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L^p–distributions on symmetric spaces

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Abstract

The notion of L^p–distributions is introduced on Riemannian symmetric spaces of noncompact type and their main properties are established. We use a geometric description for the topology of the space of test functions in terms of the Laplace–Beltrami operator. The techniques are based on a-priori estimates for elliptic operators. We show that structure theorems, similar to $\mathbb{R}^n$, hold on symmetric spaces. We give estimates for the convolutions.

1 Introduction

In this paper we will generalize the notion of L^p–distributions to the setting of symmetric spaces and establish their main properties. Basic examples are L^p-functions and their generalized derivatives. We will give an invariant description of these distributions as sums of iterated Laplace-Beltrami operators applied to L^p-functions. As consequence, we give several results on convolution and other properties of these distributions.

We use a geometric approach based on a-priori estimates involving elliptic operators. We suggest alternative definitions of test functions using the iterated powers of the Laplace–Beltrami operator associated to the Riemannian structure. Combining the regularity properties of pseudo-differential operators with the existence of fundamental solutions for elliptic invariant operators on a symmetric space, we can obtain certain uniform a-priori estimates for the elliptic invariant operators. These estimates are established in L^p spaces for $1 \leq p \leq \infty$. Similar estimates for the gradient are known for $p = \infty$ on manifolds with bounded curvature ([3]). On the other hand, estimates for $1 < p < \infty$ are well known for Euclidean spaces ([11]) and are closely related to the continuity of pseudo-differential operators of order zero, which holds only locally on general manifolds. We will give a unifying algebraic proof of global a-priori estimates in L^p–space for all $1 \leq p \leq \infty$ on symmetric spaces of the noncompact type. The estimates are uniform since one can use convolution to extend them globally. The general theory of the second order differential operators on Lie groups can be found in [8], [13], and we refer there for more detailed information.

In the last section we use these estimates to define the spaces of test functions dependent on p. Then the standard construction leads to the distribution spaces $\mathcal{D}'_{L^p}$. We relate these

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spaces to the corresponding L^p spaces and their generalized derivatives. Distribution spaces of this type are useful in a number of applications ([10], [2]). The general distribution theory on symmetric spaces can be found in [4] (as well as many general results on invariant differential operators and convolutions). However, in comparison with $\mathbb{R}^n$, we obtain the representation formulas involving the iterated Laplacian. Finally we show that the convolution properties of [10] hold. The spaces $\mathcal{D}'_{L^p}$ provide a scale of distribution spaces which leads to the tempered distributions. In [1], summable distributions ($\mathcal{D}'_{L^1}$) are used as a foundation for the theory of distributions in locally convex spaces. In the case of symmetric spaces, this leads to the distribution theory as well, since the convolution techniques are available. We hope to develop this point of view in subsequent work.

Finally we note that in this paper we consider groups with classical Lie algebras only. The exceptional cases $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ are left out because the mapping $d\pi$ induced by the canonical projection π of the symmetric space is not surjective from the center of the algebra of left invariant differential operators on the group to the algebra of left invariant differential operators on the symmetric space ([6]).

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2 Preliminary estimates

Let $M = G/H$ be a Riemannian symmetric space of the noncompact type. This means, that it can be viewed as a quotient $M = G/H$, where G is a connected semisimple Lie group with trivial center and H is its maximal compact subgroup. We will always assume that the Lie algebra $\mathfrak{g}$ of G is classical. Therefore, we exclude the exceptional cases $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, with pairs (G, H) listed in [6]. The Riemannian structure on M is supposed to be invariant under the left action of G.

We will describe first a relation between parametrix and fundamental solutions of P. This will lead to some integrability properties of fundamental solutions, which will be applied for getting uniform L^p estimates. Let P be an elliptic differential operator of order m on M. Then there exists a parametrix for P, namely a pseudo-differential operator $Q \in \Psi^{-m}(M)$ of order $-m$, such that

$$QP = I + R \tag{1}$$

with $R \in \Psi^{-\infty}(M)$. Let F be the distributional kernel of Q. This means that $F \in \mathcal{D}'(M \times M)$ satisfies $\langle Q\phi, \psi \rangle = \langle F, \phi \otimes \psi \rangle$ for all $\phi, \psi \in \mathcal{D}(M)$, where $\mathcal{D}(M)$ is the standard space of smooth compactly supported functions on M. In the sequel we will write

$$(Qu)(x) = \int_M F(x, y)u(y)d\mu(y),$$

and we view Q as a singular integral operator with kernel F, where μ is the Riemannian measure on M. Note also, that P is automatically proper supported because it is a local operator. Assume now that P is G–left invariant. Then, according to [5, Theorem 4.2], there exists a fundamental solution for P, namely a distribution $K \in \mathcal{D}'(M)$, such that

$$PK = \delta, \tag{2}$$

δ being a delta function at the origin p of M. Let $\alpha \in \mathcal{D}(M)$ be a test function, such that $\alpha(x) = 1$ for x in a small neighborhood of p. Then equality (2) implies the existence of $\beta \in \mathcal{D}(M)$, such that

$$P(\alpha K) = \delta + \beta. \tag{3}$$

In fact, taking $\beta = P(\alpha K) - \delta$ one readily verifies that $\beta \in \mathcal{D}(M)$. The only singularity of K occurs at the point p due to the ellipticity of P and we will be interested in the integrability properties of αK. An application of formula (1) to αK yields

$$\alpha K + R(\alpha K) = QP(\alpha K) = Q\delta + Q\beta, \tag{4}$$

the last equality due to (3). We have $\beta \in \mathcal{D}(M)$ and $R \in \Psi^{-\infty}(M)$ implying $R(\alpha K), Q\beta \in \mathcal{C}^{\infty}(M)$. The operators Q and R are properly supported, therefore all the functions in (4) have compact support. Let $D \in \Psi^k(M)$ be properly supported, $k < m$. The application of D to (4) and the arguments above imply

$$D(\alpha K) = DQ\delta + \psi, \tag{5}$$

with $\psi \in \mathcal{D}(M)$. Now, the operator DQ is of a negative order $k - m$ and this implies the integrability of of its integral kernel at p. This property holds locally on arbitrary smooth manifolds (cf. [11], and some related properties can be found in [9]). The latter is equal to $DQ\delta(x)$. Equality (5) implies the integrability of $D(\alpha K)$. Thus, we have proved the following

Lemma 1 *Let P be an invariant elliptic differential operator of order m on M and K its fundamental solution at p. Then, for every $D \in \Psi^k(M)$, $k < m$, DK is locally integrable $DK \in L^1_{loc}(M)$.*

We will apply this lemma to two cases, D being an invariant differential operator and P being the Laplace–Beltrami operator on M equipped with a Riemannian structure. The space of G–left invariant differential operators of order k on M will be denoted $\mathbb{D}^k(M)$. Let $\mathbb{Z}(G)$ denote the center of the algebra of the left invariant differential operators on G. Let $\pi : G \to M = G/H$ be the canonical projection and let $\mathfrak{g}$ denote the Lie algebra of G. Note, that $d\pi : \mathfrak{g} \to T_pM$ can be extended to the algebra $\mathbb{D}(G)$ of the left invariant differential operators on G.

Theorem 1 *Let the Lie algebra $\mathfrak{g}$ of G be classical and semisimple. Let $P \in \mathbb{D}^m(M)$ be elliptic, $D \in \mathbb{D}^k(M)$, $0 < k < m$ and $1 \leq p \leq \infty$. Then there exist constants A, B, such that for every $u \in L^p(M)$ with $Pu \in L^p(M)$, we have $Du \in L^p(M)$ and*

$$||Du||_p \leq A||Pu||_p + B||u||_p. \tag{6}$$

If $p = \infty$, then Du is continuous.

If $\mathfrak{g}$ is not classical (for example, when $\mathfrak{g}$ is the real form of the exceptional Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$), the statement of Theorem 1 still holds if we assume that $P \in d\pi(\mathbb{Z}(G))$.

Let X be a smooth vector field on M. X is called *bounded* if there exist a constant C such that $||X_x|| \leq C$ for every point $x \in M$, where $||\cdot|| = \langle \cdot, \cdot \rangle^{1/2}$ is the Riemannian norm on T_xM, corresponding to the Riemannian structure.

Theorem 2 *Let M be a Riemannian symmetric space as above and let Δ be the associated Laplace–Beltrami operator. Let $1 \leq p \leq \infty$ and let X be a smooth bounded vector field on M. Then there exist constants A, B, such that for every $u \in L^p(M)$ with $\Delta u \in L^p(M)$, we have that the derivative of u with respect to X is L^p–integrable, $Xu \in L^p(M)$ and*

$$||Xu||_p \leq A||\Delta u||_p + B||u||_p. \tag{7}$$

If $p = \infty$, then Xu is continuous. Moreover, A and B can be chosen independently over the set of the smooth vector fields X bounded by 1.

The statements of Theorems 1 and 2 can be improved with respect to the order k. However, we will not discuss it here since we do not need it for the applications in the next section.

To show the statements of the theorems, first, convolving (3) with $u \in \mathcal{D}'(M)$, we conclude that

$$u = u * \delta = u * P(\alpha K) - u * \beta. \tag{8}$$

For classical algebras $\mathfrak{g}$ the extension $d\pi : \mathfrak{g} \to T_pM$ to the algebra of the left invariant differential operators on G has the property $d\pi(\mathbb{Z}(G)) = \mathbb{D}(M)$, ([5, Proposition 7.4, Theorem 7.5] or [4, Remark, p.326]). This means that $P \in \mathbb{D}^m(M)$ is an image of a bi-invariant operator on G and, in particular, commutes with left and right convolution. Therefore, (8) implies

$$u = Pu * \alpha K - u * \beta. \tag{9}$$

Let us make several remarks. In general, for a separable unimodular Lie group G and a compact H, such that (G, H) is a symmetric pair, equation (9) follows from (8), for an arbitrary $P \in \mathbb{D}(G/H)$. Indeed, distribution αK is compactly supported and, according to [4, Theorem 5.5, p. 293], we have $u * P(\alpha K) = Pu * \alpha K$. In our case, (9) follows from (8) in conditions of Theorem 1 because P is the image of a bi-invariant operator on G. Note, that if the Lie group G is compact, then statements of Theorem 1 and Theorem 2 follow directly from the local estimates of the same type. Indeed, if we apply equality (1) to u and then apply D (not necessarily invariant) to it, we get

$$Du = DQPu - DRu.$$

Now, operators DQ and DR are pseudo-differential operators of strictly negative orders and, therefore, locally bounded on L^p. Therefore, we may assume that G is not compact. Applications of the next section will deal with the case of the iterated Laplace–Beltrami operator $P = \Delta^l$. According to [4, p.331], if the Riemannian structure on G is associated to the Killing form, the Laplace–Beltrami operator on $M = G/H$ is the image of the Casimir operator on G, which belongs to the center $\mathbb{Z}(G)$.

Proof of Theorem 1: An application of D to equality (9) together with an argument above imply

$$Du = Pu * D(\alpha K) - u * D\beta.$$

By Lemma 1 we have $D(\alpha K) \in L^1(M)$ and Young inequality ([7]) yields estimate (6). In case $p = \infty$ we have convolutions of the type $L^\infty * L^1$, which give continuous functions. This

completes the proof of Theorem 1.

Now we will need some notation and auxiliary results. We start with constructing an invariant Riemannian structure on G, making the projection π a Riemannian submersion. In the sequel $l_g h = gh,\ r_g h = hg,\ g, h \in G$ will denote the left and right group actions on G. The induced actions of G on M will be denoted by the same letters. The adjoint representation will be denoted by $\mathrm{Ad} : G \to GL(T_eG)$. The following lemma is standard.

Lemma 2 *There exists a G–left and H–right invariant Riemannian structure on G, such that the canonical projection $\pi : G \to M$ is a Riemannian submersion, i.e. a submersion, for which horizontal lift of vector fields preserves Riemannian norms.*

For every $g \in G$ let $K_g = \mathrm{Ker}\ d_g\pi$ ($K_g \cong T_gH \cong T_eH$). Let N_g be the orthogonal complement to K_g with respect to $\mathrm{Ad}(H)$-invariant inner product $(\cdot,\cdot)_0$ given by

$$(X, Y)_0 := \int_{\mathrm{Ad}(H)} \langle A(X), A(Y)\rangle_0 d\mu(A), \tag{10}$$

where μ is Haar measure on a compact set $\mathrm{Ad}(H)$ and $\langle\cdot,\cdot\rangle_0$ is an arbitrary inner product on T_eG. Thus we have $T_gG = K_g \bigoplus N_g$. Define an inner product $(\cdot,\cdot)_{N_e}$ on N_e for vectors $\bar{X}, \bar{Y} \in N_e$ by

$$(\bar{X}, \bar{Y})_{N_e} := \langle X, Y\rangle_{M_p}, \tag{11}$$

where $\langle\cdot,\cdot\rangle_{M_p}$ is the restriction to T_pM of the given invariant Riemannian structure on M and $X = d\pi(\bar{X}), Y = d\pi(\bar{Y}) \in T_pM$. Vectors $\bar{X}$ and $\bar{Y}$ are uniquely defined, $d\pi$ being an isomorphism between N_e and T_pM, and they are called the *horizontal lifts* of X and Y. The desired Riemannian structure on G can now be constructed by applying dl_g to

$$\langle u, v\rangle := (u|_{K_e}, v|_{K_e})_0 + (u|_{N_e}, v|_{N_e})_{N_e}, \tag{12}$$

where $u, v \in T_eG$, $u|_{K_e}, v|_{K_e}$ and $u|_{N_e}, v|_{N_e}$ are projections of u, v on K_e and N_e, respectively. The inner product in (12) is clearly $\mathrm{Ad}(H)$–invariant, the expansion is therefore G–left and H–right invariant. It follows immediately from formulas (11) and (12) that all $d_g\pi$ are partial isometries (isometries from N_g to $T_{\pi(g)}M$). We fix this inner product on G since it satisfies Lemma 2. For the sake of completeness let us list briefly some well known properties of the pullback $\sharp$ which will be necessary.

Lemma 3 (i) *Let $\phi \in \mathcal{C}_c(G)$ be a continuous compactly supported function on G. Then for $x = \pi(g)$ the function $\phi^\flat(x)$ is correctly defined by*

$$\phi^\flat(x) = \int_H \phi(gh)dh,$$

where dh is the normalized Haar measure on H. Moreover, $\phi^\flat \in \mathcal{C}_c(M)$ and mapping $\phi \to \phi^\flat$ is linear surjective from $\mathcal{C}_c(G)$ to $\mathcal{C}_c(M)$ and from $\mathcal{D}(G)$ to $\mathcal{D}(M)$.

(ii) *The transpose of* $\phi \to \phi^\flat$ *defined by*

$$\langle T^\sharp, \phi \rangle = \langle T, \phi^\flat \rangle$$

is an injective mapping from $\mathcal{D}'(M) \to \mathcal{D}'(G)$.

(iii) *Let* $S \in \mathcal{D}'(G)$. *Then* S *is right* H*-invariant if and only if there exists* $T \in \mathcal{D}'(M)$, *such that* $S = T^\sharp$. *For* $T_1, T_2 \in \mathcal{D}'(M)$ *the convolution products on* G *and* M *are related by*

$$T_1^\sharp * T_2^\sharp = (T_1 * T_2)^\sharp.$$

(iv) *Let* Y *be a horizontal lift of a vector field* X *on* M *and let* $T \in \mathcal{D}'(M)$ *be a distribution on* M. *Then* $Y(T^\sharp) = (XT)^\sharp$.

Proof of Theorem 2: The pullback of formula (9) now reads

$$u^\sharp = (\Delta u)^\sharp * (\alpha K)^\sharp - u^\sharp * \beta^\sharp. \tag{13}$$

Let X be a smooth vector field on M, bounded by one: $||X_x|| \leq 1$. Let Y denote the horizontal lift of X:

1. $d_g\pi(Y_g) = X_x$, where $x = \pi(g)$.
2. $Y_g \in N_g = (\text{Ker } d_g\pi)^\perp$.

It is not difficult to see that Y is smooth. Let $Y_1, \ldots, Y_N$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$, such that $(Y_1)_g, \ldots, (Y_n)_g \in N_g$ for all $g \in G$. Vector field Y can be decomposed with respect to the basis $Y_1, \ldots, Y_n$ at every point $g \in G$:

$$Y_g = \sum_{i=1}^{n} a_i(g) Y_{i,g} \in N_g \subset T_gG, \tag{14}$$

where $Y_{i,g} = (Y_i)_g = d_e l_g (Y_i)_e$ are values at g of the left invariant vector fields Y_i. Note, that such decomposition is only pointwise because Y need not be left invariant in general, we use that $Y_g \in N_g$ and the fact that $Y_{1,g}, \ldots, Y_{n,g}$ constitute a basis for a linear space N_g. However, it is global and functions $a_1, \ldots, a_n$ are smooth due to the smoothness of Y and $Y_1, \ldots, Y_n$.

The norm of Y_g at T_gG is $||Y_g||^2 = \sum_{i=1}^n |a_i(g)|^2$. In view of Lemma 2, $||Y_g|| = ||X_x|| \leq 1$. In particular, $|a_i(g)| \leq 1$ for all $g \in G$. Now we differentiate $u^\sharp$ in (13) with respect to the basis vector fields Y_i and the left invariance of Y_i yields:

$$Y_i u^\sharp = (\Delta u)^\sharp * Y_i(\alpha K)^\sharp + u^\sharp * Y_i \beta^\sharp. \tag{15}$$

Obviously $Y_i\beta^\sharp \in \mathcal{D}(G) \subset L^1(G)$. In view of Lemma 1 the compactly supported distribution αK and its derivatives are integrable and so are their pullbacks, the pullback mapping being an isometry of L^p spaces. Let $A_i = ||Y_i(\alpha K)^\sharp||_1$ and $B_i = ||Y_i\beta^\sharp||_1$. Application of Young inequality [7, Cor. 20.14] to (15) yields

$$||Y_i u^\sharp||_p \leq A_i ||(\Delta u)^\sharp||_p + B_i ||u^\sharp||_p.$$

Decomposition (14) together with bounds on a_i and equalities $||(\Delta u)^\sharp||_p = ||\Delta u||_p$ and $||u^\sharp||_p = ||u||_p$ imply

$$||Yu^\sharp||_p \leq A||\Delta u||_p + B||u||_p$$

with $A = \sum_{i=1}^n A_i$ and $B = \sum_{i=1}^n B_i$. By Lemma 3 we have $||Yu^\sharp||_p = ||Xu||_p$, establishing inequality (7). In case $p = \infty$, formulas (15) and (14) imply the continuity of $Yu^\sharp$. By Lemma 3, $(Xu)^\sharp = Yu^\sharp$ is continuous. The continuity of Xu now follows from the fact that M is equipped with the quotient topology, i.e. the strongest topology, for which π is a continuous mapping. This finishes the proof of Theorem 2.

3 L^p–distributions

Let M be a symmetric space as before. In this section we will apply estimates of the previous section to the particular cases of $P = \Delta^k$, to construct spaces $\mathcal{D}'_{L^p}(M)$ of L^p–distributions, invariantly on M. For $1 \leq p \leq \infty$, we consider the space

$$\mathcal{D}_{L^p}(M) = \{\phi \in C^\infty(M) : \ \Delta^k\phi \in L^p(M), \ \ \forall\, k \in \mathbb{Z}_{\geq 0}\},$$

equipped with a countable system of seminorms

$$\omega_{p,k}(\phi) = \max\{||\phi||_p, ||\Delta^k\phi||_p\}, \ k \in \mathbb{Z}_{\geq 0}.$$

They define the coarsest locally convex topology for which the maps $\Delta^k : \mathcal{D}_{L^p}(M) \to L^p(M)$ are continuous for all $k \in \mathbb{Z}_{\geq 0}$. With this topology $\mathcal{D}_{L^p}(M)$ become Fréchet spaces. For $1 \leq p < \infty$ we obviously have the continuous embeddings

$$\mathcal{D}(M) \subset \mathcal{D}_{L^p}(M) \subset \mathcal{D}'(M).$$

The first inclusion is also dense for $p < \infty$. For $p = \infty$ define $\dot{\mathcal{D}}_{L^\infty}$ to be the subspace of $\mathcal{D}_{L^\infty}$ of functions vanishing at infinity. Then $\mathcal{D}(M)$ is dense in $\dot{\mathcal{D}}_{L^\infty}$, so the spaces $\mathcal{D}_{L^p}$ for $1 \leq p < \infty$ and $\dot{\mathcal{D}}_{L^\infty}$ are normal spaces of distributions. This implies that their strong duals are the subspaces of $\mathcal{D}'(M)$. For $1 < p \leq \infty$ and q such that $1/p + 1/q = 1$, we denote by $\mathcal{D}'_{L^p}(M)$ the strong dual of $\mathcal{D}_{L^q}(M)$. For $p = 1$ we denote by $\mathcal{D}_{L^1}{}'(M)$ the strong dual of $\dot{\mathcal{D}}_{L^\infty}(M)$. In view of Theorem 2 the spaces $\mathcal{D}_{L^p}(M)$ coincide with the spaces $\mathcal{D}_{L^p}(\mathbb{R}^n)$ of L. Schwartz ([10, Ch. VI]) when $M = \mathbb{R}^n$.

Theorem 3 *A distribution T belongs to $\mathcal{D}'_{L^p}(M)$ if and only if there exist $m = m(T) \in \mathbb{N}$ and $f, g \in L^p(M)$, such that*

$$T = f + \Delta^m g. \tag{16}$$

Proof: Let $1/p + 1/q = 1$. First, if T is given by formula (16), then T is a linear continuous functional on $\mathcal{D}_{L^q}(M)$, so $T \in \mathcal{D}'_{L^p}(M)$.

Conversely, suppose that $T \in \mathcal{D}'_{L^p}(M)$ and assume that $p > 1$. Then there exists a number m such that

$$|\langle T, \phi\rangle| \leq C\omega_{q,m}(\phi), \tag{17}$$

for every $\phi \in \mathcal{D}_{L^q}(M)$. Let $\iota : \mathcal{D}_{L^q}(M) \to L^q(M) \times L^q(M)$ be an injective inclusion, $\iota(\phi) = (\phi, \Delta^m \phi)$. On the image of ι consider a linear map $L : \iota(\mathcal{D}_{L^q}(M)) \to \mathbb{C}$ defined by

$$L(\phi, \Delta^m \phi) = \langle T, \phi \rangle.$$

Inequality (17) implies

$$|L(\phi, \Delta^m \phi)| \leq C\omega_{q,m}(\phi) = C \max\{||\phi||_q, ||\Delta^m \phi||_q\},$$

which means that L is continuous if we equip $\iota(\mathcal{D}_{L^q}(M))$ with the induced topology of $L^q(M) \times L^q(M)$. Therefore, by Hahn–Banach theorem L allows a linear continuous extension to $L^q(M) \times L^q(M)$, which we also denote by L. Now, the dual of $L^q(M) \times L^q(M)$ is $L^p(M) \times L^p(M)$, implying

$$\langle T, \phi \rangle = L(\phi, \Delta^m \phi) = \int_M \phi f d\mu + \int_M (\Delta^m \phi) g \, d\mu =$$

$$\langle f, \phi \rangle + \langle g, \Delta^m \phi \rangle = \langle f, \phi \rangle + \langle \Delta^m g, \phi \rangle =$$

$$\langle f + \Delta^m g, \phi \rangle$$

for all $\phi \in \mathcal{D}_{L^q}(M)$, and some $f, g \in L^p(M)$, with integration with respect to the Riemannian measure μ. The case with $p = 1$ follows the same lines, but one should take $C_0(M) \times C_0(M)$ instead of $L^q(M) \times L^q(M)$. Then f and g are Radon measures, but a standard closed graph argument shows that f and g can be taken in L^∞. The proof is complete.

From Theorem 3 we can imagine the general structure of L^p distributions. For example, if $f \in L^p(M)$ is compactly supported and $g \in L^p(M)$ is supported "near infinity", then sums of derivatives of f and g represent elements of $\mathcal{D}'_{L^p}(M)$. There is an improvement to Theorem 3:

Corollary 1 *Let $T \in \mathcal{D}'_{L^p}(M)$. Then there exist $m = m(T) \in \mathbb{N}$ and continuous functions $f_k \in L^p(M)$, $0 \leq k \leq m$, such that*

$$T = \sum_{k=0}^{m} \Delta^k f_k.$$

Proof: Let Q_j be a right parametrix for the elliptic operator $(I - \Delta)^j$. Then we have $(I - \Delta)^j Q_j = I + R_j$, with R_j a smoothing operator. Applying this to a function $h \in L^p$, we get

$$h = (I - \Delta)^j (Q_j h) - R_j h.$$

Choosing j sufficiently large, the functions $Q_j h$ and $R_j h$ are continuous L^p-functions. This argument for functions f and g in (16) imply the corollary.

Theorem 4 *A distribution T belongs to $\mathcal{D}'_{L^p}(M)$ if and only if $\alpha * T \in L^p(M)$ for every $\alpha \in \mathcal{D}(M)$.*

Proof: Assume first that $1 < p \leq \infty$ and let q be such that $1/p + 1/q = 1$. Let T be in $\mathcal{D}'_{L^p}(M)$. The set

$$B = \{\phi \in C_c(M) : \ ||\phi||_q \leq 1\}$$

is dense in the unit ball of $L^q(M)$ since $1 \leq q < \infty$ and μ is a Radon measure. Let $\alpha \in \mathcal{D}(M)$ be fixed. Then by the Young inequality for every $\phi \in B$ we get

$$||\Delta^k(\hat{\alpha} * \phi)||_q = ||\Delta^k \hat{\alpha} * \phi||_q \leq ||\Delta^k \hat{\alpha}||_1 ||\phi||_q \leq ||\Delta^k \hat{\alpha}||_1,$$

where $\hat{\alpha} = (\widehat{\alpha^\sharp})^\flat$ and $\hat{\beta}(g) = \beta(g^{-1})$ for $g \in C_c(G)$. This implies that the set $\{\hat{\alpha} * \phi, \phi \in B\}$ is bounded in $\mathcal{D}_{L^q}(M)$ and, by using Lemma 3, we get that

$$\langle \alpha * T, \phi \rangle = \langle \alpha^\sharp * T^\sharp, \phi^\sharp \rangle = \langle T^\sharp, \widehat{\alpha^\sharp} * \phi^\sharp \rangle = \langle T, (\widehat{\alpha^\sharp})^\flat * \phi \rangle = \langle T, \hat{\alpha} * \phi \rangle$$

is bounded if we let $\phi \in B$. It follows that $\sup_{\phi \in B} |\langle \alpha * T, \phi \rangle|$ are bounded and hence $\alpha * T$ extends to a linear continuous functional on $L^q(M)$, which means that $\alpha * T \in L^p(M)$.

Conversely, suppose that $\alpha * T \in L^p(M)$ for all $\alpha \in \mathcal{D}(M)$. For a fixed $\alpha \in \mathcal{D}(M)$,

$$\langle \hat{\phi} * T, \hat{\alpha} \rangle = \langle \alpha * T, \phi \rangle$$

are bounded for $\phi \in B$ and, therefore, $\hat{\phi} * T$ are bounded in $\mathcal{D}'(M)$ for $\phi \in B$. Let us denote by $k(m)$ the smallest number k for which the fundamental solution K for Δ^k is in $C^m(M)$. Let F be a compact neighborhood of the origin of M and let $\gamma \in \mathcal{D}_F(M)$ be equal to one in a neighborhood of the origin and supported in F. Then formula (3) with $P = \Delta^k$ implies that

$$T = \delta * T = \Delta^k(\gamma K * T) - \beta * T, \tag{18}$$

where $\beta * T \in L^p(M)$ according to our assumption and $\gamma K \in C^m(M)$ is supported in F.

Now we will use the property that bounded sets of distributions are equicontinuous on compact sets in $\mathbb{R}^n$ (see [12]), which means that if $D' \subset \mathcal{D}'(\mathbb{R}^n)$ is bounded and $N \subset \mathbb{R}^n$ is compact, then there exists an integer m such that

$$|\langle T, \psi \rangle| \leq C \max_{|\nu| \leq m} ||D^\nu \psi||_q, \ \forall \psi \in \mathcal{D}_N, \ \forall T \in D'.$$

This estimate is a direct consequence of the uniform boundedness principle in the Fréchet space $\mathcal{D}_{L^q}$. Because F is compact, it follows that distributions $\hat{\phi} * T$ are equicontinuous on F, since we can apply Theorem 1 to their localizations in $\mathbb{R}^n$. Therefore $\hat{\phi} * T$ extend to $\mathcal{D}_F^{(m)}(M)$ for all $\phi \in B$ and the extensions satisfy

$$\sup_{\phi \in B} |\langle \alpha * T, \phi \rangle| < \infty,$$

in particular if we take $\alpha = \gamma K$. It follows that $\gamma K * T \in L^p(M)$ since $\gamma K \in \mathcal{D}_F^{(m)}(M)$ if K is a fundamental solution for $\Delta^{k(m)}$. The statement follows now from Theorem 3 and decomposition (18).

The case of $p = 1$ is similar, implying that $\gamma K * T$ must be a Radon measure. A standard additional closed graph argument shows that it is actually in $L^\infty(M)$. Finally we generalize some properties of convolutions.

Theorem 5 *The following holds:*

(i) *Let* $T \in \mathcal{D}'_{L^p}(M)$ *and* $\phi \in \mathcal{D}_{L^q}(M)$. *Then* $\phi T \in \mathcal{D}'_{L^r}(M)$ *if* $r \geq 1$ *and* $1/r \leq 1/p + 1/q$.

(ii) *Let* $T \in \mathcal{D}'_{L^p}(M)$ *and* $S \in \mathcal{D}'_{L^q}(M)$, *where* $1/p + 1/q \geq 1$. *Then* $T * S \in \mathcal{D}'_{L^r}(M)$ *if* $1/r = 1/p + 1/q - 1$.

(iii) *Let* $T \in \mathcal{D}'_{L^p}(M)$ *and* $\phi \in \mathcal{D}_{L^q}(M)$, *where* $1/p + 1/q \geq 1$. *Then* $\phi * T \in \mathcal{D}_{L^r}(M)$ *if* $1/r = 1/p + 1/q - 1$.

Note that the mappings induced by (i) and (iii), are separately continuous in the corresponding spaces, and the mapping in (ii) is continuous. The proof of Theorem 5 consists of application of Theorem 3 to the spaces in Theorem 5. The statement follows from the corresponding properties of the convolution ([7]). We omit the details since they are similar to those in Section VI in [10] after we reduce the problems to the Lebesgue spaces by Theorem 3.

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