Gluing Constructions of Special Lagrangian Cones *

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Abstract

We survey our recent work constructing new special Lagrangian cones in $\mathbb{C}^n$ for all $n \geqslant 3$ by gluing methods.

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1 Introduction

Let M be a Calabi-Yau manifold of complex dimension n with Kähler form ω and non-zero parallel holomorphic n-form Ω satisfying a normalization condition. Then $\mathrm{Re}\,\Omega$ is a calibrated form whose calibrated submanifolds are called special Lagrangian (SLG) submanifolds [12].

Moduli spaces of SLG submanifolds began appearing in string theory in the mid 1990s [1, 20, 21, 24, 47]. On physical grounds, Strominger, Yau and Zaslow in 1996 argued that a Calabi-Yau manifold M with its mirror partner $\widehat{M}$ admits a (singular) fibration by SLG tori, and that $\widehat{M}$ should be obtained by compactifying the dual fibration [47]. The SYZ prediction that there is an underlying differential-geometric explanation of Mirror Symmetry for Calabi-Yau 3-folds prompted many mathematicians to study the relations between special Lagrangian geometry and Mirror Symmetry.

To make the ideas from SYZ mathematically rigorous one needs control over the singularities and compactness properties of families of SLG submanifolds. In dimensions three and higher these properties are not well understood. As a result

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there has been considerable recent interest in singular SLG subvarieties [2, 3, 9–11, 13–15, 25, 41, 45, 46].

One natural class of singular SLG varieties is the class of SLG varieties with isolated conical singularities [25]. Loosely speaking, these are compact SLG varieties of Calabi-Yau manifolds which are singular at a finite number of points, near each of which they resemble asymptotically some SLG cone C in $\mathbb{C}^n$ with the origin as the only singular point of C. This motivates the recent interest in constructing SLG cones in $\mathbb{C}^n$ with an isolated singularity at the origin [4, 13, 14, 23, 42]. Until recently, few examples of such SLG cones in $\mathbb{C}^n$ were known. In the period between 2000 and 2005, many new families of examples were constructed principally using techniques from equivariant differential geometry, symplectic geometry and integrable systems [4, 13, 14, 23, 42]. In contrast, PDE techniques were not used in the construction of new special Lagrangian cones during this period (see Joyce's series of papers [27–29] on U(1)-invariant special Lagrangians in $\mathbb{C}^3$ for PDE constructions of other special Lagrangian 3-folds.)

In 2007 we used geometric PDE gluing methods to construct a plethora of new special Lagrangian cones in $\mathbb{C}^3$ [16]. Very recently we have extended these gluing methods to construct large numbers of new special Lagrangian cones in $\mathbb{C}^n$ for all $n \geqslant 3$. The current article gives a detailed survey of our recent work on gluing constructions of special Lagrangian cones [16–19]. It also serves as an announcement for the main new higher dimensional gluing results proved in [18, 19].

The particular kind of gluing methods used relates most closely to the methods developed in [31, 44], especially as they evolved and were systematized in [34–36]. (For a general discussion of the methodology see the survey [39]). The basic building blocks of our construction are provided by $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Lagrangian cones in $\mathbb{C}^n$ where $p+q=n$. Equivalently the links of these cones, that is their intersections with the unit sphere $\mathbb{S}^{2n-1}\subset\mathbb{C}^n$, are $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrian submanifolds of $\mathbb{S}^{2n-1}$. When $(p,q)=(1,2)$ these special Legendrians were first constructed in [13, 14] and studied in more detail in our work on gluing constructions of higher genus special Legendrian cones in $\mathbb{C}^3$ [16]. For general (p,q), $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrians were first considered by Castro-Li-Urbano [5] and then studied in more detail by the authors in [17].

The main results

In all cases these special Legendrian submanifolds form one-parameter smooth families. We call the parameter τ. For small τ the submanifold approximates a necklace of equatorial spheres of dimension $n-1$ connected to each other through regions we call "transition regions" or "necks". In this sense they are reminiscent of the Delaunay/Fowler metrics (rotationally invariant constant scalar curvature metrics), the Delaunay surfaces (rotationally invariant CMC surfaces in $\mathbb{R}^3$), and to some extent the Wente tori (more complicated CMC surfaces in $\mathbb{R}^3$). The above have been used successfully as building blocks in various gluing constructions where approximately spherical regions of different copies are "fused" to form more

complicated examples [30–36, 44].

The transition regions of the special Legendrian submanifolds we use as building blocks depend on (p, q). If $(p, q) = (1, n-1)$ the transition regions adjacent to an almost spherical region concentrate at two antipodal points, and their cores approximate small $n-1$-dimensional Lagrangian catenoids. If $p, q > 1$ the transition regions concentrate close to two orthogonal equatorial spheres $\mathbb{S}^{p-1}$ and $\mathbb{S}^{q-1}$. Their cores approximate the product of a unit $\mathbb{S}^{p-1}$ and a q-dimensional Lagrangian catenoid or a unit $\mathbb{S}^{q-1}$ and a p-dimensional Lagrangian catenoid. As we will see, in both cases the transition regions do not carry approximate kernel, unlike the catenoidal necks of the Delaunay surfaces, but similarly to the necks of the Delaunay/Fowler metrics.

For suitable values of τ the submanifolds "close up" to form closed submanifolds containing a large but finite number of almost spherical regions. These examples have $\tau \to 0$ as the number of the almost spherical regions goes to ∞. The gluing constructions we discuss enjoy "maximal" symmetry, have only one spherical region where fusion occurs, and the building blocks fused are copies of one of the examples with a large finite number of almost spherical regions. In this respect many of the considerable extra complications a general (as in [31, 33, 44]) construction would have, are avoided. In particular the maximal symmetry allows no approximate kernel on the "central" almost spherical region which takes care of the "balancing" and "unbalancing" conditions [39] automatically.

The price of these simplifications is that the possible configurations of the building blocks being "fused" are restricted. This reduces the topological types of special Legendrian submanifolds being constructed as well. Nevertheless we have been very successful with gluing constructions involving building blocks of types with $(p, q) = (1, n-1)$ or $p = q$. In the other cases we have transition regions concentrating close to an equator of such high dimension that it can not be moved so that it does not intersect itself, something which is necessary to attach to the central sphere a second copy of the building block. Moreover the fact that we can not attach more than one building block at the same place restricts the possible configurations even in the cases we consider.

In the case $p = q$ we can fuse an odd number of copies symmetrically placed [19]. (An even number would lead to trying to attach different copies at the same place). In the case $(p, q) = (1, n-1)$ the configurations of the building blocks we allow can be described by a set of points Δ [16, 18]. The necks attaching to the central ("fusion") sphere are close to the points of Δ or their antipodals. To avoid trying to attach different building blocks at the same points we need Δ to contain no pair of antipodal points. Because of the need of maximal symmetry in the constructions we discuss, Δ consists either of the vertices of a regular g-gon on a real two-plane in $\mathbb{C}^n$, or the vertices of a regular $(k+1)$-simplex in a real k-plane in $\mathbb{C}^n$. To avoid antipodal points we need g odd in the polygon case and we can not allow other Platonic solids in the other case. This implies that in the case $n = 3$ in particular we only obtain surfaces of odd genus or (when Δ is a tetrahedron) genus $g = 4$ [16].

Organization of the paper

In addition to the introduction there are seven more sections and an appendix. In section 2 we recall basic facts about special Lagrangian geometry in $\mathbb{C}^n$, special Lagrangian cones in $\mathbb{C}^n$ and their connection to minimal Legendrian submanifolds of $\mathbb{S}^{2n-1}$. In section 3 we discuss the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds of $\mathbb{S}^{2n-1}$, where $p+q = n$ and without loss of generality $p \leqslant q$. These submanifolds are the ones we use as the building blocks for our constructions. The geometry of these submanifolds differs according to whether $p = 1$ or not. In the latter case, the case $p = q$ is special. We pay particular attention to the symmetries of these submanifolds because of the importance of symmetries in our constructions.

In section 4 we start the discussion of our gluing constructions by carrying out the bare hands constructions of the initial submanifolds. Particular attention is given to presenting the common features of these constructions. We also define carefully the groups of symmetries each construction satisfies. In section 5 we discuss the general framework for correcting the initial submanifolds which are only Legendrian, to special Legendrian. We discuss in particular the action of the symmetries on the approximating spheres and the meridians of the building blocks. This allows us by using the results of the appendix to understand the equivariant kernel on the approximating spheres and the eigenfunctions of low eigenvalue on the meridians. These are very important in understanding the linearized equation later.

In section 6 we discuss the statement and proof of proposition 6.8. This proposition allows us to solve the linearized inhomogeneous equation with appropriate estimates with decay. We can only solve modulo the extended substitute kernel which is discussed in the same section. In section 7 we discuss the creation of extended substitute kernel. This requires modifications in the construction of the initial surfaces and substantially complicates the whole construction. All the work in section 7 leads to stating and proving proposition 7.2. Finally in section 8 we discuss the nonlinear terms and then we state and discuss the proof of the main theorem.

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2 Special Lagrangian cones and special Legendrian submanifolds of $\mathbb{S}^{2n-1}$

In this section we recall basic facts about special Lagrangian geometry in $\mathbb{C}^n$, special Lagrangian cones in $\mathbb{C}^n$ and their connection to minimal Legendrian submanifolds of $\mathbb{S}^{2n-1}$. Special Lagrangian geometry is an example of a *calibrated geometry* [12]. We begin by defining *calibrations* and *calibrated submanifolds*.

Calibrations and Special Lagrangian geometry in $\mathbb{C}^n$

Let (M, g) be a Riemannian manifold. Let V be an oriented tangent p-plane on M, *i.e.* a p-dimensional oriented vector subspace of some tangent plane T_xM to M. The restriction of the Riemannian metric to V, $g|_V$, is a Euclidean metric on V which together with the orientation on V determines a natural p-form on V, the volume form Vol_V. A closed p-form ϕ on M is a *calibration* on M if for every oriented tangent p-plane V on M we have $\phi|_V \leqslant \mathrm{Vol}_V$. Let L be an oriented submanifold of M with dimension p. L is a *ϕ-calibrated submanifold* if $\phi|_{T_xL} = \mathrm{Vol}_{T_xL}$ for all $x \in L$. There is a natural extension of this definition to singular calibrated submanifolds using the language of Geometric Measure Theory and rectifiable currents [12, §II.1]. The key property of calibrated submanifolds (even singular ones) is that they are *homologically volume minimizing* [12, Thm. II.4.2]. In particular, any calibrated submanifold is automatically *minimal*, *i.e.* has vanishing mean curvature.

Let $z_1 = x_1 + iy_1, \cdots, z_n = x_n + iy_n$ be standard complex coordinates on $\mathbb{C}^n$ equipped with the Euclidean metric. Let

$$\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge d\overline{z}_j = \sum_{j=1}^{n} dx_j \wedge dy_j,$$

be the standard symplectic 2-form on $\mathbb{C}^n$. Define a complex n-form Ω on $\mathbb{C}^n$ by

$$\Omega = dz_1 \wedge \cdots \wedge dz_n. \tag{2.1}$$

Then $\mathrm{Re}\,\Omega$ and $\mathrm{Im}\,\Omega$ are real n-forms on $\mathbb{C}^n$. $\mathrm{Re}\,\Omega$ is a calibration on $\mathbb{C}^n$ whose calibrated submanifolds we call *special Lagrangian submanifolds* of $\mathbb{C}^n$, or SLG n-folds for short. There is a natural extension of special Lagrangian geometry to any Calabi-Yau manifold M by replacing Ω with the natural parallel holomorphic $(n, 0)$-form on M. Special Lagrangian submanifolds appear to play an important role in a number of interesting geometric properties of Calabi-Yau manifolds, *e.g.* Mirror Symmetry [47, 48].

Let $f : L \to \mathbb{C}^n$ be a Lagrangian immersion of the oriented n-manifold L, and Ω be the standard holomorphic $(n, 0)$-form defined in 2.1. Then $f^*\Omega$ is a complex n-form on L satisfying $|f^*\Omega| = 1$ [12, p. 89]. Hence we may write

$$f^*\Omega = e^{i\theta}\,\mathrm{Vol}_L \quad \text{on} \quad L, \tag{2.2}$$

for some *phase function* $e^{i\theta} : L \to \mathbb{S}^1$. We call $e^{i\theta}$ the *phase of the oriented Lagrangian immersion* f. L is a SLG n-fold in $\mathbb{C}^n$ if and only if the phase function $e^{i\theta} \equiv 1$. Reversing the orientation of L changes the sign of the phase function $e^{i\theta}$. The differential $d\theta$ is a well-defined closed 1-form on L which satisfies

$$d\theta = \iota_H\omega, \tag{2.3}$$

where H is the mean curvature vector of L. In particular, 2.3 implies that a connected component of L is minimal if and only if the phase function $e^{i\theta}$ is constant. 2.3 may also be restated as

$$H = -J\nabla\theta, \tag{2.4}$$

where J and ∇ denote the standard complex structure and gradient on $\mathbb{C}^n$ respectively. For a general Lagrangian submanifold it is not possible to find a global lift of the $\mathbb{S}^1$ valued phase function $e^{i\theta}$ to a real function θ, although of course such a lift always exists locally. However, we will always be interested in Lagrangian submanifolds for which such a global lift θ does exist. In this case we call $\theta : L \to \mathbb{R}$ the *Lagrangian angle* of L. In particular, any Lagrangian submanifold which is sufficiently close to a special Lagrangian submanifold will have a globally well-defined Lagrangian angle θ.

Special Legendrian submanifolds and special Lagrangian cones

For any compact oriented embedded (but not necessarily connected) submanifold $\Sigma \subset \mathbb{S}^{2n-1}(1) \subset \mathbb{C}^n$ define the *cone on* Σ,

$$C(\Sigma) = \{tx : t \in \mathbb{R}^{\geqslant 0}, x \in \Sigma\}.$$

A cone C in $\mathbb{C}^n$ (that is a subset invariant under dilations) is *regular* if there exists Σ as above so that $C = C(\Sigma)$, in which case we call Σ the *link* of the cone C. $C'(\Sigma) := C(\Sigma) - \{0\}$ is an embedded smooth submanifold, but $C(\Sigma)$ has an isolated singularity at 0 unless Σ is a totally geodesic sphere. Sometimes it will also be convenient to allow Σ to be just immersed not embedded, in which case $C'(\Sigma)$ is no longer embedded. Then we call $C(\Sigma)$ an *almost regular* cone.

Let r denote the radial coordinate on $\mathbb{C}^n$ and let X be the Liouville vector field

$$X = \frac{1}{2} r \frac{\partial}{\partial r} = \frac{1}{2} \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y_j}.$$

The unit sphere $\mathbb{S}^{2n-1}$ inherits a natural contact form

$$\gamma = \iota_X \omega|_{\mathbb{S}^{2n-1}} = \sum_{j=1}^{n} x_j dy_j - y_j dx_j \Big|_{\mathbb{S}^{2n-1}}$$

from its embedding in $\mathbb{C}^n$.

There is a one-to-one correspondence between regular Lagrangian cones in $\mathbb{C}^n$ and Legendrian submanifolds of $\mathbb{S}^{2n-1}$. The Lagrangian angle or the phase of a Lagrangian cone in $\mathbb{C}^n$ is homogeneous of degree 0. We define the Lagrangian angle of a Legendrian submanifold Σ of $\mathbb{S}^{2n-1}$ to be the restriction to $\mathbb{S}^{2n-1}$ of the Lagrangian angle of the Lagrangian cone $C'(\Sigma)$. We call a submanifold Σ of $\mathbb{S}^{2n-1}$ *special Legendrian* if the cone over Σ, $C'(\Sigma)$ is special Lagrangian in $\mathbb{C}^n$. In other words, Σ is special Legendrian if and only if its Lagrangian phase is identically 1 or its Lagrangian angle is identically 0 modulo 2π.

A special Legendrian submanifold of $\mathbb{S}^{2n-1}$ is minimal, that is it has mean curvature $H = 0$. Conversely, up to rotation by a constant phase $e^{i\theta}$ any connected minimal Legendrian submanifold of $\mathbb{S}^{2n-1}$ is special Legendrian. Using this language the goal of our paper is to construct special Legendrian immersions of (orientable) $n-1$ manifolds Σ into $\mathbb{S}^{2n-1}$.

Since any special Legendrian curve in $\mathbb{S}^3$ is geodesic and therefore a union of great circles, the only SLG cones in $\mathbb{C}^2$ are the union of SLG 2-planes. Hence the first non-trivial SLG cones/special Legendrian submanifolds appear when $n = 3$.

3 Cohomogeneity one special Legendrian submanifolds of $\mathbb{S}^{2n-1}$

In this section we discuss the building blocks for all the gluing constructions that follow. These building blocks are the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds where $p + q = n$. When $(p, q) = (1, 2)$ these special Legendrians were first constructed in [13, 14] and studied in more detail in our work on gluing constructions of higher genus special Legendrian cones in $\mathbb{C}^3$ [16]. For general (p, q), $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrians were first considered by Castro-Li-Urbano [5] and then studied in more detail by the authors in [17] for use as building blocks in the gluing constructions detailed in [18, 19].

All these building blocks are cohomogeneity one, in the sense that the generic orbit of the group action has codimension one in the resulting special Legendrian submanifolds. Because of the symmetries imposed, $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrian submanifolds are controlled by a family of first order nonlinear ODE rather than PDE. For each (p, q) there is a real one-parameter family of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian cylinders $X_\tau : \mathbb{R} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1} \to \mathbb{S}^{2p+2q-1}$. Geometrically, the parameter τ controls the maximum of the absolute value of the curvature that occurs on the cylinder X_τ and this value tends to infinity as $\tau \to 0$. Hence as $\tau \to 0$, the family X_τ must degenerate in some way. In all our cases as $\tau \to 0$, X_τ approaches a necklace of equatorial $n - 1$ spheres. In this sense our building blocks are reminiscent of building blocks used in other gluing constructions—Delaunay surfaces in the construction of CMC surfaces in $\mathbb{R}^3$ and Delaunay/Fowler metrics in the construction of constant scalar curvature metrics. The fact that the family X_τ degenerates to a union of very simple geometric objects is fundamental to our construction.

For a given value of $p + q = n$, but different values of p and q, the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds all approach a necklace of equatorial $n-1$ spheres as $\tau \to 0$. For different values of (p, q) these give rise to different kinds of spherical necklaces, in which the geometry of the transition regions that connects two adjacent almost spherical regions (and thus the relative positioning of adjacent almost spherical regions) changes. We will describe the geometry of the different kinds of transition regions in detail below.

Another important feature of our gluing constructions is that we exploit fully the availability of various discrete symmetries to simplify the later analysis. Hence below we will give a detailed description of the symmetries (both continuous and discrete) the building blocks X_τ enjoy.

We begin with an overview of the building blocks in the case where $(p, q) = (1, 2)$. In this case we are dealing with special Legendrian surfaces and so the building blocks are easier to visualize than in the higher-dimensional cases.

Special Legendrian surfaces in $\mathbb{S}^5$

Before describing the SO(2)-invariant special Legendrian surfaces in $\mathbb{S}^5$, we recall some general results about special Legendrian surfaces in $\mathbb{S}^5$.

If Σ is a special Legendrian surface in $\mathbb{S}^5$ that is homeomorphic to S^2, then applying a result from one of Yau's very early papers [49] one can prove (see [14, Thm. 2.7]) that Σ must be totally geodesic. Hence the corresponding SL cone C is a special Lagrangian 3-plane.

The oldest (circa 1974) and simplest singular SL cone in $\mathbb{C}^3$ is the cone over a flat 2-torus in $\mathbb{S}^5$ (the so-called Legendrian Clifford torus) [15, Ex. 3.18]. In the period 2000-2005, integrable systems techniques were used by a number of authors to construct many new special Legendrian 2-tori. In the first instance, countably infinite families of special Legendrian 2-tori invariant under U(1)-subgroups of SU(3) were constructed by the first author in [13, 14] using classical integrable systems techniques. These methods yield explicit solutions in terms of elliptic functions and integrals. The existence of many special Legendrian 2-tori in $\mathbb{S}^5$, without any continuous symmetry, was proved by McIntosh and Carberry-McIntosh using more sophisticated algebro-geometric tools from integrable systems [4, 42]. Using the fact that these special Legendrian 2-tori can occur in continuous families of arbitrarily high dimension, the first author proved that these examples rapidly become geometrically very complex [15, Thms. C & D].

The integrable systems methods that are so effective in the genus one case have so far yielded no insight into the higher genus case. However, in 2007 the authors [16] used geometric PDE gluing methods to construct infinitely many closed special Legendrian surfaces in $\mathbb{S}^5$ of genus 4 and of any odd genus $g > 1$. These are currently the only known examples of special Legendrian surfaces of genus $g > 1$. The basic building blocks for the construction are the one-parameter family of SO(2)-invariant special Legendrian cylinders X_τ constructed in [13, 14] and that we now describe.

SO(2)-invariant special Legendrian surfaces in $\mathbb{S}^5$

If we write $z = (z_1, z') \in \mathbb{C} \times \mathbb{C}^2$ then the SO(2)-action in question is given by

$$A.z = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \begin{pmatrix} z_1 \\ z' \end{pmatrix},$$

for $A \in \mathrm{SO}(2)$. In other words, SO(2) acts on $\mathbb{C}^3 = \mathbb{C} \times \mathbb{C}^2$ trivially on the first factor $\mathbb{C}$ and via its standard complex linear action on the $\mathbb{C}^2$ factor.

We are interested in finding special Legendrian surfaces that are unions of SO(2) orbits. Since the surface is Legendrian we are interested only in the isotropic orbits of SO(2), i.e. the orbits on which the contact 1-form γ restricts to zero. The generic isotropic SO(2)-orbit of a point in $\mathbb{S}^5$ is a round circle, $\mathbb{S}^1(r)$, of radius $r \in (0, 1]$. There are also singular orbits that are just points (where the radius r of the circle shrinks to zero). Hence the SO(2)-action is cohomogeneity one on Legendrian surfaces—that is, the generic orbit of the action is of codimension one in the Legendrian surface.

As a result the special Legendrian PDE reduce to a first-order nonlinear system of complex ODE (see [16, 3.18]). By studying this system of ODE, one can show that up to SU(3) motions there is a one-parameter family of SO(2)-invariant special Legendrian cylinders $X_\tau : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^5$ for τ in some finite interval $[-\tau_{\max}, \tau_{\max}]$. The real parameter τ controls the radius of the smallest SO(2)-orbit that occurs in X_τ. In particular, the radius of the smallest SO(2)-orbit goes to zero as $\tau \to 0$ and hence the X_τ exhibit some geometric degeneration as $\tau \to 0$.

For each nonzero τ, the special Legendrian cylinders $X_\tau : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^5 \subset \mathbb{C}^3$ are periodic. Each fundamental domain of X_τ approaches an equatorial two-sphere as $\tau \to 0$ except in the vicinity of two antipodal points. We call each region approaching an equatorial two-sphere an *almost spherical region*. Hence each almost spherical region has an associated equatorial two-sphere. Each almost spherical region connects to the almost spherical regions in the two adjacent fundamental domains via *transition regions* located close to these two antipodal points. Each transition region contains a subregion that approaches a Lagrangian catenoid of size $\sqrt{\tau}$. This behaviour is somewhat reminiscent of the Delaunay cylinders.

The period $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau} \in \mathrm{SU}(3)$ of the cylinder has matrix

$$\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau} = \begin{pmatrix} e^{2i\widehat{\mathsf{p}}_\tau} & 0 & 0 \\ 0 & e^{-i\widehat{\mathsf{p}}_\tau} & 0 \\ 0 & 0 & e^{-i\widehat{\mathsf{p}}_\tau} \end{pmatrix}, \tag{3.1}$$

where

$$\widehat{\mathsf{p}}_\tau = \frac{\pi}{2} - \tau \log \tau + O(\tau) \tag{3.2}$$

as $\tau \to 0$.

As $\tau \to 0$, almost spherical regions tend to equatorial two-spheres, while a transition region connecting neighboring almost spherical regions tends to a point of intersection of the corresponding equatorial two-spheres. Since $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau} \to \mathrm{diag}(-1, -i, -i)$, the intersection of the equatorial two-spheres is orthogonal. This is consistent with the fact that the Lagrangian catenoids are asymptotic to two orthogonal two-planes in $\mathbb{C}^2$. When $\widehat{\mathsf{p}}_\tau$ is a rational multiple of π, X_τ factors through a special Legendrian embedding of the torus.

The gluing construction now proceeds by fusing g copies of the tori we just described to obtain closed Legendrian surfaces of genus g. These initial Legendrian surfaces are approximately special Legendrian and get corrected to be exactly special Legendrian by using PDE methods, that are described in detail later. The technical aspects of constructions of this type are significantly simplified when the maximal possible symmetry is imposed. Thus the full symmetry group of the basic building blocks X_τ plays a central role in our constructions as we will also describe in detail later.

$\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant special Legendrian surfaces in $\mathbb{S}^{2p+2q-1}$

To generalize our gluing methods to construct higher dimensional special Legendrian submanifolds, building blocks analogous to the SO(2)-invariant special

Legendrian cylinders X_τ are needed in higher dimensions. For our purposes the most natural generalizations of these special Legendrian cylinders are the $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrian cylinders in $\mathbb{S}^{2(p+q)-1}$ first studied by Castro-Li-Urbano in [5] and later, more extensively by the current authors in [17]. Because in dimension n there are $[\frac{n}{2}]$ different possible building blocks available, there are many different ways to attempt gluing constructions of higher dimensional special Legendrian submanifolds. In this article we will describe how to use both the $\mathrm{SO}(n-1)$-invariant and the $\mathrm{SO}(p)\times\mathrm{SO}(p)$-invariant special Lagrangian cones as the basic building blocks for two different gluing constructions of higher dimensional special Legendrian submanifolds [18, 19].

To generalize the $\mathrm{SO}(2)$-invariant special Legendrian cylinders X_τ it is natural to seek special Legendrian submanifolds of $\mathbb{S}^{2n-1}$ that are invariant under an appropriate action that is cohomogeneity one for Legendrian submanifolds.

When $n\geqslant 4$, very natural generalizations of the above $\mathrm{SO}(2)$-action on $\mathbb{C}^3$ are given as follows. First choose positive integers p and q (without loss of generality we assume that $p\leqslant q$) with $p+q=n$ and split $\mathbb{C}^n$ as $\mathbb{C}^p\times\mathbb{C}^q$. Then $\mathrm{SO}(p)\times\mathrm{SO}(q)$ acts on $\mathbb{C}^p\times\mathbb{C}^q$ via the product of the two standard complex linear actions of $\mathrm{SO}(p)$ and $\mathrm{SO}(q)$ on the $\mathbb{C}^p$ and $\mathbb{C}^q$ factors respectively. That is, if we write $z=(w',z')\in\mathbb{C}^p\times\mathbb{C}^q$ then $\mathrm{SO}(p)\times\mathrm{SO}(q)$ acts by

$$(A,B).z=\begin{pmatrix}A & \mathbf{0}\\ \mathbf{0} & B\end{pmatrix}\begin{pmatrix}w'\\ z'\end{pmatrix},$$

for $A\in\mathrm{SO}(p)$ and $B\in\mathrm{SO}(q)$. When $p=1$ and $q=2$ this reduces to the previous $\mathrm{SO}(2)$-action on $\mathbb{C}^3$. Clearly, $\mathrm{SO}(p)\times\mathrm{SO}(q)$ is a subgroup of $\mathrm{SO}(p+q)$ and since this is naturally a subgroup of $\mathrm{SU}(p+q)$ it preserves the holomorphic $(p+q,0)$-form Ω on $\mathbb{C}^{p+q}$.

As previously, since we are interested in Legendrian submanifolds that are unions of orbits then we are only interested in the isotropic orbits. When $p=1$ and $q=n-1\geqslant 2$ the generic isotropic $\mathrm{SO}(n-1)$-orbit for a point in $\mathbb{S}^{2n-1}(1)$ is a round sphere $\mathbb{S}^{n-2}(r)$ of radius $r\in(0,1]$. In particular, this $\mathrm{SO}(n-1)$-action is cohomogeneity one on Legendrian submanifolds. The action has singular orbits which are just points; these correspond to the $r\to 0$ limit of the spheres above.

When $p>1$, the generic isotropic $\mathrm{SO}(p)\times\mathrm{SO}(q)$-orbit for a point in $\mathbb{S}^{2p+2q-1}(1)$ is a product of two round spheres

$$\mathbb{S}^{p-1}(r_1)\times\mathbb{S}^{q-1}(r_2),$$

where

$$r_1^2+r_2^2=1.$$

In particular, again this action is cohomogeneity one for Legendrian submanifolds. When $p\neq q$, the action has two types of singular orbits: one diffeomorphic to $\mathbb{S}^{p-1}$ as the radius $r_2\to 0$ and another diffeomorphic to $\mathbb{S}^{q-1}$ as the radius $r_1\to 0$. When $p=q$, both singular orbits are diffeomorphic to $\mathbb{S}^{p-1}$ and are in fact isometric.

Since each $\mathrm{SO}(p)\times\mathrm{SO}(q)$ action has cohomogeneity one on Legendrian submanifolds of $\mathbb{S}^{2(p+q)-1}$, it is natural to expect that $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special

Legendrian submanifolds of $\mathbb{S}^{2p+2q-1} \subset \mathbb{C}^{p+q}$ should be determined by a first-order system of nonlinear ODE generalizing equation 3.18 of [16]. This is indeed the case and the equations governing $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrians were first written down and studied by Castro-Li-Urbano in [5]. Later the current authors discovered these equations independently by looking directly for higher dimensional generalizations of the ODE system of [16], before becoming aware of Castro-Li-Urbano's work.

The fundamental ODE system for $\mathbf{SO}(p)\times\mathbf{SO}(q)$-invariant special Legendrians.

Given integers p and q satisfying $1 \leqslant p \leqslant q$, and $q \geqslant 2$ we define $n = p + q$. We define round cylinders of type (p, q), $\mathrm{Cyl}^{p,q}_I$, by

$$\mathrm{Cyl}^{p,q}_I := \begin{cases} I \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}, & \text{if } p > 1; \\ I \times \mathbb{S}^{n-2}, & \text{if } p = 1, \end{cases} \tag{3.3}$$

where $I \subset \mathbb{R}$ is an interval which we omit in the notation when $I = \mathbb{R}$.

Let $\mathbf{w} = \mathbf{w}(t) = (w_1, w_2) \in \mathbb{C}^2$ be a solution of the system

$$\overline{w}_1 \dot{w}_1 = -\overline{w}_2 \dot{w}_2 = \overline{w}_1^p \overline{w}_2^q, \tag{3.4}$$

with initial condition $\mathbf{w}(0) = \mathbf{w}_0 \in \mathbb{S}^3$. Then one can prove that the map

$$X_{\mathbf{w}_0} : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2n-1}$$

given by

$$\begin{aligned} X_{\mathbf{w}_0}(t, \sigma_1, \sigma_2) &= (w_1(t)\sigma_1, w_2(t)\sigma_2), && \text{if } p > 1; \\ X_{\mathbf{w}_0}(t, \sigma) &= (w_1(t), w_2(t)\sigma), && \text{if } p = 1, \end{aligned}$$

is a $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersion. When $(p, q) = (1, 2)$, the ODEs 3.4 reduce to equation 3.18 of [16] as expected.

The one-parameter family of $\mathbf{SO}(p) \times \mathbf{SO}(q)$-invariant special Legendrian cylinders X_τ.

Recently the authors made a detailed study of the geometry of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian submanifolds with a view to using them as basic building blocks for gluing constructions of higher dimensional special Legendrian submanifolds [17].

The ODEs 3.4 have two conserved quantities $I_1 = |\mathbf{w}|$ and $I_2 = \mathrm{Im}(w_1^p w_2^q)$. If $I_1(\mathbf{w}) = 1$, then the range of values of $I_2 = \mathrm{Im}(w_1^p w_2^q)$ is $[-2\tau_{\max}, 2\tau_{\max}]$, where $2\tau_{\max} = \sqrt{\frac{p^p q^q}{n^n}}$. Solutions of 3.4 with the same values of I_1 and I_2 are the same up to translations in t and the action of M_x on $\mathbb{C}^2$ for some $x \in \mathbb{R}$, where $M_x \in \mathrm{U}(2)$ is given by

$$M_x = \begin{pmatrix} e^{ix/p} & 0 \\ 0 & e^{-ix/q} \end{pmatrix}.$$

Using this one can see that up to the action of $\mathrm{SU}(p+q)$ there is a one-parameter family of distinct $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrian immersions $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ parametrized by the value 2τ of the conserved quantity $I_2 = \mathrm{Im}(w_1^p w_2^q) \in [-2\tau_{\max}, 2\tau_{\max}]$.

Given $\tau \in [-\tau_{\max}, \tau_{\max}]$ we choose initial data $\mathbf{w}_\tau \in \mathbb{C}^2$ satisfying $I_1(\mathbf{w}_\tau) = 1$ and $I_2(\mathbf{w}_\tau) = -2\tau$, so that $\mathbf{w}_\tau$ depends smoothly on τ for $\tau \in (-\tau_{\max}, \tau_{\max})$ and satisfies $\mathbf{w}_{-\tau} = \overline{\mathbf{w}}_\tau$. We will also denote the unique solution of 3.4 with initial condition $\mathbf{w}_\tau(0) = \mathbf{w}_\tau$ by $\mathbf{w}_\tau(t)$. The one-parameter family of $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrian immersions that we obtain from $\mathbf{w}_\tau(t)$, we denote X_τ where $\tau \in [-\tau_{\max}, \tau_{\max}]$.

Since $\mathbf{w}_{-\tau} = \overline{\mathbf{w}}_\tau$ it follows that $X_{-\tau} = \overline{X}_\tau$ and hence that the image of X_0 is real. The image of $X_0 : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ is contained in the totally real equatorial sphere $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q} \subset \mathbb{C}^{p+q}$ and misses only two antipodal points (in the case $p = 1$) or two orthogonal equatorial subspheres $\mathbb{S}^{p-1}, \mathbb{S}^{q-1} \subset \mathbb{S}^{p+q-1}$ (for $p > 1$).

In order to describe how to construct our initial almost special Legendrian submanifolds we need to describe the geometry of X_τ in greater detail, in particular the (singular) limit as $\tau \to 0$. There are substantial differences between the cases $p = 1$ and $p > 1$, and the case $p = q$ is also special. Therefore we describe these cases separately.

General features of the symmetries of X_τ

The symmetries of our $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant building blocks X_τ will play a fundamental role in our gluing constructions. Analyzing these symmetries (including discrete ones) will allow us to impose certain symmetries throughout our entire gluing construction. The imposition of these symmetries allows us to simplify some of the most technical parts of the gluing construction. Hence we will describe the symmetries of X_τ in detail. In describing the family X_τ as $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant, we have been thinking of them as subsets of $\mathbb{C}^n$. For a more detailed discussion of the symmetries of X_τ it is more convenient to treat X_τ as an immersion, to consider appropriate actions on the domain and target separately and then to talk about the equivariance properties of X_τ with respect to the domain and target actions.

Let p and q be positive integers satisfying $1 \leqslant p \leqslant q$, $q \geqslant 2$ and let $n = p+q$. We define two subgroups of $\mathrm{Isom}(\mathrm{Cyl}^{p,q})$, the group of isometries of the domain of X_τ,

$$\mathrm{Sym}(X_\tau) := \{\mathsf{M} \in \mathrm{Isom}(\mathrm{Cyl}^{p,q}) \,|\, \exists\, \widetilde{\mathsf{M}} \in \mathrm{O}(2n) \text{ s.t. } \widetilde{\mathsf{M}} \circ X_\tau = X_\tau \circ \mathsf{M}\}, \tag{3.5}$$

and $\mathrm{Per}(X_\tau) \subset \mathrm{Sym}(X_\tau)$ defined by

$$\mathrm{Per}(X_\tau) := \{\mathsf{M} \in \mathrm{Isom}(\mathrm{Cyl}^{p,q}) \,|\, X_\tau \circ \mathsf{M} = X_\tau\}. \tag{3.6}$$

We also define a subgroup $\widetilde{\mathrm{Sym}}(X_\tau) \subset \mathrm{O}(2n)$ by

$$\widetilde{\mathrm{Sym}}(X_\tau) := \{\widetilde{\mathsf{M}} \in \mathrm{O}(2n) \mid \widetilde{\mathsf{M}} \circ X_\tau = X_\tau \circ \mathsf{M} \text{ for some } \mathsf{M} \in \mathrm{Sym}(X_\tau)\}. \tag{3.7}$$

Using the fact that any Legendrian submanifold of $\mathbb{S}^{2n-1}$ that is not totally geodesic is linearly full [15, Lemma 3.13], one can see that if $\widetilde{\mathsf{M}}_1, \widetilde{\mathsf{M}}_2 \in \mathrm{O}(2n)$ and $\widetilde{\mathsf{M}}_1 \circ X_\tau = \widetilde{\mathsf{M}}_2 \circ X_\tau$ for some $\tau \neq 0$, then $\widetilde{\mathsf{M}}_1 = \widetilde{\mathsf{M}}_2$. Hence given $\mathsf{M} \in \mathrm{Sym}(X_\tau)$ there exists a unique element $\widetilde{\mathsf{M}} \in \mathrm{O}(2n)$ such that $\widetilde{\mathsf{M}} \circ X_\tau = \mathsf{M} \circ X_\tau$. Therefore we have a well-defined map $\rho : \mathrm{Sym}(X_\tau) \to \widetilde{\mathrm{Sym}}(X_\tau)$ given by $\mathsf{M} \mapsto \widetilde{\mathsf{M}}$. ρ is readily seen to be a homomorphism whose kernel is exactly the subgroup $\mathrm{Per}(X_\tau)$ defined above. Hence $\mathrm{Per}(X_\tau)$ is a normal subgroup of $\mathrm{Sym}(X_\tau)$ and $\widetilde{\mathrm{Sym}}(X_\tau) \cong \mathrm{Sym}(X_\tau)/\mathrm{Per}(X_\tau)$.

Recall that $\mathrm{Isom}(\mathbb{R}) = \mathbb{R} \rtimes \mathrm{O}(1)$ where $\mathbb{R}$ acts on itself by translations. For $x \in \mathbb{R}$ we denote translation by x by T_x,

$$\mathsf{T}_x : t \mapsto t + x.$$

Also we denote the generator of $\mathrm{O}(1)$ by $\underline{\mathsf{T}}$,

$$\underline{\mathsf{T}} : t \mapsto -t,$$

and the reflection of $\mathbb{R}$ fixing x by $\underline{\mathsf{T}}_x := \mathsf{T}_{2x} \circ \underline{\mathsf{T}}$,

$$\underline{\mathsf{T}}_x : t \mapsto 2x - t.$$

If (M, h) is a Riemannian manifold then by the *metric cylinder on* M we mean the manifold $\mathbb{R} \times M$ endowed with the product metric $g = dt^2 + h$. We always have an inclusion of isometry groups $\mathrm{Isom}(\mathbb{R}) \times \mathrm{Isom}(M, h) \subseteq \mathrm{Isom}(\mathbb{R} \times M, g)$ (and usually we have equality of these groups).

For $p = 1$, the metric cylinder $\mathrm{Cyl}^{1,n-1} = \mathbb{R} \times \mathbb{S}^{n-2}$ has isometry group $\mathrm{Isom}(\mathbb{R}) \times \mathrm{O}(n-1)$. In particular, the group $\mathrm{O}(n-1)$ acts isometrically on the cylinder $\mathbb{R} \times \mathbb{S}^{n-2}$ mapping each cross-section $\{t\} \times \mathbb{S}^{n-2}$ to itself. $\mathrm{O}(n-1)$ also acts isometrically on $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ as the direct product of the trivial action on $\mathbb{C}$ and the standard complex linear action of $\mathrm{O}(n-1)$ on $\mathbb{C}^{n-1}$. It is easy to check from the definition of $X_\tau : \mathbb{R} \times \mathbb{S}^{n-2} \to \mathbb{S}^{2n-1}$ that it is equivariant with respect to these two $\mathrm{O}(n-1)$ actions. Hence $\mathrm{O}(n-1)$ is a subgroup of both $\mathrm{Sym}(X_\tau)$ and $\widetilde{\mathrm{Sym}}(X_\tau)$. This extends the $\mathrm{SO}(n-1)$-invariance used to construct X_τ in the first place.

Note that $\mathrm{O}(n-1) \subset \mathrm{O}(n) \subset \mathrm{U}(n)$, so that $\mathrm{O}(n-1)$ acts on $\mathbb{C}^n$ by holomorphic isometries, i.e. isometries that preserve the standard complex structure J on $\mathbb{C}^n$. While we have $\mathrm{SO}(n-1) \subset \mathrm{SO}(n) \subset \mathrm{SU}(n)$, $\mathrm{O}(n-1) \nsubseteq \mathrm{SU}(n)$ and so not every element in $\mathrm{O}(n-1)$ acts as a special unitary transformation of $\mathbb{C}^n$. More precisely, $\widetilde{\mathsf{M}} \in \mathrm{O}(n-1)$ acts as a special unitary transformation of $\mathbb{C}^n$ if and only if $\det(\widetilde{\mathsf{M}}) = +1$, i.e. if and only if $\widetilde{\mathsf{M}} \in \mathrm{SO}(n-1)$.

Similarly, if $p > 1$ and $p \neq q$, then the metric cylinder $\mathrm{Cyl}^{p,q}$ has isometry group $\mathrm{Isom}(\mathbb{R}) \times \mathrm{O}(p) \times \mathrm{O}(q)$. In particular, the group $\mathrm{O}(p) \times \mathrm{O}(q)$ acts isometrically on the cylinder $\mathrm{Cyl}^{p,q}$ mapping each cross-section to itself. $\mathrm{O}(p) \times \mathrm{O}(q)$ also acts isometrically on $\mathbb{C}^{p+q} = \mathbb{C}^p \times \mathbb{C}^q$ as the direct product of the standard complex linear actions of $\mathrm{O}(p)$ and $\mathrm{O}(q)$ on $\mathbb{C}^p$ and $\mathbb{C}^q$ respectively. It is easy to check from the definition of $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2p+2q-1}$ that it is $\mathrm{O}(p) \times \mathrm{O}(q)$ equivariant with

respect to these two $\mathrm{O}(p)\times\mathrm{O}(q)$ actions (extending the $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariance used to construct X_τ). Hence $\mathrm{O}(p)\times\mathrm{O}(q)$ is a subgroup of both $\mathrm{Sym}(X_\tau)$ and $\widetilde{\mathrm{Sym}}(X_\tau)$.

As in the case $p=1$, $\mathrm{O}(p)\times\mathrm{O}(q)\subset\mathrm{O}(n)\subset\mathrm{U}(n)$, so that $\mathrm{O}(p)\times\mathrm{O}(q)$ acts by holomorphic isometries on $\mathbb{C}^n$. While we have $\mathrm{SO}(p)\times\mathrm{SO}(q)\subset\mathrm{SO}(n)\subset\mathrm{SU}(n)$, $\mathrm{O}(p)\times\mathrm{O}(q)\not\subseteq\mathrm{SU}(n)$. Hence there are elements of $\mathrm{O}(p)\times\mathrm{O}(q)$ that do not act on $\mathbb{C}^n$ as special unitary transformations of $\mathbb{C}^n$. More precisely, if $\widetilde{\mathsf{M}}=(\widetilde{\mathsf{M}}_1,\widetilde{\mathsf{M}}_2)\in\mathrm{O}(p)\times\mathrm{O}(q)$ then $\widetilde{\mathsf{M}}$ acts as a special unitary transformation of $\mathbb{C}^n$ if and only if $\det(\widetilde{\mathsf{M}}_1)\det(\widetilde{\mathsf{M}}_2)=+1$.

When $p=q$, the isometry group of $\mathrm{Cyl}^{p,p}$ is larger than $\mathrm{Isom}(\mathbb{R})\times\mathrm{O}(p)\times\mathrm{O}(p)$ because of the presence of an additional isometry that exchanges the two $p-1$ spheres. This extra symmetry will be discussed in detail when we study the geometry of X_τ in the case $p=q$.

The geometry of X_τ when $p=1$.

We begin by describing the geometry of X_τ in the simplest case $p=1$. The immersion X_τ induces a metric g on the cylinder $\mathbb{R}\times\mathbb{S}^{n-2}$ given by

$$g=|w_2|^{2(n-2)}dt^2+|w_2|^2g_{\mathbb{S}^{n-2}},\tag{3.8}$$

where $\mathbf{w}_\tau=(w_1,w_2)$. If we reparametrize the curve $\mathbf{w}_\tau(t)$ by a new time parameter $\tilde{t}$ chosen so that $d\tilde{t}^2=|w_2|^{2(n-2)}dt^2$, then g becomes a warped product metric on $\mathbb{R}\times\mathbb{S}^{n-2}$.

$X_0:\mathbb{R}\times\mathbb{S}^{n-2}\to\mathbb{S}^{2n-1}$ is an embedding whose image is the totally real equatorial sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n\subset\mathbb{C}^n$ minus the two antipodal points $\pm e_1$. However, for $0<|\tau|<\tau_{\max}$, the real-valued function $|w_2|^2$ (and therefore the metric g induced by X_τ) is periodic of period $2\mathsf{p}_\tau>0$. The period $2\mathsf{p}_\tau$ is a smooth function of τ for $0<|\tau|<\tau_{\max}$ that tends to ∞ as $\tau\to0$. Moreover, the immersion X_τ is $2\mathsf{p}_\tau$-periodic in the sense that

$$\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}\circ X_\tau=X_\tau\circ\mathsf{T}_{2k\mathsf{p}_\tau},\quad\text{for }\ k\in\mathbb{Z},\tag{3.9}$$

where T_x denotes translation by x on $\mathbb{R}$ and $\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}=(\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau})^k\in\mathrm{SU}(n)$ and $\widehat{\mathsf{p}}_\tau\in\mathbb{R}$ will be defined shortly (see 3.10 and 3.11). We call $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ the *rotational period of* X_τ. Our choice of initial data for X_τ ensures that for each $k\in\mathbb{Z}$ the function $|w_2|^2$ has a maximum at $2k\mathsf{p}_\tau$, a minimum at $(2k+1)\mathsf{p}_\tau$, and is decreasing on $(2k\mathsf{p}_\tau,(2k+1)\mathsf{p}_\tau)$ and increasing on $((2k+1)\mathsf{p}_\tau,(2k+2)\mathsf{p}_\tau)$–see Figure 1.

Since X_τ depends smoothly on τ, as $\tau\to0$ the immersion X_τ approaches the equatorial embedding X_0 uniformly on compact subsets of $\mathbb{R}\times\mathbb{S}^{n-2}$. In particular, for τ small, X_τ is close to X_0 on a neighborhood $S[0]$ of $\{0\}\times\mathbb{S}^{n-2}$ in the fundamental domain $(-\mathsf{p}_\tau,\mathsf{p}_\tau)\times\mathbb{S}^{n-2}$. Hence the image of $S[0]$ under X_τ is close to the image of $S[0]$ under X_0, and the latter is the totally real equatorial sphere $\mathbb{S}^{n-1}\subset\mathbb{R}^n\subset\mathbb{C}^n$ minus the union of two small antipodal disks centered at $\pm e_1$.

More generally we define the neighborhood $S[k]$ of $\{2k\mathsf{p}_\tau\}\times\mathbb{S}^{n-2}$ in the fundamental domain $((2k-1)\mathsf{p}_\tau,(2k+1)\mathsf{p}_\tau)\times\mathbb{S}^{n-2}$ by $S[k]:=\mathsf{T}_{2k\mathsf{p}_\tau}(S[0])$. From

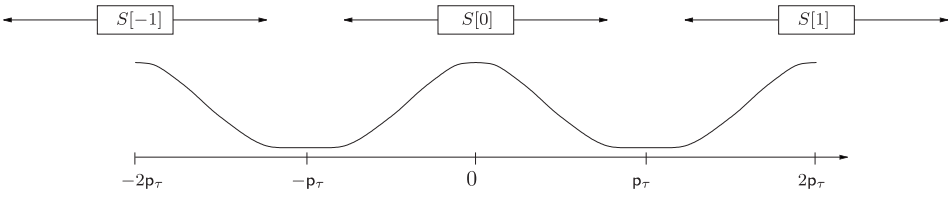


Figure 1: The integration scheme

the $2\mathsf{p}_\tau$-periodicity of X_τ, it follows that X_τ is close to $\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \circ X_0$ on $S[k]$ and its image is close to the equatorial sphere $\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}(\mathbb{S}^{n-1})$ (again minus two small antipodal disks). We call the regions $S[k]$, *almost spherical regions* of X_τ and the equatorial sphere $\mathbb{S}[k] := \widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}(\mathbb{S}^{n-1})$ the *approximating sphere* to $S[k]$.

Each almost spherical region $S[k]$ connects to the almost spherical regions $S[k-1]$ and $S[k+1]$ in the two adjacent fundamental domains via *transition regions* whose image under X_τ is located close to the antipodal points $\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}(\pm e_1)$. As $\tau \to 0$, each transition region contains a subregion approaching a Lagrangian catenoid of dimension $n-1$ and size $\tau^{1/(n-1)}$ (see the discussion following 3.33 for a description of the Lagrangian catenoid in $\mathbb{C}^n$). In particular, the transition regions that arise when $p = 1$ and $n > 3$ are the obvious generalizations of the transition regions that occurred in the SO(2)-invariant case. See Figure 2 for a schematic illustration of the intrinsic geometry of X_τ in the case $p = 1$.

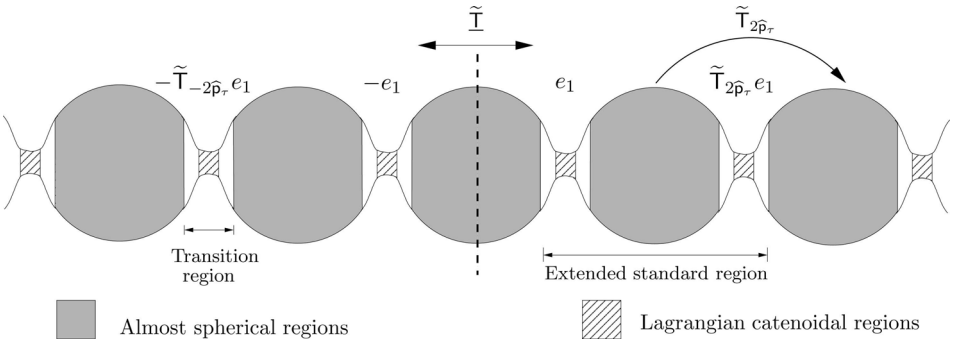


Figure 2: Schematic presentation of the intrinsic geometry of a special Legendrian cylinder X_τ with small τ and $p = 1$

The rotational period of X_τ when $p = 1$

To specify fully the rotational period of X_τ described by 3.9 we define the angular period

$$2\widehat{\mathsf{p}}_\tau := \arg w_1(2\mathsf{p}_\tau), \tag{3.10}$$

and a one-parameter subgroup $\widetilde{\mathsf{T}}_x$ of $\mathrm{SU}(n)$ by

$$\widetilde{\mathsf{T}}_x = \begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix/(n-1)}\,\mathrm{Id}_{n-1} \end{pmatrix}. \tag{3.11}$$

The angular period $2\widehat{\mathsf{p}}_\tau$ is a smooth function of τ for $0 < |\tau| < \tau_{\max}$ that satisfies

$$\lim_{\tau\to 0} \widehat{\mathsf{p}}_\tau = \frac{\pi}{2}. \tag{3.12}$$

As $\tau \to 0$, almost spherical regions tend to equatorial $n-1$ spheres, while a transition region connecting neighboring almost spherical regions tends to a point of intersection of the corresponding equatorial $n-1$ spheres. It follows from 3.11 and 3.12 that as $\tau \to 0$

$$\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \to \begin{pmatrix} -1 & 0 \\ 0 & e^{-i\pi/(n-1)}\,\mathrm{Id}_{n-1} \end{pmatrix}. \tag{3.13}$$

Hence in the $\tau \to 0$ limit, the real n-planes in $\mathbb{C}^n$ associated to the almost spherical region $S[0]$ and the almost spherical region $S[1]$ are $\mathbb{R}\oplus\mathbb{R}^{n-1}$ and $\mathbb{R}\oplus e^{-i\pi/(n-1)}\mathbb{R}^{n-1}$ respectively. This is consistent with the fact that the Lagrangian catenoid in $\mathbb{C}^{n-1}$ is asymptotic to the union of two $n-1$ planes (which up to rotation we can take to be) $\mathbb{R}^{n-1}$ and $e^{-i\pi/(n-1)}\mathbb{R}^{n-1}$.

Other symmetries of X_τ for $p = 1$.

The rotational period of X_τ is one of several discrete symmetries that X_τ enjoys. Since these discrete symmetries are fundamental to our gluing constructions we describe them all in detail. In addition to the $\mathrm{O}(n-1)$ equivariance already discussed, the immersions X_τ have discrete symmetries involving the $\mathrm{Isom}(\mathbb{R})$ factor of the isometry group of the cylinder $\mathbb{R}\times\mathbb{S}^{n-2}$. One symmetry—the rotational period of X_τ discussed in 3.9—corresponds to the isometry $\mathsf{T}_{2\mathsf{p}_\tau} \in \mathrm{Isom}(\mathbb{R})$, i.e. to translation in the cylinder $\mathbb{R}\times\mathbb{S}^{n-2}$ by a multiple of the period $2\mathsf{p}_\tau$ of the metric g induced by X_τ. A second symmetry corresponds to the reflection $\underline{\mathsf{T}} \in \mathrm{Isom}(\mathbb{R})$. A third symmetry corresponding to $\underline{\mathsf{T}}_{\mathsf{p}_\tau} \in \mathrm{Isom}(\mathbb{R})$ is a consequence of the existence of the first two symmetries.

To describe the additional discrete symmetries X_τ enjoys we define isometries $\widetilde{\underline{\mathsf{T}}}$, and $\widetilde{\underline{\mathsf{T}}}_x$ of $\mathbb{C}^n$ corresponding to the isometries $\underline{\mathsf{T}}, \underline{\mathsf{T}}_x \in \mathrm{Isom}(\mathbb{R})$ defined earlier. We define $\widetilde{\underline{\mathsf{T}}} \in \mathrm{O}(2n)$ by

$$\widetilde{\underline{\mathsf{T}}}(z_1, z_2, \cdots, z_n) = (-\overline{z}_1, \overline{z}_2, \cdots, \overline{z}_n), \tag{3.14}$$

and $\widetilde{\underline{\mathsf{T}}}_x \in \mathrm{O}(2n)$ by

$$\widetilde{\underline{\mathsf{T}}}_x = \widetilde{\mathsf{T}}_{2x} \circ \widetilde{\underline{\mathsf{T}}}. \tag{3.15}$$

Note that $\underline{\mathsf{T}}, \mathsf{T}_x \in \mathrm{Isom}(\mathbb{R})$ commute with every element in $\mathrm{O}(n-1) \subset \mathrm{Isom}(\mathbb{R}\times\mathbb{S}^{n-2})$, since they act on different factors of $\mathbb{R}\times\mathbb{S}^{n-2}$. Similarly, it is straightforward

to verify that $\widetilde{\underline{\mathsf{T}}}$ and $\widetilde{\mathsf{T}}_x$ also commute with every element in $\mathrm{O}(n-1) \subset \mathrm{O}(n) \subset \mathrm{Isom}(\mathbb{C}^n)$. Also since $\underline{\mathsf{T}}$ is reflection in the origin in $\mathbb{R}$, we have the commutation relation $\underline{\mathsf{T}} \circ \mathsf{T}_x \circ \underline{\mathsf{T}} = \mathsf{T}_{-x}$. Analogously, we find that $\widetilde{\underline{\mathsf{T}}} \circ \widetilde{\mathsf{T}}_x \circ \widetilde{\underline{\mathsf{T}}} = \widetilde{\mathsf{T}}_{-x}$. Hence $\widetilde{\underline{\mathsf{T}}}$ and $\widetilde{\mathsf{T}}_x$ together generate a subgroup of $\mathrm{O}(2n)$ isomorphic to $\mathrm{O}(2) \simeq \mathbb{S}^1 \rtimes \mathbb{Z}_2$, where $\widetilde{\underline{\mathsf{T}}}$ generates the $\mathbb{Z}_2$ factor and acts by inversion on the $\mathbb{S}^1$ factor.

Note also that $\widetilde{\underline{\mathsf{T}}}_x^*(\omega) = -\omega$, where ω is the standard symplectic form on $\mathbb{C}^n$, and therefore $\widetilde{\underline{\mathsf{T}}}_x^*(J) = -J$. Hence $\widetilde{\underline{\mathsf{T}}}_x$ acts on $\mathbb{C}^n$ as an *anti-holomorphic* isometry.

For $0 < |\tau| < \tau_{\max}$, the additional discrete symmetries that X_τ enjoys are:

$$\widetilde{\underline{\mathsf{T}}}_{k\widehat{\mathsf{p}}_\tau} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}}_{k\mathsf{p}_\tau}, \quad \text{for} \quad k \in \mathbb{Z}. \tag{3.16}$$

Notice that when $k = 0$, 3.16 specializes to

$$\widetilde{\underline{\mathsf{T}}} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}}. \tag{3.17}$$

Geometrically, 3.16 expresses the reflection symmetry that X_τ has about the $\mathrm{SO}(n-1)$ orbits of extremal radius (maximal radius when k is even and minimal radius when k is odd) in X_τ.

Structure of the symmetry groups of X_τ for $p = 1$

To describe the structure of the group of symmetries of X_τ more completely, we need to recall some elementary group theory. We will use the following notions a number of times in our discussion of symmetry groups.

Direct products, central products and semidirect products of groups.

Let $K, N \subset G$ be any two subsets of a group G. We write $KN = \{kn \,|\, k \in K, n \in N\} \subset G$. If both K and N are subgroups of G, then KN is a subgroup if and only if $KN = NK$ [22, p.22]. Moreover, N is a normal subgroup of KN if and only if $kNk^{-1} = N$ for all $k \in K$ and similarly for K. In particular, if N centralizes K, i.e. every element of N commutes with every element of K, then clearly $KN = NK$ and both K and N are normal subgroups of KN. If additionally $K \cap N \subseteq Z(G)$, where $Z(G)$ denotes the center of G, then we say that KN is the *central product of K and N identifying $K \cap N$* [8, p. 29]. If in fact $K \cap N = 1$, then KN is the *(internal) direct product of K and N*, and KN is isomorphic to the (external) direct product $K \times N$.

If K is a normal subgroup of KN then conjugation by any element of N gives a homomorphism $\psi : N \to \mathrm{Aut}(K)$ and the kernel of ψ is the centralizer of K in KN. If K is a normal subgroup of KN and $K \cap N = 1$, then KN is the *semidirect product of K by N*, $K \rtimes N$.

To apply the discussion above to study $\mathrm{Sym}(X_\tau)$, the group of all domain symmetries of X_τ, we can take K to be the subgroup of $\mathrm{Isom}(\mathbb{R})$ generated by $\underline{\mathsf{T}}$ and $\mathsf{T}_{2\mathsf{p}_\tau}$ and N to be $\mathrm{O}(n-1) \subset \mathrm{Isom}(\mathbb{R} \times \mathbb{S}^{n-2})$. K is an infinite discrete subgroup of $\mathrm{Isom}(\mathbb{R})$ isomorphic to the infinite dihedral group $\mathbf{D}_\infty \cong \mathbb{Z} \rtimes \mathbb{Z}_2$. Since

K centralizes N and $K \cap N = (\mathrm{Id})$, by the discussion above the set KN forms a group isomorphic to the direct product $K \times N \cong \mathbf{D}_\infty \times \mathrm{O}(n-1)$. Hence

$$\mathrm{Sym}(X_\tau) \cong \mathbf{D}_\infty \times \mathrm{O}(n-1).$$

The subgroup $\mathrm{Sym}_k(X_\tau) \subset \mathrm{Sym}(X_\tau)$ leaving the kth almost spherical region, $S[k]$, invariant is isomorphic to $\mathbb{Z}_2 \times \mathrm{O}(n-1)$, where $\mathbb{Z}_2$ is generated by $\underline{\mathsf{T}}_{2k\mathsf{p}_\tau}$. The subgroup of $\mathrm{Sym}(X_\tau)$ leaving *every* almost spherical region $S[k]$ invariant is equal to $\mathrm{O}(n-1) = \bigcap_{k\in\mathbb{Z}} \mathrm{Sym}_k(X_\tau)$.

To analyze the structure of $\widetilde{\mathrm{Sym}}(X_\tau) \subset \mathrm{O}(2n)$ we can take $K = \mathrm{O}(2)$ to be the subgroup of $\mathrm{O}(2n)$ generated by $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{T}}_x$ and $N = \mathrm{O}(n-1) \subset \mathrm{O}(n) \subset \mathrm{U}(n) \subset \mathrm{O}(2n)$. As we saw above, both $\underline{\widetilde{\mathsf{T}}}$ and $\underline{\mathsf{T}}_x$ commute with all elements of $\mathrm{O}(n-1)$ and hence K centralizes N. Also $K \cap N = (\mathrm{Id})$, and hence by the discussion above KN is isomorphic to $K \times N = \mathrm{O}(2) \times \mathrm{O}(n-1)$.

The group $\widetilde{\mathrm{Sym}}(X_\tau)$ contains the subgroup N but not the whole subgroup K. Instead, by 3.9 and 3.17 it contains the subgroup $\mathbf{D}$ of K generated by $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$. We will see later, that if the angular period $\widehat{\mathsf{p}}_\tau$ defined in 3.10 is a rational multiple of π, then $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ has finite order and $\mathbf{D}$ is isomorphic to a finite dihedral group $\mathbf{D}_m$ for some $m \in \mathbb{N}$. Otherwise $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ has infinite order and $\mathbf{D}$ is isomorphic to the infinite dihedral group $\mathbf{D}_\infty$ (in this case $\mathbf{D}$ is dense in K). Hence the symmetry group is

$$\widetilde{\mathrm{Sym}}(X_\tau) \cong \mathbf{D} \times \mathrm{O}(n-1).$$

It follows from the discussion above that every element $\widetilde{\mathsf{M}} \in \widetilde{\mathrm{Sym}}(X_\tau)$ can be written uniquely in the form

$$\widetilde{\mathsf{M}} = \underline{\widetilde{\mathsf{T}}}^i \circ \widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \circ A, \quad \text{where} \quad i \in \mathbb{Z}_2,\ k \in \mathbb{Z} \text{ and } A \in \mathrm{O}(n-1).$$

In particular, every element $\widetilde{\mathsf{M}} \in \widetilde{\mathrm{Sym}}(X_\tau)$ acts on $\mathbb{C}^n$ either by holomorphic (if $i = 0$) or anti-holomorphic (if $i = 1$) isometries. Also, $\widetilde{\mathsf{M}} \in \widetilde{\mathrm{Sym}}(X_\tau)$ is special unitary if and only if $i = 0$ and $A \in \mathrm{SO}(n-1)$.

Symmetries of the approximating spheres for $p = 1$

It will be important for our later analysis that the approximating sphere $\mathbb{S}[k]$ associated to the almost spherical region $S[k]$ inherits the symmetries of $S[k]$.

To make this precise, let $\mathrm{Sym}_k(X_\tau)$ denote the subgroup of the group of all domain symmetries that preserves the kth almost spherical region and let $\widetilde{\mathrm{Sym}}_k(X_\tau)$ denote the corresponding subgroup of the group of all target symmetries $\widetilde{\mathrm{Sym}}(X_\tau)$. We claim that the subgroup $\widetilde{\mathrm{Sym}}_k(X_\tau) \subset \mathrm{O}(2n)$ leaves the set $\mathbb{S}[k]$ invariant. We can see this as follows.

Recall from the discussion above that $\widetilde{\mathrm{Sym}}_k(X_\tau) = \langle \underline{\widetilde{\mathsf{T}}}_{2k\widehat{\mathsf{p}}_\tau} \rangle \times \mathrm{O}(n-1)$. In particular, $\widetilde{\mathrm{Sym}}_0(X_\tau) = \langle \underline{\widetilde{\mathsf{T}}} \rangle \times \mathrm{O}(n-1)$. Also recall the definition

$$\mathbb{S}[k] := \widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \mathbb{S}[0],$$

where $\mathbb{S}[0] = \mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$. Since $\mathrm{O}(n-1) \subset \mathrm{O}(n) \subset \mathrm{U}(2n)$ it is clear that $\mathrm{O}(n-1)$ preserves $\mathbb{S}[0]$. Also it follows immediately from the definition of $\widetilde{\underline{\mathsf{T}}}$ given in 3.14 that $\widetilde{\underline{\mathsf{T}}}$ leaves $\mathbb{S}[0]$ invariant. Hence $\widetilde{\mathrm{Sym}}_0(X_\tau) = \langle \widetilde{\underline{\mathsf{T}}} \rangle \times \mathrm{O}(n-1)$ leaves the central approximating sphere $\mathbb{S}[0]$ invariant.

Using the facts that (a) $\widetilde{\underline{\mathsf{T}}}_x$ commutes with $\mathrm{O}(n-1)$ for all $x \in \mathbb{R}$ and (b) $\mathrm{O}(n-1)$ leaves $\mathbb{S}[0]$ invariant, it follows immediately from the definition of $\mathbb{S}[k]$ that $\mathrm{O}(n-1)$ leaves $\mathbb{S}[k]$ invariant for any $k \in \mathbb{Z}$. Using the definition of $\widetilde{\underline{\mathsf{T}}}_x$ and the fact that $\widetilde{\underline{\mathsf{T}}} \circ \widetilde{\mathsf{T}}_x = \widetilde{\mathsf{T}}_{-x} \circ \widetilde{\underline{\mathsf{T}}}$, again it follows from the definition of $\mathbb{S}[k]$ that $\widetilde{\underline{\mathsf{T}}}_{2k\widehat{\mathsf{p}}_\tau}$ leaves $\mathbb{S}[k]$ invariant for any $k \in \mathbb{Z}$. Hence $\widetilde{\mathrm{Sym}}_k(X_\tau) = \langle \widetilde{\underline{\mathsf{T}}}_{2k\widehat{\mathsf{p}}_\tau} \rangle \times \mathrm{O}(n-1)$ leaves the kth approximating sphere $\mathbb{S}[k]$ invariant.

The geometry and symmetries of X_τ for $p > 1$.

The metric induced by X_τ on $\mathrm{Cyl}^{p,q}$ for $p > 1$ is

$$g = |w_1|^{2(p-1)}|w_2|^{2(q-1)}dt^2 + |w_1|^2 g_{\mathbb{S}^{p-1}} + |w_2|^2 g_{\mathbb{S}^{q-1}}, \tag{3.18}$$

where $\mathbf{w}_\tau = (w_1, w_2)$. If we reparametrize the curve $\mathbf{w}(t)$ by a new time parameter $\tilde{t}$ such that $d\tilde{t}^2 = |w_1|^{2(p-1)}|w_2|^{2(q-1)}dt^2$ then the metric g takes the form of a doubly warped product metric [43, §1.4, §3.2.4].

Now X_0 is an embedding whose image is the totally real equatorial sphere $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q} \subset \mathbb{C}^{p+q}$ minus a set S which is the union of two orthogonal equatorial subspheres $\mathcal{P} := \mathbb{S}^{p-1} \times \{0\} \subset \mathbb{R}^p \times \mathbb{R}^q \subset \mathbb{C}^{p+q}$ and $\mathcal{Q} := \{0\} \times \mathbb{S}^{q-1} \subset \mathbb{R}^p \times \mathbb{R}^q \subset \mathbb{C}^{p+q}$. As in the case $p = 1$, for $0 < |\tau| < \tau_{\max}$ the real-valued function $|w_2|^2$ (and hence the metric g induced by X_τ) is periodic of period $2\mathsf{p}_\tau > 0$. Moreover, if we define the one-parameter subgroup $\widetilde{\mathsf{T}}_x \in \mathrm{SU(p+q)}$ by

$$\widetilde{\mathsf{T}}_x = \begin{pmatrix} e^{ix/p}\,\mathrm{Id}_p & 0 \\ 0 & e^{-ix/q}\,\mathrm{Id}_q \end{pmatrix}, \tag{3.19}$$

where Id_p denotes the $p \times p$ identity matrix, and the angular period $\widehat{\mathsf{p}}_\tau$ by

$$2\widehat{\mathsf{p}}_\tau := p \arg w_1(2\mathsf{p}_\tau), \tag{3.20}$$

then the immersion X_τ is again $2\mathsf{p}_\tau$-periodic in the sense of 3.9. The period $2\mathsf{p}_\tau$ is a smooth function of τ for $0 < |\tau| < \tau_{\max}$ that tends to ∞ as $\tau \to 0$.

However, the geometry of a domain of periodicity of X_τ is more complicated than in the case $p = 1$. Recall that when $p = 1$ and τ is small, each fundamental domain of X_τ contains just one almost spherical region. We will see below that for $p > 1$, for small τ each fundamental domain of X_τ contains *two* almost spherical regions.

For $p > 1$, there are two radii $r_1 = |w_1|$ and $r_2 = |w_2|$ associated with each $\mathrm{SO}(p) \times \mathrm{SO}(q)$-orbit in X_τ, namely the radii of the spheres of dimensions $p-1$ and $q-1$ respectively (for $p = 1$, there is only one radius $r_2 = |w_2|$ associated with each $\mathrm{SO}(n-1)$-orbit that measures the radius of $\mathbb{S}^{n-2}(r_2)$). For $p > 1$, the initial data $\mathbf{w}_\tau(0)$ is chosen so that there are unique real numbers $\mathsf{p}_\tau^+, \mathsf{p}_\tau^- \in (0, \mathsf{p}_\tau)$ such that the second radius $r_2 = |w_2|$ is maximal at $t = -\mathsf{p}_\tau^-$, minimal at $t = \mathsf{p}_\tau^+$,

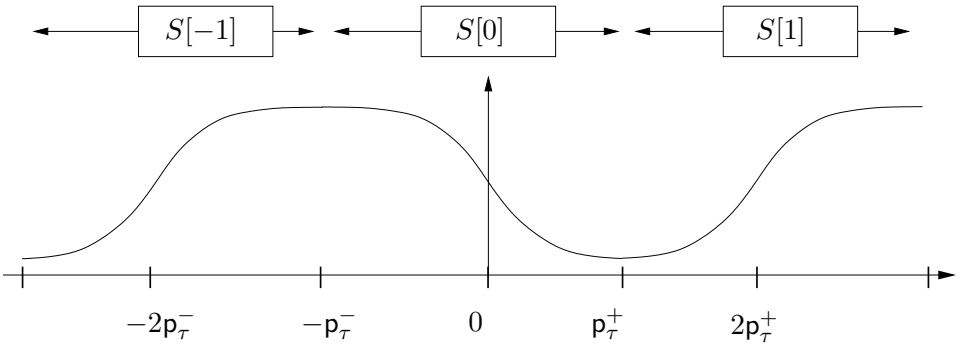


Figure 3: Profile of $|w_2|$ for $p > 1$

decreasing on $(-\mathsf{p}_\tau^-, \mathsf{p}_\tau^+)$, increasing on $(\mathsf{p}_\tau^+, \mathsf{p}_\tau + \mathsf{p}_\tau^+)$ and takes the value $\sqrt{\frac{q}{p+q}}$ at $t = 0$—see Figure 3 (for $p = 1$ the initial data $\mathbf{w}_\tau(0)$ is chosen so that the radius r_2 of $\mathbb{S}^{n-2}(r_2)$ is maximal at $t = 0$, minimal at $t = \pm\mathsf{p}_\tau$, increasing on $(-\mathsf{p}_\tau, 0)$ and decreasing on $(0, \mathsf{p}_\tau)$ as illustrated in Figure 1). We call p_τ^+ and p_τ^- the *partial-periods*, since $2\mathsf{p}_\tau = 2\mathsf{p}_\tau^+ + 2\mathsf{p}_\tau^-$. Using the last equality, it follows that the reflections $\underline{\mathsf{T}}_{\mathsf{p}_\tau^+}$ and $\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}$ satisfy

$$\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} = \mathsf{T}_{-2\mathsf{p}_\tau}, \quad \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} = \mathsf{T}_{2\mathsf{p}_\tau}. \tag{3.21}$$

Because $|w_2|$ has extrema at $t = \mathsf{p}_\tau^+$ and $t = -\mathsf{p}_\tau^-$, X_τ has associated reflectional symmetries

$$\widetilde{\underline{\mathsf{T}}}^{+} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau^+}, \tag{3.22a}$$

$$\widetilde{\underline{\mathsf{T}}}^{-} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}, \tag{3.22b}$$

for reflections $\widetilde{\underline{\mathsf{T}}}^{+}$ and $\widetilde{\underline{\mathsf{T}}}^{-} \in \mathrm{O}(2n)$ (defined below in 3.26 and 3.27) satisfying

$$\widetilde{\underline{\mathsf{T}}}^{+} \circ \widetilde{\underline{\mathsf{T}}}^{-} = \widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}, \quad \widetilde{\underline{\mathsf{T}}}^{-} \circ \widetilde{\underline{\mathsf{T}}}^{+} = \widetilde{\mathsf{T}}_{-2\widehat{\mathsf{p}}_\tau}. \tag{3.23}$$

As $\tau \to 0$, both partial-periods p_τ^+ and p_τ^- tend to ∞.

The geometry of X_τ for $p > 1$ and $p \neq q$ when τ is small

Since X_τ depends smoothly on τ, as $\tau \to 0$ the immersion X_τ approaches the equatorial embedding X_0 on compact subsets of $\mathrm{Cyl}^{p,q}$. In particular, for τ small, X_τ is close to X_0 on a neighborhood $S[0]$ of $\{0\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ in the domain $(-\mathsf{p}_\tau^-, \mathsf{p}_\tau^+) \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$. Hence for τ small the image of $S[0]$ under X_τ is close to the image of $S[0]$ under X_0, and the latter is the totally real equatorial sphere $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q} \subset \mathbb{C}^{p+q}$ minus the union of small tubular neighborhoods of the equatorial subspheres $\mathcal{P}$ and $\mathcal{Q}$. $S[0]$ is an *almost spherical region* of X_τ and $\mathbb{S}[0] = \mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q} \subset \mathbb{C}^{p+q}$ is the *approximating sphere* to $S[0]$.

Using 3.22a and 3.22b we see that

$$S[1] := \underline{\mathsf{T}}_{\mathsf{p}_\tau^+}(S[0]) \quad \text{and} \quad S[-1] := \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}(S[0])$$

are also almost spherical regions of X_τ whose approximating spheres are

$$\mathbb{S}[1] := \widetilde{\underline{\mathsf{T}}}^{+}(\mathbb{S}[0]), \quad \text{and} \quad \mathbb{S}[-1] := \widetilde{\underline{\mathsf{T}}}^{-}(\mathbb{S}[0])$$

respectively. Similarly, using the $2\mathsf{p}_\tau$-periodicity of X_τ, 3.9, we see that

$$S[2] := \mathsf{T}_{2\mathsf{p}_\tau}(S[0])$$

is another almost spherical region of X_τ whose approximating sphere is

$$\mathbb{S}[2] = \widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}(\mathbb{S}[0]).$$

Using the commutation relations 3.21 we see that we could also have defined $S[2]$ by $\mathsf{T}_{2\mathsf{p}_\tau}(S[0]) = \underline{\mathsf{T}}_{\mathsf{p}_\tau^+} \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}(S[0]) = \underline{\mathsf{T}}_{\mathsf{p}_\tau^+}(S[-1])$.

In general for any $k \in \mathbb{Z}$, we define the *almost spherical region*

$$S[2k] := \mathsf{T}_{2k\mathsf{p}_\tau}(S[0]) \quad \text{and} \quad S[2k+1] := \mathsf{T}_{2k\mathsf{p}_\tau} \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau^+}(S[0]) \tag{3.24}$$

whose *approximating spheres* are

$$\mathbb{S}[2k] := \widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}(\mathbb{S}[0]) \quad \text{and} \quad \mathbb{S}[2k+1] := \widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \circ \widetilde{\underline{\mathsf{T}}}^{+}(\mathbb{S}[0]) \tag{3.25}$$

respectively.

Hence we see that any domain of periodicity of X_τ contains *two* almost spherical regions. For example, $\mathrm{Cyl}^{p,q}_{(-2\mathsf{p}_\tau^-, 2\mathsf{p}_\tau^+)}$ contains all of the almost spherical region $S[0]$, as well as half of both of the almost spherical regions $S[-1]$ and $S[1]$. Moreover, while there is a translational symmetry relating $S[k]$ and $S[k+2]$ there is no such translational symmetry relating the adjacent almost spherical regions $S[k]$ and $S[k+1]$.

Each almost spherical region $S[k]$ of X_τ connects to the two neighboring almost spherical regions $S[k-1]$ and $S[k+1]$ via *transition regions* whose images under X_τ are located close to orthogonal equatorial subspheres of dimension $p-1$ and $q-1$. The geometry of the transition regions is also more complicated than in the case $p = 1$. When $p \neq q$, there are two fundamentally different types of transition regions.

To see this note that $\{-\mathsf{p}_\tau^-\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ corresponds to one of the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-orbits in X_τ for which the radius r_2 of the $q-1$ sphere is maximal and therefore the radius r_1 of the $p-1$ sphere is minimal. On the other hand, $\{\mathsf{p}_\tau^+\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ corresponds to an $\mathrm{SO}(p) \times \mathrm{SO}(q)$-orbit in X_τ for which the radius r_2 of the $q-1$ sphere is minimal and therefore the radius r_1 of the $p-1$ sphere is maximal. Hence for τ small, the regions where t is close to $-\mathsf{p}_\tau^-$ or to p_τ^+ are both highly curved regions of X_τ.

However, the geometry of the regions close to $t = -\mathsf{p}_\tau^-$ and $t = \mathsf{p}_\tau^+$ is different when $p \neq q$. The transition region containing $\{-\mathsf{p}_\tau^-\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ has a

subregion that for small τ approaches the product of $\mathbb{S}^{q-1}(1)$ with a p-dimensional Lagrangian catenoid of size $\tau^{1/p}$. We call this a *transition region of type* 1, since in this region the first radius r_1 is close to zero. The transition region containing $\{\mathsf{p}_\tau^+\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$, however, has a subregion that for small τ approaches the product of $\mathbb{S}^{p-1}(1)$ with a q-dimensional Lagrangian catenoid of size $\tau^{1/q}$. We call this a *transition region of type* 2, since in this transition region the second radius r_2 is close to zero.

We summarize our discussion of the geometry of X_τ for $p > 1$ as follows. For τ small, $\mathrm{Cyl}^{p,q}_{(-2\mathsf{p}_\tau^-, 2\mathsf{p}_\tau^+)}$, a domain of periodicity of X_τ, contains:

- one full almost spherical region $S[0]$—contained in $\mathrm{Cyl}^{p,q}_{(-\mathsf{p}_\tau^-, \mathsf{p}_\tau^+)}$,
- half of the almost spherical regions $S[-1]$ and $S[1]$—contained in $\mathrm{Cyl}^{p,q}_{(-2\mathsf{p}_\tau^-, -\mathsf{p}_\tau^-)}$ and $\mathrm{Cyl}^{p,q}_{(\mathsf{p}_\tau^+, 2\mathsf{p}_\tau^+)}$ respectively,
- two transition regions, one of type 1 and one of type 2—located close to $\{-\mathsf{p}_\tau^-\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ and $\{\mathsf{p}_\tau^+\} \times \mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ respectively.

The two half almost spherical regions are different because the one contained in $\mathrm{Cyl}^{p,q}_{(-2\mathsf{p}_\tau^-, -\mathsf{p}_\tau^-)}$ connects to the full almost spherical region $S[0]$ via a transition region of type 1, while the one contained in $\mathrm{Cyl}^{p,q}_{(\mathsf{p}_\tau^+, 2\mathsf{p}_\tau^+)}$ connects to $S[0]$ via a transition region of type 2.

The structure of $\mathrm{Sym}(X_\tau)$ for $p > 1$ and $p \neq q$

Recall that one presentation for the infinite dihedral group $\mathbf{D}_\infty$ is

$$\langle s, t \,|\, s^2 = 1,\, t^2 = 1 \rangle.$$

The commutation relations 3.21 for the reflection symmetries $\underline{\mathsf{T}}_{\mathsf{p}_\tau^+}$ and $\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}$ of $\mathrm{Cyl}^{p,q}$, show that $\underline{\mathsf{T}}_{\mathsf{p}_\tau^+}$ and $\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}$ together generate a subgroup of $\mathrm{Isom}(\mathbb{R})$ isomorphic to the infinite dihedral group $\mathbf{D}_\infty$. The index 2 infinite cyclic subgroup $\mathbf{C}_\infty$ of $\mathbf{D}_\infty$ generated by the element $\underline{\mathsf{T}}_{\mathsf{p}_\tau^+} \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau^-} = \mathsf{T}_{2\mathsf{p}_\tau}$ is isomorphic to $\mathbb{Z}$ and consists only of translations. Since $\mathbf{D}_\infty \subset \mathrm{Isom}(\mathbb{R})$ it is clear that $\mathbf{D}_\infty$ commutes with $\mathrm{Isom}(\mathbb{S}^p \times \mathbb{S}^q)$. Hence

$$\mathrm{Sym}(X_\tau) \cong \mathbf{D}_\infty \times \mathrm{O}(p) \times \mathrm{O}(q).$$

Since neither $\underline{\mathsf{T}}_{\mathsf{p}_\tau^+}$ nor $\underline{\mathsf{T}}_{-\mathsf{p}_\tau^-}$ preserve any of the almost spherical regions $S[k]$, the subgroup of $\mathrm{Sym}(X_\tau)$ preserving a given almost spherical region $S[k]$ is simply $\mathrm{O}(p) \times \mathrm{O}(q)$, which is also the subgroup of $\mathrm{Sym}(X_\tau)$ that preserves all almost spherical regions.

For concreteness, we give the following explicit expressions for $\underline{\widetilde{\mathsf{T}}}^+$ and $\underline{\widetilde{\mathsf{T}}}^-$

$$\underline{\widetilde{\mathsf{T}}}^+(z,w) = (e^{2i\alpha_\tau/p} e^{i\psi_1(2\mathsf{p}_\tau^+)} \overline{z}, e^{2i\alpha_\tau/q} e^{i\psi_2(2\mathsf{p}_\tau^+)} \overline{w}), \tag{3.26}$$

and

$$\underline{\widetilde{\mathsf{T}}}^-(z,w) = (e^{2i\alpha_\tau/p} e^{i\psi_1(-2\mathsf{p}_\tau^-)} \overline{z}, e^{2i\alpha_\tau/q} e^{i\psi_2(-2\mathsf{p}_\tau^-)} \overline{w}), \tag{3.27}$$

where $z \in \mathbb{C}^p$ and $w \in \mathbb{C}^q$. The precise definitions of the real valued functions α_τ, ψ_1 and ψ_2 are not important for our present discussion. Using 3.26 and 3.27 one can see immediately that $\underline{\widetilde{\mathsf{I}}}^+$ and $\underline{\widetilde{\mathsf{I}}}^-$ centralize $\mathrm{O}(p) \times \mathrm{O}(q)$. Also it follows from 3.26 and 3.27 that both $\underline{\widetilde{\mathsf{I}}}^+$ and $\underline{\widetilde{\mathsf{I}}}^-$ map the symplectic form $\omega \mapsto -\omega$ and hence both $\underline{\widetilde{\mathsf{I}}}^+$ and $\underline{\widetilde{\mathsf{I}}}^-$ are anti-holomorphic isometries. However, their product $\underline{\widetilde{\mathsf{I}}}^+ \circ \underline{\widetilde{\mathsf{I}}}^- = \widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ is special unitary.

If $\widehat{\mathsf{p}}_\tau$ is a rational multiple of π then, as we will see later, $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau} = \underline{\widetilde{\mathsf{I}}}^+ \circ \underline{\widetilde{\mathsf{I}}}^-$ has finite order and hence the group generated by the reflections $\underline{\widetilde{\mathsf{I}}}^+$ and $\underline{\widetilde{\mathsf{I}}}^-$ is isomorphic to a finite dihedral group $\mathbf{D}_m$ for some $m \in \mathbb{N}$. Otherwise $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ has infinite order and the group generated by $\underline{\widetilde{\mathsf{I}}}^+$ and $\underline{\widetilde{\mathsf{I}}}^-$ is isomorphic to the infinite dihedral group. Hence

$$\widetilde{\mathrm{Sym}}(X_\tau) \cong \mathbf{D} \times \mathrm{O}(p) \times \mathrm{O}(q),$$

where the dihedral group $\mathbf{D}$ is finite or infinite according to whether $\widehat{\mathsf{p}}_\tau$ is a rational multiple of π or not. For our gluing constructions we will always choose a value of τ so that $\widehat{\mathsf{p}}_\tau$ is a rational multiple of π and so $\widetilde{\mathrm{Sym}}(X_\tau)$ is isomorphic to $\mathbf{D}_m \times \mathrm{O}(p) \times \mathrm{O}(q)$ for some $m \in \mathbb{N}$, where $\mathbf{D}_m$ is generated by $\underline{\widetilde{\mathsf{I}}}^+$ and $\underline{\widetilde{\mathsf{I}}}^-$.

The geometry of X_τ for $p > 1$ and $p = q$.

If $p = q$, then X_τ enjoys additional discrete symmetries which we now describe.

In the first place, the isometry group of the cylinder $\mathrm{Cyl}^{p,p}$ contains an extra $\mathbb{Z}_2$ subgroup generated by exchange $\mathsf{E} \in \mathrm{Isom}(\mathbb{S}^{p-1} \times \mathbb{S}^{p-1}) \subset \mathrm{Isom}(\mathbb{R} \times \mathbb{S}^{p-1} \times \mathbb{S}^{p-1})$ of the two spherical factors

$$\mathsf{E}(t, \sigma_1, \sigma_2) = (t, \sigma_2, \sigma_1).$$

There is also a natural $\mathbb{Z}_2$ subgroup, $\langle \mathsf{E}' \rangle \subset \mathrm{Aut}(\mathrm{Isom}\,\mathbb{S}^{p-1} \times \mathrm{Isom}\,\mathbb{S}^{p-1}) = \mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p)$ generated by the involution E' defined by

$$\mathsf{E}'(A, B) = (B, A),$$

and an obvious isomorphism $\rho_E : \langle E \rangle \to \langle \mathsf{E}' \rangle \subset \mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p) \subset \mathrm{Aut}\,\mathrm{O}(2p)$ given by

$$\mathsf{E} \mapsto \mathsf{E}'.$$

It is readily checked that conjugation by E is an automorphism of $\mathrm{O}(p) \times \mathrm{O}(p)$ and in fact this automorphism is just E'. In particular, $\langle \mathsf{E} \rangle (\mathrm{O}(p) \times \mathrm{O}(p)) = (\mathrm{O}(p) \times \mathrm{O}(p)) \langle \mathsf{E} \rangle$ and hence from our previous discussions the set $\langle \mathsf{E} \rangle (\mathrm{O}(p) \times \mathrm{O}(p)) \subset \mathrm{Isom}(\mathbb{S}^{p-1} \times \mathbb{S}^{p-1})$ forms a group G. Moreover $\mathrm{O}(p) \times \mathrm{O}(p)$ is a normal subgroup of G and clearly $\mathrm{O}(p) \times \mathrm{O}(p) \cap \langle \mathsf{E} \rangle = (\mathrm{Id})$. Hence G is the semidirect product of $\mathrm{O}(p) \times \mathrm{O}(p)$ by $\langle \mathsf{E} \rangle \cong \mathbb{Z}_2$ where the twisting homomorphism $\rho : \langle \mathsf{E} \rangle \to \mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p)$ is ρ_E defined above, i.e. $G = (\mathrm{O}(p) \times \mathrm{O}(p)) \rtimes_{\rho_E} \langle \mathsf{E} \rangle$.

We have an injective homomorphism $\rho : G \subset \mathrm{Isom}(\mathbb{S}^{p-1} \times \mathbb{S}^{p-1}) \to \mathrm{O}(2p)$ determined by

$$\rho(A, B) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \rho(\mathsf{E}) = \underline{\widetilde{\mathsf{I}}},$$

where $\widetilde{\underline{\mathsf{T}}} \in \mathrm{O}(2p) \subset \mathrm{U}(2p)$ is the involution defined by

$$\widetilde{\underline{\mathsf{T}}}\,(z,w) = (w,z) \quad \text{where} \quad w,\, z \in \mathbb{C}^p. \tag{3.28}$$

Conjugation by $\widetilde{\underline{\mathsf{T}}}$ defines an order 2 element in $\mathrm{Aut}\,\mathrm{O}(2p)$ that preserves the subgroup $\mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p)$ and on this subgroup acts by exchange

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mapsto \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}.$$

We also define $\widetilde{\underline{\mathsf{S}}} \in \mathrm{O}(4p)$ by

$$\widetilde{\underline{\mathsf{S}}}\,(z,w) = e^{-i\pi/2p}(\overline{w},\overline{z}), \quad \text{where } w, z \in \mathbb{C}^p. \tag{3.29}$$

Using these new isometries we can rewrite $\widetilde{\underline{\mathsf{T}}}^{+}$ and $\widetilde{\underline{\mathsf{T}}}^{-}$ as

$$\widetilde{\underline{\mathsf{T}}}^{+} = \widetilde{\mathsf{T}}_{\widehat{\mathsf{p}}_\tau} \circ \widetilde{\underline{\mathsf{S}}} \circ \widetilde{\underline{\mathsf{T}}}, \tag{3.30}$$

and

$$\widetilde{\underline{\mathsf{T}}}^{-} = \widetilde{\mathsf{T}}_{-\widehat{\mathsf{p}}_\tau} \circ \widetilde{\underline{\mathsf{S}}} \circ \widetilde{\underline{\mathsf{T}}}, \tag{3.31}$$

respectively. When $p = q$ we also have the equalities $\mathsf{p}_\tau^- = \mathsf{p}_\tau^+ = \mathsf{p}_\tau/2$.

For $0 < |\tau| < \tau_{\max}$ the $\mathrm{O}(p) \times \mathrm{O}(p)$ equivariant special Legendrian immersion X_τ enjoys the following symmetries:

$$\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau} \circ X_\tau = X_\tau \circ \mathsf{T}_{2k\mathsf{p}_\tau}, \quad \text{for} \quad k \in \mathbb{Z}, \tag{3.32a}$$

$$\widetilde{\underline{\mathsf{T}}}^{+} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau/2}, \tag{3.32b}$$

$$\widetilde{\underline{\mathsf{T}}}^{-} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}}_{-\mathsf{p}_\tau/2}, \tag{3.32c}$$

$$\widetilde{\underline{\mathsf{T}}} \circ X_\tau = X_\tau \circ \underline{\mathsf{T}} \circ \mathsf{E}, \tag{3.32d}$$

$$\widetilde{\mathsf{T}}_{\widehat{\mathsf{p}}_\tau} \circ \widetilde{\underline{\mathsf{S}}} \circ X_\tau = X_\tau \circ \mathsf{T}_{\mathsf{p}_\tau} \circ \mathsf{E}. \tag{3.32e}$$

3.32d and 3.32e express the two additional symmetries that X_τ possesses when $p = q$ and both utilize the additional exchange isometry $\mathsf{E} \in \mathrm{Isom}(\mathrm{Cyl}^{p,p})$. 3.32d expresses an additional reflectional symmetry of X_τ about the $\mathrm{SO}(p) \times \mathrm{SO}(p)$-orbit for which the radii of both $p-1$ spheres are equal. This symmetry interchanges the type 1 and type 2 transition regions that in the $p = q$ case are actually isometric. Clearly such a symmetry cannot exist if $p \neq q$, since then type 1 and type 2 transition regions are not even diffeomorphic. The symmetry 3.32e is an immediate consequence of the symmetries 3.32b, 3.32d and the expression for $\widetilde{\underline{\mathsf{T}}}^{+}$ given in 3.30. 3.32e expresses a symmetry between one almost spherical region $S[k]$ and the adjacent almost spherical region $S[k+1]$ (as opposed to symmetry in 3.32a that relates $S[k]$ and $S[k+2]$). We see that the corresponding isometry of $\mathbb{S}^{4p-1}$ is not in $\mathrm{SU}(2p)$.

The structure of $\mathrm{Sym}(X_\tau)$ for $p = q$

It is straightforward to check that

$$\underline{\mathsf{T}}_{-\mathsf{p}_\tau/2} = (\underline{\mathsf{T}} \circ \mathsf{E}) \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \circ (\underline{\mathsf{T}} \circ \mathsf{E}).$$

It follows that all the domain symmetries described in 3.32 are generated by the two involutions $\underline{\mathsf{T}} \circ \mathsf{E}$ and $\underline{\mathsf{T}}_{\mathsf{p}_\tau/2}$. The group $\langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle$ is isomorphic to the infinite dihedral group $\mathbf{D}_\infty$. Any element of this group can be expressed as an alternating word in the two generators. The subgroup $\mathbf{D}'_\infty \subset \mathbf{D}_\infty$ generated by the two elements $\underline{\mathsf{T}}_{\mathsf{p}_\tau/2}$ and $\underline{\mathsf{T}}_{-\mathsf{p}_\tau/2} = (\underline{\mathsf{T}} \circ \mathsf{E}) \circ \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \circ (\underline{\mathsf{T}} \circ \mathsf{E})$ is also isomorphic to the infinite dihedral group, and corresponds to the symmetries that are shared with the case $p \neq q$. Alternatively, $\mathbf{D}'_\infty \subset \mathbf{D}_\infty$ is the subgroup consisting of all words containing an even number of copies of $\underline{\mathsf{T}} \circ \mathsf{E}$.

It is straightforward to see that conjugation by any element of $\mathbf{D}_\infty = \langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle \subset \mathrm{Isom}(\mathbb{R} \times \mathbb{S}^{p-1} \times \mathbb{S}^{p-1})$ preserves the subgroup $\mathrm{O}(p) \times \mathrm{O}(p) \subset \mathrm{Isom}(\mathbb{R} \times \mathbb{S}^{p-1} \times \mathbb{S}^{p-1})$. More precisely, if $\gamma \in \langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle$ then

$$\gamma(\mathsf{M}_1, \mathsf{M}_2)\gamma^{-1} = \begin{cases} (\mathsf{M}_1, \mathsf{M}_2) & \text{if } \gamma \text{ is a word containing an even number of copies of } \underline{\mathsf{T}} \circ \mathsf{E}, \\ (\mathsf{M}_2, \mathsf{M}_1) & \text{if } \gamma \text{ is a word containing an odd number of copies of } \underline{\mathsf{T}} \circ \mathsf{E}. \end{cases}$$

As in previous discussions, it follows that the set $\langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle\ \mathrm{O}(p) \times \mathrm{O}(p)$ is the same as the set $\mathrm{O}(p) \times \mathrm{O}(p)\ \langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle$ and hence that $\langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle\ \mathrm{O}(p) \times \mathrm{O}(p)$ is a group. Moreover, the subgroup $\mathrm{O}(p) \times \mathrm{O}(p)$ is a normal subgroup of this group, with $\mathrm{O}(p) \times \mathrm{O}(p) \cap \mathbf{D}_\infty = (\mathrm{Id})$.

From this we find that

$$\mathrm{Sym}(X_\tau) \cong (\mathrm{O}(p) \times \mathrm{O}(p)) \rtimes_\rho \mathbf{D}_\infty\,,$$

where the twisting homomorphism $\rho : \langle \underline{\mathsf{T}} \circ \mathsf{E}, \underline{\mathsf{T}}_{\mathsf{p}_\tau/2} \rangle \cong \mathbf{D}_\infty \to \mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p)$ is defined by

$$\rho(\gamma) = \begin{cases} \mathrm{Id} & \text{if } \gamma \text{ is a word containing an even number of copies of } \underline{\mathsf{T}} \circ \mathsf{E}, \\ \mathsf{E}' & \text{if } \gamma \text{ is a word containing an odd number of copies of } \underline{\mathsf{T}} \circ \mathsf{E}. \end{cases}$$

If follows that $\ker \rho$ is precisely the (normal) subgroup $\mathbf{D}'_\infty \subset \mathbf{D}_\infty$. Hence $\mathrm{Sym}(X_\tau)$ contains a subgroup isomorphic to $(\mathrm{O}(p) \times \mathrm{O}(p)) \times \mathbf{D}'_\infty$. This subgroup consists of the elements of the symmetry group of X_τ that can be expressed without using the exchange E, and corresponds to the symmetries that X_τ shares with the case $p \neq q$.

The subgroup $\mathrm{Sym}_k(X_\tau) \subset \mathrm{Sym}(X_\tau)$ preserving the kth almost spherical region $S[k]$ is the group generated by $\underline{\mathsf{T}}_{k\mathsf{p}_\tau} \circ \mathsf{E}$ and $\mathrm{O}(p) \times \mathrm{O}(p)$. It is straightforward to check that

$$\mathrm{Sym}_k(X_\tau) \cong (\mathrm{O}(p) \times \mathrm{O}(p)) \rtimes_\rho \mathbb{Z}_2,$$

where the homomorphism $\rho : \mathbb{Z}_2 \cong \langle \underline{\mathsf{T}}_{k\mathsf{p}_\tau} \circ \mathsf{E} \rangle \to \mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p)$ is the restriction of the homomorphism $\rho : \mathbf{D}_\infty \to \mathrm{Aut}\,\mathrm{O}(p) \times \mathrm{O}(p)$ defined above, i.e. $\rho(\underline{\mathsf{T}}_{k\mathsf{p}_\tau} \circ \mathsf{E}) = \mathsf{E}'$.

It follows that the subgroup of $\mathrm{Sym}(X_\tau)$ preserving every almost spherical region is $\mathrm{O}(p) \times \mathrm{O}(p)$.

One can check that the symmetries of the target of X_τ satisfy

$$\widetilde{\underline{\mathsf{T}}}^{-} = \widetilde{\underline{\mathsf{T}}} \circ \widetilde{\underline{\mathsf{T}}}^{+} \circ \widetilde{\underline{\mathsf{T}}}.$$

Hence the involutions $\widetilde{\underline{\mathsf{T}}}^{+}$ and $\widetilde{\underline{\mathsf{T}}}$ generate a subgroup $\mathbf{D} = \langle \widetilde{\underline{\mathsf{T}}}, \widetilde{\underline{\mathsf{T}}}^{+} \rangle \subset \widetilde{\mathrm{Sym}}(X_\tau)$. We will see later that $\widetilde{\underline{\mathsf{T}}}^{+} \circ \widetilde{\underline{\mathsf{T}}}$ has finite order if and only if $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$. In that case $\mathbf{D}$ is isomorphic to a finite dihedral group $\mathbf{D}_m$ for some $m \in \mathbb{N}$. Otherwise $\mathbf{D}$ is isomorphic to an infinite dihedral group. The set $\mathbf{D} \cdot \mathrm{O}(p) \times \mathrm{O}(p)$ coincides with the set $\mathrm{O}(p) \times \mathrm{O}(p) \cdot \mathbf{D}$, and hence forms a group, which coincides with $\widetilde{\mathrm{Sym}}(X_\tau)$. $\mathrm{O}(p) \times \mathrm{O}(p)$ is a normal subgroup of $\widetilde{\mathrm{Sym}}(X_\tau)$ and

$$\widetilde{\mathrm{Sym}}(X_\tau) \cong \mathrm{O}(p) \times \mathrm{O}(p) \rtimes_\rho \mathbf{D},$$

where the homomorphism $\rho : \mathbf{D} \to \mathrm{Aut}\, \mathrm{O}(p) \times \mathrm{O}(p)$ is given by

$$\rho(\gamma)(\mathsf{M}_1, \mathsf{M}_2) = \begin{cases} (\mathsf{M}_1, \mathsf{M}_2) & \text{if } \gamma \in \mathbf{D} \text{ is a word containing an even number of copies of } \widetilde{\underline{\mathsf{T}}}, \\ (\mathsf{M}_2, \mathsf{M}_1) & \text{if } \gamma \in \mathbf{D} \text{ is a word containing an odd number of copies of } \widetilde{\underline{\mathsf{T}}}. \end{cases}$$

Every element $\widetilde{\mathsf{M}} \in \widetilde{\mathrm{Sym}}(X_\tau)$ can be written in the form $\widetilde{\mathsf{M}} = \gamma \cdot \mathsf{M}$, where $\gamma \in \mathbf{D}$ and $\mathsf{M} \in \mathrm{O}(p) \times \mathrm{O}(p) \subset \mathrm{O}(2p)$. In particular, every element $\widetilde{\mathsf{M}} \in \widetilde{\mathrm{Sym}}(X_\tau)$ acts on $\mathbb{C}^{2p}$ either by holomorphic isometries (if γ is a word involving an even number of copies of $\widetilde{\underline{\mathsf{T}}}^{+}$) or anti-holomorphic isometries (if γ is a word involving an odd number of copies of $\widetilde{\underline{\mathsf{T}}}^{+}$). Notice that the subgroup $\widetilde{\mathrm{Sym}}_k(X_\tau) \subset \widetilde{\mathrm{Sym}}(X_\tau)$, corresponding to the subgroup $\mathrm{Sym}_k(X_\tau) \subset \mathrm{Sym}(X_\tau)$ of domain symmetries which preserve the kth almost spherical region $S[k]$, consists entirely of holomorphic isometries.

Symmetries of the approximating spheres for $p > 1$

For $p > 1$, the approximating sphere $\mathbb{S}[k]$ associated to the almost spherical region $S[k]$ of an $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant SL cylinder X_τ inherits the symmetries of $S[k]$, as was the case when $p = 1$. As previously, what we mean by this is that the subgroup $\widetilde{\mathrm{Sym}}_k(X_\tau) \subset \mathrm{O}(2n)$ leaves the set $\mathbb{S}[k]$ invariant. We can see this as follows.

First recall that

$$\mathbb{S}[k] = \begin{cases} \widetilde{\mathsf{T}}_{2l\widehat{\mathsf{p}}_\tau}\, \mathbb{S}[0], & \text{if } k = 2l,\ l \in \mathbb{Z}; \\ \widetilde{\mathsf{T}}_{2l\widehat{\mathsf{p}}_\tau} \circ \widetilde{\underline{\mathsf{T}}}^{+}\, \mathbb{S}[0], & \text{if } k = 2l+1,\ l \in \mathbb{Z}; \end{cases}$$

where $\mathbb{S}[0] = \mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q} \subset \mathbb{C}^{p+q}$.

When $p \neq q$, $\widetilde{\mathrm{Sym}}_k(X_\tau) = \mathrm{O}(p) \times \mathrm{O}(q)$ for any $k \in \mathbb{Z}$. Since $\mathrm{O}(p) \times \mathrm{O}(q) \subset \mathrm{O}(p+q) \subset \mathrm{U}(p+q)$, clearly $\mathrm{O}(p) \times \mathrm{O}(q)$ leaves the central approximating sphere

$\mathbb{S}[0]$ invariant. $\mathrm{O}(p)\times\mathrm{O}(q)$ invariance of $\mathbb{S}[k]$ for any $k\in\mathbb{Z}$ now follows using the definition of $\mathbb{S}[k]$ above and the fact that the group generated by $\underline{\widetilde{\mathrm{T}}}^{+}$ and $\underline{\widetilde{\mathrm{T}}}^{-}$ centralizes $\mathrm{O}(p)\times\mathrm{O}(q)$ (recalling that $\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}=\underline{\widetilde{\mathrm{T}}}^{+}\circ\underline{\widetilde{\mathrm{T}}}^{-}$).

When $p=q$, $\widetilde{\mathrm{Sym}}_k(X_\tau)=\mathrm{O}(p)\times\mathrm{O}(p)\rtimes\langle\underline{\widetilde{\mathrm{T}}}_{k\widehat{\mathsf{p}}_\tau}\rangle$, where $\underline{\widetilde{\mathrm{T}}}_{k\widehat{\mathsf{p}}_\tau}:=\widetilde{\mathsf{T}}_{2k\widehat{\mathsf{p}}_\tau}\circ\underline{\widetilde{\mathrm{T}}}$. Since the previous argument proves the invariance of $\mathbb{S}[k]$ under $\mathrm{O}(p)\times\mathrm{O}(p)$, it remains only to prove invariance of $\mathbb{S}[k]$ under $\underline{\widetilde{\mathrm{T}}}_{k\widehat{\mathsf{p}}_\tau}$. When $k=0$, invariance of $\mathbb{S}[0]$ under $\underline{\widetilde{\mathrm{T}}}$ follows immediately from the definition of $\underline{\widetilde{\mathrm{T}}}$ given in 3.28. For $k=2l$, $l\in\mathbb{Z}$, invariance of $\mathbb{S}[k]$ under $\underline{\widetilde{\mathrm{T}}}_{k\widehat{\mathsf{p}}_\tau}$ follows from the definition of $\mathbb{S}[k]$, the fact that $\underline{\widetilde{\mathrm{T}}}\circ\widetilde{\mathsf{T}}_x=\widetilde{\mathsf{T}}_{-x}\circ\underline{\widetilde{\mathrm{T}}}$ and the invariance of $\mathbb{S}[0]$ under $\underline{\widetilde{\mathrm{T}}}$. For $k=2l+1$, $l\in\mathbb{Z}$, invariance of $\mathbb{S}[k]$ under $\underline{\widetilde{\mathrm{T}}}_{k\widehat{\mathsf{p}}_\tau}$ follows using the facts above and additionally that $\underline{\widetilde{\mathrm{T}}}^{+}\circ\underline{\widetilde{\mathrm{T}}}^{-}=\widetilde{\mathsf{T}}_{2\widehat{\mathsf{p}}_\tau}$ and $\underline{\widetilde{\mathrm{T}}}^{-}\circ\underline{\widetilde{\mathrm{T}}}^{-}=\mathrm{Id}$. Hence $\widetilde{\mathrm{Sym}}_k(X_\tau)=\mathrm{O}(p)\times\mathrm{O}(p)\rtimes\langle\underline{\widetilde{\mathrm{T}}}_{k\widehat{\mathsf{p}}_\tau}\rangle$ leaves the kth approximating sphere $\mathbb{S}[k]$ invariant.

Lagrangian catenoids and a heuristic explanation for behaviour of X_τ as $\tau\to 0$.

In this section we give a heuristic explanation why one expects to see Lagrangian catenoids appearing in the transition regions of X_τ as $\tau\to 0$. To begin with recall the following result

Proposition 3.33 ([14, Thm A],[6, 23])**.** *Let Σ be any special Legendrian submanifold in $\mathbb{S}^{2n-1}$. For any $d\in\mathbb{R}$, let Σ_d denote the set*

$$\{zp\in\mathbb{C}^n:\ p\in\Sigma, z\in\mathbb{C}, \textit{with }\ \mathrm{Im}\, z^n=d, \arg z\in(0,\tfrac{\pi}{n})\}.$$

Then for $d\neq 0$, Σ_d is an asymptotically conical special Lagrangian submanifold of $\mathbb{C}^n$ with two ends asymptotic to the two SLG cones $C(\Sigma)$ and $e^{i\pi/n}C(\Sigma)$.

If we apply Proposition 3.33 to $\Sigma=\mathbb{S}^{n-1}\subset\mathbb{R}^n\subset\mathbb{C}^n$ then the ACSLG n-folds we obtain are known as the "Lagrangian catenoids" L. The Lagrangian catenoids L can be characterized in a number of ways. For example, they are the only non-flat SLG n-folds in $\mathbb{C}^n$ foliated by round $n-1$-spheres [6]. Also any nonsingular SLG n-fold in $\mathbb{C}^n$ invariant under the standard complex linear action of $\mathrm{SO}(n)$ on $\mathbb{C}^n\cong\mathbb{R}^n\times\mathbb{R}^n$ is (conjugate to a piece of) a Lagrangian catenoid.

The curves in $\mathbb{C}^*$ defined by $\mathrm{Im}\, z^n=d$ that appear in Proposition 3.33 can be characterized in terms of a single ODE for z. Namely, if $z:\mathbb{R}\to\mathbb{C}^*$ satisfies

$$\overline{z}\dot{z}=\overline{z}^n, \tag{3.34}$$

then $\frac{d}{dt}z^n=nz^{n-1}\dot{z}=n|z|^{2(n-1)}\in\mathbb{R}^+$ and hence $\mathrm{Im}(z^n)=d$ for some $d\in\mathbb{R}$. One can think of the ODE 3.34 as a degenerate case of the ODE system 3.4 for $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant special Legendrian submanifolds. In fact, if we consider the standard complex linear action of $\mathrm{SO}(n)$ on $\mathbb{C}^n$, then any isotropic $\mathrm{SO}(n)$-orbit can be written in the form $z.\mathbb{S}^{n-1}(1)$ for some $z\in\mathbb{C}$. Given an immersed curve $z:\mathbb{R}\to\mathbb{C}$, the immersion $X_z:\mathbb{R}\times\mathbb{S}^{n-1}\to\mathbb{C}^n$ given by $(t,\sigma)\mapsto z(t)\sigma$ is an

$\mathrm{SO}(n)$-invariant Lagrangian immersion. Moreover, the immersion X_z is special Lagrangian when z satisfies the ODE 3.34.

The above remarks are useful for understanding heuristically the behaviour of X_τ as $\tau \to 0$. When either $w_1 = 0$ or $w_2 = 0$ the $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbit $(w_1 \cdot \mathbb{S}^{p-1}, w_2 \cdot \mathbb{S}^{q-1})$ collapses from a product of spheres $\mathbb{S}^{p-1} \times \mathbb{S}^{q-1}$ to a single sphere $\mathbb{S}^{p-1}$ or $\mathbb{S}^{q-1}$ respectively. Hence the interesting singular behaviour of X_τ happens when either $|w_1|$ or $|w_2|$ gets close to zero. If $|w_1|$ is small (and hence $|w_2|$ is close to 1) the corresponding $\mathrm{SO}(p) \times \mathrm{SO}(q)$ orbit is metrically close to $\mathbb{S}^{p-1}(\epsilon) \times \mathbb{S}^{q-1}(1)$ for some small ϵ. Similarly, if $|w_2|$ is small then the orbit is close to $\mathbb{S}^{p-1}(1) \times \mathbb{S}^{q-1}(\epsilon)$ for some small ϵ.

Suppose we want to understand the behaviour of $\mathbf{w} = (w_1, w_2)$ near a point where $|w_1|$ is small. So let us suppose that at some point $w_1(0)$ is small and real and $w_2(0)$ is real and close to 1. The most interesting behaviour is captured by how w_1 changes close to $w_1(0)$ since this determines how the radius of the collapsing sphere $\mathbb{S}^{p-1}$ is changing. In fact from 3.4 it is easy to see that close to such a point w_2 changes much more slowly than w_1 (e.g. $\dot{w}_1(0) \sim \overline{w}_1^{p-1}$ whereas $\dot{w}_2(0) \sim -\overline{w}_1^{p}$.) So in a neighborhood of such a point the first approximation to $\mathbf{w}$ is that w_2 is almost constant at 1 while w_1 almost satisfies the ODE

$$\overline{w}_1 \dot{w}_1 = \overline{w}_1^p,$$

and hence $\mathrm{Im}\, w_1^p \sim d$ for some constant d. In fact, since we have $\mathrm{Im}\, w_1^p w_2^q = -2\tau$ then we must have $d \sim -2\tau$. So close to such a point metrically the curve of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-orbits approaches the product of a small Lagrangian catenoid in $\mathbb{C}^p$ with $\mathbb{S}^{q-1}(1) \subset \mathbb{R}^q \subset \mathbb{C}^q$. In the limiting picture what we see is two equatorial $\mathbb{S}^{n-1}$s intersecting along a common $\mathbb{S}^{q-1}$. In $\mathbb{C}^n$ we see the SLG cone $(\mathbb{R}^p \cup e^{i\pi/p}\mathbb{R}^p) \times \mathbb{R}^q \subset \mathbb{C}^p \times \mathbb{C}^q$, which is singular along the whole q-plane $\{0\} \times \mathbb{R}^q$.

Repeating the analysis close to a point where $|w_2|$ is small, we conclude that close to such a point the curve of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-orbits approaches the product of $\mathbb{S}^{p-1}(1) \subset \mathbb{R}^p \subset \mathbb{C}^p$ with a small Lagrangian catenoid in $\mathbb{C}^q$. In the limiting picture what we see is two equatorial $\mathbb{S}^{n-1}$s intersecting along a common $\mathbb{S}^{p-1}$.

Almost spherical regions and marked SL spheres.

In this section, we introduce the notion of a (p, q)-marked (equatorial) special Legendrian sphere. This notion is important because the approximating sphere of any almost spherical region of the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian cylinders X_τ as $\tau \to 0$ has the structure of a (p, q)-marked special Legendrian sphere.

$(1, n-1)$-marked SL spheres.

We saw that in the $\tau \to 0$ limit, the image of each almost spherical region of X_τ approaches the complement of two antipodal points p^+ and $p^- = -p^+$ inside an equatorial special Legendrian $n-1$ sphere SL; such an ordered triple (p^+, p^-, SL) we call a $(1, n-1)$-*marked special Legendrian sphere* (since X_τ has a preferred

orientation we want to keep track of the ordering of the two antipodal points p^+ and p^-). In the following subsection we will also introduce the concept of a (p,q)-marked special Legendrian sphere for general p and q. However, for the remainder of this subsection we will abbreviate $(1,n-1)$-marked as marked.

A marked special Legendrian sphere is obviously equivalent to a pair (l, SLG), where $SLG \subset \mathbb{C}^n$ is a special Lagrangian n-plane and $l = \langle p^+ \rangle_{\mathbb{R}} \subset SLG$ is an oriented real line in the n-plane SLG. For the central almost spherical region, $S[0]$, of X_τ this pair (l, SLG) consists of the standard real n-plane $\mathbb{R}^n \subset \mathbb{C}^n$ and the oriented line $l = \langle e_1 \rangle_{\mathbb{R}}$, where $e_1, \cdots, e_n$ is the standard oriented orthonormal basis of $\mathbb{R}^n$. The corresponding marked special Legendrian sphere is then clearly $(e_1, -e_1, \mathbb{S}^{n-1})$ where $\mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$. We call $\mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$ the *standard special Legendrian sphere* and $(e_1, -e_1, \mathbb{S}^{n-1})$ the *standard* $(1,n-1)$*-marked special Legendrian sphere*.

The group $\mathrm{SU}(n)$ acts transitively both on the set of all equatorial SL spheres and on the set of all marked SL spheres. In particular, every marked SL sphere is $\mathrm{SU}(n)$-equivalent to the standard marked SL sphere $(e_1, -e_1, \mathbb{S}^{n-1})$. The set of all equatorial SL $n-1$ spheres in $\mathbb{S}^{2n-1}$ is naturally identified with the special Lagrangian Grassmannian $\mathrm{SU}(n)/\mathrm{SO}(n)$. In particular, the subgroup of $\mathrm{SU}(n)$ fixing the standard SL sphere $\mathbb{S}^{n-1}$ is $\mathrm{SO}(n)$. A general element in $\mathrm{SO}(n)$ does not preserve a given marking of this standard SL sphere; the subgroup of $\mathrm{SO}(n)$ preserving $\langle e_1 \rangle_{\mathbb{R}}$ is clearly $\mathrm{SO}(n-1)$.

Hence the subgroup of $\mathrm{SU}(n)$ fixing the standard marked SL sphere $(e_1, -e_1, \mathbb{S}^{n-1})$ is the subgroup $\mathrm{SO}(n-1) \subset \mathrm{SO}(n) \subset \mathrm{SU}(n)$, as before acting trivially on the first factor $\mathbb{C}$ of $\mathbb{C}^n$ and via its standard complex linear action on $\mathbb{C}^{n-1}$ (this is the identity component of the group under which X_τ is invariant). The subgroup of $\mathrm{SU}(n)$ fixing an arbitrary marked special Legendrian sphere is then conjugate to $\mathrm{SO}(n-1)$, via any element of $\mathrm{SU}(n)$ that takes the arbitrary marked sphere to the standard one.

Since $\mathrm{SU}(n)$ acts transitively on the set of all marked SL spheres, with stabilizer conjugate to $\mathrm{SO}(n-1)$, the space of all marked SL spheres can be identified with the homogeneous space $\mathrm{SU}(n)/\mathrm{SO}(n-1)$. We can describe this homogeneous space $\mathrm{SU}(n)/\mathrm{SO}(n-1)$ as a fiber bundle over the special Lagrangian Grassmannian as follows. Given a fixed equatorial special Legendrian $n-1$ sphere SL, the set of all markings of SL can clearly be identified with points in the sphere SL itself. Hence we see that the space of all $(1,n-1)$-marked SL spheres is a S^{n-1} bundle over the special Lagrangian Grassmannian $\mathrm{SU}(n)/\mathrm{SO}(n)$. This fibration structure corresponds to the natural projection

$$\pi : \mathrm{SU}(n)/\mathrm{SO}(n-1) \to \mathrm{SU}(n)/\mathrm{SO}(n)$$

associated to the subgroup inclusions $\mathrm{SO}(n-1) \subset \mathrm{SO}(n) \subset \mathrm{SU}(n)$. Geometrically, the map π is the "forgetful map" sending a marked special Legendrian sphere (p^+, p^-, SL) to the underlying equatorial special Legendrian sphere SL. Since $\mathrm{SU}(n)/\mathrm{SO}(n-1)$ is an S^{n-1} bundle over $\mathrm{SU}(n)/\mathrm{SO}(n)$ it follows from the standard long exact sequence in homotopy associated to the fibration π that π gives an isomorphism

$$\pi_* : \pi_i(\mathrm{SU}(n)/\mathrm{SO}(n-1)) \to \pi_i(\mathrm{SU}(n)/\mathrm{SO}(n)) \quad \text{for } i = 0, \cdots, n-2.$$

In particular, since $\mathrm{SU}(n)/\mathrm{SO}(n)$ is connected and simply-connected, the space of marked SL spheres is also connected and simply-connected.

Note that the symmetry $\widetilde{\underline{\mathrm{I}}} \in \widetilde{\mathrm{Sym}}(X_\tau) \subset \mathrm{O}(2n)$ (defined in 3.14) preserves the standard special Legendrian sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$. $\widetilde{\underline{\mathrm{I}}}$ simultaneously reverses the orientation on $\mathbb{S}^{n-1}$ and exchanges the two marked points $\pm e_1$ of the standard marked special Legendrian sphere $(e_1, -e_1, \mathbb{S}^{n-1})$. Because $\widetilde{\underline{\mathrm{I}}}$ reverses both the orientation and the marking of the standard marked special Legendrian sphere it gives rise to a symmetry of the standard marked special Legendrian sphere even though $\widetilde{\underline{\mathrm{I}}} \notin \mathrm{SU}(n)$.

(p,q)-marked SL spheres for $p > 1$.

In the $\tau \to 0$ limit, the image of each almost spherical region of X_τ approaches the complement of two orthogonal oriented equatorial subspheres S_p and S_q of dimension $p-1$ and $q-1$ respectively, inside an equatorial special Legendrian sphere SL of dimension $p+q-1$ in $\mathbb{S}^{2(p+q)-1} \subset \mathbb{C}^{p+q}$. Since X_τ has a preferred orientation we want to keep track of the orientations of the two subspheres, S_p and S_q. We call such a triple (S_p, S_q, SL) of equatorial spheres a (p,q)-*marked special Legendrian sphere*.

A (p,q)-marked special Legendrian sphere is obviously equivalent to a triple of oriented real planes (P, Q, SLG), consisting of a special Lagrangian $p+q$-plane $SLG \subset \mathbb{C}^{p+q}$ and a pair of orthogonal oriented planes $P, Q \subset SLG$ of real dimension p and q respectively. Since P and Q are orthogonal planes of complementary dimensions, $Q = P^\perp$ and hence P determines Q (and vice versa). Hence each almost spherical region of X_τ determines a unique pair (P, SLG), where $P \subset SLG \subset \mathbb{C}^{p+q}$ is an oriented p-dimensional real subspace of the special Lagrangian $p+q$ plane SLG.

For the central almost spherical region of X_τ, $S[0]$, this pair (P, SLG) consists of

$$P = (\mathbb{R}^p, 0) \subset \mathbb{R}^p \times \mathbb{R}^q = \mathbb{R}^{p+q} = SLG \subset \mathbb{C}^{p+q}.$$

The corresponding (p,q)-marked special Legendrian sphere (S_p, S_q, SL) is then clearly

$$S_p = (\mathbb{S}^{p-1}, 0) \subset \mathbb{R}^p \times \mathbb{R}^q,\ S_q = (0, \mathbb{S}^{q-1}) \subset \mathbb{R}^p \times \mathbb{R}^q,\ SL = \mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q} \subset \mathbb{C}^{p+q}. \tag{3.35}$$

We call 3.35 the *standard* (p,q)*-marked special Legendrian sphere*. For the rest of this subsection marked SL sphere will be understood to mean a (p,q)-marked SL sphere.

The group $\mathrm{SU}(p+q)$ acts transitively on the set of all (p,q)-marked SL spheres (because $\mathrm{SU}(p+q)$ acts transitively on the space of special Lagrangian planes in $\mathbb{C}^{p+q}$ with stabilizer $\mathrm{SO}(p+q)$ and $\mathrm{SO}(p+q)$ acts transitively on the space of oriented p-planes in $\mathbb{R}^{p+q}$). In particular, every marked SL sphere is $\mathrm{SU}(p+q)$-equivalent to the standard marked SL sphere. The subgroup of $\mathrm{SU}(p+q)$ fixing the standard marked SL sphere is $\mathrm{SO}(p) \times \mathrm{SO}(q) \subset \mathrm{SO}(p+q) \subset \mathrm{SU}(p+q)$ (again this is the continuous symmetry group under which X_τ is invariant). The subgroup

of $\mathrm{SU}(p+q)$ fixing an arbitrary marked SL sphere is conjugate to $\mathrm{SO}(p)\times\mathrm{SO}(q)$ via any element of $\mathrm{SU}(p+q)$ taking the arbitrary marked sphere to the standard one. Hence the space of all (p,q)-marked SL spheres can be identified with the homogeneous space $\mathrm{SU}(p+q)/(\mathrm{SO}(p)\times\mathrm{SO}(q))$.

We can also describe the homogeneous space $\mathrm{SU}(p+q)/\mathrm{SO}(p)\times\mathrm{SO}(q)$ as a fiber bundle over the special Lagrangian Grassmannian as follows. As previously, the set of all equatorial SL $p+q-1$ spheres in $\mathbb{S}^{2(p+q)-1}$ is naturally identified with the special Lagrangian Grassmannian $\mathrm{SU}(p+q)/\mathrm{SO}(p+q)$. If we now fix an equatorial special Legendrian $p+q-1$ sphere SL, then the set of all (p,q)-markings of SL can be identified with the Grassmannian, $\widetilde{G}_p(SLG)$, of all oriented real p-dimensional subspaces of SLG. Hence the space of all (p,q)-marked SL spheres is a $\widetilde{G}_p(\mathbb{R}^{p+q})$ bundle over the special Lagrangian Grassmannian $\mathrm{SU}(p+q)/\mathrm{SO}(p+q)$. This fibration structure corresponds to the natural projection

$$\pi : \mathrm{SU}(p+q)/(\mathrm{SO}(p)\times\mathrm{SO}(q)) \to \mathrm{SU}(p+q)/\mathrm{SO}(p+q),$$

associated to the subgroup inclusions $\mathrm{SO}(p)\times\mathrm{SO}(q)\subset\mathrm{SO}(p+q)\subset\mathrm{SU}(p+q)$. Again, geometrically π is the forgetful map that ignores the marking of a marked SL sphere. Since the special Lagrangian Grassmannian $\mathrm{SU}(p+q)/\mathrm{SO}(p+q)$ and the real Grassmannian $\widetilde{G}_p(\mathbb{R}^{p+q})$ are both connected, it follows that the space of (p,q)-marked SL spheres is also connected.

When $p=q$ the symmetry $\widetilde{\underline{\mathrm{I}}}\in\widetilde{\mathrm{Sym}}(X_\tau)\subset\mathrm{O}(2p)\subset\mathrm{U}(2p)$ (defined in 3.28) preserves the standard special Legendrian sphere $\mathbb{S}^{2p-1}\subset\mathbb{R}^{2p}\subset\mathbb{C}^{2p}$ as a set. It follows from 3.28 that $\widetilde{\underline{\mathrm{I}}}$ preserves the symplectic form ω and sends the holomorphic volume form Ω to $(-1)^p\Omega$. Hence $\widetilde{\underline{\mathrm{I}}}\in\mathrm{SU}(2p)$ if and only if p is even. $\widetilde{\underline{\mathrm{I}}}$ sends the standard (p,p)-marked special Legendrian sphere $(S_p,S'_p,\mathbb{S}^{2p-1})$ (defined in 3.35) to the (p,p)-marked special Legendrian sphere $(S'_p,S_p,\mathbb{S}^{2p-1})$, i.e. $\widetilde{\underline{\mathrm{I}}}$ exchanges the two equatorial $p-1$ spheres S_p and S'_p in the marking. If p is even then $\widetilde{\underline{\mathrm{I}}}\in\mathrm{SU}(2p)$ and it preserves the orientation of $\mathbb{S}^{2p-1}$. If p is odd then $\widetilde{\underline{\mathrm{I}}}\in-\mathrm{SU}(2p)$ and it reverses the orientation of $\mathbb{S}^{2p-1}$.

Marked spheres, $\mathbf{SO}(p)\times\mathbf{SO}(q)$-invariant SL cylinders and attachment sets

For $p>1$, it follows from our previous discussion of the geometry of X_τ that the approximating sphere associated to any almost spherical region of the $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant SL cylinders X_τ has the structure of a (p,q)-marked special Legendrian sphere (S_p,S_q,SL). The image of an almost spherical region of X_τ approaches its approximating sphere uniformly as $\tau\to 0$, except on a small tubular neighborhood of the marked set $S_p\cup S_q$. Since any almost spherical region $S[k]$ attaches to its adjacent almost spherical regions $S[k-1]$ and $S[k+1]$ via transition regions whose image is contained in a small tubular neighborhood of the marked set $S_p\cup S_q$, we will also call the marked set of the approximating sphere, its *attachment set*.

There is also an obvious adaption of the above discussion to the case when $p=1$, where we consider the $\mathrm{SO}(n-1)$-invariant SL cylinders X_τ, and $(1,n-1)$-

markings of the approximating spheres. In this case each attachment set is a pair of antipodal points.

Any (p,q)-marked special Legendrian sphere determines a one-parameter family of special Legendrian cylinders $Y_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2n-1}$, whose central approximating sphere is the given marked special Legendrian sphere. More specifically, $Y_\tau = \widetilde{\mathsf{M}} \circ X_\tau$ where $\widetilde{\mathsf{M}} \in \mathrm{SU}(n)$ is any element taking the standard (p,q)-marked SL sphere to the given (p,q)-marked SL sphere (hence $\widetilde{\mathsf{M}}$ is determined uniquely up to the action of $\mathrm{SO}(p) \times \mathrm{SO}(q)$). Thus by choosing different (p,q)-markings on the same fixed equatorial special Legendrian sphere we obtain different (but isometric) SL cylinders sharing a common approximating sphere.

In this paper we will construct our initial Legendrian submanifolds by "fusing together" $g \in \mathbb{N}$ rotated copies of X_τ at a common approximating sphere; without loss of generality we can assume the common sphere to be the standard SL sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$. By the discussion above, such a configuration of isometric copies of X_τ will be uniquely determined by specifying g different (p,q)-markings of the standard special Legendrian sphere $\mathbb{S}^{n-1}$, one of which we may choose to be the standard marking defined earlier. Hence, for constructions using the $\mathrm{SO}(n-1)$-invariant SL cylinders X_τ as building blocks the initial submanifolds will be determined by specifying g pairs of antipodal points $p_1^{\pm}, \cdots, p_g^{\pm} \in \mathbb{S}^{n-1}$. Similarly, for constructions using the $\mathrm{SO}(p) \times \mathrm{SO}(p)$-invariant SL cylinders X_τ as building blocks the initial submanifolds will be determined by specifying g pairs of orthogonal $p-1$ subspheres of $\mathbb{S}^{2p-1}$.

Embeddedness of X_τ

Given $1 \leqslant p \leqslant q \geqslant 2$ and $\tau \in [-\tau_{\max}, \tau_{\max}]$, let $\mathbf{w}_\tau$ be the solution of the fundamental ODE system 3.4 defined earlier and let $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2p+2q-1}$ be the associated $\mathrm{SO}(p) \times \mathrm{SO}(q)$-equivariant special Legendrian immersion.

Define the *period lattice* of $\mathbf{w}_\tau$ by

$$\mathrm{Per}(\mathbf{w}_\tau) := \{x \in \mathbb{R} \,|\, \mathbf{w}_\tau \circ \mathsf{T}_x = \mathbf{w}_\tau\},$$

where as previously $\mathsf{T}_x \in \mathrm{Isom}(\mathbb{R})$ denotes translation by x. It is easy to show that $\mathrm{Per}(\mathbf{w}_\tau) \subset 2\mathsf{p}_\tau\mathbb{Z}$ where as previously $2\mathsf{p}_\tau$ denotes the period of the real-valued function $|w_2|^2$. Furthermore, for $k \in \mathbb{Z}$ we have,

$$2k\mathsf{p}_\tau \in \mathrm{Per}(\mathbf{w}_\tau) \iff k\widehat{\mathsf{p}}_\tau \in \mathrm{lcm}(p,q)\pi\mathbb{Z}, \tag{3.36}$$

where $\widehat{\mathsf{p}}_\tau$ is the angular period (defined in 3.10 for $p=1$ and in 3.20 for $p>1$). Define $k_0 \in \mathbb{Z}^+ \cup (\infty)$ by

$$k_0 := \min\{k \in \mathbb{Z}^+ \,|\, k\widehat{\mathsf{p}}_\tau \in \pi\,\mathrm{lcm}(p,q)\,\mathbb{Z}\}. \tag{3.37}$$

Then k_0 is finite if and only if $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$, in which case $\mathrm{Per}(\mathbf{w}_\tau) = 2k_0\mathsf{p}_\tau\mathbb{Z}$. If $\widehat{\mathsf{p}}_\tau \notin \pi\mathbb{Q}$, then $k_0 = +\infty$ and $\mathrm{Per}(\mathbf{w}_\tau) = (0)$.

We also define what we call the *half-periods* of $\mathbf{w}_\tau$. Define a homomorphism $\rho : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{O}(2) \subset \mathrm{U}(2)$ by

$$(i,j) \mapsto \rho_{ij} := \begin{pmatrix} (-1)^i & 0 \\ 0 & (-1)^j \end{pmatrix}.$$

The half-period lattice of $\mathbf{w}_\tau$ is defined by

$$\mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) := \{x \in \mathbb{R} \,|\, \exists\ (i,j) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \text{ such that } \rho_{ij} \circ \mathbf{w}_\tau = \mathbf{w}_\tau \circ \mathsf{T}_x\}. \quad (3.38)$$

It is easy to see that $T \in \mathrm{Per}(\mathbf{w}_\tau) \Rightarrow T \in \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$. We call $T \in \mathbb{R}$ a *strict half-period* if $T \in \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$ and $T \notin \mathrm{Per}(\mathbf{w}_\tau)$. We say a strict half-period has type $(--)$, $(-+)$, or $(+-)$ if i, j are both odd, i is odd, j is even and i is even, j is odd respectively.

We can completely describe the half-period lattice $\mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$ as follows:

- If $\widehat{\mathsf{p}}_\tau \notin \pi\mathbb{Q}$ then $\mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) = \mathrm{Per}(\mathbf{w}_\tau) = (0)$.
- Otherwise if $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$, and $k_0 \in \mathbb{Z}$ is defined as in 3.37 then

(i) k_0 is odd $\Rightarrow \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) = \mathrm{Per}(\mathbf{w}_\tau) = 2k_0\mathsf{p}_\tau\mathbb{Z}$, *i.e.* $\mathbf{w}_\tau$ has no strict half-periods.

(ii) k_0 is even $\Rightarrow \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau) = \frac{1}{2}\mathrm{Per}(\mathbf{w}_\tau) = k_0\mathsf{p}_\tau\mathbb{Z}$. Moreover, for fixed p and q exactly one type of strict half-period occurs:
   a. type $(--)$ if both $q/\operatorname{hcf}(p,q)$ and $p/\operatorname{hcf}(p,q)$ are odd,
   b. type $(-+)$ if $q/\operatorname{hcf}(p,q)$ is odd and $p/\operatorname{hcf}(p,q)$ is even,
   c. type $(+-)$ if $q/\operatorname{hcf}(p,q)$ is even and $p/\operatorname{hcf}(p,q)$ is odd.

The following lemma allows us to understand the embeddedness of X_τ and its period lattice, $\mathrm{Per}(X_\tau) = \{\gamma \in \mathrm{Isom}(\mathrm{Cyl}^{p,q}) \mid X_\tau \circ \gamma = X_\tau\}$, in terms of the period and half-period lattices of $\mathbf{w}_\tau$.

Lemma 3.39. *Let $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2(p+q)-1}$ be the $SO(p) \times SO(q)$-equivariant special Legendrian immersion with $0 < |\tau| < \tau_{max}$, and let $i = q/\operatorname{hcf}(p,q) \mod 2$ and $j = p/\operatorname{hcf}(p,q) \mod 2$. Suppose there exist triples $(t_1, \alpha_1, \beta_1), (t_2, \alpha_2, \beta_2) \in \mathrm{Cyl}^{p,q}$ such that*

$$X_\tau(t_1, \alpha_1, \beta_1) = X_\tau(t_2, \alpha_2, \beta_2). \quad (3.40)$$

Then $t_2 - t_1 \in \mathrm{Per}_{\frac{1}{2}}(\mathbf{w}_\tau)$. Moreover, if $t_2 - t_1 \in \mathrm{Per}(\mathbf{w}_\tau)$ then $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. If $t_2 - t_1$ is a strict half-period of $\mathbf{w}_\tau$ then $\alpha_1 = (-1)^i\alpha_2$ and $\beta_1 = (-1)^j\beta_2$.

Moreover we have:

(i) *If $\widehat{\mathsf{p}}_\tau \notin \pi\mathbb{Q}$ then $\mathrm{Per}(X_\tau) = (\mathrm{Id})$.*

(ii) *If $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$ and $k_0 \in \mathbb{N}$ defined in 3.37 is odd then*

$$\mathrm{Per}(X_\tau) = \{\mathsf{T}^l_{2k_0\mathsf{p}_\tau} \mid l \in \mathbb{Z}\},$$

where as previously $\mathsf{T}_{2k_0\mathsf{p}_\tau} \in \mathrm{Isom}(\mathbb{R}) \subset \mathrm{Isom}(\mathrm{Cyl}^{p,q})$ denotes translation by $2k_0\mathsf{p}_\tau$.

(iii) *If $\widehat{\mathsf{p}}_\tau \in \pi\mathbb{Q}$ and $k_0 \in \mathbb{N}$ defined in 3.37 is even then*

$$\mathrm{Per}(X_\tau) = \{\gamma^l \mid l \in \mathbb{Z}, \text{ where } \gamma = (\mathsf{T}_{k_0\mathsf{p}_\tau}, (-1)^i\,\mathrm{Id}, (-1)^j\,\mathrm{Id})\}.$$

Hence the simplest case to handle is when the integer k_0 defined in 3.37 is odd.

Embeddedness of X_τ for type $(1, n-1)$

When $p = 1$ and $q = n-1$ one can use Lemma 3.39 to prove the following result about embeddings of closed manifolds arising from X_τ for small well-chosen values of τ.

Lemma 3.41. *For any $\underline{m} \in \mathbb{N}$ sufficiently large, there exists a unique small positive number $\underline{\tau}$ satisfying*

$$\widehat{\mathsf{p}}_{\underline{\tau}} = \left(\frac{(n-1)\underline{m}}{2(n-1)\underline{m}-1}\right)\pi > \frac{\pi}{2}. \tag{3.42}$$

By choosing $\underline{m}$ sufficiently large, $\underline{\tau}$ can be chosen as close to 0 as desired. Let k_0 be the (large) positive odd integer defined in terms of $\underline{m}$ by

$$k_0 = 2(n-1)\underline{m} - 1. \tag{3.43}$$

Then $\mathrm{Per}(X_{\underline{\tau}}) = \{\mathsf{T}^l_{2k_0\mathsf{p}_{\underline{\tau}}}\}_{l\in\mathbb{Z}}$, $\widetilde{\mathsf{T}}_{k_0\widehat{\mathsf{p}}_{\underline{\tau}}} = \widetilde{\mathsf{T}}_{-k_0\widehat{\mathsf{p}}_{\underline{\tau}}}$, $\underline{\widetilde{\mathsf{T}}}_{k_0\widehat{\mathsf{p}}_{\underline{\tau}}} = \underline{\widetilde{\mathsf{T}}}_{-k_0\widehat{\mathsf{p}}_{\underline{\tau}}} = \underline{\widetilde{\mathsf{T}}}$, *and $X_{\underline{\tau}}$ factors through an embedding of* $(\mathbb{R}/2k_0\mathsf{p}_{\underline{\tau}}\mathbb{Z}) \times \mathbb{S}^{n-2}$.

Remark 3.44. When $n = 3$, 3.42 specializes to

$$\widehat{\mathsf{p}}_{\underline{\tau}} = \frac{\pi}{2}\left(\frac{4\underline{m}}{4\underline{m}-1}\right) = \frac{\pi}{2}\left(1+\frac{1}{m}\right), \quad \text{where } m = 4\underline{m}-1.$$

This agrees with equation 4.3 in [16] and hence Lemma 3.41 generalizes [16, Lemma 4.5].

The embedding of $(\mathbb{R}/2k_0\mathsf{p}_{\underline{\tau}}\mathbb{Z}) \times \mathbb{S}^{n-2}$ induced by $X_{\underline{\tau}}$ contains exactly $k_0 = 2(n-1)\underline{m} - 1$ almost spherical regions $S[0], S[1], \cdots, S[2(n-1)\underline{m}-2]$.

Embeddedness of X_τ for type (p, p)

Similarly, when $p = q$ one can use Lemma 3.39 to prove the following result about embeddings of closed manifolds arising from X_τ for small well-chosen values of τ.

Lemma 3.45. *For any $\underline{m} \in \mathbb{N}$ sufficiently large, there exists a unique small positive number $\underline{\tau}$ satisfying*

$$\widehat{\mathsf{p}}_{\underline{\tau}} = \left(\frac{p\underline{m}}{2p\underline{m}-1}\right)\pi > \frac{\pi}{2}. \tag{3.46}$$

By choosing $\underline{m}$ sufficiently large, $\underline{\tau}$ can be chosen as close to 0 as desired. Let k_0 be the (large) positive odd integer defined in terms of $\underline{m}$ by

$$k_0 = 2p\underline{m} - 1. \tag{3.47}$$

Then $\mathrm{Per}(X_{\underline{\tau}}) = \{\mathsf{T}^l_{2k_0\mathsf{p}_{\underline{\tau}}}\}_{l\in\mathbb{Z}}$, $\widetilde{\mathsf{T}}_{k_0\widehat{\mathsf{p}}_{\underline{\tau}}} = \widetilde{\mathsf{T}}_{-k_0\widehat{\mathsf{p}}_{\underline{\tau}}}$, $\underline{\widetilde{\mathsf{T}}}_{k_0\widehat{\mathsf{p}}_{\underline{\tau}}} = \underline{\widetilde{\mathsf{T}}}_{-k_0\widehat{\mathsf{p}}_{\underline{\tau}}} = \underline{\widetilde{\mathsf{T}}}$, *and $X_{\underline{\tau}}$ factors through an embedding of* $(\mathbb{R}/2k_0\mathsf{p}_{\underline{\tau}}\mathbb{Z}) \times \mathbb{S}^{p-1} \times \mathbb{S}^{p-1}$.

Precise asymptotics as $\tau \to 0$

We define functions of τ by

$$T_k(\tau) := \begin{cases} \tau^{-1+2/k}, & \text{for } k > 2; \\ \log \tau^{-1}, & \text{for } k = 2, \end{cases} \tag{3.48}$$

and introduce the notation $f_1 \sim f_2$ for functions f_1 and f_2 of τ to mean that

$$\frac{f_2(\tau)}{f_1(\tau)} \to C \quad \text{as} \quad \tau \to 0 \quad \text{for some} \quad C \in (0, \infty). \tag{3.49}$$

For $p = 1$ we have then

$$\mathsf{p}_\tau \sim T_{n-1}(\tau). \tag{3.50}$$

Similarly in the case $p > 1$ we have

$$\mathsf{p}_\tau^+ \sim T_q(\tau), \qquad \mathsf{p}_\tau^- \sim T_p(\tau). \tag{3.51}$$

For the angular period we have (recall $p \leqslant q$ and $q \geqslant 2$) that

$$\frac{d\widehat{\mathsf{p}}_\tau}{d\tau} \sim T_q(\tau), \qquad \widehat{\mathsf{p}}_\tau - \frac{\pi}{2} \sim \tau T_q(\tau). \tag{3.52}$$

Using 3.52, 3.42 and 3.46 we conclude that in the cases we are considering we have

$$\underline{m} \sim 1/(\underline{\tau} T_q(\underline{\tau})). \tag{3.53}$$

4 Construction of the initial almost special Legendrian submanifolds

Introduction

We will construct our initial almost special Legendrian immersions using the $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian cylinders $X_\tau : \mathrm{Cyl}^{p,q} \to \mathbb{S}^{2n-1}$ (for τ sufficiently small) as basic building blocks. First we have to decide the number of spherical regions we will have in the construction. Following 3.41 and 3.45 we fix a large number $\underline{m} \in \mathbb{N}$. This also determines a small $\underline{\tau} > 0$ by 3.42 or 3.46. This is a convenient choice because $X_{\underline{\tau}}$ factors then through an embedding of a closed manifold which contains an odd number of almost spherical regions (more precisely k_0 regions as in 3.43 or 3.47). Moreover because there is no single furthest away spherical region in the intrinsic geometry there are no spherical regions which enjoy special symmetries besides the central region, as we will see later. Besides using $X_{\underline{\tau}}$ as a building block we will use later X_τ for τ close to $\underline{\tau}$.

Convention 4.1. From now on $\underline{m} \in \mathbb{N}$ is assumed fixed and as large as it may be needed. $\underline{\tau}$ is fixed then by 3.42 or 3.46 and can be assumed by 3.53 as small as needed. We use C to denote positive constants which are independent of $\underline{m}$ and $\underline{\tau}$.

All our gluing constructions are based on using multiple copies of the basic building blocks that have some approximating spheres in common. The basic idea of all these constructions is that almost spherical regions corresponding to the common approximating spheres get "fused" to provide the initial submanifolds. This is very similar to a number of gluing constructions of CMC surfaces and CSC metrics [31–33, 35, 44].

General features

In this paper we will describe only the simplest possible fusing constructions. The first assumption we make is:

Assumption 1. *There is only one approximating sphere where fusion occurs. We call this sphere* the central sphere*, and denote it* $\mathbb{S}^{n-1}_{central} \subset \mathbb{S}^{2n-1}$.

We will further restrict to fusing constructions that are $\mathcal{G}$-equivariant for a suitably large group of symmetries $\mathcal{G}$. The initial almost special Legendrian immersion $Y : M \to \mathbb{S}^{2n-1}$ we construct and all perturbations of Y we consider will be equivariant under the action of $\mathcal{G}$, which will act by diffeomorphisms on M and isometrically on the ambient sphere $\mathbb{S}^{2n-1}$. Furthermore in all the constructions we discuss here all the building blocks used are in the $\mathcal{G}$-orbit of one of them, $X_{\underline{\tau}}$ (this is implied by assumption 3 below). We denote the subgroup of $\mathcal{G}$ that preserves the building block $X_{\underline{\tau}}$, or equivalently M_1 defined below, by $\mathcal{G}^X$.

In order to avoid trying to attach different building blocks at the same place we need the following:

Assumption 2. *Let A denote the attachment set of the central approximating sphere of $X_{\underline{\tau}}$, i.e. A is the marked set of the standard (p, q)-marked SL sphere. We assume there exists $\delta > 0$ so that for every $\gamma \in \mathcal{G}$ either*

a. $\gamma \in \mathcal{G}^X$*, or;*
b. $d(\gamma(A), A) > \delta$.

We proceed to describe the domain M of the immersion Y. The smooth $n-1$ manifold M has an open cover by $g+1$ sets $M_0, \cdots, M_g$ for some positive integer g determined by the symmetry group $\mathcal{G}$ of the construction. M_0 corresponds to the central sphere where fusing occurs and $M_1, \cdots, M_g$ correspond to the g copies of the building block $X_{\underline{\tau}}$ that we are fusing at the central sphere. More precisely we have the following:

1. The regions $M_1, \cdots, M_g$ are mutually disjoint and each of them is diffeomorphic to (and can be identified with a copy of) the cylinder $\mathrm{Cyl}^{p,q}_I$. I here is chosen so that the projection P to the closed manifold through which $X_{\underline{\tau}}$ factors, maps $\mathrm{Cyl}^{p,q}_I$ diffeomorphically to an open neighborhood of the complement of $P(S_{central})$. The boundary of $\bigcup_{i=1}^{g} M_i$ has $2g$ components each of which is diffeomorphic to a meridian of $\mathrm{Cyl}^{p,q}$.
2. The region M_0 is diffeomorphic to the central sphere $\mathbb{S}^{n-1}_{\mathrm{central}}$ with $2g$ disjoint connected open sets removed. These sets are small tubular neighborhoods of the connected components of the attachment sets of the building blocks. The

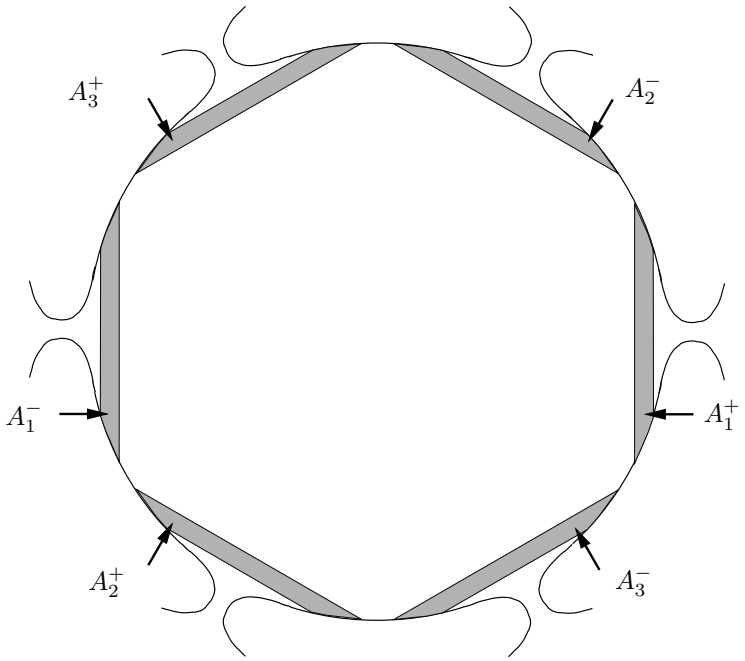


Figure 4: The central spherical region with the gluing regions shaded

boundary of each of them is diffeomorphic to a meridian of $\mathrm{Cyl}^{p,q}$. These $2g$ boundaries form the $2g$ connected components of ∂M_0.

3. For each $i > 0$, $M_0 \cap M_i$ consists of two connected components, A_i^+ and A_i^-, each diffeomorphic to $\mathrm{Cyl}^{p,q}$. A_i^+ and A_i^- are collar neighborhoods of the two boundary components of the cylinder M_i. We call these $2g$ components the *gluing regions* because, as we will see, this is where the immersion Y transits from being equatorial to the immersion of the building blocks. See Figure 4 for a schematic illustration of the central spherical region and the gluing regions in the case $p = 1$.

For all the constructions we discuss the following is satisfied:

Assumption 3. *$\mathcal{G}$ acts transitively on the $2g$ gluing regions $A_1^\pm, \cdots, A_g^\pm$, i.e. any two gluing regions are in the same $\mathcal{G}$-orbit.*

Assumption 3 implies that all building blocks are in one $\mathcal{G}$-orbit (as we mentioned before), and, moreover, that for any given building block there is an element $\underline{\gamma_i} \in \mathcal{G}$ preserving that block while exchanging its two attachment components. In particular, Assumption 3 implies that in a given construction: (i) we use building blocks $X_{\underline{\tau}}$ of one fixed type (p, q), (ii) we use only finitely many copies of this fixed building block, (iii) we use the same value of τ for each building block, and (iv) for $i > 0$ there exists $\gamma_i \in \mathcal{G}$ such that $\gamma_i(M_1) = M_i$.

We proceed now to describe the initial Legendrian immersion $Y : M \to \mathbb{S}^{2n-1}$:

1. On $M_1 \setminus (A_1^+ \cup A_1^-)$, Y agrees with $X_{\underline{\tau}}$ (subject to the identification of M_1 with some $\mathrm{Cyl}_I^{p,q}$). For $i > 1$, $\mathcal{G}$-equivariance then forces Y to be $\widetilde{\gamma}_i \circ X_{\underline{\tau}}$ on $M_i \setminus (A_i^+ \cup A_i^-)$ (subject to the identification of M_i with $\mathrm{Cyl}_I^{p,q}$). Here $\widetilde{\gamma}_i$ denotes the ambient spherical isometry corresponding to some $\gamma_i \in \mathcal{G}$ with $\gamma_i(M_1) = M_i$.
2. The restriction of Y to $M_0 \setminus \bigcup_{i=1}^g (A_i^+ \cup A_i^-)$ is an embedding whose image is the central sphere $\mathbb{S}^{n-1}_{\mathrm{central}}$ with $2g$ connected disjoint open sets removed. On neighborhoods of the two boundary components it shares with $A_1^{\pm}$, (which are then also boundary components of M_1), Y agrees with X_0 (subject to the usual identification with the cylindrical domain). By equivariance this determines Y on a neighborhood of the full boundary of $M_0 \setminus \bigcup_{i=1}^g A_i^+ \cup A_i^-$ as in (1).
3. It remains to describe Y on each gluing region $A_i^{\pm}$. By equivariance and Assumption 3 it is enough to do so on A_1^+. A_1^+ can be identified with a cylindrical domain $\mathrm{Cyl}_I^{p,q}$ for some interval I of length one. From (2), the immersion Y agrees with X_0 close to one boundary component of A_1^+ and from (1) it agrees with $X_{\underline{\tau}}$ close to the other. Because X_0 and $X_{\underline{\tau}}$ are close to each other on A_1^+, by adapting a construction used earlier for Lagrangian manifolds by Butscher [2, 3] and Lee [41], we can interpolate to obtain Y that is Legendrian and close to both X_0 and $X_{\underline{\tau}}$. In particular the Lagrangian angle θ of Y is supported on the gluing regions and satisfies

$$\|\theta : C^k(M)\| \leqslant C(k)\underline{\tau}.$$

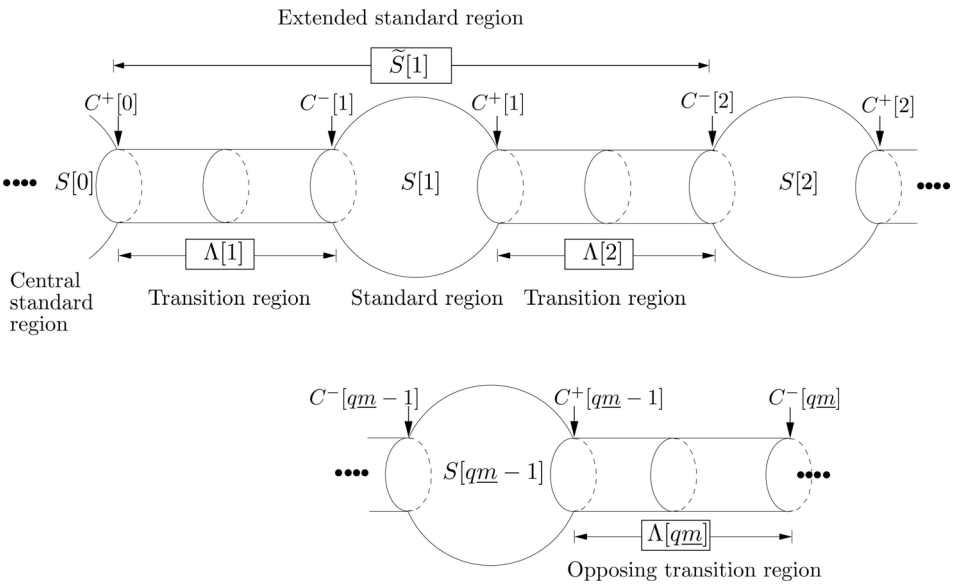


Figure 5: A schematic rendering of the standard (which appear spherical) and transition (which appear cylindrical) regions of the manifold M

The initial configurations for $p = q$

In this section we describe gluing constructions that use only the $\mathrm{SO}(p) \times \mathrm{SO}(p)$-equivariant special Legendrian cylinders $X_{\underline{\tau}}$ for $\underline{\tau}$ sufficiently small. The discussion of the previous section shows how an initial configuration of markings of the central sphere determines an immersion $Y : M \to \mathbb{S}^{4p-1}$. Therefore we concentrate on describing the initial configuration of markings and its symmetry group $\mathcal{G}$.

For any odd $g \geqslant 3$, we define an element $\widetilde{\mathsf{R}}_g \in \mathrm{SO}(2p) \subset \mathrm{SU}(2p)$ of order g. To define $\widetilde{\mathsf{R}}_g$ first we define a closed 1-parameter subgroup $\mathrm{SO}(2) \subset \mathrm{SO}(2p) \subset \mathrm{SU}(2p)$ by

$$\widetilde{\mathsf{R}}_x = \begin{pmatrix} \cos x\,\mathrm{Id} & \sin x\,\mathrm{Id} \\ -\sin x\,\mathrm{Id} & \cos x\,\mathrm{Id} \end{pmatrix}, \quad x \in \mathbb{R}, \tag{4.2}$$

where Id denotes the $p \times p$ identity matrix. Now given any $g \in \mathbb{N}$ we define $\widetilde{\mathsf{R}}_g \in \mathrm{SO}(2p)$ by

$$\widetilde{\mathsf{R}}_g := \widetilde{\mathsf{R}}_{\frac{2\pi}{g}}.$$

$\widetilde{\mathsf{R}}_g$ generates a cyclic subgroup $\mathbb{Z}_g \subset \mathrm{SO}(2) \subset \mathrm{SO}(2p)$ of order g. When g is odd, $\mathbb{Z}_g \cap \mathrm{O}(p) \times \mathrm{O}(p) = (\mathrm{Id})$ (while $\mathbb{Z}_g \cap \mathrm{O}(p) \times \mathrm{O}(p) = (\pm\,\mathrm{Id})$, if g is even). When g is odd one can verify that the images of the orthogonal pair of equatorial spheres $(\mathbb{S}^{p-1}, 0)$ and $(0, \mathbb{S}^{p-1})$ under $\widetilde{\mathsf{R}}_g^i$ are disjoint for $i = 0, \cdots, g-1$.

The relevant symmetry group in this case is the group $\mathcal{G}_g \subset \mathrm{O}(4p)$ that preserves the set $\bigcup_{i=0}^{g-1} \widetilde{\mathsf{R}}_g^i \circ X_{\underline{\tau}}$ and leaves the central sphere $\mathbb{S}^{2p-1} \subset \mathbb{R}^{2p} \subset \mathbb{C}^{2p}$ invariant. One can prove that $\mathcal{G}$ is the group generated by $\widetilde{\mathsf{R}}_g$, the element $\underline{\widetilde{\mathsf{T}}} \in \mathrm{O}(2p) \subset \mathrm{U}(2p)$, defined in 3.28, and the subgroup $\mathcal{G}' \subset \mathrm{O}(p) \times \mathrm{O}(p) \subset \mathrm{O}(2p) \subset \mathrm{U}(2p)$ by

$$\mathcal{G}' = \{M \in \mathrm{O}(p) \times \mathrm{O}(p) \mid M = (A, \pm A) \text{ for some } A \in \mathrm{O}(p)\}. \tag{4.3}$$

Since $\widetilde{\mathsf{R}}_g$, $\underline{\widetilde{\mathsf{T}}} \in \mathrm{O}(2p) \subset \mathrm{U}(2p)$ and $\mathcal{G}' \subset \mathrm{O}(2p) \subset \mathrm{U}(2p)$, the group $\mathcal{G}_g$ is a subgroup of $\mathrm{O}(2p) \subset \mathrm{U}(2p)$. Hence every element of $\mathcal{G}_g$ acts by holomorphic isometries on $\mathbb{C}^{2p}$.

The structure of the group $\mathcal{G}_g$.

We will now describe the structure of the group $\mathcal{G}_g \subset \mathrm{O}(2p) \subset \mathrm{U}(2p)$ generated by $\mathcal{G}'$, $\widetilde{\mathsf{R}}_g$, $\underline{\widetilde{\mathsf{T}}}$. Let $\mathcal{G}''$ be the diagonal subgroup of $\mathrm{O}(p) \times \mathrm{O}(p)$

$$\mathcal{G}'' = \{M \in \mathrm{O}(p) \times \mathrm{O}(p) \mid M = (A, A) \text{ for some } A \in \mathrm{O}(p)\}. \tag{4.4}$$

$\mathcal{G}''$ is a subgroup of $\mathcal{G}'$ isomorphic to $\mathrm{O}(p)$. The group $\mathcal{G}'$ is generated by the subgroup $\mathcal{G}''$ and the involution

$$\widetilde{\mathsf{N}} := \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \in \mathrm{O}(p) \times \mathrm{O}(p) \subset \mathrm{O}(2p).$$

Hence we can also describe $\mathcal{G}_g$ as the group generated by $\mathcal{G}''$, $\widetilde{\mathsf{R}}_g$, $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{N}}$. It is convenient to use this description of $\mathcal{G}_g$ to investigate its structure.

There is an injective homomorphism $\rho : \mathrm{O}(2) \to \mathrm{O}(2p) \subset \mathrm{U}(2p)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a\,\mathrm{Id}_p & b\,\mathrm{Id}_p \\ c\,\mathrm{Id}_p & d\,\mathrm{Id}_p \end{pmatrix}.$$

From now on identify $\mathrm{O}(2)$ with its image under ρ. Notice that the subgroup $\mathrm{O}(2)$ centralizes $\mathcal{G}''$, i.e. $h.g = g.h$ for all $h \in \mathrm{O}(2)$ and $g \in \mathcal{G}''$, and that $\mathrm{O}(2) \cap \mathcal{G}'' = \{\pm \mathrm{Id}\}$.

Since $\widetilde{\mathsf{R}}_x$, $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{N}}$ all belong to the subgroup $\mathrm{O}(2) \subset \mathrm{O}(2p)$, the subgroup Γ_g generated by $\widetilde{\mathsf{R}}_g$, $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{N}}$ is a subgroup of $\mathrm{O}(2)$. Recall that one presentation of the dihedral group $\mathbf{D}_n$ of order $2n$ is

$$\langle s, t \mid s^2 = 1,\, t^2 = 1,\, (st)^n = 1 \rangle.$$

Since $\widetilde{\mathsf{N}}$ and $\underline{\widetilde{\mathsf{T}}}$ satisfy

$$\widetilde{\mathsf{N}}^2 = \mathrm{Id}, \quad \underline{\widetilde{\mathsf{T}}}^2 = \mathrm{Id}, \quad (\widetilde{\mathsf{N}} \circ \underline{\widetilde{\mathsf{T}}})^4 = \mathrm{Id} \quad \text{and} \quad (\widetilde{\mathsf{N}} \circ \underline{\widetilde{\mathsf{T}}})^i \neq \mathrm{Id} \quad \text{for } i = 1, 2, 3$$

it follows that $\widetilde{\mathsf{N}}$ and $\underline{\widetilde{\mathsf{T}}}$ generate a group of order 8 isomorphic to $\mathbf{D}_4$, the symmetry group of a square. The cyclic subgroup $\mathbf{C}_4$ of $\mathbf{D}_4$, representing the orientation preserving symmetries of a square, is the (normal) subgroup generated by the element $\widetilde{\mathsf{N}} \circ \underline{\widetilde{\mathsf{T}}}$.

As above, let $\mathbb{Z}_g$ denote the cyclic group of order g generated by $\widetilde{\mathsf{R}}_g$. Recall from our previous discussion of elementary group theory that the subset $\mathbb{Z}_g\,\mathbf{D}_4 \subset \mathrm{O}(2)$ forms a group if and only if $\mathbb{Z}_g\,\mathbf{D}_4 = \mathbf{D}_4\,\mathbb{Z}_g$.

Using the commutation relations

$$\widetilde{\mathsf{N}} \circ \widetilde{\mathsf{R}}_x \circ \widetilde{\mathsf{N}} = \widetilde{\mathsf{R}}_{-x}, \quad \underline{\widetilde{\mathsf{T}}} \circ \widetilde{\mathsf{R}}_x \circ \underline{\widetilde{\mathsf{T}}} = \widetilde{\mathsf{R}}_{-x},$$

one can see that $\mathbb{Z}_g\,\mathbf{D}_4 = \mathbf{D}_4\,\mathbb{Z}_g$ and hence $\mathbb{Z}_g\,\mathbf{D}_4$ forms a finite subgroup Γ_g of $\mathrm{O}(2)$. Since the subgroup Γ_g contains $\widetilde{\mathsf{R}}_g$, $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{N}}$ it coincides with the subgroup generated by these 3 elements. The above two commutation relations also show that $\mathbb{Z}_g$ is a normal subgroup of $\Gamma_g = \mathbb{Z}_g\,\mathbf{D}_4$ and that $\mathbf{D}_4$ is not normal. The conjugation action of $\mathbf{D}_4$ on $\mathbb{Z}_g$ gives us a nontrivial homomorphism $\rho : \mathbf{D}_4 \to \mathrm{Aut}(\mathbb{Z}_g)$ given by

$$\rho(\gamma) = \begin{cases} \mathrm{Id}, & \text{if } \gamma \in \mathbf{C}_4 \subset \mathbf{D}_4; \\ \widetilde{\mathsf{R}}_g \mapsto \widetilde{\mathsf{R}}_g^{-1}, & \text{otherwise.} \end{cases}$$

It is straightforward to check that

$$\mathbb{Z}_g \cap \mathbf{D}_4 = \mathbb{Z}_g \cap \mathbf{C}_4 = \begin{cases} 1, & \text{if } g \text{ is odd;} \\ \pm \mathrm{Id}, & \text{if } g \equiv 2 \mod 4; \\ \mathbf{C}_4, & \text{if } g \equiv 0 \mod 4. \end{cases}$$

Hence if g is odd then the subgroup $\Gamma_g = \mathbb{Z}_g\,\mathbf{D}_4 \subset \mathrm{O}(2)$ generated by $\widetilde{\mathsf{R}}_g$, $\underline{\widetilde{\mathsf{T}}}$ and $\widetilde{\mathsf{N}}$, is the semidirect product of $\mathbb{Z}_g$ by $\mathbf{D}_4$, $\mathbb{Z}_g \rtimes_\rho \mathbf{D}_4$, and therefore Γ_g has order $8g$.

Since $\Gamma_g \subset \mathrm{O}(2)$ and $\mathrm{O}(2)$ centralizes $\mathcal{G}''$, then Γ_g centralizes $\mathcal{G}''$. Hence $\Gamma_g \mathcal{G}'' = \mathcal{G}'' \Gamma_g$ forms a group and is the group generated by $\mathcal{G}'$, $\widetilde{\mathsf{R}}_g$ and $\underline{\widetilde{\mathsf{T}}}$. It is also easy to see that $\Gamma_g \cap \mathcal{G}'' = \mathrm{O}(2) \cap \mathcal{G}'' = (\pm \mathrm{Id})$, and hence $\mathcal{G}_g$ is the central product of Γ_g and $\mathcal{G}''$ identifying $\Gamma_g \cap \mathcal{G}'' = (\pm \mathrm{Id}) \subset Z(\mathcal{G})$.

Consider now the subgroup $\mathcal{G}_g^X := \mathcal{G}_g \cap \widetilde{\mathrm{Sym}}(X_{\underline{\tau}})$ which is the subgroup of symmetries which fix the building block $X_{\underline{\tau}}$. It is easy to check that $\mathcal{G}_g^X$ is the subgroup generated by the involutions $\widetilde{\mathsf{N}}$, $\underline{\widetilde{\mathsf{T}}}$ and the group $\mathcal{G}''$ (i.e. by omitting the generator $\widetilde{\mathsf{R}}_g$). Hence by similar reasoning to above $\mathcal{G}_g^X$ is isomorphic to the central product of $\mathbf{D}_4$ and $\mathcal{G}''$ identifying $(\pm \mathrm{Id}) \subset Z(\mathcal{G}_g^X)$.

We consider also the subgroup $\mathcal{G}_g^{\mathrm{off}}$ of $\mathcal{G}_g^X$ and the symmetry group $\mathcal{G}_g$ consisting of those symmetries which fix the generic off-central almost spherical region of the building block $X_{\underline{\tau}}$ or equivalently the generic meridian of (the domain $\mathrm{Cyl}^{p,q}$ of) $X_{\underline{\tau}}$. Clearly $\mathcal{G}_g^{\mathrm{off}} = \mathcal{G}'$ is generated by $\widetilde{\mathsf{N}}$ and $\mathcal{G}''$ (i.e. by omitting both the generators $\widetilde{\mathsf{R}}_g$ and $\underline{\widetilde{\mathsf{T}}}$) and hence is isomorphic to the direct product $\mathbb{Z}_2 \times \mathcal{G}''$. Note that in our case we have taken care to arrange the number of almost spherical regions (recall 3.47) so that there are no almost spherical regions with extra symmetries as would be the case if there was a furthermost from the central spherical region. Similarly the only meridian with extra symmetries is the furthermost meridian from the central spherical region.

The initial configurations and their symmetry groups for $p = 1$.

In this section we describe gluing constructions that use only the $\mathrm{SO}(n-1)$-equivariant special Legendrian cylinders $X_{\underline{\tau}}$ in $\mathbb{S}^{2n-1}$ (for $\underline{\tau}$ sufficiently small).

We let Δ be either

a. a regular k-simplex in $\mathbb{R}^k$, with $3 \leqslant k \leqslant n$, or
b. a regular g-gon with $g \geqslant 3$ odd in $\mathbb{R}^2$.

Remark 4.5. Another possibility is to use a Δ which is the product of one of the above with itself, or more generally products of the above, rescaled to be in the unit sphere $\mathbb{S}^{2n-1}$. This leads to configurations which satisfy all three assumptions and moreover have no kernel in the central sphere. We are currently investigating such constructions, but we will not discuss them further in this paper.

In each case we center the polytope Δ at the origin and place one vertex at $(1, 0, \cdots, 0)$. Now we linearly embed $\mathbb{R}^k$ (for $2 \leqslant k \leqslant n$) in $\mathbb{R}^n$ by

$$(x_1, x_2, \cdots, x_k) \mapsto (x_1, x_2, \cdots, x_k, 0, \cdots, 0).$$

The subgroup $\widetilde{\mathcal{G}}_\Delta$ of $\mathrm{O}(n)$ preserving Δ is

$$\widetilde{\mathcal{G}}_\Delta = \begin{cases} \mathrm{Aut}(\Delta) \times \mathrm{O}(n-k), & \text{if } k < n; \\ \mathrm{Aut}(\Delta) & \text{if } k = n, \end{cases}$$

where $\mathrm{Aut}(\Delta)$ denotes the subgroup of $\mathrm{O}(k)$ preserving Δ and $\mathrm{Aut}(\Delta)$ is

$$\mathrm{Aut}(\Delta) = \begin{cases} \mathrm{S}_{k+1}, & \text{the symmetric group on } k+1 \text{ letters for a regular } k\text{-simplex,} \\ \mathbf{D}_g, & \text{the dihedral group of order } 2g \text{ for a regular } g\text{-gon.} \end{cases}$$

We extend the action of $\widetilde{\mathcal{G}}_\Delta$ from $\mathbb{R}^n$ to $\mathbb{C}^n$ by requiring the action be $\mathbb{C}$-linear on $\mathbb{C}^n$. In this way $\widetilde{\mathcal{G}}_\Delta$ is naturally a subgroup of $\mathrm{O}(n) \subset \mathrm{U}(n)$. Finally we want to define a larger subgroup $\mathcal{G}_\Delta$ of $\mathrm{O}(2n)$ containing $\widetilde{\mathcal{G}}_\Delta$ as a normal subgroup but which is no longer a subgroup of $\mathrm{O}(n)$. To this end we define an isometry $\underline{\widetilde{\mathsf{C}}}$ of $\mathbb{C}^n$ by

$$\underline{\widetilde{\mathsf{C}}}(\mathbf{z}) = -\overline{\mathbf{z}}, \quad \text{where } \mathbf{z} \in \mathbb{C}^n. \tag{4.6}$$

Notice that

$$\underline{\widetilde{\mathsf{C}}}^* \Omega = (-1)^n \, \overline{\Omega}, \quad \text{and} \quad \underline{\widetilde{\mathsf{C}}}^* \omega = -\omega$$

where Ω is the standard holomorphic $(n,0)$-form on $\mathbb{C}^n$ and ω is the symplectic form. Hence $\underline{\widetilde{\mathsf{C}}}$ acts on $\mathbb{C}^n$ as an anti-holomorphic isometry. It is easy to see from 4.6 that $\underline{\widetilde{\mathsf{C}}}$ commutes with every element of $\widetilde{\mathcal{G}}_\Delta$—since we started with an underlying real action every element of $\widetilde{\mathcal{G}}_\Delta$ is invariant under complex conjugation.

Let $K = \langle \underline{\widetilde{\mathsf{C}}} \rangle \subset \mathrm{O}(2n)$ be the cyclic group of order 2 generated by $\underline{\widetilde{\mathsf{C}}}$. Since $\underline{\widetilde{\mathsf{C}}}$ commutes with $\widetilde{\mathcal{G}}_\Delta$, $K\widetilde{\mathcal{G}}_\Delta = \widetilde{\mathcal{G}}_\Delta K$ forms a group $\mathcal{G}_\Delta$, and both $\widetilde{\mathcal{G}}_\Delta$, K are normal subgroups of $\mathcal{G}_\Delta := K\widetilde{\mathcal{G}}_\Delta$. Also since $K \cap \widetilde{\mathcal{G}}_\Delta = (\mathrm{Id})$, $\mathcal{G}_\Delta$ is the direct product of $\widetilde{\mathcal{G}}_\Delta$ and $K = \langle \underline{\widetilde{\mathsf{C}}} \rangle$.

As in the previous $p = q$ case we consider now subgroups $\mathcal{G}_\Delta^X$ and $\mathcal{G}_\Delta^{\mathrm{off}}$ fixing the building block $X_{\underline{\tau}}$ and its non-central spherical regions (equivalently generic meridians) respectively. $\mathcal{G}_\Delta^X = \mathcal{G}_\Delta \cap \widetilde{\mathrm{Sym}}(X_{\underline{\tau}})$ is isomorphic to $\mathbb{Z}_2 \times \mathcal{G}_{\Delta'}$ where $\mathbb{Z}_2$ is generated by $\underline{\widetilde{\mathsf{C}}}$ as defined in 4.6 and $\mathcal{G}_{\Delta'}$ denotes the subgroup of the automorphisms of the polytope Δ that fix the vertex at e_1. $\mathcal{G}_\Delta^{\mathrm{off}}$ is simply $\mathcal{G}_{\Delta'}$. As in the previous case $p = q$ we have arranged (via 3.43) to have no spherical regions with extra symmetries.

From our discussion above if follows that any element $\gamma \in \mathcal{G}_\Delta$ can be written uniquely in the form

$$\gamma = \underline{\widetilde{\mathsf{C}}}^i \circ \widetilde{\gamma}, \quad \text{for } i \in \mathbb{Z}_2 \text{ and } \widetilde{\gamma} \in \widetilde{\mathcal{G}}_\Delta \ .$$

Since $\widetilde{\mathcal{G}}_\Delta \subset \mathrm{O}(n) \subset \mathrm{U}(n)$, the element γ acts on $\mathbb{C}^n$ as a holomorphic isometry if $i = 0$ and an anti-holomorphic isometry if $i = 1$. Hence the subgroup $\mathcal{G}_\Delta^{\mathrm{off}}$ of $\mathcal{G}_\Delta$ preserving any non-central sphere consists entirely of holomorphic isometries, but $\mathcal{G}_\Delta^X$ (and of course $\mathcal{G}_\Delta$) contain both holomorphic and anti-holomorphic isometries.

5 The symmetry group and the general framework for correcting the initial surfaces

Introduction

In this section we describe the PDE framework for correcting the initial almost special Legendrian immersion to exactly special Legendrian. We also concentrate on how the general framework is affected by the symmetries imposed in the construction.

Perturbations of Legendrian immersions

Given a Legendrian immersion $X : M \to \mathbb{S}^{2n-1}(R)$ of a smooth manifold M and a C^2-small function $f : M \to \mathbb{R}$, we will construct a new Legendrian immersion $X_f : M \to \mathbb{S}^{2n-1}(R)$. This construction is an adaptation of the one used in [2, 3, 41] in the Lagrangian setting. In the case $n = 3$ this Legendrian construction is described in detail in Appendix C of our earlier paper [16].

Since the construction is local we can assume that X is an embedding. We first extend f from M to a tubular neighborhood $\mathcal{N}$ of $X(M)$ in $\mathbb{S}^{2n-1}(R)$ by requiring that if v is a normal vector to $X(M)$ at $X(p)$ then

$$f(\exp X(p), v; \mathbb{S}^{2n-1}(R)) = f(p).$$

We extend f to the cone $C(\mathcal{N})$ over $\mathcal{N}$ by requiring that f be homogeneous of degree 2. Let V be the Hamiltonian vector field on $C(\mathcal{N})$ defined by f, i.e. $V = -J\nabla f$. Provided that f is sufficiently small, we can flow X by V for unit time to an immersion $X_f : M \to \mathbb{S}^{2n-1}(R)$. Since f is homogeneous of degree 2 and the flow of V preserves the symplectic structure on $\mathbb{C}^n$ it follows that the immersion $X_f : M \to \mathbb{S}^{2n-1}(R)$ remains Legendrian. For future reference we have the following:

Definition 5.1. *We call X_f the Legendrian graph of f on X.*

$\mathcal{G}$-equivariant Legendrian perturbations

In the case that the initial Legendrian immersion X is equivariant under a group of symmetries $\mathcal{G}$ (as in the case of the initial almost special Legendrian immersions Y constructed in Section 4), the question arises: under what conditions on f is X_f also $\mathcal{G}$-equivariant?

Each element of one of our symmetry groups $\mathcal{G}$ acts on the ambient sphere $\mathbb{S}^{2n-1}$ either as a holomorphic or anti-holomorphic isometry, i.e. it either preserves the standard complex structure J on $\mathbb{C}^n$ or sends it to $-J$. For example, $\widetilde{\underline{\mathsf{C}}}$ defined in 4.6 is an anti-holomorphic isometry because it sends the symplectic form $\omega \mapsto -\omega$ and therefore $J \mapsto -J$.

If $\gamma \in \mathcal{G}$ acts on the ambient sphere $\mathbb{S}^{2n-1}$ as a holomorphic isometry, then it is straightforward to see that X_f is γ-equivariant if and only if f is preserved by the action of γ, i.e. $f = f \circ \gamma$. If however, $\gamma \in \mathcal{G}$ acts on the ambient sphere as an

anti-holomorphic isometry the condition on f that X_f be γ-equivariant is that f have odd symmetry with respect to γ, i.e. $f \circ \gamma = -f$.

Let $Y : M \to \mathbb{S}^{2n-1}$ be one of the initial almost special Legendrian immersions constructed in Section 4 and let $\mathcal{G}$ denote the group with respect to which Y is equivariant.

Definition 5.2. *We call a function $f : M \to \mathbb{R}$, $\mathcal{G}$-equivariant if for any $\gamma \in \mathcal{G}$ we have $f \circ \gamma = \pm f$, where the $\pm$ sign is determined by whether γ acts as a holomorphic or anti-holomorphic isometry on $\mathbb{S}^{2n-1}$.*

*We use the subscript "*sym*" to denote subspaces of $\mathcal{G}$-equivariant functions, for example, $C^{k,\beta}_{sym}(M)$.*

If $X : M \to \mathbb{S}^{2n-1}$ is a $\mathcal{G}$-equivariant Legendrian immersion then its Lagrangian angle is a $\mathcal{G}$-equivariant function on M. In particular, if f is a $\mathcal{G}$-equivariant function on M then the Lagrangian angles θ and θ_f of Y and Y_f respectively, are also both $\mathcal{G}$-equivariant functions on M, as is the function $\mathcal{L}f$.

The $\mathcal{G}$-equivariant linearized equation on the approximating spheres

It is well-known that the Lagrangian angle θ_f of the perturbed immersion X_f satisfies

$$\theta_f = \theta + \mathcal{L}f + Q_f,$$

where θ is the Lagrangian angle of the original Legendrian immersion X, $\mathcal{L}$ is the linearized operator and Q_f is quadratic and higher in f and its derivatives. $\mathcal{L} = \Delta + 2n$ where Δ denotes the Laplacian on M with respect to the metric induced by the immersion X. As we mentioned briefly in the introduction to the paper and as we will explain in detail in the next section, the heart of the analytic work needed in our gluing constructions is to understand the inhomogeneous linear equation

$$\mathcal{L}f := (\Delta + 2n)f = E,$$

where f and E are functions on our initial submanifold and Δ denotes the Laplacian with respect to the induced metric. Since we constructed our initial Legendrian submanifolds to be equivariant with respect to a symmetry group $\mathcal{G}$, we may try to perturb them to special Legendrian within the class of $\mathcal{G}$-equivariant Legendrian submanifolds. At the linear level this means that we should restrict the functions f and E considered to be $\mathcal{G}$-equivariant functions in the sense of 5.2.

Recall that our initial Legendrian submanifold contains finitely many copies of $X_{\underline{\tau}}$ "fused" at one central sphere and that for $\underline{\tau}$ small, $X_{\underline{\tau}}$ consists of many almost spherical regions connected by transition regions. Roughly speaking, we can compare the spectrum of $\mathcal{L}$ on the initial submanifold with the spectrum of $\mathcal{L}$ on the collection of all the approximating spheres associated with our initial submanifold. Because we are interested in the $\mathcal{G}$-equivariant spectrum of $\mathcal{L}$, a crucial fact is that each approximating sphere shares the symmetries of its corresponding almost spherical region. We will see in Section 6 that eventually we will be able to control the analytic behaviour of the linearized operator $\mathcal{L} = \Delta + 2n$ on our initial submanifold if we understand

a. the $\mathcal{G}$-equivariant kernel of $\mathcal{L}$ on the collection of approximating spheres associated to our initial Legendrian submanifold,
b. the $\mathcal{G}$-equivariant harmonics up to second order on a meridian.

Rigidity of equatorial special Legendrian spheres

Since the action of $\mathrm{SU}(n)$ preserves the special Legendrian condition, its one-parameter subgroups induce exact kernel of the linearized operator $\mathcal{L}$ on every special Legendrian submanifold of $\mathbb{S}^{2n-1}$. More concretely, given $A \in \mathfrak{su}(n)$, the function

$$f_A(z) = \omega(z, Az),$$

where $z \in \mathbb{C}^n$ and ω is the standard symplectic form on $\mathbb{C}^n$ restricts on every special Legendrian submanifold to a function in the kernel of $\mathcal{L} = \Delta + 2n$. Following Joyce [26, Defn. 6.7], we say a special Legendrian submanifold Σ of $\mathbb{S}^{2n-1}$ is *rigid*, if its only infinitesimal special Legendrian deformations arise from $\mathrm{SU}(n)$ motions as described above. In other words, Σ is rigid if

$$\mathcal{L}_\Sigma f = 0 \;\Rightarrow f = f_A|_\Sigma \quad \text{for some } A \in \mathfrak{su}(n).$$

An elementary but important fact is:

Lemma 5.3. *Any special Legendrian equatorial $n-1$ sphere in $\mathbb{S}^{2n-1}$ is rigid as a special Legendrian submanifold.*

Proof. First, it is enough to consider the standard real $n-1$ sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$, since $\mathrm{SU}(n)$ acts transitively on the set of equatorial special Legendrian $n-1$ spheres. The spectrum of the Laplacian Δ on a unit round sphere is very well-known. The k-th eigenvalue of $\Delta_{\mathbb{S}^{n-1}}$ is $\lambda_k = k(n-2+k)$. The corresponding space of k-th eigenfunctions is the space of spherical harmonics of degree k, written $\mathcal{H}^n_k$. These are the functions obtained by restricting to the sphere any degree k harmonic polynomial on $\mathbb{R}^n$. Setting $k = 2$, we see that the eigenvalue $\lambda = 2n$ of $\Delta_{\mathbb{S}^{n-1}}$ corresponds to the space of second spherical harmonics $\mathcal{H}^n_2$. It is easy to see that $\dim \mathcal{H}^n_2 = \binom{n+1}{2} - 1$. Since the standard special Legendrian sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n \subset \mathbb{C}^n$ is invariant under $\mathrm{SO}(n) \subset \mathrm{SU}(n)$ the functions f_A corresponding to $A \in \mathfrak{so}(n) \subset \mathfrak{su}(n)$ vanish identically on $\mathbb{S}^{n-1}$. Hence the dimension of the geometric contribution to the kernel of $\mathcal{L}$ on $\mathbb{S}^{n-1}$ is $\dim \mathrm{SU}(n) - \dim \mathrm{SO}(n) = n^2 - \frac{1}{2}n(n-1) - 1 = \binom{n+1}{2} - 1 = \dim \mathcal{H}^n_2$. □

This simple observation is crucial for our later work.

The $\mathcal{G}$-equivariant kernel of $\mathcal{L}$ on the approximating spheres

The symmetry group $\mathcal{G}$ imposed on our initial submanifold is shared by its collection of associated approximating spheres. Hence we can study the $\mathcal{G}$-equivariant kernel of $\mathcal{L}$ on this finite collection of equatorial special Legendrian spheres. In all our current constructions $\mathcal{G}$ acts transitively on the building blocks. Since every almost spherical region belongs to at least one of the building blocks, there is an

element $\gamma \in \mathcal{G}$ that moves a given almost spherical region to one of the almost spherical regions in $X_{\underline{\tau}}$. The same is also true for any approximating sphere. Hence it is enough to understand the $\mathcal{G}$-equivariant kernel on approximating spheres associated with almost spherical regions of the building block $X_{\underline{\tau}}$.

We are interested in the $\mathcal{G}$-equivariant kernel of $\mathcal{L}$ that is supported on single approximating spheres. If $\widehat{S}$ is such an approximating sphere we denote this $\mathcal{G}$-equivariant kernel of $\mathcal{L}$ on $\widehat{S}$ by $\mathcal{K}_{sym}[\widehat{S}]$. From Lemma 5.3 we know the full kernel of $\mathcal{L}$ on any equatorial special Legendrian. Hence the study of the $\mathcal{G}$-invariant kernel of $\mathcal{L}$ on our finite collection of equatorial special Legendrian spheres reduces to a problem of a group-theoretic character; namely, for any given approximating sphere $\widehat{S}$ we first determine the subgroup of $\mathcal{G}$ preserving $\widehat{S}$ and then determine $\mathcal{K}_{sym}[\widehat{S}]$ as the subspace of the full spherical kernel that is invariant under that subgroup of $\mathcal{G}$.

As we have mentioned earlier we have arranged that there are no off-central spheres with extra symmetries. Therefore there are only two cases to consider: central and off-central approximating spheres. In the former, the central sphere is invariant under the full group of symmetries $\mathcal{G}$ of the construction. In the latter, any off-central sphere is invariant under the subgroup $\mathcal{G}^{\text{off}}$. Moreover in our cases $\mathcal{G}^{\text{off}}$ centralizes the rotational periods $\widetilde{\mathsf{T}}_x$. Hence for each construction, it remains only to study which elements of $\ker \mathcal{L}$ on $\mathbb{S}^{n-1}$ are invariant under $\mathcal{G}$ or under $\mathcal{G}^{\text{off}}$.

The $\mathcal{G}$-equivariant kernel for $p = q$ constructions.

Recall from the discussion following 4.2 the symmetry group $\mathcal{G}_g \subset \mathrm{O}(2p)$ we imposed on our initial submanifolds in this case.

The off-central $\mathcal{G}$-invariant kernel: Every approximating sphere except the central one has symmetry group $\mathcal{G}_g^{\text{off}} \subset \mathcal{G}_g$. The subgroup $\mathcal{G}_g^{\text{off}}$ of $\mathcal{G}_g$ preserving any non-central sphere is the group $\mathcal{G}' \cong \mathbb{Z}_2 \times \mathrm{O}(p)$ defined in 4.3. Appendix A.1ii.e shows that the $\mathcal{G}'$-invariant kernel of $\mathcal{L}$ on $\mathbb{S}^{2p-1}$ is one-dimensional and is spanned by

$$f_{p,p}(x, y) = |x|^2 - |y|^2.$$

The function $f_{p,p}$ is the restriction to $\mathbb{S}^{n-1}$ of the Hamiltonian function $\widehat{f}_1 : \mathbb{C}^n \to \mathbb{R}$ defined by

$$\widehat{f}_1(\mathbf{z}) = |z|^2 - |w|^2 \tag{5.4}$$

in the notation of 3.28 and 3.29. $\widehat{f}_1$ is associated with the one-parameter subgroup $\widetilde{\mathsf{T}}_x$ associated to the angular period of $X_{\underline{\tau}}$. $\widehat{f}_1$ is also invariant under the action of the angular periods and therefore its restriction to the off-central approximating spheres spans the $\mathcal{G}$-equivariant kernel on them.

The central $\mathcal{G}$-invariant kernel: The central sphere is invariant under the full group of symmetries $\mathcal{G}_g$. In particular, the $\mathcal{G}_g$-invariant kernel of $\mathcal{L}$ is a subspace of the $\mathcal{G}'$-invariant kernel of $\mathcal{L}$ considered above. Since $f(x, y) = |x|^2 - |y|^2$ is not invariant under $\underline{\widetilde{\mathsf{I}}} \in \mathcal{G}_g$ (as defined in 3.28) , the $\mathcal{G}_g$-invariant kernel of $\mathcal{L}$ is trivial. The lack of kernel corresponding to the central sphere will imply that balancing is automatic in constructions of this type.

The $\mathcal{G}$-equivariant kernel for $p = 1$ constructions.

Recall from the discussion following 4.6 the symmetry group $\mathcal{G}_\Delta$ we imposed in this case.

The central $\mathcal{G}$-invariant kernel: The central sphere is invariant under the full group of symmetries

$$\mathcal{G}_\Delta \cong \mathbb{Z}_2 \times \widetilde{\mathcal{G}}_\Delta = \mathbb{Z}_2 \times \operatorname{Aut}\Delta \times \mathrm{O}(n-k).$$

If $k = n$, from Appendix A.1i.b, the $\widetilde{\mathcal{G}}_\Delta$-invariant kernel of $\mathcal{L}$ is trivial on $\mathbb{S}^{n-1}$. However, if $k < n$, from Appendix A.1ii.d, we have a one-dimensional $\widetilde{\mathcal{G}}_\Delta$-invariant kernel for $\mathcal{L}$ on $\mathbb{S}^{n-1}$, spanned by $f(x,y) = (n-k)|x|^2 - k|y|^2$ (where we write $x' \in \mathbb{R}^n$ as $x' = (x,y)$ with $x \in \mathbb{R}^k$ and $y \in \mathbb{R}^{n-k}$). However, the extra symmetry in the extended group $\mathcal{G}_\Delta$ arising from the anti-holomorphic isometry $\widetilde{\underline{\mathsf{C}}}$ means that there is no non-trivial $\mathcal{G}_\Delta$-invariant kernel even in the case $k < n$. The lack of kernel corresponding to the central spherical region will imply that balancing (in the sense of [39]) is automatic.

The off-central $\mathcal{G}$-invariant kernel: The off-central approximating spheres have a smaller symmetry group than the central one, namely $\mathcal{G}_\Delta^{\mathrm{off}} = \mathcal{G}_\Delta \cap \mathrm{O}(n-1)$.

Case (i): Δ is the regular k-simplex. The subgroup $\mathcal{G}_\Delta^{\mathrm{off}} = \mathrm{S}_k \times \mathrm{O}(n-k)$. Hence if $k = n$, then by Appendix A.1ii.b there is a 1-dimensional off-central kernel spanned by

$$f_1 = nx_1^2 - |x|^2 - |y|^2. \tag{5.5}$$

However, when $k < n$, then by case Appendix A.1iii.b there is a 2-dimensional off-central kernel modulo the symmetries imposed which is spanned by f_1 and

$$f_2 = (n-k)|x|^2 - k|y|^2.$$

Case (ii): Δ is a regular g-gon in $\mathbb{R}^n$. The subgroup $\mathcal{G}_\Delta^{\mathrm{off}} = \mathrm{O}(1) \times \mathrm{O}(n-2) \subset \mathrm{O}(n-1)$. Hence by Appendix A.1iii.c, there is a 2-dimensional off-central kernel modulo the symmetries imposed which is spanned by f_1 and $f_2 = (n-2)|x|^2 - 2|y|^2$.

We define now Hamiltonian functions $\widehat{f}_i : \mathbb{C}^n \to \mathbb{R}$ by

$$\widehat{f}_1(\mathbf{z}) = (n-1)|z_1|^2 - \sum_{i=2}^{n} |z_i|^2, \qquad \widehat{f}_2(\mathbf{z}) = \frac{1}{k-1}\sum_{i=2}^{k} |z_i|^2 - \frac{1}{n-k}\sum_{i=k+1}^{n} |z_i|^2. \tag{5.6}$$

$\widehat{f}_1$ is associated with the one-parameter subgroup $\widetilde{\mathsf{T}}_x$ associated to the angular period of $X_{\underline{T}}$. Both $\widehat{f}_1$ and $\widehat{f}_2$ are invariant under the action of the angular periods and therefore their restrictions (just $\widehat{f}_1$ if $k = n$) to the off-central approximating spheres span the $\mathcal{G}$-equivariant kernel on them.

Low $\mathcal{G}$-invariant harmonics of the meridians

To understand the decay and interaction between the almost spherical regions we use a Fourier decomposition along the meridians as in [35]. To achieve the decay we

need to be able to prescribe the lower harmonics of the solution on meridians close to the boundary of a transition region. In our case the meridinal harmonics up to and including degree 2 are important in ensuring the decay we need. We denote the space of such $\mathcal{G}$-equivariant harmonics on a meridian C by $\mathcal{H}[C]$. To simplify the presentation in the subsequent discussion we consider only the meridians of the building block $X_{\underline{\tau}}$ with the exception of the meridian which is furthest away from the central spherical region.

Any meridian where the building block $X_{\underline{\tau}}$ we use is of type (p,p) is the product of two round $p-1$ spheres with radii $r_1^2=|w_1|^2$ and $r_2^2=|w_2|^2$, satisfying $r_1^2+r_2^2=1$. On a transition region one of the radii is close to 1 and the other is therefore close to 0. By harmonics of degree at most two we mean those eigenfunctions which have restrictions on the factors which are harmonics of degree at most two. Any meridian where we use a building block $X_{\underline{\tau}}$ of type $(1,n-1)$ is a round $n-2$ sphere, and on a transition region this $n-2$ sphere has small radius.

The low $\mathcal{G}$-invariant harmonics on the meridians for type $(1,n-1)$ constructions

The dimension of the space of spherical harmonics of degree at most 2 on $\mathbb{S}^{n-2}\subset\mathbb{R}^{n-1}$ is

$$1+(n-1)+\left(\tfrac{1}{2}n(n-1)-1\right)=\tfrac{1}{2}n(n+1)-1,$$

which as we saw in the proof of Lemma 5.3 is also the dimension of the space of spherical harmonics of degree exactly 2 on $\mathbb{S}^{n-1}\subset\mathbb{R}^n$, *i.e.* the dimension of the kernel of $\mathcal{L}$ on $\mathbb{S}^{n-1}$ before any symmetries are imposed. However, because of the symmetries imposed many of these harmonics are not allowed.

The group of symmetries we can impose on any meridian $\mathbb{S}^{n-2}$ is precisely the same as the off-central symmetry subgroup $\mathcal{G}_\Delta^{\text{off}}=\mathcal{G}_\Delta\cap\mathrm{O}(n-1)$. Obviously, the constant function on $\mathbb{S}^{n-2}$ is $\mathcal{G}_\Delta^{\text{off}}$-invariant. Note also that the restriction of the $\mathcal{G}_\Delta^{\text{off}}$-invariant function $\widehat{f_1}\circ Y$ on a meridian is constant.

There are no $\mathcal{G}_\Delta^{\text{off}}$-invariant degree one spherical harmonics.

If $n=k$, then $\mathcal{G}_\Delta^{\text{off}}=\mathrm{S}_n\subset\mathrm{O}(n-1)$ acting on $\mathbb{S}^{n-2}$. Hence by Appendix A.1i.b there are also no $\mathcal{G}_\Delta^{\text{off}}$-invariant degree two spherical harmonics on $\mathbb{S}^{n-2}$. Hence, if $n=k$, then the only $\mathcal{G}_\Delta^{\text{off}}$-invariant spherical harmonics of degree at most two on a meridian $\mathbb{S}^{n-2}$ are the constant functions.

If $k<n$, and Δ is a regular k-simplex then $\mathcal{G}_\Delta^{\text{off}}=\mathrm{S}_k\times\mathrm{O}(n-k)\subset\mathrm{O}(n-1)\cap\mathrm{O}(k-1)\times\mathrm{O}(n-k)$ acting on $\mathbb{S}^{n-2}$. Hence by Appendix A.1ii.d, the only $\mathcal{G}_\Delta^{\text{off}}$-invariant degree two spherical harmonics on $\mathbb{S}^{n-2}\subset\mathbb{R}^{n-1}$ with coordinates written as $(x_2,\cdots,x_k,y_1,\cdots,y_{n-k})$ are multiples of

$$f(x_2,\cdots,x_k,y_1,\cdots,y_{n-k})=\frac{1}{k-1}\sum_{i=2}^{k}x_i^2-\frac{1}{n-k}\sum_{j=1}^{n-k}y_j^2.$$

Clearly then $\widehat{f_2}\circ Y$ restricted on a (generic) meridian spans the $\mathcal{G}_\Delta^{\text{off}}$-invariant degree two spherical harmonics on the meridian. A similar argument establishes that the same statement holds in the case where $k=2$ and Δ is a g-gon with g odd.

The low $\mathcal{G}$-invariant harmonics on the meridians for type (p, p) constructions.

The subgroup of $\mathcal{G}_g$ mapping any meridian to itself is the subgroup $\mathcal{G}'$ defined in 4.3. In the Appendix we prove the following

Lemma 5.7. *A function $f(x, y) : \mathbb{S}^{p-1} \times \mathbb{S}^{p-1} \to \mathbb{R}$ is invariant under the group $\mathcal{G}'$ defined in* 4.3 *if and only if $f(x, y) = g(\langle x, y\rangle)$ for some even function $g : [-1, 1] \to \mathbb{R}$.*

On the Riemannian product of two compact Riemannian manifolds (M_1, g_1) and (M_2, g_2), the spectrum of the Laplacian on functions is given by

$$\mathrm{Spec}(\Delta_{g_1 \times g_2}) = \mathrm{Spec}(\Delta_{g_1}) + \mathrm{Spec}(\Delta_{g_2}).$$

Hence on $\mathbb{S}^{p-1}(r_1) \times \mathbb{S}^{p-1}(r_2)$ the spectrum of the Laplacian is given by

$$\mathrm{Spec}(\Delta) = \{r_1^{-2} i(p - 2 + i) + r_2^{-2} j(p - 2 + j) \,|\, i, j \in \mathbb{Z},\ i, j \geqslant 0.\}$$

In any transition region of $X_{\underline{\tau}}$, one of the two radii r_1, r_2 is close to one while the other radius is close to zero. Hence, before we impose any symmetry on such a meridian, there are low lying eigenvalues of Δ very close to the low lying eigenvalues of the sphere $\mathbb{S}^{p-1}(1)$ (and these are the only low lying eigenvalues). In particular, the first nonzero eigenvalue λ_1 is very close to $p-1$ and has multiplicity p (the corresponding eigenfunctions being the product of a degree one spherical harmonic on the large sphere with the constant function 1 on the small sphere).

However, the $\mathcal{G}'$-invariant spectrum of Δ on $\mathbb{S}^{p-1}(r_1) \times \mathbb{S}^{p-1}(r_2)$ is very different. If f has the form $f(x, y) = g(\langle x, y\rangle)$ for $g : [-1, 1] \to \mathbb{R}$, then f is an eigenfunction of Δ on $\mathbb{S}^{p-1}(r_1) \times \mathbb{S}^{p-1}(r_2)$ with eigenvalue λ if and only if for each fixed $y \in \mathbb{S}^{p-1}$ the function

$$x \mapsto f(x, y) = g(\langle x, y\rangle)$$

is an eigenfunction of the Laplacian on $\mathbb{S}^{p-1}(1)$ with eigenvalue $\lambda r_1^{-2} r_2^{-2}$. This is equivalent to the fact that $x \mapsto f(x, y)$ is an O_y-invariant eigenfunction of $\mathbb{S}^{p-1}(1)$, where $O_y \cong \mathrm{O}(p-1) \subset \mathrm{O}(p)$ denotes the stabilizer of y in $\mathrm{O}(p)$.

The O_y-invariant spherical harmonics are known as the *zonal spherical harmonics*. It is well-known that for each degree k, the zonal spherical harmonics are spanned by a single element denoted $Z_{k,y}$. $Z_{k,y}(x)$ can be expressed in terms of the (one-variable) Gegenbauer (or ultraspherical harmonic) polynomials $G_k^{n/2-1}(t)$ as

$$Z_{k,y}(x) = G_k^{n/2-1}(\langle x, y\rangle).$$

In particular, for each fixed $y \in \mathbb{S}^{p-1}$, $Z_{k,y}(x)$ is an eigenfunction on $\mathbb{S}^{p-1}$ with eigenvalue $\lambda_k = k(p - 2 + k)$. Hence $f(x, y) = g(\langle x, y\rangle)$, where g is the Gegenbauer polynomial $G^{n/2-1}$, is the unique $\mathcal{G}''$-invariant (where $\mathcal{G}''$ is the subgroup of $\mathcal{G}'$ defined in 4.4) eigenfunction of Δ on $\mathbb{S}^{p-1}(r_1) \times \mathbb{S}^{p-1}(r_2)$ with eigenvalue

$r_1^{-2}r_2^{-2}k(p-2+k)$. Hence the $\mathcal{G}''$-invariant spectrum of Δ on $\mathbb{S}^{p-1}(r_1)\times\mathbb{S}^{p-1}(r_2)$ is

$$\begin{aligned}\mathrm{Spec}(\Delta^{\mathcal{G}''}_{\mathbb{S}^{p-1}(r_1)\times\mathbb{S}^{p-1}(r_2)}) &= r_1^{-2}r_2^{-2}\,\mathrm{Spec}(\Delta_{\mathbb{S}^{p-1}(1)})\\ &= \{r_1^{-2}r_2^{-2}j(p-2+j)\,|\,j\in\mathbb{Z}, j\geqslant 0\},\end{aligned}$$

and every eigenvalue is simple. The $\mathcal{G}''$-invariant function $f(x,y)=g(\langle x,y\rangle)$ is $\mathcal{G}'$-invariant if and only if the function g is even. Also the Gegenbauer polynomials G_k^λ are odd functions when k is odd and even functions when k is even. Hence the $\mathcal{G}'$-invariant spectrum is

$$\mathrm{Spec}(\Delta^{\mathcal{G}'}_{\mathbb{S}^{p-1}(r_1)\times\mathbb{S}^{p-1}(r_2)}) = \{r_1^{-2}r_2^{-2}j(p-2+j)\,|\,j\in 2\mathbb{Z}, j\geqslant 0\},$$

and again every eigenvalue is simple. In particular, on a meridian on which say r_2 is close to 1 (and hence r_1 close to 0), we have

$$\lambda_1^{\mathcal{G}'}\approx\frac{2p}{r_1^2}.$$

Hence the only functions in $\mathcal{H}[C]$ for C a meridian close to the boundary of a transition region are constants. Note that $\widehat{f}_1\circ Y$ restricts to a nonvanishing constant on such a meridian. Hence $\widehat{f}_1\circ Y$ spans $\mathcal{H}[C]$ for such a meridian in this case also. As we will see this allows us to control the decay of solutions by controlling the amount of the constants on a meridian close to the boundary of the transition region.

The spaces $\mathcal{K}_{sym}[\widehat{S}]$ and $\mathcal{H}[C]$ in conlusion

To facilitate future reference we recall now our conlusions about the spaces $\mathcal{K}_{sym}[\widehat{S}]$ and $\mathcal{H}[C]$ where $\widehat{S}$ is a non-central approximating sphere and C is a meridian close to the boundary of a transition region of the building block $X_{\underline{\tau}}$. All these spaces in each of our constructions have the same dimension which we call $\widehat{m}$. $\widehat{m}=1$ for the constructions with $p=q$ and also for the constructions with $p=1, n=k$. $\widehat{m}=2$ for the constructions with $p=1, k<n$. When $\widehat{m}=1$ all these spaces are spanned by the restriction of $\widehat{f}_1$ to $\widehat{S}$ or $\widehat{f}_1\circ Y$ to C, where $\widehat{f}_1$ is defined by 5.4 when $p=q$ or by 5.6 when $p=1$. When $\widehat{m}=2$ all these spaces are spanned by the restrictions of $\widehat{f}_1$ and $\widehat{f}_2$ to $\widehat{S}$, or $\widehat{f}_1\circ Y$ and $\widehat{f}_2\circ Y$ to C, where $\widehat{f}_1$ and $\widehat{f}_2$ are defined by 5.6.

6 The linearized equation

Introduction

The main focus of the section is on developing a sufficiently good understanding of the linearized equation and the linearized operator for functions which should correct our initial Legendrian immersions to special Legendrian immersions. In

particular we state and discuss the proof of proposition 6.8. This proposition allows us to solve the linearized inhomogeneous equation with appropriate estimates with decay. We can only solve modulo the extended substitute kernel which we also discuss in this section.

The linearized equation on the transition regions

We first study the linearized equation on the transition regions. Recall that a transition region is some $\Lambda = \mathrm{Cyl}_I^{p,q}$ on which the induced metric is given by 3.8 or 3.18. In the case $p = 1$ the radius $r := |w_2|$ varies from a maximum value $r_{\partial\Lambda}$ attained on $\partial\Lambda$, to a minimum value $r_{min} \sim \tau^{1/(n-1)}$ in the middle of Λ. Similarly in the case $p = q$, one of the radii we call r (that is $r = |w_1|$ or $r = |w_2|$ depending on the type of the neck) varies from a maximum value $r_{\partial\Lambda}$ attained on $\partial\Lambda$, to a minimum value $r_{min} \sim \tau^{1/p}$ in the middle of Λ. The other radius then is close to one (we assume $r_{\partial\Lambda}$ small independently of τ by the definition of the transition region Λ). We subdivide $\Lambda = \Lambda_+ \cup \Lambda_-$, where Λ_+ and Λ_- are the closures of the two connected components into which Λ is subdivided if we remove the middle meridian where $r = r_{min}$. We also write $\partial\Lambda = C^+ \cup C^-$ where $C^+ \subset \Lambda_+$ and $C^- \subset \Lambda_-$. C^+ and Λ_+ are chosen so that they are closer to the central spherical region than C^- and Λ_- in the intrinsic geometry of the surface, with the exception that in the case that Λ is the transition region furthest away this does not hold. In the remaining discussion we ignore this exceptional transition region to simplify the presentation.

We can understand the linearized equation on Λ by comparing the behaviour of $\mathcal{L}$ on each of $\Lambda_\pm$ to the behaviour of Δ on an annulus in $\mathbb{R}^q$ ($q = n-1$ or $q = p$ in the cases under consideration respectively). The outer radius of the annulus is $r_{\partial\Lambda}$ and the inner radius is $\sim r_{min} \sim \tau^{1/q}$. Let r' be the ratio of the inner over the outer radius, then $r' \sim \tau^{1/q}$. We fix some $\gamma \in (2, 3)$ and we work with a decay at the rate of r^γ on Λ_+ as we move away from C^+, and at the rate of $r^{2-n-\gamma}$ as we move towards C^- on Λ_-. More precisely we define Hölder norms with decay on Λ by

$$\begin{aligned} \|u\|_{k,\beta,\gamma;\Lambda} &:= \sup_{x\in\Lambda} f^{-1}(x)\, \|u : C^{k,\beta}(B_x(r/10), r^{-2}(x)\, g)\|, \\ \|u\|'_{k,\beta,\gamma;\Lambda} &:= \sup_{x\in\Lambda} r^2(x)\, f^{-1}(x)\, \|u : C^{k,\beta}(B_x(r/10), r^{-2}(x)\, g)\|, \end{aligned} \tag{6.1}$$

where $B_x(r/10)$ is the geodesic ball in Λ with respect to the induced metric g, centered at x and of radius $r/10$, the metric on the geodesic ball is the induced metric g enlarged by a factor $r^{-2}(x)$, and the weight function f on Λ is defined as follows: On Λ^+ where $r > r' r_{\partial\Lambda}$ we have $f = (r/r_{\partial\Lambda})^\gamma$. Where $r \leqslant r' r_{\partial\Lambda}$ we have $f = r'^\gamma$. On Λ^- where $r \geqslant r' r_{\partial\Lambda}$ we have $f = c r^{2-q-\gamma}$ where c is such that on $r = r' r_{\partial\Lambda}$ this agrees with the previous definition. Notice that f is continuous, $f(C^+) = 1$, and the total decay along Λ is by a factor

$$d := f(C^-) = r'^{2\gamma+q-2} \sim \tau^{1+2(\gamma-1)/q}. \tag{6.2}$$

Since we are working subject to the symmetries imposed by $\mathcal{G}$, we are only interested in functions f on Λ which satisfy $f \circ \gamma = \pm f$ for $\gamma \in \mathcal{G}_\Lambda$, where the sign

is determined as in 5.2 and $\mathcal{G}_\Lambda$ is the stabilizer of Λ by the action of $\mathcal{G}$. We use the subscript "*sym*" as before to denote subspaces of such equivariant functions on Λ.

In order to achieve appropriate decay for a solution of the linearized equation on Λ, we need to be able to modify the boundary conditions on C^+ by up to second order harmonics. Recall that we denote the space of such harmonics on C^+ which are compatible with the symmetries by $\mathcal{H}[C^+]$ and that the dimension of $\mathcal{H}[C^+]$ is $\widehat{m}$ which is one or two depending on the case at hand. We have now the following proposition:

Proposition 6.3. *There is a linear map $\mathcal{R}_\Lambda : C^{0,\beta}_{sym}(\Lambda) \to C^{2,\beta}_{sym}(\Lambda)$ such that the following hold for $E \in C^{0,\beta}_{sym}(\Lambda)$ and $V := \mathcal{R}_\Lambda E$:*
(i). *$\mathcal{L}V = E$ on Λ.*
(ii). *$V|_{C^+} \in \mathcal{H}[C^+]$ and V vanishes on C^-.*
(iii). *$\|V\|_{2,\beta,\gamma;\Lambda} \leqslant C \|E\|'_{0,\beta,\gamma;\Lambda}$.*

An obvious modification of this proposition holds with the roles of C^+ and C^- exchanged. Also this proposition can be modified to apply to solutions of the Dirichlet problem for $\mathcal{L}$ on Λ, where the given Dirichlet data on C^- vanish, and we are allowed to modify the given data on C^+ by adding second harmonics on C^+. In this case the given data control the norm of the solution which decays as before.

Now that we understand the linearized equation on the transition regions as needed, we proceed to study it on extended almost spherical regions. An extended (almost) spherical region is defined to be the union of an almost spherical region and its adjacent transition regions.

The approximate kernel

Now that we understand the linearized equation on the transition regions as needed, we proceed to study it on larger regions. Our final goal in this section is to understand and solve the linearized equation on an initial almost special Legendrian immersion $Y : M \to \mathbb{S}^{2n-1}$. In order to solve the inhomogeneous linearized equation globally on M we solve it first on appropriate regions where we understand the equation better, and then we combine these "semi-local" solutions.

One of the major obstacles to solving the linearized equation on our initial immersion Y is the existence of small eigenvalues or approximate kernel of the linearized operator $\mathcal{L}$ on any region which contains an almost spherical region, arising from the existence of nontrivial kernel for $\mathcal{L}$ on the associated approximating spheres. By "approximate kernel", we mean in general (see [30–39, 44]) the span of the eigenfunctions with small eigenvalue for the linearized operator. "Approximate" refers to the smallness of eigenvalues rather than their vanishing as for the "exact" kernel. To be more precise, we adopt for this paper the following definition:

Definition 6.4. *The approximate kernel on a domain Ω of our submanifold is defined to be the span of the eigenfunctions for the Dirichlet problem with vanishing Dirichlet data, for $\mathcal{L} = \Delta + 2n$ on Ω, and whose corresponding eigenvalues are in $(-1, 1)$.*

Remark 6.5. In all the cases considered in this paper the eigenvalues that appear in the interval $(-1,1)$ will all be very close to 0. Exactly how close to 0 depends essentially on the choice of the parameter τ in the building blocks X_τ used. This is what justifies the use of the term *approximate kernel*. The choice of the particular interval $(-1,1)$ in Definition 6.4 is not essential.

We need to understand the linearized equation on regions we call *extended almost spherical regions*. An extended almost spherical region is the union of an almost spherical region with its adjacent transition regions—$2g$ transition regions for the central almost spherical region or two transition regions otherwise. An extended almost spherical region, as well as any region containing an almost spherical region, inherits from the approximating sphere eigenfunctions corresponding to the eigenfunctions of the kernel on the approximating sphere. The advantage of using an extended spherical region is that there are no more eigenfunctions of small eigenvalue, because the transition regions do not carry low spectrum, while the region is large enough for the solution to decay along the transition regions towards the boundary as needed.

We consider now an almost spherical region S, and we denote by $\widetilde{S}$ the corresponding extended spherical region, and by $\widehat{S}$ the corresponding approximating sphere. We have already studied the kernel $\mathcal{K}_{sym}[\widehat{S}]$ on the approximating sphere $\widehat{S}$ of functions which respect the symmetries, that is they are equivariant under the action of the stabilizer $\mathcal{G}_S$ of S (equivalently of $\widehat{S}$) in the sense of 5.2. We have found that in the cases under consideration $\mathcal{K}_{sym}[\widehat{S}]$ has dimension at most two, and when nontrivial it is generated by the restrictions to $\widehat{S}$ of the functions $\{\widehat{f}_i\}_{i=1}^{\widehat{m}}$ defined in 5.4 and 5.6.

Recall also that S approximates a domain of $\widehat{S}$ whose complement is a δ-tubular neighborhood of the attachment set for some $\delta > 0$. We denote by S' a neighborhood of S which is approximated by the complement of the $\delta/2$-tubular neighborhood of the attachment set. We have then the following proposition:

Proposition 6.6. *Assuming δ and $\underline{\tau}$ small enough in terms of $\varepsilon > 0$ the following hold: The approximate kernel on $\widetilde{S}$ of functions which respect the symmetries, that is they are equivariant under the action of the stabilizer $\mathcal{G}_S$ of S (equivalently of $\widetilde{S}$) in the sense of 5.2, has the same dimension as $\mathcal{K}_{sym}[\widehat{S}]$. In particular it is trivial when S is the central spherical region and (in the cases we consider) has dimension $\widehat{m}$ if S is not the central spherical region. In the latter case it is spanned by orthonormal eigenfunctions $\{f_{i,\widetilde{S}}\}_{i=1}^{\widehat{m}}$ which satisfy*

$$\|f_{i,\widetilde{S}} - \|\widehat{f}_i \circ Y : L^2(S')\|^{-1}\, \widehat{f}_i \circ Y : C^{2,\beta}(S')\| < \varepsilon.$$

The proof of this proposition is by comparing with the limiting manifold as $\tau \to 0$: Each transition region tends to two tubular neighborhoods of a component of the attachment set, one neighborhood in $\widehat{S}$ and one neighborhood in the approximating sphere of an adjacent almost spherical region. Therefore the limiting manifold is $\widehat{S}$ (with the attachment set removed) and a collection of small tubular neighborhoods of attachment sets. The tubular neighborhoods do not carry

low spectrum and the removal of the attachment set from $\widehat{S}$ does not change the spectrum, therefore the kernel on the limiting manifold is the same as that of $\widehat{S}$.

The comparison is carried out by using the results [31, Appendix B] where instead of Sobolev inequalities we use our understanding of the linearized equation on the transition regions to obtain supremum bounds.

The extended substitute kernel

In order to solve the inhomogeneous linear equation we need to correct the inhomogeneous term so that it becomes orthogonal to the approximate kernel. By doing so we can get solutions with good estimates. If we were to use the actual approximate kernel to correct the inhomogeneous terms we would run into technical difficulties because the approximate kernel depends on the extended almost spherical region under consideration. A simple device to bypass this technical difficulty is the use of what we call the *substitute kernel* as in [31].

The substitute kernel consists of functions globally defined but vanishing on the transition regions. The space of their restrictions to an extended almost spherical region $\widetilde{S}$ has the same dimension as the approximate kernel on that region. More specifically, for each almost spherical region S we define some space $\mathcal{K}_{substitute}[S] \subset C^{\infty}_{sym}(M)$ whose functions are supported on $\mathcal{G}S$, and such that the L^2 projection of $\mathcal{K}_{substitute}[S]$ onto the approximate kernel of the corresponding extended almost spherical region $\widetilde{S}$ is invertible with an inverse whose norm is bounded independently of $\underline{\tau}$. We define then the substitute kernel $\mathcal{K}_{substitute}$ as the span of these $\mathcal{K}_{substitute}[S]$'s.

The main property of the substitute kernel is that given $E \in L^2(\widetilde{S})$ there is a unique $w \in \mathcal{K}_{substitute}[S]$ such that $E - w$ is orthogonal to the approximate kernel of the corresponding extended spherical region $\widetilde{S}$. We can solve then the Dirichlet problem

$$\mathcal{L}u = E - w \quad \text{on} \quad \widetilde{S}, \qquad u = 0 \quad \text{on} \quad \partial\widetilde{S}.$$

to obtain u which satisfies good estimates. Suppose now that S is not the central spherical region. S has then two adjacent transition regions, let Λ be the one furthest away from the central one in the intrinsic geometry. We would like u to decay fast away from S on this $\Lambda \subset \widetilde{S}$. To achieve this we need a space of functions $\mathcal{K}_v[S] \subset C^{2,\beta}_{0,sym}(\widetilde{S})$ which is mapped by $\mathcal{L}$ to $\mathcal{K}_{substitute}[S]$ and which has the following property: The map from $\mathcal{K}_v[S]$ to $\mathcal{H}[C^+]$ defined by first restricting the function to C^+ and then projecting L^2 orthogonally to $\mathcal{H}[C^+]$, is invertible and the inverse has bounded norm by a bound independent of $\underline{\tau}$.

When S is not the central spherical region such a space can be easily found by using a simple construction first used in [35, Lemma 6.6]. We can then modify the solution u to $u + v$ for some $v \in \mathcal{K}_{substitute}[S]$ so that the L^2 projection on C^+ onto $\mathcal{H}[C^+]$ can be chosen as needed, and at the same time it solves the Dirichlet problem

$$\mathcal{L}(u + v) = E - w' \quad \text{on} \quad \widetilde{S}, \qquad u + v = 0 \quad \text{on} \quad \partial\widetilde{S}.$$

for some $w' \in \mathcal{K}_{substitute}[S]$ as before. By using our understanding of $\mathcal{L}$ on Λ and the properties of $\mathcal{K}_v[S]$ we can choose the projection then so that the solution

$u + v$ has the required decay on Λ and satisfies good estimates.

In the case that S is the central spherical region $S_{central}$, $\mathcal{K}_{sym}[\widehat{S}_{central}]$ is trivial, and so $\mathcal{K}_{substitute}[S_{central}]$ is trivial as well. It is still important however to arrange decay of the solutions on $\widetilde{S}_{central}$ along the adjacent necks away from $S_{central}$. To make this possible a space $\mathcal{K}_v[S_{central}]$ is needed as before. Let C^+ be a component of $\partial S_{central}$ contained in M_1. The map from $\mathcal{K}_v[S_{central}]$ to $\mathcal{H}[C^+]$ defined by restriction to C^+ followed by L^2 projection to $\mathcal{H}[C^+]$ should have an inverse of norm independent of $\underline{\tau}$. In particular the dimension of $\mathcal{K}_v[S_{central}]$ should be the same as the dimension $\widehat{m}$ of $\mathcal{H}[C^+]$.

Such a space can be defined by solving the Dirichlet problem on the transition regions and cutting off the solutions appropriately. We can then modify the solutions to the inhomogeneous Dirichlet problem on $\widetilde{S}_{central}$, to new solutions which have appropriate decay and estimates, at the expense of modifying the given inhomogeneous term by an element of $\mathcal{L}(\mathcal{K}_v[S_{central}])$. The functions in $\mathcal{L}(\mathcal{K}_v[S_{central}])$ can be extended trivially to smooth functions on M. We define $\mathcal{K}_{extended}[S_{central}]$ to be the space consisting of these extensions, and we call it *the extended substitute kernel on* $S_{central}$. We also define the *extended substitute kernel* $\mathcal{K}_{extended}$ to be the span of $\mathcal{K}_{substitute}$ and $\mathcal{K}_{extended}[S_{central}]$.

We denote the dimension of $\mathcal{K}_{extended}$ by $\widetilde{m}$. By referring to 3.43, 3.47, and the action of $\mathcal{G}$, we see that there are $q\underline{m}$ almost spherical regions modulo the action of $\mathcal{G}$ in our initial surfaces ($q = n - 1$ or $q = p$ in the cases we discuss). Therefore we have $\widetilde{m} = q\underline{m}\widehat{m}$, where as we have determined earlier $\widehat{m} = 1$ in the cases $p = q$, or $p = 1$ with $n = k$, and $\widehat{m} = 2$ in the cases $p = 1$ with $k < n$. In all cases we can use the elements of $\mathcal{K}_{extended}$ corresponding to restrictions of the $\widehat{f}_i$'s on non-central regions or appropriate meridians for the central region to produce a basis for $\mathcal{K}_{extended}$. This basis induces as usual an isomorphism between $\mathbb{R}^{\widetilde{m}}$ and $\mathcal{K}_{extended}$. We denote by $w_{\boldsymbol{\xi}}$ the element of $\mathcal{K}_{extended}$ which corresponds by this isomorphism to $\boldsymbol{\xi} \in \mathbb{R}^{\widetilde{m}}$.

Solving the linearized equation globally

By the previous subsection we can always solve the inhomogeneous Dirichlet problem for $\mathcal{L}$ on any extended spherical region, provided we appropriately modify the inhomogeneous term by elements of $\mathcal{K}_{extended}$, to obtain solutions which satisfy good estimates and decay appropriately on the transition regions. By cutting and pasting we can combine these semi-local solutions to obtain a global approximate solution which only fails to satisfy the equation by an error term which is small compared with the initial data E. By iterating we obtain the desired solution. We provide the precise statement and estimates below in proposition 6.8. Before we do so we define the norms with decay we will be using.

Definition 6.7. *Given $u \in C^{k,\beta}_{sym}(M)$ we define $\|u\|_{k,\beta,\gamma}$ (or $\|u\|'_{k,\beta,\gamma}$) to be the maximum of $d^j\|u\|_{k,\beta,\gamma;\Lambda}$ (or $d^j\|u\|'_{k,\beta,\gamma;\Lambda}$) and $d^j\|u : C^{k,\beta}(S, g)\|$ for any transition region Λ and almost spherical region S, where the norms and d are as in 6.1 and 6.2, and j is the (minimum) number of transition regions between the region Λ or S in consideration and the central spherical region.*

The decay we have therefore is consistent with the decay we imposed along the necks as we move away from the central spherical region.

Proposition 6.8. *There is a linear map* $\mathcal{R}_M : C^{0,\beta}_{sym}(M) \to C^{2,\beta}_{sym}(M) \times \mathcal{K}_{extended}$ *such that for* $E \in C^{0,\beta}_{sym}(M)$ *and* $(\varphi, w) = \mathcal{R}_M E$ *the following hold:*
(i). $\mathcal{L}\varphi = E + w$ *on* M.
(ii). $\|\varphi\|_{2,\beta,\gamma} \leqslant C \|E\|'_{0,\beta,\gamma}$.
(iii). $\|w\|_{2,\beta,\gamma} \leqslant C \|E\|'_{0,\beta,\gamma}$.

7 Using the Geometric Principle to prescribe the extended substitute kernel

Introduction and conclusion

In this section we discuss how to prescribe elements of the extended substitute kernel as required by our general approach [35, 39]. This is necessary because the linearized problem is solved in 6.8 only modulo such elements. Recall that our construction is carried out in two main stages: In the first stage an initial submanifold is constructed. In the second stage a function on the initial submanifold is chosen and the initial submanifold is perturbed according to the information provided by the chosen function. By modifying either of these steps in a controlled way we can modify the Lagrangian angle of the final submanifold. We want the new Lagrangian angle introduced by these modifications to be any small prescribed element of the extended substitute kernel.

According to the geometric principle proposed in [34, 35], the needed modifications amount to a dislocation of the submanifold, where the standard regions are repositioned relative to each other by the flow of the Killing fields which are responsible in the first place for the approximate kernel and the low harmonics on the meridian. All these dislocations could be introduced at the first stage of the construction as in [31]. This would result in a family of initial submanifolds depending on a large number of parameters equal to the dimension of $\mathcal{K}_{extended}$. Some of these dislocations are small enough however, so that if we describe them by a function as in the second stage, the nonlinearity of the perturbation is negligible. It is simpler then to introduce these small dislocations at the second stage of the construction as in [34, 35]. This approach results in a family of initial submanifolds which depend only on a small number of parameters. In our case the number of parameters equals the dimension $\widehat{m}$ of $\mathcal{K}_{extended}[S_{central}]$ which is one or two. The rest of the parameters control a function which is added to the function used in the second stage in the construction.

More precisely the creation of extended substitute kernel is controlled by $\boldsymbol{\xi} \in \mathbb{R}^{\widetilde{m}}$. Recall that as we have already discussed $\mathbb{R}^{\widetilde{m}}$ is isomorphic to $\mathcal{K}_{extended}$ by an isomorphism induced by the basis of $\mathcal{K}_{extended}$ whose elements correspond to restrictions of the $\widehat{f}_i$'s. Under this isomorphism $\boldsymbol{\xi} \in \mathbb{R}^{\widetilde{m}}$ corresponds to $w_{\boldsymbol{\xi}} \in \mathcal{K}_{extended}$. We define a norm with appropriate decay by $\|\boldsymbol{\xi}\|_\gamma := \|w_{\boldsymbol{\xi}}\|_{0,0,\gamma}$. We

consider $\boldsymbol{\xi} \in \mathbb{R}^{\widetilde{m}}$ which satisfy the condition

$$\|\boldsymbol{\xi}\|_\gamma \leqslant \underline{c}\,\underline{\tau}, \tag{7.1}$$

where $\underline{c}$ is a constant independent of $\underline{\tau}$ and $\underline{m}$ which is chosen in the proof of the main theorem.

We denote by $\boldsymbol{\zeta} \in \mathbb{R}^{\widehat{m}}$ the vector whose coordinates are the (one or two) coordinates of $\boldsymbol{\xi}$ corresponding to $\mathcal{K}_{extended}[S_{central}]$. Clearly $|\boldsymbol{\zeta}| \leqslant \underline{c}\,\underline{\tau}$. The initial immersion $Y : M \to \mathbb{S}^{2n-1}$ is modified to $Y_{\boldsymbol{\zeta}} : M \to \mathbb{S}^{2n-1}$ which is still equivariant under the actions of $\mathcal{G}$, but a dislocation is introduced between the building blocks and the central equatorial sphere $\mathbb{S}^{n-1}$. The Lagrangian angle θ of $Y_{\boldsymbol{\zeta}}$ can be decomposed as

$$\theta = \theta_{gluing} + \theta_{dislocation},$$

where θ_{gluing} and $\theta_{dislocation}$ are supported on disjoint subsets of $S_{central}$ and satisfy estimates

$$\|\theta_{gluing}\| \leqslant C\underline{\tau}, \qquad \|\theta_{dislocation}\| \leqslant C|\boldsymbol{\zeta}|,$$

where the norms are $C^k(g)$ for some k and the constants depend only on k and are independent of $\underline{m}$ and $\underline{\tau}$. Moreover on M we construct a smooth function $\Phi_{\boldsymbol{\xi}}$ which is $\mathcal{G}$-equivariant and satisfies the following:

Proposition 7.2. *$\Phi_{\boldsymbol{\xi}}$ depends continuously on $\boldsymbol{\xi}$ and satisfies the following:*
(i). *$\mathcal{L}\Phi_{\boldsymbol{\xi}} + \theta_{dislocation} = \underline{w}_{\boldsymbol{\xi}}$ on M, where $\mathcal{L} = \Delta + 2n$ is the linearized operator with Δ the Laplacian with respect to the metric induced by $Y_{\boldsymbol{\zeta}}$, and $\underline{w}_{\boldsymbol{\xi}} \in \mathcal{K}_{extended}$ and depends continuously on $\boldsymbol{\xi}$.*
(ii). *$\|\Phi_{\boldsymbol{\xi}}\|_{2,\beta,\gamma} \leqslant C\,\|\boldsymbol{\xi}\|_\gamma$.*
(iii). *$\|\underline{w}_{\boldsymbol{\xi}} - w_{\boldsymbol{\xi}}\|_\gamma \leqslant C\,\kappa(\underline{\tau})\,\|\boldsymbol{\xi}\|_\gamma$, where $\kappa(\underline{\tau})$ is either a positive power of $\underline{\tau}$ or $-1/\log|\underline{\tau}|$ depending on p and q.*

We proceed now to discuss the constructions of $Y_{\boldsymbol{\zeta}}$ and $\Phi_{\boldsymbol{\xi}}$.

The construction of the initial immersions $Y_{\boldsymbol{\zeta}} : M \to \mathbb{S}^{2n-1}$

We first consider the case $p = q$. We have then $\widehat{m} = 1$ and $\widehat{f}_1$ is given by 5.4 which is associated with the one-parameter subgroup $\widetilde{\mathsf{T}}_x$. We can create then the appropriate dislocation by modifying the construction of the initial immersion by using the building block X_τ instead of $X_{\underline{\tau}}$. This introduces a change of the angular period to $\widehat{\mathsf{p}}_\tau$. X_τ is rotated so that the furthest away neck is symmetric with respect to $\underline{\widetilde{\mathsf{T}}}$ defined in 3.28. This is consistent with the symmetries because $\underline{\widetilde{\mathsf{T}}}$ anticommutes with the $\widetilde{\mathsf{T}}_x$'s in the sense that $\underline{\widetilde{\mathsf{T}}} \circ \widetilde{\mathsf{T}}_x \circ \underline{\widetilde{\mathsf{T}}} = \widetilde{\mathsf{T}}_{-x}$.

The dislocation introduced then is $\widetilde{\mathsf{T}}_{m'(\widehat{\mathsf{p}}_{\underline{\tau}} - \widehat{\mathsf{p}}_\tau)}$, where m' is the number of almost spherical regions between the central region and the furthest away neck. Since the desired dislocation is $\widetilde{\mathsf{T}}_{\zeta_1}$ this determines τ through equations 3.52, in particular using 3.53 and 7.1 we conclude that

$$|\tau - \underline{\tau}| \leqslant C\,\underline{c}\,\underline{\tau}^2. \tag{7.3}$$

In the case $p = 1$ we have $\widehat{f}_1$ given by 5.6 associated again to the corresponding one-parameter subgroup $\widetilde{\mathsf{T}}_x$. We can again create the appropriate dislocation by changing $\underline{\tau}$ to τ and 7.3 is satisfied again by similar arguments as in the previous case. When $k < n$ however, (as in the case of the configuration based on a polygon in the special case $p = 1$, $q = 2$ of [16]) we have $\widehat{m} = 2$. $\widehat{f}_2$ is given then by 5.6 and is associated to the one-parameter subgroup $\{\widetilde{\mathsf{Q}}_y\}$ where

$$\widetilde{\mathsf{Q}}_y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{iy/(k-1)}\,\mathrm{Id}_{k-1} & 0 \\ 0 & 0 & e^{-iy/(n-k)}\,\mathrm{Id}_{n-k} \end{pmatrix}, \tag{7.4}$$

where Id_p denotes the $p \times p$ identity matrix. Notice that $\widetilde{\mathsf{Q}}_y$ commutes with $\widetilde{\mathsf{T}}_x$ and anticommutes with $\widetilde{\underline{\mathsf{T}}}$.

We need to create a dislocation corresponding to $\widetilde{\mathsf{Q}}_{\zeta_2}$. To achieve this we would like to employ some special Legendrian modification $X_{\tau,\alpha}$ of X_τ in place of X_τ, where $X_{\tau,\alpha}$ would have rotational period $\widetilde{\mathsf{Q}}_\alpha \circ \widetilde{\mathsf{T}}_{\widehat{\mathsf{p}}_\tau}$ instead of $\widetilde{\mathsf{T}}_{\widehat{\mathsf{p}}_\tau}$. α would be determined by ζ_2 and arguing as before, we determine that (recall 3.48)

$$|\alpha| \leqslant C\,\underline{c}\,\underline{\tau}^2\,T_{n-1}(\underline{\tau}). \tag{7.5}$$

Unfortunately such an $X_{\tau,\alpha}$ is not rotationally invariant and we have to construct it as a perturbation of X_τ. Several of the symmetries of X_τ persist and the fundamental domain contains a single spherical region. Moreover the symmetries allow no approximate kernel and therefore the construction is simplified considerably. The construction of $X_{\tau,\alpha}$ is carried out in two steps as usual: First, X_τ is modified with a bare hands construction to a Legendrian immersion $X'_{\tau,\alpha}$ which satisfies the required symmetries. Second, the new immersion is corrected by finding a function $\widetilde{f}'$ which determines a perturbation $X_{\tau,\alpha}$ which is special Legendrian and respects the symmetries. $\widetilde{f}$ is determined by solving the appropriate Partial Differential Equation as usual. Because of the smallness of the required α's and the triviality of the approximate kernel the estimates we obtain are strong enough to allow easy control of the nonlinearity. Moreover we can easily establish continuous dependence on the parameters τ and α. By using $X_{\tau,\alpha}$ in place of X_τ we construct $Y_{\boldsymbol{\zeta}} : M \to \mathbb{S}^{2n-1}$ which is $\mathcal{G}$-equivariant, depends continuously on $\boldsymbol{\zeta}$, and is special Legendrian.

The construction of $\Phi_{\boldsymbol{\xi}}$

We discuss now how to construct $\Phi_{\boldsymbol{\xi}}$ so that 7.2 is satisfied. We construct functions on extended spherical regions which satisfy the required properties and then by cutting them off appropriately we combine them to a global function which almost satisfies the required properties except for some error because of the cutting-off. We can correct this error by appealing to proposition 6.8. The semi-local constructions vary depending on whether the spherical region in consideration is the central or the non-central one.

We discuss first the case of the central region. We want to construct $\phi_{\boldsymbol{\zeta}}$, a $\mathcal{G}$-equivariant function on $\widetilde{S}_{central}$ continuously dependent on $\boldsymbol{\zeta}$ and which satisfies

the following:
(i). $\mathcal{L}\phi_{\boldsymbol{\zeta}} + \theta_{dislocation} = w' \in \mathcal{K}_{extended}[S_{central}]$ on $\widetilde{S}_{central}$, where $\theta_{dislocation}$ is induced by $Y_{\boldsymbol{\zeta}}$.
(ii). $\phi_{\boldsymbol{\zeta}} = 0$ on $\partial\widetilde{S}[0]$.
(iii). $\|w_{\boldsymbol{\xi}} - w' : C^0(S_{central})\| \leqslant C\,\kappa(\underline{\tau})\,|\boldsymbol{\zeta}|$.
(iv). $\|\,\phi_{\boldsymbol{\zeta}} : C^{2,\beta}(S_{central})\,\| \leqslant C\,|\boldsymbol{\zeta}|$.
(v). $\|\,\phi_{\boldsymbol{\zeta}}\|_{2,\beta,\gamma';\Lambda} \leqslant C\,|\boldsymbol{\zeta}|$, where the transition region $\Lambda \subset \widetilde{S}_{central}$ and $\gamma' \in (\gamma, 3)$.

The construction of $\varphi_{\boldsymbol{\zeta}}$ on $\widetilde{S}_{central}$ is based on the dislocation which has been created. Ignoring negligible (quadratic in $\boldsymbol{\zeta}$) terms the argument is as follows: $\phi'_{\boldsymbol{\zeta}}$ can be constructed as the function which on $S[0]$ corrects the dislocated immersion to one which has no dislocation. On the transition region then $\phi_{\boldsymbol{\zeta}}$ amounts to the correct combination of the $\widehat{f}_i$'s. This however implies nonvanishing boundary data at $\partial\widetilde{S}_{central}$. Since we need vanishing boundary data we correct the data by solving the Dirichlet problem appropriately on the transition region. This correction introduces terms consistent with the required estimates because of the decay along the transition region. Finally we subtract the correct element of $\mathcal{K}_v[S_{central}]$ and obtain $\phi_{\boldsymbol{\zeta}}$ which satisfies the required fast decay away from $S_{central}$. For a detailed argument we refer the reader to [16, Proposition 6.3], or to [35, Proposition 6.7] where this type of argument was first used.

In the case now that S is a noncentral region we denote by Λ^- and Λ^+ the adjacent transition regions and $C^{\pm}$ the boundary circles of $\widetilde{S}$, where "$-$" is used to denote the one that is closer to the central region in the intrinsic geometry. The main step of the construction is to consider the solution u to the Dirichlet problem $\mathcal{L}u = 0$ on Λ^-, with u an appropriate element of $\mathcal{H}[C^-]$ on C^-, and $u = 0$ on the other boundary circle of Λ^-. In the language of the geometric principle this corresponds to introducing a dislocation where S is fixed and the adjacent spherical region closer to the central one is moved. u is corrected to a smooth function ϕ on $\widetilde{S}$ with $\mathcal{L}\phi \in \mathcal{K}_{substitute}[S]$ by cutting off close to S and then correcting by solving the inhomogeneous problem on $\widetilde{S}$ modulo $\mathcal{K}_{substitute}[S]$.

To confirm that the element of $\mathcal{K}_{substitute}[S]$ we obtain is appropriately close to the given one we use Green's second identity to convert its inner product with $\widehat{f}_i$ to boundary terms we understand from the construction. Green's second identity in this capacity amounts to a linearized balancing identity. We refer to [16, Lemma 6.8] for a detailed argument like this. Note also that the "dislocation" introduced to the adjacent spherical region leads to an error term when we combine these semi-local solutions to obtain $\Phi_{\boldsymbol{\xi}}$. This error term is however negligible because of the fast decay we have imposed, which is faster than the growth of u towards C^-.

8 The main results

The nonlinear terms

In order to close the argument we need to estimate the nonlinear terms to ensure that they are smaller than the linear terms. We consider then a function $V \in C^{2,\beta}_{sym}(M)$ and the Legendrian graph (recall 5.1) $(Y_{\boldsymbol{\zeta}})_V$ of V over $Y_{\boldsymbol{\zeta}}$, where we

assume that V and its derivatives are small enough so that the Legendrian graph is well defined. We define the nonlinear terms by

$$Q_V := \theta_V - \theta - \mathcal{L}V, \tag{8.1}$$

where θ_V is the Lagrangian angle of the Legendrian graph of V, θ is the Lagrangian angle of $Y_{\boldsymbol{\zeta}}$, and $\mathcal{L} = \Delta + 2n$ is the linearized operator for the Lagrangian angle. Recall the definitions of the norms we are using in 6.1 and 6.7. The geometry on the almost spherical regions stays uniform as $\underline{m} \to \infty$, so we only need to understand the decay behaviour on the necks. Let r, Λ, and $B_x(r/10)$ be as in 6.1. By scaling our immersions by a factor $1/r(x)$, we can ensure that the geometry of $Y_{\boldsymbol{\zeta}}$, which amounts to $X_{\tau,\alpha}$ by the construction, is uniformly bounded on $B_x(r/10)$. This allows us to conclude that

$$r^2(x)\,\|Q_V : C^{0,\beta}(B_x(r/10), r^{-2}(x)\,g)\| \leqslant r^{-2}(x)\,\|V : C^{2,\beta}(B_x(r/10), r^{-2}(x)\,g)\|^2. \tag{8.2}$$

Since $\gamma > 2$ we have $f \leqslant r^2$ and so we can conclude that

$$\|Q_V\|'_{0,\beta,\gamma} \leqslant C\,\|V\|^2_{2,\beta,\gamma}. \tag{8.3}$$

The same scaling argument ensures that if $\|V\|_{2,\beta,\gamma}$ is small enough independently of $\underline{\tau}$, the Legendrian graph $(Y_{\boldsymbol{\zeta}})_V$ is well defined.

The main theorem

In all the cases we are considering we have the following theorem:

Theorem 8.4. *There is a constant C independent of $\underline{m}$ and $\underline{\tau}$ such that if $\underline{m}$ is large enough (equivalently by 3.53 $\underline{\tau}$ is small enough), there is $\boldsymbol{\zeta} \in \mathbb{R}^{\widehat{m}}$ with $|\boldsymbol{\zeta}| \leqslant C\underline{\tau}$ and $V \in C^\infty_{sym}(M)$ with $\|V\|_{2,\beta,\gamma} \leqslant C\underline{\tau}$, such that the Legendrian graph $(Y_{\boldsymbol{\zeta}})_V$ is special Legendrian and $\mathcal{G}$-equivariant.*

The proof of the theorem is based on a reduction to finding a fixed point for the map $\mathcal{J}$ we define as follows: We assume given $v \in C^{2,\beta}_{sym}(M)$ such that $\|v\|_{2,\beta,\gamma} \leqslant \underline{\tau}$, and $\boldsymbol{\xi} \in \mathbb{R}^{\widetilde{m}}$ such that 7.1 is satisfied. Consider $\Phi_{\boldsymbol{\xi}}$ and $\underline{w}_{\boldsymbol{\xi}}$ provided by 7.2, and $(\varphi, w) = \mathcal{R}_M(\theta_{gluing})$ and $(u, w'') = \mathcal{R}_M(Q_{\varphi+\Phi_{\boldsymbol{\xi}}+v})$ provided by appropriate modifications of 6.8 where Y is replaced by $Y_{\boldsymbol{\zeta}}$. We define $V := \varphi + \Phi_{\boldsymbol{\xi}} + v$. We have then

$$\begin{aligned} \mathcal{L}\varphi &= \theta_{gluing} + w, \\ \mathcal{L}u &= Q_{\varphi+\Phi_{\boldsymbol{\xi}}+v} + w'', \\ \mathcal{L}\Phi_{\boldsymbol{\xi}} &= \theta_{dislocation} + \underline{w}_{\boldsymbol{\xi}}, \\ Q_V &= \theta_V - \theta - \mathcal{L}(\varphi + \Phi_{\boldsymbol{\xi}} + v). \end{aligned} \tag{8.5}$$

We define then $\mathcal{J}(v, \boldsymbol{\xi}) := (-u, \boldsymbol{\xi}')$ where $\xi' \in \mathbb{R}^{\widetilde{m}}$ corresponds by the isomorphism we have been using to $w_{\boldsymbol{\xi}} - \underline{w}_{\boldsymbol{\xi}} - w + w'' \in \mathcal{K}_{extended}$. Clearly then a fixed point $(v, \boldsymbol{\xi})$ satisfies $\theta_V = 0$ as desired.

The simplest way to find a fixed point is to employ the Schauder fixed point theorem, because we only need then to check the continuity of $\mathcal{J}$ in the appropriate topology and to estimate $\|u\|_{2,\beta,\gamma}$ and $\|\xi'\|_\gamma$. The continuity follows by

the construction. The required estimates follow by choosing $\underline{c}$ large enough, and applying then the estimates we already have available.

A Symmetries and quadratics

We introduce the notion of a k-full subgroup of $\mathrm{O}(n)$. We call a subgroup $\mathcal{G}$ of $\mathrm{O}(n)$, *k-full* if the orbit of any point $x \in \mathbb{S}^{n-1}$ is not contained in any k-dimensional linear subspace of $\mathbb{R}^n$. If $\mathcal{G} \subset \mathrm{O}(n)$ is n-full then we shall simply say that $\mathcal{G}$ is *full*. Clearly, if $\mathcal{G}$ is k-full for $k \geqslant 1$ then $\mathcal{G}$ is fixed point free and $|\mathcal{G}| \geqslant k$.

Important examples of full subgroups of $\mathrm{O}(n)$ are the symmetry groups of all (linearly full) regular polytopes in $\mathbb{R}^n$. For $n \geqslant 5$, the only regular polytopes are the regular simplex, the n-cube and the n-octahedron [7]. For the n-simplex the group of symmetries is the symmetric group on $n+1$ letters S_{n+1}. For $n = 3$ and $n = 4$, there are a few exceptional regular polytopes [7]. Later we will see that other considerations will preclude us from using any of these symmetry groups, except for that of the n-simplex. For $n = 2$, of course we have regular g-gons for any $g \geqslant 3$, with the dihedral groups as their symmetry groups.

The following proposition determines the $\mathcal{G}$-invariant subspace $\ker \mathcal{L}_{\mathcal{G}} \subset \ker \mathcal{L}$ on $\mathbb{S}^{n-1}$ for a variety of subgroups $\mathcal{G} \subset \mathrm{O}(n)$. Proposition A.1 is used in Section 4 to determine the dimension of $\ker \mathcal{L}$ invariant under the symmetry group $\mathcal{G}$ imposed on all the approximating spheres associated to our initial Legendrian submanifolds. This information is used in an important way in Section 6 to understand the analytic behaviour of the linearized operator $\mathcal{L}$ on our initial submanifolds.

Proposition A.1. *Let $e_1, \cdots, e_n$ be the standard oriented orthonormal basis for $\mathbb{R}^n$. For $x \in \mathbb{R}^n$, let its coordinates with respect to $e_1, \cdots, e_n$ be $(x_1, \cdots, x_n)$.*

(i) *For the following symmetry groups $\mathcal{G} \subset O(n)$, $\ker \mathcal{L}_{\mathcal{G}} = (0)$.*
   a. *$\mathcal{G} = O(n)$ or $\mathcal{G} = SO(n)$.*
   b. *$\mathcal{G}$ a k-full subgroup of $O(n)$ for any $k \geqslant [n/2]$.*
   c. *$\mathcal{G} = O(p) \times O(p) \times \mathbb{Z}_2 \subset O(2p)$, where $n = 2p$ and $\mathbb{Z}_2$ is generated by the involution $(x, y) \mapsto (y, x)$ and $x, y \in \mathbb{R}^p$.*

(ii) *For the following symmetry groups $\mathcal{G} \subset O(n)$, $\dim \ker \mathcal{L}_{\mathcal{G}} = 1$.*
   a. *$\mathcal{G} = O(n-1) \subset O(n)$ the subgroup of $O(n)$ fixing the vector e_1. Moreover, $\ker \mathcal{L}_{\mathcal{G}}$ is spanned by*

$$f_{1,n-1} = x_1^2 - \frac{1}{n-1} \sum_{i=2}^{n} x_i^2. \tag{A.2}$$

   b. *$\mathcal{G} = S_n \subset O(n-1)$, where S_n is the subgroup of S_{n+1}, the symmetry group of a regular n-simplex, fixing a vertex positioned at e_1. Moreover, $\ker \mathcal{L}_{\mathcal{G}}$ is spanned by the function $f_{1,n-1}$ defined in A.2.*
   c. *$\mathcal{G} = O(p) \times O(q)$, $p+q = n$. Moreover, $\ker \mathcal{L}_{\mathcal{G}}$ is spanned by the function*

$$f_{p,q} = \frac{1}{p} \sum_{i=1}^{p} x_i^2 - \frac{1}{q} \sum_{i=1}^{q} y_i^2, \tag{A.3}$$

where we write the coordinates of $x' = (x, y) \in \mathbb{R}^p \times \mathbb{R}^q$ *as* $(x_1, \cdots, x_p, y_1, \cdots, y_p)$.

d. $\mathcal{G} = S_{p+1} \times O(q) \subset O(p) \times O(q)$, *the symmetry group of a regular p-simplex (we centre the p-simplex at the origin and with one vertex located at* e_1*)* $\Delta^p \times \{0\} \subset \mathbb{R}^p \times \mathbb{R}^q$. *Moreover,* $\ker \mathcal{L}_{\mathcal{G}}$ *is spanned by the function f defined in A.3.*

e. $\mathcal{G} = \{M \in O(p) \times O(p) \mid M = (A, \pm A),\ A \in O(p)\} \cong \mathbb{Z}_2 \times O(p) \subset O(p) \times O(p)$. *Moreover,* $\ker \mathcal{L}_{\mathcal{G}}$ *is spanned by the function* $f_{p,p}$ *defined in A.3 (where we set* $q = p$*).*

(iii) *For the following symmetry groups* $\mathcal{G} \subset O(n)$, $\dim \ker \mathcal{L}_{\mathcal{G}} = 2$.

a. $\mathcal{G} = O(p-1) \times O(q) \subset O(p+q-1) \subset O(p+q)$, *where* $p + q = n$ *and* $O(p-1)$ *is the subgroup of* $O(p)$ *fixing* e_1. *Moreover,* $\ker \mathcal{L}_{\mathcal{G}}$ *is spanned by the two functions* $f_{1,n-1}$ *and* $f_{p,q}$ *defined in A.2 and A.3 respectively.*

b. $\mathcal{G} = S_p \times O(q) \subset O(p-1) \times O(q) \subset O(p+q)$, *the group of symmetries of a regular p-simplex (centred at the origin with one vertex located at* e_1*)* $\Delta^p \times \{0\} \subset \mathbb{R}^p \times \mathbb{R}^q$ *that fix the vertex* e_1. *Moreover,* $\ker \mathcal{L}_{\mathcal{G}}$ *is spanned by the two functions* $f_{1,n-1}$ *and* $f_{p,q}$ *defined in A.2 and A.3 respectively.*

Remark A.4. Notice that the function $f_{1,n-1}$ that occurs in A.2, is the Hamiltonian function associated with the one-parameter subgroup $\widetilde{\mathsf{T}}_x$ associated to the period of X_τ, where X_τ is the one-parameter family of $\mathrm{O}(n-1)$-invariant special Legendrian immersions. Similarly, the function $f_{p,q}$ occurring in A.3 is the Hamiltonian function associated with the one-parameter subgroup $\widetilde{\mathsf{T}}_x$ associated to the period of X_τ, where X_τ is the one-parameter family of $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant special Legendrian immersions.

Remark A.5. The group $\mathcal{G} = \mathrm{S}_n \subset \mathrm{O}(n-1)$ appearing in (ii)b is the symmetry group that we can impose on an off-central sphere if we use the points of the n-simplex as the points of attachment on the central sphere. $\mathcal{G}$ appears in this case because it is the subgroup of the full symmetry group that preserves the marking of the standard marked special Legendrian sphere, i.e. it is the intersection of the full symmetry group of the construction with the symmetry group $\mathrm{O}(n-1)$ of the standard marked special Legendrian sphere.

Remark A.6. If we set $p = 2$ and $q = n - 2$ then the group $\mathcal{G} = \mathrm{O}(1) \times \mathrm{O}(n-2)$ in (iii)a is the symmetry group we can impose on an off-central sphere when we arrange our $\mathrm{O}(n-1)$-invariant SLG cylinders around the central sphere using the symmetries of a regular g-gon P (with g odd) placed in the plane $\langle e_1, e_2 \rangle$ with one vertex at e_1. $\mathcal{G}$ is the subgroup of $\mathrm{O}(n)$ that leaves P invariant and fixes the vertex e_1.

Proof of Proposition A.1. (i)a. In this case the only symmetric bilinear form on $\mathbb{R}^n$ which is $\mathcal{G}$-invariant is a constant multiple of the square of the distance function, $\|x\|^2 = \sum_{i=1}^n x_i^2$, and in particular is constant on the sphere. Therefore since any function in $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is the restriction to the sphere of a homogeneous harmonic quadratic on $\mathbb{R}^n$, we see that $\ker \mathcal{L}_{\mathcal{G}} = (0)$.

(i)b. Recall that any element in $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is the restriction of some quadratic form Q on $\mathbb{R}^n$ whose restriction to $\mathbb{S}^{n-1}$ is non-constant. Let A be the real symmetric matrix associated with Q, *i.e.* $Q(x) = \langle x, Ax\rangle$. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of A and let $v_1, \cdots, v_n$ be an orthonormal basis of eigenvectors of A with $Av_i = \lambda_i v_i$. Since Q restricted to the sphere is non-constant we must have strict inequality $\lambda_1 < \lambda_n$. Let $m_- = \max_{i\in\{1,\cdots,n\}} \lambda_i = \lambda_1$ and $m_+ = \min_{i\in\{1,\cdots,n\}} \lambda_i = \lambda_n$. Then $m_- + m_+ \leqslant n$, and hence at least one of these integers is less than or equal to $[n/2]$. Let $V_- = \langle v_1, \cdots, v_{m_-}\rangle_{\mathbb{R}}$ and $V_+ = \langle v_n, \cdots, v_{n-m_++1}\rangle_{\mathbb{R}}$.

Now we consider the restriction of Q to the unit sphere in $\mathbb{R}^n$, which abusing notation we will also denote Q. Let Q_{min} and Q_{max} be the subsets of $\mathbb{S}^{n-1}$ where Q attains its minimum value and maximum value respectively. Since Q is non-constant, Q_{min} and Q_{max} are disjoint. It follows from the Rayleigh-Ritz variational characterization of eigenvalues that $Q_{min} = V_- \cap \mathbb{S}^{n-1}$ and $Q_{max} = V_+ \cap \mathbb{S}^{n-1}$.

Suppose now that Q is also $\mathcal{G}$-invariant and that $\mathcal{G}$ is k-full for some $k \geqslant [n/2]$. Without loss of generality we may assume that $m_- \leqslant m_+$ and hence $m_- \leqslant [n/2]$ (if not we can apply the argument below to a maxima of Q instead of a minima). Let $x \in \mathbb{S}^{n-1}$ be any minima of Q. Then since Q is $\mathcal{G}$-invariant, the whole orbit $\mathcal{G}.x$ belongs to $Q_{min} \subset V_-$, *i.e.* the orbit $\mathcal{G}.x$ is contained in the subspace V_- of dimension m_- . Since $m_- \leqslant [n/2]$ this contradicts the $[n/2]$-fullness assumption.

(i)c. We will see in (ii)c that the $\mathrm{O}(p)\times\mathrm{O}(p)$-invariant kernel of $\mathcal{L}$ is spanned by $f(x,y) = |x|^2 - |y|^2$. Since f is not invariant under the remaining $\mathbb{Z}_2$ action in $\mathcal{G}$, the $\mathcal{G}$-invariant kernel of $\mathcal{L}$ is trivial.

(ii)a. We write a general quadratic on $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}^{n-1}$ as

$$f = a_1 x_1^2 + \sum_{i=2}^{n} b_i x_1 x_i + \sum_{i,j=2}^{n} c_{ij} x_i x_j.$$

Then invariance under $(x_1, x_2, \cdots, x_n) \mapsto (x_1, -x_2, \cdots, -x_n)$ implies that $b_i = 0$ for $2 \leqslant i \leqslant n$. $\mathrm{O}(n-1)$ invariance on the $\mathbb{R}^{n-1}$ factor now forces the third summand to be a multiple of $\sum_{i=2}^n x_i^2$. The result now follows since f is required to be harmonic and therefore $a_1 + (n-1)c_{11} = 0$.

We could also have seen that $\dim\ker\mathcal{L}_{\mathcal{G}} = 1$ by appealing to the general result that for each degree d there is a unique (up to scale) harmonic polynomial of degree d, which is $\mathrm{SO}(n-1)$-invariant. These are often called the *zonal* spherical harmonics and can also be expressed in terms of so-called Gegenbauer or ultraspherical polynomials.

(ii)b. Since $\mathcal{G}$ is a subgroup of $\mathrm{O}(n-1)$, then by case (ii)a the function $f_{1,n-1}$ given in A.2 is clearly a $\mathcal{G}$-invariant element of $\ker\mathcal{L}_{\mathbb{S}^{n-1}}$. We claim that in fact that any $\mathcal{G}$-invariant element of $\ker\mathcal{L}_{\mathbb{S}^{n-1}}$ is a multiple of this function f. The proof is similar in spirit to that of case (i)b—the sufficiently full subgroups of $\mathrm{O}(n)$.

Since $\mathcal{G} \subset \mathrm{O}(n-1)$, $\mathcal{G}$ fixes e_1 and so $\mathcal{G}$ is not k-full for any $k \geqslant 2$. Also if $p \in (0, \mathbb{S}^{n-2}) \subset \mathbb{S}^{n-1}$ then clearly $\mathcal{G}.p \in (0, \mathbb{S}^{n-2})$ and hence the orbit of any such point is contained in the hyperplane $x_1 = 0$. However, if $p = (p_1, \cdots, p_n) \in$

$\mathbb{S}^{n-1}\setminus(\pm 1,\mathbf{0})\cup(0,\mathbb{S}^{n-2})$, then the $\mathcal{G}$ orbit of p is a regular $n-1$ simplex contained in the affine $n-1$ plane $x_1=p_1\neq 0$. Since this affine plane does not pass through the origin, the orbit of any such point p is linearly full.

Let Q be any quadratic form on $\mathbb{R}^n$, whose restriction to $\mathbb{S}^{n-1}$ is non-constant, and let A be the associated real symmetric matrix. Suppose that A has k distinct eigenvalues ordered so that $\lambda_1<\cdots<\lambda_k$ and let V_i denote the corresponding space of eigenvectors with eigenvalue λ_i. By our assumption on Q, $2\leqslant k\leqslant n$. By orthogonality of eigenspaces for distinct eigenvalues of A, V_i is orthogonal to V_j provided $i\neq j$. Let $d_i=\dim V_i$. Since $\sum_{i=1}^k d_i=n$ and $k\geqslant 2$, all the d_i satisfy $1\leqslant d_i\leqslant n-1$. Consider the restriction of Q to $\mathbb{S}^{n-1}$, which we continue to denote Q. Then the critical submanifolds of Q are the mutually orthogonal totally geodesic d_i-1 dimensional spheres $C_i=\mathbb{S}^{n-1}\cap V_i$. Clearly, Q has value λ_i on the critical submanifold C_i.

Now suppose that Q is harmonic and $\mathcal{G}$-invariant. Hence all the critical submanifolds C_i of Q are also $\mathcal{G}$-invariant. But since $d_i\leqslant n-1$ for all i, each critical submanifold C_i is contained in some hyperplane through the origin. Hence for all i we must have $C_i\subset(\pm 1,\mathbf{0})\cup(0,\mathbb{S}^{n-1})$. Since the critical submanifolds C_i are mutually orthogonal (and hence mutually disjoint) totally geodesic subspheres of $\mathbb{S}^{n-1}$ this forces $k=2$, i.e. there are only two distinct critical submanifolds C_1 and C_2 and only 2 distinct eigenvalues $\lambda_1<\lambda_2$. Hence C_1 and C_2 can be either $(\pm 1,\mathbf{0})$ or $(0,\mathbb{S}^{n-2})$. If $C_1=(\pm 1,\mathbf{0})$ and $C_2=(0,\mathbb{S}^{n-2})$, then we have $V_1=\mathbb{R}\oplus\mathbf{0}\subset\mathbb{R}^n$, $V_2=0\oplus\mathbb{R}^{n-1}\subset\mathbb{R}^n$, and hence $d_1=1$ and $d_2=n-1$. Similarly, if $C_2=(\pm 1,\mathbf{0})$ and $C_1=(0,\mathbb{S}^{n-2})$, then we have $V_2=\mathbb{R}\oplus\mathbf{0}\subset\mathbb{R}^n$, $V_1=0\oplus\mathbb{R}^{n-1}\subset\mathbb{R}^n$, and hence $d_2=1$ and $d_1=n-1$. Either choice for the pair (V_1,V_2) implies that Q is of the form

$$\lambda x_1^2+\sum_{i,j=2}^{n}a_{ij}x_ix_j.$$

Then invariance of Q under the action of S_n on the coordinates $(x_2,\cdots,x_n)$ as in (i)b forces the second summand to be a multiple of $x_2^2+\cdots+x_n^2$. Harmonicity of Q forces $\lambda+(n-1)\mu=0$ and therefore Q must be a multiple of the function f defined in A.2.

(ii)c. We write a general quadratic on $\mathbb{R}^p\times\mathbb{R}^q$ as

$$f(x,y)=\sum_{i,j}a_{ij}x_ix_j+\sum_{i,\alpha}b_{i\alpha}x_iy_\alpha+\sum_{\alpha,\beta}c_{\alpha\beta}y_\alpha y_\beta, \tag{A.7}$$

where $i,j\in\{1,\cdots,p\}$ and $\alpha,\beta\in\{1,\cdots,q\}$. Invariance under $(x,y)\mapsto(x,-y)$ forces the off-diagonal terms of f to vanish. Now the $\mathrm{O}(p)$ invariance in the $\mathbb{R}^p$ factor forces the first summand to be a multiple of $|x|^2$ and the $\mathrm{O}(q)$-invariance in the $\mathbb{R}^q$ factor forces the third summand to be a multiple of $|y|^2$. The result now follows since f being harmonic must be trace-free.

Again we could instead have appealed to more general results on spherical harmonics, namely the fact that for every even degree $2d$ there is a unique (up to scale) harmonic polynomial of degree $2d$ which is $\mathrm{SO}(p)\times\mathrm{SO}(q)$-invariant [50] (see

also [40, Thm 4.1] for a more recent reference in English). These $\mathrm{SO}(p) \times \mathrm{SO}(q)$-invariant spherical harmonic can be written in terms of the classical orthogonal

(ii)d. We write a general quadratic on $\mathbb{R}^p \times \mathbb{R}^q$ as the sum of three components as in A.7. Invariance under $(x, y) \mapsto (x, -y)$ forces the second summand of A.7 to vanish. $\mathrm{O}(q)$ invariance in the $\mathbb{R}^q$ factor forces the third summand to be a multiple of $|y|^2$. From case (i)b we see that S_{p+1} invariance in the $\mathbb{R}^p$ factor forces the first summand to be a multiple of $|x|^2$. Thus a general $\mathcal{G}$-invariant quadratic has the form $f = a|x|^2 + b|y|^2$ for some $a, b \in \mathbb{R}$. Since f is also harmonic then $ap + bq = 0$, and hence f is a multiple of A.3.

(ii)e. Since $\mathcal{G} \subset \mathrm{O}(p) \times \mathrm{O}(p)$, (ii)c shows that the function

$$f_{p,p}(x, y) = |x|^2 - |y|^2, \tag{A.8}$$

is a $\mathcal{G}$-invariant element of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$. We claim that any $\mathcal{G}$-invariant element of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is a multiple of $f_{p,p}$. As previously, we write a quadratic form f on $\mathbb{R}^p \times \mathbb{R}^p$ in the form A.7. Invariance under $(\mathrm{Id}, -\mathrm{Id})$ forces the second summand of A.7 to vanish. It now follows easily from the $\mathcal{G}$-invariance that the first and third summands of f must be a multiples of $|x|^2$ and $|y|^2$ respectively. Hence by the harmonicity of f, it must be multiple of A.8.

(iii)a. Clearly, $\mathcal{G} \subset \mathrm{O}(p) \times \mathrm{O}(q)$ and $\mathcal{G} \subset \mathrm{O}(p+q-1)$ (the subgroup of $\mathrm{O}(p+q)$ fixing e_1). Hence by cases (ii)a and (ii)c $\dim \ker \mathcal{L}_{\mathcal{G}} \geqslant 2$, since the functions $f_{1,n-1}$ and $f_{p,q}$ defined in A.2 and A.3 respectively are clearly $\mathcal{G}$-invariant. We claim that any $\mathcal{G}$-invariant element of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is in the span of these two functions.

The proof is very similar to previous cases. As previously, we write a quadratic form f on $\mathbb{R}^p \times \mathbb{R}^p$ in the form A.7. Invariance under $(x, y) \mapsto (x, -y)$ forces the second summand of A.7 to vanish. $\mathrm{O}(q)$-invariance in the $\mathbb{R}^q$ factor forces the third summand of f in A.7 to be a multiple of $|y|^2$. As in case (ii)a, $\mathrm{O}(p-1)$-invariance in the $\mathbb{R}^p$ factor forces the first summand of f to be of the form $ax_1^2 + b|x|^2$, for some $a, b \in \mathbb{R}$. Hence any $\mathcal{G}$-invariant element of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is of the form

$$ax_1^2 + b|x|^2 + c|y|^2,$$

where $a + pb + cq = 0$. By choosing $a = p + q, b = c = -1$ and $a = 0, b = 1/p, c = -1/q$ we obtain the two functions defined in A.2 and A.3 respectively.

(iii)b. Clearly $\mathcal{G}$ is a subgroup of the group considered in (iii)a and hence $\dim \ker \mathcal{L}_{\mathcal{G}} \geqslant 2$. In fact, we claim we have the same equivariant kernel as in case (iii)a. Arguing as in case (a) we see that the $\mathrm{O}(q)$-invariance forces the second summand of f in A.7 to vanish and the third summand to be a multiple of $|y|^2$. Now by (ii)b we see that the first summand of f must still be of the form $ax_1^2 + b|x|^2$, for some $a, b \in \mathbb{R}$ as in the previous case. Hence we have the same equivariant kernel as in (iii)a. □

We can generalize case (ii)c as follows. Let $(p_1, \ldots, p_k)$ be a partition of n, *i.e.* $p_i \in \mathbb{N}$ and $\sum_{i=1}^k p_i = n$. Let $\mathcal{G} = \mathrm{O}(p_1) \times \cdots \times \mathrm{O}(p_k)$ acting on $\mathbb{R}^n = \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k}$ as the direct product of the standard actions of $\mathrm{O}(p_k)$ on $\mathbb{R}^{p_k}$. Write $x \in \mathbb{R}^n = (X^1, X^2, \cdots, X^k)$, where $X^i \in \mathbb{R}^{p_i}$. There is an obvious generalization of the decomposition A.7 of a quadratic function on $\mathbb{R}^p \times \mathbb{R}^q$ into 3 summands

(the partition in this case had 2 factors) into an analogous decomposition with $\frac{1}{2}k(k+1)$ summands. Invariance under reflections in single coordinates forces all the mixed terms involving X^i and X^j for $i \neq j$ to vanish for all i and j. Hence any $\mathcal{G}$-invariant element of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is of the form

$$f(X^1, \cdots, X^k) = \sum_{i=1}^{k} a_i |X^i|^2,$$

where $a_i \in \mathbb{R}$ satisfy $\sum_{i=1}^{k} a_i p_i = 0$. Clearly, any f of this form is a $\mathcal{G}$-invariant element of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ and hence $\dim \ker \mathcal{L}_{\mathcal{G}} = k-1$. If $k=2$, we recover the case (ii)c.

Consider the special case where $n = pk$ for some $k, p \in \mathbb{N}$ and $(p_1, \cdots, p_k) = (p, \cdots, p)$. Then we may augment $\mathcal{G}$ to include symmetries of $\mathbb{R}^n = \mathbb{R}^p \times \cdots \times \mathbb{R}^p$ which permute the various factors of $\mathbb{R}^p$. If we allow all such permutations then we get an action of $\hat{\mathcal{G}} = \mathrm{S}_p \times \mathcal{G}$. Any quadratic form on $\mathbb{R}^n$ invariant under this augmented symmetry group must now be a multiple of $|x|^2$, since we must have $a_1 = \cdots = a_k$ in the decomposition above. Hence there are no $\hat{\mathcal{G}}$-invariant elements of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$.

More generally, given any partition $(p_1, \cdots, p_k)$ of n if any number occurs multiple times in this partition then we may consider an augmented symmetry group $\hat{\mathcal{G}}$ that allows the interchange of any factors of the same size. The $\hat{\mathcal{G}}$-invariant kernel can be considerably smaller than the $\mathcal{G}$-invariant one, as the previous example illustrates. If l is the number of distinct integers that occur in the given partition of n then the dimension of the space of all $\hat{\mathcal{G}}$-invariant quadratics on $\mathbb{R}^n$ is l, and the dimension of the space of all $\hat{\mathcal{G}}$-invariant elements of $\ker \mathcal{L}_{\mathbb{S}^{n-1}}$ is $l-1$.

Proof of Lemma 5.7. Let $\mathcal{G}''$ be the diagonal subgroup of $\mathrm{O}(p) \times \mathrm{O}(p)$ (defined in 4.4). Recall that the group $\mathcal{G}'$ is generated by the subgroup $\mathcal{G}''$ and the involution $\widetilde{\mathsf{N}} = (\mathrm{Id}, -\mathrm{Id}) \in \mathrm{O}(p) \times \mathrm{O}(p)$. If we can prove that any $\mathcal{G}''$-invariant function is of the form $f(x,y) = g(\langle x, y \rangle)$ for some function $g : [-1,1] \to \mathbb{R}$ then clearly f will be $\mathcal{G}'$-invariant if and only if additionally the function g is even.

It remains to prove that a real-valued function f on $\mathbb{S}^{p-1} \times \mathbb{S}^{p-1}$ is $\mathcal{G}''$-invariant if and only if $f(x,y)$ is a function only of the inner product $\langle x, y \rangle$.

Let $f(x,y)$ be a $\mathcal{G}''$-invariant function on $\mathbb{S}^{p-1} \times \mathbb{S}^{p-1}$. Suppose (x,y) and $(\tilde{x}, \tilde{y}) \in \mathbb{S}^{p-1} \times \mathbb{S}^{p-1}$ are two pairs of points with $d(x,y) = d(\tilde{x}, \tilde{y})$, where d denotes the standard spherical distance on the unit $p-1$ sphere. It is well-known that the sphere with its standard metric d is *two-point homogeneous*: that is, given two pairs (x,y) and $(\tilde{x}, \tilde{y}) \in \mathbb{S}^{p-1} \times \mathbb{S}^{p-1}$ such that $d(x,y) = d(\tilde{x}, \tilde{y})$ then there exists an isometry $A \in \mathrm{O}(p)$ such that $A(x,y) = (\tilde{x}, \tilde{y})$. Since f is $\mathcal{G}''$-invariant, we have $f(\tilde{x}, \tilde{y}) = f(Ax, Ay) = f(x,y)$. In other words, any $\mathcal{G}''$-invariant function $f(x,y)$ on $\mathbb{S}^{p-1} \times \mathbb{S}^{p-1}$ is a function only of the distance between x and y. Alternatively, since for $x, y \in \mathbb{S}^{p-1}$ the inner product $\langle x, y \rangle$ is a function only of the distance between x and y, then any such function is also a function of $\langle x, y \rangle$. Conversely, given any function $g : [-1,1] \to \mathbb{R}$ then the function $f : \mathbb{S}^{p-1} \times \mathbb{S}^{p-1} \to \mathbb{R}$ defined by $f(x,y) = g(\langle x, y \rangle)$ is clearly $\mathcal{G}''$-invariant. □

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