma_j^\pi \sigma_\pi) \vartheta_{\sigma_j^\pi}(i_1, i_2) \vartheta_{\sigma_2^\pi}(i_2, i_3) \dots \vartheta_{\sigma_{j-1}^\pi}(i_{j-1}, i_j) \vartheta_{\sigma_j^\pi} \\ & (i_j, n+1) \vartheta_{\sigma_j^\pi}(n+1, i_{j+1}) \vartheta_{\sigma_{j+1}^\pi}(i_{j+1}, i_{j+2}) \dots \vartheta_{\sigma_{n-1}^\pi}(i_{n-1}, i_n) W(i_1, i_2, \dots, i_j, \\ & n+1, i_{j+1}, \dots, i_n) + \sum_{\pi} \sigma_\pi \vartheta_\pi(i_1, i_2) \vartheta_{\sigma_2^\pi}(i_2, i_3) \dots \vartheta_{\sigma_{n-1}^\pi}(i_{n-1}, i_n) \vartheta_+ \\ & (i_n, n+1) W(i_1, i_2, \dots, i_n, n+1). \end{aligned}$$

It is easy to see that the last expression is just the expansion of the operator

$$R \{ \alpha_n, - (n+1) \}$$

so that we obtain equation (32a). The second part, equation (32b), can

be proved in an exactly similar manner.

Let us now make the following two remarks.

i) The set of Steinmann's G-operators ⁽¹⁵⁾, which are defined by

$$G(x_1, x_2, \dots, x_n) = \varphi(x_1, \uparrow x_{i_2} \downarrow \dots \downarrow x_{i_\mu}) \uparrow \varphi(x_{i_{\mu+1}} \uparrow x_{i_{\mu+1}} \downarrow \dots \downarrow x_{i_n})$$

include not only

$$\varphi(x_1, \uparrow x_{i_2} \downarrow \dots \downarrow x_{i_n}) \equiv R\{\pm i_1, \pm i_2, \dots, \pm i_n\}$$

but also other generalised retarded operators corresponding to cones having a basis of just (n-1) elements. Thus, for example, in the 5-point case, $r'_{1,23}$, where $C_{1,23}^{1-}$ is the cone ⁽¹⁷⁾

$$C_{1,2,3}^{1-} = (q_2+q_3), (q_1+q_3), (q_1+q_4), (q_1+q_3) \in V_+$$

is not of the form

$$R\{\pm i_1, \pm i_2, \pm i_3, \pm i_4, \pm i_5\},$$

but is one of the G-operators

$$r'_{12,3} = -\varphi(x_1 \uparrow x_3) \uparrow \varphi(x_5 \downarrow x_2 \uparrow x_4).$$

However, it is not clear whether the set of the G-operators contains all the generalised retarded operators which correspond to cones having a

basis of just $(n-1)$ elements

(ii) In reference (15) Steinmann states that all in G -operators correspond to cones having a basis of $(n-1)$ elements, but he does not give any explicit expression for these elements. Similarly, he does not explicitly give the suppose of these operators, although he proves that it is characterized by conditions of the form

$$x_i - x_j \in V_+, \text{ or } / \text{ and } x_i, -x_j \in V_+, \dots$$

We have given these two expressions for the special case of the operators

$$\varphi(x_{i_1} \updownarrow x_{i_2} \updownarrow \dots \updownarrow x_{i_n})$$

by (21) and (28).

Next we consider Burgoyne's ⁽²⁰⁾"special subclass" of generalised retarded operators $r_{\pi_e^+}$. These correspond to cones $C_{\pi_e^+}$ which are defined as follows :

- (i) Choose any permutation $\pi: k \rightarrow i_k$ of $(2, 3, 4, \dots, n)$.
- (ii) Choose any $e \in E$, where E is the set of all the possible ways of partitioning a sequence $(i_2, i_3, \dots, i_n)$.
- (iii) For a fixed e , let i_j belong to m^{th} partition, counting from the left. Define

$$|i_j| = (-1)^m$$

(iv) $q \in C_{\pi_e^+}$ if

$q_J \in V_+$ for all subsets J of $(2, 3, \dots, n)$ with $i_\alpha = +1$

$q_J \in V_-$ for all subsets J of $(2, 3, \dots, n)$ with $i_\alpha = -1$.

Here

$$\alpha = \min \{k, i_k \in J\}.$$

$C_{\pi_e^-}$ is defined similarly, except that the signs of V are reversed i.e. if $r_{\pi_e^+}$ is a retarded operator, $r_{\pi_e^-}$ is corresponding advanced operator.

Although, in this definition, the index "1" has been distinguished, this is purely for convenience i.e. we could have distinguished any other index, say i , and this would have lead to a different, but similar, subclass. Now if e is the partition

$$e: (i_2, i_3, \dots, i_{t_1}), (i_{t_1+1}, i_{t_1+2}, \dots, i_{t_2}), \dots, (i_{t_{l-1}}, i_{t_{l-1}+2}, \dots, i_{t_l}), t_l \equiv n$$

it is easily seen from (iv) that

$$R_{\pi_e^+} = R\{+1, -i_2, -i_3, \dots, -i_{t_1}, +i_{t_1+1}, +i_{t_1+2}, \dots, +i_{t_2}, \dots, (-1)^{i_{t_{l-1}+1}}, (-1)^{i_{t_{l-1}+2}}, \dots, (-1)^{i_{t_l}}\}$$

$$R_{\pi e^-} = R \left\{ -1, +i_2, +i_3, \dots, +i_{t_1}, -i_{t_1} + 1 - i_{t_1+2}, \dots, i_{t_2}, \dots, (-1)^{1-i_{t_{1-1}}} + 1, \right. \\ \left. (-1)^{1-i_{t_{1-1}+2}}, \dots, (-1)^{1-i_{t_1}} \right\} .$$

Burgoyne ⁽²⁰⁾ has shown that the set of operators $R_{\pi e^+}$, which in our notation is the set

$$R \left\{ \pm 1, \pm i_2, \pm i_3, \dots, \pm i_n \right\} ,$$

forms a complete set of linearly independent retarded operators. This means that an arbitrary generalised retarded operator can be expressed as a linear combination of operators of this set. Clearly, the same holds for any other set, obtained by replacing "1" by some other index. In Chapter (iv), we gave a direct proof of linear independence of this set and use a simple modification of this proof to obtain an explicit expression for the expansion coefficient a_β :

$$R_\alpha = \sum_\beta a_\beta R_\beta \quad R_\beta \in \text{the above set.}$$

6. Generating Functionals.

A convenient tool for the study of formal aspects of the theory of retarded operators is the use of generating functionals. These functionals were introduced by Schwinger ⁽²¹⁾ and have been extensively studied by LSZ ⁽²⁾

and Symanzik (22). It must be emphasised however, that, in general they do not exist in any other than a formal sense.

The functionals

$$\begin{aligned}\mathcal{T}(J) &= T \exp \left(i \int dx J(x) \phi(x) \right) \\ &\equiv \sum_{n=0} \frac{i^n}{n!} \int dx_1 dx_2 \dots dx_n J(x_1) J(x_2) \dots J(x_n) T(x_1, x_2, \dots, x_n)\end{aligned}$$

$J(x)$ being a real c-number function, generates the time ordered products in the sense that

$$T(x_1, x_2, \dots, x_n) = (-i)^n \frac{\delta^n}{\delta J(x_1) \delta J(x_2) \dots \delta J(x_n)} \mathcal{T}(J) \Big|_{J=0}$$

The Hermitian conjugate

$$\mathcal{T}^+(J) = \bar{T} \exp \left(-i \int dx J(x) \phi(x) \right)$$

similarly generates the anti-time ordered products. It can be verified that

$$\mathcal{T}^+(J) \mathcal{T}(J) = \mathcal{T}(J) \mathcal{T}^+(J) = 1$$

i.e. $\mathcal{T}(J)$ is unitary.

A functional differentiation w.r.t. $J(x)$ gives

$$\mathcal{T}_x \equiv \frac{\delta}{\delta J(x)} \mathcal{T}(J) = \frac{\delta}{\delta J(x)} T \exp \left(i \int dy J(y) \phi(y) \right)$$

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$$= i \text{ T } (\phi(x) \exp (i \int dy J(y) \phi(y)))$$

$$= i \text{ T } \exp (i \int_{x_0}^{\infty} dy J(y) \phi(y)) \phi(x) \text{ T } \exp (i \int_{-\infty}^{x_0} dy J(y) \phi(y))$$

$$\text{i.e. } \mathcal{T}_x = i \mathcal{T}_{x_0}^{\infty} \phi(x) \mathcal{T}_{-\infty}^{x_0} \quad (33a)$$

Here we have used the notation

$$\mathcal{T}_{y_0}^{y'_0} = \text{T } \exp (i \int_{y_0}^{y'_0} dx J(x) \phi(x)), \quad \mathcal{T}_{-\infty}^{\infty}(J) = \mathcal{T}(J)$$

Clearly,

$$\mathcal{T}_{y_0}^{y'_0} + \mathcal{T}_{y_0}^{y'_0} = 1 = \mathcal{T}_{y_0}^{y'_0} \mathcal{T}_{y_0}^{y'_0} + \text{and } \mathcal{T}_{y_0}^{y'_0} \mathcal{T}_{y_0''}^{y_0} = \mathcal{T}_{y_0''}^{y'_0}$$

If we replace $\mathcal{T}(J)$ by $\mathcal{T}_{y_0}^{y'_0}$ in the LHS of (33a) we will get

$$\frac{\delta}{\delta J(x)} \mathcal{T}_{y_0}^{y'_0} = i \delta (x_0 - y_0) \delta (y'_0 - x_0) \mathcal{T}_{x_0}^{y'_0} \phi(x) \mathcal{T}_{y_0}^{x_0} \quad (33b)$$

The $\mathcal{R}$ and $\mathcal{A}$ functionals ⁽²²⁾ are defined by

$$\begin{aligned} \mathcal{R}_x &= -i \mathcal{T}^+ \mathcal{T}_x \equiv i \mathcal{T}_x^+ \mathcal{T} \\ \mathcal{A}_x &= -i \mathcal{T}_x \mathcal{T}^+ \equiv i \mathcal{T} \mathcal{T}_x^+ \end{aligned}$$

and are related by

$$R_x = T_x^+ A T \quad A_x = T R_x T^+$$

Using (33), we get

$$\begin{aligned} R_{x,y} &= \frac{\delta}{\delta J(y)} R_x = \frac{\delta}{\delta J(y)} [-i T^+ \cdot i T_{x_0}^\infty \phi(x) T_{-\infty}^{x_0}] \\ &= \frac{\delta}{\delta J(y)} [T_{-\infty}^{x_0} \phi(x) T_{-\infty}^{x_0}] \\ &= i \vartheta(x-y) [-T_{-\infty}^{y_0+} \phi(y) T_{y_0}^{x_0+} \phi(x) T_{-\infty}^{x_0} + T_{-\infty}^{x_0+} \phi(x) T_{y_0}^{x_0} \\ &\quad \phi(y) T_{-\infty}^{y_0}] \end{aligned}$$

Here we have used the result

$$\frac{\delta}{J(y)} T_{-\infty}^{x_0+} = \vartheta(x-y) \cdot -i T_{-\infty}^{y_0+} \phi(y) T_{y_0}^{x_0+}$$

which follows from

$$T_{-\infty}^{x_0+} T_{-\infty}^{x_0} = 1.$$

Hence we get

$$\begin{aligned} R_{x,y} &= i \vartheta(x-y) [-T_{-\infty}^{y_0+} \phi(y) T_{-\infty}^{y_0} T_{-\infty}^{x_0+} \phi(x) T_{-\infty}^{x_0} + T_{-\infty}^{x_0+} \phi(x) \\ &\quad T_{-\infty}^{x_0} T_{-\infty}^{y_0+} \phi(y) T_{-\infty}^{y_0}] \end{aligned}$$

$$= i \vartheta(x-y) [-R_y R_x + R_x R_y]$$

$$\text{i.e. } {}^{(22)}R_{x,y} = i\vartheta(x-y)[R_x, R_y] \quad (34a)$$

Similarly, we can show that ${}^{(22)}$

$$A_{x,y} = i\vartheta(y-x)[A_x, A_y] \quad (34b)$$

Now, let H be an expression of the form

$$H(x_1, x_2, \dots, x_n) = \sum_{\pi} f_{\pi}(x_1, x_2, \dots, x_n) R_{i_1} R_{i_2} \dots R_{i_n}$$

where $R_m \equiv R_{x_m}$, π is the permutation $\pi: k \rightarrow i_k$, and f_{π} are e-number functions. Then

$$\begin{aligned} \frac{\delta}{\delta J(x_{n+1})} H(x_1, x_2, \dots, x_n) &= \sum_{\pi} f_{\pi}(x_1, x_2, \dots, x_n) \frac{\delta}{\delta J(x_{n+1})} (R_{i_1} R_{i_2} \dots R_{i_n}) \\ &= i \sum_{\pi} f_{\pi}(x_1, x_2, \dots, x_n) \sum_{j=1}^n \vartheta(i_j, n+1) R_{i_1} R_{i_2} \dots R_{i_{j-1}} [R_{i_j}, R_{n+1}] R_{i_{j+1}} \dots R_{i_n} \\ &= i \sum_{\pi} \sum_{j=1}^n \vartheta(i_j, n+1) f_{\pi}(x_1, x_2, \dots, x_n) R_{i_1} R_{i_2} \dots R_{i_{j-1}} [R_{i_j}, R_{n+1}] R_{i_{j+1}} \dots R_{i_n} \\ &= i H(x_1, x_2, \dots, x_n) \uparrow R_{n+1} \end{aligned} \quad (35a)$$

in the notation of equation (31). Similarly, if H' is of the form

$$H'(x_1, x_2, \dots, x_n) = \sum_{\pi} f_{\pi}(x_1, x_2, \dots, x_n) \mathcal{A}_{i_1} \mathcal{A}_{i_2} \dots \mathcal{A}_{i_n},$$

we will have

$$\frac{\delta}{\delta J(x_{n+1})} H'(x_1, x_2, \dots, x_n) = i H'(x_1, x_2, \dots, x_n) \mathcal{A}_{n+1} \quad (35b)$$

We are now in a position to give the generating functional for the operators

$$R \{ \pm i_1, \pm i_2, \dots, \pm i_n \}$$

In fact, we first show that

$$\mathcal{B} \{ +i_1, -i_2, \sigma_3 i_3, \sigma_4 i_4, \dots, \sigma_n i_n \} = (-i)^{n-1} \sigma_n \delta_n^{\sigma_n} \delta_{n-1}^{\sigma_{n-1}} \dots \delta_3^{\sigma_3} \delta_2^{\sigma_2} \mathcal{R}_{i_1} \quad (36)$$

$$\mathcal{B} \{ -i_1, +i_2, \sigma_3 i_3, \sigma_4 i_4, \dots, \sigma_n i_n \} = (-i)^{n-1} \sigma_n \delta_n^{\sigma_n} \delta_n^{\sigma_n-1}, \dots \delta_3^{\sigma_3} \delta_2^{\sigma_2} \mathcal{A}_{i_1}$$

Here each of σ_1 is either (+) or (-), $\sigma = \prod_{l=2}^n (-\sigma_l)$,

$$\mathcal{B} \{ \pm i_1, \pm i_2, \dots, \pm i_n \} = \mathcal{R} \{ \pm i_1, \pm i_2, \dots, \pm i_n \} \text{ or } \mathcal{A} \{ \pm i_1, \pm i_2, \dots, \pm i_n \}$$

according as σ_n is (+) or (-), $\mathcal{R}$ and $\mathcal{A}$ being the functionals obtained from

$$R \{ \pm i_1, \pm i_2, \dots, \pm i_n \}$$

by replacing $\varphi(x_1)$ by R_1 and A_1 respectively, and the δ^s are defined by

$$\frac{\sigma_1}{\delta_1} X = \frac{\delta}{\delta J(x_{i_1})} \begin{cases} X & \text{if } \sigma_{1-1} = \sigma_1 \\ \mathcal{T}_X \mathcal{T}^+ & \text{if } \sigma_{1-1} = (-), \sigma_1 = (+) \\ \mathcal{T}^+ X \mathcal{T} & \text{if } \sigma_{1-1} = (+), \sigma_1 = (-) \end{cases}$$

Consider first the equation (36a). Suppose that it is true for $n=m$, so that, if $\sigma_m = (-)$, we have

$$R\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m\} = (-i)^{m-1} \sigma \delta_m^- \delta_{m-1}^{\sigma_{m-1}} \dots \delta_3^{\sigma_3} \delta_2^- R_{i_1}.$$

As the LHS is a functional of the type H of (35a), we get

$$\begin{aligned} & (-i)^m \sigma \delta_{m+1}^- \delta_m^- \delta_{m-1}^{\sigma_{m-1}} \dots \delta_3^{\sigma_3} \delta_2^- R_{i_1} \\ &= -i \frac{\delta}{\delta J(x_{i_{m+1}})} R\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m\} \\ &= -i^2 R\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m\} \uparrow R_{i_{m+1}} \\ &= R\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m, -i_{m+1}\} \end{aligned} \tag{37a}$$

Also

$$\begin{aligned}
 & (-i)^m \cdot -\sigma \delta_{m+1}^+ \delta_m^- \delta_{m-1}^{\sigma_{m-1}} \dots \delta_3^{\sigma_3} \delta_2^- R_{i_1} \\
 &= -i \cdot \frac{-\delta}{\delta J(x_{i_{m+1}})} \mathcal{T} R\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m\} \mathcal{T}^+ \\
 &= i \frac{\delta}{\delta J(x_{i_{m+1}})} \mathcal{A}\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m\}
 \end{aligned}$$

using $\mathcal{A}_i = \mathcal{T} R_i \mathcal{T}^+$,

$$\begin{aligned}
 &= i^2 \mathcal{A}\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m\} \downarrow \mathcal{A}_{i_{m+1}} \\
 &= \mathcal{A}\{i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, -i_m, +i_{m+1}\} \quad (37b)
 \end{aligned}$$

Similarly, if $\sigma_m = (+)$, we can prove that

$$\begin{aligned}
 & (-i)^m \cdot \sigma \delta_{m+1}^- \delta_m^+ \delta_{m-1}^{\sigma_{m-1}} \delta_3^{\sigma_3} \delta_2^- R_{i_1} \\
 &= R\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, +i_m, -i_{m+1}\} \quad (37c)
 \end{aligned}$$

and

$$\begin{aligned}
 & (-1)^m \cdot -\sigma \delta_{m+1}^+ \delta_m^+ \delta_{m-1}^{\sigma_{m-1}} \dots \delta_3^{\sigma_3} \delta_2^- R_{i_1} \\
 &= \mathcal{A}\{+i_1, -i_2, \sigma_3 i_3, \dots, \sigma_{m-1} i_{m-1}, +i_m, +i_{m+1}\} \quad (37d)
 \end{aligned}$$

Equations (37a-d) show that if (36a) is true for $n=m$, it will also be true from $n=m+1$. Hence, as it is true for $n=2$ according to (34a), it follows by induction that it is true for every n . (36b) can be proved by an exactly similar method.

Now using the fact that

$$R_x \Big|_{J(x)=0} = A_x \Big|_{J(x)=0} = \varphi(x)$$

we see that (36) gives

$$R \{ +i_1, -i_2, \sigma_3 i_3, \dots, \sigma_n i_n \} = (-i)^{n-1} \sigma_n^{\sigma_n} \delta_{n-1}^{\sigma_{n-1}} \dots \delta_3^{\sigma_3} \delta_2^{\sigma_2} R_{i_1} \Big|_{J=0} \quad (38)$$

$$R \{ -i_1, +i_2, \sigma_3 i_3, \dots, \sigma_n i_n \} = (-i)^{n-1} \sigma_n^{\sigma_n} \delta_{n-1}^{\sigma_{n-1}} \dots \delta_3^{\sigma_3} \delta_2^{\sigma_2} A_{i_1} \Big|_{J=0}$$

These are the required expressions for

$$R \{ \pm i_1, \pm i_2, \pm i_3, \dots, \pm i_n \}$$

in terms of generating functionals. In the special case $\sigma_1 = (-)$, $3 \leq 1 \leq n$, we get

$$R \{ +i_1, -i_2, -i_3, \dots, -i_{n-1}, -i_n \} = (-i)^{n-1} \frac{\delta^{n-1}}{\delta J(x_{i_n}) \delta J(x_{i_{n-1}}) \dots \delta J(x_{i_2})} R_{i_1} \Big|_{J=0}$$

for the ordinary retarded products i.e. for these products (38) reduces to

the well known LSZ generating functional.

It should be noted that the generalised retarded operators defined in this chapter are quite different from those of Symanzik defined in reference (23). These latter are obtained from the generating functional

$$R_{(\bar{J}, J)} \equiv T^+ \left(J + \frac{i}{2} \bar{J} \right) T \left(J - \frac{i}{2} \bar{J} \right)$$

with two c-number source functions $J(x)$ and $\bar{J}(x)$, according to the formula

$$\begin{aligned} & R(x_1, x_2, \dots, x_m; y_1, y_2, \dots, y_n) \\ &= \frac{\delta^{m+n}}{\delta \bar{J}(x_1) \delta \bar{J}(x_2) \dots \delta \bar{J}(x_m) \delta J(y_1) \delta J(y_2) \dots \delta J(y_n)} R_{(\bar{J}, J)} \Big|_{J=\bar{J}=0} \end{aligned}$$

Similarly, they are also different from the operators introduced by Polkinghorne in ref (12) and obtainable from the generating function (24)

$$T_{(J)}^+ T_{(J+\bar{J})} T^+(\bar{J}),$$

although, they are identical with the ones defined by him in (6). But unfortunately, he discards them by saying that they are too complicated.

CHAPTER III

REDUCTION OF S-MATRIX ELEMENTS

1. Introduction and the Main Theorem.

In the last chapter we proved certain reduction formulae (equations 2.19a-d) for the generalised retarded operator, i.e. we obtained expressions for their commutators with in- and out-fields similar to those for time ordered and ordinary retarded product given by LSZ. We will now use them to reduce the S-matrix elements

$$\langle p_1, p_2, \dots, p_m \text{ out} / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle \quad (1)$$

in terms of retarded functions. Now it has been shown by Burgoyne, in his Thesis ⁽⁵⁾, that if

$$\tilde{T}(p_1, p_2, \dots, p_n) \text{ and } \tilde{r}_\alpha(p_1, p_2, \dots, p_n)$$

are the Fourier transform of the time ordered function and the generalised retarded function corresponding to the cone C_α , respectively, then the two are equal for $p \in C_\alpha$. Also we have seen in the first chapter that by using (1.2) we can reduce (1) to

$$(-i)^n \int dx_1, dx_2, \dots, dx_n e^{i \sum_{i=1}^n p_i x_i} K_{x_1}^K x_2 \dots K_{x_n}^K \langle 0 | T(x_1, x_2, \dots, x_n) | 0 \rangle$$

Hence, if we are able to carry out the reduction of (1) in terms generalised retarded functions, we expect that if $p \notin C_\alpha$, we must have

$$\langle p_1, p_2, \dots, p_m \text{ out} / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle \quad (2)$$

$$= (-i)^n \int dx_1 dx_2 \dots dx_n e^{i \sum_{i=1}^n p_i x_i} K_{x_1 x_2} \dots K_{x_{n-1} x_n} r_\alpha(x_1, x_2, \dots, x_n)$$

We shall see in the following that this is indeed the case.

First of all, we give the following lemma

LEMMA:-

$$\varphi_{\text{out}}^{p_1} \varphi_{\text{out}}^{p_2} \dots \varphi_{\text{out}}^{p_n} / K_1, K_2, \dots, K_m \text{ in} \rangle = 0$$

and

$$\varphi_{\text{in}}^{p_1} \varphi_{\text{in}}^{p_2} \dots \varphi_{\text{in}}^{p_n} / K_1, K_2, \dots, K_m \text{ out} \rangle = 0$$

if $p_1 + p_2 + \dots + p_n - K_1 - K_2 - \dots - K_m \notin V_-$ and $\neq 0$

These are generalisations of

$$\varphi_{\text{in}}^p / K \text{ out} \rangle = 0 \quad \text{if } p - K \notin V_- \text{ and } \neq 0$$

which one should expect to be true as these are no states which have their energy-momentum 4-vector outside the future light cone.

The proof runs as follows :

If P is the energy momentum operator, we have

$$\begin{aligned}
 & e^{iPx} \phi_{out}^{p_1} \phi_{out}^{p_2} \dots \phi_{out}^{p_n} /K_1, K_2, \dots, K_m \text{ in} \rangle \\
 &= (-i)^n \lim_{y_i^0 \rightarrow \infty} \int dy_1 dy_2 \dots dy_n e^{iPx} \phi(y_1) \phi(y_2) \dots \phi(y_n) \\
 & \quad \overleftrightarrow{\partial}_{y_1^0} \overleftrightarrow{\partial}_{y_2^0} \dots \overleftrightarrow{\partial}_{y_n^0} \exp(i \sum_i^n p_i y_i) /K_1, K_2, \dots, K_m \text{ in} \rangle \\
 &= (-i)^n \lim_{y_i^0 \rightarrow \infty} \int dy_1 dy_2 \dots dy_n \phi(y_1+x) \phi(y_2+x) \dots \phi(y_n+x) \overleftrightarrow{\partial}_{y_1^0} \overleftrightarrow{\partial}_{y_2^0} \dots \\
 & \quad \overleftrightarrow{\partial}_{y_n^0} \exp(i \sum_i^n p_i y_i) \exp(i \sum_i^m K_i x) /K_1, K_2, \dots, K_m \text{ in} \rangle.
 \end{aligned}$$

Changing the variables of integration from y_i to $y_i - x$ we see that this becomes

$$\begin{aligned}
 & (-i)^n \lim_{y_i^0 \rightarrow \infty} \int dy_1 dy_2 \dots dy_n \phi(y_1) \phi(y_2) \dots \phi(y_n) \overleftrightarrow{\partial}_{y_1^0} \overleftrightarrow{\partial}_{y_2^0} \dots \overleftrightarrow{\partial}_{y_n^0} \\
 & \quad \exp(i \sum_i^n p_i y_i) \exp(-i \sum_i^n p_i x) \exp(i \sum_i^m K_i x) /K_1, K_2, \dots, K_m \text{ in} \rangle \\
 &= e^{-ix (\sum_i^n p_i - \sum_i^m K_i)} \phi_{out}^{p_1} \phi_{out}^{p_2} \dots \phi_{out}^{p_n} /K_1, K_2, \dots, K_m \text{ in} \rangle
 \end{aligned}$$

Changing the variables integration from y_i to $y_i - x$ we see that this becomes

$$\begin{aligned}
 & (-i)^n \text{Li}_{y_1^0 \rightarrow \infty} \int dy_1 dy_2 \dots dy_n \phi(y_1) \phi(y_2) \dots \phi(y_n) \overleftrightarrow{\partial}_{y_1^0} \overleftrightarrow{\partial}_{y_2^0} \dots \overleftrightarrow{\partial}_{y_n^0} \\
 & \exp(i \sum_1^n p_i y_i) \exp(-i \sum_1^n p_i x) \exp(i \sum_1^m K_i x) /K_1, K_2, \dots, K_m \text{ in} \rangle \\
 & = e^{-ix (\sum_1^n p_i - \sum_1^m K_i)} \phi_{\text{out}}^{p_1} \phi_{\text{out}}^{p_2} \dots \phi_{\text{out}}^{p_n} /K_1, K_2, \dots, K_m \text{ in} \rangle.
 \end{aligned}$$

This shows that

$$\phi_{\text{out}}^{p_1} \phi_{\text{out}}^{p_2} \dots \phi_{\text{out}}^{p_n} /K_1, K_2, \dots, K_m \text{ in} \rangle$$

is an eigenstate of P corresponding to the eigenvalue

$$- \left(\sum_1^n p_i - \sum_1^m K_i \right).$$

As the energy-momentum 4-vector of every eigenstate of P is either zero or lies in the forward light cone we have

$$\phi_{\text{out}}^{p_1} \phi_{\text{out}}^{p_2} \dots \phi_{\text{out}}^{p_n} /K_1, K_2, \dots, K_m \text{ in} \rangle = 0$$

if $\left(\sum_1^n p_i - \sum_1^m K_i \right) \not\geq V_-$ and $\neq 0$.

The second part of the lemma can be proved in an exactly similar manner. Clearly, the Hermitian conjugates of these two results are

$$\begin{aligned} \langle K_1, K_2, \dots, K_m \text{ in } / \varphi_{\text{out}}^{p_1^*} \varphi_{\text{out}}^{p_2^*} \dots \varphi_{\text{out}}^{p_n^*} \rangle &= 0 \\ \langle K_1, K_2, \dots, K_m \text{ out } / \varphi_{\text{in}}^{p_1^*} \varphi_{\text{in}}^{p_2^*} \dots \varphi_{\text{in}}^{p_n^*} \rangle &= 0 \end{aligned}$$

if $(\sum_1^n p_i - \sum_1^m K_i) \notin V_-$ and $\neq 0$.

If C_α is a cone of the type

$$C_\alpha : C \left\{ \pm i_1, \pm i_2, \dots, \pm i_n \right\} \quad \text{in order } i_1, i_2, \dots, i_n$$

and we perform the reduction of (1) with $p \in C_\alpha$, then it is not difficult to see that all the reductions can be carried out with the help of this lemma, and that the final result is just r_α i.e. (2) is satisfied.

For example, let us consider the 5-point matrix element

$$S_5 = \langle p_1, p_2 \text{ out } / -p_3, -p_4, -p_5 \text{ in } \rangle$$

when

$$p \in C_{3,12}^- = C \{ +1, -3, +2, -4, -5 \}$$

so that

$$\begin{aligned} p_1, p_2, p_1+p_4, p_2+p_4, p_1+p_5, p_2+p_5 &\in V_+ \\ p_3, p_4, p_5, p_1+p_3, p_2+p_3 &\in V_- \end{aligned}$$

We have

$$\begin{aligned}
 S_5 &= \langle p_2 \text{ out} / \varphi_{\text{out}}^{p_1} / -p_3, -p_4, -p_5 \text{ in} \rangle \\
 &= -i M_1 \langle p_2 \text{ out} / \varphi(x_1) \varphi_{\text{in}}^{-p_3^*} / -p_4, -p_5 \text{ in} \rangle \\
 &= (-i)^2 M_1 M_3 \langle p_2 \text{ out} / R\{+1, -3\} / -p_4, -p_5 \text{ in} \rangle -i M_1 \langle p_2 \text{ out} / \varphi_{\text{in}}^{-p_2^*} \\
 &\quad \varphi(x_1) / -p_4, -p_5 \text{ in} \rangle
 \end{aligned}$$

$$\text{where } M_i = M_i^{p_i^*} = M_i^{-p_i} = \int dx_i e^{ip_i x_i} Kx_i,$$

and we have used (2.19a). The second term vanishes according to the lemma, as

$$-p_3 -p_2 \in V_+$$

and so we get

$$\begin{aligned}
 S_5 &= (-i)^2 M_1 M_3 \langle \circ / \varphi_{\text{out}}^{p_2} R\{+1, -3\} / -p_4, -p_5 \text{ in} \rangle \\
 &= (-i)^3 M_1 M_2 M_3 \langle \circ / R\{+1, -3, +2\} / -p_4, -p_5 \text{ in} \rangle + (-i)^2 M_1 M_3 \\
 &\quad \langle \circ / R\{+1, -3\} \varphi_{\text{out}}^{p_2} / -p_4, -p_5 \text{ in} \rangle \quad \text{using (2.19d)} \\
 &= (-i)^3 M_1 M_2 M_3 \langle \circ / R\{+1, -3, +2\} \varphi_{\text{in}}^{-p_4^*} / -p_5 \text{ in} \rangle
 \end{aligned}$$

according to the lemma, as $p_2 + p_4 + p_5 \in V_+$

$$= (-i)^4 M_1 M_2 M_3 M_4 \langle 0 / R \{+1, -3, +2, -4\} \phi_{in}^{-p_5^*} / 0 \rangle$$

$$= (-i)^5 M_1 M_2 M_3 M_4 M_5 \langle 0 / R \{+1, -3, +2, -4, -5\} / 0 \rangle$$

as required. Clearly no complications arise when $n > 5$.

If, however, C_α is of the form

$$C_\alpha : C \{ \alpha_j, \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_n \}$$

where C_{α_j} cannot be obtained from any of the $(j-1)$ - point cones, it is clear that a direct application of the above method cannot lead to the reduction of (1) with $p \in C_\alpha$, to r_α . To get some idea as to what really happens in such cases, let us consider

$$S_5 = \langle p_1, p_2 \text{ out } / -p_3, -p_4, -p_5 \text{ in} \rangle$$

when p is in one of those 5-point cones which cannot be obtained from any of the 4-point cones, say $p \in C'_{12,3}$, so that

$$p_1, p_2, p_1+p_3, p_1+p_4, p_1+p_5, p_2+p_4, p_2+p_5 \in V_+$$

$$p_3, p_4, p_5, p_2+p_3 \in V_-$$

We have

$$S_5 = -iM_1 \langle p_2 \text{ out } / \phi(x_1) / -p_3, -p_4, -p_5 \text{ in} \rangle$$

$$\begin{aligned}
&= -iM_1 \langle p_2 \text{ out } / \varphi(x_1) \varphi_{\text{in}}^{-p_1^*} / -p_4, -p_5 \text{ in} \rangle \\
&= (-i)^2 M_1 M_3 \langle p_2 \text{ out } / R\{+1, -3\} / -p_4, -p_5 \text{ in} \rangle \quad \text{as } -p_3 - p_2 \in V_+ \\
&= (-i)^3 M_1 M_3 M_4 \langle p_2 \text{ out } / R\{+1, -3, -4\} / -p_5 \text{ in} \rangle + (-i)^2 M_1 M_3 \langle p_2 \text{ out } / \\
&\quad \varphi_{\text{in}}^{-p_4^*} R\{+1, -3\} / -p_5 \text{ in} \rangle \\
&= (-i)^4 M_1 M_2 M_3 M_4 \langle o / R\{+1, -3, -4, +2\} / -p_5 \text{ in} \rangle + (-i)^3 M_1 M_3 M_5 \langle p_2 \text{ out } / \\
&\quad \varphi_{\text{in}}^{-p_4^*} R\{+1, -3, -5\} / o \rangle + (-i)^2 M_1 M_3 \langle p_2 \text{ out } / \varphi_{\text{in}}^{-p_4^*} \varphi_{\text{in}}^{-p_5^*} R\{+1, -3\} / o \rangle
\end{aligned}$$

The last term vanishes according to the lemma as

$$-p_4 - p_5 - p_2 \in V_+$$

and so we get

$$\begin{aligned}
S_5 &= (-i)^5 M_1 M_2 M_3 M_4 M_5 \langle o / R\{+1, -3, -4, +2, -5\} / o \rangle - (-i)^4 M_1 M_3 M_4 M_5 \\
&\quad \langle p_2 \text{ out } / R\{+1, -3, -4, 5\} / o \rangle + (-i)^3 M_1 M_3 M_5 \langle p_2 \text{ out } / R\{+1, -3, -5\} / -p_4 \text{ in} \rangle \\
&= (-i)^5 M_1 M_2 M_3 M_4 M_5 [\langle o / R\{+1, -3, -4, +2, -5\} / o \rangle - \langle o / R\{+1, -3, -4, -5,
\end{aligned}$$

$$+ 2 \} / o \rangle + \langle o / R \{ +1, -3, -5, +2, -4 \} / o \rangle]$$

$$= (-i)^5 M_1 M_2 M_3 M_4 M_5 \langle o / R_{12,5} - R_{12} + R_{12,4} / o \rangle$$

in Burgoyne's notation (17). Now we have Steinmann relation

$$R_{ij} - R_{ij,k} + R'_{ij,l} - R_{ij,m} = 0$$

which gives

$$R'_{12,3} = R_{12,5} + R_{12,4} - R_{12}$$

so that we finally get

$$\langle p_1, p_2 \text{ out } / -p_3, -p_4, -p_5 \text{ in} \rangle = (-i)^5 M_1 M_2 M_3 M_4 M_5 r'_{12,3}$$

$$\text{when } p \in C_{12,3}^{1-}.$$

The two important points to be noted about this reduction are

- i) The matrix element can be completely reduced, and the result of reduction is a linear combination of functions of the type

$$r\{\pm i_1, \pm i_2, \dots, \pm i_5\}.$$

- ii) This linear combination is just equal to the correct retarded function $r'_{12,3}$, as required by (2).

It is found that these two statements are true in general i.e. for the matrix element (1) when p is in any cone C_α . The first one follows from the theorem given below, while the second one is proved in the next section.

THEOREM: The matrix element

$$\langle p_1, p_2, \dots, p_m \text{ out } / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle \quad (1)$$

when $p \in C_\alpha$, an arbitrary n -point cone, can be completely reduced to a linear combination of retarded function of the form

$$r\{\pm i_1, \pm i_2, \dots, \pm i_n\}.$$

PROOF: Let us assume that the first $(k-j)$ reductions in (1) can be carried out i.e. (1) can be expressed as a linear combination of terms of the form

$$X = \text{II } M \langle i_1, i_2, \dots, i_j / R\{\pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k\} / i_{k+1}, i_{k+2}, \dots, i_n \rangle \quad (3)$$

where the product is over suitable M -operators, and we have used the notation

$$\begin{aligned} & \langle p_{i_1}, p_{i_2}, \dots, p_{i_j} \text{ out } / R\{\pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k\} / -p_{i_{k+1}}, p_{i_{k+2}}, \dots, -p_{i_n} \text{ in} \rangle \\ &= \langle i_1, i_2, \dots, i_j / R\{\pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k\} / i_{k+1}, i_{k+2}, \dots, i_n \rangle \end{aligned} \quad (4)$$

Further, let

$$-(p_{i_{k+1}}^0 + p_{i_{k+2}}^0 + \dots + p_{i_n}^0) > p_{i_1}^0 + p_{i_2}^0 + \dots + p_{i_j}^0 \quad (5)$$

Then there will exist a smallest integer l such that the sum of l of the energies

$$-p_{i_{k+1}}^0, -p_{i_{k+2}}^0, \dots, -p_{i_{k+l}}^0,$$

is greater than the RHS of (5). Without any loss of generality we may take them to be

$$-p_{i_{k+1}}^0, -p_{i_{k+2}}^0, \dots, -p_{i_{k+l}}^0 \quad (6)$$

Consider first then case $l=1$. We have (writing $\varphi_{in}^{-p_j^0} = \varphi_{in}^{j*}$)

$$\begin{aligned} X &= \sum \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k \} \varphi_{in}^{i_{k+1}^*} / i_{k+2}, i_{k+3}, \dots, i_n \rangle \\ &= \sum \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1} \} / i_{k+2}, i_{k+3}, \dots, i_n \rangle + \sum \\ &\quad \langle i_1, i_2, \dots, i_j / \varphi_{in}^{i_{k+1}^*} R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k \} / i_{k+2}, i_{k+3}, \dots, i_n \rangle \end{aligned}$$

The second term vanishes according to the lemma as

$$-p_{i_{k+1}}^0 > p_{i_1}^0 + p_{i_2}^0 + \dots + p_{i_j}^0$$

and so X is equal to a matrix element in which $(k-j+1)$ reductions have been carried out.

For the case $l=2$, we have

$$\begin{aligned}
 X &= \prod M \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k \} \varphi_{in}^{i_{k+1}^*} / i_{k+2}, i_{k+3}, \dots, i_n \rangle \\
 &= \prod M \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1} \} / i_{k+2}, i_{k+3}, \dots, i_n \rangle + \prod M \\
 &\quad \langle i_1, i_2, \dots, i_j / \varphi_{in}^{i_{k+1}^*} R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k \} / i_{k+2}, i_{k+3}, \dots, i_n \rangle \\
 &= \prod M \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1} \} / i_{k+2}, i_{k+3}, \dots, i_n \rangle + \prod M \\
 &\quad \langle i_1, i_2, \dots, i_j / \varphi_{in}^{i_{k+1}^*} R \{ \pm i_{j+1}, \pm i_{k+2}, \dots, \pm i_k, -i_{k+2} \} / i_{k+3}, i_{k+4}, \dots, i_n \rangle + \\
 &\quad \prod M \langle i_1, i_2, \dots, i_j / \varphi_{in}^{i_{k+1}^*} \varphi_{in}^{i_{k+2}^*} R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k \} / i_{k+3}, i_{k+4}, \dots, i_n \rangle
 \end{aligned}$$

The last term again vanishes according to the lemma, as now we have

$$-p_{i_{k+1}}^0 - p_{i_{k+2}}^0 \rangle p_{i_1}^0 + p_{i_2}^0 + \dots + p_{i_j}^0$$

Hence commuting $\varphi_{in}^{i_{k+1}^*}$ back in the second term, we get

$$X = \prod M \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1} \} / i_{k+2}, i_{k+3}, \dots, i_n \rangle$$

$$+ \prod M \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+2} \} / i_{k+1}, i_{k+3}, \dots, i_n \rangle$$

$$- \prod M \langle i_1, i_2, \dots, i_j / R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1}, -i_{k+2} \} / i_{k+3}, i_{k+4}, \dots, i_n \rangle$$

i.e. X is equal to a sum of terms in each of which, either $(k-j+1)$ or more reductions have been carried out. Similarly, for general $k \leq n$ it is not difficult to see that X is equal to the matrix elements between suitable states (clearly, the outstate is $(i_1, i_2, \dots, i_j)$ for every term while the in-state contain $i_{k+1+1}, i_{k+1+2}, \dots, i_n$ and possibly some other indices) of

$$\begin{aligned} & [R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1} \} + R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+2} \} + \dots + R \\ & \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+l} \}] - [R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1}, i_{k+2} \} + R \\ & \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1}, -i_{k+3} \} + \dots + R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1}, -i_{k+l} \}] \\ & + \dots + (-1)^{l-1} [R \{ \pm i_{j+1}, \pm i_{j+2}, \dots, \pm i_k, -i_{k+1}, -i_{k+2}, \dots, -i_{k+l} \}] \quad (7) \end{aligned}$$

The case when the inequality (5) is reversed can be similarly treated by committing the outstates (using 2.19d) instead of the in-states. Thus we can always express X as a linear combination of terms in each of which either $(k-j+1)$ or more reductions have been carried out, and hence

the same will also be true for the matrix element (1). This shows that if (k-j) reductions can be carried out in (1), the (k-j+1)th reduction can also be carried out.

Now we can perform the first reductions, as (1) can always be put in the form

$$-i M_1 \langle p_2, p_3, \dots, p_m \text{ out } / \phi(x_1) / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle$$

by using

$$\begin{matrix} p_1 \\ \phi \\ \text{out} \end{matrix} = \begin{matrix} p_1 \\ \phi \\ \text{in} \end{matrix} - i M_1 \phi(x_1) \equiv \begin{matrix} p_1 \\ \phi \\ \text{in} \end{matrix} - i \int dx_1 e^{ip_1 x_1} K_{x_1} \phi(x_1).$$

Hence, the proof of the theorem follows by induction.

2. Result of Reduction.

The above theorem shows that a complete reduction of (1) can always be carried out, so that the only thing that remains to be proved is that the result of this reduction is just equal to the correct retarded function as required by (2). First of all, let us introduce some new notations. Let $\mathcal{D}$ be the set of all the sets of the type

$$\{ \pm i_1, \pm i_2, \pm i_3, \dots, \pm i_n \} ,$$

the prefix of each index being either (+) or (-). Here,

$$(i_2, i_3, \dots, i_n)$$

is a permutation of

$$(2, 3, \dots, n).$$

Also let D and D' be the subsets of $\mathcal{D}$ consisting of all the elements which have the prefix of "1" as (+) or (-) respectively. Then according to the above theorem, we have

$$S_{\alpha} = M \sum_{\beta \in \mathcal{D}} b_{\beta} r_{\beta}$$

where S_{α} is the matrix element (1) with $p \in C_{\alpha}$, b_{β} 's are constants and

$$M = (-i)^n M_1 M_2 \dots M_n$$

If we start our reduction from the p_1 -state, then as it is an out-state, every term in the final result will have the prefix of "1" as (+) i.e. we will have

$$b_{\beta} = 0 \text{ for } \beta \in D',$$

and so we have only to find b_{β} for $\beta \in D$. We shall obtain its value when C_{β} is the cone, (k even)

$$C_{\beta}: C \{ \overline{+1, -2, -3, \dots, -R_2+1, +R_2+2, +R_2+3, \dots, +R_3, +R_3+1, -R_3+2, -R_3+3}, \dots, -\overline{R_{k-1}+2, -R_{k-1}+3, \dots, -R_{k-1} - R_k+1} \}, R_k \equiv n-1 \quad (8)$$

as for any other $\beta \in D$, we have just to replace

$$(2, 3, 4, \dots, n)$$

by some permutation

$$(\pi(2), \pi(3), \dots, \pi(n)).$$

Consider now the set

$$G_m : (R_{m-1}+2, R_{m-1}+3, \dots, R_m, R_m+1).$$

Let $\mathcal{H}_-^m$ be the set of all the subsets H_- of G_m which are such that

$$\sum_{i=R_m+2}^n p_i + \sum_{i \in H_-} p_i \in V_- \quad \text{for } p \in C_\alpha.$$

$\mathcal{H}_+^m$ is similarly defined. Clearly, $\mathcal{H}_-^m$ and $\mathcal{H}_+^m$ together contain all the subsets of G_m i.e. any subset of G_m is either in $\mathcal{H}_-^m$ or $\mathcal{H}_+^m$. Also let

$$H_e^f, \quad f = 1, 2, \dots, N_e^m$$

be all the subsets of G_m of e elements which are in $\mathcal{H}_-^m$.

Next, let us make the following remark about the process of reduction. If we have a matrix element of the type

$$\langle p_1, p_2, \dots, p_i \text{ out } / R \{ \dots \} / -p_{i+1}, -p_{i+2}, \dots, -p_j \text{ in} \rangle,$$

we can always perform the next reduction, which will be of the out-state or the in-state, and so will lead to the next index in $R \{ \dots \}$ with prefix (+) or (-), according as

$$p_1 + p_2 + p_3 + \dots + p_j \in V_+ \text{ or } V_- \text{ for } p \in C_\alpha \text{ respectively.}$$

Suppose now that at some stage in reduction of S_α , we have a term
(m even)

$$\begin{aligned} & \square M \langle \dots \text{out} / R \{ +1, -2, -3, \dots, -\overline{R_2+1}, \overline{R_2+2}, \overline{R_2+3}, \dots, \overline{R_3+1}, \\ & - \overline{R_3+2}, -\overline{R_3+3}, \dots, -\overline{R_{m-2}+1}, + \overline{R_{m-2}+2}, + \overline{R_{m-2}+3}, \dots, + \overline{R_{m-1}+1} \} / \dots \text{in} \rangle \\ & \equiv \square M \langle \dots \text{out} / R \{ U_m, + \overline{R_{m-1}+1} \} / \dots \text{in} \rangle \end{aligned} \quad (9)$$

It will lead to a term

$$\square M \langle \dots \text{out} / R \{ U_m, + \overline{R_{m-1}+1}, -\overline{R_{m-1}+2}, -\overline{R_{m-1}+3}, \dots, -R_m, -\overline{R_m+1}, +g \} / \dots \text{in} \rangle \quad (10)$$

with some coefficient. Let us try to find this coefficient. It will be different from zero only if

$$\sum_{i=R_{m-1}+2}^n p_i \in V_- \quad \text{and} \quad \sum_{i=R_m+2}^n p_i \in V_+ \quad \text{in } C_\alpha \quad (11)$$

for otherwise the prefix of the index after $R_{m-1}+1$ cannot be $(-)$ and that of the one after R_m+1 cannot be $(+)$. We can therefore take (10) to be true. Because of the first of these, we can write the in-states in (9) as in-fields and commute all of them with R so as to get (similarly to (7)) the matrix elements, between suitable states, of

$$\begin{aligned} & [R \{ U_m + \overline{R_{m-1}+1}, -\overline{R_{m-1}+2} \} + R \{ U_m, + \overline{R_{m-1}+1}, -\overline{R_{m-1}+3} \} + \dots + R \{ U_m, + \overline{R_{m-1}+1}, \\ & - \overline{R_m+1} \}] - [R \{ U_m, + \overline{R_{m-1}+1}, -\overline{R_{m-1}+2}, -\overline{R_{m-1}+3} \} + R \{ U_m, + \overline{R_{m-1}+1}, -\overline{R_{m-1}+2}, \end{aligned}$$

$$\begin{aligned}
 & \overline{-R_{m-1}+4} \} + \dots + R \{ U_m, + \overline{R_{m-1}+1}, -R_m, - \overline{R_m+1} \}] + \dots + (-1)^{r_m-1} [R \{ U_m, \\
 & + \overline{R_{m-1}+1}, -\overline{R_{m-1}+2}, -\overline{R_{m-1}+3}, \dots, -R_m, -\overline{R_m+1} \}] + [\text{some other terms} \\
 & R \{ U_m, + \overline{R_{m-1}+1}, -t_1, -t_2, \dots, -t \} \text{ such that there is at least} \\
 & \text{one } t \text{ which is not in } G_m]. \quad (12)
 \end{aligned}$$

Here $r_m = R_m - R_{m-1}$, $2 \leq m \leq k$, $R_1 \equiv 0$.

The terms in the last square brackets are not interesting as they cannot lead to (10). Due to the second of the conditions (11), the next reduction in the last but one square bracket of (12) will be that of an out-state i.e. the prefix of the next index in which will be (+) so that it leads to (10). Thus we may say that corresponding to the subset $G_m \in \mathcal{H}_-^m$ of r_m elements (that $G_m \in \mathcal{H}_-^m$ follows from the first of (11), there is a term (10) with coefficient $(-1)^{r_m-1}$

As $\mathcal{N}_{r_m-1}^m$ subsets

$$H_{r_m-1}^f, \quad f = 1, 2, 3, \dots, \mathcal{N}_{r_m-1}^m$$

of G_m of (r_m-1) elements are in $\mathcal{H}_-^m$, in $\mathcal{N}_{r_m-1}^m$ terms of the first square brackets in (12) the next reduction can be performed by commuting the in-fields with the retarded commutator. For example, if we consider the subset

$$(l_1, l_2, \dots, l_{r_m-1}) \in \mathcal{H}_-^m$$

where $(l_1, l_2, \dots, l_{r_m-1}) \cong$ a permutation of $(R_{m-1}+2, \dots, R_m+1)$ then the term

$$\square M \langle \dots \text{out} / R \{ U_m, +\overline{R_{m-1}+1}, -l_{r_m} \} / \dots \text{in} \rangle$$

can be reduced so as to give the matrix element of

$$\begin{aligned} & [R \{ U_m, +\overline{R_{m-1}+1}, -l_{r_m}, -l_1 \} + R \{ U_m, +\overline{R_{m-1}+1}, -l_{r_m}, -l_2 \} + \dots + R \{ U_m \\ & +\overline{R_{m-1}+1}, -l_{r_m}, -l_{r_m-1} \}] - [R \{ U_m, +\overline{R_{m-1}+1}, -l_{r_m}, -l_1, -l_2 \} + \dots + R \{ \\ & U_m, +\overline{R_{m-1}+1}, -l_{r_m}, -l_{r_m-2}, -l_{r_m-1} \}] + \dots + (-1)^{r_m-2} [R \{ U_m, +\overline{R_{m-1}+1}, \\ & -l_{r_m}, -l_1, -l_2, \dots, -l_{r_m-1} \}] + [\text{some other (unninteresting) terms}] \quad (13) \end{aligned}$$

Due to the second part of (11), the next reduction in the last but one square brackets will again be that of an out-state and so it will again lead to (10).

If the rest of the terms in the first square brackets of (12), the next reduction will be that of an out-state, so that they can not lead to (10). Thus corresponding to each of the $\mathcal{N}_{r_m-1}^m$ subsets $H_{r_m-1}^f$ of G_m of (r_m-1) elements, which are in $\mathcal{H}_-^m$, there is a term (10) with a coefficient $(-1)^{r_m-2}$ i.e. these subsets lead to a coefficient

$$(-1)^{r_m-2} \mathcal{N}_{r_m-1}^m = \sum_{i=1}^{\mathcal{N}_{r_m-1}^m} (-1)^{r_m-2}$$

of (10).

Next, as the $\mathcal{H}_{r_m-2}^m$ subsets

$$H_{r_m-2}^f, f = 1, 2, 3, \dots, \mathcal{N}_{r_m-2}^m$$

of G_m of $(r_m - 2)$ elements are in $\mathcal{H}_m^m$, there will be just $\mathcal{N}_{r_m - 2}^m$ terms in the second square brackets of (12) in which the reduction of the in-fields can be carried, and each of these will lead to one term (10) with coefficient $(-1)^{r_m - 2}$.

Further, let $d_{r_m - 1}^{f, r_m - 2}$ be the number of the subsets

$$H_{r_m - 1}^{f'}, \quad f' = 1, 2, 3, \dots, \mathcal{N}_{r_m - 1}^m$$

which contain $H_{r_m - 2}^{f, r_m - 2}$. Then in the expression (13) corresponding to each of these $d_{r_m - 1}^{f, r_m - 2}$ subsets $H_{r_m - 1}^{f'}$, there will be a term in the first square brackets, containing the two indices of G_m which are not in $H_{r_m - 2}^{f, r_m - 2}$. Then as $\mathcal{H}_{r_m - 2}^{f, r_m - 2} \in \mathcal{H}_m^m$, the reduction of in-fields can be carried out in this term, and this will lead to (10) with a coefficient $(-1)^{r_m - 3}$. Clearly, if two $H_{r_m - 2}^{f, r_m - 2}$ are included in some $H_{r_m - 1}^{f'}$, the two terms in the first square brackets of (13) corresponding to this $H_{r_m - 1}^{f'}$, in which the in-field reduction can be performed due to the two $H_{r_m - 2}^{f, r_m - 2}$, will be distinct. Thus the total number of terms in the first square brackets of $\mathcal{N}_{r_m - 1}^m$ expressions (13), which lead to (10), will be

$$\sum_{f=1}^{\mathcal{N}_{r_m - 2}^m} d_{r_m - 1}^{f, r_m - 2}.$$

As before, the rest of the terms (in the first square brackets of various expressions (13)) cannot lead to (10). Thus in all, corresponding to the

$\mathcal{N}_{r_m - 2}^m$ subsets $H_{r_m - 1}^f$, there is a coefficient

$$\begin{aligned}
 & (-1)^{r_m-2} \mathcal{N}_{r_m-2}^m + \sum_{f=1}^m (-1)^{r_m-3} d_{r_m-1}^{f, r_m-2} \\
 &= \sum_{f=1}^m \left[(-1)^{r_m-2} + (-1)^{r_m-3} d_{r_m-1}^{f, r_m-2} \right]
 \end{aligned}$$

of (10).

Similarly, we can show that corresponding to the $\mathcal{N}_{r_m-3}^m$ subsets $H_{r_m-3}^f$ of G_m of (r_m-3) elements, which are in $\mathcal{H}_-^m$, there is a coefficient

$$\sum_{f=1}^m \left[(-1)^{r_m-2} + (-1)^{r_m-3} d_{r_m-1}^{f, r_m-3} + (-1)^{r_m-4} d_{r_m-2}^{f, r_m-3} \right]$$

of (10), and so on, so that in general, the $\mathcal{N}_{r_m-u}^m$ subset $H_{r_m-u}^f$ lead to a coefficient

$$\sum_{f=1}^m \left[(-1)^{r_m-2} + (-1)^{r_m-3} d_{r_m-1}^{f, r_m-u} + (-1)^{r_m-4} d_{r_m-2}^{f, r_m-u} + \dots + (-1)^{r_m-u-1} d_{r_m-u+1}^{f, r_m-u} \right]$$

Here $d_{r_m-\nu}^{f, r_m-u}$, $u > \nu$, $u > 0$, is the number of the subsets

$$H_{r_m-\nu}^{f'}, \quad f' = 1, 2, 3, \dots, \mathcal{N}_{r_m-\nu}^m$$

which contain $H_{r_m-u}^f$.

As the terms (10) obtained above are the only ones which we can get from (9), it follows that (9) leads to (10) with a coefficient

$$(-1)^{r_m-1} + \sum_{u=1}^{r_m-1} \sum_{f=1}^{r_m-u} \sum_{v=1}^u (-1)^{r_m-v-1} d_{r_m-v+1}^{f, r_m-u} \quad (14)$$

where $d_{r_m}^{f, r_m-t} \equiv 1 \quad 1 \leq t \leq r_{m-1}$.

Let us now evaluate the numbers $d_{r_m-v}^{r, r_m-u}$, $u > v$. As the u p_i^s , $i \in G_m$ but $i \notin G_m$, are in V_- in C_α , we can choose any $(v-u)$ of these and the union of these $(v-u)$ indices and $H_{r_m-u}^f$ will be in $\mathcal{H}_-^m$ i.e. it will be one of $H_{r_m-v}^{f'}$. As there are ${}^u C_{u-v} \equiv {}^u C_v$ ways of choosing $v-u$ out of u indices, there will be ${}^u C_v$ subsets $H_{r_m-v}^{f'}$ which contain $H_{r_m-u}^f$, and clearly, these will be the only ones which do this. Thus we have

$$\begin{aligned} d_{r_m-v}^{f, r_m-u} &= {}^u C_v \\ \therefore \sum_{v=1}^u (-1)^{r_m-v-1} d_{r_m-v+1}^{f, r_m-u} &= (-1)^{r_m} \sum_{v=1}^u (-1)^{v-1} {}^u C_{v-1} \\ &= (-1)^{r_m} \sum_{v=0}^{u-1} (-1)^v {}^u C_v = (-1)^{r_m} \left[\sum_{v=0}^u (-1)^v {}^u C_v - (-1)^u {}^u C_u \right] \\ &= (-1)^{r_m} [0 + (-1)^{u+1}] = (-1)^{r_m+u-1}. \end{aligned}$$

Hence (14) becomes

$$\begin{aligned}
 & (-1)^{r_m-1} + \sum_{u=1}^{r_m-1} \mathcal{N}_{r_m-u}^m (-1)^{r_m+u+1} = (-1)^{r_m-1} + \sum_{u=1}^{r_m-1} (-1)^{r_m+u+1} \mathcal{N}_{r_m-u}^m \\
 & = \sum_{u=0}^{r_m-1} (-1)^{r_m-u-1} \mathcal{N}_{r_m-u}^m \quad \text{where } \mathcal{N}_{r_m}^m \equiv 1, \\
 & = \eta_m \quad (\text{say})
 \end{aligned}$$

The case when m is odd gives an identical result, except for the fact that $\mathcal{N}_e^m$ is this time the number of subsets of G_m of e elements which are in $\mathcal{H}_+^m$.

Consider now the matrix element S_a . It can be written as

$$-iM_1 \langle p_2, p_3, \dots, p_m \text{ out} / \varphi(x_1) / -p_{m+1}, -p_{m+2}, \dots, -p_n \text{ in} \rangle$$

and this will lead to a term

$$(-i)^{R_2+1} \eta_2 \prod \langle \dots \text{ out} / R \{ +1, -2, -3, \dots, -\overline{R_2+1} \} / \dots \text{ in} \rangle$$

which itself leads to

$$\begin{aligned}
 & (-i)^{R_3+1} \eta_2 \eta_3 \prod \langle \dots \text{ out} / R \{ +1, -2, -3, \dots, -\overline{R_2+1}, +\overline{R_2+2}, +\overline{R_2+3}, \dots, \\
 & + \overline{R_3+1} \} / \dots \text{ in} \rangle
 \end{aligned}$$

and so on, so that ultimately, we have the term

$$\eta_2 \eta_3 \dots \eta_k (-i)^n \prod_{i=1}^n M_i \langle \dots / R \{ +1, -2, -3, \dots, -\overline{R_2+1}, +\overline{R_2+2}, +\overline{R_2+3}, \dots, \dots \} \rangle$$

$$\overline{+R_3+1}, \overline{-R_4+2}, \overline{-R_4+3}, \dots, \overline{-R_{k-1}+2}, \overline{-R_{k-1}+3}, \dots, \overline{-R_k}, \overline{-R_k+1} \} / 0 \rangle$$

$$= \eta_2 \eta_3 \dots \eta_k \mathcal{M}_{r_\beta} \quad \text{where } C_\beta \text{ is the cone (8).}$$

Thus

$$b_\beta = \eta_2 \eta_3 \dots \eta_k$$

for the cone (8), and clearly, the same will be true when k is odd.

As mentioned earlier, its value for any other $\beta \in D$ can be obtained from this by replacing $(2, 3, 4, \dots, n)$ by some permutation

$$(i_1, i_3, \dots, i_n).$$

Thus we have

$$S_\alpha = \mathcal{M} \sum_{\beta \in D} b_\beta r_\beta$$

where b_β are determined as follows. Let β be the set

$$\beta : \{ +1, -i_2, -i_3, \dots, -i_{R_2+1}, +i_{R_2+2}, +i_{R_2+3}, \dots, +i_{R_3+1}, -i_{R_3+2}, \\ -i_{R_3+3}, \dots, (-1)^{k-1} i_{R_{k-1}+2}, (-1)^{k-1} i_{R_{k-1}+3}, \dots, (-1)^{k-1} i_{R_k+1} \}$$

and

$$G_m^\pi \equiv (i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_{R_m}, i_{R_m+1}) \quad (15)$$

Then

$$b_\beta = \eta_2 \eta_3 \eta_4 \dots \eta_k$$

$$\text{where } \eta_m = \sum_{u=0}^{r_m-1} (-1)^{r_m-u-1} \mathcal{N}_{r_m-u}^m$$

and $\mathcal{N}_e^m$ is the number of subsets of G_m^π of e elements which are in $\mathcal{H}_-^m$ or $\mathcal{H}_+^m$ according as m is even or odd respectively.

Now, it is shown in the next chapter that

$$\sum_{\beta} b_{\beta} r_{\beta}$$

where b_{β} is given by (15) is just equal to r_{α} . Hence we get

$$S_{\alpha} = \mathcal{M}_{r_{\alpha}}$$

which proves (2) for an arbitrary n -point cone C_{α} .

Until now we have considered only the case when all the partial sums p_j 's are timelike. We have seen that in the reduction already carried out, a matrix element of the type

$$\langle \dots \text{out} \mid R \left\{ \dots \right\}_{\substack{p_1 \\ \text{out}}}^{\substack{p_1 \\ \text{out}}} \quad \phi_{\substack{p_2 \\ \text{out}}}^{\substack{p_2 \\ \text{out}}} \quad \dots \quad \phi_{\substack{p_j \\ \text{out}}}^{\substack{p_j \\ \text{out}}} \mid -p_{j+1}, -p_{j+2}, \dots, -p_{j_1} \text{in} \rangle$$

vanishes when

$$\sum_{i=1}^1 p_i \in V_+.$$

But the lemma, proved in the first section, states that it will also vanish when $\sum_1^1 p_i$ is spacelike, so that the same process of reduction can be

carried out in this case also. Thus if a matrix element can be reduced to a retarded function when $p \in C_\alpha$, it can also be reduced (to the same retarded function) when p is such that, for each J , p_J is either in $V_\pm$ according as $q_J \in V_\pm$ for $q \in C_\alpha$, or is spacelike. This shows that (1) can be reduced to (generalised) retarded functions for every point of momentum space.

It should be noted that in our calculations, we have not used the assumption of stability of one particle states:

$$|p \text{ in}\rangle = |p \text{ out}\rangle.$$

A consequence of this assumption is that on the mass shell, we have

$$M_{n+1} \langle 0 / R \{ \alpha_1 + (n+1) \} / 0 \rangle = M_{n+1} \langle 0 / R \{ \alpha_1 - (n+1) \} | 0 \rangle \quad (16)$$

where R_α is an arbitrary n -point generalised retarded operator. This follows from

$$\langle 0 / R_\alpha(x_1, x_2, \dots, x_n) / p_{n+1} \text{ out} \rangle = \langle 0 / R_\alpha(x_1, x_2, \dots, x_n) / p_{\text{in}} \rangle .$$

CHAPTER IV.

LINEAR INDEPENDENCE AND EXPANSION

1. Linear Independence of $R_{\beta}, \beta \in \mathcal{D}$.

As mentioned earlier in the second chapter, Burgoyne has stated in his thesis ⁽⁵⁾ that the set of operators

$$R\left\{ \pm 1, \pm i_2, \pm i_3, \dots, \pm i_n \right\}, \quad (1)$$

$$(i_2, i_3, \dots, i_n) = \text{a permutation of } (2, 3, \dots, n).$$

the prefix of each index being either (+) or (-), forms a complete set of linearly independent operators i.e. an arbitrary n-point retarded operator can be expressed as a linear combination of operators of the type (1). (n-1) other sets obtained by distinguishing one of 2, 3, 4, ..., n-1, n, instead of "1" (as is done in (1)) are again complete and linearly independent.

In the following we give a direct and straight forward proof of linear independence of the set (1), and, in the next two sections, a slight modification of the same is used to obtain an explicit expression for the expansion of an arbitrary generalised retarded operator. This is done by evaluating the coefficient of any given operator of the type (1) in terms of conditions defining this operator and the one which is being expanded.

To prove the linear independence of the set (1) we have to show that

if a relation

$$\sum_{\beta \in D} a_{\beta} R_{\beta} = 0 \quad (2)$$

holds for every x , a_{β} being c-numbers independent of x , then

$$a_{\beta} = 0 \text{ for all } \beta.$$

If we now take x such that

$$x_1^0 > x_i^0 \text{ for } i = 2, 3, 4, \dots, n \quad (3)$$

we get

$$R_{\beta} = 0 \text{ for } \beta \in D',$$

so that (2) becomes

$$\sum_{\beta \in D} a_{\beta} R_{\beta} = 0. \quad (2a)$$

Expanding R_{β} in terms of W -products, we get

$$\sum_{\beta \in D} \sum_{\pi} \pm a_{\beta} \vartheta_{\pm}(i_1, i_2) \vartheta_{\pm}(i_2, i_3) \dots \vartheta_{\pm}(i_{n-1}, i_n) W(i_1, i_2, \dots, i_n) = 0$$

$$\text{or } \sum_{\pi} \sum_{\beta \in D} \pm a_{\beta} \vartheta_{\pm}(i_1, i_2) \vartheta_{\pm}(i_2, i_3) \dots \vartheta_{\pm}(i_{n-1}, i_n) W(i_1, i_2, \dots, i_n) = 0$$

where the suffix of the ϑ -functions and the overall sign of each term clearly depends on β . As this equation is true for all x (satisfying (3)) including the points where each x_i is timelike with respect to every other

x_i , the coefficient of $w(i_1, i_2, \dots, i_n)$ must vanish (since, in this case, we may assume that the W -functions are linearly independent) i.e. we must have

$$\sum_{\beta \in D} \pm a_{\beta} \vartheta_{\pm}(i_1, i_2) \vartheta_{\pm}(i_2, i_3) \dots \vartheta_{\pm}(i_{n-1}, i_n) = 0$$

with the suffix of $\vartheta(i_j, i_{j+1})$ as (+) or (-) according as q_j is in V_+ or in V_- in C_{β} , J being the set $(i_1, i_2, \dots, i_j)$. Clearly, this represents a set of $n!$ equations, one corresponding to each of the $n!$ permutations π of $(1, 2, 3, \dots, n)$. However, in the following, we need only those $(n-1)!$ of these equations which correspond to $i_1=1$, i.e. the equations

$$\sum_{\beta \in D} \pm a_{\beta} \vartheta_{+}(1, i_2) \vartheta_{\pm}(i_2, i_3) \dots \vartheta_{\pm}(i_{n-1}, i_n) = 0 \quad (4)$$

as $q_1 \in V_+$ in C_{β} when $\beta \in D$. Now let the point x be such that

$$x_{i_n}^0 - x_{i_{n-1}}^0 \in V_{\sigma_{i_n}}^0, x_{i_{n-1}}^0 - x_{i_{n-2}}^0 \in V_{\sigma_{i_{n-1}}}^0, \dots, x_{i_3}^0 - x_{i_2}^0 \in V_{\sigma_{i_3}}^0,$$

$$x_{i_2}^0 - x_1^0 \in V_{-}^0,$$

where for each j , σ_{i_j} is either (+) or (-) and $V_{\pm}^0$ are defined by

$$x_k^0 \in V_{+}^0 \text{ if } x_k^0 > 0, \quad x_k^0 \in V_{-}^0 \text{ if } x_k^0 < 0.$$

Then in every term of (4) which corresponds to a β such that in the cone C_β , we have

$$q_{i_n} + q_{i_{n-1}} + \dots + q_{i_{n-j}} \varepsilon V \sigma_{i_{n-j}}$$

$$j = 0, 1, 2, \dots, n-3, n-2, q_{i_2} \equiv (-),$$

the product to θ -functions is +1 and the overall sign is the same (either (+) or (-1), while in the rest of the terms, this product vanishes. Hence we get

$$\sum a \{+1, -j_2, \pm j_3, \dots, \pm j_n\} = 0 \quad (6)$$

where $a \{+1, -j_2, \pm j_3, \dots, \pm j_n\}$ stands for a_β when $\beta = \{+1, -j_2, \pm j_3, \dots, \pm j_n\}$ and where the summation extends over all the sets

$$\{+1, -j_2, \pm j_3, \dots, \pm j_n\}$$

in which the set of conditions (5) is satisfied. Clearly corresponding to each permutation $(i_2, i_3, \dots, i_n)$ of $(2, 3, \dots, n)$ there will be 2^{n-2} sets of conditions of the type (5), and these will lead to 2^{n-2} equations like eq. (6).

Now, corresponding to each set of conditions (5), we can construct

a unique cone, namely the cone

$$C \left\{ +1, -i_2, \sigma_{i_3} i_3, \sigma_{i_4} i_4, \dots, \sigma_{i_n} i_n \right\}$$

We shall call them (i.e. the set of conditions) the principal conditions of this cone. Similarly, (5) for any particular value of j will be called the principal condition of this cone corresponding to the index i_{n-j} .

Consider now the cone (n even)

$$C \left\{ +1, -2, +3, -4, +5, -6, \dots, +n-1, -n \right\} \quad (7)$$

Its principal conditions are

$$q_n + q_{n-1} + \dots + q_{n-j} \in \begin{cases} V_- & \text{for } j \text{ even} \\ V_+ & \text{for } j \text{ odd} \end{cases}$$

$$j = 0, 1, 2, \dots, n-3, n-2.$$

These conditions do not, in general, define a cone C_α uniquely i.e. there are many cones satisfying them. However, it can be shown that if they are to be of the form C_β , $\beta \in D$, then (7) is the only cone in which they are satisfied. For this, we need the following property of C_β . If

$$\beta = \left\{ +1, -i_2, \pm i_3, \pm i_4, \dots, \pm i_n \right\}$$

then for $k > 2$,

$$q_{i_k} + q_{j_1} + q_{j_2} + \dots + q_{j_r} \in V_+ \text{ or } V_- \text{ in } C_\beta$$

according as the prefix of i_k is (+) or (-) in β , provided that all the indices $j_1, j_2, \dots, j_r$ are on the right of i_k in it. In particular, q_{i_k} is in V_+ or in V_- in C_β according as the prefix of i_k is (+) or (-) in β .

Let C_γ be any cone, with $\gamma \in D$, which satisfies the p.c. (principal conditions) of (7). As $q_n \in V_-$, we must have $-n$ in γ . From the p.c. corresponding to n and $n-1$, we get $q_{n-1} \in V_+$ so that the prefix of $n-1$ must be (+) in γ . Also $+(n-1)$ must be on the left of $-n$, for if it were on the right, we would get $q_n + q_{n-1} \in V_-$ which contradicts the p.c. corresponding to $n-1$. Again, the p.c. corresponding to $n-1$ and $n-2$ give $q_{n-2} \in V_-$ so that $n-2$ has a prefix (-) in γ and must lie on the left of $+(n-1)$, as otherwise the p.c. corresponding to $n-2$ will be contradicted. Proceeding in this way, we see that there is just one cone C_γ with $\gamma \in D$ which satisfies the p.c. of (7), namely (7) itself.

Now as p.c. of (7) are just a set of conditions like (5), the corresponding equation (6) will be

$$a \{ +1, -2, +3, -4, \dots, +n-1, -n \} = 0.$$

Similarly, we can show that, for odd n

$$a \{ +1, -2, +3, -4, \dots, -n-1, +n \} = 0.$$

If instead of (7), we start from

$$C \{ +1, -i_2, +i_3, -i_4, \dots, (-1)^{n-2} i_{n-1}, (-1)^{n-1} i_n \} \quad (7a)$$

we will get

$$a \{ +1, -i_2, +i_3, -i_4, \dots, (-1)^{n-2} i_{n-1}, (-1)^{n-1} i_n \} = 0 \quad (8)$$

Next, let us consider the cone (1k even)

$$C \{ +1, -2, -3, \dots, -\overline{R_2+1}, \overline{+R_2+2}, \overline{+R_2+3}, \dots, \overline{+R_3+1}, -\overline{R_3+2}, -\overline{R_3+3}, \dots, -\overline{R_{k-1}+2}, -\overline{R_{k-1}+3}, \dots, -\overline{R_k}, -\overline{R_k+1} \} , R_k \equiv n-1 \quad (9)$$

and try to find all the cones C_γ , $\gamma \in D$ which satisfy its p.c.

As $q_n \equiv q_{R_{K+1}} \in V_-$ we will have $-n \equiv -\overline{R_K+1}$ in . The rest of the indices

$$R_k, R_{k-1}, R_{k-2}, \dots, R_{k-1}+2$$

must be on the right of $-\overline{R_k+1}$, for if some of them were on the left, then either the prefixes of all of those latter are (-), in which case they can be brought on the right, or else, at least one of them has a prefix (+) so that p.c. of (9) corresponding to this index is contradicted. However, once on the right of $-\overline{R_k+1}$, these indices can be in any order and can have any prefix without contradicting any of p.c. of (9). We express this by

writing

$$\gamma = \left\{ \begin{array}{l} , -n, \pi_k(\overline{+R_k}, \overline{+R_{k-1}+1}, \dots, \overline{+R_{k-1}+2}) \end{array} \right\}$$

where π_k indicates that any permutation of the indices appearing in the small brackets is permissible. The p.c. of (9) corresponding to $R_{k-1}+2$ and $R_{k-1}+1$ give $q_{R_{k-1}+1} \in V_+$ so that $R_{k-1}+1$ has a prefix (+) and lies on the left of $-n$ in γ , as otherwise, the p.c. of (9) corresponding to $R_{k-1}+1$ will be contradicted. On the other hand, the indices

$$R_{k-1}, R_{k-1}-1, R_{k-1}-2, \dots, R_{k-2}+3, R_{k-2}+2$$

can be anywhere on the right of $R_{k-1}+1$, can have any prefixes and can be in any order i.e. we have

$$\gamma = \left\{ \begin{array}{l} , \overline{+R_{k-1}+1}, \pi_{k-1}(\overline{+R_{k-1}}, \overline{+R_{k-1}-1}, \dots, \overline{+R_{k-2}+2}) \rightarrow , \\ -n, \pi_k(\overline{+R_k}, \overline{+R_{k-1}}, \dots, \overline{+R_{k-1}+2}) \end{array} \right\}$$

where the arrow in (...) indicates that any index in this bracket can be anywhere even on the right of $-n$ i.e. it is not necessarily between $+R_{k-1}+1$ and $-n$. Proceeding in this way, we see that every cone $C_\gamma, \gamma \in D$ in which p.c. of (9) are satisfied can be written as

$$C_Y: C \left\{ +1, \overline{-R_2+1}, \pi_2(\overline{\pm R_2 \pm R_2 -1}, \dots, \overline{\pm 2}) \rightarrow, \overline{R_3+1}, \pi_3(\overline{\pm R_3, \pm R_3 -1}, \dots, \overline{\pm R_2+2}) \rightarrow, \dots, \overline{+R_{k-1}+1}, \pi_{k-1}(\overline{\pm R_{k-1} \pm R_{k-1} -1}, \dots, \overline{\pm R_{k-2}+2}) \rightarrow, \dots, \pi_k(\overline{\pm R_k, \pm R_{k-1}}, \dots, \overline{\pm R_{k-1}+2}) \right\}.$$

As the p.c. of (9) are a set of conditions like (5), the corresponding equation (6) will be

$$\sum a \left\{ +1, \overline{-R_2+1}, \pi_2(\overline{\pm R_2, \pm R_2 -1}, \dots, \overline{\pm 2}) \rightarrow, \dots, \overline{+R_{k-1}+1}, \pi_{k-1}(\overline{\pm R_{k-1} \pm R_{k-1} -1}, \dots, \overline{\pm R_{k-2}+2}) \rightarrow, \overline{-R_{k-1}+1}, \pi_k(\overline{\pm R_k, \pm R_{k-1}}, \dots, \overline{\pm R_{k-1}+2}) \right\} = 0 \quad (10)$$

where the summation extends only over the distinct terms. It can be rewritten as

$$\sum_{T_2} \sum_1 a \left\{ +1, \overline{-R_2+1}, \pi_2(\overline{\pm R_2, \pm R_2 -1}, \dots, \overline{\pm 2}) \rightarrow, \overline{+R_3+1}, T_2 \right\} = 0$$

where T_2 denotes all the indices which have not been explicitly written and $\sum$ denotes the summation over all the distinct terms obtained while keeping T_2 fixed the prefixes, the permutation π_2 and the position relative to $\overline{+R_3+1}$ of the indices

$$R_2, R_{2-1}, R_{2-2}, \dots, 3, 2.$$

The rest of the summation is contained in Σ_1 . Clearly, this will itself consist of Σ_2' and Σ_2'' , where Σ_2' is the summation over all the distinct terms obtained by varying the permutation and prefixes of these indices and Σ_2'' , over those obtained by varying their positions relative to R_3+1 . Thus we have

$$\sum_{T_2} \Sigma_2' \Sigma_2'' a \left\{ +1, -\overline{R_2+1}, (\pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_2) \xrightarrow{+\overline{R_3+1}, T_2} \right\} = 0$$

where

$$\tilde{\pi}_2 (R_2, \overline{R_2-1}, \dots, 3, 2) = (j_{R_2}, j_{R_2-1}, \dots, j_3, j_2).$$

Writing the summation Σ_2'' explicitly, we get

$$\begin{aligned} & \sum_{T_2} \Sigma_2' a \left\{ +1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, +\overline{R_3+1}, T_2 \right\} \\ & + \sum_{T_2} \Sigma_2' a \left\{ +1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_n, -j_3, +\overline{R_3+1}, (\pm j_2) \xrightarrow{T_2} \right\} \\ & + \sum_{T_2} \Sigma_2' a \left\{ +1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_5, -j_4, +\overline{R_3+1}, (\pm j_2, \pm j_3) \xrightarrow{T_2} \right\} \\ & + \dots \\ & + \sum_{T_2} \Sigma_2' a \left\{ +1, -\overline{R_2+1}, +\overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_{R_2}) \xrightarrow{T_2} \right\} = 0 \end{aligned}$$

$$\text{or } \sum_{T_2} \sum'_{\pi_2} \sum_{l=1}^{R_2} a \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, +j_{1+2}, -j_{1+1}, \right. \\ \left. + \overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2 \right\} = 0 \quad (11)$$

Similarly, corresponding to the cone

$$C \left\{ +1, -i_{r+1}, -i_{r+2}, \dots, -i_{R_2}, \overline{-R_2+1}, +i_2, +i_3, \dots, +i_r, \overline{R_2+2}, \overline{R_2+3}, \dots, \right. \\ \left. + \overline{R_3+1}, \overline{-R_3+2}, \dots, \overline{-R_{k-1}+2}, \dots, -R_k, \overline{-R_k+1} \right\} \quad (12)$$

there will be an equation

$$\sum_{T_2} \sum'_{\pi_2} \sum''_2 a \left\{ +1, \overline{-R_2+1}, (\pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_{r+1}) \rightarrow, + \overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \right. \\ \left. \pm j_r) \rightarrow, T_2 \right\} = 0$$

Here, the permutations π'_2 are such that none of the indices

$$i_2, i_3, \dots, i_r$$

is taken to any of

$$i_{r+1}, i_{r+2}, \dots, i_{R_2}.$$

Writing $\sum_2'$ explicitly, we get

$$\sum_{T_2} \sum_{\pi_2}' \sum_{l=r}^{R_2} a \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_{l+2}, -j_{l+1}, + \overline{R_3+1} \right\},$$

$$(\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2 \} = 0 \quad (13)$$

There will be one such equation corresponding to each of the $C_{r-1}^{R_2-1}$ ways of choosing the $r-1$ indices $(i_2, i_3, \dots, i_r)$ from the R_2-1 indices $(2, 3, \dots, R_2)$. Summing all these equations, we get

$$\sum_{T_2} \sum_{\pi_2}' \sum_{l=r}^{R_2} C_{r-1}^{l-1} a \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_{l+2}, -j_{l+1}, \right.$$

$$\left. + \overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2 \right\} = 0 \quad (14)$$

where this time, π_2 is an arbitrary permutation of $(2, 3, \dots, R_2)$.

This is because, for any such permutation

$$(j_2, j_3, \dots, j_{R_2})$$

of these indices, the term

$$a \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_{l+2}, -j_{l+1}, + \overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2 \right\}$$

appears in all the equations (13) which correspond to those cones (12)

which are such that the set

$$(i_2, i_3, \dots, i_r) \subset (j_2, j_3, \dots, j_1),$$

and there are ${}^{l-1}C_{r-1}$ such cones, one corresponding to each of the ${}^{l-1}C_{r-1}$ ways of choosing $(r-1)$ indices

$$(i_2, i_3, \dots, i_r)$$

out of the $(l-1)$ indices

$$(j_2, j_3, \dots, j_1).$$

Clearly, (11) is the equation (14) with $r=1$. If we multiply (14) by $(-1)^{r-1}$ and sum over r from 1 to R_2 , we get

$$\sum_{r=1}^{R_2} \sum_{T_2} \sum_{\overline{T_2}}' \sum_{l=r}^{R_2} (-1)^{r-1} {}^{l-1}C_{r-1} a_{\{+1, -\overline{R_2+1}, \pm j_{R_2}, \dots, \pm j_{l+2}, -j_{l+1},$$

$$+ \overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2 \} = 0$$

$$\text{or } \sum_{T_2} \sum_{\overline{T_2}}' \sum_{l=1}^{R_2} \sum_{r=1}^1 (-1)^{r-1} {}^{l-1}C_{r-1} a_{\{+1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_{l+2},$$

$$-j_{l+1}, +\overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2 \} = 0$$

by interchanging the summation over l and r ,

$$\begin{aligned}
 & \text{or } \sum_{T_2} \sum'_{\pi_2} a\{+1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, + \overline{R_3+1}, T_2\} \\
 & + \sum_{T_2} \sum'_{\pi_2} \sum_{l=2}^{R_2} \sum_{r=1}^1 (-1)^{r-1} {}^{l-1}C_{r-1} a\{+1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \\
 & \pm j_{l+2}, -j_{l+1}, + \overline{R_3+1}, (\pm j_2, \pm j_3, \dots, \pm j_1) \rightarrow, T_2\} = 0
 \end{aligned}$$

which finally gives

$$\begin{aligned}
 & \sum_{T_2} \sum'_{\pi_2} a\{+1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, + \overline{R_3+1}, T_2\} = 0 \quad (15) \\
 & \text{as } \sum_{r=1}^1 (-1)^{r-1} {}^{l-1}C_{r-1} = 0 \text{ for } 2 \leq l \leq R_2.
 \end{aligned}$$

Now we rewrite it as

$$\begin{aligned}
 & \sum_{T_2} \sum'_{\pi_2} a\{+1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, + \overline{R_3+1}, \pi_3(\pm R_3, \pm R_3-1, \dots, \\
 & \pm \overline{R_2+2}) \rightarrow, - \overline{R_4+1}, T_3\} = 0
 \end{aligned}$$

The summation $\sum_{T_2}$ can again be broken up into three parts $\sum_{T_3}$, $\sum'_{\pi_3}$ and $\sum''_{\pi_3}$ as was done before, so as to get

$$\sum_{T_3} \sum'_{\pi_3} \sum''_{\pi_3} a\{+1, \overline{-R_2+1}, \pm j_{R_2}, \dots, \pm j_2, -j_2, + \overline{R_3+1}, (\pm j_{R_3}, \pm j_{R_3-1}, \dots,$$

$$\dots, \pm j_{R_2+3}, \pm j_{R_2+2}) \rightarrow, -\overline{R_4+1}, T_3 \} = 0$$

$$\text{or } \sum_{\pi_3} \sum_{\pi_3}' \sum_{l=1+R_2}^{R_3} \sum_{\pi_2}' a_{\{+1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, +\overline{R_3+1},$$

$$\pm j_{R_3}, \pm j_{R_3-1}, \dots, \pm j_{1+2}, +j_{1+1}, -\overline{R_4+1}, (\pm j_{R_2+2}, \pm j_{R_2+2}, \dots, \pm j_1) \rightarrow, T_3 \} = 0 \quad (16)$$

where π_3 is the permutation

$$\pi_3 : m \rightarrow j_m, \quad R_2+2 \leq m \leq R_3$$

Thus starting from the cone (9) we arrive at equation (15) so that if we start from the cone

$$\begin{aligned} & 0 \{ +1, -2, -3, \dots, -\overline{1+R_2}, +i_{s+1}, +i_{s+2}, \dots, +i_{R_3}, +\overline{R_3+1}, -i_{R_2+2}, -i_{R_2+3}, \dots, \\ & -i_s, -\overline{R_3+2}, \dots, -\overline{R_4+1}, +\overline{R_4+2}, \dots, -\overline{R_k+1} \} \end{aligned} \quad (17)$$

where

$$i_{R_2+2}, i_{R_2+3}, \dots, i_s$$

are any given $s - (R_2+1)$ of the indices

$$R_2+2, R_2+3, \dots, R_3,$$

and carry out the whole process, we will get

$$\sum_{\pi_3} \sum_{\pi_3}' \sum_{\pi_3}'' \sum_{\pi_2}' a \left\{ +1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, +\overline{R_3+1}, (\pm j_{R_3}, + \right.$$

$$j_{R_3-1}, \dots, \pm j_{s+1}) \rightarrow, -\overline{R_4+1}, (\pm j_{R_2+2}, \pm j_{R_2+3}, \dots, \pm j_s) \rightarrow, \pi_3 \left. \right\} = 0$$

or $\sum_{\pi_3} \sum_{\pi_3}' \sum_{\pi_2}' \sum_{l=s}^{R_3} a \left\{ +1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3 - j_2, +\overline{R_3+1}, \pm j_{R_3}, \pm \right.$

$$j_{R_3-1}, \dots, \pm j_{l+2}, +j_{l+1}, -\overline{R_4+1} (\pm j_{R_2+2}, \pm j_{R_2+3}, \dots, \pm j_l) \rightarrow, \pi_3 \left. \right\} = 0 \quad (18)$$

Here, the permutation

$$\pi_3' : i_m \rightarrow j_m, \quad R_2+2 \leq m \leq R_3$$

is such that it does not take any of

$$i_{R_2+2}, i_{R_2+3}, \dots, i_{s-1}, i_s$$

to any of

$$i_{s+1}, i_{s+2}, \dots, i_{R_3-1}, i_{R_3}.$$

There will be one such equation corresponding to each of the

$$r_3^{-1} C_{s-R_2-1} \quad (r_3 \equiv R_3 - R_2)$$

ways of choosing $(s-(R_2+1))$ indices

$$i_{2+R_2}, i_{2+R_3}, \dots, i_{s-1}, i_s$$

out of the (r_3-1) indices

$$R_2+2, R_2+3, \dots, R_3-1, R_3$$

Summing them up, we get (just like equation (14))

$$\sum_{T_3} \sum_{\pi_3}' \sum_{\pi_2}' \sum_{l=s}^{R_3} 1-(R_2+1) C_{s-(R_2+1)} a_{\{+1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, +\overline{R_3+1}, \pm j_{R_3}, \pm j_{R_3-1}, \dots, \pm j_{l+2}, \pm j_{l+1}, -\overline{R_4+1}, (\pm j_{R_2+2}, \pm j_{R_2+3}, \dots, \pm j_1)\}_{T_3}} =$$

(19)

T_3 being any permutation of $(R_2+2, R_2+3, \dots, R_3)$.

Clearly, (16) is again equation (19) with $s=R_2+1$. Multiplying by $(-1)^{s-(R_2+1)}$

and summing over s from R_2+1 to R_3 , we get

$$\sum_{T_3} \sum_{\pi_3}' \sum_{\pi_2}' \sum_{s=R_2+1}^{R_3} \sum_{l=s}^{R_3} (-1)^{s-(R_2+1)} 1-(R_2+1) C_{s-(R_2+1)} a_{\{+1, -\overline{R_2+1}, \pm j_{R_2}, \pm$$

$$j_{R_2-1}, \dots, \pm j_3, -j_2, \overline{+R_3+1}, \pm j_{R_3}, \pm j_{R_3-1}, \dots, \pm j_{1+2}, +j_{1+1}, \overline{-R_4+1}, (\pm$$

$$j_{R_2+2}, \pm j_{R_2+3}, \dots, \pm j_1) \rightarrow , T_3 \} = 0 .$$

Interchanging the l and s summations and writing down the term corresponding to $l = R_2+1$ explicitly, we get

$$\begin{aligned} & \sum_{T_3} \sum'_{T_3} \sum'_{T_2} a \left\{ +1, -\overline{R_2+1}, \pm j_{R_2}, \pm j_{R_2+1}, \dots, \pm j_3, -j_2, \overline{+R_3+1}, \pm j_{R_3}, \pm j_{R_3-1}, \dots \right. \\ & \left. \pm j_{R_2+3}, +j_{R_2+2}, \overline{-R_4+1}, T_3 \right\} + \sum_{T_3} \sum'_{T_3} \sum'_{T_2} \sum_{l=R_2+2}^{R_3} \frac{1}{\sum_{s=R_2+1}} (-1)^{s-(R_2+1)} \\ & 1-(R_2+1) C_{s-(R_2+1)} a \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, +\overline{R_3+1}, \pm j_{R_3}, \pm \right. \\ & \left. j_{R_3-1}, \dots, \pm j_{1+2}, +j_{1+1}, \overline{-R_4+1}, (\pm j_{R_2+2}, \pm j_{R_2+3}, \dots, \pm j_1) \rightarrow , T_3 \right\} = 0 \end{aligned}$$

$$\text{or } \sum_{T_3} \sum'_{T_3} \sum'_{T_2} a \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, \overline{+R_3+1}, \pm j_{R_3-1}, \dots, \right.$$

$$\left. \pm j_{R_2+3}, +j_{R_2+2}, \overline{-R_4+1}, T_3 \right\} = 0 \quad (20)$$

as we have

$$\sum_{s=R_2+1} \frac{1}{(-1)^{s-(R_2+1)}} 1-(R_2+1) C_{s-(R_2+1)} = 0 \text{ for } R_2+2 \leq l \leq R_3 .$$

If we now repeat the process of obtaining (15) from (10) and (20) from (15), $k-2$ times, we will get

$$\sum_{\pi} \left\{ +1, \overline{-R_2+1}, \pm j_{R_2}, \pm j_{R_2-1}, \dots, \pm j_3, -j_2, \overline{+R_3+1}, \pm j_{R_3}, \pm j_{R_3-1}, \dots, \right. \\ \left. \pm j_{R_2+3}, \pm j_{R_2+2}, \overline{-R_4+1}, \dots, \overline{-R_k+1}, \pm j_{R_k}, \pm j_{R_k-1}, \dots, \pm j_{R_{k-1}+2} \right\} = 0$$

Here $\sum_{\pi}$ sums all the distinct terms obtained by varying the prefixes and the permutation of indices, the permutation π of

$$(2, 3, 4, \dots, n)$$

being such that

$$\pi(R_u+2, R_u+3, \dots, R_{u+1}) = (j_{R_u+2}, j_{R_u+3}, \dots, j_{R_{u+1}})$$

$$\pi(R_u+1) = R_u+1 \quad (22)$$

for $u = 2, 3, 4, \dots, k$.

Had we started from the cone (k even)

$$C \left\{ +1, -i'_2, -i'_3, \dots, i'_{R_2+1}, +i'_{R_2+2}, +i'_{R_2+3}, \dots, +i'_{R_3+1}, -i'_{R_3+2}, -i'_{R_3+3}, \right. \\ \left. \dots, -i'_{R_{k-1}+2}, -i'_{R_{k-1}+3}, \dots, -i'_{R_k}, -i'_{R_k+1} \right\} \quad (23)$$

instead of (9), we would have reached (21), with the only difference that this time would have had i'_{R_u+1} instead of R_u+1 , and

$$\pi(j_{R_u+2}, j_{R_u+3}, \dots, j_{R_u+1}) = (i'_{R_u+2}, i'_{R_u+3}, \dots, i'_{R_u+1})$$

$$\pi(i'_{R_u+1}) = (i_{R_u+1}) \quad (24)$$

If we now consider (21) corresponding to the cone (23), we see that one of the following two alternatives will hold

- i) $r_u \cong R_u - R_{u-1} \leq 2$ for $u = 2, 3, \dots, k$, so that there is no summation i.e. there is only one 'a' on the LHS, namely that of (23), and it is therefore equal to zero.
- ii) There is at least one $r_u > 2$, so that the LHS has more than one terms. One of these is clearly the 'a' of (23) while the rest of them are a's corresponding to cones having their η -value greater than that of (23). Here η -value of a cone $C_{\beta, \beta \in \mathcal{D}}$ is defined as the number of changes in the prefixes of indices in β .

This shows that 'a' of a cone with a given η -value vanishes if those of the cones with greater η -value vanish. For the cone either falls in the case (i) and hence its 'a' is zero, or else it falls in the

case (ii) so that its 'a' is equal to a linear combination of a's of cones with greater η -value, each of which is equal to zero. Now, the maximum value of η is $(n-1)$ which is for the cones of the type (7a) and their 'a' is zero according to (8). Hence the 'a' of any cone with η -value $(n-2)$ also vanishes and so on, so that it follows by induction that 'a' of any cone of the type (23) is zero.

The case when k is odd i.e. when we have the cone

$$\begin{aligned} & C \left(+1, -i_2^1, -i_3^1, \dots, -i_{R_2+1}^1, +i_{R_2+2}^1, +i_{R_2+3}^1, \dots, +i_{R_3+1}^1, -i_{R_3+2}^1, -i_{R_3+3}^1, \dots, \right. \\ & \left. + i_{R_{k-1}+2}^1, +i_{R_{k-1}+3}^1, \dots, +i_{R_k}^1, +i_{R_k+1}^1 \right) \end{aligned} \quad (25)$$

instead of (23), can be treated in an exactly the same way and its 'a' proved equal to zero. Now as every $\beta \in D$ is either of the form (23) or the form (25) it follows that

$$a_\beta = 0 \quad \text{for } \beta \in D$$

Further, going back to (2), we see that if we reverse the inequalities (3), we will get

$$\sum_{\beta \in D'} a_\beta R_\beta = 0$$

instead of (2a). Proceeding now as before, we can show that

$$a_\beta = 0 \quad \text{for } \beta \in D'$$

This proves that

$$a_{\beta} = 0 \text{ for } \beta \in \mathcal{D}$$

and hence also that

$$R_{\beta}, \beta \in \mathcal{D}$$

form a linearly independent set of generalised retarded operators.

2. Expansion in terms of $R_{\beta}, \beta \in \mathcal{D}$

We now come to the problem of determining the coefficient of

$$R_{\beta}, \beta \in \mathcal{D}$$

in the expansion of an arbitrary operator R_{α} i.e. we want to find a_{β} where

$$R_{\alpha} = \sum_{\beta \in \mathcal{D}} a_{\beta} R_{\beta} \quad (26)$$

Let us assume that $q_1 \in V_+$ in C_{α} . Then for x satisfying (3), we have

$$R_{\alpha} = \sum_{\beta \in \mathcal{D}} a_{\beta} R_{\beta}$$

We will find a_{β} corresponding to the cone (9). Let D_{α} be the sub-set of $\mathcal{D}$ such that the prefix of each index in D_{α} is the same as that in C_{α} , and let $\bar{D}_{\alpha}$ be the complement of D_{α} in $\mathcal{D}$. Consider first the case when (9) is in D_{α} . Proceeding as before, we get equation (11) with the only difference that the RHS is +1 or zero according as the p.c. of (9) are or are not satisfied in C_{α} .

Now there are two possibilities.

- i) All the p.c. of (9) which correspond to R_3+1 and the indices represented by T_2 , are satisfied in C_α .
- ii) There is at least one of them which is not satisfied in C_α .

Consider the first case. Starting from the cone (12) we reach equation (13) with RHS as +1 or zero according as the p.c. of (12) corresponding to

$$2, 3, 4, \dots, R_2, R_2+1$$

are or are not satisfied in C_α . This leads to equation (14) with RHS as N_r^2 instead of zero, where N_r^2 is the number of cones of the type (12) with r and T_2 fixed but

$$(i_2, i_3, \dots, i_r)$$

varying, whose p.c. corresponding to the above indices are satisfied in C_α . It should be noted that this number N_r^2 is quite independent of the position and prefixes of the indices coming on the right of R_2+2 in (12) so long as its p.c. corresponding to R_2+2 is satisfied in C_α . Summing (14) over r from 1 to R_2 after multiplying it by $(-1)^{r-1}$, we get (15) with RHS as η_2 where

$$\eta_2 = \sum_{r=1}^{R_2} (-1)^{r-1} N_r^2 ,$$

and the same will be true of (16).

In the second case, we get (13) with RHS zero for every choice of

$$(i_2, i_3, \dots, i_r)$$

as there is always at least one p.c. of (12) which is not satisfied in C_α . This leads to equation (14) and then to equation (15) and (16) with RHS zero. Thus we reach (15) and (16) with RHS as γ_2 or zero according as the p.c. of (9) corresponding to

$$R_2 + 2 R_{2+3}, \dots, R_k, R_{k+1} \leq n$$

are or are not satisfied in C_α .

There are again two possibilities

- i) All the p.c. of (9) corresponding to $R_4 + 1$ and the indices represented by T_3 are satisfied in C_α .
- ii) There is at least one of them which is not satisfied in C_α .

In the first case, corresponding to the cone (17) there will be equation (18) with RHS γ_2 or zero, according as the p.c. of (17) corresponding to

$$R_2 + 2, R_2 + 3, \dots, R_3, R_3 + 1$$

are or are not satisfied in C_α . This gives equation (19) with RHS $\gamma_2 N_s^3$

where N_s^3 is the number of cones of the type (17) with T_3 and s fixed, but

$$i_{R_2+2}, i_{R_2+3}, \dots, i_{s-1}, i_s$$

Varying, whose p.c. corresponding to the above indices are satisfied in C_α . Clearly, N_s^3 is independent of the permutation and prefixes of indices on the right of R_3+2 and left of R_2+1 in (17), so long as its p.c. corresponding to R_3+2 is satisfied in C_α . Summing (19) over s from R_2+1 , to R_3 after multiplying it by $(-1)^{s-(R_2+1)}$ we reach (20) with RHS as $\eta_2 \eta_3$ where

$$\eta_3 = \sum_{s=R_2+1}^{R_3} (-1)^{s-(R_2+1)} N_s^3.$$

In the second case, however we get (18) with RHS zero corresponding to (17) with any choice of the indices

$$i_{R_2+2}, i_{R_2+3}, \dots, i_s,$$

as there is always at least one p.c. of (17) which is not satisfied in C_α . This leads to (19) and then to (20) with RHS as zero. Thus we have (20), again with the RHS as $\eta_2 \eta_3$ as zero according as the p.c. of (9) corresponding to

$$R_3+2, R_3+3, \dots, R_k, R_k+1$$

are or are not satisfied in C_α .

Proceeding in this way, we ultimately reach equation (21) with RHS as $\eta_2 \eta_3 \dots \eta_k$, where

$$\eta_m = \sum_{u=R_{m-1}+1}^{R_m} (-1)^{u-(R_{m-1}+1)} N_u^m$$

and N_u^m is the number of cones of the type

$$C \left\{ +1, -2, -3, \dots, \overline{-R_2+1}, \overline{+R_2+2}, \overline{+R_2+3}, \dots, (-1)^{m-2} \overline{R_{m-1}+1}, (-1)^{m-1} i_{u+1}, \right. \\ (-1)^{m-1} i_{u+1}, \dots, (-1)^{m-1} i_{R_m}, (-1)^{m-1} \overline{R_m+1}, (-1)^m i_{R_{m-1}+2}, (-1)^m i_{R_{m-1}+3}, \dots, \\ \left. (-1)^m i_u, (-1)^m \overline{R_m+2}, (-1)^m \overline{R_m+3}, \dots, \overline{+R_{k-1}+1}, \overline{-R_{k-1}+2}, \dots, \overline{-R_k}, \overline{-R_k+1} \right\}$$

obtained by varying the permutation

$$(i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_{R_m}) \text{ of } (R_{m-1}+2, R_{m-1}+3, \dots, R_m)$$

while keeping u fixed between $R_{m-1}+1$ and R_m , when the prefixes of R_m+2 and the indices on its right are such that the p.c. corresponding to them are satisfied in C_α ,

whose p.c. corresponding to the indices

$$R_{m-1}+2, R_{m-1}+3, \dots, R_m, R_m+1$$

are satisfied in C_α . As mentioned earlier for N_2^r and N_3^s , it is clear that N_m^u is also independent of the permutation and prefixes of indices on the right of R_m+2 and on the left of $R_{m-1}+1$, so long as the p.c. corresponding to R_m+2 is satisfied in C_α . Hence N_m^u can also be defined as the number of distinct cones of the type

$$\begin{aligned} & C \left\{ S_m, (-1)^{m-2} \overline{R_{m-1}+1}, (-1)^{m-1} i_{u+1}, (-1)^{m-1} i_{u+2}, \dots, (-1)^{m-1} \overline{R_m+1}, \right. \\ & \left. (-1)^m i_{R_{m-1}+2}, (-1)^m i_{R_{m-1}+3}, \dots, (-1)^m i_u, (-1)^m \overline{R_m+2}, T_m \right\} \end{aligned} \quad (27)$$

obtained by varying the permutation

$$(i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_{R_m}) \text{ of } (R_{m-1}+2, R_{m-1}+3, \dots, R_m)$$

while keeping fixed S_m, T_m and u , with T_m such that the p.c. corresponding to R_m+2 is satisfied in C_α ,

whose p.c. corresponding to the indices

$$R_{m-1}+2, R_{m-1}+3, \dots, R_m, R_m+1$$

are satisfied in C_α .

Next we come to the case when (9) is in $\bar{D}_\alpha$, so that there is at least one index whose prefix in it is different from that in C_α . We can take this index to be R_m+1 for some m $2 \leq m \leq K$, for if it were not, we could shift it to the right and bring it there. The p.c. corresponding to

$$i_{R_m+1}, i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_u$$

of any cone (27) cannot all be satisfied in C_α simultaneously. For either the p.c. corresponding to at least one of

$$i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_u$$

is not satisfied in C_α , or else, those corresponding to all of them are, but then the one corresponding to R_m+1 is not satisfied as it has different indices in (23) and in C_α . To see the validity of this last statement, we note that when

$$\sum_{t=R_{m-1}+2}^u q_{i_t} + q_{R_m+2} + \sum_{t \in T_m} q_t \in V_{-\sigma} \quad \sigma = (-1)^m$$

$q_{R_m+1} \in V$ in C_α (as it is in $V_{-\sigma}$ in (27)) gives

$$q_{R_m+1} + \sum_{t=R_{m-1}+2}^u q_{i_t} + q_{R_m+2} + \sum_{t \in T_m} q_t \in V_{-\sigma} \text{ in } C_\alpha$$

while the p.c. corresponding R_m+1 of (27) states that this last partial sum is in $V_{-\sigma}$. This shows that $N_m^u = 0$ for all u so that $\eta_m = 0$ and hence also $\eta_2 \eta_3 \dots \eta_m$ i.e. we get (21) with RHS as zero.

Now, two cases arise :

- i) Each of the numbers $r_2, r_3, \dots, r_k \leq 2$ so that there is no summation (21) and we have 'a' of (9) equal to zero.
- ii) There is at least one $r_t > 2$, so that there are more than one terms in (21). We can therefore express 'a' of (9) as a linear combination of some a_γ 's where the η -value of each of the γ 's is greater than that of (9), and for every γ , there is at least one index (namely $R_m + 1$) whose prefix in it differs from that in C_α i.e. γ 's are all in $\bar{D}_\alpha$

Had we started from (23) instead of (9) (of course with (23) in $\bar{D}_\alpha$), we would have obtained the same results for 'a' of (23) as also in the case k odd. In other words, these results are true for a_{γ_0} where γ_0 is an arbitrary element of $\bar{D}_\alpha$. Thus a_{γ_0} can be different from zero only when it falls in the second case, but then each of the $a_{\gamma'}^s$ of (ii), being an element of $\bar{D}_\alpha$, is itself equal to a linear combination of some $a_{\gamma''}$'s where γ'' 's are again in $\bar{D}_\alpha$ and have still higher η -values. Continuing this process, we see that ultimately, a_{γ_0} will be equal to a linear combination of a's of cones which are in $\bar{D}_\alpha$ and which fall in the case (i) above. As these a's vanish, the same will also be true for a_{γ_0} . Thus we have proved that $a_{\gamma_0} = 0$ for every $\gamma_0 \in \bar{D}_\alpha$.

Returning back to the case when (9) is in D_α , we have (21) with RHS as $\eta_2 \eta_3 \dots \eta_k$. If there is no summation on the LHS, we have just the 'a' of (9) equal to $\eta_2 \eta_3 \dots \eta_k$. If, on the other hand, there is a summation,

the LHS will be equal to 'a' of (9) plus a linear combination of some a_γ 's where for each γ , there is at least one index whose prefix in it is different from that in (9) and hence also from that in C_α . These γ 's are therefore in $\bar{D}_\alpha$ so that all the a_γ 's vanish (as proved above) and the LHS reduces to just the 'a' of (9). This shows that

$$a \text{ of (9) } = \eta_2 \eta_3 \dots \eta_k.$$

The same result clearly holds when k is odd, and also in the case when (9) is replaced by (23), though this time with the difference that in (27)

$$i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_{R_m}, i_{R_m+1}$$

is a permutation of

$$i'_{R_{m-1}+2}, i'_{R_{m-1}+3}, \dots, i'_{R_m}, i'_{R_m+1}, \text{ with } i_{R_m+1} = i'_{R_m+1},$$

and not of

$$R_{m-1}+2, R_{m-1}+3, \dots, R_m, R_m+1.$$

It is now ^{not} difficult to prove that a_β is zero when β is in D' . For let x be such that

$$x_1^0 < x_i^0 \quad i = 2, 3, \dots, n$$

Then $R_\alpha = 0$ (as $q_1 \in V_+$ in C_α) as well as $a_\beta = 0$ for $\beta \in D$, so that (26) becomes

$$\sum_{\beta \in D'} a_\beta R_\beta = 0$$

and this leads to $a_\beta = 0$ for $\beta \in D'$, as was seen in proving the linear independence of the set

$R_\beta, \beta \in D$. Thus we have

$$\begin{aligned} a_\beta &= \eta_2 \eta_3 \dots \eta_k \quad \text{for } \beta \in D_\alpha \\ a_\beta &= 0 \quad \text{for } \beta \notin \bar{D}_\alpha \text{ and } \beta \in D' \end{aligned} \quad (28a)$$

Till now we had assumed that $q_1 \in V_+$ for $q \in C_\alpha$. However, the same method can be used to show that when $q_1 \in V_-$ for $q \in C_\alpha$, a_β for $\beta \in D$ are given by

$$\begin{aligned} a_\beta &= \eta_2 \eta_3 \dots \eta_k \quad \text{for } \beta \in D'_\alpha \\ a_\beta &= 0 \quad \text{for } \beta \in \bar{D}'_\alpha \text{ and } \beta \in D. \end{aligned} \quad (28b)$$

3. Evaluation of η_m

We will now evaluate η_m in terms of conditions defining C_α . Consider first the case when m is even, so that (27) can be written

as

$$C \left\{ \overline{S_m, +R_{m-1}+1}, -i_{u+1}, -i_{u+2}, \dots, -i_{R_m}, -\overline{R_m+1}, +i_{R_{m-1}+2}, \right. \\ \left. + i_{R_{m-1}+3}, \dots, +i_u, +\overline{R_m+2}, T_m \right\} \quad (29)$$

where

$$(i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_{R_m})$$

is a permutation of

$$(R_{m-1}+2, R_{m-1}+3, \dots, R_m).$$

Let G_m , $\mathcal{K}_m^-$, H_e^f and $\mathcal{N}_e^m$ be as defined in the second section of the chapter 3. Also let

$$I_e^f, \quad f = 1, 2, \dots, \mathcal{N}_e^{1m}$$

be those of H_e^f which do not contain R_m+1 and

$$I_e^f, \quad f = 1, 2, \dots, \mathcal{N}_e^{2m}$$

which do contain it. Clearly

$$\mathcal{N}_e^m = \mathcal{N}_e^{1m} + \mathcal{N}_e^{2m}.$$

Now, if the set G_m is not in $\mathcal{K}_m^-$, $\mathcal{N}_u^m$ is zero for all u , so that η_m

and hence also the product $\eta_1 \eta_2 \eta_3 \dots \eta_k$ will be zero. For, if $G_m \in \mathcal{K}_+^m$, every subset of G_m is also in $\mathcal{K}_+^m$, so that, for any choice of

$$i_{u+1}, i_{u+2}, \dots, i_{R_m},$$

the p.c. of (29) corresponding to R_m+1 is not satisfied in C_α i.e. $N_u^m = 0$. As this case is not interesting, we take G_m to be in $\mathcal{K}_-^m$.

Corresponding to the set G_m , let us consider the cone

$$C\{S_m, +\overline{R_{m-1}+1}, -\overline{R_m+1}, +\overline{R_{m-1}+2}, +\overline{R_{m-1}+3}, \dots, +\overline{R_m+2}, T_m\}$$

whose p.c. corresponding to R_m+1 is satisfied in C_α . Those corresponding to the elements of the set

$$(R_{m-1}+2, R_{m-1}+2, \dots, R_m-1, R_m)$$

are or are not satisfied in C_α , according as this set is not or is in $\mathcal{K}_-^m$ i.e. according as

$$\mathcal{N}_{R_m-1}^{1_m} \text{ is } 0 \text{ or } 1.$$

Thus, we have

$$N_{R_m}^u = 1 \text{ or } 0 \text{ according as } \mathcal{N}_{R_m-1}^{1_m} = 0 \text{ or } 1.$$

As $\mathcal{N}_{R_m}^{2m}$ is just 1, we may write

$$N_{R_m}^u = \mathcal{N}_{R_m}^{2m} - \mathcal{N}_{R_m-1}^{1_m}$$

Again, corresponding to each of the $\sqrt[2m]{r_{m-1}}$ subsets $I_{r_{m-1}}^f$ say

$$(t_{R_{m-1}+2}, t_{R_{m-1}+3}, \dots, t_{R_{m-1}}, R_{m-1}+1)$$

we consider the cone

$$C \left\{ S_m, + \overline{R_{m-1}+1}, -t_{R_m}, -\overline{R_m+1}, + t_{R_{m-1}+2}, +t_{R_{m-1}+3}, \dots, +t_{R_{m-1}}, +R_m+2, T_m \right\}$$

The p.c. of all these cone which correspond to t_{R_m} and R_m+1 , are satisfied in C_α , but in $\sqrt[1m]{r_{m-2}}$ of these, those corresponding to at least one of

$$(t_{R_{m-1}+2}, t_{R_{m-1}+3}, \dots, t_{R_{m-1}})$$

are not satisfied, due to the fact that $\sqrt[1m]{r_{m-2}}$ subsets I_e^f are in $\sqrt[1m]{r_{m-2}}$. To see that each of these $\sqrt[1m]{r_{m-2}}$ subsets does in fact contradict a p.c. of the cone corresponding to one of $I_{r_{m-1}}^f$, we just note that if

$$(t_{R_{m-1}+2}, t_{R_{m-1}+3}, \dots, t_{R_{m-1}})$$

is one of $I_{r_{m-2}}^f$, then

$$t_{R_{m-1}+2}, t_{R_{m-1}+3}, \dots, t_{R_{m-1}}, R_m+1$$

will be one of $I_{r_{m-1}}^f$ and the former will contradict at least one p.c.

of the cone corresponding to the latter. Thus the number of cones of the type (29) with $u = R_{m-1}$ is

$$N_{R_{m-1}}^m = \mathcal{N}_{r_m-1}^{2m} - \mathcal{N}_{r_m-2}^{1m}.$$

Proceeding in this way, we can show that, in general, the $\mathcal{N}_{r_m-u}^{2m}$ subsets $I_{r_m-u}^{1f}$ and $\mathcal{N}_{r_m-u-1}^{1m}$ subsets $I_{r_m-u-1}^{1f}$ lead to

$$\mathcal{N}_{r_m-u}^{2m} - \mathcal{N}_{r_m-u-1}^{1m}, \quad u=0, 1, \dots, r_m-1, \quad \mathcal{N}_0^{1m} = 0$$

cones of the type (29) whose p.c. corresponding to the elements of the set G_m are satisfied in C_α i.e.

$$N_{R_m-u}^m = \mathcal{N}_{r_m-u}^{2m} - \mathcal{N}_{r_m-u-1}^{1m}$$

Hence we get

$$\begin{aligned} \eta_m &= \sum_{u=R_{m-1}+1}^{R_m} (-1)^{u-(R_{m-1}+1)} N_u^m \\ &= \sum_{u=0}^{r_m-1} (-1)^{r_m-u-1} N_{R_m-u}^m \\ &= \sum_{u=0}^{r_m-1} (-1)^{r_m-u-1} (\mathcal{N}_{r_m-u}^{2m} - \mathcal{N}_{r_m-u-1}^{1m}) \end{aligned}$$

This can be rearranged as

$$\eta_m = \sum_{u=0}^{r_m-1} (-1)^{r_m-u-1} \left(\cancel{N}_{r_m-u}^{2m} + \cancel{N}_{r_m-u-1}^{1m} \right)$$

where $\cancel{N}_{r_m}^{1m} \equiv 0,$

$$\text{i.e. } \eta_m = \sum_{u=0}^{r_m-1} (-1)^{r_m-u-1} \cancel{N}_{r_m-u}^m \quad (30)$$

The case when m is odd can be treated in an exactly the same way, and will give the result (30) with the difference that $\cancel{X}_-^m$ is interchanged with $\cancel{X}_+^m$ i.e. I_e^f and I_e^f are the subsets of G_m of e elements which are in $\cancel{X}_+^m$, and so $\cancel{N}_u^m$ is the number of subsets of G_m of u elements which are in $\cancel{X}_+^m$.

Similarly, if we consider (29) with

$$(i_{R_{m-1}+2}, i_{R_{m-1}+3}, \dots, i_{R_m})$$

as a permutation of

$$(i'_{R_{m-1}+2}, i'_{R_{m-1}+3}, \dots, i'_{R_m}),$$

we will reach equation (30) for the η 's of the cones (23) and (25) provided that G_m is now replaced by the set

$$G_m^\pi: (i'_{R_{m-1}+2}, i'_{R_{m-1}+3}, \dots, i'_{R_m}, i'_{R_{m+1}}). \quad (31)$$

Thus, we may summarise the results of the last two sections as follows. We have shown that when an arbitrary generalised retarded operator R_α is expanded in the form (26), then the expansion coefficients a_β are given by (28 a,b) while γ_m , for the cones (23) and (25), is itself given by (30), where N_u^m is the number of subsets of the set G_m^π , defined by (31), of u elements which are in $\mathcal{H}_-^m$ or $\mathcal{H}_+^m$ according as m is even or odd.

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