CANONICAL FORMALISM IN SUPERSPACE

by

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CONTENTS

Abstract

We investigate the " space "+" time " canonical decomposition of N=1 Minkowski superspace and discuss its applications in the identification of the canonical superfield variables of superspace field theories .

The canonical decomposition of Minkowski superspace in null-plane coordinates is straightforward and leads to the standard " light-cone " superfield formalism . In orthonormal coordinates , though , in order to obtain a kinematical , that is not involving time shifts , subclass of the full Minkowski supersymmetry it becomes necessary to consider a " ^{s}complex " extension of the usual or real superspace which , without disrupting the basic component field content of superfields , naturally enlarges the class of superPoincaré symmetries in an essential way .

What results from the canonical decomposition of ^{s}complex Minkowski superspace , after the dynamical θ directions have been identified and integrated over , is a reduced " canonical " superspace described by the x^m , $m = 0,1,2,3$ and two complex kinematical coordinates θ^{ξ} , $\xi = \mathrm{I,II}$. Full Minkowski superspace action integrals are thereby reduced to action integrals over canonical superspace and , in these , ∂_0 is the sole dynamical derivative . An Hamiltonian superspace formalism can then be constructed in the usual way .

The methods developed are applied to the chiral and the abelian vector supermultiplets . In each case , the canonical superspace formalism does reproduce the Hamiltonian structure of the ordinary field theory of which the supermultiplet is the supersymmetric extension .

Also , part of this thesis is dedicated to the study of the holomorphic supersymmetry algebras , an example of which is the 3 dimensional " spatial " supersymmetry algebra obtained in the canonical decomposition of Minkowski superspace .

In this respect , we propose a new constructive scheme for both the usual or real and the holomorphic SuSy algebras , within the general formalism of real Clifford algebra theory . The scheme is based on a class of real valued " Majorana " spinor scalar products , some of which , in signatures $s-t = 3 \bmod 8$, admit an extension to complex valued " holomorphic " spinor scalar products , bilinear with respect to the natural complex structure in the underlying real Clifford algebra . This mechanism of holomorphic extension then carries over to characterize the relation between real and holomorphic SuSy algebras .

Preface

The work described in this thesis was carried out in the Theoretical Physics Group at the Blackett Laboratory , Imperial College , London , between October 1982 and March 1987 , under the supervision of Dr. Kellogg S. Stelle . Unless otherwise stated , the work is original and has not been submitted before for a degree of this or any other university . The material of most of chapters 2 and 3 has been the subject of two Imperial College preprints , one of which has already been published - see references [1] and [2] .

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§ 1 General Introduction

Ever since its original insights and a priori motivations , the development of the work compiled in this thesis refers to a period of great expectations over the ultimate significance of supersymmetry as a key ingredient in a viable Quantum Gravity theory . The focus of those expectations has hence changed , from supergravity to the theory of superstrings , but supersymmetry has not ceased to be the underlying conceptual framework . This despite inexistent experimental evidence in its favour . Indeed , the arguments put forward in support of supersymmetry have invariably been of formal , though no doubt significant , nature .

The project of this thesis was born out of the idea of developing within the geometrical framework most natural to the description of supersymmetric theories , that of superspace , the no less geometrical constructions of Hamiltonian or canonical analysis . This has admitedly been a formal quest whose goal is to compatibilize the machineries of superspace and canonical formalisms and whose justification is ultimately based on the well established conceptual and practical importance of both .

Superspace supersymmetry is thus the dominant element of the work to be presented . In getting acquainted with the subject we have found references [3] and [4] particularly useful , the first group emphasizing the physical and computational side and the second group emphasizing the more mathematical and geometrical issues . The introductory notes on real superspace presented in the way of an appendix at the end of this introductory chapter provide , at an elementary level , an overview of the subject title plus a catalogue of topical notation and conventions used throughout the thesis .

Canonical methods themselves will only play a background role in most of what follows . Even in the last part of the thesis , where the canonical superspace formalism is developed along the lines of the Dirac-Bergmann algorithm for constrained systems , this is done in the context of two simple superspace models , the chiral and the abelian vector multiplets . Thus , what we have achieved with this work is to have made Hamiltonian methods available in superspace , not to have unveiled the intrinsicacies of any particular supersymmetric theory . In any case , we have listed in [5] a standard and fairly comprehensive set of references on canonical methods .

Since well before the construction of a Hamiltonian formalism with superfields as canonical variables was first addressed by ourselves [1] and others [15,16] , canonical methods had already been oftenly employed in the context of supersymmetry , notably in the Hamiltonian treatments of the relativistic superparticle [6] , supersymmetric quantum mechanics [7] , the relativistic spinning particle [8] and supergravity [9] . We should also mention here the series of articles on the general theme of energy positivy in gravity and supergravity [10] , following the energy positivity result for supergravity and its subsequent extension to conventional General Relativity via an appropriate supergravity embedding .

In all those earlier instances of the use of canonical methods in supersymmetry , the Hamiltonian treatment of the supersymmetric field theory model under consideration was performed at component field level , with at most indirect implications to the actual development of canonical superspace , which raises questions of a different nature . Consequently , from the point of view of this thesis , the significance of such earlier work resides more on the inventory of motivations that it has brough to light concerning the sort of supersymmetry related questions which a canonical analysis can help to clarify .

Let us begin with the relativistic superparticle [6] , the natural superspace extension of the ordinary relativistic point particle . Although a formulation exists in every underlying spacetime signature here we prefer to be specific and consider the familiar 4 d Minkowski case in order to stress the correspondence with the ordinary point particle formalism .

The free relativistic superparticle of arbitrary mass m in real Minkowski superspace

$\mathbb{S}^{3,1}_{R}$ is best described in terms of a superspace trajectory $(x^m(\tau),\theta^\alpha(\tau))$, $m = 0,1,2,3$ and $\alpha = 1,2,3,4$, plus an einbein field $e(\tau)$, according to the action (in real units)

$$\mathfrak{S} = -\frac{1}{2} \int (e^{-1} \omega^2 - e m^2 c^2) \, d\tau$$

$$\text{where} \qquad \omega^m \equiv \dot{x}^m - i\, \theta^\alpha (\varsigma \Gamma^m)_{\alpha\beta} \dot{\theta}^\beta$$

and τ is an arbitrary worldline parameter having the dimension of time , say . The notation is explained at length in the Appendix which concludes this introductory chapter . In summary , the Grassmann valued superspace coordinates x^m and θ^α are real and the Γ_m are orthogonal 4×4 real matrices , $\{\Gamma_m,\Gamma_n\} = 2\,\eta_{mn}\,\mathbf{I}$, $\eta_{mn} = \mathrm{diag}\,(-1,+1,+1,+1)$. Choosing $\varsigma = \Gamma_0$, each of the product matrices $(\varsigma\Gamma_m)$ is symmetric .

The einbein field $e(\tau)$ along the worldline has the dimension of inverse mass . Its equation of motion consists of the mass-shell condition

$$p^2 + m^2 c^2 = 0 \qquad , \quad \text{where} \quad p_m \equiv -\frac{\partial \mathcal{L}}{\partial \dot{x}^m} = e^{-1}\, \omega_m \ ,$$

which , when $m \neq 0$, can be solved for the einbein field e as

$$e = m^{-1} c^{-1} \sqrt{-\omega^2} \ .$$

After substituting this on-shell solution for e back into the action , one finds

$$\mathfrak{S} = m \int \sqrt{-\omega^2} \, d\tau$$

which was the action integral formulation originally proposed in [6 , Casalbuoni] . Yet , there are two reasons to prefer the presence of the einbein field e : one is the possibility of describing both the massive and the massless cases in a unified way ; the other is the explicit invariance under infinitesimal local reparametrizations $\tau \rightarrow \tau - \lambda(\tau)$,

$$\delta x^m = \lambda \dot{x}^m \quad , \quad \delta \theta^\alpha = \lambda \dot{\theta}^\alpha \quad , \quad \delta e = (\lambda e)^{\cdot} \; .$$

Note : The same technique applies to the ordinary relativistic particle case , obtained directly from the above by reducing ω^m to $\dot{x}^m$. There , when $m \neq 0$, the on-shell solution for the dimensionful einbein field e relates it with the natural dimensionless einbein field along the particle's worldline $e_w = c^{-1}\sqrt{-\dot{x}^2} = m\, e$. From this relation follows , in particular , that choosing τ as the particle's proper time , and consequently $\dot{x}^2 = -c^2$, corresponds to the " gauge fixing " $e = m^{-1}$, in which the dimensionful character of the einbein field e is made explicit . In the proper time gauge , moreover , p^m becomes precisely the standard momentum four vector $p^m = (E c^{-1}, \vec{p} = m\vec{v})$ of relativistic point particle kinematics .

The action integral $\mathfrak{I}$ for the free relativistic superparticle is invariant under the standard superPoincaré transformations and , in particular , under the global supersymmetries

$$\delta x^m = i\,\varepsilon^\alpha (\varsigma \Gamma^m)_{\alpha\beta}\, \theta^\beta \quad , \quad \delta\theta^\alpha = \varepsilon^\alpha$$

specified by infinitesimal real anticommuting parameters ε^α . In fact , ω^m itself is invariant under these transformations . A more subtle and this time local invariance , applicable only in the massless case $m = 0$, is

$$\delta x^m = i\,\theta^\alpha (\varsigma\Gamma^m)_{\alpha\beta}\, \delta\theta^\beta \quad , \quad \delta\theta^\alpha = \omega^m (\Gamma_m)^\alpha{}_\beta\, \kappa^\beta$$

$$\delta e = 4 i\, e\, \dot{\theta}^\alpha \varsigma_{\alpha\beta}\, \kappa^\beta$$

with infinitesimal real anticommuting parameters $\kappa^{\alpha}(\tau)$. This local fermionic symmetry, first noticed in [6 , Siegel] , is carefully studied in [6 , Brink , Henneaux & Teitelboim] within a general Hamiltonian analysis of the relativistic superparticle . There , the motivation is essentially the understanding of the mechanisms leading to the construction of a covariant splitting between first and second class constraints , a difficulty present in both the superparticle and the superstring theories . We remark that the relativistic superparticle has also been formulated in 2N extended real superspace - see [6 , Lukierski plus Lusanna] .

Supersymmetric quantum mechanics is a simple field theory model over $\mathbb{S}_R^{0,1}$, designed for the purposes of illustrating spontaneous supersymmetry breaking through non-perturbative effects [7] . The " wave function " of supersymmetric quantum mechanics is a real even superfield

$$\Phi(t,\theta,\bar{\theta}) = \phi(t) + i\theta\,\psi(t) + i\bar{\theta}\,\bar{\psi}(t) + \theta\bar{\theta}\,\mathcal{D}(t)$$

which propagates in time t according to the superspace action integral

$$\mathfrak{I} = \int \left(\tfrac{1}{2} \bar{D}\Phi\, D\Phi - f(\Phi) \right) dt\, d\bar{\theta}\, d\theta$$

where $f(\Phi) = \Sigma\, a_n \Phi^n$, with ordinary real number coefficients a_n , and

$$D = -\partial_\theta + i\bar{\theta}\partial_t \qquad \bar{D} = \partial_{\bar{\theta}} - i\theta\partial_t \ .$$

The superspace action integral $\mathfrak{I}$ above is invariant under the global supersymmetries

$$\delta\Phi = (\varepsilon Q - \bar{\varepsilon}\bar{Q})\Phi \ ,$$

$$Q = -(\partial_\theta + i\bar{\theta}\partial_t) \qquad \bar{Q} = \partial_{\bar{\theta}} + i\theta\partial_t$$

specified by a complex parameter ε. The corresponding supersymmetry transformation rules for the basic component fields ϕ, ψ, $\bar{\psi}$, $\mathcal{D}$ may be deduced from this, simply by interpreting the component field expansion of $\delta\Phi$ in terms of the one of Φ.

Once the θ, $\bar{\theta}$ integration is performed and the auxiliary field $\mathcal{D}$ is eliminated through its own equation of motion, the action integral $\mathfrak{I}$ of supersymmetric quantum mechanics is expressed as

$$\mathfrak{I} = \frac{1}{2} \int \left(\dot{\phi}^2 + i(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - \mathcal{W}^2(\phi) - [\bar{\psi},\psi]\, \mathcal{W}'(\phi) \right) dt$$

where $\mathcal{W}$ and $\mathcal{W}'$ denote minus the first and second derivatives of the polynomial f. Details of this derivation can be found in [7, Cooper & Freedman]. There, developing an argument from [7, Witten], the Hamiltonian picture is used to obtain a first understanding of the general relation between the form of the potential $\mathcal{W}$ and the existence or not of a degenerate ground state of non-zero energy, signalling the breaking of supersymmetry.

The spinning particle [8], or supergravity in one dimension, shared with the superparticle the original interest of providing a classical description of spin. But, while the hidden local fermionic symmetry of the superparticle was uncovered only years later, the spinning particle had from the beginning an inherent local supersymmetry which soon transformed into a convenient laboratory for supergravity techniques.

The massless spinning particle in ordinary Minkowski spacetime is described by a bosonic vector field $\varphi^m(\tau)$ describing the position, a fermionic vector field $\psi^m(\tau)$ describing spin, plus an einbein field $e(\tau)$ and its supersymmetric companion $\chi(\tau)$, the last two necessary for the gauging of supersymmetry. The fermionic field $\chi(\tau)$ is thus the analogue of the Rarita-Schwinger field in supergravity. The massive case requires a further fermion field $\psi_5(\tau)$, whose transformation properties under local supersymmetry compensate the lack of invariance of the mass term $e m^2$. Each of the fields $\varphi^m(\tau)$, $e(\tau)$ (commuting) and $\psi^m(\tau)$, $\chi(\tau)$, $\psi_5(\tau)$ (anti-commuting) is real.

The full action for the relativistic spinning particle of arbitrary mass m, in our metric conventions, is

$$\mathfrak{S} = -\frac{1}{2} \int \Big[\left(e^{-1} \dot{\varphi}^2 - i \psi \dot{\psi} - i e^{-1} \chi \psi \dot{\varphi} \right) - \\ - \left(e m^2 + i \psi_5 \dot{\psi}_5 + i m \chi \psi_5 \right) \Big] \, d\tau \ .$$

Besides the overall reparametrization invariance $\tau \to \tau - \lambda(\tau)$,

$$\delta\varphi^m = \lambda \dot{\varphi}^m \qquad \delta\psi^m = \lambda \dot{\psi}^m \qquad \delta\psi_5 = \lambda \dot{\psi}_5$$

$$\delta e = (\lambda e)^{\cdot} \qquad \delta\chi = (\lambda \chi)^{\cdot} \quad ,$$

each of the two groups of terms in the action $\mathfrak{S}$ for the spinning particle (upper group : kinetic terms describing the massless case ; lower group : additional mass related terms) is separately invariant under the local supersymmetries

$$\delta\varphi^m = i\varepsilon\psi^m \qquad \delta\psi^m = e^{-1}\varepsilon\left(\dot{\varphi}^m - \tfrac{1}{2} i \chi \psi^m\right)$$

$$\delta e = i\varepsilon\chi \qquad \delta\chi = 2\dot{\varepsilon}$$

$$\delta\psi_5 = m\varepsilon$$

specified by an infinitesimal real anticommuting parameter $\varepsilon(\tau)$. In the limit $m \to 0$, the fermion field ψ_5 decouples, with an equation of motion $\dot{\psi}_5 = 0$, and accordingly does not transform in any way.

The detailed Hamiltonian formulation of the spinning particle can be found in [8, Sundermeyer] , together with a list of references on earlier treatments of classical spin . The constraint structure of the theory is clearly revealed and , in particular , the two first class constraints rooted in reparametrization invariance

$$p^m p_m + m^2 \cong 0 \qquad\qquad \psi^m p_m + m\psi_5 \cong 0$$

mass-shell — orthogonality of spin and momentum when $m = 0$

emerge as a supersymmetric pair from the primary first class constraints $\pi_\epsilon \cong 0 \cong \pi_\chi$. As before in the superparticle case , p_m denotes minus the momentum conjugate to φ^m. We note that a simple superspace formulation of the massless spinning particle exists , inspired on the fact that there is a clear correspondence between bosonic and fermionic fields . The superspace used has , apart from τ , one extra real coordinate θ , which means that superfields are constructed from two component fields . Naturally , then , what happens in simple terms is that each of the bosonic fields incorporates its single supersymmetric partner in a superfield of the same name - see [8 , Brink , Deser , Zumino , Di Vechia & Howe] .

The last and most outstanding entry in this brief history of canonical methods in supersymmetry is the Hamiltonian formulation of supergravity [9] . Naturally , the theory of supergravity offers a much richer environment to the techniques of canonical analysis and the degree of sophistication of the subject does not compare with the one of the preceeding supersymmetric models . As a result , the intention here is merely to sketch the essential facts and leave it to the interested reader to make use of the reference list which has been organized by topics as a guide to the early literature on the subject , including the original articles , the independent Hamiltonian and superspace formulations and higher-derivative supergravity .

Supergravity has been originally formulated in different ways . In [9 , Deser &

Zumino] , the action integral is constructed from independent vierbein $e_\mu^{\ a}$ and connection coefficients $\omega_\mu^{\ ab}$, plus a four component massless Majorana Rarita-Schwinger field ψ_μ ,

$$\mathcal{S} = \frac{1}{2} \int (e\, R(e,\omega) \ - \ i \in^{\mu\nu\rho\sigma} \bar{\psi}_\mu \Gamma_5 \Gamma_\nu D_\rho \psi_\sigma) \ d^4x \ .$$

The pure gravity sector amounts to the standard Einstein-Hilbert action in first order tetrad formalism : in local spacetime coordinates x^μ , $\mu = 0,1,2,3$,

$$g_{\mu\nu} = e_\mu^{\ a} e_\nu^{\ b} \eta_{ab} \qquad\qquad \eta_{ab} = \mathrm{diag}\,(-1,+1,+1,+1)$$

$$e = \det e_\mu^{\ a} = \sqrt{-g} \qquad\qquad R = e^\mu_{\ a} e^\nu_{\ b} R_{\mu\nu}^{\ \ ab}$$

$$R_{\mu\nu}^{\ \ ab} = \partial_\mu \omega_\nu^{\ ab} - \partial_\nu \omega_\mu^{\ ab} + \omega_\mu^{\ a}{}_c\, \omega_\nu^{\ cb} - \omega_\nu^{\ a}{}_c\, \omega_\mu^{\ cb} \ .$$

In the coupled Rarita-Schwinger sector , the notation is as follows : as before , the Γ_a are 4×4 real orthogonal matrices and

$$\{\Gamma_a, \Gamma_b\} = 2\,\eta_{ab}\, I \qquad\qquad \Sigma_{ab} = \tfrac{1}{4} [\Gamma_a, \Gamma_b]$$

$$\Gamma_5 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \qquad\qquad \in^{0123} = 1 \qquad\qquad \Gamma_\mu = e_\mu^{\ a} \Gamma_a$$

$$\psi_\mu^* = \psi_\mu \qquad\qquad \bar{\psi}_\mu = \psi_\mu^T \varsigma \qquad\qquad \varsigma = \Gamma_0$$

$$D_\mu \psi_\nu = \partial_\mu \psi_\nu + \tfrac{1}{2}\, \omega_\mu^{\ ab} \Sigma_{ab} \psi_\nu \ .$$

This form of the covariant derivative , which only responds to the spin $\frac{1}{2}$ content of the Rarita-Schwinger field ψ_μ , characterizes the so-called non-minimal coupling with gravity and is crucial to the success of the theory . Moreover , if ∇_μ is the standard full covariant derivative ,

$$D_{[\mu}\psi_{\nu]} = \nabla_{[\mu}\psi_{\nu]} - S^{\ \rho}_{\mu\nu}\psi_\rho \quad , \text{ where } S^{\ \rho}_{\mu\nu} = \Gamma^{\ \rho}_{[\mu\nu]} \text{ is the torsion ,}$$

which means that the anti-symmetrized $D_{[\mu}\psi_{\nu]}$ is a proper tensor. The Hamiltonian treatment in [9 , Pilati] clarifies further the choice of non-minimal coupling.

Note : In summary , the covariant derivative D_μ acts without the terms in $\Gamma^{\ \rho}_{\mu\nu}$ present in the full ∇_μ. So , if ψ is a four component Majorana spinor and υ^μ , υ^a are the components of a vector ,

$$D_\mu\psi = \partial_\mu\psi + \tfrac{1}{2}\,\omega_\mu^{\ ab}\,\Sigma_{ab}\,\psi \quad , \quad D_\mu\upsilon^a = \partial_\mu\upsilon^a + \omega_\mu^{\ a}{}_b\,\upsilon^b \quad , \quad D_\mu\upsilon^\nu = \partial_\mu\upsilon^\nu$$

$$\text{and , since } \nabla_\mu e_\nu^{\ a} = 0 \ , \quad D_\mu e_\nu^{\ a} - D_\nu e_\mu^{\ a} = -S^{\ a}_{\mu\nu} \ .$$

Besides general coordinate and local Lorentz invariance , the action of supergravity remains unchanged under the local supersymmetries

$$\delta e_\mu^{\ a} = i\bar{\varepsilon}\Gamma^a\psi_\mu \qquad\qquad \delta\psi_\mu = 2D_\mu\varepsilon$$

$$\delta\omega_\mu^{\ ab} = \bar{\varepsilon}\,(\Psi_\mu^{\ ab} + \tfrac{1}{2}e_\mu^{\ a}\Psi^b - \tfrac{1}{2}e_\mu^{\ b}\Psi^a)$$

where the infinitesimal parameter $\varepsilon\,(x^\mu)$ is itself a four component Majorana spinor. Above, $\Psi_a^{\ \mu\nu} = i\in^{\mu\nu\rho\sigma}\Gamma_5\Gamma_a D_\rho\psi_\sigma$ and $\Psi^\mu = \Psi_a^{\ \mu\nu}e_\nu^{\ a}$.

For a general flavour of the Hamiltonian treatment of supergravity consider , for instance , the approach of [9 , Fradkin & Vasiliev] . There , the connection coefficients $\omega_\mu^{\ ab}$ are solved for through their own equations of motion and the $e_\mu^{\ a}$ plus the spinor components $\psi_\mu^{\ \alpha}$ are taken as dynamical variables . Amongst these , the four $e_0^{\ a}$ plus the four $\psi_0^{\ \alpha}$ have no momentum conjugate . There follow four + four secondary first class constraints which generate the general coordinate and the local supersymmetry transformations , respectively . The Poisson bracket algebra involving those and the primary first class constraints generating the local Lorentz transformations is explicitly computed .

Alternative approaches reveal a number of interesting features . In [9 , Deser , Kay & Stelle] , for example , the authors remark that the fermionic kinetic terms associated with the $\psi_a = e_a{}^{\mu}\psi_{\mu}$ have no dependence on either the vierbein or the connection coefficients , which renders the ψ_a more natural as dynamical variables than the original ψ_{μ} . In the same article , the extension of supergravity with mass and cosmological terms is also discussed .

Finally , in [9 , Teitelboim plus Teitelboim & Tabenski] , a general scheme to introduce spin into a physical system is presented . Relying on the fact that the constraint structure of a theory determines it entirely , the scheme introduces the dynamical variables associated with the new spin degrees of freedom by extracting the square root , à la Dirac , of the Hamiltonian constraints characterizing the original spinless system . Uniquely from these considerations , the authors are naturally led to constraint structure of supergravity .

From the geometrical point of view , the development of Hamiltonian or canonical methods in superspace depends primarily on the possibility of implementing in this a " space "+" time " canonical decomposition which extends , at superspace level , the one in the underlying spacetime . Such superspace canonical decomposition is the central topic of this thesis and we construct it for the case of N=1 Minkowski superspace .

As we have already mentioned , real Minkowski superspace $\mathbb{S}_{R}^{3,1}$ is described by real (even,odd) Grassmann coordinates (x^m,θ^α) , where $m = 0,1,2,3$ and $\alpha = 1,2,3,4$. Furthermore , there is essentially one single possible choice for ς , which is $\varsigma = \Gamma_0$ relatively to a standard set of matrices Γ_m .

This is the so-called Majorana representation for real Minkowski superspace. Later, in § 3, it will be written γ^m_M , $\varsigma_M = \gamma^M_0$ and θ^α_M , in parallel with the notation for the Weyl and the canonical representations. These others consist of complex matrices γ^m_W , ς_W and γ^m_C , ς_C and associated complex four component spinors θ^α_W and θ^α_C , each of which naturally decomposes in complex two component spinors as

$$\theta^\alpha_W = (\theta_\rho , \bar{\theta}^{\dot{\rho}}) \quad \rho = 1{,}2 \qquad \text{and} \qquad \theta^\alpha_C = (\theta^\xi , \bar{\theta}^{\dot{\xi}}) \quad \xi = \mathrm{I{,}II} \ .$$

Details can be found both in § 3.2 and in the Appendix to § 3 .

Returning to real Minkowski superspace in its defining representation (x^m,θ^α) : In order to obtain a canonical decomposition at superspace level one needs to identify the eventual "temporal" coordinates θ_T , each of which a linear combination of the θ^α , such that supersymmetries along the "spatial" hypersurfaces $\theta_T = 0$ do not involve time shifts δx^0 .

It turns out that no such "temporal" coordinates θ_T *exist* . Although the precise argument is developed later, roughly it goes as follows : say we assume the existence of one such "temporal" coordinate θ_T , which can necessarily be written as

$$\theta_T = a_\alpha \theta^\alpha \quad , \text{ the } a_\alpha \text{ being ordinary real numbers ;}$$

A supersymmetry along the "spatial" hypersurface

$$\theta_T = a_\alpha \theta^\alpha = 0$$

is characterized by supersymmetry parameters such that

$$\delta\theta_T = \varepsilon_T = a_\alpha \varepsilon^\alpha = 0 \ ;$$

Each of these two equations can be solved for one of the original variables, respectively θ^α and ε^α, in terms of the others, and when those solutions are substituted in the supersymmetry transformation law

$$\delta x^0 = i\,\varepsilon^\alpha (\varsigma\,\Gamma^0)_{\alpha\beta}\,\theta^\beta = i\,\varepsilon^\alpha\,\delta_{\alpha\beta}\,\theta^\beta \quad ,$$

one finds that the three terms of type $\varepsilon^\alpha \theta^\alpha$ which are left come with strictly positive coefficients of the type $(1 + (a_\alpha)^2)$. This implies $\delta x^0 \neq 0$ for whatever choice of θ_T and thus contradicts the original existence hypothesis for such a " temporal " cordinate.

If, however, we relax the reality condition on the superspace coordinates θ^α, and thus on the a_α, we immediately find solutions for two independent complex θ_T and, as a result, we are able to induce a " space "+" time " canonical decomposition on the θ sector of Minkowski superspace, which then extends to the whole of it. The resulting canonical superhypersurfaces are the ones defined by $\theta_T = 0$ and $x^0 =$ constant, and the canonical superfield variables or canonical superfields are functions on those.

The price to be paid is thus the complexification of the superspace coordinates θ. *This consistently implies the complexification of the coordintes* x^m *but only of their purely Grassmann or " soul " component, the underlying real number spacetime coordinate, referred to as the " body " of* x^m*, remaining the same*. The resulting superspace is referred to as *soul-complex (^{s}complex) Minkowski superspace*, to distinguish it from the original or real version. The fact that only the soul components of the superspace coordinates need be complexified is crucial, in so far as that does not affect the component field content of general superfields; in order to preserve Grassmann differentiability, these admit a unique domain extension from real to ^{s}complex variables.

One could have guessed such complications from the fact that minimal spinors in 3 Euclidean and 4 Minkowski dimensions have the same number of components, four.

This derives naturally from the theory of real Clifford algebras which will be discussed later in some detail . Alternatively , one can think of it as a manifestation of the fact that , in 3 Euclidean dimensions , there do not exist Majorana spinors . In any case , it follows that the θ sector of real 3 d Euclidean superspace can never be identified as a proper subspace of the θ sector of real 4 d Minkowski superspace , an essential requirement for the canonical decomposition of the latter .

From a perhaps more immediate point of view , the necessity of considering complex coordinates θ in order to construct an Hamiltonian formalism in superspace arises in the following way :

Suppose we are given an action integral over full real Minkowski superspace ,

$$\mathfrak{S} = \int \mathcal{L}\,[F]\; d^4x\; d^2\theta\; d^2\bar{\theta}$$

$$\text{where} \quad d^2\theta \equiv \frac{1}{2}\, d\theta^1\, d\theta^2 \quad \text{and} \quad d^2\bar{\theta} \equiv \frac{1}{2}\, d\bar{\theta}^{\dot{2}}\, d\bar{\theta}^{\dot{1}}$$

as in the two component Weyl spinor notation of [3 , Wess & Bagger]

and F denotes a full superfield . In particular , we could consider as an example the superspace action integral for the chiral multiplet , where

$$\mathcal{L}\,[\Phi] = \bar{\Phi}\Phi \quad .$$

We know that such Lagrangian density describes dynamical component fields , one scalar and one chiral spinor . Yet , at superfield level there are absolutely no explicit derivatives whatsoever , which makes it impossible to define any kind of conjugate momentum to Φ .

One could argue , though , that the diffilcuties in this case are due to the fact that the

Lagrangian density $\mathcal{L}$ is written in terms of constrained superfields, Φ and $\bar{\Phi}$. After all, even a variational principle can not be applied to $\mathcal{L}$ in that form. But suppose, then, that we write Φ and $\mathcal{L}$ in terms of an unconstrained superfield potential F as

$$\Phi \equiv \bar{D}\bar{D}F \quad \text{and} \quad \mathcal{L}[F] = DD\bar{F}\,\bar{D}\bar{D}F \ .$$

There are still no explicit time derivatives in $\mathcal{L}$ but we can make them appear using the hidden dynamical character of the spinor covariant derivatives D_ρ and $\bar{D}_{\dot{\rho}}$, which manifests itself in their anticommutators

$$\{D_1, \bar{D}_{\dot{1}}\} = 2i\,(\partial_0 - \partial_3) \quad \text{and} \quad \{D_2, \bar{D}_{\dot{2}}\} = 2i\,(\partial_0 + \partial_3) \ .$$

Thus, for instance, we can rewrite $\mathcal{L}$, modulo a total superspace divergence, as

$$\mathcal{L}[F] = \bar{D}\bar{D}\bar{F}\,DDF + 4i\,\bar{\mathbf{D}}_m\bar{F}\,\partial_m F + 16\,\partial^m\bar{F}\,\partial_m F \ ,$$

where $\bar{\mathbf{D}}_m \equiv \frac{1}{2}\bar{D}_{\dot{\rho}}\,(\bar{\sigma}_m)^{\dot{\rho}\gamma}\,D_\gamma$.

The same happens with the action integral for the abelian vector multiplet, expressed in terms of an unconstrained but real superfield potential $V = \bar{V}$,

$$\mathcal{L}[V] = \frac{1}{4}\left(W^\rho D_\rho V + \bar{W}_{\dot{\rho}}\,\bar{D}^{\dot{\rho}}V\right)$$

where $W_\rho \equiv -\frac{1}{4}\bar{D}\bar{D}\,D_\rho V$. Here the Lagrangian density can also be rewritten, modulo a total superspace divergence, as

$$\mathcal{L}[V] = \frac{1}{8}\bar{D}\bar{D}V\,DDV + \partial^m V\,\partial_m V$$

where again we have made time derivatives explicit. Clearly, this conjuring with time derivatives, made possible by the hidden dynamical character of the spinor covariant derivatives D_ρ and $\bar{D}_{\dot{\rho}}$, implies that there exists no immediate natural definition for the conjugate momenta at full superfield level.

The logical conclusion is then that one must first be able to recombine the spinor covariant derivatives in a way that only some of them are dynamical, the rest being kinematical. Once such recombination is achieved and the superspace coordinates θ associated with the dynamical directions D_θ are integrated over in the superspace action integral, the Lagrangian density that results, expressed in terms of a reduced type of superfields dependent only on the kinematical θ coordinates, contains ∂_0 as the sole dynamical derivative. From there, the conjugate momenta, themselves reduced superfields, could be defined in the usual way, without ambiguities.

Integrating over a superspace θ coordinate is equivalent, modulo a total divergence term ∂_m, and an eventual sign difference, to differentiating the integrand $\mathcal{L}$ with the corresponding spinor covariant derivative D_θ and evaluating what results at $\theta = 0$,

$$\int \mathcal{L}\, d\theta \overset{\text{mod } \partial_m}{\approx} (D_\theta \mathcal{L})\Big|_{\theta=0} .$$

This technique is of direct conceptual significance within the program of identifying and integrating over the dynamical θ coordinates, since it turns out that one can do no better than a recombination of the original spinor covariant derivatives for which

$$\{D_{\mathrm{I}}, \bar{D}_{\dot{\mathrm{I}}}\} = \{D_{\mathrm{II}}, \bar{D}_{\dot{\mathrm{II}}}\} = 2i\partial_0$$

plus kinematical anticommutators of the form $\{D,D\}$ and $\{\bar{D},\bar{D}\}$. We infer from this that no actual dynamical × kinematical decomposition is possible within real Minkowski superspace . To avoid spinor covariant derivatives anticommuting to the time derivative ∂_0 , one would have to integrate over either the θ^{ξ} or the $\bar{\theta}^{\dot{\xi}}$, $\xi = \mathrm{I,II}$, something which would ultimately involve the non-sensical operation , say

$$(\ \ )\Big|_{\bar{\theta}^{\dot{\xi}}=0}$$

while keeping $\theta^{\xi} \neq 0$. *Thus the need for ^{s}complex Minkowski superspace , in which θ^{ξ} and $\bar{\theta}^{\dot{\xi}}$ are indeed independent complex coordinates , not any more the mere complex conjugates of each other .*

The construction of ^{s}complex Minkowski superspace , plus its canonical decomposition in both the null-plane and the usual orthonormal coordinates , and the way the canonical superfield variables of a superspace field theory are then identified , are discussed in chapter § 3 of this thesis .

Chapter § 2 , on the other hand , concerns the systematic construction , for space-time signatures $s\text{-}t = 3 \bmod 4$, of what has been named as the holomorphic SuSy algebras . These correspond to a new type of supersymmetry algebras , an example of which is the 3 dimensional Euclidean " spatial " supersymmetry algebra that results from the canonical decomposition of Minkowski superspace .

The discussion here makes extensive use of the theory of real Clifford algebras , for which a comprehensive set of references is compiled in [12] .

The universal real Clifford algebra $\mathfrak{R}_{s,t}$ associated with spacetime signature (s,t) is a real associative algebra with identity **1** . $\mathfrak{R}_{s,t}$ has dimension 2^{d} , where $d = s+t$, and an orthonormal set of generators e_m , $m = 1,2,...,d$ satisfies the fundamental relations

$$\{e_m, e_n\} = 2\,\eta_{mn}\,\mathbf{1}$$

where η_{mn} is diagonal with signature (s,t) .

The Clifford algebra $\mathfrak{R}_{s,t}$ comes equipped with a copy of the original orthogonal space $\mathbb{R}^{s,t}$. This is a particular subspace of the Clifford algebra which we refer to as the *space of vectors* $\mathbf{X}$. Other crucial subspaces of $\mathfrak{R}_{s,t}$ are its minimal left ideals Θ , which we refer to as the *spaces of spinors* . In this respect , some general notes will perhaps be helpful .

Consider a real associative algebra $\mathcal{A}$. A left ideal $\mathcal{L}$ is any proper subspace of $\mathcal{A}$ which is overall invariant under the left multiplication by elements in $\mathcal{A}$,

$$\mathcal{A}\,\mathcal{L} \subset \mathcal{L} \quad .$$

Left ideals are thus natural representation spaces for the associative algebra $\mathcal{A}$, acting via left multiplication . We shall denote by $\tau_{\mathcal{L}}$ the representation of $\mathcal{A}$ carried by the left ideal $\mathcal{L}$. Naturally , the representation $\tau_{\mathcal{L}}$ is irreducible whenever $\mathcal{L}$ is minimal , i.e. contains no proper subspace which is itself a left ideal in $\mathcal{A}$.

An associative algebra $\mathcal{A}$ is simple if it has no proper two-sided ideals . Simple associative algebras of finite dimension have the remarkable property of having only one irreducible representation , modulo equivalence . This essentially unique irreducible representation of a simple associative algebra $\mathcal{A}$ must therefore coincide , again modulo equivalence , with its irreducible representation $\tau_{\mathcal{L}}$ on a arbitrary minimal left ideal $\mathcal{L}$.

According to the nature of their irreducible representations τ_{Θ} on the minimal left ideals Θ , the real Clifford algebras $\mathfrak{R}_{s,t}$ divide into three main categories :

i) Signatures **s-t = 0,1,2 mod8**

These are the so-called Majorana signatures, when the Dirac matrices admit a Majorana or real representation.

The irreducible representation τ_Θ covers the whole of End Θ, i.e. the real associative algebra of all linear maps on Θ. The commutator of the representation, that is the subalgebra of End Θ which commutes with the entire representation, consists solely of the multiples of the identity map $\mathbf{I}$, and it is thus isomorphic to the real numbers. As a result, $\mathfrak{R}_{s,t}$ is realized as the real matrix algebra $\mathbb{R}(D)$, where $D = 2^{[d/2]}$ is the real dimension of the minimal left ideals Θ. For this reason, these signatures are also referred to as the real structure signatures.

ii) Signatures **s-t = 3,7 mod8**, or **s-t = 3 mod4**

The commutator of the irreducible representation τ_Θ, which in this case does not cover the whole of End Θ, is isomorphic to the complex numbers. Indeed, the Clifford algebra element

$$e_\iota \equiv e_1 e_2 \ldots e_d \quad , \quad (e_\iota)^2 = -1$$

commutes with every other element in $\mathfrak{R}_{s,t}$ and the linear map on Θ which represents e_ι acting via left multiplication, denoted $\Gamma_\iota \equiv \tau_\Theta(e_\iota)$, plays the role of imaginary unit in the commutator of the representation. In this case, thus, $\mathfrak{R}_{s,t}$ is realized as the complex matrix algebra $\mathbb{C}(D/_2)$, where $D/_2 = 2^{[d/2]}$ is the complex dimension of the minimal left ideals Θ. For this reason, these signatures are also referred to as the complex structure signatures.

iii) Signatures **s-t = 4,5,6 mod8**

The commutator of the irreducible representation τ_Θ, which again does not cover the whole of End Θ, is now isomorphic to the quaternions. As a result, $\mathfrak{R}_{s,t}$ is realized as the quaternionic matrix algebra $\mathbb{Q}(D/_4)$, where $D/_4 = 2^{[d/2]-1}$ is the quaternionic dimension of the minimal left ideals Θ . For this reason , these signatures are also referred to as the quaternionic structure signatures .

Comment : In signatures $s\text{-}t = 1 \bmod 4$ the Clifford algebra $\Re_{s,t}$ is not simple and e_ι is represented by $\pm\mathbf{I}$. As a result , there will be two , instead of one , equivalence classes of irreducible representations of $\Re_{s,t}$, corresponding to those two possibilities . In other words , there will be two inequivalent types of minimal left ideals Θ .

We hope to have provided , with the preceding notes , a useful rough idea of the very rich structure of real Clifford algebras , together with an introduction to the kind of technical jargon that is used . With that in mind , we return to the contents of chapter 2 .

There , a constructive scheme is introduced through which appropriate types of spinor scalar products , depending on the spacetime signature (s,t) , are used to characterize both real and holomorphic SuSy algebras , making use of the underlying structure provided by the real Clifford algebra $\Re_{s,t}$.

Central to the construction are the real valued " Majorana " spinor scalar products ς , which constitute a generalization for arbitrary signature of the scalar product of Majorana spinors , defined only when $s\text{-}t = 0,1,2 \bmod 8$. *Each available Majorana spinor scalar product naturally characterizes the symmetric product law of a real Lie superalgebra , which is referred to as the real SuSy algebra associated with* ς . This real SuSy algebra is defined over the vectors and spinors in $\Re_{s,t}$, that is , over $X \oplus \Theta$, and the symmetric product law in question is of the form

$$\{\theta\,\phi\} \;:=\; -\,2\,\varsigma(\theta,\Gamma^m\phi)\,e_m$$

where θ and ϕ are spinors in a minimal left ideal Θ and the $\Gamma_m \equiv \tau_\Theta(e_m)$ represent the Clifford algebra generators e_m , acting on Θ via left multiplication .

One other crucial element in the constructive scheme we introduce is the idea of a *complex structure on a real vector space* . This has in fact already been mentioned , though only implicitly , when commenting on the complex matrix realization of the Clifford algebras in signatures $s\text{-}t = 3 \bmod 4$. It can be understood as follows :

Let $\mathcal{V}$ denote a real vector space of dimension n and let $\mathbf{T} \subset \mathrm{End}\,\mathcal{V}$ be some particular subalgebra of the full real associative algebra of linear maps on $\mathcal{V}$. Tipically , $\mathbf{T}$ consists of a class of linear transformations on $\mathcal{V}$ which are of special interest in some particular context . Suppose , then , that there exists a non-singular linear map on $\mathcal{V}$, denoted $\mathfrak{I}$, which squares to minus the identity map $\mathbf{I}$ and , moreover , which commutes with all linear maps in $\mathbf{T}$,

$$\mathfrak{I}^2 = -\mathbf{I}$$

$$\mathfrak{I}\,\mathcal{T} = \mathcal{T}\,\mathfrak{I} \quad , \text{ for all } \mathcal{T} \in \mathbf{T} \quad .$$

We say that $\mathfrak{I}$ induces a complex structure on the real vector space $\mathcal{V}$, relatively to $\mathbf{T}$.

The justification for such nomenclature is that , indeed , one can use $\mathfrak{I}$ to define a " multiplication by ι " on vectors in $\mathcal{V}$,

$$\iota v := \mathfrak{I}\, v \quad , \text{ for all } v \in \mathcal{V} \quad .$$

And since $\mathfrak{I}$ commutes with all $\mathcal{T}$ in $\mathbf{T}$,

$$\mathcal{T}(\iota v) = \iota(\mathcal{T} v)$$

as one would expect of a multiplication by ι .

Once the real vector space $\mathcal{V}$ has been equipped with a complex structure , it can also be regarded as a complex vector space. However, not all of the original $\mathbb{R}$-linear maps on $\mathcal{V}$ will necessarily be $\mathbb{C}$-linear. Among those who will, of course, are the elements of $\mathbf{T}$. An $\mathbb{R}$-linear map on $\mathcal{V}$ is also $\mathbb{C}$-linear if and only if it commutes with the complex structure map $\mathfrak{I}$.

A complex basis for $\mathcal{V}$ is a minimal set of elements in terms of which all vectors in $\mathcal{V}$ can be written as a linear combination, with complex coefficients. Naturally, a complex basis for $\mathcal{V}$ will have only $n/_2$ elements. A choice of a complex basis $\{f_\xi\}$ for $\mathcal{V}$ associates to each of its $\mathbb{C}$-linear maps, and only those, an $n/_2 \times n/_2$ complex matrix in the usual way: the ξ^{th} column of the matrix associated to $\mathcal{T}$ consists of the complex number coefficients in the complex basis expansion of $\mathcal{T} f_\xi$.

A particular instance in which such concepts are of relevance is when the original real vector space is itself an associative algebra $\mathcal{A}$, a real Clifford algebra for example. There one is naturally interested in those linear maps on $\mathcal{A}$ which correspond to the left multiplication by an element u in $\mathcal{A}$,

$$\mathcal{T}_u v := u v \quad , \text{for all } \; v \in \mathcal{A} \; .$$

In this case, the algebra of all such $\mathcal{T}_u$ plays the role of $\mathbf{T}$ and, if there exists an element u_ι in $\mathcal{A}$ for which

$$u_\iota^{\,2} = -\mathbf{1} \quad \text{and} \quad u_\iota v = v\, u_\iota \quad ,$$

then the non-singular linear map $\mathcal{T}_{u_\iota}$ plays the role of $\mathfrak{I}$.

This is precisely what happens with the real Clifford algebras $\mathfrak{R}_{s,t}$ associated with spacetime signatures $s\text{-}t = 3 \bmod 4$. There, the ordered product of all the Clifford algebra generators $e_\iota \equiv e_1 e_2 \ldots e_d$ has the required properties above, in that it commutes with all other elements in $\Re_{s,t}$ and squares to minus the identity , $(e_\iota)^2 = -\mathbf{1}$. Therefore , in signatures $s\text{-}t = 3 \bmod 4$ the Clifford algebra $\Re_{s,t}$ has a natural complex structure , induced by e_ι . Correspondingly , the minimal left ideals Θ also have a complex structure , induced by Γ_ι , relatively to the left action of $\Re_{s,t}$.

This complex structure , which is available to the real Clifford algebra in spacetime signatures $s\text{-}t = 3 \bmod 4$ *, can sometimes be incorporated in the Majorana spinor scalar products* ς *and in their associated real SuSy algebras* . Indeed , some of the Majorana spinor scalar products in signatures $s\text{-}t = 3 \bmod 4$ admit an extension to complex valued holomorphic spinor scalar products , bilinear with respect to the complex structure and of the form $\varsigma_h = \varsigma + \iota\varsigma'$. The original Majorana spinor scalar product ς not only remains the real part of its holomorphic extension ς_h , but also uniquely determines the imaginary part ς' from the requirement that ς_h be bilinear with respect to the multiplication by ι .

The SuSy algebra naturally associated with the holomorphic spinor scalar product ς_h corresponds itself to a holomorphic extension of the original real SuSy algebra associated with ς . Its product laws are bilinear with respect to the complex structure and , as a result , *the holomorphic SuSy algebra can be regarded as an algebra over the complex numbers* .

A significant consequence of this is that the holomorphic SuSy algebra can then be described , via a choice of a complex basis , with half the number of supersymmetry " charges " as the original real SuSy algebra , a fact which is of crucial importance in the context of the canonical decomposition of Minkowski superspace .

Appendix to § 1

In this appendix we provide a summary of some basic aspects of the usual or real superspace in an arbitrary spacetime signature , with emphasis on the interrelated properties of complex conjugation and differential calculus over Grassmann variables . The systematic use of a real spinor representation , a type of generalized Majorana representation available in every spacetime signature , makes our presentation of real superspace somewhat unconventional but has the merit of being better adapted to both the construction of the canonical decomposition of Minkowski superspace and the general characterization of holomorphic SuSy algebras , which are the subject of this thesis .

The brief notes that follow are by no means a review of real superspace , for many review texts are already available in the literature . The intention here is essentially that of establishing a common language , at both conceptual and practical levels , before engaging in the description of this or that particular model or result pertinent to the development of canonical methods in superspace . At the same time , we have tried to present a compact sample of the technical jargon , notation , conventions and even some of the main themes that we shall develop later . Emphasis is put on the clarity of the statements and general arguments are either replaced or accompanied by simple illustrative examples .

Introductory notes on real superspace , our notation and conventions

1 . The spacetime signature is denoted (s,t) meaning (s " plus " signs , t " minus " signs) and the spacetime dimension is $d=s+t$. We associate " spatial " directions with " plus " signs in the metric signature and " temporal " directions with " minus " signs . Therefore , a d dimensional Minkowski signature is of the form $(s=d-1, t=1)$ and a d dimensional Euclidean signature is of the form $(s=d, t=0)$. The two outstanding examples are , of course , the 4d Minkowski signature $(3,1)$, that is $(+1,+1,+1,-1)$ and the 3d Euclidean signature $(3,0)$, that is $(+1,+1,+1)$.

Note : In the text we shall frequently refer to these two particular signatures simply as Minkowski (for 4 d Minkowski) and Euclidean (for 3 d Euclidean) .

$\aleph$ is an infinite dimensional complex Grassmann algebra equipped with an anti-involutive complex conjugation denoted $(\ )^*$. Tipically , x and θ will denote elements of $\aleph$ which are respectively even and odd : $x\,x' = x'\,x$, $\theta\,\theta' = -\,\theta'\,\theta$, $x\,\theta = \theta\,x$. In these introductory notes , moreover , x and θ will mostly be taken real . Note that due to the anti-involutive character of the Grassmann algebra complex conjugation , the product of real θ and θ' turns out to be imaginary : $(\theta\,\theta')^* = (\theta')^*\,(\theta)^* = \theta'\,\theta = -\,\theta\,\theta'$.

Besides appearing as even and odd Grassmann algebra elements , x and θ have a handful of other roles in this thesis . In this respect , the interpretation of the notation follows two basic principles : 1. characters x , y always denote vector-like objects , with components x^m , y^n , while characters θ , ϕ always denote spinor-like objects , with components θ^α , ϕ^β ; 2. objects written in italic have ordinary numerical components while objects denoted by plain characters have Grassmann algebra valued components .

Here is a brief notational guide : In this introductory chapter , x^m and θ^α denote the real Grassmann coordinates of real superspace . In § 2 , we use x, y for elements of the orthogonal space $\mathbb{R}^{s,t}$ and x, y for the associated vectors in the real Clifford algebra $\mathfrak{R}_{s,t}$. In the same way , θ, ϕ are used for spinors in $\mathfrak{R}_{s,t}$. All these objects have real numerical components , denoted x^m , y^n and θ^α, ϕ^β in the case of Clifford algebra vectors and spinors . In § 3 , on the other hand , x^m and θ^α_M denote the real , first , and later soul-complex Grassmann coordinates of Minkowski superspace in the Majorana representation .

2 . The usual or real N=1 superspace $\mathbb{S}^{s,t}_R$ associated with spacetime signature (s,t) is described by real (even,odd) Grassmann coordinates (x^m,θ^α) , with $m = 1,2,\ldots,d$

and $\alpha = 1,2,\ldots,D$. The range of vector and spinor indices depends upon the spacetime signature (s,t) as

$$\boxed{d = s+t \qquad \text{and} \qquad D = \begin{cases} 2^{[d/2]} & \text{when } s-t = 0,1,2 \bmod 8 \\ 2^{[d/2]+1} & \text{otherwise} \end{cases}} \tag{1.A.1}$$

where $[d/2]$ denotes the integer part of $d/2$. Thus , for instance , the real Minkowski superspace $\mathbb{S}_{R}^{3,1}$ is described by four real even Grassmann coordinates x^m , $m = 0,1,2,3$ and four real odd Grassmann coordinates θ^α , $\alpha = 1,2,3,4$.

Another example of interest is the real superspace $\mathbb{S}_{R}^{0,1}$ of supersymmetric quantum mechanics - the original spacetime has one " temporal " coordinate only . $\mathbb{S}_{R}^{0,1}$ is described by one real even Grassmann coordinate $x^0 \equiv t$ and two real odd Grassmann coordinates θ^α , $\alpha = 1,2$. Note that the same coordinate structure appears in the N=2 version of $\mathbb{S}_{R}^{1,0}$.

3 . Superfields F are Grassmann algebra valued functions over superspace , that is $F(x^m,\theta^\alpha) \in \aleph$. In this work , superfields are always assumed to be smooth , in the sense of Grassmann differentiability - see for instance [4 , De Witt] . Superfields are also generally taken to be either even or odd , which is indicated as $(-1)^F = \pm 1$. Due to the requirement of Grassmann differentiability plus the nilpotence of the coordinates θ^α , superfields present a characteristic power expansion , affine in each of the θ^α , whose coefficients or component fields are smooth Grassmann algebra valued functions of the even superspace coordinates x^m only .

In order to illustrate the general features of differentiation , integration and complex conjugation of superfields without unnecessary generalities , let us consider the simplest real superspace of all , $\mathbb{S}_{R}^{1,0}$, described by the pair of real Grassmann coordinates (x,θ) .

Superfields on $\mathbb{S}_R^{1,0}$ have the component expansion

$$F(x,\theta) = A(x) + \theta\, B(x) = A(x) + (-1)^B B(x)\, \theta \tag{1.A.2}$$

where the parities of superfield and component fields relate as $(-1)^F = (-1)^A = -(-1)^B$.

Generally speaking , the differential and integral calculus over even real Grassmann variables reduces almost entirely to that of ordinary real variables . This is discussed in detail in § 3.2 . The case of odd real Grassmann variables , on the other hand , is much less familiar . In these notes , we follow closely the approach in [4 , De Witt] .

In each case Grassmann differentiation is defined through

$$\delta_x F = \delta x\, (\partial_x F) \qquad\qquad \delta_\theta F = \delta\theta\, (\partial_\theta F) \ . \tag{1.A.3}$$

It follows that , while the even differential operator ∂_x acts componentwise on the superfield F and satisfies the usual Leibnitz rule ,

$$\partial_x F = \partial_x A + \theta\, \partial_x B \qquad\qquad \partial_x (FG) = (\partial_x F) G + F (\partial_x G) \ , \tag{1.A.4}$$

the odd differential operator ∂_θ , in turn , extracts the θ component of the superfield F and satisfies a graded Leibnitz rule ,

$$\partial_\theta F = B \qquad\qquad \partial_\theta (FG) = (\partial_\theta F) G + (-1)^F F (\partial_\theta G) \ . \tag{1.A.5}$$

Both the differential operators ∂_x and ∂_θ satisfy the standard chain rule ,

$$\partial_{z'} F = J^{-1}\, \partial_z F \qquad , \text{ where } \ J \equiv \frac{\partial z'}{\partial z} \tag{1.A.6}$$

where z stands for either x or θ. Note that the Jacobian matrix $\mathbf{J}$ is even in each case.

Integration with respect to an even real Grassmann variable x reduces as well to componentwise integration, which can be shown to be contour independent,

$$\int_{x_1}^{x_2} F\,dx = \int_{x_1}^{x_2} A\,dx + \theta \int_{x_1}^{x_2} B\,dx \qquad \left(\begin{array}{c}\text{for consistency, dx}\\ \text{must be manipulated}\\ \text{as an even quantity}\end{array}\right) . \tag{1.A.7}$$

In the case of an odd real Grassmann variable θ, only indefinite Integration is defined. It is essentially an algebraic procedure based upon the formal rules

$$\int 1\,d\theta = 0 \qquad \int \theta\,d\theta = 1 \tag{1.A.8a}$$

from which follows that

$$\int F\,d\theta = -(-1)^F\,\partial_\theta F \qquad \left(\begin{array}{c}\text{for consistency, } d\theta\\ \text{must be manipulated}\\ \text{as an odd quantity}\end{array}\right) . \tag{1.A.8b}$$

The requirement that the formal integration rules (1.A.8a) should be preserved under a change of variable $\theta \to \theta'$, implies a non-standard transformation rule $d\theta \to d\theta'$ for the corresponding integration "measure". In fact, while in the case of an even Grassmann variable x we have the standard transformation rule

$$dx' = (\det \mathbf{J})\,dx \qquad \text{or} \qquad \int_{x_1}^{x_2} F\,(\det \mathbf{J})\,dx = \int_{x_1'}^{x_2'} F\,dx' \quad , \tag{1.A.9}$$

in the case of an odd Grassmann variable θ,

$$d\theta' = (\det \mathbf{J})^{-1}\,d\theta \qquad \text{or} \qquad \int F\,(\det \mathbf{J})^{-1}\,d\theta = \int F\,d\theta' \quad . \tag{1.A.10}$$

As an example, consider the change of variable $\theta' = c\,\theta$, where c is some invertible even Grassmann constant. Then $J = c$ and

$$\int \theta'\, d\theta' = \int (c\,\theta)\; c^{-1}\, d\theta = \int \theta\, d\theta = 1 \quad . \tag{1.A.11}$$

Though the superspace variables (x,θ) are real, superfields and component fields may be complex valued. The Grassmann algebra anti-involutive complex conjugation $(\ )^*$ induces in the usual way a complex conjugation $(\overline{\ \ })$ of Grassmann algebra valued functions and differential operators. In the first case, complex conjugation acts as

$$\bar{F}(x,\theta) = \bar{A}(x) + \bar{B}(x)\,\theta = \bar{A}(x) + (-1)^B\,\theta\,\bar{B}(x) \quad . \tag{1.A.12}$$

The complex conjugation of differential operators is perhaps less obvious. Its properties follow from the basic conventional rules

$$(\overline{\partial_x F}) = (\overline{\partial_x})\,\bar{F} \qquad\qquad (\overline{\partial_\theta F}) = (-1)^F\,(\overline{\partial_\theta})\,\bar{F} \tag{1.A.13}$$

which reflect the even and odd nature of ∂_x and ∂_θ respectively, together with the anti-involutive character of the Grassmann complex conjugation. Since differentiation with respect to the real even variable x commutes with complex conjugation, it follows from (1.A.13) that the even differential operator ∂_x is real. On the other hand, since

$$(\overline{\partial_\theta F}) = \bar{B} \;\text{ from (1.A.2)} \quad \text{and} \quad \partial_\theta \bar{F} = -(-1)^F\,\bar{B} \;\text{ from (1.A.12)} \;,$$

the conventional rule (1.A.13) implies that the odd differential operator ∂_θ is imaginary, even though θ itself is real,

$$(\overline{\partial_x}) = \partial_x \qquad\qquad (\overline{\partial_\theta}) = -\,\partial_\theta \quad . \tag{1.A.14}$$

4 . The $N=1$ real superspace $\mathbb{S}_R^{s,t}$ associated with spacetime signature (s,t) is further characterized by the vector fields

$$P_m \equiv -\partial_m \qquad \text{and} \qquad Q_\alpha \equiv -(\partial_\alpha + i\,(\varsigma\Gamma^m)_{\alpha\beta}\,\theta^\beta\,\partial_m)$$

$$\text{with} \qquad (\overline{P_m}) = P_m \quad , \quad (\overline{Q_\alpha}) = -Q_\alpha \tag{1.A.15}$$

and their real supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = -2i\,(\varsigma\,\Gamma^m)_{\alpha\beta}\,P_m \quad , \tag{1.A.16}$$

where the Γ_m are $D\times D$ real matrices satisfying (see also 8.)

$$\{\Gamma_m, \Gamma_n\} = 2\eta_{mn}\mathbf{I} \quad , \tag{1.A.17}$$

η_{mn} is diagonal with signature (s,t) and ς is a non-singular $D\times D$ real matrix , either symmetric or antisymmetric , and such that all the matrix products $(\varsigma\,\Gamma_m)$ are symmetric ,

$$\varsigma^T = \kappa\,\varsigma \ , \ \kappa=\pm1 \qquad \text{and} \qquad (\varsigma\,\Gamma_m)^T = \varsigma\,\Gamma_m \ . \tag{1.A.18}$$

Note : The fact that the general form of $N=1$ real supersymmetry algebras is indeed as (1.A.16) will be justified in § 2.4 plus , for computational details , the Appendix to § 2 .

The real superspace $\mathbb{S}_R^{0,1}$ of supersymmetric quantum mechanics is once more convenient for the illustration purposes . It corresponds to the 1 d Minkowski signature (0,1) , which implies that $D=2$ and $(\Gamma_0)^2 = -\mathbf{I}$. Accordingly , we take

$$\Gamma_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -\Gamma^0 \ . \tag{1.A.19}$$

As with all Minkowskian signatures , the conventional choice for ς relatively to a set of orthogonal matrices Γ_m is $\varsigma = \Gamma_0$. From the algebraic point of view of (1.A.18) , though , this is just one among several possible choices for ς . In the case of $\mathbb{S}_R^{0,1}$,

$$\varsigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{or} \quad \varsigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \text{ for instance ,}$$

would equally validate (1.A.18) . Behind the conventional choice $\varsigma = \Gamma_0$ lies a physical reason : in a d dimensional Minkowski signature , the matrix products $\Gamma_0 \Gamma_j$, with $j = 1,2,\ldots,d-1$, are traceless , since $\Gamma_0 \Gamma_j = - \Gamma_j (\Gamma_0 \Gamma_j) \Gamma_j$ and $(\Gamma_j)^2 = + \mathbf{I}$. Thus , taking the trace of (1.A.16) with $\varsigma = \Gamma_0$, one obtains

$$Q_\alpha Q_\alpha = - i D P_0 = D \mathcal{E} \tag{1.A.20}$$

where $\mathcal{E} \equiv i P^0 = i \partial_0$. Upon quantization , the imaginary vector field $\mathcal{E}$ becomes the hermitian Energy operator and , in much the same way , the imaginary vector fields Q_α are themselves associated with hermitian operators . Consequently , (1.A.20) expresses the Energy eigenvalues as a sum of squares of the real eigenvalues of the operators Q_α . Naturally , this means that the Energy eigenvalues are necessarily non-negative in supersymmetric field theories over d dimensional Minkowski spacetime , provided $\varsigma = \Gamma_0$!

Returning to the discussion of $\mathbb{S}_R^{0,1}$, with Γ_0 as in (1.A.19) and $\varsigma = \Gamma_0$. The real superspace coordinates are $(x^0 \equiv t , \theta^1 , \theta^2)$ and the vector fields (1.A.15) are explicitly

$$P_t = - \partial_t \quad , \quad Q_1 = - (\partial_1 + i \theta^1 \partial_t) \quad , \quad Q_2 = - (\partial_2 + i \theta^2 \partial_t) \quad . \tag{1.A.21}$$

The real supersymmetry algebra anticommutators (1.A.16) are then

$$\{Q_1,Q_1\} = -2i\,P_t = \{Q_2,Q_2\} \qquad \{Q_1,Q_2\} = 0 \ . \tag{1.A.22}$$

This precisely coincides with the $N = 2$ version (without central charges) of the real supersymmetry algebra (1.A.16) associated with $\mathbb{S}_R^{1,0}$, as we have already mentioned. We note, however, that this is so due to the choice $\varsigma = \Gamma_0$. Different choices for ς would entail real supersymmetries algebras for $\mathbb{S}_R^{0,1}$ other than (1.A.22).

§ . Consider a general superfield $F(x^m,\theta^\alpha)$ over $\mathbb{S}_R^{s,t}$ and a general infinitesimal superspace diffeomorphism δ,

$$\delta \ : \ p \approx (x^m,\theta^\alpha) \ \rightarrow \ p' = p + \delta p \approx (x^m + \delta x^m , \theta^\alpha + \delta\theta^\alpha) \tag{1.A.23}$$

where the necessarily real infinitesimal displacements δx^m and $\delta\theta^\alpha$ are, in general, smooth local functions of the real superspace coordinates x^m and θ^α. (Strictly speaking, the diffeomorphism δ is one in which the defining finite displacements δx^m and $\delta\theta^\alpha$ have both been scaled down by an infinitesimal numerical factor λ).

The diffeomorphism δ induces an infinitesimal (scalar) superfield transformation

$$\delta \ : \ F \ \rightarrow \ F' = F + \delta F \qquad \text{according to} \quad F'(p') = F(p) \quad . \tag{1.A.24}$$

This allows us to relate the original superfield F and its transformed F' at the same point p of superspace, since

$$\begin{aligned} F'(p') &= F'(x^m + \delta x^m , \theta^\alpha + \delta\theta^\alpha) = \\ &= F'(p) + \delta x^m \partial_m F'(p) + \delta\theta^\alpha \partial_\alpha F'(p) + O(\delta^2) = \\ &= F'(p) + \delta x^m \partial_m F(p) + \delta\theta^\alpha \partial_\alpha F(p) + O(\delta^2) \ . \end{aligned} \tag{1.A.25}$$

There follows that

$$\delta F(p) = F'(p) - F(p) = -(\delta x^m \partial_m + \delta\theta^\alpha \partial_\alpha) F(p) \qquad (1.A.26)$$

where the even vector field $-(\delta x^m \partial_m + \delta\theta^\alpha \partial_\alpha)$ is referred to as the generator of the diffeomorphism δ. Note that, according to the conventional rule (1.A.13) for complex conjugation, real superspace diffeomorphisms $(\delta x^m, \delta\theta^\alpha)$ are necessarily generated by real even vector fields, since

$$\begin{aligned} \overline{(\delta x^m \partial_m F)} &= (\overline{\partial_m F})(\overline{\delta x^m}) = (\partial_m \bar{F})\, \delta x^m = \delta x^m \partial_m \bar{F} \\ \overline{(\delta\theta^\alpha \partial_\alpha F)} &= (\overline{\partial_\alpha F})(\overline{\delta\theta^\alpha}) = (-\partial_\alpha \bar{F})\, \delta\theta^\alpha = \delta\theta^\alpha \partial_\alpha \bar{F} \end{aligned} \qquad (1.A.27)$$

where we have assumed F to be even, for simplicity.

The two most basic diffeomorphisms of the real superspace $\mathbb{S}^{s,t}_{\mathbb{R}}$ are the

translations and the **supersymmetry transfs.**

$$\begin{aligned} &\delta x^m = u^m \;,\; \delta\theta^\alpha = 0 && \delta x^m = i\,\varepsilon^\alpha (\varsigma\Gamma^m)_{\alpha\beta}\,\theta^\beta \;,\; \delta\theta^\alpha = \varepsilon^\alpha \\ &\therefore\; \delta F = (u^m P_m) F && \therefore\; \delta F = (\varepsilon^\alpha Q_\alpha) F \end{aligned} \qquad (1.A.28)$$

with infinitesimal global real parameters u^m (even) and ε^α (odd), respectively.

There follow the Lorentz transformations of $\mathbb{S}^{s,t}_{\mathbb{R}}$, generated from the real vector fields

$$J_{mn} = x_m \partial_n - x_n \partial_m - (\Sigma_{mn})^\alpha{}_\beta\, \theta^\beta \partial_\alpha \;,\quad \text{where}\quad \Sigma_{mn} = {}^1\!/_4\, [\Gamma_m, \Gamma_n] \;,$$

Lorentz transfs.

$$\delta x^m = v^m{}_n x^n \quad , \quad \delta\theta^\alpha = \tfrac{1}{2} v^{mn} (\Sigma_{mn})^\alpha{}_\beta\, \theta^\beta$$
$$\therefore \quad \delta F = (\tfrac{1}{2} v^{mn} J_{mn})\, F \qquad (1.A.29)$$

with infinitesimal global antisymmetric real parameters v^{mn} (even) .

Altogether , the global diffeomorphisms (1.A.28 , 29) constitute the superPoincaré algebra of transformations of real superspace , a Lie algebra generated by the real even vector fields of the form

$$u^m P_m + \varepsilon^\alpha Q_\alpha + \tfrac{1}{2} v^{mn} J_{mn} \qquad (1.A.30)$$

and whose properties follow from

$$\boxed{\begin{aligned} &[P_m, P_n] = [P_m, Q_\alpha] = 0 \\ &\{Q_\alpha, Q_\beta\} = -2i\, (\varsigma\Gamma^m)_{\alpha\beta} P_m \\ &[J_{mn}, P_q] = \eta_{nq} P_m - \eta_{mq} P_n \\ &[J_{mn}, Q_\alpha] = (\Sigma_{mn})^\beta{}_\alpha\, Q_\beta \\ &[J_{mn}, J_{pq}] = \eta_{mq} J_{np} + \eta_{np} J_{mq} - \eta_{mp} J_{nq} - \eta_{nq} J_{mp} \ . \end{aligned}} \qquad (1.A.31)$$

According to our definitions , the commutation relations $[J_{mn}, P_q]$ and $[J_{mn}, J_{pq}]$ have precisely the same form as the matrix commutators $[\Sigma_{mn}, \Gamma_q]$ and $[\Sigma_{mn}, \Sigma_{pq}]$.

To appreciate the effect of supersymmetry transformations at the level of component fields let us once more consider the elementary case of $\mathbb{S}^{1,0}_R$, see (1.A.2) . The real superspace coordinates are (x,θ) and the global supersymmetries $\delta x = i\varepsilon\theta$, $\delta\theta = \varepsilon$, specified by an infinitesimal real parameter ε, are generated by the vector fields

$$\varepsilon Q = -\varepsilon(\partial_\theta + i\theta\partial_x) \quad ; \quad \text{therefore, since} \quad F = A + \theta B ,$$
$$\delta F = (\varepsilon Q) F = -\varepsilon(B + i\theta\partial_x A) \tag{1.A.32}$$

which means that, at component field level, the supersymmetry transformation rules are

$$\delta A = -\varepsilon B \quad , \quad \delta B = i\varepsilon\partial_x A \quad . \tag{1.A.33}$$

𝕾 . The covariant derivatives of the real superspace $\mathbb{S}_R^{s,t}$, relatively to the translation and supersymmetry generators P_m and Q_α, consist of the vector fields

$$\partial_m \quad \text{and} \quad D_\alpha \equiv -(\partial_\alpha - i(\varsigma\Gamma^m)_{\alpha\beta}\theta^\beta\partial_m)$$
$$\text{with} \quad \overline{(\partial_m)} = \partial_m \quad , \quad \overline{(D_\alpha)} = -D_\alpha \tag{1.A.34}$$

whose anticommutator algebra

$$\{D_\alpha, D_\beta\} = -2i(\varsigma\Gamma^m)_{\alpha\beta}\partial_m \tag{1.A.35}$$

coincides with the real supersymmetry algebra (1.A.16) with each P_m and Q_α replaced by the corresponding covariant derivative.

The derivative operators ∂_m and D_α are covariant in the sense that they anticommute with the translation and supersymmetry generators P_m and Q_α. In the case of the spinor covariant derivatives D_α, for instance, one can easily verify that

$$\{D_\alpha, Q_\beta\} = 0 \quad . \tag{1.A.36}$$

At component field level, the covariance of the D_α means that the supersymmetry representation carried by $D_\alpha F$ is a simple derivation of the basic supersymmetry representation carried by the superfield F .

To illustrate this point, take the elementary case of $\mathbb{S}_R^{1,0}$ continued from (1.A.32) and (1.A.33). There, the spinor covariant derivative D is explicitly

$$D = -(\partial_\theta - i\theta\partial_x) \qquad \text{and, since} \qquad F = A + \theta B ,$$

$$DF = (-B) + \theta(i\partial_x A) \quad . \text{ Therefore,} \tag{1.A.37}$$

$$\delta(DF) = (\varepsilon Q)\, DF = (-i\varepsilon\partial_x A) + \theta(-i\varepsilon\partial_x B) \quad .$$

Thus, at component field level, the supersymmetry representation carried by DF is

$$\delta B = i\varepsilon\partial_x A \quad , \quad \delta(\partial_x A) = -\varepsilon\partial_x B \tag{1.A.38}$$

which is clearly derivated from the basic transformation rules (1.A.33).

7 . Real superspace action integrals have the general form

$$\mathfrak{S} = \int \mathcal{L}\; d^d x\, d^D\theta$$

$$\text{where} \quad d^d x \equiv dx^d \ldots dx^2 dx^1 \quad , \tag{1.A.39}$$

$$d^D\theta \equiv \begin{cases} d\theta^D d\theta^{D-1} \ldots d\theta^2 d\theta^1 & \text{when } D = 0,1 \bmod 4 \\ i\, d\theta^D d\theta^{D-1} \ldots d\theta^2 d\theta^1 & \text{when } D = 2,3 \bmod 4 \end{cases}$$

and the real even Lagrangian density $\mathcal{L}$ is constructed from the basic superfields of the theory plus the two covariant derivative operators ∂_m and D_α . Note that, in natural units, each odd superspace coordinate θ has mass dimension $-\frac{1}{2}$ and thus both ∂_θ and $d\theta$ have mass dimension $\frac{1}{2}$.

It is common practice in the description of superfield theory models to make use of complex recombinations of the real superspace coordinates θ^α and their associated vector fields Q_α . In other words, one deals with a complex representation of the real superspace in question, instead of the defining real representation (x^m,θ^α) . In the case of $\mathbb{S}^{0,1}_{\mathbb{R}}$, see (1.A.21, 22), the conventional complex representation $(x^0 \equiv t, \theta, \bar\theta)$ is

$$\theta \equiv \frac{1}{\sqrt{2}}(\theta^1 + i\theta^2) \qquad \bar\theta = \frac{1}{\sqrt{2}}(\theta^1 - i\theta^2)$$

$$\text{and thus} \tag{1.A.40}$$

$$\partial_\theta = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2) \qquad \partial_{\bar\theta} = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2) = -(\overline{\partial_\theta}) \ .$$

Accordingly, the spinor covariant derivatives are chosen as

$$D \equiv \frac{1}{\sqrt{2}}(D_1 - iD_2) \qquad \bar D = -\frac{1}{\sqrt{2}}(D_1 + iD_2) \tag{1.A.41}$$

$$\text{so that} \qquad \theta^\alpha D_\alpha = \theta D - \bar\theta \bar D \ ,$$

plus identical relations for the supersymmetry generators $Q_1, Q_2 \to Q, \bar Q$. Therefore,

$$D = -(\partial_\theta - i\bar\theta\partial_t) \qquad \bar D = \partial_{\bar\theta} - i\theta\partial_t \ ,$$

$$\tag{1.A.42}$$

$$Q = -(\partial_\theta + i\bar\theta\partial_t) \qquad \bar Q = \partial_{\bar\theta} + i\theta\partial_t \ .$$

The Jacobian matrix J of the coordinate change $\theta_1, \theta_2 \to \theta, \bar\theta$ consists of

$$\mathrm{J} = \begin{bmatrix} \frac{\partial\theta}{\partial\theta^1} & \frac{\partial\theta}{\partial\theta^2} \\ \frac{\partial\bar{\theta}}{\partial\theta^1} & \frac{\partial\bar{\theta}}{\partial\theta^2} \end{bmatrix} \quad , \ \det \mathrm{J} = -i \quad \text{and accordingly}$$

$$d^2\theta \equiv i\, d\theta^2 d\theta^1 = (\det \mathrm{J})^{-1}\, d\theta^2 d\theta^1 = d\bar{\theta}\, d\theta \quad \text{so that} \tag{1.A.43}$$

$$\int (\theta\,\bar{\theta})\; d\bar{\theta}\, d\theta = \int (-i\theta^1\theta^2)\; i\,d\theta^2 d\theta^1 = 1 \quad .$$

As a result, an action integral over $\mathbb{S}_{R}^{0,1}$ is expressed as

$$\mathfrak{S} = \int \mathcal{L}\; dt\, d^2\theta = \int \mathcal{L}\; i\, dt\, d\theta^2 d\theta^1 = \int \mathcal{L}\; dt\, d\bar{\theta}\, d\theta \quad . \tag{1.A.44}$$

$\mathbb{S}$. If one chooses to think of the Γ_m as being real Dirac matrices - which they strictly are only in the Majorana signatures $s\text{-}t = 0,1,2 \bmod 8$ - then ς plays the role of a real charge conjugation matrix . There exist , nevertheless , significant differences between the $D \times D$ real matrices Γ_m and the conventional $[d/_2] \times [d/_2]$ complex Dirac matrices γ_m :

While the Γ_m are necessarily linearly independent , they do not always generate the full $D \times D$ real matrix algebra . This happens only in the Majorana signatures $s\text{-}t = 0,1,2 \bmod 8$. Otherwise , the completion of the generating set consists of one ($s\text{-}t = 3,7 \bmod 8$) or two ($s\text{-}t = 4,5,6 \bmod 8$, see **Note**) extra matrices $\hat{\Gamma}$ which , together with the original Γ_m , correspond to the matrices Γ_M of the nearest higher dimensional Majorana signature ,

$$\Gamma_M : \Gamma_m , \hat{\Gamma} \qquad , \qquad \{\Gamma_M , \Gamma_N\} = 2\eta_{MN}\mathbf{I} \quad . \tag{1.A.45}$$

Consider , for instance , the signature ($s=0,t=1$) of type $s\text{-}t = 7 \bmod 8$. The nearest higher dimensional Majorana signature is ($s=1,t=1$) of type $s\text{-}t = 0 \bmod 8$. The extra $\hat{\Gamma}$ is thus of " spatial " character , that is $(\hat{\Gamma})^2 = +\mathbf{I}$.

Another interesting example is the 4 d Euclidean signature (s=4,t=0) of type s-t = 4 mod8 . The nearest higher dimensional Majorana signature is (s=4,t=2) of type s-t = 2 mod8 . The two extra $\hat{\Gamma}$ are thus of " temporal " character , that is $(\hat{\Gamma}_1)^2 = (\hat{\Gamma}_2)^2 = -\mathbf{I}$.

Note : In signatures s-t = 1 mod4 , or equivalently s-t = 1,5 mod8 , the product sequence $\Gamma_\iota \equiv \Gamma_1 \Gamma_2 ... \Gamma_d$ reduces to $\pm\mathbf{I}$. In other words , any one of the Γ_m coincides with the product of all the others , modulo a sign . In the case of signatures s-t = 5 mod8 , thus , one of the Γ_m must be left out and the completion of the generating set as in (1.A.45) made in relation to the remaining ones . Naturally , the nature of the two extra $\hat{\Gamma}$ will depend on these and thus on which particular Γ_m is left out . In summary , the procedure extracts from the original signature s-t = 5 mod8 a lower dimensional signature of type s-t = 4,6 mod8 and then looks for the nearest higher dimensional Majorana signature relatively to the latter .

The matrices Γ_m can always be chosen orthogonal or " standard " . The same holds for the eventual extra matrices $\hat{\Gamma}$. In Table I : s-t = 0,1,2 mod8 and Table II : s-t = 3,7 mod8 , presented in § 2.3 , we compute the matrices ς associated to an arbitrary set of standard matrices Γ_m . The remaining signature group s-t = 4,5,6 mod8 is left out because it is computationally more involved and of no immediate relevance to the constructions of holomorphic SuSy algebras and canonical superspace , which are the subject of this thesis .

§ 2 Holomorphic SuSy Algebras

§ 2.1 *Introduction and Preview*

To our knowledge, the terminology " holomorphic supersymmetry " has first been used in a recent overview [11] of supersymmetry algebras in flat spacetime of arbitrary signature (s,t). There, it was observed that the usual N=1 real supersymmetry algebras do sometimes admit a " holomorphic plus anti-holomorphic decomposition " in so far as one is able to find appropriate linear combinations of the original supersymmetry generators which trivialize the anticommutators $\{\bar{Q}\,Q\}$ and transfer their content to the holomorphic and anti-holomorphic anticommutators $\{Q\,Q\}$ and $\{\bar{Q}\,\bar{Q}\}$, related by complex conjugation.

This can be shown to occur for a variety of signatures, including ones associated with real (s-t = 2 mod8), complex (s-t = 3 mod8) and quaternionic (s-t = 4,5,6 mod8) structures on the real Clifford algebra $\Re_{s,t}$.

A particular example of such a decomposition which has emerged independently in work by the author on the development of Canonical Superspace [1] occurs in the 3 dimensional Euclidean case (s = 3, t = 0). The associated N=1 real supersymmetry algebra has the conventional form

$$\{\bar{Q}_{\dot{a}}, Q_b\} = i\,(A\,\sigma^j)_{\dot{a}b}\,P_j \quad , \quad j = 1,2,3 \quad , \quad a,b = 1,2$$

where $(A)_{\dot{a}b} = \mathbb{1}$, $(\sigma_j)^a{}_b$ are the Pauli matrices, $(\overline{})$ denotes complex conjugation and $\bar{P}_j = P_j$. If, however, we rearrange the supersymmetry generators as

$$\begin{aligned} Q_I &\equiv Q_1 + i\,\bar{Q}_{\dot{2}} \quad , \quad \bar{Q}_{\dot{I}} = \bar{Q}_{\dot{1}} - i\,Q_2 \\ Q_{II} &\equiv Q_2 - i\,\bar{Q}_{\dot{1}} \quad , \quad \bar{Q}_{\dot{II}} = \bar{Q}_{\dot{2}} + i\,Q_1 \end{aligned}$$

we obtain the holomorphic plus anti-holomorphic decomposition

$$\{\bar{Q}_{\dot{\xi}}, Q_{\zeta}\} = 0 \quad , \quad \xi,\zeta = I,II \tag{2.1.1a}$$

and

$$\{Q_{\xi}, Q_{\zeta}\} = -\,2i\,(\varsigma\,\sigma^{j})_{\xi\zeta}\,P_j \tag{2.1.1b}$$

$$\{\bar{Q}_{\dot{\xi}}, \bar{Q}_{\dot{\zeta}}\} = -\,2i\,(\bar{\varsigma}\,\bar{\sigma}^{j})_{\dot{\xi}\dot{\zeta}}\,P_j \quad , \tag{2.1.1c}$$

where $\varsigma_{\xi\zeta} \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $(\sigma_j)^{\xi}{}_{\zeta}$ are the Pauli matrices .

The term decomposition is suggestive but somewhat misleading , as it does not refer to an actual direct sum decomposition of the real supersymmetry algebra . This is the <u>real</u> linear span of the translation generators P (even sector) and the supersymmetry generators $Q , \bar{Q}$ (odd sector) and thus does not contain any purely holomorphic or anti-holomorphic elements , meaning linear combinations of only the Q_{ξ} or the $\bar{Q}_{\dot{\xi}}$.

The odd sector of a real supersymmetry algebra is therefore not a holomorphic plus anti-holomorphic direct sum in an essential way . Besides , the even sector does not decompose correspondingly in any sense .

The correct reference of the term decomposition is that the product laws of the real supersymmetry algebra can be fully described using only half the original number of supersymmetry generators , either just the Q_{ξ} or the $\bar{Q}_{\dot{\xi}}$. From this point of view , the occurrence of the holomorphic plus anti-holomorphic decomposition suggests the presence of a complex structure in the real supersymmetry algebra .

This suggestion is strengthened by the realization that the 3 dimensional Euclidean signature $(s = 3, t = 0)$ is actually a particular case of the $s\text{-}t = 3 \bmod 4$ signatures , for which the associated real Clifford algebras $\mathfrak{R}_{s,t}$ have a natural complex structure . Since the real SuSy algebras - the abstract models of the real supersymmetry algebras - can all be constructed within these , it is plausible that in signatures $s-t = 3 \bmod 4$ they naturally inherit the complex structure present in the underlying real Clifford algebras .

Indeed , it is the subject of the present chapter to show that the real SuSy algebras associated with signatures $s-t = 3 \bmod 4$ can in some cases be made to absorb the available complex structure in $\mathfrak{R}_{s,t}$ via an extension of their even sector which , when it exists , is unique . The product laws of the resulting holomorphic SuSy algebras can then be described with a complex basis consisting , at the actual supersymmetry algebra level , of all the original P_j but only either the Q_ξ or the $\bar{Q}_{\dot{\xi}}$.

We note , however , that our discussion of holomorphic supersymmetry algebras is restricted to signatures $s-t = 3 \bmod 4 = s-t = 3,7 \bmod 4$ and is thus not presented as providing an exhaustive understanding of the precise mechanisms underlying the holomorphic plus anti-holomorphic decomposition of [11] . It should nevertheless provide some insight into what those mechanisms are since ultimately , we believe , they must involve some kind of complex structure reduction , just as does our scheme .

The material in this chapter § 2 is organized as follows :

§ 2.2 consists of a concise and somewhat informal review of those basic constructions in real Clifford algebra theory which will be frequently used thereafter . The geometric or intrinsic aspects are gratifyingly simple and effective in structuring the main ideas in this work , and we will emphasize them together with the more computational or basis related ones . For our purposes , references [12] have been particularly useful .

§ 2.3 is devoted to the definition and computation of a class of real valued spinor scalar products ς , which we refer to as Majorana spinor scalar products . In signatures $s\text{-}t = 3 \bmod 4$, some of these admit a natural extension to complex valued spinor scalar products ς_h which are holomorphic , that is , bilinear with respect to the complex structure present in the underlying real Clifford algebra $\mathfrak{R}_{s,t}$.

In the following section , § 2.4 , we discuss the way in which real and holomorphic SuSy algebras can be associated to each available Majorana and holomorphic spinor scalar products , respectively . The constructive scheme we utilize differs significantly from the one originally proposed in [14] , being simpler in formal terms and computationally better adapted to the characterization of the holomorphic extension process .

§ 2.4 begins thus with the definition and properties of the real SuSy algebra associated with a Majorana spinor scalar product ς . Then , assuming that this ς admits holomorphic extension , we construct in an analogous way the holomorphic SuSy algebra associated with the holomorphic spinor scalar product ς_h and show that it corresponds to a holomorphic extension of the original real SuSy algebra associated to ς , that is , an extension which has incorporated the complex structure available in the underlying real Clifford algebra .

Finally , in § 2.5 , we present the detailed application of the holomorphic extension scheme to the 3 dimensional Euclidean case .

§ 2.2 *Real Clifford Algebras*

Notationwise... : (s " plus " signs , t " minus " signs) and $d = s+t$ stand for the signature and dimension of spacetime . Latin characters $m,n,p,q = 1,2,...,d$ are used for vector indices and greek characters $\alpha,\beta = 1,2,...,D$ are used for spinor indices ; D depends on the signature (s,t) as in (2.2.15) . The commutator [,] and anticommutator { , } products , with a comma , are the ones built from the underlying associative product in the usual way .

Let $\mathfrak{R}_{s,t}$ denote the real Clifford algebra for the orthogonal space $\mathbb{R}^{s,t} \equiv (\mathbb{R}^d , \eta)$, where $d=s+t$ and η is the canonical non-degenerate symmetric bilinear form on $\mathbb{R}^d$ with signature (s,t) . $\mathfrak{R}_{s,t}$ is a real 2^d dimensional associative algebra with identity $\mathbf{1}$ and comes equipped with an injective linear map

$$\rho \; : \; \mathbb{R}^{s,t} \to \mathfrak{R}_{s,t} \quad , \quad x \to \mathbf{x} \equiv \rho(x)$$

such that

$$\{\mathbf{x},\mathbf{y}\} \;=\; 2\,\eta(x,y)\,\mathbf{1} \tag{2.2.1}$$

and

$$\mathbf{X} \equiv \mathrm{Im}\,\rho \quad \text{generates the whole of } \mathfrak{R}_{s,t} \; .$$

The d dimensional subspace $\mathbf{X} \subset \mathfrak{R}_{s,t}$ is thus , through the map ρ , a copy of the orthogonal space $\mathbb{R}^{s,t}$. In particular , $\mathbf{X}$ inherits from $\mathbb{R}^{s,t}$ the metric structure η ,

$$\eta(\mathbf{x},\mathbf{y}) \;:=\; \eta(x,y) \quad .$$

Accordingly, $\mathbf{X}$ is referred to as the space of vectors $\mathbf{x},\mathbf{y} \in \mathbf{X}$. In turn , the space of real multiples of the identity $\mathbb{R}\,\mathbf{1} \subset \mathfrak{R}_{s,t}$ is referred to as the space of scalars .

Let $\{e_m , m = 1,...,d\}$ denote an orthonormal basis for $\mathbf{X}$, that is one for which the metric matrix elements $\eta_{mn} \equiv \eta(e_m,e_n)$ are diagonal and unimodular with signature (s,t) .

Then , the Clifford algebra structure relations (2.2.1) take the usual form

$$\{e_m,e_n\} = 2\,\eta_{mn}\,1 \quad . \tag{2.2.2}$$

The set B of the 2^d different irreducible ordered products of the generators e_m ,

$$B \equiv \{\ (e_1)^{k_1}\,(e_2)^{k_2} \ldots (e_d)^{k_d}\ ,\ \text{with}\ k_m = 0,1\ \} \quad , \qquad \text{where} \quad (e_m)^0 \equiv 1 \quad \text{and} \quad (e_m)^1 \equiv e_m \quad , \tag{2.2.3}$$

forms a basis for the Clifford algebra $\Re_{s,t}$. In particular , the basis set B includes the identity 1 and the longest irreducible ordered product $e_\iota \equiv e_1e_2...e_d$. This is used to construct the 1 and d dimensional subspaces $\mathbb{R}e_\iota$ and $e_\iota\mathbf{X}$, which are respectively referred to as the spaces of pseudoscalars and pseudovectors .

Let us now return for a moment to the orthogonal space of vectors **X** and examine more closely some features of its inherited metric structure η , namely the notion of dual basis to $\{e_m\}$ and the basic characterization of both group and algebra isometries of **X** :

Since η is non-degenerate , there exists in **X** a unique basis $\{e^m\}$ dual to $\{e_m\}$,

$$\text{in the sense that} \qquad \eta(e^m,e_n) \;=\; \delta^m{}_n \quad . \tag{2.2.4}$$

This dual basis $\{e^m\}$ is , of course , within the linear span of the original $\{e_m\}$. Thus , writing $e^m = h^{mn}\,e_n$ for some h^{mn} , implies

$$\eta^{mn} \;\equiv\; \eta(e^m,e^n) \;=\; h^{mp}\,\eta(e_p,e^n) \;=\; h^{mp}\,\delta_p{}^n \;=\; h^{mn}$$

$$\text{and thus} \qquad e^m = \eta^{mn}\,e_n \quad .$$

A similar argument leads to $e_m = \eta_{mn} e^n$ and , as a result , $\eta^{mp} \eta_{pn} = \delta^m{}_n$.

The orthogonal group $\mathbf{O}_{s,t}$ is the $\frac{1}{2}d(d-1)$ dimensional Lie group of non-singular linear maps

$$\Lambda \in \mathbf{O}_{s,t} : \mathbf{X} \to \mathbf{X} \qquad \text{such that}$$

$$\eta(\Lambda x, \Lambda x) = \eta(x,x) \quad \overset{x \to x+y}{\Leftrightarrow} \quad \eta(\Lambda x, \Lambda y) = \eta(x,y) \quad . \tag{2.2.5}$$

With respect to an orthonormal basis $\{e_m\}$ for $\mathbf{X}$, the matrix elements $\Lambda^m{}_n$ of the Lie group isometries Λ are defined by $\Lambda(e_n) = \Lambda^m{}_n e_m$. In a similar way , the matrix elements $\Lambda_m{}^n$ of the same Lie group isometries Λ with respect to the dual basis $\{e^m\}$ are defined by $\Lambda(e^n) = \Lambda_m{}^n e^m$. They satisfy

$$\eta(e^m, \Lambda e_n) = \eta(\Lambda^{-1} e^m, e_n) \qquad \text{and thus} \qquad \Lambda^m{}_n = (\Lambda^{-1})_n{}^m \quad . \tag{2.2.6}$$

The orthogonal algebra $\mathcal{O}_{s,t}$ is the $\frac{1}{2}d(d-1)$ dimensional Lie algebra of linear maps

$$L \in \mathcal{O}_{s,t} : \mathbf{X} \to \mathbf{X} \qquad \text{such that}$$

$$\eta(x, Lx) = 0 \quad \overset{x \to x+y}{\Leftrightarrow} \quad \eta(x, Ly) + \eta(Lx, y) = 0 \quad . \tag{2.2.7}$$

One can easily verify that if L_1 and L_2 are orthogonal algebra maps , then so is their Lie bracket commutator $[L_1, L_2] = L_1 L_2 - L_2 L_1$. The Jacobi identities follow automatically from this definition of the Lie bracket .

With respect to the orthonormal bases $\{e_m\}_{\text{domain}}$ and $\{e^m\}_{\text{counter-domain}}$, the matrix elements L_{mn} of the Lie algebra isometries L are defined by $L(e_n) = L_{mn} e^m$. They satisfy

$$\eta(e_m, Le_n) + \eta(Le_m, e_n) = 0 \qquad \text{and thus} \qquad L_{mn} = -L_{nm} \ . \tag{2.2.8}$$

The linear span of **[X,X]**, denoted $\Omega \subset \mathfrak{R}_{s,t}$, is a $\frac{1}{2}d(d-1)$ dimensional subspace of the Clifford algebra. Ω is referred to as the space of bivectors $\omega, w \in \Omega$. Moreover, the $\frac{1}{2}d(d-1)$ independent bivectors $\sigma_{mn} \equiv \frac{1}{4}[e_m, e_n]$ form a basis for Ω.

The space of bivectors Ω plays a crucial role within the Clifford algebra structure. Consider, to begin with, the commutation relations

$$[\sigma_{mn}, \sigma_{pq}] = \eta_{mq}\sigma_{np} + \eta_{np}\sigma_{mq} - \eta_{mp}\sigma_{nq} - \eta_{nq}\sigma_{mp} \ . \tag{2.2.9}$$

They mean that Ω, equipped with the commutator product [,], has the structure of a Lie algebra. Furthermore, Ω acts naturally on the space of vectors **X** through the linear maps

$$L_{\sigma_{mn}}(e_p) := [\sigma_{mn}, e_p] = \eta_{np}e_m - \eta_{mp}e_n$$
$$\text{for which} \qquad \eta(e_p, L_{\sigma_{mn}}e_p) = 0 \ . \tag{2.2.10}$$

The linear map $L_{\sigma_{mn}}$ associated to the bivector σ_{mn} is thus an orthogonal algebra isometry of **X**, that is $L_{\sigma_{mn}} \in O_{s,t}$. Moreover, the $\frac{1}{2}d(d-1)$ independent linear maps $L_{\sigma_{mn}}$ are linearly independent and satisfy commutation relations identical to (2.2.9). They must therefore span the $\frac{1}{2}d(d-1)$ dimensional orthogonal algebra $O_{s,t}$, just as the σ_{mn} span the space of bivectors Ω.

In summary, what we have constructed through (2.2.10) amounts to a Lie algebra isomorphism between the space of bivectors Ω and the orthogonal algebra $O_{s,t}$ of linear maps on **X** satisfying (2.2.7),

$$\text{bivector } \omega \in \Omega \;\leftrightarrow\; \text{linear map } L_\omega \in O_{s,t}$$

$$L_\omega(x) := [\omega, x] \qquad \text{satisfying} \tag{2.2.11}$$

$$\eta(x, L_\omega x) = 0 \;.$$

Due to universality, the linear maps $x \in \mathbf{X} \to \pm x$ extend uniquely to involutions or anti-involutions of the whole Clifford algebra $\mathfrak{R}_{s,t}$. Operationally, it suffices to understand how these involutions and anti-involutions act on the ordered products of generators e_m in the Clifford algebra basis B : involution - replaces each e_m by $\pm e_m$; anti-involution - the same, followed by an inversion of the product order.

The Clifford algebra has a natural $\mathbb{Z}_2$ grading $\mathfrak{R}_{s,t} = \mathfrak{R}^+_{s,t} \oplus \mathfrak{R}^-_{s,t}$ induced by the so-called main involution

$$(\tilde{\ }) \;:\; u \in \mathfrak{R}_{s,t} \to \tilde{u} \qquad \text{extended from} \quad e_m \to -e_m . \tag{2.2.12}$$

Also of importance are the two anti-involutions $\beta_\pm$, so-called reversion and conjugation,

$$\beta_\pm \;:\; u \in \mathfrak{R}_{s,t} \to \beta_\pm(u) \qquad \text{extended from} \quad e_m \to \pm e_m . \tag{2.2.13}$$

Examples : Consider the real Clifford algebra $\mathfrak{R}_{1,1}$ and a generic element $u \in \mathfrak{R}_{1,1}$

$$u = u_\emptyset 1 + u_1 e_1 + u_2 e_2 + u_\iota e_\iota \qquad , \text{ where } \; e_\iota \equiv e_1 e_2 \;\; ; \;\; \text{then},$$

$$\tilde{u} = u_\emptyset 1 - u_1 e_1 - u_2 e_2 + u_\iota e_\iota \;,$$

$$\beta_+(u) = u_\emptyset 1 + u_1 e_1 + u_2 e_2 - u_\iota e_\iota \;,$$

$$\beta_-(u) = u_\emptyset 1 - u_1 e_1 - u_2 e_2 - u_\iota e_\iota \;.$$

The non-isotropic vectors $\tau \in \mathbf{X}$, $\eta(\tau,\tau) = \tau^2 \neq 0$, generate a multiplicative sub-group of the Clifford algebra called the Clifford group $\Gamma_{s,t}$. Any of its elements $s \in \Gamma_{s,t}$ is expressible , not neccessarily uniquely . as a finite product of non-isotropic vectors τ_k ,

$$s = \Pi \, \tau_k \qquad , \quad (\tau_k)^2 = \eta(\tau_k,\tau_k)\, 1 \neq 0 \ . \tag{2.2.14}$$

There is a natural way to associate a scalar valued norm to each element s of the Clifford group . Given $s \in \Gamma_{s,t}$, define its norm $N(s)$ to be

$$N(s) := \beta_+(s)\, s \ . \tag{2.2.15}$$

The norm N coincides with the original norm $\eta(\cdot,\cdot)\, 1$ over the non-isotropic vectors $\tau \in \mathbf{X}$ and extends it to all other elements of the Clifford group $\Gamma_{s,t}$, which they generate . Indeed , it follows from (2.2.14) that $N(s) = \Pi\, N(\tau_k) = \Pi\, (\tau_k)^2$ and thus the norm N over the Clifford group $\Gamma_{s,t}$ is both scalar valued and non-zero . It is actually a group homomorphism from $\Gamma_{s,t}$ to the multiplicative group of non-zero scalars ,

$$N(1) = 1 \quad , \quad N(s s') = N(s)\, N(s') \quad , \quad N(s^{-1}) = N(s)^{-1} \ . \tag{2.2.16}$$

Consider now a non-isotropic vector $\tau \in \mathbf{X}$ and the associated linear map

$$\Lambda_\tau : \mathbf{X} \to \mathbf{X} \quad , \quad \Lambda_\tau(x) := \tau\, x\, \tilde{\tau}^{-1} = x - 2\frac{\eta(x,\tau)}{\eta(\tau,\tau)}\, \tau$$

$$\text{for which} \qquad \eta(\Lambda_\tau x, \Lambda_\tau x) = \eta(x,x) \ , \tag{2.2.17}$$

where we have made use of $\tau x = - x \tau + 2\eta(x,\tau)\, 1$ and $\tau^{-1} = \frac{1}{\eta(\tau,\tau)}\, \tau$. Clearly , the linear map Λ_τ performs a reflection in the hyperplane orthogonal to τ : any vector $x \in \mathbf{X}$ can be written $x = x_0 + \lambda\, \tau$, where λ is a real number and $\eta(x_0,\tau) = 0$; as a result ,

$$\Lambda_\tau(x_0 + \lambda\tau) = \Lambda_\tau(x_0) + \lambda\Lambda_\tau(\tau) = x_0 - \lambda\tau \quad . \tag{2.2.18}$$

The linear map Λ_τ associated to the non-isotropic vector τ is thus an orthogonal group isometry of $\mathbf{X}$, that is $\Lambda_\tau \in \mathbf{O}_{s,t}$. On the other hand, every orthogonal group isometry Λ in $\mathbf{O}_{s,t}$ is the composite of a finite number of hyperplane reflections Λ_{τ_k} - see for instance [12, Porteous p. 256]. In other words, every Λ in $\mathbf{O}_{s,t}$ can be regarded as the orthogonal group isometry $\Lambda_s \equiv \Pi\, \Lambda_{\tau_k}$ associated to an element $s = \Pi\, \tau_k$ of the Clifford group $\Gamma_{s,t}$.

In summary, what we have constructed through (2.2.17) amounts to a Lie group homomorphism from the Clifford group $\Gamma_{s,t}$ to the orthogonal group $\mathbf{O}_{s,t}$ of linear maps on $\mathbf{X}$ satisfying (2.2.5),

$$\begin{gathered} \text{Clifford group element } s \in \Gamma_{s,t} \;\rightarrow\; \text{linear map } \Lambda_s \in \mathbf{O}_{s,t} \\ \Lambda_s(x) := s x \tilde{s}^{-1} \qquad \text{satisfying} \\ \eta(\Lambda_s x, \Lambda_s x) = \eta(x,x) \quad . \end{gathered} \tag{2.2.19}$$

Notice that we have mentioned "homomorphism" and not "isomorphism" as in the Lie algebra case (2.2.11). In fact, all Clifford group elements differing by a non-zero real number λ engendre the same orthogonal group map, $\Lambda_{\lambda s} = \Lambda_s$. That this is all that can happen follows from the fact that

$$\text{if } \; ux = x\tilde{u} \; \text{ for all vectors } x \in \mathbf{X}, \text{ then } \; u \text{ is a scalar} \quad . \tag{2.2.20}$$

In face of this situation, one immediately thinks of imposing some sort of normalization condition on the Clifford group elements s in order to make the correspondence (2.2.19) as univocal as possible. This can be achieved straightforwardly with the help of the norm N defined in (2.2.15). Indeed, it follows from $N(\lambda s) = \lambda^2 N(s)$ that each class of Clifford group elements related by a non-zero real factor $\{\lambda s\}$ can be reduced consistently to the two elements $\pm s_0$ which have unimodular norm , $N(s_0) = N(-s_0) = \pm 1$.

The group $\mathbf{Pin}_{s,t}$ is defined to be the (normal) subgroup of $\Gamma_{s,t}$ consisting of those elements $s \in \Gamma_{s,t}$ of unimodular norm , $N(s) = \pm 1$. It follows that the Lie group homomorphism $s \in \mathbf{Pin}_{s,t} \rightarrow \Lambda_s \in \mathbf{O}_{s,t}$ has kernel $\{\pm 1\}$ and constitutes a double covering of the orthogonal group $\mathbf{O}_{s,t}$ by $\mathbf{Pin}_{s,t}$. This classical result in real Clifford algebra theory is what lies at the origin of the notion of spinors and spinor representations .

In a similar way , the group $\mathbf{Spin}_{s,t} \equiv \mathbf{Pin}_{s,t} \cap \Re^+_{s,t}$ double covers the special orthogonal group $\mathbf{SO}_{s,t}$. The connected component of $\mathbf{Spin}_{s,t}$, denoted $\mathbf{Spin}^0_{s,t}$, coincides with $\exp(\Omega)$, the group of exponentials of bivectors $\exp(\omega) \equiv \sum_{k=0}^{\infty} \omega^k/_{k!}$.

The unit norm subgroup of $\mathbf{Spin}_{s,t}$, denoted $\mathbf{Spin}^{\hat{0}}_{s,t}$,

$$\mathbf{Spin}^{\hat{0}}_{s,t} \equiv \{ s \in \mathbf{Spin}_{s,t} \ : \ N(s) = 1 \}$$

double covers the connected compoent of the special orthogonal group , $\mathbf{SO}^0_{s,t}$. Apart from the few lower dimensional cases listed below , the unit norm subgroup $\mathbf{Spin}^{\hat{0}}_{s,t}$ coincides with the connected component $\mathbf{Spin}^0_{s,t}$,

$$\mathbf{Spin}^{\hat{0}}_{0,0} = \mathbf{Spin}^{\hat{0}}_{1,0} = \mathbf{Spin}^{\hat{0}}_{0,1} = \{\pm 1\}$$

$$\mathbf{Spin}^{\hat{0}}_{1,1} = \{ a\,1 + b\,\mathrm{e}_\iota \text{ , with } a,b \in \mathbb{R} \text{ and } a = \pm\sqrt{1+b^2} \} \ . \tag{2.2.21}$$

Consider now the Euclidean signatures (s=d,t=0) : for $d \geq 2$ both $\mathbf{SO}_d$ and $\mathbf{Spin}_d$ are connected and $\mathbf{Spin}_d$ is simply-connected for $d \geq 3$. The same is true in the case of the inverse Euclidean signatures (s=0,t=d) . For proofs and further properties of the **Spin** groups we refer the reader to [12 , Porteous p. 257,268,427 and Atiyah , Bott & Shapiro p. 9] in particular .

Note : Consider the two real Clifford algebras $\mathfrak{R}_{s,t}$ and $\mathfrak{R}_{t,s}$, related by a signature inversion , with orthonormal generators e_m and e'_m . Both the even sectors $\mathfrak{R}^+_{s,t}$ and $\mathfrak{R}^+_{t,s}$ are themselves real Clifford algebras , the bivectors σ_{mn} and σ'_{mn} playing the role of orthonormal generators . Now suppose , without loss of generality , that the relative location of plus and minus signs in the diagonal metric matrices η_{mn} and η'_{mn} has been chosen in such a way that $(e_m)^2 = -(e'_m)^2$. It is then clear that $(\sigma_{mn})^2 = (\sigma'_{mn})^2$ and , as a result , we conclude that $\mathfrak{R}^+_{s,t}$ and $\mathfrak{R}^+_{t,s}$ are isomorphic . Accordingly , the even groups $\mathbf{Spin}_{s,t}$ and $\mathbf{Spin}_{t,s}$ are equally isomorphic ,

$$\mathfrak{R}^+_{s,t} \approx \mathfrak{R}^+_{t,s} \qquad \text{and} \qquad \mathbf{Spin}_{s,t} \approx \mathbf{Spin}_{t,s} \quad .$$

This is not so , however , in the case of the full real Clifford algebras $\mathfrak{R}_{s,t} \neq \mathfrak{R}_{t,s}$, as well as in the case of $\mathbf{Pin}_{s,t} \neq \mathbf{Pin}_{t,s}$.

The minimal left ideals Θ of the Clifford algebra $\mathfrak{R}_{s,t}$ all have the same dimension

$$D = \begin{cases} 2^{[d/2]} & \text{when } s\text{-}t = 0,1,2 \bmod 8 \\ 2^{[d/2]+1} & \text{otherwise} \end{cases} \quad . \tag{2.2.22}$$

A minimal left ideal Θ is referred to as a space of spinors $\theta, \phi \in \Theta$. Via left multiplication , each spinor space Θ constitutes a natural irreducible representation space for $\mathfrak{R}_{s,t}$. These irreducible representations of the Clifford algebra , denoted τ_Θ , are all faithful and equivalent when $\mathfrak{R}_{s,t}$ is simple , i.e. when $s\text{-}t \neq 1 \bmod 4$. Otherwise they are two-to-one (although still faithful over $\mathfrak{R}^{+}_{s,t}$) and divide into two equivalence classes , depending on whether e_ι is represented by $\Gamma_\iota = \pm \mathbf{I}$, plus or minus the identity transformation on Θ .

In the above spinor representations τ_Θ of the Clifford algebra $\mathfrak{R}_{s,t}$ we denote by $\Gamma_m : \theta \in \Theta \rightarrow e_m \theta$ the representative of the generator e_m acting on Θ via left multiplication . A basis $\{f_\alpha , \alpha = 1,...,D\}$ for Θ associates to Γ_m a $D{\times}D$ real matrix whose matrix elements $(\Gamma_m)^\alpha{}_\beta$ are defined by

$$\Gamma_m(f_\beta) = (\Gamma_m)^\alpha{}_\beta \, f_\alpha \quad , \tag{2.2.23}$$

that is , $(\Gamma_m)^\alpha{}_\beta$ is the component of $\Gamma_m(f_\beta)$ along f_α . We say that the basis $\{f_\alpha\}$ is standard if all the matrices Γ_m are orthogonal . Such standard bases can always be found , which is of significant practical importance .

We finish this section with a technical note on how spinor spaces $\Theta \subset \mathfrak{R}_{s,t}$ can be constructed from a given set of orthonormal generators e_m :

An idempotent $p \in \mathfrak{R}_{s,t}$, $p^2 = p$, is said to be primitive if it can not be writen as the sum of two mutually annihilating idempotents p_1 and p_2 , $p_1 p_2 = p_2 p_1 = 0$. And ...

<u>Fact</u> : if p is a primitive idempotent in $\mathfrak{R}_{s,t}$, then $\Theta \equiv (\mathfrak{R}_{s,t})p$ is a minimal left ideal .

Primitive idempotents , and thus their associated spinor spaces , can be obtained through the following technique : to every Clifford algebra element u such that $u^2 = +1$, we can associate an idempotent $p \equiv {}^1/_2(1 + u)$. Now , take B as in (2.2.3) and let $B_+ \subset B/_{\{1\}}$ consist of $(d - \log_2 D)$ mutually commuting basis elements which square to $+1$. Such subsets B_+ always exist (they are maximal when $\mathfrak{R}_{s,t}$ is simple) . Then ...

<u>Fact</u> : the product of the idempotents associated to the elements of B_+ is a primitive idempotent in $\mathfrak{R}_{s,t}$.

Summary : we have described the Clifford algebra $\mathfrak{R}_{s,t}$ and , within it ,

- the 1 dimensional space of scalars $\mathbb{R}\,\mathbf{1}$
- the d dimensional space of vectors $\mathbf{X}$
- the $\frac{1}{2}$d(d-1) dimensional space of bivectors Ω

  plus the " pseudo " version of each , constructed via left multiplication , say , with either one of the two unimodular pseudoscalars $\pm e_\iota$

- the D dimensional spaces of spinors Θ
- the Lie groups $\mathbf{Spin}^0_{s,t} \subset \mathbf{Spin}_{s,t} \subset \mathbf{Pin}_{s,t} \subset \Gamma_{s,t}$,

  with their vector and spinor representations

$$s \in \Gamma_{s,t} \rightarrow \left\langle \begin{array}{l} \Lambda_s : x \in \mathbf{X} \rightarrow x' \equiv s x \tilde{s}^{-1} \\ \\ S : \theta \in \Theta \rightarrow \theta' \equiv s\,\theta \end{array} \right. \qquad (2.2.24)$$

.

§ 2.3 *Majorana and Holomorphic Spinor Scalar Products*

Having developed the basic real Clifford algebra vocabulary we need , we can now pass to the characterization of the Majorana spinor scalar products and their holomorphic extension .

Let Θ be a spinor space in $\mathfrak{R}_{s,t}$. Spinor scalar products or ssp^s on Θ are non-degenerate bilinear forms which we shall take to be either real or complex valued .

A real valued ssp $\varsigma \, : \, (\theta,\phi) \;\rightarrow\; \varsigma(\theta,\phi) \in \mathbb{R}$ is said to be Majorana if it is either symmetric or antisymmetric

$$\varsigma(\theta,\phi) \;=\; \kappa\, \varsigma(\phi,\theta) \quad , \quad \kappa = \pm 1 \tag{2.3.1a}$$

and such that , for all vectors $x \in \mathbf{X}$, the bilinear form $\varsigma(\cdot\,,X\cdot)$ is symmetric ,

$$\varsigma(\theta,X\phi) \;=\; \varsigma(\phi,X\theta) \quad , \tag{2.3.1b}$$

where X represents the vector x acting on Θ via left multiplication .

Majorana ssp^s are objects intrinsic to the spinor space Θ . Nevertheless , it is convenient for computational purposes to consider them in matrix form and this involves , as usual , refering to some basis choice : a basis $\{f_\alpha\}$ for Θ associates to a Majorana ssp ς a real $\mathrm{D}{\times}\mathrm{D}$ non-singular matrix with $\varsigma_{\alpha\beta} \equiv \varsigma(f_\alpha,f_\beta)$ for which

$$\varsigma^T \;=\; \kappa\,\varsigma \quad , \; \kappa = \pm 1 \qquad \text{and} \qquad (\varsigma\,\Gamma_m)^T \;=\; \varsigma\,\Gamma_m \quad , \; m = 1,2,\ldots,d \;, \tag{2.3.2}$$

where , as before , the Γ_m represent an arbitrarily chosen set of orthonormal generators e_m acting on Θ via left multiplication .

Note : In signatures s-t = 0,1,2 mod8 the Γ_m constitute a Majorana representation for the Dirac matrices and thus ς plays the role of a real charge conjugation matrix .

A crucial property of Majorana ssp^s is their invariance under the action of $\mathbf{Spin}^0_{s,t}$ on the spinor space Θ . This means that the real valued Majorana scalar product $\varsigma(\theta,\phi)$ of any two spinors $\theta,\phi \in \Theta$ remains unchanged under the action of $\mathbf{Spin}^0_{s,t}$,

$$\varsigma(S\theta,S\phi) = \varsigma(\theta,\phi) \quad , \text{ for all } \; s \in \mathbf{Spin}^0_{s,t} \quad , \tag{2.3.3}$$

where , as usual , S represents the $\mathbf{Spin}^0_{s,t}$ group element s acting on Θ via left multiplication . As in the case of the Γ_m or , for that matter , of any linear map on Θ , a basis $\{f_\alpha\}$ for Θ associates to S a D×D real matrix whose matrix elements $S^\alpha{}_\beta$ are defined by $S(f_\beta) = S^\alpha{}_\beta \, f_\alpha$. To validate (2.3.3) , the $S^\alpha{}_\beta$ must satisfy

$$\varsigma(Sf_\alpha, Sf_\beta) = \varsigma(f_\alpha,f_\beta) \qquad \text{and thus} \qquad S^\gamma{}_\alpha \, \varsigma_{\gamma\delta} \, S^\delta{}_\beta = \varsigma_{\alpha\beta} \quad ,$$

$$\text{or , in matrix terms ,} \quad S^T \varsigma \, S = \varsigma \quad . \tag{2.3.4}$$

This can be proved as follows :

Let W represent the bivector w acting on Θ via left multiplication . In matrix terms , since W is of the form $W \approx \Gamma\Gamma$, we obtain from (2.3.2) that $W^T = - \varsigma \, W \, \varsigma^{-1}$. Thus , with $S \equiv \exp(W)$, there follows that $S^T = \varsigma \, S^{-1} \, \varsigma^{-1}$ or , equivalently , $S^T \varsigma \, S = \varsigma$.

In Tables I (s-t = 0,1,2 mod8) and II (s-t = 3,7 mod8) we list all the available Majorana ssp^s on an arbitrary spinor space Θ of $\Re_{s,t}$. Table I refers to the simpler case of the real structure " Majorana " signatures s-t = 0,1,2 mod8 , while Table II refers to those involved in our discussion of holomorphic SuSy algebras , the complex structure signatures s-t = 3 mod4 . In both tables an arbitrary choice of a <u>standard</u> basis $\{f_\alpha\}$ for Θ is understood and the matrices associated to the Γ_m are used to specify , modulo an arbitrary positive real normalization factor, the matrices associated to the symmetric and antisymmetric Majorana ssp[s] on Θ .

The computational scheme involved in the construction of Tables I and II is as follows :

Let Θ be a spinor space in $\mathfrak{R}_{s,t}$ and let Γ_+ (respectively Γ_-) denote any of the Γ_m which square to $+\mathbf{I}$ (respectively $-\mathbf{I}$). Furthermore, let Π_+ denote the ordered product of all the Γ_m which square to $+\mathbf{I}$, $\Pi_+ \equiv \Gamma_1 \ldots \Gamma_s$ (or $\Pi_+ \equiv \mathbf{I}$ if $s=0$) and let Π_- denote the ordered product of all the Γ_m which square to $-\mathbf{I}$, $\Pi_- \equiv \Gamma_{s+1} \ldots \Gamma_{s+t}$ (or $\Pi_- \equiv \mathbf{I}$ if $t=0$), so that $\Gamma_\iota = \Pi_+ \Pi_-$. In a standard basis $\{f_\alpha\}$ for Θ, in which the matrices associated to the Γ_m are orthogonal,

$$(\Pi_+)^T = \left\langle \begin{array}{ll} \Pi_+ & \text{when } s = 0,1 \bmod 4 \\ -\Pi_+ & \text{when } s = 2,3 \bmod 4 \end{array} \right. \quad , \quad (\Pi_-)^T = \left\langle \begin{array}{ll} \Pi_- & \text{when } t = 0,3 \bmod 4 \\ -\Pi_- & \text{when } t = 1,2 \bmod 4 \end{array} \right.$$

and the matrices associated to Majorana ssp[s] must satisfy

$$\begin{gathered} \varsigma^T = \kappa \varsigma \quad , \quad \kappa = \pm 1 \\ \varsigma \Gamma_+ = \kappa \Gamma_+ \varsigma \quad , \quad \varsigma \Gamma_- = -\kappa \Gamma_- \varsigma \qquad . \end{gathered} \tag{2.3.5}$$

We have then to search for such matrices ς :

Table I

In signatures $s\text{-}t = 0,1,2 \bmod 8$ the spinor representation of the Clifford algebra $\mathfrak{R}_{s,t}$ covers the whole of End Θ , the set of all linear maps on Θ , and therefore the matrices ς will necessarily lie within the matrix algebra generated by the Γ_m . Moreover, and since the ς matrices will have to either commute with every Γ_+ and anticommute with every

Γ_- ($\kappa = +1$) or vice-versa ($\kappa = -1$), the list of plausible candidates is restricted to Π_+ and Π_- (and linear combinations thereof). This list is further reduced in signatures s-t = 1 mod8 , when $\Gamma_\iota \sim I$, meaning they differ at most by a sign , and so $\Pi_+ \sim \Pi_-$.

The groups of four (s,t) specifications presented on the left of Tables I and II are given mod8
Each (s,t) specification has a twin consisting of (s+4,t+4)
Twins considered, each group corresponds to all possible (s,t) subcases of its s-t signature specification

s-t = 0 mod8

	$\kappa = +1$	$\kappa = -1$
(s = 0,t = 0)	$\pm\Pi_-$	---
(s = 1,t = 1)	$\pm\Pi_+$	$\pm\Pi_-$
(s = 2,t = 2)	---	$\pm\Pi_+$
(s = 3,t = 3)	---	---

s-t = 1 mod8

(s = 1,t = 0)	$\pm\Pi_+ \sim \pm\Pi_-$	---
(s = 2,t = 1)	---	$\pm\Pi_+ \sim \pm\Pi_-$
(s = 3,t = 2)	---	---
(s = 0,t = 3)	---	---

s-t = 2 mod8

(s = 2,t = 0)	$\pm\Pi_-$	$\pm\Pi_+$
(s = 3,t = 1)	---	$\pm\Pi_-$
(s = 0,t = 2)	---	---
(s = 1,t = 3)	$\pm\Pi_+$	---

Table II

In signatures s-t = 3,7 mod8 the spinor representation of the Clifford algebra $\mathfrak{R}_{s,t}$ does not cover the whole of End Θ and there always exists an <u>extra</u> real D×D orthogonal matrix $\hat{\Gamma}$,

$$\text{either } \hat{\Gamma}_+ \text{ with } (\hat{\Gamma}_+)^2 = +\mathbf{I}\text{ , when } s\text{-}t = 7 \text{ mod}8\text{ ,}$$

$$\text{or } \hat{\Gamma}_- \text{ with } (\hat{\Gamma}_-)^2 = -\mathbf{I}\text{ , when } s\text{-}t = 3 \text{ mod}8\text{ ,}$$

which anticommutes with all the Γ_m and , together with them , generates the full real D×D matrix algebra . This extra $\hat{\Gamma}$ is not unique ; indeed , if a particular matrix $\hat{\Gamma}$ has the properties above , then so do the matrices $(a\,\mathbf{I} + b\,\Gamma_\iota)\hat{\Gamma}$, with $a,b \in \mathbb{R}$, $a^2+b^2 = 1$. As far as the search for matrices ς is concerned , the effect of all this is to double the list of plausible candidates we had previously , by adding to it $\Pi_+\hat{\Gamma}$ and $\Pi_-\hat{\Gamma}$.

s-t = 3 mod8

	$\kappa = +1$	$\kappa = -1$
(s = 3,t = 0)	$\pm\Pi_-$	$(a\,\Pi_+ + b\,\Pi_-)\,\hat{\Gamma}_-$
(s = 0,t = 1)	---	$\pm\Pi_-$
(s = 1,t = 2)	$\pm\Pi_+$	---
(s = 2,t = 3)	$(a\,\Pi_+ + b\,\Pi_-)\,\hat{\Gamma}_-$	$\pm\Pi_+$

s-t = 7 mod8

(s = 0,t = 1)	$(a\,\Pi_+ + b\,\Pi_-)\,\hat{\Gamma}_+$	$\pm\Pi_-$
(s = 1,t = 2)	$\pm\Pi_+$	$(a\,\Pi_+ + b\,\Pi_-)\,\hat{\Gamma}_+$
(s = 2,t = 3)	---	$\pm\Pi_+$
(s = 3,t = 0)	$\pm\Pi_-$	---

We now wish to concentrate on signatures $s\text{-}t = 3 \bmod 4$, for which the unimodular pseudoscalars $\pm e_\iota$ not only commute with every element of the Clifford algebra $\Re_{s,t}$ but also have $(\pm e_\iota)^2 = -1$. In these signatures, then, the Clifford algebra has the natural complex structures

$$\iota u := \pm e_\iota u \quad , \quad u \in \Re_{s,t} \tag{2.3.6}$$

which we shall refer to as the holomorphic (+) and the anti-holomorphic (-) complex structures on $\Re_{s,t}$. (This is clearly not an intrinsic identification, since which complex structure is called holomorphic or anti-holomorphic depends on the choice of the e_m). In what follows we shall deal exclusively with the holomorphic complex structure, with the understanding that all constructions have analogous for the anti-holomorphic case.

Let Θ be a spinor space in $\Re_{s,t}$. Being a left ideal, it automatically inherits the complex structure on $\Re_{s,t}$. We say that a complex valued ssp $\varsigma_h : (\theta,\phi) \rightarrow \varsigma_h(\theta,\phi) \in \mathbb{C}$ is holomorphic if its real part is Majorana and the full ς_h is $\mathbb{C}$-bilinear with respect to the complex structure on Θ,

$$\begin{gathered} \varsigma_h = \varsigma + \iota\varsigma' \quad \text{where } \varsigma \text{ is Majorana} \\ \text{and} \\ \varsigma_h(\iota\theta,\phi) = \varsigma_h(\theta,\iota\phi) = \iota\varsigma_h(\theta,\phi) \quad . \end{gathered} \tag{2.3.7}$$

The holomorphic conditions (2.3.7) imply that

i) the Majorana ssp ς that forms the real part of a holomorphic ssp ς_h has to be such that Γ_ι is self-adjoint :

$$\begin{gathered} \varsigma(\Gamma_\iota\theta,\phi) = \varsigma(\theta,\Gamma_\iota\phi) \\ \text{or, in matrix terms,} \quad \Gamma_\iota^{\,T}\varsigma = \varsigma\Gamma_\iota \quad . \end{gathered} \tag{2.3.8}$$

ii) the imaginary part ς' of a holomorphic ssp ς_h is fully determined by the real part ς :

$$\varsigma'(\theta,\phi) = -\varsigma(\theta,\Gamma_\iota\phi)$$

$$\text{or, in matrix terms,} \quad \varsigma' = -\varsigma\Gamma_\iota \quad . \tag{2.3.9}$$

Moreover, ς' is then automatically a Majorana ssp itself, with the same symmetry κ as ς and also such that Γ_ι is self-adjoint.

If a Majorana ssp ς satisfies the constraint (2.3.8) we say that it admits a holomorphic extension, which we have shown to be unique, to the holomorphic ssp $\varsigma_h = \varsigma + \iota\varsigma'$, where ς' is as in (2.3.9). The holomorphic ssp^s are therefore in one-to-one correspondence with such Majorana ssp^s.

In a standard basis for a spinor space Θ, the Γ_ι self-adjointness constraint (2.3.5) becomes simply $\{\varsigma,\Gamma_\iota\} = 0$. This means that, going back to Table II, the Majorana ssp^s which admit holomorphic extension are those whose associated matrices involve the extra $\hat{\Gamma}$.

To summarize, we have started with the definition of Majorana ssp^s and, in Tables I and II, provided complete lists of those available in an arbitrary spinor space of the Clifford algebra $\mathfrak{R}_{s,t}$, for both the real (s-t = 0,1,2 mod8) and complex (s-t = 3,7 mod8) structure signatures. In this latter category we went on to define the holomorphic ssp^s, bilinear with respect to the complex structure on Θ, and proved that each of them is the holomorphic extension of one in a subclass of Majorana ssp^s, which we then identified in Table II.

One last comment is in order: since in signatures s-t = 3 mod4 we have $[e_m,e_\iota] = 0$, the whole spinor representation of the Clifford algebra $\mathfrak{R}_{s,t}$ on a spinor space Θ is in fact linear with respect to the multiplication by ι and , by definition , the holomorphic ssp^{s} are bilinear with respect to it . Therefore , a complex matrix representation is appropriate : a complex basis $\{ f_\xi \, , \xi = \mathrm{I,II,...,}{}^1\!/_2 D = 2^{[d/2]} \}$ for Θ associates to Γ_m and ς_h the ${}^1\!/_2 D \times {}^1\!/_2 D$ complex matrices whose entries $(\Gamma_m)^\xi{}_\zeta$ and $(\varsigma_h)_{\xi\zeta}$ are , as usual , defined by

$$\Gamma_m(f_\zeta) = (\Gamma_m)^\xi{}_\zeta \, f_\xi \qquad \text{and} \qquad (\varsigma_h)_{\xi\zeta} = \varsigma_h(f_\xi, f_\zeta) \ . \tag{2.3.10}$$

§ 2.4 *Real and Holomorphic SuSy Algebras*

Consider a arbitrary spinor space Θ in the real Clifford algebra $\Re_{s,t}$. One of the significant features of the Majorana ssp^s ς is that they can be used , in a very natural and direct way , to define vector valued symmetric products of spinors on Θ ,

$$\{\theta\,\phi\}_\varsigma = \{\phi\,\theta\}_\varsigma \in \mathbf{X} \qquad , \qquad \theta,\phi \in \Theta \;.$$

On the other hand , such vector valued symmetric products of spinors form the core of the product structure of real SuSy algebras , the abstract models of the real supersymmetry algebras . In summary , each available Majorana ssp on Θ naturally characterizes the product structure of a real SuSy algebra constructed over $\mathbf{X} \oplus \Theta$.

These real SuSy algebras , which are almost trivial examples of real Lie superalgebras [13] , lie thus within the real Clifford algebra $\Re_{s,t}$. Moreover , the underlying structure of $\Re_{s,t}$ intervenes in their construction . However , the product structure of the real SuSy algebras thus constructed is absolutely independent of the underlying associative Clifford product , that is , the vector $\{\theta\,\phi\}_\varsigma$ $\neq$ the spinor $\{\theta,\phi\}$!

We begin with the construction of the real SuSy algebra associated to a Majorana ssp ς and we discuss its properties . Then , in the special case of signatures $s\text{-}t = 3 \bmod 4$ and Majorana ssp^s ς which admit holomorphic extension , we construct the holomorphic SuSy algebra associated to ς_h and conclude that it corresponds to an extension of the real SuSy algebra associated to ς which has incorporated the underlying complex structure on the real Clifford algebra $\Re_{s,t}$.

Let Θ be a spinor space in $\Re_{s,t}$ and ς a Majorana ssp on Θ . The real SuSy algebra associated to ς is defined over $\mathbf{X} \oplus \Theta$ to be

$$[x\,y] := 0 \quad , \quad [x\,\theta] := 0$$
$$\text{and} \tag{2.4.1}$$
$$\{\theta\,\phi\} := -\,2\,\varsigma(\theta,\Gamma^m\phi)\,e_m \quad ,$$

where the e_m constitute some arbitrarily chosen set of orthonormal generators and the Γ^m represent the dual e^m acting on Θ via left multiplication. Notice that there is no comma in this <u>new</u> Lie graded product [} , as opposed to the Clifford commutators and anticommutators [,] and { , } .

One could appropriately ask whether the characterization of the real SuSy algebra associated to a Majorana ssp ς, as defined in (2.4.1), is dependent upon the choice of a reference set of orthonormal generators e_m, and thus of the corresponding Γ_m or Γ^m. This is not so. The vector valued non-degenerate bilinear form $\varsigma(\cdot\,,\Gamma^m\cdot)\,e_m$ remains unchanged for whatever other choice of orthonormal generators e'_m. Indeed, since the e'_m must relate to the original e_m by some orthogonal group isometry $e'_m = \Lambda\,(e_m)$,

$$\varsigma(\cdot\,,\Gamma'^m\cdot)\;\; e'_m \;=\; [\,\Lambda_p{}^m\,\varsigma(\cdot\,,\Gamma^p\cdot)\,]\;\;[\,\Lambda^q{}_m\,e_q\,] \;=$$
$$\tag{2.4.2}$$
$$=\;[\,\Lambda_p{}^m\,\varsigma(\cdot\,,\Gamma^p\cdot)\,]\;\;[\,(\Lambda^{-1})_m{}^q\,e_q\,] \;=\; \varsigma(\cdot\,,\Gamma^p\cdot)\,\delta_p{}^q\,e_q \;=\; \varsigma(\cdot\,,\Gamma^m\cdot)\;\,e_m$$

where we have used $\Lambda^q{}_m = (\Lambda^{-1})_m{}^q$ as in (2.2.6).

A crucial property of real SuSy algebras is their covariance under the simultaneous action of the connected component of $\mathbf{Spin}^0_{s,t}$ on both $\mathbf{X}$ and Θ. This means that if $x \equiv \{\theta\,\phi\}$ denotes the vector product of two spinors θ,ϕ then the action of $\mathbf{Spin}^0_{s,t}$ is such that $x' = \{\theta'\,\phi'\}$, where the primes are as in (2.2.24).

This can be proved as follows :

We already know from (2.3.3) that Majorana ssps are invariant under the action of $\mathbf{Spin}^{0}_{s,t}$ on the spinor space Θ and thus

$$\{\,\theta'\,\phi'\,\} \;=\; -\,2\,\varsigma(\,S\theta,\Gamma^{m}S\phi)\,e_{m} \;= \;\; = \;-\,2\,\varsigma(\,\theta,S^{-1}\Gamma^{m}S\phi)\,e_{m} \;. \tag{2.4.3a}$$

The $S^{-1}\Gamma^{m}S$ represent the

$$s^{-1}\,e^{m}\,s \;=\; \Lambda_{s^{-1}}(e^{m}) \;=\; \Lambda_{s}^{-1}(e^{m}) \;=\; (\Lambda_{s}^{-1})_{n}{}^{m}\,e^{n}$$

acting on Θ via left multiplication . Therefore , $S^{-1}\Gamma^{m}S \;=\; (\Lambda_{s}^{-1})_{n}{}^{m}\,\Gamma^{n}$ and

$$\{\,\theta'\,\phi'\,\} \;=\; -\,2\,\varsigma(\theta,\Gamma^{n}\phi)\,(\Lambda_{s}^{-1})_{n}{}^{m}\,e_{m} \;= \;\; = \;-\,2\,\varsigma(\theta,\Gamma^{n}\phi)\,(\Lambda_{s})^{m}{}_{n}\,e_{m} \;=\; -\,2\,\varsigma(\theta,\Gamma^{n}\phi)\,e'_{n} \;=\; \{\,\theta\,\phi\}' \tag{2.4.3b}$$

where we have used $(\Lambda_{s}^{-1})_{n}{}^{m} \;=\; (\Lambda_{s})^{m}{}_{n}$ as in (2.2.14) .

In a basis $\{f_{\alpha}\}$ for Θ , the non-trivial product law (2.4.1) reads

$$\{f_{\alpha}\,f_{\beta}\} \;=\; -\,2\,(\varsigma\,\Gamma^{m})_{\alpha\beta}\,e_{m} \;. \tag{2.4.4}$$

With the identifications $e_{m} \leftrightarrow i\,P_{m}$ and $f_{\alpha} \leftrightarrow Q_{\alpha}$, this expression becomes

$$\{Q_{\alpha},Q_{\beta}\} \;=\; -\,2i\,(\varsigma\,\Gamma^{m})_{\alpha\beta}\,P_{m} \qquad , \tag{2.4.5}$$

which coincides with the general form (1.8) of real supersymmetry algebras presented in the introductory notes to real superspace , back in § 1 . In fact , it suffices to denote $\Gamma^{m} \;\rightarrow\; \gamma^{m}_{M}$, $\varsigma \;\rightarrow\; \varsigma_{M}$ and $Q_{\alpha} \;\rightarrow\; Q^{M}_{\alpha}$ to obtain the standard Majorana representation of real supersymmetry algebras in signatures $s\text{-}t = 0,1,2 \bmod 8$,

$$\{Q^M_\alpha, Q^M_\beta\} = -2i\,(\varsigma_M\,\gamma^m_M)_{\alpha\beta}\,P_m \quad , \quad \text{see also § 3.2} \quad . \tag{2.4.6}$$

In the remaining signatures , with complex Dirac matrices γ_m , the real supersymmetry algebras have the standard form

$$\{\bar{Q}_{\dot{a}}, Q_b\} = i\,(A\,\gamma^m)_{\dot{a}b}\,P_m \quad , \quad a,b = 1,2,\ldots,{}^1/_2 D \quad ,$$
$$\text{with the } (A\,\gamma_m) \text{ hermitian and } \bar{P}_m = P_m \quad , \tag{2.4.7}$$

but can nevertheless be shown to be ultimately of the same type (2.4.5) , simply by choosing an appropriate imaginary basis for the Q sector , namely

$$(\,Q_\alpha\,) \equiv (\,i(\bar{Q}_{\dot{a}} + Q_a)\,,(\bar{Q}_{\dot{a}} - Q_a)\,) \quad , \tag{2.4.8}$$

remembering α has twice the range of a . For details , see the Appendix to § 2 .

Now , once more , we wish to concentrate on signatures $s\text{-}t = 3 \bmod 4$.
Let Θ be a spinor space in $\mathfrak{R}_{s,t}$ and ς a Majorana ssp on Θ which admits holomorphic extension to ς_h . The holomorphic SuSy algebra associated to ς_h is defined over $(X \oplus \iota X) \oplus \Theta$ to be

$$[z\,z]_h := 0 \quad , \quad [z\,\theta]_h := 0$$
$$\text{and} \tag{2.4.9}$$
$$\{\theta\,\phi\}_h := -2\,\varsigma_h(\theta,\Gamma^m\phi)\,e_m$$

for all $z,z \in (X \oplus \iota X)$ and $\theta,\phi \in \Theta$. Since both Γ_m and ς_h are linear or bilinear with respect to the multiplication by ι , the holomorphic SuSy algebra (2.4.9) possesses the complex structure available in $\Re_{s,t}$, that is ,

$$\{(\iota\,\theta)\,\phi\}_h = \{\theta\,(\iota\,\phi)\}_h = \iota\,\{\theta\,\phi\}_h \quad . \tag{2.4.10}$$

Moreover , the non-trivial product law (2.4.9) can be rewritten as

$$\{\theta\,\phi\}_h = -\,2\,\varsigma(\,\theta,\Gamma^m\,\phi)\,e_m + 2\,\varsigma(\,\theta,\Gamma_\iota\,\Gamma^m\,\phi)\,\iota\,e_m \tag{2.4.11}$$

and it is thus clear that $\{\,\theta\,\phi\}$ and $\{\,\theta\,\phi\}_h$ differ only by a pseudovector component , whose presence in the holomorphic SuSy product law makes it compatible with the underlying complex structure , as in (2.4.10) .

In a complex basis $\{f_\alpha\}$ for Θ , the non-trivial product law (2.4.9) reads

$$\{f_\xi\, f_\zeta\}_h = -\,2\,(\varsigma_h\,\Gamma^m)_{\xi\zeta}\,e_m \quad . \tag{2.4.12}$$

Since $\{\,\theta\,\phi\}$ = vector component of $\{\,\theta\,\phi\}_h$, the relations above do also describe the original real SuSy algebra associated to ς , the real part of ς_h , and yet the f_ξ are only <u>half</u> as many as the f_α .

The anti-holomorphic extensions of a Majorana ssp ς and of its associated real SuSy algebra [} can be obtained from the holomorphic ones simply by replacing ι with $-\iota$. This comes from the fact that the anti-holomorphic analogue of (2.3.9) has a " plus " sign on the right hand side and thus the holomorphic and anti-holomorphic spinor scalar product extensions of ς are the complex conjugates of each other . Nevertheless , the associated SuSy algebra extensions of [} are actually the same , simply because $\iota\,e_m$ in the holomorphic case is the same as $-\iota\,e_m$ in the anti-holomorphic one ; they both correspond to the same pseudovector $e_\iota\,e_m$.

§ 2.5 *The 3 dimensional Euclidean Example*

To finish , we will now discuss how the holomorphic extension scheme applies to the specific case of the $(s = 3, t = 0)$ signature :

Here $d = 3$, the vector indices m,n are customarily substituted by $i,j = 1,2,3$ and $\mathfrak{R}_{3,0}$ has orthonormal generators

$$(e_1)^2 = (e_2)^2 = (e_3)^2 = +1 \quad .$$

The spinor spaces Θ in $\mathfrak{R}_{3,0}$, which are all equivalent , have dimension

$$D = 2^{[3/2]+1} = 4 \quad , \quad \text{so that} \quad \alpha,\beta = 1,2,3,4 \quad .$$

From the orthonormal generators above we construct the Clifford algebra basis

$$B = \{1 , e_1 , e_2 , e_3 , e_1e_2 , e_2e_3 , e_1e_3 , e_\iota \equiv e_1e_2e_3 \} \quad ,$$

from which we choose , for instance ,

$$B_+ = \{e_3\} \quad .$$

Thus , $p \equiv \frac{1}{2}(1 + e_3)$ is a primitive idempotent in $\mathfrak{R}_{3,0}$ and $\Theta \equiv \mathfrak{R}_{3,0}p$ is a minimal left ideal .

Directly from $B\,p \equiv \{bp , b \in B\}$, we choose the following maximal subset of linearly independent elements :

$$f_1 \equiv \tfrac{1}{2}(1 + e_3) = p \quad , \quad f_2 \equiv \tfrac{1}{2}(e_1 + e_1e_3) \quad ,$$
$$f_3 \equiv \tfrac{1}{2}(e_2 + e_2e_3) \quad , \quad f_4 \equiv \tfrac{1}{2}(e_1e_2 + e_\iota) \quad .$$

This basis $\{f_\alpha\}$ for Θ associates to the Γ_j the matrices

$$\Gamma_1 = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix} \;,\; \Gamma_2 = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & -1 & & \end{bmatrix} \;,\; \Gamma_3 = \begin{bmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & -1 & 0 \\ & & 0 & 1 \end{bmatrix} ;$$

these are all orthogonal and thus the basis $\{f_\alpha\}$ is standard .

As the extra $\hat{\Gamma}_-$, we choose the orthogonal antisymmetric matrix

$$\hat{\Gamma}_- = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix} \quad , \text{ for which } \; (\hat{\Gamma}_-)^2 = -\mathbf{I} \; \text{ and } \; \{ \Gamma_j , \hat{\Gamma}_- \} = 0 \quad .$$

Then , the matrix $\varsigma \equiv \Pi_- \hat{\Gamma}_- = \mathbf{I}\, \hat{\Gamma}_- = \hat{\Gamma}_-$ is associated , in the basis $\{f_\alpha\}$, with an antisymmetric Majorana ssp on Θ which admits holomorphic extension .

The product laws (2.4.2) of the real SuSy algebra associated to ς are

$$\begin{aligned} \{f_1\, f_1\} = \{f_2\, f_2\} = - \{f_3\, f_3\} = - \{f_4\, f_4\} &= - 2\, e_2 \\ \{f_1\, f_4\} = - \{f_2\, f_3\} &= - 2\, e_1 \\ \{f_1\, f_3\} = \{f_2\, f_4\} &= 2\, e_3 \quad . \end{aligned} \tag{2.5.1}$$

The complex structure induced on Θ by e_ι is given by

$$\iota f_1 = f_4 \quad , \quad \iota f_2 = f_3 \quad .$$

The complex basis $\{f_I \equiv f_1 , f_{II} \equiv f_2\}$ for Θ associates to the Γ_j and to $\varsigma_h(\cdot,\cdot) = \varsigma(\cdot,\cdot) - \iota\,\varsigma(\cdot,\Gamma_\iota\cdot)$ the matrices

$$\Gamma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} , \quad \Gamma_2 = \begin{bmatrix} 0 & -\iota \\ \iota & 0 \end{bmatrix} , \quad \Gamma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} , \quad \varsigma_h = \begin{bmatrix} 0 & -\iota \\ \iota & 0 \end{bmatrix} .$$

The product laws (2.4.6) of the holomorphic SuSy algebra associated to ς_h are

$$\begin{aligned} \{f_I\, f_I\}_h &= -2\,(e_2 - \iota e_1) \quad , \\ \{f_{II}\, f_{II}\}_h &= -2\,(e_2 + \iota e_1) \quad , \\ \{f_I\, f_{II}\}_h &= -2\,\iota e_3 \quad . \end{aligned} \tag{2.5.2}$$

The other non-trivial product laws of this algebra , for instance $\{f_1\, f_4\}_h$, can all be obtained from the ones above simply by making use of the linearity with respect to the multiplication by ι :

$$\{f_1\, f_4\}_h = \{f_1\, (\iota f_1)\}_h = \iota\,\{f_1\, f_1\}_h = -2\,(e_1 + \iota e_2) \quad .$$

Note how , in the holomorphic extension from $[\ \ \}$ to $[\ \ \}_h$, the vector component of the product of two spinors has remained the same , only a pseudovector component has been added .

With the identifications $e_j \leftrightarrow i P_j$ and $f_\xi \leftrightarrow Q_\xi$ already mentioned , plus now $\iota \leftrightarrow i$, the product laws (2.5.2) take on precisely the form of the 3 dimensional Euclidean holomorphic supersymmetry algebra (2.1.1b) .

§ 2.6 *Summary and Conclusions*

We have set up a constructive scheme for (N=1) real SuSy algebras within the general formalism of real Clifford algebra theory. The scheme is based on a class of real valued spinor scalar products here referred to as Majorana ssp^s, all of which we have listed in the cases of signatures associated with real (s-t = 0,1,2 mod8) and complex (s-t = 3,7 mod8) structures on the Clifford algebra $\mathfrak{R}_{s,t}$.

The constructive scheme we have introduced differs in a significant way from the one originally proposed in [14], which is based on the spinor scalar products induced by the two anti-involutions $\beta_{\pm}$ [12, Lounesto 1981]. These are not necessarily Majorana - the spinor scalar product induced by β_{+} in signature (s = 3, t = 1) isn't, for instance - and it would therefore be interesting to establish the precise relation between the two schemes. In particular, one could take further our analysis by undertaking an actual classification of the real SuSy algebras we have listed here and then compare such classification with the one in [14].

The main topic of this chapter, however, was the characterization of the natural holomorphic extension which a subclass of Majorana ssp^s and their associated real SuSy algebras admit in signatures s-t = 3 mod4. The resulting holomorphic SuSy algebras were then identified with supersymmetry algebras of a new type which are relevant (see next chapter) in the development of canonical or Hamiltonian methods in superspace.

The case of signatures s-t = 4,5,6 mod8, for which the associated real Clifford algebras $\mathfrak{R}_{s,t}$ have a natural quaternionic structure, has not been specifically addressed here but constitutes the most promising and straightforward sequel to the work presented. The issue is whether in those signatures some of the available Majorana ssp^s and their associated real SuSy algebras admit a quaternionic extension along the same lines as the holomorphic extension in signatures s-t = 3,7 mod8.

Quaternionic structures in real supersymmetry algebras have indeed already been mentioned in [11] . Within the context of our scheme , however , the analysis of the quaternionic signatures $s\text{-}t = 4,5,6 \bmod 8$ offers particular technical difficulties of its own , essentially arising from the existence of not one but two extra $\hat{\Gamma}$, making the computation of the Majorana ssp^s more involved , plus also the non-commutativity of the quaternions and the associated issue of ordering .

Appendix to § 2

Let $\mathbb{R}(\mathcal{D})$ denote the real associative algebra of $\mathcal{D}\times\mathcal{D}$ real matrices. Similarly, let $\mathbb{C}(\mathcal{D})$ denote the real associative algebra of $\mathcal{D}\times\mathcal{D}$ complex matrices. The identity matrix is denoted $\mathbf{I}$. $\mathbb{R}(\mathcal{D})$ acts naturally on $\mathbb{R}^{\mathcal{D}}$, while $\mathbb{C}(\mathcal{D})$ acts naturally on $\mathbb{C}^{\mathcal{D}}$.

A typical element of $\mathbb{C}(\mathcal{D})$ is denoted

$$\mathcal{M} = M_r + i M_i \tag{2.A.1a}$$

where M_r, M_i are in $\mathbb{R}(\mathcal{D})$. On the other hand, a typical element of $\mathbb{C}^{\mathcal{D}}$ is denoted

$$\vartheta = v_r + i v_i \tag{2.A.1b}$$

where v_r, v_i are in $\mathbb{R}^{\mathcal{D}}$. The action of $\mathbb{C}(\mathcal{D})$ on $\mathbb{C}^{\mathcal{D}}$ may thus be decomposed as

$$\begin{aligned} \mathcal{M}\vartheta &= (M_r + i M_i)(v_r + i v_i) = \\ &= (M_r v_r - M_i v_i) + i(M_r v_i + M_i v_r) \ . \end{aligned} \tag{2.A.2}$$

Define the map $\mu : \mathbb{C}(\mathcal{D}) \rightarrow \mathbb{R}(2\mathcal{D})$ as being

$$\mu(\mathcal{M}) := \begin{bmatrix} M_r & -M_i \\ M_i & M_r \end{bmatrix} \ . \tag{2.A.3}$$

One can easily show that μ is a real associative algebra homomorphism from $\mathbb{C}(\mathcal{D})$ to $\mathbb{R}(2\mathcal{D})$, that is

$$\mu(r\mathcal{M}) = r\,\mu(\mathcal{M}) \quad , \quad \mu(\mathcal{M}_1 + \mathcal{M}_2) = \mu(\mathcal{M}_1) + \mu(\mathcal{M}_2) \tag{2.A.4a}$$

and

$$\mu(\mathcal{M}_1 \mathcal{M}_2) = \mu(\mathcal{M}_1)\,\mu(\mathcal{M}_2) \tag{2.A.4b}$$

for all $r \in \mathbb{R}$. Note that the map μ is certainly not surjective , since the real dimension of $\mathbb{R}(2\mathcal{D})$ is twice the one of $\mathbb{C}(\mathcal{D})$. Indeed , with

$$\mathfrak{I} \equiv \mu(i\,I) = \begin{bmatrix} & -I \\ I & \end{bmatrix} , \tag{2.A.5}$$

eq. (2.A.4b) implies that $\mathfrak{I}$ commutes with all $2\mathcal{D}{\times}2\mathcal{D}$ real matrices in the image of μ .

One can define an associated map $\hat{\mu} : \mathbb{C}^{\mathcal{D}} \to \mathbb{R}^{2\mathcal{D}}$ as being

$$\hat{\mu}(\vartheta) := \begin{bmatrix} v_r \\ v_i \end{bmatrix} . \tag{2.A.6}$$

This time , $\hat{\mu}$ is in fact a real vector space isomorphism between $\mathbb{C}^{\mathcal{D}}$ and $\mathbb{R}^{2\mathcal{D}}$. Furthermore , it follows from (2.A.3) that

$$\hat{\mu}(\mathcal{M}\,\vartheta) = \mu(\mathcal{M})\,\hat{\mu}(\vartheta) \quad . \tag{2.A.7}$$

The original map μ has yet two other important properties : given $\mathcal{M}$ non-singular , it follows from (2.A.4b) that

$$\mu(\mathcal{M}^{-1}) = \mu(\mathcal{M})^{-1} \quad , \tag{2.A.8a}$$

and , for all $\mathcal{M}$,

$$\mu(\mathcal{M}^{\dagger}) = \mu(\mathcal{M})^{T} \quad . \tag{2.A.8b}$$

Consider an arbitrary spacetime signature (s,t) and let $\mathcal{D} \equiv 2^{[d/2]}$, where $d = s+t$.

A Dirac matrix algebra $\mathbf{D}_{s,t}$ consists of $\mathbb{C}(\mathcal{D})$ equipped with a specific set of matrix generators $(\gamma_m)^a{}_b$ for which

$$\{\gamma_m,\gamma_n\} = 2\,\eta_{mn}\,\mathbf{I} \quad . \tag{2.A.9}$$

Two comments are in order :

1. The above definition for a Dirac matrix algebra $\mathbf{D}_{s,t}$ is , of course , based on the well known fact that any set of $\mathcal{D}\times\mathcal{D}$ complex matrices satisfying (2.A.9) will then automatically generate , over $\mathbb{C}$, the whole of $\mathbb{C}(\mathcal{D})$.

2. A Dirac matrix algebra $\mathbf{D}_{s,t}$ is usually referred to as a particular representation for the relations (2.A.9) , considered in abstract .

Since the Dirac matrices γ_m generate the whole of $\mathbb{C}(\mathcal{D})$, any other matrix in $\mathbb{C}(\mathcal{D})$ which commutes with all of the γ_m is necessarily proportional to the identity $\mathbf{I}$. Then , consider the matrix

$$\pi_\iota \equiv \begin{cases} \gamma_\iota \equiv \gamma_1\gamma_2\cdots\gamma_d & \text{when } s\text{-}t = 0,1 \bmod 4 \;,\quad (\gamma_\iota)^2 = +\mathbf{I} \\ i\,\gamma_\iota & \text{when } s\text{-}t = 2,3 \bmod 4 \;,\quad (\gamma_\iota)^2 = -\mathbf{I} \;. \end{cases}$$

In odd spacetime dimensions , π_ι commutes with all of the Dirac matrices γ_m and thus differs from the identity $\mathbf{I}$ at most by a sign , which we indicate $\pi_\iota \sim \mathbf{I}$. As a result of this , whereas in even spacetime dimensions all Dirac matrix algebras $\mathbf{D}_{s,t}$ are equivalent (through similarity transformations of the form $\gamma_m \to \gamma'_m = T^{-1}\,\gamma_m\,T$) , in odd spacetime dimensions there are two equivalence classes of Dirac matrix algebras $\mathfrak{D}_{s,t}$, corresponding to the two possibilities $\pi_\iota = \pm\mathbf{I}$.

We say that a Dirac matrix algebra $\mathfrak{D}_{s,t}$ is standard if all the matrices γ_m are unitary. Every Dirac matrix algebra is equivalent to a standard one.

Let $(A)_{\dot{a}b}$ be an hermitian or antihermitian matrix in $\mathbb{C}(\mathcal{D})$ for which all the matrices $(A\,\gamma^m)$ are hermitian, that is

$$A^\dagger = \kappa A \quad , \quad \kappa = \pm 1 \tag{2.A.10a}$$

and

$$(A\,\gamma^m)^\dagger = (A\,\gamma^m) \quad . \tag{2.A.10b}$$

In the Table below, we list all the possible matrices A in each signature (s,t). An arbitrary choice of a standard set of Dirac matrices γ_m is understood, in terms of which the matrices A are specified modulo an arbitrary positive real normalization factor. Since the matrices A will necessarily lie within the matrix algebra generated by the γ_m, the computational method employed in the Table below is identical to the one already used in constructing Table I, presented in § 2.3 .

Table

Let π_+ denote the ordered product of all the γ_m which square to $+\mathbf{I}$, $\pi_+ \equiv \gamma_1 ... \gamma_s$ (or $\pi_+ \equiv \mathbf{I}$ if $s=0$) and let π_- denote the ordered product of all the γ_m which square to $-\mathbf{I}$, $\pi_- \equiv \gamma_{s+1} ... \gamma_{s+t}$ (or $\pi_- \equiv \mathbf{I}$ if $t=0$), so that $\gamma_\iota = \pi_+ \pi_-$.

The groups of four (s,t) specifications presented on the left of the Table are given mod4

Each group corresponds to all possible (s,t) subcases of the its main s , t specification

s even , t even

	$\kappa = +1$	$\kappa = -1$
$(s = 0, t = 0)$	$\pm\pi_-$	$\pm i\pi_+$
$(s = 0, t = 2)$	$\pm i\pi_-$	$\pm i\pi_+$
$(s = 2, t = 0)$	$\pm\pi_-$	$\pm\pi_+$
$(s = 2, t = 2)$	$\pm i\pi_-$	$\pm\pi_+$

s even , t odd

$(s = 0, t = 1)$	---	$\pm i\pi_+ \sim \pm\pi_-$
$(s = 0, t = 3)$	---	$\pm i\pi_+ \sim \pm i\pi_-$
$(s = 2, t = 1)$	---	$\pm\pi_+ \sim \pm\pi_-$
$(s = 2, t = 3)$	---	$\pm\pi_+ \sim \pm i\pi_-$

s odd , t even

$(s = 1, t = 0)$	$\pm\pi_+ \sim \pm\pi_-$	---
$(s = 1, t = 2)$	$\pm\pi_+ \sim \pm i\pi_-$	---
$(s = 3, t = 0)$	$\pm i\pi_+ \sim \pm\pi_-$	---
$(s = 3, t = 2)$	$\pm i\pi_+ \sim \pm i\pi_-$	---

s odd , t odd

$(s = 1, t = 1)$	$\pm\pi_+$	$\pm\pi_-$
$(s = 1, t = 3)$	$\pm\pi_+$	$\pm i\pi_-$
$(s = 3, t = 1)$	$\pm i\pi_+$	$\pm\pi_-$
$(s = 3, t = 3)$	$\pm i\pi_+$	$\pm i\pi_-$

Now, consider spacetime signatures <u>other</u> than the Majorana $s\text{-}t = 0,1,2 \bmod 8$.

In those signatures, real supersymmetry algebras have the general form

$$\{\bar{Q}_{\dot{a}}, Q_b\} = i\,(A\,\gamma^m)_{\dot{a}b}\,P_m \quad , \quad a,b = 1,2,\ldots,\mathcal{D} \tag{2.A.11}$$

where, as usual, $(\overline{})$ denotes complex conjugation and $\bar{P}_m = P_m$.

Let $\varsigma \equiv \mu\,(A)$ and $\Gamma_m \equiv \mu\,(\gamma_m)$. To these $\mathbb{R}(D{=}2\mathcal{D})$ matrices we assign the index structures $(\varsigma)_{\alpha\beta}$ and $(\Gamma_m)^{\alpha}{}_{\beta}$, where $\alpha,\beta = 1,2,\ldots, D{=}2\mathcal{D}$. From (2.A.4b), we know that $\mu\,(A\,\gamma_m) = \varsigma\,\Gamma_m$. Furthermore, let

$$(\,Q_\alpha\,) \equiv (\,i(\bar{Q}_{\dot{a}} + Q_a\,)\,,\,(\bar{Q}_{\dot{a}} - Q_a\,)\,) \tag{2.A.12}$$

by which we mean that $Q_{\alpha=a} \equiv i(\bar{Q}_{\dot{a}} + Q_a)$ and $Q_{\alpha=a+\mathcal{D}} \equiv \bar{Q}_{\dot{a}} - Q_a$.

Now, it follows from (2.A.11) and (2.A.12) that

$$\begin{aligned} \{\,Q_{\alpha=a}\,,\,Q_{\beta=b}\,\} &= \{\,i(\bar{Q}_{\dot{a}} + Q_a\,)\,,\,i(\bar{Q}_{\dot{b}} + Q_b\,)\,\} = \\ &= -\,2i\,(\,(A\,\gamma_m)_{r}\,)_{\dot{a}b}\;P_m = -\,2i\,(\varsigma\,\Gamma_m)_{\alpha=a\,,\,\beta=b}\;P_m \end{aligned} \tag{2.A.13a}$$

and

$$\begin{aligned} \{\,Q_{\alpha=a}\,,\,Q_{\beta=b+\mathcal{D}}\,\} &= \{\,i(\bar{Q}_{\dot{a}} + Q_a\,)\,,\,(\bar{Q}_{\dot{b}} - Q_b\,)\,\} = \\ &= +\,2i\,(\,(A\,\gamma_m)_{\iota}\,)_{\dot{a}b}\;P_m = -\,2i\,(\varsigma\,\Gamma_m)_{\alpha=a\,,\,\beta=b+\mathcal{D}}\;P_m \end{aligned} \tag{2.A.13b}$$

where we have made use of (2.A.10b) in its version $(A\,\gamma^m)_{\dot{b}a} = ((A\,\gamma^m)_{\dot{a}b})^*$.

Similar computations show that the two remaining cases reduce to the first two ,

$$\{\, Q_{\alpha=a+\mathcal{D}} \,,\, Q_{\beta=b+\mathcal{D}} \,\} \;=\; \{\, Q_{\alpha=a} \,,\, Q_{\beta=b} \,\} \tag{2.A.13c}$$

and

$$\{\, Q_{\alpha=a+\mathcal{D}} \,,\, Q_{\beta=b} \,\} \;=\; -\;\{\, Q_{\alpha=a} \,,\, Q_{\beta=b+\mathcal{D}} \,\}\;\; . \tag{2.A.13d}$$

We conclude , thus , that (2.A.11) can be put in the form

$$\{Q_\alpha, Q_\beta\} \;=\; -\,2\,i\,(\varsigma\,\Gamma^m)_{\alpha\beta}\,P_m$$

as stated in § 2.4 , in reference to (2.4.7) and (2.4.8) .

§ 3 Canonical Superspace

§ 3.1 *Introduction and Preview*

A primary issue in the development of superspace canonical methods as regarding (N=1 say) supersymmetric field theories in 4 dimensional Minkowski spacetime is the question of whether the 3+1 canonical decomposition can actually be extended from ordinary spacetime to full superspace. Ideally, this would involve a foliation of the latter in such a way that, first, the superhypersurfaces be themselves superspace extensions of the usual constant time hypersurfaces of ordinary spacetime and, second, that the foliation be compatible with the superPoincaré group action on superspace, i.e. that the Lie algebra of translations and rotations along the superhypersurfaces be a subalgebra of the full superPoincaré Lie algebra of transformations.

Soon enough, however, one realizes that such a picture, in direct analogy with the canonical decomposition of ordinary Minkowski spacetime, is not exactly what one should expect in superspace.

In ordinary spacetime, the vector representation of the Lorentz group $\mathbf{SO}_{3,1}$ on the coordinates x^m, when restricted to $\mathbf{SO}_3$, not only reduces but also acts as the identity on time x^0, preserving its value, be it zero or otherwise. That is not so in the case of the spinor representation of $\mathbf{Spin}_{3,1}$ on the odd superspace coordinates θ^α; it either does not reduce at all when restricted to $\mathbf{Spin}_3$ (real case) or else it does reduce (complex case) but into two non-trivial representations of $\mathbf{Spin}_3$. As a result of that, the θ sector of Minkowski superspace does not admit any foliation, parametrized by some " temporal " coordinates θ_T, with respect to which $\mathbf{Spin}_3$ acts consistently in a " spatial " manner, parallel to the hypersurfaces of constant θ_T throughout the foliation.

Nevertheless, one may hope to obtain a foliation of Minkowski superspace, parametrized by x^0 and θ_T, whose model superhypersurfaces, the ones with $\theta_T = 0$, are overall invariant under spatial translations P_j, spatial rotations J_{ij} and a kinematical, i.e. not producing time shifts, subclass of supersymmetries. Here we use the term

" subclass " and not subalgebra or subgroup simply because the supersymmetry generators do not anticommute to themselves .

If such a canonical decomposition of Minkowski superspace could be achieved , one would then be able to identify the canonical superfield variables of theories with full superspace action integrals simply by integrating over the superspace coordinates θ_T , " normal " to the canonical superhypersurfaces . In the resulting Lagrangian densities , the spinor covariant derivatives would all be of the kinematical type , " parallel " to the canonical superhypersurfaces , and the only dynamical derivative would thus be ∂_0 itself .

This procedure is to be contrasted with the one adopted in [16] , in which the canonical variables , although still functions over full superspace , are identified in a supersymmetric non-covariant manner , with the consequence that all manifest supersymmetry is lost .

This chapter , which is basically about how the canonical decomposition of N=1 Minkowski superspace can indeed be implemented , is organized as follows :

In § 3.2 we briefly review some of the basic facts and formulae of the usual or real Minkowski superspace $\mathbb{S}^{3,1}_{R}$ in an arbitrary spinor representation . For our purposes , references [3,4] have been particularly useful . We also introduce soul-complex Minkowski superspace $\mathbb{S}^{3,1}_{C}$, a type of complex extension of the real version which will later (§ 3.4) be an essential tool as the appropriate arena for the canonical decomposition of superspace in Minkowski , i.e. orthonormal , coordinates .

§ 3.3 deals with the null-plane Hamiltonian approach , in which $x^+ \equiv {}^1/_{\sqrt{2}}\,(x^0 + x^3)$ is regarded as the dynamical parameter . There , the canonical decomposition of real Minkowski superspace turns out as straightforward as it could possibly be : in the Weyl spinor representation , the canonical foliation is parametrized by $(x^+ , \theta^2 , \bar{\theta}^{\dot{2}})$ and each canonical superhypersurface has coordinates $(x^1 , x^2 , x^- , \theta^1 , \bar{\theta}^{\dot{1}})$; infinitesimal canonical supersymmetry transformations are those for which $\varepsilon^2 = \bar{\varepsilon}^{\dot{2}} = 0$.

In § 3.4 we discuss the usual Hamiltonian approach , with x^0 as the dynamical parameter . In contrast with the null-plane case , here every real supersymmetry transformation produces time shifts δx^0 . It is in fact hardly surprising that no appropriate superhypersurfaces can be found with less than all the four real Grassmann spinor coordinates of real Minkowski superspace ; the reason being that the spinor representations of the real Clifford algebras $\mathfrak{R}_{3,1}$ and $\mathfrak{R}_{3,0}$ (and of the corresponding **Spin** groups) have the same real dimension 4 , implying that both the corresponding real superspaces $\mathbb{S}^{3,1}_{R}$ and $\mathbb{S}^{3,0}_{R}$ have thus four real θ coordinates .

However , essentially by exploiting the fact that $\mathfrak{R}_{3,0}$ admits a 2×2 complex matrix representation while $\mathfrak{R}_{3,1}$ does not , it is possible to construct a holomorphic version of the 3 dimensional Euclidean supersymmetry algebra , see (2.1.1b) , whose associated superspace $\mathbb{S}^{3,0}_{h}$ is described by two complex θ coordinates , instead of the four real θ coordinates of $\mathbb{S}^{3,0}_{R}$. The basic result in § 3.4 is then to identify in $\mathbb{S}^{3,1}_{C}$ an appropriate choice of two complex " temporal " coordinates θ^1_T , θ^2_T such that each of the one parameter family of constant time superhypersurfaces $\theta^1_T = \theta^2_T = 0$ is a copy of $\mathbb{S}^{3,0}_{h}$.

In this way we achieve the canonical decomposition of Minkowski superspace, even though the relevant versions of the 4 dimensional Minkowski and 3 dimensional Euclidean superspaces turn out to be not the usual $\mathbb{S}^{3,1}_{R}$ and $\mathbb{S}^{3,0}_{R}$, but instead $\mathbb{S}^{3,1}_{C}$ and $\mathbb{S}^{3,0}_{h}$.

Finally , in § 3.5 , we develop the Hamiltonian superspace formalism which results from the canonical decomposition of Minkowski superspace . This we do in the context of two simple supersymmetric models , the free chiral multiplet and the free abelian vector multiplet . In each case , the canonical superspace formalism reproduces the Hamiltonian structure of the ordinary field theory of which the model is the supersymmetric extension .

§ 3.2 Real and ^{s}Complex Minkowski Superspaces

Minkowski spacetime signature is $\eta_{mn} = \mathrm{diag}(-1,+1,+1,+1)$ and the Dirac matrix algebra is $\{\gamma_m,\gamma_n\} = 2\,\eta_{mn}\,\mathbb{1}$. Vector indices are $m,n,p,q = 0,1,2,3$ or $i,j,k = 1,2,3$, while spinor indices are $\alpha,\beta,\gamma = 1,2,3,4$. Dirac matrices have index structure $(\gamma_m)^\alpha{}_\beta$. In each representation γ_m there exists a non-singular matrix ς , unique modulo a non-zero scalar factor , such that all $(\varsigma\gamma_m)$ are symmetric matrices . Such ς has index structure $(\varsigma)_{\alpha\beta}$.

$\aleph$ stands for an infinite dimensional complex Grassmann algebra , which comes equipped with an anti-involutive complex conjugation $(\ )^*$. The even and odd sectors of $\aleph$ are denoted by $\aleph^{(h)}$, $h = 0,1$, respectively . Let also $\aleph_R$ denote the real sector of the Grassmann algebra $\aleph$.

In a Majorana (i.e. real) orthogonal representation γ^m_M , the usual or real Minkowski superspace $\mathbb{S}^{3,1}_R$ has real (even,odd) $\aleph$ valued coordinates (x^m,θ^α_M) . Let $\mathcal{F} = \oplus\,\mathcal{F}^{(h)}$ denote the space of smooth $\aleph = \oplus\,\aleph^{(h)}$ valued functions over real Minkowski superspace . Note that the term smooth is meant in the sense of Grassmann differentiability - see [4 , De Witt] . The elements of $\mathcal{F}$ are the so-called general (scalar) superfields , while those of $\mathcal{F}^{(h)}$ are referred to as even or odd (scalar) superfields .

As is well known , the requirement of Grassmann differentiability, together with the full nilpotency of the superspace coordinates θ^α_M , implies that every superfield $F \in \mathcal{F}$ is necessarily affine in each of the θ^α_M ,

$$F\,(x^m,\theta^\alpha_M) \;=\; \sum_{k_\alpha=0,1} f_{k_1k_2k_3k_4}(x^m)\;(\theta^1_M)^{k_1}\,(\theta^2_M)^{k_2}\,(\theta^3_M)^{k_3}\,(\theta^4_M)^{k_4} \quad ,$$

$$(\theta^\alpha_M)^0 \equiv \mathbf{1} \quad \text{and} \quad (\theta^\alpha_M)^1 \equiv \theta^\alpha_M \quad , \tag{3.2.1}$$

where the $\aleph$ valued Smooth functions $f_{k_1k_2k_3k_4}(x^m)$ are referred to as the component fields of the superfield F.

Vector fields over real Minkowski superspace map superfields to superfields and thus, in particular, they inherit the Even ⊕ Odd decomposition of $\mathcal{F}$. The basic Poincaré vector field superalgebra $\mathcal{G}$ of real Minkowski superspace is the $\aleph$-linear span of the even P_m, J_{mn} and odd Q^M_α vector fields

$$P_m \equiv - \partial_m \tag{3.2.2a}$$

$$Q^M_\alpha \equiv - (\partial^M_\alpha + i\,(\varsigma_M \gamma^m_M)_{\alpha\beta}\, \theta^\beta_M\, \partial_m) \tag{3.2.2b}$$

$$J_{mn} \equiv - (x_n \partial_m - x_m \partial_n + (\sigma^M_{mn})^\alpha{}_\beta\, \theta^\beta_M\, \partial^M_\alpha) \quad , \tag{3.2.2c}$$

where ς_M is chosen to be numerically the same as γ^M_0 and $\sigma_{mn} \equiv {}^1\!/_4[\gamma_m,\gamma_n]$.

It follows that,

$$[P_m,P_n] = [P_m,Q^M_\alpha] = 0 \tag{3.2.3a}$$

$$\{Q^M_\alpha,Q^M_\beta\} = -2i\,(\varsigma_M \gamma^m_M)_{\alpha\beta}\, P_m \tag{3.2.3b}$$

$$[J_{mn},P_q] = \eta_{nq}\, P_m - \eta_{mq}\, P_n \tag{3.2.3c}$$

$$[J_{mn},Q^M_\alpha] = (\sigma^M_{mn})^\beta{}_\alpha\, Q^M_\beta \tag{3.2.3d}$$

$$[J_{mn},J_{pq}] = \eta_{mq}\, J_{np} + \eta_{np}\, J_{mq} - \eta_{mp}\, J_{nq} - \eta_{nq}\, J_{mp} \quad . \tag{3.2.3e}$$

Spinor covariant derivatives are

$$D^M_\alpha \equiv -(\partial^M_\alpha - i\,(\varsigma_M \gamma^m_M)_{\alpha\beta}\,\theta^\beta_M\,\partial_m) \tag{3.2.4a}$$

so that

$$\{D^M_\alpha, D^M_\beta\} = -2i\,(\varsigma_M \gamma^m_M)_{\alpha\beta}\,\partial_m \tag{3.2.4b}$$

and

$$\{D^M_\alpha, Q^M_\beta\} = 0 \quad . \tag{3.2.4c}$$

The properties of complex conjugation in the Grassmann algebra $\aleph$ carry over to that of $\aleph$ valued functions, which we denote $(\overline{\ \ })$. Thus, complex conjugation in $\mathcal{F}$ is such that, for $F \in \mathcal{F}^{(h)}$ and $G \in \mathcal{F}^{(h')}$,

$$(\overline{F+G}) = \bar{F}+\bar{G} \quad \text{and} \quad (\overline{FG}) = \bar{G}\bar{F} = (-1)^{hh'}\,\bar{F}\bar{G} \quad . \tag{3.2.5a}$$

Moreover, let $\mathcal{E}$, $\mathcal{O}$ be even and odd vector fields, respectively. Then, for $F \in \mathcal{F}^{(h)}$, the complex conjugation of vector fields is chosen to be such that

$$(\overline{\mathcal{E}F}) = \bar{\mathcal{E}}\bar{F} \quad \text{and} \quad (\overline{\mathcal{O}F}) = (-1)^h\,\bar{\mathcal{O}}\bar{F} \quad . \tag{3.2.5b}$$

Accordingly, complex conjugation in $\mathcal{G}$ is

$$\begin{aligned} (\overline{n\,P_m}) &:= n^*\,P_m \quad , \qquad (\overline{n\,J_{mn}}) := n^*\,J_{mn} \quad , \\ (\overline{n\,Q^M_\alpha}) &:= -(-1)^h\,n^*\,Q^M_\alpha \quad , \quad \text{the same for } D^M_\alpha \quad , \end{aligned} \tag{3.2.6}$$

where $n \in \aleph^{(h)}$ and $(\ )^*$ denotes the Grassmann algebra complex conjugation.

The Lie algebra $\mathcal{G}^{(0)}_R$, the real even sector of $\mathcal{G}$, referred to as the superPoincaré

Lie algebra of real Minkowski superspace , is the set of all vector fields of the form

$$n^m P_m + n_M^\alpha Q_\alpha^M + n^{mn} J_{mn} \quad , \tag{3.2.7}$$

with $\aleph_R^{(0)}$ coefficients n^m , n^{mn} for the even vector fields P_m , J_{mn} and $\aleph_R^{(1)}$ coefficients n_M^α for the odd vector fields Q_α^M .

$\mathcal{G}_R^{(0)}$ generates the superPoincaré Lie algebra of transformations in $\mathbb{S}_R^{3,1}$:

Under infinitesimal translations and Lorentz rotations , scalar superfields $F =$ $= F(x^m, \theta_M^\alpha)$ transform as , respectively ,

$$\delta F = (u^m P_m) F \tag{3.2.8a}$$
$$\delta x^m = u^m \quad , \quad \delta \theta_M^\alpha = 0$$

and

$$\delta F = (^1/_2 v^{mn} J_{mn}) F \tag{3.2.8b}$$
$$\delta x^m = v^m{}_n x^n \quad , \quad \delta \theta_M^\alpha = {}^1/_2 v^{mn} (\sigma_{mn}^M)^\alpha{}_\beta \theta_M^\beta$$

with infinitesimal $\aleph_R^{(0)}$ global parameters u^m and v^{mn} (antisymmetric) .

Under infinitesimal supersymmetry transformations , the transformation laws are

$$\delta F = (\varepsilon_M^\alpha Q_\alpha^M) F \tag{3.2.8c}$$
$$\delta x^m = i\, \varepsilon_M^\alpha (\varsigma_M \gamma_M^m)_{\alpha\beta} \theta_M^\beta \quad , \quad \delta \theta_M^\alpha = \varepsilon_M^\alpha$$

with infinitesimal $\aleph_R^{(1)}$ global parameters ε_M^α .

A change of representation , from the Majorana γ_M to a " New " representation γ_N , via a (complex) invertible matrix $(T)^\alpha{}_\beta$, amounts to

$$\gamma_M \rightarrow \gamma_N = T^{-1}\,\gamma_M\,T \quad , \quad \varsigma_M \rightarrow \varsigma_N = T^T\,\varsigma_M\,T$$

$$\theta^\alpha_M \rightarrow \theta^\alpha_N = (T^{-1})^\alpha{}_\beta\,\theta^\beta_M \quad , \quad \text{the same for } \varepsilon^\alpha_M \ , \tag{3.2.9}$$

$$D^M_\alpha \rightarrow D^N_\alpha = (T)^\beta{}_\alpha\,D^M_\beta \quad , \quad \text{the same for } Q^M_\alpha \ ,$$

and therefore preserves the general form of eqs. (3.2.1) to (3.2.4) and (3.2.8), except of course for $\varsigma = \gamma_0$, since ς and γ_0 transform differently. We shall refer to three specific representations: Majorana (M), Weyl (W) and canonical (C) - all about them in the Appendix to § 3.

So much for the usual or real Minkowski superspace. As we shall discuss in the next section § 3.3, the full real supersymmetry (3.2.8c) does incorporate a subclass of transformations which are kinematical with respect to the null plane " time " x^+. Those kinematical supersymmetries have parameters constrained by $\varepsilon^1_M = \varepsilon^3_M$, $\varepsilon^2_M = \varepsilon^4_M$ and correspond to translations parallel to a foliation of the θ sector of real Minkowski superspace parametrized by the two real " temporal " coordinates $\theta^1_T \equiv \theta^1_M - \theta^3_M$ and $\theta^2_T \equiv \theta^2_M - \theta^4_M$.

When we come to § 3.4, however, we soon realize that Minkowski superspace in its real form can never accommodate a canonical decomposition into copies of the real 3 dimensional Euclidean superspace $\mathbb{S}^{3,0}_R$; one reason being simply that real Minkowski superspace, and specificly its θ sector, is in any event far too small. In both signatures (3,1) and (3,0) spinors have (the equivalent to) four real components and thus such must be the number of real θ coordinates of the corresponding real superspaces. So, we are confronted with the immediate obstacle that the θ sector of $\mathbb{S}^{3,0}_R$ can never be identified as a proper subspace of the θ sector of $\mathbb{S}^{3,1}_R$.

The way round the problem is two-fold :

On one hand , there is the construction of the 3 d Euclidean holomorphic supersymmetry algebra (2.1.1b) , whose corresponding superspace $\mathbb{S}_{h}^{3,0}$ is described by two complex θ coordinates instead of the four real ones describing the usual real superspace $\mathbb{S}_{R}^{3,0}$. Such construction has already been discussed in chapter § 2 and is based in that $\mathfrak{R}_{3,0}$, the real Clifford algebra associated with signature (3,0) , admits a natural complex structure (not so for $\mathfrak{R}_{3,1}$) . Then , via the essential role played by $\mathfrak{R}_{3,0}$ which serves as an underlying space in the construction of the 3 d Euclidean SuSy algebra , this can be made to incorporate the complex structure of $\mathfrak{R}_{3,0}$ and the supersymmetry transformations associated with the resulting 3 d Euclidean holomorphic supersymmetry algebra may then be described with only two vector field generators Q , associated with two complex ε parameters .

On the other hand , there is a natural way to double the real θ dimension of Minkowski superspace without enlarging the class of smooth superfields defined on it and so having no drastic effect on the degrees of freedom content of Minkowski superspace field theories . The end product of such construction , which we shall refer to as the soul-complex (^{s}complex) Minkowski superspace $\mathbb{S}_{C}^{3,1}$, will have four independent complex θ coordinates , of which a particular pair will correspond to kinematical θ directions .

In simple geometrical terms , one can imagine the θ sector of $\mathbb{S}_{C}^{3,1}$ as being a real 8 dimensional vector space with two very special real 4 dimensional subspaces : one corresponds to the θ sector of the original $\mathbb{S}_{R}^{3,1}$, whose natural embedding in its ^{s}complex extension consists of the real valued domain of the four generally complex θ coordinates ; and the other , " at an angle " with the first , corresponds to the θ sector of $\mathbb{S}_{h}^{3,0}$ and defines the 3+1 canonical decomposition of ^{s}complex Minkowski superspace .

It is with the construction of this latter $\mathbb{S}_{\mathrm{C}}^{3,1}$ that we shall deal next. Before we proceed, however, a word about nomenclature and some basic results :

The infinite dimensional complex Grassmann algebra $\aleph$ is generated by the identity $\mathbf{1}$ and the mutually anticommuting generators ζ_a, $a = 1,\ldots,\infty$. A general element n of $\aleph$ has a unique decomposition

$$n = n_B + n_S \quad , \quad n \in \aleph$$

where n_B is a complex multiple of the identity $\mathbf{1}$ (though sometimes we use n_B to denote the ordinary complex number coefficient itself) and n_S is all the rest, proportional to the generators ζ_a and their products of arbitrary degree. Following [4, De Witt], we call n_B the " body " of n and n_S its " soul ". Because we are dealing with an infinite dimensional Grassmann algebra, souls in $\aleph^{(0)}$ are not necessarily nilpotent. Odd elements of $\aleph$ are, of course, pure soul.

$\aleph_{\mathrm{R}}$, the real sector of $\aleph$, is not a subalgebra, since the product of two odd elements of $\aleph_{\mathrm{R}}$ is in fact imaginary :

$$(\theta_{\mathrm{R}}\,\theta'_{\mathrm{R}})^* = (\theta'_{\mathrm{R}})^*(\theta_{\mathrm{R}})^* = \theta'_{\mathrm{R}}\,\theta_{\mathrm{R}} = -\,\theta_{\mathrm{R}}\,\theta'_{\mathrm{R}} \quad .$$

One does get a subalgebra of $\aleph$ including $\aleph_{\mathrm{R}}$ with the set

$\aleph_{\mathrm{C}} \equiv \{\, n_{\mathrm{C}} = n_{\mathrm{R}} + \mathrm{i}\, n_{\mathrm{I}}$, where both n_{R} and n_{I} are real <u>but</u> n_{I} is pure soul$\}$.

$\aleph_{\mathrm{C}}$ is indeed a subalgebra of $\aleph$. At an abstract level, $\aleph_{\mathrm{C}}$ can be regarded [17] as a real $\mathbb{Z}_2$ graded associative algebra generated by the identity $\mathbf{1} \in \aleph_{\mathrm{C}}^{(0)}$ and the mutually anticommuting generators ζ_a, $\xi_b \in \aleph_{\mathrm{C}}^{(1)}$, together the extra product law

$$\zeta_a \zeta_b = - \xi_a \xi_b \quad .$$

The soul sector of $\aleph_C$ (set of all pure soul elements of $\aleph_C$) clearly admits the complex structure

$$i\,\zeta_a = \xi_a \quad , \quad i\,\xi_a = -\,\zeta_a$$

which takes us back to the complex Grassmann algebra origins of $\aleph_C$.

We shall refer to the elements of $\aleph_C$ as soul-complex (^{S}complex) so as to best contrast them with the real elements of $\aleph_R$.

Now we come back to introducing ^{S}complex Minkowski superspace $\mathbb{S}^{3,1}_C$: simply take eqs. (3.2.1) to (3.2.9) and allow the originally $\aleph_R$ valued coordinates x^m , θ^α_M and parameters u^m , ε^α_M, v^{mn} to assume generally ^{S}complex values . In other words , take everything that has been said from the beginning of this section up to (3.2.9) inclusively , and consistently change " real (R) " to " ^{S}complex (C) " . Only the Majorana representation γ^m_M , whose matrix elements are ordinary real numbers and not real Grassmann algebra elements , remains unchanged .

The reason why such ^{S}complex extension is well defined resides on the fact that there exists a one-to-one correspondence between smooth superfields on real and ^{S}complex Minkowski superspaces , a correspondence which commutes with differentiation and with an appropriately generalized complex conjugation of functions .

To see this we refer again to [4 , De Witt] , where it is shown that $\aleph$ valued smooth functions on $\aleph^{(0)}_R$ and $\aleph^{(1)}_R$, respectively , necessarily have the form

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k!} (x_S)^k \; f^{(k)}(x_B) \qquad (3.2.10a)$$

$$f(\theta) \;=\; n + \theta\, m \qquad (3.2.10b)$$

where $x \in \aleph_R^{(0)}$, $\theta \in \aleph_R^{(1)}$, m,n are arbitrary constant elements of $\aleph$ and $f(x_B)$ is a $\aleph$ valued smooth function on the real line $\mathbb{R}$. Eq. (3.2.10a) , in particular , implies that every $\aleph$ valued smooth function on $\mathbb{R}$ can be uniquely extended to a $\aleph$ valued smooth function on the whole of $\aleph_R^{(0)}$, in a way which amounts in fact to a Taylor expansion around x_B , in powers of x_S . This infinite series contributes , at each level of the Grassmann algebra generators ζ_a and their products , with only a finite number of terms and therefore no convergence problems arise . One can further show that the $\mathbb{R}$ to $\aleph_R^{(0)}$ domain extension process commutes with differentiation , that is

$$\frac{df}{dx}(x) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!}\, (x_S)^k\; f^{(k+1)}(x_B) \quad . \qquad (3.2.10c)$$

The basic fact behind the idea of ^{S}complex Minkowski superspace is then that in (3.2.10) it doesn't actually make any difference whether x_S and θ are real or complex . Indeed , eqs. (3.2.10) would still hold true were x and θ elements of $\aleph_C$ instead of $\aleph_R$. Note , however , that its only x_S which can be complexified in this way and not the numerical core of the variable x , its body x_B , which must remain real . This is why the properties of functions over $\aleph_C^{(0)}$ are still much the same as those of functions over $\aleph_R^{(0)}$, or indeed over $\mathbb{R}$ itself .

It is clear from (3.2.10a) that smooth functions over $\aleph_C^{(0)}$ depend solely on the variable x_C and not on its complex conjugate . This is one of a number of analogies that smooth functions over $\aleph_C^{(0)}$ have with ordinary analytic functions of one complex variable . However , for reasons which we have already pointed out , such analogies are invariably rather superficial and , when refering to functions over $\aleph_C^{(0)}$, we shall thus continue to use " differentiable " and " smooth " rather then " analytic " , which is much more appropriate in characterizing functions over the whole of $\aleph^{(0)}$ [4, De Witt].

We remark, though, that since $\aleph_C^{(1)} = \aleph^{(1)}$, functions over $\aleph_C^{(1)}$ of the form (3.2.10b) are properly analytic, which is how they are referred to in [4, De Witt]. However, as a common characterization for functions over $\aleph_C^{(h)}$, we think that " smooth " correctly emphasizes the crucial fact that x_B is still real.

To summarize, then, we have concluded that the smooth functions over either $\aleph_R^{(0)}$ or $\aleph_C^{(0)}$ are in one-to-one correspondence, through (3.2.10a), with the smooth functions on the real line $\mathbb{R}$. As a result, they are also in one-to-one correspondence with each other. For the case of an even variable x, this is expressed in

$$f(x_C) = f(x_R + i\,x_I) = \sum_{k=0}^{\infty} \frac{1}{k!}\,(i\,x_I)^k\,\frac{d^k f}{dx^k}(x_R) \qquad (3.2.11a)$$

whereas, for an odd variable, the soul is fully nilpotent and so the corresponding expression only has the first two terms,

$$f(\theta_C) = f(\theta_R + i\,\theta_I) = f(\theta_R) + (i\,\theta_I)\,\frac{df}{d\theta}(\theta_R) \quad . \qquad (3.2.11b)$$

The theory of integration over $\aleph_R^{(h)}$ can also be extended to $\aleph_C^{(h)}$ without modifications. Integration with respect to an odd $\aleph_R$ or $\aleph_C$ is a simple algebraic procedure in any case, based upon the formal rules

$$\int 1\,d\theta = 0 \quad , \quad \int \theta\,d\theta = 1$$

supplemented by the convention that $d\theta$ be manipulated as an odd quantity,

$$n\,d\theta = (-1)^h\,d\theta\,n \quad , \; n \in \aleph^{(h)} \qquad \text{and} \qquad d\theta\,d\theta' = -\,d\theta'\,d\theta \quad .$$

It follows that

$$\int f(\theta)\, d\theta \;=\; m' \quad , \quad f(\theta) = n + \theta\, m = n + m'\theta \quad . \tag{3.2.12a}$$

On the other hand, as it is shown in [4, De Witt] in the real case, integration with respect to an even $\aleph_R$ or $\aleph_C$ variable is independent of the particular contour,

$$\int_{x_1}^{x_2} f(x)\, dx \;=\; F(x_2) - F(x_1) \tag{3.2.12b}$$

where $F(x_B)$ is an indefinite integral of $f(x_B)$.

Suppose, in particular, that $(x_1)_B = -(x_2)_B = -\infty$ and that $f(x_B)$ is a smooth function vanishing at the end points. Its derivatives $f^{(k)}(x_B)$ are then likewise, and thus $f(x)$ itself is a smooth function vanishing at x_1, x_2. It follows [4, De Witt] that the integral (3.2.12b) is actually independent of the souls of the end points x_1, x_2. The contour can thus be displaced to coincide with $\mathbb{R}$, so that (3.2.12b) is after all nothing other than the real line integral

$$\int_{-\infty}^{+\infty} f(x_B)\, dx_B \qquad .$$

This is why, in the action integral of superspace theories, there is no ambiguity in passing from integrals over $\mathbb{R}$ to integrals over $\aleph_R^{(0)}$ or $\aleph_C^{(0)}$.

The ^{s}complex extensions (3.2.10) and (3.2.11) do also commute with an appropriately generalized definition of functional complex conjugation : given an $\aleph$ valued function on $\aleph_C^{(h)}$, $\quad f : \aleph_C^{(h)} \to \aleph \quad$, define

$$\overline{f}\,(n_c) := (f(n_c{}^*))^* \qquad (3.2.13)$$

where $n_c \in \aleph_c^{(h)}$. According to this definition, a real function $f = \overline{f}$ associates, as usual, real images to real values of the domain variable. However, the image under $f = \overline{f}$ of a general ^{s}complex element of $\aleph_c^{(h)}$ will in general be ^{s}complex as well.

Despite apparences, perhaps, the definition (3.2.13) is in fact quite natural when considering functions of a complex variable. Take for example a $\mathbb{C}$-linear map T on a complex vector space. As usual, T can be represented by a matrix with complex entries and one would expect its complex conjugate map to be represented by the complex conjugate of such matrix. This is precisely what results from definition (3.2.13). Another instance in which (3.2.13) is the appropriate functional complex conjugation to choose is the case of analytic functions of a complex variable. There, (3.2.13) assures that $\overline{f}$ is also an analytic function, which would not be the case had we used the more direct complex conjugation $f^*(z) := f(z)^*$, appropriate for functions of a real variable.

An important conclusion to be drawn, combining (3.2.11) and (3.2.13), is that a function on $\aleph_c^{(h)}$ is real if and only if its restriction to $\aleph_R^{(h)}$ is so. This commutativity of the ^{s}complex extension process with the generalized complex conjugation (and with differentiation too) is what justifies the validity of (3.2.6) in ^{s}complex superspace.

To illustrate what has been said, take as a simple example the real superspace $\mathbb{S}_R^{0,1}$ of supersymmetric quantum mechanics together with its ^{s}complex extension $\mathbb{S}_c^{0,1}$. The superspace coordinates $(x^0 \equiv t, \theta^1, \theta^2)$ are either $\aleph_R^{(h)}$ or $\aleph_c^{(h)}$ valued depending on whether we are considering the real or the ^{s}complex version of the superspace. Take the latter: an even, say, superfield F has component expansion

$$F(t,\theta^1,\theta^2) \;=\; A(t) + \theta^1\, B(t) + \theta^2\, C(t) + \theta^1\, \theta^2\, D(t)$$

where the component fields $A(t)$, etc are ^{s}complex extensions of the original fields $A(t_R)$, etc present in the component expansion of F as a superfield on the original real version of this superspace . The component fields in ^{s}complex superspace are thus as many, and carry as much information, as the original ones in real superspace .

Using the generalized complex conjugation (3.2.13) ,

$$\bar{F}(t,\theta^1,\theta^2) = (F(t^*,\theta^{1*},\theta^{2*}))^* = \bar{A}(t)- \theta^1\, \bar{B}(t) - \theta^2\, \bar{C}(t) - \theta^1\, \theta^2\, \bar{D}(t) \quad .$$

As one can easily verify , we would have got the same answer if we had complex conjugated while still in the original real superspace and performed the ^{s}complex extension afterwards .

Now , consider the new set of superspace variables

$$\tau\,(t,\theta^1,\theta^2) \equiv\; t - \theta^1\, \theta^2$$

$$\theta\,(\theta^1,\theta^2) \equiv \frac{1}{\sqrt{2}}\,(\theta^1 + i\, \theta^2)$$

$$\bar{\theta}\,(\theta^1,\theta^2) = (\theta(\theta^{1*},\theta^{2*}))^* = \frac{1}{\sqrt{2}}\,(\theta^1 - i\, \theta^2) \qquad .$$

Note that $\bar{\theta}$ is different from $\theta^* = {}^1/_{\sqrt{2}}\,(\theta^{1*} - i\, \theta^{2*})$; the same happens with $\bar{\tau} = t + \theta^1\, \theta^2$, which is different from $\tau^* = t^* + \theta^{1*}\, \theta^{2*}$. Only in the real sector of ^{s}complex superspace (i.e. real superspace) is it the case that $\bar{\tau} = \tau^*$ or $\bar{\theta} = \theta^*$. In general , θ and $\bar{\theta}$ are totally independent complex variables , as are θ^1 and θ^2 .

The one thing which makes all the difference between the real and ^{S}complex versions of Minkowski superspace is that superPoincaré translations and rotations of the latter are naturally much richer than the ones of real superspace. In ^{S}complex superspace, ^{S}complex transformations (3.2.8) do explore every new (imaginary) direction that has been introduced. In particular, and since the ε parameters are pure soul in any case, the ^{S}complex extension as doubled the number of supersymmetries available: apart from the original real supersymmetries, with $\varepsilon^{\alpha}_{M} = (\varepsilon^{\alpha}_{M})_{R}$, there are also the imaginary supersymmetries, with $\varepsilon^{\alpha}_{M} = i\,(\varepsilon^{\alpha}_{M})_{I}$, and indeed any combination of the two, with $\varepsilon^{\alpha}_{M} = (\varepsilon^{\alpha}_{M})_{R} + i\,(\varepsilon^{\alpha}_{M})_{I}$.

It is only within this wider Minkowski supersymmetry that a subclass of transformations kinematical with respect to time x^0 may in fact be identified; it consists of the ^{S}complex supersymmetries with parameters constrained by $\varepsilon^{1}_{M} = i\,\varepsilon^{4}_{M}$, $\varepsilon^{2}_{M} = i\,\varepsilon^{3}_{M}$. Note that this would be impossible were we restricted to $\aleph^{(1)}_{R}$ valued ε parameters.

In the end, one important question remains: is it the case after all that, given a particular real Minkowski superspace field theory, all this new symmetry is already there? The answer is yes and no: " yes " because, as we explained, the action integral of such theory can be equally written as an integral over ^{S}complex superspace and thus ^{S}complex transformations of the form (3.2.8) will necessarily leave it invariant; " no " because the basic real degrees of freedom (either at superfield or component field level) from the original real superspace field theory do not accommodate a representation of the full ^{S}complex superPoincaré symmetry of ^{S}complex Minkowski superspace, but only of the subclass generated by real vector fields, corresponding to the superPoincaré symmetry of the original real superspace.

In fact, under a general ^{S}complex superPoincaré transformation real superfields (as real component fields) get imaginary increments as well: take for example a real superfield $V = \bar{V}$ on ^{S}complex superspace; the effect of an infinitesimal ^{S}complex supersymmetry transformation is such that

$$(\overline{\delta V}) = (\overline{\varepsilon_M^\alpha\, Q_\alpha^M\, V}) = (\varepsilon_M^\alpha)^*\; Q_\alpha^M\, V \neq \delta V \quad .$$

Another way of saying it is that, for a general superfield F on ^{s}complex superspace,

$$(\overline{\delta F}) \neq \delta \overline{F} \quad .$$

As far as the component fields are concerned, the supersymmetry representations carried by F and $\overline{F}$ are still the same with respect to the real parameters $(\varepsilon_M^\alpha)_R$ but are minus one another with respect to the imaginary parameters $(\varepsilon_M^\alpha)_I$.

As a result, what one obtains after the ^{s}complex extension of a field theory in real Minkowski superspace is a ^{s}complex superspace field theory in a gauge, the real gauge, where the basic superfields are either real or appear in pairs F, $\overline{F}$. The real gauge is preserved under real superPoincaré transformations, the ones of the original real superspace, but is broken by general ^{s}complex superPoincaré transformations, in which real superfields get complex increments and F and $\overline{F}$ transform independently.

§ 3.3 *The Null-plane Canonical Decomposition*

Even though null-plane canonical superfields (usually known as light-cone superfields) are no novelty [18] , we thought it worthwhile to discuss them here in this new context of canonical superspace decomposition because the null-plane case provides a simple illustration of the methods and a good comparison material with what is to follow in § 3.4 .

The null-plane vector indices are $m,n = \pm,1,2$ or $\ell = 1,2$, where

$$v^{\pm} \equiv \frac{1}{\sqrt{2}}(v^0 \pm v^3) \quad \text{and} \quad v_{\pm} \equiv \frac{1}{\sqrt{2}}(v_0 \pm v_3)\,,$$

such that

$$v^{\pm} = - v_{\mp} \quad \text{and} \quad v^m v_m = v^+ v_+ + v^- v_- + v^{\ell} v_{\ell} = - 2\, v_+ v_- + v_{\ell} v_{\ell}\,.$$

Between the null-plane coordinates $x^{\pm}$, x^+ is conventionally chosen as the dynamical parameter .

In relation to the null-plane " time " x^+, extending the ordinary (flat) spacetime canonical decomposition to real Minkowski superspace does not involve any problems . This is because there exists a two real parameter subclass of kinematical supersymmetries whose generating vector fields , together with the ones generating the kinematical translations and rotations , span a subalgebra of the real superPoincaré vector field Lie algebra $\mathcal{G}_R^{(0)}$:

From real (3.2.6c) ,

$$\delta x^+ = \frac{i}{\sqrt{2}}(\varepsilon_M^1 - \varepsilon_M^3)(\theta_M^1 - \theta_M^3) + \frac{i}{\sqrt{2}}(\varepsilon_M^2 - \varepsilon_M^4)(\theta_M^2 - \theta_M^4)$$

which means that real supersymmetries with parameters constrained by $\varepsilon^1_T \equiv \varepsilon^1_M - \varepsilon^3_M = 0$ and $\varepsilon^2_T \equiv \varepsilon^2_M - \varepsilon^4_M = 0$ do not involve " time " shifts δx^+ throughout the whole θ sector of real Minkowski superspace, in which they induce the foliation $\theta^1_T \equiv \theta^1_M - \theta^3_M = \text{const.}$ and $\theta^2_T \equiv \theta^2_M - \theta^4_M = \text{const.}$.

The Weyl representation (see also the Appendix to § 3), with

$$\begin{aligned} \theta^1 &\equiv \frac{1}{2}(\theta^2_M + \theta^4_M) + \frac{i}{2}(\theta^1_M + \theta^3_M) = (\bar\theta^{\dot 1})^* \\ \theta^2 &\equiv -\frac{1}{2}(\theta^1_M - \theta^3_M) + \frac{i}{2}(\theta^2_M - \theta^4_M) = (\bar\theta^{\dot 2})^* \end{aligned} \tag{3.3.1}$$

is thus a good representation to work with . Note that, with this choice of notation, the derivatives $\partial/\partial x^1$ and $\partial/\partial\theta^1$ are both naturally denoted by ∂_1 ; the same happens with ∂_2. In the sequel, however, we shall have no need to use symbols ∂_1 and ∂_2 in their new role of spinor derivatives and thus they will invariably stand for $\partial/\partial x^1$ and $\partial/\partial x^2$.

In the Weyl representation, a full real supersymmetry transformation (3.2.6c) is generated by $(-\varepsilon^\rho Q_\rho + \bar\varepsilon^{\dot\rho}\bar Q_{\dot\rho})$, $\rho = 1,2$, produces $\delta\theta^\rho = \varepsilon^\rho$, $\delta\bar\theta^{\dot\rho} = \bar\varepsilon^{\dot\rho}$ and

$$\delta x^+ = i\sqrt{2}\,(\varepsilon^2\bar\theta^{\dot 2} + \bar\varepsilon^{\dot 2}\theta^2) \quad . \tag{3.3.2}$$

The basic Poincaré vector field superalgebra (3.2.2) reads, explicitly,

(non-trivial relations only)

$$\begin{aligned} &\{Q_1,\bar Q_{\dot 1}\} = 2\sqrt{2}\,i\,P_- \quad , \quad \{Q_2,\bar Q_{\dot 2}\} = 2\sqrt{2}\,i\,P_+ \quad , \\ &\{Q_1,\bar Q_{\dot 2}\} = -2\,i\,(P_1 - i\,P_2) \quad , \quad \{Q_2,\bar Q_{\dot 1}\} = -2\,i\,(P_1 + i\,P_2) \end{aligned} \tag{3.3.3a}$$

and (3.3.3b↓)

*	P_+	P_-	P_1	P_2	Q_1	Q_2	$\bar{Q}_{\dot{1}}$	$\bar{Q}_{\dot{2}}$
$[J_{12},\ ]$			$-P_2$	P_1	$-\frac{i}{2}Q_1$	$\frac{i}{2}Q_2$	$\frac{i}{2}\bar{Q}_{\dot{1}}$	$-\frac{i}{2}\bar{Q}_{\dot{2}}$
$[J_{+-},\ ]$	$-P_+$	P_-			$\frac{1}{2}Q_1$	$-\frac{1}{2}Q_2$	$\frac{1}{2}\bar{Q}_{\dot{1}}$	$-\frac{1}{2}\bar{Q}_{\dot{2}}$
$[J_{+1},\ ]$		P_1	P_+		$-\frac{1}{\sqrt{2}}Q_2$		$-\frac{1}{\sqrt{2}}\bar{Q}_{\dot{2}}$	
$[J_{+2},\ ]$		P_2		P_+	$\frac{i}{\sqrt{2}}Q_2$		$-\frac{i}{\sqrt{2}}\bar{Q}_{\dot{2}}$	
$[J_{-1},\ ]$	P_1		P_-			$-\frac{1}{\sqrt{2}}Q_1$		$-\frac{1}{\sqrt{2}}\bar{Q}_{\dot{1}}$
$[J_{-2},\ ]$	P_2			P_-		$-\frac{i}{\sqrt{2}}Q_1$		$\frac{i}{\sqrt{2}}\bar{Q}_{\dot{1}}$

and

$$[J_{+1},J_{-2}] = [J_{-1},J_{+2}] = J_{12} \quad , \quad [J_{+1},J_{-1}] = [J_{+2},J_{-2}] = -\,J_{+-}$$

$$[J_{+-},J_{\pm\ell}] = -\,(\pm\, J_{\pm\ell}) \tag{3.3.3c}$$

$$[J_{12},J_{\pm 1}] = -\,J_{\pm 2} \quad , \quad [J_{12},J_{\pm 2}] = J_{\pm 1} \quad .$$

Canonical supersymmetry transformations form a subclass of real (3.2.6c), with parameters ε^1, $\bar{\varepsilon}^{\dot{1}}$ free and $\varepsilon^2 = \bar{\varepsilon}^{\dot{2}} = 0$. They are generated by $(-\,\varepsilon^1 Q_1 + \bar{\varepsilon}^{\dot{1}}\bar{Q}_{\dot{1}})$ and produce kinematical displacements δx^ℓ, δx^- and $\delta\theta^1 = \varepsilon^1$, $\delta\bar{\theta}^{\dot{1}} = \bar{\varepsilon}^{\dot{1}}$ along the super-hypersurfaces of constant x^+, θ^2, $\bar{\theta}^{\dot{2}}$.

The kinematical subalgebra of the superPoincaré vector field Lie algebra $\mathcal{G}^{(0)}_R$ is therefore the appropriate (refer to real (3.2.5)) span of

$$P_\ell \,, P_- \,, Q_1 \,, \bar{Q}_{\dot{1}} \,, J_{12} \,, J_{-\ell} \quad ;$$

J_{+-} is out because it does generate dynamical shifts : from (3.2.6c) ,

$$\delta x^{+} = v^{+\ell} x^{\ell} - v^{+-} x^{+} \quad .$$

Once canonical superspace has been identified, it can be used to obtain the canonical variables of a superspace field theory , the basic canonical superfields and their conjugate momenta , in a way which is covariant with respect to canonical supersymmetry : let $\mathfrak{S}$ be a real Minkowski superspace action integral ,

$$\mathfrak{S} = \int \mathcal{L}[F] \; d^4x \, d^2\theta \, d^2\bar{\theta} \qquad \text{where} \quad d^2\theta \equiv \frac{1}{2} d\theta^1 d\theta^2 \quad \text{and} \quad d^2\bar{\theta} \equiv \frac{1}{2} d\bar{\theta}^{\dot{2}} d\bar{\theta}^{\dot{1}} \tag{3.3.4a}$$

and F stands for a , or a set of , Minkowski superfield(s) . If the integration with respect to the dynamical coordinates $\theta^2 , \bar{\theta}^{\dot{2}}$ is performed , the original full superspace action integral (3.3.4a) reduces to

$$\mathfrak{S} = \int -\frac{1}{4} \left(\bar{D}_{\dot{2}} D_2 \, \mathcal{L} \right) \Big|_{\theta^2 = \bar{\theta}^{\dot{2}} = 0} \, d^4x \, d\bar{\theta}^{\dot{1}} \, d\theta^1 \tag{3.3.4b}$$

where the resulting canonical superspace Lagrangian density is a kinematical functional , i.e. involving only derivatives $\partial_\ell , \partial_- , (D_1)| , (\bar{D}_{\dot{1}})|$ of the canonical superfields

$$\begin{array}{cc} (F)| & \left(\frac{1}{2} (\bar{D}_{\dot{2}} D_2 - D_2 \bar{D}_{\dot{2}}) \, F \right)| \\ (D_2 \, F)| & (\bar{D}_{\dot{2}} \, F)| \end{array} \tag{3.3.5}$$

and their " time " derivatives ∂_+ ; above and throughout this section , $(\ \ )|$ always means $(\ \ )|_{\theta^2 = \bar{\theta}^{\dot{2}} = 0}$.

Each of the canonical superfields in (3.3.5), generically denoted f, transforms covariantly under canonical supersymmetry,

$$\delta f = (-\varepsilon^1 (Q_1)| + \bar{\varepsilon}^{\dot{1}}(\bar{Q}_{\dot{1}})|)\, f \qquad , \tag{3.3.6}$$

while dynamical supersymmetry, i.e. parameters $\varepsilon^2, \bar{\varepsilon}^{\dot{2}}$, rotates the canonical superfields and their " time " derivatives amongst themselves. The precise form of the transformation rules is not essential here and can be deduced straightforwardly from (3.3.3) and the Weyl representation versions of (3.2.1), (3.2.2), (3.2.3) and (3.2.6).

The same holds for the transformation properties of the canonical superfields (3.3.5) under dynamical Lorentz rotations, i.e. the ones associated with parameters v^{+-} and $v^{+\ell}$. As for the kinematical Lorentz rotations, i.e. those associated with parameters v^{12} and $v^{-\ell}$, it suffices to say that they do not involve any " time " derivatives ∂_+.

The identification of the conjugate canonical supermomenta can be done directly from (3.3.4b) in the usual way: roughly speaking, as the coefficients of the " time " derivatives of the basic canonical superfields (3.3.5).

We finish this section with two simple examples of what canonical superspace Lagrangian densities (3.3.4b) look like: in the Weyl representation, with spinor indices $\rho,\gamma = 1,2$,

i) A chiral Minkowski superfield Φ, $\bar{D}_{\dot{\rho}}\Phi = 0$, produces only two independent canonical superfields (3.3.5),

$$\phi \equiv (\Phi)| \quad \text{and} \quad \bar{\psi} \equiv -\frac{1}{4}(D^\rho D_\rho \Phi)| = \frac{1}{2}(D_1)|\,(D_2\Phi)| \quad , \tag{3.3.7a}$$

where ϕ and ψ are both constrained canonical superfields , since

$$(\bar{D}_{\dot{1}})\big| \; \phi = (\bar{D}_{\dot{1}})\big| \; \psi = 0 \quad .$$

It is simply more convenient to use $\bar{\psi}$ than $(D_2 \Phi)\big|$ itself ; one can switch from one to the other through (3.3.7a) and its inverse

$$(D_2 \Phi)\big| = -\frac{1}{\sqrt{2}\partial_-}(\partial_1 + i\,\partial_2)\,(D_1)\big|\,\phi - \frac{i}{\sqrt{2}\partial_-}\,(\bar{D}_{\dot{1}})\big|\,\bar{\psi} \quad . \tag{3.3.7b}$$

The anti-chiral $\bar{\Phi}$, $D_\rho \bar{\Phi} = 0$, similarly only produces

$$(\bar{\Phi})\big| = \bar{\phi} \quad \text{and} \quad -\frac{1}{4}\,(\bar{D}_{\dot{\rho}}\bar{D}^{\dot{\rho}}\,\bar{\Phi})\big| = -\frac{1}{2}\,(\bar{D}_{\dot{1}})\big|\,(\bar{D}_{\dot{2}}\,\bar{\Phi})\big| = \psi \quad . \tag{3.3.7c}$$

The original Minkowski superspace action integral , as in [3 , Wess & Bagger] ,

$$\mathfrak{I} = \int \bar{\Phi}\Phi \; d^4x \; d^2\theta \; d^2\bar{\theta} \tag{3.3.8}$$

reduces to the canonical superspace action integral

$$\mathfrak{I} = \int \frac{i}{2\sqrt{2}} \left(\bar{\phi}\,\frac{\Box}{\partial_-}\,\phi \;+\; \bar{\psi}\,\frac{1}{\partial_-}\,\psi \right) d^4x \; d\bar{\theta}^{\dot{1}} \; d\theta^1 \tag{3.3.9}$$

where $\Box \equiv \partial^m \partial_m$.

Above , the canonical supermultiplet structure of the propagating and non-propagating degrees of freedom shows up very distinctively . The fact that the canonical superfields ϕ and ψ are both constrained , however , does not allow for straightforward functional differentiation ; derivatives $(D_1)|$ and $(\bar{D}_{\dot{1}})|$ must be inserted at appropriate places .

ii) The case of the real (abelian) Minkowski superfield V , $\bar{V} = V$, does not have such problem . It produces four real , but otherwise unconstrained , canonical superfields

$$u \equiv (V)| \qquad \mathcal{U} \equiv (\tfrac{1}{2}(\bar{D}_{\dot{2}}D_2 - D_2\bar{D}_{\dot{2}})\, V)| \qquad v \equiv (D_2V)| \qquad \bar{v} = (\bar{D}_{\dot{2}}V)| \tag{3.3.10}$$

and the original Minkowski superspace action integral , as in [3 , Wess & Bagger] ,

$$\mathfrak{S} = \int \tfrac{1}{4}(\, W^{\rho}\, D_{\rho}V \;+\; \bar{W}_{\dot{\rho}}\, \bar{D}^{\dot{\rho}}\, V\,)\, d^4x\, d^2\theta\, d^2\bar{\theta} \;= \;= \int (\, \tfrac{1}{8}\, \bar{D}_{\dot{\rho}}\bar{D}^{\dot{\rho}} V\, D^{\gamma}D_{\gamma}V \;+\; \partial^m V\, \partial_m V\,)\, d^4x\, d^2\theta\, d^2\bar{\theta} \tag{3.3.11}$$

reduces to the canonical superspace action integral

$$\mathfrak{S} = \int \tfrac{i}{2\sqrt{2}}\,(\, \bar{\mathcal{V}}_{p}\, \tfrac{\Box}{\partial_-}\, \mathcal{V}_{p} \;+\; \bar{\mathcal{V}}_{c}\, \tfrac{1}{\partial_-}\, \mathcal{V}_{c}\,)\, d^4x\, d\bar{\theta}^{\dot{1}}\, d\theta^1 \tag{3.3.12}$$

where the basic canonical superfield content is encoded in

$$\mathcal{V}_{p} \equiv \tfrac{1}{\sqrt{2}}(\bar{D}_{\dot{1}})|\; v \;+\; \tfrac{1}{\sqrt{2}\partial_-}(\partial_1 + i\,\partial_2)\, d_-\, u \qquad \mathcal{V}_{c} \equiv -\tfrac{1}{\sqrt{2}}(\partial_1 - i\,\partial_2)(\bar{D}_{\dot{1}})|\; v \;+\; \tfrac{1}{\sqrt{2}\partial_-}(\partial_+\partial_- - \partial_\ell\partial_\ell)\, d_-\, u \;+\; \tfrac{i}{2}\, d_-\, \mathcal{U}\;. \tag{3.3.13}$$

Above, $d_- \equiv \frac{1}{\sqrt{2}} \bar{D}_{\dot{1}})| \, (D_1)|$. Therefore, $\bar{d}_- = -\frac{1}{\sqrt{2}} (D_1)| \, \bar{D}_{\dot{1}})|$ and

$$d_- - \bar{d}_- = 2i\,\partial_- \quad .$$

This is very similar in structure to the action integral for an ordinary gauge field v_m ,

$$\mathfrak{S} = \int -\frac{1}{4} v^{mn} v_{mn} \, d^4x = \int \frac{1}{2} (\mathcal{V}_p^{\ell} \Box \mathcal{V}_p^{\ell} + \mathcal{V}_c \mathcal{V}_c) \, d^4x$$

where $v_{mn} \equiv \partial_m v_n - \partial_n v_m$, $\ell = 1,2$ and

$$\mathcal{V}_p^{\ell} \equiv \frac{1}{\partial_-} v_{-\ell} = v_{\ell} - \frac{1}{\partial_-} \partial_{\ell} v_-$$

$$\mathcal{V}_c \equiv \frac{1}{\partial_-} \partial^m v_{-m} = \partial_{\ell} v_{\ell} + \frac{1}{\partial_-} (\partial_+ \partial_- - \partial_{\ell} \partial_{\ell}) v_- - \partial_- v_+ \quad . \tag{3.3.14}$$

Comparing (3.3.13) and (3.3.14) , one concludes that the canonical superfields $v, \bar{v}$ correspond to the propagating v_{ℓ} , that u corresponds to the gauge degree of freedom v_- and $\mathcal{U}$ parallels the Lagrange multiplier v_+ .

§ 3.4 *The Minkowski Canonical Decomposition*

Now we turn to the Minkowski, i.e. orthonormal, coordinates (x^0,x^j) and again the question of whether Minkowski supersymmetry (3.2.6c) admits a corresponding kinematical subclass.

The fact is that no supersymmetries exist, either in real or ^{s}complex Minkowski superspaces, which do not involve time shifts δx^0. This is very simply because $\varsigma\gamma^0$ is, in any representation, an invertible matrix (as opposed to $\varsigma\gamma^+$, which isn't); it can be thought of as the matrix of a non-degenerate spinor scalar product, and therefore no non-zero set of supersymmetry parameters ε^α exists such that $\delta x^0 = i\,\varepsilon^\alpha\,(\varsigma\gamma^0)_{\alpha\beta}\,\theta^\beta$ is null for all possible values of the superspace coordinates θ^α.

So, we shall somewhat loosen the sense of " kinematical " and ask instead whether there exists a subclass of supersymmetries, associated with a foliation of the θ sector of Minkowski superspace into hypersurfaces θ_T = constant, involving no δx^0 shifts at least in the model hypersurface $\theta_T = 0$.

In this sense kinematical supersymmetries do exist but only in ^{s}complex superspace. To see this, take real superspace first and consider any foliation of its θ sector into hypersurfaces (of real codimension 2, say)

$$\begin{aligned} \theta^1_T &\equiv A_\alpha\,\theta^\alpha_M = \text{const.} \\ \theta^2_T &\equiv B_\alpha\,\theta^\alpha_M = \text{const.} \end{aligned} \tag{3.4.1a}$$

where the ordinary number coefficients A_α, B_α are real; in real superspace they have to, since the θ^α_M are real and we want the equations for model hypersurface, $A_\alpha\,\theta^\alpha_M =$ $= B_\alpha\,\theta^\alpha_M = 0$, to make sense. These equations must be solvable for two of the θ^α_M, say θ^1_M and θ^2_M, in terms of the other two; thus, in the model hypersurface $\theta^1_T = \theta^2_T = 0$,

$$\theta^1_M = a\,\theta^3_M + b\,\theta^4_M \qquad (3.4.1b)$$
$$\theta^2_M = c\,\theta^3_M + d\,\theta^4_M$$

where a,b,c,d are again real numbers .

The supersymmetries parallel to the foliation (3.4.1a) have parameters constrained by $\varepsilon^1_T \equiv A_\alpha\,\varepsilon^\alpha_M = 0$, $\varepsilon^2_T \equiv B_\alpha\,\varepsilon^\alpha_M = 0$, i.e. satisfying equations identical to (3.4.1b) .

Now , we have from (3.2.6c) and $\varsigma_M = \gamma^M_0$ that

$$\delta x^0 = i\,\varepsilon^\alpha_M\,\delta_{\alpha\beta}\,\theta^\beta_M$$

and therefore , under a supersymmetry along the model hypersurface , (3.4.2↓)

$$\delta x^0 = i\,(1 + a^2 + c^2)\,\varepsilon^3_M\,\theta^3_M + i\,(1 + b^2 + d^2)\,\varepsilon^4_M\,\theta^4_M + i\,(ab + cd)\,(\varepsilon^3_M\,\theta^4_M + \varepsilon^4_M\,\theta^3_M) \quad .$$

Since a,b,c,d are real , as we are in real superspace , it is clear that δx^0 must forcibly be non-zero , whatever the foliation we consider . And this would still be the case had we considered hypersurfaces of real codimension 3 .

However , what (3.4.2) implies is that kinematical supersymmetries do exist in ^{s}complex superspace ; there , the foliation (3.4.1a) can be defined with complex number coefficients A_α , B_α and a,b,c,d so that , from (3.4.2) , the general solution to $\delta x^0 = 0$ is

$$a^2 + b^2 = -1 \quad , \quad c = \pm b \quad , \quad d = -(\pm a) \quad . \qquad (3.4.3)$$

In ^{s}complex Minkowski superspace , each solution of (3.4.3) defines a particular canonical hypersurface (3.4.1b) along which ^{s}complex supersymmetry is kinematical. Yet other canonical hypersurfaces exist: they correspond to foliations (3.4.1a) for which the model hypersurface equations $\theta^1_T = \theta^2_T = 0$ can not be solved with respect to θ^1_M, θ^2_M as in (3.4.1b) but instead with respect to some other pair of coordinates θ^α_M. Nevertheless, it is straightforward to show that no canonical hypersurfaces exist with complex codimension 1.

A particularly simple solution of (3.4.3) is $b = c = i$, $a = d = 0$, for which the corresponding foliation of the θ sector of ^{s}complex Minkowski superspace is parametrized by

$$\theta^1_T \equiv \theta^1_M - i\,\theta^4_M \quad , \quad \theta^2_T \equiv \theta^2_M - i\,\theta^3_M \quad ;$$

thus, ^{s}complex supersymmetries with parameters constrained by

$$\varepsilon^1_T \equiv \varepsilon^1_M - i\,\varepsilon^4_M = 0 \quad , \quad \varepsilon^2_T \equiv \varepsilon^2_M - i\,\varepsilon^3_M = 0$$

preserve the foliation and, in the canonical hypersurface $\theta^1_T = \theta^2_T = 0$, do not involve time shifts δx^0.

The canonical representation (see also the Appendix to § 3), with

$$\theta^{I}(\theta^\alpha_M) \equiv \frac{1}{\sqrt{2}}(\theta^1_M + i\,\theta^4_M) \qquad \bar{\theta}^{\dot{I}}(\theta^\alpha_M) = \frac{1}{\sqrt{2}}(\theta^1_M - i\,\theta^4_M)$$

$$\text{and} \tag{3.4.4}$$

$$\theta^{II}(\theta^\alpha_M) \equiv \frac{1}{\sqrt{2}}(\theta^2_M + i\,\theta^3_M) \qquad \bar{\theta}^{\dot{II}}(\theta^\alpha_M) = \frac{1}{\sqrt{2}}(\theta^2_M - i\,\theta^3_M)$$

is thus an appropriate representation to use; note that we are in ^{s}complex superspace and so θ^ξ and $\bar{\theta}^{\dot{\xi}}$, $\xi = I,II$, are totally independent complex coordinates.

In the canonical representation , a full ^{s}complex supersymmetry transformation (3.2.6c) is generated by $(\varepsilon^{\xi} Q_{\xi} - \bar{\varepsilon}^{\dot{\xi}} \bar{Q}_{\dot{\xi}})$, produces $\delta\theta^{\xi} = \varepsilon^{\xi}$, $\delta\bar{\theta}^{\dot{\xi}} = \bar{\varepsilon}^{\dot{\xi}}$ and

$$\delta x^0 = i\,(\varepsilon^{I}\,\bar{\theta}^{\dot{I}} + \varepsilon^{II}\,\bar{\theta}^{\dot{II}} + \bar{\varepsilon}^{\dot{I}}\,\theta^{I} + \bar{\varepsilon}^{\dot{II}}\,\theta^{II}) \quad . \tag{3.4.5}$$

The basic Poincaré vector field superalgebra (3.2.2) reads , explicitly ,

(non-trivial relations only)

$$\begin{aligned}
\{Q_I,Q_I\} &= -\,2\,(P_1 + i\,P_2) = \{\bar{Q}_{\dot{II}},\bar{Q}_{\dot{II}}\} \\
\{Q_{II},Q_{II}\} &= 2\,(P_1 - i\,P_2) = \{\bar{Q}_{\dot{I}},\bar{Q}_{\dot{I}}\} \\
\{Q_I,Q_{II}\} &= 2\,P_3 = -\,\{\bar{Q}_{\dot{I}},\bar{Q}_{\dot{II}}\} \\
\{Q_I,\bar{Q}_{\dot{I}}\} &= \{Q_{II},\bar{Q}_{\dot{II}}\} = 2i\,P_0
\end{aligned} \tag{3.4.6a}$$

and

*	Q_I	Q_{II}	$\bar{Q}_{\dot{I}}$	$\bar{Q}_{\dot{II}}$
$[J_{12},\ ]$	$\frac{i}{2}Q_I$	$-\frac{i}{2}Q_{II}$	$-\frac{i}{2}\bar{Q}_{\dot{I}}$	$\frac{i}{2}\bar{Q}_{\dot{II}}$
$[J_{13},\ ]$	$\frac{1}{2}Q_{II}$	$-\frac{1}{2}Q_I$	$\frac{1}{2}\bar{Q}_{\dot{II}}$	$-\frac{1}{2}\bar{Q}_{\dot{I}}$
$[J_{23},\ ]$	$\frac{i}{2}Q_{II}$	$\frac{i}{2}Q_I$	$-\frac{i}{2}\bar{Q}_{\dot{II}}$	$-\frac{i}{2}\bar{Q}_{\dot{I}}$
$[J_{01},\ ]$	$\frac{i}{2}\bar{Q}_{\dot{I}}$	$-\frac{i}{2}\bar{Q}_{\dot{II}}$	$-\frac{i}{2}Q_I$	$\frac{i}{2}Q_{II}$
$[J_{02},\ ]$	$-\frac{1}{2}\bar{Q}_{\dot{I}}$	$-\frac{1}{2}\bar{Q}_{\dot{II}}$	$-\frac{1}{2}Q_I$	$-\frac{1}{2}Q_{II}$
$[J_{03},\ ]$	$-\frac{i}{2}\bar{Q}_{\dot{II}}$	$-\frac{i}{2}\bar{Q}_{\dot{I}}$	$\frac{i}{2}Q_{II}$	$\frac{i}{2}Q_I$

(3.4.6b)

plus the usual Poincaré algebra commutators (3.2.2c) and (3.2.2e) .

Comment : If we change the ^{s}complex superspace time coordinate

$$x^0 \rightarrow \tau \equiv x^0 - i\,(\theta^I \bar{\theta}^{\dot{I}} + \theta^{II} \bar{\theta}^{\dot{II}}) \tag{3.4.7}$$

where $(\theta^I \bar{\theta}^{\dot{I}} + \theta^{II} \bar{\theta}^{\dot{II}})$ is a pure soul Lorentz scalar , then a ^{s}complex supersymmetry transformation with $\bar{\varepsilon}^{\dot{\xi}} = 0$ involves no dynamical shift $\delta\tau$ anywhere in the θ sector of ^{s}complex Minkowski superspace , as can easily be deduced from (3.4.5) . This coordinate change (3.4.7) corresponds , as usual , to a change in the coset-space parametrization of ^{s}complex superspace

$$e^{x^m P_m}\, e^{(\theta^\xi Q_\xi - \bar{\theta}^{\dot{\xi}} \bar{Q}_{\dot{\xi}})}\, \mathbb{1} \rightarrow e^{(\tau P_0 + x^j P_j)}\, e^{\theta^\xi Q_\xi}\, e^{-\bar{\theta}^{\dot{\xi}} \bar{Q}_{\dot{\xi}}}\, \mathbb{1} \quad .$$

So , in ^{s}complex Minkowski superspace with time coordinate τ , the vector fields Q_ξ are truly kinematical while the $\bar{Q}_{\dot{\xi}}$ are dynamical .

The overall picture is then the following : within ^{s}complex Minkowski superspace , canonical supersymmetry transformations form a subclass of ^{s}complex (3.2.6c) , with parameters ε^ξ free and $\bar{\varepsilon}^{\dot{\xi}} = 0$. They are generated by $(\varepsilon^\xi Q_\xi)$ and produce kinematical displacements

$$\delta x^j = i\, \varepsilon^\xi\, (\varsigma\, \sigma^j)_{\xi\zeta}\, \theta^\zeta \quad , \quad \delta\theta^\xi = \varepsilon^\xi$$

along the canonical , i.e. with $\bar{\theta}^{\dot{\xi}} = 0$, superhypersurfaces of constant x^0 ; above , $\varsigma_{\xi\zeta} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ and $(\sigma_j)^\xi{}_\zeta$ are the Pauli matrices .

The kinematical subalgebra of the superPoincaré vector field Lie algebra $\mathcal{G}_c^{(0)}$ is thus the appropriate (refer to ^{s}complex (3.2.5)) span of P_j , Q_ξ , J_{ij} . In fact, making use of the notation

$$^{3}(\quad) \equiv (\quad)\Big|_{\bar{\theta}^{\dot{\xi}}=0} \quad ,$$

the vector fields ${}^{3}P_j$, ${}^{3}Q_\xi$ and ${}^{3}J_{ij}$ precisely realize the 3 d Euclidean holomorphic supersymmetry algebra (2.1.1b). Therefore, the canonical superhypersurfaces in $\mathbb{S}_c^{3,1}$ with constant x^0 are copies of the 3 d Euclidean holomorphic superspace $\mathbb{S}_h^{3,0}$.

Consider now a field theory with ^{s}complex Minkowski superspace action integral

$$\mathfrak{S} = \int \mathcal{L}[F]\ d^4x\ d^2\theta\ d^2\bar{\theta}$$

$$\text{where} \quad d^2\theta \equiv \frac{1}{2}\, d\theta^{I}\, d\theta^{II} \quad \text{and} \quad d^2\bar{\theta} \equiv \frac{1}{2}\, d\bar{\theta}^{\dot{II}}\, d\bar{\theta}^{\dot{I}} \quad . \tag{3.4.8a}$$

Despite apparences, this coincides with the action integral $\mathfrak{S}$ in (3.3.4a). In fact, the Weyl and canonical representations relate in such a way that the full $d^2\theta\, d^2\bar{\theta}$ is the same - see (3.4.12). As a result, the Lagrangian density that results from the full θ integration is essentially the same in (3.4.8a) and (3.3.4a), the sole difference being that the former is the ^{s}complex extension of the latter. But since, by construction, they coincide over the ordinary real number range of the spacetime coordinates x^m, which is all that matters in superspace action integrals as we have pointed out in § 3.2, we arrive at the desired conclusion.

Integration with respect to the dynamical $\bar{\theta}^{\dot{\xi}}$ coordinates in (3.4.8a) produces

$$\mathfrak{S} = \int -\frac{1}{4}\, (\bar{D}_{\dot{II}} \bar{D}_{\dot{I}}\, \mathcal{L})\Big|_{\bar{\theta}^{\dot{\xi}}=0}\ d^4x\ d\theta^{II}\, d\theta^{I} \quad , \tag{3.4.8b}$$

where the resulting canonical superspace Lagrangian density is a kinematical functional, i.e. involving only derivatives ∂_j, ${}^3D_\xi$ of the canonical superfields

$$\mathcal{g} \equiv (F)\big| \qquad \mathcal{G} \equiv (\tfrac{1}{2}(\bar{D}_{\dot{II}}\bar{D}_{\dot{I}} - \bar{D}_{\dot{I}}\bar{D}_{\dot{II}})\,F)\big|$$
$$f^\xi \equiv (\bar{D}_{\dot{\xi}}\ F)\big| \tag{3.4.9}$$

and their time derivatives ∂_0; throughout this section, $(\ \ )\big|$ always means $(\ \ )\big|_{\bar{\theta}^{\dot{\xi}}=0}$.

Under canonical supersymmetry, each of the canonical superfields (3.4.9), generically denoted f, transforms covariantly

$$\delta f = (\varepsilon^\xi\,{}^3Q_\xi)\,f \ , \tag{3.4.10}$$

while under dynamical supersymmetry,

$$\begin{aligned}
\delta \mathcal{g} &= \bar{\varepsilon}^{\dot{I}}(-f^{I} - d^{I}\mathcal{g}) + \bar{\varepsilon}^{\dot{II}}(-f^{II} - d^{II}\mathcal{g})\\
\delta f^{I} &= \bar{\varepsilon}^{\dot{I}}(-(\partial_1 - i\partial_2)\mathcal{g} - d^{I}f^{I}) + \bar{\varepsilon}^{\dot{II}}(-\mathcal{G} + \partial_3\mathcal{g} - d^{II}f^{I})\\
\delta f^{II} &= \bar{\varepsilon}^{\dot{I}}(\mathcal{G} + \partial_3\mathcal{g} - d^{I}f^{II}) + \bar{\varepsilon}^{\dot{II}}((\partial_1 + i\partial_2)\mathcal{g} - d^{II}f^{II})\\
\delta \mathcal{G} &= \bar{\varepsilon}^{\dot{I}}((\partial_1 - i\partial_2)f^{II} + \partial_3 f^{I} - d^{I}\mathcal{G}) + \bar{\varepsilon}^{\dot{II}}((\partial_1 + i\partial_2)f^{I} - \partial_3 f^{II} - d^{II}\mathcal{G})
\end{aligned}$$

where $d^\xi \equiv 2\,i\,\theta^\xi\,\partial_0$.

Under a ^{S}complex Lorentz rotation generated by J_{ij}, the canonical superfields $\mathcal{g}$ and $\mathcal{G}$ transform as scalars, while f^ξ transforms as a 3 dimensional spinor,

$$J_{ij}: \qquad \delta\mathcal{g} = {}^3J_{ij}\,\mathcal{g} \quad , \quad \delta\mathcal{G} = {}^3J_{ij}\,\mathcal{G}$$
$$\delta f^\xi = {}^3J_{ij}\,f^\xi + ({}^3\sigma_{ij})^\xi{}_\zeta\,f^\zeta \tag{3.4.11}$$

where ${}^{3}\sigma_{ij} \equiv \frac{1}{4}[\sigma_i,\sigma_j]$ and $(\sigma_j)^{\xi}{}_{\zeta}$ are the Pauli matrices. On the other hand, under dynamical Lorentz rotations, the transformation laws are, for example,

$$J_{01}: \qquad \delta g = \hat{J}_{01}\, g + \frac{i}{2}\theta^{I} f^{I} - \frac{i}{2}\theta^{II} f^{II}$$

$$\delta G = \hat{J}_{01}\, G + \frac{i}{2}(\partial_I f^{II} + \partial_{II} f^{I}) - i(\theta^{I}\partial_1 f^{II} - \theta^{II}\partial_1 f^{I})$$

$$\delta f^{I} = \hat{J}_{01} f^{I} - \frac{i}{2}\partial_I g + i\theta^{I}\partial_1 g - \frac{i}{2}\theta^{II} G$$

$$\delta f^{II} = \hat{J}_{01} f^{II} + \frac{i}{2}\partial_{II} g + i\theta^{II}\partial_1 g - \frac{i}{2}\theta^{I} G$$

where $\hat{J}_{mn} \equiv x_m \partial_n - x_n \partial_m$.

In the superspace field theory (3.4.8), the supermomenta conjugate to the basic canonical superfields (3.4.9) can be identified from (3.4.8b) in the usual way. They will also transform covariantly under canonical supersymmetry, as in (3.4.10).

The two simple examples of superspace field theories - the free chiral multiplet and the free abelian vector multiplet - discussed in § 3.3 in the null-plane coordinates, here go as follows :

Relating the canonical and Weyl representations, one obtains

$$\theta^{I} = \frac{1}{\sqrt{2}}(-\theta^2 + i\bar{\theta}^{\dot{1}}) \qquad \bar{\theta}^{\dot{I}} = \frac{1}{\sqrt{2}}(-\bar{\theta}^{\dot{2}} - i\theta^1)$$
$$\theta^{II} = \frac{1}{\sqrt{2}}(\theta^1 + i\bar{\theta}^{\dot{2}}) \qquad \bar{\theta}^{\dot{II}} = \frac{1}{\sqrt{2}}(\bar{\theta}^{\dot{1}} - i\theta^2) \qquad (3.4.12a)$$

so that $\theta^{I}\theta^{II}\bar{\theta}^{\dot{I}}\bar{\theta}^{\dot{II}} = \theta^1\theta^2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}$, and

$$D_I = \frac{1}{\sqrt{2}}(D_2 - i\bar{D}_{\dot{1}}) \qquad \bar{D}_{\dot{I}} = \frac{1}{\sqrt{2}}(\bar{D}_{\dot{2}} + i D_1)$$
$$D_{II} = \frac{1}{\sqrt{2}}(-D_1 - i\bar{D}_{\dot{2}}) \qquad \bar{D}_{\dot{II}} = \frac{1}{\sqrt{2}}(-\bar{D}_{\dot{1}} + i D_2) \quad . \tag{3.4.12b}$$

Relations (3.4.12a) also apply to parameters ε and spinors ψ in general, while (3.4.12b) apply equally to vector fields Q.

i) A chiral Minkowski superfield Φ, $\bar{D}_{\dot{\rho}}\Phi = 0$, satisfies

$$\bar{D}_{\dot{I}}\Phi = -i D_{II}\Phi \qquad \bar{D}_{\dot{II}}\Phi = i D_I \Phi \tag{3.4.13a}$$

and $\bar{\Phi}$, being anti-chiral, $D_\rho \bar{\Phi} = 0$, correspondingly satisfies

$$\bar{D}_{\dot{I}}\bar{\Phi} = i D_{II}\bar{\Phi} \qquad \bar{D}_{\dot{II}}\bar{\Phi} = -i D_I \bar{\Phi} \quad . \tag{3.4.13b}$$

Thus, Φ produces through (3.4.9) only one independent canonical superfield, $(\Phi)|$. This, as a canonical superfield, is unconstrained, which contrasts with the null-plane case. Similarly, $\bar{\Phi}$ only produces $(\bar{\Phi})|$. In what follows, though, we shall take as basic canonical superfields

$$\phi \equiv \frac{1}{2}\left((\bar{\Phi})| + (\Phi)| \right) \quad \text{and} \quad \tilde{\phi} \equiv \frac{i}{2}\left((\bar{\Phi})| - (\Phi)| \right) \quad . \tag{3.4.14}$$

From the component field expansion of Φ as in [3, Wess & Bagger], we obtain

$$\phi = A_R - \frac{1}{\sqrt{2}}\theta^I \psi^{II} + \frac{1}{\sqrt{2}}\theta^{II}\psi^I - \theta^I\theta^{II}(F_R - i\partial_0 A_I)$$
$$\tilde{\phi} = A_I - \frac{1}{\sqrt{2}}\theta^I \bar{\psi}^{\dot{I}} - \frac{1}{\sqrt{2}}\theta^{II}\bar{\psi}^{\dot{II}} - \theta^I\theta^{II}(F_I + i\partial_0 A_R) \quad . \tag{3.4.15}$$

There is thus one independent real degree of freedom for each component of the canonical superfields ϕ and $\tilde{\phi}$. By which we mean , not only that the original component fields as chosen in [3 , Wess & Bagger] carry as many real degrees of freedom (8) as there are components in the canonical superfields ϕ and $\tilde{\phi}$, but also that those original component fields can all be locally reconstructed from the components of the canonical superfields ϕ and $\tilde{\phi}$, and vice-versa . As a result , the variational principles applied at canonical superfield level should be equivalent to (i.e. lead to the same on-shell system configurations as) the variational principles applied at the original component field level , just as happens with full Minkowski superfields .

Here , the canonical superspace version of (3.3.8) is

$$\mathfrak{I} = \int \left(i\,\tilde{\phi}\,\partial_0\phi - i\,\phi\,\partial_0\tilde{\phi} + \phi\,{}^3\Delta\,\phi + \tilde{\phi}\,{}^3\Delta\,\tilde{\phi} \right) d^4x\, d\theta^{II}\, d\theta^{I} \tag{3.4.16}$$

where ${}^3\Delta \equiv \frac{1}{2}\,[{}^3D_{II}, {}^3D_I]$.

Note that the canonical superspace action integral $\mathfrak{I}$ above leads to a dynamical structure similar to the one of an ordinary spinor field ψ , with the characteristic first order (in ∂_0) coupled differential equations for the pair ϕ , $\tilde{\phi}$.

ii) A real Minkowski superfield V , $\bar{V} = V$, produces through (3.4.9) the four canonical superfields

$$\mathcal{u} \equiv (V)\Big| \qquad \mathcal{U} \equiv \left(\tfrac{1}{2}\,(\bar{D}_{\dot{II}}\bar{D}_{\dot{I}} - \bar{D}_{\dot{I}}\bar{D}_{\dot{II}})\, V\right)\Big|$$

$$\mathcal{v}^{\dot{\xi}} \equiv (\bar{D}_{\dot{\xi}} V)\Big| \quad . \tag{3.4.17}$$

From the component field expansion of V as in [3 , Wess & Bagger] , we obtain

$$\begin{aligned}
u &= [\,C\,] + \theta^{I}[\,\dot{\bar{\chi}}^{I}\,] + \theta^{II}[\,\dot{\bar{\chi}}^{II}\,] + \theta^{I}\theta^{II}[\,N + i\,v_0\,] \\
\tfrac{1}{2}(\,\mathcal{U} + \Delta\,u) &= [\,i\,v_0\,] - \theta^{I}[\,\dot{\bar{\lambda}}^{I} + i\,\partial_0\,\chi^{II}\,] - \theta^{II}[\,\dot{\bar{\lambda}}^{II} - i\,\partial_0\,\chi^{I}\,] - \\
&\qquad - \theta^{I}\theta^{II}[\,D - \partial_0\partial_0\,C - i\,\partial_0\,M\,] \\
v^{I} &= [\,-\chi^{I}\,] - \theta^{I}[\,M + v_3 + i\,\partial_0\,C\,] - \theta^{II}[\,v_1 - i\,v_2\,] - \\
&\qquad - \theta^{I}\theta^{II}[\,2\,\lambda^{I} + \bar{\partial}\,\chi^{II} + \partial_3\,\chi^{I}\,] \\
v^{II} &= [\,-\chi^{II}\,] - \theta^{I}[\,v_1 + i\,v_2\,] - \theta^{II}[\,M - v_3 + i\,\partial_0\,C\,] - \\
&\qquad - \theta^{I}\theta^{II}[\,2\,\lambda^{II} + \partial\,\chi^{II} - \partial_3\,\chi^{I}\,]
\end{aligned} \tag{3.4.18}$$

Again , there is one independent real degree of freedom for each component of the canonical superfields u , v^{ξ} , $\mathcal{U}$.

The canonical superspace version of (3.3.11) is here

$$\begin{aligned}
\mathfrak{S} = \int \tfrac{1}{2}\Big(&-\partial^{m}v^{I}\,\partial_{m}v^{II} + \tfrac{i}{2}\,\partial_0 v^{\xi}\,{}^{3}D_{\xi}(\mathcal{U} + {}^{3}\Delta u) + \\
&+ \tfrac{1}{4}\,\nu\,{}^{3}\Delta\nu + \tfrac{1}{4}\,(\mathcal{U} + {}^{3}\Delta u)\,{}^{3}\Delta(\mathcal{U} + {}^{3}\Delta u)\Big)\, d^{4}x\, d\theta^{II}\, d\theta^{I}
\end{aligned} \tag{3.4.19}$$

where $\nu \equiv {}^{3}D_{\zeta}\, v^{\zeta}$.

Note that , this time , the canonical superspace action integral $\mathfrak{S}$ above leads to a dynamical structure similar to the one of an ordinary vector gauge field v_m : v^{ξ} carry the propagating degrees of freedom and u , $\mathcal{U}$ play the double role of v_0 , u as the natural variable for the (partial) gauge fixing analogue to $v_0 = 0$ and $\mathcal{U}$ acting as the Lagrange multiplier for a constraint similar to $\partial^{j}\, v_{0j} = 0$.

§ 3.5 *The Canonical Superspace Formalism*

In the last section , we have described the identification of canonical superspace within the canonical decomposition of Minkowski superspace , plus the way in which field theory action integral formulations reduce from full Minkowski superspace to canonical superspace . Here , we proceed to develop the resulting canonical superspace formalism , meaning canonical formalism in superspace , as applied to the free chiral multiplet and the free abelian vector multiplet .

To begin , though , we have organized a Formularium describing , in its first part , some specific notation and formulae which we have found of practical use in manipulating spinorial expressions within canonical superspace . An introduction to the two component notation in the canonical representation can be found in the Appendix to § 3 . The second part of the Formularium presents the basic facts of Poisson bracket calculus in canonical superspace , including a discussion of Dirac δ functions and functional differentiation .

Formularium

Vector indices $i,j = 1,2,3$ are placed indistinctly contravariantly or covariantly , $v_j = v^j$, while contravariant and covariant spinor indices $\xi,\zeta = \mathrm{I,II}$ relate as follows :

$$\psi_{\mathrm{I}} = - \psi^{\mathrm{II}} \quad , \quad \psi_{\mathrm{II}} = \psi^{\mathrm{I}} \quad \text{, the same for dotted spinor indices .}$$

Henceforth , the superscript ${}^3(\;)$ will be omitted , though still remaining implicit :

$${}^3D \to D \quad \text{and} \quad {}^3\Delta \to \Delta$$

$$\partial \equiv \partial_1 + i\,\partial_2 \quad \text{and} \quad \bar{\partial} \equiv \partial_1 - i\,\partial_2$$

$$D_{\mathrm{I}} = -\,(\partial_{\mathrm{I}} - \theta^{\mathrm{I}}\,\partial + \theta^{\mathrm{II}}\,\partial_3) \quad , \quad D_{\mathrm{II}} = -\,(\partial_{\mathrm{II}} + \theta^{\mathrm{I}}\,\partial_3 + \theta^{\mathrm{II}}\,\bar{\partial})$$

$$\{D_\xi, D_\zeta\} = 2\,(\sigma^j)_{\xi\zeta}\,\partial_j$$

$$D_I D_I = -\,\partial \quad , \quad D_{II} D_{II} = \bar{\partial} \quad , \quad \frac{1}{2}\{D_I, D_{II}\} = \partial_3$$

The two differential operators

$$D^j{}_\xi \equiv (\sigma_j)^\zeta{}_\xi\, D_\zeta \qquad \text{and} \qquad \Delta \equiv \frac{1}{2} D^\xi D_\xi = \frac{1}{2}[D_{II}, D_I] \quad ,$$

reccur rather frequently . The following are useful relations involving them :

$$D_j{}^\xi\, D^k{}_\xi + (\, j \leftrightarrow k\,) = -\,4\,\eta_{jk}\,\Delta$$

$$D^j{}_\xi\, D^j{}_\zeta = D_\xi\, D_\zeta - 4\,\varepsilon_{\xi\zeta}\,\Delta$$

$$D_\xi\,\Delta = -\,\Delta\, D_\xi = \partial_j\, D^j{}_\xi \quad , \quad D^\xi\, D^j{}_\xi = -\,D_j{}^\xi\, D_\xi = 2\,\partial_j$$

$$\Delta^2 \equiv \Delta\,\Delta = \partial_j\,\partial_j \quad , \quad D_j{}^\xi\, D^j{}_\xi = -\,6\,\Delta$$

Given a general canonical superfield $\quad f = A + \theta^I B + \theta^{II} C + \theta^I \theta^{II} D \quad ,$

$$\Delta f = D + \theta^I (\,\partial_3 B + \partial C\,) + \theta^{II} (\,\bar{\partial} B - \partial_3 C\,) + \theta^I \theta^{II}\, \partial_j \partial_j\, A$$

The free field equations for a (Majorana) spinor ψ are

$(\gamma^m_c)_{\alpha\beta}\,\partial_m\,\psi^\beta_c = 0 \qquad$ or , in two component notation ,

$$\partial_0\, \bar{\psi}^{\dot{\xi}} = -\,i\,(\bar{\tau}\sigma_j)^{\dot{\xi}}{}_\zeta\,\partial_j\,\psi^\zeta \quad , \quad \partial_0\,\psi^\xi = i\,(\tau\bar{\sigma}_j)^\xi{}_{\dot{\zeta}}\,\partial_j\,\bar{\psi}^{\dot{\zeta}} \quad .$$

Dirac δ Functions and Functional Differentiation :

i) Of an even variable x

Let $f : x \to f(x)$ be an even or odd valued function .

The parity , i.e. the even or odd character , of f is indicated by the exponent of $(-1)^f$, which takes values $(-1)^{f=0,1}$ depending on whether f is even or odd , respectively . When a sum is indicated in the exponent , it is meant mod2 .

The Dirac " function " $\delta : x \to \delta(x)$ is even valued and defined so that

$$(1) : \ \delta(x) = \delta(-x) \quad \text{and} \quad (2) : \ f(x) = \int f(y)\, \delta(y - x)\, dy$$

Let $\partial\delta : x \to \partial\delta(x)$ denote the derivative of δ . Then ,

$$(1) \Rightarrow \partial\delta(x) = - \partial\delta(-x) \quad \text{and} \quad (2) \Rightarrow \partial f(x) = - \int f(y)\, \partial\delta(y - x)\, dy$$

In the proof of $(2) \Rightarrow \ldots$ above ,
we have used partial integration and the fact that $\partial^y\delta(y - x) = \partial\delta(y - x)$.
But what does $\partial^y\delta(y - x)$ mean , exactly ?

Let $z : (x,y) \to z(x,y) := x - y$ and $\hat{z} : (x,y) \to \hat{z}(x,y) := y - x$;
Then , $\partial^y\delta(y - x)$ denotes $\partial^y(\delta \circ \hat{z})$ evaluated at (x,y) , that is ,

$$\partial^y\delta(y - x) \equiv \partial^y(\delta \circ \hat{z})\,(x,y) = \partial\delta\,(\hat{z}(x,y))\ \partial^y\hat{z}\,(x,y) = \partial\delta(y - x)\,(+1) = \partial\delta(y - x)$$

which simply means $\partial\delta$ evaluated at $(y - x)$.

In the same way , one can deduce

$$\partial^y\delta(y - x) = - \partial^x\delta(x - y) = \partial\delta(y - x)$$

which is sometimes a useful variable interchange in Poisson bracket calculus .

Let $F : f \to F[f]$ be an even or odd valued functional .

Functional differentiation is defined as

$$\frac{\delta F[f]}{\delta f(x)} := \mathrm{Lim}_{\varepsilon\to 0} \frac{F[f(y)+\varepsilon\delta(y-x)] - F[f(y)]}{\varepsilon}$$

from which follows the Leibnitz rule $\quad \dfrac{\delta(FG)}{\delta f} = \dfrac{\delta F}{\delta f} G + (-1)^{fF} F \dfrac{\delta G}{\delta f}$

and $\quad \dfrac{\delta f(y)}{\delta f(x)} = \delta(y - x) \quad , \quad \dfrac{\delta\, \partial f(y)}{\delta f(x)} = \partial\delta(y - x) \quad ,$

where it shows clearly that the operator $\dfrac{\delta}{\delta f(x)}$ as the same parity as f.

It follows that

$$\frac{\delta}{\delta f(x)} \int f(y)\, g(y)\, dy = g(x) \quad , \quad \frac{\delta}{\delta f(x)} \int \partial f(y)\, g(y)\, dy = - \partial g(x)$$

where $\quad g : x \to g(x) \quad$ is an even or odd valued function .

ii) <u>Of an odd variable</u> θ

Let $\quad f : \theta \to f(\theta) = a + \theta b \quad$ be an even or odd valued function .
In any case , $\quad \partial f = \text{const.} = b$.

The Dirac function $\quad \delta : \theta \to \delta(\theta) \quad$ is odd valued and defined directly as

$$\delta(\theta) := \theta \quad .$$

There follows

$$\delta(\theta) = - \delta(-\theta) \quad \text{and} \quad f(\theta) = \int f(\phi)\, \delta(\phi - \theta)\, d\phi$$

where we have used the standard integration rules $\quad \int 1\, d\theta = 0 \;, \int \theta\, d\theta = 1 \quad .$

Furthermore , since $\quad \partial\delta = \text{const.} = 1 \quad$, we have

$$\partial\delta(\theta) = \partial\delta(-\theta) \quad \text{and} \quad \partial f(\theta) = -(-1)^f \int f(\phi)\, \partial\delta(\phi - \theta)\, d\phi \ .$$

Along the same lines as before, one obtains here the variable interchange formula

$$\partial^{\phi}\delta(\phi - \theta) = \partial^{\theta}\delta(\theta - \phi) = \partial\delta(\theta - \phi) \ .$$

Let $F : f \rightarrow F[f]$ be an even or odd valued functional.
Functional differentiation is defined as

$$\frac{\delta F[f]}{\delta f(\theta)} := \mathrm{Lim}_{\varepsilon\rightarrow 0} \frac{F[f(\phi)+\varepsilon\delta(\phi-\theta)] - F[f(\phi)]}{\varepsilon}$$

from which follows the Leibnitz rule $\displaystyle \frac{\delta(FG)}{\delta f} = \frac{\delta F}{\delta f} G + (-1)^{(f+1)F} F \frac{\delta G}{\delta f}$

and $\displaystyle \frac{\delta f(\phi)}{\delta f(\theta)} = \delta(\phi - \theta) \ , \quad \frac{\delta\, \partial f(\phi)}{\delta f(\theta)} = (-1)^{f+1}\, \partial\delta(\phi - \theta)$

where it shows clearly that f and the operator $\displaystyle \frac{\delta}{\delta f(\theta)}$ have opposite parities.

It follows that

$$\frac{\delta}{\delta f(\theta)} \int f(\phi)\, g(\phi)\, d\phi = (-1)^g g(\theta) \ , \quad \frac{\delta}{\delta f(\theta)} \int \partial f(\phi)\, g(\phi)\, d\phi = (-1)^{f+g}\, \partial g(\theta)$$

where $g : \theta \rightarrow g(\theta)$ is an even or odd valued function.

iii) <u>Of the canonical superspace variables</u> $\vec{X} \equiv (\vec{x},\vec{\theta}) \equiv (x^j,\theta^\xi)$

Let $f : \vec{X} \rightarrow f(\vec{X})$ be an even or odd valued canonical superfield.

$$\delta(\vec{X}) \equiv \delta(x^1)\delta(x^2)\delta(x^3)\delta(\theta^{I})\delta(\theta^{II})$$

$$d\vec{X} \equiv d^3x\, d\theta^{II}\, d\theta^{I}$$

Composing the individual properties of $\delta(x)$ and $\delta(\theta)$ as in i) and ii), one obtains

$$\delta(\vec{X}) = \delta(-\vec{X}) \quad \text{and} \quad f(\vec{X}) = \int f(\vec{Y})\, \delta(\vec{Y} - \vec{X})\, d\vec{Y}$$

plus also

$$\partial_j \delta(\vec{X}) = -\partial_j \delta(-\vec{X}) \quad \text{and} \quad \partial_\xi \delta(\vec{X}) = -\partial_\xi \delta(-\vec{X})$$

which we indicate, in general, as

$$\partial\delta(\vec{X}) = -\partial\delta(-\vec{X})$$

where ∂ stands for ∂_j, ∂_ξ or D_ξ.

Correspondingly, $\partial f(\vec{X}) = -(-1)^{f\partial} \int f(\vec{Y})\, \partial\delta(\vec{Y} - \vec{X})\, d\vec{Y}$.

The variable interchange formula is here

$$\partial^Y \delta(\vec{Y} - \vec{X}) = -\partial^X \delta(\vec{X} - \vec{Y}) = \partial\delta(\vec{Y} - \vec{X}) \quad .$$

Functional differentiation is defined as before. There follows

the Leibnitz rule $\quad \dfrac{\delta(FG)}{\delta f} = \dfrac{\delta F}{\delta f} G + (-1)^{fF} F \dfrac{\delta G}{\delta f}$

and $\quad \dfrac{\delta f(\vec{Y})}{\delta f(\vec{X})} = \delta(\vec{Y} - \vec{X}) \quad , \quad \dfrac{\delta\, \partial f(\vec{Y})}{\delta f(\vec{X})} = (-1)^{f\partial}\, \partial\delta(\vec{Y} - \vec{X})$

where it shows clearly that the operator $\dfrac{\delta}{\delta f(\vec{X})}$ has the same parity as f.

It follows that

$$\frac{\delta}{\delta f(\vec{X})} \int f(\vec{Y})\, g(\vec{Y})\, d\vec{Y} = g(\vec{X}) \quad , \quad \frac{\delta}{\delta f(\vec{X})} \int \partial f(\vec{Y})\, g(\vec{Y})\, d\vec{Y} = -(-1)^{f\partial}\, \partial g(\vec{X})$$

where $g : \vec{X} \rightarrow g(\vec{X})$ is an even or odd valued function .

Poisson Brackets :

Consider a system described by the basic canonical superfield variables q^e , q^o and their conjugate momenta p_e , p_o . The indices $e = 1,2,\ldots,N^e$ and $o = 1,2,\ldots,N^o$ enumerate the even and odd valued phase space variables , respectively .

Let F,G be even or odd valued functionals of the canonical superfield phase space variables (q^e,p_e,q^o,p_o) . The Hamiltonian H , even valued , is one of such functionals .

The Poisson bracket algebra , which we denote { , } , is defined to be

$$\{F,G\} := \int \Big[\Big(\frac{\delta F}{\delta q^e(\vec{Y})} \frac{\delta G}{\delta p_e(\vec{Y})} - \kappa \frac{\delta G}{\delta q^e(\vec{Y})} \frac{\delta F}{\delta p_e(\vec{Y})} \Big) +$$

$$+ \Big((-1)^F \frac{\delta F}{\delta q^o(\vec{Y})} \frac{\delta G}{\delta p_o(\vec{Y})} - \kappa (-1)^G \frac{\delta G}{\delta q^o(\vec{Y})} \frac{\delta F}{\delta p_o(\vec{Y})} \Big) \Big] \, d\vec{Y}$$

where $\kappa \equiv (-1)^{FG}$.

In this way , the Poisson bracket algebra has all the standard properties of a Lie superalgebra , that is

Graded antisymmetry $\{F,G\} = - (-1)^{FG} \{G,F\}$,

Graded linearity $\{F,G+G'\} = \{F,G\} + \{F,G'\}$

$$e\{F,G\} = \{eF,G\} = \{F,eG\} = \{F,G\}e$$

$$o\{F,G\} = \{oF,G\} = (-1)^F \{F,oG\} = (-1)^{FG} \{F,G\}o$$

where e (respectively o) is an even (respectively odd) constant ,

plus also the Graded Jacobi Identity

$$(-1)^{LG}\{L,\{F,G\}\} + (-1)^{FL}\{F,\{G,L\}\} + (-1)^{GF}\{G,\{L,F\}\} = 0$$

where the position of the brackets is always the same, the three functionals L,F,G are always in the same cyclic order and the (-1) parity factors always refer to the first and last of them.

Furthermore, the Poisson brackets satisfy a product rule

$$\{L,FG\} = \{L,F\}\,G + (-1)^{FL} F\,\{L,G\}$$

and thus $\mathfrak{D}_L : F \to \{L,F\}$ is a derivation, with the same parity as L.

The same applies to $\mathfrak{D}_R : F \to \{F,R\}$, provided that R is even valued. Indeed, one other crucial property of the Poisson brackets is

$$\partial_0 F \cong \{F,H\}$$

where a weak equation $A \cong B$ indicates that A and B differ at most by a linear combination of the constraints.

The chiral multiplet

From the canonical superspace action integral (3.4.16),

$$\mathfrak{I} = \int \mathcal{L}\ d\vec{X}$$

with (3.5.1)

$$\mathcal{L} = i\,\tilde{\phi}\,\partial_0\phi - i\,\phi\,\partial_0\tilde{\phi} + \phi\,\Delta\,\phi + \tilde{\phi}\,\Delta\,\tilde{\phi}$$

we derive the Lagrangian equations of motion

$$\partial_0\,\phi = i\,\Delta\,\tilde{\phi} \quad \text{and} \quad \partial_0\,\tilde{\phi} = -\,i\,\Delta\,\phi\ . \tag{3.5.2}$$

The conjugate momenta to ϕ and $\tilde{\phi}$ are respectively

$$\pi_{\phi} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = i\,\tilde{\phi} \equiv \pi \quad \text{and} \quad \pi_{\tilde{\phi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \tilde{\phi})} = -i\,\phi \equiv \tilde{\pi} \ . \tag{3.5.3}$$

There are thus the primary constraints

$$\varphi \equiv \pi - i\,\tilde{\phi} \quad , \quad \tilde{\varphi} \equiv \tilde{\pi} + i\,\phi \ . \tag{3.5.4}$$

The existence of primary constraints as above implies that the Hamiltonian density

$$\mathcal{H} = (\partial_0 \phi)\,\pi + (\partial_0 \tilde{\phi})\,\tilde{\pi} - \mathcal{L}$$

is not uniquely determined as a function of the phase space variables $\phi , \tilde{\phi}$ and $\pi , \tilde{\pi}$, one form of $\mathcal{H}$ differing from another by a linear combination of the primary constraints.

The general form of $\mathcal{H}$, the so-called primary Hamiltonian density $\mathcal{H}_P$, is then

$$\begin{aligned} \mathcal{H}_P &= -(\,\phi\,\Delta\,\phi + \tilde{\phi}\,\Delta\,\tilde{\phi}\,) + \lambda\,\varphi + \tilde{\lambda}\,\tilde{\varphi} = \\ &= -\,\phi\,\Delta\,\phi - \tilde{\phi}\,\Delta\,\tilde{\phi} + \lambda\,(\,\pi - i\,\tilde{\phi}\,) + \tilde{\lambda}\,(\,\tilde{\pi} + i\,\phi\,) \end{aligned} \tag{3.5.5}$$

where the Lagrange multipliers λ and $\tilde{\lambda}$ are , as yet , arbitrary functions of the basic canonical superfields $\phi , \tilde{\phi}$ and their conjugate momenta $\pi , \tilde{\pi}$.

The particular choice of a " radical " in the primary Hamiltonian density, which above was $-(\,\phi\,\Delta\,\phi + \tilde{\phi}\,\Delta\,\tilde{\phi}\,)$, is sometimes referred to as the canonical Hamiltonian density $\mathcal{H}_C$.

For systems without a local gauge symmetry , as is the case of the chiral multiplet , one particular form of Hamiltonian density assures the propagation of all the constraint equations , which is an essential consistency requirement in the Hamiltonian description . To find out what that particular form is , i.e. what is the appropriate choice for the Lagrange multipliers λ , $\tilde{\lambda}$ in the primary Hamiltonian $\mathbf{H}_p$, one evaluates the Poisson brackets of the primary constraints with the general $\mathbf{H}_p$ and , in imposing that they should at least weakly vanish, one obtains equations specifying the Lagrange multipliers .

Sometimes , though , not all such equations can be validated by an appropriate choice of the Lagrange multipliers , in which case secondary constraints are obtained . The cycle should then be repeated until a final set of constraints is reached , final in the sense that all constraints have at least weakly vanishing Poisson brackets with the Hamiltonian . For systems like the chiral multiplet , with no gauge degrees of freedom , this requires the specification of all the Lagrange multipliers , and thus of the Hamiltonian density .

The canonical superspace description of the chiral multiplet is , however , rather simple and the primary constarints are all that is needed .

Indeed , with $\mathbf{H}_p \equiv \int \mathcal{H}_p \, d\vec{x}$,

$$\partial_0 \varphi \cong \{\varphi, \mathbf{H}_p\} \cong 2 \Delta \phi - 2i \tilde{\lambda} \quad , \quad \partial_0 \tilde{\varphi} \cong \{\tilde{\varphi}, \mathbf{H}_p\} \cong 2 \Delta \tilde{\phi} + 2i \lambda \ ,$$

from which follows immediatly the complete specification of the Lagrange multipliers ,

$$\tilde{\lambda} = - i \Delta \phi \qquad \text{and} \qquad \lambda = i \Delta \tilde{\phi} \ , \tag{3.5.6}$$

guaranteeing that the constraint equations $\varphi = \tilde{\varphi} = 0$ are propagated through the time development of the system. In other words, if those constraint equations are satisfied by the phase space configuration of the system at some initial time, they will remain so at each moment in time thereafter.

With the resulting fully determined primary Hamiltonian $\mathbf{H}_P$, in which λ and $\tilde{\lambda}$ are as in (3.5.6), we obtain the Hamiltonian equations of motion

$$
\begin{aligned}
\partial_0 \phi &\cong \{\phi, H_P\} \cong i \Delta \tilde{\phi} \quad \text{and} \quad \partial_0 \pi \cong \{\pi, H_P\} \cong \Delta \phi \\
\partial_0 \tilde{\phi} &\cong \{\tilde{\phi}, H_P\} \cong -i \Delta \phi \quad \text{and} \quad \partial_0 \tilde{\pi} \cong \{\tilde{\pi}, H_P\} \cong \Delta \tilde{\phi}
\end{aligned}
\tag{3.5.7}
$$

which are, of course, ultimately redundant due to (3.5.3), and equivalent to the Lagrangian equations of motion (3.5.2).

Finally, we evaluate

$$\{\varphi(\vec{X}), \tilde{\varphi}(\vec{X}')\} = -2i\, \delta(\vec{X} - \vec{X}')$$

to conclude that both the constraints φ, $\tilde{\varphi}$ are second class.

The canonical superspace formalism of the chiral multiplet is thus, and in all respects, parallel to the Hamiltonian description of the original field theory of which it, the chiral multiplet, is the supersymmetric extension: the (free) chiral fermion, whose ordinary spacetime Lagrangian density is

$$
\begin{aligned}
\mathcal{L} = -\tfrac{1}{2} \Big(& i\, \psi^{\xi} (\tau)_{\xi\dot{\zeta}}\, \partial_0 \bar{\psi}^{\dot{\zeta}} + i\, \bar{\psi}^{\dot{\xi}} (\bar{\tau})_{\dot{\xi}\zeta}\, \partial_0 \psi^{\zeta} + \\
& + \psi^{\xi} (\sigma^j)_{\xi\zeta}\, \partial_j \psi^{\zeta} - \bar{\psi}^{\dot{\xi}} (\bar{\sigma}^j)_{\dot{\xi}\dot{\zeta}}\, \partial_j \bar{\psi}^{\dot{\zeta}} \Big) ,
\end{aligned}
$$

which may be abbreviated as

$$\mathcal{L} = -\frac{1}{2} \left(i\psi\tau\partial_0\bar{\psi} + i\bar{\psi}\bar{\tau}\partial_0\psi + \psi\sigma^j\partial_j\psi - \bar{\psi}\bar{\sigma}^j\partial_j\bar{\psi} \right) .$$

Indeed , the constraint and propagating structure of the above original field theory and of its supersymmetric extension are precisely the same when the latter is formulated in canonical superspace . In both cases , there are two primary second class constraints and , in making appropriate use of the Formularium plus of the component field expansion (3.4.15) , one can straightforwardly check that , for example , the Lagrangian equations of motion (3.5.2) for the chiral multiplet do contain the correct component equations of motion , in particular the ones for the chiral fermion .

The vector multiplet

The canonical superspace action integral (3.4.19) can be rewritten as

$$\mathfrak{I} = \int \mathcal{L} \ d\vec{X}$$

with (3.5.8a)

$$\mathcal{L} = -\frac{1}{4} \nu_m^{\ \xi} \nu^m_{\ \xi}$$

where

$$\nu_0^{\ \xi} \equiv \partial_0 v^\xi + \frac{i}{2} D^\xi (\mathcal{U} + \Delta\, u)$$

$$\nu_j^{\ \xi} \equiv \frac{1}{\sqrt{3}} (\partial_j v^\xi - \frac{1}{2} D^\xi \nu^j) \qquad (3.5.8b)$$

and $\nu^j \equiv D^j{}_\zeta\, v^\zeta$.

From the action integral $\mathfrak{I}$ above, we derive the Lagrangian equations of motion

$$\mathfrak{I}_{v^\xi} \equiv \partial^0 \boldsymbol{v}_{0\xi} + \frac{1}{\sqrt{3}} (\partial_j \boldsymbol{v}^j{}_\xi + \frac{1}{2} D^j{}_\xi D_\zeta \boldsymbol{v}_j{}^\zeta) =$$

$$= \Box\, v_\xi + \frac{1}{2} D_\xi \Delta \boldsymbol{v} - \frac{i}{2} D_\xi \partial_0 (\mathcal{U} + \Delta\, u)) = 0$$

$$\mathfrak{I}_{\mathcal{U}} \equiv D_\xi \boldsymbol{v}_0{}^\xi = \partial_0 \boldsymbol{v} - i \Delta (\mathcal{U} + \Delta\, u) = 0 \tag{3.5.9}$$

$$\mathfrak{I}_u \equiv \Delta (\mathfrak{I}_{\mathcal{U}}) = 0 \qquad , \quad \text{where} \quad \boldsymbol{v} \equiv D_\zeta v^\zeta \; .$$

The theory is invariant under local gauge transformations which, in the original full Minkowski superspace formulation, took the form $\delta V = {}^i/_2 (\bar{\Phi} - \Phi)$, for some arbitrary chiral superfield parameter Φ. Correspondingly, in canonical superspace,

$$\begin{aligned} \delta u &= \tilde{\phi} \\ \delta v^\xi &= - D^\xi \phi \\ \delta \mathcal{U} &= - 2i \partial_0 \phi - \Delta \tilde{\phi} \; . \end{aligned} \tag{3.5.10}$$

Under these gauge transformations, the $\boldsymbol{v}_m{}^\xi$ are invariant. Furthermore, the Bianchi Identities associated with the two local gauge symmetries ϕ and $\tilde{\phi}$ are

$$D^\xi (\mathfrak{I}_{v^\xi}) - \partial_0 (\mathfrak{I}_{\mathcal{U}}) = 0 \qquad \text{and} \qquad \mathfrak{I}_u = \Delta (\mathfrak{I}_{\mathcal{U}}) \; .$$

The canonical superfield u is clearly a spurious variable in the vector supermultiplet: its equation of motion is superfluous and, moreover, u could be absorbed in a simple redefinition of $\mathcal{U}$, of the form $\mathcal{U} \rightarrow \hat{\mathcal{U}} \equiv -{}^i/_2 (\mathcal{U} + \Delta\, u)$. The gauge transformations of the remaining fields v^ξ and $\hat{\mathcal{U}}$ would then depend exclusively on ϕ.

Another point of view would be to opt for the gauge fixing $u = 0$, which is consistent with the Lagrangian equations of motion, and contains the partial " radiaton " gauge fixing $v_0 = 0$. In that case, the gauge symmetry would be severely curtailed, but not fully eliminated. Indeed, an exam of (3.4.15) reveals that the condition $\tilde{\phi} = 0$ does not imply $\phi = 0$, but only restricts ϕ to be of the form

$$\phi = A_R - \theta^I \theta^{II} F_R \quad , \quad \partial_0 A_R = 0 \ .$$

This residual gauge symmetry, at component field level, corresponds precisely to

$$\delta v_j = -\partial_j A_R \quad \text{and} \quad \delta M = -F_R \ .$$

We conclude that the partial gauge fixing $u = 0$, which is consistent with canonical supersymmetry, is thus the appropriate analogue of $v_0 = 0$ in canonical superspace. In relation to the standard Wess-Zumino gauge in full Minkowski superspace, which is also partial but does break full supersymmetry, the gauge fixing $u = 0$ eliminates the same component fields except for v_0, which is also put to zero, and M, which is not.

We now pass to the Hamiltonian description of the vector multiplet in canonical superspace. The conjugate momenta to v^ξ and u, $\mathcal{U}$ are respectively

$$\pi_{v^\xi} = \frac{\partial \mathcal{L}}{\partial(\partial_0 v^\xi)} = \frac{1}{2} \nu_{0\xi} \equiv \pi_\xi$$

$$\pi_u = \frac{\partial \mathcal{L}}{\partial(\partial_0 u)} = 0 \equiv \pi \quad , \quad \pi_{\mathcal{U}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \mathcal{U})} = 0 \equiv \Pi \ . \tag{3.5.11}$$

There are thus the primary constraints

$$\varphi \equiv \pi \quad , \quad \Phi \equiv \Pi \ . \tag{3.5.12}$$

As a consequence , the Hamiltonian density

$$\mathcal{H} = (\partial_0 \vartheta^\xi)\, \pi_\xi + (\partial_0 u)\, \pi_u + (\partial_0 \mathcal{U})\, \pi_{\mathcal{U}} - \mathcal{L}$$

is once again not uniquely determined . For the canonical Hamiltonian density we choose

$$\mathcal{H}_c = \pi^\xi \pi_\xi - \frac{i}{2} D^\xi (\mathcal{U} + \Delta\, u)\, \pi_\xi + \frac{1}{4} \nu_j{}^\xi \nu^j{}_\xi$$

and thus the primary Hamiltonian density is

$$\begin{aligned} \mathcal{H}_p &= \mathcal{H}_c + \lambda_\varphi \varphi + \lambda_\Phi \Phi = \\ &= \pi^\xi \pi_\xi - \frac{i}{2} D^\xi (\mathcal{U} + \Delta\, u)\, \pi_\xi + \frac{1}{4} \nu_j{}^\xi \nu^j{}_\xi + \lambda_\varphi \pi + \lambda_\Phi \Pi \end{aligned} \tag{3.5.13}$$

where λ_φ and λ_Φ are , as yet , arbitrary functions of the basic canonical superfields ϑ^ξ , u , $\mathcal{U}$ and their conjugate momenta π_ξ , π , Π .

Contrary to the case of the chiral multiplet , though , the vector multiplet possesses gauge degrees of freedom and , in such case , when the final set of constraints is reached , with all constraints having weakly vanishing Poisson brackets with the primary Hamiltonian , this remains partly undetermined due to the persisting arbitrariness of the Lagrange multipliers associated with a subset of the constraints , the first class primary constraints .

The canonical superspace description of the vector multiplet , in particular , will accumulate those features with the presence of a secondary constraint , something that may occur for all singular systems , gauge and non-gauge , and which we have already commented on .

Proceeding as before , we evaluate

$$\partial_0 \varphi \cong \{\varphi, H_p\} \cong -\frac{i}{2} \Delta (D^{\xi} \pi_{\xi}) \quad , \quad \partial_0 \Phi \cong \{\Phi, H_p\} \cong -\frac{i}{2} (D^{\xi} \pi_{\xi})$$

only to obtain the secondary constraint

$$\chi \equiv D^{\xi} \pi_{\xi} \quad . \tag{3.5.14}$$

And when the process is repeated , we find that the Poisson bracket of χ with $\mathbf{H}_p$ weakly vanishes identically ,

$$\partial_0 \chi \cong \{\chi, H_p\} = - D^{\xi} \left(\frac{\delta H_P}{\delta v^{\xi}} \right) \cong$$

$$\cong \frac{1}{2\sqrt{3}} D^{\xi} (\partial^j \varepsilon_{\xi\zeta} + \frac{1}{2} D^j{}_{\xi} D_{\zeta}) \boldsymbol{\nu}_j{}^{\zeta} = (0) \boldsymbol{\nu}_j{}^{\zeta} = 0 \, !$$

due to $D^{\xi} D^j{}_{\xi} = 2 \, \partial_j$. The propagation of the constraint equations $\phi = \Phi = \chi = 0$ therefore requires no specifying conditions on the Lagrange multipliers λ_{φ} , λ_{Φ} . They , and with them the primary Hamiltonian $\mathbf{H}_p$, remain undetermined .

First class constraints , primary or secondary , are those which have identically zero Poisson brackets with all constraints . Since , by definition , each constraint is invariant under the canonical transformations generated by first class constraints , these do not lead off the constraint surface , the region of phase space defined by the constraint equations , and they are thus of particular relevance in the case of singular systems .

In the case of the vector multiplet , the Poisson Bracket between any two of the constraints φ, Φ, χ is clearly identically zero. Therefore, they are all three first class constraints. The canonical transformations that they generate have the form

(non-trivial transformations only)

$$\delta_\varphi u \equiv -\{\int(\lambda_\varphi \varphi), u\} \equiv \lambda_\varphi$$

$$\delta_\Phi \mathcal{U} \equiv -\{\int(\lambda_\Phi \Phi), \mathcal{U}\} \equiv \lambda_\Phi \tag{3.5.15}$$

$$\delta_\chi v^\xi \equiv -\{\int(\lambda_\chi \chi), v^\xi\} \equiv -D^\xi \lambda_\chi$$

where the Lagrange multipliers $\lambda_\varphi, \lambda_\Phi, \lambda_\chi$ play the role of independent arbitrary parameters. The original gauge transformations (3.5.10) are thus contained in the above canonical transformations generated by the first class constraints, a standard (though not necessary) feature in the relation between the Hamiltonian and Lagrangian descriptions of gauge freedom. This relation is generally somewhat ellusive and the interested reader is recommended to the discussion in [5, Sundermeyer], page 89 (generalities) and 134 (the case of the electromagnetic field).

With the primary Hamiltonian H_p, we obtain the Hamiltonian equations of motion

$$\partial_0 u \equiv \{u, H_p\} \equiv \delta_\varphi u \qquad \text{and} \qquad \partial_0 \pi \equiv \{\pi, H_p\} \equiv 0$$

$$\partial_0 \mathcal{U} \equiv \{\mathcal{U}, H_p\} \equiv \delta_\Phi \mathcal{U} \qquad \text{and} \qquad \partial_0 \Pi \equiv \{\Pi, H_p\} \equiv 0$$

$$\partial_0 v^\xi \equiv \{v^\xi, H_p\} \equiv 2\pi^\xi - \frac{i}{2} D^\xi (\mathcal{U} + \Delta u) \tag{3.5.16}$$

and

$$\partial_0 \pi_\xi \equiv \{\pi_\xi, H_p\} \equiv \frac{1}{2\sqrt{3}} (\partial_j \nu^j{}_\xi + \frac{1}{2} D^j{}_\xi D_\zeta \nu_j{}^\zeta) = \frac{1}{2} (\partial_j \partial_j v_\xi + \frac{1}{2} D_\xi \Delta \nu)$$

which , together with the constraint equations $\varphi = \Phi = \chi = 0$, are equivalent to the Lagrangian equations of motion (3.5.9) .

It is once again the case that the canonical superspace formalism for the vector multiplet essentially parallels the Hamiltonian description of Maxwell's theory of the electromagnetic field , of which the vector multiplet is the supersymmetric extension . The structure of the action integral is evidently similar in both cases , and in both one finds primary and secondary first class constraints of respectively the same types . Also , the component content of the canonical superfield equations of motion is once again correct .

True that , in the above canonical superspace description of the vector multiplet , one finds two primary constraints of the type $\pi = 0$, instead of the single one in Maxwell's theory , $\pi^0 = 0$. This is due to the presence of the spurious canonical superfield variable $\mathcal{u}$, which collects all the lower dimensional and gauge dependent auxiliary degrees of freedom , indispensable for full supersymmetry but , as we have pointed out , not so for canonical supersymmetry . Indeed , if one chooses to incorporate $\mathcal{u}$ in a redefinition of $\mathcal{U}$, the analogy between Maxwell's theory and the vector multiplet in canonical superspace becomes absolute . Otherwise , the canonical superspace formalism with both $\mathcal{u}$ and $\mathcal{U}$ present resembles somewhat the null-plane description of the electromagnetic field , where v_- and v_+ play respectively similar roles (see end of § 3.3) .

§ 3.6 *Summary and Conclusions*

We have examined how the 3+1 canonical decomposition of Minkowski spacetime can be extended to N=1 Minkowski superspace.

For that purpose we have introduced ^{s}complex Minkowski superspace $\mathbb{S}^{3,1}_{C}$, a type of complex extension of the usual or real Minkowski superspace $\mathbb{S}^{3,1}_{R}$ which contains it in a natural way : $\mathbb{S}^{3,1}_{R}$ corresponds to the real sector of $\mathbb{S}^{3,1}_{C}$, whose coordinates $(x^m, \theta^{\alpha}_{M})$ take their values in the subalgebra $\aleph_C$ of all real " bodied ", but complex " soulled " elements of an infinite dimensional complex Grassmann algebra $\aleph$. It was shown that the analysis on $\aleph_C$ is identical to the one on $\aleph_R$, the real sector of $\aleph$, and the functional extension formulas plus their implications for superspace field theories were discussed.

In the null-plane case, the dynamical parameter being x^+, real Minkowski superspace does accommodate a canonical foliation into superhypersurfaces of constant x^+, θ^2, $\bar{\theta}^{\dot{2}}$, each of which with two real θ coordinates, represented by θ^1, $\bar{\theta}^{\dot{1}}$.

When the dynamical parameter is x^0, however, the superhypersurfaces must have four real θ coordinates - so that $\mathfrak{R}_{3,0}$ and/or $\mathbf{Spin}_3$ can be represented on them - which is as much as for full real Minkowski superspace itself. Here, the way to achieve a canonical decomposition of Minkowski superspace involves considering not the usual or real 3 dimensional Euclidean superspace $\mathbb{S}^{3,0}_{R}$, but instead the holomorphic version $\mathbb{S}^{3,0}_{h}$ associated with the holomorphic supersymmetry algebra (2.1.1b), and which therefore has two complex coordinates θ^{ξ}, $\xi = \mathrm{I,II}$. The canonical decomposition consists then in identifying in ^{s}complex Minkowski superspace $\mathbb{S}^{3,1}_{C}$ an appropriate one parameter family of constant time superhypersurfaces, each of which is a copy of $\mathbb{S}^{3,0}_{h}$.

Finally, we have discussed in two simple examples, the chiral and the abelian vector supermultiplets, how the canonical decomposition can be used to identify the canonical superfield variables of a superspace field theory in a way which preserves the manifest realization of canonical supersymmetry . The method is based on decomposing each general 4 dimensional scalar superfield into three canonical superfields : two 3 dimensional scalars and one 3 dimensional spinor .

Encouragingly enough , we have found for both the test models above that their Hamiltonian structure in canonical superspace explicitly reproduces the Hamiltonian structure of the ordinary field theories of which they are the supersymmetric extensions . To achieve this , the formalism keeps some time derivatives inside the basic canonical superfields - see for example (3.4.15) - but only , it seems , up to an order for which they can still be expressed as the momenta conjugate to the basic component fields in the supermultiplet . Even though this mechanism may still require some further understanding and/or refinement in cases where component fields of mass dimension less than 1 are present , as in the vector supermultiplet , it clearly has the merit of not producing undesirable higher derivative theories (in ∂_0) at the canonical superfield level for the supersymmetric extension of perfectly well behaved ordinary field theories .

Although we have dealt exclusively with flat superspace , a vector space , there should be no serious obstruction to the ^{s}complex extension of non-trivial real supermanifolds (here , by " supermanifold " we implicitly mean smooth DeWitt supermanifold , as also " manifold " will stand for smooth manifold) : the standard extension construction of all real supermanifolds over a particular ordinary or body real manifold M - see for example [4 , Rogers 1985] and references therein - essentially relies on the way the original transition functions uniquely extend from the real line to $\aleph_R^{(0)}$, and this would work as well for $\aleph_C^{(0)}$; therefore , we expect every real supermanifold to be the real sector of a corresponding unique ^{s}complex supermanifold . In particular , then , the methods of canonical superspace analysis developed in this work should be applicable to superspace supergravity theories with non-trivial body manifolds , as long as these admit a canonical decomposition themselves .

Appendix to § 3

$$\eta_{mn} = \mathrm{diag}\,(-1,+1,+1,+1) \quad , \quad \{\gamma_m,\gamma_n\} = 2\,\eta_{mn}\,\mathrm{I} \quad , \quad m,n = 0,1,2,3$$

Majorana representation (M) , real orthogonal

$$\text{both} \quad \varsigma_M \quad \text{and} \quad \gamma_0^M = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ -1 & 0 & & \\ 0 & 1 & & \end{bmatrix} \qquad \gamma_3^M = \begin{bmatrix} 1 & 0 & & \\ 0 & -1 & & \\ & & -1 & 0 \\ & & 0 & 1 \end{bmatrix}$$

$$\gamma_1^M = \begin{bmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{bmatrix} \qquad \gamma_2^M = \begin{bmatrix} & & 1 & 0 \\ & & 0 & -1 \\ 1 & 0 & & \\ 0 & -1 & & \end{bmatrix}$$

$$\partial_m \equiv \partial/\partial x^m \quad , \quad \partial_\alpha^M \equiv \partial/\partial\theta_M^\alpha = -\,(\overline{\partial_\alpha^M})$$

$$\text{In real superspace (only)} \quad (\theta_M^\alpha)^* = \theta_M^\alpha$$

Weyl representation (W) , complex unitary

Change of representation (M) $\rightarrow$ (W) via complex unitary matrix $\Omega = \frac{1}{2}\begin{bmatrix} 1 & -i & i & -1 \\ i & 1 & 1 & i \\ -1 & -i & i & 1 \\ -i & 1 & 1 & -i \end{bmatrix}$

$$\Omega^{-1} = \Omega^\dagger \qquad \det\Omega = 1$$

$$\varsigma_W = \begin{bmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & i \\ & & -i & 0 \end{bmatrix} \quad \text{and} \quad \gamma_0^W = \begin{bmatrix} & i\mathbb{1} \\ i\mathbb{1} & \end{bmatrix} \qquad \gamma_j^W = \begin{bmatrix} & i\sigma_j \\ -i\sigma_j & \end{bmatrix}$$

where σ_j , j=1,2,3 , are the Pauli matrices $\begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$, $\begin{bmatrix} & -i \\ i & \end{bmatrix}$, $\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}$.

Two component notation in the Weyl representation , as in [3 , Wess & Bagger] :

The identity 1 is grouped with the Pauli matrices σ_j in the 4 dimensional Pauli matrices

$$(\sigma_m)_{\rho\dot{\gamma}} \equiv (1 , \sigma_j) \quad \text{whose complex conjugates are} \quad (\bar{\sigma}_m)_{\dot{\rho}\gamma} \equiv ((\sigma_m)_{\rho\dot{\gamma}})^*$$

where $\rho,\gamma = 1,2$ and the following general notational rule is employed :

Rule †
"Complex conjugation interchanges barred and unbarred , dotted and undotted".

Spinor indices are lowered and raised , always on the left , with

$$\varepsilon_{\rho\gamma} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \varepsilon^{\rho\gamma} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \text{ so that} \quad \varepsilon^{\rho\tau}\,\varepsilon_{\tau\gamma} = \delta^{\rho}{}_{\gamma} \ .$$

Thus $\psi_\rho = \varepsilon_{\rho\gamma}\,\psi^\gamma$, $\psi^\rho = \varepsilon^{\rho\gamma}\,\psi_\gamma$ and $\psi^\rho\,\lambda_\rho = -\,\psi_\rho\,\lambda^\rho$.

Exactly the same applies to dotted spinor indices ,
i.e. $\bar{\varepsilon}_{\dot{\rho}\dot{\gamma}} = \varepsilon_{\rho\gamma}$ and $\bar{\varepsilon}^{\dot{\rho}\dot{\gamma}} = \varepsilon^{\rho\gamma}$.

Thus , in particular , one can show that $(\bar{\sigma}_m)^{\dot{\rho}\gamma} = \bar{\varepsilon}^{\dot{\rho}\dot{\tau}}\,\varepsilon^{\gamma\eta}\,(\bar{\sigma}_m)_{\dot{\tau}\eta} = (1 , -\sigma_j)$.

The basic four component index structure of the Dirac matrices , $(\gamma^{w}_{m})^{\alpha}{}_{\beta}$,
is replaced by

$$\gamma^{w}_{m} = \begin{bmatrix} & i\,(\sigma_m)_{\rho\dot{\gamma}} \\ i\,(\bar{\sigma}_m)^{\dot{\rho}\gamma} & \end{bmatrix}$$

and correspondingly

$$\theta_W = \begin{bmatrix} \theta_W^1 \\ \theta_W^2 \\ \theta_W^3 \\ \theta_W^4 \end{bmatrix} \rightarrow \begin{bmatrix} -\theta^2 \\ \theta^1 \\ \bar{\theta}^{\dot{1}} \\ \bar{\theta}^{\dot{2}} \end{bmatrix} = \begin{bmatrix} \theta_\rho \\ \bar{\theta}^{\dot{\rho}} \end{bmatrix}$$

$$D^W = [\, D_1^W \; D_2^W \; D_3^W \; D_4^W \,] \rightarrow [\, D_2 \; -D_1 \; \bar{D}_{\dot{1}} \; \bar{D}_{\dot{2}} \,] = [\, D^\rho \; \bar{D}_{\dot{\rho}} \,] \; .$$

Idem for parameters ε and spinor fields ψ in general (same as for θ) and for vector fields Q (same as for D).

Since $\theta_W^\alpha = (\Omega^{-1})^\alpha{}_\beta \, \theta_M^\beta$,

the coordinate change $\theta_M^1 , \theta_M^2 , \theta_M^3 , \theta_M^4 \rightarrow -\theta^2 , \theta^1 , \bar{\theta}^{\dot{1}} , \bar{\theta}^{\dot{2}}$

has Jacobian matrix

$J_W = \Omega^{-1}$ and thus $\det J_W = 1$

It follows that

$$\theta_M^1 \, \theta_M^2 \, \theta_M^3 \, \theta_M^4 = (-\theta^2) \, \theta^1 \, \bar{\theta}^{\dot{1}} \, \bar{\theta}^{\dot{2}}$$

$$d^4\theta_M \equiv d\theta_M^4 \, d\theta_M^3 \, d\theta_M^2 \, d\theta_M^1 = d\bar{\theta}^{\dot{2}} \, d\bar{\theta}^{\dot{1}} \, d\theta^1 \, d(-\theta^2) = -4 \, d^2\bar{\theta} \, d^2\theta$$

$$\partial_\rho \equiv \partial/\partial\theta^\rho \quad , \quad (\overline{\partial_\rho}) = -\partial/\partial\bar{\theta}^{\dot{\rho}}$$

In real superspace (only) $(\theta^\rho)^* = \bar{\theta}^{\dot{\rho}}$

<u>**Canonical representation**</u> (C) , complex unitary (not the Dirac rep.!)

Change of representation (M) $\rightarrow$ (C) via complex unitary matrix $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -i & 0 & i \\ -i & 0 & i & 0 \end{bmatrix}$

$$H^{-1} = H^{\dagger} \qquad \det H = 1$$

$$\varsigma_C = \begin{bmatrix} 0 & -i & & \\ i & 0 & & \\ & & 0 & i \\ & & -i & 0 \end{bmatrix} \quad \text{and} \quad \gamma_0^C = \begin{bmatrix} & & 0 & i \\ & & -i & 0 \\ 0 & -i & & \\ i & 0 & & \end{bmatrix} \qquad \gamma_j^C = \begin{bmatrix} \sigma_j & \\ & \sigma_j^* \end{bmatrix}$$

<u>Two component notation</u> in the canonical representation :

The Pauli matrices σ_j are given the index structure $(\sigma_j)^{\xi}{}_{\zeta}$, where $\xi, \zeta = \mathrm{I,II}$.

Accordingly , the complex conjugated Pauli matrices σ_j^* have index structure $(\bar{\sigma}_j)^{\dot{\xi}}{}_{\dot{\zeta}}$.

Furthermore , we define

$$\tau^{\xi}{}_{\dot{\zeta}} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \bar{\tau}^{\dot{\xi}}{}_{\zeta} \quad .$$

The conventions to lower and raise spinor indices are identical to the ones in the Weyl representation , that is

$$\varepsilon_{\xi\zeta} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \varepsilon^{\xi\zeta} \equiv \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad , \quad \varepsilon^{\xi\chi}\,\varepsilon_{\chi\zeta} = \delta^{\xi}{}_{\zeta} \quad ,$$

$$\psi_{\xi} = \varepsilon_{\xi\zeta}\,\psi^{\zeta} \quad , \quad \psi^{\xi} = \varepsilon^{\xi\zeta}\,\psi_{\zeta} \quad , \quad \psi^{\xi}\,\lambda_{\xi} = -\,\psi_{\xi}\,\lambda^{\xi}$$

and idem for dotted spinor indices .

The basic four component index structure of the Dirac matrices , $(\gamma^{c}_{m})^{\alpha}{}_{\beta}$,
is replaced by

$$\gamma^{c}_{0} = \begin{bmatrix} & -i\tau^{\xi}{}_{\dot\zeta} \\ i\bar\tau^{\dot\xi}{}_{\zeta} & \end{bmatrix} \quad , \quad \gamma^{c}_{j} = \begin{bmatrix} (\sigma_j)^{\xi}{}_{\zeta} & \\ & (\bar\sigma_j)^{\dot\xi}{}_{\dot\zeta} \end{bmatrix}$$

and correspondingly

$$\theta_c = \begin{bmatrix} \theta^1_c \\ \theta^2_c \\ \theta^3_c \\ \theta^4_c \end{bmatrix} \rightarrow \begin{bmatrix} \theta^{I} \\ \theta^{II} \\ \bar\theta^{\dot I} \\ \bar\theta^{\dot{II}} \end{bmatrix} = \begin{bmatrix} \theta^{\xi} \\ \bar\theta^{\dot\xi} \end{bmatrix}$$

$$D^c = [\, D^c_1 \; D^c_2 \; D^c_3 \; D^c_4 \,] \rightarrow [\, D_I \; D_{II} \; -\bar D_{\dot I} \; -\bar D_{\dot{II}} \,] = [\, D_\xi \; -\bar D_{\dot\xi} \,] \; .$$

Idem for parameters ε and spinor fields ψ in general (same as for θ)
and for vector fields Q (same as for D) .

Since $\quad \theta^{\alpha}_{c} = (H^{-1})^{\alpha}{}_{\beta}\, \theta^{\beta}_{M}$,

the coordinate change $\quad \theta^1_M , \theta^2_M , \theta^3_M , \theta^4_M \rightarrow \theta^{I} , \theta^{II} , \bar\theta^{\dot I} , \bar\theta^{\dot{II}}$

has Jacobian matrix

$J_c = H^{-1}$ and thus $\det J_c = 1$

It follows that

$$\theta^1_M \theta^2_M \theta^3_M \theta^4_M = \theta^{I} \theta^{II} \bar\theta^{\dot I} \bar\theta^{\dot{II}}$$

$$d^4\theta_M \equiv d\theta^4_M\, d\theta^3_M\, d\theta^2_M\, d\theta^1_M = d\bar\theta^{\dot{II}}\, d\bar\theta^{\dot I}\, d\theta^{II}\, d\theta^{I} = -4\, d^2\bar\theta\, d^2\theta$$

From the Pauli matrices $(\sigma_j)^{\xi}{}_{\zeta}$, we construct

$$(\sigma^j)_{\xi\zeta} = \varepsilon_{\xi\chi}\,(\sigma_j)^{\chi}{}_{\zeta} = \begin{bmatrix} -1 & \\ & 1 \end{bmatrix} , \begin{bmatrix} -i & \\ & -i \end{bmatrix} , \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$$

$$(\sigma_j)^{\xi\zeta} = \varepsilon^{\zeta\chi}\,(\sigma_j)^{\xi}{}_{\chi} = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} , \begin{bmatrix} -i & \\ & -i \end{bmatrix} , \begin{bmatrix} & -1 \\ -1 & \end{bmatrix}$$

$$(\sigma^j)_{\xi}{}^{\zeta} = \varepsilon_{\xi\chi}\,(\sigma_j)^{\chi\zeta} = \varepsilon^{\zeta\chi}\,(\sigma^j)_{\xi\chi} = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} , \begin{bmatrix} & i \\ -i & \end{bmatrix} , \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} .$$

Thus , $(\sigma^j)_{\xi\zeta}$ and $(\sigma_j)^{\xi\zeta}$ are symmetric matrices and the $(\sigma^j)_{\xi}{}^{\zeta}$ correspond to the Pauli matrices transposed :

$$(\sigma^j)_{\xi\zeta} = (\sigma^j)_{\zeta\xi} \quad , \quad (\sigma_j)^{\xi\zeta} = (\sigma_j)^{\zeta\xi} \quad , \quad (\sigma^j)_{\xi}{}^{\zeta} = (\sigma_j)^{\zeta}{}_{\xi} \quad .$$

$$\partial_{\xi} \equiv \partial/\partial\theta^{\xi} \quad , \quad \overline{(\partial_{\xi})} = -\,\partial/\partial\bar{\theta}^{\dot{\xi}}$$

In real superspace (only) $(\theta^{\xi})^* = \bar{\theta}^{\dot{\xi}}$

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