# Optimisation of Hardy-type inequalities with applications in infinite dimensional Geometry

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A thesis presented for the degree of
Doctor of Philosophy




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*To my family and friends,*

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## Abstract

The following thesis concerns the generalisation of Hardy's integral inequality to multiple dimensions. The structure involves three main papers, two published in journals and one paper in preparation.

In the first article, we make use of the classical Hardy inequality away from a point and a multi-dimensional Hardy inequality associated to the Laplacian operator. The inequality's domain is extended to an infinite case and the behaviour of systems is studied when multiple particles are introduced as a concept. The result we obtain resembles the classical Hardy inequality, but, as an extension, we introduce new vector fields and boundary approximations on a many-particle quantum system.

The second article presents the interest in the refinement of Hardy inequalities using Fourier expansions applied to a new Hopf system of coordinates. As a starting point, we configured the coordinates in four dimensions for block - radial functions such that our argument could be easier understood. In comparison to previous research, our result behaves promising in the sense that the inequality discovered is true for functions on the whole space. Moreover, by introducing a new system of coordinates, we are able to apply our result to the extension of the Hopf fibration to dimension eight.

Last but not least, in article three we look at discrete Hardy Inequalities and their link to the continuous Hardy inequalities discovered so far. Considering that Hardy inequalities are more difficult to analyse for discrete operators, we make use of two important results in order to establish whether for the discrete Hardy inequality discovered there is a sharp constant on a closed disk in $\mathbb{Z}^2$. The results involve a specific type of Aharonov-Bohm magnetic potential which facilitates the analysis of the constant's behaviour, which is present in the new discrete Hardy inequality.

# Declaration

The work in this thesis is based on research carried out at the Imperial College London through the Department of Mathematical Sciences. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

*Dorian Guzu*

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# Contents


**1 Introduction** **1**

1.0.1 First article . . . 3

1.0.2 Second article . . . 4

1.0.3 Third article . . . 4

1.1 Discrete Hardy inequalities . . . 6

1.2 Continuous Hardy inequalities . . . 8

1.2.1 $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ spaces . . . 10

1.2.2 Multidimensional Hardy Inequalities . . . 11

1.2.2.1 Bounded domains . . . 12

1.2.2.2 Convex domains . . . 15

1.2.2.3 Hardy inequality for general domains . . . 18

1.2.3 Multiparticle Hardy Inequalities . . . 21

1.2.3.1 The uncertainty principle . . . 21

1.2.3.2 Systems of particles . . . 23

1.3 Outline of the thesis . . . 28

**2 Theoretical background** **31**

2.1 Improved many-particle Hardy inequalities . . . 31

2.2 Sharp multiplicative Hardy inequalities . . . 36

2.3 Discrete Aharonov-Bohm Hardy inequalities . . . 37

**3 On Many-Particle Hardy inequality** **40**

3.1 Introduction . . . 40

3.2 Preliminary results . . . 41

3.3 Main results . . . 43

3.4 Proof of main results . . . 46

3.5 Alternative findings . . . 50

**4 Sharp Hardy inequalities for block radial functions** **59**

4.1 Introduction . . . 59

4.2 Main results . . . 61

4.3 Proof of main results . . . 62

4.4 Block radial Hardy inequality applications . . . 63

4.4.1 3D case . . . 63

4.4.2 4D case . . . 64

4.4.3 Hopf coordinates in $\mathbb{R}^8$ . . . 69

**5 Discrete inequalities for magnetic potential terms** **72**

5.1 Introduction . . . 72

5.1.1 Magnetic Laplacian . . . 73

5.1.2 Magnetic fields and Schrödinger operators . . . 74

5.1.3 Aharonov - Bohm magnetic field . . . 75

5.2 Main Result . . . 76

5.3 Proof of main result . . . 77

5.3.1 Polar coordinates in $\mathbb{Z}^2$ space . . . 77

5.3.2 Proof of Theorem 5.2.1 . . . 80

5.4 The $\mathbb{S}_{n-1}$ and $\mathbb{S}_{n+1}$ cases . . . 83

**6 Future developments** **87**

6.1 Hardy inequalities for the volume of simplices . . . 87

6.1.1 Spherical Hardy inequality . . . 89

6.2 Maximum ionization of atoms . . . 90

6.3 Fermionic Hardy inequalities . . . 91

6.4 On the continuous versions of Hardy inequality . . . 94

**7 Conclusion** **96**

**Bibliography** **99**

# Chapter 1

# Introduction

Since 1911 the study of Hardy inequalities has had a great impact in several branches of mathematical analysis and physics, facilitating the research on Schrödinger operators. In the field of functional analysis, G. H. Hardy used his famous inequality in a proof of Hilbert's inequality and there has been a constant development regarding the original inequalities on $(0, \infty)$.

As we specified, one main reason to study Hardy inequalities is their connection to the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{x},t) = \left[\nabla^2 + V(\mathbf{x})\right]\Psi(\mathbf{x},t), \quad \hbar - \text{Planck}'\text{s constant}$$

and the case that provides a description for the stationary states

$$-\nabla^2\Psi(\mathbf{x},t) - V(\mathbf{x})\Psi(\mathbf{x},t) = 0$$

which are considered the foundation of classical quantum mechanics.

It is imperative to add that Hardy inequalities are considered to be $L_p$ inequalities and they are also essential in the study of $p$-Schrödinger equation

$$\nabla\cdot\left(|\nabla u|^{p-2}\nabla u\right) - V(\mathbf{x})u = 0.$$

For introductory purposes, in the following lines we shall present a few key remarks regarding the discrete and continuous Hardy inequality after which we proceed to more in depth notions related to the topic.

Let us consider the Hardy inequality statements:

- For $\{a_j\}_{j\geq 1} \in \mathbb{R}_+$ and $p > 1$, then the following **discrete inequality** holds

$$\left(\frac{p-1}{p}\right)^p \sum_{d=1}^{\infty}\left(\frac{1}{d}\sum_{j=1}^{d} a_j\right)^p \leq \sum_{d=1}^{\infty}|a_d|^p. \tag{1.0.1}$$

- If $p > 1$ and $f \in L^p(0,\infty)$, then $f$ is integrable over the interval $(0,x), \forall x \in \mathbb{R}_+$ and the following **continuous inequality** holds

$$\left(\frac{p-1}{p}\right)^p \int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \int_0^\infty |f(x)|^p dx. \tag{1.0.2}$$

The remarks are as follows:

($\mathbf{R}_1$) The constant involved in both identities is considered **sharp**. More specifically, there does not exist a larger constant for which (1.0.1) and (1.0.2) hold for all functions and quantities.

($\mathbf{R}_2$) Identities (1.0.1) and (1.0.2) are considered standard forms of the Hardy inequalities. These are found in various textbooks on analysis and were first emphasized in the book *Inequalities* by Hardy, Pólya and Littlewood [55].

($\mathbf{R}_3$) By studying (1.0.1) and (1.0.2), we can derive the following *weak forms*:

(a) If $\sum_{d=1}^{\infty} a_d^p < \infty$ and $a_d \geq 0$, then $\sum_{d=1}^{\infty}\left(\frac{1}{d}\sum_{j=1}^{d} a_j\right)^p < \infty$

(b) If $\int_0^\infty f(x)^p dx < \infty$ and $f(x) \geq 0$, then $\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < \infty$

($\mathbf{R}_4$) The spectral theory of Schrödinger operators and theory of differential equations have used numerous generalized versions of inequalities (1.0.1) and (1.0.2). A large number of these developments, generalizations, and applications have been discussed in books [55], [66] and [70].

($\mathbf{R}_5$) The sharpness of the constant involved in the structure of (1.0.1) and (1.0.2) along with the aforementioned identities imply that:

(a) The *discrete Hardy operator*

$$h_{dis}(\{a_j\}) := \sum_{j=1}^{d} a_j$$

maps $l_p$ into $l_p$ for $p > 1$ with norm $\frac{p}{p-1}$.

(b) The *continuous Hardy operator*

$$h_{cont}(f(x)) := \frac{1}{x}\int_0^x f(t)dt$$

maps $L_p$ into $L_p$ for $p > 1$ with norm $\frac{p}{p-1}$.

We have to specify that $l_p$ and $L_p$ are the Lebesgue spaces consisting of all sequences $a = \{a_n\}$ of real numbers and all measurable functions $f$ on $(0, \infty)$. Briefly introduced in the above lines and considering that this type of spaces is highly used in the present manuscript, more detailed information shall be provided in the following chapters.

Extending the classical Hardy inequality to systems of particles in multiple dimensions has been of great interest for researchers throughout time. This fact drove us in the search for optimal Hardy inequalities which can be applied to simplices, block radial functions and magnetic fields.

This thesis consists of three main papers. Two of these were reviewed and published, while one is still being reviewed. In the following lines, we shall briefly describe each result, highlighting the outcome accordingly.

### 1.0.1 First article

The geometrical Hardy inequality which describes the behaviour of particles in a quantum system with a dimension larger than two. Namely, for $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,d}) \in \mathbb{R}^d$, $j = 1, \dots N$ and

$$r_{ij} := |\mathbf{x}_i - \mathbf{x}_j| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2},$$

the following theorem holds

**Theorem 1.0.1** *For $N \geq 4$ and $d \geq 3$ the following Hardy many-particle inequality holds*

$$\int_{\mathbb{R}^{Nd}} |\nabla u|^2 d\mathbf{x} \geq \mathcal{C}(d, N) \int_{\mathbb{R}^{Nd}} \sum_{k<j}^{N} \frac{1}{|\mathbf{x}_k - \mathbf{x}_j|^2} |u|^2 d\mathbf{x}, \tag{1.0.3}$$

*where*

$$\mathcal{C}(d, N) = \frac{2\sqrt{N}\left[(N-1)d^2 - 2Nd + 4\right]}{N\sqrt{N}(N-1) + 6} - \frac{2N\left[(N-1)d - 2\right]^2}{\left[N\sqrt{N}(N-1) + 6\right]^2}.$$

The sharpness of our constant has yet to be established, but using numerical analysis techniques

$$\begin{aligned}
&\mathcal{C}(3, N) \approx 0.497, \quad when \quad N = 4,\\
&\mathcal{C}(3, N) \approx 0.493, \quad when \quad N = 5,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mathcal{C}(3, N) \approx 0.39, \quad when \quad N = 10,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

it is possible to predict a certain behaviour for particles in various dimensions. This implies that the problem of finding the sharp value of $\mathcal{C}(d, N)$ is strongly related to the problem of obtaining an optimal configuration of points(i.e. particles).

We drew our inspiration from paper [119], where the author concluded that for $N = 3, u \in C_0^\infty(\mathbb{R}^{3d})$ and $d \geq 3$, the following holds

$$\begin{aligned}
\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \geq \left[2c_1(d-2) - 2c_1^2\right] \int_{\mathbb{R}^{3d}} \sum_{i<j}^{3} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|^2} |u|^2 d\mathbf{x} - c_1^2 \int_{\mathbb{R}^{3d}} \frac{1}{R^2} |u|^2 d\mathbf{x} +\\
+ 3(d - 1 - 3c_1)^2 \int_{\mathbb{R}^{3d}} \frac{1}{\rho^2} |u|^2 d\mathbf{x}, \quad \forall c_1 \in \mathbb{R},
\end{aligned}$$

where $R$ is defined as the circumradius of the simplex in $3d$ dimensions.

Our result regarding the many-particle Hardy inequality provides an insight in the field of Geometrical combinatorics and it can be considered a founding stone in the research of optimal configurations for a quantum system.

### 1.0.2 Second article

The result relates to the study of sharp Hardy inequalities with weight functions with singularities on the unit sphere. More specifically, for $\mathbf{x} = (r, \vartheta) \in \mathbb{R}^d$ polar coordinates and $d \geq 3$. Let $-\Delta_\vartheta$ be the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$.

Assume that $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$, $\Phi \geq 0$, and that the Schrödinger operator $H$ in $L^2(\mathbb{S}^{d-1})$

$$H = -(\nabla_\vartheta)^2 - \tau\Phi = -\Delta_\vartheta - \tau\Phi, \quad \tau \in \mathbb{R},$$

is bounded from below. Denote

$$\lambda(\tau) = \inf_{\psi \in \mathcal{H}^1(\mathbb{S}^{d-1})\setminus\{0\}} \frac{\int_{\mathbb{S}^{d-1}} (|\nabla_\vartheta \psi|^2 - \tau\Phi\,|\psi|^2)\, d\vartheta}{\int_{\mathbb{S}^{d-1}} |\psi|^2\, d\vartheta}, \tag{1.0.4}$$

where $\mathcal{H}^1(\mathbb{S}^{d-1})$ is the Sobolev space on the sphere. Then the following inequality holds

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{x}} u(\mathbf{x})|^2\, d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\, |u(\mathbf{x})|^2\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d),$$

with sharp $\tau = \tau(\Phi)$ and under specific conditions on $\Phi$.

The applications of this result are various and we shall describe one main usage. The papers by L.Skrzypczak and C.Tintarev [106] and [107] present a constant in the case of block radial functions and our result introduces a sharper estimate for the Hardy constant involved. For example, when $d = 4$ then

$$\int_{\mathbb{R}^4} |\nabla_{\mathbf{x}} u(\mathbf{x})|^2\, d\mathbf{x} \geq \tau \int_{\mathbb{R}^4} \frac{|u(\mathbf{x})|^2}{r_1(\mathbf{x}) r_2(\mathbf{x})}\, d\mathbf{x}, \tag{1.0.5}$$

with $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $r_1(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$, $r_2(\mathbf{x}) = \sqrt{x_3^2 + x_4^2}$. It is imperative to highlight the fact that paper [106] presents inequality (1.0.5) only for the case when $u = u(r_1, r_2)$. Our result holds for general $u \in H^1(\mathbb{R}^4)$.

In this situation, we introduce Hopf coordinates in $\mathbb{S}^3$ and reduce the problem of finding a sharp constant to solving a minimisation problem for the Legendre type equation on the interval $\left[0, \frac{\pi}{2}\right]$.

Our proof for the above result includes a method presented in [58] which we shall apply on a Keller-Lieb-Thirring inequality. This gives an estimate for the first eigenvalue of a magnetic Schrödinger operator on $\mathbb{S}^{d-1}$ in [31].

### 1.0.3 Third article

The last but not least main result of the present thesis relates to discrete Hardy inequalities for $d \geq 2$. Considering that a Hardy inequality does not exist for two dimensions, our curiosity drives us to find new types of discrete Hardy inequalities. This new type involves magnetic potential terms which are defined for the particular case when dimension is equal to two.

For example, the paper by Laptev and Weidl [74] presents the continuous case for a magnetic potential

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{F})u|^2\, d\mathbf{x} \geq \min_{k\in\mathbb{Z}}(k-\alpha)^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x},$$

where $\alpha \in \mathbb{R}$ and the potential

$$\mathbf{F} = (F_1, F_2) = \alpha \left( \frac{-x_2}{|\mathbf{x}|^2}, \frac{x_1}{|\mathbf{x}|^2} \right)$$

for $\mathbf{x} = (x_1, x_2)$ in $\mathbb{R}^2$.

As a comparison, let us consider $\mathbb{1}_i \in \mathbb{Z}^2$ to be the unit vector in the direction $i$. Consequently, the discrete partial derivative of the function $u$ is $u(\mathbf{x}) - u(\mathbf{x} - \mathbb{1}_i)$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$.

In [72], the authors obtained, for $d \geq 3$ the following discrete inequality

$$\sum_{\mathbf{x} \in \mathbb{Z}^d} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2} \leq C \sum_{\mathbf{x} \in \mathbb{Z}^d} \sum_{i=1}^{d} |u(\mathbf{x}) - u(\mathbf{x} - \mathbb{1}_i)|^2,$$

where

$$C \leq \frac{\pi^2}{4} \left[ \frac{2}{d-2} + \frac{\sqrt{2}(d-4)^2}{(d-2)(\sqrt{2}d - 4\sqrt{d-2})\pi} \right]$$

and $u$ defined on $\mathbb{Z}^d$. Hardy inequalities for discrete operators are harder to analyse and considering the constant involved, it is not clear if sharpness is achieved.

Also, in the same paper [72], for a closed disk $D_R = \{x \in \mathbb{Z}^2 : ||\mathbf{x}|| \leq R\}$, its boundary, for an integer $n > 0$, is the following

$$\partial D_n = \{\mathbf{x} \in \mathbb{Z}^2 : |x_1| = n, -n \leq x_2 \leq n, \quad \text{or} \quad |x_2| = n, -n \leq x_1 \leq n\},$$

where

$$||\mathbf{x}|| = \max\{|x_1|, |x_2|\}$$

is the $l^\infty$ norm of $\mathbf{x}$. Our result is embodied in the following theorem

**Theorem 1.0.2** *Suppose that $u \in l^2(\mathbb{Z}^2)$ such that $u(0,0) = u(-1,0) = u(0,-1) = u(1,0) = u(0,1) = 0$. Then the following Hardy inequality holds*

$$\sum_{\substack{-\infty \\ 1 \leq i \leq 2}}^{\infty} |u(\mathbf{x} + \mathbb{1}_\mathbf{i}) - u(\mathbf{x}) + \mathbf{A}_i u(\mathbf{x})|^2 \geq \mathcal{C}_\alpha \sum_{\mathbf{x} \in \mathbb{Z}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}, \tag{1.0.6}$$

*for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, the constant*

$$\mathcal{C}_\alpha := \frac{1}{8\pi} \min_{l \in \mathbb{Z}_+} |l - \alpha|^2$$

*and potential*

$$\mathbf{A}_\alpha = (\mathbf{A}_1(\mathbf{x}), \mathbf{A}_2(\mathbf{x})) = (1 - e^{2\pi i \alpha \varphi_1}, 1 - e^{2\pi i \alpha \varphi_2}),$$

*where the function $\varphi_i$ is defined on a discrete square of radius $n$.*

It is not clear whether the constant is sharp, but we can make use of further numerical analysis to predict the behaviour. The results in [72] involved logarithmic weights for compactly supported functions, but our identity holds for a larger spectrum of functions $u$. The proof of our result consists of manipulating the concept of polar coordinates. More specifically, it involves non-linear operators applied on squares with certain radial and angular components.

In comparison, the article [39] presents a different version of the inequality (1.0.6). However, the authors did not make usage of the standard quadratic form on $\mathbb{Z}^2$ and considered a discretised cylinder $\mathbb{R} \times [0, 2\pi]$ described by $\{\frac{j}{m} : j \in \mathbb{Z}\} \times \{\frac{2\pi j}{n} : j \in \{1, 2, \dots, n\}\}$.

The operators are intended such that we can preserve as many familiar formulas from the continuous case to the discrete one as possible. For exactness purposes, we want our discrete inequality case to be described in a similar manner as

$$\int_{\mathbb{R}^2} |\nabla u|^2 d\mathbf{x} = \int_{\mathbb{R}^2} \left( |\partial_r u|^2 + \frac{1}{r^2} |\partial_\theta u|^2 \right) d\mathbf{x} = \int_0^\infty \int_0^{2\pi} \left( r|\partial_r u|^2 + \frac{1}{r} |\partial_\theta u|^2 \right) dr d\theta.$$

The estimate which gives the constant $C_\alpha$ in (1.0.6) provides a new insight in the discrete second dimensional case for a Hardy inequality.

We shall proceed by giving a brief introduction on variations of the *discrete* and *continuous* Hardy inequalities such that the transition to the outline and key results of the thesis can be smoother. References shall also be provided in such a manner that the reader can discover more about the discussed identities or ideas as we will not dwell on extensive proofs at the moment.

## 1.1 Discrete Hardy inequalities

In this section we shall briefly discuss Hardy's motivation towards the discovery of his well known discrete inequality and a few generalisations of it. To be more specific, the main source of inspiration for Hardy's research was David Hilbert's inequality.

The basic form of *Hilbert's inequality* reads:
If $\sum_{m=1}^{\infty} a_m^2 < \infty$ and $\sum_{n=1}^{\infty} b_n^2 < \infty$ with $a_m \geq 0$ and $b_n \geq 0$, then the double series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n}$$

converges. More accurately, the inequality

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} \leq \pi \left( \sum_{m=1}^{\infty} a_m^2 \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} b_n^2 \right)^{\frac{1}{2}}$$

holds, where $\pi$ being a sharp constant.

It is obviously clear that Hardy's motivation when he began his research was proving the weak form of the Hilbert inequality. Hence, in his proof, he stated and applied the following theorem:

**Theorem 1.1.1** *Let $a_n \geq 0$ and $A_n := \sum_{k=1}^{n} a_k$. The convergence of any of the three series*

$$(i)\ \sum_{n=1}^{\infty} \frac{a_n A_n}{n} \qquad ii)\ \sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^2$$

$$iii)\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m a_n}{m+n}$$

*implies that of the others.*

**Proof**: We shall not provide the full proof, but as an outline, Hardy estimated (iii) by (i) using similar techniques as in [54], more specifically:

$$\begin{aligned}\sum_{n=1}^{M}\sum_{m=1}^{M} \frac{a_m a_n}{m+n} &\leq 2\sum_{n=1}^{M}\sum_{m=1}^{n} \frac{a_m a_n}{m+n} = 2\sum_{n=1}^{M} a_n \sum_{m=1}^{n} \frac{a_m}{m+n} \\ &\leq 2\sum_{n=1}^{M} a_n \frac{1}{n}\sum_{m=1}^{n} a_m = 2\sum_{n=1}^{M} a_n \frac{A_n}{n}. \quad \square\end{aligned}$$

Taking into consideration the above proof, Hardy established the following theorem, which is a precursor of (1.0.1) when $p = 2$:

**Theorem 1.1.2** *The convergence of the series $\sum_{n=1}^{\infty} a_n^2$ with $a_n \geq 0$ implies the convergence of $\sum_{n=1}^{\infty} \left(\frac{A_n}{n}\right)^2$, where $A_n := \sum_{k=1}^{n} a_k$.*

**Proof**: We consider the following identity

$$\begin{aligned}\left(\frac{A_n}{n}\right)^2 &= \left(a_n + \frac{A_n}{n} - a_n\right)^2 \leq 2a_n^2 + 2\left(\frac{A_n}{n} - a_n\right)^2 \\ &= 4a_n^2 + 2\left(\frac{A_n}{n}\right)^2 - \frac{4a_n A_n}{n} \\ \Longrightarrow \sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 &\leq 4\sum_{n=1}^{N} a_n^2 + 2\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 - 4\sum_{n=1}^{N}\frac{a_n A_n}{n}.\end{aligned}$$

It is obvious that $-2a_n A_n = -(A_n^2 - A_{n-1}^2) - a_n^2 \leq -(A_n^2 - A_{n-1}^2)$, hence

$$\begin{aligned}-2\sum_{n=1}^{N}\frac{a_n A_n}{n} &\leq -\sum_{n=1}^{N}\frac{-(A_n^2 - A_{n-1}^2)}{n} \\ &\leq -\sum_{n=1}^{N}\frac{A_n^2}{n(n+1)}.\end{aligned}$$

By using this substitution we obtain

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 &\leq 4\sum_{n=1}^{N} a_n^2 + 2\sum_{n=1}^{N}\left(\frac{A_n}{n}\right)^2 - 2\sum_{n=1}^{N}\frac{A_n^2}{n(n+1)} \\ &= 4\sum_{n=1}^{N} a_n^2 + 2\sum_{n=1}^{N}\frac{A_n^2}{n^2(n+1)},\end{aligned}$$

which generates

$$\sum_{n=1}^{N}\left(1-\frac{2}{n+1}\right)\left(\frac{A_n}{n}\right)^2 \leq 4\sum_{n=1}^{N}a_n^2. \tag{1.1.1}$$

As we can see, inequality (1.1.1) can be used to derive (1.0.1) for $p=2$ with the sharp constant 4. □

Through numerous letter exchanges between Marcel Riesz and himself (the reader can analyse the proofs from paper [102]), Hardy proceeded to refine **Theorem 1.1.2** and extend it to a general $p$. But only later, more specifically, in 1924, he managed to give a substantial proof with the sharp constant $\left(\frac{p}{p-1}\right)^p$.

It is worth mentioning that Hardy also specified that the best constant in (1.0.1) can not be strictly less than $\left(\frac{p}{p-1}\right)^p$, but did not wish to publish the fact the discrete case could hold for that specific constant. Moreover, he received letters from E. Landau, who provided reassuring results towards the usage of the above constant. The reader should check papers [68] and [69] for a direct view on these results, considering that they involve polynomial approximation and sequence analysis.

Also, papers [54], [55], [67], [68] and [69] provide specific examples of discrete Hardy inequalities that can help the reader understand the concepts that we used so far.

For the following lines, we shall turn our attention to the continuous case of the Hardy inequality.

## 1.2 Continuous Hardy inequalities

By using similar techniques as in the previous proofs for the discrete case, Hardy established a different manner in which inequality (1.0.1) reached its form.

Assuming that we are dealing with a decreasing sequence $\{a_n\}_{n\geq 1}$ as Prof. E. Landau communicated to Hardy in [68], we observe that the function

$$\frac{1}{x}\int_0^x f(t)dt = \frac{\sum_{k=1}^{n-1}a_k + [a_n\cdot(x-n+1)]}{x}$$

decreases on the interval $[n-1,n]$, where

$$\begin{cases} f(x)=a_1, & 0\leq x<1 \\ f(x)=a_2, & 1\leq x<2 \\ \dots \end{cases}.$$

Hence, we obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{\sum_{k=1}^{n}a_k}{n}\right)^p &\leq \sum_{n=1}^{\infty}\int_{n-1}^{n}\left(\frac{\sum_{k=1}^{n}a_k+[a_n\cdot(x-n+1)]}{x}\right)^p dx \\ &= \int_0^{\infty}\left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p\int_0^{\infty}f(x)^p dx \\ &= \left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p.\end{aligned}$$

Taking this in consideration, Hardy proved the famous inequality (1.0.2) as follows:

**Theorem 1.2.1** *Let $p > 1$ and let $f(x) \geq 0$ be $p$-integrable on $(0, \infty)$. Then $F(x) = \int_0^x f(t)dt < \infty$ for every $x > 0$ and*

$$\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx, \tag{1.2.1}$$

*where $\left(\frac{p}{p-1}\right)^p$ is sharp.*

We have to specify that Hardy's original proof contains various explanations and details, but we shall present a proof which resembles his original ideas, but does not contain a high amount of technical details.

**Proof**: Observing that

$$\frac{d}{dx}\left(F(x)^p\right) = pF(x)^{p-1}f(x)$$

holds for almost every $x$ in $(0, \infty)$, for arbitrary $\alpha$ and $A$ such that $0 < \alpha < A < \infty$ we obtain

$$\begin{aligned}\int_\alpha^A \left(\frac{F(x)}{x}\right)^p dx &= -\frac{1}{p-1}\int_\alpha^A F(x)^p \frac{d}{dx}\left(x^{1-p}\right) dx \\ &= \frac{\alpha^{1-p}}{p-1}F(\alpha)^p - \frac{A^{1-p}}{p-1}F(A)^p + \frac{1}{p-1}\int_\alpha^A x^{1-p}\frac{d}{dx}\left(F(x)^p\right) dx \\ &\leq \frac{\alpha^{1-p}}{p-1}F(\alpha)^p + \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)}{x}\right)^{p-1} f(x)dx.\end{aligned}$$

Using the continuous version of the Hölder inequality, we observe that

$$\int_\alpha^A \left(\frac{F(x)}{x}\right)^{p-1} f(x)dx \leq \left(\int_\alpha^A f(x)^p dx\right)^{\frac{1}{p}} \left(\int_\alpha^A \left(\frac{F(x)}{x}\right)^p dx\right)^{\frac{p-1}{p}}.$$

If we take a $\beta$ such that $\alpha < \beta < A$ and using the substitution $F(x) - F(\alpha)$ for $F(x)$, we find

$$\begin{aligned}\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx &\leq \frac{p}{p-1}\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^{p-1} f(x)dx \\ &\leq \frac{p}{p-1}\left(\int_\alpha^A f(x)^p dx\right)^{\frac{1}{p}} \left(\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{\frac{p-1}{p}}.\end{aligned}$$

It follows that

$$\left(\int_\alpha^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_\alpha^A f(x)^p dx\right)^{\frac{1}{p}}.$$

In the above inequality, if we allow $\alpha$ to tend to $0^+$, observing that $F(x) - F(\alpha)$ increases to $F(x)$ (by the definition of $F(x)$), we reach

$$\left(\int_\beta^A \left(\frac{F(x)-F(\alpha)}{x}\right)^p dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty f(x)^p dx\right)^{\frac{1}{p}}.$$

The proof is completed, if we let $A$ tend to $\infty$ and $\beta$ to $0^+$. □

We should also mention that various alternative proofs have been provided by numerous writers, for example Broadbent [17], Hardy, Littlewood, Pólya [55] and Elliot [34].

It is clear that G. H. Hardy had good contact with other well known mathematicians who were also interested in the subject and who were willing to pass along significant information, either via private or formal letters. Moreover, he synthesized the knowledge obtained from numerous sources in a concise manner such that his theory would emerge itself as innovative and interesting.

### 1.2.1 $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ spaces

Since the theory of Hardy inequality is strongly intertwined with the existence of Sobolev spaces, we shall provide a brief introduction into aspects of the specified topic. Let $\Omega$ be a non-empty subset of $\mathbb{R}^n, n \geq 1$ with closure $\overline{\Omega}$ and boundary $\partial\Omega$. Considering $p \in [1, \infty], k \in \mathbb{N}$, the points $\mathbf{x} = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$ and $n$-tuples $\alpha = (\alpha_1, \alpha_2, \dots \alpha_n) \in \mathbb{N}_0^n$, we shall write

$$|\mathbf{x}| = \left( \sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}, \quad |\alpha| = \sum_{j=1}^{n} \alpha_j, \quad \partial^\alpha = \prod_{j=1}^{n} \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}},$$

while taking the derivatives in the weak sense. Let us recall the fact that a function $f$ is in the space of integrable functions on all compact subsets of $\Omega$ and $f$ has weak derivative $g := \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} f$, if $g \in L^1_{loc}(\Omega)$ and

$$\int_\Omega f(\mathbf{x}) \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} \varphi(\mathbf{x}) d\mathbf{x} = -\int_\Omega g(\mathbf{x}) \varphi(\mathbf{x}) d\mathbf{x},$$

for all $\varphi \in C_0^\infty(\Omega)$.

Taking the above into consideration, we can define the Sobolev space as being

$$W^{k,p} := \{u : \partial^\alpha u \in L^p(\Omega) \quad \text{for} \quad |\alpha| \leq k\},$$

endowed with the norm

$$||u||_{k,p,\Omega} = \begin{cases} \left( \sum_{0 \leq |\alpha| \leq k} ||\partial^\alpha||_{p,\Omega}^p \right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \left( \sum_{0 \leq |\alpha| \leq k} ||\partial^\alpha||_{\infty,\Omega}^p \right), & p = \infty \end{cases}$$

Here $||\cdot||_{p,\Omega}$ represents the standard $L^p$ norm,

$$||u||_{p,\Omega} := \left( \int_\Omega |u|^p d\mathbf{x} \right)^{\frac{1}{p}}.$$

As a result, we have that for $k = 1$ and $1 \leq p < \infty$,

$$||u||_{1,p,\Omega}^p = ||u||_{p,\Omega}^p + ||\nabla u||_{p,\Omega}^p,$$

where

$$||\nabla u||_{p,\Omega} = ||\,|\nabla u|\,||_{p,\Omega}, \quad |\nabla u| := \left( \sum_{i=1}^{n} \left| \frac{\partial}{\partial x_i} u \right|^2 \right)^{\frac{1}{2}}.$$

Transferring our attention to the space $C_0^\infty(\Omega)$, we need to highlight the fact that its closure in the space $W^{k,p}(\Omega)$ is known as $W_0^{k,p}(\Omega)$. Broadly speaking, $W^{k,p}(\Omega) \neq W_0^{k,p}(\Omega)$, but on $\mathbb{R}^n$,

$$W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n).$$

Comprehensive ideas and information about Sobolev spaces can be found in [1], [3], [9], [15], [18], [24], [49] and [78].

### 1.2.2 Multidimensional Hardy Inequalities

As we saw, original versions of the famous inequality were defined in a straight forward manner, but in general it is quite hard to provide a precise definition of what a Hardy–type inequality is, mainly because of several possibilities of generalizing it, see [2], [4], [11], [48], [61] and [62]. An adequately precise definition that is valid for many variations can be found in [66]:

**Definition 1.2.2 (Hardy–type inequality in $\mathbb{R}^n$)** *The following inequality holds*

$$\left(\int_\Omega |\nabla u(\mathbf{x})|^p v(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{p}} \geq C \left(\int_\Omega |u(\mathbf{x})|^q w(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}}$$

*for $\Omega \subset \mathbb{R}^n$ and $w(\mathbf{x}), v(\mathbf{x})$ being real–valued weight functions with $0 < q \leq \infty, p \in [0, \infty]$.*

It is important to highlight the fact that for some $C > 0$ we might have that this type of inequality does not hold. There must exist necessary and sufficient conditions such that these inequalities hold firstly in one dimension and these conditions have been thoroughly studied in the past thirty years. Most often the conditions involve integral criteria for the weight functions $w$ and $v$. As specified before, the material available concerning Hardy inequalities is extensive.

For example the classical generalization of Hardy inequality to higher dimensions may be considered one other precise definition as follows

**Proposition 1.2.1** *The following inequality holds*

$$\int_{\mathbb{R}^n} |\nabla u|^p d\mathbf{x} \geq \left|\frac{n-p}{p}\right|^p \int_{\mathbb{R}^n} \frac{|u|^p}{|\mathbf{x}|^p} d\mathbf{x}, \tag{1.2.2}$$

*with a sharp constant.*

Here $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$ represents the gradient of function $u$, while the $x_i$ elements for $1 \leq i \leq n$ describe our vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. The constant is sharp and various mathematical publications, such as [5], [7] and [59] use this fact to prove numerous other interesting results that will be presented in a later chapter.

A proof for (1.2.2) can be found, for example, in the book by Opic and Kufner [66]. Moreover, a classic book that provides an inspiration for numerous mathematicians, but not only, is the book by G.H. Hardy himself along with Pólya , [55], where they presented the original form of a Hardy inequality.

Making another reference to (1.2.2), it is rather easy to observe that by allowing $p = 2$, the result is exactly the *uncertainty principle*, see [36]. If we instead want to obtain the uncertainty principle in two dimensions, this can not be done unless there exist some weak singularities. The weak amendments are introduced by adding a logarithmic term and extra conditions on function $u$.

As an example, one could obtain

$$\int_{\mathbb{R}^2} |\nabla u|^2 d\mathbf{x} \geq C \int_{\mathbb{R}^2} \frac{|u|^2}{|\mathbf{x}|^2 \left(1 + \ln^2 |\mathbf{x}|\right)} d\mathbf{x}, \quad \text{for} \quad \int_{|\mathbf{x}|=1} u(\mathbf{x}) = 0$$

and

$$\int_{\mathbb{R}^2} |\nabla u|^2 d\mathbf{x} \geq C \int_{\mathbb{R}^2} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad \text{for} \quad \int_{|\mathbf{x}|=r} u(\mathbf{x}) = 0 \quad \forall r > 0$$

as it can be seen in [26], [86], [99], [100] and [122].

#### 1.2.2.1 Bounded domains

Consider $\Omega$ to be an open, connected set in $\mathbb{R}^n$ such that the boundary $\partial\Omega$ is non-empty and $1 < p < \infty$. Let us write that for $\delta(\mathbf{x})$ being the distance function from a point $\mathbf{x} \in \Omega$ to the boundary $\partial\Omega$, we have

$$\delta(\mathbf{x}) := \inf\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in \mathbb{R}^n \setminus \Omega\}.$$

We can introduce the Hardy inequality for a domain $\Omega$ as follows

$$\int_\Omega |\nabla u(\mathbf{x})|^p d\mathbf{x} \geq C_{p,n,\Omega} \int_\Omega \frac{|u(\mathbf{x})|^p}{\delta(\mathbf{x})^p} d\mathbf{x}, \quad u \in C_0^\infty(\Omega). \tag{1.2.3}$$

We shall highlight the fact that the inequality will be *mathematically valid*, if the positive constant $C_{p,n,\Omega}$ exists and depends on $p, n$ and $\Omega$, but not on the function $u$. For example, in [75] it was shown that if $n < p < \infty$, inequality (1.2.3) holds for any proper subset of $\mathbb{R}^n$.

Therefore, a proper question to ask ourselves once we established whether the inequality is valid, is how we can find the best constant, provided there exists. The aim is to determine

$$\mu_p(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p d\mathbf{x}}{\int_\Omega \frac{|u|^p}{|\delta|^p} d\mathbf{x}}, \tag{1.2.4}$$

if it exists. It is evident that inequality (1.2.3) holds for $\mu_p(\Omega) > 0$ and such the best possible constant will be $\mu_p(\Omega)$. More specific, for an interval it was shown in [54] and [55] for one dimension that the optimal constant is $\frac{(p-1)^p}{p^p}$, but for larger dimensions, the situation becomes more complicated. As an example, in [84] and [85] it is proven that for a convex smooth bounded domain $\Omega$ with boundaries of class $C^2$

$$\mu_p(\Omega) \leq C_{p,n,\Omega} := \frac{(p-1)^p}{p^p}.$$

For the $1 < p \leq n$ case, we encounter difficulties since geometrical assumptions on the boundary are necessary.

What if we assume that $\Omega$ is a punctured disc in $\mathbb{R}^2$ and $p = 2$ ?

Article [84] provides an alternative in the sense that if we evaluate $\mu_p(\Omega)$ over radially symmetric functions and $\Omega = \mathbb{R}^n \setminus \{0\}$ then

$$\mu_p(\Omega) = \left|1 - \frac{n}{p}\right|^p.$$

Hence the following more general result holds

$$\int_\Omega |\nabla u|^p d\mathbf{x} \geq \left|\frac{n-p}{p}\right|^p \int_\Omega \frac{|u|^p}{|\mathbf{x}|^p} d\mathbf{x},$$

where the boundedness does not change the sharpness of the constant involved.

Various boundary conditions were introduced in the literature, but we shall briefly provide a number of results from different authors which contributed on this topic.

As examples, we shall look at [25], [26] and [27] where it was proven that a bounded region $\Omega \subset \mathbb{R}^n$ has the Hardy property, if there exist $a, c \in \mathbb{R}_+$ such that

$$\int_\Omega \frac{|u|^2}{d(\mathbf{x})^2} dvol \leq c \int_\Omega |\nabla u|^2 dvol + \int_\Omega |u|^2 dvol \tag{1.2.5}$$

for all $u \in W_0^{1,2}(\Omega)$ where

$$d(\mathbf{x}) := \min\{\rho_{Riemann}(\mathbf{x}, \mathbf{y}) : \mathbf{y} \in \partial\Omega\}.$$

and $\rho_{Riemann}(\mathbf{x}, \mathbf{y})$ being the extension of the Riemannian distance function to $\overline{\Omega} \times \overline{\Omega}$.

Another application corresponding to higher order operators(i.e. bi-harmonic operators) can be observed in the following version of the Rellich's inequality, see [62].

**Theorem 1.2.3 (Rellich)** *Let $n > 4$ and $\Omega$ be a bounded domain in $\mathbb{R}^n$, then*

$$\int_\Omega |\Delta u|^2 d\mathbf{x} \geq \frac{n^2(n-4)^4}{16} \int_\Omega \frac{u^2}{|\mathbf{x}|^4} d\mathbf{x}, \quad \forall u \in W_0^{2,2}(\Omega). \tag{1.2.6}$$

One step further in the study of Hardy inequalities is the discovery of **improved optimal Hardy inequalities**. Provided that the best possible constant has been found, one should commence adding positive reminder terms containing singularities to the said inequality. Considering the weight function that is used in the inequality, we could choose a various type of remainder terms.

Let us make use of (1.2.2). If we want to improve this inequality and its extensions, it is imperative to add remainder terms similar to the main term, only with different exponents. As a proof, this has already been done in [48] as follows:

**Theorem 1.2.4 (Gazzola, Grunau, Mitidieri)** *Let $\Omega \subset \mathbb{R}^n$ be a sufficiently smooth domain and let $k \in \mathbb{N}$ with $2k \leq n$. Then there exist constants $c_1, c_2, \ldots, c_k$ depending only on $k, n$ and $|\Omega|$, such that for every $u \in W^{2,k}(\Omega)$ with $\Delta^j u|_{\partial\Omega} = 0$ for $j \in \mathbb{N}_0$ and $2j < k$ there holds:*

$$\int_\Omega |\Delta^m u|^2 d\mathbf{x} \geq \frac{1}{4^{2m}} \left( \prod_{j=1}^{2m} [n + 4(m-j)]^2 \right) \int_\Omega \frac{|u|^2}{|\mathbf{x}|^{4m}} d\mathbf{x} + \sum_{l=1}^{2m} c_l \int_\Omega \frac{|u|^2}{|\mathbf{x}|^{4m+2}} d\mathbf{x},$$

*if $k = 2m, m \in \mathbb{N}$ and*

$$\int_\Omega |\nabla \Delta^m u|^2 d\mathbf{x} \geq \frac{1}{4^{2m+1}} \left( \prod_{j=1}^{2m+1} [n + 4(m-j) + 2]^2 \right) \int_\Omega \frac{|u|^2}{|\mathbf{x}|^{4m+2}} d\mathbf{x} + \sum_{l=1}^{2m+1} c_l \int_\Omega \frac{|u|^2}{|\mathbf{x}|^{4m+2-2l}} d\mathbf{x},$$

*if $k = 2m + 1$ with $m \in \mathbb{N}_0$.*

It is worth mentioning that the techniques used in the proof of the above inequalities do, indeed, involve reduction of dimension.

One other interesting example of a Hardy inequality is an infinite series improvement of (1.2.2). It involves logarithmic weight functions and it is presented in [42]:

**Theorem 1.2.5 (Filippas, Tertikas)** *Let*

$$X_1(t)=\frac{1}{1-\ln(t)},\dots,X_k(t)=X_1(X_{k-1}(t)),\quad k=2,3,\dots$$

*and* $D\geq\sup_{x\in\Omega}|\mathbf{x}|$. *Then the following inequality holds for any* $u\in W_0^{1,2}(\Omega)$ *:*

$$\int_\Omega|\nabla u|^2d\mathbf{x}\geq\frac{(n-2)^2}{n^2}\int_\Omega\frac{|u|^2}{|\mathbf{x}|^2}d\mathbf{x}+\frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega\frac{1}{|\mathbf{x}|^2}X_1^2\left(\frac{|\mathbf{x}|}{D}\right)X_2^2\left(\frac{|\mathbf{x}|}{D}\right)\dots X_i^2\left(\frac{|\mathbf{x}|}{D}\right)|u|^2d\mathbf{x}.$$

*Also, for each* $k=2,3,\dots$ *the constant* $\frac{1}{4}$ *is optimal for the corresponding* $k$*–improved Hardy inequality, hence*

$$\frac{1}{4}=\inf_{u\in W_0^{1,2}(\Omega)}\frac{\int_\Omega|\nabla u|^2d\mathbf{x}-\frac{(n-2)^2}{n^2}\int_\Omega\frac{|u|^2}{|\mathbf{x}|^2}d\mathbf{x}-\frac{1}{4}\sum_{i=1}^{\infty}\int_\Omega\frac{1}{|\mathbf{x}|^2}X_1^2\left(\frac{|\mathbf{x}|}{D}\right)X_2^2\left(\frac{|\mathbf{x}|}{D}\right)\dots X_i^2\left(\frac{|\mathbf{x}|}{D}\right)|u|^2d\mathbf{x}}{\int_\Omega\frac{1}{|\mathbf{x}|^2}X_1^2\left(\frac{|\mathbf{x}|}{D}\right)X_2^2\left(\frac{|\mathbf{x}|}{D}\right)\dots X_i^2\left(\frac{|\mathbf{x}|}{D}\right)|u|^2d\mathbf{x}}$$

Article [48] provides numerous new types of remainder terms, which improve (1.2.2):

**Theorem 1.2.6 (Gazzola, Grunau, Mitidieri)** *Consider* $n>p$ *and* $\Omega\subset\mathbb{R}^n$ *a bounded domain. Let* $e_n=|B(0,1)|$ *and* $|\Omega|$ *be the* $n$*–dimensional Lebesgue measure of the unit ball and of the domain* $\Omega$ *respectively.*

- *If* $p\geq 2$*, then for every* $u\in W_0^{1,p}(\Omega)$ *we have*

$$\int_\Omega|\nabla u|^pd\mathbf{x}\geq\left(\frac{n-p}{p}\right)^p\int_\Omega\frac{|u|^p}{|\mathbf{x}|^p}d\mathbf{x}+\frac{c_{p,n}}{|\Omega|^{\frac{p}{n}}}\int_\Omega|u|^pd\mathbf{x};$$

- *If* $1<p<2$*, then there exists a constant* $C>0$ *such that for every* $u\in W_0^{1,p}(\Omega)$ *the following inequality holds:*

$$\int_\Omega|\nabla u|^pd\mathbf{x}\geq\left(\frac{n-p}{p}\right)^p\int_\Omega\frac{|u|^p}{|\mathbf{x}|^p}d\mathbf{x}+C\frac{e_n^{\frac{p}{n}}}{|\Omega|^{\frac{p}{n}}}\int_\Omega|u|^pd\mathbf{x},$$

*where*

$$C=\frac{1}{2}p(p-1)^{1+\frac{2-p}{3-p}}2^{\frac{(p-1)(p-2)}{p}}\left[\left(\frac{n-p}{2-p}\right)^p+1\right]^{\frac{p-2}{p}}C_0(p).$$

In terms of the last constant, $C_0(p)$ the same article contains a lower bound which is written as

$$C_0(p)\geq(2\pi)^{-\frac{1}{8}}\approx 0.8.$$

This type of remainder terms can be observed in other Hardy inequalities that improved a result from E.B.Davies, for example:

**Theorem 1.2.7 (Gazzola, Grunau, Mitidieri)** *Consider* $p\geq 2$, $\Gamma>0$ *constant and let* $\Omega\subset\mathbb{R}^n$ *be a bounded domain and let* $e_n=|B(0,1)|=Vol(B(0,1))$*, with* $n\geq 2p$ *and* $B(0,1)$ *being the unit ball.*

*Then for any* $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ *it follows that*

$$\int_\Omega|\Delta u|^pd\mathbf{x}\geq\left(\frac{n(n-2p)(p-1)}{p^2}\right)^p\int_\Omega\frac{|u|^p}{|\mathbf{x}|^{2p}}d\mathbf{x}+\frac{A(n,p)}{p^{2p-1}}\gamma\left(\frac{e_n}{|\Omega|}\right)^{\frac{2}{n}}\int_\Omega\frac{|u|^p}{|\mathbf{x}|^{2p-2}}d\mathbf{x}+\Gamma\left(\frac{e_n}{|\Omega|}\right)^{\frac{2p}{n}}\int_\Omega|u|^pd\mathbf{x},$$

*where*

$$A(n,p)=\frac{4(p-1)^p(n-2p)^{p-1}n^{p-1}}{p^{2p-1}}.$$

We have to specify that the value of $\gamma$ is independent of both $p$ and $n$. It was used in [48] as the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions on the unit disk $B \subset \mathbb{R}^2$.

#### 1.2.2.2 Convex domains

As we saw in (1.2.3), the optimal constant satisfies $\mu_p(\Omega) \leq C_{p,n,\Omega} := \left(\frac{p-1}{p}\right)^p$. Book [9] contains a few theorems and definitions relevant to the topic of this section. We shall provide these statements in the following lines.

**Definition 1.2.8** *The mean distance function $\delta_M$ is defined by*

$$\delta_M(\mathbf{x})^{-2} := n \int_{\mathbb{S}^{n-1}} \rho_\nu(\mathbf{x})^{-2} d\omega(\nu),$$

*where $\rho_\nu(\mathbf{x})$ is the least distance from $\mathbf{x} \in \Omega$ to $\partial\Omega$ in the direction of either $\nu$ or $-\nu$ while $d\omega(\nu)$ is the normalized measure on $\mathbb{S}^{n-1}$.*

Moreover, we shall not present the full extent of the proof, yet we are going to highlight a few key points made by the authors and let the reader experience the joy of understanding the full proof by accessing book [9].

**Theorem 1.2.9** *If $\Omega$ is a convex domain, then*

$$\delta_{M,p} \leq \delta(\mathbf{x}),$$

*hence*

$$\int_\Omega |\nabla u|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \frac{|u|^p}{\delta(\mathbf{x})^p} d\mathbf{x}.$$

Using the above theorem, the following holds

**Theorem 1.2.10** *Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a convex domain and suppose there is a point $P \in \partial\Omega$ such that $\Omega \in C^2$ in a neighbourhood of $P$. Then*

$$\mu_p(\Omega) := \inf_{u \in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p d\mathbf{x}}{\int_\Omega \left|\frac{u}{\delta}\right|^p d\mathbf{x}} = C_{p,n,\Omega}.$$

The proof is an application of **Theorem 1.2.9** and it is quite interesting to analyse and dissect the abundance of ideas presented. More specifically, in order to prove the result of **Theorem 1.2.10**, it is sufficient to prove that $\mu_p \leq C_{p,n,\Omega}$. From the very definition of a sharp constant the sufficiency is justified, since once we find a higher constant for which the inequality holds, we found our needed estimator.

As an example, paper [16] expresses a remarkable interest towards **Theorem 1.2.9**, where we have a convex domain, but in this case $p = 2$. The result of the aforementioned theorem is proven in the following statement:

**Theorem 1.2.11 (Brezis, Marcus)** *For every smooth bounded domain $\Omega \in \mathbb{R}^n$, there exists $\lambda_\Omega \in \mathbb{R}$ such that*

$$\int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \int_\Omega \frac{|u|^2}{\delta^2(\mathbf{x})} d\mathbf{x} + \lambda_\Omega \int_\Omega |u|^2 d\mathbf{x}, \quad u \in W_0^{1,2}(\Omega). \tag{1.2.7}$$

The optimal value of $\lambda_\Omega$ takes the form

$$\lambda_\Omega^* := \inf_{u\in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 d\mathbf{x} - \frac{1}{4}\int_\Omega \frac{|u|^2}{\delta^2(\mathbf{x})} d\mathbf{x}}{\int_\Omega |u|^2 d\mathbf{x}}$$

and it has been established in the article by Avkadiev and Wirths [8]. From literature, it was discovered that for there exist bounded domains such that $\lambda_\Omega^* < 0$, but for a category of convex domains with a $C^2$ boundary, it was shown that

$$\lambda_\Omega^* \geq \frac{1}{4\text{diam}^2(\Omega)}.$$

Taking the previous results into consideration and denoting $D_\Omega := \text{diam}(\Omega)$ inequality (1.2.7) becomes

$$\int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4}\int_\Omega \frac{|u|^2}{\delta^2(\mathbf{x})} d\mathbf{x} + \frac{1}{4D_\Omega^2}\int_\Omega |u|^2 d\mathbf{x}, \quad u \in W_0^{1,2}(\Omega).$$

In addition, curiosity rose as to whether one could replace the diameter with the volume of $\Omega$ and such to have the following identity

$$\lambda_\Omega^* \geq \alpha(\text{vol}_\Omega)^{\frac{2}{n}},$$

for a constant $\alpha > 0$. As it was observed, the proof arrived by using the distance function presented in Definition 1.2.8. Paper [85] presents an innovative approach towards proving this statement and the reader can check [100] and [122] for a more extensive reading regarding the above inequality.

**Remarks**:

1. For $\Omega_1$ and $\Omega_2$ being two bounded Lipschitz domains in $\mathbb{R}^n$ such that $\overline{\Omega}_1 \subset \Omega_2$, we are able to define a set $\Omega_3 := \mathbb{R}^n \setminus \Omega_1$ and $\Omega := \Omega_3 \cap \Omega_2$ with $\Omega^k := \Omega_3 \cap (k\Omega_2)$, where $k \in \mathbb{N}$.

   Moreover, if $0 \in \Omega$, then

   $$\mu_p(\Omega) \leq \min\left\{C_{p,n,\Omega}, \left|\frac{n-p}{p}\right|^p\right\}$$

   and such $\mu_n(\Omega_3) = 0$. Hence, there does not exist a valid Hardy inequality of the form

   $$\int_{\mathbb{R}^n\setminus\{0\}} |\nabla u(\mathbf{x})|^n d\mathbf{x} \geq c\int_\Omega \frac{|u(\mathbf{x})|^n}{\delta(\mathbf{x})^n} d\mathbf{x}.$$

2. One last remark, regarding $\Omega$ sets, could be that if $\Omega$ is defined to be $\mathbb{R}^n \setminus B(0,R), B(0,R)\setminus B(0,r)$ for $0 < r < R$ or $B(0,R)\setminus\{0\}$, then

   $$\mu_2(\Omega) = \begin{cases} 0, & n = 2, \\ \frac{1}{4}, & n \geq 3. \end{cases}$$

The following theorem includes a refined version of the Theorem 1.2.11, where we have the weights $\delta(\mathbf{x})^\sigma$.

**Theorem 1.2.12** *For any* $u \in C_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4}\int_\Omega \frac{|u(\mathbf{x})|^2}{\delta_M(\mathbf{x})^2} d\mathbf{x} + \frac{3}{2}K(n)\int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{2}{n}}} d\mathbf{x} \tag{1.2.8}$$

*where* $K(n) := n\left[\frac{s_{n-1}}{n}\right]^{\frac{2}{n}}, s_{n-1} := |\mathbb{S}^{n-1}|$ *and*

$$\Omega_{\mathbf{x}} := \{\mathbf{y} \in \Omega : \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in \Omega, \forall t \in [0,1]\}.$$

*If* $\Omega$ *is convex, then* $\Omega_{\mathbf{x}} = \Omega$ *and for any* $u \in C_0^1(\Omega)$

$$\int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4}\int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + \frac{3K(n)}{2|\Omega|^{\frac{2}{n}}}\int_\Omega |u(\mathbf{x})|^2 d\mathbf{x} \tag{1.2.9}$$

It can be seen in [9] that, as a proof, the authors consider defining new least distances such as

$$\tau_\nu(\mathbf{x}) := \min\{s > 0 : \mathbf{x} + s\nu \notin \Omega\};$$
$$\rho_\nu(\mathbf{x}) := \min\{\tau_\nu(\mathbf{x}), \tau_{-\nu}(\mathbf{x})\};$$
$$\mu_\nu(\mathbf{x}) := \max\{\tau_\nu(\mathbf{x}), \tau_{-\nu}(\mathbf{x})\};$$
$$D_\nu(\mathbf{x}) := \tau_\nu(\mathbf{x}) + \tau_{-\nu}(\mathbf{x});$$

and then proceed to rewriting the gradient in terms of partial derivatives on the sphere, after which Cauchy-Schwarz inequality is used.

It is clear to observe that Hardy inequalities on convex domains can be improved by adding various weight functions. Another result of S.Filippas and Tertikas [42] estimates the constant from Theorem 1.2.11 in terms of the interior diameter $D_{int}(\Omega) := 2\sup_{\mathbf{x}\in\Omega}\delta(\mathbf{x})$. More specifically, they prove that for a convex $\Omega$ then

$$\lambda_\Omega^* \geq \frac{3}{D_{int}(\Omega)^2}.$$

One other class of inequalities which extend the usual Hardy inequality is called Hardy - Sobolev - Maz'ya inequalities. As an example, one could make use of the result in [42], which states that if $\Omega$ has a finite interior diameter, $D_{int}(\Omega)$, and $\Delta\delta \geq 0$, for $p = 2$, then

$$\int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4}\int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + C_\Omega\left(\int_\Omega |u(\mathbf{x})|^{\frac{2n}{n-2}} d\mathbf{x}\right)^{\frac{n-2}{n}}.$$

This result holds for $n \geq 3$ and $u \in C_0^\infty(\Omega)$.

One could mention also that for a convex domain $\Omega \in \mathbb{R}^n$ and any compact subset $K$ of $\Omega$, there exists a convex $n$-dimensional polytope $T$ such that $K \subset \text{int}T \subset \Omega$; hence, for any $u \in C_0^\infty(\Omega)$, there exists a convex $n$- dimensional polytope $T$ such that

$$\text{supp u} \subset \text{int}T \subset \Omega.$$

Using this aspect, the process of proving various $n$-dimensional Hardy inequalities will be reduced to the application inequalities in one dimension. In [2], [5] and [7], one could see that the results presented can be used in the study of spectral properties of elliptic differential operators with a certain degree of degeneracy at $\partial\Omega$ measured in terms of $\delta(\mathbf{x})$.

#### 1.2.2.3 Hardy inequality for general domains

Since we observed a variety of important results related to the classical Hardy inequality on bounded and convex domains, we should turn our attention to more general aspects of Hardy inequalities such as $L^p$ and half spaces.

As a proof of the *Hardy-Rellich inequality* in the whole space, article [19] provides the following theorem

**Theorem 1.2.13 (Whole space Hardy-Rellich)** *Assume $d \geq 3$. Then, for any $u \in C_c^\infty(\mathbb{R}^d)$ it holds*

$$\int_{\mathbb{R}^d} |\Delta u|^2 d\mathbf{x} \geq C(d) \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|\mathbf{x}|^2} d\mathbf{x}, \tag{1.2.10}$$

*where*

$$C(d) := \begin{cases} \frac{d^2}{4}, & d \geq 5 \\ 3, & d = 4 \\ \frac{25}{36}, & d = 3. \end{cases}$$

As a comparison, we shall consider results that resemble the fact that for functions $u \in H^1(\mathbb{R}^d)$, where $d \geq 3$, the following holds

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x}. \tag{1.2.11}$$

Of course, the literature regarding higher order analogues of Hardy inequalities is extensive and we shall provide a large list of references at the end of this document, but as an introduction we present a few well known results from which we shall create our research interest.

Another version of (1.2.5), but for the general domain $\mathbb{R}^n$ was proved in [118] and presented as follows

**Theorem 1.2.14 (Maz'ya generalisation)** *For $u \in C_0^\infty(\mathbb{R}^n_+), p > 1, \tau > 0$ and $\alpha(p, \tau)$ a positive constant, the following inequality holds*

$$\int_{\mathbb{R}^n_+} |\nabla u|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{|x_n|^p} d\mathbf{x} + \alpha(p,\tau) \int_{\mathbb{R}^n_+} \frac{|u|^p}{|x_n|^{p-\tau}(x_{n-1}^2 + x_n^2)^{\frac{\tau}{2}}} d\mathbf{x} \tag{1.2.12}$$

The above inequality was considered an open problem and a generalisation to the following theorem found in [86]

**Theorem 1.2.15 (Maz'ya)** *For $u \in C_0^\infty(\mathbb{R}^n)$, the following holds*

$$\int_{\mathbb{R}^n} |x_n|^{p-1} |\nabla u|^p d\mathbf{x} \geq \left(\frac{1}{2p}\right)^p \int_{\mathbb{R}^n} \frac{|u|^p}{(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} d\mathbf{x} \tag{1.2.13}$$

*and if $p = 2$ when $u = |x_n|^{-\frac{1}{2}} v(\mathbf{x})$, then*

$$\int_{\mathbb{R}^n} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \int_{\mathbb{R}^n} \frac{|v|^2}{x_n^2} d\mathbf{x} + \frac{1}{16} \int_{\mathbb{R}^n} \frac{|v|^2}{|x_n|(x_{n-1}^2 + x_n^2)^{\frac{1}{2}}} d\mathbf{x}. \tag{1.2.14}$$

Moreover, in his paper [127], Prof. Yafaev introduced fractional order generalizations of (1.2.2) and discovered the best constants in higher order by using the Fourier transform:

**Theorem 1.2.16 (Yafaev)** *Let $u \in W^{l,2}(\mathbb{R}^n)$, then*

$$C_l \int_{\mathbb{R}^n} |\xi|^{2l} |u(\xi)|^2 d\xi \geq \int_{\mathbb{R}^n} |\mathbf{x}|^{-2l} |\hat{u}(\mathbf{x})|^2 d\mathbf{x}, \quad \text{if} \quad l < \frac{n}{2}$$

*and*

$$C_l \int_{\mathbb{R}^n} |\xi|^{2l} |u(\xi)|^2 d\xi \geq \int_{\mathbb{R}^n} |\mathbf{x}|^{-2l} |u(\mathbf{x}) - \sum_{|\alpha| \leq m} (\alpha!)^{-1} (\partial^\alpha u)(0) \mathbf{x}^\alpha|^2 d\mathbf{x},$$

*if*

$$l > \frac{n}{2}, \quad l - \frac{n}{2} \notin \mathbb{Z} \quad \text{and} \quad m = \left[ l - \frac{n}{2} \right].$$

The paper presents the best constants as being

$$C_l = 2^{-2l} \max \left\{ \frac{\Gamma^2\left(\frac{n-2l}{4}\right)}{\Gamma^2\left(\frac{n+2l}{4}\right)}, \frac{\Gamma^2\left(\frac{n-2l+2}{4}\right)}{\Gamma^2\left(\frac{n+2l+2}{4}\right)} \right\}$$

and in the case if $l$ is an integer, then the constants become

$$C_l = 2^{2l} (n-2l)^{-2} (n-2l+4)^{-2} \dots (n+2l-4)^{-2}.$$

The author made use of function decomposition into spherical harmonics in order for the Mellin transform and diagonalization processes to be applied on a one dimensional inequality.

Before introducing other inequalities, we should make use of a definition given in [9]

**Definition 1.2.17** *Suppose $\Omega$ does not contain a half-space and the set*

$$N(\mathbf{x}) := \{\mathbf{y} \in \partial\Omega : \delta(\mathbf{x}) = |\mathbf{x} - \mathbf{y}|\}$$

*to be the **near set** of $\mathbf{x}$ on $\partial\Omega$. For $\mathbf{x} \in \Omega, \mathbf{y} \in N(\mathbf{x})$ and*

$$\lambda := \sup \{t \in (0, \infty) : \mathbf{y} \in N(\mathbf{y} + t[\mathbf{x} - \mathbf{y}])\},$$

*the point $p(\mathbf{x}) := \mathbf{y} + \lambda(\mathbf{x} - \mathbf{y})$ is called the **ridge point** of $\mathbf{x}$ in $\Omega$ and the set $\mathcal{R}(\Omega) = \{p(\mathbf{x}) : \mathbf{x} \in \Omega\}$ is called the **ridge** of $\Omega$.*

Let us consider some examples where we shall highlight the connection between the curvature of the boundary $\partial\Omega$ and the distance function $\delta(\mathbf{x})$.

**Proposition 1.2.2** *Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a domain with ridge $\mathcal{R}(\Omega)$ and a sufficiently smooth boundary for Green's theorem to hold. Consider $\delta(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \mathbb{R}^n \setminus \Omega)$. Then for all $f \in C_0^\infty(\Omega \setminus \overline{\mathcal{R}(\Omega)})$ and $p \in (1, \infty)$ the following inequality holds*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p d\mathbf{x} \geq \left( \frac{p-1}{p} \right)^p \int_\Omega \left( 1 - \frac{p\delta\Delta\delta}{p-1} \right) \frac{|f|^p}{\delta^p} d\mathbf{x}. \tag{1.2.15}$$

In comparison to the above example, the below inequalities will introduce a few requirements that are necessary to obtain more general results.

**Proposition 1.2.3** *Let $\Omega \subset \mathbb{R}^n, n \geq 2$ satisfy the conditions of Proposition 1.2.2 and furthermore, assume that*

*(i)* $\sum = \overline{\mathcal{R}(\Omega)}$ *has Lebesgue measure zero and is the intersection of a decreasing family of open neighbourhoods* $\{S_\epsilon : \epsilon > 0\}$ *with smooth boundaries and a unit inward normal* $\eta_\epsilon(\mathbf{x}), \mathbf{x} \in \partial S_\epsilon$*;*

*(ii) for all sufficiently small* $\epsilon$

$$(\nabla\delta \cdot \eta_\epsilon)(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \partial S_\epsilon$$

*and*

*(iii)*

$$\frac{p-1}{p} \geq [\delta\Delta\delta](\mathbf{x}), \quad \mathbf{x} \in \Omega \setminus \overline{\mathcal{R}(\Omega)}.$$

*Then, for all* $u \in C_0^\infty(\Omega)$ *and* $p \in (1, \infty)$*,*

$$\int_\Omega |\nabla\delta \cdot \nabla f|^p d\mathbf{x} \geq \left(\frac{p-1}{p}\right)^p \int_\Omega \left(1 - \frac{p\delta\Delta\delta}{p-1}\right) \frac{|f|^p}{\delta^p} d\mathbf{x}.$$

The proof of this proposition includes the application of monotone convergence theorem and taking into the account the contribution of the boundary of $S_\epsilon$, see [9].

We shall introduce a few results of Dr. Tidblom from which our research was highly inspired by. More specific, he proves in [116] that in the upper half-space in $\mathbb{R}^n$, for $u \in C_0^\infty(\mathbb{R}^n)$, the inequality holds

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 d\mathbf{x} \geq \sum_{k=0}^{n-1} \int_{\mathbb{R}^n_+} \frac{|u|^2}{4(x_{n-k}^2 + \cdots + x_n^2)} d\mathbf{x}$$

where $\frac{1}{4}$ is the optimal constant.

Another important result of Dr. Tidblom includes higher order differential forms and relates to the distance to the boundary of a set $\Omega$.

**Theorem 1.2.18** *For* $u \in C_0^\infty(\Omega)$ *and* $\delta(\mathbf{x}) = \operatorname{dist}(\mathbf{x}, \partial\Omega_\alpha)$*, the following inequality holds*

$$\int_{\Omega_\alpha} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \int_{\Omega_\alpha} \frac{|u|^2}{\delta(\mathbf{x})^2} d\mathbf{x}$$

*where* $\Omega_\alpha$ *is a conical sector of angle* $\alpha$ *in* $\mathbb{R}^2$*.*

This was analysed numerically in [26] and the value of $\alpha$ is approximately 4.856. In terms of the exact value of the angle, the same article proves that

$$\begin{aligned} \alpha &= \pi + 4 \arctan\left(\frac{4\Gamma\left(\frac{3}{4}\right)^2}{\Gamma\left(\frac{1}{4}\right)^2}\right) \\ &= \pi + 4 \arctan\left(\frac{1}{2} \cdot \frac{3^2-1}{3^2} \cdot \frac{5^2-1}{5^2} \cdot \ldots\right). \end{aligned}$$

One other important result provides a corner stone to the theory of existence and uniqueness of solutions of the stationary Navier - Stokes equations

$$\begin{cases} \nu\Delta\mathbf{v} + \sum_{k=1}^{n} v_k \frac{\partial \mathbf{v}}{x_k} = -\nabla p + \mathbf{f}(\mathbf{x}) \\ \text{div } \mathbf{v} = 0, \end{cases}$$

where $\mathbf{f}(\mathbf{x})$ is a given vector valued function and $\mathbf{v}(\mathbf{x})$ and $p(\mathbf{x})$ are to be found.

The theory of Hardy inequalities has been developed on a rigorous analysis background and the reader can access this information by inspecting [1], [3], [9], [10], [13], [15], [18], [23], [24], [33], [37], [38], [43], [49], [64], [78], [86], [87], [88], [97], [110], [114] and [121]. It is to be specified that a brief background on the tools, which we shall use in this thesis to prove the results obtained, shall be provided, but we will also provide citations towards all the relevant material that can ease the learning process.

If the reader is also interested in further reading about the process in which sharp constants for multi-dimensional Hardy inequalities are found, then papers [5], [7], [28], [29], [30], [41], [48], [72], [73], [75], [77], [82], [84], [85], [116], [117], [118] and [127] provide substantial information.

### 1.2.3 Multiparticle Hardy Inequalities

The theory behind multiparticle Hardy inequalities has been extensively worked upon in the last decades and it had as motivation the Schrödinger operators describing many particle systems. Results are obtained in papers [58], [59], [82] and [117] for three dimensional particles from which the question whether it is possible to extend them to higher dimensions arises.

#### 1.2.3.1 The uncertainty principle

Let us consider the state of a system consisting of a single particle in $\mathbb{R}^3$. This physical quantity can be described by a wave function $\psi \in L^2(\mathbb{R}^3)$ that satisfies

$$\int_{\mathbb{R}^3} |\psi(\mathbf{x})|^2 d\mathbf{x} = 1.$$

It is a well known fact that the probability density function of the position of our particle is expressed by $|\psi|^2$. Moreover, the probability of the particle being in a set $N$ can be written as

$$\int_{\mathbb{R}^3} |\psi(\mathbf{x})|^2 \mathcal{X}_N d\mathbf{x},$$

where $\mathcal{X}_N$ is the characteristic function of $N$. We should mention that by normalising Planck's constant, the momentum of the particle will become $-i\nabla\psi(\mathbf{x})$.

If we consider the particle's momentum to be $\mathbf{p}$, then, by making use of Plancherel's theorem, we obtain

$$\int_{\mathbb{R}^3} |\hat{\psi}(\mathbf{p})|^2 d\mathbf{p} = \int_{\mathbb{R}^3} |\psi(\mathbf{x})|^2 d\mathbf{x} = 1,$$

where $|\hat{\psi}(\mathbf{p})|^2$ will represent the probability density of the momentum.

Making use of the notations above, we can present **Heisenberg's uncertainty principle**. The notable result states that the position $\mathbf{x}$ of the particle and the momentum $\mathbf{p}$ can not be determined in the same time. More specifically, by using the commutator identity

$$[\mathbf{p}\cdot\mathbf{a}, \mathbf{x}\cdot\mathbf{b}] = (\mathbf{p}\cdot\mathbf{a})(\mathbf{x}\cdot\mathbf{b}) - (\mathbf{x}\cdot\mathbf{b})(\mathbf{p}\cdot\mathbf{a}) = -i(\mathbf{a}\cdot\mathbf{b}), \tag{1.2.16}$$

where $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$, we can deduce that $|\mathbf{x}|$ and $|\mathbf{p}|$ can not be simultaneously small.

This important result has a description in the following Hardy inequality

$$\begin{aligned}\int_{\mathbb{R}^3} |\mathbf{p}\hat{\psi}(\mathbf{p})|^2 d\mathbf{p} &= \int_{\mathbb{R}^3} |\nabla\psi(\mathbf{x})|^2 d\mathbf{x} \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3} \frac{|\psi(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x}.\end{aligned}$$

With an appropriate choice for $\mathbf{a}$ and $\mathbf{b}$ and a function $\psi \in C_0^\infty(\mathbb{R}^3)$, book [9] presents a simple, yet not intuitive consequence, which states that if $\int_{\mathbb{R}^3} |\psi(\mathbf{x})|^2 d\mathbf{x} = 1$, then

$$\left(\int_{\mathbb{R}^3} |\psi(\mathbf{x})|^2 |\mathbf{x}|^2 d\mathbf{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\mathbf{p}\hat{\psi}(\mathbf{p})|^2 d\mathbf{p}\right)^{\frac{1}{2}} \geq \frac{3}{2}. \tag{1.2.17}$$

On $\mathbb{R}^n$, for $n \geq 3$ we have that

$$\left(\int_{\mathbb{R}^n} |\psi(\mathbf{x})|^2 |\mathbf{x}|^2 d\mathbf{x}\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |\mathbf{p}\hat{\psi}(\mathbf{p})|^2 d\mathbf{p}\right)^{\frac{1}{2}} \geq \frac{n}{2}, \tag{1.2.18}$$

provided that $\int_{\mathbb{R}^n} |\psi(\mathbf{x})|^2 d\mathbf{x} = 1$. Moreover, the above inequalities known as **Heisenberg's inequality** are a direct consequence of Hardy's inequality, the only difference being that the Heisenberg inequalities have a smaller constant in their structure.

As an example for applications in physics apart from the one we saw above, consider the results from J.Dolbeault, M.J. Esteban and Séré. They proved an optimal Hardy inequality using Spinors and Pauli matrices, [32].

**Theorem 1.2.19 (Dolbeaut, Esteban, Séré)** *Let $\sigma = (\sigma_i)_{i=1,2,3}$ be the Pauli-matrices:*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

*Then*

$$\int_{\mathbb{R}^3} \left( \frac{|\sigma\cdot\nabla u|^2}{1+\frac{1}{|\mathbf{x}|}} + |u|^2 \right) d\mathbf{x} \geq \int_{\mathbb{R}^3} \frac{|u|^2}{|\mathbf{x}|} d\mathbf{x},$$

*for all $u \in W^{1,2}(\mathbb{R}^3)$.*

A specific area where Hardy inequalities are of particular use is in the context of exactly solvable quantum systems, hence the introduction of many-particle Hardy inequalities.

#### 1.2.3.2 Systems of particles

We shall focus now on systems with many quantum particles, such as electrons in solid matter or atoms with the number of protons larger than one. Considering that identical particles cannot be distinguished in the quantum world, the following figure shall provide a brief example of what type of particle systems we consider in our research.

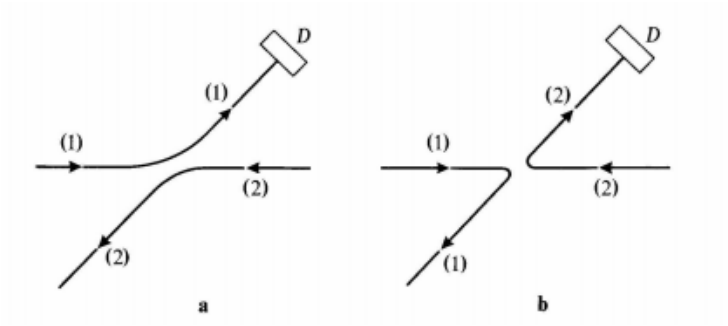



Figure 1.1: Two types of paths that the system of two colliding particles could have followed.

For example, in [59], the authors prove three important results regarding systems of two or more particles, but before we proceed in presenting them, we shall have some introductory notions.

Let us pick $N$ as being a positive integer and consider the number of particles in our system to be equal to $N$. As a consequence, we work with a vector $\mathbf{x} \in \mathbb{R}^{dN}$, such that $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,d}) \in \mathbb{R}^d$.

Define $r_{ij}$ as

$$r_{ij} := |\mathbf{x}_i - \mathbf{x}_j| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2},$$

using the usual Euclidean norm on $\mathbb{R}^d$.

**Remark**: We write $\Delta_i = \sum_{k=1}^{d} \frac{\partial^2}{\partial x_{i,k}^2}$ such that $\Delta = \sum_{i=1}^{N} \Delta_i$. In a similar manner, the gradient for each $i^{\text{th}}$ particle shall be written as $\nabla_i$.

As a first result of the above mentioned article, we state the following:

**Theorem 1.2.20 (Standard Hardy inequality for many particles)** *For $d \geq 3, u \in H^1(\mathbb{R}^{dN})$ and constant $\mathcal{C}(d,N)$ with $c_1 N^{-1} \leq \mathcal{C}(d,N) \leq c_2 N^{-1}$, where $c_1, c_2 > 0$, the following inequality holds*

$$\sum_{j=1}^{N} \int_{\mathbb{R}^{dN}} |\nabla_{x_j} u|^2 d\mathbf{x} \geq \mathcal{C}(d,N) \sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{ij}^2} d\mathbf{x} \tag{1.2.19}$$

**Remark**: For one dimensional particles, inequality (1.2.19) will hold, if the function $u$ is zero on the diagonals $\mathbf{x}_i = \mathbf{x}_j$. In this case, the optimal constant will be $\mathcal{C}(1,N) = \frac{1}{2}$.

**Theorem 1.2.21** *Let*

$$\mathcal{N}_N = \{\mathbf{x} = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N \,|\, x_i = x_j \quad \text{for some} \quad i \neq j\}.$$

*Suppose* $u \in W_0^{1,2}(\mathbb{R}^N \setminus \mathcal{N}_N)$, *then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{2} \int_{\mathbb{R}^N} \sum_{1 \leq i < j \leq N} \frac{|u|^2}{r_{ij}^2} d\mathbf{x}.$$

A proof for this statement is presented in [59], which we shall extensively use in Chapter 3.

**Remark**:

Using the classical Hardy inequality

$$\int_0^\infty \left| \frac{df}{dx} \right|^2 dx \geq \frac{1}{4} \int_0^\infty \frac{|f|^2}{|x-y|^2} dx$$

for a function $f \in W_0^{1,2}(\mathbb{R} \setminus \{y\})$ and the number of particles $N = 2$, one obtains

$$-\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \geq \frac{1}{2} \frac{1}{r_{12}^2}$$
$$\implies \sum_{i=1}^2 \int_{\mathbb{R}^2} \left| \frac{\partial f}{\partial x_i} \right|^2 d\mathbf{x} \geq \int_{\mathbb{R}^2} \frac{1}{4} \frac{|f|^2}{|x_1 - x_2|^2} d\mathbf{x},$$

for $f \in H_0^1(\mathbb{R}^2 \setminus \mathcal{N}_2)$.

By adding the above identity in the quadratic form sense for functions that resemble Theorem 1.2.21, we would replace the $\frac{1}{2}$ with $\frac{1}{2(N-1)}$ which tends to 0 as $N \to \infty$.

One other example of many-particle Hardy inequality was studied in [82] and it involves ground state representations(GSR). We have to specify that some of these results shall be referred in the next chapter to establish the motivation for the present thesis.

**Proposition 1.2.4 (GSR)** *Let* $\Omega$ *be an open set in* $\mathbb{R}^n$ *and let* $f : \Omega \to \mathbb{R}_+$ *be twice differentiable. Then, for any* $u \in C_0^\infty(\Omega)$ *and* $\alpha \in \mathbb{R}$,

$$\int_\Omega |\nabla u|^2 d\mathbf{x} = \int_\Omega \left[ \alpha(1-\alpha) \frac{|\nabla f|^2}{f^2} + \alpha \left( \frac{-\Delta f}{f} \right) \right] |u|^2 d\mathbf{x} + \int_\Omega |\nabla \tau|^2 f^{2\alpha} d\mathbf{x} \geq 0$$

*where* $\tau := f^{-\alpha} u$.

***Proof**: Since* $u = f^\alpha \tau$, *then* $\nabla u = \alpha f^{\alpha-1} (\nabla f) \tau + f^\alpha \nabla \tau$. *Therefore*

$$|\nabla u|^2 = \alpha^2 f^{2(\alpha-1)} |\nabla f|^2 |\tau|^2 + \alpha f^{2\alpha-1} (\nabla f) \cdot \nabla |\tau|^2 + f^{2\alpha} |\nabla \tau|^2.$$

*By integrating the expression over* $\Omega$, *for the term* $\alpha f^{2\alpha-1} (\nabla f) \cdot \nabla |\tau|^2$ *we get*

$$-\alpha \int_\Omega \nabla \cdot (f^{2\alpha-1} \nabla f) |\tau|^2 d\mathbf{x}.$$

*Furthermore, if we use the identity*

$$\nabla \cdot (f^{2\alpha-1} \nabla f) = (2\alpha - 1) f^{2\alpha-2} |\nabla f|^2 + f^{2\alpha-1} \Delta f,$$

*we get that*

$$\begin{aligned}\int_\Omega |\nabla u|^2 d\mathbf{x} &= \int_\Omega \Big(\alpha^2 f^{2(\alpha-1)}|\nabla f|^2|\tau|^2 + \alpha f^{2\alpha-1}(\nabla f)\cdot\nabla|\tau|^2 + f^{2\alpha}|\nabla\tau|^2\Big)d\mathbf{x}\\ &= \int_\Omega \Big(\alpha^2 f^{2(\alpha-1)}|\nabla f|^2|\tau|^2 + \alpha f^{2\alpha-1}(\nabla f)\cdot\nabla|\tau|^2\Big)d\mathbf{x} + \int_\Omega f^{2\alpha}|\nabla\tau|^2 d\mathbf{x}\\ &= \int_\Omega \alpha^2 f^{2(\alpha-1)}|\nabla f|^2|\tau|^2 d\mathbf{x} - \alpha\int_\Omega \Big((2\alpha-1)f^{2\alpha-2}|\nabla f|^2 + f^{2\alpha-1}\Delta f\Big)|\tau|^2 d\mathbf{x} + \int_\Omega f^{2\alpha}|\nabla\tau|^2 d\mathbf{x}\\ &= \int_\Omega \Big[(-\alpha^2+\alpha)f^{2\alpha-2}|\nabla f|^2 - \alpha f^{2\alpha-1}\Delta f\Big]|\tau|^2 d\mathbf{x} + \int_\Omega f^{2\alpha}|\nabla\tau|^2 d\mathbf{x}. \qquad \square\end{aligned}$$

Subsequently, $f$ is seen as the *ground state* and $\alpha$ as the *GSR weight*, while the potential term created in the above inequality

$$\alpha(1-\alpha)\frac{|\nabla f|^2}{f^2} + \alpha\left(\frac{-\Delta f}{f}\right)$$

as the *GSR potential*.

Paper [82] considers various values for the weight constant such that the inequality which results from calculations is the best possible. Moreover, the $\alpha$ values will provide aid in finding appropriate ground states for computations. The author specifies that the main advantage for working with ground state representations is that it provides an easier process of finding the sharpness of constants and allows further improvements of inequalities.

As a generalisation of inequality (3.5) from **Lemma 3.6** in [59], the same paper [82] provides the following theorem

**Theorem 1.2.22 (GSR for $N \geq 2$ particles)** *Let $\Omega := \mathbb{R}^{dN} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N\}$. Suppose the ground state function is $\rho(\mathbf{x})^2 := \sum_{i<j} |\mathbf{x}_i - \mathbf{x}_j|^2$, then*

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - N\left(\frac{N-1}{2}d - 1\right)^2 \int_\Omega \frac{|u|^2}{\rho^2} d\mathbf{x} = \int_\Omega |\nabla \rho^{-2\alpha} u|^2 \rho^{4\alpha} d\mathbf{x} \geq 0, \tag{1.2.20}$$

*for all $u \in C_0^\infty(\Omega)$, with the optimal weight $\alpha := -\frac{(N-1)d-2}{4}$.*

This result holds on $H^1(\mathbb{R}^{dN})$, unless $d = 1$ and $N = 2$, because $\operatorname{codim}(\Omega^c) = dN - d = d(N-1)$.

The same author presents a Hardy inequality for circumradii of pairs and triplets of particles in higher dimensions.

**Theorem 1.2.23 (Separation for circumradii)** *Consider $\Omega := \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \in \mathbb{R}^{dN} : \mathbf{x}_i \neq \mathbf{x}_j,\ \forall i \neq j\}$.*

*By taking the ground state $f(\mathbf{x}) := \prod_{j<k} |\mathbf{x}_j - \mathbf{x}_k|^{-(d-2)}$, we obtain*

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - (d-2)^2 \int_\Omega \left(2\alpha(1-\alpha)\sum_{1\leq i<j\leq N}\frac{1}{r_{ij}^2} - \alpha^2 \sum_{i<j<k\leq N}\frac{1}{R_{ijk}^2}\right)|u|^2 d\mathbf{x} = \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} d\mathbf{x} \geq 0, \tag{1.2.21}$$

*where $R_{ijk}$ is the circumradius associated to three non-coincident particles.*

The result holds for all $u \in C_0^\infty(\Omega)$ and $\alpha \in \mathbb{R}$. Moreover, for upper bound

$$K_{d,N} := \sup_{x\in\Omega} \frac{\sum_{i<j<k\leq N} \frac{1}{R_{ijk}^2}}{\sum_{i<j} \frac{1}{r_{ij}^2}} \leq N-2 < \infty$$

and optimal weight $\alpha := \frac{1}{2+K_{d,N}}$, (1.2.21) becomes

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - \frac{(d-2)^2}{2+K_{d,N}} \int_\Omega \sum_{i<j} \frac{|u|^2}{r_{ij}^2} d\mathbf{x} \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} d\mathbf{x} \geq 0.$$

If one assumes that $\alpha(1-\alpha) \geq 0$ and uses the bound term for $r_{ij}$, then

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - \frac{(d-2)^2}{K_{d,N}(2+K_{d,N})} \int_\Omega \sum_{i<j<k} \frac{|u|^2}{R_{ijk}^2} d\mathbf{x} \geq \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} d\mathbf{x} \geq 0.$$

More specifically, the $R_{ijk}$ term represents the radius of the circle which particles $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$ inscribe. As an extreme case, for collinear particles, the radius will tend to infinity hence

$$\lim_{R_{ijk}\to\infty} \frac{1}{2R_{ijk}^2} = 0.$$

After using geometric proprieties for circumradii, paper [59] presents the case when $N = 3$ and the following result is obtained

$$\int_\Omega |\nabla u|^2 d\mathbf{x} \geq \frac{(d-1)^2}{3} \binom{3}{3}^{-1} \frac{3}{3} \int_\Omega \sum_{i<j<k} \frac{|u|^2}{R_{ijk}^2} d\mathbf{x}.$$

Combining this with (1.2.21), the maximizing constant for the term $\frac{\alpha(1-\alpha)}{1+c\alpha^2}$ becomes

$$c := \frac{3(d-2)^2}{2(d-1)^2}(N-1)(N-2).$$

Hence

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - (1+\sqrt{1+c})^{-1}(d-2)^2 \int_\Omega \sum_{i<j} \frac{|u|^2}{r_{ij}^2} d\mathbf{x} \geq \frac{1}{1+c(1+\sqrt{1+c})^{-2}} \int_\Omega |\nabla f^{-\alpha} u|^2 f^{2\alpha} d\mathbf{x} \geq 0.$$

All the above results hold for the full space $H^1(\mathbb{R}^{dN})$ because of the fact that $\operatorname{codim}(\Omega^c) = dN - d - Nd + 2d = d \geq 3$. If we take $N = 2$, (1.2.21) becomes (1.2.20) with $\alpha = \frac{1}{2}$ and the sharp constant $\frac{(d-2)^2}{2}$.

One interesting question to ask ourselves is whether a Hardy inequality for two dimensions can be found. This is a popular topic because it has been extensively researched upon and it is related to the magnetic Schrödinger operator and the corresponding magnetic form

$$H_{\mathbf{a}}[u] = \int_{\mathbb{R}^n} |(-i\nabla - \mathbf{a})u|^2 d\mathbf{x},$$

for $u \in C_0^1(\mathbb{R}^n), n \geq 2$, where $\mathbf{a}$ represents a vector of real-valued functions from $L^2(\mathbb{R}^n)$.

As application, article [74] proves that the inequality

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{A})u|^2 d\mathbf{x} \geq C \int_{\mathbb{R}^2} \frac{|u|^2}{1+|\mathbf{x}|^2} d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^2)$$

holds, where the magnetic vector potential $\mathbf{A}$ gives the constant $C$. One particular case of the above inequality is also proved and it states that

$$\int_{\mathbb{R}^2} |(i\nabla + \mathbf{A})u|^2 d\mathbf{x} \geq C \int_{\mathbb{R}^2} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$$

holds for a constant $C > 0$. As it can be seen, the extra term added by the magnetic field provides a better inequality than the ordinary Hardy inequality, considering that the classical case does not exist in $\mathbb{R}^2$. There exist various magnetic fields that create such inequalities and they are known as *Aharonov-Bohm* type magnetic potentials.

The optimal constant was also proved to have the following form

$$C = \min_{n \in \mathbb{Z}} |\Phi - n|^2,$$

where

$$\Phi = \frac{1}{2\pi} \int B(\mathbf{x}) d\mathbf{x}, \quad B = \operatorname{curl} \mathbf{A},$$

defines the total flux of the magnetic field $B$.

Can this be generalized to $N$ particles ? Article [59] provides the answer!

The following theorem presents a generalisation of Hardy inequality for magnetic Dirichlet forms with Aharonov - Bohm type vector potentials. The new potential vector fields are associated to each particle in the quantum system.

**Theorem 1.2.24 (Aharonov-Bohm many-particle inequality)** *For* $\mathbf{x}_j = (x_{j1}, x_{j2}) \in \mathbb{R}^2, j = 1, 2, \dots N, \alpha \in \mathbb{R}$ *and the vector field*

$$\mathbf{F}_j = \alpha \left( -\sum_{k \neq j} \frac{x_{j2} - x_{k2}}{r_{jk}^2}, \sum_{k \neq j} \frac{x_{j1} - x_{k1}}{r_{jk}^2} \right),$$

*the inequality holds*

$$\int_{\mathbb{R}^{2d}} \sum_{j=1}^{N} |(i\nabla_{x_j} + \mathbf{F}_j)u|^2 d\mathbf{x} \geq D_{N,\alpha} \int_{\mathbb{R}^{2d}} |u|^2 \sum_{k \neq j} \frac{1}{r_{kj}^2} d\mathbf{x}. \tag{1.2.22}$$

*where*

$$D_{N,\alpha} := \min_{l=1,\dots,N-1} \left( \frac{\min_{k \in \mathbb{Z}} |k - l\alpha|}{l} \right)^2.$$

**Remark**: It is imperative to highlight the fact that the explicit value of $D_{N,\alpha}$, will depend on the "degree of rationality" of the magnetic flux $\alpha$.

This example of a magnetic case is relevant for models of anyons in two dimensions and it was improved in various other papers. The improvements led to the discovery of accurate bounds for the energy of the anyon gas.

Moreover, Prof. Hoffmann-Ostenhof and Prof. Laptev, as we saw above, considered the many-particle generalisation of classical Hardy inequalities in arbitrary dimensions. Recently, they studied various Hardy inequalities for bosonic and fermionic particles, which are symmetric or antisymmetric wave functions. Their research involves, among other topics, the magnetic interaction between particles in two dimensions, considering that there is no Hardy inequalities for such case.

## 1.3 Outline of the thesis

Considering the variety of results presented so far, we have to specify that throughout this thesis we shall focus on identities which resolve around many-particle Hardy inequalities and later on consider Hardy inequalities on spheres. Most generalisations have been considered in [59], [116] and [118] and we aim to introduce new techniques and methods to discover new Hardy inequalities. As a setting, we aim to start from simple cases in one or three dimensions and build a general framework for multiple particles and dimensions.

The present thesis has the following structure. Chapter 2 presents the preliminary setup which allows an uncomplicated and systematic derivation of the results. The setup involves the presentation of key ideas regarding Hardy inequalities based on which the main discoveries of this thesis had their roots.

In Chapter 3 we present a new Hardy identity valid for finitely many particles by analysing the inequalities presented in **Lemma 3.6** in [59] and **Theorem 4** in [82]. The new inequalities hold true for any dimension greater or equal to three. More specifically, the authors prove in [59] a geometrical Hardy inequality. The result involves vector fields and circumradii of simplices in dimension higher or equal to three. Our curiosity begins when the constant determined in the inequality has its sharpness yet to be discovered. Suitable vector fields are considered and through calculations, we discover a more general result that contains a specific bound for the circumradii terms in **Lemma 3.6**.

To be more precise, we extend the quadratic form identity

$$-\Delta \geq C(N) \sum_{i \neq j}^{N} \frac{1}{|x_i - x_j|^2}$$

which uses particles in $\mathbb{R}^3$ to particles in $d \geq 3$ dimensions. From previous results,

$$C(N) = \frac{1}{4 + 4K(N)},$$

where

$$K(N) := \sup \frac{\sum_{i \neq k, i, k \neq j}^{N} \frac{1}{R_{ijk}^2}}{2 \sum_{i \neq j}^{N} \frac{1}{r_{ij}^2}}.$$

It is known that

$$K(N) \leq \frac{N-2}{2}$$

and such the optimal $C(N)$ satisfies

$$C(N) \geq \frac{1}{2N}.$$

However, studying the behaviour of optimal constants corresponding to the large number of particles $N$ is very complicated due to the geometrically nature of our terms. We do obtain a certain behaviour for our constant, but future work is needed to determine whether it is sharp or not.

Chapter 4 present a shift of our attention to paper [106], which provides sharp pointwise estimates for functions with radial symmetry. The authors consider a simple example of Hardy inequality for functions $u = (r_1(\mathbf{x}), r_2(\mathbf{x}))$,

where $r_1(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$ and $r_2(\mathbf{x}) = \sqrt{x_3^2 + x_4^2}$ for the subspace $\dot{H}^{1,2}_{(2,2)}(\mathbb{R}^4)$ of the Sobolev space $\dot{H}^{1,2}(\mathbb{R}^4)$, radially symmetric in variables $(x_1, x_2)$ and in variables $(x_3, x_4)$:

$$\int_{\mathbb{R}^4} \frac{|u|^2}{r_1(\mathbf{x}) r_2(\mathbf{x})} d\mathbf{x} \leq C \int_{\mathbb{R}^4} |\nabla u|^2 d\mathbf{x}.$$

We make use of the results in the above paper and derive new Hardy inequalities with weights which are homogeneous functions with respect to the radial variable. Our results establish a sharp constant $\tau$ in the case of the block radial potential used by L. Skrzypczak and C. Tintarev. But this is possible only after introducing a new system of Hopf coordinates for dimension larger than four. Firstly we consider the usual Hopf coordinates in $\mathbb{S}^3$ and reduce the problem of finding the constant $\tau$ to finding the first eigenvalue $\lambda_1$ of the Neumann problem for the classical Legendre type equation on the interval $[0, \pi/2]$. We then find $\lambda_1$ by numerical approximation.

In Chapter 5 we analyse paper [72], which studies a number of Hardy inequalities for continuous and discrete operators and paper [74]. We are interested more in the discrete Hardy inequality, but, as the authors mention, the inequalities for discrete operators are more complicated to study. Considering that neither a continuous or discrete classical Hardy inequality exist for dimension two, our interest is to work upon this case by considering a specific magnetic potential field. This is possible by using similar techniques to those used in [72] and finding a magnetic potential term resembling the Aharonov Bohm potential presented in [74] for dimension two.

As an example, we start from the classical discrete Hardy inequality

$$\sum_{n=1} \left( \frac{1}{n} \sum_{k=1}^{n} u_k \right)^p < \left( \frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} u_n^p, \quad p > 1,\ u_k \geq 0.$$

Its extension from the continuous cases to discrete quadratic forms seems to be not easy and such we make use results presented in [72]. The authors obtained some Hardy type inequalities for the discrete Laplacian on $\mathbb{Z}^d$, $d \geq 2$ by using a version of polar coordinates and the constants were far from being optimal. We introduce a discrete version of Aharonov - Bohm vector field on $\mathbb{Z}^2$ and, as in [74], to obtain a Hardy inequality on $\mathbb{Z}^2$ using "polar coordinate" introduced in [72].

Finally, Chapter 6 includes future developments on our research about Hardy inequalities on the $\mathbb{S}^{n-1}$ sphere and for $n$ dimensional simplices. More specifically, for $n+1$ points in $\mathbb{R}^{n(n+1)}$, we consider the simplex

$$\mathcal{T}_n = \text{conv}(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1})$$

and the corresponding volume

$$|T_n| = \frac{1}{n!} \left| \det \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} & 1 \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n+1,1} & x_{n+1,2} & \ldots & x_{n+1,n} & 1 \end{pmatrix} \right|.$$

Our interest is to weaken the conditions imposed on a function $u$ such that the following theorem holds:

**Theorem 1.3.1 (Simplex volume)** *Let $\mathcal{T}_n$ be a simplex in $\mathbb{R}^{n(n+1)}$, then*

$$-\Delta \geq \frac{1}{4} \frac{|\nabla T_n|^2}{|T_n|^2} = \frac{1}{4n^2} \frac{\sum_{i=1}^{n+1} F_i^2}{|T_n|^2}.$$

We introduce a new domain which contains locally integrable functions on the space $\mathbb{R}^{n(n+1)}$ and analyse whether it is possible to find a function $u$ for our theorem to produce a valid result.

Also, we study the improvements upon constants in a version of the continuous Hardy inequality resembling one of our main results. The approach makes use of bound approximations and function optimisation. More specifically, in [59], the authors discovered that

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \geq \frac{(d-1)^2}{3} \int_{\mathbb{R}^{3d}} \frac{|u|^2}{R^2} d\mathbf{x}$$

for $R(x)$ the circumradius of the triangle with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and $u \in C_0^\infty(\mathbb{R}^{3d} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3\})$.

The sharpness of the above mentioned constant is not yet established, hence it would be interesting to analyse the constant's behaviour for multiple dimensions and particles.

Another interesting research idea is introduced in [119] where the author investigates possibilities on bounding a Hardy constant in the 2D Fermionic Hardy inequality. Our interest is to refine those bounds for three particles in two dimensions, using antisymmetric function in $H^1$. More specifically, for inequality

$$\int_{\mathbb{R}^{2\cdot 3}} |\nabla u|^2 d\mathbf{x} \geq C \cdot \frac{4}{3} \int_{\mathbb{R}^{2\cdot 3}} \left( \sum_{i<j}^{3} \frac{1}{|z_i - z_j|^2} \right) |u|^2 d\mathbf{x}$$

it is known that the lower bound for constant $C$ is 1 and we research upon one idea that 2 is an upper bound. One approach consists in inserting a function which has cut off's at the singularity $z_1 = z_2 = z_3$. This results in the function being dependent on two variables, $\alpha$ and $k$. Once $k$ tends to infinity we shall allow $\alpha$ approach a certain critical value and such obtain a suitable function of compact support.

Last but not least, as one other curiosity for future research, we consider $\mathbb{S}^{n-1}$ and make use of [9] and [71]. Hence, the following theorem holds

**Theorem 1.3.2** *For $f \in C_0^\infty(\mathbb{R}^n)$ , $r \in \mathbb{R}$ and $W : S^{n-1} \to \mathbb{R}$, $u \in C_0^\infty(\mathbb{R}^n)$, then*

$$\int_{\mathbb{R}^n} |\nabla u|^2 d\mathbf{x} \geq C(W) \int_{\mathbb{R}^n} \frac{W|u|^2}{r^2} d\mathbf{x} + \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{|f|^2}{r^2} d\mathbf{x}.$$

Extra geometrical assumptions on the $C(W)$ quantity are needed such that we can study the validity of the Hardy inequality on this particular space. We use these assumptions on a Riemannian manifold, but it is not conclusive yet whether the constant involved is optimal.

# Chapter 2

# Theoretical background

This chapter includes a few key ideas and concepts which we use in our research. We analyse interesting papers which contain valuable techniques which we modify in order to obtain new Hardy inequalities.

## 2.1 Improved many-particle Hardy inequalities

As it was specified in the introduction chapter, our research resolves more around many-particle Hardy inequalities and various other generalisations. Paper [59] is our main point of interest in studying many-particle cases. As a starting point, let us recall the following theorem

**Theorem 2.1.1 (Hardy inequality for 1D particles)** *Let*

$$\mathcal{N}_N = \{\mathbf{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N | \, x_i = x_j \quad \text{for some} \quad i \neq j\}.$$

*Suppose* $u \in W_0^{1,2}(\mathbb{R}^N \setminus \mathcal{N}_N)$*, then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{2} \int_{\mathbb{R}^N} \sum_{1 \leq i < j \leq N} \frac{|u|^2}{r_{ij}^2} d\mathbf{x}.$$

The authors obtain a better bound for the general constant when the number of particles tends to infinity. This is possible by noting that for any fixed $\mathbf{y} \in \mathbb{R}^d$, $d \geq 3$,

$$-\Delta \geq \frac{(d-2)^2}{4} \frac{1}{|\mathbf{x} - \mathbf{y}|^2},$$

one obtains

$$-\Delta_i - \Delta_j \geq \frac{(d-2)^2}{4} \frac{1}{r_{ij}^2}$$

in the quadratic form. By adding the terms

$$-\Delta \geq \frac{(d-2)^2}{2N-2} \sum_{i<j}^{N} \frac{1}{r_{ij}^2},$$

where $r_{ij}$ terms are defined as such

$$r_{ij} := |\mathbf{x}_i - \mathbf{x}_j| = \sqrt{\sum_{k=1}^{d}(x_{i,k} - x_{j,k})^2}.$$

The result involving the better bound was assigned to the following theorem

**Theorem 2.1.2 (Hardy constant for $d$ - dimensional particles)** *Consider $d \geq 3$, $N \geq 2$ and $u \in H^1(\mathbb{R}^{dN})$. Define*

$$\mathcal{C}(d,N) := \inf_{u \in H^1(\mathbb{R}^{dN})} \left( \frac{\int_{\mathbb{R}^{dN}} |\nabla u|^2 d\mathbf{x}}{\sum_{1 \leq i < j \leq N} \int_{\mathbb{R}^{dN}} \frac{|u|^2}{r_{ij}^2} d\mathbf{x}} \right). \tag{2.1.1a}$$

*Then*

$$\mathcal{C}(d,N) \geq (d-2)^2 \max\left\{ \frac{1}{N}, \frac{1}{1+\sqrt{1+\frac{3(d-2)^2}{2(d-1)^2}(N-1)(N-2)}} \right\}. \tag{2.1.1b}$$

The proof for this statement considers a number of different vector fields which produce convenient Hardy inequalities. The resulting inequalities are grouped together such that each three particle coordinates have a correspondent. This process produces estimates from above for the constant $\mathcal{C}(d,N)$. More details about the proof itself can be found in the paper [59].

Regarding the results presented in Theorem 2.1.2, we have to add a number of observations

**Remarks**:

(i) Hardy inequalities of the type presented do not hold for general function $u \in H^1(\mathbb{R}^{dN})$ when $d = 1, 2$

(ii) In the case when $d \leq 6$ and for $N$ relatively large, the maximum for quantity (2.1.1b) is given by the second term

(iii) The bounds for $\mathcal{C}(d,N)$ are not sharp. For the lower bound, the authors of [59] consider only the derivation of the case for three particles. It is not clear how to obtain explicit bounds and more improvement is susceptible. For example, let us consider $R(x,y,z)$ to be the circumradius of the triangle with vertices $x, y$ and $z$. Suppose that the optimal asymptotic configuration of points can be described by a probability measure $\mu$ on $\mathbb{R}^d$. Let

$$K = \sup_{\mu} \left( \frac{\int\int\int R^{-2}(x,y,z) d\mu(x) d\mu(y) d\mu(z)}{\int\int |x-y|^{-2} d\mu(x) d\mu(y)} \right).$$

Then one can obtain a better estimate for the constant $\mathcal{C}(d,N)$ for a large $N$ given by

$$\lim_{N \to \infty} N\mathcal{C}(d,N) \geq \frac{(d-2)^2}{2+K}.$$

(iv) Finding the best configuration of points is revealed to be quite difficult considering that if we integrate over the whole real space we come across degenerate cases of triangles with area 0.

One other main result considered in [59], which defines part of our research is the following two version Hardy inequality for three particles. This new identity involves different aspects from multiple particles to circumradius of triangles. Also, since the identity is a Hardy inequality, different constants are present in its structure, so our main concern is to determine whether the quantities can be generalised for larger number of particles.

**Lemma 2.1.3 (Hoffmann - Ostenhof, Laptev, Tidblom)** Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{R}^{3d}, u \in C_0^\infty(\mathbb{R}^{3d} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3\})$ with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^d, d \geq 2$ and let

$$\rho^2 = r_{12}^2 + r_{23}^2 + r_{13}^2.$$

Then

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \geq 3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u|^2}{\rho^2} d\mathbf{x}. \tag{2.1.2a}$$

Furthermore, if $R(x)$ is the circumradius of the triangle with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, then

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \geq \frac{(d-1)^2}{3} \int_{\mathbb{R}^{3d}} \frac{|u|^2}{R^2} d\mathbf{x}. \tag{2.1.2b}$$

**Proof**:

Before proceeding with the proof of the lemma, let us make use of the helpful identity in [59]

**Lemma 2.1.4** Let $u \in H^1(\mathbb{R}^d), d \geq 1$ and let $\mathcal{F} = (\mathcal{F}_1(\mathbf{x}), \ldots, \mathcal{F}_d(\mathbf{x}))$ be a vectorfield whose components and their first derivatives are uniformly bounded in $\mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \frac{(\int_{\mathbb{R}^d} |u|^2 \mathrm{div}(\mathcal{F}) d\mathbf{x})^2}{\int_{\mathbb{R}^d} |u|^2 |\mathcal{F}|^2 d\mathbf{x}}. \tag{2.1.3}$$

**Proof of lemma:**

We will provide a simple and straight forward proof using Cauchy-Schwarz inequality and partial integration. Hence,

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} |u|^2 \mathrm{div}(\mathcal{F}) d\mathbf{x} \right| &= 2 \left| \mathrm{Re} \left[ \int_{\mathbb{R}^d} \langle \mathcal{F}, \nabla u \rangle \overline{u} d\mathbf{x} \right] \right| \\
&\leq 2 \left( \int_{\mathbb{R}^d} |u|^2 |\mathcal{F}|^2 d\mathbf{x} \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \right)^{\frac{1}{2}} \\
\implies \left( \int_{\mathbb{R}^d} |u|^2 \mathrm{div}(\mathcal{F}) d\mathbf{x} \right)^2 &\leq 4 \int_{\mathbb{R}^d} |u|^2 |\mathcal{F}|^2 d\mathbf{x} \int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x}.
\end{aligned}$$

By rearranging terms, we get to (2.1.3) and such complete the proof. □

If we want to use Lemma 2.1.4 for the vector field $\mathcal{F}$

$$\mathcal{F}_\epsilon = \frac{1}{\rho^2 + \epsilon^2} (2\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{x}_3, 2\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{x}_3, 2\mathbf{x}_3 - \mathbf{x}_1 - \mathbf{x}_2) \tag{2.1.4}$$

we have to calculate the divergence and the vector modulus squared. Hence, by tedious calculations, we obtain

$$\begin{cases}
\mathrm{div}\mathcal{F}_\epsilon = \dfrac{6(d-1)}{\rho^2 + \epsilon^2} \\
|\mathcal{F}_\epsilon|^2 = \dfrac{3}{\rho^2 + \epsilon^2}.
\end{cases}$$

Then for a function $u \in H^1(\mathbb{R}^{3d})$ with the vector field $\mathcal{F}_\epsilon : \mathbb{R}^d \to \mathbb{R}^d$, one can apply Lemma 2.1.4 and obtain

$$\begin{aligned}\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} &\geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^{3d}} |u|^2 \frac{6(d-1)}{\rho^2+\epsilon^2} d\mathbf{x}\right)^2}{\int_{\mathbb{R}^{3d}} |u|^2 \frac{3}{\rho^2+\epsilon^2} d\mathbf{x}} \\ &\geq \frac{36(d-1)^2}{12} \frac{\left(\int_{\mathbb{R}^{3d}} |u|^2 \frac{1}{\rho^2+\epsilon^2} d\mathbf{x}\right)^2}{\int_{\mathbb{R}^{3d}} |u|^2 \frac{1}{\rho^2+\epsilon^2} d\mathbf{x}} \\ &\geq 3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u|^2}{\rho^2+\epsilon^2} d\mathbf{x}.\end{aligned}$$

Furthermore, since $C_0^\infty(\mathbb{R}^{3d})$ is dense in $H^1(\mathbb{R}^{3d})$ it can be concluded that when $\epsilon$ tends to 0 our inequality converges to (2.1.2a). For the second part of the proof, it is recommended to introduce the following lemma

**Lemma 2.1.5 (Hoffmann, Laptev, Tidblom)** Let $R$ be the circumradius of a triangle with side lengths a, b, c. Then

$$\frac{1}{R^2} \leq \frac{9}{a^2+b^2+c^2} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{2.1.5}$$

**Proof of Lemma:**

Observing that $R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\eta}$ where the angles $\alpha, \beta, \eta$ correspond to the corner opposite to the sides with side lengths $a, b$ and $c$, we can show that

$$4R^2(\sin^2\alpha + \sin^2\beta + \sin^2\eta) \leq 9R^2.$$

It is already known that for any angles in a triangle their sum is equal to $\pi$. Hence, one possibility which gives us that $\sin^2\alpha + \sin^2\beta + \sin^2\eta \leq \frac{9}{4}$ is when $\alpha = \pi, \beta = \eta = 0$ and the first inequality is proved.

For the second inequality, we should see that $\forall a, b, c \in \mathbb{N}^*$ the following inequalities hold

$$\begin{cases} \frac{a^2}{b^2} + \frac{b^2}{a^2} \geq 2 \\ \frac{a^2}{c^2} + \frac{c^2}{a^2} \geq 2 \\ \frac{c^2}{b^2} + \frac{b^2}{c^2} \geq 2 \end{cases}$$

and such

$$\begin{aligned}3 + \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{b^2}{c^2} &= \frac{1}{a^2}a^2 + \frac{1}{c^2}c^2 + \frac{1}{b^2}b^2 + \frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{a^2}{c^2} + \frac{c^2}{a^2} + \frac{c^2}{b^2} + \frac{b^2}{c^2} \\ &= \left(\frac{1}{b^2} + \frac{1}{a^2} + \frac{1}{c^2}\right)(a^2+b^2+c^2) \geq 9.\end{aligned}$$

Hence, the second inequality is satisfied. □

Now, we can make use of Lemma 2.1.5 with $r_{12}^2, r_{23}^2, r_{13}^2$ instead of $a, b, c$ as side lengths of a triangle.

Hence, for $R$ being the circumradius of the triangle with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, one has that

$$\begin{aligned}\frac{1}{R^2} &\leq \frac{9}{r_{12}^2 + r_{23}^2 + r_{13}^2} \\ \implies \rho^2 &\leq 9R^2.\end{aligned}$$

And such

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \geq 3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u|^2}{\rho^2} d\mathbf{x}$$
$$\geq 3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u|^2}{9R^2} d\mathbf{x} = \frac{(d-1)^2}{3} \int_{\mathbb{R}^{3d}} \frac{|u|^2}{R^2} d\mathbf{x}.$$

□

**Remarks**:

(i) If we consider one-dimensional particles in our calculations, then the circumradius is equal to infinity, hence (2.1.2b) produces a trivial result

(ii) The constant presented in (2.1.2b) is not considered sharp. If we consider different vector fields we could obtain a Hardy inequality for the $R^{-2}$ term directly

Recalling (1.2.20), D.Lundholm provides an interesting extended formula for the constant in (2.1.2b). The proof of this statement can be found in [82] along with a helpful proposition which claims the sharpness of the constant involved.

**Proposition 2.1.1** *Let $\Omega$ be an open set in $\mathbb{R}^d$ such that $\overline{\Omega} = \mathbb{R}^d$ and let $f : \Omega \to \mathbb{R}_+$ be twice differentiable. Moreover,*

*(i) $\int_{\mathbb{R}^d} f^{1-\delta} e^{-|\mathbf{x}|} d\mathbf{x}$ is uniformly bounded for small values of $\delta > 0$,*

*(ii) $\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|\mathbf{x}|} d\mathbf{x} < \infty$ for small values of $\delta > 0$ and tends to infinity when $\delta \to 0$.*

*Then, for every $\epsilon > 0$, there exists $u_\delta := f^{\frac{-\delta}{2}} f^{\frac{1}{2}} e^{-\frac{1}{2}|\mathbf{x}|} \in H^1(\mathbb{R}^d)$, with $\delta > 0$, such that*

$$\int_{\mathbb{R}^d} |\nabla u_\delta|^2 d\mathbf{x} \leq \left(\frac{1}{4} + \epsilon\right) \int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f^2} |u_\delta|^2 d\mathbf{x}.$$

**Proof of proposition:**

Using the fact that $u_\delta \in C^1(\Omega \setminus \{0\})$ and

$$|\nabla u_\delta| = \left| \frac{\nabla f}{2f} u_\delta - \frac{\delta \nabla f}{2f} u_\delta - \frac{\nabla |\mathbf{x}|}{f} u_\delta \right|$$
$$\leq \frac{1+|\delta|}{2} \frac{|\nabla f|}{f} |u_\delta| + \frac{1}{2} |u_\delta|$$

on $\Omega \setminus \{0\}$, along with (i) and (ii), we find that $u_\delta \in H^1(\mathbb{R}^d)$ for sufficiently small $\delta > 0$.

Moreover,

$$\|\nabla u_\delta\|_{L^2} \leq \frac{1+|\delta|}{2} \left\| \frac{\nabla f}{f} u_\delta \right\|_{L^2} + \frac{1}{2} \|u_\delta\|_{L^2}$$
$$\leq \left( \frac{1}{2} + \frac{|\delta|}{2} + \frac{\sqrt{M}}{2} \left\| \frac{\nabla f}{f} u_\delta \right\|_{L^2}^{-1} \right) \left\| \frac{\nabla f}{f} u_\delta \right\|_{L^2},$$

where $M > 0$ is the bound of (i). Then by taking $\delta \to 0$, the result follows.

It is compulsory to specify, also, that conditions (i) and (ii) hold when $f$ is of the form $f \sim \delta_\Omega^{-(k-2)}$ near the boundary of $\Omega$, where $\delta_\Omega$ is the distance to $\Omega^c$ and $k$ the codimenstion of $\Omega^c$. □

By using this proposition, the author of [82] claims that the sharpness of the general constant in (1.2.20) is proven.

Results (2.1.2b) and (2.1.2a) represent an important step in the study of many-particle Hardy inequalities and one of the main discoveries of this thesis involves a Hardy inequality for terms $\frac{1}{r_{ij}^2}$. Similarly by using Lemma 2.1.5 for $d$ dimensional sides, we derive a formula for the circumradius of a simplex and obtain a new Hardy inequality. The curiosity arises from the idea of having a more general geometrical inequality which is valid for multiple dimensions with finitely many particles. A practical book that presents key concepts about $d$ dimensional geometry is [110]. Results about many-particle Hardy inequalities have been obtained in [4], [57], [59], [75] and [82], using ground state representations. We introduce a new technique involving a generalised vector field. This approach uses bound approximations along with studying the generalisation possibilities of many-particle inequalities.

## 2.2 Sharp multiplicative Hardy inequalities

Moving to our next research idea, we stop at article [58]. The paper presents sharp Hardy inequalities with weight functions. These functions may also admit singularities on the unit sphere, hence it is imperative to understand how these singularities can be avoided. The authors work with the classical Hardy inequality for the Laplacian in $\mathbb{R}^d$

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad d \geq 3. \tag{2.2.1}$$

It is known that this inequality intertwines with Heisenberg's uncertainty principle, see (1.2.18), and provides an insight in the spectral theory of Schrödinger operators. The inequality can also be written in the quadratic form

$$-\Delta - \frac{(d-2)^2}{4} \frac{1}{|\mathbf{x}|^2} \geq 0,$$

which shows that for $d \geq 3$, one can subtract a positive operator from the Laplacian in such a way for the difference to remain non-negative. Hence, paper [58] considers a different class of homogeneous functions of degree $-2$, instead of the usual $1/|\mathbf{x}|^2$ term. More specifically, the following result is proved

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |u(\mathbf{x})|^2 d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^d), \quad d \geq 3, \tag{2.2.2}$$

where $\tau > 0$ and $\Phi$ belong to a class of measurable functions defined on $\mathbb{S}^{d-1}$.

Our contribution towards the application of the result (2.2.2) regards block radial potentials which are present in the work of L.Skrzypczak and C.Tintarev [106] and [107]. We make use of various theorems which provide better bound approximations that we shall use in our argument regarding the new Hopf coordinate system.

**Theorem 2.2.1 (Bound approximation)** *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where*

$$p \geq \frac{(d-2)^2}{2(d-1)} + 1.$$

*Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |u(\mathbf{x})|^2 d\mathbf{x} \tag{2.2.3a}$$

*for* $u \in C_0^\infty(\mathbb{R}^d)$ *and*

$$\tau = \frac{(d-2)^2}{4} |\mathbb{S}^{d-1}|^{\frac{1}{p}} \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}. \tag{2.2.3b}$$

This theorem provides conditions on $p$ that can be weakened. Hence the authors also present the following result

**Theorem 2.2.2** *Let* $d \geq 3$ *and* $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, *where*

$$\begin{cases} p \in \left(1, \frac{5}{4}\right) & \text{for} \quad d = 3 \\ p \in \left[\frac{d-1}{2}, \frac{(d-2)^2}{2(d-1)} + 1\right) & \text{for} \quad d \geq 4 \end{cases}.$$

*Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq (1-\nu_0) \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x} + \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |u(\mathbf{x})|^2 d\mathbf{x}, \tag{2.2.4}$$

*for*

$$\nu_0 = \frac{2(d-1)(p-1)}{(d-2)^2} < 1,$$

*and*

$$\tau = \nu_0 \frac{(d-2)^2}{4} |\mathbb{S}^{d-1}|^{\frac{1}{p}} \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}. \tag{2.2.5}$$

**Remarks**:

(i) Inequality (2.2.4) is sharp and can be achieved for functions $\Phi \equiv$ *constant*

(ii) For

$$p \in \left(\frac{d-1}{2}, \frac{(d-2)^2}{2(d-1)} + 1\right)$$

the optimal $\Phi$ does not involve constant terms anymore.

## 2.3 Discrete Aharonov-Bohm Hardy inequalities

Considering the last main result of our research, we relate to the paper by L. Kapitanski and A. Laptev [72]. In this paper, the authors consider discrete and continuous Hardy inequalities for dimensions $d \geq 2$. They present techniques which allow the attainment of inequalities with multiple singularities also. For example, inequalities with multiple singularities provide a better and worse result in comparison to the one derived by adding standard Hardy inequalities at various different points. Knowing that in two dimensions Hardy inequality fails, the authors introduce a vector potential $A(\mathbf{x}) \in \mathbb{R}^d$ which will determine the type of inequality. More specifically,

$$\begin{aligned} \int_{\mathbb{R}^d} \sum_{j=1}^d (|\partial_j u(\mathbf{x}) - \lambda A_j(\mathbf{x}) u(\mathbf{x})|^2) d\mathbf{x} &= \int_{\mathbb{R}^d} (|\nabla u|^2 - 2\lambda \sum_{j=1}^d A_j u \partial_j u + \lambda^2 \sum_{j=1}^d |A_j|^2 |u|^2) d\mathbf{x} \\ &= \int_{\mathbb{R}^d} [|\nabla u|^2 + (\lambda \mathrm{div} A + \lambda^2 \sum_{j=1}^d |A_j|^2) |u|^2] d\mathbf{x}, \end{aligned}$$

where $\lambda \in \mathbb{R}$ and $u(\mathbf{x})$ sufficiently smooth and decaying. Hence

$$\sup_{\lambda}\left[-\int_{\mathbb{R}^d}(\lambda \mathrm{div}A+\lambda^2\sum_{j=1}^{d}|A_j|^2)|u|^2d\mathbf{x}\right]\leq\int_{\mathbb{R}^d}|\nabla u|^2d\mathbf{x}.$$

After maximizing the left hand side term with respect to $\lambda$, the authors obtain that

$$2\lambda|A|^2+\mathrm{div}A=0,\quad \forall \mathbf{x}\in\mathbb{R}^d.$$

This result leads to the restriction

$$\frac{\mathrm{div}A}{|A(\mathbf{x})|^2}=constant$$

and by rescaling the potential $A$ such that $A=\nabla\Psi$, for a function $\Psi$, the following is obtained

$$\Delta\Psi+|\Delta\Psi|^2=0.$$

This implies that the function $\omega=e^{\Psi}\geq 0$ is harmonic and positive valued.

As an example, for a dimension larger than three, the following result follows

**Theorem 2.3.1** *Let $d\geq 3$ and consider $a_k=(a_{k1},a_{k2},\ldots,a_{kd})\in\mathbb{R}^d$ for $k=1,2,\ldots,n$. Then*

$$\int_{\mathbb{R}^d}|\nabla u|^2d\mathbf{x}\geq\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{\sum_{j=1}^{d}(\sum_{k=1}^{d}\frac{x_j-a_{kj}}{|\mathbf{x}-a_k|^d})^2}{\omega^2}|u|^2d\mathbf{x}. \tag{2.3.1}$$

*is the general Hardy inequality with multiple singularities.*

Now, if one adds together the standard Hardy inequalities at different points, then

$$\int_{\mathbb{R}^d}|\nabla u|^2d\mathbf{x}\geq\frac{1}{n}\frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\sum_{k=1}^{d}\left(\frac{1}{|\mathbf{x}-a_k|^2}\right)|u|^2d\mathbf{x}. \tag{2.3.2}$$

Hence, if we look at the Hardy weight in (2.3.2), we can conclude that it behaves

$$\frac{1}{n}\frac{(d-2)^2}{4}\frac{1}{|\mathbf{x}|^2},\quad as\quad \mathbf{x}\to\infty$$

and

$$\frac{1}{n}\frac{(d-2)^2}{4}\frac{1}{|\mathbf{x}-a_k|^2},\quad as\quad \mathbf{x}\to a_k.$$

The authors conclude that inequality (2.3.1) behaves better asymptotically, since

$$|A|^2\approx\frac{(d-2)^2}{4}\frac{1}{|\mathbf{x}|^2},\quad as\quad \mathbf{x}\to\infty$$

and

$$|A|^2\approx\frac{(d-2)^2}{4}\frac{1}{|\mathbf{x}-a_k|^2},\quad as\quad \mathbf{x}\to a_k.$$

The results in [72] provide examples for the continuous Hardy inequalities for 1D case, 2D case and for when $d \geq 3$, but our interest consists in whether there exists an optimal potential vector field $A$ for the discrete Hardy inequality. The same paper provides an answer to this problem by introducing a different form of the Poincaré - Friedrichs inequality that includes *sine* terms, but we would like to check, if this is true also for Aharonov - Bohm effect on a square in $\mathbb{Z}^2$.

As we see, there are multiple theorems and lemmas which our research consists on and in the following chapters we shall present our main results.

# Chapter 3

# On Many-Particle Hardy inequality

## 3.1 Introduction

Throughout the past century, Hardy inequalities have been presented in various forms in the mathematical literature. As highlighted before, they play a crucial role in analysis and mathematical physics(for example, see book [9] and papers [4], [85], [93]). The standard Hardy inequality (away from a point) associated to the Laplacian in $\mathbb{R}^d$ for $d \geq 3$ is

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x}. \tag{3.1.1}$$

The constant is sharp in this case and function $u \in H^1(\mathbb{R}^d)$. The inequality describes the ability of the Laplacian $-\Delta$ on $\mathbb{R}^d$ through the quadratic form to be non-negative and also to be bounded below by a potential term. The potential increases in strength without being bounded as the distance to the origin tends to zero.

This chapter includes variations of inequality (3.1.1) as seen in [59], [82] and [116], but also starting from geometrical extensions we shall introduce a new many-particle generalisation. This new identity studies the dependence of sharp values of the constants involved to the number of particles.

More specifically, we study the improvements upon constants in a version of many-particle Hardy inequality. This approach involves new vector fields and bound approximations along with studying the generalisation possibilities of these type of many-particle inequalities.

Let $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,d}) \in \mathbb{R}^d$, $j = 1, \dots N$ and $\rho^2(x) := \sum_{k<j} |\mathbf{x}_k - \mathbf{x}_j|^2$.

In [82] the author obtains the following inequality

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - N\left(\frac{N-1}{2}d - 1\right)^2 \int_\Omega \frac{|u|^2}{\rho^2} d\mathbf{x} \geq 0,$$

where

$$\Omega := \mathbb{R}^{dN} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N\}$$

and $u \in C_0^\infty(\mathbb{R}^{dN})$. Regarding this inequality, it was proved that the constant $N\left(\frac{N-1}{2}d - 1\right)^2$ is sharp.

Chapter 3 contains a version of the Hardy inequality, where instead of $\frac{1}{\rho^2}$ we consider $\sum_{k<j}^{N} \frac{1}{|\mathbf{x}_k - \mathbf{x}_j|^2}$. Namely, we prove the following result

**Theorem 3.1.1** *For a function $u \in C_0^\infty(\mathbb{R}^{dN})$, $N \geq 4$ and $d \geq 3$ the following Hardy many-particle inequality holds*

$$\int_{\mathbb{R}^{Nd}} |\nabla u|^2 d\mathbf{x} \geq \mathcal{C}(d,N) \int_{\mathbb{R}^{Nd}} \sum_{k<j}^{N} \frac{1}{|\mathbf{x}_k - \mathbf{x}_j|^2} |u|^2 d\mathbf{x}, \tag{3.1.2}$$

*where*

$$\mathcal{C}(d,N) = \frac{2\sqrt{N}\left[(N-1)d^2 - 2Nd + 4\right]}{N\sqrt{N}(N-1)+6} - \frac{2N\left[(N-1)d-2\right]^2}{(N\sqrt{N}(N-1)+6)^2}.$$

It is not clear if the constant is sharp, but from further numerical analysis

$$\begin{aligned}
&\mathcal{C}(3,N) \approx 0.497, \quad when \quad N = 4,\\
&\mathcal{C}(3,N) \approx 0.493, \quad when \quad N = 5,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\mathcal{C}(3,N) \approx 0.39, \quad when \quad N = 10,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

it is possible to predict a certain pattern of the constant's behaviour in various dimensions. In comparison to Theorem 2.1.2, the new estimates from (3.1.2) are better.

Using the approach suggested in [59] we find an auxiliary lemma that reflects a rather geometrical flavour of the Hardy inequality. This result involves different aspects from multiple particles to circumradius of triangles.

Also, since the identity is a Hardy inequality, different constants are present in its structure, so our main concern is to determine whether the quantities can be generalised for larger number of particles.

## 3.2 Preliminary results

Before we present the main results, we shall provide an insight on helpful identities used in the proof of our result.

As a starting point, let us recall the following lemma

**Lemma 3.2.1 (Hoffmann, Laptev, Tidblom)** Let $u \in H^1(\mathbb{R}^d), d \geq 1$ and let $\mathcal{F} = (\mathcal{F}_1(\mathbf{x}), \ldots, \mathcal{F}_d(\mathbf{x}))$ be a vectorfield whose components and their first derivatives are uniformly bounded in $\mathbb{R}^d$. Then

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \frac{\left(\int_{\mathbb{R}^d} |u|^2 \mathrm{div}(\mathcal{F}) d\mathbf{x}\right)^2}{\int_{\mathbb{R}^d} |u|^2 |\mathcal{F}|^2 d\mathbf{x}}. \tag{3.2.1}$$

Moreover, the inequality above admits a more straight forward version of the form

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \int_{\mathbb{R}^d} \left[\mathrm{div}(\mathcal{F}) - |\mathcal{F}|^2\right] |u|^2 d\mathbf{x}.$$

For $\mathbf{x}_j = (x_{j,1}, \dots, x_{j,d}) \in \mathbb{R}^d$, $j = 1, \dots N$ recall

$$r_{ij} := |\mathbf{x}_i - \mathbf{x}_j| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2}.$$

The next result is considered to be related to the Melnikov-Menger curvature and is to be found in [59].

**Lemma 3.2.2** Define three points $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k \in \mathbb{R}^d$ with

$$b_{ijk} = \frac{\langle \mathbf{x}_i - \mathbf{x}_j, \mathbf{x}_i - \mathbf{x}_k \rangle}{r_{ij}^2 r_{ik}^2} + \frac{\langle \mathbf{x}_j - \mathbf{x}_i, \mathbf{x}_j - \mathbf{x}_k \rangle}{r_{ij}^2 r_{jk}^2} + \frac{\langle \mathbf{x}_k - \mathbf{x}_i, \mathbf{x}_k - \mathbf{x}_j \rangle}{r_{ik}^2 r_{jk}^2}.$$

Let $R$ be the circumradius of the triangle with corners $\mathbf{x}_i, \mathbf{x}_j, \mathbf{x}_k$. Then

$$\begin{cases} 2b_{ijk} = \dfrac{1}{R_{ijk}^2} = \displaystyle\sum_{j=1}^{3} \sum_{i \neq k, i, k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{x}_j - \mathbf{x}_k)}{r_{ij}^2 r_{jk}^2}, & \text{if} \quad d \geq 2 \\ b_{ijk} = 0, \quad \text{if} \quad d = 1. \end{cases}$$

If $d = 1$, then $x_i, x_j, x_k \in \mathbb{R}$ need not be equal such that $R_{ijk} = \infty$.

**Proof of lemma:**

Let $a = \mathbf{x}_i - \mathbf{x}_j$ and $b = \mathbf{x}_i - \mathbf{x}_k$. Then

$$\begin{aligned} b_{ijk} &= \frac{(a \cdot b)}{|a|^2 |b|^2} - \frac{a \cdot (b - a)}{|a|^2 |b - a|^2} - \frac{b \cdot (a - b)}{|b|^2 |b - a|^2} \\ &= \frac{2(|a|^2 |b|^2 - (a \cdot b)^2)}{|a|^2 |b|^2 |b - a|^2} \\ &= \frac{2 \sin^2 \theta}{r_{jk}^2}, \end{aligned}$$

for $\theta$ being the angle between $a$ and $b$. We establish the relation between the circumradius and the angle by using the sine-theorem and hence conclude our proof. □

As another important result, we shall also recall the following

**Lemma 3.2.3** Let $R$ be the circumradius of a triangle with side lengths a, b, c. Then

$$\frac{1}{R^2} \leq \frac{9}{a^2 + b^2 + c^2} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{3.2.2}$$

The next statement can be checked through straight forward computation.

**Lemma 3.2.4** Let $\mathbf{x}_j \in \mathbb{R}^d, j = 1, 2, 3$.Then

$$r_{12}^2 + r_{23}^2 + r_{13}^2 = 2 \left( \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle \right).$$

**Proof of lemma:**

Knowing the definition of $r_{ij}$ in $\mathbb{R}^d$ we have that

$$\begin{cases} r_{12}^2 = \sum_{k=1}^{d} (x_{1,k} - x_{2,k})^2 \\ r_{13}^2 = \sum_{k=1}^{d} (x_{1,k} - x_{3,k})^2 \\ r_{23}^2 = \sum_{k=1}^{d} (x_{2,k} - x_{3,k})^2 \end{cases}$$

and

$$\begin{cases} \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle = \sum_{k=1}^{d} (\mathbf{x}_1 - \mathbf{x}_2)_k (\mathbf{x}_1 - \mathbf{x}_3)_k \\ \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle = \sum_{k=1}^{d} (\mathbf{x}_2 - \mathbf{x}_1)_k (\mathbf{x}_2 - \mathbf{x}_3)_k \\ \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle = \sum_{k=1}^{d} (\mathbf{x}_3 - \mathbf{x}_1)_k (\mathbf{x}_3 - \mathbf{x}_2)_k. \end{cases}$$

By expanding the sums and rearranging the terms, we obtain what is to be proved. □

The final lemma is needed to rewrite various Laplacian terms along our research also, hence we shall present the following results

**Lemma 3.2.5** Let $\mathbf{x}_j \in \mathbb{R}^d$, $j = 1, 2, \ldots, N$. Then

$$N \sum_{j=1}^{N} \Delta_{\mathbf{x}_j} = \sum_{1 \le j < k \le N} (\nabla_{\mathbf{x}_j} - \nabla_{\mathbf{x}_k})^2 + \left( \sum_{j=1}^{N} \nabla_{\mathbf{x}_j} \right)^2.$$

These identities have been presented in the purpose of highlighting what ideas and concepts we use in the proof of our results. Since we managed to explain and understand the theoretical concepts that are needed to proceed in dealing with our problem, we move forward to the main results of this chapter.

## 3.3 Main results

Using the approach suggested in [59] we find an auxiliary lemma that reflects a rather geometrical flavour of the Hardy inequality. This result involves different aspects from multiple particles to circumradius of triangles.

Also, since the identity is a Hardy inequality, different constants are present in its structure, so our main concern is to determine whether the quantities can be generalised for larger number of particles.

**Lemma 3.3.1** Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in \mathbb{R}^{3d}, u \in C_0^\infty(\mathbb{R}^{3d} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3\})$ with $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^d, d \ge 2$ and let

$$\rho^2 = r_{12}^2 + r_{23}^2 + r_{13}^2.$$

Then

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \ge 3(d-1)^2 \int_{\mathbb{R}^{3d}} \frac{|u|^2}{\rho^2} d\mathbf{x}. \tag{3.3.1a}$$

Furthermore, if $R(x)$ is the circumradius of the triangle with vertices $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, then

$$\int_{\mathbb{R}^{3d}} |\nabla u|^2 d\mathbf{x} \ge \frac{(d-1)^2}{3} \int_{\mathbb{R}^{3d}} \frac{|u|^2}{R^2} d\mathbf{x}. \tag{3.3.1b}$$

The authors provided an interesting and quite simple proof of this statement, but also in paper [82], there exists a proof of the sharpness of constant $3(d-1)^2$. More specifically, the following theorem holds

**Theorem 3.3.2** *Let* $\Omega := \mathbb{R}^{dN} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \cdots = \mathbf{x}_N\}$. *Suppose the ground state function is* $\rho(\mathbf{x})^2 := \sum_{i<j} |\mathbf{x}_i - \mathbf{x}_j|^2$, *then*

$$\int_\Omega |\nabla u|^2 d\mathbf{x} - N \left( \frac{N-1}{2} d - 1 \right)^2 \int_\Omega \frac{|u|^2}{\rho^2} d\mathbf{x} = \int_\Omega |\nabla \rho^{-2\alpha} u|^2 \rho^{4\alpha} d\mathbf{x} \ge 0, \tag{3.3.2}$$

*for all* $u \in C_0^\infty(\Omega)$, *with the optimal weight* $\alpha := -\frac{(N-1)d-2}{4}$.

**Proof**:

Consider

$$\nabla_k \rho^2 = 2\sum_{j\neq k}(\mathbf{x}_k - \mathbf{x}_j),$$

hence

$$\Delta\rho^2 = 2\sum_k \sum_{j\neq k} \nabla_k \cdot (\mathbf{x}_k - \mathbf{x}_j) = 2N(N-1)d,$$

and

$$|\nabla\rho^2|^2 = 8\sum_{i<j} r_{ij}^2 + 8\sum_k \sum_{i<j} (\mathbf{x}_k - \mathbf{x}_i)\cdot(\mathbf{x}_k - \mathbf{x}_j) = 4N\rho^2,$$

where for the last step it was used the fact that

$$\sum_k \sum_{i<j} (\mathbf{x}_k - \mathbf{x}_i)\cdot(\mathbf{x}_k - \mathbf{x}_j) = \frac{N-2}{2}\sum_{i<j}|\mathbf{x}_i - \mathbf{x}_j|^2.$$

Making use of Proposition 1.2.4, we have that

$$\alpha(1-\alpha)\frac{|\nabla\rho^2|^2}{\rho^4} - \alpha\frac{\Delta\rho^2}{\rho^2} = 4N\alpha\left(\frac{2-(N-1)d}{2} - \alpha\right)\frac{1}{\rho^2}$$

and using optimization, the theorem follows. □

**Proof of sharpness:**

Let us consider the set $W := \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^d : \mathbb{R}^{3d} \setminus \{\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3\}\}$. It is a linear subspace of $\mathbb{R}^{3d}$, which one can parametrize by $\mathbf{x}_3 \in \mathbb{R}^d$. Moreover, let us define our function $f$ in Proposition 2.1.1 as

$$f(\mathbf{x}) := \left(\sum_{i<3}|\mathbf{x}_i - \mathbf{x}_3|^2 + \sum_{i<j<3}|\mathbf{x}_i - \mathbf{x}_j|^2\right)^{1-d}.$$

Consequently, knowing that

$$|Y|^2 := \sum_{i<3}|\mathbf{x}_i - \mathbf{x}_3|^2 \leq \rho^2 \leq \sum_{i<3}|\mathbf{x}_i - \mathbf{x}_3|^2 + \sum_{i<j<3}(|\mathbf{x}_i - \mathbf{x}_3|^2 + |\mathbf{x}_j - \mathbf{x}_3|^2) \leq B|Y|^2,$$

for $B$ being a positive constant, we obtain

$$\begin{aligned}\int_{\mathbb{R}^{3d}} f^{1-\delta} e^{-|\mathbf{x}|} d\mathbf{x} &= \int_{\mathbb{R}^{3d}} \rho^{(2-2d)(1-\delta)} e^{-|\mathbf{x}|} d\mathbf{x} \\ &\leq B\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}} e^{-\sqrt{|\mathbf{x}'|^2+|\mathbf{x}_3|^2}} |Y|^{(2-2d)(1-\delta)} dY d\mathbf{x},\end{aligned}$$

(where $\mathbf{x} = (\mathbf{x}', \mathbf{x}_3)$), which is uniformly bounded when $0 < \delta$.

Hence

$$\int_{\mathbb{R}^d} \frac{|\nabla f|^2}{f^2} f^{1-\delta} e^{-|\mathbf{x}|} d\mathbf{x} \leq B\int_{\mathbb{R}^d}\int_{\mathbb{R}^{2d}} e^{-\sqrt{|(\mathbf{x}_1,\mathbf{x}_2)|^2+|\mathbf{x}_3|^2}} |Y|^{-2+(2-2d)(1-\delta)} dY d\mathbf{x},$$

is finite for $0 < \delta$ sufficiently small, but tends to infinity for $\delta \to 0$. Therefore, by Proposition 2.1.1, the constant in (3.3.1a) is sharp. □

Since the results in research paper [82] provide a smooth and understandable idea about the sharpness of the constant in (3.3.1a), one can ask what happens to the constant when the number of particles becomes finitely large and, moreover, if there exists a similar inequality for the $\frac{1}{R^2}$ and $\sum_{i<j}^{N}\frac{1}{r_{ij}^2}$ terms.

Consider the straight forward version of Lemma 3.2.1. We shall use

$$\int_{\mathbb{R}^{Nd}}|\nabla u|^2 d\mathbf{x}\geq\int_{\mathbb{R}^{Nd}}\left[\operatorname{div}(\mathcal{F})-|\mathcal{F}|^2\right]|u|^2 d\mathbf{x},$$

for $u\in H^1(\mathbb{R}^{Nd})$ and a vector field $\mathcal{T}:\mathbb{R}^{Nd}\to\mathbb{R}^{Nd}$ with the same proprieties as in the Lemma 3.2.1 which we already know.

To be more specific, from [119], it is known that the following result holds

**Lemma 3.3.3** For $N=3, u\in C_0^\infty(\mathbb{R}^{3d})$ and $d\geq 3$, the following holds

$$\begin{aligned}\int_{\mathbb{R}^{3d}}|\nabla u|^2 d\mathbf{x}\geq\left[2c_1(d-2)-2c_1^2\right]\int_{\mathbb{R}^{3d}}\sum_{i<j}^{3}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^2}|u|^2 d\mathbf{x}-c_1^2\int_{\mathbb{R}^{3d}}\frac{1}{R^2}|u|^2 d\mathbf{x}+\\ +3(d-1-3c_1)^2\int_{\mathbb{R}^{3d}}\frac{1}{\rho^2}|u|^2 d\mathbf{x},\ \forall c_1\in\mathbb{R}.\end{aligned}$$

This new identity provides a fresh point of view about the Hardy inequality for 3 particles. Now, we managed to extend this result for $N>3$ through approximation techniques and thorough calculations as can be seen bellow.

**Theorem 3.3.4** *For $N>3, u\in C_0^\infty(\mathbb{R}^{Nd})$ and $d\geq 3$, the following holds*

$$\begin{aligned}\int_{\mathbb{R}^{Nd}}|\nabla u|^2 d\mathbf{x}\geq\left[2c_1(d-2)-2c_1^2\right]\int_{\mathbb{R}^{Nd}}\sum_{i<j}^{N}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^2}|u|^2 d\mathbf{x}-c_1^2\int_{\mathbb{R}^{Nd}}\sum_{i<j<k}^{N}\frac{1}{R_{ijk}^2}|u|^2 d\mathbf{x}+\\ +\frac{N}{4}[(N-1)d-2-c_1N(N-1)]^2\int_{\mathbb{R}^{Nd}}\frac{1}{\rho^2}|u|^2 d\mathbf{x},\ \forall c_1\in\mathbb{R}.\end{aligned}\tag{3.3.3}$$

Moreover, we also have the following theorem:

**Theorem 3.3.5** *The following Hardy Inequality holds*

$$\int_{\mathbb{R}^{Nd}}|\nabla u|^2 d\mathbf{x}\geq\mathcal{C}(d,N)\int_{\mathbb{R}^{Nd}}\sum_{i<j}^{N}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^2}|u|^2 d\mathbf{x},\tag{3.3.4}$$

*where*

$$\mathcal{C}(d,N)=\frac{2\sqrt{N}\left[(N-1)d^2-2Nd+4\right]}{N\sqrt{N}(N-1)+6}-\frac{2N(Nd-2-d)^2}{(N\sqrt{N}(N-1)+6)^2}.$$

One is entitled to think that a better constant might be obtained by using a clever estimate, but we make the assumption that the best constant in (3.3.4) is less than $\frac{1}{2}$ when $d=3$.

**Remark**: Considering the choice of vector field $\mathcal{F}$, we cannot theoretically obtain an inequality better than

$$\int_{\mathbb{R}^{3N}}|\nabla u|^2 d\mathbf{x}\geq T\int_{\mathbb{R}^{3N}}\sum_{i<j}^{N}\frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^2}|u|^2 d\mathbf{x},\tag{3.3.5}$$

where

$$T\approx 0.49705113832.$$

## 3.4 Proof of main results

**Proof of Theorem 3.3.4:**

For $\mathbf{x} \in \mathbb{R}^{dN}$, where N is a positive integer, consider N particles. This allows our $\mathbf{x}$ to be described as $\mathbf{x} = (\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N})$, with $\mathbf{x_i} = (x_{i,1}, \ldots, x_{i,d}) \in \mathbb{R}^d$ and $1 \leq i \leq N$.

Apart from this, recall

$$r_{ij} := |\mathbf{x}_i - \mathbf{x}_j| = \sqrt{\sum_{k=1}^{d} (x_{i,k} - x_{j,k})^2},$$

where $1 \leq i \leq N$ and $1 \leq j \leq N$.

Consider the vector field $\mathcal{F} = (\mathcal{F}_1(x), \ldots, \mathcal{F}_N(x))$ where

$$\mathcal{F}_j = \sum_{k \neq j}^{N} \frac{\mathbf{x}_j - \mathbf{x}_k}{|\mathbf{x}_j - \mathbf{x}_k|^2}$$

and another vector field

$$\mathcal{G} = \frac{1}{\rho^2}((N-1)\mathbf{x}_1 - \mathbf{x}_2 - \ldots - \mathbf{x}_N, (N-1)\mathbf{x}_2 - \mathbf{x}_1 - \ldots - \mathbf{x}_N, \ldots, (N-1)\mathbf{x}_N - \mathbf{x}_1 - \ldots - \mathbf{x}_{N-1})$$

with

$$\rho^2 := \sum_{i<j}^{N} |\mathbf{x}_i - \mathbf{x}_j|^2 = \sum_{i<j}^{N} r_{ij}^2.$$

We want to apply the generalised version of Lemma 3.2.1 and so define the following vector field

$$\mathcal{T} := c_1\mathcal{F} + c_2\mathcal{G}$$

with divergence and square modulus

$$\begin{aligned} \mathrm{div}(\mathcal{T}) &= c_1 \mathrm{div}(\mathcal{F}) + c_2 \mathrm{div}(\mathcal{G}) \\ &= 2c_1(d-2) \sum_{i<j}^{N} \frac{1}{r_{ij}^2} + \frac{N[d(N-1)-2]c_2}{\rho^2}, \\ |\mathcal{T}|^2 &= |c_1\mathcal{F} + c_2\mathcal{G}|^2 \\ &= c_1^2|\mathcal{F}|^2 + c_2^2|\mathcal{G}|^2 + 2c_1c_2\mathcal{F} \cdot \mathcal{G} \end{aligned}$$

and constants $c_1, c_2 \in \mathbb{R}$.

Computing the divergence and square modulus of each of vector fields $\mathcal{F}$ and $\mathcal{G}$, we obtain

$$|\mathcal{F}|^2 = 2\sum_{i<j}^{N} \frac{1}{r_{ij}^2} + \sum_{j=1}^{N} \sum_{i \neq k, i, k \neq j} \frac{(\mathbf{x}_j - \mathbf{x}_i) \cdot (\mathbf{x}_j - \mathbf{x}_k)}{r_{ij}^2 r_{jk}^2}$$

and

$$\begin{aligned}|\mathcal{G}|^2 &= \frac{N}{\rho^2},\\ \mathcal{F}\cdot\mathcal{G} &= \frac{1}{\rho^2}\left[N(N-1)+\sum_{j=1}^{N}\sum_{i\neq j,k\neq j}\frac{(\mathbf{x}_j-\mathbf{x}_i)\cdot(\mathbf{x}_j-\mathbf{x}_k)}{|\mathbf{x}_j-\mathbf{x}_i|^2}\right]\\ &= \frac{1}{\rho^2}\left[N(N-1)+\frac{N(N-1)(N-2)}{2}\right]\\ &= \frac{N^2(N-1)}{2\rho^2}.\end{aligned}$$

Moreover, by Lemma 3.2.2, we arrive at

$$\sum_{i<j<k}^{N}\frac{1}{R_{ijk}^2} := \sum_{j=1}^{N}\sum_{i\neq k,i,k\neq j}\frac{(\mathbf{x}_j-\mathbf{x}_i)\cdot(\mathbf{x}_j-\mathbf{x}_k)}{r_{ij}^2 r_{jk}^2}.$$

Therefore,

$$\begin{aligned}\operatorname{div}(\mathcal{T})-|\mathcal{T}|^2 = \left[2c_1(d-2)-2c_1^2\right]\sum_{i<j}^{N}\frac{1}{r_{ij}^2}+\\ +\frac{N[d(N-1)-2]c_2-c_2^2N-c_1c_2N^2(N-1)}{\rho^2}-\sum_{i<j<k}^{N}\frac{c_1^2}{R_{ijk}^2},\end{aligned}$$

for $R_{ijk}$ being the circumradius of the triangle with corners on the points $\mathbf{x}_i, \mathbf{x}_j$ and $\mathbf{x}_k$.

For the optimisation part, we want to determine for which value of $c_2$ the expression in front of $\rho^{-2}$ attains its maximum.

Therefore, denote $f(c_2) := -c_2^2N + N[d(N-1)-2]c_2 - c_1c_2N^2(N-1)$. The derivative of this function becomes

$$\frac{df}{dc_2} = -2c_2N + \left[N[d(N-1)-2] - N^2(N-1)c_1\right].$$

We want to determine where this derivative vanishes and so $\frac{df}{dc_2}=0 \iff$

$$\begin{aligned}&-2c_2N+\left[N[d(N-1)-2]-N^2(N-1)c_1\right]=0\\ \iff\ & c_2 = \frac{N[d(N-1)-2]-N^2(N-1)c_1}{2N}.\end{aligned}$$

By analysing $f(c_2)$ we have

| Behaviour of f($c_2$) | | | |
|---|---|---|---|
| Values of $c_2$ | $-\infty$ | $\frac{N[d(N-1)-2]-N^2(N-1)c_1}{2N}$ | $+\infty$ |
| $f'(c_2)$ | $+\infty + + + + + + + + + + + 0$ | - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - | $-\infty$ |
| $f(c_2)$ | $-\infty \nearrow\nearrow\nearrow\nearrow\nearrow\nearrow$ | $\frac{N[(N-1)d-2-N(N-1)c_1]^2}{4}$ | $\searrow\searrow\searrow\searrow\searrow\searrow\searrow\searrow\searrow\searrow\searrow\searrow -\infty$ |

As we can see from the table, we have a maximum point and a maximum value for which our identity in (3.2.1)

becomes

$$\begin{aligned}\int |\nabla u|^2 d\mathbf{x} &\geq \int \left[\operatorname{div}(\mathcal{T}) - |\mathcal{T}|^2\right] |u|^2 d\mathbf{x} \\ &\geq \left[2c_1(d-2) - 2c_1^2\right] \int \sum_{i<j}^{N} \frac{1}{r_{ij}^2} |u|^2 d\mathbf{x} + \\ &+ \frac{N[(N-1)d - 2 - N(N-1)c_1]^2}{4} \int \frac{|u|^2}{\rho^2} d\mathbf{x} \\ &- c_1^2 \int \sum_{i<j<k}^{N} \frac{1}{R_{ijk}^2} |u|^2 d\mathbf{x}. \quad \forall c_1 \in \mathbb{R}\end{aligned}$$

which completes the proof. □

The above proof and theorem present a new Hardy Inequality with a variety of weight terms which are described as a dependency on the number of particles in a quantum system.

**Proof of Theorem 3.3.5:**

Using Lemma 3.2.3 with a slight generalisation, one can deduce the following

$$\sum_{i<j<k}^{N} \frac{1}{R_{ijk}^2} < \frac{9}{\sum_{i<j}^{N} r_{ij}^2} = \frac{9}{\rho^2}.$$

This will give that

$$\left[2c_1(d-2) - 2c_1^2\right] \sum_{i<j}^{N} \frac{1}{r_{ij}^2} + \frac{N[(N-1)d - 2 - N(N-1)c_1]^2}{4} \frac{1}{\rho^2} - c_1^2 \sum_{i<j<k}^{N} \frac{1}{R_{ijk}^2} \geq \left[2c_1(d-2) - 2c_1^2\right] \sum_{i<j}^{N} \frac{1}{r_{ij}^2} + \\ + \frac{N[(N-1)d - 2 - N(N-1)c_1]^2 - 36c_1^2}{4\rho^2}.$$

Consider the coefficient in front of $\frac{1}{\rho^2}$ to be a function of $c_1$

$$g(c_1) := \frac{N[(N-1)d - 2 - c_1 N(N-1)]^2 - 36c_1^2}{4},$$

one has solve $g(c_1) = 0$ in order to determine for which values of $c_1$ our function $g$ vanishes, hence

$$\begin{aligned} g(c_1) = 0 &\implies \frac{N[(N-1)d - 2 - c_1 N(N-1)]^2 - 36c_1^2}{4} = 0 \\ &\implies N[(N-1)d - 2 - c_1 N(N-1)]^2 - 36c_1^2 = 0 \\ \implies N[(N-1)^2 d^2 + 4 + &c_1^2 N^2 (N-1)^2 - 4(N-1)d - 2c_1 dN(N-1)^2 + 4c_1 N(N-1)] - 36c_1^2 = 0 \\ \implies c_1^2[N^3(N-1)^2 - 36] + &c_1[4N^2(N-1) - 2dN^2(N-1)^2] + [N(N-1)^2 d^2 + 4N - 4N(N-1)d] = 0. \end{aligned}$$

We can calculate the discriminant as follows

$$\begin{aligned}\Delta_{c_1} &= [4N^2(N-1) - 2dN^2(N-1)^2]^2 - 4[N(N-1)^2 d^2 + 4N - 4N(N-1)d][N^3(N-1)^2 - 36] \\ &= [16N^4(N-1)^2 + 4d^2 N^4 (N-1)^4 - 16dN^4(N-1)^3] - \\ &-4[N^4(N-1)^4 d^2 - 36N(N-1)^2 d^2 + 4N^4(N-1)^2 - 4 \cdot 36N - 4N^4(N-1)^3 d + 4 \cdot 36N(N-1)d] \\ &= 4 \cdot 36N[(N-1)d - 2]^2.\end{aligned}$$

And so the solutions are

$$\begin{aligned} c_{1_{1,2}} &= \frac{-[4N^2(N-1) - 2dN^2(N-1)^2] \pm \sqrt{\Delta_{c_1}}}{2[N^3(N-1)^2 - 36]} \\ &= \frac{-[4N^2(N-1) - 2dN^2(N-1)^2] \pm \sqrt{4 \cdot 36N[(N-1)d-2]^2}}{2[N^3(N-1)^2 - 36]} \\ &= \frac{2N^2(N-1)[-2 + d(N-1)] \pm 2 \cdot 6\sqrt{N}[(N-1)d-2]}{2[N^3(N-1)^2 - 36]} \\ &= \frac{[(N-1)d-2][N^2(N-1) \pm 6\sqrt{N}]}{[N^3(N-1)^2 - 36]}, \quad \text{when} \quad (N-1)d-2 > 0 \end{aligned}$$

If we use these values of $c_1$ in Theorem 3.3.4, then the largest value of the leading coefficient $2c_1(d-2) - 2c_1^2$ can be found when

$$c_1 = \frac{\sqrt{N}[(N-1)d-2]}{N\sqrt{N}(N-1)+6},$$

which is just $c_{1_2}$ without rationalizing the denominator.

The value needed is

$$\begin{aligned} \mathcal{C}(d,N) &= 2(c_1(d-2) - c_1^2) \\ &= 2\left[\frac{\sqrt{N}[(N-1)d-2](d-2)}{[N\sqrt{N}(N-1)+6]} - \frac{N[(N-1)d-2]^2}{[N\sqrt{N}(N-1)+6]^2}\right] \\ &= 2\left[\frac{\sqrt{N}[(N-1)d-2](d-2)[N\sqrt{N}(N-1)+6] - N[(N-1)d-2]^2}{[N\sqrt{N}(N-1)+6]^2}\right] \\ &= \frac{[(N-1)d-2]\left[2\sqrt{N}(d-2)[N\sqrt{N}(N-1)+6] - 2N[(N-1)d-2]\right]}{[N\sqrt{N}(N-1)+6]^2} \\ &= \frac{2(N-1)[(N-1)^2N + 6\sqrt{N}]d^2}{[N\sqrt{N}(N-1)+6]^2} + \frac{8[N^2(N-1) + 6\sqrt{N} - N]}{[N\sqrt{N}(N-1)+6]^2} - \\ &- \frac{4d[N^2(N-1) + 6\sqrt{N} + N^2(N-1)^2 + 6\sqrt{N}(N-1) - 2N(N-1)]}{[N\sqrt{N}(N-1)+6]^2} > 0. \end{aligned}$$

□

Judging from this form, one can set a generic behaviour for the expressions in $N$, when $N \to \infty$, by calculating the order of every coefficient of the powers of $d$ as follows

$$\begin{aligned} &\mathcal{O}\left(\frac{2(N-1)[(N-1)^2N + 6\sqrt{N}]}{[N\sqrt{N}(N-1)+6]^2}\right) = \frac{4}{5}, \\ &\mathcal{O}\left(\frac{4[N^2(N-1) + 6\sqrt{N} + N^2(N-1)^2 + 6\sqrt{N}(N-1) - 2N(N-1)]}{[N\sqrt{N}(N-1)+6]^2}\right) = \frac{4}{5}, \\ &\mathcal{O}\left(\frac{8[N^2(N-1) + 6\sqrt{N} - N]}{[N\sqrt{N}(N-1)+6]^2}\right) = \frac{3}{5}. \end{aligned}$$

Hence, for $N \to \infty$,

$$\frac{2(N-1)[(N-1)^2N+6\sqrt{N}]}{[N\sqrt{N}(N-1)+6]^2} \approx \frac{2}{N},$$
$$\frac{4[N^2(N-1)+6\sqrt{N}+N^2(N-1)^2+6\sqrt{N}(N-1)-2N(N-1)]}{[N\sqrt{N}(N-1)+6]^2} \approx \frac{4}{N}$$
$$\frac{8[N^2(N-1)+6\sqrt{N}-N]}{[N\sqrt{N}(N-1)+6]^2} \approx \frac{8}{N^2}.$$

**Remark**: It is safe to assume that for a large number of particles $N$, our $\mathcal{C}(d,N)$ will have a behaviour similar to $\frac{1}{N}$, hence as a precision remark, one can add that for

$$\mathcal{C}(3,4) \approx 0.497, \quad \text{when} \quad N=4,$$
$$\mathcal{C}(3,5) \approx 0.490, \quad \text{when} \quad N=5,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathcal{C}(3,10) \approx 0.39, \quad \text{when} \quad N=10,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

concluding that our constant will decrease, when $N$ is significantly large.

This result is quite promising, if we take also into considering the lower bound of (2.1.1b). More specifically, even for $d=3$ our result is trivially better than the estimate

$$\frac{1}{1+\sqrt{1+\frac{3(d-2)^2}{2(d-1)^2}(N-1)(N-2)}} \leq \mathcal{C}(3,N) \leq \frac{1}{2}.$$

Also, this is useful in the search of optimality of the quantum system involved. For example, three particles' optimality implies that the configuration should be an equilateral triangle, while four particles define a tetrahedron. Finding the optimal configuration of points (i.e. particles) might be tricky, but in the situations described above, one could intertwine the theory of Hardy inequalities and Geometrical combinatorics and obtain new Hardy identities for simplices, as we shall see in the following chapters.

## 3.5 Alternative findings

From the previous section, we see that a potential optimal constant is not far from being obtained for the $\sum_{i<j}^{N} \frac{1}{|\mathbf{x}_i-\mathbf{x}_j|^2}$ term and such we wanted to check whether constant improvements can be found in terms of the $\sum_{i<j<k}^{N} \frac{1}{R_{ijk}^2}$ quantity, presented in (3.3.1b).

We take each term of the $b_{ijk}$ description from Lemma 3.2.2 and calculate the gradient in terms of each component.

For simplicity, consider calculations when $d = 2$ and $N = 3$. Hence, for $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{R}^2$, we have that

$$\mathbf{x}_1 = (x_{11}, x_{12})$$
$$\mathbf{x}_2 = (x_{21}, x_{22})$$
$$\mathbf{x}_3 = (x_{31}, x_{32})$$

and such our $b_{ijk}$ term becomes

$$b_{123} = \frac{\langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle}{r_{12}^2 r_{13}^2} + \frac{\langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle}{r_{12}^2 r_{23}^2} + \frac{\langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle}{r_{13}^2 r_{23}^2}.$$

Now, we already know that the gradient of the term $b_{123}$ is in $\mathbb{R}^6$ and can be written as follows

$$\nabla b_{123} = (\nabla_1 b_{123}, \nabla_2 b_{123}, \nabla_3 b_{123}),$$

where

$$\nabla_s = \left( \frac{\partial}{\partial \mathbf{x}_s} \right) = \left( \frac{\partial}{\partial x_{s1}}, \frac{\partial}{\partial x_{s2}} \right), \text{for} \quad s \leq N.$$

We begin by computing the gradient for each term in the expression of $b_{123}$ and afterwards sum them. Hence, the first term of our sum is expressed as

$$\begin{aligned} b_{123_1} :&= \frac{(x_{11} - x_{21}, x_{12} - x_{22})(x_{11} - x_{31}, x_{12} - x_{32})}{[(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2][(x_{11} - x_{31})^2 + (x_{12} - x_{32})^2]} \\ &= \frac{(x_{11} - x_{21})(x_{11} - x_{31}) + (x_{12} - x_{22})(x_{12} - x_{32})}{[(x_{11} - x_{21})^2 + (x_{12} - x_{22})^2][(x_{11} - x_{31})^2 + (x_{12} - x_{32})^2]} \end{aligned} \tag{3.5.1}$$

and by analogy the following two terms are

$$\begin{aligned} b_{123_2} :&= \frac{(x_{21} - x_{11}, x_{22} - x_{12})(x_{21} - x_{31}, x_{22} - x_{32})}{[(x_{21} - x_{11})^2 + (x_{22} - x_{12})^2][(x_{21} - x_{31})^2 + (x_{22} - x_{32})^2]} \\ &= \frac{(x_{21} - x_{11})(x_{21} - x_{31}) + (x_{22} - x_{12})(x_{22} - x_{32})}{[(x_{21} - x_{11})^2 + (x_{22} - x_{12})^2][(x_{21} - x_{31})^2 + (x_{22} - x_{32})^2]}, \end{aligned} \tag{3.5.2}$$

$$\begin{aligned} b_{123_3} :&= \frac{(x_{31} - x_{11}, x_{32} - x_{12})(x_{31} - x_{21}, x_{32} - x_{22})}{[(x_{31} - x_{11})^2 + (x_{32} - x_{12})^2][(x_{31} - x_{21})^2 + (x_{32} - x_{22})^2]} \\ &= \frac{(x_{31} - x_{11})(x_{31} - x_{21}) + (x_{32} - x_{12})(x_{32} - x_{22})}{[(x_{31} - x_{11})^2 + (x_{32} - x_{12})^2][(x_{31} - x_{21})^2 + (x_{32} - x_{22})^2]}. \end{aligned} \tag{3.5.3}$$

If we apply $\nabla_1$ to the components of $b_{123}$ we get that:

$$\nabla_1 b_{123} = \nabla_1 b_{123_1} + \nabla_1 b_{123_2} + \nabla_1 b_{123_3}.$$

And such

$$\begin{aligned}
\frac{\partial}{\partial x_{11}} b_{123_1} &= \frac{(2x_{11} - x_{31} - x_{21})r_{12}^2 r_{13}^2 - \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle [2(x_{11} - x_{21})r_{13}^2 + 2(x_{11} - x_{31})r_{12}^2]}{r_{12}^4 r_{13}^4}, \\
\frac{\partial}{\partial x_{12}} b_{123_1} &= \frac{(2x_{12} - x_{32} - x_{22})r_{12}^2 r_{13}^2 - \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle [2(x_{12} - x_{22})r_{13}^2 + 2(x_{12} - x_{32})r_{12}^2]}{r_{12}^4 r_{13}^4}, \\
\frac{\partial}{\partial x_{11}} b_{123_2} &= \frac{-(x_{21} - x_{31})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle 2(x_{21} - x_{11})r_{23}^2}{r_{12}^4 r_{23}^4}, \\
\frac{\partial}{\partial x_{12}} b_{123_2} &= \frac{-(x_{22} - x_{32})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle 2(x_{22} - x_{12})r_{23}^2}{r_{12}^4 r_{23}^4}, \\
\frac{\partial}{\partial x_{11}} b_{123_3} &= \frac{-(x_{31} - x_{21})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle 2(x_{31} - x_{11})r_{23}^2}{r_{13}^4 r_{23}^4}, \\
\frac{\partial}{\partial x_{12}} b_{123_3} &= \frac{-(x_{32} - x_{22})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle 2(x_{32} - x_{12})r_{23}^2}{r_{13}^4 r_{23}^4}.
\end{aligned}$$

Hence

$$\begin{aligned}
\nabla_1 b_{123} &= \left( \frac{\partial}{\partial x_{11}} b_{123_1} + \frac{\partial}{\partial x_{11}} b_{123_2} + \frac{\partial}{\partial x_{11}} b_{123_3}, \frac{\partial}{\partial x_{12}} b_{123_1} + \frac{\partial}{\partial x_{12}} b_{123_2} + \frac{\partial}{\partial x_{12}} b_{123_3} \right) \\
&= \Bigg( \frac{(2x_{11} - x_{31} - x_{21})r_{12}^2 r_{13}^2 - \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle [2(x_{11} - x_{21})r_{13}^2 + 2(x_{11} - x_{31})r_{12}^2]}{r_{12}^4 r_{13}^4} + \\
&+ \frac{-(x_{21} - x_{31})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle 2(x_{21} - x_{11})r_{23}^2}{r_{12}^4 r_{23}^4} + \\
&+ \frac{-(x_{31} - x_{21})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle 2(x_{31} - x_{11})r_{23}^2}{r_{13}^4 r_{23}^4}, \\
&\frac{(2x_{12} - x_{32} - x_{22})r_{12}^2 r_{13}^2 - \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle [2(x_{12} - x_{22})r_{13}^2 + 2(x_{12} - x_{32})r_{12}^2]}{r_{12}^4 r_{13}^4} + \\
&+ \frac{-(x_{22} - x_{32})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle 2(x_{22} - x_{12})r_{23}^2}{r_{12}^4 r_{23}^4} + \\
&+ \frac{-(x_{32} - x_{22})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle 2(x_{32} - x_{12})r_{23}^2}{r_{13}^4 r_{23}^4} \Bigg).
\end{aligned} \tag{3.5.4}$$

For $\nabla_2 b_{123}$ by analogy, we have that

$$\nabla_2 b_{123} = \nabla_2 b_{123_1} + \nabla_2 b_{123_2} + \nabla_2 b_{123_3}.$$

Hence,

$$\begin{aligned}
\frac{\partial}{\partial x_{21}} b_{123_1} &= \frac{-(x_{11} - x_{31})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle 2(x_{11} - x_{21})r_{13}^2}{r_{12}^4 r_{13}^4}, \\
\frac{\partial}{\partial x_{22}} b_{123_1} &= \frac{-(x_{12} - x_{32})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3 \rangle 2(x_{12} - x_{22})r_{13}^2}{r_{12}^4 r_{13}^4}, \\
\frac{\partial}{\partial x_{21}} b_{123_2} &= \frac{(2x_{21} - x_{31} - x_{11})r_{12}^2 r_{23}^2 - \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle [2(x_{21} - x_{11})r_{23}^2 + r_{12}^2 2(x_{21} - x_{31})]}{r_{12}^4 r_{23}^4}, \\
\frac{\partial}{\partial x_{22}} b_{123_2} &= \frac{(2x_{22} - x_{32} - x_{12})r_{12}^2 r_{23}^2 - \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3 \rangle [2(x_{22} - x_{12})r_{23}^2 + 2(x_{22} - x_{32})r_{12}^2]}{r_{12}^4 r_{23}^4}, \\
\frac{\partial}{\partial x_{21}} b_{123_3} &= \frac{-(x_{31} - x_{11})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle 2(x_{31} - x_{21})r_{13}^2}{r_{13}^4 r_{23}^4}, \\
\frac{\partial}{\partial x_{22}} b_{123_3} &= \frac{-(x_{32} - x_{12})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2 \rangle 2(x_{32} - x_{22})r_{13}^2}{r_{13}^4 r_{23}^4}
\end{aligned}$$

and such

$$\begin{aligned}\nabla_2 b_{123} &= \left(\frac{\partial}{\partial x_{21}} b_{123_1} + \frac{\partial}{\partial x_{21}} b_{123_2} + \frac{\partial}{\partial x_{21}} b_{123_3}, \frac{\partial}{\partial x_{22}} b_{123_1} + \frac{\partial}{\partial x_{22}} b_{123_2} + \frac{\partial}{\partial x_{22}} b_{123_3}\right) \\ &= \Bigg(\frac{(x_{11}-x_{31})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3\rangle 2(x_{11}-x_{21})r_{13}^2}{r_{12}^4 r_{13}^4} + \\ &+ \frac{(2x_{21}-x_{31}-x_{11})r_{12}^2 r_{23}^2 - \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3\rangle [2(x_{21}-x_{11})r_{23}^2 + r_{12}^2 2(x_{21}-x_{31})]}{r_{12}^4 r_{23}^4} + \\ &+ \frac{-(x_{31}-x_{11})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2\rangle 2(x_{31}-x_{21})r_{13}^2}{r_{13}^4 r_{23}^4}, \\ &\frac{-(x_{12}-x_{32})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3\rangle 2(x_{12}-x_{22})r_{13}^2}{r_{12}^4 r_{13}^4} + \\ &+ \frac{(2x_{22}-x_{32}-x_{12})r_{12}^2 r_{23}^2 - \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3\rangle [2(x_{22}-x_{12})r_{23}^2 + 2(x_{22}-x_{32})r_{12}^2]}{r_{12}^4 r_{23}^4} + \\ &+ \frac{-(x_{32}-x_{12})r_{13}^2 r_{23}^2 + \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2\rangle 2(x_{32}-x_{22})r_{13}^2}{r_{13}^4 r_{23}^4}\Bigg).\end{aligned} \tag{3.5.5}$$

The $\nabla_3 b_{123}$ term can be described as

$$\nabla_3 b_{123} = \nabla_3 b_{123_1} + \nabla_3 b_{123_2} + \nabla_3 b_{123_3}.$$

Thus,

$$\begin{aligned}\frac{\partial}{\partial x_{31}} b_{123_1} &= \frac{-(x_{11}-x_{21})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3\rangle 2(x_{11}-x_{31})r_{12}^2}{r_{12}^4 r_{13}^4}, \\ \frac{\partial}{\partial x_{32}} b_{123_1} &= \frac{-(x_{12}-x_{22})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3\rangle 2(x_{12}-x_{32})r_{12}^2}{r_{12}^4 r_{13}^4}, \\ \frac{\partial}{\partial x_{31}} b_{123_2} &= \frac{-(x_{21}-x_{11})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3\rangle 2(x_{21}-x_{31})r_{12}^2}{r_{12}^4 r_{23}^4}, \\ \frac{\partial}{\partial x_{32}} b_{123_2} &= \frac{-(x_{22}-x_{12})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3\rangle 2(x_{22}-x_{32})r_{12}^2}{r_{12}^4 r_{23}^4}, \\ \frac{\partial}{\partial x_{31}} b_{123_3} &= \frac{(2x_{31}-x_{11}-x_{21})r_{13}^2 r_{23}^2 - \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2\rangle [2(x_{31}-x_{11})r_{23}^2 + 2(x_{31}-x_{21})r_{13}^2]}{r_{13}^4 r_{23}^4}, \\ \frac{\partial}{\partial x_{32}} b_{123_3} &= \frac{(2x_{32}-x_{12}-x_{22})r_{13}^2 r_{23}^2 - \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2\rangle [2(x_{32}-x_{12})r_{23}^2 + 2(x_{32}-x_{22})r_{13}^2]}{r_{13}^4 r_{23}^4}\end{aligned}$$

which brings $\nabla_3 b_{123}$ to the form

$$\begin{aligned}\nabla_3 b_{123} &= \left(\frac{\partial}{\partial x_{31}} b_{123_1} + \frac{\partial}{\partial x_{31}} b_{123_2} + \frac{\partial}{\partial x_{31}} b_{123_3}, \frac{\partial}{\partial x_{32}} b_{123_1} + \frac{\partial}{\partial x_{32}} b_{123_2} + \frac{\partial}{\partial x_{32}} b_{123_3}\right) \\ &= \Bigg(\frac{-(x_{11}-x_{21})r_{12}^2 r_{13}^2 + \langle \mathbf{x}_1 - \mathbf{x}_2, \mathbf{x}_1 - \mathbf{x}_3\rangle 2(x_{11}-x_{31})r_{12}^2}{r_{12}^4 r_{13}^4} + \\ &\frac{-(x_{21}-x_{11})r_{12}^2 r_{23}^2 + \langle \mathbf{x}_2 - \mathbf{x}_1, \mathbf{x}_2 - \mathbf{x}_3\rangle 2(x_{21}-x_{31})r_{12}^2}{r_{12}^4 r_{23}^4} + \\ &+ \frac{(2x_{31}-x_{11}-x_{21})r_{13}^2 r_{23}^2 - \langle \mathbf{x}_3 - \mathbf{x}_1, \mathbf{x}_3 - \mathbf{x}_2\rangle [2(x_{31}-x_{11})r_{23}^2 + 2(x_{31}-x_{21})r_{13}^2]}{r_{13}^4 r_{23}^4},\end{aligned}$$

$$\frac{-(x_{12}-x_{22})r_{12}^2r_{13}^2+\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle 2(x_{12}-x_{32})r_{12}^2}{r_{12}^4r_{13}^4}+$$
$$+\frac{-(x_{22}-x_{12})r_{12}^2r_{23}^2+\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle 2(x_{22}-x_{32})r_{12}^2}{r_{12}^4r_{23}^4}+ \tag{3.5.6}$$
$$+\frac{(2x_{32}-x_{12}-x_{22})r_{13}^2r_{23}^2-\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle[2(x_{32}-x_{12})r_{23}^2+2(x_{32}-x_{22})r_{13}^2]}{r_{13}^4r_{23}^4}\Big).$$

Using (3.5.4), (3.5.5) and (3.5.6) we obtain that

$$\begin{aligned}&\nabla b_{123}=(\nabla_1 b_{123},\nabla_2 b_{123},\nabla_3 b_{123})\\ \Longrightarrow\ &\nabla b_{123}=(M_1,M_2,M_3,M_4,M_5,M_6).\end{aligned} \tag{3.5.7}$$

The following lines present each $M_i$ term for $i\leq 6$.

More specifically,

$$\begin{aligned}M_1:=&\frac{(2x_{11}-x_{31}-x_{21})r_{12}^2r_{13}^2-\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle[2(x_{11}-x_{21})r_{13}^2+2(x_{11}-x_{31})r_{12}^2]}{r_{12}^4r_{13}^4}+\\ &+\frac{-(x_{21}-x_{31})r_{12}^2r_{23}^2+\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle 2(x_{21}-x_{11})r_{23}^2}{r_{12}^4r_{23}^4}+\\ &+\frac{-(x_{31}-x_{21})r_{13}^2r_{23}^2+\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle 2(x_{31}-x_{11})r_{23}^2}{r_{13}^4r_{23}^4}\\ \Longrightarrow\ M_1:=&\frac{1}{r_{12}^4r_{13}^4r_{23}^4}\Big[(2\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3)_1r_{12}^2r_{13}^2r_{23}^4-2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle((\mathbf{x}_1-\mathbf{x}_2)_1r_{13}^2+\\ &+(\mathbf{x}_1-\mathbf{x}_3)_1r_{12}^2)+2r_{23}^2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_1)_1-(\mathbf{x}_2-\mathbf{x}_3)_1r_{12}^2r_{23}^2r_{13}^4+\\ &+2r_{23}^2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_1)_1-(\mathbf{x}_3-\mathbf{x}_2)_1r_{13}^2r_{23}^2r_{12}^4\Big].\end{aligned}$$

Here, by using the form

$$(2\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3)_w,\quad \text{for}\quad w\leq 2$$

we intend to highlight the $w^{th}$ component of the vector in use.

For the other $M_j$ terms the calculations are as follows:

$$\begin{aligned}M_2:=&\frac{(2x_{12}-x_{32}-x_{22})r_{12}^2r_{13}^2-\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle[2(x_{12}-x_{22})r_{13}^2+2(x_{12}-x_{32})r_{12}^2]}{r_{12}^4r_{13}^4}+\\ &+\frac{-(x_{22}-x_{32})r_{12}^2r_{23}^2+\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle 2(x_{22}-x_{12})r_{23}^2}{r_{12}^4r_{23}^4}+\\ &+\frac{-(x_{32}-x_{22})r_{13}^2r_{23}^2+\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle 2(x_{32}-x_{12})r_{23}^2}{r_{13}^4r_{23}^4}\\ \Longrightarrow\ M_2:=&\frac{1}{r_{12}^4r_{13}^4r_{23}^4}\Big[(2\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3)_2r_{12}^2r_{13}^2r_{23}^4-2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle((\mathbf{x}_1-\mathbf{x}_2)_2r_{13}^2+\\ &+(\mathbf{x}_1-\mathbf{x}_3)_2r_{12}^2)+2r_{23}^2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_1)_2-(\mathbf{x}_2-\mathbf{x}_3)_2r_{12}^2r_{23}^2r_{13}^4+\\ &+2r_{23}^2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_1)_2-(\mathbf{x}_3-\mathbf{x}_2)_2r_{13}^2r_{23}^2r_{12}^4\Big],\end{aligned}$$

$$M_3 := \frac{-(x_{11}-x_{31})r_{12}^2r_{13}^2+\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle 2(x_{11}-x_{21})r_{13}^2}{r_{12}^4r_{13}^4}+$$
$$+\frac{(2x_{21}-x_{31}-x_{11})r_{12}^2r_{23}^2-\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle[2(x_{21}-x_{11})r_{23}^2+r_{12}^2 2(x_{21}-x_{31})]}{r_{12}^4r_{23}^4}+$$
$$+\frac{-(x_{31}-x_{11})r_{13}^2r_{23}^2+\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle 2(x_{31}-x_{21})r_{13}^2}{r_{13}^4r_{23}^4}$$
$$\implies M_3 := \frac{1}{r_{12}^4r_{13}^4r_{23}^4}\Big[(2\mathbf{x}_2-\mathbf{x}_1-\mathbf{x}_3)_1r_{12}^2r_{23}^2r_{13}^4-2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle((\mathbf{x}_2-\mathbf{x}_1)_1r_{23}^2+$$
$$+(\mathbf{x}_2-\mathbf{x}_3)_1r_{12}^2)+2r_{13}^2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_2)_1-(\mathbf{x}_1-\mathbf{x}_3)_1r_{12}^2r_{13}^2r_{23}^4+$$
$$+2r_{13}^2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_2)_1-(\mathbf{x}_3-\mathbf{x}_1)_1r_{13}^2r_{23}^2r_{12}^4\Big],$$

$$M_4 := \frac{-(x_{12}-x_{32})r_{12}^2r_{13}^2+\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle 2(x_{12}-x_{22})r_{13}^2}{r_{12}^4r_{13}^4}+$$
$$+\frac{(2x_{22}-x_{32}-x_{12})r_{12}^2r_{23}^2-\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle[2(x_{22}-x_{12})r_{23}^2+2(x_{22}-x_{32})r_{12}^2]}{r_{12}^4r_{23}^4}+$$
$$+\frac{-(x_{32}-x_{12})r_{13}^2r_{23}^2+\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle 2(x_{32}-x_{22})r_{13}^2}{r_{13}^4r_{23}^4}$$
$$\implies M_4 := \frac{1}{r_{12}^4r_{13}^4r_{23}^4}\Big[(2\mathbf{x}_2-\mathbf{x}_1-\mathbf{x}_3)_2r_{12}^2r_{23}^2r_{13}^4-2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle((\mathbf{x}_2-\mathbf{x}_1)_2r_{23}^2+$$
$$+(\mathbf{x}_2-\mathbf{x}_3)_2r_{12}^2)+2r_{13}^2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_2)_2-(\mathbf{x}_1-\mathbf{x}_3)_2r_{12}^2r_{13}^2r_{23}^4+$$
$$+2r_{13}^2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_2)_2-(\mathbf{x}_3-\mathbf{x}_1)_2r_{13}^2r_{23}^2r_{12}^4\Big],$$

$$M_5 := \frac{-(x_{11}-x_{21})r_{12}^2r_{13}^2+\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle 2(x_{11}-x_{31})r_{12}^2}{r_{12}^4r_{13}^4}+$$
$$+\frac{-(x_{21}-x_{11})r_{12}^2r_{23}^2+\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle 2(x_{21}-x_{31})r_{12}^2}{r_{12}^4r_{23}^4}+$$
$$+\frac{(2x_{31}-x_{11}-x_{21})r_{13}^2r_{23}^2-\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle[2(x_{31}-x_{11})r_{23}^2+2(x_{31}-x_{21})r_{13}^2]}{r_{13}^4r_{23}^4}$$
$$\implies M_5 := \frac{1}{r_{12}^4r_{13}^4r_{23}^4}\Big[(2\mathbf{x}_3-\mathbf{x}_1-\mathbf{x}_2)_1r_{13}^2r_{23}^2r_{12}^4-2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle((\mathbf{x}_3-\mathbf{x}_1)_1r_{23}^2+$$
$$+(\mathbf{x}_3-\mathbf{x}_2)_1r_{13}^2)+2r_{12}^2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_3)_1-(\mathbf{x}_2-\mathbf{x}_1)_1r_{12}^2r_{23}^2r_{13}^4+$$
$$+2r_{12}^2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_3)_1-(\mathbf{x}_1-\mathbf{x}_2)_1r_{13}^2r_{12}^2r_{23}^4\Big],$$

with the last term being

$$M_6 := \frac{-(x_{12}-x_{22})r_{12}^2r_{13}^2+\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle 2(x_{12}-x_{32})r_{12}^2}{r_{12}^4r_{13}^4}+$$
$$+\frac{-(x_{22}-x_{12})r_{12}^2r_{23}^2+\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle 2(x_{22}-x_{32})r_{12}^2}{r_{12}^4r_{23}^4}+$$
$$+\frac{(2x_{32}-x_{12}-x_{22})r_{13}^2r_{23}^2-\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle[2(x_{32}-x_{12})r_{23}^2+2(x_{32}-x_{22})r_{13}^2]}{r_{13}^4r_{23}^4}$$

$$\implies M_6 := \frac{1}{r_{12}^4 r_{13}^4 r_{23}^4}\Big[(2\mathbf{x}_3-\mathbf{x}_1-\mathbf{x}_2)_2 r_{13}^2 r_{23}^2 r_{12}^4 - 2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle((\mathbf{x}_3-\mathbf{x}_1)_2 r_{23}^2 + \\ + (\mathbf{x}_3-\mathbf{x}_2)_2 r_{13}^2) + 2r_{12}^2 r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_3)_2 - (\mathbf{x}_2-\mathbf{x}_1)_2 r_{12}^2 r_{23}^2 r_{13}^4 + \\ + 2r_{12}^2 r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_3)_2 - (\mathbf{x}_1-\mathbf{x}_2)_2 r_{13}^2 r_{12}^2 r_{23}^4\Big].$$

Now, (3.5.7) becomes:

$$\nabla b_{123} := \frac{1}{r_{12}^4 r_{13}^4 r_{23}^4}\Big((2\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3)_1 r_{12}^2 r_{13}^2 r_{23}^4 - 2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle((\mathbf{x}_1-\mathbf{x}_2)_1 r_{13}^2 + \\ + (\mathbf{x}_1-\mathbf{x}_3)_1 r_{12}^2) + 2r_{23}^2 r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_1)_1 - (\mathbf{x}_2-\mathbf{x}_3)_1 r_{12}^2 r_{23}^2 r_{13}^4 + \\ + 2r_{23}^2 r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_1)_1 - (\mathbf{x}_3-\mathbf{x}_2)_1 r_{13}^2 r_{23}^2 r_{12}^4, \\ (2\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3)_2 r_{12}^2 r_{13}^2 r_{23}^4 - 2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle((\mathbf{x}_1-\mathbf{x}_2)_2 r_{13}^2 + (\mathbf{x}_1-\mathbf{x}_3)_2 r_{12}^2) + \\ + 2r_{23}^2 r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_1)_2 - (\mathbf{x}_2-\mathbf{x}_3)_2 r_{12}^2 r_{23}^2 r_{13}^4 + \\ + 2r_{23}^2 r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_1)_2 - (\mathbf{x}_3-\mathbf{x}_2)_2 r_{13}^2 r_{23}^2 r_{12}^4, \\ (2\mathbf{x}_2-\mathbf{x}_1-\mathbf{x}_3)_1 r_{12}^2 r_{23}^2 r_{13}^4 - 2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle((\mathbf{x}_2-\mathbf{x}_1)_1 r_{23}^2 + (\mathbf{x}_2-\mathbf{x}_3)_1 r_{12}^2) + \\ + 2r_{13}^2 r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_2)_1 - (\mathbf{x}_1-\mathbf{x}_3)_1 r_{12}^2 r_{13}^2 r_{23}^4 + \\ + 2r_{13}^2 r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_2)_1 - (\mathbf{x}_3-\mathbf{x}_1)_1 r_{13}^2 r_{23}^2 r_{12}^4, \\ (2\mathbf{x}_2-\mathbf{x}_1-\mathbf{x}_3)_2 r_{12}^2 r_{23}^2 r_{13}^4 - 2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle((\mathbf{x}_2-\mathbf{x}_1)_2 r_{23}^2 + (\mathbf{x}_2-\mathbf{x}_3)_2 r_{12}^2) + \\ + 2r_{13}^2 r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_2)_2 - (\mathbf{x}_1-\mathbf{x}_3)_2 r_{12}^2 r_{13}^2 r_{23}^4 + \\ + 2r_{13}^2 r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_2)_2 - (\mathbf{x}_3-\mathbf{x}_1)_2 r_{13}^2 r_{23}^2 r_{12}^4, \\ (2\mathbf{x}_3-\mathbf{x}_1-\mathbf{x}_2)_1 r_{13}^2 r_{23}^2 r_{12}^4 - 2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle((\mathbf{x}_3-\mathbf{x}_1)_1 r_{23}^2 + (\mathbf{x}_3-\mathbf{x}_2)_1 r_{13}^2) + \\ + 2r_{12}^2 r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_3)_1 - (\mathbf{x}_2-\mathbf{x}_1)_1 r_{12}^2 r_{23}^2 r_{13}^4 + \\ + 2r_{12}^2 r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_3)_1 - (\mathbf{x}_1-\mathbf{x}_2)_1 r_{13}^2 r_{12}^2 r_{23}^4, \\ (2\mathbf{x}_3-\mathbf{x}_1-\mathbf{x}_2)_2 r_{13}^2 r_{23}^2 r_{12}^4 - 2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle((\mathbf{x}_3-\mathbf{x}_1)_2 r_{23}^2 + (\mathbf{x}_3-\mathbf{x}_2)_2 r_{13}^2) + \\ + 2r_{12}^2 r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_3)_2 - (\mathbf{x}_2-\mathbf{x}_1)_2 r_{12}^2 r_{23}^2 r_{13}^4 + \\ + 2r_{12}^2 r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_3)_2 - (\mathbf{x}_1-\mathbf{x}_2)_2 r_{13}^2 r_{12}^2 r_{23}^4\Big).$$

By gathering all the components together, we obtain

$$\nabla b_{123} := \frac{1}{r_{12}^4 r_{13}^4 r_{23}^4}\Big((2\mathbf{x}_1-\mathbf{x}_2-\mathbf{x}_3) r_{12}^2 r_{13}^2 r_{23}^4 - 2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2, \mathbf{x}_1-\mathbf{x}_3\rangle((\mathbf{x}_1-\mathbf{x}_2) r_{13}^2 + (\mathbf{x}_1-\mathbf{x}_3) r_{12}^2) + \\ + 2r_{23}^2 r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1, \mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_1) - \\ - (\mathbf{x}_2-\mathbf{x}_3) r_{12}^2 r_{23}^2 r_{13}^4 + 2r_{23}^2 r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1, \mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_1) - \\ - (\mathbf{x}_3-\mathbf{x}_2) r_{13}^2 r_{23}^2 r_{12}^4,$$

$$
\begin{aligned}
(2\mathbf{x}_2-\mathbf{x}_1-\mathbf{x}_3)r_{12}^2r_{23}^2r_{13}^4 - 2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle((\mathbf{x}_2-\mathbf{x}_1)r_{23}^2+(\mathbf{x}_2-\mathbf{x}_3)r_{12}^2)+\\
+2r_{13}^2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_2)-\\
-(\mathbf{x}_1-\mathbf{x}_3)r_{12}^2r_{13}^2r_{23}^4+2r_{13}^2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle(\mathbf{x}_3-\mathbf{x}_2)-\\
-(\mathbf{x}_3-\mathbf{x}_1)r_{13}^2r_{23}^2r_{12}^4,\\
(2\mathbf{x}_3-\mathbf{x}_1-\mathbf{x}_2)r_{13}^2r_{23}^2r_{12}^4 - 2r_{12}^4\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle((\mathbf{x}_3-\mathbf{x}_1)r_{23}^2+(\mathbf{x}_3-\mathbf{x}_2)r_{13}^2)+\\
-(\mathbf{x}_2-\mathbf{x}_1)r_{12}^2r_{23}^2r_{13}^4+2r_{12}^2r_{13}^4\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle(\mathbf{x}_2-\mathbf{x}_3)-\\
+2r_{12}^2r_{23}^4\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle(\mathbf{x}_1-\mathbf{x}_3)-\\
-(\mathbf{x}_1-\mathbf{x}_2)r_{13}^2r_{12}^2r_{23}^4\Big).
\end{aligned}
$$

We started by mentioning that we want a specific form for $\frac{1}{R^2}$, hence by Lemma 3.2.2 for $\mathbb{R}^2$,

$$
\begin{aligned}
2b_{123} &= \frac{1}{R^2}\\
\implies \nabla(2b_{123}) &= \nabla\left(\frac{1}{R^2}\right) = \frac{-2\nabla R}{R^3}\\
\implies R\nabla b_{123} &= \frac{-\nabla R}{R^2}.
\end{aligned}
$$

Factorising the left hand side provides an insight on whether the resulting vector may be appropriate for determining an optimal constant.

Hence

$$
\begin{aligned}
b_{123} &= \frac{\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle}{r_{12}^2r_{13}^2}+\frac{\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle}{r_{12}^2r_{23}^2}+\frac{\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle}{r_{13}^2r_{23}^2}=\frac{1}{2R^2}\\
\implies \frac{1}{R} &:= \sqrt{2b_{123}} = \sqrt{2}\sqrt{\frac{\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle}{r_{12}^2r_{13}^2}+\frac{\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle}{r_{12}^2r_{23}^2}+\frac{\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle}{r_{13}^2r_{23}^2}}\\
&= \sqrt{2}\sqrt{\frac{r_{23}^2\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle+r_{13}^2\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle+r_{12}^2\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle}{r_{13}^2r_{23}^2r_{12}^2}}\\
&= \frac{\sqrt{2}}{r_{13}r_{23}r_{12}}\sqrt{r_{23}^2\langle \mathbf{x}_1-\mathbf{x}_2,\mathbf{x}_1-\mathbf{x}_3\rangle+r_{13}^2\langle \mathbf{x}_2-\mathbf{x}_1,\mathbf{x}_2-\mathbf{x}_3\rangle+r_{12}^2\langle \mathbf{x}_3-\mathbf{x}_1,\mathbf{x}_3-\mathbf{x}_2\rangle}\\
&= \frac{\sqrt{2}\sqrt{2}}{r_{13}r_{23}r_{12}}\sqrt{A}\\
&= \frac{2}{r_{13}r_{12}r_{23}}\sqrt{A},
\end{aligned}
$$

where

$$
\begin{aligned}
A &:= r_{23}^2(x_{11}-x_{21},x_{12}-x_{22})(x_{11}-x_{31},x_{12}-x_{32})+r_{13}^2(x_{21}-x_{11},x_{22}-x_{12})(x_{21}-x_{31},x_{22}-x_{32})+\\
&\quad +r_{12}^2(x_{31}-x_{11},x_{32}-x_{12})(x_{31}-x_{21},x_{32}-x_{22})\\
&= x_{22}(x_{11}-x_{31})^2+x_{12}(x_{21}-x_{31})^2-2x_{12}x_{22}(x_{11}x_{21}+x_{31}^2-x_{31}x_{21})+\\
&\quad +2x_{12}x_{32}(x_{11}-x_{21})(x_{21}-x_{31})-\\
&\quad -2x_{22}x_{32}(x_{11}^2-x_{11}x_{21}+x_{11}x_{31}+x_{21}x_{31})+\\
&\quad +x_{32}^2[(x_{11}-x_{21})^2-2x_{21}x_{31}].
\end{aligned}
$$

The form of $A$ as it stands from the above calculations does not provide a clear response to our optimal question regarding the constant in (3.3.1b). We are still searching for a better approach towards simplifying $A$. If a simpler form is established, then it is possible to extend the concept towards the $N$ particle Hardy inequality extension which we saw in Theorem 3.3.4.

# Chapter 4

# Sharp Hardy inequalities for block radial functions

## 4.1 Introduction

The chapter includes key notes regarding the weighted Hardy inequalities presented in article [51]. The sharp constants involved depend on the lowest eigenvalues of auxiliary Schrödinger operators on a sphere. For block radial weights such sharp constants are given in terms of the lowest eigenvalue of a Legendre type equation.

Let us consider the classical Hardy inequality for the Laplacian in $\mathbb{R}^d$.

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{|u|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad d \geq 3. \tag{4.1.1}$$

The literature devoted to various types of Hardy inequalities is vast and in article [58] authors are considering more general classes of homogeneous functions of degree $-2$ which may have singularities along rays starting at the origin. The general classes are used instead of spherical symmetrical weights $1/|\mathbf{x}|^2$ present in the integral of (4.1.1). Namely,

Let $\mathbf{x} \in \mathbb{R}^d$, $d \geq 3$ and $\mathbf{x} = (r, \vartheta)$ be polar coordinates in $\mathbb{R}^d$. Let $-\Delta_\vartheta$ be the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$.

Assume that $\Phi : \mathbb{S}^{d-1} \to \mathbb{R}$, $\Phi \geq 0$, and that the Schrödinger operator $H$ in $L^2(\mathbb{S}^{d-1})$

$$H = -(\nabla_\vartheta)^2 - \tau\Phi = -\Delta_\vartheta - \tau\Phi, \quad \tau \in \mathbb{R}, \tag{4.1.2a}$$

is bounded from below. Denote

$$\lambda(\tau) = \inf_{\psi \in \mathcal{H}^1(\mathbb{S}^{d-1}) \setminus \{0\}} \frac{\int_{\mathbb{S}^{d-1}} (|\nabla_\vartheta \psi|^2 - \tau\Phi\,|\psi|^2)\, d\vartheta}{\int_{\mathbb{S}^{d-1}} |\psi|^2\, d\vartheta}, \tag{4.1.2b}$$

where $\mathcal{H}^1(\mathbb{S}^{d-1})$ is the Sobolev space on which function $u$ is defined.

If $d \geq 3$, then the classical Hardy inequality in $\mathbb{R}^d$ is obtained

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{x}} u(\mathbf{x})|^2\, d\mathbf{x} \geq \left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d).$$

The main result of this chapter presents that for $d \geq 3$ and under some conditions imposed on $\Phi$

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{x}} u(\mathbf{x})|^2 \, d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} \, |u(\mathbf{x})|^2 \, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d),$$

with sharp $\tau = \tau(\Phi)$.

If $d \geq 3$ and $0 \leq V \in L^{\frac{d}{2},\infty}$ then in [87] and [86] the authors obtained the inequality

$$\int_{\mathbb{R}^d} V(\mathbf{x}) \, |u(\mathbf{x})|^2 \, d\mathbf{x} \leq \left( \frac{2}{d-2} \right)^2 d^{\frac{2}{d}} |\mathbb{S}^{d-1}| \, \|V\|_{\frac{d}{2},\infty} \int_{\mathbb{R}^d} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x}, \ u \in \mathcal{H}^1(\mathbb{R}^d),$$

where $||\cdot||_{\frac{d}{2},\infty}$ is the Lorentz norm.

If $V(\mathbf{x}) = \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}$, $\Phi \in L^{\frac{d}{2}}(\mathbb{S}^{d-1})$, $d \geq 3$, then

$$\int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} \, |u(\mathbf{x})|^2 \, d\mathbf{x} \leq \left( \frac{2}{d-2} \right)^2 |\mathbb{S}^{d-1}| \, \|\Phi\|_{\frac{d}{2}} \int_{\mathbb{R}^d} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x}.$$

Article [58] presents a stronger inequality in the form of the following theorem

**Theorem 4.1.1** *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$, where*

$$p \geq \frac{(d-2)^2}{2(d-1)} + 1.$$

*Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} \, |u(\mathbf{x})|^2 d\mathbf{x}$$

*for $u \in C_0^\infty(\mathbb{R}^d)$ and*

$$\tau = \frac{(d-2)^2}{4} |\mathbb{S}^{d-1}|^{\frac{1}{p}} \, \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}.$$

Therefore a sufficient condition for the operator $H$, in (4.1.2a), to be bounded is $\Phi \in L^p(\mathbb{S}^{d-1})$, $\Phi \geq 0$ with $p > 1$ for $d = 3$ and $p = \frac{d-1}{2}$ for $d \geq 4$. By using polar coordinates and Sobolev's inequality, we have

$$\int_{\mathbb{S}^{d-1}} \Phi(\vartheta) |\psi(\vartheta)|^2 \, d\vartheta \leq \varepsilon \int_{\mathbb{S}^{d-1}} |\nabla_\vartheta \psi(\vartheta)|^2 \, d\vartheta + C_\varepsilon \int_{\mathbb{S}^{d-1}} |\psi(\vartheta)|^2 \, d\vartheta, \tag{4.1.3}$$

for some $\varepsilon > 0$.

The last inequality is equivalent to the compact embedding of the form $\int_{\mathbb{S}^{d-1}} \Phi(\vartheta) |\psi(\vartheta)|^2 \, d\vartheta$ into the Sobolev space $H^1(\mathbb{R}^d)$. As a consequence, the spectrum of the operator $H$ from (4.1.2a) is discrete.

A main application of the result presented in this chapter consists of establishing a sharp constant $\tau$ in the case of block radial potential considered by L.Skrzypczak and C.Tintarev [106] and [107]. For example, when $d = 4$, we have

$$\int_{\mathbb{R}^4} |\nabla_{\mathbf{x}} u(\mathbf{x})|^2 \, d\mathbf{x} \geq \tau \int_{\mathbb{R}^4} \frac{|u(\mathbf{x})|^2}{r_1(\mathbf{x}) r_2(\mathbf{x})} \, d\mathbf{x}, \tag{4.1.4}$$

for $\mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $r_1(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}$, $r_2(\mathbf{x}) = \sqrt{x_3^2 + x_4^2}$. It is important to emphasise that in [106] the authors obtained inequality (4.1.4) only for a particular class of functions $u$. More specifically, the paper requires that $u = (r_1, r_2)$. Our result holds for general $u \in H^1(\mathbb{R}^4)$. Introducing Hopf coordinates in $\mathbb{S}^3$ reduces the problem of finding a sharp constant in (4.1.4) to a minimisation problem related to a Legendre type equation on the interval $\left[0, \frac{\pi}{2}\right]$.

Proving the main results involves a method used in [58] for a Keller-Lieb-Thirring inequality for estimating a first eigenvalue of a magnetic Schrödinger operator on $\mathbb{S}^{d-1}$ in [31].

## 4.2 Main results

Considering the information presented in the previous section we can proceed to presenting the main results. We shall provide the necessary proofs in the following section such that it is easier to follow our arguments throughout the section.

**Theorem 4.2.1** *Let $d \geq 3$, $\Phi \geq 0$ satisfy the inequality (4.1.3) and $\lambda(\tau)$, defined by the quotient (4.1.2b), satisfy the equation*

$$\lambda(\tau_0) = -\left(\frac{d-2}{2}\right)^2 \quad \text{for some } \tau_0. \tag{4.2.1a}$$

*Then*

$$\int_{\mathbb{R}^d} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geq \tau_0 \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |u(\mathbf{x})|^2 \, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^d). \tag{4.2.1b}$$

**Remark**: Finding the value of the constant $\tau_0$ is not easy. Consider the equation

$$-\Delta_\vartheta \psi(\vartheta) - \tau \, \Phi(\vartheta)\psi(\vartheta) = \lambda \psi(\vartheta). \tag{4.2.2}$$

Article [31] presents a proof which states that for any non-trivial $\Phi \geq 0$, $-\lambda = -\lambda(\tau)$ represents an increasing convex function of $\tau$. Clearly $\lambda(0) = 0$ and $-\lambda(\tau) \to +\infty$ as $\tau \to \infty$. Therefore there exists an unique $\tau = \tau(\lambda)$ satisfying (4.2.2). Namely, there is an unique $\tau_0$ and non-trivial solution $\psi$ of (4.2.2) such that $\lambda(\tau_0) = -\frac{(d-2)^2}{4}$.

The next statement involves the sharpness of inequality (4.2.1b).

**Theorem 4.2.2** *If $\Phi \geq 0$ satisfy the inequality (4.1.3) and for some $\tau_0$ the value $\lambda(\tau_0) = -\frac{(d-2)^2}{2}$, then the Hardy inequality (4.2.1b) is sharp.*

**Remark**: As an important side note, we have to mention that there does not exist a minimiser for inequality (4.2.1b).

Making reference to article [58], the following theorem provides an estimate for $\tau_0$:

**Theorem 4.2.3** *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$ with $p \geq \frac{d-1}{2} + \frac{1}{2(d-1)}$. Then*

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, d\mathbf{x} \geq \tau \int_{\mathbb{R}^d} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)|u|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad u \in \mathcal{H}^1(\mathbb{R}^d). \tag{4.2.3}$$

**Corollary 4.2.4** Under the conditions given in Theorem 4.2.3 we have

$$\tau_0 \geq \frac{(d-2)^2}{4} |\mathbb{S}^{d-1}|^{\frac{1}{p}} \|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}.$$

**Remark**: If $d = 3$, then the lowest possible value of $p$ that is allowed in Theorem 4.2.3 equals $\frac{5}{4}$.

The proof of the last theorem was also provided in [58] and it involves the usage of the following result:

**Theorem 4.2.5 (Increasing weight)** *Let $d \geq 3$ and $0 \leq \Phi \in L^p(\mathbb{S}^{d-1})$ and $p \in \left(\frac{d-1}{2}, +\infty\right)$. Then there exists an increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R}$*

$$\alpha(\mu) = \mu \quad \text{for any} \quad \mu \in \left[0, \frac{d-1}{2}(p-1)\right] \tag{4.2.4a}$$

*and convex, if $\mu \in \left(\frac{d-1}{2}(p-1), +\infty\right)$, such that*

$$|\lambda_1(-\Delta_\vartheta - \Phi)| \leq \alpha\left(\frac{1}{|\mathbb{S}^{d-1}|^{\frac{1}{p}}}||\Phi||_{L^p(\mathbb{S}^{d-1})}\right). \tag{4.2.4b}$$

*The estimate (4.2.4b) is optimal in the sense that there exists a non-negative function $\Phi$, such that*

$$|\lambda_1(-\Delta_\vartheta - \Phi)| = \alpha\left(\frac{1}{|\mathbb{S}^{d-1}|^{\frac{1}{p}}}||\Phi||_{L^p(\mathbb{S}^{d-1})}\right),$$

*for any $\mu \in \left(\frac{d-1}{2}(p-1), +\infty\right)$. If $\mu \leq \frac{d-1}{2}(p-1)$, then equality in (4.2.4b) is achieved for constants.*

*For large values of $\mu$ we have*

$$\alpha(\mu)^{p-\frac{d-1}{2}} = L^1_{p-\frac{d-1}{2},d-1}\mu^p(1+o(1)),$$

*where $L^1_{\gamma,d-1}$ are the Lieb - Thirring constants appearing in [79] in the inequality for the lowest eigenvalue of a Schrödinger operator in $L^2(\mathbb{R}^{d-1})$.*

*If $p = \frac{d-1}{2}$, for $d \geq 4$, then (4.2.4b) is satisfied with $\alpha(\mu) = \mu$ for $\mu \in \left[0, \frac{(d-1)(d-3)}{3}\right]$.*

## 4.3 Proof of main results

**Proof of Theorem 4.2.1:**

Consider the polar coordinates $\mathbf{x} = (r, \vartheta)$. Then

$$\int_{\mathbb{R}^d}\left(|\nabla u(\mathbf{x})|^2 - \tau_0\frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}|u|^2\right)d\mathbf{x} = \int_0^\infty\int_{\mathbb{S}^{d-1}}\left(|u_r'|^2 + \frac{1}{r^2}\left(|\nabla_\vartheta u|^2 - \tau_0\Phi(\vartheta)|u|^2\right)\right)r^{d-1}\,d\vartheta\,dr.$$

Moreover, since

$$\int_0^\infty |u_r'|^2\,r^{d-1}\,dr \geq \frac{(d-2)^2}{4}\int_0^\infty \frac{|u|^2}{r^2}\,r^{d-1}\,dr,$$

then

$$\int_{\mathbb{R}^d}\left(|\nabla u(\mathbf{x})|^2 - \tau_0\frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}|u|^2\right)d\mathbf{x} \geq \int_0^\infty\int_{\mathbb{S}^{d-1}}\left(\frac{(d-2)^2}{4} + \lambda(\tau_0)\right)\frac{|u|^2}{r^2}\,r^{d-1}\,d\vartheta\,dr = 0.$$

The last equality was obtained by using (4.1.2b) and (4.2.1a). □

Let us proceed to the proof of sharpness of (4.2.1b).

**Proof of Theorem 4.2.2:**

The sharpness of the classical Hardy inequality implies that for any $\delta > 0$ there exists a non-trivial $\varphi_\delta \in C_0^\infty(0,\infty)$ such that

$$\int_0^\infty\left|\frac{d}{dr}\varphi_\delta(r)\right|^2 r^{d-1}\,dr \leq \left[\frac{(d-2)^2}{4} + \delta\right]\int_0^\infty\frac{|\varphi_\delta(r)|^2}{r^2}\,r^{d-1}\,dr.$$

Consider $u_\delta(r,\vartheta)=\varphi_\delta(r)\psi_\delta(\vartheta)$, where $\psi_\delta$ is given by the equation

$$-\Delta_\vartheta\psi_\delta(\vartheta)-\tau_\delta\,\Phi(\vartheta)\psi_\delta(\vartheta)=-\left[\frac{(d-2)^2}{4}+\delta\right]\psi_\delta(\vartheta).$$

As a consequence

$$\int_{\mathbb{S}^{d-1}}\left[|\nabla_\vartheta\psi_\delta(\vartheta)|^2+\left[\frac{(d-2)^2}{4}+\delta\right]|\psi_\delta(\vartheta)|^2\right]d\vartheta=\tau_\delta\int_{\mathbb{S}^{d-1}}\Phi(\vartheta)|\psi_\delta(\vartheta)|^2\,d\vartheta$$

and such

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}|\nabla_{\mathbf{x}}u_\delta|^2d\mathbf{x}&=\int_0^\infty\int_{\mathbb{S}^{d-1}}\left[\left|\frac{d}{dr}\varphi_\delta(r)\right|^2|\psi_\delta(\vartheta)|^2+|\varphi_\delta(r)|^2\frac{|\nabla_\vartheta\psi_\delta(\vartheta)|^2}{r^2}\right]r^{d-1}d\vartheta dr\\&\le\int_0^\infty\int_{\mathbb{S}^{d-1}}\frac{|\varphi_\delta(r)|^2}{r^2}\left(|\nabla_\vartheta\psi_\delta(\vartheta)|^2+\left[\frac{(d-2)^2}{4}+\delta\right]|\psi_\delta(\vartheta)|^2\right)d\vartheta dr\\&=\int_0^\infty\int_{\mathbb{S}^{d-1}}\tau_\delta\Phi(\vartheta)\frac{|\varphi_\delta(r)\psi_\delta(\vartheta)|^2}{r^2}\,r^{d-1}\,d\vartheta dr\\&=\tau_\delta\int_{\mathbb{R}^d}\tau_0\Phi(\mathbf{x}/|\mathbf{x}|)\frac{|u_\delta(\mathbf{x})|^2}{|\mathbf{x}|^2}\,d\mathbf{x}.\end{aligned}$$

Since $\tau_\delta$ is a continuous function at $\delta$ and $\tau_\delta\to\tau_0$ as $\delta\to0$, the proof is completed. □

## 4.4 Block radial Hardy inequality applications

Let $\mathbf{x}\in\mathbb{R}^d$ and let $r=|\mathbf{x}|$, $r_1=(x_1^2+\cdots+x_k^2)^{\frac12}$ and $r_2=(x_{k+1}^2+\cdots+x_d^2)^{\frac12}$, $1\le k\le d-1$. More specifically, article [106] presents Theorem **1** which states that if $u$ is a block radial function such that $u(\mathbf{x})=u(r_1,r_2)$, then

$$\int_{\mathbb{R}^d}|\nabla u(\mathbf{x})|^2\,d\mathbf{x}\ge C\int_{\mathbb{R}^d}\frac{|u(\mathbf{x})|^2}{r_1^\alpha r_2^{2-\alpha}}\,d\mathbf{x},\quad u\in H^1(\mathbb{R}^d),\tag{4.4.1}$$

with $0<\alpha<2$ with some constant $C=C(k)$.

As we provided the proofs to our main results of this chapter, we shall proceed in presenting a few applications. In particular, we aim to provide examples of inequality (4.4.1) not only for block radial functions $u$ and describe the sharp value of constant $C$. We shall make use of potential function $\Phi$ which satisfies all the conditions from Theorems 4.2.1 and 4.2.2.

### 4.4.1 3D case

For the usual three dimensional case, we can make use of the standard spherical coordinates in $\mathbb{R}^3$. Consider

$$x_1=r\sin\eta,\quad x_2=r\cos\xi\cos\eta,\quad x_3=r\sin\xi\cos\eta$$

with $\xi\in[0,2\pi]$, $\eta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, $r>0$. We want to determine the Jacobian of the mapping $\mathbf{x}\to(r,\vartheta)$ for which $\vartheta=(\xi,\eta)$. After computations, we obtain

$$d\mathbf{x}=r^2\cos\eta\,drd\xi d\eta.$$

If we denote $r_1 = |x_1|$ and $r_2 = \sqrt{x_2^2 + x_3^2}$ we have $|\mathbf{x}|^2 = r^2 = r_1^2 + r_2^2$. Considering the definition of our potential function $\Phi$ to be

$$\Phi(r_1, r_2) = \frac{r_1^2 + r_2^2}{r_1^\alpha r_2^{2-\alpha}}, \qquad 0 < \alpha < 1,$$

we compute the following integral:

$$\int_{\mathbb{R}^3} |\nabla u|^2 \, d\mathbf{x} = \int_0^\infty \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( |u_r'|^2 + \frac{|u_\xi'|^2}{r_2^2} + \frac{|u_\eta'|^2}{r^2} \right) r^2 \cos\eta \, d\eta d\xi dr.$$

The version of the quotient (4.1.2b) for defining the sharp constant $\tau_0$ becomes

$$\inf_{\psi \in \mathcal{H}^1\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus \{0\}} \frac{\int_{-\pi/2}^{\pi/2} \left( |\psi_\eta'|^2 - \frac{\tau_0}{\sin^\alpha \eta \, \cos^{2-\alpha} \eta} |\psi|^2 \right) \cos\eta \, d\eta}{\int_0^{\pi/2} |\psi|^2 \cos\eta \, d\eta} = -\frac{1}{4} \tag{4.4.2}$$

and the following result follows

**Theorem 4.4.1** *Let $0 < \alpha < 1$ and let $\tau_0$ is defined by the minimising problem (4.4.2). Then for*

$$\int_{\mathbb{R}^3} |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geq \tau_0 \int_{\mathbb{R}^3} \frac{|u(\mathbf{x})|^2}{r_1^\alpha r_2^{2-\alpha}} \, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^3).$$

*The constant $\tau_0 = \tau_0(\alpha)$ in the latter inequality is sharp.*

### 4.4.2 4D case

As we saw previously, the 3D case is up to some extent self explanatory and it provides interesting results. In this section we shall use the so called Hopf fibrations acting on the sphere $\mathbb{S}^3$.

Hopf fibrations are frequently used in quantum theory and other various subsections of abstract algebra. Hence, they are used in the description of rotational 3-space, see [83].

In order to demonstrate our idea related to Theorems 4.2.1 and 4.2.2, we shall switch our attention to the case when $d = 4, \alpha = 1$ and $k = 2$. Note that in this case $\frac{(d-2)^2}{4} = 1$ and $r_1 = \sqrt{x_1^2 + x_2^2}$, $r_2 = \sqrt{x_3^2 + x_4^2}$, $r^2 = r_1^2 + r_2^2$. Introduce

$$\Phi(r_1, r_2) = \frac{r^2}{r_1 \, r_2} = \frac{r_1^2 + r_2^2}{r_1 \, r_2} \geq 0.$$

Applying Theorem 4.2.1 we find

$$\int_{\mathbb{R}^4} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geq \tau_0 \int_{\mathbb{R}^4} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2} |u(\mathbf{x})|^2 \, d\mathbf{x} = \tau_0 \int_{\mathbb{R}^4} \frac{1}{r_1 r_2} |u(\mathbf{x})|^2 \, d\mathbf{x},$$

where $\tau_0$ is defined by (4.2.1a). Thus (4.2.1a) becomes

$$\lambda(\tau_0) = \inf_{\psi \in \mathcal{H}^1(\mathbb{S}^3) \setminus \{0\}} \frac{\int_{\mathbb{S}^3} (|\nabla_\vartheta \psi|^2 - \tau_0 \Phi \, |\psi|^2) \, d\vartheta}{\int_{\mathbb{S}^3} |\psi|^2 \, d\vartheta} = -1. \tag{4.4.3}$$

In order to make sure that our constant $\tau_0$ satisfies the inequality presented in Corollary 4.2.4, we check the converge of the integral $\int_{\mathbb{R}^4} \frac{1}{r_1 r_2} |u(\mathbf{x})|^2 \, d\mathbf{x}$. Hence we shall analyse the convergence of the second term in the product

$$\int_{\mathbb{R}^4} \frac{1}{r_1 r_2} |u|^2 d\mathbf{x} = \int_{\mathbb{R}^4} \frac{|u|^2}{r_1^2 + r_2^2} \cdot \frac{r_1^2 + r_2^2}{r_1 r_2} d\mathbf{x}. \tag{4.4.4}$$

Let us introduce the following coordinates system

$$\begin{aligned} x_1 &= r\cos(\theta_1) \\ x_2 &= r\sin(\theta_1)\cos(\theta_2) \\ x_3 &= r\sin(\theta_1)\sin(\theta_2)\cos(\theta_3) \\ x_4 &= r\sin(\theta_1)\sin(\theta_2)\sin(\theta_3), \end{aligned}$$

for

$$r := \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

and $\theta_1, \theta_2 \in [0, \pi]$ and $\theta_3 \in [0, 2\pi]$.

Taking this into consideration

$$\begin{aligned} \int_{\mathbb{R}^4} \frac{r_1^2 + r_2^2}{r_1 r_2} d\mathbf{x} &= \int_{\mathbb{R}^4} \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{\sqrt{(x_1^2 + x_2^2)(x_3^2 + x_4^2)}} d\mathbf{x} \\ &= \int_{\mathbb{R}^4} \frac{r^2}{\sqrt{x_1^2 x_3^2 + x_1^2 x_4^2 + x_2^2 x_3^2 + x_2^2 x_4^2}} d\mathbf{x}. \end{aligned}$$

Now, the denominator term needs special consideration because each term from the sum represents a long expression in $r, \theta_1, \theta_2, \theta_3$

$$\begin{aligned} x_1^2 x_3^2 &= r^4 \cos^2(\theta_1)\sin^2(\theta_1)\sin^2(\theta_2)\cos^2(\theta_3) \\ x_1^2 x_4^2 &= r^4 \cos^2(\theta_1)\sin^2(\theta_1)\sin^2(\theta_2)\sin^2(\theta_3) \\ x_2^2 x_3^2 &= r^4 \sin^4(\theta_1)\cos^2(\theta_2)\sin^2(\theta_2)\cos^2(\theta_3) \\ x_2^2 x_4^2 &= r^4 \sin^4(\theta_1)\cos^2(\theta_2)\sin^2(\theta_2)\sin^2(\theta_3) \end{aligned}$$

and such

$$\begin{aligned} \sum_{i<j, i\neq 3}^{4} x_i x_j &= r^4 \big[\cos^2(\theta_1)\sin^2(\theta_1)\sin^2(\theta_2) + \sin^4(\theta_1)\cos^2(\theta_2)\sin^2(\theta_2)\big] \\ &= r^4 \sin^2(\theta_1)\sin^2(\theta_2)[\cos^2(\theta_1) + \sin^2(\theta_1)\cos^2(\theta_2)]. \end{aligned}$$

Knowing that the topological space of the sphere $\mathbb{S}^3$ is homeomorphic to the set of points in 4-dimensional Euclidean space, we can rewrite our integral as such

$$\int_{\mathbb{S}^3} \frac{r_1^2 + r_2^2}{r_1 r_2} d\mathbf{x} = \int_0^1 \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} \frac{r^2 \sin^2(\theta_1)\sin(\theta_2)}{r^2 |\sin(\theta_1)\sin(\theta_2)| \sqrt{\cos^2(\theta_1) + \sin^2(\theta_1)\cos^2(\theta_2)}} dr d\theta_1 d\theta_2 d\theta_3.$$

We use the fact that $p = \frac{5}{3}$ when $d = 4$ and such our norm in Theorem 4.2.3, becomes

$$\begin{aligned} \|\Phi\|_{L^p(\mathbb{S}^{d-1})} &= \left( \int_{\mathbb{S}^3} \left| \frac{r_1^2 + r_2^2}{r_1 r_2} \right|^{5/3} d\mathbf{x} \right)^{3/5} \\ &= \left( \int_0^1 \int_0^{2\pi} \int_0^{\pi} \int_0^{\pi} \frac{\sin^2(\theta_1)\sin(\theta_2)}{\left| |\sin(\theta_1)\sin(\theta_2)| \sqrt{\cos^2(\theta_1) + \sin^2(\theta_1)\cos^2(\theta_2)} \right|^{5/3}} dr d\theta_1 d\theta_2 d\theta_3 \right)^{3/5} \end{aligned}$$

After numerically analysing the expression with Matlab simulations, we discover that

$$\left(\int_{\mathbb{R}^4}\left|\frac{r_1^2+r_2^2}{r_1r_2}\right|^{5/3}d\mathbf{x}\right)^{3/5}\simeq 18.8413,$$

hence

$$\begin{aligned}\|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})} &= \frac{1}{\left(\int_{\mathbb{R}^4}\left|\frac{r_1^2+r_2^2}{r_1r_2}\right|^{5/3}d\mathbf{x}\right)^{3/5}}\\ &\simeq \frac{1}{18.8413}\\ &\simeq 0.053<\infty\end{aligned}$$

which shows that our integral converges. This helps in the proof of existence of the optimal constant $\tau_0$, since

$$\tau_0\geq\frac{(d-2)^2}{4}|\mathbb{S}^{d-1}|^{1/p}\|\Phi\|^{-1}_{L^p(\mathbb{S}^{d-1})}.$$

In our case

$$\begin{aligned}\tau_0 &\geq 1\cdot|\mathbb{S}^3|^{3/5}0.053\\ &\geq\left(\frac{1}{2}\pi^2r^4\right)^{3/5}\cdot 0.053\\ &\geq 0.053\cdot\left(\frac{1}{2}\pi^2\right)^{3/5}.\end{aligned}$$

Since we saw that the integral is finite, let us proceed to the main argument of this example.

On $\mathbb{S}^3$ we introduce the classical Hopf coordinates that are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1\to\mathbb{S}^3\to\mathbb{S}^2.$$

If $\mathbf{x}\in\mathbb{R}^4$, then

$$\begin{aligned}x_1&=r\cos\xi_1\sin\eta, & x_2&=r\sin\xi_1\sin\eta,\\ x_3&=r\cos\xi_2\cos\eta, & x_4&=r\sin\xi_2\cos\eta.\end{aligned}$$

Here $\eta$ runs over the range 0 to $\frac{\pi}{2}$ and $\xi_1$ and $\xi_2$ can take any values between 0 and $2\pi$. For any fixed value of $\eta\in\left[0,\frac{\pi}{2}\right]$ the coordinates $(\xi_1,\xi_2)$ parametrise a 2-dimensional torus. In the degenerate cases, when $\eta$ equals 0 or $\frac{\pi}{2}$ these coordinates describe a circle.

The Jacobian of the mapping $\mathbf{x}\to(r,\vartheta)$, $\vartheta=(\xi_1,\xi_2,\eta)$ is defined by

$$d\mathbf{x}=r^3\sin\eta\cos\eta\,drd\xi_1d\xi_2d\eta$$

and $r_1=r\sin\eta$, $r_2=r\cos\eta$. Then the function $\Phi$ in these coordinates equals

$$\Phi(r_1,r_2)=\frac{1}{\sin\eta\cos\eta}\geq 0,\quad \eta\in\left(0,\frac{\pi}{2}\right).$$

Rather simple computations imply that for a function $\psi$ on $\mathbb{S}^3$ in Hopf coordinates we have

$$\int_{\mathbb{S}^3} |\nabla_\vartheta \psi|^2 \, d\mathbf{x} = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^{2\pi} \left( \frac{|\psi'_{\xi_1}|^2}{\sin^2 \eta} + \frac{|\psi'_{\xi_2}|^2}{\cos^2 \eta} + |\psi'_\eta|^2 \right) \sin\eta \, \cos\eta \, d\xi_1 d\xi_2 d\eta. \tag{4.4.5}$$

What is important to observe is that the class of functions in (4.4.5) satisfy periodic boundary conditions with respect to the variables $\xi_1$ and $\xi_2$ on the interval $(0, 2\pi)$.

Hence consider $\psi_{nm}(\eta)$ to be the Fourier coefficients of $\psi$ with respect to $\xi_1$ and $\xi_2$

$$\psi_{nm}(\eta) = \int_0^{2\pi} \int_0^{2\pi} e^{-i(n\,\xi_1 + m\,\xi_2)} \psi(\xi_1, \xi_2, \eta) \, d\xi_1 d\xi_2 \, d\eta.$$

Using Parseval's identity, we can rewrite the expression (4.4.5) in the following form

$$\int_{\mathbb{S}^3} |\nabla_\vartheta \psi|^2 \, d\mathbf{x} = (2\pi)^2 \sum_{n,m}^{\infty} \int_0^{\frac{\pi}{2}} \left( \frac{n^2 |\psi_{nm}(\eta)|^2}{\sin^2(\eta)} + \frac{m^2 |\psi_{nm}(\eta)|^2}{\cos^2(\eta)} + |\psi_{nm}(\eta)'|^2 \right) d\eta.$$

We observe that the infimum of the expression is achieved at $n = m = 0$. Therefore the terms with $\psi'_{\xi_1}$ and $\psi'_{\xi_2}$ in (4.4.5) can be omitted and thus the infimum in (4.4.3) is reduced to a class of functions $\psi$ depending only on $\eta$

$$\inf_{\psi \in \mathcal{H}^1((0,\pi/2), d\rho) \setminus \{0\}} \frac{\int_0^{\pi/2} \left( |\psi'_\eta|^2 - \frac{\tau_0}{\sin\eta \, \cos\eta} |\psi|^2 \right) \sin\eta \, \cos\eta \, d\eta}{\int_0^{\pi/2} |\psi|^2 \sin\eta \cos\eta \, d\eta} = -1, \tag{4.4.6}$$

where

$$d\rho(\eta) = \sin\eta \cos\eta \, d\eta$$

and

$$\mathcal{H}^1(\left(0, \frac{\pi}{2}\right), d\rho) = \{\psi \in L^2((0,\pi/2), d\rho) : \, \psi' \in L^2((0,\pi/2), d\rho)\}.$$

Note that for $\psi$ defined on $\mathbb{S}^3$ and depending only on $\eta$ we have

$$\|\psi\|_{L^2(\mathbb{S}^3)} = 4\pi^2 \int_0^{\pi/2} |\psi|^2 \sin\eta \, \cos\eta \, d\eta.$$

Due to identity (4.2.2), $\tau_0$ in (4.4.6) is uniquely achieved. The quadratic form

$$\int_0^{\frac{\pi}{2}} (|\psi'(\eta)|^2 \sin\eta \cos\eta - \tau |\psi(\eta)|^2) \, d\eta, \quad \tau > 0, \quad \psi \in H^1((0,\pi/2), d\rho)$$

defines a singular self-adjoint Sturm-Liouville operator

$$A_\tau \psi(\eta) = -\psi''(\eta) \sin 2\eta - 2\psi'(\eta) \cos 2\eta - 2\tau\psi(\eta),$$

acting in $\mathcal{H}^2((0,\pi/2), d\rho)$.

Therefore finding $\psi$ and $\tau_0$ in (4.4.6) is reduced to finding the minimal value of $\tau_0 > 0$ satisfying the equation

$$A_{\tau_0} \psi(\eta) = -\psi(\eta) \sin 2\eta. \tag{4.4.7}$$

**Theorem 4.4.2** *Let $\tau = \tau_0$ be the minimal value for which the equation (4.4.7) has a non-trivial solution $\psi$. Then*

$$\int_{\mathbb{R}^4} |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geq \tau_0 \int_{\mathbb{R}^4} \frac{|u(\mathbf{x})|^2}{r_1 r_2}\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^4). \tag{4.4.8}$$

*The constant $\tau_0$ in (4.4.8) is sharp.*

**Proof**: By Theorem 4.2.1 we have that

$$\begin{aligned}\int_{\mathbb{R}^4} |\nabla u(\mathbf{x})|^2\, d\mathbf{x} &\geq \tau_0 \int_{\mathbb{R}^4} \frac{\Phi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\, |u(\mathbf{x})|^2\, d\mathbf{x} \\ &\geq \tau_0 \int_{\mathbb{R}^4} \frac{|u(\mathbf{x})|^2}{r_1 r_2}\, d\mathbf{x}.\end{aligned}$$

The latter identity is established by using form (4.4.4).

Moreover, the sharpness of constant $\tau_0$ is proved by applying Theorem 4.2.2 for the following definition of $\Phi$

$$\Phi(r_1, r_2) = \frac{r^2}{r_1\, r_2} = \frac{r_1^2 + r_2^2}{r_1\, r_2}.$$

□

**Remarks**: It is important to observe that inequality (4.4.8) differs from that standard Hardy inequality in $\mathbb{R}^4$ since

$$\frac{1}{r_1 r_2} \geq \frac{2}{r_1^2 + r_2^2}$$

and it has singularities not just at zero but also at $r_1 = 0$ and $r_2 = 0$.

In the following theorems, a few results shall be presented which use a similar reduction used previously due to the periodicity of the test functions on the intervals $(0, 2\pi)$.

We present the next statement which relates to the inequality (4.4.1) in dimension four with $\alpha$ not necessarily equal to one. Let

$$\Phi_\alpha(r_1, r_2) = \frac{r^2}{r_1^\alpha r_2^{2-\alpha}} = \frac{1}{\sin^\alpha \eta \cos^{2-\alpha} \eta}, \qquad \eta \in \left(0, \frac{\pi}{2}\right),$$

where $0 < \alpha < 2$. In this case the quotient (4.4.6) becomes

$$\inf_{\psi \in \mathcal{H}^1(0,\pi/2)\setminus\{0\}} \frac{\int_0^{\pi/2} \left(|\psi_\eta'|^2 - \frac{\tau_0}{\sin^\alpha \eta \cos^{2-\alpha} \eta} |\psi|^2\right) \sin\eta \cos\eta\, d\eta}{\int_0^{\pi/2} |\psi|^2 \sin\eta \cos\eta\, d\eta} = -1. \tag{4.4.9}$$

Therefore as in (4.4.7) the problem is reduced to finding $\tau_0$ appearing in the equation

$$-\psi''(\eta)\sin 2\eta - 2\psi'(\eta)\cos 2\eta - \frac{2\tau_0}{\sin^{\alpha-1}\eta \cos^{1-\alpha}\eta}\psi(\eta) = -\psi(\eta)\sin 2\eta \tag{4.4.10}$$

and we have the following statement:

**Theorem 4.4.3** *Let $\tau_0$ be the minimal value for which the equation (4.4.10) has a non-trivial solution. Then if $0 < \alpha < 2$*

$$\int_{\mathbb{R}^4} |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geq \tau_0 \int_{\mathbb{R}^4} \frac{|u(\mathbf{x})|^2}{r_1^\alpha r_2^{2-\alpha}}\, d\mathbf{x}, \quad u \in H^1(\mathbb{R}^4).$$

*The constant $\tau_0$ in the latter inequality is sharp.*

### 4.4.3 Hopf coordinates in $\mathbb{R}^8$

Since we treated the case of $\mathbb{R}^4$ and $\mathbb{R}^3$ using Hopf bundles embedded on unit spheres, we move forward to present a new result regarding Hardy inequalities using dimensions higher than four.

It is already known that on $\mathbb{S}^3$, one can introduce the classical Hopf coordinates with the usual radius $r$ and the angle variables $\xi_1, \xi_2, \eta$, where $\xi_1, \xi_2 \in [0, 2\pi]$ and $\eta \in [0, \frac{\pi}{2}]$. The 3-sphere can be described using the Hopf bundle

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$$

and such we relate our result to the classical Hopf bundle

$$\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4.$$

If $\mathbf{x} \in \mathbb{R}^8$ then we introduce the coordinates

$$\begin{aligned}
x_1 &= r\cos\xi_1\cos\eta_2\sin\eta_1, & x_2 &= r\sin\xi_1\cos\eta_2\sin\eta_1,\\
x_3 &= r\cos\xi_2\sin\eta_2\sin\eta_1, & x_4 &= r\sin\xi_2\sin\eta_2\sin\eta_1,\\
x_5 &= r\cos\xi_3\cos\eta_3\cos\eta_1, & x_6 &= r\sin\xi_3\cos\eta_3\cos\eta_1,\\
x_7 &= r\cos\xi_4\sin\eta_3\cos\eta_1, & x_8 &= r\sin\xi_4\sin\eta_3\cos\eta_1,
\end{aligned}$$

where $\xi_j \in [0, 2\pi]$, $j = 1, \dots, 4$ and $\eta_j \in \left[0, \frac{\pi}{2}\right]$, $j = 1, 2, 3$. Hence,

$$\begin{aligned}
\sum_{j=1}^{8} x_j^2 &= r^2(\cos^2\xi_1\cos^2\eta_2\sin^2\eta_1 + \sin^2\xi_1\cos^2\eta_2\sin^2\eta_1 + \cos^2\xi_2\sin^2\eta_2\sin^2\eta_1 +\\
&\quad + \sin^2\xi_2\sin^2\eta_2\sin^2\eta_1 + \cos^2\xi_3\cos^2\eta_3\cos^2\eta_1 + \sin^2\xi_3\cos^2\eta_3\cos^2\eta_1 +\\
&\quad + \cos^2\xi_4\sin^2\eta_3\cos^2\eta_1 + \sin^2\xi_4\sin^2\eta_3\cos^2\eta_1)\\
&= r^2(\cos^2\eta_2\sin^2\eta_1 + \sin^2\eta_2\sin^2\eta_1 + \cos^2\eta_3\cos^2\eta_1 + \sin^2\eta_3\cos^2\eta_1)\\
&= r^2(\sin^2\eta_1 + \cos^2\eta_1)\\
&= r^2.
\end{aligned}$$

It can also be seen that

$$\begin{aligned}
\xi_1 &= \arctan\left(\frac{x_2}{x_1}\right)\\
\xi_2 &= \arctan\left(\frac{x_4}{x_3}\right)\\
\xi_3 &= \arctan\left(\frac{x_6}{x_5}\right)\\
\xi_4 &= \arctan\left(\frac{x_8}{x_7}\right)
\end{aligned}$$

and

$$\eta_2 = \arctan \sqrt{\frac{x_4^2 + x_3^2}{x_1^2 + x_2^2}}$$
$$\eta_3 = \arctan \sqrt{\frac{x_8^2 + x_7^2}{x_5^2 + x_6^2}}$$
$$\eta_1 = \arctan \sqrt{\frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{x_8^2 + x_7^2 + x_6^2 + x_5^2}}.$$

Using these definitions, we can compute the Jacobian of the mapping $\mathbf{x} \to (r, \xi, \eta)$ which has the following form

$$d\mathbf{x} = \frac{1}{8} r^7 \ \sin^3(2\eta_1) \sin 2\eta_2 \sin 2\eta_3 \ dr \, d\xi \, d\eta,$$

where $\xi = (\xi_1, \dots \xi_4)$ and $\eta = (\eta_1, \eta_2, \eta_3)$.

If if $Q_1 = (0, 2\pi)^4$ and $Q_2 = \left(0, \frac{\pi}{2}\right)^3$ then

$$\int_{\mathbb{R}^8} |\nabla u(\mathbf{x})|^2 d\mathbf{x} = \frac{1}{8} \int_0^\infty \int_{Q_1} \int_{Q_2} [|u_r'|^2 + \frac{1}{r^2 \cos^2 \eta_1} \left( \frac{|u_{\xi_1}'|^2}{\cos^2 \eta_2} + \frac{|u_{\xi_2}'|^2}{\sin^2 \eta_2} + |u_{\eta_2}'|^2 \right) +$$
$$+ \frac{1}{r^2 \sin^2 \eta_1} \left( \frac{|u_{\xi_3}'|^2}{\cos^2 \eta_3} + \frac{|u_{\xi_4}'|^2}{\sin^2 \eta_3} + |u_{\eta_3}'|^2 \right) + |u_{\eta_1}|^2] r^7 \sin^3 2\eta_1 \sin 2\eta_2 \sin 2\eta_3 \ d\eta \, d\xi dr.$$

Consider

$$r_1^2 = x_1^2 + x_2^2, \quad r_2^2 = x_3^2 + x_4^2, \quad r_3^2 = x_5^2 + x_6^2, \quad r_4^2 = x_7^2 + x_8^2.$$

Then

$$\Phi(r_1, r_2, r_3, r_4) = \frac{r^2}{\left(\prod_{j=1}^4 r_j\right)^{1/2}} = \frac{4}{\sin(2\eta_1) \sqrt{\sin(2\eta_2) \sin(2\eta_3)}}.$$

Define

$$d\mu = \sin^3 2\eta_1 \sin 2\eta_2 \sin 2\eta_3 \ d\eta.$$

The following theorem follows:

**Theorem 4.4.4** *We have*

$$\int_{\mathbb{R}^8} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geq \tau_0 \int_{\mathbb{R}^8} \frac{|u(\mathbf{x})|^2}{\left(\prod_{j=1}^4 r_j\right)^{1/2}} \, d\mathbf{x}, \tag{4.4.11}$$

*where $\tau_0$ is defined by the equation*

$$\inf_{\psi \in \mathcal{H}^1(Q_2) \setminus \{0\}} \frac{\int_{Q_2} \left( \frac{|\psi_{\eta_2}'|^2}{\cos^2 \eta_1} + \frac{|\psi_{\eta_3}'|^2}{\sin^2 \eta_1} + |\psi_{\eta_1}'|^2 - \tau_0 \frac{4 \, |\psi|^2}{\sin(2\eta_1) \sqrt{\sin(2\eta_2) \sin(2\eta_3)}} \right) d\mu}{\int_{Q_2} |\psi|^2 \, d\mu} = -9.$$

**Proof**: The reduction given in this theorem can be obtained similarly to the reduction of (4.4.3) to (4.4.6). □

The interest in inequalities of the type (4.4.11) is related to Hardy inequalities for $N$ 2-dimensional particles. More specifically, it would be interesting to check whether in $\mathbb{R}^{2N}$ the Hardy-type inequality

$$\int_{\mathbb{R}^{2N}} |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geq C_N \int_{\mathbb{R}^{2N}} \frac{|u(\mathbf{x})|^2}{\left(\prod_{j=1}^{2N} r_j\right)^{2/N}} \, d\mathbf{x},$$

holds for some $C_N > 0$ and how $C_N$ depends on the number of particles $N$.

We considered the case when there is an even number of particles(i.e. $N = 4$ ), but for example for an odd number of particles, the construction of $r_j$ terms becomes more difficult. Moreover, we already saw that in order to define a similar inequality to the case in (4.4.11), we need the norm in $L^p$ to be finite. Hence in the case when dimension is $2N$, the only acceptable manner in which we can define the $r_j$ terms is

$$r_1 := \sqrt{x_1^2 + x_2^2}$$
$$r_l := \sqrt{x_{l+1}^2 + x_{l+2}^2},$$

for $N - 2 \geq l \geq 2$.

This facilitates the process of checking the dependence of constant $p$ defined in article [58] on the number of particles $N$.

# Chapter 5

# Discrete inequalities for magnetic potential terms

## 5.1 Introduction

This chapter shall present new approaches towards the two dimensional discrete Hardy's inequality. More specific, we shall provide key results presented in article [52]. Hence, we introduce a discrete version of Aharonov - Bohm vector field on $\mathbb{Z}^2$ and, subsequently, obtain a discrete Hardy inequality on $\mathbb{Z}^2$ using an innovative system of coordinates.

Recall the discrete and continuous Hardy inequalities

$$\sum_{n=1}\left(\frac{1}{n}\sum_{k=1}^{n}u_k\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty}u_n^p, \quad p>1,\ u_k\geq 0. \tag{5.1.1}$$

and

$$\int_0^\infty \left(\frac{1}{x}\int_0^x u(t)\,dt\right)^p < \left(\frac{p}{p-1}\right)^p \int_0^\infty v(x)^p\,dx, \quad p>1,\ v(x)\geq 0.$$

The classical Hardy inequality in $\mathbb{R}^d$ with $p=2$ states

$$\left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d}\frac{u(x)^2}{|\mathbf{x}|^2}\,d\mathbf{x} < \int_{\mathbb{R}^d}|\nabla u(\mathbf{x})|^2\,d\mathbf{x}, \qquad d\geq 3. \tag{5.1.2}$$

The constant $\left(\frac{d-2}{2}\right)^2$ is sharp but not achieved and could be approximated by functions depending on $|\mathbf{x}|$. Note that the case $p=2$, $d\geq 3$ plays an important role in theory of Schrödinger operator expressing the uncertainty principle in quantum mechanics that follows from the inequality

$$\left(\frac{d-2}{2}\right)^2 \frac{1}{|\mathbf{x}|^2} < -\Delta,$$

namely, for a function $u$ and the Fourier transform $\widehat{u}$ such that

$$\|u\|_{L^2(\mathbb{R}^d)} = \|\widehat{u}\|_{L^2(\mathbb{R}^d)} = 1$$

we have

$$\begin{aligned}\frac{1}{(2\pi)^{2d}}\left(\frac{d-2}{2}\right)^2 &= \frac{1}{(2\pi)^{2d}}\left(\frac{d-2}{2}\right)^2 \int_{\mathbb{R}^d} |\xi|^2\,|\widehat{u}|^2\,d\xi \int_{\mathbb{R}^d} \frac{|\widehat{u}|^2}{|\xi|^2}\,d\xi \\ &< \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} |\xi|^2\,|\widehat{u}|^2\,d\xi \int_{\mathbb{R}^d} |\nabla \widehat{u}|^2\,d\xi \\ &= \int_{\mathbb{R}^d} |\xi|^2\,|\widehat{u}|^2\,d\xi \int_{\mathbb{R}^d} |x|^2 |u|^2\,d\xi.\end{aligned}$$

This implies that both $u$ and $\widehat{u}$ cannot be concentrated near the origin.

A version of Hardy's inequality in dimension two was obtained by M.Solomyak in [108], [109]

$$\int_{\mathbb{R}^2} |\nabla u|^2\,d\mathbf{x} \geq \int_{\mathbb{R}^2} \frac{|u(\mathbf{x}) - \overline{u}(|\mathbf{x}|)|^2}{|\mathbf{x}|^2}\,d\mathbf{x} + \frac{1}{4}\int_{\mathbb{R}^2} \frac{u(|\mathbf{x}|)^2}{|\mathbf{x}|^2 \log^2 |\mathbf{x}|}\,d\mathbf{x}, \tag{5.1.3}$$

where $u \in C_0^\infty\left(\mathbb{R}^2 \setminus (\{0\} \cup \{|\mathbf{x}| = 1\})\right)$ and where $\overline{u}$ is the average of $u$ over the unit circle

$$\overline{u}(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\theta)\,d\theta, \qquad \mathbf{x} = (r,\theta).$$

The classical Hardy inequality in dimension two fails on radial functions and in [74] this problem was solved by considering a magnetic gradient.

The main result of this chapter involves the following theorem

**Theorem 5.1.1** *For a magnetic discrete quadratic form in $l^2(\mathbb{Z}^2)$ we have*

$$\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{j=1,2} |u(\mathbf{x}+e_j) - u(\mathbf{x}) + A_\alpha(\mathbf{x})\,u(\mathbf{x})|^2 \geq \frac{1}{8\pi}\min_{l\in\mathbb{Z}_+} |l-\alpha|^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{0\}} \frac{|u(\mathbf{x})|^2}{\|\mathbf{x}\|^2}, \tag{5.1.4}$$

$$A_\alpha(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x})) = \left(1 - e^{2\pi i\alpha\,\varphi_1}, 1 - e^{2\pi i\alpha\,\varphi_2}\right),$$

where function $\varphi_i$ is defined on a discrete square of radius $n$.

Since this chapter presents ideas such as magnetic Laplacians and Aharonov-Bohm magnetic fields, we shall provide a few theoretical notions for the reader to understand our results in a better manner.

### 5.1.1 Magnetic Laplacian

In order to have a better grasp towards the Aharonov - Bohm effect, it is compulsory to understand that once magnetic fields are introduced as a notion, changes must be to the momentum and kinetic energy of a charged particle.

The expressions are

$$\begin{aligned}\mathbf{p_A} &= \frac{1}{i}\nabla_{\mathbf{A}} \\ &:= \frac{1}{i}(\partial_1 + iA1, \ldots, \partial_3 + iA_3) \\ -\Delta_{\mathbf{A}} &:= -\sum_{j=1}^{3}(\partial_j + iA_j)^2,\end{aligned}$$

where $\mathbf{A} = (A_1, A_2, A_3) : \mathbb{R}^3 \to \mathbb{R}^3$ is the vector potential, determining the magnetic field $\mathbf{B}$, by the means of the following equation

$$\mathbf{B}(\mathbf{x}) = \text{curl}\mathbf{A}(\mathbf{x}). \tag{5.1.5}$$

As an extension, we shall consider that in $n$ - dimensions our vector potential is

$$\mathbf{A} = (A_1, A_2, \ldots, A_n) : \mathbb{R}^n \to \mathbb{R}^n$$

and

$$\nabla_{\mathbf{A}} = (\partial_1 + iA1, \ldots, \partial_n + iA_n)$$

for the derivative with respect to $\mathbf{A}$.

As a consequence, this introduces a new Sobolev space in dimension $n$, which depends on the vector potential $\mathbf{A}$. Hence, for $\Omega \subseteq \mathbb{R}^n$, $A_j \in L^p_{loc}(\Omega)$ and $1 < p < \infty$, define the space $H^{1,p}_{0,\mathbf{A}}(\Omega)$ as the completion of $C^\infty_0(\Omega)$ with respect to the norms determined by

$$||f||^p_{H^{1,p}_{0,\mathbf{A}}(\Omega)} := \sum_{j=1}^{n} \int_{\mathbb{R}^n} |(\partial_j + iA_j)f|^p d\mathbf{x} + \int_{\mathbb{R}^n} |f|^p d\mathbf{x}.$$

### 5.1.2 Magnetic fields and Schrödinger operators

Classical mechanics provides a unique manner to describe the motion of charged particles using Maxwell's equations:

$$\begin{cases} \nabla \cdot \mathbf{E} = 4\pi\rho \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{B} = 4\pi\mathbf{J} + \frac{\partial \mathbf{E}}{\partial t}. \end{cases}$$

Moreover, these relations determine the connections between the classical electric field $\mathbf{E}$, magnetic field $\mathbf{B}$, the electric current density $\mathbf{J}$ and the charge density $\rho$. This allows us to observe that the second Maxwell equation allows the magnetic field to be written as

$$\mathbf{B} = \nabla \times \mathbf{A}$$

for some magnetic vector potential $\mathbf{A}$. In the same manner, the third Maxwell equation becomes

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

for a scalar potential function $\phi$.

It is known that with the development of quantum mechanics, the pair of potentials $(\phi, \mathbf{A})$ started to play important roles. This is mainly to the fact that the Schrödinger equation contains the scalar potential $\phi$ and the magnetic potential $\mathbf{A}$. Now, if the magnetic field $\mathbf{B}$, defined above, is considered, then the free magnetic Hamiltonian $H_{\mathbf{A}} := -\Delta_{\mathbf{A}}$ becomes the non-negative, self-adjoint operator associated to the closure $h_{\mathbf{A}}$ of the form

$$(H_{\mathbf{A}}\psi,\psi) = \int_{\mathbb{R}^n} |\nabla_{\mathbf{A}}\psi|^2 d\mathbf{x}, \quad \psi \in C^\infty_0(\mathbb{R}^n).$$

This is valid only when $A_j \in L^2_{loc}(\mathbb{R}^n)$, $j = 1, \dots, n$. Moreover, the domain of the form of $h_{\mathbf{A}}$ is

$$\mathcal{D}(h_{\mathbf{A}}) := \left\{ f : f \in H^1_{0,\mathbf{A}}(\mathbb{R}^n), -\Delta_{\mathbf{A}} f \in L^2(\mathbb{R}^n) \right\}$$

which is a dense subspace of $H^1_{0,\mathbf{A}}(\mathbb{R}^n)$. As a consequence, the following Hardy inequality holds

$$h_{\mathbf{A}}[u] \geq \int_{\mathbb{R}^n} |\nabla f(\mathbf{x})|^2 d\mathbf{x} \geq C(n, \mathbb{R}^n) \int_{\mathbb{R}^n} \frac{|f(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x}, \tag{5.1.6}$$

with constant $C(n, \mathbb{R}^n) = \frac{(n-2)^2}{4}$ being optimal for $n > 2$. Also, for $n = 2$, there is no Hardy inequality and this prompts to deduce from (5.1.6) that $h_{\mathbf{A}} \geq 0$. This situation is bound to be improved, considering that a magnetic field is present in the inequality, as we shall see in the following sections. On more introductory notions regarding the aspects discussed so far, we recommend [9], [12], [14], [23], [24], [49], [56], [87] and [113].

### 5.1.3 Aharonov - Bohm magnetic field

The history of quantum mechanics provided substantial results throughout time, but in 1959, Yakir Aharonov and doctoral advisor David Bohm suggested a new experiment to determine the role and significance of potential fields in quantum mechanics. Their research predicted that a wave function could acquire an extra observable phase once it travels through non-simple connected domains. These domains do not have electromagnetic fields, but do present non-zero potentials $(\phi, \mathbf{A})$. This phase shift was named the **Aharonov - Bohm** effect.

Recalling the fact that there is no Hardy inequality in $\mathbb{R}^2$ and that (5.1.6) provides already known information, researchers started to question whether magnetic potentials could provide a result of the form

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} u|^2 d\mathbf{x} \geq c \int_{\mathbb{R}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x}, \quad u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\}),$$

when the corresponding magnetic field $\mathbf{B}$ is bounded on $\mathbb{R}^2 \setminus \{0\}$. This mathematical curiosity was answered by Laptev and Weidl in [74]. More specific, the following theorem holds

**Theorem 5.1.2 (Magnetic potential)** *Let*

$$\mathbf{A} = \frac{\psi(\theta)}{r} \mathbf{e}_\theta,$$

*where* $\psi \in L^1(0, 2\pi)$ *and*

$$\Psi(\mathbf{A}) = \Psi = \frac{1}{2\pi} \int_0^{2\pi} \psi(\theta) d\theta.$$

*Then, for all* $\varphi \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$,

$$\int_{\mathbb{R}^2} |(\nabla + i\mathbf{A}(\mathbf{x}))u(\mathbf{x})|^2 d\mathbf{x} \geq (\min_{k \in \mathbb{Z}} |k + \Psi|)^2 \int_{\mathbb{R}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2} d\mathbf{x} \tag{5.1.7}$$

*with* $(\min_{k \in \mathbb{Z}} |k + \Psi|)^2$ *being sharp.*

**Remark**:

If the flux $\Psi$ is considered to be an integer, then operator $H_{\mathbf{A}}$ defined by Theorem 5.1.2 is equivalent to $-\Delta$ and a Hardy inequality does not exist.

Next section shall present our main result regarding the discrete Hardy inequality for $\mathbb{Z}^2$ and the usage of magnetic potentials in our proof.

## 5.2 Main Result

Consider $\mathcal{A}$ a vector potential such that

$$\mathcal{A} = \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

As it was seen before Theorem 5.1.2 states that for $u \in H^1(\mathbb{R}^2 \setminus \{0\})$

$$\int_{\mathbb{R}^2} |i\nabla u + \alpha\, \mathcal{A}\, u|^2 \, d\mathbf{x} \geq \min_{k\in\mathbb{Z}} |k-\alpha|^2 \int_{\mathbb{R}^2} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2}\, d\mathbf{x}, \tag{5.2.1}$$

where $\alpha$ is the magnetic flux for the magnetic vector field $\alpha\, \mathcal{A}$.

The continuous case proves to be already a breakthrough in the research of Hardy inequalities for dimension two, but the extension from the continuous cases (5.1.7) and (5.2.1) to discrete quadratic forms seems not to be easy. For example, in [72], authors obtained for $d \geq 3$ the following discrete inequality

$$\sum_{\mathbf{x}\in\mathbb{Z}^d} \frac{|u(\mathbf{x})|^2}{|\mathbf{x}|^2} \leq C \sum_{\mathbf{x}\in\mathbb{Z}^d} \sum_{i=1}^{d} |u(\mathbf{x}) - u(\mathbf{x} - \mathbb{1}_i)|^2,$$

where

$$C \leq \frac{\pi^2}{4} \left( \frac{2}{d-2} + \frac{\sqrt{2}(d-4)^2}{(d-2)(\sqrt{2}d - 4\sqrt{d-2})\pi} \right).$$

and $u$ defined on $\mathbb{Z}^d$. Hardy inequalities for discrete operators are harder to analyse and considering the constant involved, it is not clear if sharpness is achieved.

In order to present our results, it is convenient to introduce the norm

$$\|\mathbf{x}\| = \max\{|x_1|, |x_2|\}, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2.$$

We denote by $\partial\mathbb{S}_n$ the boundary of the square $\mathbb{S}_n$ defined by a fixed $\|\mathbf{x}\| = n$. Hence

$$\partial\mathbb{S}_n = \{\mathbf{x} \in \mathbb{Z}^2 : |x_1| = n, -n \leq x_2 \leq n, \quad \text{or} \quad |x_2| = n, -n \leq x_1 \leq n\} \tag{5.2.2}$$

Let us define

$$\varphi(\mathbf{x}) = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x})), \qquad \mathbf{x} \in \mathbb{S}_n, \tag{5.2.3}$$

where

$$\begin{cases} \varphi_1(\mathbf{x}) & = -\frac{x_2}{8\|\mathbf{x}\|^2} = -(8n)^{-1}, \quad x_2 = n, \ -n \leq x_1 \leq n, \\ \varphi_2(\mathbf{x}) & = -\frac{x_1}{8\|\mathbf{x}\|^2} = -(8n)^{-1}, \quad x_1 = -n, \ -n < x_2 \leq n, \end{cases}$$

and

$$\begin{cases} \varphi_1(\mathbf{x}) & = \frac{x_2}{8\|\mathbf{x}\|^2} = (8n)^{-1}, \quad x_2 = -n, \ -n \leq x_1 \leq n, \\ \varphi_2(\mathbf{x}) & = \frac{x_1}{8\|\mathbf{x}\|^2} = (8n)^{-1}, \quad x_1 = n, \ -n \leq x_2 < n. \end{cases}.$$

Then, our choice of discrete Aharonov-Bohm magnetic vector field equals

$$A_\alpha(\mathbf{x}) = (A_1(\mathbf{x}), A_2(\mathbf{x})) = \left(1 - e^{2\pi i \alpha\, \varphi_1}, 1 - e^{2\pi i \alpha\, \varphi_2}\right), \tag{5.2.4}$$

where $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. If we consider the Taylor expansion of $A_1(\mathbf{x})$ with $x_2 = n$ and $-n \leq x_1 \leq n$ then

$$1 - A_1(\mathbf{x}) = 2\pi i\alpha \frac{x_2}{8\|\mathbf{x}\|^2} + O(n^{-2}).$$

Let us consider $e_j \in \mathbb{Z}^2$ to be the unit vector in the direction $j$, namely $e_1 = (1,0)$ and $e_2 = (0,1)$. As a consequence, the discrete partial derivative of the function $u$ is $-u(\mathbf{x}) + u(\mathbf{x} + e_j)$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$.

The main result of the chapter is:

**Theorem 5.2.1** *For a magnetic discrete quadratic form in $l^2(\mathbb{Z}^2)$ we have*

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} \sum_{j=1,2} |u(\mathbf{x} + e_j) - u(\mathbf{x}) + A_\alpha(\mathbf{x})\, u(\mathbf{x})|^2 \geq \frac{1}{8} \left(\operatorname{dist}(\alpha, \mathbb{Z})\right)^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{0\}} \frac{|u(\mathbf{x})|^2}{\|\mathbf{x}\|^2}.$$

It is not clear whether the constant involved is sharp, but we can make use of further numerical analysis to predict the behaviour.

In [39], the authors obtained a different version of the inequality (5.2.1) using polar coordinate. However, they did not use the standard quadratic form on $\mathbb{Z}^2$, but considered a discretised cylinder $\mathbb{R} \times [0, 2\pi]$ described by $\{\frac{j}{m} : j \in \mathbb{Z}\} \times \{\frac{2\pi j}{n} : j \in \{1, 2, \ldots, n\}\}$.

## 5.3 Proof of main result

Before we start presenting the proof of our result, we must provide the notions of a rather new system of coordinates used in the argument of our proof.

### 5.3.1 Polar coordinates in $\mathbb{Z}^2$ space

Let us introduce the following non-linear operator $U$ which is moving one step counter-clockwise along the square $\mathbb{S}_n$ defined in (5.2.2). If $\mathbf{x} \in \mathbb{S}_n$ then

$$U(\mathbf{x}) = \begin{cases} (x_1, x_2 + 1) = \mathbf{x} + e_2 & \text{if } x_1 > 0 \text{ and } -|x_1| \leq x_2 < |x_1|, \\ \\ (x_1 - 1, x_2) = \mathbf{x} - e_1 & \text{if } x_2 > 0 \text{ and } -|x_2| < x_1 \leq |x_2|, \end{cases}$$

and

$$U(\mathbf{x}) = \begin{cases} (x_1, x_2 - 1) = \mathbf{x} - e_2 & \text{if } x_1 < 0 \text{ and } -|x_1| < x_2 \leq |x_1|, \\ \\ (x_1 + 1, x_2) = \mathbf{x} + e_1 & \text{if } x_2 < 0 \text{ and } -|x_2| \leq x_1 < |x_2|. \end{cases}$$

For the case when $n = 0$, we set $U(0) = 0$. Note that $U$ is invertible on $\mathbb{S}_n$ for $n > 0$.

Also, it is important to highlight that if the radius of our "square" is $n$, then the identity permutation would be $U^{4\cdot 2\cdot n}$, since the corners of our shape correspond to $U^{0\cdot 2\cdot n}, U^{1\cdot 2\cdot n}, U^{2\cdot 2\cdot n}, U^{3\cdot 2\cdot n}$ and $U^{4\cdot 2\cdot n}$.

We shall call $(n, m)$ the polar coordinates of $\mathbb{Z}^2 \ni \mathbf{x} = U^m((n, n))$, where $(n, n)$ is the upper right corner of $\mathbb{S}_n$ and

$$\mathbf{x} = U^m \begin{bmatrix} n \\ n \end{bmatrix} = U^m(ne_1 + ne_2) = (n, m).$$

As before, the corners of the square $\mathbb{S}_n$ become

$$U^0(ne_1 + ne_2),\ U^{2n}(ne_1 + ne_2),\ U^{4n}(ne_1 + ne_2) \text{ and } U^{6n}(ne_1 + ne_2).$$

For an arbitrary point $\mathbf{x} = U^m((n, n)) \in \mathbb{S}_n$ we have

$$m = 2sn + t, \quad s = 0, \ldots, 3; \quad 0 < t < 2n$$

and as an abbreviation $U^m(n) = U^m((n, n))$. Here, the $s$ value indicates which side of the square a point lies onto while the $t$ value highlights which point we take on that particular side.

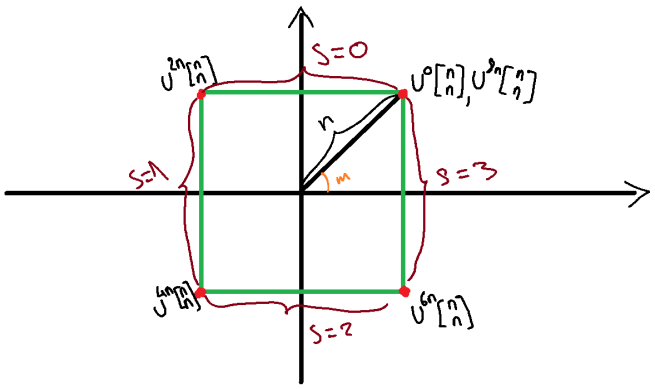



Figure 5.1: $n$ radius square with angular component $m$

We desire to obtain a discrete analogue of

$$\int_{\mathbb{R}^2} |\nabla u|^2 d\mathbf{x} = \int_{\mathbb{R}^2} |\partial_r u|^2 + \frac{1}{r^2} |\partial_\theta u|^2 \, d\mathbf{x} = \int_0^\infty \int_0^{2\pi} r \, |\partial_r u|^2 + \frac{1}{r} |\partial_\theta u|^2 \, dr \, d\theta, \tag{5.3.1}$$

hence the angular and radial derivatives of a function $u$ are needed.

In the discrete space, we can introduce the polar angle function

$$\theta(U^m(n)) = 2\pi \frac{m}{8n},$$

hence $\theta(U^{m+1}(n)) - \theta(U^m(n)) = \frac{2\pi}{8n}$ and $\theta(U^{8n}(n)) - \theta(U^0(n)) = 2\pi$. In particular, $\partial_\theta \theta = 1$. The angular derivative of a function $u$ at $\mathbf{x} = U^m(n)$ is

$$\partial_\theta \, u(U^m(n)) = \frac{8n}{2\pi} \left( u\left(U^{m+1}(n)\right) - u\left(U^m(n)\right) \right). \tag{5.3.2}$$

Here $\frac{8n}{2\pi}$ represents the deviation of the arcs between $u(U^{m+1}(n))$ and $u(U^m(n))$.

Next, we shall provide a simple extension to the Parseval theorem in order to describe the left hand-side term of (5.1.4) easily.

Consider the scalar product in $l^2(\mathbb{S}_n)$

$$\langle u, v\rangle = \sum_{m=0}^{8n-1} u\left(U^m(n)\right)\overline{v\left(U^m(n)\right)}$$

and let $g_{n,l}(m) = \frac{1}{\sqrt{8n}}\, e^{2\pi i \frac{ml}{8n}}$. Then

$$\langle g_{n,l}, g_{n,k}\rangle = \delta_{l,k}, \qquad k, l = 0, 1, \ldots, n-1.$$

As a consequence, the discrete Fourier transform with respect to the angular variable $m$ becomes

$$\widehat{u}(n,l) = \langle u, g_{n,l}\rangle = \sum_{m=0}^{8n-1} u\left(U^m(n)\right)\,\overline{g_{n,l}(m)}. \tag{5.3.3}$$

The inverse Fourier transform becomes

$$u\left(U^m(n)\right) = \sum_{l=0}^{8n-1} \widehat{u}(n,l)\, g_{n,l}(m)$$

which follows the Parseval formula

$$\langle u, v\rangle = \langle \widehat{u}, \widehat{v}\rangle\,.$$

This can be easily seen, since

$$\begin{aligned}
\partial_\theta g_{n,k}(m) &= \frac{8n}{2\pi}\left(\frac{1}{\sqrt{8n}}\, e^{2\pi i \frac{(m+1)k}{8n}} - \frac{1}{\sqrt{8n}}\, e^{2\pi i \frac{mk}{8n}}\right)\\
&= \frac{8n}{2\pi}\,\frac{1}{\sqrt{8n}}\, e^{2\pi i \frac{mk}{8n}}\left(e^{2\pi i \frac{k}{8n}} - 1\right)\\
&= \frac{8n}{2\pi}\left(e^{2\pi i \frac{k}{8n}} - 1\right) g_{n,k}(m).
\end{aligned}$$

This highlights the fact that the eigenfunctions and eigenvalues of operator $\partial_\theta$ on $\mathbb{S}_n$ are $g_{n,k}$ and $\frac{8n}{2\pi}\left(e^{2\pi i \frac{k}{8n}} - 1\right)$ respectively. Note that

$$\lim_{n\to\infty} \frac{8n}{2\pi}\left(e^{2\pi i \frac{k}{8n}} - 1\right) = i\,k.$$

Also the set of functions $\{g_{n,k}\}_{k=0}^{n-1}$ is an orthonormal basis in the $n$-dimensional space $\mathbb{S}_n$ with respect to the scalar product $\langle\cdot,\cdot\rangle$. For any $u \in l^2(\mathbb{Z}^2)$ we have

$$u\left(U^m(n)\right) = \sum_{l=0}^{8n-1} \widehat{u}(n,l) g_{n,l}(m),$$

where $\widehat{u}(n,l)$ are Fourier coefficients defined in (5.3.3). When introducing polar coordinates we consider all points from $\mathbb{Z}^2$. Therefore

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x})|^2 = \sum_{n=0}^{\infty}\sum_{m=0}^{8n-1} \left|u\left(U^m(n)\right)\right|^2 = \sum_{n=0}^{\infty}\sum_{l=0}^{8n-1} |\widehat{u}(n,l)|^2.$$

Now that we explained how the new coordinates system aids us, we can provide one more useful statement:

**Proposition 5.3.1** *Let $\alpha$ be any non-integer real number. Let $N \geq 1$ be an integer. Then*

$$\inf_{k\in\mathbb{Z}} \left|\sin\left(\pi\,\frac{k-\alpha}{N}\right)\right| \geq \frac{2\,\mathrm{dist}(\alpha,\mathbb{Z})}{N}\,. \tag{5.3.4}$$

**Proof**: :

As a first step, write $\alpha = \lfloor\alpha\rfloor + \{\alpha\}$, where $\{\alpha\}$ is the fractional part of $\alpha$, $0 < \{\alpha\} < 1$. It is sufficient to prove that (5.3.4) holds for $\{\alpha\}$ instead of $\alpha$. Thus, we assume from the beginning that $0 < \alpha < 1$. Also, instead of (5.3.4) we shall prove

$$\min_{k=0,1,\ldots,N-1} \sin\left(\pi\,\frac{k+\alpha}{N}\right) \geq \frac{2\,\mathrm{dist}(\alpha,\mathbb{Z})}{N}\,. \tag{5.3.5}$$

If $0 < \alpha \leq \frac{1}{2}$, then $k = 0$ gives the minimum value $\sin\left(\pi\,\frac{\alpha}{N}\right)$. If $\frac{1}{2} \leq \alpha < 1$, then $k = N-1$ provides the minimum value $\sin(\pi\,\frac{1-\alpha}{N})$. Thus,

$$\min_{k=0,1,\ldots,N-1} \sin\left(\pi\,\frac{k+\alpha}{N}\right) = \sin\left(\pi\,\frac{\mathrm{dist}(\alpha,\mathbb{Z})}{N}\right).$$

Moreover, if $0 \leq x \leq \frac{1}{2}$ we have

$$\sin(\pi x) \geq 2x.$$

Since in our case $\frac{\mathrm{dist}(\alpha,\mathbb{Z})}{N} \leq \frac{1}{2}$ we have

$$\sin\left(\pi\,\frac{\mathrm{dist}(\alpha,\mathbb{Z})}{N}\right) \geq \frac{2\,\mathrm{dist}(\alpha,\mathbb{Z})}{N}.$$

□

### 5.3.2 Proof of Theorem 5.2.1

Consider the following quadratic form

$$\sum_{\mathbf{x}\in\mathbb{Z}^2}\sum_{j=1,2} \left|u(\mathbf{x}+e_j) - u(\mathbf{x}) + A_\alpha(\mathbf{x})\,u(\mathbf{x})\right|^2. \tag{5.3.6}$$

In order to recreate (5.3.1) in discrete form, we need to evaluate the partial derivatives of the operator $U$ by considering the following sum

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} \left[|u(\mathbf{x}+e_1) - u(\mathbf{x})|^2 + |u(\mathbf{x}+e_2) - u(\mathbf{x})|^2\right].$$

Without loss of generality, consider also that $u(0,0) = u(-1,0) = u(0,-1) = u(1,0) = u(0,1) = 0$. Let us compute the partial derivatives

$$U^{2sn+t}\begin{bmatrix} n \\ n \end{bmatrix} + e_1 = \begin{cases} U^{2\cdot 0\cdot n+t-1}(n) & \text{if } s=0,\ t=1,\ldots,2n \\ U^{2(n-1)+t}(n-1) & \text{if } s=1,\ t=1,\ldots,2n-1 \\ U^{4n+t}(n) & \text{if } s=2,\ t=0,\ldots,2n-1 \\ U^{6(n+1)+t-1}(n+1) & \text{if } s=3,\ t=0,\ldots,2n \end{cases}. \tag{5.3.7}$$

Also,

$$U^{2sn+t}\begin{bmatrix} n \\ n \end{bmatrix} + e_2 = \begin{cases} U^{t+1}(n+1) & \text{if } s=0,\ t=0,\dots,2n \\ U^{2n+t-1}(n) & \text{if } s=1,\ t=1,\dots,2n \\ U^{4(n-1)+t-1}(n-1) & \text{if } s=2,\ t=1,\dots,2n-1 \\ U^{6n+t+1}(n) & \text{if } s=3,\ t=0,\dots,2n-1 \end{cases}. \tag{5.3.8}$$

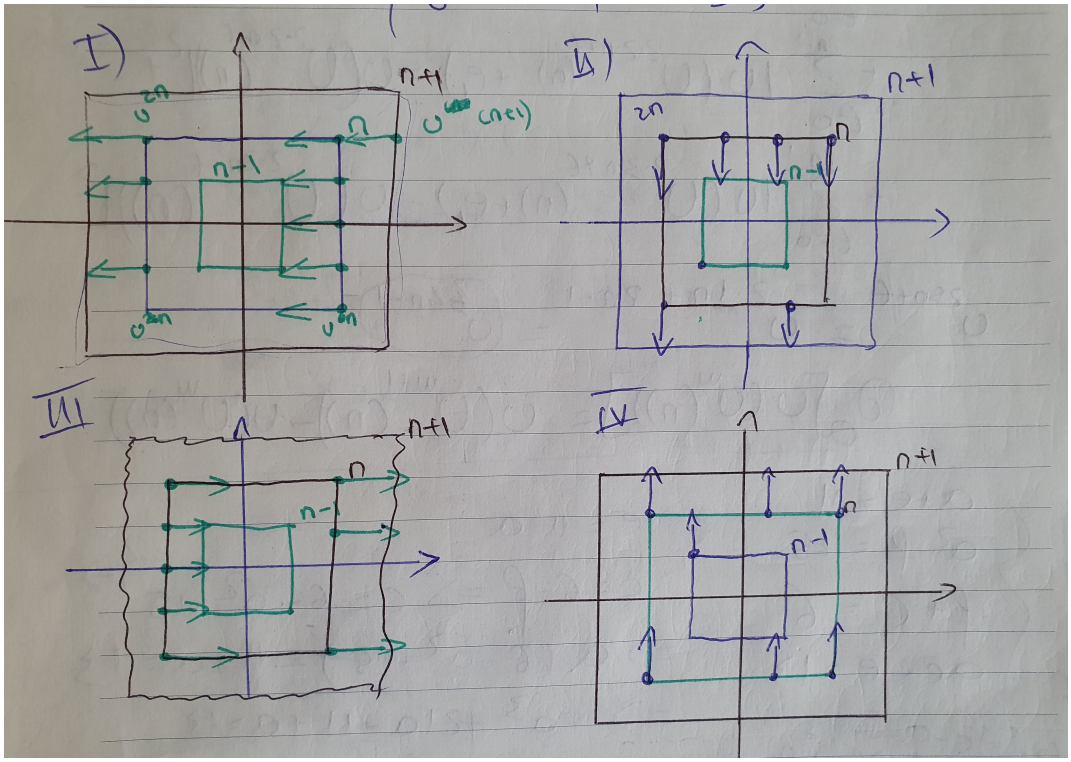



Figure 5.2: Magnetic flow depending on radial and angular component

Making use only of the angular derivatives, we find

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_1) - u(\mathbf{x}) + A_1(\mathbf{x})\,u(\mathbf{x})|^2 + \sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_2) - u(\mathbf{x}) + A_2(\mathbf{x})\,u(\mathbf{x})|^2 \geq \\ \geq \sum_{n=1}^{\infty}\sum_{m=1}^{2n-1} \left| u\left(U^{m-1}(n)\right) - e^{2\pi i\alpha\varphi_1(U^m(n))} u\left(U^m(n)\right)\right|^2 \\ + \sum_{n=1}^{\infty}\sum_{m=2n}^{4n-1} \left| u\left(U^{m-1}(n)\right) - e^{2\pi i\alpha\varphi_2(U^m(n))} u\left(U^m(n)\right)\right|^2 \\ + \sum_{n=1}^{\infty}\sum_{m=4n}^{6n-1} \left| u\left(U^{m+1}(n)\right) - e^{2\pi i\alpha\varphi_1(U^m(n))} u\left(U^m(n)\right)\right|^2 \\ + \sum_{n=1}^{\infty}\sum_{m=6n}^{8n-1} \left| u\left(U^{m+1}(n)\right) - e^{2\pi i\alpha\varphi_2(U^m(n))} u\left(U^m(n)\right)\right|^2. \end{aligned} \tag{5.3.9}$$

It can be seen that in the last inequalities we do not include the term corresponding to $m=0$. This particular case corresponds to the corner point with coordinates $(n,n)$. By using the periodicity of $U$ the term with $m=0$ would

be equal to

$$\left|u\left(U^{-1}(n)\right)-e^{2\pi i\alpha\varphi_1\left(U^0(n)\right)}u\left(U^0(n)\right)\right|^2=\left|u\left(U^{8n-1}(n)\right)-e^{2\pi i\alpha\varphi_1\left(U^0(n)\right)}u\left(U^0(n)\right)\right|^2.$$

Howbeit, the quadratic form (5.3.6) does not have such a term since in the case $\mathbf{x}=(n,n)$, the points $\mathbf{x}+e_1$ and $\mathbf{x}+e_2$ belong to $\mathbb{S}_{n+1}$. We shall treat this case separately in the next section. Note that due to the periodicity of $U$ and the property $-\varphi_1\left(U^0(n)\right)=\varphi_2\left(U^{8n-1}(n)\right)$ we obtain

$$\left|u\left(U^{8n-1}(n)\right)-e^{2\pi i\alpha\varphi_1\left(U^0(n)\right)}u\left(U^0(n)\right)\right|=\left|u\left(U^{n}(n)\right)-e^{2\pi i\alpha\varphi_2\left(U^{8n-1}(n)\right)}u\left(U^{8n-1}(n)\right)\right|.$$

The term already exists in (5.3.9). As a consequence, if we include the term with $m=0$ and loosing a factor of 2 in our Hardy constant, we write

$$\begin{aligned}\sum_{\mathbf{x}\in\mathbb{Z}^2}|u(\mathbf{x}+e_1)-u(\mathbf{x})+A_1(\mathbf{x})\,u(\mathbf{x})|^2+\sum_{\mathbf{x}\in\mathbb{Z}^2}|u(\mathbf{x}+e_2)-u(\mathbf{x})+A_2(\mathbf{x})\,u(\mathbf{x})|^2\geq\\ \geq\frac{1}{2}\Big(\sum_{n=1}^{\infty}\sum_{m=0}^{2n-1}\left|u\left(U^{m-1}(n)\right)-e^{2\pi i\alpha\varphi_1(U^m(n))}u\left(U^m(n)\right)\right|^2\\ +\sum_{n=1}^{\infty}\sum_{m=2n}^{4n-1}\left|u\left(U^{m-1}(n)\right)-e^{2\pi i\alpha\varphi_2(U^m(n))}u\left(U^m(n)\right)\right|^2\\ +\sum_{n=1}^{\infty}\sum_{m=4n}^{6n-1}\left|u\left(U^{m+1}(n)\right)-e^{2\pi i\alpha\varphi_1(U^m(n))}u\left(U^m(n)\right)\right|^2\\ +\sum_{n=1}^{\infty}\sum_{m=6n}^{8n-1}\left|u\left(U^{m+1}(n)\right)-e^{2\pi i\alpha\varphi_2(U^m(n))}u\left(U^m(n)\right)\right|^2\Big).\end{aligned}\tag{5.3.10}$$

By recalling the properties of $\varphi$ defined in (5.2.3), we obtain that if $0\leq m<4n$, then

$$\begin{aligned}u\left(U^{m-1}(n)\right)-e^{-2\pi i\frac{\alpha}{8n}}\,u\left(U^m(n)\right)&=\sum_{l=0}^{8n-1}\left(\widehat{u}(n,l)\,g_{n,l}(m-1)-e^{-2\pi i\frac{\alpha}{8n}}\widehat{u}(n,l)g_{n,l}(m)\right)\\ &=\sum_{l=0}^{8n-1}\widehat{u}(n,l)\,g_{n,l}(m)\,\left(e^{-2\pi i\frac{l}{8n}}-e^{-2\pi i\frac{\alpha}{8n}}\right)\\ &=\sum_{l=0}^{8n-1}\widehat{u}(n,l)\,g_{n,l}(m)\,e^{-\pi i\frac{l+\alpha}{8n}}\,2i\,\sin\left(\pi\frac{\alpha-l}{8n}\right).\end{aligned}$$

Similarly the special case is treated as such

$$\begin{aligned}u\left(U^{1}(n)\right)-e^{2\pi i\frac{\alpha}{8n}}\,u\left(U^0(n)\right)&=\sum_{l=0}^{8n-1}\left(\widehat{u}(n,l)\,g_{n,l}(1)-e^{-2\pi i\frac{\alpha}{8n}}\widehat{u}(n,l)g_{n,l}(0)\right)\\ &=\sum_{l=0}^{8n-1}\widehat{u}(n,l)\,g_{n,l}(0)\,\left(e^{2\pi i\frac{l}{8n}}-e^{2\pi i\frac{\alpha}{8n}}\right)\\ &=\sum_{l=0}^{8n-1}\widehat{u}(n,l)\,g_{n,l}(0)\,e^{\pi i\frac{l+\alpha}{8n}}\,2i\,\sin\left(\pi\frac{l-\alpha}{8n}\right).\end{aligned}$$

For the case when $4 \leq m < 8n-1$ we find

$$\begin{aligned} u\left(U^{m+1}(n)\right) - e^{2\pi i \frac{\alpha}{8n}}\, u\left(U^{m}(n)\right) &= \sum_{l=0}^{8n-1} \left(\widehat{u}(n,l)\, g_{n,l}(m+1) - e^{2\pi i \frac{\alpha}{8n}} \widehat{u}(n,l) g_{n,l}(m)\right) \\ &= \sum_{l=0}^{8n-1} \widehat{u}(n,l)\, g_{n,l}(m) \left(e^{2\pi i \frac{l}{8n}} - e^{2\pi i \frac{\alpha}{8n}}\right) \\ &= \sum_{l=0}^{8n-1} \widehat{u}(n,l)\, g_{n,l}(m)\, e^{\pi i \frac{l+\alpha}{8n}}\, 2i \sin\left(\pi \frac{l-\alpha}{8n}\right). \end{aligned}$$

If we use Parseval's identity, we obtain

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_1) - u(\mathbf{x}) + A_1(\mathbf{x})\, u(\mathbf{x})|^2 + \sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_2) - u(\mathbf{x}) + A_2(\mathbf{x})\, u(\mathbf{x})|^2 \geq \\ \geq 4\, \frac{1}{2} \sum_{n=1}^{\infty} \sum_{l=0}^{8n-1} |\widehat{u}(n,l)|^2\, |\sin\left(\pi \frac{l-\alpha}{8n}\right)|^2. \end{aligned} \tag{5.3.11}$$

By Proposition 5.3.1 when $N = 8n$ we have

$$\left|\sin \frac{\pi(l-\alpha)}{8n}\right| \geq \frac{\min(\alpha, \mathbb{Z})}{4n}. \tag{5.3.12}$$

In conclusion, by using (5.3.11) and (5.3.12) and the fact that $n \geq 1$ we obtain

$$\begin{aligned} \sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_1) - u(\mathbf{x}) + A_1(\mathbf{x})\, u(\mathbf{x})|^2 + \sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_2) - u(\mathbf{x}) + A_2(\mathbf{x})\, u(\mathbf{x})|^2 \\ \geq \frac{1}{8} \left(\min(\alpha, \mathbb{Z})\right)^2 \sum_{n=1}^{\infty} \sum_{l=0}^{8n-1} \frac{1}{n^2} |\widehat{u}(n,l)|^2 \\ = \frac{1}{8} \left(\min(\alpha, \mathbb{Z})\right)^2 \sum_{\mathbf{x}\in\mathbb{Z}^2\setminus\{0\}} \frac{|u(k)|^2}{\|k\|^2}, \end{aligned}$$

which finishes the proof. □

**Remark**:

Due to the structure of the discrete derivative at $\mathbf{x} = (n,n)$ we are forced to put the extra constant $\frac{1}{2}$ in front of the terms which give the derivatives at all points on $\mathbb{S}_n \setminus (n,n)$. We consider that there should be a possible manner in which one could avoid this and improve the constant in (5.1.4) at least by 2.

## 5.4 The $\mathbb{S}_{n-1}$ and $\mathbb{S}_{n+1}$ cases

Recall that

$$U^{2sn+t}\begin{bmatrix} n \\ n \end{bmatrix} + e_1 = \begin{cases} U^{2\cdot 0\cdot n+t-1}(n) & \text{if } s=0,\ t=1,\dots,2n \\ U^{2(n-1)+t}(n-1) & \text{if } s=1,\ t=1,\dots,2n-1 \\ U^{4n+t}(n) & \text{if } s=2,\ t=0,\dots,2n-1 \\ U^{6(n+1)+t-1}(n+1) & \text{if } s=3,\ t=0,\dots,2n \end{cases}. \tag{5.4.1}$$

Also,

$$U^{2sn+t}\begin{bmatrix} n \\ n \end{bmatrix} + e_2 = \begin{cases} U^{t+1}(n+1) & \text{if } s=0,\ t=0,\dots,2n \\ U^{2n+t-1}(n) & \text{if } s=1,\ t=1,\dots,2n \\ U^{4(n-1)+t-1}(n-1) & \text{if } s=2,\ t=1,\dots,2n-1 \\ U^{6n+t+1}(n) & \text{if } s=3,\ t=0,\dots,2n-1 \end{cases}. \tag{5.4.2}$$

In order to explain the special case when $m=0$, we need to make reference to Figure 5.2 where it can be seen that the magnetic flow has a certain pattern of movement. Our case considers just the square with "radius" $n$, but in order to obtain all terms for our sums, we observed that once the particle starts in the corner $(0,0)$, it will exit the square towards a smaller one of "radius" $n-1$ or a larger one of "radius" $n+1$.

In this situation, let us consider the discretisation of the term $|\nabla u|^2$:

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_1)-u(\mathbf{x})|^2 = \sum_{n=1}^{\infty}\sum_{s=0}^{3}\sum_{t=0}^{2n-1} |u(U^{2sn+t}(n)+e_1)-u(U^{2sn+t}(n))|^2 \tag{5.4.3}$$

and

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_2)-u(\mathbf{x})|^2 = \sum_{n=1}^{\infty}\sum_{s=0}^{3}\sum_{t=0}^{2n-1} |u(U^{2sn+t}(n)+e_2)-u(U^{2sn+t}(n))|^2 \tag{5.4.4}$$

It implies that the inner terms of (5.4.3) and (5.4.4) become

$$\begin{aligned}\sum_{s=0}^{3}\sum_{t=0}^{2n-1} |u(U^{2sn+t}(n)+e_1)-u(U^{2sn+t}(n))|^2 = &\sum_{t=1,\dots,2n} |u(U^{2\cdot 0\cdot n+t}(n)+e_1)-u(U^{2\cdot 0\cdot n+t}(n))|^2 + \\ &+ \sum_{t=0,\dots,2n-1} |u(U^{2\cdot 2\cdot n+t}(n)+e_1)-u(U^{2\cdot 2\cdot n+t}(n))|^2 + \\ &+ \sum_{t=1,\dots,2n-1} |u(U^{2\cdot 1\cdot n+t}(n)+e_1)-u(U^{2\cdot 1\cdot n+t}(n))|^2 \\ &+ \sum_{t=0,\dots,2n} |u(U^{2\cdot 3\cdot n+t}(n)+e_1)-u(U^{2\cdot 3\cdot n+t}(n))|^2,\end{aligned}$$

respectively

$$\begin{aligned}\sum_{s=0}^{3}\sum_{t=0}^{2n-1} |u(U^{2sn+t}(n)+e_2)-u(U^{2sn+t}(n))|^2 = &\sum_{t=0,\dots,2n} |u(U^{2\cdot 0\cdot n+t}(n)+e_2)-u(U^{2\cdot 0\cdot n+t}(n))|^2 + \\ &+ \sum_{t=1,\dots,2n-1} |u(U^{2\cdot 2\cdot n+t}(n)+e_2)-u(U^{2\cdot 2\cdot n+t}(n))|^2 + \\ &+ \sum_{t=1,\dots,2n} |u(U^{2\cdot 1\cdot n+t}(n)+e_2)-u(U^{2\cdot 1\cdot n+t}(n))|^2 \\ &+ \sum_{t=0,\dots,2n-1} |u(U^{2\cdot 3\cdot n+t}(n)+e_2)-u(U^{2\cdot 3\cdot n+t}(n))|^2.\end{aligned}$$

We consider the case when our square has fixed "radius" $n$ and relating this to (5.4.1) and (5.4.2) we shall separate our sums only with regards to the angular derivative. Hence,

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_1)-u(\mathbf{x})+A_1u(\mathbf{x})|^2 = \underbrace{\sum_{n=1}^{\infty}\sum_{s=0}^{3}\sum_{t=0}^{2n-1} |u(U^{2sn+t}(n)+e_1)-u(U^{2sn+t}(n))+A_1(U^{2sn+t}(n))u(U^{2sn+t}(n))|^2}_{S_{e_1}},$$

where the inner sum $S_{e_1}$ becomes

$$\begin{aligned}
S_{e_1} = &\sum_{t=1}^{2n} |u(U^{2\cdot 0\cdot n+t}(n)+e_1) - u(U^{2\cdot 0\cdot n+t}(n)) + A_1(U^{2\cdot 0\cdot n+t}(n))u(U^{2\cdot 0\cdot n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{2\cdot 2\cdot n+t}(n)+e_1) - u(U^{2\cdot 2\cdot n+t}(n)) + A_1(U^{2\cdot 2\cdot n+t}(n))u(U^{2\cdot 2\cdot n+t}(n))|^2 + \\
&+ \sum_{t=1}^{2n-1} |u(U^{2\cdot 1\cdot n+t}(n)+e_1) - u(U^{2\cdot 1\cdot n+t}(n)) + A_1(U^{2\cdot 1\cdot n+t}(n))u(U^{2\cdot 1\cdot n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n} |u(U^{2\cdot 3\cdot n+t}(n)+e_1) - u(U^{2\cdot 3\cdot n+t}(n)) + A_1(U^{2\cdot 3\cdot n+t}(n))u(U^{2\cdot 3\cdot n+t}(n))|^2.
\end{aligned}$$

Recalling figure (5.2), it can be seen that the first two sums involve the top and bottom similar sides of the "square" and the last two sums involve the left and right similar sides of the "square", hence

$$\begin{aligned}
S_{e_1} = &\sum_{t=1}^{2n} |u(U^{t-1}(n)) - u(U^t(n)) + A_1(U^t(n))u(U^t(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{4n+t+1}(n)) - u(U^{4n+t}(n)) + A_1(U^{4n+t}(n))u(U^{4n+t}(n))|^2.
\end{aligned}$$

Similarly, for the $e_2$ term we have

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_2) - u(\mathbf{x}) + A_2u(\mathbf{x})|^2 = \sum_{n=1}^{\infty} \underbrace{\sum_{s=0}^{3}\sum_{t=0}^{2n-1} |u(U^{2sn+t}(n)+e_2) - u(U^{2sn+t}(n)) + A_2(U^{2sn+t}(n))u(U^{2sn+t}(n))|^2}_{S_{e_2}},$$

with the inner sum $S_{e_2}$ described as

$$\begin{aligned}
S_{e_2} = &\sum_{t=1}^{2n} |u(U^{2\cdot 1\cdot n+t}(n)+e_2) - u(U^{2\cdot 1\cdot n+t}(n)) + A_2(U^{2\cdot 1\cdot n+t}(n))u(U^{2\cdot 1\cdot n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{2\cdot 3\cdot n+t}(n)+e_2) - u(U^{2\cdot 3\cdot n+t}(n)) + A_2(U^{2\cdot 3\cdot n+t}(n))u(U^{2\cdot 3\cdot n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n} |u(U^{2\cdot 0\cdot n+t}(n)+e_2) - u(U^{2\cdot 0\cdot n+t}(n)) + A_2(U^{2\cdot 0\cdot n+t}(n))u(U^{2\cdot 0\cdot n+t}(n))|^2 + \\
&+ \sum_{t=1}^{2n-1} |u(U^{2\cdot 2\cdot n+t}(n)+e_2) - u(U^{2\cdot 2\cdot n+t}(n)) + A_2(U^{2\cdot 2\cdot n+t}(n))u(U^{2\cdot 2\cdot n+t}(n))|^2.
\end{aligned}$$

Likewise, we observe that the first two sums involve the left and right similar sides of the "square" and the last two sums involve the bottom and top similar sides of the "square", hence

$$\begin{aligned}
S_{e_2} = &\sum_{t=1}^{2n} |u(U^{2n+t-1}(n)) - u(U^{2n+t}(n)) + A_2(U^{2n+t}(n))u(U^{2n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{6n+t+1}(n)) - u(U^{6n+t}(n)) + A_2(U^{6n+t}(n))u(U^{6n+t}(n))|^2.
\end{aligned}$$

As a consequence, the sum for the angular part is

$$\begin{aligned}
S_{ang}\begin{bmatrix} n \\ n \end{bmatrix} &= S_{e_1} + S_{e_2} \\
&= \sum_{t=1}^{2n} |u(U^{t-1}(n)) - u(U^t(n)) + A_1(U^t(n))u(U^t(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{4n+t+1}(n)) - u(U^{4n+t}(n)) + A_1(U^{4n+t}(n))u(U^{4n+t}(n))|^2 + \\
&+ \sum_{t=1}^{2n} |u(U^{2n+t-1}(n)) - u(U^{2n+t}(n)) + A_2(U^{2n+t}(n))u(U^{2n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{6n+t+1}(n)) - u(U^{6n+t}(n)) + A_2(U^{6n+t}(n))u(U^{6n+t}(n))|^2 = \\
&= \sum_{t=0}^{2n-1} |u(U^{t+1}(n)) - u(U^t(n)) + A_1(U^{t+1}(n))u(U^t(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{4n+t+1}(n)) - u(U^{4n+t}(n)) + A_1(U^{4n+t}(n))u(U^{4n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{2n+t+1}(n)) - u(U^{2n+t}(n)) + A_2(U^{2n+t+1}(n))u(U^{2n+t}(n))|^2 + \\
&+ \sum_{t=0}^{2n-1} |u(U^{6n+t+1}(n)) - u(U^{6n+t}(n)) + A_2(U^{6n+t}(n))u(U^{6n+t}(n))|^2.
\end{aligned}$$

Taking everything into consideration, the angular part of our sum with regards to the choice of our magnetic field becomes

$$\sum_{\mathbf{x}\in\mathbb{Z}^2} |u(\mathbf{x}+e_i) - u(\mathbf{x}) + A_i u(\mathbf{x})|^2 = \sum_{n=1}^{\infty} S_{ang}\begin{bmatrix} n \\ n \end{bmatrix} = \sum_{n=1}^{\infty}\sum_{m=0}^{8n-1} |u(U^{m+1}(n)) - u(U^m(n)) + A(U^m(n))u(U^m(n))|^2. \tag{5.4.5}$$

The last equation could potentially be used in the argument of the proof considering that it avoids the problem points where the particle exists towards the squares of different radius.

This new form produces the extra factor of 2 needed, but future research shall provide a definite answer whether it is conclusive enough to consider this form instead of (5.3.10).

For the past three chapters we presented our main results related to Hardy inequality. In the following chapter, we shall present our future developments which we consider at the moment. They can be seen as interesting ideas or open problems and the outcome of our future research shall tell the impact they have on the vast topics of Functional Analysis and Quantum Mechanics.

# Chapter 6

# Future developments

In past years substantial attention has been given to Hardy Inequalities and considerable improvement has been done towards this topic, but there are, still, inequalities that are not easily improved. Nevertheless, these inequalities play an important role in the theory of partial differential equations and in the study of the stability of eigenvalues in elliptic problems, along with spectral theory related to the Schrödinger operators.

As the reader shall see in this chapter, we include future research ideas which proved to be interesting and of potential to shine new lights on what Hardy inequalities have to offer to Mathematics and Mathematical Physics.

## 6.1 Hardy inequalities for the volume of simplices

Our curiosity pushes us to study whether we can develop a Hardy inequality which overcomes the issue of degeneracy. More specifically, we see it possible to discover a new inequality with a geometrical flavour which can be integrated over the whole domain without it failing for cases such as when the points lie on the same line, for example.

In this sense, let us suppose we have $G : \mathbb{R}^m \to \mathbb{R}^m$ a real valued vectorfield, locally defined on $L^2$ such that $\mathrm{div}G \in L^2_{loc}$. Moreover, for $\beta \in \mathbb{R}$ consider

$$\int_{\mathbb{R}^m} |\nabla u + \beta G u|^2 d\mathbf{x} \geq 0.$$

If we assume that $u \in C_0^\infty(\mathbb{R}^m)$, we can apply partial integration and obtain

$$||\nabla u||^2 \geq \int_{\mathbb{R}^m} (\beta \mathrm{div}G - \beta^2 |G|^2)|u|^2 d\mathbf{x}.$$

By optimising with respect to $\beta$, we shall have

$$\int_{\mathbb{R}^m} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \frac{\int_{\mathbb{R}^m} |u|^2 \mathrm{div}G d\mathbf{x}}{\int_{\mathbb{R}^m} |Gu|^2 d\mathbf{x}}. \tag{6.1.1}$$

In addition, denote $\varphi : \mathbb{R}^m \to \mathbb{R}$ to be a harmonic function. In this situation, we want to apply (6.1.1) for a specific $G$ that has the following property

$$\mathrm{div}G = |G|^2 \in L^2_{loc}.$$

Hence, define $G$ as

$$G = -\frac{\nabla\varphi}{\varphi}.$$

It is possible to apply (6.1.1) now and such it follows that

$$\int_{\mathbb{R}^m} |\nabla u|^2 d\mathbf{x} \geq \frac{1}{4} \frac{\int_{\mathbb{R}^m} |u|^2 |\nabla\varphi|^2 d\mathbf{x}}{\int_{\mathbb{R}^m} |u\varphi|^2 d\mathbf{x}}. \tag{6.1.2}$$

Let us consider $n+1$ points in $\mathbb{R}^n, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}$ with $\mathbf{x}_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n})$. Furthermore denote the closed convex set $\mathcal{T}_n$ (i.e. the simplex in $\mathbb{R}^n$) with

$$\mathcal{T}_n = conv(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}).$$

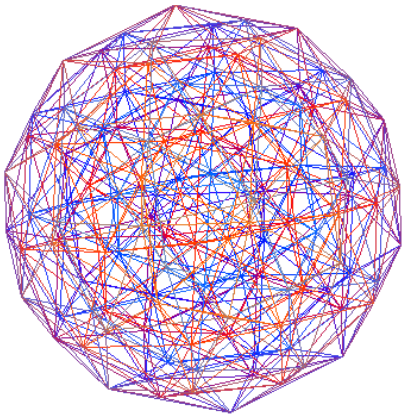


Figure 6.1: Standard $n$ simplex.

Hence, for the function $T_n : \mathbb{R}^{n(n+1)} \to \mathbb{R}$ defined by

$$T_n = \tfrac{1}{n!} det \begin{pmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} & 1 \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ x_{n+1,1} & x_{n+1,2} & \ldots & x_{n+1,n} & 1 \end{pmatrix},$$

the volume of simplex $\mathcal{T}_n$ is given by $|T_n|$. It is easy to observe that

$$\Delta T_n = 0$$

considering that $T_n$ is a harmonic homogeneous of degree $n$ polynomial. Here we introduce $m = n(n+1)$ and such (6.1.2) gives us

$$-\Delta \geq \frac{1}{4} \frac{|\nabla T_n|^2}{|T_n|^2}$$

on the domain

$$Q_n = \left\{ u \in W_0(\mathbb{R}^{n(n+1)}) : \frac{|\nabla T_n|}{|T_n|} u \in L^2_{loc}(\mathbb{R}^{n(n+1)}) \right\}.$$

If the vertices of the simplex $\mathcal{T}_n$ are to be $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \mathbf{x}_{n+1}$, one can denote by $f_i, 1 \leq i \leq n+1$ the facets of $T_n$ which are opposite to the $\mathbf{x}_i$. Hence, let $F_i$ be the $(n-1)$-dimensional volumes of the facets $f_i$.

One interesting idea which we have is to maximise the volume of our simplex and in this manner to obtain a formula for the term $|\nabla T_n|^2$. Our findings state that for fixed $\mathbf{x}_k$ and $k \neq i$, the $n$ - dimensional vector

$$\nabla \frac{T_n}{n!}$$

is pointing in the direction by which the volume increases or decreases mostly. Since we want to maximise this, we shall only move $\mathbf{x}_i$. Hence, we move it orthogonally further from the facet $f_i$. This new rate of change which equals the absolute value of the gradient becomes the area divided by the dimension. More specifically, this is the case since the volume of a $n$ - dimensional pyramid is $\frac{b \cdot h}{n}$, where $b$ is the base area, $h$ is the height and we are taking the derivative with respect to $h$.

### 6.1.1 Spherical Hardy inequality

Consider the function $W : \mathbb{S}^{n-1} \to \mathbb{R}$ with $W \geq 0$. Assume that

$$A(W) := \left( \int_{\mathbb{S}^{n-1}} W d\mathbf{x} \right)^{-1} > 0,$$

with

$$\|W\|_{\gamma + \frac{n-1}{2}} := \|W\|_{L^{\gamma + \frac{n-1}{2}}(\mathbb{S}^{n-1})} \tag{6.1.3}$$

and

$$\begin{cases} \gamma > 0, \text{ for } n = 3 \\ \gamma \geq 0, \text{ for } n \geq 4. \end{cases} \tag{6.1.4}$$

Further, if for $u \in C_0^\infty(\mathbb{R}^n)$ we define

$$u_0 = \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} u \, d\mathbf{x}$$

and consider that for all real-valued functions $f$

$$u = u_0 + f,$$

then

$$\int_{\mathbb{S}^{n-1}} u_0 f d\mathbf{x} = 0.$$

Moreover, write $-\Delta = p_r^2 + r^2 \mathcal{L}^2$ for all $r > 0$. We shall denote the radial part of $-\Delta$ with $p_r^2$ and with $\mathcal{L}^2$ the minus Laplace - Beltrami operator on the sphere. If $W$ and $\gamma$ satisfy (6.1.3) and (6.1.4), we desire to show that there exists a constant $B(W)$ such that

$$\int_{\mathbb{S}^{n-1}} |\mathcal{L} f|^2 d\mathbf{x} \geq B(W) \int_{\mathbb{S}^{n-1}} W |f|^2 d\mathbf{x}. \tag{6.1.5}$$

For now, define a linear orthogonal projection operator $Q$ on $L^2(\mathbb{S}^{n-1})$ by

$$Q\varphi := \varphi - \varphi_0.$$

Hence, $Q$ commutes with $-\Delta$ and $\int_{\mathbb{R}^n} u_0 \Delta f = 0$. Here we used the fact that the operator can be extended to a function $\varphi \in L^2(\mathbb{R}^n)$.

Consider

$$\begin{aligned} -\int_{\mathbb{R}^n} (u_0+f)\Delta(u_0+f)d\mathbf{x} &= -\int_{\mathbb{R}^n} u_0\Delta u_0 d\mathbf{x} - \int_{\mathbb{R}^n} f\Delta f d\mathbf{x} \\ &= \int_{\mathbb{S}^{n-1}} u_0 p_r^2 u_0 r^{n-1} dr d\omega - \int_{\mathbb{R}^n} f\Delta f d\mathbf{x}, \end{aligned}$$

where $d\omega$ is seen as the integration over the unit sphere. Applying the classical Hardy inequality on $\mathbb{R}^n$ implies that

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} u_0 p_r^2 u_0 r^{n-1} dr d\omega &\geq \frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{|u_0|^2}{r^2} r^{n-1} dr d\omega \\ &= A(W)\frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{W|u_0|^2}{r^2} d\mathbf{x}. \end{aligned}$$

Hence

$$\int_{\mathbb{R}^n} u p_r^2 u \geq \frac{(n-2)^2}{4}\left(A(W)\int_{\mathbb{R}^n}\frac{W|u_0|^2}{r^2}d\mathbf{x} + \int_{\mathbb{R}^n}\frac{|f|^2}{r^2}d\mathbf{x}\right).$$

If we apply (6.1.5), we then get

$$-\int_{\mathbb{R}^n} u\Delta u \geq \int_{\mathbb{R}^n}\frac{W}{r^2}\left(\frac{(n-2)^2}{4}A(W)|u_0|^2 + B(W)|f|^2\right)d\mathbf{x} + \frac{(n-2)^2}{4}\int_{\mathbb{R}^n}\frac{|f|^2}{r^2}d\mathbf{x}.$$

It can be observed that a potential estimation for a Hardy constant can be

$$C = \frac{1}{2}\min\left[\frac{(n-2)^2}{4}A(W), B(W)\right],$$

where $B(W)$ can be further optimised.

## 6.2 Maximum ionization of atoms

Another interesting research road was considered when we switched our attention towards the ionisation of atoms. As we saw in [82], the author introduces the concept of ground state under the behaviour of a function.

Throughout history the mathematical description of quantum mechanics has been quite an adventure for researchers. Several important spectral inequalities for Schrödinger operators are reviewed and their applications in the spectral analysis of quantum systems has been discussed for years.

For example, our interest begins with a particle (i.e. a photon or an electron) which is described by a wave function $\psi : \mathbb{R}^3 \to \mathbb{C}$. Now, to function $\psi$ researchers usually affiliate the quantity $|\psi(\mathbf{x})|^2$, which is the density of the wave function. In this circumstance, this quantity represents the probability of finding our particle at position $\mathbf{x}$.

Considering that in nature there exist larger quantum systems, starting from atoms or molecules, we define our $\psi$ as many-body wave function $\psi = \psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$, with $\mathbf{x}_j \in \mathbb{R}^3, 1 \leq j \leq N$. Moreover, when researchers work with many-particle quantum systems, statistics is also present. It is known that quantum particles are thought to

be indistinguishable, hence the quantity $|\psi(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)|^2$ corresponding to the probability of finding particle $j$ at position $\mathbf{x}_j$ , could potentially be seen as invariant under particle interchanging process. This leads to certain symmetry requirements for the wave function. More information related to the Quantum Theory aspects of this topic could be found in [92], [93], [94] and [95].

We are interested in the following operator on $L^2(\mathbb{R}^{3N})$, for $N \geq 3$,

$$\mathcal{H} = H_{N,z} = -\frac{1}{2}\sum_{j=1}^{N} \Delta_{\mathbf{x}_j} - z\sum_{j=1}^{N} \frac{1}{|\mathbf{x}_j|} + \sum_{1\leq j<k\leq N} \frac{1}{|\mathbf{x}_j - \mathbf{x}_k|}.$$

Denoting also $\sigma(\mathcal{H})$ to be the spectrum of our operator $H_{N,z}$ and $\sigma_{ess}(\mathcal{H}) \subset \sigma(\mathcal{H})$ to be the essential spectrum, we want to show that we have a reverse inclusion as well.

Namely, we would like to show that

$$\sigma(\mathcal{H}) \subset \sigma_{ess}(\mathcal{H}).$$

We start by assuming that there exists a ground state $\psi$ with $||\psi|| = 1$ such that

$$(\mathcal{H} - E)\psi = 0,$$

where $E = E_{N,z}$ is the corresponding eigenvalue of $\mathcal{H}$. It is easy to observe that

$$\begin{aligned}\int_{\mathbb{R}^{3N}} \left(|\nabla(f\psi)|^2 - V|f\psi|^2\right) d\mathbf{x} - E||f\psi||^2 &= \int_{\mathbb{R}^{3N}} \left(|\nabla f|^2|\psi|^2 + 2\mathrm{Re}(f\overline{\nabla f}\nabla\psi\overline{\psi}) + |f|^2|\nabla\psi|^2 - V|f\psi|^2\right) d\mathbf{x} - E||f\psi||^2 \\ &= \int_{\mathbb{R}^{3N}} \left(|\nabla f|^2|\psi|^2 + \mathrm{Re}(\nabla|f|^2\nabla\psi\overline{\psi}) + |f|^2|\nabla\psi|^2 - V|f\psi|^2\right) d\mathbf{x} - E||f\psi||^2 \\ &= \int_{\mathbb{R}^{3N}} |\nabla f|^2|\psi|^2 d\mathbf{x}.\end{aligned}$$

If $f = \sqrt{|\mathbf{x}_1|}$, where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N)$, then

$$\left(\sqrt{|\mathbf{x}_1|}\psi, (\mathcal{H} - E)\sqrt{|\mathbf{x}_1|}\psi\right) = \frac{1}{8}\int_{\mathbb{R}^{3N}} \frac{|\psi|^2}{|\mathbf{x}_1|} d\mathbf{x}.$$

Rewriting the operator $\mathcal{H}$

$$H_{N,z} = H_{N-1,z} - \frac{1}{2}\Delta_{\mathbf{x}_1} - \frac{z}{|\mathbf{x}_1|} + \sum_{j=2}^{N} \frac{1}{|\mathbf{x}_1 - \mathbf{x}_j|} - E(N, z),$$

we obtain that

$$H_{N-1,z} - E_{N,z} \geq 0.$$

This result looks promising for the time being as we are still researching potential formulas and identities which could push our inequality further. At this point, we can apply the classical Hardy inequality and the symmetry of $\psi$.

## 6.3 Fermionic Hardy inequalities

Nature presents us a 3D world with more than 3 particles interacting to each other. Most physical systems involve interaction of many particles, from electrons in a solid to atoms in a gas. Hence, the notion of **fermions** comes to existence. All elementary particles are classified as **fermions** and **bosons**.

Particles with half-integer spin are fermions and their wave function must be antisymmetric under particle exchange(e.g. electron, positron, neutron, proton).

Studying [119] we came across the following inequality:

$$\int_{\mathbb{R}^{2\cdot 3}} |\nabla u|^2 d\mathbf{z} \geq C \cdot \frac{4}{3} \int_{\mathbb{R}^{2\cdot 3}} \left( \frac{1}{|\mathbf{z}_1 - \mathbf{z}_2|^2} + \frac{1}{|\mathbf{z}_1 - \mathbf{z}_3|^2} + \frac{1}{|\mathbf{z}_2 - \mathbf{z}_3|^2} \right) |u|^2 d\mathbf{z}$$

for all antisymmetric functions $u \in H^1(\mathbb{R}^6)$, where $\mathbf{z}_i = (x_i, y_i)$, $i = 1, 2, 3$. Our curiosity is to show that 2 is an upper bound for $C$.

First, let

$$h := \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = x_2y_3 + y_2x_1 + y_1x_3 - x_2y_1 - y_2x_3 - y_3x_1,$$

$$\rho^2 := \sum_{1 \leq j < k \leq N} |\mathbf{z}_i - \mathbf{z}_j|^2$$

and

$$\phi(t) := \begin{cases} 0, & t < \frac{1}{k^2} \\ 1 + \frac{\ln(kt)}{\ln k}, & \frac{1}{k^2} \leq t < \frac{1}{k} \\ 1, & t \geq \frac{1}{k} \end{cases} .$$

Note that $h$ introduces the antisymmetry and it is a harmonic polynomial of degree two. Also, if we consider that we can form a triangle $A$ with corners $\mathbf{z}_1, \mathbf{z}_2$ and $\mathbf{z}_3$, we observe that $|h| = 2A$.

Define $\psi(t)$ to be any positive, even function of compact support. Our trial function in $H^1$ becomes

$$u := h \cdot \rho^\alpha \cdot \phi(\rho) \cdot \psi(|\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3|),$$

for $\alpha, k$ constants on the real line. The role of $\rho^\alpha$ then is to obtain extra freedom to improve the upper bound of $C$, while the purpose of $\phi(\rho)$ is to cut off the function at the singularity $\mathbf{z}_1 = \mathbf{z}_2 = \mathbf{z}_3$. In the same manner, $\psi(|\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3|)$ integrates "away" the direction $\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3$ from the problem point.

As an extra measure of security, we show that $\nabla h \cdot (\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3) = 0$ and $\nabla \rho \cdot (\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3) = 0$, which translates into the fact that $h$ and $\rho$ are independent of the centre of gravity of the triangle.

Recalling an interesting property for any domain $\Omega$ and an integrable function $f$, we have that

$$\underbrace{\int_\Omega |\nabla(hf)|^2 d\mathbf{z}}_{I} = \int_\Omega h^2 |\nabla f|^2 d\mathbf{z} + \int_{\partial\Omega} f^2 h \nabla h \cdot \eta dS.$$

Inserting our trial function into the $I$ term, we get

$$I = \int_\Omega h^2 |\rho^\alpha \cdot \phi(\rho) \cdot \psi(|\mathbf{z}_1 + \mathbf{z}_2 + \mathbf{z}_3|)|^2 d\mathbf{z} = \int_\Omega h^2 |\alpha \rho^{\alpha-1} \phi\psi\nabla\rho + \rho^\alpha \psi \nabla\phi + \rho^\alpha \phi \nabla\psi|^2 d\mathbf{z} \tag{6.3.1}$$

By further computations, we obtain that

$$\begin{cases} |\nabla\rho|^2 = 3, \\ \nabla\phi(\rho) = \frac{1}{\ln k}\frac{\nabla\rho}{\rho} \implies |\nabla\phi(\rho)|^2 = \frac{1}{(\ln k)^2}\frac{3}{\rho^2}, \\ \nabla\rho\cdot\nabla\phi(\rho) = \frac{1}{\ln k}\frac{3}{\rho} \\ \nabla\rho\cdot\nabla\phi = 0 \\ \nabla\phi(\rho)\cdot\psi = 0 \\ |\nabla\psi|^2 = D(\psi'(|\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3|))^2 \end{cases}$$

for some constant $D$. Hence, from (6.3.1) that

$$\begin{aligned} I = 3\alpha^2\int_{\rho\geq k^2} h^2\rho^{2\alpha-2}\phi^2\psi^2 d\mathbf{z} + \frac{3}{(\ln k)^2}\int_{\frac{1}{k^2}\leq\rho\leq\frac{1}{k}} h^2\rho^{2\alpha-2}\psi^2 d\mathbf{z} + \\ + \frac{6\alpha}{\ln k}\int_{\frac{1}{k^2}\leq\rho\leq\frac{1}{k}} h^2\rho^{2\alpha-2}\phi\psi^2 d\mathbf{z} \\ + D\int_{\rho\geq\frac{1}{k^2}} h^2\rho^{2\alpha}\phi^2(\psi')^2 d\mathbf{z}. \end{aligned}$$

As an assumption $-4 < \alpha < -3$ and this will make the last integral bounded for all $k$.

Consider also the following definition

$$d_{ij}^2 := \frac{|\mathbf{z}_i - \mathbf{z}_j|}{\sqrt{2}}, \quad 1\leq i<j\leq 3$$

with the observation that $|\nabla d_{ij}|^2 = 1$. Moreover,

$$\begin{cases} \nabla h\cdot\nabla d_{ij} = \frac{h}{d_ij}, \\ \nabla\rho\cdot\nabla d_{ij} = \frac{2d_{ij}}{\rho}, \\ \nabla\psi\cdot\nabla d_{ij} = 0, \\ \Delta d_{ij} = \frac{1}{d_{ij}} \end{cases}$$

while

$$\nabla\phi\cdot\nabla d_{ij} = \begin{cases} \frac{3}{\ln k}\frac{d_{ij}}{\rho^2}, & \text{if } \frac{1}{k^2}\leq\rho\leq\frac{1}{k} \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, we compute the following term:

$$\begin{aligned} \int\frac{h^2\rho^{2\alpha}\phi^2\psi^2}{d_{ij}^2}d\mathbf{z} &= \int\frac{h^2\rho^{2\alpha}\phi^2\psi^2}{d_{ij}^2}\nabla d_{ij}\cdot\nabla d_{ij}d\mathbf{z} \\ &= -2\int h\rho^{2\alpha}\phi^2\psi^2\nabla h\cdot\frac{\nabla d_{ij}}{d_{ij}}d\mathbf{z} - 2\alpha\int h\rho^{2\alpha-1}\phi^2\psi^2\nabla\rho\frac{\nabla d_{ij}}{d_{ij}}d\mathbf{z} - \\ &- 2\int h\rho^{2\alpha}\phi\psi^2\nabla\phi\cdot\frac{\nabla d_{ij}}{d_{ij}}d\mathbf{z} - 2\int h\rho^{2\alpha}\phi^2\psi\nabla\psi\cdot\frac{\nabla d_{ij}}{d_{ij}}d\mathbf{z} - \\ &- \int h\rho^{2\alpha}\phi^2\psi^2\frac{\Delta d_{ij}}{d_{ij}}d\mathbf{z} + 2\int\frac{h^2\rho^{2\alpha}\phi^2\psi^2}{d_{ij}^2}d\mathbf{z}. \end{aligned}$$

Using the computations for the $d_{ij}$ term, we get that

$$\int \frac{h^2\rho^{2\alpha}\phi^2\psi^2}{d_{ij}^2}d\mathbf{z} = -3\alpha\int h^2\rho^{2\alpha-2}\phi^2\psi^2 d\mathbf{z} - \frac{3}{\ln k}\int h^2\rho^{2\alpha-2}\phi\psi^2 d\mathbf{z}$$

If we consider all the calculations done so far, we can conclude that

$$\frac{4}{3}\int_{\mathbb{R}^{2\cdot 3}}\left(\frac{1}{|\mathbf{z}_1-\mathbf{z}_2|^2}+\frac{1}{|\mathbf{z}_1-\mathbf{z}_3|^2}+\frac{1}{|\mathbf{z}_2-\mathbf{z}_3|^2}\right)|u|^2 d\mathbf{z} = -6\alpha\int h^2\rho^{2\alpha-2}\phi^2\psi^2 d\mathbf{z} - \frac{6}{\ln k}\int h^2\rho^{2\alpha-2}\phi\psi^2 d\mathbf{z} \quad (6.3.2)$$

Let us call $C_{\alpha,k}$ the upper bound for the constant $C$, then from (6.3.1) and (6.3.2)

$$C_{\alpha,k} = \frac{I}{-6\alpha\int h^2\rho^{2\alpha-2}\phi^2\psi^2 d\mathbf{z} - \frac{6}{\ln k}\int h^2\rho^{2\alpha-2}\phi\psi^2 d\mathbf{z}}$$

We shall study the quotient for the case when $k\to\infty$. We believe this can be done by calculating the coarea formula over the level sets of $\rho$. It is worth observing that we only need to obtain the value of the denominator up to a multiplicative constant which is not dependent on $k$.

## 6.4 On the continuous versions of Hardy inequality

Let us consider $\mathbf{x} = (\mathbf{x}_1,\dots,\mathbf{x}_N)\in\mathbb{R}^{Nd}$ and $\mathbf{x}_j = (x_{j,1},\dots,x_{j,d})\in\mathbb{R}^d$, $j = 1,\dots N$.

The result considered in this section is a trial version of one other of our curiosities in the sense that our first main article obtains a Hardy inequality for terms as $\frac{1}{r_{ij}^2}$. Paper [59] presents the case when

$$\int_{\mathbb{R}^{3d}}|\nabla u|^2 d\mathbf{x} \geq \frac{(d-1)^2}{3}\int_{\mathbb{R}^{3d}}\frac{|u|^2}{R^2}d\mathbf{x}$$

for $R$ being the circumradius for the triangle with $\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3$ as vertices and $u\in C_0^\infty(\mathbb{R}^{3d}\setminus\{\mathbf{x}_1=\mathbf{x}_2=\mathbf{x}_3\})$. For the case when the number of particles $N$ is larger than 3, we obtained the following theorem, but there are still doubts on whether the configuration of points chosen is accurate enough to produce this result.

**Theorem 6.4.1** *For a function $u\in C_0^\infty(\mathbb{R}^{Nd})$ and $c_1\in[0,d-2]$ the following Hardy inequality holds*

$$\int_{\mathbb{R}^{Nd}}|\nabla u|^2 d\mathbf{x} \geq \mathcal{G}(c_1,d,N)\int_{\mathbb{R}^{Nd}}\sum_{i<j<k}^{N}\frac{1}{R_{ijk}^2}|u|^2 d\mathbf{x}, \quad (6.4.1)$$

*where*

$$\mathcal{G}(c_1,d,N) = \frac{N^3(N-1)^2-108}{36}c_1^2 + \left[2(d-2) - \frac{N^2(N-1)((N-1)d-2)}{18}\right]c_1 + \\ + \frac{((N-1)d-2)^2 N}{36}.$$

Considering that for the above value of $c_1$ our function has a global minimum, our result for $N>3$ is used to check whether the values, for different points, behave nicely.

For example, since we work on the interval $[0, d-2]$ we can have the following estimates

$$\mathcal{G}(0,3,3) \approx 1.33333$$

$$\mathcal{G}(0,3,4) \approx 5.44444$$

$$\mathcal{G}(0,3,5) \approx 13.8889$$

$$\mathcal{G}(1,3,6) \approx 47.1667$$

$$\dots\dots\dots\dots\dots$$

$$\mathcal{G}(1,3,10) \approx 1172.61$$

$$\dots\dots\dots\dots\dots$$

$$\mathcal{G}(1,3,100) \approx 2.56267 \cdot 10^{8}$$

As we expected, if we increase the number of particles our constant becomes bigger and bigger in 3 dimensions.

Moreover, since our function has a minimum we could choose to evaluate the function when $c_1$ reaches the boundaries of the interval in which it is defined(i.e. either 0 or $d-2$) and obtain the result stated in [59].

# Chapter 7

# Conclusion

The thesis has presented theoretical and applied key concepts which allowed the discovery of various Hardy type inequalities. These concepts have been approached and proved in three important instances throughout our research.

The theoretical section involved an extensive presentation of what the classical Hardy inequality represents and how it can be applied into Quantum Mechanics. Initially, we considered the theory behind the Schrödinger operators and the research obtained so far. We, then, described the discrete Hardy inequality and a few characteristic properties which explained how we can optimise such inequalities. Since the theory behind Hardy inequalities involves the existence of Sobolev spaces, we presented a few ideas on how norms and convergence of functions can be manipulated such that the results obtained become optimal. As part of the introduction, we saw that researchers have discovered many other Hardy type inequalities with vast applications in the real world. Starting from multidimensional Hardy inequalities to bounded domains and half space inequalities, we observed that it is difficult, but in the same time, interesting to understand and apply different solving techniques to form our results. We highlighted our points of interest regarding various important articles which treated the many-particle case Hardy inequalities.

Regarding the results obtained, they are accompanied by the structure: §3 studies the standard Hardy inequality away from a point, which is associated to the Laplacian operator in $n$ dimensions. The constant involved is sharp and the inequality on its whole describes the capacity of the Laplacian to be non-negative and bounded below. We obtain a variation of the classical Hardy inequality by introducing new vector fields and boundary approximations on a many-particle quantum system. Firstly, we put our attention in establishing a reasonably easy to compute vector field for which its divergence and modulus squared are finite. We needed these quantities to be finite due to the preliminary lemmas which we used. The lemma involves different aspects from multiple particles to circumradius of triangles. Also, since the identity in the lemma is considered a Hardy inequality, different constants are present in its structure, so our main concern is to determine whether the quantities can be generalised for an infinite amount of particles, firstly in three dimensions and later on for infinite dimensions. Our approach then followed an optimisation side with regards to the constant functions which were present in the structure of the potential sharp function. This involved MATLAB computations and gradient studies that allowed us to obtain a close to optimal

constant with a behaviour which can be easily predicted through numerical analysis. It is important to highlight that the constant involved in our new inequality is better than other constants presented in various research papers. After the result was established, we discovered also a trial version of a Hardy inequality regarding the circumradius of a simplex in $n$ dimensions. It is still a trial version considering the fact that the vector field involved might not be the optimal one to use. This is the case because the terms used may behave nicely for relatively high dimensions, but if we intend to allow the dimension tend to infinity, it is not quite clear what the behaviour would be.

In §4, we are looking for new Hardy-type inequalities motivated by a recent paper of L. Skrzypczak and C. Tintarev [106]. We considered dimension four as a starting point in order for our configuration of points to be easily observable. Our main points of interest were what the best possible value for the constant involved could be and whether the inequality discovered is true for functions on the whole space and not just for block-radial functions. The first question has a rather complicated answer which consists in a minimiser. This is obtained by introducing a non negative test function on the sphere. One more important condition on this function was established to be that it is bounded and by using polar coordinates and the Sobolev inequality, we were able to show also that the inequality is equivalent to the compact embedding into the Sobolev space $H^1(\mathbb{R}^n)$. The second question was answered positively by considering another polar coordinate system for which an optimal configuration of points in $\mathbb{R}^n$ was used. Once this was accomplished, we checked whether the inequality obtained has a finite norm such that we can take the infimum over the whole quantity and, in this manner, to establish the value of the optimal constant. We moved forward to applying our result into an abstract field known as Hopf fibrations. Using the configuration of points introduced previously, we developed a new system of coordinates resembling the Hopf fibration for dimension four. The classical Hopf fibration for four dimensions provided a good starting point from which we treated the next Hopf fibration for eight dimensions. Afterwards, our new inequality was translated into Hopf coordinates on the $\mathbb{S}^7$ sphere while still checking that the norm in the usual Lebesgue space is finite.

Reaching the last main result chapter, in §5 we treated the case when the Hardy inequality can be defined on dimension two. More specifically, how we can investigate the existence of a discrete Hardy inequality in two dimensions. It is known that the classical Hardy inequality fails in two dimensions due to the structure of the terms and the dependence of the constant on dimension. Our interest, hence, consists in finding a suitable magnetic potential field for which the quantities in the discrete inequality are properly defined. The problem was formed around the existence of a magnetic Hamiltonian involving a magnetic field and the flux of this field. This is clearly applied in Quantum Mechanics and we used the Aharonov - Bohm magnetic effect to create our potential field. This effect was also used by Laptev and Weidl in [74] to discover a two dimensional improved version of Hardy inequality. For our case, once the Hamiltonian was formed, we considered a special polar coordinate system which explains how the magnetic flow spins around the origin of a square of radius $n$. The polar coordinates system was used with a non linear operator which moves counter-clockwise along the square hence resembling the spin of the magnetic flow. We discretised the continuous gradient on $\mathbb{R}^2$ using the non linear operator and preserving as many terms as possible. Through various and numerous computations, we were able to reach a concise form of the new inequality which contains a constant depending on the magnetic potential field chosen. Once the magnetic field

was introduced we were able to determine the value of the constant in the resulting discrete Hardy inequality by using Parseval's theorem and the periodicity of the non-linear operator on the space used. The inequality obtained provides a refined version of what the reader could find in the work [39] and [72] in the sense that our inequality was researched upon by using the standard quadratic form on $\mathbb{Z}^2$.

Since we manifested an interest in Hardy inequalities for multiple dimensions and particles we also have curiosities which seems worthy of our attention. In §6 we revisited the case when the Hardy inequality was defined for the circumradius of a simplex in $n$ dimensions with a large number of particles. This created a snowball effect and we found ourselves into a whole new different stage. Once we managed to find one other inequality for the circumradius, we had an impulse on defining and discovering a more pedantic manner to determine a Hardy inequality for the volume of a $n$ dimensional simplex. The usual manner in which one can define the volume of a simplex is through the determinant of an $n$ by $n+1$ matrix. Moreover, another interesting idea would be to maximise the volume of the simplex involved and such obtain a formula for the gradient term involving the description of this determinant.

Tracing back the exceeding possibilities to extend the results obtained so far in the theory of Hardy inequalities, the present thesis has laid the foundation stone for their research. Continuous research involving these aspects holds enormous promise towards the improvement of future results.

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