

**MATHEMATICAL MODELLING OF JET-FORCED
CIRCULATION IN RESERVOIRS AND HARBOURS**

by

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**To My Future Wife Nicola
and To My Parents**

ABSTRACT

In the planning and design of reservoirs and harbours questions relating to water quality are assuming increasing importance. With this in mind, the objective of this research has been to develop a mathematical model capable of predicting the depth averaged velocity field in a reservoir or harbour caused by the incoming momentum of a river, pump-jet or tide.

The time dependent non-linear equations of motion were formulated to include the effects of bottom roughness, wind action, the earth's rotation and the turbulent diffusion of momentum. These equations were integrated over the depth and then analysed to determine the mechanism by which circulation and vorticity can be both generated and maintained. A comparison of the numerical properties of a simplified set of the equations allowed the most suitable finite difference scheme to be selected. The final difference scheme chosen was of the Alternating Direction Implicit (ADI) type, with the problem of non-linear instability being overcome by a novel time centred representation of the convective accelerations. While the equations derived may be solved for any planform of flow, the mathematical model has been run for a circular reservoir with a diametrically opposite inlet and outlet, and for a rectangular harbour with an asymmetric entrance.

To verify the mathematical model a comparison has been made between the predicted and experimentally determined velocities (using floats) for hydraulic model studies of a reservoir and harbour. The agreement, in general, has been satisfactory.

Further numerical experiments have been carried out using typical prototype data. These experiments have included a comparison of the different velocity fields predicted by individually neglecting various terms of the momentum equations. The geometric and dynamic boundary conditions of the model were also varied, illustrating the flexibility of mathematical models and providing useful engineering design information.

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LIST OF SYMBOLS

a	wave amplitude
a_{jet}	cross-sectional area of jet
A	planform area of harbour or reservoir
A_j	Fourier series component
B	breadth of harbour
C	Chézy coefficient
C^*	wind resistance coefficient
C_d	coefficient of discharge for weir
C_f	skin friction coefficient
C_n	numerical wave celerity
C_p	physical wave celerity or propagation velocity
d	entrance width of harbour
D	horizontal diffusion coefficient
e	kinetic energy per unit mass
e	difference between exact and numerical water elevation solutions
f	Coriolis parameter ($2\omega \sin \phi$)
f	difference between exact and numerical velocity solutions
$f(x)$	single-valued function of x having continuous derivatives
F_r	Froude number
g	acceleration of gravity
h	mean depth
H	exact water elevation solution
j,k	finite difference grid point co-ordinates in x,y directions
k	von Kármán coefficient
l, l_m	mixing length

L	length scale
L	wave length
L	mean diameter of reservoir
L	length of harbour
M, N	upper boundary grid points in x, y directions
n	Manning's roughness coefficient
n	time step level
P	pressure
P_a	atmospheric pressure
Q	discharge between two adjacent grid points
Q_{in}, Q_{out}	unsteady reservoir inflow, outflow
Q_{jet}	jet discharge
Re	local equivalent pipe Reynolds number
s	concentration of a solute
t	time
T	wave period
u, v, w	temporal mean velocity components in x, y, z directions
u_n, v_n, w_n	surface velocity components in x, y, z directions
u_h, v_h, w_h	bed velocity components in x, y, z directions
$\tilde{u}, \tilde{v}$	departure in velocity u, v from depth averaged value U, V
u_i	instantaneous velocity fluctuation in x_i direction
U, V	depth averaged velocity components in x, y directions
U_d	desired velocity for characteristic boundary relationship
U_o	constant local velocity or maximum velocity
U_*	line of errors for velocity vector U
U_*, V_*	shear velocity components in x, y directions

V_{jet}	jet velocity
Vol	volume of water stored in reservoir
W	absolute wind velocity
W_x, W_y	wind velocity components in x,y directions
x, y, z	horizontal, lateral and vertical co-ordinates
x_i	co-ordinate ($x_1, x_2, x_3 = x, y, z$)
X, Y	depth averaged convection effects in x,y directions
α	weighting function
α	eigenvalue
β	wave number ($2\pi/L$)
γ	stability number ($U_0 \Delta t / \Delta x$)
$\Delta x, \Delta y$	grid size in x,y directions
Δt	time step
ϵ	eddy viscosity
η	water elevation above or below mean depth
η_d	desired water elevation for characteristic boundary relationship
θ	grid size per wavelength ($\pi \Delta x / L$)
λ	eigenvalue
μ	Courant number ($\Delta t \sqrt{gh} / \Delta x$)
ν	kinematic viscosity
ρ	fluid density
ρ_a	air density
σ or ω	wave frequency ($2\pi/T$)
τ_{ij}	Reynolds stress acting in x_i direction and on x_j plane
τ_{xb}, τ_{yb}	bottom shear stress components in x,y directions

τ_{xw}, τ_{yw} wind stress components in x,y directions

ϕ geographical latitude

ω angular velocity of earth's rotation

$\bar{\omega}$ depth averaged vorticity

$| \quad |$ magnitude

$O(\quad)$ order of approximation of series

arg. ($\quad$) argument of ($\quad$)

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INTRODUCTION

In recent years there has been a growing concern as to the water quality characteristics of shallow reservoirs and harbours, especially in the planning and design stages. It is therefore not surprising that in the current engineering literature there have been frequent case studies, for both reservoirs and harbours, where the improvement of the overall water quality has been the main objective. As far as the hydraulic engineer is concerned, the quality of water within any particular basin is chiefly assumed to be dependent upon the overall flow patterns, the mixing processes and the basin's retention time or flushing rate.

In water supply reservoirs, the problem of water quality has arisen due to a growing demand being placed on reservoirs and their increasing use as a means of primary treatment of the water. In the case of deep valley reservoirs selective withdrawal allows a choice to be made of the required water quality. However, in the case of broad shallow storage reservoirs selective withdrawal has a far less significant effect and artificial methods of mixing the water have to be adopted, e.g. the use of jets.

Water quality problems that arise in harbours are generally due to poor overall flushing characteristics and internal circulation patterns. In the past, it has been noticed that one of the causes of poor water quality characteristics has been the dominance of large stagnant regions within the planform area of the harbour. Consequently, harbour design engineers now have to face the additional task of improving the internal flow patterns within existing harbours and avoiding such

problems in future designs.

Three typical examples are considered where the main objective has been to improve the overall water quality characteristics. The solutions have, in each case, been arrived at by using a physical model of the particular flow situation. One of the earliest studies into the problem of stagnation in large shallow bodies of water was initiated by the Metropolitan Water Board in 1945. White et al (120) were commissioned to advise on the hydraulic design of a proposed new reservoir at Walton-on-Thames which, on completion in 1962, was named the Queen Elizabeth II Reservoir. The report describes an experimental study to investigate how far the physical conditions leading to stagnation and thermal stratification in relatively large reservoirs can be controlled by the suitable arrangement of jet inlets. The effects of several jets interacting at different locations, orientations and flow rates were investigated to determine the optimum overall flow patterns. Two further reservoirs have since been brought into operation by the Thames Water Authority (Metropolitan Water Division), namely Wraysbury 1972 and Datchet 1974, where, during the design stages, particular attention was paid to the original recommendations of White's report.

In a more recent but similar study, Mayer (72) was commissioned by the Georgia Power Company to undertake a large scale hydraulic model study of its Wallace Dam Pumped Storage project. The principle objectives of this study were to determine, from a water quality standpoint, the most suitable relative locations of the power house and the spillway, and to define factors pertinent to the locations of four reversible pump-turbine units within the power house. It was assumed that, if the spillway and pump-turbine units were located in an ideal position then, during pumping, a greater degree of control would be possible over

reservoir stratification and circulation. The experimental investigation involved the analysis of a number of differing jet locations and orientations, with dye injections and time-lapse colour photography being used to study the jet geometries and reservoir mixing patterns. The basic conclusions of the hydraulic model study were that, from the standpoint of enhanced water quality, the pump-turbine units should discharge horizontally and be in the centre of the power house.

The water quality problem in harbours, which is mainly concerned with the flushing out of pollutants at each tide and so avoiding the storage of stagnant polluted water, has been studied in some detail. Nece et al (76, 77, 78) have been concerned with improving the overall flushing characteristics of six, existing or proposed, small boat harbours in the Puget Sound waters of the State of Washington. Hydraulic model studies have been carried out on the harbours of Des Moines, Edmonds, Lagoon Point, Lake Crockett, Penn Cove and East Bay. In each case, the influence of the tidal range, entrance configuration(s) and internal geometry on the gross flushing characteristics and internal circulation patterns have been directly related to an exchange coefficient and a water quality parameter. Several criteria have been developed which the design engineer could use to prepare the best possible initial design when evaluating proposals for a new harbour project.

All these flow situations are similar in that they are equivalent to a relatively wide jet flowing into, or out of, a confined body of water. In the first two cases the wide jet is produced by the interaction of the small jets and the surrounding water, while in the harbour the broad entrance and the oscillating tidal flow produces the jet.

Having generalised the flow condition it should be possible to develop a mathematical model capable of predicting the jet-forced

circulation in all three situations. A mathematical model, as compared to a physical model, has advantages in being more flexible, adaptable and economical. From a literature survey at the start of this project it appears that such a model has not been attempted, possibly due to the non-uniform flow properties of jets and the corresponding non-linear stability problems associated with computer numerical techniques. Consequently, the principal objectives of this study have been to develop a mathematical model which would predict the depth averaged velocity field of a confined jet. At this stage any numerical results obtained can only be expected to give gross characteristics of the flow. However, it was thought that even this information would be useful to the engineer faced with minimising the water quality problems in any such design of the future, as it would enable him to avoid constructing basins which are liable to produce large stagnant regions within the flow field. This thesis describes the construction and testing of a mathematical model that has been developed for this purpose.

With present day computers having limited storage it is not yet possible, or even economical, to analyse the three-dimensional unsteady motion of a large body of water. It was therefore first necessary to decide whether to reduce the modelling to that of a three-dimensional steady flow, or, to one which was two-dimensional, depth averaged and unsteady. The disadvantage with opting for a solution of the depth averaged velocity field is that the effects of stratification and the mixing associated with secondary currents are immediately eliminated. However, a two-dimensional depth averaged mathematical model can be used to suggest an ideal planform geometry which might then be subjected to flow visualisation tests in a laboratory, hopefully to give some idea of the stratification effects. As most shallow reservoirs tend to

have flows that are predominantly two-dimensional and horizontal, and also, the fact that reservoirs have an unsteady supply and demand, it was thought more appropriate to solve for the latter flow situation. Another advantage of this decision is that whereas the properties of two-dimensional turbulence are well documented, the properties of three-dimensional turbulence, particularly in the presence of density layers, are less well known.

Having chosen the type of physical flow situation to be modelled, the first part of this thesis is concerned with the formulation of the depth averaged equations of motion. These equations include the effects of bottom friction, wind action, the earth's rotation and the horizontal diffusion of turbulent momentum. Various turbulence models are discussed in some detail allowing the most suitable representation of the turbulent diffusion to be implemented into the equations. Following this, the corresponding vorticity transport equation is analysed such that the terms leading to changes in vorticity, or circulation, can be clearly identified from the outset.

The differential equations are then rewritten in a number of finite difference forms, with the numerical properties of approximation, consistency, convergence, stability and accuracy being investigated for each proposed scheme. The analytical investigation of the different representations of the equations allows a choice to be made as to the most suitable finite difference form. Also, it allows a better understanding of the numerical limitations of the resulting mathematical model. Having chosen the most appropriate finite difference scheme and knowing the boundary conditions, documentation is then presented concerning both the implementation of these equations into the computer model and secondly, the general solution technique being adopted.

All mathematical models need verification, whether it be with prototype measurements or experimental results. The purpose of such comparisons is to ensure that the physical boundary value problem is being solved mathematically with sufficient accuracy to provide useful numerical results. Unfortunately no measurements are available for the velocity fields in prototypes of either reservoirs or harbours. Consequently, the mathematical model had to be tested with small scale physical hydraulic models. Direct comparisons were made between the experimental and mathematical model results for similar physical and geometric properties and similar dynamic boundary conditions.

An experimental programme was carried out with a 1.5 m diameter, flat-bottomed, circular reservoir with a full depth jet inlet and a diametrically opposite weir outlet. Circulation paths were carefully observed for a steady flow and depth averaged velocity measurements were recorded using floats. The results of a previous experimental programme, carried out by the Author (29) on a flat-bottomed rectangular harbour, with adjustable sidewalls, were also available. The asymmetric harbour entrance was located in one of the corners and perpendicular to the direction of the simulated sinusoidal tides. Circulation paths were observed at different intervals of the tidal cycle, for varying length, breadth and entrance width ratios, while the planform cross-sectional area and the tidal range were kept constant. In conjunction with observing the circulation paths, overall flushing characteristics were determined for each different harbour configuration. For both experimental programmes the agreement with the corresponding mathematical model was fairly good, showing that at least in these cases the mathematical model was satisfactorily solving for the depth averaged velocity field.

To investigate the performance of the mathematical model for

prototype conditions, numerical experiments were undertaken for the jet-forced circulation in a reservoir and the tidal flow into a harbour. Although the mathematical model could be used to solve for any planform geometry or depth, for the sake of simplicity both these parameters were idealised in the numerical tests. The reservoir was assumed to be flat-bottomed, circular and with a diametrically opposite inlet and outlet. To economise in computer time both the inlet and outlet widths were assumed to be relatively large, e.g. $1/19$ th and $3/19$ ths of the reservoir diameter respectively. In the case of the harbour, the planform geometry was assumed to be rectangular with an asymmetric entrance and a horizontal bed. The numerical experiments were carried out for typical prototype conditions with realistic geometric, kinematic and dynamic properties. Apart from running the model for different physical shapes and boundary flow conditions, the actual format of the differential equations was varied such that the influence of the individual terms could be directly compared. In all cases results were obtained having the resemblance of reality in such a form which, it is hoped, will provide useful information to the design engineer.

DERIVATION OF THE DEPTH AVERAGED DIFFERENTIAL EQUATIONS OF MOTION

2.1 FORMULATION OF THE BASIC HYDRODYNAMIC EQUATIONS

The equations of motion for any two-dimensional depth averaged flow can be obtained by integrating the three-dimensional hydrodynamic equations over the depth. Using the Eulerian Cartesian Co-ordinate system shown in Fig. 1, the basic equations representing an incompressible turbulent flow on a rotating earth can be written, see Lamb (59) and Schlichting (93), as:-

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv + \frac{1}{\rho} \frac{\partial P}{\partial x} \\ - \frac{1}{\rho} \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) = 0 \end{aligned} \quad (2.1)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{\partial (vu)}{\partial x} + \frac{\partial (v^2)}{\partial y} + \frac{\partial (vw)}{\partial z} + fu + \frac{1}{\rho} \frac{\partial P}{\partial y} \\ - \frac{1}{\rho} \left(\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) = 0 \end{aligned} \quad (2.2)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{\partial (wu)}{\partial x} + \frac{\partial (wv)}{\partial y} + \frac{\partial (w^2)}{\partial z} + \frac{1}{\rho} \frac{\partial P}{\partial z} + g \\ - \frac{1}{\rho} \left(\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) = 0 \end{aligned} \quad (2.3)$$

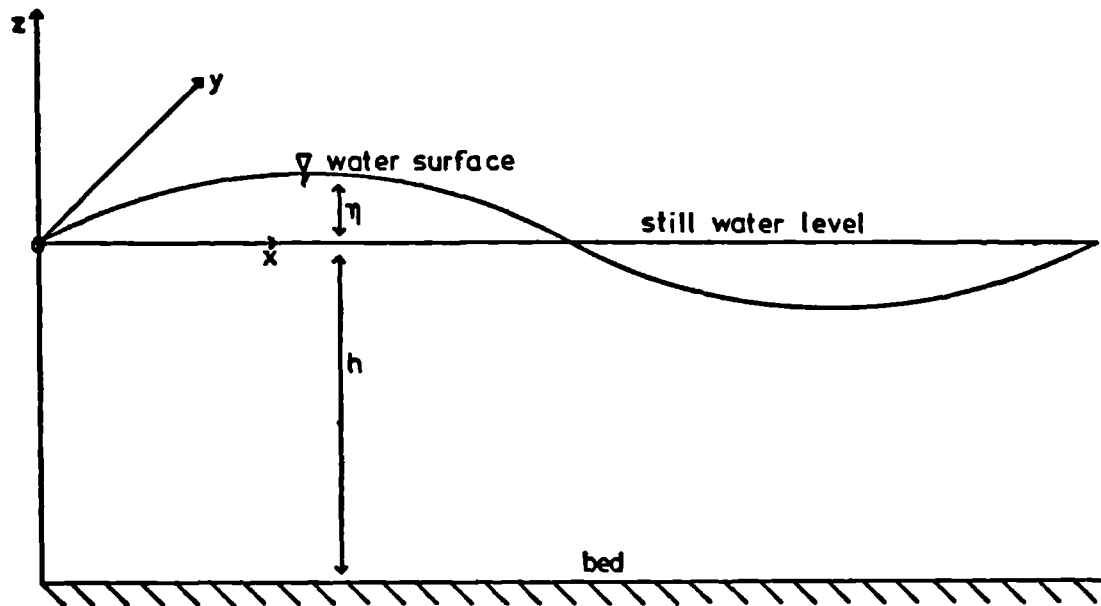


FIGURE 1 Schematic Drawing Showing the Co-ordinate System

x, y, z = co-ordinates of a right handed system with z upwards

u, v, w = time averaged velocity components in the x, y, z directions respectively

t = time

h = distance between the bed and the undisturbed water level

η = distance between the undisturbed water level and the water surface elevation

P = pressure

g = acceleration due to gravity

ρ = fluid density

f = Coriolis parameter $2\omega \sin \phi$ where ω = angular velocity of the earth's rotation and ϕ = geographical latitude

τ_{ij} = Reynolds stresses acting in the i direction and on the j plane, where $i = x, y, z$ and $j = x, y, z$

The equation of continuity for an incompressible flow is:-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (2.4)$$

In any predominantly horizontal flow, the vertical accelerations due to hydrodynamic processes are generally regarded as being small in comparison with gravity effects. Consequently, in Eq. (2.3), the vertical accelerations and shear stresses can be neglected so that, in the vertical direction, a hydrostatic pressure distribution results:-

$$\frac{\partial P}{\partial z} + \rho g = 0 \quad (2.5)$$

Integrating Eq. (2.5) gives the linear pressure distribution with depth:-

$$P(z) = \int -\rho g \, dz = -\rho g z + c \quad (2.6)$$

Using the boundary condition that the pressure at the free surface, $z = \eta$, is equal to the atmospheric pressure, P_a , then Eq. (2.6) becomes:-

$$P(z) = \rho g(\eta - z) + P_a \quad (2.7)$$

The derivatives of the pressure in any horizontal direction can now be expressed in terms of the water elevations:-

$$\frac{\partial P}{\partial x} = \rho g \frac{\partial \eta}{\partial x} + \frac{\partial P_a}{\partial x} \quad (2.8)$$

$$\frac{\partial P}{\partial y} = \rho g \frac{\partial \eta}{\partial y} + \frac{\partial P_a}{\partial y} \quad (2.9)$$

In the scale of motion being considered, the atmospheric pressure gradient constitutes an insignificant part of the hydrodynamic pressure gradient and can be neglected in Eqs. (2.8) and (2.9).

In determining the two-dimensional depth averaged equations, use is made of the kinematic boundary conditions. The free surface boundary condition, see Lamb (59) is:-

$$\frac{\partial \eta}{\partial t} + u_{\eta} \frac{\partial \eta}{\partial x} + v_{\eta} \frac{\partial \eta}{\partial y} - w_{\eta} = 0 \quad (2.10)$$

and at the bed:-

$$u_{-h} \frac{\partial (-h)}{\partial x} + v_{-h} \frac{\partial (-h)}{\partial y} - w_{-h} = 0 \quad (2.11)$$

where u_{η} , v_{η} and w_{η} are the respective velocity components at the surface, and similarly u_{-h} , v_{-h} and w_{-h} are the velocity components at the bed.

2.2 THE TWO-DIMENSIONAL DEPTH AVERAGED EQUATIONS

The two-dimensional depth averaged equations, defining the mean horizontal flow, are obtained by integrating Eqs. (2.1), (2.2) and (2.4) between the bed, $z = -h(x,y)$, and the surface, $z = \eta(x,y)$.

Assuming $\frac{\partial P_a}{\partial x} = 0$ (eq. 2.8), the momentum equation in the x direction for the full depth of fluid becomes:-

$$\begin{aligned}
& \int_{-h}^{\eta} \left[\frac{\partial u}{\partial t} + \frac{\partial (u^2)}{\partial x} + \frac{\partial (uv)}{\partial y} + \frac{\partial (uw)}{\partial z} - fv + g \frac{\partial \eta}{\partial x} \right. \\
& \left. - \frac{1}{\rho} \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \right] dz = 0 \quad (2.12)
\end{aligned}$$

The integrals may be expanded using Leibnitz rule, as derived by Sokolnikoff and Redheffer (101):-

$$\begin{aligned}
& \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx = \frac{\partial}{\partial \alpha} \int_a^b f(x, \alpha) dx \\
& - f(b, \alpha) \frac{\partial b}{\partial \alpha} + f(a, \alpha) \frac{\partial a}{\partial \alpha} \quad (2.13)
\end{aligned}$$

Eq. (2.12) becomes:-

$$\begin{aligned}
& \frac{\partial}{\partial t} \int_{-h}^{\eta} u dz + \frac{\partial}{\partial x} \int_{-h}^{\eta} u^2 dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} uv dz \\
& - f \int_{-h}^{\eta} v dz + g(\eta + h) \frac{\partial \eta}{\partial x} - \frac{1}{\rho} \frac{\partial}{\partial x} \int_{-h}^{\eta} \tau_{xx} dz \\
& - \frac{1}{\rho} \frac{\partial}{\partial y} \int_{-h}^{\eta} \tau_{xy} dz - \left[u_{\eta} \frac{\partial \eta}{\partial t} + u_{\eta}^2 \frac{\partial \eta}{\partial x} + u_{\eta} v_{\eta} \frac{\partial \eta}{\partial y} - u_{\eta} w_{\eta} \right] \\
& + \left[u_{-h} \frac{\partial (-h)}{\partial x} + u_{-h} v_{-h} \frac{\partial (-h)}{\partial y} - u_{-h} w_{-h} \right] \\
& + \frac{1}{\rho} \left(\tau_{xx_{\eta}} \frac{\partial \eta}{\partial x} + \tau_{xy_{\eta}} \frac{\partial \eta}{\partial y} - \tau_{xz_{\eta}} - \tau_{xx_{-h}} \frac{\partial (-h)}{\partial x} \right. \\
& \left. - \tau_{xy_{-h}} \frac{\partial (-h)}{\partial y} + \tau_{xz_{-h}} \right) = 0 \quad (2.14)
\end{aligned}$$

The depth averaged velocity components may be defined as Hansen (43):-

$$U = \frac{1}{(h + \eta)} \int_{-h}^{\eta} u \, dz$$

$$V = \frac{1}{(h + \eta)} \int_{-h}^{\eta} v \, dz$$
(2.15)

and the wind and bottom friction stress components as:-

$$\tau_{xw} = - \tau_{xx}_{\eta} \frac{\partial \eta}{\partial x} - \tau_{xy}_{\eta} \frac{\partial \eta}{\partial y} + \tau_{xz}_{\eta}$$
(2.16)

$$\tau_{xb} = - \tau_{xx}_{-h} \frac{\partial (-h)}{\partial x} - \tau_{xy}_{-h} \frac{\partial (-h)}{\partial y} + \tau_{xz}_{-h}$$
(2.17)

It is now possible to substitute for u in terms of the depth averaged velocity in the convective acceleration terms:-

$$u = U + (u - U)$$

$$v = V + (v - V)$$
(2.18)

then, using the kinematic boundary conditions i.e. Eqs. (2.10) and (2.11), and dividing by $(h + \eta)$, Eq. (2.14) becomes:-

$$\frac{\partial U}{\partial t} + \frac{1}{(h + \eta)} \left[\frac{\partial [(h + \eta) U^2]}{\partial x} + \frac{\partial [(h + \eta) UV]}{\partial y} \right] - fV$$

$$+ g \frac{\partial \eta}{\partial x} - \frac{(\tau_{xw} - \tau_{xb})}{\rho(h + \eta)} - \frac{1}{\rho(h + \eta)} \left[\frac{\partial}{\partial x} \int_{-h}^{\eta} [\tau_{xx} - \rho(u - U)^2] \, dz \right]$$

$$\begin{aligned}
& - 2\rho U(u - U)]dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} [\tau_{xy} - \rho(u - U)(v - V) \\
& - \rho\{V(u - U) + U(v - V)\}]dz \Big] = 0
\end{aligned} \tag{2.19}$$

The terms in square brackets constitute the effective stresses; the first term in each bracket represents the Reynolds stress and the other terms represent the momentum transfer due to the non-uniformity of the vertical velocity profiles. In both square brackets the last term can be directly integrated over the depth and equated to zero. In general the horizontal variations in depth will be small and can often be neglected for the convective acceleration terms.

The effects of wind action can be represented as, see Neumann and Pierson (80),:-

$$\tau_{xw} = \rho_a C^* W W_x \tag{2.20}$$

where ρ_a = density of air

W_x, W_y = components of wind velocity

W = absolute wind velocity

C^* = a resistance coefficient which, on the basis of sea slope observations, Neumann suggested $C^* = 9 \times 10^{-3} W^{-\frac{1}{2}}$ where W is in metres per second. In a similar manner the bottom friction may be expressed as, see Dronkers (25),:-

$$\tau_{xb} = \frac{\rho g U |U + V|}{C^2} \tag{2.21}$$

where C = Chézy coefficient.

Using these modifications, Eq. (2.19) becomes:-

$$\begin{aligned}
& \frac{\partial U}{\partial t} + \frac{\partial U^2}{\partial x} + \frac{\partial UV}{\partial y} - fV + g \frac{\partial \eta}{\partial x} - \frac{\rho_a C^* W_x}{\rho(h+\eta)} \\
& + \frac{gU|U+V|}{(h+\eta)C^2} - \frac{1}{\rho(h+\eta)} \left[\frac{\partial}{\partial x} \int_{-h}^{\eta} [\tau_{xx} - \rho(u-U)^2] dz \right. \\
& \left. + \frac{\partial}{\partial y} \int_{-h}^{\eta} [\tau_{xy} - \rho(u-U)(v-V)] dz \right] = 0 \quad (2.22)
\end{aligned}$$

By a similar process of substitution the depth averaged momentum equation in the y direction becomes:-

$$\begin{aligned}
& \frac{\partial V}{\partial t} + \frac{\partial VU}{\partial x} + \frac{\partial V^2}{\partial y} + fU + g \frac{\partial \eta}{\partial y} - \frac{\rho_a C^* W_y}{\rho(h+\eta)} \\
& + \frac{gV|U+V|}{(h+\eta)C^2} - \frac{1}{\rho(h+\eta)} \left[\frac{\partial}{\partial x} \int_{-h}^{\eta} [\tau_{yx} - \rho(u-U)(v-V)] dz \right. \\
& \left. + \frac{\partial}{\partial y} \int_{-h}^{\eta} [\tau_{yy} - \rho(v-V)^2] dz \right] = 0 \quad (2.23)
\end{aligned}$$

Finally integration of the continuity equation (2.4) over the depth gives:-

$$\frac{\partial \eta}{\partial t} + \frac{\partial \{(h+\eta)U\}}{\partial x} + \frac{\partial \{(h+\eta)V\}}{\partial y} = 0 \quad (2.24)$$

2.3 REPRESENTATION OF THE TURBULENT REYNOLDS STRESSES

The three depth averaged equations of motion, (2.22), (2.23) and (2.24) contain nine unknown functions, namely:- $U, V, \eta, \tau_{xx}, \tau_{xy}, \tau_{yx}, \tau_{yy}, u, v$. This problem of having more unknowns than equations can generally be overcome by neglecting the effective stresses. This approximation

may be justifiable when the horizontal velocity gradients are small, as Tennekes and Lumley (111) have shown that, for this case, the effective stresses only alter the velocity at a point by a few per cent. However, in this particular study the horizontal velocity gradients are not always small and so, it was thought more appropriate to retain these stresses in the equations and to try to obtain some insight as to their influence on the flow pattern.

To evaluate the effective stresses, the first step is to derive or assume a relationship between the velocity and the Reynolds stress - this being the "closure problem" of turbulence modelling.

One of the earliest proposed turbulence models was the mixing length hypothesis of Prandtl, see Goldstein (37). Prandtl based his hypothesis on the kinetic theory of gases and assumed a direct relationship between the Reynolds stress and the velocity field of the form:-

$$\tau_{ij} = -\rho \overline{u_i' u_j'} = \rho \ell_m^2 \left| \frac{\partial u_i}{\partial x_j} \right| \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.25)$$

where ℓ_m = a mixing length

u_i' = instantaneous velocity fluctuations with the bar denoting time averaged values.

Detailed experimental measurements have since shown that provided a good choice is made of the value of ℓ_m , the mixing length hypothesis allows realistic predictions to be made of the velocity and shear distributions. The major difficulty of this approach is determining the appropriate value of the mixing length ℓ_m .

Prandtl (85) realising the shortcomings of the mixing length hypothesis, went on to propose that instead of relating the random velocity fluctuations (u_i') of turbulence to the mean velocity gradient,

they should have been related to some property of the turbulence. The turbulent kinetic energy, e , was chosen and the proposed equation was:-

$$\epsilon = L \sqrt{e} \quad (2.26)$$

where ϵ = eddy viscosity

L = an appropriate length scale

$e = \frac{1}{2}(\overline{u_i^2})$ = kinetic energy per unit mass

Using this theory the closure problem was then solved by introducing one further equation e.g. the exact differential energy equation, which can be derived from the Navier-Stokes equations, see Reynolds (89). Because this method only requires one additional equation, this type of turbulence model has since been classified by Launder and Spalding (60) as a "one-equation hydrodynamic model of turbulence". More recently other workers in this field have developed similar one-equation turbulence models, notably among them:- Bradshaw et al 1967 (14), Nee and Kovaszny 1969 (79), Gawain and Pritchett 1970 (36), Vagar and Kagan 1971 (113). The major disadvantage with these particular models is that like the mixing length hypothesis, the length scale L does not have a precise definition and some empirical relationship has to be assumed.

Much experimental work in connection with turbulence modelling has led to the conclusion that the Prandtl-type energy model gives a good prediction of the velocity distribution when compared with the other one-equation models - see Launder and Spalding (60). It has also been shown experimentally that in most circumstances the one-equation models are only marginally superior to the original, but simpler, mixing length hypothesis.

To overcome the fundamental disadvantage with the one-equation

models, several authors have derived a further transport equation to determine the length scale L and so remove the empiricism of the previous theories. These particular turbulence models have become categorised as "two-equation models of turbulence", some of the most notable models being proposed by:- Harlow and Nakayama 1967 (46), Gosman et al 1969 (38), Runchal 1969 (92), Rodi and Spalding 1970 (91), Jones and Launder 1972 (56), Launder and Spalding 1972 (60). The elimination of the length scale L is achieved by using some functional relationship of the form:-

$$Z = f(L, e) \quad (2.27)$$

where Z is the property to be transported and L and e are as defined previously. The transported property most frequently analysed is the dissipation:-

$$Z = e^{3/2} L \quad (2.28)$$

The form of the transport equation for such a property Z is given as, see Reynolds (89):-

$$\begin{aligned} \frac{DZ}{Dt} = & \underbrace{\frac{\partial}{\partial y} \left[\frac{\epsilon}{\sigma_z} \frac{\partial Z}{\partial y} \right]}_{(1)} + Z \left[\underbrace{c_1 \frac{\epsilon}{e} \left(\frac{\partial u}{\partial y} \right)^2}_{(2)} - \underbrace{c_2 \frac{e}{\epsilon}}_{(3)} \right] \\ & + \underbrace{\dot{S}_z}_{(4)} \end{aligned} \quad (2.29)$$

where σ_z = diffusivity ratio ≈ 1 for most quantities

c_1 and c_2 = constants depending on the form of Z , generally

$$|c_1| \approx 0.1 \text{ and } |c_2| \approx 1.$$

Equation (2.29) expresses the time rate of change of the property Z by diffusion (term 1), interaction of the turbulence with the mean flow (term 2), the turbulent energy (term 3) and a secondary source (term 4), see Launder and Spalding (60). The main advantage of using a two-equation model is that it provides a direct means of calculating the evolution of the turbulent energy and the length scale L , unlike the mixing length and one-equation models which relied greatly on empiricism. Two-equation models enable three-dimensional flows to be analysed with more certainty but at the cost of considerably greater computational effort.

Even two-equation turbulence models do not provide a completely satisfactory solution to the closure problem. The equations do not include the normal stresses needed for the realistic calculation of secondary flows and also, the dissipation has been related to the shear stresses in an empirical manner, e.g. Eq. (2.28). As a consequence, recent research in this field has been directed towards the development of transport equations for the individual Reynolds stresses. Even in the simple case of two-dimensional boundary layer flow this type of modelling involves the solution of four additional equations. Two typical 'multi-equation models of turbulence' have been proposed by:- Daly and Harlow 1970 (22), Hanjalic and Launder 1972(41). The actual equations used in these models can be derived in much the same manner as for the turbulent energy equation. As an example, the transport equation for the Reynolds stress $\overline{u'v'}$ can be shown to be, see Reynolds (89):-

$$\begin{aligned}
 \frac{D \overline{u'v'}}{Dt} &= - \overline{v'^2} \frac{\partial u}{\partial y} - \frac{\partial}{\partial y} \left[v' (u'v' + \frac{p'}{\rho}) \right] \\
 &\quad + \frac{p'}{\rho} \left(\frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} \right) - \epsilon_{12}
 \end{aligned}
 \tag{2.30}$$

where p' = instantaneous pressure fluctuations.

In words, equation (2.30) expresses the time rate of change of the Reynolds stress by mean flow production of turbulent energy (term 1), dissipation due to pressure and shear (term 2), redistribution due to velocity fluctuation interactions (term 3) and viscous dissipation (term 4). The complexity of this type of model is obvious. In the more complicated turbulent flows where any form of prediction of a characteristic length scale L is an impossible task then the use of such a multi-equation turbulence model might well be justified; but in general, the use of such higher order models is limited due to the high cost of programming and computing.

For the particular turbulent flow that is to be modelled by Eqs. (2.22), (2.23) and (2.24), a comparison of the advantages and disadvantages of each type of turbulence model reviewed suggests that the classical mixing length hypothesis of Prandtl will be the most appropriate turbulence model. The fact that the velocity field being analysed has to be averaged over depth implies an oversimplification of the turbulence structure and there would seem little justification for using a higher order turbulence theory of the two or multi-equation type. Secondly, no advantage can be gained by using a one-equation model as a length scale would still have to be empirically determined. Finally, the mixing length hypothesis has the overwhelming advantage of simplicity and requires the minimum of programming and computational effort.

A generalised version of Prandtl's mixing length hypothesis, applicable to cases where neither the mean nor the turbulent motions are confined to two dimensions, can be found in Goldstein (37). This version suggests that, when one particular component of the mean rate of

deformation tensor is much greater than the others, the unknown Reynolds stresses of Eqs. (2.22) and (2.23) can be approximated by:-

$$\begin{aligned}\tau_{xx} &= -\rho \overline{u'^2} = 2\rho l^2 J \frac{\partial U}{\partial x} \\ \tau_{xy} &= \tau_{yx} = -\rho \overline{u'v'} = \rho l^2 J \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \\ \tau_{yy} &= -\rho \overline{v'^2} = 2\rho l^2 J \frac{\partial V}{\partial y}\end{aligned}\quad (2.31)$$

$$\begin{aligned}\text{where } J^2 &= 2\left(\frac{\partial U}{\partial x}\right)^2 + 2\left(\frac{\partial V}{\partial y}\right)^2 + 2\left(\frac{\partial W}{\partial z}\right)^2 \\ &+ \left(\frac{\partial W}{\partial y} + \frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\right)^2\end{aligned}\quad (2.32)$$

As the horizontal flow being considered is relatively shallow, i.e. the breadth of flow far exceeds the depth, then two approximations can be made. Firstly, the velocities in the vertical direction can be assumed negligible and secondly, apart from the region close to a jet, the velocity gradients in the horizontal plane will always be small compared to those in the vertical direction. Using these approximations Eq. (2.32) simplifies to:-

$$J^2 = \left[\left(\frac{\partial V}{\partial z}\right)^2 + \left(\frac{\partial U}{\partial z}\right)^2 \right] \quad (2.33)$$

The velocity profiles in the vertical direction can be adequately represented by the logarithmic velocity distribution which, using the co-ordinate system of Fig. 1, gives:-

$$u = \frac{U_*}{k} \ln (h + z) + U_* \text{ Const.}$$

(2.34)

$$v = \frac{V_*}{k} \ln (h + z) + V_* \text{ Const.}$$

where U_* = shear velocity, $\frac{\sqrt{g} U}{C}$ and similarly

$$V_* = \frac{\sqrt{g} V}{C}$$

k = von Kármán's universal constant equal to 0.41 for all homogeneous fluids

Differentiating Eq. (2.34) and substituting in Eq. (2.33) gives:-

$$J = \frac{\sqrt{g} |U + V|}{Ck(h + z)} \quad (2.35)$$

Since the largest velocity gradients have been assumed to be those in the vertical plane then the corresponding mixing length is related to the depth rather than some horizontal length. For a linear stress distribution and a logarithmic velocity profile throughout the depth, Reynolds (89) shows that, using the co-ordinate system in Fig. 1,:-

$$l = k(h + z) \left[1 - \left(\frac{h + z}{h + \eta} \right) \right]^{\frac{1}{2}} \quad (2.36)$$

Jobson and Sayre (55) confirmed this approximation for the mixing length with detailed measurements in an open channel.

Substituting Eqs. (2.35) and (2.36) in Eq. (2.31) and integrating over the depth $(h + \eta)$, the final depth averaged Reynolds stresses become:-

$$\begin{aligned}
\int_{-h}^{\eta} \tau_{xx} &= \frac{2\rho k(h + \eta)^2}{6} \frac{\sqrt{g}}{C} |U + V| \frac{\partial U}{\partial x} \\
\int_{-h}^{\eta} \tau_{xy} &= \int_{-h}^{\eta} \tau_{yx} = \frac{\rho k(h + \eta)^2}{6} \frac{\sqrt{g}}{C} |U + V| \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x} \right) \\
\int_{-h}^{\eta} \tau_{yy} &= \frac{2\rho k(h + \eta)^2}{6} \frac{\sqrt{g}}{C} |U + V| \frac{\partial V}{\partial y}
\end{aligned} \tag{2.37}$$

2.4 REPRESENTATION OF THE CONVECTION EFFECTS BY THE MEAN FLOW

Before the equations of motion can be written in a convenient form for solution, the remaining terms contributing to the effective stresses need to be represented in terms of the mean velocity field. For this purpose two of the previous assumptions are again used; namely, that the flow field is turbulent everywhere and secondly, that the logarithmic velocity distribution satisfactorily defines the vertical variation of velocity.

For the depth averaged convection effects in the x direction, the terms in Eq. (2.22) to be included are defined as:-

$$X = \frac{\partial}{\partial x} \int_{-h}^{\eta} (u - U)^2 dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} (u - U)(v - V) dz \tag{2.38}$$

where u and v are the velocity components at depth z and U and V are the mean velocities. Integrating the logarithmic velocity distribution, Eq. (2.34), over the depth gives the mean velocity to be:-

$$\begin{aligned}
U &= \frac{U_*}{k} (\ln(h + \eta) - 1) + U_* \text{ Const.} \\
V &= \frac{V_*}{k} (\ln(h + \eta) - 1) + V_* \text{ Const.}
\end{aligned}
\tag{2.39}$$

Substituting Eqs. (2.39) and (2.34) in (2.38) gives:-

$$\begin{aligned}
X &= \frac{\partial}{\partial x} \int_{-h}^{\eta} \frac{U_*^2}{k^2} \left(\ln \left(\frac{h+z}{h+\eta} \right) + 1 \right)^2 dz \\
&+ \frac{\partial}{\partial y} \int_{-h}^{\eta} \frac{U_* V_*}{k^2} \left(\ln \left(\frac{h+z}{h+\eta} \right) + 1 \right)^2 dz
\end{aligned}
\tag{2.40}$$

Integrating Eq. (2.40) by parts and expressing the shear velocity in its previously defined form, i.e. $U_* = \sqrt{g} U/C$, results in:-

$$X = \frac{\partial}{\partial x} \left(\frac{g}{C^2 k^2} (h + \eta) U^2 \right) + \frac{\partial}{\partial y} \left(\frac{g}{C^2 k^2} (h + \eta) UV \right) \tag{2.41}$$

and similarly, for the y direction:-

$$Y = \frac{\partial}{\partial x} \left(\frac{g(h + \eta) VU}{C^2 k^2} \right) + \frac{\partial}{\partial y} \left(\frac{g(h + \eta) V^2}{C^2 k^2} \right) \tag{2.42}$$

Eqs. (2.41) and (2.42) could now be substituted into the equations of motion (2.22) and (2.23) by adding them to the depth averaged convective accelerations; this would result in each accelerative term being multiplied by a coefficient $(1 + g/C^2 k^2)$. However, these non-linear terms pose complications in their numerical representation and restrain the scope of the mathematical model to be constructed - this

will be discussed later in Chapter 3. If the frictional effects are large, i.e. C is relatively small, then these complications are amplified due to the increased magnitude of the above coefficient, and for this reason a more suitable representation of Eq. (2.38) is necessary. If the velocities $\tilde{u}$, $\tilde{v}$ are defined as the departures from the spatial mean values U , V in the vertical direction, then:-

$$\tilde{u} = u - U \quad (2.43)$$

and $\tilde{v} = v - V$

Substituting in Eq. (2.38):-

$$X = \frac{\partial}{\partial x} \int_{-h}^{\eta} \tilde{u}^2 dz + \frac{\partial}{\partial y} \int_{-h}^{\eta} \tilde{u} \tilde{v} dz \quad (2.44)$$

Indicating average values over a cross-section by means of an over-bar, Eq. (2.44) becomes:-

$$X = \frac{\partial}{\partial x} \overline{\tilde{u}^2 (h + \eta)} + \frac{\partial}{\partial y} \overline{\tilde{u} \tilde{v} (h + \eta)} \quad (2.45)$$

The similarity between the terms in Eq. (2.45) and the corresponding Reynolds stresses in Eq. (2.31) is obvious. Therefore, it would seem possible to assume that the above terms may be represented in a similar horizontal diffusive manner as for the Reynolds stresses in section 2.3:-

$$\overline{u_i u_j} = -D_j \frac{\partial \overline{u_i}}{\partial x_j} \quad (2.46)$$

Unlike the mixing length theory, an estimation of the order of magnitude of D_j is not so straightforward. Elder (27), in analysing the dispersion of matter in a two-dimensional turbulent shear flow, arrived at similar terms in his depth averaged transport equation. For a solute, s , Elder assumed:-

$$\overline{su_j} = - D_\lambda \frac{\partial s}{\partial x_j} \quad (2.47)$$

and found the value of D_λ to be:-

$$D_\lambda = 5.86(h + \eta) \left[\bar{u}_* \right] \quad (2.48)$$

At first glance it seems reasonable to assume that the solute s and the velocity defect $\tilde{u}_i$ are transmitted in an identical manner, i.e. $s = \tilde{u}_i$ and $D_\lambda = D_j$. However, in comparing the mass transport and momentum equations, there are no terms in the former equation which incorporate any transfer due to pressure gradients. On the other hand, the analogy between momentum and mass transfer for the region near a wall has been well established, see Elder (27), Raudkivi and Callender (88). As this is the region where the departures in velocity from the spatial mean will be largest, it follows that this will also be the region where most of the corresponding momentum transfer will occur. Therefore, due to pressure gradient effects, D_λ and D_j are unlikely to be exactly equal but they should, at least, be of an equal order of magnitude. Using this assumption, the depth averaged convection effects in the x direction become:-

$$\begin{aligned}
X = & -5.86 \left[\frac{\partial}{\partial x} \left[(h + \eta)^2 |U_*| \frac{\partial U}{\partial x} \right] \right. \\
& \left. + \frac{\partial}{\partial y} \left[(h + \eta)^2 |V_*| \frac{\partial U}{\partial y} \right] \right]
\end{aligned} \tag{2.49}$$

and similarly, in the y direction:-

$$\begin{aligned}
Y = & -5.86 \left[\frac{\partial}{\partial x} \left[(h + \eta)^2 |U_*| \frac{\partial V}{\partial x} \right] \right. \\
& \left. + \frac{\partial}{\partial y} \left[(h + \eta)^2 |V_*| \frac{\partial V}{\partial y} \right] \right]
\end{aligned} \tag{2.50}$$

Approximating the convection effects of Eq. (2.38) in this manner tends to improve, rather than complicate, the computational difficulties associated with the numerical representation of the non-linear terms. This will be clarified in more detail in Chapter 3.

As the mean flow convection effects and the turbulent Reynolds stresses have been represented in a similar diffusive manner, their coefficients and terms may be added, in an approximate manner, to give the required effective stresses. Assuming that for Eq. (2.37) $\partial U / \partial y \approx \partial V / \partial x$ and that the spatial variations of the effective stresses are small, the two-dimensional depth averaged equations of motion can now be written in the form in which they will be solved. Substituting the Reynolds stresses, Eq. (2.37), and the mean flow convection effects, Eqs. (2.49) and (2.50), into Eqs. (2.22) and (2.23), the x direction momentum equation approximates to:-

$$\frac{\partial U}{\partial t} + \frac{\partial U^2}{\partial x} + \frac{\partial UV}{\partial y} - fV + g \frac{\partial \eta}{\partial x} - \frac{\rho_a C^* W W_x}{\rho (h + \eta)}$$

$$\begin{aligned}
& + \frac{gU|U+V|}{(h+\eta)c^2} - \frac{6\sqrt{g}(h+\eta)}{c} \left[|U| \frac{\partial^2 U}{\partial x^2} \right. \\
& \left. + |V| \frac{\partial^2 U}{\partial y^2} \right] = 0
\end{aligned} \tag{2.51}$$

and similarly, for the y direction momentum equation:-

$$\begin{aligned}
& \frac{\partial V}{\partial t} + \frac{\partial VU}{\partial x} + \frac{\partial V^2}{\partial y} + fU + g \frac{\partial \eta}{\partial y} - \frac{\rho_a C^* W_y}{\rho(h+\eta)} \\
& + \frac{gV|U+V|}{(h+\eta)c^2} - \frac{6\sqrt{g}(h+\eta)}{c} \left[|U| \frac{\partial^2 V}{\partial x^2} \right. \\
& \left. + |V| \frac{\partial^2 V}{\partial y^2} \right] = 0
\end{aligned} \tag{2.52}$$

For completeness, the continuity equation can be written as before:-

$$\frac{\partial \eta}{\partial t} + \frac{\partial [(h+\eta)U]}{\partial x} + \frac{\partial [(h+\eta)V]}{\partial y} = 0 \tag{2.53}$$

2.5 ANALYSIS OF THE DEPTH AVERAGED VORTICITY TRANSFER

Before attempting to solve the equations of motion and predict the circulation produced by a confined jet, it is first worthwhile fully understanding the hydrodynamic mechanism by which circulation can be both generated and maintained. For this purpose the depth averaged transport equation for vorticity, about a vertical axis, can be analysed in a manner similar to that used by Kuipers and Vreugdenhil (58), and Vreugdenhil (115).

Using Vreugdenhil's derivation, which can be simply outlined as:-

$$\begin{aligned}
\text{Vorticity Eq.} &= \frac{\partial}{\partial y} [\text{Eq. (2.51)} - \frac{U \times \text{Eq. (2.53)}}{(h + \eta)}] \\
&- \frac{\partial}{\partial x} [\text{Eq. (2.52)} - \frac{V \times \text{Eq. (2.53)}}{(h + \eta)}] \quad (2.54)
\end{aligned}$$

the vorticity equation becomes:-

$$\begin{aligned}
&\underbrace{\frac{\partial \omega}{\partial t}}_1 + \underbrace{\frac{\partial U \omega}{\partial x} + \frac{\partial V \omega}{\partial y}}_2 + \underbrace{(\omega - f) \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right)}_3 \\
&- \underbrace{\left[\frac{\partial}{\partial y} \left(\frac{\tau_{xw}}{\rho(h + \eta)} \right) - \frac{\partial}{\partial x} \left(\frac{\tau_{yw}}{\rho(h + \eta)} \right) \right]}_4 \\
&+ \underbrace{\left[\frac{\partial}{\partial y} \left(\frac{\tau_{xb}}{\rho(h + \eta)} \right) - \frac{\partial}{\partial x} \left(\frac{\tau_{yb}}{\rho(h + \eta)} \right) \right]}_5 \\
&- \frac{\partial}{\partial y} \underbrace{\left[\frac{6\sqrt{g}(h + \eta)}{c} \left(|U| \frac{\partial^2 U}{\partial x^2} + |V| \frac{\partial^2 U}{\partial y^2} \right) \right]}_6 \\
&+ \frac{\partial}{\partial x} \underbrace{\left[\frac{6\sqrt{g}(h + \eta)}{c} \left(|U| \frac{\partial^2 V}{\partial x^2} + |V| \frac{\partial^2 V}{\partial y^2} \right) \right]}_7 = 0 \quad (2.55)
\end{aligned}$$

where $\omega = \left(\frac{\partial U}{\partial y} - \frac{\partial V}{\partial x} \right)$

The terms in the above equation are defined by Vreugdenhil as follows:-

term 1 represents the time rate of change of vorticity, term 2 the convection of vorticity by the mean flow, term 3 the convergence or divergence of vorticity by the mean flow, term 4 the curl of the wind stress, term 5 the effects of bottom friction, and lastly, terms 6 and 7 represent the moments of the effective stresses about a vertical axis.

The actual generation of vorticity is only affected by the latter five of these terms, with the bottom friction effects, i.e. term 5, tending to dissipate any vorticity generated. Thus in the original equations of motion (2.51) and (2.52) there are only three sources of generating circulation, firstly the non-linear convective accelerations, secondly the wind stress and thirdly, the effective stresses in the vertical plane.

Vreugdenhil went on to show that if the velocity profiles are changing significantly over short distances then, not only the inertial terms but also the effective stresses are important. Consequently, as this research is concerned with the mathematical modelling of velocity fields where the velocities change appreciably over short distances, both these terms are included in the model and particular care has been taken in their numerical representation.

DEVELOPMENT OF THE FINITE DIFFERENCE EQUATIONS OF MOTION

3.1 SELECTING THE SOLUTION PROCEDURE

With the depth averaged differential equations of motion having been fully derived, the next stage in attempting to solve for the required velocity field is to select a procedure for transposing the differential equations into the form most suitable for numerical solution. Two solution procedures are at present available for this type of problem, they are the finite difference method and secondly, the finite element method.

In the case of the finite difference method space and time are represented, throughout the velocity field, by a set of grid points usually at equal distances in each direction (Δx and Δy) and at a chosen time step Δt . The behaviour of a variable at each grid point is described as a function of the variables at other grid points in such a manner that in the limit of zero increment these equations, called finite difference equations, are consistent with the differential equations to be solved.

For the finite element method, the domain enclosing the velocity field is sub-divided into a number of finite elements, usually triangular in shape. The dependent variables (U, V, η) are located at the corners, or nodal points, of each element and are inter-related by means of a set of linear simultaneous equations. The unknown coefficients of these simultaneous equations are found from the equations of motion and the boundary conditions for the enclosed velocity field.

For free surface flow problems the finite difference method of

solution has the advantage that it requires less computer time than the finite element method. Weare (117), in a study of the relative economics of both methods, concluded that the explicit and implicit finite difference methods require approximately 40 and 50 units of work per grid point per time step respectively; whereas, the explicit and implicit finite element methods require approximately $18\ m$ and $(\frac{27}{2} m^2 + 18\ m)$ units of work per nodal point per time step respectively - where m is the largest difference in node numbers in any element and i is the number of iterations per time step. Even though the finite element method generally requires fewer nodal points than corresponding grid points, Weare's conclusions show that for a satisfactory spatial representation of any enclosed body of water the finite difference method is nearly always cheaper.

With the finite difference method the approximation of the differential equations is relatively straightforward, although treatment of the boundary conditions is not so simple. The opposite is true with the finite element method, which is therefore well suited to approximate irregular boundaries but which produces difficulties in the approximation of the differential equations. Another advantage of the finite difference method is that it is easier to extend to the unsteady three-dimensional problem, which would tend to be the obvious extension of this research - see Spalding (105).

Bearing in mind the properties of the two methods briefly summarised above, it was decided early in the project to adopt the finite difference solution procedure. A further reason was the relative novelty of the finite element method in the field of hydraulics which would have required more time to be spent on the development of the method rather than its application. There have been no developments during the work which would suggest that the finite element method of solution would have any previously unforeseen advantages.

3.2 INTRODUCTION TO THE ANALYSIS OF FINITE DIFFERENCE SCHEMES

Although the conservation of mass and momentum for any confined velocity field may be represented by one set of partial differential equations, these equations can be written in many different finite difference forms. Consequently, for any given set of differential equations, it is desirable to be able to compare and contrast these finite difference schemes such that the most suitable representation of the original differential equations of motion, (2.51), (2.52) and (2.53), can be made.

Each scheme will have different numerical properties and they may be summarised under the headings of:-

- a. approximation (section 3.3)
- b. consistency (section 3.4)
- c. convergence (section 3.5)
- d. stability (sections 3.6 and 3.7)
- e. accuracy (section 3.8)

To briefly illustrate the properties of each scheme only the one-dimensional form of the equations of continuity and momentum will be considered giving, for continuity:-

$$\frac{\partial \eta}{\partial t} + h \frac{\partial U}{\partial x} = 0 \quad (3.1)$$

and for momentum:-

$$\begin{aligned} \frac{\partial U}{\partial t} + U_o \frac{\partial U}{\partial x} + g \frac{\partial \eta}{\partial x} - P W^2 + Q U|U| - R|U| \frac{\partial^2 U}{\partial x^2} \\ = 0 \end{aligned} \quad (3.2)$$

where $P = \frac{\rho_a C^*}{\rho(h + \eta)}$ represents the wind stress

$Q = \frac{g}{(h + \eta)C^2}$ represents the bottom friction

$R = \frac{6\sqrt{g}(h + \eta)}{C}$ represents the effective stress

U_0 = a constant local value of the velocity U

W = wind velocity

Most of the differences between the finite difference schemes are still apparent even if the frictional, diffusive and convective acceleration effects are ignored. For the purposes of comparison of the different schemes, the momentum equation in the form:-

$$\frac{\partial U}{\partial t} + g \frac{\partial \eta}{\partial x} = 0 \quad (3.3)$$

will be used. Equations (3.1) and (3.3) plus the initial values and boundary conditions completely describe wave motion in one dimension.

3.3 APPROXIMATION

The numerical property of approximation is a precise mathematical method for determining how well the finite difference equations approximate to the differential equations. The mathematical method used for this purpose is the Taylor series expansion of the derivatives of the differential equations. Any finite difference scheme will only consider a limited number - usually one - of the terms of the Taylor expansion thereby introducing a truncation error. If the finite difference equation is subtracted from the series expansion then an order of magnitude estimate of the leading truncation error can be made.

3.3.1 THE FINITE DIFFERENCE APPROXIMATION FOR ONE TERM

If a function $f(x)$ has single-valued, finite and continuous derivatives with respect to x , then Taylor's theorem gives:-

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2f''(x) + \frac{1}{6}h^3f'''(x) + \dots \quad (3.4)$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2f''(x) - \frac{1}{6}h^3f'''(x) + \dots \quad (3.5)$$

Three simple approximations of the first derivative are possible, the forward difference, the backward difference and the central difference - see Fig. 2.

From Eq. (3.4) the forward difference formula can be written as:-

$$f'(x) = \frac{df(x)}{dx} \approx \frac{1}{h} [f(x+h) - f(x)] \quad (3.6)$$

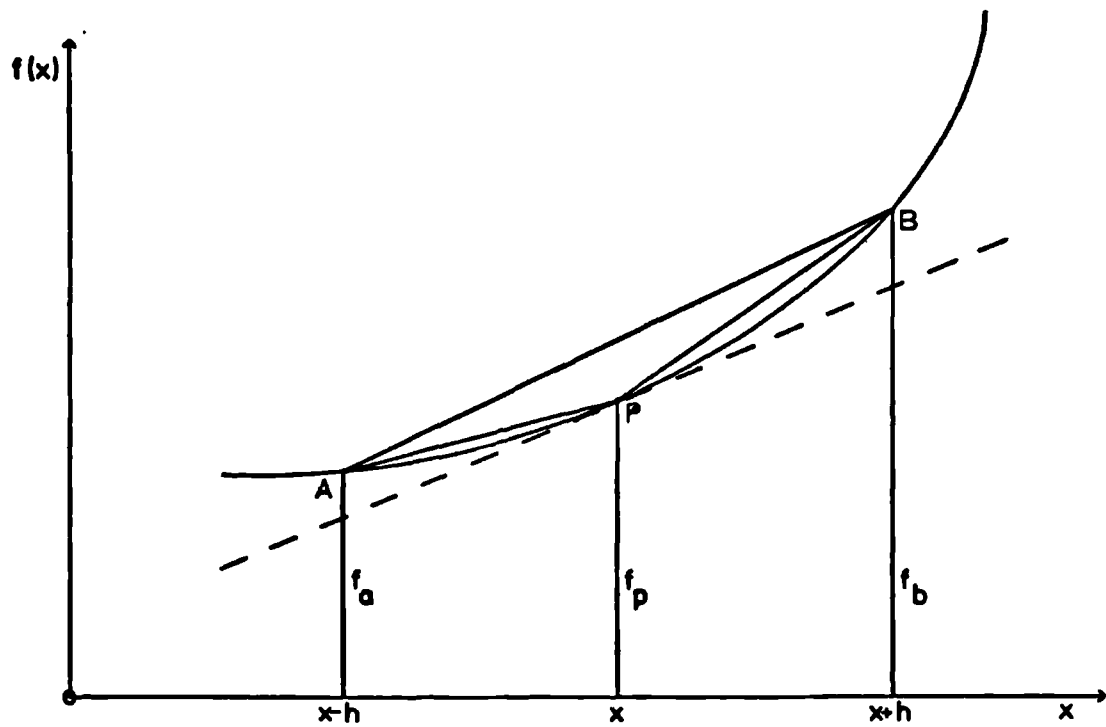
with a leading error on the right hand side of order h , i.e. $O(h)$.

Similarly, from Eq. (3.5), the backward difference formula can be written as:-

$$f'(x) = \frac{df(x)}{dx} \approx \frac{1}{h} [f(x) - f(x-h)] \quad (3.7)$$

again with a leading error $O(h)$. Subtracting Eq. (3.5) from (3.4) gives the central difference formula:-

$$f'(x) = \frac{df(x)}{dx} \approx \frac{1}{2h} [f(x+h) - f(x-h)] \quad (3.8)$$



Forward Difference Form With Respect to P:-

$$\left. \frac{df(x)}{dx} \right|_P \approx \frac{f_b - f_p}{h} + O(h)$$

Backward Difference Form With Respect to P:-

$$\left. \frac{df(x)}{dx} \right|_P \approx \frac{f_p - f_a}{h} + O(h)$$

Central Difference Form With Respect to P:-

$$\left. \frac{df(x)}{dx} \right|_P \approx \frac{f_b - f_a}{2h} + O(h^2)$$

FIGURE 2 Principle Finite Difference Forms

where the leading error is now $O(h^2)$. By using weighted combinations of these three difference formula it is possible to derive other, more complicated, difference approximations.

A leading error $O(h^2)$ indicates that a change in h to $\frac{1}{2}h$ will reduce the error to $\frac{1}{4}$ of its original value. Consequently, in any finite difference scheme central differences are preferable due to the associated smaller truncation errors. Figure 2 shows that for the curve $f(x)$, the tangential slope at P is better approximated by the central difference chord AB than either the forward or backward difference chords PB and AP respectively.

3.3.2 THE FINITE DIFFERENCE APPROXIMATION FOR THE WAVE EQUATIONS

With the three basic finite difference formula having been derived, it is now possible to express the orders of approximation of four finite difference schemes commonly used to represent the shallow water wave equations (3.1) and (3.3). The schemes include, the forward and central explicit and the backward and central implicit representations - the backward explicit and forward implicit schemes do not exist.

In an explicit method the unknown value is expressed directly in terms of known values. For the time dependent one-dimensional problem defined by Eqs. (3.1) and (3.3) this means that an unknown value at the advanced time level $(n + 1)\Delta t$ can be calculated directly from values at the known, present time level $n\Delta t$ and from values already calculated at $(n - 1)\Delta t$.

If the water level η and the velocity U are defined at the grid points j and $j + \frac{1}{2}$ respectively, the forward difference explicit representation of Eqs. (3.1) and (3.3) can be expressed as:-

$$\frac{1}{\Delta t} \left(\eta_j^{n+1} - \eta_j^n \right) + \frac{h}{\Delta x} \left(u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n \right) = 0 \quad (3.9)$$

$$\frac{1}{\Delta t} \left(u_{j+\frac{1}{2}}^{n+1} - u_{j+\frac{1}{2}}^n \right) + \frac{g}{\Delta x} \left(\eta_{j+1}^n - \eta_j^n \right) = 0 \quad (3.10)$$

and is illustrated in Fig. 3(a). In both these equations the finite difference approximations of the time and space derivatives are at the different time levels $n + \frac{1}{2}$ and n respectively, i.e. the equations are not time centred. Consequently, the scheme has a first order accuracy with a leading truncation error $O(\Delta t)$.

The central difference, or leap frog, explicit scheme can be written as:-

$$\frac{1}{\Delta t} \left(\eta_j^{n+1} - \eta_j^n \right) + \frac{h}{\Delta x} \left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0 \quad (3.11)$$

$$\frac{1}{\Delta t} \left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j+\frac{1}{2}}^{n-\frac{1}{2}} \right) + \frac{g}{\Delta x} \left(\eta_{j+1}^n - \eta_j^n \right) = 0 \quad (3.12)$$

and has a physical representation as shown in Fig. 3(b). In this case the difference approximations of the time and space derivatives are both fully centred, giving the scheme a second order accuracy and a truncation error of $O(\Delta t^2) + O(\Delta x^2)$.

For implicit schemes the finite difference structure is somewhat different. The value of the computed solution at any particular grid point depends on the solution at the adjacent grid points and for the same time level, unlike explicit schemes where the unknown value is directly expressed in terms of known values. Because of this difference

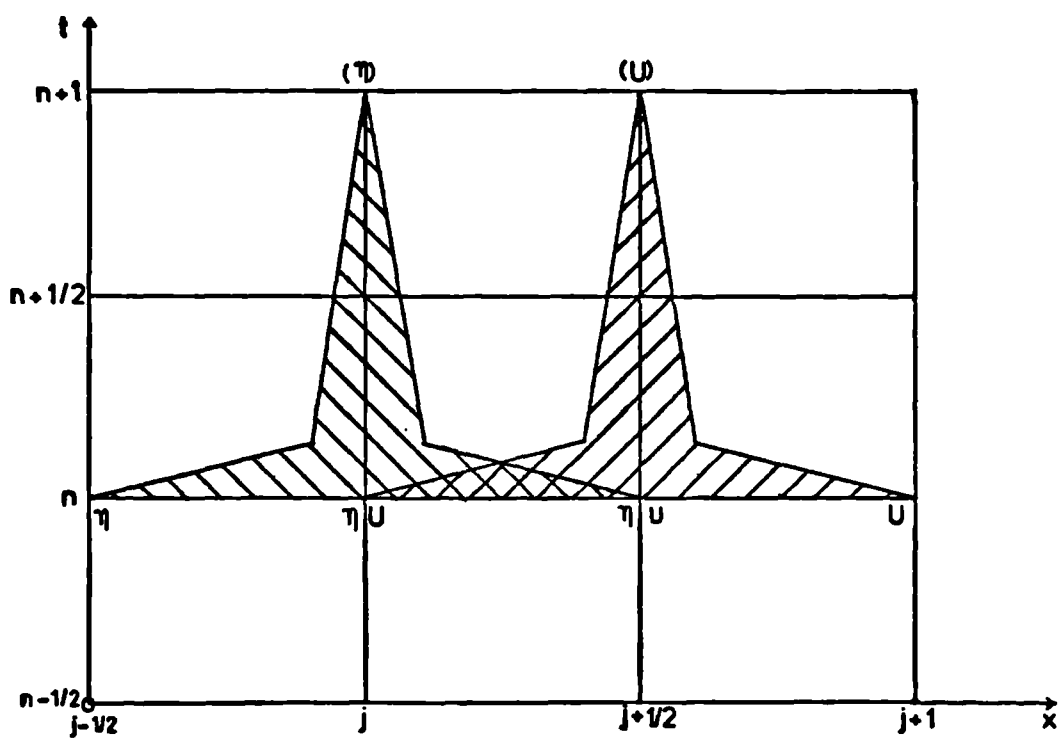


FIGURE 3(a) Forward Explicit Scheme

() UNKNOWN TERM

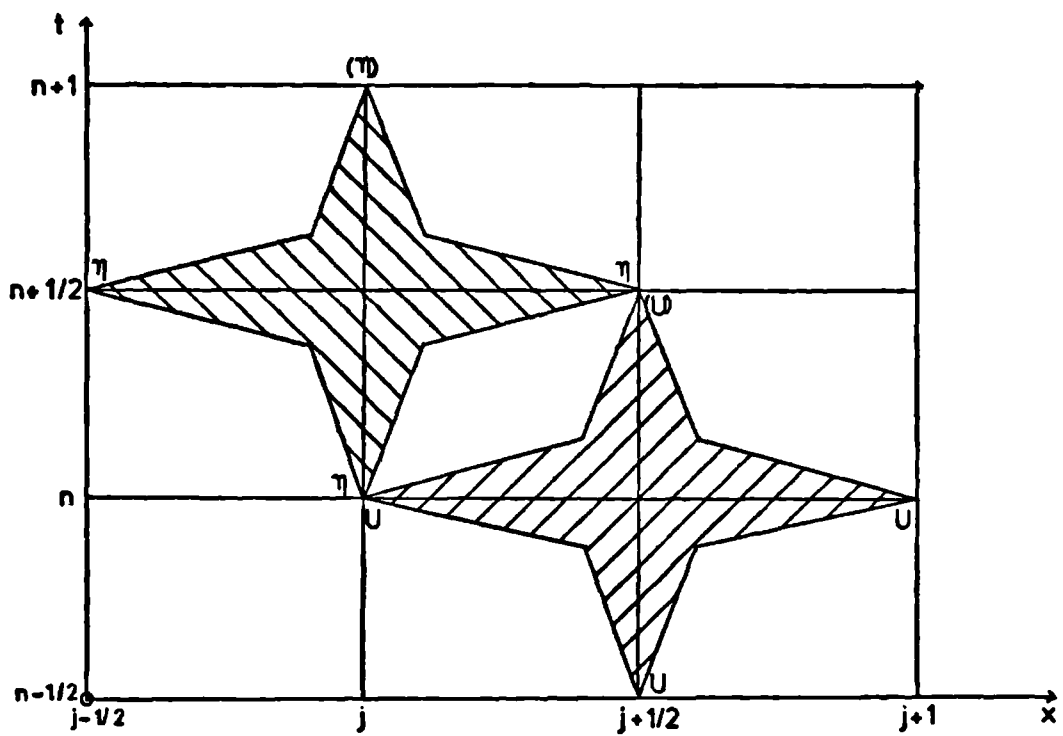


FIGURE 3(b) Central Explicit Scheme

FIGURE 3 Structure of Four Finite Difference Schemes

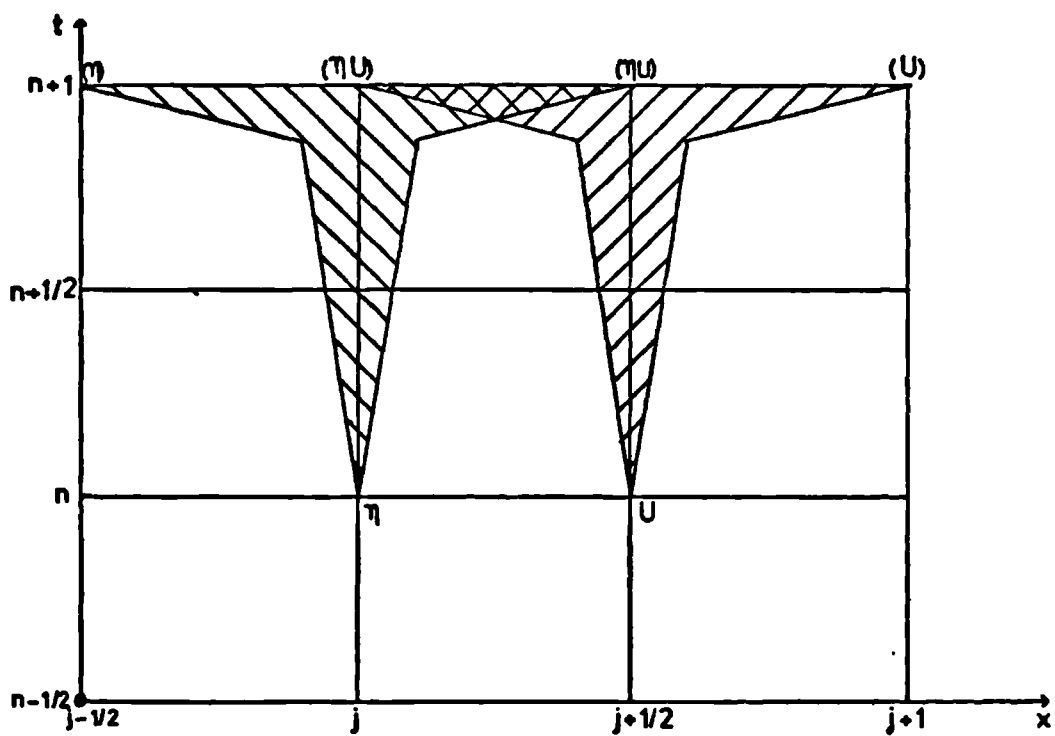


FIGURE 3(c) Backward Implicit Scheme

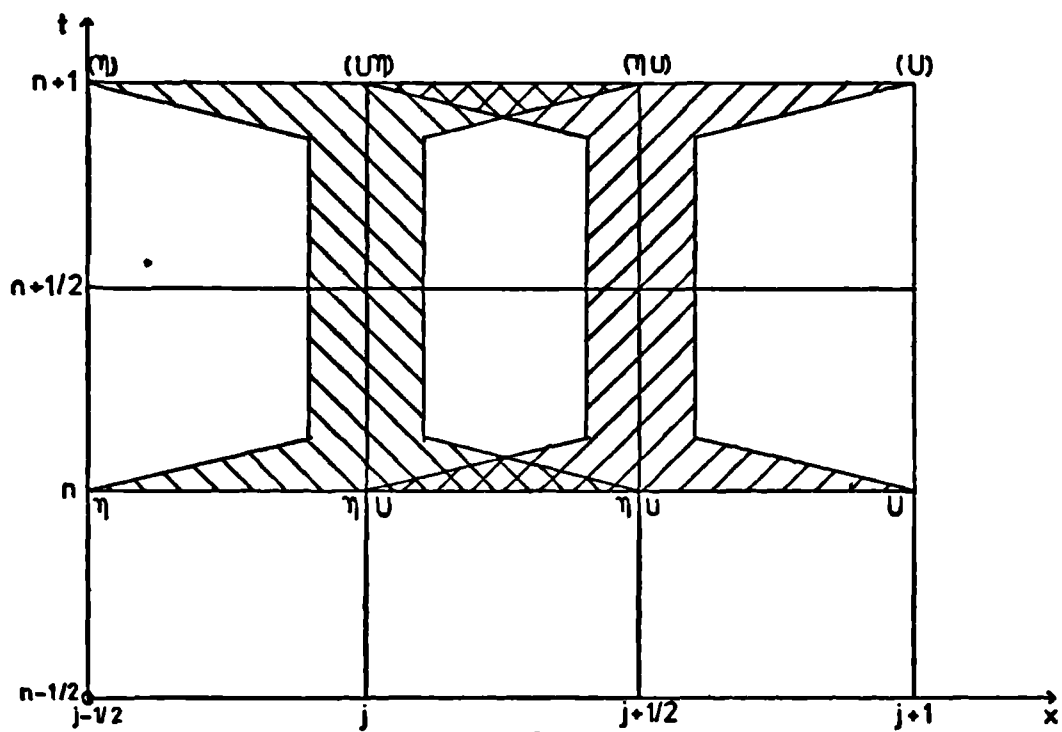


FIGURE 3(d) Central Implicit Scheme

FIGURE 3 Continued

the equations of motion for an implicit scheme cannot be solved directly for the required solutions of η and U , as with explicit schemes, but must be solved simultaneously. This, more complex, method of solution accounts for the higher computational cost per grid point per time step concluded by Weare (117) and discussed previously in section 3.1.

The backward difference implicit scheme, representing Eqs. (3.1) and (3.3), illustrates this differing representation:-

$$\frac{1}{\Delta t} \left(\eta_j^{n+1} - \eta_j^n \right) + \frac{h}{\Delta x} \left(U_{j+\frac{1}{2}}^{n+1} - U_{j-\frac{1}{2}}^{n+1} \right) = 0 \quad (3.13)$$

$$\frac{1}{\Delta t} \left(U_{j+\frac{1}{2}}^{n+1} - U_{j+\frac{1}{2}}^n \right) + \frac{g}{\Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} \right) = 0 \quad (3.14)$$

which has a space-time structure as shown in Fig. 3(c). Again the difference approximations of the time derivatives are not centred, resulting in a first order scheme with truncation error $O(\Delta t)$.

Finally, the central difference, or Crank-Nicolson, implicit scheme representing Eqs. (3.1) and (3.3) can be written as:-

$$\frac{1}{\Delta t} \left(\eta_j^{n+1} - \eta_j^n \right) + \frac{h}{2\Delta x} \left(U_{j+\frac{1}{2}}^{n+1} - U_{j-\frac{1}{2}}^{n+1} + U_{j+\frac{1}{2}}^n - U_{j-\frac{1}{2}}^n \right) = 0 \quad (3.15)$$

$$\frac{1}{\Delta t} \left(U_{j+\frac{1}{2}}^{n+1} - U_{j+\frac{1}{2}}^n \right) + \frac{g}{2\Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} + \eta_{j+1}^n - \eta_j^n \right) = 0 \quad (3.16)$$

which is a combination of the forward explicit and backward implicit schemes and is illustrated in Fig. 3(d). As the finite difference

equations are time centred by averaging the spatial differences between the forward and backward times, this scheme has a second order accuracy with a truncation error $O(\Delta t^2) + O(\Delta x^2)$.

It has already been shown that in order to represent a differential equation as accurately as possible by finite differences then a second order, i.e. fully centred, difference scheme should be used. Consequently, at this stage one is led to conclude that the central difference explicit and implicit, or some form of either scheme, are most suitable for accurately representing the shallow water wave equations (3.1) and (3.3).

3.4 CONSISTENCY

For any finite difference scheme with a centre point, j,n say, it is usually desirable that in the limit as $\Delta x, \Delta t \rightarrow 0$ the difference equation should contract, about the centre point, to the original differential equation, i.e. the truncation error terms should disappear. If this requirement is satisfied the difference scheme is said to be consistent with the differential equation, or unconditionally consistent should the error terms tend to zero regardless of how $\Delta x, \Delta t \rightarrow 0$. On the other hand, if the error terms do not disappear as $\Delta x, \Delta t \rightarrow 0$ then the solution to the inconsistent difference scheme will be the solution to some different, and not the original, differential equation.

In some cases, particularly where the physical problem cannot be adequately represented by the appropriate differential equations - usually due to appreciable energy losses - the use of inconsistent finite difference schemes can often be advantageous. A deliberately inconsistent scheme can be constructed such that, under limiting conditions, residual terms arise which act in a pseudo-viscous manner to dissipate

the energy of high frequency disturbances. A classic example of the frequent use of inconsistent difference schemes is in computing the floodwave profile of a breaking dam, see Liggett and Cunge (69).

However, in solving the shallow water wave equations (3.1) and (3.3), where energy losses are relatively small, the use of an inconsistent finite difference scheme is undesirable, particularly if mass and momentum are to be conserved. From the outset, it is therefore essential to ensure that any difference schemes considered should be consistent with the differential equations that they are to represent. Using the central difference, or leap-frog, explicit scheme of section 3.3 as an example in checking for consistency, the Eqs. (3.11) and (3.12) can be rewritten as:-

$$\eta_j^{n+1} = \eta_j^n - \frac{h\Delta t}{\Delta x} \left(U_{j+\frac{1}{2}}^{n+\frac{1}{2}} - U_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \quad (3.17)$$

$$U_{j+\frac{1}{2}}^{n+\frac{1}{2}} = U_{j+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{g\Delta t}{\Delta x} \left(\eta_{j+1}^n - \eta_j^n \right) \quad (3.18)$$

The individual terms in Eq. (3.17) do not correspond to the centre point $j, n+\frac{1}{2}$ and therefore need to be expanded, using Taylor's series, in terms of their corresponding values at the point $j, n+\frac{1}{2}$. The terms when expanded give:-

$$\begin{aligned} \eta_j^{n+1} &= \eta_j^{n+\frac{1}{2}} + \frac{1}{2} \left(\frac{\partial \eta}{\partial t} \right)_j^{n+\frac{1}{2}} \Delta t + \frac{1}{4} \left(\frac{\partial^2 \eta}{\partial t^2} \right)_j^{n+\frac{1}{2}} \frac{\Delta t^2}{2!} \\ &+ \frac{1}{8} \left(\frac{\partial^3 \eta}{\partial t^3} \right)_j^{n+\frac{1}{2}} \frac{\Delta t^3}{3!} + \dots \end{aligned}$$

$$\begin{aligned}
\eta_j^n &= \eta_j^{n+\frac{1}{2}} - \frac{1}{2} \left(\frac{\partial \eta}{\partial t} \right)_j^{n+\frac{1}{2}} \Delta t + \frac{1}{4} \left(\frac{\partial^2 \eta}{\partial t^2} \right)_j^{n+\frac{1}{2}} \frac{\Delta t^2}{2!} \\
&- \frac{1}{8} \left(\frac{\partial^3 \eta}{\partial t^3} \right)_j^{n+\frac{1}{2}} \frac{\Delta t^3}{3!} + \dots \\
U_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= U_j^{n+\frac{1}{2}} + \frac{1}{2} \left(\frac{\partial U}{\partial x} \right)_j^{n+\frac{1}{2}} \Delta x + \frac{1}{4} \left(\frac{\partial^2 U}{\partial x^2} \right)_j^{n+\frac{1}{2}} \frac{\Delta x^2}{2!} \\
&+ \frac{1}{8} \left(\frac{\partial^3 U}{\partial x^3} \right)_j^{n+\frac{1}{2}} \frac{\Delta x^3}{3!} + \dots \\
U_{j-\frac{1}{2}}^{n+\frac{1}{2}} &= U_j^{n+\frac{1}{2}} - \frac{1}{2} \left(\frac{\partial U}{\partial x} \right)_j^{n+\frac{1}{2}} \Delta x + \frac{1}{4} \left(\frac{\partial^2 U}{\partial x^2} \right)_j^{n+\frac{1}{2}} \frac{\Delta x^2}{2!} \\
&- \frac{1}{8} \left(\frac{\partial^3 U}{\partial x^3} \right)_j^{n+\frac{1}{2}} \frac{\Delta x^3}{3!} + \dots
\end{aligned}$$

Substituting in Eq. (3.17) and dividing by Δt , this gives:-

$$\begin{aligned}
\left(\frac{\partial \eta}{\partial t} \right)_j^{n+\frac{1}{2}} + \frac{1}{4} \left(\frac{\partial^3 \eta}{\partial t^3} \right)_j^{n+\frac{1}{2}} \frac{\Delta t^2}{3!} + \dots &= -h \left[\left(\frac{\partial U}{\partial x} \right)_j^{n+\frac{1}{2}} \right. \\
&\left. + \frac{1}{4} \left(\frac{\partial^3 U}{\partial x^3} \right)_j^{n+\frac{1}{2}} \frac{\Delta x^2}{3!} + \dots \right]
\end{aligned} \tag{3.19}$$

which can be written as:-

$$\begin{aligned}
\left[\frac{\partial \eta}{\partial t} + h \frac{\partial U}{\partial x} \right] + \left[\frac{1}{4} \left(\frac{\partial^3 \eta}{\partial t^3} \right) \frac{\Delta t^2}{3!} + \dots \right. \\
\left. + \frac{h}{4} \left(\frac{\partial^3 U}{\partial x^3} \right) \frac{\Delta x^2}{3!} + \dots \right] = 0 \text{ at } j, n+\frac{1}{2}
\end{aligned} \tag{3.20}$$

Equation (3.20) is simply another way of writing equation (3.17), with the second square bracketed term being the truncation error. In the limit as $\Delta x, \Delta t \rightarrow 0$, regardless of the manner in which they do so, the truncation error will also tend to zero. Consequently, as the truncation error tends to zero the difference equation (3.17) will converge, about the centre point $j, n+\frac{1}{2}$, to the original differential equation (3.1). In a similar manner the difference equation (3.18) can be shown to converge, about the centre point $j+\frac{1}{2}, n$, to the differential equation (3.3). With both these difference equations converging to their corresponding differential equations, this central difference explicit scheme satisfies the requirement of being unconditionally consistent with the original shallow water wave equations (3.1) and (3.3).

The three other finite difference schemes discussed in section (3.3) can also be shown to be unconditionally consistent with the original differential equations (3.1) and (3.3). Thus, from the standpoint of consistency, any one of these four basic difference schemes may be used to represent the shallow water wave equations without any residual terms arising through the truncation errors.

3.5 CONVERGENCE

Although a difference scheme may be consistent with the corresponding differential equations, the solution to the former need not necessarily converge to that of the latter as $\Delta x, \Delta t \rightarrow 0$. It is possible, for certain values of the time step Δt and the grid size Δx , that the numerical solution will either become unstable or may even converge to an incorrect analytical solution. To ensure that this does not happen any proposed numerical scheme must be analytically tested such that in the limit, as $\Delta x, \Delta t \rightarrow 0$, the convergence properties of its solution are known.

The manner in which convergence is generally tested for can be best illustrated by again analysing the central difference explicit scheme of section 3.3. Writing the equations as before, this gives:-

$$\eta_j^{n+1} = \eta_j^n - \frac{h\Delta t}{\Delta x} \left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \quad (3.21)$$

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} = u_{j+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{g\Delta t}{\Delta x} \left(\eta_{j+1}^n - \eta_j^n \right) \quad (3.22)$$

If the exact solutions to the differential equations (3.1) and (3.3) and the difference equations (3.21) and (3.22) are denoted by H, u and η, U respectively, then the corresponding differences between these solutions can be defined as $e = H - \eta$ and $f = u - U$. These differences between the analytical and the numerical solutions are called discretization errors and can be substituted into Eqs. (3.21) and (3.22) to give:-

$$\begin{aligned} e_j^{n+1} = e_j^n + \left(H_j^{n+1} - H_j^n \right) - \frac{h\Delta t}{\Delta x} \left(f_{j+\frac{1}{2}}^{n+\frac{1}{2}} - f_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right. \\ \left. - u_{j+\frac{1}{2}}^{n+\frac{1}{2}} + u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} f_{j+\frac{1}{2}}^{n+\frac{1}{2}} = f_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j+\frac{1}{2}}^{n-\frac{1}{2}} \right) - \frac{g\Delta t}{\Delta x} \left(e_{j+1}^n - e_j^n \right. \\ \left. - H_{j+1}^n + H_j^n \right) \end{aligned} \quad (3.24)$$

Expanding the analytical solutions in terms of their corresponding Taylor's series, as for section 3.4, and substituting into Eqs. (3.23) and (3.24), this gives:-

$$\begin{aligned}
e_j^{n+1} &= e_j^n - \frac{h\Delta t}{\Delta x} \left(f_{j+\frac{1}{2}}^{n+\frac{1}{2}} - f_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \\
&+ \Delta t \left[\frac{\partial H^{n+\frac{1}{2}}}{\partial t_j} + \dots + \frac{h\partial u^{n+\frac{1}{2}}}{\partial x_j} + \dots \right] \quad (3.25)
\end{aligned}$$

and:-

$$\begin{aligned}
f_{j+\frac{1}{2}}^{n+\frac{1}{2}} &= f_{j+\frac{1}{2}}^{n-\frac{1}{2}} - \frac{g\Delta t}{\Delta x} \left(e_{j+1}^n - e_j^n \right) \\
&+ \Delta t \left[\frac{\partial u^n}{\partial t_{j+\frac{1}{2}}} + \dots + \frac{g\partial H^n}{\partial x_{j+\frac{1}{2}}} + \dots \right] \quad (3.26)
\end{aligned}$$

Substituting Eq. (3.26) in Eq. (3.25) to eliminate $f_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ leads to:-

$$\begin{aligned}
e_j^{n+1} - h \frac{\Delta t}{\Delta x} f_{j-\frac{1}{2}}^{n+\frac{1}{2}} &= \left(1 - gh \frac{\Delta t^2}{\Delta x^2} \right) e_j^n + gh \frac{\Delta t^2}{\Delta x^2} e_{j+1}^n \\
&- h \frac{\Delta t}{\Delta x} f_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \Delta t \left[\frac{\partial H^{n+\frac{1}{2}}}{\partial t_j} + \dots + h \frac{\partial u^{n+\frac{1}{2}}}{\partial x_j} + \dots \right] \\
&- h \frac{\Delta t^2}{\Delta x} \left[\frac{\partial u^n}{\partial t_{j+\frac{1}{2}}} + \dots + g \frac{\partial H^n}{\partial x_{j+\frac{1}{2}}} + \dots \right] \quad (3.27)
\end{aligned}$$

If $gh \frac{\Delta t^2}{\Delta x^2} \leq 1$ then all the coefficients of e , as well as f , will be positive such that Eq. (3.27) can be written as:-

$$\begin{aligned}
\left| e_j^{n+1} \right| - h \frac{\Delta t}{\Delta x} \left| f_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right| &\leq \left(1 - gh \frac{\Delta t^2}{\Delta x^2} \right) \left| e_j^n \right| \\
&+ gh \frac{\Delta t^2}{\Delta x^2} \left| e_{j+1}^n \right| - h \frac{\Delta t}{\Delta x} \left| f_{j+\frac{1}{2}}^{n-\frac{1}{2}} \right| + \Delta t M \\
&- h \frac{\Delta t^2}{\Delta x} N \quad (3.28)
\end{aligned}$$

where M and N denote the moduli of the maximum values of the respective square bracketed expressions in Eq. (3.27). If E^n and $F^{n-\frac{1}{2}}$ represent the modulus of the maximum error at the n and $n-\frac{1}{2}$ time level respectively, then Eq. (3.28) simplifies to:-

$$\begin{aligned} \left| e_j^{n+1} \right| - h \frac{\Delta t}{\Delta x} \left| f_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right| &\leq \left(1 - gh \frac{\Delta t^2}{\Delta x^2} \right) E^n \\ &+ gh \frac{\Delta t^2}{\Delta x^2} E^n - h \frac{\Delta t}{\Delta x} F^{n-\frac{1}{2}} + \Delta t M - h \frac{\Delta t^2}{\Delta x} N \\ &\leq E^n - h \frac{\Delta t}{\Delta x} F^{n-\frac{1}{2}} + \Delta t M - h \frac{\Delta t^2}{\Delta x} N \end{aligned}$$

As this is true for all time levels, then:-

$$\begin{aligned} E^{n+1} - h \frac{\Delta t}{\Delta x} F^{n+\frac{1}{2}} &\leq E^n - h \frac{\Delta t}{\Delta x} F^{n-\frac{1}{2}} + \Delta t M - h \frac{\Delta t^2}{\Delta x} N \\ &\leq E^{n-1} - h \frac{\Delta t}{\Delta x} F^{n-\frac{3}{2}} + 2\Delta t M - 2h \frac{\Delta t^2}{\Delta x} N \end{aligned}$$

etc., from which it follows:-

$$\begin{aligned} E^n - h \frac{\Delta t}{\Delta x} F^{n-\frac{1}{2}} &\leq E^0 - h \frac{\Delta t}{\Delta x} F^0 + n\Delta t M \\ &- nh \frac{\Delta t^2}{\Delta x} N \end{aligned} \quad (3.29)$$

With the previous assumption that $\frac{\Delta x}{\Delta t} \geq \sqrt{gh}$, then Δt cannot tend to zero faster than Δx . Consequently, in the limit as $\Delta x, \Delta t \rightarrow 0$ so too will $h \frac{\Delta t}{\Delta x}$ and $h \frac{\Delta t^2}{\Delta x}$ such that the expressions M and N reduce to:-

$$\left(\frac{\partial H}{\partial t} + h \frac{\partial u}{\partial x} \right)_j^{n+\frac{1}{2}} \quad \text{and} \quad \left(\frac{\partial u}{\partial t} + g \frac{\partial H}{\partial x} \right)_{j+\frac{1}{2}}^n \quad \text{respectively, which,}$$

on substituting in Eqs. (3.1) and (3.3), can both be equated to zero.

This simplifies Eq. (3.29) to:-

$$E^n \leq E^0 \quad (3.30)$$

However, as $e_j^n = H_j^n - \eta_j^n$ then $E^n \geq |H_j^n - \eta_j^n|$ so that Eq. (3.30) can be rewritten in the form:-

$$|H_j^n - \eta_j^n| \leq |H_j^0 - \eta_j^0| \quad (3.31)$$

This proves that as $\Delta x, \Delta t \rightarrow 0$, when $gh \frac{\Delta t^2}{\Delta x^2} \leq 1$, then the discretization error $(H - \eta)$ must decay with time, i.e. the numerical solution η must converge to the analytical solution H . Substituting Eq. (3.30) into Eq. (3.29) and multiplying throughout by $\frac{\Delta x}{h\Delta t}$ it can also be shown that U converges to u for the same set of conditions.

Applying the same analytical treatment of convergence to the three other basic finite difference schemes discussed in section (3.3), leads to the conclusion that the forward difference explicit scheme does not converge, whereas the two remaining schemes do - unconditionally.

If the method of estimating the convergence used previously were applied to the full equations, including the diffusive and convective acceleration terms, the analysis would be extremely laborious. For this reason, the various finite difference schemes considered in deciding how best to numerically represent the differential equations (3.1) and (3.2) were not analytically treated for convergence in this way. Instead, use was made of the Equivalence theorem (see Ames (7)), which states that

If the difference scheme is stable and consistent with the differential equation then its solution will converge to that of the differential equation.

3.6 STABILITY

In the previous section on convergence mention was made of the exact numerical solution U , which, in theory, could only be determined if it were possible to carry out all the numerical calculations to an infinite number of decimal places. However, in practice each calculation is evaluated to a finite number of decimal places, or significant figures, and so a round-off error has to be introduced for each individual calculation. This results in the computed solution being different from the exact numerical solution U , and it is the behaviour of this difference that gives rise to the study of the stability of finite difference schemes, see Richtmeyer and Morton (90). Generally, a set of difference equations will be stable if the cumulative effect of all the rounding errors is negligible. If, on the other hand, the difference scheme is such that the round-off errors are amplified as the computation progresses, then the correct solution to the unstable scheme will be swamped by the exponential growth of these errors. Clearly, this leads to an unsatisfactory solution and it is therefore necessary that all finite difference schemes should be checked for stability from the commencement.

To investigate analytically the stability of the finite difference schemes available to represent the shallow water wave equations (3.1) and (3.2), two standard methods can be used. The one method involves expressing the equations of motion in matrix form and examining the

eigenvalues of the associated amplification matrix; in the other method the eigenvalues are determined by using a finite Fourier series.

Although the first method is the more rigorous, in that it includes the boundary conditions, the latter approach is much simpler, and for this reason has been the method used to test for stability throughout the development of the present mathematical model.

3.6.1 STABILITY OF THE SIMPLIFIED WAVE EQUATIONS

Application of the Fourier stability method, introduced by von Neumann in 1951 (90), can be illustrated by applying it to the central difference explicit scheme. For convenience the appropriate finite difference equations (3.11) and (3.12) can be rewritten giving:-

$$\begin{pmatrix} \eta_j^{n+1} - \eta_j^n \end{pmatrix} + h \frac{\Delta t}{\Delta x} \begin{pmatrix} u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \end{pmatrix} = 0 \quad (3.32)$$

$$\begin{pmatrix} u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j+\frac{1}{2}}^{n-\frac{1}{2}} \end{pmatrix} + g \frac{\Delta t}{\Delta x} \begin{pmatrix} \eta_{j+1}^n - \eta_j^n \end{pmatrix} = 0 \quad (3.33)$$

The first step in determining whether and under what conditions this particular scheme is stable is to assume that there exists some initial error for both η and U at every grid point between $x = 0$ and $x = \ell$ at time $t = 0$. These error lines η_* and U_* can then be respectively decomposed into a finite, complex exponential, Fourier series as follows (see Sokolnikoff and Redheffer (101)):-

$$\eta_{*j}^n = \sum_{j=0}^N A_j e^{i\beta_j \Delta x}, \quad (j = 0, 1, 2 \dots N) \quad (3.34)$$

where N = total number of grid points

A_j = coefficients of the Fourier series

β = wavenumber = $2\pi/L$, where L is the wavelength

$i = \sqrt{-1}$

and $j\Delta x$ gives the co-ordinates of each point as $j = 0, 1, 2 \dots N$. A similar series relationship can be developed for U_{*j}^n .

As the finite difference equations (3.32) and (3.33) are linear, and therefore their separate solutions additive, consideration need only be given to the propagation of the error due to a single term, such as $A_j e^{i\beta j\Delta x}$, see Smith (95). Also, if the constant coefficient A_j is expressed purely in terms of the corresponding error that governs its value, the Eq. (3.34) may be simplified to give:-

$$\eta_{*j}^n = \eta_* e^{i\beta j\Delta x} \quad (3.35)$$

To investigate the propagation of this error function as the time t increases, another finite Fourier series can be developed such that when $t = 0$ the solution reduces to that of Eq. (3.35). Using the same procedure as before, the final time and space dependent error function η_{*j}^n reduces to:-

$$\eta_{*j}^n = \eta_* e^{i\beta j\Delta x} e^{i\sigma n\Delta t} \quad (3.36)$$

where σ = wave frequency = $2\pi/T$, where T is the wave period, and $n\Delta t$ gives the time level at $n = 0, 1, 2 \dots$ etc.

The line of errors η_{*j}^n , defined by Eq. (3.36), and U_{*j}^n will only

decay with time if the modulus of the so-called amplification factor λ , or $e^{i\sigma t}$, is always less than or equal to unity. This is the von Neumann condition for stability, which mathematically states that a difference scheme will only be stable if:-

$$|\lambda| = |e^{i\sigma t}| \leq 1 \quad (3.37)$$

As the error functions η_{*j}^n and U_{*j}^n will satisfy the same difference equations as η_j^n and U_j^n then substituting Eq. (3.36), and a similar expression for U_{*j} , into Eqs. (3.32) and (3.33) leads to:-

$$\begin{aligned} & \left(e^{i\sigma(n+1)\Delta t} - e^{i\sigma n\Delta t} \right) e^{i\beta j\Delta x} \eta_{*} + h \frac{\Delta t}{\Delta x} \left(e^{i\beta(j+\frac{1}{2})\Delta x} \right. \\ & \left. - e^{i\beta(j-\frac{1}{2})\Delta x} \right) e^{i\sigma(n+\frac{1}{2})\Delta t} U_{*} = 0 \end{aligned} \quad (3.38)$$

$$\begin{aligned} & \left(e^{i\sigma(n+\frac{1}{2})\Delta t} - e^{i\sigma(n-\frac{1}{2})\Delta t} \right) e^{i\beta(j+\frac{1}{2})\Delta x} U_{*} + g \frac{\Delta t}{\Delta x} \left(e^{i\beta(j+1)\Delta x} \right. \\ & \left. - e^{i\beta j\Delta x} \right) e^{i\sigma n\Delta t} \eta_{*} = 0 \end{aligned} \quad (3.39)$$

Dividing Eq. (3.38) by $e^{i(\beta j\Delta x + \sigma n\Delta t)}$ and Eq. (3.39) by $e^{i(\beta(j+\frac{1}{2})\Delta x + \sigma(n+\frac{1}{2})\Delta t)}$ and substituting λ - the amplification factor - for $e^{i\sigma\Delta t}$ gives:-

$$(\lambda - 1)\eta_{*} + \sqrt{\lambda} U_{*} h \frac{\Delta t}{\Delta x} 2i \sin \theta = 0 \quad (3.40)$$

$$(\lambda - 1)U_{*} + \sqrt{\lambda} \eta_{*} g \frac{\Delta t}{\Delta x} 2i \sin \theta = 0 \quad (3.41)$$

where $\theta = \frac{1}{2}\beta\Delta x$. Eliminating η_* from Eq. (3.40) and Eq. (3.41) gives the following quadratic equation for λ :-

$$\lambda^2 - (2 - 4\mu^2\sin^2\theta)\lambda + 1 = 0 \quad (3.42)$$

where $\mu = \sqrt{gh} \frac{\Delta t}{\Delta x}$ - is the Courant number. Thus the amplification factor for the scheme is, solving Eq. (3.42):-

$$\lambda = (1 - 2\mu^2\sin^2\theta) \pm 2i\mu\sin\theta(1 - \mu^2\sin^2\theta)^{\frac{1}{2}} \quad (3.43)$$

from which it can be shown that if $\mu\sin\theta \leq 1$ then $|\lambda| \leq 1$ and the scheme is stable. Since stability is necessary for all possible wavelengths of the error function, then this constraint must hold true for all values of θ . The maximum value of $\sin\theta$ occurs when $\theta = \pi/2$, which corresponds to a wavelength of $2\Delta x$, i.e. the shortest possible wavelength. Therefore, the round-off errors introduced in solving these central difference explicit equations will only decay with time, and so remain stable, if the following stability requirement is satisfied:-

$$\mu \leq 1 \text{ or } \frac{\Delta t}{\Delta x} \sqrt{gh} \leq 1 \quad (3.44)$$

This stability requirement is rather severe for the type of flow fields to be mathematically modelled. For example, to model the tidal flow in a rectangular harbour with a mean depth h of 10 m and a grid size Δx of 100 m, a maximum time step of only 10 seconds can be used. To represent a complete tidal cycle, of period $12\frac{1}{2}$ hours say, this would require 4500 time steps. Consequently, this particular difference

scheme tends to be computationally expensive and for this reason is not an attractive method of solving the shallow water wave equations.

For the forward difference explicit scheme, given by Eqs. (3.9) and (3.10) in section 3.3, the amplification factor λ , for the term $e^{i\sigma\Delta t}$, can be determined in a similar manner giving:-

$$\lambda = 1 \pm 2i\mu \sin \theta \quad (3.45)$$

where:-

$$|\lambda| = (1 + 4\mu^2 \sin^2 \theta)^{\frac{1}{2}} \quad (3.46)$$

As the term $4\mu^2 \sin^2 \theta$ in Eq. (3.46) will always be positive then $|\lambda| \geq 1$, and the stability requirement $|\lambda| \leq 1$, given by Eq. (3.37), will only be true for the limiting case where $\Delta x = 0$. Consequently, this explicit scheme is always unstable and can therefore be disregarded as a means of representing the original differential equations (3.1) and (3.3).

In the case of the backward difference implicit scheme, given by Eqs. (3.13) and (3.14) in section 3.3, the amplification factor λ can be obtained in the same way to give:-

$$\lambda = (1 \pm 2i\mu \sin \theta)^{-1} \quad (3.47)$$

where:-

$$|\lambda| = (1 + 4\mu^2 \sin^2 \theta)^{-\frac{1}{2}} \quad (3.48)$$

Consequently, as $4\mu^2 \sin^2 \theta$ will always be positive then $|\lambda| \leq 1$, and so this scheme will be stable for all values of the Courant number μ , i.e.

for any grid size and time step. With the amplification factor always being less than unity, except for $\Delta x = 0$, the original line of errors introduced into the difference equations will be continuously damped as time progresses.

Finally, for the central difference implicit (Crank-Nicolson) scheme, given by Eqs. (3.15) and (3.16), the amplification factor λ can readily be shown to be:-

$$\lambda = \frac{(1 - \mu^2 \sin^2 \theta) \pm 2i\mu \sin \theta}{(1 + \mu^2 \sin^2 \theta)} \quad (3.49)$$

where:-

$$|\lambda| = 1$$

for all wavelengths. This scheme is therefore unconditionally stable, with the original round-off errors neither growing nor decaying with time.

On comparing the stability properties of these four separate difference schemes, definite conclusions can be made in determining the most suitable difference representations of the shallow water wave equations (3.1) and (3.3). Firstly, the forward difference explicit scheme can be disregarded due to its unstable characteristics for all Courant numbers, i.e. μ values. Secondly, the central difference explicit scheme, with its stability constraint $\mu \leq 1$, can be disregarded purely on economic grounds. Even though explicit schemes require approximately 20% less computational effort per grid point per time step than implicit schemes, see section 3.1, the small time step that this explicit scheme requires makes it uneconomical in comparison with the implicit schemes and their

unlimited time step size. As for the backward and central difference implicit schemes, although both methods are unconditionally stable, the latter is preferable because of its second order accuracy - as opposed to the first order accuracy of the former, see section 3.3.

3.6.2 STABILITY OF THE COMPLETE WAVE EQUATIONS

So far, this stability analysis has been applied to the simplified wave equations (3.1) and (3.3), where the momentum equation (3.3) has only involved terms defining the dominant dynamic properties of waves. At this stage, it is desirable to investigate how the stability of the preferred central difference implicit scheme is affected by each individual diffusive term of the more complete momentum equation (3.2). Deferring the important non-linear convective accelerations to the next section, the remaining terms include the effects of wind action, bottom friction and the turbulent diffusion of momentum.

For the stability effect of wind action on the numerical representation of the wave equations (3.1) and (3.3), the appropriate momentum equation can be written as:-

$$\frac{\partial U}{\partial t} + g \frac{\partial \eta}{\partial x} - PW^2 = 0 \quad (3.50)$$

where $P =$ coefficient defined as $\frac{\rho_a C^*}{\rho(h + \eta)}$

$W =$ absolute wind velocity

As the term representing the wind effect has been assumed independent of the water velocity U , then the central difference implicit equations will be the same as before, except for the inclusion of the constant term PW^2

In the momentum equation (3.16). For convenience the appropriate difference equations are rewritten giving:-

$$\eta_j^{n+1} - \eta_j^n + \frac{h\Delta t}{2\Delta x} \left(u_{j+\frac{1}{2}}^{n+1} - u_{j-\frac{1}{2}}^{n+1} + u_{j+\frac{1}{2}}^n - u_{j-\frac{1}{2}}^n \right) = 0 \quad (3.51)$$

$$u_{j+\frac{1}{2}}^{n+1} - u_{j+\frac{1}{2}}^n + \frac{g\Delta t}{2\Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} + \eta_{j+1}^n - \eta_j^n \right) - PW^2 = 0 \quad (3.52)$$

which, on equating for η_* as before, gives the following quadratic equation for the amplification factor λ :-

$$(1 + \mu^2 \sin^2 \theta) \lambda^2 + (2\mu^2 \sin^2 \theta - 2 - 2\gamma) \lambda + (1 + \mu^2 \sin^2 \theta - 2\gamma) = 0 \quad (3.53)$$

where $2\gamma = PW^2 U_*^{-1} e^{-i\theta(2j+1)}$

Solving Eq. (3.53) for λ gives:-

$$\lambda = \frac{(1 - \mu^2 \sin^2 \theta + \gamma) \pm (4(\gamma+1)\mu^2 \sin^2 \theta - \gamma^2)^{\frac{1}{2}}}{(1 + \mu^2 \sin^2 \theta)} \quad (3.54)$$

where:-

$$|\lambda| = \left(1 + \frac{2\gamma}{(1 + \mu^2 \sin^2 \theta)} \right)^{\frac{1}{2}} \quad (3.55)$$

For the scheme to be stable, i.e. $|\lambda| \leq 1$, then the following requirement is necessary:-

$$\frac{2\gamma}{(1 + \mu^2 \sin^2 \theta)} \quad \text{or} \quad \frac{PW^2}{(1 + \mu^2 \sin^2 \theta) U_* e^{i\theta(2j+1)}} \leq 0$$

which reduces to:-

$$\frac{\rho_a C^* W^2}{\rho(h + \eta)} \leq 0 \quad (3.56)$$

As all the terms on the left hand side of Eq. (3.56) will always be positive, then this stability requirement will never be satisfied. This means that the wind action will always tend to have an unstabilising effect on the implicit scheme considered. However, for the type of flow fields to be modelled, where the wind effects will be at least one order of magnitude less than the dynamic head, this instability will generally tend to be relatively weak and will have little effect on the numerical solution. When the two-dimensional wave equations are considered, it can be shown that the Coriolis terms - which take into account the earth's rotation - introduce a similar weak instability into the solution.

To determine the stability effect of the bottom friction on the difference wave equations (3.15) and (3.16), the appropriate momentum equation can be written as:-

$$\frac{\partial U}{\partial t} + g \frac{\partial \eta}{\partial x} + Q|U|U = 0 \quad (3.57)$$

where Q = coefficient defined as $\frac{g}{(h + \eta)C^2}$ and $|U|$ can be treated as

a constant value for the speed U . Expressing the friction term in a fully centred implicit (Crank-Nicolson) manner and the remaining terms as before, then the corresponding difference representation for Eq. (3.57) can readily be shown to be:-

$$\begin{aligned}
 & U_{j+\frac{1}{2}}^{n+1} - U_{j+\frac{1}{2}}^n + \frac{g\Delta t}{2\Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} + \eta_{j+1}^n - \eta_j^n \right) \\
 & + \frac{Q|U|\Delta t}{2} \left(U_{j+\frac{1}{2}}^{n+1} + U_{j+\frac{1}{2}}^n \right) = 0
 \end{aligned} \quad (3.58)$$

Equating (3.58) with the corresponding central difference implicit continuity equation (3.51), the quadratic equation for the amplification factor becomes:-

$$\begin{aligned}
 & (1 + \mu^2 \sin^2 \theta + \delta) \lambda^2 + (2\mu^2 \sin^2 \theta - 2) \lambda + (1 + \mu^2 \sin^2 \theta - \delta) \\
 & = 0
 \end{aligned} \quad (3.59)$$

where $\delta = \frac{Q|U|\Delta t}{2} = \frac{g|U|\Delta t}{2(h + \eta)C^2}$

Solving Eq. (3.59) for λ gives:-

$$\lambda = \frac{(1 - \mu^2 \sin^2 \theta) \pm i(4\mu^2 \sin^2 \theta - \delta^2)^{\frac{1}{2}}}{(1 + \mu^2 \sin^2 \theta + \delta)} \quad (3.60)$$

where:-

$$|\lambda| = \left[\frac{(1 + \mu^2 \sin^2 \theta)^2 - \delta^2}{(1 + \mu^2 \sin^2 \theta)^2 + \delta(2 + 2\mu^2 \sin^2 \theta + \delta)} \right]^{\frac{1}{2}} \quad (3.61)$$

As δ is always positive, then $\delta(2 + 2\mu^2\sin^2\theta + \delta) > -\delta^2$ such that $|\lambda| < 1$, i.e. the scheme will always be stable and the round-off errors continuously damped.

Therefore, in solving the more complete shallow water wave equations (3.1) and (3.2), the bottom friction term assists in the stability of the difference scheme and imposes no constraints on the values of the grid size Δx and the time step Δt . This stability improvement plays a useful role in the computation, in that it tends to counteract any instabilities introduced by wind action or Coriolis effects, particularly since in most cases the bed shear will be much greater than the wind shear.

Finally, the influence of the effective turbulent stress on the stability of the difference equations (3.15) and (3.16) can be determined by analysing the following momentum equation:-

$$\frac{\partial U}{\partial t} + g \frac{\partial \eta}{\partial x} - R|U| \frac{\partial^2 U}{\partial x^2} = 0 \quad (3.62)$$

where R = coefficient defined as $\frac{6\sqrt{g}(h + \eta)}{c}$

In this case the difference representation of the turbulent diffusive term is not so straightforward as for the bottom friction because of the second derivative. If the diffusive term is represented in a fully centred implicit manner as before, i.e.:-

$$R|U| \frac{\partial^2 U}{\partial x^2} = \frac{R|U|}{2\Delta x^2} \left[U_{j+\frac{3}{2}}^{n+1} + U_{j+\frac{3}{2}}^n - 2 \left(U_{j+\frac{1}{2}}^{n+1} + U_{j+\frac{1}{2}}^n \right) + U_{j-\frac{1}{2}}^{n+1} + U_{j-\frac{1}{2}}^n \right]$$

then, at the new time level $n+1$, not only is the unknown velocity $U_{j+\frac{1}{2}}^{n+1}$ included but so too is the unknown velocity $U_{j+\frac{3}{2}}^{n+1}$. This difference representation of the diffusive term complicates the solution procedure to be used, in that, it requires the setting up of a five diagonal matrix array of coefficients rather than a three diagonal matrix as required by the other terms. The use of a larger diagonalised matrix in this manner significantly increases the computational cost/grid point/time step and for this reason makes the above difference representation unattractive. To overcome this problem the term $U_{j+\frac{3}{2}}^{n+1}$ needs to be represented at some previous time level, i.e. in an explicit manner.

To satisfy this requirement, the most suitable difference representation of the diffusive term appears to be the Du-Fort and Frankel fully centred explicit scheme - see Table 8.1 of Richtmeyer and Morton (90). However, although this scheme has a second order accuracy, it does have the disadvantage of not being consistent with the original differential equation when Δx and Δt tend to zero at the same rate, i.e. if $\Delta x/\Delta t$ remains constant as $\Delta x, \Delta t \rightarrow 0$ see Abbott (1). This means that, in adopting this scheme, care needs to be taken when selecting the values of the grid size and time step for the mathematical model. Thus, the difference representation used for the momentum equation (3.62) can be written as:-

$$\begin{aligned}
 U_{j+\frac{1}{2}}^{n+1} &= U_{j+\frac{1}{2}}^n + \frac{g\Delta t}{2\Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} + \eta_{j+1}^n - \eta_j^n \right) \\
 &- \frac{R|U|\Delta t}{\Delta x^2} \left(U_{j+\frac{3}{2}}^{n+\frac{1}{2}} - 2 \left(U_{j+\frac{1}{2}}^{n+1} + U_{j+\frac{1}{2}}^n \right) + U_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) \\
 &= 0
 \end{aligned} \tag{3.63}$$

If α denotes the amplification factor $e^{i\sigma\Delta t}$ for the continuity equation (3.51), and λ the amplification factor $e^{i\sigma\Delta t/2}$ for Eq. (3.63), then equating (3.51) and (3.63) for η_* , the quadratic equation for λ becomes:-

$$[(\alpha - 1)(2\phi + 1) + \mu^2(\alpha + 1)\sin^2\theta]\lambda^2 - 2\phi \cos 2\theta\lambda + [(\alpha - 1)(2\phi - 1) + \mu^2(\alpha + 1)\sin^2\theta] = 0 \quad (3.64)$$

where $\phi = \frac{R|U|\Delta t}{2} = \frac{6\sqrt{g}(h + \eta)|U|\Delta t}{2C \Delta x^2}$

Solving Eq. (3.64) for λ in terms of α gives:-

$$\lambda = \left[\phi \cos 2\theta \pm i \left\{ -(\phi \cos 2\theta)^2 + [(\alpha - 1)(2\phi + 1) + (\alpha + 1)\mu^2\sin^2\theta][(\alpha - 1)(2\phi - 1) + (\alpha + 1)\mu^2\sin^2\theta]^{\frac{1}{2}} \right\} \right] / [(\alpha - 1)(2\phi + 1) + (\alpha + 1)\mu^2\sin^2\theta] \quad (3.65)$$

which, on reduction gives:-

$$\lambda^2 = \left[\frac{(\alpha - 1)(2\phi - 1) + (\alpha + 1)\mu^2\sin^2\theta}{(\alpha - 1)(2\phi + 1) + (\alpha + 1)\mu^2\sin^2\theta} \right] \quad (3.66)$$

But $\alpha = e^{i\sigma\Delta t} = \lambda^2$, and on substituting in Eq. (3.66) leads to:-

$$(2\phi + 1 + \mu^2\sin^2\theta)\lambda^4 - 4\phi\lambda^2 + (2\phi - 1 - \mu^2\sin^2\theta) = 0 \quad (3.67)$$

which simplifies to:-

$$\lambda^4 = \frac{4\phi^2 + (1 + \mu^2 \sin^2 \theta)^2}{(2\phi + 1 + \mu^2 \sin^2 \theta)^2} \quad (3.68)$$

Consequently, as the value of ϕ cannot be negative then the right hand side of Eq. (3.68) will always be less than unity, this means that $|\lambda| < 1$ and the scheme is always stable.

The turbulent effective stress therefore has the same effect on the stability of the central difference implicit wave equations (3.15) and (3.16) as the bottom friction term. Again the stability of the difference scheme is improved with no constraints being imposed on the values of Δx and Δt , apart from the consistency requirement that Δx and Δt should not tend to zero at the same rate.

In summarising the results of this linear stability analysis, the most suitable difference representation of the shallow water wave equations (3.1) and (3.3) is the fully centred (Crank-Nicolson) implicit scheme. By itself this scheme is unconditionally stable with an amplification factor of unity, however, if the bottom friction and the effective stress terms are included, then the overall stability of the difference equations is improved. The effects of the wind stress and the Coriolis term are to introduce weak instabilities into the scheme, but for most practical purposes these instabilities will have a negligible influence on the numerical solution.

3.7 NON-LINEAR INSTABILITY

In most of the mathematical models of horizontal flows documented

in the recent literature, the convective acceleration terms have either been neglected or have had little effect on the value of the final numerical solutions, e.g. Hansen (42,43), Leendertse (63,64) etc. This has generally been due to the fairly uniform nature of the flow fields being modelled, e.g. wide estuaries, and the dominance of such hydrodynamic features as pressure gradients, oscillatory flows, wind effects etc. However, for the type of flow fields being considered in this thesis, i.e. a jet issuing into a circular reservoir and a tide flowing into a narrow entranced harbour, the convective accelerations, associated with the sudden changes in the lateral cross-sectional area, are of extreme importance. Hence care has to be taken to ensure that the non-linear accelerative terms in the depth averaged equations (2.51) and (2.52) do not give rise to instabilities.

The occurrence of non-linear instabilities, in many of the numerical methods used to solve the equations of motion, is discussed by Phillips (84) in terms of the physical process of transferring turbulent energy from large to small scales through the non-uniform characteristics of the flow. In the numerical representation of this process, energy cannot be transferred to scales smaller than twice the grid size, and for this reason tends to accumulate at the corresponding wavelength, i.e. $2\Delta x$. This results in an explosive growth of oscillations which, after some time, can completely swamp the correct numerical solution.

One of the most common approaches used to suppress the growth of energy at these small wavelengths is to introduce an artificial viscosity type energy dissipation process into the mathematical model. This has the effect of simulating the transfer of turbulent energy to scales smaller than twice the grid size by artificially smoothing the numerical

solution. The procedure generally adopted for this purpose is to spatially average the computed velocities, at the end of each time step, by using a weighting factor α and an equation of the following form:-

$$\bar{u}_{j+\frac{1}{2}}^{n+1} = (1 - \alpha) u_{j+\frac{1}{2}}^{n+1} + \frac{\alpha}{2} \left(u_{j+\frac{3}{2}}^{n+1} + u_{j-\frac{1}{2}}^{n+1} \right) \quad (3.69)$$

where $u_{j+\frac{1}{2}}^{n+1}$ = the computed velocity

$\bar{u}_{j+\frac{1}{2}}^{n+1}$ = the averaged velocity

with α being typically of the order 0.01, see Kuipers and Vreugdenhil (58). However, there are two basic disadvantages with using this type of method. Firstly, there is the difficulty in selecting the most appropriate value for the weighting factor α ; secondly, it is impossible to determine how much of the smoothing process is used up in damping out non-linear instabilities, rather than diffusing the small scale turbulent energy.

Due to these disadvantages and the relatively high degree of turbulent energy transport expected in the flow fields being considered, this artificial energy dissipation process has not been included in the mathematical model. Instead, a numerically stable representation of the non-linear terms has been sought such that the accumulation of short wavelength energy might be damped by the difference scheme itself, i.e. in the same way as the round-off errors were damped in the previous section. Although this numerical process of dissipating energy is analogous to the physical dissipation of small scale turbulence by viscous friction, the former is not a correct mathematical representation of the

latter. Consequently, in the mathematical model energy will not strictly be conserved.

To determine the influence of the non-linear terms on the stability of a finite difference scheme, it is not analytically correct to consider the propagation of only one Fourier component as before. This is because different components of the Fourier series will interact with one another through the non-linear terms. However, it is usually possible to get some idea of the stability of a non-linear difference scheme by considering a linearised version of the appropriate equations. This can be achieved by splitting the non-linear terms into linear components with constant coefficient, see Richtmeyer and Morton (90). Thus, for the shallow water wave equations being modelled, the corresponding quasi-linearised momentum equation can be written as:-

$$\frac{\partial U}{\partial t} + U_0 \frac{\partial U}{\partial x} + g \frac{\partial \eta}{\partial x} = 0 \quad (3.70)$$

where U_0 is a constant local value of the velocity U .

In the literature, most of the mathematical models for the wave equations (3.1) and (3.70) have included the convective accelerations in an explicit form, even though the other terms may have been expressed implicitly, e.g. Leendertse (63,64) etc. Combining this explicit representation of the accelerative term with the fully centred implicit scheme so far considered, the difference equation for the momentum becomes:-

$$\begin{aligned} U_{j+\frac{1}{2}}^{n+1} - U_{j+\frac{1}{2}}^n + U_0 \frac{\Delta t}{\Delta x} \left(U_{j+1}^n - U_j^n \right) \\ + \frac{g \Delta t}{2 \Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} + \eta_{j+1}^n - \eta_j^n \right) = 0 \end{aligned} \quad (3.71)$$

Equating (3.71) with the continuity equation (3.51), leads to the following quadratic equation for the amplification factor λ :-

$$(1 + \mu^2 \sin^2 \theta) \lambda^2 + 2(-1 + \mu^2 \sin^2 \theta + \gamma \sin \theta) \lambda + (1 + \mu^2 \sin^2 \theta - 2\gamma \sin \theta) = 0 \quad (3.72)$$

where $\gamma = \frac{U_0 \Delta t}{\Delta x}$

Solving Eq. (3.72) for λ gives:-

$$\lambda = \frac{(1 - \mu^2 \sin^2 \theta - \gamma \sin \theta) \pm [4(1 - \gamma \sin \theta) \mu^2 \sin^2 \theta + \gamma^2 \sin^2 \theta]^{\frac{1}{2}}}{(1 + \mu^2 \sin^2 \theta)} \quad (3.73)$$

where:-

$$|\lambda^2| = 1 + \frac{4\gamma^2 \sin^2 \theta}{(1 + \mu^2 \sin^2 \theta)^2} \quad (3.74)$$

With $\gamma^2 \sin^2 \theta$ being positive then, apart from $\theta = 0, \pi, \text{etc.}$, $|\lambda| > 1$ and so the scheme will always be unstable. Consequently, as a stable scheme is being sought to counteract the growth of turbulent energy at short wavelengths, the convective acceleration term needs to be represented in some other form.

Considering a fully time centred representation of this term, the finite difference momentum equation can be rewritten as:-

$$\begin{aligned}
& u_{j+\frac{1}{2}}^{n+1} - u_{j+\frac{1}{2}}^n + \frac{u_o \Delta t}{\Delta x} \left(u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}} \right) \\
& + \frac{g \Delta t}{2 \Delta x} \left(\eta_{j+1}^{n+1} - \eta_j^{n+1} + \eta_{j+1}^n - \eta_j^n \right) = 0
\end{aligned} \tag{3.75}$$

If α denotes the amplification factor $e^{i\sigma\Delta t}$ for Eq. (3.51) and λ the amplification factor $e^{i\sigma\Delta t/2}$ for Eq. (3.75), then equating for η_* , the quadratic equation for λ becomes:-

$$\begin{aligned}
& [(\alpha + 1)\mu^2 \sin^2 \theta + (\alpha - 1)]\lambda^2 + 2i(\alpha - 1)(\sin \theta)\lambda \\
& + [(\alpha + 1)\mu^2 \sin^2 \theta - (\alpha - 1)] = 0
\end{aligned} \tag{3.76}$$

Solving Eq. (3.76) for λ in terms of α gives:-

$$\lambda = \frac{[-\gamma i(\alpha-1)\sin \theta \pm i\{\gamma^2(\alpha-1)^2 \sin^2 \theta + (\alpha+1)^2 \mu^4 \sin^4 \theta - (\alpha-1)^2\}]^{\frac{1}{2}}}{[(\alpha + 1)\mu^2 \sin^2 \theta + (\alpha - 1)]} \tag{3.77}$$

which simplifies to:-

$$\lambda^2 = \frac{2\gamma^2(\alpha-1)^2 \sin^2 \theta + (\alpha+1)^2 \mu^4 \sin^4 \theta - (\alpha-1)^2}{2(\alpha^2-1)\mu^2 \sin^2 \theta + (\alpha+1)^2 \mu^4 \sin^4 \theta + (\alpha-1)^2} \tag{3.78}$$

But $\lambda^2 = \alpha = e^{i\sigma\Delta t}$, and on substituting in Eq. (3.78) the relevant modulus of α becomes:-

$$|\alpha| = \left[\frac{-1 + \frac{2\gamma^2 \sin^2 \theta}{1 + 2\mu^2 \sin^2 \theta} + \frac{\mu^4 \sin^4 \theta}{\mu^4 \sin^4 \theta}}{1 + \frac{2\gamma^2 \sin^2 \theta}{2\mu^2 \sin^2 \theta} + \frac{\mu^4 \sin^4 \theta}{\mu^4 \sin^4 \theta}} \right]^{\frac{1}{2}} \tag{3.79}$$

Provided $-1 + 2\gamma^2 \sin^2 \theta \leq 1$ then, since $2\mu^2 \sin^2 \theta$ is always positive, $|\alpha| < 1$ or $|\lambda^2| < 1$ and the scheme is stable. This imposes the constraint:-

$$\gamma \sin \theta \leq 1 \quad (3.80)$$

and with stability being necessary for all wavelengths, Eq. (3.80) must be satisfied for all values of θ . For the limiting case when $\theta = \pi/2$, i.e. $L = 2\Delta x$ - the shortest possible wavelength, then the requirement for stability reduces to:-

$$U_0 \frac{\Delta t}{\Delta x} \leq 1$$

$$\text{i.e. } |U| \leq \frac{\Delta x}{\Delta t}$$

For the type of flow fields to be mathematically modelled this stability requirement is not too severe. For example, to model the jet induced circulatory flow in a reservoir with a grid size Δx of 100 m and a jet velocity of 1 m/sec., a maximum time step of 100 secs. may be used. This limiting time step is far less severe than the 10 secs. required, for a 10 m depth, by the explicit stability constraint (3.44). Consequently, this novel finite difference representation of the non-linear convective acceleration term seems to be most suitable for the purpose of continuously damping the production of small scale turbulent energy. With this term being represented in a numerically stable manner, the corresponding round-off errors will also be continuously damped and the stability of the difference scheme as a whole further improved.

Another problem that frequently arises with non-linear terms is

their numerical representation at the boundaries of the model, see Kuipers and Vreugdenhil (58). However, the effects of the boundaries cannot be predicted analytically and for this reason have to be investigated by numerical experiments.

3.8 ACCURACY

Although a finite difference scheme may be stable and consistent with the original differential equations, the sum of the truncation errors may still be large enough to provide a relatively inaccurate numerical solution. Consequently, it is desirable to be able to obtain some analytical description of the accuracy of each finite difference scheme.

In the numerical representation of the shallow water wave equations (3.1) and (3.3) the usual measure of accuracy is a comparison of the celerities of the numerical and the physical wave. The manner in which this accuracy is generally determined can be best illustrated by again analysing the central difference explicit scheme of section 3.3. For convenience, the difference equations can be written as before:-

$$\eta_j^{n+1} - \eta_j^n + \frac{h\Delta t}{\Delta x} \left(u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right) = 0 \quad (3.81)$$

$$u_{j+\frac{1}{2}}^{n+\frac{1}{2}} - u_{j+\frac{1}{2}}^{n-\frac{1}{2}} + \frac{g\Delta t}{\Delta x} \left(\eta_{j+1}^n - \eta_j^n \right) = 0 \quad (3.82)$$

with $\mu \leq 1$ for stability and an amplification factor of:-

$$\lambda = (1 - 2\mu^2 \sin^2 \theta) \pm 2i\mu \sin \theta (1 - \mu^2 \sin^2 \theta)^{\frac{1}{2}} \quad (3.83)$$

Defining the numerical wave celerity as:-

$$c_n = \frac{L}{T} = \frac{\sigma}{\beta} \quad (3.84)$$

and with the shallow water wave velocity being:-

$$c_p = \sqrt{gh} \quad (3.85)$$

then the measure of accuracy with which the numerical wave is propagated is given by the ratio:-

$$\frac{c_n}{c_p} = \frac{\sigma}{\beta\sqrt{gh}} \quad (3.86)$$

However, from the definition of the amplification factor, i.e. $\lambda = e^{i\sigma\Delta t}$, the wave frequency σ can be represented in the following complex notational form:-

$$\sigma = \frac{1}{\Delta t} \arg.\lambda \quad (3.87)$$

giving, for the measure of accuracy:-

$$\frac{c_n}{c_p} = \frac{\arg.\lambda}{2\mu\theta} \quad (3.88)$$

where, as before, $\theta = \frac{\beta\Delta x}{2}$ and $\mu = \frac{\Delta t\sqrt{gh}}{\Delta x}$.

The argument of the amplification factor can readily be evaluated as:-

$$\arg.\lambda = \sigma\Delta t = \sin^{-1}[(1-\mu^2\sin^2\theta)^{\frac{1}{2}}2\mu\sin\theta] \quad (3.89)$$

Simplifying this equation, the terms on the right can be approximated by expressing $\sin\theta$ and $\sin^{-1}[(1-\mu^2\sin^2\theta)^{\frac{1}{2}}2\mu\sin\theta]$ in their corresponding series forms. Since, for stability, $\mu \leq 1$ and for the type of flows being considered the wavelength L will always be appreciably larger than the grid size Δx , i.e. $\theta (= \frac{\pi\Delta x}{L}) < 1$, then terms of higher orders than $\mu^3\theta^3$ may be neglected. Introducing the appropriate series expansions, as given by Hughes and Gaylord (53), Eq. (3.87) can be reduced to:-

$$\arg.\lambda = 2\mu\theta \left(1 - \frac{\theta^2}{6} (1 - \mu^2 + \dots) \right) \quad (3.90)$$

Substituting in Eq. (3.88), the measure of accuracy C_n/C_p becomes:-

$$\frac{C_n}{C_p} = \left(1 - \frac{\theta^2}{6} (1 - \mu^2 + \dots) \right) \quad (3.91)$$

where $\mu\theta < 1$. A graphical representation of this equation, for differing values of θ and μ , is given in Fig. 4.

For the central difference implicit scheme, given by Eqs. (3.15) and (3.16), the amplification factor λ can be written as before, i.e. Eq. (3.49):-

$$\lambda = \frac{(1 - \mu^2\sin^2\theta) \pm 2i\mu\sin\theta}{(1 + \mu^2\sin^2\theta)} \quad (3.92)$$

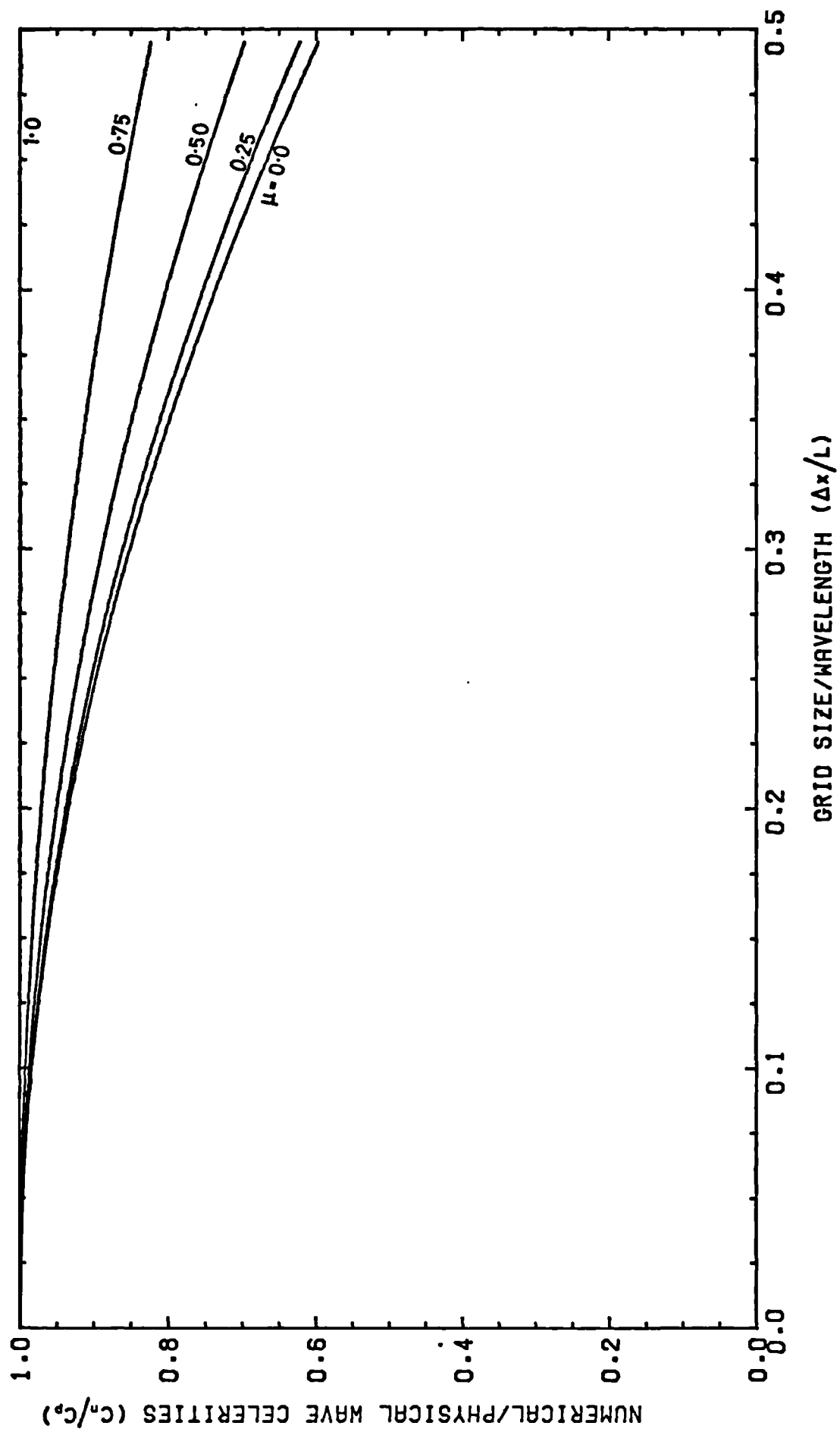


FIGURE 4 Measure of Accuracy for the Central Difference Explicit Scheme

with an argument most suitably expressed as:-

$$\arg.\lambda = \tan^{-1} \left(\frac{2\mu \sin \theta}{1 - \mu^2 \sin^2 \theta} \right) \quad (3.93)$$

Substituting the appropriate series expansions, as for the central difference explicit scheme, Eq. (3.93) reduces to:-

$$\arg.\lambda = 2\mu\theta \left(1 - \frac{\theta^2}{6} (1 + 2\mu^2 + \dots) \right) \quad \text{for } \mu\theta \leq \frac{1}{2}$$

and (3.94)

$$\arg.\lambda = \frac{\pi}{2} - \frac{1}{2\mu\theta \left(1 - \frac{\theta^2}{6} + \mu^2\theta^2 + \dots \right)} + \dots$$

for $\mu\theta > \frac{1}{2}$

This gives, for the measure of accuracy:-

$$\frac{c_n}{c_p} = \left(1 - \frac{\theta^2}{6} (1 + 2\mu^2 + \dots) \right) \quad \text{for } \mu\theta \leq \frac{1}{2}$$

and (3.95)

$$\frac{c_n}{c_p} = \frac{\pi}{4\mu\theta} - \frac{1}{4\mu^2\theta^2 \left(1 - \frac{\theta^2}{6} + \mu^2\theta^2 + \dots \right)} + \dots$$

for $\mu\theta > \frac{1}{2}$

and can be graphically represented as shown in Fig. 5.

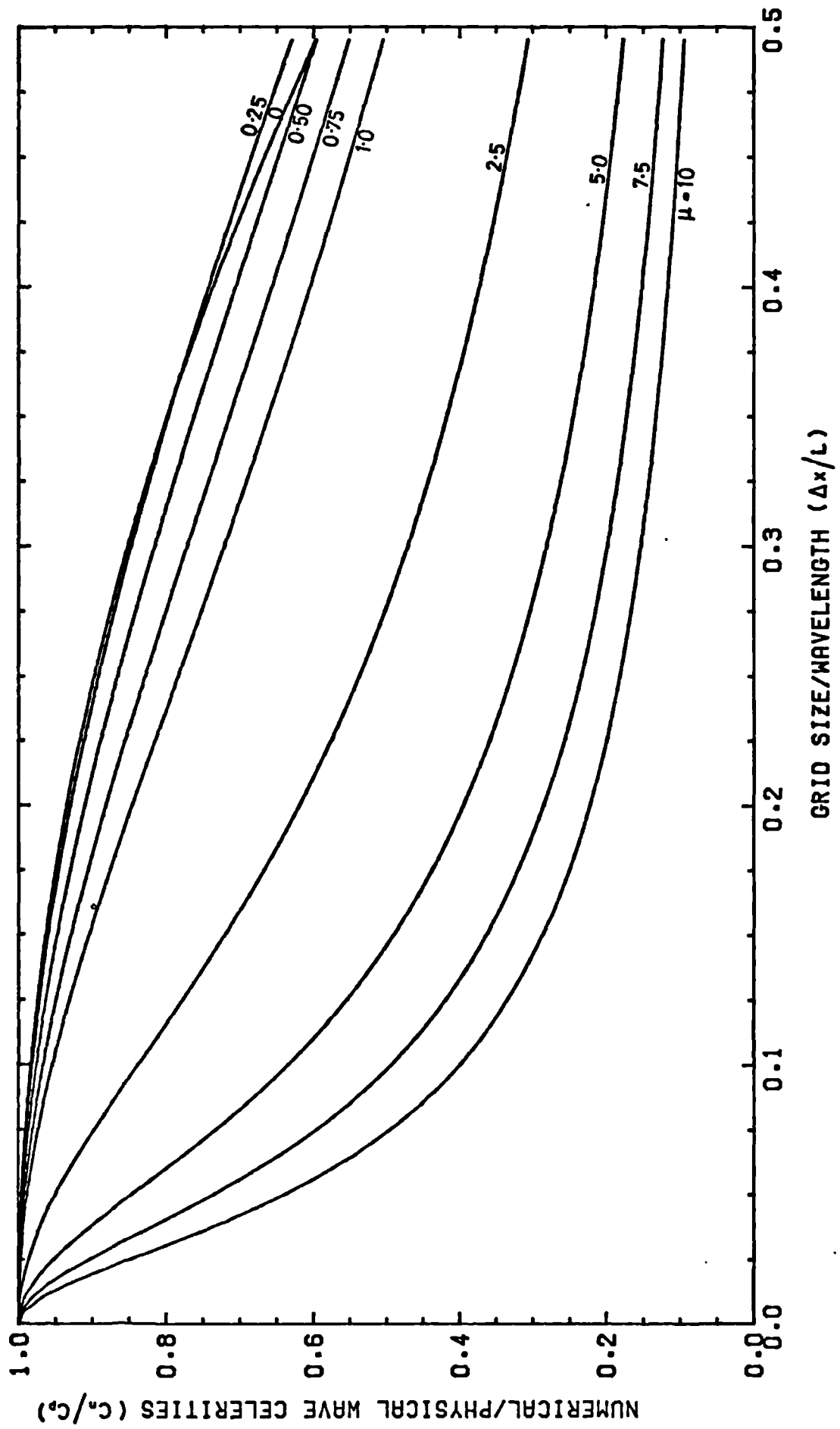


FIGURE 5 Measure of Accuracy for the Central Difference Implicit Scheme

Similarly, for the backward difference implicit scheme, given by Eqs. (3.13) and (3.14) with an amplification factor:-

$$\lambda = (1 \pm 2i\mu \sin \theta)^{-1} \quad (3.96)$$

the corresponding measure of accuracy can be evaluated as:-

$$\frac{c_n}{c_p} = \left(1 - \frac{\theta^2}{6} (1 + 8\mu^2 + \dots) \right) \quad \text{for } \mu\theta \leq \frac{1}{2}$$

and (3.97)

$$\frac{c_n}{c_p} = \frac{\pi}{4\mu\theta} - \frac{1}{4\mu^2\theta^2 \left(1 - \frac{\theta^2}{6} + \dots \right)} + \dots \quad \text{for } \mu\theta > \frac{1}{2}$$

The accuracy of this particular implicit scheme is illustrated in Fig. 6 and, on comparing with Fig. 5 (or Eq. 3.95), is clearly the less accurate of the two implicit schemes considered, i.e. the c_n/c_p ratio is always smaller. This result is not surprising, bearing in mind that the leading truncation error of the backward implicit scheme is of the first order whereas that of the central implicit scheme is of the second order - see section 3.3.2. Due to this greater degree of inaccuracy, this backward difference scheme is not given further consideration.

In comparing the ratio of the numerical and physical wave celerities for both the explicit and implicit central difference schemes (Figs. 4 and 5) a number of general conclusions can be made. Firstly, as the grid size Δx is continuously reduced the accuracy of both schemes

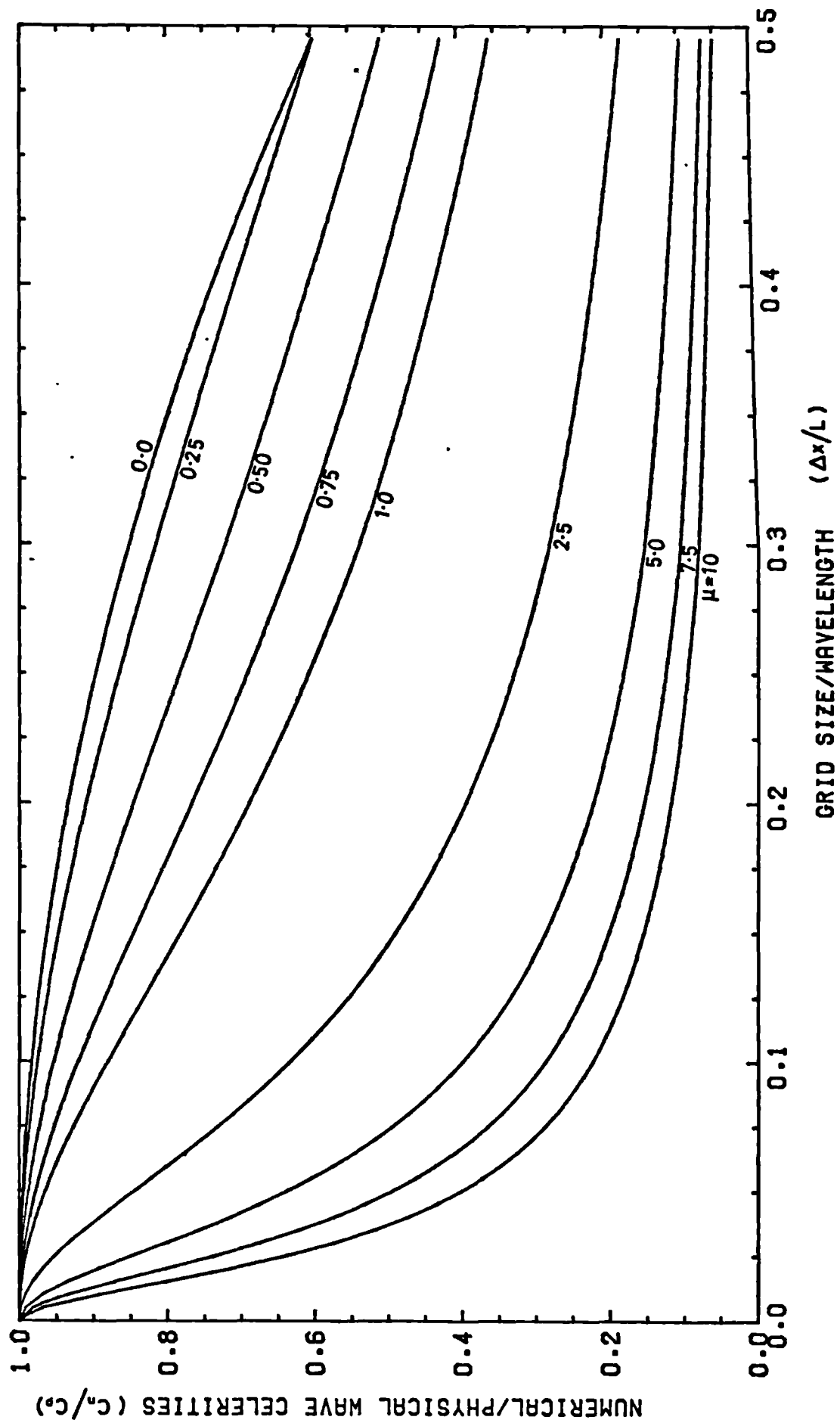


FIGURE 6 Measure of Accuracy for the Backward Difference Implicit Scheme

improves. On the other hand, if the time step Δt is reduced, i.e. $\mu \rightarrow 0$, then the accuracy of the explicit scheme deteriorates while that of the implicit scheme improves. At the stability limit of the explicit scheme, $\mu = 1$, the ratio C_n/C_p is unity for all wavelengths and, under these conditions, the scheme represents the differential equations exactly. However, this is not true for the implicit scheme when $\mu = 1$.

Another important property which can be deduced from the graphs is that the ratio C_n/C_p is less than unity - except for the explicit scheme when $\mu = 1$. This means that for all values of μ and $\Delta x/L$ the numerical wave travels more slowly than the physical wave and is due to the numerical damping associated with the truncation error.

Finally, for the same grid spacing and time step, i.e. the same $\Delta x/L$ and μ values, comparison of Fig. 4 and 5 leads to the conclusion that the explicit scheme is always more accurate than the implicit scheme. However, the advantage of using an implicit scheme is that a much larger time step may be used and the computational effort thereby reduced.

For the type of flow fields being considered, where jet diffusion introduces appreciable velocity changes over short distances, the grid spacing needs to be relatively small. Due to the long periodic unsteady nature of the flows, e.g. the tidal flow, it is necessary to choose a time step large enough to ensure that a realistic period of unsteadiness can be modelled at a reasonable computational cost. To satisfy these requirements it is desirable to select the implicit scheme, even though the spatical variations will be represented somewhat less accurately.

In summarising the development of the most suitable finite difference representation of the wave equations (3.1) and (3.2), Table 1 shows a comparison of the fundamental numerical properties of the difference schemes considered in this section.

PROPERTY SCHEME	APPROXIMATION	CONSISTENCY	CONVERGENCE	STABILITY	ACCURACY
FORWARD DIFFERENCE EXPLICIT	FIRST ORDER $O(\Delta t)$	UNCONDITIONAL	NOT CONVERGENT	UNSTABLE	_____
CENTRAL DIFFERENCE EXPLICIT	SECOND ORDER $O(\Delta t)^2 + O(\Delta h)^2$	UNCONDITIONAL	CONVERGENT PROVIDED $\frac{\Delta t}{\Delta x} \sqrt{gh} \leq 1$	STABLE PROVIDED $\frac{\Delta t}{\Delta x} \sqrt{gh} \leq 1$	$C_n/C_p =$ $1 - \frac{\theta^2}{6}(1 - \mu^2)$
BACKWARD DIFFERENCE IMPLICIT	FIRST ORDER $O(\Delta t)$	UNCONDITIONAL	UNCONDITIONAL	UNCONDITIONAL	$C_n/C_p =$ $1 - \frac{\theta^2}{6}(1 + 8\mu^2)$ for $\mu\theta \leq 1$
CENTRAL DIFFERENCE IMPLICIT	SECOND ORDER $O(\Delta t)^2 + O(\Delta h)^2$	UNCONDITIONAL	UNCONDITIONAL	UNCONDITIONAL	$C_n/C_p =$ $1 - \frac{\theta^2}{6}(1 + 2\mu^2)$ for $\mu\theta \leq 1$
DU-FORT, FRANKEL EXPLICIT (DIFFUSION)	SECOND ORDER $O(\Delta t)^2 + O(\Delta h)^2$	CONDITIONAL (PROVIDED $\frac{\Delta t}{\Delta x} \rightarrow 0$, as $\Delta x \rightarrow 0$)	CONDITIONAL	UNCONDITIONAL	_____

TABLE 1 Comparison of the Numerical Properties for Various Finite Difference Schemes

NUMERICAL PROCEDURE FOR SOLVING THE FINITE DIFFERENCE EQUATIONS

4.1 THE TWO-DIMENSIONAL DIFFERENCE EQUATIONS

A comparison of the numerical properties of the various finite difference schemes analysed in Chapter 3 has shown that, wherever possible, the terms of the shallow water wave equations (3.1) and (3.2) should be represented in a central difference implicit manner. Also, to avoid the problems associated with non-linear instability, the convective acceleration term should be centred in time, particularly if no artificial viscosity-type effects are to be included in the solution. As this requires the velocity U to be expressed at three different time levels, the second derivative of the effective stress term can be readily represented in the Du-Fort and Frankel explicit form. This has the advantage of reducing the computational cost. By applying these conclusions to the two-dimensional depth averaged equations of motion (2.51), (2.52) and (2.53), the most suitable finite difference equations can be developed to include all the grid points $1 \rightarrow M$ and $1 \rightarrow N$, in the x and y directions respectively.

The corresponding difference equations are expressed in an Alternating Direction Implicit (ADI) manner such that for each computing cycle two successive time level operations are performed. For the first operation, from time $n\Delta t$ to $(n + \frac{1}{2})\Delta t$, the derivatives and terms involving U and η of Eqs. (2.51) and (2.53) are approximated in an implicit form, while those of V are approximated explicitly. Application of the resulting finite difference equations to include the M grid points along a row parallel to the x direction, gives $(M - 1)$ continuity and momentum equations for the corresponding unknown values $U^{n+\frac{1}{2}}$ and $\eta^{n+\frac{1}{2}}$. With

(N - 1) unknown grid points parallel to the x direction, the advancement of the equations over the whole domain incorporates the solution of (N - 1) independent sets of equations, each set containing (M - 1) unknowns. Similarly, for the second operation, from time $(n + \frac{1}{2})\Delta t$ to $(n + 1)\Delta t$, the derivatives and terms involving V and η of Eqs. (2.52) and (2.53) are approximated implicitly, while the previous implicit values for U are now approximated explicitly. Again, the continuity and momentum equations are written for each non-boundary grid point along rows parallel to the y direction. This gives (M - 1) independent sets of equations, with each set containing (N - 1) unknown values of V^{n+1} and η^{n+1} .

To represent the equations in two dimensions the computing grid illustrated in Fig. 7 is used, together with the following notation shown for η , see Leendertse (63):-

$$\eta_{j,k} = \eta(j\Delta x, k\Delta y, n\Delta t), \quad \Delta x = \Delta y = \Delta s$$

where $(j\Delta x, k\Delta y)$ is a spatial grid point with

$$j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

$$k = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

$$n = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots$$

and, for average values at the grid points:-

$$\bar{\eta}_{j,k}^x = \frac{1}{2} \left(\eta_{j+\frac{1}{2},k} + \eta_{j-\frac{1}{2},k} \right)$$

$$\bar{\eta}_{j,k}^y = \frac{1}{2} \left(\eta_{j,k+\frac{1}{2}} + \eta_{j,k-\frac{1}{2}} \right)$$

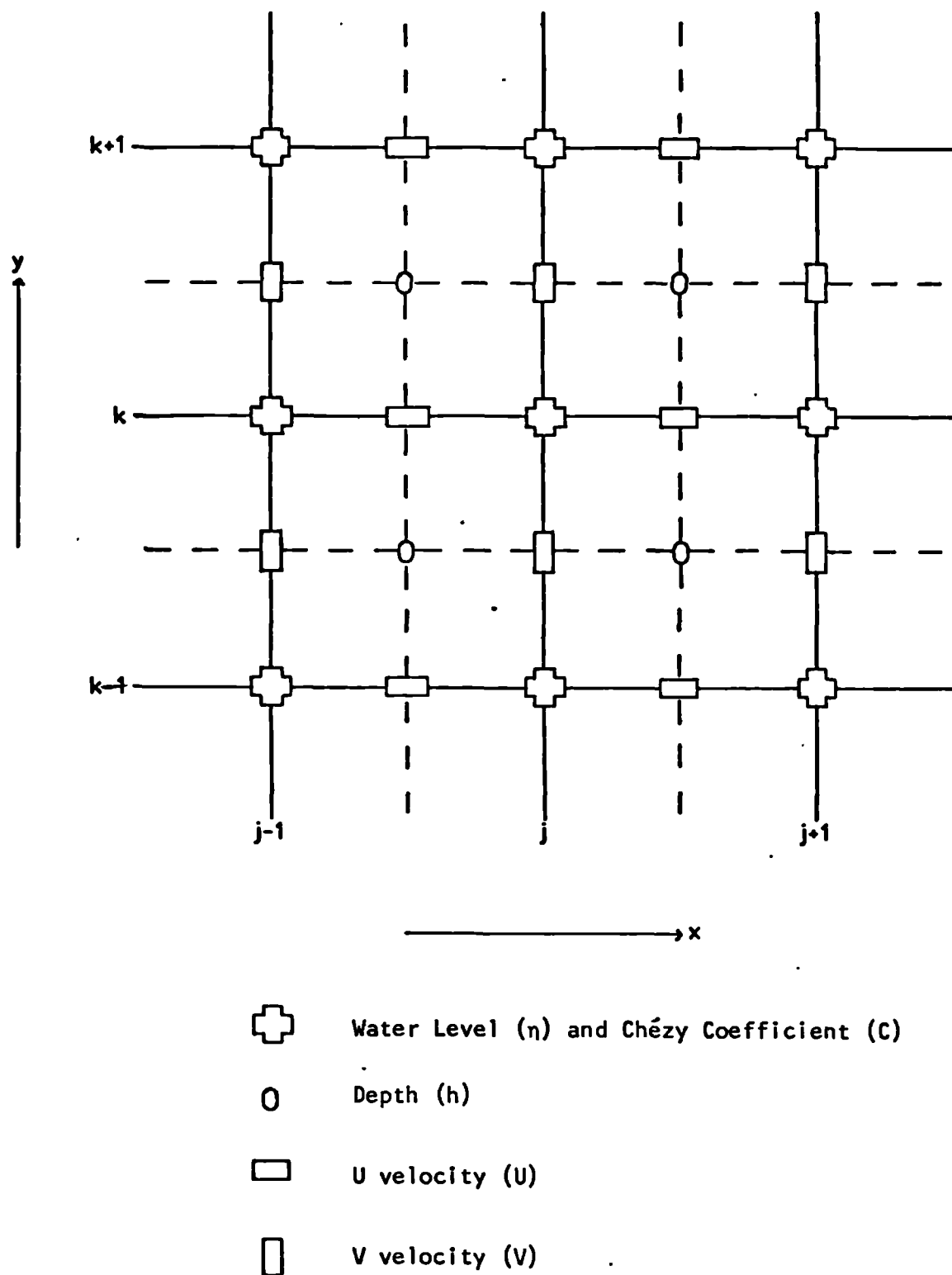


FIGURE 7 The Computing Grid

$$\bar{\eta}_{j,k} = \frac{1}{4} \left(\eta_{j-\frac{1}{2},k-\frac{1}{2}} + \eta_{j-\frac{1}{2},k+\frac{1}{2}} + \eta_{j+\frac{1}{2},k-\frac{1}{2}} + \eta_{j+\frac{1}{2},k+\frac{1}{2}} \right)$$

Thus, for the first operation, the continuity equations (2.53) along a grid line in the x direction reads:-

$$\begin{aligned} & \left(\eta_{j,k}^{n+\frac{1}{2}} - \eta_{j,k}^n \right) + \frac{\Delta t}{2\Delta s} \left[\left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k} u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} \right. \\ & - \left. \left(\bar{h}^y + \bar{\eta}^x \right)_{j-\frac{1}{2},k} u_{j-\frac{1}{2},k}^{n+\frac{1}{2}} \right] + \frac{\Delta t}{2\Delta s} \left[\left(\bar{h}^x + \bar{\eta}^y \right)_{j,k+\frac{1}{2}} v_{j,k+\frac{1}{2}}^{n-1} \right. \\ & - \left. \left(\bar{h}^x + \bar{\eta}^y \right)_{j,k-\frac{1}{2}} v_{j,k-\frac{1}{2}}^{n-1} \right] = 0 \end{aligned} \quad (4.1)$$

and for the x direction momentum equation:-

$$\begin{aligned} & \left(u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} - u_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right) + \frac{2\Delta t}{\Delta s} \left[\left(\bar{u}_{j+1,k}^x \right)^2 - \left(\bar{u}_{j,k}^x \right)^2 \right]^{n-\frac{1}{2}} \\ & + \frac{2\Delta t}{\Delta s} \left[u_{j+\frac{1}{2},k+\frac{1}{2}+p}^{n-\frac{1}{2}} \left(\bar{v}_{j+\frac{1}{2},k+\frac{1}{2}}^x \right)^n - u_{j+\frac{1}{2},k-\frac{1}{2}+q}^{n-\frac{1}{2}} \left(\bar{v}_{j+\frac{1}{2},k-\frac{1}{2}}^x \right)^n \right] \\ & - 2\Delta t f \bar{v}_{j+\frac{1}{2},k}^n + \frac{\Delta t}{\Delta s} g \left[\eta_{j+1,k}^{n+\frac{1}{2}} + \eta_{j+1,k}^{n-\frac{1}{2}} - \eta_{j,k}^{n+\frac{1}{2}} - \eta_{j,k}^{n-\frac{1}{2}} \right] \\ & + \Delta t g \left[\left(u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} + u_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right) \left[\left(u_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right)^2 + \left(\bar{v}_{j+\frac{1}{2},k}^n \right)^2 \right]^{\frac{1}{2}} \right. \\ & \left. \left[\left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k} \left(\bar{c}^x \right)_{j+\frac{1}{2},k}^2 \right] \right] - 2\Delta t \rho_a C^* W W_x / \left[\rho \left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k} \right] \end{aligned}$$

$$\begin{aligned}
& - 12 \frac{\Delta t}{\Delta s^2} \sqrt{g} \frac{(\bar{h}^y + \bar{\eta}^x)_{j+\frac{1}{2},k}}{\bar{c}_{j+\frac{1}{2},k}^x} \left[\left| U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right| \left(U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} - U_{j+\frac{1}{2},k}^{n+\frac{1}{2}} \right. \right. \\
& - \left. U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} + U_{j-\frac{1}{2},k}^{n-\frac{1}{2}} \right) + \left| \bar{V}_{j+\frac{1}{2},k}^n \right| \left(U_{j+\frac{1}{2},k+1}^{n-\frac{1}{2}} - U_{j+\frac{1}{2},k}^{n+\frac{1}{2}} \right. \\
& \left. \left. - U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} + U_{j+\frac{1}{2},k-1}^{n-\frac{1}{2}} \right) \right] = 0 \quad (4.2)
\end{aligned}$$

where $p = \left(\frac{-\bar{V}^x}{2|\bar{V}^x|} \right)_{j+\frac{1}{2},k+\frac{1}{2}}$ and $q = \left(\frac{-\bar{V}^x}{2|\bar{V}^x|} \right)_{j+\frac{1}{2},k-\frac{1}{2}}$

For the second operation the equations are:-

$$\begin{aligned}
& \left(\eta_{j,k}^{n+1} - \eta_{j,k}^{n+\frac{1}{2}} \right) + \frac{\Delta t}{2\Delta s} \left[(\bar{h}^y + \bar{\eta}^x)_{j+\frac{1}{2},k} U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right. \\
& - \left. (\bar{h}^y + \bar{\eta}^x)_{j-\frac{1}{2},k} U_{j-\frac{1}{2},k}^{n-\frac{1}{2}} \right] + \frac{\Delta t}{2\Delta s} \left[(\bar{h}^x + \bar{\eta}^y)_{j,k+\frac{1}{2}} V_{j,k+\frac{1}{2}}^{n+1} \right. \\
& - \left. (\bar{h}^x + \bar{\eta}^y)_{j,k-\frac{1}{2}} V_{j,k-\frac{1}{2}}^{n+1} \right] = 0 \quad (4.3)
\end{aligned}$$

and:-

$$\begin{aligned}
& \left(V_{j,k+\frac{1}{2}}^{n+1} - V_{j,k+\frac{1}{2}}^{n-1} \right) + \frac{2\Delta t}{\Delta s} \left[\left(\bar{V}_{j,k+\frac{1}{2}}^y \right)^2 - \left(\bar{V}_{j,k}^y \right)^2 \right]^n \\
& + \frac{2\Delta t}{\Delta s} \left[V_{j+\frac{1}{2}+p,k+\frac{1}{2}}^n \left(\bar{U}_{j+\frac{1}{2},k+\frac{1}{2}}^y \right)^{n+\frac{1}{2}} - V_{j-\frac{1}{2}+q,k+\frac{1}{2}}^n \left(\bar{U}_{j-\frac{1}{2},k+\frac{1}{2}}^y \right)^{n+\frac{1}{2}} \right] \\
& + 2\Delta t f \bar{U}_{j,k+\frac{1}{2}}^{n+\frac{1}{2}} + \frac{\Delta t}{\Delta s} g \left[\eta_{j,k+1}^{n+1} + \eta_{j,k+1}^n - \eta_{j,k}^{n+1} - \eta_{j,k}^n \right] \\
& + \Delta t g \left[\left(V_{j,k+\frac{1}{2}}^{n+1} + V_{j,k+\frac{1}{2}}^{n-1} \right) \left[\left(\bar{U}_{j,k+\frac{1}{2}}^{n+\frac{1}{2}} \right)^2 + \left(V_{j,k+\frac{1}{2}}^n \right)^2 \right]^{\frac{1}{2}} \right]
\end{aligned}$$

$$\begin{aligned}
& \left[\left(\bar{h}^x + \bar{\eta}^y \right)_{j,k+\frac{1}{2}} \left(\bar{c}^y \right)_{j,k+\frac{1}{2}}^2 \right] - 2\Delta t \rho_a C^* W W_y / \left[\rho \left(\bar{h}^x + \bar{\eta}^y \right)_{j,k+\frac{1}{2}} \right] \\
& - 12 \frac{\Delta t}{\Delta s^2} \sqrt{g} \frac{\left(\bar{h}^x + \bar{\eta}^y \right)_{j,k+\frac{1}{2}}}{\bar{c}_{j,k+\frac{1}{2}}^y} \left[\left| \bar{u}_{j,k+\frac{1}{2}}^{n+\frac{1}{2}} \right| \left(v_{j+1,k+\frac{1}{2}}^n - v_{j,k+\frac{1}{2}}^{n+1} \right. \right. \\
& - \left. \left. v_{j,k+\frac{1}{2}}^{n-1} + v_{j-1,k+\frac{1}{2}}^n \right) + \left| v_{j,k+\frac{1}{2}}^n \right| \left(v_{j,k+\frac{1}{2}}^n - v_{j,k+\frac{1}{2}}^{n+1} \right. \right. \\
& - \left. \left. v_{j,k+\frac{1}{2}}^{n-1} + v_{j,k-\frac{1}{2}}^n \right) \right] = 0 \quad (4.4)
\end{aligned}$$

where $p = \left(\frac{-\bar{u}^y}{2|\bar{u}^y|} \right)_{j+\frac{1}{2},k+\frac{1}{2}}$ and $q = \left(\frac{-\bar{u}^y}{2|\bar{u}^y|} \right)_{j-\frac{1}{2},k+\frac{1}{2}}$

The only terms requiring special mention at this stage are the convective accelerations expressing the lateral transport of momentum in the x and y directions respectively, i.e. the UV products. These terms are represented in such a way that their spatial locations are dependent upon the direction of the velocity component perpendicular to the axis direction being considered. For example, the grid location of the product UV when considered in the x direction depends on the sign of V. This scheme allows the momentum ρU , and similarly ρV , to be evaluated more nearly at the position where it originates. Thus, momentum is more rigorously conserved in the finite difference sense - see Williams and Holmes (121).

For the difference momentum equations (4.2) and (4.4) to be linked by the continuity equations in the manner shown, it is not possible for the derivatives of the water elevation to be fully centred in time. However, even though these terms are represented in the forward difference form, they are averaged throughout the equations to give a reduced leading

truncation error of $O(\Delta t/2)$. The use of central differences in space (for all the terms) gives truncation errors of the second order for Δs , i.e. $O(\Delta s^2)$. The remaining numerical properties of the equations (4.1) to (4.4) are the same as those for the central difference implicit scheme of Chapter 3, together with the time centred convective accelerations and the Du-Fort and Frankel representation of the effective stresses. The stability of the equations is therefore governed by the non-linear accelerative terms and hence reads:-

$$|U| + |V| \leq \frac{\Delta s}{\Delta t} \quad (4.5)$$

Care needs to be taken in choosing values for Δs and Δt such that, in conjunction with the open boundary conditions, the inconsistent behaviour of the Du-Fort and Frankel scheme has a negligible effect on the final solution.

4.2 SOLVING THE EQUATIONS BY THE GAUSSIAN ELIMINATION METHOD

For the type of problem at hand requiring the solution of a large number of simultaneous equations, the method of Gaussian Elimination and Backward Substitution is by far the most appropriate numerical solution procedure. The alternatives, chiefly involving iterative techniques, are generally too expensive (in the computational sense) for the number of unknown terms typically involved, see Weare (117).

To solve the equations (4.1) and (4.2) by Gaussian Elimination, the unknown implicit terms first need to be separated from the known explicit and constant terms. Using Leendertse's notation (63) this gives, for the continuity equation (4.1),:-

$$- r_{j-\frac{1}{2}}^{n+\frac{1}{2}} u_{j-\frac{1}{2},k}^{n+\frac{1}{2}} + \eta_{j,k}^{n+\frac{1}{2}} + r_{j+\frac{1}{2}}^{n+\frac{1}{2}} u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} = A_{j,k}^{n-\frac{1}{2}} \quad (4.6)$$

where:-

$$r_{j-\frac{1}{2}} = \frac{\Delta t}{2\Delta s} \left(h_{j-\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \eta_{j-1,k}^n + \eta_{j,k}^n \right)$$

$$r_{j+\frac{1}{2}} = \frac{\Delta t}{2\Delta s} \left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \eta_{j+1,k}^n + \eta_{j,k}^n \right)$$

$$A_{j,k}^{n-\frac{1}{2}} = \eta_{j,k}^n + \frac{\Delta t}{2\Delta s} \left(h_{j+\frac{1}{2},k-\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \eta_{j,k-1}^n + \eta_{j,k}^n \right) v_{j,k-\frac{1}{2}}^{n-1} \\ - \frac{\Delta t}{2\Delta s} \left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k+\frac{1}{2}} + \eta_{j,k+1}^n + \eta_{j,k}^n \right) v_{j,k+\frac{1}{2}}^{n-1}$$

and for the momentum equation (4.2):-

$$- s_j \eta_{j,k}^{n+\frac{1}{2}} + s_{j+\frac{1}{2}} u_{j+\frac{1}{2},k}^{n+\frac{1}{2}} + s_{j+1} \eta_{j+1,k}^{n+\frac{1}{2}} = B_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \quad (4.7)$$

where

$$s_j = s_{j+1} = g \frac{\Delta t}{\Delta s}$$

$$s_{j+\frac{1}{2}} = 1 + \Delta t g \left[\left(u_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right)^2 + \left(\bar{v}_{j+\frac{1}{2},k}^n \right)^2 \right]^{\frac{1}{2}} /$$

$$\left[\left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k} \left(\bar{c}^x \right)_{j+\frac{1}{2},k}^2 \right]$$

$$+ 12 \frac{\Delta t}{\Delta s^2} \sqrt{g} \frac{\left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k}}{\bar{c}_{j+\frac{1}{2},k}^x} \left[\left| u_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right| + \left| \bar{v}_{j+\frac{1}{2},k}^n \right| \right]$$

$$\begin{aligned}
B_{j+\frac{1}{2},k}^{n-\frac{1}{2}} &= U_{j+\frac{1}{2},k}^{n-\frac{3}{2}} - \frac{2\Delta t}{\Delta s} \left[\left(\bar{U}_{j+1,k}^x \right)^2 - \left(\bar{U}_{j,k}^x \right)^2 \right]^{n-\frac{1}{2}} \\
&- \frac{2\Delta t}{\Delta s} \left[U_{j+\frac{1}{2},k+\frac{1}{2}+p}^{n-\frac{1}{2}} \left(\bar{V}_{j+\frac{1}{2},k+\frac{1}{2}}^x \right)^n - U_{j+\frac{1}{2},k-\frac{1}{2}+q}^{n-\frac{1}{2}} \left(\bar{V}_{j+\frac{1}{2},k-\frac{1}{2}}^x \right)^n \right] \\
&+ 2\Delta t f \bar{V}_{j+\frac{1}{2},k}^n - \frac{\Delta t}{\Delta s} g \left[\eta_{j+1,k}^{n-\frac{1}{2}} - \eta_{j,k}^{n-\frac{1}{2}} \right] \\
&- \Delta t g \left(U_{j+\frac{1}{2},k}^{n-\frac{3}{2}} \right) \left[\left(U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right)^2 + \left(\bar{V}_{j+\frac{1}{2},k}^n \right)^2 \right]^{\frac{1}{2}} / \\
&\left[\left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k} \left(\bar{c}^x \right)_{j+\frac{1}{2},k}^2 \right] + 2\Delta t \rho_a C^* W_x \sqrt{\rho \left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k}} \\
&+ 12 \frac{\Delta t}{\Delta s^2} \sqrt{g} \frac{\left(\bar{h}^y + \bar{\eta}^x \right)_{j+\frac{1}{2},k}}{\bar{c}_{j+\frac{1}{2},k}^x} \left[\left| U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \right| \left(U_{j+\frac{3}{2},k}^{n-\frac{1}{2}} - U_{j+\frac{1}{2},k}^{n-\frac{3}{2}} + U_{j-\frac{1}{2},k}^{n-\frac{1}{2}} \right) \right. \\
&\left. + \left| \bar{V}_{j+\frac{1}{2},k}^n \right| \left(U_{j+\frac{1}{2},k+1}^{n-\frac{1}{2}} - U_{j+\frac{1}{2},k}^{n-\frac{3}{2}} + U_{j+\frac{1}{2},k-1}^{n-\frac{1}{2}} \right) \right]
\end{aligned}$$

When all the grid points $1 \rightarrow M$ are considered along the row, k , parallel to the x direction, the Eqs. (4.6) and (4.7) constitute a tridiagonal matrix of the form:-

$$\begin{bmatrix}
 1 & r_{3/2} & 0 & 0 & 0 & \text{-----} & 0 \\
 -s_1 & s_{3/2} & s_2 & 0 & 0 & \text{-----} & 0 \\
 0 & -r_{3/2} & 1 & r_{5/2} & 0 & \text{-----} & 0 \\
 0 & 0 & -s_2 & s_{5/2} & s_3 & \text{-----} & 0 \\
 \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\
 0 & 0 & \text{-----} & s_{M-1} & s_{M-1/2} & s_M & U_{M-1/2} \\
 0 & 0 & \text{-----} & 0 & -r_{M-1/2} & 1 & \eta_M
 \end{bmatrix}
 \begin{bmatrix}
 \eta_1 \\
 U_{3/2} \\
 \eta_2 \\
 U_{5/2} \\
 \eta_3 \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 U_{M-1/2} \\
 \eta_M
 \end{bmatrix}^{n+1/2}
 =
 \begin{bmatrix}
 A_1 \\
 B_{3/2} \\
 A_2 \\
 B_{5/2} \\
 A_3 \\
 \vdots \\
 \vdots \\
 \vdots \\
 \vdots \\
 B_{M-1/2} \\
 A_M
 \end{bmatrix}^{n-1/2}
 +
 \begin{bmatrix}
 r_{1/2} U_{1/2} \\
 0 \\
 0 \\
 0 \\
 0 \\
 \vdots \\
 \vdots \\
 \vdots \\
 0 \\
 -r_{M+1/2} U_{M+1/2}
 \end{bmatrix}^{n+1/2}$$

(4.8)

where the known velocities $U_{1/2}^{n+1/2}$ and $U_{M+1/2}^{n+1/2}$ provide the lower and upper boundary conditions respectively. Knowing the boundary conditions, whether they are open or closed (i.e. $U^{n+1/2}$ being finite or zero), the values of the velocities and water levels of the vector $(\eta_1, U_{3/2}, \eta_2, U_{5/2}, \eta_3, \dots, U_{M-1/2}, \eta_M)^{n+1/2}$ can be solved for by a process of elimination of unknowns. Starting with the first equation of the matrix, the water level $\eta_1^{n+1/2}$ can be written in the form:-

$$\eta_1^{n+1/2} = -P_1 U_{3/2}^{n+1/2} + Q_1 \quad (4.9)$$

where $P_1 = r_{3/2}$

$$Q_1 = A_j^{n-1/2} + r_{1/2} U_{1/2}^{n+1/2}$$

Equation (4.9) can now be substituted into the second equation of the matrix to eliminate the water level $\eta_1^{n+1/2}$, this gives:-

$$\begin{aligned}
& - s_1 \left(-P_1 U_{3/2}^{n+1/2} + Q_1 \right) + s_{3/2} U_{3/2}^{n+1/2} + s_2 \eta_2^{n+1/2} \\
& = B_{3/2}^{n-1/2}
\end{aligned} \tag{4.10}$$

or, expressing $U_{3/2}^{n+1/2}$ in terms of $\eta_2^{n+1/2}$, yields:-

$$U_{3/2}^{n+1/2} = -R_1 \eta_2^{n+1/2} + S_1 \tag{4.11}$$

where

$$\begin{aligned}
R_1 &= \frac{s_2}{\left(s_{3/2} + s_1 P_1 \right)} \\
S_1 &= \frac{B_{3/2}^{n-1/2} + s_1 Q_1}{\left(s_{3/2} + s_1 P_1 \right)}
\end{aligned}$$

Similarly, the velocity $U_{3/2}^{n+1/2}$ can be eliminated by substituting Eq. (4.11) into the third equation of the matrix, thereby expressing the water level $\eta_2^{n+1/2}$ in terms of $U_{5/2}^{n+1/2}$:-

$$\eta_2^{n+1/2} = -P_2 U_{5/2}^{n+1/2} + Q_2 \tag{4.12}$$

where $P_2 = \frac{r_{5/2}}{\left(1 + r_{3/2} R_1 \right)}$

$$Q_2 = \frac{A_2^{n-1/2} + r_{3/2} S_1}{\left(1 + r_{3/2} R_1 \right)}$$

Substituting Eq. (4.12) into the fourth matrix equation to eliminate $\eta_2^{n+1/2}$ and so on, gives the following general recurring coefficients:-

$$U_{J-\frac{1}{2}}^{n+\frac{1}{2}} = -R_{J-1} \eta_J^{n+\frac{1}{2}} + S_{J-1} \quad (4.13)$$

$$\eta_J^{n+\frac{1}{2}} = -P_J U_{J+\frac{1}{2}}^{n+\frac{1}{2}} + Q_J \quad (4.14)$$

$$\text{where } R_{J-1} = \frac{s_J}{(s_{J-\frac{1}{2}} + s_{J-1} P_{J-1})}$$

$$S_{J-1} = \frac{B_{J-\frac{1}{2}}^{n-\frac{1}{2}} + s_{J-1} Q_{J-1}}{(s_{J-\frac{1}{2}} + s_{J-1} P_{J-1})}$$

$$P_J = \frac{r_{J+\frac{1}{2}}}{(1 + r_{J-\frac{1}{2}} R_{J-1})}$$

$$Q_J = \frac{A_J^{n-\frac{1}{2}} + r_{J-\frac{1}{2}} S_{J-1}}{(1 + r_{J-\frac{1}{2}} R_{J-1})}$$

These recurring coefficients can be calculated for all the matrix coefficients $J = 1, 3/2, 2, 5/2, \dots, M$ until finally, the last but one equation can be used to eliminate $U_{M-\frac{1}{2}}^{n+\frac{1}{2}}$ from the last equation giving one equation with only one unknown $\eta_M^{n+\frac{1}{2}}$. Similarly, by continuing this process of back substitution, all the unknown water levels and velocities of the vector $(\eta_M, U_{M-\frac{1}{2}}, \dots, \eta_3, U_{5/2}, \eta_2, U_{3/2}, \eta_1)^{n+\frac{1}{2}}$ can be found in turn. This numerical solution procedure is then repeated for the $k + 1$ row etc., until all the N rows parallel to the x direction have been solved. When completed, the values of the water levels $\eta^{n+\frac{1}{2}}$ and velocities $U^{n+\frac{1}{2}}$ are known throughout the domain.

During the second operation from time $(n + \frac{1}{2})\Delta t$ to $(n + 1)\Delta t$, the computation of the velocities $V_{k+\frac{1}{2}}^{n+1}$ and water levels η_k^{n+1} are carried out in the same manner as described for Eqs. (4.1) and (4.2). Equations (4.3) and (4.4) are first modified and put into matrix form as before. From

this matrix equation similar recurring coefficients are developed which, when expressed for the row j parallel to the y direction, read:-

$$V_{k-\frac{1}{2}}^{n+1} = -R_{k-1} \eta_k^{n+1} + S_{k-1} \quad (4.15)$$

$$\eta_k^{n+1} = -P_k U_{k+\frac{1}{2}}^{n+1} + Q_k \quad (4.16)$$

where

$$R_{k-1} = \frac{s_k}{\left(s_{k-\frac{1}{2}} + s_{k-1} P_{k-1} \right)}$$

$$S_{k-1} = \frac{B_{k-\frac{1}{2}}^n + s_{k-1} Q_{k-1}}{\left(s_{k-\frac{1}{2}} + s_{k-1} P_{k-1} \right)}$$

$$P_k = \frac{r_{k+\frac{1}{2}}}{\left(1 + r_{k-\frac{1}{2}} R_{k-1} \right)}$$

$$Q_k = \frac{A_k^n + r_{k-\frac{1}{2}} S_{k-1}}{\left(1 + r_{k-\frac{1}{2}} R_{k-1} \right)}$$

and

$$\begin{aligned} A_{j,k}^n &= \eta_{j,k}^{n+\frac{1}{2}} + \frac{\Delta t}{2\Delta s} \left(h_{j-\frac{1}{2},k+\frac{1}{2}} + h_{j-\frac{1}{2},k-\frac{1}{2}} + \eta_{j-1,k}^{n+\frac{1}{2}} \right. \\ &+ \left. \eta_{j,k}^{n+\frac{1}{2}} \right) U_{j-\frac{1}{2},k}^{n-\frac{1}{2}} - \frac{\Delta t}{2\Delta s} \left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} + \eta_{j+1,k}^{n+\frac{1}{2}} \right. \\ &+ \left. \eta_{j,k}^{n+\frac{1}{2}} \right) U_{j+\frac{1}{2},k}^{n-\frac{1}{2}} \end{aligned}$$

$$B_{j,k+\frac{1}{2}}^n = V_{j,k+\frac{1}{2}}^{n-1} - 2 \frac{\Delta t}{\Delta s} \left[\left(\bar{V}_{j,k+1}^y \right)^2 - \left(\bar{V}_{j,k}^y \right)^2 \right]^n$$

$$\begin{aligned}
& - \frac{2\Delta t}{\Delta s} \left[v_{j+\frac{1}{2}+p, k+\frac{1}{2}}^n \left(\bar{u}_{j+\frac{1}{2}, k+\frac{1}{2}}^y \right)^{n+\frac{1}{2}} - v_{j-\frac{1}{2}+q, k+\frac{1}{2}}^n \left(\bar{u}_{j-\frac{1}{2}, k+\frac{1}{2}}^y \right)^{n+\frac{1}{2}} \right] \\
& - 2\Delta t f \bar{u}_{j, k+\frac{1}{2}}^n - \frac{\Delta t}{\Delta s} g \left[\eta_{j, k+1}^n - \eta_{j, k}^n \right] \\
& - \Delta t g \left(v_{j, k+\frac{1}{2}}^{n-1} \right) \left[\left(\bar{u}_{j, k+\frac{1}{2}}^{n+\frac{1}{2}} \right)^2 + \left(v_{j, k+\frac{1}{2}}^n \right)^2 \right]^{\frac{1}{2}} / \\
& \left[\left(\bar{h}^x + \bar{\eta}^y \right)_{j, k+\frac{1}{2}} \left(\bar{c}^y \right)_{j, k+\frac{1}{2}}^2 \right] + 2\Delta t \rho_a C^* W_y / \left[\rho \left(\bar{h}^x + \bar{\eta}^y \right)_{j, k+\frac{1}{2}} \right] \\
& + 12 \frac{\Delta t}{\Delta s^2} \sqrt{g} \frac{\left(\bar{h}^x + \bar{\eta}^y \right)_{j, k+\frac{1}{2}}}{\bar{c}_{j, k+\frac{1}{2}}^y} \left[\left| \bar{u}_{j, k+\frac{1}{2}}^{n+\frac{1}{2}} \right| \left(v_{j+1, k+\frac{1}{2}}^n - v_{j, k+\frac{1}{2}}^{n-1} \right. \right. \\
& \left. \left. + v_{j-1, k+\frac{1}{2}}^n \right) + \left| v_{j, k+\frac{1}{2}}^n \right| \left(v_{j, k+\frac{1}{2}}^n - v_{j, k+\frac{1}{2}}^{n-1} + v_{j, k-\frac{1}{2}}^n \right) \right]
\end{aligned}$$

Again, both closed or open boundaries can be specified at the left and right hand extremities of the computing grid.

4.3 THE BOUNDARY CONDITIONS

In applying Equations (4.1) to (4.4) and the outlined solution procedure to the type of flow fields being considered, two difficulties associated with the closed and open boundaries need special comment.

Concerning the closed boundaries, the differential quotients in Eqs. (4.2) and (4.4) are dependent upon velocity values at grid points outside the actual boundaries. For example, the second term of Eq. (4.2), at the uppermost grid point $j, k = M, N$, can be expanded to give:-

$$\frac{2\Delta t}{\Delta s} \left[\left(u_{M+\frac{1}{2}, N}^{n-\frac{1}{2}} + u_{M+\frac{1}{2}, N}^{n-\frac{1}{2}} \right)^2 - \left(u_{M+\frac{1}{2}, N}^{n-\frac{1}{2}} + u_{M-\frac{1}{2}, N}^{n-\frac{1}{2}} \right)^2 \right]$$

where the velocity $U_{M+\frac{1}{2},N}$ lies outside the boundary, as shown in Fig. 8. The usual method of overcoming this difficulty is to use backward differences at the upper boundary and forward differences at the lower boundary. This results in the above expression being rewritten as:-

$$\frac{2\Delta t}{\Delta s} \left[\left(U_{M+\frac{1}{2},N}^{n-\frac{1}{2}} \right)^2 - \left(U_{M-\frac{1}{2},N}^{n-\frac{1}{2}} \right)^2 \right]$$

Kuipers and Vreugdenhil (58) found that for secondary flow computations this difference representation resulted in physically unrealistic water surface slopes, particularly for curved streamlines. In the present scheme this difficulty is removed by extrapolating the velocities outside the boundaries from velocities at interior grid points. In the actual mathematical model two different sets of extrapolated velocities are used depending on whether the condition of free-slip or no-slip is assumed at the closed boundaries, see Williams and Homes (121). For the free-slip condition the velocities extrapolated in the x direction, shown for the upper boundary, are (see Fig. 8(a)):-

$$V_{M+1,N+\frac{1}{2}} = V_{M,N+\frac{1}{2}}$$

$$V_{M+1,N-\frac{1}{2}} = V_{M,N-\frac{1}{2}}$$

(4.17)

$$U_{M+\frac{1}{2},N} = 0$$

$$U_{M+\frac{1}{2},N} = - U_{M-\frac{1}{2},N}$$

and similarly for the y direction. For the no-slip condition the

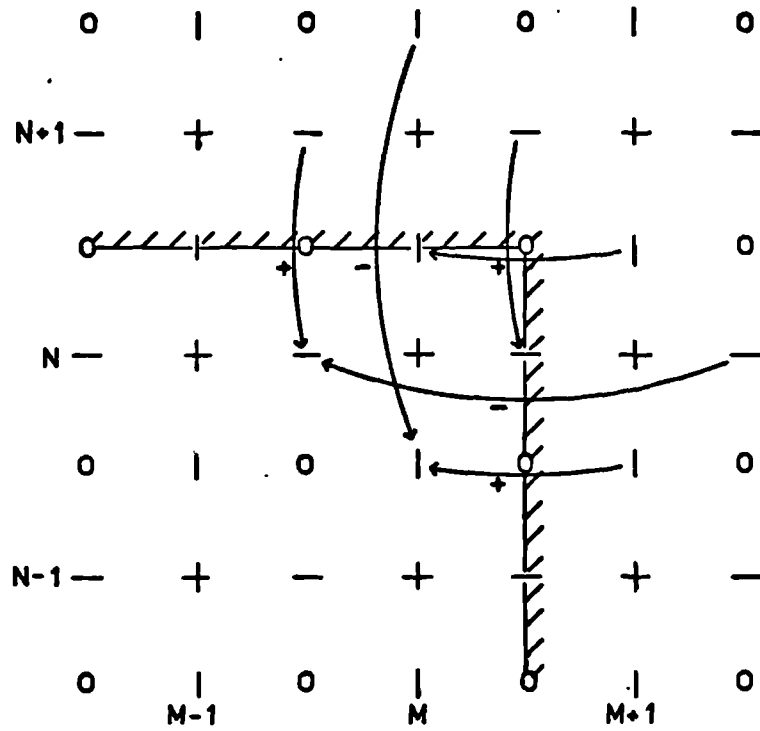


FIGURE 8(a) Free-Slip Boundary Conditions

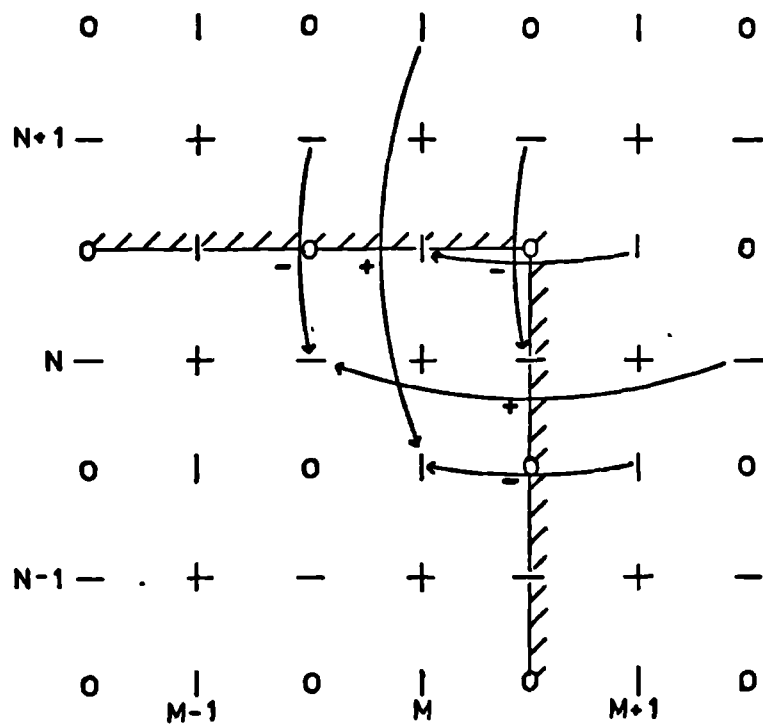


FIGURE 8(b) No-Slip Boundary Conditions

+ Water Level (η) 0 Depth (h)
 - U velocity U | V velocity V

FIGURE 8 Velocity Representation at the Boundaries

corresponding velocities are (see Fig. 8(b)):-

$$\begin{aligned}
 V_{M+1, N+\frac{1}{2}} &= -V_{M, N+\frac{1}{2}} \\
 V_{M+1, N-\frac{1}{2}} &= -V_{M, N-\frac{1}{2}} \\
 U_{M+\frac{1}{2}, N} &= 0 \\
 U_{M+\frac{1}{2}, N} &= U_{M-\frac{1}{2}, N}
 \end{aligned}
 \tag{4.18}$$

again with similar values in the y direction. Although the behaviour of difference schemes at the boundaries cannot be analytically predicted, see Chapter 3 (section 3.6.2), numerical experiments have indicated that this treatment of the closed boundary conditions does give satisfactory numerical results.

In the case of open boundaries, the numerical solution procedure illustrated in section 4.2 assumes that the given flow conditions are provided in the form of velocities, e.g. the terms $U_{\frac{n+1}{2}}$ and $U_{M+\frac{n+1}{2}}$ of the matrix equation (4.8). However, it is often more convenient to be able to specify these conditions in terms of discharges (as for the reservoir) or water levels (as for the harbour).

Discharges can be most suitably included in the mathematical model by expressing them in terms of the mean velocity across each corresponding grid spacing, see Kuipers and Vreugdenhil (58). For a known lower boundary discharge $Q_{j+\frac{n+1}{2}, k}^{\frac{n+1}{2}}$ between the grid points $j+\frac{1}{2}, k+\frac{1}{2}$ and $j+\frac{1}{2}, k-\frac{1}{2}$, the relationship used reads:-

$$\begin{aligned}
U_{j+\frac{1}{2},k}^{n+\frac{1}{2}} &= 2Q_{j+\frac{1}{2},k}^{n+\frac{1}{2}} / \left(h_{j+\frac{1}{2},k+\frac{1}{2}} + h_{j+\frac{1}{2},k-\frac{1}{2}} \right. \\
&\quad \left. + \frac{1}{2} \left(3\eta_{j+1,k}^n - \eta_{j+2,k}^n + \eta_{j+1,k+1}^n + \eta_{j+1,k-1}^n \right) \right) \quad (4.19)
\end{aligned}$$

and for the upper boundary:-

$$\begin{aligned}
U_{M+\frac{1}{2},k}^{n+\frac{1}{2}} &= 2Q_{M+\frac{1}{2},k}^{n+\frac{1}{2}} / \left(h_{M+\frac{1}{2},k+\frac{1}{2}} + h_{M+\frac{1}{2},k-\frac{1}{2}} \right. \\
&\quad \left. + \frac{1}{2} \left(3\eta_{M,k}^n - \eta_{M-1,k}^n + \eta_{M,k+1}^n + \eta_{M,k-1}^n \right) \right) \quad (4.20)
\end{aligned}$$

with similar relationships being used for the velocity components in the y direction.

To represent water levels rather than velocities at the open boundaries, the first and last continuity equations are removed from the matrix (4.8) of the solution procedure. This results in the first equation of the matrix being expressed as:-

$$U_{\frac{3}{2}}^{n+\frac{1}{2}} = -R_1 \eta_2^{n+\frac{1}{2}} + S_1 \quad (4.21)$$

$$\text{where } R_1 = \frac{s_2}{s_{\frac{3}{2}}}$$

$$S_1 = \frac{B_{\frac{3}{2}}^{n-\frac{1}{2}} + s_1 \eta_1^{n+\frac{1}{2}}}{s_{\frac{3}{2}}}$$

while the other matrix equations remain unchanged. Again a similar relationship can be developed for the y direction.

4.4 THE COMPUTER PROGRAM

The computer program ROGER used to solve the finite difference equations (4.1) to (4.4) is fully listed, including a typical output, in Appendix 1. This listing illustrates the computation for the velocities and water levels in an approximated circular reservoir, with a uniform depth and a diametrically opposite shallow jet inlet and weir outlet. The structure of this program is based on that developed by Leendertse (63); however, although the method of data input is similar, the system of equations being solved is quite different.

The manner in which the computational model solves the actual difference equations follows the same lines as the solution procedure outlined in section 4.2. Firstly, the necessary data and initial conditions are introduced into the main program by calling the subroutines KURIH, DIVE, FIND, DEPTH and CHEZY respectively. These subroutines are basically the same as those used by Leendertse and introduce the following information:-

SUBROUTINE KURIH - this subroutine describes the time dependent open boundary conditions at the various grid points by the terms XIA(K), XIB(K) etc., where K represents the number of half time steps. For the listing given, the flows specified by XIA and XIB are those of a sine wave between $\pi/2$ and $3\pi/2$ followed by a steady state situation at the wave peak. XIB gives the flow through the incoming jet, of width one grid space, while XIA gives the flow over the weir for each of three grid spacings.

SUBROUTINE DIVE - this subroutine sets up the physical boundaries of the velocity field to be computationally modelled. Each grid point is considered in turn with a 1 or 0 being introduced as data, depending on whether the grid point lies inside or outside the boundary. For the

program listed, the circular shape of the reservoir is approximated as closely as possible by a square grid containing 21 x 21 grid points.

SUBROUTINE FIND - this subroutine processes the DIVE data by tabulating the lower and upper boundary grid points for each column and row in the x and y directions respectively. This information is then used in the main program to set up the matrix equations, described in section 4.2, and to control the lower and upper bounds of the computation.

SUBROUTINE DEPTH - this subroutine reads in the water depths, between the bed and the still water level, for all of the grid points within the physical boundaries. For the example given, the reservoir depth is assumed to be uniform at 24 m with the shallow incoming jet being 4.3 m deep.

SUBROUTINE CHEZY - this subroutine reads in Chézy coefficients which represent an average value for the bottom roughness over a grid spacing. The Chézy value C is determined by using Manning's formula which, when expressed in metres, reads:-

$$C = \frac{(h + \eta)^{1/6}}{n} \quad (4.22)$$

where n = Manning's roughness coefficient. For the example shown, Manning's n has been taken as 0.03 giving a Chézy value of 56.6 m^{1/2}/sec.

Apart from the information supplied by these subroutines, a number of constant parameters need to be specified at the beginning of the program. These parameters, together with the values used in the listing, are as follows:-

ANGLAT = Latitude in degrees of the centre of the area of computation
(51.5°).

AG = acceleration due to gravity (9.81 m/sec²).

AL = grid size in metres between adjacent water levels (80 m).
 AT = number of seconds per half time step (30 secs).
 FW = friction coefficient for the air-water interface (0.03).
 RHOD = ratio of the density of air to that of water (0.0012).
 WX = wind velocity in the x direction (zero).
 WY = wind velocity in the y direction (7.5 m/sec).

With all the necessary data being specified, the numerical solution procedure begins by executing the computations for the first half time step AT. This involves solving the equations (4.1) and (4.2), in their programmed form, to give the velocities $U_P (\equiv U_{j+\frac{1}{2},k}^{n+\frac{1}{2}})$ and the water levels $SEP (\equiv \eta_{j,k}^{n+\frac{1}{2}})$, throughout the domain specified by the SUBROUTINE DIVE. At the end of the first half time step these variables are transferred to the corresponding lower time level arrays UM and SEM. Similarly, for the second half time step, the computational solution procedure is repeated for the equations (4.3) and (4.4) to give the velocities $V_P (\equiv V_{j,k+\frac{1}{2}}^{n+1})$ and the water levels $SEP (\equiv \eta_{j,k}^{n+1})$. Again the higher time level values of these variables are transferred to those of the lower time level VM and SEM respectively. The computing cycle then progresses to the next time step, and so on, until the input data, specified by NPRINT, requests a printout of the water levels $SEM (\equiv \eta_{j,k}^{n+\frac{1}{2}})$ and the velocities $UM (\equiv U_{j+\frac{1}{2},k}^{n+\frac{1}{2}})$ and $\frac{1}{2}(VP + VM) (\equiv \frac{1}{2}(V_{j,k+\frac{1}{2}}^{n+1} + V_{j,k+\frac{1}{2}}^n))$. For certain time steps of particular interest, e.g. at mean tide, the computed velocities are more readily interpreted by calling the SUBROUTINE PLOTRAF. This subroutine first determines the vectors of the velocity components $U_{j,k}^{n+\frac{1}{2}}$ and $V_{j,k}^{n+\frac{1}{2}}$ according to the relationship:-

$$\begin{aligned} \bar{V}_{j,k}^{n+\frac{1}{2}} &= \frac{i}{2} \left(U_{j+\frac{1}{2},k}^{n+\frac{1}{2}} + U_{j-\frac{1}{2},k}^{n+\frac{1}{2}} \right) \\ &+ \frac{1}{4} \left(V_{j,k+\frac{1}{2}}^{n+1} + V_{j,k+\frac{1}{2}}^n + V_{j,k-\frac{1}{2}}^{n+1} + V_{j,k-\frac{1}{2}}^n \right) \end{aligned} \quad (4.23)$$

with the vector $\bar{V}_{j,k}^{n+\frac{1}{2}}$ being denoted by the mnemonic HEIGHT in the subroutine. Using a Kingmatic Drafting Machine the resulting velocity vectors are then plotted to scale - as shown in Appendix 11.

The program was run on a CDC 6400 computer under a KRONOS operating system, and used approximately 6 minutes of central processor time to compute the flow field for 400 time steps.

VERIFICATION AND APPLICATION OF THE MATHEMATICAL MODEL

5.1 TESTS OF A MODEL CIRCULAR RESERVOIR

The most appropriate method of testing the reliability of the mathematical model (described in Chapter 4) was to compare the computed and the measured velocity fields for a typical prototype reservoir. However, this objective was not readily attainable due to the general difficulties associated with measuring prototype flows and the corresponding lack of suitable data in the literature. As an alternative, a laboratory model reservoir was constructed and tested, and the measured velocities compared to those of the mathematical model.

As the purpose of the laboratory tests was to verify the mathematical model, rather than simulate the flow field of a particular prototype reservoir, it was convenient to idealise the hydraulic model geometry. The principal aims in the design were that the model should be simple to construct and the depth averaged velocities should be easy to measure. Consequently, the geometry of the reservoir was assumed to be circular, flat-bottomed, with a full depth jet inlet and a diametrically opposite weir outflow. The advantage of having a full depth inflow was that the resulting mean velocities represented those of the depth averaged velocity field more accurately. As the mathematical model had not been developed to represent flow features that occurred for less than the grid size Δx , it was advantageous to assume that this was also the width of the jet. Apart from conserving the jet momentum, in the finite difference sense, this assumption allowed a direct comparison to be made between the computed and the measured velocity fields.

To economise in computer time a coarse grid (19 x 19) was used, with the jet width and weir crest being 1/19th and 3/19ths of the reservoir diameter respectively, as shown in Fig. 9.

Although the actual geometry of the reservoir model had already been specified, it would have been preferable to arrange the flow conditions to simulate those of a prototype as closely as possible. This would have ensured that the laboratory tests were being carried out for realistic velocity fields and proportionate geometric dimensions. However, it was not possible to attain perfect dynamic similarity in the model and it was therefore necessary to choose between either a Froude or a Reynolds number scaling. For the type of flow fields being considered, where the incoming jet momentum had to raise the water level (against gravity) over the sharp crested weir, it was essential that the Froude number should be kept constant. Expressing the Froude number Fr in terms of the jet velocity, this gave:-

$$F_r = \frac{V_{jet}^2}{gh} = \text{constant} \quad (5.1)$$

where V_{jet} = the jet velocity, assumed to act over the entire depth
 g = acceleration due to gravity
 h = the mean initial depth

The prototype reservoir chosen to define the required model parameters was the Thames Water Authority's Datchet Reservoir. This was thought to be an ideal prototype for this purpose due to its almost circular, flat-bottomed topographical features, see Sobey (98). Although the jets in Datchet Reservoir have a typical velocity of 4.4 m/sec and

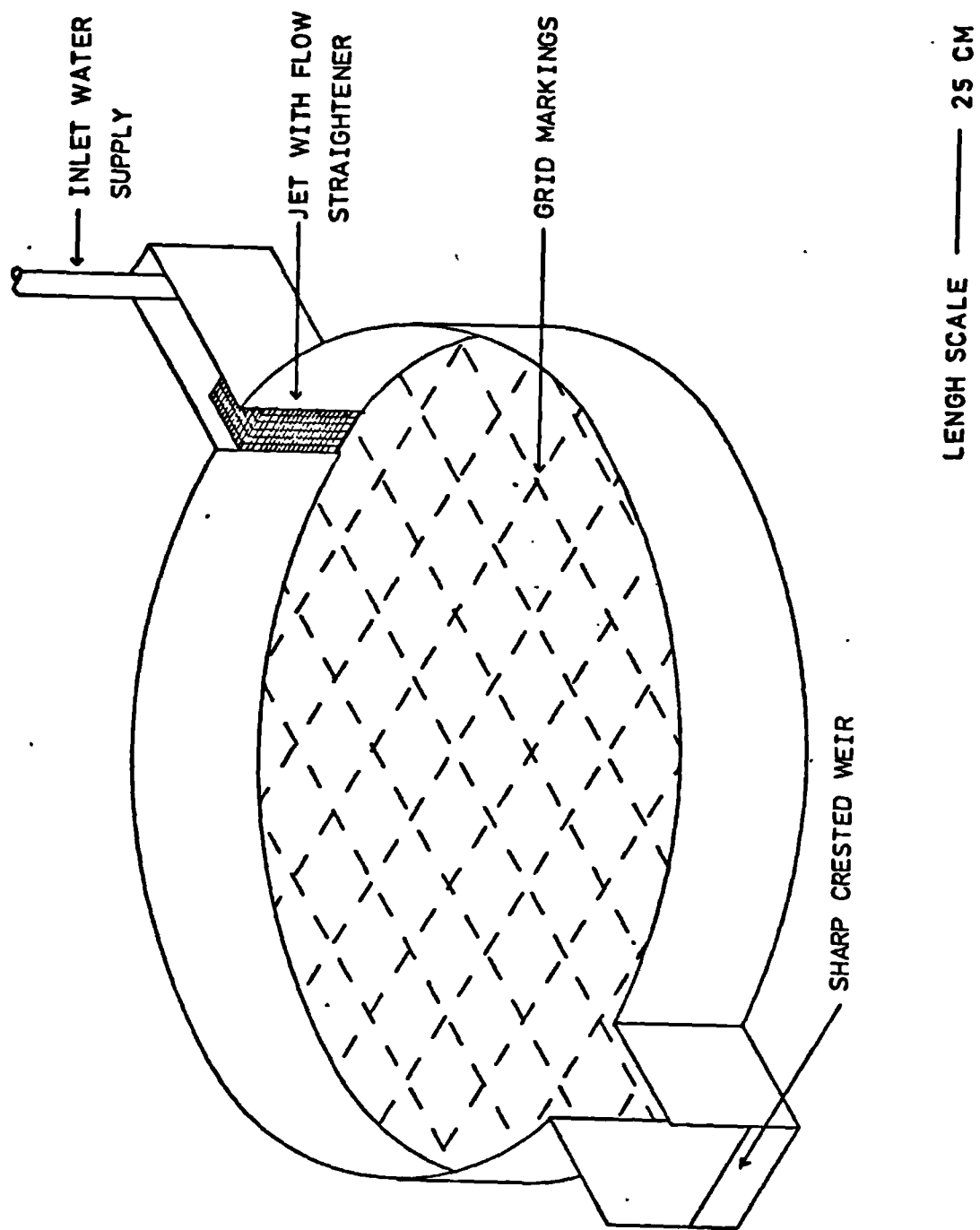


FIGURE 9 The Model Reservoir

are elevated at 22.5° , it was assumed that the horizontal component of the jet velocity acted uniformly throughout the entire depth, i.e. $V_{\text{jet}} = 1.68 \text{ m/sec}$ ($= 4.4 \times \sin 22.5^\circ$). For the corresponding hydraulic model, the horizontal and vertical length scales were assumed (where possible) to be 1 : 1000 and 1 : 300 respectively, giving a compatible velocity scale of $1 : 300^{\frac{1}{2}}$ or $h^{\frac{1}{2}}$. However, as the jet area in the model had already been conveniently exaggerated, the model inflow was much larger than that governed by these length scales, i.e. the model and prototype were not kinematically similar. Nevertheless this discrepancy was advantageous in that, not only were the higher velocities in the model easier to measure, but the behaviour of the mathematical model could be more stringently tested for highly non-uniform flows, i.e. for non-linear stability. The resulting prototype and model data can be seen in Table 2, with the correctly scaled model data shown in brackets.

TABLE 2 Typical Prototype Reservoir Data and Corresponding Model Data

	PROTOTYPE DATA	MODEL DATA
Capacity	$37.7 \times 10^6 \text{ m}^3$	$1.54 \times 10^4 \text{ cm}^3$
Mean Initial Depth	24.0 m	8.5 cm
Mean Diameter	1500 m	150 cm
No. of Inlets	2 (at 22.5°)	1 (at 0°)
Jet Area	1.24 m^2	68 cm^2 (0.041 cm^2)
Mean Jet Discharge	$5.43 \text{ m}^3/\text{sec}$	$750 \text{ cm}^3/\text{sec}$ ($0.41 \text{ cm}^3/\text{sec}$)
Mean Jet Velocity	4.4 m/sec	10 cm/sec
Latitude	51.4°N	—

The experiments carried out in the hydraulic model enabled the

mean velocity field for a steady flow to be measured. The reservoir was first filled with water and the jet inlet supply continually adjusted until a steady discharge of $750 \text{ cm}^3/\text{sec}$ was obtained (as determined by weighing the outflow). The model reservoir was left for approximately one hour so that the resulting pair of symmetric counter-rotating gyres were given sufficient time to acquire a constant angular velocity. The mean velocities were then determined by using 8.5 cm draught floats whose motion was assumed to correspond with the local depth averaged velocity. The floats were tracked and timed over a measured distance centred, as accurately as possible, over the intersecting points of a marked grid. This grid was arranged so that the intersection points coincided with alternate grid points in the mathematical model. After the mean velocity had been determined at each intersecting point, the velocities at corresponding locations on either side of the jet axis were averaged and plotted to scale using the computer. The measured velocity field for these flow conditions is shown in Fig. 15 of Appendix 11. Two further experimental tests were carried out for different flow conditions, allowing further comparisons to be made between the mathematical and the hydraulic model. In the first of these experiments the discharge was reduced to $540 \text{ cm}^3/\text{sec}$ and the jet velocity was $V_{\text{jet}} = 3.0 \text{ cm/sec}$. In the second experiment, with the original discharge of $750 \text{ cm}^3/\text{sec}$, the weir was raised by 3.5 cms giving an initial depth of 12 cms. As the jet had a free surface, increasing the depth reduced the jet velocity at the entrance to 7.0 cm/sec . The measured velocity fields for both these experiments are shown in Figs. 16 and 17 of Appendix 11 respectively.

The mathematical model corresponding to the hydraulic model

reservoir consisted of a domain of 19 x 19 grid points with an equidistant grid spacing of 8 cms. The jet inlet was represented at the central grid space of the x-direction lower boundary, while the weir outflow was represented at the diametrically opposite three adjacent grid spaces of the upper boundary. The circular configuration of the reservoir was approximated on the square grid in the same manner as that used by Kuipers and Vreugdenhil (58). The circular shape of the reservoir was located within the boundaries of the grid, and the water depths H at grid points outside the curve were then given appropriate values such that the cross-sectional area of each boundary grid space was equal to that defined by the actual curved surface, see Fig. 10. The open boundary conditions were read in as steady discharges and, for the weir outflow, the upper boundary water level was determined by using the sharp crested weir equation given by Henderson (47). The finite difference representation of this equation was expressed as:-

$$\eta_{M,k}^{n+\frac{1}{2}} = \left(\frac{3 q_{M+\frac{1}{2},k}^{n+\frac{1}{2}}}{2\sqrt{2g} \Delta s Cd} \right)^{\frac{2}{3}} \quad (5.2)$$

where $q_{M+\frac{1}{2},k}^{n+\frac{1}{2}}$ = the upper boundary discharge between the grid points $M+\frac{1}{2}, k+\frac{1}{2}$ and $M+\frac{1}{2}, k-\frac{1}{2}$

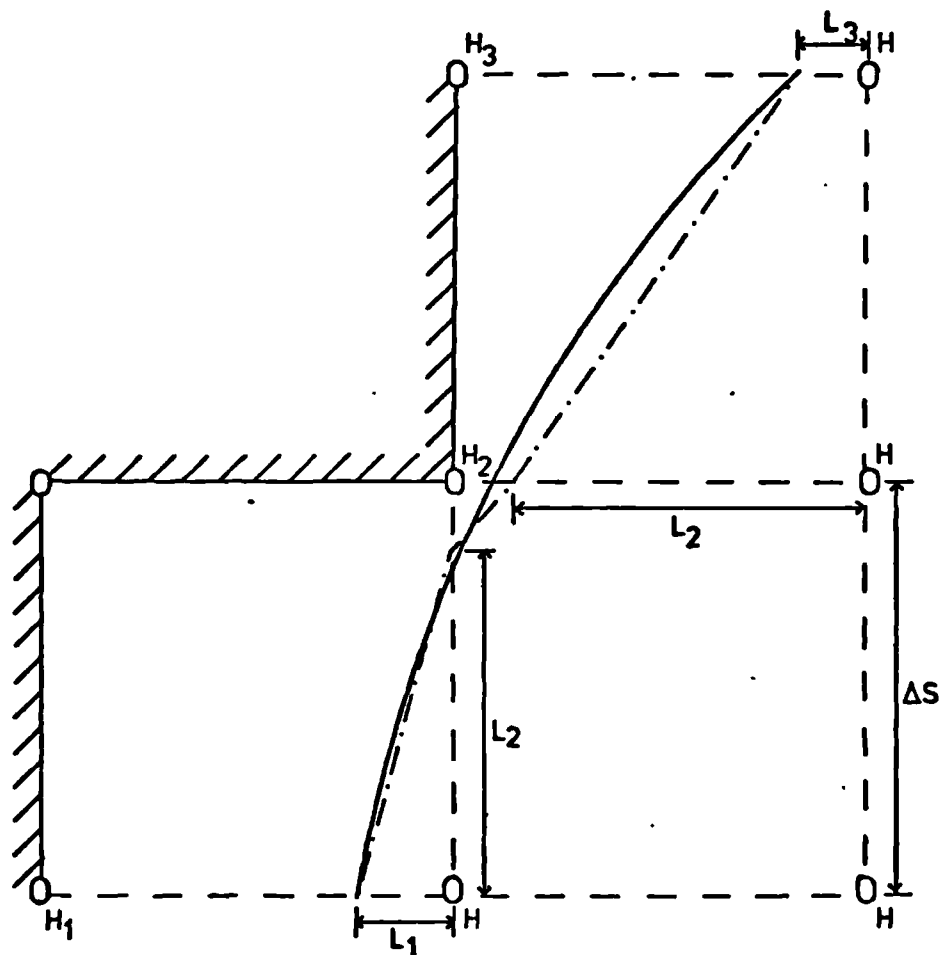
Δs = the grid spacing

Cd = the dimensionless weir discharge coefficient ≈ 0.6

The computational process was started from an initial state of rest with:-

$$\eta_{j,k} = u_{j+\frac{1}{2},k} = v_{j,k+\frac{1}{2}} = 0 \quad (5.3)$$

where $j = k = 0, \frac{1}{2}, 1, \frac{3}{2}, 2 \dots$ etc. Preliminary runs of the mathematical



O DEPTH (H)
 — CIRCLE-ARC (DESIRED BOUNDARY)
 - - - - BOUNDARY DEFINED BY $H_i = H \left(\frac{2L_i}{\Delta S} - 1 \right)$

FIGURE 10 Closed Boundary Representation for a Curve After Kuipers and Vreugdenhil

model showed that the rate of convergence, from rest to the final numerical solution, was accelerated by introducing the downstream open boundary conditions in characteristic form, see Wurtele et al (122). For a boundary with a positive propagation velocity, i.e. a velocity directed out of the model, the relationship used reads (see Kuipers and Vreugdenhil (58)):-

$$U - U_d = \frac{g}{c_p} (\eta - \eta_d) \quad (5.4)$$

where c_p = the propagation velocity = $\sqrt{g(h + \eta)}$

U_d = the desired velocity

η_d = the desired water level

Combining this equation with the weir equation (5.2), the resulting finite difference approximation of the downstream upper boundary condition gave:-

$$U_{M+\frac{1}{2},k}^{n+\frac{1}{2}} = \left(\frac{g}{\frac{-y}{h_{M+\frac{1}{2},k}} + \eta_{M,k}^n} \right)^{\frac{1}{2}} \left[\eta_{M,k}^{n+\frac{1}{2}} - \left(\frac{3 q_{M+\frac{1}{2},k}^{n+\frac{1}{2}}}{2\sqrt{2g} \Delta s C_d} \right)^{\frac{2}{3}} \right] + \left(\frac{q_{M+\frac{1}{2},k}^{n+\frac{1}{2}}}{\frac{-y}{h_{M+\frac{1}{2},k}} + \eta_{M,k}^n} \right) \quad (5.5)$$

The last equation of the x-direction tridiagonal matrix in section 4.2, i.e. $\eta_{M,k}^{n+\frac{1}{2}} = -P_M U_{M+\frac{1}{2},k}^{n+\frac{1}{2}} + Q_M^{n+\frac{1}{2}}$, was then substituted into Eq. (5.4) to give the upper boundary velocity $U_{M+\frac{1}{2},k}^{n+\frac{1}{2}}$. This boundary representation hardly reflected any incident waves and a steady state was obtained in a

shorter time. The computations were executed for a time step of 0.36 secs until, after approximately 400 time steps, the initial disturbances had completely disappeared and the computed velocities remained steady. A printout of the velocity field was produced for every 25th consecutive time step up to the 400th, when the final velocity vectors were plotted to scale. As the effect of the earth's rotation was not included in the hydraulic model the Coriolis parameter was set equal to zero; as was the wind velocity.

The velocity field was first computed using the following conditions:-

1. a free-slip at the boundary,
2. a discharge of $750 \text{ cm}^3/\text{sec}$,
3. an initial depth of 8.5 cm,

to correspond with the hydraulic model tests of Fig. 15. The resulting computations were non-linearly stable (see Fig. 18) and when compared to the corresponding measured velocity field (Fig. 15), the overall agreement of the flow patterns coincided reasonably well - particularly with respect to the diffusing jet. However, the magnitudes of the velocities in the symmetrical counter-rotating gyres were noticeably different. For the mathematical model, the velocities in the gyres were smaller than those measured experimentally and the peak velocities did not always occur adjacent to the curved boundaries - as in the hydraulic model. This discrepancy in the velocity magnitudes was probably due to the relatively large grid size of the mathematical model and the resulting imprecise representation of the the pronounced horizontal velocity gradients (or momentum change), see Deardorff (24).

In an attempt to predict the gyre velocities more accurately, other than by increasing the number of grid points and the computational expense,

the computed velocity field was determined with a no-slip closed boundary representation, see Fig. 19. It was thought that in this way vorticity would be more accurately conserved and that the velocities in the gyres might therefore be increased. However, in comparing Figs. 18 and 19 it was clear that this was not the case and, apart from predicting even lower velocities, the mean diameter of the gyres was also reduced.

In the hydraulic model experiment, flow visualisation studies (using potassium permanganate dye) indicated that near the centre of the gyres the flow appeared to be laminar, i.e. there was no obvious mixing. However, the mathematical model assumed that the flow was turbulent everywhere and that the Chezy coefficient was a function of the depth only. Consequently, it was thought that the measured velocities might be better predicted by the mathematical model if the Chezy coefficient C was varied with the local Reynolds number (corresponding to $(R_e = 4|U + V|(h + \eta)/\nu)$ for the open channel), and according to the friction factor relationships for a smooth pipe, see Reynolds (89). The resulting computed velocities, shown in Fig. 20, were determined by evaluating C - after every 5th consecutive time step - using Chezy's equation:-

$$C = \sqrt{2g/C_f} \quad (5.6)$$

where C_f = the friction factor, defined in the model as:-

for laminar flow i.e. $R_e \leq 2300$. $C_f = 16/R_e$

for transitional flow $C_f = 1.765 \times 10^{-6} R_e + 2.93 \times 10^{-3}$

for turbulent flow i.e. $R_e \geq 4000$ $C_f = 0.079(R_e)^{-0.25}$

In comparing Figs. 20 and 18 the two computed velocity fields appeared almost identical. This implied that the bottom roughness and the effective stresses hardly influenced the flow pattern for these particular hydraulic conditions, and that there was no advantage in using a more complicated representation of the bottom friction.

In deriving a suitable representation of the effective stresses in sections (2.3) and (2.4), the x and y directional components of the shear velocity magnitude $|U_* + V_*|$ were taken as being $\sqrt{g}|U|/C$ and $\sqrt{g}|V|/C$ respectively. However, some authors, e.g. Kuipers and Vreugdenhil (58) etc., have used the resultant velocity magnitude for both the x and y components of the shear velocity, i.e. $\sqrt{g}|U + V|/C$. In order to examine which of these two representations constitutes a better approximation of the horizontal diffusion of the turbulent momentum, the computed velocity field was determined as before but with the velocity magnitude being used for the components of the shear velocity, see Fig. 21. The results illustrated that for this particular mathematical model the damping of the turbulent momentum was excessive.

In summarising the comparisons of the measured and the various computed velocity fields, i.e. Fig. 15 and Figs. 18 to 21, the overall flow patterns were most accurately and economically predicted by using the free-slip closed boundary representation and the original mathematical model. For the two other hydraulic model experiments, i.e. with a lower discharge, Fig. 16, and with an increased depth, Fig. 17, the mathematical model was further tested by comparing the measured and the corresponding computed velocity fields - shown in Figs. 22 and 23 respectively. Again in both cases the overall flow patterns coincided reasonably well. Also, with the initial free-slip computed velocity field (Fig. 18), it was noticed that the mathematical model seemed to

have predicted the centre of the gyres in about the right place - at least to within one grid point. This result was particularly encouraging, bearing in mind that the centre of the gyres was a region of low velocities, or laminar flows.

5.2 TESTS OF A MODEL RECTANGULAR HARBOUR

As experimental results were available for the flows in model rectangular harbours with a tidal input (see Falconer (29) and Nece et al (76, 77, 78)), these were used to further check the performance of the mathematical model. In this case the observed mean water level flow patterns were compared to the corresponding computed velocity fields for various planform geometries, e.g. with different length to breadth ratios. The harbour was assumed to be rectangular, flat-bottomed, with vertical sidewalls and an asymmetric entrance.

The laboratory investigations were treated as idealised model studies of a number of prototype harbours, existing or proposed, in the Puget Sound water of the State of Washington, U.S.A., see Nece et al (78). The scale ratios used in the hydraulic models were 1 : 500 horizontally and 1 : 50 vertically, i.e. the models were vertically distorted with a 10 : 1 distortion ratio. The tides were assumed to be sinusoidal and repetitive, with a dynamic similarity governed by the Froude Law scaling relationship - as for the model reservoir. The typical prototype data used in this study, together with the corresponding model specifications can be seen in Table 3.

The experimental programme was conducted in a laboratory tidal tank containing a generator which produced constant period, constant amplitude tides. The harbour model tested in the basin had three adjustable sidewalls, allowing the overall flow patterns and the gross

TABLE 3 Typical Prototype Harbour Data and Corresponding Model Data

	PROTOTYPE DATA	MODEL DATA
Planform Area	$1.625 \times 10^5 \text{m}^2$	$4.65 \times 10^3 \text{cm}^2$
Length:Breadth (L/B=1.16)	367m : 317m	73cm : 63cm
(L/B=0.58)	260m : 448m	52cm : 90cm
(L/B=1.80)	457m : 254m	91cm : 51cm
(L/B=3.20)	610m : 191m	122cm : 38cm
Entrance Width d_1	80m	16cm
d_2	120m	24cm
Mean Depth	6.87m	13.74cm
Tidal Range-Mean	3.08m	6.16cm
Neap	1.54m	3.08cm
Spring	4.62m	9.24cm
Tidal Period	12.4 hrs	634 secs

flushing characteristics to be determined for different harbour lengths, breadths and entrance widths, see Fig. 11. For the first of the laboratory studies the harbour walls were adjusted to give the desired entrance width d and the initial length to breadth ratio (L/B) - to be consistent with the constant planform area given in Table 3. The tidal tank was filled with water and the sinusoidal oscillating generator weir was set at the low water level, from which point the tidal cycles were always commenced. The tide generator was then switched on and during the passage of five complete tidal cycles, the internal circulation patterns were observed and sketched at the mean water level for the ebb and flood tides. Ink dye was used as the colouring agent, in conjunction with polystyrene particles which were sprinkled over the water surface. In an independent test, the gross flushing characteristics were determined by measuring the changes in concentration of the fluorescent dye Rhodamine - WT. Knowing the initial and final dye concentrations, before and after five tidal cycles, the average per cycle exchange ratio X

was determined according to the relationship (see Nece et al (77)):-

$$X = 1 - (C_i/C_o)^{1/i} \quad (5.7)$$

where i = the number of tidal cycles

C_i = the final concentration

C_o = the initial concentration

These experiments were then repeated for each of a number of different combinations of the harbour lengths, breadths and entrance widths. The resulting observed velocity fields for the entrance width of 16 cms and for the L/B ratios of 1.16, 0.58, 1.8 and 3.2 respectively, are shown in Fig. 24 of Appendix 11.

The main conclusion from the observed velocity fields and the experimental data, see Falconer (29) and Nece et al (76), was that the maximum internal circulation occurred for a square harbour configuration. When the harbour length to breadth ratios (L/B) exceeded 2 or were less than 2/3 rds, several circulation cells existed within the harbour and the average gross flushing characteristics were reduced. However, within these limits there was little variation in the average per cycle exchange ratio X . The laboratory results also showed that a smaller entrance width gave rise to increased velocities within the harbour and so improved the internal mixing.

The four harbour model configurations shown in Fig. 24, with L/B ratios of 1.16, 0.58, 1.8 and 3.2, were most suitably represented in the mathematical model by domains of 19 x 19, 11 x 19, 19 x 11 and 19 x 5 grid points respectively. The resulting L/B ratios were 1.0, 0.58, 1.73 and 3.8 and the corresponding grid sizes were 4 cms, 5.3 cms, 5.3 cms and 8 cms. For the open boundary conditions, located at the

asymmetric harbour entrance, the repetitive tidal flow was represented in terms of the unknown water elevations as specified by the equation:-

$$\eta = a \sin (\omega t + \pi/2) \quad (5.8)$$

where a = the amplitude of 3.08 cms

ω = the wave number $2\pi/T$ where T is the wave period of 634 secs

t = time, given by the product of the time step ΔT and the time step level n .

In the case of the closed boundary conditions, the free-slip boundary representation was used for all of the mathematical models. Again, the computational processes were started from an initial state of rest, see Eq. (5.3), with the open boundary conditions commencing at the high tide water level, see Eq. (5.8). The computations were executed for a time step of 0.4 secs, with a printout being requested at every 24th increment of the tidal cycle, up to the termination time step level $n = 1188$, i.e. after three quarters of the tidal cycle. At the mean water level, for both the ebb and flood tides, the computed velocity fields were plotted to scale in the same manner as before. To coincide with the hydraulic model the effects of Coriolis and wind action were again ignored in the mathematical models.

The computed velocity fields were first determined for the square harbour configuration ($L/B = 1$) with an entrance width of 16 cms, see Fig. 25. In comparing these velocities with the corresponding observed flow patterns of Fig. 24a, the agreement was reasonably good - particularly since the former were determined during the first, as opposed to the fifth, tidal cycle. Although the finer flow details were not predicted by the mathematical model, e.g. vortex streets and small corner

eddies, the flood tide separation streamline (at the harbour entrance) and the unsteady circulation cell induced by this separation phenomenon, were well portrayed. Similarly, the velocity fields were then computed for the same harbour configuration, i.e. $L/B = 1$, but for the wider entrance width of 24 cms, see Fig. 26. In this case, apart from again giving good agreement with the observed flow patterns - see Falconer (29), the computed velocities confirmed the experimental conclusion that the wider entrance width gave rise to lower circulation velocities. For the next harbour configuration, with $L/B = 0.58$, the agreement between the computed and the observed flow fields, Figs. 27 and 24b respectively, was less acceptable. The observed flow patterns clearly indicated the existence of two similar counter-rotating circulation cells. Whereas, for the computed velocity fields, although the main circulation cell induced by the separation streamline was well predicted, the secondary cell - located at the rear end of the harbour - was not illustrated. This implied that the mathematical model was incapable of solving for the existence of several circulation cells, induced by the moments of the effective stresses about different origins. These conclusions were further substantiated by comparing the computed velocity fields for the harbour configuration with $L/B = 1.73$ to the corresponding observed flow patterns, i.e. Figs. 28 and 24c respectively. Again, only one main circulation cell was predicted rather than the two that occurred in the hydraulic model. For the final harbour configuration with $L/B = 3.8$, the agreement between the computed and the observed flow fields was exceptionally good - apart from some minor flow features, see Figs. 29 and 24d. This agreement was thought to have been due to the dominance of the adverse pressure gradients rather than the strong influence of the effective stresses as in the previous configurations.

Although the mathematical model did not predict the existence of several circulation cells, the general agreement between the computed and the observed flow fields was better than had been expected. This was particularly encouraging since the flow in the hydraulic model was always laminar whereas that in the mathematical model was assumed to be turbulent. However, for the mathematical model, some doubt arose as to the influence of the open boundary conditions on the internal circulation patterns. This was particularly true for the ebb tide where, in the laboratory tests, a pronounced cross flow occurred at the harbour entrance; this was not so obvious in the mathematical model. To provide some form of a check as to the effects of the entrance boundary conditions, the computed velocity fields of Fig. 28 were re-determined for a long uniform entrance channel, see Fig. 30. On comparing Figs. 28 and 30 the internal circulation patterns showed little difference, indicating that they were not strongly dependent upon the entrance conditions.

5.3 THE PROTOTYPE CIRCULAR RESERVOIR TESTS

With the comparisons between the computed and the measured velocity fields agreeing reasonably well for the model reservoir, the mathematical model was then applied to various flow conditions for the prototype reservoir - detailed in Table 2. The main purpose of these applications was to be able to compare the computed velocity fields for different flow conditions, boundary conditions, jet locations etc., so that useful engineering design information could be obtained.

Although the mathematical model was capable of predicting the velocity field for any planform geometry or local depth, for the sake of simplicity the prototype reservoir was assumed to be geometrically similar to that of the model tests, see section 5.1 and Fig. 9. This meant that the corresponding reservoir had a circular configuration of

diameter 1520 m, an initial constant depth of 24 m and a diametrically opposite jet inlet and weir outflow; in addition, the Manning's roughness coefficient was assumed to be 0.03, giving a Chezy coefficient C of $56.6 \text{ m}^{1/2}/\text{sec}$. In the mathematical model the reservoir was represented by a domain of 19×19 grid points with an equidistant grid spacing of 80 m. The circular closed boundary was approximated on the square grid by using the previously mentioned method of Kuipers and Vreugdenhil (58), see Fig. 10, and the jet inlet and weir outflow were again represented by one and three grid spacings respectively. In general, the throughflow was assumed to be steady, at 240 cumec, with the downstream open boundary conditions being determined by the weir equation (5.2), and the rate of convergence being accelerated by the characteristic relationship (5.4). In order to avoid non-linear instabilities the time step had to be severely constrained according to the relationship derived in Section 3.7, i.e.:-

$$\Delta t \leq \Delta x / U_0 \quad (5.9)$$

where U_0 = the maximum flow velocity, i.e. the throughflow per unit area of the jet

As before, a printout of the velocity field was produced for every 25th consecutive time step, with the velocity vectors being plotted to scale (apart from those greater than four grid spaces) and the computations terminated after the 400th time step. The computed velocity field was first determined for the reservoir with a full depth (two-dimensional) jet inlet, and for no Coriolis force or wind effects, i.e. as for the model tests. The resulting velocity field, see Fig. 31, clearly indicated that the jet momentum (or velocity) was insufficient to cause

any noticeable horizontal circulation. Apart from the region close to the jet centreline, the reservoir velocities were extremely small and the associated mixing could be assumed negligible. From the standpoint of enhancing the gross water quality characteristics it would appear that these flow conditions were generally poor. However, in reality the occurrence of a jet covering the full depth is unlikely, and the mathematical model was therefore modified to determine the velocity field for a shallow jet inlet having the same throughflow. The corresponding jet was approximated to that of a typical river inflow, in that, the depth was assumed to diverge uniformly from an initial value of 4.3 m to that of the full depth, see Fig. 12. In introducing these shallow jet specifications into the mathematical model it was not possible to use the simplified form of the momentum equations which assumed that the horizontal variations in depth were small, i.e. Eqs. (2.22) and (2.23). Instead, it was necessary to solve for the momentum equations in their original complete form, i.e. with the depth averaged convective accelerations being expressed as:-

$$\frac{1}{(h + \eta)} \left[\frac{\partial(h + \eta)U^2}{\partial x} + \frac{\partial(h + \eta)UV}{\partial y} \right] - \text{for the x-direction} \quad (5.10)$$

and

$$\frac{1}{(h + \eta)} \left[\frac{\partial(h + \eta)UV}{\partial x} + \frac{\partial(h + \eta)V^2}{\partial y} \right] - \text{for the y-direction} \quad (5.11)$$

These expressions were then represented in their appropriate finite difference form and replaced the corresponding accelerative terms in the solution procedure equations (4.2) and (4.4) respectively. The resulting velocity field with its higher momentum input, see Fig. 32,

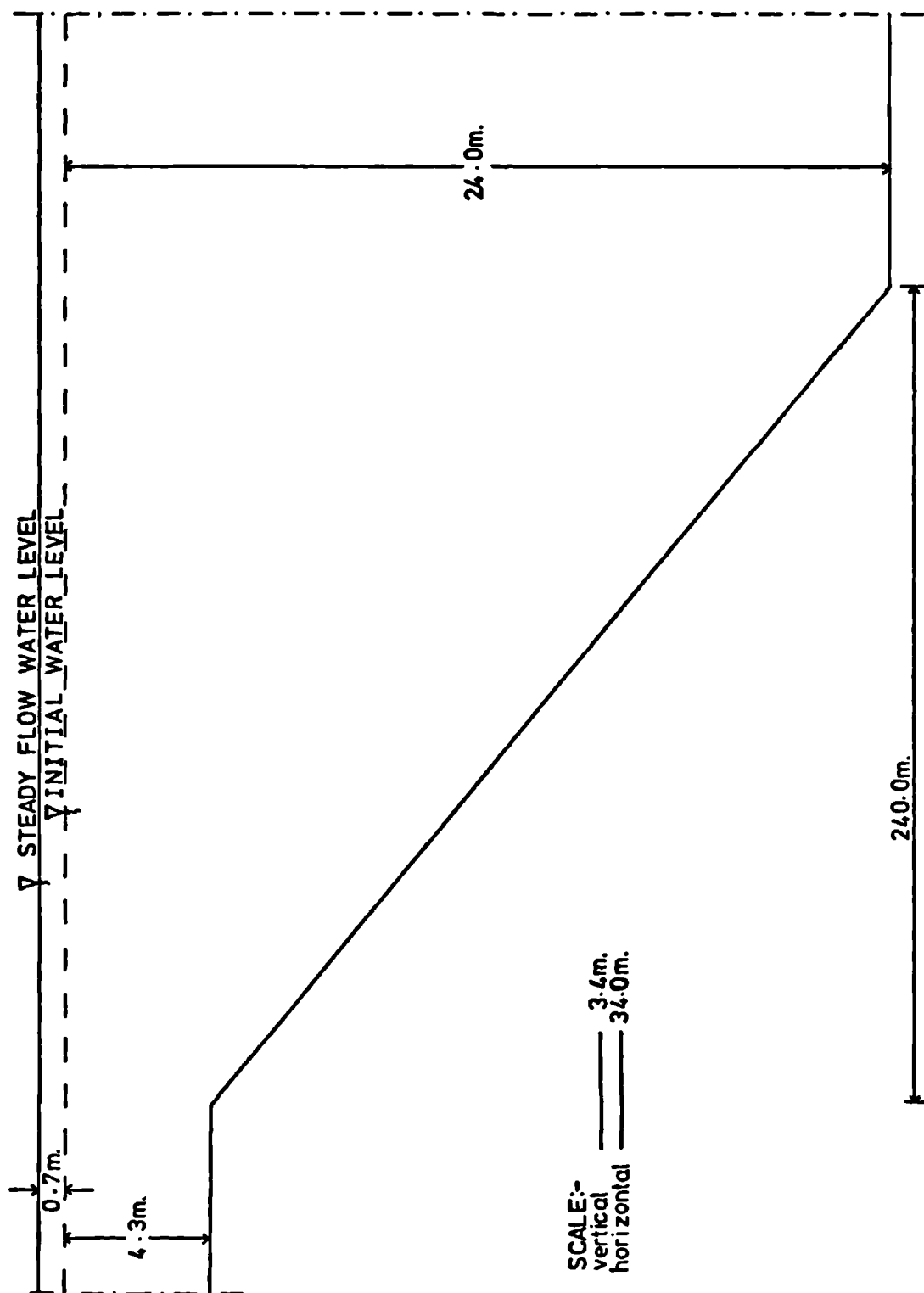


FIGURE 12 Longitudinal Cross-Section of the Shallow Jet Inlet

showed a significant increase in the overall reservoir velocities with the presence of two pronounced counter-rotating symmetrical gyres. This implied that, in general the gross water quality characteristics were much improved and the overall flow conditions far superior.

To study the effects of the convective accelerations, the effective stresses and the bottom roughness stress on the velocity field shown in Fig. 32, the appropriate terms of the equations of motion were individually omitted to give the corresponding velocity fields shown in Figs. 33, 34 and 35. In comparing these various velocity fields with the original (Fig. 32), the most apparent difference was the substantial influence of the convective accelerations. From Fig. 33 it can be seen that without these accelerative terms the existence of the counter-rotating gyres was not predicted and the velocity field showed little similarity to that of Fig. 32. This means that for highly non-uniform flows of this nature, not only must these non-linear terms be included but, extreme care needs to be taken to ensure that they are represented as accurately as possible in their finite difference form. On the other hand, Figs. 34 and 35 clearly showed that the effective stresses and the bottom roughness stress had a far less dominant influence in predicting the required velocity field. This indicated that the choice of Manning's roughness coefficient n , giving Chezy's C , had little effect on the final numerical solution. Although the influence of both these stresses on the circulation patterns was small, it was possible to observe that their individual exclusion from the momentum equations resulted in slightly lower and higher gyre velocities respectively. This observation supported the analysis of the depth averaged vorticity transport equation in section 2.5 (see Kuipers and Vreugdenhil (58)), in that the effective stresses tended to generate vorticity whereas the

bottom roughness stress had a dissipative effect.

In addition, the effect of varying the depth in Fig. 32, was considered for two different types of bed configurations. Firstly, the velocities were determined for a 16 m, as opposed to a 24 m, initial depth (i.e. a one-third reduction), see Fig. 36; and secondly, the reservoir bed was assumed to be V-shaped with the mean depth again being reduced to 16 m, see Fig. 37. These reductions in depth (Figs. 36 and 37) tended to increase the horizontal circulation and could therefore be assumed to improve the gross water quality characteristics. This result was not unexpected as, for the same momentum input, the reduced volume of water required less linear momentum to produce the same degree of angular momentum. Of the two differing bed forms studied, the second appeared to be less satisfactory due to the greater centreline depth and the associated lower velocity gradients giving rise to a lower lateral transport of momentum.

The velocity field was then determined as for Fig. 32 but with the inclusion of the Coriolis force in the two-dimensional depth averaged equations of motion, see Fig. 38. The resulting computations showed that, although the water level distribution did not change significantly, the horizontal velocity structure of the gyres was noticeably altered. The net effect of the Coriolis force appeared to give rise to an increase in the circulation velocities (or vorticity) in the anticlockwise gyre, while the velocities in the clockwise gyre were reduced. Although this resulted in a distinct asymmetry of the gyre velocities, the direction of the axial jet-flow was not appreciably deflected. This asymmetry of the gyre velocities was due to the streamline curvature associated with the vorticity generated by the vertical accelerations of the diverging jet over the first three grid points, see Batchelor (10).

The effect of a wind stress on the depth averaged velocity field in Fig. 38 was determined for a northerly (Fig. 39), southerly (Fig. 40) and westerly (Fig. 41) wind velocity of 7.5 m/sec. The resulting velocity fields further verified the analysis of section 2.5 in that, the circulation velocities (or vorticity) were increased for all three wind directions - as compared to the zero wind velocity in Fig. 38. Consequently, by generating vorticity the wind can be assumed to increase the overall lateral mixing, thereby improving the corresponding water quality characteristics. In comparing Figs. 39 and 40 it is interesting to note that, although the wind directions were opposite, the computed velocity fields were almost identical. This implied that the increase in vorticity was independent of the wind direction. Also, for these two particular computed velocity fields, the increase in vorticity has resulted in a more distinct deflection of the axial jet flow. For the westerly wind, Fig. 41 shows that there has been a tendency to partially compensate for the effects of the Coriolis force, with the degree of asymmetry of the counter-rotating gyre velocities being reduced. However, for the full depth jet inlet, the same westerly wind seems to have had little effect on the low momentum velocity field, as shown in Fig. 42.

In most prototype reservoirs it is rare for the throughflow to be steady for a long period of time - particularly when the inflow conditions are specified by a river inlet. Consequently, in order to determine the resulting velocity field for an unsteady flow, the mathematical model was modified to include, as open boundary conditions, the discharges given by the inflow-outflow hydrographs shown in Fig. 13. The outflow hydrograph was first determined by specifying a time dependent sinusoidal discharge relationship, with a period of 16 hours and a wave height of

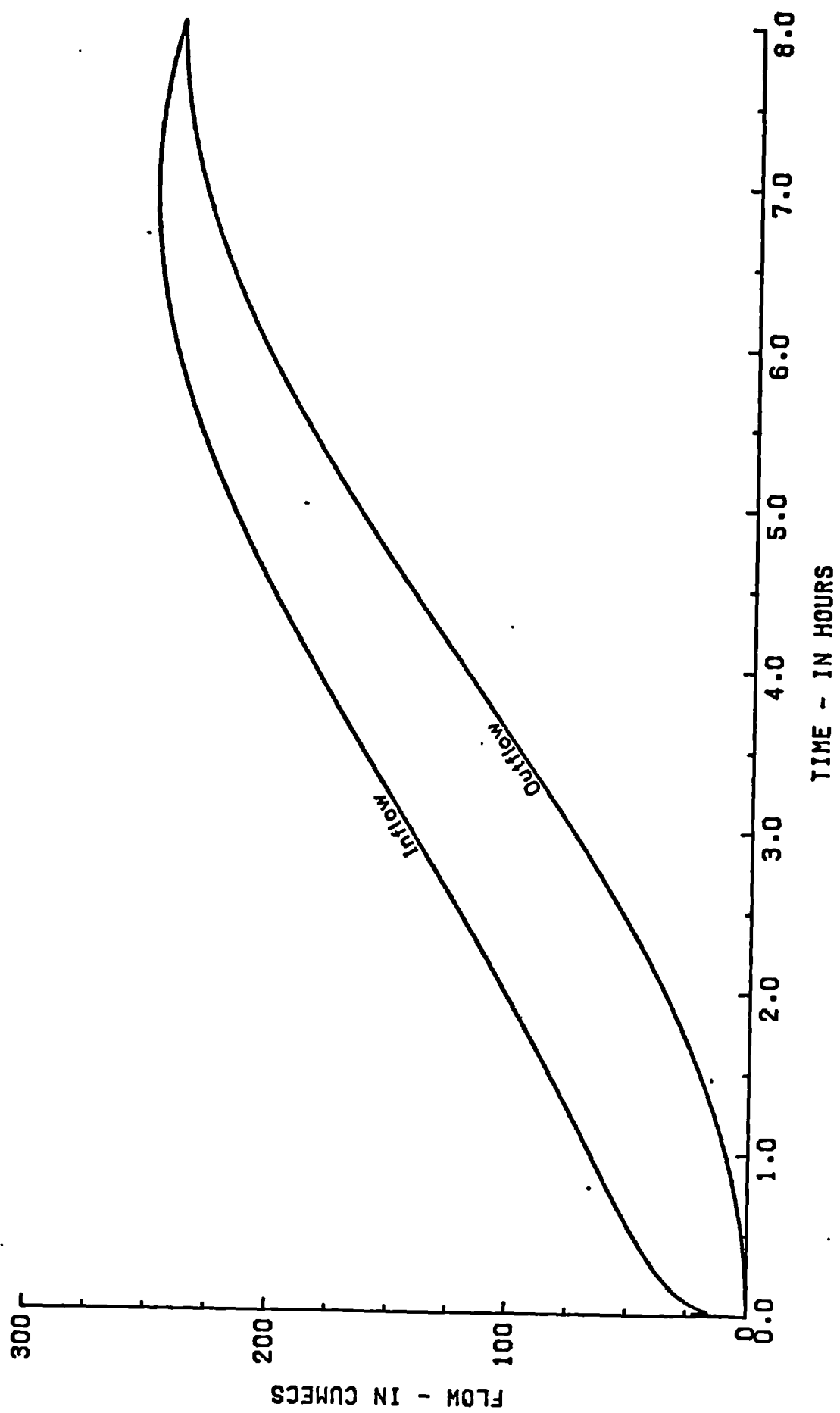


FIGURE 13 Open Boundary Conditions for the Unsteady Throughflow

240 cumec (as for the steady flow), of the form:-

$$Q_{out} = 120 \left(1 + \sin\left(\frac{\pi t}{8} - \frac{3\pi}{2}\right) \right) \quad (5.12)$$

where Q_{out} = the outflow
 t = time

Similarly, to determine the corresponding inflow hydrograph, the storage routing continuity equation (see Henderson (47)) was used, giving:-

$$Q_{in} - Q_{out} = \frac{dVol}{dt} \quad (5.13)$$

where Q_{in} = the inflow
 Vol = the volume stored in the reservoir

For the equation (5.13), the time rate of change of the reservoir volume was evaluated from the product of the reservoir planform area and the time derivative of the water elevation, specified by the weir equation (5.2), thus:-

$$\frac{dVol}{dt} = A \frac{d\eta}{dt} = \frac{A Q_{out}^{-1/3}}{\sqrt{2g} \Delta s C_d} \quad (5.14)$$

where A = the planform area of the reservoir

This equation assumes that the rate of rise of the water elevation was the same everywhere. With the known unsteady discharge relationships (Q_{in} and Q_{out}) being introduced into the mathematical model by the equations (4.19) and (4.20), the computed velocity fields were plotted

for the mean and peak outflow discharge levels, i.e. after 4 and 8 hours respectively, see Figs. 43a and 43b. In comparing these computed velocity fields, some idea of the time taken for the circulation cells, or gyres, to become fully established could be made; after 4 hours no circulation was apparent and at the peak discharge level, after 8 hours, the circulation was still noticeably less than that of the comparable steady flow in Fig. 38.

Finally, a number of velocity fields were determined to show the flexibility and ease of applicability of the mathematical model for different boundary configurations. With the same throughflow as before (240 cumec) these boundary configurations included:- the jet inlet with a west-east barrier (Fig. 44), a circumferential jet inlet (Fig. 45), a circumferential jet inlet with an east-west barrier (Fig. 46), two jet inlets at right angles (Fig. 47) and two adjacent jet inlets at right angles (Fig. 48). In comparing the computed velocity fields in Figs. 44, 45 and 46 with the original in Fig. 38, no general conclusions about the overall mixing and the gross water quality characteristics could be made - although it did appear that the barriers in Figs. 44 and 46 showed some improvement. However, for both the two jet inlet configurations in Figs. 47 and 48, the overall velocities within the reservoir were generally lower than those in Fig. 38 - indicating that the single jet was preferable for better mixing. This result was not surprising since, although the mass throughflow remained the same, the lower jet velocities and the larger total jet area resulted in a reduction of the net momentum input.

Apart from providing reasonable predictions of the prototype velocity fields, the mathematical model was seen to be non-linearly stable as long as the stability constraint of Eq. (5.9) was satisfied.

This was particularly encouraging considering the importance of the highly non-uniform flow properties - as shown by ignoring the convective accelerations (Fig. 33) - and the relatively high velocities associated with the shallow jet inlet.

5.4 THE PROTOTYPE SQUARE HARBOUR TESTS

With the growing demand for improvements in the tidal flushing characteristics of harbours, see Falconer (29), and the previous lack of satisfactory numerical modelling procedures for this purpose, see Nece et al (76), the mathematical model was finally applied to an idealised prototype harbour. It was expected that the turbulent equations of motion in the mathematical model would be a better approximation to the prototype flows, than the laminar flow occurring in the physical models.

For the sake of simplicity, the prototype harbour was assumed to have a square configuration, a flat bottom, vertical sidewalls and an asymmetric entrance, i.e. similar to that shown in Fig. 11. Using the data given in Table 3, the harbour was represented in the mathematical model by a domain of 19×19 grid points with a grid spacing of 20 m. This meant that the planform area was $1.44 \times 10^5 \text{m}^2$ and the entrance, covering 4 grid spaces, had a width of 80 m. The Manning's roughness coefficient n was assumed to be 0.03 and for the mean depth of 6.87 m the corresponding Chezy coefficient C , given by Eq. (4.22), was $44.7 \text{ m}^{1/2}/\text{sec}$. As in the case of the model harbour, the open boundary conditions were defined in terms of the water elevations for repetitive sinusoidal tides, as given by Eq. (5.8). For a mean tidal range of 3.08 m and with the mathematical model starting from the high water level, the specified open boundary conditions are shown in Fig. 14. The computations were executed from an initial state of rest and for a time step of 30 secs,

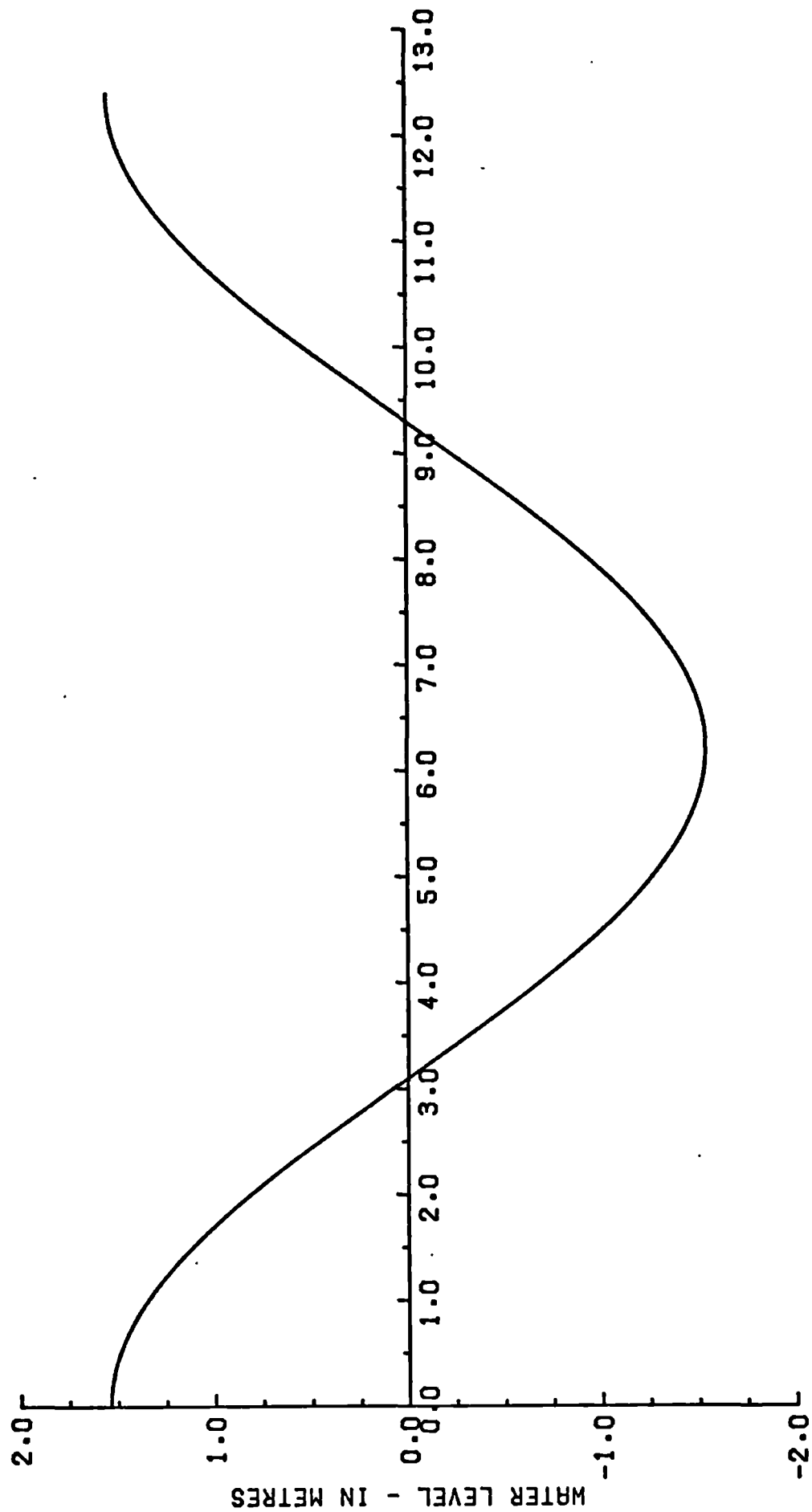


FIGURE 14 Open Boundary Conditions for the Mean Tidal Range

to the termination time step level $n = 1116$. As before, a printout was requested for every 24th increment of the tidal cycle, with the velocity fields being plotted to scale at the mean water level for the ebb and flood tides.

The computed velocity fields, including the effects of the Coriolis force, were first determined for the prototype harbour with the specified flow conditions, see Fig. 49. The resulting flood tide velocities showed a fairly uniform circulatory flow pattern across the basin, which suggested that the overall mixing and the gross flushing characteristics were reasonably good, i.e. there were no obvious stagnant regions. Also, in comparing these computed velocities to those of the corresponding harbour model in Fig. 25, the comparable velocity fields showed kinematic similarity, i.e. $V_{\text{prototype}} = h^{\frac{1}{2}} V_{\text{model}}$ where h is the mean depth. This meant that the vastly different scales of turbulence, between the model and prototype, had little influence on the resulting flow field (as in the reservoir).

To be able to compare the effects of some typical differing flow conditions, for the same harbour size, the computed velocity fields were then determined for:- twice the mean depth (Fig. 50), twice the harbour perimeter (Fig. 51), and the neap and spring tidal ranges of 1.54 m (Fig. 52) and 4.62 m (Fig. 53) respectively. A comparison of Figs. 49 and 50 showed that, by doubling the mean depth, the flood tide circulation cell had been completely eliminated and the depth averaged velocities were noticeably reduced. Therefore, in attempting to maximise the overall flushing and the water quality characteristics of any harbour, the mean depth needs to be kept as shallow as possible. However, for the harbour with twice the perimeter, see Fig. 51, even though the velocities were generally double those of the original (Fig. 49),

the circulatory flow patterns showed little difference, i.e. they were kinematically similar. This result further confirmed the model-prototype similarity, which indicated that the overall flushing and water quality characteristics were generally independent of the harbour size. In comparing the computed velocity fields for the neap, mean and spring tides, see Figs. 52, 49 and 53 respectively, the anticipated improvements in the overall circulation patterns, with increasing tidal ranges, were well illustrated. For the neap flood tide (Fig. 52), no circulation cell was apparent and an overall lack of mixing and flushing seemed to occur. On the other hand, the spring flood tide (Fig. 53) showed a noticeable increase in the degree of circulation generated (or vorticity), and clearly indicated that the gross water quality characteristics were much better than those of the mean and neap tides.

Finally, as for the prototype reservoir tests, the flexibility of the mathematical model was illustrated by determining the computed velocity fields for four different boundary configurations. With the same harbour size and flow conditions as before (Fig. 49), these boundary configurations included the individual positioning of a barrier, or shear wall, located at the centre and perpendicular to each of the four harbour sidewalls - as shown in Figs. 54 to 57 respectively. In comparing these computed velocity fields with those of the original in Fig. 49, it was impossible to make any general conclusions about the overall flushing and gross water quality characteristics. However, as for the reservoir tests, it appeared that the barrier showed some improvement in the general water quality characteristics - but particularly for the locations shown in Figs. 56 and 57.

Again, the mathematical model appears to have provided reasonable predictions of the prototype velocity fields, particularly since the

corresponding model-prototype velocities were kinematically similar. From the standpoint of avoiding non-linear instability, the much lower velocities in the harbour - as compared to the reservoir - did not result in the time step being constrained by Eq. (5.9) to anywhere near the same extent as before. However, subsidiary tests of the mathematical model showed that the use of a large time step resulted in an inaccurate prediction of the velocity field. This was due to the importance of the pressure gradient terms and their associated first order finite difference approximation. A comparison of the results of these subsidiary tests indicated that, to obtain a relatively accurate numerical prediction of the tidal flow, the Courant number μ should not exceed 15 - as satisfied for all the harbour tests. This upper limit of the Courant number therefore imposes the following time constraint:-

$$\Delta t \leq 15 \Delta x / \sqrt{gh} \quad (5.15)$$

where h is the mean depth.

A summary of the basic physical and numerical data used in the four different mathematical model tests of section 5.1, 5.2, 5.3 and 5.4 respectively, can be seen in Table 4.

Data	Mean Depth h	Chezy Coefficient C	Maximum Velocity U _o	Grid Size Δx	Time Step Δt	Non-Linear Stability No. $\frac{U_o \Delta t}{\Delta x}$	Courant No. μ $\frac{\Delta t}{\Delta x} \sqrt{gh}$
Model Tests							
Model Reservoir	8.50 cm	438 cm ^{1/2} /sec	10 cm/sec	8.0 cm	0.36 secs	0.45	4.1
Model Harbour	13.74 cm	405 cm ^{1/2} /sec	0.94cm/sec	4.0 cm	0.40 secs	0.10	11.6
Prototype Reservoir	24.0 m	56.6 m ^{1/2} /sec	0.6 m/sec	80.0 m	36 secs	0.27	6.9
Prototype Harbour	6.87 m	44.7 m ^{1/2} /sec	0.056m/sec	20.0 m	30 secs	0.08	12.3

TABLE 4 Basic Physical and Numerical Data for the Four Mathematical Models

CONCLUDING REMARKS

It has been shown that the jet-forced circulation in reservoirs and harbours can be successfully modelled by solving the finite difference equations developed. The resulting mathematical model - which includes the effects of bottom friction, wind action, the earth's rotation and the horizontal diffusion of turbulent momentum - was capable of computing the water elevations and the two-dimensional depth averaged velocity components. The model has been tested by comparing the computed velocity fields with those measured and observed for two hydraulic model studies. These studies included a jet flow penetrating a circular reservoir and a sinusoidal tide entering a rectangular harbour; in both cases the comparisons proved satisfactory. Similarly, the mathematical model was applied to typical prototype reservoirs and harbours so that useful engineering design information could be obtained.

With respect to the numerical aspects the following conclusions could be drawn:-

1. A study of the basic numerical properties of four finite difference schemes showed that the most suitable difference representation of the shallow water wave equations, including the bottom friction, was the fully centred (Crank-Nicolson) implicit scheme. However, in order to avoid excessive computational costs, the second derivatives of the effective stresses were best represented by the conditionally consistent explicit method of Du-Fort and Frankel. Although the effects of the Coriolis and the wind stress terms were to introduce weak instabilities into the original difference scheme, the dominance of the unconditionally stable bottom friction and effective stresses generally damped these disturbances.

2. In attempting to suppress the occurrence of non-linear instabilities, an analytical analysis of the quasi-linearised equations of motion indicated that the convective accelerations should be represented in the novel time centred form. Although this representation imposed the time constraint:-

$$\Delta t \leq \Delta x/U_0$$

it was still preferable to using an artificial viscosity term. The mathematical model tests confirmed these analytical conclusions, in that, provided this stability constraint was satisfied the numerical solution was always stable - even for the highly non-uniform flows.

3. In the mathematical model the finite difference equations were most suitably represented in an Alternating Direction Implicit form, and were solved by the method of Gaussian Elimination. These equations were interlinked through the water elevations which were not therefore fully centred in time. Preliminary tests of the model showed that, momentum was most rigorously conserved (in the finite difference sense) by expressing the convective accelerations in such a way that the lateral transport of momentum was evaluated more nearly at the position where it originated.

4. The numerical problems generally associated with the treatment of the velocity gradients at the closed boundaries were successfully overcome by adopting the same technique as that used in the Marker-and-Cell method, see Williams et al (121). This involved equating the velocities required outside the boundaries to the corresponding velocities inside. The sign with which these velocities were equated depended on whether the fluid-wall interface was assumed to be a free-slip or a no-slip boundary.

5. Although the implicit difference equations in the model did not impose any stability constraint on the value of the Courant number, subsidiary tests for the prototype harbour showed that no horizontal circulation, or vorticity, was reproduced if this number exceeded approximately 15. This result was principally due to the numerical inaccuracies associated with the water elevation terms not being fully centred in time. In order to represent circulatory flows satisfactorily it was concluded that, as well as the stability requirement of the non-linear terms, the following time constraint should also be imposed:-

$$\Delta t \leq 15\Delta x/\sqrt{gh}$$

However, even with this additional time constraint, these difference equations were still preferable to using those of the explicit type where, for stability, the Courant number had to be less than unity.

6. The accuracy with which horizontal gyres or circulation cells were reproduced also depended upon the number of grid points covering their diameters. For the prototype harbour (with a maximum possible gyre diameter of 19 grid points) the predicted gyres appeared to be more realistically represented and showed a smoother circulation than those of the reservoir (with a maximum possible gyre diameter of 9 grid points). Preliminary computational tests for the reservoir, with a domain of 31 x 31 grid points, verified this point by reproducing the counter-rotating gyres more smoothly. However, this numerical improvement was offset by a corresponding increase in the computational cost.

7. In applying the mathematical model to a steady throughflow situation, numerical experiments have shown that the rate of convergence to the final numerical solution was accelerated by using characteristic

relationships at the open boundaries. These relationships absorbed the initial disturbances and hardly reflected any incident waves. This resulted in a noticeable reduction of the computational cost.

With respect to the physical processes, the following conclusions could be drawn:-

8. The results of the mathematical model have shown that a realistic reproduction of the flow patterns for a jet-forced circulation in reservoirs and harbours was possible. Although small scale features such as vortex streets and corner eddies were not reproduced, the general shape and size of the gyres (or circulation cells) together with the separation streamlines were well predicted. This implied that the corresponding depth averaged equations of motion, including a simple turbulence model (i.e. Prandtl's mixing length hypothesis), adequately reproduced the mean flow properties and the physical processes involved.

9. The convective acceleration terms were shown to be extremely important in the generation of gyres (or circulation cells), and hence led to improvements in the overall mixing and gross water quality characteristics. This result implied that, in an attempt to maximise the water quality of a reservoir or harbour, the cross-sectional area of any jet inlet should be made as small as possible, giving a higher jet velocity (or momentum input) and a correspondingly higher non-uniformity of the flow.

10. The generation of gyres (or circulation cells) and the assumed improvements in the water quality were also shown to be enhanced, to a lesser extent, by the effective stresses and a wind stress. On the other hand, both increasing the depth and the effects of the bottom roughness tended to dissipate the vorticity associated with the gyres.

11. The influence of the Coriolis force on the velocity field for the

two-dimensional full depth jet inlet was shown to be negligible. However, for the shallow jet inlet and the associated three-dimensional divergence of the jet, the effect of the Coriolis force was to cause a distinct asymmetry of the velocity field. This resulted in the gyre velocities to the left of the jet axis being noticeably greater than those to the right. Consequently, although the Coriolis force did not have much affect on the fully two-dimensional horizontal flow, it substantially influenced the velocity field where the vertical accelerations were not negligible.

12. Finally, although no water quality parameters were simulated in the mathematical model, comparison of the various velocity fields indicated that the suitable location of barriers could be advantageous in improving the overall mixing, flushing and gross water quality characteristics in reservoirs and harbours.

RECOMMENDATION FOR FURTHER STUDY

Although the main objectives of this research have been satisfactorily accomplished (i.e. the numerical modelling of the depth averaged velocity field for a confined jet), the mathematical model developed has limitations. Any attempt to eliminate these limitations would therefore provide an obvious extension of this research and can only lead to a better understanding of the jet-forced circulation in reservoirs and harbours. These restrictions, and the corresponding recommendations for further study, are discussed in largely the same order as that of the text.

The neglect of stratification effects, or density differences, and the influence of secondary currents are perhaps the most serious restrictions of the mathematical model. To determine the effects of these flow features on the mean velocity field the mathematical model would have to be re-developed, in a manner similar to that described, for the three-dimensional steady flow. This would provide useful engineering information, particularly in the case of deep reservoirs where stratification effects are often severe.

The inclusion of a simple turbulence model in the equations of motion (Prandtl's mixing length hypothesis) and the corresponding assumption that the horizontal velocity gradients were small compared to those in the vertical plane, has resulted in an appreciable simplification of the turbulence structure. Consequently, further numerical experiments involving the inclusion of more sophisticated, computer orientated, turbulence models might provide more accurate numerical predictions of the required velocity fields - particularly near the boundaries.

The finite difference equations of motion, which were solved for in the mathematical model, did not include the water elevation terms in a fully time centred form. This meant that the truncation error was not completely of the second order and the resulting numerical solutions less accurate than they might have been. Therefore, further investigations involving more accurate finite difference representations of the water elevation terms would be advantageous - particularly for velocity fields where the pressure gradients are substantial.

The assumption that the width of the jet needed to be represented by at least one grid space, could not always be realistically applied to prototype boundary configurations. Studies into the representation of such boundary configurations, that are smaller than an economic grid size, would be most useful in the numerical prediction of a diffusing jet.

The present form of the mathematical model is such that a rise (or fall) of the water elevation over a sloping bed would not result in any change of the planform wetted area. Clearly this is a poor representation of many prototype flow fields, e.g. the flooding (or drawdown) of a reservoir with a V-shaped bed, and therefore needs to be further investigated if the model is to have a more general applicability.

The mathematical model, as such, does not include any quantitative predictions of a water quality parameter, e.g. temperature, concentration, salinity etc. However, with the depth averaged velocity field being satisfactorily computed, the inclusion of the convective-diffusion equation in the mathematical model (and its corresponding solution), would provide useful qualitative information to the design engineer. This information would be of particular interest in

determining whether or not a barrier, for example, does actually improve the gross water quality characteristics.

Apart from these limitations of the mathematical model, there is a need for more detailed experimental research and a more accurate method of measuring the low velocities typical of those in the laboratory models. These results would allow the computed velocity fields to be more accurately checked and compared with those measured, particularly when the flows in the hydraulic model are fully turbulent. Even better, however, would be an attempt to measure the velocities in some typical prototype.

Finally, further applications of the mathematical model, to different planform geometries and flow conditions, would provide further information to the design engineer faced with improving any existing or anticipated water quality problems in a reservoir or harbour.

APPENDIX 1

**FORTRAN LISTING OF THE
COMPUTER PROGRAM ROGER**

PROGRAM LISTING FOR THE
JET-FORCED CIRCULATION IN A
PROTOTYPE CIRCULAR RESERVOIR

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C
C      ROGER ALEXANDER FALCONER,
C      DEPT. OF CIVIL ENGINEERING,
C      IMPERIAL COLLEGE,
C      SOUTH KENSINGTON, LONDON SW7.
C
C      MATHEMATICAL MODELLING OF JET-FORCED CIRCULATION
C      IN RESERVOIRS AND HARBOURS.
C
C      PROGRAM ROGER (INPUT,OUTPUT, TAPE5=INPUT, TAPE6=OUTPUT, TAPE62)
C      SET DIMENSIONS OF SYSTEM
C      DIMENSION A(21), B(21), P(21), Q(21), R(21), S(21), NH(21),
1    INPLOT(20), NPRINT(80), KONVRT(21)
C      COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1    UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2    NRD(40), MBD(40), MOBD(3), NUBD(3), TITL(12), XIA(980), XIB(980),
3    FLOW(21,21)
1007 WRITE(6,1007)
C      FORMAT(1HS)
C      NMAX=21
C      MMAX=21
C      PI=3.1416
C      ANGLAT=51.5
C      AG=9.81
C      AL=80.0
C      AT=30.0
C      MAXST=400
C      SEINV=0.0
C      MORO(1)=0111110
C      MORO(2)=2110121
C      MINDO=3
C      NINDO=1
C      NSECT=21
C      NCARD=970
C      IP=1
C      IW=1
C
C      SET OPEN BOUNDS AS FUNCTIONS OF TABLEVALUES (XIA(K),ETC.)
C      OR AS FUNCTIONS OF HALF TIMESTEP NUMBER (K)
C
C      GO TO 87
89 CONTINUE
C      FLOW(11,1)=XIB(K)
C      FLOW(10,21)=XIA(K)
C      FLOW(11,21)=XIA(K)
C      FLOW(12,21)=XIA(K)
C      IF (ISTEP.EQ.1)GO TO 96
C      GO TO 301
87 CONTINUE
C      READ(5,4) (TITL(J),J=1,7)

```

```

4 FORMAT(7A10)
  READ(5,9) (NPLOT(N),N=1,4)
9 FORMAT(4I4)
  FW=0.003
  RHOD=0.0012
  WX=0.0
  WY=7.5
  FF=PI*SIN(ANGLAT*PI/180.0)/21600.0
  CALL KURIH(NCARD,PI,AL,AG,AT)
  NST=0
  C1=AT*AG/AL
  C2=AT/AL
  C3=AT
  C4=8.0*AT*AG
  C5=AT/(AL**2)
  D1=2.0*C1
  D2=2.0*C2
  D3=2.0*C3
  D4=2.0*C4
  D5=2.0*C5
  DO 8 N=1,NMAX
  DO 8 M=1,MMAX
  SE(N,M)=0.0
  SEM(N,M)=0.0
  SEP(N,M)=0.0
  U(N,M)=0.0
  UM(N,M)=0.0
  UP(N,M)=0.0
  V(N,M)=0.0
  VM(N,M)=0.0
  VP(N,M)=0.0
  C(N,M)=0.0
  H(N,M)=0.0
  FLOW(N,M)=0.0
8 CONTINUE
  CALL DIVE(NMAX,MMAX)
  CALL FIND(MIND,NIND,MMAX,NMAX,MINDO,NINDO,NSECT)
  CALL DEPTH(NMAX,MMAX)
  CALL CHEZY(NMAX,MMAX)
  READ(5,25) (NPRINT(N),N=1,16)
  READ(5,25) (NPRINT(N),N=17,32)
  READ(5,25) (NPRINT(N),N=33,48)
  READ(5,25) (NPRINT(N),N=49,64)
  READ(5,25) (NPRINT(N),N=65,80)
25 FORMAT(16I4)
  NUM=1
7 IF(NUM.EQ.NIND) GO TO 3
  NSRCH=NRD(NUM)/1000000
  N=NRD(NUM)/10000-NSRCH*100
  MF=NRD(NUM)/100-NSRCH*10000-N*100
  L=NRD(NUM)-NSRCH*1000000-N*10000-MF*100
  NN=N-1
  K=MF
  DO 2 M=K,L
  SE(N,M)=SEINV
  SEM(N,M)=SEINV
  SEP(N,M)=SEINV
2 CONTINUE
  NUM=NUM+1
  GO TO 7
3 CONTINUE
C
C
  ISTEP=2
  GO TO 500
C

```

```

C
C      COMPUTE UP AND SEP ON ROW N (FIRST HALF TIMESTEP)
C
88 ISTEP=1
   NST=NST+1
   K=2*NST-1
   IF (NST.GT.MAXST) STOP
C
C      SET OPEN BOUNDS
GO TO 89
96 NUM=1
100 IF(NUM.EQ.NIND) GO TO 190
   NSRCH=NRD(NUM)/1000000
   N=NRD(NUM)/10000-NSRCH*100
   MF=NRD(NUM)/100-NSRCH*10000-N*100
   L=NRD(NUM)-NSRCH*1000000-N*10000-MF*100
   MFF=MF-1
   MMF=MF+1
   NNN=N+1
   NN=N-1
   LLL=L+1
   LL=L-1
   IT=1
   A(MF)=0.0
   P(MF)=0.0
   Q(MF)=0.0
   UP(N,MFF)=0.0
   TEMP1=SEM(NNN,MF)
   IF(TEMP1.EQ.0.0) TEMP1=2.0*SEM(N,MF)-SEM(NN,MF)
   TEMP2=SEM(NN,MF)
   IF(TEMP2.EQ.0.0) TEMP2=2.0*SEM(N,MF)-SEM(NNN,MF)
   IF(NSRCH.LT.10.OR.NSRCH.GT.11) GO TO 99
   ABC=0.5*(H(N,MFF)+H(NN,MFF))+0.25*(3.0*SEM(N,MF)-SEM(N,MMF)+
1 TEMP1+TEMP2)
   UP(N,MFF)=FLOW(N,MFF)/(ABC*AL)
99 CONTINUE
   A(MF)=SEM(N,MF)-0.5*C2*(H(N,MF)+H(N,MFF)+TEMP1+SEM(N,MF))*V(N,MF)
1 +0.5*C2*(H(NN,MF)+H(NN,MFF)+TEMP2+SEM(N,MF))*V(NN,MF)
   P(MF)= 0.5*C2*(H(N,MF)+H(NN,MF)+SEM(N,MMF)+SEM(N,MF))
   Q(MF)=(A(MF)+0.5*C2*(H(N,MFF)+H(NN,MFF)+3.0*SEM(N,MF)-SEM(N,MMF))
1 *UP(N,MFF))
   K=MF
101 DO 102 M=K,L
   MM=M-1
   MMM=M+1
   TEMP9=SEM(N,M)
   TEMP3=SEM(N,MMM)
   IF(TEMP3.EQ.0.0) TEMP3=2.0*SEM(N,M)-SEM(N,MM)
   TEMP5=SEM(N,MMM)
   IF(TEMP5.EQ.0.0) TEMP5=2.0*SEM(N,M)-SEM(N,MM)
   IF(M.EQ.K) GO TO 107
   TEMP1=SEM(NNN,M)
   IF(TEMP1.EQ.0.0) TEMP1=2.0*SEM(N,M)-SEM(NN,M)
   TEMP2=SEM(NN,M)
   IF(TEMP2.EQ.0.0) TEMP2=2.0*SEM(N,M)-SEM(NNN,M)
   TEMP4=SEM(N,MM)
   IF(TEMP4.EQ.0.0) TEMP4=2.0*SEM(N,M)-SEM(N,MMM)
   A(M)=SEM(N,M)-0.5*C2*(H(N,M)+H(N,MM)+TEMP1+TEMP9)*V(N,M) +0.5*C2*
1 (H(NN,M)+H(NN,MM)+TEMP2+TEMP9)*V(NN,M)
   P(M)= 0.5*C2*(H(N,M)+H(NN,M)+TEMP3+TEMP9)/(1.0+ 0.5*C2*(H(N,MM)+
1 H(NN,MM)+TEMP4+TEMP9)*R(MM))
   Q(M)=(A(M)+ 0.5*C2*(H(N,MM)+H(NN,MM)+TEMP4+TEMP9)*S(MM))/(1.0+
1 0.5*C2*(H(N,MM)+H(NN,MM)+TEMP4+TEMP9)*R(MM))
107 CONTINUE
   TEMP10=UM(N,MMM)
   TEMP11=UM(N,M)

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TEMP12=UM(N,MM)
TEMP13=VM(N,MMM)
TEMP14=VM(N,M)
TEMP15=VM(NN,MMM)
TEMP16=VM(NN,M)
TEMP17=UM(NNN,M)
TEMP18=UM(NN,M)
IF(TEMP11.NE.0.0) GO TO 110
IF(TEMP10.NE.0.0) GO TO 111
TEMP10=-UM(N,MM)
TEMP13=VM(N,M)
TEMP15=VM(NN,M)
111 CONTINUE
IF(TEMP12.NE.0.0) GO TO 110
TEMP12=-UM(N,MMM)
TEMP14=VM(N,MMM)
TEMP16=VM(NN,MMM)
110 CONTINUE
IF(TEMP17.EQ.0.0) TEMP17=UM(N,M)
IF(TEMP18.EQ.0.0) TEMP18=UM(N,M)
TEMP19=0.25*(TEMP13+TEMP14+TEMP15+TEMP16)
TEMP31=H(N,M)+H(NN,M)+TEMP3+TEMP9
TEMP20=2.0*((TEMP10+UM(N,M))*2)*(0.5*(H(N,M)+H(N,MMM)+H(NN,M)+
1H(NN,MMM))+TEMP3*2.0)-((UM(N,M)+TEMP12)*2)*(0.5*(H(N,M)+H(N,MM)+
2H(NN,M)+H(NN,MM))+TEMP9*2.0))/(4.0*TEMP31)
TEMP23=TEMP17
IF((TEMP13+TEMP14).GT.0.0) TEMP23=UM(N,M)
TEMP24=TEMP18
IF((TEMP15+TEMP16).LT.0.0) TEMP24=UM(N,M)
TEMP21=2.0*(TEMP23*(TEMP13+TEMP14)*(H(N,M)*2.0+TEMP3+TEMP9)-TEMP24
1*(TEMP15+TEMP16)*(H(NN,M)*2.0+TEMP3+TEMP9))/(2.0*TEMP31)
TEMP30=SQR1(TEMP11**2+TEMP19**2)
TEMP32=C(N,M)+C(N,MMM)
TEMP33=TEMP30/(TEMP31*(TEMP32**2))
EDDX=SQR1(AG*((6.00*TEMP11)**2+(0.23*TEMP30)**2))*TEMP31/TEMP32
EDDY=SQR1(AG*((6.00*TEMP19)**2+(0.23*TEMP30)**2))*TEMP31/TEMP32
B(M)=U(N,M)-D2*(TEMP20+TEMP21)+2.0*U3*FF*TEMP19-D1*(TEMP5-SE(N,M))
1-D4*U(N,M)+TEMP33+D5*2.0*(EDDX*(TEMP10-U(N,M)+TEMP12)+EDDY*(TEMP17
2-U(N,M)+TEMP18))+4.0*D3*F**KHOU*SQR1((WX-TEMP11)**2+(WY-TEMP19)**2
3)*(WX-TEMP11)/TEMP31
TEMP1=1.0+D4*TEMP33+D2*AG*P(M)+D5*2.0*(EDDX+EDDY)
R(M)=D1/TEMP1
S(M)=(B(M)+D1*Q(M))/TEMP1
102 CONTINUE
IF(NSRCH.EQ.1.OR.NSRCH.EQ.11) GO TO 104
UP(N,L)=0.0
GO TO 105
104 CONTINUE
TEMP1=SEM(NNN,LLL)
IF(TEMP1.EQ.0.0) TEMP1=2.0*SEM(N,LLL)-SEM(NN,LLL)
TEMP2=SEM(NN,LLL)
IF(TEMP2.EQ.0.0) TEMP2=2.0*SEM(N,LLL)-SEM(NNN,LLL)
DEF=0.5*(H(N,LLL)+H(NN,LLL))+0.25*(3.0*SEM(N,LLL)-SEM(N,L)+
1TEMP1+TEMP2)
UP(N,LLL)=FLOW(N,LLL)/(DEF*AL)
A(LLL)=SEM(N,LLL)-0.5*C2*(H(N,LLL)+H(N,L)+TEMP1+SEM(N,LLL))*
1V(N,LLL)+0.5*C2*(H(NN,LLL)+H(NN,L)+TEMP2+SEM(N,LLL))*V(NN,LLL)
P(LLL)=0.5*C2*(H(N,LLL)+H(NN,LLL)+3.0*SEM(N,LLL)-SEM(N,L))/
1(1.0+0.5*C2*(H(N,L)+H(NN,L)+SEM(N,L)+SEM(N,LLL))*R(L))
Q(LLL)=(A(LLL)+0.5*C2*(H(N,L)+H(NN,L)+SEM(N,L)+SEM(N,LLL))*S(L))/
1(1.0+0.5*C2*(H(N,L)+H(NN,L)+SEM(N,L)+SEM(N,LLL))*R(L))
SEP(N,LLL)=-P(LLL)*UP(N,LLL)+Q(LLL)
UP(N,L)=-R(L)*SEP(N,LLL)+S(L)
105 CONTINUE
M=L
K=MF+1

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```

      DO 106 J=K,L
      MM=M-1
      SEP(N,M)=-P(M)*UP(N,M)+U(M)
      UP(N,MM)=-R(MM)*SEP(N,M)+S(MM)
106  M=M-1
      SEP(N,MF)=-P(MF)*UP(N,MF)+U(MF)
      NUM=NUM+1
      GO TO 100
190  CONTINUE
      DO 292 N=1,NMAX
      DO 292 M=1,MMAX
      U(N,M)=UM(N,M)
      UM(N,M)=UP(N,M)
      SE(N,M)=SEM(N,M)
      SEM(N,M)=SEP(N,M)
292  CONTINUE
      NUM=1
C
C
C      PRINT INSTRUCTIONS
C
500  IF(ISTEP-2)297,296,297
296  CONTINUE
C
      IF(INST.EQ.NPLOT(IQ)) GO TO 520
      IF(INST.EQ.U) GO TO 520
      RNST=FLOAT(NSI)
      TIME=RNST*AI*2.0
      TIME=TIME/3600.0
      CALL PLOTKAF(NMAX,MMAX,NIND,NINDO,MINDO,PI,TIME,IQ)
      IF(IQ.EQ.4) CALL ENPLOT
      IQ=IQ+1
      IF(NPLOT(IQ).EQ.U) CALL ENPLOT
520  CONTINUE
      IF(INST.EQ.NPRINT(IP)) GO TO 295
      GO TO 297
295  IP=IP+1
5020 FORMAT(1H1,///7A10,///10X,*WATER ELEVATIONS IN HUNDREDTHS OF A METR
1E FOR THE TIMESTEP*,I5,/)
1157 FORMAT(/4X,*M      1      2      3      4      5      6      7      8
1 9      10      11      12      13      14      15      16      17      18      19
220      21*/ )
      WRITE(6,5020) (ITL(J),J=1,7),NST
      WRITE(6,1157)
      DO 6000 M=1,MMAX
      DO 6006 N=1,NMAX
6006  KONVRT(N)=SE(N,M)*100.0
6000  WRITE(6,6001) M,(KONVRT(N),N=1,NMAX)
6001  FORMAT(/3X,I2,1X,21I6)
5022 FORMAT(1H1,///7A10,///10X,*U VELOCITY COMPONENTS IN THOUSANDTHS OF
1A METRE PER SECOND FOR THE TIMESTEP*,I5,/)
      WRITE(6,5022) (ITL(J),J=1,7),NST
      WRITE(6,1157)
      DO 6004 M=1,MMAX
      DO 6008 N=1,NMAX
6008  KONVRT(N)=UM(N,M)*1000.0
6004  WRITE(6,6001) M,(KONVRT(N),N=1,NMAX)
5021 FORMAT(1H1,///7A10,///10X,*V VELOCITY COMPONENTS IN THOUSANDTHS OF
1A METRE PER SECOND FOR THE TIMESTEP*,I5,/)
      WRITE(6,5021) (ITL(J),J=1,7),NST
      WRITE(6,1157)
      DO 6003 M=1,MMAX
      DO 6007 N=1,NMAX
6007  KONVRT(N)=(V(N,M)+VP(N,M))*500.0
6003  WRITE(6,6001) M,(KONVRT(N),N=1,NMAX)
297  NUM=1

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        GO TO (299,d8),ISTEP
299 ISTEP=2
        K=2*NST
C      SET OPEN BOUNDS
        GO TO 89
C
C      COMPUTE VP AND SEP ON COLUMN M (SECOND HALF TIMESTEP)
C
301 IF (NUM.EQ.MIND) GO TO 390
        MSRCH=MBD(NUM)/1000000
        M=MBD(NUM)/10000-MSRCH*100
        NF=MBD(NUM)/100-MSRCH*10000-M*100
        L=MBD(NUM)-MSRCH*1000000-M*10000-NF*100
        NFF=NF-1
        NNF=NF+1
        MMM=M+1
        MM=M-1
        LLL=L+1
        LL=L-1
        IT=1
        A(NF)=0.0
        P(NF)=0.0
        Q(NF)=0.0
        VP(NFF,M)=0.0
        TEMP1=SEM(NF,MMM)
        IF (TEMP1.EQ.0.0) TEMP1=2.0*SEM(NF,M)-SEM(NF,MM)
        TEMP2=SEM(NF,MM)
        IF (TEMP2.EQ.0.0) TEMP2=2.0*SEM(NF,M)-SEM(NF,MMM)
        IF (MSRCH.LT.10.OR.MSRCH.GT.11) GO TO 399
        ABC=0.5*(H(NFF,M)+H(NFF,MM))+0.25*(3.0*SEM(NF,M)-SEM(NNF,M)+
        1TEMP1+TEMP2)
        VP(NFF,M)=FLOW(NFF,M)/(ABC*AL)
399 CONTINUE
        A(NF)=SEM(NF,M)-0.5*C2*(H(NF,M)+H(NFF,M)+TEMP1+SEM(NF,M))*U(NF,M)
        1+0.5*C2*(H(NF,MM)+H(NFF,MM)+TEMP2+SEM(NF,M))*U(NF,MM)
        P(NF)= 0.5*C2*(H(NF,M)+H(NF,MM)+SEM(NNF,M)+SEM(NF,M))
        Q(NF)=(A(NF)+0.5*C2*(H(NFF,M)+H(NFF,MM)+3.0*SEM(NF,M)-SEM(NNF,M))
        1*VP(NFF,M))
        K=NF
303 DO 302 N=K,L
        NN=N-1
        NNN=N+1
        TEMP9=SEM(N,M)
        TEMP3=SEM(NNN,M)
        IF (TEMP3.EQ.0.0) TEMP3=2.0*SEM(N,M)-SEM(NN,M)
        TEMP5=SEM(NNN,M)
        IF (TEMP5.EQ.0.0) TEMP5=2.0*SEM(N,M)-SEM(NN,M)
        IF (N.EQ.K) GO TO 307
        TEMP1=SEM(N,MMM)
        IF (TEMP1.EQ.0.0) TEMP1=2.0*SEM(N,M)-SEM(N,MM)
        TEMP2=SEM(N,MM)
        IF (TEMP2.EQ.0.0) TEMP2=2.0*SEM(N,M)-SEM(N,MMM)
        TEMP4=SEM(NN,M)
        IF (TEMP4.EQ.0.0) TEMP4=2.0*SEM(N,M)-SEM(NNN,M)
        A(N)=SEM(N,M)-0.5*C2*(H(N,M)+H(NN,M)+TEMP1+TEMP9)*U(N,M) +0.5*C2*
        1H(N,MM)+H(NN,MM)+TEMP2+TEMP9)*U(N,MM)
        P(N)= 0.5*C2*(H(N,M)+H(N,MM)+TEMP3+TEMP9)/(1.0+ 0.5*C2*(H(NN,M)+
        1H(NN,MM)+TEMP4+TEMP9)*R(NN))
        Q(N)=(A(N)+ 0.5*C2*(H(NN,M)+H(NN,MM)+TEMP4+TEMP9)*S(NN))/(1.0+
        1 0.5*C2*(H(NN,M)+H(NN,MM)+TEMP4+TEMP9)*R(NN))
307 CONTINUE
        TEMP10=VM(NNN,M)
        TEMP11=VM(N,M)
        TEMP12=VM(NN,M)
        TEMP13=UM(NNN,M)
        TEMP14=UM(N,M)

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TEMP15=UM(NNN,MM)
TEMP16=UM(N,MM)
TEMP17=VM(N,MMM)
TEMP18=VM(N,MM)
IF(TEMP11.NE.0.0) GO TO 310
IF(TEMP10.NE.0.0) GO TO 311
TEMP10=-VM(NN,M)
TEMP13=UM(N,M)
TEMP15=UM(N,MM)
311 CONTINUE
IF(TEMP12.NE.0.0) GO TO 310
TEMP12=-VM(NNN,M)
TEMP14=UM(NNN,M)
TEMP16=UM(NNN,MM)
310 CONTINUE
IF(TEMP17.EQ.0.0) TEMP17=VM(N,M)
IF(TEMP18.EQ.0.0) TEMP18=VM(N,M)
TEMP19=0.25*(TEMP13+TEMP14+TEMP15+TEMP16)
TEMP31=H(N,M)+H(N,MM)+TEMP3+TEMP9
TEMP20=2.0*((TEMP10+VM(N,M))*2)*(0.5*(H(N,M)+H(NNN,M)+H(N,MM)+
1H(NNN,MM))+TEMP3*2.0)-((VM(N,M)+TEMP12)*2)*(0.5*(H(N,M)+H(NN,M)+
2H(N,MM)+H(NN,MM))+TEMP9*2.0))/(4.0*TEMP31)
TEMP23=TEMP17
IF((TEMP13+TEMP14).GT.0.0) TEMP23=VM(N,M)
TEMP24=TEMP18
IF((TEMP15+TEMP16).LT.0.0) TEMP24=VM(N,M)
TEMP21=2.0*(TEMP23*(TEMP13+TEMP14)*(H(N,M)*2.0+TEMP3+TEMP9)-TEMP24
1*(TEMP15+TEMP16)*(H(N,MM)*2.0+TEMP3+TEMP9))/(2.0*TEMP31)
TEMP30=SQRT(TEMP11**2+TEMP19**2)
TEMP32=C(N,M)+C(NNN,M)
TEMP33=TEMP30/(TEMP31*(TEMP32**2))
EDDX=SQRT(AG*((6.00*TEMP19)**2+(0.23*TEMP30)**2))*TEMP31/TEMP32
EDDY=SQRT(AG*((6.00*TEMP11)**2+(0.23*TEMP30)**2))*TEMP31/TEMP32
B(N)=V(N,M)-D2*(TEMP20+TEMP21)-2.0*D3*FF*TEMP19-D1*(TEMP5-SE(N,M))
1-D4*V(N,M)*(TEMP33+D5*2.0*(EDDX*(TEMP17-V(N,M)+TEMP18)+EDDY*(TEMP10
2-V(N,M)+TEMP12))+4.0*D3*FW*RII00*SQRT((WX-TEMP19)**2+(WY-TEMP11)**2
3)*(WY-TEMP11)/TEMP31
TEMP1=1.0+D4*TEMP33+D2*AG*P(N)+D5*2.0*(EDDX+EDDY)
R(N)=D1/TEMP1
S(N)=(B(N)+D1*Q(N))/TEMP1
302 CONTINUE
IF(MSRCH.EQ.1.OR.MSRCH.EQ.11) GO TO 304
VP(L,M)=0.0
GO TO 305
304 CONTINUE
TEMP1=SEM(LLL,MMM)
IF(TEMP1.EQ.0.0) TEMP1=2.0*SEM(LLL,M)-SEM(LLL,MM)
TEMP2=SEM(LLL,MM)
IF(TEMP2.EQ.0.0) TEMP2=2.0*SEM(LLL,M)-SEM(LLL,MMM)
DEF=0.5*(H(LLL,M)+H(LLL,MM))+0.25*(3.0*SEM(LLL,M)-SEM(L,M)+
1TEMP1+TEMP2)
VP(LLL,M)=FLOW(LLL,M)/(DEF*AL)
A(LLL)=SEM(LLL,M)-0.5*C2*(H(LLL,M)+H(L,M)+TEMP1+SEM(LLL,M))*
1U(LLL,M)+0.5*C2*(H(LLL,MM)+H(L,MM)+TEMP2+SEM(LLL,M))*U(LLL,MM)
P(LLL)=0.5*C2*(H(LLL,M)+H(LLL,MM)+3.0*SEM(LLL,M)-SEM(L,M))/(1.0+
10.5*C2*(H(L,M)+H(L,MM)+SEM(L,M)+SEM(LLL,M))*R(L))
Q(LLL)=(A(LLL)+0.5*C2*(H(L,M)+H(L,MM)+SEM(L,M)+SEM(LLL,M))*S(L))/
1(1.0+0.5*C2*(H(L,M)+H(L,MM)+SEM(L,M)+SEM(LLL,M))*R(L))
SEP(LLL,M)=-P(LLL)*VP(LLL,M)+Q(LLL)
VP(L,M)=-R(L)*SEP(LLL,M)+S(L)
305 CONTINUE
N=L
K=NF+1
DO 306 J=K,L
NN=N-1
SEP(N,M)=-P(N)*VP(N,M)+Q(N)

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      VP(NH,M)=-R(NN)*SEP(N,M)+S(NN)
306 N=N-1
      SEP(NF,M)=-P(NF)*VP(NF,M)+u(NF)
      NUM=NUM+1
      GO TO 301
390 CONTINUE
      DO 492 N=1,NMAX
      DO 492 M=1,MMAX
      V(N,M)=VM(N,M)
      VM(N,M)=VP(N,M)
      SE(N,M)=SEM(N,M)
      SEM(N,M)=SEP(N,M)
492 CONTINUE
      NUM=1
      GO TO 500
      END

```

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C
SUBROUTINE KURIH(NCARD,PI,AL,AG,AT)

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C
C      SUBROUTINE KURIH HANDLES THE READING OF ALL TABLE VALUES OF
C      THE OPEN BOUNDARIES IN STORAGE

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      COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1 UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2 NBD(40), MBD(40), MOBD(3), NOBD(3), TITL(12), XIA(980), XIB(980),
3 FLOW(21,21)
      READ(5,10) U1,U2
10 FORMAT(2F10.5)
      T=120.0
      CEL=SQRT(AG*24.0)
      TCEL=(AL*20.0)/(CEL*AT)
      KT=TCEL+0.5
      DO 30 K=1,NCARD
      SUM=FLOAT(K)
      IF(SUM.GT.T) GO TO 31
      XIB(K)=(U2/2.0)*(1.0+SIN((1.5+(SUM/T))*PI))
      GO TO 32
31 XIB(K)=U2
32 KN=K-KT
      IF(KN.LT.1) KN=1
      XIA(K)=(U1/U2)*XIB(KN)
30 CONTINUE
      WRITE(6,9) (TITL(J),J=1,7)
      WRITE(6,11)
      WRITE(6,12) (K,XIA(K),XIB(K), K=1,NCARD)
      RETURN
9 FORMAT(1H1,///7A10///,10X,*,DISCHARGES AT BOUNDARY POINTS A AND B IN
1 CUBIC METRES PER SECOND*//)
11 FORMAT((7X,4(* K XIA XIB *)/))
12 FORMAT((7X,4(I4,2X,F8.1,2X,F8.1,4X)))
      ENH

```

```

C
SUBROUTINE DIVE(NMAX,MMAX)

```

```

C
C      SUBROUTINE DIVE READS IN LOCATIONS OF WATER LEVEL COMPUTATION

```

```

      COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1 UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2 NBD(40), MBD(40), MOBD(3), NOBD(3), TITL(12), XIA(980), XIB(980),
3 FLOW(21,21)
      WRITE(6,5) (TITL(J),J=1,7)
      WRITE(6,1157)
      DO 2 M=1,MMAX
      READ(5,3) (NBD(N),N=1,NMAX)
      WRITE(6,4) M,(NBD(N),N=1,NMAX)
      DO 2 N=1,NMAX

```

```

2 H(N,M) = FLOAT(NBD(N))
RETURN
5 FORMAT(1H1,///7A10///,10X,*WATER LEVELS IN FIELD*/)
1157 FORMAT(/4X,*M      1      2      3      4      5      6      7      8
1 9      10      11      12      13      14      15      16      17      18      19
220      21*/ )
3 FORMAT(21I2)
4 FORMAT(/3X,12,1X,21I6)
END

C
SUBROUTINE FIND(MIND,NIND,MMAX,NMAX,MINDO,NINDO,NSECT)
C
C      SUBROUTINE FIND PROCESSES DIVE DATA AND THE MOBD AND NOBD
C      VALUES AND PRODUCES TWO TABLES (NBD AND MBD) WHICH ARE USED
C      TO CONTROL THE COMPUTATION
C

LOGICAL START
COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1 UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2 NBD(40), MBD(40), MUBD(3), NOBD(3), TITL(12), XIA(980), XIB(980),
3 FLOW(21,21)
DO 1 J=1,NSECT
NBD(J)=0
1 MBD(J)=0
MIND=1
NIND=1
DO 2 N=2,NMAX
START=.TRUE.
DO 3 M=2,MMAX
IF(.NOT.START) GO TO 4
IF(H(N,M).EQ.0.0) GO TO 3
NBD(NIND)=M*100+NBD(NIND)
START=.FALSE.
GO TO 3
4 IF(H(N,M).NE.0.0) GO TO 5
NBD(NIND)=M-1+NBD(NIND)+10000*N
GO TO 6
5 IF(M.NE.MMAX) GO TO 3
NBD(NIND)=M+NBD(NIND)+10000*N
6 NIND=NIND+1
START=.TRUE.
3 CONTINUE
2 CONTINUE
DO 12 M=2,MMAX
START=.TRUE.
DO 13 N=2,NMAX
IF (.NOT.START) GO TO 14
IF (H(N,M).EQ.0.0) GO TO 13
MBD(MIND)=N*100+MBD(MIND)
START=.FALSE.
GO TO 13
14 IF (H(N,M).NE.0.0) GO TO 15
MBD(MIND)=N-1+MBD(MIND)+10000*M
GO TO 16
15 IF(N.NE.NMAX) GO TO 13
MBD(MIND)=N+MBD(MIND)+10000*M
16 MIND=MIND+1
START=.TRUE.
13 CONTINUE
12 CONTINUE
NUM=1
100 IF(NUM.EQ.NIND) GO TO 300
N=NBD(NUM)/10000
MF=NBD(NUM)/100-N*100
L=NBD(NUM)-N*10000-MF*100
MFLEF=MF-1

```

```

      LRIG=L+1
      NA=1
200 IF (NA.EQ.MIND) GO TO 210
      M=MOBD(NA)/100000
      NBOT=MOBD(NA)/1000-M*100
      NTOP=MOBD(NA)/10-M*10000-NBOT*100
      NBERN=MOBD(NA)-M*100000-NBOT*1000-NTOP*10
      IF(((N.GE.NBOT).AND.(N.LE.NTOP)).AND.(MFLEF.EQ.M)) NBD(NUM)=NBD(NU
1M)+10000000
      IF(((N.GE.NBOT).AND.(N.LE.NTOP)).AND.(LRIG.EQ.M)) NBD(NUM)=NBD(NUM
1)+1000000
      NA=NA+1
      GO TO 200
210 NUM=NUM+1
      GO TO 100
300 CONTINUE
      NUM=1
101 IF(NUM.EQ.MIND) GO TO 301
      M=MBD(NUM)/10000
      NF=MBD(NUM)/100-M*100
      L=MBD(NUM)-M*10000-NF*100
      NFBOT=NF-1
      LTOP=L+1
      NA=1
201 IF(NA.EQ.NIND) GO TO 211
      N=NOBD(NA)/100000
      MLEF=NOBD(NA)/1000-N*100
      MRIG=NOBD(NA)/10-N*10000-MLEF*100
      MBERN=NOBD(NA)-N*100000-MLEF*1000-MRIG*10
      IF(M.GE.MLEF.AND.M.LE.MRIG.AND.NFBOT.EQ.N) MBD(NUM)=MBD(NUM)+
110000000
      IF(M.GE.MLEF.AND.M.LE.MRIG.AND.LTOP.EQ.N) MBD(NUM)=MBD(NUM)+
110000000
      NA=NA+1
      GO TO 201
211 NUM=NUM+1
      GO TO 101
301 CONTINUE
      WRITE(6,20) (TITL(J),J=1,7)
      WRITE(6,21)(J,NBD(J),MBD(J),J=1,NSECT)
      RETURN
20 FORMAT((1H1,///7A10///,3X,4(4X,*NUM*,7X,*NBD*,9X,*MBD*,2X)))
21 FORMAT(///3X,4(3X,I4,3X,I9,3X,I9))
      END

```

C

SUBROUTINE DEPTH(NMAX,MMAX)

C

C

C

SUBROUTINE DEPTH READS THE WATER DEPTHS

```

      COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1 UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2 NHD(40), MBD(40), MOBD(3), NOBD(3), TITL(12), XIA(980), XIB(980),
3 FLOW(21,21)
      NO=15
      IF(NMAX.LT.NO) NO=NMAX
      DO 10 M=1,MMAX
      READ(5,3) (H(N,M),N=1,NO)
10 CONTINUE
      IF(NMAX.LE.NO) GO TO 12
      NW=NO+1
      DO 11 M=1,MMAX
      READ(5,3) (H(N,M),N=NW,NMAX)
11 CONTINUE
12 CONTINUE
      WRITE(6,9) (TITL(J),J=1,7)
      WRITE(6,115/)

```

```

      DO 15 JA=1,MMAX
15  WRITE(6,14) JA,(H(N,JA),N=1,NMAX)
      RETURN
      3  FORMAT(15F5.1)
      9  FORMAT(1H1,///7A10,///10X,*MEAN INITIAL WATER DEPTHS IN FIELD IN ME
1157 1TKES*/)
      1  2  3  4  5  6  7  8
      ,  1  9  10  11  12  13  14  15  16  17  18  19
      220 21*/)
      14  FORMAT(/3X,12,1X,21F6.1)
      END
C
      SUBROUTINE CHEZY (NMAX,MMAX)
C
C      SUBROUTINE CHEZY READS ALL CHEZY COEFFICIENTS WHICH REPRESENT
C      AN AVERAGE VALUE FOR THE BOTTOM ROUGHNESS
C
      COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1  UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2  NRD(40), MBD(40), MOBD(3), NOBD(3), TITL(12), XIA(980), XIB(980),
3  FLOW(21,21)
      NU=15
      IF(NMAX.LT.NU) NU=NMAX
      DO 10 M=1,MMAX
      READ(5,3) (C(N,M),N=1,NU)
10  CONTINUE
      IF(NMAX.LE.NU) GO TO 12
      NU=NU+1
      DO 11 M=1,MMAX
      READ(5,3) (C(N,M),N=NU,NMAX)
11  CONTINUE
12  CONTINUE
      WRITE(6,9) (TITL(J),J=1,7)
      WRITE(6,1157)
      DO 15 JA=1,MMAX
15  WRITE(6,14) JA,(C(N,JA),N=1,NMAX)
      RETURN
      3  FORMAT(15F5.1)
      9  FORMAT(1H1,///7A10,///10X,*MEAN INITIAL CHEZY COEFFICIENTS IN METRI
1157 1C UNITS*/)
      1  2  3  4  5  6  7  8
      ,  1  9  10  11  12  13  14  15  16  17  18  19
      220 21*/)
      14  FORMAT(/3X,12,1X,21F6.0)
      END
C
      SUBROUTINE PLOTRAF(MMAX,MMAX,NIND,NINDO,MINDO,PI,TIME,IQ)
C
C      SUBROUTINE PLOTRAF PLOTS THE VELOCITY VECTORS IN GRAPHICAL
C      FORM
C
      COMMON SE(21,21), SEM(21,21), SEP(21,21), U(21,21), UM(21,21),
1  UP(21,21), V(21,21), VM(21,21), VP(21,21), C(21,21), H(21,21),
2  NRD(40), MBD(40), MOBD(3), NOBD(3), TITL(12), XIA(980), XIB(980),
3  FLOW(21,21)
      DIMENSION XPAGE(21), YPAGE(21)
      CALL START(2)
      IF(IQ.EQ.2) CALL PLOT(0.0,15.12,-3)
      IF(IQ.EQ.3) CALL PLOT(21.68,0.0,-3)
      IF(IQ.EQ.4) CALL PLOT(21.68,15.12,-3)
      DELTAX=0.475
      DELTAY=0.475
      VELSCAL=DELTAX/0.08
      PLOTMAX=DELTAX*4.0
      DO 50 M=1,MMAX
      DO 50 N=1,NMAX

```

```

      RM=FLOAT(M)
      RN=FLOAT(N)
      XPAGE(M)=2.75+(RM+0.5)*DELTAX
      YPAGE(N)=1.2+RN*DELTAY
50  CONTINUE
      NUM=1
600  IF(NUM.EQ.NIND) GO TO 601
      NSRCH=NBD(NUM)/1000000
      N=NB0(NUM)/10000-NSRCH*100
      MF=NBD(NUM)/100-NSRCH*10000-N*100
      L=NB0(NUM)-NSRCH*1000000-N*10000-MF*100
      NN=N-1
      K=MF
      DO 603 M=K,L
      MM=M-1
      HEIGHT=SQRT(((UM(N,M)+UM(N,MM))*0.5)**2+((V(N,M)+V(NN,M)+VP(N,M)+
1VP(NN,M))*0.25)**2)
      IF(HEIGHT.EQ.0.0) GO TO 603
      HEIGHT=HEIGHT*VELSCAL
      IF(HEIGHT.GI.PLOI MAX) GO TO 603
      ANGLE=ATAN2(((V(N,M)+V(NN,M)+VP(N,M)+VP(NN,M))*0.5),(UM(N,M)+
1UM(N,MM)))
      XCORD=XPAGE(M)-HEIGHT*SIN(ANGLE)*6.0/20.0
      YCORD=YPAGE(N)+HEIGHT*COS(ANGLE)*6.0/20.0
      ANGLE=(ANGLE*180.0/PI)+270.0
      CALL SYMBOL(XCORD,YCORD,HEIGHT,83,ANGLE,-1)
603  CONTINUE
      NUM=NUM+1
      GO TO 600
601  CONTINUE
      NA=1
610  IF(NA.EQ.MINDU) GO TO 611
      M=MURD(NA)/100000
      NBOT=MOBD(NA)/1000-M*100
      NTOP=MOBD(NA)/10-M*10000-NBOT*100
      DO 612 N=NBOT,NTOP
      NN=N-1
      HEIGHT=SQRT(UM(N,M)**2+((V(N,M)+V(NN,M)+VP(N,M)+VP(NN,M))*0.25)
1**2)
      IF(HEIGHT.EQ.0.0) GO TO 612
      HEIGHT=HEIGHT*VELSCAL
      IF(HEIGHT.GI.PLOI MAX) GO TO 612
      ANGLE=ATAN2(((V(N,M)+V(NN,M)+VP(N,M)+VP(NN,M))*0.25),UM(N,M))
      XCORD=XPAGE(M)-HEIGHT*SIN(ANGLE)*6.0/20.0
      YCORD=YPAGE(N)+HEIGHT*COS(ANGLE)*6.0/20.0
      ANGLE=(ANGLE*180.0/PI)+270.0
      CALL SYMBOL(XCORD,YCORD,HEIGHT,83,ANGLE,-1)
612  CONTINUE
      NA=NA+1
      GO TO 610
611  NA=1
620  IF(NA.EQ.NINDU) GO TO 621
      N=NURD(NA)/100000
      MLEF=NOBD(NA)/1000-N*100
      MRIG=NOBD(NA)/10-N*10000-MLEF*100
      DO 622 M=MLEF,MRIG
      MM=M-1
      HEIGHT=SQRT(((UM(N,M)+UM(N,MM))*0.5)**2+((V(N,M)+VP(N,M))*0.5)**2)
      IF(HEIGHT.EQ.0.0) GO TO 622
      HEIGHT=HEIGHT*VELSCAL
      IF(HEIGHT.GI.PLOI MAX) GO TO 622
      ANGLE=ATAN2((V(N,M)+VP(N,M)),(UM(N,M)+UM(N,MM)))
      XCORD=XPAGE(M)-HEIGHT*SIN(ANGLE)*6.0/20.0
      YCORD=YPAGE(N)+HEIGHT*COS(ANGLE)*6.0/20.0
      ANGLE=(ANGLE*180.0/PI)+270.0
      CALL SYMBOL(XCORD,YCORD,HEIGHT,83,ANGLE,-1)

```

```

622 CONTINUE
  NA=NA+1
  GO TO 620
621 CONTINUE
  WIDSCAL=0.12
  CORUXEN=XPAGE(MMAX)+(DELTAX/2.0)+WIDSCAL+0.6
  CORDYEN=YPAGE(1)+1.5*DELTAY
  CALL SYMBOL(CORUXEN,CORDYEN,WIDSCAL,66HLENGTH SCALE      80 M
1      VELOCITY      0.08 M/SEC,90.0,66)
  SCALEN=CORDYEN+13.0*WIDSCAL
  DELTAB=DELTAX*10.0/6.0
  DORUXEN=CORUXEN-(WIDSCAL/2.0)
  CALL SYMBOL(DORUXEN,SCALEN,DELTAB,16,90.0,-1)
  FORDXEN=DORUXEN-DELTAX*6.0/20.0
  SCALVEL=CORDYEN+50.0*WIDSCAL
  CALL SYMBOL(FORDXEN,SCALVEL,DELTAX,83,0.0,-1)
  CURDXEN=CORUXEN+3.0*WIDSCAL
  CALL SYMBOL(CURDXEN,CORDYEN,WIDSCAL,66HINITIAL DEPTH      24.0 M
1      CHEZY VALUE      56.6 METRIC,90.0,66)
  CORDXEN=CORUXEN+3.0*WIDSCAL
  CALL SYMBOL(CORDXEN,CORDYEN,WIDSCAL,66HJET INFLOW      CUMEC
1      WEIR OUTFLOW      CUMEC,90.0,66)
  ENFLOW=0.0
  M=1
  DO 20 N=1,NMAX
    ENFLOW=ENFLOW+FLOW(N,M)
20 CONTINUE
  CORDIN=CORDYEN+14.0*WIDSCAL
  CALL NUMBER(CORDXEN,CORDIN,WIDSCAL,ENFLOW,90.0,-1)
  OUTFLOW=0.0
  M=MMAX
  DO 21 N=1,NMAX
    OUTFLOW=OUTFLOW+FLOW(N,M)
21 CONTINUE
  CORDOUT=CORUYEN+56.0*WIDSCAL
  CALL NUMBER(CORUXEN,CORDOUT,WIDSCAL,OUTFLOW,90.0,-1)
  CORDXEN=CORUXEN+3.0*WIDSCAL
  CALL SYMBOL(CORUXEN,CORDYEN,WIDSCAL,66HJET VELOCITY      0.60 M/SEC
1      DEPTH OF JET      5.0 M,90.0,66)
  J=(NMAX-1)/2
  WIDTITL=0.2
  XBEGIN=XPAGE(1)-(DELTAX/2.0)
  YBEGIN=YPAGE(1)-(DELTAY/2.0)
  CORDY=YPAGE(J)-14.5*WIDTITL
  CORDX=DELTAX+1.7
  CALL SYMBOL(CORDX,CORDY,WIDTITL,30HJET CIRCULATION IN A RESERVOIR,
190.0,30)
  CORDYN=YPAGE(J)+2.0
  CORDXN=CORDX+0.5
  CALL SYMBOL(CORDXN,CORDYN,WIDSCAL,12HTIME IN HRS ,90.0,12)
  CORDYN=CORDYN+WIDSCAL*12.0
  CALL NUMBER(CORDXN,CORDYN,WIDSCAL,TIME,90.0,2)
  K=(MMAX+1)/2
  L=(NMAX+1)/2
  XCIKOK=XPAGE(K)
  YCIKOK=YPAGE(L)
  DIAM=FLOAT(NMAX-2)*DELTAX
  RADIUS=0.5*DIAM
  THETA1=ATAN2(0.75,4.75)*180.0/PI
  THETA2=180.0-(ATAN2(0.25,4.75)*180.0/PI)
  THETA3=360.0-THETA1
  THETA4=360.0-THETA2
  CALL DASHON(4)
  CALL CIRARC(XCIKOK,YCIKOK,RADIUS,THETA1,THETA2)
  CALL CIRARC(XCIKOK,YCIKOK,RADIUS,THETA3,THETA4)
  CALL DASHOF

```

```

DO 51 M=1,MMAX
XPAGE(M)=XPAGE(M)+(DELTAX/2.0)
51 CONTINUE
DO 52 N=1,NMAX
YPAGE(N)=YPAGE(N)+(DELTAY/2.0)
52 CONTINUE
CALL PLOT(XBEGIN,YPAGE(11),3)
CALL PLOT(XPAGE(1),YPAGE(11),2)
CALL PLOT(XPAGE(1),YPAGE(15),2)
CALL PLOT(XPAGE(2),YPAGE(15),2)
CALL PLOT(XPAGE(2),YPAGE(17),2)
CALL PLOT(XPAGE(3),YPAGE(17),2)
CALL PLOT(XPAGE(3),YPAGE(18),2)
CALL PLOT(XPAGE(4),YPAGE(18),2)
CALL PLOT(XPAGE(4),YPAGE(19),2)
CALL PLOT(XPAGE(6),YPAGE(19),2)
CALL PLOT(XPAGE(6),YPAGE(20),2)
CALL PLOT(XPAGE(15),YPAGE(20),2)
CALL PLOT(XPAGE(15),YPAGE(19),2)
CALL PLOT(XPAGE(17),YPAGE(19),2)
CALL PLOT(XPAGE(17),YPAGE(18),2)
CALL PLOT(XPAGE(18),YPAGE(18),2)
CALL PLOT(XPAGE(18),YPAGE(17),2)
CALL PLOT(XPAGE(19),YPAGE(17),2)
CALL PLOT(XPAGE(19),YPAGE(15),2)
CALL PLOT(XPAGE(20),YPAGE(15),2)
CALL PLOT(XPAGE(20),YPAGE(12),2)
CALL PLOT(XPAGE(21),YPAGE(12),2)
CALL PLOT(XPAGE(21),YPAGE(9),3)
CALL PLOT(XPAGE(20),YPAGE(9),2)
CALL PLOT(XPAGE(20),YPAGE(6),2)
CALL PLOT(XPAGE(19),YPAGE(6),2)
CALL PLOT(XPAGE(19),YPAGE(4),2)
CALL PLOT(XPAGE(18),YPAGE(4),2)
CALL PLOT(XPAGE(18),YPAGE(3),2)
CALL PLOT(XPAGE(17),YPAGE(3),2)
CALL PLOT(XPAGE(17),YPAGE(2),2)
CALL PLOT(XPAGE(15),YPAGE(2),2)
CALL PLOT(XPAGE(15),YPAGE(1),2)
CALL PLOT(XPAGE(6),YPAGE(1),2)
CALL PLOT(XPAGE(6),YPAGE(2),2)
CALL PLOT(XPAGE(4),YPAGE(2),2)
CALL PLOT(XPAGE(4),YPAGE(3),2)
CALL PLOT(XPAGE(3),YPAGE(3),2)
CALL PLOT(XPAGE(3),YPAGE(4),2)
CALL PLOT(XPAGE(2),YPAGE(4),2)
CALL PLOT(XPAGE(2),YPAGE(6),2)
CALL PLOT(XPAGE(1),YPAGE(6),2)
CALL PLOT(XPAGE(1),YPAGE(10),2)
CALL PLOT(XBEGIN,YPAGE(10),2)
BORDEX=16.55+DELTAX
YORIGIN=0.0
BORDEY=11.69
CALL PLOT(DELTAX,YORIGIN,3)
CALL PLOT(DELTAX,BORDEY,2)
CALL PLOT(BORDEX,BORDEY,2)
CALL PLOT(BORDEX,YORIGIN,2)
CALL PLOT(DELTAX,YORIGIN,2)
RETURN
END

```

JET INDUCED CIRCULATION IN RESERVOIRS

80.000 240.000

0.0 0.0 0.0 4.0 0.0 0.0 0.0 0.0 0.0 0.0 42.5 0.0 0.0 0.0 0.0

JET INDUCED CIRCULATION IN RESERVOIRS

WATER ELEVATIONS IN HUNDRETHS OF A METRE FOR THE TIMESTEP 400

M	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	67	67	67	67	68	66	67	67	67	0	0	0	0	0	0
3	0	0	0	0	67	67	67	67	67	67	67	66	67	67	67	67	67	0	0	0	0
4	0	0	0	67	67	67	67	67	67	66	68	68	67	67	67	67	67	67	0	0	0
5	0	0	67	67	67	67	67	67	70	67	67	64	67	67	67	67	67	67	67	0	0
6	0	0	67	67	67	67	67	67	66	67	67	68	67	67	67	67	67	67	67	0	0
7	0	67	67	67	67	67	67	67	66	67	67	69	67	67	67	67	67	67	67	67	0
8	0	67	67	67	67	67	67	67	67	69	67	64	66	67	67	67	67	67	67	67	0
9	0	67	67	67	67	67	67	67	66	67	67	70	68	67	67	67	67	67	67	67	0
10	0	67	67	67	67	67	67	67	67	69	67	65	66	67	67	67	67	67	67	67	0
11	0	67	67	67	67	67	67	67	66	67	67	68	67	67	67	67	67	67	67	67	0
12	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0
13	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0
14	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0
15	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0
16	0	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0	0
17	0	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0	0
18	0	0	0	67	67	67	67	67	67	67	67	67	67	67	67	67	67	67	0	0	0
19	0	0	0	0	67	67	67	67	67	67	67	67	67	67	67	67	67	0	0	0	0
20	0	0	0	0	0	0	67	67	67	67	67	67	67	67	67	67	67	0	0	0	0
21	0	0	0	0	0	0	0	0	67	67	67	67	67	67	67	67	67	0	0	0	0

JET INDUCED CIRCULATION IN RESERVOIRS

U VELOCITY COMPONENTS IN THOUSANDTHS OF A METRE PER SECOND FOR THE TIMESTEP 400

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	0	0	0	0	0	0	0	0	0	603	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	-16	-15	-9	24	369	22	-10	-16	-17	0	0	0	0	0	0
3	0	0	0	0	-12	-12	-25	-25	-13	40	338	36	-15	-27	-28	-14	-14	0	0	0	0
4	0	0	0	-10	-20	-21	-31	-30	-13	55	287	49	-16	-33	-34	-23	-22	-10	0	0	0
5	0	0	-4	-13	-22	-24	-33	-32	-11	68	286	70	-15	-35	-36	-26	-25	-14	-5	0	0
6	0	0	-6	-13	-22	-25	-33	-34	-15	77	284	78	-18	-37	-36	-27	-24	-15	-7	0	0
7	0	-0	-5	-12	-20	-24	-33	-36	-19	85	282	76	-24	-39	-35	-26	-22	-14	-7	-0	0
8	0	-0	-5	-11	-18	-23	-32	-37	-23	80	274	84	-24	-38	-34	-24	-20	-13	-6	-0	0
9	0	-0	-5	-10	-17	-21	-30	-35	-19	84	257	78	-23	-37	-32	-23	-19	-12	-6	-1	0
10	0	-1	-5	-10	-16	-20	-28	-32	-16	78	236	80	-16	-33	-29	-22	-18	-12	-7	-2	0
11	0	-2	-5	-10	-14	-18	-25	-28	-10	75	214	74	-11	-29	-26	-20	-16	-11	-7	-3	0
12	0	-2	-5	-9	-13	-16	-22	-24	-6	69	191	70	-6	-25	-23	-18	-15	-10	-6	-2	0
13	0	-1	-4	-8	-11	-14	-19	-20	-3	64	169	65	-3	-21	-20	-15	-13	-9	-6	-2	0
14	0	-1	-3	-6	-9	-12	-16	-17	-1	58	148	59	-1	-17	-17	-13	-11	-8	-4	-1	0
15	0	0	-2	-5	-8	-9	-13	-13	-0	52	128	54	0	-14	-14	-10	-9	-6	-3	0	0
16	0	0	-1	-3	-6	-7	-10	-10	0	47	108	48	0	-11	-11	-8	-7	-4	-1	0	0
17	0	0	0	-1	-4	-4	-8	-7	0	43	98	44	0	-8	-8	-5	-4	-2	0	0	0
18	0	0	0	0	-2	-2	-5	-5	0	40	72	41	0	-5	-6	-2	-2	0	0	0	0
19	0	0	0	0	0	0	-3	-3	-0	39	56	39	-0	-3	-3	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	40	41	40	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	40	41	40	0	0	0	0	0	0	0	0	0

JET INDUCED CIRCULATION IN RESERVOIRS

V VELOCITY COMPONENTS IN THOUSANDTHS OF A METRE PER SECOND FOR THE TIMESTEP 400

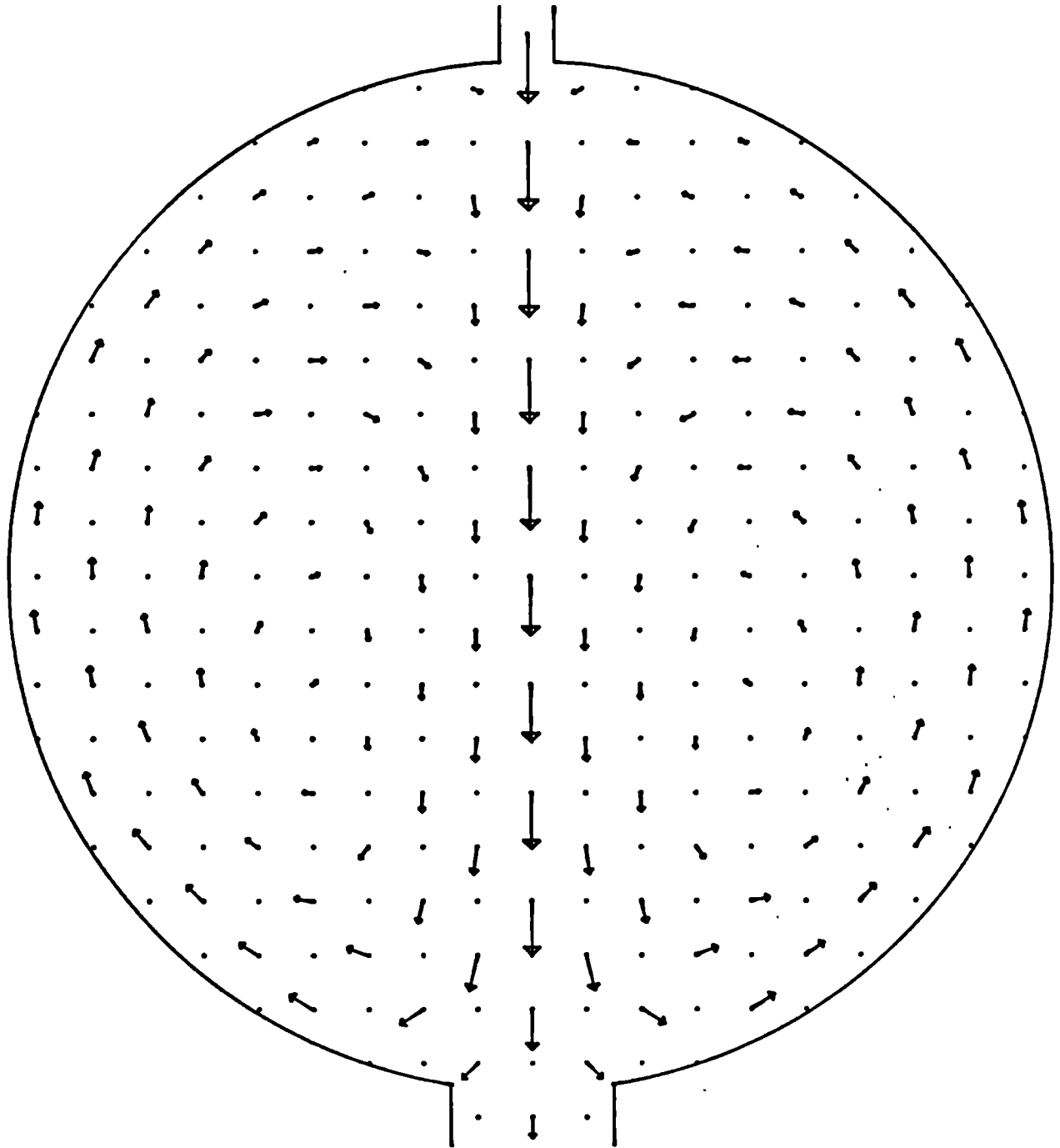
M	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	16	32	41	70	-84	-43	-33	-17	0	0	0	0	0	0	0
3	0	0	0	0	13	26	35	45	50	57	-67	-55	-49	-38	-27	-13	0	0	0	0	0
4	0	0	0	9	16	24	30	35	34	17	-28	-42	-41	-35	-29	-19	-11	0	0	0	0
5	0	0	4	8	10	13	15	16	15	-1	3	-15	-16	-15	-13	-10	-8	-5	0	0	0
6	0	0	1	1	0	0	0	1	4	-5	-0	-9	-6	-4	-3	-2	-3	-2	0	0	0
7	0	0	-0	-1	-2	-3	-3	-1	3	1	6	2	5	6	5	3	1	0	-0	0	0
8	0	-0	-0	-2	-4	-6	-7	-8	-7	-8	2	2	4	4	3	2	0	-0	-0	0	0
9	0	0	0	0	-0	-1	-3	-4	-6	-3	13	13	9	6	3	2	0	-0	-0	0	0
10	0	0	0	0	-1	-2	-5	-9	-15	-13	8	10	6	2	0	-0	-1	-1	-1	0	0
11	0	0	0	0	-0	-2	-4	-8	-12	-8	14	17	11	6	3	1	0	-0	-0	0	0
12	0	-0	-0	-1	-2	-4	-7	-12	-16	-11	10	15	11	7	4	2	1	0	-0	0	0
13	0	-0	-1	-2	-3	-6	-9	-12	-15	-9	11	17	13	9	6	4	2	1	0	0	0
14	0	-0	-1	-3	-5	-7	-10	-13	-16	-10	11	16	14	11	7	5	3	2	0	0	0
15	0	-1	-2	-3	-5	-8	-10	-14	-15	-9	10	16	15	11	8	6	4	2	1	0	0
16	0	0	-1	-3	-5	-7	-10	-13	-14	-9	10	15	14	11	8	6	3	1	0	0	0
17	0	0	-1	-3	-5	-7	-10	-13	-13	-8	9	14	14	11	8	6	3	1	0	0	0
18	0	0	0	-1	-4	-6	-9	-11	-11	-8	9	12	12	10	7	4	2	0	0	0	0
19	0	0	0	0	-2	-4	-7	-9	-9	-7	8	10	10	8	5	2	0	0	0	0	0
20	0	0	0	0	0	0	-3	-6	-6	-7	7	6	6	3	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0

APPENDIX 11

VELOCITY FIELDS FOR THE
JET CIRCULATION IN A
RESERVOIR AND THE TIDAL
CIRCULATION IN A HARBOUR

JET CIRCULATION IN A RESERVOIR

EXPERIMENTAL RESULTS



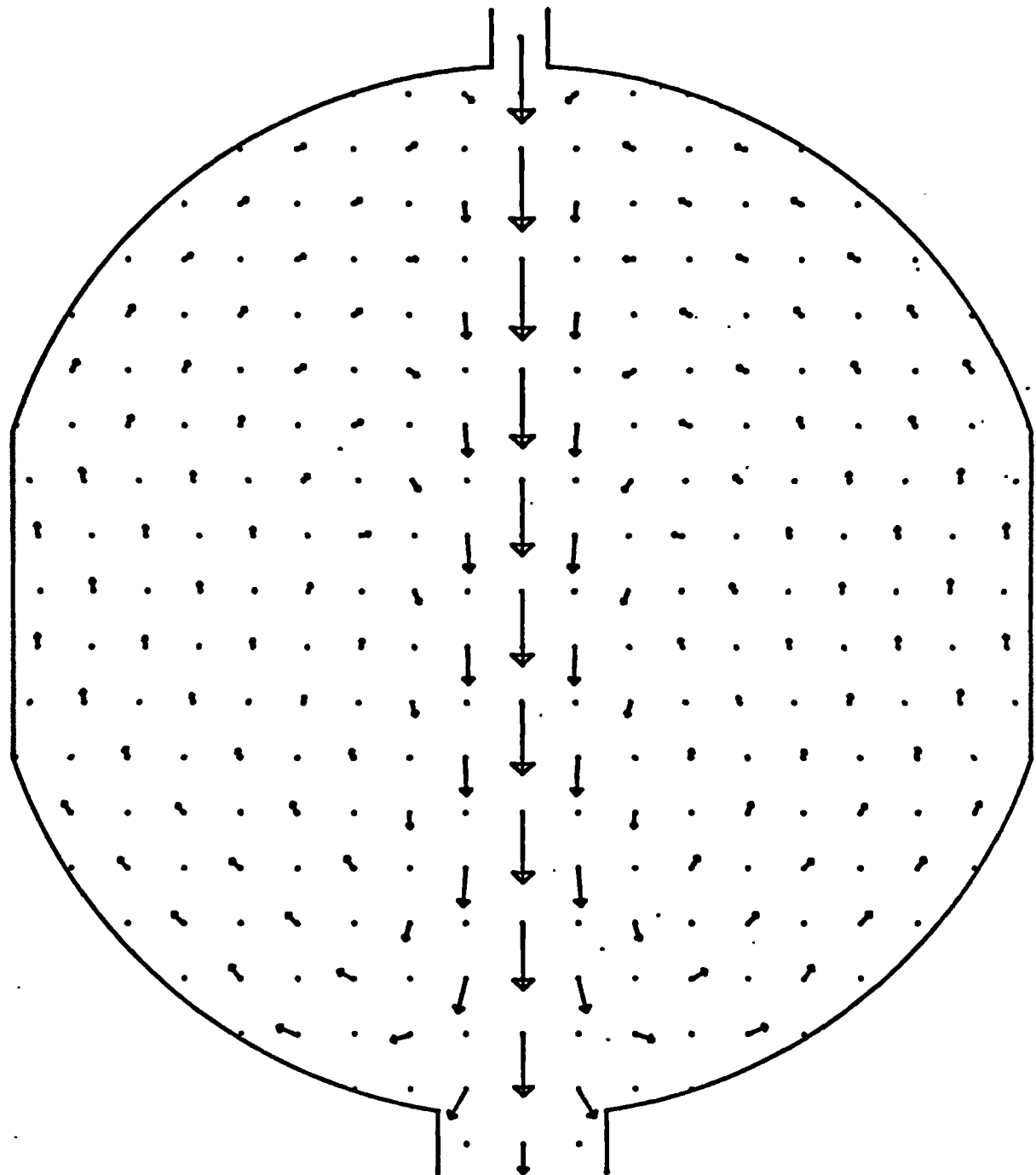
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 8.5 CM
JET INFLOW 750 CC/SEC
JET VELOCITY 10.0 CM/SEC

VELOCITY → 8.0 CM/SEC
ROUGHNESS - PLANED WOOD
WEIR OUTFLOW 750 CC/SEC
DEPTH OF JET 9.3 CM

FIGURE 15 Measured Velocity Field for the High Flow Jet Inlet

JET CIRCULATION IN A RESERVOIR

EXPERIMENTAL RESULTS



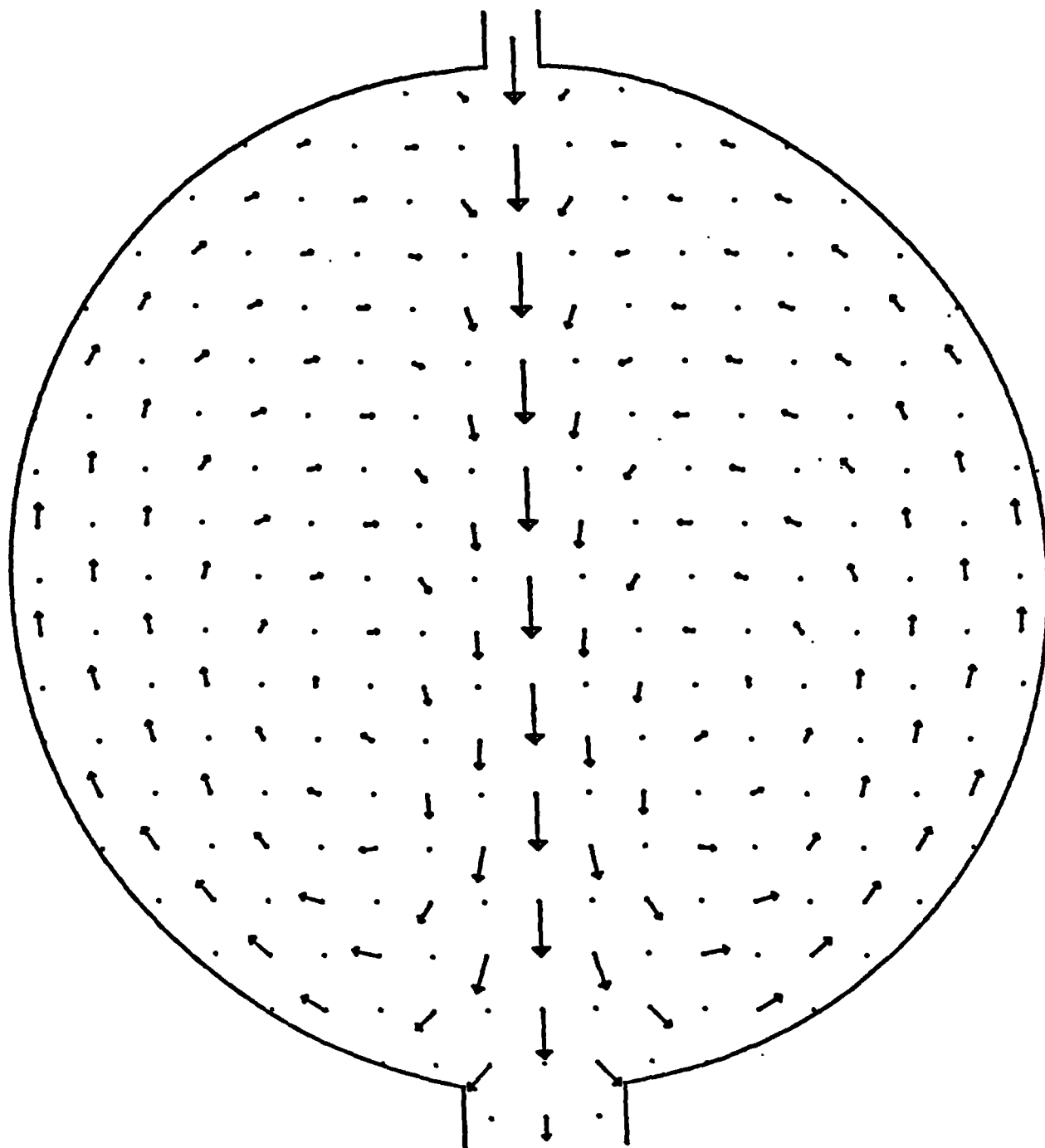
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 8.5 CM
JET INFLOW 210 CC/SEC
JET VELOCITY 3.0 CM/SEC

VELOCITY → 2.0 CM/SEC
ROUGHNESS - PLANED WOOD
WEIR OUTFLOW 210 CC/SEC
DEPTH OF JET 9.0 CM

FIGURE 16 Measured Velocity Field for the Low Flow Jet Inlet

JET CIRCULATION IN A RESERVOIR

EXPERIMENTAL RESULTS



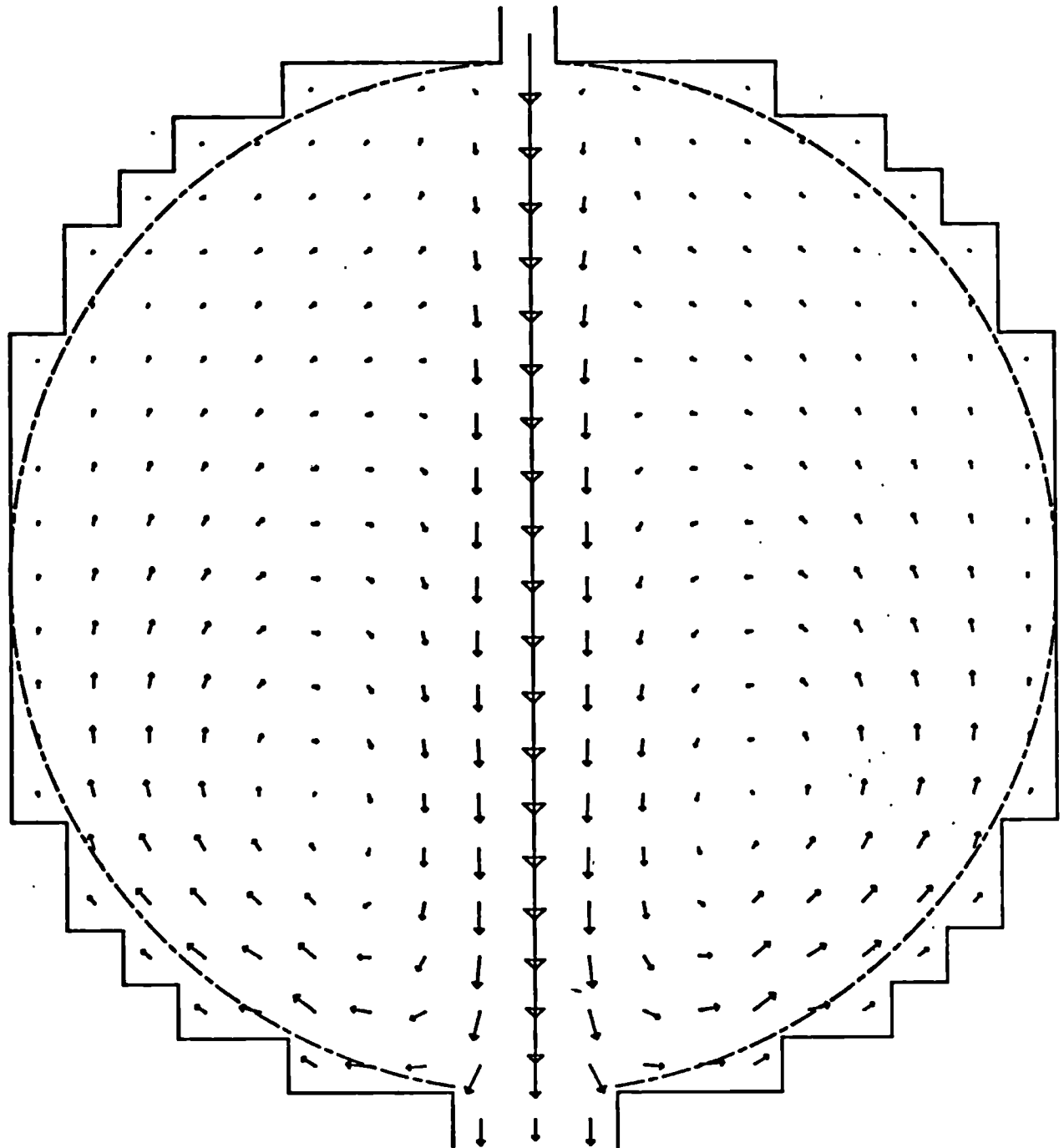
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 12.0 CM
JET INFLOW 750 CC/SEC
JET VELOCITY 7.0 CM/SEC

VELOCITY → 6.0 CM/SEC
ROUGHNESS - PLANED WOOD
WEIR OUTFLOW 750 CC/SEC
DEPTH OF JET 13.3 CM

FIGURE 17 Measured Velocity Field for the High Flow Jet Inlet
with an Increased Depth

JET CIRCULATION IN A RESERVOIR

TIME IN SECS 144



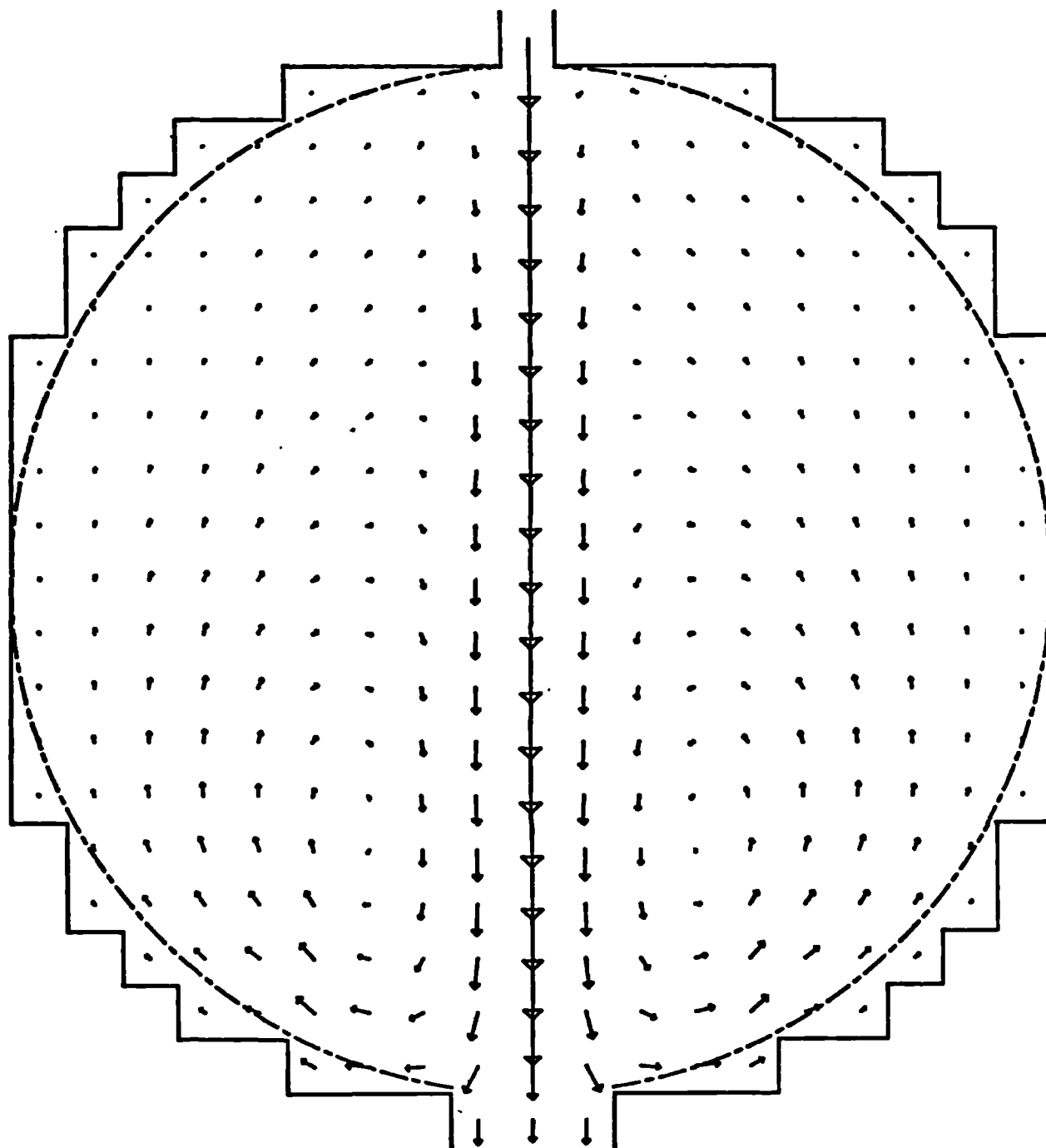
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 8.5 CM
JET INFLOW 750 CC/SEC
JET VELOCITY 10.0 CM/SEC

VELOCITY → 8.0 CM/SEC
CHEZY VALUE 438 SI UNITS
WEIR OUTFLOW 750 CC/SEC
DEPTH OF JET 9.3 CM

FIGURE 18 Computed Velocity Field for the High Flow Jet Inlet

JET CIRCULATION IN A RESERVOIR

TIME IN SECS 144



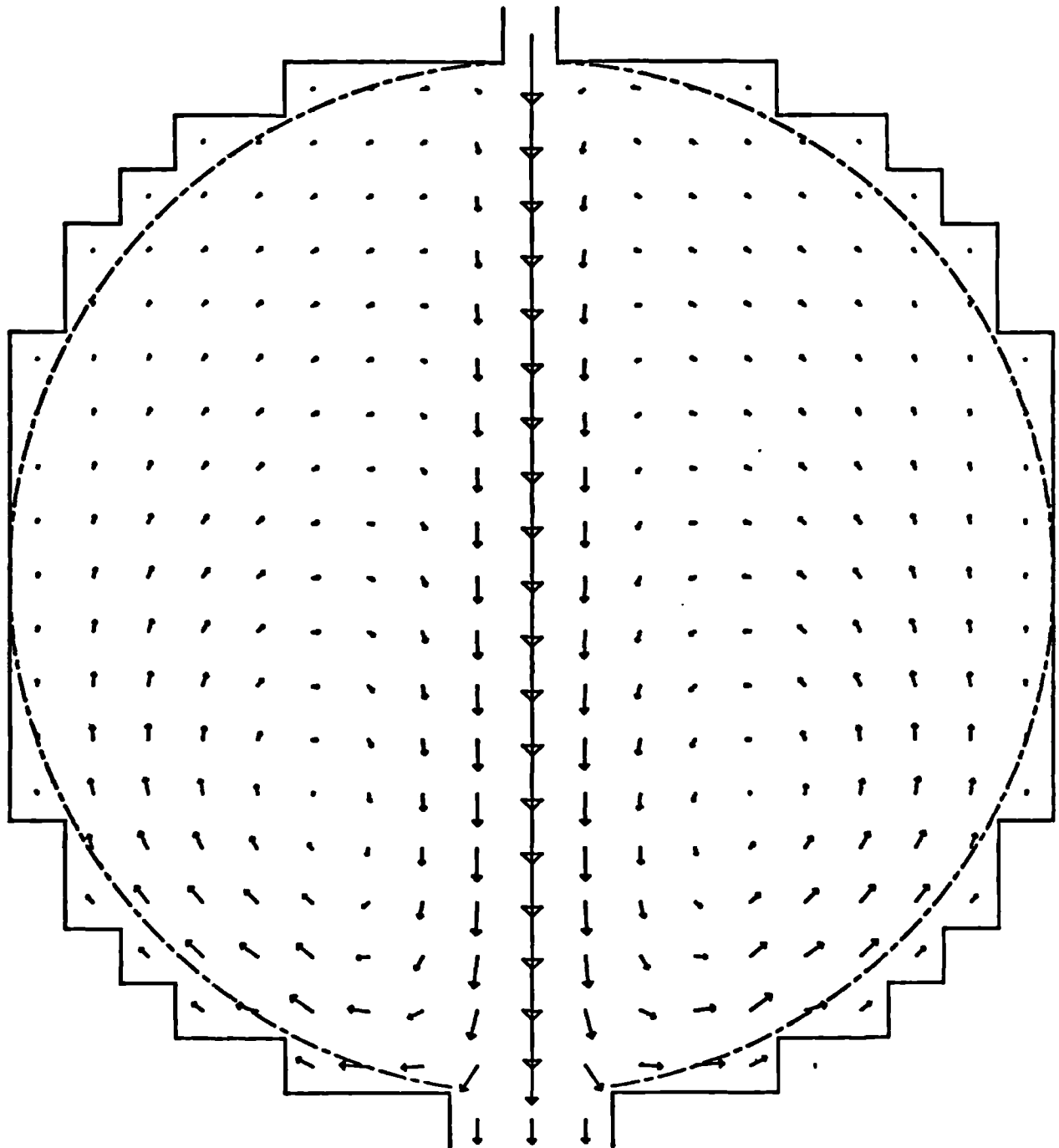
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 8.5 CM
JET INFLOW 750 CC/SEC
JET VELOCITY 10.0 CM/SEC

VELOCITY → 8.0 CM/SEC
CHEZY VALUE 438 SI UNITS
WEIR OUTFLOW 750 CC/SEC
DEPTH OF JET 9.3 CM

FIGURE 19 Computed Velocity Field for the High Flow Jet Inlet with a No-Slip Boundary Condition

JET CIRCULATION IN A RESERVOIR

TIME IN SECS 144



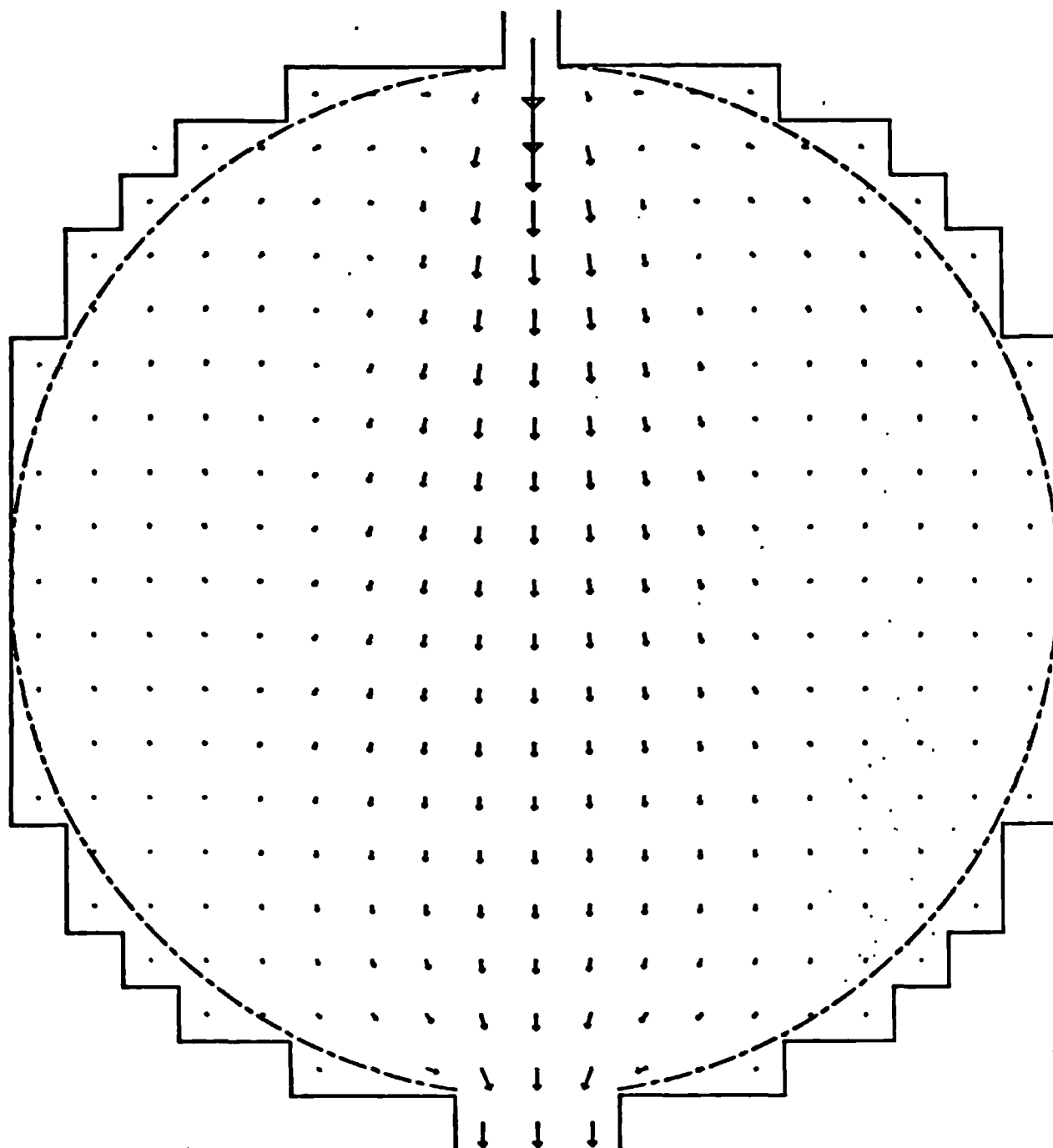
LENGTH SCALE — 8.0 CM
 INITIAL DEPTH 8.5 CM
 JET INFLOW 750 CC/SEC
 JET VELOCITY 10.0 CM/SEC

VELOCITY → 8.0 CM/SEC
 CHEZY VALUE MOODY DIAGRAM
 WEIR OUTFLOW 750 CC/SEC
 DEPTH OF JET 9.3 CM

FIGURE 20 Computed Velocity Field for the High Flow Jet Inlet with the Chezy C being a Function of the Local Reynolds Number

JET CIRCULATION IN A RESERVOIR

TIME IN SECS 144



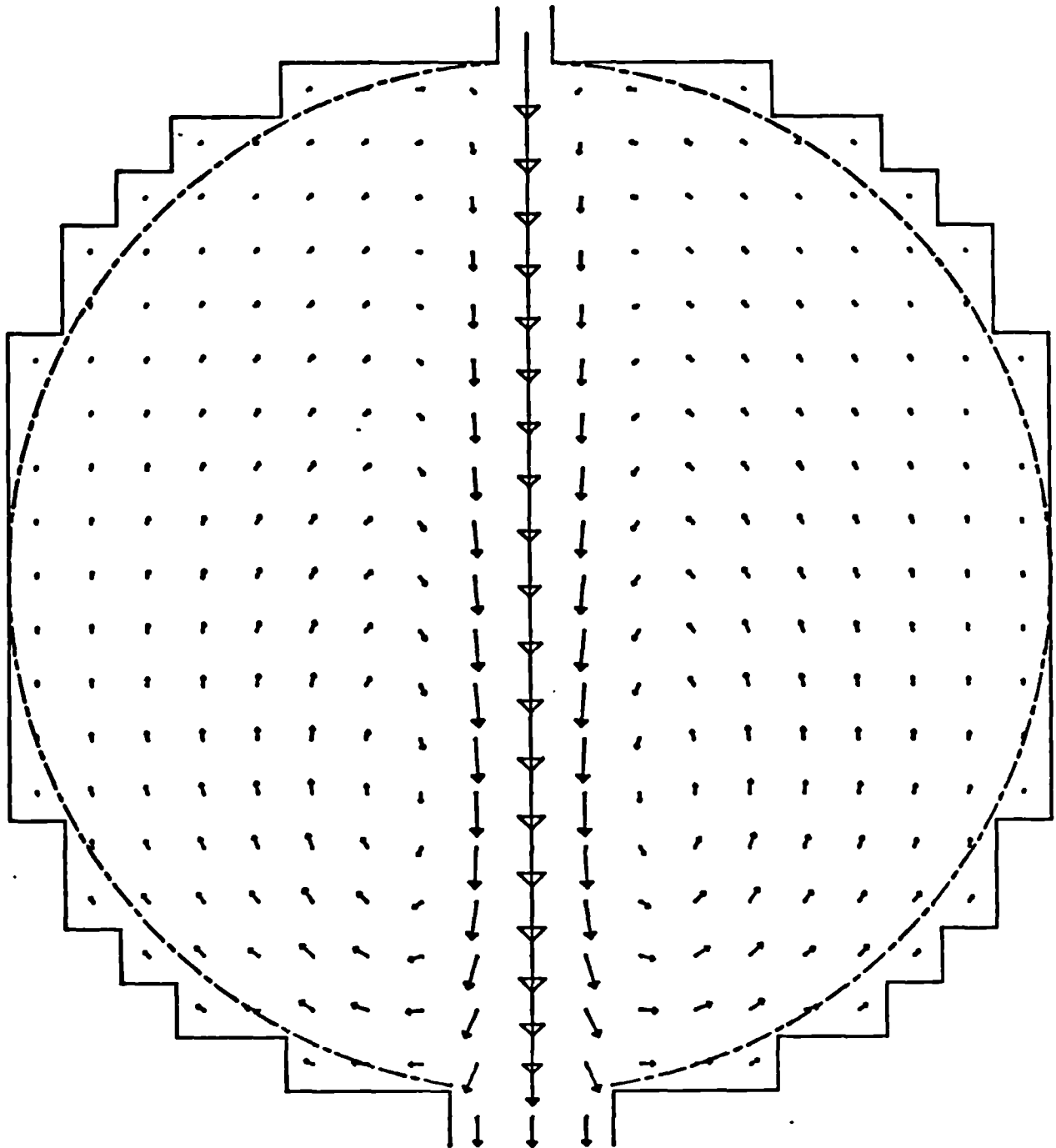
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 8.5 CM
JET INFLOW 750 CC/SEC
JET VELOCITY 10.0 CM/SEC

VELOCITY → 8.0 CM/SEC
CHEZY VALUE 438 SI UNITS
WEIR OUTFLOW 750 CC/SEC
DEPTH OF JET 9.3 CM

FIGURE 21 Computed Velocity Field for the High Flow Jet Inlet Using the Velocity Magnitude for the Components of the Shear Velocity.

JET CIRCULATION IN A RESERVOIR

TIME IN SECS 144



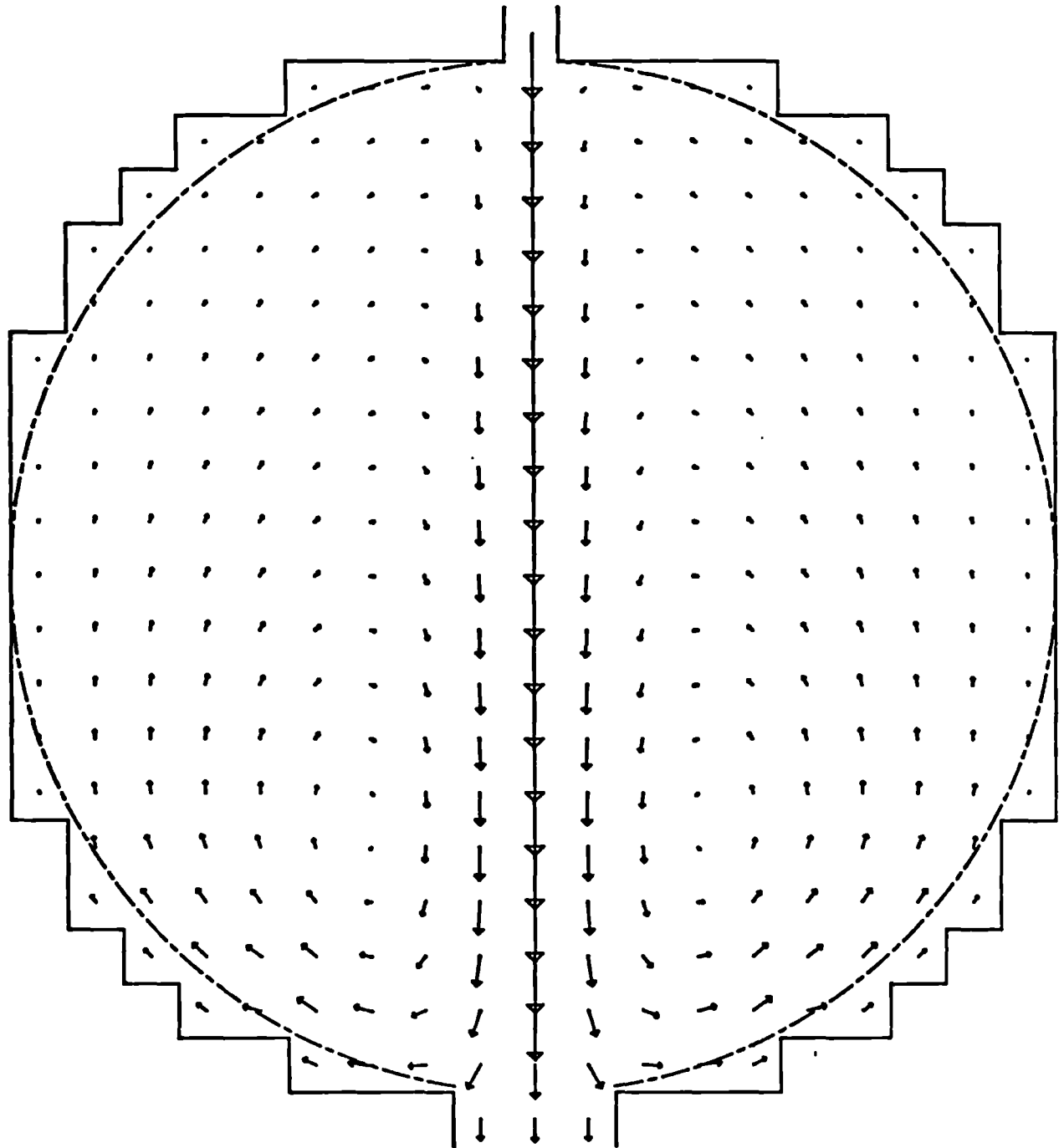
LENGTH SCALE — 8.0 CM
INITIAL DEPTH 8.5 CM
JET INFLOW 210 CC/SEC
JET VELOCITY 3.0 CM/SEC

VELOCITY → 2.0 CM/SEC
CHEZY VALUE 438 SI UNITS
WEIR OUTFLOW 210 CC/SEC
DEPTH OF JET 9.0 CM

FIGURE 22 Computed Velocity Field for the Low Flow Jet Inlet

JET CIRCULATION IN A RESERVOIR

TIME IN SECS 144



LENGTH SCALE — 8.0 CM

INITIAL DEPTH 12.0 CM

JET INFLOW 750 CC/SEC

JET VELOCITY 7.0 CM/SEC

VELOCITY → 6.0 CM/SEC

CHEZY VALUE 468 SI UNITS

WEIR OUTFLOW 750 CC/SEC

DEPTH OF JET 19.3 CM

FIGURE 23 Computed Velocity Field for the High Flow Jet Inlet
with an Increased Depth

TIDAL CIRCULATION IN A HARBOUR

EXPERIMENTAL RESULTS

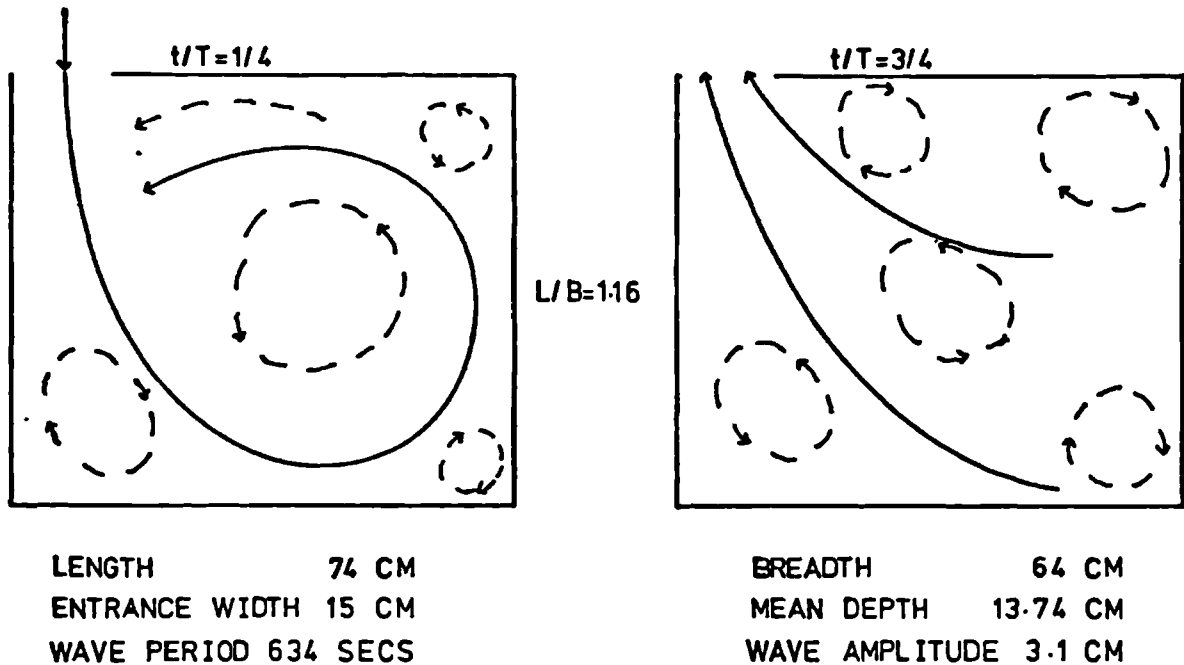


Figure 24a

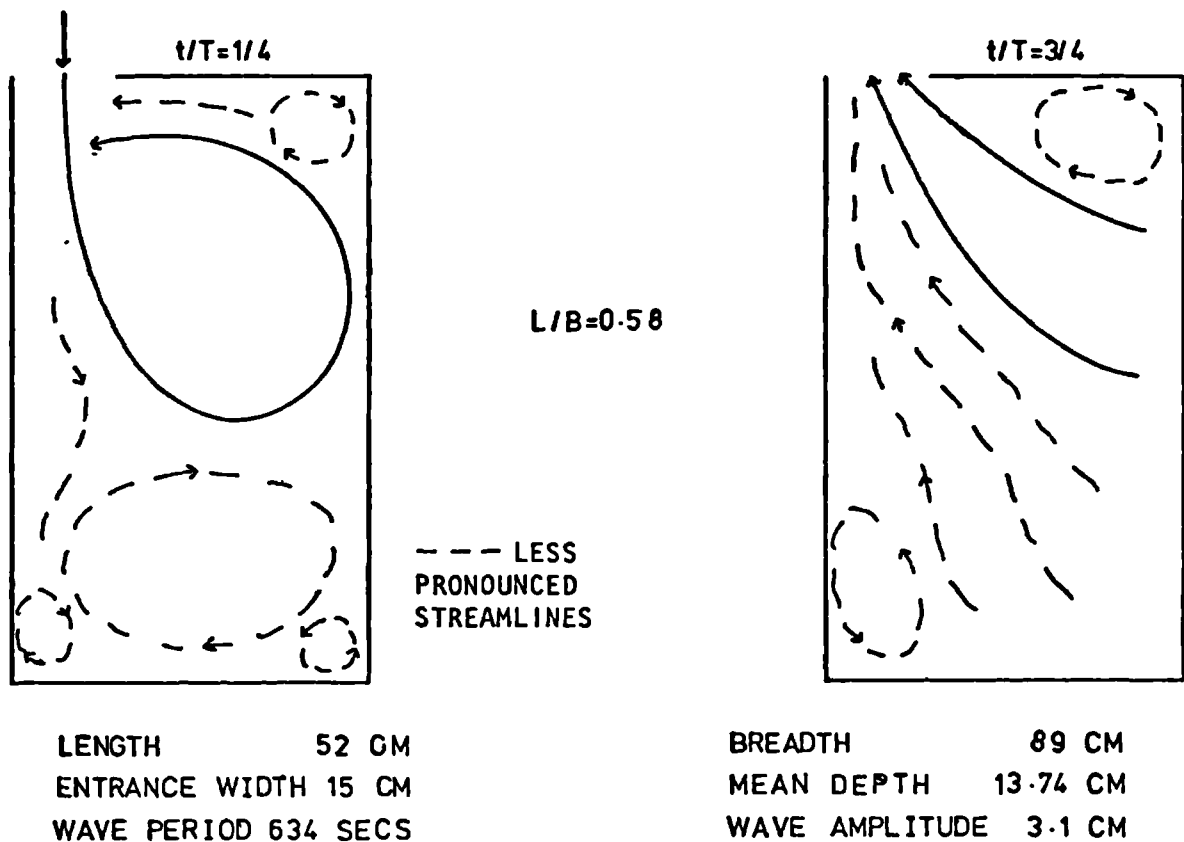
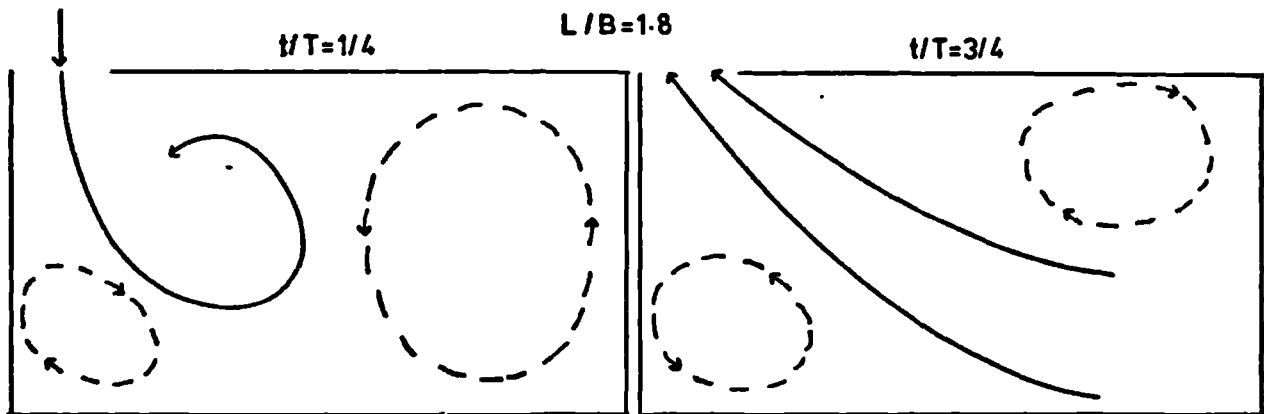


Figure 24b

FIGURE 24 Observed Velocity Fields for Different L/B Ratios

TIDAL CIRCULATION IN A HARBOUR

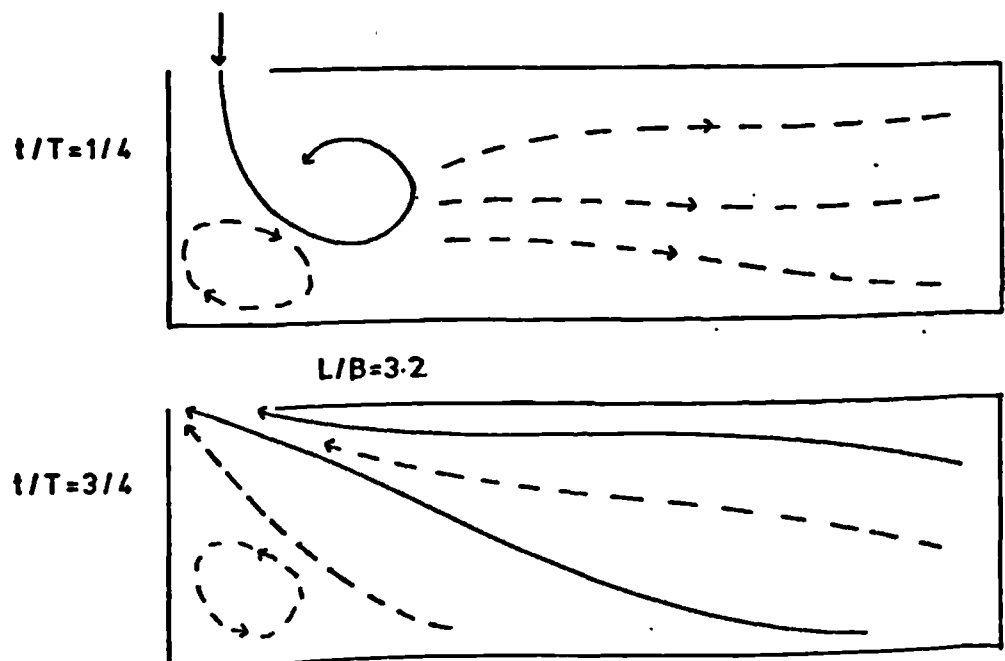
EXPERIMENTAL RESULTS



LENGTH 91 CM
ENTRANCE WIDTH 15 CM
WAVE PERIOD 634 SECS

BREADTH 50 CM
MEAN DEPTH 13.74 CM
WAVE AMPLITUDE 3.1 CM

Figure 24c



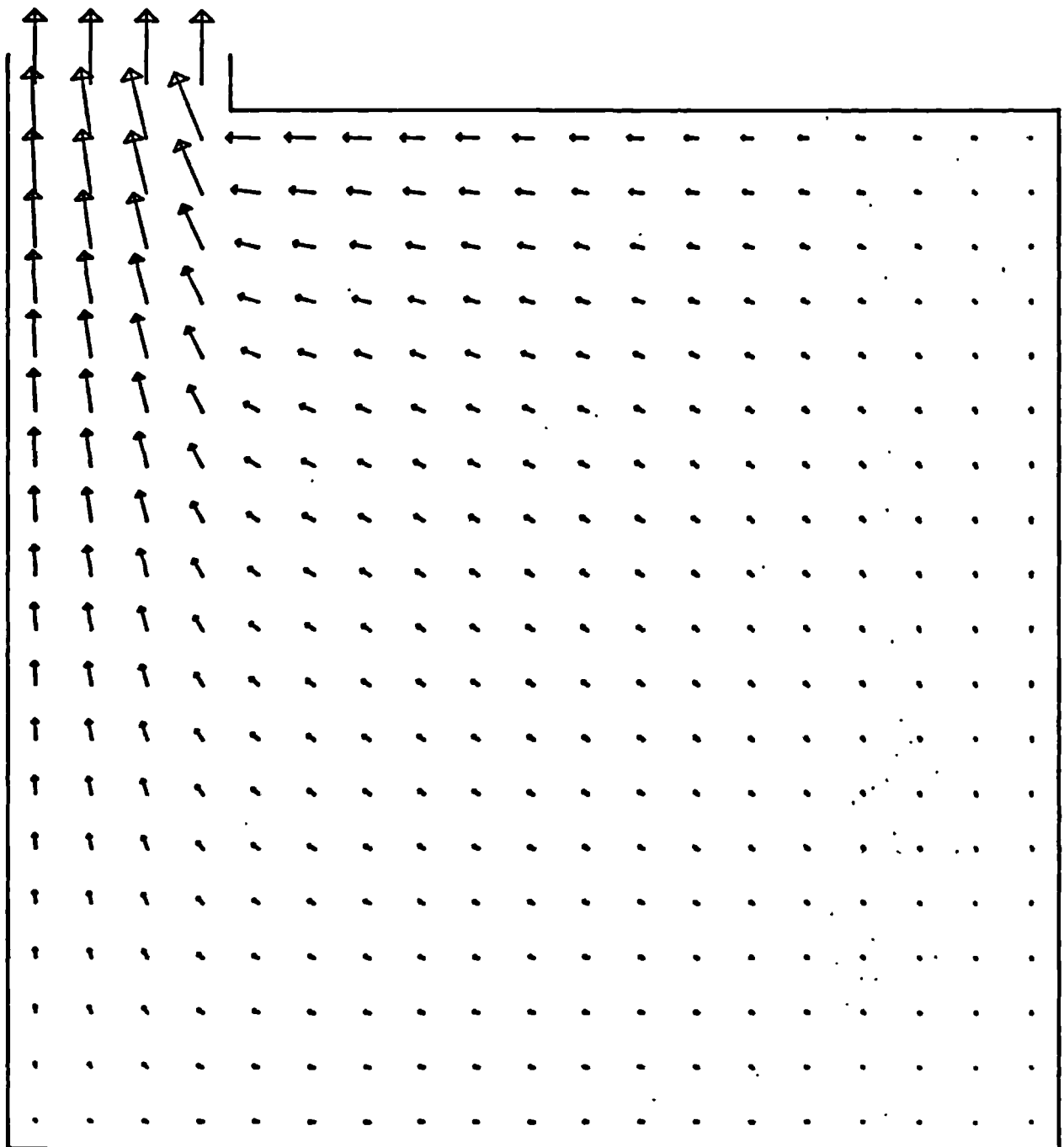
LENGTH 122 CM
ENTRANCE WIDTH 15 CM
WAVE PERIOD 634 SECS

BREADTH 38 CM
MEAN DEPTH 13.74 CM
WAVE AMPLITUDE 3.1 CM

Figure 24d

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 158



LENGTH SCALE — 4.0 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

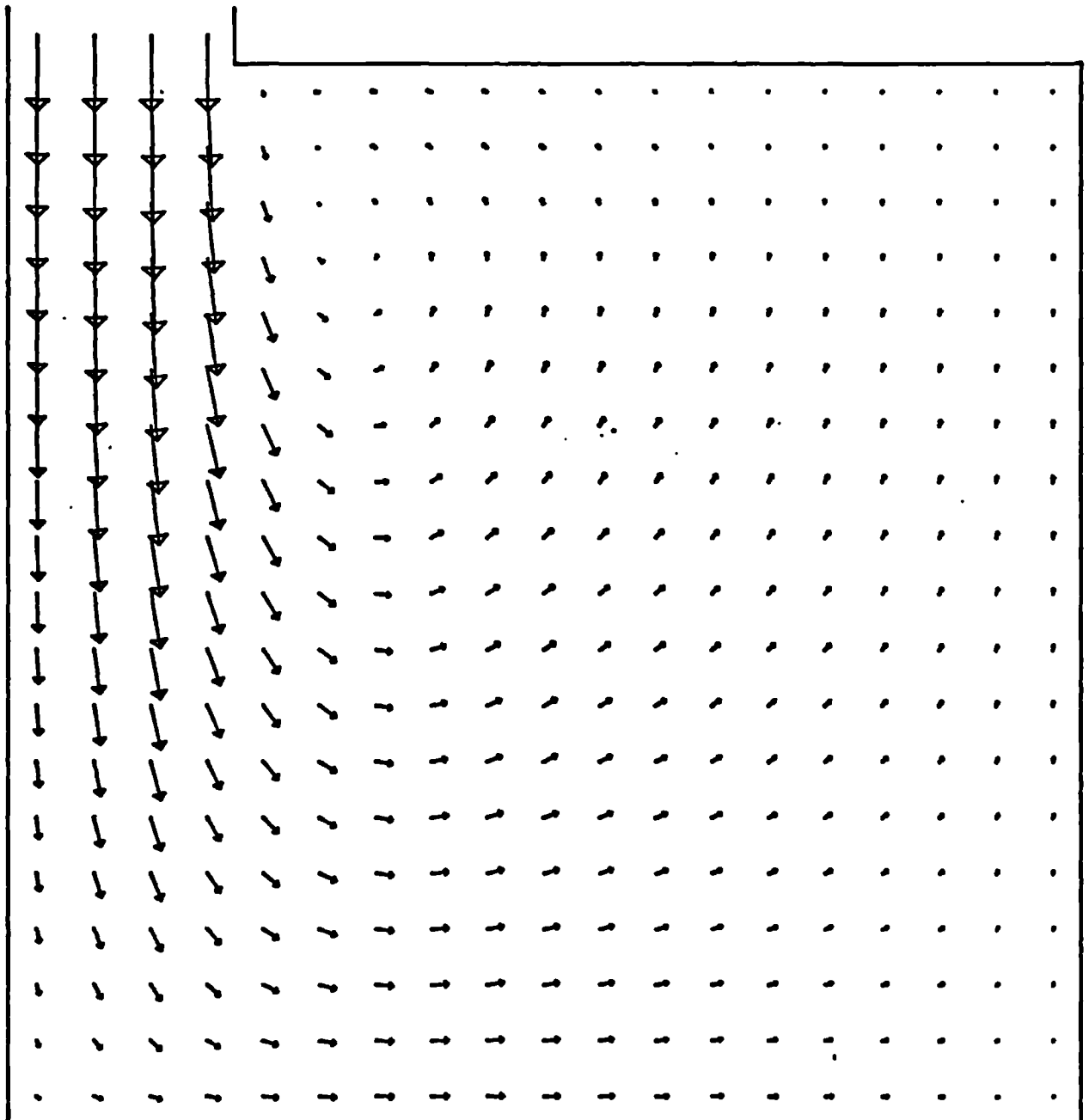
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 25a Computed Velocity Field for the Ebb Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 475



LENGTH SCALE — 4.0 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

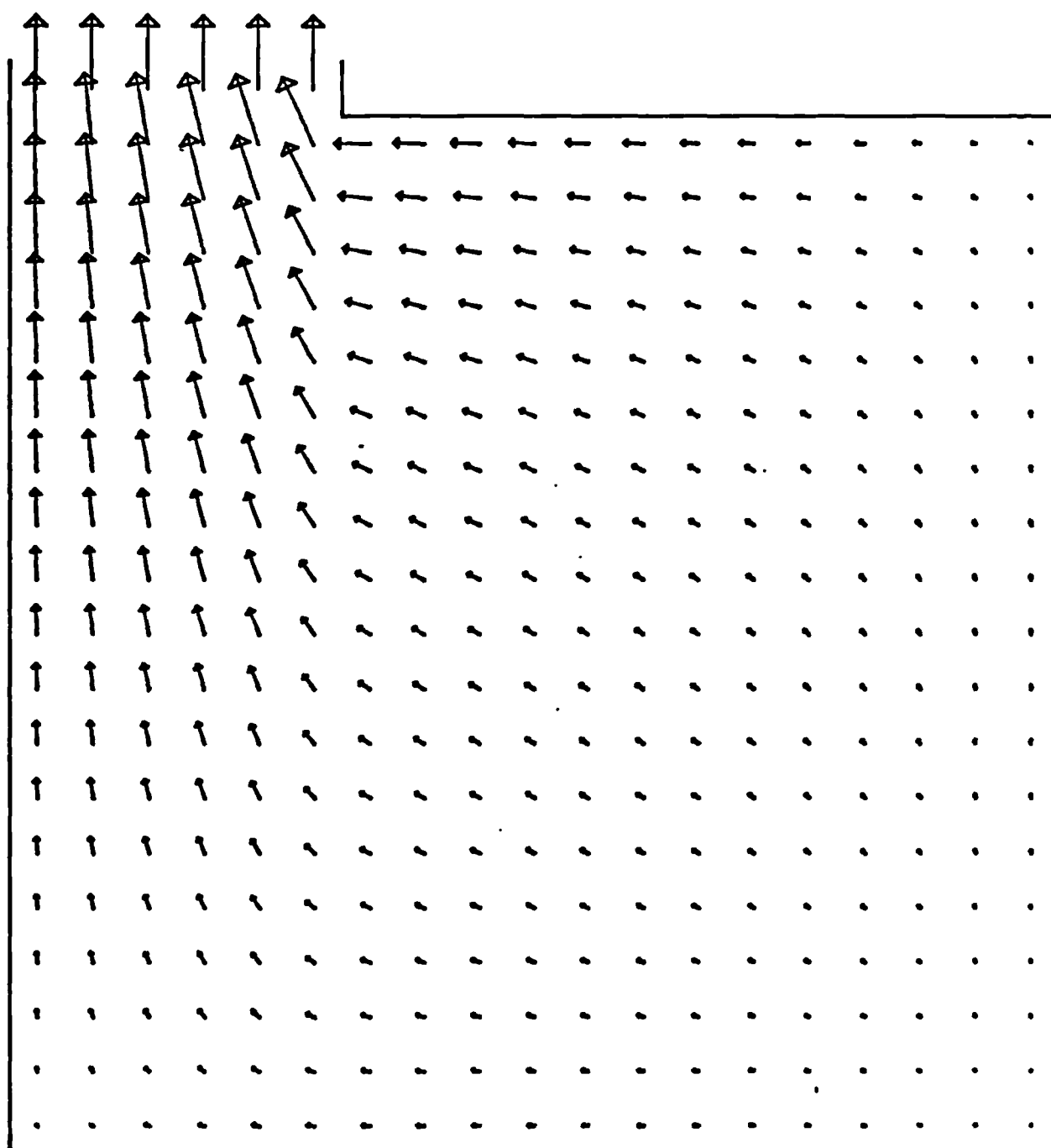
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 25b Computed Velocity Field for the Flood Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 158



LENGTH SCALE — 4.0 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY → 0.4 CM/SEC

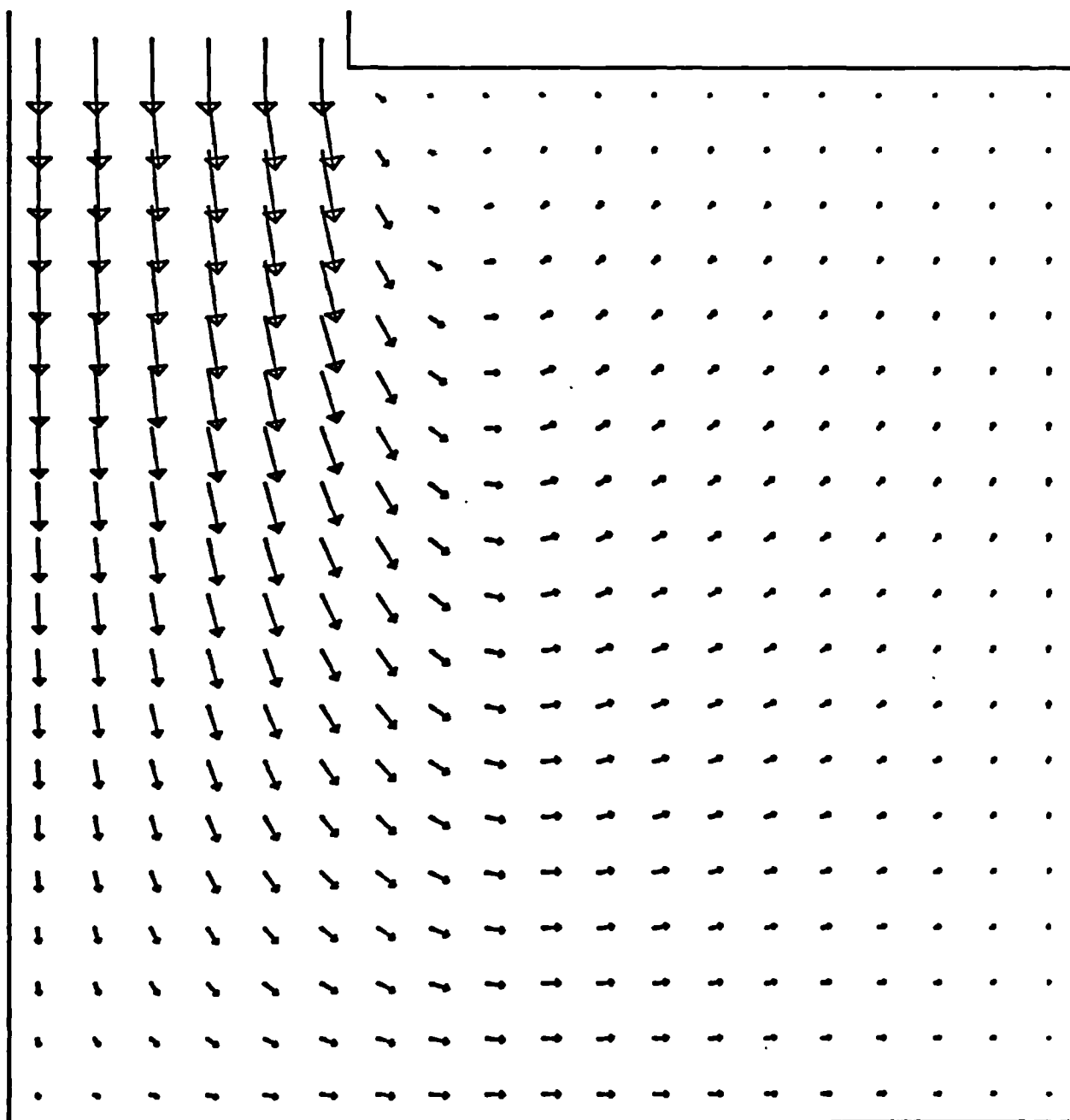
CHEZY VALUE 422 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 26a Computed Velocity Field as in Figure 25a with a Wider Entrance

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 475



LENGTH SCALE — 4.0 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.4 CM/SEC

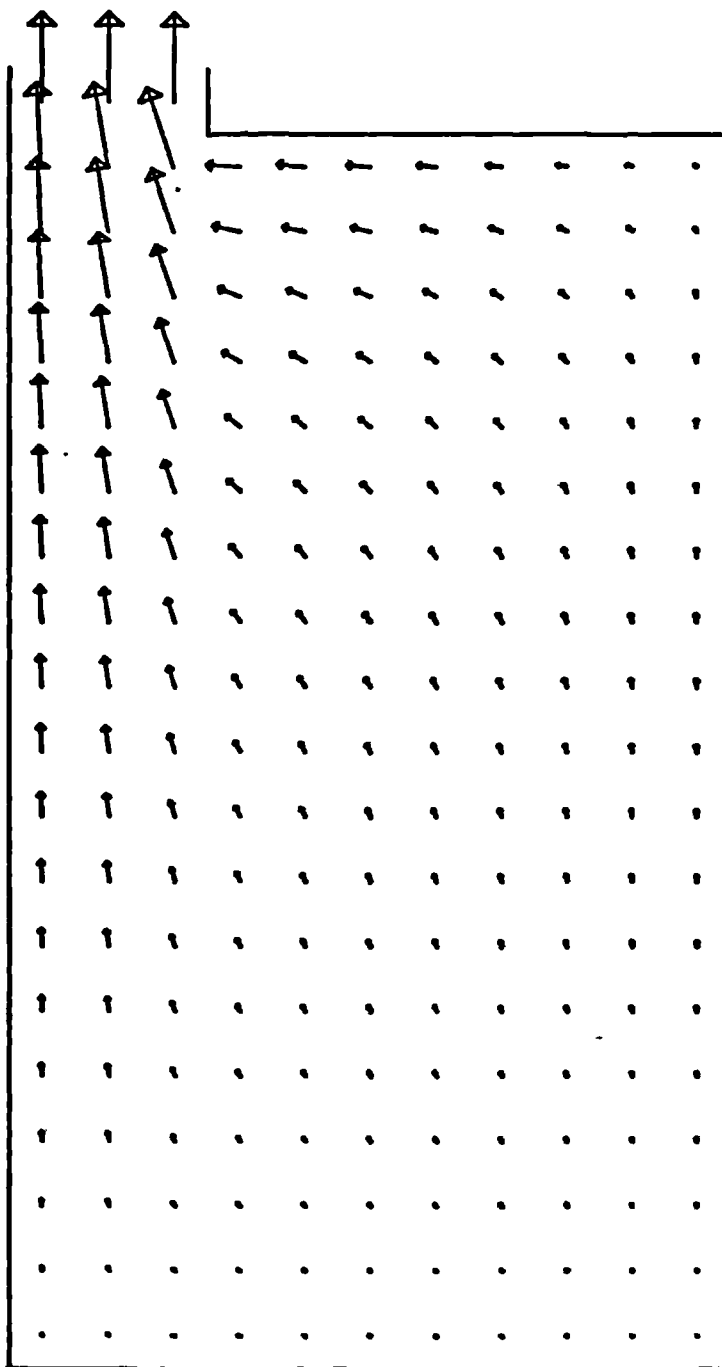
CHEZY VALUE 422 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 26b Computed Velocity Field as in Figure 25b with a Wider Entrance

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 158



LENGTH SCALE — 5.3 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

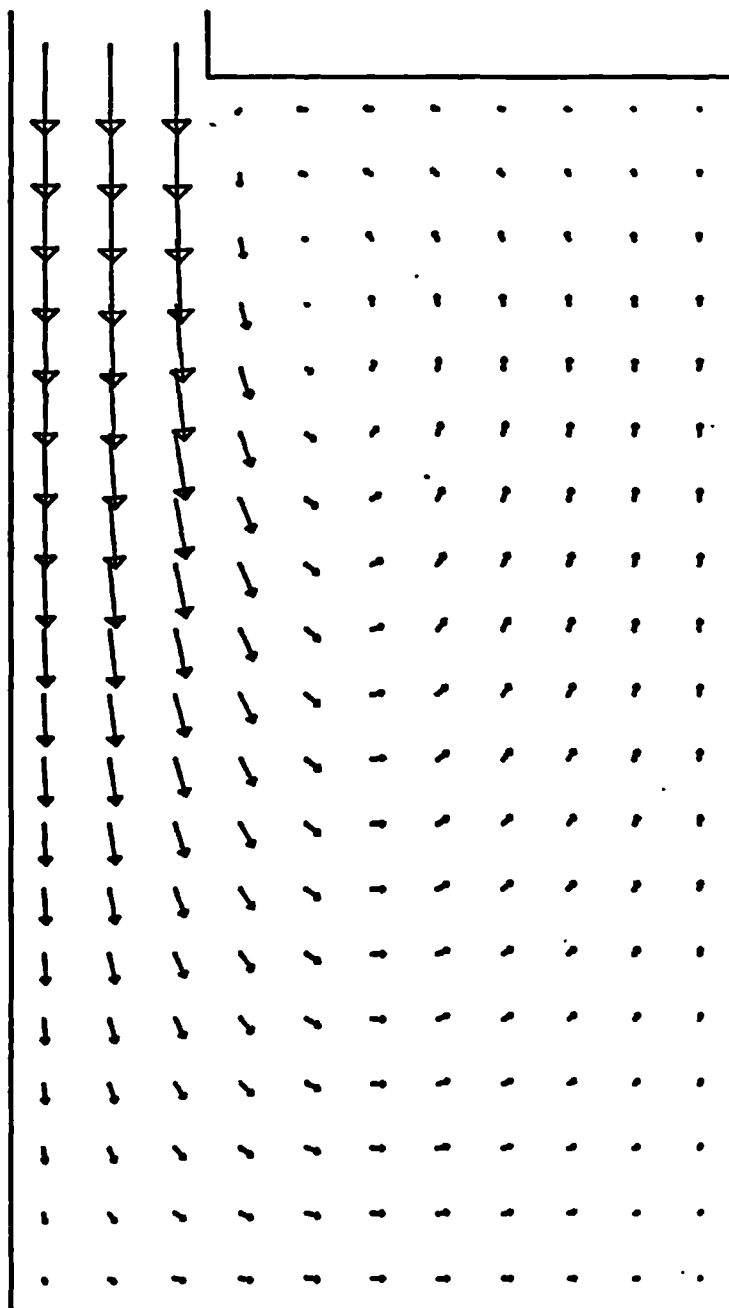
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 27a Computed Velocity Field for the Ebb Tide with $L/B = 0.58$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 475



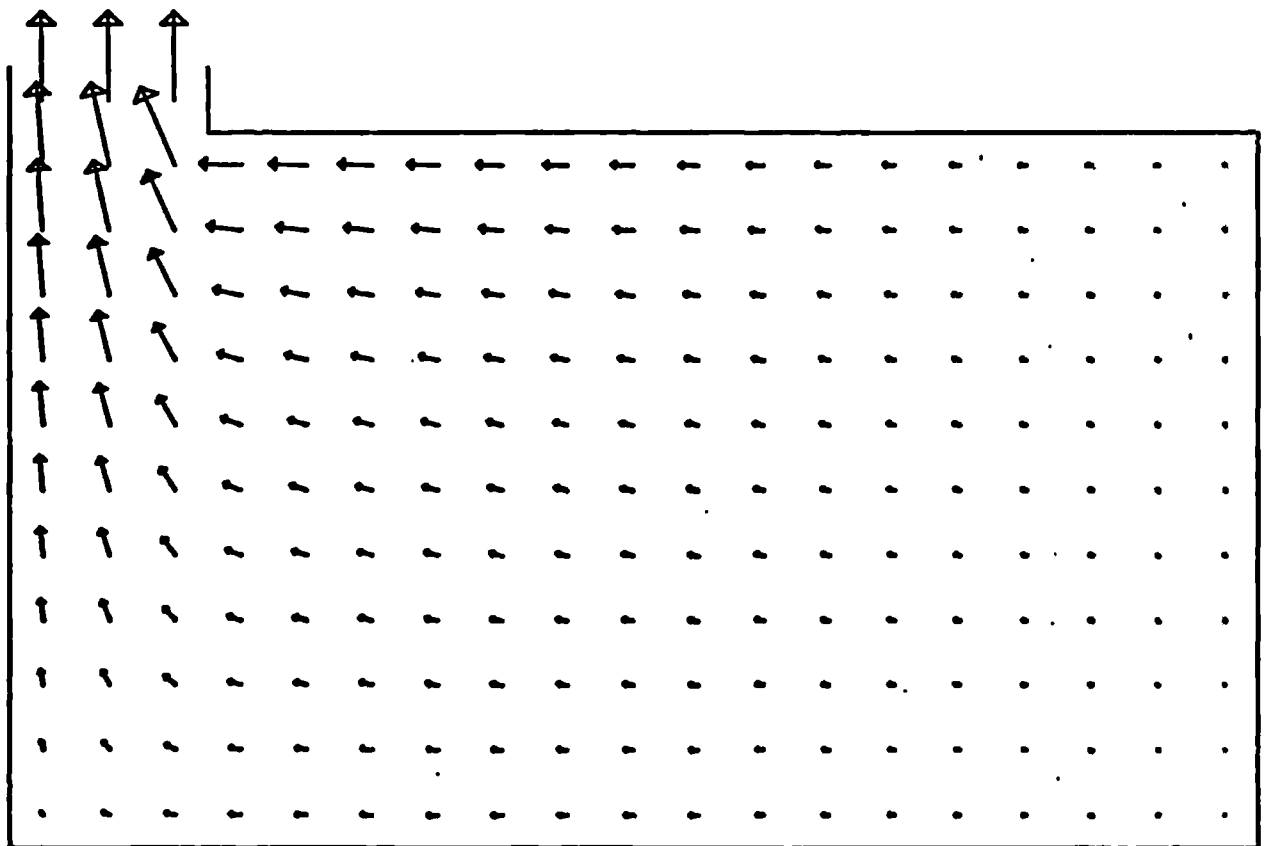
LENGTH SCALE — 5.3 CM
 MEAN DEPTH 19.74 CM
 WAVE PERIOD 634 SECS

VELOCITY → 0.6 CM/SEC
 CHEZY VALUE 405 SI UNITS
 WAVE AMPLITUDE 3.08 CM

FIGURE 27b Computed Velocity Field for the Flood Tide with $L/B = 0.58$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 158



LENGTH SCALE — 5.3 CM

MEAN DEPTH 19.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

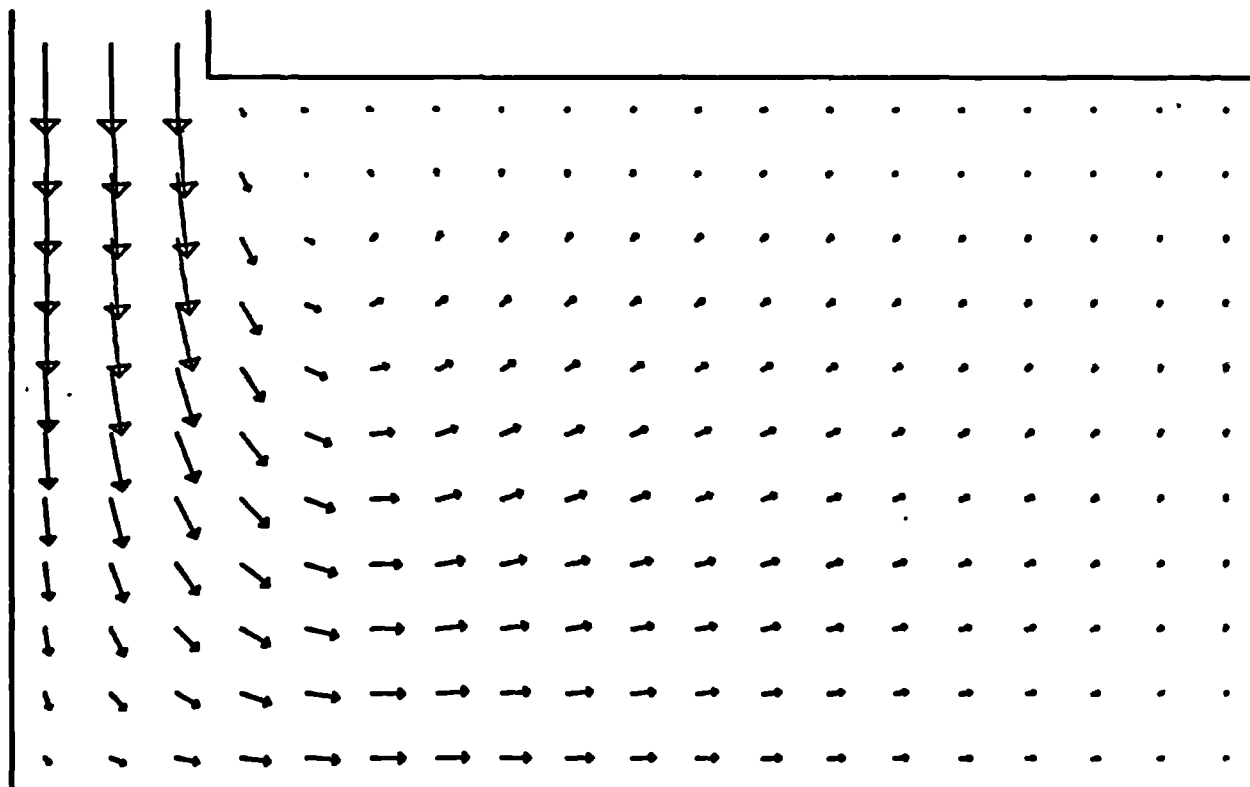
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 28a Computed Velocity Field for the Ebb Tide with $L/B = 1.73$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 475



LENGTH SCALE — 5.3 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

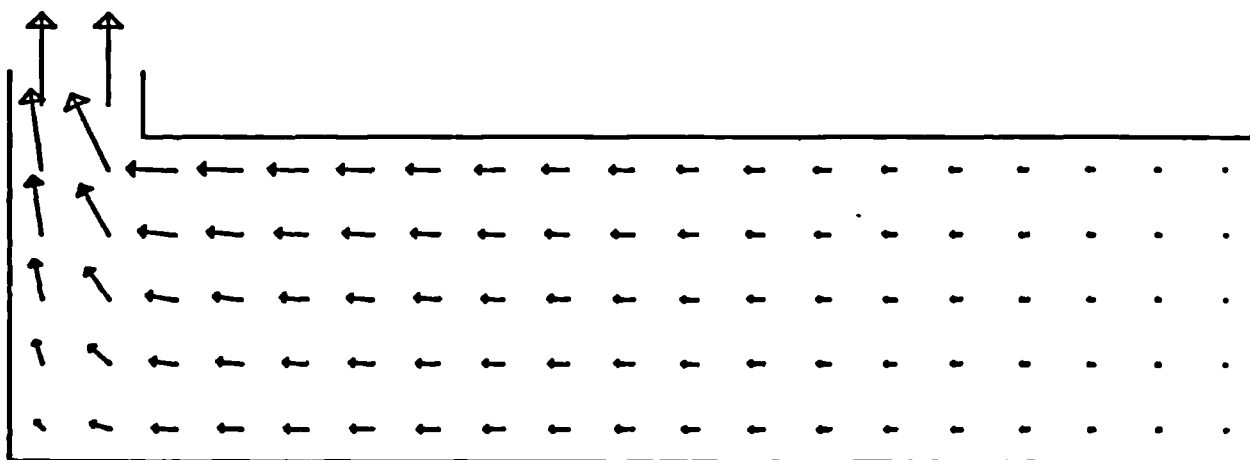
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 28b Computed Velocity Field for the Flood Tide with $L/B = 1.73$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 158



LENGTH SCALE — 8.0 CM

MEAN DEPTH 19.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

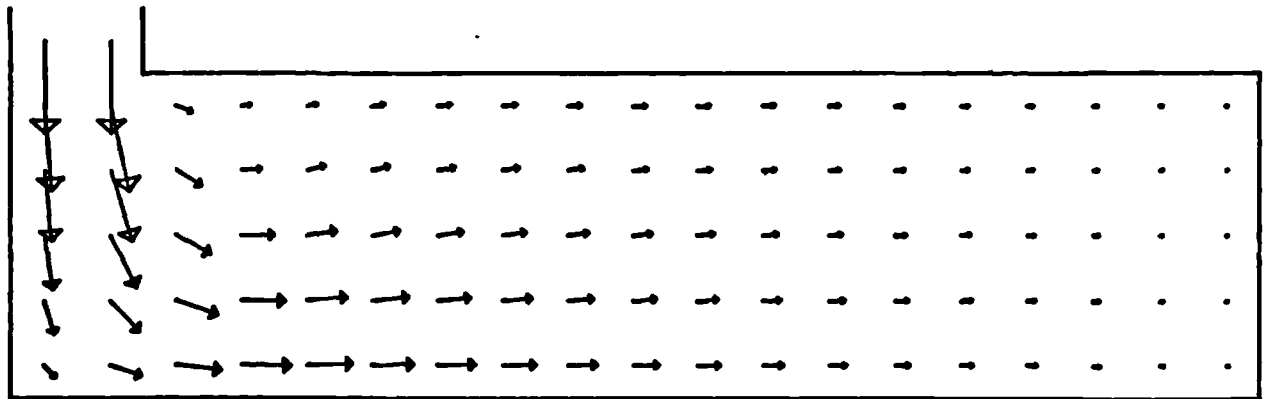
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 29a Computed Velocity Field for the Ebb Tide with $L/B = 3.8$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 475



LENGTH SCALE — 8.0 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC

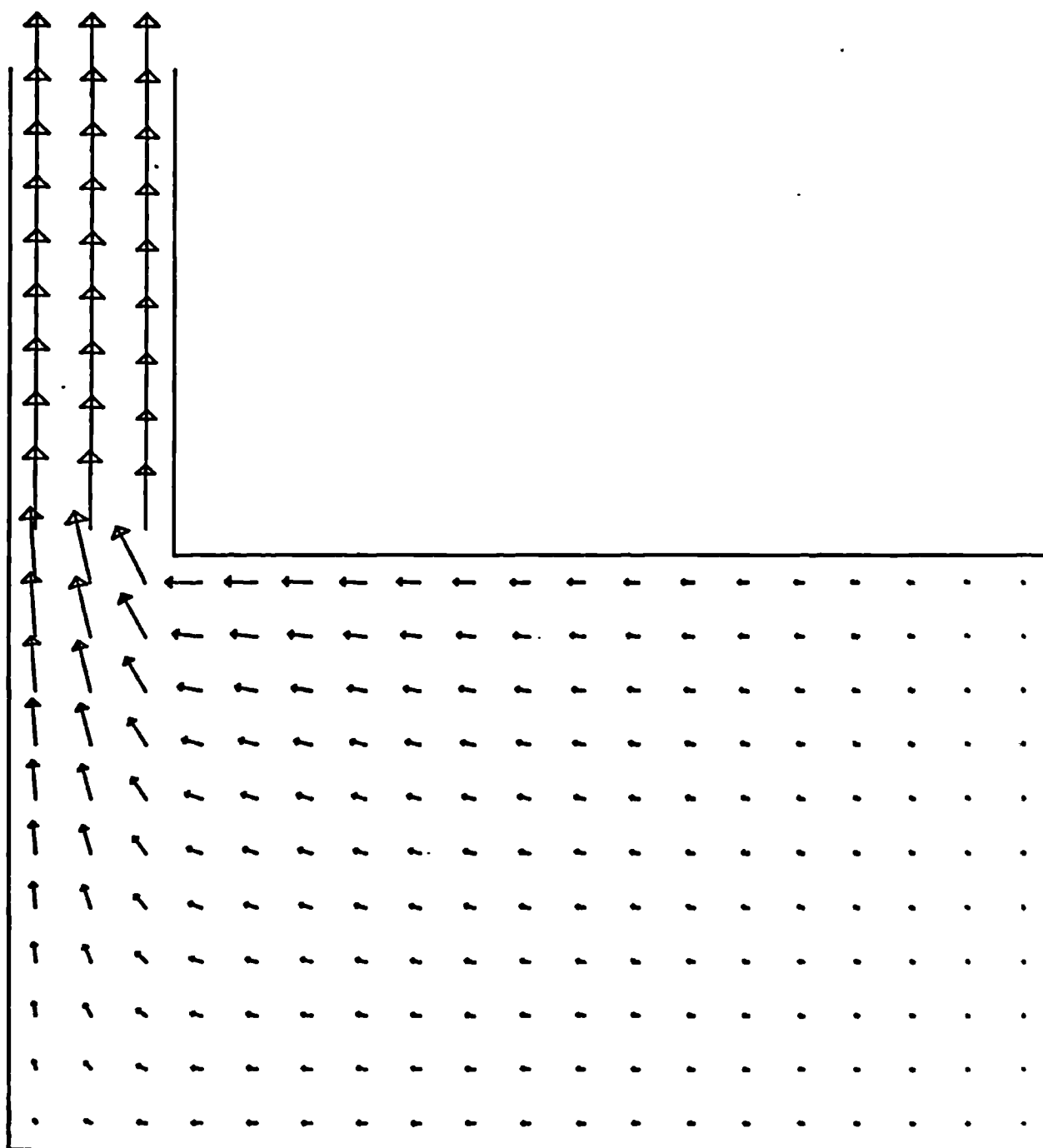
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 29b Computed Velocity Field for the Flood Tide with $L/B = 3.8$

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 158



LENGTH SCALE — 5.3 CM

MEAN DEPTH 13.74 CM

WAVE PERIOD 634 SECS

VELOCITY → 0.6 CM/SEC

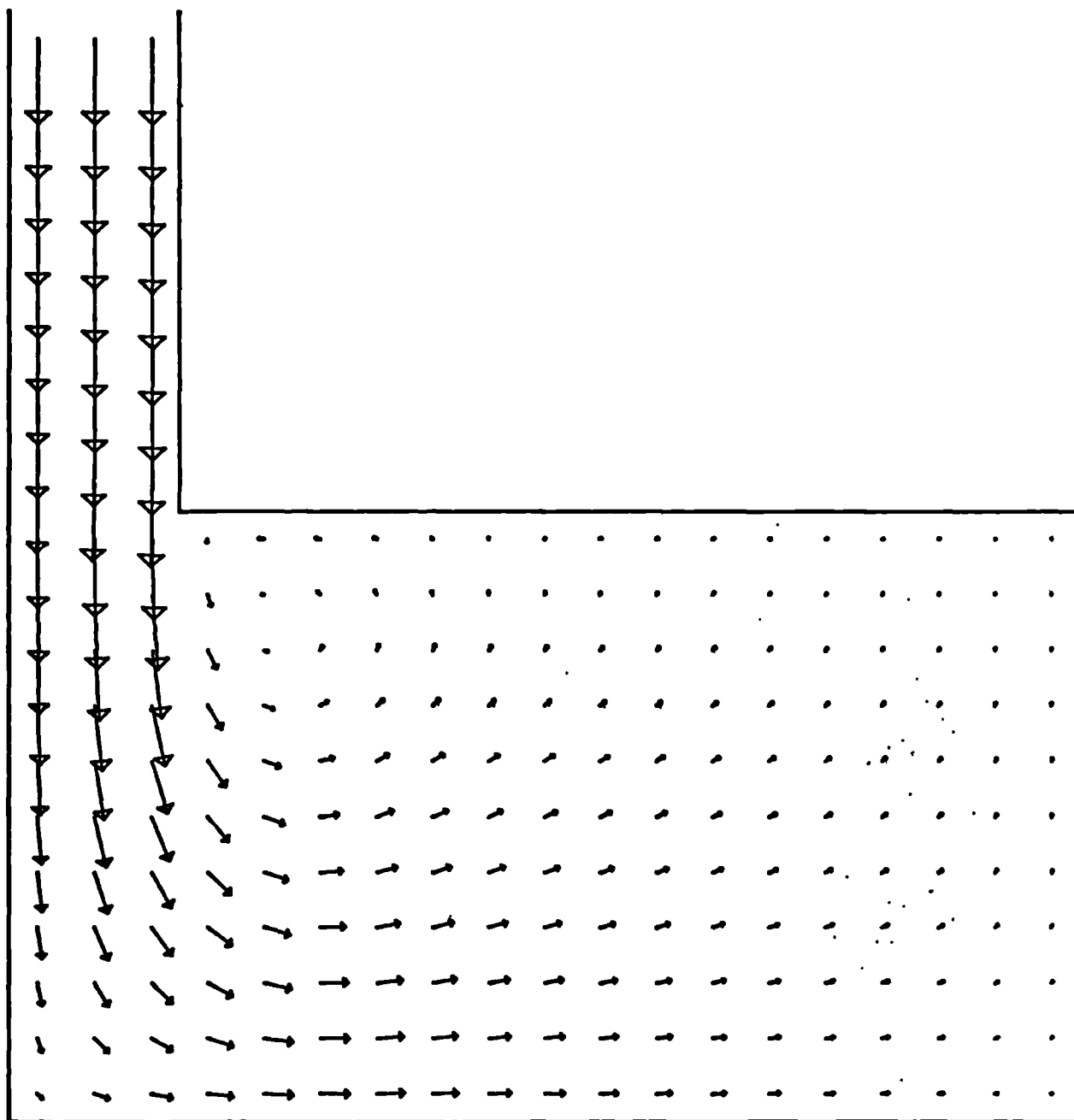
CHEZY VALUE 405 SI UNITS

WAVE AMPLITUDE 3.08 CM

FIGURE 30a Computed Velocity Field with a Channel Entrance

TIDAL CIRCULATION IN A HARBOUR

TIME IN SECS 475



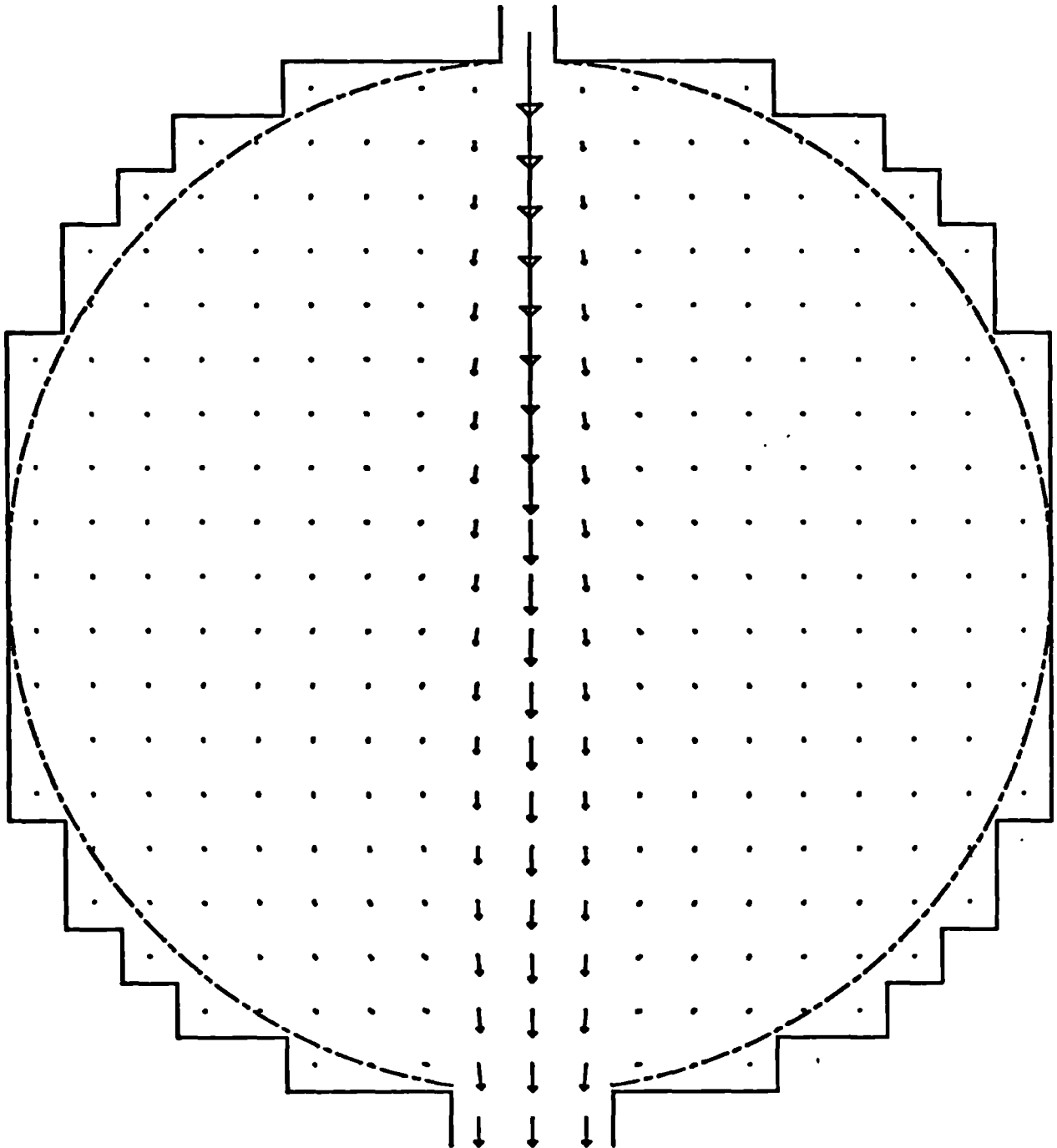
LENGTH SCALE — 5.3 CM
 MEAN DEPTH 13.74 CM
 WAVE PERIOD 634 SECS

VELOCITY —→ 0.6 CM/SEC
 CHEZY VALUE 405 SI UNITS
 WAVE AMPLITUDE 3.08 CM

FIGURE 30b Computed Velocity Field with a Channel Entrance

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



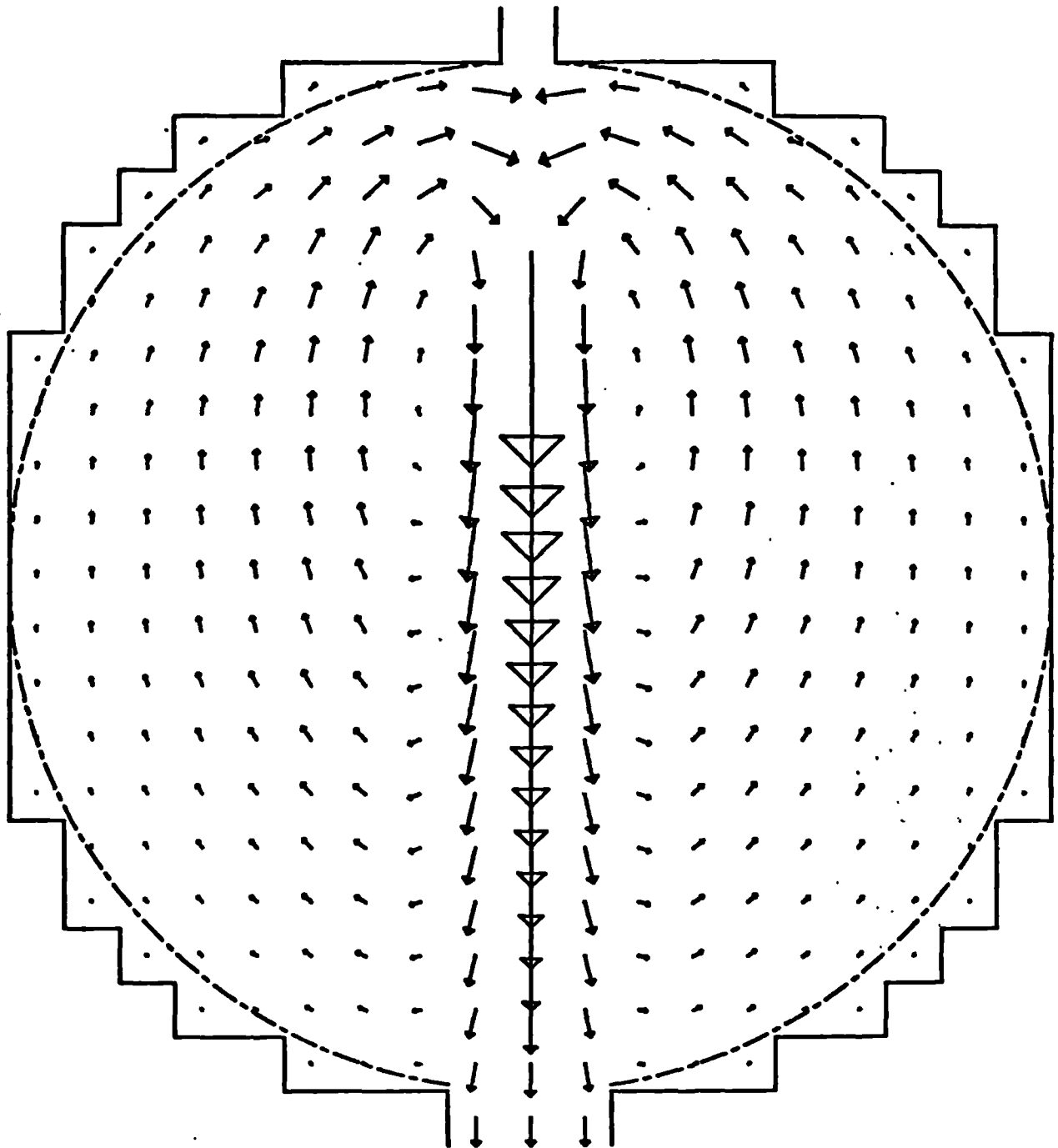
LENGTH SCALE — 80 M
INITIAL DEPTH 24.0 M
JET INFLOW 240 CUMEC
JET VELOCITY 0.12 M/SEC

VELOCITY → 0.08 M/SEC
CHEZY VALUE 56.6 METRIC
WEIR OUTFLOW 240 CUMEC
DEPTH OF JET 24.68 M

FIGURE 31 Computed Velocity Field for the Full Depth Jet Inlet
(No Coriolis)

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



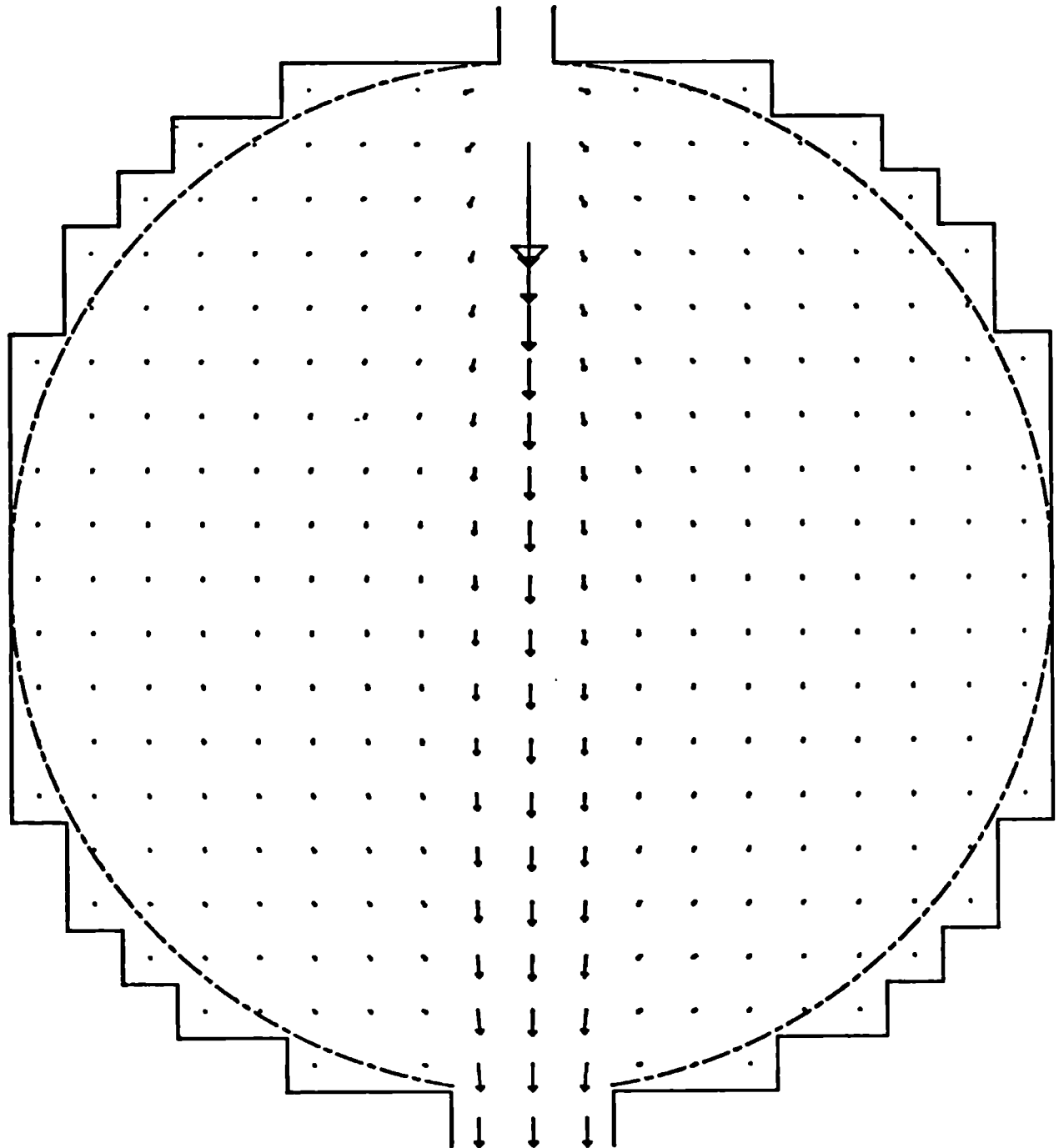
LENGTH SCALE — 80 M
INITIAL DEPTH 24.0 M
JET INFLOW 240 CUMEC
JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC
CHEZY VALUE 56.6 METRIC
WEIR OUTFLOW 240 CUMEC
DEPTH OF JET 5.0 M

FIGURE 32 Computed Velocity Field for the Shallow Jet Inlet
(No Coriolis)

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



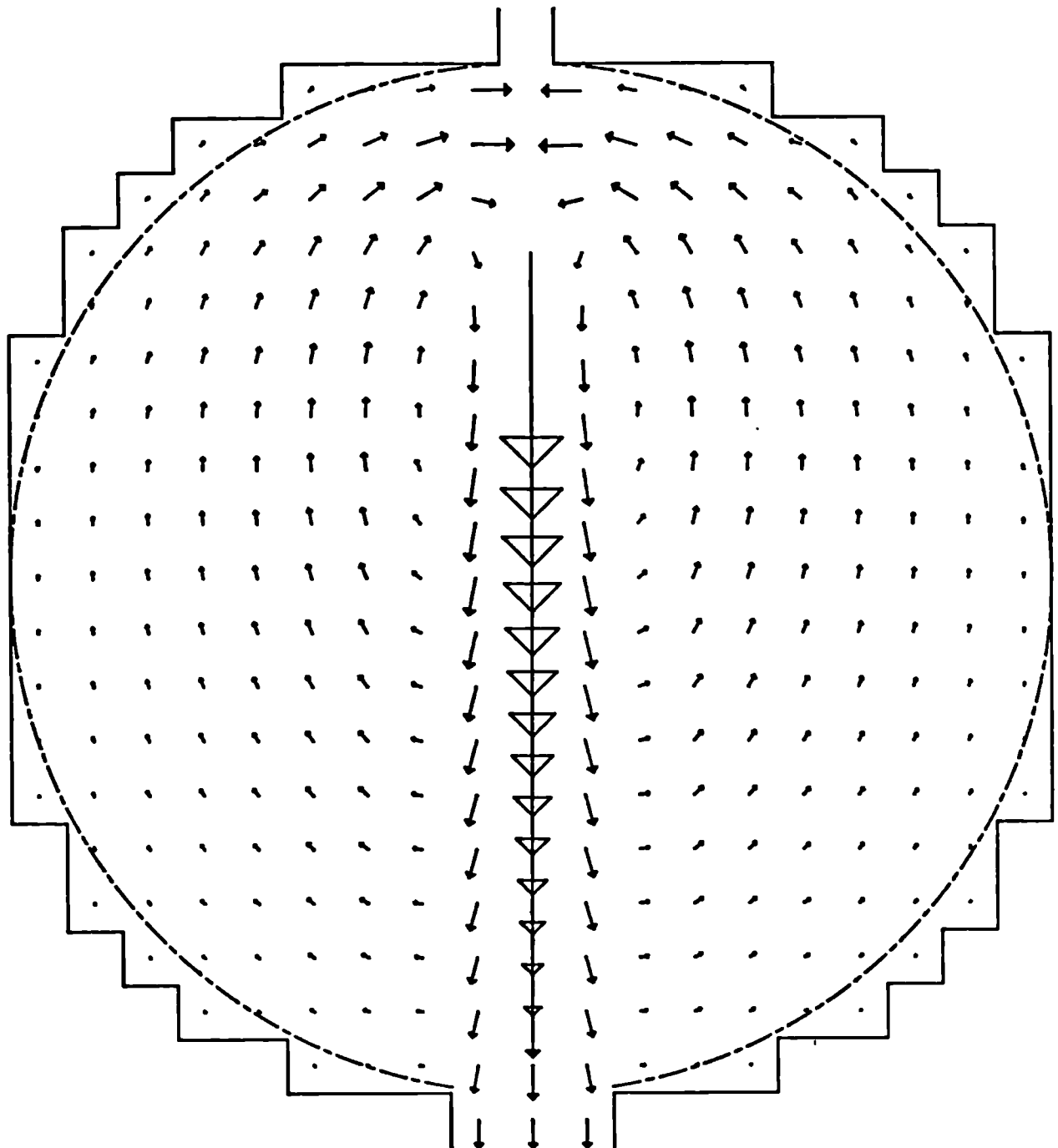
LENGTH SCALE — 80 M
INITIAL DEPTH 24.0 M
JET INFLOW 240 CUMEC
JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC
CHEZY VALUE 56.6 METRIC
WEIR OUTFLOW 240 CUMEC
DEPTH OF JET 5.0 M

FIGURE 33 Computed Velocity Field as in Figure 32 Excluding the Convective Accelerations

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 24.0 M

JET INFLOW 240 CUMEC

JET VELOCITY 0.60 M/SEC

VELOCITY —→ 0.08 M/SEC

CHEZY VALUE 56.6 METRIC

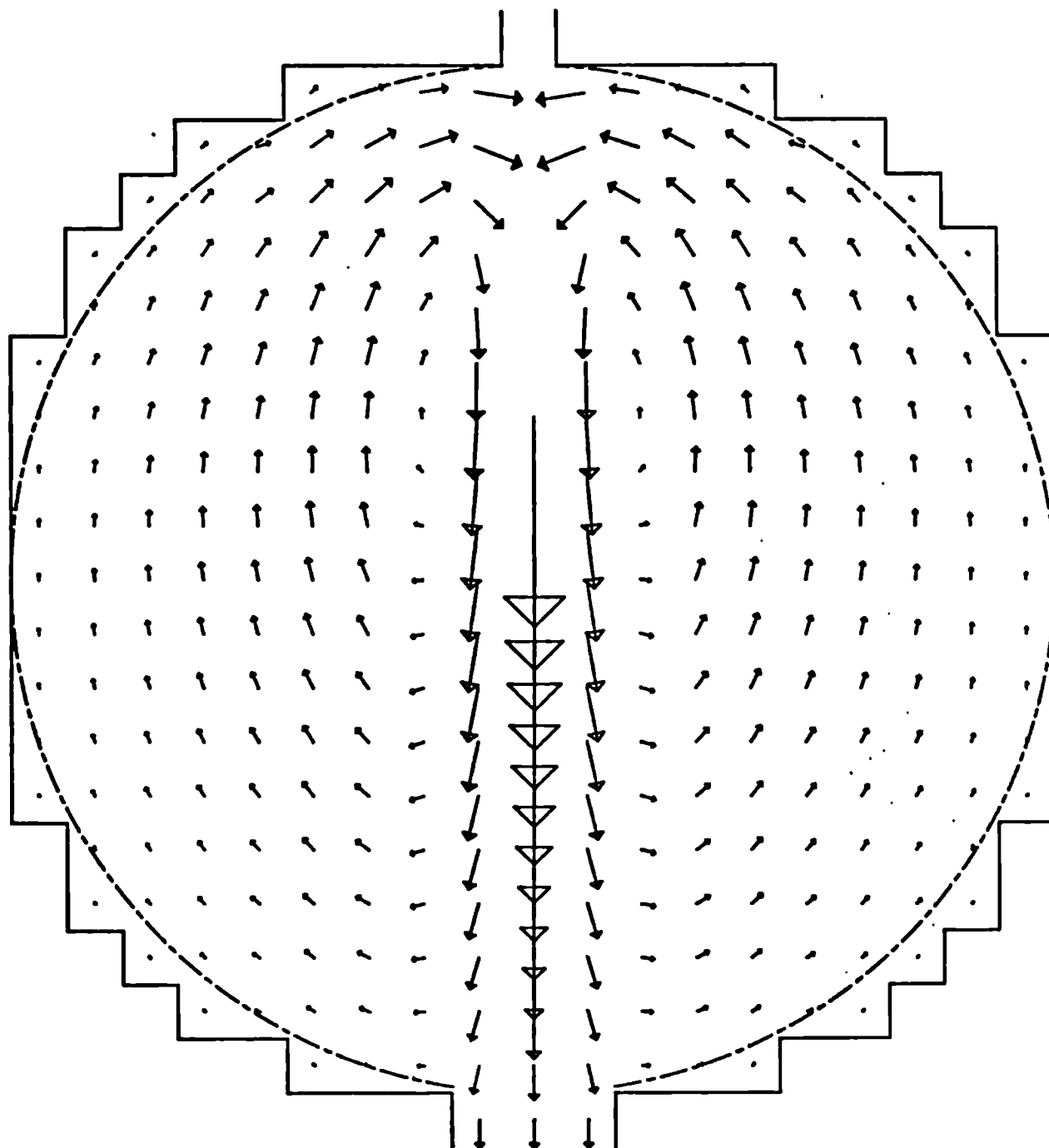
WEIR OUTFLOW 240 CUMEC

DEPTH OF JET 5.0 M

FIGURE 34 Computed Velocity Field as in Figure 32 Excluding the Effective Stresses

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



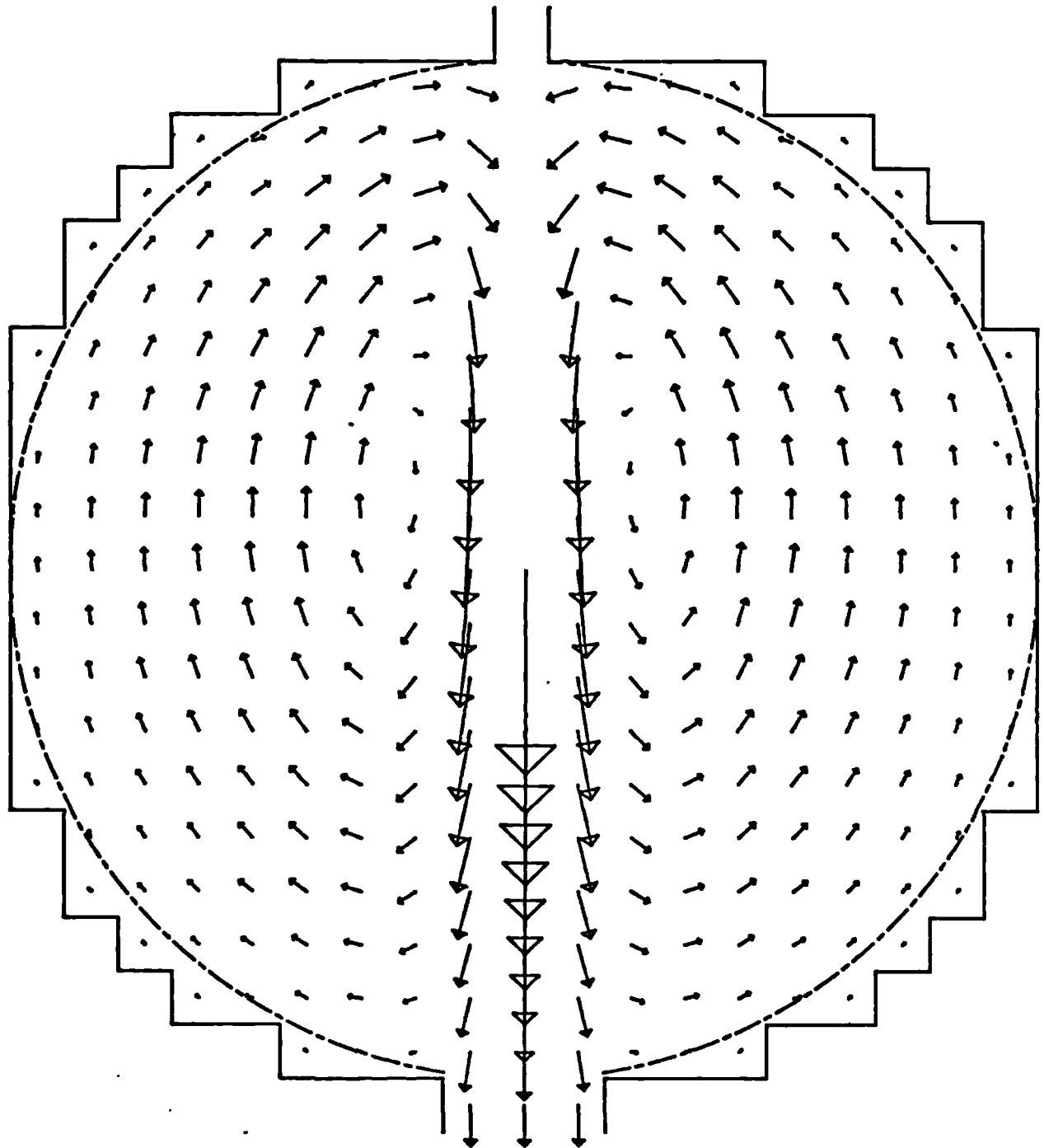
LENGTH SCALE — 80 M
 INITIAL DEPTH 24.0 M
 JET INFLOW 240 CUMEC
 JET VELOCITY 0.60 M/SEC

VELOCITY —→ 0.08 M/SEC
 CHEZY VALUE 56.6 METRIC
 WEIR OUTFLOW 240 CUMEC
 DEPTH OF JET 5.0 M

FIGURE 35 Computed Velocity Field as in Figure 32 Excluding the Bottom Roughness Stress

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



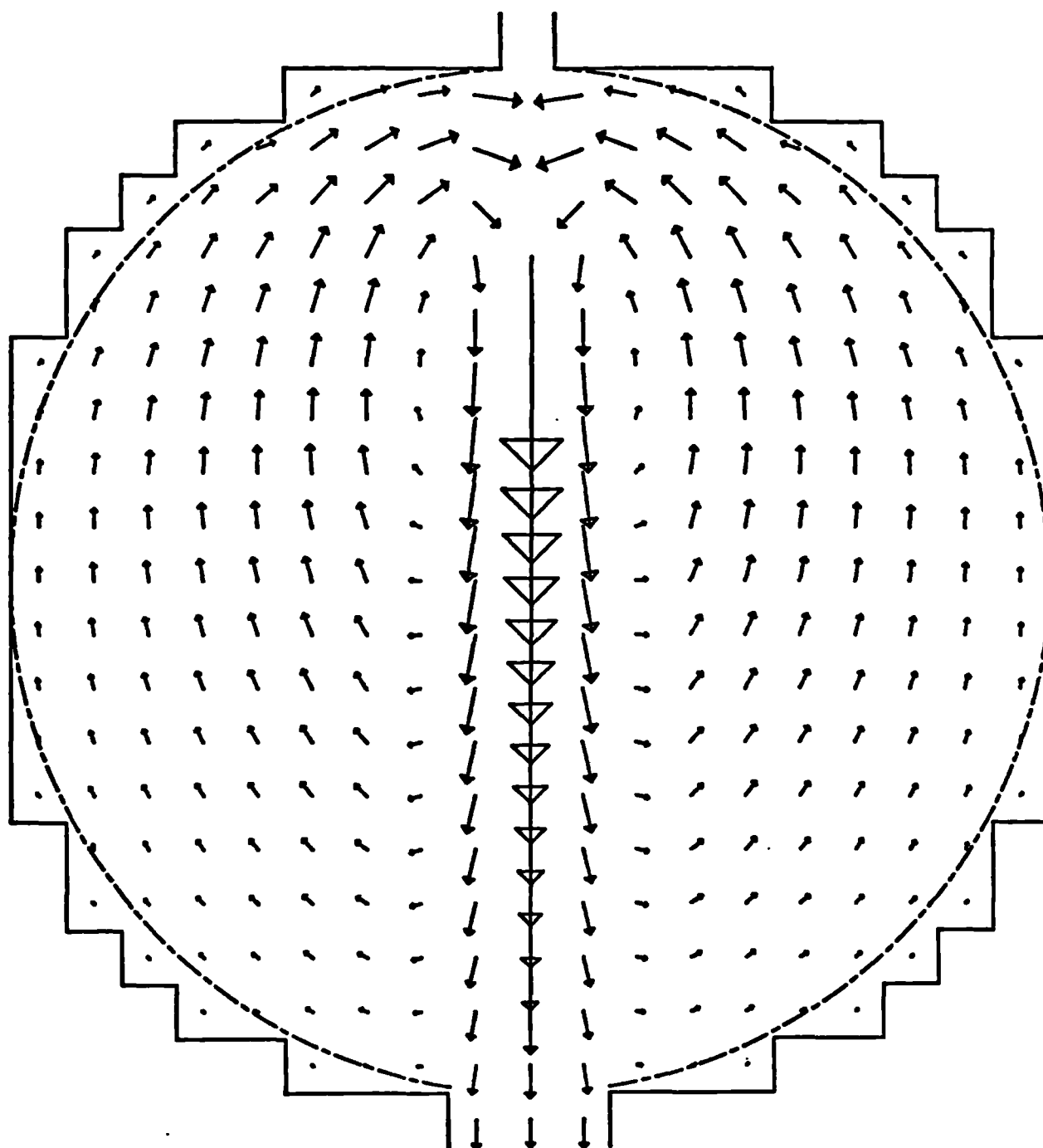
LENGTH SCALE — 80 M
 INITIAL DEPTH 16.0 M
 JET INFLOW 240 CUMEC
 JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC
 CHEZY VALUE 52.9 METRIC
 WEIR OUTFLOW 240 CUMEC
 DEPTH OF JET 5.0 M

FIGURE 36 Computed Velocity Field as in Figure 32 with a Reduced Depth

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 16.0 M

JET INFLOW 240 CUMEC

JET VELOCITY 0.60 M/SEC

VELOCITY —→ 0.08 M/SEC

CHEZY VALUE 52.9 METRIC

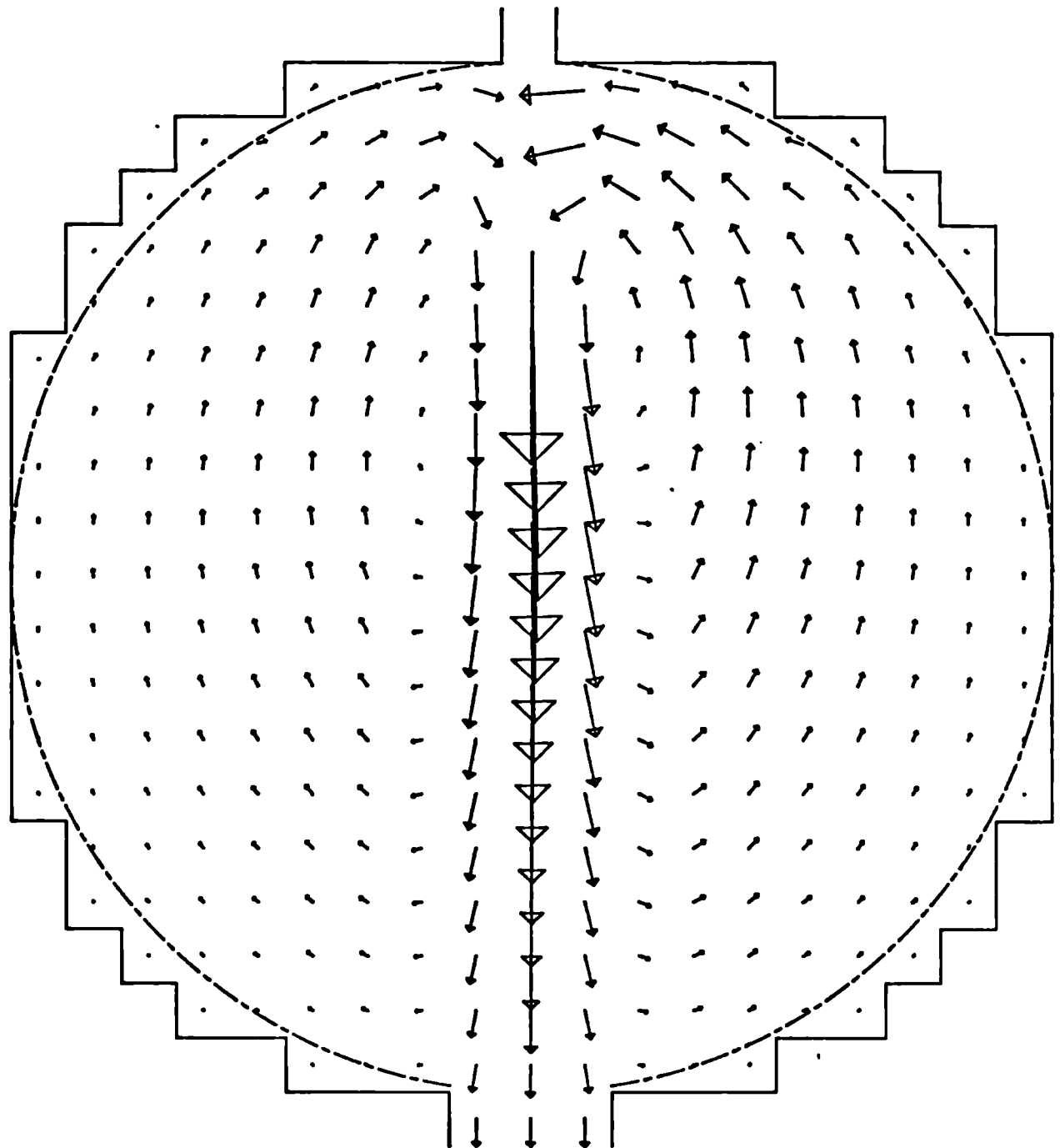
WEIR OUTFLOW 240 CUMEC

DEPTH OF JET 5.0 M

FIGURE 37 Computed Velocity Field as in Figure 32 with a V-Shaped Reservoir Bed

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M
INITIAL DEPTH 24.0 M
JET INFLOW 240 CUMEC
JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC
CHEZY VALUE 56.6 METRIC
WEIR OUTFLOW 240 CUMEC
DEPTH OF JET 5.0 M

FIGURE 38 Computed Velocity Field for the Shallow Jet Inlet Including the Coriolis Force

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00

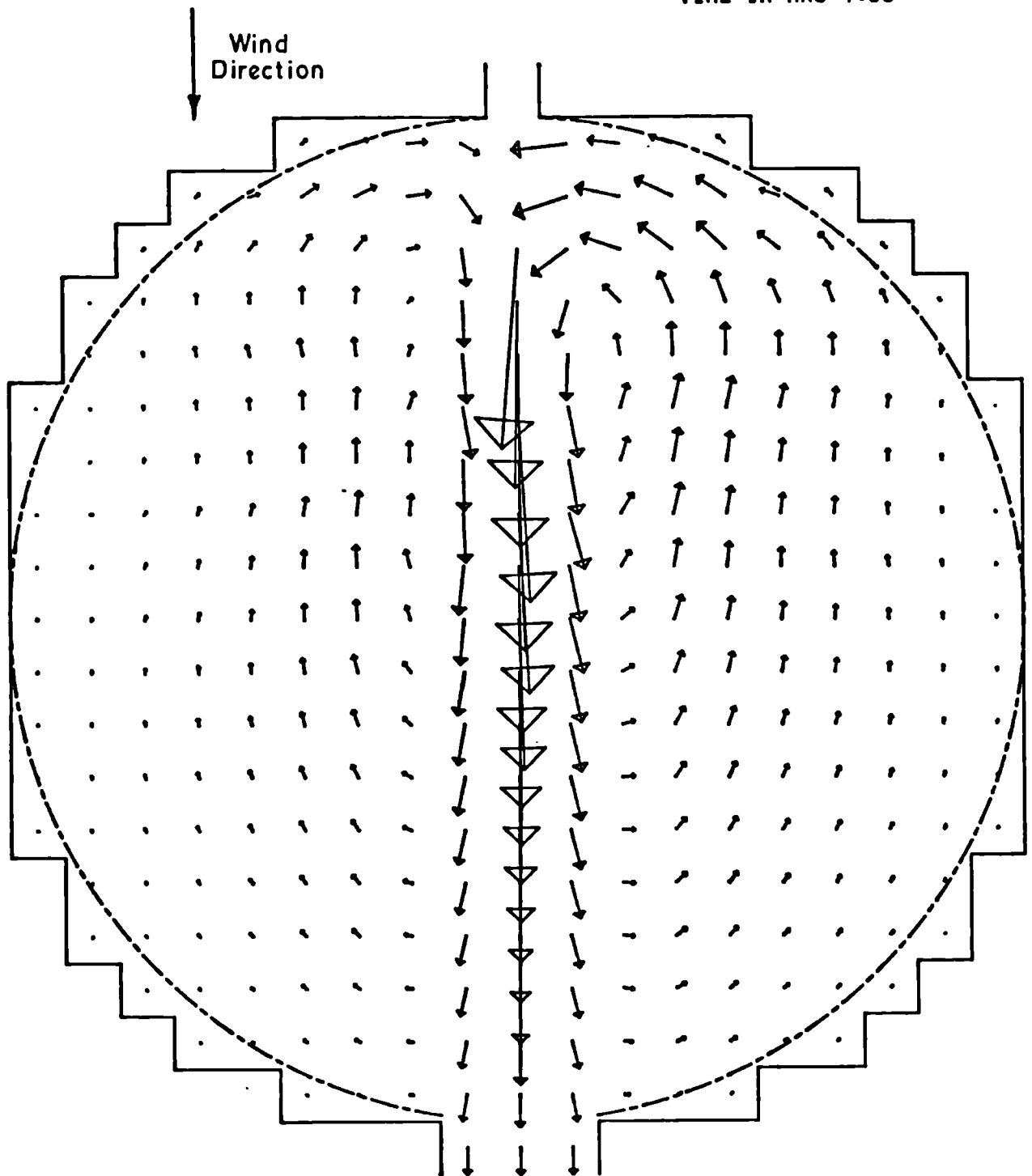
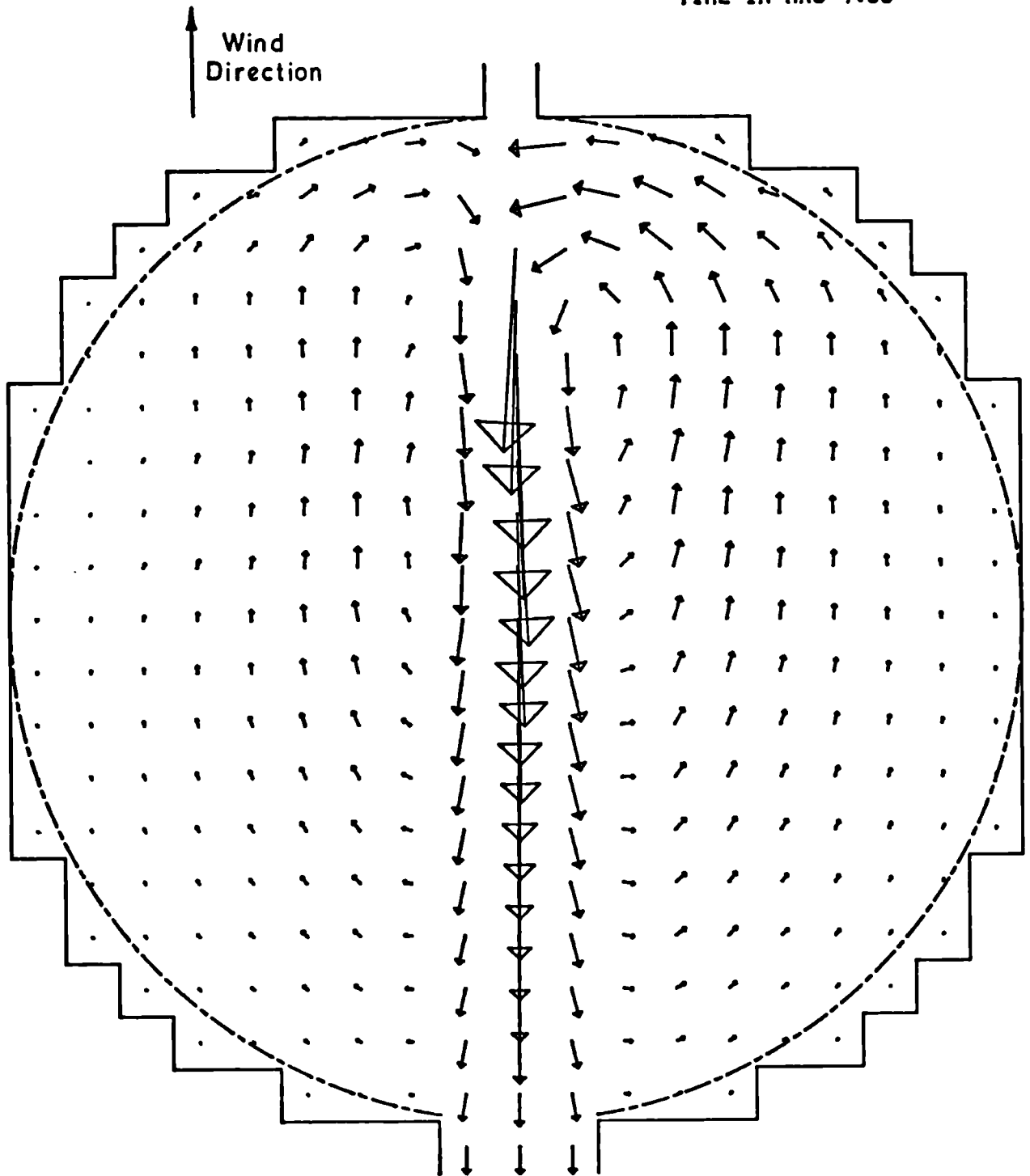


FIGURE 39 Computed Velocity Field as in Figure 38 with a 7.5 m/sec Northerly Wind

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



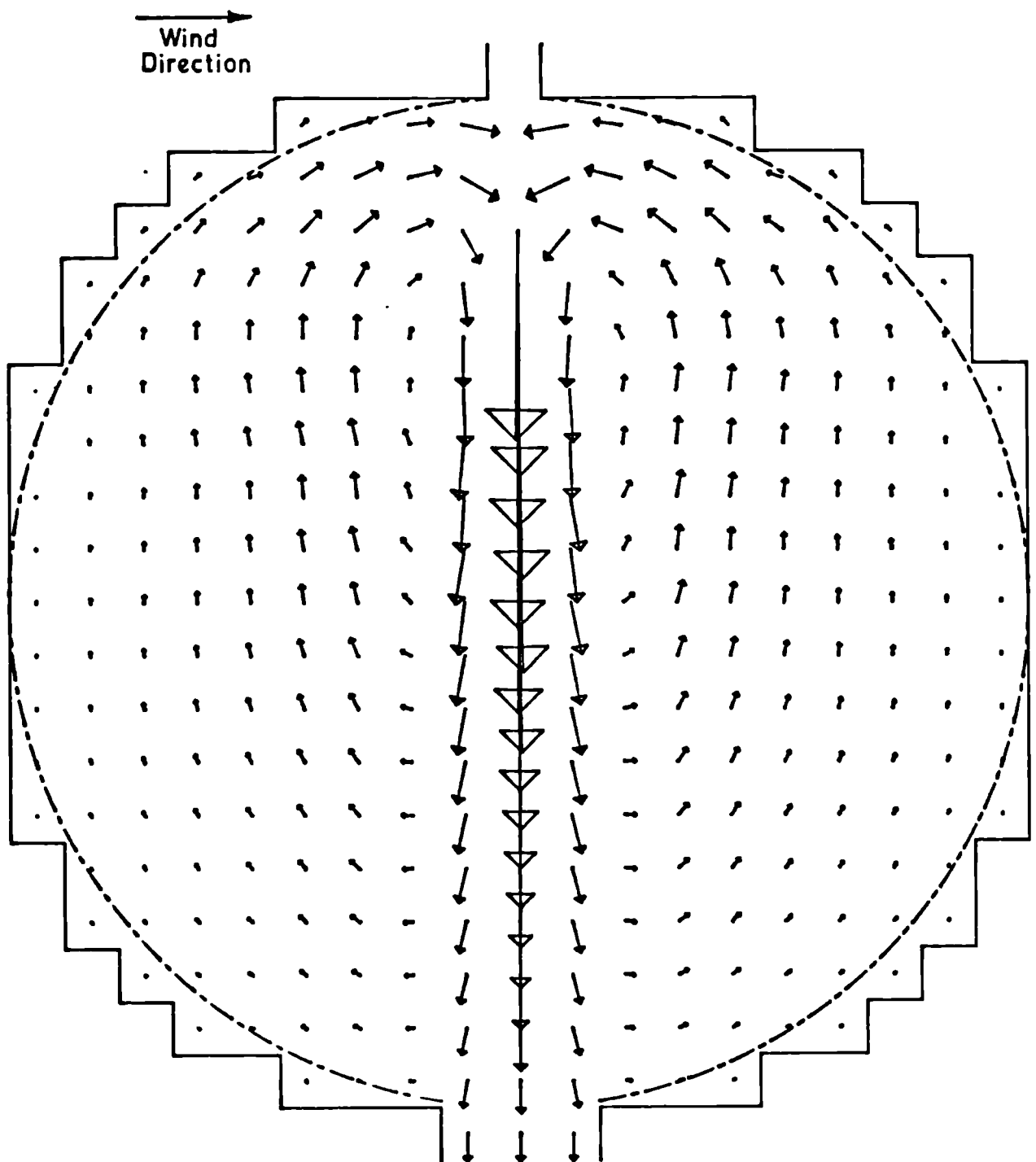
LENGTH SCALE — 80 M
 INITIAL DEPTH 24.0 M
 JET INFLOW 240 CUMEC
 JET VELOCITY 0.60 M/SEC

VELOCITY —→ 0.08 M/SEC
 CHEZY VALUE 56.6 METRIC
 WEIR OUTFLOW 240 CUMEC
 DEPTH OF JET 5.0 M

FIGURE 40 Computed Velocity Field as in Figure 38 with a 7.5 m/sec Southerly Wind

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



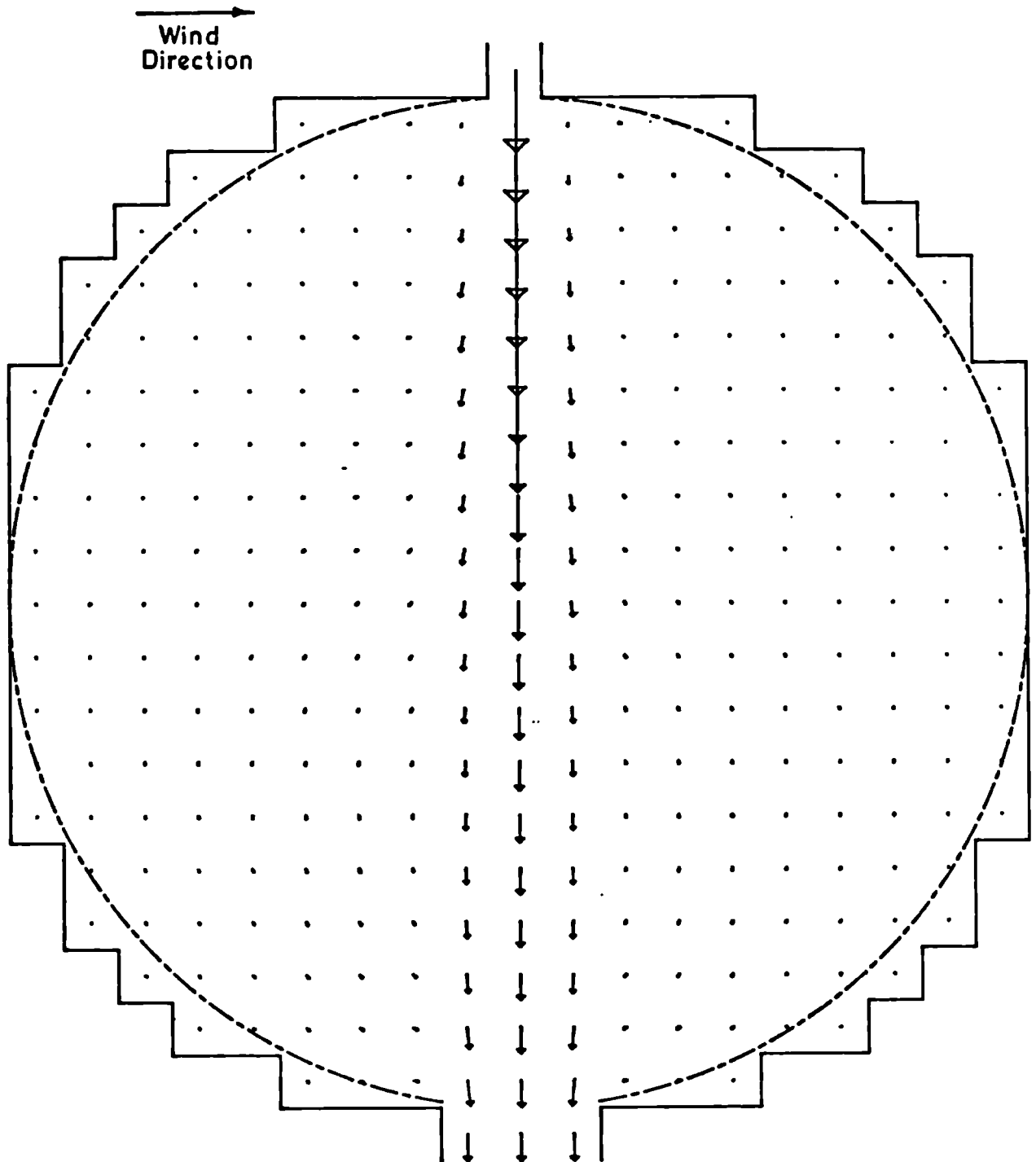
LENGTH SCALE — 80 M
 INITIAL DEPTH 24.0 M
 JET INFLOW 240 CUMEC
 JET VELOCITY 0.60 M/SEC

VELOCITY — 0.08 M/SEC
 CHEZY VALUE 56.6 METRIC
 WEIR OUTFLOW 240 CUMEC
 DEPTH OF JET 5.0 M

FIGURE 41 Computed Velocity Field as in Figure 38 with a 7.5 m/sec Westerly Wind

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 24.0 M

JET INFLOW 240 CUMEC

JET VELOCITY 0.12 M/SEC

VELOCITY —→ 0.08 M/SEC

CHEZY VALUE 56.6 METRIC

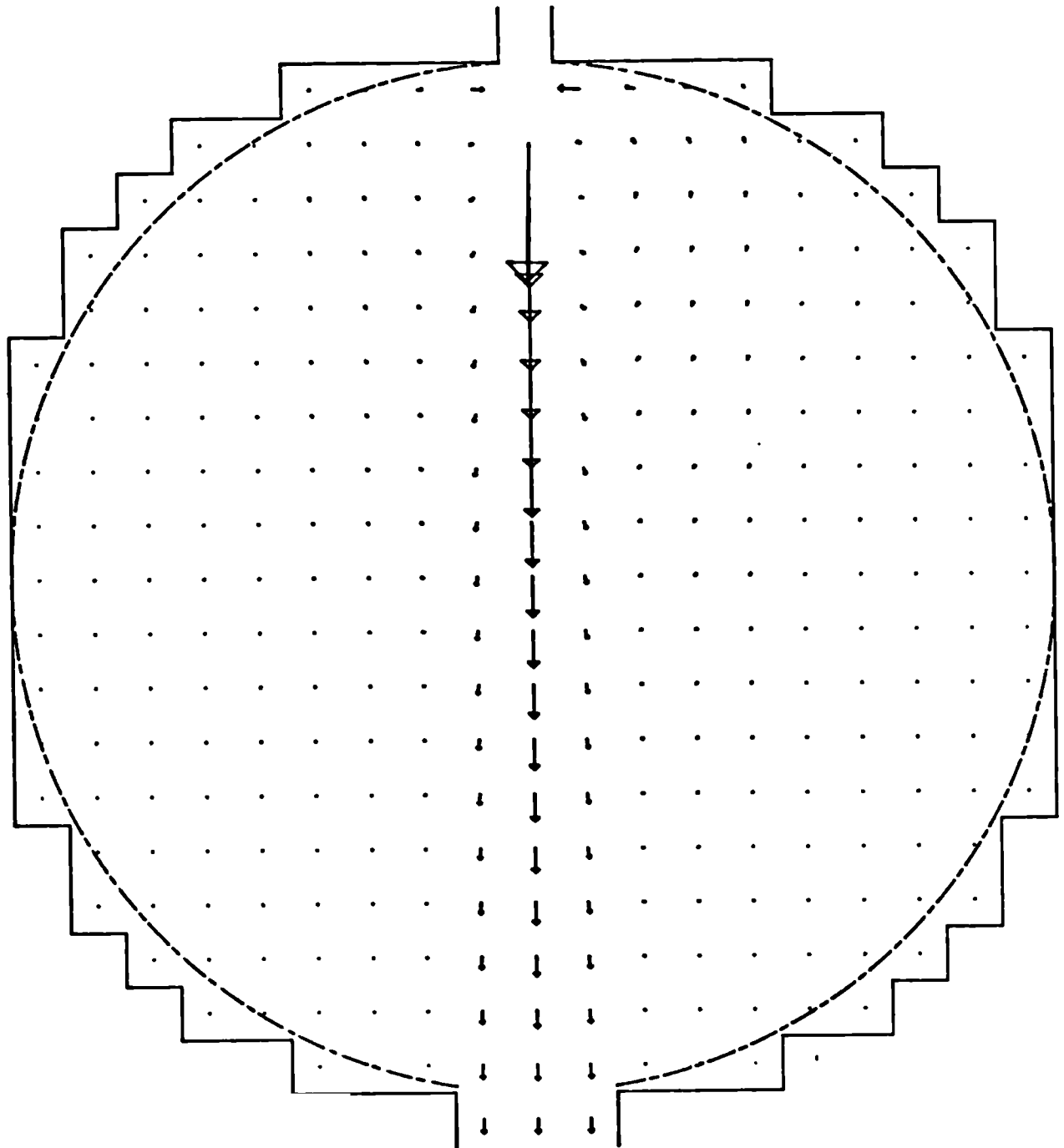
WEIR OUTFLOW 240 CUMEC

DEPTH OF JET 24.68 M

FIGURE 42 Computed Velocity Field for the Full Depth Jet Inlet with a 7.5 m/sec Westerly Wind

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 24.0 M

JET INFLOW 180 CUMEC

JET VELOCITY 0.48 M/SEC

VELOCITY → 0.08 M/SEC

CHEZY VALUE 56.6 METRIC

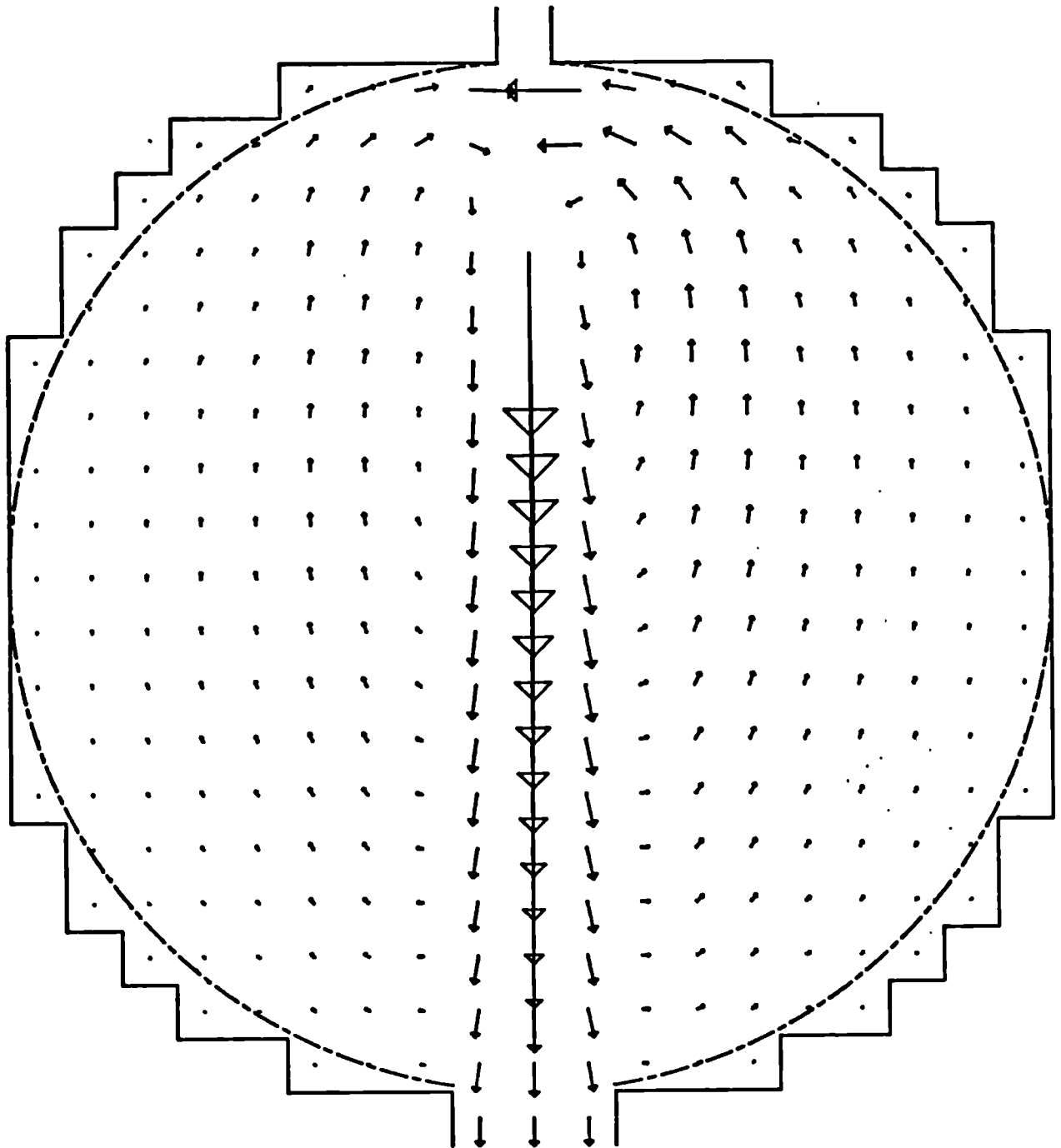
WEIR OUTFLOW 118 CUMEC

DEPTH OF JET 4.7 M

FIGURE 43a Computed Velocity Field for the Sinusoidal Weir Outflow at the Mean Discharge Level

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 8.00



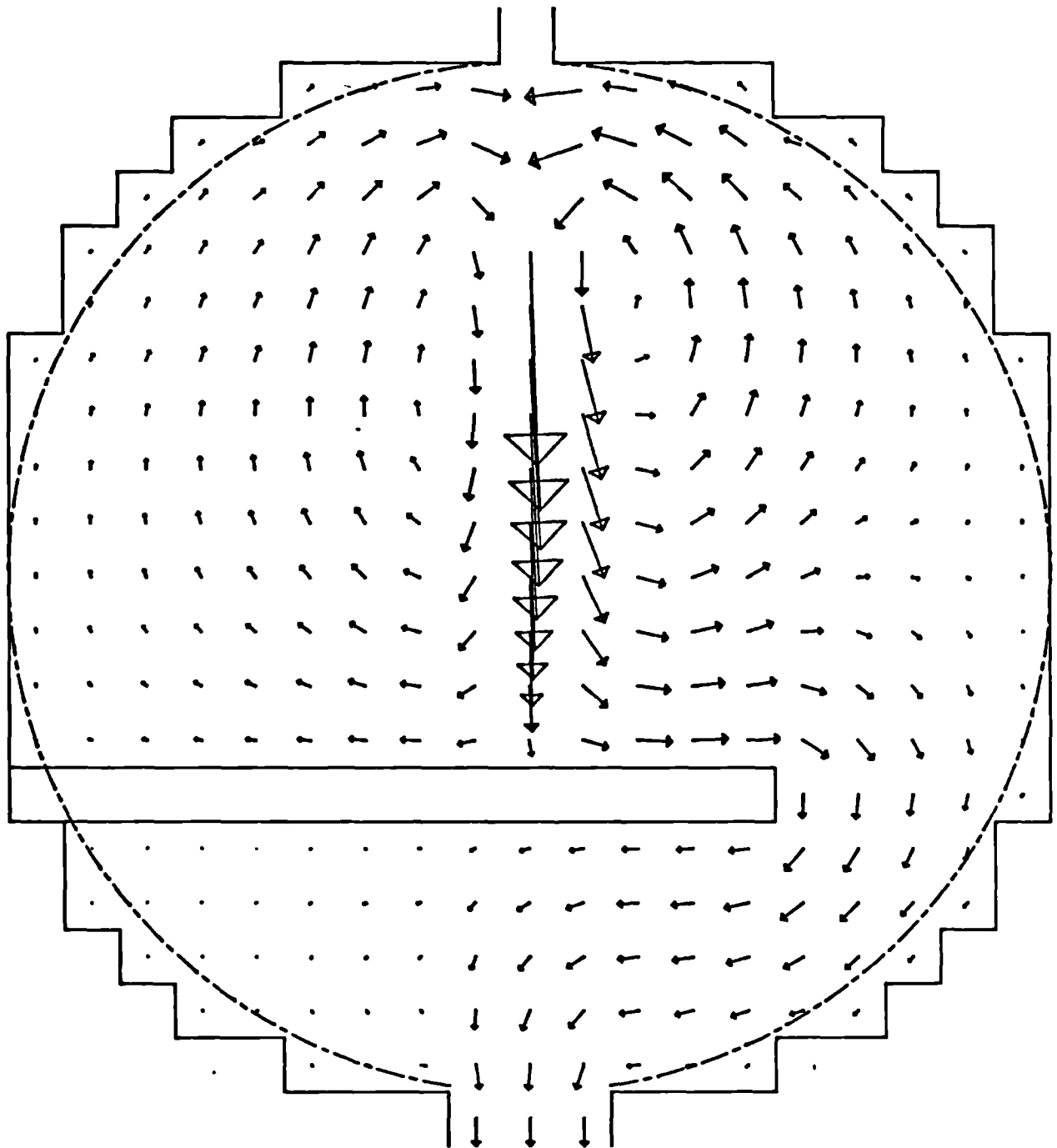
LENGTH SCALE — 80 M
INITIAL DEPTH 24.0 M
JET INFLOW 242 CUMEC
JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC
CHEZY VALUE 56.6 METRIC
WEIR OUTFLOW 240 CUMEC
DEPTH OF JET 5.0 M

FIGURE 43b Computed Velocity Field for the Sinusoidal Weir Outflow at the Peak Discharge Level

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



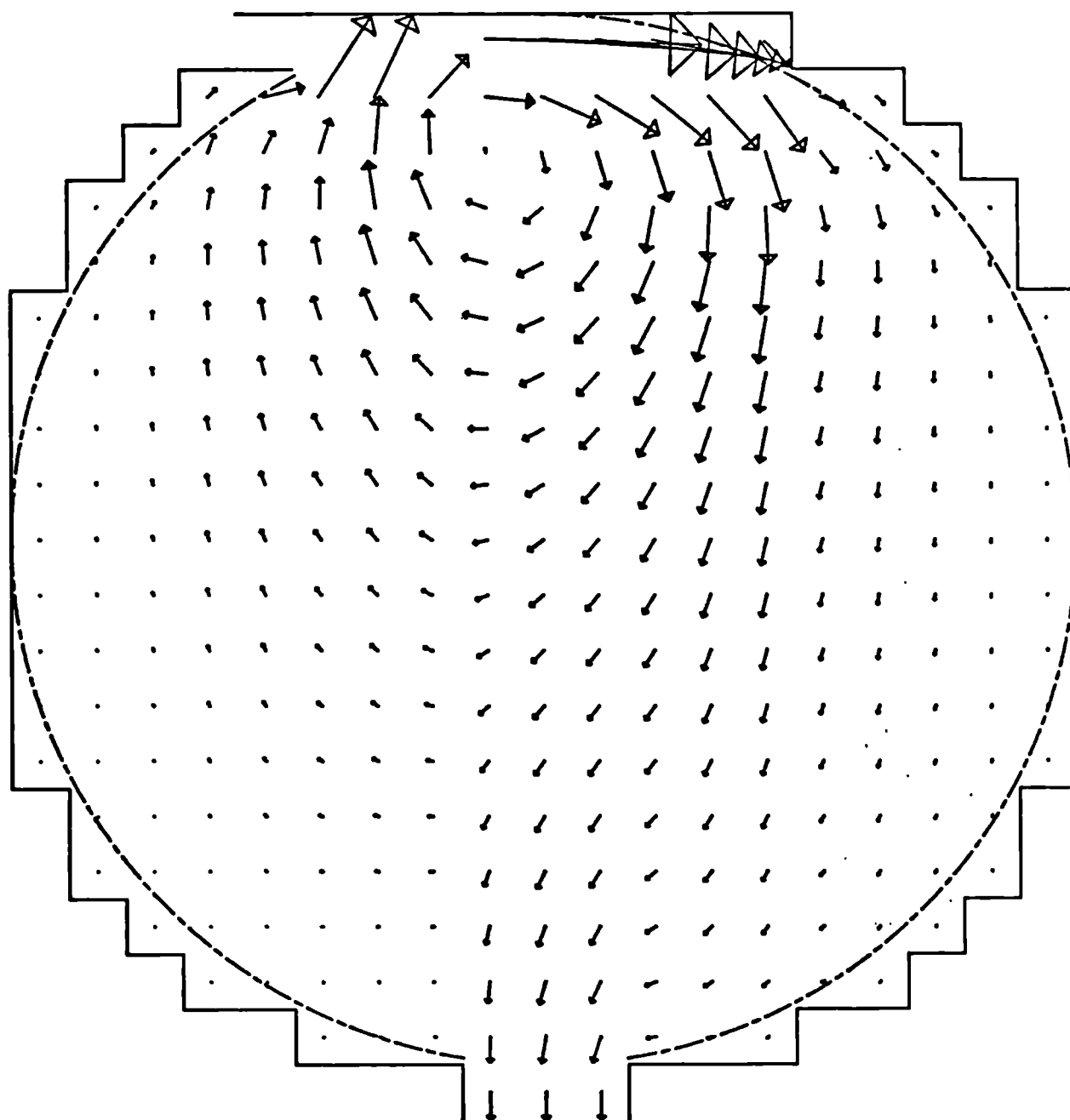
LENGTH SCALE — 80 M
 INITIAL DEPTH 24.0 M
 JET INFLOW 240 CUMEC
 JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC
 CHEZY VALUE 56.6 METRIC
 WEIR OUTFLOW 240 CUMEC
 DEPTH OF JET 5.0 M

FIGURE 44 Computed Velocity Field for the Shallow Jet Inlet with a Barrier

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 24.0 M

JET INFLOW 240 CUMEC

JET VELOCITY 0.60 M/SEC

VELOCITY → 0.08 M/SEC

CHEZY VALUE 56.6 METRIC

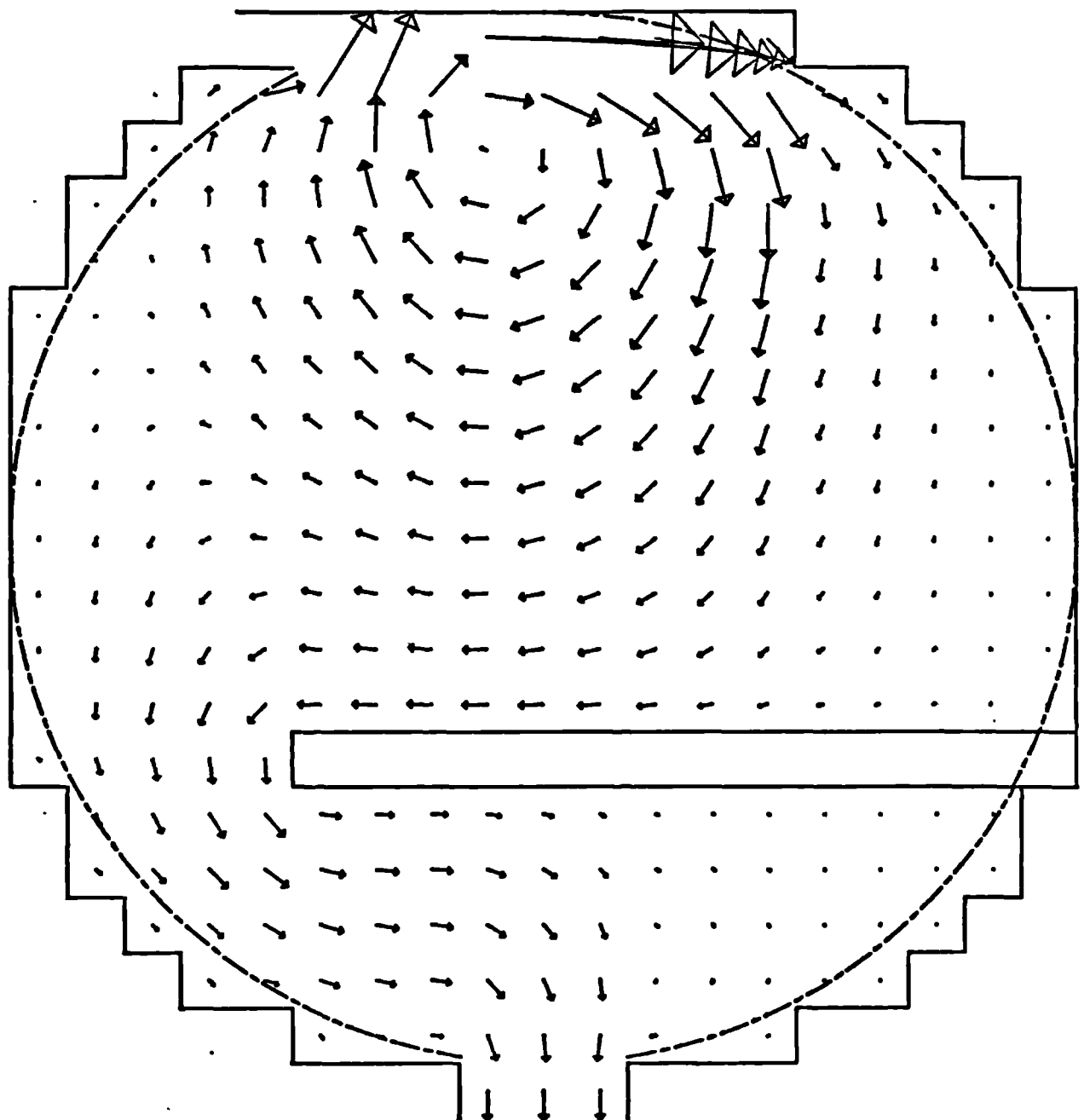
WEIR OUTFLOW 240 CUMEC

DEPTH OF JET 5.0 M

FIGURE 45 Computed Velocity Field for the Shallow Boundary Jet Inlet

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



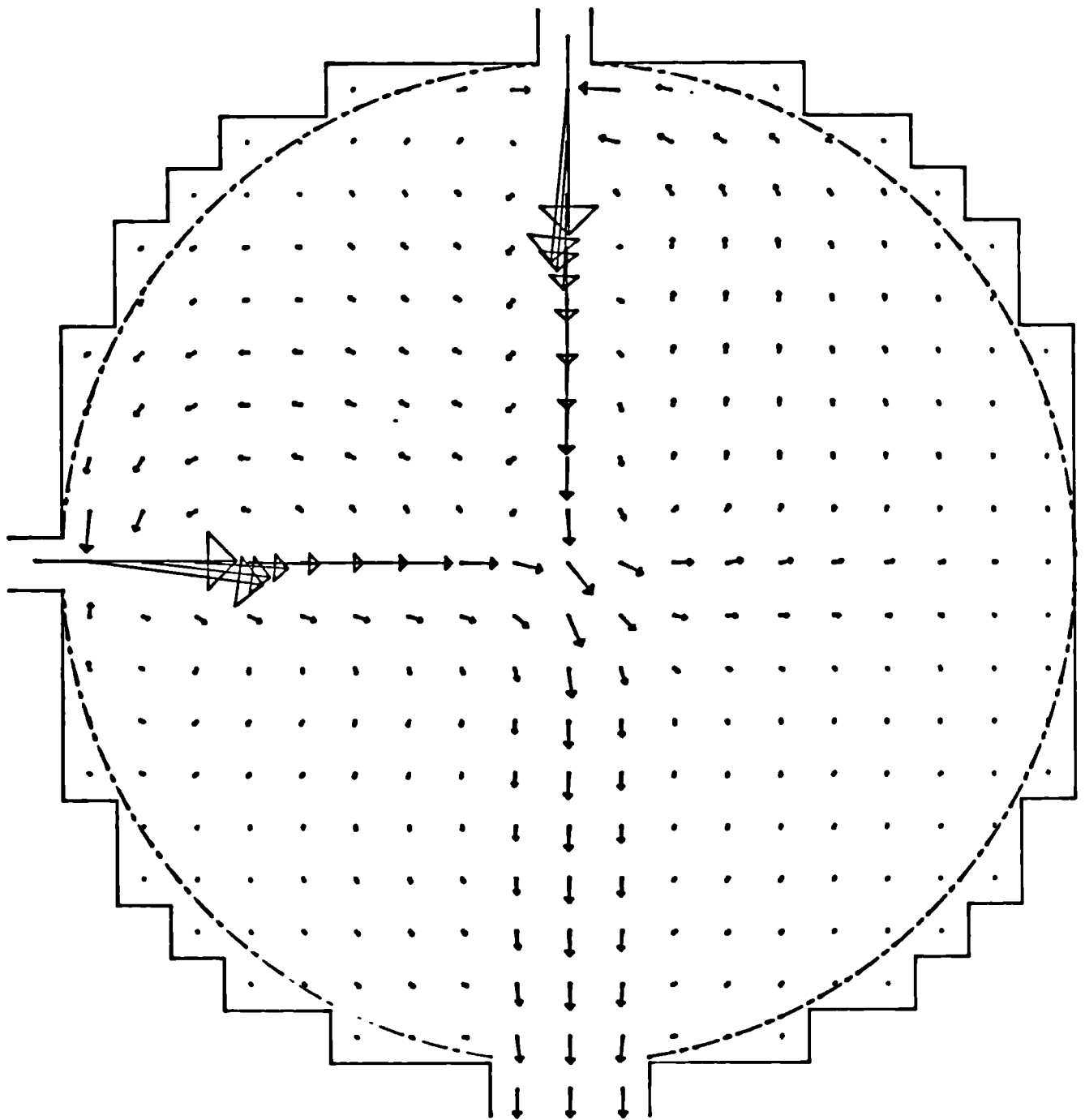
LENGTH SCALE — 80 M
 INITIAL DEPTH 24.0 M
 JET INFLOW 240 CUMEC
 JET VELOCITY 0.60 M/SEC

VELOCITY —→ 0.08 M/SEC
 CHEZY VALUE 56.6 METRIC
 WEIR OUTFLOW 240 CUMEC
 DEPTH OF JET 5.0 M

FIGURE 46 Computed Velocity Field as in Figure 45 with a Barrier

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 24.0 M

JET INFLOW 120 CUMEC

JET VELOCITY 0.90 M/SEC

VELOCITY —→ 0.08 M/SEC

CHEZY VALUE 56.6 METRIC

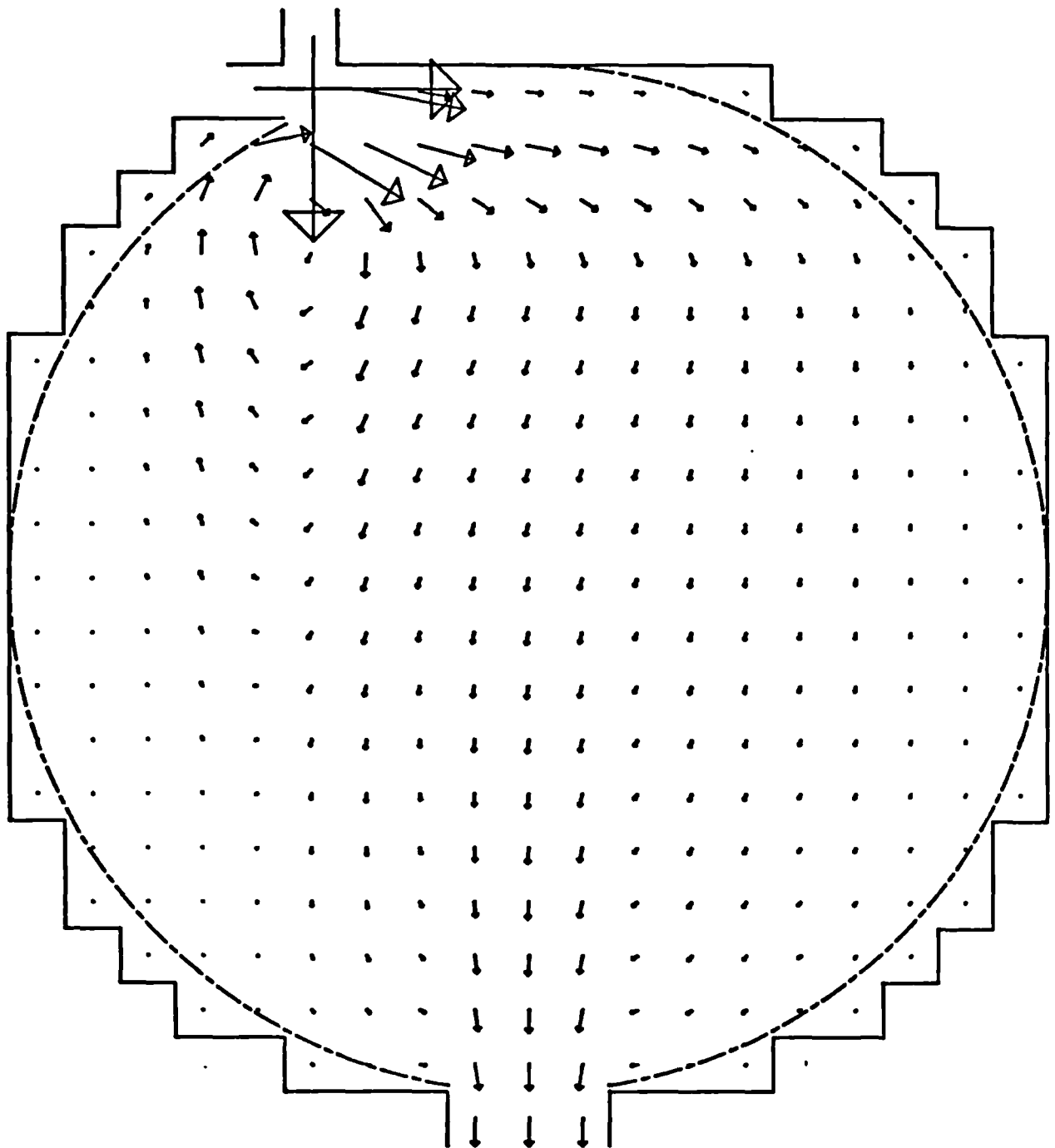
WEIR OUTFLOW 240 CUMEC

DEPTH OF JET 5.0 M

FIGURE 47 Computed Velocity Field for Two Shallow Jet Inlets at Right Angles

JET CIRCULATION IN A RESERVOIR

TIME IN HRS 4.00



LENGTH SCALE — 80 M

INITIAL DEPTH 24.0 M

JET INFLOW 120 CUMEC

JET VELOCITY 0.30 M/SEC

VELOCITY —→ 0.08 M/SEC

CHEZY VALUE 56.6 METRIC

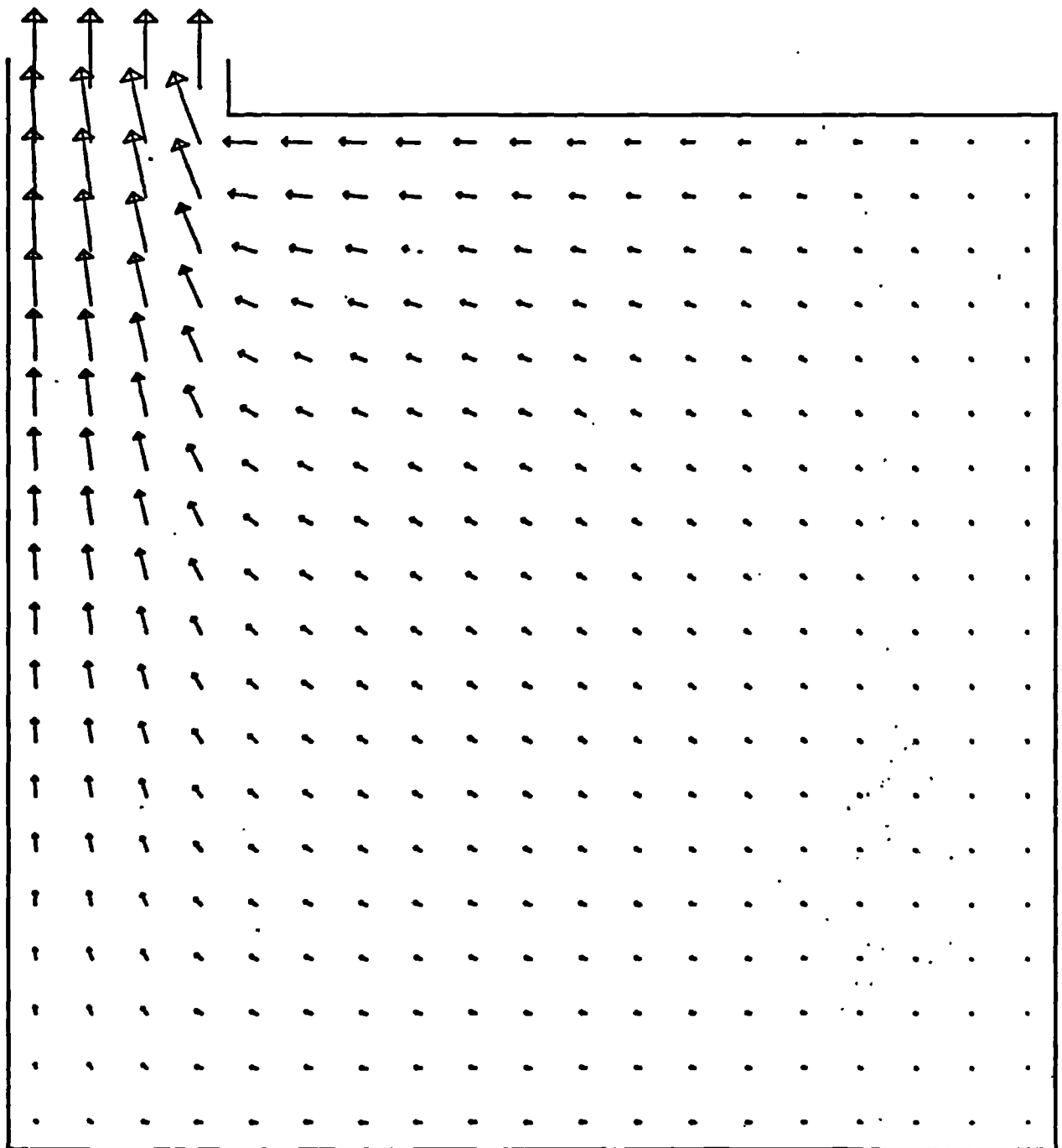
WEIR OUTFLOW 240 CUMEC

DEPTH OF JET 5.0 M

FIGURE 48 Computed Velocity Field for two Adjacent Shallow Jet Inlets at Right Angles

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



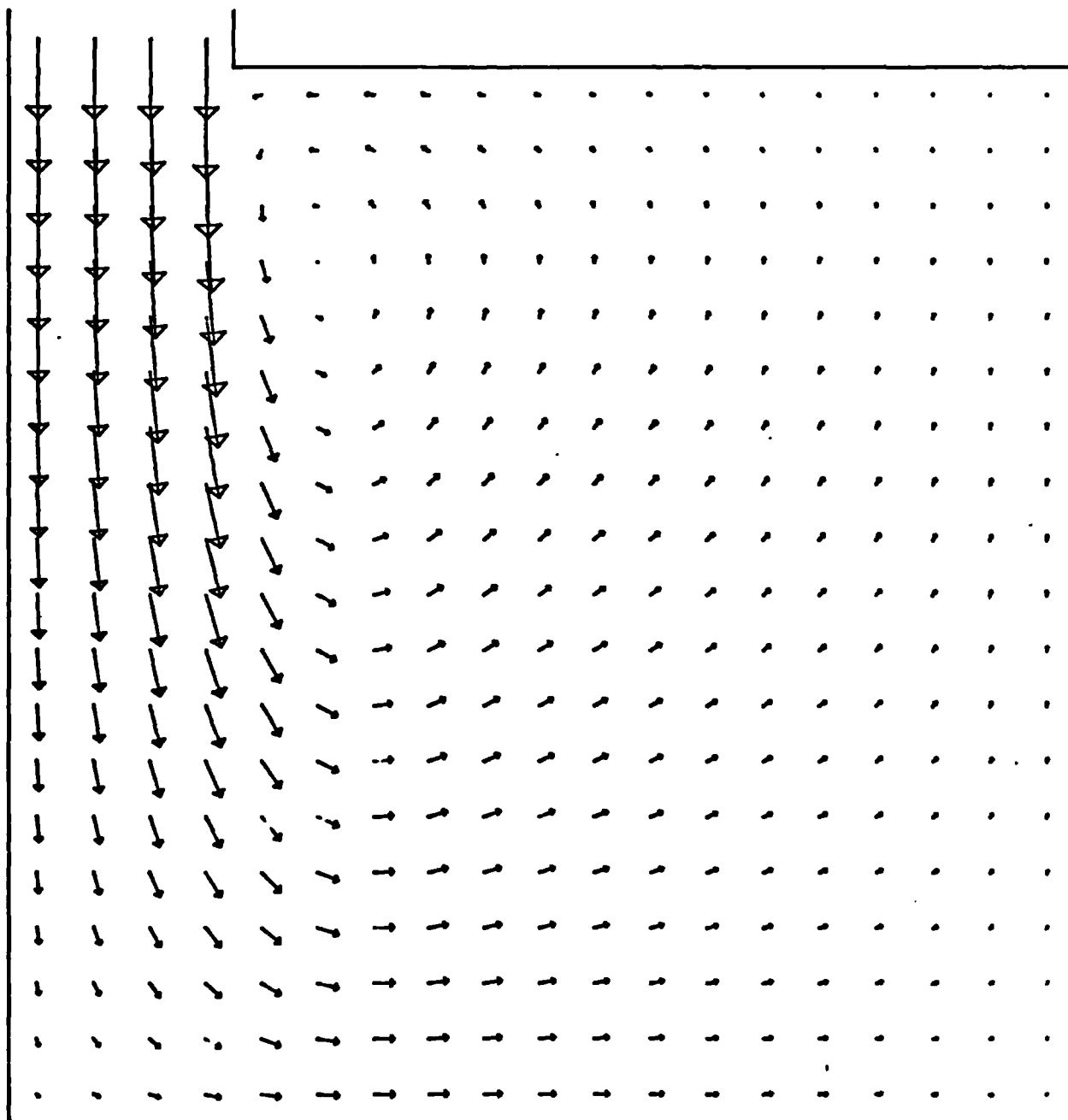
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 49a Computed Velocity Field for the Ebb Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



LENGTH SCALE — 20 M

MEAN DEPTH 6.87 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC

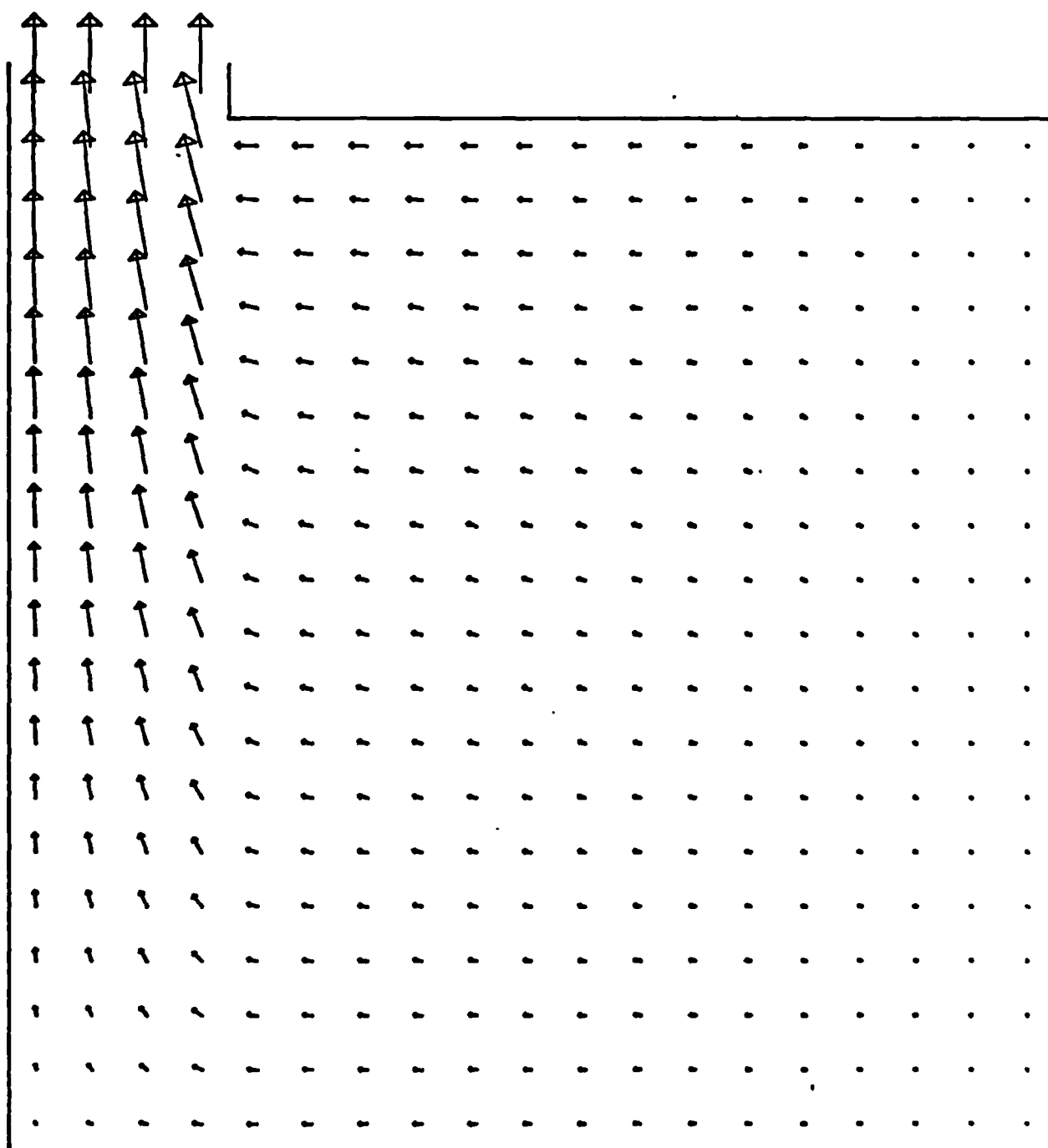
CHEZY VALUE 44.7 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 49b Computed Velocity Field for the Flood Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



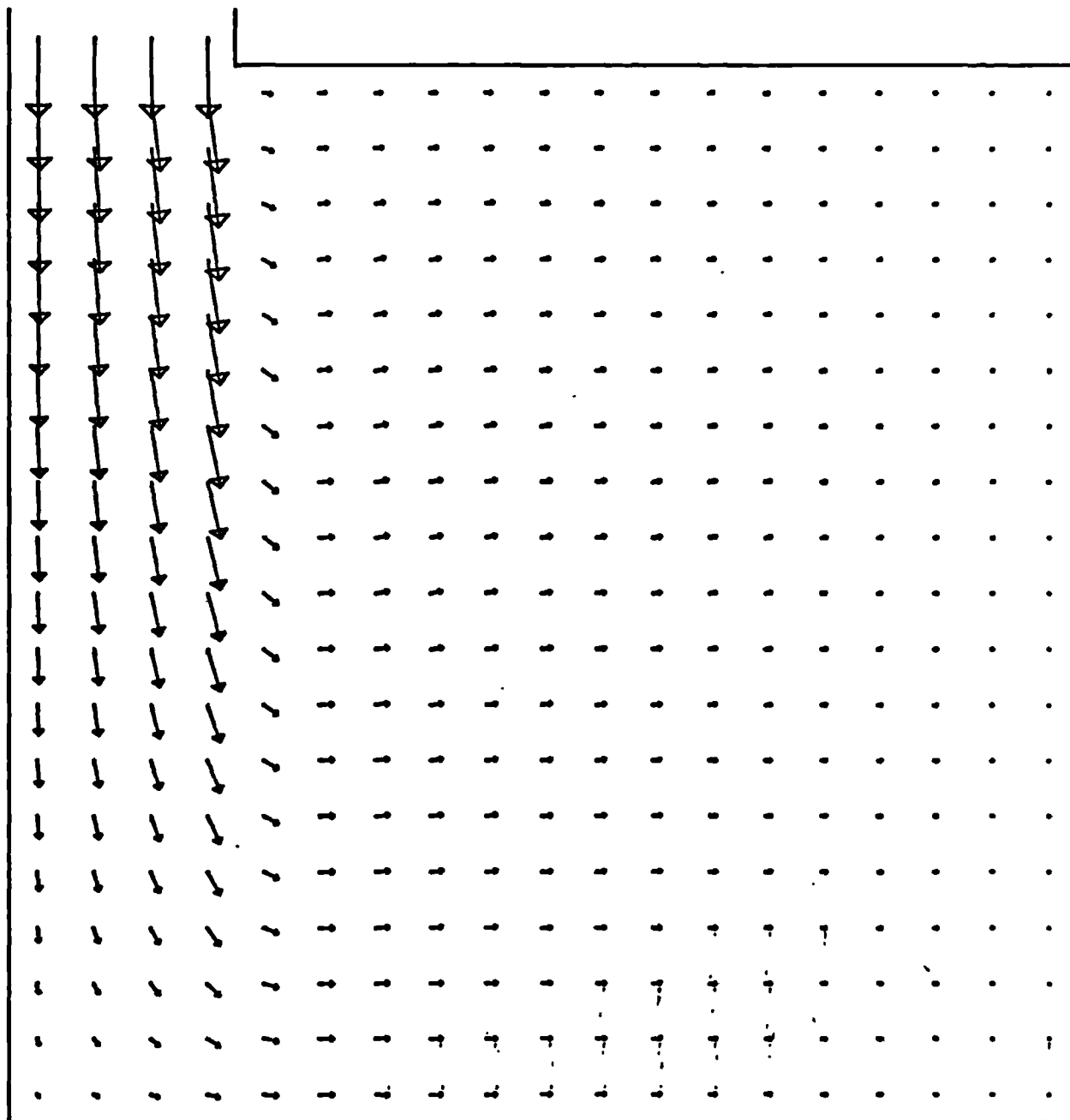
LENGTH SCALE — 20 M
 MEAN DEPTH 13.74 M
 WAVE PERIOD 12.4 HRS

VELOCITY → 0.02 M/SEC
 CHEZY VALUE 49.1 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 50a Computed Velocity Field as in Figure 49a with Twice the Depth

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



LENGTH SCALE — 20 M

MEAN DEPTH 13.74 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.02 M/SEC

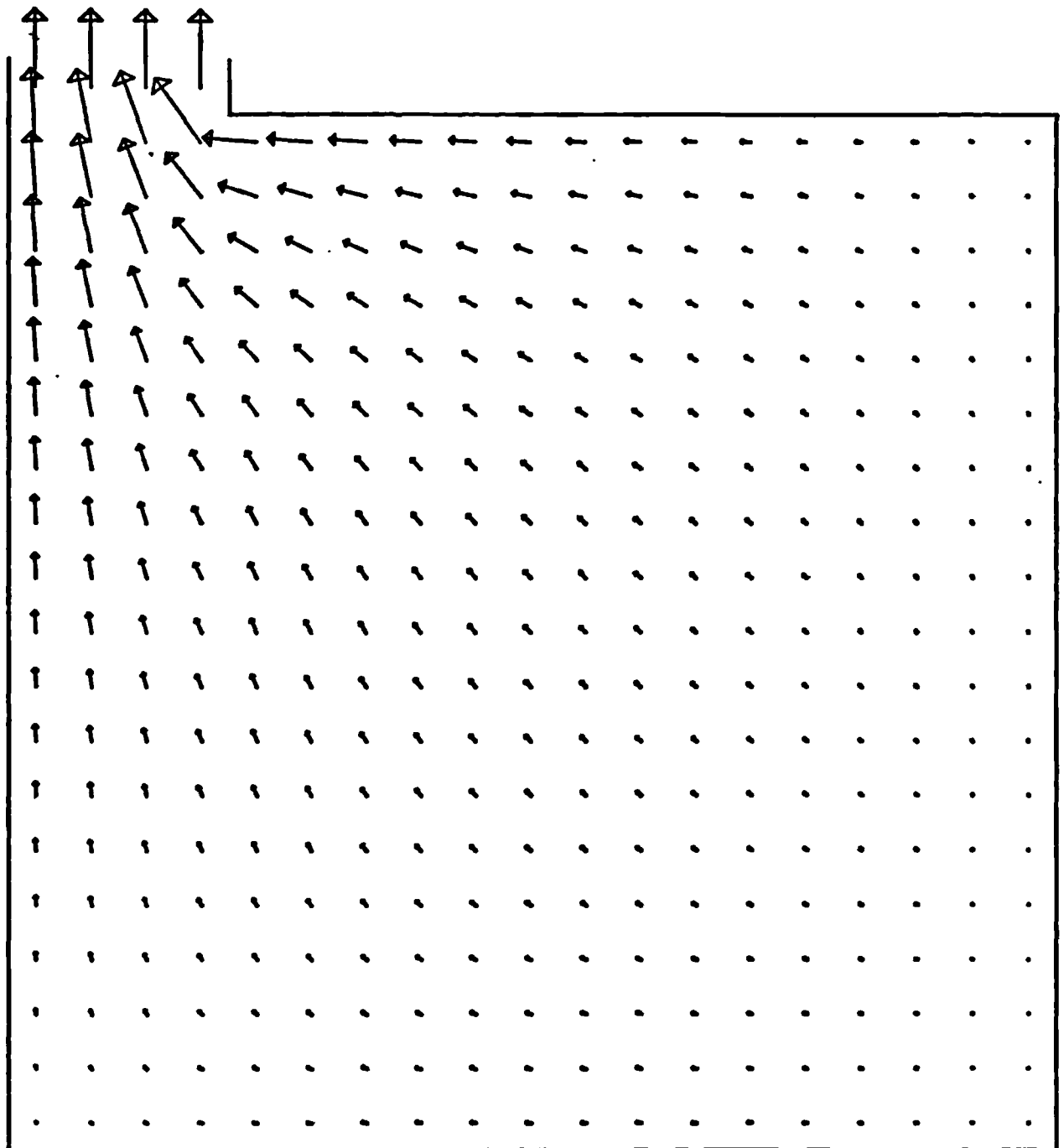
CHEZY VALUE 49.1 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 50b Computed Velocity Field as in Figure 49b with Twice the Depth

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



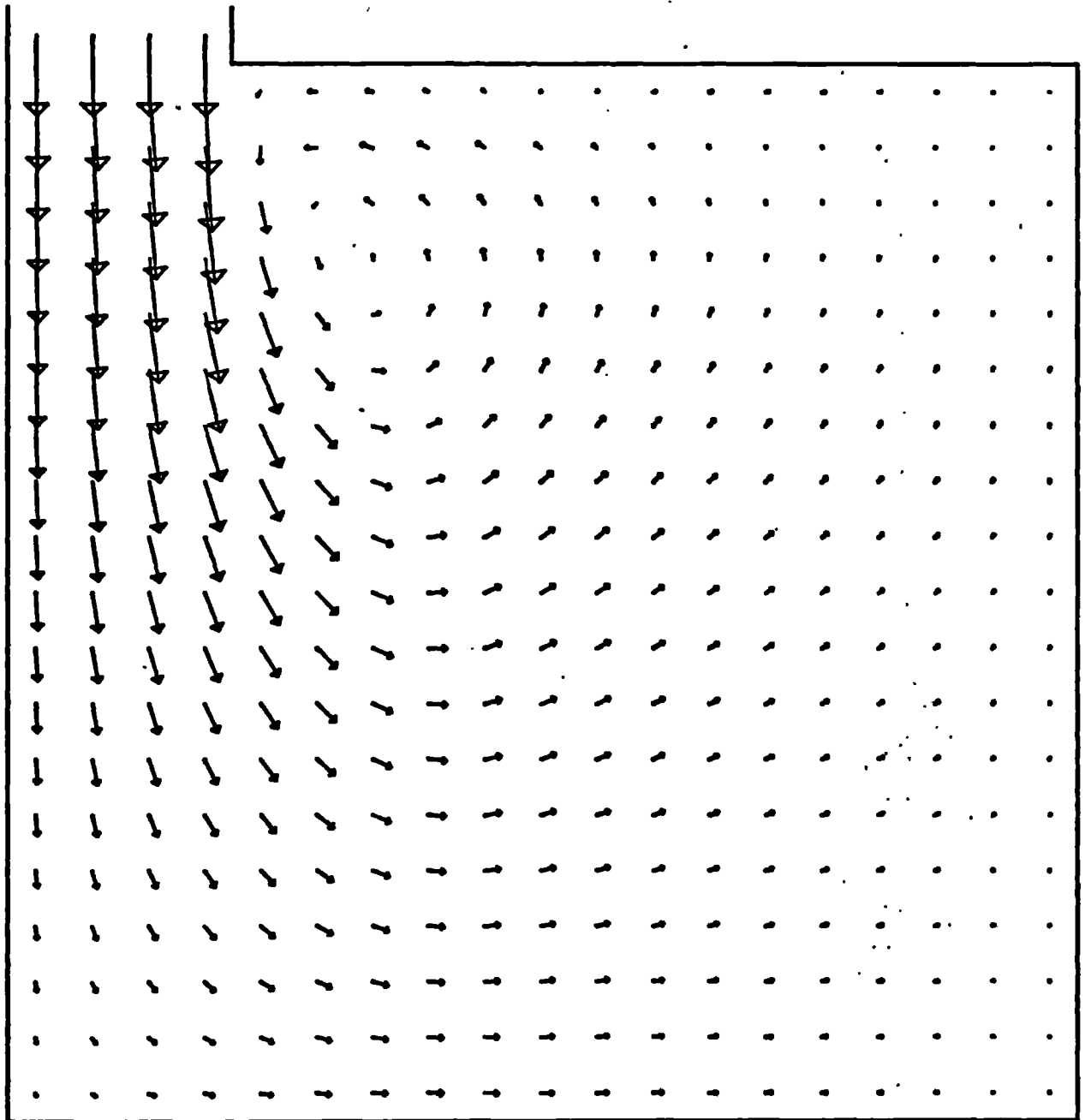
LENGTH SCALE — 40 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY → 0.08 M/SEC
 CHEZY VALUE 45.3 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 51a Computed Velocity Field as in Figure 49a with Twice the Length and Breadth

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



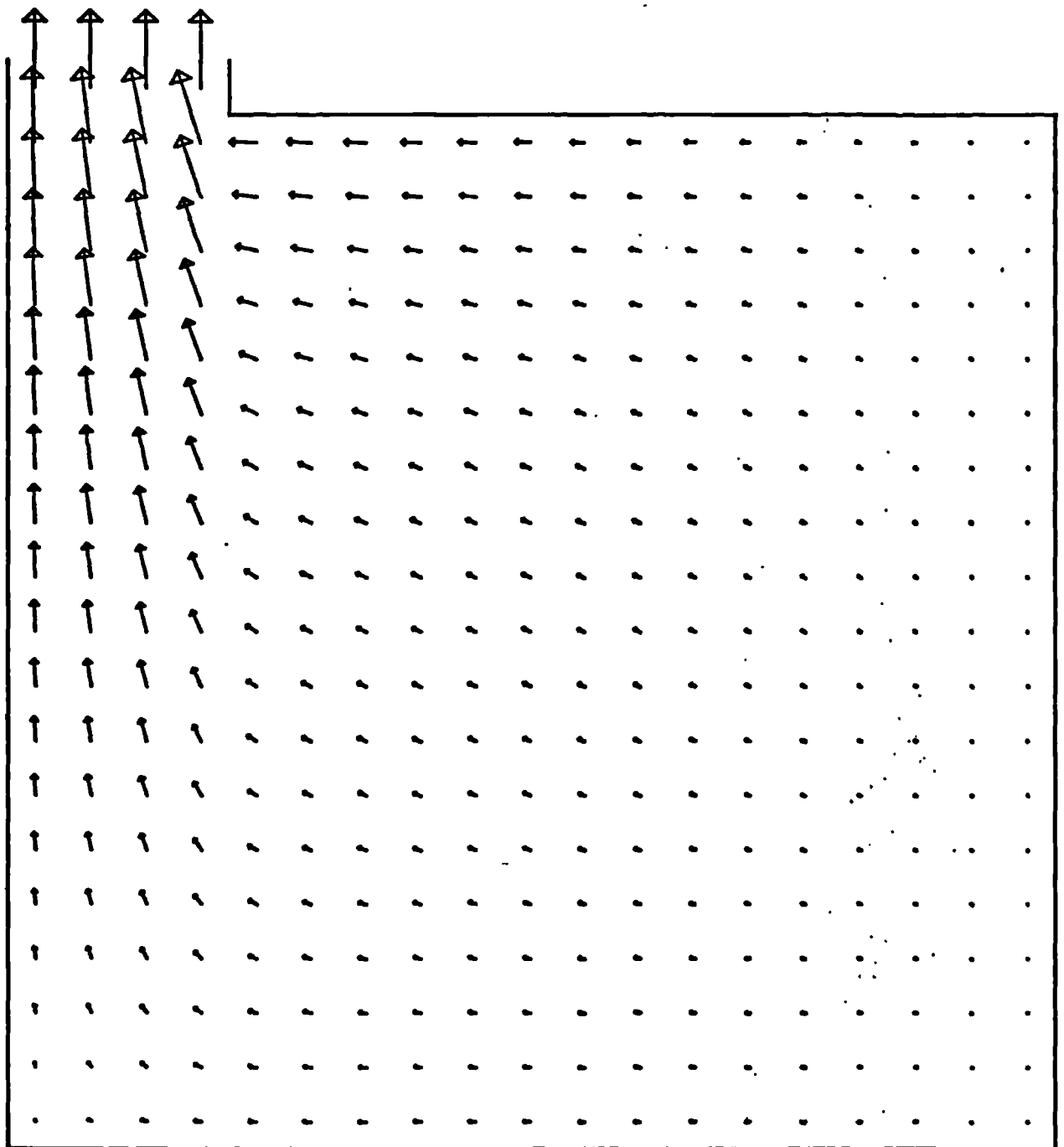
LENGTH SCALE — 40 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.08 M/SEC
 CHEZY VALUE 45.3 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 51b Computed Velocity Field as in Figure 49b with Twice the Length and Breadth

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



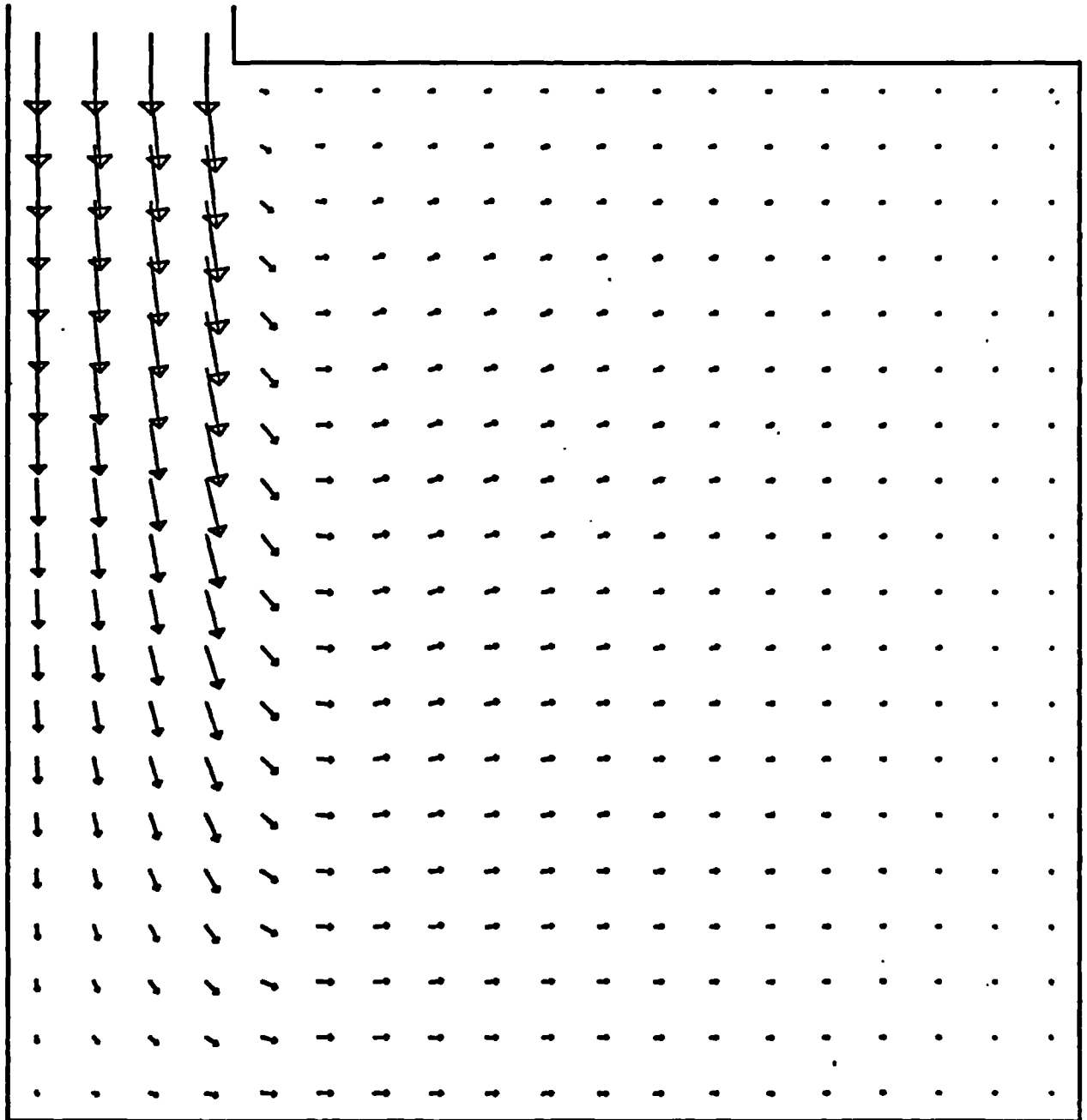
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY → 0.02 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 0.77 M

FIGURE 52a Computed Velocity Field for the Neap Ebb Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



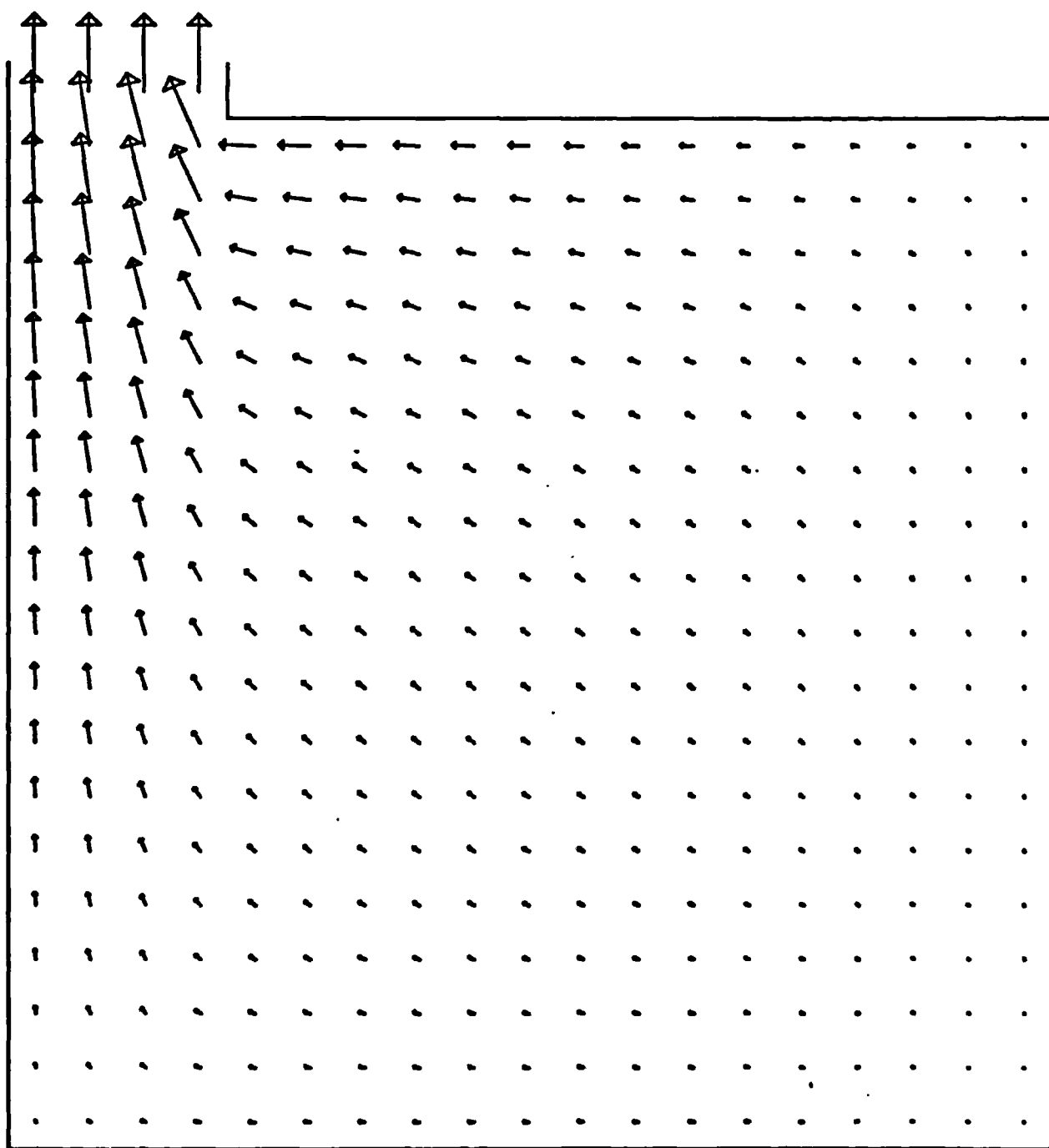
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY → 0.02 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 0.77 M

FIGURE 52b Computed Velocity Field for the Neap Flood Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



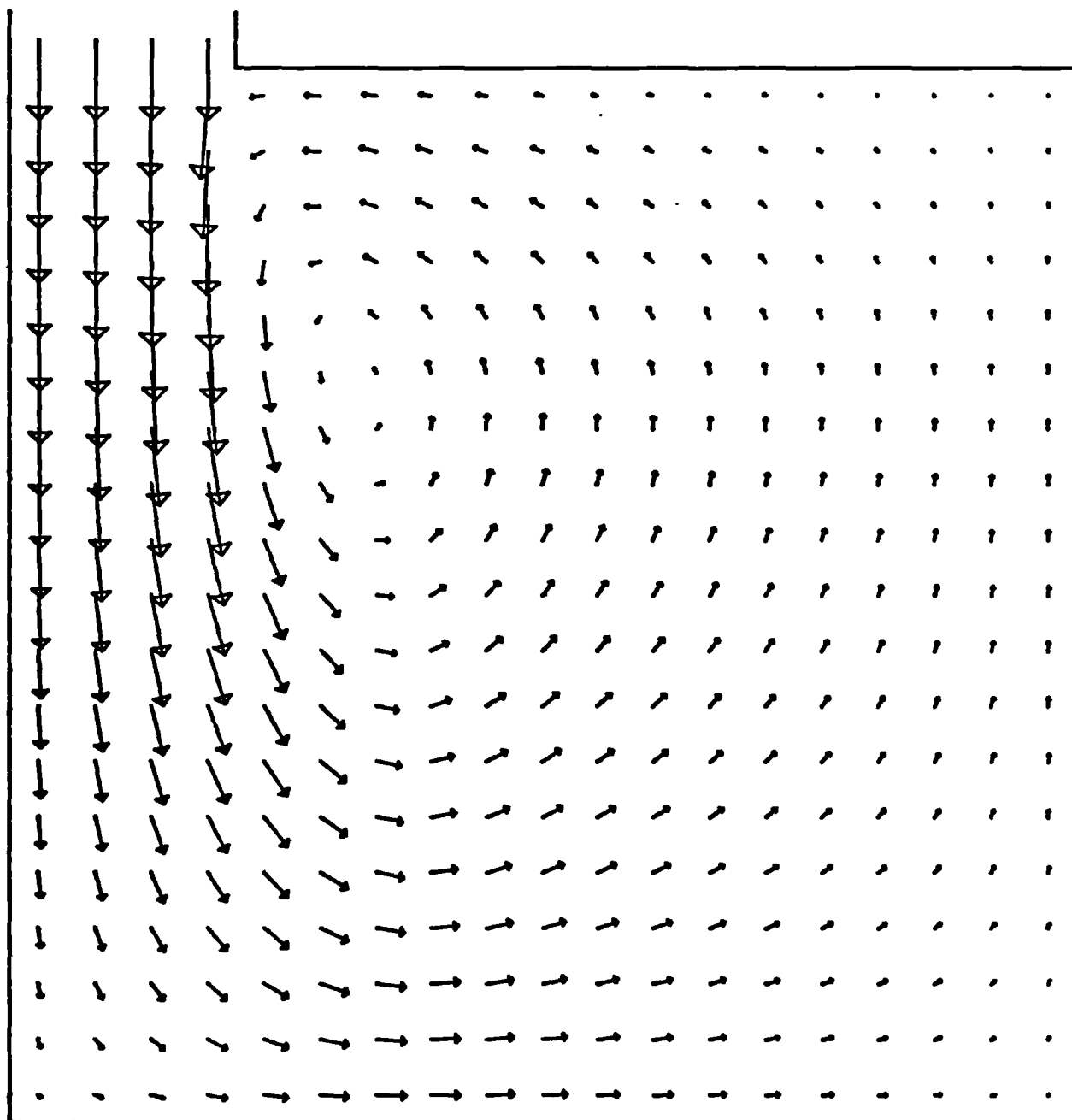
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.06 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 2.31 M

FIGURE 53a Computed Velocity Field for the Spring Ebb Tide with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



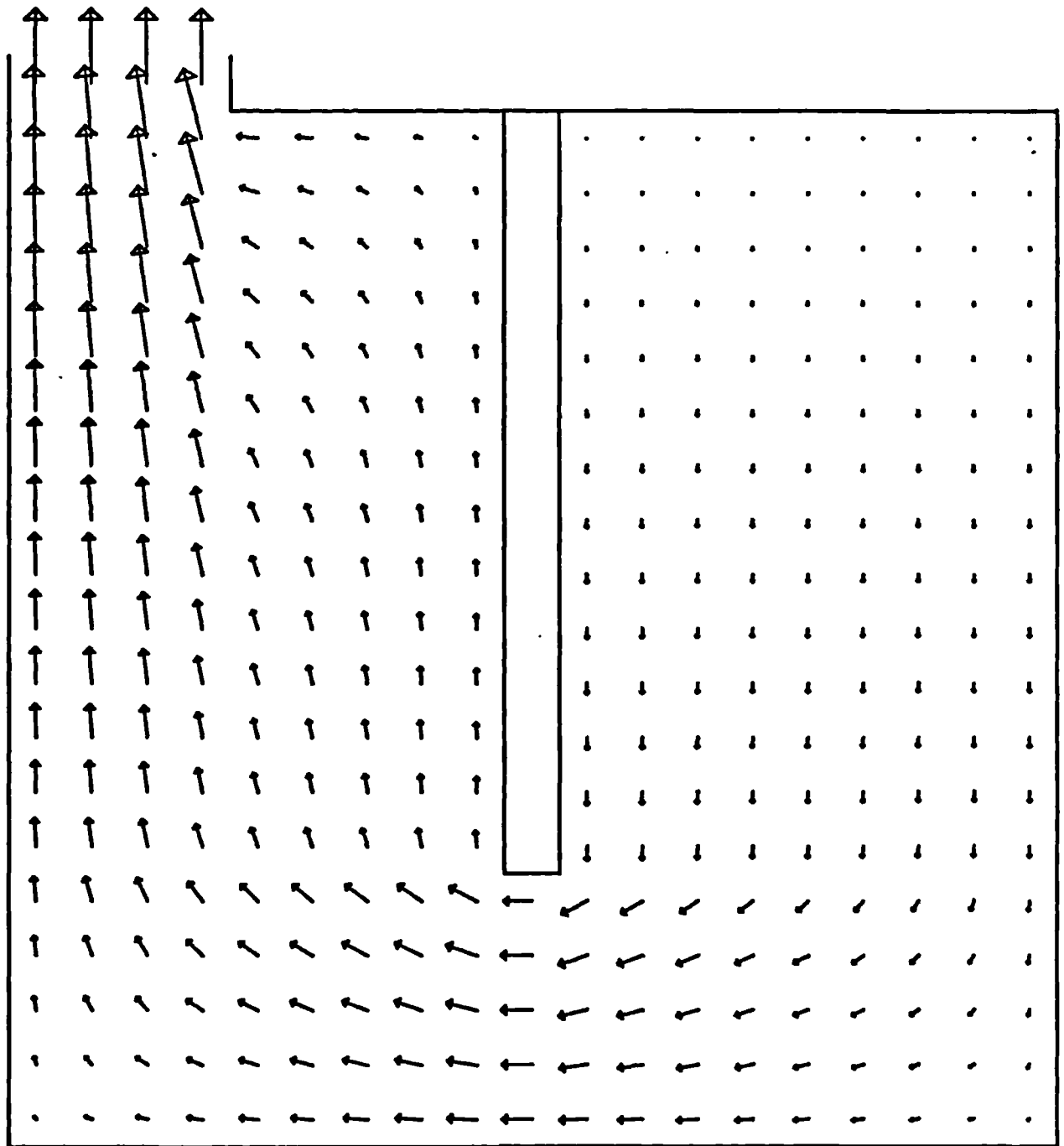
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.06 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 2.31 M

FIGURE 53b Computed Velocity Field for the Spring Flood Tide with
 with $L/B = 1.0$

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



LENGTH SCALE — 20 M

MEAN DEPTH 6.87 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC

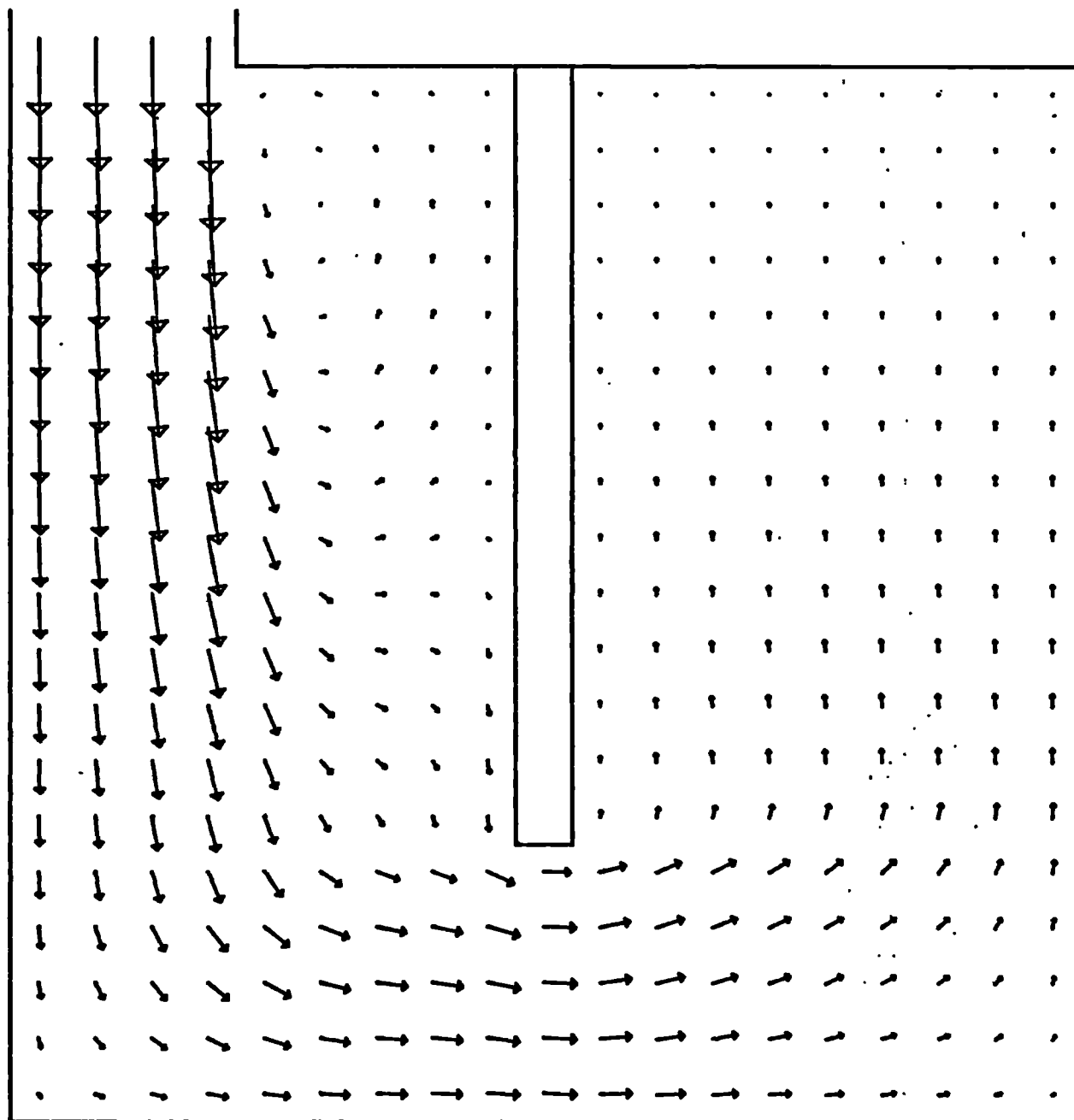
CHEZY VALUE 44.7 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 54a Computed Velocity Field for the Ebb Tide with a North-South Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



LENGTH SCALE — 20 M

MEAN DEPTH 6.87 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC

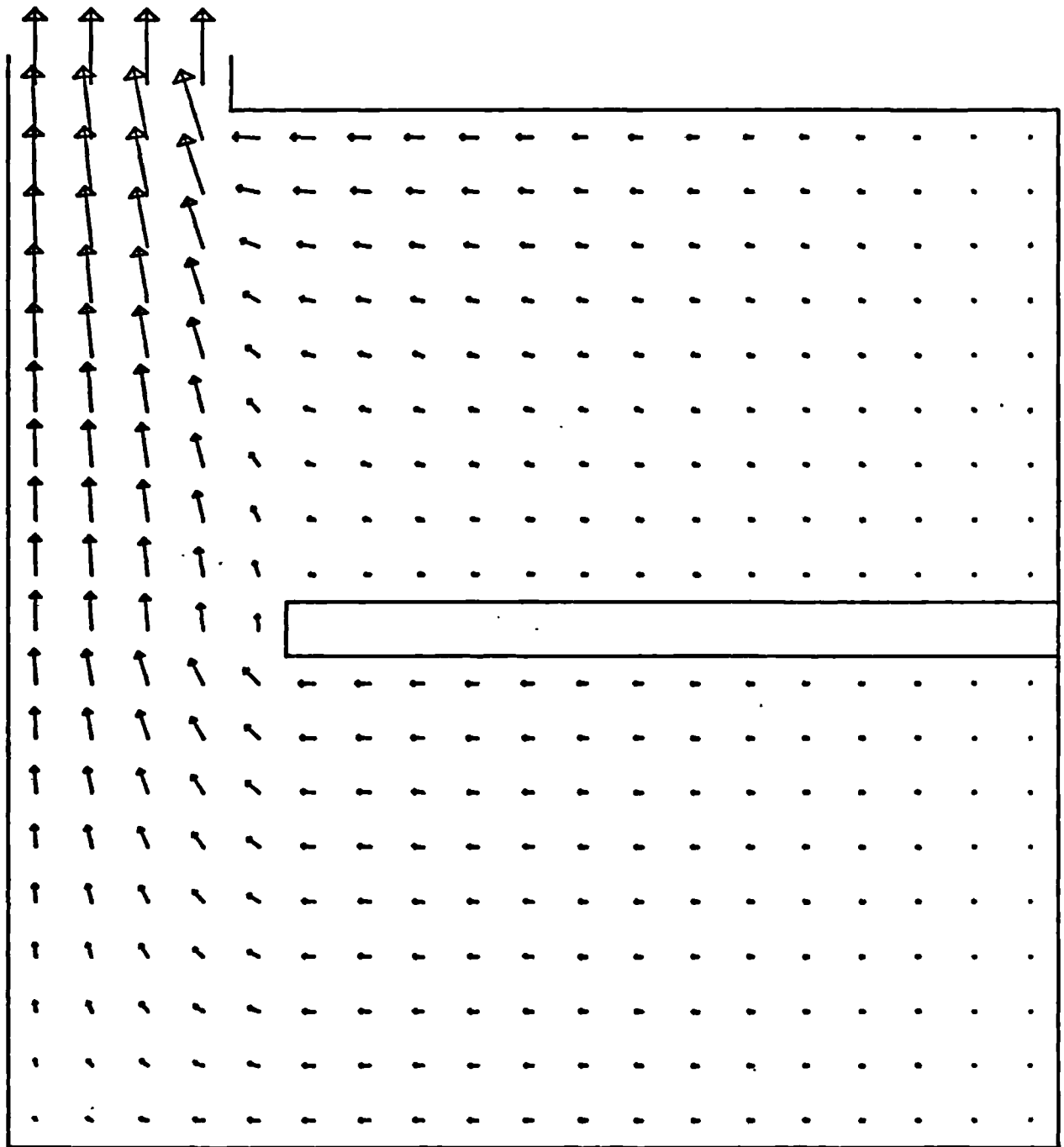
CHEZY VALUE 44.7 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 54b Computed Velocity Field for the Flood Tide with a North-South Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



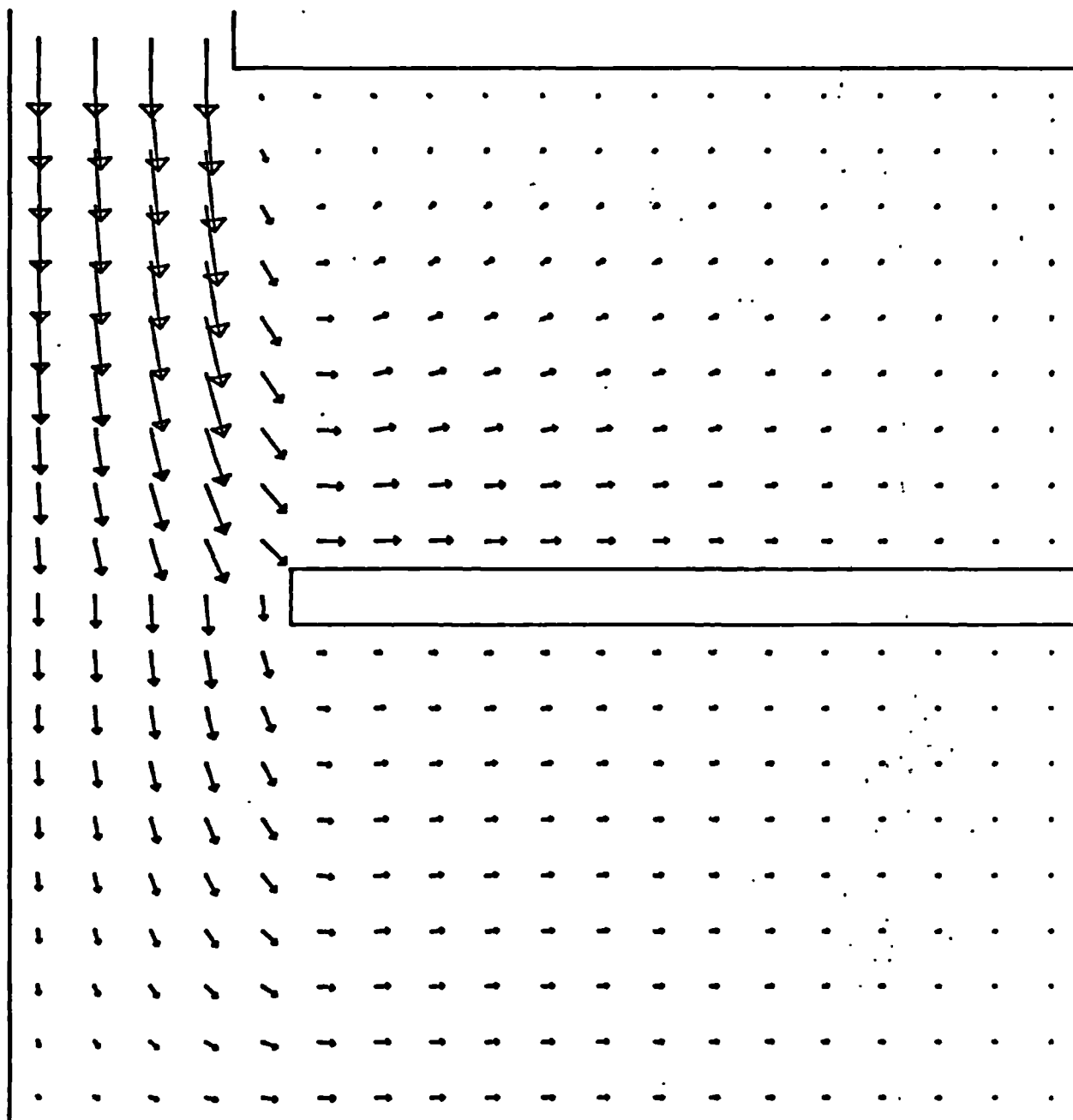
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 55a Computed Velocity Field for the Ebb Tide with an East-West Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



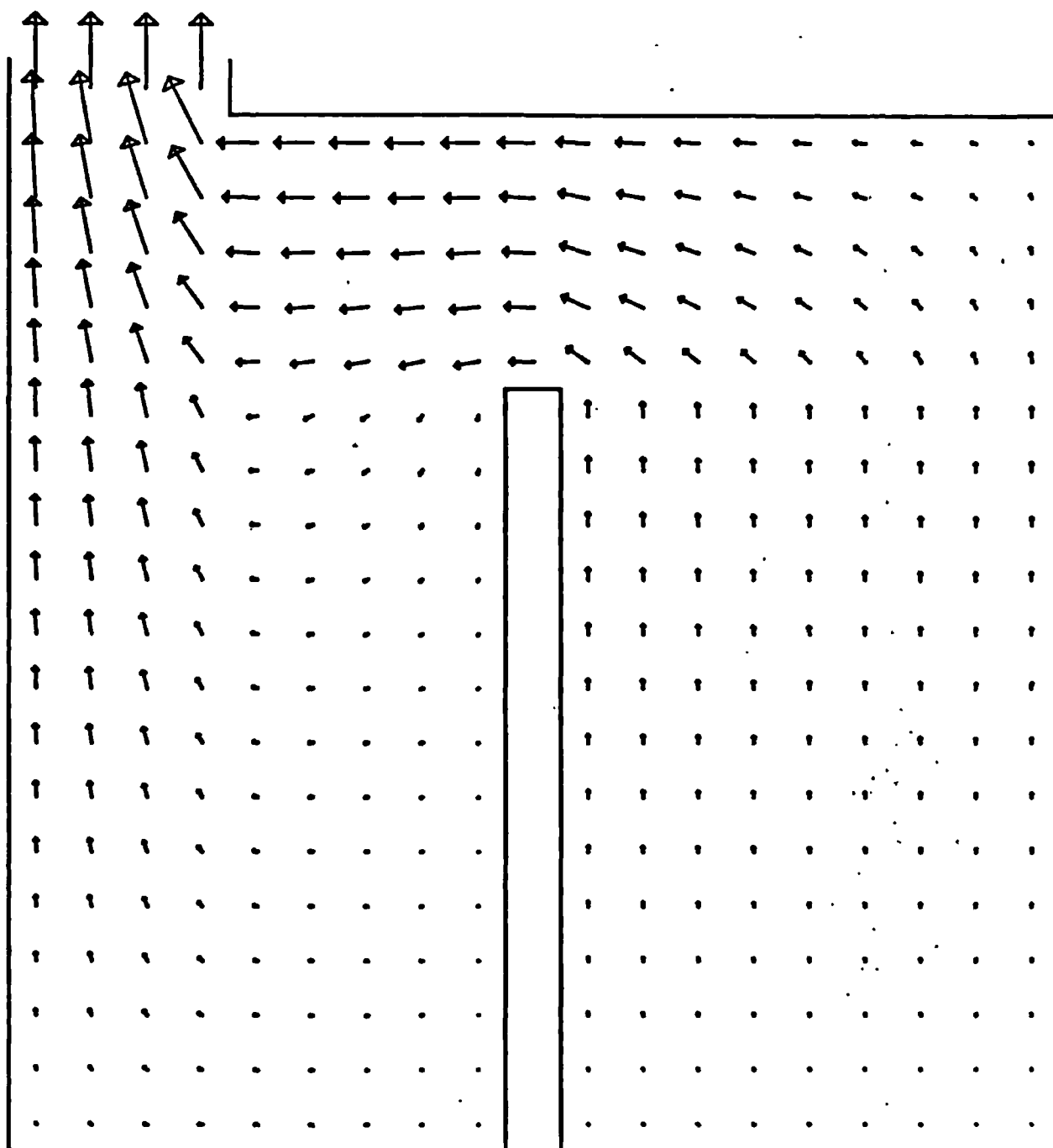
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 55b Computed Velocity Field for the Flood Tide with an East-West Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



LENGTH SCALE — 20 M

MEAN DEPTH 6.87 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC

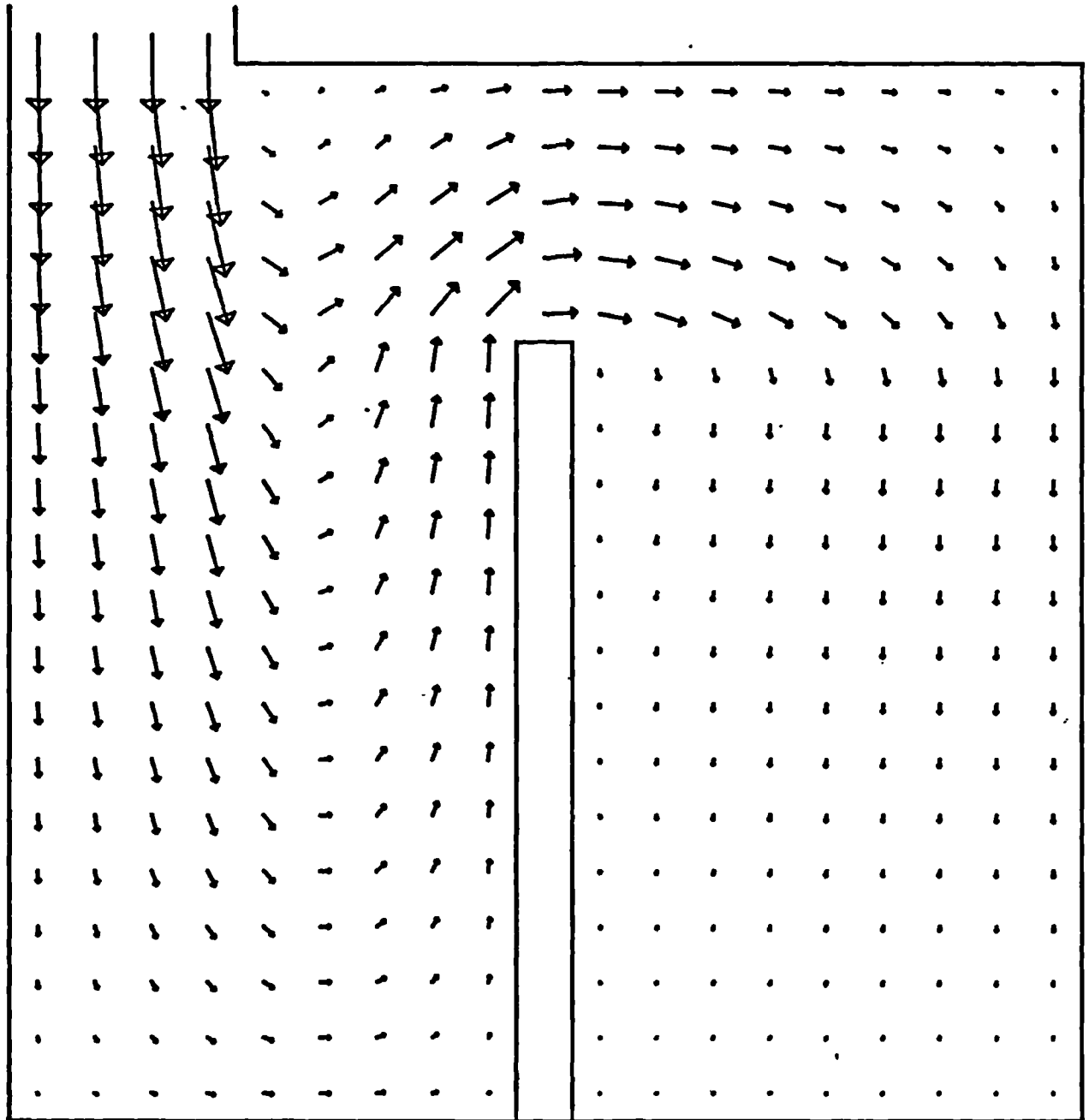
CHEZY VALUE 44.7 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 56a Computed Velocity Field for the Ebb Tide with a South-North Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



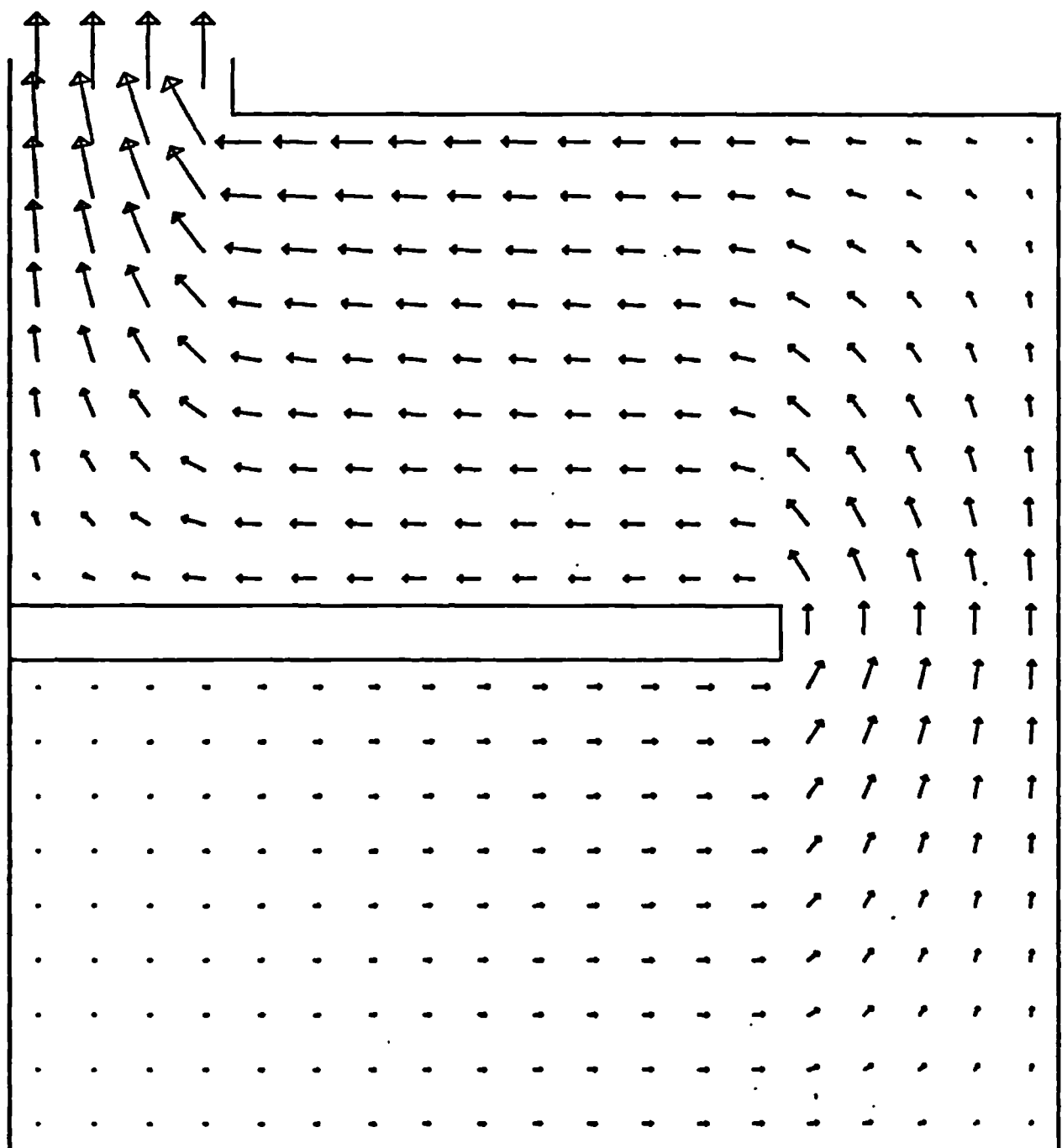
LENGTH SCALE — 20 M
 MEAN DEPTH 6.87 M
 WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC
 CHEZY VALUE 44.7 METRIC
 WAVE AMPLITUDE 1.54 M

FIGURE 56b Computed Velocity Field for the Flood Tide with a South-North Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 3.10



LENGTH SCALE — 20 M

MEAN DEPTH 6.87 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC

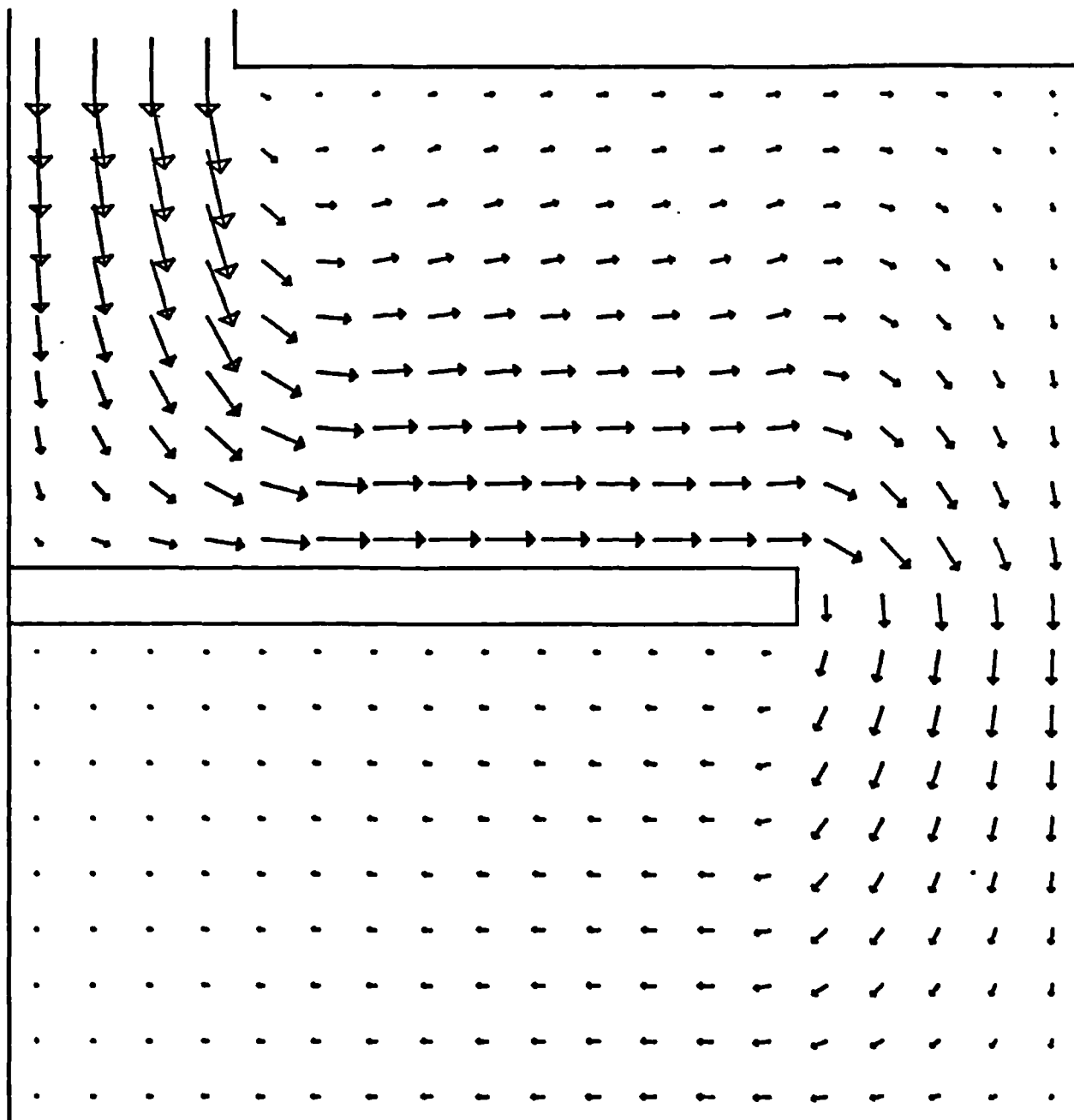
CHEZY VALUE 44.7 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 57a Computed Velocity Field for the Ebb Tide with a West-East Barrier

TIDAL CIRCULATION IN A HARBOUR

TIME IN HRS 9.30



LENGTH SCALE — 20 M

MEAN DEPTH 6.87 M

WAVE PERIOD 12.4 HRS

VELOCITY —→ 0.04 M/SEC

CHEZY VALUE 44.7 METRIC

WAVE AMPLITUDE 1.54 M

FIGURE 57b Computed Velocity Field for the Flood Tide with a West-East Barrier

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SUBSIDIARY MATERIAL

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PLANFORM INFLUENCE ON FLUSHING AND CIRCULATION IN SMALL HARBORS

by

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ABSTRACT

Laboratory data showing the influence of planform geometry on the tidal flushing characteristics of small harbors of simple surface shape. The tide ranges, water depths, and planform areas are typical of those encountered in small-boat marinas in Puget Sound, Washington. Each harbor investigated had a single, asymmetric entrance.

Flushing and circulation patterns within such harbors depend strongly upon the characteristics of the angular momentum established within the basin and upon the effective penetration distance into the basin of the stream of ambient water entering the harbor on the flood tide.

Experimental results confirm that best gross tidal flushing occurs when rectangular harbors have an aspect ratio L/B near unity, and that rounding interior corners of the basin has little effect on the gross tidal flushing but does improve local exchange. Aspect ratios L/B less than 3 lead to the creation of more than one circulation cell (gyre) within the basin.

INTRODUCTION

This paper reports on a laboratory study of tide-induced circulation in small, constructed harbors. Effects of wind and waves upon water motions within the harbors are not considered.

The study was keyed to the continuing demand for more small-boat marinas in Puget Sound, Washington. A common construction approach is to dredge the marina basin from a tideflat area, with the dredged material providing fill for parking lots, dry land work and storage areas, etc. A breakwater is provided on the seaward side, usually with a single asymmetric entrance located to provide best wave protection. Surface areas typically range from 10 to 30 acres, with provisions for from 200 to 800 boat moorages. Such marinas can truly be classified as "small" harbors.

Questions related to water quality are of considerable importance in

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the planning and design of such harbors. Of particular concern in the Puget Sound area is the potential impact of water quality in marinas on fish, particularly migrating juvenile salmon. Poor tidal flushing can lead to detrimental algal growths and low dissolved oxygen contents, dangerous to the migrating fish. It becomes imperative in the design of new marinas, and in the design evaluation leading to the legal permission to proceed with construction, that both the gross tidal flushing and the local circulation patterns within the proposed basins be considered. The particular concern of the present study is the question of how much the basin planform geometry affects the overall exchange and the internal circulation patterns.

The approach followed has been to utilize physical models. Numerical modeling procedures have been employed to study flow fields in existing small harbors (e.g., Grubert (1)), but in order to predict actual water exchanges a convective-diffusion solution would have to be superimposed on the hydrodynamic solution. Efficacy of the numerical techniques depends upon a valid determination of the separation streamline of the flood currents around the edges of the harbor entrance and on the time-and-space unsteady circulation cell which is initially engendered by this separation phenomenon, as well as upon the correct representation of diffusion coefficients. For designed but not built basins, the physical model approach has been considered the more economical and graphical approach to a combination of design tool and screen (2).

The laboratory tests described here utilized single-fluid vertically distorted models with a 10:1 distortion ratio. Such models do not properly scale diffusion-dispersion processes (3); therefore, the assumption must be made the convective transport is the dominant mode of water exchange and that diffusion per se is of less significance, allowing the use of tracer dyes in the model. This assumption has been used in a number of specific model studies of tidal exchange in small harbors subject to fairly large tides, where tidal currents are the dominant circulation feature (2). Field verification studies of water exchange are still needed; the models do reproduce depth-averaged velocities in the small harbors (4).

Because the tidal currents are indeed quite strong in the basins modeled it is safe to ignore Coriolis as well as wind and wave effects, and because the basins are relatively short and deep the absence of a friction calibration is not viewed as a major concern. Results are restricted to non-stratified harbors having no fresh water inflow; this is indeed the situation for the well-mixed water bodies which have been investigated so far in the Puget Sound marina studies. These limitations are listed here because they do restrict the generality of the results.

RANGE OF TESTS

The laboratory studies were treated as model studies of idealized basin shapes. The scale ratios were 1:500 horizontal, 1:50 vertical. Tides were restricted to repetitive, sinusoidal tides of constant period of 10.52 minutes in the model corresponding, via Froude law scaling relations, to a 12.4-hour semi-diurnal tide.

Gross water exchange (flushing) was determined by measuring changes in the concentration of fluorescent dye Rhodamine-WT in the basin over a number of cycles. Ink dye was used for visual observation of internal circulation patterns. Laboratory techniques and details of the tidal model basin used have been described previously (2,4).

Flushing data are presented in terms of E, the average per-cycle exchange coefficient, defined as that fraction of the water in the basin at high water slack which is removed from the basin and replaced by ambient water during the cycle ending at the next high water slack. The choice of selecting the period from high-water to high-water is arbitrary, but leads to a simple comparison with the tidal prism ratio, defined conventionally as

$$\text{Tidal Prism Ratio (TPR)} = \frac{(\text{Basin Vol. at High Tide}) - (\text{Basin Vol. at Low Tide})}{\text{Basin Volume at High Tide}}$$

where the numerator is known as the "tidal prism." A "flushing efficiency" is defined as

$$\eta = \frac{E}{\text{TPR}} \times 100, \text{ percent}$$

and compares measured water exchanges with those predicted by the simple tidal prism theory.

Three planforms--rectangular, rectangular with rounded corners, and circular--were investigated. For simplicity, vertical walls and horizontal basin bottoms were used in all cases. The basin planforms and notation used are shown in Figure 1. Semi-circular "jetties" were used at the entrance in each case.

The basin area was maintained at 1.25×10^6 square feet (28.7 acres). The equivalent L/B = 1.00 basin (which was not tested) has a side length of 1180 feet, or approximately 0.2 mi. The mean depth d within the harbor was 16 feet, taken at mean water level. The three tide ranges used in the tests were: neap, 4 feet; mean, 8 feet; spring, 12 feet. These values are representative of small boat basins in Puget Sound. The range of L/B in the rectangular basin tests varied from 0.20 to 5.00, more than spanning the usual range of aspect ratios found in such small harbors with single entrances aligned with one side wall. Entrance widths w were 125, 250, and 500 feet; the first is the most representative of field dimensions, and is emphasized most in following results. Data are presented in terms of equivalent prototype dimensions.

TIDAL FLUSHING

Essentials of the flushing (water exchange) test results are summarized in Figure 2 on which are shown data for three entrance widths w, three tides E, for rectangular basins over the aspect ratio L/B range 0.2 to 5.0. No data points are shown on Figure 2; a representative set is shown on Figure 3, illustrating the degree of experimental scatter which can be obtained in the rather sensitive exchange tests.

A few conclusions are apparent. Over the range of L/B from $2/3$ to 3.2 there is relatively little variation in exchange coefficient E for a given tide range H . The overall flushing falls off markedly for L/B less than $1/3$ and greater than 3 . This trend holds for the narrow entrance widths w . The dip in the E -curve near $L/B = 1.0$ is confirmed by the test points in Figure 3. At the larger tides H , the narrower entrance generally produces better overall flushing. As H decreases, the flushing efficiency η increases; this behavior has been noted on model studies of specific marinas having the very short entrance channel lengths used in the present tests.

Effects of L/B and w noted above are linked to the angular momentum of the complex flow within the harbor. The preservation of this angular momentum, associated with the deflection of flood tide flows through the entrance, is especially important to the internal circulation and, hence, harbor flushing. Circulation cells, or gyres, are created on the flood tide flow by moments of effective stresses associated with flow separation past the breakwater. These cells grow in both circulation strength and in size until they occupy a significant part of the basin, and may persist well into the ebb phase of the tidal cycle. As a consequence of the angular momentum established within the basin, even in the absence of longshore currents ebb flows are often skewed instead of leaving the basin in a direction parallel to the entrance axis. Higher velocities occur near the basin perimeter, and lower velocities near its center. These conditions in some respects are comparable to those occurring when tangential entrance jets of small size are used to produce a slow-moving large-scale circulation in water supply reservoirs (5). This angular momentum allows the inflowing ambient water to sweep past a major portion of the interior boundaries of the basin without losing its identity through diffusion. Thus, factors which contribute to increased angular momentum would be expected to improve overall flushing.

A preliminary simple angular momentum control volume analysis was made which considered shear forces on the walls and bottom of the basin and pressure forces on the walls; these would enter calculations of angular momentum about a vertical axis at the basin center. While details of these forces were not considered because interior velocities were not known, it was demonstrated that the optimum rectangular planform for maximizing the interior velocities at any instant of time is a square, $L/B = 1.0$ (6). For a constant surface area A the square planform has the minimum boundary perimeter, so that for a given H , w combination there is the greatest possibility in such small harbors that the ambient water entering as a two-dimensional jet will be able to circumnavigate the basin. Water which so completes the circuit around the boundary has a velocity component along the breakwater side of the basin as it approaches the entrance from within (clockwise, in Figure 1); if this not fully mixed ambient stream reaches the entrance at about high water slack some of it will be deflected back into the basin but some will exit the basin and not accomplish any effective exchange. The dips on the E -curves near $L/B = 1.0$ on Figures 2 and 3 reflect this phenomenon.

The 125-foot entrance width w is narrow enough to lend itself to the above reasoning over the full L/B range tested, and the 250-foot width nearly so. For the 500-foot width, tested for the 8-foot tide only, at small L/B the entrance width approached L and for large L/B for the constant surface area the width w approached B , and in neither case do the foregoing angular momentum considerations apply. Actual entrance widths are seldom much less than

the smallest (125 feet) valve tested, due to navigation considerations.

The reason for the fall-off in E for L/B less than $1/3$ and greater than 3 will be discussed in the next section.

Figure 4 shows there is little difference in gross flushing performance between the rectangular and rounded-corner basins of equal aspect ratio. The benefit of the rounded interior corners is that local separation zones, hence areas of relatively stagnant water, are eliminated; this is a point of concern to fisheries resource managers.

Figure 5 reflects the data of Figure 2, and indicates more clearly that the flushing efficiency for these short-entrance basins increases as the tide range H decreases. This is contrary to what can happen if entrance channels are longer or if exterior constraints such as a detached breakwater located athwart the channel axis impede the free exchange of basin and ambient waters, so that at the smaller tide ranges some of water exhausted from the basin on the ebb is returned to the basin on the flood.

Figures 6 and 7 indicate very little difference in the flushing performance of rectangular and rounded-corner basins with $L/B = 1.0$ and of a circular basin of the same surface area. At the "mean" tide of 8.0 feet flushing coefficients are nearly identical, and all exchange coefficients are above 80 percent. Extrapolation of these results to tide ranges of less than 4 feet may not be safe. It also is pointed out again that the combination of A , H , and d tested apply to limited geographical areas.

CIRCULATION PATTERNS IN HARBOR

Insight into the mechanisms of the tidal flushing process can be gained by considering the path and distance penetrated into the harbor on the flood tide by the inflow stream of ambient water and by relating this distance to the planform geometry of the basin. This section deals with a simple correlation of these quantities.

Experimental results are shown in Figure 8 for two tidal ranges for each of four harbor geometries. Small tubular drogues, weighted to provide adequate vertical penetration so that the drogues move with the local depth-averaged velocity, were released at the center of the entrance at the time of mean water level crossing and were tracked through the basin for the subsequent one-quarter cycle to high water slack. The locations of the drogues at one-eighth and one-fourth of the tidal period T after release are indicated, as well as the paths followed. The effect of doubling the range H from 4 feet to 8 feet is apparent.

For the elongated basins of $L/B = 3.20$ and $L/B = 0.31$, the drogue paths indicate how two circulation cells are developed in the harbors. For the inlet orientation shown the cell near the harbor entrance is clockwise in each case, while that in the rear portion of the harbor is counterclockwise. The drogue pathlines for these two configurations do not show that part of the ambient water stream does not follow the plotted pathline into the rear section of the harbor but rather moves into the gyre in the front section of the harbor. Figure 8 indicates pathlines for water entering at the time of mean

water level crossing. Ambient water entering the basin prior to this time on the flood tide has followed much the same path after initial development of the cell inside the harbor entrance; as a consequence, at the time of high water slack there are currents moving along the walls of the harbor and toward the mouth.

For purposes of comparison, a penetration distance p has been defined as the distance traveled along its pathline by a particle of water which enters the basin at the time of the mean sea level crossing, which for practical purposes in an entrance of the type used in the present study is the time of maximum velocity of the flood tide current. The flood current has maximum or near-maximum momentum at this arbitrarily selected but convenient time, and from consideration of conservation of momentum of the inflow "jet" it possesses the capability of penetrating a maximum distance into the basin in a finite time. The following simple analysis considers the penetration distance p at high water slack, a time $T/4$ after water entrance into the basin. Notation is shown in the definition sketch, Figure 9. The rounded jetties used in the tests minimized contraction of the inflow as it passed through the inlet, so that a one-dimensional condition of uniform velocity V_0 over the entrance area wd could be a reasonable assumption.

The inflow stream initially flows parallel to one wall in the harbor planforms studied, but in all cases ultimately deflects from its initial direction. The choice remains, in a simple momentum correlation, whether to employ a wall-jet analysis or a slot-jet analysis. In either case, neither of which matches the physics of the tidal currents in the small harbors, the expression for the velocity v along the streamline of maximum velocity is

$$\frac{v}{V_0} = C \left(\frac{x}{w} \right)^{-1/2}$$

Values of the constant C are 3.45 for the wall-jet (7) and 2.28 for the slot-jet (8), respectively. Assuming that the steady flow relations can be applied to a water particle moving along this streamline, integration of the expression $v = dx/dt$ along the initial direction of motion, x , gives

$$x = \left(\frac{3}{2} C \right)^{2/3} V_0^{2/3} w^{1/3} t^{2/3}$$

It remains to relate V_0 to the tidal range and to the basin geometry. Applying the simple one-dimensional continuity equation for a sinusoidal tide of range H and period T to the basin of planform area A and inlet cross-sectional area wd at the mean water level, V_0 at this time is given by

$$V_0 = \frac{\pi H}{T} \frac{A}{wd}$$

Interchanging the penetration distance p for the distance along the x -direction gives

$$p = \left(\frac{3}{2} \pi C \right)^{2/3} \frac{H^{2/3}}{w^{1/3}} \left(\frac{A}{d} \right)^{2/3} \left(\frac{\pi T}{T} \right)^{2/3}$$

where n = the fractional part of the tidal cycle after $t = 0$ at which p is evaluated.

The results of Figure 8 have been replotted in Figure 10, in which the maximum value $n = 1/4$ represents conditions at high water slack. The slot-jet value of $C = 2.28$ was used because it correlates more closely with the experimental observations; the wall-jet empirical constant C over-predicts the penetration. Numerical agreement with the jet models considered could not be expected, but the test results do show agreement in the form of the penetration distance vs. time relationships. Therefore, the form of the last equation given above can be used to predict changes in the ambient-water penetration into a harbor with variations in tidal range H or in harbor geometry parameters A , d , and w .

CONCLUSIONS

The effect of planform geometry upon the tidal flushing characteristics of small harbors with single asymmetric entrances has been studied for simple, rectangular basins in which wind and stratification effects are negligible and tidal action is the dominant flow-producing mechanism. When harbor aspect ratios exceed 3 or are less than $1/3$, multiple circulation cells exist within the harbor and the basin-average tidal flushing is reduced. So long as the aspect ratio is kept within these limits there is relatively little variation in the tidal exchange coefficient, E , with L/B for a given tidal range H , although best flushing occurs when the aspect ratio is close but not necessarily equal to unity, the value predicted by simple considerations of angular momentum of the internal tidal currents.

The experimental data indicate that rounding interior corners of the harbor does not improve the gross flushing. The elimination of sharp interior corners does, however, eliminate local stagnation zones and thus improves local water exchange and local water quality. This latter consideration is important to the safety of fish which pass through or temporarily reside in small-boat basins of the size range tested. In the Puget Sound area of Washington, this point is of particular concern in the safe migration of juvenile salmon. Figure 11 illustrates changes in typical marina planform shapes for small-boat basins in the Puget Sound region which have in some part been brought about by considerations of tidal flushing and internal circulation patterns, particularly the latter.

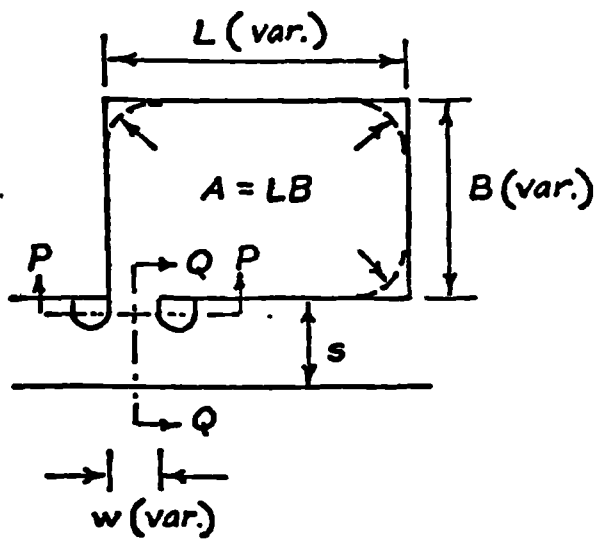
Results of this study may be used to answer some of the hydraulic questions considered in the design of such harbors and may reduce the need for individual model studies on a case-by-case basis.

ACKNOWLEDGMENT

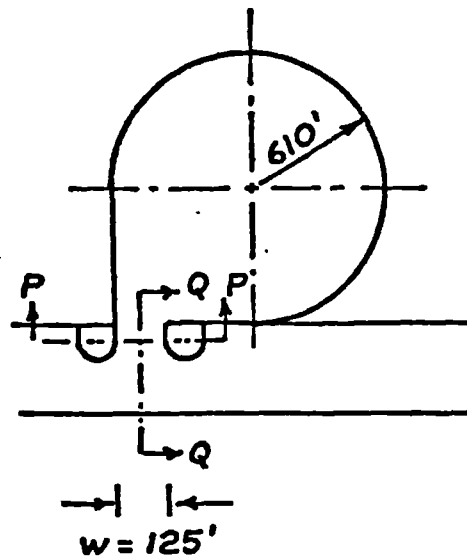
The experiments reported were conducted in the C. W. Harris Hydraulics Laboratory, University of Washington, Seattle, U.S.A.

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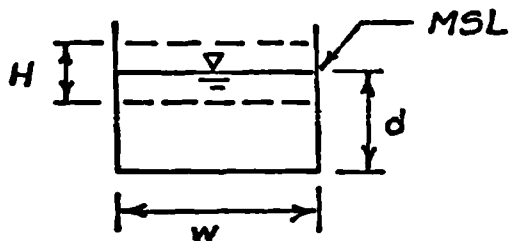
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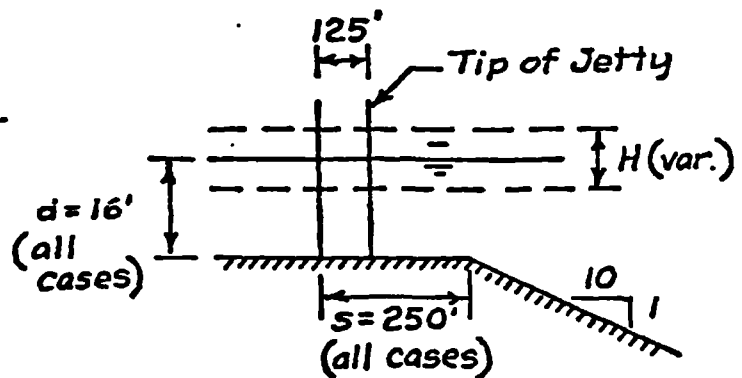
Plan - Rectangular Basin
(Note: $r = 250'$, all cases,
Rounded Corner Basin)



Plan - Circular Basin
(Note: Surface Area = LB)



Sect. P-P



Sect. Q-Q

Vertical Wall, Level Bottom Basin:

$$TPR = \frac{H}{d + \frac{H}{2}}$$

$$\eta = \frac{E}{TPR} \times 100$$

FIGURE 1. Definition and notation diagram.

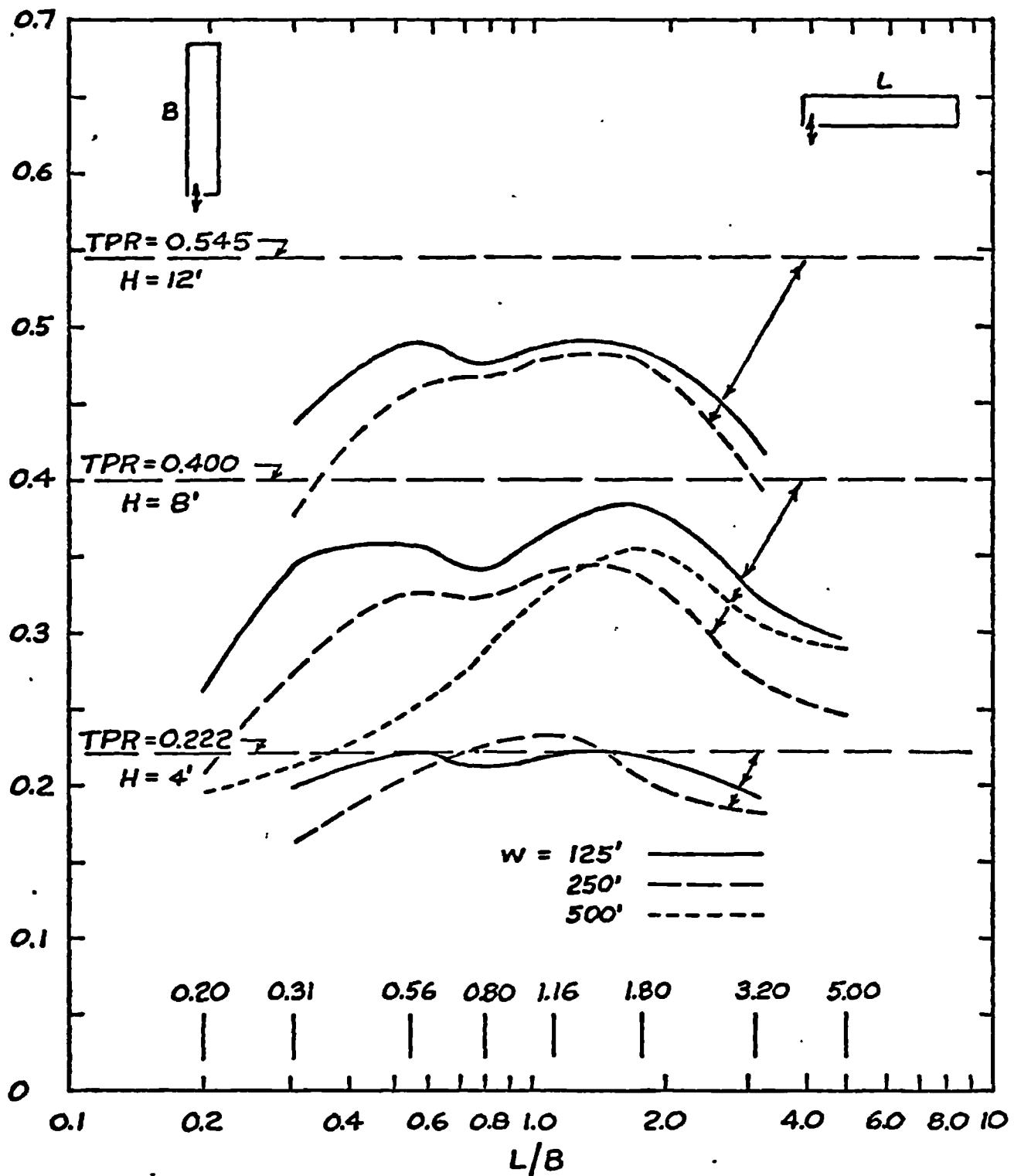


FIGURE 2. Flushing data summary, rectangular harbor.

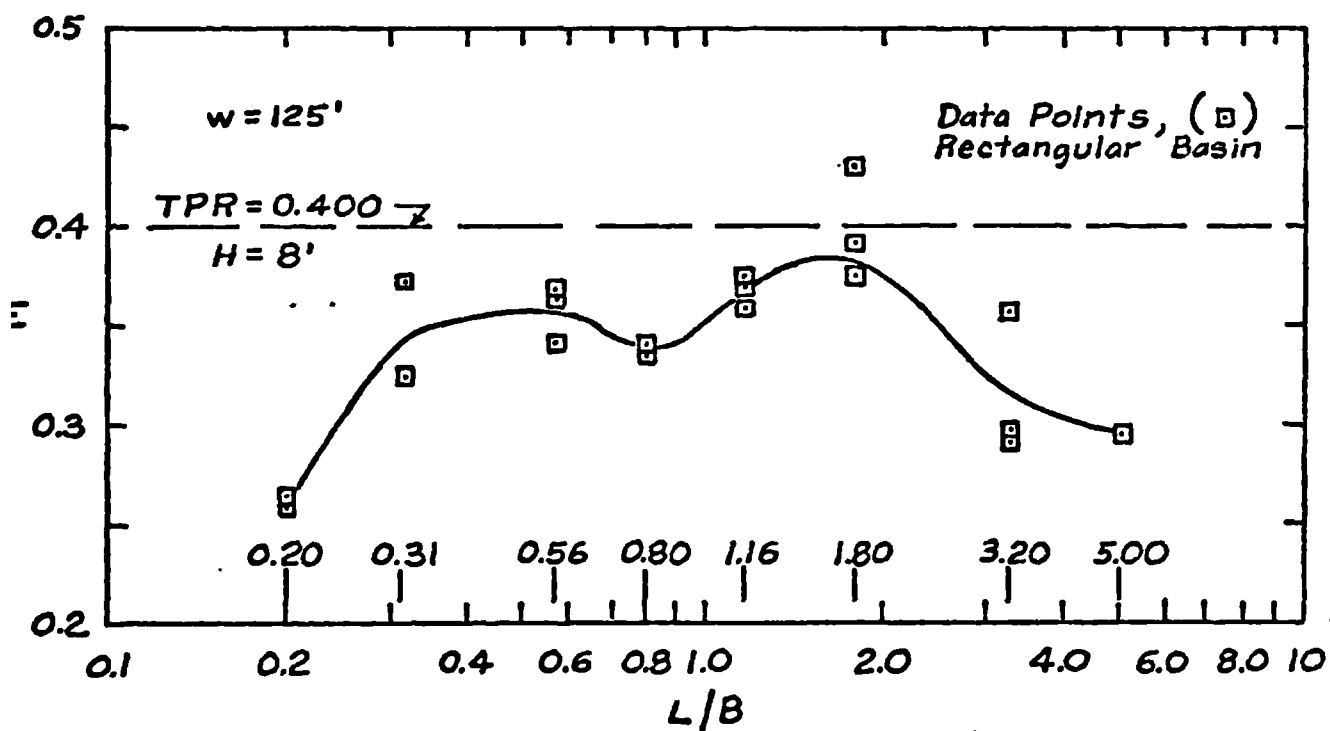


FIGURE 3. Representative data set, rectangular harbor,
 $H = 8$ feet, $w = 125$ feet.

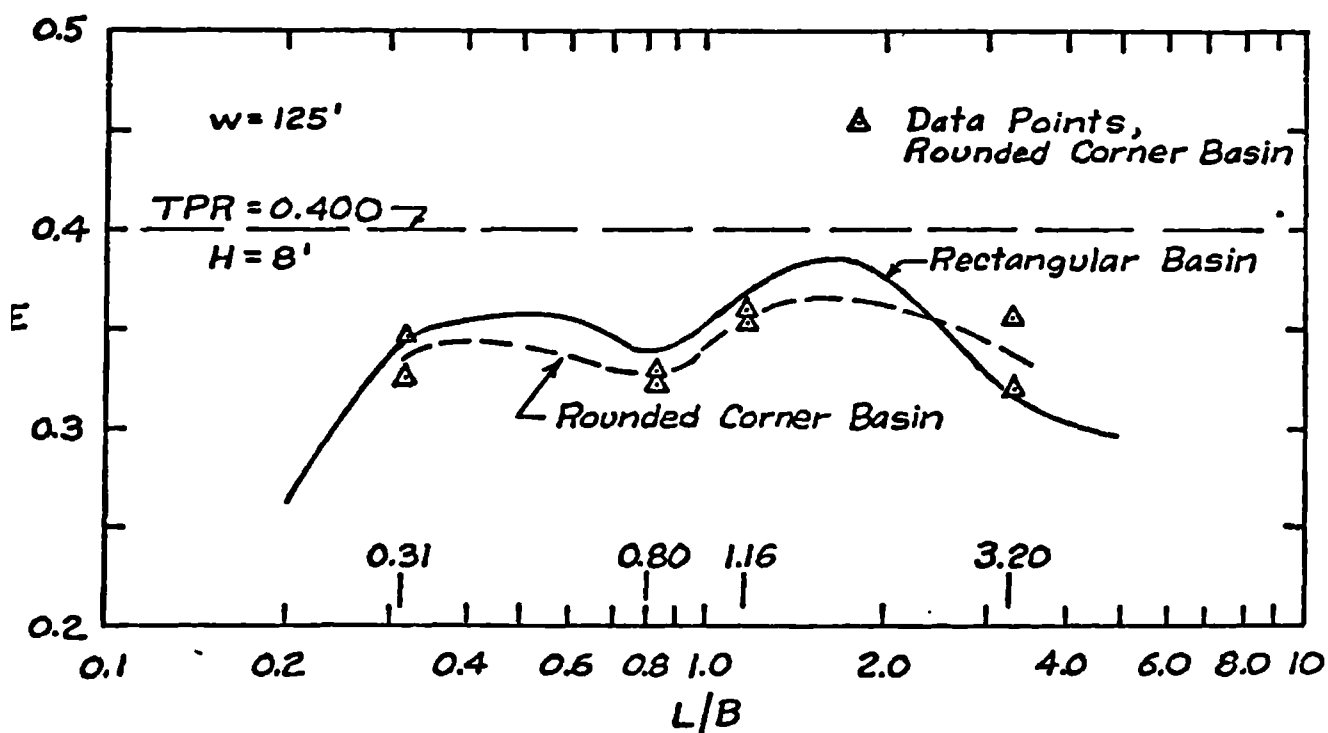


FIGURE 4. Comparison of flushing performance, rectangular
 basin and rounded corner basin, $H = 8'$, $w = 125'$.

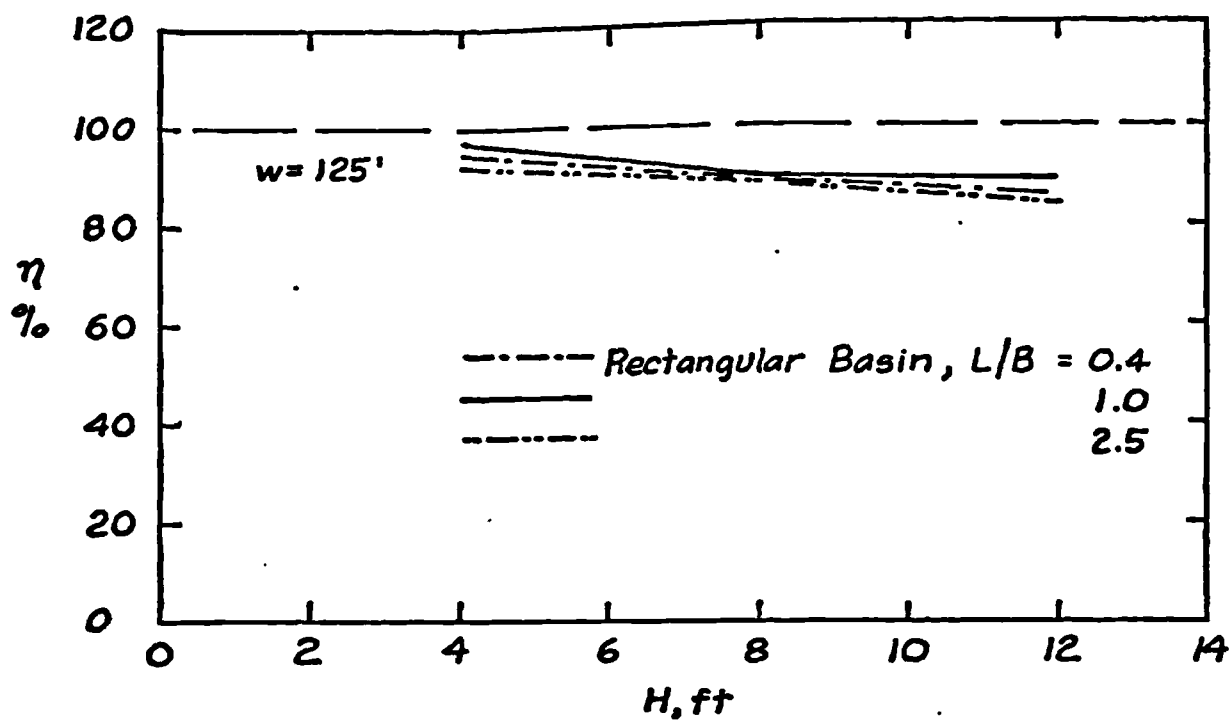


FIGURE 5. Variation of flushing efficiency with tide range, regular basin, $w = 125$ feet.

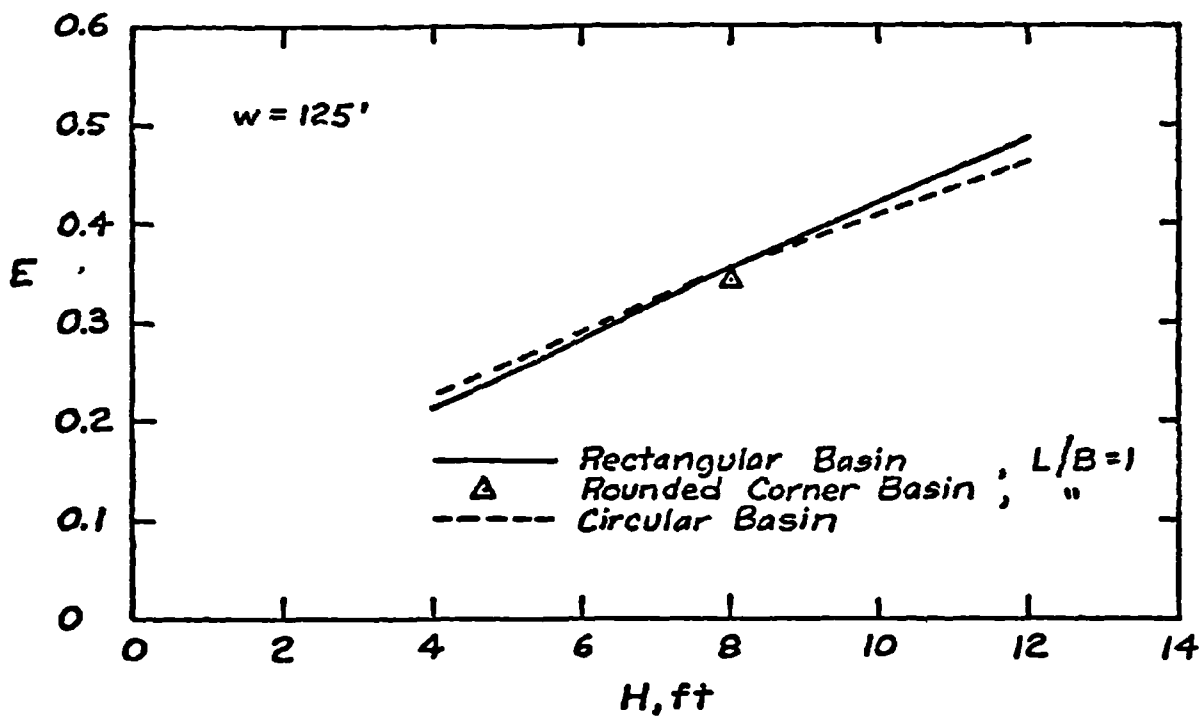


FIGURE 6. Comparison of flushing performance, circular and "square" basins.

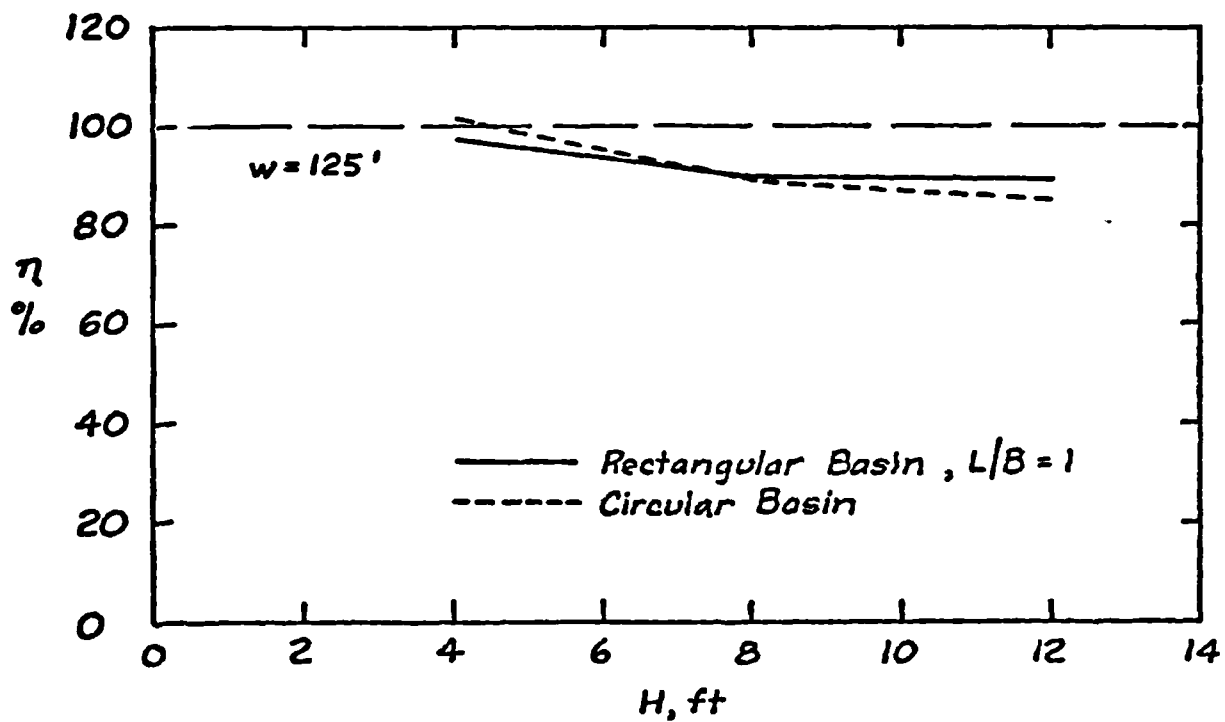


FIGURE 7. Flushing efficiency, circular and "square" basins.

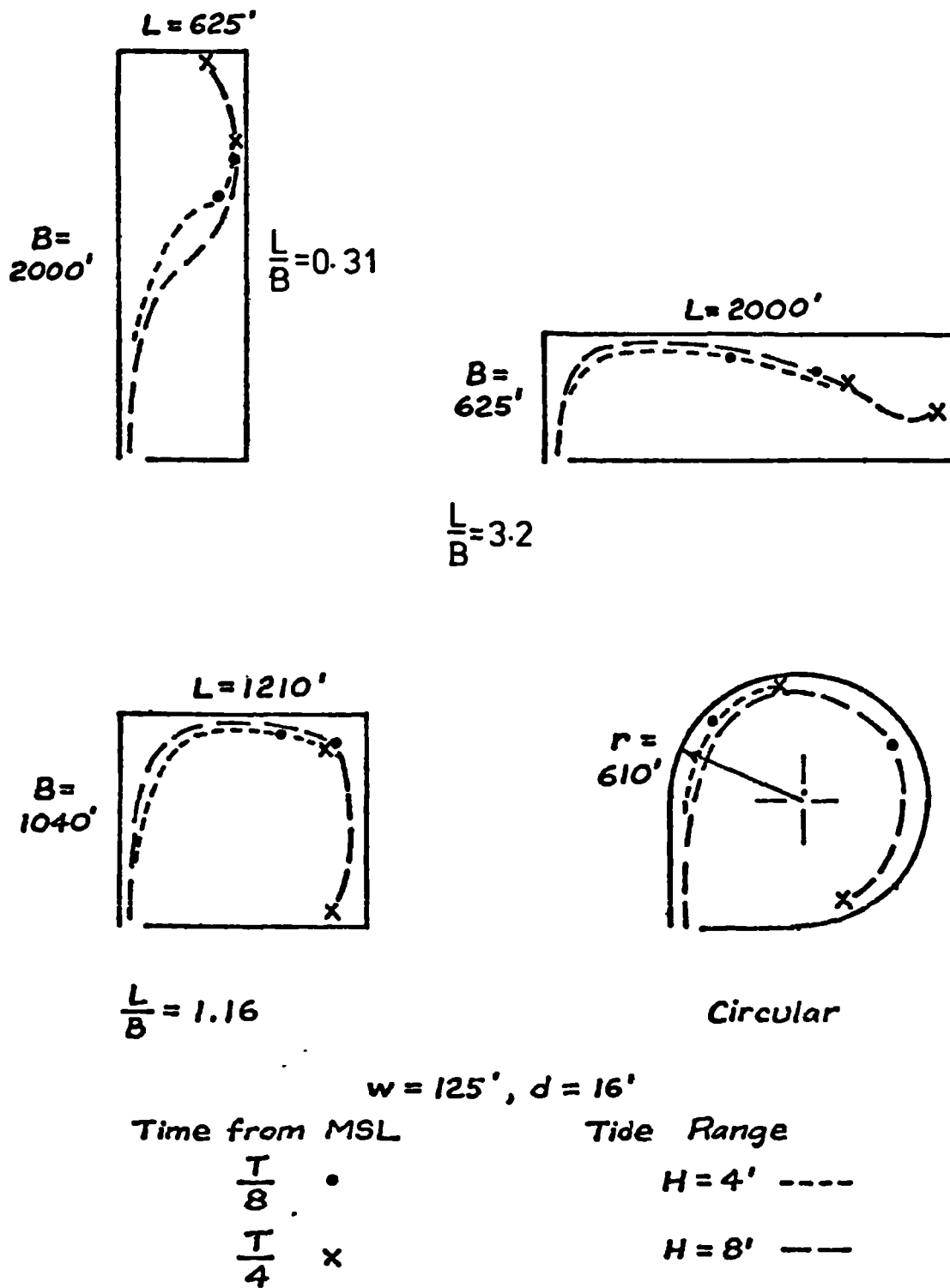


FIGURE 8. Inlet jet paths and penetration distances for four harbor planforms.

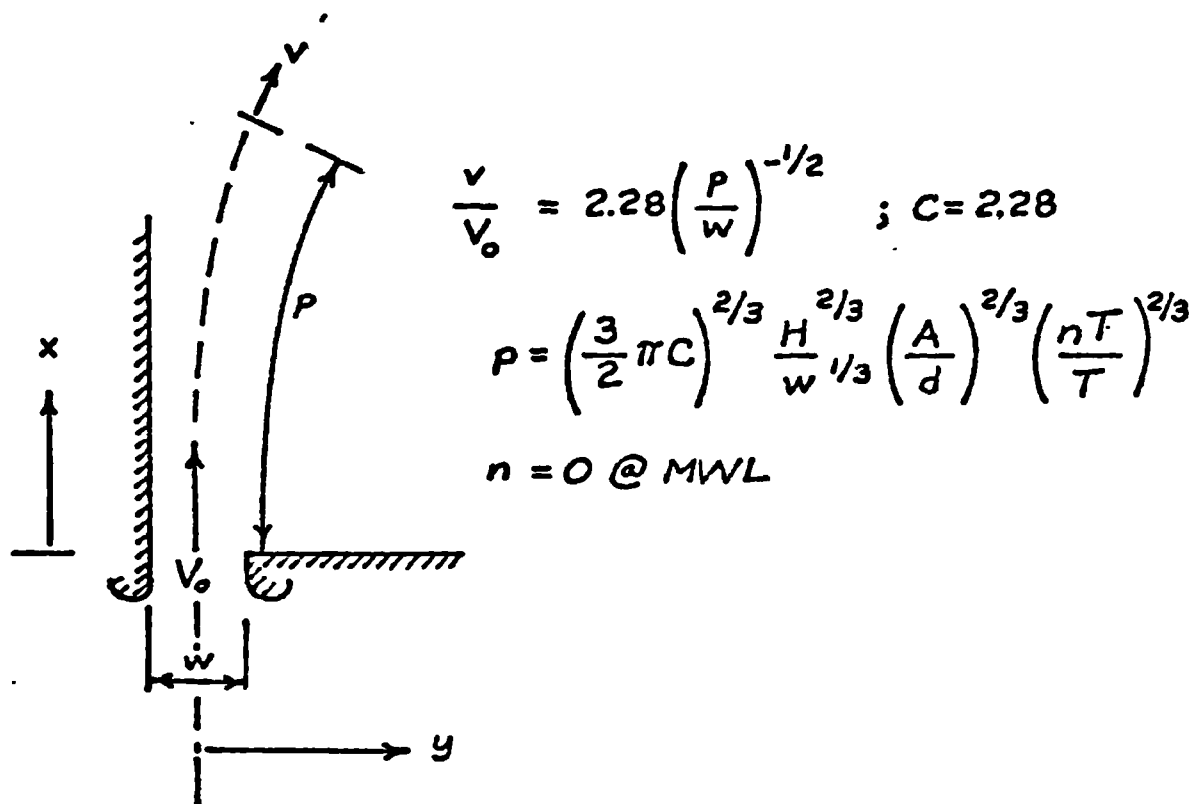


FIGURE 9. Definition diagram, jet penetration analysis.

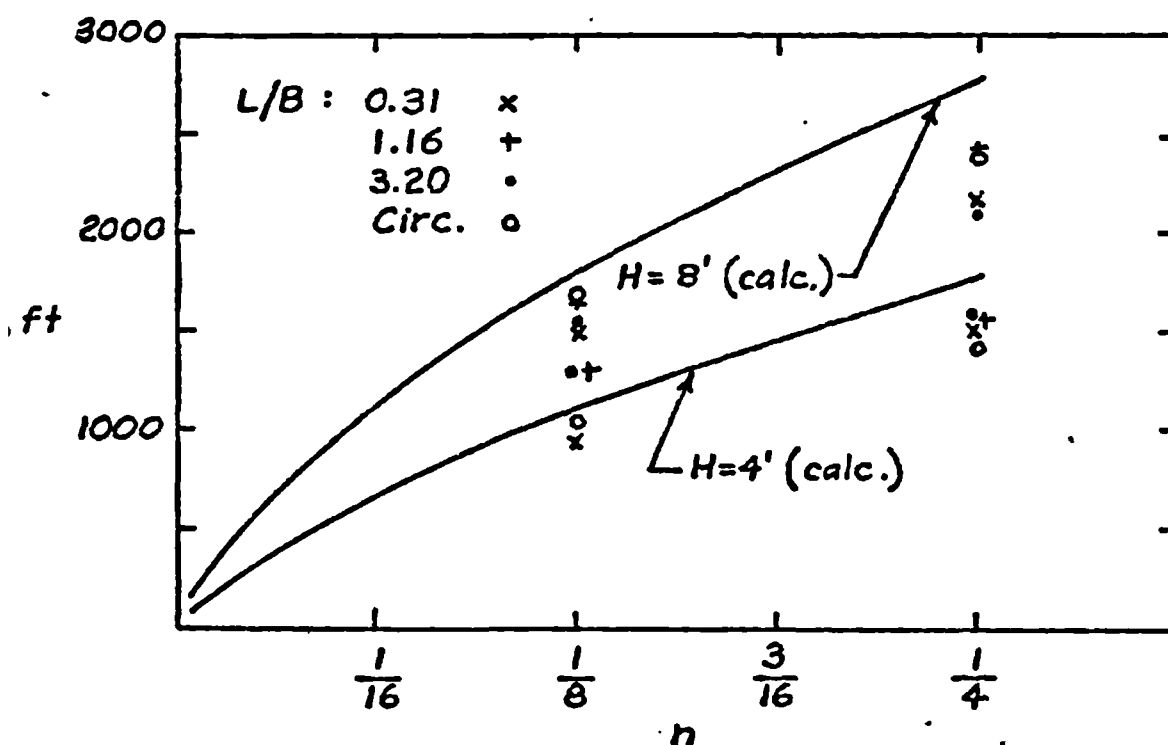


FIGURE 10. Jet penetration measurement data.

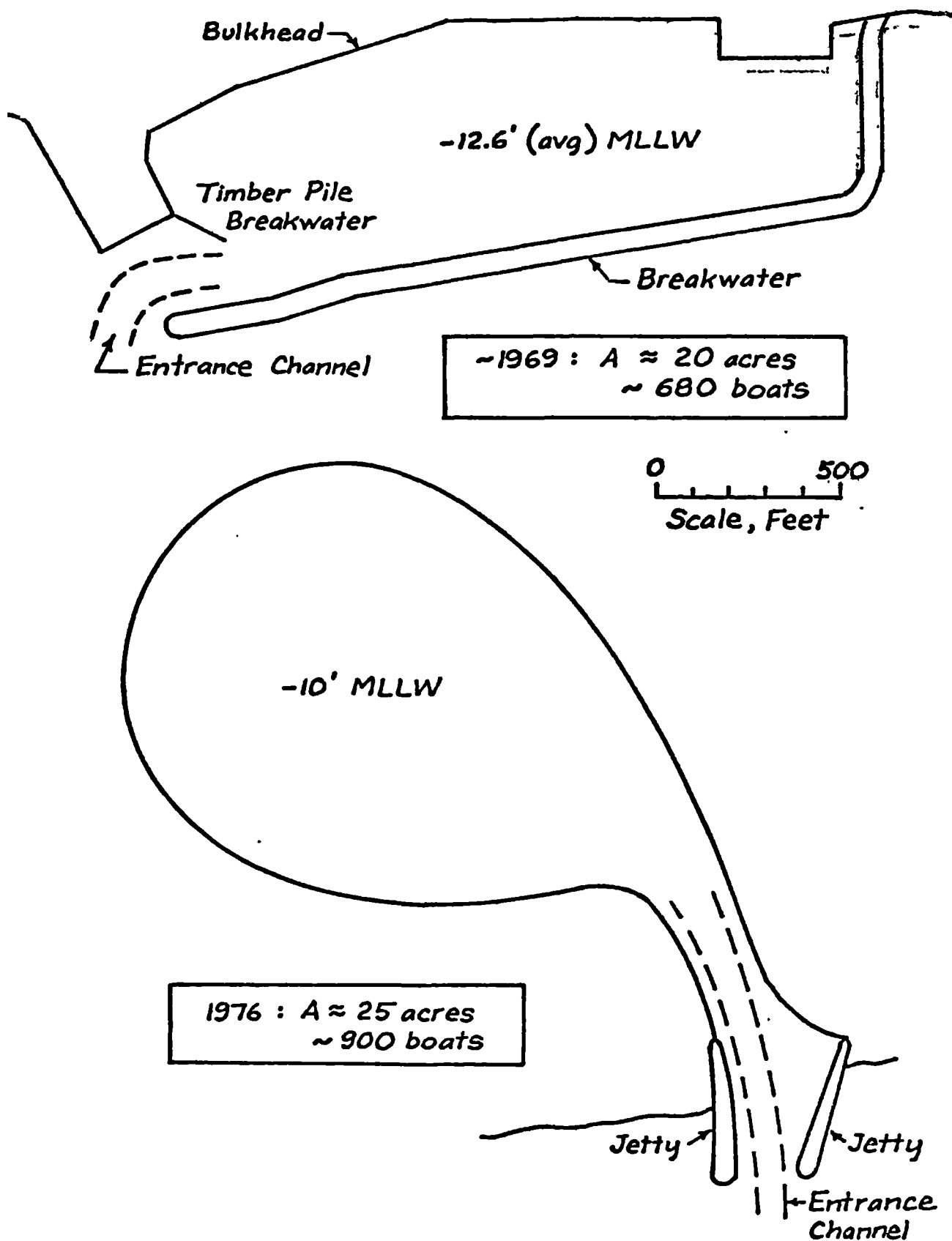


FIGURE 11. Changes in marina planform design to achieve improved internal circulation.