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OL-NE for LQ differential games: A Port-Controlled Hamiltonian system perspective and some computational strategies[☆]

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ABSTRACT

Linear Quadratic differential games and their Open-Loop Nash Equilibrium (OL-NE) strategies are studied with a threefold objective. First, it is shown that the state/costate lifted system (arising from the application of Pontryagin's Minimum Principle) is such that its behaviour restricted to the *equilibrium subspace* can be interpreted as the (non-power-preserving) interconnection of two cyclo-passive Port-Controlled Hamiltonian systems. Such PCH systems constitute the *best response* generators for each player, thus mimicking and extending the corresponding interpretation of (single-player) optimal control problems. Second, by realizing that the behaviour of the lifted dynamics *off* the equilibrium subspace is "irrelevant" for generating the equilibrium strategies, it is shown that such an invariant subspace can be rendered, via a suitably constructed virtual input, externally asymptotically stable while preserving the OL-NE. Finally, based on these premises we provide a closed-form gradient-descent method to solve the asymmetric coupled Riccati equations characterizing the OL-NE strategies.

1. Introduction

Differential games provide an ideal framework to model situations in which several decision-makers, referred to as *players*, influence the evolution of a dynamical system. Such problems have played, following the pioneering works of Isaacs (1999), Nash (1950, 1951) and Starr and Ho (1969), a crucial role in economics, military and defence, and management (see *e.g.* Dockner et al., 2000; Isaacs, 1999 and references therein). Differential games, which can be viewed as a multi-player generalization of (single-player) optimal control problems, also naturally appear across the field of control engineering, for instance, in the context of robust control, cyber–physical systems, robotics and power systems (see, for example, (Başar & Bernhard, 2008; Cappello et al., 2021; Li et al., 2015; Mylvaganam et al., 2017; Saad et al., 2012) and references therein). Despite the close relation to optimal control, the multi-player setting of differential games leads to several intricacies that are entirely absent in the former. These intricacies become immediately apparent even in the two-player case (*i.e.* as soon as the number of decision makers exceeds one). For instance, several solution concepts exist in the context of differential games (see, *e.g.* Basar & Olsder, 1982). The most common solution concept is that of Nash equilibrium strategies, that can be further categorized into open-loop strategies (OL-NE) and feedback strategies, depending on the information available to each player.

We consider linear quadratic (LQ) differential games and their OL-NE solutions. Such solutions are associated with a certain invariant submanifold (that we refer to as the *equilibrium subspace*) of the state/costate lifted dynamics, which are obtained via Pontryagin's Minimum Principle, and that is further characterized by the solution of a set of (asymmetric) coupled Algebraic Riccati equations (AREs). Differently from the case of the ARE arising in the context of optimal control and H_∞-control (see, *e.g.* Zhou et al., 1996), obtaining solutions of the coupled AREs remains a challenging task in general (see, *e.g.* Engwerda, 2005 and references therein).

The main contributions of this paper are threefold. First, we demonstrate that the OL-NE strategies, which can be associated

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with a certain invariant subspace of the state/costate dynamics (referred to as the equilibrium subspace), can be interpreted as an interconnection of two cyclo-passive Port-Controlled Hamiltonian (PCH) systems. The PCH systems can be interpreted as the individual *best response* generators. This interpretation, while mimicking the corresponding optimal control interpretation, highlights also the differences that occur due to the multi-player setting, *e.g.* the interconnection is not power-preserving. Second, while the equilibrium subspace is externally unstable, we demonstrate that the dynamics *off* the equilibrium subspace can be modified – via a *virtual control input* – to render the equilibrium subspace externally asymptotically stable, without modifying the dynamics *on* the equilibrium subspace. It is worth observing that the two previous aspects are tightly connected, although being somewhat complementary. In fact, while it has been well established that OL-NE solutions are intimately related to certain state/costate dynamics restricted on invariant subspaces, herein knowledge is expanded along two directions, in terms of abstract analysis as well as effective synthesis. On one hand, much deeper insights on the nature of the restricted dynamics are provided, which may prove instrumental for understanding the interaction of competing players in terms of energy exchange. On the other hand, recognizing that the behaviour outside the equilibrium subspace is irrelevant for generating OL-NE solutions, it is shown that such behaviour can be arbitrarily *shaped*. Such a design has two far-reaching consequences. In fact, this not only enables a (numerically) robust implementation of the OL-NE strategies, but also enables the final contribution of the paper, which is a closed-form gradient-descent method to solve the asymmetric coupled AREs that characterize OL-NE solutions. Finally, note that while we present these computational strategies in the two-player setting, the results can be readily extended to the N-player case at the cost of more cumbersome notation.

The first two contributions of the paper builds on the preliminary results presented in Sassano et al. (2021), in which it has been observed that a specially structured class of LQ differential games (satisfying stringent structural assumptions) can be analysed in terms of a certain interconnection of two PCH systems. Therein, it is also demonstrated that given a solution of the asymmetric coupled AREs associated with the considered game, the implementation of the corresponding OL-NE solution could be made robust via the addition of a virtual control input (shaping the dynamics of the costates). However, such solutions are, in general, difficult to obtain (see, *e.g.* Engwerda, 2005). Herein, we provide results for *general* LQ differential games (eliminating the limiting structural assumptions of Sassano et al., 2021) and we also provide a (computationally efficient) strategy to bypass the latter challenge. In the context of optimal control it has been demonstrated that the relationship between the value function (stemming from Dynamic Programming arguments) and the costate (stemming from Pontryagin's Minimum Principle) can be exploited to modify the dynamics of the costate in a manner similar in spirit to what is proposed in this paper (see also Sassano & Astolfi, 2020; Sassano et al., 2021, 2023). However, a major difference arises in the current game context due to the fact that there is no direct relationship between the value functions that relate to feedback Nash equilibrium strategies and the costate variables, that arise in the context of OL-NE solutions.

The remainder of the paper is organized as follows. Some preliminaries are provided in Section 2. The three main contributions of the paper, *i.e.* the interpretation of the game as an interconnection of PCH systems, the external stabilization of the equilibrium subspace, and a strategy to compute solutions of the asymmetric coupled AREs, are presented in the following three sections. An illustrative example is provided in Section 6. Finally, we provide some concluding remarks in Section 7.

Notation: The set of real numbers is denoted by $\mathbb{R}$. $\mathbb{C}^-$ and $\mathbb{C}^+$ denote the sets $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0\}$ and $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$, respectively. Given a square matrix A, its spectrum is denoted by $\sigma(A)$. Moreover, $\bar{\sigma}(A)$ and $\underline{\sigma}(A)$ describe its maximal and minimal singular values, respectively. Given a full column rank, non-square, matrix $B \in \mathbb{R}^{n\times n}$, the matrix $B^\dagger \in \mathbb{R}^{m\times n}$ denotes its left pseudo-inverse, namely, it is such that $B^\dagger B = I_m$, and it is defined as $B^\dagger = (B^\top B)^{-1}B^\top$. Given matrices $A_1, \dots, A_N$, $\mathrm{blkdiag}(A_1, \dots, A_N)$ denotes a block diagonal matrix with $A_1, \dots, A_N$ aligned along the diagonal. Given a matrix M, $\mathrm{im}(M)$ denotes its image. Given vectors $x_1 \in \mathbb{R}^{n_1}$, ..., $x_N \in \mathbb{R}^{n_N}$, whenever it does not create confusion the shorthand notation $(x_1, \dots, x_N)$ describes the stack vector $[x_1^\top, \dots, x_N^\top]^\top$.

2. Preliminaries and problem statement

Consider the class of LQ differential games in which two *non-cooperating players* seek to minimize individual cost functionals described by

$$J_i(u_1, u_2, x_0) = \frac{1}{2}\int_0^\infty (\|x(\tau)\|^2_{Q_i} + \|u_i(\tau)\|^2_{R_i})d\tau\,, \tag{1}$$

with $Q_i = Q_i^\top \succeq 0$, $R_i = R_i^\top \succ 0$, for $i = 1, 2$, by influencing the behaviour of a *shared resource* governed by the differential equation

$$\dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t)\,, \qquad x(0) = x_0\,, \tag{2}$$

with $x : \mathbb{R} \to \mathbb{R}^n$ and $u_i : \mathbb{R} \to \mathbb{R}^m$, $i = 1, 2$, describing the state of the plant and the control actions available to each player, respectively. To avoid trivial cases it is assumed that the matrices B_i are full column rank, for $i = 1, 2$. The competitive nature of the problem emerges due to the dynamics (2) since, while each player seeks to minimize its individual cost functional via the selection of its control law, its outcome depends also on the strategy selected by the opponent via the time evolution of (2). While several notions of solution exist (see, *e.g.* Basar & Olsder, 1982) we consider here the concept of Open-Loop Nash Equilibrium (OL-NE) strategy, as detailed below.

Definition 1. Consider the LQ differential game described by the cost functionals (1), $i = 1, 2$, and the dynamics (2). Let $x_0 \in \mathbb{R}^n$ be given. A pair of strategies $(u_1^\star, u_2^\star)$ is a OL-NE for the game if the inequalities

$$J_1(u_1^\star, u_2^\star, x_0) \leqslant J_1(u_1, u_2^\star, x_0) \tag{3a}$$

$$J_2(u_1^\star, u_2^\star, x_0) \leqslant J_2(u_1^\star, u_2, x_0) \tag{3b}$$

hold for any pair $(u_1, u_2^\star) \in \mathcal{U}(x_0) := \{(u_1, u_2) \in \mathcal{L}_2[0, \infty) \times \mathcal{L}_2[0, \infty) : J_i(u_1, u_2, x_0), i = 1, 2, \text{ exist for all } t \in \mathbb{R}_{>0}, \text{ and } \lim_{t\to\infty} x(t) = 0\}$ in the case of (3a) and for any $(u_1^\star, u_2) \in \mathcal{U}(x_0)$ in the case of (3b). ∘

Remark 1. A brief review of classical results concerning (single-player) optimal control problems is needed at this stage to put the following derivations into perspective. Thus, consider a dynamic optimization problem that consists in minimizing the cost functional

$$J(u, x_0) = \frac{1}{2}\int_0^\infty (\|x(\tau)\|^2_Q + \|u(\tau)\|^2_R)d\tau\,, \tag{4}$$

with $Q = Q^\top \succeq 0$, $R = R^\top \succ 0$, together with the dynamic constraint

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0\,. \tag{5}$$

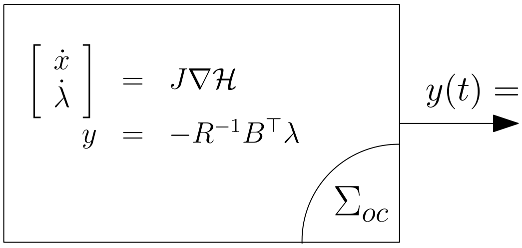

Fig. 1. Schematic representation of the optimal strategy in single-player dynamic optimization problems.

The optimal control problem is associated with a (minimized) Hamiltonian function $\mathcal{H} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$\mathcal{H}(x, \lambda) = \frac{1}{2}x^\top Qx + \lambda^\top Ax - \frac{1}{2}\lambda^\top BR^{-1}B^\top\lambda \,. \tag{6}$$

Provided that the pairs (A, B) and (A, Q) are reachable and detectable, respectively, the unique optimal control strategy $u^\star$ is produced as the output

$$u^\star(t) = -R^{-1}B^\top\lambda^\star(t)\,, \tag{7}$$

of the *Hamiltonian* system Σ_{oc}

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix} = J\nabla\mathcal{H}(x, \lambda)\,, \tag{8}$$

restricted to the (invariant) subspace $\mathcal{V} := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n : \lambda - Px = 0\}$, with $P = P^\top \succeq 0$ satisfying the ARE

$$0 = Q + A^\top P + PA - PBR^{-1}B^\top P\,. \tag{9}$$

Therefore, in (single-player) optimal control problems the behaviour over time of the *decision-maker* can be interpreted (see also the schematic in Fig. 1) as the output of an *autonomous Hamiltonian system* with the canonical skew-symmetric interconnection matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}, \tag{10}$$

and with the Hamiltonian function (6). ▲

The assumption below, which is stipulated to hold throughout the rest of the paper, ensures that each player is capable of minimizing the individual cost functional (1) in the absence of the opponent, similarly to the discussion in Remark 1, and it is further instrumental towards the computation of a OL-NE according to Definition 1.

Assumption 1. The pairs (A, B_i) and (A, Q_i) are reachable and observable, respectively, for $i = 1, 2$. ◦

It has been shown (see, *e.g.*, Engwerda, 2005 for a comprehensive discussion and further insight) that the existence and the structure of OL-NE strategies rely upon the so-called (asymmetric) coupled AREs

$$\begin{aligned} 0 &= Q_1 + P_1A + A^\top P_1 - P_1S_1P_1 - P_1S_2P_2 =: \mathcal{R}_1(P), \\ 0 &= Q_2 + P_2A + A^\top P_2 - P_2S_2P_2 - P_2S_1P_1 =: \mathcal{R}_2(P), \end{aligned} \tag{11}$$

where $S_i := B_iR_i^{-1}B_i^\top$, for $i = 1, 2$, in the variables $P_i \in \mathbb{R}^{n\times n}$, $i = 1, 2$. Note that, differently from (9), since $\mathcal{R}_i(P)$, $i = 1, 2$, are not symmetric, the matrices P_i are not required to be symmetric. The existence of a pair of stabilizing solutions $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ of (11), namely with the property that $\sigma(A - S_1P_1 - S_2P_2) \subset \mathbb{C}^-$, implies, together with Assumption 1, that the LQ differential game defined by (1), (2) admit OL-NE strategies for every initial condition $x_0 \in \mathbb{R}^n$, see Engwerda (2005, Thm. 7.13). The following structural assumption, borrowed from (Engwerda, 2005), characterizes the class of two-player LQ differential games that possess a *unique* pair of OL-NE strategies, admitting a feedback synthesis, for every initial condition $x_0 \in \mathbb{R}^n$. To this end, let the matrix H be given by

$$H = \left[\begin{array}{c|cc} A & -S_1 & -S_2 \\ \hline -Q_1 & -A^\top & 0 \\ -Q_2 & 0 & -A^\top \end{array}\right]. \tag{12}$$

Assumption 2. The matrix H in (12) possesses n eigenvalues, counted with their algebraic multiplicity, in $\mathbb{C}^-$ and $2n$ eigenvalues, counted with their algebraic multiplicities, in $\mathbb{C}^+$. Moreover, letting $\mathcal{V}^-$ denote the n-dimensional stable invariant subspace of H, the subspaces $\mathcal{V}^-$ and

$$\mathrm{im}\left(\begin{bmatrix} 0 & 0 \\ I_n & 0 \\ 0 & I_n \end{bmatrix}\right)$$

are complementary. ◦

As a consequence, provided Assumptions 1 and 2 hold, the LQ differential game defined by (1), (2) admits a unique pair of OL-NE strategies for every $x_0 \in \mathbb{R}^n$ described by the outputs

$$u_i^\star(t) = -R_i^{-1}B_i^\top\lambda_i^\star(t)\,, \tag{13}$$

$i = 1, 2$, of the *extended state/costate system* Σ

$$\dot{x} = Ax - B_1R_1^{-1}B_1^\top\lambda_1 - B_2R_2^{-1}B_2^\top\lambda_2\,, \tag{14a}$$
$$\dot{\lambda}_1 = -Q_1x - A^\top\lambda_1\,, \tag{14b}$$
$$\dot{\lambda}_2 = -Q_2x - A^\top\lambda_2\,, \tag{14c}$$

initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1x_0, P_2x_0)$, where the matrices $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ denote the unique strongly stabilizing solutions of (11), see Engwerda (2005, Thm. 7.10), namely with the property that the eigenvalues of the closed-loop system (14a) with $\lambda_i = P_ix$ coincide with the n stable ones of H (see Assumption 2).

While the proofs of the claims mentioned above are not reported due to reason of space (the interested reader is referred to Engwerda, 2005), it is worth mentioning that the system (14) is derived by initially considering the individual, classical, *preminimized* Hamiltonian functions $\mathcal{H}_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{2m}$, $i = 1, 2$, associated to each player according to

$$\mathcal{H}_i(x, \lambda_i, u) = \frac{1}{2}x^\top Q_ix + \frac{1}{2}u^\top\bar{R}_iu + \lambda_i^\top Ax + \lambda_i^\top\bar{B}u\,, \tag{15}$$

with $u = (u_1, u_2)$, $\bar{R}_1 = \mathrm{blkdiag}(R_1, 0)$, $\bar{R}_2 = \mathrm{blkdiag}(0, R_2)$, and $\bar{B} = [B_1, B_2]$. It is then not surprising that the dynamics (14a), (14a) and (14b), (14c) separately *resemble* the familiar Hamiltonian structure that underlies the solution to an open-loop optimal control problem recalled in Remark 1 (see (8) and, for instance, Liberzon, 2011 for more details). However, the peculiar nature of the minimization process associated with the computation of the Nash equilibrium strategy, namely

$$\begin{aligned} u_1^\star &= \arg\min_v \mathcal{H}_1(x, \lambda_1, (v, u_2^\star))\,, \\ u_2^\star &= \arg\min_v \mathcal{H}_2(x, \lambda_2, (u_1^\star, v))\,, \end{aligned} \tag{16}$$

in which each player minimizes its own Hamiltonian function *in the presence* of the action of the opponent, is such that neither the classic Hamiltonian structure of the overall system (14), nor some of the particularly appealing properties resulting from such a structure (*e.g.* the split spectrum of its eigenvalues, see Zhou et al. (1996) for more details) are preserved. It is therefore worth establishing whether also the overall system (14) possesses a certain Hamiltonian interpretation of its dynamics, with a specific emphasis on the perspective of each individual player and their interconnected behaviour.

Finally, it is worth observing that a consequence of Assumption 2 is that the matrix H possesses an n-dimensional stable graph subspace. Note, however, that Assumption 2 is introduced mainly for ease of presentation and it may be relaxed (by assuming instead that H possess *at least* n eigenvalues in $\mathbb{C}^-$) provided the following statements are adapted to hold for *any* admissible OL-NE of the underlying game, which may then admit multiple Nash equilibrium strategies.

3. OL-NE strategies as PCH systems

The main purpose of this section consists in showing that, in the class of LQ differential games characterized by (1), $i = 1, 2$, and (2), the *best response* over time of each player to the action of the opponent is generated as the output of a PCH system. Therefore, as a consequence, *any* pair of OL-NE strategies can be interpreted as the output of the interconnection of two PCH systems, one for each player, restricted to a specific invariant subspace.

To provide a concise statement of the following result, consider the family of (virtual) *energy functions*

$$\hat{\mathcal{H}}_i(x_i, \lambda_i) = -\lambda_i^\top B_i R_i^{-1} B_j^\dagger x_i + \frac{\alpha_i}{2}\|\lambda_i\|^2 , \tag{17}$$

parameterized with respect to $\alpha_i \in \mathbb{R}_{>0}$ and with $(x_i, \lambda_i) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, 2$. Consider then the PCH system associated to the *canonical* interconnection matrix J defined in (10), namely

$$\Sigma_i := \left\{ \begin{array}{rcl} \dot{z}_i &=& J\nabla\hat{\mathcal{H}}_i(z_i) + G_i v_i , \\ y_i &=& G_i^\top \nabla\hat{\mathcal{H}}_i(z_i) , \end{array} \right. \tag{18}$$

with state $z_i = (x_i, \lambda_i)$, input $v_i \in \mathbb{R}^{m+2n}$ and output $y_i \in \mathbb{R}^{m+2n}$, and with the matrix $G_i \in \mathbb{R}^{2n\times(m+2n)}$ defined as

$$G_i := \left[\begin{array}{c|cc} B_j & I_n & 0 \\ 0 & 0 & I_n \end{array} \right] , \tag{19}$$

$i = 1, 2$, $j = 1, 2$, $j \neq i$. The partition of the matrix G_i in (19), inherited (via the transposition of the matrix G_i) by the corresponding output y_i in (18), naturally induces a similar partition on y_i, thus distinguishing between a set of outputs, denoted by $y_{e,i} \in \mathbb{R}^m$, which are shown below to be intimately related to the *equilibrium strategy*, and a set of outputs, denoted by $y_{g,i} \in \mathbb{R}^{2n}$, which are instead linked to the contribution of each player on the interaction that defines the underlying game. Finally, define the *interconnection structure* $\mathcal{I}_c$, which prescribes the pattern of *energy flow* among the ports of the PCH systems, according to

$$\left[\begin{array}{c} v_1 \\ \hline v_2 \end{array} \right] = \left[\begin{array}{cc|cc} 0 & 0 & I_m & 0 \\ 0 & \Gamma_1 & 0 & 0 \\ \hline I_m & 0 & 0 & 0 \\ 0 & 0 & 0 & \Gamma_2 \end{array} \right] \left[\begin{array}{c} y_{e,1} \\ y_{g,1} \\ \hline y_{e,2} \\ y_{g,2} \end{array} \right] =: \mathcal{I}_c \left[\begin{array}{c} y_1 \\ \hline y_2 \end{array} \right] , \tag{20}$$

with the matrices $\Gamma_i \in \mathbb{R}^{2n\times 2n}$ specified in the statement below. In the following result, the PCH systems (18) are employed to interpret the (collective and individual) behaviour of the players in LQ differential games.

Theorem 1. *Consider solutions P_i, $i = 1, 2$, to the asymmetric coupled AREs (11) and suppose that P_i, $i = 1, 2$, are nonsingular. Let*

$$\alpha_i^\star := \frac{\bar{\sigma}(B_i R_i^{-1} B_j^\dagger)}{\underline{\sigma}(P_i)} , \tag{21}$$

and consider the feedback interconnection $\bar{\Sigma}$ of Σ_1 and Σ_2 in (18) according to the interconnection in (20), with

$$\Gamma_i = -J + \left[\begin{array}{c} A - B_i R_i^{-1} B_i^\top P_i \\ -Q_i - A^\top P_i \end{array} \right] \left[\begin{array}{c} (-B_i R_i^{-1} B_j^\dagger)^\top P_i \\ -B_i R_i^{-1} B_j^\dagger + \alpha_i P_i \end{array} \right]^\dagger \tag{22}$$

for all $\alpha_i > \alpha_i^\star$, $i = 1, 2$. Then

(i) *the subspace $\mathcal{M} := \{(z_1, z_2) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} : x_1 = x_2, \lambda_1 = P_1 x_1, \lambda_2 = P_2 x_2\}$ is invariant for $\bar{\Sigma}$, for all $x_1 \in \mathbb{R}^n$ and $x_2 \in \mathbb{R}^n$;*

(ii) *the outputs $y_{e,i}$ of the interconnected system $\bar{\Sigma}$, initialized at $(x_i(0), \lambda_i(0)) = (x_0, P_i x_0)$ are such that $(z_1(t), z_2(t)) \in \mathcal{M}$ and $y_{e,i}(t) = u_i^\star(t)$ for all $t \geq 0$.* ◇

Proof. Before commencing on the proof note that

$$\nabla\hat{\mathcal{H}}_i(z_i) = \left[\begin{array}{cc} 0 & -(B_j^\dagger)^\top R_i^{-1} B_i^\top \\ -B_i R_i^{-1} B_j^\dagger & \alpha_i I \end{array} \right] \left[\begin{array}{c} x_i \\ \lambda_i \end{array} \right] \tag{23}$$

and hence, by recalling the partition of the input matrix G_i in (19), the output $y_{e,i}$ is such that

$$\begin{aligned} y_{e,i} = \left[\begin{array}{cc} B_j^\top & 0 \end{array} \right] \nabla\hat{\mathcal{H}}_i(z_i) &= -B_j^\top (B_j^\dagger)^\top R_i^{-1} B_i^\top \lambda_i \\ &= -R_i^{-1} B_i^\top \lambda_i , \end{aligned} \tag{24}$$

for all $(x_i, \lambda_i) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, 2$. To demonstrate the first claim, consider the PCH system Σ_1 defined by (18) with $i = 1$. By the structure of G_i in (19) and the underlying interconnection structure $\mathcal{I}_c$ defined in (20), one has that

$$\begin{aligned} G_1 v_1 &= \left[\begin{array}{c} B_2 \\ 0 \end{array} \right] y_{e,2} + \Gamma_1 y_{g,1} \\ &= \left[\begin{array}{c} -B_2 R_2^{-1} B_2^\top \lambda_2 \\ 0 \end{array} \right] + \Gamma_1 \nabla\hat{\mathcal{H}}_1(z_1) \end{aligned} \tag{25}$$

where the last equality follows from (24) with $i = 2$ and by noting that $y_{g,1} = \nabla\hat{\mathcal{H}}_1(z_1)$. Therefore, the interconnected system Σ_1 becomes, for the first player,

$$\left[\begin{array}{c} \dot{x}_1 \\ \dot{\lambda}_1 \end{array} \right] = (J + \Gamma_1)\nabla\hat{\mathcal{H}}_1(z_1) + \left[\begin{array}{c} -B_2 R_2^{-1} B_2^\top \\ 0 \end{array} \right] \lambda_2 . \tag{26}$$

The restriction of the first term on the right-hand side to the subspace $\mathcal{M}$, characterized by the condition $\lambda_1 = P_1 x_1$, yields

$$(J + \Gamma_1) \left[\begin{array}{c} -(B_2^\dagger)^\top R_1^{-1} B_1^\top P_1 \\ -B_1 R_1^{-1} B_2^\dagger + \alpha_1 P_1 \end{array} \right] = \left[\begin{array}{c} A - B_1 R_1^{-1} B_1^\top P_1 \\ -Q_1 - A^\top P_1 \end{array} \right] \tag{27}$$

with the equality obtained by the construction of Γ_1 in (22), which is well defined since the condition $\alpha_1 > \alpha_1^\star$, with the latter provided in (21), ensures that the matrix right-multiplying $(J+\Gamma_1)$ has full column rank. Expressions similar to (25)–(27) can be immediately obtained also for the second player, namely with $i = 2$, $j = 1$. Thus, by replacing (27) into (26), the PCH systems Σ_i are such that

$$\left.\frac{d}{dt}(x_i(t))\right|_{\mathcal{M}} = (A - B_1 R_1^{-1} B_1^\top P_1 - B_2 R_2^{-1} B_2^\top P_2) x_i(t) , \tag{28}$$

for $i = 1, 2$, and hence

$$\left.\frac{d}{dt}(x_1(t) - x_2(t))\right|_{\mathcal{M}} = 0 . \tag{29}$$

Furthermore, from (26) and (27) it is also straightforward to see that

$$\left.\frac{d}{dt}(\lambda_i(t))\right|_{\mathcal{M}} = (-Q_i - A^\top P_i) x_i(t) ,$$

and hence

$$\left.\frac{d}{dt}(\lambda_i(t) - P_i x_i(t))\right|_{\mathcal{M}} = 0 , \tag{30}$$

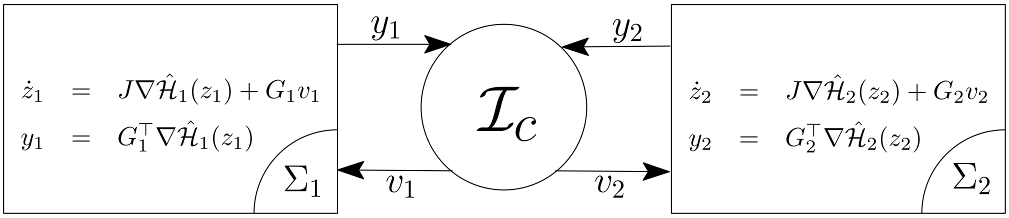

Fig. 2. Schematic representation of the open-loop Nash equilibrium strategy in two-player dynamic optimization problems.

for $i = 1, 2$, by (11) and (28). It follows from (29) and (30) that $\mathcal{M}$ is an invariant subspace. The second claim is demonstrated by noting that the initial condition $(z_1(0), z_2(0)) \in \mathcal{M}$ and thus, by invariance of the subspace $\mathcal{M}$, it follows that $(z_1, z_2) \in \mathcal{M}$ for all $t \geq 0$. Recall that the OL-NE strategies are given by (13), for $i = 1, 2$, with $\lambda_1^\star$ and $\lambda_2^\star$ the trajectories of the costates evolving according to the dynamics (14b) and (14c), respectively, initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1x_0, P_2x_0)$. On the subspace $\mathcal{M}$, the dynamics of the interconnected system are equivalent to the dynamics given in (14). Since the initial condition is such that $x_i(0) = x_0$ and $\lambda_i(0) = P_ix_0$ and since $z_i \in \mathcal{M}$ for all $t \geq 0$, the final part of the claim follows by (24). □

Upon inspection, the definition of $\alpha_i^\star$ in (21) unveils the following intuitive observation: the relevance of the squared norm of the costate variable in the virtual energy function (17) is *inversely proportional* to the cost (outcome) incurred by each player. In other words, the higher the cost for a player (*i.e.* large $\underline{\sigma}(P_i)$), the smaller the value of α_i is allowed to be in the corresponding energy function (17).

Remark 2. As schematically illustrated by Fig. 1, in (single-player) optimal control problems the behaviour of the decision-making agent (*best action*) can be interpreted and explained as the output of an autonomous Hamiltonian system with the canonical skew-symmetric interconnection matrix J in (10), together with the (minimized) Hamiltonian function (6). On the other hand, extending similar ideas to the context of multi-player dynamic *optimization* problems, the statement of Theorem 1 entails that, instead, the *individual behaviour* of each player in differential games (*best response*) is governed by the output of a PCH system generated by the (virtual) energy function $\hat{\mathcal{H}}_i$ in (17) that captures the interplay between the control authorities of each player, as shown in Fig. 2. Note that v_i summarizes the interaction of player i with the other player in the game and, in this perspective, it is not surprising that the *generator* of the most desirable behaviour for the player i, namely Σ_i, is a *non-autonomous* system capable of reacting to the interaction with the opponent. ▲

Remark 3. The interconnection pattern $\mathcal{I}_c$ defined in (20) does *not* constitute a Dirac structure, hence it is not *power-preserving*. As a consequence the energy interaction of players within a differential game may be characterized by outflow or inflow of energy from the environment. Therefore, unlike optimal control in which the (minimized) Hamiltonian function is preserved along optimal trajectories, in the case of multi-player optimization problems the (virtual) energy $\hat{\mathcal{H}}_i(x_0, P_ix_0)$ initially possessed by each player is not preserved. Finally, it is worth observing that the *full-authority* requirement for the players on the differential game, needed in Sassano et al. (2021), is removed in Theorem 1 by relying on two main factors. On one hand, the virtual energy function $\hat{\mathcal{H}}_i$ is allowed to contain also the (weighted) squared norm of the costate variable. On the other hand, and more importantly, the individual PCH systems for each players are interconnected via the more general structure $\mathcal{I}_c$ in (20) (compared to the standard positive feedback of Sassano et al. (2021, Thm. 1)). ▲

Remark 4. To shed some light on the invertibility assumption on the solutions P_i, $i = 1, 2$, of the asymmetric coupled AREs (11), a necessary condition for *singularity* is discussed here. To this end, suppose that there exists an integer $i \in \{1, 2\}$ with the property that $\ker(P_i) \neq \{0\}$, namely P_i possesses an eigenvalue $\lambda = 0$. Let then $w \in \mathbb{R}^n$, $w \neq 0$ and $v \in \mathbb{R}^n$, $v \neq 0$, denote left and right, respectively, eigenvectors of P_i associated to $\lambda = 0$, namely $w^\top P_i = 0$ and $P_iv = 0$. Then, by left and right multiplying the ith equation of (11) by $w^\top$ and v, respectively, one obtains

$$\begin{aligned} 0 &= w^\top Q_i v + w^\top (P_iA + A^\top P_i - P_iS_iP_i - P_iS_jP_j)v \\ &= w^\top Q_i v = \sum_{k=1}^n \lambda_k(Q_i) w^\top \varpi_k \varpi_k^\top v \\ &= \sum_{k=1}^n \lambda_k(Q_i)\gamma_k\delta_k \,, \end{aligned} \tag{31}$$

with $\lambda_k(Q_i) \geq 0$, in which the last equality is obtained via the *spectral decomposition* of the real symmetric matrix Q_i and by defining the scalar quantities $\gamma_k := w^\top \varpi_k$ and $\varpi_k^\top v$. In conclusion, a necessary condition for P_i to be singular is that the latter sum in (31) is zero. Noting that v and w are functions of P_i (and therefore also of Q_i) it is not straightforward to exclude, in general, that P_i may be singular (in which case (21) is not well-defined). Nevertheless, in many cases P_i is, in fact, non-singular. For instance, in the scalar example considered in Section 6 the invertibility assumption is trivially satisfied. ▲

The rest of this section is devoted to providing an interpretation of the *best response* generator of each player with respect to the action of the opponent, namely the dynamics (18) with $\hat{\mathcal{H}}_i$ defined in (17). Towards this end, the notion of *cyclo-passivity* is discussed herein specifically in the case of dynamics as in (18), since a detailed discussion about this property would be beyond the scope of the paper.[1]

Definition 2. The system (18) is said to be *cyclo-passive* if the dissipation inequality

$$\oint y_i(t)^\top v_i(t)dt \geq 0 \tag{32}$$

holds along the trajectories of (18) for all $T > 0$ and for any input v_i such that $z_i(T) = z_i(0)$. ∘

According to Definition 2 a cyclo-passive system dissipates energy when returning to its initial configuration. This notion enables insightful interpretations of the behaviour of several thermodynamical systems (with *entropy* typically qualifying as a storage function not necessarily bounded from below), as well as electrical networks with inductors. Similarly to the notion of passivity, the property of cyclo-passivity may be ensured by the existence of a *storage function* $\mathcal{S}_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that satisfies,

[1] The interested reader is referred to Willems (1973) and van der Schaft (2020).

provided it is continuously differentiable, an infinitesimal version of the dissipation inequality (32), namely

$$\frac{d}{dt}\mathcal{S}_i(x_i(t), \lambda_i(t)) \leqslant y_i(t)^\top v_i(t)\,, \tag{33}$$

along the "closed" trajectories of (18). Although not particularly evident from (33), the key difference between the notions of passivity and cyclo-passivity consists in the property that in the latter the storage function $\mathcal{S}_i$ is not required to be *bounded from below* (see, *e.g.* van der Schaft, 2020; Willems, 1972) and that $x(T) = x(0)$, for some $T > 0$.

Theorem 2. *Consider the LQ differential game described by the cost functionals (1) and the dynamics (2). Let $x_0 \in \mathbb{R}^n$ be given. The* best response *of each player to the action of the opponent is generated by the output of a* cyclo-passive PCH system. $\diamond$

Proof. The claim follows immediately by noting that the energy function $\hat{\mathcal{H}}_i$ in (17), which is not bounded from below, is a candidate storage function that satisfies the dissipation inequality (32) (in its infinitesimal form) by skew-symmetry of the matrix J in (10). $\square$

Theorem 2 can be seen as a "multi-player counterpart" of the relationship between dissipativity and (single-player) optimal control (see *e.g.* Faulwasser & Kellett, 2021; van der Schaft, 2000). Thus, the observation made in Theorem 2, while interesting in its own right, may be utilized as an instrumental tool, *e.g.* in the context of inverse differential games. Such considerations are, however, beyond the scope of this paper. We now turn our attention back towards OL-NE solutions and towards utilizing the results presented in this section to develop strategies to construct OL-NE solutions for LQ differential games.

4. External stabilization of the equilibrium subspace

The results discussed in Section 3 implicitly entail that, towards the construction of an equilibrium strategy for each player, the behaviour of the dynamics (14) is relevant only restricted to the invariant *equilibrium subspace*

$$\mathcal{M}_\circ := \{(x, \lambda_1, \lambda_2) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \lambda_i = P_i x,\ i = 1, 2\}\,. \tag{34}$$

Furthermore, while $\mathcal{M}_\circ$ is an externally unstable invariant subspace, the behaviour *external* to $\mathcal{M}_\circ$ is *not* related to the equilibrium strategy and, hence, it may be modified, as pursued in the rest of this section, with the aim of rendering $\mathcal{M}_\circ$ externally asymptotically stable.

Lemma 1. *Consider the LQ differential game described by the cost functionals (1), $i = 1, 2$, and the dynamics (2). Let $x_0 \in \mathbb{R}^n$ be given. Suppose that Assumption 2 holds. Then the* equilibrium costate *is given by*

$$\lambda_i^\star(t) = P_i e^{(A - S_1P_1 - S_2P_2)t} x_0\,, \tag{35}$$

for all $t \geqslant 0$, and $i = 1, 2$, with P_1, P_2 solutions of (11). $\circ$

Proof. Let $z := (x, \eta) = (x, \lambda_1, \lambda_2)$ and define the change of coordinates $\hat{z} = (\hat{x}, \hat{\eta}) = T^{-1}z$ with

$$T^{-1} = \begin{bmatrix} I & 0 & 0 \\ -P_1 & I & 0 \\ -P_2 & 0 & I \end{bmatrix}, \quad T = \begin{bmatrix} I & 0 & 0 \\ P_1 & I & 0 \\ P_2 & 0 & I \end{bmatrix}. \tag{36}$$

It is straightforward to show that

$$\begin{aligned} T^{-1}HT &= \left[\begin{array}{c|cc} A_{\rm cl} & -S_1 & -S_2 \\ \hline 0 & -(A - S_1P_1)^\top & P_1S_2 \\ 0 & P_2S_1 & -(A - S_2P_2)^\top \end{array}\right] \\ &:= \left[\begin{array}{c|c} A_{\rm cl} & -S \\ \hline 0 & A_u \end{array}\right], \end{aligned} \tag{37}$$

with $A_{\rm cl} := A - S_1P_1 - S_2P_2$, $S := [S_1\ S_2]$ and $A_{\rm u}$ representing the remaining bottom-right block, where the structural zeros have been obtained by recalling that P_1 and P_2 solve the asymmetric coupled AREs (11). Moreover, $\sigma(A_u) \subset \mathbb{C}^+$ by Assumption 2. Therefore, in the transformed coordinates, one has

$$\begin{aligned} \hat{x}(t) &= e^{A_{\rm cl}t}\left(\hat{x}(0) - \int_0^t e^{-A_{\rm cl}\tau} S e^{A_u\tau} d\tau\, \hat{\eta}(0)\right) \\ &:= e^{A_{\rm cl}t}\left(\hat{x}(0) - M(t)\hat{\eta}(0)\right), \end{aligned} \tag{38a}$$

$$\hat{\eta}(t) = e^{A_u t}\hat{\eta}(0)\,. \tag{38b}$$

By noting that $\lambda_i(t) = \hat{\eta}_i(t) + P_i\hat{x}(t)$, for $i = 1, 2$, since $z = T\hat{z}$ with T as defined in (36), it follows that

$$\lambda_i(t) = P_i e^{A_{\rm cl}t}x_0 + \left(\Pi_i e^{A_u t} - P_i e^{A_{\rm cl}t}M(t)\right)\hat{\eta}(0)\,, \tag{39}$$

with $\Pi_1 = [I_n\ \ 0]$, $\Pi_2 = [0\ \ I_n]$ and $\hat{\eta}(0) = (\lambda_1(0) - P_1x_0, \lambda_2(0) - P_2x_0)$. The proof is then concluded by recalling that $\lambda_i(0) = P_ix_0$, hence $\hat{\eta}(0) = 0$, and $\lambda_i(t)|_{\hat{\eta}_0 = 0} = \lambda^\star(t) = P_i e^{A_{\rm cl}t}x_0$. $\square$

Consider now instead the *controlled state/costate system*

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = H\begin{bmatrix} x \\ \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \nu := Hz + L\nu\,, \tag{40}$$

where $\nu : \mathbb{R} \to \mathbb{R}^{2n}$, $\nu = (\nu_1, \nu_2)$, denotes a *virtual* control input that allows to *steer* the costate dynamics, as discussed below.

Lemma 2. *Consider the* controlled state/costate system *(40) in closed loop with*

$$\nu_1 = (A^\top - P_1S_1 + F_1)(\lambda_1 - P_1x) - P_1S_2(\lambda_2 - P_2x) \tag{41a}$$

$$\nu_2 = (A^\top - P_2S_2 + F_2)(\lambda_2 - P_2x) - P_2S_1(\lambda_1 - P_1x) \tag{41b}$$

with $F_i \in \mathbb{R}^{n\times n}$, $i = 1, 2$, arbitrary Hurwitz matrices and P_1, P_2 solutions of (11). Then the controlled costate *is given by*

$$\lambda_i^c(t) = e^{F_i t}\xi_i + P_i e^{A_{\rm cl}t}\left(x_0 - \sum_{j=1}^{2} M_j^c(t)\xi_j\right)\,, \tag{42}$$

for $i = 1, 2$, with $\xi_j = \lambda_j(0) - P_jx_0$ and $M_j^c(t) = \int_0^t e^{-A_{\rm cl}\tau}S_j e^{F_j\tau}d\tau$. $\circ$

Proof. The claim is shown by relying on arguments identical to those in the proof of Lemma 1. In fact, note that the control inputs ν_i given in (41) become

$$\nu_i = (A - S_iP_i)^\top\hat{\eta}_i - P_iS_j\hat{\eta}_j + F_i\hat{\eta}_i \tag{43}$$

for $i = 1, 2$, $j = 1, 2$, $j \neq i$, in the transformed coordinates, where $\hat{\eta}$ is defined before (36) and $\hat{\eta} := (\hat{\eta}_1, \hat{\eta}_2)$. Therefore, by recalling the structure of the matrix H in the transformed coordinates, namely (37), and by observing that $T^{-1}L = L$, it is straightforward to note that in the transformed coordinates the system (40) in closed loop with (41) is described by the dynamic matrix (44) (see Box I overleaf). The structure of the solution (42) then follows immediately by mimicking the constructions leading to (38), (39), although with respect to the matrix (44) (see Box I overleaf) in place of (37). $\square$

A few crucial observations regarding the control inputs ν_i, $i = 1, 2$, in (41) are in order. First, the behaviour on the subspace $\mathcal{M}_\circ$ of (14) is preserved in (40), (41), *i.e.* on the subspace $\mathcal{M}_\circ$ the dynamics of (40), (14) and (41) coincide. Second, the costate dynamics, in the transformed coordinates ($\hat{z} = Tz$) and in closed loop, *i.e* (40), (41), are decoupled and asymptotically attracted by $\mathcal{M}_\circ$.

$$\begin{bmatrix} A_{\mathrm{cl}} & -S_1 & -S_2 \\ 0 & -(A-S_1P_1)^\top & P_1S_2 \\ 0 & P_2S_1 & -(A-S_2P_2)^\top \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & (A-S_1P_1)^\top + F_1 & -P_1S_2 \\ 0 & -P_2S_1 & (A-S_2P_2)^\top + F_2 \end{bmatrix} = \begin{bmatrix} A_{\mathrm{cl}} & -S_1 & -S_2 \\ 0 & F_1 & 0 \\ 0 & 0 & F_2 \end{bmatrix} \tag{44}$$

Box I.

For simplicity of presentation of the following result suppose that the matrices F_i, $i = 1, 2$, are selected as $F_1 = F_2 = -\sigma_F I_n$ with $\sigma_F > 0$. Moreover, to provide a concise statement, let $c_1 \in \mathbb{R}_{>0}$ and $c_2 \in \mathbb{R}_{>0}$ be such that $\|e^{-A_{\mathrm{cl}}t}\| < c_1 e^{c_2 t}$ and let $d_1 \in \mathbb{R}_{>0}$ and $d_2 \in \mathbb{R}_{>0}$ be such that $\|e^{A_{\mathrm{cl}}t}\| < d_1 e^{-d_2 t}$.

Proposition 1. *Consider the state/costate dynamics (14), with $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1x_0, P_2x_0)$, together with the* controlled *state/costate dynamics (40) in closed loop with (41). Fix any $\bar{T} \in \mathbb{R}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$ and $\mu \in \mathbb{R}_{>0}$ and let*

$$\sigma_F > \max\left\{\frac{2c_1d_1\mu\beta(\|S_1\|+\|S_2\|)}{\varepsilon} + c_2, \frac{1}{\bar{T}}\log\left(\frac{\varepsilon}{2\mu}\right)\right\},$$

where $\beta = \max\{\|P_1\|, \|P_2\|\}$. Let $e_i(t) := \lambda_i^\star(t) - \lambda_i^c(t)$, $i = 1, 2$. Then

$$\|e_i(t)\| \leqslant \varepsilon\,, \tag{45}$$

for all $t > \bar{T}$ and for all $(x_0, \lambda_1(0), \lambda_2(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\|(x_0, \xi_1, \xi_2)\| < \mu$. ◦

Proof. By the definitions of the functions $\lambda_i^\star$ and λ_i^c given in (35) and (42), respectively, it follows immediately that

$$e_i(t) = P_i e^{A_{\mathrm{cl}}t}\sum_{j=1}^{2} M_j^c(t)\xi_j - e^{F_i t}\xi_i\,, \tag{46}$$

for $i = 1, 2$ and for all $t \geqslant 0$. Consider first the norm of the matrix-valued function M_i^c, namely $t \mapsto \int_0^t e^{-A_{\mathrm{cl}}\tau}S_i e^{F_i\tau}d\tau$. In particular,

$$\begin{aligned}\|M_i^c(t)\| &= \left\|\int_0^t e^{-A_{\mathrm{cl}}\tau}S_ie^{F_i\tau}d\tau\right\| \leqslant \int_0^t \|e^{-A_{\mathrm{cl}}\tau}S_ie^{F_i\tau}\|d\tau \\ &\leqslant \|S_i\|\int_0^t \|e^{-A_{\mathrm{cl}}\tau}\|\,\|e^{F_i\tau}\|d\tau \\ &\leqslant c_1\|S_i\|\int_0^t e^{(c_2-\sigma_F)\tau}d\tau = \frac{c_1\|S_i\|}{c_2-\sigma_F}\left[e^{(c_2-\sigma_F)\tau}\right]_0^t \\ &\leqslant \frac{c_1\|S_i\|}{\sigma_F - c_2}\,,\end{aligned}$$

where the latter inequality holds by the selection of σ_F, since $\sigma_F > c_2$, for $i = 1, 2$. Therefore,

$$\begin{aligned}&\|e_1(t)\| \leqslant \|P_1\|\|e^{A_{\mathrm{cl}}t}\|(\|M_1^c(t)\|\|\xi_1\| + \|M_2^c(t)\|\|\xi_2\|) \\ &+ \|e^{Ft}\|\|\xi\| \leqslant \frac{c_1d_1(\|S_1\|+\|S_2\|)\|P_1\|}{\sigma_F - c_2}\mu + e^{-\sigma_F t}\mu \\ &\leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leqslant \varepsilon\,,\end{aligned}$$

for all $t > \bar{T}$, where the third inequality follows from the selection of σ_F. An identical sequence of inequalities established for e_2 concludes the proof. □

Remark 5. Preliminary results, similar to those provided in this section, have been reported in Sassano et al. (2021), without the complete analysis provided herein. However, it should be noted that due to the novel interpretation provided in Section 3 (via the virtual energy functions (17), $i = 1, 2$), the results provided herein hold for general LQ differential games. This is in stark contrast to the preliminary results provided in Sassano et al. (2021) that require rather limiting structural assumptions (namely that $m = n$ and that $\mathrm{rank}(B_i) = n$, for $i = 1, 2$). ▲

Differently from the computation of a solution of the classic ARE arising in optimal control problems, for which numerically efficient techniques have been developed in the past decades, the systematic solution of the coupled AREs arising in the context of differential games largely remains an open issue and, hence, determining sufficiently accurate solutions to (11) is a challenging task. Furthermore, an inaccurate solution to (11) may have seriously detrimental consequences whenever the state of the plant can be measured by the players only at the initial time, namely in OL-NE strategies. This relevant aspect is discussed in the following remark, which suggests interesting differences between the computation of the costate variables in (35) and in (42). More precisely, the main objective of Remark 6 consists in showing that the architecture based on the controlled state/costate dynamics (40) endows the computation of OL-NE strategies with a certain degree of robustness, *even if approximate or inaccurate solutions to (11) are employed for the constructions* (namely in (41) and for the initial conditions of (40)). Finally, the intuition behind the constructions discussed in the following remark suggests, as a byproduct, a computationally effective strategy to tackle the solution to (11), which is the topic of the subsequent Section 5.

Remark 6. The practical computation of the matrices P_i that solve the AREs (11) are inevitably affected by numerical errors — *i.e.* the matrices $\tilde{P}_i = P_i + \Delta P_i$, $i = 1, 2$, are typically employed in the above architecture in place of P_i. Therefore, the forward integration of (14) initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, \tilde{P}_1x_0, \tilde{P}_2x_0)$ yields the open-loop control laws $u_i(t) = -R_i^{-1}B_i^\top\tilde{\lambda}_i(t)$, with

$$\tilde{\lambda}_i(t) = \lambda_i^\star(t) + (\Pi_i e^{A_u t} - P_i e^{A_{\mathrm{cl}}t}M(t))\begin{bmatrix}\Delta P_1 \\ \Delta P_2\end{bmatrix}x_0,$$

which is such that $\lim_{t\to\infty}\|u_i(t)\| = \infty$, for any $x_0 \in \mathbb{R}^n$, even for arbitrarily small ΔP_i. In fact $\sigma(A_u) \subset \mathbb{C}^+$ by Assumption 2. On the other hand, by employing the solution of the controlled state/costate dynamics (40) in closed loop with $\tilde{\nu}_i = \nu_i|_{P_i=\tilde{P}_i}$, with ν_i as in (41) (although with inaccurate solutions to (11)), one has that $u_i^c(t) = -R_i^{-1}B_i^\top\tilde{\lambda}_i^c(t) = -R_i^{-1}B_i^\top(P_i + \Delta P_i)\tilde{x}(t) - R_i^{-1}B_i^\top\tilde{\eta}_i(t)$, where $\tilde{z} = (\tilde{x}, \tilde{\eta}_1, \tilde{\eta}_2)$ evolves according to the dynamics

$$\begin{aligned}\dot{\tilde{z}} &= \left[\begin{array}{c|cc} A_{\mathrm{cl}} & -S_1 & -S_2 \\ \hline -\mathcal{R}_1(P+\Delta P) & F_1 & 0 \\ -\mathcal{R}_2(P+\Delta P) & 0 & F_2\end{array}\right]\tilde{z} \\ &:= H_C(P+\Delta P)\tilde{z}\,,\end{aligned} \tag{47}$$

where $P + \Delta P = [P_1^\top + \Delta P_1^\top \;\; P_2^\top + \Delta P_2^\top]^\top$. Note that $\tilde{z}$ and its evolution described by (47) is immediately obtained via a change of coordinates identical to (36) with $\tilde{P}_i$ in place of P_i, $i = 1, 2$. Hence, since the eigenvalues of H_C are continuous functions of the entries of the uncertain matrices ΔP_i and $\sigma(H_C(0)) \subset \mathbb{C}^-$ by construction, there exists $\varepsilon > 0$ such that the open-loop control laws u_i^c remain bounded for all time for all numerical errors such that $\max\{\|\Delta P_1\|, \|\Delta P_2\|\} < \varepsilon$. ▲

The rationale behind the discussion in Remark 6 can be summarized by the following observation: OL-NE strategies based on (14) heavily rely on *invariance* of the equilibrium subspace, which is a *fragile property*, whereas OL-NE strategies obtained via (40), (41) hinges upon (external) *asymptotic stability* of the equilibrium subspace, which is an intrinsically robust property. Apart from providing a *robust* mechanism to compute OL-NE, the dynamics (47) reveals an additional property of the controlled state/costate system (14), summarized in the following statement, the proof of which borrows tools from the theory of geometric control, see *e.g.* Basile and Marro (1992).

Lemma 3. *Suppose that the solution of system (40) is forward propagated from the initial condition* $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, \tilde{P}_1 x_0, \tilde{P}_2 x_0)$ *in closed loop with* $v_i|_{P=\tilde{P}}$ *for arbitrary* $x_0 \in \mathbb{R}^n$ *and* $\tilde{P}_i \in \mathbb{R}^{n\times n}$*. Then the following statements are equivalent:*

(i) the resulting trajectory is such that $\lambda_i(t) = \tilde{P}_i x(t)$ *for all* $t \geq 0$*;*
(ii) $\tilde{P}_i$ *coincide with a solution* P_i *of (11),* $i = 1, 2$. ∘

Proof. The claim is an immediate consequence of the notion of invariant subspace for linear time-invariant systems. To this end, recall the structure of the change of coordinates $\tilde{z} = \tilde{T}^{-1} z$, where $\tilde{T}$ is defined as in (36) with P_i replaced by $\tilde{P}_i$. Then, item (i) is equivalent to requiring invariance of the subspace

$$\tilde{\mathcal{M}}_\circ := \operatorname{im}\left(\begin{bmatrix} I_n \\ \mathbf{0}_{2n\times n} \end{bmatrix}\right), \tag{48}$$

along the trajectories of (47). The latter invariance property tantamounts (see e.g. Basile & Marro, 1992) to the inclusion

$$H_{\mathcal{C}}(\tilde{P})\tilde{\mathcal{M}}_\circ \subseteq \tilde{\mathcal{M}}_\circ \tag{49}$$

hence (by slightly mixing matrix and subspace notation)

$$\left[\begin{array}{c|c} A_{\mathrm{cl}} & -S \\ \hline -\mathcal{R}(\tilde{P}) & F \end{array}\right] \operatorname{im}\left(\begin{bmatrix} I_n \\ \mathbf{0}_{2n\times n} \end{bmatrix}\right) \subseteq \operatorname{im}\left(\begin{bmatrix} I_n \\ \mathbf{0}_{2n\times n} \end{bmatrix}\right) \tag{50}$$

with $S := [S_1 \;\; S_2]$, $F := \operatorname{blkdiag}\{F_1, F_2\}$ and $\mathcal{R}(\tilde{P}) := [\mathcal{R}_1(\tilde{P})^\top \;\; \mathcal{R}_2(\tilde{P})^\top]^\top$. Then, if item (ii) holds, and hence $\mathcal{R}(\tilde{P}) = 0$, it is straightforward to note that (50) is verified, which in turn is equivalent to item (i). To show the converse implication, *i.e.* (i) ⇒ (ii), suppose by contradiction that $\tilde{P}$ is not a solution to (11) and hence $\mathcal{R}(\tilde{P})$ has at least one non-zero element. As a consequence one can always select a vector $v_\circ \in \tilde{\mathcal{M}}_\circ$ such that $H_{\mathcal{C}}(\tilde{P})v_\circ = [U_1^\top \;\; U_2^\top]^\top$, for some non-zero $U_2 \in \mathbb{R}^{2n}$. Thus, an absurd is obtained since $H_{\mathcal{C}}(\tilde{P})v_\circ \notin \tilde{\mathcal{M}}_\circ$ and the hypothesis (i), or equivalently (50), is violated. □

The above feature is employed in the following section in which we provide a closed-form gradient descent scheme that allows computing the solution to the asymmetric coupled AREs (11).

5. Gradient descent for the asymmetric coupled Riccati equations

The main purpose of this section is the exploitation of the structural property discussed in Lemma 3 to provide an iterative strategy with guaranteed convergence, at least locally, to a solution to the asymmetric coupled AREs.[2] To this end, define the auxiliary cost function $\Upsilon : \mathbb{R}^{2n\times n} \to \mathbb{R}_{>0}$ as

$$\Upsilon(P) = \frac{1}{2}\int_0^{t_f} \|\Pi e^{H_{\mathcal{C}}(P)t} z_\circ\|^2 dt \tag{51}$$

[2] Note that the constructions of this section are carried out in the context of LQ differential games specifically defined by (1) (2) and the related assumptions for consistency with the framework considered in the previous sections. Nonetheless, the results of Section 5 may be immediately extended to the more general setting in which $u_i \in \mathbb{R}^{m_i}$, $i = 1, 2$, and $m_1 \neq m_2$.

with respect to the variable $P = [P_1^\top \;\; P_2^\top]^\top$, where $t_f \in \mathbb{R}_{>0}$ is an arbitrary terminal time, $z_\circ = (x_0, 0, 0) \in \tilde{\mathcal{M}}_\circ$ and the matrix $\Pi \in \mathbb{R}^{2n\times 3n}$ is defined as

$$\Pi = \begin{bmatrix} \mathbf{0}_{2n\times n} & I_{2n} \end{bmatrix}. \tag{52}$$

By the conclusions of Lemma 3, the cost function Υ is equal to zero, *for any* $t_f > 0$, if and only if P_1 and P_2 are solutions to the asymmetric coupled AREs (11). Finally, to provide a concise statement of the following result, let $\pi_s \in \mathbb{R}^{1\times 3n}$ denote the sth row of the matrix Π, $s = 1, \ldots, 2n$, $S = [S_1 \;\; S_2]$,

$$\bar{A} := \begin{bmatrix} A^\top & 0 \\ 0 & A^\top \end{bmatrix}, \tag{53}$$

and $E_{i,j} \in \mathbb{R}^{2n\times n}$ denote the zero matrix with only the (i, j) entry equal to 1, for $i = 1, \ldots, 2n$, $j = 1, \ldots, n$.

Lemma 4. *Fix* $x_0 \in \mathbb{R}^n$ *and* $\hat{P} \in \mathbb{R}^{2n\times n}$*. Then the gradient of* Υ *with respect to* P *at* $P = \hat{P}$ *is given by (54)* (overleaf)

$$\left.\frac{\partial \Upsilon}{\partial P}\right|_{P=\hat{P}} = \int_0^{t_f} \sum_{s=1}^{2n} \pi_s e^{H_{\mathcal{C}}(\hat{P})t} z_\circ \begin{bmatrix} \pi_s h_{1,1}(t) \ldots & \pi_s h_{1,n}(t) \\ \vdots & \ddots & \vdots \\ \pi_s h_{2n,1}(t) & \ldots & \pi_s h_{2n,n}(t) \end{bmatrix} dt \tag{54}$$

with

$$h_{i,j}(t) := \int_0^t e^{H_{\mathcal{C}}(\hat{P})(t-\tau)} \Psi_{i,j} e^{H_{\mathcal{C}}(\hat{P})\tau} z_\circ d\tau\,, \tag{55}$$

where

$$\Psi_{i,j} := \begin{bmatrix} -SE_{i,j} & \mathbf{0}_{n\times 2n} \\ -\bar{A}E_{i,j} - E_{i,j}A + E_{i,j}S\hat{P} + \hat{P}SE_{i,j} & \mathbf{0}_{2n\times 2n} \end{bmatrix}. \tag{56}$$

∘

Proof. The claim follows by noting that the cost function Υ may be equivalently written as

$$\Upsilon(P) = \frac{1}{2}\int_0^{t_f} z(t)^\top \Pi^\top \Pi z(t) dt = \frac{1}{2}\int_0^{t_f} \sum_{s=1}^{2n} (\pi_s z(t))^2 dt$$

where z denotes the solution to the ordinary differential equation $\dot{z} = H_{\mathcal{C}}(P)z$ together with the initial condition $z(0) = z_\circ$. Therefore

$$\frac{\partial}{\partial P}\Upsilon(P) = \int_0^{t_f} \sum_{s=1}^{2n} \pi_s z(t) \frac{\partial}{\partial P}(\pi_s z(t)) dt$$

$$= \int_0^{t_f} \sum_{s=1}^{2n} \pi_s z(t) \begin{bmatrix} \pi_s \dfrac{\partial z}{\partial p_{1,1}}(t) & \ldots & \pi_s \dfrac{\partial z}{\partial p_{1,n}}(t) \\ \vdots & \ddots & \vdots \\ \pi_s \dfrac{\partial z}{\partial p_{2n,1}}(t) & \ldots & \pi_s \dfrac{\partial z}{\partial p_{2n,2n}}(t) \end{bmatrix} dt\,,$$

where $p_{i,j}$ denotes the entry in position (i, j) of the (composite) matrix P, $i = 1, \ldots, 2n$, $j = 1, \ldots, n$. Finally, the claim is obtained by relying on the notion of *sensitivity equations* that imply that the derivative of the solution to the differential equation with respect to the parameter $p_{i,j}$ appearing in the (linear) vector-field satisfies

$$\frac{d}{dt}\frac{\partial z}{\partial p_{i,j}} = H_{\mathcal{C}}(P)\frac{\partial z}{\partial p_{i,j}} + \frac{\partial H_{\mathcal{C}}(P)}{\partial p_{i,j}} z(t),$$

initialized at $(\partial z/\partial p_{i,j})(0) = 0$, which immediately yields (55) by integration and by observing that the partial derivative of the matrix $H_{\mathcal{C}}(P)$, defined in (47), with respect to the entry $p_{i,j}$ of P is provided by $\Psi_{i,j}$ in (56). □

Finally, the solutions to the asymmetric coupled AREs (11) can be obtained via finite-dimensional gradient-based techniques as detailed in the following statement.

Proposition 2. *Consider any pair of solutions $P_1 \in \mathbb{R}^{n\times n}$ and $P_2 \in \mathbb{R}^{n\times n}$ of the asymmetric coupled AREs (11). There exist $\mu > 0$ and a sequence of steps α_k, $k \in \mathbb{N}$, such that the iteration*

$$\hat{P}^{k+1} = \hat{P}^k - \alpha_k \frac{\partial \Upsilon}{\partial P}\bigg|_{P=\hat{P}^k} \tag{57}$$

with the gradient given by (54)–(56), converges to P for any $\hat{P}^0$ such that $\|\hat{P}^0 - P\| < \mu$. ◦

By Lemma 4, where it is established that the gradient of Υ with respect to P indeed exists, the claim is derived immediately from standard arguments on gradient methods and the properties demonstrated in Lemma 3, hence the proof is omitted. Moreover, since (57) behaves as a standard gradient descent algorithm, once the gradient of (51) is computed, the vast literature related to gradient-based algorithms may be subsumed, for instance in the selection of the step size α_k (possibly via the use of advanced line-search methods).

Remark 7. The iterations (57) provides a strategy, alternative to those already available in the literature, to solve the asymmetric coupled AREs (11). In fact, since the time integrals required for the computation of the gradient (54) may be effectively approximated by means of summations of finitely many terms, it is interesting to observe that each iteration of (57) relies only on simple operations, namely sums and products. Thus, the strategy suggested by (11) entirely circumvents the need for matrix inversion or for computing the roots of higher-order polynomials. Furthermore, as a future perspective, by inspecting the (*trajectory-based*) nature of (51) it is immediate to observe that (57) may be recast into an *online* iteration method to estimate OL-NE strategies during the evolution of the game. ▲

6. Numerical example

A numerical example is provided to demonstrate the main "computational results" presented in the paper. Namely, we demonstrate that the result of Lemma 2 can be utilized to obtain an OL-NE solution of a game in a robust fashion. We also demonstrate how the gradient descent algorithm proposed in Proposition 2 can be used to compute (without requiring the solution to (11)) an OL-NE solution of a game. Towards this end, consider a (scalar) system (2), with $A = -1$, $B_1 = 1$, $B_2 = 2$ and the cost functionals (1), $i = 1, 2$, with $Q_1 = 1$, $Q_2 = 5$ and $R_1 = R_2 = 1$. Note that the corresponding matrix H in (12) satisfies Assumption 2, *i.e.* its spectrum includes two eigenvalues with positive real parts, and the (asymmetric) coupled AREs (11) admit a (unique, stabilizing) solution $P_1 = 0.1757$ and $P_2 = 0.8787$. The initial condition of the state is taken to be $x_0 = 5$.

Recall that the OL-NE is given by (13), $i = 1, 2$, with $\lambda_i^\star$ given by the trajectory of the extended state/costate system (14) initialized at

$$z(0) = (x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1x_0, P_2x_0) \triangleq z_0^\star.$$

The specific trajectory of the extended state/costate system, obtained by numerical integration, is shown in Fig. 3 (dashed, grey line). It can be seen that the trajectory diverges due to (unavoidable) numerical errors (as expected due to Assumption 2).

Next, consider the controlled state/costate dynamics (40) in closed-loop with (41), with $F_1 = F_2 = -20$. The trajectory of the system initialized at $z(0) = z_0^\star$ is shown in Fig. 3 (solid, black line). The result in Lemma 2 can clearly be appreciated: the virtual input externally stabilizes the invariant equilibrium subspace $\mathcal{M}_\circ$. Thus, the trajectory that is initialized on $\mathcal{M}_\circ$ remains there, despite the presence of numerical errors. The same trajectory is shown in Fig. 4 (solid, black line) along with the trajectory of the controlled state/costate dynamics *incorrectly initialized* at $z(0) = (x_0, (P_1 + \varDelta P_1)x_0, (P_2 + \varDelta P_2)x_0)$, with $\varDelta P_1 = \varDelta P_2 = 1$ (dotted, red line). As discussed in Remark 6 it can be seen that the trajectory converges to $\mathcal{M}_\circ$, despite the incorrect initialization.

Finally, for the above simulations we have utilized the solution of the coupled AREs (11). However, the insights provided in this paper also play an instrumental role (as detailed in Proposition 2) in constructing the gradient descent algorithm (57). Letting $F_1 = F_2 = -20$ (as before), $t_f = 0.5$, $\hat{P}^0 = [0\ 0]^\top$ and $\alpha_k = 0.1$, for all $k \geq 0$, the evolution of the $\hat{P}^k$ (as specified in (57)) is shown in Fig. 5, where the black and grey solid lines represent the first and second components of $\hat{P}^k$, respectively, whereas the black and grey dashed lines indicate the solutions P_1 and P_2 of (11), respectively.

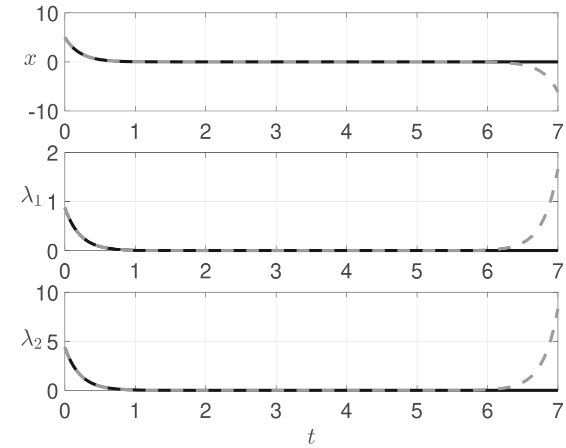

Fig. 3. Trajectory of the extended state/costate system (dashed, grey line) and the controlled state/costate system (solid, black line) initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = z_0^\star$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

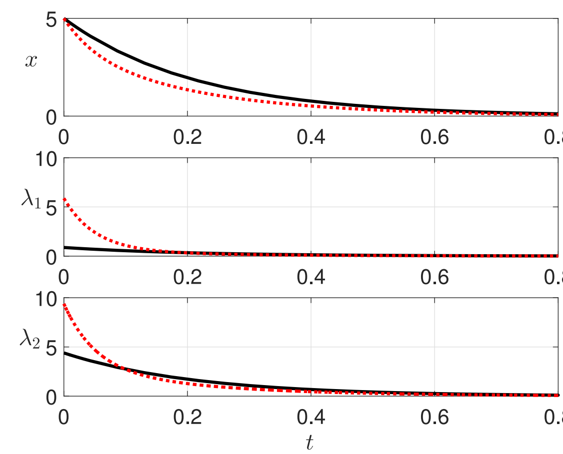

Fig. 4. Trajectory of the controlled state/costate system *correctly* initialized (solid, black line) and the controlled state/costate system *incorrectly* initialized (dotted, red line). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

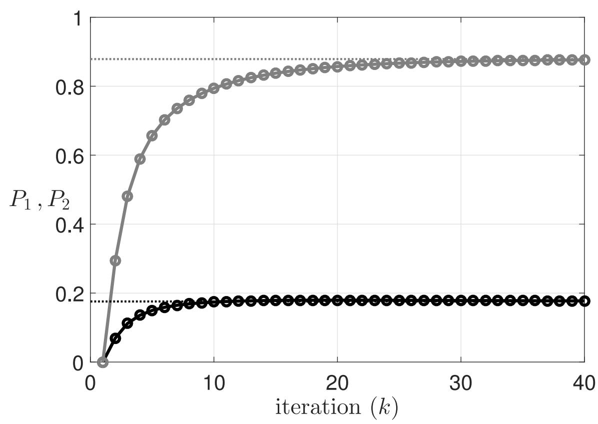

Fig. 5. The evolution of the first (solid, black line) and second (solid, grey line) entry of $\hat{P}^k$. The dashed lines indicate the solutions P_1 (black line) and P_2 (grey line) of (11). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

7. Conclusions and further extensions

Two-player LQ differential games and their solutions in terms of OL-NE strategies have been considered. It has been demonstrated that any such strategies are generated by the output of an interconnection of two PCH systems, restricted to a certain invariant subspace. Then, it has been demonstrated that such invariant subspace can be rendered externally asymptotically stable, via a virtual input directly influencing the costate dynamics, which in turn facilitates the robust computation of OL-NE strategies. Further exploiting properties of the "controlled" state/costate dynamics, a gradient descent algorithm is proposed to determine OL-NE solutions without requiring the explicit solution of the coupled (asymmetric) AREs characterizing the OL-NE. Finally, it would be interesting and insightful to categorize classes of LQ differential games or even directly of specific equilibria (since the game may, in general, admit several OL-NE solutions) on the basis of the energy exchange pattern that is established between the players at the equilibrium, and to develop an online iteration method to obtain OL-NE solutions (as discussed in Remark 7).

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