

STOCHASTIC MODELS FOR VALUATION AND RISK MANAGEMENT OF CREDIT-SENSITIVE HYBRID DERIVATIVES

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

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Stochastic Models for Valuation and Risk Management of Credit-Sensitive Hybrid Derivatives

ABSTRACT

We investigated different ways to model the dependence between the credit and other market risk components in hybrid derivatives.

To do so, we used both structural and reduced-form frameworks for credit risk modelling. In particular, we applied results from Analytically Tractable First-Passage (AT1P) model — a model belonging to the family of structural approaches — to the pricing of Contingent Conversion bonds and a reduced-form approach to the pricing of quanto Credit Default Swaps (CDS).

With respect to the former problem, we proposed a method to incorporate regulatory capital information into an AT1P-based model.

With respect to the latter, we showed how to derive coupled two-dimensional PDE systems to price quanto CDSs in reduced form approaches. Furthermore, we investigated the impact of jumps to default on the FX/credit dependence structure, arguing that this is necessary mechanism to explain the observed quanto CDS spreads on Italian Republic during the Euro-debt crisis of 2011–2012. We proved that an invariance property of the FX rate with respect to replacing it with its reciprocal rate is satisfied by our proposed, jump-to-default / diffusion, model.

Finally, we applied both approaches to *Credit Valuation Adjustment* (CVA) estimation, where we discussed the modelling-choice impact on the wrong-way risk estimation.

To Pietro — don't stop asking whys.

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LIST OF FIGURES

1.1	Protection on a given reference entity is bought by A from B. The stream of premium payments is indicated by the solid arrow, while the dashed arrow is used for the contingent payment.	10
1.2	Protection on a portfolio of reference entities is bought by A from B. Differently from the case of a single-name CDS (illustrated in Figure 3.2), multiple loss-given-defaults can be paid (one for on any underlying credit's default).	17
1.3	Hazard rate in PWCHR model from a flat CDS term-structure. .	35
1.4	Hazard rate in PWCHR model from an arbitrageable CDS term-structure.	36
2.1	Scatter plot and OLS estimation of Tier 1 Capital Ratio against asset/equity ratio on December 2009 for class rating C.	52
2.2	Evolution of α and β for the C rating class	52
2.3	Comparison between using the AT1P model and the (widely spread) Black-Scholes formula. Top charts shows B&S results, where $dV_t^{BS} = rV_t^{BS} dt + \sigma_1 V_t^{BS} dW_t$ with W a Brownian motion under $\mathbb{Q}$. Bottom charts show results where V_t follows the process from Eq. (1.23) with $q = 0$	57
2.4	Lloyds' bond-yields term structure on 15 December 10.	61
2.5	Empirical distributions of the equity value at conversion, of the conversion time, and of the the default time.	71
3.1	In the top chart, 5Y par spread time series for USD-denominated CDSs, S_{USD}^{5Y} , and EUR-denominated CDSs, S_{EUR}^{5Y} , on Italy. The difference between the two par-spreads is showed in the bottom chart. Source: Markit	75

3.2	Protection on a given reference entity can be bought by A from B in different currencies. The stream of payments in Eq. (3.1) is indicated by the solid arrow, while the dashed arrow is used for the contingent payment in Eq. (3.2). The <i>LGD</i> payment, albeit settled in different currencies, is the same percentage of the notional in the two contracts.	81
3.3	5Y CDS par-spread impact vs ρ and γ . The reference value for the par-spread is calculated using the parameters' values in Table 3.2.	108
3.4	5Y CDS par-spread impact vs σ^Z and σ^Y . The reference value is produced using the parameters' values in Table 3.3.	109
3.5	5 years par-spread impact vs ρ and σ^Y . The reference value is produced using the parameters' values in Table 3.4.	110
3.6	Maximum quanto CDS basis achieved by correlation moves, $S(1, \sigma^Y) - S(-1, \sigma^Y)$, for different values of the intensity volatility, σ^Y	110
3.7	Comparison of $1 - q_0^f(T)/1 - q_0(T)$ for different maturities in a low spreads scenario, $Y_0 = y^L$ (CDS spreads at 100 bps).	112
3.8	Comparison of $1 - q_0^f(T)/1 - q_0(T)$ for different maturities in a high spreads scenario, $Y_0 = y^H$ (CDS spreads at 740 bps).	112
3.9	Difference between $q_0^f(T; \gamma, \rho = -0.9)$ and $q_0^f(T; \gamma, \rho = 0.9)$ for different values of devaluation jump size, γ , and maturity, T	114
3.10	The top chart shows the calibrated γ through $\mathcal{T}$. The middle chart shows the calibrated ρ through $\mathcal{T}$. The bottom chart shows the time series of corresponding quanto CDS par-spreads.	118
3.11	Scatter plot comparing the calibrated γ^Z in ordinates with a relative basis spread in abscissas. As discussed in Section 3.4.4, the chart shows that the relative basis spread is a good estimate of the devaluation rate if the spreads have short maturities	119
3.12	Scatter plot comparing the product of the calibrated ρ, σ^Y, σ^Z and the model-implied difference between risky annuities in ordinates with a difference of relative basis spread in abscissas (we used $A_T := \frac{S_{EUR}(T) - S_{USD}(T)}{S_{USD}(T)}$).	121
3.13	Implied and historical correlation between EURUSD FX rate and Italy's CDS spread.	122

4.1	Comparison between Monte Carlo and Finite Difference Method-based solvers in the case $\rho = 0.4$, $\gamma^Z = -0.3$, $\sigma^Y = 0.4$, $y = -5$. The lines $\text{MC} \pm 2\text{std}_{\text{MC}}$ indicate a 95%–confidence interval on the MC–result.	134
4.2	Comparison between Monte Carlo and Finite Difference Method-based solvers in the case $\rho = -0.5$, $\gamma^Z = -0.1$, $\sigma^Y = 0.5$, $y = -4.5$. The lines $\text{MC} \pm 2\text{std}_{\text{MC}}$ indicate a 95%–confidence interval on the MC–result.	134
4.3	Correlation impact in the structural AT1P approach versus the reduced–form expOU approach. The CVA values obtained in case of credit–market independence (highlighted in both charts) are the same.	143
4.4	Devaluation jump impact in reduced–form (BK) approach.	145
4.5	Correlation impact on FX vega in a reduced–form model (BK).	146

LIST OF TABLES

1.1	CDS quotations on junior debt issued by Lloyds on 15 December 2010 (Source: Fitch Solutions).	25
1.2	Calibrated volatilities and correspondent CDS spread relative errors under different initial guesses for H ($H^{(0)}$). Here ΔS is the difference between the model spreads and the market spreads S^M	27
1.3	Calibrated values for H with their corresponding starting points.	28
2.1	Coordinates of the points calculated as output of the simulated annealing algorithm ($x^{(0)}$) and Levenberg–Marquardt (p). These have been obtained using rating class C, $\bar{\alpha} = 14.09\%$, $\bar{\beta} = -0.21$	60
2.2	Financial features of the CoCo bond.	61
2.3	Price, 95%-confidence interval and annualized yield-to-maturity dependence on the sampling frequency Δt . The market price bid–ask spread is (1.014; 1.02875).	62
2.4	Price variation given different q values and different η values. $r = 0.54\%$ p.a. We recall that the market traded price at the same date was (1.014; 1.02875)	65
2.5	Financial features of the PDB with ISIN code XS0497187640	66
2.6	Price of the PDB with ISIN code XS0497187640	66
2.7	Coordinate of the point associated to the market data set that includes the CDS–bond basis.	68
2.8	Calibration error associated to the set of parameters p^b	68
2.9	Price of the PDB with ISIN code XS0497187640 under the set of parameters p^b	69
2.10	Price variation given different q values and different η values using the set of parameters p^b from Table 2.7. $r = 0.54\%$ p.a. We recall that the market traded price at the same date was (1.014; 1.02875)	69
2.11	Financial features of the CoCo – PDB.	70

2.12	Prices of the CoCo – PDB, case $q = 0, \eta = 1$. The analogous price with conversion features is reported in the last column	70
2.13	Prices of the CoCo bond under stressed market data input. The percentage change has to be intended as negative when referred to the equity spot price and positive and uniformly applied to the whole term structure when referred to CDS. The unstressed price is 1.02351	72
3.1	Currencies involved in the pricing of the test cases detailed in Section 3.2.1.	83
3.2	Parameters used to produce the par-spreads impact in Figure 3.3	108
3.3	Parameters used to produce the results shown in Figure 3.4. . . .	109
3.4	Parameters used to produce the results shown in Figure 3.5. . . .	109
3.5	Comparison between $\tilde{k}$ and k with $\rho = 0$	113
3.6	Comparison between $\tilde{k}$ and k with $\rho = -0.9$	113
3.7	Comparison between $\tilde{k}$ and k with $\rho = 0.9$	114
3.8	Difference between $q_0^f(T; \gamma, \rho = -0.9)$ and $q_0^f(T; \gamma, \rho = 0.9)$ for different values of devaluation jump size, γ , and maturity, T	114
4.1	Parameters used in the first test of the FDM solver.	132
4.2	Parameters used in the second test of the FDM solver.	133
4.3	Summary of the models/asset classes tested in Section 4.4.	141

CONTENTS

0	INTRODUCTION	1
0.1	Summary of the Main Results	4
0.2	Preprints and Published Material	5
1	CREDIT RISK MODELLING	7
1.1	Vanilla Credit Derivatives	8
1.1.1	CDS and Survival Probabilities	8
1.1.2	CDS Indices	16
1.1.3	CDS Options and Credit Volatility	18
1.2	Structural Approach	21
1.2.1	The Merton and Black–Cox Models	22
1.2.2	AT1P Model: Calibration to the Whole CDS Term Structure	23
1.3	Reduced–form Approach	26
1.3.1	Cox Process Construction	26
1.3.2	Piece–wise Constant Hazard Rate Model	31
1.3.3	The Credit Triangle Formula	33
1.3.4	Detecting Arbitrage	35
2	CoCo BONDS VALUATION WITH EQUITY AND CREDIT CALI- BRATED FIRST PASSAGE STRUCTURAL MODELS	38
2.1	Introduction	38
2.2	Contingent Capital (CoCo)	45
2.2.1	Conversion Price Determined at Conversion Date	45
2.2.2	Conversion Price Fixed at the CoCo’s Inception	46
2.2.3	Transparency and Inputs	46

2.3	Adapting AT1P to CoCos	47
2.3.1	The Payoff of a CoCo	48
2.3.2	A Proxy for c_t	50
2.3.3	How to Evaluate the Stock-price in AT1P	55
2.3.4	Calibration Procedure for a CoCo Bond in AT1P	58
2.4	Results	59
2.4.1	Calibration	60
2.4.2	Pricing Application	61
2.4.3	Price of a Plain Defaultable Bond (PDB)	66
2.4.4	Price of the CoCo Bond as a Plain Defaultable Bond	70
2.4.5	Stressing Input Data	71
2.5	Conclusions	72
3	MULTI CURRENCY CREDIT DEFAULT SWAPS: QUANTO EFFECTS AND FOREIGN EXCHANGE RATE DEVALUATION JUMPS	74
3.1	Introduction	74
3.1.1	Previous Literature	77
3.1.2	Quanto CDS	79
3.2	Model Description	81
3.2.1	The Roles of the Currencies	82
3.2.2	Two Markets Measures	83
3.2.3	Modeling Framework for the Quanto CDS Correction	88
3.2.4	A Diffusive Correlation Model: Exponential OU / GBM	90
3.2.5	A Jump-to-Default Framework	95
3.3	Finite Difference Schemes	105
3.3.1	Douglas ADI Methods	105
3.4	Results	107
3.4.1	Quanto CDS Par-Spreads Parameters Dependence	107
3.4.2	Test on the Impact of Tenor and Credit Worthiness on the Quanto Correction	111

3.4.3	Correlation Impact on the Short Term Versus Long Term	112
3.4.4	Model Calibration to Market Data for 2011–2013	115
3.5	Conclusions and Further Work	123
4	EXAMPLES OF WRONG–WAY RISK IN CVA INDUCED BY DEVALUATIONS ON DEFAULT	124
4.1	Introduction	124
4.1.1	Overview of the Modelling Framework	125
4.2	A PDE Approach for both FX–driven and Equity–driven WWR .	127
4.2.1	FX	127
4.2.2	Equity	133
4.3	A Structural Approach for Equity/Credit WWR	137
4.3.1	AT1P	137
4.3.2	Introducing WWR	140
4.4	Results	141
4.4.1	Models Calibrations	141
4.4.2	Models Parameters and Performance	142
4.4.3	Equity WWR: Correlation Impact	143
4.4.4	Equity WWR: Devaluation Impact	144
4.4.5	FX WWR: FX Vega	144
4.5	Conclusions	146
5	CONCLUSION	148
	REFERENCES	159
	APPENDIX A SOME RESULTS ON COX MODELLING	160
A.1	Proof of Theorem 7	160
A.2	Proof of Corollary 9	162
	APPENDIX B SOME RESULTS ON AT1P MODEL	164

B.1	How to evaluate the survival probability in AT1P	164
B.2	How to evaluate the stock-price in AT1P	169



INTRODUCTION

In recent years, the challenge of correctly modelling credit risk in a framework where multiple risk-factors must be accounted for has become more and more important.

Following the financial crisis that reached its apex in 2008 and that led — as its most notorious effect — to the collapse of Lehman Brothers, regulators and financial operators alike awoke to the fact that no bank or financial institution, no matter how big or influential, is exempt from credit risk.

As a result, the global set of regulations that rule the financial industry has since then undergone a massive revision. The process has taken several years to complete: the internationally agreed set of measures developed by the Basel Committee on Banking Supervision — better known as *Basel III* — w^{as} started being discussed in the aftermath of the 2008 financial crisis and it has only recently (2017) been finalised. The measures contained in it aim to strengthen the regulation, supervision, and risk management of banks. In order to do that, capital requirements for investment banks have been made more demanding and severe.

One of the consequences of imposing more stringent capital rules on banks

has been of making the presence in trading books of financial derivatives — especially the ones with complex payoffs — more and more expensive. In this sense, a simplification of the global derivatives market has effectively been achieved as the liquidity of derivatives with exotic payoffs is sensibly smaller today with respect to what it was before the crisis.

From a different perspective, however, the complexity has not disappeared from financial markets, but has rather been pushed along other “directions”. For the sake of the present thesis, we are mainly interested in three of these directions.

The first one is the introduction of the regulatory capital itself as of a new risk-factor, directly affecting the prices of certain financial instruments. In the Basel III framework, banks are allowed to meet part of their capital requirement through qualified *Contingent Conversion* (CoCo) bonds. These are a type of hybrid bonds designed to facilitate automatic bail-ins, that is, an automatic deleveraging of distressed institutions. They achieve this purpose by introducing a conversion into equity triggered when the regulatory capital of the issuer reaches a certain, pre-determined, level. CoCo bonds and a modelling framework to price and risk-manage them are the subjects of Chapter 2.

The same capital rules that have made uneconomic to enter into derivatives with sophisticated payoffs, have also forced financial operators to analyse more deeply and from multiple angles what were once considered to be simple, plain vanilla, derivatives.

On the one hand, this has happened in an indirect way, because of the profound modification undertaken by some “market data”, as, for example, certain basis spreads that were negligible up until 2008, but that have started increasing in the aftermath of the crisis. One of the first areas where the need for a new, post-2008, valuation framework was more evident was the interest-rate derivatives market. There, as early as in 2009, financial operators came to realise that there was not such thing as a single, risk-free, interest rate term-structure that could be used both for discounting and for estimation of the forward rate levels. This resulted in the arising of the multi-curve

framework for pricing interest rate derivatives (see, for example, Ametrano and Bianchetti¹, Mercurio⁵⁶, Morini⁵⁸). The novelty of the new paradigm, effectively, lied in the introduction of an expanded set of risk-factors to price and risk-manage even seemingly simple interest rate derivatives as vanilla *Interest Rate Swaps* (IRS). Together with the LIBOR rates — that were used as the only risk-factor in the pre-crisis framework — were now taken into account overnight rates (like OIS) and all the basis spreads between IRSs with different floating leg frequency.

In a similar way, in the credit derivatives market, it is after a financial crisis — in this case, the Euro-debt crisis of 2011–2012 — that the basis spread between *Credit Derivative Swaps* (CDS) denominated in different currencies became both more sizable and, therefore, more relevant from a modelling perspective. In Chapter 3, we discussed the limitations of using purely diffusive modelling approaches to price the quanto CDS basis where a high quanto CDS basis is observed in the market. In the same chapter, we present our proposal for a modelling approach that is instead able to handle the observed quanto CDS basis in such cases.

A second, more direct, way in which revised global financial regulations have changed the way to look at derivatives pricing is through the increased focus on counterparty risk. The calculation of *Credit Valuation Adjustment* (CVA), in particular, poses new challenges with respect to valuation, as, on the one hand, it requires a hybrid setting even for derivatives that would not normally be hybrids. For example, there are assumptions under which the CVA of an IRS can be calculated as the price of defaultable interest-rate swaption. This makes this estimation particularly sensitive to the choice of the dependency structure between interest rates and credit risk. Different modelling approaches in cases where the dependency between market risk-factors (in the IRS example, this would be the interest rate process) and the credit worthiness of a given reference entity is considered crucial are discussed in Chapter 4.

Underlying all the modelling approaches that we will be showing during this thesis is the problem of how to correctly model the dependence between credit and market risk-factors. The prices of all the hybrid derivatives that

will be studied in the next chapters can be decomposed in elements of type

$$\mathbb{E} [\phi(X_T)\mathbb{1}_{\{\tau>T\}} + \bar{\phi}(X_\tau)\mathbb{1}_{\{\tau\leq T\}}]$$

where we used $(X_t, t \geq 0)$ to denote a market risk-factor (e.g. the driver of the interest rates term-structure, or an equity asset, or an FX rate), τ to denote the default time of a given reference entity, and $\phi(\cdot)$ and $\bar{\phi}(\cdot)$ to denote the discounted payoff-upon-survival and payoff-upon-default functions, respectively, that depend on market risk-factors realisation at time T or τ .

In Chapter 1, we will be laying the foundations for different ways to tackle this problem, by illustrating the two main families of credit risk models that can be used to model τ , namely, structural and reduced-form approaches. The choice whether to use one approach or the other strongly constrains the possible approaches to use to model the credit-market correlation dependence. Furthermore, we will be showing that different asset classes are more naturally modelled in one of these two frameworks than in the other.

0.1 SUMMARY OF THE MAIN RESULTS

In Chapter 2, we will be illustrating how to effectively fix the over-parameterization that can arise in AT1P model (a model belonging to the family of structural approaches) when this is calibrated to CDS data alone. In that case, the solution of the calibration problem is strongly dependent on the initial parameters choice. We will be showing how, by including the spot equity price into the calibration problem, this dependence disappears. In the same chapter, we will be presenting an original pricing model for CoCo bonds. Given the CoCo bond price dependence on regulatory capital ratios, we will be presenting a way to incorporate this information into an AT1P-based model. Furthermore, we will be investigating the model-implied correlation between conversion time and default time and show how to extend the model in cases where a given correlation between the two random times is needed.

In Chapter 3, we will be presenting two models to consistently handle the

currency basis that can arise when CDSs on the same entity are available in multiple denominations. In doing so, we will be showing the importance of devaluation jumps in explaining the observed basis. We will be solving the problem through a PDE approach and we will be illustrating how to derive a two-spatial dimension PDE system for the problem of valuing the quanto-adjusted survival probability. Moreover, we will be investigating a symmetry property of the FX rate component with respect to replacing it with its reciprocal rate and we will prove that our proposed jump-diffusion model retains such property. Finally, we will be presenting an approximated formula to estimate the jump magnitude parameter in our model. We will be testing the approximation in a backtest exercise, where we calibrated the model to real market data on Italian quanto CDSs on each day in a three-year time window that included the Euro-debt crisis.

The problem of correctly assessing the wrong-way risk component in the context of Credit Valuation Adjustment estimation is the subject of Chapter 4. This will give us the opportunity to link the works from Chapters 2 and 3. We will be using both AT1P-based approaches and reduced-form approaches with jump-diffusion dynamics to price the CVA of the same portfolio. As in the case of the models presented in Chapter 3, we will be using a PDE approach to calculate CVA values in the reduced form model. In contrast to the PDE systems discussed in Chapter 3, we will be showing how, in this second case, the CVA is given by the solution of two coupled two-dimensional PDE systems.

0.2 PREPRINTS AND PUBLISHED MATERIAL

Most of the material presented in this thesis has been either published or submitted for publication to peer-reviewed journals. Specifically:

- Chapter 2 is based on the work published in Brigo et al.²¹;
- Chapter 3 is based on the work available as a pre-print on SSRN, Brigo et al.²². The content of it has also been submitted for publication to *Mathematics of Operations Research* and it is currently awaiting for a

response from the journal's editor. In July 2016, it has been presented to the 9th World Congress of the Bachelier Finance Society in New York.

- The content of Chapter 4 has been presented to the *Innovations in Insurance, Risk— and Asset Management* conference in Munich in April 2017. Further to that, it has been accepted as a chapter of the proceedings book derived from the conference, Brigo and Pede¹⁶, and it is currently awaiting publication.

1

CREDIT RISK MODELLING

In this chapter, we present the two families of credit risk models that will be used in the rest of the work. The first approach to be discussed (see Section 1.2), is also known as *firm-value*, *structural*, or *equity-based* default model and is inspired on the classic work of Merton⁵⁷ and Black and Cox⁷, both building on the original framework of Black and Scholes⁸. The second approach is presented in Section 1.3 and is known as *intensity* or *reduced-form* credit model. The seminal work this framework is based upon is due to Duffie and Singleton²⁹. Further to the above mentioned groundbreaking works, there is a number of books that we refer the interested reader to for comprehensive and detailed illustrations of the main models discussed in this chapter: Lando⁵³, Brigo and Mercurio¹⁵, Jeanblanc et al.⁴⁹, Bielecki and Rutkowski³, Bluhm et al.⁹.

Before going into details of the two frameworks, we describe in the next section the vanilla credit derivatives that will be used to calibrate any chosen credit-risk model.

1.1 VANILLA CREDIT DERIVATIVES

In this section we describe the elemental single-name credit derivatives normally used to calibrate default risk models.

Survival probabilities are usually inferred from the prices of Credit Default Swaps (CDSs) and from prices of CDS Indices, and these instruments are described in Section 1.1.1. Bonds can be used as well for this purpose, but, in this case, particular care must be taken in trying to decouple survival probabilities from interest rates and funding components. Indeed, the credit spread paid on a bond issued by a given reference entity is usually quite different from the CDS spread that would be paid to buy protection on its default. This difference is referred to as *bond-CDS basis*. Given the funded nature of the bond against the un-funded nature of the CDS, the spreads inferred from the prices of bonds are expected to be affected not only by the credit-worthiness of the issuer, but also by its funding cost risk, whereas the CDS spread is regarded as a cleaner indicator of its credit risk only. A wide literature is available about bond-CDS basis, the financial reasons behind it, and ways to estimate it — we refer the interested reader to Haworth et al.⁴⁴, Elizalde et al.³⁶ for more details about it.

Credit volatility is instead calibrated to the prices of options on CDSs or to the prices of options on CDS Indices, and both these instruments are described in Section 1.1.3.

We collectively refer to the two classes of credit derivatives just mentioned as of *vanilla*, as opposed to exotic, or hybrid credit derivatives. In the chapters to come, we will show how to use them to calibrate the credit component in more complicated models that are introduced to price hybrid securities.

1.1.1 CDS AND SURVIVAL PROBABILITIES

The first building block that any credit model is based upon is represented by the survival probabilities. The preferred way to calculate them is to extract them from the market price of *Credit Default Swaps* (CDS).

CDSs are bilateral derivative contracts in which one party pays a fixed coupon to another party with, usually, quarterly frequency. The payments are interrupted upon default of a given reference entity. When this happens, the party receiving the quarterly payments will pay to the other party a fraction, called *recovery rate*, of the contract's notional amount.

The party making the quarterly payments until the default event is called *protection buyer*, whilst the party making the final, contingent on default, payment is referred to as *protection seller*.

In order to describe the cashflows of a CDS, let us consider a set of payment dates $\mathcal{T} := \{T_0, \dots, T_N\}$, with $T_0 < T_1 < \dots < T_N$. The market convention for the payment dates in a standard CDS is to be quarterly spaced, and to take one of the following values: March 20th, June 20th, September 20th, or December 20th. Dates built in this way are referred to as *IMM dates*, where IMM stands for “International Monetary Market”. We further define the set of payment dates happening before a given time, t , as

$$\mathcal{T}(t) := \{T \in \mathcal{T} : T < t\},$$

and the index of the last payment date before a given time, t , as

$$\kappa(t) := \begin{cases} \sup\{i \in \mathbb{N} : T_i \in \mathcal{T}(t)\}, & \text{if } \mathcal{T}(t) \neq \emptyset, \\ -1, & \text{otherwise.} \end{cases}$$

Given the notation set above, the discounted stream of cashflows of the premium leg, Π^{Premium} (as seen from the protection seller's perspective), can be expressed as a function of the valuation time, the contractual dates, and the contractual spread as

$$\begin{aligned} \Pi^{\text{Premium}}(t, \mathcal{T}, S^C) = S^C \sum_{i=\kappa(t)+1}^{N-1} & \left(\mathbb{1}_{\{\tau > T_{i+1}\}} D(t, T_{i+1}) \Delta(T_i, T_{i+1}) \right. \\ & \left. + \mathbb{1}_{\{\tau > T_{\kappa(t)+1}\}} D(t, T_{\kappa(t)+1}) \Delta(t, T_{\kappa(t)+1}) + D(t, \tau) \Delta(T_{\kappa(\tau)}, \tau) \mathbb{1}_{\{\tau \leq T\}} \right) \end{aligned} \quad (1.1)$$

where:

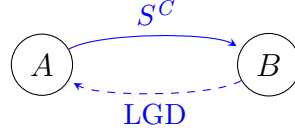


Figure 1.1: Protection on a given reference entity is bought by A from B. The stream of premium payments is indicated by the solid arrow, while the dashed arrow is used for the contingent payment.

- $D(t, T)$ is the stochastic discount factor at time t for maturity T ;
- S^C is the contractual annualised spread or premium rate;
- $\Delta(T_i, T_{i+1})$ is the year fraction between the two times T_i and T_{i+1} , if $T_{i+1} > T_i$ — for standard CDSs, this is calculated using the *Act/360* day count convention. If, instead, $T_{i+1} \leq T_i$, $\Delta(T_i, T_{i+1}) = 0$.
- τ is the default time of the reference entity.

The first term in the sum represents the regular stream of protection payments, while the second term accounts for the potential (non-null only if $t > T_0$) accrued amount with respect to the previous payment date, and the third term for the potential accrued amount if default happens on a date that is not a payment date.

The discounted stream of cashflows of the protection leg is, instead, made of a single cash flow. This is paid upon default of the reference entity. The cashflow can be written as a function of the valuation time, maturity time, and recovery rate as

$$\Pi^{\text{Protection}}(t, T_N; R) = (1 - R) \mathbb{1}_{\{\tau \leq T_N\}} D(t, \tau). \quad (1.2)$$

The factor $1 - R$ is sometimes referred to as *loss given default*, and denoted by LGD.

Figure 1.1 provides a schematic representation of the derivative contract just described.

CDS PRICING

In this section the pricing formula for a standard CDS is derived. In order to do that, we will use a probability space $(\Omega, \mathcal{G}, \mathbb{Q}, (\mathcal{G}_t, t \geq 0))$ satisfying the usual hypotheses. In particular $(\mathcal{G}_t, t \geq 0)$ is a filtration under which the dynamics of the risk factors are adapted and under which the default time of the reference entity is a stopping-time.

Let us start by writing the survival probability as a parametric function of valuation time and maturity time:

$$\mathbb{Q}_t(\tau > T) =: q_t(T; x), \quad T \geq t, \quad x \in \mathcal{X}, \quad (1.3)$$

where $\mathbb{Q}_t$ is the risk-neutral probability conditioned to $\mathcal{G}_t$, $\mathcal{X}$ is the parameters space, and where x is an element in it.

In many cases, expressions of this type are available in closed-form — this is the case, for example, of reduced-form models driven by hazard rate specified as an *affine processes* or of some structural models. For more details on affine processes modelling, we refer to Duffie and Singleton²⁹, Filipovic³⁹, Duffie et al.³⁰, and to Brigo and Mercurio¹⁵, while we refer to Lando⁵³, Bielecki and Rutkowski³, and to Chapter 2 for more references and discussion on structural approach to credit modelling. The existence of closed-form expressions, however, is not a necessary condition for modelling credit or credit-hybrid derivatives. Especially in view of the computation power available in contemporary processors, both pricing and calibration of models where such expressions are not available — and where, therefore, they must be calculated using nested numerical algorithms — are indeed operations that can be carried out in acceptable amounts of time for most practical applications. As a matter of fact, we will be showing examples of calibration of credit-hybrid models where survival probabilities must be calculated resorting to *finite-difference methods* in Chapter 3.

Before applying the knowledge of the survival probability to CDS pricing, let us first consider a CDS contract in which protection is sold on a reference entity whose recovery rate is R . The contract starts in T_0 , has maturity T_N ,

and annualized premium rate S^C . Its stream of discounted cash-flows at time $t \leq T_0$ can be derived by putting together equations (3.1)–(3.2). Let us denote it by $\Pi^{\text{CDS}}(t, \mathcal{T}, S^C, R) = \Pi^{\text{Premium}}(t, \mathcal{T}, S^C) - \Pi^{\text{Protection}}(t, T_N; R)$ or, in extended form:

$$\begin{aligned} \Pi^{\text{CDS}}(t, \mathcal{T}, S^C, R) = S^C \sum_{i=\kappa(t)+1}^{N-1} & \left(\mathbb{1}_{\{\tau > T_{i+1}\}} D(t, T_{i+1}) \Delta(T_i, T_{i+1}) \right. \\ & + \mathbb{1}_{\{\tau > T_{\kappa(t)+1}\}} D(t, T_{\kappa(t)+1}) \Delta(t, T_{\kappa(t)+1}) + D(t, \tau) \Delta(T_{\kappa(\tau)}, \tau) \mathbb{1}_{\{\tau \leq T\}} \Big) \\ & - (1 - R) \mathbb{1}_{\{\tau \leq T_N\}} D(0, \tau). \end{aligned} \quad (1.4)$$

Its price at time t is given by the expected value of the above discounted stream of cash-flows calculated using the measure $\mathbb{Q}$ and conditional on $\mathcal{G}_t$:

$$V^{\text{CDS}} = \mathbb{E}_t \left[\Pi^{\text{CDS}}(t, \mathcal{T}, S^C, R) \right]$$

where $\mathbb{E}_t[\cdot] := \mathbb{E}[\cdot | \mathcal{G}_t]$.

Throughout this work, we will be assuming deterministic interest rates and, therefore, deterministic discount factors. Under this assumption, the terms containing contingent payments — that is, the one that accounts for the accrual between a payment date and the default time, and the last term, the protection leg — can be evaluated precisely as Stieltjes integrals (see, for example, Brigo and Tarenghi¹⁷) and then trivially discretized for practical implementation.

The value of a contingent payment is given by

$$\mathbb{E}_t \left[D(t, \tau) \mathbb{1}_{\{\tau \leq T\}} \right] = \int_t^\infty D(t, s) \mathbb{1}_{\{s \leq T\}} F_\tau(ds; t)$$

where $F_\tau(\cdot; t)$ is the cumulative distribution function of τ , conditioned to $\mathcal{G}_t$. By definition, τ 's c.d.f. is given by

$$F_\tau(s; t) = \mathbb{Q}_t(\tau < s), \quad (1.5)$$

so that

$$F_\tau(ds; t) = \mathbb{Q}_t(\tau < s + ds) - \mathbb{Q}_t(\tau < s).$$

Given the last equation, the fair value of contingent payments can be written in terms of survival probabilities, once one has chosen a discretization grid upon which to calculate them.

In the case of the accrual term (third term in RHS of Eq. (1.4)), a widely used approximation⁶¹ is to set it as half the full premium due to be paid at the end of each period. This leads to

$$\begin{aligned} \mathbb{E}_t \left[\Pi^{\text{Premium}}(t, \mathcal{T}, S^C) \right] = \\ S^C \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) \Delta(T_i, T_{i+1}) \left(\mathbb{Q}_t(\tau > T_i) + \frac{1}{2} (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)) \right). \end{aligned} \quad (1.6)$$

With respect to the protection leg, we will be using the premium leg's time-grid as a discretization grid for the time integral:

$$\mathbb{E}_t \left[\Pi^{\text{Protection}}(t, T_N; R) \right] = (1-R) \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)). \quad (1.7)$$

The fair value of a CDS can be calculated using equations (1.6)–(1.7) together with Eq. (1.4). One obtains

$$\begin{aligned} \mathbb{E}_t \left[\Pi^{\text{CDS}}(t, \mathcal{T}, S^C, R) \right] = \\ S^C \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) \Delta(T_i, T_{i+1}) \left(\mathbb{Q}_t(\tau > T_i) + \frac{1}{2} (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)) \right) \\ - (1-R) \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)). \end{aligned} \quad (1.8)$$

Definition 1 (CDS par-spread). *The CDS par-spread evaluated at time t for a contract with maturity $T > t$, $S_t^*(T)$ is the spread such that the value,*

calculated at time t , of the contract is null. In formulas:

$$\mathbb{E}_t \left[\Pi^{CDS}(t, \mathcal{T}, S_t^*(T), R) \right] = 0. \quad (1.9)$$

The model-implied par-spreads can be calculated by inversion from Eq. (1.8) through

$$S_t^*(T) = \frac{(1 - R) \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i))}{\sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) \Delta(T_i, T_{i+1}) \left(\mathbb{Q}_t(\tau > T_i) + \frac{1}{2} (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)) \right)}. \quad (1.10)$$

Definition 2 (Risky PV01). *The quantity at the denominator in Eq. (1.10) represents the expected value of a stream of unitary payments, weighted by the probability of the reference entity's survival. In case of (default-free) interest rate derivatives pricing, such a quantity goes under the name of PV01. Given the fact that, in this case, the coupon payments were weighted using the survival probabilities, a risky PV01 (RPV01) is introduced via the following definition:*

$$\begin{aligned} \text{RPV01}_t(T_0, T_N) := & \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) \Delta(T_i, T_{i+1}) \left(\mathbb{Q}_t(\tau > T_i) \right. \\ & \left. + \frac{1}{2} (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)) \right). \end{aligned}$$

The Risky PV01 just defined can be used to express the price of the protection leg (the numerator in Eq. (1.10)) as a function of the CDS's par-spread, $S_t^*(T)$, so that:

$$(1 - R) \sum_{i=\kappa(t)+1}^{N-1} D(t, T_i) (\mathbb{Q}_t(\tau > T_{i-1}) - \mathbb{Q}_t(\tau > T_i)) = S_t^*(T) \text{RPV01}_t(T_0, T_N). \quad (1.11)$$

Given a contractual spread, S^C , the relation above can be used to evaluate

the CDS's price as

$$\left(S^C - S_t^*(T)\right) \text{RPV01}_t(T_0, T_N).$$

Remark 1. *Eq. (1.8) shows that, under the assumption of independence between rates and credit, the CDS's price is a function of the set of survival probabilities calculated on the premium leg payment times. From Eq. (1.10), the same applies to the CDS's par-spread. Therefore, in cases where the survival probabilities are parametric functions of valuation time and maturity time, the par-spread, too, can be expressed as a function of the model parameters, x*

$$S_t^*(T) = s_t(T; x).$$

MARKET DATA AVAILABILITY For a given reference entity, it is usually possible to find a set of CDSs with different maturities ranging from maturities as short as one year or, in some case, even six month, to 15 or even 20 years. The market price of such contracts is expressed either in the form of par-spreads, S^M (where we use the superscript M to denote a *market* quote, as opposed to a model-implied one), or in the form of an upfront cost.

The existence of a set of quoted spreads

$$S^M(T_0, T), \quad T \in \{1Y, 2Y, \dots, 20Y\},$$

for different maturities, T , allows for an easy way to calibrate a credit risk model as one can look for the parameters that allow to exactly match a set of market quoted spreads.

In general, and especially for more complicated cases, where hybrid models will be calibrated to a set of instruments including not only CDSs, it seems convenient to state the following definition:

Definition 3 (Calibration procedure). *Let us consider a state space $\mathcal{X}$, market observable values $\phi^M = (\phi_1^M, \dots, \phi_{N_M}^M)$ and their correspondent model calculated quantities $\phi(x) = (\phi_1(x), \dots, \phi_{N_M}(x))$ where $x \in \mathcal{X}$ are model*

parameters. The calibrated model parameters element p is such that

$$C(p) = \min_{x \in \mathcal{X}} C(x)$$

where $C(x)$ is a distance between ϕ^M and $\phi(x)$ (in the following $C(\cdot)$ can also be called cost function). In those cases where an iterative algorithm will be used, we will refer to the i -th point in the iterative process as $x^{(i)}$.

Unless differently specified, in the following the cost function will be specified as

$$C(x) = \sum_{i=1}^{N_M} w_i \left(\frac{\phi_i(x) - \phi_i^M}{\phi_i^M} \right)^2,$$

with $w_i = \frac{1}{N_M}$.

Remark 2. In the case of credit risk calibration to CDS spread quotes, for example, the definition above can be applied with

$$\begin{aligned} \phi_i^M &= S^M(0, T_i), & i &= 1, \dots, N_M, \quad T_i \in \{1Y, \dots, 20Y\}, \\ \phi_i(x) &= s_0(T_i; x), & i &= 1, \dots, N_M, \quad T_i \in \{1Y, \dots, 20Y\}, \\ w_i &= \frac{1}{N_M}, & i &= 1, \dots, N_M. \end{aligned}$$

1.1.2 CDS INDICES

CDS indices are the most liquid *multi-name* credit derivatives. Like single-name CDSs, they are bilateral contracts where one party buys protection from a another party and where protection is paid in the form of quarterly coupons. Differently from a single-name CDS, however, the protection buyer in a CDS index is not insuring himself/herself against the potential loss on a reference entity, but on the loss on a portfolio of reference entities.

The index components of the most widespread indices, like iTraxx and CDX, are refreshed every six months. This process is referred to as *index roll* and it consists on selecting the components with the most liquid CDSs in Europe (iTraxx) or US (CDX).

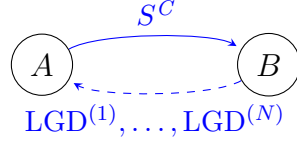


Figure 1.2: Protection on a portfolio of reference entities is bought by A from B. Differently from the case of a single-name CDS (illustrated in Figure 3.2), multiple loss-given-defaults can be paid (one for on any underlying credit's default).

Let us consider premium leg payments to be done on a time-grid $\mathcal{T} = \{T_0, \dots, T_N\}$ with $T_0 < T_1 < \dots < T_N$ and a reference portfolio made of N_c components. Let us further denote by $\tau_1, \dots, \tau_{N_c}$ the default times of the individual portfolio components and by $R_1, \dots, R_{N_c}$ their recovery rates. We present a schematic representation of the index mechanics in Figure 1.2. Its discounted cash-flows are given by:

$$\begin{aligned} \Pi^{\text{Index}}(t, S^C, \mathcal{T}) = & \sum_{i=\kappa(t)+1}^{N-1} \left(S^C \left[\Delta(T_i, T_{i+1}) \sum_{j=1}^{N_c} \mathbb{1}_{\{\tau_j > T_i\}} + D(t, \tau) \Delta(T_{\kappa(\tau)}, \tau) \sum_{j=1}^{N_c} \mathbb{1}_{\{T_{i-1} < \tau_j \leq T_i\}} \right] \right. \\ & \left. - \sum_{i=\kappa(t)+1}^{N-1} \left(D(t, \tau) (1 - R_j) \sum_{j=1}^{N_c} \mathbb{1}_{\{T_{i-1} < \tau_j \leq T_i\}} \right) \right). \quad (1.12) \end{aligned}$$

Price and par-spread can be calculated taking the expected value of the expression above. In particular, given the linearity of the expectation, we can bring the sums over the index components outside, so that the price of the index is given by

$$\mathbb{E}_t \left[\Pi^{\text{Index}}(t, S^C, \mathcal{T}) \right] = \frac{1}{N_c} \sum_{j=1}^{N_c} \left(S^C - S_j^* \right) \text{RPV01}_t^j(T_0, T_N),$$

where we denoted by S_j^* and $\text{RPV01}_t^j(T_0, T_N)$ the par-spread and RPV01, respectively, of the j -th index component. This price is usually expressed in terms of an index par-spread, S_t^* , and an index-RPV01, $\text{RPV01}_t^I(T_0, T_N; c)$, defined on a flat par-spreads term-structure with spreads level c . The index

par-spread can be calculated by solving the following equation:

$$\left(S^C - S_I^*\right) \text{RPV01}_t^I(T_0, T_N; S_I^*) = \frac{1}{N_c} \sum_{j=1}^{N_c} \left(S^C - S_j^*\right) \text{RPV01}_t^j(T_0, T_N). \quad (1.13)$$

1.1.3 CDS OPTIONS AND CREDIT VOLATILITY

Prices of options on credit events are needed if one wants to extract risk-neutral credit volatility information from the market. Options on single-name CDSs would provide the ideal instrument to use for such purpose — they would effectively act in the same way as interest-rate swaptions act in interest rates volatility estimation. Unfortunately, in recent times, liquidity on this type of derivatives decreased dramatically, so that their quotes are difficult to find and, even in cases where they can be somehow obtained, they are not fully trustworthy. Options on CDS indices, on the other hand, are still liquid enough, at least for certain expiries and strikes (see <http://www.dtcc.com/repository-otc-data>). For this reason, our approach in this thesis (see Chapter 3) will be to rely on the volatilities of index options as a rough market data, and to use them as proxies for the volatilities of single-names.

SINGLE-NAME OPTION'S MECHANICS We will use the following notation to denote two relevant contractual dates in a single-name CDS option:

- T^e , with $T^e \geq t$, is the option's expiry date;
- T , with $T \geq T^e$, is the option's underlying CDS' maturity date.

With respect to credit exposure direction, options on single-name CDSs can be of two types: *payer* options or *receiver* options. In both cases, the option buyer purchases the right of entering into a CDS starting at the option expiry T^e and ending at the option maturity T . In a payer (resp. receiver) option, the option buyer, if s/he decides to exercise his option, will be paying (resp. receiving) the premium K .

Knock-out (KO) options cancels out at no cost in case of a reference's entity default between time t and time T^e . These type of options is considered standard practice outside US⁶¹. The price of a KO option, V^{KO} , can be expressed as:

$$V^{\text{KO}}(t; T^e, T, K) = \mathbb{E}_t \left[\mathbb{1}_{\{\tau \geq T^e\}} (S_{T^e}^*(T^e, T) - K)^+ \text{RPV01}_{T^e}(T^e, T) \right]. \quad (1.14)$$

In non-KO options, on the other hand, the contract does not cancel out in case of early default. Rather, the holder of a payer swaption is delivered the long-protection CDS at expiry. That entitles him/her to a net gain worth $1 - R$ on the expiry date T^{e*} , where R is the recovery rate of the credit underlying the option. The price of a non-KO option, $V^{\text{non-KO}}$ can be expressed as

$$V^{\text{non-KO}}(t; T^e, T, K) = \mathbb{E}_t \left[\mathbb{1}_{\{\tau < T^e\}} (1 - R) D(t, T^e) + \mathbb{1}_{\{\tau \geq T^e\}} (S^*(T^e, T) - K)^+ \text{RPV01}_{T^e}(T^e, T) \right]. \quad (1.15)$$

The difference between the two types of options is referred to as *front-end protection*:

$$\text{FEP}(t, T) := \mathbb{E}_t \left[\mathbb{1}_{\{\tau < T\}} (1 - R) D(t, T) \right], \quad T > t.$$

INDEX OPTION'S MECHANICS Differently from the single-name case, an option on a CDS index can only be of non-KO type. This is because the CDS index itself still exists if one of its component defaults and investors can still buy or sell protection on the remaining index components.

The expiry of a CDS index option is constrained by the fact that indices are rolled with biannual frequency, so that, in practice, it is only possible to find options with expiries of at most six months.

There are four important contractual dates to consider:

⁶¹Likewise, the holder of a receiver swaption would be delivered a short-protection CDS. His/her option would therefore be worthless in case of early default.

- i) T_0^I is the underlying index issue date;
- ii) T_0 is the option initiation date. The option strike, K , and the index underlying the option are both set at this date. This, in turn, determines the *exercise price* that will be paid upon expiry (see Eq. (1.16)). The option premium (see Eq. (1.18)) is paid.
- iii) T_1 with $T_0 < T_1$ is the expiry date. It is the date on which the decision whether to exercise or not the option is taken. Options on credit indices are usually of European type, so this is the only date when they can be exercised.
- iv) T is the maturity of the underlying index — indices with 5Y maturity are by far the most widely traded ones.

To illustrate the pricing of a CDS index option, we will consider the case of a payer option, that is, an option where the option buyer has the right to enter into a long protection position in the index. On expiry date, T_1 , if the option buyer decides to exercise his or her right, s/he must pay the exercise price, given by the difference between the index coupon, C , and the option strike, K :

$$G(K) = \left(K(T_0, T) - C(T_0^I, T) \right) \text{RPV01}_{T_1}(T_1, T; K), \quad (1.16)$$

where $\text{RPV01}_{T_1}(T_1, T; K)$ is the index-RPV01 defined through Eq. (1.13). Once the option is exercised, the long protection position of the option buyer will be worth

$$\left(S^*(T_1, T) - C(T_0^I, T) \right) \text{RPV01}_{T_1}(T_1, T; S^*(T_1, T)). \quad (1.17)$$

Finally, all the payments related to defaults of index components that happened between T_0 and T_1 will be settled at T_1 . The net position for the

option buyer at T_1 is therefore given by:

$$V_{T_1}^{\text{payer}} = \left(\frac{1}{M} \sum_{m=1}^M \mathbb{1}_{\{\tau_m \leq T_1\}} (1 - R_m) - \left(K(T_0, T) - C(T_0^I, T) \right) \text{RPV01}_{T_1}(T_1, T; K) + \left(S^*(T_1, T) - C(T_0^I, T) \right) \text{RPV01}_{T_1}(T_1, T; S^*(T_1, T)) \right)^+. \quad (1.18)$$

In order to price CDS Index options, it is standard market practice (see O’Kane⁶¹, Pedersen⁶²), to make a distributional assumption on $S^*(T_1, T)$. Specifically, the spread is assumed to be log-normal. In this way, CDS Index options can be quoted in terms of Black-volatilities. Market quotes for CDS Index options — expressed as implied volatility — will be used in Chapter 3 to calibrate a model able to handle the quanto-effect in CDS market.

1.2 STRUCTURAL APPROACH

In this section we will focus on Merton-like models and we refer to Garcia and Goossens⁴⁰, Bielecki and Rutkowski³, and Brigo et al.¹⁹ and the references therein for details.

In structural models, default occurs when a latent variable, the firm’s asset value, breaches a barrier, typically the debt book value of the company. Assumptions for the asset value process and for the capital structure of a company are, therefore, typically needed for models adopting this framework. In the following, we will be considering a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with a standard Brownian motion defined on it, $(W_t^P, t \geq 0)$, and the filtration generated by the Brownian motion, $(\mathcal{G}_t, t \geq 0)$. Let us also denote also by V_t the value of the company at time t , by S_t its equity price at time t , and by $D(t)$ the value of its outstanding debt at time t . Additionally, D is the par or notional value of the debt at maturity T .

1.2.1 THE MERTON AND BLACK-COX MODELS

The Merton model only checks for default at the final debt maturity T . The default time is the maturity if the firm value is below the debt par level D , and is time infinity (meaning no default) in the other case.

The default time is defined as

$$\tau = T\mathbb{1}_{\{V_T < D\}} + \infty\mathbb{1}_{\{V_T \geq D\}},$$

while the value V_t of the company is given by

$$V_t = S_t + D(t), \quad 0 \leq t \leq T \quad (1.19)$$

whereas, at maturity T , outstanding debt, firm value and debt at maturity are related by

$$B_T = \min(D, V_T) = D - (D - V_T)^+$$

so that

$$S_T = (V_T - D)^+. \quad (1.20)$$

Under Merton's assumptions the market is composed by V_t and by a money-market account,

$$B(t) = B(0)e^{rt}.$$

V_t is a geometric Brownian motion so that its dynamics is described by the following stochastic differential equation

$$dV_t = \mu V_t dt + \sigma V_t dW_t^P, \quad V_0 = v,$$

where v , μ , and σ are constant parameters. As usual, V_t is expressed as Geometric Brownian motion in the equivalent risk neutral martingale measure $\mathbb{Q}$, so that

$$dV_t = rV_t dt + \sigma V_t dW_t^Q, \quad V_0 = v, \quad (1.21)$$

where W_t^Q is a Brownian motion in the probability space $(\Omega, \mathcal{G}, \mathbb{Q})$.

In the original Merton model the value of the company should not fall below

the outstanding debt at maturity. In this case we have from Eq. (1.20) and Eq. (1.21) that the value of the equity of a company at maturity T is given by the Black–Scholes formula.

An important limitation of the Merton model is the assumption that default may only occur at final maturity T . This limitation was addressed first in Black and Cox⁷, where early default is introduced, resulting from the breaching of safety covenants. Safety covenants are modeled according to a continuous time default barrier $H(t)$ and the default time as the hitting time of the firm value process on the barrier H

$$\tau = \inf\{t \geq 0 : V_t \leq H(t)\}, \quad \inf \emptyset := +\infty. \quad (1.22)$$

Notice that τ is a random time with respect to $\{\mathcal{G}_t\}_{t \geq 0}$. The Black-Cox model improves the Merton model but still maintains limitations. The shape of the default barrier can only be exponential, and the parameters r, q and σ need to be constant. The next model we consider removes such limitation and provide the structural models with enough flexibility to compete with the reduced form models in the single name calibration space.

1.2.2 AT1P MODEL: CALIBRATION TO THE WHOLE CDS TERM STRUCTURE

Brigo and Tarenghi¹⁷ have extended the original Merton and Black-Cox setting in two important ways: by considering a deterministic non-flat barrier and by allowing for a time-dependent volatility for the firm value process.

This approach has recently been used by²³ for pricing Lehman equity swaps taking into account counterparty risk. Notice also that these authors have considered random default barriers associated with misreporting and risk of fraud. The related research is summarized for example in²³, where the random barrier framework is applied to Lehman’s default. In this work, we will be using the first version of the model, having deterministic non-flat barriers.

The reason for imposing a particular shape for the barrier is to bring feasibil-

ity / practicality to the calibration and pricing process. That is, the survival probabilities of the firm can be recovered using closed form formulas. Additionally, it is possible to consider different maturities for the outstanding debt of the firm. Finally, the main achievement of the AT1P generalization relies in the possibility of calibrating the model to the whole CDS term structure of the firm, as illustrated by the case studies on Parmalat's and Lehman's collapses discussed in the above references.

We present here a formulation of the model in the simplified setting in which dividends and the risk free short rate are constant. The firm value process is assumed to have the form

$$dV_t = (r - q)V_t dt + \sigma(t)V_t dW_t^Q, \quad V_0 = v, \quad (1.23)$$

where $\sigma(\cdot)$ is a deterministic function of time, and the barrier is given by

$$\hat{H}(t) = H e^{(r-q)t - B \int_0^t \sigma^2(s) ds}. \quad (1.24)$$

Under this choice of the barrier shape, the survival probabilities can be expressed in closed-form. This is proved in the following theorem.

Theorem 4 (Survival probabilities in AT1P.). *Let us consider a dynamics for the asset-value, $(V_t, t \geq 0)$, given by Eq. (1.23). Let us further consider a deterministic barrier, $\hat{H}(\cdot)$, given by Eq. (1.24) and let us define the default time as Eq.(1.22).*

For any given maturity $T \geq 0$, the survival probability estimated at time 0, $\mathbb{Q}_0(\tau > T)$ can be expressed as:

$$\mathbb{Q}_0(\tau > T) = \Phi(d_1) - \left(\frac{H}{V_0}\right)^{2B-1} \Phi(d_2), \quad (1.25a)$$

$$d_1 := \frac{\log \frac{V_0}{H} + \frac{2B-1}{2} \int_0^T \sigma(s)^2 ds}{\left(\int_0^T \sigma(s)^2 ds\right)^{1/2}}, \quad (1.25b)$$

$$d_2 := d_1 - \frac{2 \log \frac{V_0}{H}}{\left(\int_0^T \sigma(s)^2 ds\right)^{1/2}}. \quad (1.25c)$$

Table 1.1: CDS quotations
on junior debt issued by Lloyds
on 15 December 2010 (Source:
Fitch Solutions).

Term	CDS Rate (bps)
1Y	347.9934
2Y	373.1248
3Y	396.6364
4Y	417.8327
5Y	436.3855
7Y	441.1132
10Y	445.8688

A proof of this formula is presented in Appendix B.1.

The model is easily generalized to time dependent r and q , so as to be able to fit the term structure of interest rates and safety covenants, see Brigo and Tarenghi¹⁷ for the fully general model. In the following we will make the same choices proposed in Brigo and Tarenghi¹⁷ about $\sigma(\cdot)$, using a piecewise constant function to model the evolution of the firm value's volatility. We will partition the time-domain of the volatility function in a number of time-steps equal to the number (n) of CDS contracts used for calibration. This choice provides the model with enough flexibility to be able to fit the whole CDS term-structure. In the following we will therefore be considering the set of model parameters $p := (B, H, \sigma_1, \dots, \sigma_n)$.

In the next section we are showing how (1.25) is used in the pricing of CDS's.

ON THE PARAMETERS B AND H

Let us consider CDS par spreads on junior debt issued by Lloyd's on 15 December 2010. They are shown in Table 1.1.

The number of parameters to describe $\sigma(\cdot)$ is equal to the number of CDS contracts included in the calibration. The volatility-linked parameters $\sigma_1, \dots, \sigma_n$ can provide enough flexibility to fit the (n) CDS par-spreads on their own, no matter the values of B and H . This leaves us with two additional degrees

of freedom. Table 1.2 shows the results of four different calibrations. In all these four calibrations B has been constrained to be equal to zero. The calibrations differ from each other due to different choices of the initial value for H — they are indicated as $H^{(0)}$.

Table 1.2 shows that the four calibration problems converged with similarly good fit to the market observed spreads. That is the four calibrations result in a model satisfactorily near to the benchmark market quantities. However, different starting points for the barrier level resulted in different calibrated values both for sigmas and for H (see Table 1.3 for values for calibrated H), due to the over-parametrization for the model. This issue will be addressed later on, when we use the AT1P model to price a CoCo bond.

1.3 REDUCED-FORM APPROACH

The second family of models that is going to be used in this work is usually referred to as *reduced-form* or *intensity* credit models. Differently from what happens in structural models, there is no attempt in this case of modelling the “true” source of default — like the reference entity’s asset and liabilities. Rather, the object of modelling is the intensity of a Poisson process whose first jump triggers the default event. This way of constructing a default time is referred to as *Cox* (or *doubly stochastic Poisson*) process construction.

As we will show in this section, the choice of hiding the true default mechanics behind such a construction, rewards us with a framework where survival probabilities come in a highly tractable form and where, in general, a lot of well established techniques from short-rate modelling can be applied (see Brigo and Mercurio¹⁵, Lando⁵³ for in depth discussions about this point).

1.3.1 COX PROCESS CONSTRUCTION

For the Cox process construction of the default time, we will be considering as given a probability space $(\Omega, \mathcal{F}, \mathbb{Q}, (\mathcal{F}_t, t \geq 0))$ satisfying the usual hypotheses.

Table 1.2: Calibrated volatilities and correspondent CDS spread relative errors under different initial guesses for H ($H^{(0)}$). Here ΔS is the difference between the model spreads and the market spreads S^M .

Term	$H^{(0)} = 0.2$		$H^{(0)} = 0.4$		$H^{(0)} = 0.6$		$H^{(0)} = 0.8$	
	σ	$\Delta S/S^M$ ($10^{-14} \times$)	σ	$\Delta S/S^M$ ($10^{-14} \times$)	σ	$\Delta S/S^M$ ($10^{-14} \times$)	σ	$\Delta S/S^M$ ($10^{-14} \times$)
1Y	0.3019	0.0	0.2316	-0.3775	0.1737	0.3775	0.1330	-0.4219
2Y	0.1171	0.1554	0.0938	-0.0222	0.0729	-0.0888	0.0572	-0.0666
3Y	0.1992	0.1110	0.1558	-0.0999	0.1189	-0.0777	0.0922	-0.1998
4Y	0.1886	-0.0222	0.1495	-0.1110	0.1154	0.0666	0.0902	-0.0888
5Y	0.2088	-0.0666	0.1663	-0.0333	0.1289	0.0444	0.1011	-0.0333
7Y	0.1946	-0.0222	0.1565	-0.0666	0.1225	0.0444	0.0967	0.0222
10Y	0.2182	0.0222	0.1780	-0.1221	0.1653	0.0222	0.1126	-0.0333

Table 1.3: Calibrated values for H with their corresponding starting points.

	$H^{(0)} = 0.2$	$H^{(0)} = 0.4$	$H^{(0)} = 0.6$	$H^{(0)} = 0.8$
Calibrated H	0.5584	0.6431	0.7205	0.7794

In general, we consider processes of state variables $(X_t, t \geq 0)$ taking values in $\mathbb{R}^d$ and $(\mathcal{F}_t, t \geq 0)$ -adapted. They model market moves, for example, or any other exogenous variable that might affect the default of a reference entity. We then define the intensity process $(\lambda_t, t \geq 0)$ as an $(\mathcal{F}_t, t \geq 0)$ -adapted process where $\lambda(\cdot)$ is a measurable function mapping realizations of $(X_t, t \geq 0)$ in $\mathbb{R}^+$:

$$\lambda : \mathbb{R}^d \rightarrow \mathbb{R}^+.$$

Given a realization of the intensity process, we define the *hazard function* as its integral over time

$$\Lambda(t) := \int_0^t \lambda_s \, ds. \quad (1.26)$$

Let us now consider a r.v. Θ constructed on Ω , independent of $(\mathcal{F}_t, t \geq 0)$ and with the exponential law of parameter 1, that is

$$\mathbb{Q}(\Theta < x) = 1 - e^{-x}. \quad (1.27)$$

We are now ready to define the default time in this framework — this is given by

$$\tau := \inf \{t \geq 0 : \Lambda(t) > \Theta\}$$

In particular, this definition allows to write the survival event in terms of hazard function as

$$\{\tau \leq t\} = \{\Lambda(t) > \Theta\}$$

so that

$$\mathbb{E} \left[\mathbb{1}_{\{\tau > t\}} \middle| \mathcal{F}_t \right] = \mathbb{Q}(\tau > t | \mathcal{F}_t) = \mathbb{Q}(\Lambda(t) < \Theta | \mathcal{F}_t) = e^{-\Lambda(t)} \quad (1.28)$$

ENLARGEMENT OF FILTRATIONS

Let us now consider the default process $(D_t, t \geq 0)$. This is defined through

$$D_t = \mathbb{1}_{\{\tau \leq t\}}, \quad t \geq 0. \quad (1.29)$$

Let us further introduce the sigma-algebra generated by it, $\mathcal{D}_t := \sigma(D_s : s \leq t)$, and let us consider the filtration $(\mathcal{G}_t, t \geq 0)$ given by $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$.

The next lemma provides a way to switch from conditional expectations taken with respect to $\mathcal{G}_t$, to conditional expectations taken with respect to $\mathcal{F}_t$.

Lemma 5 (Filtration switching formula). *Let Y be an integrable r.v.. Then,*

$$\mathbb{1}_{\{\tau > t\}} \mathbb{E}[Y | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} \frac{\mathbb{E}[Y \mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t]}{\mathbb{E}[\mathbb{1}_{\{\tau > t\}} | \mathcal{F}_t]} = \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t)} \mathbb{E}[Y | \mathcal{F}_t].$$

Proof. For a proof of this result, we refer to Jeanblanc et al.⁴⁹, chapter 7. $\square$

From this Lemma, we can derive the following corollary.

Corollary 6. *If X is an integrable $\mathcal{F}_T$ -measurable r.v., then, for $t < T$*

$$\mathbb{E}[X \mathbb{1}_{\{T < \tau\}} | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t)} \mathbb{E}[X e^{-\Lambda(T)} | \mathcal{F}_t].$$

Proof. We follow Jeanblanc et al.⁴⁹, chapter 7. The conditional expectation is equal to 0 on the set $\{\tau \leq t\}$.

$$\mathbb{E}[X \mathbb{1}_{\{T < \tau\}} | \mathcal{F}_t] = \mathbb{E}[X \mathbb{1}_{\{T < \tau\}} | \mathcal{F}_T | \mathcal{F}_t] = \mathbb{E}[X e^{-\Lambda(T)} | \mathcal{F}_t],$$

where the last equation is given by Eq. (1.28). The result follows from Lemma 5. $\square$

We are now able to prove that the intensity process $(\lambda_t, t \geq 0)$ is in fact the intensity of the jump process $(D_t, t \geq 0)$ — and so to give a justification for

this framework's name. This is proven in the next Theorem, where we build the compensator of the default process.

Theorem 7. *The process $(M_t, t \geq 0)$, where $M_t = D_t - \int_0^{\tau \wedge t} \lambda_s ds$ is a $(\mathcal{G}_t, t \geq 0)$ -martingale.*

Proof. See Appendix A.1. □

Remark 3 (Positiveness of the intensity). *Let us consider an intensity defined as a deterministic function of time, $\lambda(\cdot)$. We can write the expression for the survival probability until time T , using the result from Corollary 6:*

$$\mathbb{Q}(\tau > T | \mathcal{G}_t) = \mathbb{E} \left[\mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t \right] = \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t)} \mathbb{E} \left[e^{-\Lambda(T)} | \mathcal{F}_t \right] = \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T)}$$

Let us consider the total derivative in T of the last term in the previous equation. We obtain

$$\frac{d}{dT} \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T)} = -\lambda(T) \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T)},$$

from which we can express the hazard-rate at time T as

$$\lambda(T) = -\frac{\frac{d}{dT} \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T)}}{\mathbb{E} \left[\mathbb{1}_{\{\tau > T\}} | \mathcal{G}_t \right]}. \quad (1.30)$$

The numerator of this expression can be re-written as

$$\begin{aligned} -\frac{d}{dT} \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T)} &= -\lim_{\Delta T \rightarrow 0} \left(\frac{\mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T + \Delta T)} - \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t) - \Lambda(T)}}{\Delta T} \right) \\ &= -\lim_{\Delta T \rightarrow 0} \left(\frac{\mathbb{Q}(\tau > T + \Delta T | \mathcal{G}_t) - \mathbb{Q}(\tau > T | \mathcal{G}_t)}{\Delta T} \right) \\ &= \lim_{\Delta T \rightarrow 0} \left(\frac{\mathbb{Q}(T < \tau < T + \Delta T | \mathcal{G}_t)}{\Delta T} \right). \end{aligned}$$

The intensity from Eq. (1.30) can then be interpreted through Bayes rule as a conditional probability over time. This is the reason why it is required to be modelled as a process taking positive values.

1.3.2 PIECE-WISE CONSTANT HAZARD RATE MODEL

Single-name CDSs as well as CDS indices are usually priced using a deterministic intensity assumption. The intensity in this context is referred to a *hazard-rate*. More specifically, the hazard rate is assumed to be *piece-wise constant* (hence the acronym we will be using for it — PWCHR), and, like in many other cases where a certain quantity can be expressed as a deterministic function of times, we may refer to it as a *term-structure*. Let us consider a set of times $0 = T_0^h, \dots, T_N^h$, the hazard rate term-structure is given by a function of time

$$\lambda : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \quad (1.31)$$

specified through a set of positive real values $h_1, \dots, h_N$ as

$$\lambda(t) = \sum_{i=1}^N \mathbb{1}_{\{T_{i-1}^h \leq t < T_i^h\}} h_i. \quad (1.32)$$

Let us further consider a function mapping each time to the corresponding interval in the hazard rate term-structure

$$\begin{aligned} k : \mathbb{R}^+ &\rightarrow \mathbb{N} \\ t &\mapsto \sup \left(i \in \{0, \dots, N\} \text{ s.t. } T_i^h \leq t \right) \end{aligned} \quad (1.33)$$

In this setting, the survival probabilities can be expressed as

$$\mathbb{Q}_t(\tau > T) = e^{-H(k(T)) - h_{k(T)}(t - T_k^h)} \quad (1.34)$$

where we have further defined a discrete hazard function as a function of the last index it includes as

$$H(k) := \sum_{i=1}^k h_i (T_i^h - T_{i-1}^h) \quad (1.35)$$

Remark 4. *It is clear from Eq. (1.34) and Eq. (1.35) that, in this model, for a fixed $T \in (T_{i-1}^h, T_i^h)$, the survival probability $\mathbb{Q}_t(\tau > T)$ only depends on $h_1, \dots, h_i$.*

BOOTSTRAP

When linked to credit risk modelling, the word *bootstrap* refers to a specific calibration procedure that can be used when the model to be calibrated is a PWCHR. $T_1^h, \dots, T_N^h$ are chosen as the maturities of the quoted CDS spreads for a given reference entity.

In this setting, the model parameters to be calibrated are the values $h_1, \dots, h_N$.

The fact that there is one h -parameter for each and any quoted CDS contract, and that the T_k -maturity CDS's price only depends on the first k parameters $h_1, \dots, h_k$ (see Remark 4), provides a straightforward strategy to calibrate a PWCHR model:

1. set $k = 1$;
2. given that the only variable in formula for the k -maturity CDS par-spread is h_k , use a root-solver to find the value h_k such that the model-implied CDS par-spread for maturity T_k^h matches the quoted one;
3. if $k \neq N$, increment it by one, and go back to step 2.

Remark 5 (Bootstrapping is a local calibration procedure). *It is obviously possible to calibrate a PWCHR model using the standard procedure defined in 3, by setting $x = (h_1, \dots, h_N)$. Given the fact that in the latter case, all the parameters are calibrated at once using the information from all the quoted CDS contract, whereas exactly one CDS is used to calibrate each model parameter in bootstrapping (see step 2 above), we refer to the bootstrapping as of a local calibration procedure.*

In the author's experience, no material differences have been found between results produced by using the two different procedures to calibrate a PWCHR model. Speed of execution is not a concern either given that, in general, no more than a dozen CDSs can be found for a given reference entity on a given valuation date.

Implementing a bootstrap procedure is, however, materially simpler than implementing a minimization algorithm as the one described in Definition 3.

This, together with the fact that, under certain implementation choices, detecting arbitrage might be more straightforward while bootstrapping than while running a minimization algorithm, makes usually prefer using a bootstrap to calibrate a PWCHR model.

1.3.3 THE CREDIT TRIANGLE FORMULA

For many applications, it is useful to derive an approximated relation between the hazard rate and a CDS par-spread. One way to establish that, is given by the following proposition.

Proposition 8 (Triangle relation). *Let us consider a stream of CDSs where the premium is paid continuously and a PWCHR model for the hazard rate. Then, the shortest maturity CDS par-spread, can be approximated through*

$$S^*(T^{\min}) = h_1(1 - R).$$

Proof. Let us consider a short-protection CDS with maturity T , and where the premium, c , is paid continuously over time. The price at time t , with $t \leq T$, $V_t(T, c)$, of the instrument can be calculated under a PWCHR assumption using the following expression:

$$V_t(T, c) = \int_t^T cQ(t, s)D(t, s) ds - (1 - R) \int_t^T D(t, s) d_s F_\tau(t, s), \quad (1.36)$$

where we used the definition $Q(x, y) := \mathbb{Q}_x(\tau \geq y)$.

Let us further consider a hazard-rate term structure built on a time-grid with points $0, T^{\min}, \dots, T_i, \dots$. In particular, let h_1 denote the hazard-rate's value for any time between 0 and $T^{\min}$.

The cumulative distribution function of the default time for time horizons shorter than $T^{\min}$ can be evaluated through

$$F_\tau(t, s) = 1 - Q(t, s) = 1 - e^{-(s-t)h_1}, \quad s \leq T^{\min}$$

and its differential with respect to s as

$$d_s F_\tau(t, s) = h_1 e^{-(s-t)h_1} ds, \quad s \leq T^{\min}.$$

The last two expressions can be used in Eq. (1.36) to calculate the spread, c^* , such that $V(T, c^*) = 0$. This is given by the solution to

$$\int_0^{T^{\min}} c^* e^{-sh_1} D(t, s) ds = (1 - R) \int_0^{T^{\min}} D(t, s) h_1 e^{-sh_1} ds,$$

that is, by

$$c^* = (1 - R)h_1.$$

□

This result is called *triangle relation* because it establishes a relation between the three variables c^* , h_1 and R .

The following corollary is an important one for further analysis done in the next chapters as it provides the hypothesis under which a flat hazard rate is to be expected.

Corollary 9. *In case of CDS par-spreads term-structures $T \mapsto S^M(T)$ where*

$$S^M(t) = \bar{S}, t \in \{T_0, \dots, T_N = \bar{T}\},$$

and where all the corresponding CDSs have the same recovery rate, R , a PWCHR model $T \mapsto \lambda(T) = \sum_{i=1}^N \mathbb{1}_{\{T_{i-1} < T < T_i\}} h_i$ can be calibrated in $t < \bar{T}$ by setting

$$h_i = \frac{\bar{S}}{1 - R}, \quad i = 1, \dots, N. \tag{1.37}$$

Proof. See Appendix A.2.

□

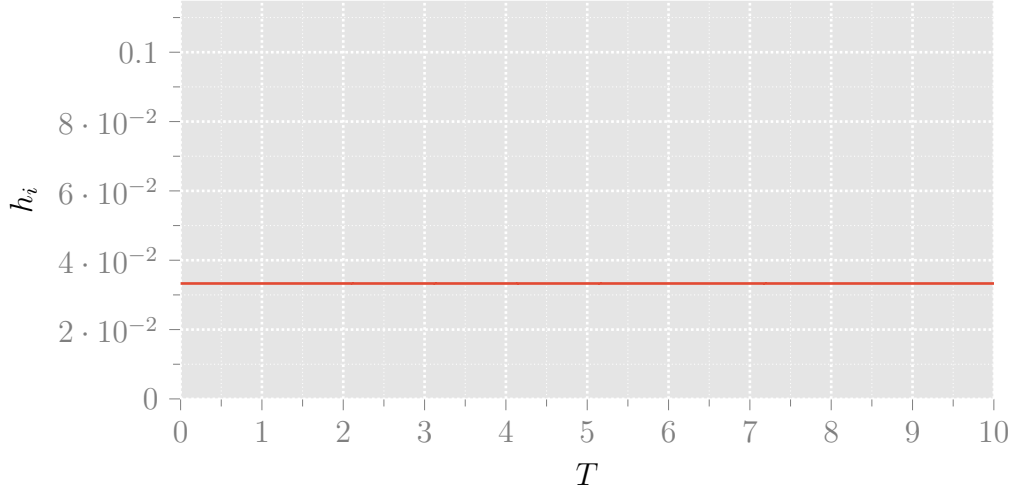


Figure 1.3: Hazard rate in PWCHR model from a flat CDS term–structure.

1.3.4 DETECTING ARBITRAGE

Let us consider a stylised example where we consider deterministic discount factors (and, therefore, deterministic ZCB) given by

$$P_0(T) = e^{-r_0 T}, \quad T \geq 0,$$

with $r_0 = 1\%$ and where quoted CDS spreads for a given reference entity exist and are flat

$$S^M(0, T) \equiv S_0, \quad T \in \{1Y, 2Y, 3Y, 4Y, 5Y, 7Y\}.$$

Recovery rates are assumed to be set at 40% for all the quoted CDS contracts.

The bootstrapped hazard rate term–structure is shown in Figure 1.3 and, consistently with the flat CDS par–spreads term–structure, is flat. Moreover, it is remarkably close to the value it would have according to the Corollary 9 ($0.\bar{3}$). The small discrepancies with respect to the theoretical value are caused by the fact that we used actual market conventions with respect to premium leg frequency (quarterly), day count convention (Act/360), and dates generation (IMM dates), whereas the corollary was proved under the hypothesis of continuous payments.

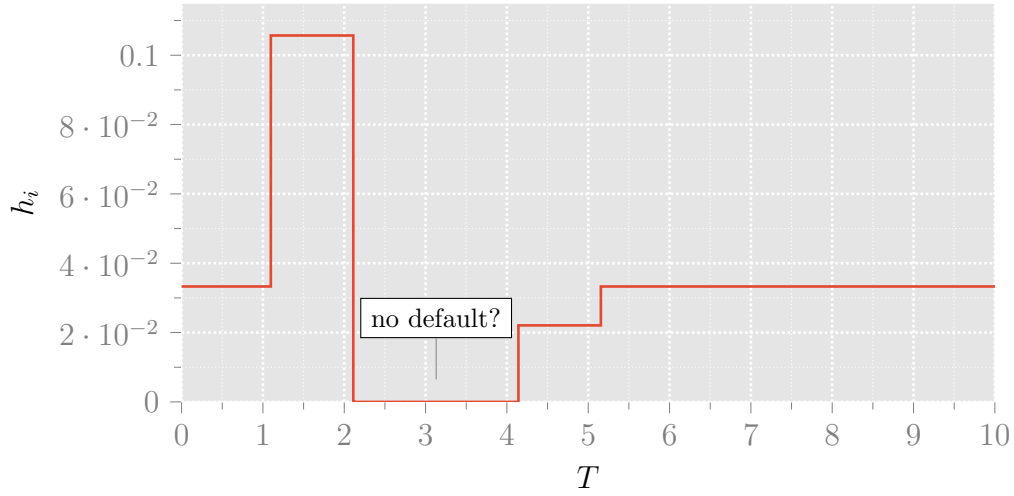


Figure 1.4: Hazard rate in PWCHR model from an arbitrageable CDS term-structure.

Arbitrage can be introduced, for example, by setting the quoted par-spread for a CDS with maturity 2 years to, say, 4%, while keeping all the other market quotes from the previous scenario. Given the resulting par-spread term-structure, an investor could decide to create the following strategy:

- s/he buys protection via a three year CDS,
- s/he sells protection via a two year CDS.

His or her resulting net position would be equivalent to buy protection against a default happening between two and three years. The cost of such protection, however, would amount to pay the difference between the quoted par-spreads for two years and three years maturity — that is, to receive 2% p.a. for the first two years, and to pay 2% p.a. for the remaining one year, resulting in a negative net cost and, therefore, in an arbitrage.

The inconsistency of such a market scenario can be easily highlighted through a bootstrapping procedure, as shown by the results presented in Figure 1.4, where no positive solutions were found in the time range in between two and four years. In the chart, the hazard rate has been set to zero for that time period.

This example highlights the two different aspects through which arbitrages

can manifest themselves in credit derivatives market data. In addition to producing the trading strategy itself — in this case, a negative protection cost — arbitrageable CDS spreads imply a null probability of default in a certain time range for the CDS's reference entity.

2

CoCo BONDS VALUATION WITH EQUITY AND CREDIT CALIBRATED FIRST PASSAGE STRUCTURAL MODELS

2.1 INTRODUCTION

The credit crisis that began in the summer of 2007 has led the financial industry to reconsider deeply ingrained concepts. One of those ideas is the role of capital and leverage in financial institutions. The term *capital* is related to the amount of assets and cash an institution is supposed to set aside to prevent its net asset value from falling below the level that could affect its business or hinder the pursuit of its strategic objectives. Capital has been the focus of regulation. There are broadly two notions of capital: *regulatory capital* (RC) and *economic capital* (ECAP). The former is the capital mandated by the regulatory authorities for the financial institution to be considered safe. The latter is an institution's internal estimate of

the needed capital and is supposed to reflect more closely the market and economic conditions in which the company operates.

REGULATORY CAPITAL ASSUMPTIONS: GRANULARITY AND ADDITIVITY

Up until the credit crisis, the regulatory capital requirement for a position in the portfolio of a financial institution was based on two key technical assumptions (see Gordy⁴²): *portfolio invariance* and *additivity*. The first means that the cost of capital should depend on the *position* that is added to a portfolio and *not* on the portfolio composition. This requirement implied a *ratings* based approach to capital management. The second requirement is that the total cost of capital is given by the sum of the capital costs of the individual positions. The second requirement has been traditionally implemented by resorting to an assumption of infinite-granularity and to the use of the one-factor Gaussian copula. This approach features a global single systemic risk factor. Within this framework the regulatory capital of a certain position depends ultimately on the rating of the position and on the capital cost associated with the rating class. Since the latter has been time invariant, the Basel regulatory capital of a portfolio has been a quite *static* measure by construction. For a critique see for example Blundell-Wignall and Atkinson¹⁰.

ECONOMIC CAPITAL

The economic capital of a portfolio, on the other hand, is supposed to depend on assumptions concerning the economic activity and conditions at the time of calculation. Additionally, the capital associated to a certain position depends on the specific portfolio in which the position is located. This point becomes much more relevant when the instrument in the portfolio is a securitization note. *Asset backed securities* (ABS) have been in the market for at least 25 years, and the origin of that asset class has its roots in the US, where it has appeared as an attempt to address part of the economic problems caused by mortgages (see e.g. Kothari⁵¹). The economic capital of a portfolio depends on the shape of the tail of the loss distribution, and

this shape is very much affected by the portfolio assets *correlations* (or, better, dependence; here we will use the word “correlation” also to denote the more general and well-posed notions of statistical dependence). The concept of correlation and its calibration depends significantly on the adopted *assumptions* for the determination of the loss distribution ^{*}.

CONTINGENT CAPITAL BONDS (CoCos)

As a consequence of the credit crisis, very large financial institutions are still under-capitalized at a time when economic development is most needed. To solve the issues of low capital levels [†] the industry has recently proposed a new concept of capital called *contingent capital (CoCo)*. This instrument has a hybrid nature and it is similar to a convertible bond, with some relevant caveats. Similarly to usual convertible bonds, CoCo bonds can be converted into equity. However, the CoCo bond conversion is not triggered by an event linked to the level of the stock price of the issuing institution. The triggering event is instead related to its capital situation and it is usually linked to a capital ratio falling below a predetermined threshold. In order to compensate the bond-holder for the risk of a conversion at a possible distressed time, the CoCo bond would pay a more generous coupon than a similar bond without CoCo’s features. A crucial question is “how much more generous”?

CoCo PRICING, DEFAULT MODELS AND CONVERSION

A very important issue is then what would be the price that would cover that risk and how it would be determined. CoCos are driven by conversion, but conversion in turn may be driven by credit quality and default risk, so that we need to consider a model for the credit and the default time of the issuing institution. Generally speaking, there are two distinct ways to model the default process.

^{*}The details are out of the scope of this work. We refer to Garcia and Goossens⁴⁰ and Brigo et al.¹⁸ for details.

[†]The use of taxpayer money to bail out the financial system has become, expectedly, a matter of very intense debate.

The first default modeling area is the one we will adopt in the present work. This is known as the *firm value* models area and it is generally attributed to Merton⁵⁷ and Black and Cox⁷, and underpins the Black–Scholes formula. In these models a company defaults when a latent variable, the firm asset value, breaches some barrier, typically the debt value and safety covenants. One then needs assumptions for the asset value process and for the capital structure of the company. Since the original work of Merton and Black–Cox there have been many extensions to it, and we refer to Bielecki and Rutkowski³ and the references therein for details. In this chapter, we will use the *analytically tractable first passage* model (AT1P)[‡] developed by Brigo and Tarengi¹⁷ to price a contingent capital bond. The model is calibrated to the implied risk neutral probabilities extracted from the CDS market term structure by the use of analytical formulas for barrier options, but can easily be calibrated to a term structure of corporate bonds for the same name. The peculiarity of this model, compared to otherwise more traditional models such as for example Briys and de Varenne²⁴, allowing for stochastic interest rates, is that AT1P allows for an *exact* calibration of the term structure of credit spreads across maturities, a feat that is not attainable with typical firm value models and is rather the specialty of “reduced form models”. It would not be difficult to extend AT1P to stochastic interest rates when these are assumed to be independent of the Brownian motion driving the firm value, but this would distract us from the immediate focus on credit and CoCos, so that we focus on the basic AT1P model here.

As we mentioned earlier, the second default modeling area is termed “reduced form models” or “intensity model”, see also Section 1.3.1. In this case one renounces modeling the default from an economic point of view and just assumes that the default time is the first jump of a time-inhomogeneous Poisson process, where the intensities of default (or instantaneous credit spreads) are usually calibrated to prices of the CDS or corporate bonds market. This has been the preferred approach by market practitioners interested in fitting (by construction) the implied risk neutral probability of default from the CDS

[‡]The model has been developed by Brigo and Tarengi¹⁷ for pricing equity swap under counterpart risk and by Brigo et al.²³ for the same problem in the context of an analysis of the collapse of Lehman Brothers.

market. In this context, the market is often assuming that the intensity is deterministic and that there is no credit spread volatility. This assumption is completely unrealistic in the light of the sizes of both implied and historical volatility for CDS, see for example Brigo¹¹. Furthermore, when coupled with static copula functions for default correlation, the lack of spread volatility may cause counterintuitive and dangerous patterns and results in counterparty risk modeling, see for example Brigo and Chourdakis¹³. It is possible to include credit spread volatility in a tractable way while preserving CDS calibration. Indeed, models with credit spread volatility can still be tractable even for CDS options and can calibrate the CDS market quickly and precisely, see for example the works Brigo and Alfonsi¹² and Brigo and El-Bachir¹⁴.

The reason why we choose the first family of models is due to its appeal in terms of linking model features (a default barrier) with economic content (asset returns determine financial conditions). In intensity models default strikes out of the blue, following a totally unpredictable exponential random time, that is hard to connect to economic variables. In firm value models instead default is the result of a process involving firm value and debt, so that a clear link with economics is given.

We will postulate a simple relationship between the conversion trigger and the asset/equity ratio (AER), a quantity available in our firm value model. We then improve the relationship by allowing a given amount of de-correlation between the conversion trigger and the AER.

EARLIER LITERATURE

Given capital concerns in banks, there has been much interest in CoCos in the literature. We may roughly classify previous research in two areas. The first area deals with the pricing and risk management of CoCos while the second area deals with investigating the impact of the introduction of CoCos on the optimal capital structure of the issuer. The work done for this chapter lies in the first area.

Concerning the second area of research, Barucci and Del Viva² extends an

approach originally proposed by Goldstein et al.⁴¹ where the underlying state variable is the claim to earnings before interest and taxes (EBIT) and where straight debt, contingent capital, equity and bankruptcy costs are modeled as perpetual assets, thus allowing for closed-form valuation formulas. The default barrier level, the trigger level and the optimal capital structure are then calculated such as to maximize the equity value and the net value of the company. Barucci and Del Viva² shows as the optimal capital structure strategy and the bankruptcy costs can change depending on the introduction of CoCos and on the type of trigger event and rule for equity conversion that are adopted.

Going back to the first area of pricing and risk management of CoCos, we refer the reader to Wilkens and Bethke⁶⁸ and references therein for a fairly comprehensive review of the three main previous approaches to CoCo bonds valuation, which we may briefly describe as structural-default based, reduced-form based and equity modeling based.

We further mention here the work of Li, J. and Metzler, A. and Mark Reesor, R.⁵⁴ that has been developed independently and appeared after the initial version of the present chapter in Brigo et al.²¹ since both works use a structural model framework for CoCo bond valuation. Differently from the present work, however, the model shown in Li, J. and Metzler, A. and Mark Reesor, R.⁵⁴ seems to be limited to a calibration to balance account data.

The structural-default approach models the issuer default via the first passage time of the firm value process through some threshold barrier associated with debt and safety covenants. The application of such models to CoCos pricing relies on the same idea and considers two different thresholds, the first one triggering equity conversion and the second one triggering default. In reduced-form based models, on the other hand, the trigger time is the first arrival time of a Poisson process with constant intensity, and the relevant intensity should be calibrated to the probability that the equity of the issuer falls below a certain threshold. A possible issue in reduced form models may be to model consistently equity conversion and default stopping times, since there is no economics guiding principle helping with coordination of the two times.

Equity-based models decompose the CoCo payoff in terms of plain bonds, knock-in equity forward and a strip of knock-out equity puts. The knock-in and knock-out equity levels substitute the capital ratio trigger. For an introduction to CoCos and for a discussion of the different approaches to valuation see De Spiegeleer and Schoutens²⁸.

To the best of our knowledge, none of the existing approaches has modeled explicitly and at the same time the bond-equity conversion time, the equity price at conversion, and the default time.

Without one or more of these three features it is not possible to calibrate a model to credit markets and equity market at the same time. This implies that one is discarding important information. Moreover, the set of instruments to which the CoCo pricing model is calibrated affects the set of considered risk factors and, hence, hedging strategies. Models able to only capture the credit-component of this hybrid instrument, will only allow to hedge it through credit derivatives, while models able to only handle the equity component will only allow to use equity instruments. A CoCo bond price, however, could in principle react to information coming from both markets.

Our work adds to the literature in few important ways. Firstly, we show how to use the AT1P firm value model to price CoCo bonds. Secondly, we describe a new calibration procedure, how to implement it, and finally we show the numerical results of our pricing approach, comparing our outputs with market quotes for a traded instrument. Thirdly, we analyze stress tests on our framework to see the impact on prices during periods of financial distress.

This chapter is organized as follows. In Section 2.2 we describe the CoCo bond instrument and its use in practice, inclusive of the motivation underlying its creation. The AT1P model used in our pricing approach is summarised in Section 1.2.2 while the discussion on its calibration for the specific problem of modelling regulatory capital, too, is shown in Section 2.3. Results and a related discussion are presented in Section 2.4. Finally, in Section 2.5 we summarize our conclusions.

2.2 CONTINGENT CAPITAL (CoCo)

As mentioned in Section 2.1, CoCos are a new form of capital. They have been created with the main objective of increasing capital levels of systemically important large banks. These instruments' equity like feature may be attractive for banks in their effort to comply with the new regulatory capital rules required by the Basel III framework.

Generally speaking, the CoCo instrument takes the form of a convertible bond where the conversion trigger event is a bank capital ratio falling below a certain level known at the issuance of the instrument. In case the event is triggered, the CoCo bond will be converted into equity at a certain conversion price. More precisely, in case the core capital ratio fell below a certain level, the financial institution would issue new shares (equity) to be bought at a certain contract pre-determined price. The way the conversion price is determined is specified in the contract and is an important part of the CoCo bond features.

2.2.1 CONVERSION PRICE DETERMINED AT CONVERSION DATE

Indeed, an important feature of a CoCo is its *conversion price* (CP). There are two important and contrasting possibilities underlying the CP. In the first case the conversion price is determined at the conversion date. If so, the price might be far too low, possibly resulting in significant dilution for the shareholders. In that case, shareholders may certainly be tempted to sell their equity holdings as soon as they sense an increase in risk of the capital ratio reaching the barrier (triggering conversion). This *rush to sell* may put the stock into a downward spiral that will potentially be self fulfilling. This approach, however, is quite convenient to the bond holders, given that they will be acquiring the shares at distressed prices. One can say that the approach will affect bond holders less at the expense of the equity holders. Depending on the equity price at the time of the conversion, the bond holders might become a quite large portion of the institution shareholders. Furthermore, depending on the created amount of equity, this might be seen as good

news for the stock.

2.2.2 CONVERSION PRICE FIXED AT THE CoCo's INCEPTION

If the conversion price, on the other hand, is pre-determined, there are different considerations to make. To begin with, it is important to understand what a reasonable initial value for the conversion price would be, in order to fix it at inception. In this sense, if the conversion price is determined as the spot price at the issuance date of the CoCo, then the bond holder will be acquiring newly issued shares for a price that might be much higher than the actual equity price at the moment the conversion trigger kicks in. In this case, the dilution in shares will be relatively small while the losses for the bond holders will be large. The bond holder might request a higher coupon, given the risk of having to acquire equity for a considerably higher value than the market price (in case of conversion). This might seem a good solution for the equity holders, at the cost of penalizing the bond holders. The latter might be tempted to short the stock or buy put options. Bond holders will do so especially if they sense conversion as unavoidable and intend to hedge their losses at that event. Again there is a high possibility of downside pressure on the stock and of a self fulfilling "downward spiral".

A different possibility would be to align the conversion price with the percentage in fall associated with the conversion trigger. That is, if the reduction trigger in capital amounts to say 40%, then the equity conversion trigger would also amounts to a 40% reduction in the spot equity price with respect to the known equity price at inception. From the equity holder perspective this might be a less radical solution than the first one, but in any case would still imply a considerable dilution effect.

2.2.3 TRANSPARENCY AND INPUTS

A second important point on CoCos has to do with *transparency*. The instrument is strongly dependent on the capital ratios of a financial institution. Essentially, for pricing purposes, one needs to have access to the portfolio

composition of the issuing institution. That is, one needs to have access to information on the portfolio compositions of the financial institution, or at least its aggregate percentage distribution in terms of sectors (and regions) and types of instruments. A portfolio composition however is a very hard information to obtain. In the next section we will present a model that does not depend on the portfolio composition while depending on the path of the equity return of the company.

2.3 ADAPTING AT1P TO CoCos

Given the hybrid equity/credit nature of a CoCo bond, we will be using a structural model of default to price it. As explained in Section 1.2, in structural models default occurs when a latent variable, the asset value, breaches a barrier, typically the debt book value of the company. Furthermore, structural models can naturally link the credit and the equity components as the value of the equity can be represented as an option on the same latent variable that drives the asset value.

Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ with a standard Brownian motion defined on it, W_t^Q , and the filtration generated by the Brownian motion, $\{\mathcal{F}_t\}_{t \geq 0}$. Denote also by $(V_t, t \geq 0)$ the value of the company, and by $H(\cdot)$ a deterministic barrier. The default event is defined in this setting as:

$$\tau = \inf\{t \geq 0 : V_t \leq H(t)\}, \quad \inf \emptyset := +\infty. \quad (2.1)$$

The firm-value process is assumed to have the form

$$dV_t = (r - q)V_t dt + \sigma(t)V_t dW_t^Q, \quad V_0 = v, \quad (2.2)$$

and the barrier is given by

$$\hat{H}(t) = H e^{(r-q)t - B \int_0^t \sigma^2(s) ds}. \quad (2.3)$$

When dealing with a contingent conversion bond, in addition to the default

time τ we need to model a second random time. This second time accounts for the conversion of the CoCo bond. Assume τ_c to be the (random) conversion time that is defined in the contract as

$$\tau_c = \inf\{t \geq 0 : c_t \leq \bar{c}\}, \quad \inf \emptyset := \infty$$

where c_t is the *regulatory capital* at time t and $\bar{c}$ is the threshold level below which the CoCo bond is converted into equity. In the current set up, c_t is an exogenous variable whose proxy is shown in Section 2.3.2. An alternative approach would involve the direct modeling of c_t . This is left for further research.

2.3.1 THE PAYOFF OF A CoCo

Given that τ and τ_c are not independent, a natural question is whether a CoCo bond can default before being triggered. In this regard we can make some considerations. These instruments are designed specifically to avoid that, in case of troubles caused by capital issues, banks will need to go through expensive new stocks issuances. In this sense, we would expect a CoCo to be triggered before the issuer's default. As for the value of the capital ratio, it makes sense to assume a null capital ratio at default, that is $c_\tau = 0$. A final consideration concerns the data availability as regulatory capital information is neither continuously (say daily in this context) updated, nor it is as transparent as liquidly traded quantities. This capital information is expected to be updated at most twice a year, on the balance account dates of June and December.

Given the low frequency with which c_t is updated, in practice it can certainly happen that default occurs between two updating dates. However, let us assume for the time being that issuers and regulators alike will use the conversion option as a last resort *before* any technical default[§]. Under this hypothesis the relation between τ and τ_c , default and CoCo conversion time

[§]As it will become clear in Section 2.3.2 one does not need this assumption as it is in fact a *consequence* of the proxy assumed for c_t (see 2.9 for detail).

respectively, is constrained by

$$\mathbb{P}(\tau < \tau_c \mid \tau < T) = \mathbb{Q}(\tau < \tau_c \mid \tau < T) = 0. \quad (2.4)$$

This implies that, before T , the default of the bond will always follow the conversion time. This way, since at conversion the payoff is converted into equity, the payoff of the CoCo bond depends explicitly on τ_c and equity only, with no explicit dependence on τ . Of course equity will depend on τ implicitly.

Consider now two times t and T — namely the evaluation and the maturity dates respectively. Suppose $D(t, T) = B_t/B_T$ to be the risk free discount factor at time t for the period $[t, T]$, where, as in the rest of this work, we assume deterministic interest rates. Using such discount factors, let us assume $\Pi_c(t, T)$ and $\Pi(t, T)$ to be the discounted payoff of the CoCo bond and that of an identical risk-free bond without the conversion feature, respectively. Additionally, Let E_t be the value of the stock price at time t . We can then write:

$$\Pi_c(t, T) = \mathbb{1}_{\{\tau_c > T\}} \Pi(t, T) + \mathbb{1}_{\{\tau_c \leq T\}} \left(\Pi(t, \tau_c) + N E_{\tau_c} D(t, \tau_c) \right), \quad (2.5)$$

where the number of shares N obtained in case of conversion is set at the CoCo issuance date in a way that $N E^* = \mathbb{E}_0 [\Pi_c(0, T)]$. Here E^* denotes the stock conversion price, and Eq. (2.5) can be rewritten as

$$\Pi_c(t, T) = \mathbb{1}_{\{\tau_c > T\}} \Pi(t, T) + \mathbb{1}_{\{\tau_c \leq T\}} \left(\Pi(t, \tau_c) + \mathbb{E}_0 [\Pi_c(0, T)] \frac{E_{\tau_c}}{E^*} D(t, \tau_c) \right). \quad (2.6)$$

In passing we observe that the following analogy is possible with traditional CDS models. Usually the CoCo bond is issued at par, meaning that $\mathbb{E}_0 [\Pi_c(0, T)] = 1$, so that

$$\Pi_c(t, T) = \mathbb{1}_{\{\tau_c > T\}} \Pi(t, T) + \mathbb{1}_{\{\tau_c \leq T\}} \left(\Pi(t, \tau_c) + \frac{E_{\tau_c}}{E^*} D(t, \tau_c) \right). \quad (2.7)$$

In order to establish an analogy with corporate bonds pricing, it might be

useful to interpret the conversion time as a default time, and E_{τ_c}/E^* as a recovery rate. In doing so, we note that in reality recovery rates never exceed one, whereas in this case this can happen. Nevertheless, this expedient can be used to suggest an alternative approach to the valuation of these instruments via a reduced-form approach. That is, we could use the first jump of a Poisson process as a model for the conversion time. However, a full analogy with the reduced-form approach used to price CDS contracts, would mean that τ_c would no longer be a $\{\mathcal{F}_t\}_{t \geq 0}$ stopping-time. Moreover, the intensity of the underlying Poisson process could be made random and linked with other variables (rates and equity/recovery), but the exponential component of the random time would be independent of all Brownian motions. This may lead to a weak dependency between conversion time and other market drivers. It is worth stressing that in the AT1P framework we can have strong dependency between τ_c and E_t as they are driven by the same factor.

2.3.2 A PROXY FOR c_t

Consider *total equity* to be given by the difference between total asset and total liabilities. A common indicator of the leverage of a company is the ratio between total assets and total equity (also known as asset/equity ratio (AER)). We assume here that this ratio is the driver for the change in capital ratios. To have an idea of how the capital ratio behaves with respect to the AER, we selected from *Fitch's Financial Delivery Service (FDS)* ¶ all the end-of-year balance account dates. We filtered at these dates all the entities by their *individual ratings* § and, for every individual rating class i and for every end-of-year balance account date, we went through a cross sectional ordinary least square (OLS) estimation of the following statistical model

$$c_i = \alpha + \beta X_i + \epsilon_i, \quad (2.8)$$

¶FDS is a proprietary financial database that includes banks and insurance companies.

§The individual/viability rating is a concept developed by Fitch Ratings and we refer to the company's website for details.

where

$$X_i := \frac{A_i}{A_i - L_i}.$$

Let us make few comments on the individual ratings to understand why we used them to filter and aggregate financial institutions. First, individual ratings do not assume that financial institutions are inherently protected by an underlying government support. Moreover, they are based on a database spanning more than 20 years of observations. This means that they cover at least three business cycles. Finally, although each financial institution might invest according to its own idiosyncratic strategy, we assume that financial institutions may be aggregated in homogeneous groups for capital purposes. That is, we are assuming that the individual ratings capture the long term aspect of the institutions that are supposed to be reflected in the price of the instrument.

In Figures 2.1–2.2 we show some results for the OLS estimation. In particular, Figure 2.1 contains the scatter plot and the simple linear regression for the balance account date of 31 December 09 and for the individual rating class C. Figure 2.2 shows the evolution, year by year, of parameters α and β for the same rating class.

For every individual rating class, γ , we chose a simple average of the evolution over time of the two parameters α and β . Let us call the two averaged values $\bar{\alpha}_\gamma$ and $\bar{\beta}_\gamma$. We estimated a proxy for c_t in terms of V_t and $\hat{H}(t)$ by identifying them as total assets and total liabilities respectively.

This makes possible to express our proxy for the regulatory capital ratio as $c_t = c^\gamma(V_t, \hat{H}(t))$, with

$$c^\gamma(V_t, \hat{H}(t)) = \begin{cases} \bar{\alpha}_\gamma + \bar{\beta}_\gamma \frac{V_t}{V_t - \hat{H}(t)} & V_t \geq V_{\min}(\gamma), \\ 0 & V_t < V_{\min}(\gamma), \end{cases} \quad (2.9)$$

where

$$V_{\min}(\gamma) = \hat{H}(t) \cdot \max \left(\frac{\bar{\alpha}_\gamma}{\bar{\alpha}_\gamma + \bar{\beta}_\gamma}, 1 \right).$$

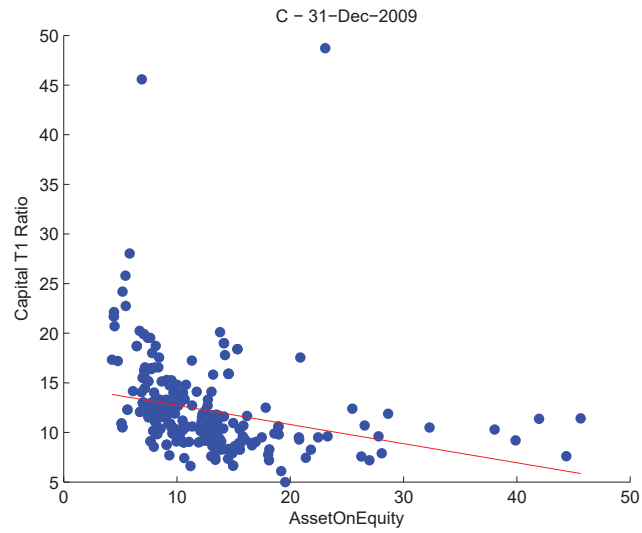


Figure 2.1: Scatter plot and OLS estimation of Tier 1 Capital Ratio against asset/equity ratio on December 2009 for class rating C.

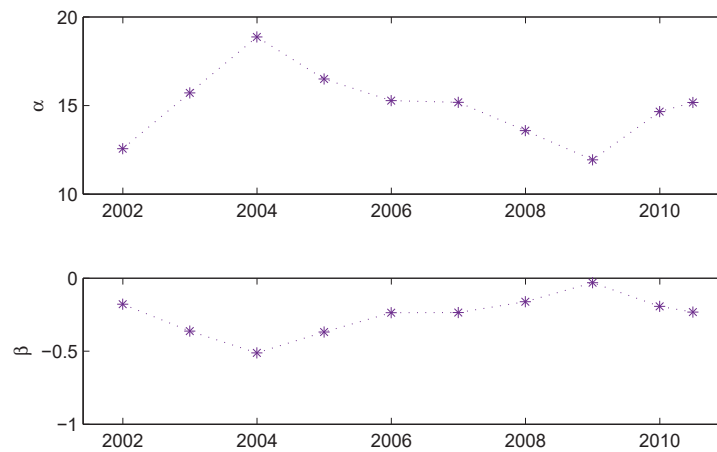


Figure 2.2: Evolution of α and β for the C rating class

Notice that, once the model has been calibrated, Eq. (2.9) allows us to evaluate the price of a CoCo bond via a Monte Carlo simulation using the payoff as in Eq. (2.7).

The following proposition specifies under which conditions the proxy in Eq. (2.9) implies the condition in Eq. (2.4).

Proposition 10. *Let us define $f(X_t) = \alpha + \beta X_t$ and assume $f(X_0) > \bar{c}$ and $\beta < 0$. Then the triggering event occurs before default, namely*

$$\mathbb{Q}(\tau_c < \tau) = 1.$$

Proof. We have easily $X \rightarrow \infty$ as $V \rightarrow \hat{H}^+$ and, given that $\beta < 0$, $f \rightarrow -\infty$ as $X \rightarrow \infty$. This means that in the limit of a default occurring we would have $f \rightarrow -\infty$. Observing that f is a continuous function completes the proof, since f would have to pass by $\bar{c}$ before reaching the default barrier. $\square$

The relation between c and X for the individual rating C is shown in Figure 2.1. In terms of our proxy we can infer from Figure 2.2 that the value of $\bar{\beta}_\gamma$ is negative. Eventually, as the proxy starts from a value higher than the contractual trigger $\bar{c}$ it becomes relevant that we add it to the calibration procedure requiring that $c^\gamma(V_0, \hat{H}(0))$ matches the last reported capital ratio. We will address this point in more detail in Section 2.3.4.

DECORRELATING CAPITAL RATIO AND TOTAL ASSET

We chose a setting which implies extreme values for the correlation between the capital ratio and the process AER. Indeed a straightforward calculation shows that

$$\text{corr}(c_t^\gamma, X_t) = \text{sign}(\beta),$$

where, for every random variables X, Y , we used the following definitions:

$$\begin{aligned}\text{corr}(X, Y) &:= \frac{\mathbb{E}_0 [(X - \mathbb{E}_0 [X])(Y - \mathbb{E}_0 [Y])]}{\text{std}(X) \text{std}(Y)}, \\ \text{std}(X) &:= \text{var}(X)^{1/2}, \\ \text{var}(X) &:= \mathbb{E}_0 [(X - \mathbb{E}_0 [X])^2].\end{aligned}$$

We can however obtain intermediate values for the correlation considering an alternative shape for the capital ratio. We propose

$$\mathcal{C}_t^\gamma = \bar{\alpha}_\gamma + \bar{\beta}_\gamma \text{std}(X_t) \left(\eta \frac{X_t}{\text{std}(X_t)} + \sqrt{1 - \eta^2} \epsilon_t^\gamma \right) \quad (2.10)$$

where we've introduced a second random shock, independent of the Brownian motion underlying the total asset,

$$\epsilon_t^\gamma \sim \mathcal{N}(0, 1) \quad \forall t$$

and a new parameter $\eta \in [0, 1]$.

Model (2.10) is a generalization of the previous capital ratio in the sense that

$$\mathcal{C}_t^\gamma \rightarrow c_t^\gamma \quad \text{for } \eta \rightarrow 1.$$

Moreover, this model change doesn't affect the variance of the capital ratio

$$\text{var}(\mathcal{C}_t^\gamma) = \text{var}(c_t^\gamma).$$

Furthermore, the new shape of the capital ratio provides an easy way to control its correlation with the AER process as it holds

$$\text{corr}(\mathcal{C}_t^\gamma, X_t) = \eta \text{sign}(\beta).$$

It is in view of the above equation that we will often refer to η as the correlation parameter. Finally, we will refer to the time step Δt for checking the capital ratios as to the "sampling frequency".

2.3.3 HOW TO EVALUATE THE STOCK-PRICE IN AT1P

As explained in Section 2.3, in AT1P an entity can default at any time and not only at maturity. If we assume that the debt still has a clear single final maturity T and that early default is given by safety covenants, it is not unreasonable to model equity as an option on the firm value with maturity T that is killed if the default barrier is reached before T . See also the equity chapter in Brigo et al.¹⁹. Namely, we calculate the stock price E_t (in this framework) as a down-and-out European call option, that is

$$E_t = \beta(t)\mathbb{E}_t \left[\frac{\left(V_T - \hat{H}(T) \right)^+ \mathbb{1}_{\{\tau > T\}}}{\beta(T)} \right] = f(t, V_t). \quad (2.11)$$

We can use Eq. (B.25) to calculate the stock price both inside a simulation of Eq. (2.7) and in the calibration procedure.

A closed form solution for the price of the option is given for example in Rapisarda⁶⁵, see also the equity chapter in Brigo et al.¹⁹.

The formula is as follows:

$$f = D(t, T) \left(V_t e^{\int_t^T (v(s) + \frac{\sigma(s)^2}{2}) ds} (1 - \Phi(d_3)) - \hat{H}(T) (1 - \Phi(d_4)) \right. \\ \left. - \hat{H}(t) \left(\frac{\hat{H}(t)}{V_t} \right)^{2B} e^{\int_t^T (v(s) + \frac{\sigma(s)^2}{2}) ds} (1 - \Phi(d_5)) + \hat{H}(T) \left(\frac{\hat{H}(t)}{V_t} \right)^{2B-1} (1 - \Phi(d_6)) \right),$$

where

$$\begin{aligned}
v(t) &= r - q - \frac{\sigma(t)^2}{2}, \\
d_3 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{V_t}{H} - \int_t^T (v(s) + \sigma(s)^2) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}, \\
d_4 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{V_t}{H} - \int_t^T (v(s)) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}, \\
d_5 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{\hat{H}(t)^2}{HV_t} - \int_t^T (v(s) + \sigma(s)^2) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}, \\
d_6 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{\hat{H}(t)^2}{HV_t} - \int_t^T (v(s)) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}.
\end{aligned}$$

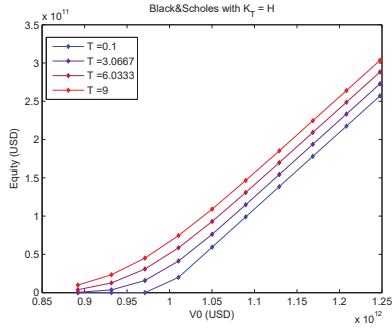
SOME REMARKS ON THE STOCK PRICE EVALUATION

In this section we will try to capture the effects of some of the AT1P model features in the stock price evaluation.

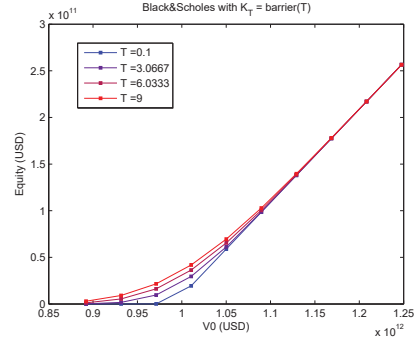
We reported in Figure 2.3 a series of plots illustrating a comparison between the solution of Eq. (B.25) and the evaluation of an European call option under the classic Black–Scholes framework. We used a set of parameters calibrated to the relevant market prices and such that $B = 0$.

Observe from Eq. (1.24) that, in case $B = 0$, the growth rate of $\hat{H}(\cdot)$ is the same as the one of the mean value of V_t .

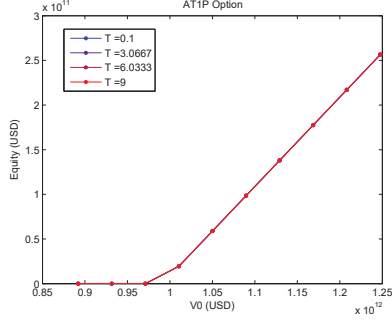
Main points we wanted to highlight by showing Figure 2.3 are the following ones. First of all the comparison between the plots in Figures 2.3a and 2.3b shows the impact of considering a fixed strike K against a time-dependent strike $K(t) = \hat{H}(t)$ that follows the safety covenants barrier. The latter choice for the strike results in an effect that is similar to what we would obtain by considering a null drift for the underlying process. The comparison between



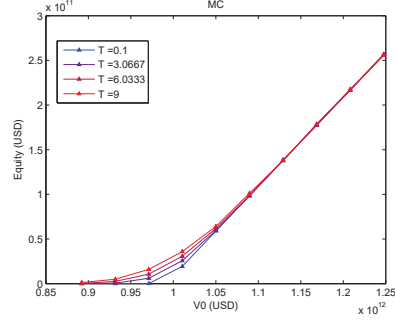
(a) $V_0^{BS} \mapsto D(t, T) \mathbb{E}_t \left[(V_T^{BS} - K)^+ \right]$
with
 $K = \hat{H}(0)$.



(b) $V_0^{BS} \mapsto D(t, T) \mathbb{E}_t \left[(V_T^{BS} - K(T))^+ \right]$
with
 $K(t) = \hat{H}(t)$.



(c) equity spot price in AT1P.



(d) $V_0 \mapsto D(t, T) \mathbb{E}_t \left[(V_T - \hat{H}(T))^+ \right]$

Figure 2.3: Comparison between using the AT1P model and the (widely spread) Black–Scholes formula. Top charts shows B&S results, where $dV_t^{BS} = rV_t^{BS} dt + \sigma_1 V_t^{BS} dW_t$ with W a Brownian motion under $\mathbb{Q}$. Bottom charts show results where V_t follows the process from Eq. (1.23) with $q = 0$.

plot in Figure 2.3b and the one in Figure 2.3d shows that the same behavior is attainable in the AT1P framework when we modify the payoff in Eq. (B.25) by considering a plain call option rather than a down-and-out call option, that is by removing the indicator function inside the expectation^{**}. Finally, the comparison between charts in Figure 2.3d and Figure 2.3c shows that adding the down-and-out feature results in a sensible decrease of the diffusion effect, effectively making the option price fairly similar to a forward in the region where it is in-the-money.

It is particularly interesting to note that this last point is a confirmation of a result found in Chapter 8 in Brigo et al.¹⁹. We state it in the next proposition.

Proposition 11 (Equity spot with $B = 0$). *In an AT1P model where $B = 0$, the equity spot price $(E_t, t \geq 0)$ is given by*

$$E_t = V_t - \hat{H}(t), \quad t > 0.$$

We refer to Appendix B.2 for a sketch of the proof of this result.

2.3.4 CALIBRATION PROCEDURE FOR A CoCo BOND IN AT1P

We followed the calibration procedure described in Definition 3 to determine the model's parameters. In addition to the quoted CDS spreads, we used the results from Sections 2.3.2 and 2.3.3 to include the spot value of the stock price and the most recent available regulatory capital ratio in the calibration procedure. The inclusion of these two additional market observables was needed to fix the indetermination in the values of the remaining model parameters (e.g. H), shown by results in Tables 1.2 and 1.3.

For this calibration, we used in sequence simulated annealing and Levenberg–Marquardt algorithms to perform the minimization needed for this calibration.

In particular, considering n quoted CDS par spreads for different maturities

^{**}We resorted to a MonteCarlo engine to calculate the call prices in AT1P.

$(S^{T_1}, \dots, S^{T_n})$, the last published tier 1 capital ratio, c , and the spot price of the equity E_0 , the calibration works as follows: first simulated annealing is used to set a starting point, $x^{(0)}$, in the region $\mathcal{X} := (0, 5) \times (0, 1)^{n+1}$. In the second step of the calibration procedure, we started from $x^{(0)}$ and we used Levenberg–Marquardt to find the optimum point p , in the state space $\mathcal{X} := \mathbb{R}_+ \times (0, 1) \times \mathbb{R}_+^n$. For both the minimization problems we considered the set of market observables $\phi^M = (S^M(0, 1Y), \dots, S^M(0, 10Y), c, E_0)$.

2.4 RESULTS

In this section we will show numerical results for the two-step calibration procedure described above. Additionally, we will illustrate the algorithm by using it to price a Lloyds issued CoCo bond.

Before we move forward an important point to keep in mind is the following. As we have seen in Section 2.3, the AT1P model is a *Gaussian* based model and the advantage of it is certainly mathematical tractability. This happens in the form of analytical formulas that make the model very handy for the calibration process. However, this model also suffers from a number of problems. The basic AT1P suffers from lack of short term credit spreads, common to more basic firm value models, see for example Brigo et al.¹⁹. It happens however that many processes seen in finance not only have jumps (discontinuities, see for example Brigo et al.¹⁸) but also present the feature of long term memory (see e.g. Garcia and Goossens⁴⁰ for a more extensive discussion on these issues). In that case one might be tempted to address the issue of model risk when using AT1P. For this reason we have added a section on stress tests to get a feeling of what might happen with the model output in case input parameters have significant changes. Further analysis could consider the extensions of the AT1P model with random default barrier and possibly random volatility, see for example Brigo et al.²³ and Brigo et al.¹⁹.

Table 2.1: Coordinates of the points calculated as output of the simulated annealing algorithm ($x^{(0)}$) and Levenberg–Marquardt (p). These have been obtained using rating class C, $\bar{\alpha} = 14.09\%$, $\bar{\beta} = -0.21$.

Point	B	H	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
$x^{(0)}$	0.0002	0.8174	0.1030	0.0328	0.0007	0.2981	0.1961	0.2989	0.5006
p	0.0000	0.9533	0.0260	0.0119	0.0187	0.0187	0.0212	0.0207	0.0250

2.4.1 CALIBRATION

The coordinates of the point x_0 found using simulated annealing and of the point p found using Levenberg–Marquardt are reported in Table 2.1.

The optimum set of parameters, p , resulted in the following relative errors with respect to the chosen set of market instruments

$$\phi^M = (S^{1Y}, \dots, S^{10Y}, c, E_0) :$$

ϕ_i^M	$\Delta\phi_i/\phi_i^M$
S^{1Y}	-1.433×10^{-12}
S^{2Y}	-2.532×10^{-12}
S^{3Y}	-1.877×10^{-12}
S^{4Y}	-1.997×10^{-12}
S^{5Y}	-2.464×10^{-12}
S^{7Y}	-2.533×10^{-12}
S^{10Y}	-2.781×10^{-12}
c	-7.860×10^{-2}
E_0	4.120×10^{-2}

We point out that the global optimization we used is quite coarse, so that stopping our calibration at that stage would hardly be satisfactory. However, we use this global step simply to identify a region for the parameter spaces where it is good to start a more local-type optimization. We have tried several other calibration schemes and the final results we obtain are always close to the optimum parameters in the Levenberg–Marquardt algorithm above.

Table 2.2: Financial features of the CoCo bond.

ISIN ID:	XS0459088877
Level of subordination	Lower Tier 2 (LT2)
Issue date	Dec – 09
Maturity date	Mar – 20
Coupon	11.04% p.a. paid twice per year, on Mar and Sept
Trigger event	Core Capital Tier 1 Ratio falling below 5%
Conversion Price	fixed on 27th Sept 2009 at 59 GBp per share

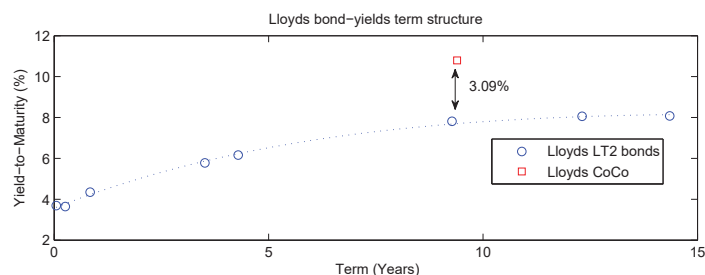


Figure 2.4: Lloyds' bond-yields term structure on 15 December 10.

2.4.2 PRICING APPLICATION

For the pricing exercise of this chapter we have used the Lloyds Bank issued CoCo bond shown in Table 2.2.

The pricing date for this example is 15 December 10. The instrument is actively traded in the market and on the trading date of our example had a bid-offer price per unit of nominal of (1.014; 1.02875), corresponding to a *yield-to-maturity* (YTM) of (10.52%; 10.28%).

Figure 2.4 shows the YTM term structure for Lower Tier 2 bonds issued by Lloyds on 15 December 2010. Observe that the YTM of the CoCo bond is

Table 2.3: Price, 95%-confidence interval and annualized yield-to-maturity dependence on the sampling frequency Δt . The market price bid-ask spread is (1.014; 1.02875).

Δt (Years)	Estimated value	Confidence interval 95%	Annualized YtM
1/2	1.04405	1.03488; 1.05321	0.100424
1/10	1.03041	1.02196; 1.03887	0.102567
1/20	1.01775	1.00972; 1.02578	0.104589
1/500	1.02351	1.01599; 1.03102	0.103666

3.09% above the LT2 term structure. An interpretation for it is the following: 3.09% is the additional spread that the market considers fair to pay to account for the additional risks embedded in a CoCo bond.

SENSITIVITY TO THE SAMPLING FREQUENCY

In general capital ratios are supposed to be made public by companies twice per year. However, as soon as the sampling is not continuously performed, the condition (2.4) is not satisfied anymore. As trigger and default time can be observed only on discrete times, observing simultaneity between the two becomes more likely. Simultaneity of trigger and default time impacts on the price in two ways in the current framework: on one hand it increases the number of coupons received by bondholders, since default is postponed to the next discrete date; on the other hand it brings to zero the value of the “conversion recovery rate”.

With respect to the second point, this is due to the very definition of default we have in AT1P. At default $V_\tau = \hat{H}(\tau)$ so that the value of the barrier option through which we calculate the stock price is zero^{††}.

^{††}Even though the institution might review capital ratios in an annual or bi-annual basis, Rating Agency analysts and Regulators will certainly follow it much more closely. In that case either a rating agency might downgrade an institution, thus triggering a chain of events that will call the conversion trigger, or a regulatory authority might come forward and request the conversion to happen. This sort of events have not been explicitly modeled in our approach.

In Table 2.3 we show how the choice of the sampling frequency impacts the price of the CoCo bond. We fix the payout ratio $q = 0$ and the correlation parameter $\eta = 1$. We resort to Monte Carlo simulation to calculate prices. We also report the 95%-confidence interval and the YTM. Observe that varying the sampling frequency causes a variation in price and YTM of about 2.5% and 4% respectively.

The change in price that is due to the change of the sampling frequency points to the same sort of model risk that has been observed e.g. in Garcia and Goossens⁴⁰ (Ch.15 and the references therein) when analyzing CPDO's using the continuous Gaussian process. This suggests the necessity of additional research on the AT1P model this time using jump / Levy processes. Furthermore, a richer version of AT1P, called SBTv, with a random default barrier and possible inclusion of misreporting or fraud is introduced in Brigo et al.²³, see also Brigo et al.¹⁹ and references therein.

A HEURISTIC CRITERION TO SET THE SAMPLING FREQUENCY

Let us consider a bond paying a fixed rate c per annum on a time schedule $T_1^b, \dots, T_N^b$. Assume a null recovery rate in case of default. We can calculate this bond price per unit notional at time t with $t < T_0^b < T_1^b$ (where T_0^b is the first reset date) as

$$P(t) = c \sum_{i=1}^N \alpha_i^b D(t, T_i) \mathbb{Q}_t(\tau > T_i^b) + D(t, T_N^b) \mathbb{Q}_t(\tau > T_N^b), \quad \alpha_i^b := T_i^b - T_{i-1}^b. \quad (2.12)$$

The AT1P model allows us to calculate analytically expressions as the one in Eq. (2.12) by using Eq. (1.25). We can use this fact to establish a criteria to set the time step of the Monte Carlo time grid that we use to price the CoCo bond via simulation. We may request the grid to be such that the price of plain defaultable bonds obtained through the Monte Carlo engine agrees with the one calculated through the AT1P closed-form formula. Following this criteria, we found that by using $\Delta t = 1/500$ the price calculated through Eq. (2.12) always lies in the 95%-confidence interval of the Monte Carlo

engine. Therefore, our choice for the sampling frequency is $1/500$.

SENSITIVITY TO DRIFT AND CORRELATION

For the calculations in this subsection, we fixed the sampling frequency to $\Delta t = 1/500$. It is worth stressing the extreme difficulty in calibrating a crucial parameter: the payout ratio q . Since a sound and robust estimation for this parameter does not appear to be immediate, we decided to run scenarios on it. The same approach has been followed for the correlation parameter η . We calibrated our model to different values of the payout ratio q and for every model we obtained in this way, we calculated the price of the CoCo bond under different correlation conditions. Results are shown in Table 2.4.

On one hand, with maximum η and given that $\beta < 0$, when X grows then c decreases more than with lower η . In the limit case where $\eta = 0$ and if we consider $\text{std}(X_t)$ as constant, the capital ratio does not depend on X_t . In this case, it is more likely that the capital ratio does not cross the trigger ratio at all, even in scenarios where V_t approaches $\hat{H}_t$. In those cases, however, conversion is triggered by the default of the bond as per Eq. (2.4). The effect of lowering η is an increase in the duration of the bond together with a decrease of the equity price at conversion. Table 2.4 illustrates how, for low values of η , the price monotonically decreases as η increases. This holds until η reaches a level such that the trigger time is actually given by the capital ratio falling below the contractual threshold rather than the firm value crossing the barrier. In those cases one can notice that the duration is less sensitive to η , and thus the fact that the equity price at conversion is increasing in η becomes more relevant. Overall, it is also important to notice that negative payout ratios are not very realistic, and hence it is good in our case that the true market price is captured by the first two rows of the table, using non-negative values for the payout ratio q .

Table 2.4: Price variation given different q values and different η values. $r = 0.54\%$ p.a. We recall that the market traded price at the same date was (1.014; 1.02875)

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$q = r$	1.0476	1.0273	1.0092	1.0091	1.0095
$q = 0$	1.0618	1.0221	1.0167	1.0184	1.02351
$q = -r$	1.0391	1.0271	1.0174	1.0208	1.0398

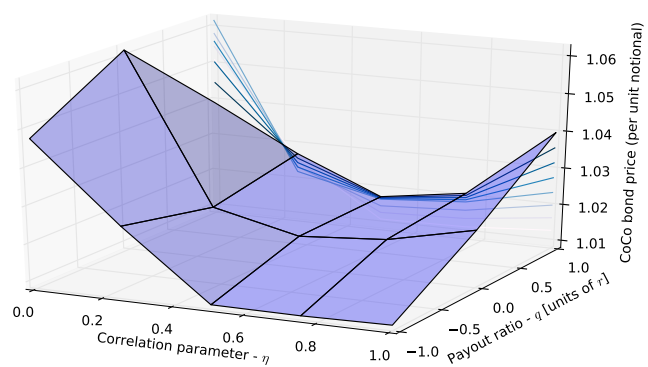


Table 2.5: Financial features of the PDB with ISIN code XS0497187640

ISIN code	XS0497187640
Level of subordination	Lower Tier 2 (LT2)
Issue date	Mar – 10
Maturity date	Mar – 20
Coupon	fixed, 6.5% per year paid once per year, on Mar
Trigger event	n.a.
Conversion Price	n.a.

Table 2.6: Price of the PDB with ISIN code XS0497187640

Market Price	Estimated value	Confidence interval 95%
0.904517	0.922531	0.916056; 0.929007

2.4.3 PRICE OF A PLAIN DEFAULTABLE BOND (PDB)

We inferred default probabilities from the CDS market. It is indeed possible to price a plain defaultable bond using only such default probabilities and deterministic interest rates. As we have pointed out in Section 2.3, the payoff $\Pi_d(t, T)$ of the PDB with unit notional 1 can be written as:

$$\Pi_d(t, T) = \mathbb{1}_{\{\tau > T\}} \Pi(t, T) + \mathbb{1}_{\{\tau \leq T\}} (\Pi(t, \tau) + RD(t, \tau)1), \quad (2.13)$$

where R is a deterministic recovery rate, and $\Pi(t, T)$ the discounted payoffs of an identical risk free bond.

We used Monte Carlo simulation to value $\mathbb{E}_t [\Pi_d(t, T)]$ and we chose a PDB with characteristics (except possibly for the conversion feature of the CoCo) that was as close as possible to the CoCo bond we valued in Section 2.4.2. Its features are shown in Table 2.5.

In Table 2.6 we show our estimation for the price of this PDB (for convenience purposes we also show the bid market price).

The difference between our model price and the market price of this security is an example of CDS–bond basis. There is a wide literature on this subject, and the basis itself can be measured in different ways. We refer the interested reader to Haworth et al.⁴⁵ and Elizalde et al.³⁵ for an extensive discussion

and for a survey on credit spread measures for bonds.

INCLUDING THE BOND–CDS BASIS

We may consider a correction to our CoCo bond price that takes into account the CDS–bond basis. As we hinted at in the introduction, given an entity on which a CDS and a corporate bond are traded, one could extract credit spreads both from the CDS or the corporate bond from the same name. Given that the bond is a funded instrument, the difference between the credit spread implied by the bond prices term structure and the credit spread implied by CDS quotes term structure is often interpreted as a funding liquidity basis. The difference between the market implied credit spread for a bond and a CDS on the same name is called CDS–bond basis. One way to express the basis is to calculate the *par-equivalent CDS spread* (PECS, see for example Elizalde et al.³⁵) associated to the PDB price for the same name as the CDS we are considering. This is defined as the par-spread of the CDS that should be used in calculating a CDS–implied bond price if this value has to agree with the bond market price.

The decision on whether one should use bond or CDS quotes as calibration inputs in handling the CoCo bond, and hence on whether or not one should include the basis, depends on how one plans to hedge the CoCo bond. If one is planning to hedge the credit part of the bond with CDS, then no basis should be added, in line with our earlier results. If however one is planning to hedge the CoCo bond with ordinary defaultable bonds, then the basis should be added to CDS quotes before the AT1P calibration or, equivalently, bond prices should be used in calibrating AT1P.

To cover both the CDS and the bond hedging cases, we include in the discussion the case where the CDS bond basis entered the model calibration.

We used the basis and the notion of PECS to measure the shift in par-CDS spreads needed to recover the observed market price of the PDB. Under market conditions observed on 15 December 2010, the price of the PDB translates in a parallel shift of the CDS spreads term structure of 40 bps.

Table 2.7: Coordinate of the point associated to the market data set that includes the CDS–bond basis.

Point	B	H	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7
p^b	0.0000	0.9533	0.0267	0.0128	0.0196	0.0198	0.0226	0.0225	0.0275

Table 2.8: Calibration error associated to the set of parameters p^b .

ϕ_i^M	$\Delta\phi_i^M/\phi_i^M$
S^{1Y}	5.942×10^{-11}
S^{2Y}	-1.046×10^{-12}
S^{3Y}	-1.118×10^{-12}
S^{4Y}	-1.133×10^{-12}
S^{5Y}	-1.280×10^{-12}
S^{7Y}	-1.267×10^{-12}
S^{10Y}	-1.663×10^{-12}
c	-7.860×10^{-2}
E_0	4.119×10^{-2}

This market information can be embedded in our model by considering this synthetically shifted CDS term structure as input of the AT1P calibration process in place of the original CDS spreads. We associate to this new set of market data a new point in parameters space, p^b , where we use b to indicate that we are including the CDS–bond basis. The calibration routine in this case was only Levenberg–Marquardt and we set p as its initial point. The new set of parameters is showed in Table 2.7.

The calibration error associated to this new set of market data and model parameters is reported in Table 2.8. It is similar in order of magnitude to the calibration error obtained for the set of parameters that was not accounting for the CDS–bond basis, p .

By definition, the CDS—bond basis correction made the model price of the PDB agree with its observed market price, as shown in Table 2.9.

The price of the CoCo bond calculated using this new set of parameters is showed in Table 2.10. As expected, as a result of the positive shift in credit

Table 2.9: Price of the PDB with ISIN code XS0497187640 under the set of parameters p^b .

Market Price	Estimated value	Confidence interval 95%
0.904517	0.900579	0.894121; 0.907036

Table 2.10: Price variation given different q values and different η values using the set of parameters p^b from Table 2.7. $r = 0.54\%$ p.a. We recall that the market traded price at the same date was (1.014; 1.02875)

	$\eta = 0$	$\eta = 0.25$	$\eta = 0.5$	$\eta = 0.75$	$\eta = 1$
$q = r$	1.0096	0.9779	0.9761	0.988	0.994
$q = 0$	1.0034	1.0026	0.9825	0.9863	1.0128
$q = -r$	1.0068	0.9954	0.9924	1.0044	1.0267

spreads, each point in the $(q-\eta)$ -table has been shifted down with respect to the corresponding point in Table 2.4. Now the table no longer catches the exact market price we observed, differently from the previous CDS based case, but still gets close. Note that a negative payout ratio is not very realistic, so that the last row of the table should be thought of as a stress test for interest rates but not as a realistic pricing scenario. We focus then on the first two rows. We could reconcile the difference in the first two rows prices with the market price either by arguing that the hedging strategy may often resort to CDS in cases where corporate bonds are not liquid enough, or by pointing out that our model may be detecting a slightly overpriced market for the specific CoCo we are looking at. Our model seems to suggest that the market CoCo price is more in line with CDS implied credit information than with bond implied one, although we recognize this evidence to be circumstantial, given the number of inputs, modeling and calibration choices we made, and a broader analysis across a panel of market CoCo price should be conducted before stating general conclusions. This analysis is difficult because market quotes of traded CoCos are hard to obtain.

Table 2.11: Financial features of the CoCo – PDB.

Level of subordination	Lower Tier 2 (LT2)
Issue date	Dec – 09
Maturity date	Mar – 20
Coupon	11.04% p.a., paid twice per year, on Mar and Sept
Trigger event	–
Conversion Price	–

Table 2.12: Prices of the CoCo – PDB, case $q = 0, \eta = 1$. The analogous price with conversion features is reported in the last column

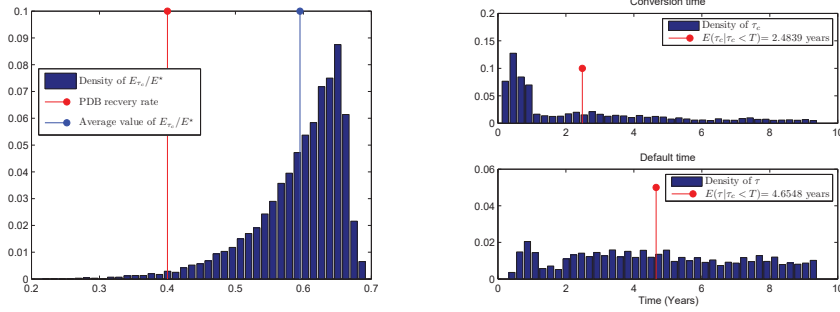
Parameters	PDB value	95% Conf. Interval	YtM (p.a.)	CoCo value
p	1.21666	1.20807; 1.22525	0.076	1.01775
p^b	1.1875	1.17887; 1.19613	0.0798	1.00781

2.4.4 PRICE OF THE CoCo BOND AS A PLAIN DEFAULTABLE BOND

In this section we estimate the impact of the conversion option embedded in the CoCo bond. We calculate this impact in terms of price/yield by pricing the same instrument but eliminating the conversion feature. That is, we price a plain defaultable bond (PDB) with the characteristics presented in Table 2.11.

With a slight abuse of notation, we can use Eq. (2.13) to write the discounted payoff of this instrument. Results are shown in Table 2.12. As expected the inclusion of the conversion option reduces the price of the instrument. Conversely, removing the optionality from the CoCo, we get a higher price for the instrument.

Observe that an important difference between a PDB and a CoCo is what the holder receives in case of default (for the first) or conversion (for the second). The concept of *recovery rate* in the case of the CoCo is the ratio between equity at trigger time (E_τ) and the contractually-stated conversion price (E^*). In Figure 2.5a we show the distribution of the mentioned ratio (that is $\frac{E_\tau}{E^*}$) resulted of our Monte Carlo simulation. For comparison purposes we also plotted the widespread assumption of recovery rate in the CDS market (40%). Notice that in our MC (*gaussian* based) simulation the average price



(a) Ratio between equity value at conversion (E_T) and contractual conversion price (E^*). $q = 0$ and $\eta = 1$. (b) Conversion and default times.

Figure 2.5: Empirical distributions of the equity value at conversion, of the conversion time, and of the the default time.

at conversion is 50% of the price set in the contract. The higher recovery rate is compensated by a sensibly smaller value for the conversion time compared to the default time as shown in Figure 2.5b.

2.4.5 STRESSING INPUT DATA

In order to address possible issues linked to the model risk mentioned in Section 2.4.2 we decided to keep the model as is, but to stress its parameters ^{‡‡}.

We have tested the price changes for variations of the CDS curve and the equity value of the company. The results are shown in Table 2.13. We moved CDS spreads up and equity prices down by 10% and 30% for both cases. Observe that under the model the sensitivity to equity price moves is significantly higher than to CDS spread moves. Or in another way, moves of 10% and 30% on equity would be equivalent to higher moves on CDS spreads. Indeed observe that a move of 10% in equity causes a larger impact than a 30% upper move on CDS spreads.

^{‡‡}One may certainly come up with jump diffusion models. In that case however, complexity increases significantly as to our knowledge there is no closed form solutions available for calibration purpose.

Table 2.13: Prices of the CoCo bond under stressed market data input. The percentage change has to be intended as negative when referred to the equity spot price and positive and uniformly applied to the whole term structure when referred to CDS. The unstressed price is 1.02351

Market data	Percentage change			
	10%		30%	
	Price	Conf. Interval	Price	Conf. Interval
Spot equity price	0.99156	0.98359; 0.99953	0.91609	0.90852; 0.9237
CDS	1.00439	0.99638; 1.01241	0.97059	0.96282; 0.97836

2.5 CONCLUSIONS

In this chapter we have made a detailed study of an equity / Merton based approach to price a contingent capital / CoCo bond. The model is based on the seminal work of Brigo and Tarengi¹⁷. As a Gaussian based model it has the advantage of analytical formulas readily available for calibration purposes. On the data side we have also used (for calibration purposes) CDS and capital ratio estimations based on (the) proprietary database available from Fitch Solutions. Another advantage of the model is exactly the fact one does not need to have access to the whole portfolio of the institution (to have an estimation of an instrument that is in fact capital dependent).

On the technical side a possible weakness of the model would be the fact that it is Gaussian based. For addressing that issue we have come up with the sensitivity of the prices to stressed model parameters. In addition, we have considered a simple and intuitive analysis that may be helpful in highlighting poor outcomes. A possible continuation of this work is an analysis of the hedging performance of the model. This requires a dataset of CoCo prices in the market in time. Given that our model has closed form solutions for CDS and bonds and has a closed form formula for equity, a hedge analysis based on credit and equity products is straightforward. In further work we plan to analyze the model hedging performance also in comparison with other works, including Schaefer and I.A.⁶⁶ and Wilkens and Bethke⁶⁸. Further future points of investigation will be adopting more realistic firm value models, possibly with jumps, and more complex models for the conversion time and its

relationship with the default time. Included in the latter point are different assumptions in the dependence between regulatory capital and firm-value (see Eq. (2.9)).

3

MULTI CURRENCY CREDIT DEFAULT SWAPS: QUANTO EFFECTS AND FOREIGN EXCHANGE RATE DEVALUATION JUMPS

3.1 INTRODUCTION

The need for quanto default modeling arises naturally when pricing credit derivatives offering protection in multiple currencies.

Reasons for entering into *Credit Default Swaps* (CDS) in different currencies can come from financial, economic, or even legislative considerations: they range from the composition of the portfolio that has to be hedged to the accounting rules in force in the country where the investor is based. In case the reference entity is sovereign, economic reasons play a major role since for an investor it might be more appealing to buy protection on, for example,

Republic of Italy's default in USD rather than in EUR. Indeed, in the latter case the currency value itself is strongly related with the reference entity's default.

Figure 3.1 shows the time series of par-spreads for USD-denominated and EUR-denominated CDSs on Republic of Italy from the beginning of 2011 until the end of 2013. The time range has been chosen so to include the 2011 Euro-debt crisis.

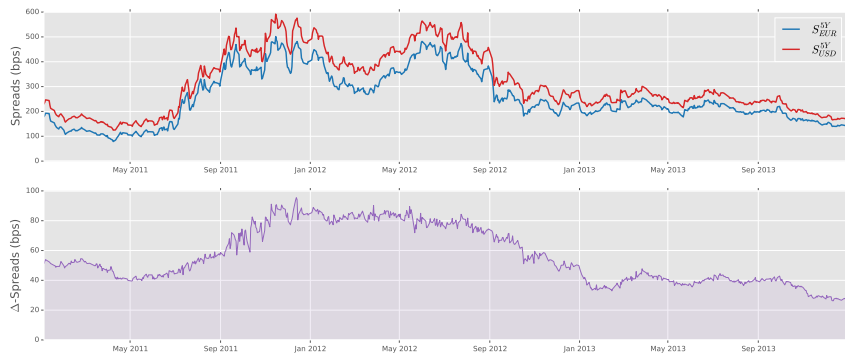


Figure 3.1: In the top chart, 5Y par spread time series for USD-denominated CDSs, S_{USD}^{5Y} , and EUR-denominated CDSs, S_{EUR}^{5Y} , on Italy. The difference between the two par-spreads is showed in the bottom chart. Source: Markit

The difference between the par-spreads for USD-denominated and EUR-denominated CDSs is shown in the bottom chart. In order to build a model which accounts for the default information and generate the spreads in the two currencies, the joint evolution of the obligors default's intensity and of the FX rate between the two currencies must be modelled.

In the present chapter, we show two ways to model the joint dynamics of credit and FX rates. In the first approach the interaction between the credit and the FX component is entirely explained by an instantaneous correlation between the Brownian motions driving the stochastic intensity and the FX rate. In a second, more sophisticated modelling approach, a further source of dependence between the two components is introduced in the form of a conditional devaluation jump of the FX rate upon default of the reference obligor.

The diffusive approach emphasizes the limitations of delegating the whole credit/market interaction to instantaneous correlation between intensity and market risk factors. As shown by comparing the model-implied quanto spreads in Figure 3.5 with the observed quanto spreads in Figures 3.1–3.6, instantaneous correlation alone is not able to explain the observed quanto spread. This phenomenon is somehow related to what happens when pricing credit correlation instruments. In that context it is well known that instantaneous correlation between intensities is unable to generate the sufficient level of dependence to generate the market spreads of index tranches (see, for example, Brigo et al.¹⁹, Brigo and Mercurio¹⁵, Cherubini et al.²⁷).

Using the latter modelling approach we will show how the introduction of jump-to-default effects achieves a much stronger FX/credit dependence than correlated Brownian motions. In particular, we will be showing in Section 3.4.4 that the addition of FX jumps allows to recover both the EUR and the USD spreads. Furthermore, in the same section, we will be testing a powerful, yet simple, way of extracting the magnitude of currency devaluation upon default from the CDS market data that is derived in Section 3.2.5.

In addition to multi-currency CDSs, the quanto effect in credit modelling finds a natural application in the *Credit Valuation Adjustment* (CVA) space. CVA is an adjustment to the fair value of a derivative contract that accounts for the expected loss due to the counterparty's default. One of the main challenges in calculating CVA is the lack of liquid CDS market data to calibrate model parameters. The calibration and approximation techniques showed in this chapter to connect currency devaluation with multi-currency CDS par-spreads can as well be applied to CVA modelling — for example, to better reflect right-way or wrong-way risk. The resulting FX/credit cross modelling improvement is crucial, especially in those cases where the interaction between the counterparty credit and the FX component is strong, i.e. with emerging market credits and systemically relevant counterparties.

We refer the interested reader to Brigo et al.¹⁹ for a comprehensive overview of CVA modelling and to Cherubini²⁶ for specific discussions about collateral modelling. Modelling the dependence between credit and market risk factors is crucial to accurately calculate the CVA charge. This is the subject of

Chapter 4.

In Section 3.2.5, we show how the introduction of default-driven FX jumps changes the dynamics of the stochastic default's intensity after a measure change. This happens because, from a mathematical perspective, the FX rate is a component of the Radon–Nikodym derivative that links the risk neutral probability measures associated to two different currencies. As stated by Girsanov Theorem (see , for example, Jeanblanc et al.⁴⁹), the dynamics of the compensated default process under different risk-neutral measures differ in their drift component. Such drift depends on the quadratic covariation between the FX rate and the default process (and it is zero when such covariation is null) and can be interpreted as the stochastic default's intensity of the reference entity.

The above result is strongly linked to another aspect of FX rate modelling, which we will refer to as FX symmetry throughout this document (see the discussion in section 3.2.2). Consistency between an FX rate process and its reciprocal is not guaranteed under every possible distributional assumptions made on its dynamics. For example, in case of stochastic volatility FX modelling, the reciprocal FX rate would not necessarily have the same dynamics that one would expect given that the reciprocal FX rate is also a Radon–Nikodym derivative. For geometric Brownian motions, however, this consistency is guaranteed. Due to the change in the intensity in the second pricing measure induced by the jump-to-default feature of the FX rate/Radon–Nikodym derivative process, we prove in Section 3.2.5 that the symmetry is preserved also for our specific FX model.

3.1.1 PREVIOUS LITERATURE

We refer to Bielecki et al.⁴ for an overview of the general problem of deducing a PDE to price defaultable claims and to Bielecki et al.⁵ for the specific problem of CDS hedging in a reduced-form framework.

For an introduction to the joint modelling of credit and FX in a reduced-form framework with application to Quanto–CDS pricing, we refer to Ehlers and

Schönbucher³³, El-Mohammadi, R.³⁴. Ehlers and Schönbucher³³ propose the idea to link FX and default's intensity by considering a jump-diffusion model for the FX-rate process where the jump happens at the default time. Differently from the present work, no explicit derivation of the PDE is presented, as the focus is on affine processes modelling.

The same idea is presented and developed in El-Mohammadi, R.³⁴. In that work it is shown how to calculate quanto-corrected survival probabilities using a PDE-based approach. In order to do that, the author deduces a Fokker-Planck equation for the joint distribution of FX and intensity.

The approach we present in Section 3.2 below is based on the same jump-to-default framework as the one used in the references above. In our case, however, the calculation of the quanto-corrected survival probabilities depends on solving a Feynman-Kac equation, the solution of which is a price, while in El-Mohammadi, R.³⁴ a probability density distribution was calculated. At implementation level, the difference between the two approaches lies in the fact that in the latter case an additional integration step would be required to calculate a price. Additionally, the way we work out our main pricing equation makes clear what instruments and in what amounts one would need to effectively implement a delta-hedging strategy.

The techniques showed in this chapter seem particularly relevant for long-maturity trades, where the effects of idiosyncratic jump-to-default components on counterparty risk can be more pronounced and where, therefore, they can have a big impact on wrong way risk estimation. For a relevant example of CVA calculations related to long-maturity trades, we refer to Biffis et al.⁶, where the cost of CVA and collateralization are calculated for longevity swaps.

The use of Lévy processes with local volatility to price options on defaultable assets has been recently explored in Lorig et al.⁵⁵, where a family of asymptotic expansions for the transition density of the underlying is derived. Differently from the approach presented in this chapter, in that case a single stochastic process drives both the default intensity and the option's underlying. On the other hand, being able to account for the implied volatility skew

is a feature currently missing from the framework presented in Section 3.2 and that will be explored in future works.

With respect to the Republic of Italy's test case that is presented in the results Section 3.4, we note that the Euro-area situation presents interesting problems that go beyond the mere credit-FX interaction which is the focus of the present work. An additional layer of complexity is provided in this case by the interconnectedness between the credit risk of the different currencies.

Empirically, Germany quanto CDS basis tends to be more pronounced than the Greece one (see Pykhtin and Sokol⁶⁴), reflecting higher correlation between EUR/USD and Germany intensity of default and higher EUR/USD devaluation upon Germany default.

3.1.2 QUANTO CDS

Quanto CDSs are designed to provide protection upon default of a certain entity in a given currency. There are cases, like for sovereign entities or for systemically important companies, when an investor might prefer to buy protection on a currency other than the one in which the assets of the reference entity are denominated. A typical reason for entering this type of trades would be to avoid the FX risk linked to the devaluation effect associated to the reference entity's default.

Alternatively, protection might be needed in a different currency from the one in which the assets of the reference entity are denominated because it serves as a hedge on a security denominated in that specific currency.

The discounted cashflows of the premium leg, $\Pi_Q^{\text{Premium}}(ccy)$, can be written as a generalization of the single-currency case presented in Section 1.1.1:

$$\Pi_Q^{\text{Premium}}(ccy) = S^C \sum_{i=0}^N \mathbb{1}_{\{\tau > T_i\}} D^{ccy}(0, T_i) + D_0^{ccy}(\tau) \Delta(T_{\kappa(\tau)}, \tau) \mathbb{1}_{\{\tau \leq T\}} \quad (3.1)$$

where

- valuation time is assumed to be at $t = 0$;

- $\{T_0; \dots; T_N\}$ with $0 < T_0 < \dots < T_N$ is the set of quarterly spaced payment times;
- $D^{ccy}(t, T)$ is the stochastic discount factor for currency ccy at time t for maturity T ;
- S^C is the contractual spread;
- τ is the default time of the reference entity.

The only difference between this function and the one used for single-currency CDSs is that in the quanto CDS case we allow for different denominations (through the argument ccy), whereas in the single currency CDS case, we left the payment currency unspecified, as that would be determined by the risk-neutral measure used for pricing.

The protection leg is made of a single cash flow, $\Pi_Q^{\text{Protection}}(ccy)$, paid upon default of the reference entity on a reference obligation:

$$\Pi_Q^{\text{Protection}}(ccy) = \text{LGD} \mathbb{1}_{\{\tau \leq T_N\}} D^{ccy}(0, \tau), \quad (3.2)$$

where LGD is the *loss given default* related to the contract.

The spread S^C that makes the expected value of the cash-flows in Eq. (3.1) equal to the expected value of the cash-flow in Eq. (3.2) is referred to as *par-spread* and we will usually use $S^*(ccy)$ to denote it. The existence of CDSs on the same reference entity whose premium and protection cashflows are paid in different currencies creates a basis spread between the par-spreads of these contracts. Figure 3.2 provides a schematic representation of two possible contracts settled in two different currencies.

We refer to Elizalde et al. ³⁷ and references therein for an overview on quanto CDS markets and for a thorough exposition of the rules governing these contracts. We note here that

- the standard contracts for sovereign CDS are denominated in USD. This means in particular that for countries of the EUR zone, like Italy,



Figure 3.2: Protection on a given reference entity can be bought by A from B in different currencies. The stream of payments in Eq. (3.1) is indicated by the solid arrow, while the dashed arrow is used for the contingent payment in Eq. (3.2). The LGD payment, albeit settled in different currencies, is the same percentage of the notional in the two contracts.

Greece or Germany, the modeling set up to use when including a devaluation approach is the one detailed in Section 3.2.5;

- upon default of the reference entity, a common auction sets the loss given default (LGD). The LGD so defined is valid for all the CDSs, irrespectively of the currency they are denominated in.

3.2 MODEL DESCRIPTION

We begin by considering a probability space $(\Omega, \mathcal{G}, \mathbb{Q}, (\mathcal{G}_t, t \geq 0))$ satisfying the usual hypotheses. In particular $(\mathcal{G}_t, t \geq 0)$ is a filtration under which the dynamics of the risk factors are adapted and under which the default time of the reference entity is a stopping-time. Depending on the specific examples, we will also consider spaces with a different equivalent measure, for example the risk neutral measure associated to the liquid money market or the risk neutral measure associated to the contractual currency money market.

We will be using a Cox process model for the default component and we will refer to the stochastic intensity of the default event simply as *hazard rate* or *intensity*, using the two terms interchangeably.

Unlike the usual approach followed in the reduced-form framework for credit risk modelling (for example in Lando⁵³, Brigo and Mercurio¹⁵), we do not introduce a second filtration with respect to which only the stochastic processes driving the market risk-factors are measurable. The total filtration $(\mathcal{G}_t, t \geq 0)$, inclusive of market and default risk, is the only filtration we will consider (see also Section 1.3.1). We note that the practical reason for

considering this second filtration is because that allows to apply theoretical results developed to price interest rates derivative to credit risk derivatives pricing. Due to the specific model choices we make in the following, however, this would not present any real advantage, while, as shown in sections 3.2.4 and 3.2.5, working with a single filtration gives us the possibility to calculate the quanto adjustment using a PDE approach.

3.2.1 THE ROLES OF THE CURRENCIES

In this section we set up some definitions concerning the role of the currencies that will be used in our modelling approach.

For of any quanto CDS pricing, we will be considering the following two relevant currencies:

- *contractual currency* — This currency is a contract’s attribute: it is the currency in which both premium leg and protection leg payments are settled. When considering applications to quanto CDS, for a given reference entity, CDSs are available in at least two different contractual currencies;
- *liquid currency* — This is the contractual currency of the most liquidly traded CDS on a given entity. It is used to define a risk–neutral measure used to price and calibrate the model.

We list here two examples to illustrate the use of the contractual and liquid currencies.

1. the pricing in USD–measure of a CDS on Republic of Italy settled in EUR;
2. the pricing in USD–measure of a CDS on Republic of Italy settled in USD.

We specified the values of the two currencies for each of these test cases in Table 3.1. We chose the test cases so that for all of them USD is the

Table 3.1: Currencies involved in the pricing of the test cases detailed in Section 3.2.1.

	Test case 1	Test case 2
Contractual currency	EUR	USD
Liquid currency	USD	USD

the liquid currency, but this is not necessarily true for all CDS available in multiple currencies. It is worth noting that the test case 2 can be priced using a usual single currency approach. Test cases 1 and 2 will be used in Section 3.4.4 to illustrate the capability of the model specified in Section 3.2.5 to explain the currency basis observed in the market.

3.2.2 TWO MARKETS MEASURES

In this section we summarize known results about change of measure in presence of FX effects. This is mostly done to establish notation and set the scene for the following original developments.

Contractual and liquid currencies are *absolute* attributes of a CDS instrument and they are needed to correctly setup the calibration and pricing problems. An alternative currency classification is *relative* to the chosen pricing measure. From this perspective, we will be using *domestic* and *foreign* currency to denote the currency linked to the risk-neutral pricing measure and the currency of the contract we want to price, respectively.

Let us consider the corresponding money market accounts as the numeraires for both the economies. We will use a superscript ^f, to denote variables in the foreign economy, so that, for example, the two numeraires are $(B_t, t \geq 0)$ for the domestic economy and $(B_t^f, t \geq 0)$ for the foreign economy. The money market accounts' dynamics are given by

$$dB_t = r_t B_t dt, \quad B_0 = 1, \quad (3.3)$$

$$dB_t^f = r_t^f B_t^f dt, \quad B_0^f = 1, \quad (3.4)$$

where $(r_t, t \geq 0)$ and $(r_t^f, t \geq 0)$ are the stochastic processes describing the

short rates in the two economies.

Let us also consider an exchange rate $(Z_t^f, t \geq 0)$ between the currencies of the two economies. Z_t^f is defined as the price of one unit of the liquid currency expressed as units of the contractual/foreign currency in a spot exchange at time t .

We are interested in finding an expression for the Radon–Nikodym derivative that changes the probability measure from $\mathbb{Q}^f$ to $\mathbb{Q}$. This can be worked out by using the Change of Numeraire technique and a generic payoff denominated in the contractual currency, represented by the function ϕ_T^f . To do so, we consider, as said above, the contractual–currency money market account, $(B_t^f, t \geq 0)$, as a numeraire for the measure $\mathbb{Q}^f$, while for the measure $\mathbb{Q}$ we still use the liquid–currency money market account, but with value denominated in the contractual currency, $((Z^f B)_t, t \geq 0)$. The price of the contractual currency payoff ϕ^f can be expressed in the two measures as:

$$\mathbb{E}_0^f \left[\frac{1}{B_T^f} \phi_T^f \right] = \mathbb{E}_0 \left[\frac{Z_0^f}{B_T Z_T^f} \phi_T^f \right]. \quad (3.5)$$

The $\mathbb{E}_t^f[\cdot]$ expectation on the left-hand side, on the other hand, can be written as

$$\mathbb{E}_0^f \left[\frac{1}{B_T^f} \phi_T^f \right] = \mathbb{E}_0^f \left[\frac{B_T Z_T^f}{B_T^f Z_0^f} \frac{Z_0^f}{B_T Z_T^f} \phi_T^f \right] \quad (3.6)$$

and the two expressions above can be used to obtain the Radon–Nikodym derivative that defines the change of measure from $\mathbb{Q}^f$ to $\mathbb{Q}$:

$$L_T^f := \frac{d\mathbb{Q}}{d\mathbb{Q}^f} \Big|_{\mathcal{G}_T} = \frac{B_T Z_T^f}{Z_0^f B_T^f}, \quad L_0^f = 1. \quad (3.7)$$

Remark 6. *As in the rest of the work, in the following, we will be considering deterministic interest rates both for the liquid–currency and for the contractual–currency economy. This means that the money market accounts*

will be described by

$$dB_t = r(t)B_t dt, \quad B_0 = 1, \quad (3.8)$$

$$dB_t^f = r^f(t)B_t^f dt, \quad B_0^f = 1, \quad (3.9)$$

in place of (3.3) and (3.4). To lighten the notation, in most cases we will drop the t -dependency for $r(t)$ and $r^f(t)$ in the following equations.

The process defined in Eq. (3.7) has to be a martingale in the foreign measure. This, together with the assumption of deterministic interest rates, condition can be used to determine the drift of $(Z_t^f, t \geq 0)$. Due to Itô's formula, the dynamics of $(L_t^f, t \geq 0)$ can be written as

$$dL_t^f = d\left(\frac{B_t Z_t^f}{B_t^f X_0}\right) = \frac{B_t}{B_t^f Z_0^f} (dZ_t^f + rZ_t^f dt - r^f Z_t^f dt), \quad L_0^f = 1. \quad (3.10)$$

If for example we assume a lognormal dynamics for the FX rate

$$dZ_t^f = \mu^f Z_t^f dt + \sigma Z_t^f dW_t^f, \quad Z_0^f = x, \quad (3.11)$$

then asking that $(L_t, t \geq 0)$ in Eq. 3.10 is a martingale brings to the familiar condition

$$\mu^f = r^f - r. \quad (3.12)$$

This result can be proven under slightly more general terms, that is if we consider a $\mathbb{Q}^f$ -martingale, $(I_t^f, t \geq 0)$, in place of a Wiener process and a predictable process $(\nu_t, t \geq 0)$ as the diffusive term (σZ_t^f in our case) in the $(Z_t^f, t \geq 0)$ -dynamics.

Proposition 12. *Let us consider a FX rate $(X_t, t \geq 0)$ whose dynamics is given by*

$$dX_t = \mu^X X_t dt + \nu_t dI_t^f, \quad X_0 = x,$$

where $(I_t^f, t \geq 0)$ is a $\mathbb{Q}^f$ -martingale, and where $(\nu_t, t \geq 0)$ is an adapted and left-continuous process.

Let us further consider the money market accounts for the two economies, $(B_t, t \geq 0)$ and $(B_t^f, t \geq 0)$ as defined by equations (3.3) and (3.4), respec-

tively.

Then $\mu^X = r^f - r$ for all $t \geq 0$.

Proof. We want to prove that

$$x = \mathbb{E}_0 \left[\frac{B_t X_t}{B_t^f} \right], \quad t \geq 0.$$

In order to do that, it is useful to define the process $(Y_t, t \geq 0)$ as $Y_t := B_t X_t / B_t^f$ and to calculate its expected value. This can be done using Fubini theorem as:

$$\mathbb{E}_0 [Y_t] = \mathbb{E}_0 \left[\int_0^t dY_s \right] = \int_0^t \mathbb{E}_0 [dY_s].$$

The differential of $(Y_t, t \geq 0)$ can be easily calculated through Itô expansion:

$$\begin{aligned} dY_t &= X_t d\frac{B_t}{B_t^f} + \frac{B_t}{B_t^f} dX_t \\ &= X_t \frac{r B_t B_t^f - B_t r^f B_t^f}{B_t^{f2}} dt + \frac{B_t}{B_t^f} (\mu^X X_t dt + \nu dI_t^f) \\ &= X_t \frac{B_t}{B_t^f} (r - r^f + \mu^X) dt + \frac{B_t}{B_t^f} \nu dI_t^f, \\ &= Y_t (r - r^f + \mu^X) dt + \frac{B_t}{B_t^f} \nu dI_t^f, \quad Y_0 = x, \end{aligned} \tag{3.13}$$

where we used the fact that the dynamics of the money market accounts $(B_t, t \geq 0)$ and $(B_t^f, t \geq 0)$ are given by equations (3.3) and (3.4), respectively.

From Eq. (3.13), and the deterministic rates assumption, we have that

$$\mathbb{E}_0 [dY_s] = (r - r^f + \mu^X) \mathbb{E}_0 [Y_s] dt, \tag{3.14}$$

as the expected value of the second term on the RHS — being a predictable process integrated with respect to the $\mathbb{Q}^f$ -martingale $(I_t^f, t \geq 0)$ — is zero (see Protter⁶³, Chapter IV).

We can now apply, again, a Fubini argument to the LHS of Eq. (3.14):

$$\mathbb{E}_0 [dY_t] = d\mathbb{E}_0 [Y_t].$$

Let us further define $y(t) := \mathbb{E}_0[Y_t]$. By definition, $y(0) = x$. Eq. (3.14) can then be re-written as

$$dy(t) = (r - r^f + \mu^X)y(t) dt, \quad y(0) = x. \quad (3.15)$$

$y(\cdot)$ is then given by the solution of the ODE above. Straightforward calculations lead to

$$y(t) = xe^{(r-r^f+\mu^X)t}.$$

From the equation above, the only value of μ^X such that $\mathbb{E}_0[Y_t] = y(t) = x$ for $t \geq 0$ is given by $\mu^X = r^f - r$. $\square$

An equivalent argument would lead, starting from the foreign-currency measure and going to the pricing-currency one, to set a drift condition for the process $(Z_t, t \geq 0)$ defined as $Z_t = \frac{1}{Z_t^f}$. We can define it along the same lines of what was done with $(Z_t^f, t \geq 0)$, as a geometric Brownian motion with a drift to be determined through arbitrage considerations

$$dZ_t = \mu Z_t dt + \sigma Z_t dW_t, \quad Z_0 = 1/x.$$

The Radon-Nikodym measure in this case would be given by

$$L_t = \frac{Z_t B_t^f}{Z_0 B_t}, \quad L_0 = 1. \quad (3.16)$$

Requiring that $(L_t, t \geq 0)$ has to be a martingale under the pricing-currency measure, would set the drift term as

$$\mu = r - r^f. \quad (3.17)$$

Remark 7 (Symmetry). *Alternatively, one could deduce the dynamics for $(Z_t, t \geq 0)$ in $\mathbb{Q}$ starting from $(Z_t^f, t \geq 0)$, whose dynamics is known in $\mathbb{Q}^f$. Then, it is possible to deduce the dynamics of $(Z_t, t \geq 0)$ in $\mathbb{Q}^f$ by applying Itô's formula to the process given by $Z_t = f(Z_t^f)$ where $f(x) = 1/x$. Once its dynamics is known, the form of the driving martingales under $\mathbb{Q}$ can be worked out using Girsanov Theorem. Under the log-normal dynamics chosen for the FX rates, this latter approach and the one starting from the*

Radon–Nikodym derivative in Eq. (3.16) lead to the same result. A detailed calculation in case the dynamics of the FX rate is subject also to jump–to–default effect, is presented in Section 3.2.5 below.

There are cases, for example Constant Elasticity of Variance or stochastic volatility FX rate models, where the consistency between the arbitrage–free dynamics obtained under the two different specifications is not guaranteed. In these models, if one starts from $(Z_t^f, t \geq 0)$ as a primitive modelling quantity, and then implies the distribution of $(Z_t, t \geq 0)$ at some time t from the law of Z_t^f , what will be obtained can be a different distribution from the one that one would have had by starting from $(Z_t, t \geq 0)$ as a primitive modelling quantity based on the same dynamical properties as $(Z_t^f, t \geq 0)$.

In applications to quanto CDS pricing, where the FX rate is used in Eq. (3.5), and where, depending on the circumstances, we might be interested in pricing or calibrating either under the liquid–currency measure or under the contractual–currency measure, there is a degree of arbitrariness in using one specification or the other. Having consistency between the two specifications is a desirable property to avoid results that depend on the aforementioned choice.

3.2.3 MODELING FRAMEWORK FOR THE QUANTO CDS CORRECTION

In this section we derive model–independent formulas to price contingent claims where contractual currency is different from the liquid currency used to define the pricing measure. In the next sections we will show the application of these formulas under different dynamics assumptions for the main risk factors.

Let us start by calculating the value of a defaultable zero–coupon bond; it will be then used as a building block to calculate CDS values. To do so, we choose a payoff function $\phi_T^f = \mathbb{1}_{\{\tau > T\}}$ in Eq (3.6) and write

$$V_t^f(T) = \mathbb{E}_t^f \left[\frac{B_t^f}{B_T^f} \mathbb{1}_{\{\tau > T\}} \right] = \mathbb{E}_t \left[\frac{B_t^f}{B_T^f} \mathbb{1}_{\{\tau > T\}} \frac{dQ^f}{dQ} \right]. \quad (3.18)$$

Using a Radon–Nikodym derivative derived following the same techniques from Section 3.2.2, the price of the contingent claim in the contractual (foreign) currency economy can be calculated by taking the expectation in the domestic currency economy:

$$V_t^f(T) = \frac{B_t}{Z_t} \mathbb{E}_t \left[\frac{Z_T}{B_T} \mathbb{1}_{\{\tau > T\}} \right].$$

Under the assumption of deterministic interest rates, the above can be rewritten as

$$V_t^f(T) = \frac{D(t, T)}{Z_t} \mathbb{E}_t [Z_T \mathbb{1}_{\{\tau > T\}}], \quad (3.19)$$

where $D(t, T) = B_t/B_T$ is the discount factor from time T to time $t \leq T$.

It might be useful* to define the foreign currency survival probabilities as

$$q_t^f(T) := \frac{V_t^f(T)}{D^f(t, T)}. \quad (3.20)$$

Let us now consider the price, expressed in liquid currency, of the defaultable zero-coupon bond settled in the contractual currency, U . This is given by:

$$U_t(T) = V_t^f(T) Z_t = D(t, T) \mathbb{E}_t [Z_T \mathbb{1}_{\{\tau > T\}}].$$

Being the $\mathbb{Q}$ -price of a tradable asset, the drift of the process $(U_t, t \geq 0)$ has to be given by $r(t)U_t dt$. Therefore, we can write a Feynman–Kac equation to calculate $U_t(T)$. Once $U_t(T)$ is known, $q_t^f(T)$ can be calculated as

$$q_t^f(T) = \frac{U_t(T)}{Z_t D^f(t, T)}.$$

*Mostly for computational reasons because such definition would easily allow CDS pricers defined for single currency calculations to be re-used for quanto CDS pricing without further modifications.

3.2.4 A DIFFUSIVE CORRELATION MODEL: EXPONENTIAL OU / GBM

In this section we present a specific model to calculate U . We will be working with an intensity process and a FX rate process which are defined and calibrated in the liquid measure.

Let us denote by $(\lambda_t, t \geq 0)$ a stochastic process given by $\lambda_t = e^{Y_t}$ where $(Y_t, t \geq 0)$ is an Ornstein–Uhlenbeck process defined as the solution of

$$dY_t = a(b - Y_t) dt + \sigma^Y dW_t^Y, \quad Y_0 = y, \quad (3.21)$$

where the parameters $(a, b, \sigma^Y, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$. Let us also consider a GBM process for the FX rate

$$dZ_t = \mu^Z Z_t dt + \sigma^Z Z_t dW_t^Z \quad Z_0 = z, \quad (3.22)$$

where μ^Z is set by no arbitrage considerations and it is given in this case by Eq. (3.17), and where $(\sigma^Z, z) \in \mathbb{R} \times \mathbb{R}^+$.

The dependence between FX rate and the default's intensity can be specified in this model through the instantaneous correlation between the two driving Brownian motions, $\rho \in [-1, 1]$,

$$d\langle W^Y, W^Z \rangle_t = \rho dt.$$

Let finally $(D_t, t \geq 0)$ be the default process $D_t = \mathbb{1}_{\{\tau < t\}}$.

Remark 8. *Due to the symmetry relation holding for FX rates that are modeled as geometric Brownian motions that was stated in Remark 7, it does not matter if we choose to model $(Z_t, t \geq 0)$, or $(Z_t^f, t \geq 0)$, as the two dynamics are consistent.*

Remark 9. *The choice of the (exponential OU and GBM) dynamics has been mainly driven by the need for the intensity process to stay positive (see Remark 3). However, different intensities dynamics, possibly with local volatilities, can easily be accounted for using the same framework presented below as far as they only driven by Wiener processes and no jump processes are*

involved. Extensive literature has been produced on the use of square root processes for default intensity, mostly due to their tractability in obtaining closed form solutions for Bonds, CDS and CDS options, see for example Brigo and El-Bachir¹⁴ and Brigo and Alfonsi¹², where exact and closed form calibration to CDS curves is also discussed. For the FX rate dynamics, instead, there is no such freedom of choice as the drift is given by no-arbitrage conditions, and introducing local or stochastic volatilities might break the symmetry relation between the FX rate and its reciprocal.

HAZARD RATE'S DYNAMICS IN THE $\mathbb{Q}^F$ MEASURE

We are assuming that the intensity process dynamics is known in $\mathbb{Q}$. Knowing the Radon–Nikodym derivative between measure $\mathbb{Q}$ and measure $\mathbb{Q}^f$ would allow us to write the dynamics of the intensity in $\mathbb{Q}^f$. That can be obtained by using Girsanov's Theorem, from which

$$dW^{Y,f}_t = dW_t^Y - \frac{d\langle W^Y, Z \rangle_t}{Z_t} = dW_t^Y - \rho\sigma^Z dt$$

so that

$$dY_t = a(b - Y_t) dt - \sigma^Y \rho \sigma^Z dt + \sigma^Y dW^{Y,f}_t. \quad (3.23)$$

PRICING EQUATION

In this section we deduce a pricing equation to calculate the value of U . We follow the approach used in Bielecki et al.⁴. Given the strong Markov property of all the processes defined so far, $U_t(T)$ can be expressed as a function of t , Z_t , Y_t and D_t . Let us denote its value at t for $Z_t = z$, $Y_t = y$ and $D_t = d$ by $f(t, z, y, d)$. f is a function depending on both continuous and jump processes, and its Ito differential can be written as (see , for example,

Jeanblanc et al.⁴⁹⁾

$$\begin{aligned} df_t = & \partial_t f dt + \partial_z f \left(\mu^Z z dt + \sigma^Z z dW_t^Z \right) + \partial_y f \left(a(b - Y_t) dt + \sigma^Y dW_t^Y \right) \\ & + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f dt + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f dt + \rho \sigma^Z \sigma^Y z \partial_{zy} f dt + \Delta f dD_t, \end{aligned} \quad (3.24)$$

where, with some abuse of notation, we have defined the jump-to-default term as

$$\Delta f := f(t, Z_{t-}, Y_t, D_{t-} + \Delta D_t) - f(t, Z_{t-}, Y_t, D_{t-}). \quad (3.25)$$

DEFINITION OF Δf . It must be noted that $(D_t, t \geq 0)$ starts at 0 and jumps to 1 at single time, τ , upon default. This means, in particular, that $D_{t-} + \Delta D_t$ takes a value different from zero only upon default, and that, for all the times previous to that, the following equation holds:

$$D_t = 0, \quad t < \tau. \quad (3.26)$$

The first term in Eq.(3.25) can then be rewritten as

$$f(t, Z_{t-}, Y_t, D_{t-} + \Delta D_t) = f(t, Z_{t-}, Y_t, \Delta D_t)$$

and the equation for Δf can be written as

$$\Delta f = f(t, Z_{t-}, Y_t, 1) - f(t, Z_{t-}, Y_t, 0).$$

COMPENSATED MARTINGALE FOR $(D_t, t \geq 0)$. A compensator for $(D_t, t \geq 0)$ in the measure $\mathbb{Q}$ is defined as the process $(A_t, t \geq 0)$ such that $D_t - A_t$ is a $\mathbb{Q}$ -martingale with respect to $(\mathcal{G}_t, t \geq 0)$. The compensator for $(D_t, t \geq 0)$ is given by (see Theorem 7)

$$dA_t = \mathbb{1}_{\{\tau > t\}} \lambda_t dt, \quad A_0 = 0. \quad (3.27)$$

We define the resulting martingale as $(M_t, t \geq 0)$, where

$$M_t = D_t - A_t. \quad (3.28)$$

Consequently, the compensator of the term $\Delta f \Delta D_t$ in Eq.(3.24) can be written as

$$(1 - D_t)e^{Y_t}\Delta f \, dt,$$

which, conditional on $\mathcal{G}_t$, $D_t = d$, $Z_{t-} = z$, and $Y_t = y$, is equal to

$$(1 - d)e^y (f(t, z, y, 1) - f(t, z, y, 0)) \, dt.$$

It is possible to write a Feynman–Kac type PDE to compute the value of $U_t(T)$. Indeed $(U_t, t \geq 0)$ is a $\mathbb{Q}$ –price and, as such, it must locally grow at the rate r . Therefore, its drift must satisfy the following equation

$$\begin{aligned} \partial_t f + \mu^Z z \partial_z f + a(b - Y_t) \partial_y f + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f \\ + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f + \rho \sigma^Z \sigma^Y z \partial_{zy} f + e^y (1 - d) \Delta f = r f \, dt, \end{aligned}$$

where the explicit dependence of f on the state variables (x, y, t, d) has been omitted for clarity of reading. If it wasn't for the last term, this would be the typical PDE for default-free payoffs. Incidentally, this jump-to-default term is also the only term of the equation where the values $f(t, z, y, 0)$ and $f(t, z, y, 1)$ appear together. In fact, by conditioning first on $d = 1$ and then on $d = 0$ we can decouple the two functions

$$\begin{aligned} u(t, z, y) &:= f(t, z, y, 1) \\ v(t, z, y) &:= f(t, z, y, 0) \end{aligned}$$

and calculate them by solving iteratively two separate PDE problems. We first solve for u , as for $d = 1$ the last term does not appear in the equation, and, once u has been calculated, we use it to solve for v . Final conditions for the two functions are respectively given by

$$\begin{aligned} v(T, z, y) &= f(T, z, y, 0) = z; \\ u(T, z, y) &= f(T, z, y, 1) = 0. \end{aligned}$$

The PDE problem that must be solved to obtain u is then given by

$$\begin{aligned} \partial_t u &= ru - \mu^Z z \partial_z u - a(b - y) \partial_y u - \frac{1}{2} \left(\sigma^Z x \right)^2 \partial_{zz} u \\ &\quad - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} u - \rho \sigma^Z \sigma^Y z \partial_{zy} u \end{aligned} \quad (3.29)$$

$$u(T, z, y) = 0. \quad (3.30)$$

The solution to this problem is $u \equiv 0$, therefore in this case one can solve directly the PDE for v , which is then given by

$$\begin{aligned} \partial_t v &= rv - \mu^Z z \partial_z v - a(b - y) \partial_y v - \frac{1}{2} \left(\sigma^Z x \right)^2 \partial_{zz} v \\ &\quad - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} v - \rho \sigma^Z \sigma^Y z \partial_{zy} v + e^y v \end{aligned} \quad (3.31a)$$

$$v(T, z, y) = z. \quad (3.31b)$$

Remark 10 (Interpretation of u and v). *The functions u and v account for the post-default value and pre-default value, respectively, of a derivative with payoff $\phi(x, y, d)$. The price of this derivative can be written as*

$$V_t = \mathbb{1}_{\{\tau > t-\}} \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = d],$$

where, due to the strong Markov property of the processes $(X_t, t \geq 0)$, $(Y_t, t \geq 0)$, and $(D_t, t \geq 0)$, the expected value on the right-hand side can be written as

$$f(t, x, y, d) = \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = d].$$

This can be decomposed as $f(t, x, y, d) = \mathbb{1}_{\{d=1\}} u(t, x, y) + \mathbb{1}_{\{d=0\}} v(t, x, y)$ where

$$\begin{aligned} v(t, x, y) &:= \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 0], \\ u(t, x, y) &:= \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 1], \end{aligned}$$

in fact

$$\begin{aligned}
f(t, x, y, d) &= \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = d] \\
&= \mathbb{1}_{\{\tau > t\}} \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 0] \\
&\quad + \mathbb{1}_{\{\tau \leq t\}} \mathbb{E}_t [\phi(X_T, Y_T, D_T) | X_t = x, Y_t = y, D_t = 1] \\
&= \mathbb{1}_{\{\tau > t\}} v(t, x, y) + \mathbb{1}_{\{\tau \leq t\}} u(t, x, y)
\end{aligned}$$

as both $\mathbb{1}_{\{\tau > t\}}$ and $\mathbb{1}_{\{\tau \leq t\}}$ are measurable in the $\mathcal{G}_t$ filtration. The derivative price can then be written as

$$V_t = \mathbb{1}_{\{\tau > t\}} v(t, X_t, Y_t) + \Delta D_t u(t, X_t, Y_t),$$

where we defined

$$\Delta D_t := \mathbb{1}_{\{\tau > t\}} - \mathbb{1}_{\{\tau > t-\}}.$$

3.2.5 A JUMP-TO-DEFAULT FRAMEWORK

The exponential OU-based model described in Section 3.2.4 can be extended by incorporating a devaluation mechanism in the FX rate dynamics. By linking the devaluation to the default event, it is possible to introduce a further source of dependence between $(\lambda_t, t \geq 0)$ and $(Z_t, t \geq 0)$. In Section 3.4 it will be shown that this will prove to be a more suitable mechanism to model the basis spread for quanto-CDS.

This section is organised as follows: in the first subsections, from Section 3.2.5 to Section 3.2.5, we will discuss in general how the dynamics of the risk factors are affected by the introduction of a jump-to-default effect on the FX component. Given that the Radon-Nikodym derivative depends on the FX rate, this change is expected to have an impact on all the risk factors whose dynamics has to be written in a measure different from the one in which they have been originally calibrated and, potentially, on the FX symmetry discussed in Remark 7. This is proven to hold true also in this new, more general, framework (see Proposition 14). In Section 3.2.5 we will apply the general results from the first subsections to the pricing of quanto CDS.

RISK FACTORS DYNAMICS

Let us then consider a jump–diffusion process for the FX rate in place of Eq. (3.22), while we will be keeping the same model choice for the intensity $\lambda_t = e^{Y_t}$:

$$dY_t = a(b - Y_t) dt + \sigma^Y dW_t^Y, \quad Y_0 = y, \quad (3.32)$$

$$dZ_t = \bar{\mu}Z_t dt + \sigma^Z Z_t dW_t^Z + \gamma^Z Z_{t-} dD_t, \quad Z_0 = z, \quad (3.33)$$

$$d\langle W^Y, W^Z \rangle_t = \rho dt \quad (3.34)$$

where, as before, the parameters $(a, b, \sigma^Y, y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, $(\sigma^Z, z) \in \mathbb{R} \times \mathbb{R}^+$, $\rho \in [-1, 1]$, and where $\gamma^Z \in [-1, \infty)$ is the devaluation/revaluation rate of the FX process. From a mathematical point of view, γ^Z could lie in $[-1, \infty)$, but in this model we restrict it to be non-positive for financial reasons. The typical case in which this devaluation factor is used is for reference entities whose default can negatively impact the value of their local currency. As an example, we expect the value of EUR expressed in USD to fall in case of Italy's default.

We leave unspecified the drift term of $(Z_t, t \geq 0)$ and we simply use $\bar{\mu}$ for it in order to distinguish it from μ^Z . It will be shown in Section 3.2.5 that the introduction of the jump term will lead to a result different from Eq. (3.17) if we want the process defined in Eq. (3.16) to still be a martingale.

Remark 11 (Jumps). *The jump term in SDE for jump–diffusion processes can be described equivalently using $(D_t, t \geq 0)$ or the compensated process $(M_t, t \geq 0)$, the effect of using one term or the other being just a change in the drift term. We prefer using the non–compensated term when introducing the FX process in order to highlight the jump structure and hence the additional source of dependence between the FX and the credit component. On the other hand, the description in terms of the compensated martingale $(M_t, t \geq 0)$ will arise naturally every time the Fundamental Theorem of Asset Pricing will be used to derive no arbitrage drift conditions, e.g. when Eq. (3.10) is used to deduce Eq. (3.42) below and, as it will be shown in Section 3.2.5, to deduce the main pricing equation.*

HAZARD RATE'S AND FX RATE'S DYNAMICS IN $\mathbb{Q}^F$

Given the dependence of $(L_t, t \geq 0)$ on $(D_t, t \geq 0)$ via $(Z_t, t \geq 0)$, in this case the change of measure modifies not only the expected value of $(W_t, t \geq 0)$, but also the expected value of $(M_t, t \geq 0)$ which was originally given by $dM_t = dD_t - (1 - D_t)\lambda_t dt$ in $\mathbb{Q}$. However, Girsanov's Theorem provides the adjustments for each of these processes needed to obtain a martingale in the new measure.

$$dW_t^f = dW_t - \frac{d\langle W, Z \rangle_t}{Z_t} = dW_t - \sigma^Z dt, \quad (3.35a)$$

$$dM_t^f = dM_t - (1 - D_t)\gamma^Z \lambda_t dt. \quad (3.35b)$$

The Wiener process decomposition in $\mathbb{Q}^f$ is given by the same formula used in Section 3.2.4, while we derive the martingale decomposition for $(D_t, t \geq 0)$ as a result of the following Proposition.

Proposition 13. *Let $(M_t, t \geq 0)$ be the martingale associated to the default process $(D_t, t \geq 0)$ in the domestic currency measure*

$$dM_t = dD_t - (1 - D_t)\lambda_t dt,$$

then an application of the Girsanov Theorem allows to write the correspondent martingale in the foreign measure $(M_t^f, t \geq 0)$ as

$$\begin{aligned} dM_t^f &= dM_t - \frac{d\langle M, L \rangle_t}{L_t} = dM_t - d\langle D, \gamma^Z D \rangle_t \\ &= dM_t - (1 - D_t)\gamma^Z \lambda_t dt \end{aligned} \quad (3.36)$$

$$= dD_t - (1 - D_t)(1 + \gamma^Z)\lambda_t dt \quad (3.37)$$

where the dynamics of $(L_t, t \geq 0)$ is defined by Eq. (3.16) and Eq. (3.33). Eq. (3.37) states that the intensity of the Poisson process driving the default event in the foreign currency is given by

$$\lambda_t^f := (1 + \gamma^Z)\lambda_t. \quad (3.38)$$

Proof. Integration by parts gives

$$\begin{aligned}
d(M_t^f L_t) &= L_t dM_t^f + M_t^f dL_t + d[M^f, L]_t \\
&= L_t dM_t^f + M_t^f dL_t + \gamma^Z L_t dD_t \\
&= L_t(dM_t - (1 - D_t)\gamma^Z \lambda_t dt) + M_t^f d\hat{L}_t + \gamma^Z L_t dD_t \\
&= L_t dM_t + M_t^f dL_t + \gamma^Z L_t dM_t
\end{aligned}$$

so the process $((LM^f)_t, t \geq 0)$ is a martingale in the domestic measure as it can be written as a sum of stochastic integrals on local martingales. As a consequence, the process $(M_t^f, t \geq 0)$ is a local martingale in the foreign measure. $\square$

Remark 12 (CDS par-spreads approximation). *In all the cases where the well known approximation*

$$\lambda \approx \frac{S}{1 - R} \quad (3.39)$$

between hazard rates, CDS par-spreads, S , and recovery rates, R , holds, the relation in Eq. (3.38) can be written in terms of CDS par-spreads rather than hazard rates as

$$S^f = (1 + \gamma^Z)S. \quad (3.40)$$

This happens, for example, when the hazard rate is constant in time and when the premium leg's cash-flows can be approximated by a stream of continuously compounded payments (see Corollary 9).

FX RATES DYNAMICS IN THE TWO MEASURES AND SYMMETRY

The FX rate in this model is a jump-diffusion process, whose jumps are given by (see Eq. (3.33))

$$\Delta Z_t = \gamma^Z Z_{t-} \Delta D_t. \quad (3.41)$$

Notice that also this specification of the FX rate is subject to arbitrage constraints such that the Radon-Nikodym derivative defined by Eq. (3.16) be a martingale. The condition equivalent to Eq. (3.17) in the case where

the FX dynamics is given by Eq. (3.33) is provided by

$$\bar{\mu} = \mu^Z - \lambda_t \gamma^Z \mathbb{1}_{\{\tau > t\}} = r - r^f - \lambda_t \gamma^Z \mathbb{1}_{\{\tau > t\}}. \quad (3.42)$$

It is considered to only take values in $[-1, 0]$, although it could in principle lie in $[-1, \infty)$, due to economic reasons, the expectation being that Z would drop in case of default of the reference entity.

Despite the introduction of the jump in the FX rate dynamics, the consistency highlighted in Remark 7 between $(Z_t^f, t \geq 0)$ and $(Z_t, t \geq 0)$ is maintained. From a practical point of view this means that we do not need to worry about which FX rate we use, as one can be obtained as a transformation of the first one and it is guaranteed to satisfy the no-arbitrage relations for the associated Radon–Nikodym derivative. This is proved in the next

Proposition 14 (FX rates symmetry under devaluation jump to default). *Let us consider an FX rate process whose dynamics in the domestic measure $\mathbb{Q}$ is specified by Eq. (3.33) and whose drift is given by Eq. (3.42). Then the dynamics of the process $(Z_t^f, t \geq 0)$ where $Z_t^f = 1/Z_t$ in the foreign measure $\mathbb{Q}^f$ is given by*

$$dZ_t^f = (r^f - r)Z_t^f dt - \sigma^Z Z_t^f dW_t^{Z,f} + Z_{t-}^f \gamma^f dM_t^f, \quad Z_0^f = \frac{1}{z}, \quad (3.43)$$

where the devaluation rate for $(Z_t^f, t \geq 0)$ is given by

$$\gamma^f = -\frac{\gamma^Z}{1 + \gamma^Z}. \quad (3.44)$$

In particular, (3.43) is such that the Radon–Nikodym derivative defined by Eq. (3.7) is a $\mathbb{Q}^f$ -martingale.

Proof. The relation between Z and Z^f is given by $Z_t^f = \phi(Z_t)$ where $\phi(x) = 1/x$. By applying Itô's rule for jump–diffusion processes (see, for example,

Jeanblanc et al.⁴⁹, Protter⁶³), we obtain:

$$\begin{aligned}
dZ_t^f &= d\phi(Z_t) = \phi'(Z_{t-}) dZ_t + \frac{1}{2}\phi''(Z_{t-}) d[Z]_t^c \\
&\quad + \sum_{s \leq t} \left((\phi(Z_{s-} + \Delta Z_{s-}) - \phi(Z_{s-})) - \phi(Z_{s-})\Delta Z_{s-} \right) \\
&= d\left(\frac{1}{Z_t}\right) = -\frac{dZ_t}{Z_t^2} + \frac{d[Z]_t^c}{Z_t^3} + \left(\frac{1}{Z_{t-} + \gamma^Z Z_{t-}} - \frac{1}{Z_{t-}}\right) dD_t + \frac{\Delta Z}{Z_t^2} \\
&= -\bar{\mu} \frac{1}{Z_t} dt - \sigma^Z \frac{1}{Z_t} dW_t^Z - \gamma^Z \frac{1}{Z_{t-}} dD_t + (\sigma^Z)^2 \frac{1}{Z_t} dt \\
&\quad + \frac{1}{Z_{t-}} \left(\frac{1}{1 + \gamma^Z} - 1 \right) dD_t + \frac{\gamma^Z}{Z_{t-}} dD_t \\
&= -\bar{\mu} \frac{1}{Z_t} dt - \sigma^Z \frac{1}{Z_t} dW_t^Z + (\sigma^Z)^2 \frac{1}{Z_t} dt + \frac{1}{Z_{t-}} \gamma^X dD_t \\
&= -\bar{\mu} Z_t^f dt - \sigma^Z Z_t^f dW_t^Z + (\sigma^Z)^2 Z_t^f dt + Z_{t-}^f \gamma^f dD_t
\end{aligned}$$

where we used γ^X to denote the jumps of $(Z_t^f, t \geq 0)$, given by

$$\gamma^f = -\frac{\gamma^Z}{1 + \gamma^Z}. \quad (3.45)$$

We can now use Girsanov's Theorem in the form of Eq. (3.35a) for $(W_t^Z, t \geq 0)$ and Eq. (3.37) for $(D_t, t \geq 0)$ to decompose $(Z_t^f, t \geq 0)$ in a sum of local martingales in the new measure $\mathbb{Q}^f$. As a result

$$\begin{aligned}
dZ_t^f &= -\bar{\mu} Z_t^f dt - \sigma^Z Z_t^f (dW_t^{Z,f} - \sigma^Z dt) + (\sigma^Z)^2 Z_t^f dt + Z_{t-}^f \gamma^f (dM_t^f \\
&\quad + (1 + \gamma^Z)(1 - D_t)\lambda_t dt) \\
&= -(\bar{\mu} - \gamma^f(1 + \gamma^Z)(1 - D_t)\lambda_t) Z_t^f dt - \sigma^Z Z_t^f dW_t^{Z,f} + Z_{t-}^f \gamma^f dM_t^f
\end{aligned}$$

Reminding that $\bar{\mu}$ is given by (see Eq. (3.12) and (3.42)) $r - r^f - (1 - D_t)\gamma^Z \lambda_t$, the $\mathbb{Q}^f$ -dynamics of $(Z_t^f, t \geq 0)$ can be written as

$$\begin{aligned}
dZ_t^f &= (r^f - r + \gamma^Z(1 - D)\lambda_t + \gamma^f(1 + \gamma^Z)(1 - D_t)\lambda_t) Z_t^f dt \\
&\quad - \sigma^Z Z_t^f dW_t^{Z,f} + Z_{t-}^f \gamma^f dM_t^f \\
&= (r^f - r) Z_t^f dt - \sigma^Z Z_t^f dW_t^{Z,f} + Z_{t-}^f \gamma^f dM_t^f. \quad (3.46)
\end{aligned}$$

□

Alternatively, a representation where the jumps are highlighted can be used for the $\mathbb{Q}^f$ -dynamics of $(Z_t^f, t \geq 0)$

$$dZ_t^f = \left(r^f - r - (1 - D_t)\gamma^f\lambda_t \right) Z_t^f dt - \sigma^Z Z_t^f dW_t^{Z,f} + Z_{t-}^f \gamma^f dD_t, \quad Z_0^f = \frac{1}{\tilde{z}}. \quad (3.47)$$

PRICING EQUATION

In this section, we consider the case where liquid currency and pricing currency coincide and are different from the contractual currency. As discussed in Section 3.1.2, this is the typical setup arising to price in the USD-market measure CDSs written on European Monetary Union countries, as the standard currency for them is USD. If one wants to price a EUR denominated contract for such reference entities in the USD measure, one has first to calibrate the default's intensity to USD-denominated contracts and then the pricing can be carried out using the equations derived in this section. This is also the procedure followed to produce the results showed in Section 3.4.4 below.

Without loss of generality, we will study the case of liquid currency and pricing currency associated to the domestic measure $\mathbb{Q}$.

$$dY_t = a(b - Y_t) dt + \sigma^Y dW_t^Y, \quad (3.48)$$

$$dZ_t = \bar{\mu}_Z Z_t dt + \sigma^Z Z_t dW_t^Z + \gamma^Z Z_t dD_t \quad (3.49)$$

$$d\langle W^Y, W^Z \rangle_t = \rho dt \quad (3.50)$$

with

$$dM_t = dD_t - (1 - D_t)\lambda_t dt$$

so that the no-arbitrage drift is given by (see Eq. (3.42))

$$\bar{\mu}_Z = r - r^f - \gamma^Z(1 - D_t)\lambda_t.$$

An application of the generalized Itô formula (see, for example, Jeanblanc et al.⁴⁹) allows us to write the $\mathbb{Q}$ -dynamics of $(U_t, t \geq 0)$. Using $U_t = f(t, Z_t, Y_t, D_t)$:

$$\begin{aligned} df = & \partial_t f dt + \partial_z f \left(\bar{\mu}_Z Z_{t-} dt + \sigma^Z Z_t dW_{t-}^Z + \gamma^Z Z_{t-} dD_t \right) \\ & + \partial_y f \left(a(b - Y_t) dt + \sigma^Y dW_t^Y \right) + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f dt + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f dt \\ & + \rho \sigma^Z \sigma^Y z \partial_{zy} f dt + \Delta f dD_t - \partial_z f \Delta Z_t. \end{aligned}$$

The pricing equation could be deduced by the f dynamics in the same way discussed in Section 3.2.4. In this case the jump-to-default term is defined as:

$$\Delta f := f(t, Z_{t-} + \Delta Z_t, Y_t, D_{t-} + \Delta D_t) - f(t, Z_{t-}, Y_t, D_{t-}). \quad (3.51)$$

DEFINITION OF Δf . Differently from the pure diffusive case from Section 3.2.4, Δf depends in this case on both the jumps of $(Z_t, t \geq 0)$ and $(D_t, t \geq 0)$. The two jump components, however, are driven by a common jump driver $((D_t, t \geq 0)$ itself, see Eq. (3.33)), and the jumps in the FX rate dynamics are given by Eq. (3.41).

Consequently, the equation for Δf can be written as

$$\Delta f = f(t, (1 + \gamma^Z) Z_{t-}, Y_t, 1) - f(t, Z_{t-}, Y_t, 0),$$

and the final conditions are given by:

$$\begin{aligned} v(T, z, y) &= f(T, z, y, 0) = z; \\ u(T, z, y) &= f(T, (1 + \gamma^Z) z, y, 1) = 0. \end{aligned}$$

As for the pure diffusive case, $u(t, x, y) \equiv 0$, so that there is only one PDE

system for v to solve:

$$\begin{aligned} \partial_t v = & rv - (r - r^f)z\partial_z v - a(b - y)\partial_y v - \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} v \\ & - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} v - \rho \sigma^Z \sigma^Y z \partial_{zy} v + e^y (v - \gamma^Z z \partial_z v) \end{aligned} \quad (3.52a)$$

$$v(T, z, y) = z. \quad (3.52b)$$

INFERRING DEFAULT PROBABILITY DEVALUATION FACTOR FROM THE FX RATE DEVALUATION FACTOR

It is possible to link the FX rate devaluation factor introduced in Eq. (3.33) with a probability rescaling factor. This is done in the following Proposition.

Proposition 15 (Default probabilities devaluation). *Under the following hypotheses:*

i) small tenors:

$$T \rightarrow 0,$$

ii) deterministic hazard rate, $\lambda(\cdot)$,

the ratio of the quanto-corrected and single-currency default probabilities can be approximated through

$$\frac{1 - q_0^f(T)}{1 - q_0(T)} \approx 1 + \gamma^Z. \quad (3.53)$$

Proof. Using the definition of quanto survival probability (see Eq. (3.20))

$$\begin{aligned}
1 - q_0^f(T) &= 1 - \frac{\mathbb{E}_0 [Z_T \mathbb{1}_{\{\tau > T\}}]}{Z_0} \frac{B(T)}{B^f(T)} \\
&= \frac{Z_0 - \mathbb{E}_0 [Z_T \mathbb{1}_{\{\tau > T\}}]}{Z_0} \frac{B(T)/B^f(T)}{1} \\
&= \frac{\mathbb{E}_0 [Z_0 \mathbb{1}_{\{\tau \leq T\}} + Z_0 \mathbb{1}_{\{\tau > T\}} - B(T)/B^f(T) Z_T \mathbb{1}_{\{\tau > T\}}]}{Z_0} \\
&= \mathbb{E}_0 [\mathbb{1}_{\{\tau \leq T\}}] + \frac{\mathbb{E}_0 [\mathbb{1}_{\{\tau > T\}} (Z_0 - B(T)/B^f(T) Z_T)]}{Z_0} \quad (3.54)
\end{aligned}$$

The second term can be evaluated through an application of Bayes' formula:

$$\begin{aligned}
&\frac{\mathbb{E}_0 [\mathbb{1}_{\{\tau > T\}} (Z_0 - B(T)/B^f(T) Z_T)]}{Z_0} \\
&= \frac{\mathbb{E}_0 \left[\left(Z_0 - B(T)/B^f(T) Z_T \right) \middle| \mathbb{1}_{\{\tau > T\}} \right]}{Z_0} \mathbb{E}_0 [\mathbb{1}_{\{\tau > T\}}] \\
&= \left(1 - \frac{B(T) \mathbb{E}_0 [Z_T | \mathbb{1}_{\{\tau > T\}}]}{B^f(T) Z_0} \right) \mathbb{E}_0 [\mathbb{1}_{\{\tau > T\}}].
\end{aligned}$$

Let us now consider the term $\mathbb{E}_t [Z_T | \mathbb{1}_{\{\tau > T\}}]$. Under the assumption of deterministic hazard rate, this can be evaluated as

$$\mathbb{E}_t [Z_T | \mathbb{1}_{\{\tau > T\}}] = Z_0 e^{(r-r^f)(T-t) - \gamma^Z \int_t^T \lambda(s) ds}$$

so that

$$\frac{B(T) \mathbb{E}_0 [Z_T | \mathbb{1}_{\{\tau > T\}}]}{B^f(T) Z_0} = e^{-\gamma^Z \int_0^T \lambda(s) ds}.$$

Eq.(3.54) can then be re-written as

$$\frac{1 - q_0^f(T)}{1 - q_0(T)} = e^{-\gamma^Z \int_0^T \lambda(s) ds}.$$

When the integral $\int_t^T \lambda(s) ds$ is small enough, the above can be approximated as γ^Z . This happens, in particular, in the limit of small maturities, from

which the thesis follows. □

Remark 13. *Although the above approximation is only proven in the case of deterministic hazard-rate, results from the later sections suggest that, for small tenors and up to a certain level of credit risk, it can be applied to the case of stochastic processes as well.*

We tested the behaviour of this approximation with respect to different levels of credit risk in Section 3.4.2 on results generated using the model (3.32)–(3.34).

3.3 FINITE DIFFERENCE SCHEMES

Given the lack of closed form solutions for the PDE-system (3.52), we resorted to numerical techniques — to *finite difference methods* (FDM), specifically — to calculate quanto-adjusted survival probabilities.

Results calculated in this way have been used both for direct calculations — to estimate the impact of diffusive correlation versus devaluation jumps as in Section 3.4.1 — and for the calibration problems described later in the chapter in Section 3.4.4. For single-currency calculations, we used Crank–Nicolson method to discretise the pricing equations, whereas for multi-currency problems, we used an ADI scheme, described in the next section.

For a general introduction to finite difference methods, and in particular for details on how to prove the consistency, stability, and convergence of Crank–Nicolson scheme, we refer to Morton and Mayers⁵⁹ and Natalini⁶⁰. Finally, for specific introduction to applications of FDM to finance, we refer the interested reader to Duffy³².

3.3.1 DOUGLAS ADI METHODS

The scheme that we implemented to solve the PDE system (3.52) is a finite-difference method belonging to the family of *alternating-direction implicit* (ADI) schemes.

These are a family of schemes used to solve multi-dimensional problems. They do so by splitting the problem in a series of simpler, uni-dimensional, problems. For an introduction to time-splitting methods in ODE/PDE equations can be found in Hundsdorfer⁴⁶.

In general, they can be applied to solve initial value problems of type

$$\begin{aligned}\frac{d}{dt}u(t, x) &= L[t, u(t, x)] \\ u(0, x) &= u_0(x)\end{aligned}$$

where L is a differential operator.

It is often possible to decompose the operator L into a number of simpler components, n^c ;

$$L[t, u] = L_0[t, u] + L_1[t, u] + \dots, L_{n^c}[t, u].$$

The idea behind Douglas ADI method is that the simpler components are chosen so to be “mono-dimensional”. In the Stabilising-Corrections scheme for a two-dimensional spatial problem⁴⁷, the choice is to have

$$\begin{aligned}L_0 &:= \partial_{xy} \\ L_1 &:= \partial_x + \partial_{xx} \\ L_2 &:= \partial_y + \partial_{yy}\end{aligned}$$

ALGORITHM Let us consider the calculation at time t_0 of a quanto-adjusted survival probability for a maturity T . The discretization in time is given through a time-grid $t_0, \dots, t_{N^T} = T$, where $t_0 < t_1 < \dots < t_{N^T}$.

The algorithm to follow to produce a numerical approximation of $u(t_0, \cdot, \cdot)$ is as follows:

1. First, let us consider

$$u_{N^T} = u(T, x, y),$$

where u_{N^T} is, in the two-dimensional case we are considering, a two-

dimensional grid.

2. Then, set $n = N^T$ and calculate the time split step

$$\begin{aligned} v_0 &= u_n + L[t_n, u_n] \Delta t \\ v_j &= v_{j-1} + \theta \Delta t_n (L_j[t_n, v_j] - L_j[t_n, u_n]), \quad j = 1, 2, \\ u_{n-1} &= v_2 \end{aligned}$$

3. If $n \neq 1$, decrease n by one and repeat step 2.

At the end of this time-loop, the grid u_0 contains the numerical approximation of $(x, y) \mapsto u(t_0, x, y)$.

BOUNDARY CONDITIONS The PDE system (3.52) consists of a pricing PDE and of a terminal condition. In order to apply the chosen scheme to such PDE systems, we also have to specify boundary conditions. In contrast from the works quoted above, where Neumann or Dirichlet conditions are used, we chose to set the second derivative of the solution to zero on the boundaries.

CONVERGENCE In the case of $\theta = \frac{1}{2}$ and $L_0 = 0$, the order of convergence of this scheme in time is 2; it is 1 otherwise. We implemented the scheme with the choice $\theta = \frac{1}{2}$.

Comparisons between a numerical approximation obtained through FDM and one obtained using Monte Carlo are shown, in the more general case of *Credit Valuation Adjustment*, in Section 4.2.1.

3.4 RESULTS

3.4.1 QUANTO CDS PAR-SPREADS PARAMETERS DEPENDENCE

In this section we show how the quanto-corrected CDS par-spreads are affected by changing the value of some of the parameters.

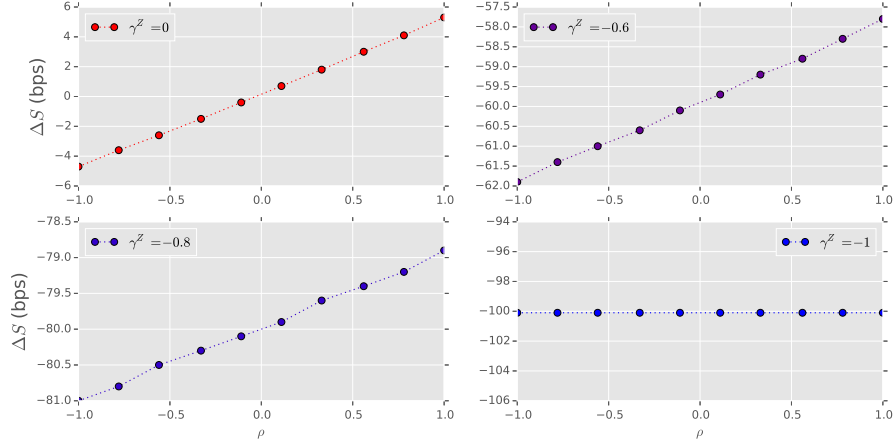


Figure 3.3: 5Y CDS par-spread impact vs ρ and γ . The reference value for the par-spread is calculated using the parameters' values in Table 3.2.

Table 3.2: Parameters used to produce the par-spreads impact in Figure 3.3

z	μ	σ^Z	a	b	y	σ^Y	T
0.8	0.0	0.1	0.0001	-210.0	-4.089	0.2	5.0

The parameter which affected the most the value of the spreads in this analysis is, as one expects, the devaluation rate, γ^Z (see Figure 3.3). For the chosen value of the parameters, a change in the instantaneous correlation from its extreme values, -1 and 1 , can usually move the par spread of less than 10bps, while moving the devaluation rate to its extreme value, -1 , can bring to zero the level of the par spread. In such a degenerate case, there would be no sensitivity to instantaneous correlation.

Figure 3.4 shows that par-spreads' sensitivity to the volatility of the FX rate process is slightly weaker than the one to the log-intensity's volatility for the chosen ranges of parameters' values. In our example, a 5Y par-spread can change of around 10 bps with σ^Z ranging from 1% to 20%, while it can range up to 30 bps with σ^Y going from 20% to 70% and with σ^Z fixed at 20%.

In Figure 3.5 we showed the sensitivity of par-spreads to the value of diffusive correlation ρ for different values of the intensity volatility, σ^Y . Let us indeed

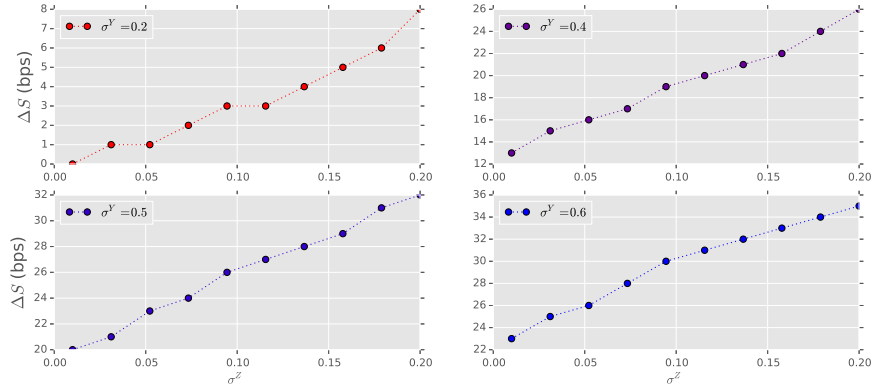


Figure 3.4: 5Y CDS par-spread impact vs σ^Z and σ^Y . The reference value is produced using the parameters' values in Table 3.3.

Table 3.3: Parameters used to produce the results shown in Figure 3.4.

z	μ	ρ	a	b	y	T
0.8	0.0	0.5	0.0001	-210.0	-4.089	5.0

consider the model-implied quanto basis as a function of σ^Y and ρ , $S(\rho, \sigma^Y)$ and let us define the maximum variation that correlation can achieve in this, that is $S(1, \sigma^Y) - S(-1, \sigma^Y)$. The dependence of par spreads on the correlation is extremely weak for values of σ^Y in the range of 20%. Around this level of log-intensity volatility, the maximum change that correlation can produce on the quanto par-spreads is 10 bps. The maximum variation achievable through correlation moves is summarised in Figure 3.6. As shown in the chart, a more realistic value of σ^Y of 60% is required to observe an impact of around 30 bps on the 5Y par-spread. This highlights the limits of a purely diffusive correlation model in explaining large differences between domestic and quanto-corrected CDS par-spreads.

Table 3.4: Parameters used to produce the results shown in Figure 3.5.

z	μ	σ^Z	a	b	y	T
0.8	0.0	0.1	0.0001	-210.0	-4.089	5.0

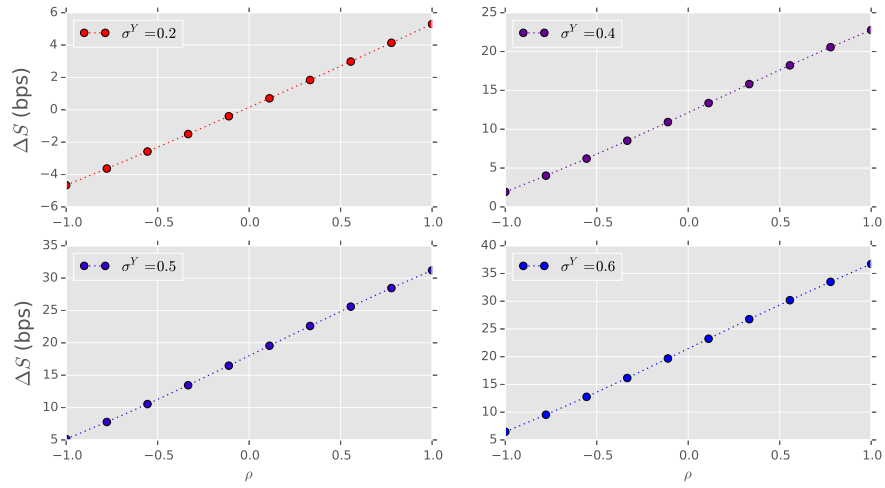


Figure 3.5: 5 years par-spread impact vs ρ and σ^Y . The reference value is produced using the parameters' values in Table 3.4.

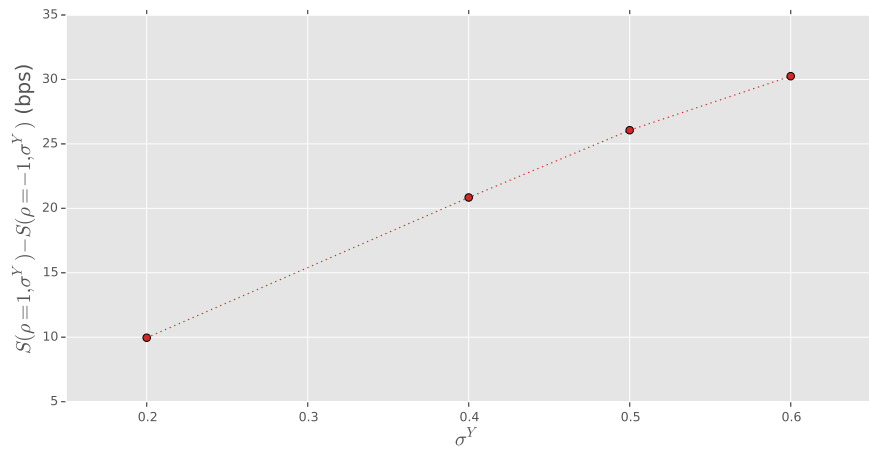


Figure 3.6: Maximum quanto CDS basis achieved by correlation moves, $S(1, \sigma^Y) - S(-1, \sigma^Y)$, for different values of the intensity volatility, σ^Y .

There are circumstances where the basis between par-spreads of CDSs in different currencies can be sensibly higher than these values. In those cases, a purely diffusive model for the default's intensity is not sufficient to explain the observed basis and an approach where dependence is induced by devaluation jumps is required. As an example of an historical occurrence of such a wide basis, we show in Section 3.4.4 results of model calibrations to the time series of par-spreads for EUR-denominated and USD-denominated 5Y CDSs on the Italian Republic.

In the different context of impact of dependence on CDS credit valuation adjustments, even under collateralization, Brigo et al Brigo and Chourdakis¹³, Brigo et al.^{20, 19} show that a copula function on the jump to default exponential thresholds may be necessary to obtain sizable effects when looking at credit-credit dependence, pure diffusive correlation not being enough.

3.4.2 TEST ON THE IMPACT OF TENOR AND CREDIT WORTHINESS ON THE QUANTO CORRECTION

To test the relation given in Eq. (3.53), we set the diffusive correlation to zero and we chose the following set of log-intensity parameters:

$$a = 1.00e - 004, \quad b = -210.45 \quad \sigma = 0.2,$$

whereas we have produced low spread scenarios and high spread scenarios by choosing two different values for Y_0 , the first one, $y^L = -4.089$, such that the resulting CDS par spread term structure is flat at around 100 bps, and the second one, $y^H = -2.089$ such to produce a flat CDS par spread term structure with a value of around 740 bps.

Figures 3.7 and 3.8 show, in line with the nature of the approximation (3.53), that the approximation is less accurate for higher maturities, as evident in both charts by comparing blue lines (short maturities) with red lines (long maturities). It is also less accurate and for higher values of CDS spreads, as highlighted by the comparison between Figure 3.8 (high spreads) and Figure 3.7 (low spreads).

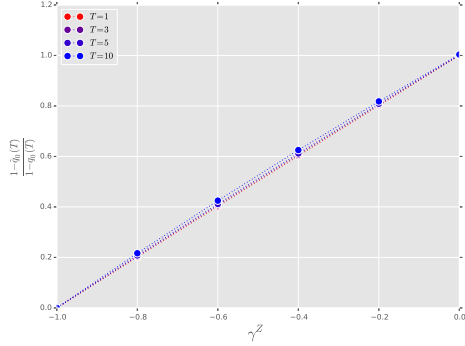


Figure 3.7: Comparison of $1 - q_0^f(T)/1 - q_0(T)$ for different maturities in a low spreads scenario, $Y_0 = y^L$ (CDS spreads at 100 bps).

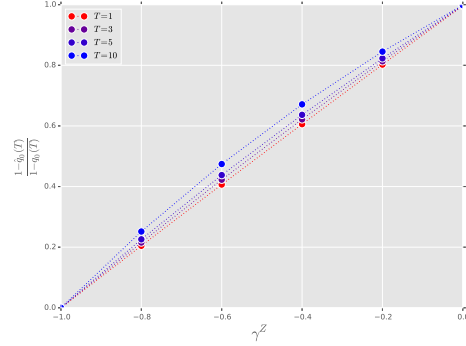


Figure 3.8: Comparison of $1 - q_0^f(T)/1 - q_0(T)$ for different maturities in a high spreads scenario, $Y_0 = y^H$ (CDS spreads at 740 bps).

3.4.3 CORRELATION IMPACT ON THE SHORT TERM VERSUS LONG TERM

We checked numerically the robustness of the theoretical relation between survival probabilities and γ^Z that was shown in Eq. (3.53). We calculated the ratio between the local and the quanto-corrected survival probability returned by the exponential OU model for different maturities and for different values of ρ . We can express this value as a function of the devaluation jump size, γ^Z , and of the maturity, T , and denote it by $k(\gamma, t)$. We then checked how this is affected by changes in γ and we compared k with the limit-case value provided by Eq. (3.53):

$$\tilde{k}(\gamma) := 1 + \gamma.$$

We used a set of parameters such to produce a flat CDS par-spread term-structure:

$$a = 1.00e - 004, \quad b = -210.45 \quad \sigma = 0.2, \quad Y_0 = -4.089.$$

The results are reported in Tables 3.5–3.7. They show that the approximation becomes less and less effective with increasing values of correlation or maturities. In the case of 10Y survival probabilities, $\tilde{k}$ doesn't provide a

Table 3.5: Comparison between $\tilde{k}$ and k with $\rho = 0$.

γ	$\tilde{k}(\gamma)$	$k(\gamma, T)$			
		$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	1.0	0.996	0.997	0.998	0.999
-0.2	0.8	0.802	0.813	0.823	0.845
-0.4	0.6	0.606	0.622	0.637	0.671
-0.6	0.4	0.407	0.423	0.438	0.474
-0.8	0.2	0.205	0.215	0.226	0.252
-1.0	0.0	0.000	0.000	0.000	0.000

Table 3.6: Comparison between $\tilde{k}$ and k with $\rho = -0.9$.

γ	$\tilde{k}(\gamma)$	$k(\gamma, T)$			
		$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	1.0	0.989	0.980	0.971	0.956
-0.2	0.8	0.796	0.798	0.800	0.806
-0.4	0.6	0.601	0.610	0.619	0.638
-0.6	0.4	0.404	0.415	0.425	0.449
-0.8	0.2	0.203	0.211	0.219	0.238
-1.0	0.0	0.000	0.000	0.000	0.000

good approximation of k not even in case of null correlation. Conversely, for $\rho = -0.9$, the approximation fails to provide accurate values of k even in the case of 1Y maturity. This is in line with the discussion from Section 3.2.5, as in both cases the hypotheses under which the approximation was deduced are clearly not valid.

Additionally, let us consider the value of quanto-adjusted survival probability obtained using different values of maturities, T , devaluation jump sizes, γ , and correlation ρ , and let us denote it by $q_0^f(T; \gamma, \rho)$. In Table 3.8 and in Figure 3.9 we reported the difference between the values of the quanto-corrected survival probabilities obtained using extreme values of correlation, $(T, \gamma) \mapsto q_0^f(T; \gamma, -0.9) - q_0^f(T; \gamma, 0.9)$.

They show, as hinted in El-Mohammadi, R.³⁴, that correlation has a smaller impact on short term survival probabilities: moving the correlation between

Table 3.7: Comparison between $\tilde{k}$ and k with $\rho = 0.9$.

γ	$\tilde{k}(\gamma)$	$k(\gamma, T)$			
		$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	1.0	1.004	1.014	1.021	1.029
-0.2	0.8	0.808	0.827	0.841	0.867
-0.4	0.6	0.610	0.632	0.650	0.685
-0.6	0.4	0.410	0.429	0.446	0.481
-0.8	0.2	0.206	0.219	0.230	0.253
-1.0	0.0	0.000	0.000	0.000	0.000

Table 3.8: Difference between $q_0^f(T; \gamma, \rho = -0.9)$ and $q_0^f(T; \gamma, \rho = 0.9)$ for different values of devaluation jump size, γ , and maturity, T .

γ	$q_0^f(T; \gamma, -0.9) - q_0^f(T; \gamma, 0.9)$			
	$T = 1$	$T = 3$	$T = 5$	$T = 10$
0.0	0.13%	0.80%	1.76%	4.28%
-0.2	0.10%	0.66%	1.45%	3.60%
-0.4	0.08%	0.50%	1.11%	2.77%
-0.6	0.05%	0.34%	0.75%	1.85%
-0.8	0.03%	0.17%	0.37%	0.90%
-1.0	0.00%	0.00%	0.00%	0.00%

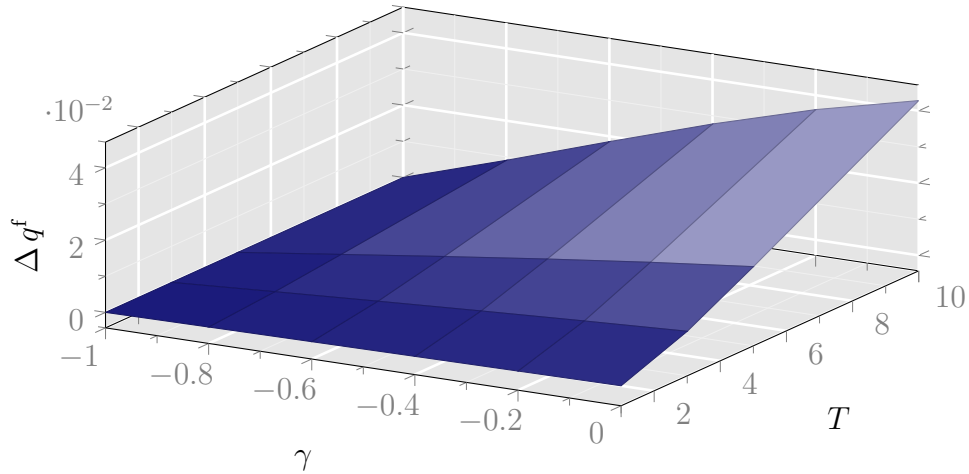


Figure 3.9: Difference between $q_0^f(T; \gamma, \rho = -0.9)$ and $q_0^f(T; \gamma, \rho = 0.9)$ for different values of devaluation jump size, γ , and maturity, T .

the values of -0.9 and 0.9 has an absolute impact of 0.13% in the case of zero devaluation jumps and for 1Y survival probabilities, whereas the impact for 10Y survival probabilities — also considered in the case of no devaluation jumps — can achieve 4.28% . They also show, as suggested by the results presented in Section 3.4.1, that with increasing values of the devaluation jump size, the correlation impact tends to become less and less material until it vanishes for the value $\gamma = -1$.

3.4.4 MODEL CALIBRATION TO MARKET DATA FOR 2011–2013

In this section we present the results of the calibration of the model described in Section 3.2.5, where pricing currency and liquid currency coincide and are USD, and where we considered two contractual currencies, EUR and USD.

We used the observed CDSs spreads on Italy, both the USD-denominated ones and the EUR-denominated ones, to calibrate the model parameters. Given the current model parameterization (i.e. constant parameters), the model is not able to fit CDS par-spreads term-structures containing an arbitrary number of points. For this reason, we calibrated it using two USD-denominated and two EUR-denominated CDSs, choosing only the most liquid tenor points for this task (five and ten years). The model could be re-parameterized in a way to fit more granular CDS term-structures — for example, by introducing a time-dependency on the long-term mean log-intensity parameter, b .

In principle, also single-name CDS swaptions could be used in this calibration process (see Brigo and Alfonsi¹²), but, given the lack of liquidity on this instrument, we preferred proxying them with the at-the-money implied volatilities quoted for options on CDX.

MARKET DATA DESCRIPTION

On each business day in the time range 2011–2013, we used a Levenberg Marquardt algorithm to calibrate the model to the market data. Let $\mathcal{T} = \{t_0 \dots, t_N\}$ denote the dates in this sample period. We made the following

assumptions on the market data:

- i) we consider the CDS par-spreads on Republic of Italy with 5 years and 10 years tenor, both in USD and in EUR;
- ii) we use the same short rate for domestic and foreign currency

$$r(t_i) = r^f(t_i) = r, \quad t_i \in \mathcal{T},$$

- iii) on every $t_i \in \mathcal{T}$ we assign the value of the at-the-money Black-volatility from an option with 6 months expiry to σ^Z ;
- iv) we keep the speed of mean reversion a of $(Y_t, t \geq 0)$ flat at the level 1×10^{-4} ;
- v) on every $t_i \in \mathcal{T}$ we calibrated σ^Y to the at-the-money option Black volatility for expiry one month.

Denoting by $p^Y := (b, y_0)$ the parameters to be calibrated for $(Y_t, t \geq 0)$ that are needed in single currency CDS pricing, and by $p := (b, y_0, \rho, \gamma)$ the set of parameters needed to price a quanto CDS, we adopted the following procedure to calibrate the model in Eq. (3.48)–(3.50):

- i) first we calibrated p^Y to the USD-denominated par-spread for the given date. We kept the parameters a and σ^Y fixed at a level of 1×10^{-4} and 50% respectively;
- ii) we calibrated σ^Y to the CDS index option, keeping the p^Y at the level calibrated at the previous step. In order to do that, we looked for the value of σ^Y such that a single-name CDS options, would have a price, calculated using a Crank Nicolson scheme, with Black volatility matching the one quoted for the CDS Index option. Single-name and Index CDS options are described in Section 1.1.3.
- iii) we used the calibrated value of p^Y as a starting point in the iterative routine carried out to calibrate the set of model parameters p to both

the EUR-denominated and the USD-denominated CDSs. The starting guess point to calibrate p can be written in terms of the calibrated point p^Y as $p_0 = (p_1^Y, p_2^Y, \gamma_0, \rho_0)$, where p_i^Y is the i -th component of p^Y and where γ_0 and ρ_0 are the guess values for γ and ρ , respectively. We kept σ^Y fixed at the level calibrated at the previous step.

BACKTEST RESULTS

In this section we show the results of the repeated daily calibrations to the 3 years of data contained in $\mathcal{T}$. The calibrated γ and ρ are showed in Figure 3.10 together with the relevant market data used in calibration, EUR-denominated and USD-denominated CDS par spreads for 5 years maturities, S_{EUR}^{5Y} and S_{USD}^{5Y} , and for 10 years maturities, S_{EUR}^{10Y} and S_{USD}^{10Y} .

It is important to note that, even in periods of high quanto CDS basis spreads as the one taken in consideration, the value of γ^Z is always far from reaching its extreme limit, -1 . This avoids the identification problem that would otherwise arise when trying to calibrate ρ . As noted when commenting the results presented in Figure 3.3, under such an extreme γ^Z scenario, there would be no sensitivity to ρ , and, therefore, it would be not possible to calibrate it.

The aim of this section is to interpret the calibrated parameters in terms of market data. To do so, we will be relying on the theoretical results from the previous section.

INTERPRETATION OF THE DEVALUATION FACTOR γ^Z For the devaluation rate, γ^Z , we exploited the results from Section 3.2.5, and we used the relative basis spreads as an approximation:

$$\gamma^Z \approx \frac{S_{\text{EUR}} - S_{\text{USD}}}{S_{\text{USD}}}. \quad (3.57)$$

This relation is similar to the one proved in Proposition 13 for hazard-rates. As noted in Remark 12, the result from that proposition could be trans-

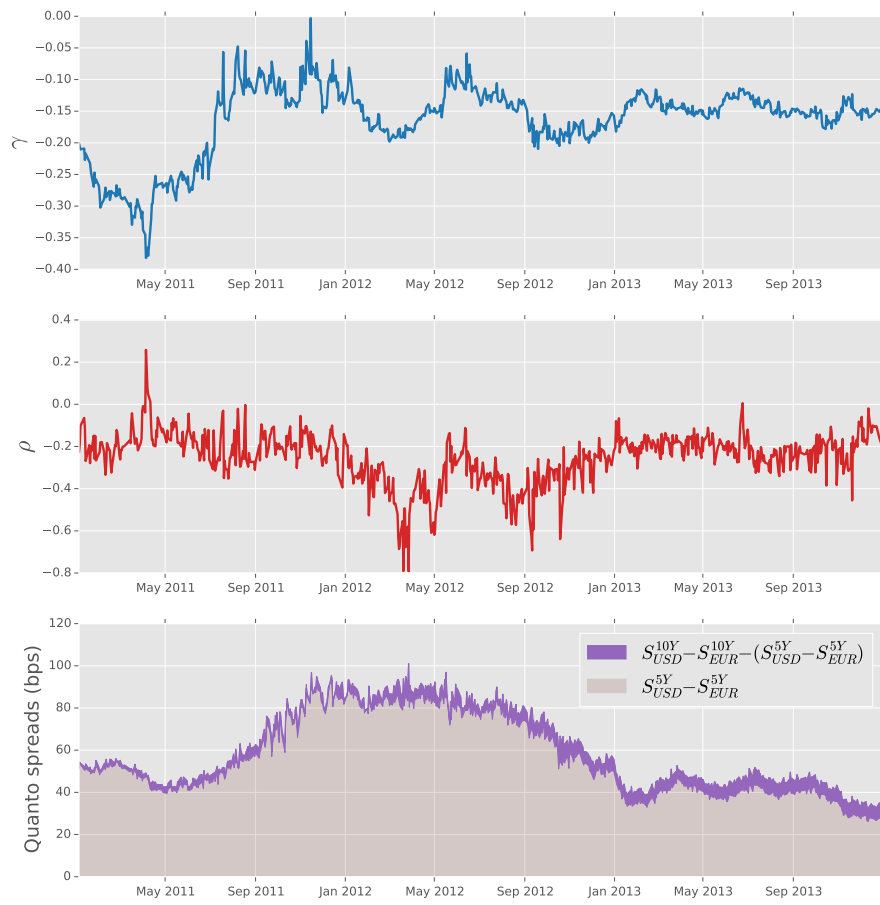
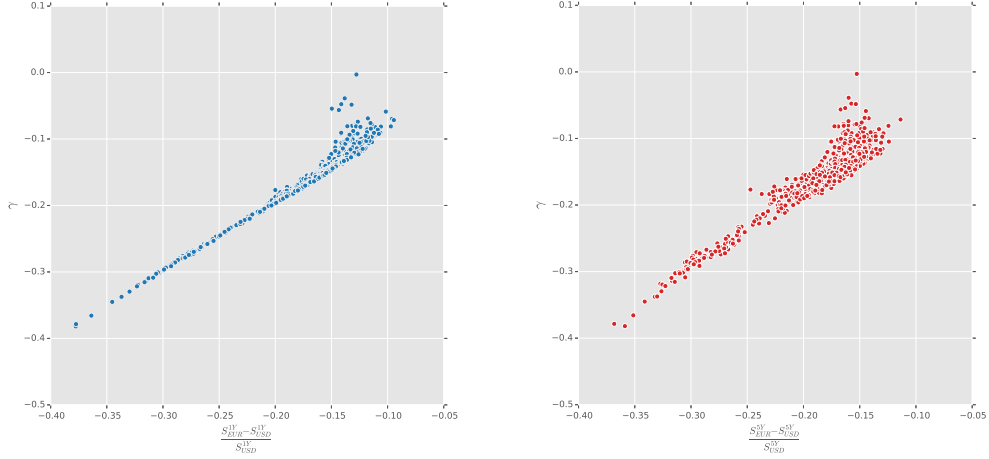


Figure 3.10: The top chart shows the calibrated γ through $\mathcal{T}$. The middle chart shows the calibrated ρ through $\mathcal{T}$. The bottom chart shows the time series of corresponding quanto CDS par-spreads.



(a) Relative basis spread for 1Y maturity CDSs.

(b) Relative basis spread for 5Y maturity CDSs.

Figure 3.11: Scatter plot comparing the calibrated γ^Z in ordinates with a relative basis spread in abscissas. As discussed in Section 3.4.4, the chart shows that the relative basis spread is a good estimate of the devaluation rate if the spreads have short maturities

posed in terms of CDS par-spreads — and therefore be written as Eq. (3.57) — in cases where the stream of the premium leg’s cash-flows could be approximated as continuously compounded stream of payments and in a setting where the default’s intensity was modelled as a deterministic function of time and where the CDS par-spread term structure was flat (see Corollary 9).

We expect such conditions to be more easily satisfied by short-term CDS par-spread. Due to liquidity reasons, however, we had to use CDS par-spreads with 5 years and 10 years tenor for calibration, and these maturity values can be too large. Therefore we used model-implied par-spreads for this test; in this way we have been able to use also short maturities, like one year, that are usually not very liquid in the market.

The comparison between γ^Z and its market-data approximation is showed in Figure 3.11. In line with our expectations, the left-hand chart, where 1Y-spreads have been used to build the relative basis spread, shows a remarkably good agreement between the two variables. The same agreement does not hold for the right-hand chart, where 5Y-spreads were instead used.

Remark 14. *A quanto CDS could be priced using a model where the FX rate jumps upon default of the reference entity, but where — differently from our working assumption — the intensity of default is deterministic.*

Beside allowing for different dependence assumption comparisons both in within the reduced form family (pure correlation against devaluation jumps) and between different credit risk modelling families (reduced form against structural approach — see next chapter), the introduction of spread dynamics — and, hence, correlation between the risk factors — provides the flexibility needed to account for time-dependency in the basis term-structure. Without it, the proportional adjustment between spreads would be the same for all maturities (given in Eq. (3.57)).

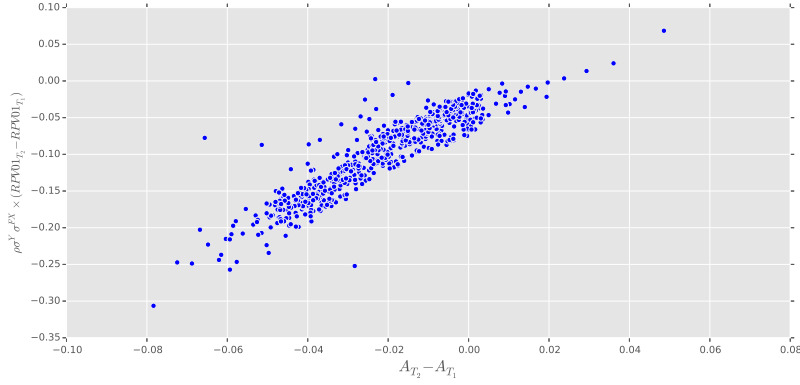
INTERPRETATION OF THE INSTANTANEOUS CORRELATION PARAMETER ρ
In order to provide a similar assessment on the parameter ρ , we relied on some heuristic results derived in Elizalde et al.³⁷. In that technical report, a simplified pricing formula based on cost of hedging arguments is presented for pricing quanto CDS. Their result can be written in terms of the variables defined in our framework as

$$\frac{S_{\text{EUR}}(T) - S_{\text{USD}}(T)}{S_{\text{USD}}(T)} \approx \gamma^Z + \sigma^Y \sigma^Z \rho \text{RPV01}(T), \quad (3.58)$$

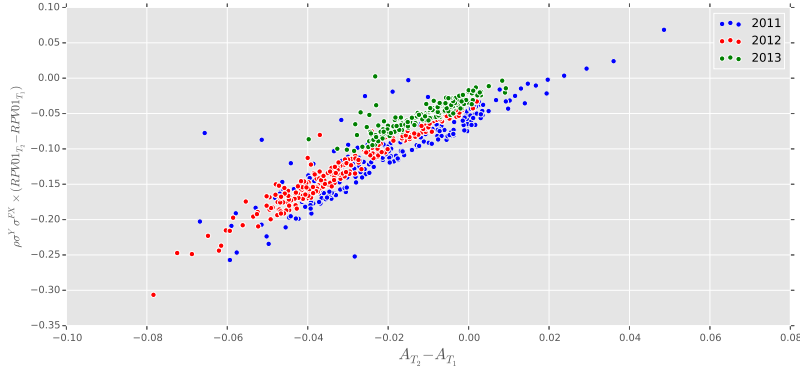
where $\text{RPV01}(t)$ is the risky annuity of a CDS with tenor t years (see Definition 2). We applied the formula above to two tenor points T_1 and T_2 obtaining two equations, one for each tenor. In order to test the values of ρ that we obtained in calibration, we worked out a single equation as a difference between the equations for the two tenor points:

$$\begin{aligned} \frac{S_{\text{EUR}}(T_2) - S_{\text{USD}}(T_2)}{S_{\text{USD}}(T_2)} - \frac{S_{\text{EUR}}(T_1) - S_{\text{USD}}(T_1)}{S_{\text{USD}}(T_1)} \\ \approx \sigma^Y \sigma^Z \rho (\text{RPV01}(T_2) - \text{RPV01}(T_1)). \end{aligned} \quad (3.59)$$

Specifically, we chose $T_1 = 1$, $T_2 = 10$ and we used the model-implied values of $S_{\text{EUR}}(T_1)$, $S_{\text{EUR}}(T_2)$, $S_{\text{USD}}(T_1)$, $S_{\text{USD}}(T_2)$, $\text{RPV01}(T_1)$ and $\text{RPV01}(T_2)$. We further used the values σ^Z coming from the market while the values of



(a) Scatter plot for the time-range 2011–2013



(b) Split by year

Figure 3.12: Scatter plot comparing the product of the calibrated ρ, σ^Y, σ^Z and the model-implied difference between risky annuities in ordinates with a difference of relative basis spread in abscissas (we used $A_T := \frac{S_{\text{EUR}}(T) - S_{\text{USD}}(T)}{S_{\text{USD}}(T)}$).

σ^Y and ρ are the ones obtained in calibration and discussed in Section 3.4.4. The results are presented in Figure 3.12 and they show a scatterplot of the proposed relation between model parameters and market data. The data are reported for the whole time-range 2011–2013 in Figure 3.12a, while Figure 3.12b contains the year-by-year plot. Due to the empirical nature of the Eq. (3.59), we didn't expect to find an exact relation between ρ and other model parameters and market data. Nonetheless, a clear pattern is exhibited and this gives some confidence that such relation can be used to produce at least rough approximations for ρ by using observable market data.

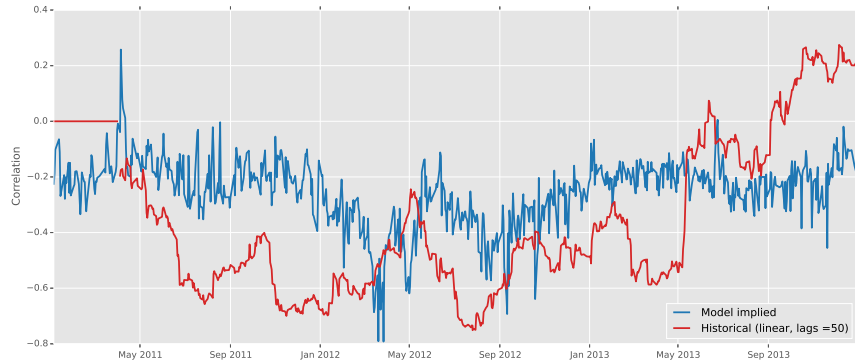


Figure 3.13: Implied and historical correlation between EURUSD FX rate and Italy's CDS spread.

MODEL-IMPLIED VS HISTORICAL CORRELATION In Figure 3.13 we reported a comparison between the correlation parameter we obtained in calibration, ρ , and a historical estimator of correlation between daily log-returns of CDS par-spreads for one-year tenor contracts and daily log-returns of the FX spot rate. For assets where the market correctly prices gamma and cross-gamma risks, the basis between implied and realised covariance terms can be actually traded. This happens, for example, for implied and historical volatilities on equity indices.

In times where the values of implied and realised covariance terms diverge, the effect of such trading strategies is usually to bring them closer. We interpret the lack of evident convergence between implied and realised correlation in the chart in Figure 3.13 as a signal of the lack of an efficient market for this correlation risk.

The fact that the implied correlation is generally smaller than the realised one in absolute value is consistent with our modeling choices and with the estimator used to calculate the realised correlation. The historical correlation has been estimated on a 50 days time-window using log-returns of the FX rate and this would neglect the impact of the jump term on its instantaneous volatility. Such an underestimation of the instantaneous volatility of the jump-diffusion process used in our modelling approach would result in an overestimation of the correlation with the credit component.

3.5 CONCLUSIONS AND FURTHER WORK

In this chapter, we investigated the impact on quanto CDS basis coming from different modelling assumptions about the FX rate / reference entity's credit worthiness dependence structure. In particular, we discussed the effectiveness of the introduction of default-driven FX devaluation jumps to explain large basis spreads. We used CDSs on Republic of Italy denominated in EUR and USD — and the big change in quanto basis observed in the market between them in 2011–2013 — as a testing case for our proposed modelling approach. We found that the jump mechanism allows to explain the size of the quoted basis, whereas pure shock correlation between FX rates and credit spread does not. Moreover, we derived an approximated formula to calibrate the size of the FX devaluation jump from quoted CDS spreads and we checked its performance in the same Republic of Italy testing case for the time range 2011–2013. Finally, we investigated a symmetry property of FX rate modelling in the case of our proposed jump-to-default approach, proving that such model is invariant under inversion of the FX rate direction.

Further lines of research we may consider in future works include studying modelling approaches embedding a consistent FX skew modelling and multiple FX rates / reference entities settings. Among the potential modelling tools that could be considered for the latter task, we mention here Hawkes processes (see Hawkes⁴³).

4

EXAMPLES OF WRONG-WAY RISK IN CVA INDUCED BY DEVALUATIONS ON DEFAULT

4.1 INTRODUCTION

Credit Valuation Adjustment (CVA) is a risk adjustment to the fair value of a portfolio of derivative contracts that reflects the credit risk of the common counterparty with which such contracts have been agreed. It accounts for the potential losses incurred due to the default of the counterparty before the contracts' expiries, and, as such, it heavily depends on the correct modelling of the credit risk factors, the market risk factors, and of the interaction between them. One of the main challenges in calculating CVA is indeed constituted by the lack of liquid market data to be used to infer risk-neutral credit/market joint distributions.

The calibration and approximation techniques showed in this chapter can be used, for example, to connect currency devaluation with multi-currency CDS par-spreads (see previous chapter for more details) and that, in turn, allows

to calculate CVA more accurately. The resulting FX/credit cross modelling improvement is crucial where the interaction between the counterparty credit and the FX is strong, i.e. with emerging market credits and systemically relevant counterparties where the right/wrong way-ness is more relevant.

Throughout this work, we will refer to the interaction between market and credit risk factors as *Wrong-Way Risk* (WWR).

4.1.1 OVERVIEW OF THE MODELLING FRAMEWORK

We will be using *unilateral*, as opposed to bilateral, CVA to illustrate the impact of WWR modelling. 'Unilateral' in this context means that we will be assuming only the counterparty to be a defaultable entity, while we neglect our own default risk. This assumption effectively amounts to considering ourselves as a default-free entity. On the one hand, the assumptions made in a bilateral framework are more realistic (both counterparties are subject to default risk), but, on the other hand, they introduce additional complexity in the form of the default time / default time interaction that, we think, might obfuscate the main points with respect to WWR modelling that this article wants to illustrate.

We will use a probability space $(\Omega, \mathcal{G}, \mathbb{Q}, (\mathcal{G}_t, t \geq 0))$ satisfying the usual hypotheses. In particular $(\mathcal{G}_t, t \geq 0)$ is a filtration under which the dynamics of the risk factors are adapted and under which the default time of the reference entity is a stopping-time.

In the setting just described, CVA can be represented through the following formula:

$$\text{CVA}_t = (1 - R)\mathbb{E}_t \left[\left(\phi(\tau, X_\tau^{(0)}, \dots, X_\tau^{(N)}) \right)^+ \frac{B_t}{B_\tau} \mathbb{1}_{\{\tau \leq T\}} \right]$$

where

- $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot | \mathcal{G}_t]$ is the expected value calculated with respect to the global filtration $\mathcal{G}_t$;

- $(t, x_0, \dots, x_N) \mapsto \phi(t, x_0, \dots, x_N)$ is the value of the portfolio at time t and for a realisation $x_0, \dots, x_N$ of the market risk-factors;
- τ is the counterparty's default time;
- $(B_t, t \geq 0)$ is the numeraire associated to the pricing measure;
- R is the recovery rate associated to the reference entity. Given our focus is on investigating the impact of different risk factors dependence structures, in the following we will neglect this parameter and we will be assuming $R = 0$.

For the same reason, and given that the main examples shown throughout this work focus on equity/credit WWR and FX/credit WWR, we will neglect the randomness in interest rates and just assume deterministic interest rate term-structures. We will generically refer to the FX and equity related risk-factors as the *market risk-factors*.

The plan of the work is the following: in Section 4.2 we will present a PDE approach based on reduced-form framework for credit risk modelling. We will show how to handle the case where the market risk-factor is a jump-diffusion process and how to link the jump times to the default time of the counterparty. In Section 4.4 we will show how this approach is able to provide a more effective way to model WWR.

For the equity case, in Section 4.3 we will present an alternative credit modelling framework called AT1P and we will show how it naturally links equity and credit risk factors. In Section 4.4 we will compare this approach to the jump-diffusion approach in reduced-form framework.

We refer to Brigo et al.¹⁹ and Cesari and Gordon²⁵ for a general overview on CVA modelling with applications to multiple asset classes. In Brigo et al.¹⁹, both unilateral and bilateral CVA calculation frameworks are described. Techniques for calculations of extreme CVA values, in the context of bilateral CVA modelling, have been recently shown in Scherer and Schulz⁶⁷. An algorithm to calculate CVA using a fixed-point approach has been proposed in Kim and Leung⁵⁰.

4.2 A PDE APPROACH FOR BOTH FX-DRIVEN AND EQUITY-DRIVEN WWR

In this section we present a modelling approach to handle the case where the counterparty's default intensity is stochastic and where one additional market risk-factor is modelled as a jump-diffusion process whose only jump occurs upon the counterparty's default.

This approach can be effectively applied to both the case where the market risk-factor is an FX rate and to the case where the market risk-factor is an equity asset or index. Test cases with respect to both the examples are presented in Section 4.4.

4.2.1 FX

Let us consider the existence of multiple risk-neutral pricing measures, each of them linked to a specific currency. In this context, we will denote by $(B(t), t \geq 0)$ the money market account denominated in the (arbitrarily chosen) domestic currency, while we will denote by $(B^f(t), t \geq 0)$ the money market account denominated in another foreign currency. Both of them are assumed deterministic. Furthermore, we will denote by Z the spot FX rate expressing the cost of one unit of foreign currency in the domestic currency and we will be using $(D_t, t \geq 0)$ for the default process:

$$D_t = \mathbb{1}_{\{\tau \leq t\}}, \quad t \geq 0.$$

Let us consider the following specification for the dynamics of $(Z_t, t \geq 0)$ and

of the counterparty's default intensity, $(\lambda_t, t \geq 0)$:

$$dY_t = a(b - Y_t) dt + \sigma^Y dW_t^Y, \quad t \geq 0, \quad (4.1)$$

$$Y_0 = y, \quad (4.2)$$

$$dZ_t = \mu^Z Z_t dt + \sigma^Z Z_t dW_t^Z + \gamma^Z Z_{t-} dD_t, \quad t \geq 0, \quad (4.3)$$

$$Z_0 = z, \quad (4.4)$$

$$d\langle W^Y, W^Z \rangle_t = \rho dt, \quad t \geq 0, \quad (4.5)$$

$$\lambda_t = e^{Y_t}, \quad t \geq 0. \quad (4.6)$$

An application of the generalized Itô formula (see Jeanblanc et al.⁴⁹) allows us to write the $\mathbb{Q}$ -dynamics of $(\text{CVA}_t, t \geq 0)$. Using $f(t, Z_t, Y_t, D_t) = \text{CVA}_t$:

$$\begin{aligned} df = & \partial_t f dt + \partial_z f \left(\mu^Z z dt + \sigma^Z z dW_t^Z + \gamma^Z z dD_t \right) \\ & + \partial_y f \left(a(b - Y_t) dt + \sigma^Y dW_t^Y \right) + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f d[Z, Z]_t \\ & + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f d[Y, Y]_t + \rho \sigma^Z \sigma^Y z \partial_{zy} f d[Z, Y]_t \\ & + \Delta f \Delta D_t - \partial_z f \Delta Z_t \end{aligned} \quad (4.7)$$

where, with some abuse of notation, we have defined the jump-to-default term as

$$\Delta f := f(t, Z_{t-} + \Delta Z_t, Y_t, D_{t-} + \Delta D_t) - f(t, Z_{t-}, Y_t, D_{t-}). \quad (4.8)$$

DEFINITION OF Δf . For the same reasons exposed in Section 3.2.4, Δf can be re-written as

$$\Delta f = f(t, (1 + \gamma^Z)Z_{t-}, Y_t, 1) - f(t, Z_{t-}, Y_t, 0).$$

COMPENSATED MARTINGALE FOR Δf . The compensator of the term $\Delta f \Delta D_t$ in Eq.(4.7) was derived in Section 3.2.4. It is equal to

$$(1 - D_t) e^{Y_t} \Delta f dt,$$

which, conditional on $\mathcal{G}_t$, $D_t = d$, $Z_{t-} = z$, and $Y_t = y$, is equal to

$$(1 - d)e^y \left(f(t, z(1 + \gamma^Z), y, 1) - f(t, z, y, 0) \right) dt.$$

NO-ARBITRAGE DRIFT FOR THE MARKET RISK-FACTOR (FX)

Any specification of the FX rate dynamics is subject to arbitrage constraints. One way to formulate them is by requiring that the Radon–Nikodym derivative defined by

$$L_t = \frac{Z_t B_t^f}{Z_0 B_t}, \quad L_0 = 1. \quad (4.9)$$

is a martingale. The drift specification that satisfies such condition is provided by

$$\mu^Z = r(t) - r^f(t) - \lambda_t \gamma^Z \mathbb{1}_{\{\tau > t\}} = r(t) - r^f(t) - \lambda_t \gamma^Z (1 - D_t), \quad (4.10)$$

where $r(t)$ and $r^f(t)$ are the — assumed deterministic — domestic and foreign short rates, respectively.

It is now possible to write a Feynman–Kac PDE to compute the value of $CVA_t(T)$. Indeed, $(CVA_t, t \geq 0)$ is a $\mathbb{Q}$ -price and, as such, it must locally grow at the rate r . Therefore, its drift must satisfy the following equation:

$$\begin{aligned} \partial_t f + \left(r - r^f - \lambda_t \gamma^Z (1 - d) \right) z \partial_z f + a(b - Y_t) \partial_y f \\ + \frac{1}{2} \left(\sigma^Z z \right)^2 \partial_{zz} f + \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} f + \rho \sigma^Z \sigma^Y z \partial_{zy} f \\ + e^y (1 - d) \Delta f = r f dt, \end{aligned}$$

where the explicit dependence of f on the state variables (x, y, t, d) has been omitted for clarity of reading. It is worth noting that, if it wasn't for the last term, this would be the typical PDE for default-free payoffs. Incidentally, this jump-to-default term is also the only term of the equation where the two default-specific components $f(t, (1 + \gamma^Z)z, y, 1)$ and $f(t, z, y, 0)$ appear together. In fact, by conditioning first on $d = 1$ and then on $d = 0$ we can

decouple the two functions

$$\begin{aligned} u(t, z, y) &:= f(t, (1 + \gamma^Z)z, y, 1) \\ v(t, z, y) &:= f(t, z, y, 0) \end{aligned}$$

and calculate them by solving iteratively two separate — lower dimension — PDE problems. We first solve for u , as for $d = 1$ the last term does not appear in the equation, and, once u has been calculated, we use it to solve for v .

FINAL CONDITIONS — CVA PAYOFF

The final conditions for functions u and v depend on the portfolio for which the CVA is going to be calculated. For the sake of explanation, let us consider a stylised portfolio where we expect to receive a single — deterministic and constant in time — cash-flow payment from our counterparty at maturity $T > 0$. The payment will be settled in a different currency from the one used to determine the risk-neutral pricing measure. Furthermore, we consider null interest rates. Under these assumptions, the CVA is given by:

$$\text{CVA}_0 = \mathbb{E}_0 \left[Z_\tau \mathbb{1}_{\{\tau \leq T\}} \right]. \quad (4.11)$$

The final conditions for the two functions can be written as:

$$\begin{aligned} u(T, z, y) &= f(T, (1 + \gamma^Z)z, y, 1) = (1 + \gamma^Z)z, \\ v(T, z, y) &= f(T, z, y, 0) = 0. \end{aligned}$$

Remark 15 (Terminal condition for u). *In order to get a better understanding of the conditions set above, it might be useful to write the terminal condition for u in terms of conditioned expected values:*

$$u(T, z, y) = f(T, (1 + \gamma^Z)z, y, 1) = \mathbb{E} \left[Z_\tau \mathbb{1}_{\{\tau \leq T\}} \mid \mathcal{G}_T, \tau \leq T \right] = Z_\tau.$$

It is also worth recalling that $u(t, z, y)$ is only needed in solving v in the term representing the jump-on-default component Δf , that is, the change in value

given to a default happening at t (see Remark 10).

The PDE problem that must be solved to obtain u is then given by

$$\begin{aligned} \partial_t u = & -(r - r^f)z\partial_z u - a(b - y)\partial_y u - \frac{1}{2}(\sigma^Z x)^2 \partial_{zz} u \\ & - \frac{1}{2}(\sigma^Y)^2 \partial_{yy} u - \rho\sigma^Z \sigma^Y z \partial_{zy} u \end{aligned} \quad (4.12a)$$

$$u(T, z, y) = (1 + \gamma^Z)z. \quad (4.12b)$$

Once the solution to this problem has been calculated, it can be used to solve the PDE for v , which is then given by

$$\begin{aligned} \partial_t v = & -(r - r^f)z\partial_z v - a(b - y)\partial_y v - \frac{1}{2}(\sigma^Z z)^2 \partial_{zz} v \\ & - \frac{1}{2}(\sigma^Y)^2 \partial_{yy} v - \rho\sigma^Z \sigma^Y z \partial_{zy} v \\ & + e^y (v - u - \gamma^Z z \partial_z v) \end{aligned} \quad (4.13a)$$

$$v(T, z, y) = 0. \quad (4.13b)$$

Remark 16. *The PDE system (4.13) above explains why it makes sense to define $u(t, z, y) := f(t, (1 + \gamma^Z)z, y, 1)$ rather than, for example, $u(t, z, y) := f(t, z, y, 1)$. It is indeed the former that is directly used as a component of the Δf -term.*

NUMERICAL SOLUTION

In order to calculate the solutions for the two systems (4.12)–(4.13), we used the same scheme described in Section 3.3 with the same time and space discretization grids for both systems. In particular, we applied the Stabilising-Corrections scheme to solve system (4.12). Once the numerical approximation of u was calculated and stored for each time in the common discretization grid, we were able to apply the same numerical scheme to system (4.13), thus obtaining a numerical approximation of v .

Table 4.1: Parameters used in the first test of the FDM solver.

Model parameters		FDM parameters	
Parameter	Value	Parameter	Value
a	0.001	z_0	0.5
b	3.000	z_{N^Z}	1.5
σ^Y	0.400	N^Z	40.0
y	-5.000	y_0	-5.5
μ^Z	0.020	y_{N^Y}	-4.0
σ^Z	0.200	N^Y	100.0
z	1.000	t_0	0.0
ρ	0.400	t_{N^T}	1.0
γ^Z	-0.300	N^T	300.0

NUMERICAL BEHAVIOUR We tested the behaviour of the Finite-Difference Method (FDM) by comparing its results with the ones coming from a Monte Carlo implementation of the same model. With respect to the FDM algorithm, this has been implemented on a discretization grid for the processes $(Z_t, t \geq 0)$ and $(Y_t, t \geq 0)$:

$$(z_m, y_n), \quad m = 0, \dots, N^Z, \quad n = 0, \dots, N^Y.$$

The Monte Carlo solver was used with the same time-discretization of the FDM-solver, $t_0, \dots, t_{N^T}$. Euler schemes were used to discretise the two processes $(Z_t, t \geq 0)$ and $(Y_t, t \geq 0)$. We didn't implement any variance reduction technique.

We tested two different configurations of the credit and of the dependence parameters:

1. one where $\rho = 0.4$, $\gamma^Z = -0.3$, $\sigma^Y = 0.4$, and $y = -5$ (the full set of model and solver parameters is reported in Table 4.1);
2. one where $\rho = -0.5$, $\gamma^Z = -0.1$, $\sigma^Y = 0.5$, and $y = -4.5$ (the full set of model and solver parameters is reported in Table 4.2).

Table 4.2: Parameters used in the second test of the FDM solver.

Model parameters		FDM parameters	
Parameter	Value	Parameter	Value
a	0.001	z_0	0.5
b	3.000	z_{N^Z}	1.5
σ^Y	0.500	N^Z	40.0
y	-4.500	y_0	-5.5
μ^Z	0.020	y_{N^Y}	-4.0
σ^Z	0.200	N^Y	100.0
z	1.000	t_0	0.0
ρ	-0.500	t_{N^T}	1.0
γ^Z	-0.100	N^T	300.0

The results of the tests are been reported in Figures 4.1 and 4.2. They show a convergence of the solutions calculated using a Monte Carlo solver to the level calculated using the Finite-Difference Method approximation.

4.2.2 EQUITY

We can use the same modelling approach presented in Section 4.2.1 to calculate the CVA of an equity portfolio. Similarly to the previous case, we consider an exponential Ornstein–Uhlenbeck process for the stochastic intensity and a geometric Brownian motion with a deterministic relative jump occurring upon the counterparty’s default for the other relevant risk-factor — in this case, equity, rather than FX.

We specify the model as:

$$\begin{aligned}
dS_t &= \mu^S S_t dt + \sigma^S S_t dW_t^S + \gamma^S S_{t-} dD_t, \\
S_0 &= s_0, \\
dY_t &= a(b - Y_t) dt + \sigma^Y dW_t^Y, \\
Y_0 &= y_0, \\
d\langle W^S, W^Y \rangle_t &= \rho dt,
\end{aligned}$$

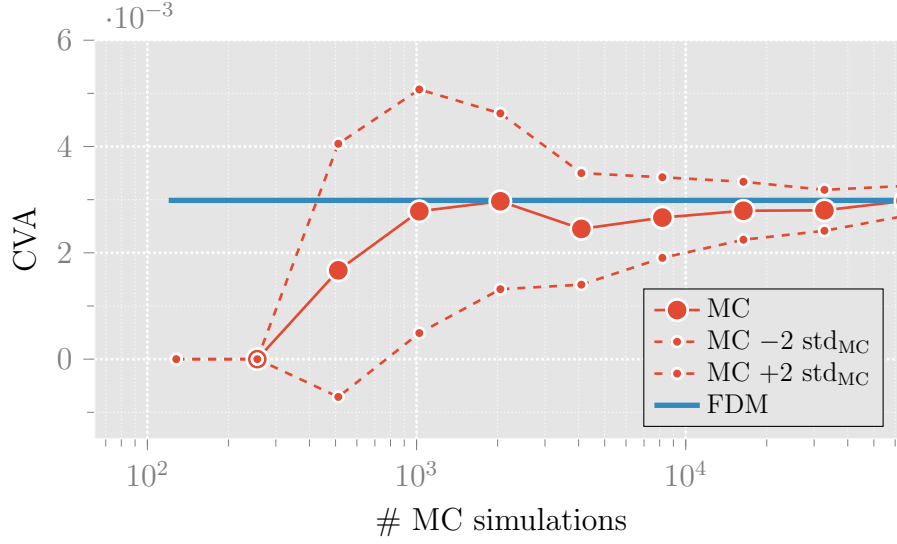


Figure 4.1: Comparison between Monte Carlo and Finite Difference Method-based solvers in the case $\rho = 0.4$, $\gamma^Z = -0.3$, $\sigma^Y = 0.4$, $y = -5$. The lines $\text{MC} \pm 2\text{std}_{\text{MC}}$ indicate a 95%-confidence interval on the MC-result.

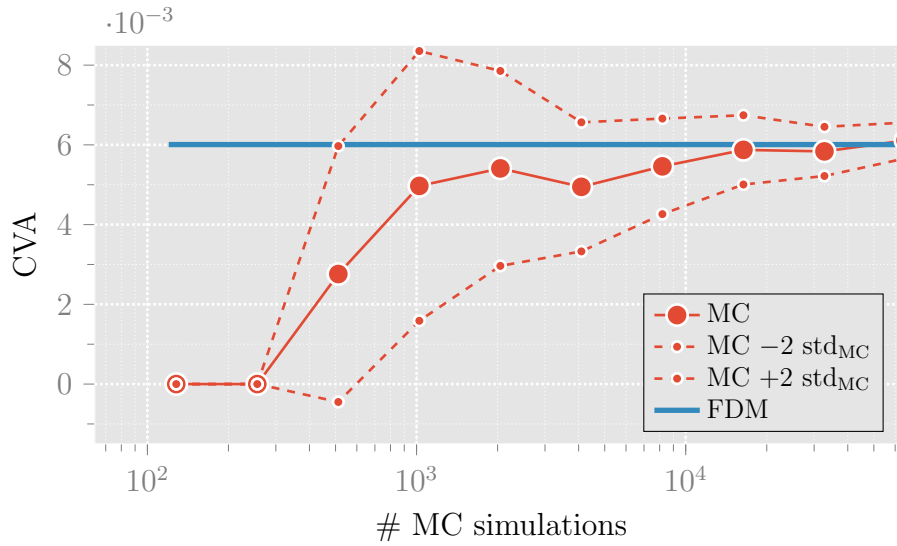


Figure 4.2: Comparison between Monte Carlo and Finite Difference Method-based solvers in the case $\rho = -0.5$, $\gamma^Z = -0.1$, $\sigma^Y = 0.5$, $y = -4.5$. The lines $\text{MC} \pm 2\text{std}_{\text{MC}}$ indicate a 95%-confidence interval on the MC-result.

where the stochastic intensity of default $(\lambda_t, t \geq 0)$ is given — as in the FX/credit case — by

$$\lambda_t = e_t^Y.$$

NO-ARBITRAGE DRIFT FOR THE MARKET RISK-FACTOR (EQUITY)

The no-arbitrage condition on $(S_t, t \geq 0)$'s drift is given by requiring that its discounted price is a martingale. In formulas:

$$\mathbb{E}_t \left[\frac{S_T}{B_T} \right] = \frac{S_t}{B_t}.$$

Under the assumption of deterministic interest rates, that translates into

$$\mu^S = r(t) - \gamma^S(1 - D_t)\lambda_t. \quad (4.14)$$

FINAL CONDITIONS — CVA PAYOFF

In this case, the prototype portfolio we are interested in studying is made of a single put option and it can be written as

$$V_t = (1 - R)\mathbb{E}_t \left[(K - S_\tau)^+ \mathbb{1}_{\{\tau \leq T\}} \right].$$

This is a classic example of a position carrying a high level of WWR, and therefore highly sensitive to correlation assumptions. In order to get some intuition about the correlation effect in this type of trade, it might be useful to consider the limit case where the equity option's underlying is the counterparty's stock. In case of, for example, financial problems of the reference entity, these will be reflected in their balance account, and, arguably, in their stock price (that will decrease) and in their credit quality (that will decrease). Both these changes will affect the CVA of the position in the same direction (it will increase) and, moreover, they can reinforce each other. Therefore, taking in account their joint effect can have a dramatic effect on our calculations.

Given the strong Markov property of all the processes involved, the CVA can

be assumed to be a function of their values at valuation time:

$$f(t, s, y, d) := (1 - R)\mathbb{E}_t \left[(K - S_\tau)^+ \mathbb{1}_{\{\tau \leq T\}} \right].$$

The PDE system that must be solved to calculate the value above can be deduced using the same techniques showed in Section 4.2.1 in the FX/credit case. Let us then define the conditioned on default and on survival values of f :

$$\begin{aligned} u(t, s, y) &:= f(t, s(1 + \gamma^S), y, 1), \\ v(t, s, y) &:= f(t, s, y, 0) \end{aligned}$$

so that the final conditions on the newly defined functions are given by

$$u(T, s, y) := f(T, s, y, 1) = (1 - R)(K - s), \quad (4.15)$$

$$v(T, s, y) := f(T, s, y, 0) = 0. \quad (4.16)$$

The same equations can therefore be used, with two important differences to take into account:

- i) the no-arbitrage drift of the market risk-factor (equity in this case, rather than FX) is given by Eq.(4.14) rather than Eq.(4.10);
- ii) the terminal conditions are given by Equations (4.15) and (4.16).

The resulting PDE system is then given by:

$$\begin{aligned} \partial_t u &= -rs\partial_s u - a(b - y)\partial_y u - \frac{1}{2} \left(\sigma^S x \right)^2 \partial_{ss} u \\ &\quad - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} u - \rho \sigma^S \sigma^Y s \partial_{sy} u \end{aligned} \quad (4.17a)$$

$$u(T, s, y) = (1 - R)(K - s). \quad (4.17b)$$

Once the solution to this problem has been calculated, it can be used to solve

the PDE for v , which is then given by

$$\begin{aligned} \partial_t v = & -(r)s\partial_s v - a(b-y)\partial_y v - \frac{1}{2} \left(\sigma^S s \right)^2 \partial_{ss} v \\ & - \frac{1}{2} \left(\sigma^Y \right)^2 \partial_{yy} v - \rho \sigma^S \sigma^Y s \partial_{sy} v + e^y (v - u - \gamma^S s \partial_s v) \end{aligned} \quad (4.18a)$$

$$v(T, s, y) = 0. \quad (4.18b)$$

4.3 A STRUCTURAL APPROACH FOR EQUITY/CREDIT WWR

In contrast to the reduced-form approach presented in Section 4.2.2, the model that we show in this section is based on a *structural* approach to credit modelling. Structural models seem a particularly good candidate to model WWR in equity as they naturally link credit risk and stock prices through their knowledge of balance account quantities. Furthermore, in the equity case, differently from the FX rate case, we don't have the constraint provided by the existence of quanto CDS basis. On the one hand, this means that we have more freedom on modelling the underlying process dynamics. For the sake of illustration, we will be using a simple *Geometric Brownian Motion* (GBM) to do that. On the other hand, by doing that, we lose one “correlation” parameter (the devaluation-on-default parameter γ), thus delegating the whole WWR effect to the instantaneous correlation between the equity process and the credit process. The effects on WWR of this modelling choice are compared with reduced-form based models both in a purely diffusive case and in a jump-diffusion setting in Section 4.4 showing that, from a WWR perspective, a structural approach lies in between the other two modelling approaches.

4.3.1 AT1P

We will be using an approach based on *Analytically Tractable First-Passage* (AT1P) to model the dependence between equity and credit in this section. AT1P was first presented in Brigo and Tarenghi¹⁷ where the authors extended the original Merton and Black Cox setting

We introduced AT1P in Section 1.2.2, so we refer to that section for more details about the model and its related literature. For the sake of convenience, we repeat here the model formulation in the simplified setting where — in addition to the debt barrier being deterministic — the dividends and the risk free short rate are constant. The firm-value process is specified by the following SDE:

$$dV_t = (r - q)V_t dt + \sigma(t)V_t dW_t^V, \quad V_0 = v,$$

while a time-dependent barrier is parameterised as

$$\hat{H}(t) = H e^{(r-q)t - B \int_0^t \sigma^2(s) ds}, \quad \hat{H}(0) = H_0. \quad (4.19)$$

Default time in this framework is defined as

$$\tau = \inf\{t \geq 0 : V_t \leq H(t)\}, \quad \inf \emptyset := +\infty. \quad (4.20)$$

Next, we look separately at how this model can be used to model both the credit risk and the equity component when calculating CVA for an equity trade.

CREDIT RISK

Survival probabilities are given in closed-form expression as

$$\mathbb{Q}(\tau > T) = \Phi(d_1) - \left(\frac{H}{V_0}\right)^{2B-1} \Phi(d_2), \quad (4.21)$$

where τ is defined in Eq. (1.22) and where

$$d_1 := \frac{\log \frac{V_0}{H} + \frac{2B-1}{2} \int_0^T \sigma(s)^2 ds}{\left(\int_0^T \sigma(s)^2 ds\right)^{1/2}},$$

$$d_2 := d_1 - \frac{2 \log \frac{V_0}{H}}{\left(\int_0^T \sigma(s)^2 ds\right)^{1/2}}.$$

Closed-form formulas for survival probabilities are sufficient for CDS calibration in a single-currency framework and when credit is assumed to be independent of interest rates (see Section 1.1.1).

EQUITY PRICE

In AT1P an entity can default at any time and not only at its debt maturity. If we assume that the debt still has a clear single final maturity T and that early default is given by safety covenants, it is not unreasonable to model equity as an option on the firm value with maturity T that is killed if the default barrier is reached before T . We calculate the stock price E_t (in this framework) as a down-and-out European call option, that is

$$E_t = \beta(t) \mathbb{E}_t \left[\frac{\left(V_T - \hat{H}(T) \right)^+ \mathbb{1}_{\{\tau > T\}}}{\beta(T)} \right] = f(t, V_t). \quad (4.22)$$

A closed form solution for the price of the option is given for example in Rapisarda⁶⁵, see also the Equity chapter in Brigo et al.¹⁹. The formula is as follows:

$$\begin{aligned} f = \frac{\beta(t)}{\beta(T)} & \left(V_t e^{\int_t^T (v(s) + \frac{\sigma(s)^2}{2}) ds} (1 - \Phi(d_3)) - \hat{H}(T) (1 - \Phi(d_4)) \right. \\ & - \hat{H}(t) \left(\frac{\hat{H}(t)}{V_t} \right)^{2B} e^{\int_t^T (v(s) + \frac{\sigma(s)^2}{2}) ds} (1 - \Phi(d_5)) \\ & \left. + \hat{H}(T) \left(\frac{\hat{H}(t)}{V_t} \right)^{2B-1} (1 - \Phi(d_6)) \right), \end{aligned} \quad (4.23)$$

where

$$\begin{aligned}
v(t) &= r - q - \frac{\sigma(t)^2}{2}, \\
d_3 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{V_t}{H} - \int_t^T (v(s) + \sigma(s)^2) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}, \\
d_4 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{V_t}{H} - \int_t^T (v(s)) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}, \\
d_5 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{\hat{H}(t)^2}{HV_t} - \int_t^T (v(s) + \sigma(s)^2) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}, \\
d_6 &= \frac{\left(\log \frac{\hat{H}(T)}{H}\right)^+ - \log \frac{\hat{H}(t)^2}{HV_t} - \int_t^T (v(s)) ds}{\left(\int_t^T \sigma(s)^2 ds\right)^{1/2}}.
\end{aligned}$$

4.3.2 INTRODUCING WWR

We introduce WWR in AT1P by allowing the option's underlying process, $(S_t, t \geq 0)$, to be correlated to the firm's stock price process. Similarly to the case analysed in Section 4.2.2 — but with the crucial difference that in this case we don't assume a jump-to-default component — the dynamics of the option's underlying process is specified as a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma^S S_t dW_t^S, \quad S_0 = s_0.$$

The instantaneous correlation between processes $(S_t, t \geq 0)$ and $(V_t, t \geq 0)$ is in principle difficult to estimate, because it mixes together information of fundamentally different nature: a stock price S , that is observable and traded, and a firm-value V , that is not traded, and that is only observable at most quarterly or at any planned balance account public disclosure. The link provided by firm-value and stock price in AT1P Eq.(4.23), however, proves

Test case	Reduced-form model		Structural model
	expOU	expOU+J	AT1P
FX	✓	✓	
Equity	✓	✓	✓

Table 4.3: Summary of the models/asset classes tested in Section 4.4.

useful in this respect, because it allows, as a first approximation, to use the empirical correlation between the firm’s stock price and the underlying equity process as a firm–value/underlying equity’s correlation estimate. In formulas, we will be assuming that:

$$d \langle W^S, W^V \rangle = d \langle W^S, W^E \rangle.$$

4.4 RESULTS

In this section, we show results produced by all the models described in the previous sections and we compare the different “WWR power” that each model is able to provide. From an asset class perspective, we will present tests on FX / credit interaction and on equity / credit interaction. From a modelling perspective, we will test impact on WWR of instantaneous correlation in both reduced-form model and in structural model (the latter only in equity case). We will also test the devaluation jump impact on WWR. We will use acronyms to refer to models. We will use *exponential Ornstein–Uhlenbeck* (expOU) for the reduced-form models, and we will denote by a (+J) the addition of jumps to default on the market risk factor. We will use AT1P for the structural model. A test/asset class summary is presented in Table 4.3.

4.4.1 MODELS CALIBRATIONS

For the tests presented in this section we considered a dummy set of market data. Specifically, we used

- a flat term-structure or CDS par-spreads (200 bps);
- a flat term-structure of zero rates (100 bps);
- we fixed the lognormal volatility of both equity and FX rate to 20%;
- in the reduced-form approach, we fixed the the normal volatility of the OU process driving the stochastic intensity to 50% and its speed of mean reversion speed to 0.001.

For the AT1P calibration, we refer to^{23, 17} for further discussions and examples around calibration. We also refer to²¹ for a more recent example and for a detailed description on how to use balance account information to fix the barrier level H (see Eq.(4.19)).

For the expOU model calibration, we refer in particular to²² where calibration is discussed extensively and where 3 year daily calibration outputs relying on quanto CDS spreads are presented.

4.4.2 MODELS PARAMETERS AND PERFORMANCE

We used the same numerical setup for all the tests presented in this section. Specifically, the Monte Carlo tests were performed using:

- Euler discretizations for the market risk factors and:
 - for $(Y_t, t \geq 0)$ in case of reduced-form based tests;
 - for $(V_t, t \geq 0)$ in case of firm-value models based tests.
- 10,000 simulations.
- No variance reduction techniques.

Finite-Difference methods were implemented using:

- a market risk factor grid $x_0, \dots, x_{N^x}$ with $x_0 = -0.8$, $x_{N^x} = 1.2$, and $N^y = 40$;

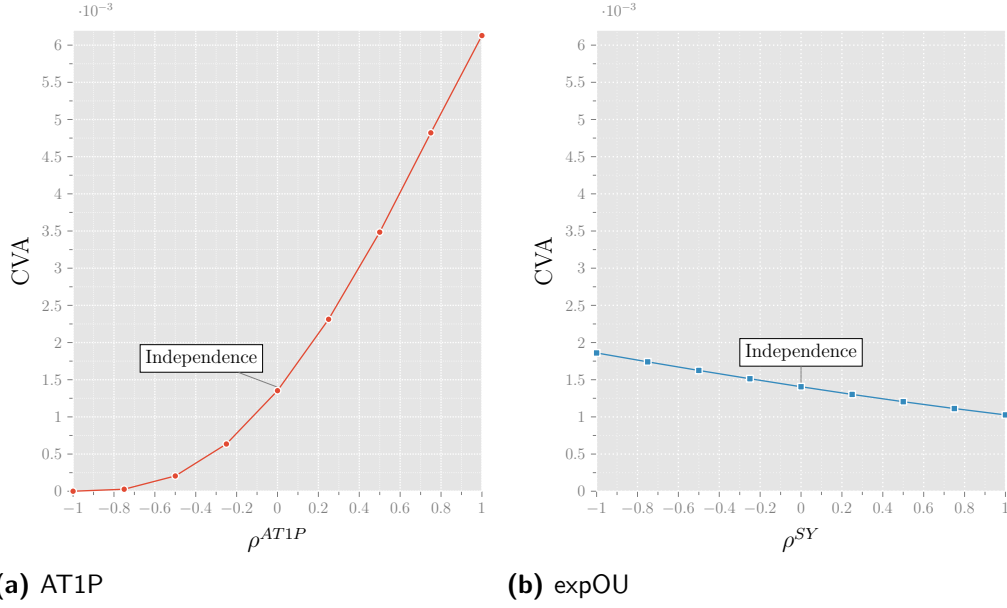


Figure 4.3: Correlation impact in the structural AT1P approach versus the reduced-form expOU approach. The CVA values obtained in case of credit-market independence (highlighted in both charts) are the same.

- a log-intensity grid $y_0, \dots, y_{N^y}$ with $y_0 = -4$, $y_{N^y} = 1.2$, and $N^y = 40$;
- a time grid $t_0, \dots, t_{N^t}$ with $t_0 = 0$, $t_{N^t} = 1$, and $N^t = 300$.

Using a CPU processor Intel Core i5-3320M 2.6 GHz our implementations are such to produce a single price in around 0.35 seconds in the case of the Monte Carlo one, and in around 0.8 seconds in the Finite-Difference one. It is worth stressing that only the Monte Carlo implementation is parallelised, using the four logical processors available on the CPU, whereas the Finite-Difference implementation runs on a single core.

4.4.3 EQUITY WWR: CORRELATION IMPACT

In Figure 4.3 we present the correlation impact on CVA calculation of a portfolio consisting of a single trade: an ATM put option with expiry 1 year.

The agreement between the expOU and AT1P graphs on the zero correlation case provides a good safety check on the two model implementations.

It is worth highlighting that we used a common x-axis for the two charts, as indeed both the CVA calculated using expOU and using AT1P depend on a correlation parameter. The correlation parameter, however, has a very different interpretation in the two cases. As discussed in Section 4.3.2, in AT1P a possible proxy for it is provided by the equity/equity correlation between the counterparty and the option's underlying. In expOU, instead, the correlation to be used is the one between the counterparty's credit and the option's underlying. Unsurprisingly, the two parameters have opposite impact on CVA. Given the fact that the stochastic factor driving the asset value in AT1P is more directly linked to the counterparty's default time (through Eq.(4.20)) than the stochastic intensity in expOU, AT1P's correlation parameter has a stronger impact than expOU's correlation on WWR.

4.4.4 EQUITY WWR: DEVALUATION IMPACT

The impact of the devaluation factor γ^S on the CVA of a portfolio constituted of a single 1-year expiry at-the-money equity put option is showed in Figure 4.4. The maximum CVA values that could be produced through correlation both in this reduced-form framework (for negative equity/credit correlation) and in AT1P (for positive equity/equity correlation) are highlighted in the chart.

The chart illustrates an unsurprising behaviour of CVA — and that applies to credit modelling in general: the more a parameter is directly linked to the default time definition, the higher its impact in terms of WWR. This phenomenon is akin to tranche pricing, where correlation on stochastic defaults intensities turns up being a much less effective mechanism to price CDO tranche than a copula approach (see, for example, ¹⁵).

4.4.5 FX WWR: FX VEGA

In this section, we show a less intuitive, yet of high practical importance, effect that correlation has on CVA pricing. For this example, we considered a portfolio constituted of a single cash payment that is settled in a cur-

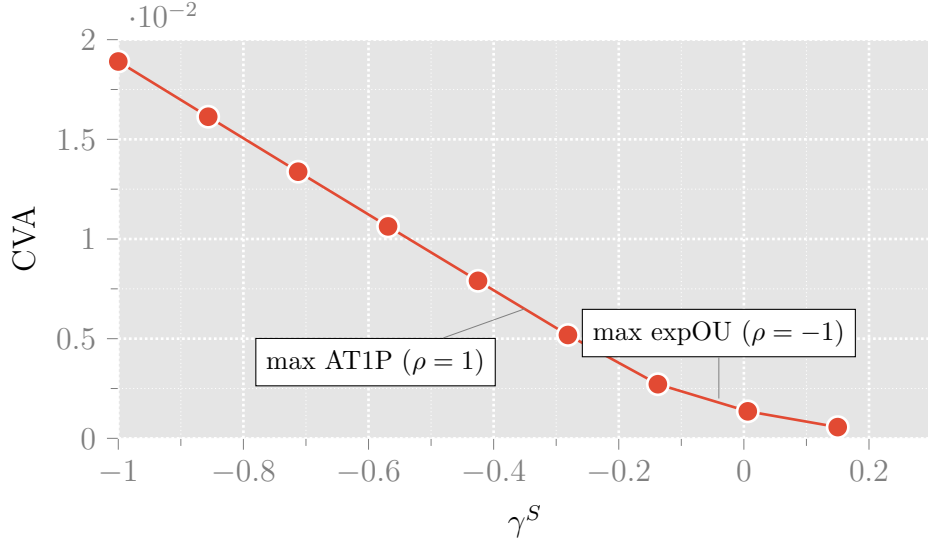


Figure 4.4: Devaluation jump impact in reduced-form (BK) approach.

rency different from the one in which the numeraire is denominated (like in Eq.(4.11)). We used a reduced-form approach to calculate the CVA with no devaluation jumps on the FX rate. The correlation impact on FX vega has been plot on Figure 4.5, showing that, when FX and credit are independent, the CVA has no FX Vega. This is not the case when we introduce correlation between FX and credit. The practical impact of having or not having correlation/dependence is in this case quite relevant as it would drastically change the hedging strategy to be used to manage the CVA risk.

We can explain the results above in formulas: when FX and credit are independent — and considering zero interest rates for simplicity — the CVA from Eq. (4.11) can be written as:

$$\text{CVA}_0 = \mathbb{E}_0 [X_\tau \mathbb{1}_{\{\tau \leq T\}}] = \int_0^T \mathbb{E}_0 [X_s] \mathbb{Q}(s \leq \tau < s + ds) = X_0 \mathbb{Q}(\tau \leq T)$$

showing in fact no dependence on $(X_t, t \geq 0)$ dynamics. It is worth nothing that the same would happen also for more complicated derivatives — as far as their only dependence on the FX rate is given by a mismatch in the pricing/payment currencies. Let us consider a derivative paying off an amount $\phi(Y_T)$ at T if the counterparty has not defaulted before T , where $(Y_t, t \geq 0)$

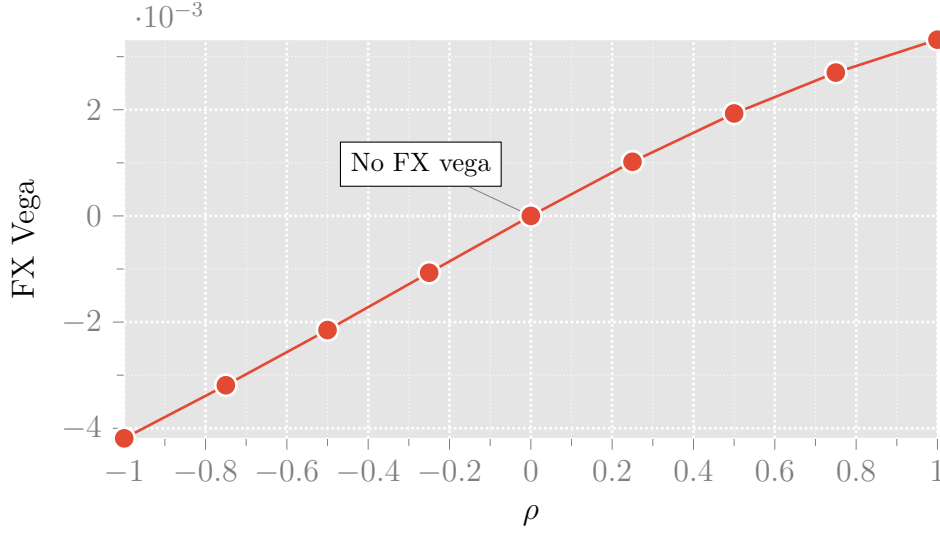


Figure 4.5: Correlation impact on FX vega in a reduced-form model (BK).

is a non-FX market risk factor. CVA can be written in this case as:

$$\begin{aligned}
 \text{CVA}_0 &= \mathbb{E}_0 [\phi(Y_T) X_\tau \mathbb{1}_{\{\tau \leq T\}}] = \int_0^T \mathbb{E}_0 [\phi(Y_s) X_s] \mathbb{Q}(s \leq \tau < s + ds) \\
 &= X_0 \int_0^T \mathbb{E}_0^f [\phi(Y_s)] \mathbb{Q}(s \leq \tau < s + ds)
 \end{aligned}$$

where we denoted as $\mathbb{E}_0^f[\cdot]$ the expected value calculated in the payment currency pricing measure. The formula shows, again, no dependence on $(X_t, t \geq 0)$ dynamics.

4.5 CONCLUSIONS

In the present chapter, we investigated different modelling approaches to account for the credit/market dependence in CVA pricing. We focused on two distinct cases, one where FX was the only market risk-factor and one where equity was the only market risk-factor. In both cases, we showed how to calculate CVA in a PDE framework using a reduced-form approach for credit modelling and we investigated the impact on WWR coming from the inclusion of a default-driven devaluation jump on the market risk-factor

dynamics. In the equity case, furthermore, we showed how a structural approach to credit modelling can also be effectively used to model WWR.

Consistently with our intuition, the more directly the credit risk component is linked to the market risk component, the higher is the variation in WWR that can be achieved. In particular, delegating the whole credit/market dependence to instantaneous correlation provides the weakest form of WWR, while having a jump-to-default effect on the market risk factor provides the strongest one, with the structural approach lying somewhere in between these two extreme cases, in the equity case.

In future works, we plan to include testing with more realistic portfolios, where impacts coming from variations on moneyness can be studied and, possibly, we will introduce a modelling approach that could consistently account for the observed skew in the market risk factor.

5

CONCLUSION

Different modelling approaches to price and risk-manage hybrid derivatives with a component of credit risk have been investigated from multiple angles.

In particular, we discussed the problem of correctly modelling the dependence between equity and credit risk in the context of CoCo bonds pricing in Chapter 2. This gave us the opportunity to study in detail, on the one hand, the calibration problem of the AT1P model, and, on the other hand, the opportunity to build an AT1P-based model that is also able to handle regulatory capital as one of its inputs. We investigated the model-implied correlation between default time and conversion time and showed how to extend the model to account for a given correlation, if needed.

The problem of modelling the dependence between a reference entity's credit risk and an FX rate has been tackled in the context of quanto CDS pricing in Chapter 3. Market quotes for CDSs on Italy are available (in the form of par-spreads) for different paying currencies. After the Euro-debt crisis that happened in 2012, the basis between EUR-denominated and USD-denominated par-spreads has considerably increased. We showed that a reduced form model where the only source of credit/market dependence structure is represented by the instantaneous correlation between the Brownian motions

driving the FX rate and the hazard rate is not able to account for the more extreme values of the observed currency basis. In doing so, we derived the relevant pricing equations and solved them numerically using a Finite Difference Method approach. The resulting Cauchy problem is three-dimensional (in space), with state variables given by the FX rate realisation, the hazard rate realisation, and the current default/non-default state. We used a dimension reduction technique to replace such three-dimension system with two, coupled, two-dimension PDE systems. However, given the specific structure of the payoff involved in this problem, the solution to one of the two PDE systems is identically null, therefore, only one of such two-dimensional systems is effectively used.

Furthermore, it has been showed how to use the same technique to derive the pricing equation for a model where the FX-rate jumps upon the default of the reference entity. In this second modelling approach, the link between the jump time of the FX-rate and the hazard rate provides a much stronger form of dependence between market and credit risk-factors. We back-tested this model using three years of market data for quanto CDSs on Italy, with the time-window starting in 2011 and ending in 2013. For each day in this time range, we calibrated the jump-diffusion FX/credit model to real market data, showing that, differently from the purely diffusive approach described before, this model is always able to explain the observed currency basis. Moreover, we derived an approximated formula to link the magnitude of the FX-rate jump to the observed basis, and we tested it using the same time series of data, showing that it provides a good approximation if used with short maturity CDSs.

While developing the hybrid FX/credit models discussed above, we studied a symmetry property of FX-rate models. An FX-rate is naturally linked to the Radon-Nikodym derivative that provides the change of measure between the risk-neutral probability measures linked to the two currencies that are part of the FX-rate pair. In the case of deterministic rates, the whole stochasticity of the Radon-Nikodym derivative comes from the FX-rate dynamics. There exist FX-rate dynamics specifications such that:

- i) the Radon–Nikodym derivative that is used to change from risk–neutral measure $\mathbb{A}$ to $\mathbb{B}$ is a $\mathbb{A}$ –martingale;
- ii) the dynamics of the reciprocal rate — as derived from the original one — is such that the corresponding to the Radon–Nikodym derivative (that is used to change from risk–neutral measure $\mathbb{B}$ to $\mathbb{A}$) is a $\mathbb{B}$ –martingale.

Such invariance property is desirable because it enforces consistency in FX–rate modelling. In a deterministic–rates setting, it is satisfied, for example, by FX–rates whose dynamics is given by Geometric Brownian Motions (GBM). It is not satisfied, instead, by FX–rates whose dynamics is given by a Constant Elasticity of Variance (CEV) process or by many stochastic volatility specifications. We proved that an FX–rate specified as a GBM that jumps upon default of a reference entity satisfies said property.

Recently, the work presented in Chapter 3 has been extended to include jump–to–defaults in interest rates, too. The work is available as a pre–print, see Itkin et al. ⁴⁸.

Finally, in Chapter 4, we linked the techniques used in the previous chapters for the purpose of *wrong–way risk* (WWR) modelling in the context of unilateral CVA. In our opinion, this is one of the more relevant applications in the industry for the solutions proposed throughout the rest of this thesis. This is because the currency linked to the risk–neutral pricing measure used for CVA calculations does not always coincide with the currency in which CDSs of the counterparties are denominated. A way of consistently modelling quanto CDSs is therefore needed if one wants to transfer the information provided by market CDS quotes into CVA prices. We showed how, even in the case of simple portfolios, and without taking into account more sophisticated frameworks like bilateral CVA or more realistic closeout conventions or collateral modelling techniques, the range of values that a CVA can take is extremely broad. Moreover, in the case of reduced–form approaches, we showed the full application of the dimension reduction technique illustrated in Chapter 3. By “full” we mean that in this second application the two PDE systems are both needed as neither of their solutions vanishes by construction as it instead happens in the case of quanto CDS pricing.

The experiments carried out in Chapter 4 confirmed our intuition in terms of “wrong-way risk power” of the modelling approaches taken into account: the more directly a given model links the default event to the portfolio’s underlying, the wider is the range of wrong-way risk values that can be produced by it (by suitably changing its main dependence-regulating parameter). In particular, in our test results, the smallest variability of WWR was achieved by a reduced-form approach where both market and credit risk-factors have continuous trajectories and where they are only linked through an instantaneous correlation between Brownian motions. AT1P, being a model where equity and the default event depend on the same risk-factor, showed a greater range of WWR values. The biggest variability of WWR has been showed when we adopted a jump-upon-default approach to model the market risk-factor.

The work started in this thesis can be extended in several ways. We mention here what we think are the three main areas that can be immediately explored further:

- i) from a theoretical standpoint, a setup involving multiple FX rates could be considered. This would give the opportunity to study the impact of jump-to-defaults on multiple invariance properties of the underlying FX models. In addition to the invariance under inversion explored in the present work, every triplet of FX rates, X_1 , X_2 , and X_3 , involving three currencies is expected to satisfy a “triangle” rule, that is $X_1 X_2 = X_3$. A potential research theme would be to investigate if and how this translates into dynamics constraint on the stochastic processes used to model the FX rates.
- ii) In all the modelling approaches illustrated throughout this work, we neglected an important aspect of the market risk-factors (either equity or FX rates in our case): the skew modelling. More realistic volatility assumptions can be made for the equity or FX rate components, trying to capture the marginal distributions implied by the corresponding vanilla options prices.
- iii) Alternative ways to enhance the dependence between credit and market risk-factors can be investigate. In particular, allowing both credit and

market risk-factors to be jump-diffusion processes (as opposed to only the market risk-factors) could represent an alternative source of covariation between them. Jumps with common jump times are expected to provide the maximum co-dependence in such approach. A more general case would be the one where each risk-factor has its own Poisson component and where the intensities of the Poisson processes are stochastic and correlated. The impact of correlation between the stochastic intensities of their jumps could be studied.

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SOME RESULTS ON COX MODELLING

The content of this chapter is a review of some results originally presented in Jeanblanc et al. ⁴⁹

A.1 PROOF OF THEOREM 7

Theorem. *If $(h_t, t \geq 0)$ is an $(\mathcal{F}_t, t \geq 0)$ -predictable (bounded) process, then the process $(M_t, t \geq 0)$, where $M_t = D_t - \int_0^{\tau \wedge t} \lambda_s \, ds$ is a $(\mathcal{G}_t, t \geq 0)$ -martingale.*

Proof. Let us consider a payout upon default that is adapted with respect to $(\mathcal{F}_t, t \geq 0)$, that is $h_\tau = \mathbb{1}_{\{\tau > t\}} B_t$, with $B_t \in \mathcal{F}_t$, for every $t > 0$. Let us

also consider a case where $T > s > t$. It holds:

$$\begin{aligned}
\mathbb{E}[h_\tau | \mathcal{F}_t] &= \mathbb{E}[\mathbb{1}_{\{\tau > s\}} B_s | \mathcal{F}_t] = \mathbb{E}\left[\mathbb{E}[\mathbb{1}_{\{\tau > s\}} B_s | \mathcal{F}_T] \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[B_s \mathbb{E}[\mathbb{1}_{\{\tau > s\}} | \mathcal{F}_T] \middle| \mathcal{F}_t\right] = \mathbb{E}\left[B_s e^{-\Lambda(s)} \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[B_s \int_s^\infty \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_t\right] = \mathbb{E}\left[B_s \int_0^\infty \mathbb{1}_{\{u > s\}} \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_0^\infty B_s \mathbb{1}_{\{u > s\}} \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_0^\infty h_u \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_t\right].
\end{aligned}$$

Due to Corollary 6, we have that

$$\mathbb{E}[h_\tau | \mathcal{G}_t] = \mathbb{1}_{\{\tau > t\}} e^{\Lambda(t)} \mathbb{E}\left[\int_0^\infty h_u \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_t\right] + \mathbb{1}_{\{\tau \leq t\}} h_\tau. \quad (\text{A.1})$$

Let us now apply the above result to $h_\tau := \Lambda(t \wedge \tau)$. Let us consider its conditional expectation with respect to $\mathcal{G}_s$ with $s \leq t$. We have:

$$\mathbb{E}[\Lambda(t \wedge \tau) | \mathcal{G}_s] = \mathbb{E}[\Lambda(t) \mathbb{1}_{\{t < \tau\}} | \mathcal{G}_s] + \mathbb{E}[\Lambda(\tau) \mathbb{1}_{\{t \geq \tau\}} | \mathcal{G}_s]$$

The first term in the RHS can be evaluated applying Corollary 6, while the second term can be evaluated using Eq.(A.1). We obtain:

$$\begin{aligned}
\mathbb{E}[\Lambda(t \wedge \tau) | \mathcal{G}_s] &= \mathbb{1}_{\{\tau > s\}} e^{\Lambda(s)} \mathbb{E}\left[\Lambda(t) e^{-\Lambda(t)} \middle| \mathcal{F}_s\right] \\
&\quad + \mathbb{1}_{\{\tau > s\}} e^{\Lambda(s)} \mathbb{E}\left[\int_s^t \Lambda(u) \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_s\right] + \mathbb{1}_{\{\tau \leq s\}} \Lambda(\tau).
\end{aligned}$$

Let us now consider the expected value, conditioned to $\mathcal{G}_s$, of the difference

$\Lambda(t \wedge \tau) - \Lambda(s \wedge \tau)$:

$$\begin{aligned} \mathbb{E} \left[\Lambda(t \wedge \tau) - \Lambda(s \wedge \tau) \middle| \mathcal{G}_s \right] &= \mathbb{1}_{\{\tau > s\}} e^{\Lambda(s)} \mathbb{E} \left[\Lambda(t) e^{-\Lambda(t)} \middle| \mathcal{F}_s \right] \\ &\quad + \mathbb{1}_{\{\tau > s\}} e^{\Lambda(s)} \mathbb{E} \left[\int_s^t \Lambda(u) \lambda_u e^{-\Lambda(u)} du \middle| \mathcal{F}_s \right] - \mathbb{1}_{\{\tau > s\}} \Lambda(s). \end{aligned}$$

The integral term inside the second expectation can be evaluated as

$$\begin{aligned} \int_s^t \Lambda(u) \lambda_u e^{-\Lambda(u)} du &= \int_{\Lambda(s)}^{\Lambda(t)} \Lambda e^{-\Lambda} d\Lambda \\ &= \Lambda e^{-\Lambda} \Big|_{\Lambda(s)}^{\Lambda(t)} - \int_{\Lambda(s)}^{\Lambda(t)} e^{-\Lambda} d\Lambda \\ &= \Lambda(s) e^{-\Lambda(s)} - \Lambda(t) e^{-\Lambda(t)} + e^{-\Lambda(s)} - e^{-\Lambda(t)}, \end{aligned}$$

so that

$$\mathbb{E} \left[\Lambda(t \wedge \tau) - \Lambda(s \wedge \tau) \middle| \mathcal{G}_s \right] = \mathbb{1}_{\{\tau > s\}} \left(1 - e^{\Lambda(s)} \mathbb{E} \left[e^{-\Lambda(t)} \middle| \mathcal{F}_s \right] \right).$$

Due to Corollary 6, the right-hand side of this equation is also equal to $\mathbb{E} [D_t - D_s | \mathcal{G}_s]$. This proves the theorem's claim. $\square$

A.2 PROOF OF COROLLARY 9

Corollary. *In case of CDS par-spreads term-structures $T \mapsto S^M(T)$ where*

$$S^M(t) = \bar{S}, t \in \{T_0, \dots, T_N = \bar{T}\},$$

and where all the corresponding CDSs have the same recovery rate, R , a PWCHR model $T \mapsto \lambda(T) = \sum_{i=1}^N \mathbb{1}_{\{T_{i-1} < T < T_i\}} h_i$ can be calibrated in $t < \bar{T}$ by setting

$$h_i = \frac{\bar{S}}{1 - R}, \quad i = 1, \dots, N. \tag{A.2}$$

Proof. We can prove Eq. (A.2) by induction. The base case follows from the

previous Proposition, with $T^{\min} = T_0$:

$$h_1 = \frac{\bar{S}}{1 - R}.$$

The inductive step can be proved by writing the price equation of the T_n -maturity CDS as a function of the T_{n-1} -maturity one and recalling that the price of the T_{n-1} -maturity is given by

$$V(T_{n-1}, \bar{S}) = \bar{S} \int_0^{T_{n-1}} Q(0, s) D(0, s) \, ds - (1 - R) \int_0^{T_{n-1}} D(0, s) \, d_{\mathbb{S}} Q(0, s).$$

The above is worth zero by definition of par-spread. The price of the T_n -maturity CDS can be written as

$$\begin{aligned} V(T_n, \bar{S}) &= \bar{S} \int_0^{T_n} Q(0, s) D(0, s) \, ds - (1 - R) \int_0^{T_n} D(0, s) \, d_{\mathbb{S}} Q(0, s) \\ &= \bar{S} \int_0^{T_{n-1}} Q(0, s) D(0, s) \, ds + \bar{S} \int_{T_{n-1}}^{T_n} Q(0, s) D(0, s) \, ds \\ &\quad - (1 - R) \int_0^{T_{n-1}} D(0, s) \, d_{\mathbb{S}} Q(0, s) - (1 - R) \int_{T_{n-1}}^{T_n} D(0, s) \, d_{\mathbb{S}} Q(0, s) \\ &= V(T_{n-1}, \bar{S}) + \bar{S} \int_{T_{n-1}}^{T_n} Q(0, s) D(0, s) \, ds - (1 - R) \int_{T_{n-1}}^{T_n} D(0, s) \, d_{\mathbb{S}} Q(0, s). \end{aligned}$$

Given that $V(T_{n-1}, \bar{S}) = 0$, and that $d_{\mathbb{S}} Q(0, s) = h_n Q(0, s) \, ds$ for $s \in (T_{n-1}, T_n)$, the above can only be zero if $h_n = \frac{\bar{S}}{1 - R}$. $\square$

B

SOME RESULTS ON AT1P MODEL

The content of this chapter is a review of some results originally presented in Rapisarda⁶⁵ and Brigo et al.¹⁹, and it is mainly based on the material found in Duffy³¹, Evans³⁸, and Kwok⁵².

B.1 HOW TO EVALUATE THE SURVIVAL PROBABILITY IN AT1P

Proof. We can split the proof in several steps.

First, we express the survival probability as

$$\mathbb{Q}_t(\tau > T) = \mathbb{E}_t \left[\mathbb{1}_{\{\tau > T\}} \right]. \quad (\text{B.1})$$

Given the strong Markov property of $(V_t, t \geq 0)$, the expected value on the right-hand side of the above equation can be expressed as a function of the asset-value at time t , $p(t, V_t)$.

the discounted survival probability, $p(t, V_t)/B(t)$, is a conditioned expectation

and a discounted price and, as such, must be a martingale, therefore:

$$\partial_t p = \mu_V V \partial_V p - \frac{1}{2} V^2 \sigma^2 \partial_{VV} p + r p = 0, \quad t \in [0, T], V \in \mathbb{R}^+ \quad (\text{B.2a})$$

$$p(t, \hat{H}(t)) = 0, \quad t \in [0, T], \quad (\text{B.2b})$$

$$p(T, V) = \mathbb{1}_{\{(V > \hat{H}(T))\}}, \quad V \in \mathbb{R}^+. \quad (\text{B.2c})$$

the above PDE system can be made linear in t and V by applying the following change of coordinates

$$\tau(t, V) = T - t, \quad (\text{B.3})$$

$$x(t, V) = \log \frac{V}{\hat{H}(t)}. \quad (\text{B.4})$$

This is an invertible system of equation, and the old coordinates can be expressed in terms of the new one by using the mappings

$$t = \hat{t}(\tau, x) = T - \tau, \quad (\text{B.5})$$

$$V = \hat{V}(\tau, x) = \hat{H}(T - \tau) e^x. \quad (\text{B.6})$$

Let us denote by g the price of the discounted survival probability expressed using the new coordinates. This is given by:

$$g(\tau, x) := p(\hat{t}, \hat{V}) = p(T - \tau, \hat{H}(T - \tau) e^x). \quad (\text{B.7})$$

From (B.7), we can calculate the partial derivatives of g with respect the new set of coordinates in terms of the partial derivatives of p with respect the old set of coordinates:

$$\begin{aligned} \partial_\tau g &= \frac{\partial \hat{t}}{\partial \tau} \partial_t p + \frac{\partial \hat{V}}{\partial \tau} \partial_V p = -\partial_t p - \hat{H}' e^x \partial_V p \\ &= -\partial_t p - (\mu_V - B\sigma^2(t)) V \partial_V p, \end{aligned} \quad (\text{B.8})$$

$$\partial_x g = \frac{\partial \hat{V}}{\partial x} \partial_V p = V \partial_V p, \quad (\text{B.9})$$

$$\partial_{xx} g = \partial_x \hat{V} \partial_V p + V \partial_x \hat{V} \partial_{VV} p = V \partial_V p + V^2 \partial_{VV} p. \quad (\text{B.10})$$

from which $V^2 \partial_{VV} p = \partial_{xx} g - \partial_x g$, therefore (B.2) becomes

$$\partial_\tau g = \frac{1}{2} \sigma^2 \partial_{xx} g + \left(B - \frac{1}{2} \right) \sigma^2 \partial_x g - r g \quad (\text{B.11a})$$

$$g(\tau, 0) = 0 \quad (\text{B.11b})$$

$$g(0, x) = \mathbb{1}_{\{V > \hat{H}(T)\}} = \mathbb{1}_{\{\hat{H}(T)e^x > \hat{H}(T)\}} = \mathbb{1}_{\{x > 0\}} \quad (\text{B.11c})$$

We can now calculate the Green's function associated to the system (B.11). Let us denote by L the differential operator $L = \partial_\tau - 1/2 \sigma^2 \partial_{xx} - (B - 1/2) \partial_x + r \mathbb{1}$, then its *free-space* solution $G(\tau, x, x_0)$ solves

$$L[G] = 0 \quad (\tau, x, x_0) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \quad (\text{B.12a})$$

$$G(0, x, x_0) = \delta(x - x_0) \quad (x, x_0) \in \mathbb{R} \times \mathbb{R}_+ \quad (\text{B.12b})$$

The solution of (B.12) is given by

$$G_0(\tau, x, x_0) = \frac{e^{-r\tau}}{\sigma \sqrt{2\pi\tau}} e^{-\frac{\left(x + \left(B - \frac{1}{2}\right)\sigma^2\tau - x_0\right)^2}{2\sigma^2\tau}}. \quad (\text{B.13})$$

The *method of images*^{31,52} can be used to find the solution satisfying the boundaries conditions. The Green's function we are looking for is defined only for $x \in \mathbb{R}_+$ and must satisfy the following boundary condition

$$G(\tau, 0, x_0) = 0, \quad (\tau, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \quad (\text{B.14})$$

while G_0 was defined for x values in the whole $\mathbb{R}$ and obviously violates (B.14). In fact $G_0(\tau, \cdot, x_0)$ is a non-null function along the whole real line. It is, however, symmetric with respect to the point $x = 1/4 - x_0/2\sigma^2\tau$ and we can exploit this feature, together with the fact that we are dealing with a linear problem, to work out a solution satisfying the required boundary conditions.

Given the linearity of the problem, we can look for a solution $G = G_0 + \tilde{G}_0$

where G_0 is the free-space solution of the problem in (B.12), while G solves

$$L[G(\tau, x, x_0)] = 0, \quad (\tau, x, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \quad (\text{B.15a})$$

$$G(0, x, x_0) = \delta(x - x_0) \quad (x, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad (\text{B.15b})$$

$$G(\tau, 0, x_0) = 0 \quad (\tau, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+. \quad (\text{B.15c})$$

To do so, we must look for a solution $\tilde{G}_0$ such that the resulting distribution G satisfies the desired boundary conditions, that is

$$G(\tau, 0, x_0) = G_0(\tau, 0, x_0) + \tilde{G}_0(\tau, 0, x_0) = 0, \quad (\tau, x_0) \in \mathbb{R}_+ \times \mathbb{R}_+ \quad (\text{B.16})$$

A natural candidate for such a correction is given by the solution of

$$L[\tilde{G}_0(\tau, x, x_0)] = 0, \quad (\tau, x, x_0) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \quad (\text{B.17a})$$

$$\tilde{G}_0(0, x, x_0) = -\delta(x + x_0) \quad (x, x_0) \in \mathbb{R} \times \mathbb{R}_+, \quad (\text{B.17b})$$

that is given by

$$\tilde{G}_0(\tau, x, x_0) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} e^{-\frac{\left(x + \left(B - \frac{1}{2}\right)\sigma^2\tau + x_0\right)^2}{2\sigma^2\tau}}. \quad (\text{B.18})$$

Indeed, G satisfies (B.15a) due to the linearity of the operator L ,

$$L[G] = L[G_0 + \tilde{G}_0] = L[G_0] + L[\tilde{G}_0] = 0, \quad (\text{B.19})$$

and, given that we restricted ourselves to the positive semi-axis, G also satisfies the initial condition (B.15b). The boundary condition is satisfied in $\tau = 0$ due to the symmetry of the problem. For later times, however, the distribution $\tilde{G}_0(\tau, \cdot, x_0)$ drifts away and it is not equal to $-G_0(\tau, \cdot, x_0)$ anymore.

$\tilde{G}_0$ can be corrected in a way that G still satisfies Eq (B.19) (that is affected only by the dependence of G on τ and x) and the initial condition by multiplying it by a function $\tilde{g}_0(x_0)$. The boundary condition in this case

reads

$$0 = G(\tau, 0, x_0) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \left(e^{-\frac{\left((B-\frac{1}{2})\sigma^2\tau-x_0\right)^2}{2\sigma^2\tau}} - \tilde{g}_0(x_0)e^{-\frac{\left((B-\frac{1}{2})\sigma^2\tau+x_0\right)^2}{2\sigma^2\tau}} \right) \quad (\text{B.20})$$

$$= \frac{e^{-r\tau-\frac{x_0^2}{2\sigma^2\tau}-(B-\frac{1}{2})\sigma^2\tau}}{\sigma\sqrt{2\pi\tau}} \left(e^{(B-\frac{1}{2})x_0} - \tilde{g}_0(x_0)e^{(B-\frac{1}{2})x_0} \right) \quad (\text{B.21})$$

from which $\tilde{g}_0(x_0) = e^{(2B-1)x_0}$ so that

$$G(\tau, x, x_0) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \left(e^{-\frac{\left(x+(B-\frac{1}{2})\sigma^2\tau-x_0\right)^2}{2\sigma^2\tau}} - e^{-\frac{\left(x+(B-\frac{1}{2})\sigma^2\tau+x_0\right)^2}{2\sigma^2\tau}+(2B-1)x_0} \right). \quad (\text{B.22})$$

The solution to the system (B.11) can now be calculated through a convolution product between the boundary condition and the Green function just calculated:

$$g(\tau, x) = \int_{-\infty}^{\infty} \mathbb{1}_{\{\hat{V}(0, x_0) > \hat{H}(T)\}} G(\tau, x, x_0) dx_0 \quad (\text{B.23})$$

The first term in the integral is equal to

$$\mathbb{1}_{\{\hat{V}(0, x_0) > \hat{H}(T)\}} = \mathbb{1}_{\{\hat{H}(T)e^{x_0} > \hat{H}(T)\}} = \mathbb{1}_{\{x_0 > 0\}}$$

so that Eq. (B.29) can be written as

$$\begin{aligned} g(\tau, x) &= \int_0^{\infty} G(\tau, x, x_0) dx_0 \\ &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} \left(e^{-\frac{\left(x+(B-\frac{1}{2})\sigma^2\tau-x_0\right)^2}{2\sigma^2\tau}} - e^{-\frac{\left(x+(B-\frac{1}{2})\sigma^2\tau+x_0\right)^2}{2\sigma^2\tau}+(2B-1)x_0} \right) dx_0 \end{aligned} \quad (\text{B.24})$$

□

B.2 HOW TO EVALUATE THE STOCK-PRICE IN AT1P

As explained in Section 2.3, in AT1P an entity can default at any time and not only at maturity. If we assume that the debt still has a clear single final maturity T and that early default is given by safety covenants, it is not unreasonable to model equity as an option on the firm value with maturity T that is killed if the default barrier is reached before T . See also the Equity chapter in ¹⁹. Namely, we calculate the stock price E_t (in this framework) as a down-and-out European call option, that is

$$E_t = \beta(t) \mathbb{E}_t \left[\frac{\left(V_T - \hat{H}(T) \right)^+ \mathbb{1}_{\{\tau > T\}}}{\beta(T)} \right] = f(t, V_t). \quad (\text{B.25})$$

We can use (B.25) to calculate the stock price both inside a simulation of (2.7) and in the calibration procedure.

A closed form solution for the price of the option is given for example in ⁶⁵, see also the Equity chapter in ¹⁹.

In case $B = 0$ (and for simplicity $q = 0$), then

$$f = (V - \hat{H}(t))^+. \quad (\text{B.26})$$

Proof. The proof of the result above can be obtained by following the same steps from Appendix B.1. The only two differences with respect to the proof outlined there lie in the value of B , now set to 0, and in a different terminal condition. As a result, the PDE system that gives the solution of

$$g(\tau, x) := p(\hat{t}, \hat{V}) = p(T - \tau, \hat{H}(T - \tau)e^x).$$

is given by

$$\partial_\tau g = -\mu_V \partial_x g + \mu_V \partial_x g + \frac{1}{2} \sigma^2 (\partial_{xx} g - \partial_x g) - r g \quad (\text{B.27a})$$

$$g(\tau, 0) = 0 \quad (\text{B.27b})$$

$$g(0, x) = (V - \hat{H}(T))^+ = (\hat{H}(T)e^x - \hat{H}(T))^+ = \hat{H}(T)(e^x - 1)^+ \quad (\text{B.27c})$$

The Green function associated to it is given by

$$G(\tau, x, x_0) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \left(e^{-\frac{(x-\frac{1}{2}\sigma^2\tau-x_0)^2}{2\sigma^2\tau}} - e^{-\frac{(x-\frac{1}{2}\sigma^2\tau+x_0)^2}{2\sigma^2\tau}-x_0} \right) \quad (\text{B.28})$$

so that Convolution to get the price.

$$g(\tau, x) = \int_{-\infty}^{\infty} (\hat{V}(0, x_0) - \hat{H}(T))^+ G(\tau, x, x_0) dx_0 \quad (\text{B.29})$$

The first term in the integral is equal to

$$(\hat{V}(0, x_0) - \hat{H}(T))^+ = (\hat{H}(T)e^{x_0} - \hat{H}(T))^+ = \hat{H}(T)(e^{x_0} - 1)^+ \quad (\text{B.30})$$

so that it is non-zero only for $x_0 \geq 0$ and (B.29) can be written as

$$\begin{aligned} g(\tau, x) &= \int_0^{\infty} \hat{H}(T)(e^{x_0} - 1) G(\tau, x, x_0) dx_0 \\ &= \hat{H}(T) \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_0^{\infty} (e^{x_0} - 1) \left(e^{-\frac{(x-\frac{1}{2}\sigma^2\tau-x_0)^2}{2\sigma^2\tau}} - e^{-\frac{(x-\frac{1}{2}\sigma^2\tau+x_0)^2}{2\sigma^2\tau}-x_0} \right) dx_0 \end{aligned} \quad (\text{B.31})$$

Straightforward, but somehow long, calculations lead to the final result:

$$g(\tau, x) = \hat{V}(\tau, x) - \hat{H}(T - \tau). \quad (\text{B.32})$$

□