A MATHEMATICAL ELUCIDATION OF THE BASES OF KEPLER'S LAWS

by

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Abstract of thesis

Throughout this analysis I have followed the Keplerian usage, based ultimately on considerations of geometrical constructibility and cohesion, by contrast with the Newtonian usage: in particular, for the discussion of a single planet, time has been designated by mean anomaly, which invariably possesses some geometrical representation.

I have considered separately the observational verifications of Law I and Law II in Astronomia Nova. By stipulating an observation-controlled level of accuracy I prove that Kepler distinguished the correct (Sun-focused) ellipse at the quadrant from both the eccentric circle and the small-grade curve which he first proposed, and found a parametric-polar equation (in terms of eccentric anomaly) which specified it completely. Concurrently he reformulated the mean anomaly in terms of area, also involving the eccentric anomaly as parameter. While these two formulae in conjunction produce the correct orbit, I demonstrate that, by the application of his characteristic geometrical method, Kepler could have arrived at no other solution.

The proofs of Law I and Law II in Epitome are again treated separately: Kepler used these laws to derive expressions leading to the exact modern formulae for radial and transverse motion respectively. By stipulating a level of analytical exactitude I have confirmed that his methods were mathematically sound, achieving results tantamount to those nowadays obtained via integration and differentiation. The two distinct faculties of the Sun supposed to produce these motions were the magnetism (radially) and the driving force (transversely); thus Kepler's account of the physical causes was invalid in modern terms owing to his defective (Aristotelian) understanding of the concept of force.

Moreover there is a quantifiable error in Kepler's kinematical conclusions that vitiates a potential proof of Law III (interconnecting the planetary system) which I have discovered in Harmonice Mundi; he undoubtedly established the approximate truth of this law from numerical calculations.

CONTENTS

INTRODUCTION 2

Index of Terminology and Notation 7

PROLEGOMENA 9

PART A: THE DISCOVERY OF LAW I AND LAW II 22

CHAPTER I: The common foundations of the analysis of Astronomia 23
Appendices I 55

CHAPTER II: The discovery of the curve 60
Appendices II 102

CHAPTER III: The discovery of the orbit 119
Appendices III 170

CHAPTER IV: Further consideration of the two rôles of the Sun: developments in the intervening years 197
Appendix IV 211

PART B: THE DISCOVERY OF THE RADIAL AND TRANSVERSE COMPONENTS OF MOTION WITH THEIR DISTINCT PHYSICAL CAUSES: AND THE CONSEQUENCES 212

CHAPTER V: The common foundations of the analysis of Epitome and Harmonice 213

CHAPTER VI: The radial motion explained by the magnetism of the Sun: proof of Law I 230
Appendices VI 267

CHAPTER VII: The transverse motion produced by the driving force of the Sun: proof of Law II 276
Appendix VII 315

CHAPTER VIII: Correlations and some consquences 320
Appendices VIII 358

CONCLUSION 366

BIBLIOGRAPHY 370

INTRODUCTION

Sarton once suggested a three-dimensional reference system whereby the prospective student of the history of science could classify his areas of interest into three mutually perpendicular cross-sections. The intersection of these planes (by a corollary which Sarton did not state) ought then to give the 'co-ordinates' of the particular topic for specialization, in terms of subject, time, and language.[1] Unfortunately, I did not come across this mechanism of selection-criteria until after I had embarked on this research, and might, had I done so, have made a wiser choice in one respect: but it is doubtful whether I could ever have chanced on problems which I would have found so uniquely fascinating as those encountered in the analysis of Kepler's astronomical work.

The application of this three-dimensional representation to the present context subsists in its provision, by analogy, of a relationship between the three constituents of Kepler's astronomy: geometrical, kinematical, physical. These three interact in pairs, as shown in Sarton's diagram, while the point in space at the intersection of all the planes can be taken to signify Kepler's eventual synthesis. I am aware that the physical and kinematical constituents have formerly been regarded as synonymous, but I shall distinguish them here on the grounds that the physical considerations additionally involve questions of direction, and therefore possess geometrical as well as kinematical significance. Thus geometrical and kinematical considerations will dominate the subsequent discussion since their mathematical quantification is immediate, and Kepler's physical opinions will only crop up from time to time as a stimulus to further progress. We shall also trace the development of these opinions quite independently (and finally they too will be expressed in mathematical terms on Kepler's behalf).

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1. The Study of the History of Science pp. 27-8 and diagram p. 29: he gives the third coordinate as nationality/race, but, in the example he mentions, this comes down to a question of language skills.

It will eventually turn out that the theoretical basis of Law III (the interconnection of the planetary system) depends on an assessment of one (comparatively minor) aspect of Kepler's synthesis for a single planet, which comes to light almost incidentally at the end of this study. Accordingly, we shall concentrate attention on the first two laws, which govern the orbit of a single primary planet.[2.] The plan to be adopted is that their discovery in Astronomia will be the subject of Part A, and their confirmation in Epitome (supported by Harmonice) that of Part B. Consequently, the investigations in Part A will entail the stipulation of an observation-controlled level of accuracy, while the proofs in Part B will be founded on a stipulated level of analytical exactitude: the latter discussion is inevitably as much concerned with Kepler's contribution to mathematics as to astronomy.

Within these broad divisions, Law I and Law II have been examined separately—for the first time, it seems: this duality of treatment is justified in each part in turn. It has involved a box-and-coxing approach to Astronomia, rather than a chronological one: such an account by topics has previously been tentatively applied to Epitome, but is here carried through with some conviction because I have found it possible to link the topics by means of common, or similar, diagrams. This is a simple idea but nevertheless it appears that it has not been tried before.

It should perhaps be remarked that this study is hardly historical in the usual sense, for I have not endeavoured to put Kepler's work into perspective against a background of contemporary knowledge. Instead, believing it to be generally accepted that Kepler pioneered new ground when he broke the 'spell of circularity',[3.] I have examined only those parts of his earlier work which contributed to that great leap forward. Because his achievements are to be assessed by modern standards, I have, in four Prolegomena, gathered together for reference such geometrical and

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2. Thus we formally exclude any mention of Kepler's work on lunar astronomy, the subject of Astronomia C. 37 and Epitome IV, Part II Section 6 and Part III Section 6.

3. Koyré: La révolution astronomique, p. 119.

kinematical formulae as will be required, some of which are not readily accessible elsewhere.[4] However, the kinematical standards are in practice those set by Newton, and thus many of the comparisons made are between Kepler and his great successor: a reversal of the normal procedure. Nevertheless, there is in fact no question of comparisons: indeed it is (three) specific contrasts between the usages of the two people that will be highlighted continually in what follows.[5]

I take this early opportunity to state that in this study we shall be concerned solely with dimensions of length[6] and time, and the absence of a dimension of mass is neither here nor there. Indeed the concept of mass is obviously in no way pertinent to the investigation of a single planet, and should, moreover, have been inappropriate in considering a system of planets[7] at least when, for a first approximation, we make the simplifying assumption that the mass of every planet is negligible by comparison with that of the Sun.[8] Hence we shall totally disregard the mass-correction factor introduced by Newton in Principia III to interconnect the planetary system; the kinematical standards referred to above will be those of Principia I Sections II and III.

Kepler made many contradictory statements in the course of his work, partly due to the inadequacy of his knowledge as to the way things worked, coupled with the fertility of his ideas as to how they might. Perhaps the most notable instance is cited by Caspar in his biography,[9] in a description of opposing theories of the cause of the tides,[10] as evidence that Kepler failed to discipline himself to a consistent set of beliefs. By contrast, the contradictions with which we shall be concerned in this study arise in relation

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4. Partly because Kepler's approach was not that adopted nowadays.
5. Once the necessary terminology has been defined.
6. Of which Kepler possessed a sound grasp, as we shall demonstrate when occasion arises.
7. Though Kepler did, in one instance, introduce a concept of what we shall, when the time comes, describe as the variable planetary massiveness.
8. As we shall see, this is preferable to the assumption that the mass of every planet is constant.
9. Kepler, pp. 280-1.
10. Introduction to Astronomia contra Harmonice IV C. 7: this topic does not concern us here.

to his views on the causes of planetary motion and are simply a consequence of the development of those views with maturity and greater expertise—though it seems to me that no commentator except Caspar[11.] has taken into account the possibility that Kepler's opinions in Epitome might differ from those expressed in Astronomia (as they do), in that those preliminary opinions have frequently been quoted with no indication that they were not final. However, since I could not hope to deal with every one of Kepler's statements on this subject, I have restricted myself, by intent, to examining those views which are either mathematically quantifiable, or, at the least, diagrammatically represented.

In this connection I remark that Kepler's letters have to be treated with caution, to the extent that they may record transient attitudes, later to be superseded. Additionally, Caspar has detected the use of two amanuenses in the letter most frequently quoted with reference to Kepler's technical astronomical work,[12.] and has concluded that: 'These transcribers, lacking comprehension, have been responsible for a series of errors in the text'.[13.] Therefore, by these self-imposed terms of reference, I have quoted from Kepler's letters very sparingly, on the rare occasions when they reiterate, and clarify, the text of his published works, without ambiguity.

One feature of Kepler's approach which I believe has not previously been sufficiently acknowledged was what will be described as his tendency to 'universalization'. This term has been chosen in preference to 'generalization' in order to emphasize that the latter permits the consideration of special cases, whereas it was Kepler's intention to eliminate them, by regarding the Earth as just another planet, on a par with all the rest. We shall subsequently highlight both the power and the limitation of this concept.

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11. Nevertheless I do not agree even with Caspar's interpretation in every respect.

12. Letter 358 to Fabricius (11.10.1605): G.W. XV especially p. 247 line 204-p. 253 line 518.

13. G.W. XV, pp. 527-8.

Turning now to the technical details of the methods to be adopted in this study, I have included only an occasional vivid work or phrase in Kepler's own language, and have refrained from giving the Latin originals of the extracts which I have needed to quote, in the expectation that this will be helpful to those who, like myself, read Latin only laboriously; in any case, reference to the context is frequently desirable, and so my account is intended to be read in conjunction with Astronomia, Epitome, and Harmonice, of which Caspar's superbly accurate modern editions are readily available.[14] Moreover, the freshness of Kepler's approach was manifest in the way he expressed himself, and he gradually began to write in terms which are recognizably modern: I have hoped to strike the correct balance in those short passages that I have translated.

It has been necessary to restrict the sense in which some words are employed in the course of the present analysis: such specialized terminology has been underlined where it first occurs and is defined. Additionally, it has seemed more efficient to insist on a consistent scheme of notation, if only to save continual reiteration: indeed, failure to do so previously has in fact led to errors in interpretation (I have found two, to be noted when they arise): then, too, I must deprecate the present-day practice of taking the radius of the eccentric circle as unity; not only does this detract from a theoretical appreciation of the order of linear dimensions involved, but it has, on one occasion, introduced confusion when the orbits of the Earth and Mars have to be examined together, in an observational context, as I shall demonstrate. In fact I shall follow Kepler by setting out an index of terminology and notation as a starting-point, as he did in Astronomia: meanwhile, though his own lettering was utterly unsystematic at first, we can detect the beginnings of uniformity in Epitome, in the use of a consistent designation for the line of apsides.[15]

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14. Except in the case of Epitome V Part I, where mistakes have been noted in the appropriate context: I point out that this was the only one of Kepler's major works that Caspar had not translated prior to editing it.

15. And at the end of Astronomia Part IV as well, A for the Sun, B for the centre of the eccentric—which we shall adopt here.

INDEX OF STANDARD TERMINOLOGY AND NOTATION

$\underline{a}$ is the radius of the eccentric circle and the major semiaxis of the orbital ellipse

$\underline{b}$ is the minor semiaxis of the orbital ellipse

$\underline{c}$ is the dimension-parameter involved in various time-hypotheses or their equivalents

$\underline{e}$ is the eccentricity

$\underline{i}$ is an intermediate value of the range 1, 2...n

$\underline{n}$ is a positive integer

$\underline{q}_0, \underline{q}_1 \ldots$ are coefficients in the polar equation

$\underline{r}$ is the radius vector

$\underline{s}$ is the arc length

$\underline{t}$ is the time

$\underline{v}$ is the (tangential) velocity in orbit

$\underline{A}$ is the area

$\underline{T}$ is the periodic time

α is the (completed) mean anomaly: defined in Chapter I, § 2

β is the eccentric anomaly: defined in Chapter I, § 2

γ is the central anomaly: defined in Chapter I, § 2

δ denotes an increment or observational uncertainty

Δ denotes a theoretical defect

ϵ is the eccentricity of the Auxiliary Ellipse

θ is the true anomaly: defined in Chapter I, § 2

λ is the (completed) libration

π as always

ϕ is the angle between the radius vector and the tangent

Notes to this Index of Terminology and Notation.

1. Any non-standard notation which is introduced for a special purpose, is to be understood to be restricted to the particular section or Appendix in which it is defined: it has not been listed here.

2. According to normal mathematical practice, the use of a suffix denotes a particular interpretation of the quantity or point under consideration, but will not affect its underlying definition.

3. Letters in parenthesis—e.g. [Q]-are used to indicate that my standard notation has been employed for clarity to denote a point given a different designation by Kepler.

4. The notation $O(e^n)$ means 'involving terms of degree e^n and higher' and is used as a concise way of specifying accuracy: to save unnecessary reiteration, once this has been stated in the first line of each calculation, it will be omitted until the final line is reached.

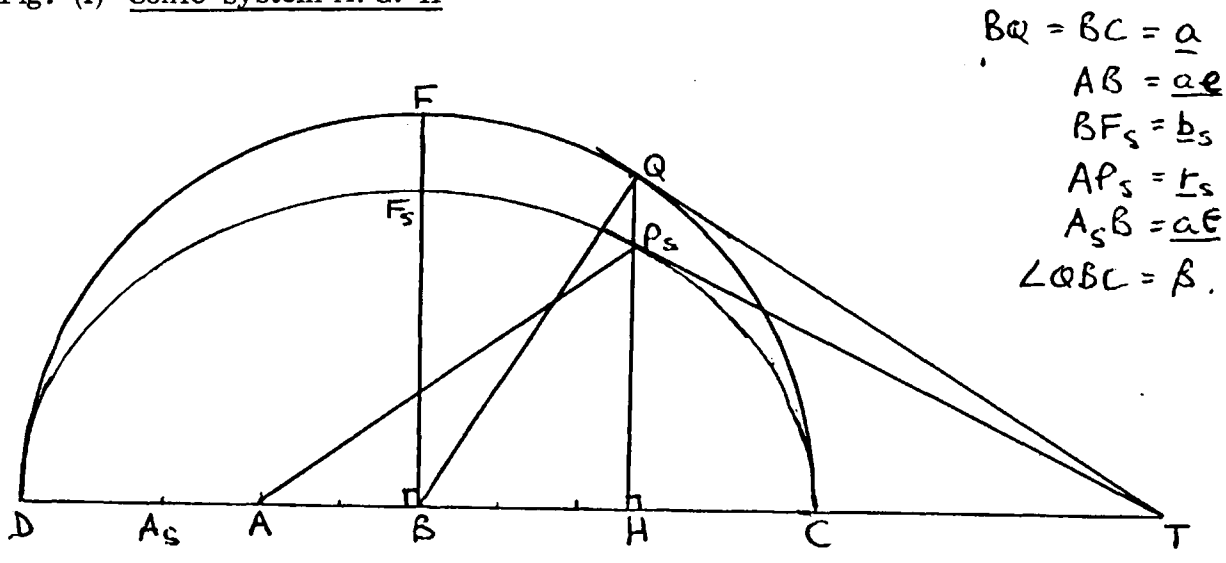
5. The barred notation—e.g. $\bar{\alpha}$—is used for an erroneous value when it is to be compared with a correct one—e.g. α.

6. A dash applied to a letter—e.g. Q'—is used to mark a point symmetrically positioned in relation to Q: that is, Q' will denote either the reflection of Q in the minor axis, or the point that lies diametrically opposite Q.

PROLEGOMENON I

CONIC-SYSTEM A.G. II

Fig. (i) Conic-system A.G. II



I refer to my earlier article: Systems of Conics in Kepler's Work[1]. where I defined an astronomical system as a collection of semiconfocal conics whose common focus was fixed at the Sun. In particular, members of the system labelled A.II possess a major semiaxis of common length a, and it is this one which will be applied to the analysis of Kepler's work. We first identify the eccentric circle of centre B and diameter CD=2a (the line of apsides), on which the Sun A occupies an eccentric position given by AB=ae, the eccentricity e being supposed known. Then we apply to system A.II what is nowadays known as a translation, drawing all the ellipses with common centre B, so that their major axes will be superimposed and coincident with CD. This new astronomical-geometrical arrangement is conic-system A.G. II: we shall be concerned here with its ellipse members only, and its outer bound, the eccentric circle: one of these will be the unique orbital ellipse whose focus falls at the Sun A, which functions as the pole for them all.

Thus when we consider any specific member of this system chosen at random we shall not expect its focus to be at A, but at A_s, as illustrated.

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1. Included as the General Appendix to this study.

We shall choose to examine typical points of this ellipse and the eccentric circle defined by the common ordinate QP_sH, supported by the well-known fact of conical geometry (which is not otherwise of relevance here) that the tangents at P_s and Q cut the line of apsides at a common point T. This will be known as the geometrical synthesizing-property of conic-system A.G. II.

We define any ellipse by the lineal ratio-property:[2] as a curve whose ordinates are in a constant ratio to those of the eccentric circle:

$$\frac{P_sH}{QH} = \frac{BF_s}{BF} = \frac{b_s}{a},$$

where b_s is the minor semiaxis of the ellipse, and may be regarded as its shape-parameter. Indeed, it is evident, from symmetry, that all properties of the ellipse referred to the axes (that is, nonfocal properties) will differ from those of the circle only in respect of this shape-parameter. Thus, the area-parameter of this ellipse is ab_s.

We simplify the kinematical analysis by representing time in radian measure α (the mean anomaly) as Kepler did, where, in modern terms:

$$\frac{d\alpha}{dt} = \frac{2\pi}{T}.$$

When the ellipse is described as an orbit, the essential condition (which will elsewhere be termed the area-hypothesis, Newtonian style) is:

$$\frac{d\alpha}{d\theta} = \frac{r_s^2}{ab_s}.$$

Now from the diagram,

$$\tan\theta_s = \frac{P_sH}{AH} = \frac{b_s \sin\beta}{a(\cos\beta+e)},$$

and $$\sec^2\theta_s = \frac{r_s^2}{a^2(\cos\beta+e)^2}$$

---

2. This is very easily deduced from the fundamental property of an ellipse as a section of a cone as proved by Apollonius: see Heath: Apollonius of Perga, Proposition 3, pp. 11-12.

Differentiating,

$$\frac{d\theta}{d\beta} = \frac{1}{\sec^2\theta_s} \cdot \frac{b_s}{a} \cdot \frac{\cos\beta(\cos\beta+e)+\sin^2\beta}{(\cos\beta+e)^2} = \frac{ab_s}{r_s^2}(1+e\cos\beta).$$

$$\therefore \frac{d\alpha}{d\beta} = \frac{d\alpha}{d\theta} \cdot \frac{d\theta}{d\beta} = 1+e\cos\beta.$$

$$\therefore \underline{\alpha = \beta + e\sin\beta.}$$

This is the kinematical synthesizing-property of conic-system A.G. II: it is especially to be emphasized that it depends only on the position of the centre of force or pole (the Sun at A, where AB=ae): it is independent of the individual member of the conic-system considered and therefore does not involve knowledge of the position of the focus.

Nevertheless it will be necessary to specify the position of the focus A_s in order to determine the equations of this ellipse: we follow Apollonius who defined it by the property:[3.]

$$BF_s^2 = CA_s \cdot A_sD$$

or in modern terms,

$$b_s^2 = a^2(1-\epsilon^2).$$

Now
$$r_s^2 = AH^2 + P_sH^2$$
$$= a^2(\cos\beta+e)^2 + b_s^2\sin^2\beta$$

$$\therefore \underline{r_s^2 = a^2\{1+2e\cos\beta+(e^2-\epsilon^2\sin^2\beta)\}, \text{ exactly.}}$$

$$\therefore \underline{r_s = a\{1+e\cos\beta-\tfrac{1}{2}(\epsilon^2-e^2)\sin^2\beta+\tfrac{1}{2}e(\epsilon^2-e^2)\sin^2\beta\cos\beta\}+O(e^4).}$$

Also, since $a\cos\beta = r_s\cos\theta_s - ae$ and $b_s\sin\beta = r_s\sin\theta_s$,

$$b_s^2(r_s\cos\theta_s - ae)^2 + a^2r_s^2\sin^2\theta_s = a^2b_s^2$$

$$\therefore \underline{r_s^2(1-\epsilon^2\cos^2\theta_s) - 2ae(1-\epsilon^2)r_s\cos\theta_s - a^2(1-e^2)(1-\epsilon^2) = 0, \text{ exactly.}}$$

$$\therefore \underline{r_s = a\{1+e\cos\theta_s - \tfrac{1}{2}(\epsilon^2+e^2)\sin^2\theta_s - \epsilon^2e\sin^2\theta_s\cos\theta_s\}+O(e^4).}$$

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3. Heath: Apollonius of Perga pp. 113-14.

This polar equation is to be identified with the encapsulating polar equation set out in Appendix I(1) which is supposed to include all the curves relevant to Kepler's work:[4.]

$$r = a(1+e\cos\theta - q_o e^2 \sin^2\theta - q_1 e^3 \sin^2\theta\cos\theta) + O(e^4).$$

Hence we obtain

$$\epsilon^2 = (2q_o - 1)e^2.$$

Hence, $r_s^2 = a^2\{1+2e\cos\beta + e^2 - (2q_o-1)e^2\sin^2\beta\}$, exactly.

$$\therefore\ r_s = a\{1+e\cos\beta - (q_o-1)e^2\sin^2\beta + (q_o-1)e^3\sin^2\beta\cos\beta\} + O(e^4).$$

Thus we find $q_1 = 2q_o - 1$:[5.] and indeed it is evident, from the fact that the exact quadratic contains q_o alone, that all successive coefficients are capable of being expressed in terms of q_o.

Again from the diagram,

$$\cos\theta_s = \frac{a(\cos\beta+e)}{r_s}$$

$$= (\cos\beta+e)\{1-e\cos\beta + (q_o-1)e^2\sin^2\beta + e^2\cos^2\beta\} + O(e^3)$$

$$= \cos\beta + e\sin^2\beta + (q_o-2)e^2\sin^2\beta\cos\beta + O(e^3).$$

This is satisfied by:

$$\theta_s = \beta - e\sin\beta + \tfrac{1}{2}\left(\frac{3}{2} - q_o\right)e^2\sin 2\beta + O(e^3).$$

Hence, $\beta = \theta_s + e\sin\theta_s + \frac{1}{2}(q_o - \frac{1}{2})e^2\sin 2\theta_s + O(e^3)$.

We combine this with the kinematical synthesizing-property to obtain (as in Appendix III(4), more generally):

$$\alpha = \theta_s + 2e\sin\theta_s + \tfrac{1}{2}(q_o + \tfrac{1}{2})e^2\sin 2\theta_s + O(e^3)$$

$$\therefore\ \theta_s = \alpha - 2e\sin\alpha + \tfrac{1}{2}\left(\frac{7}{2} - q_o\right)e^2\sin 2\alpha + O(e^3).$$

---

4. The possibility of a situation in which this is not the case is mentioned in Prolegomenon IV below.

5. As expected, since this is shown, in Appendix I(1), to be a (first) condition for a curve to possess a second axis of symmetry.

The three members of conic-system A.G. II which are significant in relation to Kepler's work will now be treated separately, in the following order:

The orbital ellipse: the mediate (third) prototype

The eccentric circle: the first prototype

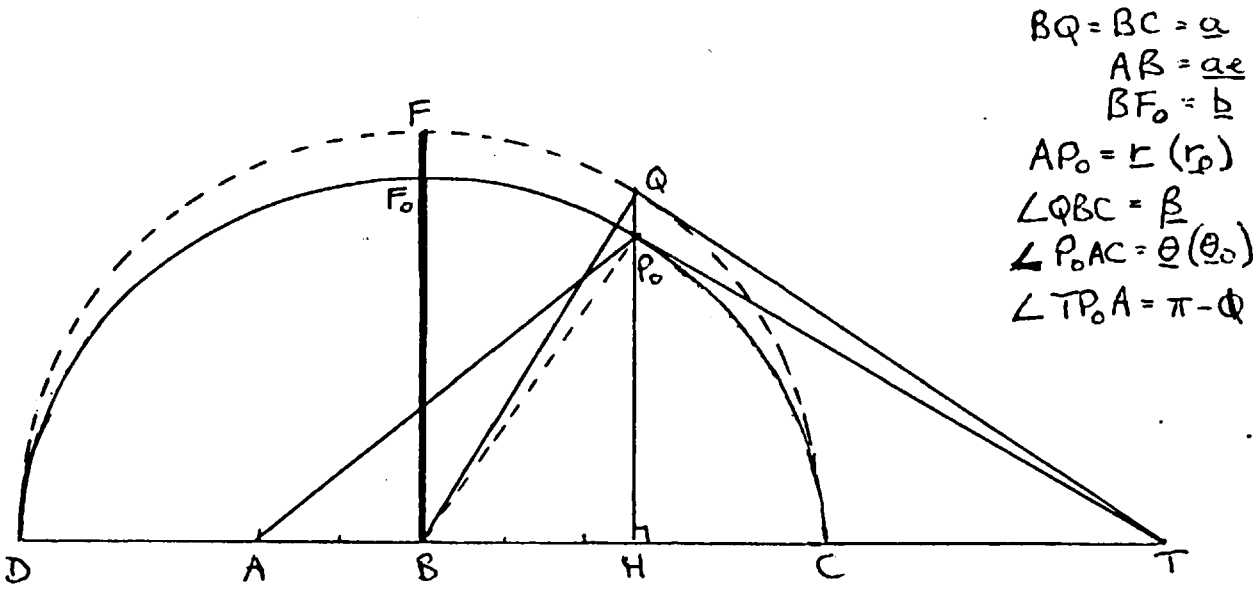
The (notional) Auxiliary Ellipse: the second prototype.

The term prototype simply denotes a perfect ellipse; the numbering refers to the three stages of Kepler's work in Astronomia: only the orbital ellipse is of relevance to Epitome and Harmonice.

PROLEGOMENON II

THE ELLIPSE WITH ONE FOCUS AT THE POLE

Fig. (i) The orbital ellipse.



This is the orbital ellipse for which we write s≡o. Then $b_o \equiv b$, $\epsilon \equiv e$, and hence $q_o = 1$, $q_1 = 1$, and $\underline{b^2 = a^2(1-e^2)}$.

A. Geometrical properties.

$r = a(1+e\cos\beta)$: the Secant Law

$$\therefore \frac{dr}{d\beta} = -ae\sin\beta.$$

$$r = \frac{b^2}{a(1-e\cos\theta)} = \frac{a(1-e^2)}{1-e\cos\theta}\ ; \qquad \text{the polar equation}^{1}.$$

$$\therefore \frac{dr}{d\theta} = -\frac{b^2 e\sin\theta}{a(1-e\cos\theta)^2} = -\frac{r^2}{b^2}\ .\ ae\sin\theta.$$

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1. Nowadays, this is generally expressed in terms of l, the semilatus rectum, wnere $l = a(1-e^2) = \frac{b^2}{a}$: but we do not do so here, because Kepler never quantified this constant.

By the lineal ratio-property,

$$\frac{r\sin\theta}{a\sin\beta}=\frac{b}{a}.$$

$$\therefore \sin\beta=\frac{r}{b}\sin\theta=\frac{b\sin\theta}{a(1-e\cos\theta)}$$

$$\sin\theta=\frac{b}{r}\sin\beta=\frac{b\sin\beta}{a(1+e\cos\beta)}$$

$$\cos\beta=\frac{\cos\theta-e}{1-e\cos\theta}$$

$$\cos\theta=\frac{\cos\beta+e}{1+e\cos\beta}$$

Hence we determine that:

$$\frac{d\theta}{d\beta}=\frac{b}{r}.$$

Now $\tan\phi=r\frac{d\theta}{dr}$ for any curve.[2.]

$$\therefore \tan\phi=\frac{b^2}{r.ae\sin\theta}=\frac{b}{ae\sin\beta}.$$

$$\therefore \sin\phi=\frac{b}{\sqrt{(b^2+a^2e^2\sin^2\beta)}}=\frac{b}{a\sqrt{(1-e^2\cos^2\beta)}}$$

Also $\frac{ds}{d\theta}=\frac{r}{\sin\phi}$ for any curve.

$$\therefore \frac{ds}{d\beta}=\frac{b}{r}\frac{ds}{d\theta}=\frac{b}{\sin\phi}=a\sqrt{(1-e^2\cos^2\beta)}.$$[3.]

Additionally, when $2P_oB$ is a diameter of this ellipse, we have:

$$P_oB^2=BH^2+P_oH^2$$
$$=a^2\sin^2\beta+b^2\cos^2\beta$$

$$\therefore P_oB=a\sqrt{(1-e^2\sin^2\beta)}.$$

---

2. The formulae valid 'for any curve' are proved in textbooks of calculus.

3. It will thus be clear why Kepler was never able to quantify the arc length $\underline{\delta s}$ of this ellipse.

B. Kinematical properties.

$$\frac{dr}{d\alpha} = \frac{dr}{d\theta}\cdot\frac{d\theta}{d\alpha} = \frac{a^2}{b}\cdot e\sin\theta$$ from Prolegomenon I

or, $\frac{dr}{d\alpha} = \frac{dr}{d\beta}\cdot\frac{d\beta}{d\alpha} = \frac{a^2 e\sin\beta}{r}$: the radial motion

$\therefore$ $\frac{dr}{dt} = \frac{2\pi}{T}\cdot\frac{a^2 e\sin\beta}{r}$: the radial velocity.

$r\frac{d\theta}{d\alpha} = \frac{ab}{r}$: the transverse motion

$\therefore$ $r\frac{d\theta}{dt} = \frac{2\pi}{T}\cdot\frac{ab}{r}$: the transverse velocity.

For completeness we find the resultant velocity:

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2$$

$$= \left(\frac{2\pi}{T}\right)^2\left(\frac{a^4e^2\sin^2\beta}{r^2} + \frac{a^2b^2}{r^2}\right)$$

$$= \left(\frac{2\pi}{T}\right)^2\cdot\frac{a^4}{r^2}(1-e^2\cos^2\beta)$$

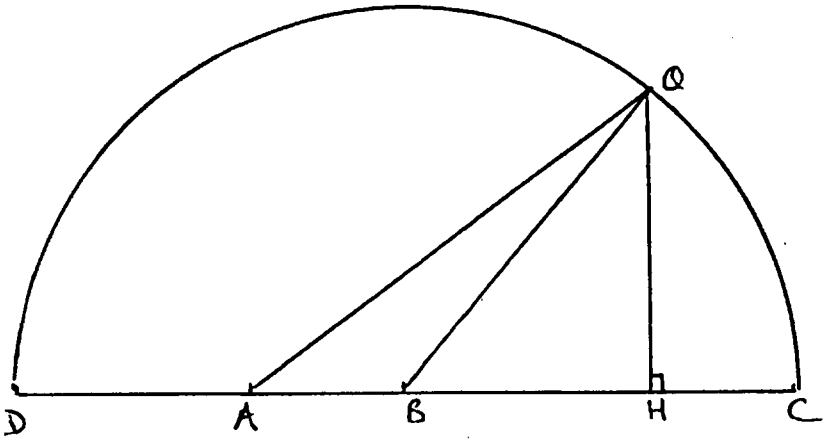
$$= \left(\frac{2\pi}{T}\right)^2\cdot\frac{a^3}{r}(1-e\cos\beta)$$

$\therefore$ $v = \frac{2\pi}{T}\cdot a\sqrt{\left(\frac{2a}{r} - 1\right)}$.

PROLEGOMENON III

THE ECCENTRIC CIRCLE

Fig. (i) The eccentric circle.



$BQ = BC = a$
$AB = ae$
$AQ = r\ (r_c)$
$\angle QBC = \beta$
$\angle QAC = \theta\ (\theta_c)$

For this circle—the outer bound of conic-system A.G.II—we write $s \equiv c$.

Then $b_c \equiv a$, $\epsilon \equiv o$, and hence $q_o = \frac{1}{2}$, $q_1 = O$.

A. Geometrical properties.

From ΔAQB,

$$r^2 = a^2(1+e^2) + 2a^2 e\cos\beta$$

$$\therefore\ r = a(1+e\cos\beta + \tfrac{1}{2}e^2\sin^2\beta - \tfrac{1}{2}e^3\sin^2\beta\cos\beta) + O(e^4).$$

Also, $r^2 - 2aer\cos\theta - a^2(1-e^2) = O$

$$\therefore\ r = a\{e\cos\theta + \sqrt{(1-e^2\sin^2\theta)}\}$$ in the first quadrant

$$\therefore\ r = a(1+e\cos\theta - \tfrac{1}{2}e^2\sin^2\theta) + O(e^4).$$

Again from ΔAQB,

$$\frac{\sin\beta}{r} = \frac{\sin\theta}{a} = \frac{\sin(\beta-\theta)}{ae}$$

$$\therefore\ \angle AQB = \beta - \theta = \sin^{-1}\left(\frac{e\sin\beta}{r}\right)$$

$$= e\sin\beta(1+e\cos\beta)^{-1} + O(e^3)$$

$$= e\sin\beta - \tfrac{1}{2}e^2\sin 2\beta + O(e^3).$$

B. Kinematical properties.

When this circle is described as an orbit, it is known as the Contemporaneous Circle since, by the geometrical synthesizing-property of the system, its motion parallel to theline of apsides keeps pace with the motion of the orbital ellipse.[1.] Moreover, because of the symmetry remarked above, the difference in relative motion perpendicular to the line of apsides will depend only on the difference in their shape-parameters (that is, it will be proportional to QP_o). Hence, as illustrated in Prolegomenon II, the resultant velocity in the orbital ellipse may be derived from the resultant velocity in the Contemporaneous Circle as represented by ΔQTP_o acting as the triangle of velocities. However, this is only of interest as a modern comparison.

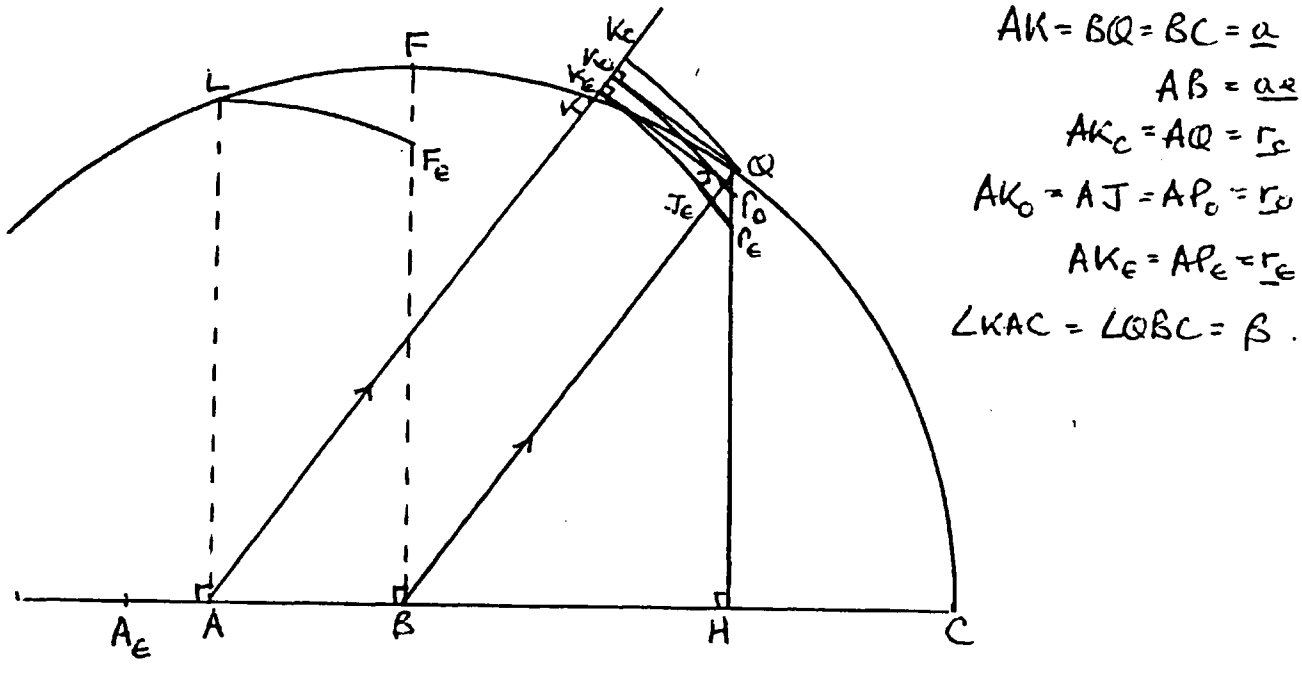
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1. True also, in fact, for all ellipses of the system.

PROLEGOMENON IV

THE ELLIPSE WITH ITS FOCUS NOT AT THE POLE

Fig. (i) The regular second prototype.



I confirm that there is a Euclidean construction by which one can construct an ellipse of the dimensions needed as the inner bound of the range considered by Kepler: this construction was suggested to me by his own characteristic method, as we shall see. The instructions for performing it are as follows:

Take a point Q on the eccentric, draw AKK_c parallel to BQ, and drop a perpendicular from Q onto AK_c to meet it at K_o. With radius AK_o, centre A, draw an arc to cut BQ at J, QH at P_o. Next drop a perpendicular from J onto AK_c to meet it at K_ϵ and finally with radius AK_ϵ, centre A, draw an arc to meet QH at P_ϵ.[1] Then $AP_\epsilon = r_\epsilon$ is the radius vector of

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1. The arc K_cQ has been included for comparison.

the required ellipse, as we shall now show:

$$r_\epsilon^2 = AK_\epsilon^2 = AJ^2 - JK_\epsilon^2$$

$$= AK_o^2 - JK_\epsilon^2$$

$$= AQ^2 - QK_o^2 - JK_\epsilon^2$$

$$= r_c^2 - 2a^2e^2\sin^2\beta \quad \text{since } QK_o = JK_\epsilon = ae\sin\beta$$

$$= a^2\{1+2e\cos\beta+(1-2\sin^2\beta)e^2\}, \quad \text{exactly (from Prolegomenon II).}$$

This is in agreement with the general formula stated in Prolegomenon I when we write

$$\epsilon^2 \equiv 2e^2,$$

so that $\underline{b_\epsilon^2 = a^2(1-2e^2)}$.

Hence this ellipse is defined by $q_o = \frac{3}{2}$, $q_1 = 2$. This will be termed the regular second prototype.

We now digress to mention the existence of an irregular second prototype, produced by setting:

$$\bar{b}_\epsilon = a(1-e^2)$$

$$\therefore \bar{b}_\epsilon^2 = a^2(1-2e^2+e^4)$$

$$\therefore \bar{\epsilon}^2 = 2e^2 - e^4.$$

This ellipse undoubtedly possesses the geometrical and kinematical synthesizing-properties common to all members of conic-system A.G. II (as set out in Prolegomenon I) because these are indifferent to the position of the focus; yet, we cannot determine g_o—alternatively, an attempt to work out g_o by finding the equation of $\bar{r}_\epsilon$ by the method of Prolegomenon I apparently produces $q_o = \frac{3}{2}$ as above, which is of course impossible, since the two versions of the second prototype are evidently distinct; and thus we must conclude that this ellipse cannot be represented by the encapsulating polar equation of Appendix I(1).

This complication has arisen because Kepler omitted to draw the second prototype (which is consequently described as the notional Auxiliary Ellipse):[2] and I have shown that, had he attempted to do so, the ways available to him might have produced either of these two versions. Since, however, they differ so minimally, we shall choose to represent the notional Auxiliary Ellipse as the regular second prototype in due course, pointing out that it is a member of a subset of conic-system A.G. II, whose focal distances are derived from that of the unique ellipse whose focus lies at the centre of force, by the relationship:

$$\epsilon^2 = (2q_o - 1)e^2, \quad \text{where } q_o \text{ is rational.}$$

2. I shall suggest that his failure to do so was due to the fact that he knew of no Euclidean method of constructing its radius vector (as described above).

PART A

THE DISCOVERY OF LAW I AND LAW II

CHAPTER I

THE COMMON FOUNDATIONS OF THE ANALYSIS OF ASTRONOMIA

§ 1 The two rôles of the Sun and the consequent duality of treatment

§ 2 Definitions of terminology and the contrast between the Keplerian and Newtonian usages

§ 3 Stipulation of the observation-controlled level of accuracy

§ 4 The prototypes and the three stages of Kepler's progress: consequences of the stipulated level of accuracy

§ 5 The geometrical frame of reference and Kepler's characteristic construction, founded on the invariant distantiary angle

Appendix I(1) Encapsulating formula: polar equation of an oviform $r(\theta)$

Appendix I(2) Formulae for any oviform involving the eccentric anomaly

Appendix I(3) Formulae for any oviform involving the central anomaly

§ 1 The two rôles of the Sun and the consequent duality of treatment.

The purpose of Part A is to analyze how Kepler discovered the laws governing the orbit of a single planet: they are stated here in the imprecise form which represented the state of Kepler's knowledge when he had completed Astronomia:

Law I : the orbit is a particular ellipse

Law II : the time is measured by a certain area in that ellipse.

I reject the approach by which the two laws have formerly been treated simultaneously, and shall split up the enquiry in a way which has up to now only been implied:

Orbit = curve + time-hypothesis.

Then the determination of the correct curve will be the subject of Chapter II, and the question of time-measurement will be discussed exclusively in Chapter III. These investigations will each be carried out in three (common) stages, to be classified in § 4 below.

The fundamental premise underlying all Kepler's work was that the Sun was fixed in position at the centre of his universe;[1.] and quite early in his work he proved the pair of propositions :[2.] that each planetary motion takes place in a distinct plane, and that these planes all pass through the Sun.[3.] The first of these enabled him to concentrate on the representation of the orbit by means of a plane curve: however, one cannot embark on any curve-finding without an origin, or pole of coordinates; and the second of these propositions placed this pole firmly at the Sun A, as Kepler explained in the Introduction to Astronomia (G.W. III p. 20 lines 23-6):

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1. The importance of the Sun's rotation on its axis is discussed in Chapter IV.
2. In Astronomia C.13-14: confirmed in C.52.
3. I am in agreement with those who believe that the importance of this combined result is such that it ought to have been given the status of one of 'Kepler's Laws', in addition to the other three: see Small: The Astronomical Discoveries of Kepler, pp. 168-9 and Dreyer: Planetary Systems, p. 410.

> 'For the purpose of investigating ... the motions, the first step is in fact to show that the conjunction of [the planes of] the eccentric circles does not occur at some point near the Sun, but at the centre of the body of the Sun itself, contrary to what Copernicus and Brahe believed.'

I take this opportunity to confirm that the curve was also assumed to be closed (that is, repeating): this was achieved by making corrections to the observations where necessary to compensate for the small secular changes. Accordingly, in geometrical terms, Kepler envisaged the planetary orbit as a continuous, planar, closed curve, everywhere concave to the origin (the Sun), having its line of apsides as an axis of symmetry.[4.] Such a curve is one of an infinity of what is to be termed, generically, an oviform,[5.] defined mathematically in Appendix I(1) (to include every curve proposed by Kepler, as Whiteside has pointed out).[6.]

As well as acting geometrically as the origin of coordinates, the second rôle of the Sun was to function as the centre of force and source of motion, in conformity with Kepler's fundamental physical principle, stated as the title to Astronomia C.33: 'The force that moves the planets is located in the body of the Sun'. Indeed, a precisely quantified version of this principle had already been stated as the title to C.32:[7.] 'The force that moves the planet round decreases with distance from the source': thus, denoting this force by F,[8.] we have Kepler's force-hypothesis:

$$F \propto \frac{1}{r}.$$

---

4. Kepler checked this in Astronomia C.51, but otherwise assumed it; and so, in Part A, we shall concern ourselves only with the first half of the orbit—the first and second quadrants.

5. The description 'oval' will be avoided in this study because it is sometimes taken by those not mathematically trained to mean an ellipse.

6. Keplerian Planetary Eggs, p. 15.

7. Prior to this, the point A had generally been described (in relation to every planet except the Earth) as locus oculi.

8. virtus in both titles: the question of terminology becomes of great importance in Part B, as explained in Chapter V, § 1.

Now the interpretation of force current in Kepler's day had remained unchanged since the time of the Greeks, summarized in what I shall hereafter refer to as the Aristotelian Equivalence:

Force $\propto$ velocity.

Hence Kepler deduced:

Velocity $\propto \frac{1}{r}$.

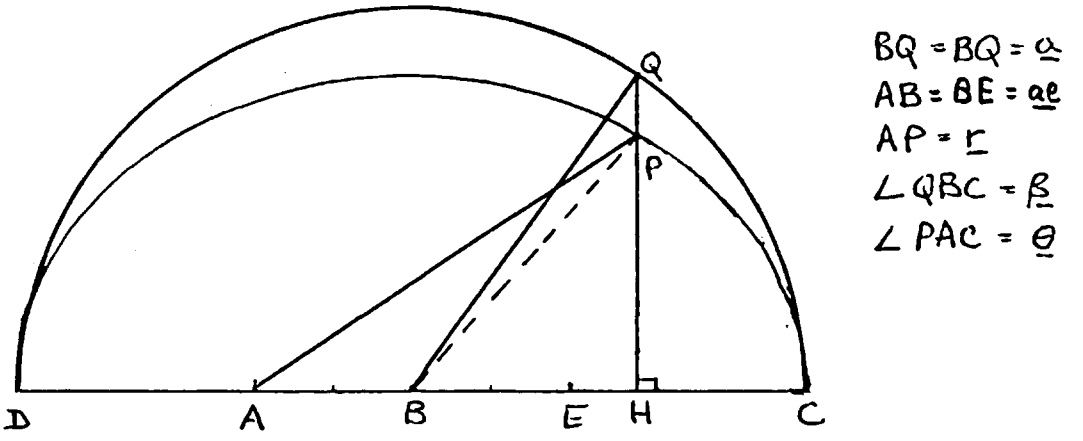
Since he never questioned the Aristotelian Equivalence, he believed that force and velocity were dimensionally synonymous, and thus his analysis of physical quantities was bound to be unsound with respect to the dimensions of time. However, this will not concern us here[9.] because the deduced relationship above is purely kinematical and will be the one from which we shall start, in Chapter III, to derive new types of time-hypotheses never before applied in astronomy. Indeed, no further mention of force will be necessary until Chapter IV, § 2—except to remark that, as it turns out, one consequence of Kepler's defective understanding of the concept was altogether fortunate, in that it involved no second derivatives (differential coefficients)—which he would undoubtedly have been unable to cope with.

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9. It is the subject of Chapter VIII, § 2.

§ 2 Definitions of terminology and the contrast between the Keplerian and Newtonian usages.

Fig. (i) Any oviform.



Though three of the four anomalies to be defined here were employed in ancient astronomy, we have naturally illustrated them after the transposition to heliocentricity: this phrase has been chosen to emphasize the well-known fact that such a transposition from geocentricity is a purely technical matter which can be carried out by parallels so that the sizes of the angles with which we are now concerned remain unchanged. Nevertheless, we shall have to update these definitions where necessary so that they can be applied to the noncircular curves which Kepler himself was responsible for introducing into astronomy.

The true anomaly $\angle PAC = \theta$.

This term describes the angle at the pole (which is to be taken always as the Sun A, as stated in §1) between the line of apsides CD and the radius vector to the actual position P of the planet (whether the curve is circular or noncircular) measured from aphelion. This angle is denoted in general by θ but I have used θ_s to refer to the true anomaly of a specific member of conic-system A.G. II (defined in Prolegomenon I) for the geometrical comparisons which are necessary from time to time throughout Chapters I, II and III.[1].

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1. At the first, second, and third stages, $s \equiv c, \epsilon, o$ respectively as set out in § 4 below.

The eccentric anomaly $\angle$ QBC=β.

The original definition.

In ancient astronomy, this term referred to the angle at the centre B of the eccentric circle between the line of apsides and the radius BQ to the position Q of the planet on the eccentric, measured from aphelion: it is denoted by β. We take the opportunity to define the precise values of particular significance in the subsequent discussions (recalling that in Part A we shall be dealing only with the first half of the orbit):

At the apsides, $\beta=0$
At the quadrant, $\beta=90^{o}$ (in Chapter II)
At the octants, $\beta=45^{o}$, 135^{o} (in Chapter III).

This interpretation is straightforward whenever the orbit is taken to be circular, as it was at the first stage of Kepler's work.

The updated definition.

However, when the proposed position P of the planet lies on a curve within the circle, as it did at the second and third stages, we have to update this definition for a noncircular curve, and we naturally choose to follow the Keplerian instructions given in Astronomia C.47 (G.W. III p. 300 lines 39–41) by erecting a perpendicular HP to the line of apsides through P to cut the eccentric circle at Q. Then the eccentric anomaly of P is again given by $\angle$ QBC=β. The corollary—which Kepler did not state, but certainly appreciated, as we shall see—is that if, from a given point Q on the eccentric circle, one drops a perpendicular QH to the line of apsides, then every point on QH possesses common eccentric anomaly given by $\angle$ QBC=β.

The Keplerian procedure provides an entirely satisfactory definition of eccentric anomaly which will be adopted in this study for the purpose of any modern analysis which is carried out for comparison: in such cases the suffix employed to denote theparticular eccentric anomaly will always correspond to the point which gives rise to it; while the zero suffix will be reserved to denote an angle β_0 of considerable importance in Kepler's own approach, to be introduced in § 5.

The central anomaly $\angle PBC=\gamma$.

This term describes the angle at the centre B between the line of apsides and the line BP drawn to the actual position P of the planet. This angle did not appear in ancient astronomy (because it coincides with the eccentric anomaly when the curve is a circle), and has been formally distinguished from the eccentric anomaly—by being named—only quite recently, by Whiteside.[2] I point out that this is the sole occasion on which I have found it desirable to use different notation for two quantities which differ (as I shall prove) merely at the level $O(e^2)$.[3] The problem that occurs at the third stage certainly makes it essential for us to distinguish these two anomalies in order to give a satisfactory analysis of Kepler's reasoning in that situation—but in modern terms alone: I shall account for it otherwise in Kepler's terms, in due course. Indeed, a consequence of this study has been to demonstrate Kepler's dependence upon the concept of eccentric anomaly as the basis of his progress and therefore I most emphatically do not accept that he formulated the concept of central anomaly by intent: that is, I cannot believe, as does Whiteside, that it would ever have been 'natural' for him to have introduced this concept. (Nevertheless, he did in fact make an estimate of the size of the central anomaly for the orbital ellipse: this formula is listed among others in Appendix IV(1), but it is clear that in that context it made no contribution to an increase in his understanding.)

The modern series-expansions, expressed in ascending powers of the eccentricity e, giving the radius vector in terms of these three anomalies, as well as the relationships between them, are set out for reference in Appendices I(1), I(2), I(3).

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2. Keplerian Planetary Eggs, p. 12, illustrated on p. 13.

3. In all other circumstances I use suffixes to distinguish quantities which differ at this level, as explained in the Notes to the Index of Standard Terminology.

The mean anomaly.

This quantity was described by Kepler in Astronomia as the measurement used by astronomers to designate time (C. 40: G.W. III p. 263 lines 32-3; C.49: p. 312 lines 34-5). In ancient astronomy it was represented by $\angle$QEC (this was the Ptolemaic equant hypothesis, analyzed in modern terms in Appendix III(1))—while its reformulation by Kepler, to produce a representation of mean anomaly α which is correct for a planetary orbit by present-day standards, is the theme of Chapter III.

Meanwhile we define the diurnal mean angular motion, whose value is evidently constant for each individual planet:

$\delta\alpha_m = \frac{2\pi}{T}$, where T is the periodic time.

In Part A, this is regarded as an angle of standard size, its diurnality merely implicit (and in numerical applications, radians are replaced by sexagesimal measure).

The Keplerian usage versus the Newtonian usage.

We make this contrast between Kepler's original methods in Astronomia[4]. and those of Newton (which are equivalent, of course, to the approach employed nowadays), in order to pinpoint the extent to which, and where, in his later work (in relation to the orbital ellipse) Kepler began to adopt the modern usages. We shall catalogue three aspects of this contrast.

The form of the hypothesis (kinematical only).

Keplerian usage: time is the dependent variable (direct form)

Newtonian usage: time is the independent variable (inverse form).

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4. At the first stage, and to which he returned at the third: various attempts to adopt Newtonian usages at the second stage are recorded in Chapter III, §5 .

The designation of time.

Keplerian usage: mean anomaly α

Newtonian usage: time t.

Because Kepler's reformulation depended on the fact that time possessed direct geometrical representation (as we shall see), we follow him in retaining its designation as mean anomaly throughout the analysis of Astronomia and, by a prochronism, will simplify the results achieved by Newton—the modern formulae we require for comparison—by expressing them in terms of mean anomaly, on the grounds that it is mathematically, undoubtedly—and must be historically—justifiable, to assume less sophisticated versions of established results.

The style of the hypothesis.

Keplerian usage: the variable involved is the eccentric anomaly

Newtonian usage: the variable involved is the true anomaly.

Combining all these considerations, when the dependent variable is geometrical, we define:

Velocity when the independent variable is t.[5]

Motion when the independent variable is α

Variation when the independent variable is θ or β.

The observational uncertainties versus the theoretical defects.

The symbol δ is commonly used to indicate an increment—in particular in the definitions of the time-measures introduced in Chapter III—which is capable of becoming infinitesimal,[6] and this symbol will also denote an

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5. Not to be mentioned again in connection with Astronomia—that is, not in Part A—as we have explained above.

6. In Chapter V, §3, it will become necessary to define this term precisely, in order to interpret what I believe to be Kepler's intentions in Epitome.

uncertainty which can arise from the application of an observational method because this too can become negligible under favourable conditions (the subject of investigation in § 3(c) below). By contrast, the symbol Δ will denote a defect which is theoretically determinate and irreducible, because it is the consequence of a wrong choice of curve, or an erroneous mean anomaly hypothesis (or both): such defects are to be calculated subsequently, as appropriate. A further important distinction is the fact that the theoretical defects are of fixed sign, while the increments or uncertainties may occur in either sense.

It can easily be checked, from the various Appendices, that all the geometrical relationships, and all the mean anomaly hypotheses, possess similar structures, and so, to a first approximation, we find:

$$\delta\theta = \delta\beta = \delta\alpha$$

$$\delta r = (-)\, ae \sin\beta . \delta\beta .$$

These results establish that, along a given curve, a change $O(e^n)$ in the size of any particular anomaly will produce a change at the same level in any other anomaly, but the value of $\underline{r}$ will be affected at the level $O(e^{n+1})$ only. This is of great importance in what follows, in enabling us to confirm that, in particular, even when a proposed orbit is erroneous in anomaly at the level $O(e^2)$, the error in its radius vector is merely of value $O(e^3)$, which thus cannot distort our account of Kepler's progress from stage to stage.[7.]

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7. Because these errors do not therefore alter the group to which the erroneous orbit belongs—as defined in § 4 below, where we show that each group corresponds to a particular stage of Kepler's work.

§ 3 Stipulation of the observation-controlled level of accuracy

Astronomia C.51–4.

(a) The accuracy of two fundamental elements of the orbit.

Whether or not the problem had been assigned to him by Tycho, Kepler must sooner or later have decided for himself that his best chance of success—in his overall aim of determining the planetary orbits—would lie in tackling the motion of the planet with the highest eccentricity which was observable at all points of its orbit. It so happened that the nearest outer planet fulfilled both these criteria and thus began Kepler's struggle with Mars.

Kepler invariable used the value 100000 as his standard of comparison throughout Astronomia: in our terms, he worked correct to five decimal places (this fact will be of vital importance in enabling us to stipulate the level of analytical exactitude required for Part B): this figure originally represented the radius $\underline{a}_t$ of the eccentric circle of the Earth[1.] (whose actual value is irrelevant); while the eccentricity $\underline{e}_t$ of the Earth was investigated in Astronomia C.22–8, and was eventually taken to be 1800. Then the Mars-distances were in the first instance expressed in terms of $a_t = 100000$. We list the two main alternative sets of values of the elements $\underline{a}$ and $\underline{e}$ employed in Astronomia Part IV: after C.55 Kepler abandoned the first, and the second set was adopted thereafter, to appear finally in Tabulae Rudolphinae.[2.]

It is with this justification that the sufficient accuracy of the fundamental constants $\underline{a}$ and $\underline{e}$ (the Keplerian values of the second column) will be taken as established for the purpose of this study.

Nevertheless, when considering the orbit of Mars in isolation, Kepler worked in terms of the radius of its own eccentric circle as standard, writing $a = 100000$: it will obviously be necessary to distinguish these two sets of figures as carefully as Kepler himself did. Meanwhile, it will be

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1. The suffix '$\underline{t}$' is invariably used for all quantities relating to the Earth.
2. G.W. X Part II, Tables 62–4.

helpful also to list the various values of e used, for reference in Chapters II and III.

Tabulation I The constants of the orbit of Mars

	Keplerian values		Modern value †	Percentage level of of inaccuracy
	C.54 p. 343	C.54 p. 344		
Eccentric radius a	1.52342	1.52350	1.52369	0.01%
Eccentricity e	0.09300	0.09265 ††	0.09337	0.8%
Aphelion distance $a(1+e)$	1.66510	1.66465	1.66596	0.08%
Perihelion distance $a(1-e)$	1.38173	1.38234	1.38142	0.07%

† From Smart: Spherical Astronomy, Appendix C.

†† Sometimes given as 0.09264: this variant will be ignored throughout.

Tabulation II Values of the eccentricity of Mars used in Astronomia

e	$\frac{1}{2}e^2$	Principal appearances
0.09265	0.00429	C.42, C.43; C.47, C.50; C.52, C.54, C.56 and thereafter
0.09300	0.00432	C.43; C.54, C.55
0.09230	0.00426	Temporarily at the second stage (C.47 onwards:
0.09165	0.00420	rejected in C.50)
0.09337	0.00436	Modern

Now when the radius of either eccentric circle is taken as 100000, we have:

1 radian = 100000 Keplerian units.

Then as a consequence of the modern mathematical treatment to be undertaken here, for which we employ series-expansions in ascending powers of e, it is desirable to set out the relative sizes of the approximations we shall require, expressed as powers of e (and e_t, when appropriate).

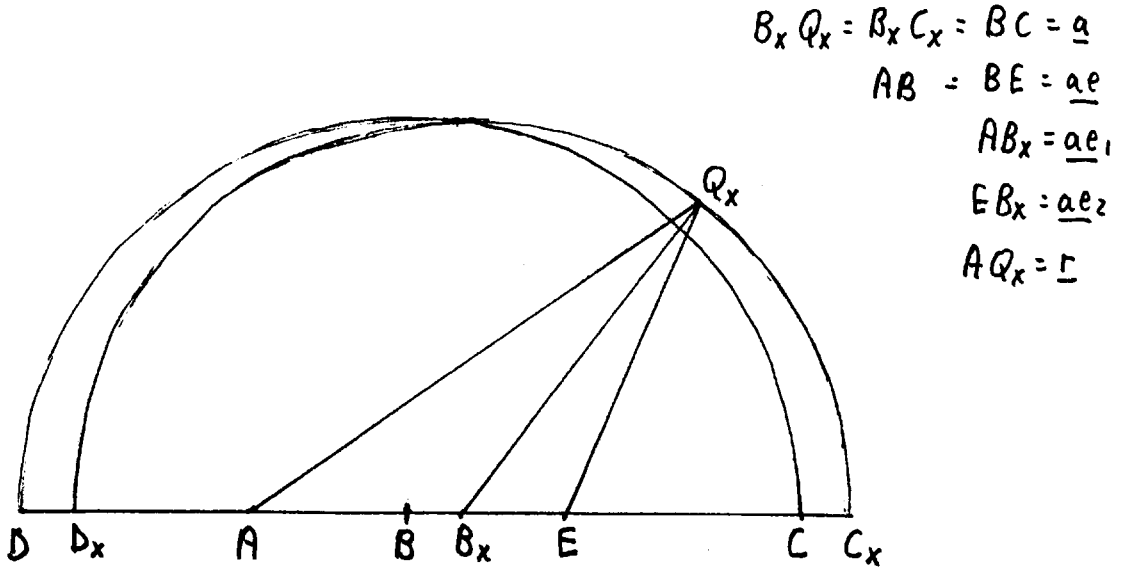
Tabulation III Conversion-table of equivalences.

Mars (a=100000)	Earth (a_t=100000)	Keplerian units (0.00001^c)	Sexagesimal measure
e		9265	$5^\circ 18\frac{1}{2}'$
	e_t	1800†	$1^\circ 2'$
		1745	1°
e^2		858	$\simeq 30'$
$\frac{1}{2}e^2$		429	$\simeq 15'$
$\frac{1}{4}e^2$		215	$7\frac{1}{2}'$-$8'$
$\frac{1}{6}e^2$		150	$6'$
$\frac{1}{8}e^2$		100	$3\frac{1}{2}'$-$4'$
e^3		80	$\simeq 3'$
$\frac{1}{3}e^3$	e_t^2	29	$1'$
$2e^4$	$\frac{1}{2}e_t^2$	15	$\frac{1}{2}'$
e^4	$\frac{1}{4}e_t^2$	8	$\frac{1}{4}'$
$O(e^5)$		$\frac{1}{2}$	$1''$

† The modern value is $e_t = 0.01673$.

(b) The Vicarious Hypothesis and other mean anomaly hypotheses involved in Kepler's early work.

Fig. (ii) The Vicarious Hypothesis.



In ancient astronomy, as we have said, time was designated by mean anomaly and was geometrically represented by the angle at the equant E (nowadays identified as the empty focus of the correct elliptic orbit). Kepler demonstrated, in Astronomia C.21, that the correct positioning of the equant—AE = 2 ae—is of vital importance to the accuracy of the theory, but that the centre of the particular circle in which the planet is actually supposed to move may be adjusted so that the inaccuracy becomes negligible (as in (3) below). This has been clearly explained—and proved otherwise—by Curtis Wilson,[3.] to whose work I am indebted for inducing me to take a fresh look at the question of observational accuracy—though unfortunately as a consequence I have discovered a serious mathematical error in his subsequent analysis (to be corrected in subsection (c) below).

Then because Kepler took mean anomaly hypotheses of the equant type (transposed to heliocentricity) as his starting-points, we shall give a brief background survey of the variations employed over the centuries, according to the position of the centre B_x of the orbital circle assumed. Thus we shall follow Caspar[4.] in setting $AB_x = ae_1$, $EB_x = ae_2$ for a general position B_x, while the fixed length AE=2ae will be described, for clarity, as 'the eccentricity of the equant', or, with Keplerian authority (G.W. III p. 37 line 23) as 'the double eccentricity'.

Since all such hypotheses are to be used here in connection with an observational method, the time (mean anomaly) α of the observation is known, and we shall therefore adopt the Newtonian usage as to both style and form, so that the value of the true anomaly produced by an erroneous hypothesis will itself be erroneous, denoted by $\bar{\theta}$. Then we follow Whiteside[5.] to obtain, from ΔAQ_xB_x and ΔB_xQ_xE:

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3. Kepler's Derivation of the Elliptical Path Part 1, pp. 8-9.
4. Neue Astronomie pp. 61*-62* (2).
5. Keplerian Planetary Eggs, pp. 7-8.

$$\frac{\sin\bar{\theta}}{a} = \frac{\sin(\angle B_x - \bar{\theta})}{ae_1} \quad \text{and} \quad \frac{\sin\alpha}{a} = \frac{\sin(\alpha - \angle B_x)}{ae_2},$$

and hence $\bar{\theta} = \alpha - 2e\sin\alpha + e_1 e \sin 2\alpha + O(e^3)$.

Now from Prolegomenon I ,

$$\theta = \alpha - 2e\sin\alpha + \frac{5}{4}e^2 \sin 2\alpha + O(e^3).$$

Thus we evaluate the theoretical angular defect:

$$\Delta\theta = (e_1 - \frac{5}{4}e)e \sin 2\alpha + O(e^3).$$

This defect is to be assessed for various sets of values of e_1 and e_2, using the Table of Equivalences given in (a) above.

(1) The doubly-eccentric equant hypothesis.

We set $e_1 = 2e$, $e_2 = 0$, so that $B_x \equiv E$.

Then

$$\Delta\theta = \frac{3}{4}e^2 \sin 2\alpha + O(e^3).$$

Since this rises to a maximum of about 22' in the case of Mars, it was inevitable that, in due course, this theory was superseded for the planets whose eccentricity is comparatively high. However, in the case of the Earth, the defect has an entirely undetectable maximum value of $\frac{3}{4}'$, and so this simple hypothesis was retained to represent the Earth-Sun motion, and constituted Tycho's 'Solar theory'.

(2) The Ptolemaic equant hypothesis.

We set $e_1 = e_2 = e$, so that $B_x \equiv B$.

Then $\Delta\theta = -\frac{1}{4}e^2 \sin 2\alpha + O(e^3)$.

This was the major improvement introduced by Ptolemy for the representation of all the planetary orbits (excluding the Earth-Sun motion), which entailed the bisection of the [double] eccentricity of the equant, so that the

apsidal distances AC = a(1+e) and AD = a(1-e) were more correctly related to each other (Kepler checked this in Astronomia C.42, and again in C.54): we shall ignore all refinements to this theory instigated by Ptolemy, as Kepler did: its bare bones are shown in his diagram of Astronomia C.4 (G.W. III p. 73).

Thus the above defect has a maximum of approximately $7\frac{1}{2}'$ for Mars, and it was this that Kepler set out to cure—as described in Chapter III; the Ptolemaic equant hypothesis itself is applied, in Appendix III(1), to the noncircular curves he subsequently proposed. Meanwhile we point out that it was also Kepler who was responsible for applying this improved hypothesis to the orbit of the Earth[6.]—so that he was the first to treat the Earth in practice, as well as in theory, as just another planet—this is an instance of his 'universalizing' approach, to which we return in Chapter III, § 3. Under this hypothesis, the angular defect in relation to the orbit of the Earth is:

$$\Delta\theta_t = -\frac{1}{4}e_t^2 \sin 2\alpha + O(e_t^3): \text{ maximum} \simeq \frac{1}{4}'.$$

Furthermore, using Prolegomena III and IIA (since both $\Delta\theta_t = 0$ and $r_t = a_t$ to a first approximation), we evaluate the proportional defect in distance:

$$\frac{\Delta r_t}{r_t} = \tfrac{1}{2}e_t^2 \sin^2\theta_t + O(e_t^3): \text{ maximum} \simeq \tfrac{1}{2}'.$$

Both these in turn will be disregarded in their contribution to the observational uncertainty to be discussed in subsection (c) below.

(3) The Vicarious Hypothesis Astronomia Part II C.16.

As we have explained above, Kepler was aware that there was an optimum position for the centre of the orbital circle of the planet, at which the angular defect would be minimized. From the general formula for $\Delta\theta$ above, it is clear that the defect will reduce to $O(e^3)$ when $e_1 = \frac{5}{4}e$: and indeed Kepler

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6. He tested it in Astronomia C.22-28.

achieved a near approximation to this by setting $e_1 \simeq 1.22e$ (G.W. III p. 168 lines 1-3). With either of these values the maximum defect—which occurs at the quadrant—is less than $\underline{e}^3$, and the Keplerian value gives a maximum of about 2' in the positive sense.[7.] Kepler recognized the deficiencies of this Vicarious Hypothesis—the apsidal distances, in particular, are wrong by about 2%—whose advantage was that it gave values of the true anomaly $\underline{\theta}$, for known times $\underline{\alpha}$, which were correct to $O(e^2)$, and could therefore be treated as 'equivalent to observations' at this level. Again, at the second stage, Kepler made two attempts to apply the Vicarious Hypothesis to a noncircular curve; these are assessed in Appendix III(5).

It should be remarked that Kepler also invented a method by which he could find both the distance of Mars and the true anomaly $\underline{\theta}$ for a given time $\underline{\alpha}$ independently of the Vicarious Hypothesis: indeed he used this method—which depended on paired observations of Mars taken at an interval which was a multiple of its periodic time (687 days)—in C.28, as clearly explained by Small.[8.] Rather naturally, since such precisely tallying observations were sparse, the method was not of general application, so I have not attempted to determine the inaccuracy involved in this case.

(c) The accuracy of Kepler's observational method.

It is not my intention in this study to evaluate the errors that can occur in the course of gathering and processing raw observations: these are matters on which there are other commentators better-informed than I. Errors in the Tychonic values could have been due to observer aberration or inadequacy (careless time-keeping, in addition to the fallibility of the human eye), or to instrumental breakdown, or have resulted from mis-computed allowances for parallax or refraction. Further inaccuracies could have arisen when Kepler carried out the necessary technical refinements, in making allowance for the inclination of the plane of the orbit to the plane of the ecliptic, as well as taking account of the small secular

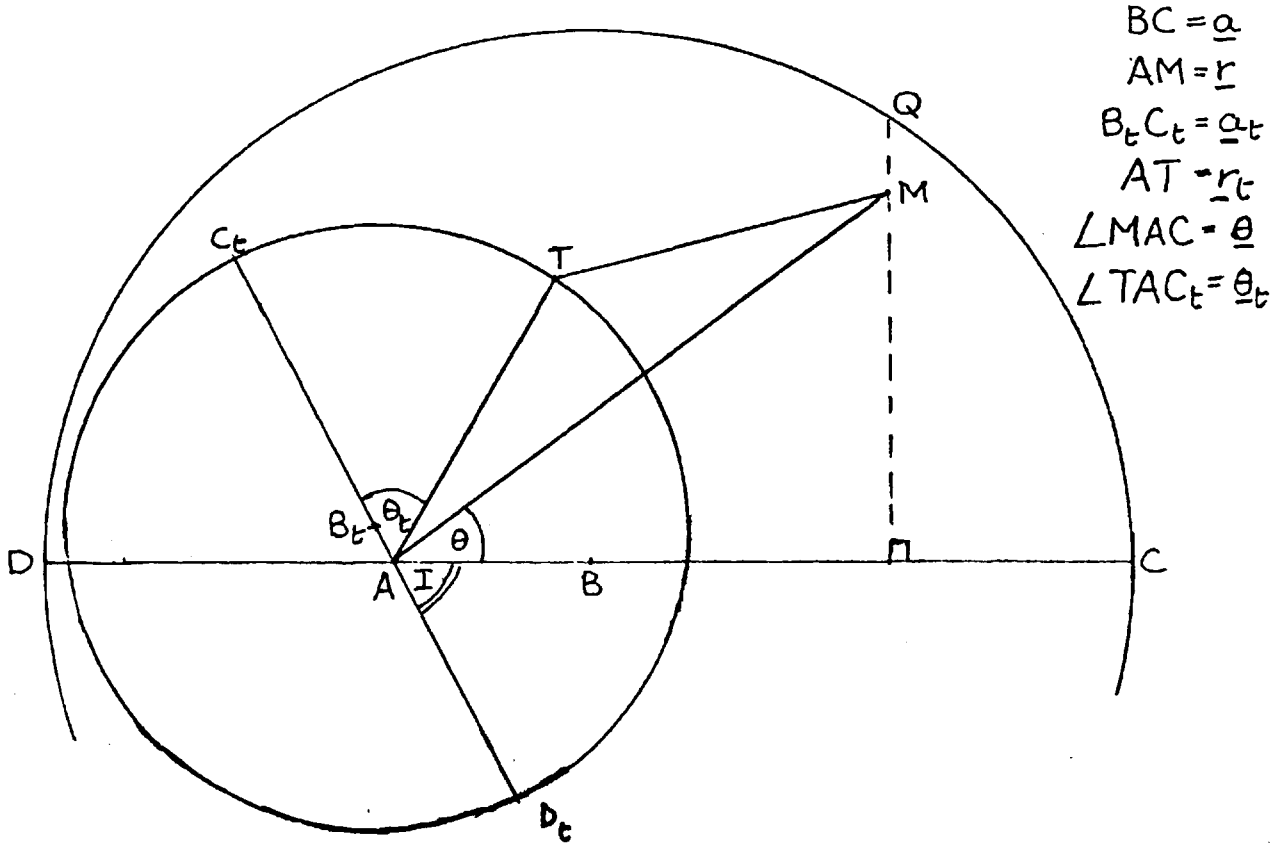
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7. See Whiteside: Keplerian Planetary Eggs, p. 8.

8. The Astronomical Discoveries of Kepler § 135, pp. 207-8, and Note S.

changes. My concern here is rather, by setting an arbitrary figure to represent the total effect on the observed angle of the errors from all these sources, to assess some reasonable minimum level of observational uncertainty imposed on the distance-determinations by the use of the method of triangulation—and thereafter to abide by its consequences in mathematical terms.

Fig. (iii) The triangulation diagram: Astronomia C.52.



The suffix 't' is used for all quantities connected with the Earth.

The lines of apsides are: CBD for the orbit of Mars

$C_tB_tD_t$ for the orbit of the Earth.

∠ I is the acute angle between these lines of apsides: for the present purpose it is to be taken as accurately known. I have here illustrated intermediate positions of the Earth and Mars, on the same foundation as that of Kepler's

own most sophisticated complete diagram given in C.52 (G.W. III p. 336). This lettering (ΔATM) will be employed whenever we consider an observational method, involving positions of the Earth and Mars simultaneously: while it is to be proved (in Chapter II), that, in relation to the orbit of Mars taken on its own, $M \equiv P_o$ in our standard notation. In ΔATM, it is the angle at T which is directly measured from an observation made at a known time $\underline{\alpha}$, and $\underline{\delta T}$ is the angular uncertainty which is alone subject to the possible errors listed above—which may be positive or negative. At the given time $\underline{\alpha}$ we can find the true anomaly from the Vicarious Hypothesis, whose (positive) theoretically determinate defect has been evaluated in (b) above; and since $\underline{\Delta\theta_t}$ is to be disregarded, as also established there, it is clear from the diagram that the defect $\underline{\Delta A}$ will be negative. We also point out that because $\angle M$ cannot be directly observed, $\underline{\delta M}$ cannot be independently determined: we have:

$$\angle M + \angle T + \angle A = \pi$$

$$\therefore \delta M = -(\delta T + \Delta A).$$

The standard triangulation-formula, from which the distance of Mars is calculated, is:

$$r = r_t \cdot \frac{\sin T}{\sin M}$$

$$\therefore \log r = \log r_t + \log \sin T - \log \sin M.$$

Differentiating, we have, to a first approximation:

$$\frac{\delta r}{r} = \frac{\Delta r_t}{r_t} + \cot T \cdot \delta T - \cot M \cdot \delta M.$$

Before evaluating this correct formulation of the proportional distance-uncertainty, we digress to consider the 'estimate' proposed by Curtis Wilson:[9].

$$\frac{\delta r}{r_t} = \frac{\Delta r_t}{r_t} + \delta T + \delta M.$$

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9. Kepler's Derivation of the Elliptical Path, Part 2, pp. 11-2 (expressed in the standard notation of this study).

This, however, is faulty on several counts: firstly, the proportional inaccuracy in the distance of Mars has been expressed in terms of the distance of the Earth; indeed this mistake recurs, in a different context, when Curtis Wilson investigates the difference in the radius vector of Mars corresponding to 1° change in anomaly:[10.] the values he states are too large by a factor $\frac{3}{2}$ (the approximate ratio of the radii of the two orbits).[11.] The insidiousness of this error must serve to vindicate my insistence throughout this study on a literal representation (as $\underline{a}$ or $\underline{a}_t$), rather than a numerical one (as 100000) of the unit under consideration.

Even if this were put right, Curtis Wilson's formula cannot be identified with the correct version unless, in the latter, one considers the special case $\cot T = \cot M = 1$, which is obviously impossible, since this entails $AM = AT$, with the fascinating consequence that Mars would then be moving in the same orbit as the Earth. Nor does the erroneous estimate lend itself to minimization by an optimum choice of viewing-position, as does the correct formula: moreover, Curtis Wilson applied it, by setting $\delta M \doteq \delta T$, as if the inaccuracy in $\angle M$ were on a par with that in $\angle T$—whereas we have already shown that δM is dependent on $\underline{\delta T}$ and $\underline{\Delta A}$.

Returning to the correct formula, we express it in terms of the independent quantities $\underline{\delta T}$ and $\underline{\Delta A}$; and at this juncture we shall disregard the term $\frac{\Delta r_t}{r_t}$, as justified in (b) above, to obtain:

$$\underline{\frac{\delta r}{r} = (\cot M + \cot T)\,\delta T + \cot M . \Delta A .}$$

Now it is evident from the geometry of the figure that $\angle M$ possesses an absolute upper bound, which occurs when $\angle T = 90^{\circ}$, and so we find:

$$\text{Max. } \angle M = \sin^{-1}\left(\frac{\text{max. } AT}{\text{min. } AM}\right) < 47\tfrac{1}{2}^{\circ}.$$

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10. Op. cit. Part 2, p. 14, note 44.

11. The values stated by Curtis Wilson are given by Kepler himself (or easily deduced), in Astronomia C.51 (G.W. III p. 324 lines 31–4 and p. 330 line 2): but it is quite clear from those calculations that he needed these values in terms of the radius of the Earth as unit.

Hence there exist some (few) favourable arrangements (subject to the availability of Tycho's observations) for which $47\frac{1}{2}^{o} > \angle M > 45^{o}$, and then $\cot M \leq 1$.

I shall have occasion in Chapter II, § 3(a) (supported by Appendix II(3a)), to demonstrate that Kepler himself fully understood the theory of the relative positionings of the Earth and Mars necessary to ensure that the observational distance-uncertainty is an absolute minimum:[12.] he got it exactly right at the quadrant (and the apsides), but for other positions of Mars in its orbit his method was less satisfactory, for a reason to be explained there. Meanwhile we shall consider the special arrangement—at quadratures—which Kepler investigated in some detail in C.51 (G.W. III pp. 332–4): we set $\angle T = 90^{o}$, $\angle M = 45^{o}$ (or as nearly as possible), so that $\cot T = 0, \cot M = 1$, to give:

$$\frac{\delta r}{r} = \delta T + \Delta A.$$

Since Kepler described this situation as 'very favourable' (p. 333 line 6), we may treat it as representing an approximation to the optimum accuracy he regarded as feasible, and evaluate the distance-uncertainty accordingly. The value of ΔA depends on the position of Mars in its orbit but (as proved in (b) above) can rise to 2'; and we have to make some assessment of the cumulative effect of the possible errors from all the sources listed above on the observable angle at T: quite arbitrarily, I shall suggest setting $\delta T \leq 2'$. Thus (at least when δT is in the same sense as ΔA), even supposing the availability of many similarly favourable combinations of the relative positions of the Earth and Mars, it seems that the size of the proportional distance-uncertainty cannot be expected to fall below 4', which will produce, from the Table of Equivalences above:

An observational distance-uncertainty δr of about 100 Keplerian units.

This corresponds to an uncertainty at $\frac{1}{8}e^{2}$ when the distance (or the angle) is expressed in modern terms as a series-expansion.

Kepler himself showed that he had a clear understanding of the irreducible size of the distance-uncertainties when he commented, in C.51 (G.W. III

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12. From the details of Kepler's method given in Appendix II(3a), we deduce the correct values $\cot M = \frac{3}{2}$, $\cot T = -\frac{2}{3}$.

p. 326 lines 13-14) that he would be extremely satisfied if he could keep their values always within 100 (Keplerian) units: it would have represented a pinnacle of attainment. Indeed, the fact that he managed to do so sometimes was a tribute not only to the reliability of Tycho's observations and to the discrimination he himself showed in picking out favourable combinations, but also to the development of his skill in handling the necessary astronomical refinements: with the consequence that, during the period under review, his triangulation-calculations converged steadily to the true distances, in parallel with the confluence of theory which led to the discovery of the correct orbit.

Nevertheless, to reach a theoretical conclusion, we shall need to decide the level at which the size of differences in distances or angles is genuinely significant, irrespective of the quirks of observational uncertainty. For this Keplerian analysis the observation-controlled level of accuracy stipulated is set at $O(e^2)$. Hence, by reference to the Table of Equivalences in (a) above, we automatically neglect terms $O(e^3)$ in the series-expansions (corresponding to distance-differences of less than 80 units, angle-differences lower than 3'); equally, distance- and angle-differences of 200 units, $7\frac{1}{2}'$-8' respectively—corresponding to terms involving $\frac{1}{4}e^2$ in the expansions—will emphatically be of theoretical significance: while it so happens that no terms involving smaller fractions of e^2 occur in the course (of a modern transcription) of Kepler's own working;[13.] so we are able to avoid investigating the cases involving terms in $\frac{1}{8}e^2$ whose reliability is questionable.

It may perhaps seem somewhat disappointing that, despite the great increase in the skill with which Kepler handled the observations, as remarked above—notably in Astronomia Part IV—we have in practice reverted to dependence on the 'reformation of astronomy' figure of 8' which was the level of accuracy already available in Part II.[14.] However, it will turn out that the level now stipulated is just sufficient to distinguish the separate stages, and so we anticipate that it was precisely what Kepler required in order firstly to discern that the orbit was noncircular, and then to determine its correct dimensions.

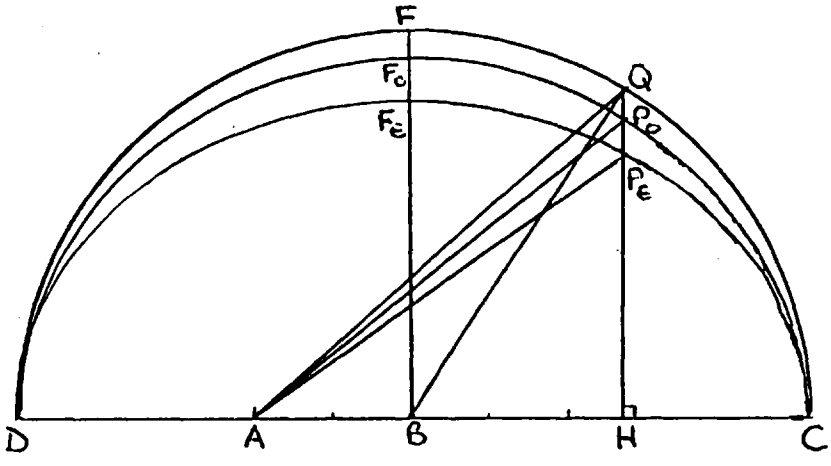
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13. Terms involving $\frac{1}{8}e^2$ do arise in some of the series-expansions introduced for comparison—but these could not be tested observationally.

14. C. 19 (G.W. III p. 178 lines 1-12).

§ 4 The prototypes and the three stages of Kepler's progress: consequences of the stipulated level of accuracy.

Fig. (iv) Conic-system A.G. II: the prototypes.



The general formulation of conic-system A.G. II has been given in modern terms in Prolegomenon I: we here pick out the three members which serve to distinguish the stages of Kepler's progress. At the beginning he supposed the orbit to be the eccentric circle, and thereafter every oviform he proposed approximated to one or other of the exact ellipses illustrated: each of these three is to be named a prototype (curve or orbit). Thus the outer bound, the eccentric circle through Q, is the first prototype; the inner bound, the ellipse passing through P_ϵ, will be known as the second prototype; and the ellipse passing through P_o will for the present be described as the mediate prototype.[1.] It is to be emphasized that we must avoid ~~anoitipation~~ anticipation of the fact that this mediate ellipse will turn out to be the correct orbit, and, notably, the unique Sun-focused ellipse of the system:[2.] for we shall confirm,

1. The relevant terminology is to be distinguished by the suffixes c, ϵ, o in turn.
2. The uniqueness of the ellipse with the Sun at one focus is not invariably appreciated even nowadays, as we shall see.

in Chapter IV, § 1, that Kepler did not consider the position of the focus—he did not mention the term—until much later in his work, and therefore, throughout Astronomia, he treated the two ellipses on a par, defined by the lineal ratio-property—stated in C.47 (G.W. III p. 300 line 41-p. 301 line 1) and reiterated as Protheorema I of C.59:

$$\frac{P_sH}{QH} = \frac{b_s}{a}, \text{ where } s \equiv c, \text{ or } e, \text{ or } o.^{3.}$$

Hence we are justified, in Keplerian terms, in representing the three typical points of this system as lying on the common ordinate QH—that is, possessing common eccentric anomaly (the significance of this will become apparent in § 5 below).[4.]

The radius vector of each prototype is derived from the exact quadratic expression worked out in Prolegomenon I in terms of a single coefficient q_o:[5.]

$$r_s^2 = a^2[1+2e\cos\beta+e^2-(2q_o-1)e^2\sin^2\beta]$$

$$\therefore\ r_s = a[1+e\cos\beta+(1-q_o)e^2\sin^2\beta] + O(e^3).$$

From Prolegomena III and IV we find that the first and second prototypes are specified by $q_o = \frac{1}{2}, \frac{3}{2}$, respectively, and hence we find:

Distance-divergency between first and second prototype

$$\Delta r = ae^2\sin^2\beta + O(e^3),$$

and, using the formula for true anomaly also given in Prolegomenon I, since the two differences are interconnected, we have:

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3. The suffix 's' is to be employed hereafter to denote any specific prototype of the three (or sometimes, of the first two only): thus $P_c \equiv Q$, $b_c \equiv a$; and $b_o \equiv b$.

4. And any foreknowledge of its consequent kinematical synthesizing-property must be firmly excluded, on the grounds that Kepler's struggle to prove this is the subject of Chapter III.

5. That is, we direct attention to the regular second prototype alone: see Prolegomenon IV.

Appurtenant angular separation between first and second prototype

$$\Delta\theta = \tfrac{1}{2}e^2 \sin 2\beta + O(e^3).$$

These theoretical values are required (because of their importance), to confirm the observational results of Chapter II: the latter also appears in Chapter III.

Meanwhile, we shall use the fact that each prototype is specified by a particular value of q_o (that for the mediate prototype, naturally, is yet to be discovered) to formulate a precise definition of the groups which correspond to the stages of Kepler's progress. Each group will consist of one of the three prototypes, together with (an infinity of) associated oviforms, whose radius vectors are each expressed in terms of the value of q_o which specifies the particular prototype, and successive arbitrary constants q_1, q_2.... Then it is an automatic consequence of the expression of the radius vector by expansions in ascending powers of e, assumed for the basis of this study (see Appendices I(1) and I(2)), that the theoretical difference in radius vector will exhibit the following features:

Difference between distinct groups: of value $O(e^2)$
Difference within a single group: of value $O(e^3)$ at most.

For completeness we anticipate that in Chapter III all the individual mean anomaly hypotheses have been expressed as series-expansions of similar structure, once more involving the coefficients q_o, q_1.... (see Appendices III(1) to III(4)); consequently again the theoretical angular difference[6.] when any one hypothesis is consistently applied to various curves will exhibit the same features:

Difference between distinct groups: of value $O(e^2)$
Difference within a single group: of value $O(e^3)$ at most.

Then by reference to the observation-controlled level of accuracy stipulated in §3 it is clear that we can expect distance-determinations to indicate differences between distinct groups, while corresponding angle-determinations should produce detectable differences when a given hypothesis is applied to

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6. This is the subject of Chapter III, §2, where the appurtenant angular separation is identified as the defect in (updated) optical equation (in the opposite sense).

curves of distinct groups.[7] Indeed, it will be our purpose in the next two chapters to confirm that observations will supply both geometrical and kinematical criteria—enough and to spare—by which Kepler could decide the dimensions of the correct curve. This was all he needed to know, to embark on the third stage (that is, until the end of Chapter II, §4, and Chapter III, §7, respectively): while the question of the exact shape of the curve—it should go without saying—can never be resolved by observational means (neither from distance- nor from angle-determinations),[8] however rigorous the level of accuracy stipulated; and therefore this topic is discussed separately, in Chapter II, §5, and Chapter III, §8.

Nevertheless Kepler was capable of recognizing the presence of differences in shape within a single group when they arose as a result of the construction. Thus, in C.47, he demonstrated his grasp of the relative levels of accuracy involved, by neglecting the effect of differences in shape, of value $O(e^3)$ at most,[9] by comparison with the difference between groups, at $O(e^2)$. It is with this Keplerian justification that we shall frequently, up to the end of the second stage, work correct to $O(e^2)$:[10] in practice, we confine consideration to the prototypes alone. However, this is a wholly desirable simplification only as long as it does not lead to any unwarranted assumption as to the shape of the correct curve—which we do not begin to consider until the third stage.

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7. Or equivalently, when a range of hypotheses is applied to a given curve: evidently, the stipulated level of accuracy demands that all proposed hypotheses exhibit differences at the level $O(e^2)$—as is the case—otherwise Kepler would have been unable to distinguish them.

8. Though a misapprehension in this respect is noted in Chapter III, §7.

9. By the introduction of the notional Auxiliary Ellipse—which is the subject of Chapter II, §2 and Chapter III, §6.

10. Though naturally, some calculations in the text have been carried out to a higher level for interest; and in various Appendices where I have aimed to provide the foundation for a modern assessment.

We may categorize the oviforms alternatively, by sorting them, like eggs, into large, small, and standard grades which correspond to the groups already defined: this down-to-earth terminology reinforces the idea that minor differences in shape within these major gradings are to be disregarded. (The egg-joke originated with Kepler himself, as Koestler has pointed out,[11] and the metaphor has been interestingly employed by Whiteside in his paper Keplerian Planetary Eggs: I refrain from stretching it beyond the present application.)

We now set out a chronological classification of Astronomia Part III[12] and Part IV,[13] to which continual reference will be made throughout Part A, for it will enable every statement to be placed in context in the development of Kepler's thought. (The names of the groups have been chosen primarily for distinctiveness: we shall account for them in due course.)

The stages of Kepler's progress.

Stage 1. C. 39-44: the circle 'group': $q_o = \frac{1}{2}$
First prototype: the eccentric circle (large grade)

Stage 2. C.45-50: the Ovoid group: $q_o = \frac{3}{2}$
Second prototype: the Auxiliary Ellipse[14] (small grade)

Stage 3. C.55-6, C.58-60: the 'via buccosa' group
Mediate prototype: to be determined (standard grade).

Hence, $\frac{1}{2} \leq q_o \leq \frac{3}{2}$ defines the Keplerian range.

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11. The Sleepwalkers, p. 330.

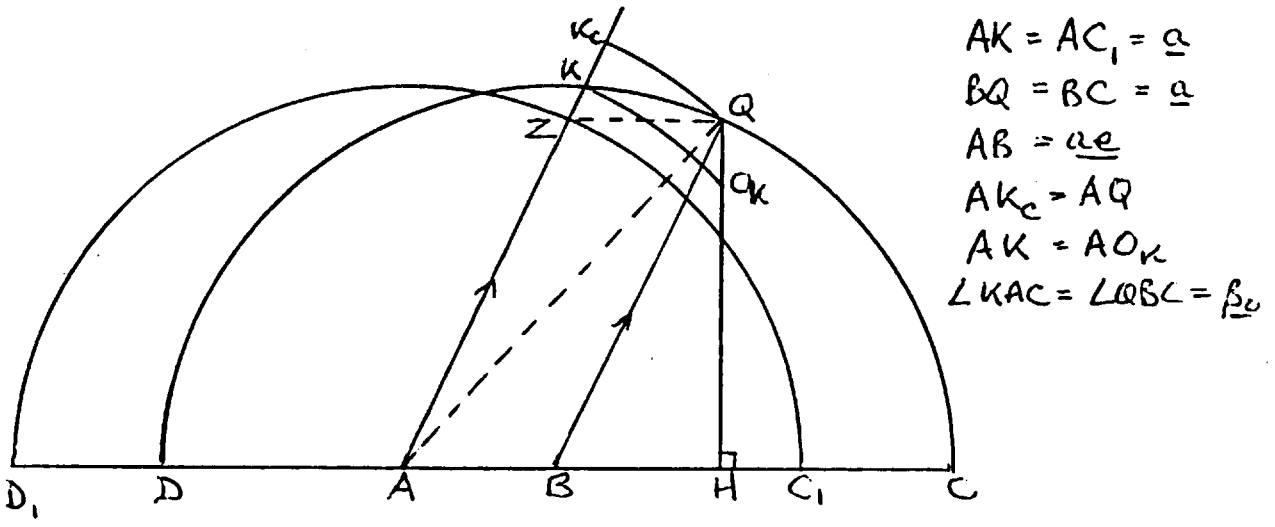
12. Apart from C.39 and C.40, as listed, only C.32 receives detailed attention in this study—though we shall refer to other chapters for background information.

13. C.51-54 are concerned with observational accuracy, the subject of § 3 above: C.57 is mentioned in Appendix II(2), and in Chapter IV, fundamentally as an introduction to Epitome; it is treated as such in Chapter VI of Part B.

14. This term describes Kepler's knowledge of the second prototype, to be discussed in Chapter II, § 2 and Chapter III, § 6.

§ 5 The geometrical frame of reference and Kepler's characteristic construction, founded on the invariant distantiary angle.

Fig. (v) Kepler's characteristic construction.



I here set out the technique by which Kepler arranged the geometrical comparability of the intermediate points[1.] of every curve he successfully proposed for consideration as a planetary orbit. Subject to certain permanent exceptions,[2.] this assertion can, naturally, and will, be checked in every individual instance. Unfortunately Kepler automatically assumed that these points would be contemporaneous as well, which was of course quite wrong because of the geometrical representation (as mean anomaly) which he adopted for the designation of time. However he cannot be too severely criticized for this misapprehension since it has only been noticed quite recently.[3.] Moreover,

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1. Not at the apsides, quadrant, or octants: these special cases will be examined separately as occasion arises.

2. 'Ovoids' II, III, IV of C.50 are excluded because they turn out to be geometrical 'throwbacks' to a circular orbit, as proved in Appendix III(5): similarly the triplet of orbits not mentioned in Astronomia is excluded since it constitutes a 'throwback' in kinematical terms to an equant theory: see Chapter III § 9(b).

3. By Whiteside (1964): it is of great importance, but mainly in discussion of the third stage: see Chapter III, § 8.

now that it has been recognized, we can employ the exactitude of his geometrical comparison to assess the extent of the noncontemporaneity in each instance, as we shall see.

We reproduce the geometrical skeleton shown in Kepler's diagrams of Astronomia C.45 and C.58 (G.W. III p. 288 and p. 365), by taking K on the eccentric circle, drawing BQ parallel to AK to meet the eccentric circle at Q, and dropping the perpendicular from Q onto the line of apsides at H.[4.]

Nevertheless, to achieve the necessary geometrical comparability, the rigidity of this skeleton must be ensured, and I have found the authority for this in Kepler's own work, at the beginning of Astronomia C.50, by discovering that he introduced an anomalia distantiaria, described as 'only for the purpose of determining the distances' (G.W. III p. 314 second marginal note and lines 32-3). He identified it as $\angle$KAC (p. 314 lines 33-5—here adopting our standard notation) in two separate diagrams, one of which was repeated from C.46 (G.W. III p. 294), and it is with this justification[5.] that I shall extend its applicability retrospectively to every curve of the second stage, as well as prospectively into the third stage. However, the way he used it suggests that it was to be regarded as a special 'one-off' value, not as a concept, and so we shall denote it by β_0, and make a slight adjustment of terminology, to write:

The distantiary angle = β_0 = $\angle$KAC = $\angle$QBC (since QB $\|$ AK).

Then for the discussion of Chapters II and III (except for the special cases at the apsides, quadrant, and octants which are treated separately) we shall retain and depend on the invariancy of β_0 to provide the desired geometrical cohesion, by ensuring the rigidity of the geometrical frame of reference which will be universally invoked throughout the analysis of Astronomia, whenever an intermediate point is involved.

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4. When AZ cuts the concentric at Z, we have ZQ and AB both equal and parallel, and the diagram can be used to establish the equivalence of the eccentric to the concentric-with-epicycle, known to the Greeks, and reconfirmed by Kepler in Astronomia C.2.

5. Supported, rather obliquely, by references in C.47 to $\angle$KBC in this same connection (since $\angle$KBC depends on $\angle$KAC, as we confirm in Chapter III, § 5(c)).

To construct an intermediate point of each proposed curve in turn, Kepler started from the line AK, henceforth determined by the invariant distantiary angle β_o. He took a particular length $\underline{r}$ along AK (produced if necessary), and then, with centre A, described an arc of that radius towards aphelion. Because its effect was to shift the direction of the radius vector $\underline{r}$, this procedure will be termed Kepler's characteristic (radius-swinging) construction. Its origins have been traced to the first stage, by two different routes: in Chapter IV, § 2, we shall demonstrate that a combination of the two physical causes, suggested in C.39, produced this result;[6.] while in Chapter II, § 1(b), we shall show how the method was used in C.40 for the construction of the first 'conchoid' (a curve whose main interest is geometrical but which, for Kepler, had kinematical connotations). Thus, though he obviously did not draw arc K_cQ in order to construct the point Q of the eccentric circle (it has been included merely to complete the analogy), neither is its introduction a non-Keplerian anachronism. Meanwhile we recognize the Euclidean criterion which is implied in the use of this characteristic construction: the requirement that, in every case, the radius of the arc—the length $\underline{r}$ of the radius vector—must be constructible by Euclidean methods. Thence we strictly define the typical point of any curve proposed as an orbit as a point lying on such a characteristic arc and uniquely defined by some additional geometrical property.

Now because—as stated in § 4—the prototype will on many occasions replace an oviform which is associated with it, it becomes essential to subject the points Q , P_ϵ , P_o , as illustrated there, to careful scrutiny. Evidently, there will be no loss of generality in setting:

$$\beta_o = \angle \text{QBC},$$

so that the distantiary angle will function as the common eccentric anomaly of the typical point of every prototype. Then we construct AK parallel to BQ and take lengths AK_c, AK_o, AK_ϵ along AK (produced), describing arcs with these radii in turn to cut QH at Q, P_ϵ , P_o respectively. Of these radii, $AK_c = AQ$ is determined by the frame of reference, while we show in Chapter II, § 4, that Kepler discovered two ways of constructing AK_o ; and lastly, I have found

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6. And similarly at the second stage, in fact, as explained there.

a Euclidean construction for AK_ϵ (though Kepler himself could not have done so, since it depends on the prior construction of AK_O).[7.] I therefore confirm, with hindsight, that all three points can be constructed by Kepler's characteristic method using radii which satisfy the Euclidean criterion, and thus Q, P_ϵ , P_O may properly be regarded as typical points of the prototypes, as defined above.

Thus the Keplerian range, as illustrated in the diagram, is determined by:

$$AK_C = AQ \geq r \geq AK = AO_K .$$

This highlights the fact that Kepler did not know how to construct $AK_\epsilon = AP_\epsilon$ and so took $AK = AO_K$ as the length of the radius vector of all the small-grade curves at the second stage.[8.] In correlation we point out, however, that the two lengths differ only at $O(e^3)$, so that $AK = AO_K$ is given by $q_o = \frac{3}{2}$, in agreement with §4 above (this is proved in Chapter III, § 5).

Indeed the exceptional aptness of the description distantiary (distance-determining) angle will become apparent in the course of this study as, using modern methods, we show, stage by stage, that every radius vector that occurs in Kepler's work can be formulated mathematically in terms of distantiary angle β_o.[9.] Moreover, in the end it will be demonstrated that this Keplerian-inspired approach involves no loss of generality nor distortion of the mathematics: for the conclusions reached are consistent with the results obtained by other commentators using Newtonian methods.

I attribute Kepler's unquestioning dependence on his characteristic method of construction for the typical point of every curve proposed as an orbit[10.] to the fact that he was, more fundamentally than anything else, a sound geometer; though this is in no way intended to detract from appreciation of his genius in other respects. It has not previously been remarked that the conventions of geometry in his day—and indeed within living memory—demanded that no length

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7. It is set out, for completeness, in Prolegomenon IV.

8. The point O_K is in fact the typical point of yet another Ovoid, as mentioned—but not analyzed—in Chapter III § 6(a).

9. On the grounds that β_o now functions as the common eccentric anomaly of the typical points of the prototypes whose radius vectors are given by the formulae set out in § 4 above.

10. Continuing to exclude those instances listed in the footnote at the beginning of this section.

be introduced into a problem until it had been visibly constructed and physically transferred to where it was needed. This is the Euclidean 'ruler-and-compasses' requirement. There are several other instances of its application which crop up in the course of Kepler's astronomical work: in Astronomia, the construction of a line to represent $\frac{r^2}{a}$ in C.50 (see Appendix III(5)), and in some of the Protheoremata of C.59, where the difference of squares was represented by a gnomon; in Epitome, a representation of $\underline{a^{3/2}}$ was constructed geometrically.[11.] These examples provide supporting evidence for my contention that Kepler chose for the radius vector of the curves at each stage only those lengths which were constructible according to this Euclidean criterion: AK, at the second stage, was obviously determined by the eccentric circle, while AK_o, at the third stage, may be constructed in either of two ways which are equivalent, as mentioned above. It is moreover my conviction that had the radius vector of the correct curve not been amenable to Euclidean constructibility, then Kepler would not have been the person who discovered the exact result—though he would undoubtedly have reached a close approximation to it.

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11. Needed for Law III, to be found in Epitome IV Part II, Section 4 (G.W. VII p. 307 lines 23-4). The working is verified analytically by Caspar in his note G.W. VII pp. 589-90—but he has not mentioned that the construction is a straightforward application of Euclid II, Prop. XIV.

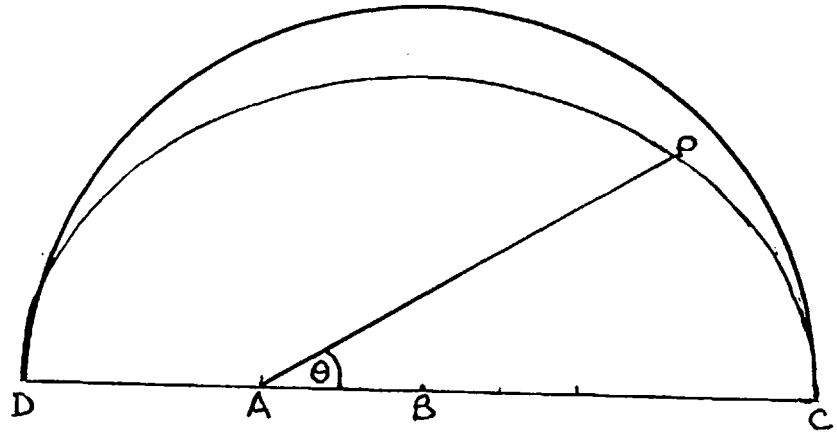
APPENDIX I (1)

ENCAPSULATING FORMULA: POLAR EQUATION OF AN OVIFORM $r(\theta)$

with acknowledgements to Whiteside: Keplerian Planetary Eggs p. 15

Fig. (i) Any proposed orbit.

$BC = \underline{a}$
$AB = \underline{ae}$
$AP = \underline{r}$



We assume that the orbit proposed is a continuous closed curve possessing at least one axis of symmetry, CD. The midpoint of this will be called the centre B, where BC = BD = $\underline{a}$, and thus there will be a circle centre B, on CD as diameter, associated with all the orbits under consideration. This will be known as the eccentric circle. In the general case such a curve will be generically described as an oviform (this includes curves which also possess a second axis of symmetry).

We set out to find the polar equation of the oviform determined by the point P as shown, with respect to A as pole of coordinates, A lying on the axis of symmetry and such that AB = ae, while AC is taken as the initial line.

We assume a series-expansion for $\underline{r}$ in terms of ascending powers of $\underline{e}$ and of powers of sines and/or cosines of the polar angle. Since the line $\theta = O$ is an axis of symmetry, only even powers of $\sin\theta$ can appear in the expansion, while at C and D, where $\cos\theta = \pm 1$, $r = a(1 \pm e)$, so that all subsequent terms must contain a factor $\sin^2\theta$, to ensure that they vanish at C and D. Then we have the polar equation of an oviform:

$$r = a(1 + e\cos\theta - q_o e^2 \sin^2\theta - q_1 e^3 \sin^2\theta\cos\theta) + O(e^4),$$

in which the coefficients q_n are rational.

APPENDIX I(2)

FORMULAE FOR ANY OVIFORM INVOLVING THE ECCENTRIC ANOMALY

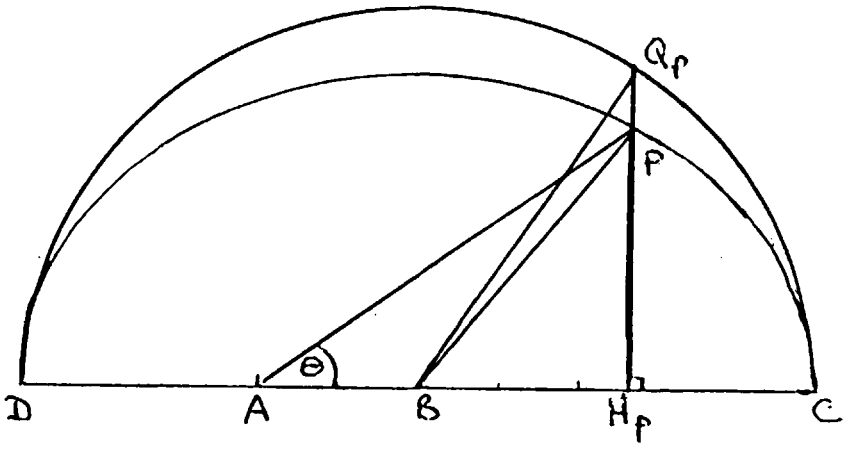
Fig. (i) Any proposed orbit.

$BQ_p = BC = \underline{a}$

$AB = \underline{ae}$

$AP = \underline{r}$

$\angle Q_p BC = \underline{\beta}$



The eccentric anomaly of a general point P is constructed as described in §2, so that $\angle Q_p BC = \underline{\beta}$.

From the polar equation of Appendix I(1),

$$r = a(1 + e\cos\theta - q_o e^2 \sin^2\theta - q_1 e^3 \sin^2\theta\cos\theta) + O(e^4).$$

From the geometry of the figure,

$$a\cos\beta = r\cos\theta - ae$$

$$\therefore\ \cos\beta = \cos\theta - e\sin^2\theta - q_o e^2 \sin^2\theta\cos\theta - q_1 e^3 \sin^2\theta\cos^2\theta + O(e^4).$$

This is satisfied by:

$$\underline{\beta = \theta + e\sin\theta + \tfrac{1}{2}(q_o - \tfrac{1}{2})e^2 \sin 2\theta - (q_o - q_1 - \tfrac{1}{2})e^3 \sin\theta + (q_o - q_1 - \frac{1}{3})e^3 \sin^3\theta + O(e^4).}$$

Inverting,

$$\theta = \beta - e\sin[\beta - e\sin\theta - (q_o - \tfrac{1}{2})e^2 \sin\beta\cos\beta]$$

$$- (q_o - \tfrac{1}{2})\sin(\beta - e\sin\beta)\cos(\beta - e\sin\beta) + (q_o - q_1 - \tfrac{1}{2})e^3 \sin\beta - (q_o - q_1 - \frac{1}{3})e^3 \sin^3\beta + O(e^4)$$

$$\theta = \beta - e\sin\beta(1-\tfrac{1}{2}e^2\sin^2\beta) + e\cos\beta[e(\sin\beta - e\sin\beta\cos\beta) + (q_o - \tfrac{1}{2})e^2\sin\beta\cos\beta]$$
$$-(q_o-\tfrac{1}{2})(\sin\beta - e\sin\beta\cos\beta)(\cos\beta + e\sin^2\beta) + (q_o - q_1 - \tfrac{1}{2})e^3\sin\beta - (q_o - q_1 - \frac{1}{3})e^3\sin^3\beta$$

$$\therefore \theta = \beta - e\sin\beta + \tfrac{1}{2}(\frac{3}{2} - q_o)e^2\sin 2\beta + (3q_o - q_1 - \frac{5}{2})e^3\sin\beta - (4q_o - q_1 - \frac{10}{3})e^3\sin^3\beta + O(e^4).$$

This pair of equations, $\beta(\theta)$ and $\theta(\beta)$, will jointly be known as Relation B_2.

Now we substitute in the polar equation of Appendix I(1):

$$r = a(1 + e\cos\beta(1-\tfrac{1}{2}e^2\sin^2\beta) + e^2\sin^2\beta - (\frac{3}{2} - q_o)e^3\sin^2\beta\cos\beta$$
$$- q_o e^2(\sin\beta - e\sin\beta\cos\beta)^2 - q_1 e^3\sin^2\beta\cos\beta] + O(e^4)$$

$$\therefore r = a[1 + e\cos\beta - (q_o - 1)e^2\sin^2\beta + (3q_o - q_1 - 2)e^3\sin^2\beta\cos\beta[+ O(e^4).$$

This will be referred to as Relation B_1.

We take the opportunity to state the general formula for the updated optical equation for any oviform (which, of course, reduces to the Ptolemaic-inspired optical equation when the point P lies on the eccentric circle):

$$\beta - \theta = e\sin\beta - \tfrac{1}{2}(\frac{3}{2} - q_o)e^2\sin 2\beta - (3q_o - q_1 - \frac{5}{2})e^3\sin\beta + (4q_o - q_1 - \frac{10}{3})e^3\sin^3\beta + O(e^4).$$

It is to be emphasized that accuracy at the level $O(e^3)$ is necessary only in connection with the modern analysis of the third stage, carried out in Appendix III(7): for the investigation of the first two stages in Keplerian terms, on the basis set out in Chapter III, §2, where we confine consideration to the prototypes alone, it is sufficient to assess inaccuracy at $O(e^2)$.

Now from ΔAPB,

$$BP^2 = r^2 + a^2e^2 - 2ae\,r\cos\theta$$
$$= r^2 + a^2e^2 - 2ae(a\cos\beta + ae)$$
$$= r^2 - a^2e^2 - 2a^2e\cos\beta$$
$$= a^2[1 - (2q_o - 1)e^2\sin^2\beta + 2(2q_o - q_1 - 1)e^3\sin^2\beta\cos\beta] + O(e^4)$$

By considering the value of BP'^2 given by the supplementary value of β, it is easy to see that:

When $2q_o - 1 > q_1$, the oviform is (more) pointed at perihelion

When $2q_o - 1 < q_1$, the oviform is (more) pointed at aphelion.

Hence the condition for a curve to be indistinguishable from an exact ellipse, correct to $O(e^3)$, is:

$$\underline{q_1 = 2q_o - 1.}$$

Moreover, we require every proposed curve to coincide with, or lie within, the eccentric circle, which necessitates $BP^2 \leq a^2$ for all β, and thus

$$q_o \geq \tfrac{1}{2}.$$

APPENDIX I(3)

FORMULAE FOR ANY OVIFORM INVOLVING THE CENTRAL ANOMALY

The central anomaly of a general point P is defined in §2, so that $\angle PBC=\chi$, as illustrated in Appendix I(2), while $\angle Q_pBC=\beta$.

From the geometry of the figure,

$$BP\cos\gamma=a\cos\beta.$$

$$\therefore\ \cos\gamma=\cos\beta[1-(2q_o-1)e^2\sin^2\beta]^{-\frac{1}{2}}+O(e^3)$$

$$=\cos\beta+\tfrac{1}{2}(2q_o-1)e^2\sin^2\beta\cos\beta.$$

This is satisfied by:

$$\left.\begin{aligned}\gamma&=\beta-\tfrac{1}{2}(q_o-\tfrac{1}{2})e^2\sin2\beta+O(e^3)\\ \beta&=\gamma+\tfrac{1}{2}(q_o-\tfrac{1}{2})e^2\sin2\gamma+O(e^3)\end{aligned}\right\}\quad \underline{\text{Relation } C_2}$$

Substituting in Relation B_1 of Appendix I(2) we obtain:

$$\underline{r=a[1+e\cos\gamma-(q_o-1)e^2\sin^2\gamma+(2q_o-q_1-\tfrac{1}{2})e^3\sin^2\gamma\cos\gamma]+O(e^4).}$$

This will be referred to as <u>Relation C_1</u>.

Now using Relation B_2 of Appendix I(2) we find:

$$\underline{\gamma-\theta=e\sin\beta-\tfrac{1}{2}e^2\sin2\beta+O(e^3).}$$

This is named <u>the optical angle</u>: it is essentially a nonKeplerian quantity, as is the central anomaly itself; in the sense explained in §2; but it has had to be quantified since the concept is needed in connection with a modern assessment of one aspect of Kepler's proof of Law I (discussed in Chapter VI, §4). Otherwise, the reasons for confusion that can arise between the various optical concepts—particularly around the quadrant—have been set out in Appendix II(4), where we show that Kepler failed to distinguish the optical angle from the (updated) optical equation for the correct ellipse in a geometrical investigation—even though the two evidently differ at $O(e^2)$.

CHAPTER II

THE DISCOVERY OF THE CURVE

§ 1 The origins of the Secant Law

§ 2 The first and second prototype at the quadrant

§ 3 Kepler's determination of the quadrant-distance for the correct curve

§ 4 Kepler's first flash of enlightenment: the discovery of the provisional Secant Law

§ 5 Kepler's second flash of enlightenment: the discovery of the (specific) Secant Law

§ 6 'Nihil...nisi ellipsis': the geometrical justification for Kepler's choice

Appendix II(1) Kepler's 'conchoids'

Appendix II(2) Kepler's trigonometrical summations of Astronomia

Appendix II(3) The optimum choice of viewing-position

Appendix II(4) Background information relating to the optical concepts

§1 The origins of the Secant Law Astronomia C.39; C.40, C.43.

Our purpose in this chapter is to show how, almost exclusively for geometrical reasons, Kepler arrived at the conclusion that an exact ellipse—the mediate prototype—was observationally satisfactory. Moreover we anticipate that he discovered a parametric-polar equation for its radius vector—in fact this will specify the curve completely—which is in §4 to be termed the Secant Law. The origins of this formula are to be found in the first stage of his work, when he still believed the orbit to be circular; obviously, in this situation, the concept is premature, so this description will not be employed in the present section.

In this chapter we will invoke the frame of reference set up in Chapter I, § 5, for the consideration of typical points, so that the distantiary angle β_o will function as the eccentric anomaly of the typical point Q of the eccentric circle, at the first stage, and subsequently, as the eccentric anomaly of any point on QH, by the updated Keplerian definition stated in Chapter I, §2.

* * * * * *

The two Keplerian investigations to be examined here have their source in completely separate physical causes: their geometrical basis, however, is identical—as we emphasize by treating them together. It has been easy to pinpoint the connection with his anticipated discovery of C.56 because Kepler took one diagram from each investigation to illustrate his reasoning there (see §4).

In this theoretical context, the Earth is to be regarded as just another planet.[1.] Then the geometrical correlation is achieved by the choice of common (invariant) distantiary angle β_o functioning as the eccentric anomaly of the typical point Q of the eccentric circle, together with its supplement

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1. Thus it would be as inappropriate in this situation as it is vital in the observational context, to distinguish the Earth by special notation.

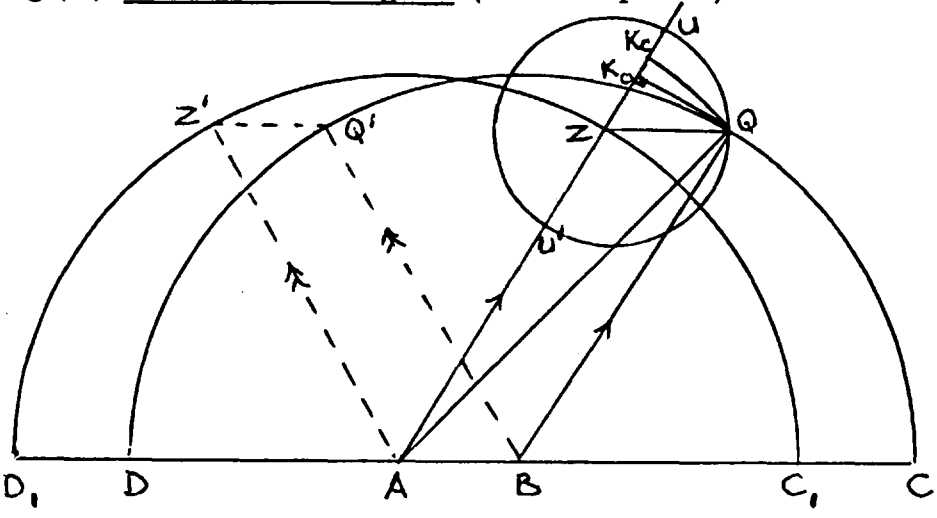
determining Q'. The terms we shall use in this analysis are Keplerian: though I cannot find them defined until C.57 (G.W. III p. 348 lines 6-8), there is a mention of the second one in C.56 (G.W. III p. 347 lines 38-9) which refers back to C.39, to which Kepler himself evidently traced them. In any case, they are illustrated in all the diagrams:

The circumferential distance is the radius vector for the circle, denoted by $r_c = AQ = AK_c = QR_c$

The diametral distance crops up in both investigations for comparison, represented by $AK_o = QR = AJ_1 = AJ$.

(a) The libration Astronomia C.39 (G.W. III p. 259 line 3-p. 260 line 4).

Fig (ia) The eccentric diagram (G.W. III p. 257).

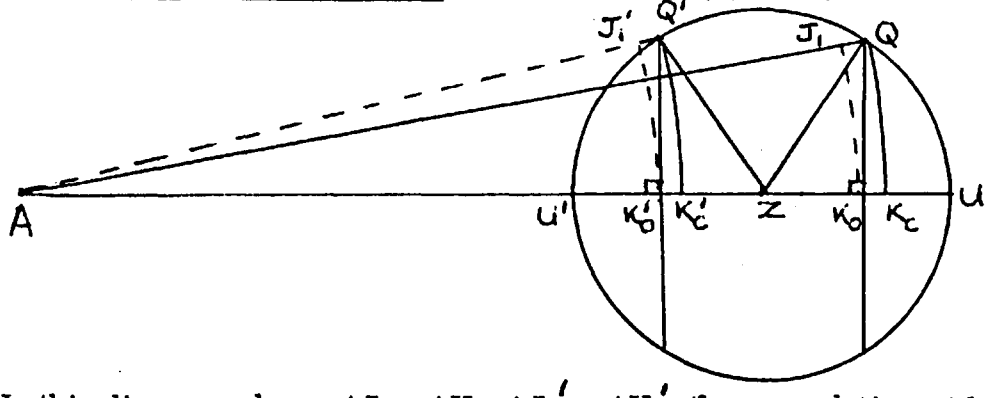


In both diagrams:

$AZ = BQ = BC = a$
$AB = ZQ = ZU = ae$
$AQ = AK_c = r_c$
$AQ' = AK_c' = r_c'$

$\angle QBC = \angle QZU = \beta_o$
$\angle Q'BC = \angle Q'ZU = \beta_o' = \pi - \beta_o$
$\angle QAB = \theta_c = \angle AQZ$

Fig (ib) The epicycle diagram (G.W. III p. 257, p. 261).



In this diagram alone, $AJ_1 = AK_o$, $AJ_1' = AK_o'$ (for correlation with § 5).

The suggestion that the planet librated on the diameter of the epicycle was the third possibility mentioned in C.39 by which (together with the motion round the Sun) Kepler hoped to account for motion in the eccentric circle (all three possibilities are listed in Chapter IV, §2, where we give a brief survey of Kepler's physical opinions): for the moment, we merely point out that its particular interest is in connection with the variation in distance from the Sun, since the diameter of the epicycle is necessarily directed precisely towards the Sun (p. 259 lines 4-6).

While the term libration had certainly been employed by earlier astronomers, including Copernicus—evidently coined by association with the up-and-down motion of the scales—we shall use it throughout this study only in the technical sense introduced by Kepler in C.39 (p. 259 line 5 and marginal note);[2] as the reduction of the diametral distance (or its equivalent) from the aphelion distance AU=a(1+e)—to be denoted by λ.

The geometrical interest subsists in the paired diagrams which, though not drawn on the same scale, are correlated by the fact that the identical triangle AZQ appears in both (their construction is described on pp. 256-7). Though the representation of true anomaly θ_c hardly suggests strict correspondence between them, this is, by contrast, strongly affirmed with respect to β_o, functioning as the eccentric anomaly, and its supplement β_o'.[3] We take the opportunity here to quantify the optical equation for the eccentric circle at a typical point:

$$\beta_o - \theta_c = \angle AQB = \angle K_o AQ.$$

In fact only its quadrant value, represented by $\angle AFB = \tan^{-1} e$ (illustrated in §2 below) is required in this chapter (because of its importance at the third stage, as we shall see in §4): a more general discussion of this and other concepts derived from ancient astronomy is deferred to Chapter III, §2.

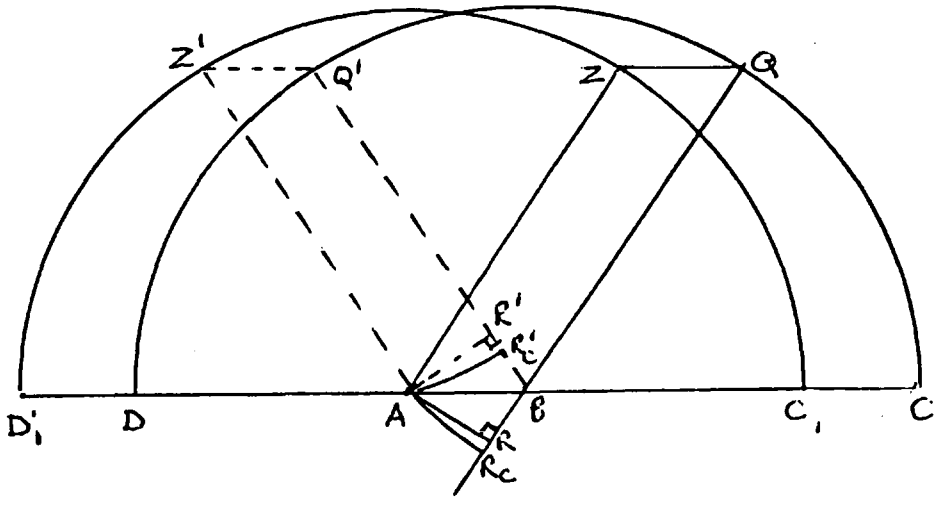
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2. The necessity for more precise terminology will become apparent in Part B.

3. Confirmed p. 256, lines 28-9, 32-3.

(b) Kepler's 'conchoids' Astronomia C.40 (G.W. III p. 268 line 12-p.269 line 26);
C.43 (G.W. III p. 264 line 11-p. 265 line 11).

Fig. (iia) The construction of the two 'conchoids'.



In both diagrams:

$AZ = AZ' = BQ = BQ' = BC = a$

$AB = ZQ = Z'Q' = ae$

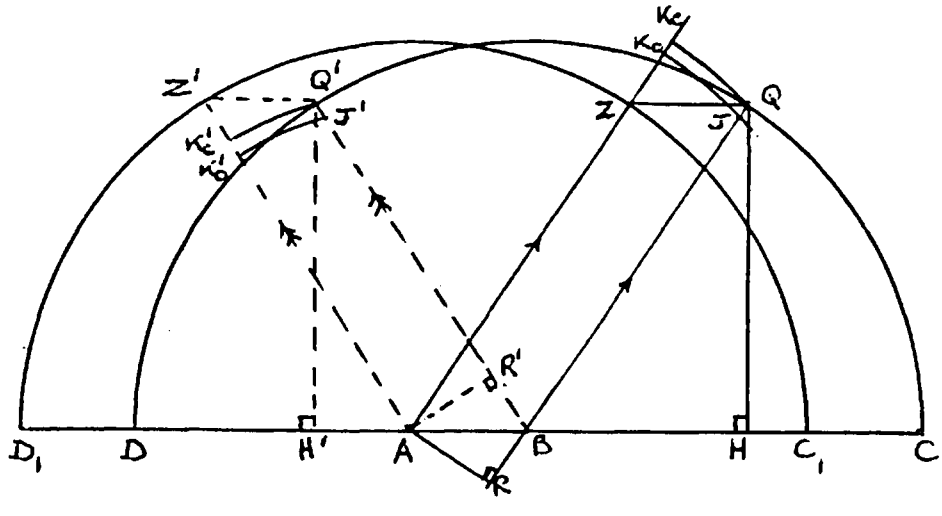
$AQ = AK_c = r_c$

$AQ' = AK_c' = r_c'$

$\angle QBC = \beta_0$

$\angle Q'BC = \beta_0' = \pi - \beta_0$

Fig. (iib) The eccentric diagram (G.W. III p. 264 and p. 267).



In this diagram alone, $AJ = AK_0 = QR$, $AJ' = AK_0' = Q'R'$ (for correlation in § 5).

The name 'conchoid' for the pair of curves drawn by Kepler on p. 268 and p. 284[4.] was chosen because, as he explained in a marginal note on p. 268, they appeared to be of a somewhat similar shell-shape to the conchoid of Nicostratus.[5.] However, the question of their exact identification has no bearing on the present issues, and so has been set out, for information only, in Appendix II(1). Indeed, since we are concerned here only with the geometrical significance of Kepler's 'conchoids', for simplicity I illustrate merely a detail from his diagrams, to show, by reference to the typical point Q of the eccentric circle, the construction of a particular pair of lengths (AQ, RQ) which were employed as the ordinates of the first and second 'conchoids' respectively.

In C.40, describing the construction of his first 'conchoid' (p. 268 lines 17-24), Kepler quoted Archimedes' result for the area of a circle, though he adopted a rather different approach,[6.] envisaging the circle as composed of a number of very small triangles with their apexes at the centre, having common altitude $\underline{a}$ while their bases (of size $a\delta\beta$) form the circumference. When the bond at the centre is cut, the circumference can be unrolled into a straight line of length πa, and then the other semicircle is inserted into the gaps, to give a rectangle twice the area of one semicircle. Kepler's innovation was to extend this method to an eccentric distance AQ by swinging it round to pass through B, so that QR_c became perpendicular to the base (the circumference) and thus determined an ordinate of the first 'conchoid';[7.] that for the second 'conchoid' was obtained by dropping a perpendicular from A onto QB to meet it at R (p. 268 line 36—p. 269 line 1). These two lengths are shown in the first of my diagrams:

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4. We do not include the apparently similar diagram of C.47 (p. 300), for the reason given in Appendix II(1).

5. I have used the same justification in describing the special set of small-grade oviforms (ova) which arise in C.45-50, 'Ovoids'.

6. Archimedes' method is described by Heath: Greek Mathematics II pp. 50-1. Kepler's method of dealing with the circle—though not the subsequent deductions—may be traced to the work of Nicholas of Cusa.

7. Hence these ordinates—and those for the second conchoid similarly—are erected at equal intervals of eccentric anomaly.

the second illustrates how this would have led Kepler to the idea of his characteristic (radius-swinging) method which he then went on to employ for the construction of typical points of noncircular curves, as we shall see in due course. It is particularly to be emphasized that this investigation, like the first, is based on (invariant) β_o, again functioning as the eccentric anomaly of the typical point Q, and its supplement β_o' determining Q' (p. 268 lines 17-19 taken with p. 264 lines 12-13).

It is clear from Kepler's statements in both C.40 and C.43 that he invented the first 'conchoid' in order to represent distance-sums as areas—with the second 'conchoid' for comparison. Hence, the kinematical significance of this investigation—by contrast with that in (a) above—is in connection with motion round the Sun. An unexpected consequence was that it provided Kepler with the incentive to embark on the first of a collection of summations of sines of angles at discrete intervals: he eventually discovered (in C.43) that this way was wrong and that he should have summed squares of sines instead, as we demonstrate in Appendix II(1)—and also, that he misinterpreted the distance-sums—so it was fortunate that the method proved sterile and was soon abandoned. Nevertheless, that summation of simple sines initiated an important step towards integration, as we shall see in Chapter VI. Meanwhile, all such trigonometrical summations are gathered together, for convenience, in Appendix II(2).

(c) Geometrical correlation of the two investigations — Astronomia C.39, C. 40, C.43.

We reiterate that this correlation is made possible by the choice of common (invariant) supplementary values β_o and β_o', functioning as the eccentric anomalies of Q and Q', as the basis of the comparison: this has been confirmed above for each investigation in turn. We now list Kepler's own statements as to the symmetries and asymmetries involved, adding the corresponding relation from the other investigation when Kepler did not do so: the amounts of the inequalities can easily be checked in modern terms from Appendix I(2).

In C.39		In C.40
$UK_o = U'K_o'$ (p. 259 line 23)	Equalities	$BR = BR'$ (p. 269 line 3)
$ZK_o = ZK_o'$ (by deduction)		$ZK_o = Z'K_o'$ (by deduction)
$UK_c < U'K_c'$ (p. 259 lines 23-4)	Inequalities at $O(e^2)$	$BR_c > BR_c'$ (by deduction)
$ZK_c > ZK_c'$ (by deduction)		$ZK_c > Z'K_c'$ (by deduction)
$K_cK_o < K_c'K_o'$ (by deduction)	Inequality at $O(e^3)$	$R_cR < R_c'R'$ (p. 269 line 22)

The quantification of the equalities in particular is essential for future reference:

$$UK_o = ae(1-\cos\beta_o) = U'K_o'$$
$$BR = ae\cos\beta_o = BR'.$$

When, in §4, a noncircular curve is proposed, the first pair will correctly represent the libration λ: meanwhile, we highlight the symmetry of this quantity with respect to the centre of the epicycle, and similarly, the symmetry of the second pair with respect to the centre of the eccentric. This consideration is of fundamental importance in the discussion of § 5.

* * * * *

Though there is no evidence that Kepler exhibited any interest in the value either of the libration or of the diametral distance at the quadrant, in C.43 he stated the relationship between the ordinates of the first and second 'conchoids', in terms of the optical equation,[8] in that special case (using the notation of fig. (ii) of §2):

$$a = BF = \frac{AF}{\sec AFB} \quad \text{(G.W. III p. 284 lines 13-16).}$$

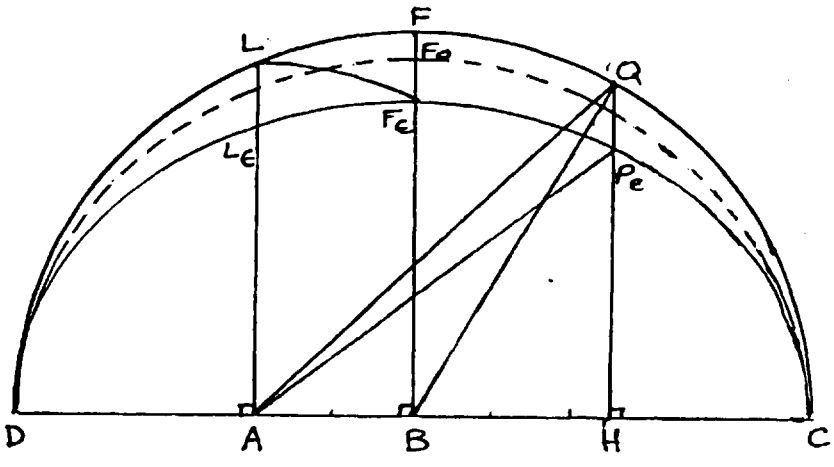
We will find that this pointed the way to the formulation of the general law of distances in C.56 as translated and discussed in §4: but its significance was entirely missed at this stage because Kepler was not then ready to

8. (Wrongly) described as maximum there: see §4.

accept that the proposed orbit should be noncircular (so that its radius vector at the quadrant, measured from A, could be of length a). I shall demonstrate in §3 (supported by §2) how he came to be persuaded of this—and thus that the above relationship constituted an important link in the geometrical chain of reasoning which led to the Secant Law, set out in §4.

§ 2 The first and second prototype at the quadrant Astronomia C.43, C.47.

Fig (iii) The first and second prototypes.



$BQ = BC = \underline{a}$
$AB = \underline{ae}$
$AQ = \underline{r_c}$
$AP_\epsilon = \underline{r_\epsilon}$
$\angle QAC = \underline{\theta_c}$
$\angle P_\epsilon AC = \underline{\theta_\epsilon}$
$\angle QBC = \underline{\beta_o}$
$AL = AF_\epsilon$

We first take the opportunity to define some Keplerian terminology of relevance in C.47 and subsequently.

The lunula is the crescent-shaped area between the eccentric semicircle and any noncircular semicurve. Its width may be measured in three ways:

(1) Along the radius vector, when it is then known, distinctively, as the curtation of the radius vector
(2) Along the radius of the eccentric
(3) Along the ordinate.

It is only the last two measures which will exhibit symmetry, if it exists: for instance, when the curve is an exact ellipse: this consideration is of great significance, in relation to the second measure, in § 5 below.

Mean distance = arithmetic mean of apsidal distances
= radius of eccentric a.

The position of the radius vector of this particular length obviously changes with the curve under consideration: for the circle, it is represented by theline from A to the point of intersection of the eccentric and concentric circles (G.W. III p. 297 lines 19-21). However, we shall ignore the

designation of mean longitude also given there (according to the usage of the Arabs),[1] to identify:

Mean longitude ≡ quadrant-position: where $\beta = 90^{o}$.

The second prototype has been introduced in modern terms in Chapter I, § 4, and we have confirmed in § 5 there that Kepler would never have managed to construct its typical point by means of his characteristic radius-swinging method. This may well be the reason why he never actually drew the curve he had in mind for his standard of comparison at the second stage, so we will in future describe it as the notional Auxiliary Ellipse. By contrast, he would certainly have been able to construct a comparable point[2] by applying the lineal ratio-property by which he defined an ellipse,[3] to reduce the ordinate determined by the typical point Q on the eccentric circle in a fixed ratio—thus ensuring that the distantiary angle β_o will function as the eccentric anomaly of the ellipse point as well (by the updated definition of eccentric anomaly stated in Chapter I, § 2). The construction is performed by dividing BQ=BF in the given ratio and then QH is likewise divided, by Euclid VI, Prop. X. However, this is subject to the Euclidean requirement that the fixed ratio (of incommensurables) be physically represented as the length of two lines: in this case, the minor semiaxis of the ellipse must first be determined. It is at this point that the difficulty arises, for Kepler gave no indication of the dimensions of the ellipse he was envisaging. Therefore, I suggest that he could have applied his characteristic construction as shown in the diagram, with radius AL swung round to cut the ordinate BF (which here coincides with the radius of the eccentric) at F_ϵ. This

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1. This was certainly unsuitable for Kepler's purpose, since it refers only to a circular orbit, as he recognized (p. 297, marginal note).

2. This will indeed turn out to be a typical point, as we prove below—but only in modern terms.

3. Stated in C.47 (p. 300 line 41-p. 301 line 1), repeated as Protheorema I of C.59: this property ensures symmetry along the ordinate.

is a special case of the construction shown in Chapter I, § 5, where I have anticipated that Kepler used the characteristic arc KO_k to determine the typical points of all the small-grade curves at the second stage. Then, in modern terms, we find:

$$AF_\epsilon^2 = AL^2 = a^2(1-e^2)$$

$$\therefore BF_\epsilon^2 = a^2(1-2e^2).$$

$$\therefore BF_\epsilon = a(1-2e^2)^{\frac{1}{2}} = a(1-e^2) + O(e^4).$$

Hence Kepler could have constructed an exact ellipse whose ordinates were in the constant ratio $(1-2e^2)^{\frac{1}{2}}$ to those of the circle: and it is easy to check, from Prolegomenon IV, that this is indeed identical with the (regular) second prototype set out in modern terms in Chapter I, §4 (possessing eccentricity $\sqrt{2}e$).

I digress to explain that this exactness of identification is theoretically essential, because Kepler also (p. 297 line 11-p. 298 line 12) found a representation of the length ae^2, exactly,[4.] and thence could evidently have constructed an exact ellipse whose ordinates were in the constant ratio $(1-e^2)$ to those of the circle. However, this produces an irregular second prototype, as also mentioned in Prolegomenon IV, and is therefore to be rejected forthwith (as I am sure Kepler would have done, on aesthetic grounds, had he been able to discriminate between the two).

Having made this decision, we revert to a more practical investigation of the state of Kepler's own knowledge of the prototypes at the quadrant, by reference to the observation-controlled level of accuracy stipulated in Chapter I, § 3. Some of the facts tabulated below are taken (selectively) from Kepler's own comparative tabulation of hypotheses in C.47 (p. 302),[5.] while references (only) for the mediate (unknown) curve have been included for completeness.

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4. It is quite irrelevant that this investigation was undertaken to estimate the area of the lunula, not in connection with the quadrant-distance, for which the eccentric anomaly ($\simeq 87\frac{1}{5}^\circ$) was far too inaccurate.

5. The full range of hypotheses and their general application are discussed in Chapter III, § 7.

The basis of comparison in the present tabulation can be explained by following the chronological order of the references. For the circle (the first prototype), Kepler examined the situation for eccentric anomaly $\beta=90^{o}$—which defines 'the quadrant'—and then worked out the mean anomaly at this particular point from the relationship:

$$\alpha = \beta + e \sin\beta$$ [6].

$$= 95^{o}18'\,28'' \text{ when } \beta = 90^{o}.$$

By contrast, for the notional Auxiliary Ellipse (now identified with the second prototype) he set $\alpha = 95^{o}18'28''$ and afterwards checked that this corresponded (near enough) to the quadrant-position given by $\beta=90^{o}$. [7] It was this two-way argument, I believe, which convinced him that the observed position would be at (or very near) the quadrant at this common time, and thus provides the justification for the diagram of § 3 below, in which the observed position is represented as both contemporaneous with the two prototypes, and also geometrically comparable with them via common eccentric anomaly: nevertheless I emphasize that there is to be no suggestion that any theory of time-measure for the correct curve is involved in this observational context, in which the assumption of kinematical and geometrical cohesion simply struck Kepler as reasonable. It seems probable that this opinion was reinforced when he determined the true anomaly, equivalent to 'observations', given by the Vicarious Hypothesis, again at this common time: while he recognized that this hypothesis was least reliable at or near the quadrant, the value found still lay between the values of the true anomaly calculated for the first and second prototypes in turn, as he would have expected.

The purpose of this tabulation is to enable us to pick out and affirm Kepler's knowledge of the quantifiable differences between the first and second prototypes at the quadrant. In particular,

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6. See Chapter III, § 4: this is of course the modern version of what is described there as the area-hypothesis (Keplerian style).

7. Again using the area-hypothesis (Keplerian style): see Chapter III, § 6.

Quadrantal distance-divergency between prototypes $FF_{\mathfrak{C}} = ae^2 + O(e^4)$

Quadrantal appurtenant angular separation between prototypes $< 3' \backsimeq e^3$.

Kepler certainly appreciated that the former was maximized while the latter was observationally insignificant (the separation between either prototype and 'observations' is even smaller). Thus, he treated the defect $\Delta\theta$ as zero in this special case (as we shall do also, with modern justification) in the geometrical investigation to be examined next: the consequences of this simplification were entirely favourable to his progress, as we shall see.

Kepler's knowledge of the prototype curves at the quadrant.

	Eccentric anomaly $\beta = 90^{\circ}$	Mean anomaly α	Minor semiaxis	Radius vector	True anomaly
Eccentric circle C. 43	p. 283 line 1	$95^{\circ}18'28''$ p. 283 line 8 p. 302 (tabulated)	$BF = a$ p. 284 lines 14-15	$AF \doteq a(1+\frac{1}{2}e^2)$ p. 284 lines 14-15	$\theta_c = 84^{\circ}42'26''$ p. 283 line 9 p. 302 (tabulated)
Auxiliary Ellipse C.47	p. 301 line 20, lines 24-5	$95^{\circ}18'28''$ p. 302 lines 17-18 p. 302 (tabulated)	$BF_\epsilon \doteq a(1-e^2)$ p. 301 line 26 p. 302 line 14	$AF_\epsilon \doteq a(1-\frac{1}{2}e^2)$	$\theta_\epsilon = 84^{\circ}39'42''$ p. 301 line 28 p. 302 (tabulated)
Vicarious Hypothesis 'observations'	Not applicable	$95^{\circ}18'28''$ p. 302 (tabulated)	Not applicable		$\theta = 84^{\circ}42'2''$ p. 283 lines 9-11 p. 301 lines 28-9 p. 302 (tabulated)
Mediate curve	C.56 p. 346 lines 5,6,8 p. 347 line 4	$95^{\circ}18'30''$ G.W. XV Letter 358 p. 250 lines 404-408	C.59 Protheorema VII see § 6	C.56 p. 346 lines 7-8 see § 4	C.60 p. 377 lines 13-26 see § 6†

† Kepler's failure to identify the true anomaly for the correct curve is the practical consequenceof his problem in C.58, discussed in § 5: by the time he discovered a formula for it, there was certainly no need to evaluate it at the quadrant.

§ 3 Kepler's determination of the quadrant-distance for the correct curve.

Unfortunately I have never seen Kepler's 'Leningrad Notebooks'; but Gingerich[1.] indicates that Volume XIV alone is supplementary to Astronomia, and states that it covers the period 1600-01, finishing before any non-circular curve was introduced. Thus there is apparently no further evidence in existence to provide information as to exactly how Kepler determined the quadrant-distance for the correct curve: the fact that he did so seems to me incontrovertible, when we examine his own statements quoted below— my doubt only arises over the question of the degree of reliance he may have placed on observation-based calculation (my first suggestion) relative to the method involving kinematical considerations, explained subsequently.

(a) The observation-based method Astronomia C. 53, C. 55.

I prefer to believe that Kepler first and foremost obtained the result from a calculation based on direct observation, as he stated in C. 55 (G.W. III p. 344 lines 30-3 —my italics):

> '[In C. 44] the distances computed from the theory for the circle are longer at the quadrant than those observed; the distances worked out from the hypothesis which makes the orbit of the planet an Ovoid [C. 45-50] are shorter.'

Also in C. 55 (p. 344 line 33—p. 345 line 1):

> 'Therefore it seems that the path of the planet is not the circle, nor.... an Ovoid, but a curve that lies in the middle.'

And again in C. 55 (p. 345 lines 6-8):

> 'There [at the quadrant] the Ovoid of C. 45 is wrong, being too small by 660 parts, by as much as theperfect circle is wrongly too large: which can produce an apparent effect of 20' or more.'

We digress briefly to remark that the involvement of the figure 660 confirms that the method used by Kepler just here was undoubtedly observation-based, since

$$660 = 0.00432 \times 152350 = \tfrac{1}{2}ae^2:$$

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1. A complete list of these manuscripts is set out in Kepler, Johannes, p. 309.

that is, this figure is expressed in terms of the radius of the Earth (a_t=100000), as well as of the radius of Mars which is then a=152350 (C. 54, G.W. III p. 344 line 13). Thus Kepler's letter to Fabricius of 18.12.1604[2.] correctly described the situation at the quadrant with the circle greater by '800 or 900 parts' (i.e. $0.00864 = ae^2$) than the Auxiliary Ellipse (or any Ovoid), which is itself too short by '400 or more parts (i.e. $0.00432 = \frac{1}{2}ae^2$)—here in terms of the radius of Mars taken as base (a=100000), as Kepler said. Two further letters,[3.] one earlier, one later, unfortunately create confusion[4.] by suggesting that the correct curve lay somewhere between the two prototypes, but not, as in the evidence just cited, exactly 'in the middle'. We circumvent this difficulty by assuming at least the possibility of mistranscription or omission (some general grounds for this opinion, in relation to Kepler's letters of this period, is given in the Introduction): otherwise, Kepler's statements in these two instances are at odds with all the rest, and I cannot account for them.

Disregarding this aberration, our present purpose is to marshall the argument for Kepler's conviction that the correct curve lay precisely 'in the middle' by justifying his assessment quoted above, that the prototypes can be distinguished, and thence rejected, because of the observable difference between them at the quadrant: that is, the theoretical results can fall more than 10' on either side of the observed—correct—value. To account for this figure we introduce the diagram justified in § 2 above, in which the observed position M is assumed to lie contemporaneously at the quadrant.

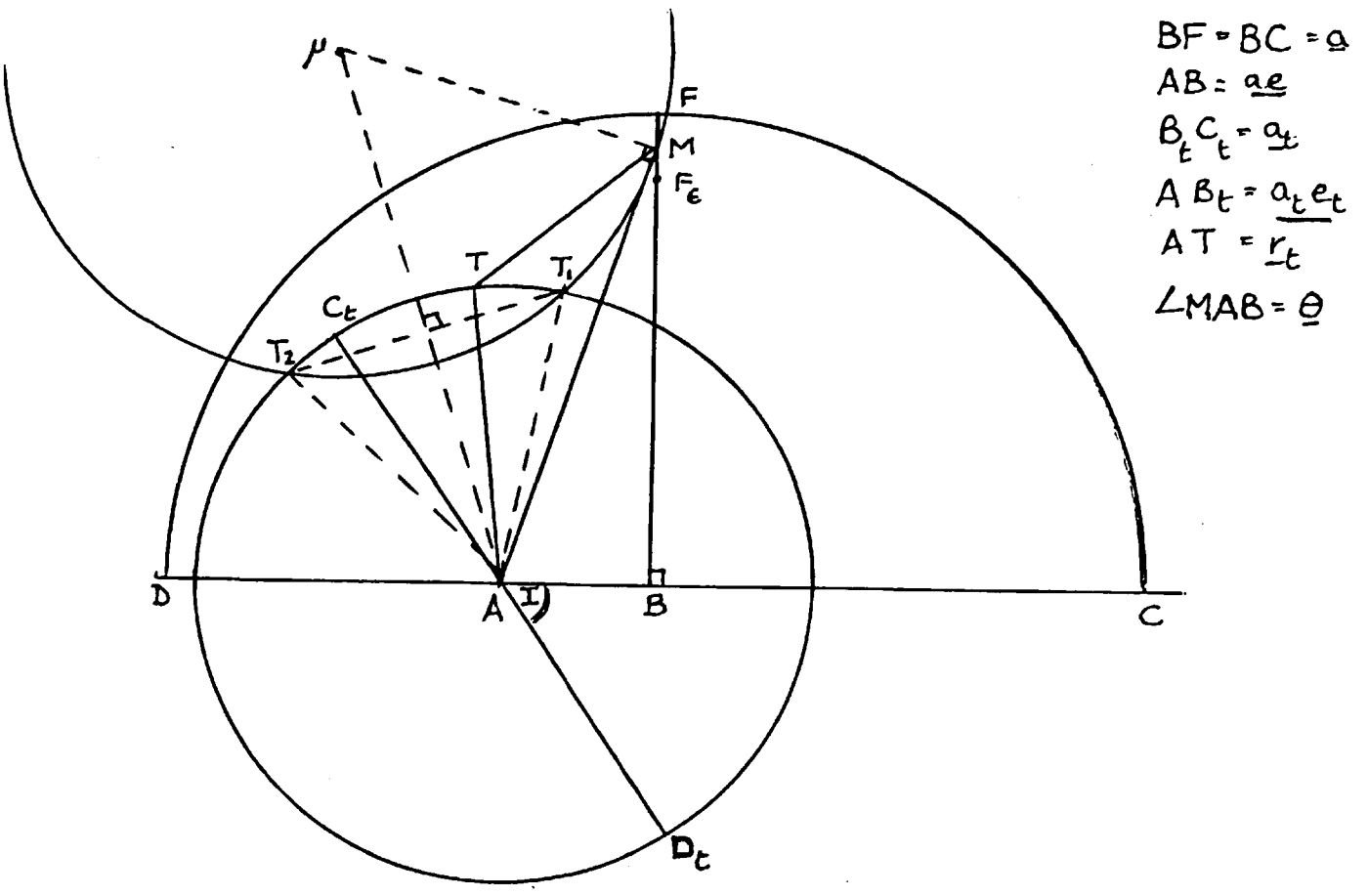
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2. G.W. XV Letter 308, p. 79 lines 70-1.

3. G.W. XV Letter 305 (p. 73 lines 32-8) to Maestlin, 14.12.1604. Letter 323 (p. 141 lines 253-5) to Longomontanus, early 1605.

4. Curtis Wilson: Kepler's Derivation of the Elliptical Path, part 2, p. 13, comment (1), lumps the three letters together, neglecting to consider the differing bases used (a repetition of his recurrent mistake noted in Chapter I § 3 (c)).

Fig. (iv) The triangulation diagram (Mars at the quadrant).



We return to the formula found in Chapter I, § 3(c), now applying it to the theoretically determinate defects between either the first or second prototype and 'observations'. As evaluated in Chapter I, § 4, these are:

Distance-divergency $\Delta r = \pm \frac{1}{2} ae^2 \sin^2\beta + O(e^3)$

Appurtenant angular separation $\Delta\theta\ (= -\Delta A^{5.}) = \pm \frac{1}{4} e^2 \sin 2\beta + O(e^3)$.

Then the formula is written:

$$\frac{\Delta r}{r} = (\cot M + \cot T)\,\Delta T - \cot M . \Delta\theta$$

$$\therefore (\cot M + \cot T)\,\Delta T = \frac{\Delta r}{r} + \cot M . \Delta\theta = \tfrac{1}{2} e^2 \sin^2\beta + \cot M . \tfrac{1}{4} e^2 \sin 2\beta ,$$

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5. Since $\angle A = \pi - \theta - \theta_t - I$, where $\angle I$ is supposed known (fixed) and $\Delta\theta_t$ is negligible, just as in Chapter I, § 3.

It has also been remarked previously that there are only a few relative positions of the Earth and Mars for which $\angle M > 45^{\circ}$, so we may take $\cot M \geq 1$ in general, to recognize that the defect ΔT at the Earth is more strongly affected by the angular separation in the first and fourth octants, and beyond, while the distance-divergency only becomes predominant around the quadrant. Thus it is not unless Mars is near—or preferably at—the quadrant that we can hope to determine the visible effect of the distance-divergency free of any contribution from the angular separation (because of the importance of this to my argument that Kepler undoubtedly distinguished the grade of the correct curve at the quadrant, I have already proved numerically—in § 2 above—that $\Delta\theta = O(e^3)$ may be neglected there—and that Kepler knew this). Hence, at the quadrant alone:

$$\Delta T = \angle FTM = \angle F_{\epsilon} TM$$

$$= \frac{1}{\cot M + \cot T} \cdot \tfrac{1}{2}e^2 .$$

Now $\frac{1}{2}e^2 \doteq 15'$, from the Table of Equivalences in Chapter I, § 3(a).

$$\therefore \Delta T = \frac{1}{\cot M + \cot T} \cdot 15' .$$

Thus for the justification of Kepler's own assessment, $\Delta T \geq 10'$, we require:

$$\cot M + \cot T \leq \frac{3}{2} .$$

It has already been demonstrated, in Chapter I, § 3(c), that this is possible around quadratures, but it seems to me desirable to confirm the existence of a reasonably wide range of viewing-positions from which Kepler could have observed $\Delta T \geq 10'$, as he said. We will exhibit this by a geometrical method derived from one used by Kepler himself in Astronomia C. 53, involving the approximation whereby the orbit of the Earth is taken to be a circle centred at the Sun A, so that we set $r_t = a_t = \frac{2}{3}a$.[6] Then we erect a perpendicular $\mu M = \frac{2}{3}a$ to AM and construct the circle with centre μ and radius $\frac{2}{3}a$ to cut the orbit of the Earth (now assumed to be an equal circle) at T_1 and T_2. Thus from § 2,

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6. Only for the purpose of the present geometrical investigations: this approximation is quite inadmissible when we are concerned with evaluating the observational uncertainties.

$$\angle F\mu F\epsilon = \frac{FF\epsilon}{2/3a} = \frac{3}{2}e^2 = 45'$$

$\therefore \angle FT_1F_\epsilon = \angle FT_2F_\epsilon > 20'$ by Euclid III, Prop. XXI

$\therefore \angle FT_1M = \angle F_\epsilon T_1M > 10'$ and similarly for T_2.

Also, the observable discrepancy at every position of the Earth within the arc T_1T_2 of its orbit is greater still (by Euclid I, Prop. XXI)[7].

Now because the circles have been assumed equal, purely as a convenient simplification, we have:

$$A\mu = \sqrt{(AM^2 + \mu M^2)} = \frac{\sqrt{13}}{3}a \quad \text{when } AM=a,\ \mu M=\frac{2}{3}a.$$

$$\therefore \angle T_1A\mu = \angle T_2A\mu = \cos^{-1}\left(\frac{\frac{1}{2}A\mu}{AT_1}\right) = \cos^{-1}\left(\frac{\sqrt{13}}{4}\right) = \cos^{-1}(0.9015)$$

$$> 25^\circ.$$

$$\therefore \angle T_1AT_2 > 50^\circ.$$

Hence there is one sector[8] of the Earth's orbit[9] of at least 50° from which the discrepancy between theory and observations is too high to be disregarded: indeed, it is larger than the 'reformation of astronomy' figure of 8'.[10]

Again because of its importance to my argument, I have, in Appendix II (3b), set out a modern analysis of this theory, in support of an account, in Appendix II (3a), of Kepler's own method of Astronomia C.53—which we have followed above—of determining the optimum viewing-position to maximize the observable discrepancy. Since the Keplerian method is entirely sound by modern standards,[11] we deduce that the minimum value

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7. Both these Euclidean propositions were cited by Kepler in his preliminary discussion of Astronomia C.6.

8. I have not considered the full range of viewing-positions because it has been my intention to demonstrate only that the available range is sufficient.

9. An angle of 20' exactly is subtended in a circle of radius 3/4a, which would make arc T_1T_2 larger: this is worked out by a more accurate method in Appendix II(3c).

10. Astronomia C.19 (G.W. III P.178 lines 1-12).

11. Though only at the quadrant, by reason of the great difficulty of disentangling the effect of the angular separation in other positions, as explained above.

of (cot M+cot T) is $\frac{5}{6}$, to give a maximum for ΔT as high as 18'.

Armed with this new information, Kepler could have obtained the quadrant-distance in the correct curve from a triangulation-calculation with far higher accuracy than he had previously been able to attain. However, I think it much more likely, particularly from the third extract quoted above, that he made the direct assumption that the radius vector of the correct curve itself lay precisely in the middle. This interpretation is to be preferred because it accounts for the exactness of the result stated in C.56 (G.W. III p. 346 lines 7-8).

This achievement firmly refutes Curtis Wilson's opinion[12.] that the distance-determinations were ill suited to lead Kepler to the dimensions of the correct ellipse—though of course I agree that they could never have indicated to him the shape of the curve: that it was an exact ellipse rather than an oviform (in any case I believe that this idea had been at the back of Kepler's mind since C.47). Thus we must be thankful that no doubts as to the efficacy of such a procedure inhibited Kepler from searching out observations which must have seemed to him quite the most obvious ones by which to test the accuracy of the first two prototypes. His instinct, as usual, was sound, and so he was able to deduce with some confidence that the quadrant-distance in the correct curve is the mean distance a.

12. Kepler's Derivation of the Elliptical Path, p. 15 (end of Part 2).

(b) The kinematical method: the diurnal apparent angular motion Astronomia C.42.

For his determination of the quadrant-distance, I nevertheless believe that Kepler used all the information at his disposal. It will be demonstrated in Chapter III, especially § 7, that the defect in the eccentric equation is useless as a criterion to discriminate geometrically between curves (since it can only do so when we already know that the time-measure is right, which is the prior consideration in that case): so, we abandon all attempt to apply that criterion in this geometrical context—the more readily because we are concerned here with variations which Kepler himself could detect, at the level $O(e^2)$, and the eccentric equation is only effective as a criterion at or around the octants at that level: its defect reduces to $O(e^3)$ at the quadrant (this is proved in Chapter III, § 2).

However, investigation of the diurnal apparent motion does provide a kinematical criterion which is effective at the quadrant: indeed, Kepler was only capable of calculating its values there and at the apsides. The complete set of references are listed here, and it is evident that this criterion was on almost every occasion mentioned in close conjunction with that of the defect in the eccentric equation.[13.] Though the latter is to be disregarded, as we have explained, this circumstance may account for the scant attention that the question of the diurnal apparent motion has received from previous commentators, who have treated it simply as a consequence of Kepler's assessment of the eccentric equation: when, in the circular orbit, at the first octant, the planet appears ahead of where it should be,[14.] it must have already travelled too fast at aphelion, and be about to travel too slowly at the quadrant, since the periodic time T is fixed (and the reverse for the Ovoids which are associated with the Auxiliary Ellipse). This common-sense interpretation is, of course, quite correct and I would endorse it myself were it not for the fact that I can justify Kepler's reliance on comparison of the values of the diurnal apparent motion by showing that it operates as an independent criterion.

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13. The references to the eccentric equation have been tabulated in Chapter III, § 7.

14. Compared, as always, with the position given by the Vicarious Hypothesis, or observations.

Tabulation I: Comparison of diurnal apparent motions at the apsides and quadrant

Stage 1: Circle	Stage 2: Auxiliary Ellipse/Ovoid
C.43 p. 283 lines 31-2	C.47 p. 303 lines 8-9
C.44 p. 287 lines 11-13	C.48 p. 310 lines 1-5
†{C.50 'Ovoid' II p. 317 lines 1-3	C.49 p. 313 lines 35-6
†{C.50 'Ovoid' III p. 318 lines 15-16	C.50 Ovoid I p. 316 lines 28-9
	C.50 Ovoid VI p. 321 lines 18-20
Conclusions	Conclusions
{Too fast at apsides	{Too fast at quadrant
{Too slow at quadrant	{Too slow at apsides

† These 'Ovoids' are misplaced because they turn out to be throwbacks to a circular orbit: see Appendix III(5).

As in (a) above, by implication, we confine consideration to the prototypes alone, and introduce the area-hypothesis (of Newtonian form and style) from Chapter III, § 2, to determine the values correct in modern terms, for comparison below:

$$\frac{\delta\theta}{\delta\alpha} = \frac{c_s}{r_s^2} \quad \text{(where } s \equiv c,\ \epsilon \text{ or } o\text{)}$$

$$= \frac{a^2[1-(q_o-\frac{1}{2})e^2]}{a^2[1+2e\cos\beta+e^2-(2q_o-1)e^2\sin^2\beta]} + O(e^4), \text{ from Chapter I, § 4.}$$

$$\therefore \delta\theta = [1-2e\cos\beta-(1-2\sin^2\beta)q_o e^2 - (\tfrac{3}{2}-5\cos^2\beta)e^2]\delta\alpha + O(e^3). \quad 15.$$

Then we will need special notation to distinguish the values at the apsides and quadrants:

Diurnal quadrantal apparent (angular) motion $\underline{\delta\theta m}$

Diurnal $\left\{\begin{matrix}\text{aphelion}\\ \text{perihelion}\end{matrix}\right\}$ apparent (angular) motion $\left\{\begin{matrix}\underline{\delta\theta a}\\ \underline{\delta\theta p}\end{matrix}\right.$

while, as already defined in Chapter I, § 2,

Diurnal mean (angular) motion $\delta\alpha_m = \frac{2\pi}{T} = 31'\,27''$ for Mars. 16.

15. This expression may be checked, after differentiation and substitution, with that given by Whiteside: Keplerian Planetary Eggs p. 18, note 23(b): this inverse form is not otherwise needed for the approach adopted in this study.

16. This is correct to the nearest second: such exactness is not surprising since the periodic time T was by then very accurately known.

Thus we deduce the correct formulae for the diurnal apparent motions in these special cases: the numerical values worked out for each prototype in turn will be found in the tabulation below:

At the apsides,

$$\delta\theta_a = [1-2e + (\tfrac{7}{2}-q_o)e^2]\,\delta\alpha_m$$

$$\delta\theta_p = [1+2e + (\tfrac{7}{2}-q_o)e^2]\,\delta\alpha_m$$

At the quadrant,

$$\delta\theta_m = [1-(\tfrac{3}{2}-q_o)e^2]\,\delta\alpha_m.$$

The facts known to Kepler are set out in C.42, where he stated what will, in Chapter III, §1, be identified as Time-axiom III (inverse):

$$\delta\theta \propto r^{-2} \quad (\delta\alpha \text{ constant}) \text{ (G.W. III p. 281 line 3).}$$

It was in fact wrongly stated as a direct proportion, but it is quite obvious that this was merely a mistranscription, since the figures mentioned (line 4) are clearly in the inverse proportion.[17.] Recognizing, nevertheless, that it is impossible to get quantifiable results from a proportion-statement, I have surmised that Kepler applied it as the exact expression:

$$\delta\theta = \frac{a^2}{r^2}\cdot\delta\alpha.$$

This, of course, is the Newtonian area-hypothesis, specific for the circle, and indeed it would have been perfectly correct to employ it in the context of C.42, at the first stage (though it may be that Kepler happened to formulate the expression merely from consideration of the linear dimensions—of which he had a sound grasp, as we demonstrate throughout this study—by including the right power of the only constant available to him at that stage). However, I suggest that he continued to apply this relationship, without modification, to the noncircular curves he subsequently proposed. At the quadrant, in particular, expressed in modern terms, again from Chapter I, §4, it produces:

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17. This was corrected by Small: The Astronomical Discoveries of Kepler, p. 232, without comment.

$$\bar{\delta\theta}_m = [1-2(1-q_o)e^2]\delta\alpha_m \quad [18.] \quad \text{when } q_o \neq \tfrac{1}{2}.$$

From this, the erroneous Keplerian values listed in the tabulation are calculated.

The values of the apsidal apparent motions are accounted for differently. Those stated by Kepler in C.42 (G.W. III p. 281 line 4) are of extreme accuracy (within 1") for the correct orbit, and I have found the same values for Mars (changed by only 1") listed in Harmonice V C.4, Table V, prefaced by the words: 'We learn from Tycho's astronomy...' (G.W. VI p. 312 line 19). Thus we must conclude—improbable though it may seem—that these figures were genuinely observational, albeit arrived at by averaging the results over many years. Additionally, Kepler could have obtained the modern apsidal values for the circle by (correctly) applying the Newtonian area-hypothesis above; and he could subsequently have inferred that, for the second prototype (because the periodic time is fixed), the motions at the apsides would be too slow, being too fast at the quadrant—quite possibly by the same amount at each stage. Thus this argument, summarized in the following tabulation,[19.] has avoided the disadvantage inherent in the Keplerian formulation—that it gives values of the diurnal apparent motion at the apsides that are identical for every curve.

The figures available were barely sufficient to provide the basis for this deduction: yet, after I had completed it, I found that the identical kinematical relationship had been explicitly quantified—and applied specifically to the mediate ellipse—in the course of the harmonic discussion of Harmonice V. This is to be confirmed in Chapter VII, § 3, but meanwhile the unlikelihood of such a coincidence occurring by chance in two contexts so widely separated has given me reason to believe that I have correctly analyzed Kepler's method, and therefore to regard the Keplerian values listed in the tabulation as well-authenticated. That being so, I point out that the effect of the error involved was on this occasion conducive to Kepler's progress, since it produced a gap between the values for the two prototypes at the quadrant which was twice as wide as it should have been, thereby permitting greater tolerance of inaccuracies in calculation.

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18. Where $\bar{\delta\theta}_m$ denotes an erroneous value of $\delta\theta_m$.

19. In which the Keplerian values underlined are those correct in modern terms.

Tabulation II: The diurnal apparent motion calculated as suggested

	Apsides	Quadrant	
	Aphelion $\delta\theta_a$ / Perihelion $\delta\theta_p$	Keplerian value $\bar{\delta\theta}_m$	Modern value $\delta\theta_m$
First prototype $q_o=\frac{1}{2}$	$(1\pm e)^{-2}\delta\alpha_m = \begin{cases} 26'21'' \\ 38'10'' \end{cases}$	$(1-e^2)\delta\alpha_m = 31'11''$	$(1-e^2)\delta\alpha_m = 31'11''$
Correct curve 'observations'	$\left.\begin{matrix} 26'13'' \\ 38'2'' \end{matrix}\right\}$ from observations† (p. 281 line 4)	$\delta\alpha_m = 31'27''$ (p. 281 line 5)	$(1-\frac{1}{2}e^2)\delta\alpha_m = 31'19''$
Second prototype $q_o=\frac{3}{2}$	$\left.\begin{matrix} 26'5'' \\ 37'54'' \end{matrix}\right\}$ inferred	$(1+e^2)\delta\alpha_m = 31'43''$	$\delta\alpha_m = 31'27''$

† 26'14", 38'1" respectively in Harmonice V C. 4 Table V.

As far as the diurnal quadrantal apparent motion in the correct curve is concerned, Kepler did not attempt to observe it, but instead evidently made the unwarranted assumption that it was to be identified with the diurnal mean motion:

$$\bar{\delta\theta}_m = \delta\alpha_m \quad \text{(p. 281 line 5).}$$

We return to interpret this error geometrically in Chapter VII, §4. Nevertheless the conjunction of errors (this misevaluation together with the faulty kinematical formulation) led him to the result he required:

$$\bar{\delta\theta}_m = \delta\alpha_m \text{ only when } r=a, \text{ precisely.}$$

Thus Kepler could have felt convinced, on these (spurious) theoretical grounds, as well as from the observational considerations set out in (a), that the quadrant-distance in the correct unknown curve is exactly the mean distance a: we shall next see how this crucial value led him to discover a general formula for its radius vector: the Secant Law.

§ 4 Kepler's first flash of enlightenment: the discovery of the provisional Secant Law Astronomia C.56.

Despite the availability—and suitability—of the diametral distance as a formula to give the length of the radius vector in a noncircular curve, foreshadowed in § 1, it is one of the ironies that we find in the course of Kepler's work that when, only one chapter later,[1] he finally decided that the orbit was not a circle, the idea of the diametral distance was allowed to remain in abeyance. Instead, as he wrote in C.45 (G.W. III p. 288 lines 23-8), with scathing hindsight: 'blinded by a craving for results[2] ... I became involved in new labyrinths': this refers to the saga of the Ovoids, which is the subject of Chapter III, § 5—with the exception of C.47, which has supplied us with the grounds for introducing the notional Auxiliary Ellipse. But for this exemption, the work done at the second stage turns out to be absolutely irrelevant to the investigation of the third—on which we now embark—whose roots lie in the first stage.

The situation at the quadrant turns out to be the key to the maze. Thereafter, for the investigation of the typical point, the geometrical correlation with § 1, affirmed by the common diagrams,[3] is achieved by again choosing invariant distantiary angle β_o to function as the eccentric anomaly of the typical point Q on the eccentric circle, with its supplement β_o' determining Q'. Indeed the working will be shown to depend on the geometry of the eccentric circle, involving its optical equation, whose quadrant-value is of particular significance, as anticipated in § 1:

$$\sec \mathrm{AFB} = (1+e^2)^{\frac{1}{2}} \simeq 1+\tfrac{1}{2}e^2 .$$

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1. At the end of C.44 the circle was abandoned for ever as a proposed orbit.
2. cupiditas.
3. G.W. III p. 346 line 9 (the eccentric diagram from C.40: fig. (iib)) and line 12 (the epicycle diagram from C.39: fig. (ib)).

Consequently I take the opportunity to mention a mis-statement made by Kepler at the vital point of C.56, in describing the value of the optical equation as maximum at the quadrant. The inaccuracy is minor,[4] and had no practical effect in the outcome, but I have thought the topic worth brief comment—in Appendix II(4)—partly on account of the importance of the context and partly because it was the recognition of this wrongly-attributed maximum which provided the stimulus for my original enquiry into the basis of Law I, after which one thing led to another.

Kepler announced his breakthrough in C.56 (G.W. III p. 346 lines 3–4): 'quasi e somno experge factus, et novam lucem intuitus, sic coepi rationcinari.' Whiteside[5] has described this flash of enlightenment as 'celebrated, if still inadequately understood', while Dreyer[6] identifies it in effect as the point of discovery of the correct ellipse, as does Koestler[7] —but actually this is premature as I shall explain in § 6 below. For the moment we concentrate on C.56, where Kepler determined a formula for the length of the radius vector: fortunately (for us, since we can thereby investigate each question in turn) he happened to defer consideration of the true anomaly (the specification of the direction of the radius vector) until C.58. Thus the formula found here will turn out to define a curve of the correct grade—the 'via buccosa' group—but not necessarily an exact ellipse.

Kepler's own detailed account of his discovery follows (p. 345 line 25–p. 346 line 13):

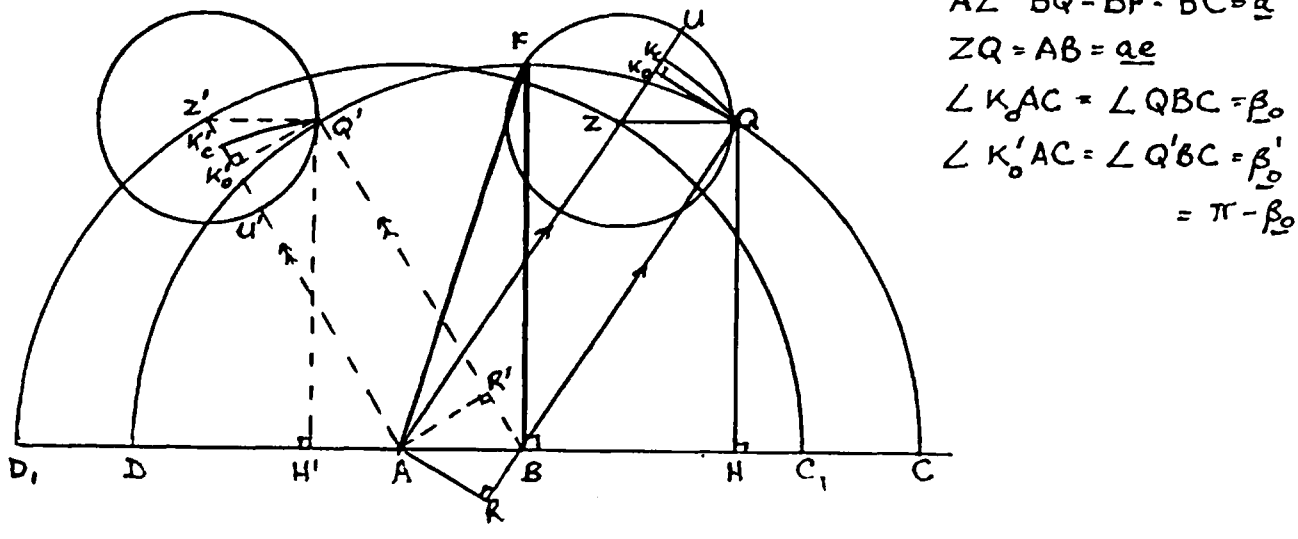
> While I was thinking and worrying...by great good fortune I came across the secant of the angle of 5°18', which is the size of the maximum optical equation...I noticed that it was 1.00429...it is at the quadrants that the optical equation is maximum. At the quadrants the [width of the] lunula (that is, the curtation of the distances) is maximum, and such that its value is the amount by

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4. At the level of $1\frac{1}{2}' \approx \frac{1}{2}e^3$ only.
5. Keplerian Planetary Eggs, p. 17.
6. Planetary Systems, pp. 391-2.
7. The Sleepwalkers, p. 332.

which the secant[8.] of the maximum optical equation 1.00429 a exceeds the radius a. Therefore if the secant [hypotenuse] is replaced by the radius at the quadrant, it will turn out that this is in agreement with the observations. And we may make the same change at other points of the eccentric, replacing AQ by QR etc. ...And equivalently, in the epicycle diagram of C.39 we take AK_o instead of AQ or AK_c, and AK_o' instead of AQ' or AK_c'.[9.]

Fig. (v) Eccentric diagram from C.40 combined with epicycle diagram from C.39.



The basis of this account is obviously the fact that the quadrant-distance for the correct curve was known to be a[10.] (the ways by which Kepler could have come to this conclusion have been detailed in § 3)—and that, therefore, the curtation in radius vector required, by comparison with that of the circle at the quadrant, was of value $\frac{1}{2}ae^2$. (We point out that two distinct values

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8. Strictly speaking, the secant is taken as representing the hypotenuse of ΔAFB here.

9. This equivalence is proved in § 1: Koyré: La révolution astronomique, note 51, p. 425 to p. 256, in denying its existence, is mistaken: he is probably referring to directions, not lengths.

10. This discovery was made before 5.3.1605 since it appears in Kepler's letter to Maestlin of that date: G.W. XV, Letter 335, p. 173, lines 126, 132-5.

of the optical equation at the quadrant were implied in C.56, $5^{\circ}18'$ in the extract quoted, and $5^{\circ}19'$ deduced from the previous paragraph: these arose from the use of different values of $\underline{e}$[11.]—but their effect on the value of $\frac{1}{2}\underline{ae}^2$, now the object of our interest, was altogether negligible and will be ignored.) Then it is my argument that Kepler's flash of enlightenment was sparkled by the recognition that this figure $\frac{1}{2}\underline{ae}^2$ had already cropped up in C.43, as the secant of the optical equation at the quadrant. Thereafter, I believe that Kepler synthesized the result by the following chain of reasoning which can be traced backwards through the diagrams:

Required quadrant-distance (quantified in C.55, p. 345 lines 6-7)

→ its previous appearance in C.43 (p. 284 lines 13-16), in connection with the 'maximum' optical equation

→ <u>eccentric diagram</u> showing 'maximum' and typical optical equation in C.40 (p. 267)

→ earlier diagram showing typical optical equation in C.39 (p. 257)

→ <u>epicycle diagram</u> illustrating libration in C. 39 (p. 257)

The facts involved have already been mentioned individually as part of the build-up to the discovery: the two diagrams specially marked are the pair Kepler actually used to illustrate C.56. It also seems likely that the second link in the chain was reinforced in Kepler's mind by the presence of the 'conchoid' diagram in C.43 (p. 284), which originated in C.40 (p. 268). Moreover, this historical explanation in terms of linked diagrams contributes to the understanding of Kepler's progress because it demonstrates the logical structure of his argument, as well as emphasizing the geometrical basis of his approach.

Two separate formulations of the discovery are to be found in the passage quoted. At the quadrant, Kepler took $\underline{a}$ as the correct radius vector, to replace AF (measured, naturally, from A), where:

$$a = \frac{AF}{\sec AFB} .$$

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11. $e = 0.09265$ gives $\frac{1}{2}e^2 = 0.00429$
$e = 0.09300$ gives $\frac{1}{2}e^2 = 0.00432$
} respectively: see Chapter I, § 3(a).
The latter value of $\underline{e}$ disappears without trace from here on.

Then he took the inductive leap to assume that the radius vector for the correct curve (denoted by $\underline{r_o}$) would be given in general by the expression

$$\underline{r_o = \frac{AQ}{\sec AQB} = QR.}$$

Hence, from the diagram, we obtain the second version in two equivalent ways:

$$r_o = \begin{cases} AK_o = AZ + ZK_o & \text{from C.39} \\ QR = BQ + BR & \text{from C.40} \end{cases}$$

$$\therefore \underline{r_o = a(1+e\cos\beta_o).}$$

The future importance of this relationship justifies me in terming it the Secant Law. This name has been chosen principally as a tribute to Kepler's original formulation, but also for distinctiveness, by contrast with the second version, which happens to be the way the Secant Law is expressed nowadays.[12.] Moreover the latter indicates two methods (independent in origin) for constructing $\underline{r_o}$ in accordance with the Euclidean requirement described in Chapter I, § 5, as a convention which played a fundamental role in Kepler's progress. Indeed it is my opinion that Kepler's recognition of the numerical congruity relating the quadrant-distances and the secant of the 'maximum' optical equation, would have instigated no fresh train of thought if the formula for the correct distance had not been amenable to Euclidean constructibility.

So it was that the diametral distance AK_o cropped up in a new guise—this time identified with the formula which gave the length QR of the radius vector $\underline{r_o}$ of the curve of the correct grade. Kepler immediately realized (G.W. III p. 346 lines 16-18, 20-6) that this involved a geometrical correlation with the libration on the diameter AZU of the epicycle, denoted by $\underline{\lambda}$, where:

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12. However it is pleasant to record that the corresponding relation for the hyperbola: $\underline{r_o = a(e\sec\beta_o - 1)}$ provides the archetype (when the initial direction is towards perihelion, essential for this curve) where β_o is the analogous 'eccentric anomaly.'

$$r_o = AK_o = AU - UK_o = a(1+e) - \lambda$$

$$\therefore \quad \underline{\lambda = ae\,(1-\cos\beta_o).}$$

The amount of libration is unambiguous, and it seems to me that Kepler regarded the diametral distance (or the libration) as supplying independent corroboration of the newly-discovered formula for the radius vector because of their distinct origins: I reiterate that the value of the libration at the quadrant had never been specially investigated. However, no interpretation of the libration in relation to a noncircular curve had yet been attempted, and in fact this was the gist of Kepler's difficulty in C.58, to be discussed in §5 below; at this juncture, Kepler had established only that the point of the curve to be found lay somewhere on the arc whose radius had now been constructed: it is this length AK_o which he described in C.58: 'citra controversiam erit justa distantia' (G.W. III p. 365 lines 9-10). Obviously, the formula for the radius vector in these circumstances is insufficient on its own to define a particular curve—so we shall, until more information becomes available, refer to it as the provisional Secant Law.

Because the interpretation of the libration in this new situation turns out to be purely geometrical in its effect, as we shall see, it will be unnecessary to examine Kepler's extended attempt to account for its formulation, in C.57, which is more appropriately treated in conjunction with certain sections of Epitome, in Chapter VI.[13]. Indeed I suggest that C.57 is out of place in its present position in Astronomia as published, since its length (17 pages) is likely to detract from our appreciation of the logical connection between C.56 and C.58, which I now underline, by omitting all mention of C.57 (both here and in Chapter III), to move at once to consider the problems which beset Kepler in C.58.

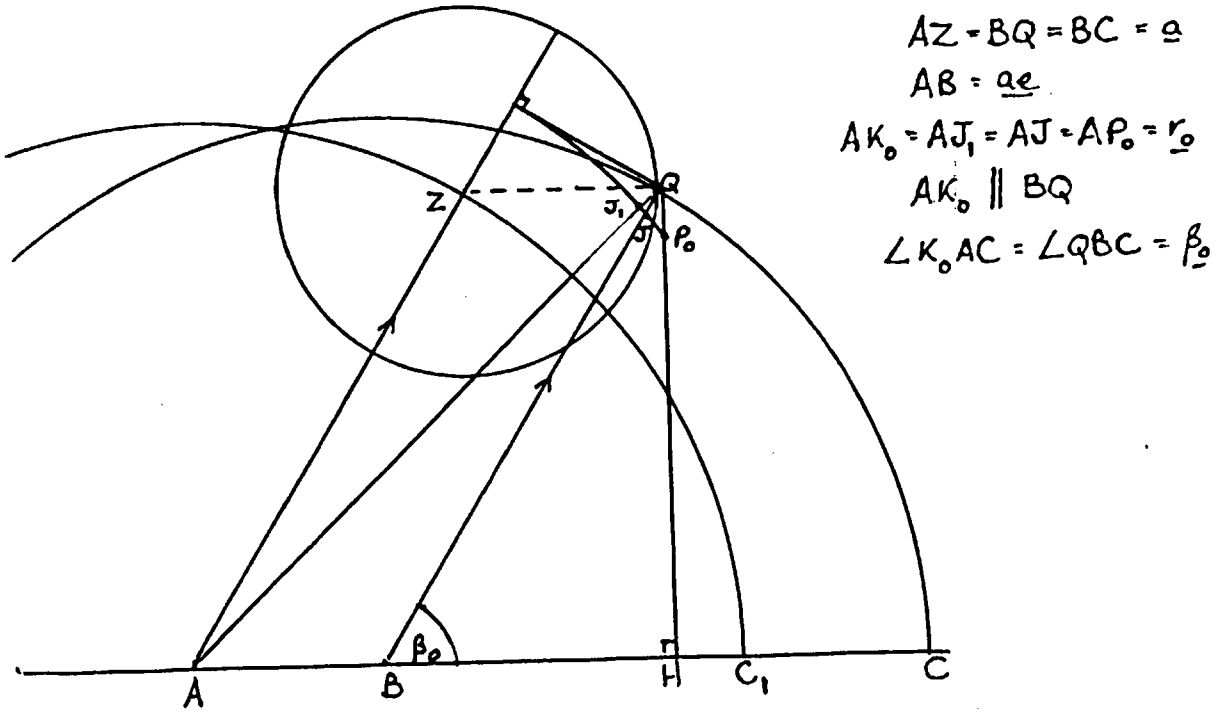
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13. In Chapter VI, §5, we point out that Kepler's views in C.57 were in any case unacceptably inaccurate: this is not unexpected with a preliminary investigation, but they should not be regarded as his final opinions.

§ 5 Kepler's second flash of enlightenment: the discovery of the (specific) Secant Law Astronomia C.58.

In C.58 (G.W. III p. 366 lines 30-3) Kepler vividly described how 'pene usque ad insaniam' he debated with himself the question of the correct choice for the planetary position: he could not decide whether this should be governed by the libration, or whether it should lie on a 'perfect ellipse' (G.W. III p. 366 line 38). The crux of the argument was that both alternatives could be justified by what seemed to him to be compelling reasons, but that these appeared to be contradictory. In order to deal with one thing at a time, we will in this section assume—as will be confirmed in §6—that the point P_o in which the characteristic arc of radius AK_o cuts QH is indeed the typical point of the exact (mediate) ellipse: Kepler must have become aware of this in the course of writing C.58, though he did not prove it until C.59. Thus we will be able to concentrate entirely on how the interpretation of the libration was able to govern the selection of the planetary position.

Fig. (vi) The oviform of C.56 and the oviform of C.58.



The oviform of C.56 has typical point J_1 lying on AQ. The oviform of C.58 has typical point J lying on BQ. The (assumed) exact ellipse has typical point P_o lying on QH. These three possible positions are certainly typical points since they additionally lie on the characteristic arc determined by the provisional Secant Law, whose radius is geometrically constructible. While the equations of the first two curves are of no relevance to the current discussion, it is perhaps of interest to remark that the equations of their radius vectors are identical, at the level $O(e^3)$, since their angular difference is $O(e^3)$, which affects the radius vector only at $O(e^4)$, as explained in Chapter I, §2. It is for the same reason that we are able to confirm that all three curves are members of the <u>via buccosa</u> group, given by $q_o=1$. Neither is it of any significance whatsoever in this present context that the distantiary angle β_o functions (approximately or exactly) as the <u>central anomaly</u> for the first two curves, since the effect of this is to make the representation noncontemporaneous, and thus it is the subject of Chapter III, §8, where this topic will be investigated in modern terms.[1.]

As far as I am aware, the reason for Kepler's conviction—that J was the point governed by the libration—has never been explicitly stated:[2.] yet it is easy to appreciate, by reference to the discussion of §1, especially fig. (ia), that he believed that the libration had to be directed along BQ because it was parallel to AK_o, the diameter of the epicycle. This was sufficient grounds for him to reject J_1 in favour of J—a step that has not previously been accounted for—and so to conduct his argument of C.58 as a straight choice between J and P_o. However, this exposes Kepler's original misunderstanding of the way the libration operated in relation to a noncircular curve: he evidently began by regarding it merely as a geometrical device which would achieve the necessary change from the eccentric to the required (standard-grade) curve. Thus he naturally expected that the symmetry of the libration with respect to supplementary values of eccentric anomaly (established from the considerations listed in C.39 and C.40, as anticipated in §1) would be reproduced in the new

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1. Supported by Appendix III(7), where the polar equation of the oviform of C. 58 is set out.

2. Koyré seems to regard the position J as a gratuitous assumption: see <u>La révolution astronomique</u>, p. 263.

curve, so that the width of the lunula with respect to radii BQ and BQ' of the eccentric would also be symmetrical. It was when he found that this was wrong—QJ <Q'J' in fact (G.W. III p. 366 lines 6-22)—that his ideas were thrown into the disarray already described, until the solution came to him and he castigated his own denseness with the exclamation: 'How stupid of me!.' (p. 366 line 34). I pinpoint this as Kepler's second flash of enlightenment, and identify it as his realization of the physical significance of the libration—the fact that it should be directed precisely towards the Sun. In that case, the planetary position governed by the libration had to possess the symmetry exhibited by the libration itself—and the choice of curve, inevitably, to be an exact ellipse. Indeed this new interpretation of libration—now referred to theplanet itself rather than to the centre of an imaginary epicycle—is of great importance to Kepler's later understanding of the component motions, as we shall see. Meanwhile we anticipate that the angle through which its direction of operation has now been changed:

$$\angle K_0AP_0=\beta_0-\vartheta_0,$$

will crop up again in Chapter VI, § 4, in connection with the angle of deflection of the magnetic fibres which Kepler invented to account for radial motion.

We further mention a minor consideration of symmetry by noting the position of all three possibilities around the quadrant: it is clear that J_1 lies on AF, as described in C.56 (p. 346 line 10), while J and P_0 coincide at F_0 on BF. Thus Kepler must also have rejected J_1 on the grounds of its asymmetry in this special case. (It is perhaps of interest that I have found that the changeover from the oviform of C.56 to the oviform of C.58 occurred in the middle of C.57 (p. 358 lines 10-12) where the quadrant-point is stated to lie on BF.)

In the course of his determination of the asymmetry of the oviform of C.58, as described above—which he in fact showed to be pointed at perihelion[3]—Kepler nevertheless referred to it throughout that paragraph as a

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3. As confirmed in Appendix III(7).

via buccosa. This term designates the shape drawn by Kepler in his letter to Fabricius[4] of 11.10.1605; Kepler described the curve obtained (G.W. XV p. 249 lines 380-5) as outside the ellipse at 'the octants', coinciding with it at the quadrants and (naturally) at the apsides. This seems to be due to a simple preliminary mistake in that Kepler had checked the position at the first octant but had not then done so for the third (an omission which he remedied in Astronomia C.58, as we have noted). However the whole sense of this passage strongly emphasized the criterion of symmetry about the minor axis, and so we may speculate, alternatively, that Kepler was attempting to illustrate a curve which would possess such symmetry, but yet by contrast was not an ellipse. Be that as it may, the description via buccosa—even though misapplied by Kepler to oviform J of C.58, as we have demonstrated—has been chosen, for its distinctiveness, to name the whole group of standard-grade curves which includes the correct curve.

One of my principal purposes in this chapter has been to emphasize the great extent to which Kepler relied on the exclusively geometrical criterion of symmetry. This is not to ignore the fact that the oviform of C.58 has also been correctly classified as a proposed orbit because the first criterion Kepler used to test the viability of J as a possible planetary position was a kinematical one, which involved a misapprehension. The importance of this is recognized, and it is accorded some detailed discussion in Chapter III, § 8: but nevertheless it is my opinion that had Kepler not introduced kinematical considerations he would still have rejected any oviform in favour of an ellipse at this stage; and indeed he would not have laid himself open to modern criticism had he simply stated—as was clearly the case—that the ellipse was his choice for the planetary orbit on the grounds of symmetry, in preference to any other which could apparently satisfy the observations at the stipulated level of accuracy.

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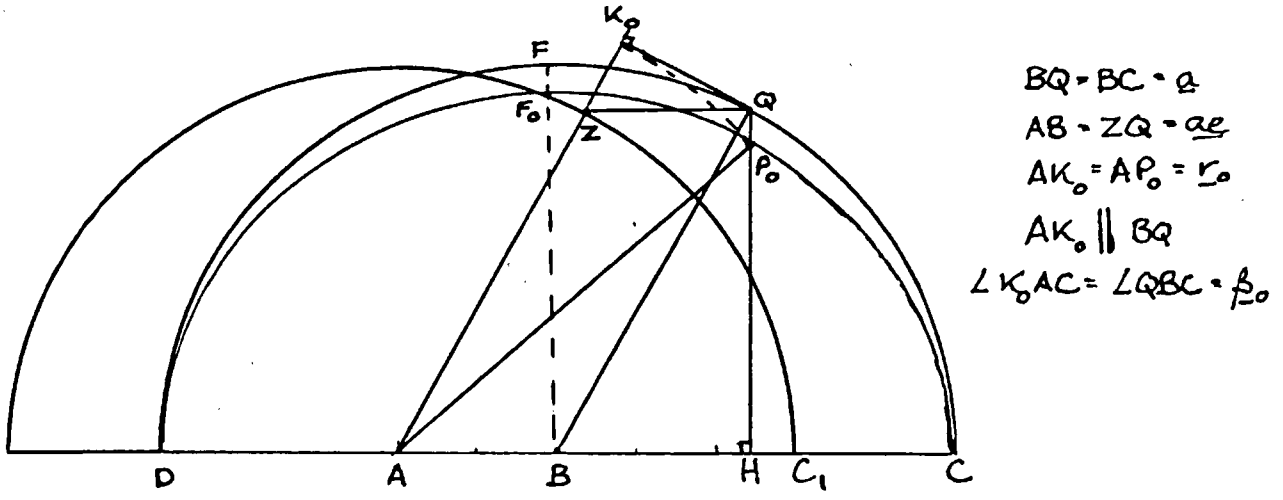
4. G.W. XV, Letter 358, p. 249. This curve is not an oviform as defined in Appendix I (1), which does not include curves possessing a cusp (or a discontinuity): nor was it a proposed curve, in the sense of Chapter I, § 5, since Kepler gave no instructions for constructing it.

As a result of the foregoing discussion we must conclude that both Kepler's twin sparks of illumination were essential in order to create the synthesis: the first produced the multiple-choice situation of the provisional Secant Law, and the second narrowed it down to a specific point. In Astronomia, Kepler continued to refer to the formula as 'the diametral distance': in here renaming the final version simply the Secant Law, we acknowledge that it completely defines a new, noncircular curve, and thus recognize that it represents a giant leap into modern astronomy, from which there can be no return.

§ 6 'Nihil...nisi ellipsis': the geometrical justification for Kepler's choice

Astronomia C.58, C.59, C.60.

Fig. (vii) The exact ellipse.



At this juncture, we have arrived at the geometrical analogue of the situation envisaged by Whiteside[1.] (as discussed in Chapter III, § 8): both depend on the fact that Kepler was incapable of detecting angular discrepancies at $O(e^3)$, and thus could in practice only check his solution against observations correct to $O(e^2)$—this is of course a consequence of the level of accuracy stipulated in Chapter I, § 3. In the present geometrical context, there has as yet been no confirmation that P_0 was the typical point of a 'perfect ellipse'; the point determined by the Secant Law might well have turned out to be a 'highly accurate variant'[2.] which, whether Kepler had recognized it or not, would nonetheless have destroyed the exactness of his synthesis.

It therefore remains to demonstrate unequivocally—as Kepler did in C.59—that there is an exact ellipse whose typical point P_0 possesses a

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1. Keplerian Planetary Eggs, pp. 12-14.
2. Compare the point Q_k, discussed in Chapter III, § 6.

radius vector determined by the Secant Law. As anticipated in Chapter I, § 4, Kepler defined an ellipse by means of the lineal ratio-property:

$$\frac{P_oH}{QH} = \frac{BF_o}{BF} : \text{constant (Protheorema I)}$$

$$\therefore P_oH = BF_o \sin\beta_o.$$

Also, $QH^2 - P_oH^2 = (BF^2 - BF_o^2)\sin^2\beta_o$ (Protheorema VI)

$$= AB^2 \sin^2\beta_o \quad \text{(Protheorema VII)}.$$

Now $AP_o^2 = AH^2 + P_oH^2$

$$= AH^2 + QH^2 - AB^2\sin^2\beta_o$$

$$= a^2[(\cos\beta_o + e)^2 + \sin^2\beta_o - e^2\sin^2\beta_o]$$

$$\therefore r_o = a(1+e\cos\beta_o) \quad \text{(Protheorema XI)}.$$

Thus Kepler confirmed that the radius vector of the exact mediate ellipse is determined by the Secant Law—and thus can be constructed by the use of his characteristic radius-swinging method, in accordance with the Euclidean criterion set out in Chapter I, § 5. This proof indeed constitutes the culmination of Kepler's geometrical knowledge[3] in relation to the ellipse which, he was now satisfied, represented the curve in which the planetary orbit was described: alternatively, he could have obtained the same result more shortly by the converse reasoning:

$$P_oH^2 = AP_o^2 - AH^2$$

$$= a^2[(1+e\cos\beta_0)^2 - (\cos\beta_o + e)^2]^4.$$

$$= a^2(1-e^2)\sin^2\beta_o.$$

$$\therefore \frac{P_oH}{QH} = \sqrt{(1-e^2)} : \text{constant},$$

so that P_o satisfies the lineal ratio-property which defines an exact ellipse.

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3. The rest of the Protheoremata of C.59 are either obviously concerned with kinematical considerations or have kinematical connotations, and are thus the subject of Chapter III.

4. AH was quantified in C. 60 (G.W. III p. 377 lines 22-4).

We emphasize that this curve is (still) to be termed the exact (mediate) ellipse since, even at this stage, Kepler referred to the position of the Sun as the punctum eccentricum. This oddity is the subject of Chapter IV, § 1: nevertheless, for the rest of this section, the ellipse under consideration is, by implication, the unique member of conic-system A.G. II whose pole is at the focus.

We return briefly to the provisional Secant Law:

$$r_o = a(1+e\cos\beta_o)$$

to point out that β_o, on which it depends, functioned as the eccentric anomaly of the typical point Q of the eccentric circle when the formula was originally mooted in C.39 and C.40. However, in relation to the oviforms whose typical points are J_1 and J it possessed no such representation (it functioned as the central anomaly for J, while its connection with J_1 was remote). By contrast, in relation to P_o, which lies on QH by construction, β_o retains its function as the eccentric anomaly of the typical point of the ellipse: this is the updated definition of eccentric anomaly necessitated by Kepler's introduction of noncircular curves (see Chapter I, § 2): his own affirmation of this is to be found in C.60 (G.W. III p. 377 lines 6-9):

> the eccentric anomaly is the curve described by the planet from aphelion,[5] namely arc P_oC of the ellipse, measured by arc QC [of the eccentric circle].

This new formulation of the eccentric anomaly will be adopted as Keplerian hereafter: it enabled him also in C.60, to work out a formula for the corresponding true anomaly (p. 377 lines 13-26):

$$\cos\theta_o = \frac{AH}{AP_o} = \frac{\cos\beta_o + e}{1+e\cos\beta_o} \quad 6.$$

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5. Not apogee: the advantage of the standard notation adopted throughout this study becomes apparent here, since Kepler used two different sets in C.58 and C. 59-60.

6. Other possibilities are mentioned in Appendix IV(1).

This, taken with the formula for r_o (the Secant Law), is nowadays regarded as the simplest available pair of parametric-polar equations for this particular ellipse: yet, one or other of them is actually superfluous—no more than an automatic consequence of the updated definition of eccentric anomaly. It is very simple to verify this fact by considering the construction of the typical point: given eccentric anomaly fixes both QH and also QB ,along which we can determine the length $QR = AK_o$ (as Kepler did in C.56), and then with centre A draw an arc of that radius to cut QH at the (unique) point P_o.

This analysis has been set out in detail, elementary though it be, in order to explain the disparate views of previous commentators. When Dreyer and Koestler stated that the Secant Law of C.56 defined an exact ellipse, as cited in §4 above, they merely assumed that since β_o started as the eccentric anomaly (for the eccentric circle), it retained that function in the case of the point to be determined—a somewhat misleading simplification which however turns out to be absolutely accurate in the end: they have thereby restricted attention to the specific Secant Law alone. On the other hand Curtis Wilson propounds the provisional Secant Law, stating, quite rightly, that the formula $r_o = a(1 + e\cos\beta_o)$ does not have to denote the Sun-planet distances in an ellipse, though it can, and this then involves a 'new geometrical relationship'.[7.] We are left to guess that he refers to some formula for true anomaly (in terms of eccentric anomaly), but he does not mention the uniqueness of the final identification of P_o, which would certainly not be obvious to a non-mathematical, or superficial, reader—and hence, as on other occasions, Kepler has not received the credit which is his due for the exactness he achieved.

Now that the specificity of the Secant Law has been ensured on the basis of eccentric anomaly, we will take this opportunity to abandon dependence on the invariancy of the distantiary angle β_o and the necessity to consider

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7. Kepler's Derivation of the Elliptical Path, pp. 19-20: Curtis Wilson defines the eccentric anomaly for the eccentric circle, but has not updated it for a noncircular curve as the circumstances now require.

the typical point,[8.] to state the general formula for the radius vector at any point:

$$r = a(1 + e\cos\beta),$$

and also to confirm that Kepler was entirely justified in concentrating his efforts, in Epitome, on the proof of this single formula, via the libration with which it is closely connected, as explained in §4:

$$\lambda = ae(1 - \cos\beta).$$

This is the subject of Chapter VI.

8. That is, for the investigation of Epitome and Harmonice: we reintroduce this consideration in Chapter III where we start again from the first stage.

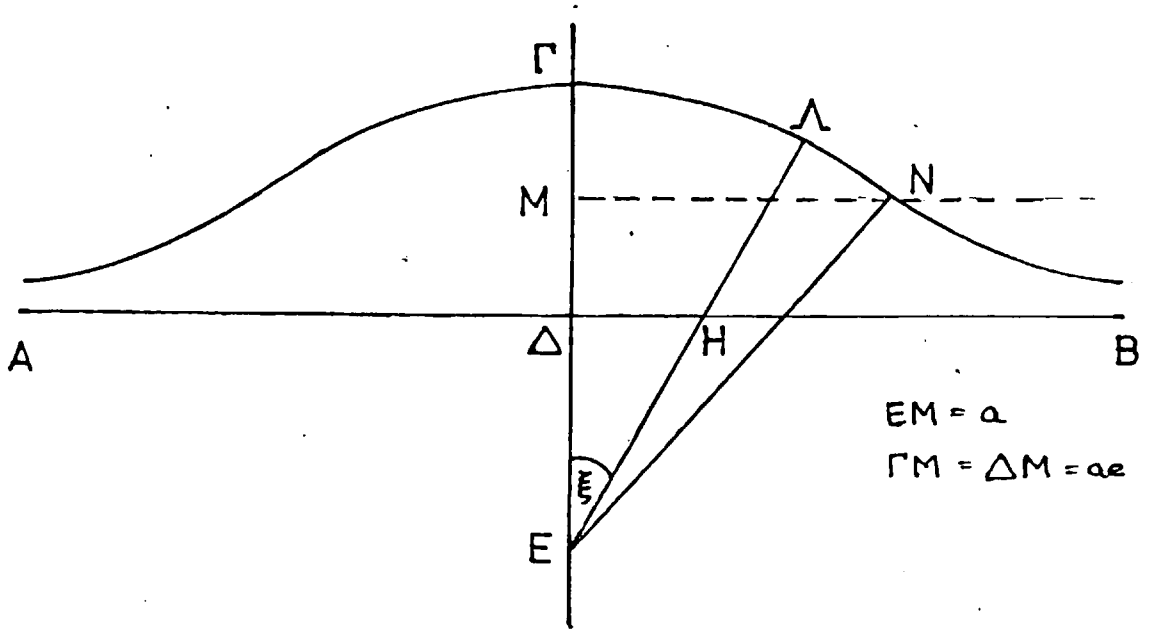
APPENDIX II(1) to §1

KEPLER'S 'CONCHOIDS' ASTRONOMIA C. 40, C.47

The notation in this Appendix is NOT standard

Nicostratus has been identified by Frisch as Nicomedes (born between 280 and 264 B.C.), probably because he was the Greek geometer who invented this curve, according to Heath[1.]—though variations were described later by Pappus, as was the original conchoid (in Greek times also called a cochloid), whose construction is given here for comparison.

Fig. (i) Nicomedes' conchoid.



Let AB and ΓΔE be fixed perpendicular lines, intersecting at Δ: AB is called the ruler, E the pole. The defining-property of the curve is that any radius vector EΛ from the pole, making an angle ξ with EΔΓ, the axis of symmetry, and cutting AB at H, is such that the intercept HΛ between the ruler and the curve, is constant. Then to operate Kepler's analogy, we consider the right-hand half of the curve only, where HΛ=ΔΓ=2ae.

Then the Cartesian equations of the original conchoid are:

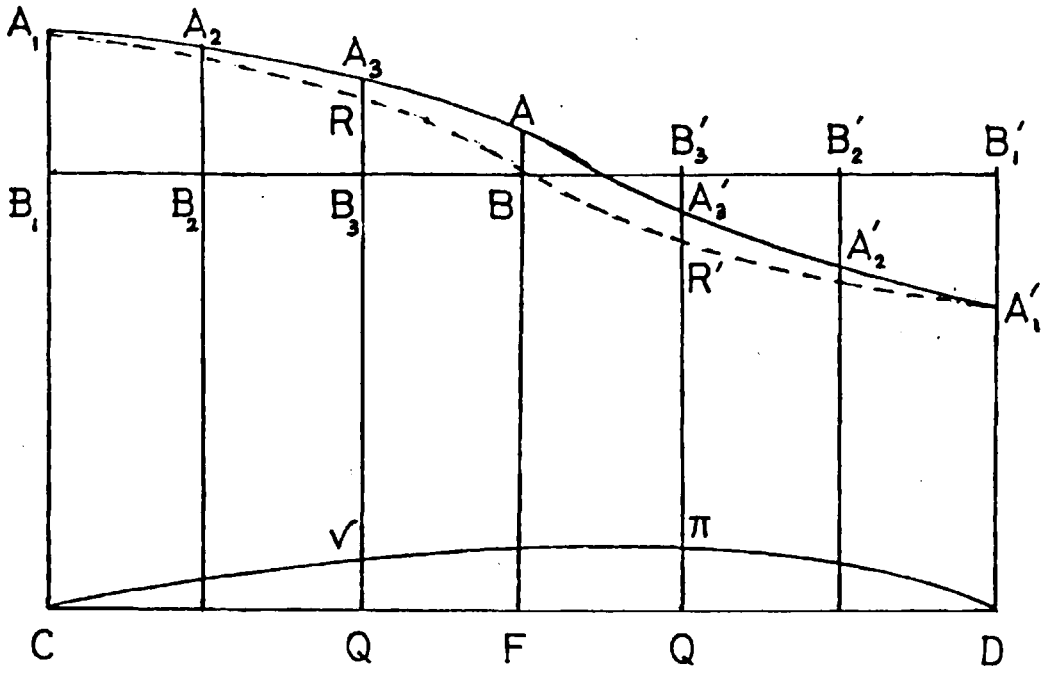
$$x = (a-ae)\tan\xi + 2ae\sin\xi$$

$$y = (a-ae) + 2ae\cos\xi .$$

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1. Greek Mathematics I pp. 238-9.

Fig. (ii) Kepler's 'conchoids'.



These curves were obtained by the unrolling process described in §1, in each case with respect to equal intervals of eccentric anomaly.

Kepler's 'first conchoid'

The ordinate of this curve is the circumferential distance—the radius vector $\underline{r}_c$ of the eccentric circle. Then its Cartesian equations are:

$$x_1 = a\beta$$
$$y_1 = r_c = a(1+e\cos\beta+\tfrac{1}{2}e^2\sin^2\beta-\tfrac{1}{2}e^3\sin^2\beta\cos\beta)+O(e^4).$$

Since a change of sign occurs in the $\underline{e}^3$ term for supplementary values of β, the curve is not symmetrical about BF. Kepler was aware of this and correctly determined that $A_3'R'>A_3R$ (G.W. III p. 269 line 22).

Kepler's 'second conchoid'

The ordinate of this curve is the diametral distance which is eventually identified as the radius vector $\underline{r}_o$ of the exact mediate ellipse. Then its Cartesian equations are:

$$x_2 = a\beta$$
$$y_2 = r_o = a(1+e\cos\beta).$$

This curve is symmetrical about BF.

The kinematical purpose for which Kepler introduced these curves was to represent distance-sums as areas, so that:

$$\text{Area under 'first conchoid'} = \int_o^{\pi} r_c \, d\beta$$

$$\text{Area under 'second conchoid'} = a\int_o^{\pi} r_o \, d\beta.$$

These evaluations will be discussed in Chapter III, § 9, because the latter eventually turns out to be an instance of Time-measure II applied to the mediate ellipse:[2] meanwhile, I shall indicate how he attempted to employ these curves to make further deductions.

We follow Kepler in naming the area between the first and second conchoids the spaciolum (G.W. III p. 269 lines 6-7), Then, by modern methods,

$$\begin{aligned}\text{Spaciolum} &= a\int_o^{\pi}(r_c - r_o)d\beta \\ &= a^2\int_o^{\pi}\tfrac{1}{2}e^2\sin^2\beta\, d\beta \quad \text{from Appendix I(2)} \\ &= \tfrac{1}{4}a^2e^2\int_o^{\pi}(1-\cos 2\beta)d\beta \\ &= \tfrac{1}{4}a^2e^2\pi.\end{aligned}$$

Kepler never managed to evaluate this even approximately—as we mention in passing in Appendix II(2).

The curve C$\nu\pi$D was introduced in C.47 (G.W. III p. 300 lines 2-8), and it seems that its widths $Q\nu$, $Q'\pi$ were intended to represent the differences between the radius vectors of the eccentric circle and those of the curve with which Kepler was at that stage concerned: the Auxiliary Ellipse (which he might have drawn as the 'third conchoid'). With this interpretation, the areola between curve C$\nu\pi$D and CFD (G.W. III p. 300 lines 21-2) would be analogous to the spaciolum in representing the area between the first and absent 'third conchoid'. Hence, similarly,

$$\text{Areola} = a\int_o^{\pi}(r_c - r_e)d\beta = \tfrac{1}{2}a^2e^2\pi.$$

However, Kepler believed that the areola equivalently represented the area of the lunula between the circle and the Auxiliary Ellipse, which he had already quantified as πa^2e^2 (G.W. III p. 299 lines 11-12). This misapprehension amounts to a confusion between the (modified) distance-sum hypothesis he intended to apply, and the area-hypothesis: this is entirely fortunate that (owing to an insufficiency of geometrical information) he was forced to abandon this line of enquiry.

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2. Appendix III(3) is anticipated here.

APPENDIX II(2) to §1.

KEPLER'S TRIGONOMETRICAL SUMMATIONS OF ASTRONOMIA C. 43 and C. 57

In Astronomia C.43 Kepler first summed a series of simple sines, mistakenly believing that this was what he required in order to determine the conchoid-areas (see Appendix II(1)); but in the course of that chapter, he suddenly realized that what he actually needed was the sum of a series of squares of sines. The latter topic has, for clarity, been deferred to subsection (b), while in (a) we have pursued the discussion of the former series by reference to C.57: in order to simplify comparison of the results, all dimensions have been reduced to zero (despite the fact that the enquiry of C.43 arose in connection with areas). Moreover, the summations have been carried out in terms of eccentric anomaly, for consistency, even though summations in terms of true anomaly were also involved in C.57, as confirmed in Chapter VI: indeed, the discussion of Chapter VI, §5- §6, may be regarded, in many ways, as the continuation of subsection (a).

(a) Summation of simple sines of angles in arithmetic progression.

We first perform this summation of sines in the general case by a modern method, to provide the standard of comparison (the regrettable necessity for this will shortly become evident). However, this method (and also that used for the summation of cosines, below) is based on expressions for the cosines (and sines) of the sum and difference of two angles, which are to be found in the work of Ptolemy.[1] Thence we derive the relationship:

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1. Heath: Greek Mathematics II, pp. 278-80, p. 281, shows how Ptolemy obtained two of these expansions: the other two can be worked out similarly, from the same diagrams.

$$2\sin i\delta\beta \sin \tfrac{1}{2}\delta\beta = \cos(i-\tfrac{1}{2})\delta\beta - \cos(i+\tfrac{1}{2})\delta\beta.$$

Then $2\tan\tfrac{1}{2}\delta\beta \sum_{1}^{n} \sin i\delta\beta$

$$= \frac{1}{\cos\frac{1}{2}\delta\beta}\left[2\sin\delta\beta\sin\tfrac{1}{2}\delta\beta + \ldots\ldots + 2\sin n\delta\beta\sin\tfrac{1}{2}\delta\beta\right]$$

$$= \frac{1}{\cos\frac{1}{2}\delta\beta}\left[\cos\tfrac{1}{2}\delta\beta - \cos\tfrac{3}{2}\delta\beta + \ldots\ldots + \cos(n-\tfrac{1}{2})\delta\beta - \cos(n+\tfrac{1}{2})\delta\beta\right]$$

$$= \frac{1}{\cos\frac{1}{2}\delta\beta}\left[(1-\cos n\delta\beta)\cos\tfrac{1}{2}\delta\beta + \sin n\delta\beta\sin\tfrac{1}{2}\delta\beta\right]$$

$$= (1-\cos n\delta\beta) + \sin n\delta\beta\tan\tfrac{1}{2}\delta\beta.$$

Hence, since $\sin 0^{\circ} = 0$, we have:

$$2\tan\tfrac{1}{2}\delta\beta \sum_{0}^{n} \sin i\delta\beta = (1-\cos n\delta\beta) + \sin n\delta\beta.\tan\tfrac{1}{2}\delta\beta.$$

This version is obviously equivalent to the one given, in effect, by Archimedes:[2.]

$$2\sin\tfrac{1}{2}\delta\beta \sum_{0}^{n} \sin i\delta\beta = (1-\cos n\delta\beta)\cos\tfrac{1}{2}\delta\beta + \sin n\delta\beta.\sin\tfrac{1}{2}\delta\beta.$$

Taking particular values of n, and setting $\delta\beta = 1^{\circ}$, we can use either version as a modern check for the formula (or the evaluation) which Kepler himself stated in two special cases.

(1) When $n = 180$. In C.43, Kepler applied 'Cardan's Rule' to find:

$$\sum_{0}^{180} \sin i\delta\beta = \sec 89^{\circ} + \tan 89^{\circ} \quad \text{(G.W. III p. 284 lines 20-9)}$$

$$= \operatorname{cosec} 1^{\circ} + \cot 1^{\circ}$$

$$= \frac{1+\cos 1^{\circ}}{\sin 1^{\circ}}$$

$$= \cot \tfrac{1}{2}^{\circ}$$

$$= 114.589.[3.]$$

---

2. On the Sphere and Cylinder I, Prop. 22: as explained by Heath: Greek Mathematics II, pp. 38-9, p. 41.

3. Kepler gave the figure 114.58869: but we shall correct all such results to an appropriate number of decimal places, anticipating the level of exactitude to be stipulated in Chapter V.

(2) When n = 90. In C.57 Kepler simply stated a numerical value, and it is not possible to discover whether or not he knew this second formula, though I have reconstructed a method by which he would have been capable of deducing it:

$$\sum_{0}^{180} \sin i\delta\beta = \sum_{0}^{90} \sin i\delta\beta + \sum_{90}^{180} \sin i\delta\beta - \sin 90^{\circ} \quad [4.]$$

$$\therefore \sum_{0}^{90} \sin i\delta\beta = \tfrac{1}{2}\left[\sum_{0}^{180} \sin i\delta\beta + \sin 90^{\circ}\right]$$

$$= \tfrac{1}{2}\cot \tfrac{1}{2}^{\circ} + \tfrac{1}{2} \text{ by the first special case}$$

The numerical value is 57.794, not 57.894, as Kepler stated (G.W. III p. 354 lines 7-8).[5.]

Subsequently, in C.57, Kepler tackled similar sums over other ranges—but since he calculated these by performing term-by-term summations, there can be no doubt that he never managed to derive either version of the general formula stated above. We revert to this topic in Chapter VI, §5.

We shall take this opportunity also to work out the general formula for the sum of cosines of angles in arithmetic progression by a similar Ptolemaic-based method, using the relationship:

$$2 \cos i\delta\beta \sin \tfrac{1}{2}\delta\beta = \sin (i+\tfrac{1}{2})\delta\beta - \sin (i-\tfrac{1}{2})\delta\beta.$$

Then
$$2 \tan \tfrac{1}{2}\delta\beta \sum_{1}^{n} \cos i\delta\beta$$

$$= \frac{1}{\cos \frac{1}{2}\delta\beta}\left[\sin \frac{3}{2}\delta\beta - \sin \tfrac{1}{2}\delta\beta + \ldots + \sin (n+\tfrac{1}{2})\delta\beta - \sin (n-\tfrac{1}{2})\delta\beta\right]$$

$$= \frac{1}{\cos \frac{1}{2}\delta\beta}\left[\sin n\delta\beta \cos \tfrac{1}{2}\delta\beta + \cos n\delta\beta \sin \tfrac{1}{2}\delta\beta - \sin \tfrac{1}{2}\delta\beta\right]$$

$$= \sin n\delta\beta + (\cos n\delta\beta - 1)\tan \tfrac{1}{2}\delta\beta.$$

Hence, since $2 \tan \frac{1}{2}\delta\beta \cos 0^{\circ} = 2 \tan \frac{1}{2}\delta\beta$,

$$2 \tan \tfrac{1}{2}\delta\beta \sum_{0}^{n} \cos i\delta\beta = \sin n\delta\beta + (1 + \cos n\delta\beta)\tan \tfrac{1}{2}\delta\beta.$$

---

4. By contrast with the erroneous relationship stated by Frisch, as mentioned below.

5. If Kepler did not know the correct formula, the value 57.894 stated might have been a mistranscription for 57.294 instead: we discuss this possibility further in Chapter VIII §1.

This result is employed in subsection (b) below, while it is of interest subsequently to make the general observation that the summation of cosines at discrete intervals is neither more nor less difficult than the corresponding summation of sines.

* * * * * * *

The inefficiency of previous discussion of Kepler's work, remarked from time to time throughout this study, is nowhere more apparent than in the treatment of this topic by the following commentators:

Frisch: Opera Omnia III (1860), note p. 501 to p. 391

Günther: Über eine merkwürdige Beziehung zwischen Pappus und Kepler (1883), p. 86

Braunmühl: Vorlesungen über Geschichte der Trigonometrie II (1903), p. 23.

Frisch made two errors in his note, stating the general formula incorrectly—in respect of the last term—and also giving the wrong answer for the second special case.[6] These results were reproduced by Günther and reiterated by Braunmühl, while Günther's article provided the basis of a further account:

Cantor: Vorlesungen über Geschichte der Mathematik II (1892), pp. 756-7.

Such errors could have been avoided had any of the commentators mentioned taken the trouble to check these elementary trigonometrical formulae for themselves: they have indeed been corrected in the only sound account of this topic that I have been able to find:

Enestrom (2): Über die angebliche Integration einer trigonometrischen Funktion bei Kepler (1912).

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6. He apparently failed to check the number of terms involved, since:

$$\sum_{0}^{180} \sin i\delta\beta = \sum_{0}^{90} \sin i\delta\beta + \sum_{91}^{180} \sin i\delta\beta \neq 2\sum_{0}^{90} \sin i\delta\beta, \text{ as he stated.}$$

Nevertheless Enestrom's criticism (especially p. 240) is of specific errors and does not extend to pointing out that the further opinions of Günther and Cantor—on the question of whether Kepler's summation could eventually be regarded as tantamount to integration[7]—are of doubtful worth, on the grounds that they themselves could not properly have derived a method of integration on the false foundation of the formula stated or implied in their accounts.

(b) Summation of squares of sines of angles in arithmetic progression.

To perform this summation in the general case by a modern method, we apply a further relationship, involving half-angles, also to be found in the work of Ptolemy:[8]

$$\sin^2 \tfrac{1}{2} i\delta\beta = \tfrac{1}{2}(1 - \cos i\delta\beta).$$

$$\therefore \sum_0^n \sin^2 i\delta\beta = \tfrac{1}{2}\sum_0^n 1 - \tfrac{1}{2}\sum_0^n \cos 2i\delta\beta$$

Then we adapt the general formula for the sum of cosines, found above, to obtain:

$$\sum_0^n \cos 2i\delta\beta = \sin 2n\delta\beta . \tfrac{1}{2}\cot\delta\beta + (1 + \cos 2n\delta\beta)$$

$$\therefore \sum_0^n \sin^2 i\delta\beta = \tfrac{1}{2}(n+1) - \frac{1}{4}\sin 2n\delta\beta \cot\delta\beta - \frac{1}{4}(1 + \cos 2n\delta\beta).$$

However, Kepler only applied this calculation to the whole semicurve, so we simply require the answer in the special case when $n = 180$, taking $\delta\beta = 1^\circ$. Then,

$$\sum_0^{180} \sin^2 i\delta\beta = 90.$$

Hence I cannot account for the figure $\frac{7000}{432} \simeq 16$ which appears in C.43 (p. 285 line 7) to represent Kepler's estimate of this quantity.

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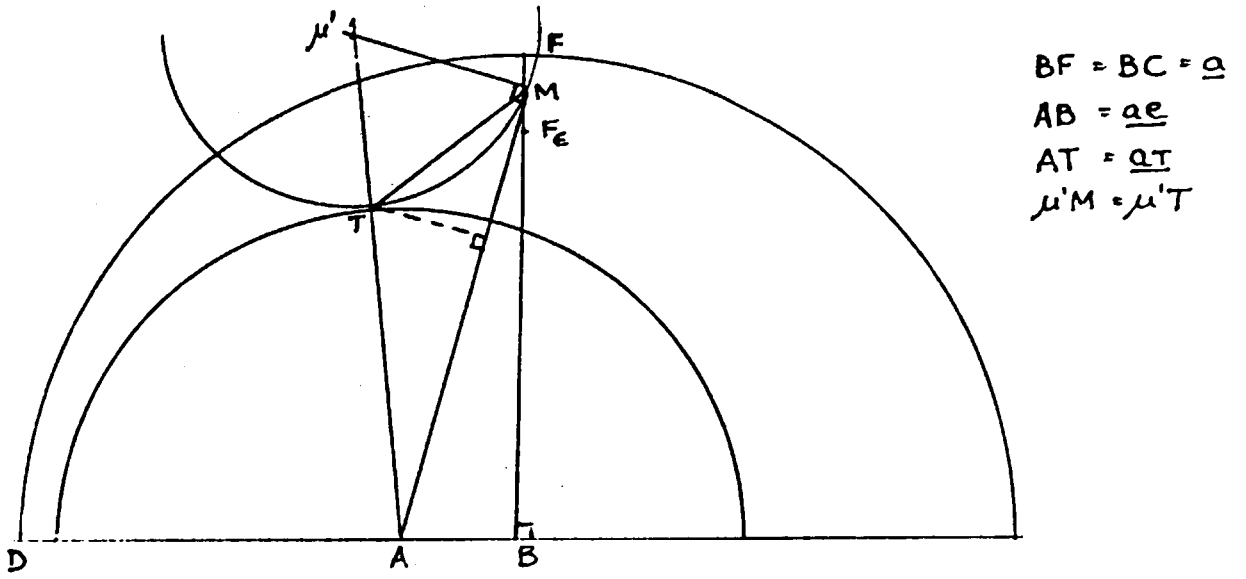
7. The affirmative answer to this question is the consequence of the investigations of Chapter VI, § 5 onwards while further views of commentators on this subject are set out in Appendix VI (3), which constitutes a sequel to the present discussion in this respect.

8. Heath: Greek Mathematics II, p. 280.

APPENDIX II(3) to §3(a)

THE OPTIMUM CHOICE OF VIEWING-POSITION

Fig. (1) The simplified triangulation diagram (Mars at the quadrant).



This diagram is taken from Astronomia C.53 (G.W. III p. 338), but here represents the special case when Mars is at the quadrant: indeed, both methods to be set out below are not valid unless $\Delta\theta$ can be disregarded (around the apsides and the quadrant) as proved in §3(a). The simplification is the same as that adopted later in §3(a), by assuming the Earth's orbit to be centred at the Sun, so that $r_t = a_t = \frac{2}{3}a$.

(a) Kepler's own method of C.53.

Kepler erected a perpendicular $\mu'M$ to AM and then, with centre μ', drew a circle which touched both AM at M and the orbit of the Earth at T (there is a Euclidean construction for μ',[1] but Kepler evidently determined its position by trial and error). This method is only sound when AF and AF_ϵ are also approximately tangential to the circle centre μ'—which entails $\Delta\theta$ negligible, as we have said: though Kepler himself failed to appreciate this. He had constructed this circle much earlier, in Astronomia C.6 (G.W. III p. 93 lines 20–41), where he used the two Euclidean propositions cited in §3(a) to confirm that it identified the optimum viewing-position, by maximizing the discrepancy observable at T. Subsequently, in C.53 (G.W. III p. 338 lines 32–4), he proved the relationship:

$$\frac{TM \sin M}{AT} = \frac{AM - AT \cos A}{AM} \qquad \text{(G.W. III p. 338 lines 24–6).}$$

This is, of course, exact in modern terms, and we should ignore the approximation Kepler unnecessarily introduced. Thence, using the simplification $AT = \frac{2}{3}a$, $AM = a$, it would have been easy for him to deduce that, at the quadrant, in this optimum position:

$$\frac{TM \sin M}{2/3a} = \frac{TM \cos M}{a}$$

$$\therefore \quad \underline{\cot M = \frac{3}{2}.}$$

$$\therefore \quad \sin M = 2/\sqrt{13} \quad \text{and} \quad \cos M = 3/\sqrt{13}.$$

Moreover, other properties of the diagram are:

$$\angle ATM = 180^\circ - \angle \mu' TM = 180^\circ - (90^\circ - M) = 90^\circ + M$$

$$A = 90^\circ - \angle \mu' = 90^\circ - 2M.$$

$$\therefore \quad \cot T = -\tan M = -\frac{2}{3}$$

$$\cot A = \tan 2M = \frac{12}{5}.$$

---

1. By producing $\mu'M$ 'backwards' a distance a_t from M, and thence constructing the isosceles triangle to which $\Delta \mu' MT$ is similar.

(b) Maximization of the observable discrepancy by a modern method.

From §3(a), at the quadrant, we have:

$$\Delta T = \frac{1}{\cot M + \cot T} \cdot \frac{\Delta r}{r}$$

since $\Delta\theta = O(e^3)$ can be disregarded there.

Then with the simplification $a_t = \frac{2}{3}a$, we obtain from ΔATM:

$$TM = \frac{\frac{2}{3}\, a \sin A}{\sin M} = \frac{a \sin A}{\sin T}$$

$$\therefore\ TM^2 = \frac{2}{3}a^2 \sin A \cdot \frac{\sin(M+T)}{\sin M \sin T} = \frac{2}{3}a^2 \sin A(\cot M + \cot T).$$

Also $TM^2 = a^2 + \frac{4}{9}a^2 - \frac{4}{3}a^2 \cos A$

$$\therefore\ \cot M + \cot T = \frac{13 - 12\cos A}{6 \sin A}.$$

$$\text{Now } \frac{d}{dA}\left(\frac{13-12\cos A}{6\sin A}\right) = \frac{12\sin^2 A - \cos A(13 - 12\cos A)}{6\sin^2 A}.$$

It may easily be checked that it is the minimum which occurs when

$$12\sin^2 A - 13\cos A + 12\cos^2 A = O$$

$$\therefore\ \cos A = \frac{12}{13}.$$

Hence $\sin A = \frac{5}{13}$, $TM^2 = \frac{25}{117}a^2$, and we can confirm the values deduced from Kepler's work:

$$\cot A = \frac{12}{5} \quad \therefore\ \angle A = 23^\circ$$

$$\cot M = \frac{3}{2} \quad \therefore\ \angle M = 33\tfrac{1}{2}^\circ$$

$$\cot T = -\frac{2}{3} \quad \therefore\ \angle T = 123\tfrac{1}{2}^\circ.$$

Moreover, the maximum value of $\frac{1}{\cot M + \cot T}$ or $\frac{6 \sin A}{13 - 12 \cos A}$ is $\frac{6}{5}$, and this will of course maximize ΔT, to give, in especially favourable viewing-positions, a discrepancy between theory and observations which can rise as high as $\frac{6}{5}.15' = 18'$.

(c) The range of available viewing-positions.

In § 3(a) we showed that the existence of a satisfactory viewing-position involved the inequality:

$$\cot M + \cot T \leq \frac{3}{2}$$

or, $13 - 12 \cos A \leq 9 \sin A$, by (b) above.

When the equality holds,

$$144 \cos^2 A = 81 \sin^2 A - 2.117 \sin A + 169$$

$$225 \sin^2 A - 2.117 \sin A + 25 = 0$$

$$\sin A = \frac{1}{225} (117 \pm \sqrt{13689 - 5625}) = \frac{1}{225} (117 \pm 89.8)$$

$$= \frac{206.8}{225} \quad \text{or} \quad \frac{27.2}{225}.$$

$\therefore \angle A = 66^\circ 48'$ and $6^\circ 56'$.

These values determine the limits of the range. However, since, as proved in (b) above, $\angle A = 23^\circ$ is the optimum position, it is clear that the range must be:

$$7^\circ \leq \angle A \leq 66^\circ.$$

Thus we have established that there is one sector of the Earth's orbit of almost 60° from which Kepler could have found a discrepancy between theory and observation of at least $10'$, as he stated.

APPENDIX II(4) to §4 and §5

BACKGROUND INFORMATION RELATING TO THE OPTICAL CONCEPTS

I take the opportunity to distinguish the concepts:

Optical equation = $\beta - \theta$

Optical angle $= \gamma - \theta$.

These have been evaluated in Appendices I(2) and I(3) respectively, but theoretical confusion between them can arise because of the considerations:

(1) For the circle, optical equation ≡ optical angle

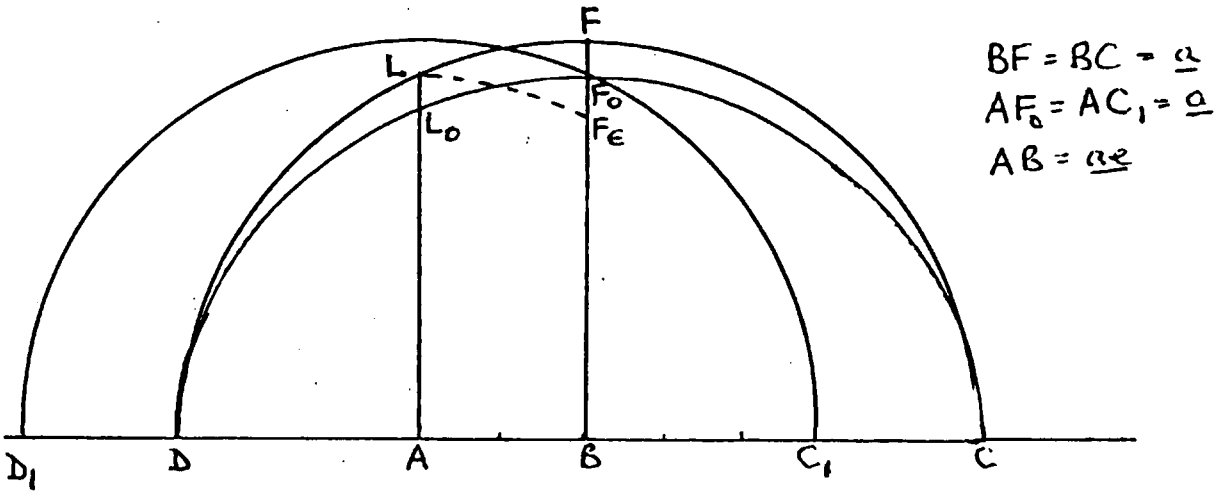
(2) At the quadrant, optical equation = optical angle for any oviform

(3) The optical angle does not vary with the grade of the curve, correct to $O(e^2)$, and so it has the same value as the optical equation for the circle, by (1), for common β.

In what follows it will be sufficient (except in one instance) to restrict attention to the application of these concepts to the correct ellipse—the mediate prototype—and to the circle, for comparison.

(a) Some importance special values.

Fig. (i) The concentric, the eccentric, and the ellipse.



$$AF = a\sqrt{(1+e^2)} = a(1+\tfrac{1}{2}e^2) + O(e^4)$$

$$AF_o = BF = BL = a$$

$$\begin{cases} AF_\epsilon = AL = BF_o = a\sqrt{(1-e^2)} = a(1-\tfrac{1}{2}e^2) + O(e^4) \\ BL_o = a\sqrt{(1-e^2+e^4)} = a(1-\tfrac{1}{2}e^2) + O(e^4) \end{cases}$$

$$\begin{cases} BF_\epsilon = a\sqrt{(1-2e^2)} = a(1-e^2) + O(e^4) \\ AL_o = a(1-e^2), \text{ exactly.} \end{cases}$$

We shall need the positions of the various maxima:

The maximum of the optical equation for the eccentric circle,

In Prolegomenon III it was found that:

$$\angle AQB = \beta - \theta_c = \tan^{-1}\left(\frac{e\sin\beta}{1+e\cos\beta}\right).$$

It is easy to show, by differentiation, that this is maximum at L, where $\cos\beta = -e$, and then its exact maximum value is $\angle ALB = \sin^{-1}e$.

Kepler certainly knew this, as he stated in Astronomia C.29 (on the authority of Ptolemy) and repeated in Epitome V Part II Section 4.[1.] I suggest

1. G.W. III p. 227 lines 38-40 and G.W. VII p. 392 lines 21-2 respectively.

that his apparently unaccountable mispositioning of the maximum at F, in Astronomia C.56, may have been due to his identification of the width of the lunula with the curtation of the radius vector at the instant of discovery (G.W. III p. 346 line 5), leading him temporarily to believe that all the quantities involved were maximum at the quadrant. (Of course, the effect of this mistaken attribution is minimal in practice, since—as indicated in the tabulation below—the values of the optical equations at F and L differ only at $O(e^3)$.)

A similar misattribution of the position of the maximum is noted in connection with the question of the maximum deflection of the fibres which constitute Kepler's magnetic explanation for Law I: this is discussed in Chapter VI, § 4, where reference will be made to this Appendix.

The maximum of the optical equation updated for the ellipse.

$$\frac{d(\beta-\theta_o)}{d\beta} = 1 - \frac{d\theta_o}{d\beta} = 1 - \frac{b}{r} \text{ from Prolegomenon IIA.}$$

Hence $(\beta-\theta_o)$ is maximum when r=b,[2] and then $\sin\theta_o = \sin\beta$, so that β and θ_o are supplementary. It is again easy to show that this maximum falls at the point of the ellipse whose ordinate divides AB in the ratio b:a—that is, slightly nearer L_o than F_o[3]—and that its value is $2\sin^{-1}\left(\frac{ae}{a+b}\right)$ Kepler stated that it fell at the point whose ordinate bisected AB, which is near-enough correct, but this was the result of two mistakes, the subject of (b) below.

The maximum of the optical angle for the ellipse.

There is no exact specification of the position of this maximum, but I have been able to show that it falls at the point whose eccentric anomaly is approximately $\cos^{-1}(-e+e^3)$, so that it lies within arc $F_o L_o$, but very close

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2. Incidentally, this is the mean distance for the ellipse calculated with respect to true anomaly, as listed in the tabulation of Chapter III, § 9.

3. On the diagram, it is represented by the point in which arc LF_ϵ cuts the ellipse.

to L_o, and its value will therefore be slightly greater than $\angle AL_oB = \sin^{-1}\frac{e}{\sqrt{(1-e^2+e^4)}}$: we return to this in (b) below.

Special values stated by Kepler.

Angle	References	Identification
$\angle AFB = \tan^{-1} e \simeq 5^o18'$	Astronomia C.43 (G.W. III p. 283 lines 2-3) Astronomia C.56 (G.W. III p. 346 lines 1-2)	Quadrant-value of the optical equation for the circle
$\angle ALB = \sin^{-1} e$ $[\simeq 5^o19']$	Evaluated as in Astronomia C.29 Epitome V Part II Section 4 (G.W. VII p. 392 lines 21-2)	Maximum optical equation for the circle (evaluated only for the Earth)
$\angle AF_\epsilon B \simeq \tan^{-1}\frac{e}{1-e^2} \simeq 5^o20'$	Astronomia C.47 (G.W. III p. 301 lines 26-7)	Quadrant-value of both the optical equation updated for the second prototype, and the optical angle
$\angle AF_oB = \sin^{-1} e \simeq 5^o19'$	Tabulae Rudolphinae Part II (G.W. X Tables 62-4)	Quadrant-value of both the optical equation updated for the mediate prototype, and the optical angle

For completeness I have tabulated all the special values stated by Kepler: his results are corrected to the nearest minute, because this most clearly exhibits their pattern. Also, it must be borne in mind that these were worked out using different values of e (as listed in Chapter I, § 3(a)).[4.]

4. In particular, in Astronomia C.56, Kepler stated $\angle AFB = 5^o18'$ using $e = 0.09265$, but had he continued to use $e = 0.09300$, as in the passage before, he would have obtained $\angle AFB = 5^o19'$.

(b) Kepler's failure to distinguish the optical equation updated for the ellipse from its optical angle.

Kepler ought to have been capable of distinguishing the two concepts because he clearly differentiated the eccentric anomaly β from the central anomaly γ, by finding a formula which connected the two (we here anticipate Astronomia C.60: the formula is set out, with others relating to the focal geometry of the ellipse, in Appendix IV(1)). Unfortunately, even at the end of his astronomical work, in Epitome V Part II Section 4, though from the context it is clear that he intended the optical equation (updated),[5.] he misrepresented it by $\angle AP_oB$, the optical angle (G.W. VII p. 391 lines 11-12): this undoubtedly occurred because he lacked the technique to represent it other than geometrically; and theupdated optical equation only appears as the angle between BQ and AP_o, whose intersection had not seemed important enough to name.[6.]

Thus, Kepler's attempt in Epitome V Part II Section 4 (G.W. VII p. 392 lines 25-36) to identify the maximum of the updated optical equation was misguided in that he was finding the maximum of the optical angle instead: then he carried this out incorrectly,[7.] and so determined that the maximum would fall midway between F_o and L_o.[8.] However, I point out that this particular piece of erroneous reasoning was almost certainly arranged to give him the answer he expected, for he must have known by that stage that the (actual) updated optical equation was maximum approximately midway between F_o and L_o, as evidenced by the figures given in Tabulae Rudolphinae Part II (G.W. X Table 63) for the true anomalies corresponding to eccentric anomalies ranging from 90^o to 95^o.

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5. As confirmed in Chapter III, §2.
6. We return to this representation in Chapter VI, §4.
7. Of the two pairs of lengths he compared, both differ only at $O(e^4)$, not just one pair, as he stated—so this method would not have been suitable for his purpose in any case.
8. Rather than very near L_o, as established above.

CHAPTER III

THE DISCOVERY OF THE ORBIT

§ 1 Definitions of kinematical terminology: the time-measures

§ 2 The criteria for the assessment of angular defects, derived from the concepts of ancient astronomy

§ 3 Proof of the pairs of time-axioms

§ 4 The crucial innovation in the representation of mean anomaly: the first prototype orbit

§ 5 The second-stage semi-equant hypothesis: the Ovoid group

§ 6 A further breakthrough in the representation of mean anomaly: the second prototype orbit

§ 7 The pattern of Kepler's progress to the third stage

§ 8 Nihil ... nisi ellipsis: the kinematical justification for Kepler's choice

§ 9 Further kinematical considerations

Appendix III(1) The Ptolemaic equant hypothesis

Appendix III(2) The original distance-sum hypothesis, from Time-measure I

Appendix III(3) The modified distance-sum hypothesis, from Time-measure II

Appendix III(4) The area-hypothesis for any oviform, from Time-measure III

Appendix III(5) The Ovoid Group

Appendix III(6) The rogue mean anomaly hypothesis of C.58

Appendix III(7) Three versions of via buccosa W

Appendix III(8) The triplet of equant hypotheses

§ 1 Definitions of kinematical terminology: the time-measures.

In this chapter we will trace Kepler's route to the successful formulation of the time-hypothesis appropriate to a planetary orbit; then—in conjunction with the (mediate) ellipse which was established in Chapter II to be the preferred curve—this will finally synthesize the correct orbit, to complete the proof of both Law I and Law II.

Neither the phrase 'the area hypothesis' nor 'the distance [-sum] hypothesis' will be employed without mathematical quantification in this chapter; both are otherwise of such vagueness as to be useless in any investigation of Law II which purports to be either detailed or thorough.[1.] Indeed, for the method of analysis to be adopted here I am indebted to Whiteside, who has instigated a mathematical treatment of these hypotheses in terms of the coefficients in the polar equation of the oviform, so that each may be consistently applied to every curve proposed: their incompatibilities are thereby so clearly demonstrated that it is obvious that blanket phrases cannot be adequate to distinguish them.

As stated in Chapter I, § 2, I shall retain the Keplerian, rather than the Newtonian, usage in relation to the designation of time, and shall refer hereafter to the mean anomaly hypotheses. In ancient astronomy, it was only feasible to represent the mean anomaly geometrically, by means of an angle; but, from Kepler's work, we find that (three) analytical formulations became possible simply as a consequence of the explanation in terms of physical causes which he was responsible for introducing into astronomy: each of these will be described as a time-measure, and be defined as a quotient of increments. Then, each will be split into a pair of time-axioms,[2.] of direct and inverse form (with α the dependent or

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1. The confusion that can arise from the use of these descriptions will become apparent in the subsequent discussion.

2. I point out that 'time-measures' and 'time-axioms' can be correctly so described because they are proportion-relations in which the changeover from mean anomaly to time (for a single planet) can be achieved merely by adjusting the constant involved, as explained in Chapter I, § 2.

independent variable respectively), in continuance of the contrast between Keplerian and Newtonian usage; while finally the hypotheses, evaluated (by integration, in modern terms, in Appendices III(2), III(3), III(4)) have been set out in both Keplerian and Newtonian style (with eccentric anomaly or true anomaly the respective independent variable) for comparison with those ancient hypotheses which are not derivable from time-measures.

In fact, in the following survey of hypotheses which are relevant to Kepler's work, the only such hypothesis we need seriously to consider is the Ptolemaic equant hypothesis—analyzed in Appendix III(1)—because at every stage it acted as a standard of comparison, to indicate the level of impermissible error, as far as Kepler was concerned.[3.] It is therefore somewhat surprising to find that this hypothesis, in conjunction with the correct (mediate) ellipse, as the simplest possible 'empty focus' hypothesis, was apparently regarded as a suitable approximation as late as 1654[4.]— since, as proved in that Appendix, this combination reintroduced the 'reformation of astronomy' inaccuracy of 8' which Kepler had successfully eliminated at the cost of so much effort several decades earlier. We revert to this in §5(c).

We next return to the consequence of Kepler's fundamental physical principle, taken with the Aristotelian Equivalence, as deduced in Chapter I, §1, but now stated in the Keplerian usage:

$$\text{Motion} \propto \frac{1}{r}.$$ [5.]

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3. At the first and second stages, its error was quantified in C.43 and C.47 respectively (see §4 and §6): it was considered at the third stage—and rejected on conceptual grounds—in Epitome V Part I Section 5 (G.W. VII p. 380 lines 8-30).

4. See Whiteside: Newton's Early Thoughts on Planetary Motion, pp. 121-3, especially note 20 and the first figure on p. 123.

5. As explained above, this proportion-relation remains valid since the change-over to mean anomaly from time merely requires an adjustment of the constant involved.

Expressed in Keplerian form, this is quantified as:

$$\frac{\delta\alpha}{\delta s} = \frac{r}{c_{\mathrm{I}}}: \text{ Time-measure I.}^{6.}$$

Hence we have the original distance-sum hypothesis, evaluated in Appendix III(2):

$$\alpha = \frac{1}{c_{\mathrm{I}}} \int r ds,$$

where $c_{\mathrm{I}} = a^2[1-(q_o-\frac{3}{4})e^2]+O(e^4)$;[7.] each hypothesis will be found to involve such a constant, which is to be termed its dimension-parameter.

Then we follow Kepler's treatment by splitting the time-measure into axioms:

Time-axiom I (direct): $\delta\alpha \propto r$ (δs constant)

Time-axiom I (inverse): $\delta s \propto r^{-1}$ ($\delta\alpha$ constant).

Evidently, Kepler's attempts to apply Time-measure I to noncircular curves were bound to fail because he was quite unable to evaluate arc lengths (even for the mediate ellipse). In practice, therefore, he ensured the equality of the small increments he needed (for the independent variable, in modern terms) by working with respect to intervals of some angle (anomaly) $\delta\theta$ or $\delta\beta$ or $\delta\alpha$: in C.40 (G.W. III p. 263 lines 26-7)—and reiterated from time to time throughout Astronomia, notably in C.57—Kepler defined 'the very smallest parts' as 1^{o}:[8.] in fact their precise size is a matter of indifference in Part A, except in C.32 (the subject of § 3 below),[9.] for we are concerned here to show that the time-measures Kepler

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6. **s** denotes arc length.

7. See Whiteside: Keplerian Planetary Eggs, p. 18, note 23, series (a).

8. Or, uniquely, in C.42, the lesser values corresponding to the diurnal mean motion $\delta\alpha_m = 31'\,27''$: see Chapter II §3(b).

9. It is however of great significance in Part B, where the discussion depends on the level of analytical exactitude implicit throughout Astronomia: see Chapter V, § 3.

actually applied were transmutations of Time-measure I, derived by setting, in turn:

$$\delta s = a\delta\beta, \quad \delta s = r\delta\theta.$$

Both these relationships are valid for a circle, at the apsides alone—which was the situation in which Kepler set out to prove the individual axioms, in C.32: hence, the transmuted time-measures were initially strictly equivalent to Time-measure I—though they took on an independent existence very soon afterwards.

Then first writing $\delta s = a\delta\beta$, we obtain from Time-measure I:

$$\frac{\delta\alpha}{a\delta\beta} = \frac{r}{c_{II}} \quad : \text{ Time-measure II.}$$

Hence we have the modified distance-sum hypothesis, evaluated in Appendix III(3):

$$\alpha = \frac{a}{c_{II}} \int r d\beta,$$

where $c_{II} = a^2[1-\frac{1}{2}(q_o-1)e^2] + O(e^4)$ is the appropriate dimension-parameter: this is expressed in areal dimensions (somewhat artificially) as a consequence of its derivation—but with the advantage that we are thus enabled to compare it with the others.

Once more we split the time-measure into its component axioms:

Time-axiom II (direct) : $\delta\alpha \propto r$ ($\delta\beta$ constant)

Time-axiom II (inverse) : $\delta\beta \propto r^{-1}$ ($\delta\alpha$ constant).

Since $\delta s = a\delta\beta$ at all points of a circle, Time-measure II is evidently identical to Time-measure I at the first stage:[10] we shall suppose that it was the former that was actually applied in calculations, while the latter remained the better expression of Kepler's physical opinions. The derived hypothesis has been so named following Aiton, who described it as 'the modified distance law';[11] however, he quantified it only in relation to the correct mediate ellipse: I have, by contrast, felt compelled to give

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10. When $q_o = \frac{1}{2}$, $c_I = c_{II} = a^2(1+\frac{1}{4}e^2)+O(e^4)$.

11. Kepler's Second Law of Planetary Motion, especially p. 79, p. 86; also Infinitesimals and the Area Law and The Vortex Theory of Planetary Motions.

it the full status of an independent hypothesis on account of its importance at the third stage, and also because it was applied (on one occasion) to a curve at the second stage.

Then we write $\delta s = r\delta\theta$, to derive, likewise from Time-measure I:

$$\frac{\delta\alpha}{\delta\theta} = \frac{r^2}{c_{III}} \quad : \underline{\text{Time-measure III.}}$$

$$\alpha = \frac{1}{c_{III}} \int r^2 d\theta.$$

This will be recognized as the area-hypothesis (Newtonian style). It has been evaluated in Appendix III(4) where we find $c_{III} = a^2[1-(q_o-\frac{1}{2})e^2] + O(e^4)$:[12] this will be termed its area-parameter.

Again we split the time-measure into its component axioms:

Time-axiom III (direct) : $\delta\alpha \propto r^2$ ($\delta\theta$ constant)

Time-axiom III (inverse) : $\delta\theta \propto r^{-2}$ ($\delta\alpha$ constant).

At the end of this section I have tabulated what I hope is a complete list of the axioms enunciated in Astronomia: this is supplementary to a similar tabulation relation to Epitome IV and V and Harmonice V, which is set out in Chapter VII, § 3. Though the present tabulation includes at least one component axiom from each of the three time-measures, it is to be emphasized that no axiom can be applied to give a quantifiable result: for this we require the measures themselves. However, as explained above, Time-measure I cannot be actually applied in calculation; this is borne out by the absence of any numerical value in this case alone among all the measures and hypotheses Kepler tested in the course of Astronomia[13] by evaluating their effect over the whole curve (which amounts to an evaluation of the appropriate dimension-parameters: this comparative tabulation is given in § 9). Thus it is as the source of both Time-measures II and III that Time-measure I possesses practical significance in Kepler's work.

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12. See Whiteside: Keplerian Planetary Eggs, p. 18, note 23, series (b).

13. In addition to the three time-measures, I have found three perverted hypotheses propounded in connection with the Ovoid group.

I have found two occasions on which an axiom was enunciated wrongly.[14] As the fourth axiom at the beginning of C.39 (p. 256 lines 15-17), Kepler said: 'if a particular planet were found at two different distances in turn in the course of one revolution, then the...times would be proportional to the squares of the distances'. This, of course, would have been perfectly correct as a statement of Time-axiom III (direct), except that, instead of 'times', Kepler wrote 'periodic times'—which does not make sense in relation to a single planet.[15] I therefore suggest that he had in mind the two apsidal distances, which occur at each half-period: he certainly stated this axiom correctly later, as cited in the tabulation. Additionally, in C.42 (p. 281 line 3), Kepler said: 'the diurnal [apsidal] apparent motions are proportional to the squares of the distances.' Because this was used correctly (as demonstrated in Chapter II, § 3,) we can be sure that this was a mere mistranscription. Thus neither mistake had any effect on the outcome.

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14. Neither has been noted by Caspar: I have checked, from an original edition of Astronomia, that these were Kepler's own mistakes.

15. Thus I cannot agree with Gingerich: The Origins of Kepler's Third Law, p. 598, that this statement can be relevant to the subject of his article, since Law III deals with the interconnection of a system of planets.

Time-axioms stated in Astronomia. (Valid at all points of the orbit unless otherwise noted.)

Hypothesis	Split into axioms	Reference in G.W. III
Time-measure I $\frac{\delta\alpha}{\delta s} = \frac{r}{c_{I}}$	Direct form: $\delta\alpha \propto r$ (δs constant) Inverse form: $\delta s \propto r^{-1}$ ($\delta\alpha$ constant)	C.46 (p. 292 lines 17-19) C.48 (p. 308 lines 21-2)
Time-measure II $\frac{\delta\alpha}{a\delta\beta} = \frac{r}{c_{II}}$	Direct form: $\delta\alpha \propto r$ ($\delta\beta$ constant) Inverse form: $\delta\beta \propto r^{-1}$ ($\delta\alpha$ constant)	C.33 (p. 236 lines 8-11)† C.40 (p. 263 lines 17-18) C.50 (p. 315 lines 30-1)
Time-measure III $\frac{\delta\alpha}{\delta\theta} = \frac{r^2}{c_{III}}$	Direct form: $\delta\alpha \propto r^2$ ($\delta\theta$ constant) Inverse form: $\delta\theta \propto r^{-2}$ ($\delta\alpha$ constant)	C.39 (p. 256 lines 15-17): apsides†† C.50 (p. 320 lines 38-9) C.42 (p. 281 line 3): apsides and quadrant††

† This axiom was proved in C.32 at the apsides alone, but was generalized in the reference cited.

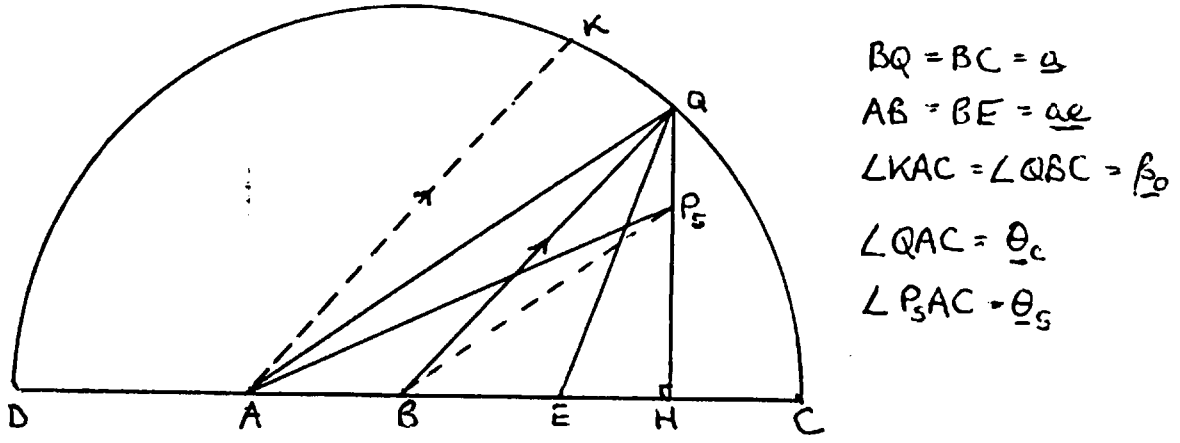
†† These axioms were mis-stated in various ways, as discussed separately.

§ 2 The criteria for the assessment of angular defects, derived from the concepts of ancient astronomy.

In this chapter (as in Chapter II) we will invoke the geometrical frame of reference, set up in Chapter I, § 5, for the consideration of typical points, so that the invariant distantiary angle β_o will function as the eccentric anomaly of the typical point Q of the eccentric circle, and subsequently, as the eccentric anomaly of any point on QH, by the updated Keplerian definition stated in Chapter I, § 2.

In the present section we shall provide the foundation for the investigation of observable angular defects, the subject of § 7 below. Just as when, in Chapter II, up to the end of the second stage, we were concerned with observable divergencies in the distance-determinations, so here, again by reference to the level of accuracy stipulated in Chapter I, § 3, we shall work correct to $O(e^2)$:[1] in practice, we confine consideration to the prototypes alone.

Fig. (i) The Ptolemaic equant hypothesis.



1. Working to a higher level is carried out in Appendix III(4) in order to supply the information for a more detailed assessment at the third stage.

I take the opportunity to set out the three concepts which Kepler took over from ancient astronomy, automatically transposed to heliocentricity, and illustrated in his diagrams of Astronomia C.4 and C.31 (G.W. III p. 73, p. 232):

The eccentric equation is $\angle$ AQE
The physical equation is $\angle$ BQE
The optical equation is $\angle$ AQB.

These representations are the basis of what we have described as the Ptolemaic equant hypothesis, analyzed in Appendix III(1), where their quantifications are to be found. Nevertheless, in re-expressing these concepts in a way which takes account of the innovations that Kepler himself was responsible for introducing into astronomy, we acknowledge his continuing dependence on their fundamental property of interconnectedness:

Eccentric equation = physical equation + optical equation.

Hence we have:

Eccentric equation = mean anomaly - true anomaly
Physical equation = mean anomaly - eccentric anomaly
$\therefore$ Updated optical equation = eccentric anomaly - true anomaly.

This last updated expression is evidently the consequence of Kepler's own updating of the concept of eccentric anomaly in relation to a noncircular curve, stated in Chapter I, § 2. Nevertheless he did not formally define it, even though its numerical evaluation can be immediately worked out from the corresponding eccentric and true anomalies he himself listed: in Astronomia C.47 for the second prototype,[2.] and in Tabulae Rudolphinae for the correct ellipse.[3.] Unfortunately, up to the end of his work, he never appreciated that the updated optical equation for the correct ellipse

2. G.W. III p. 302: see § 6(c) below.
3. G.W. X Part II, Tables 62-4 (for Mars).

is measured geometrically by $\angle$ KAP$_o$, and instead misrepresented it by $\angle$ AP$_o$B, the optical angle.[4.] This latter, derived from the central anomaly (see Appendix I(3)), is an entirely nonKeplerian concept, mentioned only for the avoidance of confusion:[5.] it is to be eliminated from consideration in the present context. By contrast, however, Kepler's discovery of the geometrical representation of the physical equation constituted a cornerstone of the 'new astronomy', as we shall see.

The prototype orbits.

These are the prototype curves to which we now apply the area-hypothesis, Newtonian style, to provide a modern standard of comparison—particularly the mediate ellipse, regarded as equivalent to 'observations'. From Prolegomenon I,

$$\frac{\delta\alpha}{\delta\theta} = \frac{r_s^2}{ab_s} = \frac{r_s^2}{c_s} \qquad \text{where } c_s = a^2[1-(2q_o-1)e^2]^{\frac{1}{2}}.$$

Thence, as in Appendix III(4),

$$\underline{\alpha = \theta_s + 2e\sin\theta_s + \tfrac{1}{2}(q_o+\tfrac{1}{2})e^2\sin 2\theta_s + O(e^3).}$$

Again from Prolegomenon I,

$$\theta_s = \beta_o - e\sin\beta_o + \tfrac{1}{2}(\tfrac{3}{2}-q_o)e^2\sin 2\beta_o + O(e^3).$$

$$\therefore\ \alpha - \theta_s = 2e\sin(\beta_o - e\sin\beta_o) + \tfrac{1}{2}(q_o+\tfrac{1}{2})e^2\sin 2\beta_o + O(e^3)$$

$$= 2e\sin\beta_o - \tfrac{1}{2}(\tfrac{3}{2}-q_o)e^2\sin 2\beta_o + O(e^3).$$

Hence,

The eccentric equation for any prototype:

$$\alpha - \theta_s = 2e\sin\beta_o - \tfrac{1}{2}(\tfrac{3}{2}-q_o)e^2\sin 2\beta_o + O(e^3)$$

And for the mediate prototype,

$$\alpha - \theta_o = 2e\sin\beta_o - \frac{1}{4}e^2\sin 2\beta_o + O(e^3).$$

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4. This is of interest simply in connection with Kepler's magnetic explanation for Law I: see Chapter VI, § 4.

5. See Appendix II(4).

The (updated) optical equation for any prototype;

$$\beta_o - \theta_s = e \sin\beta_o - \tfrac{1}{2}(\tfrac{3}{2} - q_o)e^2 \sin 2\beta_o + O(e^3)$$

And for the mediate prototype,

$$\beta_o - \theta_\sigma = e \sin\beta_o - \tfrac{1}{4}e^2 \sin 2\beta_o + O(e^3).$$

$\therefore$ The physical equation for every prototype:

$$\alpha - \beta_o = e \sin\beta_o .$$

(This is known to be exact, in modern terms alone, from Prolegomenon I.)

Then we shall assess the defects in any proposed curve or mean anomaly hypothesis by means of the following criteria (we shall use $\bar{\alpha}$, as elsewhere, to denote an erroneous value of the mean anomaly):

Criterion I: the defect in optical equation $= (\beta_o - \theta_s) - (\beta_o - \theta_\sigma) = \theta_o - \theta_s$ [6].

$$= \tfrac{1}{2}(q_o - 1)e^2 \sin 2\beta_o + O(e^3).$$

This is obviously geometrical in origin, due to the fact that the proposed curve is of the wrong grade.

Criterion II: the defect in physical equation $= (\bar{\alpha} - \beta_o) - (\alpha - \beta_o) = \bar{\alpha} - \alpha$.

This is purely kinematical, clearly arising from a fault in the mean anomaly hypothesis chosen (it is zero, by definition, whenever the area-hypothesis is applied).

Criterion III: the defect in eccentric equation $= (\bar{\alpha} - \theta_s) - (\alpha - \theta_o) = (\bar{\alpha} - \alpha) + (\theta_o - \theta_s)$.

Thus the defect in the eccentric equation is the sum of the defects in the optical and physical equations.

Now the defect in eccentric equation is the sole observable criterion, so there is no way one can tell which bit is which—how much of the defect is attributable to a fault in the curve and how much to a fault in the hypothesis. I suggest that Kepler was probably aware of this; and we shall in § 7 demonstrate how he solved the difficulty.

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6. This is half the angular separation between first and second prototypes, found in Chapter I, § 4.

§ 3 Proof of the pairs of time-axioms Astronomia C.32.

In Astronomia C.22-28 Kepler had devoted much effort to solving the problem of the motion of the Earth and had finally satisfied himself that it could properly be represented by the Ptolemaic equant hypothesis, which was the theory currently proposed for the rest of the planets (G.W. III p. 233 lines 13-21)—the orbits, at this first stage, being supposed circular. This is the first instance of his 'universalizing' approach (explained elsewhere) which put the Earth on a par with every other planet.

Since Kepler applied Time-measure II (direct form) and Time-measure III (both forms) in the course of Astronomia (references are given in the Tabulation at the end of § 1), I have felt justified in employing an adaptation of his method to prove both pairs of Time-axioms II and III,[1.] and have extended it to include the optical measure, split likewise into direct and inverse axioms—which will not be mentioned again in Part A, but will return to prominence in Chapter VII, § 3: for the present we may regard it simply as a correlating relationship which crops up in the course of the proofs.

The common properties of the three figures are as follows:

The eccentric circle diameter CD
The equant circle diameter C_2D_2 } are both of radius a.

By the Ptolemaic bisection of the (double) eccentricity, AB=BE=ae.

$$\therefore \frac{AC}{AD} = \frac{ED}{EC} = \frac{1+e}{1-e}.$$

For these preliminary proofs Kepler confined consideration to points around the apsides, which he described as arcus minimae (p. 234 line 10). As explained in § 1, we shall interpret his intentions in more concrete terms in relation to increments of angles (it is for this reason that Time-axioms II have been substituted for Time-axioms I as the subject of these proofs),[2.] anticipating the standard subsequently common throughout

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1. The exception, Time-axiom II (inverse), is included anyway because its proof provides the foundation for that of Time-axiom III (inverse).

2. Since we know that Time-measures I and II are identical in the case of the circle, the distinction is purely theoretical.

Astronomia by taking pairs of equal angles (different in each figure) of size 1^{o}: the consequence of this restriction is to ensure that it would have been perfectly proper for Kepler to have assumed, as he required:

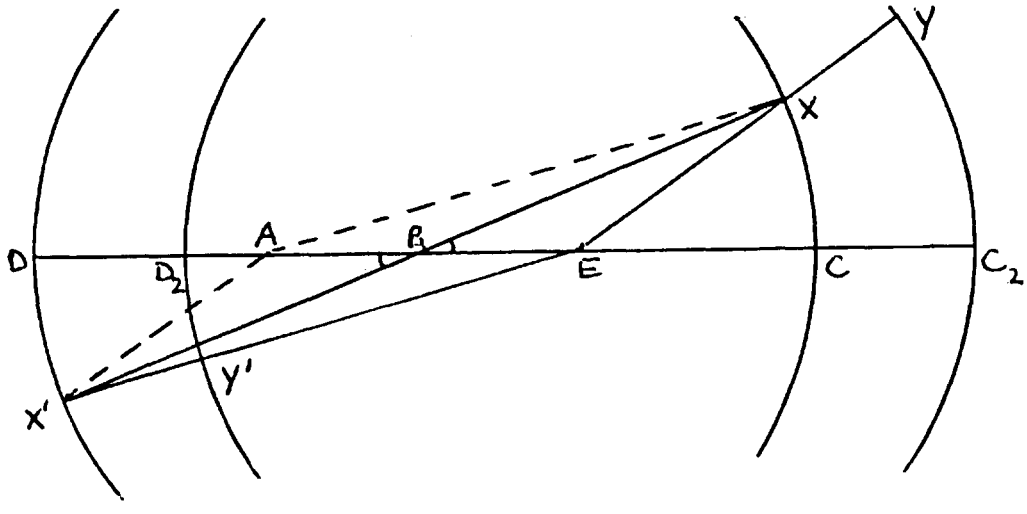
$$\tan \delta\theta \simeq \delta\theta, \ \tan \delta\beta \simeq \delta\beta, \ \tan \delta\alpha \simeq \delta\alpha.$$ [3.]

It is thus clear that Kepler was quite right to treat the circular arcs as straight lines and the angles made by them with the line of apsides as right angles (p. 234 lines 11-12). However I suggest that, at this early stage, he did not feel sufficiently confident that it was also valid to regard the other arcs as circular or—what is the same thing—to assume the pairs of similar triangles:

$$\Delta EC_2Y \text{ and } \Delta ECX$$
$$\Delta ED_2Y' \text{ and } \Delta EDX',$$

as we have done in the present simplification of his working: this has been necessary since Kepler's attempt to compensate for his nonrecognition of the similarity of the triangles involved an approximation which is unacceptably inaccurate, and has rightly been criticized by Small[4.]—who was also the first to point out that the simplified method can be applied to prove more than one of the axioms that Kepler subsequently required.

Fig. (iia)



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3. This is to be confirmed in Chapter V, §3.

4. The Astronomical Discoveries of Kepler pp. 210-13: there is therefore no need for me to discuss it further here.

XBX' is a straight line so that $\delta\beta$ is constant (p. 235 line 8).

$\therefore$ XC = X'D.

Then, $\frac{\delta\theta}{\delta\theta'} = \frac{XC}{AC} \cdot \frac{AD}{X'D} = \frac{AD}{AC}$

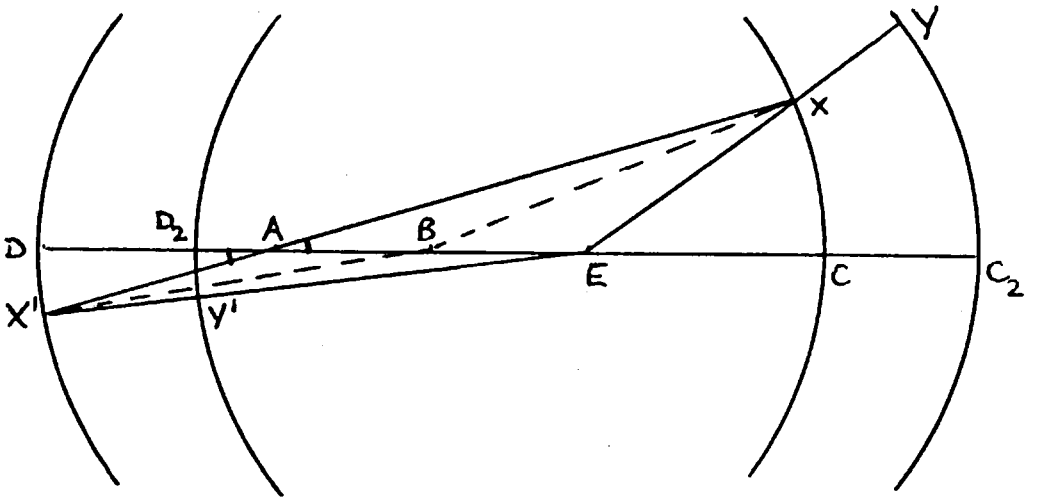
$\therefore$ $\delta\theta \propto r^{-1}$ ($\delta\beta$ constant): Optical axiom (direct).

Also, $\frac{\delta\alpha}{\delta\alpha'} = \frac{YC_2}{Y'D_2} = \frac{XC}{EC} \cdot \frac{ED}{X'D} = \frac{ED}{EC} = \frac{AC}{AD}$ (p. 235 lines 14-16)

$\therefore$ $\delta\alpha \propto r$ ($\delta\beta$ constant): Time-axiom II (direct).

This is the axiom that Kepler himself clearly intended to prove.

Fig. (iib)



XAX' is a straight line so that $\delta\theta$ is constant (p. 234 lines 8-9).[5.]

$\therefore$ $\frac{XC}{X'D} = \frac{AC}{AD}$ (p. 234 lines 12-13)

Then, $\frac{\delta\beta}{\delta\beta'} = \frac{XC}{X'D} = \frac{AC}{AD}$

$\therefore$ $\delta\beta \propto r$ ($\delta\theta$ constant): Optical axiom (inverse).

Also, $\frac{\delta\alpha}{\delta\alpha'} = \frac{YC_2}{Y'D_2} = \frac{XC}{EC} \cdot \frac{ED}{X'D} = \left(\frac{AC}{AD}\right)^2$

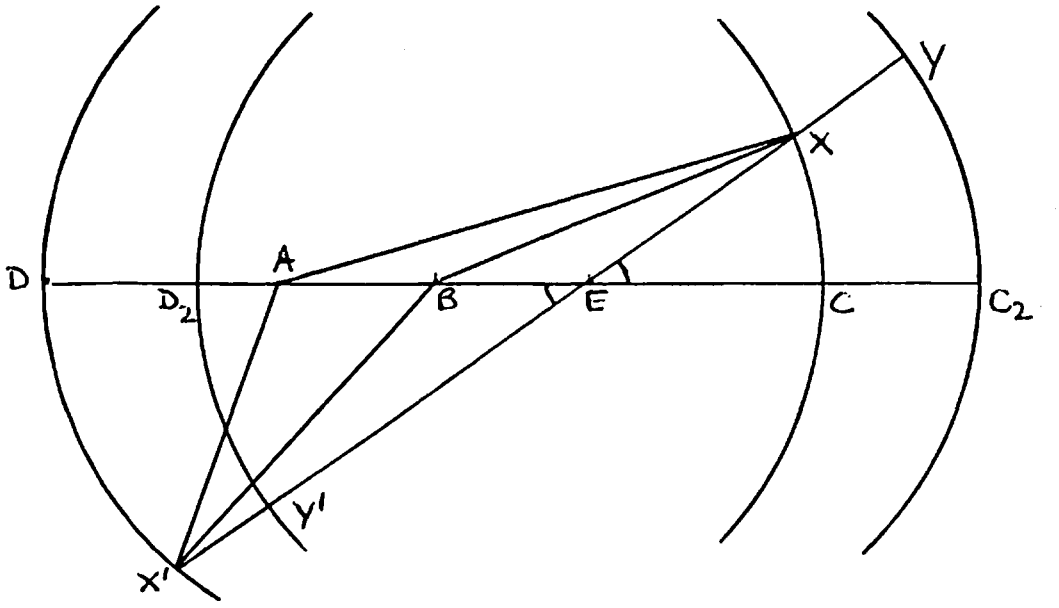
$\therefore$ $\delta\alpha \propto r^2$ ($\delta\theta$ constant):[6.] Time-axiom III (direct).

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5. This was the situation illustrated by Kepler himself (G.W. III p. 235).

6. This was mis-stated as $\delta\alpha \propto r^2$ ($\delta\beta$ constant) (p. 235 lines 10-11).

Fig. (iic)



XEX' is a straight line so that $\delta\alpha$ is constant (p. 234 lines 16-17).

$\therefore$ $YC_2 = Y'D_2$.

Then, $\frac{\delta\beta}{\delta\beta'} = \frac{XC}{X'D} = \frac{EC \cdot YC_2}{ED \cdot Y'D_2} = \frac{EC}{ED} = \frac{AD}{AC}$

$\therefore$ $\delta\beta \propto r^{-1}$ ($\delta\alpha$ constant): Time-axiom II (inverse).

Also, $\frac{\delta\theta}{\delta\theta'} = \frac{XC}{AC} \cdot \frac{AD}{X'D} = \left(\frac{AD}{AC}\right)^2$

$\therefore$ $\delta\theta \propto r^{-2}$ ($\delta\alpha$ constant): Time-axiom III (inverse).

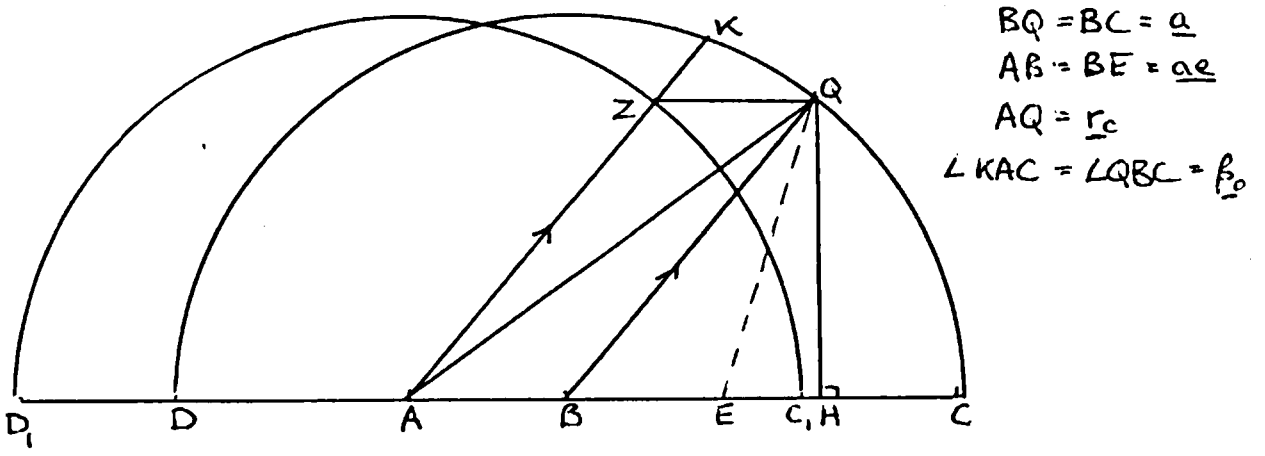
Nevertheless, as pointed out in §1, the axioms cannot be applied to give quantifiable results and therefore what matters is that Kepler here established the truth of Time-measures II and III, specifically for a circular orbit, and only at the apsides. The suggested generalization of Time-measure III to the quadrant of this orbit and thence erroneously[7.] to special points of other proposed orbits has been discussed in Chapter II, §3(b);

7. We recognize that Time-measure III is the area-hypothesis (Newtonian style) but if the area-parameter is not evaluated correctly it will produce an erroneous kinematical formulation.

while its application to intermediate points of a proposed orbit of the Ovoid group is established in § 5 below: our present interest is to remark that Kepler immediately went on (at the beginning of C.33) to generalize Time-measure II (or Time-measure I, to which it is identical) for all points of the circular orbit (p. 236 lines 8-11)—ready for application in C.40, the subject of our next enquiry.

§ 4 The crucial innovation in the representation of mean anomaly: the first prototype orbit Astronomia C.40, C.43.

Fig. (iii) The eccentric circle.



In this theoretical context (as in Chapter II, § 1) the Earth is to be treated as just another planet, and it will again be inappropriate to distinguish it by special notation.[1.] Nevertheless we must recognize, as Kepler stated in C.40 (G.W. III p. 263 lines 12-15), that the assumption of a circular orbit is accurate enough for the Earth (discussed in C.40), but not for Mars (in C.43)—and therefore it is only in the latter case that we shall expect to find defects of observable size.

(a) Introduction of a distance-sum hypothesis C.40.

As we have already pointed out, Time-measures I and II are identical when applied to a circle, and we shall here refer to the latter alone because, in practice, Kepler depended on small intervals of eccentric anomaly (G.W. III p. 263 lines 26-7): though nevertheless we should concede that it must have been Time-measure I, in principle, which acted as the catalyst for the vital reformulation of mean anomaly which immediately followed.

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1. By contrast with the observational context, where such a distinction is vital, as already demonstrated.

Thus he first stated Time-axiom II (direct):

δαcr (δβ constant) (p. 263 lines 17-18),

before applying it to the evaluation of distance-sums, via the measure expressed in modern terms:

$$\alpha \propto a \int r_c d\beta^2.$$

Kepler himself could only tackle this summation by performing 180 individual distance calculations: even he found the method unacceptably tedious and gladly abandoned it when he happened to think of an easier, apparently almost equivalent procedure. However, I suggest that he would probably in any case have rejected this measure on theoretical grounds, for he tested it over the whole curve[3.] and discovered that the total came to more than $\underline{a}^2$ (p. 267 lines 10-19)[4.]—also represented by the area under the first 'conchoid'—by contrast with the area under the second 'conchoid' (both are illustrated in Appendix II (1)), which seems with hindsight to have operated as a standard.[5.] This turns out to have been the way by which he initially checked every proposed hypothesis, as we shall see in § 9, where all such results have been evaluated and tabulated. Thus, Kepler was again ready to embark on something completely new.

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2. Where $\underline{r}_c$ is the circumferential distance as defined in Chapter II, §1.
3. Hence it is only of interest in modern terms to evaluate the result of applying this hypothesis at an intermediate point (the typical point Q), to find, from Appendices III(2) or III(3):

$$\alpha = \beta_o + e \sin\beta_o - \frac{1}{8}e^2 \sin 2\beta_o + O(e^3).$$

4. Reiterated in Protheorema VIII of C.59.
5. The distance-sum represented by the area under the second 'conchoid' is the subject of §9, but we note that both conchoids are constructed with respect to common eccentric anomaly, as confirmed in Chapter II, § 1.

(b) Introduction of the area-hypothesis C.40, C.43.

It was on the authority of Archimedes, as Kepler explained in C.40 (G.W. III p. 264 lines 3-7), that he felt justified in replacing distance-sums by the corresponding areas, thereby obtaining:

$\alpha \propto$ area QAC (p. 265 lines 5-7)

$\therefore \alpha \propto$ sector QBC + $\triangle$AQB (p.265 lines 8-10)

$\therefore \underline{\frac{1}{2}c_u\alpha = \frac{1}{2}a^2\beta_o + \frac{1}{2}a^2 e \sin\beta_o, \text{ where } c_u = a^2.}$

This is the first appearance of the area-hypothesis, Keplerian style: the constant c_u will be named the universal area-parameter;[6.] it has been inserted on Kepler's behalf simply because he lacked the technique to do so—he was always well aware of the necessity to get the (linear) dimensions right.

In C.43 (p. 282 lines 36-7), Kepler stated that this new theory, in its entirety, was to be applied to Mars. This orbit has been named the Contemporaneous Circle, the outer bound of conic-system A.G. II, as set out in Chapter I, § 4: while for Kepler, it constituted the first prototype orbit.

The especial significance of this innovation is that, for the first time, mean anomaly has been expressed in Keplerian form, with eccentric anomaly as the independent variable.

(c) The end of the first stage.

Additional evidence that we have correctly interpreted Kepler's intentions is obtained from his calculations of the mean anomaly for three special values of the eccentric anomaly:

At the quadrant, $\beta = 90^\circ$ gives $\alpha = 95^\circ 18' 28''$ (p. 283 lines 1-9)[7.]

At the octants, $\left.\begin{matrix}\beta = 45^\circ \text{ gives } \alpha = 48^\circ 45' 12'' \\ \beta = 135^\circ \text{ gives } \alpha = 138^\circ 45' 12''\end{matrix}\right\}$ (p. 283 lines 13-18).

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6. For contrast, later, with $c_s = ab_s$, the area-parameter specific to a particular ellipse.

7. This value has already been used in discussion of the quadrant-position in Chapter II, § 2.

It is easy to check that these have been worked out from the general formulation:

$$\tfrac{1}{2}c_u\alpha = \tfrac{1}{2}a^2\beta + \tfrac{1}{2}a^2 e\sin\beta, \text{ where } c_u = a^2.$$

It is with this justification that, in §7 below, we present a modern comparison of the defects in eccentric equation produced by various mean anomaly hypotheses expressed in terms of the distantiary angle β_o, which functions as the eccentric anomaly of the typical point Q of the eccentric circle. This comparison will in fact show that the area-hypothesis applied to the circle is no more accurate than the ancient one—the Ptolemaic equant hypothesis—which it was supposed to supersede (the defects are equal, but of opposite sign). However, it seems that Kepler had by then decided to repudiate a circular orbit on the independent grounds that his exhaustive efforts to find a circle which would pass through more than threeof the best-authticated observed positions were ultimately unsuccessful. Thus at the end of C.44 Kepler finally concluded: 'the orbit of the planet is not a circle but an oval shape'.

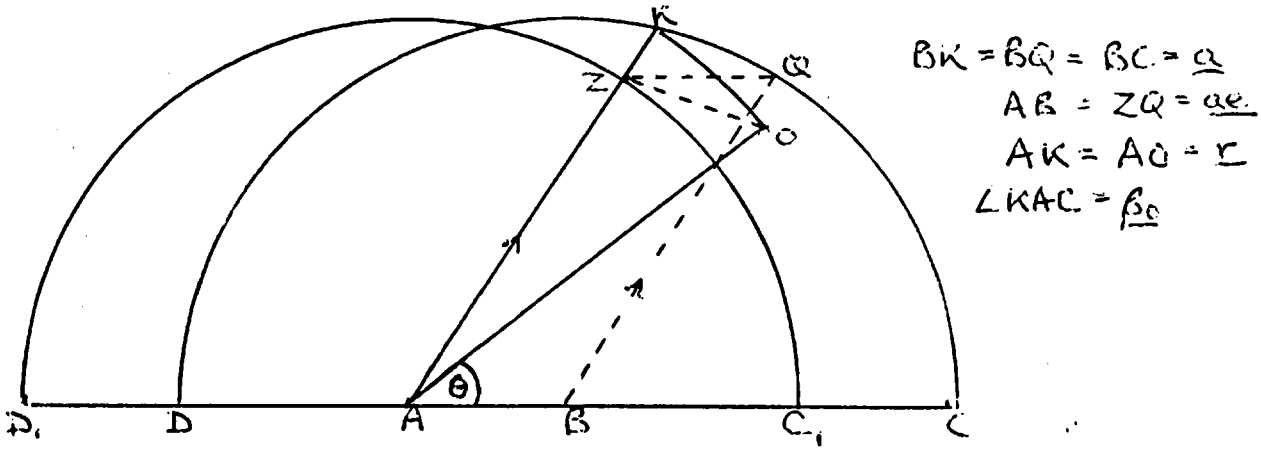
§ 5 The second-stage semi-equant hypothesis: the Ovoid group

Astronomia C.45, C.46, C.48, C.49, C.50.

I have chosen the name Ovoid for these proposed orbits following Kepler's own description in C.46 (G.W. III p. 295 lines 24-5), with his analogy explained in C.40 (G.W. III p. 268, marginal note). They form a distinct subset which is to be treated in isolation, in conjunction with a mean anomaly hypothesis which does not appear elsewhere in Kepler's work; geometrically they are associated with the second prototype (given by $q_o = \frac{3}{2}$). In Appendix III(5) I have enumerated a total of twelve members of this group, of which three ('Ovoids' II, III, IV of C.50) are to be immediately eliminated from consideration because the first two are 'throwbacks' to a circle, while Kepler himself described the third as 'a monstrosity'. We shall first set out the extent of the common structure of the remaining nine ovoids, before investigating the two most interesting members which make up the composite Ovoid V.

(a) The geometrical and kinematical structure of the general Ovoid.

Fig. (iv) The common geometrical structure of every Ovoid.



The circular arc KO—which I have designated Kepler's characteristic arc—appears without exception in all the diagrams of C.46, C.48, C.49, C.50 which illustrate the typical point of an Ovoid, to confirm that, for every member of this group, Kepler partially determined the typical point

by applying his radius-swinging construction.[1] Therefore, we are justified in setting:

$$AK = AO = \underline{r}.$$

However, we make no attempt to determine the point completely, since the second geometrical property required differs from one member to another (as set out in Appendix III (5)).

Now it seems that Kepler intended to achieve cohesion within this isolated group by introducing a common representation of mean anomaly, to be denoted by $\underline{\alpha_k}$: individual references to support this assertion have been cited in Appendix II (5):

$$\underline{\alpha_k = \angle KBC.}$$

This determines the point K on the eccentric—and thus it may be regarded as a matter of convenience to identify:

$$\underline{\angle KAC = \beta_o, \text{ the distantiary angle,}}$$

as Kepler did at the beginning of C.50 (as explained in Chapter I, § 5). Then from $\triangle AKB$ we find, in turn:

$$\underline{r^2 = a^2(1+2e\cos\alpha_k + e^2), \text{ exactly}}$$

$$\therefore r = a(1+e\cos\alpha_k + \tfrac{1}{2}e^2\sin^2\alpha_k - \tfrac{1}{2}e^3\sin^2\alpha_k\cos\alpha_k) + O(e^4).$$

Either version will subsequently be described as $\underline{\text{Relation } A_k}$.

Also, $$\underline{r^2 - 2ae\cos\beta_o - a^2(1-e^2) = O, \text{ exactly}}$$

$$\therefore r = a[e\cos\beta_o + (1-e^2\sin^2\beta_o)^{\frac{1}{2}}], \text{ exactly}$$

$$\therefore r = a(1+e\cos\beta_o - \tfrac{1}{2}e^2\sin^2\beta_o) + O(e^4).$$

Any of these versions will be known as $\underline{\text{Relation } B_k}$: and from Appendix I(2) we check that $q_o = \frac{3}{2}$ for the Ovoid group, as anticipated in Chapter I, § 5.

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1. It is just because it can be shown that construction was not used to produce 'Ovoids' II, III, IV of C.50 that these turned out to be 'throwbacks' to a circular orbit—and as such, have already been eliminated from consideration.

Tabulation of the Ovoid group: a synopsis of the information from Appendix III(5)

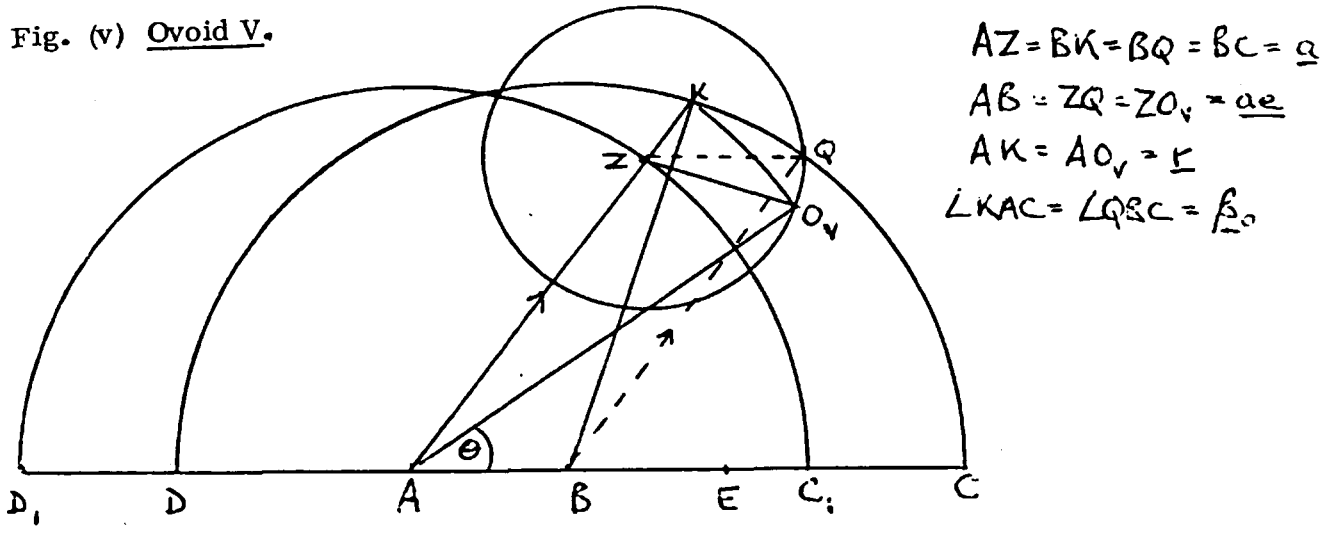
Reference in G.W. III	Second geometrical property	Extra representation of mean anomaly	Polar equation $q_0=\frac{3}{2}$ common	Conclusion
Ovoid I of C.46 pp. 291-3	Undefined	Time-axiom I (inverse) Incompatible	Indeterminate	Geometrically uncertain Kinematically unsound
Ovoid II of C.46 p. 293	On BQ	Rogue mean anomaly hypothesis See Appendix III(6)	$q_1=\frac{3}{2}$	Ovoid-analogue of oviform J
Ovoid III of C.46 pp. 294-5	On AQ_x where $EQ_x \parallel BK$ Vicarious	$\angle Q_xEC=\underline{\alpha}_k$ Compatible at $O(e^2)$	$q_1 \simeq 2$	Physically unsatisfactory
Ovoid IV of C.46 p. 295	On line through $E \parallel BK$ Ptolemaic equant	$\angle OEC=\underline{\alpha}_k$ Compatible at $O(e^2)$	$q_1=\frac{5}{2}$	Ovoid V Correct to $O(e^2)$
Ovoid of C.48 pp. 304-5, p. 308	Undefined	Time-axiom I (inverse) Incompatible	Indeterminate	Geometrically uncertain Kinematically unsound
Ovoid of C.49 pp. 311-13	On epicycle	$\angle KZO_v=\underline{\alpha}_k$	$q_1=\frac{5}{2}$	Ovoid V †
Ovoid of C.50 pp. 315-16	On BQ	Rogue mean anomaly hypothesis See Appendix III(6)	$q_1=\frac{3}{2}$	Ovoid analogue of oviform J
Ovoid V of C.50 pp. 319-20	On AQ_x where $EQ_x \parallel BK$ Vicarious	$\angle Q_xEC=\underline{\alpha}_k$ Compatible at $O(e^2)$	$q_1 \simeq 2$	Physically unsatisfactory
Ovoid VI of C.50 pp. 320-1	On epicycle (by deduction)	Time-measure III Compatible	$q_1=\frac{5}{2}$	Ovoid V

† In identifying the Ovoid of C.49 as Ovoid V we ignore the redundant considerations also mentioned.

In order to check the validity of this pair of relationships in general, it has been necessary to examine every individual Ovoid in modern terms. This working has been set out in Appendix III(5)—in the hope that it will obviate the need for anyone else to reiterate this 'mechanical and tedious' analysis[2.]—and the results alone are stated in the accompanying tabulation, to demonstrate that four Ovoids should at once be disregarded (either because they are indeterminate or because of a previously-unsuspected fault in their mean anomaly).[3.] Of the remaining five, three are inaccurate at $O(e^3)$ and for only two are the above relationships exactly compatible. These two Ovoids make up the composite Ovoid V, the subject of our next enquiry.

(b) The composite Ovoid V: a properly-constituted orbit.

Fig. (v) Ovoid V.



(1) The Ovoid of C.49.

The title to this chapter describes this Ovoid as founded on the principles of C.45,[4.] where Kepler set out his preliminary opinions as to the physical causes which could produce a noncircular orbit: he assumed that the planet moved uniformly in the epicycle:[5.] this, with the application of his characteristic

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2. Kepler's apt phrase, which he used in another context in C.40.

3. The rogue mean anomaly hypothesis implicitly involved is of significance at the third stage, where it gives rise to oviform J, whose ovoid-analogues are treated incidentally, in Appendix III(6).

4. We disregard the extra considerations mentioned in Appendix III(5).

5. In Chapter IV we survey the development of Kepler's physical opinions throughout Astronomia: in fact this is the only Ovoid for which this assumption was explicitly stated.

construction (shown in his diagram of G.W. III p. 311), completely determined the position of the planet, while it also entailed the kinematical relationship:

$\angle KZO_v = \underline{\alpha}_k$ (G.W. III p. 310 lines 21-2).

Then in the triangles AZO_v, AKB we have:

$AZ = BK = \underline{a}$

$ZO_v = AB = \underline{ae}$

$AO_v = AK = \underline{r}$

$\therefore$ ΔAZO_v and ΔAKB are congruent.

$\therefore$ $\angle KZO_v = \angle KBC = \alpha_k$,

and Relation A_k holds, as in (a) above.

Now we invoke the invariant frame of reference to provide the starting-point for Kepler's characteristic construction by setting:

$\angle KAC = \underline{\beta}_o$, the distantiary angle.

$\therefore$ $\angle ZO_vA = \angle KAB = \beta_o$

and also, $\angle O_vAZ = \angle AKB$.

Again from the diagram,

$\angle O_vAZ = \beta_o - \theta$ and $\angle AKB = \alpha_k - \beta_o$

$\therefore$ $\alpha_k - \beta_o = \beta_o - \theta$ [6.]

$\therefore$ $\beta_o = \frac{1}{2}(\alpha_k + \theta)$, exactly.

It is thus confirmed that the introduction of the distiantiary angle reproduces Kepler's own intentions, forhe himself defined it in C.50 (G.W. III p. 315 lines 29-33) as the arithmetic mean of the mean anomaly and the true anomaly—though I have no evidence that he was aware that the relationship was theoretically sound in this case.

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6. It can be checked from § 2 above that the physical and optical equations are equal, correct to $O(e^2)$, for every second-stage orbit: but even in the present case this is inexact at $O(e^3)$, because of the presence of a discrepancy in the eccentric anomaly, as we shall see.

Additionally from the congruent triangles we find:

$$\frac{\sin AKB}{ae} = \frac{\sin\beta_o}{a} = \frac{\sin\alpha_k}{r} = \frac{\sin O_v AZ}{ae}$$

$$\therefore \quad \sin(\alpha_k - \beta_o) = e\sin\beta_o = \frac{ae}{r}\sin\alpha_k = \sin(\beta_o - \theta).$$

Hence, $\sin\alpha_k\cos\beta_o - \cos\alpha_k\sin\beta_o = e\sin\beta_o$

$$\therefore \quad \frac{r}{a}\cos\beta_o = \cos\alpha_k + e.$$

Similarly, $\sin\beta_o\cos\theta - \cos\beta_o\sin\theta = e\sin\beta_o$.

Also from the diagram,

$$r\sin\theta = a\sin\beta_o - ae\sin QZO_v$$

$$= a\sin\beta_o - ae\sin(\alpha_k - \beta_o)$$

$$= a(1-e^2)\sin\beta_o.$$

Substituting, $\frac{a(1-e^2)}{r}\cos\beta_o = \cos\theta - e.$

$$\therefore \quad \frac{a^2}{r^2}(\cos\alpha_k + e) = \frac{\cos\theta - e}{1-e^2}$$

Using Relation A_k,

$$(1-e^2)(\cos\alpha_k + e) = (\cos\theta - e)(1 + 2e\cos\alpha_k + e^2)$$

$$\therefore \quad \cos\alpha_k = \frac{(1+e^2)\cos\theta - 2e}{1-2e\cos\theta + e^2}.$$

Then in substituting in Relation A_k, we obtain:

$$r^2 = a^2(1+e^2) + \frac{2a^2e[(1+e^2)\cos\theta - 2e]}{1-2e\cos\theta+e^2}$$

$$= a^2 \cdot \frac{(1+e^2)^2 - 4e^2}{1-2e\cos\theta+e^2}$$

$$\therefore \quad r^2 = \frac{a^2(1-e^2)^2}{1-2e\cos\theta+e^2} : \quad \underline{\text{the exact polar equation of Ovoid V.}}$$

$$\therefore \quad r = a(1 + e\cos\theta - \tfrac{3}{2}e^2\sin^2\theta - \tfrac{5}{2}e^3\sin^2\theta\cos\theta) + O(e^4),$$

whence we confirm that $q_o = \frac{3}{2}$, while $q_1 = \frac{5}{2}$, to give $2q_o - q_1 - 1 = -\frac{1}{2} < 0$, so that this curve is pointed at aphelion.

(2) Ovoid VI of C.50.

Kepler actually applied an area-hypothesis to the last of his Ovoids,[7] stating it as Time-axiom III (direct) (G.W. III p. 320 lines 38-9):

$$\delta\alpha \propto r^2 \quad (\delta\theta \text{ constant}).$$

In fact, however, it must have been used as a measure in the inverse form since Kepler had previously calculated the squares of the distances in terms of mean anomaly via Relation A_k; thus mean anomaly will here be the independent variable, denoted by α. Then,

$$\theta = c_v \int \frac{1}{r^2}\, d\alpha$$

$$= \frac{c_v}{a^2} \int_0^{\alpha_k} \frac{1}{1+2e\cos\alpha+e^2}\, d\alpha$$

$$= \frac{2c_v}{a^2(1-e^2)} \tan^{-1}\left(\frac{1-e}{1+e}\tan\tfrac{1}{2}\alpha_k\right)$$

The technique by which this integration is performed is set out in textbooks of calculus, but the possibility of deriving this exact result was first noticed by Fladt,[8] as Whiteside has pointed out. Hence,

Area-parameter $c_v = a^2(1-e^2)$ exactly,

and the result may be concisely expressed:

$$\tan\tfrac{1}{2}\theta = \frac{1-e}{1+e}\tan\tfrac{1}{2}\alpha_k.$$

$$\therefore \quad \cos\alpha_k = \frac{1-\tan^2\frac{1}{2}\alpha_k}{1+\tan^2\frac{1}{2}\alpha_k} = \frac{(1-e)^2\cos^2\frac{1}{2}\theta - (1+e)^2\sin^2\frac{1}{2}\theta}{(1-e)^2\cos^2\frac{1}{2}\theta + (1+e)^2\sin^2\frac{1}{2}\theta}$$

$$\therefore \quad \cos\alpha_k = \frac{(1+e^2)\cos\theta - 2e}{1+e^2-2e\cos\theta}.$$

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7. This was not merely an approximation, as Whiteside seems to suggest in his detailed treatment of this Ovoid: Keplerian Planetary Eggs, p. 9.

8. Das Keplersche Ei, p. 74: however, the subsequent analysis is disappointing because Fladt's aim seems to be to associate this orbit with the mediate ellipse: he does not classify it as a small-grade curve.

This is the same value as that obtained from the Ovoid of C.49, and thus the two are coincident, as Kepler believed (p. 321 lines 21-2 and p. 322 lines 13-17)—though he could have had no idea that the agreement would turn out to be theoretically precise—specified by the common exact polar equation:

$$r^2 = \frac{a^2(1-e^2)^2}{1-2e\cos\theta+e^2}.$$

Hence this composite Ovoid V possesses all the properties, both geometrical and kinematical, already found—and in particular, the description of the curve is regulated by an area-hypothesis (of Newtonian style in this case): it is for this reason that we term Ovoid V a <u>properly-constituted</u> orbit.[9.]

Meanwhile, this Keplerian approach has highlighted the <u>noncontemporaneity</u> of Ovoid V in relation to the prototypes, which, though minimal (it is to be quantified in (c) below) is of some significance in principle—which is not so immediately evident from the modern (Newtonian) treatments by Fladt and Whiteside, referred to above.

The eccentric anomaly β_V of this Ovoid has been evaluated, for comparison, in Appendix III(5), to show that its discrepancy is only $O(e^3)$, as anticipated above: even at the quadrant, where the maximum difference ($\frac{1}{2}e^3 \simeq 1\frac{1}{2}'$) between its eccentric anomaly and that of the prototype occurs, it was negligible in Kepler's terms. Nevertheless, we remark in correlation that it is clear from the polar equation that Ovoid V does not pass through F_ϵ and so this curve could not have been employed to determine the quadrant-distance, and hence the dimensions, of the notional Auxiliary Ellipse—the problem discussed (and solved otherwise) in Chapter II, § 2.

(c) <u>The semi-equant hypothesis</u>.

We account for this name by considering one of the two measures of mean anomaly stated for Ovoid V:

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9. For interest, I point out that it is easy to establish, by modern methods, that the central force which will produce this orbit is the combination of a direct-linear force and an inverse-cube force.

$$\alpha_k = \angle KBC$$

or, $\alpha_k \propto$ sector KBC

$\therefore\ \alpha_k \propto$ sector QBC + sector KBQ

$\therefore\ \alpha_k \propto \frac{1}{2}a^2\beta_o + \frac{1}{2}a^2 \sin^{-1}(e \sin\beta_o)$

since $\sin KBQ = \sin ABK = e \sin\beta_o$ from $\triangle AKB$.

Hence when mean anomaly is represented by $\angle KBC$, and if this is supposed to vary uniformly, then this is an equant hypothesis with the equant point at the centre B of the eccentric; on the other hand, when mean anomaly is represented by an area, and the eccentric anomaly is supposed to vary uniformly, this is an area-hypothesis somewhat similar to that stated for the prototypes[10]. (even though the area here is not that actually described by the planet, it is closely connected with it by geometrical construction): the particular interpretation of this (Keplerian style) measure depends on the independent variable adopted, and thus consists of a contrast of Keplerian and Newtonian form.

As always throughout this study, we select the Keplerian form to quantify this measure in terms of invariant β_o:

$$\alpha_k = \beta_o + \sin^{-1}(e \sin\beta_o), \text{ exactly}$$

$$\therefore\ \alpha_k = \beta_o + e \sin\beta_o + \frac{1}{6}e^3 \sin^3\beta_o + O(e^5).$$

Therefore, as anticipated in (b) above, the amount of noncontemporaneity of Ovoid V by comparison with a prototype (of maximum value $\frac{1}{6}e^3 \approx \frac{1}{2}'$) is altogether negligible. Indeed, it appears that Kepler had stumbled on an extremely sophisticated equant hypothesis—both accurate and simple—that ought immediately to have ensured the obsolescence of the Ptolemaic equant theory (whose defect, even when applied to a mediate curve, rises to 8') which was in the event not finally abandoned as a working hypothesis until the middle of that century (see § 1).

However, Kepler entirely failed to appreciate the potential accuracy of this hypothesis because it seemed to him to be inextricably bound up with the

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10. In § 4 for the first prototype, in § 6 following for the second prototype.

physical cause which gave rise to small-grade curves of the second stage (specifically, Ovoid V), expressed via the other measure of mean anomaly:

$$\alpha_k = \angle \mathrm{KZO}_v.$$

Consequently, he also regarded the measure $\alpha_k = \angle$KBC as connected with the determination of the (wrong) distances, as he emphasized in C.47 (p. 301, first marginal note, and lines 4-6, lines 8-9 - in parenthesis).[11.] In § 9 we shall present further evidence of his mistaken assessment of this hypothesis, by showing that he misguidedly tried to 'correct' it.

(d) Assessment of the Ovoid group.

An analysis of all twelve members of this group (including the three subsequently eliminated) had to be undertaken to make sure that nothing of importance in Kepler's work has been missed. A few grains of positive information (among an enormous pile of chaff, as Kepler remarked at the beginning of C.50) relate to the extent of his success in dealing with the time-measures whose application at the third stage is the subject of § 9. In particular it is to be emphasized that Time-axiom II (direct) was stated in C.50 (G.W. III p. 315 line 31, lines 25-6), so confirming the independent existence of the modified distance-sum hypothesis,[12.] as anticipated in § 1.

As far as Time-measure I is concerned, since the arc δs (and indeed, the curve itself) was unknown, Kepler tried to use the inverse axiom: $\delta s \propto r^{-1}$ ($\delta\alpha$ constant), which he twice stated correctly, as cited in the tabulation of § 1. Even so, he could in practice sum distances only at equal intervals of some angle—in this case $\delta\alpha$—and thus he ended up (in

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11. These references to the semi-equant hypothesis are confusingly included in a discussion of the area-hypothesis (see § 6(b)). In fact, they provide additional support for my interpretation of Kepler's attitude to the distantiary angle as 'for the purpose of determining the distances': see Chapter I, § 5.

12. Caspar has drawn attention to this, in his note p. 410 to Neue Astronomie p. 295—though he has denoted the distantiary angle by a separate symbol, whereas I have shown that it is justified to neglect the discrepancy in eccentric anomaly in this connection.

C.49) by working out the mean value of the radius vector with respect to mean anomaly[13.] (which in fact he did not require), as has been pointed out by Caspar.[14.] This difficulty led Kepler to make a nonsensical misstatement of principle—in C.48 (p. 304 lines 33-8)[15.]—which may fortunately be ignored, since he did not attempt to employ it in calculation. Hence it seems probable that Kepler would not have been altogether unwilling to abandon Time-measure I when the time came to apply it to the correct curve (see § 9).

Thus—despite the intrinsic interest of Ovoid V as a properly-constituted orbit—the foregoing considerations alone are relevant at the third stage, for I can find no other information in the Ovoid group which could have contributed to Kepler's progress.[16.] Indeed, the results of my investigation have confirmed that, since the divergency in the radius vector occurred at $O(e^3)$ (which Kepler could sense but not quantify), and because he had no means of recognizing the angular deviations, also at $O(e^3)$, Ovoid V may be treated as identical to the second prototype in Keplerian terms: and in the event all the Ovoids were rejected with it, en masse, for the reasons set out in Chapter II, § 3, supported by the criterion to be discussed in § 7.

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13. Evaluated in the tabulation of § 9—where this relationship is described as Perverted Hypothesis I'.

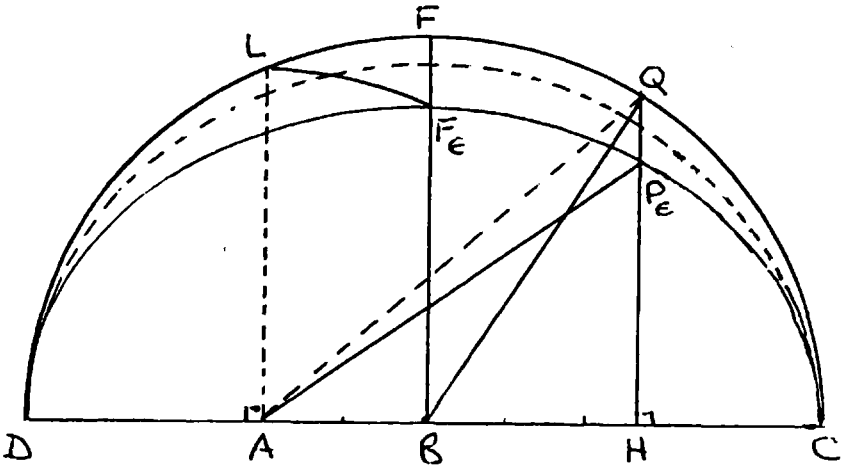
14. Neue Astronomie, note p. 409 to p. 294 (G.W. III p. 313).

15. G.W. III p. 473, note to p. 304.

16. The wide range of assorted hypotheses mentioned is tabulated separately in § 9: they were formulated only to be immediately discarded.

§ 6 A further breakthrough in the representation of mean anomaly: the second prototype orbit Astronomia C.47.

Fig. (vi) The second prototype.



(a) The notional Auxiliary Ellipse.

Kepler introduced the concept of a perfect ellipse in C.47 (G.W. III p. 297 line 6, p. 300 lines 30-1), in order (justifiably, as we have shown in § 5) to replace the vagaries of shape of the various Ovoids by a determinate curve, which we have described as the second prototype, analyzed in modern terms in Chapter I, §4: he confirmed that the effect of such a substitution was negligible:[1.] parum enim differt (p. 297 line 11, p. 306 line 39). We have explained, in Chapter I, § 5, that Kepler could not have managed to construct the typical point of this ellipse by means of his characteristic radius-swinging method—and this may be why he never actually drew it. Nonetheless, he envisaged that a comparable point[2.] would be constructed

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1. We have pointed out there that this amounts to a recognition that differences in shape ($O(e^3)$ at most) are to be disregarded by comparison with the effect of the difference between grades, at $O(e^2)$.

2. The fact that P_ϵ is indeed a typical point, as strictly defined in Chapter I, § 5, can only be established by modern methods.

by applying the lineal ratio-property—by which he defined an ellipse [3.]— to reduce the ordinate determined by the typical point Q on the eccentric circle in a fixed ratio:

$$\frac{P_\epsilon H}{QH} = \frac{b_\epsilon}{a}$$ (p. 300 line 41-p. 301 line 1).

This ensures that the distantiary angle $\angle\beta_0$ will function as the eccentric anomaly of the ellipse point as well (so that we can invoke the geometrical frame of reference) by the updated (Keplerian) definition of eccentric anomaly set out in Chapter I, § 2—which was actually stated in this particular connection (p. 300 lines 39-41).

The fixed ratio has been evaluated, in Chapter II, § 2, by determining the minor semiaxis of this ellipse:

$$BF_\epsilon = b_\epsilon = a(1-2e^2)^{\frac{1}{2}}$$

a matter of some theoretical difficulty—because it was necessary to demonstrate that Kepler was capable of constructing such a length, precisely—but of no consequence in practice, since he used the approximation $b_\epsilon \simeq a(1-e^2)$ in all calculations. However it may be of interest to remark that the characteristic construction by which F_ϵ was determined—using an arc of radius $AL = AF_\epsilon = (1-e^2)$—would, if applied to the construction of a non-quadrantal point, have produced yet another Ovoid, whose typical point is O_k:[4.] this has not been followed up because it does not seem to contribute to our understanding of Kepler's progress.

(b) Return to the area-hypothesis.

As earlier, in C.40 (see §4(b)), Kepler once more referred to the authority of Archimedes by citing the areal ratio-property proved by him for any ellipse:[5.]

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3. Restated as Protheorema I of C.59—for the mediate ellipse.

4. As illustrated in Chapter I, §5: we now know that the typical point of every Ovoid lies on the characteristic arc KO_k which acts as the inner bound of the Keplerian range.

5. On Conoids and Spheroids Props 4-6: see Heath: Greek Mathematics II pp. 57-8: restated—for the mediate ellipse—as Protheorema II of C.59.

$$\frac{\text{Segment } P_\epsilon HC}{\text{Segment } QHC} = \frac{b_\epsilon}{a} \quad \text{(p. 301 line 2)},$$

while also,

$$\frac{\Delta P_\epsilon AH}{\Delta QAH} = \frac{b_\epsilon}{a} \quad \text{(p. 301 lines 2-3)}.^{6.}$$

Hence,

$$\frac{\text{Sector } P_\epsilon AC}{\text{Sector } QAC} = \frac{b_\epsilon}{a} \quad \text{(p. 301 line 3)}.$$

I suggest that this derivation was probably Keplerian, on the grounds that there had previously been no reason to investigate the sector of an ellipse taken at an eccentric point (indeed, there had been no incentive to take an eccentric point as pole, as we remark elsewhere): also, Kepler reproved it—but not Archimedes' own proposition—for the mediate ellipse, as Protheorema III of C.59. Then he stated:

$$\alpha \propto \text{sector } QAC \text{ (p. 301 lines 7-9)}^{7.}$$

$$\therefore \quad \alpha \propto \text{sector } QBC + \Delta AQB \text{ (p. 301 lines 9-12)}$$

$$\therefore \quad \underline{\tfrac{1}{2}c_u\alpha = \tfrac{1}{2}a^2\beta_o + \tfrac{1}{2}a^2 e \sin\beta_o, \text{ where } c_u = a^2.}$$

This is the second appearance of <u>the area-hypothesis, Keplerian style, with universal area-parameter c_u</u>: its especial significance is that, for the first time, the hypothesis has been applied to a <u>perfect ellipse</u>. Indeed, Kepler must have realized that this result is valid for every prototype (in particular for the mediate ellipse): but we firmly defer discussion of any such extended application to the third stage: as we emphasize in § 7, Kepler's prior objective was to discover the grade of the correct curve.

We digress to continue the deduction that Kepler obviously intended, but did not complete until the end of his astronomical work:

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6. Kepler proved this, in a different context, in C.40 (G.W. III p. 266 lines 1-16).

7. The remark in parenthesis, in conjunction with p. 301, first marginal note, and lines 4-6, confusingly introduced a reference to the semi-equant hypothesis, as discussed in § 5(c) above.

$\alpha \propto$ sector $P_\epsilon AC$

$\therefore\ \alpha \propto \frac{b_\epsilon}{a}$. sector QAC

$\therefore\ \underline{\frac{1}{2}c_\epsilon \alpha = \frac{1}{2}ab_\epsilon \beta_o + \frac{1}{2}ab_\epsilon e \sin\beta_o}$, where $\underline{c_\epsilon = ab_\epsilon}$.

Here we have the area-hypothesis, Keplerian style, with $c_\epsilon = ab_\epsilon$ the area-parameter specific to the particular ellipse under investigation. However, we point out that this considerable step forward is of no practical relevance to Kepler's achievements in Astronomia, where we are concerned only to show that he proved the proportionality of mean anomaly to area.[8]

(c) Towards the end of the second stage.

We list the evidence of Kepler's calculations of the mean anomaly for three special values of the eccentric anomaly:

At the quadrant, $\beta = 90^\circ$ gives $\alpha = 95^\circ 18'28''$ (p. 301 lines 17-28)[9].

At the octants $\beta = 45^\circ$ gives $\alpha = 48^\circ 45'12''$ (p. 302 lines 5-17)

$\beta = 135^\circ$ gives $\alpha = 138^\circ 45'12''$ (p. 302 lines 19-26).

It is easy to check (as in §4(c) above) that these have been worked out from the general formulation:

$$\underline{\frac{1}{2}c_u \alpha = \frac{1}{2}a^2\beta + \frac{1}{2}a^2 e \sin\beta, \text{ where } c_u = a^2}.$$

This serves to confirm that, in §7, we have correctly interpreted Kepler's intentions by presenting his comparison of various mean anomaly hypotheses in terms of distantiary angle $\angle_o$, which already functions as the eccentric anomaly of the typical point Q of the eccentric circle, and now, as the eccentric anomaly of the point P_ϵ on the Auxiliary Ellipse as well. This consistent geometrical basis will enable us to give a coherent account of Kepler's progress from stage to stage.

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8. We return to the question of the specific area-parameter in §9(c).

9. This value has already been used in discussion of the quadrant-position in Chapter II, § 2.

§ 7 The pattern of Kepler's progress to the third stage Astronomia C.47.

It is not until now, after consideration of the second stage, that we can usefully introduce the defect in eccentric equation as a criterion by which to assess the accuracy of a proposed orbit, since it is obvious that it will operate most powerfully when the possibilities are bounded in both directions. By contrast with the differences in diurnal apparent angular motion—maximum at the quadrant—all references to which are tabulated in Chapter II, § 3(b),[1.] this criterion is effective around the octants,[2.] as we emphasize by tabulating the corresponding set of references (in which those values that are out of line with the rest may be disregarded because the Ovoids in those cases were kinematically unsound, as explained in Appendix III(5)).

Tabulation I Comparison of defects in eccentric equation around the octants.
(The references underlined are those that lead to the conclusions stated.)

Stage 1: Circle	Stage 2: Auxiliary Ellipse/Ovoid
C.43 p. 283 lines 13-22	C.47 p. 302 lines 6-27
† C.50 'Ovoid' II p. 317 lines 20-1 C.50 'Ovoid' III p. 318 lines 25,27	C.48 p. 309 lines 37, 39 C.49 p. 313 lines 31,33-4 C.50 Ovoid I p. 316 lines 24,26 C.50 Ovoid VI p. 321 lines 17-18
Conclusion	Conclusion
Defect at first octant $\simeq -\frac{1}{4}e^2$	Defect at first octant $\simeq +\frac{1}{4}e^2$
Defect at third octant $\simeq +\frac{1}{4}e^2$	Defect at third octant $\simeq -\frac{1}{4}e^2$

† These 'Ovoids' are misplaced because they turn out to be throwbacks to a circular orbit: see Appendix III(5)

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1. In this discussion, as well as that of the quadrant-distances in Chapter II, § 3(a),the possibilities are similarly bounded in both directions.

2. Not always at the octants, exactly, because the structure of some of the Ovoids necessitated making the comparison by setting either true anomaly, or mean anomaly, as 45°, 135° in turn.

Kepler himself carried out a comparison of a selection of mean anomaly hypotheses, tabulating the results towards the end of C.47 (G.W. III p. 302):

(1) The doubly-eccentric equant hypothesis, briefly described in Chapter I, § 3(b). Its maximum defect was found there to be $\frac{3}{4}e^2$, too high to merit serious consideration in the present context.

(2) An equant hypothesis, in which it was assumed that the eccentric equation was either twice the physical equation, or twice the optical equation. Kepler had accepted this hypothesis as adequate in the case of the Earth (in C.40), but stated that it was unsatisfactory for Mars (G.W. III p. 265 lines 35-7): it also receives no further consideration.[3.]

(3) The Ptolemaic equant hypothesis, applied to the eccentric circle (the first prototype): Kepler could also have checked its application to the second prototype—represented, approximately, by Ovoid IV of C.46—but I have no evidence that he did this.[4.]

(4) The Vicarious Hypothesis, to be taken here, since inaccuracies at $O(e^3)$ are to be disregarded, as equivalent to 'observations'—in fact, the application of the area-hypothesis to the mediate prototype.

(5) and (6) The area-hypothesis—described here as 'physical'—applied to the perfect circle and a perfect ellipse (the first and second prototypes) in turn.

The two distance-sum hypotheses were, notably, not tested here, but have been included in the summary of results given below, for completeness.

On the face of it, Kepler carried out this assessment of the defects in eccentric anomaly in terms of common mean anomaly: but we have already confirmed, in both § 4(c) and § 6(c) that the three values he stated are

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3. There is a peculiarity here, in that the maximum defects $\underline{e}^2$, equal and of opposite sign, are arrived at by a comparison with the Ptolemaic equant hypothesis: if the area-hypothesis had been used, the defects are smaller, and unequal for the two alternatives. Thus it is entirely fortunate (in eliminating a possible source of confusion) that this hypothesis was rejected early on.

4. Nor did he need to, as we demonstrate below.

actually derived from special values of the eccentric anomaly which define positions at the quadrant and the octants, precisely. Therefore, we do not distort his results by tabulating our transcription of their salient features, more generally, for typical points (only in the first quadrant, for clarity) in terms of distantiary angle β_o, which functions as their eccentric anomaly, when we confine consideration—as Kepler did in this context—to the prototypes alone.[5.]

Tabulation II The pattern of the defects in eccentric equation.
The entries underlined are those worked out by Kepler himself in C.47 (G.W. III p. 302)—though at the quadrant and octants alone. The present quantifications, correct to $O(e^2)$, are taken from Appendices III(1) to III(4).

	Area-hypothesis	Ptolemaic equant hypothesis	Original distance-sum hypothesis	Modified distance-sum hypothesis
First prototype	$-\frac{1}{4}e^2\sin 2\beta_o$	$+\frac{1}{4}e^2\sin 2\beta_o$	$-\frac{3}{8}e^2\sin 2\beta_o$	$-\frac{3}{8}e^2\sin 2\beta_o$
Second prototype	$+\frac{1}{4}e^2\sin 2\beta_o$	$+\frac{1}{4}e^2\sin 2\beta_o$	$+\frac{1}{8}e^2\sin 2\beta_o$	$+\frac{3}{8}e^2\sin 2\beta_o$
Mediate prototype	0	$+\frac{1}{4}e^2\sin 2\beta_o$	$-\frac{1}{8}e^2\sin 2\beta_o$	0

Now it has been established in §2 above that the defect in the eccentric equation (the sole observable defect) is the sum of the defect in the optical equation (measuring the amount by which the grade of the curve is wrong) and the defect in the physical equation (the amount by which the hypothesis is wrong). Hence, the most that can be achieved by assessment of the defect in the eccentric equation is to highlight the amount by which a particular orbit is wrong, either geometrically, or kinematically, or a combination of both (the direction of the necessary correction being indicated by the sign of the defect). Hence, unless either a common curve, or a common hypothesis, is consistently assumed, the defect in eccentric equation cannot operate as an

5. In any case, it has been explained in Chapter I, § 2, that, to a first approximation, quantifications may be indifferently expressed in terms of any anomaly: thus the defect in the Ptolemaic equant hypothesis may, at this level, be expressed in terms of β_o for comparison, as is confirmed in Appendix III(1).

efficient criterion: as an instrument of analysis it is too blunt by far.

I therefore radically disagree with the interpretations of Kepler's argument proposed by Curtis Wilson[6.] and Gingerich,[7.] which contain a logical flaw if, as it appears, they believe that a selection of random defects, produced by the application of inconsistent hypotheses,[8.] would have led Kepler to the curve of the correct grade (the dimensions of the correct curve). Nor have these views been challenged subsequently, and so I have to regret the absence of any satisfactory account of Kepler's reasoning up to the beginning of the third stage. The truth is that, for the purpose of his comparison of the defects in C.47, the procedure Kepler actually adopted was sound: he set aside the Ptolemaic equant hypothesis, and ignored the distance-sum hypotheses, to apply the area-hypothesis consistently to the first and second prototypes: then the fact that the defects found were approximately equal, and of opposite sign, by comparison with the Vicarious Hypothesis, equivalent to 'observations'[9.] enabled him to infer (in a footnote to his tabulation of p. 302): 'veritatem esse exacte in harum medio'.

It must be emphasized that the area-hypothesis was selected for this comparison as a matter of convenience, simply because it was the only one Kepler had managed satisfactorily to apply to curves of both grades, and also that this investigation, being observation-based, could only detect defects at the level $O(e^2)$. Indeed, far from concluding that the correct curve was thus inevitably an exact ellipse, we shall start afresh at the third stage, in §8 below—as in Chapter II, §5—assuming only knowledge

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6. Kepler's Derivation of the Elliptical Path, Part (3), especially pp. 17-18.

7. Johannes Kepler and the New Astronomy, pp. 357-8.

8. Curtis Wilson mentions either 'the area rule' or some form of 'the radius rule' (op. cit. p. 18) while Gingerich suggests the Ptolemaic equant hypothesis or (by implication) the area-hypothesis (op. cit.).

9. This is most clearly evident at the octants, both because the defects are maximum there, and also because the Vicarious Hypothesis is more reliable, as Kepler knew.

of the correct grade of the curve, to solve the question of its shape.[10] The justification for this approach has been explained in Chapter I, §4, on the grounds that the grade of the correct curve involves a difference at $O(e^2)$, which can be observationally detected (both from distance- and angle-determinations), while the shape can never be so established. Hence Curtis Wilson[11] is further mistaken in suggesting that the observable defect could have been of relevance in deciding the shape of the curve: the fact is that he has placed on the criterion of the defect in eccentric equation a great deal more weight than it can properly sustain.

Moreover—though the availability of this extra information has only been noticed very recently (see Chapter II, §3)—Kepler possessed independent knowledge of the quadrant-distance of the correct curve, which immediately specified its mediate grade. Therefore, disregarding the complications of the Ovoid group (much effort to little purpose, as demonstrated in §5), I state unequivocally that the overall pattern of Kepler's progress, up to the third stage, was altogether orderly. In fact it is reminiscent of a game of celestial noughts-and-crosses, played against an imaginary Ptolemaic or counter-progressive opponent, in which Kepler first started right, and thereafter made the right move each time.[12] It has been my aim in this study to demonstrate that, as anticipated in Chapter I, §3, his grasp of theoretical considerations developed throughout Astronomia in parallel with his skill in handling raw observations—backed up by an increasing awareness both of the sources of errors and how to correct them—and was sufficient to account for his progress from stage to stage: so I am inclined to believe that the element of luck involved was minimal.

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10. And not until then is it of significance that the area-hypothesis applies precisely to every prototype: though this had already been proved in C.47 (see §6(b)).

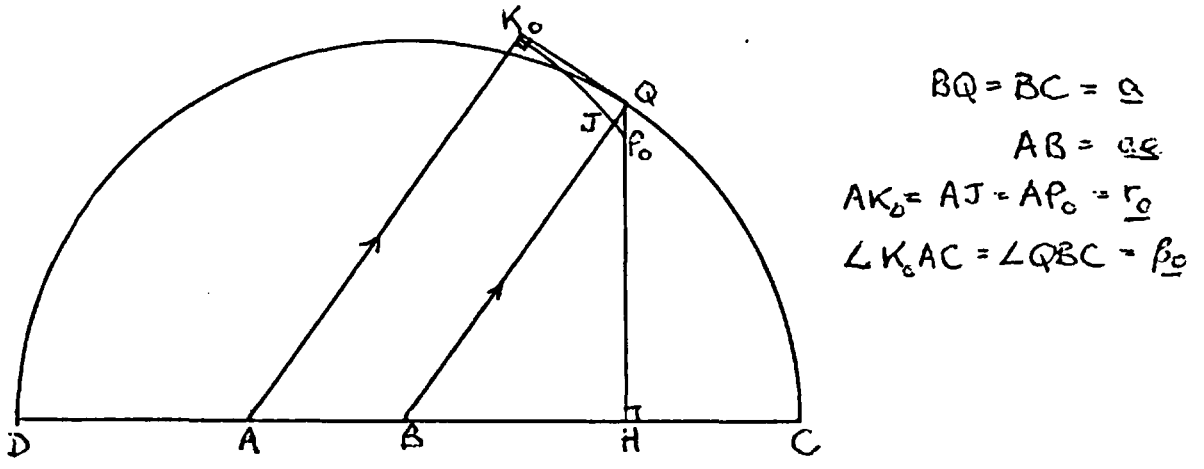
11. Kepler's Derivation of the Elliptical Path, p. 18.

12. Unfortunately I cannot illustrate this idea because it would involve a mathematical distortion: some of the erroneous positions need to be represented contemporaneously, some on a basis of geometrical comparability.

§ 8 Nihil....nisi ellipsis: the kinematical justification for Kepler's choice

Astronomia C.58.

Fig. (vii) The oviform of C.58 and the mediate ellipse.



Having now reached the third stage—the curve of mediate grade whose quadrant-distance is a, exactly—we return to the situation discussed from a geometrical point of view in Chapter II, § 5, by which the argument was reduced to a straight choice between the oviform of C.58 and the perfect mediate ellipse: that is, between J[1.] and P_0[2.] both lying on the characteristic arc drawn in accordance with the provisional Secant Law. Then our present concern is with the kinematical error—at $O(e^2)$—which arose when J was temporarily proposed as the typical point of an orbit.

This error has been exposed only quite recently by Whiteside:[3.] it is insidious to the extent that Kepler himself assumed (as also have some

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1. We recall that the oviform of C.56 was abandoned by then because Kepler regarded J, not J_1, as the point determined by the libration.

2. As we have demonstrated in Chapter II, § 6, Kepler checked that P_0 did indeed define a perfect ellipse.

3. Newton's Early Thoughts on Planetary Motion, p. 129, footnote 42 (1964): discussed more fully in Keplerian Planetary Eggs, pp. 12-15.

influential commentators)[4]. that he had applied the area-hypothesis to oviform J as well as to the perfect ellipse—so that J and P_o would be contemporaneous points. However the fact is that he actually applied to J what I describe (because it was unintentional) as a rogue mean anomaly hypothesis, which entailed a kinematical error, while the visual difference he attributed to the mispositioning of J on BQ is entirely fallacious: the wrong time, not the wrong place. To emphasize this, I have, in Appendix III(6), analyzed the rogue mean anomaly hypothesis—which was formulated in terms of the central anomaly instead of, correctly, in terms of the eccentric anomaly—and have there assessed its defects (in common with those of all other hypotheses considered in this study) by applying it to the prototypes, taking the area-hypothesis as standard: it is perhaps of interest that, had Kepler formally recognized this hypothesis, he would have found its observable defect (in the eccentric equation) equal and opposite to that produced by the Ptolemaic equant hypothesis.[5]. Then, despite the fact that Kepler's rejection of J was spurious, he did not go on to consider any of the several points W on the oviform which are strictly comparable with P_o. Such points differ from P_o at the level $O(e^3)$ only, and would therefore have been indistinguishable from P_o in Kepler's terms (in accordance with the level of accuracy stipulated in Chapter I, § 3) with respect to every anomaly: though certainly Kepler could have recognized the difference in radius vector at $O(e^3)$ arising from the construction; indeed, as we have pointed out in Chapter II, § 5, he investigated the shape of the oviform and found that it was pointed at perihelion. The orbits resulting from the application to this oviform of Time-measures I, II, III, in turn, will be comprehensively described as via buccosa W: because they are of significance only in modern terms, they are treated separately in Appendix III(7).

Adopting a modern point of view, Whiteside criticizes Kepler for his failure to take account of (any of) these highly accurate variant orbits: indeed,

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4. Conspicuously, Caspar: Neue Astronomie note p. 411 to p. 344 (see G.W. III p. 365 lines 27-32).

5. In fact, the defects stated by Kepler (G.W. III p. 365 lines 31-2) were noticeably smaller than they should have been: see Caspar: Neue Astronomie p. 411 as cited above.

by the same standard, it must be regarded as reprehensible if, more generally, Kepler overlooked the potential existence of the infinity of possible orbits, all observationally indistinguishable (differing at $O(e^3)$ at most). Yet, rather than convict him of 'ignorance and oversight',[6.] I shall elect to confirm that Kepler's preferred orbit was both a deliberate and informed choice,[7.] the expression of some of his deepest convictions. Moreover, no other preNewtonian astronomer could have defended his preference as cogently as Kepler, had he been called upon to do so.

It was established in Chapter II that Kepler selected the ellipse on the grounds of its symmetry: of course, the ellipse is not the only curve with this property,[8.] but Kepler ensured that it was unique by adding the condition that its radius vector be constructible by Euclidean methods: we have seen that these two considerations were fundamental to his progress all along (so he could hardly have abandoned them at this stage). Then, in this chapter, we have shown how Kepler proved that the area-hypothesis, Keplerian style, applied exactly to every prototype, and therefore, in particular, to this (mediate) ellipse.[9.] It is thus clear that Kepler's personal criteria for his preferred orbit were that it should possess both geometrical and kinematical precision: and they could be satisfied by the ellipse alone.

Some other astronomers advocated curves of a particular shape (Huygens preferred an oviform more pointed at perihelion,[10.] for instance) for theoretical reasons, but since the mechanisms used to represent each proposed orbit were equant hypotheses[11.] the criteria by which they were

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6. Keplerian Planetary Eggs, p. 15.
7. Informed to the extent that he knew that it could not be contradicted by observations at the level of accuracy to which he had access.
8. Except insofar as it is the only symmetrical curve whose polar equation is of the form set out in Appendix I(1), assumed for the basis of this study.
9. In C.47 (see §6(b)), repeated in Protheorema III of C.59.
10. See Whiteside: Newton's Early Thoughts on Planetary Motion, pp. 128-9.
11. Kepler's own equant hypotheses are to be discussed in §9(b) below.

judged were essentially pragmatic: convenience in operation, combined with the requirement that they be at least as accurate as the standard of observations of the day, against which they had continually to be checked. (It was only Kepler, by his choice, who successfully freed himself from this burdensome necessity.) I suggest that it is because the two sets of criteria are qualitatively so disparate that Kepler's opinions made very little impression on his contemporaries and successors (even Newton, at first).[12]

I am convinced that, with this precise, determinate choice, Kepler believed that he had reached the end of his search for the true planetary orbit, and that nothing thereafter could have changed his mind (he would, in due course, have been able additionally to point out that he had managed to account for it in terms of physical causes—this is the subject of Part B). Consequently I must conclude that, had he been wrong, his work would have been ranked as one of the most spectacular failures in the history of astronomy.

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12. Whiteside: Newton's Early Thoughts on Planetary Motion.

§9 Further kinematical considerations.

(a) A complete survey of time-measure and other hypotheses Astronomia C. 40, C.45-50, C.59.

As far as I am aware, it has not been remarked by anyone previously that Kepler evolved a way of deciding whether or not a hypothesis would be likely to turn out satisfactory by testing it, in the first instance, over the whole curve—so that, in practice, he evaluated its dimension-parameter. Neither have I come across any appreciation of the fact that, somehow or other, Kepler managed to consider every possible hypothesis within a certain range. Unfortunately he did not do so systematically—as I have classified them in the following tabulation—and the majority of these instances are to be found in the Ovoid group,[1.] where it is perhaps not surprising that they have been overlooked. Those hypotheses which are not the time-measures listed in §1 (and quantified in modern terms in Appendices II(2), III(3), III(4)) are described as Perverted Hypotheses I', II', III' (also quantified in those Appendices, to which they have been allocated by reference to the independent variable involved). On checking, most of these hypotheses were immediately eliminated from further consideration[2.] merely by calculating their respective dimension-parameters.[3.] I have identified the value used for the standard of comparison as the sum of the diametral distances, worked out in C. 40—and represented there as the area under the second 'conchoid' (illustrated in Appendix II(1))—where Kepler stated the exact result:

$$\frac{a}{\pi}\int_0^{\pi} r_o \, d\beta = a^2 \quad \text{(G.W. III p. 269 lines 1-2).}$$

Evidently, expression of the result in these terms would have been quite inappropriate at that stage (when the orbit was still assumed to

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1. Further complicated by the fact that 'Ovoids' II and III of C.50 have had to be classified as circles—as justified in Appendix III(5).
2. However, the rejection of Perverted Hypothesis II' (applied to Ovoid I of C.50) cannot be explained on the same grounds.
3. This is not to underrate the sheer slog involving in calculating 180 distances (or their squares) each time, before summing them.

be a circle) and so has been deferred until, in the context of the third stage, Kepler established (in Protheorema XI of C.59, set out in Chapter II, §6), that the diametral distance coincided with the radius vector $\underline{r}_o$ of the perfect mediate ellipse. Hence, confirmation of the above result, in Protheorema IX of C.59 (p. 369 lines 29-31) became, in Protheorema XII (p. 371 lines 22-4), a proof of the validity of the modified distance-sum hypothesis in relation to the perfect ellipse. It was by the application of this standard that Kepler was persuaded finally to reject the original distance-sum hypothesis (Time-measure I, as quantified in modern terms in the Tabulation)—which highlighted the remaining unresolved problem of Astronomia, to be discussed in subsection (c) below.

It is to be emphasized that the dimensional compatibility between all these 'distance-sum' hypotheses and the area-hypothesis, here set out in modern terms, is nonetheless Keplerian: he undoubtedly visualized the former in terms of area and indeed recognized that in such circumstances the area was doubled (G.W. III p. 268 lines 23-4) and so the distances must be summed over the semi-curve. We shall return to this confirmation of his firm grasp of the order of linear dimensions in Part B.

Incidentally we remark that while the Perverted Hypotheses are evidently unsuitable as mean anomaly hypotheses, they may (when the dimensions are adjusted, if necessary) be regarded as mean value determinations: indeed, that is the only possible interpretation of Perverted Hypotheses I'.

Comparison of hypotheses by assessment of their dimension-parameters.

Hypothesis	Dimension-parameter for specific prototypes	References in G.W. III
Time-measure II see Appendix III(3) $c_{II} = \frac{1}{\pi}\int_o^\pi r.ad\beta$	First: $a^2(1+\frac{1}{4}e^2)$	C.40 (p. 267 lines 10-19) 'Ovoid' III of C.50 (p. 318 lines 2-3) C. 59 Protheorema VIII
	Second: $a^2(1-\frac{1}{4}e^2)$	Ovoid I of C.50 (p. 315 lines 1-2, 22)
	Mediate: a^2 the standard	C. 40 (p. 269 lines 1-2) C.59 Protheoremata IX, XII
Perverted Hypothesis III' see Appendix III(4) $c'_{III} = \frac{1}{\pi}\int_o^\pi r.ad\theta$	First: $a^2(1-\frac{1}{4}e^2)$	C.40 (p. 267 line 36-p.268 line 2)
	Second: $a^2(1-\frac{3}{4}e^2)$	Ovoid I of C.50 (p. 316 lines 5-16) Ovoid V of C.50 (p. 319 line 15)
	Mediate: $a^2(1-\frac{1}{2}e^2)$	[=ab, exactly]
Time-measure I see Appendix III(2) $c_I = \frac{1}{\pi}\int_o^\pi rds$	First: $a^2(1+\frac{1}{4}e^2)$	as for Time-measure II
	Second: $a^2(1-\frac{3}{4}e^2)$	
	Mediate: $a^2(1-\frac{1}{4}e^2)$	C.59 Protheorema X (unquantified)
Perverted Hypothesis I' see Appendix III(2) $\dot{c}'_I = \frac{1}{\pi}\int_o^\pi r.ad\alpha$	First: $a^2(1+\frac{3}{4}e^2)$	
	Second: $a^2(1+\frac{1}{4}e^2)$	Ovoid of C.48 (p. 304 lines 23-5) Ovoid of C.49 (p. 312 lines 40-1)
	Mediate: $a^2(1+\frac{1}{2}e^2)$	
Perverted Hypothesis II' see Appendix III(3) $c'_{II} = \frac{1}{\pi}\int_o^\pi r^2 d\beta$	First: $a^2(1+e^2)$	'Ovoid' III of C.50 (p. 318 line 4)
	Second: a^2	Ovoid I of C.50 (p. 315 line 3)
	Mediate: $a^2(1+\frac{1}{2}e^2)$	
Time-measure III see Appendix III(4) $c_{III} = \frac{1}{\pi}\int_o^\pi r^2 d\theta$	First: a^2	'Ovoid' II of C.50 (p. 317 line 23)
	Second: $a^2(1-e^2)$	Ovoid VI of C.50 (p. 321 lines 4-6)
	Mediate: $a^2(1-\frac{1}{2}e^2)$	[=ab, exactly]

Note: Those references underlined are where Kepler himself stated a numerical value: I have checked that these are all recognizably in agreement with the formulae stated, with the exception of 'Ovoid' III of C. 50 (p. 318 line 4).

(b) The fate of Kepler's equant hypotheses.

It was in a letter to Fabricius[4.] that Kepler outlined a triplet of equant orbits, whose defects have been assessed, as for all other orbits, in Appendix III(8): they comprised one curve of each grade, and thus would hardly have fitted into the deductive scheme of Astronomia where, as we have demonstrated, orbits of each grade were considered separately in turn. Their inaccuracy, as it turns out, would make them of little interest—but I draw attention to the fact that Kepler believed that the small-grade orbit approximated to Ovoid V (which it did not, as I prove), and it is possible, because of inadequacies of the diagram, that Kepler thought that he had proposed the semi-equant hypothesis, and was misguidedly attempting to 'correct' it in relation to a curve of mediate grade. If this is so, it provides confirmation of my view, expressed in §5(c), that Kepler never realized the extreme accuracy of that hypothesis, because he regarded it as applicable to the Ovoid group alone. Thus he was spared another distraction in his progress to an exact formulation of the mean anomaly.

(c) The modified distance-sum hypothesis and the remaining problem Astronomia C. 59.

It will have become evident from the conclusions of Chapter II—reconfirmed in §8 above—that I believe that the perfect ellipse was Kepler's deliberate choice, and that, by the end of C. 58 he had intentionally excluded consideration of all other possible curves.[5.] Accordingly, now we are to be concerned with the orbital ellipse alone, I shall take the opportunity finally to abandon dependence on the invariant distantiary angle, since it will be appropriate to state the formulae more generally, in terms of eccentric anomaly. My view that the problem of C. 59 was the purely kinematical one to be set cut here is, however, in fundamental disagreement with that of Aiton who, in giving a detailed survey of all fifteen Protheoremata,[6.] suggests that in C. 59 Kepler was still attempting to prove both the geometrical and the kinematical properties of the orbit, using each in turn as a working hypothesis.

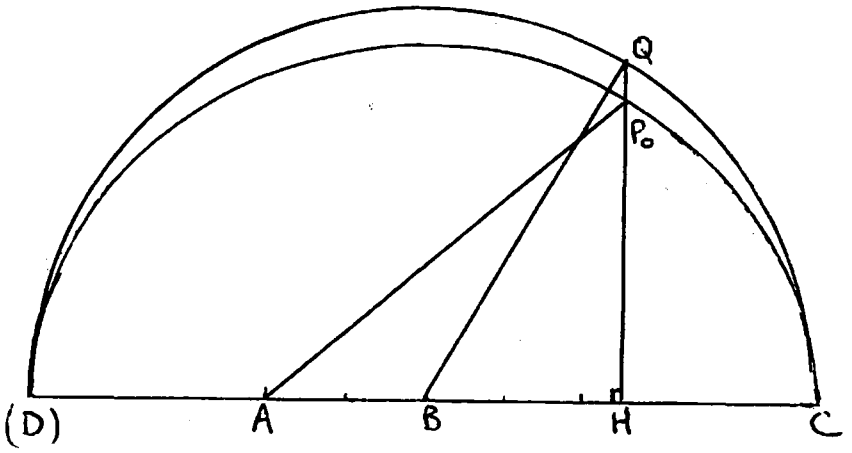
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4. G.W. XV Letter 358 lines 330-64.

5. Therefore I interpret the mentions of point 'J' (as illustrated in §8) in Protheoremata XIII and XV merely as references back to his former erroneous opinion of C. 58.

6. Kepler's Second Law of Planetary Motion, pp. 83-7; Infinitesimals and the Area Law, pp. 300-3.

Fig. (viii) The perfect (mediate) ellipse.



$BQ = BC = a$
$AB = ae$
$AP_0 = r_0$
$\angle QBC = \beta$
$\angle P_0AC = \theta$

It has been pointed out in (a) above that Kepler tested the validity of the modified distance-sum hypothesis by working out its dimension-parameter over the whole curve: only when this turned out to be the 'satisfactory' value $\underline{a}^2$ did he go on to apply the hypothesis to the ellipse by sectors to obtain, also in Protheorema XII:

$$a\int_0^{\beta} r_0 d\beta = \text{area of circular sector QAC (p. 371 lines 32-5).}$$

This was reaffirmed even more strongly in the concluding Protheorema XV (p. 375 lines 14-23) and again in C. 60 (p. 376 lines 21-34):

$$\alpha \propto a\int_0^{\beta} r_0 d\beta \propto \tfrac{1}{2}a^2(\beta + e\sin\beta).$$

Hence Kepler established that the modified distance-sum hypothesis coincides with the area-hypothesis when it is applied specifically to the exact mediate ellipse, the area of a sector of that ellipse being measured by the sector of the circle cut off by the common ordinate: this is equivalent to adopting the universal area-parameter $c_u = a^2$, rather than the area-parameter $c_0 = ab$ specific to this ellipse. Yet he was undoubtedly aware of the value of the latter for, earlier in the same chapter (having, in Protheorema II, cited Archimedes' result for the area of the whole ellipse) he went on to prove, in Protheorema III, that:

$$\text{Area of elliptic sector} = \frac{b}{a}\,(\text{area of circular sector})$$
$$= \tfrac{1}{2}ab(\beta+e\sin\beta).$$ [7.]

This was not reiterated (in final correlation with the mean anomaly, there) until Epitome V Part I Section 5:[8.] it is notable (as we also anticipate in Chapter IV, §1) that, in the interim, the constant b did not appear in his account of the focal geometry of the ellipse. We shall attempt to explain this in Chapter VII, §4.

Then because—as mentioned in (a) above—the original distance-sum hypothesis had, by Protheorema X, been decisively dismissed from consideration, the problem of C. 59 will be pinpointed as Kepler's inability (at that stage) to prove that what we shall now describe as:

$$\delta\alpha \propto r: \quad \text{the indeterminate time-axiom}$$

could properly be identified with:

$$\delta\alpha \propto r(\delta\beta \text{ constant}): \quad \text{Time-axiom II (direct)}.$$

That is, he could not demonstrate that measuring with respect to equal arcs $a\delta\beta$ of the eccentric circle would be compatible with taking equal arcs of the orbit, which he thought was what was required, in accordance with his original physical opinions. The problem was not resolved (almost satisfactorily, at any rate) until Epitome V Part I Section, by the introduction of a new technique to supplement the two already employed in C. 59: this is the subject of Chapter VII, §1. Meanwhile—though the interpretation of a 'small arc' will continue to be a major difficulty in Part B—we shall start there from the now firmly-established representation of mean anomaly for the perfect ellipse:

$$\tfrac{1}{2}c_u\alpha=\tfrac{1}{2}a^2\beta+\tfrac{1}{2}a^2e\sin\beta \quad (c_u=a^2).$$

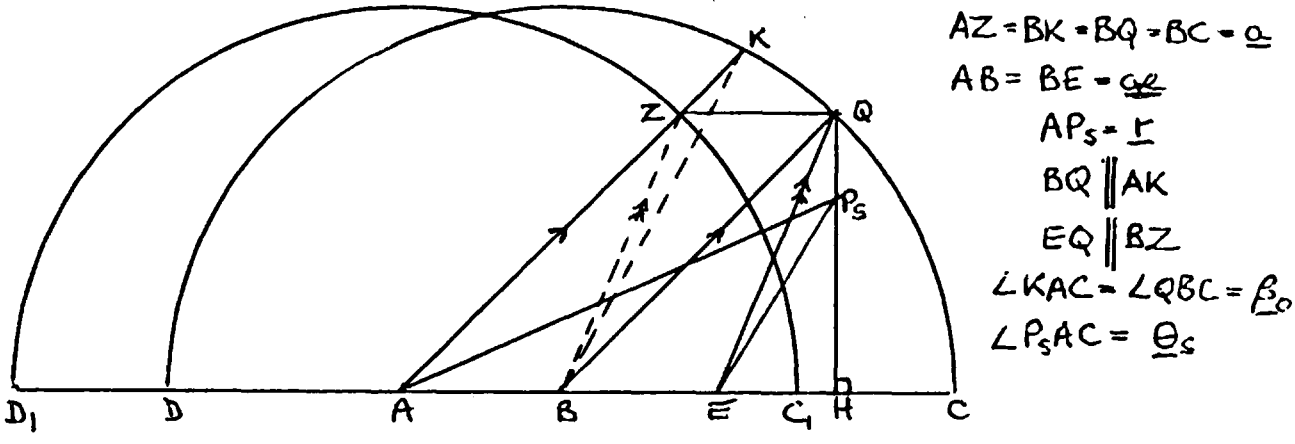
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7. Compare the implied quantification of the area-parameter $c_\varepsilon=ab_\varepsilon$ for the second prototype, as set out in §6(b).

8. G.W. VII pp. 379-81.

APPENDIX III(1)

THE PTOLEMAIC EQUANT HYPOTHESIS



This hypothesis was defined in Chapter I, §3(b), as the simplest possible representation of Ptolemaic planetary theory, transposed to heliocentricity. In that context, obviously, the orbit was supposed to be circular; but since Kepler—and some of his successors—applied it to noncircular curves (as cited in §1), we shall now analyze the updated version which is appropriate for the whole Keplerian range of orbits. However, with hindsight we shall find that it is sufficient to confine consideration to the prototypes alone—for whose typical points P_s, β_o functions as the common eccentric anomaly.

The updated definition of the equant is:

$$\angle P_s E C = \alpha.$$

Then from ΔAP_sE we have:

$$\frac{\sin(\alpha-\theta_s)}{2ae} = \frac{\sin\alpha}{r}.$$

$$\therefore\ \alpha - \theta_s = \frac{2ae\sin\alpha}{r} + O(e^3)$$

$$= 2e\sin\alpha(1 + e\cos\theta_s)^{-1} \quad \text{from Appendix I(1)}$$

$$\therefore\ \alpha = \theta_s + 2e[\sin(\theta_s + 2e\sin\theta_s)](1 - e\cos\theta_s) \quad \text{from Appendix III(4)}$$

$$= \theta_s + 2e(\sin\theta_s + 2e\sin\theta_s\cos\theta_s)(1 - e\cos\theta_s)$$

$$\therefore\ \underline{\alpha = \theta_s + 2e\sin\theta_s + e^2\sin 2\theta_s + O(e^3)}: \text{ Newtonian style.}$$

Now from Appendix I(2),

$$\theta_s = \beta_o - e\sin\beta_o + \tfrac{1}{2}\left(\tfrac{3}{2} - q_o\right)e^2\sin 2\beta_o + o(e^3)$$

$$\therefore\ \underline{\alpha = \beta_o + e\sin\beta_o + \tfrac{1}{2}\left(\tfrac{3}{2} - q_o\right)e^2\sin 2\beta_o + O(e^3)}: \text{ Keplerian style.}$$

We will assess the defects of this hypothesis at $O(e^2)$ against the area-hypothesis as standard, using the criteria set out in §2:

I Defect in optical equation $= +\tfrac{1}{2}(q_o - 1)e^2\sin 2\beta_o + O(e^3)$.

II Defect in physical equation $= +\tfrac{1}{2}\left(\tfrac{3}{2} - q_o\right)e^2\sin 2\beta_o + O(e^3)$.

This evidently becomes negligible when the hypothesis is applied to the second prototype (or to one of its associated Ovoids, as Kepler did in C.46—see Appendix III(5)), which is itself a curve of the wrong grade.

$\therefore$ III Defect in eccentric equation $= +\frac{1}{4}e^2\sin 2\beta_o + O(e^3)$.

Thus this hypothesis produces an observable defect of absolute value, which cannot be cured by any choice of curve, since it is independent of q_o.

The foregoing comparison has evidently been undertaken on a geometrical basis, where the typical point P_s of each prototype in turn is represented within the invariant frame of reference. However, from a comparison on the basis of common mean anomaly (which has, in fact, been carried out in Chapter I, §3(b)), we find:

Defect in eccentric equation $= +\frac{1}{4}e^2\sin 2\alpha + O(e^3)$.

We confirm that the two evaluations are the same, to a first approximation: this justification will be required in connection with the comparison of hypotheses in §7.

APPENDIX III(2)

THE ORIGINAL DISTANCE-SUM HYPOTHESIS, FROM TIME-MEASURE I

$$\frac{\delta\alpha}{\delta s} = \frac{r}{c_I} \quad : \text{ Time-measure I.}$$

$$\therefore \alpha = \frac{1}{c_I} \int r\,ds \quad : \text{ The original distance-sum hypothesis.}$$

By modern analysis we obtain:

$$\begin{aligned}
\delta s &= [r^2 (\delta\theta)^2 + (\delta r)^2]^{\frac{1}{2}} \\
&= r\,\delta\theta \left[1 + \frac{1}{r^2}\left(\frac{\delta r}{\delta\theta}\right)^2\right]^{\frac{1}{2}} \\
&= r\,\delta\theta \left[1 + \frac{1}{2r^2}\left(\frac{\delta r}{\delta\theta}\right)^2 - \frac{1}{8r^4}\left(\frac{\delta r}{\delta\theta}\right)^4 + \ldots.\right].
\end{aligned}$$

Then from the polar equation of Appendix I(1),

$$\frac{dr}{d\theta} = -a(e\sin\theta + 2q_o e^2 \sin\theta\cos\theta) + O(e^3)$$

$$\left(\frac{dr}{d\theta}\right)^2 = a^2(e^2\sin^2\theta + 4q_o e^3 \sin^2\theta\cos\theta) + O(e^4)$$

$\left(\frac{dr}{d\theta}\right)^4$ is $O(e^4)$ which is to be neglected.

$$\therefore \delta s = \left[r + \frac{a^2}{2r}(e^2\sin^2\theta + 4q_o e^3 \sin^2\theta\cos\theta)\right]\delta\theta + O(e^4).$$

$$\begin{aligned}
\alpha &= \frac{1}{c_I}\int r\,ds \\
&= \frac{a^2}{c_I}\int_o^\theta (1 + e\cos\theta - q_o e^2\sin^2\theta)^2 + \tfrac{1}{2}e^2\sin^2\theta]\,d\theta + O(e^3) \\
&= \frac{a^2}{c_I}\int_o^\theta [1 + 2e\cos\theta - (q_o - \tfrac{3}{4})e^2 + (q_o + \tfrac{1}{4})e^2\cos 2\theta]\,d\theta \\
&= \frac{a^2[1-(q_o - 3/4)e^2]}{c_I}\,[\theta + 2e\sin\theta + \tfrac{1}{2}(q_o + \tfrac{1}{4})e^2\sin 2\theta]
\end{aligned}$$

$$\therefore \alpha = \theta + 2e\sin\theta + \tfrac{1}{2}(q_o + \tfrac{1}{4})e^2\sin 2\theta + O(e^3), \text{ Newtonian style,}^{1.}$$

where $c_I = a^2[1 - (q_o - \frac{3}{4})e^2] + O(e^4)$.

---

1. Whiteside: Keplerian Planetary Eggs, p. 18, note 23, series (a), gives the inverse (Newtonian) form.

Now from Relation B_2 of Appendix I(2),

$$\alpha = \beta - e\sin\beta + \tfrac{1}{2}(\tfrac{3}{2} - q_o)e^2 \sin 2\beta + 2e\sin\beta - 2e^2 \sin\beta\cos\beta + \tfrac{1}{2}(q_o + \tfrac{1}{4})e^2 \sin 2\beta + O(e^3)$$

$$\therefore \alpha = \beta + e\sin\beta - \tfrac{1}{8}e^2 \sin 2\beta + O(e^3), \quad \text{Keplerian style.}$$

Since our comparison in this chapter will be conducted in Keplerian terms, we confine consideration to the prototypes alone—for which β_o functions as the eccentric anomaly of the typical points—to assess the defects in this hypothesis at $O(e^2)$, against the area-hypothesis as standard, using the criteria set out in § 2:

I Defect in optical equation $= + \tfrac{1}{2}(q_o - 1)e^2 \sin 2\beta_o + O(e^3)$.

II Defect in physical equation $= -\tfrac{1}{8}e^2 \sin 2\beta_o + O(e^3)$.

In this case the defect is of absolute value—it cannot be cured by a suitable choice of curve since it is independent of q_o, as Whiteside has also pointed out in the note cited.

$\therefore$ III Defect in eccentric equation $= -\tfrac{1}{2}(\tfrac{5}{4} - q_o)e^2 \sin 2\beta_o + O(e^3)$.

Thus there does exist a curve to which Time-measure I can be applied, to produce a result in agreement with observations, correct to $O(e^2)$. However, this curve is given by $q_o = \frac{5}{4}$, which does not arise anywhere in Kepler's work—though it does in fact lie within the 'Keplerian range'.

Perverted Hypothesis I′: $\frac{a}{c_I'} \int r\,d\alpha$

For our survey of § 9 we shall need to evaluate the dimension-parameter c_I' of this perverted hypothesis, using $r(\alpha)$ from Appendix III(4):

$$c_I' = \frac{a}{\pi}\int_o^\pi r\,d\alpha$$

$$= \frac{a^2}{\pi}\int_o^\pi [1 + e\cos\alpha - (q_o - 2)e^2 \sin^2\alpha]\,d\alpha + O(e^3)$$

$$= \frac{a^2}{\pi}\int_o^\pi [1 + e\cos\alpha - \tfrac{1}{2}(q_o - 2)e^2 + \tfrac{1}{2}(q_o - 2)e^2 \cos 2\alpha]\,d\alpha$$

$$c_I' = \frac{a^2[1-(\frac{1}{2}q_o-1)e^2]}{\pi}[\alpha + e\sin\alpha + \frac{1}{4}(q_o-2)e^2\sin 2\alpha]_o^\pi$$

$$\therefore\ c_I' = a^2[1-(\tfrac{1}{2}q_o-1)e^2] + O(e^4).$$

APPENDIX III(3)

THE MODIFIED DISTANCE-SUM HYPOTHESIS, FROM TIME-MEASURE II

$$\frac{d\alpha}{a d\beta} = \frac{r}{c_{II}}: \text{ Time-measure II.}$$

$$\therefore \alpha = \frac{a}{c_{II}} \int r d\beta: \text{ The modified distance-sum hypothesis.}$$

Then from Relation B_1 of Appendix I(2),

$$\alpha = \frac{a^2}{c_{II}} \int_o^\beta [1 + e\cos\beta - (q_o - 1)e^2 \sin^2\beta]d\beta + O(e^3)$$

$$= \frac{a^2[1 - \frac{1}{2}(q_o - 1)e^2]}{c_{II}} \int_o^\beta [1 + e\cos\beta + \tfrac{1}{2}(q_o - 1)\cos 2\beta]d\beta$$

$$\therefore \alpha = \beta + e\sin\beta + \frac{1}{4}(q_o - 1)e^2 \sin 2\beta + O(e^3), \text{ Keplerian style,}$$

where $c_{II} = a^2[1 - \frac{1}{2}(q_o - 1)e^2] + O(e^4)$.

Now from Relation B_2 of Appendix I(2),

$$\alpha = \theta + e\sin\theta + \tfrac{1}{2}(q_o - \tfrac{1}{2})e^2 \sin 2\theta + e\sin\theta + e^2 \sin\theta\cos\theta + \frac{1}{4}(q_o - 1)e^2 \sin 2\theta + O(e^3)$$

$$\therefore \alpha = \theta + 2e\sin\theta + \frac{3}{4}q_o e^2 \sin 2\theta + O(e^3), \text{ Newtonian style.}$$

For the comparison in Keplerian terms to be carried out in this chapter, we confine consideration to the prototypes alone- for which β_o functions as the eccentric anomaly of the typical points—again to assess the defects in this hypothesis at $O(e^2)$, against the area-hypothesis as standard:

I Defect in optical equation $= +\frac{1}{2}(q_o - 1)e^2 \sin 2\beta_o + O(e^3)$.

II Defect in physical equation $= +\frac{1}{4}(q_o - 1)e^2 \sin 2\beta_o + O(e^3)$.

$\therefore$ III Defect in eccentric equation $= +\frac{3}{4}(q_o - 1)e^2 \sin 2\beta_o + O(e^3)$.

Thus these three all vanish, at $O(e^2)$, when $q_o=1$—for the mediate prototype: we show by modern methods, in § 9, that this hypothesis is in fact exact for the correct ellipse; Kepler did not discover this until much later (in Epitome V Part I Section 4).

Perverted Hypthesis II′ : $\frac{1}{c_{II'}} \int r^2 d\beta$.

For our survey of § 9 we shall need to evaluate the dimension-parameter c_{II}' of this further perverted hypothesis:

$$
\begin{aligned}
c_{II}' &= \frac{1}{\pi} \int_o^\pi r^2 d\beta \\
&= \frac{a^2}{\pi} \int_o^\pi [1 + e\cos\beta - (q_o - 1)e^2 \sin^2\beta]^2 d\beta + O(e^3) \\
&= \frac{a^2}{\pi} \int_o^\pi [1 + 2e\cos\beta + e^2\cos^2\beta - 2(q_o - 1)e^2 \sin^2\beta] d\beta \\
&= \frac{a^2}{\pi} \int_o^\pi [1 + 2e\cos\beta - (q_o - \tfrac{3}{2})e^2 + (q_o - \tfrac{1}{2})e^2 \cos 2\beta] d\beta \\
&= \frac{a^2 [2 - (q_o - \frac{1}{2})e^2]}{\pi} [\beta + 2e\sin\beta + \tfrac{1}{2}(q_o - \tfrac{1}{2})e^2 \sin 2\beta]_o^\pi
\end{aligned}
$$

$$\therefore c_{II}' = a^2 [1 - (q_o - \tfrac{1}{2})e^2] + O(e^4).$$

APPENDIX III(4)

THE AREA-HYPOTHESIS FOR ANY OVIFORM, FROM TIME-MEASURE III

$$\frac{\partial\alpha}{\partial\theta} = \frac{r^2}{c_{III}} \quad : \text{ Time-measure III}$$

$$\therefore\ \alpha = \frac{1}{c_{III}} \int r^2 d\theta \quad : \text{ The area-hypothesis.}$$

From Appendix I(1),

$$\alpha = \frac{a^2}{c_{III}} \int_o^\theta (1 + e\cos\theta - q_o e^2 \sin^2\theta - q_1 e^3 \sin^2\theta\cos\theta)^2 d\theta + O(e^4)$$

$$= \frac{a^2}{c_{III}} \int_o^\theta [1 + 2e\cos\theta + e^2\cos^2\theta - 2q_o e^2 \sin^2\theta - 2(q_o + q_1)e^3 \sin^2\theta\cos\theta]\, d\theta$$

$$= \frac{a^2}{c_{III}} \int_o^\theta [1 + 2e\cos\theta - (q_o - \tfrac{1}{2})e^2 + (q_o + \tfrac{1}{2})e^2\cos 2\theta - 2(q_o + q_1)e^3 \sin^2\theta\cos\theta]\, d\theta$$

$$= \frac{a^2[1 - (q_o - \frac{1}{2})e^2]}{c_{III}} \quad [1 + 2e\cos\theta + (q_o + \tfrac{1}{2})e^2\cos 2\theta + (2q_o - 1)e^3\cos\theta$$
$$- 2(q_o + q_1)e^3 \sin^2\theta\cos\theta]\, d\theta$$

$$\therefore\ \underline{\alpha = \theta + 2e\sin\theta + \tfrac{1}{2}(q_o + \tfrac{1}{2})e^2 \sin 2\theta + (2q_o - 1)e^3 \sin\theta - \frac{2}{3}(q_o + q_1)e^3 \sin^3\theta + O(e^4),}$$

$$\underline{\text{where } c_{III} = a^2[1 - (q_o - \tfrac{1}{2})e^2] + O(e^4).}$$

This is the area-hypothesis, Newtonian style.

Now from Relation B_2 of Appendix I(2),

$$\theta = \beta - e\sin\beta + \tfrac{1}{2}(\tfrac{3}{2} - q_o)e^2 \sin 2\beta + (3q_o - q_1 - \tfrac{5}{2})e^3 \sin\beta - (4q_o - q_1 - \tfrac{10}{3})e^3 \sin^3\beta + O(e^4).$$

$$\therefore \alpha = \beta - e\sin\beta + \tfrac{1}{2}(\tfrac{3}{2} - q_o)e^2 \sin 2\beta + (3q_o - q_1 - \tfrac{5}{2})e^3 \sin\beta - (4q_o - q_1 - \tfrac{10}{3})e^3 \sin^3\beta$$
$$+ 2e\sin\beta(1 - \tfrac{1}{2}e^2\sin^2\beta) - 2e\cos\beta[e\sin\beta - (\tfrac{3}{2} - q_o)e^2 \sin\beta\cos\beta]$$
$$+ (q_o + \tfrac{1}{2})e^2(\sin\beta - e\sin\beta\cos\beta)(\cos\beta + e\sin^2\beta) + (2q_o - 1)e^3\sin\beta$$
$$- \frac{2}{3}(q_o + q_1)e^3\sin^3\beta + O(e^4)$$

$$\therefore\ \underline{\alpha = \beta + e\sin\beta + (2q_o - q_1 - 1)e^3\sin\beta - \frac{1}{3}(2q_o - q_1 - 1)e^3\sin^3\beta + O(e^4).}$$

This is the area-hypothesis, Keplerian style.

I now quote the inverse form, Newtonian style, given by Whiteside: Keplerian Planetary Eggs, p. 18, note 23, series (b):[1.]

$$\theta = \alpha - 2e\sin\alpha + (\tfrac{7}{4} - \tfrac{1}{2}q_o)e^2\sin 2\alpha + O(e^3).$$

We apply this to the polar equation of Appendix I(1):

$$r = a[1 + e\cos\theta - q_o e^2\sin^2\theta - q_1 e^3\sin^2\theta\cos\theta] + O(e^4)$$

$$= a\{1 + e\cos\alpha(1 - 2e^2\sin^2\alpha) + e\sin\alpha\,[\,2e\sin\alpha - (\tfrac{7}{2} - q_o)e^2\sin\alpha\cos\alpha]$$

$$- q_o e^2(\sin\alpha - 2e\sin\alpha\cos\alpha)^2 - q_1 e^3\sin^2\alpha\cos\alpha\} + O(e^4)$$

$$\therefore r = a[1 + e\cos\alpha + (2 - q_o)e^2\sin^2\alpha + (5q_o - q_1 - \tfrac{11}{2})e^3\sin^2\alpha\cos\alpha] + O(e^4).$$

This will be known as Relation A—for comparison, subsequently, with the closely approximating Relation A_K which crops up in connection with the Ovoid group (see §5).

Perverted Hypothesis III′: $\frac{a}{c'_{III}}\int r\,d\theta$.

For our survey of §9 we shall need to evaluate the dimension-parameter c'_{III} of yet another perverted hypothesis:

$$c'_{III} = \frac{a}{\pi}\int_o^{\pi} r\,d\theta$$

$$= \frac{a^2}{\pi}\int_o^{\pi}(1 + e\cos\theta - q_o e^2\sin^2\theta)\,d\theta + O(e^3)$$

$$= \frac{a^2}{\pi}\int_o^{\pi}(1 + e\cos\theta - \tfrac{1}{2}q_o e^2 + \tfrac{1}{2}q_o e^2\cos 2\theta)\,d\theta$$

$$= \frac{a^2[1 - \frac{1}{2}q_o e^2]}{\pi}[\theta + e\sin\theta + \tfrac{1}{4}q_o e^2\sin 2\theta]_o^{\pi}$$

$$\therefore c'_{III} = a^2[1 - \tfrac{1}{2}q_o e^2] + O(e^4).$$

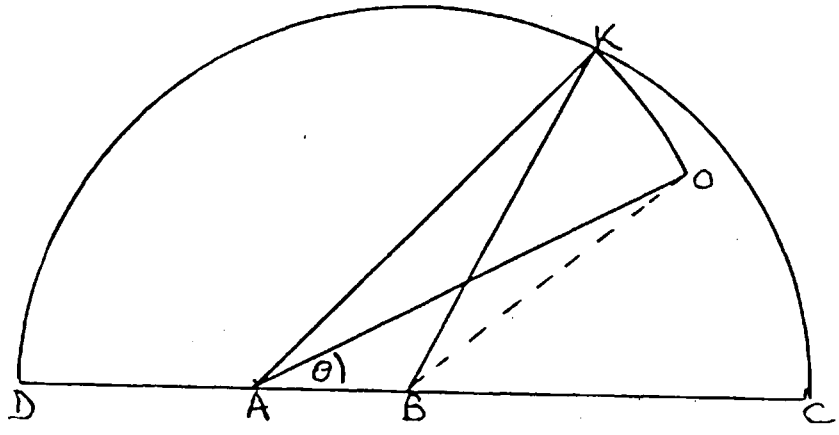
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1. Stated here for convenience in the standard notation of this study (with c_{III} replacing Whiteside's c_2).

APPENDIX III(5) to § 5

THE OVOID GROUP

Ovoid I of C.46 G.W. III pp. 291-2.



BK = BC = a
AB = ae
AK = AO = r
∠KAC = β_o

This diagram is taken from the one on p. 291— the first instance of the use of Kepler's characteristic construction in connection with a proposed orbit—which also illustrates two other attempts in C.46. This Ovoid possesses the common Ovoid-properties:

$\angle$ KBC = α_k (p. 292, marginal note).

Then $\angle$KAC = β_o, the distantiary angle, ensures geometrical cohesion.

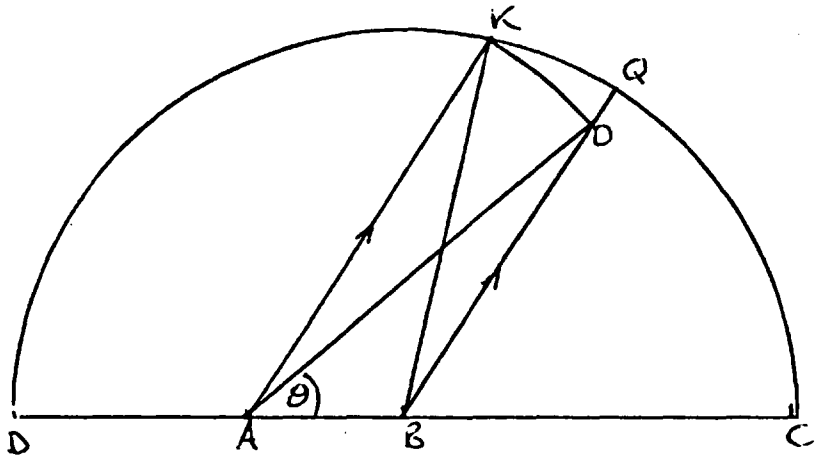
The typical point lies on the characteristic arc KO but is otherwise geometrically undefined.

Kepler tried to specify the position of O by introducing what he believed to be an alternative representation of mean anomaly via Time-axiom I (inverse): $r\delta s = c_I\, \delta\alpha$ ($\delta\alpha$ constant) (p. 292 lines 17-19): of course this is incompatible, as proved in Appendix III(2)—but in any case Kepler misapplied it, to obtain a geometric mean statement which he expressed approximately as an arithmetic mean:

Area KAC - area KBC = area KBC - area OBC (p. 292 lines 19-21) which entails ΔAKB = area KBO (p. 293 lines 11-12).

This is obviously impossible when O lies within the circle, and provides an even more compelling reason than the four stated by Kepler (p. 292 line 27-p. 293 line 17) for rejecting this attempt as geometrically unsatisfactory (p. 293 lines 18-20).

Ovoid II of C.46 G.W. III p. 293.



$BK = BQ = BC = a$

$AB = ae$

$AK = AO = r$

$\angle KAC = \angle QBC = \beta_o$

The diagram is taken from the one of p. 291, and this Ovoid possesses the common Ovoid-properties:

$\angle KBC = \alpha_k$ (p. 293 lines 21-2).

Then $\angle KAC = \beta_o$, the distantiary angle, ensures geometrical cohesion. Additionally, Kepler suggested that the sum of the distances was equal to sector KBC (p. 293 lines 26-8), from which we may deduce that he intended:

$\Delta AOB = \Delta KBO$, at least approximately.

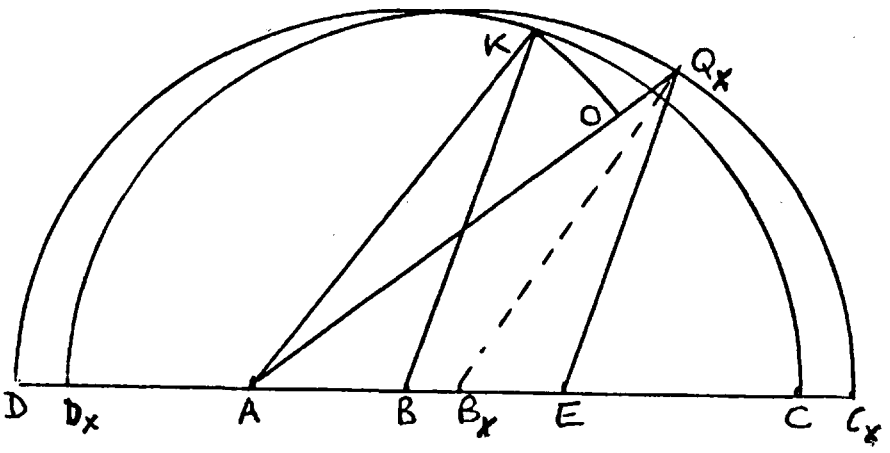
We will assume for simplicity that this relationship is exact, in which case BO||AK, and thus O lies on BQ, where BQ||AK in accordance with the geometrical frame of reference: then this attempt will coincide with Ovoid I of C.50, in which O was completely defined in this way. However, the distantiary angle is thereby identified β_o with the central anomaly γ, which entails the application of the rogue Keplerian mean anomaly hypothesis, exclusively the subject of Appendix III(6), so that this Ovoid becomes a constituent of Ovoid J.

Also from Relation B_k we have (since $\gamma \equiv \beta_o$):

$$r = a(1 + e\cos\gamma - \tfrac{1}{2}e^2\sin^2\gamma) + O(e^4),$$

and by comparison with Relation C_1 in Appendix I(3) we find that $q_o = \frac{3}{2}$, $q_1 = \frac{3}{2}$, and hence this Ovoid is pointed at perihelion.

Ovoid III of C.46 (Vicarious) G.W. III pp. 294-5.



$B_x C_x = BK = BC = B_x Q_x = \underline{a}$

$AB = BE = \underline{ae}$

$AB_x = \underline{ae}_1$

$B_x E = \underline{ae}_2$

$AK = AO = \underline{r}$

$\angle KAC = \underline{\beta}_o$

This diagram is given on p. 294 and illustrates a return to the Vicarious theory of Astronomia C.16 (set out in Chapter I, § 3(b)), here applied to an Ovoid. The adaptation is extremely ingenious, since Kepler managed to retain the advantage of the Vicarious Hypothesis by leaving the relationship $\theta(\alpha)$ unaltered—so that the results remain 'equivalent to observations'—while removing the main geometrical objection to it by adjusting the distances—so that, most notably, the apsidal distances are now given correctly. The typical point of this Ovoid is completely defined and we deduce that it possesses the common Ovoid-properties:

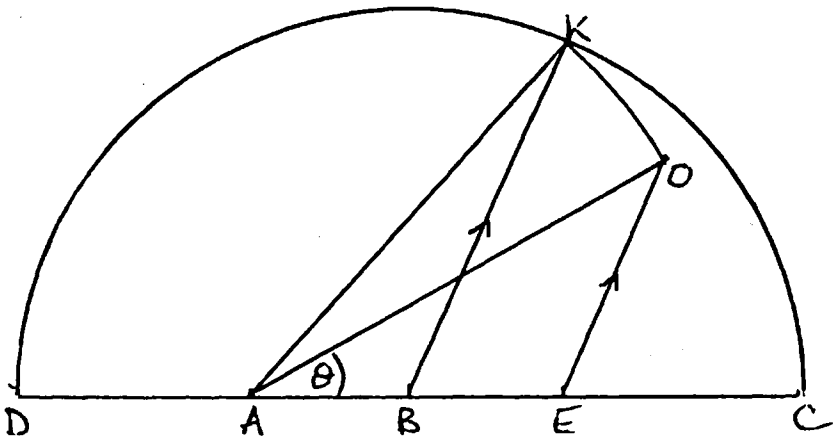
$\angle Q_x EC$ is the mean anomaly, by the Vicarious Hypothesis

But, $BK \| EQ_x$ (p. 294 line 40)

$\therefore \angle Q_x EC = \angle KBC = \alpha_k$

Then $\angle KAC = \beta_o$, the distantiary angle, ensures geometrical cohesion. Thus we have followed Kepler in making the implicit assumption that the time represented by $\angle KBC$ was in fact the true time measured by the Vicarious Hypothesis—as indeed it is, to within $\frac{1}{6}e^3$, as we establish in § 5(b): while this representation certainly reflects Kepler's intentions, as I confirm in connection with the identical Ovoid V of C.50.

Ovoid IV of C.46 G.W. III p. 295



$BK = BC = a$
$AB = BE = ae$
$AK = AO = r$
$\angle KAC = \beta_o$

This diagram is again taken from that on p. 291, but here illustrates a return to the Ptolemaic equant hypothesis, applied to an Ovoid. Its typical point is completely defined and it apparently possesses the common Ovoid-properties:

$\angle$ OEC is the mean anomaly (p. 295 line 14)

But, BK||EO (p. 295 lines 14-15)

$\therefore \angle OEC = \angle KBC = \alpha_k$

Then $\angle KAC = \beta_o$, the distantiary angle, ensures geometrical cohesion.

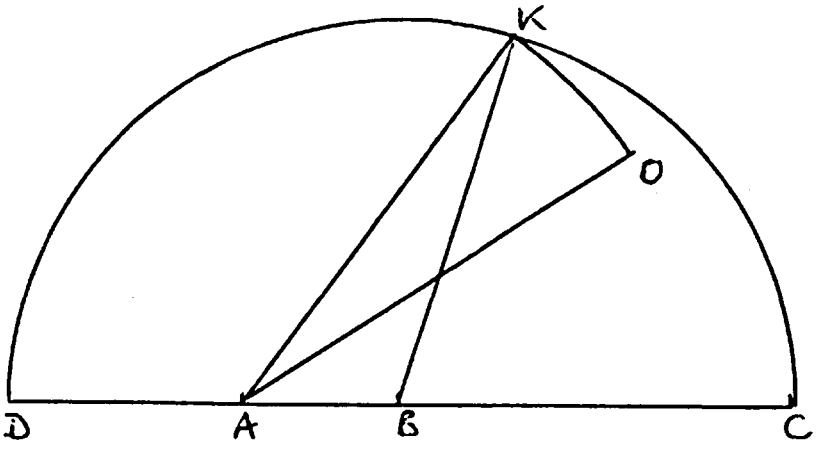
Also $\alpha_k = \theta + 2e\sin\theta + e^2\sin 2\theta + O(e^3)$ from Appendix III(1), and from Relation A_k the polar equation of this Ovoid is:

$$r = a(1 + e\cos\theta - \tfrac{3}{2}e^2\sin^2\theta - \tfrac{5}{2}e^3\sin^2\theta\cos\theta) + O(e^4).$$

At this level, this is also the polar equation of Ovoid V: but it can be shown that the two true anomalies differ at $O(e^3)$, and hence this Ovoid can be identified with Ovoid V only in Keplerian terms, at the level $O(e^2)$. More importantly, we have again followed Kepler in making the implicit assumption that the time represented by $\angle$ KBC was actually the true time, and thus there is a (minimal) distortion of the equant hypothesis in this application.

This was the only attempt overlooked by Small: The Astronomical Discoveries of Kepler, at the end of § 152.

The Ovoid of C.48 G.W. III pp. 304-5, p. 308



$BK = BC = \underline{a}$
$AB = \underline{ae}$
$AK = AO = \underline{r}$
$\angle KAC = \underline{\beta_e}$

The diagram is a simplified version of that on p. 304 and p. 307 (the rest of it is concerned with estimating the total length of the curve). This Ovoid possesses the common Ovoid-properties:

$\angle$ KBC = α_k (p. 304 lines 12-13).

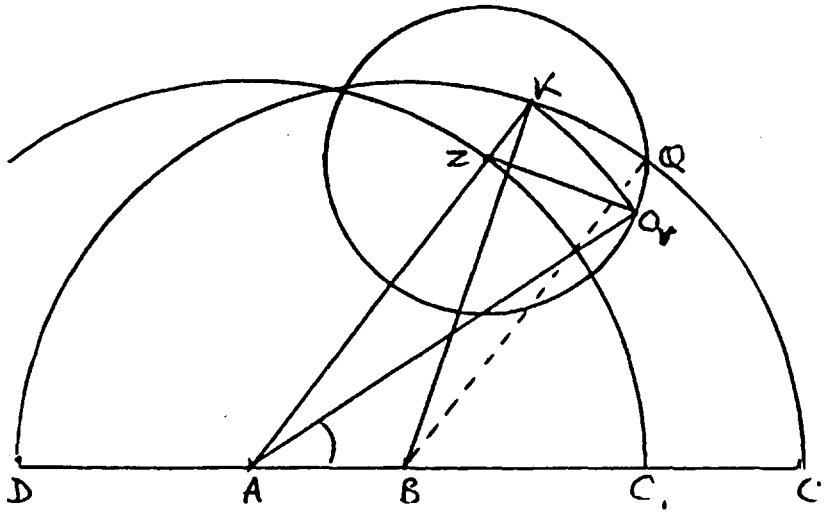
Then $\angle$ KAC = β_o, the distantiary angle, ensures geometrical cohesion. The typical point lies on the characteristic arc KO but is otherwise geometrically undefined.

As for Ovoid I of C.46, Kepler tried to specify the position of O by introducing the supposedly alternative representation of mean anomaly via Time-axiom I (inverse): $\delta s \propto r^{-1}$ ($\delta\alpha$ constant) (p. 308 lines 21-2)—but again he misapplied it. Additionally, however, he here formulated a new hypothesis (p. 304 lines 10-27):

$\frac{a}{c_I'} \int r d\alpha$: Perverted Hypothesis I',

whose accuracy he checked simply by evaluating its dimension-parameter c_I' (p. 304 lines 23-5). The significance of this—and two subsequent perverted hypotheses—is to be discussed in § 9.

The Ovoid of C.49 (G.W. III pp. 311-13.



$AC_1 = AZ = BK = BC = a$

$AB = ZQ = ZO_v = ae$

$AK = AQ_v = r$

$\angle KAC = \angle QBC = \beta_o$

This diagram is taken from that on p. 311, which itself represents the construction stated, but not illustrated, in C.45. It has been proved, in § 5(b), that this Ovoid possesses the common Ovoid-properties:

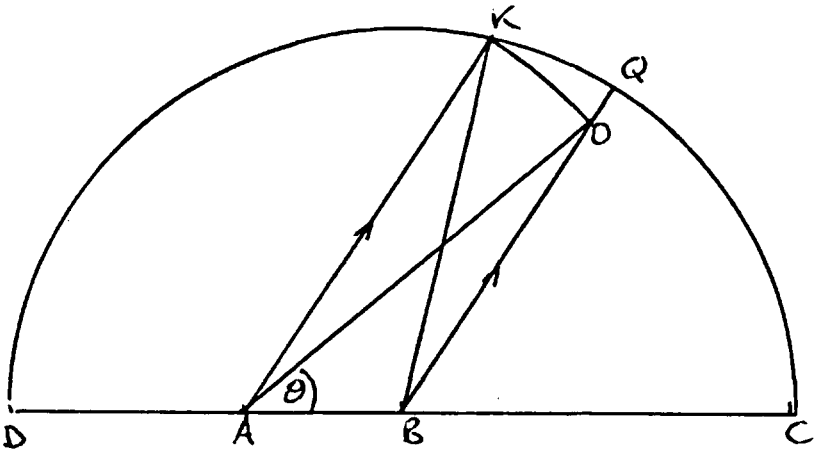
$\angle KZO = \alpha_k$

$\therefore \angle KBC = \alpha_k$ by the congruent triangles AZO_v, AKB.

Then $\angle KAC = \beta_o$, the distantiary angle, ensures geometrical cohesion. The typical point lies both on the characteristic arc KO and on the epicycle, so it is completely defined, to give Ovoid V.

Unfortunately, as in Ovoid I of C.46 and the Ovoid of C.48, Kepler once more introduced a misapplication of Time-measure I (which is redundant here anyway): while at the same time he restated Perverted Hypothesis I' (p. 312 lines 34-41) and again evaluated its dimension-parameter c_I' (p. 312 lines 40-1), explained in connection with the Ovoid of C.48. However, he also applied the hypothesis correctly (p. 313 line 2) to find the mean value of r: we return to discuss this in § 9.

Ovoid I of C.50 G.W. III pp. 315-6.



$BK = BQ = BC = a$

$AB = ae$

$AK = AO = r$

$\angle KAC = \angle QBC = \beta_o$

This diagram is taken from that on p. 315 but repeats that illustrating Ovoid II of C.46, since BO||AK (p. 316 lines 8-9, 18-19), so that O lies on BQ. Hence the typical point of this Ovoid is completely defined and it possesses the common Ovoid-properties:

$\angle KBC = \alpha_k$ (p. 315, marginal note).

Then $\angle KAC = \beta_o$, the distantiary angle, ensures geometrical cohesion. However, the distantiary angle is again identified with the central anomaly γ, which entails the application of the rogue Keplerian mean anomaly hypothesis, and so this Ovoid becomes another constituent of Ovoid J, the Ovoid-analogue of 'via buccosa J'.

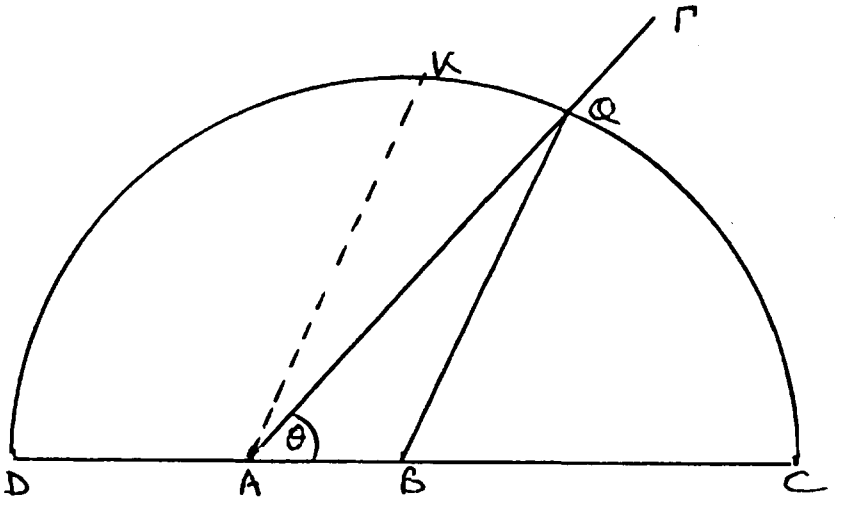
Additionally, Kepler took this opportunity to formulate further hypotheses, which he tested simply by evaluating their respective dimension-parameters: the results are tabulated in § 9:

From $\frac{a}{c_{II}} \int r d\beta$ (p. 315 lines 1-12), the modified distance-sum hypothesis, he determined c_{II} (p. 315 lines 1-2, 22);

from $\frac{1}{c'_{II}} \int r^2 d\beta$ (p. 315 lines 1-16), Perverted Hypothesis II', he determined c'_{II} (p. 315 line 3);

from $\frac{1}{c'_{III}} \int r d\theta$ (p. 316 lines 5-16), Perverted Hypothesis III', he determined $c_{III'}$.

'Ovoids' II and III of C.50: two throwbacks G.W. III pp. 316-18.



$BQ = BC = a$
$AB = ae$
$AQ = r$
$\angle QBC = \beta_c$
$A\Gamma = \frac{r^2}{a}$

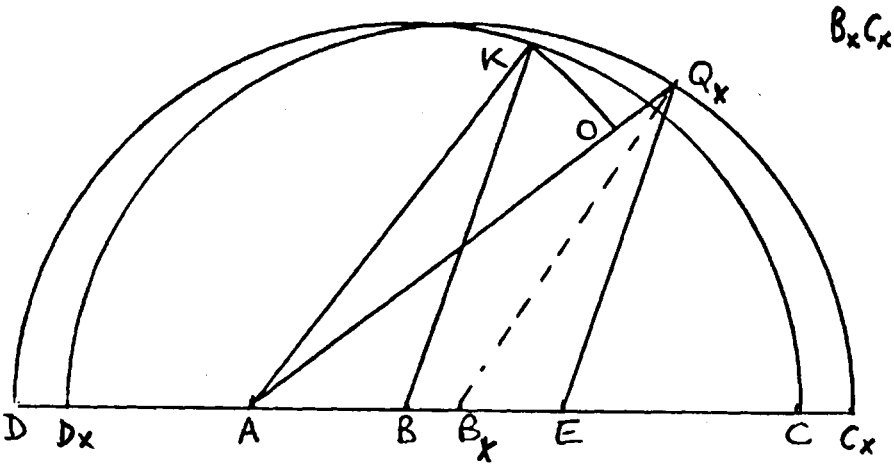
The unusual absence of Kepler's radius-swinging construction on this occasion makes the diagram look unfamiliar, while its consequence was to produce a return to a circular orbit, as Kepler realized (p. 316 line 40, p. 318 line 13); in confirmation, ∠ QBC was identified as the eccentric anomaly, ∠QAC as the true anomaly (p. 316, lower marginal note, p. 318, upper marginal note): it is for this reason that these two are considered together and afterwards excluded for ever as proposed orbits. (Ovoid IV, as described in the lower marginal note on p. 318, is so strange that it cannot even be usefully illustrated.) However, the hypotheses that Kepler formulated at this juncture are of some significance (as we shall see in § 9), even though he tested them merely by calculating their respective dimension-parameters:

In 'Ovoid' II he evaluated c_{II} from $\frac{1}{c_{\text{II}}} \int r^2 d\theta$ (p. 316 lines 33–40); the area-hypothesis.

In 'Ovoid' III he evaluated c_{III} (p. 318 lines 2–3) from $\frac{a}{c_{\text{III}}} \int r d\beta$ (p. 317 lines 29–30): the modified distance-sum hypothesis; and c'_{II} (p. 318 line 4) from $\frac{1}{c'_{\text{II}}} \int r^2 d\beta$ (p. 317 lines 29–30): Perverted Hypothesis II'.

It is also of interest that, before $\underline{r}^2$ was used in these calculations, the length AΓ was first physically represented in Kepler's diagrams (p. 315, p. 319), described as a third proportional (p. 314 lines 36-8)—and thus implicitly constructible by Euclid VI, Prop. XI: see Chapter I, § 4.

Ovoid V of C.50 (Vicarious) G.W. III pp. 319-20.



$B_xC_x = BK = BC = B_xQ_x = a$

$AB = BE = ae$

$AB_x = ae_1$

$B_xE = ae_2$

$AK = AO = r$

$\angle KAC = \beta_o$

This diagram is repeated on p. 314 from p. 294 where it illustrated Ovoid III of C.46: the two are evidently identical, since the typical point of the present Ovoid is also completely defined and possesses the common Ovoid-properties:

$\angle Q_xEC = \alpha_k$ (p. 319 line 8)

$\angle KBC = \alpha_k$ (p. 320, second marginal note).

Then $\angle KAC = \beta_o$, the distantiary angle, ensures geometrical cohesion.

Also from Chapter I, § 3(b), we have:

$$\theta = \alpha_k - 2e\sin\alpha_k + ee_1\sin 2\alpha_k + O(e^3).$$

Inverting, $\alpha_k = \theta + 2e\sin\theta + (2e^2 - ee_1)\sin 2\theta + O(e^3)$.

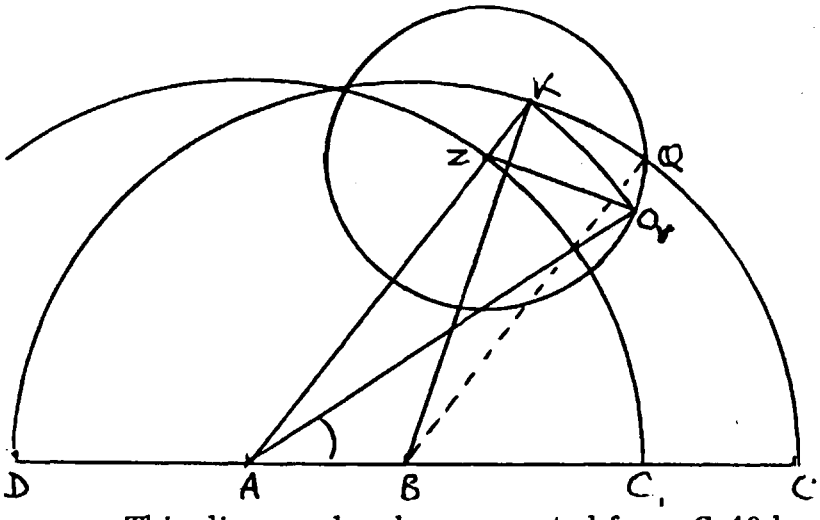
Substituting in Relation A_k (from § 5(a)) we obtain:

$$r = a[1 + e\cos\theta - \tfrac{3}{2}e^2\sin^2\theta - (\tfrac{9}{2}e^2 - 2ee_1)e\sin^2\theta\cos\theta] + O(e^4),$$

in agreement with Whiteside: Keplerian Planetary Eggs, p. 10, who also finds $q_o = \frac{3}{2}$, $q_1 \simeq 2.06$ when $e_1 \simeq 1.22e$ (from C.16).

We remark that Kepler here formulated Perverted Hypothesis III': $\frac{a}{c'_m}\int r d\theta$ (p. 319 lines 9-11) and evaluated its dimension-parameter c'_{III} (p. 319 line 15).

Ovoid VI of C.50 G.W. III pp. 320-1.



$AC_1 = AZ = BK = BC = a$

$AB = ZQ = ZO_V = ae$

$AK = AQ_1 = r$

$\angle KAC = \angle QBC = \beta_o$

This diagram has been repeated from C.49 because we have proved in §5(b) that these two Ovoids are identical in both geometrical and kinematical structure and hence make up the composite Ovoid V (which also includes Ovoid IV of C.46, at the level of accuracy stipulated in this study).

We record that Kepler evaluated the dimension-parameter c_{III} (p. 319 line 15, p. 321 lines 4-6) of the area-hypothesis $\frac{1}{c_{III}}\int r^2 d\theta$ (p. 320 lines 38-9).

We also take the opportunity to evaluate the eccentric anomaly β_V of Ovoid V, for a modern comparison with β_o which functions as the eccentric anomaly of the typical point of the Auxiliary Ellipse (its associated prototype). Using the equalities from the congruent triangles AZO_V, AKB, we find that:

$$a\cos\beta_V = a\cos\beta_o + ZO_V\cos QZO_V - ae$$

$$\therefore \cos\beta_V = \cos\beta_o - e + e\cos(\beta_o - \theta)$$

$$= \cos\beta_o - e + e\sqrt{(1 - e^2\sin^2\beta_o)} \text{ since } \sin(\beta_o - \theta) = e\sin\beta_o$$

$$= \cos\beta_o - \tfrac{1}{2}e^3\sin^2\beta_o + O(e^5).$$

This is satisfied by:

$$\beta_V = \beta_o + \tfrac{1}{2}e^3\sin\beta_o + O(e^5).$$

APPENDIX III(6) to § 8

THE ROGUE MEAN ANOMALY HYPOTHESIS OF C.58

This hypothesis arises from a misformulation of the area-hypothesis in terms of central anomaly γ (instead of eccentric anomaly β):

$$\alpha = \gamma + e \sin \gamma .$$

We will here apply it to the prototypes alone, for which β_o functions as the eccentric anomaly of their typical points. Then from Appendix I(3):

$$\gamma = \beta_o - \tfrac{1}{2}(q_o - \tfrac{1}{2})e^2 \sin 2\beta_o + O(e^3).$$

$$\therefore \quad \alpha = \beta_o + e \sin \beta_o - \tfrac{1}{2}(q_o - \tfrac{1}{2})e^2 \sin 2\beta_o + O(e^3).$$

Hence we assess its defects against the area-hypothesis as standard:

Defect in optical equation $= +\tfrac{1}{2}(q_o - 1)e^2 \sin 2\beta_o + O(e^3)$.

Defect in physical equation $= -\tfrac{1}{2}(q_o - \tfrac{1}{2})e^2 \sin 2\beta_o + O(e^3)$.

$\therefore$ Defect in eccentric equation $= -\frac{1}{4}e^2 \sin 2\beta_o + O(e^3)$.

This defect is equal and opposite to that produced by the Ptolemaic equant hypothesis (see Appendix III(1)). Since its value is absolute, it is the observable effect to be found when we consider the mediate curve, as indeed Kepler did (G.W. III p. 365 lines 30-2), though he misattributed it to the incorrect positioning of the typical point, as we have explained in § 8.[1].

I confirm that there is no inaccuracy, at $O(e^2)$, in applying this hypothesis to the prototypes, as I have done here, rather than as Kepler did, to the oviforms whose typical points were determined by the points in which the characteristic arc cut BQ: not only oviform J of C.58,

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1. This may be further emphasized by pointing out that there can be no question of any defect in the optical equation since this, at $O(e^3)$, is negligible for all curves of standard grade, as is obvious from the formula given in § 2.

the subject of § 8, but also Ovoid II of C.46 and Ovoid I of C.50, as tabulated in § 5 and illustrated in Appendix III(5).[2.]

I digress to mention that this misformulation has been promulgated by the authority of Caspar:[3.] if it were to be regarded as a consequence of aberrant notation alone, I have discovered a further instance—in connection with his analysis of Kepler's estimation of the variation in diameter of the Sun[4.]—in which Caspar denoted the true anomaly by $\underline{\alpha}$ (which he habitually used to designate mean anomaly) and then proceeded to treat $\underline{\alpha}$ as if it were the mean anomaly as usual. The misapprehensions resulting in either case serve to vindicate my insistence on uniformity of notation throughout this study.

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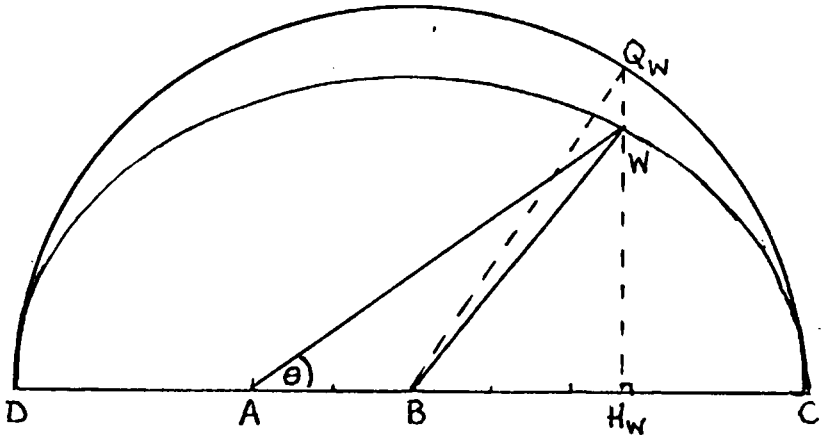
2. In that case, the appropriate formulation was the near approximation $\alpha = \gamma + \sin^{-1}(e \sin\gamma)$.

3. Neue Astronomie p. 411, note to p. 344 (G.W. III p. 365 line 27).

4. Neue Astronomie p. 411, note to p. 337 (G.W. III p. 358 line 9): this topic, of some interest in connection with Astronomiae Pars Optica C. XI, is not relevant to the present enquiry.

APPENDIX III(7) to § 8

THREE VERSIONS OF VIA BUCCOSA W



BC = a
AB = ae
AW = r

This curve is defined geometrically by the relation:

$$r = a(1 + e\cos\gamma),$$ [1.]

where $\gamma = \angle WBC$ is the central anomaly, as explained in § 8 (in Chapter II, § 5, the same curve was described as the oviform of C.58). By comparison with Relation C_1 of Appendix I(3), we find that $q_0=1$, $q_1=\frac{1}{2}$, to confirm that this curve is pointed at perihelion. Hence, also, its polar equation is:

$$r = a(1 + e\cos\theta - e^2\sin^2\theta - \tfrac{1}{2}e^3\sin^2\theta\cos\theta) + O(e^4).$$

Then from the updated optical equation of Appendix I(2), putting $q_0=1$, we have:

$$\beta - \theta = e\sin\beta - \tfrac{1}{4}e^2\sin 2\beta - (\tfrac{1}{2} - q_1)e^3\sin\beta + (\tfrac{2}{3} - q_1)e^3\sin^3\beta + O(e^4).$$

But $q_1=1$ for the exact ellipse, and now $q_1=\frac{1}{2}$ for via buccosa W.

$\therefore$ Defect in optical equation = $-\frac{1}{2}e^3\sin\beta + \frac{1}{2}e^3\sin^3\beta + O(e^4)$.

This will be used to correlate the pairs of kinematical defects to be found below when we apply the time-measures in turn: it will be recalled (as pointed out in Chapter I) that to a first approximation, $\delta\theta \simeq \delta\beta \simeq \delta\alpha$ and thus

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1. See also Whiteside: Keplerian Planetary Eggs, pp. 12-13.

all the evaluations at $O(e^3)$ may be expressed interchangeably in terms of any of these anomalies at that level: the signs are changed as commonsense dictates.

The following three orbits have been analyzed in order to gather together material scattered among footnotes in the work of Whiteside and Aiton; I have here adopted Whiteside's approach in consistently applying each hypothesis both to the oviform and to the exact ellipse. Nevertheless—even if the defects, at $O(e^3)$, had not been undetectable in Kepler's terms—I believe this discussion to be essentially non-Keplerian in concept, as I have argued in §8.

Assessment of W_I by applying Time-measure I.

From Appendix III(2), now working correct to $O(e^3)$, putting $q_o=1$, and using Appendix I(1), we have:

$$\alpha = \frac{1}{c_I}\int rds \quad \text{where } c_I = a^2(1-\tfrac{1}{4}e^2)+O(e^4)$$

$$= (1-\tfrac{1}{4}e^2)^{-1}\int_o^\theta\left(\frac{r^2}{a^2}+\tfrac{1}{2}(e^2\sin^2\theta+4e^3\sin^2\theta\cos\theta)\right)d\theta+O(e^4)$$

$$= (1-\tfrac{1}{4}e^2)^{-1}\int_o^\theta(1+2e\cos\theta+e^2\cos^2\theta-2e^2\sin^2\theta-2e^3\sin^2\theta\cos\theta$$
$$-2q_1e^3\sin^2\theta\cos\theta+\tfrac{1}{2}e^2\sin^2\theta+2e^3\sin^2\theta\cos\theta)d\theta$$

$$= \int_o^\theta(1+2e\cos\theta+\tfrac{5}{4}e^2\cos2\theta+\tfrac{1}{2}e^3\cos\theta-2q_1e^3\sin^2\theta\cos\theta)d\theta$$

$$\therefore\ \alpha=\theta+2e\sin\theta+\tfrac{5}{8}e^2\sin2\theta+\tfrac{1}{2}e^3\sin\theta-\tfrac{2}{3}q_1e^3\sin^3\theta+O(e^4).$$

Now $q_1=1$ for the exact ellipse, $q_1=\frac{1}{2}$ for via buccosa W. Hence when the two orbits are compared geometrically,

Defect in eccentric equation $= \frac{1}{3}e^3\sin^3\theta+O(e^4)$.

Alternatively when the two are compared contemporaneously,

$$\theta_w-\theta_o=-\tfrac{1}{3}e^3\sin^3\alpha+O(e^4),$$

in agreement with the result obtained from the inverse (Newtonian) form worked out by Whiteside: Keplerian Planetary Eggs, p. 18, note 23, series (a).

Consideration of the defect in physical equation is inappropriate in this case, since this hypothesis is wrong at $O(e^2)$ even for the exact ellipse itself, as proved in Appendix III(2).

Assessment of W_{II} by applying Time-measure II.

From Appendix III(3), now working correct to $O(e^3)$, putting $q_o=1$, andusing Relation B_2 of Appendix I(2), we have:

$$\alpha = \frac{a}{c_{II}} \int r d\beta \quad \text{where } c_{II} = a^2 + O(e^4)$$

$$= \int_o^\beta [1 + e\cos\beta + (1-q_1)e^3 \sin^2\beta\cos\beta]d\beta + O(e^4)$$

$$\therefore \alpha = \beta + e\sin\beta + \frac{1}{3}(1-q_1)e^3\sin^3\beta + O(e^4).$$

Now $q_1=1$ for the exact ellipse, $q_1=\frac{1}{2}$ for via buccosa W.

$\therefore$ Defect in physical equation $= \frac{1}{6}e^3\sin^3\beta + O(e^4)$.

Then using Relation B_2 of Appendix I(2) we obtain:

$$\alpha = \theta + e\sin\theta + \frac{1}{4}e^2\sin 2\theta - (\tfrac{1}{2}-q_1)e^3\sin\theta + (\frac{2}{3}-q_1)e^3\sin^3\theta + \frac{1}{3}(1-q_1)e^3\sin^3\theta$$

$$+ e\sin\theta(1-\tfrac{1}{2}e^2\sin^2\theta) + e\cos\theta\,(e\sin\theta + \tfrac{1}{2}e^2\sin\theta\cos\theta) + O(e^4)$$

$$\therefore \alpha = \theta + 2e\sin\theta + \frac{3}{4}e^2\sin 2\theta + q_1 e^3\sin\theta - \frac{4}{3}q_1 e^3\sin^3\theta + O(e^4).$$

Hence when the two orbits are compared geometrically,

Defect in eccentric equation $= -\frac{1}{2}e^3\sin\theta + \frac{2}{3}e^3\sin^3\theta + O(e^4)$.

These two results are evidently compatible with the defect in optical equation, evaluated above. Additionally, when the two orbits are compared contemporaneously,

$$\theta_w - \theta_o = \tfrac{1}{2}e^3\sin\alpha - \frac{2}{3}e^3\sin^3\alpha + O(e^4),$$

in agreement with the result stated by Whiteside in Studies in History and Philosophy of Science, note 2, pp. 391-2.

Assessment of W_{III} by applying Time-measure III.

By the area-hypothesis, Newtonian style, as worked out in Appendix III(4), putting $q_o=1$, we have:

$$\alpha=\theta+2e\sin\theta+\frac{3}{4}e^2\sin2\theta+e^3\sin\theta-\frac{2}{3}(1+q_1)e^3\sin^3\theta+O(e^4),$$

where $c_{III}=a^2(1-\frac{1}{2}e^2)+O(e^4)$.

Now $q_1=1$ for the exact ellipse, $q_1=\frac{1}{2}$ for via buccosa W.

$\therefore$ Defect in eccentric equation $=\frac{1}{3}e^3\sin^3\theta+O(e^4)$.

When the two orbits are compared contemporaneously,

$$\theta_w-\theta_o=-\frac{1}{3}e^3\sin^3\alpha+O(e^4),$$

in agreement with the result obtained from the inverse (Newtonian) form worked out by Whiteside: Keplerian Planetary Eggs, p. 18, note 23, series (b), as originally stated by him in Newton's Early Thoughts on Planetary Motion, p. 129, note 42.

Equivalently by the area-hypothesis, Keplerian style, also from Appendix III(4), we have:

$$\alpha=\beta+e\sin\beta+(1-q_1)e^3\sin\beta-\frac{1}{3}(1-q_1)e^3\sin^3\beta+O(e^4).$$

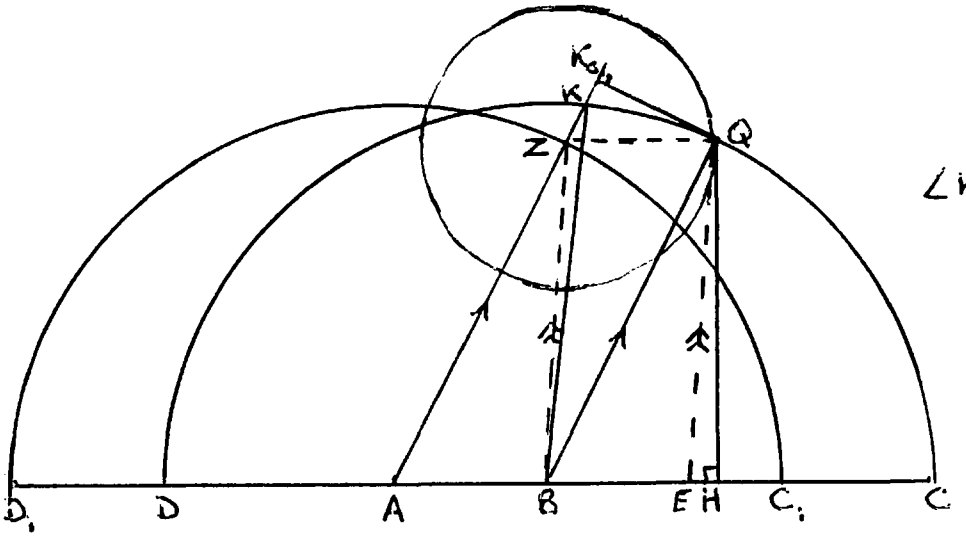
$\therefore$ Defect in physical equation $=\frac{1}{2}e^3\sin\beta-\frac{1}{6}e^3\sin^3\beta+O(e^4)$.

These two results are again compatible with the defect in optical equation, as expected.

APPENDIX III(8) to §9

THE TRIPLET OF EQUANT HYPOTHESES

Reference: Letter to Fabricius 11.10.1605: G.W. XV Letter 358 lines 330-64.



AZ = BQ = BC = a
BE = AB = ZQ = ae
∠KAC = ∠QBC = β
∠KBC = α

I suggest that the reason for the formulation of this triplet of orbits was Kepler's hope that they would supply a smooth transition from the Ptolemaic equant hypothesis for a circle, via some member of the Ovoid group, to give a representation of mean anomaly appropriate to some curve of mediate grade. Whiteside, in Keplerian Planetary Eggs, pp. 11-12, was the first to direct attention to the existence of this triplet, which was not mentioned in Astronomia, and has pointed out that the figure drawn by Caspar (G.W. XV p. 248) is badly out in its proportions (p. 19 footnote 32), so he has reproduced Kepler's intentions, as stated in the text: we recall from the Introduction that it was with reference to this particular letter that Caspar deplored the series of errors for which, presumably, two amanuenses were responsible. However, after some further investigation I find that it is not possible to arrive at a more sensible interpretation by assuming the presence of mistranscriptions in the text, and therefore the underlying fault must be attributed to Kepler himself, in that, for all three grades of orbit, he proposed:

$\alpha = \angle ZBC$ (lines 337,347,355)

$= \angle QEC$, since $BZ \| EQ$.

as in the Ptolemaic equant hypothesis for a circle. Hence, as set out in Appendix III(1) (putting $q_o = \frac{1}{2}$ for the circle) we have:

$\alpha = \beta_o + e\sin\beta_o + \frac{1}{2}e^2\sin 2\beta_o + O(e^3)$, Keplerian style.

We assess the defects of this hypothesis against the area-hypothesis as standard, using the criteria stated in §2:

I Defect in optical equation $= +\frac{1}{2}(q_o - 1)e^2\sin 2\beta_o + O(e^3)$.

II Defect in physical equation $= +\frac{1}{2}e^2\sin 2\beta_o + O(e^3)$.

$\therefore$ III Defect in eccentric equation $= +\frac{1}{2}q_o e^2\sin 2\beta_o + O(e^3)$.

Thus it has not been necessary to show Kepler's construction for the typical points of the subsequent oviforms: they were supposed to lie on the epicycle centre Z, the radius of the epicycle drawn to each being such that it made an angle α with, successively, BZ produced, AZ produced, and the line joining the midpoint of AB to Z, produced: to give Q itself and two points lying inside the eccentric as required.[1] For completeness, however, I remark that, to the first order, the defect in eccentric equation is of size $\frac{1}{2}q_o e^2\sin 2\theta + O(e^3)$, and when we add this to the correct value found in Appendix III(4) we obtain:

$\alpha = \theta + 2e\sin\theta + \frac{1}{2}(q_o + \frac{3}{2})e^2\sin 2\theta + O(e^3)$, Newtonian style,

which is compatible with the expression (in inverse form) found by Whiteside.

Consequently it should be remarked that, since the defect in eccentric equation for the small-grade curve is $\frac{3}{4}e^2\sin 2\beta_o + O(e^3)$, this does not approximate to Ovoid V^2.[2] as Kepler believed (lines 349-51): [redacted] while the corresponding defect for the mediate curve is, at $\frac{1}{2}e^2\sin 2\beta_o + O(e^3)$, equal and of course exclusively due to the original and absolute defect in physical equation, as Kepler recognized (lines 357-9)-which was certainly too high for him to ignore.

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1. Whiteside has quantified their positions using successive values of 1, 0, ½, of the ratio into which AB is divided (measured from A): for compatibility we note that this ratio is equal to $\frac{3}{2} - q_o$.
2. Because that orbit is consistent with the area-hypothesis (as proved in §5), its defect is that in optical equation alone, of value $\frac{1}{4}e^2\sin 2\beta_o + O(e^3)$.

CHAPTER IV

FURTHER CONSIDERATION OF THE TWO RÔLES OF THE SUN: DEVELOPMENTS IN THE INTERVENING YEARS

§ 1 The focal geometry of the ellipse

§ 2 Kepler's physical opinions in Astronomia and beyond

Appendix IV(1) Formulae of focal geometry synthesized by Kepler

§1 The focal geometry of the ellipse.

The first rôle of the Sun listed in Chapter I, §1, was as the pole of coordinates: and it was sometime between finishing Astronomia and writing Epitome V that Kepler discovered its even more crucial significance in relation to the special grade of ellipse he had chosen as his preferred planetary orbit—the fact that the Sun occupied a focus of that ellipse: he confirmed this in Epitome V Part I Section 3 (G.W. VII p. 372 lines 16-17, 33); and it is my purpose here to examine the implications of this discovery. However it should perhaps first be unequivocally stated that this discussion is founded on the mathematical verity that the focal distance of a given ellipse is uniquely determinate—that is, of all the members of conic-system A.G. II, there is only one whose focus will coincide with the Sun at A; and the mediate ellipse turns out to be the one. Of course this goes without saying to anyone who has been mathematically trained, but it is necessary to mention it because there have, in the past, been some strange mis-statements in which the focus of the Auxiliary Ellipse has been said to lie at the Sun: Small was guilty of this aberration,[1.] and has been followed by at least one historical commentator; so I have thought it opportune to nail this error once and for all.[2.]

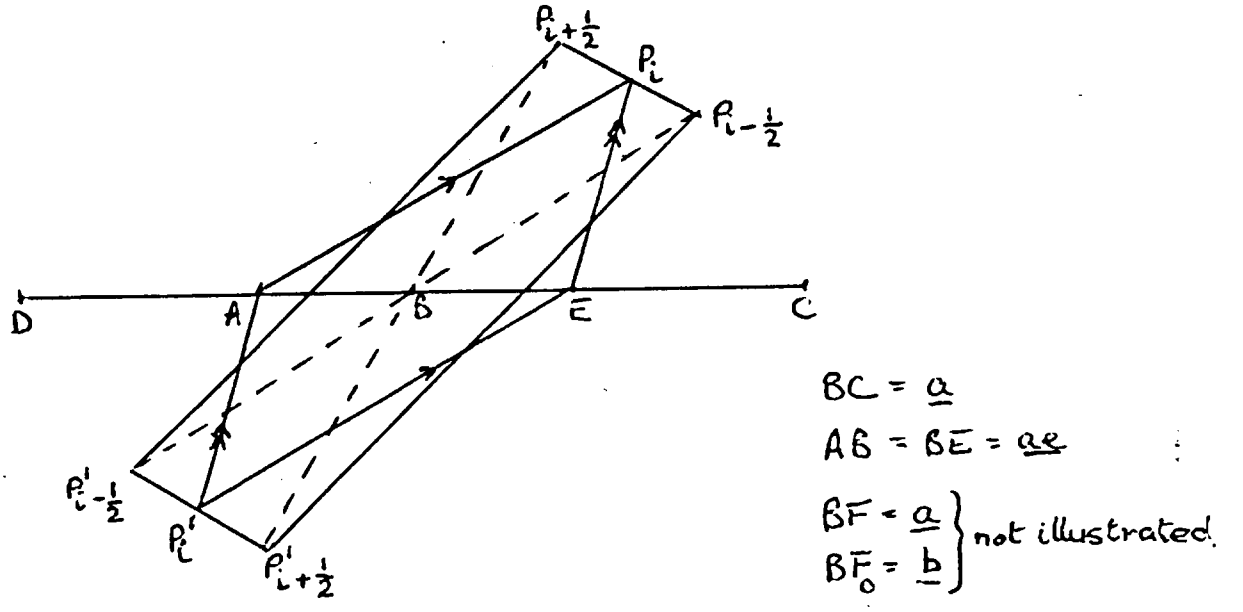
We have explained in Chapter I, §4, supported by Chapter II, §6, and Chapter III, §6, that Kepler initially defined any perfect ellipse by its possession of the lineal ratio-property, and we set out his views in Astronomia C.58 by rounding off the quotation used as the title to the sections (Chapter II, §6 and Chapter III, §8) which were concerned with separate aspects of his decision to adopt the mediate ellipse as his preferred orbit: 'The curve midway between the circle and the [Auxiliary] ellipse can be nothing except another ellipse' (G.W. III p. 366 lines 3-4); and, even more strongly: 'The circle and the [Auxiliary] ellipse are curves of the same class...between elliptic curves lies nothing except another ellipse' (G.W. XV, p. 247 lines

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1. The Astronomical Discoveries of Kepler, § 154, p. 250.
2. The converse at least can be easily appreciated from the property of the constant sum of pairs of radius vectors, as explained below.

309-12).[3.] These statements summarize Kepler's own synthesis of conic-system A.G. II, and thus it seems that, at the time of completing Astronomia, he believed the mediate ellipse to be just another member of that system, in no way remarkable. Indeed, in Protheorema IX of C.59 (G.W. III p. 369 line 27), and subsequently in that chapter, he described the position of the Sun as punctum eccentricum; this constitutes the conic-dichotomy which was the subject of an earlier article (to be cited below) in which I highlighted the lack of correlation between Kepler's work in Astronomia and the geometrical investigations only shortly before, in Astronomiae Pars Optica, in the course of which he not only invented the term focus, but also employed the concept most effectively. I still cannot account for the dichotomy, but shall present some further information on this topic, starting with a synopsis of the relevant work of Apollonius.

Fig. (i) An intermediate pair of small sectors of the Sun-focused ellipse.



3. Letter 358 to Fabricius 11.10.1605: it is with some reservations that I present this evidence, as explained in the Introduction; in fact this is one of the rare occasions I have come across when Kepler's correspondence reinforces the text of Astronomia, on a technical matter, without ambiguity.

The notable contribution of Apollonius to the focal geometry of central conics consists of eight propositions: we here cite their clear transcriptions as set out by Heath.[4.] They were arranged so that Apollonius deduced the optical property (Prop. 71) and the property of the constant sum of pairs of radius vectors (Prop. 73) using a general relationship (that is, not involving the foci) of the tangents at opposite ends of a diameter (Prop. 66) together with a definition of the foci A and E (which we state for A alone):

$$BF_o^2 = AC.AD = (BC+AB)(BC-AB) = BC^2 - AB^2.$$

Expressed in modern notation, this will be named the defining-property of the minor semiaxis $BF_o = b$:

$$b^2 = a^2(1-e^2).$$

It was precisely this property which Kepler derived, independently, from the Secant Law, in Protheorema VII of C.59 (as already implied in Chapter II, § 6, where it was used in the course of establishing Protheorema XI): yet, it was in this particular context that he described the position of the Sun as 'the point whence the eccentricity arises' (in Protheorema XII). Hence we must conclude that Kepler had temporarily forgotten a salient fact from the work of Apollonius, with which he was already familiar, as we confirm below.

It is because of the inadequacy—or absence—of diagrammatic representation of small sectors (as we shall reiterate with regret in Part B) that we can never be sure how much of Kepler's potential knowledge became actual, However, it will be proved in Chapter V, § 3—with reference to Epitome—that the level of exactitude stipulated there will ensure that arcs may properly be treated as straight lines. Therefore, we illustrate an intermediate pair of small sectors of the Sun-focused ellipse, on Kepler's behalf,[5.] in order to indicate the probable extent of his knowledge, first in the earlier context of Astronomiae Pars Optica C.IV Section 4, where he defined the foci in two distinct ways:[6.]

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4. Apollonius of Perga, pp. 113-18, p. 111: we disregard their application to the hyperbola.

5. We adopt the notation to be introduced formally in Part B.

6. Again we disregard the application to the hyperbola.

By the optical property (there used as the basis of the semi-confocal conic-system G.III):[7]

$$\left.\begin{array}{l} \angle AP_iP_{i+\frac{1}{2}} = \angle EP_iP_{i-\frac{1}{2}} \\ \angle AP_i'P_{i-\frac{1}{2}}' = \angle EP_i'P_{i+\frac{1}{2}}' \end{array}\right\} \quad \text{(G.W. II p. 91 lines 13-16).}$$

By the constant sum of pairs of radius vectors (there used as the basis of conic-system G.IV):[8]

$AP_i + EP_i = 2a$ (G.W. II p. 92 lines 38-40, p. 93 lines 6-12).

Kepler gave no authority for the first property, but for the second, he cited Apollonius—so we know that he must have been acquainted with this work, as anticipated above.

In Epitome V Part I Section 3, Kepler both restated the second property, and extended it:

$$\left.\begin{array}{l} AP_i + EP_i = 2a \\ AP_i + AP_i' = 2a \end{array}\right\} \quad \text{(G.W. VII p. 372 lines 17-19).}$$

I suggest that the proof of this extension was derived from Astronomia C.59, as follows:

$QR + Q'R = 2a$[9] (Protheorema IX, G.W. III p. 369 lines 34-5)

But, $QR = AP_i$, $Q'R = AP_i'$ (Protheorema XII, G.W. III p. 371 lines 29-31)

$\therefore AP_i + AP_i' = 2a$ (not stated in Astronomia).

This extension should be more precisely described as the constant sum of pairs of radius vectors at diametrically opposite points—the consideration which determined the choice of small sectors to be paired (in connection with the

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7. Systems of Conics in Kepler's Work, p. 677, p. 681.

8. Systems of Conics in Kepler's Work, pp. 681-2. The 'string-construction' described provides an easy way of appreciating that given foci determine a unique ellipse, as mentioned above.

9. QQ' is a diameter of the eccentric circle and R is constructed as shown in the diagrams of Chapter II, § 1 and §4 (in accordance with the standard notation of this study).

proof of Law II, referred to below). Then from the juxtaposition of both versions in Epitome it seems almost inevitable that Kepler would have recognized that:

$$AP_i = EP_i' \text{ and } AP_i' = EP_i$$

and hence

$$AP_i \| EP_i' \text{ and } AP_i' \| EP_i,$$

because AP_iEP_i' is a parellelogram (by the Scholium to Euclid I Prop. XXXIV): alternatively he might have deduced this by noticing that P_iP_i' and AE bisect each other at B. Hence, despite the absence of any concrete evidence, I shall assume that Kepler was aware of the existence of these parallels, while pointing out that the optical property was not included among those listed in Epitome V Part I Section 3. Thus in Chapter VII, Kepler's failure to achieve a proof of Law II (tackled in Epitome V Part I Section 4) which was satisfactory by modern standards[10.] will again be attributed to temporary forgetfulness of earlier knowledge—of the optical property, on this occasion.

Nevertheless I shall award Kepler the credit for having originated an analytical treatment of the focal geometry of the ellipse, on the grounds that there had never previously been any necessity to take an eccentric point as pole (nor, correspondingly, to consider areas of sectors with respect to such a point). As evidence of this, I have gathered together, in Appendix IV(1), a number of formulae that Kepler proved in Astronomia C.60 and subsequently, which are valid only when the pole coincides with the focus (even though Kepler himself did not then realize this limitation). This classification does not claim to be exhaustive, but is intended to provide some indication of his capabilities in this field—though in fact some of the formulae listed make no positive contribution to his progress.[11.]

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10. We shall eventually show that such a proof requires both the focal properties mentioned, as well as a symmetry-property of the ellipse.

11. With hindsight we may in particular regret that he dissipated his energy on evaluating the central anomaly rather than tackling the identification and properties of the punctum eccentricum.

Notwithstanding Kepler's lamentable loss of memory in relation to the optical property, he did not omit to confirm the defining-property of the minor semiaxis in Epitome V Part I Section 3:

$AB^2 = (BF-BF_o)(BF+BF_o)$ (G.W. VII p. 374 line 8)

$\therefore a^2e^2 = a^2-b^2$

$\therefore \underline{b=a\sqrt{(1-e^2)}}$.

This quantity was not given in a name by Kepler until Epitome V Part II Section 1, when he designated it the (semi-)diacentral length (G.W. VII p. 385 lines 5-7).[12] I suggest that it was because the eccentric circle had initially generated the ellipse and continued to act as the mainspring for his discoveries (as we shall further emphasize in Part B), that Kepler never appreciated the essential function of the constant b in the geometry of the ellipse, so that his eventual quantifications (the subject of Chapter VII) were expressed in terms of the radius a of the eccentric (even though he was aware of the involvement of b in the area of the ellipse, in determining what we have described as its specific area-parameter $c_o=ab$). We conclude that the gaps in his knowledge of the focal geometry of the ellipse are as remarkable as that knowledge itself.

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12. Confusingly, he first referred to it as 'the shorter diameter or latus rectum'.

§ 2 Kepler's physical opinions in Astronomia and beyond.

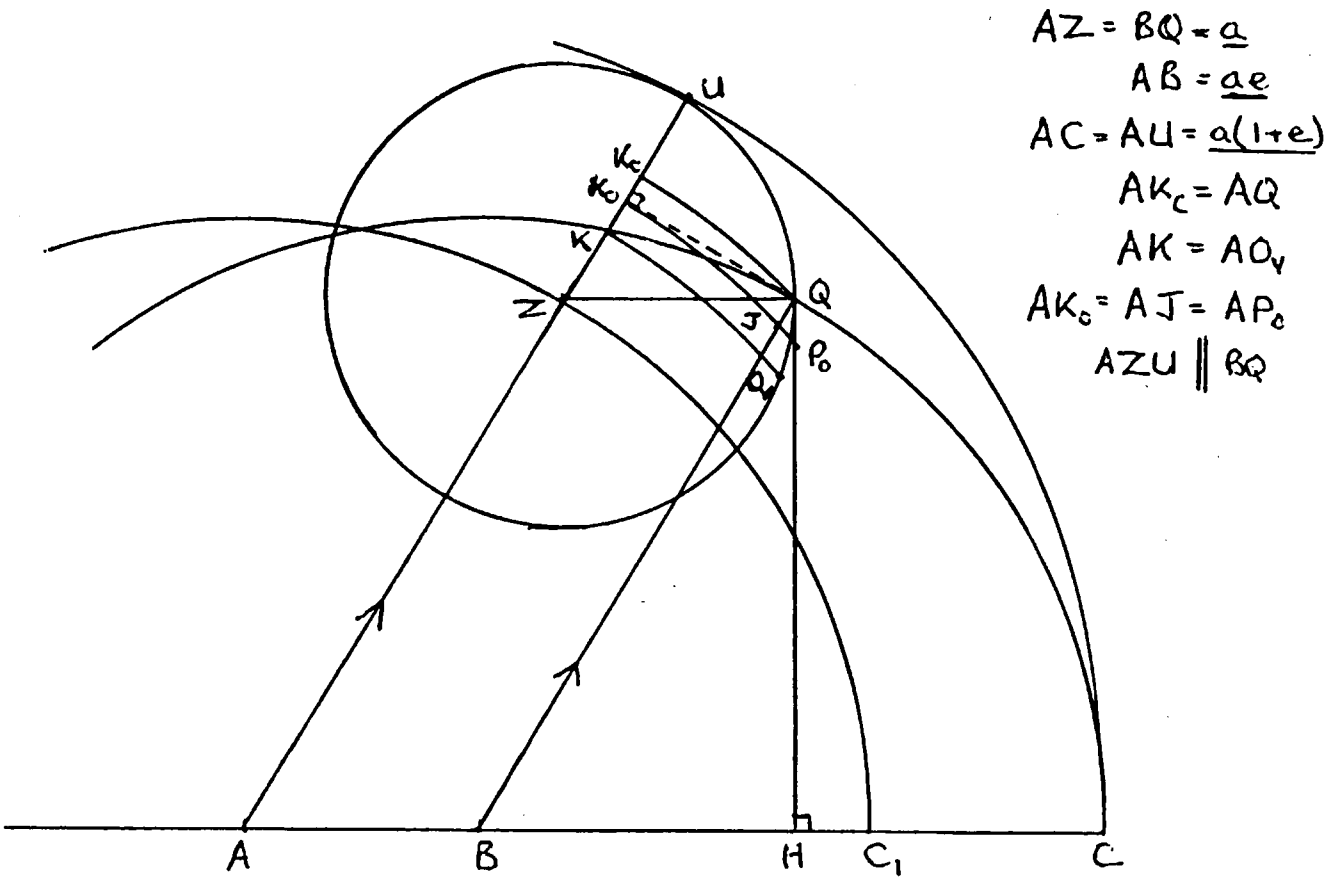
The second rôle of the Sun, again listed in Chapter I, §1, was as the source (fons) of a force F (virtus)—stated in the title to Astronomia C.32—that drove the planets round (circumagens in C.34: G.W. III p. 242 line 15)[1]. as it rotated on its axis (stated in the title to C.34: Kepler made this assumption some years before there was any question of such a rotation being established observationally).[2] This force F was said (also in the title to C.32) to be inversely proportional to distance; and this quantification has been termed Kepler's force-hypothesis. Then since the problem was that the planets did not move uniformly in circles round the Sun, Kepler had to invent ways of varying the distance to obtain a variable force F which would then produce their irregular motion.[3] His speculations (in C.34) as to the presence of magnetism in the Sun[4] and the planets were not clarified until C.57 and will therefore be discussed in that context. Alternatively (or, if necessary, additionally), Kepler envisaged that each planet independently possessed its own inherent power (vis insita in C.33: p. 236 line 15), which in practice was available to be responsible for everything otherwise unaccounted-for.

I now present a brief factual survey of the shifts in Kepler's opinions as to how various combinations of these physical causes could give rise to the orbits he successively proposed. I shall confine attention to those combinations whose effect can be quantified (if not mathematically, then at least by illustration).

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1. Or, that whirled them round (turbinatio) in Epitome IV Part II Sections 3 and 4: G.W. VII p. 297 line 6; p. 301 lines 1,2 .
2. The period of this rotation supposed (three days: G.W. III p. 245 line 2) is only of interest in connection with Kepler's numerological speculations: he adopted the observationally-determined value in due course.
3. This may seem self-evident but was stated by Kepler as an axiom in C.39 (G.W. III p. 256 lines 13-15).
4. Stated also in the title to C.34.

Fig. (ii) The three stages of Kepler's progress.



At the first stage, the orbit was assumed to be the eccentric circle, and in C.39 Kepler suggested three alternative ways of achieving this by various interpretations of the Sun's force F and some proposed vis insita:

(i) By nonuniform motion produced by the variable force from the Sun—as the distance reduced from AU=AC to AK_c—combined with some vis insita which turned the planet on the epicycle through ∠UZQ (G.W. III p. 257 line 9-p. 258 line 12). However, this involved inconsistencies, particularly that, because ∠UZQ= ∠ZAC, the two actions could not be independent as required (p. 257 lines 16-18).

Nevertheless, we remark that this method of combining motions as if in the concentric and in the epicycle, when translated into a geometrical construction, provides a physical foundation for Kepler's characteristic radius-swinging method applied to produce a circle, as anticipated in Chapter I, § 5.

(ii) Again by nonuniform motion produced by the variable force from the Sun (as the distance reduced), together with some vis insita to ensure that the planet maintained a constant distance from the centre B of the eccentric (p. 258 line 13-p. 259 line 2). Kepler thought this unlikely, not least because the centre of the eccentric was empty (p. 258 lines 17-19).[5.]

(iii) By a 'libratory' motion[6.] of the planet on the diameter of the epicycle—due to some vis insita—reducing the distance from AU to AK_c (p. 259 line 3-p. 260 line 8): the geometrical consequences of this investigation have been set out in Chapter II, §1(a). Though Kepler did not mention the force from the Sun in this connection, its variation evidently depended on the reduction of the distance from AC to AQ, referred to the eccentric orbit itself. This was obviously unsatisfactory, but Kepler remarked that the more absurd it seemed, the more acceptable would be his change to a noncircular orbit in due course (p. 260 lines 6-8).

At the second stage, Kepler adapted the opinion set out as alternative (i) at the first stage, to propose that the vis insita produced uniform motion in the epicycle (as explained in C.45 and repeated in C.49: p. 310 lines 21-2);[7.] it has been proved in Chapter III, § 5, that the geometrical properties of the Ovoids in general—and Ovoid V precisely—were determined by this physical requirement. Meanwhile, the variable force from the Sun gave rise to non-uniform motion of the planet (O_v), so that the reduction of the distance from AC to AO_v referred to the orbit itself.

At the beginning of the third stage, where Kepler reintroduced the libration on the diameter of the epicycle,[8.] he accordingly reverted to another opinion of C.39 (alternative (iii)), to believe that the motion to be ascribed to the Sun' s force took place in the eccentric circle, which in conjunction with some other cause, would produce the noncircular orbit.

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5. Compare C.57 (p. 359 lines 8-9).

6. The result of this motion is not that quantified as the libration, evaluated in Chapter II, § 4, at the third stage.

7. Where it was illustrated (p. 311), uniquely.

8. Now correctly quantified: see Chapter II §4.

His first suggestion, in C.57, was a boating analogy, by which he regarded the Sun's force as represented by a circular current in the eccentric circle (p. 349 lines 9-10), flowing nonuniformly (p. 349 line 28), while the planet was likened to a dinghy steered by the boatman's oar acting as a rudder (p. 349 lines 7-26): Gingerich[9.] has traced this analogy to Kepler's observation of a boatman in a Prague amusement park, described in C.38 (p. 255 lines 20-30). The oar headed the boat always inwards, towards the line of apsides, as shown in Kepler's own diagram (p. 349), and at first sight this arrangement appears to work qualitatively to produce the required noncircular orbit (except for the peculiarities at the apsides), when the steering function is attributed to some vis insita. However, closer investigation shows that these two causes are totally inadequate for defining the directions of component motions which could, in modern terms, give even an approximation to resultant motion in the ellipse.

Indeed, Kepler himself must eventually have recognized that libration on the diameter of the epicycle entailed motion away from the Sun in the second half of the orbit, and his revised suggestion, also in C.57, was certainly consistent with this: a theory of magnetism, in combination with the Sun's force F (still) responsible for motion in the eccentric circle, and the concept of vis insita temporarily in abeyance.[10.] The reason for Kepler's choice of this particular theory will be given below: meanwhile we remark that, rather confusingly, he illustrated both the boating analogy and the magnetic theory on the same diagram repeated throughout C.57 (p. 349, p. 353, p. 356, p. 360); whereas, we have now confirmed, they should be treated as mutually exclusive.

At this juncture the motion in the eccentric and the motion on the diameter of the epicycle combined to give the oviform of C.58 (whose typical point is J) as demonstrated in Chapter II, § 5,[11.] where we have also drawn attention

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9. Johannes Kepler and the New Astronomy, p. 358.

10. We return to this in connection with Kepler's final synthesis, in Chapter VIII, § 2.

11. The oviform of C.56 (whose typical point is J_1) was undoubtedly abandoned in C.57—perhaps even at the beginning—as we have confirmed there.

to Kepler's second flash of enlightenment, which occurred in the course of writing C.58, when he suddenly realized that because the libration represented the reduction of the radius vector from its original length AU=AC, it should therefore be measured, not in the direction BQ||AZU, but along its own length. We have already emphasized that this left Kepler free to select the perfect ellipse as his preferred orbit; another consequence was that he returned to an interpretation of libration directed precisely towards the Sun, the centre of force. Thus by the end of the third stage, Kepler had arrived at an understanding of radial variation.[12] However, his physical opinions developed no further in Astronomia, since he did not happen to consider the corresponding reinterpretation of the Sun's force (that it needed no longer to be regarded as driving the planet round the eccentric circle): unfortunately I can offer no evidence as to how, or when, he came to appreciate the significance of the concomitant transverse component required to complete his synthesis of the motions and their distinct physical causes—as he eventually set it out in Epitome V Part I Section 4.

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Kepler decided on magnetism, rather than gravity, as his explanation for the libration, even though both concepts could account for action at a distance. He had set out his common theory of both in the Introduction to Astronomia (G.W. III p. 25 lines 21-2, lines 32-4), which referred to cognate bodies possessing a mutual tendency to conjunction (depending on their relative distance): yet, it seemed to him unthinkable that the Sun could be included in this classification, to be affected by the planets. This at once ruled out any consideration of gravity as far as he was concerned, for it can only act mutually; however, the concept of magnetism was more flexible because it allowed for differentiation between an activating magnet and a passive assimilation of magnetism, and so could be adapted to the extra-terrestrial situation, as we shall see.

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12. Since this was not reached until C.58, it seems that the discussion of C.57 could more usefully have followed it, to reinforce my opinion (of Chapter II, §4) that C.57 is misplaced in its present position.

We have already mentioned (in Chapter III, §3) one instance of the fruitfulness of Kepler's 'universalizing' approach, whereby he supposed the Earth to be no different from any other planet. This indeed led him, in the present context, to the important insight that because the Earth was known to be a magnet,[13] so likewise was every other planet[14].—but it simultaneously inhibited him from anticipating Newton by taking a truly 'universal' view. Nevertheless, in Epitome, Kepler managed to formulate a theory of magnetism which gave an explanation of radial variation satisfactory in his terms: the ramifications of this are the subject of Chapter VI.

Hence it will be clear that I fundamentally disagree with those previous commentators who have suggested that there is an implied analogy between gravity and magnetism in Kepler's work: I believe that the two concepts were incompatible and that, by adopting a magnetic explanation, Kepler ensured that any consideration of gravity was automatically excluded.

Further, with regard to the two faculties of the Sun: its driving force F, and its magnetism: it seems to me not unexpected that Kepler's opinions in this previously unexplored field should undergo modification in the course of his work. However, only one commentator has made allowance for this: the rest (too numerous to mention) have, explicitly or implicitly, suggested that Kepler's 'motive force' was 'magnetic', on the evidence of his earliest views set out in Astronomia C.34. Caspar, alone, describes the former cause as 'rotating rays of force'[15].—a modern phrase that nevertheless precisely expresses Kepler's intentions—which 'non est attractoria sed promotoria' (G.W. XV p. 174 lines 94-5)[16]. This 'moving onwards' was supposed, at

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13. He cited the work of Gilbert: see Chapter VI, §1.

14. Stated originally in C.34 (G.W. III p. 246, third marginal note), assumed in C.57 (G.W. III p. 350 lines 6-8 and thereafter).

15. Kraftstrahlen in his note G.W. VII p. 589 to p. 304 (Epitome IV Part II Section 3).

16. Letter 335 to Maestlin 5.3.1605: again this is put forward with reservations, but it occurs in the course of a passage which summarizes Kepler's more mature views.

the end of Astronomia, to take place round the eccentric (as we have explained above), and only later did Kepler realize that it should be identified with transverse motion: but in no circumstances did he envisage it as 'tangential' as Dreyer has mistakenly written.[17.] In fact I am convinced that the separation of the two faculties was foreshadowed in Astronomia C.57, but it was not until Epitome that Kepler stated unequivocally that they were distinct—and so used them, as the bases of his explanations of Laws I and II in turn. We return to give a complete systematization of Kepler's physical opinions in Chapter VIII, § 2.

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17. Planetary Systems, p. 398.

APPENDIX IV(1) to §1

FORMULAE OF FOCAL GEOMETRY SYNTHESIZED BY KEPLER

In §1 we have explained that, though Kepler himself was at first unaware of this, the mediate ellipse he chose as his preferred planetary orbit was in fact the unique member of conic-system A.G. II whose focus lies at the pole—and all the information to be listed here related to this particular ellipse, and therefore must be regarded as formulae of focal geometry.

We repeat, from Chapter II, §6, the pair of parametric-polar equations by which he originally identified this ellipse:

$r = a\,(1+e\cos\beta)$: the Secant Law

$\cos\theta = \dfrac{\cos\beta + e}{1+e\cos\beta}$ (G.W. III p. 377 lines 13-26, G.W. VII p. 390 lines 7-15).

The elimination of the parameter β produces the polar equation, whatever the value of the parameter involved: but when β is correctly identified as the eccentric anomaly, as emphasized there, one or other of these formulae is strictly redundant.

Otherwise, Kepler also found:

$\tan\theta = \dfrac{b\sin\beta}{a\cos\beta + ae}$ Letter 358 to Fabricius (11.10.1605) (G.W. XV p. 250 lines 420-3).

$\cos\beta = \dfrac{\cos\theta - e}{1-e\cos\theta}$ (G.W. III p. 380 lines 19-37).[1.]

Additionally, he made an approximate evaluation of the central anomaly γ:

$\gamma = \beta - \frac{1}{4}e^2\sin 2\beta + O(e^3)$ (C. 60, Protheoremata I, II, III).

I hope to have included all such formulae which occur in Astronomia and Epitome; though the list is not expected to be comprehensive, for there may be some I have not discovered in Stereometria Doliorum, or Messekunst Archimedis, as there certainly are in Tabulae Rudolphinae.[2.]

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1. This was explained by a numerical example, from which the general statement can easily be obtained.

2. On the authority of Hammer, the editor of that volume, who particularly mentioned G.W. X C.20.

PART B

THE DISCOVERY OF THE RADIAL AND TRANSVERSE COMPONENTS OF MOTION WITH THEIR DISTINCT PHYSICAL CAUSES: AND THE CONSEQUENCES

CHAPTER V

THE COMMON FOUNDATIONS OF THE ANALYSIS OF EPITOME AND HARMONICE

§1 The two faculties of the Sun and the continued duality of treatment

§2 Some common features of the mathematics involved

§3 Stipulation of the level of analytical exactitude

§4 The fibre-mechanisms

§1 The two faculties of the Sun and the continued duality of treatment.

Our aim in Part B is to show how Kepler affirmed the two laws governing the motion of a single planet by accounting for them separately in physical terms. We recall (from Chapter IV, §1) that, by the time of writing Epitome, he had identified the position of the Sun with the focus of the orbital ellipse (G.W. VII p. 372 line 27),[1.] so the laws can now be stated in their modern versions:[2.]

Law I : the orbit is the ellipse with the Sun at one focus: as established in Chapter II, §6, this curve is completely specified by one or other of the equivalent relationships:[3.]

$$r = a(1+e\cos\beta)$$

$$\lambda = ae(1-\cos\beta).$$

Law II : the time is measured by the area of the focal sector described in that ellipse: as established in Chapter III, it is expressed in terms of mean anomaly by:

$$\alpha\propto \text{area } A \propto \beta + e\sin\beta.$$

Accordingly, the verification of these formulae—in the sense to be explained in §2 below—which will subsequently be referred to as 'the proof of Law I' and 'the proof of Law II', will be considered independently, in Chapter VI and Chapter VII respectively, just as was their discovery in Part A. Moreover, since the two formulae are interrelated parametrically by common eccentric anomaly β (strictly defined as in Chapter I, §2), it will not be unexpected to find that these proofs are conducted in Keplerian form: while we take this opportunity to confirm that the Keplerian style, once adopted (or

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1. In Epitome V Part I Section 3 (G.W. VII p. 372 lines 16-17), Kepler reintroduced this term, invented in Astronomiae Pars Optica (G.W. II p. 91 line 17), and used in Stereometria (G.W. IX p. 38 lines 20-30).
2. Hereafter, all notation will refer to this curve alone, except on the rare occasions when we need to consider the eccentric circle for comparison, when we shall revert to the distinction by suffixes defined in Chapter I, §4.
3. In what follows, the libration will replace the radius vector as the subject of enquiry, and it is therefore essential to appreciate that the libration is defined specifically for the orbital ellipse alone, measured along the radius vector.

reinstated) was retained by Kepler to the end of his astronomical work.[4]

Nevertheless Kepler also visualized the orbit in a different way—as if determined by a combination of component motions in 'altitude'[5] and 'longitude'[6]—the subjects (stated as the titles) of Epitome IV Part III Sections 3 and 2 in turn. These are evidently intended to represent mutually perpendicular variations in the coordinates of distance and true anomaly, so that the first is immediately recognized as the radial motion, obtainable from Law I, while the second is the angular motion,[7] associated with the transverse motion, but, initially, less clearly with Law II as stated above. In any event, their evaluation will not be tackled until after the completion of the proofs of Law I and Law II, referred to above, because investigation of motions will obviously entail an eventual change from Keplerian to Newtonian form (though the Keplerian style was never abandoned, as stated above):[8] further, when it becomes necessary (in Chapter VIII, §2), we shall make the transition from the Keplerian usage (mean anomaly α) to the Newtonian usage (time t) for the designation of time, using the modern relationship:

$$\frac{\delta\alpha}{\delta t} = \frac{2\pi}{T}$$

to turn radial and transverse motions into the corresponding velocities.[9]

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4. The evidence for this assertion is to be found in Tabulae Rudolphinae (G.W. X) Part II, in which the distances and true anomalies for each planet were tabulated in terms of (at equal intervals of) eccentric anomaly β.

5. This Keplerian description is in fact entirely apt when we regard the heliocentric distance r as the height of the planet viewed by an observer at the Sun.

6. We warn that neither this, nor the 'latitude' mentioned in the next footnote, have their present-day meanings in spherical astronomy.

7. A third component was also suggested: the motion in 'latitude'—which was due to a Keplerian misapprehension in connection with the constant inclination of the plane of the orbit, to be sorted out in §4 below.

8. Except temporarily, in an unsuccessful attempt to formulate the Newtonian time-measure: see Chapter VII, §3.

9. Kepler himself considered only diurnal velocities, using the relationship $\delta\alpha_m = \frac{2\pi}{T}$ (where δt = 1 day): see Chapter II, §3(b).

So it is that the duality of treatment is justified in the end, for we find that Kepler accounted for the velocity (or motion) finally to be associated with each Law in turn, by attributing its cause to a distinct faculty of the Sun, as he stated unequivocally in Epitome V Part I Section 4 (G.W. VII p. 377 lines 33-6):

> ...in order to obtain [the orbit], two components of motion are combined; one is the [transverse] motion round the Sun [produced] by one faculty of the Sun, the other is the libration [radial motion] towards the Sun [resulting] from another faculty of the Sun distinct from the first.

In this extract I have translated virtus (twice) as faculty—a term whose connotation is intended to be neutral—for the first faculty mentioned here is the driving force F of the Sun, to be interpreted here, as in Chapter IV, §2, as 'rotating rays of force'; whereas Kepler categorically denied that the second, the Sun's magnetism, could function as a force:[10.] this is established in Chapter VI, §2, where, however, the confusion of terminology is compounded because Kepler described both the driving force, and the magnetism, as vis. By contrast, we recall—again as stated in Chapter IV, §2—that in Astronomia C. 32-C.33 Kepler employed what we shall adopt as the standard designations: virtus ≡ force, vis ≡ power;[11.] but in the course of the subsequent discussion this distinction became blurred, first, in C.34, by the use of vis to describe the 'non-force', magnetism (G.W. III p. 244 line 26, p. 246 lines 8, 13, 17); while immediately afterwards, notably in Astronomia C. 57, and even more chaotically in Epitome, he seems to have referred to virtus, or vis, quite haphazardly. Thus it becomes impossible to give a consistent translation of this pair of words which will at the same time lead to a sensible interpretation:[12.] the solution adopted throughout Part B has been to choose the term 'force', or 'faculty', or to propose some concept of magnetism, as appropriate, in order to give a coherent expression of Kepler's intentions in each particular context.

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10. Despite the implication in this extract that the two faculties are conceptually on a par: see Chapter VIII, §2.
11. vis insita the inherent power of the planet.
12. The grammatical defectiveness of vis introduces a further minor complication in that Kepler had, upon occasion, to use the plural when the singular would have been correct (for instance G.W. VII p. 366 lines 13-18, 21-3, as translated in Chapter VI, §2).

§2 Some common features of the mathematics involved.

In order to emphasize such analogies as exist, we shall extend the applicability of the terms Kepler used in connection with the libration:

$\lambda = ae(1-\cos\beta)$: the completed libration[1].

$A \propto \alpha = \beta + e\sin\beta$: the completed area or mean anomaly.

These are invariably to be distinguished from the increment[2] of libration $\delta\lambda$[3] and the increment of area δA (or of mean anomaly $\delta\alpha$): unfortunately neither Kepler himself, nor his subsequent commentators, kept scrupulously enough to this exactness of terminology, as we shall see.[4]

For Kepler, what we shall refer to as 'the proof of Law I' and 'the proof of Law II' subsisted in the respective determination of the increments of libration and of mean anomaly. In the case of Law I, he tried out successive formulations of the increment of libration until he discovered the one whose correctness could be checked from the known completed libration, retroversely, by performing a summation: we shall demonstrate, in Chapter VI, §6, that this summation was tantamount to definite integration. In the case of Law II, however, he invented an independent method of deducing the increment of area which he used to quantify the increment of mean anomaly as he required (see Chapter VII, §1); he did not check that an analogous summation (of either increment) would produce the corresponding completed quantity. For interest, the latter summation will be examined in Chapter VIII, §1, in order to indicate that it would have been beyond Kepler's capabilities

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1. libratio peracta from Astronomia C. 57 (G.W. III pp. 353-4), and in the title to Epitome V Part I Section 2.
2. incrementum librationis from Epitome V Part I Section 2 and in the title to Section 1. Indeed to lessen confusion we shall substitute this expression for all the others (celeritas, fortitudo, velocitas) which Kepler variously used to describe the increment.
3. Kepler unquestionably appreciated that $\delta r = -\delta\lambda$.
4. This terminology will later be extended, notably to distinguish the completed sagitta from its increment (this quantity will be defined in §3 below).

to perform it—which is unfortunate because this would, I believe, have constituted a precedent result: while the former summation was, from a modern point of view, no more than had been achieved many centuries earlier by Archimedes.[5] Thus there is no suggestion that the two mathematical investigations which arose in the course of Kepler's astronomical work in Epitome[6] broke any new ground in relation to the conclusions reached; yet, a contribution was made to the development of the infinitesimal calculus in this context which is not inconsiderable, despite being a more subtle matter of technique.[7] The achievement was to express the typical term of the latent summation[8]—and eventually, by inference, the increment itself—in a way which does indeed supply a sound foundation for a modern treatment. This has enabled me to pinpoint certain prerequisites for integration (and differentiation) which, it seems, have not previously been explicitly categorized. For the moment I shall define only the one which is logically prior to all the rest:

Prerequisite I : the presence in the increment of a small interval.

It will be obvious that this must be satisfied, for otherwise we would be considering the summation of a series of unconnected terms—which could never provide a basis from which we could proceed to integration:[9] two other prerequisites will be defined in Chapter VIII, §1, for it is not until then that we shall have the information at our disposal to appreciate their significance (and the consequences if they were not satisfied). Meanwhile we introduce the generic term precisely structured to describe an expression which, when analyzed by present-day standards, with hindsight, will turn out to be amenable to integration in every respect.

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5. As cited in Chapter VI, §5.
6. We are not concerned here with the summations carried out in Stereometria (G.W. IX)—of which a comparably detailed secondary account is unfortunately lacking.
7. An assessment of my ground for suggesting that this technique may have originated with Kepler is deferred to Chapter VIII, §1.
8. Such a typical term will involve i, preparatory to summation for integer values of i ranging from 0 or 1 up to n.
9. Nor could a quotient of increment and interval be formed as a first step towards differentiation.

We first confirm that all the summations of Astronomia were precisely structured,[10] despite Kepler's own belief that some were not—evidenced by his statement, in Epitome V Part I Section 4 (G.W. VII p. 379 lines 4-9) that, in Astronomia C. 59, lines had been mistakenly substituted for areas.[11] We prove that this is quite wrong by reference to the tabulation of hypotheses given in Chapter III, §9, from which it is clear that every one contained some small interval with respect to which the summation was performed—and it is of great significance in what follows that this interval was invariably of linear dimensions so that the summation would produce an area. Thus the obscurities of C. 59 should rather have been attributed to confusion over the interpretation of 'the arc of the orbit', which we discuss further below.[12]

In Epitome V, likewise, the proof of Law I (initially) and the proof of Law II (throughout), were founded on the supposition that they were to be carried out 'in equal small intervals' or, as we should say nowadays 'with respect to equal small intervals'. Then, as we have anticipated in Chapter III, §9, it is in the identification of these equal (small) intervals that we encounter a major source of confusion for, prior to embarking on the proof of Law I, Kepler variously wrote (G.W. VII p. 365 line 26-p. 366 line 18): 'equal arcs of the orbit', 'equal arcs of the eccentric' or even 'equal arcs of the eccentric orbit'. It is suggested that the first phrase represented an obligatory gesture of reference in Kepler's part to his original distance-sum hypothesis (and thence back to his fundamental force-hypothesis): but it has been proved, again in Chapter III, §9, that this distance-sum hypothesis (derived from Time-measure I) had long ago, in Astronomia C. 59, been decisively dismissed in relation to the orbital ellipse; and therefore we will be justified in eliminating all mention of the small arc δs of the elliptic orbit from now on (except

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10. For the present we except the summation of Astronomia C. 57, which is to receive special consideration in Chapter VIII, §1.

11. Unfortunately this Keplerian misassessment has been repeated uncritically by Caspar, in his note p. 481 to G.W. III p. 371; and by Aiton: Kepler's Second Law of Planetary Motion, p. 87.

12. And any erroneous results, incidentally, simply indicated that a hypothesis was improperly formulated for its intended purpose.

for the purpose of modern comparisons), to regard all the above phrases as tentative approaches to the correct interpretation which only became clearly developed in the course of the proof of Law II—and was not explicitly stated until Epitome V Part II Section 1 (G.W. VII p. 384 lines 25-9): the 'completed' arc of the ellipse is measured by the arc of the eccentric cut off between aphelion and a common ordinate; small arcs of the ellipse are defined by equal small arcs of the eccentric aδβ cut off between successive common ordinates.[13.]

We shall find that the difficulty which is peculiar to the proof of Law I arises from the fact that Kepler continued to regard this interval as linearly dimensioned (a consideration which had formerly been essential in the summations of Astronomia, as we have remarked above). Nevertheless, the interval was invariably measured by reference to some anomaly (angle), so that its size can be strictly controlled, with the important consequences to be explained next.

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13. This technique was foreshadowed in Protheorema IV of Astronomia C. 59 and is illustrated in Chapter VII, §1.

§3 Stipulation of the level of analytical exactitude.

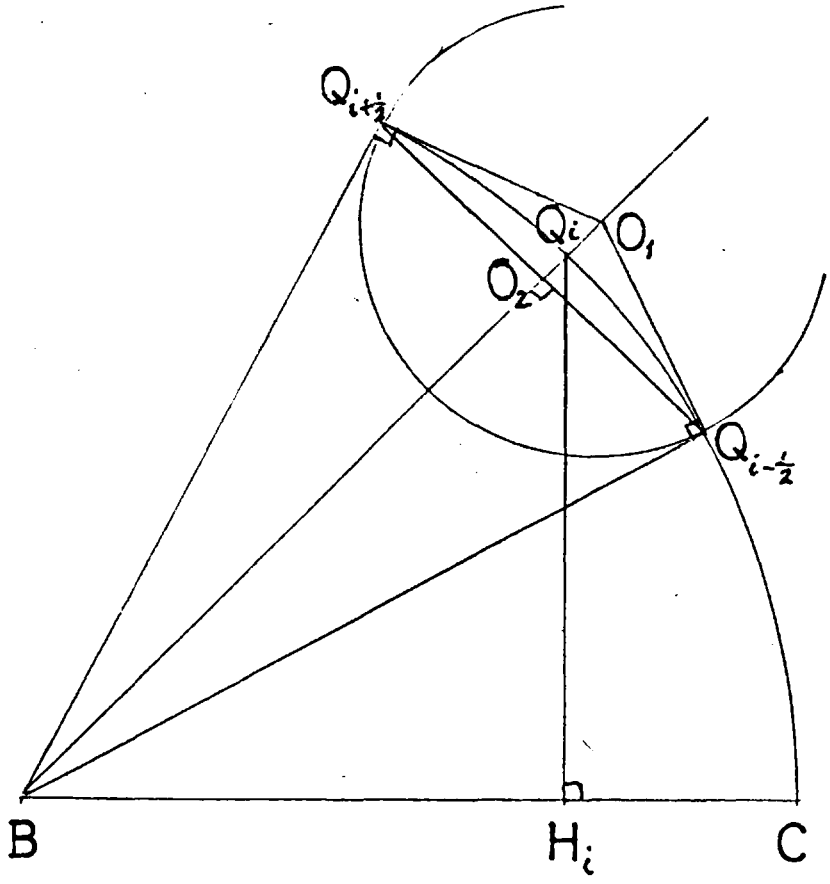
We define the level of exactitude as that beyond which any value is regarded as theoretically zero—which is now to be derived from Kepler's own work. In Astronomia Parts III and IV all distances were expressed as integers in terms of a base-unit of 10^5, while fractions were ignored: this was equivalent, in modern terms, to correcting all values to five decimal places. We shall therefore elect to follow Kepler by stipulating the level of analytical exactitude as set at five decimal places. (Corespondingly, areas were expressed in terms of a base-unit of 10^{10}, and I digress to point out that it is this distinction of size that has enabled me to confirm that distance-sums were in fact represented as areas too, as mentioned in §2 above.)[1].

This analytical criterion is to be treated as absolute for the subsequent analysis undertaken in Part B; it involves neglecting any value less than ½ Keplerian unit, by contrast with the level of observational accuracy stipulated in Part A, where values below 100 Keplerian units were disregarded in practice (see Chapter I, §3): so the present level is 200 times as rigorous. We shall investigate its consequences when successive restrictions are imposed on the size of the interval: these will inevitably be expressed analytically in the first instance (and will therefore be applicable to small intervals of both eccentric and true anomaly); but it should be emphasized that Kepler was only capable of recognizing their geometrical interpretation, which we exhibit by depicting a small arc of the eccentric circle.

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1. Other commentators have been inhibited from demonstrating this by their practice of taking the radius of the eccentric circle as unity: as deplored in the Introduction.

Fig. (i) An intermediate sector of the eccentric circle.



$BQ_{i-\frac{1}{2}}=BQ_i=BQ_{i+\frac{1}{2}}=a$
$\angle\ Q_iBC=\beta$
$\angle\ Q_{i-\frac{1}{2}}BQ_i=\angle\ Q_{i+\frac{1}{2}}BQ_i=\frac{1}{2}\delta\beta$

We take the opportunity to confirm Kepler's exclusively geometrical point of view by making the comprehensive assertion that every trigonometrical function he investigated was represented as a quantity which possessed linear dimensions: the ordinate was invariably substituted for the sine (and indeed the arc for the angle itself, as pointed out in §2 above); while we digress to remark that the present diagram supplies an illustration of the Keplerian term sagitta[2]. which, following him, we shall employ in Chapter VI, §6—meticulously:

Completed sagitta $= CH_i = a(1-\cos\beta)$.[3].

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2. More generally, the sagitta is the part of the radius cut off between a circular arc and the chord subtended by the same angle.

3. And thus in due course the distance $H_{i-\frac{1}{2}}H_{i+\frac{1}{2}}$ between successive ordinates (not illustrated here) will be described as the increment of sagitta.

Hence this quantity is the linearly-dimensioned representation of the trigonometrical function sinus versus $\beta = (1-\cos\beta)$; and so, in the subsequent analysis, we must regard every non-dimensioned (pure) trigonometrical function as an anachronism in Keplerian terms.[4.]

(i) The least rigorous restriction $\delta\beta \leqq 1°$.

It was in Astronomia C. 32 (G.W. III p. 234 line 10), I believe, that Kepler introduced the consideration of small intervals: arcus minimi whose size was unspecified; then in C. 40 (p. 263 lines 26-7) he quantified minimae particulae as 1°—and this interval was also mentioned in the course of the second stage (C. 45-50); while in C. 57 (p. 354 lines 7-19) the term-by-term summations of sines were carried out at intervals of 1° (see Chapter VIII, §1). Hence this choice for the size of the interval is well-authenticated: we convert to circular measure for the purpose of our modern comparison, to set:

$$\delta\beta \leqq 0.01745.$$

Then from the diagram:

$$Q_{i-\frac{1}{2}}Q_i = a.\tfrac{1}{2}\delta\beta$$

$$Q_{i-\frac{1}{2}}O_2 = a \sin \tfrac{1}{2}\delta\beta = a\{\tfrac{1}{2}\delta\beta - \frac{1}{6}(\tfrac{1}{2}\delta\beta)^3 + \ldots\}$$

$$Q_{i-\frac{1}{2}}O_1 = a \tan \tfrac{1}{2}\delta\beta = a\{\tfrac{1}{2}\delta\beta + \frac{1}{3}(\tfrac{1}{2}\delta\beta)^3 + \ldots\}.$$

$$\therefore\ Q_{i-\frac{1}{2}}Q_i - Q_{i-\frac{1}{2}}O_2 = a(\tfrac{1}{2}\delta\beta - \sin\tfrac{1}{2}\delta\beta) < a.\frac{1}{48}\delta\beta^3$$

$$< a.\ 0.000005 \quad \text{when} \quad \delta\beta \leqq 1°.$$

Unfortunately since the series for $\tan\frac{1}{2}\delta\beta$ is steadily increasing, the corresponding result cannot be proved, but it is easy to check from six-figure tables that:

$$Q_{i-\frac{1}{2}}O_i - Q_{i-\frac{1}{2}}Q_i = a(\tan\tfrac{1}{2}\delta\beta - \tfrac{1}{2}\delta\beta)$$

$$< a.\ 0.000005 \text{ when } \delta\beta \leqq 1°.$$

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4. Analogously, sinus versus $\theta = (1-\cos\theta)$: but this was meaningless to Kepler unless (as in Astronomia C. 57) θ was illustrated as the angle at the centre of a circle.

This is interpreted geometrically as establishing the equality of the three lengths:

$$\underline{Q_{i-\frac{1}{2}}O_1Q_{i+\frac{1}{2}} = \text{arc } Q_{i-\frac{1}{2}}Q_iQ_{i+\frac{1}{2}} = Q_{i-\frac{1}{2}}O_2Q_{i+\frac{1}{2}}}\ .$$

It is interpreted analytically as a deduction from a wellknown inequality to give the equality:

$$\underline{2\tan\tfrac{1}{2}\delta\beta = \delta\beta = 2\sin\tfrac{1}{2}\delta\beta} \text{ when } \delta\beta \leqq 1^\circ.$$

We point out that it is in fact possible to prove, or to check, similarly, that:

$$\sin\delta\beta = \delta\beta \text{ when } \delta\beta \leqq 1^\circ$$
$$\tan\delta\beta = \delta\beta \text{ when } \delta\beta \leqq 59',$$

and so to verify, as promised in Chapter III, §3, that it would have been perfectly proper for Kepler to have replaced tangents by the angles themselves, in Astronomia C. 32, provided only that he considered angles of size 1° (or less), as I have surmised.

(ii) The slightly more rigorous restriction $\delta\beta \leqq \frac{1^\circ}{4}$.

The Keplerian authority for this size of interval is to be found in Epitome, in an entirely different context, and therefore its justification is to be deferred to Chapter VIII, §1, backed up by Appendix VIII (1). Meanwhile, again converting to circular measure, we have:

$$\delta\beta \leqq 0.00436.$$

Once more from the diagram:

$$Q_iO_2 = a(1-\cos\tfrac{1}{2}\delta\beta) = a\{1-[1-\tfrac{1}{2}(\tfrac{1}{2}\delta\beta)^2+\ldots]\}$$
$$< a.\frac{1}{8}\delta\beta^2 < a.\ 0.000005 \text{ when } \delta\beta \leqq \frac{1^\circ}{4}.$$
$$Q_iO_1 = a(\sec\tfrac{1}{2}\delta\beta - 1)$$
$$< a.\frac{1}{8}\delta\beta^2 < a.\ 0.000005 \text{ when } \delta\beta \leqq \frac{1^\circ}{4}.$$

This is interpreted geometrically as establishing the coincidence of the three points $\underline{O_1 \equiv Q_i \equiv O_2}$. It is interpreted analytically as the equality:

$\cos \frac{1}{2}\delta\beta = 1$ when $\delta\beta \leqq \frac{1^\circ}{4}$.

Hence the imposition of this only slightly more rigorous restriction has ensured that the circular arc may be regarded as indistinguishable from a straight line in position as well as in length. Such an assumption was implicit in Kepler's work in Epitome, as we shall find; and in the proof of Law II he also treated the elliptic arc as a straight line. The fact that this too is ensured, when the identical restriction is imposed, is demonstrated in Appendix VII(1), where we encounter the many more geometrical equivalences which can be established specifically for the ellipse by the imposition of this same restriction—and so, to this extent, that Appendix constitutes a continuation of the present discussion.

In correlation we turn to Harmonice V (C. 4 onwards), to find that the values of the (diurnal) anomalies considered by Kepler were expressed correct to the nearest second,[5] where:

$1'' = 0.0000048^c \cong 0.000005^c$.

It seems extremely fortunate that this is, as accurately as is feasible, identical to the level of exactitude appropriate for Astronomia and Epitome, which, being now well-authenticated, will be invoked when required throughout Part B.

While I have no means of knowing whether this fresh approach originated with Kepler, it deserves remark on the grounds that the stipulation of any level of exactitude whatsoever constituted a decisive break away from the Greek tradition of completing a summation by applying the method of exhaustion—because it involved control of the interval rather than control of the sum. However, this approach should only be regarded as intermediate in a progression to a modern point of view for, once a level of exactitude—absolute, numerically, by definition—had been stipulated, it would

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5. One exception is cited below.

have been unnecessary for Kepler to have supposed that the interval vanished to zero.[6] It is perhaps of interest to suggest that the correct interpretation of a Keplerian infinitesimal may have been a discrete interval (anomaly) of less than one second—because, as above:

$$1'' \cong 0.000005^{c}:$$

that is, itself undetectable, or vanishingly small, as far as Kepler was concerned, yet such that the sum of a finite number would be macroscopic (non-negligible); in any case, emphatically not an 'indivisible', since there was no lower bound to the size of the interval selected as long as it remained discrete.[7] Thus the summation performed by Kepler (in connection with the proof of Law I) can at best be described as potentially-infinite, but nonetheless tantamount to integration in the sense that the correct result was reached, and by a method that is amenable to a treatment which is sufficiently rigorous by present-day standards: it was hardly to be expected that Kepler would have achieved these standards at the outset.

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6. Kepler himself apparently did so, on one occasion, using the phrase divisio in infinita: but this was simply to get round a temporary difficulty, as I explain in Chapter VI, §6.

7. For instance higher orders of a degree: employed in Harmonice V C. 4 Tabulation I (G.W. VI p. 307): the exception mentioned above.

§4 The fibre-mechanisms.

We here survey the mechanisms which Kepler invented to account for every aspect of planetary theory, supplementing the driving force of the Sun, while ensuring the functioning of its magnetism—this is the only mechanism with which we shall subsequently be concerned, once we have cleared up the Keplerian misapprehension mentioned in §1 above. Some of these aspects were known only in relation to the Earth, but were assumed by Kepler to be common to all the (primary) planets, in accordance with his tendency to 'universalization', to which we have drawn attention in Part A. The mechanisms he proposed were sets of parallel fibres, described, in Astronomia C. 57, as 'quasi-physical lines' in the body of the planet[1.] (G.W. III p. 352 line 32), each set possessing a unique fixed direction in space,[2.] and being supposed to function independently (by analogy with waves on the surface of water due to current and tide and wind).[3.] We will now categorize the sets and the motion, or its limitation, for which each was said to be responsible.

Fibres Set I: a rectilinear three-dimensional set parallel to the axis of rotation was supposed, as a limitation of motion, in order to maintain the constant orientation of that axis in space.

Fibres Set II: a circular set was assumed, to account for the axial rotation.

Both these sets were originally proposed for the Earth in Epitome I Part V (G.W. VII pp. 87-90), and then universalized, in Epitome IV Part II Section 6 (G.W. VII p. 316 lines 20-6, p. 318 lines 7-17):

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1. Not 'filaments', as here—a word which was later used in relation to the rays of the Sun: though it is not surprising that the two terms were confused in the preliminary investigations of Astronomia Part III and C.57.
2. It is not clear how Kepler accounted for this, since he apparently rejected the idea that the constant direction was ensured by the fibres being 'locked on' to some fixed star (G.W. VII p. 355 lines 12-21).
3. Epitome IV Part III Section 4 (G.W. VII p. 346 lines 14-24): Kepler also suggested an analogy with the work of Galen, who had classified the stomach-fibres into coexistent sets (one rectilinear, one circular, and one oblique, in fact).

because it was an extension of the whirling-force theory that planets with satellites had to rotate to drive those satellites round; and there could be no reason why all the rest of the primary planets, with yet-undiscovered satellites, should not do the same.[4.]

Fibres Set III (magnetic): Kepler invented a rectilinear set which was supposed to be permanently magnetized in relation to the Sun. He stated that they would fill the planetary body (G.W. VII p. 367 lines 19-21) but in fact this was by no means essential—it was only necessary for them to be a two-dimensional set, lying in the plane of the of the orbit, consisting of one diametral (great circle) fibre of maximum length, and other (lateral) fibres which were shorter. It is this set alone on which we shall rely in Chapter VI in connection with Kepler's magnetic explanation, meanwhile anticipating the existence of two mutually exclusive possibilities: these fibres were at first supposed to be magnetized in a fixed direction, perpendicular to the line of apsides, and later, to act like a compass needle, affected by the magnetism of the Sun. The consequences of these two alternatives are discussed in Chapter VI, §3 and §4 respectively.

Fibres Set IV: as we have remarked in Chapter I, §1, Kepler proved early in his career that each planetary motion was confined to a particular plane, and that all such planes pass through the Sun:[5.] therefore, to ensure the constant inclination to the plane of the ecliptic of each orbit, he proposed a further rectilinear set of fibres lying in that plane and parallel to the line of limits—that is, in the direction of the (maximum) inclination. Their operation was described in Epitome IV Part III Section 4 and again in Epitome V Part I Section 6: we first dispose of an unfortunate confusion of terminology noted by Kepler himself in Epitome V Part II Section 5

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4. Though Kepler quite properly remarked that he had no direct evidence of the axial rotation even for those planets which possessed satellites (G.W. VII p. 318 lines 32-4): nor, if this were established, could he then know the direction of their axes of rotation (G.W. VII p. 347 lines 11-12).

5. Restated in Epitome IV Part III Section 4 (G.W. VII p. 342 lines 41-3, p. 343 line 4 respectively): these later references are cited because, as we have already found, Kepler's opinions in Astronomia did not always agree with his more mature views in Epitome (in this case in fact they do).

(G.W. VII p. 394 lines 22-3): that he had referred to 'latitude' instead of, correctly, to 'inclination'. However, it is of more importance to point out that he wrongly conceived the variation in inclination as a motion (in the title to the first section cited), rather than as a limitation of motion, as he should have. Hence it is of no consequence that the fibres acting as he supposed could never have made the planet start to move 'down' the plane when it reached the upper limit (and vice versa); he should, of course, have recognized that he had completely accounted for the orbital motion of the planet by the two components specified in §1 (and analyzed in Chapters VI and VII), and that all that was needed was a set of fibres in the plane of the orbit to ensure that the planet remained in that plane. Then, optimally, he would have appreciated that this function was already fulfilled by the magnetic fibres (Set III above)—whether two- or three-dimensional: and hence, Set IV fibres were redundant, and should never have been introduced, in accordance with the Principle of Economy. Thus the 'motion in latitude' is to be regarded as a Keplerian misapprehension, to be excluded from further consideration, together with Set IV fibres supposed responsible for producing it.[6.]

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6. Both Koyré: La révolution astronomique, p. 439 and note 43 and p. 440 note 63, and Aiton: Kepler's Second Law of Planetary Motion, p. 82 list the first three sets of fibres, but fibres Set IV (rightly, as it turns out) do not receive even a passing mention.

CHAPTER VI

THE RADIAL MOTION EXPLAINED BY THE MAGNETISM OF THE SUN: PROOF OF LAW I

§1 The properties of extra-terrestrial magnetism

§2 The cause of the increment of libration: the concept of magnetic impulsion

§3 The increment of libration associated with the fibres of fixed direction

§4 The increment of libration associated with the deflected fibres

§5 Temporary confusion arising from Kepler's preliminary opinions

§6 The proof of Law I

§7 The radial motion

Appendix VI(1) Three erroneous versions of the angle of deflection of the fibres

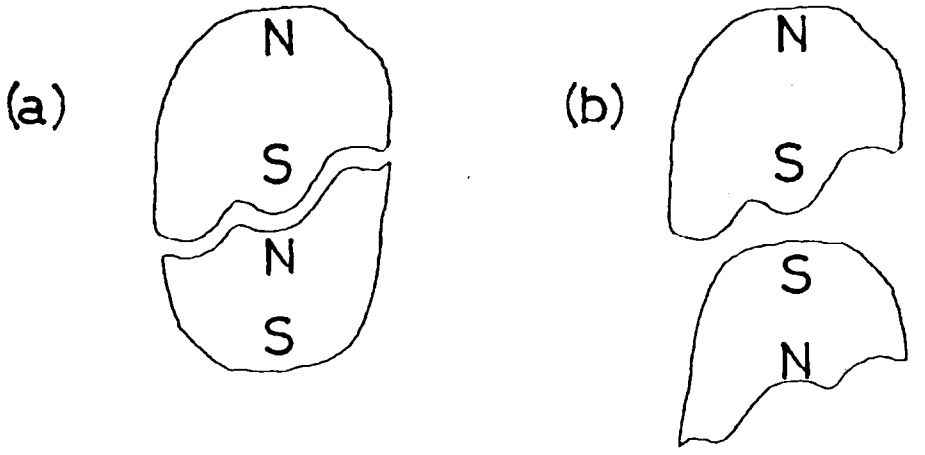
Appendix VI(2) The pair of associated mis-statements in Kepler's derivation of the completed libration

Appendix VI(3) The controversy as to whether Kepler's summation was tantamount to integration

§ 1 The properties of extra-terrestrial magnetism

Epitome IV Part IV Part II Section 3 and Part III Section 3.

Fig. (i) The lodestone.

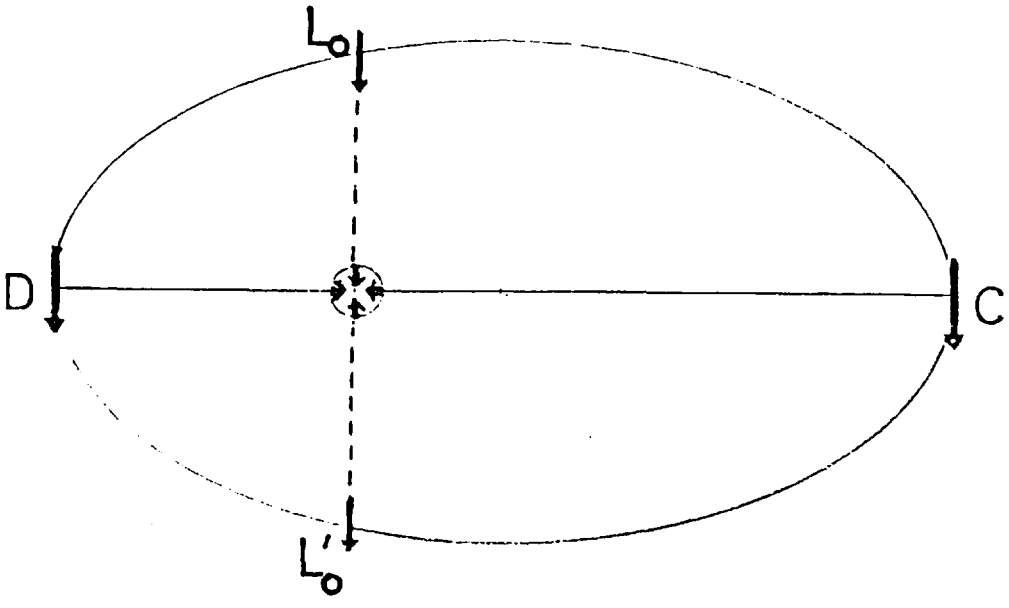


The way in which Kepler extended the known properties of terrestrial magnetism expounded by Gilbert[1] is set out in Epitome IV Part II Section 3 (G.W. VII pp. 300-1), starting with a statement of the properties of the lodestone itself as shown in the diagram of p. 300, reproduced above, on which they are illustrated, by different arrangements of the same lodestone broken into two pieces. I have marked their polarity in the usual way (Kepler himself used the term polus,[2] but not in the present context). Now Kepler undoubtedly said (p. 300 lines 21-9) that in case (b) the pieces attract one another and in case (a) repel one another, which Caspar justly notes (G.W. VII p. 589) is wrong: but Kepler discussed the

1. Kepler acknowledged this debt in both Astronomia and Epitome.
2. Astronomia C. 57 (G.W. III p. 353 line 4), and elsewhere in Epitome.

second diagram first, which might have occasioned confusion; moreover, he went on immediately to apply the theory to the planetary situation, and so I am convinced that this was not an error of principle but a simple mistranscription in which the words 'attract' and 'repel' were inadvertently interchanged.

Fig. (ii) The orbital ellipse.



Nevertheless Kepler's theory of extra-terrestrial magnetism required the imagination to envisage the Sun as a magnet of a unique kind—for which there was no terrestrial analogy—since it was supposed to possess one pole at its centre and the other spread over its whole surface (p. 300 lines 35-6). The diagram given here reproduces Kepler's own repeated figure (G.W. VII p. 301 and p. 337), in which the magnetic directions are (as subsequently in this chapter) marked by solid arrows, those of the fibres of the planet[3]. (in its elliptical orbit) being as illustrated by Kepler, those within the body of the Sun being deduced from his description: in the beginning (at L_o) the part of the planet facing the Sun was similarly polarized to the centre of the Sun, and attracted to the Sun, while the part facing away from the Sun (at L_o) was similarly

3. Set III, as listed in Chapter V, §4. This is the only set with which we will be concerned in this chapter and hereafter.

polarized to the surface of the Sun, and when turned towards it later (at L_o'), was repelled (p. 300 lines 36-40). Thus the situation at L_o corresponds to case (a) for the lodestone, and the situation at L_o' to case (b), so we must accept that Kepler understood the fundamentals of magnetic attraction and repulsion, as enunciated by Gilbert and his predecessors.

Kepler further supposed that the planetary fibres could only be activated by the magnetism of the Sun: there is an implied quantification here—of the maximum which can be assimilated by an individual planet, which will be termed the planetary magnetic capacity—but the evaluation of this (with Keplerian authority) will be deferred to the correlating discussion of Chapter VIII, §2. The contrast between the active and passive functions, however, was emphasized in Epitome IV Part III Section 3 (G.W. VII p.336 lines 6-36) in order—as I have suggested in Chapter IV, §2—unequivocally to deny that the Sun could be affected by the planets and moved from its mandatory position at the centre of the universe.[4.] The directions of the fibres themselves were originally taken to be fixed in space (perpendicular to the line of apsides), and subsequently, said to be subject to some deflection by the Sun. This pair of mutually exclusive alternatives provides a compelling reason for distinguishing the separate investigations to be undertaken in §3 and §4—though another will come to light in what follows. Meanwhile the question of the direction in which the activation of the planetary fibres took place was the subject of an initial Keplerian misapprehension, to be clarified in §2(b).

It will of course be appreciated that the very existence of extra-terrestrial magnetism was a pure speculation, for which neither Kepler, nor anyone else at that time, possessed any concrete evidence. Indeed this theory was proposed simply in order to allow him to arrive at a predetermined result, and thus the properties to be attributed to it were entirely arbitrary. This will account for Kepler's quite uncharacteristically cavalier attitude to matters which were in fact amenable to, and should have received, accurate analysis, as we shall see (up to the end of §4). It is not until the end of this chapter that it will become possible to rationalize the magnetic explanation—in Kepler's terms: this will be carried out as part of a survey of his final physical opinions, in Chapter VIII, §2.

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4. And so to rule out any consideration of gravity in connection with planetary motion.

§2 The cause of the increment of libration: the concept of magnetic impulsion Epitome V Part I Section 1 (G.W. VII pp. 364-6).

The cause of the increment of libration was here split into its active and passive constituents (p. 365 lines 26-7), each to be investigated in turn.

(a) The active cause (G.W. VII p. 365 line 32-p. 366 line 18).

This was the effect of the driving force F of the Sun (which should therefore have been termed virtus throughout this subsection—as indicated in Chapter V§1—but Kepler consistently designated it vis). Since his argument was no more than a means to an end, there is no necessity to assess its accuracy, and in particular we will not be concerned with his inability, as yet, correctly to identify the 'equal small arcs' on which it was founded (the variety of descriptions used have been mentioned already in Chapter V, §2). Kepler believed that only three factors could contribute to this cause, and we now transcribe his view on each, numbered in the order in which he originally listed them (p. 365 lines 33-4).

Factor (ii): the length of the arc.

This factor was neutral in its effect, since 'equal small arcs' (however identified) were supposed (p. 366 lines 1-4).

Factor (i): the distance of the arc.

$$F \propto \frac{1}{r} \quad \text{(p. 365 lines 36-7).}$$

This was, of course, Kepler's force-hypothesis (now applied specifically to the orbital ellipse).

Factor (iii): the time spent in the arc.

$$\delta t \propto r \quad \text{(p. 366 lines 7-8).}$$

This was the indeterminate time-axiom (direct) which, when the equal small arcs are not identified, could represent either Time-axiom I or Time-axiom II (both direct): see Chapter III, §1.[1] Additionally,

$$F \propto \frac{1}{\delta t} \quad \text{(p. 366 lines 11-13).}$$

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1. It was only as a result of the proof of Law II that Kepler was able to quantify this time-axiom: see Chapter VII, §1.

This was the Aristotelian Equivalence—when, again, 'equal small arcs' were supposed.

Then Kepler concluded (p. 366 lines 13-18):

> Therefore the weakness of the force [vis] is compensated by the length of time for which the planet is subject to that force [vis], both being in the same ratio—that of the distance from the Sun: so the final result is that, in 'equal small arcs', the measure of the cause of the increment of libration is indeed absolutely constant with respect to the driving force of the Sun.[2]

This is quantified as follows:

$$F.\delta t \propto \frac{1}{r}.r \quad \text{from (i) and (iii) above}$$

$$\therefore \quad F.\delta t \text{ is independent of distance.}$$

In Kepler's terms this reasoning was certainly internally consistent, and its peculiarity subsists in its physical significance (our account of this, however, is to be deferred to Chapter VIII, §2). Meanwhile it seems incontrovertible that the argument was designed simply to enable Kepler to arrive at this result, eliminating the active cause, so that the increment of libration could be attributed to a passive cause alone.

Nevertheless it seems quite remarkable that not one of all the previous commentators whose work I have consulted, while emphasizing the importance to Kepler of the consideration of force and its kinematical consequence, has discovered that he himself excluded it, by cancellation, in this situation—which comprises half of his eventual synthesis.

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2. This last phrase is a transcription, for Kepler actually wrote: 'the measure of the force [vis] of libration is given out by the Sun and with respect to itself, as agent, is indeed absolutely equal'.

(b) The passive cause (G.W. VII p. 366 lines 19-28, p. 367 lines 12-27).

Since Kepler had just categorically denied the involvement of force—in favour of the magnetic explanation now to be expounded—it is extremely confusing that he continued to use the description vis in this fresh context;[3] so I have had to define a new concept in order to clarify his intentions: the magnetic impulsion from the Sun will be adopted as the correct transcription of vis hereafter in Epitome V Part I Section 1.

Moreover it has to be emphasized that the completed libration, and its increment, are strictly defined only in relation to the orbital ellipse (from Chapter II, §4, reiterated in Chapter V, §1). This is vital because, in the investigations of §3 and §4 (G.W. VII pp. 366-7 and pp. 369-70 respectively), we find that what was described in the text as 'the orbit' was drawn as the eccentric circle.[4] I explain this major ambivalence by suggesting that, in anticipating his eventual need to sum the increments (to obtain the completed libration) Kepler had realized that he could not manage this except in relation to a circle (see §6). Yet the representation of 'the orbit' by the eccentric circle is in no way conducive to this aim—the circle is no more advantageous than the orbital ellipse[5]—and so in the investigations of both §3 and §4 we shall depict the orbit correctly as the ellipse, on the grounds that Kepler's commitment to that ellipse had long been irrevocable, added to the fact that it is essential to do so in order to ensure that the increment of libration can be expressly identified with its cause. Kepler stated (p. 366 lines 23-8):

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3. Employed wrongly in (a) above, for a different reason, as we have already noted.

4. I have checked this from an original edition of Epitome, p. 649 (see also G.W. VII p. 366) and p. 654.

5. Because we shall find that the angle with respect to which the summation is performed has to lie at the centre of the circle: for instance, the angle at the centre of the circle which represents the planetary body in Astronomia C. 57 (G.W. III p. 354 lines 32-3), illustrated on p. 353 and in fig. (iii) below.

> Since the actual causes of the [increment of] libration are the ray of the Sun and the magnetic fibres of the planetary body, these being two physical lines; the increment of libration is appropriately measured by the angle between these lines, that is, by the sine [of its complement].[6]

Hence, in the orbital ellipse,

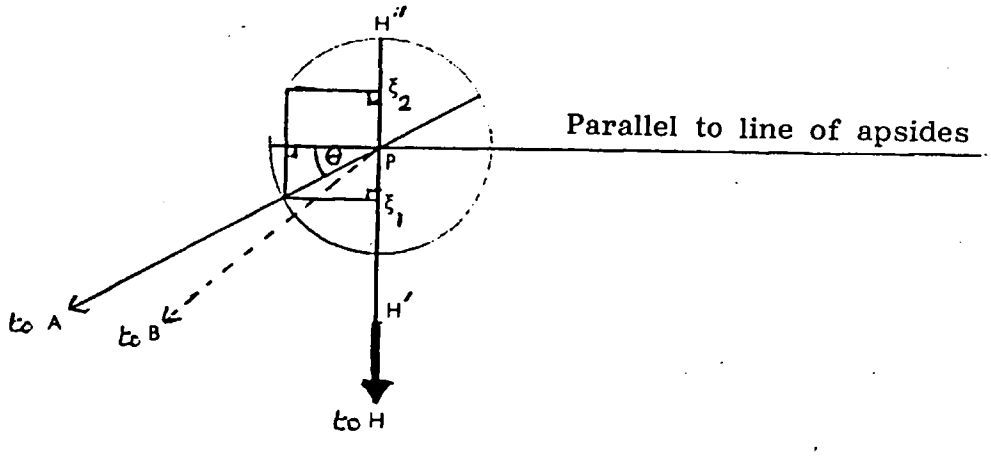
Increment of libration $\propto$ cosine of angle between fibres and solar ray.

Just earlier in the same passage, Kepler had stated (p. 366 lines 21-3):

> We consider the angle that the ray of the Sun makes with the magnetic fibres of the planet. Then the sine of the complement of this angle measures the amount of the magnetic impulsion from the Sun assimilated.[7]

This is clearly a statement of the principle of resolution, but unfortunately it involves a serious misapprehension on Kepler's part (which is not altogether surprising, since its discovery was comparatively recent) which we shall demonstrate from the physical analogies which he introduced, hoping that they would support his argument.

Fig. (iii) The body of the planet.



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6. Kepler wrote 'by its sine': this is obviously a mistranscription, from the sense of the whole passage: but it constitutes an additional source of confusion, to which Caspar unwittingly contributed, by an unusually ungrammatical statement referring to the angle and its complement, in his note p. 595 to G.W. VII p. 370.

7. admissus.

The diagram shows the body of the planet P—greatly enlarged—on which I have temporarily named points H', H", ξ_1, ξ_2, on PH symmetrically placed with respect to P, as illustrated.[8.]

Kepler first proposed an analogy with the heating effect of a solar ray on a plane parallel to the line of apsides (p. 367 lines 12-18), by comparison with the effect when the ray falls perpendicularly on the plane. Then:

$$\text{Heat assimilated} \propto \frac{\xi_1 P}{H'P} = \cos \text{HPA} = \sin \Theta$$

Secondly, he propounded a 'more elegant' analogy involving the fibres themselves (p. 367 lines 19-27), suggesting that the effect of the magnetic impulsion should be measured by the ratio of the length of a 'lateral' fibre to its maximum 'great circle' length,[9.] whence:

$$\text{Magnetic impulsion assimilated} \propto \frac{\xi_1 \xi_2}{H'H''} = \sin \Theta.$$

These two analogies are sufficient to indicate what Kepler had in mind, and it will be unnecessary to examine the two further ones he discussed: the scales (libra) and the steelyard (statera):[10.] we remark in passing that these were in the nature of thought-experiments, and may be described as 'all-purpose' to the extent that they were somewhat optimistically invoked regardless of context.[11.]

Thus the above quantifications confirm that the magnetic impulsion from the Sun was supposed constant, and the amount assimilated was due to its component perpendicular to the line of apsides, so that the planetary fibres were activated in their own direction.[12.] This situation could only be interpreted physically if Kepler had still envisaged a combination of motion round the eccentric circle together with some

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8. We are not for the moment concerned with the orbit in which the planet is moving.
9. These terms were introduced in Chapter V, §4.
10. Strangely, Caspar did not discriminate between the two in his notes to Epitome, nor even in direct translation of Astronomia C. 57 (Neue Astronomie p. 332) where he renders both libra and statera as die Waage (or die Wage).
11. Notably in connection with Law II in Epitome IV Part III Section 2.
12. The argument applies also to the case of the deflected fibres, though the physical analogies do not then make sense.

other motion which would change that circle into the orbital ellipse.[13.] However, this tentative physical opinion had been abandoned as a consequence of the second flash of enlightenment in Astronomia C.58 (see Chapter IV, §2), and hence I cannot accept that such was Kepler's view at this late stage: I prefer to believe that his apprehension of the operation of the principle of resolution was initially defective. I shall therefore revise the last sentence of the second statement quoted above in accordance with what I am convinced Kepler must have meant (p. 366 lines 22-3): 'The sine of the complement of this angle [between the ray of the Sun and the magnetic fibres of the planet] measures the effect on the planet—in the direction of the Sun—of the magnetic impulsion from the Sun'. With this revision, it is evidently the planetary magnetic capacity which is supposed constant and, as far as the planet is concerned, the magnetic impulsion from the Sun is equivalent to the component of its own magnetic capacity activated in the direction of the Sun, so that:

Magnetic impulsion $\propto$ cosine of angle between fibres and solar ray.

Hence, the magnetic impulsion is identified as the cause of the increment of libration.

Indeed, from the juxtaposition of the two statements in the passage translated above, it is quite clear that this result was what Kepler intended, but it has been demonstrated in the foregoing discussion that the conclusion is valid (assuming the reality of the fibres) only when the orbit is correctly represented as the ellipse, and when the operation of the principle of resolution is revised as suggested. Thus the separate investigations of §3 and §4 will be carried out on the basis that both these conditions are fulfilled. This being so, the physical background of each was sound (in Kepler's terms) and the contrast between them will depend on geometrical considerations alone: we shall

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13. A change from motion in the eccentric to motion in the ellipse can be achieved by motion down the ordinate, in modern terms, as indicated at the end of Prolegomenon III—but this involves consideration of the (tangential) velocities in orbit, which was most emphatically non-Keplerian.

defer further examination of Kepler's physical opinions to the correlating discussion of Chapter VIII §2.

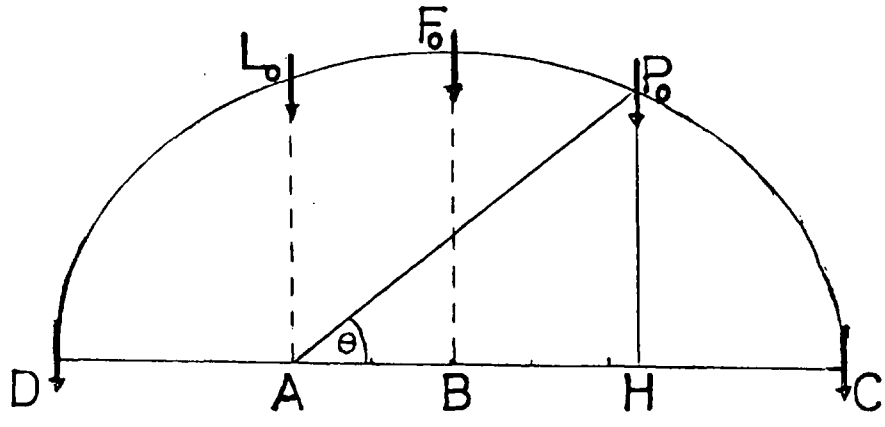
Finally, we remark that the description, in both statements, of the angle 'between the solar ray and the magnetic fibres of the planet' is sufficiently vague to be applicable whether that angle is the complement of the true anomaly (as here, and again in §3) or of the eccentric anomaly (as in §4): that is, these two investigations involve a further pair of mutually exclusive alternatives distinguished by the style—Newtonian or Keplerian, respectively—of the angle that 'measures' the 'arc of the orbit',[14.] in terms of which they are conducted.

14. This phraseology will be adopted because it was Kepler's own (p. 370 lines 6-7).

§3 The increment of libration associated with the fibres of fixed direction

Epitome V Part I Section 1 (G.W. VII pp. 366-7).

Fig. (iv) The orbital ellipse.



BC = $\underline{a}$
AB = $\underline{ae}$
AP_o = $\underline{r}$

Additional confirmation that this figure is correctly drawn as the orbital ellipse may be obtained from its connection—via the fixed direction of the fibres (perpendicular to the line of apsides)—with the repeated diagram of Epitome IV Part II Section 3 and Part III Section 3 (G.W. VII p. 301 and p. 337), reproduced in §1 above, which unquestionably depicted that ellipse. We have inserted solid arrows, indicating the direction of the fibres, both to strengthen this connection, and to emphasize the contrast in that respect with the following investigation of §4, as mentioned in §1. The further contrast, anticipated in §2(b), is in the style in which each investigation in conducted. Here, in Newtonian style, we have:

The 'arc of the orbit' is said to be 'measured' by Θ.

The following series of statements, which are merely repetitive in the present situation, are set out because they provide a supposed foundation for the reasoning in the investigation of §4. From p. 366 lines 23-8, translated in §2(b), we have:

Increment of libration $\propto$ cosine of angle between fibres and solar ray.

But, from the diagram,

Angle between fibres and solar ray = complement of 'arc of orbit'
(p. 367 lines 2-3)

$\therefore$ Increment of libration $\propto$ cosine of complement of 'arc of orbit'

$\therefore$ Increment of libration $\propto$ sine of 'arc of orbit' (p. 367 lines 4-6).[1.]

$\therefore$ Increment of libration $\propto \sin\theta$.

This is simply a reiteration of the incorrect formulation of the increment of libration which had already been suggested in the course of Kepler's preliminary enquiry of Astronomia C. 57. We shall return in §5 to evaluate the ensuing disparity when this formulation was used to calculate the completed libration.

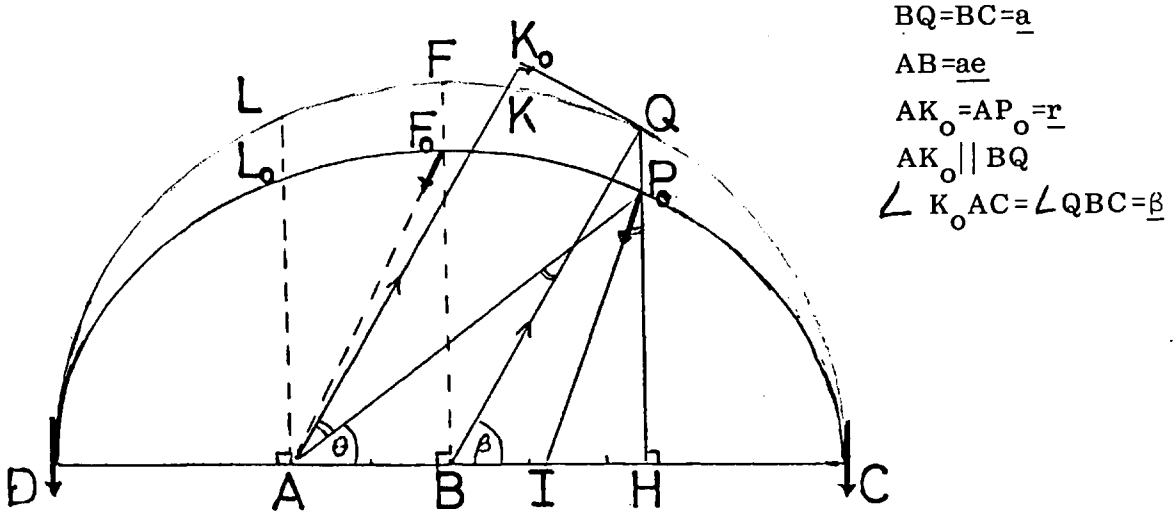
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1. From Prolegomenon IIA, we find that this proportion is invalid, for it should correctly have included the non-constant factor $\frac{aer^2}{b^2}$, and hence it is fortunate that Kepler now abandoned this formulation.

§4 The increment of libration associated with the deflected fibres

Epitome IV Part III Section 3; Epitome V Part I Section 1 (G.W. VII pp. 369-70).

Fig. (v) The orbital ellipse.



In Epitome IV Part III Section 3 Kepler introduced the supposition that the fibres should be subject to some (slight) deflection by the Sun (G.W. VII p. 339 lines 1-11), just as a magnetized needle is affected by an external source of magnetism (p. 339 lines 27-32): the deflection is measured from the fixed direction (perpendicular to the line of apsides) supposed in §3. In the present discussion we shall confine our attention to the geometrical consequences of this, deferring consideration of its physical implications to Chapter VIII, §2.

The question of the diagram is further complicated by the fact that the figure reproduced by Caspar (G.W. VII p. 369) is a duplication of that on p. 366 (where it should have occurred uniquely):[1].

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1. Caspar has remarked this misplacement in his note p. 594 to G.W. VII p. 370.

that given by Kepler[2]. was a repetition of the figure that illustrated Epitome IV Part III Section 3 (G.W. VII p. 339 and p. 341), showing the deflected fibres. Moreover, though drawn by Kepler on all three occasions as the eccentric circle (I have checked), it is represented here as the orbital ellipse, on the grounds set out in §2(b), supported by textual evidence (G.W. VII p. 340 lines 6-7, p. 341 lines 10-11) where Kepler stated (in our standard notation) $AF_o = BC = a$—which is the defining-property of the ellipse with one focus at A.

(a) Quantification of the angle of deflection that Kepler required.

As well as the contrast in the direction of the fibres, the present investigation is to be further distinguished from that of §3 by the style in which it is conducted. Kepler wrote (p. 369 lines 21, 24) of some 'alteration' of the angle under consideration—but it is only in retrospect, by its consequence, that we can discover that he intended to reinstate the Keplerian style, by which:

β 'measures' the (restyled) 'arc of the orbit'.[3].

Then the statements made in connection with §3 were repeated as if they constituted an argument:[4]. to provide, in reality, a dubious post-justification:

Again from p. 366 lines 23-8, translated in §2(b),

Increment of libration $\propto$ cosine of angle between fibres and solar ray

And as in §3 (from the diagram, in that case):

Angle between fibres and solar ray $\propto$ complement of (restyled) 'arc of orbit'

$\therefore$ Increment of libration $\propto$ sine of (restyled) 'arc of orbit'.[5].

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2. P. 654 in the original edition of Epitome.
3. Not formally confirmed until Epitome V Part II—as anticipated in Chapter V, §2.
4. It is of interest that this same 'argument' is employed, with a different interconnecting term, to give the same result in the case of erroneous version (I'), discussed separately in Appendix VI(1).
5. The constant involved is correctly ae in either proportion: compare version (I') in Appendix VI(1).

This was the result Kepler required in order to embark on the derivation of the completed libration (the subject of §6 below), for which he started from the formulation which is indeed correct in modern terms (see Prolegomenon IIA):

Increment of libration $\propto \sin\beta$.

Nevertheless it will turn out to be essential to appreciate that Kepler habitually envisaged a sine as represented by an ordinate (possessing linear dimensions). Therefore we take the opportunity to confirm that this formulation (in Keplerian style) remains valid whether he had in mind the ordinate of the ellipse ($P_oH = b \sin\beta$), as supposed here, or the ordinate of the eccentric circle ($QH = a \sin\beta$), which he employed in the derivation of §6. Hence the ambivalence, highlighted in §2(b) (between the curve drawn as the eccentric but described as 'the orbit') becomes of no further consequence in practice.

Since this is the value of the increment of libration on which the eventual synthesis will be founded, it will be fruitful to identify the angle of deflection which is compatible with this result:

Angle between fibres and solar ray $= \frac{\pi}{2} - \beta = \angle IP_oA$ from the diagram.

$\therefore$ Angle of deflection $\angle IP_oH = \left(\frac{\pi}{2} - \theta\right) - \left(\frac{\pi}{2} - \beta\right)$

$\therefore$ Angle of deflection $= \beta - \theta$

This is the optical equation updated for the ellipse, evaluated from Appendix I(2):

Angle of deflection $\angle IP_oH = e \sin\beta - \frac{1}{4}e^2 \sin 2\beta + O(e^3)$.

(b) Erroneous versions of the angle of deflection.

Since the fibres themselves were imaginary, their deflection must, from a modern point of view, be regarded as entirely arbitrary, chosen, as above, simply to be compatible with the desired increment of libration. Nevertheless, Kepler quite misguidedly attempted to justify the angle of deflection, and this has produced no less than three erroneous versions. Because these are mathematically quantifiable, they have had to be examined in order to ensure that nothing of importance in Kepler's work has been missed-though to avoid engendering further confusion, this has been carried out in Appendix VI(1), where they have been separately illustrated, with the results alone tabulated below for comparison: however, a brief discussion of the spurious reasoning involved which will throw some light on Kepler's thought-process is to be undertaken here. The points I', I", I''' are used to distinguish the three versions: they mark the intersection of the supposed direction of the deflected fibre with the line of apsides in each case, to indicate that every version is different, and that none of them coincides with the correct version denoted by I.

It has been emphasized throughout the analysis of Astronomia (in Part A) that Kepler invariably tried to arrange geometrical representation for every innovation he proposed, before beginning any calculations. So I suggest that it was because version (I') was not, in general, amenable to such representation[6]. that it was abandoned, and Kepler began instead to think in terms of an optical concept.[7]. His choice of the optical equation for the eccentric circle—version (I")—is altogether to be regretted, for it necessitated a temporary regression to the consideration of a circular 'orbit'—which, however, does not modify my opinion (expressed in §2(b)) of his current commitment to the orbital ellipse, because of the

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6. Only at the quadrant (except in modern terms) as indicated in Appendix VI(1).

7. This description is not actually used in either Epitome IV Part III Section 3 or in Epitome V Part I Section 1, but the optical equation is mentioned in Epitome IV Part III Section 4 (G.W. VII p. 348 lines 6-7) in an implied reference back to Section 3.

disregard for accuracy displayed in this particular piece of analysis, of which Kepler must have been well aware.[8] Nonetheless this choice is of some historical interest since he stated in Epitome IV Part III Section 3 (G.W. VII p. 341 lines 14-20), and evidently continued to believe, that the maximum deflection occurred precisely at the quadrant (when the fibre pointed directly towards the Sun): but, the optical equation is not maximum at the quadrant.[9] This error, though minimal, intriguingly recalls the situation in Astronomia C.56 (discussed in Chapter II, §4) where the position of the maximum optical equation was similarly misattributed. To account for this coincidence, I remark that the first of Kepler's two flashes of enlightenment—all we know of—came to him in connection with that optical equation, and led to the discovery of the radius vector of the orbital ellipse and thence to a formula for the libration. Thus there is a possibility that the dependence of the libration on the optical equation for the circle may have developed into a minor fixation—though this is something that can never be proved.

Version (I'''), on the other hand, is what Kepler might have been tempted to choose had he drawn the orbit correctly as the ellipse—because it is amenable to direct geometrical representation—and he might not have realized that it involved the central anomaly—with which it was not natural for him to work, as pointed out in Chapter I,§2—rather than the eccentric anomaly, on which his progress always depended. Certainly it has to be acknowledged that Kepler never successfully sorted out the distinction between the updated optical equation and the optical angle (at $O(e^2)$), even though he recognized the difference between the central anomaly and the eccentric anomaly, also at this level: these facts are confirmed in Appendices II(4) and IV(1) respectively.

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8. G.W. VII p. 369 line 41—p. 370 line 3, including the fabricated equivalence (p. 369 lines 43-4) would have been better omitted.

9. The quadrant-values and the positions of the maxima have been tabulated for every version, in order to present a complete comparison.

Now adopting a Keplerian standpoint, it was extremely unfortunate that he was prevented—by depicting 'the orbit' as a circle—from illustrating the angle of deflection he required, as evaluated in (a) above. Had he drawn the correct diagram, he would have realized that this angle is in fact the change in direction of libration which resulted from the second flash of enlightenment in Astronomia C. 58: that is, the angle through which the radius vector had to be swung to construct a point of the orbital ellipse by Kepler's characteristic method (see Chapter II, §5), as marked in fig. (v) above:

$\angle K_0AP_0$ = angle between BQ and AP_0 = $\beta-\theta$.

Had he been capable of recognising it, Kepler would have been delighted with this correlation, which would have seemed to him to have supplied overwhelming evidence of the validity of his magnetic explanation.

However, there is no indication at all that Kepler ever managed to identify the angle of deflection he required, and by reference to the tabulation we see that all the erroneous versions were inaccurate at $O(e^2)$—by $\frac{1}{4}e^2$ at maximum, round the octants, though only minimally so, at $O(e^3)$, at the quadrant. This residual flaw is of no significance in his eventual synthesis: meanwhile it appears that the purpose of the present investigation was an attempt to justify the introduction of a new formulation from which Kepler could derive the completed libration:

Increment of libration $\propto \sin\beta$.

COMPARISON OF THE CORRECT AND ERRONEOUS VERSIONS OF THE ANGLE OF DEFLECTION

	Angle between fibres and solar ray	Angle of deflection of fibres: General formulae from Appendices I(2) and I(3)	Angle of deflection of fibres: Quadrant-value	Position of maximum from Appendix II(4)
Correct version (I)	$\angle IP_oA=\frac{\pi}{2}-\beta$	$\angle IP_oH=\angle K_oAP_o$ $=e\sin\beta-\frac{1}{4}e^2\sin 2\beta+O(e^3)$ Optical equation updated for ellipse	$\angle AF_oB=\sin^{-1}e$ †	Between F_o and L_o (slightly nearer L_o)
Erroneous version (I ')	$\angle I'P_oA=(\frac{\pi}{2}-\theta)-\sin^{-1}(e\sin\beta)$	$\angle I'P_oH=\angle AKB=\angle KBC$ $=\sin^{-1}(e\sin\beta)$ $=e\sin\beta+O(e^3)$ Not an optical concept	$\angle AF_oB=\sin^{-1}e$ (maximum)	$\angle AF_oB=\sin^{-1}e$ (at ^{o}quadrant)
Erroneous version (I ")	$\angle I''QA=\frac{\pi}{2}-\beta$	$\angle I''QH=\angle AQB$ $=e\sin\beta-\frac{1}{2}e^2\sin^2\beta+O(e^3)$ Optical equation for eccentric circle	$\angle AFB=\tan^{-1}e$	At L, of value $\angle ALB=\sin^{-1}e$ ††
Erroneous version (I "')	$\angle I'''P_oA=\frac{\pi}{2}-\gamma$	$\angle I'''P_oH=\angle AP_oB$ $=e\sin\beta-\frac{1}{2}e^2\sin 2\beta+O(e^3)$ Optical angle for ellipse	$\angle AF_oB=\sin^{-1}e$ †	Very near L_o

† denotes the quadrant-value stated by Kepler in Epitome IV Part III Section 3 (G.W. VII p. 339 lines 43-5, p. 340 lines 10-11), repeated in Epitome V Part I Section 1 (G.W. VII p. 369 lines 30-3), since he believed that the fibres pointed directly towards the Sun there.

†† denotes the maximum value stated by Kepler in Epitome IV Part III Section 3 (G.W. VII p. 341 lines 16-20).

§5 Temporary confusion arising from Kepler's preliminary opinions Astronomia C. 57.

(a) The irreducible disparity between the alternative formulations.

We have just been confronted with two mutually exclusive formulations of the increment of libration, which are most conveniently distinguished by their style: respectively in terms of the sine of the true anomaly θ (Newtonian style, from §3) and the sine of the eccentric anomaly β (Keplerian style, from §4), both in relation to the same point P_0. In this section we shall concentrate on the trigonometrical relationships involved: the necessary linear constant of proportionality is identified in §6. Then in order to appreciate the problems of Astronomia C. 57—and their solution in Epitome V Part I, we will assess the disparity between the two formulations by comparing their sums—without, for the moment, concerning ourselves with the question of which is right and which wrong: applying modern methods, by integration we express each sum of sines as the corresponding sinus versus,[1] to find:

$$\begin{aligned}\text{Disparity between alternatives} &= \text{sinus versus } \beta - \text{sinus versus } \theta \\ &= (1-\cos\beta)-(1-\cos\theta) \\ &= \cos\theta - \cos\beta \\ &= \cos(\beta - e\sin\beta)-\cos\beta+O(e^2) \\ &= e\sin^2\beta+O(e^2) \\ &\text{or } e\sin^2\theta+O(e^2) \text{ similarly.}^{2}\end{aligned}$$

Evidently, this was far too large for Kepler to ignore when he encountered it; and moreover we point out that this disparity must be classed as irreducible, for it can only be eradicated by getting the theory right: that is, by choosing the second of the two alternatives because it is the one which is correct in modern terms. It has been anticipated in §4 that this is what Kepler eventually did, making the switch abruptly, without any explanation, in the course of Epitome V Part I Section 1.

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1. This term was defined in Chapter V, §3.
2. Since the difference between the true anomalies of the eccentric circle and the orbital ellipse is only $O(e^2)$, this will not affect the value of the disparity, and so it is of no consequence that, in Astronomia C. 57, the true anomaly of the circle was substituted for that of the ellipse: see Chapter II, §5 and Chapter III, §8.

(b) The ephemeral disparity in the summation.

Quite early in Astronomia C. 57, Kepler realized (G.W. III p. 348 lines 20-2) that the increment of libration was dependent on the sine of some anomaly, and that therefore an expression proportional to the completed libration ought to be obtained by summing a series of those sines. Such summations were undertaken for angles at intervals of 1°, and it has been proved, in Chapter V, §3, that at the level of exactitude stipulated in this study, this entails:

$$2 \tan \tfrac{1}{2}\delta\beta = \delta\beta = 2 \sin \tfrac{1}{2}\delta\beta .^{3.}$$

We apply this restriction in Appendix II (2a) to deduce the modern formula we shall need as a standard of comparison:

$$\sum_{o}^{n} \sin i\delta\beta . \delta\beta = (1-\cos\beta) + \tfrac{1}{2}\sin\beta . \delta\beta .^{4.}$$

It is to be emphasized that Kepler did not know this general formula for the sum of sines of angles in arithmetic progression: I have no evidence that he was acquainted with the trigonometrical relationships stated by Ptolemy (upon which we base our modern treatment). In particular—for a conceptual reason to be explained in §6 below—it would have been impossible for him to have derived it from the work of Archimedes: On the Sphere and Cylinder I,[5.] though it is clear, from an earlier statement in Astronomia C. 36,[6.] that he was familiar with the relevant set of propositions.[7.] So it was that he was forced to carry out laborious term-by-term summations (G.W. III p. 354 lines 8-19), and consequently found that every result was

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3. Any purely trigonometrical relationship stated, for convenience, in terms of β, is to be regarded as valid also in terms of θ: in the summations of this section, the two are distinguished simply by taking different values for the upper limit.

4. We write $n\delta\beta \equiv \beta$, and later $n'\delta\theta \equiv \theta$, the 'completed' anomalies.

5. The availability of this derivation from a modern point of view has been set out with great clarity by Heath: Greek Mathematics II, pp. 38-9 and especially p. 41.

6. G.W. III p. 249 lines 32-4.

7. And he further referred to them in Stereometria (G.W. IX)-before writing Epitome.

inexplicably greater than the corresponding sinus versus:[8] the amount of this inequality has been quantified by modern methods, above, as:

$\frac{1}{2}\sin\beta.\delta\beta$: the ephemeral disparity,

so called because it becomes relatively less, the greater the number of sines summed (as Kepler himself observed in Astronomia C. 57) or, preferably, can be decreased—as he later realized (see §6)—by reducing the size of the intervals.

For our subsequent comparison of the two disparities we remark that, when $\delta\beta=1°$, then for Mars, $\frac{1}{2}\delta\beta\cong e^2$, so that for this particular planet (which was alone under consideration in Astronomia C. 57), the size of the ephemeral disparity is approximately $e^2\sin\beta$, or $e^2\sin\theta$, as appropriate.

(c) Kepler's initial inability to cope with these disparities.

As things stood in Astronomia, C. 57, Kepler could only account for the increment of libration expressed as the sine of the true anomaly θ, on the grounds that this angle could be 'perceived' by the planet merely by reference to the Sun and the fixed stars,[9] whereas the eccentric anomaly β was an angle with no physical significance whatsoever, since the centre of the orbit was empty.[10] We have indicated, in (b) above, that he expected the sum of any series of sines to approximate to the corresponding sinus versus: that is, to the sinus versus of the true anomaly θ: yet he knew, without question (as stated in Chapter V, §1, reiterated from Part A), that the completed libration was measured

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8. That is, after the essential reduction-factor had been applied: this technique is to be explained in Chapter VIII, §1.

9. And also because the physical analogies adduced in support were directly applicable only when the true anomaly was considered: see §2(b).

10. G.W. III p. 358, marginal note, and p. 359, second marginal note.

by the sinus versus of the eccentric anomaly β. So it was that the irreducible disparity was brought to his attention; and the major part of C. 57 consists of his extended attempt to justify the lack of correlation, which is summarised as follows:

When $\beta = n\delta\beta$ and $\theta = n'\delta\theta$ are the respective anomalies of the same point P_o, then obviously:

$$\sum_{o}^{n} \sin i\delta\beta.\delta\beta > \sum_{o}^{n'} \sin i\delta\theta.\delta\theta \quad \text{(G.W. III p. 354 lines 33-6)}$$

and sinus versus β > sinus versus θ.

Now because of the ephemeral disparity, he found:

$$\left.\begin{array}{l} \sum_{o}^{n'} \sin i\delta\theta.\delta\theta > \text{sinus versus } \theta \\ \sum_{o}^{n} \sin i\delta\beta.\delta\beta > \text{sinus versus } \beta \end{array}\right\} \quad \begin{array}{l} \text{p. 354 lines 8-19} \\ \text{p. 355 lines 7-8.}^{11.} \end{array}$$

Then from the first and fourth inequalities taken together, as well as from the second and third, Kepler evidently felt able to conclude, provisionally, that:

$$\sum_{o}^{n'} \sin i\delta\theta.\delta\theta \cong \text{sinus versus } \beta \text{ (p. 354, fourth marginal note).}$$

Of course, this reasoning is entirely spurious, and, from the evaluations in (a) and (b), we can see that the irreducible disparity is, in general, much greater than the ephemeral disparity,[12.] and so the first term of the above 'approximation' is likely to be much smaller than the last. Nevertheless, at this early stage, Kepler was prepared to believe that the ephemeral disparity would cancel out the irreducible disparity, and thus the lack of correlation might in practice be glossed over.

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11. Small: The Astronomical Discoveries of Kepler, note Aa, p. 362, gives details of the calculations and sets out Kepler's reasoning-process.

12. In the case of Mars, which was the only planet under consideration in Astronomia Part IV.

(d) The failure of subsequent commentators to appreciate these problems or their solution.

It is the temporary expression of optimism just cited which has been picked out by Aiton to represent, by implication, Kepler's ultimate conclusion in relation to the determination of the completed libration: 'Kepler regarded the difference [between a formulation in terms of the true anomaly and one in terms of the eccentric anomaly] as negligible.'[13]. Since Aiton is the only commentator to have embarked on a technical account of this topic, it is particularly unfortunate that he has merely stated the result of Kepler's tentative preliminary work and has omitted to consider the mature opinions of Epitome V, where (as we have anticipated in §4), Kepler distinguished the alternative formulations, and thereby eliminated the source of the irreducible disparity. Thus Aiton's analysis has had the unintentional effect of denigrating Kepler's achievement.

Indeed it seems that the sudden switch from the first to the second alternative which took place in the middle of Epitome V Part I Section 1 has not been understood by any previous commentator—though Kepler himself is to blame for this since he entirely failed to explain what he was doing. I therefore take the opportunity to draw attention to some prevalent misapprehensions which are likely to be those most widely read. Koyré is the only author to have reproduced the diagrams on which Kepler based his reasoning in that section (fig. (iv) and fig. (v) of §3 and §4 respectively, in the reverse order);[14]. his account repeats the ambivalence in Kepler's own treatment—the diagrammatic representation by circles, the textual references to 'the orbit'—while Koyré overlooked the existence of the mathematical disparities, consequently remaining unaware of any problem, so he was unable to interpret the way in which Kepler eventually got round these difficulties. Moreover, Caspar's single technical note on that section (G.W. VII pp. 594-5 to p. 370) is altogether unhelpful

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13. The Vortex Theory of Planetary Motions, p. 17, reiterating a similar view expressed in Kepler's Second Law of Planetary Motion, p. 83, and in Infinitesimals and the Area Law, p. 298. Though the last two accounts mentioned are concerned mainly with Astronomia, in the first Aiton surveyed Epitome IV and V as well.

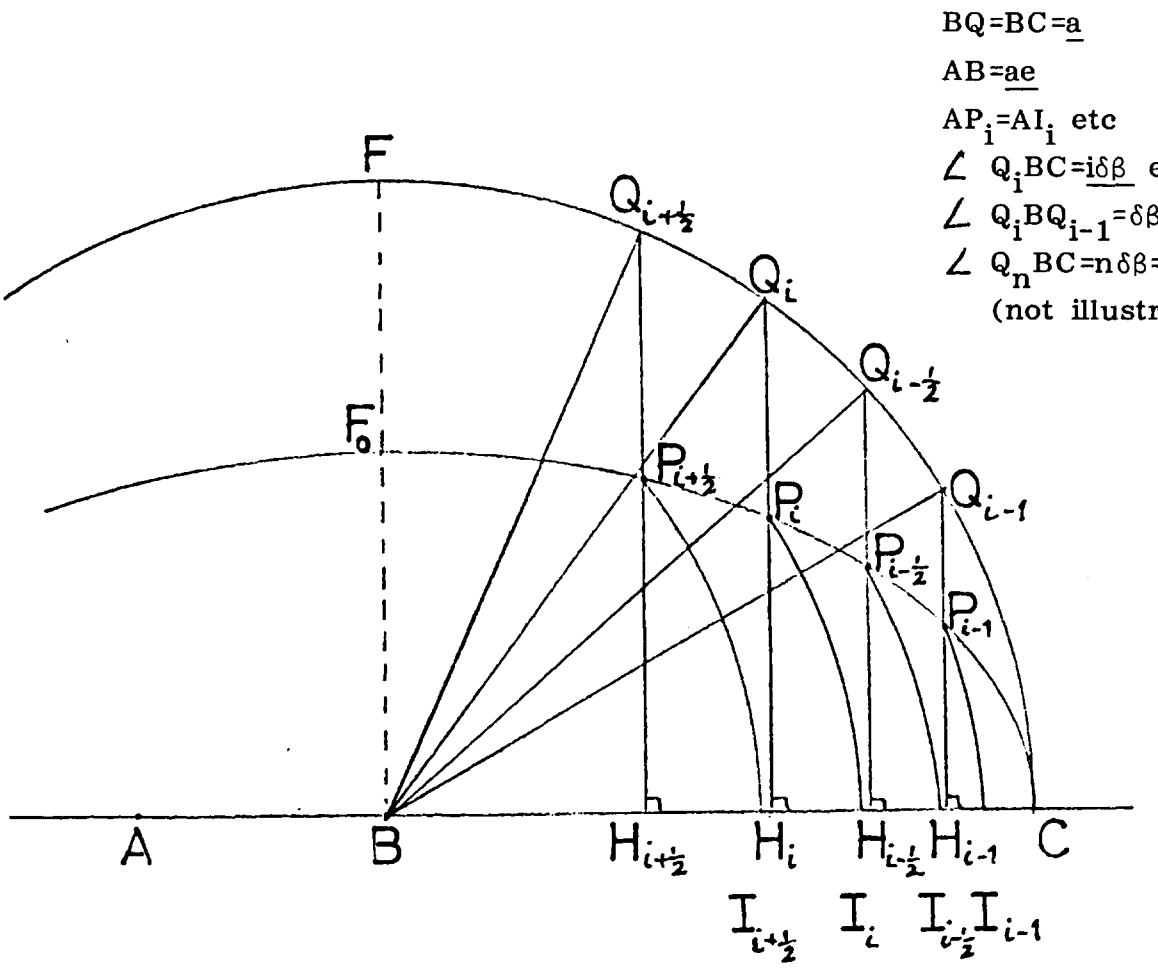
14. La révolution astronomique p. 308: discussed pp. 307-11.

for the same reason, and in that case the confusion is compounded by the misplaced diagram (see §4), as well as by his references to 'libration' when 'increment of libration' is correct, four times on p. 595. The views of these commentators will have had a great deal of influence and it would be hardly surprising if their failure to clarify Kepler's many inaccuracies and inconsistencies had inhibited further detailed enquiry—but naturally I deprecate the fact that his capabilities have been so underestimated.

The derivation of the completed libration to be examined next, turns out to be the source of yet more mistranscriptions and misinterpretations which similarly have to be disposed of before we can assess the extent of Kepler's eventual success.

§6 The proof of Law I Epitome V Part I Section 2.

Fig. (vi) The sectionings of the eccentric circle.



Preliminary remarks.

The intermediate small sections now to be examined were determined by equal small arcs $a\delta\beta$ of the eccentric circle, which Kepler initially described as partes minimae (G.W. VII p. 370 line 23).[1] Obviously their size is in practice measured and controlled by the interval of eccentric anomaly $\delta\beta$, and in the first instance it will be sufficient to set $\delta\beta \leq 1°$—or, upon occasion, to impose the slightly more rigorous restriction $\delta\beta \leq \frac{1°}{4}$. Then one of the major complications to be encountered arises in

1. By contrast with the phrases partes minutae, or minutissimae, employed in the proof of Law II.

connection with notational considerations. Kepler chose to investigate the small section identified by the intermediate arc illustrated here as $Q_{i-1}Q_i$, in conjunction with the named ordinate Q_iH_i at its upper boundary: this will be referred to as the variant positioning of the ordinate, because it turns out that when a modern treatment is undertaken, the ordinate falls at the midpoint of the section. Also, when such comparisons become necessary, we shall contrast the Keplerian choice of sectioning $Q_{i-1}Q_i$, whose midpoint ordinate is $Q_{i-\frac{1}{2}}H_{i-\frac{1}{2}}$= a $\sin(i-\frac{1}{2})\delta\beta$, with the preferred choice of sectioning $Q_{i-\frac{1}{2}}Q_{i+\frac{1}{2}}$, in which the midpoint ordinate Q_iH_i=a sin $i\delta\beta$ coincides with that named by Kepler (this will also be adopted—by preference—for the proof of Law II in Chapter VII); the change from one to the other is simply a matter of adjusting the notation.

The completed sagitta has been defined in Chapter V, §3, and here, by symmetry, denotes the length cut off along the line of apsides by any ordinate, measured from aphelion C. Hence we shall follow Kepler in using the term increment of sagitta as a convenient abbreviation for 'the distance between successive ordinates'—although it will be necessary to distinguish (more carefully than he did) between the increment and the completed quantity, just as in the case of the libration.

In the interest of generality we shall ignore the fact that Kepler stated the condition $e=\frac{1}{2}$ (G.W. VII p. 370 lines 22-3), and drew the diagram accordingly. Fortunately, this special value does not crop up anywhere in the subsequent working.

Step (1): The increment of libration (G.W. VII p. 371 lines 1-3, 15-16).[2]

Kepler started from the relationship he had tried misguidedly to justify at the end of Epitome V Part I Section 1 (as described in §4):

Increment of libration $I_{i-1}I_i \propto Q_iH_i$[3] or a sin $i\delta\beta$ (p. 371 lines 1-3).

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2. The statement on line 15 was a mistranscription, and is to be analyzed separately in Appendix VI (2), together with another to be noted in Step (2).

3. We recall from §4 that the sine, there proportional to the ordinate of the orbital ellipse, is also proportional to the ordinate of the eccentric circle, as introduced here.

Unfortunately, this expression is apparently imprecisely structured, for it does not contain any interval with respect to which it may (later) be summed (see Chapter V, §2). However I shall argue (and again in Step (2)) that, as stated in the preliminary remarks, these increments were in fact referred to equal small arcs, and therefore to this extent their precise structuring is implicit.

Step (2): Kepler and Pappus (G.W. VII p. 371 lines 16-17,22-33).[4.]

Kepler temporarily regarded the eccentric circle as the cross-section of a sphere divided into zones by planes through the ordinates perpendicular to the line of apsides, in order to apply a proposition proved by Pappus:[5.]

$$\frac{\text{Surface area of spherical cap}}{\text{Surface area of remainder}} = \frac{\text{Diameter cut off by cap}}{\text{Remainder of diameter}}.$$

It must have been Kepler himself who made the simple inferences:

$$\frac{\text{Surface area of spherical cap}}{\text{Total surface area}} = \frac{\text{Diameter cut off by cap}}{\text{Total diameter}}$$

and hence, by subtraction:

$$\text{Surface area of zone} \propto \left\{ \begin{matrix} \text{Distance between ordinates} \\ \text{Increment of sagitta } H_{i-1}H_i \end{matrix} \right\} \quad \text{(p.371 lines 25-8).}$$

Next, by treating the zones as 'circles without width' (p. 371 line 30), he fallaciously deduced:

Surface area of zone $\propto Q_iH_i$ or a sin $i\delta\beta$ (p. 371 lines 29-32),

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4. The statement on line 16 was another mistranscription and is to be analyzed separately in Appendix VI(2), together with the one already noted in Step (1)

5. Pappi Alexandrini Collectionis quae supersunt V, Prop. 36.

which apparently enabled him to obtain a relationship to match that found in Step (1):

Increment of sagitta $H_{i-1}H_i \propto Q_iH_i$ or a sin $i\delta\beta$ (p. 371 lines 32-3).

Now it seems at first sight quite remarkable that Kepler cited that particular proposition from Pappus when the correct result had been stated several times just previously[6.] (and also by Archimedes, to whom Pappus referred):[7.]

Surface area of zone $\propto Q_iH_i.Q_{i-1}Q_i$[8.].

However I am convinced that Kepler deliberately elected not to follow Archimedes, on dimensional grounds—because he required a length rather than an area. It was Kepler's common practice not only to represent a sine by an ordinate, and the sinus versus by the sagitta,[9.] but also a small interval by a small arc—like them, possessing a linear dimension. Indeed, this had been vital to his success in all the summations of Astronomia (see Chapter V, §2), and was again in the proof of Law II—since every one of these involved an area—but in the present situation he needed intervals of zero dimensions. Undoubtedly he realized this, but instead of introducing intervals of angle (eccentric anomaly $\delta\beta$, here)—he was probably deterred because, at this juncture, he envisaged them as measured in degrees—he resolved the difficulty by the fallacious deduction set out above. This was entirely reprehensible by modern standards, for it involved the belief—or at least the hope—that when the interval of arc became vanishingly small, the unwanted extra dimension would disappear as well. Its undesirable consequence was that the expression once more appears to be imprecisely structured: but we shall counter this by pointing out that these increments of sagitta were, like the increments of libration in Step (1), actually

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6. Op. cit.: in the course of Props 19-35.
7. Kepler was in any case thoroughly familiar with Archimedes' work, as confirmed in §5(b).
8. Here adopting the variant positioning of the ordinate, to ensure comparability with Kepler's approach in this respect: the Greeks actually employed the ordinate at the midpoint of the section.
9. See §4, and Chapter V, §3 respectively.

referred to equal small arcs (and also by recognizing, in modern terms, that one cannot reduce an interval to vanishing size unless it was present in the first place): hence, in this respect, their precise structuring is implicit.

Steo (3): Kepler's verification of the completed libration, (G.W. VII p. 371 lines 11-14, 19-21).

We first establish the connection between the increments in modern terms:

$$\begin{aligned} I_{i-1}I_i &= AI_{i-1}-AI_i \\ &= AP_{i-1}-AP_i \text{ by construction (p. 371 lines 5-10)} \\ &= a\{1+e\cos(i-1)\delta\beta\}-a\{1+e\cos i\delta\beta\} \\ &= ae\{\cos(i-1)\delta\beta-\cos i\delta\beta\} \\ &= e(BH_{i-1}-BH_i) \\ &= eH_{i-1}H_i. \end{aligned}$$

Kepler's own derivation followed the lines:

$$\frac{\text{Increment of libration}}{\text{Increment of sagitta}} = \frac{I_{i-1}I_i}{H_{i-1}H_i} : \text{ constant (p. 371 lines 19-20).}$$

In Steps (1) and (2) Kepler had determined to his own satisfaction that the two increments were proportional to the same quantity and thus their ratio was constant, as we have shown that it is in modern terms also. Hence he was justified in applying the addition-property of equal ratios to obtain:

$$\frac{\text{Completed libration}}{\text{Completed sagitta}} = \frac{\text{Total libration}}{\text{Total sagitta}} \text{ (p. 371 lines 12-14, 20-1)}^{10}.$$

But,

$$\frac{\text{Total libration}}{\text{Total sagitta}} = \frac{2AB}{CD} = e \text{ (p. 371 lines 11-12)}$$

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10. Kepler wrote 'sagittas' and 'increments of libration' on lines 12, 13— the latter at least does not make sense; but he actually named the individual completed quantities: compare the pair of (reverse)mis-statements examined in Appendix VI(2), as well as the confusion mentioned in §5(d).

and,

$$\frac{\text{Completed libration}}{\text{Completed sagitta}} = \frac{\sum_{1}^{n} I_{i-1}I_i}{\sum_{1}^{n} H_{i-1}H_i} = \frac{CI_n}{CH_n} \quad 11.$$

$\therefore\ CI_n = e.CH_n = e(BC-BH_n)$

$\therefore$ Completed libration = $ae(1-\cos\beta)$ (p. 370 lines 15-17).

It has been emphasized, from Chapter V, §1 onwards, that this exact result was known in advance and was undoubtedly what Kepler intended to verify. Nevertheless this Keplerian derivation has glossed over the two principal problems which have arisen in the course of this enquiry. The crucial one is the fact that, if the (matching) increments were imprecisely structured, there is no way the ensuing summations could be regarded as tantamount to integration (most notably because they would then contain no provision for progress to the limit). Moreover, the increments would in that case be noncomparable with the corresponding modern values which are—naturally—precisely structured, and it would then be impossible to determine the errors in the Keplerian evaluations which are the source of the second problem to be encountered. Hence we shall be compelled to make explicit the precise structuring of the Keplerian increments preparatory to summation:

$$\left.\begin{array}{l}\text{Increment of libration } I_{i-1}I_i \\ \text{Increment of sagitta } H_{i-1}H_i\end{array}\right\} \propto \sin i\delta\beta.\delta\beta.$$

Opinions will differ over the extent to which this has been justified by the arguments of Steps (1) and (2): I shall prefer to rely on its total vindication via a technique by which Kepler actually achieved the precise structuring of a linearly-dimensioned summation[12].

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11. For consistency of notation we write $I_o \equiv H_o \equiv C$, while $\angle\ Q_nBC = n\delta\beta \equiv \beta$ defines I_n and H_n (not illustrated).
12. It is not suggested, however, that he himself appreciated the implications of what he had done.

—though because of its wider significance, demonstration of this is deferred to Chapter VIII,§1. Meanwhile we shall direct our attention to confirming that—despite the idiosyncracies of his notation—the precisely structured summation performed by Kepler was indeed tantamount to integration. A brief survey of the previous commentators who have written on this issue—albeit a matter of mathematical history—has been set out in Appendix VI (3): the conclusion reached by Eneström, which as far as I know has remained unchallenged till now, was that Kepler's method of summation was invalid. To rebut this erroneous appraisal it will be necessary to embark on a detailed analysis involving modern comparisons, in order to determine the errors in the Keplerian evaluations: the second problem alone.

Step (4): The summation tantamount to integration (G.W. VII p. 371 lines 4, 17-18).

We return to one of the (Ptolemaic) relationships set out in Appendix II (2a), to apply it to the correct evaluation of the increments $\frac{1}{ae} I_{i-1}I_i = \frac{1}{a} H_{i-1}H_i$ arising from the Keplerian choice of sectioning:

$$\cos(i-1)\delta\beta - \cos i\delta\beta = \sin(i-\tfrac{1}{2})\delta\beta \cdot 2 \sin \tfrac{1}{2}\delta\beta$$
$$= \sin(i-\tfrac{1}{2})\delta\beta \cdot \delta\beta \text{ when } \delta\beta \leq 1°.$$ [13]

That is,

$$\frac{1}{ae} I_{i-1}I_i = \frac{1}{a} H_{i-1}H_i = \frac{1}{a} Q_{i-\frac{1}{2}} H_{i-\frac{1}{2}} \cdot \delta\beta.$$

$\therefore$ Error in the Keplerian evaluation of the increment of libration

$$= \{\sin i\delta\beta - \sin(i-\tfrac{1}{2})\delta\beta\}\delta\beta.$$

Hence we have a summation:

$$\sum_1^n \sin(i-\tfrac{1}{2})\delta\beta \cdot \delta\beta = \sum_1^n \{\cos(i-1)\delta\beta - \cos i\delta\beta\}$$

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13. Because this is a formula exact by modern standards, it inevitably turns out to be precisely structured, as anticipated above.

$$\therefore \quad \sum_{1}^{n} \sin(i-\tfrac{1}{2})\delta\beta.\delta\beta = 1-\cos\beta.$$

We repeat this procedure for the correct evaluation of the increments $\frac{1}{ae}I_{i-\frac{1}{2}}I_{i+\frac{1}{2}} = \frac{1}{a}H_{i-\frac{1}{2}}H_{i+\frac{1}{2}}$ which would have been required had the preferred choice of sectioning been adopted: this is arranged merely by an adjustment of the notation:

$$\cos(i-\tfrac{1}{2})\delta\beta-\cos(i+\tfrac{1}{2})\delta\beta = \sin i\delta\beta.2\sin\tfrac{1}{2}\delta\beta$$
$$= \sin i\delta\beta.\delta\beta \quad \text{when } \delta\beta \leq 1^\circ.$$

That is, $\frac{1}{ae}I_{i-\frac{1}{2}}I_{i+\frac{1}{2}} = \frac{1}{a}H_{i-\frac{1}{2}}H_{i+\frac{1}{2}} = \frac{1}{a}Q_iH_i.\delta\beta.$

$\therefore$ Error in the Keplerian evaluation of the increment of sagitta

$$= \{\cos(i-1)\delta\beta-\cos i\delta\beta\}-\{\cos(i-\tfrac{1}{2})\delta\beta-\cos(i+\tfrac{1}{2})\delta\beta\}$$
$$= \sin(i-\tfrac{1}{2})\delta\beta.2\sin\tfrac{1}{2}\delta\beta-\sin i\delta\beta.2\sin\tfrac{1}{2}\delta\beta$$
$$= \{\sin(i-\tfrac{1}{2})\delta\beta-\sin i\delta\beta\}.\delta\beta \quad \text{when} \quad \delta\beta \leq 1^\circ.$$

Since these two Keplerian errors are equal and opposite they will be termed duplicated errors. Meanwhile, we have a further summation:[14]

$$\sum_{1}^{n} \sin i\delta\beta.\delta\beta = \sum_{1}^{n} \{\cos(i-\tfrac{1}{2})\delta\beta-\cos(i+\tfrac{1}{2})\delta\beta\}$$
$$= \cos\tfrac{1}{2}\delta\beta-\cos(n+\tfrac{1}{2})\delta\beta$$
$$= \cos\tfrac{1}{2}\delta\beta.(1-\cos\beta) + \sin\beta.\sin\tfrac{1}{2}\delta\beta.$$

When we impose the slightly more rigorous restriction $\delta\beta \leqq \frac{1^\circ}{4}$, then $\cos\frac{1}{2}\delta\beta=1$, as well as $\sin\frac{1}{2}\delta\beta=\frac{1}{2}\delta\beta$, at the stipulated level of exactitude, and so:

$$\sum_{0}^{n} \sin i\delta\beta.\delta\beta = (1-\cos\beta)+\tfrac{1}{2}\sin\beta.\delta\beta.$$ [15]

The second of these results represents what Kepler was aiming to do (the summation of the increments of libration), while the first represents what in fact he obtained (the summation of the increments of sagitta). Comparing the two, we find:

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14. This reproduces the summation used in §5(b) as a modern standard of comparison, itself derived from the one actually performed in that Appendix.

15. Since $\sin 0^\circ=0$.

$\sum_{1}^{n} \sin i\delta\beta.\delta\beta - \sum_{1}^{n} \sin(i-\frac{1}{2})\delta\beta.\delta\beta = \frac{1}{2} \sin\beta.\delta\beta$: the ephemeral disparity.[16]

Thus Kepler's method must be regarded as unsatisfactory, judged by modern standards, insofar as he bypassed the proper procedure by failing to quantify this disparity, as he should have, before minimizing it, which he did. In Astronomia C. 57 he noticed that the disparity became relatively small when the number of sines summed was large, and it may even then have occurred to him that such a favourable situation could be arranged equivalently by reducing the size of the intervals. However it was in Epitome that he came to realize that the disparity could be made to vanish altogether when the interval was determined by a process described as divisio in infinita (p. 371 lines 4, 17-18, 29).[17] With this justification, the summation performed by Kepler ultimately coincides with the common limiting value of both the modern versions, expressed as:

$$\sum_{o}^{n} \sin i\delta\beta.\delta\beta = 1-\cos\beta$$

when $\delta\beta$ is sufficiently small and n correspondingly large. Then this potentially-infinite summation is tantamount to definite integration, formally expressed as:

$$\int_{o}^{\beta} \sin\beta d\beta = 1-\cos\beta.$$

Indeed we should observe, more succinctly, that this outcome was mathematically inevitable, for it is an essential property of an integrable function that the limiting sum is uniquely defined and does not depend on the positioning of the ordinates within the sections:[18] hence no merely technical matter of idiosyncratic notation can invalidate the summation, and consequently the conclusion reached by Enestrŏm—mentioned in Step (3)—was bound to be wrong in principle.

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16. In correlation, we point out that this summation represents the sum of either of the individual duplicated errors, which therefore vanish with the ephemeral disparity, as justified below.

17. This phrase was employed, I surmise, because Kepler wrongly believed that he required sections of zero width for Step (2)—see p. 371 line 30. Otherwise—as we have pointed out in Chapter V, §3, he only needed to set $\delta\beta < 1''$ for $\frac{1}{2}\sin\beta.\delta\beta$ to become undetectable in his terms.

18. See, for instance, Lamb: Infinitesimal Calculus §90, pp. 207-9: the sine-function is integrable in the sense required because it is continuous and increases or decreases steadily within given ranges.

§7 The radial motion.

We should not lose sight of the fact that (as anticipated in Chapter V, §2) the underlying purpose of Kepler's verification of the completed libration was, retroversely, to determine the increment of libration:

$$\delta\lambda = ae \sin\beta . \delta\beta$$ [1].

We shall now go on to a further deduction, showing both that Kepler was unquestionably capable of carrying it out, and that it would not have occurred to him to proceed in this way.

In some circumstances at least (as demonstrated in Chapter VII, §3) Kepler was accustomed to dealing with quotients of increments, so we shall write:

$$\frac{\delta\lambda}{\delta\beta} = ae \sin\beta .$$

Moreover in Epitome V Part I Section 4 (as proved in Chapter VII, §1) he established the kinematical relationship, specific to the orbital ellipse, which will be termed:

$$\frac{\delta\beta}{\delta\alpha} = \frac{a}{r}$$: the Keplerian Time-measure. [2].

Then, because Kepler possessed the ability to combine such quotients of increments (this is also confirmed in Chapter VII, §3) he could have obtained:

$$\frac{\delta\lambda}{\delta\alpha} = \frac{a^2 e \sin\beta}{r}$$

and, since he knew that $\delta r = -\delta\lambda$,

$$\frac{\delta r}{\delta\alpha} = -\frac{a^2 e \sin\beta}{r}$$: the radial motion.

This evaluation is of course correct by modern standards (see Prolegomenon IIB); and we shall return to consider this result in a physical setting in Chapter VIII, §2. Meanwhile we explain that this deduction is nonKeplerian because, as a consequence of the duality of

---

1. The precise structuring is here explicitly stated on Kepler's behalf.

2. This is the special name to be given to Time-measure II, applied to the orbital ellipse alone: it is obtained from the quantification of the indeterminate time-axiom which is a result of the proof of Law II.

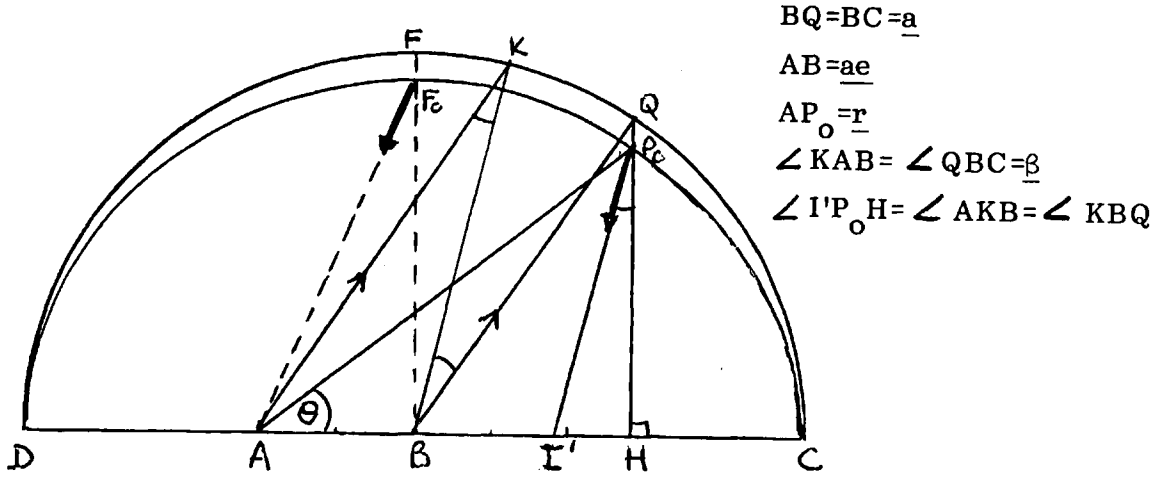
treatment which has been adopted in Part B, on Keplerian authority (see Chapter V, §1), it seems almost certain that he would have regarded the Keplerian Time-measure as relevant to transverse motion alone (in connection with which it was synthesized, as we shall see) and would have been unable to appreciate that the same relationship could be admissible in the present context of radial motion.

APPENDIX VI(*1*) to § 4

THREE ERRONEOUS VERSIONS OF THE ANGLE OF DEFLECTION OF THE FIBRES

Version (I'): Epitome IV Part III Section 3 (G.W. VII p. 339 line 33—p. 340 line 24; p. 340 line 31—p. 341 line 25).

Fig. (i) The elliptic orbit.



Kepler here introduced the deflection of the fibres and attempted a preliminary quantification of the increment of libration. Though his statements are somewhat ambiguous, I deduce that he intended:

Increment of libration ∝ sine of angle of deflection[1].
(p. 340 lines 11-13)

Sine of angle of deflection ∝ sine of arc of orbit (p. 341 lines 6-8)

∴ Increment of libration ∝ sine of arc of orbit.[2]

---

1. In this proportion alone, the constant implied is simply a, while for the other two it is ae: we shall use this distinction in what follows.

2. It is of interest to note the exact correspondence between this argument and that of § 4(a), save only for the change in the interconnecting term: 'the cosine of the angle between the fibres and the solar ray' (multiplied by e) replaces 'the sine of the angle of the deflection' as a consequence of Kepler's more sophisticated opinion on the way magnetism functioned, as set out in § 2(b).

Now the involvement of the deflection entails the restyling of the 'arc of the orbit', and hence:

Increment of libration $\propto \sin\beta$

Sine of angle of deflection $\propto \sin\beta = e\sin\beta$

$\therefore$ Angle of deflection $\angle I'P_oH = \sin^{-1}(e\sin\beta) = e\sin\beta + O(e^3)$.

In fact this version was discussed in relation to the quadrant alone, where:

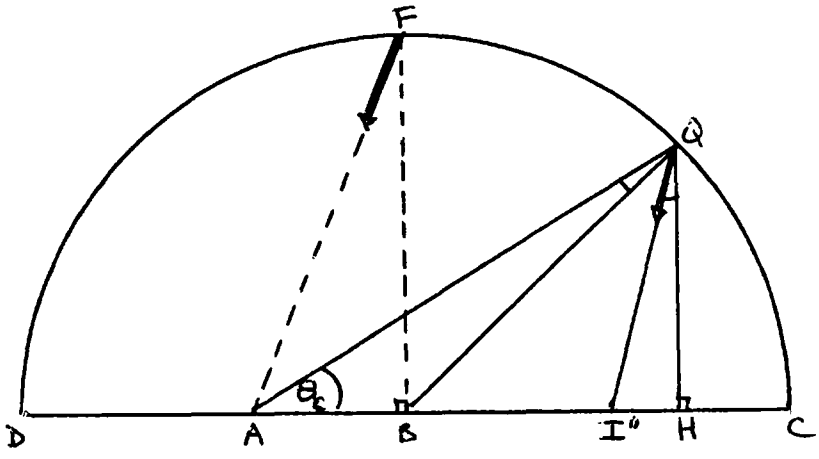
Angle of deflection $\angle AF_oB = \sin^{-1}e$.

This can of course be identified with either the optical equation updated for the ellipse (the required version evaluated in §4(a)), or the optical angle (see version (I''')) in this special case. In general, however, it cannot be represented as an optical concept- and I believe that Kepler abandoned it for this reason. Nevertheless, I have found that it can be geometrically represented, since $\angle I'P_oH = \angle AKB = \angle KBQ$: though it is not suggested that Kepler himself could have realized this since it reintroduces the point K which appeared in his work only in connection with the Ovoid group (see Chapter III, §5).

The fact that the orbit is indeed correctly drawn as the ellipse is confirmed by the property $AF_o = BC = a$ (G.W. VII p. 340 lines 6-7, p. 341 lines 10-11), as already emphasized in §4(a).

Version (I"): Epitome V Part I Section 1 (G.W. VII p. 369 line 41-p. 370 line 8).

Fig. (ii) The circular 'orbit'.



BC=BF=BQ=a

AB=ae

∠ QBC=β

AQ=r_c

∠ I"QH=∠ AQB

We remark that Kepler could scarcely have adopted a more confusing approach, for we have pointed out, in § 4 (and § 3 as well), that he was well aware that the orbit was an ellipse, and have emphasized the need to rectify the ambivalence when he drew it as a circle—yet, in this instance, we will find that Kepler treated the orbit as a circle for the purpose of his calculations. I explain this temporary regression by surmising that he was not at first certain that he was justified in restyling the arc of the orbit (measured by eccentric anomaly β, as explained in § 4 (a))—though he undoubtedly became convinced of this by the end of Epitome V Part I Section 1—so meanwhile, he applied this restyling to an assumed circular 'orbit', as an interim step.

We consider Kepler's reasoning in the reverse order, for greater clarity.

From the Sine Rule applied to ΔQAB and ΔFAB in turn:

$$\frac{\sin AQB}{\sin QAB} = \frac{ae}{BQ}$$ (p. 369 line 45—p. 370 line 1)

$$\frac{\sin AFB}{\sin FAB} = \frac{ae}{BF}$$ (p. 370 lines 1-2).

But BQ=BF (p. 370 line 1)—thus confirming that the orbit was indeed supposed circular—and in that case this pair of ratios are equal.

$$\therefore \frac{\sin AQB}{\sin AFB} = \frac{\sin QAB}{\sin FAB} \quad \text{(p. 369 lines 44-5, p. 370 lines 2-3).}$$

Then by reference back to Epitome IV Part III Section 3, Kepler stated:

$$\frac{\sin I''QH}{\sin AFB} = \frac{QH}{FB} \quad \text{(p. 369 lines 41-2).}^{3.}$$

(This is certainly an adequate approximation to Version (I') since we obtain

$$\sin I''QH = \frac{ae}{AF}\sin\beta = \frac{e}{(1+e^2)^{\frac{1}{2}}}\sin\beta = e\sin\beta + O(e^3),$$

which becomes exact when the orbit is put right.)

However Kepler avoided a return to Version (I') for he next suggested:

$$\frac{\sin QAB}{\sin FAB} = \frac{QH}{FB} \quad \text{(p. 369 lines 43-4).}$$

This relationship is wildly wrong,[4.] as Kepler himself must certainly have been aware: we shall therefore describe it as a fabricated equivalence, recognizing that it was propounded solely in order to enable him to equate the ratios in pairs and so identify a further angle of deflection:

$$\angle I''QH = \angle AQB \quad \text{(p. 370 line 4)}$$

$$\angle I''QA = \angle HQB \quad \text{(p. 370 line 5)}$$

But, $\angle HQB = \frac{\pi}{2} - \beta$ (p. 370 line 6)

$$\therefore \angle I''QA = \frac{\pi}{2} - \beta \quad \text{(p. 370 lines 6-7)}$$

$\therefore$ Increment of libration $\propto \cos I''QA = \sin\beta$ (p. 370 lines 7-8).

Thus, this approach has produced the version:

$$\underline{\text{Angle of deflection} = \angle I''QH = \beta - \theta_c}.$$

This is the optical equation for the circle, geometrically represented by $\angle I''QH = \angle AQB$.

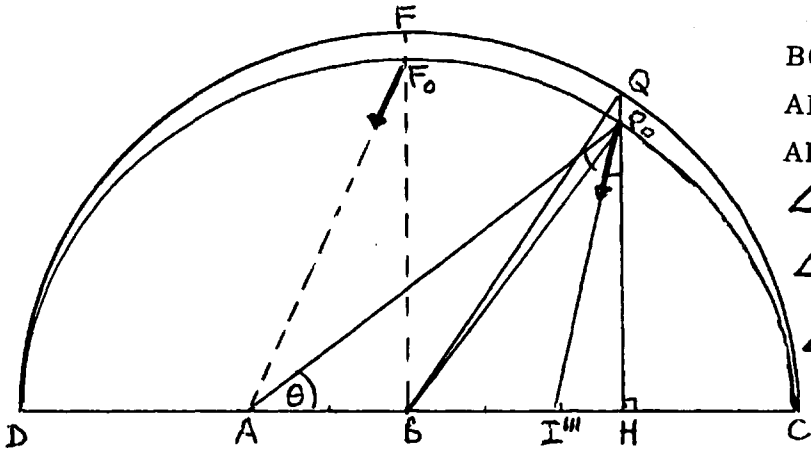
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3. Using Caspar's correction of the original text in which angles were mis-transcribed for their sines.

4. I believe that it arose from a misconceived attempt to restyle the arcs of the orbit: nevertheless, since $\frac{\sin QAB}{\sin FAB} = \frac{AF\sin\theta_c}{a}$ and $\frac{QH}{FB} = \frac{r_c\sin\theta_c}{a}$ their supposed equality would require $r_c = AF$ (constant).

Version (I'''): Epitome V Part I Section 1 (G.W. VII p. 370 lines 4-8 only).

Fig. (iii) The elliptic orbit.



$BQ=BC=\underline{a}$

$AB=\underline{ae}$

$AP_0=\underline{r}$

$\angle QBC=\underline{\beta}$: the eccentric anomaly

$\angle P_0BC=\underline{\gamma}$: the central anomaly

$\angle I'''P_0H=\angle AP_0B$

We derive yet another version from a literal quantification of Kepler's statements by direct reference to his own diagram—which will once more be assumed to represent the elliptic orbit.

$\angle I'''P_0H=\angle AP_0B$ (p. 370 line 4)

$\angle I'''P_0A=\angle HP_0B$ (p. 370 line 5)

But $\angle HP_0B=\frac{\pi}{2}-\gamma$ since $\angle P_0BH=\gamma$[5]. (p. 370 line 6)

$\therefore\ \angle I'''P_0A=\frac{\pi}{2}-\gamma$ (p. 370 lines 6-7)

$\therefore$ Increment of libration$\propto\cos I'''P_0A=\sin\gamma$ (p. 370 lines 7-8).

Thus this approach has produced the version:

Angle of deflection$=\angle I'''P_0H=\gamma-\theta$.

This is the optical angle for the ellipse, as defined in Appendix I(3): it is geometrically represented, since the orbit is an ellipse, by the identification $\angle I'''P_0H=\angle AP_0B$.

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5. γ is the central anomaly (see Chapter I, § 2).

APPENDIX VI(2) to §6

THE PAIR OF ASSOCIATED MIS-STATEMENTS IN KEPLER'S DERIVATION OF THE COMPLETED LIBRATION

Reference: Epitome V Part I Section 2 (G.W. VII p. 371).

An analytical comparison of the two versions.

Incorrect version		Kepler's intended version
$\frac{CI_1}{X_1H_1} = \frac{CI_2}{X_2H_2}$ (line 15)	Step (1)	$\frac{CI_1}{X_1H_1} = \frac{I_1I_2}{X_2H_2}$ (line 3)
$\frac{CH_1}{X_1H_1} = \frac{CH_2}{X_2H_2}$ (line 16)	Step (2)	$\frac{CH_1}{X_1H_1} = \frac{H_1H_2}{X_2H_2}$ (line 22)
$\frac{CI_1}{CH_1} = \frac{CI_2}{CH_2}$	$\therefore$ Step (3)	$\frac{CI_1}{CH_1} = \frac{I_1I_2}{H_1H_2} = \frac{CI_2}{CH_2}$ by addition of ratios.

This is an instance of a common cause of confusion between the completed libration and the corresponding completed sagitta, which, in the first version, have incorrectly replaced the increment of libration and also the increment of sagitta, respectively.[1] Indeed the wrong ratios are specious, though the one at Step (2) may be pinpointed by observing that it entails similar triangles CH_1X_1, CH_2X_2, so that CX_1X_2 must be a straight line, and X_2 cannot then lie on the eccentric circle as required.

Moreover, by a coincidence, the 'proof' in the case of the incorrect version is slightly shorter—which no doubt accounts for the fact that this pair of fundamental mis-statements of theory have been overlooked until now.[2]

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1. In Step (3) of §6 it is shown that the confusion also occurred in reverse: see also the instances listed in §5(d).

2. Except by Enestrŏm (see Appendix VI (3)) who noted the incorrect ratio of Step (2)—the only one with which he was concerned: neither of them was mentioned by Caspar.

APPENDIX VI (3) to §6

THE CONTROVERSY AS TO WHETHER KEPLER'S SUMMATION WAS TANTAMOUNT TO INTEGRATION

We present a synopsis of a controversy extending over the period 1860-1912, on the issue of whether Kepler effectively achieved the integration of a trigonometrical function. In none of these accounts was there any mention of the problem of ensuring the precise structuring of the expression, which was simply assumed quite arbitrarily when necessary: though of course I shall agree (as a consequence of the discussion of Chapter VIII, §1) that this approach was justified in the event.[1].

(a) In Appendix II(2a) we have drawn attention to the error in the general formula for the sum of sines of angles in arithmetic progression (most probably copied from Frisch) involved in the following accounts:

Günther: Über eine merkwürdige Behiehung zwischen Pappus und Kepler (1883), pp. 82-6

Cantor: Vorlesungen über Geschichte der Mathematik II (1892), pp. 756-7).

Since these commentators set out to analyze Kepler's work from an incorrect basis, their views may be dismissed from consideration. This is extremely fortunate because (as a consequence of a misreproduction of Kepler's diagram) the angle they employ in their account of Kepler's magnetic theory leading up to the summation of sines) is not the true anomaly but its complement, so that all the trigonometrical ratios mentioned are wrong.

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1. However it is with some surprise that I have noticed that in some cases angles have been measured in degrees or radians entirely at random: this is also the subject of Chapter VIII, §1.

(b) In §6, Step (3), it was shown that Kepler proved:

$$\text{Completed libration} = ae(1-\cos\beta).$$

If we then accept without more ado (omitting any enquiry into the question of notational considerations) that:

$$\text{Increment of libration} = ae \sin i\delta\beta.\delta\beta,$$

and,
$$\text{Completed libration} = ae \sum_{o}^{n} \sin i\delta\beta.\delta\beta,$$

thus it would appear that integration has effectively been achieved merely by overlooking all the complications. This was the over-simplified interpretation favoured by:

Enestrŏm (1): Sur un théorème de Kepler equivalent à l'intégration d'une fonction trigonométrique (1889), p. 66.

Zeuthen: Geschichte der Mathematik im XVI und XVII Jahrhundert (1903), p. 255.

Enestrŏm subsequently recognized that this approach was too slick, and undertook a more detailed investigation.

(c) Enestrŏm (2): "Über die angebliche integration einer trigonometrischen Funktion bei Kepler (1912) especially pp. 238-9.

Though Enestrŏm carried out the most detailed analysis of this topic that I have seen, unfortunately he examined only the error in Step (2), and failed to consider that in Step (1), so he remained unaware that by these duplicated errors Kepler achieved a constant ratio which could properly be summed, as we have seen. Thus Enestrŏm believed that Kepler attempted to sum a nonconstant ratio, and he easily established that this was unsound, by equating the sum of n ratios to the 'sum' of the first one alone, using intervals of $1°$.[2] However, as he pointed out, Kepler knew, by the time of writing Epitome (G.W. VII p. 371 lines 34-9) that this interval was not small enough, and so I suggest that Enestrŏm ought rather to have been concerned to assess Kepler's results under the more sophisticated condition:

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2. In Astronomia C. 57 (G.W. III p. 354 lines 8-11), Kepler found that this 'first sum' was twice the value expected.

divisio in infinita. Had he done so, he would have been unlikely to decide, as he did, that Kepler's method was invalid. This conclusion is wrong in principle, as we have pointed out in Step (4): unfortunately, this assessment seems to have ended the controversy, and since its consequence is to detract from Kepler's attainments, I have taken the opportunity here to put the record straight.

CHAPTER VII

THE TRANSVERSE MOTION PRODUCED BY THE DRIVING FORCE OF THE SUN: PROOF OF LAW II

§1 The proof of Law II

§2 The outcome of Kepler's failure to recall the optical property

§3 The internal consistency of Kepler's kinematical reasoning, incorporating the sequential error

§4 The problem of the constants of the orbit

Appendix VII(1) The positioning of the modulated arc

§1 The proof of Law II Epitome V Part I Section 4.

Preliminary remarks.

The small sectors to be examined were determined by equal small arcs $a\delta\beta$ of the eccentric circle, which Kepler initially described as partes minutae (G.W. VII p. 376 line 7).[1] Once more, their size is in practice measured and controlled by the interval of eccentric anomaly $\delta\beta$, and we anticipate that it will be necessary to impose the slightly more rigorous restriction $\delta\beta \leq \frac{1^\circ}{4}$, in order to ensure, at the level of exactitude stipulated, that not only will the small circular arcs coincide with straight lines (see Chapter V, §3), but so also will the elliptic arcs (this is established formally in Appendix VII (1)).

In this proof (unlike that of Law I) we shall be investigating increments of area, requiring that the small intervals be linearly dimensioned, which was in fact how Kepler invariably expressed them, adopting a geometrical rather than an analytical point of view—as we have emphasized in Chapter V, §3. Thus there is no question but that these increments were, and remained, precisely structured (as defined in Chapter V, §2), and so no difficulty in this respect will occur in the present discussion: we return to this topic in Chapter VIII, §1.

I draw attention to two related techniques employed in the proof, whose source can be traced to Astronomia C. 59:

(i) In the repeated diagram there (G.W. III p. 367, p. 370, p. 373) we encounter the first Keplerian illustration of diametrically opposite points of the ellipse, determined symmetrically from a diameter of the eccentric circle by dropping perpendiculars (ordinates) to the line of apsides—so that the eccentric anomalies of those points of the ellipse differed by π.

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1. Subsequently, divisiones minutissimae (p. 376 line 2), particulae minutissimae (p. 377 lines 27-8).

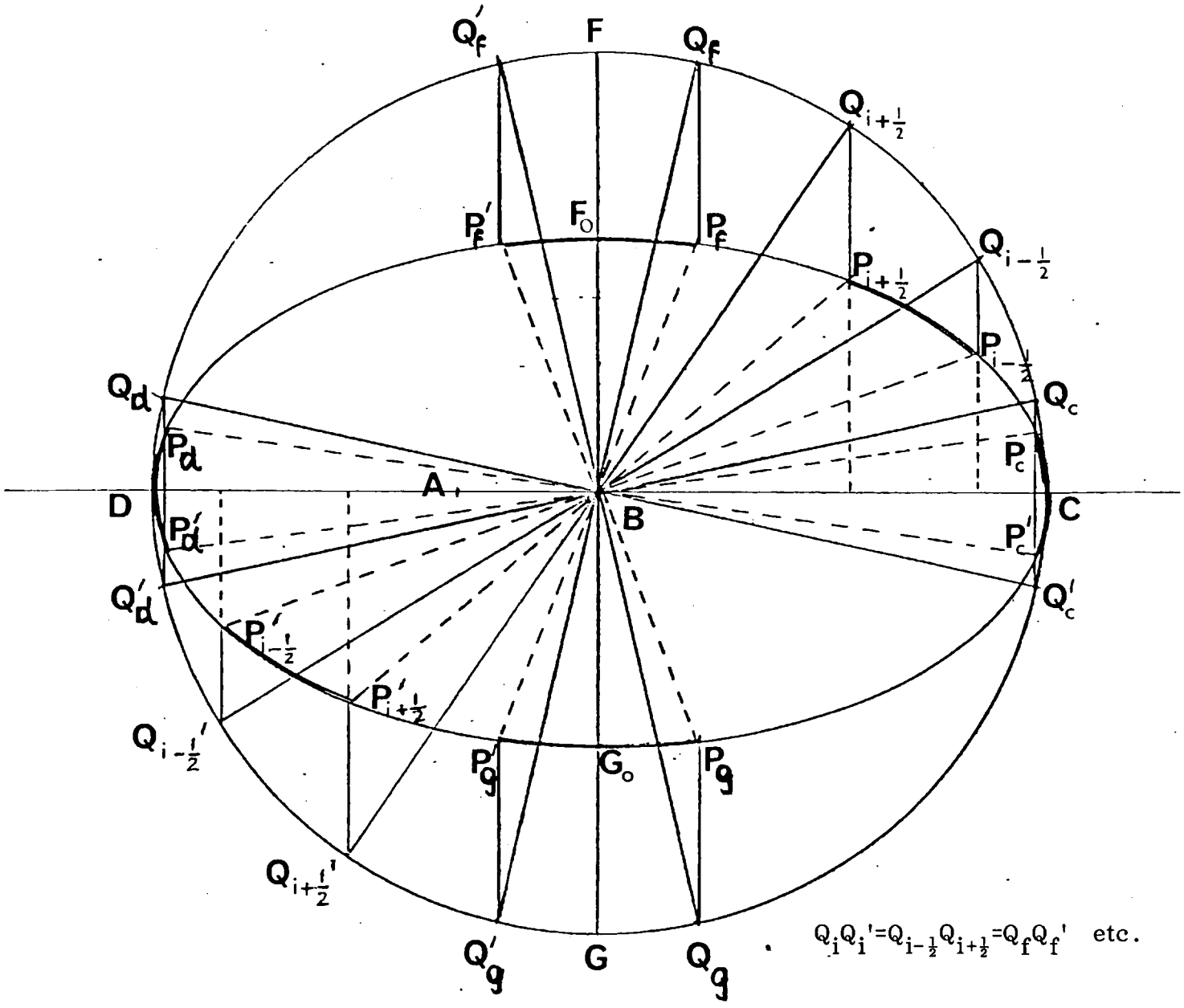
(ii) Equal small arcs aδβ of the eccentric were used to determine unequal arcs δs of the ellipse by dropping successive perpendiculars (ordinates) to the line of apsides, as foreshadowed in Protheorema IV (G.W. III p. 368 lines 11-12).

Thus the concept of diametrically opposite points will be extended to the consideration of diametrically opposite paired sectors, upon which Kepler's measure of success depended, in conjunction with a third technique, entirely new, to be introduced in Step (3).

Unfortunately we shall find that this proof is vitiated by the absence of any attempt to establish the result for intermediate small sectors: Kepler made a gesture towards generalization only by considering the pair 'next' to those at the apsides, and even so, somewhat as an afterthought. However, a satisfactory proof even for this pair has to be as strict as for an intermediate pair and therefore I have undertaken this on his behalf, to indicate the point at which his method would probably have broken down. Yet it is because Kepler himself did not illustrate the intermediate sectors, that I have been free to adopt the preferred choice of sectioning to represent them (thereby avoiding the complication which arose in the proof of Law I on account of the Keplerian choice of sectioning), and indeed I have employed it consistently for the whole diagram: I point out that Kepler himself (naturally) used the Keplerian choice for the special cases he considered, at the apsides and quadrants—but since the results he stated were exact, as obtained from my representation, I do not feel that this has distorted his intentions.

Step (1): change of bases of each pair of triangles, retaining the common vertex at B (G.W. VII p. 376 lines 15-25, p. 377 lines 3-12).

Fig. (i) The sectoring of the eccentric circle.



$Q_iQ_i' = Q_{i-\frac{1}{2}}Q_{i+\frac{1}{2}} = Q_fQ_f'$ etc.

We start from pairs of diametrically opposite sectors of the eccentric circle of combined area $a^2\delta\beta$, obtained by dividing the circle into equal arcs $a\delta\beta$: when we set $\delta\beta \leq \frac{1^\circ}{4}$, as anticipated above, these sectors may properly be regarded, equivalently, as pairs of isosceles triangles.

At the apsides,

$$\frac{P_cP_c'}{Q_cQ_c'} = \frac{b}{a} \quad \text{(p. 376 lines 22-3)}$$

$$\therefore \frac{\Delta P_cBP_c'}{\Delta Q_cBQ_c'} = \frac{b}{a} \quad \text{since BC = BD = a.}$$

Also $\Delta P_cBP_c' = \Delta P_dBP_d'$ (p. 377 lines 4-5).

$$\therefore \Delta P_cBP_c' + \Delta P_dBP_d' = \frac{b}{a}(\Delta Q_cBQ_c' + \Delta Q_dBQ_d').$$

At the quadrants,

$$\frac{BF_o}{BF} = \frac{b}{a}$$

$$\therefore \frac{\Delta P_fBP_f'}{\Delta Q_fBQ_f'} = \frac{b}{a} \quad \text{since } P_fP_f' = Q_fQ_f' \quad \text{(p. 376 line 20).}$$

Also $\Delta P_fBP_f' = \Delta P_gBP_g'$ (p. 377 lines 5-6).

$$\therefore \Delta P_fBP_f' + \Delta P_gBP_g' = \frac{b}{a}(\Delta Q_fBQ_f' + \Delta Q_gBQ_g').$$

Kepler himself unfortunately treated these two special cases together, to derive merely:

$$\Delta P_cBP_c' + \Delta P_dBP_d' = \Delta P_fBP_f' + \Delta P_gBP_g' \quad \text{(p. 377 lines 11-12).}$$

We shall have occasion subsequently to regret his failure to evaluate any of these quantities separately and precisely (particularly at the apsides), for it would have been well within his power to do so, as we have now shown.

For an intermediate paired sector, Kepler would have been quite capable of carrying out the following deduction. By a method similar to that used in Astronomia C. 59, Protheorema III, reiterated in Epitome V Part I Section 3 (G.W. VII p. 374 lines 35-9),[2] he could have obtained:

$$\text{Ellipse sector } P_{i-\frac{1}{2}}BC = \frac{b}{a} \cdot \text{circle sector } Q_{i-\frac{1}{2}}BC$$

$$\text{Ellipse sector } P_{i+\frac{1}{2}}BC = \frac{b}{a} \cdot \text{circle sector } Q_{i+\frac{1}{2}}BC$$

$$\therefore \quad \Delta P_{i-\frac{1}{2}}BP_{i+\frac{1}{2}} = \frac{b}{a} \cdot \Delta Q_{i-\frac{1}{2}}BQ_{i+\frac{1}{2}}$$

and $$\Delta P_{i-\frac{1}{2}}'BP_{i+\frac{1}{2}}' = \frac{b}{a} \cdot \Delta Q_{i-\frac{1}{2}}'BQ_{i+\frac{1}{2}}' \quad \text{similarly}$$

$$\therefore \quad \Delta P_{i-\frac{1}{2}}BP_{i+\frac{1}{2}} + \Delta P_{i-\frac{1}{2}}'BP_{i+\frac{1}{2}}' = \frac{b}{a}(\Delta Q_{i-\frac{1}{2}}BQ_{i+\frac{1}{2}} + \Delta Q_{i-\frac{1}{2}}'BQ_{i+\frac{1}{2}}').$$

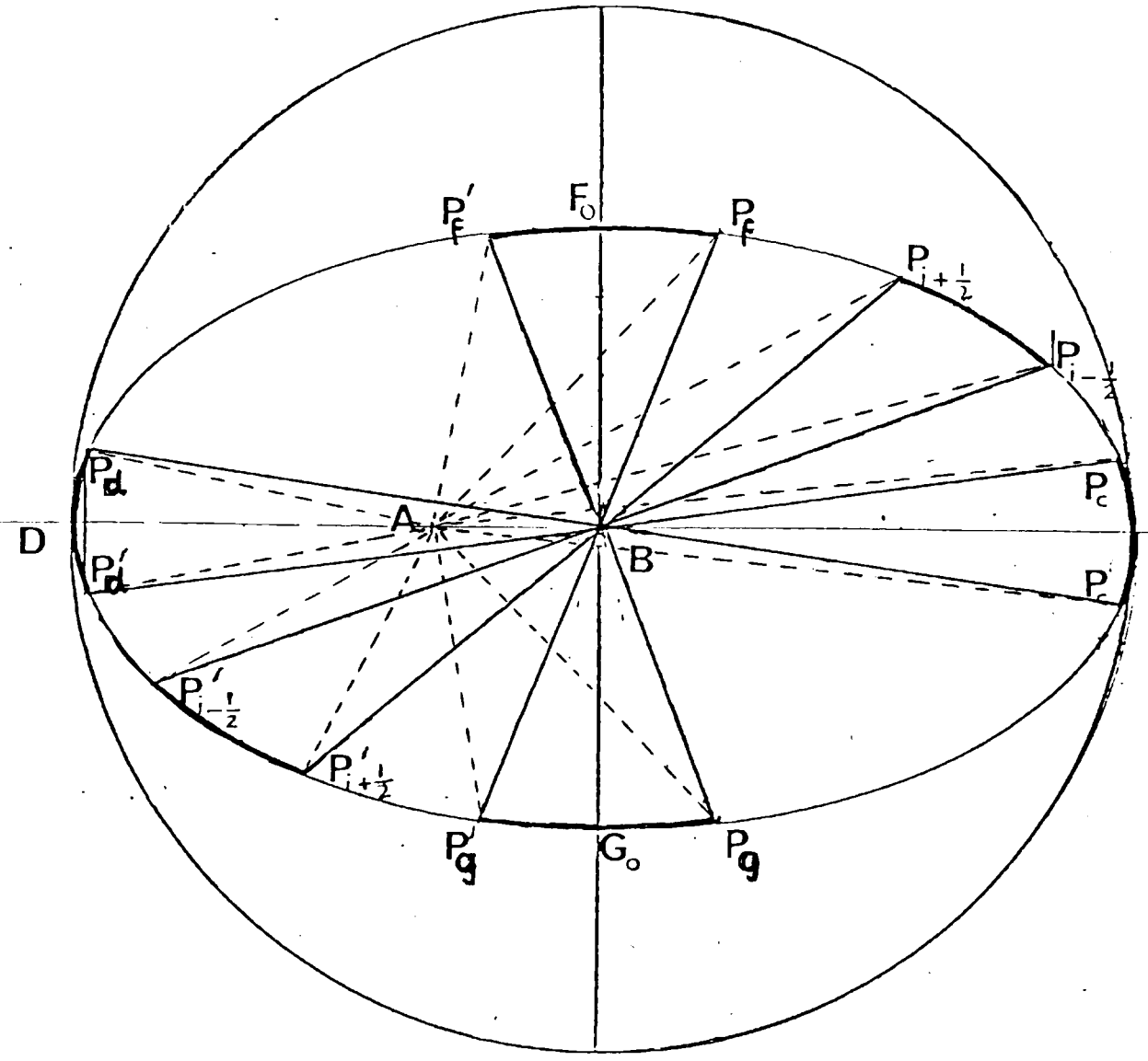
By this step Kepler changed the bases of each pair of triangles to the arcs δs of the ellipse, thereby reducing the sum of every pair of areas in the ratio b : a.

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2. Where the sectors were referred to the focus A, rather than to the centre B, as he needed here.

Step (2): change of vertex of each pair of triangles, leaving their bases unchanged (G.W. VII p. 377 lines 12-20, p. 378 lines 18-19).

Fig. (ii) The sectoring of the ellipse.



At the apsides,

$$\Delta P_c AP'_c + \Delta P_d AP'_d = \Delta P_c BP_c' + \Delta P_d BP_d' \text{ (p. 377 line 12),}$$

since each pair of triangles has equal, parallel bases and CD is their common altitude-sum (p. 377 line 13).

At the quadrants,

$$\Delta P_f AP_f' + \Delta P_g AP_g' = \Delta P_f BP_f' + \Delta P_g BP_g' \text{ (p. 377 line 14),}$$

since in this case all four triangles are individually equal in

area (p. 377 lines 14-16): though in fact it is more helpful to demonstrate the analogy with the situation at the apsides by pointing out that each pair of triangles has equal, parallel bases and F_oG_o is their common altitude-sum.

These two special cases have involved the application of a proposition which Kepler evidently deduced from the Scholium to Euclid I, Prop. XLI (which indeed provides the foundation for the whole of the present discussion) now to be labelled, for future reference:

The conservation-proposition for pairs of triangles:

Given Equal (parallel) bases
Altitude-sum fixed
Conclusion sum of pairs of areas is conserved.

For an intermediate paired sector,

$$\Delta P_{i-\frac{1}{2}}AP_{i+\frac{1}{2}} + \Delta P'_{i-\frac{1}{2}}AP'_{i+\frac{1}{2}} = \Delta P_{i-\frac{1}{2}}BP_{i+\frac{1}{2}} + \Delta P'_{i-\frac{1}{2}}BP'_{i+\frac{1}{2}}.$$

A proof of this even for the partial generalization stated in relation to the pair of sectors 'next' to the apsides (p. 378 lines 18-19) strictly demands the same check that the requirements necessary for the application of the conservation-proposition are again fulfilled: Kepler could easily have satisfied himself that the altitude-sum (the perpendicular distance between the arcs) is fixed;[3] while a recognition that the arcs $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}$ and $P'_{i-\frac{1}{2}}P'_{i+\frac{1}{2}}$ are both equal and parallel requires no knowledge of the geometry of the ellipse except a general appreciation of its symmetry about the centre, on which Kepler had relied throughout his astronomical work. Alternatively, he could have derived these properties from the work of Apollonius, who proved that $P_{i-\frac{1}{2}}P_{i-\frac{1}{2}}'$ and $P_{i+\frac{1}{2}}P_{i+\frac{1}{2}}'$ bisect each other at the

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3. He could not have managed to determine the length of this perpendicular, but in fact it is quite unnecessary to do so.

centre B,[4] so that $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}'P_{i+\frac{1}{2}}'$ is a parallelogram. With this justification it will be assumed in what follows that Kepler possessed this particular information, and his probable failure satisfactorily to complete the proof, as we shall see, will be attributed to another cause.

By this step Kepler changed the vertex of each pair of triangles from B to A, while ensuring that the sum of every pair of areas was conserved.

Step (3): change of base of each individual triangle, retaining the common vertex at A: modulation of the arc

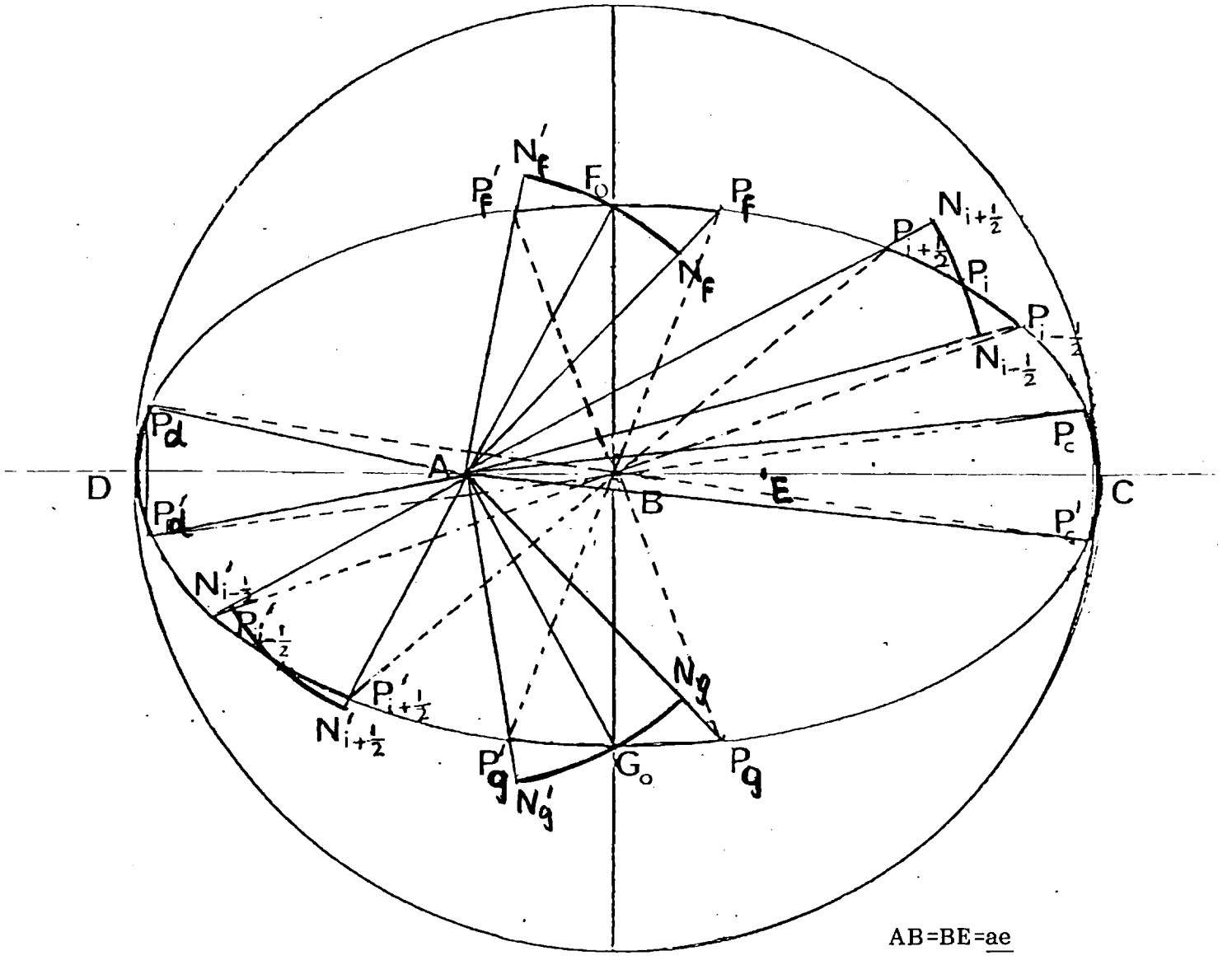
(G.W. VII p. 377 line 24-p. 378 line 13).

We introduce the term modulation to describe the technique that Kepler invented to enable him to make further progress, which involved the replacement of each small elliptic arc by a small circular arc centred at A, to be named the modulated arc[5]—or equivalently, since each may be regarded as a straight line, it will be referred to as the modulated base (of an isosceles triangle).

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4. Heath: Apollonius of Perga , Prop. 10, p. 21.
5. This technique has been clearly explained by Aiton, in Kepler's Second Law of Planetary Motion, p. 88—which is therefore the source of the analysis to be undertaken here.

Fig. (iii) The modulated arcs.



At the apsides, there was no need to modulate the arcs since they were already circular with respect to A (p. 378 lines 11-13)— and, of course, equal, being of the value that Kepler regarded as a standard:

$P_cP_c'=P_dP_d'$ (p. 377 line 4)

and hence, $P_cP_c'+P_dP_d'=2P_cP_c'$, also standard.

At the quadrants, the positioning of the modulated arcs was determined by a process of compensation of areas:

$$\Delta P_f F_o N_f = \Delta P_g' G_o N_g' \quad \text{(p. 378 lines 4-5)},$$

and presumably $\Delta P_f' F_o N_f' = \Delta P_g G_o N_g$ by symmetry,

so that $\Delta N_f A N_f' + \Delta N_g A N_g' = \Delta P_f A P_f' + \Delta P_g A P_g'$.

However, this particular compensation-process is too complicated to be applied in general—it does not give sufficient information as to the positioning of each individual modulated arc—so we will determine the position of each in its own sector in turn by employing other attributes which Kepler also mentioned in relation to the quadrants alone. Meanwhile we remark that, though Kepler did not state it, he must have been aware that, in this special case, the modulated arcs are equal:

$$N_f N_f' = N_g N_g'.$$

The fact that he did not draw them symmetrically, to emphasize this, is evidence only that he never attempted actually to evaluate them: we shall return to regret this subsequently.

For an intermediate paired sector, we first construct the modulated circular arc $N_{i-\frac{1}{2}} N_{i+\frac{1}{2}}$ with centre A, to meet $AP_{i-\frac{1}{2}}$ at $N_{i-\frac{1}{2}}$, $AP_{i+\frac{1}{2}}$ at $N_{i+\frac{1}{2}}$ (produced if necessary), and to cut $P_{i-\frac{1}{2}} P_{i+\frac{1}{2}}$ at P_i, whose position is now to be specified.

(i) We define a point M_1 which bisects the elliptic arc $P_{i-\frac{1}{2}} P_{i+\frac{1}{2}}$

(ii) We define a point M_2 which bisects the modulated arc $N_{i-\frac{1}{2}} N_{i+\frac{1}{2}}$.

These two distinct points are illustrated in Appendix VII(1) where it is established, by modern analysis, that, when $\delta\beta \leq \frac{1^\circ}{4}$,[6] $M_1 M_2$ is evanescent, and so we may write $M_1 \equiv M_2 \equiv P_i$: we also prove that,

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6. In fact we work in terms of true anomaly there: but when $\delta\beta \leq \frac{1^\circ}{4}$, $\delta\theta < \frac{1^\circ}{4}$ in the first half of the orbit—and these restrictions are reversed in the second half.

with this restriction, AP_i is equal to both the arithmetic and the geometric means of the radius vectors $AP_{i-\frac{1}{2}}$ and $AP_{i+\frac{1}{2}}$, as Kepler stated (p. 377 lines 36-8); while AM_2—and therefore AP_i—is evidently perpendicular to the modulated arc, as Kepler additionally required (p. 378 lines 6-7). Hence P_i possesses the aggregated attributes of all these points, so we can write:

$$AP_i = r = a(1+e\cos\beta).$$

And when $N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}'$ is subsequently constructed similarly, since $P_{i-\frac{1}{2}}$ and $P_{i-\frac{1}{2}}'$, $P_{i+\frac{1}{2}}$ and $P_{i+\frac{1}{2}}'$ are diametrically opposite points, we have similarly:

$$AP_i' = r' = a(1-e\cos\beta).$$

Moreover, $\Delta P_{i-\frac{1}{2}}P_iN_{i-\frac{1}{2}}$ and $\Delta P_{i+\frac{1}{2}}P_iN_{i+\frac{1}{2}}$ can then be regarded as congruent, and so we establish the process of compensation of areas appropriate in general:

$$\Delta N_{i-\frac{1}{2}}AN_{i+\frac{1}{2}} = \Delta P_{i-\frac{1}{2}}AP_{i+\frac{1}{2}}$$

and similarly $\Delta N_{i-\frac{1}{2}}'AN_{i+\frac{1}{2}}' = \Delta P_{i-\frac{1}{2}}'AP_{i+\frac{1}{2}}'$.

$$\therefore\ \Delta N_{i-\frac{1}{2}}AN_{i+\frac{1}{2}} + \Delta N_{i-\frac{1}{2}}'AN_{i+\frac{1}{2}}' = \Delta P_{i-\frac{1}{2}}AP_{i+\frac{1}{2}} + \Delta P_{i-\frac{1}{2}}'AP_{i+\frac{1}{2}}'.$$

Thus when the technique of modulation was carried out in accordance with Kepler's intentions, by this step each scalene triangle (an elliptic sector) was replaced by an isosceles triangle (a circular sector) to which it was individually equal in area, so ensuring that the sum of every pair of areas was automatically conserved.

Step (4): the determination of the individual modulated bases for an intermediate paired sector: correction of mis-statements (G.W. VII p. 378 lines 14-22, 22-9).

Kepler next introduced a focal property so that the discussion would refer specifically to the orbital ellipse: the constant sum of radius vectors at diametrically opposite points:

$$AP_i + AP_i' = 2a \quad \text{(p. 378 lines 16-18)},$$

repeated from Section 3 (p. 372 lines 17-18), probably proved as anticipated in Chapter IV, §1,[7] involving recognition of the parallelogram AP_iEP_i', as illustrated there, and again in §2 below. These radius vectors $AP_i = r$ and $AP_i' = r'$ are the altitudes of the pair of isosceles triangles which are now, as a consequence of Step (3), to replace the pair of scalene triangles (the intermediate double elliptic sector) as the subject of our investigation. Then we are to imagine that these isosceles triangles were rotated about A,[8] to arrange their altitudes collinear, so that the two modulated bases would become parallel, and perpendicularly bisected by the line of altitudes. Accordingly we have arrived at a situation potentially commensurate with that at the apsides, where the pair of triangles had the same constant altitude-sum—but also the bases were known to be equal, of standard size P_cP_c': whereas nothing whatsoever was known about the size of the intermediate bases except what could be deduced from the application of the conservation of the sums of pairs of areas, as established by Step (3), in conjunction with the constant altitude-sum. Therefore we will set out a brief survey of the range of validity of some converses of the conservation-proposition stated in Step (2), in an attempt to discover how Kepler may have proceeded. Nowadays these arguments are more conveniently presented algebraically, but I

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7. While in modern terms, from the Secant Law, we have immediately:
$$AP_i + AP_i' = a(1+e\cos\beta) + a(1-e\cos\beta) = 2a.$$

8. Another instance of this technique is to be found in an entirely different context, in a diagram of Epitome IV Part II Section 3 (G.W. VII p. 305).

have checked that each one can be expressed geometrically, as Kepler would have tackled it: the latter approach entails a comparison with the bases of standard size (these have to be drawn in order to illustrate the area to be conserved), and in turn compels us to distinguish two representations which are, in general, mutually exclusive.

The modern representation[9]. when the bases are first supposed unequal, and either both greater, or both less, than the standard value P_cP_c' (and, of course, their sum repectively greater or less than $2P_cP_c'$). Then, algebraically or geometrically, it is easy to demonstrate a (double) converse of the conservation-proposition:

Given Altitude-sum fixed

Unequal (parallel) bases either both greater or both less than P_cP_c'

Conclusion Conservation of sums of pairs of areas is impossible.

We subsequently introduce what will be labelled condition I: the bases are still supposed both greater, or both less, than P_cP_c', but:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}'$$

Indeed this condition has no effect for the conservation of pairs of areas remains impossible.

Hence because the sum of pairs of areas is certainly conserved—as proved by Step (3)—it follows logically that the bases are neither both greater nor both less than the standard value (whether equal or not), but are both of that standard size P_cP_c'.

The Keplerian representation when one base is greater, and one less, than the standard value P_cP_c', and we first suppose their sum to be either greater or less than $2P_cP_c'$. Then it is correspondingly simple, by taking numerical examples, or by a

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9. This description will be accounted for below.

geometrical method, to demonstrate the (double) counter-proposition:

Given Altitude-sum fixed

Sum of parallel bases either greater or less than $2P_cP_c'$

Conclusion Conservation of sums of pairs of areas is possible.

We subsequently introduce what will be labelled condition II : the bases are still supposed unequal, but:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} + N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}' = 2P_cP_c'.$$

The effect of this is to make the conservation of pairs of areas once more impossible. And since this condition was stated by Kepler (p. 378 lines 21-2) it is clear that he certainly considered this representation at least. Hence, again, because the sum of pairs of areas is certainly conserved, Kepler could logically have deduced that—when their sum is constant—the bases cannot be unequal, and therefore, since their sum has the standard value $2P_cP_c'$, they are individually of standard size P_cP_c'. However, it will in due course become obvious that he should not have reasoned along these lines.

To assess the significance of these distinct representations we make an apparent digression to consider, with modern hindsight, an extension of the Keplerian method set out here, applied to any ellipse: that is, to a member of conic-system A.G. II whose focus lies elsewhere on the major axis than at the Sun A, which is still to function as the pole or centre of force. It is straightforward to check that we may proceed by Step (1), and also Step (2)—since the change of vertex involved did not depend on the fact that A was then the focus—to establish the conservation of sums of pairs of areas and thence, by Step (3), to evaluate the modulated bases, using the notation of Chapter I, §4 and formulae from Prolegomenon I:

At the apsides,

Equal modulated bases = $b_s\delta\beta$ (the standard).

For an intermediate paired sector,

Modulated base at $P_s = r_s\delta\theta = \dfrac{ab_s(1+e\cos\beta)}{r_s}\delta\beta = \dfrac{b_sr_o}{r_s}\delta\beta$

Modulated base at $P_s' = r_s'\delta\theta_s' = \dfrac{b_sr_o'}{r_s'}\delta\beta$ similarly.

We next specify the grade of the ellipse, again by reference to Chapter I, §4, which can now be defined more generally:

A large-grade ellipse (e.g. the Contemporaneous Circle) has its focus nearer the centre than the pole A, and $r_s > r_o$, $r_s' > r_o'$. Thus such an ellipse has unequal modulated bases both less than the standard value, while its altitude-sum is greater than 2a.

A small-grade ellipse (e.g. the Auxiliary Ellipse) has its focus farther from the centre than the pole A, and $r_s < r_o$, $r_s' < r_o'$. Thus such an ellipse has unequal modulated bases both greater than the standard value, while its altitude-sum is less than 2a.

It is certainly possible to continue this argument, on the lines of Step (5) below), to show that the Keplerian method provides a perfectly sound proof of Law II for an orbit in any ellipse of eccentricity ε about a centre of force (the pole) at a distance ae from its centre B—not, in general, at its focus: that is, we interchange the role of the pole and the focus.[10.] However, since Kepler was certainly incapable of determining the necessary geometrical formulae, this would only be of historical interest, in highlighting the latent power of his method, which has undoubtedly never been fully appreciated. Our present purpose is, rather, to demonstrate that it is the modern representation alone which is appropriate to any ellipse, taken in conjunction with a variable altitude-sum; from which the standard value of the modulated bases for the orbital ellipse may be deduced by continuity, when its focal property is specified simply by taking the altitude-sum constant.

A full classification of the validity of every possible converse of the conservation-proposition—eight instances in all, doubled

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10. The law of force is then:

$$F \propto r \left\{ \frac{e[\varepsilon^2 r^2 - a^2(1-\varepsilon^2)(\varepsilon^2-e^2)]^{\frac{1}{2}} - a(\varepsilon^2-e^2)}{e^2 r^2 - a^2(1-e^2)(\varepsilon^2-e^2)} \right\}^3$$

I have not come across this formula in any textbook.

when we drop the requirement that the altitude-sum is constant—constitutes a circumscribed topic which is not mathematically fruitful,[11.] and therefore has not been undertaken here; for we have now established that only the modern representation was appropriate in this context, and, had Kepler adopted it, his deduction, though incomplete, could have been shown to have been fundamentally sound by modern standards: while if he did not, his argument in this step must be judged defective. Unfortunately we have no information on this point,[12.] merely evidence (cited above) that he must inadvisedly have adopted the Keplerian representation (named accordingly), since he needed to assume condition II (and consequently his erroneous belief that he had proved this condition is strictly irrelvant—though it will be the subject of separate discussion below). However, it is fair to recognize the speciousness of the misconception involved in relying on condition II, for it is tempting to expect that the existence of a constant sum for the two bases is a less demanding property, correspondingly easier to prove, than the suggestion of their individual equality (condition I)—that is, until we appreciate the mutual exclusiveness of the two representations, which demonstrates that the two conditions are of equivalent status in relation to their respective representations, the fact being that neither can be proved independently: they are fulfilled together—only when the two representations coincide. And in that particular situation, the modulated bases possess the common standard value:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}' = P_cP_c'.$$

As we shall emphasize in Step (5), Kepler somehow satisfied himself that he had established this, though (because of the lack of any evidence that he argued via the modern representation)

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11. Except in connection with the extension to conic-system A.G. II: but the necessity for such an approach has been long since superseded.

12. This is inevitable, to the extent that, as we have indicated, there would have been no necessity for him to have stated condition I.

we have to conclude that he probably never completed an unobjectionable proof of Law II, this step being the weak link in his reasoning. However, it will be reinforced when, in §2 below, we show that there was knowledge potentially available to him which would have enabled him, by a different route, to have deduced the common equality of the modulated arcs in all generality, to provide what may be regarded as an amended version of Step (4).

* * * *

Correction of a rash of erroneous statements (p. 378 lines 22-9).

As we have already remarked, Kepler attempted a partial generalization only for sectors 'next' to the apsides, but we shall continue to transcribe his statements as if they referred to an intermediate double sector, as illustrated. We return, as promised, to his belief that he had proved condition II: that the modulated bases have a fixed sum (p. 378 lines 21-2): this, he said, was because

$$\frac{r}{r'} = \frac{\delta\theta'}{\delta\theta} \text{ (p. 378 lines 22-4).}$$

However, we point out that this entails the relationship which in fact Kepler was hoping subsequently to establish:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = r\delta\theta = r'\delta\theta' = N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}'.$$

I am sure that Kepler did not realize that he had begged the question at this juncture, but in fact this statement introduces a passage (p. 378 lines 24-9) which is totally incompatible with all the rest. In particular, he wrote:

$$\angle N_{i-\frac{1}{2}}AN_{i+\frac{1}{2}} + \angle N_{i-\frac{1}{2}}'AN_{i+\frac{1}{2}}' = \angle P_cAP_c' + \angle P_oAP_o' \text{ (p. 378 lines 24-5, 27).}$$

This is, of course, quite wrong,[13.] though true of the similarly-named pairs of triangles: consequently, I recommend that the word 'angles' should be replaced (four times, explicitly or implicitiy) by 'triangles' throughout p. 378 lines 24-8; and, for consistency, the phrase 'form the bases of these triangles' be substituted for 'subtend these angles' in lines 28-9 following.

It is because Kepler reverted in mid-sentence to writing of 'areas' (p. 378 line 29), and of 'triangles' (p. 378 line 35 and thereafter), that I have felt justified in assessing these aberrations as mistranscriptions on Kepler's part,[14.] rather than as the fundamental misrepresentations of theory which, taken at face valye, they undoubtedly are.[15.] I attribute this uncharacteristic carelessness to the fact that Kepler's attention had been temporarily diverted to the consideration of the true anomaly—with which he ought not to have concerned himself, since it did not enable him to prove condition II (which is unprovable, apart from being redundant, as we have demonstrated in the previous discussion). Indeed, with the corrections suggested, the passage becomes merely repetitious, so the reasoning would have been unaffected—or improved by an increase in clarity—had the section between line 21 and line 29 been omitted altogether.

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13. In modern terms, from Prolegomenon IIA, $\delta\theta = \frac{b}{r}\delta\beta$, $\delta\theta' = \frac{b}{r'}\delta\beta$

$$\therefore\ \delta\theta + \delta\theta' = \frac{2a}{rr'}.\ b\delta\beta = \frac{2b}{a(1-e^2\cos^2\beta)}\delta\beta:\ \text{not constant.}$$

14. I have checked that they occur in the first edition of Epitome.

15. We return in §2 to the oddity that these errors have been remarked neither by Caspar, in his notes to G.W. VII, nor by Koyré, who, in La révolution astronomique p. 321, has translated the entire passage.

Step (5): completion of the proof, and the quantification of the indeterminate time-axiom (G.W. VII p. 378 line 34-p. 379 line 3).

(a) 'Area pro mensura temporis constituitur' (p. 378 line 31).

For this final step it is essential to consider individual triangles, as Kepler realized—though he referred only to the triangle 'next' to the one at aphelion—stating that its base was equal to P_cP_c' 'in effect':[16] that is, after modulation. However, we shall express this in all generality, on the grounds that Kepler had certainly demonstrated the technique (of modulation) from which the existence of the common, standard value can be inferred by modern methods; and that moreover, he had undoubtedly satisfied himself that he had managed to establish the result:

$$\begin{aligned} \Delta P_{i-\frac{1}{2}}AP_{i+\frac{1}{2}} &= \Delta N_{i-\frac{1}{2}}AN_{i+\frac{1}{2}} \quad \text{by Step (3)} \\ &= \tfrac{1}{2}r.N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} \\ &= \tfrac{1}{2}r.P_cP_c'. \end{aligned}$$

Thus he could at last deduce, for every individual area, that:

$$\delta A \propto r.$$

This relationship was what he had been aiming for to correlate with the indeterminate time-axiom (direct):

$$\delta\alpha \propto r \text{ (p. 377 lines 28-9, p. 378 lines 38-40),}^{17}$$

to obtain

$$\delta\alpha \propto \delta A.$$

$\therefore$ Completed time $\propto$ completed area.[18]

Hence Kepler concluded (p. 378 line 47-p. 379 line 3):

> the total area of the ellipse, divided into triangles at A, is distributed among the arcs [those triangles] in the same proportion in which the periodic time is distributed among them them. Therefore the individual triangles are most exact measures of their individual arcs [times].[19]

This was the proof of Law II that he required.

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16. potestate (p. 378 line 44).

17. We continue to designate time by mean anomaly simply in order to avoid a proliferation of constants, by neglecting those whose consideration can properly be left in abeyance until Chapter VIII, §2.

18. Because of the difference in methods, no integration is involved, as it was in the case of Law I: but see Chapter VIII, §1.

19. The first mention of 'arcs' is confusing, the second quite wrong, from the sense of the whole passage: so we shall assume two (further) mistranscriptions.

(b) Derivation of the Keplerian Time-measure.

To complete the above proof it was evidently sufficient merely to take the common value of every modulated base, or arc, to be the standard value P_cP_c'. Deferring, to §3 and §4, discussion of two versions of the constant involved in a precise evaluation, we shall meanwhile adopt a neutral interpretation, emphasizing that Kepler himself was well aware that the modulated arcs were dependent on small intervals of eccentric anomaly,[20.] by writing:

$$P_cP_c' \propto a\delta\beta.$$

$$\therefore\ \delta A \propto \tfrac{1}{2}r.\ a\delta\beta = \tfrac{1}{2}a^2.\frac{r}{a}\,\delta\beta$$

$$\therefore\ \delta A \propto \tfrac{1}{2}c_u.\frac{r}{a}\,\delta\beta,$$

where $c_u=a^2$ is the universal area-parameter employed throughout Chapter III. And since $\delta A \propto \delta\alpha$, as proved above, it is dimensionally sound to write:

$$\delta A \propto \tfrac{1}{2}c_u\delta\alpha.$$

$$\therefore\ \delta\alpha = \frac{r}{a}\,\delta\beta$$

$$\therefore\ \frac{\delta\alpha}{\delta\beta} = \frac{r}{a} : \text{ the Keplerian Time-measure.}$$

This piece of reasoning has been so expressed with modern hindsight, for all the proportion-statements become valid equations when the modulated arc is correct—which entails the replacement of c_u by $c_o=ab$, the area-parameter specific to the orbital ellipse.[21.]

The name for this time-measure has been chosen to indicate that it is the formulation Kepler was aiming for, by which he resolved the problem of the indeterminate time-axiom, cited in (a) above. It is, of course, Time-measure II applied specifically to the orbital ellipse, for which it is exact: thus we have obtained independent confirmation that the modified distance-sum hypothesis is identical to the area-hypothesis (Keplerian style) for this particular curve alone, as anticipated in Chapter III, §9.[22.] However, though

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20. Though not his subsequent commentators, as we remark in §2 below.

21. As defined in Chapter III, §9, implied in Astronomia C. 59 Protheoremata II, III, confirmed in Epitome V Part I Section 5.

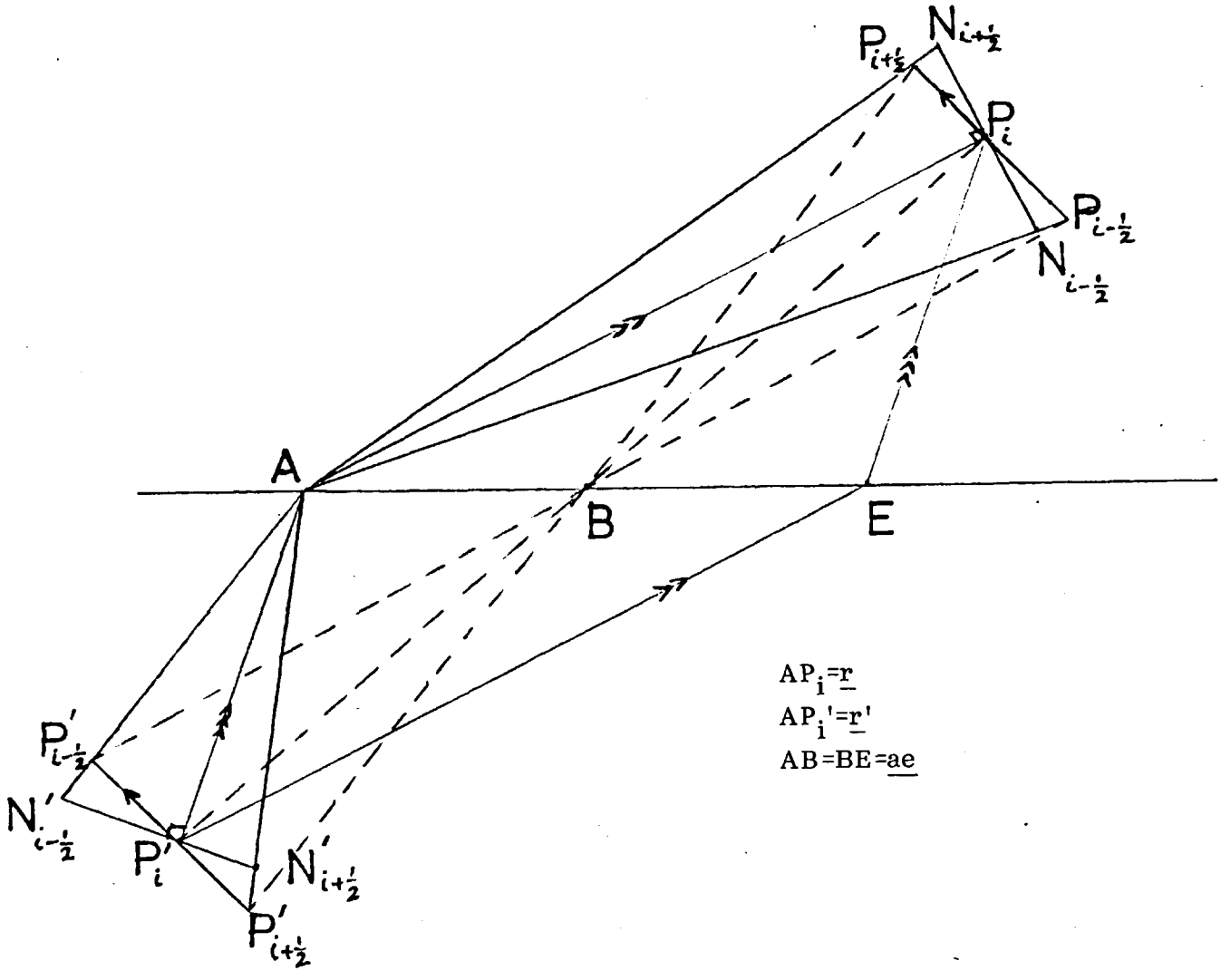
22. And so of course in this very special situation the time is given by a distance-sum hypothesis after all.

its expression is Keplerian (direct) in form as well as in style, it is the one of Newtonian (inverse) form which is to be inferred from Harmonice, in §3 below. Thus, though its derivation was probably tempered by luck as far as the constant was concerned (as discussed in §4), it will be with Keplerian authority that we shall conclude that he managed in the end to quantify the indeterminate time-axiom, as:

$$\frac{\delta \beta}{\delta \alpha} = \frac{a}{r}.$$

§2 The outcome of Kepler's failure to recall the optical property.

Fig. (iv) An intermediate pair of sectors.



The diagram is an elaboration of that given in Chapter IV, §1, to which we shall refer for the sources of the focal properties to be employed here. We first point out that confirmation of the accuracy of the diagram is to be found in §1—backed up by the detailed analysis of Appendix VII (1)—where it was shown that, at the level of exactitude stipulated for this study, it was only necessary to set $\delta\beta \leqq \frac{1^\circ}{4}$ (or $\delta\theta \leqq \frac{1^\circ}{4}$ as appropriate) to

ensure that both the elliptic arcs, and the (circular) modulated arcs, may properly be replaced by straight lines (and so to validate Kepler's frequent description of the sectors as triangles):[1.] this same restriction also ensures that P_i and P_i' may simultaneously be taken as the midpoints of their respective elliptic and modulated arcs, with:

$$AP_i \perp N_{i-\frac{1}{2}}N_{i+\frac{1}{2}},\ AP_i' \perp N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}'.$$

I here put forward the tentative opinion that Kepler implicitly identified the elliptic arc (represented as a straight line) with the tangent, as well as with the chord. Be that as it may, it is undoubtedly true in modern terms at least that the arcs may be treated as tangents or as chords quite indifferently.[2.] Thus we will now regard $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}$, $P_{i-\frac{1}{2}}'P_{i+\frac{1}{2}}'$ as the tangents at P_i, P_i' respectively and adopt the notation standard for this study to name the (acute) angles between each radius vector and the tangent:

$$\angle P_{i+\frac{1}{2}}P_iA = \phi$$

$$\angle P_{i-\frac{1}{2}}'P_i'A = \phi'.$$

$$\therefore \angle P_{i+\frac{1}{2}}P_iN_{i+\frac{1}{2}} = \angle P_{i-\frac{1}{2}}P_iN_{i-\frac{1}{2}} = \frac{\pi}{2} - \phi$$

$$\angle P_{i-\frac{1}{2}}'P_i'N_{i-\frac{1}{2}}' = \angle P_{i+\frac{1}{2}}'P_iIN_{i+\frac{1}{2}}' = \frac{\pi}{2} - \phi'.$$

Hence by resolution we find:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}\cos\left(\frac{\pi}{2}-\phi\right) = P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}\sin\phi$$

and $N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}' = P_{i-\frac{1}{2}}'P_{i+\frac{1}{2}}'\sin\phi'$ similarly.

We have already mentioned (in Chapter IV, §1) that the proof of the focal property of the constant sum of radius vectors at diametrically opposite points (used in Step (4) of §1 as the fixed

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1. There are even more instances when we correct his erroneous statements as suggested at the end of Step (4) of §1.

2. This can in fact, be deduced from the Greek definition of a conjugate diameter (since P_i is the midpoint of the chord) and of a tangent: see Heath: Apollonius of Perga, Props. 5, 11. It is not suggested that Kepler did this, but only that he sensed their coincidence.

altitude-sum) would have involved recognition of the parallelogram AP_iEP_i', so we first suppose that Kepler realized that:

$$AP_i \,||\, EP_i' \quad \text{and} \quad EP_i \,||\, AP_i'.$$

Next, it has been suggested in Step (2) of §1 that—had Kepler extended his investigation to consider the intermediate double sector illustrated here—he would have found little difficulty in appreciating the general property of its symmetry about the centre, which would in all likelihood have involved recognition of the parallelogram $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}P_{i-\frac{1}{2}}'P_{i+\frac{1}{2}}'$, but in any case we shall suppose that he would have known that:

$$P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} \,||\, P_{i-\frac{1}{2}}'P_{i+\frac{1}{2}}' \quad \text{and} \quad P_{i-\frac{1}{2}}P_{i+\frac{1}{2}} = P_{i-\frac{1}{2}}'P_{i+\frac{1}{2}}' = \delta s.$$

Then from the pairs of parallels in turn, we have:

$$\angle P_{i+\frac{1}{2}}P_iA = \angle P_{i+\frac{1}{2}}'P_I'E$$

$$\angle P_{i-\frac{1}{2}}'P_i'A = \angle P_{i-\frac{1}{2}}P_iE,$$

while

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = \sin\phi \,.\, \delta s$$

$$N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}' = \sin\phi' \,.\, \delta s.$$

Thus the pair of modulated bases are apparently unequal, and the equal angles produced by the parallels make no contribution to proving their equality, until we recall a further focal property: the optical property (also cited in Chapter IV, §1) not mentioned in Epitome V Part I Sections 3 or 4:

$$\left.\begin{aligned} \angle P_{i+\frac{1}{2}}P_iA &= \angle P_{i-\frac{1}{2}}P_iE \\ \angle P_{i-\frac{1}{2}}'P_i'A &= \angle P_{i+\frac{1}{2}}'P_i'E \end{aligned}\right\}$$

When these relationships are combined with those above, we find (repeatedly) that:

$$\phi' = \phi$$

to establish the equality of the modulated bases:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}' = \sin\phi \,.\, \delta s.$$

Thence we have merely to compare the application of the conservation-proposition (stated in Step (2) of §1) to this pair of triangles and to the pair at the apsides, to conclude that the modulated bases are of the common, standard value:

$$N_{i-\frac{1}{2}}N_{i+\frac{1}{2}} = N_{i-\frac{1}{2}}'N_{i+\frac{1}{2}}' = P_c P_c'.$$

Thus the foregoing argument constitutes an amended version of Step (4) of §1, and satisfyingly demonstrates that Kepler unquestionably possessed the potential knowledge to complete the proof of Law II in all generality, while his failure to do so in this more efficient way is attributed to a lamentable lapse of memory in respect of the optical property. It will be clear that I have adopted this explanation as a consequence of the tentative opinion expressed above, which entails the rejection of the alternative possibilities: either that Kepler happened to forget the optical property just because of the absence of the representation of a tangent to draw his attention to it; or that he made a deliberate decision to repudiate the property because it did nt seem to him appropriate to identify the tangent with the arc.

In pursuance of my interpretation, I set out opposing views held by the previous commentators who have examined Kepler's proof of Law II in any detail. Caspar's editorial note to G.W. VII (p. 597) praises the clear annunciation of the law, while the following extended note (pp. 597-8) refers mainly to Astronomia, and contains no illumination as to the originality of Kepler's method in Epitome: then, evidently identifying the tangent with the arc, Caspar states, as his version of the result achieved, the Newtonian conclusion:[3] 'the (tangential) motion is inversely proportional to the perpendicular dropped onto the tangent'. Kepler certainly never drew that perpendicular; and besides, as

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3. Precisely that of Principia I Section II Proposition I Corollary 1: but here specific to the orbital ellipse alone.

we have anticipated in Chapter IV, §2, and emphasized throughout Part B, his approach provided the foundation from which one can deduce the radial and transverse components of motion in accordance with modern standards;[4] while we may feel sure that he never tackled the determination of the resultant—tangential—motion in orbit (not least because he did not need to): hence, Caspar's view is utterly anachronous. Indeed it has been criticized for the same reason by Koyré—who, in his turn, identifies the chord with the arc,[5] but goes on to remark that Kepler's conceptions are false: yet if, as it seems, he means that Kepler's resolution of the motion is invalid, then this view too is wrong. Moreover, neither commentator has shown any awareness that the essence of Kepler's proof was that it was conducted in terms of eccentric anomaly, nor, perhaps because of this, has noticed the series of Keplerian errors mentioned at the end of Step (4) of §1, arising from the introduction of the true anomaly (at first sight gross misrepresentations of theory, though corrected in the event as mistranscriptions). Therefore I reluctantly conclude—as in the case of the proof of Law I (see Chapter VI, §5 (d))—that these commentators display some lack of understanding of Kepler's aims in Epitome, as well as of the methods he invented to carry them out.

By contrast if, as I suggest, we accept merely that Kepler identified the tangent with the arc, this is actually compatible with the fact that he did not concern himself with tangential motion, for its investigation then becomes dependent on evaluation of the length of the elliptic arc and determination of its direction: that is, another aspect of the problem that he had already abandoned as insoluble—in his terms[6]—in Astronomia C. 59. This interpretation, too, will be of support when we come to account for the absence of any attempt on Kepler's part to evaluate the modulated arcs in the special case at the quadrants (discussed in §4 below).

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4. See Chapter VI, §7 and Step (5) of §1 above, respectively: the correctness of these results is confirmed in Prolegomenon IIB.

5. La révolution astronomique, pp. 322-4 and note 115, p. 444.

6. The reason for this is obvious by reference to the formulae for δs and $\sin \phi$ (in terms of eccentric anomaly) set out in Prolegomenon IIA. Indeed, since $\frac{N_{i-\frac{1}{2}}N_{i+\frac{1}{2}}}{P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}} = \frac{r\delta\theta}{\delta s} = \sin\phi$, this may be defined, in modern terms, as the function effecting modulation.

§3 The internal consistency of Kepler's kinematical reasoning, incorporating the sequential error

Epitome IV, Part III, Section 2; Harmonice V C. 3 Propositions VI, IX and X, XI and XIII.

It is with some satisfaction that I have found it possible to demonstrate that Kepler's kinematical reasoning was internally consistent, as he explained in Epitome IV Part III Section 2 and Harmonice V C. 3 Proposition VI, directly and inversely in turn: these references are therefore listed first in each category in the accompanying tabulation. Nevertheless, its coherence can only be confirmed mathematically when the relationships are expressed as measures, not just as axioms—so it will be our purpose in this section to show that to do so is genuinely Keplerian, not anachronously modern. For completeness, I have hoped to include every instance which occurs in Epitome IV and V and Harmonice V, and thus the tabulation may be regarded as supplementary to that in Chapter III, §1, which classified every appearance of a time-axiom in Astronomia (while, of course, each of the present measures is specific to the orbital ellipse alone).

In what follows, up to the end of this chapter (that is, in §4 as well), we shall be concerned exclusively with the situation at the apsides and quadrants[1]—because Kepler was only capable of formulating his reasoning in these special cases:[2] this has already been pointed out in Chapter II, §3(b), from which we repeat the definitions of notation, and add some more:

Diurnal quadrantal apparent (angular) motion $\delta\theta_m$

Diurnal $\left\{\begin{matrix}\text{aphelion}\\ \text{perihelion}\end{matrix}\right\}$ apparent (angular) motion $\begin{cases}\delta\theta_a\\ \delta\theta_p\end{cases}$

Diurnal mean (angular) motion $\delta\alpha_m = \frac{2\pi}{T}$

Aphelion distance r_a

Perihelion distance r_p.

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1. And with the apsides alone in Chapter VIII, §3, which constitutes a continuation of the current discussion.

2. Though in fact the statements made may validly be generalized, as we shall see.

The outstanding result in this connection is stated in Harmonice V C. 3 in the course of Proposition XI (G.W. VI p. 303 lines 13-16), in a somewhat convoluted way;[3.] however, it is reiterated straight-forwardly in Proposition XIII (G.W. VI p. 304 lines 23-5), the oddity being that, while both propositions are concerned superficially with comparisons between a pair of planets, this previously unremarked result[4.] refers to a single planet, relating quantities at the apsides (for example aphelion) and the quadrant:

$$\frac{\delta\theta_a}{\delta\alpha_m} = \frac{a^2}{r_a^2}$$

Though Kepler did not say so, this clearly came from Proposition VI (G.W. VI p. 300 lines 38-9), which stated that the diurnal apparent motions were inversely proportional to the squares of the distances—apparently restricted to the apsides there,[5.] but since the relationship is valid at all points in modern terms, we may certainly extend its application to the quadrants, to write (as in Chapter III, §1):

$$\delta\theta_a = \frac{c_m}{r_a^2}\,\delta\alpha_m, \quad \delta\theta_m = \frac{c_m}{a^2}\,\delta\alpha_m,$$

to obtain: $$\frac{\delta\theta_a}{\delta\theta_m} = \frac{a^2}{r_a^2}.$$

Thus the relationship stated by Kepler has entailed the identification of $\delta\theta_m$ with $\delta\alpha_m$[6.]—quite incorrect for the orbital ellipse[7.]—involving the alternatives:

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3. Involving the geometric mean of the apsidal and mean motion: I warn that this has been mistranslated by Wallis in Great Books of the Western World, Vol. 16.

4. Mentioned three times, but not analyzed, by Caspar in his notes (G.W. VI pp. 547-50) to Propositions XI-XIII: see a subsequent footnote.

5. Said by Kepler to have been demonstrated in Astronomia: if he meant C.32, it was proved only for a circle (at the apsides): but he may have been referring to the argument set out in Chapter II, §3(b) as discussed below.

6. Since Caspar failed to distinguish the two, his notes, cited above, are unsound—though extremely helpful in another respect.

7. Correct for the Auxiliary Ellipse in fact—as indicated in the second tabulation of Chapter II, §3(b).

Either $\delta\bar{\theta}_m = \delta\alpha$ when $\delta\alpha_m = \frac{2\pi}{T}$

or $\delta\theta_m = \delta\bar{\alpha}_m$.[8.]

The former is to be assumed for the rest of this chapter (it will be represented geometrically in §4 below), and the latter is considered in Chapter VIII, §3. Meanwhile we set out, and name, a generalization of the Keplerian relationship (even though it is to be applied only at the apsides and quadrants):

$$\frac{\delta\theta}{\delta\alpha} = \frac{a^2}{r^2} : \quad \underline{\text{the Composite Time-measure}},$$

for comparison with:

$$\frac{\delta\theta}{\delta\alpha} = \frac{ab}{r^2} : \quad \underline{\text{the Newtonian Time-measure}}.^{9.}$$

Hence,

$$\text{Proportional error (in } r^2\frac{\delta\theta}{\delta\alpha}\text{)} = \frac{a^2-ab}{ab} = \frac{a-b}{b}.$$

This will be termed:

$$\text{The sequential error} = \frac{a}{b} - 1 = (1-e^2)^{-\frac{1}{2}} - 1 \simeq \tfrac{1}{2}e^2.$$

The identification of $\delta\theta_m$ with $\delta\alpha_m$, which is the reason for this error, may of course be viewed as a simple confusion of the <u>quadrantal apparent motion</u> and the <u>mean motion</u> (both diurnal). However, it will be recalled that, in Chapter II, §3(b) (where it was described as an unwarranted assumption), I surmised that this identification was used to prove that the <u>quadrantal distance</u> in the orbital ellipse coincided with its <u>mean distance</u> <u>a</u>: the first focal property to come to Kepler's attention in connection with his astronomical work. Therefore, while in any case confirming the foundation of the deduction there, as promised, we may also perhaps account for the misidentification of the analogous motions by suggesting that Kepler supposed that this, too, was a property of the orbital ellipse. With this interpretation, the sequential error has its source in a defect of theory.

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8. Where $\delta\bar{\theta}_m$, $\delta\bar{\alpha}_m$ again denote erroneous values of $\delta\theta_m$, $\delta\alpha_m$.
9. Time-measure III, applied specifically to the orbital ellipse: with which is associated the area-hypothesis (Newtonian style).

Then it is because of the coherence of Kepler's reasoning, as set out in the two passages cited above (more precise references are given in the tabulation), that we will regard the Composite Time-measure as derived by a process of composition of ratios from two constituents, of which only one obvious choice will ensure that the linear dimensions are got right:

$\frac{\delta\beta}{\delta\alpha} = \frac{a}{r}$: the Keplerian Time-measure

$\frac{\delta\theta}{\delta\beta} = \frac{a}{r}$: the optical measure.[10]

Hence it appears that all three measures have been so expressed with Keplerian authority—to be further confirmed by the subsequent discussion—and have therefore been entered in the tabulation as such, not merely as axioms.

It is the optical measure that is corroborated independently in Harmonice V C. 3 Propositions IX and X (G.W. VI p. 302 lines 35-41, p. 303 lines 1-7), both of which seem, on the face of it, to refer to comparisons between a pair of planets, but can, quite properly, be applied to two positions of a single planet simply by taking a common mean distance a, to obtain:

Modulated arc = $r\delta\theta = a\delta\beta$.

It is certain that this is in accordance with Kepler's intentions in Harmonice, on the evidence of the fourth tabulation of C.4 (G.W. VI p. 311), which contains the values of what Kepler termed the diurnal apsidal true motions $\delta\beta$, and the corresponding arcs $a\delta\beta$, for each individual planet in turn: because these are not observable, they were calculated from the above formula using the apsidal distances and the diurnal apsidal apparent motions—all being observable quantities—listed respectively in the third and fifth tabulations of C. 4 (G.W. VI p. 309, p. 312).

This formula is now to be compared with the relationship from Prolegomenon IIA:

$\frac{\delta\theta}{\delta\beta} = \frac{b}{r}$: the geometrical measure

$\therefore$ Modulated arc = $b\delta\beta$, correctly.

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10. This name has been taken from Kepler's own description in Epitome IV Part III Section 2 (G.W. VII p. 331 line 33), as anticipated in Chapter III, §3, in discussion of Astronomia C.32, where indeed it made sense, in relation to apsidal arcs of the eccentric circle.

Hence,

$$\text{Proportional error in modulated arc} = \frac{a\delta\beta - b\delta\beta}{b\delta\beta} = \frac{a-b}{b},$$

reiterating:

$$\underline{\text{The sequential error} = \frac{a}{b} - 1 \tilde{=} \tfrac{1}{2}e^2.}$$

It is this manifestation of the sequential error, regarded geometrically as a misevaluation of the modulated arc, on which we shall concentrate for the rest of this chapter.[11.] Meanwhile we remark for interest that, because the fourth tabulation of C. 4, mentioned above, was constructed in connection with Kepler's harmonic investigations, the sequential error can be traced independently to that source as well as to the defect of theory suggested above. However, because of the internal consistency of Kepler's reasoning, the question of its attribution becomes of no further consequence, only the fact of its presence, and that it was theoretically of a size which—by reference to the level of accuracy stipulated in Part A—Kepler would never have been prepared to overlook in principle. Yet, it is of as much significance to assess the error in practice by reference to the level of exactitude stipulated in Part B, recalling (from Chapter V, §3) that the figures in the tabulations of Harmonice were given correct to the nearest second. Thus, Kepler was quite wrong to disregard the sequential error which was by no means negligible at this level ($\cong 8''$ for Mars), even though the modulated (diurnal) arcs themselves were small ($\cong \frac{1}{2}°$ for Mars). Moreover this is the sole occasion, in Kepler's astronomical work, where he made an approximation which was not subsequently justified.

We digress to distinguish two versions of the transverse motion, which will be needed for the systematization of Kepler's final physical opinions, the subject of Chapter VIII, §2. Since the modulated arc was defined as circular with respect to the Sun, its length represented distance travelled round the Sun, and so, applying the Keplerian Time-measure in each case, we have:

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11. By contrast with the misevaluation of the diurnal mean motion, the subject of Chapter VIII, §3.

When the modulated arc is of length $a\delta\beta$, explicit in Harmonice, as proved above,

$$\text{Transverse motion (Harmonice version)} = \frac{a\delta\beta}{\delta\alpha} = \frac{a^2}{r} = \frac{c_u}{r} .$$

When the modulated arc is of length $b\delta\beta$, implicit in Epitome, as demonstrated in §4 below—and correct by modern standards,

$$\text{Transverse motion (Epitome version)} = \frac{b\delta\beta}{\delta\alpha} = \frac{ab}{r} = \frac{c_o}{r} .$$

Thus the distinction between the two depends on the choice of area-parameter:[12.] it is a matter of the constants alone, the subject of §4 below.

* * * * *

The Composite Time-measure has been so described to emphasize that it is derived by a process of composition of ratios which are here quotients of intervals (of anomalies):

$$\frac{\delta\theta}{\delta\beta}\cdot\frac{\delta\beta}{\delta\alpha} = \frac{\delta\theta}{\delta\alpha} .$$

Certainly in this context the intervals were diurnal—of fixed size—but this requirement is in no way essential and the operation could have been applied to any discrete intervals, its effect being to arrange a change of independent variable which, in modern terms, after proceeding to the limit, leads to the formula for differentiation of a function of a function. Thus, while it cannot be suggested that Kepler possessed any formal understanding of this technique, we have now confirmed that he was capable, in theory at least, of applying the Keplerian Time-measure to the increment of libration to produce the radial motion, as indicated in Chapter VI, §7 (it was not necessary to do so in the case of the transverse motion, which is obtained more directly, as above).

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12. As does the derivation of the Keplerian Time-measure: see Step (5) of §1.

The constituents of Kepler's kinematical reasoning

Individual constituents	References
Optical measure (inverse form) $\delta\beta \propto r$ ($\delta\theta$ constant) $\therefore \frac{\delta\beta}{\delta\theta} = \frac{r}{a}$	Epitome IV Part III Section 2 (G.W. VII p. 331 lines 37-8)
Keplerian Time-measure (direct form) $\delta\alpha \propto r$ ($\delta\beta$ constant)† $\therefore \frac{\delta\alpha}{\delta\beta} = \frac{r}{a}$	Epitome IV Part III Section 2 (G.W. VII p. 331 lines 39-40) Epitome IV Part II Section 3 (G.W. VII p. 305 lines 7-11, 19-20) Epitome V Part I Section 4: as cited in §1, Step (5) Harmonice V C.3 Prop. V(1) (G.W. VI p. 300 lines 9-10)
Composite Time-measure (direct form) $\delta\alpha \propto r^2$ ($\delta\theta$ constant)† $\therefore \frac{\delta\alpha}{\delta\theta} = \frac{r^2}{a^2}$	Epitome IV Part III Section 2 (G.W. VII p. 331 lines 41-3) Epitome IV Part II Section 3 (G.W. VII p. 298 lines 9-11)
Optical measure (direct form) $\delta\theta \propto \frac{1}{r}$ ($\delta\beta$ constant) $\therefore \frac{\delta\theta}{\delta\beta} = \frac{a}{r}$	Harmonice V C.3 Prop. VI (G.W. VI p. 300 lines 31-5) Epitome IV Part III Section 2 (G.W. VII p. 331 lines 36-7)
Keplerian Time-measure (inverse form) $\delta\beta \propto \frac{1}{r}$ ($\delta\alpha$ constant) $\therefore \frac{\delta\beta}{\delta\alpha} = \frac{a}{r}$	Harmonice V C.3 Prop. VI (G.W. VI p. 300 lines 35-7) Harmonice V C.3 Prop. V(2) (G.W. VI p. 300 lines 11-13)
Composite Time-measure (inverse form) $\delta\theta \propto \frac{1}{r^2}$ ($\delta\alpha$ constant)†† $\therefore \frac{\delta\theta}{\delta\alpha} = \frac{a^2}{r^2}$	Harmonice V C.3 Prop. VI (G.W. VI p. 300 lines 38-9) Harmonice V C.3 Prop. XI (G.W. VI p. 303 lines 13-16) Harmonice V C.3 Prop. XIII (G.W. VI p. 304 lines 23-5)

† Axioms stated at both the first and second stages in Astronomia
†† Axiom stated at the first stage alone in Astronomia

} as tabulated in Chapter III, §1.

§4 The problem of the constants of the orbit.

Any laxity of approach permitted in Harmonice (appropriate to his purpose there) was in strong contrast to the disciplined mathematical treatment adopted by Kepler in Epitome V, and consequently I at least would not have expected his conspicuous failure to check (or re-evaluate) the modulated arcs in the special cases which arose in the course of the proof of Law II, most notably at the apsides, where he could so easily have managed it, merely by a glance back at Step (1) of §1:

$$P_cP_c' = \frac{b}{a} Q_cQ_c' \quad \text{(G.W. VII p. 376 lines 22-3)}$$
$$= \frac{b}{a} \cdot a\delta\beta$$
$$= b\delta\beta.$$

In this special case the modulated arc was equal to the elliptic arc, because no modulation was necessary (see Step (3) of §1), and so Kepler's omission of this result meant that he retained the arc of the eccentric instead of replacing it by the elliptic arc—which I find almost inexplicable, in view of the importance of the proof and how prominently the length P_cP_c' featured in it (yet I shall have to attempt to explain this, in due course).

However, in §2 I have suggested that Kepler may have felt inhibited from attempting to evaluate the modulated arc at the quadrant because it would have involved investigating the small elliptic arc and thus, I have surmised, concerning himself with tangential motion, which he was convinced that he could not tackle. We can be sure that he did not examine this special case on the grounds—pointed out in Step (3) of §1—that he did not draw the modulated arcs symmetrically at the quadrants to highlight their equality, as the essential preliminary to their individual evaluation which he could quite easily have worked out, again from Step (1) of §1, and now also from Step (3):

$$P_fP_f' = Q_fQ_f' \quad \text{(G.W. VII p. 376 line 20)}$$
$$= a\delta\beta.$$

Then, $N_fN_f' = P_fP_f' \cos P_f'F_oN_f'$

$= P_fP_f' \sin P_f'F_oA$ since $AF_o \perp N_fN_f'$ (p. 378 lines 6-7)

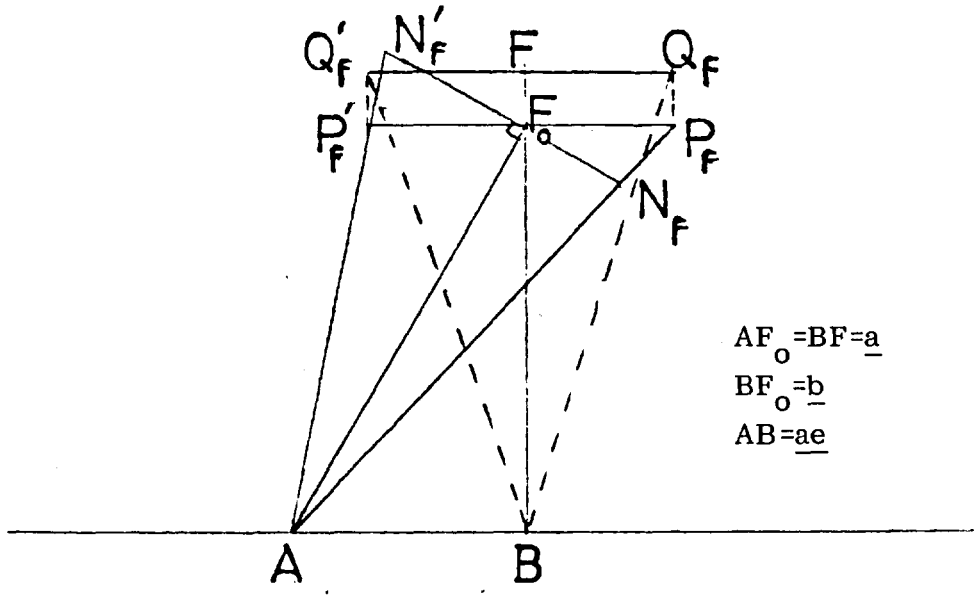
$= P_fP_f' \cos AF_oB$ since $P_fP_f' \parallel AB$,

where $\angle AF_oB$ is the quadrant-value of the optical equation updated for the ellipse,[1.] and hence:

$$N_fN_f' = \frac{b}{a} \cdot a\delta\beta = b\delta\beta.$$

In this case the reduction of the arc is due exclusively to the effect of modulation, which is maximum at the quadrant.[2.] By contrast, the updated optical equation is not maximum there,[3.] and so its appearance is purely fortuitous: it has been mentioned for interest because it is a particular instance of a concept encountered elsewhere in Kepler's Kepler's work.[4.]

Fig. (v) A quadrantal sector of the ellipse.



1. This quadrant-value crops up in entirely different circumstances in Chapter VIII, §2.
2. From §2 above, the function effecting modulation is $\sin\phi$, which, by Prolegomenon IIA, is minimum at the quadrant.
3. See Appendix II(4): at the quadrant, the updated optical equation is identical to the optical angle, and therefore it is of no consequence that, in general, Kepler failed to distinguish these two concepts— as cited in the following footnote.
4. Notably in connection with Law I: see Chapter VI, §4.

Next, temporarily regarding the diagram as depicting the diurnal motions at the quadrant, we take the opportunity to provide the geometrical representation of the misidentification of the quadrantal apparent motion with the mean motion, as promised in §3, noting that the sequential error in both the optical measure and the Composite Time-measure will be accounted for when we set:

$$\delta\bar{\theta}_m = \delta\alpha_m \quad \text{where} \quad \delta\alpha_m = \frac{2\pi}{T}$$

Applying the Keplerian Time-measure:

$$\delta\bar{\theta}_m = \delta\alpha_m = \angle Q_f B Q_f' \ 5.$$

whereas, from the Newtonian Time-measure,

$$\delta\theta_m = \frac{b}{a}\,\delta\alpha_m = \angle\, P_f A P_f'.^{6}$$

$$\therefore \quad \delta\bar{\theta}_m = \frac{a}{b}\,\delta\theta_m.$$

Inevitably, $\delta\bar{\theta}_m$ once more manifests the sequential error.

The sequential error has been so named in anticipation of what is to be demonstrated in Chapter VIII, §3—that its ramifications adversely affected Kepler's eventual synthesis: and therefore it has now become crucial to account for the fact that it was not corrected in Epitome—where there was both the opportunity and the information available—by checking the length of the modulated arcs (even if only at the apsides). In the last resort, this will oblige us to recognize that Kepler appears to have treated the specific values of the implied constants as a matter of indifference. In this connection I present some pertinent considerations whose relative significance, however, I am unable to decide.

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5. Since $AF_o = a$, $\delta\alpha_m = \delta\beta_m$ where $\delta\beta_m$ is the diurnal quadrantal eccentric motion.

6. And hence $\delta\bar{\alpha}_m = \delta\theta_m = \frac{b}{a}\delta\alpha_m$ when we are concerned with mean motion alone—in Chapter VIII, §3.

Because of his firm grasp of the order of linear dimensions, Kepler unquestionably envisaged the appropriate power of the constant in his proportion-statements (which he always handled confidently) as the way of getting the dimensions right. In continuing the approach adopted in Astronomia by employing the radius of the eccentric a as if it were the only linearly-dimensioned constant available to him,[7] it may be that he was unconsciously acknowledging the dominant rôle played in his discoveries by the geometry of the eccentric circle—which had turned out in every respect to provide a suitable measure for the focal geometry of the ellipse (not only in the formula for the radius vector, but in expressing its area as well),[8] so that his dependence on a alone may perhaps be interpreted as an (unfortunate) fixation.[9] Be that as it may, his discovery of Law III, by which the periodic times were expressed in terms of the mean distance a[10]—also the quadrant-distance, as we have emphasized in §3 above—must have greatly reinforced any underlying conviction that a was the constant with which he should be exclusively concerned.

On the other hand, Kepler was obviously greatly hampered by a lack of suitable notation, and the inevitable cumbersomeness of the terminology available to him. In particular, a direct representation of the value $b\delta\beta$ of the modulated arcs would have entailed the description 'a small arc of the circle on the minor axis of the ellipse as diameter', and this circle was certainly never drawn: though in fact this difficulty ought not to have deterred him, since he could have expressed this quantity otherwise as 'a small arc of the eccentric reduced in the ratio of the minor to the major semiaxis of the ellipse', for he was accustomed to refer to b as 'the shorter [minor] semidiameter [semiaxis] of the ellipse'. We remark that it is nevertheless some indication of his lack of appreciation of the essential function of the constant b in the geometry of the ellipse (anticipated in Chapter IV, §1) that it was not until

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7. Which it was, of course, until near the end of Astronomia Part IV.

8. The confirmation of the specific area of the ellipse in Epitome IV Part I Section 5 was undoubtedly something of an afterthought.

9. We note in passing that it should probably be regarded as sheer chance (as far as the constant was concerned) that the Keplerian Time-measure was correct by modern standards (see Step (5) of §1).

10. See Chapter VIII, §2 and §3.

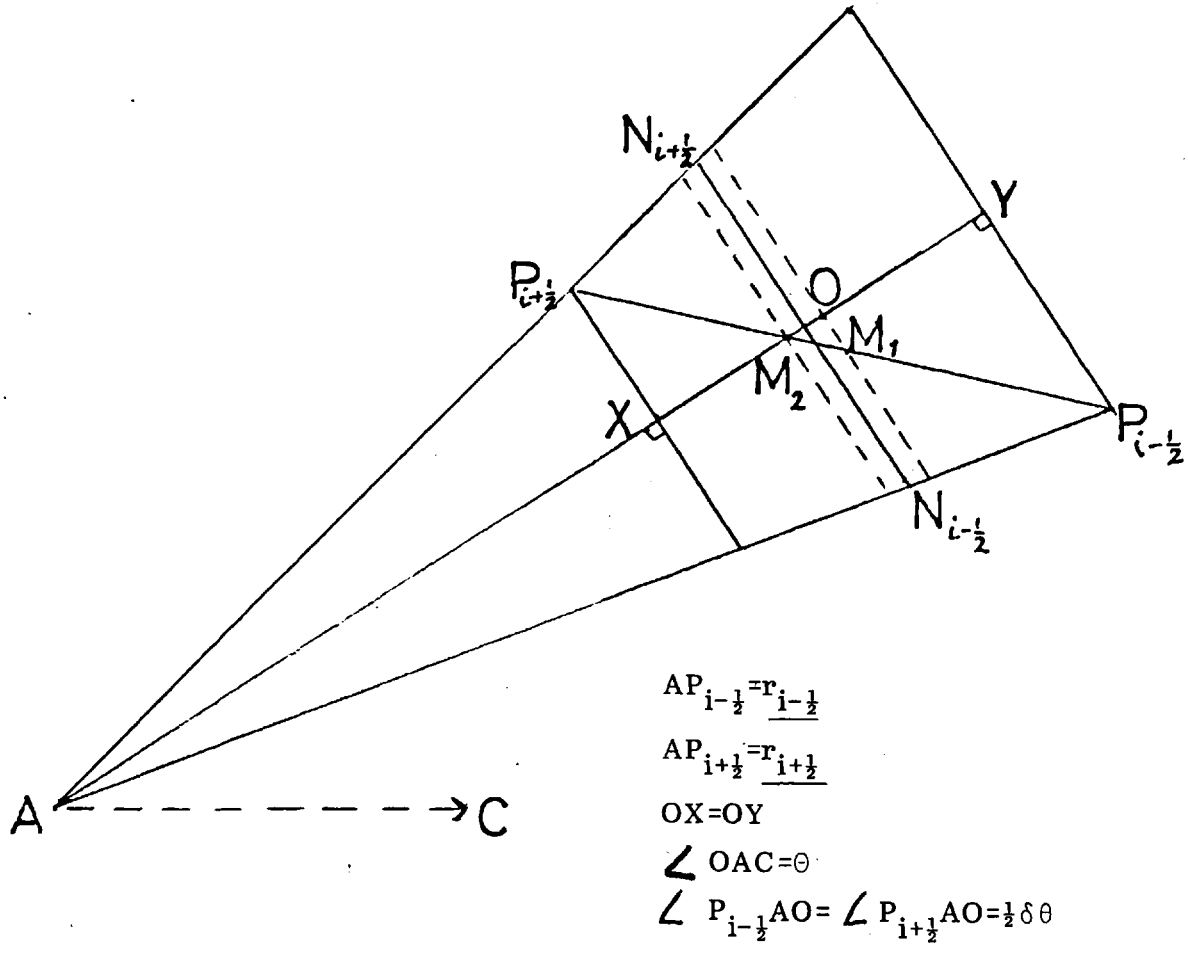
Epitome V Part II Section 1 that he chose to name it the semi-diacentral length: a designation whose redundancy is underlined by the circumstance that he concurrently introduced the term semidihelion to denote the semilatus rectum, a quantity which was totally irrelevant in his treatment.

Thus, whether because of the inadequacy of Kepler's methods for coping with the specification of the constants, or through some conviction that it was unnecessary to try to do so, the upshot was that b did not appear in his work: as we shall see in Chapter VIII, §3, this initiated a chain-reaction; while it is of interest that the kinematical error to be exposed there originated in what is, fundamentally, a geometrical fault.

APPENDIX VII(1) to §1

THE POSITIONING OF THE MODULATED ARC

Fig. (i) An intermediate elliptic sector.



In continuance of the discussion of Chapter V, §3, we shall here incidentally establish a number of geometrical equivalences specifically relating to the orbital ellipse, which are simply consequences of the level of exactitude stipulated throughout Part B, when we impose the slightly more rigorous restriction $\delta\theta\leq\frac{1^\circ}{4}$, which entails $\cos\frac{1}{2}\delta\theta=1$

analytically, as explained there. The stated purpose of this investigation, however, is to examine every point M_n (n=1,2,3) which has some claim to be regarded as the point P_i of the ellipse in which the modulated arc $N_{i-\frac{1}{2}}N_{i+\frac{1}{2}}$ cuts the elliptic arc $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}$—and so in the end to determine the position of that modulated arc: for simplicity, I have merely indicated on the diagram what turn out to be the extreme positions M_1 and M_2. I begin by defining two lengths as standards of comparison: they cannot appropriately be illustrated (for a technical reason), but will be evaluated by modern methods using the polar equations of the points $P_{i-\frac{1}{2}}$, $P_{i+\frac{1}{2}}$ of the ellipse.

AG the geometric mean of $AP_{i-\frac{1}{2}}$ and $AP_{i+\frac{1}{2}}$: $AG^2 = r_{i-\frac{1}{2}}r_{i+\frac{1}{2}}$.

$$\therefore \quad AG^2 = \frac{b^4}{a^2}\left\{\frac{1}{[1-e\cos(\theta-\frac{1}{2}\delta\theta)][1-e\cos(\theta+\frac{1}{2}\delta\theta)]}\right\}$$

$$= \frac{b^4}{a^2}\left\{\frac{1}{1-2e\cos\theta\cos\frac{1}{2}\delta\theta+\frac{1}{2}e^2(\cos 2\theta+\cos\delta\theta)}\right\}$$

$$= \frac{b^4}{a^2}\left\{\frac{1}{1-2e\cos\theta\cos\frac{1}{2}\delta\theta+\frac{1}{2}e^2(2\cos^2\theta-1+2\cos^2\frac{1}{2}\delta\theta-1)}\right\}$$

$$\therefore \quad AG^2 = \frac{b^4}{a^2}\left\{\frac{1}{1-2e\cos\theta\cos\frac{1}{2}\delta\theta+e^2(\cos^2\theta+\cos^2\frac{1}{2}\delta\theta-1)}\right\}.$$

AM the arithmetic mean of $AP_{i-\frac{1}{2}}$ and $AP_{i+\frac{1}{2}}$: $AM = \frac{1}{2}(r_{i-\frac{1}{2}}+r_{i+\frac{1}{2}})$.

$$\therefore AM = \frac{b^2}{2a}\left\{\frac{1}{1-e\cos(\theta-\frac{1}{2}\delta\theta)} + \frac{1}{1-e\cos(\theta+\frac{1}{2}\delta\theta)}\right\}$$

$$= \frac{b^2}{2a}\left\{\frac{2-e[\cos(\theta-\frac{1}{2}\delta\theta)+\cos(\theta+\frac{1}{2}\delta\theta)]}{[1-e\cos(\theta-\frac{1}{2})][1-e\cos(\theta+\frac{1}{2}\delta\theta)]}\right\}$$

$$= \frac{b^2}{a}\left\{\frac{1-e\cos\theta\cos\frac{1}{2}\delta\theta}{1-2e\cos\theta\cos\frac{1}{2}\delta\theta+e^2(\cos^2\theta+\cos^2\frac{1}{2}\delta\theta-1)}\right\} \quad \text{as above.}$$

We also take this opportunity to evaluate $\delta r = r_{i-\frac{1}{2}} - r_{i+\frac{1}{2}}$.

$$\delta r = \frac{b^2}{a}\left\{\frac{1}{1-e\cos(\theta-\frac{1}{2}\delta\theta)} - \frac{1}{1-e\cos(\theta+\frac{1}{2}\delta\theta)}\right\}$$

$$= \frac{b^2 e}{a}\left\{\frac{\cos(\theta-\frac{1}{2}\delta\theta)-\cos(\theta+\frac{1}{2}\delta\theta)}{[1-e\cos(\theta-\frac{1}{2}\delta\theta)][1-e\cos(\theta+\frac{1}{2}\delta\theta)]}\right\}$$

$$\therefore \quad \delta r = \frac{ae}{b^2} r_{i-\frac{1}{2}} r_{i+\frac{1}{2}} . \sin\theta . 2\sin\tfrac{1}{2}\delta\theta .$$

We shall return to this in Chapter VIII, §1: meanwhile we remark that:

$$\frac{1}{4}\delta r^2 = \frac{1}{4}(r_{i-\frac{1}{2}} - r_{i+\frac{1}{2}})^2$$

$$= \frac{1}{4}\left\{(r_{i+\frac{1}{2}} + r_{i+\frac{1}{2}})^2 - 4r_{i-\frac{1}{2}} r_{i+\frac{1}{2}}\right\}$$

$$= AM^2 - AG^2 .$$

Now we recall, from Chapter V, §3, that the slightly more rigorous restriction $\delta\theta \leq \frac{1^\circ}{4}$ entails $\cos\frac{1}{2}\delta\theta = 1$, as well as $\sin\frac{1}{2}\delta\theta = \frac{1}{2}\delta\theta$. Accordingly, we find:

$$\left.\begin{aligned} AG &= \frac{b^2}{a}\left(\frac{1}{1-2e\cos\theta+e^2\cos^2\theta}\right)^{\frac{1}{2}} \\ AM &= \frac{b^2}{a}\left(\frac{1-e\cos\theta}{1-2e\cos\theta+e^2\cos^2\theta}\right) \end{aligned}\right\} = \frac{b^2}{a}\left(\frac{1}{1-e\cos\theta}\right).$$

Hence, with this restriction, the arithmetic and the geometric means coincide to give the length we shall define as $AP_i \equiv r$. Moreover,

$$P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}^2 = r_{i-\frac{1}{2}}^2 + r_{i+\frac{1}{2}}^2 - 2r_{i-\frac{1}{2}} r_{i+\frac{1}{2}}\cos\delta\theta$$

$$= (r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}})^2 - 2r_{i-\frac{1}{2}} r_{i+\frac{1}{2}}(1+\cos\delta\theta)$$

$$= 4AM^2 - 4AG^2\cos^2\tfrac{1}{2}\delta\theta$$

$$P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}{}^2 = \delta r^2 + AG^2\delta\theta^2 \quad \text{when} \quad \delta\theta \leqq 1^\circ$$

$$= \delta r^2 + r^2\delta\theta^2 \quad \text{with the more rigorous restriction } \delta\theta \leqq \frac{1^\circ}{4}.$$

$$= \delta s^2 \quad \text{for any curve.}$$

Thus the identical restriction which ensures that any circular arc is indistinguishable from a straight line will also ensure that the elliptic arc may properly be regarded as a straight line (as anticipated in Chapter V, §3).

Now we define M_1 as the midpoint of the elliptic arc $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}$: that is, M_1 will be the midpoint of the line $P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}$, constructed as indicated on the diagram (by Euclid VI, Prop. II): and AM_1 will be the median of $\Delta P_{1-\frac{1}{2}}AP_{i+\frac{1}{2}}$.

$$\therefore AM_1{}^2 = \tfrac{1}{2}(r_{i-\frac{1}{2}}{}^2 + r_{i+\frac{1}{2}}{}^2) - \tfrac{1}{4}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}{}^2$$

$$= \tfrac{1}{2}(r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}})^2 - r_{i-\frac{1}{2}}r_{i+\frac{1}{2}} - \tfrac{1}{4}P_{i-\frac{1}{2}}P_{i+\frac{1}{2}}{}^2$$

$$= 2AM^2 - AG^2 - AM^2 + AG^2\cos^2\tfrac{1}{2}\delta\theta \quad \text{from above}$$

$$\therefore AM_1{}^2 = AM^2 - \tfrac{1}{4}AG^2\delta\theta^2 \quad \text{when } \delta\theta \leqq 1^\circ.$$

$$\therefore AM_1{}^2 = r^2(1 - \tfrac{1}{4}\delta\theta^2) \quad \text{when } \delta\theta \leqq \frac{1^\circ}{4}$$

$$\therefore AM_1 = r(1 - \tfrac{1}{8}\delta\theta^2) \quad \text{to a first approximation.}$$

However, $\frac{1}{8}\delta\theta^2 = 0$ when $\cos\frac{1}{2}\delta\theta = 1$ (see Chapter V, §3).

$$\therefore AM_1 = r \quad \text{when } \delta\theta \leqq \frac{1^\circ}{4}.$$

Next we define M_2 as the midpoint of the modulated arc: that is, AM_2 will be the perpendicular bisector of the modulated arc as well as the bisector of $\angle P_{i-\frac{1}{2}}AP_{i+\frac{1}{2}}$, as shown on the diagram.

Then $\frac{M_2P_{i-\frac{1}{2}}}{M_2P_{i+\frac{1}{2}}} = \frac{AP_{i-\frac{1}{2}}}{AP_{i+\frac{1}{2}}} = \frac{r_{i-\frac{1}{2}}}{r_{i+\frac{1}{2}}}$ by Euclid VI, Prop. III

But, $\frac{M_2Y}{M_2X} = \frac{M_2P_{i-\frac{1}{2}}}{M_2P_{i+\frac{1}{2}}}$ by similar triangles

$\therefore \frac{M_2Y}{XY} = \frac{r_{i-\frac{1}{2}}}{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}}$ by the addition-property of ratios.

Now $AY = r_{i-\frac{1}{2}}\cos\frac{1}{2}\delta\theta$ and $XY = (r_{i-\frac{1}{2}} - r_{i+\frac{1}{2}})\cos\frac{1}{2}\delta\theta$

$\therefore AM_2 = AY - M_2Y = r_{i-\frac{1}{2}}\left(1 - \frac{r_{i-\frac{1}{2}} - r_{i+\frac{1}{2}}}{r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}}}\right)\cos\frac{1}{2}\delta\theta$

$\therefore AM_2 = \frac{r_{i-\frac{1}{2}}r_{i+\frac{1}{2}}}{2(r_{i-\frac{1}{2}} + r_{i+\frac{1}{2}})}\cos\frac{1}{2}\delta\theta = \frac{AG^2}{AM}\cos\frac{1}{2}\delta\theta.$

$\therefore AM_2 = \frac{b^2\cos\frac{1}{2}\delta\theta}{a(1-e\cos\theta\cos\frac{1}{2}\delta\theta)}$

$\therefore AM_2 = r$ when $\cos\frac{1}{2}\delta\theta = 1$ i.e. $\delta\theta \leq \frac{1^\circ}{4}$.

We finally define M_3 as the point at which the exact compensation of areas is achieved, as described in Step (3). It is not necessary to identify its position except to point out that, because of the vertically opposite angles at M_3, the small triangles will be equal in area as required when M_3 is as nearly as possible the bisector both of the elliptic arc and the modulated arc. Thus M_3 will certainly lie between M_1 and M_2, and we have:

$AM_3 = r$ when $\delta\theta \leq \frac{1^\circ}{4}$.

Hence we are undoubtedly justified in setting

$$AP_i = r = \frac{b^2}{a(1-e\cos\theta)}$$

as required in Step (3), where P_i can be regarded as possessing the aggregated properties listed here, bisecting both the elliptic and the modulated arc as well as achieving the compensation of areas, with M_1M_2 treated as evanescent.

CHAPTER VIII

CORRELATIONS AND SOME CONSEQUENCES

§1 Kepler's possible contributions to the development of the infinitesimal calculus: the three prerequisites

§2 Assessment of Kepler's success in achieving a mechanistic systematization of planetary motion

§3 Examination of the theoretical foundation of Law III

Appendix VIII(1) Kepler's archetypal number D which delineated the universe; and other constants

Appendix VIII(2) The latency of gravity

§1 Kepler's possible contributions to the development of the infinitesimal calculus: the three prerequisites.

My reason for embarking on this excursion into the history of mathematics is that there are some matters of technique which I have come across simply by examining Kepler's astronomical work. Though they may be regarded as minor (to the extent that no new results were synthesized), their introduction constitutes progress towards a modern outlook; and because I have not found them mentioned in the specialist histories of the subject that I have consulted I feel that they should be recorded, whether or not it was Kepler who was responsible for these innovations. Indeed this is a question that I am unable to answer, since it has not been feasible to become as thoroughly familiar as I would have liked with the advances in the infinitesimal calculus made by Kepler's contemporaries or his precursors in the sixteenth century: therefore, in what follows,I shall merely indicate my grounds for believing that such techniques were likely to have been original contributions by Kepler himself. Of these, the one of most general significance, as we shall see, is the fact that he employed polar—or, preferably, parametric-polar—coordinates,[1] whereas it seems that his near predecessors, at least, were concerned mainly with centre of gravity calculations, which would have demanded the use of Cartesian coordinates.

(a) The prerequisites and the mathematical analogy between the proofs of Law I and Law II.

I here present a strictly modern analysis of the proofs of Law I and Law II (in the sense explained in Chapter V, §2) in order to exhibit an analogy between them which was not brought out by Kepler's different treatment of each: this has the advantage of providing actual instances of the individual satisfaction of the three prerequisites whose combined effect was to produce an increment which has been described throughout Chapters VI and VII as precisely structured. They will now be formally defined:

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1. Properly described as coordinate-systems only with reference to the proof of Law II.

Prerequisite I: the presence in the increment of a small interval

Prerequisite II: the selection of an appropriate common independent variable

Prerequisite III: the rendering of the interval as an element.

I remark that Prerequisite I has been included because it is logically prior: as far as I know, up to Kepler's time, the range of summations undertaken was confined to determinations of areas and volumes and their extensions (all geometrically represented)—situations in which Prerequisite I would have been automatically satisfied, at least implicitly.[2] When this is so, the increment will consist of the product of a function (in our terminology) and an interval: for clarity, these have been separated by parenthesis in what follows. We shall find that the other two prerequisites do not have to be satisfied in the order given, which has been adopted because, in accordance with present day mathematical practice, it is considered desirable, whenever possible, that the expression should be in its final form before proceeding to the limit.

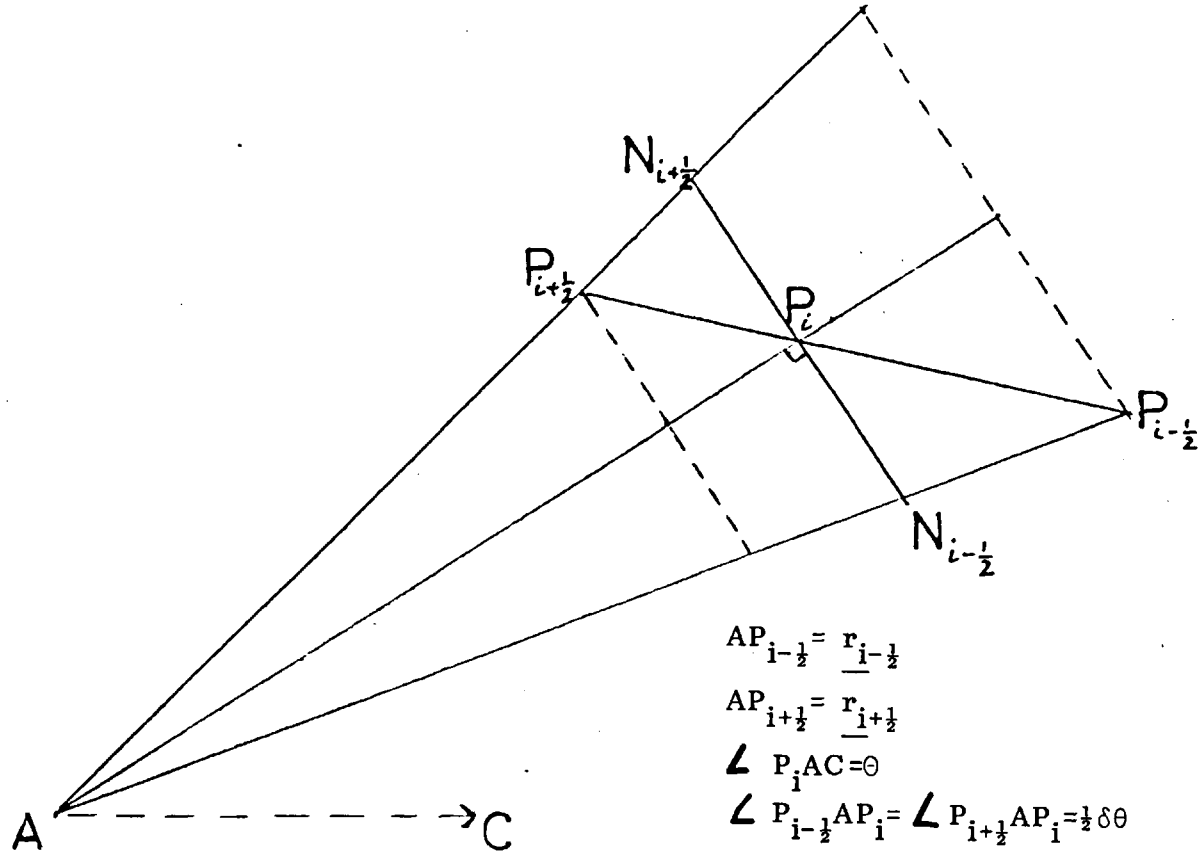
By contrast we emphasize the underlying similarity of purpose of the two proofs (as anticipated in Chapter V, §2), to the extent that each was designed to compensate, in a different way, for the fact that Kepler was quite unable to work out the increments by direct deduction from the known completed quantities, which would have been a process tantamount to differentiation[3]. (his compelling incentive to evaluate the increments, of course, arose from a need to determine the motions—the subject of §2 below). Therefore, it is unfortunately of theoretical interest alone that the three prerequisites are rightly so described in relation to an eventual process of quotient-formation tantamount to differentiation as well as to that of summation tantamount to integration.

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2. Thus the first doubt as to its satisfaction arose in Kepler's own work, in connection with the proof of Law I, as we mention further below: we are not concerned with the subsequent controversy arising in connection with the concept of indivisibles.

3. It is thus of no consequence that in any case his dependence on polar, or parametric-polar, coordinates would have precluded him from acquiring any understanding of the geometrical interpretation of differentiation.

Fig. (i) An intermediate elliptic sector



The proof of Law I

In Appendix VII(1) (measuring positively towards the Sun) it was shown that:

$$\delta r = r_{i-\frac{1}{2}} - r_{i+\frac{1}{2}}$$

$$\therefore \quad \delta r = \left(\frac{ae}{b^2} r_{i-\frac{1}{2}} r_{i+\frac{1}{2}} \sin\theta\right)\left(2\sin\tfrac{1}{2}\delta\theta\right). \quad 4.$$

It can easily be checked that this is true whether or not $\delta\theta$ is small:

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4. This is the formula that Kepler was unable to find, as mentioned in Chapter VI, §3: nor, indeed, had he done so, could he have arranged that Prerequisite II was satisfied, since he did not know the polar equation.

when it is, however, the presence of the small interval $2\sin\frac{1}{2}\delta\theta$ ensures that Prerequisite I is satisfied.

We continue by introducing a formula which is again easy to prove, from the focal geometry of the ellipse set out in Prolegomenon IIA, for corresponding values of eccentric and true anomaly:[5.]

$$\frac{\sin\frac{1}{2}(\theta_2\pm\theta_1)}{\sin\frac{1}{2}(\beta_2\pm\beta_1)} = \frac{b}{\sqrt{(r_1r_2)}}.$$

For the present situation we set:

$\theta_1 = \theta-\frac{1}{2}\delta\theta$, $\beta_1=\beta-\frac{1}{2}\delta\beta$ at the point $P_{i-\frac{1}{2}}$

$\theta_2=\theta+\frac{1}{2}\delta\theta$, $\beta_2=\beta+\frac{1}{2}\delta\beta$ at the point $P_{i+\frac{1}{2}}$,

to derive:

$$\frac{\sin\theta}{\sin\beta} = \frac{b}{\sqrt{(r_{i-\frac{1}{2}}r_{i+\frac{1}{2}})}} = \frac{\sin\frac{1}{2}\delta\theta}{\sin\frac{1}{2}\delta\beta},$$

so that

$$\delta r = ae.\ \frac{r_{i-\frac{1}{2}}r_{i+\frac{1}{2}}}{b^2}\cdot\frac{b\sin\beta}{\sqrt{(r_{i-\frac{1}{2}}r_{i+\frac{1}{2}})}}\cdot\frac{b.2\sin\frac{1}{2}\delta\beta}{\sqrt{(r_{i-\frac{1}{2}}r_{i+\frac{1}{2}})}}$$

$$\therefore\quad \delta r = ae(\sin\beta)(2\sin\tfrac{1}{2}\delta\beta).$$

Thus both the function and the small interval have been expressed in terms of the common independent variable β, and hence Prerequisite II is satisfied.

We finally impose the restriction $\delta\beta\leqq\frac{1}{4}^{\circ}$[6.] to obtain:

$$\delta r = ae(\sin\beta)(\delta\beta).$$

Then Prerequisite III is satisfied.

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5. Since $\sin\frac{1}{2}\theta=\sqrt{\{\frac{1}{2}(1-\cos\theta)\}}$, $\cos\frac{1}{2}\theta=\sqrt{\{\frac{1}{2}(1+\cos\theta)\}}$, with $\cos\theta=\frac{1}{e}(1-\frac{b^2}{ar})$, and similarly for β, with $\cos\beta=\frac{1}{e}(\frac{r}{a}-1)$.

6. Selected for compatibility with the proof of Law II, below—though it is slightly more rigorous than is actually required in the present context.

As we have shown in Chapter VI, lacking the capability to calculate this increment directly, Kepler reached it by inspired guesswork, and it was for this reason that he was obliged to confirm the validity of the result, retroversely, by performing a summation (tantamount to definite integration) to demonstrate that this would indeed produce the known completed quantity. It is notable that Kepler here displayed a practical grasp of the two-way inversity of integration and differentiation as operations—though of course this is still a far cry from its expression in terms of functions which is nowadays recognized as the fundamental theorem of the calculus.[7.]

The proof of Law II

We assume the modern formula for the area of a triangle (in which the angle $\delta\theta$ must from the outset be taken as small, so that the triangle will approximate to a sector of the ellipse):

$$\delta A = \tfrac{1}{2} r_{i-\frac{1}{2}} r_{i+\frac{1}{2}} \sin \delta\theta$$

$$\therefore \quad \delta A = \tfrac{1}{2}(r_{i-\frac{1}{2}} r_{i+\frac{1}{2}} \cos \tfrac{1}{2}\delta\theta)(2\sin \tfrac{1}{2}\delta\theta).$$

The presence of the small interval $2\sin\frac{1}{2}\delta\theta$ ensures that Prerequisite I is satisfied.

We next impose the formal restriction $\delta\theta \leqq \frac{1^\circ}{4}$,[8.] so that, from Appendix VII(1), we have:

$$r_{i-\frac{1}{2}} r_{i+\frac{1}{2}} = r^2.$$

and also $\cos\frac{1}{2}\delta\theta = 1$, as proved in Chapter V, §3, to obtain:

$$\delta A = \tfrac{1}{2}(r^2)(\delta\theta).$$

Thus we confirm that Prerequisite III is satisfied; while, when we write this as $\delta A = \frac{1}{2}(r)(r\delta\theta)$, it becomes clear that, by adopting the modulated arc $r\delta\theta$ to replace the elliptic sector by the circular sector, in Step (3) of his proof (see Chapter VII, §1), Kepler was effectively using the modern formula for area expressed in polar

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7. This is in any case not so immediate in the polar, or parametric-polar, coordinates used by Kepler.

8. It is to be understood, in what follows, that this restriction applies to whichever is the greater of $\delta\theta$ and $\delta\beta$.

coordinates (though he was not responsible for originating this approach, as we shall point out subsequently).

Finally we impose this same restriction on the geometrical formula stated in connection with Law I, to find:

$$\frac{\delta\theta}{\delta\beta} = \frac{b}{r}.$$ [9.]

Then, $\delta A = \frac{1}{2}b(r)(\delta\beta)$.

As we have seen in Chapter VII, §1, Kepler himself arrived at this result by direct deduction (though, as we have said, his independent method can in no way be described as an approach to differentiation), and therefore he rightly did not feel obliged to confirm its validity by attempting the corresponding summation. The fact is that in the course of his work he had established both the formula for δA or $\delta\alpha$[10.] (we here employ the Secant Law to ensure the explicit satisfaction of Prerequisite II):

$$\text{Increment} \propto r\delta\beta \propto (1+e\cos\beta)(\delta\beta)$$

and, in Astronomia, the formula for completed area or time:[11.]

$$\text{Completed quantity} \propto \beta + e\sin\beta.$$

Then we take the opportunity to highlight the mathematical analogy between the two proofs by pointing out that, had the retroverse method been applied in the case of Law II, it would have entailed the summation of a series of cosines analogous to the summation of sines necessitated in the case of Law I.[12.] However it is quite certain that Kepler could never have managed such a summation of cosines,[13.] since they cannot be geometrically represented

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9. This is, of course, the geometrical measure, now arrived at by a different route: it is not relevant to the present discussion that, as we have shown, Kepler probably relied on the optical measure.

10. We are concerned here with the analytical expression for either quantity, their equivalence having been established via the indeterminate time-axiom $\delta\alpha \propto r$: see Step (5) of Chapter VII, §1.

11. The former proved geometrically, the latter verified observationally, as explained in Chapter III.

12. Both similarly tantamount to integration, because Prerequisite III has been satisfied.

13. Despite the fact, remarked in Appendix II(2a), that when the modern (Ptolemaic) relationships are used, the summation of cosines is neither more nor less difficult than the summation of sines.

(even inadequately, like the sines, as we shall see) in the eccentric circle.[14.] Hence it may be regarded either as entirely fortunate, or as inevitable, that Kepler should have invented an independent method for the proof of Law II—and of interest that, as a consequent, the answer to that summation is implicit, from the juxtaposition of the two formulae above: while it would obviously be wrong to award Kepler any credit for the discovery of this precedent result, of which he himself undoubtedly remained unaware.

Using these two proofs as examples we will now make some general comments on the functioning of the three prerequisites. Prerequisite I being prior, as we have said, it is to be expected that a suitable interval will crop up naturally in the course of the mathematics. Then nowadays we apply technical skill to achieve a change of variable which, in Kepler's case, corresponded to the change of style which caused such confusion (because unannounced) in the proof of Law I, and can also, with hindsight, be identified as a major problem overcome in the proof of Law II: pinpointed at last as the change from polar to parametric-polar coordinates entailed by the fact that Kepler did not know the polar equation of the ellipse (in any case too complicated for him to handle), but only the expression of the radius vector in terms of eccentric anomaly given by the Secant Law. Then the necessity for the satisfaction of Prerequisite II which arose in Kepler's work could have been new only if nobody before him had had occasion to use more than one system of coordinates.

I have only recently become aware that the technique of modulation, explained in Step (3) of Chapter VII, §1, is to be found in the work of Archimedes, who effectively used polar coordinates, and replaced sectors of the curve by circular sectors, in order to determine the area of his spiral.[15.] Thereafter,

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14. Nowadays it is easy, taking the complementary angle, to work out the results for cosines from that for sines: but the consideration of signs, and limits, required is sophisticated.

15. On Spirals, Props. 21-8: see Heath: Greek Mathematics II, pp. 72-4.

we encounter a divergence of treatment, for Archimedes employed two sets of circular arcs as inner and outer bounds, and proceeded to the result via the method of exhaustion applied to control the sum, whereas Kepler achieved the coincidence of a single circular arc with a straight line by controlling the size of the interval, whence, by invoking the level of exactitude, the function of the interval was identified with the interval itself: concretely, taking the level appropriate to Kepler's astronomical work—as stipulated in Chapter V, §3—and imposing the restriction $\delta\beta \leqq \frac{1^\circ}{4}$ (in both proofs) we have $2\sin\frac{1}{2}\delta\beta = \delta\beta$. Hence Prerequisite III is satisfied—its significance being that only when the interval is rendered as an element of the variable can we properly introduce modern standards by reducing its size so that it becomes vanishingly small: that is, ultimately zero, or, at least, undetectable ($\delta\beta < 1''$)—which may be the correct interpretation of a Keplerian infinitesimal, as suggested also in Chapter V, §3. It is thus clear that Kepler was able to disperse with the method of exhaustion: while making no claim that he was the person responsible for this fresh approach, I point out that it seems more natural for it to have originated in connection with polar coordinates, for then the circular arc to be identified with the straight line is easily quantifiable.[16.]

However, because of Kepler's exclusively geometrical point of view, by which every trigonometrical quantity was treated as if it were linearly dimensioned (see Chapter V, §3), the foregoing discussion of the satisfaction of Prerequisite III, at least, strictly refers to the proof of Law II alone—that is, when an area was involved,[17.] for then it was quite correct to take the interval as of linear dimensions, measured by an arc. This was a consequence of Kepler's firm grasp of the order of linear dimensions—generally conducive to his progress, as we have emphasized throughout this study—yet in relation to the proof of Law I (in Epitome V Part I Section 2) it was the cause of a

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16. Even nowadays it is more complicated to justify the identification of the actual arc of the curve with a straight line; contrast the working of Chapter V, §3 with that of Appendix VII(1).

17. And, with hindsight, in all the summations of Astronomia (tabulated in Chapter III, §9).

rigidity of approach which prevented him from regarding the summation appropriate there as anything but areally-dimensioned, represented by the surface area of a sphere, rather than as linearly-dimensioned (to give the sagitta) as he required. It was this difficulty that led him temporarily to repudiate the satisfaction of Prerequisite I—and naturally, of Prerequisite III, by implication—as we have described in Chapter VI, §6. On the other hand, it may be regarded as a notable breakthrough that Kepler even attempted summation in that situation. However, despite his signal failure to manage a changeover to an analytical point of view when he really needed it, he did achieve this in practice, in other circumstances, as we shall show next.

(b) The conversion from degrees to radians achieved by Kepler Astronomia C. 57.

We resume our account of the term-by term summations of sines at intervals of 1° undertaken in Astronomia C. 57, to point out that, representing the sines, as always, by ordinates of the eccentric circle, Kepler proposed a linearly-dimensioned summation, in the form:

$$\frac{\sum_{o}^{n} a \sin i\delta\beta}{\sum_{o}^{90} a \sin i\delta\beta} = \frac{a(1-\cos\beta)}{a} \quad \text{(G.W. III p. 354 lines 2-5)}$$

Kepler clearly stated that this was merely an approximation, and there are in fact two sources of inaccuracy: the irreducible disparity caused by his failure to distinguish eccentric from true anomaly, and the ephemeral disparity due to the fact that the intervals were not small enough; these have been discussed in Chapter VI, §5(a) and §5(b) respectively, and are here to be disregarded, to concentrate attention exclusively on the significance of the reduction-factor $\sum_{o}^{90} \sin i\delta\beta$ applied to the general summation.

We first repeat, for reference, two special results worked out in Appendix II(2a): Kepler himself only stated the numerical values, in Astronomia C. 43 and C. 57:

$$\sum_{o}^{180} \sin i\delta\beta = \cot\tfrac{1}{2}^\circ$$
$$= 114.589 \text{ (G.W. III p. 284 lines 20-9)}$$

$$\sum_{o}^{90} \sin i\delta\beta \doteq \tfrac{1}{2}\cot\tfrac{1}{2}^\circ + \tfrac{1}{2}$$
$$= 57.894 \text{ (G.W. III p. 354 lines 7-8).}$$

It will be found that the theoretical significance of the reduction-factor is best appreciated when it is taken to denote 'half the sum of sines up to 180°' instead of 'the sum of sines up to 90°' as Kepler described it (G.W. III p. 354 lines 4, 7-8).[18.] Then we obtain:

$$\frac{\sum_{o}^{n} \sin i\delta\beta}{\frac{1}{2}\sum_{o}^{180} \sin i\delta\beta} = \frac{\sum_{o}^{n} \sin i\delta\beta}{\frac{1}{2}\cot\frac{1}{2}\delta\beta} \pm 2\tan\tfrac{1}{2}\delta\beta \cdot \sum_{o}^{n} \sin i\delta\beta$$

$$\doteq \sum_{o}^{n} \sin i\delta\beta . 2\tan\tfrac{1}{2}\delta\beta .$$ [19.]

Hence, the inclusion of the small interval $2\tan\frac{1}{2}\delta\beta$ has ensured the satisfaction of Prerequisite I.

We digress to remark that the summation has been expressed in terms of eccentric anomaly β purely for consistency: in actuality any symbol may be used to denote the angle at the centre of a circle whose ordinates are summed with respect to equal intervals of that same angle: only then is the satisfaction of Prerequisite II ensured. (Thus the hybrid formulations mentioned by Aiton,[20.] which provide instances of circumstances in which Prerequisite II has not yet been satisfied, would—rightly—have seemed irrelevant to Kepler.)

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18. The difference between the two sums is in fact ephemeral—but it does not vanish while the intervals remain discrete: we return to this below.

19. We have here emphasized theoretical considerations by writing $\delta\beta \leqq 1^\circ$.

20. $\Sigma\sin\beta.\delta\alpha$ in Kepler's Second Law of Planetary Motion, p. 83, footnote 51; $\Sigma\sin\theta.\delta\alpha$ in Infinitesimals and the Area Law, p. 298: I do not know why the former was proposed, since, as a formula for the libration, it is wrong in modern terms (see Prolegomenon IIB).

Moreover, each summation was carried out at intervals of 1°, and it has been proved, in Chapter V, §3, that at the stipulated level of exactitude, this entails the analytical identity $2\tan\frac{1}{2}\delta\beta=\delta\beta$. Hence the summation can finally be expressed in its modern form:

$$\sum_{o}^{n} \sin i\delta\beta.\delta\beta.$$

This has ensured the satisfaction of Prerequisite III.

It is because Kepler was inhibited from adopting this analytical interpretation of the restriction of the interval that we shall turn to examine the numerical significance of the reduction-factor. The modern values are:

$$\tfrac{1}{2}\sum_{o}^{180} \sin i\delta\beta = 57.296^{21}.$$

$$\sum_{o}^{90} \sin i\delta\beta = 57.796 \text{ (Kepler stated 57.894).}$$

Thus since the value he gave contained what must have been a mis-transcription anyway (in the first decimal place), there is no reason why we should not replace it, on his behalf, by the correct value 57.296, denoting 'half the sum of sines up to 180°',[22] to express the summation he actually performed, as:

$$\frac{1}{57.296}\sum_{o}^{n} a\sin i\delta\beta = \sum_{o}^{n} a\sin i\delta\beta.0.01745.$$

Therefore we may conclude that in this situation, after all, Kepler was not considering a series of unconnected terms, but was undertaking a precisely structured summation in which the interval, for the first and only time, had been converted to radians, for:

$$1° = \frac{\pi c}{180} = 0.01745 .$$

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21. In what follows, we correct, where appropriate, to five significant figures rather than to five decimal places.

22. Except that it involves the supposition that Kepler believed the sum of 180 sines to be twice the sum of 90 sines, which is not true when the interval is discrete: but this mistake was made by several of his commentators (with far less justification) as listed in Appendix II(2a).

I believe this to be an original contribution by Kepler himself on the grounds that nobody before him had had his compelling reason to be concerned with the expected result. Moreover, as I have suggested, it constituted a step outside the mainstream of the work then being done in the infinitesimal calculus, and thus, as far as I know, utterly failed to influence its development. Though unfortunately I have been unable to find out when this technique was rediscovered,[23] I feel that it should be put on record that, probably in 1605, Kepler had, quite unwittingly, demonstrated a way of converting from degrees to radians; which is nowadays recognized as an essential preliminary to applications of the calculus when we encounter trigonometrical functions to be treated analytically, stripped of their geometrical representation.[24]

In conclusion, we correlate the figure employed in this conversion with the Keplerian constants which cropped up in entirely different contexts in Epitome and Harmonice, their names justified in Appendix VIII (1a) and (1b):

$H = 720$: Kepler's harmonic number

$D = \frac{720}{\pi}$ or $\frac{H}{\pi} \cong 229.18$: Kepler's archetypal number.

$\therefore \frac{1}{D} = \frac{\pi}{720}$ or $\frac{\pi}{H} \cong 0.00436$.

We take this opportunity to point out that, at the stipulated level of exactitude, we have:

$$\sin \tfrac{1}{4}^\circ = 0.00436 = \tan \tfrac{1}{4}^\circ \ [25].$$

and $\operatorname{cosec} \tfrac{1}{4}^\circ = 229.18 = \cot \tfrac{1}{4}^\circ$.[26]

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23. It could hardly have been formalized until the symbol for π came into general use: see Appendix VIII(1c).

24. Though, as I have noted with some surprise, this requirement was not appreciated by any of the commentators mentioned in Appendix VI(3), some of whom themselves used degrees or radians apparently at random.

25. This is the Keplerian authority for the introduction of the slightly more rigorous restriction $\delta\beta \leq \frac{1}{4}^\circ$, as promised in Chapter V, § 3.

26. Here temporarily applying the stipulated level as if it referred to five significant figures.

Then we remark that:

$$\frac{4}{D} = 0.01745 = \frac{\pi}{180} \text{ or } \frac{4\pi}{H}$$

while

$$\frac{D}{4} = 57.296 = \frac{180}{\pi} \text{ or } \frac{H}{4\pi}.$$ [27]

Hence the conversion from degrees to radians may be expressed as $\frac{4}{D}$, where, as demonstrated in that Appendix, D is the archetypal number which delineated the Keplerian universe.

27. Using the values 229.183, 0.004363 in order to obtain final results which are correct in modern terms, at the stipulated level.

§2 Assessment of Kepler's success in achieving a mechanistic systematization of planetary motion.

(a) A partially speculative systematization of the orbital motion of a single planet.

The modernity of Kepler's outlook was such that it was his ultimate purpose to account for every sensible aspect of the motion of a primary planet[1.] by active, or activating faculties of the Sun alone, supported by some assumptions as to the internal structure of the planetary body (the sets of fibres listed in Chapter V, §4), and certain suppositions as to the characteristics of the individual planet, which was intended to remain essentially passive: this approach has been described by Dijksterhuis[2.] as Kepler's abandonment of an animistic in favour of a mechanistic conception. We shall here confine our attention to consideration of the orbital motion of a single planet (that is, its motion through space)—and, in (b) below, to the inter-related motion of the system as a whole—to define the faculties of the Sun as continuously variable, depending for their effect on the instantaneous distance r of the planet, while the characteristics of the planet will be measured by the constants of its particular orbit. Then because Kepler's opinions were (potentially) quantifiable—as I shall demonstrate here for the first time—it can be shown that the passivity of the planet was not absolute (though it can certainly be regarded as no longer contributing to its own motion through space);[3.] nevertheless, the faculties proposed were capable (in Kepler's terms)

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1. The small secular changes have been excluded from consideration in Chapter I, §1: though Kepler mentioned them in Epitome IV Part III Sections 3 and 4, he fortunately decided that their sizes were too imprecisely known for it to be feasible to determine their causes.

2. The Mechanization of the World Picture pp. 303-23, especially p. 310: it will be obvious (from Part A) that I do not agree with his analysis of Astronomia, with which most of the discussion is concerned.

3. This is to be investigated separately below: it will be assessed as a minor flaw.

of producing the component velocities correctly (in modern terms) and hence, to this extent, his scheme was both coherent and complete, and he must be judged to have been successful in achieving his aim: a mechanistic systematization of the orbital motion of a single planet, founded on Aristotelian principles.

However, I cannot feel that sufficient effort has been made, in the past, to give Kepler credit for possessing a fundamental understanding of the way force would operate, as distinct from his Aristotelian approach to its kinematical interpretation. I propound the following axioms, which make sense whichever interpretation is adopted: stated here in Keplerian terms, they are valid in modern terms when we simply substitute 'acceleration' for 'velocity'.

Firstly, zero force entails zero velocity, and conversely: hence, the particular interpretation of force adopted may be regarded as controlling the concept of inertia.[4.] Thus it was an inevitable consequence of his Aristotelian principles that Kepler viewed inertia as a property of (planetary) matter by which it persisted in a state of rest until compelled to change that state.[5.]

Secondly, a determinate velocity entails a determinate force, and conversely: moreover, the velocity and the force producing it will be in the same direction (we shall return to the significance of this below). Meanwhile, since orbital velocity can be split into only two independent component velocities, it follows that (at most) two faculties will be required to produce them. Consequently, there is no slot into which one can fit a third physical cause: I refer, in particular, to gravity. This concept has already been excluded from consideration in principle (as stated in Chapter IV, §2) on the grounds that the Sun had to be exempted from being affected by the planets.[6.] Nevertheless, not all commentators are

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4. This term was apparently coined by Kepler and was employed quite early in Epitome: for instance G.W. VII p. 89.

5. Epitome IV Part II Section 2 (G.W. VII) p. 296 lines 30-6): Kepler's unavoidably ineffective attempt to develop the concept by treating inertia as a characteristic of the individual planet is discussed in (b) below.

6. See Epitome IV Part III Section 3 (G.W. VII p. 336 lines 26-36): the same argument applies at least as strongly to gravity as to magnetism which was under discussion there.

convinced of this, and so it has seemed desirable to confirm (in Appendix VIII (2)) that not only was gravity not to be classed as a faculty of the Sun, it was to be regarded as a latent effect in every astronomical context, even in relation to the Earth-Moon system. Hence we can categorically assert that Kepler did not in this respect undermine the coherence and completeness of his systematization by introducing a redundant physical cause.

In undertaking this quantification of Kepler's physical opinions on his behalf, we will adopt the Keplerian standpoint, applying the Aristotelian Equivalence as if it represented reality, and even (with some difficulty) believing in the magnetic explanation and the existence of the planetary fibres. It must first be said that the results obtained are modern in the respect that Kepler did not concern himself with the values of the implied constants (as discussed in Chapter VII, §4).[7.] However, the quantification possesses Keplerian authority to the extent that it exhibits the continued underlying duality of treatment appropriate to Epitome, as emphasized in Chapter V, §1. Thus, in the accompanying tabulation, the right-hand upper quarter, and the left-hand lower quarter, are strictly Keplerian, while the remaining quarters have been completed by analogy. Obviously, because of the Aristotelian interpretation of force, we cannot expect to determine a universal force-constant,[8.] and therefore, to avoid the unclarity of recurrent proportion-statements, dimensionally equivalent quantities—despite being qualitatively different—have been expressed as equalities.

I have permitted myself to indulge freely in speculation on this one occasion in order to complete the tabulation, because I suspect that such a coherent and potentially quantified systematization of celestial physics,[9.] founded on Aristotelian principles, is

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7. Which also explains why, exceptionally, the characteristics of the planet listed in (b) below were explicitly quantified.

8. Nevertheless it will turn out to be possible to evaluate such a constant for comparison with the constant of gravitation, using kinematical methods: see §3 below.

9. It was in Epitome that Kepler made good the promise implied in the title to Astronomia.

probably unique, and thus of extreme interest, even though it was quite soon to be superseded.

* * * * *

The two faculties of the Sun: the upper half of the tabulation.

We can apply the modern relationship:

$$\frac{\delta\alpha}{\delta t} = \frac{2\pi}{T}$$ [10.]

to (either version of) the transverse motion and to the radial motion found in Chapter VII, §3 and Chapter VI, §7, respectively, to derive:

Transverse velocity (Harmonice version) $= \frac{r\delta\theta}{\delta t} = \frac{a\delta\beta}{\delta t} = \frac{2\pi}{T} \cdot \frac{a^2}{r}$ [11.]

Radial velocity $= \frac{\delta r}{\delta t} = \frac{2\pi}{T} \cdot \frac{a^2 e \sin\beta}{r}$. [12.]

Now we invoke the Aristotelian Equivalence, which Kepler never questioned (see Chapter I, §1), [13.] to obtain:

Driving force (virtus) $F = \frac{2\pi}{T} \cdot \frac{a^2}{r}$.

Thus at last we have confirmed the correctness, in his terms, of Kepler's (Aristotelian) force-hypothesis.

Then by analogy:

Magnetic faculty $= \frac{2\pi}{T} \cdot \frac{a^2 e \sin\beta}{r}$,

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10. Kepler himself applied this relationship only to the determination of diurnal angular velocities, using $\delta\alpha_m$, where δt = 1 day.

11. This version is the more justifiable because it depends on the optical measure, which Kepler actually used. The modern version, involving the geometrical measure (merely implicit in Epitome) is correctly:
Transverse velocity (Epitome version) $= \frac{b\delta\beta}{\delta t} = \frac{2\pi}{T} \cdot \frac{ab}{r}$.

12. This is, of course, correct in modern terms (see Prolegomenon IIB). For convenience, we measure r positively towards the Sun.

13. Except as mentioned in passing in (b) below.

where we have assumed that this faculty is subject to the Aristotelian Equivalence—because I believe it would have been in accordance with Kepler's intentions—despite the fact that it is not otherwise to be regarded as a force: this qualitative distinction is highlighted below.

We recall the second axiom above, by which the velocity and the force producing it are bound to be in the same direction. (Incidentally, this explains why Kepler was incapable of envisaging tangential motion.) Kepler knew that the transverse velocity was instantaneously circular (see Step (3) of Chapter VII, §1), and since it was caused by the rays of the Sun, rotating 'like a vortex'—as he stated in Epitome IV Part II Section 3 (G.W. VII p. 299 lines 39-46)[14.]—we should visualize the planet being instantaneously impelled round the Sun perpendicular to each ray as it passed:[15.] hence the whirling-force must depend on action by contact—at least quasi-physical. Yet this obvious deduction (never made explicit by Kepler himself) has been skirted by every previous commentator, including Caspar, even though he described the Sun's action as 'rotating rays of force' (see Chapter IV, §2).

In deriving the magnetic faculty analogously as above, we emphasize the dimensional parity of the two faculties of the Sun, implied in the passage from Epitome V Part I Section 4, translated in Chapter V, §1: and it is of interest that, being at first sight inversely proportional to distance,[16.] it is of a form that Kepler would have recognized as a force (though it is not suggested that he himself would have managed to evaluate it—see Chapter VI, §7). Nevertheless, as cited in Chapter VI, §2, Kepler categorically denied that magnetism could function

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14. I warn that in other respects Kepler's physical opinions of Epitome IV Part II Sections 2, 3, 4 are generally unsound—as we discuss in (b) below.

15. Because the period of the Sun's axial rotation, whether obtained by speculation or, later, by observation (see Appendix VIII(1)) was (much) less than the period of revolution of the fastest planet.

16. In modern terms a more complicated function of the distance, since $ae \sin \beta = \sqrt{\{a^2e^2-(r-a)^2\}}$.

as a force, while in any event I shall insist that a qualitative distinction between the two faculties be rigidly maintained, on the grounds that the Sun's magnetism did not itself attract (or repel) to produce radial motion, but instead activated the planetary fibres to do so: such behaviour is not normally that of a force. And there is the additional oddity that the planet did not then move in the direction of its fibres, but towards the Sun,[17.] which suggests a foreshadowing of the Newtonian idea of motion under constraint.[18.] Moreover, magnetism was presumed—as under terrestrial conditions—to be capable of action at a distance. This particular contrast indicates that there is no way the two faculties can be lumped together into a single vortex theory.[19.]

A minor flaw.

Additionally, we have to point out that the peculiarities of the magnetic explanation were such that it involved a certain inconsistency, for it must surely be wrong to suppose (despite his initial statement)[20.] that Kepler believed that the Sun's magnetism could manage to activate the planetary fibres as described, having, first, rotated them through the necessary angle—which has been evaluated, in Chapter VI, §4, as the optical equation updated for the ellipse:[21.]

Angle of deflection = $\beta-\theta$.

Hence, in modern terms, the velocity of angular deflection is given by:

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17. It is because of these oddities that we cannot strictly justify our extension of the Aristotelian Equivalence to the magnetic faculty—except as assumed above.

18. Principia I Section VIII Proposition XL and Corollary 1.

19. Emphatically not the 'magnetic vortex' described by Aiton: The Vortex Theory of Planetary Motions, pp. 12-19.

20. In Epitome IV Part III Section 3: see Chapter VI, §4.

21. Whose quadrant-value crops up below in a different connection.

$$\frac{d(\beta-\theta)}{dt} = \frac{d\alpha}{dt} \cdot \frac{d\beta}{d\alpha} \cdot \frac{d}{d\beta}(\beta-\theta)$$

$$= \frac{2\pi}{T} \cdot \frac{a}{r}\left(1-\frac{b}{r}\right) \text{ as in Appendix II (4),}$$

in which r may itself be expressed in terms of true anomaly θ via the polar equation.[22.]

Because the fibres were magnetic, it is clear that if this effect is not produced by the Sun's magnetism it cannot be due to the Sun at all—and thus we are obliged to return to dependence on some vis insita, or inherent power, of the planet itself. Unfortunately, it was just this concept—involving action on the part of the planet—which Kepler had intended to eliminate from his scheme: indeed, he thought that he had done so at the end of Astronomia, as we have remarked in Chapter IV, §2. However, had the necessity for this contributory cause been brought to his attention, Kepler might have been able to convince himself that it was not a radical fault in his scheme, on the grounds that the proposed vis insita did not have to be responsible for moving the planet bodily through space, as formerly,[23.] but merely for rotating its magnetic fibres about their own axis within the planet—a much less demanding requirement.[24.]—and so the presence of a vis insita of such limited scope might be regarded as no more than a minor flaw. Be that as it may, we have to conclude that, although Kepler's systematization of the planetary motions was coherent, as we have demonstrated, his account of their underlying causes was less than satisfactory.

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22. In Astronomia C. 57 Kepler pointed out that θ is known by reference to the fixed stars: see Chapter VI, §5.

23. For instance, moving it in an epicycle of radius ae: see Chapter IV, §2.

24. Since Kepler never determined the deflection—let alone its variation—he could not have appreciated that putting it into operation would have entailed a fairly sophisticated 'planetary computer'.

A SYNOPSIS OF KEPLER'S 'CELESTIAL PHYSICS'.

Radially	Concept	Transversely
THE FACULTIES OF THE SUN		
Magnetic faculty $= \frac{2\pi}{T} \cdot \frac{a^2 e \sin \beta}{r}$	Faculty = velocity	Driving force $F = \frac{2\pi}{T} \cdot \frac{a^2}{r}$
Magnetic impulsion $= \frac{a^2 e \sin \beta}{r} \delta\alpha$ $= ae \sin \beta . \delta\beta$	Impulsion (Faculty x time)	Impulsion of driving force $= \frac{a^2}{r}\delta\alpha$ $= a\delta\beta$
CHARACTERISTICS OF THE PLANET		
Capacity to assimilate magnetism $= ae\delta\beta$	Capacity	Capacity to respond to driving force $= a\delta\beta$
Coefficient of potency of fibres $= e$	Coefficient $= \frac{\text{Capacity}}{\text{Arc}}$	Coefficient of responsiveness $= 1$.

Characteristics of the planet: the lower half of the tabulation.

It is because of the lack of a universal force-constant that it is necessary to suppose that each planet possessed individual (essentially passive) characteristics[25]. which would enable it to regulate the amount of magnetism and force emitted by the Sun:

$$\text{Magnetism} \propto \frac{1}{r}, \quad \text{Force} \propto \frac{1}{r},$$

to the determinate requirements, specific to each, as quantified above.

In the case of magnetism we repeat Kepler's opinions as set out in Chapter VI, §2(b) (recalling that the increment of libration was evaluated as a result of Chapter VI, §6):

$$\text{Magnetic impulsion} = \text{increment of libration} = ae \sin\beta \cdot \delta\beta.$$

(We take this opportunity to draw attention to the modern connotation—even more appropriate in the case of the driving force—which was intentionally introduced by the terminology employed, since impulsion is the product of faculty and time: linearly dimensioned, of course, because the dimensions of the faculty, by the Aristotelian Equivalence, were those of velocity.) Thence (having revised the principle of resolution, in accordance with Kepler's intentions) it appeared that, as far as the planet was concerned, the magnetic impulsion was equivalent to the component of its own (maximum) magnetic capacity (as defined in Chapter VI, §1) activated by, and in the direction of, the Sun. Thus:

$$\text{Capacity of fibres to assimilate magnetism} = ae\,\delta\beta.$$

This (maximum) was measured with respect to small equal arcs $a\delta\beta$[26]. and hence, in order to exhibit its constancy, as expected, we introduce a further concept, on Kepler's own authority: in

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25. With some Keplerian authority, as we shall see, both here and in (b) below.

26. To be interpreted as modulated arcs, misevaluated by the optical measure: see Chapter VII, §3.

Epitome IV Part III Section 3, in the sole reference in this work to the value of the eccentricities, he remarked on their randomness, and stated that this was to be ascribed to the potency (fortitudo) of the fibres of each individual planet (G.W. VII p. 338 lines 36-40),[27] so we write:

Coefficient of magnetic potency of fibres = e.

When we start from the transverse motion (Harmonice version) the analogous quantities are derived as shown in the tabulation, and it appears, since the 'coefficient of responsiveness' is unity, that there was in this case no need for the planet to regulate the driving force of the Sun. However, when instead we start from the modern (Epitome) version we obtain:

Impulsion of driving force = $\frac{ab}{r}\delta\alpha = b\delta\beta$

Capacity to respond to driving force = $b\delta\beta$

$\therefore$ Coefficient of responsiveness = $\frac{b}{a} = \sqrt{(1-e^2)}$.

Because this approach was not explicitly Keplerian, I shall not attempt to interpret this coefficient, beyond suggesting that it could be visualized as some characteristic of the individual planetary matter: that is, an alternative specification of whichever of those listed in (b) below that Kepler regarded as the most fundamental.[28]

Of course, this whole piece of analysis—particularly in relation to the 'capacity' measured in small equal arcs—is totally chimerical by modern standards. Nevertheless, we point out that had Kepler managed to discover the modern (Epitome) version of the transverse velocity—and possessed the technique to follow through the derivation as we have here on his behalf—he would have been delighted with the correlation achieved, for both coefficients are expressible as trigonometrical ratios (sine

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27. Dreyer: Planetary Systems, p. 397, uses the term 'intensity of magnetism'—but I have not followed him because this seems to suggest a property of the Sun's magnetism rather than the planet's responsiveness to it.

28. The present coefficient, too, is 'random', depending on shape rather than on distance from the Sun, as do those proposed by Kepler himself.

and cosine respectively) of the quadrant-value of the optical equation updated for the ellipse,[29] a particular instance of a concept which was of significance from time to time throughout his work—as, for example, in connection with the angle of deflection of the fibres mentioned above.

(b) The total failure of Kepler's mechanistic explanation of the motion of the planetary system Epitome IV Part II Sections 2,3,4.

In the sections cited, Kepler's principal aim was to establish the interderivability of Law II and Law III: the latter had already been stated in Harmonice V C. 3 Proposition VIII (G.W. VI p. 302 lines 21-2), as a relationship between any pair of planets:

Law III: Periodic time $T \propto (\text{mean distance})^{3/2}$.

Hoping to achieve this, Kepler proposed characteristics of the planetary matter to function as if they were subsidiary physical causes (though, being passive—defined by the constants of the individual orbit—they could not wreck Kepler's scheme for the motion of a single planet). In this connection we must in general deplore Kepler's enthusiasm for tackling physical considerations before he had subjected himself to the discipline of resolving the mathematical problems (in Epitome V Part I)—this is reminiscent of Astronomia C. 57 where he made a similar digression prior to discovering the specific orbital ellipse (see Chapter II, §4). Yet, in the present instance, it must be recognized that Kepler would never have been able to work out, and so to allow for, the effect on the periodic times either of the inverse-linear (Aristotelian) driving force of the Sun,[30] or of the supposedly-

29. The cosine of this quadrant-value has previously turned up in an entirely different context: see Chapter VII, §4, where it is illustrated.

30. In the simplest case when the orbits are taken as circles with the force at the centre, producing uniform motion—this is not irrelevant, as we shall see in §3 below—the corresponding 'Law III' would be $T \propto a$, as Newton proved in Principia I Section II Proposition IV Corollary 7 (setting n=1).

equivalent inverse-linear transverse velocity,[31] in order to determine the amount of any extra, subsidiary, physical cause which he would need to introduce.

In an effort to derive Law III from Law II, Kepler attempted to develop the concept of inertia (defined in (a) above) to supplement the driving force of the Sun, as he described in Section 3 (G.W. VII p. 302 lines 14-19), by interpreting it as resistance to motion already taking place. This has been pointed out by Rosen, who translates the earlier passage in Section 2 where Kepler first identified inertia as proportional to weight (G.W. VII p. 296 lines 42-3)—and thus as a characteristic of each individual planet[32]—before discussing its supposed application to the axial rotation of a single planet, and to the revolution of the system about the Sun, in turn.[33]

With respect to axial rotation (G.W. VII p. 296 line 40-p. 297 line 2), Kepler said that if there were no inertia (or resistance) in the planetary body, no force would be needed to move it: once started, it would move with infinite velocity (p. 296 lines 42-5). Apart from being utterly incompatible with his previous interpretation of inertia, set out in (a) above, this seems to me to suggest 'the Aristotelian Equivalence for resisted motion'—which is not encountered elsewhere in Kepler's astronomical work:

$$\text{Velocity} \propto \frac{\text{Force}}{\text{Resistance}} .$$

In the case of revolution round the Sun (G.W. VII p. 297 lines 3-10)—that is, motion through space—Kepler stated that, if the planets possessed no inertia (or resistance), they would revolve in a common period time (lines 5-7),[34] not with infinite

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31. With the simplification of the preceding footnote, the corresponding 'Law III' would be $T \propto a^2$: op.cit. (setting n=2). This has been pointed out by Dreyer: Planetary Systems, pp. 393-4.

32. Weight itself was quantified as listed below: we do not need its value here.

33. Rosen: Kepler's Somnium, Appendix I: Kepler's Concept of Inertia, pp. 222-3, has only quoted the former application, but this is not at all clear, for he has wrongly translated verteretur and conversiones as if they referred to revolution round the Sun.

34. The period of axial rotation of the Sun.

velocity.[35] While it is of no consequence that this is quite irreconcilable with the interpretation above, the fact is that Kepler made no proposal whatsoever as to how inertia should operate in this situation, and so we must record his failure to incorporate it into his scheme—merely remarking that this statement confirmed his inability to calculate the effect of the inverse-linear driving force or the periodic times as mentioned above, by here ignoring it altogether.[36]

Finally, in Section 4, Kepler made an effort conversely to derive Law II from Law III. We first confirm that (as also in the derivation above) he was concerned with motion round the Sun alone, by reference to the diagram at the end of Section 3 (G.W. VII p. 305)—said to represent 'one and the same planet' in different parts of its orbit (p. 305 line 7)[37]—which is identical to that of Section 4 (G.W. VII p. 307) representing a pair of planets in different orbits. Since each was drawn by Kepler as circles centred on the Sun, it is clear that he intended to depict transverse motion, while the question of radial motion was entirely excluded.

For this purpose Kepler specified certain characteristics of each planet in terms of its distance from the Sun: since these characteristics are undoubtedly constant for each individual planet we shall clearly be following Kepler's intentions in taking them to depend on the mean distance a. Their quantifications were stated in Epitome IV Part I Section 4 (G.W. VII p. 281 line 27-p. 284 line 47)—where they were said to be founded on 'archetypal reasons'—and repeated in Epitome IV Part II Section 4 (G.W. VII pp. 306-8):

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35. This must have been rejected because it would have meant that the planets were not controlled by the Sun, but governed by their own inertia.

36. In the simplified situation envisaged in two preceding footnotes, we find that a common periodic time would require a direct-linear force: see Principia I Section II Proposition IV Corollary 7 (setting n=0).

37. Swung round the Sun to arrange comparability: see Chapter VII, §1, Step (4).

Massiveness[38] (pondus, or copia materiae) $\propto a^{\frac{1}{2}}$

(p. 284 lines 6-7, p. 307 lines 10-12)

Volume (moles) $\propto a$ (p. 282 lines 28-9, p. 307 lines 28-9)

$\therefore$ Density (densitas) $\propto a^{-\frac{1}{2}}$ (p. 284 lines 9-10).

In Epitome IV Part II Section 4 Kepler went on to make some entirely speculative partial-variation statements as to the effect of these and other quantities on the periodic time T (even his reasoning was faulty, in fact, since not all the causes were treated as independent):

$T \propto$ Massiveness (p. 307 lines 12-15)

$T \propto \frac{1}{\text{Volume}}$ (p. 307 lines 1-2, 4-6)

$T \propto$ Length of path (p. 307 lines 15-16)

$T \propto \frac{1}{\text{Force}}$ (p. 307 lines 2-4).

When these are put together, we obtain:

$$T \propto \frac{\text{Length of path X Massiveness}}{\text{Driving force X Volume}} .$$

Then, introducing the characteristics supposed above:

$$T \propto \frac{\text{Length of path}}{\text{Driving force}} Xa^{-\frac{1}{2}}.$$

Before commenting on this relationship with reference to a system, I shall digress to examine its validity when Kepler applied it, later in Section 4, as anticipated at the end of Section 3 (G.W. VII p. 308 lines 7-22 and p. 305 lines 7-20 respectively), to 'the varying times of one and the same planet at different distances', by assuming equal (small) arcs $a\delta\beta$[39] as 'the path' (p. 308 lines 11-12, 19-20, p. 305 line 8), which enabled him, quite spuriously, to 'deduce':

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38. This term has been introduced to indicate that Kepler had no real understanding of the concept of mass.

39. Despite the fact that these were described as 'arcs of the eccentric', they will be interpreted, as above, as modulated arcs which, when misevaluated by the optical measure, were the same length, but circular with respect to the Sun.

$\delta t \propto r$: the indeterminate time-axiom,[40.]

as he required (p. 308 lines 9-10, p. 305 lines 19-20), only when:

$F \propto \frac{1}{r}$: Kepler's (Aristotelian) force-hypothesis.

It was this apparent self-consistency which convinced Kepler that his reasoning as a whole was sound; and which, when we return to the application of the above relationship to the system, will account for the oddity (which has not previously been remarked) that he set:

Length of path $\propto$ a (p. 307 lines 7-8),[41.]

to enable him to make the fallacious 'derivation':

$T \propto a^{3/2}$ (p. 306 lines 34-6, p. 307 lines 22-3).

Of course the foregoing argument may be regarded as no more than a fine example of post-justification, in which not only all the speculations and archetypal reasons, but also the values of the characteristics, expressed as powers of the constant a,[42.] were arranged merely in order to reach this predetermined result. Nevertheless, the enormity of the fallacy becomes apparent when we realize that it entails:

Force $\propto \frac{1}{a}$: constant:

for, if we take Kepler's physical opinions seriously, it becomes obvious that the idea of a constant force was a perfect absurdity, in his own terms. This has indeed been pointed out by Aiton,[43.] but with insufficient cogency, I feel, because the fallacy can be better appreciated when we take advantage of modern notation—not available to Kepler himself—to distinguish the variable radius vector r from the radius of the eccentric a.[44.] Thus Gingerich,

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40. This problem was not resolved until Epitome V Part I Section 4: see Chapter VII, §1, Step (5).

41. His approximate formula for the perimeter of the ellipse, stated in Astronomia C. 59, Protheorema V, showed that it was proportional to (a+b).

42. Subject to the remarks in Chapter VII, §4.

43. The Vortex Theory of Planetary Motions, p. 19.

44. Failure to do this recalls the alternative present-day practice of setting a=1 (as deprecated in the Introduction), whose consequences have been analyzed earlier in this study.

by using $\underline{r}$ to denote the planetary characteristics (as well), has been led, wrongly, to describe this Keplerian argument as a 'theoretical derivation' of Law III,[45] not recognizing that it amounted to a fundamental denial of the force-hypothesis which had—successfully, to the extent demonstrated in (a) above—provided the foundation for all Kepler's discoveries. Hence we are obliged to conclude that in his efforts to account for Law III Kepler's enthusiasm overrode his normal checking procedures, and that this is one instance where optimism triumphed over exactness.

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45. The Origins of Kepler's Third Law, pp. 599-600.

§3 Examination of the theoretical foundation of Law III

Harmonice V C. 3 Propositions VIII, XI, XII, XIII;
C. 9 Proposition XLVIII.

In §2(b) above we had occasion to annunciate Law III as a proportional relationship between a pair of planets. However, for our present purpose we shall need its modern statement:

$$\frac{a^3}{T^2} = \frac{\mu}{4\pi^2} \ .$$

I confirm that, as anticipated in the Introduction, we shall adopt the approach of Principia I, by which μ will be treated as an absolute constant, to be described as 'the gravitational constant': that is, we shall totally disregard the mass-correction factor introduced by Newton in Principia III, which is equivalent to treating the mass of every planet as negligible by comparison with that of the Sun—indeed, the view most consistent with Kepler's conviction, emphasized throughout this study, that the Sun was too great (in both magnitude and status) to be comparable to—or influenced by—the planets.

Further, as Kepler knew:

$$\delta\alpha_m = \frac{2\pi}{T}$$

$$\therefore \quad a^3 \delta\alpha_m{}^2 = \mu \ : \ \text{Law III.}$$

This was in fact stated by Kepler in Harmonice V C. 9 (G.W. VI p. 358 lines 12-13): it is to be preferred to the original version in C. 3 Proposition VIII (G.W. VI p. 302 lines 21-2)—where Kepler announced his discovery—because, in the 'astronomical' units he used, the Earth being standard, the numerical value of the constant ought to have been unity, its presence merely a dimensional necessity.

Kepler undoubtedly checked the truth of this relationship using Tycho's observation-based data listed in Tabulations I and IV of C. 4 (G.W. VI p. 307, p. 311): these figures have been set out by Gingerich[1] and it is clear that the agreement would have seemed

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1. The Origins of Kepler's Third Law, p. 598.

eminently satisfactory to Kepler (bearing in mind that the values of the mean distances were not then known with the same accuracy as those of the periodic times).

However there is more information to be obtained from Harmonice V when we concentrate exclusively on a single planet—a situation which has not been examined by previous commentators, but which has already, in Chapter VII, §3 and §4, enabled us to formulate the Composite Time-measure, applied by Kepler in place of the Newtonian Time-measure. It was pointed out there that because the former is wrong, either the apsidal motions are not those observed, or the mean motion is incorrect (but not both). In the present context we shall assume the second alternative, with the erroneous Keplerian value of the diurnal mean motion denoted by $\delta\bar{\alpha}_m$, while the apsidal apparent motions $\delta\theta_a, \delta\theta_p$, are to be treated as if observed[2]. (since in pursuit of his harmonic consonances Kepler confined his consideration to apsidal motions and distances alone, so too now shall we).

We repeat for reference the contrasting kinematical theories, to be applied in the two special cases:

Either $\frac{\delta\theta_a}{\delta\bar{\alpha}_m} = \frac{a^2}{r_a^2}$ and $\frac{\delta\theta_p}{\delta\bar{\alpha}_m} = \frac{a^2}{r_p^2}$: the Composite Time-measure

or $\frac{\delta\theta_a}{\delta\alpha_m} = \frac{ab}{r_a^2}$ and $\frac{\delta\theta_p}{\delta\alpha_m} = \frac{ab}{r_p^2}$: the Newtonian Time-measure.

From this juxtaposition it is clear that:

$$\delta\bar{\alpha}_m = \frac{b}{a}\delta\alpha_m .^{3}$$

We first employ a relationship which is true whichever time-measure is applied:

$$\frac{r_p}{r_a} = \frac{\sqrt{\delta\theta_a}}{\sqrt{\delta\theta_p}} .$$

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2. Though harmonic adaptations in fact, as we remark below.
3. Where $\delta\bar{\alpha}_m = \delta\theta_m$, as illustrated in Chapter VII, §4.

Then, $a = \frac{1}{2}(r_a+r_p) \propto \frac{1}{2}(\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p})$

$$ae = \tfrac{1}{2}(r_a-r_p) \propto \tfrac{1}{2}(\sqrt{\delta\theta_p}-\sqrt{\delta\theta_a})$$

$$\therefore \quad e = \frac{r_a-r_p}{r_a+r_p} = \frac{\sqrt{\delta\theta_p}-\sqrt{\delta\theta_a}}{\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p}}$$

and, $$\frac{b^2}{a^2} = 1-e^2 = \frac{4\sqrt{\delta\theta_a}\sqrt{\delta\theta_p}}{(\sqrt{\delta\theta_a}+\delta\theta_p)^2}.$$

Hence, these results are all correct in modern terms.[4.]

I here acknowledge a debt to Caspar, whose notes (G.W. VI pp. 547-50) to Propositions XI, XII, XIII I have found invaluable, for without them I doubt whether I could have managed to make sense of the result Kepler stated in Proposition XII (G.W. VI pp. 303-4)—being wrong in modern terms, and an approximation anyway—since he gave no inkling of how he himself arrived at it. Nevertheless, his method must have been sound, as I now show, applying the Composite Time-measure:[5.]

$$2a=(r_a+r_p)=a\sqrt{\delta\bar{\alpha}_m}\left(\frac{1}{\sqrt{\delta\theta_a}}+\frac{1}{\sqrt{\delta\theta_p}}\right)$$

$$=a\sqrt{\delta\bar{\alpha}_m}\left(\frac{\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p}}{\sqrt{\delta\theta_a}\sqrt{\delta\theta_p}}\right)$$

$$\therefore \quad \delta\bar{\alpha}_m = \left(\frac{2\sqrt{\delta\theta_a}\sqrt{\delta\theta_p}}{\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p}}\right)^2 = \frac{4\delta\theta_a\delta\theta_p}{(\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p})^2}.$$

Accordingly, I follow Caspar in setting:

$$A=\tfrac{1}{2}(\delta\theta_a+\delta\theta_p), \quad G=\sqrt{\delta\theta_a}\sqrt{\delta\theta_p}.$$

$$\therefore \quad 2(A+G)=\delta\theta_a+\delta\theta_p+2\sqrt{\delta\theta_a}\sqrt{\delta\theta_p}=(\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p})^2.$$

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4. The first three, and the fourth by implication, were used to construct Table I of C. 9—to be discussed further below.

5. The presence of this time-measure, in relation to a single planet, was first drawn to my attention by Caspar's notes—though, as we have remarked earlier, he neither noticed this nor realized that the relationship was wrong.

Then the above formula becomes:

$$\delta\bar{\alpha}_m = \frac{2G^2}{A+G}.$$

Hence, as Caspar indicates (p. 549), we may derive the approximation stated by Kepler:

$$\delta\bar{\alpha}_m \cong G - \tfrac{1}{2}(A-G).$$

It is perhaps of interest to demonstrate that there is a corresponding correct formula:

$$\delta\bar{\alpha}_m^{\,2} = \frac{a^2}{b^2}\,\delta\bar{\alpha}_m^{\,2} \text{ as established above}$$

$$= \frac{A+G}{2G} \cdot \left(\frac{2G^2}{A+G}\right)^2$$

$$\therefore \quad \delta\alpha_m^{\,2} = \frac{2G^3}{A+G}$$

$$\text{or,} \quad \delta\alpha_m = \frac{2(\sqrt{\delta\theta_a}\sqrt{\delta\theta_p})^{3/2}}{\sqrt{\delta\theta_a}+\sqrt{\delta\theta_p}}.$$

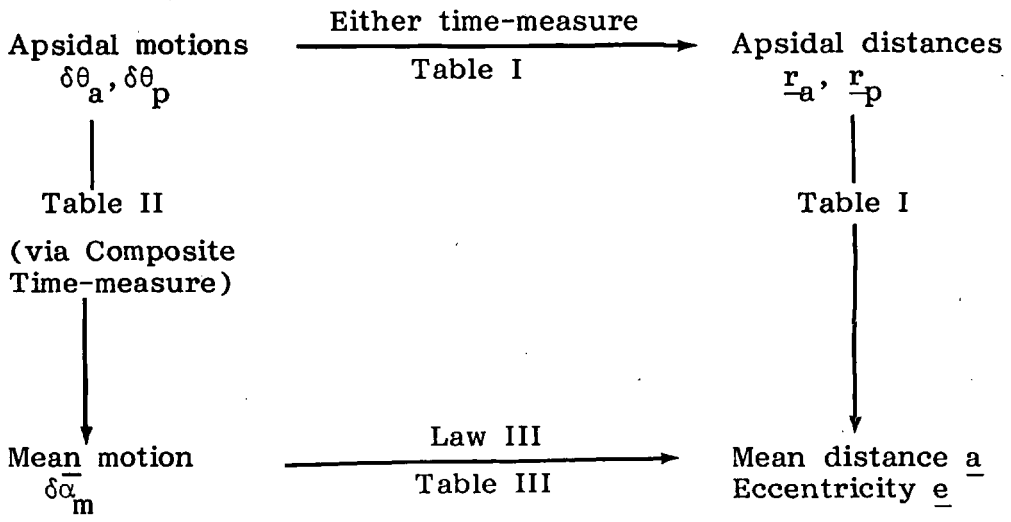
Then the corresponding approximation would have been:

$$\delta\alpha_m = G - \frac{1}{4}(A-G).$$

It was in Table II of C. 9 (G.W. VI p. 357) that Kepler calculated the values of $\delta\bar{\alpha}_m$ for the individual planets, preparatory to completing his reasoning in Table III (G.W. VI p. 358). Though C. 9 contains no new information of astronomical value, these three tables are significant in that their construction was the purpose for which the discoveries of C. 3 were made. Their analysis, however, has been unwarrantedly complicated by the fact that (to avoid fractions) Kepler introduced numerical multipliers, different for each planet and separately in each table (as well as abandoning his level of exactitude so that there was no control over the sizes of the values involved); therefore, it has been necessary to reiterate his working[6].

6. I have greatly appreciated the help of my son, R. C. L. Davis, who carried out these tiresome checking calculations.

to make sure that the theoretical argument propounded here reproduces Kepler's actual procedure—as indeed it has turned out—which I now set out schematically:



This presentation emphasizes how urgent must have been Kepler's need to discover Law III, because it was the one piece of information he required in order to complete the linkage. Successful at last, the calculations he performed to get from the first to the third column of Table III may be expressed:[7]

$$\left(\frac{1}{\delta\bar{\alpha}_m}\right)^{\frac{2}{3}} = a.$$

It is most direct to assess the validity of this relationship using the error in $\delta\bar{\alpha}_m$ already known.[8] Then:

$$a^3(\delta\bar{\alpha}_m)^2 = a^3\left(\frac{b}{a}\delta\alpha_m\right)^2 = \frac{b^2}{a^2}a^3\,\delta\alpha_m{}^2$$

$$= \frac{b^2}{a^2}\mu\ .$$

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7. The result for each planet must additionally be reduced for comparability with the Earth, but it does not matter whether this is done at the beginning or at the end of the calculation.

8. Confirmed, of course, by applying the Newtonian instead of the Composite Time-measure wherever it occurs.

Consequently we will define:

$\mu_e=(1-e^2)\mu$: Kepler's gravitational pseudo-constant.[9.]

Then,

$$\frac{\mu-\mu_e}{\mu} = e^2 \quad : \quad \text{the residual error.}$$

Once more this is of a size that Kepler would not have been prepared to ignore,[10.] had he managed to detect it: however, there was no chance of this because—apart from the numerical complications mentioned above—the values taken for the apsidal motions (the starting-point for Tables I and II) were harmonic adaptations of the observational ones and themselves quite considerably inaccurate.

As we said at the outset, Kepler must certainly have established Law III (in Proposition VIII of C. 3) using observation-based data available to him (the values for the period times T, or $\delta\alpha_m$, being notably sound by modern standards). What he did subsequently, in Proposition XII of C. 3 (put into practice, merely, in C. 9), was to give it a theoretical foundation—which was inevitably erroneous, due to the defect of his kinematical synthesis, highlighted in Chapter VII, §3 and §4, in that the application of the Composite Time-measure has reintroduced the effect of the sequential error to produce the residual error determined above (so called because it remained uncorrected at the end of Kepler's astronomical work): and thus, as we have remarked there, to be attributed, fundamentally, to a geometrical fault.

Hence it is ironic that, in theory, I can only demonstrate the disproof of Law III, on the same grounds that, had the formula for the diurnal mean motion been correct in modern terms, I would have claimed its proof—while recognizing that the status of this assessment is not unobjectionable, since it cannot be carried out until the true law is known: nevertheless even this limited achievement is contrary to expectations, when we appreciate that we have examined the situation relation to a single planet alone.

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9. Despite the fact that Kepler would have regarded this as a misnomer.
10. By reference to the level of accuracy stipulated in Part A.

The significance of the pseudo-constant is, of course, that it is variable, dependent on the eccentricity of the individual planet, and hence does not provide the foundation from which one can synthesize the system.

* * * *

It is because Kepler's pseudo-constant varied with the eccentricity that we shall now examine the special case obtained by setting:

$e=o$ so that $\mu_e \equiv \mu$.

It cannot be sufficiently emphasized that not only does each individual orbit considered become a circle with the force at the centre, but this entails that the force be constant, and the orbit described with uniform motion: thus a collection of such concentric circles in no way resembles a system of actual planetary orbits. Yet, it is for this case alone that Kepler's proof of Law III was valid. This was indeed the situation generalized by Newton in Principia I Section II Proposition IV Corollaries 1 and 2, where he proved:

$$F \propto \frac{u^2}{a}$$ [11.], $$F \propto \frac{a}{T^2}$$ [12.],

from which, adopting Kepler's Law III (validly proved as just confirmed), he deduced the inverse-square law of force $F \propto a^{-2}$ (stated as Corollary 6). Thus, as Whiteside has pointed out, any claim in this respect as to Newton's deduction from Kepler should be interpreted with reference to this very limited situation:[13.] it must be stressed that, even nowadays, one cannot go on to demonstrate the necessity for the inverse-square law of force in relation to real elliptic planetary orbits until Kepler's Law II has been established, which of course was what Newton did subsequently.[14.]

Nevertheless it is of interest in correlation to draw attention to the consequence of an application of the Keplerian method of Astronomia (even though we will here adopt a modern treatment). It has been

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11. We highlight the distinction by introducing u to denote uniform velocity (which is both transverse and tangential).

12. Apparently these formulae for uniform circular motion were first discovered by Huygens: see Whiteside: Newton's Early Thoughts on Planetary Motion, p. 129 footnote 43.

13. Op. cit., p. 120 footnote 13.

14. As Proposition I: op.cit. pp. 117-8.

proved in Prolegomenon I, where we investigated conic-system A.G. II (ellipses possessing a common eccentric circle) that the 'completed' time depends on the distance of the centre of force from the centre of the eccentric (by the formula described there as the kinematical synthesizing-property): thence we now infer that the period (total) time is independent of that distance, so that we can alter the centre of force without affecting the periodic time.[15] Accordingly, we envisage a collection of (six) concentric circles whose common centre is to be named S, and whose radii are those of the actual planetary orbits, of which a typical one will have radius a_i, and $T_i=a_i^{3/2}$ (so that an inverse-square force at S is implied) to be paired with the real planetary ellipse with its centre at S, and centre of force at its focus A_i, whose geometrical elements are (a_i,e_i). Then, as we have said, a change of centre of force for the circle from S to A_i will leave the periodic time T_i unaltered, and since this change produces a two-member conic-system A.G. II with centre of force at A_i, the periodic time in that ellipse will also be given by $T_i=a_i^{3/2}$. Finally, the ellipse must be subjected to a translation (equal and opposite to that supposed in Prolegomenon I) to bring its focus into coincidence with the Sun at S: and the pair then constitutes a two-member conic-system A.II with common centre of force at the focus S, while the periodic time T_i remains common.[16] Thus when this procedure is carried out for each of the six concentric circles paired with the Keplerian planetary ellipses, we arrive at the interconnection of the periodic times for the actual planetary system, in accordance with Law III.

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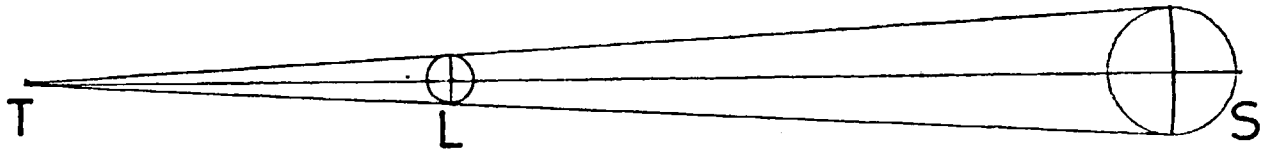
15. Certain conditions must be satisfied, but these are not relevant in the present circumstances.

16. Confirmed and illustrated in Systems of Conics in Kepler's Work, pp. 683-4.

APPENDIX VIII(1)

KEPLER'S ARCHETYPAL NUMBER D WHICH DELINEATED THE UNIVERSE; AND OTHER CONSTANTS

Fig.(i) Earth, Moon, Sun.



T ≡ Earth

L ≡ Moon

S ≡ Sun

In the course of Epitome IV Part I Section 4 Kepler set out his belief that the Earth, the home of mankind, and the Sun, the source of light, being first in order of creation, had been so placed in the heavens that the diameter of the Sun, at its maximum (apogeal) distance, would define $\frac{1}{720}$ of a complete circle (G.W. VII p. 277 lines 30-5); moreover, for the benefit of man the observer (mensurans creatura), the architect of the universe had arranged that the Moon (again at its apogeal distance) should exactly eclipse the Sun (G.W. VII p. 279 lines 45-p. 280 line 2): this situation is shown in Kepler's own diagram (G.W. VII p. 278, p. 280), reproduced above. Thence we have the observable relationships:[1.]

$$\left.\begin{array}{ll} \text{(I)} & \dfrac{\text{Earth-Sun distance}}{\text{Radius of Sun}} \\ \text{(II)} & \dfrac{\text{Earth-Moon distance}}{\text{Radius of Moon}} \end{array}\right\} = D$$

1. See Tabulae Rudolphinae (G.W. X) Part III, Table 92-3, Table 98(2) for the apparent angular radius of the Sun and the Moon, respectively.

where, as we shall demonstrate, D may justly be termed Kepler's archetypal number, when we write:

$$D = \frac{720}{\pi}[229.18]^2.$$ (G.W. VII p. 278 line 44, p. 279 lines 1,4, p. 281 line 10).

(a) Kepler's archetypal number D Epitome IV Part I Section 4.

Despite the primacy of the Earth-Sun arrangement, it seems certain that Kepler started from the secondary Earth-Moon situation, in order to ensure that his conclusions were consistent with Brahe's observations, as he confirmed (G.W. VII p. 281 lines 2-5):[3] so we shall here set out his argument in a more logical order.

(III) $\frac{\text{Earth-Moon distance}}{\text{Radius of Earth}} = D^{3/4}$ [<59][4]. (p. 281 lines 18-20).

Then from the observational result (II) above:

(IV) $\frac{\text{Radius of Earth}}{\text{Radius of Moon}} = \frac{D}{D^{3/4}} = D^{1/4}$ [3.8908] (p. 281 lines 8-12).

Now because of his firm grasp of the order of linear dimensions, Kepler knew that the volumes of the bodies were proportional to the cubes of their radii (p. 279 lines 6-8), and so:

$$\frac{\text{Volume of Earth}}{\text{Volume of Moon}} = D^{3/4}$$ [<59] (p. 281 lines 12-15)

$$\therefore \frac{\text{Volume of Earth}}{\text{Volume of Moon}} = \frac{\text{Earth-Moon distance}}{\text{Radius of Earth}}$$ (p. 280 lines 23-5).

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2. In what follows the numerical values actually stated by Kepler are given in parenthesis—corrected, where appropriate, to five figures.

3. Indeed, he must have used similar reasoning to obtain the ratio of the volumes of the Earth and Moon that he needed for his theory of gravitation in the Introduction to Astronomia (G.W. III p. 25 lines 38-40)- but there using Tycho's value for the perigeal distance.

4. More exactly, $\sqrt{3469.3}=58.901$ (see p. 279 lines 4-5).

All these are comparable with modern values: it was when Kepler came to tackle the size and distance of the Sun that there was no observational evidence available (except for the result (I) stated above) and so he had to rely on analogy (p. 280 line 19) to propose:

$$\frac{\text{Volume of Sun}}{\text{Volume of Earth}} = \frac{\text{Earth-Sun distance}}{\text{Radius of Earth}}$$ [5]. (p. 278 lines 37-40, p. 280 lines 20-2).

$$\therefore \frac{(\text{Radius of Sun})^2}{(\text{Radius of Earth})^2} = \frac{\text{Earth-Sun distance}}{\text{Radius of Sun}} = D \text{ [229.18]},$$

by the observational result (I) above. Hence:

(V) $$\frac{\text{Radius of Sun}}{\text{Radius of Earth}} = D^{\frac{1}{2}} \text{ [>15]}$$ [6]. (p. 279 lines 8-9).

$$\therefore \frac{\text{Volume of Sun}}{\text{Volume of Earth}} = D^{3/2} \text{ [3469.3]}$$ (p. 279 lines 9-10, 4-5)

(VI) $$\frac{\text{Earth-Sun distance}}{\text{Radius of Earth}} = D^{3/2} \text{ [3469.3]}$$ (p. 279 lines 3-5).

Then finally, from (IV) and (V), (III) and (VI) respectively:

(VII) $$\frac{\text{Radius of Sun}}{\text{Radius of Moon}}$$

(VIII) $$\frac{\text{Earth-Sun distance}}{\text{Earth-Moon distance}}$$

$$= D^{3/4} \text{ [<59]}$$ (p. 281) lines 23-5).

These eight results provide the foundation from which any further deductions (for instance that of p. 280 lines 38-42) may be made. In this connection I remark that Kepler possessed greater facility in manipulating fractional powers that has probably been suspected hitherto—and thus it would have been simple for him to carry out the numerical calculations to check Law III.[7].

Kepler then went on to consider the bounds of the universe, proposing the relationship:

$$\frac{\text{Distance of stars}}{\text{Saturn-Sun distance}} = \frac{\text{Saturn-Sun distance}}{\text{Radius of Sun}}$$ (p. 286 lines 6-8).

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5. If the analogy had been precise, we would have described D as 'Kepler's analogical number': but such a relationship would have produced the radius of the Sun less than four times ($D^{1/4}$) greater than that of the Earth.
6. More exactly, since $D^{-\frac{1}{2}}=0.06606$ (p. 279 line 3), $D^{\frac{1}{2}}=15.138$.
7. Despite Koyré: La révolution astronomique p. 451 note 29.

Using the approximation:

Saturn-Sun distance < 10.Earth-Sun distance[8]. (p. 286 lines 1-2),

he derived, from (I) above:

Distance of stars $\cong 100D^2$.Radius of Sun by (I)
$\cong 100D^{5/2}$.Radius of Earth by (V).[9].

I digress to mention the relationship Kepler introduced to account for the period of rotation of the Sun—which he invented before it could have been verified (by observations of sunspots)—as he explained in Astronomia C. 34 (G.W. III p. 244 line 31-p. 245 line 2):

$$\frac{\text{Radius of Sun}}{\text{Mercury-Sun distance}} = \frac{\text{Radius of Earth}}{\text{Earth-Moon distance}}$$

and 'hence' by analogy:

$$\frac{\text{Period of rotation of Sun}}{\text{Period of revolution of Mercury}} = \frac{\text{Period of rotation of Earth}}{\text{Period of revolution of Moon}}$$

which gives:

Period of rotation of Sun $\cong$ 3 days.

It seems to me surprising that Kepler did not choose instead to make the assumption (preferable, because direct):

$$\frac{\text{Period of rotation of Sun}}{\text{Period of revolution of Earth}} = \frac{\text{Period of rotation of Earth}}{\text{Period of revolution of Moon}}.$$

$\therefore$ Period of rotation of Sun $= \frac{365}{30} = 12+$ days.

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8. The more exact observed value is to be found in Harmonice V C. 4 tabulation IV (G.W. VI p. 311).

9. It is of interest that Archimedes proposed the relationship:

$$\frac{\text{Distance of stars}}{\text{Earth-Sun distance}} = \frac{\text{Earth-Sun distance}}{\text{Radius of Earth}}$$

(Heath: Greek Mathematics II pp. 81-5), so that, using the Keplerian results, Distance of stars $\cong D^3$.Radius of Earth. (With hindsight, it appears that Archimedes took a much more realistic view of the Earth-Sun distance than did Kepler in fact.)

Though this is pure speculation on my part, it would have confirmed the figure mentioned by Kepler in Epitome IV Part II Section 3 (G.W. VII p. 298 line 21)—which was, however, half the true value,[10] itself stated earlier in Epitome IV Part II Sections 1 and 2 (G.W. VII p. 290 lines 16-17, p. 296 lines 41-2) as '25 or 26 days'. But I cannot suggest any route by which Kepler could have managed to deduce this correct value from the above scheme.

(b) Kepler's harmonic number H Harmonice III C. 7, C. 14.

Additionally Kepler felt that he had to account for the number 720 of the parts into which the Sun's diameter divided a complete circle: because, as he stated in Epitome IV Part I Section 4 (G.W. VII p. 277 line 37-p. 278 line 8), it is impossible to construct a regular polygon of 720 sides, since this is derived by repeated bisection from one of 45 sides, which is itself not constructible by Euclidean methods. Thus the source of that number was to be found from harmonic considerations, for it occurred naturally when the ratios which defined the notes of both the minor and the major scales were expressed in the simplest possible whole numbers. By reference to Harmonice III C. 7 (G.W. VI p. 140) we find that the minor scale—what is described nowadays as the melodic (descending) form—could be expressed in terms of the number 144, while the version of the major scale given there required the number 720; it was in C. 14 (G.W. VI p. 165) that the same minor scale was reproduced as the second alternative, while the alternative set out for the major scale had the sharpened seventh, which we can then recognize as the modern version—and which, it is easy to check, requires only the number 180. However, 720 remains the lowest common multiple of 144 and 180, and thus of the combined scales, as Kepler stated, so it is with this justification that we name:

H=720: Kepler's harmonic number.

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10. Apparently due to Kepler's misidentification of two separate diametrically opposite groups of sunspots.

(c) The symbol for π

We must distinguish the expression of an angle as a fraction of a complete circle—obviously the concept of the greatest antiquity—from its measurement in degrees (which seems to have been introduced in response to the practical needs of astronomers)[11.] and in terms of radians. This latter measure was far too cumbersome to be of use until its adoption became crucial, in connection with the infinitesimal calculus, as we now recognize. The symbol itself first appeared in its presentday sense in 1647, and came into general use on the authority of Euler, who adopted it in 1737.[12.] Thus its employment in relation to Kepler's work is a (convenient) anachronism, though I point out that he possessed a numerical estimate of its value of greater accuracy than he needed, worked out at the end of the sixteenth century by Adrianus Romanus, as Kepler acknowledged in Astronomia C. 40 (G.W. III p. 266 lines 21-2):

$$\pi = 3.1415926536.^{13.}$$

11. Its origin is discussed by Heath: Greek Mathematics II, pp. 213-16.

12. See Smith: History of Mathematics II, p. 312.

13. Correct to 10 decimal places because he needed it in connection with areas (because of dimensional considerations, as explained in Chapter V, §3).

APPENDIX VIII (2) to §2

THE LATENCY OF GRAVITY

From the Introduction to Astronomia (G.W. III p. 25 lines 21-2, 32-4)—as anticipated in Chapter IV, §2—we discover that Kepler envisaged gravity as a property of cognate bodies, described as their mutual tendency (affectio) to conjunction. We first confirm that, at its most general, this definition related only to planetary matter, and hence gravity was not supposed to exist between the Sun and the planets: it was not a faculty of the Sun.

Kepler's quantification of gravity can be obtained from that Introduction (G.W. III p. 25 lines 32-41), or from a letter to Fabricius,[1.] translated by Rosen,[2.] who has written: 'Kepler measured the force of gravitation, as the product of a weight times [sic] a distance', so contradicting his earlier statement—which is obviously correct, by reference to what I have called Kepler's 'all-purpose' analogy with the steelyard (statera)[3.]—that this product is constant: it may thus be suitably termed 'gravitational moment'. However, Rosen has also rightly pointed out that:

$$\text{Force of gravity} \propto \frac{1}{r},$$

which is indeed of a form that Kepler would have recognized as a force (had he needed it), in accordance with his Aristotelian principles. Moreover—because the product is constant—this is compatible with Kepler's statement (see line 66 of the letter cited) that gravity possessed the same dimensions as the bodies, and so we deduce Kepler's intention to have been:

$$\text{Force of gravity} \propto \text{weight}.$$[4.]

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1. G.W. XV Letter 358 lines 65-75.
2. Kepler's Somnium, Appendix H: Kepler's Concept of Gravity, especially p. 221.
3. In this situation the analogy was both more appropriate (in his terms) and more straightforward than when it was invoked in support of the proof of either Law I or Law II: see Chapter VI, §2(b).
4. Pondus, but also moles, the term used in Epitome to denote volume: there is no inconsistency because the cognate bodies were assumed to be of common density (G.W. III p. 25 lines 40-1).

(I cannot decide to what extent the measurement of gravity by weight, in the terrestrial situation, constituted a foreshadowing of the Newtonian view, since Kepler had no apprehension of the concept of mass required to interpret it in such modern terms.)

Kepler's originality in visualizing a 'gravitational field' surrounding the Earth, which reached at least to the Moon (and vice versa) was bound to have stimulated speculation on the part of others. However, as far as he was concerned, his extension of the concept of gravity to the Earth-Moon situation was designed simply to enable him to formulate a theory of the tides (not to be considered here): for the question of the mutual conjunction of the bodies themselves was discussed, in the letter already cited, on the assumption that the bodies were exempt from all other motions (lines 69-70);[5] and in Astronomia (p. 25 lines 37-8) he stated what would occur 'if the Moon and the Earth were not retained, each in its orbit'.[6] Thus while Kepler determined what could happen, he firmly repudiated the possibility that it would. It is with this justification that I assess Kepler's concept of gravity as latent in astronomical terms: in practice, it was to be regarded as a purely local phenomenon.

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5. Where the Moon was an example of a body of non-negligible size compared with the Earth (lines 67-9).

6. We disregard the strange vis animalis by which the bodies were said to be so retained: it may be taken as an inappropriate reference to the combined causes of planetary motion which were then still inadequately systematized.

CONCLUSION

This enquiry into the bases of Kepler's laws has demonstrated how close is the interrelation between the three constituents of his astronomy: geometrical, kinematical, physical. Concomitantly it has highlighted the strong geometrical linkage between the three laws: I draw particular attention to the fact that the residual error (e^2) in Law III can be traced back to a geometrical fault (the absence of the constant b of the orbit), while confirming that this is the single flaw in Kepler's ultimate synthesis—there remain none of undetectable size, at $O(e^3)$. Indeed, this common geometrical foundation was to be expected as a result of Kepler's discovery of Law II, which effectively replaces one geometrical representation of time (angular) by another (areal). Nevertheless this amounted to a conceptual breakthrough of the greatest significance, as Newton indicated (when he had proved it in all generality) by the position of prominence he gave it, as Proposition I of Principia I.

There has been a tendency among recent commentators (post-Caspar) to suggest that Kepler reached his results more by luck than good judgement; the present study is aimed at countering this impression, for I believe that the intervention of luck—or fate—simply ensured that Kepler was in the right place at the right time to take advantage of Tycho's store of observations; thereafter, given Kepler's combination of perspiration and inspiration—coupled with the limitations of his (Euclidean) methods—it was inevitable that he would arrive at the Sun-focused ellipse as his preferred choice (no matter whether in the end it would prove to be the correct orbit, or merely observationally indistinguishable from it), this being a curve amenable to analysis in terms of eccentric anomaly: which was the nub of his treatment, and turned out to be the root of his success, both in discovering the orbit and in determining the component motions, as we have seen.

My fresh approach to Harmonice was dictated by the fact that I was in search of more information relating to the motion of a

single planet—which is indeed to be found there, as I have shown. Then, to my surprise, it became possible to use the formula for the diurnal mean motion stated by Kepler, to assess the soundness of Law III on a theoretical basis, to demonstrate that, while his formula was not compatible with the interconnection of an actual system of elliptic planetary orbits, it could validly have been applied to the simpler situation of a collection of concentric circles described with uniform velocity.[1.] However, it was Kepler's underlying purpose in that book to exhibit the harmonic consonances—of the apsidal motions—which he believed to be the final cause of the randomness of the planetary eccentricities (Harmonice V, title to C. 9).[2.] Yet it should be recognized that any adaptation of the apsidal motions—such as was proposed in C. 9 Propositions I-XLVII (G.W. VI pp. 330-56)—entailed a distortion of the well-authenticated observational figures on which he had built his life's work. This effect is particularly notable when the periodic times obtained in the second column of Table III in C. 9 (p. 358) are compared with those of Table I in C. 4 (p. 307)—the facts which Kepler concurrently described as 'the most certain in all astronomy' (p. 360 lines 2-3).[3.] Therefore, it seems inexplicable that the approximations of C. 9 should have given him such euphoric satisfaction: nonetheless, it must be acknowledged that his attempted harmonization of the planetary motions provided Kepler with the stimulus to discover Law III.

When we examine the development of Kepler's views on the physical causes of planetary motion, it is all too easy to follow him into some inconsistency, as others before me have found: I have been able to avoid this only at the expense of confining consideration to those parts of his work that can be expressed

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1. Which is what Newton did in the first instance, before he himself had proved Law III: he must, at that stage, have accepted its approximate numerical verification.

2. Repeated in Epitome IV Part III Section 3 (G.W. VII p. 338 lines 41-4)—where the potency of the magnetic fibres was said to be the instrumental cause (p. 338 line 40).

3. With reference to the periodic time of Mercury, whose inaccuracy, ironically, at about 2%, is the highest of them all.

mathematically, ignoring any contradictory statements which occurred in the course of speculative discussions. The quantification of his physical opinions has had to be carried out on Kepler's behalf because he himself did not possess the techniques to do this: in one respect the analysis has been pressed just beyond what Kepler himself would probably have felt justifiable—[4] simply in order to demonstrate the coherence of his systematization and to confirm that, even if he himself could not manage it, he had indeed accomplished his aim of giving a (near-perfect) mechanistic explanation of the planetary motions via the two faculties of the Sun: its magnetism and the whirling force of its rays. Based as this was on Aristotelian principles, it seems inherently unlikely to modern eyes—until we adopt a pre-Newtonian standpoint, in which the Aristotelian Equivalence represented reality: indeed, at first encounter, it requires a considerable effort of historical imagination—or the type of mental exercise recommended to Alice by the White Queen.

I count myself extremely fortunate to have been the first to analyze Epitome V Part I mathematically (following Kepler, again essentially in terms of eccentric anomaly) and thence to demonstrate the extent to which Kepler actually proved Law I and Law II. The importance of this as the culmination of Kepler's astronomical innovations can scarcely be sufficiently emphasized—yet Sections 1-4 have been almost entirely overlooked by previous commentators, despite the fact that an English translation has been available for nearly thirty years,[5] and extended passages have been translated by Koyré[6] who, however, has been able to shed little illumination on the specialized mathematical problems which are the crux of those sections. It has been pointed out earlier that Aiton[7] has considered only Kepler's

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4. In quantifying the magnetic faculty—as noted in context.
5. By Wallis, in Great Books of the Western World, Vol. 16.
6. La révolution astronomique, pp. 307-24.
7. The Vortex Theory of Planetary Motions, p. 17: see also Chapter VI, §5(d).

preliminary work leading up to the proof of Law I, and so has wrongly set out the results of Astronomia as if they represented Kepler's final opinions; while Caspar's notes[8.] do not clarify the proof of either Law I or Law II—in the latter case, he refers only to Kepler's clear statement of the law, as does Gingerich,[9.] by implication. These undervaluations of Kepler's later work have been reinforced by the frequent description of Epitome as a textbook—by connotation, more elementary than the 'technical' Astronomia. On the contrary, I have found that a thorough investigation of the former is at least as difficult as that of the latter; and I must urge that this book—and Harmonice too, with respect to its astronomical content—receive more detailed attention from now on, in order that Kepler may be awarded the credit for the full range of his attainments which it has been my purpose in this study to show that he so richly deserves. The essence of his originality has been caught by Koestler:[10.] 'The Keplerian discoveries were not of the kind which are "in the air" of a period, and which are usually made by several people independently; they were quite exceptional one-man achievements. That is why the way he arrived at them is particularly interesting'—and stressed by Heath:[11.] 'The Copernican hypothesis was actually...[and] a system precisely the same as that of Tycho Brahe is also, represented in Greek astronomy...It was Kepler that the Greeks failed to anticipate'.

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8. G.W. VII pp. 594-6, pp. 597-8 respectively: we recall that the former is confused because of the misplaced diagram on p. 369.

9. Kepler, Johannes p. 304.

10. The Sleepwalkers, p. 334.

11. Greek Astronomy, Introduction, p. lv.

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11.5.

Systems of Conics in Kepler's Work

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Summary

This is an attempt to put Kepler's invention of the first "non-cone-based" system of conics into historical perspective; to highlight the dichotomy in Kepler's thought which kept this system forever unconnected with his contemporaneous work in astronomy; and, in passing, to describe the other conic-systems which arise solely from the application of his planetary laws. The discussion of his conic-system also involves critical consideration of his Principle of Analogy/Geometrical Continuity, which is fundamental to modern synthetic geometrical thinking.

The *Astronomia Nova*[1] is an account of Kepler's struggle to overcome his difficulties in arriving at the concept of a system of ellipses for the planetary orbits. This has always seemed to me a problem of great interest because he was contemporaneously developing a system of conics for use in his optical work: in 1603, in the course of writing *Astronomiae Pars Optica*,[2] he needed to find a surface that would turn into parallels all rays emanating from a fixed point. He acknowledged the work of his predecessors, mentioning Apollonius and Vitelo, but decided that he would have to make a study of the conic sections for himself, from "a mechanical, analogical, and popular point of view". This resulted in an entirely new system of conics (Fig. 11.14). Earlier methods of constructing the conic sections are illustrated in Figs. 11.12 and 11.13,[3] so that they may be compared and contrasted with Kepler's system.

[1] *G.W.* III.
[2] *G.W.* II.
[3] The Greeks did not synthesize the sections as shown in my diagrams.

In order to make the comparison as direct as possible, we choose to define a *geometrical system of conics* as a group of conics having one common vertex P, together with some other property peculiar to each system. For additional simplicity, in the case of the first two systems which are derived from a generating cone, only a right circular cone will be considered. Such a cone, where it exists, has a semi-vertical angle α and its apex V is distinct from P. The angle of section σ is the angle at P made by the cutting plane with the perpendicular to the axis of the cone.

It may be helpful to point out that the type or shape of a conic depends on its eccentricity, denoted by e.[4] In particular:

$e = 0$	gives a circle,
$0 < e < 1$	gives an ellipse,
$e = 1$	gives a parabola,
$1 < e < \infty$	gives a hyperbola,

and there are also the degenerate and limiting forms of the ellipse, the parabola, and the hyperbola, when they have collapsed into straight lines (either along the major/transverse axis, or along the minor/conjugate axis). Examples of these will arise in the course of this paper.

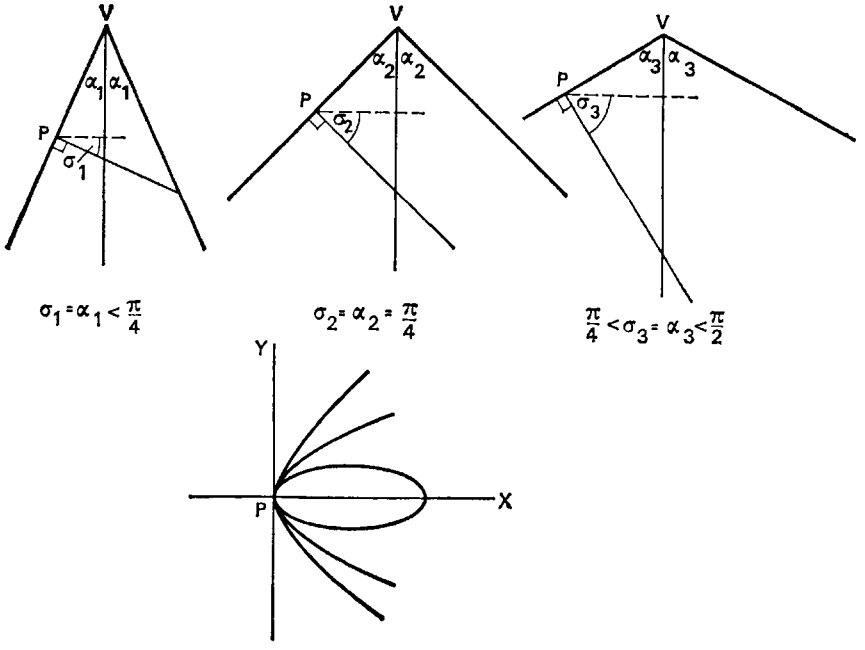


FIG. 11.12

[4] The sense in which this term is used will be explained shortly.

A description of the first method of construction was ascribed by Pappus to Aristaeus,[5] and appears to be the first instance of curves being generated from cones: to produce different conics, the vertical angle 2α of the cone varies, as illustrated in Fig. 11.12, while the plane of section is defined as passing through a fixed point P perpendicular to the generator VP, and this fact provides the other data: the angle of section σ (as marked), under this condition, is such that $\sigma = \alpha$.

The system which results is:

Point circle → Ellipse → Parabola → Hyperbola → Line-pair (along conjugate axis PY).

This will be referred to as System G.I.

Apollonius was principally responsible for the second method. It was discussed by Heath[6] in his edition of Apollonius' treatise (1896) and by Taylor[7] in 1881. All members of the system of curves are now given by a single cone of fixed vertical angle 2α as shown in Fig. 11.13; to obtain different conics, the angle of section σ (as marked) is allowed to vary, and the first diagram shows how an ellipse, a parabola, a hyperbola arise in turn as σ increases, while the second emphasizes that the system falls entirely within the asymptotes as shown.

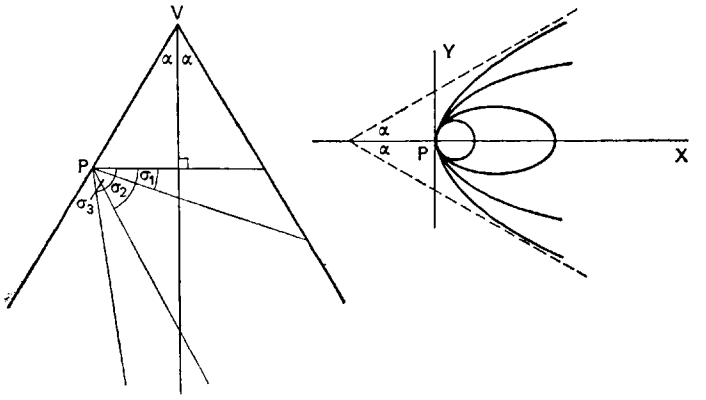


FIG. 11.13

The system which results is:

Circle → Ellipse → Parabola → Hyperbola → Degenerate hyperbola (along transverse axis PX).

This will be referred to as System G.II.

[5] Heath, *Apollonius*, p. xxxi, and Taylor, *Geometry of Conics*, p. xxxiii. Menaechmus is said to have been "the discoverer of the conic sections", but neither Heath nor Taylor offers much evidence of his work on conics other than the parabola and the rectangular hyperbola, which were necessary to his solution of the problem of the duplication of the cube.

[6] Heath, *Apollonius*, esp. pp. 8–14.

[7] Taylor, *Geometry of Conics*, p. 195.

Since both systems arise as the plane sections of a cone, common formulae for G.I. and G.II can be derived (by elementary trigonometry), using a figure such as that in Heath's edition of Apollonius[8] (restricted to a right circular cone, for our purpose, and employing the lettering of Figs. 11.12 and 11.13). Then

$$d = VP \cdot \frac{\sin\alpha \cos\alpha}{\cos(\alpha + \sigma)}\left(1 - \frac{\sin\sigma}{\cos\alpha}\right)$$

is the general formula[9] for the focus-vertex distance d which is the characteristic of special interest in Kepler's system (he used the term *sagitta* for this distance, as does Taylor). Then in particular, after simplification:
For G.I.,

$$d = VP \cdot \frac{\sin\alpha}{\sin\alpha + \cos\alpha}, \text{ since } \sigma = \alpha, \text{ with } \alpha \text{ variable,}$$

and for G.II.,

$$d = VP \cdot \frac{\sin\alpha(\cos\alpha - \sin\sigma)}{\cos(\alpha + \sigma)} \text{ with } \alpha \text{ constant, } \sigma \text{ variable.}$$

It is obvious that in neither system is the sagitta constant for all conics. However, this analysis indicates that, in G.II., d decreases as σ increases up to the value $(\pi/2) + \alpha$, and it becomes zero for the degenerate hyperbola. This fact will be needed later.

I have just mentioned the focus-vertex distance, and will make a digression to discuss the origin of the concept of the focus; this is obviously involved with the term *eccentricity* which needs explanation: in this paper it is used in a sense derived from the work of Apollonius,[10] in which the eccentricity is defined indirectly for ellipses and hyperbolas, using an area construction: the foci are identified for these conics, unnamed, but referred to as "puncta ex applicatione facta".[11] The definition adopted here is:

$$\text{Eccentricity} = \frac{\text{Focus-centre distance}}{\text{Vertex-centre distance}}.$$

This ratio provides a satisfactory definition when $0 \leqslant e < 1$ and when $1 < e < \infty$; and since a parabola (degenerate or not) occurs between ellipses and hyperbolas in every system considered here, this will be taken to be the curve corresponding to $e = 1$, thus closing the gap. (What happens when $e \to \infty$ is under discussion in this paper.) It is precisely the failure of the above ratio to give a clear definition when $e = 1$ which accounts for the fact that: "The focus of a *parabola* is not used or mentioned by Apollonius."[12] It is not very widely known that the focus-directrix property

[8] Heath, *Apollonius*, p. 11.
[9] In the case of the parabola this formula becomes $d = VP.\ \sin^2\alpha$.
[10] Heath, *Apollonius of Perga*, pp. 113–14.
[11] Taylor, *Geometry of Conics*, p. xliv.
[12] Heath, *Apollonius of Perga*, p. 114.

was proved for all three conics by Pappus. Heath, who discusses this point in detail, believes that Pappus had found a statement of the property in a work of Euclid (now lost), who may in turn have known of the proof from another source, possibly Aristaeus.[13] Nowadays, the focus-directrix property is used to define the eccentricity, and naturally the value of e for a particular conic is identical to that obtained from the present definition, but it is worth noting that the one I have adopted arises historically in astronomy from the preoccupation with eccentric circles (and in that context—and for various ovals—needs to be distinguished from the formal term eccentricity used, as here, exclusively for conics).

The advance in theory due to Pappus made little impact, and the later Greeks and the Arabs concentrated on the focal properties and their practical applications: the justification of string-constructions (at least for the ellipse), and the "optical property" for all conics (while treating the parabola separately because of its special characteristics). It will appear from our subsequent discussion of Kepler's system that his was an original attempt, in a new direction, to correlate and generalize the concept of foci for all conic sections.

As far as the name "focus" is concerned, Kepler certainly believed that he had invented it, and this is true as far as I have been able to discover. The evidence is in *Astronomiae Pars Optica*, Chapter IV, Section 4,[14] where he referred to points which were uniquely defined, but had as yet no name, and said: "we will call them Foci". The word "focus", of course, meaning "hearth" or "burning-point", must have arisen from the Arabic investigation described above—principally by Ibn al-Haitham (Alhazen)—into the parabolic burning-mirror, mentioned by Kepler in Section 3 of the same chapter—though he attributed it to Vitelo.

The stage at which the term "focus" was introduced, moreover, is of particular interest because of the concurrent, or immediately subsequent, composition of the *Astronomia Nova*, in which the same point is called the "punctum eccentricum". And, this first appearance of the term is not widely known: even Koyré[15] dated it 18 years too late, in Book V, Part I, Section 3 of the *Epitome*,[16] though it has been correctly placed, notably by Cambridge mathematicians who had read Taylor's book of 1881,[17] or his later, detailed analysis of this particular section of Kepler's work,[18] which is the only such analysis that I have seen. Of course, the correct date of introduction of the term was known to Caspar, who edited the *Epitome*, and to Hammer, who edited the

[13] Heath, *Apollonius of Perga*, pp. xxxvi–xxxviii. Heath gives a transcription of Pappus' proof for the ellipse and the hyperbola, while that for the parabola is easy.

[14] *G.W.* II, p. 91.

[15] Koyré, *La Révolution astronomique*, p. 443, note 93 to p. 314.

[16] *G.W.* VII, p. 372. It was here that Kepler defined the *position* of the foci of an ellipse by means of the sum of the focal distances.

[17] Taylor, *Geometry of Conics*.

[18] Taylor, *Geometry of Kepler and Newton*. He reproduces the complete Latin text of Section 4 from the edition of 1604.

optical work. For completeness, however, I should add that in the "Supplementum ad Archimedem" section of the *Stereometria*[19] Kepler used the term "focus" in conjunction with a diagram which is similar to Fig. 11.14, elaborated "for the generation of solids". He also mentioned, later in this section, the fact that the second focus and centre of a parabola are at infinity: as far as I know this reference has not been recorded previously except by Taylor, and by Hammer, the editor of this volume. It is of interest that Kepler's own quantified definition of eccentricity (for an ellipse only) remained quite separate as part of his astronomical work: in the *Astronomia Nova* C.LIX, Proposition VII,[20] he arrived (by an extremely roundabout route via his "Secant Law" of C.LVI) at the relation $a^2e^2 = a^2 - b^2$.[21] This situation provides an effective illustration of a dichotomy in his thought.

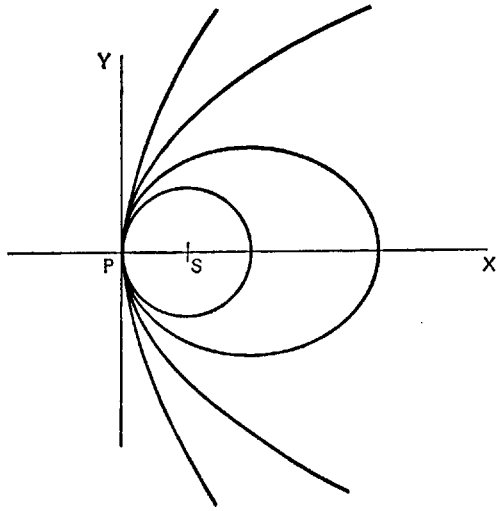


FIG. 11.14

Returning to the systems of conics, Fig. 11.14 illustrates the one originated by Kepler, which will be labelled G.III. It occurs in the same Section 4 of Chapter IV of *Astronomiae Pars Optica* referred to above, from which the diagram has been copied,[22] though the lettering has been changed to strengthen the comparison with the earlier systems (and the second branch of the hyperbola has been omitted, as irrelevant to the present analysis). Interpretation of the figure, however, cannot be attempted without some preliminary discussion.

[19] *G.W.* IX.
[20] *G.W.* III, pp. 368–9.
[21] a and b are the major and minor semi-axes.
[22] *G.W.* II, p. 91.

The section in which this figure appears is entitled "De Coni sectionibus"—"On the sections of a [single] cone"—but Kepler did not consider the question of finding the particular way of sectioning a cone continuously to obtain his system. (The reason for this omission will be given later.) Certainly the requirement "continuous" is essential, since this piece of geometry was the first, as far as is known, in which the Principle of Analogy (which is the archetype of the Principle of Geometrical Continuity) was explicitly stated. Taylor[23] says that the Principal of Analogy "expresses the inner resemblance of contrasted figures A and B, which are connected by innumerable intermediate forms; whilst [the Principle of Geometrical Continuity] expresses the possibility of passing through those intermediate forms from A to B, without any change *per saltum*". Throughout the section under consideration Kepler repeatedly stated his reasoning to be *per analogiam*, but in fact it appears rather to be based on the Principle of (Geometrical) Continuity, as in the following description[24] of his system, which "passes from the straight line through an infinite number of hyperbolas to the parabola, thence through an infinite number of ellipses to the circle".

Kepler went on to apply his principle to investigate his new system: two deductions which deserve comment are both concerned with the position of the foci. In the first case, having explained how the single focus of the circle and the two of the ellipse are related, he continued:[25] "In the parabola one focus lies within the curve, while the other is represented either outside or within it on its axis at an infinite distance from the first, so far that a line drawn from that *blind* focus [at either end] to every point of the curve is parallel to the axis." Taylor[26] points out that here Kepler formulated two related concepts—as special cases—which later became underlying principles in Projective Geometry. These are:

(1) Coincident points at infinity occur at both ends of a straight line.
(2) Parallel lines meet in a point at infinity.

Desargues is popularly credited with the origination of these concepts,[27] which were fundamental to his work on projective geometry, but Taylor believes that he must have been acquainted with Kepler's work: at least he adopted the term "foyer".

The second deduction is concerned with the position of the focus of the straight line (limiting hyperbola) and in that case Kepler reached a wrong conclusion, as we shall see. He said:[28]

> It follows by analogy that, for the straight line, the focus (as we will use the term, in relation to the straight line, without precedent, to complete the analogy) in both cases

[23] Taylor, *Geometry of Conics*, p. lviii.
[24] *G.W.* II, p. 90.
[25] *G.W.* II, p. 92. For simplicity, in this extract, I have omitted the lettering which refers to Kepler's original diagram. (The italics are mine.)
[26] Taylor: *Geometry of Conics*, p. lix (3) and (4).
[27] See, for example, the article on Desargues in *A Biographical Dictionary of Scientists.*
[28] *G.W.* II, p. 92.

> coincides with the line itself, so that there is only one focus, as for the circle. Then in the circle the focus is at the centre itself—the point furthest from the circumference—in the ellipse it is less far from the circumference, and in the parabola much less, in the end being the minimum distance away: that is, it falls on the line itself.

Kepler would, of course, have been right in saying that the focus S *appears* to move nearer to the vertex P in successive curves,[29] but in writing it down he wrongly chose this apparent characteristic as the one to which his Principle of Continuity applied. (This type of mistake crops up from time to time elsewhere in Kepler's work.) My own choice of interpretation is based on Kepler's diagram—rather than his explanation of it—which suggests a constant focus-vertex distance $PS = d$ for *all* conics of the system. With this condition, it can easily be proved that the Principle of Continuity will apply to the semi-latus rectum, which increases steadily (from the value d to infinity), as does the eccentricity, which takes all positive values, now including $e \to \infty$. I consider that this is what Kepler intended, and it can indeed be shown to be a mathematically consistent system (by more modern methods) with $d = a(1 \sim e)$, where a is the major semi-axis of the conic. Then the parabola and the line-pair arise naturally as the curves open up—Kepler used the phrase "become more obtuse"; and the limiting hyperbola still has the fixed sagitta d, since $a \to 0$ as $e \to \infty$.

In the next paragraph Kepler dealt with the ratio of *sagitta* to *semi-latus rectum*[30] for each successive type of conic. From the formulae quoted here,

$$\frac{\text{Sagitta}}{\text{Semi-latus rectum}} = \frac{d}{d\,(1 + e)} = \frac{1}{1 + e},$$

so that the value of this ratio decreases steadily from unity to zero, as $e \to \infty$, just as Kepler described,[31] "until [the ratio] vanishes for the straight line". But his reasoning was wrong, repeating the mistake made in the previous paragraph, for he continued this sentence: "where the focus falls on the line itself, . . . the sagitta vanishes, and at the same time the latus rectum becomes infinite, in fact coinciding with its arc, if we may so describe it in the case of a straight line". Thus he deduced the correct result from a ratio having a zero numerator, rather than from one with (non-zero) numerator d (both ratios have a denominator whose value tends to infinity and hence both have a common limiting value zero). Yet, while Kepler was right, according to my interpretation, in saying that the *latus rectum* is infinite (because the line through S perpendicular to PX in Fig. 11.14 is parallel to PY), he was again wrong in saying that the *latus rectum* coincides with the line PY itself. Taylor,[32] on the other hand, accepts without question Kepler's original statement that for the limiting hyperbola the focus

[29] This is actually true, relative to the centre, only for the graduation from a circle to a parabola.

[30] The *semi-latus rectum* l, known to Kepler as the *semichorda*, is the distance from the focus to the curve along the perpendicular to the common axis PX. In modern terms, $l = a(1 \sim e^2) = d(1 + e)$.

[31] *G.W.* II, p. 92.

[32] Taylor, *Geometry of Kepler and Newton*, p. 202, and also *Geometry of Conics*, p. lviii.

falls on the line itself, and then proceeds, logically, to criticize Kepler's determination of the value of the ratio, stating that in the line-pair the *sagitta* and the *latus rectum*

"are in the finite ratio $1/[2\ (1 + e)]$ and vanish together".

Here, in juxtaposition, we have two apparently inconsistent values for the limiting value of the ratio: finite: infinite and zero: zero.

My reason for rejecting Taylor's analysis on this point is not primarily because it leads to confused mathematics. His assumption as to Kepler's meaning requires that the sagitta d should decrease steadily to zero (unless a *jump* is to be permitted at the limit, contrary to the Principle of Continuity). It has already been pointed out that such is the behaviour of d in G.II, and the only suitable coherent system having this property that I have been able to invent occurs as A.I, illustrated later in this paper (Fig. 11.16). Clearly neither Fig. 11.13 nor Fig. 11.16 at all represent Kepler's intentions as he drew them in Fig. 11.14 (in the first case successive hyperbolas collapse instead of opening up, and in the second, the vertex is variable for all conics). Therefore, abandoning this possibility, I believe that my interpretation is correct and that Kepler originated the first system of conics with a single common focus: we may call this a *semi-confocal* system. Then the complete system is:

Circle → Ellipse → Parabola → Hyperbola → line pair along conjugate axis PY.

The question, mentioned earlier, of the absence of a cone from which the conics as shown in Kepler's diagram could be cut, appears to have been insoluble by the methods then available. The simplest construction that I can find was suggested to me by Mrs. Nancy Wilson, of the Department of Mathematics, Southampton University, after this paper was read,[33] using an adaptation of the theory of focal spheres;[34] by circumscribing a cone with one fixed generating line and a variable apex to a fixed sphere. It is an extension of G.I. While the concept was only introduced by Dandelin (*ca.* 1822), this solution did not necessarily exceed Kepler's capabilities, only his current interests. As such, it is historically irrelevant, but has been added for completeness.

In the course of preparing this paper for publication I have also realized that there is another system of conics G.IV (a geometrical system, though different from those already defined, because this has a fixed centre, not a fixed vertex) which is implied, though not synthesized, at the end of this same section.[35] Kepler concludes his study of this topic by setting out the string-construction for the ellipse, preceded by one for the hyperbola (using two strings)—which may or may not have been original. These

[33] October 22, 1971.
[34] Described in Taylor, *Geometry of Conics*, pp. 192–3 and pp. 196–205.
[35] *G.W.* II, C.IV, Section 4, p. 93.

are reproduced in Fig. 11.15 in which A and B are the fixed foci and the pencil passes through a ring at Q.

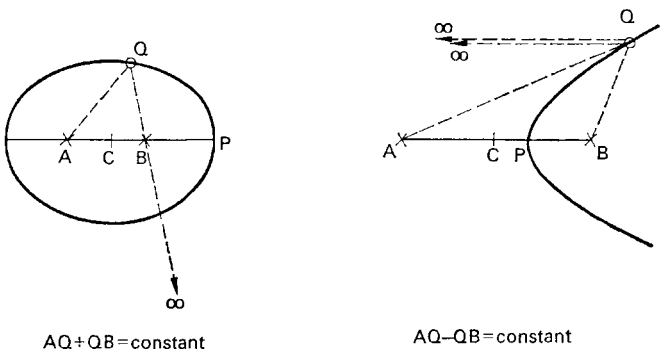


FIG. 11.15 (different scales: AB equal in both sketches).

These constructions are followed by one for the parabola, which Kepler evidently thought that he had invented, because he stated that he had worked it out by analogy, having long regretted the absence of a mechanical construction for this curve.

This connection which Kepler discovered, however, could, I suggest, have been made by a different route, if we suppose the basic elements—the two foci to which strings are attached—to be fixed. This fixes the centre and means that ae is constant—a being the major semi-axis as usual—and hence, since e increases steadily from zero to infinity, a will decrease steadily from infinity to zero; when $e = 1$, each vertex will coincide with a fixed focus. Then the intermediate "curve", when $e = 1$, is simply the x-axis: a degenerate parabola, which is derived as a limiting case from the constructions shown in Fig. 11.15. The part of the degenerate parabola between A and B is obtained from the ellipse-construction, when the single string (supposed to be potentially of infinite length) attached to A and B, is restricted so that $AQ + QB = AB$; while the part of the x-axis outside A and B comes from the hyperbola-construction, by letting $AQ - QB = AB$. The changeover from the ellipse to the hyperbola can be effected, when the pencil is at one of the foci (say B), by detaching the single string from B and picking up a second infinite string attached at B. (The first string remains attached at A.)

Taken to its limits, the resulting system is:

Infinite circle → Ellipse → Degenerate parabola → Hyperbola → Line-pair through centre.

It cannot usefully be illustrated, for obvious reasons. Kepler had no part in the composition of such a system, but since it is a direct deduction from his work, based solely on his principles of analogy/continuity reiterated constantly in this section, it therefore seems right to list it here.

Turning now to the applications of conics in astronomy, the systems available will be restricted by the fact that the one constant in the solar system is the Sun fixed at a common focus.[36] We will therefore define an *astronomical system of conics* as a semi-confocal system,[37] having one other property given, different for each. It will also be necessary to ignore Newton's mass-correction (both for simplicity, and because it did not appear in Kepler's work), so that all orbits are treated as if they are described by bodies of the same mass, and so arranged as to have a common line of apsides.

The system to be labelled A.I. is illustrated in Fig. 11.16. It is obtained by applying Kepler's Second Law in the restrictive sense that, not only is the areal velocity constant for any one orbit, but it is to be fixed at this value for all conics of the system.

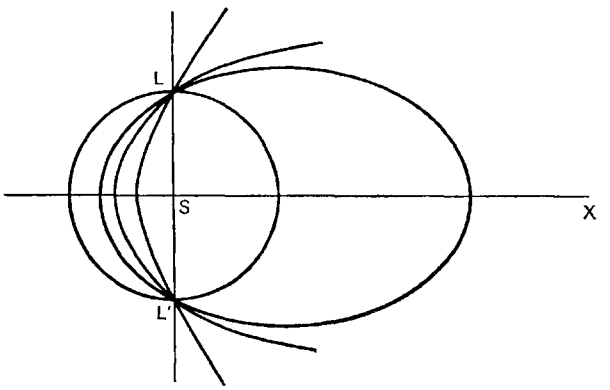


FIG. 11.16.

Then for all conics

$$r^2 . \frac{dv}{dt} = h \text{ which is given, constant, but } h^2 = \mu_0 l,$$

(where h represents the areal velocity, μ_0 is a gravitational constant,[38] and l is the *semi-latus* rectum as before) so this method gives the *latus rectum* constant—surprisingly enough, reverting to a geometrical condition—and the resulting system is:

Circle → Ellipse → Parabola → Hyperbola → Line-pair along *latus rectum.*

A different system, to be labelled A.II, is derived from the application of Kepler's Third Law, again restricted so that the period T has a fixed value for all conics of the system. Then the Third Law determines a, the major semi-axis, which must also be

[36] As introduced by Kepler in *G.W.* III, generalized in *G.W.* VII.

[37] Since we consider only conic systems, Kepler's First Law is implied throughout.

[38] $\mu_0 = G(M + m_0)$, where G is the constant of gravitation, M the mass of the Sun, and m_0 the mass of a planetary body, which we are assuming to be constant throughout.

fixed for all conics (in magnitude, but not in position). This is sketched in Fig. 11.17a in which the major axis of each conic is of length $2a$, where $PS = SA = AA' = a$.

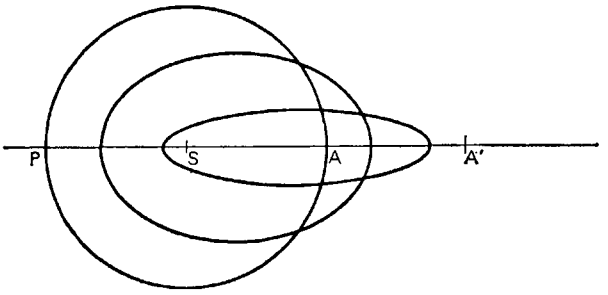


FIG. 11.17a.

Notice that the resulting system, which consists of:

Circle → Ellipse → Degenerate ellipse (on major axis)

is incomplete—though it can be extended geometrically to hyperbolas—because since T is fixed, only closed curves will satisfy this *kinematical* condition.

Another distinct system arises from Kepler's Third Law, by fixing the area at a common value for all conics of the system. This will be named A.III, and is shown in Fig. 11.17b:

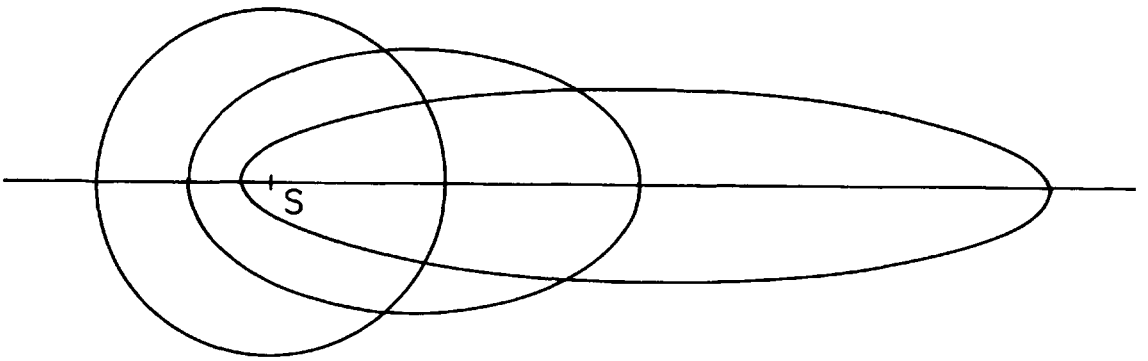


FIG. 11.17b.

The system produced is incomplete kinematically, but, like A.II, can be completed geometrically. It is:

Circle → Ellipse → Degenerate parabola.

The most interesting astronomical system is, however, one which can be derived —geometrically—merely by making an adjustment to every orbit (a little achronistically, to include comets) so that all the conics have a common vertex, the perihelion *P*. Then a system with a constant focus-vertex distance is obtained, as in Fig. 11.14: precisely Kepler's system G.III. There is certainly no record of Kepler ever having tried to systematize the ellipses occuring in his astronomical work, even though it was he who first identified them—to us—as a semi-confocal set. This situation exemplifies the irony of the lack of cross-reference—which occurs throughout Kepler's work in astronomy—which I described in the first paragraph as a fascinating problem. At the same time it provides some evidence for my opinion that Kepler's excursions into pure geometry were of more far-reaching importance than is generally realized, and that, while his physical speculation can frequently in my view be more properly described as metaphysical (as far as his astronomical work is concerned), his geometrical thinking provided a firm foundation for it, up to the limit of his mathematical ability. There is much evidence for this in the *Astronomia Nova*, which I intend to discuss elsewhere: at this time I wished only to draw attention to Kepler's original work in a comparatively little-known field, which seemed a suitable topic for this commemorative occasion.

Acknowledgement

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