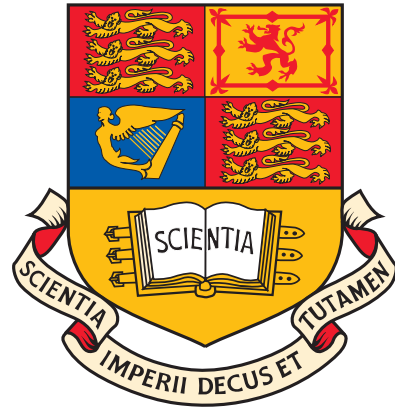


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RECENT ADVANCES IN CLASSICAL ELECTROMAGNETIC THEORY

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Abstract

The early Sections of the present Thesis utilise a metric-free and connection-free approach so as to derive the foundations of classical electrodynamics. More specifically, following a tradition established by Kottler [65], Cartan [14] and van Dantzig [137], Maxwell's theory is introduced without making reference to a notion of distance or parallel transport. With very few exceptions, the relevant concepts are derived from first principles. Indeed, Maxwell's theory is constructed starting from three experimentally justified axioms: (i) electric charge is conserved, (ii) the force acting on a test charge due to the electromagnetic field is the standard Lorentz one, (iii) magnetic flux is conserved. To be precise, a strictly deductive approach requires that three further postulates are introduced, as explained in the manual [41] by Hehl and Obukhov. Nevertheless, a shortened formalism is observed to be adequate for the purpose of this work. In nearly all cases, the electromagnetic medium is demanded to be local and linear. Moreover, the propagation of light is studied in the approximate geometrical optics regime. Lindell's astute derivation of the dispersion equation [80] is reformulated in the widespread mathematical language of tensor indices. The method devised in Ref. [80] is integrated with the analysis due to Dahl [16] of the space encompassing the physically viable polarisations. As a result, the geometry associated with the dispersion equation is investigated with considerable rigour. From the literature it is known that, to a great extent, the notion of distance can be viewed as a by-product of Maxwell's theory. In fact, imposing that the constitutive law is electric-magnetic reciprocal and skewon-free determines, albeit non-uniquely, a Lorentzian metric. A novel proof of this statement is examined. In addition, the unimodular forerunner of electric-magnetic reciprocity, defined in earlier works by Lindell [79] and Perlick [112], is shown to preserve the energy-momentum tensor.

Dedicata al babbo.



Umberto e Mansueto Favaro in visita alla tomba di Petrarca (1930 circa).

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Statement of Originality: The material presented hereinafter that is not attributed to a different source is the result of my own research in electromagnetism.

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Chapter 1

Introduction

1.1 Some history

The concept that gravity can affect electromagnetic propagation is often attributed to Einstein. Yet, as early as 1784, Cavendish investigated the bending of light due to Newton’s gravitational force [145]. In 1801 von Soldner calculated, under similar assumptions, the deflection angle for a light ray grazing the Sun’s surface. A deviation of 0.84 arcseconds was obtained [59]. This result was not known to Einstein, whose 1911 paper [26] elegantly derived a matching figure of 0.83 arcseconds (with an implicit Newtonian approximation). Four years later, Einstein could rely on a well-formed General Theory of Relativity, and thus double the deflection angle to 1.7 arcseconds, since he took care of the spatial curvature of spacetime [27]. This prediction was verified by the measurements that Eddington *et al.* collected during the solar eclipse of May 1919 [24]¹. Henceforward, the existence of an interaction between gravity and light became an established fact – even in popular culture.

The above works considered the coupling of Electromagnetism to gravity (effectively) within the realm of Ray Optics. In 1916 Einstein went beyond this approx-

¹Earman and Glymour, amongst others, argue that Eddington’s data analysis was biased [25]. This view is opposed, for instance, by Kennefick [63]. Regardless of the controversy’s outcome, modern observations confirm Einstein’s ray tracing with an accuracy of about 0.02% (Ref. [74]).

imation, casting Maxwell’s equations in a form compatible with General Relativity [28]. Albeit implicitly, one crucial insight emerged: *Maxwell’s equations could in fact be formulated as independent of the gravitational potential, that is, the metric.* Equivalently, gravity could affect the propagation of light, but only through the constitutive law (which will soon be defined). An identical conclusion was tacitly implied by Weyl’s 1918 treatment (Ref. [142], Ch. 25).

The statement that Maxwell’s equations were provided with a metric-free form became explicit with an article by Murnaghan (Ref. [100], Yr. 1921). However, it was Kottler who first studied this property in depth. In 1922, he showed that the foundations of Electrodynamics could be derived without reference to the gravitational potential *or even the affine structure* [65]. One year later, Cartan’s influential paper [14] discussed the same concept, but in an innovative way. Maxwell’s equations were represented in terms of *differential (exterior) forms*, which conveniently required no metric or connection at all. Cartan’s exterior calculus was a refined and modern instrument, but did not prove popular at first. Many pioneering works in Electromagnetism thus continued to successfully exploit the tensor formalism.

The investigation on the metric-free and connection-free structure of Maxwell’s equations found a capable advocate in van Dantzig. In his 1934 article [137] he commented: “Maxwell’s equations are totally independent of any metrical relation either Riemannian or conformal. It is remarkable that not only no fundamental tensor or tensor-density, but also no connection, neither Riemannian nor projective nor conformal, is needed for writing down the equations”. Subsequent research on this topic was carried out by Schrödinger (Ref. [120], Yr. 1950), Schouten (Ref. [119], Yr. 1951), Truesdell and Toupin (Ref. [135], Yr. 1960), Post (Ref. [113], Yr. 1962).

Undoubtedly, the developments leading to the work of Post had their roots in Einstein’s 1916 paper. This fact notwithstanding, some earlier concepts due to Mie played an equally important role. In 1910, Mie proposed to organise the electromagnetic fields in two classes [97]². The first class was formed by the intensive quantities, namely the electric field strength E and the magnetic field strength B ; the second class encompassed the extensive quantities, namely the magnetic excita-

²To the best of my knowledge, no English translation of this reference exists. As an alternative, one may read Whittaker [143] (Chapter 5) or Weyl [142] (Chapter 26 of the 1952 Dover edition).

tion $\mathcal{H}$ and the electric excitation $\mathcal{D}$ ³. Crucially, both the pair $F = (E, B)$ and its counterpart $H = (\mathcal{H}, \mathcal{D})$ were recognised to have *independent* experimental definitions. Accordingly, Maxwell's equations were most appropriately expressed in terms of the *separate* quantities H and F . As many authors later noticed, this observation immediately de-coupled Maxwell's equations from the metric. In fact, the gravitational potential could still appear only in the constitutive law, that is, only in the *relation* between H and F .

The actual *origin* of metric- and connection-freedom in Electromagnetics was first discussed by Kottler (Ref. [65], Yr. 1922). Taking a top-down approach, he *derived* Maxwell's equations from two conservation laws – namely, the continuity equation for electric charge and the closure of magnetic flux lines. These conservation statements, in turn, trivially held true whether or not a metric or affine structure was defined. Maxwell's equations, then, directly inherited the same property. The modern reformulation of Kottler's deductions, including the role of counting procedures, will soon be introduced.

1.2 A short comparison of two important books

The present Thesis is strongly influenced by the joint work of Hehl and Obukhov, and by the work of Lindell. With some exceptions, the numerous papers on electromagnetism due to these authors are summarised, respectively, in the monographs [41] and [79]. It can be argued that the differences between such books are technical, even more than conceptual. As a matter of fact, the notation and formalism of Hehl and Obukhov is often significantly distant from that of Lindell. One is thus motivated to compare the mathematical languages used in Ref. [41] and Ref. [79]. The examination of the two symbolic approaches, in turn, proves helpful so as to clarify where the present Thesis stands relative to the mentioned books.

Hehl and Obukhov perform many of their calculations in a component-free way. Nonetheless, when this choice entails a burden of notational trickery, they make use

³The ISO names are: electric field for E , magnetic flux density for B , magnetic field for $\mathcal{H}$, and electric flux density for $\mathcal{D}$. Notably, such convention does not match the way the electromagnetic fields are grouped in relativity. Hence, this work follows the terminology of Sommerfeld [128] or Hehl and Obukhov [41]. As a result, the intensive and extensive quantities are aptly distinguished.

of an arbitrary vector basis. More in detail, the key quantities are encoded as follows:

- The electromagnetic fields are viewed predominantly as differential forms. However, in some cases, the tensor-index notation is employed. For example, the magnetic field strength is considered mostly as a 2-form B . If necessary, though, Hehl and Obukhov represent it as $B_{ab} = -B_{ba}$, where $\{a, b = 1, 2, 3\}$.
- The local and linear media are primarily described in tensor components, e.g. $\kappa_{\alpha\beta}^{\gamma\delta}$ with $\{\alpha, \dots, \delta = 0, \dots, 3\}$. But, on occasion, they are written as maps κ .
- The algebra involving the medium is, by and large, carried out using indices.

Lindell's formalism is purely component-free. As a matter of fact, the tensor-index notation is almost absent from Ref. [79]. Instead, the author relies on an approach that merges exterior and dyadic calculus. The dual objects of differential forms, called multivectors, are also encompassed in such mathematical language. A tentative summary of how the basic quantities are represented by Lindell could be:

- The electromagnetic fields are viewed as exterior forms in nearly all scenarios.
- The local and linear media are encoded as dyadics. Every one of these objects is a symbolic sum, wherein each term is constructed by juxtaposing two vectors. Further information is provided in the classic manual [34] by Gibbs and Wilson.
- The algebra associated with the medium is developed by extending that of vectors, differential forms and multivectors. Indeed, the possibility of generalising familiar operations is one of the merits of dyadic calculus. An instructive example is the *double-wedge* product arising from the standard *wedge* product.

Remarkably, the mathematical language of Lindell allows one to derive powerful identities in a simple fashion. Nonetheless, it does require some additional effort to learn. Most likely, this is the reason why the formalism of Ref. [79] is understood only in a specific community of electrical engineers and physicists. Along with the notation, some of Lindell's findings are not as widely known as they deserve to be.

Both in terms of calculus, and in terms of content, this Thesis follows the approach of Hehl and Obukhov. Nevertheless, the present work does aim to review

and generalise at least certain techniques and discoveries due to Lindell. Some examples are: the use of multivectors in Sec. 2.3.3, the method of discontinuities in Sec. 2.4.1, the elegant derivation of the Fresnel equation in Sec. 2.4.2, the P -media and Q -media in Sec. 2.5, the unimodular reciprocity introduced in Sec. 3.2, and the mixed closure relation discussed in Sec. 3.3.

1.3 What material in the present Thesis is new?

Hereafter, a brief account is given of the material in the present Thesis which is substantially new. In particular, a number of minor findings and technical improvements are not mentioned, despite the fact the advanced reader may find these interesting. The key novelties are summarised in a list:

1. In the manual [41], Hehl and Obukhov illustrate the following property. Provided the whole of spacetime can be decomposed in sheets of constant time, the two covariant Maxwell equations can be split globally in four familiar laws: the Ampère-Maxwell law, Gauss's law for electricity, Faraday's law, and Gauss's law for magnetism. The analysis of Section 2.2.1 establishes that the *converse* statement is true. In order to extract the mentioned 3-dimensional laws from the 4-dimensional Maxwell equations globally, it is essential that spacetime admits a foliation into sheets of constant time. This proposition is subject to the topological requirement that all 1-dimensional cycles are also boundaries.
2. As discussed in Ref. [41], a local and linear medium is naturally expressed as an operator such that $\kappa(a\Phi + b\Psi) = a\kappa(\Phi) + b\kappa(\Psi)$, where $\{a, b\}$ are scalars and $\{\Phi, \Psi\}$ are 2-forms. Specifically, applying κ on itself multiple times and taking the trace gives rise to a family of invariants $\{\text{tr}(\kappa), \text{tr}(\kappa^2), \dots\}$, some of whose members play an important role in electromagnetism. In Section 2.3.2, the Cayley-Hamilton theorem for the medium operator is examined. Remarkably, a new set of invariants is thus defined, which contains the determinant $\det(\kappa)$.
3. With the help of the transpose operation and of two Poincaré isomorphisms, the adjoint $\bar{\kappa}$ of the constitutive map κ is introduced for the first time. Such adjoint proves useful so as to retrieve the standard decomposition of every

local and linear medium in a principal part, a skewon part and an axion part (Section 2.3.4). Once the map $\bar{\kappa}$ is available, it is also possible to generalise the requirement for a constitutive law to be electric-magnetic reciprocal in such a way that four closure equations are obtained, see Chapter 3 for further details.

4. Lindell's neat derivation of the electromagnetic dispersion relation [80] is reformulated using the widespread mathematical language of tensor indices, combined with some exterior algebra. The elegant procedure developed in Ref. [80] is further integrated with analysis due to Dahl [16] of the space encompassing the physically viable polarisations. As a result, the geometry associated with the dispersion equation is investigated with unprecedented rigour (Sec. 2.4.2).
5. The principal, skewon and axion parts corresponding to an arbitrary P -medium [86], and to an arbitrary Q -medium [79, 92] are evaluated in detail, Section 2.5.
6. Unless one dictates the absence of electric charges, or the existence of magnetic monopoles, the electric-magnetic reciprocity transformation is physically important for the following two reasons only [41]. First, it leaves the energy-momentum tensor invariant and, second, it is a symmetry of the Maxwell-Lorentz constitutive law for vacuum. In Section 3.2, the unimodular forerunner of electric-magnetic reciprocity, studied in earlier works by Lindell [79] and Perlick [112], is demonstrated to preserve the energy-momentum tensor too. Hence, specific calculations are performed in the direction of finding all local and linear media that are symmetric under a given unimodular transformation.
7. The requirement for a constitutive law to be electric-magnetic reciprocal is generalised, whereby a set of four closure equations is attained. A thorough analysis of the formal properties of these relations is carried out in Section 3.4.
8. Following the tradition described in Section 1.1, and following the approach of Hehl and Obukhov [41], the present Thesis examines several topics in electrodynamics from a metric- and connection-free viewpoint. Remarkably, the fact that the notion of distance is not assumed from the start motivates one to investigate whether such concept can be derived from Maxwell's theory itself. Expressed in other words, the problem consists in identifying requirements of

electromagnetic nature which specify an appropriate Lorentzian metric. For instance, demanding that the constitutive law is not only local and linear but also skewon-free and electric-magnetic reciprocal generates a valid notion of distance. Toupin [134], Schönberg [118] and Jadczyk [58] are seemingly the first to verify this important observation. In the book [41], Hehl and Obukhov analyse in great detail *two* routes leading from electric-magnetic reciprocity and the proviso of zero skewon to a Lorentzian metric. A *third*, novel derivation is studied in Chapter 3 hereinafter.

1.4 The author's publication and conference list

Peer-reviewed publications

- ◇ I. Lindell, L. Bergamin and **A. Favaro**, “Decomposable medium conditions in four-dimensional representation”, IEEE Transactions on Antennas and Propagation 60 (2012), 367–376.
- ◇ I. Lindell, A. Sihvola, L. Bergamin and **A. Favaro**, “Realization of the $D'B'$ boundary condition”, IEEE Antennas and Wireless Propagation Letters 10 (2011), 643–646.
- ◇ P. Kinsler and **A. Favaro**, “Comment on: Reply to comment on: Perfect imaging without negative refraction”, New J. Phys. 13 (2011), 028001.
- ◇ I. Lindell, L. Bergamin and **A. Favaro**, “The class of electromagnetic P -media and its generalization”, Progress In Electromagnetic Research B, PIER B 28 (2011), 143–162.
- ◇ **A. Favaro** and L. Bergamin, “The non-birefringent limit of all linear, skewonless media and its unique light-cone structure”, Ann. Phys. (Berlin) 523 (2011), 383–401.
- ◇ M. McCall, P. Kinsler and **A. Favaro**, “Reply to comment on: What is negative refraction?”, J. Mod. Optic. 57 (2010), 2103–2108.

- ◇ M. McCall, **A. Favaro**, P. Kinsler and A. Boardman, “A spacetime cloak, or a history editor”, *J. Opt.* 13 (2010), 024003.
- ◇ P. Kinsler, **A. Favaro** and M. McCall, “Four Poynting theorems”, *Eur. J. Phys.* 30 (2009), 983–993.
- ◇ **A. Favaro**, M. McCall and P. Kinsler, “Comment on: Correct definition of the Poynting vector in electrically and magnetically polarizable medium reveals that negative refraction is impossible”, *Optics Express* 17 (2009), 15167-15169.

Conference proceedings

- ◇ M. McCall, P. Kinsler, and **A. Favaro**, “All kinds of cloaks, all kinds of transformations”, in “Metamaterials: fundamentals and applications IV”, A. Boardman, N. Engheta, M. Noginov and N. Zheludev Eds., *Proceedings of SPIE* 8093 (2011), 31.
- ◇ L. Bergamin and **A. Favaro**, “Negative index of refraction, perfect lenses and transformation optics – some words of caution”, *Proceedings of URSI-EMTS* (2010), 760–763.
- ◇ M. McCall, P. Kinsler, **A. Favaro** and D. Censor, “What is negative refraction?”, in “Metamaterials: fundamentals and applications II”, M. Noginov, N. Zheludev, A. Boardman and N. Engheta Eds., *Proceedings of SPIE* 7392 (2009), 57.

Conference contributions

- ◇ **A. Favaro**, L. Bergamin, I. Lindell and Y. Obukhov, “Pre-metric electrodynamics, electric-magnetic duality and closure relations”, 2nd GIF Workshop, Jerusalem, Israel, February 2012 (Invited Talk).
- ◇ I. Lindell, A. Sihvola, L. Bergamin and **A. Favaro**, “Realization of the $D'B'$ boundary in terms of a metamaterial”, *Metamaterials* 2011, Barcelona, Spain, October 2011.
- ◇ P. Kinsler, M. McCall and **A. Favaro**, “All kinds of cloaks, all kinds of transformations: a general theory of Transformation Mechanics”, *Metamaterials* 2011,

Barcelona, Spain, October 2011 (Poster).

- ◇ I. Lindell, L. Bergamin and **A. Favaro**, “Differential forms and decomposable media”, Progress In Electromagnetic Research Symposium (PIERS), Suzhou, China, September 2011.
- ◇ P. Kinsler, M. McCall and **A. Favaro**, “All kinds of cloaks, all kinds of transformations: a general theory of Transformation Mechanics”, SPIE 2011, S. Diego, USA, August 2011.
- ◇ I. Lindell, L. Bergamin and **A. Favaro**, “The class of decomposable media in four-dimensional representation”, URSI-GASS 2011, Istanbul, Turkey, August 2011.
- ◇ **A. Favaro** and L. Bergamin, “Skewonless media with no birefringence”, 475th Heraeus Seminar, Bad Honnef, Germany, March 2011.
- ◇ M. McCall, **A. Favaro**, P. Kinsler and A. Boardman, “A new cloaking paradigm: the space-time cloak”, Metamaterials 2010, Karlsruhe, Germany, September 2010.
- ◇ M. McCall, **A. Favaro**, P. Kinsler and A. Boardman, “A new cloaking paradigm: the space-time cloak”, Photon’10, Southampton, UK, August 2010.
- ◇ **A. Favaro** and L. Bergamin, “Negative index of refraction, perfect lenses and transformation optics – some words of caution”, URSI-EMTS 2010, Berlin, Germany, August 2010.
- ◇ M. McCall, **A. Favaro**, P. Kinsler and A. Boardman, “Dynamic electromagnetic cloaking: external field control, magneto-optics, and novel designs”, SPIE 2010, S. Diego, USA, August 2010.
- ◇ P. Kinsler, **A. Favaro** and M. McCall, “Dispersion, causality, and negative materials”, Auxetics 2010, Islands of Gozo and Malta, Malta, July 2010.
- ◇ **A. Favaro**, M. McCall and P. Kinsler, “Coordinate-free negative phase velocity”, META’10, Cairo, Egypt, February 2010.
- ◇ P. Kinsler, **A. Favaro** and M. McCall, “Dressed for success: a Poynting vector for each season”, CLEO Pacific Rim, Shanghai, China, August/September 2009.

- ◇ P. Kinsler, **A. Favaro** and M. McCall, “Poynts of the compass: a flux vector that’s going your way”, Metamaterials 2009, London, UK, August 2009.
- ◇ **A. Favaro**, P. Kinsler and M. McCall, “A coordinate-free criterion for negative phase velocity propagation”, Metamaterials 2009, London, UK, August 2009 (Poster).

Chapter 2

Foundations

2.1 Introduction

The present Chapter briefly describes how the structure of classical Electromagnetism can be derived from first principles. The order and manner in which the concepts are discussed is largely based on the book [41], wherein Hehl and Obukhov deduce Maxwell's theory from few experimentally justified axioms. Of course, the exposition given in this Chapter differs from that of Ref. [41] in some aspects, and is less systematic. The emphasis on first principles, however, remains very pronounced. Most assumptions are thus introduced only when necessary, and not earlier. Similarly, the metric structure is defined only later in the presentation.

The exterior calculus of differential forms is used throughout this Chapter, as it facilitates the task of analysing the foundations of Electrodynamics. In order to start implementing exterior calculus, one has to assign a differential form to each basic Electromagnetic quantity. Conveniently, the process of associating physical fields to exterior forms is governed by a simple observation: every crucial Electromagnetic quantity is related to a specific integral. Accordingly, each field is characterised by the typical dimension of an integration region. For instance, the electric charge density ρ is naturally integrated over a volume, so as to evaluate the corresponding charge content. One may thus conclude that ρ is associated to an integration

Symbol	Name	p -form	Character
j	Electric current density	2-form	Extensive
ρ	Electric charge density	3-form	Extensive
$\mathcal{H}$	Magnetic excitation	1-form	Extensive
$\mathcal{D}$	Electric excitation	2-form	Extensive
E	Electric field strength	1-form	Intensive
B	Magnetic field strength	2-form	Intensive

Table 2.1: The electromagnetic fields can be easily expressed in terms of differential forms. The extensive/intensive nature of all entries is specified (see Chap. 1 and Mie [97]).

domain of dimension 3. As a further example, the electric current density j is typically integrated over an area, such as the cross-section of a wire. Consequently, j is clearly related to an integration region of dimension 2. At this point, it is important to notice that a differential p -form is naturally integrated over a p -dimensional domain [101]. By virtue of this prescription, one can obviously assign a differential p -form to each basic Electromagnetic quantity, as required. In particular, the electric charge density ρ must correspond to a 3-form, while the electric current density j must be a 2-form. Table 2.1 specifies how exterior calculus encodes the fields $\{j, \rho, \mathcal{H}, \mathcal{D}, E, B\}$. The p -form description of most entries can be checked straightforwardly. For instance, the electric field strength is typically integrated along some segment of electronic circuit, so as to obtain the associated voltage difference. The integration of E along a curve, then, dictates that the electric field strength is certainly a 1-form (cf. Table 2.1). Interesting information on this topic can be found in the papers [132] and [133] by Tonti.

The mathematics utilised in this Thesis, including exterior calculus, is built upon the properties of a 4-dimensional smooth manifold, denoted X_4 . In particular, X_4 is used as the model for a metric- and connection-free spacetime. The chosen manifold is also required to satisfy an additional condition: more explicitly, X_4 is assumed to admit a slicing (a *foliation*) into sheets of constant time [16, 41]. The following discussion briefly illustrates how the manifold's foliation is used, and clarifies the definition of the fields in Table 2.1. As a starting point, one may introduce on X_4 a differential p -form Ψ . Then, given any vector n and any 1-form ν , the interior product $\lrcorner$ satisfies

$$n \lrcorner (\nu \wedge \Psi) = (n \lrcorner \nu) \Psi - \nu \wedge (n \lrcorner \Psi) , \quad (2.1)$$

where the symbol $\wedge$ denotes exterior multiplication; re-ordering the terms yields

$$(n \rfloor \nu) \Psi = \nu \wedge (n \rfloor \Psi) + n \rfloor (\nu \wedge \Psi) . \quad (2.2)$$

Remarkably, an identity for Ψ alone is readily achieved. In fact, for any point where

$$(n \rfloor \nu) \neq 0 , \quad (2.3)$$

it is possible to divide (2.2) by the scalar $(n \rfloor \nu)$, so as to obtain the expression

$$\Psi = \frac{\nu \wedge (n \rfloor \Psi)}{(n \rfloor \nu)} + \frac{n \rfloor (\nu \wedge \Psi)}{(n \rfloor \nu)} . \quad (2.4)$$

The structure of this formula can be made more explicit by casting (2.4) in the form

$$\Psi = {}^\perp \Psi + \underline{\Psi} , \quad (2.5)$$

$${}^\perp \Psi := \frac{\nu \wedge (n \rfloor \Psi)}{(n \rfloor \nu)} , \quad (2.6)$$

$$\underline{\Psi} := \frac{n \rfloor (\nu \wedge \Psi)}{(n \rfloor \nu)} , \quad (2.7)$$

see Mielke and Wallner [98] or Hehl and Obukhov [41, 44]. At this point, it is useful to interpret ${}^\perp(\cdot)$ and $\underline{(\cdot)}$ as two linear maps. Crucially, one can easily prove that

$${}^\perp {}^\perp \Psi = {}^\perp \Psi , \quad \underline{\underline{\Psi}} = \underline{\Psi} , \quad {}^\perp(\underline{\Psi}) = \underline{({}^\perp \Psi)} = 0 , \quad (2.8)$$

whence ${}^\perp(\cdot)$ and $\underline{(\cdot)}$ are two projection operators forming a complete set. Accordingly, the split (2.5) is a well-posed *decomposition* for all locations where (2.3) holds true. Since $n \rfloor \underline{\Psi}$ vanishes identically, the component $\underline{\Psi}$ is said to be transverse to n . Correspondingly, ${}^\perp \Psi$ is called the longitudinal part with respect to n . The shorthand $\Psi_\perp := (n \rfloor \nu)^{-1} (n \rfloor \Psi)$ is often convenient, and allows one to write

$$\Psi = {}^\perp \Psi + \underline{\Psi} = \nu \wedge \Psi_\perp + \underline{\Psi} . \quad (2.9)$$

Leaving the decomposition (2.9) momentarily aside, one may now prescribe that the manifold X_4 admits a slicing (a foliation) into 3-dimensional sheets. More ac-

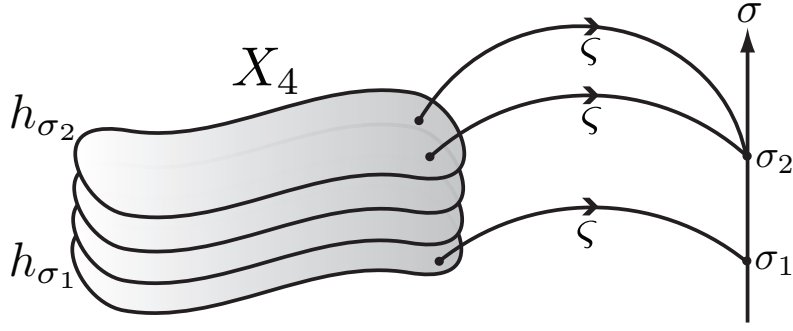


Figure 2.1: The foliation of X_4 into 3-dimensional submanifolds: there exists a smooth function $\sigma = \varsigma(p)$ such that each leaf corresponds to a hypersurface h_σ of constant σ .

curately, one may now demand the existence of a set of connected, embedded, 3-dimensional submanifolds – called sheets, leaves or folia. Any intersection of sheets is required to be empty, while the union of all leaves coincides with X_4 . Moreover, each folium must be precisely one level set h_σ of the smooth function

$$\varsigma : X_4 \rightarrow \mathbb{R} \quad \text{such that} \quad p \mapsto \sigma = \varsigma(p), \quad (2.10)$$

see Figure 2.1. Physically, if σ is viewed as a time-label with the appropriate dimension $[\sigma] = t$, the submanifolds h_σ can be interpreted as a simple model of 3-space¹. Although this statement is entirely adequate for the scope of the present Thesis, it is worth mentioning that the concept of a “point in space” is specified in full only *after* the world-lines defining an extended observer are known. The construction of the space manifold explained by Kurz and Auchmann in Ref. [67] is more rigorous, and takes due care of the subtlety just alluded to.

Operating on the function ς with the exterior derivative $d(\cdot)$ immediately yields the 1-form $d\varsigma$. A standard abuse of notation is also implemented, whence $d\varsigma$ is hereafter referred to as “ $d\sigma$ ”. Remarkably, the definition of the manifold’s foliation ensures that the 1-form $d\sigma$ is non-zero for all points on X_4 (see Exercise 5.6 of Lee’s

¹The following remarks are somewhat interesting: 1. Strictly speaking, it is possible to define a foliation even if the leaves are immersed, but not embedded. In this scenario, however, the 1-form $d\varsigma$ vanishes at the critical points of ς [75]. 2. The time dimension t is to be understood as a segment in *affine geometry*. When this definition is employed, the concept of t is meaningful even in the absence of a metric. An observation is of help in clarifying these statements. Despite the fact that a metric is not available, one can compare two vectors v_1 and v_2 inasmuch as they are projected along the time direction. More explicitly, one can compare the scalars $v_1 \rfloor d\varsigma$ and $v_2 \rfloor d\varsigma$. Resorting to similar notions, it is possible to introduce the dimension of length ℓ in a metric-free guise [41].

book [75]). Accordingly, it is always possible to find a vector n such that $(n \rfloor d\sigma)$ is nowhere vanishing. Since this statement matches the condition (2.3) globally, the split (2.9) can be utilised throughout X_4 for the specific choice $\nu = d\sigma$,

$$\Psi = {}^\perp\Psi + \underline{\Psi} = d\sigma \wedge \Psi_\perp + \underline{\Psi} = \frac{d\sigma \wedge (n \rfloor \Psi)}{(n \rfloor d\sigma)} + \frac{n \rfloor (d\sigma \wedge \Psi)}{(n \rfloor d\sigma)} . \quad (2.11)$$

The significance of σ as a time label provides a clear physical meaning for the assignment $\nu = d\sigma$: indeed, the expression (2.11) is a true *space+time* decomposition for all differential p -forms. Crucially, the fields listed in Table 2.1 are obtained by applying (2.11). For example, given the charge current density 3-form J , one has

$$J = d\sigma \wedge J_\perp + \underline{J} = -d\sigma \wedge j + \rho . \quad (2.12)$$

In a similar way, the field excitation 2-form H and the field strength 2-form F satisfy

$$H = d\sigma \wedge H_\perp + \underline{H} = +d\sigma \wedge \mathcal{H} + \mathcal{D} , \quad (2.13)$$

$$F = d\sigma \wedge F_\perp + \underline{F} = -d\sigma \wedge E + B . \quad (2.14)$$

As demonstrated in the article [56] by Itin and Hehl, the signs preceding the quantities $\{j, \rho\}$ and $\{\mathcal{H}, \mathcal{D}\}$ in (2.12) and (2.13) originate from well-established conventions. For example, it is required that the outward direction from a bounded spatial volume is positive. The overall sign of F is derived from an accepted convention as well. In electrostatics, the Lorentz force on a positive charge is agreed to be in the direction of the applied electric field. Ergo, only the relative sign of $\{E, B\}$ in (2.14) demands an additional physical input to be specified, namely *Lenz's law*. To recap, the manifold X_4 is assumed to support a foliation into 3-dimensional leaves². This introduces the prototypes (σ, h_σ) of time and space, with respect to which (2.11) decomposes all p -forms. Hence, one can formulate the expansions (2.12)–(2.14), whereby the correct experimental definitions of $\{j, \rho, \mathcal{H}, \mathcal{D}, E, B\}$ are achieved [41].

²According to Delphenich [19], the existence of a foliation by hypersurfaces should be a derived concept, and not a foundational assumption. This suggestion could be the topic of further work.

2.2 Charge, force and flux

2.2.1 Electric charge is conserved

The statement that electric charge is conserved has been verified in a number of experiments³. For example, Klapdor-Kleingrothaus *et al.* [64] considered a decay forbidden by charge conservation, namely $e^- \rightarrow \nu_e + \gamma$, and measured that the corresponding mean life time is larger than 1.22×10^{26} years. Accordingly, it is reasonable to expect that any decay that requires electric charge to be created or destroyed must be extremely rare, or impossible.

The equality in magnitude of the electron and proton charge is *related* to charge conservation [72], and is furthermore a well-tested fact. One of the best experimental setups, in this context, is due to Dylla and King [23]. A resonator filled with gas is driven by a time-dependent electric field, so that a sound is generated. In turn, if a finite acoustic signal is recorded at the driving frequency, the electron and proton charges do not agree in modulus. Recently, Bressi *et al.* [11] revisited this procedure, thus establishing the upper limit

$$\left| \frac{q_e + q_p}{q_e} \right| \lesssim 10^{-21} . \quad (2.15)$$

Here, the (signed) electron and proton charges are denoted q_e and q_p respectively. Clearly, the electron-proton charge asymmetry is either very small, or exactly zero.

Given the appropriate caveats, a slow time-variation of the fine-structure constant could imply a gradual change in the value of the electron's charge (see Bekenstein [5]). Charge conservation would then be violated, albeit over the course of a long period. In any system of units (cf. Hehl, Itin and Obukhov [40, 43, 45]), the fine-structure constant α_f can be written as

$$\alpha_f = \frac{\Omega_0}{2R_K} , \quad (2.16)$$

where Ω_0 is the vacuum impedance and R_K is Klitzing's constant – that is, the

³The introduction provided here is based on the article [72] by Lämmerzahl, Macías and Müller.

quantum Hall resistance. If α_f has a time-dependence and Ω_0 does not, the quantity

$$R_K = \frac{h}{q_e^2} , \quad (2.17)$$

will change slowly. Thereby, provided Planck's constant h does not fluctuate, a non-zero variation $(\dot{\alpha}_f/\alpha_f)$ can indeed be interpreted as a non-vanishing $(\dot{q}_e/q_e)$, since

$$\left(\frac{\dot{\alpha}_f}{\alpha_f} \right) = 2 \left(\frac{\dot{q}_e}{q_e} \right) . \quad (2.18)$$

Crucially, at least for the moment [136], only one experimental setup has confirmed that α_f is evolving in time (namely the Keck-Hires system [141], which analyses the spectrum emitted by distant quasars). Moreover, even if the fine-structure constant *was* observed to change, the electron charge could still be invariant. In fact, the alleged fluctuations of α_f could be due to a time-dependent Ω_0 [40, 43, 45], whereby charge conservation would hold true unaffected.

In what follows, the discussion is given a *fresh start*. After some restrictions are imposed on the spacetime manifold, charge conservation is introduced as the first axiom of electromagnetism. The experimental evidence provided so far thus becomes fundamental, as it guarantees that charge is indeed conserved on all physical scales.

The first axiom of classical electromagnetism

In order to develop the theory of electromagnetism from first principles, it is necessary to define an appropriate model of spacetime. Following Lee [75], a 4-dimensional smooth manifold X_4 consists of a topological space X that obeys these conditions:

1. For any two distinct points $p_1 \in X$ and $p_2 \in X$, one can find two open sets $V_1 \subset X$ and $V_2 \subset X$ which respectively contain p_1 and p_2 , but have no intersection $V_1 \cap V_2 = \emptyset$. In technical terms, X is required to be ‘‘Hausdorff’’.
2. Let $\mathcal{E}$ be a collection of open sets in X such that every open set in X is equal to the union of one or more elements in $\mathcal{E}$. Accordingly, the family $\mathcal{E}$ is said to be a basis for the topology of X (see [76]). Here, X is demanded to have a basis $\mathcal{E}$ which is a countable set. In other words, X must be ‘‘second-countable’’.

3. At any point $p \in X$, there exists a pair (U, φ) such that $U \subset X$ is an open set containing p , while φ is a homeomorphism⁴ mapping U to an open set $\varphi(U)$ in $\mathbb{R}^4$. Since φ assigns four coordinates $\varphi(p) = \{x^0(p), \dots, x^3(p)\} \in \mathbb{R}^4$ to each point $p \in U$, the pair (U, φ) is called a coordinate chart on X . In brief, the topological space X is thus prescribed to be “locally homeomorphic to $\mathbb{R}^4$ ”.
4. Whenever two charts (U_1, φ_1) and (U_2, φ_2) have a finite overlap $U_1 \cap U_2 \neq \emptyset$, it is possible to define a transition map $\psi_{12} := \varphi_1 \circ \varphi_2^{-1}$ from $\varphi_2(U_1 \cap U_2) \subset \mathbb{R}^4$ to $\varphi_1(U_1 \cap U_2) \subset \mathbb{R}^4$. More explicitly, given a point $p \in U_1 \cap U_2$, one can always specify a coordinate change ψ_{12} from $\varphi_2(p) = \{y^0(p), \dots, y^3(p)\}$ to $\varphi_1(p) = \{x^0(p), \dots, x^3(p)\}$. A collection of charts $\{(U_i, \varphi_i) | i \in I\}$ is called an atlas of X if the set $\{U_i | i \in I\}$ satisfies $\bigcup_{i \in I} U_i = X$, and is thereby an open cover of X . Crucially, we impose that the topological space X has an atlas $\{(U_i, \varphi_i)\}$ which is also smooth – for all finite overlaps $U_i \cap U_j \neq \emptyset$, the corresponding transition maps $\psi_{ij} := \varphi_i \circ \varphi_j^{-1}$ have to be “ C^∞ -differentiable”.

Throughout this Thesis, the spacetime continuum is represented by a 4-dimensional smooth manifold X_4 – subject to three additional restrictions. *Firstly*, spacetime is assumed to be “connected”, that is, every two points on X_4 can be linked by a continuous path. *Secondly*, the chosen manifold is constrained to be “orientable”. In other words, one can always find an atlas $\{(U_i, \varphi_i)\}$ of X_4 such that all transition maps have a strictly positive Jacobian determinant,

$$J(\psi_{ij}) = \det \left(\frac{\partial x^\mu}{\partial y^\nu} \right) > 0, \quad (\mu, \nu = 0, \dots, 3) \quad (2.19)$$

where $\varphi_i = \{x^0, \dots, x^3\}$ and $\varphi_j = \{y^0, \dots, y^3\}$. Crucially, the condition (2.19) implies that a single orientation can be extended across all overlaps $U_i \cap U_j \neq \emptyset$. Here, X_4 is immediately demanded to be connected and orientable for simplicity: these two properties can be enforced gradually, but the corresponding discussion is more involved (cf. Dahl [16]). *Finally*, the spacetime continuum is required to support a foliation into 3-dimensional submanifolds of constant “time”, see Sec. 2.1.

At this point, it is possible to introduce the *first axiom* of electromagnetism, which can be formulated as follows [41]. There exists a 3-form J , defined to have

⁴A homeomorphism φ is a continuous invertible map, whose inverse φ^{-1} is also continuous [101].

dimension of charge $[J] = q$, whose integral over *any* 3-dimensional cycle vanishes

$$\oint_{C_3} J = 0 , \quad \partial C_3 = 0 . \quad (2.20)$$

Notably, $\partial(\cdot)$ is the boundary operator, and the integration region $C_3 \subset X_4$ is said to be a cycle precisely because $\partial C_3 = 0$ [101]. The axiom (2.20), concerning the charge current density 3-form J , dictates that electric charge is a conserved quantity. One ought to mention that, in SI units: $[J] = q \stackrel{\text{SI}}{=} \text{C}$. The International System of Units (SI) is defined in the SI Brochure [12]. Furthermore, Appendix A is of interest.

It is useful to observe that the cycle C_3 could coincide with the boundary $\partial \Xi_4$ of a 4-dimensional domain $\Xi_4 \subset X_4$. Indeed, every p -dimensional boundary $\partial \Xi_{p+1}$ is also a cycle, as the operator $\partial(\cdot)$ is nilpotent: $\partial(\partial \Xi_{p+1}) = \partial^2 \Xi_{p+1} \equiv 0$. By contrast, *not* all p -dimensional cycles can, in general, be written as the boundary of a $(p+1)$ -dimensional region. Accordingly, one can set C_3 to equal $\partial \Xi_4$ for some domain $\Xi_4 \subset X_4$, but this corresponds to a particular choice – unless otherwise specified. Selecting the integration cycle in (2.20) to take the form $C_3 = \partial \Xi_4$ yields

$$0 = \oint_{\partial \Xi_4} J = \int_{\Xi_4} dJ , \quad (2.21)$$

where the last equality makes use of Stokes’s theorem [75, 101]. Clearly, the quantity $\int_{\Xi_4} dJ$ must vanish for *any* 4-dimensional region Ξ_4 . As a consequence, one obtains

$$dJ = 0 , \quad (2.22)$$

which is the differential version of charge conservation. In summary, (2.20) prescribes that the integral of J over an arbitrary “3-cycle” is zero, and (2.22) states that J is a “closed” 3-form. Equivalently, since de Rham’s map is one-to-one (see Frankel [33]), one can guarantee the existence of a 2-form H such that

$$dH = J . \quad (2.23)$$

The inhomogeneous Maxwell equation is thus attained, that is, the current density J is shown to be an “exact” 3-form⁵. Following Hehl and Obukhov [41], H is called the field excitation 2-form. The dimension of H is evidently charge, $[H] = q \stackrel{\text{SI}}{=} \text{C}$. It is also worth noting that the field excitation is defined up to a closed 2-form K ,

$$H' := H + K \quad \text{with} \quad dK = 0 \quad \Rightarrow \quad dH' = dH + dK = dH = J. \quad (2.24)$$

Conveniently – however – the simple assumption that the response field H vanishes when the applied field F is zero dictates that $K = 0$ [41]. It is important to remark that the excitation 2-form H is a measurable, extensive quantity [43, 97, 132, 133].

Longitudinal+transverse decomposition of (2.23)

The following derivation aims to decompose the inhomogeneous Maxwell equation (2.23) with respect to (2.9). For future reference, the split (2.9) is *initially* applied to the general case $\Phi = d\Psi$, where Ψ is some exterior p -form. The decomposition of $J = dH$ is *then* retrieved by setting $p = 2$, and assigning $\{\Phi, \Psi\} \rightarrow \{J, H\}$.

In the procedure below, most difficulties emerge when the operators ${}^\perp(\cdot)$ and $(\cdot)$ in (2.9) interact with the exterior derivative. It is therefore appropriate to assume that the vector n , the 1-form ν , and the quantity $\check{n} := (n \rfloor \nu)^{-1} n$ are *smooth* fields. Hence, for all points where $\check{n}$ is neither zero nor divergent, the $(p+1)$ -form Φ reads

$$\Phi = {}^\perp\Phi + \underline{\Phi} = \nu \wedge \Phi_\perp + \underline{\Phi} = \nu \wedge (\check{n} \rfloor \Phi) + \check{n} \rfloor (\nu \wedge \Phi). \quad (2.25)$$

The projection ${}^\perp\Phi = \nu \wedge \Phi_\perp$ is dealt with first. Recalling that Φ is exact leads to

$${}^\perp\Phi = {}^\perp(d\Psi) = \nu \wedge (\check{n} \rfloor d\Psi). \quad (2.26)$$

In the contraction $(\check{n} \rfloor d\Psi)$, a key identity can be used to exchange the operators $\check{n} \rfloor(\cdot)$ and $d(\cdot)$. More explicitly, the Lie derivative $\mathcal{L}_{\check{n}}$ obeys the property (see [41])

$$\mathcal{L}_{\check{n}}\Psi = (\check{n} \rfloor d\Psi) + d(\check{n} \rfloor \Psi), \quad (2.27)$$

⁵As an aside, it is instructive to replace the axiom (2.20) with the weaker condition $\oint_{\partial \Xi_4} J = 0$. In this scenario, equation (2.22) still holds true – but it does *not* necessarily imply (2.23). More accurately, every closed 3-form J is also exact *if and only if* each 3-cycle constitutes a 3-boundary.

and thus: $(\check{n} \rfloor d\Psi) = \mathcal{L}_{\check{n}}\Psi - d(\check{n} \rfloor \Psi)$. Substituting this result into (2.26) implies that

$${}^\perp\Phi = \nu \wedge [\mathcal{L}_{\check{n}}\Psi - d(\Psi_\perp)] . \quad (2.28)$$

Notably, (2.28) is defined up to the longitudinal component ${}^\perp(\Lambda)$ of *some* p -form Λ ,

$$\nu \wedge [\mathcal{L}_{\check{n}}\Psi - d(\Psi_\perp) + {}^\perp(\Lambda)] = \nu \wedge [\mathcal{L}_{\check{n}}\Psi - d(\Psi_\perp)] = {}^\perp\Phi . \quad (2.29)$$

Given any differential form Θ , one can easily prove that $\nu \wedge \Theta \equiv \nu \wedge \underline{\Theta}$. Correspondingly, it is possible to eliminate the freedom illustrated in (2.29) by imposing

$${}^\perp\Phi = \nu \wedge [(\mathcal{L}_{\check{n}}\underline{\Psi}) - \underline{d(\Psi_\perp)}] . \quad (2.30)$$

One can further invoke the split $\Psi = {}^\perp\Psi + \underline{\Psi}$, and thereby achieve the expression

$${}^\perp\Phi = \nu \wedge [\mathcal{L}_{\check{n}}({}^\perp\Psi + \underline{\Psi}) - \underline{d(\Psi_\perp)}] = \nu \wedge [\mathcal{L}_{\check{n}}(\underline{\Psi}) - \underline{d(\Psi_\perp)} + \mathcal{L}_{\check{n}}(\nu \wedge \underline{\Psi_\perp})] . \quad (2.31)$$

Crucially, the Lie derivative has a Leibniz rule with respect to the exterior product (cf. [41]), whence: $\mathcal{L}_{\check{n}}(\nu \wedge \underline{\Psi_\perp}) = \mathcal{L}_{\check{n}}(\nu) \wedge \underline{\Psi_\perp} + \nu \wedge \mathcal{L}_{\check{n}}(\underline{\Psi_\perp})$. Accordingly, one finds

$${}^\perp\Phi = \nu \wedge [\mathcal{L}_{\check{n}}(\underline{\Psi}) - \underline{d(\Psi_\perp)} + \mathcal{L}_{\check{n}}(\nu) \wedge \underline{\Psi_\perp} + \nu \wedge \mathcal{L}_{\check{n}}(\underline{\Psi_\perp})] . \quad (2.32)$$

The quantity $\nu \wedge \mathcal{L}_{\check{n}}(\underline{\Psi_\perp})$ vanishes identically, as $\nu \wedge \underline{\Theta} \equiv 0$ for any exterior form Θ . Moreover, the same theorem that was used in (2.27) can be implemented to attain $\mathcal{L}_{\check{n}}\nu = (\check{n} \rfloor d\nu) + d(\check{n} \rfloor \nu) = (d\nu)_\perp + d(1) = (d\nu)_\perp$. As an outcome, (2.32) turns into

$${}^\perp\Phi = \nu \wedge [\mathcal{L}_{\check{n}}(\underline{\Psi}) - \underline{d(\Psi_\perp)} + \underline{(d\nu)_\perp \wedge \Psi_\perp}] . \quad (2.33)$$

By virtue of the properties of the interior product (see Section A.1.7 of [41]), one can show that $\underline{(d\nu)_\perp \wedge \Psi_\perp} = (d\nu)_\perp \wedge \underline{\Psi_\perp}$. Consequently, equation (2.33) is simplified to

$${}^\perp\Phi = \nu \wedge [\mathcal{L}_{\check{n}}(\underline{\Psi}) - \underline{d(\Psi_\perp)} + (d\nu)_\perp \wedge \underline{\Psi_\perp}] . \quad (2.34)$$

At this point, it is useful to notice that $(\mathcal{L}_{\check{n}}\Theta)_\perp = \check{n} \rfloor (\mathcal{L}_{\check{n}}\Theta) = \mathcal{L}_{\check{n}}(\check{n} \rfloor \Theta) = \mathcal{L}_{\check{n}}(\Theta_\perp)$,

for every differential form Θ . In fact, one can re-cast the term $\underline{\mathcal{L}_{\check{n}}(\Psi)}$ in (2.34) as

$$\underline{\mathcal{L}_{\check{n}}(\Psi)} = \mathcal{L}_{\check{n}}(\underline{\Psi}) - \nu \wedge [\mathcal{L}_{\check{n}}(\underline{\Psi})]_{\perp} = \mathcal{L}_{\check{n}}(\underline{\Psi}) - \nu \wedge \mathcal{L}_{\check{n}}[(\underline{\Psi})_{\perp}] = \mathcal{L}_{\check{n}}(\underline{\Psi}). \quad (2.35)$$

When the equality $\underline{\mathcal{L}_{\check{n}}(\Psi)} = \mathcal{L}_{\check{n}}(\underline{\Psi})$ is substituted in (2.34), and the convenient abbreviation $\underline{d}(\cdot) := \underline{d}(\cdot)$ is introduced, an *important result* is obtained – namely,

$$^{\perp}\Phi = \nu \wedge \Phi_{\perp} = \nu \wedge [\mathcal{L}_{\check{n}}(\underline{\Psi}) - \underline{d}(\Psi_{\perp}) + (d\nu)_{\perp} \wedge \Psi_{\perp}] . \quad (2.36)$$

One can now consider the component $\underline{\Phi}$, which is transverse to $\check{n}$. Neglecting for a moment the shorthand $\underline{d}(\cdot) := \underline{d}(\cdot)$, one may start by performing the manipulation

$$\underline{\Phi} = \underline{d\Psi} = \underline{d(^{\perp}\Psi + \Psi_{\perp})} = \underline{d\Psi} + \underline{d(^{\perp}\Psi)} = \underline{d\Psi} + \underline{d(\nu \wedge \Psi_{\perp})} . \quad (2.37)$$

For every differential form Θ , the exterior derivative satisfies an anti-Leibniz rule of the form: $d(\nu \wedge \Theta) = (d\nu) \wedge \Theta - \nu \wedge (d\Theta)$, see [41]. Hence, equation (2.37) becomes

$$\underline{\Phi} = \underline{d\Psi} + \underline{(d\nu) \wedge \Psi_{\perp}} - \underline{\nu \wedge (d\Psi_{\perp})} , \quad (2.38)$$

where the last term is zero (cf. the remark following (2.32)). By exploiting the laws of the interior product (cf. the remark following (2.33)), one can prove that $\underline{(d\nu) \wedge \Psi_{\perp}} = \underline{d\nu} \wedge \Psi_{\perp}$. As a consequence, a *crucial expansion* is achieved – that is:

$$\underline{\Phi} = \underline{d(\Psi)} + \underline{d(\nu)} \wedge \Psi_{\perp} , \quad (2.39)$$

where the definition $\underline{d}(\cdot) := \underline{d}(\cdot)$ is re-implemented. In (2.36) and in (2.39), one may assign $\{\Phi, \Psi\} \rightarrow \{J, H\}$. Equivalently, the Maxwell equation (2.23) is split as

$$J_{\perp} = \mathcal{L}_{\check{n}}(\underline{H}) - \underline{d}(H_{\perp}) + (d\nu)_{\perp} \wedge H_{\perp} , \quad (2.40)$$

$$\underline{J} = \underline{d}(\underline{H}) + \underline{d}(\nu) \wedge H_{\perp} , \quad (2.41)$$

for all points on the manifold where the vector field $\check{n}$ is neither zero nor divergent. The expansions (2.36) and (2.39) agree with the formula (4.10) reported in the paper [30] by Fecko. In that article, however, the vector $\check{n}$ and the 1-form ν are considered

to be related via the metric. Here, such assumption is eliminated, for it is in contrast with the principle that the notion of distance ought to be invoked only when needed.

Space+time decomposition of (2.23)

If $\nu = d\sigma$, and $\check{n}$ is neither zero nor divergent *everywhere*, (2.9) becomes the global space+time decomposition (2.11). In this scenario, ν is *exact* and therefore *closed*:

$$d\nu = 0 . \quad (2.42)$$

Accordingly, both the 1-form $(d\nu)_\perp$ and the 2-form $\underline{d}(\nu) := \underline{(d\nu)}$ obviously vanish, and the general expressions (2.36) and (2.39) are simplified to

$$\Phi_\perp = \mathcal{L}_{\check{n}}(\underline{\Psi}) - \underline{d}(\Psi_\perp) , \quad (2.43)$$

$$\underline{\Phi} = \underline{d}(\underline{\Psi}) . \quad (2.44)$$

By replacing $\{\Phi, \Psi\}$ with $\{J, H\}$, one obtains the required space+time split of (2.23),

$$J_\perp = \mathcal{L}_{\check{n}}(\underline{H}) - \underline{d}(H_\perp) , \quad (2.45)$$

$$\underline{J} = \underline{d}(\underline{H}) . \quad (2.46)$$

Following Tonti [132, 133] and Hehl+Obukhov [41], (2.45) and (2.46) can be recognised to be the equations used to experimentally define $\mathcal{H}$ and $\mathcal{D}$ (cf. Table 2.1). Equivalently, (2.45) and (2.46) are identified to be the 3-dimensional Maxwell laws

$$j = \underline{d}\mathcal{H} - \mathcal{L}_{\check{n}}\mathcal{D} , \quad (\text{Ampère-Maxwell law}), \quad (2.47)$$

$$\rho = \underline{d}\mathcal{D} . \quad (\text{Gauss's law for } \mathcal{D}). \quad (2.48)$$

Here, it is instructive to recall the space+time decompositions of J and H , namely

$$J = d\sigma \wedge J_\perp + \underline{J} = -d\sigma \wedge j + \rho , \quad (2.49)$$

$$H = d\sigma \wedge H_\perp + \underline{H} = d\sigma \wedge \mathcal{H} + \mathcal{D} , \quad (2.50)$$

see Sec. 2.1. Since the dimension of J and H is charge, while $[\mathrm{d}\sigma] = t \stackrel{\text{SI}}{=} \text{s}$, one finds

$$[j] = [\mathcal{H}] = q/t \stackrel{\text{SI}}{=} \text{A} , \quad [\rho] = [\mathcal{D}] = q \stackrel{\text{SI}}{=} \text{C} . \quad (2.51)$$

Remarkably, at any one time instant σ , the quantities $\{j, \rho, \mathcal{H}, \mathcal{D}\}$ can be viewed as smooth fields on the 3-dimensional submanifold h_σ . In particular, it is possible to construct a basis $\{v^1, v^2, v^3\}$ for the co-tangent space of h_σ , and consequently write⁶

$$j = \frac{1}{2} j_{ab} (v^a \wedge v^b) , \quad \rho = \frac{1}{6} \rho_{abc} (v^a \wedge v^b \wedge v^c) , \quad (2.52)$$

$$\mathcal{H} = \mathcal{H}_a v^a , \quad \mathcal{D} = \frac{1}{2} \mathcal{D}_{ab} (v^a \wedge v^b) . \quad (2.53)$$

One can always demand that $\{v^1, v^2, v^3\}$ all have dimension of length, that is, $[v^a] = \ell \stackrel{\text{SI}}{=} \text{m}$. As a result, the so-called *physical* dimension of $\{j, \rho, \mathcal{H}, \mathcal{D}\}$ is attained:

$$[j_{ab}] = q t^{-1} \ell^{-2} \stackrel{\text{SI}}{=} \text{A m}^{-2} , \quad [\rho_{abc}] = q \ell^{-3} \stackrel{\text{SI}}{=} \text{C m}^{-3} , \quad (2.54)$$

$$[\mathcal{H}_a] = q t^{-1} \ell^{-1} \stackrel{\text{SI}}{=} \text{A m}^{-1} , \quad [\mathcal{D}_{ab}] = q \ell^{-2} \stackrel{\text{SI}}{=} \text{C m}^{-2} . \quad (2.55)$$

An accurate dimensional analysis of classical electromagnetism can be found in Schouten [119], Post [113], or Hehl and Obukhov [43]. Appendix A is also relevant.

To summarise, the manifold X_4 is provided with a foliation into constant-time sheets, whereby one can select ν to coincide with $\mathrm{d}\sigma$. When the 1-form ν is chosen to be exact, the condition (2.42) is fulfilled identically. Therefore, provided $\check{n}$ is well-behaved for all $p \in X_4$, (2.40) and (2.41) become the *space+time* laws (2.47) and (2.48). Remarkably, this chain of reasoning can be reversed – to some extent. In fact, the key results (2.47) and (2.48) emerge from (2.40) and (2.41) solely when

$$(\mathrm{d}\nu)_\perp \wedge H_\perp = 0 , \quad (2.56)$$

$$\underline{\mathrm{d}}(\nu) \wedge H_\perp = 0 . \quad (2.57)$$

Since H is an *arbitrary* solution of (2.23), the requirements (2.56) and (2.57) impose that $(\mathrm{d}\nu)_\perp$ and $\underline{\mathrm{d}}(\nu)$ both vanish, whence $\mathrm{d}\nu = \nu \wedge (\mathrm{d}\nu)_\perp + \underline{\mathrm{d}}(\nu) = 0$. In other words,

⁶From this point onwards, Einstein's summation convention is implemented. Those indices that range from 0 to 3 are represented by Greek letters. Latin letters, instead, take the values $\{1, 2, 3\}$. The (co-)bases used in this work are assumed to be *non-holonomic*, except where stated otherwise.

(2.56) and (2.57) prescribe that the 1-form ν is closed – but not necessarily exact. In the *specific case* where the manifold's 1-dimensional cycles are all boundaries, de Rham's theorem [33] does indeed guarantee that ν is exact, that is, $\nu = d\sigma$. If, in addition, the laws (2.47)-(2.48) and the associated decompositions (2.49)-(2.50) are to hold true *globally*, one is forced to conclude that $\nu = d\sigma \neq 0$ everywhere. Accordingly, by virtue of the regular level set theorem [75], the spacetime X_4 must support a foliation whose leaves are the embedded 3-dimensional submanifolds h_σ .

Counting procedures and twisted forms

In the calculation below, the differential version (2.22) of the first axiom (2.20) is shown to imply a well-known statement of charge conservation – namely: let $(\Gamma_3)_\sigma$ be a simply-connected volume in space ($(\Gamma_3)_\sigma \subset h_\sigma$), the net current flowing out of $(\Gamma_3)_\sigma$ is equal to the decrease rate of the total charge within $(\Gamma_3)_\sigma$. As a starting point, it is useful to note that every 4-form Υ satisfies $\Upsilon = {}^\perp(\Upsilon)$. Thus, (2.22) reads

$$dJ = {}^\perp(dJ) = \nu \wedge (dJ)_\perp = 0 . \quad (2.58)$$

The 4-form $dJ = 0$ is obviously a sub-case of the $(p+1)$ -form $d\Psi = \Phi$ analysed previously. Hence, the identity (2.36) can be used to expand ${}^\perp(dJ)$. One finds that

$${}^\perp(dJ) = \nu \wedge [\mathcal{L}_{\check{n}}(\underline{J}) - \underline{d}(J_\perp) + (d\nu)_\perp \wedge J_\perp] = 0 , \quad (2.59)$$

$$\Rightarrow (dJ)_\perp = \mathcal{L}_{\check{n}}(\underline{J}) - \underline{d}(J_\perp) + (d\nu)_\perp \wedge J_\perp = 0 . \quad (2.60)$$

When the choice $\nu = d\sigma$ is enforced, $(d\nu)_\perp \wedge J_\perp$ is zero. Therefore, (2.60) becomes

$$\underline{d}(J_\perp) = \mathcal{L}_{\check{n}}(\underline{J}) , \quad (2.61)$$

which is *clearly* the 3-dimensional charge continuity equation, in differential version:

$$\underline{d}j = -\mathcal{L}_{\check{n}}\rho , \quad (2.62)$$

cf. (2.49). Integrating both sides of (2.62) over a simply-connected $(\Gamma_3)_\sigma \subset h_\sigma$ yields

$$\int_{(\Gamma_3)_\sigma} \underline{d}j = - \int_{(\Gamma_3)_\sigma} \mathcal{L}_{\check{n}} \rho . \quad (2.63)$$

In order to achieve an alternative expression for the term $\int_{(\Gamma_3)_\sigma} \mathcal{L}_{\check{n}} \rho$, some preparation is necessary⁷. The Lie derivative $\mathcal{L}_{\check{n}} \Theta$ of an arbitrary p -form Θ is defined as:

$$\mathcal{L}_{\check{n}} \Theta := \left[\frac{d}{d\varepsilon} N_\varepsilon^* (\Theta) \right]_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{N_\varepsilon^* \Theta - N_0^* \Theta}{\varepsilon} . \quad (2.64)$$

If ε is a real parameter sufficiently close to zero, N_ε is a diffeomorphism from some neighborhood of each $p \in X_4$ to some neighborhood of $N_\varepsilon(p) \in X_4$. More specifically, the family of points $N_{[0,\varepsilon]}(p)$ constitutes an integral curve of the vector field $\check{n}$. Indeed, N_ε is said to be the *flow* generated by $\check{n}$, and obeys three intuitive properties:

$$N_\varepsilon^{-1} = N_{-\varepsilon} , \quad N_{\varepsilon_1} \circ N_{\varepsilon_2} = N_{\varepsilon_1 + \varepsilon_2} , \quad N_0 = \text{Id} , \quad (2.65)$$

see [75]. Since N_0 coincides with the identity map, the trajectory $N_{[0,\varepsilon]}(p)$ starts at $p \in X_4$. One final aspect of (2.64) is still to be clarified, namely the meaning of pull-back operator N_ε^* . Given two vector fields u and v , providing the directional derivatives $u(f)$ and $v(f)$ for any C^∞ -function f , one may try and select v so that

$$v(f) = u(f \circ N_\varepsilon) . \quad (2.66)$$

Conveniently, when $|\varepsilon|$ is small enough, N_ε is a local diffeomorphism. Hence, for each vector field u , one can guarantee (cf. Lee [75]) the existence and uniqueness of a matching v that satisfies (2.66). In the standard terminology, the vector field v is said to be the push-forward of u by $N_{\varepsilon*}$. Equivalently, one can write $v = N_{\varepsilon*}(u)$. The contraction of the exterior form Θ with p vectors $\{u_1, \dots, u_p\}$ produces a scalar, $\Theta(u_1, \dots, u_p)$. One can thus introduce the pull-back of Θ by N_ε^* from the condition

$$(N_\varepsilon^* \Theta)(u_1, \dots, u_p) = \Theta(N_{\varepsilon*} u_1, \dots, N_{\varepsilon*} u_p) , \quad (2.67)$$

⁷The fact that (2.62) is globally valid demands that the field $\check{n}$ has no zeros and no singularities. In what follows, these restrictions are momentarily lifted, and $\check{n}$ is merely assumed to be smooth.

see [101]. On account of (2.67), the Lie derivative (2.64) is now properly defined. Then, it is easy to observe that the integral of $\mathcal{L}_{\check{n}}\Theta$ over any p -dimensional Ξ_p obeys

$$\int_{\Xi_p} \mathcal{L}_{\check{n}}\Theta = \int_{\Xi_p} \left[\frac{d}{d\varepsilon} N_\varepsilon^*(\Theta) \right]_{\varepsilon=0} = \left[\frac{d}{d\varepsilon} \int_{\Xi_p} N_\varepsilon^*(\Theta) \right]_{\varepsilon=0} = \left[\frac{d}{d\varepsilon} \int_{N_\varepsilon(\Xi_p)} \Theta \right]_{\varepsilon=0}, \quad (2.68)$$

as reported by Mullen *et al.* [99]. The formula (2.68) allows one to express the right-hand side of (2.63) in an alternative way. In fact, the term $\int_{(\Gamma_3)_\sigma} \mathcal{L}_{\check{n}}\rho$ reads

$$\int_{(\Gamma_3)_\sigma} \mathcal{L}_{\check{n}}\rho = \left[\frac{d}{d\varepsilon} \int_{N_\varepsilon[(\Gamma_3)_\sigma]} \rho \right]_{\varepsilon=0}. \quad (2.69)$$

Since $\check{n} := (n \rfloor \nu)^{-1}n$ and $\nu = d\sigma$, a simple calculation shows that $(\check{n} \rfloor d\sigma) = 1$. Crucially, this property implies that $N_\varepsilon[(\Gamma_3)_\sigma]$ is indeed a subset of the leaf $h_{\sigma+\varepsilon}$, where a proof can be constructed as follows. Given that $N_\varepsilon^*(f) := f \circ N_\varepsilon$ for any function f [33], the Lie derivative $\mathcal{L}_{\check{n}}f$ can be considered to be a sub-case of (2.64):

$$\mathcal{L}_{\check{n}}f := \left[\frac{d}{d\varepsilon} N_\varepsilon^*(f) \right]_{\varepsilon=0} = \left[\frac{d}{d\varepsilon} f \circ N_\varepsilon \right]_{\varepsilon=0} = \lim_{\varepsilon \rightarrow 0} \frac{f \circ N_\varepsilon - f}{\varepsilon}. \quad (2.70)$$

Because of the standard notational abuse affecting the 1-form $d\sigma$ (see Section 2.1), one can write $(\check{n} \rfloor d\sigma) = (\check{n} \rfloor d\varsigma) = 1$. The theorem (2.27) further prescribes that $\mathcal{L}_{\check{n}}\varsigma = (\check{n} \rfloor d\varsigma)$, whence the result $\mathcal{L}_{\check{n}}\varsigma = 1$ is achieved. Accordingly, (2.70) leads to

$$\lim_{\varepsilon \rightarrow 0} \left[\frac{\varsigma \circ N_\varepsilon(p) - \varsigma(p)}{\varepsilon} \right] = 1, \quad (2.71)$$

for all $p \in X_4$. By performing a trivial rearrangement, one obtains the key identity

$$\varsigma \circ N_\varepsilon(p) = \varsigma(p) + \varepsilon, \quad (|\varepsilon| \ll 1). \quad (2.72)$$

Remarkably, the condition $|\varepsilon| \ll 1$ can be relaxed somewhat. In order to illustrate this idea, it is useful to introduce a set of *infinitesimal* parameters $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$. The flow of the vector field $\check{n}$ obviously specifies a matching family of points, namely:

$$\{p_1, p_2, \dots, p_k\} = \{N_{\varepsilon_1}(p), N_{\varepsilon_2}(p_1), \dots, N_{\varepsilon_k}(p_{k-1})\}. \quad (2.73)$$

By invoking the second property described in (2.65), one can immediately note that

$$p_k = N_{\Sigma_{a=1}^k \varepsilon_a}(p) . \quad (2.74)$$

It is important to recall that each parameter ε_a is infinitesimal, that is, $|\varepsilon_a| \ll 1$ for $a = 1, \dots, k$. On account of this condition, it is possible to apply (2.72) iteratively,

$$\varsigma(p_k) = \varsigma \circ \left[N_{\Sigma_{a=1}^k \varepsilon_a}(p) \right] = \varsigma \circ \left[N_{\Sigma_{a=1}^{k-1} \varepsilon_a}(p) \right] + \varepsilon_k = \dots = \varsigma(p) + \sum_{a=1}^k \varepsilon_a . \quad (2.75)$$

Assigning $\varepsilon' := \sum_{a=1}^k \varepsilon_a$ further reveals that (2.75) has the same structure as (2.72),

$$\varsigma \circ [N_{\varepsilon'}(p)] = \varsigma(p) + \varepsilon' , \quad (2.76)$$

where $\varepsilon' := \sum_{a=1}^k \varepsilon_a$ does *not* necessarily satisfy $|\varepsilon'| \ll 1$. More accurately, in (2.76) the quantity $|\varepsilon'|$ must be small enough for the flow $N_{\varepsilon'}(\cdot)$ to be defined. However, it can occur that the parameter ε' is *not* infinitesimal. Since $(\Gamma_3)_\sigma \subset h_\sigma$, (2.76) yields

$$\varsigma \circ N_{\varepsilon'}[(\Gamma_3)_\sigma] = \varsigma[(\Gamma_3)_\sigma] + \varepsilon' = \sigma + \varepsilon' . \quad (2.77)$$

The desired property is thus verified: the normalisation $(\check{n} \rfloor d\sigma) = 1$ entails that the flow of the vector field $\check{n}$ maps the region in space $(\Gamma_3)_\sigma$ to a domain $N_{\varepsilon'}[(\Gamma_3)_\sigma]$ that is a *subset* of the leaf $h_{\sigma+\varepsilon'}$. To stress this point, one can make use of the notation $(\Gamma_3)_{\sigma+\varepsilon'} := N_{\varepsilon'}[(\Gamma_3)_\sigma]$ which is encountered, for example, in the article [20] by Deschamps. With the help of the abbreviation just described, (2.69) turns into:

$$\int_{(\Gamma_3)_\sigma} \mathcal{L}_{\check{n}} \rho = \left[\frac{d}{d\varepsilon} \int_{(\Gamma_3)_{\sigma+\varepsilon}} \rho \right]_{\varepsilon=0} = \frac{d}{d\sigma} \left[\int_{(\Gamma_3)_\sigma} \rho \right] . \quad (2.78)$$

Upon substituting the result (2.78) in the conservation law (2.63), one achieves that

$$\int_{(\Gamma_3)_\sigma} \underline{d}j = - \frac{d}{d\sigma} \left[\int_{(\Gamma_3)_\sigma} \rho \right] . \quad (2.79)$$

Here, the operator $\underline{d}(\cdot)$ is simply the exterior derivative associated with the “spatial” manifold h_σ [41]. Therefore, it is possible to implement Stokes’s theorem, and write

$$\oint_{\partial(\Gamma_3)_\sigma} j = -\frac{d}{d\sigma} \left[\int_{(\Gamma_3)_\sigma} \rho \right]. \quad (2.80)$$

Accordingly, the net current $\oint_{\partial(\Gamma_3)_\sigma} j$ exiting $(\Gamma_3)_\sigma$ is equal to the decrease rate of the total charge $\int_{(\Gamma_3)_\sigma} \rho$, as expected. In other words, for every charge leaving $(\Gamma_3)_\sigma$, the region $(\Gamma_3)_{\sigma+|\varepsilon|}$ must contain one charge less. This last statement invokes the simple act of counting, which one can perform without referring to a notion of distance or parallel transport. Indeed, the law (2.80) is both metric- and connection-free, precisely because it is based on a pure *counting procedure*. The same conclusion holds true for the axiom (2.20), and for the entire formalism discussed so far [41, 44].

Hereafter, an argument is presented which verifies that the charge current density J and the field excitation H are *twisted* differential forms. In fact, under a change of 4-dimensional co-basis $\vartheta^{\alpha'} = L_\alpha^{\alpha'} \vartheta^\alpha$, the components of these quantities transform as

$$J_{\alpha'\beta'\gamma'} = \text{sgn}[\det(L_{\rho'}^\rho)] L_{\alpha'}^\alpha L_{\beta'}^\beta L_{\gamma'}^\gamma J_{\alpha\beta\gamma}, \quad (2.81)$$

$$H_{\alpha'\beta'} = \text{sgn}[\det(L_{\rho'}^\rho)] L_{\alpha'}^\alpha L_{\beta'}^\beta H_{\alpha\beta}, \quad (2.82)$$

where $L_\alpha^{\mu'} L_{\mu'}^\beta = \delta_\alpha^\beta$. Furthermore, δ_α^β is the Kronecker delta symbol, taking the value one if $\alpha = \beta$ and zero otherwise. In accord with the nomenclature used by Burke [13] or Hehl and Obukhov [41], the factor $\text{sgn}[\det(L_{\rho'}^\rho)]$ is called the twist. The objective of the analysis below is to demonstrate on physical grounds that the twist found in (2.81) and (2.82) is mandatory. From the transformation rules just described it is possible to establish that the components of ρ shown in (2.52) obey:

$$\rho_{a'b'c'} = \text{sgn}[\det(L_r^r)] L_{a'}^a L_{b'}^b L_{c'}^c \rho_{abc}. \quad (2.83)$$

under a change of 3-dimensional co-basis $v^{a'} = L_a^{a'} v^a$. One observes that J and H are twisted if and only if the electric charge density ρ is also twisted. As a result, the investigation can focus on this last quantity. It is useful to examine an arbitrary region $(\Sigma_3)_\sigma$ belonging to the folium h_σ , that is, an arbitrary volume in space. Evidently, the total charge contained in such domain is given by $Q = \int_{(\Sigma_3)_\sigma} \rho$. To

make the discussion easier, $(\Sigma_3)_\sigma$ is assumed to fit within the overlap of two spatial charts, giving rise to the coordinates $\{x^1, x^2, x^3\}$ and $\{x^{1'}, x^{2'}, x^{3'}\}$. As explained above, one can determine the total charge inside a volume by using a simple counting procedure. Therefore, evaluating the integral $Q = \int_{(\Sigma_3)_\sigma} \rho$ with respect to $\{x^1, x^2, x^3\}$ or $\{x^{1'}, x^{2'}, x^{3'}\}$ must yield the same outcome. In particular, the quantity Q cannot depend on the relative orientation of the two charts. Stated in brief, the expressions

$$Q = \int_{(\Sigma_3)_\sigma} \frac{1}{3!} \rho_{abc} dx^a \wedge dx^b \wedge dx^c = \int_{(\Sigma_3)_\sigma} \rho_{123} dx^1 \wedge dx^2 \wedge dx^3, \quad (2.84)$$

$$Q = \int_{(\Sigma_3)_\sigma} \frac{1}{3!} \rho_{a'b'c'} dx^{a'} \wedge dx^{b'} \wedge dx^{c'} = \int_{(\Sigma_3)_\sigma} \rho_{1'2'3'} dx^{1'} \wedge dx^{2'} \wedge dx^{3'}, \quad (2.85)$$

are interchangeable. To proceed, one restricts the attention to the scenario wherein the coordinates are linked via $(x^{1'}, x^{2'}, x^{3'}) = (+x^1, +x^2, -x^3)$. Applying this orientation reversing transformation to the volume form of the integral (2.85) specifies that

$$Q = \int_{(\Sigma_3)_\sigma} (-\rho_{1'2'3'}) dx^1 \wedge dx^2 \wedge dx^3. \quad (2.86)$$

Except for the requirement introduced when defining the charts, the portion of space $(\Sigma_3)_\sigma \subset h_\sigma$ is arbitrary. Thus, upon equating (2.86) and (2.84), one concludes that $\rho_{1'2'3'} = -\rho_{123}$. Because of the minus sign appearing in this result, it is established that the charge density ρ is twisted, as prescribed in (2.83). Accordingly, J and H are twisted too, see (2.81) and (2.82). One technical comment is in order: although the argument relies on two holonomic bases, the conclusions remain true when two anholonomic bases are examined [41].

2.2.2 The Lorentz 4-force density

The axiom (2.20), prescribing charge conservation, is certainly important. However, it is not sufficient to construct a full-fledged theory of electromagnetism. Some key building blocks are still missing. Most notably, the field strength F (cf. (2.14)) has not been defined rigorously – yet. In order to clarify the meaning of F , it is useful to consider how the electric field strength $E := -F_\perp$ is often introduced. A number of authors define E in the context of electrostatics, as the force exerted on a point particle of unit charge. More specifically, the field emitted by the particle itself is

neglected, and the force is said to act on a *test* charge (see Jackson [57], or Griffiths [37]). In brief, the concept of electric field strength is derived from that of force. Accordingly, it appears natural to introduce the 2-form F in a similar way.

But before our discussion is extended beyond electrostatics, it is appropriate to modify the definition of E so that every reference to point-like charges is eliminated. As a matter of fact, the first axiom does not mention discrete particles, and the charge current density J is modelled as a smooth field.⁸ The total electrostatic force on a charged body – occupying the finite volume $(\Gamma_3)_\sigma \subset h_\sigma$ – is easily found to be

$$\mathfrak{F}_i := \int_{(\Gamma_3)_\sigma} \mathfrak{f}_i = \int_{(\Gamma_3)_\sigma} E_i \rho , \quad (2.87)$$

where ρ must correspond to a smooth distribution of *test* charge. Every single quantity $\{\mathfrak{f}_1, \mathfrak{f}_2, \mathfrak{f}_3\}$ is a force density, typically integrated over a 3-dimensional region

$$\mathfrak{F}_1 := \int_{(\Gamma_3)_\sigma} \mathfrak{f}_1 , \quad \mathfrak{F}_2 := \int_{(\Gamma_3)_\sigma} \mathfrak{f}_2 , \quad \mathfrak{F}_3 := \int_{(\Gamma_3)_\sigma} \mathfrak{f}_3 . \quad (2.88)$$

As explained in Section 2.1, this last remark implies that $\mathfrak{f}_1, \mathfrak{f}_2$ and $\mathfrak{f}_3$ are all 3-forms. The treatment below aims to establish whether or not these 3-forms carry a twist. Given a basis $\{v^1, v^2, v^3\}$ for the co-tangent space of h_σ , each $\mathfrak{f}_i$ can be expanded as

$$\mathfrak{f}_i = \frac{1}{6} \mathfrak{f}_{pqri} (v^p \wedge v^q \wedge v^r) . \quad (2.89)$$

The specific identity operator, mapping any 3-form on the manifold h_σ to itself, has components $\delta_{pqr}^{stu} = \epsilon^{stu} \hat{\epsilon}_{pqr}$. More explicitly, ϵ^{pqr} and $\hat{\epsilon}_{pqr}$ are the Levi-Civita symbols

$$\left. \begin{matrix} \epsilon^{pqr} \\ \hat{\epsilon}_{pqr} \end{matrix} \right\} = \begin{cases} +1 & \text{if } pqr \text{ is an even permutation of } 123 , \\ 0 & \text{if } pqr \text{ includes a repeated index ,} \\ -1 & \text{if } pqr \text{ is an odd permutation of } 123 . \end{cases} \quad (2.90)$$

⁸The fact that (2.80) was analysed in terms of discrete particles and a counting procedure entails no contradiction: one can view each point-like charge as the suitable limit of a *smooth* distribution.

When the change of basis $v^{i'} = L_i^{i'} v^i$ is performed, the objects ϵ^{pqr} and $\hat{\epsilon}_{pqr}$ satisfy

$$\epsilon^{p'q'r'} = [\det(L_{k'}^k)]^{w_1} L_p^{p'} L_q^{q'} L_r^{r'} \epsilon^{pqr} , \quad (2.91)$$

$$\hat{\epsilon}_{p'q'r'} = [\det(L_{k'}^k)]^{w_2} L_{p'}^p L_{q'}^q L_{r'}^r \hat{\epsilon}_{pqr} , \quad (2.92)$$

where $L_i^{k'} L_{k'}^j = \delta_i^j$, and $w_1 = -w_2 = 1$. One may therefore say that ϵ^{pqr} and $\hat{\epsilon}_{pqr}$ are two tensor densities, of respective weight $w_1 = +1$ and $w_2 = -1$ (see Ref. [41]). Substituting the identity $\mathbf{f}_{pqri} = (1/6) \delta_{pqr}^{stu} \mathbf{f}_{stui} = (1/6) \epsilon^{stu} \mathbf{f}_{stui} \hat{\epsilon}_{pqr}$ into (2.89) yields

$$\mathbf{f}_i = \left[\frac{1}{6} \epsilon^{stu} \mathbf{f}_{stui} \right] \left[\frac{1}{6} \hat{\epsilon}_{pqr} (v^p \wedge v^q \wedge v^r) \right] = \check{\mathbf{f}}_i \hat{\epsilon} , \quad (2.93)$$

provided one sets $\check{\mathbf{f}}_i := (1/6) \epsilon^{stu} \mathbf{f}_{stui}$ and $\hat{\epsilon} := (1/6) \hat{\epsilon}_{pqr} (v^p \wedge v^q \wedge v^r)$. Crucially, the quantity $\check{\mathbf{f}}_i$ is comparable with the force density that emerges from the Euler-Lagrange equations (cf. Krause [66]). Hence, $\check{\mathbf{f}}_i$ must be a twisted covector density

$$\check{\mathbf{f}}_{i'} = \text{sgn}[\det(L_{k'}^k)] [\det(L_{k'}^k)]^w L_{i'}^i \check{\mathbf{f}}_i , \quad (2.94)$$

of weight $w = +1$. A simple calculation then shows that the 3-forms $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ are twisted, and obey $\mathbf{f}_{i'} = L_{i'}^i \mathbf{f}_i$. Equivalently, the mechanical force density $\mathbf{f}_i$ is found to be a *twisted* covector-valued 3-form (cf. Bossavit [10], or Hehl and Obukhov [41]). Now that the properties of $\mathbf{f}_i$ have been thoroughly discussed, one can utilise the law

$$\mathbf{f}_i = E_i \rho , \quad (2.95)$$

as the required electrostatic definition of E . Of course, (2.95) is a trivial consequence of (2.87), ergo ρ is still assumed to be a *test* charge density. The 3-form $\mathbf{f}_i$ is known to carry a twist; on account of (2.81), the same statement holds true for ρ . Accordingly,

$$E_{i'} = L_{i'}^i E_i , \quad (2.96)$$

that is, the electric field strength is a 1-form without twist. Returning to the topic of electrodynamics in X_4 , the structure of (2.95) suggests the following *second axiom*.

The Lorentz “4-force” density f_α , acting on a *test* charge current density J , reads:

$$f_\alpha = (e_\alpha \rfloor F) \wedge J , \quad (2.97)$$

where $\{e_\alpha\}$ is a vector basis [41, 56]. On the left-hand side of (2.97), the 4-force density is a twisted covector-valued 4-form, whose dimension is given by $[f_\alpha] = h[e_\alpha]$. In this formula, h is the dimension of action, not to be confused with Planck’s constant⁹. The second axiom (2.97) provides an operational definition for the field strength 2-form F . In particular, one can readily obtain the dimension of F through

$$[F] = [f_\alpha][e_\alpha]^{-1}[J]^{-1} = h[e_\alpha][e_\alpha]^{-1}q^{-1} = hq^{-1} \stackrel{\text{SI}}{=} \text{Wb} . \quad (2.98)$$

Under the frame change $e_{\alpha'} = L_{\alpha'}^\alpha e_\alpha$, the components of F transform according to

$$F_{\mu'\nu'} = L_{\mu'}^\mu L_{\nu'}^\nu F_{\mu\nu} , \quad (2.99)$$

with $L_\alpha^{\mu'} L_{\mu'}^\beta = \delta_\alpha^\beta$. In other words, the field strength F is found to be a 2-form *without* twist (see (2.96)). At this point, it is useful to recall the space+time split¹⁰

$$F = -d\sigma \wedge E + B , \quad (2.100)$$

see Section 2.1. Indeed, since it is known that $[F] = hq^{-1}$ and $[d\sigma] = t$, one achieves

$$[E] = hq^{-1}t^{-1} \stackrel{\text{SI}}{=} \text{V} , \quad [B] = hq^{-1} \stackrel{\text{SI}}{=} \text{Wb} . \quad (2.101)$$

At any one time instant σ , both E and B are smooth fields on h_σ , and accordingly

$$E = E_a v^a , \quad B = \frac{1}{2} B_{ab} (v^a \wedge v^b) . \quad (2.102)$$

⁹One can harmlessly denote the dimension of action and Planck’s constant with the same letter. In fact, a dimensional equation is always easy to recognise, and furthermore $[\text{Planck’s constant}] = h$.

¹⁰In Section 2.2.1, the decomposition (2.9) was considered first, and the limit (2.11) was calculated at a later stage. Here, the space+time split (2.11) is implemented straight away, for brevity.

Hence, by imposing that $[v^a] = \ell$, one attains the *physical* dimensions of E and B ,

$$[E_a] = hq^{-1}t^{-1}\ell^{-1} \stackrel{\text{SI}}{=} \text{V m}^{-1} , \quad [B_{ab}] = hq^{-1}\ell^{-2} \stackrel{\text{SI}}{=} \text{Wb m}^{-2} =: \text{T} . \quad (2.103)$$

By virtue of (2.100), the contraction $(e_\alpha \rfloor F)$ appearing in (2.97) can be expanded as

$$(e_\alpha \rfloor F) = -e_\alpha \rfloor (d\sigma \wedge E) + (e_\alpha \rfloor B) = (e_\alpha \rfloor E)d\sigma + (e_\alpha \rfloor B) - (e_\alpha \rfloor d\sigma)E . \quad (2.104)$$

Substituting (2.49) and (2.104) into the second axiom $f_\alpha = (e_\alpha \rfloor F) \wedge J$ implies that

$$f_\alpha = d\sigma \wedge [(e_\alpha \rfloor E)\rho + j \wedge (e_\alpha \rfloor B) - (e_\alpha \rfloor d\sigma) j \wedge E] = {}^\perp(f_\alpha) , \quad (2.105)$$

$$(f_\alpha)_\perp = (e_\alpha \rfloor E)\rho + j \wedge (e_\alpha \rfloor B) - (e_\alpha \rfloor d\sigma) j \wedge E . \quad (2.106)$$

On the sheet h_σ , one can always construct a vector basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In turn – when the choice $(e_0, e_a) = (\check{n}, \mathbf{e}_a)$ is enforced – one obtains that $(e_\alpha \rfloor d\sigma) = (1, 0, 0, 0)$ and:

$$\mathbf{f}_0 := (f_0)_\perp = -j \wedge E , \quad (2.107)$$

$$\mathbf{f}_a := (f_a)_\perp = (\mathbf{e}_a \rfloor E)\rho + j \wedge (\mathbf{e}_a \rfloor B) . \quad (2.108)$$

Here, $\mathbf{f}_a$ is the Lorentz “3-force” density, which generalises the electrostatic quantity (2.95). Moreover, the 3-form $\mathbf{f}_0$ is easily recognised to be the electric power density. Selecting a frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ characterised by $[\mathbf{e}_a] = \ell^{-1}$, allows one to observe that

$$[\mathbf{f}_0] = ht^{-2} \stackrel{\text{SI}}{=} \text{W} , \quad [\mathbf{f}_a] = h\ell^{-1}t^{-1} \stackrel{\text{SI}}{=} \text{N} . \quad (2.109)$$

The *physical* dimensions of $\mathbf{f}_0$ and $\mathbf{f}_a$ merely include an extra factor of ℓ^{-3} (cf. (2.89)). Two comments are in order: firstly, the field strength 2-form F is an intensive quantity; secondly, the axiom (2.97) does *not* depend on a metric or an affine structure.

Active charge, passive charge

Problems in electrodynamics are typically classified in three broad classes, according to the required solution strategy, see Jackson [57]. In the *first class* of problems, a distribution of charge and current is given, and the resulting electromagnetic fields

are calculated. Equivalently, the electric charge is modelled solely in its source (active) role. In the *second class* of problems, some applied fields are specified, and the consequent charge motion is computed. Hence, only the test (passive) role of the electric charge is taken into account. Finally, there exists a *third class* of problems, where one must analyse a complicated interaction: the charge motion affects the electromagnetic fields, and vice versa. Usually, a problem that falls in the first or in the second class does so in an approximate sense. Therefore, strictly speaking, most questions in electrodynamics actually belong to the third class.

When the electric charge is regarded solely in its active role, the axiom (2.97) has no relevance. Conversely, if the electric charge is considered only in its passive nature, $f_\alpha = (e_\alpha \rfloor F) \wedge J$ is the total electromagnetic 4-force density. Ergo, both in the first and in the second class of electrodynamic problems, the interpretation of (2.97) is simple. By contrast, when the mutual interaction of charge motion and electromagnetic fields is examined, the meaning of the second axiom is not as trivial. Given a problem of the third class, one may assume that the solution emerges from the interplay of two steps – respectively assigning a source and a test role to the electric charge. Then, *for the duration of the passive step*, the familiar formula $f_\alpha = (e_\alpha \rfloor F) \wedge J$ is still fundamental. This idea is illustrated well by Jackson's calculation of the self-force acting on a charged sphere (Chapter 16 of Ref. [57]).

2.2.3 Magnetic flux is conserved

The magnetic field strength B is a 2-form, achieved from the space+time decomposition (2.100). Correspondingly, B is typically integrated over an oriented surface $(\Gamma_2)_\sigma \subset h_\sigma$, in order to obtain the *magnetic flux* $\int_{(\Gamma_2)_\sigma} B$. For instance, when analysing a single-loop alternator, one integrates B across the rotating loop's area, so as to evaluate the oscillatory magnetic flux. Crucially, given an oriented volume in space $(\Gamma_3)_\sigma \subset h_\sigma$, the integral $\oint_{\partial(\Gamma_3)_\sigma} B$ is invariably zero. This statement is of central importance to electrodynamics, and is known as the law of magnetic flux *conservation*. Intuitively, the vanishing of $\oint_{\partial(\Gamma_3)_\sigma} B$ demands that any magnetic flux line entering $(\Gamma_3)_\sigma$ via $\partial(\Gamma_3)_\sigma$ must subsequently exit $(\Gamma_3)_\sigma$ through $\partial(\Gamma_3)_\sigma$. Hence, one can easily show that the magnetic flux is conserved if and only if magnetic charges do *not* exist. To date, no detection of magnetic monopoles has ever been con-

firmed. The experimental search for magnetic charges is ongoing, and continuously improving its accuracy (cf. Nakamura *et al.* [102]). For example, the production of Dirac monopoles was recently investigated in particle collisions of the hadron-hadron [2, 62], lepton-lepton [1] and hadron-lepton [3] type. Accordingly, the conservation of magnetic flux is a frequently tested law, which one can reasonably adopt as the third axiom of classical electrodynamics.

The third axiom of classical electromagnetism

The *third axiom* of electromagnetism can be formulated as follows. The integral of the field strength 2-form F over *any* oriented 2-dimensional cycle C_2 is zero [41, 56]:

$$\oint_{C_2} F = 0 , \quad \partial C_2 = 0 , \quad (2.110)$$

where (2.110) expresses the law of magnetic flux conservation in the spacetime X_4 . Since the operator $\partial(\cdot)$ is nilpotent, the boundary $\partial\Xi_3$ of each 3-dimensional domain Ξ_3 is identically a cycle: $\partial(\partial\Xi_3) = \partial^2\Xi_3 \equiv 0$. Thus, whilst there may exist some 2-cycle which is not a boundary, one can always choose C_2 so that it coincides with $\partial\Xi_3$. Implementing this selection in (2.110) – and using Stokes’s theorem – leads to

$$0 = \oint_{\partial\Xi_3} F = \int_{\Xi_3} dF . \quad (2.111)$$

Given that Ξ_3 is an arbitrary 3-dimensional region, (2.111) immediately implies that

$$dF = 0 . \quad (2.112)$$

The homogeneous Maxwell equation is therefore attained, that is, the field strength F is found to be a *closed* 2-form. Remarkably, by considering the properties (2.110) and (2.112) in combination, one can further demonstrate that F is an *exact* 2-form,

$$F = dA , \quad (2.113)$$

because de Rham's map is one-to-one [33]¹¹. The field A is called the *electromagnetic potential*, and is a 1-form carrying *no* twist. From (2.113), one can also deduce that

$$[A] = [F] = hq^{-1} \stackrel{\text{SI}}{=} \text{Wb} . \quad (2.114)$$

It is important to note that the potential A is only defined up to a closed 1-form χ ,

$$A' := A + \chi \quad \text{with} \quad d\chi = 0 \quad \Rightarrow \quad dA' = dA + d\chi = dA = F , \quad (2.115)$$

where the excitation 2-form H was observed, in (2.24), to have a similar behaviour. Besides this remark, there is a fundamental difference between A and H . As explained in Section 2.2.1, the assumption that $H = 0$ when $F = 0$ specifies H uniquely. By contrast, no such argument is available so as to constrain A [41]. Accordingly, the excitation H is a measurable quantity, while the potential A is *not*.

Space+time split of (2.112) and (2.113)

The process of decomposing $dF = 0$ in space+time is simplified by the previous analysis performed on $d\Psi = \Phi$. Indeed, by replacing $\{\Phi, \Psi\}$ with $\{0, F\}$, one can readily apply the identities (2.43)-(2.44) to (2.112). The split of $dF = 0$ thus reads¹²

$$\mathcal{L}_{\vec{n}}(\underline{F}) - \underline{d}(F_{\perp}) = 0 , \quad \underline{d}(\underline{F}) = 0 . \quad (2.116)$$

Now, it is possible to substitute (2.100) in (2.116) and obtain the familiar equations

$$\underline{d}E + \mathcal{L}_{\vec{n}}B = 0 , \quad (\text{Faraday's law}) \quad (2.117)$$

$$\underline{d}B = 0 , \quad (\text{Gauss's law for } B) \quad (2.118)$$

where (2.118) is the crucial law of magnetic flux conservation, in differential version. As one might expect, (2.113) can be decomposed in space+time by making use of

¹¹As an aside, one could replace the axiom (2.110) with the weaker condition $\oint_{\partial\Xi_3} F = 0$. In this scenario, (2.112) would still hold true. Yet, (2.113) would not necessarily retain its global validity.

¹²In Section 2.2.1, the decomposition (2.9) was invoked first, and the true *space+time* split was calculated at a later stage. Here, the intermediate step (2.9) is not discussed, for the sake of brevity.

(2.43) and (2.44). In particular, $F = dA$ has the same structure as $\Phi = d\Psi$, whence

$$F_{\perp} = \mathcal{L}_{\check{n}}(\underline{A}) - \underline{d}(A_{\perp}) , \quad \underline{F} = \underline{d}(\underline{A}) . \quad (2.119)$$

Implementing (2.100) in (2.119) allows one to interpret the projections $A_{\perp}$ and $\underline{A}$ as

$$A = A_{\perp} d\sigma + \underline{A} = -\varphi d\sigma + \mathcal{A} , \quad (2.120)$$

where the fields φ and $\mathcal{A}$ are the 3-dimensional *scalar* and *covector* potentials, respectively. By virtue of (2.119), E and B can be expressed in terms of φ and $\mathcal{A}$,

$$E = -\underline{d}\varphi - \mathcal{L}_{\check{n}}\mathcal{A} , \quad (2.121)$$

$$B = \underline{d}\mathcal{A} . \quad (2.122)$$

The dimensions of φ and $\mathcal{A}$ are straightforwardly derived from the expansion (2.120),

$$[\varphi] = [A]t^{-1} = hq^{-1}t^{-1} \stackrel{\text{SI}}{=} \text{V} , \quad [\mathcal{A}] = [A] = hq^{-1} \stackrel{\text{SI}}{=} \text{Wb} . \quad (2.123)$$

Given a basis $\{v^a\}$ for the co-tangent space of h_{σ} – with $[v^a] = \ell$ – one finds that:

$$\mathcal{A} = \mathcal{A}_a v^a \quad \Rightarrow \quad [\mathcal{A}_a] = hq^{-1}\ell^{-1} \stackrel{\text{SI}}{=} \text{Wb m}^{-1} . \quad (2.124)$$

Integral laws and counting procedures

In what follows, the integral versions of (2.117) and (2.118) are calculated. One can begin the manipulation of (2.117) by considering an oriented surface $(\Gamma_2)_{\sigma} \subset h_{\sigma}$, across which $\int_{(\Gamma_2)_{\sigma}} \underline{d}E + \int_{(\Gamma_2)_{\sigma}} \mathcal{L}_{\check{n}}B = 0$. The useful identity (2.68) then implies that

$$\int_{(\Gamma_2)_{\sigma}} \underline{d}E = - \left[\frac{d}{d\varepsilon} \int_{N_{\varepsilon}[(\Gamma_2)_{\sigma}]} B \right]_{\varepsilon=0} . \quad (2.125)$$

In the analysis following (2.69) it is demonstrated that, as a result of the normalisation $(\check{n}]d\sigma) = 1$, the flow N_{ε} maps points in h_{σ} to points that strictly belong to $h_{\sigma+\varepsilon}$. Applying this observation to the domain of integration on the right-hand side of (2.125) verifies that the area $N_{\varepsilon}[(\Gamma_2)_{\sigma}]$ is a *subset* of the leaf $h_{\sigma+\varepsilon}$. One is thus

motivated to introduce the notation $(\Gamma_2)_{\sigma+\epsilon} := N_\epsilon[(\Gamma_2)_\sigma]$, and to express (2.125) as

$$\int_{(\Gamma_2)_\sigma} \underline{d}E = - \left[\frac{d}{d\epsilon} \int_{(\Gamma_2)_{\sigma+\epsilon}} B \right]_{\epsilon=0} = - \frac{d}{d\sigma} \left[\int_{(\Gamma_2)_\sigma} B \right]. \quad (2.126)$$

By invoking Stokes's theorem in the leftmost integral $\int_{(\Gamma_2)_\sigma} \underline{d}E$, it is established that

$$\oint_{\partial(\Gamma_2)_\sigma} E = - \frac{d}{d\sigma} \left[\int_{(\Gamma_2)_\sigma} B \right], \quad (2.127)$$

which is Faraday's law in the well-known integral version. Clearly, (2.118) leads to

$$\int_{(\Gamma_3)_\sigma} \underline{d}B = 0, \quad (2.128)$$

for any oriented $(\Gamma_3)_\sigma \subset h_\sigma$. By invoking Stokes's theorem, one can therefore write

$$\oint_{\partial(\Gamma_3)_\sigma} B = 0, \quad (2.129)$$

whence the integral version of Gauss's law for B is obtained. Equation (2.129) demands that every magnetic flux line entering $(\Gamma_3)_\sigma$ via $\partial(\Gamma_3)_\sigma$ must then exit $(\Gamma_3)_\sigma$ through $\partial(\Gamma_3)_\sigma$. In other words, magnetic flux is always a *conserved* quantity.

The following example illustrates the properties of Gauss's law for B . A spherical sample of some type-II superconductor is placed in the middle of a very long solenoid. The temperature of the sphere T is kept constant, below the critical value T_c for the chosen superconductor. Initially, the current flowing through the solenoid is zero. However, it is subsequently increased by a small amount, so that a weak magnetic field is applied on the superconducting sample. Under these conditions, provided the radius of the sphere is large [146], a considerable fraction of the magnetic field is expelled from the sample's interior. This phenomenon is known as the Meissner-Ochsenfeld effect. When a stronger current is run through the solenoid, and the resulting field overcomes a certain threshold, quantised magnetic flux lines begin to pierce the sphere (cf. Figure 2.2). Equation (2.129) then prescribes that the *number* of flux lines entering the sample's surface must be equal to the *number* of flux lines leaving the sample's surface. Evidently – in the proposed example – the conservation of magnetic flux can be verified through a *counting* procedure, which is

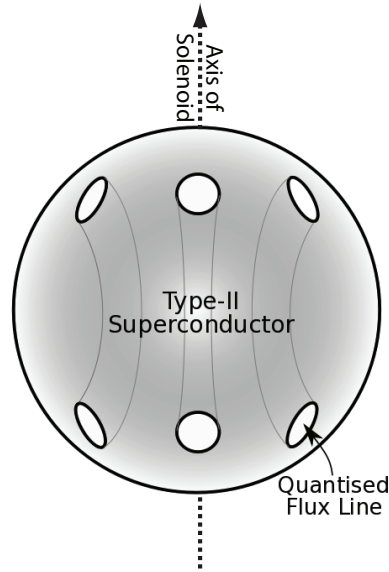


Figure 2.2: When the magnetic field applied on a type-II superconductor exceeds a critical value, thin filaments of material can re-gain a finite electric resistance. As a result, *quantised* magnetic flux lines are allowed to pierce through the sample. The spherical geometry depicted here is endowed with a number of interesting properties, see Ref. [146].

not related to the concept of distance or parallel transport. This is a manifestation of the fact that the axiom (2.110) is both metric- and connection-free [41]. The behaviour of a spherical type-II superconductor, subject to a uniform magnetic field, was investigated numerically by Xu, Milošević and Peeters [146]. It may be argued that the choice of a spherical geometry forces the discussion to be rather abstract. As a matter of fact, a literature search appears to indicate that the setup just considered has never been tested in a real experiment. Hence, it is not possible to supplement the above example with laboratory data such as the critical temperature or the value of the magnetic field. By contrast, such information is readily available when a slab geometry is investigated, as in Hehl and Obukhov’s manual [41].¹³ Having mentioned these drawbacks, the spherical shape is convenient for analysing a remarkable effect. The current across the solenoid is reduced, whereby the magnetic field is decreased below the threshold value again. Ergo, the quantised flux lines are banned from the sample’s interior due to the Meissner-Ochsenfeld phenomenon. Notably, for a field line to be expelled, two opposite locations where the surface of the sphere is pierced have to migrate towards the equator. There, the entrance and exit points *annihilate*.

¹³Another problem is that, in order to define a sphere, one needs to refer to a metric. Thankfully, as regards the counting of flux lines invoked in an earlier argument, such observation is unimportant.

2.3 The constitutive relation

For reasons of brevity, the course of this Thesis now departs from the axiomatic structure put forward by Hehl and Obukhov [41, 44]. A substantial effort has been made so as to ensure that this choice does not affect the rigour of the presentation.

Section 2.2.3 brought the derivation of Maxwell's equations to a successful end. One can therefore take a step forward¹⁴, and attempt to solve $\mathrm{d}H = J$ and $\mathrm{d}F = 0$ as part of an *initial value* problem. For instance, given a configuration of $\mathcal{D}$ and B on the folium h_σ , one can set about calculating the electromagnetic fields over some neighborhood of h_σ . In carrying out this task, the Maxwell laws (2.48) and (2.118) are found to serve a different purpose as opposed to (2.47) and (2.117). Specifically,

$$\mathrm{d}\mathcal{D} = \rho \quad \text{and} \quad \mathrm{d}B = 0 \quad (2.130)$$

are used on the leaf h_σ , so as to constrain the initial data for $\mathcal{D}$ and B ; by contrast,

$$\mathrm{d}\mathcal{H} - \mathcal{L}_{\tilde{n}}\mathcal{D} = j \quad \text{and} \quad \mathrm{d}E + \mathcal{L}_{\tilde{n}}B = 0 \quad (2.131)$$

are used in order to evolve the fields away from the sheet h_σ . Once the equations in (2.131) have been applied, it is necessary to ensure that the conditions in (2.130) are still satisfied: remarkably, provided electric charge is always conserved ($\mathrm{d}j = -\mathcal{L}_{\tilde{n}}\rho$), one can guarantee that this is the case. At this point, it is essential to observe that the present discussion is subject to a crucial caveat. As a matter of fact, unless further restrictions are introduced, the Maxwell system (2.130)–(2.131) is underdetermined. More explicitly, while the laws in (2.131) amount to $2 \times 3 = 6$ *evolution* equations, the fields $\{\mathcal{H}, \mathcal{D}, E, B\}$ consist of $4 \times 3 = 12$ independent components. Accordingly, there are $12 - 6 = 6$ variables that need to be specified further [41]. The additional elimination of 6 entries is achieved by defining the electromagnetic

¹⁴The introduction provided here is largely based on a recent article by Perlick (see Ref. [112]).

response of spacetime, that is, the *spacetime relation*. Two important examples are

$$\begin{cases} \mathcal{H} = \mathcal{H}(\mathcal{D}, B), \\ E = E(\mathcal{D}, B), \end{cases} \quad (2.132)$$

$$\begin{cases} \mathcal{H} = \mathcal{H}(E, B), \\ \mathcal{D} = \mathcal{D}(E, B), \end{cases} \quad (2.133)$$

where (2.132) and (2.133) are not mutually exclusive. Indeed, some specific response laws can be expressed, equally well, in the form (2.132) or in the form (2.133). The spacetime relations (2.132) and (2.133) are said to be *local* [41], respectively because

$$\begin{cases} \mathcal{H} \text{ at } p \in X_4 \text{ depends on the fields } \mathcal{D} \text{ and } B \text{ at the same } p \in X_4, \\ E \text{ at } p \in X_4 \text{ depends on the fields } \mathcal{D} \text{ and } B \text{ at the same } p \in X_4, \end{cases} \quad (2.134)$$

$$\begin{cases} \mathcal{H} \text{ at } p \in X_4 \text{ depends on the fields } E \text{ and } B \text{ at the same } p \in X_4, \\ \mathcal{D} \text{ at } p \in X_4 \text{ depends on the fields } E \text{ and } B \text{ at the same } p \in X_4. \end{cases} \quad (2.135)$$

Following a different terminology, if the electromagnetic response of spacetime is local, one can assert that there is *no dispersion*¹⁵. Both the property that a spacetime relation can be formulated as (2.132) and the possibility of writing (2.133) have useful consequences: on the one hand, only when one obtains $\mathcal{H} = \mathcal{H}(\mathcal{D}, B)$ and $E = E(\mathcal{D}, B)$, there exists a chance that the initial value problem (2.130)–(2.131) is well-posed [112]; on the other hand, if one achieves $\mathcal{H} = \mathcal{H}(E, B)$ and $\mathcal{D} = \mathcal{D}(E, B)$, the electromagnetic response of spacetime takes on a truly 4-dimensional meaning,

$$H = H(F) . \quad (2.136)$$

It is also worth noting that the laws (2.132) and (2.133) are not necessarily linear¹⁶.

¹⁵Mashhoon [95] proposed that the observer's acceleration could induce dispersion *in vacuum*.

¹⁶The literature investigating non-linear effects *in vacuum* is dominated by two alternative models, the Born-Infeld theory [8] and the Heisenberg-Euler theory [49]. The Born-Infeld and Heisenberg-Euler spacetime relations are special cases of Plebanski's more general ansatz (cf. [41]).

2.3.1 Fixing some assumptions

The spacetime relation is henceforward assumed to be *local*, with the specific form

$$H = \kappa(F) . \quad (2.137)$$

Of course, the operator $\kappa(\cdot)$ maps 2-forms at the point $p \in X_4$ to twisted 2-forms at the same location. Furthermore, (2.137) is obviously a re-statement of (2.136) with an improved notation. The response of spacetime is hereby restricted to be *linear*. Equivalently, given any two functions $a : X_4 \rightarrow \mathbb{R}$ and $b : X_4 \rightarrow \mathbb{R}$, one requires that

$$\kappa(a\Psi_1 + b\Psi_2) = a\kappa(\Psi_1) + b\kappa(\Psi_2) , \quad (2.138)$$

where Ψ_1 and Ψ_2 are arbitrary 2-forms. Dispersionless linear laws of the type (2.137) are often studied in the electrodynamics of laboratory media. As a matter of fact, they constitute a valid approximation for the material response under specific types of illumination. More explicitly, (i) the continuous medium picture demands that the applied fields vary slowly-enough in space; (ii) the material's polarisation and magnetisation properties are non-trivial solely when the applied fields vary slowly-enough in time; (iii) non-linear phenomena can only be neglected provided the light intensity is sufficiently low; (iv) the effect of dispersion can be disregarded almost exclusively if a plane-wave is considered. Admittedly, these restrictions are rather severe. Nonetheless, naively interpreting $H = \kappa(F)$ as the electromagnetic response for a linear laboratory medium allows one to derive several key results. Accordingly, equation (2.137) is now assumed to represent a spacetime relation or a material relation, depending on the context. The “neutral” names *constitutive law* and *medium response* are used to indicate $H = \kappa(F)$ in general. One final comment is in order: the conditions (ii) and (iv) limit the analysis of light propagation in materials to the steady-state regime. Correspondingly, one is obliged to study a largely *static* situation, which contains no information about the *evolution* and the well-posedness of the initial-value problem. This remark partly justifies why, in subsequent work, the constitutive law (2.137) is never required to take the form (2.132).

2.3.2 Properties of the constitutive tensor

Let $\{e_\alpha\}$ be a vector frame on the manifold X_4 , whose dual $\{\vartheta^\alpha\}$ satisfies $e_\alpha \lrcorner \vartheta^\beta = \delta_\alpha^\beta$. Because the map κ is defined to be local and linear, one can achieve the expansion

$$\kappa = \frac{1}{4} \kappa_{\alpha\beta}{}^{\mu\nu} (\vartheta^\alpha \wedge \vartheta^\beta) \otimes (e_\mu \wedge e_\nu) , \quad (2.139)$$

where $\otimes$ is the tensor product [101], and the components $\kappa_{\alpha\beta}{}^{\mu\nu}$ obey the symmetries

$$\kappa_{\alpha\beta}{}^{\mu\nu} = -\kappa_{\beta\alpha}{}^{\mu\nu} = -\kappa_{\alpha\beta}{}^{\nu\mu} . \quad (2.140)$$

The excitation H and field-strength F can be easily expressed in terms of $\{\vartheta^\alpha\}$, as

$$H = \frac{1}{2} H_{\alpha\beta} (\vartheta^\alpha \wedge \vartheta^\beta) , \quad (2.141)$$

$$F = \frac{1}{2} F_{\alpha\beta} (\vartheta^\alpha \wedge \vartheta^\beta) . \quad (2.142)$$

Hence, by substituting (2.139), (2.141) and (2.142) into (2.137) one readily attains

$$H_{\alpha\beta} = \frac{1}{2} \kappa_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu} . \quad (2.143)$$

Under the frame change $e_{\alpha'} = L_{\alpha'}^\alpha e_\alpha$, the laws (2.82) and (2.99) then prescribe that

$$\kappa_{\alpha'\beta'}{}^{\mu'\nu'} = \text{sgn}[\det(L_{\rho'}^\rho)] L_{\alpha'}^\alpha L_{\beta'}^\beta L_{\mu'}^{\mu'} L_{\nu'}^{\nu'} \kappa_{\alpha\beta}{}^{\mu\nu} , \quad (2.144)$$

with $L_{\alpha'}^{\mu'} L_{\mu'}^\beta = \delta_{\alpha'}^\beta$. Accordingly, the map $\kappa(\cdot)$ is found to carry a twist, as expected. By observing that $H_{\alpha\beta} = -H_{\beta\alpha}$ and $F_{\alpha\beta} = -F_{\beta\alpha}$, one can show that $H_{\alpha\beta}$ and $F_{\alpha\beta}$ only contain 6 independent entries. In turn, these can be arranged in two columns:

$$H_I = \begin{bmatrix} H_{01} \\ H_{02} \\ H_{03} \\ H_{23} \\ H_{31} \\ H_{12} \end{bmatrix} , \quad F_I = \begin{bmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{23} \\ F_{31} \\ F_{12} \end{bmatrix} . \quad (2.145)$$

Hereafter, capitalised Latin indices $\{I, J, \dots\}$ are assumed to represent pairs of anti-symmetric spacetime indices $\{[\alpha\beta], [\mu\nu], \dots\}$. In particular, the identification rule

$$\{[\alpha\beta] = [01], [02], [03], [23], [31], [12]\} \rightarrow \{I = 1, 2, 3, 4, 5, 6\}, \quad (2.146)$$

is used throughout. As an example, one can re-state (2.143) in the alternative form

$$H_I = \kappa_I^J F_J, \quad (2.147)$$

whereby it appears evident that the medium response $\kappa_{\alpha\beta}^{\mu\nu}$ consists of $6 \times 6 = 36$ independent components. The indices $\{I, J, \dots\}$ are also useful to study the powers

$$\kappa^0 := \text{Id}, \quad \Leftrightarrow \quad (\kappa^0)_I^J := \delta_I^J, \quad (2.148)$$

$$\kappa^1 := \kappa, \quad \Leftrightarrow \quad (\kappa^1)_I^J := \kappa_I^J, \quad (2.149)$$

$$\kappa^2 := \kappa \circ \kappa, \quad \Leftrightarrow \quad (\kappa^2)_I^J := \kappa_I^{I_1} \kappa_{I_1}^J, \quad (2.150)$$

$$\kappa^3 := \kappa \circ \kappa \circ \kappa, \quad \Leftrightarrow \quad (\kappa^3)_I^J := \kappa_I^{I_1} \kappa_{I_1}^{I_2} \kappa_{I_2}^J, \quad (2.151)$$

$$\vdots$$

$$\vdots$$

with $\delta_I^J = 1$ if $I = J$ and $\delta_I^J = 0$ if $I \neq J$. It is interesting to note that the operators $\{\kappa^1, \kappa^3, \kappa^5, \dots\}$ are endowed with a twist, whilst the maps $\{\kappa^0, \kappa^2, \kappa^4, \dots\}$ are not. Calculating the *trace* of the expressions (2.148)–(2.151) leads to a family of invariants

$$\text{tr}(\kappa^0) = 6, \quad (2.152)$$

$$\text{tr}(\kappa^1) = \kappa_I^I, \quad (2.153)$$

$$\text{tr}(\kappa^2) = \kappa_I^{I_1} \kappa_{I_1}^I, \quad (2.154)$$

$$\text{tr}(\kappa^3) = \kappa_I^{I_1} \kappa_{I_1}^{I_2} \kappa_{I_2}^I, \quad (2.155)$$

$$\vdots$$

where $\{\text{tr}(\kappa^0), \text{tr}(\kappa^2), \text{tr}(\kappa^4), \dots\}$ and $\{\text{tr}(\kappa^1), \text{tr}(\kappa^3), \text{tr}(\kappa^5), \dots\}$ are, respectively, true and twisted scalars. As an aside, the Cayley-Hamilton theorem dictates that:

$$\sum_{a=0}^6 (-1)^a c_a \kappa^{6-a} = c_0 \kappa^6 - c_1 \kappa^5 + \dots - c_5 \kappa^1 + c_6 \kappa^0 = 0, \quad (2.156)$$

for a set of real coefficients $\{c_0, c_1, \dots, c_6\}$ that can be calculated through the scheme

$$c_0 = 1, \quad (2.157)$$

$$c_1 = c_0 \operatorname{tr}(\kappa^1), \quad (2.158)$$

$$c_2 = \frac{1}{2} [c_1 \operatorname{tr}(\kappa^1) - c_0 \operatorname{tr}(\kappa^2)], \quad (2.159)$$

$$\vdots$$

$$c_b = \frac{1}{b} \sum_{a=1}^b (-1)^{a+1} c_{b-a} \operatorname{tr}(\kappa^a), \quad (b \geq 1), \quad (2.160)$$

cf. Serre [124]. More explicitly, one can successfully obtain $\{c_0, c_1, \dots, c_6\}$ by applying the formulae (2.157)–(2.160) in *strict order*. It is trivial to demonstrate that the invariants $\{c_1, c_3, c_5\}$ are twisted, whilst the invariants $\{c_0, c_2, c_4, c_6\}$ are not. Since

$$c_6 = \det(\kappa) = \det(\kappa_I^J), \quad (2.161)$$

evaluating the determinant of the 6×6 matrix κ_I^J clearly generates a true scalar. From (2.137), the constitutive map κ is found to have dimensions of inverse resistance

$$[\kappa] = [H][F]^{-1} = h^{-1}q^2 = (hq^{-2})^{-1} \stackrel{\text{SI}}{=} \Omega^{-1}, \quad (2.162)$$

see Appendix A. Therefore, upon inspecting (2.148)–(2.160), one can conclude that

$$[\kappa^b] = [\operatorname{tr}(\kappa^b)] = [c_b] = (hq^{-2})^{-b} \stackrel{\text{SI}}{=} \Omega^{-b}, \quad (b \geq 0). \quad (2.163)$$

It is essential to remark that the existence of a medium response does not *necessarily* imply a notion of distance. In fact, it will soon be shown that a metric can be deduced from the constitutive law (2.137) *solely* if severe restrictions are imposed.

p	p -multivector	Typical basis set
$p = 1$	vectors	$\{e_0, e_1, e_2, e_3\}$
$p = 2$	bivectors	$\{e_0 \wedge e_1, e_0 \wedge e_2, e_0 \wedge e_3, e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2\}$
$p = 3$	trivectors	$\{e_1 \wedge e_2 \wedge e_3, e_0 \wedge e_3 \wedge e_2, e_0 \wedge e_1 \wedge e_3, e_0 \wedge e_2 \wedge e_1\}$
$p = 4$	quadrivectors	$\{e_0 \wedge e_1 \wedge e_2 \wedge e_3\}$

Table 2.2: A summary of the multivector types in 4-dimensions. Each entry is equipped with an example of basis set. Whenever p is known, it is preferable to use the specific names “vector” ($p=1$), “bivector” ($p=2$), “trivector” ($p=3$) and “quadrivector” ($p=4$).

2.3.3 The 4-dimensional Levi-Civita symbols

In what follows, several manipulations make use of the Levi-Civita symbols [41, 113]

$$\left. \begin{array}{l} \epsilon^{\alpha\beta\gamma\delta} \\ \hat{\epsilon}_{\alpha\beta\gamma\delta} \end{array} \right\} := \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation of } 0123, \\ 0 & \text{if } \alpha\beta\gamma\delta \text{ includes a repeated index,} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation of } 0123, \end{cases} \quad (2.164)$$

defined in 4-dimensions. When a change of frame is performed, $\epsilon^{\alpha\beta\gamma\delta}$ and $\hat{\epsilon}_{\alpha\beta\gamma\delta}$ obey

$$\epsilon^{\alpha'\beta'\gamma'\delta'} = [\det(L_{\rho'}^{\rho})]^{w_1} L_{\alpha}^{\alpha'} L_{\beta}^{\beta'} L_{\gamma}^{\gamma'} L_{\delta}^{\delta'} \epsilon^{\alpha\beta\gamma\delta}, \quad (2.165)$$

$$\hat{\epsilon}_{\alpha'\beta'\gamma'\delta'} = [\det(L_{\rho'}^{\rho})]^{w_2} L_{\alpha'}^{\alpha} L_{\beta'}^{\beta} L_{\gamma'}^{\gamma} L_{\delta'}^{\delta} \hat{\epsilon}_{\alpha\beta\gamma\delta}, \quad (2.166)$$

with $w_1 = -w_2 = 1$. Thus, according to the terminology of Hehl and Obukhov [41], $\epsilon^{\alpha\beta\gamma\delta}$ and $\hat{\epsilon}_{\alpha\beta\gamma\delta}$ are tensor densities of weight $w_1 = +1$ and $w_2 = -1$, respectively. It is worth noting that the 4-dimensional Levi-Civita symbols can be expressed in terms of the indices $\{I, J, \dots\}$. Referring back to (2.164), one can easily prove that

$$\left. \begin{array}{l} \epsilon^{IJ} \\ \hat{\epsilon}_{IJ} \end{array} \right\} = \left[\begin{array}{c|c} \mathbb{O}_{3 \times 3} & \mathbb{I}_{3 \times 3} \\ \hline \mathbb{I}_{3 \times 3} & \mathbb{O}_{3 \times 3} \end{array} \right], \quad (2.167)$$

with $\mathbb{O}_{3 \times 3}$ and $\mathbb{I}_{3 \times 3}$ being the 3×3 null and unit matrices. Hereafter, the so-called multivector fields (cf. Table 2.2) are employed. A detailed introduction to these objects can be found in Lindell [79]. Every p -multivector field can be expanded as

$$C = \frac{1}{p!} C^{\alpha_1 \dots \alpha_p} (e_{\alpha_1} \wedge \dots \wedge e_{\alpha_p}), \quad (2.168)$$

where exchanging any two indices in $C^{\alpha_1 \dots \alpha_p}$ generates a sign flip, that is, $C^{\alpha_1 \dots \alpha_p}$ is fully antisymmetric. Moreover, given any transformation $\vartheta^{\alpha'} = L_{\alpha}^{\alpha'} \vartheta^{\alpha}$, one has that

$$C^{\alpha'_1 \dots \alpha'_p} = L_{\alpha_1}^{\alpha'_1} \dots L_{\alpha_p}^{\alpha'_p} C^{\alpha_1 \dots \alpha_p} . \quad (2.169)$$

As one might expect, *twisted* multivectors and multivector *densities* do exist – see Burke [13], Jancewicz [60] and Schouten [119]. For such fields, (2.169) is replaced by:

$$C^{\alpha'_1 \dots \alpha'_p} = \text{sgn}[\det(L_{\rho'}^{\rho})] L_{\alpha_1}^{\alpha'_1} \dots L_{\alpha_p}^{\alpha'_p} C^{\alpha_1 \dots \alpha_p} , \quad (2.170)$$

$$C^{\alpha'_1 \dots \alpha'_p} = [\det(L_{\rho'}^{\rho})]^w L_{\alpha_1}^{\alpha'_1} \dots L_{\alpha_p}^{\alpha'_p} C^{\alpha_1 \dots \alpha_p} . \quad (2.171)$$

Returning to the topic of Levi-Civita symbols, one is now in the position to show that $\epsilon^{\alpha\beta\gamma\delta}$ defines an operator, mapping each p -form to a $(4-p)$ -multivector density. Starting from $\Theta = \frac{1}{p!} \Theta_{\beta_1 \dots \beta_p} (\vartheta^{\beta_1} \wedge \dots \wedge \vartheta^{\beta_p})$, it is instructive to form the contraction

$$\check{\Theta}^{\alpha_1 \dots \alpha_{4-p}} := \frac{1}{p!} \epsilon^{\alpha_1 \dots \alpha_{4-p} \beta_1 \dots \beta_p} \Theta_{\beta_1 \dots \beta_p} . \quad (2.172)$$

Indeed, by invoking (2.165) and assuming $\Theta_{\beta'_1 \dots \beta'_p} = L_{\beta_1}^{\beta'_1} \dots L_{\beta_p}^{\beta'_p} \Theta_{\beta_1 \dots \beta_p}$, one obtains

$$\check{\Theta}^{\alpha'_1 \dots \alpha'_{4-p}} = [\det(L_{\rho'}^{\rho})] L_{\alpha_1}^{\alpha'_1} \dots L_{\alpha_{4-p}}^{\alpha'_{4-p}} \check{\Theta}^{\alpha_1 \dots \alpha_{4-p}} , \quad (2.173)$$

whence a $(4-p)$ -multivector density of weight $+1$ is immediately specified, namely

$$\check{\Theta} = \frac{1}{(4-p)!} \check{\Theta}^{\alpha_1 \dots \alpha_{4-p}} (e_{\alpha_1} \wedge \dots \wedge e_{\alpha_{4-p}}) . \quad (2.174)$$

Reconsidering (2.172), it therefore appears evident that $\epsilon^{\alpha\beta\gamma\delta}$ does induce an operator

$$\check{\Theta} = \diamond_p(\Theta) , \quad (2.175)$$

as anticipated; the label of $\diamond_p$ obviously indicates that the map's input is a p -form¹⁷. In order to better understand equation (2.175), it is useful to consider the expansions

$$\diamond_0 = \frac{1}{4!} \epsilon^{\alpha\beta\gamma\delta} (e_\alpha \wedge e_\beta \wedge e_\gamma \wedge e_\delta), \quad (2.176)$$

$$\diamond_1 = \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} (e_\alpha \wedge e_\beta \wedge e_\gamma) \otimes e_\delta, \quad (2.177)$$

$$\diamond_2 = \frac{1}{4} \epsilon^{\alpha\beta\gamma\delta} (e_\alpha \wedge e_\beta) \otimes (e_\gamma \wedge e_\delta), \quad (2.178)$$

$$\diamond_3 = \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} e_\alpha \otimes (e_\beta \wedge e_\gamma \wedge e_\delta), \quad (2.179)$$

$$\diamond_4 = \frac{1}{4!} \epsilon^{\alpha\beta\gamma\delta} (e_\alpha \wedge e_\beta \wedge e_\gamma \wedge e_\delta). \quad (2.180)$$

Crucially, the identities (2.176)–(2.180) suggest that an additional family of operators is available. In fact, one can exploit the Levi-Civita symbol $\hat{\epsilon}_{\alpha\beta\gamma\delta}$, and construct:

$$\hat{\diamond}_0 = \frac{1}{4!} \hat{\epsilon}_{\alpha\beta\gamma\delta} (\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta), \quad (2.181)$$

$$\hat{\diamond}_1 = \frac{1}{3!} \hat{\epsilon}_{\alpha\beta\gamma\delta} (\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma) \otimes \vartheta^\delta, \quad (2.182)$$

$$\hat{\diamond}_2 = \frac{1}{4} \hat{\epsilon}_{\alpha\beta\gamma\delta} (\vartheta^\alpha \wedge \vartheta^\beta) \otimes (\vartheta^\gamma \wedge \vartheta^\delta), \quad (2.183)$$

$$\hat{\diamond}_3 = \frac{1}{3!} \hat{\epsilon}_{\alpha\beta\gamma\delta} \vartheta^\alpha \otimes (\vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta), \quad (2.184)$$

$$\hat{\diamond}_4 = \frac{1}{4!} \hat{\epsilon}_{\alpha\beta\gamma\delta} (\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta), \quad (2.185)$$

where $\hat{\diamond}_p(\cdot)$ relates any p -multivector to a $(4-p)$ -form density of weight -1 . The operators (2.176) and (2.180) or (2.181) and (2.185), as represented in terms of a (co)-frame, pairwise display the same structure. Besides this fact, the maps $\diamond_0$ and $\diamond_4$ or $\hat{\diamond}_0$ and $\hat{\diamond}_4$ are not equivalent, in that they accept different inputs. The identity operator, mapping each exterior form in spacetime to itself, has components

$$\delta_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_p} = \frac{1}{(4-p)!} \epsilon^{\alpha_1 \dots \alpha_p \gamma_1 \dots \gamma_{4-p}} \hat{\epsilon}_{\beta_1 \dots \beta_p \gamma_1 \dots \gamma_{4-p}}. \quad (2.186)$$

¹⁷It is important to remark that the space of 0-forms coincides with the set of all C^∞ -functions.

In the standard terminology, $\delta_{\beta_1 \dots \beta_p}^{\alpha_1 \dots \alpha_p}$ is said to be the *generalised* Kronecker delta symbol (cf. [41, 75]). Two manipulations based on (2.186) lead to the key formulae

$$\hat{\diamond}_{4-p} \circ \diamond_p = (-1)^{p(4-p)} \text{Id} , \quad (2.187)$$

$$\diamond_{4-p} \circ \hat{\diamond}_p = (-1)^{p(4-p)} \text{Id} . \quad (2.188)$$

Following Greub's book [36], which provides a rigorous treatment, $\diamond_p$ and $\hat{\diamond}_p$ are called the *Poincaré isomorphisms*. As Hehl and Obukhov [41] or Kurz and Heumann [68] observed, the operators (2.176)–(2.185) are entirely metric- and connection-free.

2.3.4 The principal-skewon-axion decomposition

Every linear map can be expanded by means of the tensor product¹⁸. This simple statement is illustrated well by the decompositions (2.139) and (2.176)–(2.185). One can thus attain the *transpose* of a linear operator merely by interchanging the basis elements that appear on each side of $\otimes$, see Lindell [79]. For instance, one has that

$$\kappa = \frac{1}{4} \kappa_{\alpha\beta}^{\mu\nu} (\vartheta^\alpha \wedge \vartheta^\beta) \otimes (e_\mu \wedge e_\nu), \quad \Rightarrow \quad \kappa^{\mathbf{t}} = \frac{1}{4} \kappa_{\alpha\beta}^{\mu\nu} (e_\mu \wedge e_\nu) \otimes (\vartheta^\alpha \wedge \vartheta^\beta), \quad (2.189)$$

where $\kappa^{\mathbf{t}}$ maps each twisted bivector to a true bivector¹⁹. An alternative example is:

$$\diamond_3^{\mathbf{t}} = \frac{1}{3!} \epsilon^{\alpha\beta\gamma\delta} (e_\beta \wedge e_\gamma \wedge e_\delta) \otimes e_\alpha = -\frac{1}{3!} \epsilon^{\beta\gamma\delta\alpha} (e_\beta \wedge e_\gamma \wedge e_\delta) \otimes e_\alpha = -\diamond_1 . \quad (2.190)$$

Remarkably, the identity (2.190) can be generalised, so as to achieve (cf. Greub [36])

$$\diamond_p^{\mathbf{t}} = (-1)^{p(4-p)} \diamond_{4-p} , \quad (2.191)$$

$$\hat{\diamond}_p^{\mathbf{t}} = (-1)^{p(4-p)} \hat{\diamond}_{4-p} . \quad (2.192)$$

¹⁸Since no infinite-dimensional vector spaces are considered, the ensuing difficulties are excluded.

¹⁹Given two vector spaces U and V , let $K : U \rightarrow V$ be a linear operator. One can easily demonstrate that $K^{\mathbf{t}}$ is a map from the dual space of V to the dual space of U (see Nering [103]).

In addition, one can combine (2.187)–(2.188) with (2.191)–(2.192), and prove that

$$\diamond_p^{\mathbf{t}} = \hat{\diamond}_p^{-1}, \quad (2.193)$$

$$\hat{\diamond}_p^{\mathbf{t}} = \diamond_p^{-1}. \quad (2.194)$$

The notion of taking the transpose of a linear map has been discussed in sufficient detail. It is therefore possible to construct the operator $\bar{\kappa} := \hat{\diamond}_2 \circ \kappa^{\mathbf{t}} \circ \diamond_2$, which has the *same* domain and co-domain as κ . Ergo, the maps $\bar{\kappa}$ and κ are said to be *compatible*. The motivation for defining $\bar{\kappa}$ is to introduce a crucial decomposition of the medium tensor. Hereafter, the properties of $\bar{\kappa}$ are analysed more in depth. By applying (2.178), (2.183) and (2.189), one can obtain the expansion in components

$$\bar{\kappa}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{4} \hat{\epsilon}_{\alpha\beta\gamma\delta} (\kappa_{\zeta\eta}{}^{\gamma\delta}) \epsilon^{\zeta\eta\mu\nu}. \quad (2.195)$$

Given the sum $\kappa' = a\kappa_1 + b\kappa_2$, where a and b are two scalars, the operator $\bar{\kappa}'$ obeys

$$\bar{\kappa}' = a\bar{\kappa}_1 + b\bar{\kappa}_2. \quad (2.196)$$

Moreover, it is easy to demonstrate that the map $\bar{\bar{\kappa}} = \hat{\diamond}_2 \circ \bar{\kappa}^{\mathbf{t}} \circ \diamond_2$ coincides with κ ,

$$\bar{\bar{\kappa}} = \kappa. \quad (2.197)$$

When κ' consists of two operators being composed ($\kappa' = \kappa_1 \circ \kappa_2$), one can show that

$$\bar{\kappa}' = \bar{\kappa}_2 \circ \bar{\kappa}_1. \quad (2.198)$$

Finally, $\bar{\kappa}$ is a bijection if and only if κ is a bijection too. In particular, the relation

$$(\bar{\kappa})^{-1} = (\hat{\diamond}_2 \circ \kappa^{\mathbf{t}} \circ \diamond_2)^{-1} = \hat{\diamond}_2 \circ (\kappa^{-1})^{\mathbf{t}} \circ \diamond_2 = \overline{(\kappa^{-1})}, \quad (2.199)$$

holds true for every invertible medium response. On account of the fact that the maps κ and $\bar{\kappa}$ are *compatible*, one can meaningfully construct the important identity

$$\kappa = \frac{\kappa + \bar{\kappa}}{2} + \frac{\kappa - \bar{\kappa}}{2} = {}^{(+)}\kappa + {}^{(-)}\kappa. \quad (2.200)$$

As a result, the constitutive operator κ is found to split in two pieces, which satisfy

$$^{(+)}\bar{\kappa} = \frac{\bar{\kappa} + \bar{\bar{\kappa}}}{2} = +\frac{\kappa + \bar{\kappa}}{2} = +^{(+)}\kappa, \quad (2.201)$$

$$^{(-)}\bar{\kappa} = \frac{\bar{\kappa} - \bar{\bar{\kappa}}}{2} = -\frac{\kappa - \bar{\kappa}}{2} = -^{(-)}\kappa, \quad (2.202)$$

by virtue of (2.196) and (2.197). The trace of $\bar{\kappa}$ coincides with that of the standard κ ,

$$\text{tr}(\bar{\kappa}) := \frac{1}{2}\bar{\kappa}_{\alpha\beta}{}^{\alpha\beta} = \frac{1}{8}\hat{\epsilon}_{\alpha\beta\gamma\delta}(\kappa_{\zeta\eta}{}^{\gamma\delta})\epsilon^{\zeta\eta\alpha\beta} = \frac{1}{4}\delta_{\gamma\delta}^{\zeta\eta}\kappa_{\zeta\eta}{}^{\gamma\delta} = \frac{1}{2}\kappa_{\gamma\delta}{}^{\gamma\delta} =: \text{tr}(\kappa), \quad (2.203)$$

where the formulae (2.195) and (2.186) are employed. In turn, (2.203) demands that

$$\text{tr}({}^{(+)}\kappa) = \frac{\text{tr}(\kappa) + \text{tr}(\bar{\kappa})}{2} = \text{tr}(\kappa), \quad (2.204)$$

$$\text{tr}({}^{(-)}\kappa) = \frac{\text{tr}(\kappa) - \text{tr}(\bar{\kappa})}{2} \equiv 0. \quad (2.205)$$

One is thus motivated to dissect $^{(+)}\kappa$ into a traceless contribution and a residue, as

$$^{(+)}\kappa = {}^{(1)}\kappa + {}^{(3)}\kappa = \left[{}^{(+)}\kappa - \frac{1}{6}\text{tr}(\kappa)\text{Id} \right] + \frac{1}{6}\text{tr}(\kappa)\text{Id}. \quad (2.206)$$

After (2.206) is substituted into (2.200), $^{(-)}\kappa$ is assigned the new name $^{(2)}\kappa$, so that

$$\kappa = {}^{(1)}\kappa + {}^{(2)}\kappa + {}^{(3)}\kappa. \quad (2.207)$$

Equation (2.207) illustrates the crucial decomposition of the medium response into a

principal part, a *skewon* part and an *axion* part. Specifically, the elements ${}^{(1)}\kappa_{\alpha\beta}{}^{\mu\nu}$, ${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu}$ and ${}^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu}$ encompass $20 + 15 + 1 = 36$ independent entries, respectively.

The derivation leading to (2.207) can be summarised by re-stating the key properties

$${}^{(1)}\bar{\kappa} = +{}^{(1)}\kappa, \quad \text{tr}({}^{(1)}\kappa) = 0, \quad (2.208)$$

$${}^{(2)}\bar{\kappa} = -{}^{(2)}\kappa, \quad \text{tr}({}^{(2)}\kappa) \equiv 0, \quad (2.209)$$

$${}^{(3)}\bar{\kappa} = +{}^{(3)}\kappa, \quad \text{tr}({}^{(3)}\kappa) = \text{tr}(\kappa), \quad (2.210)$$

see Rubilar *et al.* [117]. Notably, the skewon component and the axion component ($^{(2)}\kappa$ and $^{(3)}\kappa$) can be represented in an alternative way. Indeed, one can prove that

$$^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu} = \delta_{\alpha\beta\rho}^{\mu\nu\sigma} \mathcal{S}_\sigma{}^\rho, \quad \text{with} \quad \mathcal{S}_\alpha{}^\beta := -\frac{1}{2} {}^{(2)}\kappa_{\alpha\rho}{}^{\beta\rho}, \quad (2.211)$$

$$^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu} = \alpha \delta_{\alpha\beta}^{\mu\nu}, \quad \text{with} \quad \alpha := \frac{1}{12} {}^{(3)}\kappa_{\rho\sigma}{}^{\rho\sigma}, \quad (2.212)$$

[47, 114]. Hereafter, the quantities $\mathcal{S}_\alpha{}^\beta$ and α are called, respectively, *skewon field* and *Abelian axion field*. Furthermore, the slash notation $\mathcal{S}_\alpha{}^\beta$ is used to indicate that the skewon field is traceless. As a matter of fact, by virtue of (2.209), one has that

$$\mathcal{S}_\rho{}^\rho = -\frac{1}{2} {}^{(2)}\kappa_{\rho\sigma}{}^{\rho\sigma} = -{}^{(2)}\kappa_I{}^I = -\text{tr}({}^{(2)}\kappa) \equiv 0. \quad (2.213)$$

In particular, the identity (2.213) guarantees that $\mathcal{S}_\alpha{}^\beta$ appropriately contains as many independent entries as $^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu}$ (15 parameters). So far, nothing was said about the possibility of measuring a non-zero skewon or axion part²⁰.

In 1962, Post [113] suggested that the requirement $^{(3)}\kappa = 0$ was valid for all spacetime relations and naturally occurring materials. Ten years later, in the light of novel experiments [51], the same author recognised that a finite *axion* piece was in fact allowed for some crystals [114]. More than two decades after this, Lakhtakia and Weiglhofer considered the condition $^{(3)}\kappa = 0$, and proposed to adopt it as a universal law [69]. Accordingly, they argued against the existence of an axion component in spacetime relations, natural materials and artificial media; the dogma $^{(3)}\kappa = 0$ was also given a name, “Post’s constraint” [71]. Various objections to the paper [69] were raised by Sihvola [125], thus motivating Lakhtakia and Weiglhofer to write a rebuttal [70]. Nowadays, there is enough observational evidence to conclude that, for certain crystals, the axion part is *non-vanishing* – see de Lange and Raab [18] or Hehl, Obukhov, Rivera and Schmid [46]. As far as man-made samples are concerned, Lindell and Sihvola [90] recently put forward the idea of a pure-axion medium $\kappa = ^{(3)}\kappa$, whose behaviour is that of a perfect electromagnetic conductor (PEMC). The metamaterial design examined in [91] gives rise to a PEMC interface, and is within reach of laboratory fabrication. As an aside, the pure-axion medium

²⁰It is of course understood that most optical media have a non-vanishing principal component.

of Lindell and Sihvola is strictly related to the so-called gyrator, an electric network element invented by Tellegen in 1948 [131] and built by Hogan in 1953 [50]. To summarise, Post's constraint remains a viable hypothesis solely in the study of exotic spacetime relations. In fact, no measurement testing a matter-free region, and signalling a finite $^{(3)}\kappa$ has been confirmed to date. The existence of an Abelian axion field in spacetime was first suggested by Dicke [21] and Ni [104]. Promising experimental results were reported in 2006 by the PVLAS collaboration [147], however, the findings were later attributed to instrumental artifacts [102, 148]. Some of the crucial theoretical aspects of this topic are discussed by Wilczek [144] and Itin [52].

The earliest articles to consider a material response with $^{(2)}\kappa \neq 0$ were written by Nieves and Pal [105, 106] in 1989 and 1994. Some years later – in 2002 – Hehl, Obukhov and Rubilar carried out a more comprehensive study [47]. Since then, various works analysing the properties of the *skewon* piece have appeared (e.g. Lindell [81, 83], Sihvola [85] and Bergamin [6], or the papers [48, 109]). Nonetheless, a literature search seemingly indicates that the quest to *detect* a non-zero $^{(2)}\kappa$ has yet to begin.

As a final remark, it is worth noticing that the fields $\$_{\alpha}^{\beta}$ and α carry a *twist*. By combining the equality $\alpha = \frac{1}{6}\text{tr}(\kappa)$ with the result (2.163), one can also observe that:

$$[\alpha] = [\text{tr}(\kappa)] = (hq^{-2})^{-1} \stackrel{\text{SI}}{=} \Omega^{-1}. \quad (2.214)$$

2.3.5 The medium response and $\chi^{\alpha\beta\mu\nu}$

In previous Sections, the field excitation was encoded by means of the twisted 2-form H . As an alternative, and with no loss of generality, numerous authors make use of

$$\check{H} := \diamond_2(H), \quad \Leftrightarrow \quad \check{H}^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} H_{\mu\nu}. \quad (2.215)$$

For example, barring some obvious notational differences, Schouten [119] and Post [113] formulate the theory of electromagnetism in terms of $\check{H}^{\alpha\beta}$ and $F_{\alpha\beta}$. In this equivalent convention, the local and linear constitutive law (2.137) clearly becomes

$$\check{H} = \diamond_2 \circ \kappa(F) = \chi(F), \quad \Leftrightarrow \quad \check{H}^{\alpha\beta} = \frac{1}{4}\epsilon^{\alpha\beta\rho\sigma} \kappa_{\rho\sigma}{}^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\chi^{\alpha\beta\mu\nu} F_{\mu\nu}. \quad (2.216)$$

The quantity $\chi^{\alpha\beta\mu\nu} := \frac{1}{2}\epsilon^{\alpha\beta\rho\sigma}\kappa_{\rho\sigma}{}^{\mu\nu}$, which represents the properties of the medium, is a twisted tensor density of weight +1 [41]. Indeed, (2.144) and (2.165) imply that

$$\chi^{\alpha'\beta'\mu'\nu'} = |\det(L_{\rho'}^{\rho})| L_{\alpha}^{\alpha'} L_{\beta}^{\beta'} L_{\mu}^{\mu'} L_{\nu}^{\nu'} \chi^{\alpha\beta\mu\nu}. \quad (2.217)$$

It appears evident that the constitutive law can be cast in the form $\check{H}^I = \chi^{IJ} F_J$. Hence, one finds that $\chi^{\alpha\beta\mu\nu}$ correctly encompasses as many independent entries as $\kappa_{\alpha\beta}{}^{\mu\nu}$ (36 parameters). Invoking the decomposition (2.207) then generates the split

$$\begin{aligned} \chi^{\alpha\beta\mu\nu} &= {}^{(1)}\chi^{\alpha\beta\mu\nu} + {}^{(2)}\chi^{\alpha\beta\mu\nu} + {}^{(3)}\chi^{\alpha\beta\mu\nu}, \\ 36 &= 20 \oplus 15 \oplus 1, \end{aligned} \quad (2.218)$$

where the elements $\{{}^{(1)}\chi^{\alpha\beta\mu\nu}, {}^{(2)}\chi^{\alpha\beta\mu\nu}, {}^{(3)}\chi^{\alpha\beta\mu\nu}\}$ are characterised by the symmetries

$${}^{(1)}\chi^{IJ} = +{}^{(1)}\chi^{JI}, \quad {}^{(1)}\chi^{[\alpha\beta\mu\nu]} = 0, \quad (2.219)$$

$${}^{(2)}\chi^{IJ} = -{}^{(2)}\chi^{JI}, \quad {}^{(2)}\chi^{[\alpha\beta\mu\nu]} \equiv 0, \quad (2.220)$$

$${}^{(3)}\chi^{IJ} = +{}^{(3)}\chi^{JI}, \quad {}^{(3)}\chi^{[\alpha\beta\mu\nu]} = \chi^{[\alpha\beta\mu\nu]}, \quad (2.221)$$

cf. (2.208)–(2.210). The square brackets appearing, for instance, in the statement ${}^{(1)}\chi^{[\alpha\beta\mu\nu]} = 0$ denote index alternation [113]. To conclude, one can prove that [41, 47]

$${}^{(2)}\chi^{\alpha\beta\mu\nu} = \epsilon^{\alpha\beta\rho[\mu} \not{F}_{\rho}{}^{\nu]} - \epsilon^{\mu\nu\rho[\alpha} \not{F}_{\rho}{}^{\beta]}, \quad {}^{(3)}\chi^{\alpha\beta\mu\nu} = \alpha \epsilon^{\alpha\beta\mu\nu}. \quad (2.222)$$

2.3.6 Space+time split of the medium response

When probing a material for engineering applications, one normally considers a single frame of reference. More accurately, a fixed slicing of the manifold X_4 into leaves of constant σ is employed. In this context, it is therefore natural to express the constitutive law (2.137) in the *space+time* decomposed way (2.133). Hereafter, the local and linear medium response $H_{\alpha\beta} = \frac{1}{2}\kappa_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}$ is taken as a starting point, and the procedure leading to the split (2.133) is described. One can demonstrate that, in the co-frame $\{d\sigma, v^a\}$, the identifications $H_{0a} = \mathcal{H}_a$ and $H_{ab} = \mathcal{D}_{ab}$ hold true.

As an immediate outcome, the constitutive law is separated in *two* relations, namely

$$\mathcal{H}_a = H_{0a} = \frac{1}{2} \kappa_{0a}{}^{\mu\nu} F_{\mu\nu} = \kappa_{0a}{}^{0c} F_{0c} + \frac{1}{2} \kappa_{0a}{}^{cd} F_{cd}, \quad (2.223)$$

$$\mathcal{D}_{ab} = H_{ab} = \frac{1}{2} \kappa_{ab}{}^{\mu\nu} F_{\mu\nu} = \kappa_{ab}{}^{0c} F_{0c} + \frac{1}{2} \kappa_{ab}{}^{cd} F_{cd}. \quad (2.224)$$

Expanding the field strength 2-form with respect to the co-basis $\{d\sigma, v^a\}$, one attains the equalities $F_{0a} = -E_a$ and $F_{ab} = B_{ab}$. Accordingly, (2.223) and (2.224) become:

$$\mathcal{H}_a = -\kappa_{0a}{}^{0c} E_c + \frac{1}{2} \kappa_{0a}{}^{cd} B_{cd}, \quad (2.225)$$

$$\mathcal{D}_{ab} = -\kappa_{ab}{}^{0c} E_c + \frac{1}{2} \kappa_{ab}{}^{cd} B_{cd}. \quad (2.226)$$

It appears evident that (2.225) and (2.226) provide a true *space+time* decomposition of the medium response $H_{\alpha\beta} = \frac{1}{2} \kappa_{\alpha\beta}{}^{\mu\nu} F_{\mu\nu}$. In order to link the present discussion with Lindell's treatment of the topic [79], one can introduce the following definitions

$$\beta_a{}^c := -\kappa_{0a}{}^{0c}, \quad (\mu^{-1})_a{}^{cd} := \kappa_{0a}{}^{cd}, \quad (2.227)$$

$$\varepsilon'_{ab}{}^c := -\kappa_{ab}{}^{0c}, \quad \alpha_{ab}{}^{cd} := \kappa_{ab}{}^{cd}. \quad (2.228)$$

Here, $\varepsilon'_{ab}{}^c$ is the permittivity, $(\mu^{-1})_a{}^{cd}$ is the inverse permeability, while $\beta_a{}^c$ and $\alpha_{ab}{}^{cd}$ are the magneto-electric couplings. The constitutive law is then formulated as [79]

$$\mathcal{H}_a = \beta_a{}^c E_c + \frac{1}{2} (\mu^{-1})_a{}^{cd} B_{cd}, \quad (2.229)$$

$$\mathcal{D}_{ab} = \varepsilon'_{ab}{}^c E_c + \frac{1}{2} \alpha_{ab}{}^{cd} B_{cd}. \quad (2.230)$$

It is important to remark that the name and symbol for $(\mu^{-1})_a{}^{cd}$ are a mere historical artifact. In other words, one need *not* prescribe that the map $(\mu^{-1})_a{}^{cd}$ has an inverse.

Now, the *alternative* expression for the medium response $\check{H}^{\alpha\beta} = \frac{1}{2} \chi^{\alpha\beta\mu\nu} F_{\mu\nu}$ is decomposed in space+time. The assignments $\check{H}^{0a} =: \check{\mathcal{D}}^a$ and $\check{H}^{ab} =: \check{\mathcal{H}}^{ab}$ imply that:

$$\check{\mathcal{D}}^a = -\chi^{0a0c} E_c + \frac{1}{2} \chi^{0acd} B_{cd}, \quad (2.231)$$

$$\check{\mathcal{H}}^{ab} = -\chi^{ab0c} E_c + \frac{1}{2} \chi^{abcd} B_{cd}. \quad (2.232)$$

Drawing an analogy with (2.227)–(2.228), it is appropriate to define the quantities

$$\check{\varepsilon}'^{ac} := -\chi^{0a0c} = \frac{1}{2}\epsilon^{apq}\varepsilon'_{pq}{}^c, \quad \check{\alpha}^{acd} := \chi^{0acd} = \frac{1}{2}\epsilon^{apq}\alpha_{pq}{}^{cd}, \quad (2.233)$$

$$\check{\beta}^{abc} := -\chi^{ab0c} = \epsilon^{abp}\beta_p{}^c, \quad (\mu^{-1})^{abcd} := \chi^{abcd} = \epsilon^{abp}(\mu^{-1})_p{}^{cd}. \quad (2.234)$$

where the Levi-Civita symbol ϵ^{abc} was introduced in (2.90). One thus obtains that

$$\check{\mathcal{D}}^a = \check{\varepsilon}'^{ac}E_c + \frac{1}{2}\check{\alpha}^{acd}B_{cd}, \quad (2.235)$$

$$\check{\mathcal{H}}^{ab} = \check{\beta}^{abc}E_c + \frac{1}{2}(\mu^{-1})^{abcd}B_{cd}. \quad (2.236)$$

It is easy to show that, upon selecting co-frame $\{d\sigma, v^a\}$, the arrays H_I and F_J read

$$H_I = \begin{bmatrix} H_{01} \\ H_{02} \\ H_{03} \\ H_{23} \\ H_{31} \\ H_{12} \end{bmatrix} = \begin{bmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{D}_{23} \\ \mathcal{D}_{31} \\ \mathcal{D}_{12} \end{bmatrix}, \quad F_J = \begin{bmatrix} F_{01} \\ F_{02} \\ F_{03} \\ F_{23} \\ F_{31} \\ F_{12} \end{bmatrix} = \begin{bmatrix} -E_1 \\ -E_2 \\ -E_3 \\ B_{23} \\ B_{31} \\ B_{12} \end{bmatrix}. \quad (2.237)$$

One can substitute the expansions (2.237) in the constitutive law $H_I = \kappa_I^J F_J$, and compare the outcome with (2.229)–(2.230). As a consequence, κ_I^J is found to satisfy

$$\kappa_I^J = \left[\begin{array}{ccc|ccc} -\beta_1^1 & -\beta_1^2 & -\beta_1^3 & (\mu^{-1})_1^{23} & (\mu^{-1})_1^{31} & (\mu^{-1})_1^{12} \\ -\beta_2^1 & -\beta_2^2 & -\beta_2^3 & (\mu^{-1})_2^{23} & (\mu^{-1})_2^{31} & (\mu^{-1})_2^{12} \\ -\beta_3^1 & -\beta_3^2 & -\beta_3^3 & (\mu^{-1})_3^{23} & (\mu^{-1})_3^{31} & (\mu^{-1})_3^{12} \\ \hline -\varepsilon'_{23}{}^1 & -\varepsilon'_{23}{}^2 & -\varepsilon'_{23}{}^3 & \alpha_{23}^{23} & \alpha_{23}^{31} & \alpha_{23}^{12} \\ -\varepsilon'_{31}{}^1 & -\varepsilon'_{31}{}^2 & -\varepsilon'_{31}{}^3 & \alpha_{31}^{23} & \alpha_{31}^{31} & \alpha_{31}^{12} \\ -\varepsilon'_{12}{}^1 & -\varepsilon'_{12}{}^2 & -\varepsilon'_{12}{}^3 & \alpha_{12}^{23} & \alpha_{12}^{31} & \alpha_{12}^{12} \end{array} \right]. \quad (2.238)$$

It is obvious that χ^{IJ} can be represented in a similar way: one can demonstrate that

$$\chi^{IJ} = \left[\begin{array}{ccc|ccc} -\check{\varepsilon}'^{11} & -\check{\varepsilon}'^{12} & -\check{\varepsilon}'^{13} & \check{\alpha}^{123} & \check{\alpha}^{131} & \check{\alpha}^{112} \\ -\check{\varepsilon}'^{21} & -\check{\varepsilon}'^{22} & -\check{\varepsilon}'^{23} & \check{\alpha}^{223} & \check{\alpha}^{231} & \check{\alpha}^{212} \\ -\check{\varepsilon}'^{31} & -\check{\varepsilon}'^{32} & -\check{\varepsilon}'^{33} & \check{\alpha}^{323} & \check{\alpha}^{331} & \check{\alpha}^{312} \\ \hline -\check{\beta}^{231} & -\check{\beta}^{232} & -\check{\beta}^{233} & (\mu^{-1})^{2323} & (\mu^{-1})^{2331} & (\mu^{-1})^{2312} \\ -\check{\beta}^{311} & -\check{\beta}^{312} & -\check{\beta}^{313} & (\mu^{-1})^{3123} & (\mu^{-1})^{3131} & (\mu^{-1})^{3112} \\ -\check{\beta}^{121} & -\check{\beta}^{122} & -\check{\beta}^{123} & (\mu^{-1})^{1223} & (\mu^{-1})^{1231} & (\mu^{-1})^{1212} \end{array} \right]. \quad (2.239)$$

When the medium response is *skewon*-free, equations (2.219)–(2.221) require that the matrix (2.239) is symmetric. Then, $\{\check{\varepsilon}'^{ab}, \check{\alpha}^{abc}, \check{\beta}^{abc}, (\mu^{-1})^{abcd}\}$ must comply with

$$\check{\varepsilon}'^{ab} = \check{\varepsilon}'^{ba}, \quad \check{\alpha}^{abc} = -\check{\beta}^{bca}, \quad (\mu^{-1})^{abcd} = (\mu^{-1})^{cdab}. \quad (2.240)$$

When (instead) the *axion* part vanishes, solely the off-diagonal blocks in the tableau (2.239) are restricted. More accurately, one demands that their traces sum to zero,

$$\check{\alpha}^{123} + \check{\alpha}^{231} + \check{\alpha}^{312} - \check{\beta}^{231} - \check{\beta}^{312} - \check{\beta}^{123} = 0, \quad \Leftrightarrow \quad \check{\alpha}^{[abc]} - \check{\beta}^{[abc]} = 0. \quad (2.241)$$

2.4 The dispersion equation

The aim of this Section is to compute the propagation of electromagnetic fields in a general local and linear medium. From experience it is known that, even if the constitutive law is relatively simple, the task of deriving an *exact* solution of Maxwell's equations is often arduous. Nevertheless, without constraining the electromagnetic response $H_{\alpha\beta} = \frac{1}{2}\kappa_{\alpha\beta}^{\mu\nu}F_{\mu\nu}$ any further, it is indeed possible to satisfy $dH = J$ and $dF = 0$ in an *approximate* sense. More explicitly, one can always invoke the *geometrical optics* limit as a coarse method of solution. This amounts to following either of two mathematical procedures, which are known to be equivalent in all situations:

- a) The field strength F is set to be equal to a sum of plane waves. All terms are assumed to share the same phase $\varphi = \lambda\tilde{\varphi}$, for some positive constant λ . By contrast, the amplitudes of successive contributions are weighted by increasing powers of λ^{-1} . After substituting this ansatz into Maxwell's equations, the

problem is restricted to the analysis of high frequencies, $\lambda \rightarrow +\infty$. Therefore, a first approximation towards solving $dH = J$ and $dF = 0$ is achieved, namely, *geometrical optics* [112]. Higher order corrections are also obtainable [16]. In the approach just described, $\tilde{\varphi}$ is a slowly-varying function called the “eikonal”.

- b) A hypersurface S is considered, across which the electromagnetic fields are discontinuous. In particular, it is demanded that Maxwell’s equations are obeyed both on and in the vicinity of S . Crucially, a set of requirements is thus derived, which governs the propagation of electromagnetic fronts. Hence, the *geometrical optics* limit is easily retrieved. The so-called method of weak discontinuities was developed by Hadamard [39], in 1903. A detailed treatment can be found in Section 2.4.1, or in the paper [88] by Lindell and Jancewicz.

When studying the properties of a laboratory material, one should apply the procedures a) and b) with great care. As a matter of fact, they both dictate that the illumination must change *rapidly* in space and time. It is important to remember that the conditions (i) and (ii) in Section 2.3.1 called for the fields to be *slowly*-varying. Accordingly, if one is not vigilant, several difficulties can arise.

2.4.1 Hadamard’s method of weak discontinuities

Two charge density 3-forms J_1 and J_2 are *introduced*, which are separately conserved

$$dJ_1 = 0, \quad dJ_2 = 0. \quad (2.242)$$

Following the discussion in Section 2.2.1, a pair of field excitations is thereby defined

$$J_1 = dH_1, \quad J_2 = dH_2. \quad (2.243)$$

Moreover, it is *assumed* that the field strength 2-forms F_1 and F_2 exist, and satisfy

$$dF_1 = 0, \quad dF_2 = 0. \quad (2.244)$$

Now, $\{H_1, H_2\}$ and $\{F_1, F_2\}$ are implemented so as to build the linear superpositions

$$H := \gamma_1 H_1 + \gamma_2 H_2, \quad (2.245)$$

$$F := \gamma_1 F_1 + \gamma_2 F_2, \quad (2.246)$$

where γ_1 and γ_2 are smooth functions. By differentiating (2.245)–(2.246), and invoking the properties of the exterior derivative [101], one readily attains the expressions

$$dH = d(\gamma_1 H_1 + \gamma_2 H_2) = (d\gamma_1) \wedge H_1 + (d\gamma_2) \wedge H_2 + \gamma_1 J_1 + \gamma_2 J_2, \quad (2.247)$$

$$dF = d(\gamma_1 F_1 + \gamma_2 F_2) = (d\gamma_1) \wedge F_1 + (d\gamma_2) \wedge F_2, \quad (2.248)$$

on account of (2.243) and (2.244). Hereafter, the quantities dH and dF are subject to different treatments. As far as equation (2.247) is concerned, one trivially assigns

$$J := dH = \gamma_1 J_1 + \gamma_2 J_2 + R, \quad (2.249)$$

$$R := (d\gamma_1) \wedge H_1 + (d\gamma_2) \wedge H_2. \quad (2.250)$$

By contrast, a new *requirement* is imposed on the 2-form $F := \gamma_1 F_1 + \gamma_2 F_2$, namely:

$$dF = (d\gamma_1) \wedge F_1 + (d\gamma_2) \wedge F_2 = 0. \quad (2.251)$$

At this point, one *demand*s that $\gamma_1 + \gamma_2 = 1$. Hence, by virtue of (2.245) and (2.246), the fields H and F become weighted averages of $\{H_1, H_2\}$ and $\{F_1, F_2\}$. Upon noting that $d\gamma_1 = d(1 - \gamma_2) = -d\gamma_2$, one can formulate (2.250) and (2.251) as

$$R := (d\gamma_2) \wedge (H_2 - H_1), \quad (2.252)$$

$$0 = (d\gamma_2) \wedge (F_2 - F_1). \quad (2.253)$$

It appears evident that γ_2 can be written as $\gamma_2 = \eta \circ \varphi$, for some smooth functions

$$\varphi : X_4 \rightarrow \mathbb{R} \quad \text{such that} \quad p \mapsto \phi = \varphi(p), \quad (2.254)$$

$$\eta : \varphi(X_4) \rightarrow \mathbb{R} \quad \text{such that} \quad \phi \mapsto \eta(\phi). \quad (2.255)$$

If $\eta'(\phi)$ is the ordinary derivative of $\eta(\phi)$, a simple proof further demonstrates that

$$d\gamma_2 = (\eta' \circ \varphi) d\varphi, \quad (2.256)$$

cf. Lee [75]. Crucially, equation (2.256) allows one to represent (2.252) and (2.253) as

$$R := (\eta' \circ \varphi) [d\varphi \wedge (H_2 - H_1)], \quad (2.257)$$

$$0 = (\eta' \circ \varphi) [d\varphi \wedge (F_2 - F_1)]. \quad (2.258)$$

whence it is possible to introduce an additional *constraint*: the 1-form $d\varphi$ must be nowhere vanishing. By definition, the level set $S_\phi \subset X_4$ includes those points in spacetime such that $\varphi(p) = \phi$, for a specific $\phi \in \varphi(X_4)$. Accordingly, one has that²¹

$$S_{\phi_1} \cap S_{\phi_2} = \emptyset, \quad (2.259)$$

whenever two distinct elements of $\varphi(X_4) \subset \mathbb{R}$ are considered, $\phi_1 \neq \phi_2$. Moreover, it is easy to observe that the union of all level sets coincides with the manifold X_4 ,

$$\bigcup_{\phi \in \varphi(X_4)} S_\phi = X_4, \quad (2.260)$$

since the domain of φ is the entire spacetime. The condition that $d\varphi$ does not vanish anywhere on X_4 now plays a crucial role, as it guarantees that $d\varphi \neq 0$ throughout $S_\phi \subset X_4$. Remarkably, this fact dictates that each level set S_ϕ is a 3-dimensional embedded submanifold (see Corollary 5.24 in Ref. [75]). On account of the properties (2.259) and (2.260), it is then clear that the function φ determines a foliation of X_4 , cf. Figure 2.3. The next step towards deriving Hadamard's method is to *assume* that

$$\eta(\phi) = \eta_\lambda(\phi - \phi_0), \quad (2.261)$$

where λ is a real parameter, ϕ_0 is a fixed element of $\varphi(X_4)$, and $\eta_\lambda(\phi - \phi_0)$ satisfies

$$\lim_{\lambda \rightarrow 0} [\eta_\lambda(\phi - \phi_0)] = \mathfrak{S}(\phi - \phi_0), \quad (2.262)$$

²¹To avoid pathological complications, every map discussed in the present work is single-valued.

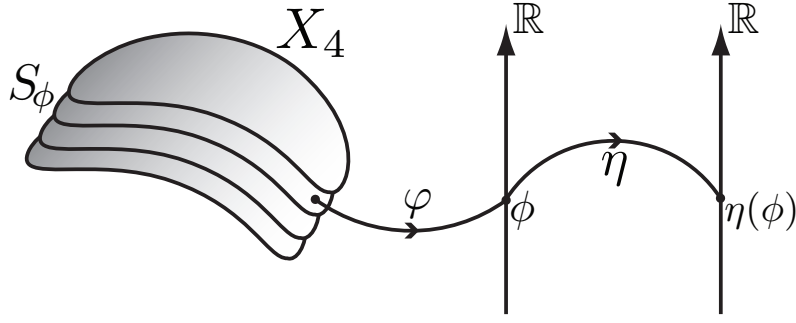


Figure 2.3: The function $\eta \circ \varphi$ is constructed by applying φ first and η last. Since $d\varphi$ is non-zero for all points on X_4 , the manifold must support a foliation into hypersurfaces S_ϕ .

provided $\mathfrak{S}$ is the Heaviside step function. For example, one can appropriately select

$$\eta_\lambda(\phi - \phi_0) = \frac{1}{2} \left[1 + \tanh \left(\frac{\phi - \phi_0}{\lambda} \right) \right], \quad (2.263)$$

$$\eta_\lambda(\phi - \phi_0) = \exp \left\{ -\exp \left[- \left(\frac{\phi - \phi_0}{\lambda} \right) \right] \right\}, \quad (2.264)$$

or any of the numerous existing alternatives. Equation (2.261) trivially implies that

$$R|_p = \eta'_\lambda(\varphi(p) - \phi_0) [d\varphi \wedge (H_2 - H_1)]|_p, \quad (2.265)$$

$$0 = \eta'_\lambda(\varphi(p) - \phi_0) [d\varphi \wedge (F_2 - F_1)]|_p, \quad (2.266)$$

at every spacetime point $p \in X_4$. In particular, when the limit $\lambda \rightarrow 0$ is *enforced*, $\eta'_\lambda(\phi - \phi_0)$ approaches the Dirac delta function $\delta(\phi - \phi_0) = \mathfrak{S}'(\phi - \phi_0)$, and therefore

$$R|_p = \delta(\varphi(p) - \phi_0) [d\varphi \wedge (H_2 - H_1)]|_p, \quad (2.267)$$

$$0 = \delta(\varphi(p) - \phi_0) [d\varphi \wedge (F_2 - F_1)]|_p. \quad (2.268)$$

In fact, one can always write $R|_p = \delta(\varphi(p) - \phi_0) [(d\varphi) \wedge L]|_p$, so that (2.267) becomes

$$\delta(\varphi(p) - \phi_0) [d\varphi \wedge (H_2 - H_1 - L)]|_p = 0, \quad (2.269)$$

with L being a twisted 2-form. Since $\delta(\varphi(p) - \phi_0) \neq 0$ solely if $p \in S_{\phi_0}$, one achieves

$$[d\varphi \wedge (H_2 - H_1 - L)]|_{S_{\phi_0}} = 0, \quad (2.270)$$

which is the first cornerstone of Hadamard's method; the formalism is completed by

$$[\mathrm{d}\varphi \wedge (F_2 - F_1)]|_{S_{\phi_0}} = 0. \quad (2.271)$$

A closer look at the calculation leading to (2.270) and (2.271) reveals that the electromagnetic fields are discontinuous across the folium S_{ϕ_0} . More precisely, they undergo an abrupt transition from $\{H_1, F_1\}$ to $\{H_2, F_2\}$, where (2.243) and (2.244) ensure that Maxwell's equations are obeyed for $\varphi(p) < \phi_0$ and $\varphi(p) > \phi_0$. In a similar way, the requirements (2.270) and (2.271) guarantee that $\mathrm{d}H = J$ and $\mathrm{d}F = 0$ are satisfied on the leaf S_{ϕ_0} .

The hypersurface S_{ϕ_0} can represent a boundary between two different media. In such situation, (2.270) and (2.271) are the conditions ruling light transmission and reflection at the interface S_{ϕ_0} . The charge and current densities located on the boundary are encoded by means of the 2-form L . Further information can be found in Luneburg [93, 94], Costen [15], Burke [13], Warnick [140], Kurz and Heumann [68], or Itin [54]. Hereafter, an alternative physical scenario is considered: the folium S_{ϕ_0} is taken to correspond to a sharp electromagnetic front, whose propagation is governed by (2.270) and (2.271). Accordingly, one can interpret φ as the phase function, $q := \mathrm{d}\varphi$ as the familiar wave-covector, and (2.270)–(2.271) as the key laws of *geometrical optics*. The restriction that $q := \mathrm{d}\varphi$ is nowhere zero forbids the occurrence of optical singularities (vortices) [107]. As an aside, Huygens's principle employs the 2-form L to describe the “secondary” sources located on the signal's edge S_{ϕ_0} [88]. Since this application is limited in scope, it is customary to set $L = 0$.

2.4.2 Electromagnetic fronts and the dispersion equation

By construction, the field jumps $(H_2 - H_1)|_{S_{\phi_0}}$ and $(F_2 - F_1)|_{S_{\phi_0}}$ are defined on and throughout the folium S_{ϕ_0} . With this understanding, it is possible to make the reference to the electromagnetic front S_{ϕ_0} implicit by virtue of the compact notation

$$h := (H_2 - H_1)|_{S_{\phi_0}}, \quad (2.272)$$

$$f := (F_2 - F_1)|_{S_{\phi_0}}. \quad (2.273)$$

After L is set to zero, the geometrical optics equations (2.270)–(2.271) thus become

$$q \wedge h = 0, \quad (2.274)$$

$$q \wedge f = 0. \quad (2.275)$$

It is straightforward to prove that the only solutions to (2.274) and (2.275) are [41, 79]

$$h = q \wedge b, \quad (2.276)$$

$$f = q \wedge a, \quad (2.277)$$

where the 1-form a is the *polarisation* that is measured in all standard experiments. Furthermore, the twisted 1-form b is the corresponding quantity associated with the jump h . One observes that, if H_1 and H_2 are respectively equal to $\kappa(F_1)$ and $\kappa(F_2)$,

$$\kappa(f) = \kappa(F_2 - F_1)|_{S_{\phi_0}} = [\kappa(F_2) - \kappa(F_1)]|_{S_{\phi_0}} = (H_2 - H_1)|_{S_{\phi_0}} = h, \quad (2.278)$$

provided the medium is linear, see (2.138). Substituting (2.278) into (2.274), yields

$$q \wedge \kappa(f) = q \wedge \kappa(q \wedge a) = 0, \quad \Leftrightarrow \quad \frac{1}{2} q_{[\alpha} \kappa_{\rho\sigma]}{}^{\mu\nu} f_{\mu\nu} = q_{[\alpha} \kappa_{\rho\sigma]}{}^{\mu\nu} q_{\mu} a_{\nu} = 0, \quad (2.279)$$

on account of (2.277). Hence, if the last expression in (2.279) is contracted by $\frac{1}{3!} \epsilon^{\alpha\rho\sigma\beta}$,

$$\frac{1}{3!} \epsilon^{\alpha\rho\sigma\beta} q_{[\alpha} \kappa_{\rho\sigma]}{}^{\mu\nu} q_{\mu} a_{\nu} = \frac{1}{2} \epsilon^{\alpha\beta\rho\sigma} \kappa_{\rho\sigma}{}^{\mu\nu} q_{\alpha} q_{\mu} a_{\nu} = \chi^{\alpha\beta\mu\nu} q_{\alpha} q_{\mu} a_{\nu} = 0. \quad (2.280)$$

Upon introducing the map $(W_q)^{\beta\nu} := \chi^{\alpha\beta\mu\nu} q_{\alpha} q_{\mu}$, one obtains the compact equation,

$$(W_q)^{\beta\nu} a_{\nu} = 0, \quad \Leftrightarrow \quad W_q(a) = 0. \quad (2.281)$$

Notably, the operator W_q is *linear* in the sense that, for any two scalars ζ_1 and ζ_2 ,

$$W_q(\zeta_1 x + \zeta_2 y) = \zeta_1 W_q(x) + \zeta_2 W_q(y). \quad (2.282)$$

Given a specific choice of wave-covector q , it is possible to use (2.281) so as to determine the polarisation 1-form a . Nonetheless, selecting a fixed q evidently pro-

vides too little information about the problem. A more in-depth analysis is thus mandatory: to begin with, (2.281) states that a must belong to the *kernel* of W_q ,²²

$$a \in \ker(W_q) . \quad (2.283)$$

If ζ is an arbitrary function, one can readily show that $W_q(\zeta q)$ vanishes identically,

$$(W_q)^{\beta\nu} \zeta q_\nu = \zeta [\chi^{\alpha\beta\mu\nu} q_\alpha q_\mu q_\nu] \equiv 0 , \quad (2.284)$$

because the indices $\{\mu, \nu\}$ are antisymmetric. Therefore, ζq lies within $\ker(W_q)$, and

$$\dim(\ker W_q) > 0 . \quad (2.285)$$

Then, rather than considering an individual q , it appears more fruitful to identify all wave-covectors for which $\dim(\ker W_q) > 1$. In other words, one constrains q according to the requirement that there exists a polarisation a such that $W_q(a) = 0$ and $f = q \wedge a \neq 0$ (cf. Dahl [16]). The following calculation aims to establish that:

$$\dim(\ker W_q) > 1 \text{ if and only if } q \text{ satisfies the } \textit{dispersion equation}. \quad (2.286)$$

Leaving (2.286) momentarily aside, it is worth observing that, provided $a \in \ker(W_q)$,

$$a' = a + \zeta q \in \ker(W_q) , \quad (2.287)$$

that is, $a' = a + \zeta q$ is also a solution of (2.281). In particular, the remark (2.287) is closely related with the *gauge* freedom (2.115). To indicate that two elements of $\ker(W_q)$ differ by a 1-form proportional to q , one can employ the notation $a' \sim a$. As a matter of fact, the link established via $\sim$ is *reflective*, *symmetric* and *transitive*,

$$a \sim a , \quad (2.288)$$

$$a' \sim a \Rightarrow a \sim a' , \quad (2.289)$$

$$a'' \sim a' \ \& \ a' \sim a \Rightarrow a'' \sim a , \quad (2.290)$$

²²The concepts of linear algebra invoked hereafter are explained well in Nering's book [103].

whence $\sim$ is a so-called *equivalence relation*, dictating a partition of $\ker(W_q)$ into mutually disjoint subsets [101]. Each subset is an *equivalence class* $[a]$, encompassing every solution a' of (2.281) such that $a' - a \in \text{span}(q)$. Remarkably, when all classes are brought together, a linear vector space V_q is achieved, which includes the *physically relevant* polarisation directions. This statement is illustrated well by the property $\dim(V_q) = \dim(\ker W_q) - 1$, and by the consequent reformulation of (2.286),

$$\dim(V_q) > 0 \text{ if and only if } q \text{ satisfies the dispersion equation.} \quad (2.291)$$

Returning to the proof of (2.286), it is convenient to define the *exterior p^{th} power* of W_q , denoted $\bigwedge^p W_q$: one prescribes that, for any selection of 1-forms $\{\alpha_1, \dots, \alpha_p\}$,

$$\bigwedge^p W_q(\alpha_1 \wedge \dots \wedge \alpha_p) := W_q(\alpha_1) \wedge \dots \wedge W_q(\alpha_p), \quad (2.292)$$

see Flanders [31] and Lindell [79]²³. The highest value of p such that $\bigwedge^p W_q \neq 0$ coincides with the *rank* of W_q , that is, with the dimension of the operator's *image*. For example, a direct calculation reveals that the 4th exterior power of W_q vanishes

$$\bigwedge^4 W_q \equiv 0, \quad (2.293)$$

whereby the map W_q cannot be of full rank, $\dim(\text{im } W_q) < 4$. Moreover, since [103]

$$\dim(\ker W_q) + \dim(\text{im } W_q) = 4, \quad (2.294)$$

the observation (2.285) is immediately retrieved. Now, by virtue of theorem (2.294), the condition $\dim(\ker W_q) > 1$ in (2.286) can be expressed as $\dim(\text{im } W_q) < 3$; thus,

$$\dim(\ker W_q) > 1, \quad \Leftrightarrow \quad \bigwedge^3 W_q = 0. \quad (2.295)$$

²³According to the nomenclature of [79], one would call $\bigwedge^p W_q$ the “double-wedge power” of W_q .

Decomposing the 3rd exterior power of W_q with respect to a frame $\{e_\alpha\}$ then leads to

$$\begin{aligned}\bigwedge^3 W_q &= \frac{1}{3!} (W_q)^{\alpha_1 \beta_1} (W_q)^{\alpha_2 \beta_2} (W_q)^{\alpha_3 \beta_3} (e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) \otimes (e_{\beta_1} \wedge e_{\beta_2} \wedge e_{\beta_3}) \\ &= \frac{1}{3!} \chi^{\alpha \alpha_1 \beta \beta_1} (W_q)^{\alpha_2 \beta_2} (W_q)^{\alpha_3 \beta_3} q_\alpha (e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) \otimes q_\beta (e_{\beta_1} \wedge e_{\beta_2} \wedge e_{\beta_3}) \\ &= \frac{1}{4!} \chi^{\alpha \alpha_1 \beta \beta_1} (W_q)^{\alpha_2 \beta_2} (W_q)^{\alpha_3 \beta_3} q_{[\alpha} (e_{\alpha_1]} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) \otimes q_{[\beta} (e_{\beta_1]} \wedge e_{\beta_2} \wedge e_{\beta_3}), \quad (2.296)\end{aligned}$$

on account of $(W_q)^{\alpha\beta} := \chi^{\alpha\mu\beta\nu} q_\mu q_\nu$. A simple, but lengthy, calculation implies that

$$q_{[\alpha} (e_{\alpha_1]} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) = 2q_{[\alpha} (e_{\alpha_1} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) + \dots, \quad (2.297)$$

$$q_{[\beta} (e_{\beta_1]} \wedge e_{\beta_2} \wedge e_{\beta_3}) = 2q_{[\beta} (e_{\beta_1} \wedge e_{\beta_2} \wedge e_{\beta_3}) + \dots, \quad (2.298)$$

where the ellipses indicate terms that do not contribute to (2.296) because of the identity $(W_q)^{\alpha\beta} q_\alpha = (W_q)^{\beta\alpha} q_\alpha \equiv 0$. The index alternations on the right-hand sides of (2.297) and (2.298) can be replaced by two generalised Kronecker delta symbols,

$$q_{[\alpha} (e_{\alpha_1]} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) = \frac{1}{12} \delta_{\alpha\alpha_1\alpha_2\alpha_3}^{\mu\mu_1\mu_2\mu_3} q_\mu (e_{\mu_1} \wedge e_{\mu_2} \wedge e_{\mu_3}) + \dots, \quad (2.299)$$

$$q_{[\beta} (e_{\beta_1]} \wedge e_{\beta_2} \wedge e_{\beta_3}) = \frac{1}{12} \delta_{\beta\beta_1\beta_2\beta_3}^{\nu\nu_1\nu_2\nu_3} q_\nu (e_{\nu_1} \wedge e_{\nu_2} \wedge e_{\nu_3}) + \dots. \quad (2.300)$$

Further making use of the $p = 4$ sub-case of equation (2.186) allows one to achieve

$$\begin{aligned}q_{[\alpha} (e_{\alpha_1]} \wedge e_{\alpha_2} \wedge e_{\alpha_3}) &= \frac{1}{12} \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \epsilon^{\mu\mu_1\mu_2\mu_3} q_\mu (e_{\mu_1} \wedge e_{\mu_2} \wedge e_{\mu_3}) + \dots \\ &= \frac{1}{2} \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \epsilon_q + \dots, \quad (2.301)\end{aligned}$$

$$\begin{aligned}q_{[\beta} (e_{\beta_1]} \wedge e_{\beta_2} \wedge e_{\beta_3}) &= \frac{1}{12} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} \epsilon^{\nu\nu_1\nu_2\nu_3} q_\nu (e_{\nu_1} \wedge e_{\nu_2} \wedge e_{\nu_3}) + \dots \\ &= \frac{1}{2} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} \epsilon_q + \dots, \quad (2.302)\end{aligned}$$

where $\epsilon_q := (1/6) \epsilon^{\gamma_1 \gamma_2 \gamma_3} q_\gamma (e_{\gamma_1} \wedge e_{\gamma_2} \wedge e_{\gamma_3})$ is a trivector density of weight +1. Hence, it is appropriate to substitute (2.301) and (2.302) into (2.296), so as to observe that

$$\bigwedge^3 W_q = \frac{1}{4 \cdot 4!} \mathcal{G}(q) (\epsilon_q \otimes \epsilon_q), \quad (2.303)$$

$$\begin{aligned}\mathcal{G}(q) &:= \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} \chi^{\alpha\alpha_1\beta\beta_1} (W_q)^{\alpha_2\beta_2} (W_q)^{\alpha_3\beta_3} \\ &= \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} \chi^{\alpha\alpha_1\beta\beta_1} \chi^{\alpha_2\rho\beta_2\sigma} \chi^{\alpha_3\tau\beta_3\nu} q_\rho q_\sigma q_\tau q_\nu. \quad (2.304)\end{aligned}$$

By implementing the formula (2.303) in (2.295), one completes the proof of (2.286),

$$\dim(\ker W_q) > 1, \quad \Leftrightarrow \quad \mathcal{G}(q) = 0. \quad (2.305)$$

Accordingly, if and only if the *dispersion equation* (the *Fresnel equation*) is satisfied

$$\mathcal{G}(q) := \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} \chi^{\alpha\alpha_1\beta\beta_1} \chi^{\alpha_2\rho\beta_2\sigma} \chi^{\alpha_3\tau\beta_3\nu} q_\rho q_\sigma q_\tau q_\nu = 0, \quad (2.306)$$

one or more polarisations do exist such that $W_q(a)=0$ and $f=q\wedge a$ is non-vanishing. With respect to the wave-covector, the quantity $\mathcal{G}(q)$ is a *quartic* homogeneous polynomial, whose coefficients depend in a *cubic* manner on the medium parameters. In addition, one can easily show that $\mathcal{G}(q)$ is a twisted scalar density of weight $+1$. A long computation, first carried out by Obukhov (see the Appendix of Ref. [53]), demonstrates that (2.304) can be written in several *equivalent* ways. The expression

$$\mathcal{G}(q) = \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_1\beta} \chi^{\alpha\alpha_1\beta_3\rho} \chi^{\sigma\alpha_2\beta_2\tau} \chi^{\nu\alpha_3\beta_1\beta} q_\rho q_\sigma q_\tau q_\nu \quad (2.307)$$

is particularly widespread in the literature [41, 115, 117, 121]. Formulating (2.304) as

$$\mathcal{G}(q) = \mathcal{G}^{\rho\sigma\tau\nu} q_\rho q_\sigma q_\tau q_\nu \quad (2.308)$$

demands that a suitable twisted tensor density of weight $+1$ is introduced – namely:

$$\mathcal{G}^{\rho\sigma\tau\nu} := \frac{1}{4!} \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} \chi^{\alpha\alpha_1\beta\beta_1} \chi^{\alpha_2(\rho\sigma|\beta_2} \chi^{\alpha_3|\tau\nu)\beta_3}, \quad (2.309)$$

where the round brackets denote index mixing [119]. A “fourth order metric” akin to $\mathcal{G}^{\rho\sigma\tau\nu}$ was considered in Tamm’s 1925 article [130]. In 2002, Rubilar [116] accomplished a *complete* and *rigorous* derivation of (2.307). Correspondingly, the structure $\mathcal{G}^{\rho\sigma\tau\nu}$ is called the *Tamm-Rubilar* tensor density [41]. The procedure employed above in order to attain the dispersion equation is based on a paper by Lindell [80]. Further information concerning the quartic Fresnel polynomial can be found in Obukhov, Fukui and Rubilar [108], Itin [53], Perlick [112], or Dahl [16]. By combining the split $\chi^{\alpha\beta\mu\nu} = {}^{(1)}\chi^{\alpha\beta\mu\nu} + {}^{(2)}\chi^{\alpha\beta\mu\nu} + {}^{(3)}\chi^{\alpha\beta\mu\nu}$ with the result (2.304),

one can establish that

$$\mathcal{G}(q) = \mathcal{G}^{(1)}[\chi](q) + \left[\left({}^{(1)}\chi^{\mu\rho\nu\sigma} \mathcal{F}_\mu{}^\tau \mathcal{F}_\nu{}^v \right) q_\rho q_\sigma q_\tau q_v \right], \quad (2.310)$$

$$\mathcal{G}^{(1)}[\chi](q) := \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} {}^{(1)}\chi^{\alpha\alpha_1\beta\beta_1} {}^{(1)}\chi^{\alpha_2\rho\beta_2\sigma} {}^{(1)}\chi^{\alpha_3\tau\beta_3v} q_\rho q_\sigma q_\tau q_v. \quad (2.311)$$

It is straightforward to observe that, if ${}^{(1)}\chi^{\alpha\beta\mu\nu} = 0$, the dispersion equation vanishes identically. Moreover, the axion part ${}^{(3)}\chi^{\alpha\beta\mu\nu}$ does *not* contribute to $\mathcal{G}(q)$ at all [116].

2.5 Important examples

2.5.1 Q -media

Following Lindell and Wallén [92], the constitutive law of a Q -medium is encoded by

$$\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q^{\rho\mu} Q^{\sigma\nu}, \quad (\mathfrak{X} \neq 0), \quad (2.312)$$

where $Q^{\alpha\beta}$ obeys the transformation rule $Q^{\alpha'\beta'} = L_{\alpha}{}^{\alpha'} L_{\beta}{}^{\beta'} Q^{\alpha\beta}$, and is not necessarily invertible or symmetric. In addition, $\mathfrak{X} \neq 0$ is a twisted scalar density of weight +1. The calculation performed hereafter identifies the principal, skewon and axion parts of (2.312). As a starting point, one can evaluate the tensor $\bar{\kappa}_{\alpha\beta}{}^{\mu\nu}$ for any Q -medium,

$$\begin{aligned} \bar{\kappa}_{\alpha\beta}{}^{\mu\nu} &= \frac{1}{4} \hat{\epsilon}_{\alpha\beta\rho\sigma} \left(\mathfrak{X} \hat{\epsilon}_{\eta\theta\xi\zeta} Q^{\xi\rho} Q^{\zeta\sigma} \right) \epsilon^{\eta\theta\mu\nu} \\ &= \frac{\mathfrak{X}}{2} \hat{\epsilon}_{\alpha\beta\rho\sigma} \delta_{\xi\zeta}^{\mu\nu} Q^{\xi\rho} Q^{\zeta\sigma} = \mathfrak{X} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q^{\mu\rho} Q^{\nu\sigma}. \end{aligned} \quad (2.313)$$

Then, it is convenient to implement the decomposition $Q^{\alpha\beta} = {}^{[s]}Q^{\alpha\beta} + {}^{[a]}Q^{\alpha\beta}$, with:

$${}^{[s]}Q^{\alpha\beta} := \frac{1}{2} (Q^{\alpha\beta} + Q^{\beta\alpha}), \quad \Rightarrow \quad {}^{[s]}Q^{\alpha\beta} = {}^{[s]}Q^{\beta\alpha}, \quad (2.314)$$

$${}^{[a]}Q^{\alpha\beta} := \frac{1}{2} (Q^{\alpha\beta} - Q^{\beta\alpha}), \quad \Rightarrow \quad {}^{[a]}Q^{\alpha\beta} = -{}^{[a]}Q^{\beta\alpha}. \quad (2.315)$$

When the split of $Q^{\alpha\beta}$ is substituted in equations (2.312)–(2.313), one achieves that

$$\begin{aligned}\kappa_{\alpha\beta}{}^{\mu\nu} &= \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[s]}Q^{\sigma\nu} + [a]Q^{\rho\mu[a]}Q^{\sigma\nu} \right) \\ &\quad + \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[a]}Q^{\sigma\nu} + [a]Q^{\rho\mu[s]}Q^{\sigma\nu} \right),\end{aligned}\quad (2.316)$$

$$\begin{aligned}\bar{\kappa}_{\alpha\beta}{}^{\mu\nu} &= \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\mu\rho[s]}Q^{\nu\sigma} + [a]Q^{\mu\rho[a]}Q^{\nu\sigma} \right) \\ &\quad + \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\mu\rho[a]}Q^{\nu\sigma} + [a]Q^{\mu\rho[s]}Q^{\nu\sigma} \right) \\ &= \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[s]}Q^{\sigma\nu} + [a]Q^{\rho\mu[a]}Q^{\sigma\nu} \right) \\ &\quad - \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[a]}Q^{\sigma\nu} + [a]Q^{\rho\mu[s]}Q^{\sigma\nu} \right),\end{aligned}\quad (2.317)$$

Accordingly, the contributions ${}^{(+)}\kappa_{\alpha\beta}{}^{\mu\nu}$ and ${}^{(-)}\kappa_{\alpha\beta}{}^{\mu\nu}$ defined in (2.200) take the form

$${}^{(+)}\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[s]}Q^{\sigma\nu} + [a]Q^{\rho\mu[a]}Q^{\sigma\nu} \right), \quad (2.318)$$

$${}^{(-)}\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[a]}Q^{\sigma\nu} + [a]Q^{\rho\mu[s]}Q^{\sigma\nu} \right). \quad (2.319)$$

Thus, it is easy to conclude that the principal, skewon and axion parts are given by

$${}^{(1)}\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[s]}Q^{\sigma\nu} + [a]Q^{\rho\mu[a]}Q^{\sigma\nu} \right) + \frac{1}{12}\mathfrak{X}(\hat{\epsilon}_{\eta\theta\xi\zeta}{}^{[a]}Q^{\eta\theta[a]}Q^{\xi\zeta})\delta_{\alpha\beta}^{\mu\nu}, \quad (2.320)$$

$${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X}\hat{\epsilon}_{\alpha\beta\rho\sigma} \left([s]Q^{\rho\mu[a]}Q^{\sigma\nu} + [a]Q^{\rho\mu[s]}Q^{\sigma\nu} \right), \quad (2.321)$$

$${}^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu} = -\frac{1}{12}\mathfrak{X}(\hat{\epsilon}_{\eta\theta\xi\zeta}{}^{[a]}Q^{\eta\theta[a]}Q^{\xi\zeta})\delta_{\alpha\beta}^{\mu\nu}. \quad (2.322)$$

In order to compute the fields $\$_{\alpha}{}^{\beta}$ and α , one can make use of (2.211) and (2.212),

$$\begin{aligned}\$_{\alpha}{}^{\beta} &= -\frac{1}{2}\mathfrak{X}\hat{\epsilon}_{\alpha\eta\rho\sigma} \left([s]Q^{\rho\beta[a]}Q^{\sigma\eta} + [a]Q^{\rho\beta[s]}Q^{\sigma\eta} \right) \\ &= -\frac{1}{2}\mathfrak{X}\hat{\epsilon}_{\alpha\rho\sigma\eta} \left([s]Q^{\rho\beta[a]}Q^{\sigma\eta} + [a]Q^{\rho\beta[s]}Q^{\sigma\eta} \right) \\ &= -\frac{1}{2}\mathfrak{X}(\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3}{}^{[s]}Q^{\beta\alpha_1[a]}Q^{\alpha_2\alpha_3}),\end{aligned}\quad (2.323)$$

$$\alpha = -\frac{1}{12}\mathfrak{X}(\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3}{}^{[a]}Q^{\alpha\alpha_1[a]}Q^{\alpha_2\alpha_3}). \quad (2.324)$$

The determinant of a 4-dimensional alternating bilinear form, such as ${}^{[a]}Q$, satisfies

$$\det({}^{[a]}Q) = [\text{Pf}({}^{[a]}Q)]^2, \quad (2.325)$$

where $\text{Pf}([a]Q)$ is the so-called *Pfaffian*. In the case being considered, one has that:

$$\text{Pf}([a]Q) = \frac{1}{8} \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} [a]Q^{\alpha\alpha_1} [a]Q^{\alpha_2\alpha_3}. \quad (2.326)$$

Hence, it appears evident the Pfaffian of $[a]Q$ is a scalar density of weight -1 [124]. Remarkably, the axion piece is closely related to the quantity (2.326), more explicitly

$${}^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu} = -\frac{2}{3} \mathfrak{X} \text{Pf}([a]Q) \delta_{\alpha\beta}^{\mu\nu}, \quad \alpha = -\frac{2}{3} \mathfrak{X} \text{Pf}([a]Q). \quad (2.327)$$

In a similar manner, one can formulate the principal component of a Q -medium as

$${}^{(1)}\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X} \hat{\epsilon}_{\alpha\beta\rho\sigma} ([s]Q^{\rho\mu} [s]Q^{\sigma\nu} + [a]Q^{\rho\mu} [a]Q^{\sigma\nu}) + \frac{2}{3} \mathfrak{X} \text{Pf}([a]Q) \delta_{\alpha\beta}^{\mu\nu}. \quad (2.328)$$

At this point, it is appropriate to compute the dispersion equation for (2.312). Since

$$\chi^{\alpha\beta\mu\nu} = \frac{1}{2} \mathfrak{X} \epsilon^{\alpha\beta\rho\sigma} \hat{\epsilon}_{\rho\sigma\eta\theta} Q^{\eta\mu} Q^{\theta\nu} = \mathfrak{X} \delta_{\eta\theta}^{\alpha\beta} Q^{\eta\mu} Q^{\theta\nu} = \mathfrak{X} (Q^{\alpha\mu} Q^{\beta\nu} - Q^{\beta\mu} Q^{\alpha\nu}), \quad (2.329)$$

the operator $(W_q)^{\beta\nu} = \chi^{\alpha\beta\mu\nu} q_\alpha q_\mu$, introduced in Section 2.4.2, is readily expanded as

$$(W_q)^{\beta\nu} = \mathfrak{X} (Q^{\alpha\mu} Q^{\beta\nu} - Q^{\beta\mu} Q^{\alpha\nu}) q_\alpha q_\mu = \mathfrak{X} [Q(q, q) Q^{\beta\nu} - (Q^{\beta\mu} q_\mu) (Q^{\alpha\nu} q_\alpha)], \quad (2.330)$$

with $Q(q, q) := Q^{\alpha\beta} q_\alpha q_\beta$. The next step consists in combining (2.330) with (2.306). Yet, instead of analysing the Fresnel equation in its entirety, the partial contraction

$$\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} (W_q)^{\alpha_2\beta_2} (W_q)^{\alpha_3\beta_3} =: M_{\alpha\alpha_1\beta\beta_1} \quad (2.331)$$

is examined in isolation. By employing the Q -medium expression for W_q , one obtains

$$\begin{aligned} M_{\alpha\alpha_1\beta\beta_1} = & \mathfrak{X}^2 Q(q, q)^2 \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_2\beta_2} Q^{\alpha_3\beta_3} \\ & - \mathfrak{X}^2 Q(q, q) \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_2\beta_2} (Q^{\alpha_3\rho} q_\rho) (Q^{\sigma\beta_3} q_\sigma) \\ & - \mathfrak{X}^2 Q(q, q) \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_3\beta_3} (Q^{\alpha_2\rho} q_\rho) (Q^{\sigma\beta_2} q_\sigma) \\ & + \mathfrak{X}^2 \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} (Q^{\alpha_2\rho} q_\rho) (Q^{\sigma\beta_2} q_\sigma) (Q^{\alpha_3\eta} q_\eta) (Q^{\theta\beta_3} q_\theta), \end{aligned} \quad (2.332)$$

where the last term is zero because of a sum involving opposite index symmetries. A simple re-labelling $\{\alpha_2, \beta_2\} \leftrightarrow \{\alpha_3, \beta_3\}$ in the third line of (2.332) promptly leads to

$$\begin{aligned} M_{\alpha\alpha_1\beta\beta_1} &= \mathfrak{X}^2 Q(q, q)^2 \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_2\beta_2} Q^{\alpha_3\beta_3} \\ &\quad - 2\mathfrak{X}^2 Q(q, q) \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_2\beta_2} (Q^{\alpha_3\rho} q_\rho) (Q^{\sigma\beta_3} q_\sigma). \end{aligned} \quad (2.333)$$

In addition, factorising the quantity $\mathfrak{X}^2 Q(q, q) \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_2\beta_2}$ allows one to write

$$\begin{aligned} M_{\alpha\alpha_1\beta\beta_1} &= \mathfrak{X}^2 Q(q, q) \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha_2\beta_2} \times \\ &\quad \times [Q(q, q) Q^{\alpha_3\beta_3} - 2(Q^{\alpha_3\rho} q_\rho) (Q^{\sigma\beta_3} q_\sigma)]. \end{aligned} \quad (2.334)$$

To extract the dispersion equation from the formula (2.334) it is sufficient to observe that $\mathcal{G}(q) = \chi^{\alpha\alpha_1\beta\beta_1} M_{\alpha\alpha_1\beta\beta_1} = 2\mathfrak{X} M_{\alpha\alpha_1\beta\beta_1} Q^{\alpha\beta} Q^{\alpha_1\beta_1}$. A direct expansion verifies that

$$\begin{aligned} \mathcal{G}(q) &= 2\mathfrak{X}^3 Q(q, q) \hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} Q^{\alpha\beta} Q^{\alpha_1\beta_1} Q^{\alpha_2\beta_2} \times \\ &\quad \times [Q(q, q) Q^{\alpha_3\beta_3} - 2(Q^{\alpha_3\rho} q_\rho) (Q^{\sigma\beta_3} q_\sigma)], \end{aligned} \quad (2.335)$$

where the first factor is proportional to the transpose of the *classical adjoint* of Q ,

$$(\text{adj } Q)_{\mu\nu} := \frac{1}{3!} \hat{\epsilon}_{\nu\mu_1\mu_2\mu_3} \hat{\epsilon}_{\mu\nu_1\nu_2\nu_3} Q^{\mu_1\nu_1} Q^{\mu_2\nu_2} Q^{\mu_3\nu_3}. \quad (2.336)$$

By virtue of the property $(\text{adj } Q)_{\mu\rho} Q^{\rho\nu} = (\det Q) \delta_\mu^\nu$, one can reach the key conclusion,

$$\begin{aligned} \mathcal{G}(q) &= 12Q(q, q) \mathfrak{X}^3 (\text{adj } Q)_{\nu\mu} [Q(q, q) Q^{\mu\nu} - 2(Q^{\mu\rho} q_\rho) (Q^{\sigma\nu} q_\sigma)] \\ &= 12Q(q, q) \mathfrak{X}^3 (\det Q) [4Q(q, q) - 2Q(q, q)] = 4! \mathfrak{X}^3 (\det Q) [Q(q, q)]^2; \end{aligned} \quad (2.337)$$

the procedure used hereinbefore to derive (2.337) was based on Lindell's paper [80].

As an aside, since $^{[a]}Q(q, q) = ^{[a]}Q^{\alpha\beta} q_\alpha q_\beta$ vanishes identically, it is easy to note that

$$\mathcal{G}(q) = 4! \mathfrak{X}^3 (\det Q) (^{[s]}Q(q, q))^2, \quad (\mathfrak{X} \neq 0). \quad (2.338)$$

When $Q^{\alpha\beta}$ is assumed to be invertible, $\mathcal{G}(q)$ is proportional to the square of a quadratic polynomial in q , via a non-zero factor. This important property defines the electromagnetic media that are *non-birefringent* (cf. Favaro and Bergamin [29]).

Returning to the case where the rank of $Q^{\alpha\beta}$ is arbitrary, it is instructive to show that the constitutive law $\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q^{\rho\mu} Q^{\sigma\nu}$ satisfies the equation $\kappa \circ \bar{\kappa} = \frac{1}{6} \text{tr}(\kappa \circ \bar{\kappa}) \text{Id}$. By substituting (2.312)–(2.313) in the component representation of $\kappa \circ \bar{\kappa}$ one achieves

$$\frac{1}{2} \kappa_{\alpha\beta}{}^{\rho\sigma} \bar{\kappa}_{\rho\sigma}{}^{\mu\nu} = \frac{\mathfrak{X}^2}{2} \hat{\epsilon}_{\alpha\beta\eta\theta} Q^{\eta\rho} Q^{\theta\sigma} \hat{\epsilon}_{\rho\sigma\xi\zeta} Q^{\mu\xi} Q^{\nu\zeta} = \frac{\mathfrak{X}^2}{2} \hat{\epsilon}_{\alpha\beta\eta\theta} \hat{\epsilon}_{\rho\sigma\xi\zeta} Q^{\eta\rho} Q^{\theta\sigma} Q^{\mu\xi} Q^{\nu\zeta}. \quad (2.339)$$

The contraction engaging the indices of $\hat{\epsilon}_{\rho\sigma\xi\zeta}$ is simplified by employing the identity

$$\hat{\epsilon}_{\mu\mu_1\mu_2\mu_3} R^{\nu\mu} R^{\nu_1\mu_1} R^{\nu_2\mu_2} R^{\nu_3\mu_3} = (\det R) \epsilon^{\nu\nu_1\nu_2\nu_3}, \quad (2.340)$$

which holds true for every tensor $R^{\alpha\beta}$. As a direct outcome, (2.339) is expressed as

$$\frac{1}{2} \kappa_{\alpha\beta}{}^{\rho\sigma} \bar{\kappa}_{\rho\sigma}{}^{\mu\nu} = \frac{1}{2} \mathfrak{X}^2 (\det Q) \hat{\epsilon}_{\alpha\beta\eta\theta} \epsilon^{\eta\theta\mu\nu}. \quad (2.341)$$

One may convert the quantity $\frac{1}{2} \hat{\epsilon}_{\alpha\beta\eta\theta} \epsilon^{\eta\theta\mu\nu}$ into a generalised Kronecker delta symbol,

$$\frac{1}{2} \kappa_{\alpha\beta}{}^{\rho\sigma} \bar{\kappa}_{\rho\sigma}{}^{\mu\nu} = \mathfrak{X}^2 (\det Q) \delta_{\alpha\beta}^{\mu\nu}. \quad (2.342)$$

Hence, it is concluded that the class of electromagnetic Q -media is indeed a *solution* of $\kappa \circ \bar{\kappa} = \frac{1}{6} \text{tr}(\kappa \circ \bar{\kappa}) \text{Id}$. A calculation similar to (2.339)–(2.342) further reveals that:

$$\frac{1}{2} \bar{\kappa}_{\alpha\beta}{}^{\rho\sigma} \kappa_{\rho\sigma}{}^{\mu\nu} = \mathfrak{X}^2 (\det Q) \delta_{\alpha\beta}^{\mu\nu}. \quad (2.343)$$

One can therefore observe that the Q -media also obey the *alternative* equation $\bar{\kappa} \circ \kappa = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) \text{Id}$. As explained in Chapter 3, the so-called mixed closure relations

$$\kappa \circ \bar{\kappa} = \frac{1}{6} \text{tr}(\kappa \circ \bar{\kappa}) \text{Id}, \quad (2.344)$$

$$\bar{\kappa} \circ \kappa = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) \text{Id}, \quad (2.345)$$

do not have the same solution set, in general. More accurately, there exist some electromagnetic responses that satisfy only one of (2.344)–(2.345). The fact that each Q -medium complies with both mixed closure relations is a *special* property.

Computing the trace on both sides of (2.342), and on both sides of (2.343) leads to

$$\mathrm{tr}(\kappa \circ \bar{\kappa}) = 6\mathfrak{X}^2(\det Q), \quad (2.346)$$

$$\mathrm{tr}(\bar{\kappa} \circ \kappa) = 6\mathfrak{X}^2(\det Q), \quad (2.347)$$

whenever $\kappa_{\alpha\beta}{}^{\mu\nu}$ is encoded by (2.312). It is important to remark that, given *any* local and linear constitutive law, the scalars $\mathrm{tr}(\kappa \circ \bar{\kappa})$ and $\mathrm{tr}(\bar{\kappa} \circ \kappa)$ coincide trivially

$$\mathrm{tr}(\kappa \circ \bar{\kappa}) := \frac{1}{4} \kappa_{\eta\theta}{}^{\rho\sigma} \bar{\kappa}_{\rho\sigma}{}^{\eta\theta} = \frac{1}{4} \bar{\kappa}_{\rho\sigma}{}^{\eta\theta} \kappa_{\eta\theta}{}^{\rho\sigma} =: \mathrm{tr}(\bar{\kappa} \circ \kappa). \quad (2.348)$$

To stress that (2.346) and (2.347) are related by the identity (2.348), one may write

$$\mathrm{tr}(\kappa \circ \bar{\kappa}) \equiv \mathrm{tr}(\bar{\kappa} \circ \kappa) = 6\mathfrak{X}^2(\det Q). \quad (2.349)$$

It is interesting to split (2.349) in two equations, concerning the modulus and sign of $\mathrm{tr}(\kappa \circ \bar{\kappa}) \equiv \mathrm{tr}(\bar{\kappa} \circ \kappa)$ respectively. Since $\mathfrak{X}$ is assumed to be non-zero, one attains

$$|\mathrm{tr}(\kappa \circ \bar{\kappa})| \equiv |\mathrm{tr}(\bar{\kappa} \circ \kappa)| = 6\mathfrak{X}^2 |\det Q|, \quad (2.350)$$

$$\mathrm{sgn}[\mathrm{tr}(\kappa \circ \bar{\kappa})] \equiv \mathrm{sgn}[\mathrm{tr}(\bar{\kappa} \circ \kappa)] = \mathrm{sgn}[(\det Q)]. \quad (2.351)$$

Thus, $Q^{\alpha\beta}$ is a tensor of full rank if and only if $\mathrm{tr}(\kappa \circ \bar{\kappa}) \equiv \mathrm{tr}(\bar{\kappa} \circ \kappa)$ is non-vanishing. Moreover, one can easily deduce the sign of $(\det Q)$ from that of $\mathrm{tr}(\kappa \circ \bar{\kappa}) \equiv \mathrm{tr}(\bar{\kappa} \circ \kappa)$.

If α and β are two arbitrary covectors, the *dimension* of the bilinear form Q is defined by the ratio $[Q(\alpha, \beta)]/[\alpha][\beta] =: [Q]$. Once this concept is established, it is possible to demonstrate that $[\det Q] = [Q]^4$. Ergo, because of (2.349), $\mathfrak{X}$ must satisfy

$$[\mathfrak{X}] = ([\det Q]^{-1} [\mathrm{tr}(\kappa \circ \bar{\kappa})])^{\frac{1}{2}} = ([Q]^{-4} [\kappa]^2)^{\frac{1}{2}} = [Q]^{-2} [\kappa] = [Q]^{-2} (hq^{-2})^{-1}. \quad (2.352)$$

Although it appears natural to demand that Q is dimensionless, pre-metric electrodynamics provides no simple physical motivation for imposing $[Q] = 1$. In other words, besides the fact that attaining $[\mathfrak{X}] = [\kappa] \stackrel{\text{SI}}{=} \Omega^{-1}$ is desirable, there is no obvious reason to fix the dimension of $\mathfrak{X}$ in a specific way, see (2.352). One additional comment is in order – the mixed closure relations are insensitive to the sign of $\mathfrak{X}$.

2.5.2 Reducing the Q -medium to a Hodge star

In the present Section, the Q -medium (2.312) is analysed at a point in spacetime such that $Q^{\alpha\beta}$ is invertible. Equivalently, a location $p \in X_4$ is considered, for which

$$\det(Q|_p) \neq 0. \quad (2.353)$$

As a consequence of (2.353) and of the requirement $\mathfrak{X} \neq 0$, one is allowed to define

$$\Upsilon|_p := |\det(Q|_p)|^{-\frac{1}{2}} \mathfrak{X}|_p^{-1}. \quad (2.354)$$

The constitutive law (2.312), as expressed in terms of the new parameter $\Upsilon|_p$, reads

$$(\kappa|_p)_{\alpha\beta}{}^{\mu\nu} = \Upsilon|_p^{-1} (\check{\kappa}|_p)_{\alpha\beta}{}^{\mu\nu}, \quad (2.355)$$

$$(\check{\kappa}|_p)_{\alpha\beta}{}^{\mu\nu} := |\det(Q|_p)|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q|_p^{\rho\mu} Q|_p^{\sigma\nu}. \quad (2.356)$$

It is important to note that the operator $\check{\kappa}|_p$ has the following features:

- (i) Given a change of basis in the tangent space of X_4 at p , the quantity $\Upsilon|_p$ transforms as a true scalar. Accordingly, $\check{\kappa}|_p$ is just a “normalised” copy of $\kappa|_p$.
- (ii) The map $\check{\kappa}|_p$ is dimensionless, that is to say, one can show that $[\Upsilon] = [\kappa]^{-1} \stackrel{\text{SI}}{=} \Omega$.
- (iii) Since the electromagnetic response (2.312) satisfies both mixed closure relations, the same property is inherited by $\check{\kappa}|_p$. More specifically, the scaling (2.355) gives rise to the compact formulae $\check{\kappa}|_p \circ \check{\kappa}|_p = \pm \text{Id}$ and $\check{\kappa}|_p \circ \check{\kappa}|_p = \pm \text{Id}$.
- (iv) The operator $\check{\kappa}|_p$ is invariant under the replacement $Q|_p \rightarrow \zeta Q|_p$ for any $\zeta \in \mathbb{R}_0$.

Hereafter, the statement (i) is verified by means of a direct strategy. In fact, the transformation behaviour of $\Upsilon|_p$ is derived from that of its constituents, namely $\mathfrak{X}|_p^{-1}$ and $|\det(Q|_p)|^{-\frac{1}{2}}$. From the treatment of Section 2.5.1, it is *known* that the parameter $\mathfrak{X}^{-1}$ is twisted scalar density of weight -1 . Hence, one is left with the task of analysing the quantity $|\det(Q|_p)|^{-\frac{1}{2}}$. As a first step, it is useful to recall that

$$Q|_p^{\mu'\nu'} = L_\mu^{\mu'} L_\nu^{\nu'} Q|_p^{\mu\nu}. \quad (2.357)$$

In particular, by computing the determinant on both sides of (2.357), the expression

$$\det(Q|_p^{\mu'\nu'}) = [\det(L_{\eta'}^{\eta'})]^2 \times \det(Q|_p^{\mu\nu}) = [\det(L_{\eta'}^{\eta})]^{-2} \times \det(Q|_p^{\mu\nu}), \quad (2.358)$$

is obtained. Thus, $|\det(Q|_p)|^{-\frac{1}{2}}$ is found to be a twisted scalar density of weight +1,

$$\begin{aligned} |\det(Q|_p^{\mu'\nu'})|^{-\frac{1}{2}} &= |\det(L_{\eta'}^{\eta})| \times |\det(Q|_p^{\mu\nu})|^{-\frac{1}{2}} \\ &= \operatorname{sgn}[\det(L_{\eta'}^{\eta})] \times [\det(L_{\eta'}^{\eta})] \times |\det(Q|_p^{\mu\nu})|^{-\frac{1}{2}}. \end{aligned} \quad (2.359)$$

To complete the proof, one invokes (2.354) and observes that $\Upsilon|_p$ is a *true scalar*. Correspondingly, the maps $\kappa|_p$ and $\check{\kappa}|_p$ are related by a simple “normalisation” factor. Our discussion may now proceed to the issue of demonstrating that (ii) holds true. The definition (2.354) and the identity $[\det Q] = [Q]^4$ lead to the dimensional formula

$$[\Upsilon] = [\det Q]^{-\frac{1}{2}}[\mathfrak{X}]^{-1} = [Q]^{-2}[\mathfrak{X}]^{-1}. \quad (2.360)$$

Employing (2.352) allows one to expand $[\mathfrak{X}]^{-1}$, so that a key conclusion is achieved

$$[\Upsilon] = [Q]^{-2}[\mathfrak{X}]^{-1} = [Q]^{-2}[Q]^2[\kappa]^{-1} = [\kappa]^{-1} = hq^{-2} \stackrel{\text{SI}}{=} \Omega. \quad (2.361)$$

On account of (2.355), the operator $\check{\kappa}|_p$ must evidently be *dimensionless*. As an aside, it is worth noting that the dimension of $\Upsilon|_p$ does not depend on the choice of $[Q]$. This property should be compared with equation (2.352), and with the remarks concerning $[\mathfrak{X}]$. In what follows, proposition (iii) is verified. An appropriate starting point is to substitute the “normalisation” (2.355) in the results (2.342) and (2.343),

$$\Upsilon|_p^{-2}(\check{\kappa}|_p \circ \check{\kappa}|_p) = \mathfrak{X}|_p^2 \det(Q|_p) \operatorname{Id}, \quad \Rightarrow \quad \check{\kappa}|_p \circ \check{\kappa}|_p = \Upsilon|_p^2 \mathfrak{X}|_p^2 \det(Q|_p) \operatorname{Id}, \quad (2.362)$$

$$\Upsilon|_p^{-2}(\check{\kappa}|_p \circ \check{\kappa}|_p) = \mathfrak{X}|_p^2 \det(Q|_p) \operatorname{Id}, \quad \Rightarrow \quad \check{\kappa}|_p \circ \check{\kappa}|_p = \Upsilon|_p^2 \mathfrak{X}|_p^2 \det(Q|_p) \operatorname{Id}. \quad (2.363)$$

Then, the right-hand sides of (2.362) and (2.363) are simplified by means of (2.354),

$$\check{\kappa}|_p \circ \check{\kappa}|_p = \operatorname{sgn}[\det(Q|_p)] \operatorname{Id} = \pm \operatorname{Id}, \quad (2.364)$$

$$\check{\kappa}|_p \circ \check{\kappa}|_p = \operatorname{sgn}[\det(Q|_p)] \operatorname{Id} = \pm \operatorname{Id}. \quad (2.365)$$

Ergo, the statement (iii) is indeed correct, and (iv) is the only item still missing a proof. Conveniently, this gap is easily filled. Given a non-vanishing parameter ζ , one can make use of the scaled tensor $Q'|_p^{\mu\nu} := \zeta Q|_p^{\mu\nu}$ so as to construct a new map,

$$\begin{aligned} (\check{\kappa}'|_p)_{\alpha\beta}{}^{\mu\nu} &:= |\det(Q'|_p)|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q'|_p^{\rho\mu} Q'|_p^{\sigma\nu} \\ &= \zeta^2 |\det(\zeta Q|_p)|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q|_p^{\rho\mu} Q|_p^{\sigma\nu}, \end{aligned} \quad (2.366)$$

which is a suitably modified version of $\check{\kappa}|_p$. It is important to observe that, by virtue of the familiar identity $\det(\zeta Q|_p) \equiv \zeta^4 \det(Q|_p)$, the operator (2.366) is expressed as

$$(\check{\kappa}'|_p)_{\alpha\beta}{}^{\mu\nu} = |\det(Q|_p)|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q|_p^{\rho\mu} Q|_p^{\sigma\nu} =: (\check{\kappa}|_p)_{\alpha\beta}{}^{\mu\nu}. \quad (2.367)$$

One thereby concludes that $\check{\kappa}|_p$ is *invariant* under the transformation $Q|_p^{\mu\nu} \rightarrow \zeta Q|_p^{\mu\nu}$, for every non-zero scalar ζ . Hereinbefore, the properties (i)–(iv) were demonstrated by means of four independent derivations. Nonetheless, it is possible to take a different approach. For example, one may view (ii) as a *consequence* of (iii) or (iv).

The analysis of the specific Q -media that obey (2.353) is completed by the following remarks. It is easy to show that the response (2.355) gives rise to the equations

$$\kappa|_p \circ \bar{\kappa}|_p = \text{sgn}[\det(Q|_p)] \Upsilon|_p^{-2} \text{Id} = \pm \Upsilon|_p^{-2} \text{Id}, \quad (2.368)$$

$$\bar{\kappa}|_p \circ \kappa|_p = \text{sgn}[\det(Q|_p)] \Upsilon|_p^{-2} \text{Id} = \pm \Upsilon|_p^{-2} \text{Id}. \quad (2.369)$$

Applying the trace on both sides of (2.368), and invoking the identity (2.348) leads to

$$\text{tr}(\kappa|_p \circ \bar{\kappa}|_p) \equiv \text{tr}(\bar{\kappa}|_p \circ \kappa|_p) = 6 \text{sgn}[\det(Q|_p)] \Upsilon|_p^{-2}. \quad (2.370)$$

It is instructive to split (2.370) in two formulae, encoding the sign and modulus of $\text{tr}(\kappa|_p \circ \bar{\kappa}|_p) \equiv \text{tr}(\bar{\kappa}|_p \circ \kappa|_p)$. Accordingly, one obtains that

$$\text{sgn}[\text{tr}(\kappa|_p \circ \bar{\kappa}|_p)] \equiv \text{sgn}[\text{tr}(\bar{\kappa}|_p \circ \kappa|_p)] = \text{sgn}[\det(Q|_p)], \quad (2.371)$$

$$|\text{tr}(\kappa|_p \circ \bar{\kappa}|_p)| \equiv |\text{tr}(\bar{\kappa}|_p \circ \kappa|_p)| = 6 \Upsilon|_p^{-2}, \quad (2.372)$$

where (2.371) is a restatement of (2.351). By contrast, (2.372) allows one to determine the absolute value of $\Upsilon|_p$ from the constitutive map alone,

$$|\Upsilon|_p| = \sqrt{6} |\text{tr}(\kappa|_p \circ \bar{\kappa}|_p)|^{-\frac{1}{2}} \equiv \sqrt{6} |\text{tr}(\bar{\kappa}|_p \circ \kappa|_p)|^{-\frac{1}{2}}. \quad (2.373)$$

It is worth noting that (2.370) does *not* indicate whether $\Upsilon|_p$ is positive or negative.

In what follows, the special Q -media that obey (2.353) are subject to an additional restriction. More explicitly, it is demanded that, at the usual location $p \in X_4$,

$$^{[a]}Q|_p = 0, \quad \text{that is,} \quad Q|_p = ^{[s]}Q|_p. \quad (2.374)$$

It is important to observe that, when the requirement $\det(Q|_p) \neq 0$ is combined with the symmetry (2.374), the map $Q|_p$ becomes a *metric* (cf. Wald [139]). Correspondingly, a notion of distance makes its appearance in the metric-free theory developed so far. While this is a crucial observation, there are good reasons not to be alarmed. *Firstly*, the metric $Q|_p$ is not assumed from the start. Rather, it is a derived quantity, obtained by gradually constraining the medium response. *Secondly*, the notion of distance enters the formalism in one precise place, namely, the constitutive law. The evident conceptual clarity that follows cannot be achieved in the traditional approach. For instance, when Maxwell's equations are expressed in terms of vector calculus, a metric is employed in several “well-hidden” places. *Lastly*, the notion of distance is hereby invoked as part of a didactic example. In fact, the whole of the present Section is to be regarded as a pedagogic detour from the strict development of metric-free electrodynamics.

Under the condition $^{[a]}Q|_p = 0$, the medium encoded by (2.355)–(2.356) turns into

$$(\kappa|_p)_{\alpha\beta}{}^{\mu\nu} = \Upsilon|_p^{-1} (*|_p)_{\alpha\beta}{}^{\mu\nu} \quad (2.375)$$

$$(*|_p)_{\alpha\beta}{}^{\mu\nu} := |\det(^{[s]}Q|_p)|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} ^{[s]}Q|_p{}^{\rho\mu} ^{[s]}Q|_p{}^{\sigma\nu}, \quad (2.376)$$

where $*|_p$ is the so-called *Hodge star* map, regarded in its action from 2-forms to twisted 2-forms, see Flanders [31]. Equivalently, if $Q|_p$ is both invertible and *symmetric*, the operator $\check{\kappa}|_p$ becomes the Hodge star commonly encountered in differential geometry. It follows that the properties of $\check{\kappa}|_p$ are inherited by $*|_p$, so that the

remarks (i)–(iv) apply unaltered to the Hodge star. In addition, one can verify that:

- (v) The skewon and axion components of $*|_p$ are *zero*. When the observation that the skewon part vanishes is combined with (iii), one obtains that $*|_p \circ *|_p = \pm \text{Id}$.
- (vi) It is always possible to find a basis of the tangent space at p which is *orthonormal*. More accurately, there exists a basis such that $Q|_p^{\mu\nu} = {}^{[s]}Q|_p^{\mu\nu}$ is a diagonal matrix, whose non-zero entries are equal to ± 1 . Provided r is the number of $+1$ signs, and s is the number of -1 signs, the difference $t = r - s$ is the *signature* of $Q|_p = {}^{[s]}Q|_p$. A simple calculation indicates that the signature of a 4-dimensional metric is restricted to the values $\{0, \pm 2, \pm 4\}$. Hence, the expression (2.376) defines three “kinds” of Hodge star, associated with the three allowed choices of $|t|$. One further comment is in order. The property (iv) dictates that all conformally related metrics (${}^{[s]}Q'|_p = \zeta {}^{[s]}Q|_p$) give rise to the same Hodge star, $*'|_p = *|_p$. Upon selecting an orthonormal basis, one can show the *converse* statement: all coinciding Hodge star maps ($*'|_p = *|_p$) are induced by conformally related metrics, ${}^{[s]}Q'|_p = \zeta {}^{[s]}Q|_p$. A rigorous proof of this theorem is available in the paper [22] by Dray, Kulkarni and Samuel.
- (vii) The scalar $\Upsilon|_p$ relating $\kappa|_p$ and $*|_p$ should be interpreted as a *signed impedance*.

For the purpose of analysing (v), it is convenient to return to a previous point in the discussion. In particular, one momentarily suspends the condition that $Q|_p$ is symmetric, and reconsiders the constitutive law (2.355)–(2.356). Since the electromagnetic response $(\kappa|_p)_{\alpha\beta}{}^{\mu\nu} = \Upsilon|_p^{-1}(\check{\kappa}|_p)_{\alpha\beta}{}^{\mu\nu}$ is a specific type of Q -medium, it is possible to compute the skewon and axion fields from (2.323) and (2.327). The quantities $(\mathcal{S}|_p)_\alpha{}^\beta$ and $\alpha|_p$, as derived from the constitutive law (2.355)–(2.356), read

$$(\mathcal{S}|_p)_\alpha{}^\beta = -\frac{1}{2}\Upsilon|_p^{-1}|\det(Q|_p)|^{-\frac{1}{2}}(\hat{\epsilon}_{\alpha\alpha_1\alpha_2\alpha_3}{}^{[s]}Q|_p^{\beta\alpha_1[a]}Q|_p^{\alpha_2\alpha_3]}), \quad (2.377)$$

$$\alpha|_p = -\frac{2}{3}\Upsilon|_p^{-1}|\det(Q|_p)|^{-\frac{1}{2}}\text{Pf}([a]Q|_p). \quad (2.378)$$

When the antisymmetric part of $Q|_p$ is set to zero again, cf. (2.374), the medium response $(\kappa|_p)_{\alpha\beta}{}^{\mu\nu} = \Upsilon|_p^{-1}(\check{\kappa}|_p)_{\alpha\beta}{}^{\mu\nu}$ turns into (2.375)–(2.376). At the same time, the skewon and axion fields (2.377)–(2.378) vanish trivially. It follows that the constitutive law $\kappa|_p = \Upsilon|_p^{-1}*|_p$ is formed solely of a principal contribution, $\kappa|_p = {}^{(1)}\kappa|_p$.

Since this statement holds true for the Hodge star map itself, the first remark in (v) is evidently correct. The next step consists in deducing the relation $*|_p \circ *|_p = \pm \text{Id}$. Given that the Hodge star is free of skewon, the properties (2.208)–(2.210) imply that

$$\bar{*}|_p = *|_p. \quad (2.379)$$

Moreover, it is known that $*|_p$ satisfies (iii), whereby one is allowed to replace $\check{\kappa}|_p$ with $*|_p$ in equations (2.364) and (2.365). As an outcome, it is readily verified that

$$*|_p \circ \bar{*}|_p = \text{sgn}[\det(^{[s]}Q|_p)] \text{Id} = \pm \text{Id}, \quad (2.380)$$

$$\bar{*}|_p \circ *|_p = \text{sgn}[\det(^{[s]}Q|_p)] \text{Id} = \pm \text{Id}. \quad (2.381)$$

By substituting (2.379) either in (2.380) or in (2.381), one attains the desired relation

$$*|_p \circ *|_p = \text{sgn}[\det(^{[s]}Q|_p)] \text{Id} = \pm \text{Id}. \quad (2.382)$$

As an aside, it is worth noting that the electromagnetic response $\kappa|_p = \Upsilon|_p^{-1} *|_p$ obeys

$$\kappa|_p \circ \kappa|_p = \text{sgn}[\det(^{[s]}Q|_p)] \Upsilon|_p^{-2} \text{Id}, \quad (2.383)$$

in view of (2.382). Therefore, calculating the trace on both sides of (2.383) leads to

$$\text{tr}(\kappa|_p \circ \kappa|_p) = 6 \text{sgn}[\det(^{[s]}Q|_p)] \Upsilon|_p^{-2}. \quad (2.384)$$

One may split (2.384) in two formulae, encoding the sign and modulus of $\text{tr}(\kappa|_p \circ \kappa|_p)$,

$$\text{sgn}[\text{tr}(\kappa|_p \circ \kappa|_p)] = \text{sgn}[\det(^{[s]}Q|_p)], \quad (2.385)$$

$$|\text{tr}(\kappa|_p \circ \kappa|_p)| = 6 \Upsilon|_p^{-2}. \quad (2.386)$$

The result (2.386) is useful in determining the absolute value of $\Upsilon|_p$ from $\kappa|_p$ alone,

$$|\Upsilon|_p| = \sqrt{6} |\text{tr}(\kappa|_p \circ \kappa|_p)|^{-\frac{1}{2}}. \quad (2.387)$$

By contrast, it is important to remark that the *sign* of $\Upsilon|_p$ is not constrained at all.

Now that the treatment of (v) is complete, it is appropriate to discuss the remark (vi). The existence of an *orthonormal basis* is guaranteed by the following two facts. Firstly, $Q|_p$ is assumed to be a metric, that is to say, a non-degenerate (2.353) and symmetric (2.374) bilinear form. Secondly, a crucial theorem in linear algebra states that every symmetric matrix can be diagonalised over the real numbers (Serre [124]).

More accurately, in view of the condition $Q|_p = {}^{[s]}Q|_p$, it is always possible to find a basis $\{f_0, \dots, f_3\}$ of the tangent space at p such that $Q|_p^{\mu\nu}$ is a diagonal matrix

$$\begin{aligned} Q|_p &= \text{diag}(Q|_p^{00}, Q|_p^{11}, Q|_p^{22}, Q|_p^{33}) \\ &= Q|_p^{00} f_0 \otimes f_0 + Q|_p^{11} f_1 \otimes f_1 + Q|_p^{22} f_2 \otimes f_2 + Q|_p^{33} f_3 \otimes f_3. \end{aligned} \quad (2.388)$$

In addition, because of the requirement $\det(Q|_p) \neq 0$, the entries $\{Q|_p^{00}, \dots, Q|_p^{33}\}$ are non-zero. Therefore, one is allowed to scale the vectors $\{f_0, \dots, f_3\}$ according to

$$\begin{aligned} f_0 &= |Q|_p^{00}|^{-\frac{1}{2}} f'_0, & f_1 &= |Q|_p^{11}|^{-\frac{1}{2}} f'_1, \\ f_2 &= |Q|_p^{22}|^{-\frac{1}{2}} f'_2, & f_3 &= |Q|_p^{33}|^{-\frac{1}{2}} f'_3. \end{aligned} \quad (2.389)$$

Crucially, when the map $Q|_p$ is expressed in terms of $\{f'_0, \dots, f'_3\}$, it takes the form

$$\begin{aligned} Q|_p &= + \text{sgn}(Q|_p^{00}) f'_0 \otimes f'_0 + \text{sgn}(Q|_p^{11}) f'_1 \otimes f'_1 \\ &\quad + \text{sgn}(Q|_p^{22}) f'_2 \otimes f'_2 + \text{sgn}(Q|_p^{33}) f'_3 \otimes f'_3, \end{aligned} \quad (2.390)$$

which, in a simplified notation, reads $Q|_p = \text{diag}(\pm 1, \dots, \pm 1)$. It follows that an *orthonormal basis* $\{f'_\alpha\}$ for the tangent space of X_4 at p does indeed exist, see (vi). As an aside, it is worth noting that the vectors $\{f'_0, \dots, f'_3\}$ are characterised by the same dimension, $[f'_0] = [f'_1] = [f'_2] = [f'_3] = [Q]^{\frac{1}{2}}$. One route towards demonstrating this property starts from the formula (2.388), which is employed so as to infer that

$$\begin{aligned} [Q^{00}] &= [f_0]^{-2} [Q], & [Q^{11}] &= [f_1]^{-2} [Q], \\ [Q^{22}] &= [f_2]^{-2} [Q], & [Q^{33}] &= [f_3]^{-2} [Q]. \end{aligned} \quad (2.391)$$

The equalities in (2.389) demand that the elements of the orthonormal basis satisfy

$$\begin{aligned} [f'_0] &= [Q^{00}]^{\frac{1}{2}}[f_0], & [f'_1] &= [Q^{11}]^{\frac{1}{2}}[f_1], \\ [f'_2] &= [Q^{22}]^{\frac{1}{2}}[f_2], & [f'_3] &= [Q^{33}]^{\frac{1}{2}}[f_3]. \end{aligned} \quad (2.392)$$

Ergo, upon substituting (2.391) in (2.392), one achieves the desired conclusion – i.e.

$$[f'_0] = [f'_1] = [f'_2] = [f'_3] = [Q]^{\frac{1}{2}}. \quad (2.393)$$

At this point, some further aspects of (vi) are examined. It is important to recall that, if the expansion (2.390) contains r plus signs and s minus signs, the difference $t = r - s$ is the signature of the metric $Q|_p$. Given that $Q|_p$ is assumed to be invertible, the non-negative integers r and s obey the equation $r + s = 4$. Consequently, one deduces that the allowed values of t are $\{0, \pm 2, \pm 4\}$. The signature of a metric does not depend on the choice of orthonormal basis [139], whereby t is said to be an invariant [31, 75]. As a matter of fact, all metrics are legitimately *classified* according to their signature. It is also possible to show that, on account of the formula (2.376), each value of $|t|$ is associated with one “kind” of Hodge star operator. The absolute value appearing in this statement is a specific manifestation of the property (iv), which dictates that $+Q|_p$ and $-Q|_p$ give rise to the same map $*|_p$. In order to clarify the existence of three “kinds” of Hodge star, it is convenient to inspect the cases $|t| \in \{0, 2, 4\}$ individually.

Hereafter, the option $|t| = 0$ is illustrated by employing the typical representative

$$Q|_p = f'_0 \otimes f'_0 + f'_1 \otimes f'_1 - f'_2 \otimes f'_2 - f'_3 \otimes f'_3 = \text{diag}(+1, +1, -1, -1). \quad (2.394)$$

More specifically, the electromagnetic response $(\kappa|_p)_{\alpha\beta}{}^{\mu\nu} = \Upsilon|_p^{-1}(*|_p)_{\alpha\beta}{}^{\mu\nu}$ arising from the metric (2.394) is calculated. After the normal form (2.394) is implemented in the formulae (2.375)–(2.376), the resulting constitutive law is expressed as a 6×6 matrix, in the manner described in (2.238). Hence, the following tableau is achieved,

$$(\kappa|_p)_I^J = \Upsilon|_p^{-1}(*|_p)_I^J = \Upsilon|_p^{-1} \left[\begin{array}{c|c} \mathbb{O}_{3 \times 3} & \mathbb{K}_{3 \times 3} \\ \hline \mathbb{K}_{3 \times 3} & \mathbb{O}_{3 \times 3} \end{array} \right], \quad (2.395)$$

where $\mathbb{O}_{3 \times 3}$ is the 3×3 null matrix, and $\mathbb{K}_{3 \times 3} := \text{diag}(+1, -1, -1)$. A direct comparison of (2.395) with (2.238) allows one to identify the permittivity, the inverse permeability and the magneto-electric couplings at $p \in X_4$. It is observed that, for the medium being analysed, the tensors β_a^c and α_{ab}^{cd} vanish. Such property is due to the fact that the basis adopted in (2.394) sets the components $Q|_p^{0i} = Q|_p^{i0}$ to zero. In the tableau (2.395), the permittivity and inverse permeability are encoded, up to a sign, by the bottom-left and top-right 3×3 blocks. Each of these blocks is found to be *diagonal*. More accurately, the tensors $\varepsilon'_{ab}{}^c$ and $(\mu^{-1})_a{}^{cd}$ at $p \in X_4$ read:

$$\varepsilon'_{ab}{}^c = \begin{bmatrix} \varepsilon'_{23}{}^1 & \varepsilon'_{23}{}^2 & \varepsilon'_{23}{}^3 \\ \varepsilon'_{31}{}^1 & \varepsilon'_{31}{}^2 & \varepsilon'_{31}{}^3 \\ \varepsilon'_{12}{}^1 & \varepsilon'_{12}{}^2 & \varepsilon'_{12}{}^3 \end{bmatrix} = \begin{bmatrix} -\Upsilon|_p^{-1} & 0 & 0 \\ 0 & +\Upsilon|_p^{-1} & 0 \\ 0 & 0 & +\Upsilon|_p^{-1} \end{bmatrix}, \quad (2.396)$$

$$(\mu^{-1})_a{}^{cd} = \begin{bmatrix} (\mu^{-1})_1{}^{23} & (\mu^{-1})_1{}^{31} & (\mu^{-1})_1{}^{12} \\ (\mu^{-1})_2{}^{23} & (\mu^{-1})_2{}^{31} & (\mu^{-1})_2{}^{12} \\ (\mu^{-1})_3{}^{23} & (\mu^{-1})_3{}^{31} & (\mu^{-1})_3{}^{12} \end{bmatrix} = \begin{bmatrix} +\Upsilon|_p^{-1} & 0 & 0 \\ 0 & -\Upsilon|_p^{-1} & 0 \\ 0 & 0 & -\Upsilon|_p^{-1} \end{bmatrix}. \quad (2.397)$$

In order to obtain the Fresnel polynomial corresponding to the constitutive law (2.395), it is sufficient to make use of the formula (2.338), which holds true for any Q -medium. The electromagnetic response being considered, gives rise to the non-birefringent dispersion equation

$$\mathcal{G}(q) = 4! \Upsilon|_p^{-3} (q_0^2 + q_1^2 - q_2^2 - q_3^2)^2 = 0. \quad (2.398)$$

In the situation where (2.395) describes a laboratory material, being illuminated by a plane wave, it is possible to interpret q_0 as the angular frequency. Similarly, one is allowed to regard q_i as the spatial wave-covector. As a consequence, the ratio (q_i/q_0) must represent the inverse phase velocity. When studying a laboratory material, the occurrence $q_0 = 0$ is rarely of interest, as it prescribes the formation of standing waves. Because of this fact, one assumes that q_0 is non-vanishing, and divides (2.398) by the angular frequency. An equation governing the inverse phase velocity is thus achieved (see Figure 2.4),

$$-\left(\frac{q_1}{q_0}\right)^2 + \left(\frac{q_2}{q_0}\right)^2 + \left(\frac{q_3}{q_0}\right)^2 = 1. \quad (2.399)$$

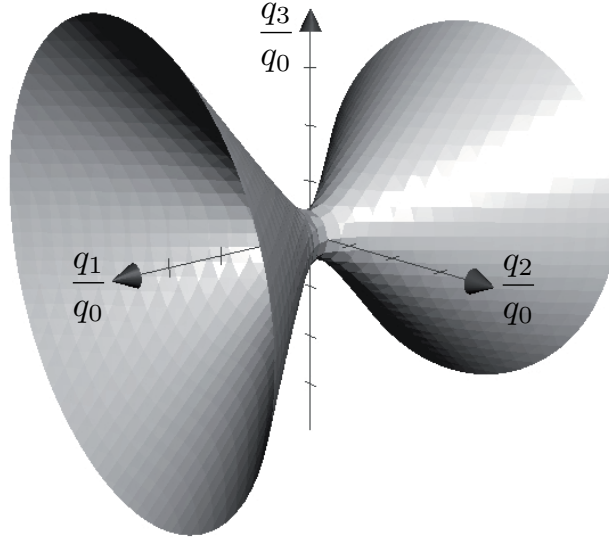


Figure 2.4: The “modified” Fresnel surface encoding the inverse phase velocity for the material (2.395). It appears evident that (2.399) defines a hyperboloid of one sheet, whose axis of symmetry is (q_1/q_0) . Aside: this plot was generated using the software Grapher[®].

Above, the Hodge star map induced by a metric of signature $t = 0$ is analysed by making use of the typical representative (2.394). However, one could select an alternative normal form, in which the $+1$ and -1 signs along the diagonal are ordered differently. For example, it is equally appropriate to choose:

$$Q|_p = \text{diag}(+1, -1, -1, +1). \quad (2.400)$$

Conveniently, when (2.400) is employed instead of (2.394), it is not necessary to re-derive the expressions (2.396)–(2.398). Indeed, one can follow a more efficient strategy, and re-label the elements of the orthonormal basis. As a first step, it is important to observe that, if the basis vectors f'_1 and f'_3 are exchanged, the typical representative (2.394) transforms into (2.400). Hence, the same scheme is applied to (2.396) and (2.397). Accordingly, one implements the switch $1 \leftrightarrow 3$ to the tensor

indices, and obtains the permittivity and inverse permeability arising from (2.400):

$$\varepsilon'_{ab}{}^c = \begin{bmatrix} \varepsilon'_{23}{}^1 & \varepsilon'_{23}{}^2 & \varepsilon'_{23}{}^3 \\ \varepsilon'_{31}{}^1 & \varepsilon'_{31}{}^2 & \varepsilon'_{31}{}^3 \\ \varepsilon'_{12}{}^1 & \varepsilon'_{12}{}^2 & \varepsilon'_{12}{}^3 \end{bmatrix} = \begin{bmatrix} -\Upsilon|_p^{-1} & 0 & 0 \\ 0 & -\Upsilon|_p^{-1} & 0 \\ 0 & 0 & +\Upsilon|_p^{-1} \end{bmatrix}, \quad (2.401)$$

$$(\mu^{-1})_a{}^{cd} = \begin{bmatrix} (\mu^{-1})_1{}^{23} & (\mu^{-1})_1{}^{31} & (\mu^{-1})_1{}^{12} \\ (\mu^{-1})_2{}^{23} & (\mu^{-1})_2{}^{31} & (\mu^{-1})_2{}^{12} \\ (\mu^{-1})_3{}^{23} & (\mu^{-1})_3{}^{31} & (\mu^{-1})_3{}^{12} \end{bmatrix} = \begin{bmatrix} +\Upsilon|_p^{-1} & 0 & 0 \\ 0 & +\Upsilon|_p^{-1} & 0 \\ 0 & 0 & -\Upsilon|_p^{-1} \end{bmatrix}, \quad (2.402)$$

at $p \in X_4$. Given that the exchange $1 \leftrightarrow 3$ leaves the property $Q|_p^{0i} = Q|_p^{i0} = 0$ unaltered, the magneto-electric terms $\beta_a{}^c$ and $\alpha_{ab}{}^{cd}$ remain vanishing. In addition, the re-labelling of basis vectors being enforced converts the Fresnel equation (2.398) into

$$\mathcal{G}(q) = 4!\Upsilon|_p^{-3}(q_0^2 - q_1^2 - q_2^2 + q_3^2)^2 = 0. \quad (2.403)$$

A laboratory medium characterised by the constitutive law (2.401)–(2.402) is examined in the article [127] by Smith and Schurig, see Figure 2 therein. More specifically, they consider an interface between the material (2.401)–(2.402) and vacuum, for the purpose of investigating the refraction of light²⁴. An interesting feature is thus observed: the options $\Upsilon > 0$ and $\Upsilon < 0$ are associated, respectively, with *positive* and *negative* refraction. Determining the extent to which this statement depends on the choice of interface will be the topic of future research. In Ref. [127], Smith and Schurig also explain how an appropriate metamaterial structure can generate the permittivity and inverse permeability tensors (2.401) and (2.402). As a matter of fact, it is demonstrated that the required medium parameters can be accomplished by means of a suitable array of metal rods and split rings. Notably, a similar metamaterial design can be used to realise the medium (2.396)–(2.397) in the laboratory.

In what follows, the second “kind” of Hodge star is examined, that is to say, the operator $\ast|_p$ is analysed in the case where the signature of $Q|_p$ obeys $|t| = 2$. For the sake of simplicity, the calculations are carried out with respect to the normal form:

$$Q|_p = f'_0 \otimes f'_0 - f'_1 \otimes f'_1 - f'_2 \otimes f'_2 - f'_3 \otimes f'_3 = \text{diag}(+1, -1, -1, -1). \quad (2.404)$$

²⁴Of course, when solving an interface problem, one takes into account more than a *single* $p \in X_4$!

As a first task, one evaluates the 6×6 constitutive matrix $(\kappa|_p)_I^J = \Upsilon|_p^{-1}(*|_p)_I^J$ generated by the typical representative (2.404). Ergo, the metric (2.404) is substituted in the expressions (2.375)–(2.376), and the ensuing medium response is encoded as

$$(\kappa|_p)_I^J = \Upsilon|_p^{-1}(*|_p)_I^J = \Upsilon|_p^{-1} \left[\begin{array}{c|c} \mathbb{O}_{3 \times 3} & +\mathbb{I}_{3 \times 3} \\ \hline -\mathbb{I}_{3 \times 3} & \mathbb{O}_{3 \times 3} \end{array} \right], \quad (2.405)$$

where $\mathbb{I}_{3 \times 3}$ is the 3×3 unit matrix. A direct comparison of (2.405) with (2.238) allows one to recognise the permittivity, the inverse permeability and the magneto-electric couplings at $p \in X_4$. It is thus observed that the tensors β_a^c and α_{ab}^{cd} vanish, consistently with the fact that the basis adopted in (2.404) sets the components $Q|_p^{0i} = Q|_p^{i0}$ to zero. The permittivity and inverse permeability, as represented by 3×3 blocks, are found to be *diagonal*. More explicitly, at the location p in spacetime,

$$\varepsilon'_{ab}{}^c = \begin{bmatrix} \varepsilon'_{23}{}^1 & \varepsilon'_{23}{}^2 & \varepsilon'_{23}{}^3 \\ \varepsilon'_{31}{}^1 & \varepsilon'_{31}{}^2 & \varepsilon'_{31}{}^3 \\ \varepsilon'_{12}{}^1 & \varepsilon'_{12}{}^2 & \varepsilon'_{12}{}^3 \end{bmatrix} = \begin{bmatrix} \Upsilon|_p^{-1} & 0 & 0 \\ 0 & \Upsilon|_p^{-1} & 0 \\ 0 & 0 & \Upsilon|_p^{-1} \end{bmatrix}, \quad (2.406)$$

$$(\mu^{-1})_a{}^{cd} = \begin{bmatrix} (\mu^{-1})_1{}^{23} & (\mu^{-1})_1{}^{31} & (\mu^{-1})_1{}^{12} \\ (\mu^{-1})_2{}^{23} & (\mu^{-1})_2{}^{31} & (\mu^{-1})_2{}^{12} \\ (\mu^{-1})_3{}^{23} & (\mu^{-1})_3{}^{31} & (\mu^{-1})_3{}^{12} \end{bmatrix} = \begin{bmatrix} \Upsilon|_p^{-1} & 0 & 0 \\ 0 & \Upsilon|_p^{-1} & 0 \\ 0 & 0 & \Upsilon|_p^{-1} \end{bmatrix}. \quad (2.407)$$

In order to obtain the Fresnel polynomial arising from the constitutive law (2.405), it is sufficient to make use of (2.338). Obviously, the resulting dispersion equation,

$$\mathcal{G}(q) = -4!\Upsilon|_p^{-3}(q_0^2 - q_1^2 - q_2^2 - q_3^2)^2 = 0, \quad (2.408)$$

is non-birefringent. In (2.408), it is possible to interpret q_0 as the angular frequency of a plane wave. Correspondingly, one can regard q_i as the spatial wave-covector, and (q_i/q_0) as the inverse phase velocity. When q_0 is zero, the electromagnetic fields appear to be stationary in space, that is to say, one observes the formation of standing waves. In the context of the present discussion, this scenario is not of interest. Accordingly, one assumes that q_0 is non-vanishing, and divides (2.408) by the angular frequency – thus obtaining an equation for the inverse phase velocity,

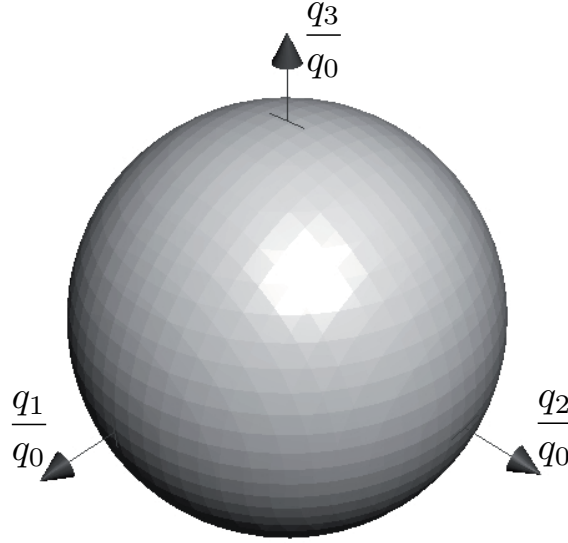


Figure 2.5: The “modified” Fresnel surface defined by (2.409), which encodes the inverse phase velocity for the medium (2.405). Aside: the plot was generated using Grapher[®].

$$\left(\frac{q_1}{q_0}\right)^2 + \left(\frac{q_2}{q_0}\right)^2 + \left(\frac{q_3}{q_0}\right)^2 = 1, \quad (2.409)$$

see Figure 2.5. It is now appropriate to consider some examples of media whose electromagnetic response is of the type (2.405)²⁵. Provided one demands that $\Upsilon > 0$, the tableau (2.405) may represent a simple dielectric, or the familiar Maxwell-Lorentz vacuum. In the latter case, the parameter Υ is merely the impedance of free space,

$$\Omega_0 \overset{\text{SI}}{\approx} 137 \, \Omega. \quad (2.410)$$

When – instead – one demands that $\Upsilon < 0$, the constitutive law (2.405) gives rise to the negative refracting medium put forward by Veselago [138] in 1968. The first experimental group to realise an artificial material with the optical properties described in Ref. [138] was that of Smith and co-workers [126], who achieved this result in 2000. Few months later, Pendry [110] suggested that a slab-shaped lens, built of Veselago’s negative refracting medium, is capable of imaging objects which are smaller than the diffraction limit. In 2004, Grbic and Eleftheriades [35] verified Pendry’s conclusions by successfully producing a lens with sub-wavelength resolution

²⁵In analysing such examples, one assumption is relaxed: for the sake of simplicity, we neglect the fact that all formulae developed in this Section refer to a *single* point in spacetime, $p \in X_4$.

(at microwave frequencies).

In (2.404), a typical representative is selected, whose signature is $t = -2$. Nevertheless, it is indeed possible to employ an alternative normal form, such that every sign along the diagonal is reversed, and $t = +2$. As stated previously, making use of this “sign-conjugate” typical representative is devoid of physical consequences. In fact, because of the property (iv), the metrics $+Q|_p$ and $-Q|_p$ generate the same Hodge star, and thus the same Fresnel equation. The following remark concerns the signatures $t = -2$ and $t = +2$ examined individually. Among the normal forms with $t = -2$, one is allowed to choose any representative obtained by permuting the diagonal elements of (2.404). It appears evident that a corresponding statement applies, *mutatis mutandis*, to the option $t = +2$. To summarise, one has that, for example:

$$Q|_p = \text{diag}(-1, -1, -1, +1), \quad (2.411)$$

is a valid alternative to (2.404). Conveniently, when (2.411) is employed instead of (2.404), it is not necessary to re-derive the matrix κ_I^J and the dispersion equation from first principles. As a matter of fact, by exchanging the basis vectors f'_0 and f'_3 , one achieves the required expressions in an efficient way. It is interesting to note that a suitable metamaterial can reproduce the permittivity and inverse permeability tensors emerging from (2.411). A detailed explanation is available in the paper [127] by Smith and Schurig, where the relevant medium is dubbed “never-cutoff”.

Hereafter, the third and last “kind” of Hodge star is examined. In other words, the map $*|_p$ is analysed under the assumption that the signature of $Q|_p$ obeys $|t| = 4$. Consistently with the usual plan of action, we base our treatment on a normal form,

$$Q|_p = -f'_0 \otimes f'_0 - f'_1 \otimes f'_1 - f'_2 \otimes f'_2 - f'_3 \otimes f'_3 = \text{diag}(-1, -1, -1, -1). \quad (2.412)$$

The first objective is to calculate the 6×6 matrix $(\kappa|_p)_I^J = \Upsilon|_p^{-1}(*|_p)_I^J$ generated by the typical representative (2.412). For this purpose, the metric (2.412) is substituted in the expressions (2.375)–(2.376), and the ensuing medium response is encoded as

$$(\kappa|_p)_I^J = \Upsilon|_p^{-1}(*|_p)_I^J = \Upsilon|_p^{-1} \left[\begin{array}{c|c} \mathbb{O}_{3 \times 3} & \mathbb{I}_{3 \times 3} \\ \hline \mathbb{I}_{3 \times 3} & \mathbb{O}_{3 \times 3} \end{array} \right]. \quad (2.413)$$

A direct comparison of (2.413) with (2.238) allows one to recognise the permittivity, the inverse permeability and the magneto-electric couplings at $p \in X_4$. It is thus observed that the tensors β_a^c and α_{ab}^{cd} are zero, while $\varepsilon'_{ab}{}^c$ and $(\mu^{-1})_a{}^{cd}$ are diagonal,

$$\varepsilon'_{ab}{}^c = \begin{bmatrix} \varepsilon'_{23}{}^1 & \varepsilon'_{23}{}^2 & \varepsilon'_{23}{}^3 \\ \varepsilon'_{31}{}^1 & \varepsilon'_{31}{}^2 & \varepsilon'_{31}{}^3 \\ \varepsilon'_{12}{}^1 & \varepsilon'_{12}{}^2 & \varepsilon'_{12}{}^3 \end{bmatrix} = \begin{bmatrix} -\Upsilon|_p^{-1} & 0 & 0 \\ 0 & -\Upsilon|_p^{-1} & 0 \\ 0 & 0 & -\Upsilon|_p^{-1} \end{bmatrix}, \quad (2.414)$$

$$(\mu^{-1})_a{}^{cd} = \begin{bmatrix} (\mu^{-1})_1{}^{23} & (\mu^{-1})_1{}^{31} & (\mu^{-1})_1{}^{12} \\ (\mu^{-1})_2{}^{23} & (\mu^{-1})_2{}^{31} & (\mu^{-1})_2{}^{12} \\ (\mu^{-1})_3{}^{23} & (\mu^{-1})_3{}^{31} & (\mu^{-1})_3{}^{12} \end{bmatrix} = \begin{bmatrix} \Upsilon|_p^{-1} & 0 & 0 \\ 0 & \Upsilon|_p^{-1} & 0 \\ 0 & 0 & \Upsilon|_p^{-1} \end{bmatrix}. \quad (2.415)$$

To obtain the Fresnel polynomial arising from the constitutive law (2.413), it is sufficient to make use of (2.338). Since the resulting dispersion equation is bi-quadratic,

$$\mathcal{G}(q) = 4!\Upsilon|_p^{-3}(-q_0^2 - q_1^2 - q_2^2 - q_3^2)^2 = 0, \quad (2.416)$$

the medium (2.413) is non-birefringent, at least in the mathematical sense of the term. Besides this fact, it is not possible for the components of the 4-dimensional wave-covector to be all real, and to simultaneously fulfill (2.416). More explicitly, the constitutive law (2.413) does not support the *propagation* of electromagnetic waves. The solutions of (2.416) may be regarded as physical only in specific cases, and at the price of ignoring several conceptual problems. As an example, it is useful to consider an interface between vacuum and the material (2.413), subject to plane-wave illumination. One can show that the radiation incoming from empty-space is reflected without loss. Correspondingly, the fields inside the material are observed to decay in an exponential guise. These *evanescent* waves are described by a covector $q_\alpha = (q_0, q_1, q_2, q_3)$ which is a solution of (2.416), and whose entries are not all real.

It is now appropriate to examine some artificial and naturally occurring materials whose electromagnetic response is of the type (2.413) – see Footnote 25. Provided one demands that $\Upsilon > 0$, the constitutive law (2.413) may represent an ionised gas (a “plasma”) driven by monochromatic light at a frequency below the critical value (the “plasma frequency”). Remarkably, across a particular region of the spectrum, metals behave like plasmas. In addition, the same statement holds true for a 3-

dimensional array of conducting rods [111]. As a consequence, the electromagnetic response (2.413) with $\Upsilon > 0$ also describes a metal or a lattice of wires. The medium parameters originating from the option $\Upsilon < 0$ require a suitable metamaterial to be achieved. More accurately, the constitutive law (2.413) is successfully mimicked by a cubic arrangement of split-ring resonators [126].

In (2.412), a typical representative is selected, whose signature is $t = -4$. Nevertheless, it is indeed possible to employ an alternative normal form, such that every sign along the diagonal is reversed, and $t = +4$. As stated previously, making use of this “sign-conjugate” typical representative is devoid of physical consequences. In fact, because of the property (iv), the metrics $+Q|_p$ and $-Q|_p$ generate the same Hodge star, and thus the same Fresnel equation.

It was previously observed that, given a metric $Q|_p$, an orthonormal basis of the tangent space at p is always available. Before concluding the analysis of (vi), a related theorem ought to be mentioned: provided a smooth metric is defined at all points in X_4 , an orthonormal frame does exist – if not globally, over some neighborhood of each spacetime location. A proof dealing with the signatures $t = \pm 4$ is found in the book [75] by Lee. The remaining cases are discussed by Michor [96]. One technical comment is in order. In general, it is not possible to construct a *holonomic* frame which is also orthonormal over an open set.

In what follows, the remark (vii) is examined. More explicitly, it is demonstrated that the modulus of $\Upsilon|_p$ is the familiar *impedance*, while $\text{sgn}(\Upsilon|_p)$ is a *free* parameter. The first element in support of this concept is that the absolute value of $\Upsilon|_p$ has the appropriate physical dimensions, see (2.361). In addition, by inspecting the tensors (2.396)–(2.397) or (2.406)–(2.407) or (2.414)–(2.415), one can easily recognise that:

$$|\Upsilon|_p = |\varepsilon'^1_{23}(\mu^{-1})_1^{23}|^{-\frac{1}{2}} = |\varepsilon'^2_{31}(\mu^{-1})_2^{31}|^{-\frac{1}{2}} = |\varepsilon'^3_{12}(\mu^{-1})_3^{12}|^{-\frac{1}{2}}. \quad (2.417)$$

Crucially, equation (2.417) is the elementary impedance formula $(\mu/\varepsilon)^{\frac{1}{2}}$ in disguise. As a result, the modulus of $\Upsilon|_p$ does indeed coincide with the well-known *impedance*. In view of the examples described earlier in this Section, it appears evident that the cases $\Upsilon|_p > 0$ and $\Upsilon|_p < 0$ are both physically viable. Furthermore, a brief analysis of the calculations performed so far reveals that the sign of $\Upsilon|_p$ is still *unconstrained*.

$ t $	$Q _p$	$\Upsilon _p$	Description of medium	Artificial?
0	$\text{diag}(+1, +1, -1, -1)$	> 0	Wave surface is one-sheet hyperboloid (Fig. 2.4)	Yes
		< 0	Wave surface is one-sheet hyperboloid (Fig. 2.4)	Yes
0	$\text{diag}(+1, -1, -1, +1)$	> 0	Smith & Schurig's "anti-cutoff" medium [127]	Yes
		< 0	Smith & Schurig's "anti-cutoff" medium [127]	Yes
2	$\text{diag}(+1, -1, -1, -1)$	> 0	Simple dielectric or Maxwell-Lorentz vacuum	No
		< 0	Veselago's negative refracting medium [138]	Yes
2	$\text{diag}(-1, -1, -1, +1)$	> 0	Smith & Schurig's "never-cutoff" medium [127]	Yes
		< 0	Smith & Schurig's "never-cutoff" medium [127]	Yes
4	$\text{diag}(-1, -1, -1, -1)$	> 0	Plasma or metal driven below critical frequency	No
		< 0	Cubic arrangement of split-ring resonators [126]	Yes

Table 2.3: A summary of the media examined in the present Section that originate from the broader constitutive law (2.375)–(2.376). The rightmost column specifies whether a given optical material is artificial or naturally occurring. Note, however, that the penultimate electromagnetic response also describes a *man-made* array of conducting rods [111].

According to the observation (i), the parameter $\Upsilon|_p$ is a true scalar. Hence, no change of basis can switch the sign this quantity, even in the case where the orientation of the tangent space is not preserved, $\det(L_\rho^\rho) < 0$. Given a constitutive law of the type (2.405), the choice of $\text{sgn}(\Upsilon|_p)$ marks the difference between a simple dielectric and Veselago's negative refracting medium. As a consequence, it is not possible to transform such materials into each other by employing an (orientation-reversing) change of basis, cf. Bergamin and Favaro [7]. The opposite, incorrect statement is reported in several publications – amongst others, those of Leonhardt and Philbin [77, 78] or Schurig, Pendry and Smith [122].

As an idea for future research, it may be interesting to compare the notions discussed in this Section with the article [55] by Itin and Friedman. Table 2.3 is a résumé of the constitutive laws just analysed that ensue from the Hodge star map.

2.5.3 P -media

The Q -media, as originally formulated in (2.312), extend the simple constitutive law described by a Hodge star map. In what follows, a different kind of electromagnetic response is examined, which generalises the pure axion case $\kappa = {}^{(3)}\kappa$. The P -media,

first introduced by Lindell, Bergamin and Favaro [86], are characterised by the law

$$\kappa_{\alpha\beta}{}^{\mu\nu} = 2Y P_{[\alpha}{}^{\mu} P_{\beta]}{}^{\nu} = Y \{ P_{\alpha}{}^{\mu} P_{\beta}{}^{\nu} - P_{\beta}{}^{\mu} P_{\alpha}{}^{\nu} \}, \quad (Y \neq 0) \quad (2.418)$$

where $P_{\alpha}{}^{\beta}$ is subject to *only one* constraint, namely the tensor transformation rule

$$P_{\alpha'}{}^{\beta'} = L_{\alpha'}{}^{\alpha} L_{\beta}{}^{\beta'} P_{\alpha}{}^{\beta}. \quad (2.419)$$

Upon comparing (2.418) and (2.419) with (2.144), it appears evident that the quantity $Y \neq 0$ is a twisted scalar. An investigation on the properties of P -media can begin by computing the axion field α and the skewon field $\mathcal{S}_{\alpha}{}^{\beta}$. With the help of the definitions $\text{tr}(P) := P_{\alpha}{}^{\alpha}$ and $(P^2)_{\alpha}{}^{\beta} := P_{\alpha}{}^{\rho} P_{\rho}{}^{\beta}$, one can immediately show that:

$$\alpha = \frac{1}{12} \kappa_{\rho\sigma}{}^{\rho\sigma} = \frac{1}{12} Y [(P_{\rho}{}^{\rho} P_{\sigma}{}^{\sigma}) - (P_{\sigma}{}^{\rho} P_{\rho}{}^{\sigma})] = \frac{1}{12} Y [(\text{tr} P)^2 - \text{tr}(P^2)], \quad (2.420)$$

by applying (2.210) and (2.212) to the ansatz (2.418). An efficient way to derive the field $\mathcal{S}_{\alpha}{}^{\beta}$ is to proceed by “difference”. More explicitly, one starts from the identity

$${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu} = \kappa_{\alpha\beta}{}^{\mu\nu} - {}^{(1)}\kappa_{\alpha\beta}{}^{\mu\nu} - {}^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu}, \quad (2.421)$$

and performs a trivial substitution in (2.211). Thus, the skewon $\mathcal{S}_{\alpha}{}^{\beta}$ is expressed as

$$\mathcal{S}_{\alpha}{}^{\beta} = -\frac{1}{2} (\kappa_{\alpha\rho}{}^{\beta\rho} - {}^{(1)}\kappa_{\alpha\rho}{}^{\beta\rho} - {}^{(3)}\kappa_{\alpha\rho}{}^{\beta\rho}). \quad (2.422)$$

Given an arbitrary electromagnetic response $\kappa_{\alpha\beta}{}^{\mu\nu}$, it is possible to demonstrate that

$${}^{(1)}\kappa_{\alpha\rho}{}^{\beta\rho} \equiv 0, \quad (2.423)$$

cf. Hehl and Obukhov [41]. As an outcome, equation (2.422) is readily simplified to

$$\mathcal{S}_{\alpha}{}^{\beta} = -\frac{1}{2} \kappa_{\alpha\rho}{}^{\beta\rho} + \frac{1}{2} {}^{(3)}\kappa_{\alpha\rho}{}^{\beta\rho} = -\frac{1}{2} \kappa_{\alpha\rho}{}^{\beta\rho} + \frac{3}{2} \alpha \delta_{\alpha}^{\beta}. \quad (2.424)$$

In the case of a P -medium one obtains that $\kappa_{\alpha\rho}{}^{\beta\rho} = Y[(\text{tr}P)P_{\alpha}{}^{\beta} - (P^2)_{\alpha}{}^{\beta}]$. Therefore:

$$\begin{aligned}\$_{\alpha}{}^{\beta} &= -\frac{1}{2}Y[(\text{tr}P)P_{\alpha}{}^{\beta} - (P^2)_{\alpha}{}^{\beta}] + \frac{1}{8}Y[(\text{tr}P)^2 - \text{tr}(P^2)]\delta_{\alpha}^{\beta} \\ &= -\frac{1}{2}Y(\text{tr}P)\left[P_{\alpha}{}^{\beta} - \frac{1}{4}(\text{tr}P)\delta_{\alpha}^{\beta}\right] + \frac{1}{2}Y\left[(P^2)_{\alpha}{}^{\beta} - \frac{1}{4}\text{tr}(P^2)\delta_{\alpha}^{\beta}\right].\end{aligned}\quad (2.425)$$

Since the terms in square brackets are both traceless, the property $\$_{\rho}{}^{\rho} = 0$ is manifestly satisfied. For some purposes, it is convenient to cast the result (2.425) in an alternative form. As a matter of fact, by completing a square, one achieves that

$$\$_{\alpha}{}^{\beta} = \frac{1}{2}Y\left\{\left[P_{\alpha}{}^{\rho} - \frac{1}{2}(\text{tr}P)\delta_{\alpha}^{\rho}\right]\left[P_{\rho}{}^{\beta} - \frac{1}{2}(\text{tr}P)\delta_{\rho}^{\beta}\right] - \frac{1}{4}\text{tr}(P^2)\delta_{\alpha}^{\beta}\right\}.\quad (2.426)$$

By virtue of (2.420) and (2.425)–(2.426), the P -medium's fields α and $\$_{\alpha}{}^{\beta}$ are now known. Hence, the contributions ${}^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu}$ and ${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu}$ may be determined from (2.212) and (2.211), respectively. For the axion part, one finds a simple expression:

$${}^{(3)}\kappa_{\alpha\beta}{}^{\mu\nu} = \frac{1}{12}Y[(\text{tr}P)^2 - \text{tr}(P^2)]\delta_{\alpha\beta}^{\mu\nu}.\quad (2.427)$$

By contrast, as far as the skewon piece ${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu}$ is concerned, substituting (2.425) or (2.426) into (2.211) leads to a *complicated* outcome. In view of this problem, one is motivated to follow a different strategy. The constitutive law (2.418) implies that

$$\begin{aligned}\bar{\kappa}_{\alpha\beta}{}^{\mu\nu} &= \frac{1}{4}Y\hat{\epsilon}_{\alpha\beta\rho\sigma}(P_{\eta}{}^{\rho}P_{\theta}{}^{\sigma} - P_{\theta}{}^{\sigma}P_{\eta}{}^{\rho})\epsilon^{\eta\theta\mu\nu} \\ &= \frac{1}{2}Y\hat{\epsilon}_{\alpha\beta\rho\sigma}\epsilon^{\mu\nu\eta\theta}P_{\eta}{}^{\rho}P_{\theta}{}^{\sigma} = \frac{1}{2}Y\delta_{\alpha\beta\rho\sigma}^{\mu\nu\eta\theta}P_{\eta}{}^{\rho}P_{\theta}{}^{\sigma}.\end{aligned}\quad (2.428)$$

Ergo, by recalling that ${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu}$ coincides with ${}^{(-)}\kappa_{\alpha\beta}{}^{\mu\nu}$, it is possible to conclude that

$${}^{(2)}\kappa_{\alpha\beta}{}^{\mu\nu} = \frac{1}{2}\left(\kappa_{\alpha\beta}{}^{\mu\nu} - \bar{\kappa}_{\alpha\beta}{}^{\mu\nu}\right) = Y\left(P_{[\alpha}{}^{\mu}P_{\beta]}{}^{\nu} - \frac{1}{4}\delta_{\alpha\beta\rho\sigma}^{\mu\nu\eta\theta}P_{\eta}{}^{\rho}P_{\theta}{}^{\sigma}\right).\quad (2.429)$$

In a similar manner, one can show that the principal component ${}^{(1)}\kappa_{\alpha\beta}{}^{\mu\nu}$ is given by

$${}^{(1)}\kappa_{\alpha\beta}{}^{\mu\nu} = Y\left(P_{[\alpha}{}^{\mu}P_{\beta]}{}^{\nu} + \frac{1}{4}\delta_{\alpha\beta\rho\sigma}^{\mu\nu\eta\theta}P_{\eta}{}^{\rho}P_{\theta}{}^{\sigma}\right) - \frac{1}{12}Y[(\text{tr}P)^2 - \text{tr}(P^2)]\delta_{\alpha\beta}^{\mu\nu}.\quad (2.430)$$

When P_α^β is proportional to the Kronecker delta symbol δ_α^β , the electromagnetic response (2.418) reduces to a pure axion. Equivalently, the constitutive law $\kappa = {}^{(3)}\kappa$ is a sub-case of the P -medium. As illustrated in (2.310), the dispersion relation of a pure axion vanishes identically. Therefore, it is not surprising that the Fresnel equation of a P -medium is trivially zero too. Hereafter, this statement is verified rigorously. By representing (2.418) as $\chi^{\alpha\beta\mu\nu} = Y\epsilon^{\alpha\beta\rho\sigma}P_\rho^\mu P_\sigma^\nu$, one easily observes that

$$(W_q)^{\beta\nu} = \chi^{\alpha\beta\mu\nu} q_\alpha q_\mu = Y(\epsilon^{\alpha\beta\rho\sigma} P_\rho^\mu q_\alpha q_\mu) P_\sigma^\nu = Y(\epsilon^{\beta\sigma\rho\alpha} P_\rho^\mu q_\alpha q_\mu) P_\sigma^\nu, \quad (2.431)$$

where the last step makes use of the symmetry properties of the Levi-Civita symbol. In order to express (2.431) in a compact way, it is convenient to define the quantity

$$[\mathcal{P}(q, q)]^{\beta\sigma} := (\epsilon^{\beta\sigma\rho\alpha} P_\rho^\mu q_\alpha q_\mu), \quad (2.432)$$

which may be viewed as a bivector density of weight +1. Evidently, one obtains that

$$(W_q)^{\beta\nu} = Y[\mathcal{P}(q, q)]^{\beta\sigma} P_\sigma^\nu. \quad (2.433)$$

Upon performing a simple manipulation, the polynomial (2.304) is cast in the form

$$\mathcal{G}(q) = 2\kappa_{\alpha_2\alpha_3}^{\beta\beta_1} \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} (W_q)^{\alpha_2\beta_2} (W_q)^{\alpha_3\beta_3}, \quad (2.434)$$

so that it is possible to implement the P -medium laws (2.418) and (2.431) *directly*:

$$\begin{aligned} \mathcal{G}(q) &= 4Y \hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} P_{\alpha_2}^\beta P_{\alpha_3}^{\beta_1} (W_q)^{\alpha_2\beta_2} (W_q)^{\alpha_3\beta_3} \\ &= 4Y^3 (\hat{\epsilon}_{\beta\beta_1\beta_2\beta_3} P_{\alpha_2}^\beta P_{\alpha_3}^{\beta_1} P_\xi^{\beta_2} P_\zeta^{\beta_3}) [\mathcal{P}(q, q)]^{\alpha_2\xi} [\mathcal{P}(q, q)]^{\alpha_3\zeta}. \end{aligned} \quad (2.435)$$

Here, the contraction over the indices of $\hat{\epsilon}_{\beta\beta_1\beta_2\beta_3}$ motivates one to invoke the identity

$$\hat{\epsilon}_{\mu\mu_1\mu_2\mu_3} S_\nu^\mu S_{\nu_1}^{\mu_1} S_{\nu_2}^{\mu_2} S_{\nu_3}^{\mu_3} = (\det S) \hat{\epsilon}_{\nu\nu_1\nu_2\nu_3}, \quad (2.436)$$

valid for any tensor S_α^β , cf. (2.340). On account of (2.436), equation (2.435) becomes

$$\begin{aligned}\mathcal{G}(q) &= +4Y^3(\det P)\hat{\epsilon}_{\alpha_2\alpha_3\xi\zeta}[\mathcal{P}(q, q)]^{\alpha_2\xi}[\mathcal{P}(q, q)]^{\alpha_3\zeta} \\ &= -4Y^3(\det P)\hat{\epsilon}_{\alpha_2\xi\alpha_3\zeta}[\mathcal{P}(q, q)]^{\alpha_2\xi}[\mathcal{P}(q, q)]^{\alpha_3\zeta} \\ &= -4Y^3(\det P)\hat{\epsilon}_{\mu\mu_1\nu\nu_1}[\mathcal{P}(q, q)]^{\mu\mu_1}[\mathcal{P}(q, q)]^{\nu\nu_1}.\end{aligned}\quad (2.437)$$

So, the Fresnel polynomial of an arbitrary P -medium is closely related to the Pfaffian of $[\mathcal{P}(q, q)]^{\alpha\beta}$. As a matter of fact, by comparing (2.437) with (2.326), one finds that

$$\mathcal{G}(q) = -32Y^3(\det P)\text{Pf}[\mathcal{P}(q, q)]. \quad (2.438)$$

Since the Levi-Civita symbol is fully antisymmetric, it is possible to remark that $[\mathcal{P}(q, q)]^{\alpha\beta}q_\beta = (\epsilon^{\alpha\beta\mu\nu}P_\mu^\lambda q_\lambda)q_\beta q_\nu \equiv 0$. More explicitly, the wave-covector always belongs to the kernel of $[\mathcal{P}(q, q)]^{\alpha\beta}$, which – as a result – cannot be an empty set. From this observation it follows that the rank of $[\mathcal{P}(q, q)]^{\alpha\beta}$ is strictly less than 4, and thus

$$\det[\mathcal{P}(q, q)] = (\text{Pf}[\mathcal{P}(q, q)])^2 \equiv 0, \quad (2.439)$$

see (2.325). When (2.439) is combined with (2.438), a key conclusion is achieved: for every P -medium, the Fresnel polynomial $\mathcal{G}(q)$ vanishes *trivially*, as anticipated.

In preparation for the analysis of Chapter 3, it is interesting to show that the constitutive law $\kappa_{\alpha\beta}^{\mu\nu} = 2Y P_{[\alpha}^\mu P_{\beta]}^\nu$ satisfies the equation $\kappa \circ \bar{\kappa} = \frac{1}{6}\text{tr}(\kappa \circ \bar{\kappa})\text{Id}$. By substituting (2.418) and (2.428) in the component representation of $\kappa \circ \bar{\kappa}$, one obtains

$$\frac{1}{2}\kappa_{\alpha\beta}^{\rho\sigma}\bar{\kappa}_{\rho\sigma}^{\mu\nu} = \frac{1}{2}Y^2 P_\alpha^\rho P_\beta^\sigma \delta_{\rho\sigma\xi\zeta}^{\mu\nu\eta\theta} P_\eta^\xi P_\theta^\zeta = \frac{1}{2}Y^2(\hat{\epsilon}_{\rho\sigma\xi\zeta} P_\alpha^\rho P_\beta^\sigma P_\eta^\xi P_\theta^\zeta)\epsilon^{\mu\nu\eta\theta}, \quad (2.440)$$

where the last manipulation invokes the split $\delta_{\rho\sigma\xi\zeta}^{\mu\nu\eta\theta} = \hat{\epsilon}_{\rho\sigma\xi\zeta}\epsilon^{\mu\nu\eta\theta}$. It is appropriate to note that (2.436) readily leads to $\frac{1}{2}\kappa_{\alpha\beta}^{\rho\sigma}\bar{\kappa}_{\rho\sigma}^{\mu\nu} = \frac{1}{2}Y^2(\det P)\hat{\epsilon}_{\alpha\beta\eta\theta}\epsilon^{\mu\nu\eta\theta}$. Hence, upon expressing $\frac{1}{2}\hat{\epsilon}_{\alpha\beta\eta\theta}\epsilon^{\mu\nu\eta\theta}$ as a generalised Kronecker delta symbol, one can verify that

$$\frac{1}{2}\kappa_{\alpha\beta}^{\rho\sigma}\bar{\kappa}_{\rho\sigma}^{\mu\nu} = Y^2(\det P)\delta_{\alpha\beta}^{\mu\nu}. \quad (2.441)$$

Accordingly, the electromagnetic P -media are indeed solutions of $\kappa \circ \bar{\kappa} = \frac{1}{6}\text{tr}(\kappa \circ \bar{\kappa})\text{Id}$, with the specific assignment $\text{tr}(\kappa \circ \bar{\kappa}) = 6Y^2(\det P)$. It is easy to demonstrate that

the constitutive law (2.418) also obeys the alternative equation $\bar{\kappa} \circ \kappa = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) \text{Id}$. Following the terminology mentioned in Section 2.5.1, each P -medium is then said to comply with *both* mixed closure relations; one may regard this as a *special* property. It is trivial to observe that the electromagnetic response (2.418) yields (cf. (2.348))

$$\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa) = 6Y^2(\det P). \quad (2.442)$$

Hereafter, (2.442) is split in two equations, concerning the modulus and sign of $\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa)$, respectively. Since Y is assumed to be non-zero, one finds that

$$|\text{tr}(\kappa \circ \bar{\kappa})| \equiv |\text{tr}(\bar{\kappa} \circ \kappa)| = 6Y^2 |\det P|, \quad (2.443)$$

$$\text{sgn}[\text{tr}(\kappa \circ \bar{\kappa})] \equiv \text{sgn}[\text{tr}(\bar{\kappa} \circ \kappa)] = \text{sgn}[(\det P)]. \quad (2.444)$$

Ergo, P_α^β is a tensor of full rank solely if $\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa)$ is non-vanishing. Moreover, one can promptly deduce the sign of $(\det P)$ from that of $\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa)$.

Chapter 3

Four closure relations

3.1 Introduction

One of the key achievements of metric-free electrodynamics is the discovery that, to a large extent, the notion of distance can be viewed as an offshoot of Maxwell's theory. In fact, requiring that the medium has some given electromagnetic properties allows one to determine, albeit non-uniquely, a tensor $g^{\alpha\beta}$ that is a valid candidate for the metric of spacetime. For example, if the medium is not only local and linear but also skewon-free and invariant under electric-magnetic reciprocity, the constitutive law is described by a Hodge operator. Such map, in turn, generates the appropriate class of metrics $g^{\alpha\beta}$. Toupin [134], Schönberg [118], and Jadczyk [58] are seemingly the first to investigate the above technique for extracting a notion of distance from electromagnetic requirements. In the treatise [41], Hehl and Obukhov examine *two* routes leading from electric-magnetic reciprocity and the absence of skewon to a non-unique metric. A *third*, novel derivation is studied in the present Chapter. The mentioned works are generalised too. For instance, the test $\kappa \circ \kappa = -\zeta^2 \text{Id}$ for a linear medium $H = \kappa(F)$ to be electric-magnetic reciprocal is considered as part of

a set of *four* equations:

$$\kappa \circ \kappa = \frac{1}{6} \text{tr}(\kappa \circ \kappa) \text{Id}, \quad (\text{“Pure”}) \quad (3.1)$$

$$\bar{\kappa} \circ \bar{\kappa} = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \bar{\kappa}) \text{Id}, \quad (\text{“Pure”}) \quad (3.2)$$

$$\kappa \circ \bar{\kappa} = \frac{1}{6} \text{tr}(\kappa \circ \bar{\kappa}) \text{Id}, \quad (\text{“Mixed”}) \quad (3.3)$$

$$\bar{\kappa} \circ \kappa = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) \text{Id}. \quad (\text{“Mixed”}) \quad (3.4)$$

These are the *closure* relations. Given that (3.1) and (3.2) respectively include κ and $\bar{\kappa}$ only, such equations are called *pure*. Likewise, (3.3) and (3.4) are called *mixed* in that they include both κ and $\bar{\kappa}$. One should observe that the trace factors $\{\text{tr}(\kappa \circ \kappa), \dots, \text{tr}(\bar{\kappa} \circ \kappa)\}$ in (3.1)–(3.4) are allowed to be positive, negative, or even zero. To understand the origin of the name “closure relations” it is appropriate to examine (3.1). Composing an operator κ that fulfills (3.1) with itself yields, up to a scalar, the identity map. Hence, in simple terms, the action of κ closes [41].

The study of the first *pure* closure equation is motivated by the fact that (3.1) is linked to an important symmetry of the energy-momentum tensor ${}^k\mathcal{T}_\alpha{}^\beta$. Such transformation leaving ${}^k\mathcal{T}_\alpha{}^\beta$ unchanged is the *unimodular* ancestor of electric-magnetic reciprocity. More explicitly, provided few simple assumptions are introduced, the task of finding all media that are invariant under the new unimodular reciprocity generates a closure relation like (3.1).

The key argument inducing one to analyse the second *mixed* closure equation is that, if and only if the constitutive tensor satisfies (3.4), the electromagnetic fields are endowed with a certain geometrical property. In fact, whenever the medium is local and linear, (3.4) is the necessary and sufficient condition for the invariant $H \wedge H$ to be proportional to $F \wedge F$ regardless of how H and F are selected. As an aside, the occurrence of $H \wedge H \propto F \wedge F$ demands that, provided $E \wedge B$ is zero, $\mathcal{H} \wedge \mathcal{D}$ vanishes too. Figure 3.2 aims to clarify this statement by illustrating a suitable configuration of fields. The *other two* closure equations, (3.2) and (3.3), are investigated for obvious reasons of completeness.

In brief, the Chapter is organised as follows. Section 3.2 demonstrates how a symmetry of the energy-momentum tensor gives rise to an equation of the form (3.1). The statement that (3.4) is linked to a geometrical constraint on the fields is

discussed in Section 3.3. Next, the closure equations are analysed from a broader perspective. Indeed, Section 3.4 gives an overview on the fundamental properties of (3.1)–(3.4). A derivation of all *invertible* solutions of the mixed closure requirements is examined in Section 3.5. Removing the skewon part from such invertible roots leads to a number of key findings, Section 3.6. In accord with the guiding principles of this Thesis, the mathematics employed below is both metric- and connection-free.

3.2 Unimodular reciprocity and the relation (3.1)

The *energy-momentum* of the electromagnetic fields is described by an object ${}^k\Sigma_\alpha$, which is a generalised potential for the Lorentz force density (2.97). More accurately, in view of the properties of the exterior differential, it is possible to write f_α as the sum of an *exact* form and a *residue*:

$$f_\alpha = d^k\Sigma_\alpha + X_\alpha, \quad (3.5)$$

$${}^k\Sigma_\alpha := \frac{1}{2} [F \wedge (e_\alpha \rfloor H) - H \wedge (e_\alpha \rfloor F)], \quad (3.6)$$

$$X_\alpha := \frac{1}{2} [H \wedge (\mathcal{L}_{e_\alpha} F) - F \wedge (\mathcal{L}_{e_\alpha} H)]. \quad (3.7)$$

Both the covector-valued 3-form ${}^k\Sigma_\alpha$ and the covector-valued 4-form X_α are twisted. It is trivial to verify that the quantities (3.6) and (3.7) also share the same dimension,

$$[{}^k\Sigma_\alpha] = [X_\alpha] = h[e_\alpha]. \quad (3.8)$$

As mentioned above, the current ${}^k\Sigma_\alpha$ is obtained by expressing the Lorentz force density (2.97) in an alternative way. Therefore, it appears evident that (3.6) is not derived from a Lagrangian. On account of this feature, the quantity ${}^k\Sigma_\alpha$ is called the *kinematic* energy-momentum 3-form, see Ref. [41].

The statement that the decomposition (3.5) identifies the potential (3.6) uniquely is a *conjecture*. In principle, specifying the Lorentz force density f_α and the residue X_α determines the energy-momentum current ${}^k\Sigma_\alpha$ up to an arbitrary closed 3-form,

$${}^k\tilde{\Sigma}_\alpha := {}^k\Sigma_\alpha + Y_\alpha \quad \text{with} \quad dY_\alpha = 0 \quad \Rightarrow \quad d^k\tilde{\Sigma}_\alpha = d^k\Sigma_\alpha + dY_\alpha = d^k\Sigma_\alpha. \quad (3.9)$$

However, it seems unlikely that the fields $\{H, F, e_\alpha\}$ can define a non-trivial Y_α such that $dY_\alpha = 0$. One is thus led to speculate that the potential (3.6) is indeed unique. Hehl and Obukhov deal with this problem by selecting ${}^k\Sigma_\alpha$ as the *correct* energy-momentum 3-form for electromagnetic fields in spacetime, see Axiom 4 in Ref. [41]. Additional references that analyse the conservation law (3.5) by means of exterior calculus are Lindell and Jancewicz [89], Segev [123], or Kaiser [61].

A metric-free representation of the energy-momentum current is also available in the language of tensors. In particular, the 3-form ${}^k\Sigma_\alpha$ is usually “translated” as:

$${}^k\mathcal{T}_\alpha{}^\beta = \frac{1}{4}\epsilon^{\beta\lambda\mu\nu}(H_{\alpha\lambda}F_{\mu\nu} - F_{\alpha\lambda}H_{\mu\nu}), \quad (3.10)$$

where (3.10) should be compared with (3.6). A brief calculation demonstrates that ${}^k\mathcal{T}_\alpha{}^\beta$ is a twisted tensor density of weight +1, see Post [113]. In view of this fact, it is appropriate to refer to (3.10) as the Minkowski energy-momentum tensor *density*. The precise link between the quantities ${}^k\mathcal{T}_\alpha{}^\beta$ and ${}^k\Sigma_\alpha$ is the formula, see Ref. [41]:

$${}^k\mathcal{T}_\alpha{}^\beta = \diamond_4(\vartheta^\beta \wedge {}^k\Sigma_\alpha). \quad (3.11)$$

Consistently with the notation employed in previous Sections, $\{\vartheta^\alpha\}$ is the co-frame.

It is important to note that the energy-momentum 3-form ${}^k\Sigma_\alpha$ is invariant under *unimodular reciprocity*. More accurately, the following map between field quantities,

$$\begin{bmatrix} H' \\ F' \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} \begin{bmatrix} H \\ F \end{bmatrix}, \quad \text{with} \quad \det(C) = C_{00}C_{11} - C_{01}C_{10} = +1, \quad (3.12)$$

leaves the current (3.6) unchanged. As a first step towards confirming this property, one introduces the 3-form ${}^k\Sigma'_\alpha := \frac{1}{2}[F' \wedge (e_\alpha \lrcorner H') - H' \wedge (e_\alpha \lrcorner F')]$. Hence, by virtue of (3.12), the newly defined energy-momentum current is expressed in terms of H and F . In the ensuing formula, several contributions cancel pairwise. Thereby, one can factor out the scalar $\det(C) = (C_{00}C_{11} - C_{01}C_{10})$, and achieve the desired result

$${}^k\Sigma'_\alpha = (C_{00}C_{11} - C_{01}C_{10}){}^k\Sigma_\alpha = {}^k\Sigma_\alpha. \quad (3.13)$$

In the derivation just considered, the proviso of unit determinant is invoked solely

at the end, cf. (3.13). Thus, one arrives at an interesting conclusion: a linear map from $\{H, F\}$ to $\{H', F'\}$ leaves the kinematic energy-momentum 3-form invariant *if and only if* the transformation matrix is unimodular, $\det(C) = 1$. As an aside, it is possible to show that the scalars $\{C_{01}, C_{10}\}$ and $\{C_{00}, C_{11}\}$ are respectively twisted and free of twist. In addition, by examining (3.12) from a dimensional point of view,

$$[C_{01}] = [C_{10}]^{-1} = (hq^{-2})^{-1} \stackrel{\text{SI}}{=} \Omega^{-1}, \quad \text{and} \quad [C_{00}] = [C_{11}] = 1. \quad (3.14)$$

The *space+time* split of the unimodular reciprocity map follows from the appropriate decompositions of H , F , and their primed equivalents. Specifically, the field excitation and the field strength are expanded according to (2.50) and (2.100). The same procedure is applied to H' and F' . Ergo, the decomposed field quantities are substituted in (3.12), so that *two* equations involving 1-forms and *two* equations involving 2-forms are obtained. Each of these pairs is best represented as a single matrix relation, whereby the *space+time* split of the map (3.12) is summarised as

$$\begin{bmatrix} E' \\ \mathcal{H}' \end{bmatrix} = \begin{bmatrix} C_{11} & -C_{10} \\ -C_{01} & C_{00} \end{bmatrix} \begin{bmatrix} E \\ \mathcal{H} \end{bmatrix}, \quad \begin{bmatrix} \mathcal{D}' \\ B' \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} \begin{bmatrix} \mathcal{D} \\ B \end{bmatrix}. \quad (3.15)$$

The calculation leading from (3.12) to (3.15) does not require one to assume that $\det(C) = 1$. Nevertheless, the condition of unit determinant does prove useful in order to express the transformation of E and $\mathcal{H}$ in an alternative guise. As a matter of fact, provided the reciprocity matrix is unimodular, its inverse is straightforward,

$$\begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}^{-1} = \begin{bmatrix} C_{11} & -C_{01} \\ -C_{10} & C_{00} \end{bmatrix}. \quad (3.16)$$

Thus, by comparing (3.16) with the first equation in (3.15), one readily obtains that

$$\begin{bmatrix} E' \\ \mathcal{H}' \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix}^{-\text{t}} \begin{bmatrix} E \\ \mathcal{H} \end{bmatrix}. \quad (3.17)$$

In analogy with Section 2.3.4, the superscript $(\cdot)^{\text{t}}$ indicates the matrix transpose.

As an important example, it is worth analysing the space+time decomposition

(3.15) in the specific case where C_{00} and C_{11} vanish. One may start by observing that the condition of unit determinant reduces to $C_{01}C_{10} = -1$. Hence, if the twisted scalar C_{01} is assigned the new name ζ , the counterpart C_{10} must be equal to $-\zeta^{-1}$. To summarise, when C_{00} and C_{11} are both zero, one can express (3.15) according to

$$\begin{bmatrix} E' \\ \mathcal{H}' \end{bmatrix} = \begin{bmatrix} 0 & \zeta^{-1} \\ -\zeta & 0 \end{bmatrix} \begin{bmatrix} E \\ \mathcal{H} \end{bmatrix}, \quad \begin{bmatrix} \mathcal{D}' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \zeta \\ -\zeta^{-1} & 0 \end{bmatrix} \begin{bmatrix} \mathcal{D} \\ B \end{bmatrix}. \quad (3.18)$$

Remarkably, (3.18) operates by converting the *electric* fields into *magnetic* ones, and the *magnetic* fields into *electric* ones. Because of this feature, the unimodular map

$$\begin{bmatrix} H' \\ F' \end{bmatrix} = \begin{bmatrix} 0 & \zeta \\ -\zeta^{-1} & 0 \end{bmatrix} \begin{bmatrix} H \\ F \end{bmatrix}, \quad \text{with} \quad \zeta \neq 0, \quad (3.19)$$

arising from (3.12) by virtue of $C_{00} = C_{11} = 0$, is called *electric-magnetic* reciprocity.

The matrix relation (3.12) was discussed in detail by Lindell [82, 84]. More precisely, the condition of unit determinant was studied in the book [79]. Perlick [112] also examined the topic of unimodular reciprocity, and suggested that a group theory approach could be fruitful. The article [112] further demonstrated the invariance of the Fresnel polynomial under the transformation (3.12). An extensive analysis of the electric-magnetic reciprocity (3.19) was carried out by Hehl and Obukhov [41, 42].

In what follows, the unimodular reciprocity map (3.12) is applied on the local and linear constitutive law (2.137). As a first step, the 2-forms H and F are expressed in terms of H' and F' . By making use of the identity (3.16), it is possible to show that

$$H = +C_{11}H' - C_{01}F', \quad (3.20)$$

$$F = -C_{10}H' + C_{00}F'. \quad (3.21)$$

Next, (3.20) and (3.21) are substituted in the medium response $H = \kappa(F)$. More specifically, invoking the fact that the constitutive map is linear allows one to achieve

$$(C_{10}\kappa + C_{11}\text{Id})(H') = (C_{00}\kappa + C_{01}\text{Id})(F'). \quad (3.22)$$

Equation (3.22) is the *transformed* medium response, linking the electromagnetic

fields derived from unimodular reciprocity, i.e. H' and F' .

As compared with the constitutive laws analysed in previous Sections, (3.22) is endowed with some unusual properties. For instance, some choices of field strength may correspond to no field excitation. In other words, given a specific F' , there may exist no H' such that (3.22) is fulfilled. Alternatively, it can occur that a single field strength determines multiple field excitations. In this scenario, one can find two or more H' that comply with (3.22) for a particular F' .

The task of studying (3.22) in full generality appears to be rather complicated. Nevertheless, it is often possible to invoke a simple *theorem*: for every 2-form F' , there exists and is unique a twisted 2-form H' such that (3.22) is satisfied if and only if $\det(C_{10}\kappa + C_{11}\text{Id})$ is non-zero. A *proof* is attained as follows: one starts by examining the right-hand side of (3.22). In particular, it is observed that the constitutive map $\kappa(\cdot)$ and the identity map $\text{Id}(\cdot)$ are single valued. Accordingly, for each 2-form F' , there *exists* and is *unique* a twisted 2-form y such that

$$y = (C_{00}\kappa + C_{01}\text{Id})(F'). \quad (3.23)$$

Now, the left-hand side of (3.22) is considered, for the purpose of applying some notational adjustments. More explicitly, the operator $C_{10}\kappa + C_{11}\text{Id}$ is called $\mathcal{A}$, and the field excitation H' is termed x . Hence, the constitutive law (3.22) is written as

$$\mathcal{A}(x) = y, \quad \text{that is,} \quad \mathcal{A}_I{}^J x_J = y_I. \quad (3.24)$$

From basic linear algebra, it is known that for every y_I there exists and is *unique* a vector x_J such that (3.24) is satisfied if and only if the 6×6 matrix $\mathcal{A}_I{}^J$ is full rank,

$$\det(\mathcal{A}) = \det(C_{10}\kappa + C_{11}\text{Id}) \neq 0. \quad (3.25)$$

In view of the result (3.25), one concludes that the proposed theorem is indeed *true*.

As a *corollary*, it is possible to show that (3.25) is the necessary and sufficient condition for the medium response (3.22) to turn into an explicit map of the form:

$$H' = \kappa'(F'). \quad (3.26)$$

Whenever (3.25) is satisfied, the operator κ' can be seen as the “image” of κ under the reciprocity (3.12). The exact link between these constitutive maps is found to be

$$\kappa' = (C_{10}\kappa + C_{11}\text{Id})^{-1} \circ (C_{00}\kappa + C_{01}\text{Id}), \quad (3.27)$$

as an inspection of (3.22) and (3.25) demonstrates. Notably, the formula (3.27) is also reported in Lindell [79, 84]. Barring exceptions, calculating the inverse of a 6×6 matrix is a difficult process. As a consequence, (3.27) is predominantly useful for abstract manipulations.

The requirement (3.25) is akin to the characteristic equation for the map $-C_{10}\kappa$. Because of this property, the theorem just discussed can be stated in an alternative manner: for every 2-form F' , there exists and is unique a twisted 2-form H' such that (3.22) is satisfied if and only if C_{11} is *not* an eigenvalue of the operator $-C_{10}\kappa$.

A local and linear medium $H = \kappa(F)$ is *invariant* under some unimodular reciprocity transformation if, given a specific choice of parameters $\{C_{00}, C_{01}, C_{10}, C_{11}\}$,

$$H' = \kappa(F'), \quad (3.28)$$

for any field strength F' . The constitutive laws that retain their form as described in (3.28) share an important symmetry with the kinematic energy-momentum current, see (3.13). Hence, one is prompted to investigate the medium relations $H = \kappa(F)$ whose structure is unchanged for an appropriate reciprocity map of the type (3.12). As a starting point, (3.28) is employed in the constitutive law (3.22) so as to achieve

$$[C_{10}\kappa^2 + (C_{11} - C_{00})\kappa - C_{01}\text{Id}](F') = 0. \quad (3.29)$$

More explicitly, it is demanded that F' is an arbitrary 2-form. Ergo, (3.29) leads to:

$$C_{10}\kappa^2 + (C_{11} - C_{00})\kappa - C_{01}\text{Id} = 0. \quad (3.30)$$

The quadratic equation (3.30) characterises all local and linear media $H = \kappa(F)$ that are invariant under the reciprocity (3.12) for some choice of $\{C_{00}, C_{01}, C_{10}, C_{11}\}$. Given that the determinant $(C_{00}C_{11} - C_{01}C_{10})$ is required to be +1, in order to solve the quadratic equation (3.30) it is necessary to consider the following *three* cases.

Case 1: $C_{10} = 0$, $C_{00} = C_{11} = \pm 1$, and $C_{01} = 0$. The map (3.12) simplifies to plus or minus the identity, $H' = \pm H$ and $F' = \pm F$. As an outcome, all local and linear media $H = \kappa(F)$ are invariant under the transformation being examined. This is in agreement with that fact that (3.30) does *not* constrain the operator κ .

Case 2: $C_{10} = 0$, $C_{00} \neq \{0, \pm 1\}$, $C_{11} = C_{00}^{-1}$, and C_{01} is arbitrary. In other words, the reciprocity map is described by a special kind of upper triangular matrix,

$$\begin{bmatrix} H' \\ F' \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ 0 & C_{00}^{-1} \end{bmatrix} \begin{bmatrix} H \\ F \end{bmatrix}, \quad \text{with} \quad C_{00} \neq \{0, \pm 1\}, \quad (3.31)$$

cf. (3.12). The only constitutive law that is not modified by the transformation (3.31) is a pure axion. More accurately, when the parameters $\{C_{00}, C_{01}, C_{10}, C_{11}\}$ are selected as in (3.31), it is possible to demonstrate that (3.30) has a *unique* root:

$$\kappa = - \left[\frac{C_{01} C_{00}^{-1}}{1 - C_{00}^{-2}} \right] \text{Id}. \quad (3.32)$$

Case 3: $C_{10} \neq 0$, C_{00} and C_{11} are arbitrary, while $C_{01} = (C_{00} C_{11} - 1) C_{10}^{-1}$. It is assumed that the reciprocity matrix (3.12) is unimodular, but *not* upper triangular. The local and linear media that are invariant under this transformation comply with

$$\kappa^2 + \left(\frac{C_{11} - C_{00}}{C_{10}} \right) \kappa + \left(\frac{1 - C_{00} C_{11}}{C_{10}^2} \right) \text{Id} = 0, \quad (3.33)$$

as one can prove by substituting the relevant choice of $\{C_{00}, C_{01}, C_{10}, C_{11}\}$ in (3.30). Finding all roots of (3.33) is beyond the scope of the present work. Nevertheless, it is established that (3.33) is equivalent to a pure closure relation of the type (3.1). To verify this fact, one needs to apply the familiar method of completing the square. Hence, an appropriate term is added and subtracted on the left-hand side of (3.33),

$$\kappa^2 + 2 \left(\frac{C_{11} - C_{00}}{2C_{10}} \right) \kappa + \left(\frac{C_{11} - C_{00}}{2C_{10}} \right)^2 \text{Id} - \left(\frac{C_{11} - C_{00}}{2C_{10}} \right)^2 \text{Id} + \left(\frac{1 - C_{00} C_{11}}{C_{10}^2} \right) \text{Id} = 0. \quad (3.34)$$

After the square is identified, the residual quantities are combined together, yielding

$$\left[\kappa + \frac{(C_{11} - C_{00})}{2C_{10}} \text{Id} \right]^2 = \left[\frac{(C_{11} + C_{00})^2 - 4}{4C_{10}^2} \right] \text{Id}. \quad (3.35)$$

The desired conclusion is attained. Provided one introduces an operator $\mathring{\kappa}$ such that

$$\mathring{\kappa} = \kappa + \left(\frac{C_{11} - C_{00}}{2C_{10}} \right) \text{Id}, \quad \text{and} \quad \text{tr}(\mathring{\kappa} \circ \mathring{\kappa}) = \frac{3}{2} \left[\frac{(C_{11} + C_{00})^2 - 4}{C_{10}^2} \right], \quad (3.36)$$

the requirement (3.33) is expressed as a closure relation: $\mathring{\kappa} \circ \mathring{\kappa} = \frac{1}{6} \text{tr}(\mathring{\kappa} \circ \mathring{\kappa}) \text{Id}$. From a more general point of view, the unimodular transformation (3.12) is a symmetry of the energy-momentum current, cf. (3.13). In addition, when $C_{10} \neq 0$, all local and linear media $H = \kappa(F)$ that are invariant under the reciprocity (3.12) correspond to some operator $\mathring{\kappa}$ satisfying (3.1). As an outcome, it is observed that the symmetry (3.13) of the energy-momentum 3-form is closely linked to the pure closure relation (3.1). This is the main physical motivation for analysing (3.1) in detail, see Sec. 3.4.

As an aside, it is worth noting that the trace factor on the right-hand side of $\mathring{\kappa} \circ \mathring{\kappa} = \frac{1}{6} \text{tr}(\mathring{\kappa} \circ \mathring{\kappa}) \text{Id}$ can be positive, negative, or even zero. More explicitly, (3.36) prescribes that the sign of $\text{tr}(\mathring{\kappa} \circ \mathring{\kappa})$ depends on C_{00} and C_{11} only. Hence, one can construct a 2-dimensional plot specifying for which values of C_{00} and C_{11} the trace factor is e.g. negative (Figure 3.1). The same plot indicates whether or not the maps $\mathring{\kappa}$ and κ are equal. In fact, as a simple inspection of (3.36) demonstrates, $\mathring{\kappa}$ and κ coincide if and only if $C_{00} = C_{11}$. Notably, the origin of the diagram ($C_{00} = C_{11} = 0$) corresponds to the electric-magnetic reciprocity (3.19). When both the parameters C_{00} and C_{11} vanish, (3.35) turns into a pure closure relation for the operator κ itself:

$$\kappa \circ \kappa = \frac{1}{6} \text{tr}(\kappa \circ \kappa) \text{Id}, \quad \text{with} \quad \text{tr}(\kappa \circ \kappa) = -6C_{10}^{-2} = -6\zeta^2 < 0. \quad (3.37)$$

A medium is said to be *electric-magnetic reciprocal* if there exists a non-zero parameter ζ such that the constitutive law is invariant under the transformation (3.19). It is trivial to demonstrate that (3.37) is the necessary and sufficient condition for a local and linear medium $H = \kappa(F)$ to be electric-magnetic reciprocal [41, 79, 134].

Hereafter, the discussion is momentarily specialised, for the purpose of examining the skewon-free roots of (3.37). An electromagnetic response is considered such that:

- (a) The constitutive map is proportional to a Hodge star via the true scalar $\pm|\zeta|$,

$$\kappa_{\alpha\beta}{}^{\mu\nu} = \Upsilon^{-1} |\det(g^{\eta\theta})|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} g^{\rho\mu} g^{\sigma\nu}, \quad \text{with} \quad |\Upsilon| = |\zeta|^{-1}. \quad (3.38)$$

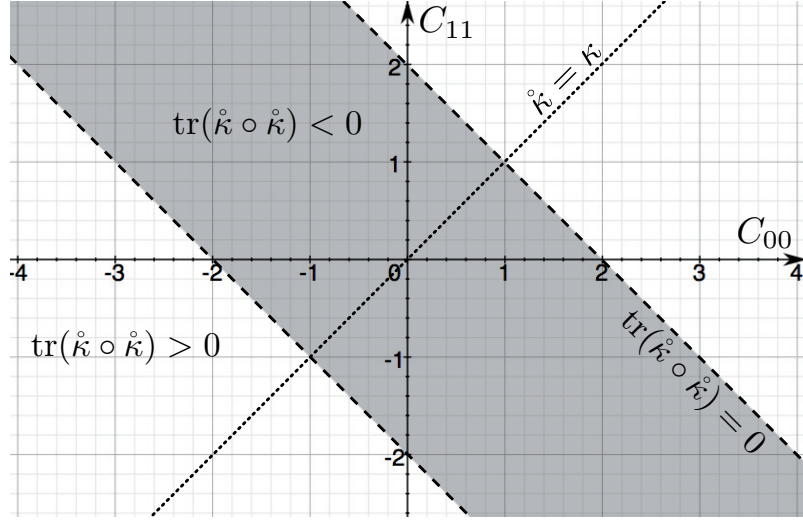


Figure 3.1: Provided $C_{10} \neq 0$, the shaded area describes all transformations (3.12) such that the corresponding invariant media obey $\text{tr}(\dot{\kappa} \circ \dot{\kappa}) < 0$. Analogously, the *dashed* lines characterise those unimodular reciprocity maps that are associated with $\text{tr}(\dot{\kappa} \circ \dot{\kappa}) = 0$. The other regions of the plot represent the occurrence of $\text{tr}(\dot{\kappa} \circ \dot{\kappa}) > 0$. In order to understand the meaning of the *dotted* line, it is important to recall that $\dot{\kappa} = \bar{\kappa}$ if and only if $C_{00} = C_{11}$. More specifically, the origin of the diagram $C_{00} = C_{11} = 0$ encodes the electric-magnetic reciprocity (3.19). As an aside, the figure was generated using the software Grapher[®].

It is demanded that $g^{\alpha\beta}$ is a symmetric invertible tensor, that is to say, a metric.

- (b) The determinant of $g^{\alpha\beta}$ is negative. By choosing an orthonormal basis, one can prove that $\det(g^{\eta\theta}) < 0$ exclusively if the signature of $g^{\alpha\beta}$ is Lorentzian, $t = \pm 2$.

Ergo, the analysis carried out in Section 2.5.2 is invoked. Upon recalling the property (v) of the Hodge star, it is observed that, if the conditions (a) and (b) are fulfilled,

- (c) The skewon part of the electromagnetic response is zero. Equivalently, $\kappa = \bar{\kappa}$.
- (d) The medium being examined is invariant under electric-magnetic reciprocity. In other words, the constitutive map κ satisfies the pure closure relation (3.1) with the quantity $\text{tr}(\kappa \circ \kappa)$ being equal to the negative scalar $-6\zeta^2$, cf. (3.37).

To summarise, the property (v) in Section 2.5.2 dictates that the requirements (a) and (b) imply (c) and (d). Remarkably, the converse statement is also true [41]. Provided a constitutive law is (c) skewon-free and (d) electric-magnetic reciprocal, the map κ is (a) proportional to the Hodge star via a suitable constant and (b) linked (non-uniquely) to a metric of Lorentzian signature.

It is thus obtained that two simple electromagnetic constraints on the medium, namely the absence of skewon and the invariance under the reciprocity (3.19), determine (non-uniquely) a symmetric invertible tensor $g^{\alpha\beta}$ that is a viable candidate for the metric of spacetime¹. The discovery that the notion of distance can, to a large extent, be viewed as an *offshoot* of local and linear electrodynamics is an important achievement of the metric-free approach to Maxwell's theory [41]. One *caveat* applies: although the requirements (c) and (d) define the Hodge star (3.38) uniquely, the metric $g^{\alpha\beta}$ is specified up to a conformal factor only. In order to confirm this fact, it is sufficient to recall the properties (iv) and (vi) discussed in Section 2.5.2.

Toupin [134], Schönberg [118] and Jadczyk [58] are seemingly the first to derive a class of conformal metrics from relations like $\kappa = \bar{\kappa}$ and (3.37). In the monograph [41], Hehl and Obukhov formulate, with unprecedented rigour, two alternative ways to extract a notion of distance from (c) and (d). Part of the analysis carried out in Ref. [41] is based on the findings of Gross, Fukui and Rubilar [38, 108, 116]. A novel path leading from electric-magnetic reciprocity and zero skewon to a conformal metric is described in Section 3.6.

3.3 From the field invariants to the relation (3.4)

By making use of the 2-forms H and F , it is possible to build three *invariants* of the electromagnetic field. More explicitly, one can implement the field strength and the field excitation so as to construct three 4-forms,

$$I_1 := F \wedge H, \quad (3.39)$$

$$I_2 := F \wedge F, \quad (3.40)$$

$$I_3 := H \wedge H. \quad (3.41)$$

There are two reasons why the quantities I_1 , I_2 and I_3 are called invariants of the electromagnetic field. Firstly, the expressions (3.39)–(3.41) are explicitly covariant. Secondly, in a 4-dimensional spacetime, the 4-forms such as $\{I_1, I_2, I_3\}$ have a single

¹As a prerequisite for $g^{\alpha\beta}$ to be viewed as a candidate for the metric of *spacetime*, one has to ensure that the domain being tested is free of matter. In fact, it is necessary to rule out the situation where the metric $g^{\alpha\beta}$ originates from a *material* having an appropriate constitutive law.

independent component. It appears evident that no metric or connection is required in order to define the invariants (3.39)–(3.41). By recalling that the field excitation and the field strength have dimensions of charge and flux respectively, $[H] = q$ and $[F] = h/q$, one can establish that

$$[I_1] = h \stackrel{\text{SI}}{=} \text{J s}, \quad [I_2] = (h/q)^2 \stackrel{\text{SI}}{=} \text{Wb}^2, \quad [I_3] = q^2 \stackrel{\text{SI}}{=} \text{C}^2, \quad (3.42)$$

cf. Appendix A. In addition, it is possible to demonstrate that (3.39) is a twisted quantity, whereas (3.40) and (3.41) are not. For the purpose of clarifying the meaning of the invariants $\{I_1, I_2, I_3\}$, each of these 4-forms is decomposed in space+time. Substituting the relevant expansions for H and F in the quantities (3.39)–(3.41), that is to say, making use of (2.50) and (2.100), leads to

$$I_1 = d\sigma \wedge (B \wedge \mathcal{H} - E \wedge \mathcal{D}), \quad (3.43)$$

$$I_2 = -2d\sigma \wedge (E \wedge B), \quad (3.44)$$

$$I_3 = +2d\sigma \wedge (\mathcal{H} \wedge \mathcal{D}). \quad (3.45)$$

When the first invariant I_1 is zero, the electric energy density is equal to the magnetic energy density. Indeed, as a consequence of the decomposition (3.43), one finds that

$$I_1 = 0 \quad \Leftrightarrow \quad \frac{1}{2}B \wedge \mathcal{H} = \frac{1}{2}E \wedge \mathcal{D} \quad \text{i.e.} \quad \frac{1}{2}\vec{B} \cdot \vec{H} = \frac{1}{2}\vec{E} \cdot \vec{D}. \quad (3.46)$$

On the very right of (3.46), the statement $I_1 = 0$ is translated in the language of vector calculus. It is worthwhile to recall that such formalism is not appropriate for investigating metric-free electrodynamics. The expansions (3.44)–(3.45) imply that

$$I_2 = 0 \quad \Leftrightarrow \quad E \wedge B = 0 \quad \text{i.e.} \quad \vec{E} \cdot \vec{B} = 0, \quad (3.47)$$

$$I_3 = 0 \quad \Leftrightarrow \quad \mathcal{H} \wedge \mathcal{D} = 0 \quad \text{i.e.} \quad \vec{H} \cdot \vec{D} = 0. \quad (3.48)$$

Accordingly, in the limit where a metric exists and vector calculus can be used, if I_2 is zero, $\vec{E}$ and $\vec{B}$ are perpendicular. Likewise, provided I_3 vanishes, $\vec{H}$ and $\vec{D}$ are orthogonal. It is important to observe that $I_1 = 0$, $I_2 = 0$, and $I_3 = 0$ are *frame-invariant* statements about the geometry of the electromagnetic fields [41].

For example, when the twisted 4-form I_1 is zero, the electric energy density and the magnetic energy density are equal in any frame of reference.

The *closure relation* (3.4) emerges from the requirement that I_3 is proportional to I_2 independently of how the electromagnetic fields are configured. As an obvious first step towards verifying this fact one imposes that

$$I_3 = \psi I_2, \quad \text{i.e.} \quad H \wedge H = \psi F \wedge F, \quad \text{for all } H \text{ and } F. \quad (3.49)$$

The parameter ψ is a true scalar with dimension of inverse resistance squared, $[\psi] = (hq^{-2})^{-2} \stackrel{\text{SI}}{=} \Omega^{-2}$. Introducing the customary local and linear medium response entails that all physically viable field excitation 2-forms can be written as $H = \kappa(F)$. Accordingly, the requirement (3.49) turns into

$$\kappa(F) \wedge \kappa(F) = \psi F \wedge F, \quad \text{for all } F. \quad (3.50)$$

In order to derive an alternative expression for the right-hand side of (3.50), it is convenient to analyse the quantity $\psi F \wedge F$ on its own. By expanding the field strength with respect to an arbitrary co-frame $\{\vartheta^\alpha\}$, it is trivial to show that

$$\psi F \wedge F = \frac{1}{4} \psi F_{\mu\mu_1} F_{\mu_2\mu_3} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}), \quad (3.51)$$

$$= \frac{1}{4} \psi F_{[\mu\mu_1} F_{\mu_2\mu_3]} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}). \quad (3.52)$$

Then, the index alternation is replaced by a generalised Kronecker delta symbol, as:

$$\psi F \wedge F = \frac{1}{4 \cdot 4!} \psi \delta^{\nu\nu_1\nu_2\nu_3}_{\mu\mu_1\mu_2\mu_3} F_{\nu\nu_1} F_{\nu_2\nu_3} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}). \quad (3.53)$$

Further making use of equation (2.186), in its $p = 4$ sub-case, allows one to achieve

$$\psi F \wedge F = \left(\frac{1}{4} \psi \epsilon^{\nu\nu_1\nu_2\nu_3} F_{\nu\nu_1} F_{\nu_2\nu_3} \right) \left[\frac{1}{4!} \hat{\epsilon}_{\mu\mu_1\mu_2\mu_3} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}) \right]. \quad (3.54)$$

When the Poincaré isomorphism (2.181) is applied on the unit scalar, a 4-form density of weight -1 is obtained. Such quantity coincides with the second factor on

the right-hand side of (3.54). Hence, it is possible to express $\psi F \wedge F$ as

$$\psi F \wedge F = \left(\frac{1}{4} \psi \epsilon^{\nu\nu_1\nu_2\nu_3} F_{\nu\nu_1} F_{\nu_2\nu_3} \right) [\hat{\diamond}_0(1)]. \quad (3.55)$$

The result (3.55) concludes the analysis of $\psi F \wedge F$. As an aside, it is appropriate to recall the objective of the calculation at hand, namely, to demonstrate that (3.49) is equivalent to the mixed closure relation (3.4). Substituting (3.55) into (3.50) yields

$$\kappa(F) \wedge \kappa(F) = \left(\frac{1}{4} \psi \epsilon^{\nu\nu_1\nu_2\nu_3} F_{\nu\nu_1} F_{\nu_2\nu_3} \right) [\hat{\diamond}_0(1)], \quad \text{for all } F. \quad (3.56)$$

The decomposition of the field strength with respect to the co-frame $\{\vartheta^\alpha\}$ is employed each time F appears in $\kappa(F) \wedge \kappa(F)$. Ergo, given that the map κ is linear, (3.56) may be rephrased as

$$\kappa(\vartheta^\mu \wedge \vartheta^{\mu_1}) \wedge \kappa(\vartheta^{\mu_2} \wedge \vartheta^{\mu_3}) F_{\mu\mu_1} F_{\mu_2\mu_3} = \psi [\hat{\diamond}_0(1)] \epsilon^{\mu\mu_1\mu_2\mu_3} F_{\mu\mu_1} F_{\mu_2\mu_3}, \quad \text{for all } F. \quad (3.57)$$

Obvious notational reasons motivate one to define the twisted bivector-valued 2-form

$$\kappa^{\mu\nu} := \kappa(\vartheta^\mu \wedge \vartheta^\nu) = \frac{1}{2} \kappa_{\alpha\beta}^{\mu\nu} (\vartheta^\alpha \wedge \vartheta^\beta), \quad (3.58)$$

whose use was pioneered in the article [48] by Hehl, Obukhov, Rubilar and Blagojević. The second equality in (3.58) can be understood as the expansion of $\kappa^{\mu\nu}$ with respect to an arbitrary co-frame $\{\vartheta^\alpha\}$. By implementing (3.58) in (3.57), one attains

$$\left\{ \kappa^I \wedge \kappa^J - \psi [\hat{\diamond}_0(1)] \epsilon^{IJ} \right\} F_I F_J = 0, \quad \text{for all } F. \quad (3.59)$$

The quantity in braces is *symmetric* under the exchange of I and J . In addition, it is required that (3.59) holds true for *every* field strength. Hence, one deduces that:

$$\kappa^I \wedge \kappa^J = \psi [\hat{\diamond}_0(1)] \epsilon^{IJ}. \quad (3.60)$$

At this point, it is convenient to analyse the left-hand side of (3.60) in isolation. For each value of I and J , the product $\kappa^I \wedge \kappa^J$ generates a 4-form similar to $\psi F \wedge F$. Correspondingly, one can derive an alternative expression for $\kappa^I \wedge \kappa^J$ by repeating

the steps (3.51)–(3.55). First, $\kappa^{\mu\nu}$ is decomposed in terms of the co-frame $\{\vartheta^\alpha\}$, second the index alternation is replaced by a generalised Kronecker delta symbol, and last $\delta_{\mu\mu_1\mu_2\mu_3}^{\nu\nu_1\nu_2\nu_3}$ is expanded as $\epsilon^{\nu\nu_1\nu_2\nu_3}\hat{\epsilon}_{\mu\mu_1\mu_2\mu_3}$. In other words, it is observed that:

$$\begin{aligned}
 \kappa^{\alpha\beta} \wedge \kappa^{\gamma\delta} &= \frac{1}{4} \kappa_{[\mu\mu_1}^{\alpha\beta} \kappa_{\mu_2\mu_3]}^{\gamma\delta} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}) \\
 &= \frac{1}{4 \cdot 4!} \delta_{\mu\mu_1\mu_2\mu_3}^{\nu\nu_1\nu_2\nu_3} \kappa_{\nu\nu_1}^{\alpha\beta} \kappa_{\nu_2\nu_3}^{\gamma\delta} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}) \\
 &= \left(\frac{1}{4} \epsilon^{\nu\nu_1\nu_2\nu_3} \kappa_{\nu\nu_1}^{\alpha\beta} \kappa_{\nu_2\nu_3}^{\gamma\delta} \right) \left[\frac{1}{4!} \hat{\epsilon}_{\mu\mu_1\mu_2\mu_3} (\vartheta^\mu \wedge \vartheta^{\mu_1} \wedge \vartheta^{\mu_2} \wedge \vartheta^{\mu_3}) \right] \\
 &= \left(\frac{1}{4} \epsilon^{\nu\nu_1\nu_2\nu_3} \kappa_{\nu\nu_1}^{\alpha\beta} \kappa_{\nu_2\nu_3}^{\gamma\delta} \right) [\hat{\diamond}_0(1)].
 \end{aligned} \tag{3.61}$$

When (3.61) is substituted in (3.60), the quantity $\hat{\diamond}_0(1)$ drops out, thus leading to

$$\epsilon^{MN} \kappa_M^I \kappa_N^J = \psi \epsilon^{IJ} \quad \Leftrightarrow \quad (\hat{\epsilon}_{KI} \kappa_M^I \epsilon^{MN}) \kappa_N^J = \psi \delta_K^J. \tag{3.62}$$

As soon as (2.195) is compared with (3.62), it appears evident that the desired result is achieved. Given a local and linear electromagnetic medium, $H = \kappa(F)$, the condition (3.49) is satisfied if and only if the map κ obeys the mixed closure relation

$$\bar{\kappa} \circ \kappa = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) \text{Id}, \quad \text{with} \quad \text{tr}(\bar{\kappa} \circ \kappa) = 6\psi. \tag{3.63}$$

In the above discussion, no requirement is imposed on the true scalar ψ . Equivalently, the trace factor $\text{tr}(\bar{\kappa} \circ \kappa) = 6\psi$ can be positive, negative, or even zero.

If the map κ is a solution of (3.4), the invariant I_3 is proportional to I_2 for any configuration of electromagnetic fields. Thus, whenever I_2 is zero, I_3 vanishes too. By virtue of (3.47) and (3.48), this statement can be reformulated in an alternative manner: provided κ is a root of (3.4), whenever $E \wedge B$ is zero, $\mathcal{H} \wedge \mathcal{D}$ vanishes too. Such implication is described in Figure 3.2. At the high price of assuming a metric, it is also possible to observe that, if κ satisfies (3.4) and the product $\vec{E} \cdot \vec{B}$ is zero, the vectors $\vec{H}$ and $\vec{D}$ are orthogonal. Importantly, the conclusions just examined hold true in any frame of reference.

The property (3.49) is a valid physical justification for analysing the mixed closure equation (3.4) in detail. Hereafter, a supplementary reason is discussed, whose nature is more technical. To be precise, the following argument motivates one to investigate *either* of the mixed relations (3.3) and (3.4). When the skewon part of

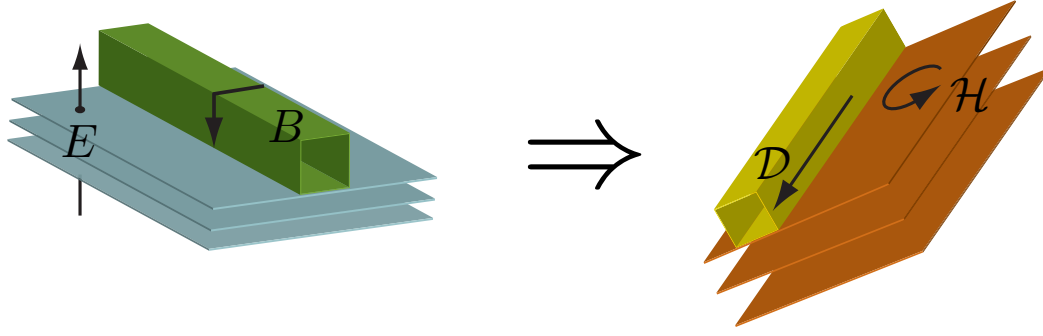


Figure 3.2: Given a local and linear constitutive law that fulfills (3.4), the occurrence of $E \wedge B = 0$ implies that $\mathcal{H} \wedge \mathcal{D} = 0$. The 1-forms E and $\mathcal{H}$ are depicted as stacks of planes. By contrast, the 2-forms B and $\mathcal{D}$ are represented as tubes. The arrows governing the orientation of E and $\mathcal{H}$ are “dissimilar” so as to indicate that these 1-forms are, respectively, twist-free and twisted. A corresponding remark explains why the 2-forms B and $\mathcal{D}$ are oriented in a fundamentally different guise. Provided $E \wedge B = 0$, the tubes of B are tangent to the planes of E . The scenario $\mathcal{H} \wedge \mathcal{D} = 0$ is depicted in the same way. Further information on the geometrical meaning of exterior forms is available in Ref. [60].

an electromagnetic response vanishes, the maps κ and $\bar{\kappa}$ coincide. Ergo, the four equations (3.1)–(3.4) become one and the same. The unique requirement emerging from (3.1)–(3.4) by virtue of $\bar{\kappa} = \kappa$ is named *the* closure relation for skewonless media. In order to solve such equation, there are two viable methods:

- (A) Any of the *pure* closure relations is selected, and the corresponding roots are evaluated. For each electromagnetic response that is obtained, the component $^{(2)}\kappa$ is set to zero. This enables one to identify all roots of the closure equation for skewon-free media. In Chapters D.3 and D.4 of the book [41], Hehl and Obukhov make use of the procedure just described with one extra assumption.
- (B) Any of the *mixed* closure relations is selected, and the corresponding roots are evaluated. For each electromagnetic response that is obtained, the component $^{(2)}\kappa$ is set to zero. This enables one to identify all roots of the closure equation for skewon-free media. Additional details are examined in Sections 3.5 and 3.6.

It may be argued that, by following the route (A), one gains more physical insight. As a matter of fact, the method (A) starts from a pure closure equation, and is thereby linked to the unimodular reciprocity (3.12). Nonetheless, the route (B) is considerably easier to pursue. Hence, one is prompted to investigate the mixed closure relations *as well as* the pure ones.

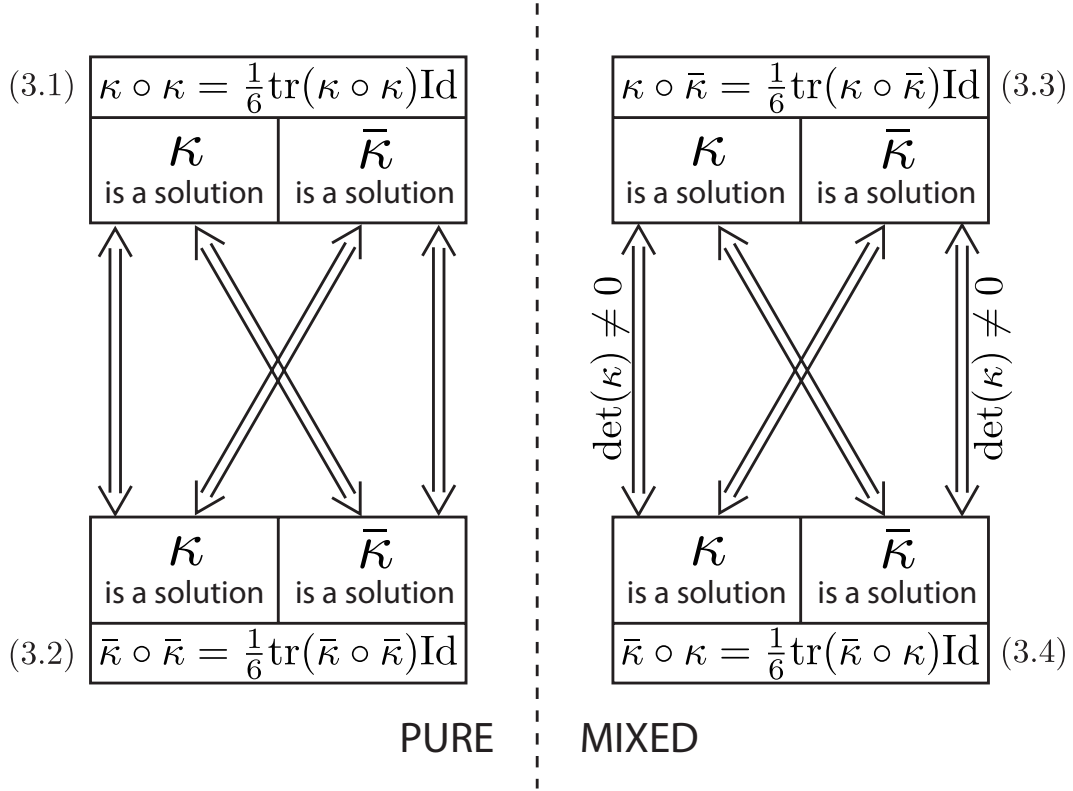


Figure 3.3: A diagram summarising the properties of the closure relations (3.1)–(3.4).

3.4 Properties of the four closure relations

In the present Section the most important features of the closure relations (3.1)–(3.4) are analysed. Figure 3.3 is a summary of the properties that are investigated. More explicitly, each arrow linking two closure equations represents a statement to be discussed. For instance, on the “mixed” side of the diagram, the arrow oriented from bottom-left to top-right prescribes that: the map κ is a root of (3.4) if and only if $\bar{\kappa}$ is a solution of (3.3). It is appropriate to recall that the trace factors $\{\text{tr}(\kappa \circ \kappa), \dots, \text{tr}(\bar{\kappa} \circ \kappa)\}$ in (3.1)–(3.4) are allowed to be positive, negative, or zero. The treatment described in this Section relies on two classes of identities, respectively valid for all maps κ and for all solutions of a given closure relation. For the purpose

of aiding future reference, the identities of the first kind are collected in a list:

$$\bar{\bar{\kappa}} = \kappa, \quad (3.64)$$

$$\overline{\kappa_1 \circ \kappa_2} = \bar{\kappa}_2 \circ \bar{\kappa}_1, \quad (3.65)$$

$$\mathrm{tr}(\kappa \circ \kappa) = \mathrm{tr}(\bar{\kappa} \circ \bar{\kappa}), \quad (3.66)$$

$$\mathrm{tr}(\kappa \circ \bar{\kappa}) = \mathrm{tr}(\bar{\kappa} \circ \kappa), \quad (3.67)$$

where (3.64), (3.65) and (3.67) are mentioned earlier in (2.197), (2.198) and (2.348). Each identity of the second kind holds true provided κ is a root of one closure equation. In particular, one can demonstrate that

$$\text{if } \kappa \text{ satisfies (3.1),} \quad \left| \mathrm{tr}(\kappa \circ \kappa) \right| = 6 \left| \det(\kappa) \right|^{\frac{1}{3}}, \quad (3.68)$$

$$\text{if } \kappa \text{ satisfies (3.2),} \quad \left| \mathrm{tr}(\bar{\kappa} \circ \bar{\kappa}) \right| = 6 \left| \det(\kappa) \right|^{\frac{1}{3}}, \quad (3.69)$$

$$\text{if } \kappa \text{ satisfies (3.3),} \quad \left| \mathrm{tr}(\kappa \circ \bar{\kappa}) \right| = 6 \left| \det(\kappa) \right|^{\frac{1}{3}}, \quad (3.70)$$

$$\text{if } \kappa \text{ satisfies (3.4),} \quad \left| \mathrm{tr}(\bar{\kappa} \circ \kappa) \right| = 6 \left| \det(\kappa) \right|^{\frac{1}{3}}. \quad (3.71)$$

From (3.68), it is possible to deduce an interesting result: the necessary and sufficient condition for a root of (3.1) to obey $\mathrm{tr}(\kappa \circ \kappa) = 0$ is that the map κ is not full-rank. By applying the same logic, three analogous propositions follow from (3.69)–(3.71).

Hereafter, the left-hand side of Figure 3.3 is examined. The arrows forming a cross dictate that: κ is a solution of one pure closure relation if and only if $\bar{\kappa}$ is a root of the other pure closure equation. As a first step towards confirming this property, one makes use of (3.64) so as to establish the double implication (cf. (3.1))

$$\kappa \circ \kappa = \frac{1}{6} \mathrm{tr}(\kappa \circ \kappa) \mathrm{Id} \quad \Leftrightarrow \quad \bar{\kappa} \circ \bar{\kappa} = \frac{1}{6} \mathrm{tr}(\bar{\kappa} \circ \bar{\kappa}) \mathrm{Id}. \quad (3.72)$$

Next, $\bar{\kappa}$ is assigned the new name $\varkappa$, and (3.72) is formulated in a more explicit way,

$$\kappa \circ \kappa = \frac{1}{6} \mathrm{tr}(\kappa \circ \kappa) \mathrm{Id} \quad \Leftrightarrow \quad \varkappa \circ \varkappa = \frac{1}{6} \mathrm{tr}(\varkappa \circ \varkappa) \mathrm{Id}. \quad (3.73)$$

It is easy to check that (3.73) is one of the desired conclusions. The remark (cf. (3.2))

$$\bar{\kappa} \circ \bar{\kappa} = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \bar{\kappa}) \text{Id} \quad \Leftrightarrow \quad \varkappa \circ \varkappa = \frac{1}{6} \text{tr}(\varkappa \circ \varkappa) \text{Id} , \quad (3.74)$$

brings the proof to a successful end. It is thereby appropriate to consider the vertical arrows on the left-hand side of Figure 3.3 – which demand that the pure closure relations are equivalent. To verify this fact, one makes use of a procedure consisting of two steps. First, it is demonstrated that, if (3.1) is obeyed, (3.2) is also satisfied:

$$\bar{\kappa} \circ \bar{\kappa} = \overline{(\kappa \circ \kappa)} = \frac{1}{6} \text{tr}(\kappa \circ \kappa) \overline{\text{Id}} = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \bar{\kappa}) \text{Id} . \quad (3.75)$$

As an aside, the manipulations shown in (3.75) rely on the identities (3.65)–(3.66). Hence, the converse statement is analysed. One observes that, when (3.2) is fulfilled,

$$\kappa \circ \kappa = \overline{(\bar{\kappa} \circ \bar{\kappa})} = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \bar{\kappa}) \overline{\text{Id}} = \frac{1}{6} \text{tr}(\kappa \circ \kappa) \text{Id} . \quad (3.76)$$

It is obvious that the identities (3.64)–(3.66) are employed. Furthermore, because of (3.75) and (3.76), the proof is complete. Joining together each vertical arrow with a diagonal one generates, in total, two novel paths. As a result, it is established that: κ is a root of one pure closure equation if and only if $\bar{\kappa}$ is a solution too.

The analysis now turns to the right-hand side of Figure 3.3. As a simple inspection demonstrates, the crossed arrows encode the following property: a necessary and sufficient condition for κ to be a root of one mixed closure equation is that $\bar{\kappa}$ is a root of the other mixed closure equation. To prove this statement, one employs the same method used to justify the diagonal arrows linking (3.1) and (3.2). Before we proceed, it is important to remark that Figure 3.3 is *not* symmetric. As a matter of fact, the mixed closure equations are not equivalent in general. More explicitly, the vertical arrows connecting (3.3) and (3.4) are valid for a restricted set of cases. Hereafter, a local and linear medium is considered which fulfills (3.3), but not (3.4):

$$\kappa_{\alpha\beta}{}^{\mu\nu} = \Phi_{\alpha\beta} S^{\mu\nu} . \quad (3.77)$$

In particular, it is demanded that the 2-form $\Phi_{\alpha\beta} \neq 0$ and the bivector $S^{\alpha\beta} \neq 0$ satisfy

$$\frac{1}{2}(\check{\Phi}^{\mu\nu}\Phi_{\mu\nu}) \neq 0, \quad \text{with} \quad \check{\Phi}^{\alpha\beta} := \frac{1}{2}\epsilon^{\alpha\beta\mu\nu}\Phi_{\mu\nu}, \quad (3.78)$$

$$\frac{1}{2}(S^{\mu\nu}\check{S}_{\mu\nu}) = 0, \quad \text{with} \quad \check{S}_{\alpha\beta} := \frac{1}{2}\hat{\epsilon}_{\alpha\beta\mu\nu}S^{\mu\nu}. \quad (3.79)$$

The expression (2.195) allows one to calculate the tensor $\bar{\kappa}_{\alpha\beta}{}^{\mu\nu}$ emerging from (3.77),

$$\bar{\kappa}_{\alpha\beta}{}^{\mu\nu} = \frac{1}{4}\hat{\epsilon}_{\alpha\beta\rho\sigma}S^{\rho\sigma}\Phi_{\eta\theta}\epsilon^{\eta\theta\mu\nu} = \check{S}_{\alpha\beta}\check{\Phi}^{\mu\nu}. \quad (3.80)$$

By implementing (3.80) and the conditions (3.78)–(3.79), it is trivial to observe that

$$\frac{1}{2}\kappa_{\alpha\beta}{}^{\rho\sigma}\bar{\kappa}_{\rho\sigma}{}^{\mu\nu} = \frac{1}{2}(S^{\rho\sigma}\check{S}_{\rho\sigma})\Phi_{\alpha\beta}\check{\Phi}^{\mu\nu} = 0, \quad (3.81)$$

$$\frac{1}{2}\bar{\kappa}_{\alpha\beta}{}^{\rho\sigma}\kappa_{\rho\sigma}{}^{\mu\nu} = \frac{1}{2}(\check{\Phi}^{\rho\sigma}\Phi_{\rho\sigma})\check{S}_{\alpha\beta}S^{\mu\nu} \neq 0. \quad (3.82)$$

From the result (3.81), one deduces that the medium (3.77) is a root of (3.3), as anticipated. Specifically, the constitutive law being examined obeys the closure equation (3.3) with $\text{tr}(\kappa \circ \bar{\kappa}) = 0$. Ergo, if it were true that (3.77) is a solution of (3.4) too, the map $\bar{\kappa} \circ \kappa$ would vanish because of the identity (3.67). Given that (3.82) rules out this option, the medium being considered does not fulfill (3.4), as expected. In summary, the tensor $\kappa_{\alpha\beta}{}^{\mu\nu} = \Phi_{\alpha\beta}S^{\mu\nu}$ is a root of one mixed closure relation only. Thus, (3.3) and (3.4) are not equivalent in general.

The determinant of the operator $\kappa_{\alpha\beta}{}^{\mu\nu} = \Phi_{\alpha\beta}S^{\mu\nu}$ is zero. As a matter of fact, the identity (3.70) prescribes that the solutions of (3.3) with $\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa) = 0$ are not invertible. Notably, provided one excludes the media – such as (3.77) – that are characterised by $\det(\kappa) = 0$, the relations (3.3) and (3.4) are indeed equivalent. In other words, if κ is full-rank, the vertical double implications ($\Updownarrow$) on the right-hand side of Figure 3.3 hold true. To verify this statement, one observes that, when κ is invertible, it is possible to cast the equations (3.3) and (3.4) in the same form:

$$\kappa^{-1} \circ [\kappa \circ \bar{\kappa}] = \kappa^{-1} \circ \left[\frac{1}{6}\text{tr}(\kappa \circ \bar{\kappa})\text{Id} \right] \Leftrightarrow \bar{\kappa} = \frac{1}{6}\alpha\kappa^{-1}, \quad (3.83)$$

$$[\bar{\kappa} \circ \kappa] \circ \kappa^{-1} = \left[\frac{1}{6}\text{tr}(\bar{\kappa} \circ \kappa)\text{Id} \right] \circ \kappa^{-1} \Leftrightarrow \bar{\kappa} = \frac{1}{6}\alpha\kappa^{-1}. \quad (3.84)$$

Here, the symbol α denotes the factor $\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa)$. Combining each vertical

arrow that links (3.3) and (3.4) with a diagonal implication generates two novel paths (Figure 3.3). It is thus established that: κ is an invertible root of one mixed closure equation if and only if $\bar{\kappa}$ is a full-rank solution too.

It is interesting to note that the electromagnetic Q -media and P -media satisfy *both* mixed closure relations whether or not $\det(\kappa) = 0$. To confirm this fact, two equalities are derived, which hold true for the constitutive laws (2.312) and (2.418) respectively. Substituting (2.350) in (3.70) or (3.71) entails that, for any Q -medium,

$$|\det(\kappa)|^{\frac{1}{3}} = \mathfrak{X}^2 |\det Q|. \quad (3.85)$$

Implementing (2.443) in (3.70) or (3.71) further demonstrates that all P -media obey

$$|\det(\kappa)|^{\frac{1}{3}} = Y^2 |\det P|. \quad (3.86)$$

As explained in Section 2.5.1 and in Section 2.5.3, the constitutive laws (2.312) and (2.418) are roots of both (3.3) and (3.4), independently of how $(\det Q)$ and $(\det P)$ are selected. Because of (3.85) and (3.86), the required statement is indeed correct: Q - and P -media satisfy both mixed closure equations whether or not κ is full-rank. This property is not the only reason why the constitutive laws (2.312) and (2.418) are special. In fact, the Q -media with $(\det Q) \neq 0$ and the P -media with $(\det P) \neq 0$ are the sole invertible roots of (3.3) and (3.4), cf. Section 3.5.

If κ_1 and κ_2 are roots of a mixed equation, $\kappa_1 \circ \kappa_2$ is also a root. For instance, as regards the composition of two maps that obey (3.3), the formula (3.65) entails that

$$(\kappa_1 \circ \kappa_2) \circ \overline{(\kappa_1 \circ \kappa_2)} = \kappa_1 \circ \kappa_2 \circ \bar{\kappa}_2 \circ \bar{\kappa}_1 = \frac{1}{36} \text{tr}(\kappa_2 \circ \bar{\kappa}_2) \text{tr}(\kappa_1 \circ \bar{\kappa}_1) \text{Id}. \quad (3.87)$$

It is thus verified that $\kappa_1 \circ \kappa_2$ does fulfill (3.3). One can replicate the analysis for the operator $\kappa_1 \circ \kappa_2$ generated by two solutions of (3.4). The claim just discussed holds true whether or not κ_1 and κ_2 are full-rank.

3.5 All invertible roots of the mixed equations

The aim of the present Section is to determine all roots of the mixed closure equations such that $\det(\kappa) \neq 0$. In other words, the local and linear media that obey (3.3) and (3.4) are calculated, under the assumption that $\text{tr}(\kappa \circ \bar{\kappa}) \equiv \text{tr}(\bar{\kappa} \circ \kappa)$ does not vanish, see (3.70) and (3.71). By invoking the fact that (3.4) and (3.49) are equivalent, it is also possible to observe that: every local and linear constitutive law is identified for which $I_3 = H \wedge H$ is proportional to $I_2 = F \wedge F$ via a non-zero factor, and regardless of the field distribution.

The Q -media with $(\det Q) \neq 0$, and the P -media with $(\det P) \neq 0$ are the only invertible roots of (3.3) and (3.4). Such property is described in the article [87] by Lindell, Bergamin and Favaro, where the result is established using non-covariant mathematics. Hereafter, a frame-independent proof is examined, which validates the conclusions of Ref. [87]. As displayed in Figure 3.3, the mixed closure equations share all invertible roots. Thus, if $\det(\kappa) \neq 0$, it is possible to solve only one of (3.3) and (3.4) with no loss of generality. In the treatment below, the latter relation is selected. One can derive the invertible roots of (3.4) by finding all maps κ such that

$$\kappa^{\alpha\beta} \wedge \kappa^{\mu\nu} = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) [\hat{\diamond}_0(1)] \epsilon^{\alpha\beta\mu\nu}, \quad \text{and} \quad \text{tr}(\bar{\kappa} \circ \kappa) \neq 0. \quad (3.88)$$

Indeed, from the analysis of Section 3.3, it is known that (3.4) is equivalent to (3.60). Expressing the Levi-Civita symbol as a 6×6 matrix allows one to convert (3.88) into

$$\kappa^I \wedge \kappa^J = \frac{1}{6} \text{tr}(\bar{\kappa} \circ \kappa) \left[\begin{array}{c|c} \mathbb{O}_{3 \times 3} & \mathbb{I}_{3 \times 3} \\ \hline \mathbb{I}_{3 \times 3} & \mathbb{O}_{3 \times 3} \end{array} \right] \hat{\diamond}_0(1), \quad \text{with} \quad \text{tr}(\bar{\kappa} \circ \kappa) \neq 0, \quad (3.89)$$

cf. (2.167). A 2-form that is equal to the exterior product of two 1-forms is called *simple*. In four dimensions, not every 2-form can be represented as the exterior product of two 1-forms. More accurately, a 2-form Φ is simple if and only if $\Phi \wedge \Phi = 0$. The condition (3.89) dictates that the invariants $\kappa^I \wedge \kappa^J$ with $I = J$ must be zero. Hence, the twisted 2-forms $\{\kappa^{\mu\nu}\} = \{\kappa^{01}, \kappa^{02}, \kappa^{03}, \kappa^{23}, \kappa^{31}, \kappa^{12}\}$ are required to be *simple*. A broader inspection of (3.89) demonstrates that two scenarios are allowed:

- **Possibility 1:** $\{\kappa^{01}, \kappa^{02}, \kappa^{03}\}$ are simple and *all* have a 1-form in common. The

entries $\{\kappa^{23}, \kappa^{31}, \kappa^{12}\}$ are simple too, but have a 1-form in common *pairwise*.

- **Possibility 2:** $\{\kappa^{23}, \kappa^{31}, \kappa^{12}\}$ are simple and *all* have a 1-form in common. The entries $\{\kappa^{01}, \kappa^{02}, \kappa^{03}\}$ are simple too, but have a 1-form in common *pairwise*.

With some effort, one can establish that the two possibilities describe, respectively:

$$\kappa^{\mu\nu} = Y(P^\mu \wedge P^\nu), \quad (3.90)$$

$$\kappa^{\mu\nu} = \mathfrak{X} \hat{\diamond}_2(Q^\mu \wedge Q^\nu). \quad (3.91)$$

Here, $\{P^\alpha\}$ is a co-frame and $Y \neq 0$ is a twisted scalar. Also, $\{Q^\alpha\}$ is a frame and $\mathfrak{X} \neq 0$ is a twisted scalar density of weight +1. The map $\hat{\diamond}_2$ is the Poincaré isomorphism (2.183). In order to link (3.90) and (3.91) to the P -media and Q -media, the 1-forms P^α and vectors Q^α are expanded as

$$P^\beta = P_\alpha{}^\beta \vartheta^\alpha, \quad (3.92)$$

$$Q^\beta = Q^{\alpha\beta} e_\alpha. \quad (3.93)$$

At each point in X_4 , it is assumed that $\{P^\alpha\}$ and $\{\vartheta^\alpha\}$ are valid bases for the cotangent space. Ergo, the tensor $P_\alpha{}^\beta$ must always be full-rank. Similarly, the properties of $\{Q^\alpha\}$ and $\{e_\alpha\}$ demand that $Q^{\alpha\beta}$ is invertible at each location in X_4 . One substitutes (3.92)–(3.93) in (3.90)–(3.91), and translates the ensuing formulae in components. It is thereby obtained that the response laws that fulfill (3.88) are:

$$\kappa_{\alpha\beta}{}^{\mu\nu} = 2Y P_{[\alpha}{}^\mu P_{\beta]}{}^\nu \quad \text{with } Y \neq 0 \quad \text{and} \quad (\det P) \neq 0, \quad (3.94)$$

$$\kappa_{\alpha\beta}{}^{\mu\nu} = \mathfrak{X} \hat{\epsilon}_{\alpha\beta\rho\sigma} Q^{\rho\mu} Q^{\sigma\nu} \quad \text{with } \mathfrak{X} \neq 0 \quad \text{and} \quad (\det Q) \neq 0. \quad (3.95)$$

The P -media with $(\det P) \neq 0$ and the Q -media with $(\det Q) \neq 0$ are indeed the only roots of the mixed closure equations such that $\det(\kappa) \neq 0$. As one can notice from (2.438)–(2.439) and (2.337), the Fresnel polynomial of (3.94) is trivially zero, while that of (3.95) is not. Hence, the two kinds of invertible solutions to (3.3) and (3.4) have no overlap. More explicitly, if $P_\alpha{}^\beta$ and $Q^{\alpha\beta}$ are full-rank, the classes of P -media and Q -media are disjoint.

The roots of (3.88) are derived for the first time in an article by Schuller, Witte

P -medium	Tensor P_α^β	Defining property	Axion α	$(\det P)$
$(+)\kappa_{\alpha\beta}^{\mu\nu} = YL^2\delta_{\alpha\beta}^{\mu\nu}$	$P_\alpha^\beta = L\delta_\alpha^\beta$	δ_α^β is the identity	$\alpha = +YL^2$	> 0
$(+)\kappa_{\alpha\beta}^{\mu\nu} = 2YL^2\psi_{[\alpha}^\mu\psi_{\beta]}^\nu$	$P_\alpha^\beta = L\psi_\alpha^\beta$	$\psi_\alpha^\rho\psi_\rho^\beta = \delta_\alpha^\beta, \psi_\rho^\rho = 0$	$\alpha = -\frac{1}{3}YL^2$	> 0
$(+)\kappa_{\alpha\beta}^{\mu\nu} = 2YL^2J_{[\alpha}^\mu J_{\beta]}^\nu$	$P_\alpha^\beta = LJ_\alpha^\beta$	$J_\alpha^\rho J_\rho^\beta = -\delta_\alpha^\beta$	$\alpha = +\frac{1}{3}YL^2$	> 0

Table 3.1: All invertible solutions of (3.96) that emerge from the constitutive law (3.94). Equivalently, the Table describes all skewon-free P -media with $(\det P) \neq 0$.

Q -medium	Tensor $Q^{\alpha\beta}$	Signature	Axion α	$(\det Q)$
$(+)\kappa_{\alpha\beta}^{\mu\nu} = \Upsilon^{-1}[\det([s]Q)]^{-\frac{1}{2}}\hat{\epsilon}_{\alpha\beta\rho\sigma}^{[s]}Q^{\rho\mu}Q^{\sigma\nu}$	$Q^{\alpha\beta} = [s]Q^{\alpha\beta}$	0	$\alpha = 0$	> 0
$(+)\kappa_{\alpha\beta}^{\mu\nu} = \Upsilon^{-1}[\det([s]Q)]^{-\frac{1}{2}}\hat{\epsilon}_{\alpha\beta\rho\sigma}^{[s]}Q^{\rho\mu}Q^{\sigma\nu}$	$Q^{\alpha\beta} = [s]Q^{\alpha\beta}$	± 2	$\alpha = 0$	< 0
$(+)\kappa_{\alpha\beta}^{\mu\nu} = \Upsilon^{-1}[\det([s]Q)]^{-\frac{1}{2}}\hat{\epsilon}_{\alpha\beta\rho\sigma}^{[s]}Q^{\rho\mu}Q^{\sigma\nu}$	$Q^{\alpha\beta} = [s]Q^{\alpha\beta}$	± 4	$\alpha = 0$	> 0
$(+)\kappa_{\alpha\beta}^{\mu\nu} = \Upsilon^{-1}[\det([a]Q)]^{-\frac{1}{2}}\hat{\epsilon}_{\alpha\beta\rho\sigma}^{[a]}Q^{\rho\mu}Q^{\sigma\nu}$	$Q^{\alpha\beta} = [a]Q^{\alpha\beta}$	n/a	$\alpha \neq 0$	> 0

Table 3.2: All invertible solutions of (3.96) that emerge from the constitutive law (3.95). Equivalently, the Table describes all skewon-free Q -media with $(\det Q) \neq 0$.

and Wohlfarth [121]. However, the property that one solution branch in Ref. [121] is equivalent to the Q -media is a novel result. It is also observed that (3.88) is just a different expression of the closure (3.4) with $\det(\kappa) \neq 0$. One further merit of the analysis reported here is the consistent use of ordinary and twisted forms. The solutions of (3.88) are identified, as well, in the paper [17] by Dahl. Notably, the author encodes the roots in terms of simple operators composed with the P -medium.

3.6 Invertible roots of the skewon-free equation

When the skewon part of an electromagnetic response vanishes, the maps κ and $\bar{\kappa}$ coincide. Ergo, the four equations (3.1)–(3.4) become one and the same. The unique requirement emerging from (3.1)–(3.4) by virtue of $\bar{\kappa} = \kappa$ is *the* closure relation for skewonless media:

$$({})\kappa \circ ({}^+)\kappa = \frac{1}{6} \text{tr} [({}^+)\kappa \circ ({}^+)\kappa] \text{Id}. \quad (3.96)$$

As described in Section 2.3.4, $({}^+)\kappa$ is the sum of the principal and axion elements. There are at least *two* viable methods of solving (3.96). Either all constitutive laws that obey a *pure* closure equation are derived, and for each root it is imposed that $({}^2)\kappa = 0$. Or, the same treatment is employed starting from a *mixed* closure equation. In Section 3.3, these two possibilities are labelled (A) and (B).

Table 3.1 and Table 3.2 encompass all *invertible* solutions of the only closure requirement for skewon-free media. Equivalently, the linear principal-axion constitutive operators are summarised which fulfill (3.96) and $\text{tr}[(^{+)}\kappa \circ (^{+)}\kappa] \neq 0$. In practical terms, the entries in Table 3.1 and Table 3.2 are obtained by following the method (B), and by demanding at each step that the electromagnetic response is full-rank. From the analysis of Section 3.5, it is known that the P -media with $(\det P) \neq 0$ and the Q -media with $(\det Q) \neq 0$ are the sole invertible roots of the mixed closure equations. Therefore, the entries in Table 3.1 and Table 3.2 are identified by prescribing that the contributions $(^2)\kappa$ of (3.94) and (3.95) vanish. As for the P -media with $(\det P) \neq 0$, the skewon part is removed by imposing that the tensor (2.426) is zero, $\$_{\alpha}^{\beta} = 0$. More in detail, a quadratic equation is solved under the condition that P_{α}^{β} is full-rank, thus generating the entries of Table 3.1. The procedure used for the Q -media with $(\det Q) \neq 0$ is similar, and is based the formula (2.323) encoding the relevant tensor $\$_{\alpha}^{\beta}$. Obviously, it gives rise to the constitutive laws examined in Table 3.2. With this result, all invertible solutions of (3.96) are determined for good.

The roots of (3.96) which fulfill $\text{tr}[(^{+)}\kappa \circ (^{+)}\kappa] \neq 0$, and that belong to the class of P -media are analysed first. By invoking (3.86), and the fact that these solutions are full-rank, one retrieves the constraints $Y \neq 0$ and $(\det P) \neq 0$. In turn, the proviso that P_{α}^{β} is invertible entails that the true scalar L defined in Table 3.1 is non-zero. The simplest P -medium root of (3.96) with $\det[(^{+)}\kappa] \neq 0$ is the pure axion law corresponding to $P_{\alpha}^{\beta} = L\delta_{\alpha}^{\beta}$. An additional solution is obtained if P_{α}^{β} is proportional to an involutory traceless $\mathcal{V}_{\alpha}^{\beta}$. In other words, the square of the tensor $\mathcal{V}_{\alpha}^{\beta}$ is the identity, and the contraction $\mathcal{V}_{\rho}^{\rho}$ must vanish (cf. Table 3.1). The third and last invertible P -medium root of (3.96) is generated by demanding that $P_{\alpha}^{\beta} = LJ_{\alpha}^{\beta}$ where J_{α}^{β} is an almost complex tensor. Hence, as prescribed in Table 3.1, the square of J_{α}^{β} is equal to minus the identity. In order to evaluate the axion α for the solutions just discussed, one can employ the formula (2.420). More specifically, alongside the properties mentioned in Table 3.1, it is necessary to know that J_{α}^{β} is traceless (Chapter 8.2.3 in Ref. [101]). Given that Y and L are both non-zero, a remarkable conclusion is attained: the full-rank P -media never consist of a principal part only. Such observation is easily verified by noting that Table 3.1

describes all skewon-free P -media with $(\det P) \neq 0$.

Hereafter, the solutions of (3.96) that obey $\text{tr}[(^{(+)}\kappa \circ (^{(+)}\kappa)] \neq 0$, and which belong to the class of Q -media are examined. Equivalently, the skewon-free constitutive laws of Table 3.2 are investigated. It is understood that $^{[s]}Q^{\alpha\beta}$ and $^{[a]}Q^{\alpha\beta}$ are the symmetric and antisymmetric parts of the tensor $Q^{\alpha\beta}$, cf. (2.314) and (2.315). Since the roots of (3.96) being analysed are invertible, the formula (3.85) guarantees that $\mathfrak{X} \neq 0$ and $(\det Q) \neq 0$. Ergo, one can make use of the true scalar Υ , defined previously in (2.354). The entries of Table 3.2 generated by a full-rank and symmetric $Q^{\alpha\beta}$ correspond to the three “kinds” of Hodge star $*_{\alpha\beta}{}^{\mu\nu}$ discussed in Section 2.5.2. More accurately, the signature t of a 4-dimensional metric is restricted to the values $\{0, \pm 2, \pm 4\}$, and each choice of $|t|$ is associated with one kind of Hodge star. As a result, one obtains the first three solutions in Table 3.2, for which $^{(+)}\kappa_{\alpha\beta}{}^{\mu\nu} = \Upsilon^{-1} *_{\alpha\beta}{}^{\mu\nu}$. Because of the distinctive form of these media, it is possible to view Υ as the customary electromagnetic impedance, up to a sign. The last invertible root of (3.96) is based on an antisymmetric $Q^{\alpha\beta}$ with $(\det Q) \neq 0$. Due to the fact that $Q^{\alpha\beta} = ^{[a]}Q^{\alpha\beta}$, the concept of signature is not applicable, and the matching entry in Table 3.2 reads “n/a”. Furthermore, one is not allowed to regard Υ as a standard type of impedance. It is now appropriate to consider the axion α for each of the solutions just described. According to the remark (v) in Section 2.5.2, the Hodge star map is purely principal. So, the first three entries of Table 3.2 have a vanishing α . By contrast, the root originating from $(\det Q) \neq 0$ and $Q^{\alpha\beta} = ^{[a]}Q^{\alpha\beta}$ is always endowed with a finite axion part. The relevant expression for α is achieved directly from (2.378). In spite of what the result $^{(3)}\kappa \neq 0$ appears to imply, one cannot derive a pure axion from this constitutive law. Such observation agrees with the property that the full-rank P -media and Q -media are disjoint. As a matter of fact, the electromagnetic response $^{(+)}\kappa_{\alpha\beta}{}^{\mu\nu} = \alpha \delta_{\alpha\beta}^{\mu\nu}$ is included in Table 3.1. Hence, it is not possible to encode an isolated axion by way of a Q -medium.

Given an invertible root of (3.96), one method to establish whether $\text{tr}[(^{(+)}\kappa \circ (^{(+)}\kappa)]$ is positive or negative is to calculate the sign of $(\det P)$ or $(\det Q)$. Indeed, since the

media being analysed fulfill $\kappa = {}^{(+)}\kappa$, the equations (2.444) and (2.351) prescribe that

$$\operatorname{sgn} \left\{ \operatorname{tr} \left[{}^{(+)}\kappa \circ {}^{(+)}\kappa \right] \right\} = \operatorname{sgn} [(\det P)] , \quad (3.97)$$

$$\operatorname{sgn} \left\{ \operatorname{tr} \left[{}^{(+)}\kappa \circ {}^{(+)}\kappa \right] \right\} = \operatorname{sgn} [(\det Q)] . \quad (3.98)$$

Notably, (3.97) and (3.98) demonstrate the importance of the rightmost columns in Table 3.1 and Table 3.2. It is now useful to mention an identity that is valid for all tensors $S_\alpha{}^\beta$ [124]:

$$(\det S) = \frac{1}{24} \left[(\operatorname{tr} S)^4 - 6(\operatorname{tr} S)^2 \operatorname{tr}(S^2) + 8(\operatorname{tr} S) \operatorname{tr}(S^3) + 3(\operatorname{tr}(S^2))^2 - 6\operatorname{tr}(S^4) \right]. \quad (3.99)$$

Specifically, contracting $S_\alpha{}^\beta$ with itself p times gives rise to the power $(S^p)_\alpha{}^\beta$. When a full-rank P -medium solution of (3.96) is examined, one can implement (3.99) so as to verify if $(\det P)$ is positive or negative. As a matter of fact, substituting the appropriate “defining property” in Table 3.1 leads to the required outcome. The first three invertible Q -medium roots of (3.96) emerge from the condition that $Q^{\alpha\beta}$ is a metric. Hence, it is possible to establish the sign of $(\det Q)$ by using an orthonormal frame. A trivial calculation shows that employing an orthonormal frame results in no loss of generality. The formula (2.325) entails that the determinant of an antisymmetric tensor with two contravariant indices is positive or zero. Ergo, for the last solution in Table 3.2, one finds that $(\det {}^{[a]}Q) > 0$.

Provided the medium is not only local and linear but also skewon-free, the constraint of electric-magnetic reciprocity (3.37) is equivalent to (3.96) with $\operatorname{tr} [{}^{(+)}\kappa \circ {}^{(+)}\kappa]$ being negative. One is thus motivated to inspect Table 3.1 and Table 3.2, and to look for solutions having a factor in the rightmost column which is less than zero. Because the analysis picks out a unique root of (3.96), a crucial statement is verified: the only local and linear medium $H = \kappa(F)$ that is both electric-magnetic reciprocal and skewon-free takes the form

$$\kappa_{\alpha\beta}{}^{\mu\nu} = \Upsilon^{-1} |\det(g^{\eta\theta})|^{-\frac{1}{2}} \hat{\epsilon}_{\alpha\beta\rho\sigma} g^{\rho\mu} g^{\sigma\nu}. \quad (3.100)$$

It is demanded that $g^{\alpha\beta}$ is a metric of Lorentzian signature ± 2 . As a further requirement, the absolute value of Υ^{-1} coincides with that of the parameter $\zeta \neq 0$ in (3.19).

Even if the property just established is known from the literature [41, 58, 118, 134], the derivation followed here is novel.

In brief, a viable candidate $g^{\alpha\beta}$ for the metric of spacetime is obtained from two assumptions of *electromagnetic* nature. So, the notion of distance can, to a large extent, be viewed as an offshoot of Maxwell's theory. One caveat is important: the method above specifies $g^{\alpha\beta}$ up to a conformal factor only.

Chapter 4

Outlook

The research underpinning this Thesis may evolve in several directions. Hereafter, the ideas for future work that appear to be of considerable interest are examined, following the order in which related topics are discussed in previous Chapters.

4.1 Topology’s role in the first and third axioms

The first axiom of electromagnetism as laid out in Hehl and Obukhov’s manual [41] and in equation (2.20) states that the integral of the charge current density J over any 3-dimensional cycle C_3 is zero. It implies, with no additional requirements, that J is an exact form. A proof this fact is described in (2.21)–(2.23), and is made up of two steps. *First*, the consequences of the axiom are tested in the scenario where C_3 is the boundary of an arbitrary 4-dimensional region. As a result of Stokes’s theorem, it is obtained that the charge current density must be a closed form. *Second*, de Rham’s map is recognised to be injective [33], thus leading to the appropriate outcome. To sum up, the first axiom of electromagnetism defined in (2.20) and in the book [41] prescribes that the Maxwell equation $J = dH$ holds true *globally*. In the work [19], Delphenich replaces the postulate just analysed with a less severe condition, that is: the integral of J over the boundary of any 4-dimensional region Ξ_4 vanishes. Although this modified axiom does specify that the charge current density is a closed form, it does not guarantee that J is exact. Indeed, repeating the first step

in the above proof establishes that the differential conservation law (2.22) holds true, as before. However, the fact that de Rham's map is injective no longer entails that one can write $J = dH$ globally. To achieve a better understanding of this negative statement, Poincaré's lemma is employed [101]. Accordingly, it is observed that the Maxwell equation (2.23) is valid at least in *some* neighborhood of each spacetime point $p \in X_4$. One can now examine the scenario wherein the topology of the manifold X_4 implies that all 3-dimensional cycles are also boundaries. In this specific case, Hehl and Obukhov's first axiom (2.20) and Delphenich's alternative requirement $\oint_{\partial\Xi_4} J = 0$ become identical. Moreover, provided a differential 3-form is closed, it is exact too. The analysis carried out so far suggests an interesting direction for future research: to test the two versions of the first axiom by means of a spacetime such that not all 3-cycles are boundaries. It is worth noting that topological considerations similar to the ones examined above indicate that the postulate (2.110) may have to be revised as well¹. Ergo, the mentioned idea of comparing the axioms $\oint_{C_3} J = 0$ and $\oint_{\partial\Xi_4} J = 0$ is found to be too limited in scope. An exhaustive investigation appears to be required, whose aim is to determine how Maxwell's theory ought to be formulated in certain exotic spacetimes, and in general. The article [32] by Frankel may turn out to be helpful in such project for future work.

4.2 Geometrical optics and singular directions

As part the metric-free analysis of geometrical optics carried out in Ref. [16], Dahl considers the linear vector space V_q of all physically relevant polarisations. The treatment of Section 2.4.2 offers two refinements. *First*, V_q is introduced with greater mathematical accuracy by using an equivalence relation, denoted $\sim$. More explicitly, if two polarisations a and a' differ by a multiple of the wave covector q , it is stipulated that $a \sim a'$. *Second*, a component-free method is investigated that allows one to evaluate the dimension of the vector space defined by the physically relevant polarisations. The technique is based on the finding that the exterior power specified in the manuals by Flanders [31] and Lindell [79] can be employed so as to calculate

¹The generalisation of the field strength F that is defined in Yang-Mills theories plays the role of a curvature, and is *not* globally exact. It is therefore reasonable to expect that the third axiom, as specified in (2.110) or in Hehl and Obukhov's treatise [41], must indeed be corrected to $\oint_{\partial\Xi_3} F = 0$.

the rank of the operator $(W_q)^{\alpha\beta} = \chi^{\alpha\mu\beta\nu} q_\mu q_\nu$. More in detail, as soon as this piece of information is known, the desired value of $\dim(V_q)$ is extracted from the formula

$$\dim(V_q) = 3 - \dim(\text{im}(W_q)) = 3 - \text{rank}(W_q). \quad (4.1)$$

Leaving momentarily aside the problem of characterising the vector space V_q in a rigorous way, it is appropriate to recall some facts about the geometrical optics of dielectric crystals. A complete treatment of such topic is available in Section 14.3 of the book [9] by Born and Wolf. If a medium of the type just indicated is birefringent, the propagation of light is governed by two separate phase velocities in nearly all directions. Equivalently, for almost every choice of *spatial* wave-covector, the Fresnel equation has two disjoint roots. To be accurate, there exist individual directions such that the propagation of light is controlled by a single phase velocity. In other words, provided the spatial wave-covector is fixed in a certain manner, the dispersion equation has a unique solution. Notably, each direction in which this degeneracy of roots takes place is said to be an *optical axis*. Returning to the general case, it is important to observe that nearly all wave-covectors that satisfy the Fresnel equation for a dielectric crystal are linked to a vector space of relevant polarisations which is 1-dimensional. The complementary statement is that some individual roots $q = K$ of the dispersion equation are associated with the occurrence of $\dim(V_K) = 2$. One can demonstrate that the wave-covectors K correspond precisely to the optical axes. Nevertheless, it is to be established whether such identification holds true for all birefringent local and linear media. The tools that are necessary to answer this question are studied at length in the existing literature and in Section 2.4.2. As a matter of fact, the quartic dispersion equation of Tamm [130] and Rubilar [116] allows one to investigate with unprecedented rigour the geometry characterising the optical axes. In addition, the aforesaid method for establishing the dimension of V_q can be used so as to identify the critical wave-covectors $q = K$. Observing that the Fresnel equation is based on a polynomial suggests an extension of the idea for future work just described. It is proposed to analyse the wave surface generated by the dispersion relation from the viewpoint of *algebraic geometry*. To be precise, the intended line of research may focus on the task of classifying the singularities of the

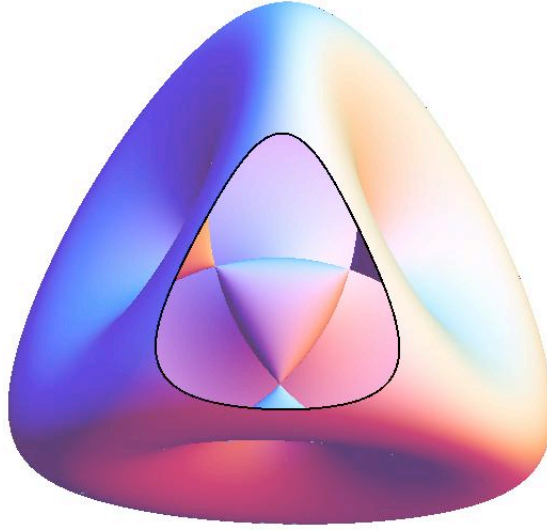


Figure 4.1: A Kummer surface, illustrated with the help of Mathematica[®]. The front part of the plot is cut out, so that *three* singular points become visible, and *one* remains hidden. In other words, there are *four* directions along which no birefringence is observed.

wave surface. Some important discoveries in this context are long-established. For instance, Bateman’s 1910 article [4] demonstrates that the propagation of light in a local and linear skewon-free medium is governed by a Kummer surface (Figure 4.1). The historical works such as Ref. [4] are an excellent starting point for the derivation of novel results. Of course, the key is to implement modern geometrical techniques.

4.3 Group theory investigation of reciprocity maps

It is appropriate to recall a few concepts discussed in Section 3.2. The unimodular reciprocity (3.12) is the most general linear map from $\{H, F\}$ to $\{H', F'\}$ that preserves the energy-momentum tensor ${}^k\mathcal{T}_\alpha{}^\beta$. For every specific choice of $\{C_{00}, C_{01}, C_{10}, C_{11}\}$, a local and linear medium $H = \kappa(F)$ is invariant under the transformation (3.12) if and only if the quadratic equation (3.30) is fulfilled. An exploratory attempt to solve (3.30) reveals that three “Cases” must be investigated. Each scenario is associated with a family of maps, defined by imposing a condition on the entries of (3.12). The electric-magnetic reciprocity (3.19), arising from the statement $C_{00} = C_{11} = 0$, is one of the transformations examined as part of Case 3. Following the article [112] by Perlick, one identifies that the general unimodular

map (3.12) is based on a representative of the special linear group $\text{SL}(2, \mathbb{R})$. In the light of such observation, it is proposed to investigate the three families of transformations associated with the mentioned Cases from a group theory perspective. A decomposition of $\text{SL}(2, \mathbb{R})$ may prove useful to this end. The idea is that of expressing the reciprocity (3.12) in terms of simpler matrices, whose group of provenance is known. More explicitly, the information thus derived facilitates the analysis of the three Cases originating from (3.30). Among the several possible decompositions of $\text{SL}(2, \mathbb{R})$, the one due to Bruhat is likely to be the most fitting for the task. Lang's book [73] illustrates in detail various techniques whereby a representative of the special linear group is expressed as a product of straightforward matrices. In order to clarify the plan for the suggested group theory investigation, it is worthwhile to look at the Bruhat decomposition of the electric-magnetic reciprocity (3.19), that is

$$\begin{bmatrix} 0 & \zeta \\ -\zeta^{-1} & 0 \end{bmatrix} = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \quad (4.2)$$

As in all previous occurrences, the twisted scalar ζ is demanded to be nonzero. The first and second matrices obtained by expanding (3.19) are representatives of the indefinite special orthogonal group $\text{SO}(1, 1)$ and of the Weyl group W . Equivalently, the Bruhat decomposition allows one to formulate the electric-magnetic reciprocity transformation as a squeeze map operating from the left on an almost-complex map.

4.4 From the roots of (3.3) and (3.4) to a group

The notions of group theory employed below are explained in the manual [129] by Szekeres. From the analysis of Section 3.4 it is known that, provided κ_1 and κ_2 are roots of a mixed closure equation, $\kappa_1 \circ \kappa_2$ is a root too. In other words, if $\mathcal{M}$ and $\mathcal{N}$ are the sets that comprise the solutions of (3.3) and (3.4) respectively, one has that:

$$\begin{aligned} \circ : \mathcal{M} \times \mathcal{M} &\rightarrow \mathcal{M}, & \text{and} & & \circ : \mathcal{N} \times \mathcal{N} &\rightarrow \mathcal{N}, \\ (\kappa_1, \kappa_2) &\mapsto \kappa_1 \circ \kappa_2, & & & (\kappa_1, \kappa_2) &\mapsto \kappa_1 \circ \kappa_2. \end{aligned} \quad (4.3)$$

Hence, the operator $\circ$ is a *binary law of composition* on each set defined by the roots of a mixed closure equation. In view of the property (4.3), it is tempting to conclude that $\mathcal{M}$ and $\mathcal{N}$ are two groups. Such intuition is found to be exact up to an additional caveat, namely, that the medium operators with $\det(\kappa) = 0$ are ruled out. Turning the statement in a positive form, one can demonstrate that the invertible roots of (3.3) and (3.4) form a group. Notably, after the solutions with a vanishing determinant are excluded, the mixed closure relations become equivalent. In particular, this fact is described by the vertical arrows on the right-hand side of Figure 3.3. Accordingly, when the condition $\det(\kappa) \neq 0$ is imposed, the families of roots $\mathcal{M}$ and $\mathcal{N}$ define the *same* group $\mathcal{O}$. Proving explicitly that the invertible solutions of (3.3) and (3.4) form a group is not difficult. Given that $\circ$ is known to be a law of composition, one needs to verify that three standard axioms are fulfilled. As an aside, the adjoint of the constitutive map $\bar{\kappa}$ inherits the properties (2.196)–(2.199) from the transposed operator κ^t . It follows that the group $\mathcal{O}$, specified by the roots of the mixed closure equations with $\det(\kappa) \neq 0$, and the orthogonal group $O(6, \mathbb{R})$ are tightly linked.

If the electromagnetic response is demanded not only to be local and linear but also to satisfy (3.4), whenever the product $I_2 = F \wedge F$ is zero, the 4-form $I_3 = \kappa(F) \wedge \kappa(F)$ vanishes too. This observation is discussed in Section 3.3. Evidently, the condition that the medium is a root of (3.4) implies that, whenever F is simple, $\kappa(F)$ is simple too. As a matter of fact, in 4-dimensions, a 2-form Φ can be expressed as the wedge product of two 1-forms if and only if $\Phi \wedge \Phi = 0$. It is thus verified that all linear operators that obey (3.4) map simple 2-forms to simple 2-forms. Establishing this property is the first step towards demonstrating that the group $\mathcal{O}$ has a *left action* on the set of simple 2-forms. Such action is nothing but the contraction of an invertible operator fulfilling (3.3) or (3.4) with a simple 2-form. More precisely, in order to conclude that $\mathcal{O}$ is endowed with the mentioned action, it is necessary to check that two axioms are satisfied. First, one is demanded to observe that the associative property $(\kappa_1 \circ \kappa_2)(F) = \kappa_1(\kappa_2(F))$ holds true. Second, one is required to note that an identity map indeed exists. The left action of $\mathcal{O}$ defines some *orbits*, that is, some equivalence classes of simple 2-forms. Computing such orbits may not be easy. Nonetheless, it does represent an interesting task for

future work.

From the analysis of Section 3.5 it is known that each P -medium with $(\det P) \neq 0$ is an invertible root of the mixed closure equations. Hence, operating with a full-rank P -medium κ_P on an element κ of the group $\mathcal{O}$ yields an electromagnetic response $\kappa_P \circ \kappa$ that belongs to $\mathcal{O}$ as well. The expression (2.418) illustrates that constitutive map κ_P is based on a tensor P_α^β which is a representative of the general linear group $\text{GL}(4, \mathbb{R})$. By utilising these observations, and by proving that two additional axioms are fulfilled, one can show that $\text{GL}(4, \mathbb{R})$ has a *left action* on the set $\mathcal{O}$. Such action is nothing but the composition of a P -medium obeying $(\det P) \neq 0$ with an invertible root of (3.3) and (3.4), that is, $\kappa_P \circ \kappa$. Remarkably, owing to the mentioned left action of $\text{GL}(4, \mathbb{R})$, it is possible to sort the elements of $\mathcal{O}$ in equivalence classes named *orbits*. This classification of the invertible media that satisfy the mixed closure equations appears to coincide with that established in Proposition 8 of the article [17] by Dahl. It is thereby suggested to examine whether this conclusion is right. An additional idea for future work may be to evaluate the dispersion relation generated by each of the aforesaid orbits. The ensuing results could be relevant to the algebraic geometry investigation put forward in Section 4.2.

Appendix A

SI units, in order of appearance

Name of unit	Symbol for unit	Examples
coulomb	C	$J, H, \rho, \mathcal{D}$
second	s	σ
ampere	A	$j, \mathcal{H}$
metre	m	v^a
weber	Wb	B
volt	V	E
tesla	T	B_{ab}
watt	W	$\mathfrak{f}_0$
newton	N	$\mathfrak{f}_a$
ohm	Ω	$\kappa, \text{tr}(\kappa)$
joule	J	$[I_1] \stackrel{\text{SI}}{=} \text{J s}$

Table A.1: SI units used in this Thesis, in order of appearance.

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