



Optimizing membrane–substrate buckling to control surface deformation pattern

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ABSTRACT

Buckling of membrane–substrate structures can result in complex deformation patterns on their top surfaces. Traditionally, control over these patterns has relied on altering the material or thickness ratios of the system, which imposes considerable constraints on design flexibility. Inspired by studies on Winkler foundation optimization, this work presents a novel framework for tailoring membrane–substrate buckling modes via topology optimization. In this approach, the material distribution within the substrate is optimized to tune the mechanical response, while filtering and projection techniques are incorporated to enhance manufacturability. Post-processing of the optimized layout yields a physically realizable structure that preserves the desired mechanical behavior. A two-dimensional case study demonstrates that the optimized design successfully generates the prescribed deformation pattern without modifying material properties or thickness ratios, thus enabling precise control over buckling-induced surface morphologies. Experimental validation using fabricated prototypes subjected to compression further confirms that the observed buckling modes closely match the targeted patterns, underscoring the practical effectiveness of the proposed method.

1. Introduction

A membrane–substrate system typically refers to a bilayer structure composed of a thin, high-stiffness membrane as the upper layer, bonded to a thick, low-stiffness substrate as the lower layer (Bowden et al., 1998). The elastic buckling of this structure when subjected to biaxial compressive loading gives rise to various complex surface deformation patterns, which have already been exploited in several applications. For instance, buckling induced dimple patterns on membrane–substrate structures can be used to delay flow separation and reduce aerodynamic drag like a golf ball, while morphable skins that switch between smooth and rough states enable on-demand drag control (Lin et al., 2021; Terwagne et al., 2014). In soft robotics, thin stiff membranes on compliant robotic bodies can be programmed to wrinkle or dimple under actuation, creating friction-tunable skins and textured contact pads that support locomotion and gripping (Trinh et al., 2019; Suzuki and Ohzono, 2016). Moreover, programmable wrinkles in thin membranes bonded to soft substrates can also be used to tailor optical appearance, wetting and frictional properties of morphing surfaces (Tan et al., 2020; Wang and Xiao, 2017). Extensive research over the past decades has focused on analytically and experimentally establishing relationships among buckling modes, material properties, and the thickness ratio between the membrane and the substrate. These studies have enabled the

design of membrane–substrate structures tailored to specific applications by tuning the key parameters that govern their buckling-induced morphology (Audoly and Boudaoud, 2008; Li and Sun, 2022).

Specifically, in the classical theory of a membrane–substrate buckling, the primary buckling is mainly governed by the membrane–substrate modulus ratio and the membrane thickness. Linear stability analyses show that, under equi-biaxial compression, the critical strain scales as $\eta_c \sim (E_s/E_m)^{2/3}$, while the wrinkle wavelength scales as $\lambda \sim h(E_m/E_s)^{1/3}$, where E_m , h and E_s are the membrane modulus, membrane thickness and substrate modulus, respectively. At larger compressive strains and for different biaxial loading ratios, secondary instabilities of this classical cylindrical pattern can generate undulating, checkerboard, and hexagonal morphologies, so that the loading path and anisotropy select between competing patterns, while the characteristic wavelength remains essentially fixed by the global material and thickness ratios (Audoly and Boudaoud, 2008).

Beyond tuning global modulus and thickness, Zhang et al. achieved different buckling of membranes by designing micro-pillar and groove patterns on substrates (Zhang et al., 2019). Wu et al. controlled wrinkle wavelength and orientation by patterning curved cracks in the membrane, thereby redistributing stresses prior to buckling (Wu et al., 2014). Chan et al. exploited pre-patterned surface wrinkles for smart

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adhesion by tailoring the membrane–substrate interface and loading path (Chan et al., 2008). These examples have shown that surface buckling can be steered by prescribing geometric features in the membrane or at the interface. Graded substrates have also been used to tune the buckling of membrane–substrate systems. Studies in this area typically prescribe spatial modulus profiles to examine how these gradients affect the critical strain and the buckling modes (Chen et al., 2018; Zhao et al., 2017). More recent models of membranes on graded half-spaces show that modifying the gradient profile produces continuously varying buckling patterns and degrees of localization (Liu et al., 2024). These patterning and graded-substrate studies illustrate effective geometric and topological control strategies for surface buckling. Nonetheless, the design process remains essentially forward and faces restrictions when material parameters and thickness ratios are fixed, or when the desired surface morphologies are too complex to be encoded by a small number of variables.

In this work we instead pose a novel inverse mode-shape design problem: for a given material pair and thickness ratio, we seek a substrate topology that produces a target surface pattern with a prescribed wavelength/wavenumber at the onset of buckling. To achieve this, we propose a density-based topology optimization framework (Bendsoe and Sigmund, 2013; Jihong et al., 2021; Wu et al., 2021) that modifies the local geometry of the substrate and redistributes the substrate stiffness field to obtain the desired buckling mode. Topology optimization has been widely applied to buckling problems, but almost always with buckling treated as a constraint, where designs are optimized for stiffness or compliance while lower bounds on critical load factors obtained from eigenvalue analysis are imposed (Dahlberg et al., 2023; Dunning et al., 2016; Ferrari and Sigmund, 2019). More recently, eigenvector aggregate constraints have been introduced to bound the shape of buckling modes, still in the context of conventional load bearing components and with the primary goal of avoiding instability (Li and Kennedy, 2024). In contrast, we treat the buckling mode shape itself as the design objective and optimize the substrate topology so that the buckling pattern on the membrane surface matches the prescribed wavelength.

The paper is organized as follows. We begin by investigating the Winkler foundation (Bazant and Cedolin, 2010) in Section 2, a simple analogy to the membrane–substrate structure, to contextualize the feasibility of buckling mode optimization. The methodology for optimizing membrane–substrate buckling is introduced in Section 3, followed by numerical examples in Section 4 that demonstrate and discuss the validation of the optimization framework. Based on the results of Sections 4, 5 presents physical models of the optimized structures.

2. Optimization of the Winkler foundation buckling mode

Before introducing the optimization of membrane–substrate systems, we first consider a simple analogy: the Winkler foundation. Conceptually analogous to a stiff membrane attached to a soft substrate, the Winkler foundation model considers a stiff beam supported along its length by an elastic foundation (Fig. 1). An axial force P is applied at both ends of the beam, and c denotes the foundation stiffness (Winkler modulus) per unit length along the beam. The distributed reaction of the foundation on the beam is written as $p_r(x) = -c w(x)$, where $w(x)$ is the beam deflection along the x -axis. In the Winkler model, the reaction $p_r(x)$ at a point x depends only on the local deflection $w(x)$ and is independent of the deflections at neighboring points. To optimize the buckling mode of the Winkler foundation, the optimization problem can be defined as minimizing the error between a predefined target mode shape $w_{\text{target}}(x)$ and the actual buckling mode $w_{\text{actual}}(x, c(x))$, which can be formulated as

$$\min_{c(x)} J = \int_0^l [w_{\text{target}}(x) - w_{\text{actual}}(x, c(x))]^2 dx \quad \text{s.t. } c(x) \geq 0 \quad (1)$$

By modeling the foundation as a system of finite discretized springs, the Winkler foundation can generate a target buckling mode from

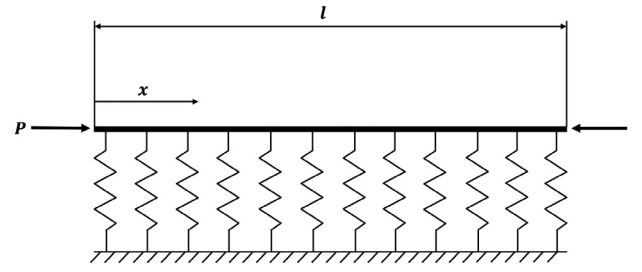


Fig. 1. Winkler foundation: an elastic support modeled as independent springs that act locally in proportion to displacement.

an appropriate stiffness distribution $c(x)$ along the beam. The actual buckling mode $w_{\text{actual}}(x, c(x))$ can be obtained by solving an eigenvalue problem in a finite element formulation expressed as

$$(\mathbf{K} + P_{\text{cr}} \mathbf{K}_G) \mathbf{w}_{\text{cr}} = \mathbf{0} \quad (2)$$

where P_{cr} and $\mathbf{w}_{\text{cr}}$ represent the critical load (eigenvalue) and the corresponding buckling mode (eigenvector), while $\mathbf{K}$ and $\mathbf{K}_G$ are the global stiffness matrix and global geometric stiffness matrix, respectively, whose local discrete forms K and K_G are given as

$$K = \int_0^l \left[EI \left(\frac{d^2 w_{\text{actual}}}{dx^2} \right)_i \left(\frac{d^2 w_{\text{actual}}}{dx^2} \right)_j + c(x) (w_{\text{actual}})_i (w_{\text{actual}})_j \right] \delta_{ij} dx \quad (3)$$

$$K_G = \int_0^l \left(\frac{dw_{\text{actual}}}{dx} \right)_i \left(\frac{dw_{\text{actual}}}{dx} \right)_j \delta_{ij} dx \quad (4)$$

where l is the length of the beam, EI is the bending stiffness of the beam section, and δ_{ij} is the Kronecker delta.

By assembling $\mathbf{K}$ and $\mathbf{K}_G$ from K and K_G , the eigenmode $\mathbf{w}_{\text{cr}}$ can be extracted which corresponds to w_{actual} . To address the buckling problem, the buckling mode w_{actual} is expressed by a Fourier series as

$$w_{\text{actual}}(x, c(x)) = \sum_{n=0}^{N-1} A_n \sin \left[\frac{(n+1)\pi x}{l} \right] \quad (5)$$

where the sinusoidal function naturally enforces boundary conditions with no displacement at both ends, and A_n are the N undetermined coefficients.

Since the eigenmode obtained from eigenvalue analysis has an arbitrary amplitude and varies across optimization iterations, the buckling mode w_{actual} is normalized using the L^2 -norm to make the optimization independent of the amplitude. Similarly, to maintain consistency, a predefined target buckling shape is normalized using the same method to facilitate comparison. In this example, the number of springs is set arbitrarily to 40, the bending stiffness EI of the beam is set to 1, and the length of the beam l is set to 10. The initial stiffness is set to be a constant value $c(x) = c_1 = 10$ along the beam. The initial normalized mode shape is shown in Fig. 2(a), exhibiting three sinusoidal waves, while Fig. 2(b) shows a predefined target shape after normalization, characterized by a symmetric sine wave with 1.5 waves. The matrix eigenvalue problem is solved with SciPy (Virtanen et al., 2020), and the optimization is conducted with the quasi-Newton interior-point method as implemented in Ipopt (Wächter and Biegler, 2006). The only constraint during optimization is the non-negativity of the spring stiffness, ensuring physical realizability.

As shown in Fig. 2(d), the final normalized buckling mode, represented by the orange dashed curve, closely matches the normalized target shape illustrated in green. The resulting stiffness distribution in Fig. 2(c) satisfies the non-negativity constraint and exhibits an overall symmetric pattern, as expected. The minor asymmetry observed is

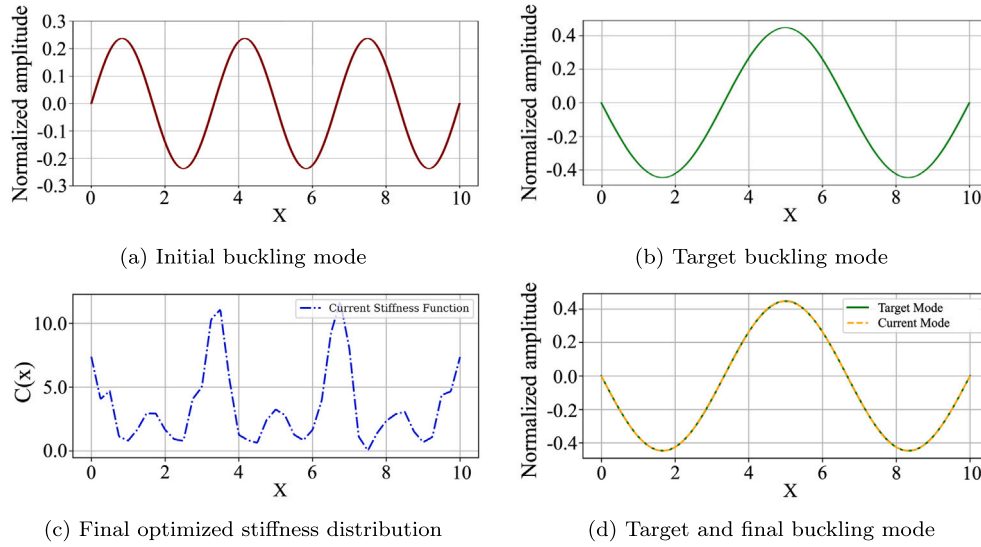


Fig. 2. Optimization of foundation stiffness to achieve a target buckling mode.

attributed to the finite convergence tolerance inherent in the nonlinear optimization process. Successive objective value difference falls below 0.77%, meeting the prescribed convergence criterion of 1%, thereby validating the feasibility of the proposed optimization approach and establishing a solid foundation for further investigations. In the final design, the largest deflections and peak bending of the buckled beam occur at positions where $c(x)$ is small, whereas regions with large $c(x)$ exhibit much smaller bending. This pattern indicates that the optimizer places soft springs at the intended wave crests and stiff springs elsewhere: low-stiffness springs provide little resistance and allow large-amplitude buckling, while high-stiffness springs resist lateral motion and suppress bending. Overall, this one-dimensional example illustrates how tailoring the stiffness distribution $c(x)$ can tune the buckling deformation along the beam.

3. Topology optimization of membrane–substrate system for buckling pattern

Having demonstrated the concept of buckling mode optimization in the Winkler foundation by tuning the foundation stiffness distribution, we now extend this idea to the membrane–substrate system. Similar to optimizing the stiffness distribution of the Winkler foundation, the stiffness in the membrane–substrate system can also vary across the entire substrate domain although now in two dimensions. A density-based topology optimization approach is employed, where the design variables are the nodal pseudo-densities ρ_i assigned to each node of the discretized mesh domain (Bendsoe and Sigmund, 2013); their collection forms the design vector ρ . These densities are used to modify the local stiffness within the substrate, until the optimized distribution enables the structure to achieve the desired buckling pattern.

The membrane–substrate system in this work is modeled as a 2D structure (Fig. 3(a)), 150 mm wide and 51 mm high, with 50 mm allocated to the substrate and 1 mm to the membrane. To facilitate experimental validation, the membrane and substrate are modeled using two silicone rubbers: Easycomposites AS-40 (Easy Composites Ltd., UK) for the membrane, with a Young's modulus of 1.69 MPa, and Ecoflex 00-10 (Smooth-On, Inc.) for the substrate, with a Young's modulus of 0.0076 MPa. The Poisson's ratio for both is set to 0.49 to simulate the incompressible behavior of the material.

3.1. Membrane–substrate buckling simulation

To obtain the buckling pattern on the top of the membrane–substrate structure, the bottom surface (Surface D) of the model in Fig. 3(a) is constrained to move only along the x -direction but restricted in the y -direction. Two opposite side surfaces (Surfaces A and B) are subjected to displacement-controlled compression, while the top surface (Surface C) is left free to deform and develop buckling patterns during compression.

A sequential process composed of an eigenvalue analysis and a geometrically nonlinear simulation is adopted to capture the accurate nonlinear large deformation of the structure. Specifically, an eigenvalue analysis is first performed to determine the minimum eigenvalue $\lambda_{\min}$ by solving the eigenvalue problem

$$\mathbf{K} + \lambda \mathbf{K}_G(\mathbf{u}_0) = \mathbf{0} \quad (6)$$

where λ is the eigenvalue of the model, and $\mathbf{K}$ and $\mathbf{K}_G(\mathbf{u}_0)$ are the linear elastic stiffness matrix and geometrical stiffness matrix, respectively. The displacement vector field $\mathbf{u}_0$ under an initial stress σ_0 can be solved by the governing equation of linear elasticity, whose variational form is

$$\int_{\Omega} \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx + \int_{\partial\Omega} \mathbf{T} \cdot \mathbf{v} ds \quad (7)$$

where $\sigma(\mathbf{u})$ is the stress tensor and $\mathbf{u}$ is the displacement vector field, $\epsilon(\mathbf{v})$ is the strain tensor, $\mathbf{v}$ is the vector test function, $\mathbf{f}$ is the body force per unit volume and $\mathbf{T}$ is the traction vector at the boundary. Once the displacement vector field $\mathbf{u}_0$ under a specific initial stress σ_0 is obtained by Eq. (7), $\mathbf{K}_G(\mathbf{u}_0)$ can be assembled, then a generalized eigenvalue problem $\mathbf{A}\mathbf{x} = \lambda\mathbf{B}\mathbf{x}$ with here $\mathbf{A} = \mathbf{K}$ and $\mathbf{B} = -\mathbf{K}_G(\mathbf{u}_0)$ is solved to obtain the minimum eigenvalue $\lambda_{\min}$. To capture a more realistic buckling mode, a geometrically nonlinear simulation is conducted. Since the objective of this study is to achieve a desired buckling mode with a target wavenumber/wavelength at the onset of instability rather than the post-buckling deformation, the applied compression is set just above the first eigenvalue $\lambda_{\min}$ (approximately +3%) to trigger the initial buckling while avoiding post-buckling evolution. For a materially linear but geometrically nonlinear model, the variational form is given as

$$\int_{\Omega} \sigma(\mathbf{u}) : \epsilon(\mathbf{v}) dx + \int_{\Omega} \sigma(\mathbf{u}) : (\nabla \mathbf{u}^T \cdot \nabla \mathbf{v}) dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} dx + \int_{\partial\Omega} \mathbf{T} \cdot \mathbf{v} ds \quad (8)$$

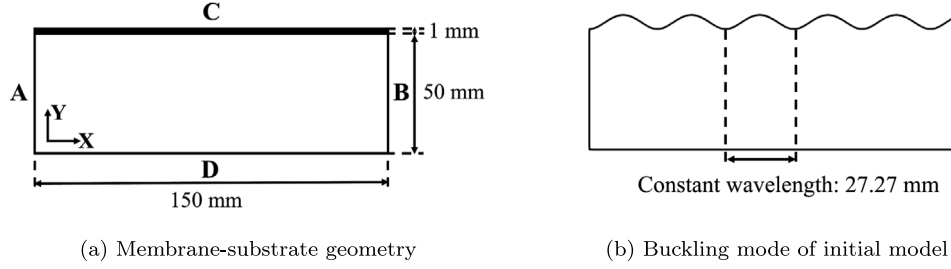


Fig. 3. 2D membrane-substrate model geometry and initial buckling mode.

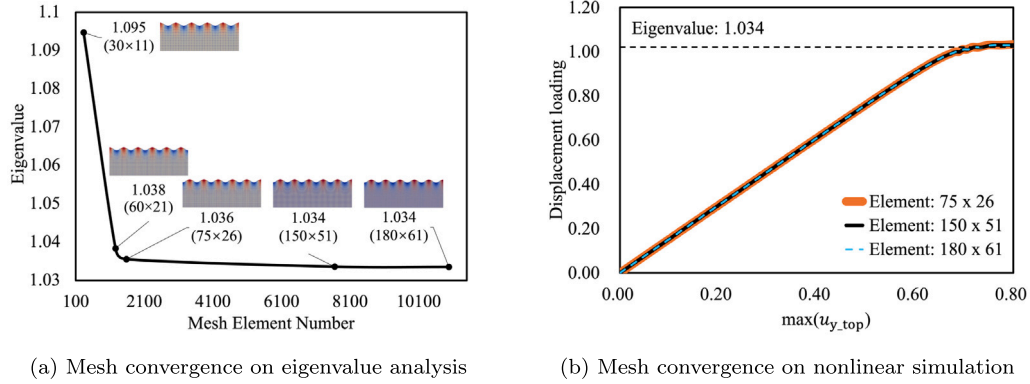


Fig. 4. Mesh convergence study and simulation validation.

For the defined 2D model, the finite element implementation is carried out using the Python library Firedrake, which is used to solve the weak form of the governing partial differential equations and is widely employed in computational research (Rathgeber et al., 2016). Linear systems are solved with PETSc (Balay et al., 2019), eigenvalue problems with slepc4py/SLEPc (Hernandez et al., 2005), and nonlinear problems with an adaptive Newton method featuring line search and step-length control based on residual reduction, with solver tolerances set to 1×10^{-8} . All computations are performed on an Apple Silicon M1 Pro, parallelized across 8 CPU cores via MPI in PETSc.

To assess the physical fidelity of the implementation, the simulations are compared with analytical solutions (Audoly and Boudaoud, 2008) for the critical buckling load η_{critical} and wavelength N of membrane-substrate buckling, which are given by

$$\eta_{\text{critical}} = W \left[\frac{3(1 - \nu_s)(1 - \nu_m^2)E_s}{2(1 + \nu_s)(3 - 4\nu_s)E_m} \right]^{\frac{2}{3}} \quad (9)$$

$$N = \frac{W}{2\pi h} \left[\frac{12(1 - \nu_s)(1 - \nu_m^2)E_s}{(1 + \nu_s)(3 - 4\nu_s)E_m} \right]^{\frac{1}{3}} \quad (10)$$

where W and h are the width and membrane thickness of the model, and E_m , E_s , ν_m , and ν_s denote the Young's modulus and Poisson's ratio of the membrane and substrate, respectively. Based on the simulated geometry and material parameters, the analytical solutions for the critical buckling load and the wavenumber are $\eta_{\text{critical}} = 1.065$ and $N = 5.684$, respectively.

The model uses a 2D plane formulation discretized with second-order quadrilateral elements on a structured rectangular grid. Mesh convergence was examined on five grids, as shown in Fig. 4(a). The smallest eigenvalue decreased monotonically with refinement and approached 1.034, while the associated wavenumber stabilized at 5.5; the resulting deviations from the analytical predictions are 2.91% for the critical load and 3.23% for the wavenumber, indicating adequate accuracy of the eigenvalue solver. For the nonlinear response, the adaptive Newton solver converged robustly on the three finest meshes

(Fig. 4(b)): buckling initiated near the linear critical load, the load-displacement curves were essentially superposed, and the post-buckling mode matched the 5.5-wave linear eigenmode, confirming the reliability of the nonlinear solver. A 150×51 grid was therefore chosen as a compromise between accuracy and computational cost for the subsequent simulations and optimizations. Although the chosen material is silicone rubber, the strains at buckling onset in the present configurations remain small. A comparative study with a modified Neo-Hookean hyperelastic formulation shows that, in this near-onset regime, material nonlinearity has only a minor influence on the critical load and the buckling mode shape. Therefore, the chosen linear elastic model with geometric nonlinearity is appropriate for the present study, as it is sufficient to capture the relevant buckling response while keeping the constitutive description simple and computationally efficient.

As shown in Fig. 3(b). The initial buckling mode is characterized by a sinusoidal wave with a constant wavelength of 27.27 mm (5.5 waves). This original buckling pattern is optimized into another sinusoidal wave with a predefined constant wavelength through the procedure introduced later, in order to achieve a target buckling mode.

3.2. Material interpolation

To integrate the design variables into the buckling simulation, the elemental Young's modulus E_i is interpolated using the SIMP method (Bendsøe and Sigmund, 2013) by

$$E_i = \rho_i^p (E_s - E_v) + E_v, \quad (11)$$

where ρ_i is the pseudo-density of the i th element; E_s denotes the Young's modulus of the solid substrate material, while $E_v = 10^{-5}$ Pa represents that of the void. This value is chosen to be small but nonzero to prevent numerical instabilities (i.e., the occurrence of a singular stiffness matrix). The penalty parameter p is set to 3 (Bendsøe and Sigmund, 1999). In this formulation, the pseudo-density field is denoted by $\rho = \rho(x, y)$ and, through the SIMP interpolation, directly defines the spatial distribution of substrate stiffness $E = E(x, y)$. In the discrete

optimization problem, the nodal values $\{\rho_i\}$ are collected in the design vector ρ .

In this work, a materially linear model that accounts for geometric stiffness is adopted. The mechanical response is governed by

$$\lambda_i = \frac{E_i \nu}{(1 - \nu)(1 - 2\nu)}, \quad (12)$$

$$\mu_i = \frac{E_i}{2(1 + \nu)}, \quad (13)$$

where λ_i and μ_i are the interpolated Lamé parameters used to compute the stress field $\sigma(\mathbf{u})$ in Eqs. (7) and (8).

To address the checkerboard effect, a Helmholtz filter (Lazarov and Sigmund, 2011) is employed to smooth the density distribution by applying a Helmholtz-type PDE to the design variable ρ , given by

$$\bar{\rho} - l^2 \nabla^2 \bar{\rho} = \rho, \quad (14)$$

where $\bar{\rho}$ is the filtered and smoothed pseudo-density field and the length scale l is set to 1.25 times the discretized mesh size to ensure a mesh-independent design. Achieving a binary material distribution is essential to ensure physical feasibility. To further alleviate the gray transition naturally introduced by the Helmholtz filtering process, the intermediate densities are processed by a projection filter to obtain a 'black and white' design (i.e., a binary density distribution containing predominantly $\rho = 0$ or $\rho = 1$). A threshold projection function is proposed to push the densities towards the bounds of the design variable, 0 and 1, respectively (Wang et al., 2011), defined as

$$\bar{\rho} = \frac{\tanh(\beta\eta) + \tanh(\beta(\bar{\rho} - \eta))}{\tanh(\beta\eta) + \tanh(\beta(1 - \eta))}, \quad (15)$$

where η is the threshold parameter set to 0.5 in this work, and $\bar{\rho}$ is the projected design variable. The 'black and white' distribution is achieved by projecting filtered values $\bar{\rho}$ below 0.5 to 0, and values above 0.5 to 1. The parameter β controls the sharpness of the projection; a larger β encourages a more binary design. In the discrete setting, the corresponding nodal vectors are denoted by $\bar{\rho}$ and $\bar{\rho}$.

Therefore, at each optimization iteration, the buckling simulation is performed on the model with an updated stiffness matrix to obtain the top surface deformation $\mathbf{u}(\bar{\rho})$.

3.3. Target pattern and objective problem formulation

The displacement vector field is denoted by $\mathbf{u} = (u_x, u_y)$. Since only the vertical displacement on the top surface is controlled, we denote the y -component of the actual and target top-surface displacements by $u_{\text{actual},y}$ and $u_{\text{target},y}$, respectively.

To optimize the top surface deformation $u_{\text{actual},y}(\bar{\rho})$, the target pattern $u_{\text{target},y}$ is defined as a sinusoidal wave with a predefined wavelength. This is achieved by interpolating trigonometric functions onto the mesh nodes along the x -direction of the model's upper surface. The left end of the top surface is considered the origin point, and the movement of mesh nodes in the y -direction after interpolation constitutes the target pattern. The interpolation formula is expressed as

$$u_{\text{target},y}(x) = -A \cos\left(\frac{2\pi x}{L}\right), \quad (16)$$

where A is the amplitude of the wave, x is the coordinate of the mesh nodes in the x -direction, and L is the controllable wavelength. Thus, the objective functional J can be formulated as the root mean square error between $u_{\text{actual},y}(\bar{\rho})$ and $u_{\text{target},y}$, which is given by

$$\min_{\rho} J = \sqrt{\frac{1}{n} \sum_{i=1}^n [u_{\text{actual},y}(\bar{\rho})_i - u_{\text{target},y,i}]^2}. \quad (17)$$

Here, $u_{\text{actual},y}(\bar{\rho})_i$ and $u_{\text{target},y,i}$ denote the y -displacement at the i th node on the top surface for the actual and target patterns, respectively.

Table 1

Continuation parameters used in the optimization process.

Parameter	Initial value	Update rule	Final value
β	1	$\beta_{i+1} = \beta_i + 1$	5
ϕ_0	0.5	$\phi_{0,i+1} = \phi_{0,i} - 0.1$	0.1

To prevent the presence of unrealistic intermediate density in the design, a supplementary constraint is applied by constraining the grayness metric ϕ , according to

$$\phi = \int_{\Omega} [4\bar{\rho}(1 - \bar{\rho})] d\Omega \leq \phi_0, \quad (18)$$

where ϕ_0 places an upper bound on the level of grayness (Thillaithevan et al., 2024). The design variable field ρ is constrained to lie between ρ_0 and 1, where a small but nonzero lower bound $\rho_0 = 1 \times 10^{-6}$ is used. Owing to the E_v term in Eq. (11), the actual void stiffness is 10^{-5} (for $p = 3$) and as such ρ_0 is not strictly required. Therefore, the optimization problem in discrete form can be formulated as

$$\min_{\rho} J(\bar{\rho}, u_{\text{actual},y}(\bar{\rho})) \quad (19)$$

$$\text{s.t. } R(\bar{\rho}, \mathbf{u}(\bar{\rho}); \nu) = 0 \quad (20)$$

$$\phi \leq \phi_0 \quad (21)$$

$$\rho_0 \leq \rho_i \leq 1, \quad i = 1, \dots, N_d, \quad (22)$$

where R is the residual of the variational form in Eq. (8) and N_d is the number of design variables.

3.4. Sensitivity analysis

As a gradient-based optimization algorithm is utilized in this work, the gradient information of the objective with respect to each design variable is required, which is a process known as sensitivity analysis. Specifically, in our case, the chain rule is utilized to obtain the gradient of the objective function with respect to the design vector ρ as

$$\frac{dJ}{d\rho} = \frac{\partial J}{\partial \bar{\rho}} \cdot \frac{d\bar{\rho}}{d\bar{\rho}} \cdot \frac{d\bar{\rho}}{d\rho}, \quad (23)$$

where $dJ/d\rho$ denotes the objective sensitivity with respect to the design variables, and $\bar{\rho}$ and $\bar{\rho}$ are the filtered and projected density vectors, respectively.

3.5. Material distribution initialization and optimization procedure

To ensure that the starting buckling mode in the optimization always corresponds to the original one, the design variables are initialized with a constant value $\rho_{\text{init}} = 1$, i.e., $\rho_i = 1$ for all i . Moreover, a continuation scheme (Thillaithevan et al., 2024) is employed during the optimization process for both the threshold projection parameter β and the upper bound on the level of grayness ϕ_0 , as shown in Table 1. The initial values of β and ϕ_0 are set to 1 and 0.3, respectively. In subsequent iterations, they are incrementally increased by a fixed amount at regular intervals. This scheme facilitates the smooth convergence of the objective function while gradually guiding the design towards a binary material distribution, ultimately resulting in a clear black-and-white configuration. The complete workflow is shown in Fig. 5.

4. Numerical examples of optimizing membrane-substrate buckling pattern

To validate the feasibility of the buckling mode optimization framework, two different optimization objectives are considered: one targeting a pattern with a wavelength of 50 mm (3 waves), and the other targeting a pattern with a wavelength of 75 mm (2 waves), as shown

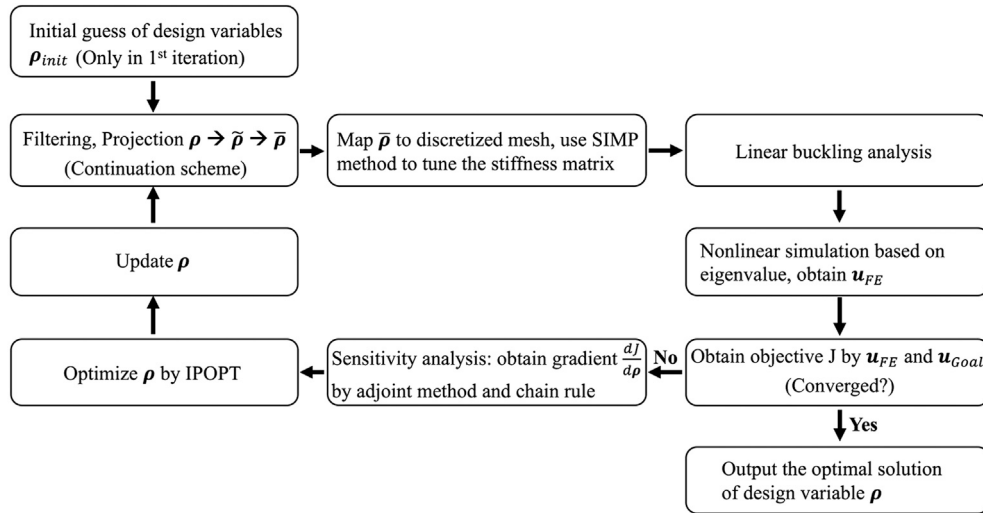
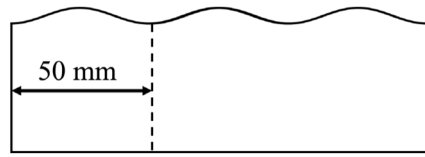
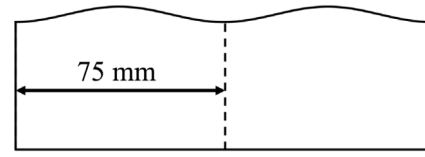


Fig. 5. Workflow for topology optimization of membrane-substrate buckling mode.

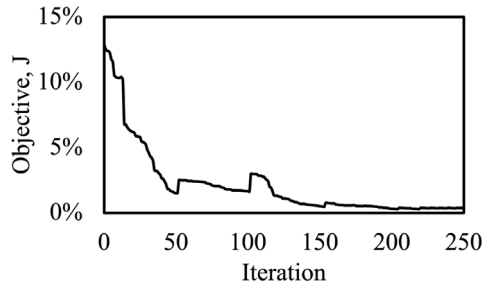


(a) Target pattern with 50 mm wavelength

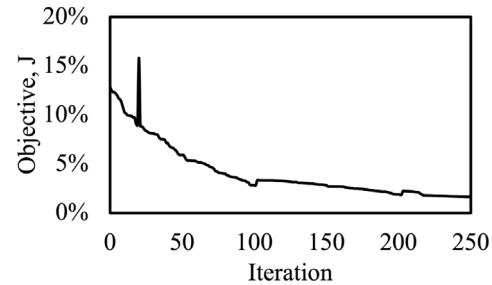


(b) Target pattern with 75 mm wavelength

Fig. 6. Target patterns of the 2D membrane-substrate model with different goal wavelengths.



(a) Objective convergence for the 50 mm wavelength (3-wave) case



(b) Objective convergence for the 75 mm wavelength (2-wave) case

Fig. 7. Convergence of the objective in both optimization cases.

in Figs. 6(a) and 6(b). The final optimized designs are post-processed for practical validation, with the discussion focusing on verifying the buckling simulation of the post-processed structures. The finite element library Firedrake (Rathgeber et al., 2016) is used for the buckling simulations, while pyadjoint (Mitsch et al., 2019) provides an efficient means to compute sensitivities through automatic differentiation. The gradient-based interior-point algorithm Ipopt (Wächter and Biegler, 2006) is employed to solve the optimization problems.

4.1. Optimization results and post-processing

The convergence histories of the objective function for both cases are presented in Fig. 7, demonstrating the robustness and effectiveness of the optimization process. The root mean square errors are reduced to 0.38% for the 50 mm case and 1.61% for the 75 mm case. These results confirm that the optimized structures effectively achieve the desired buckling modes with high accuracy.

The final material distributions for both cases exhibit a distinct binary characteristic, with red regions corresponding to fully solid material and blue regions denoting voids. In the 75 mm wavelength case, an isolated solid region fully enclosed by voids is observed, which is a common issue when objectives other than structural stiffness are considered and can be solved by applying physics-based or geometric connectivity constraints during optimization (Cool et al., 2025). To simplify the problem, no additional constraints, not even a volume constraint, are applied here. As a result, isolated solid regions emerge and persist throughout the optimization. The surrounding voids effectively disconnect these regions from load paths, preventing compressive stress transfer and rendering them mechanically inactive. Such regions can be safely removed during post-processing without affecting the structural response. In both the 3-wave and 2-wave designs, large-amplitude bulges of the membrane at the onset of buckling occur above void regions of the substrate, whereas regions above solid substrate show much smaller displacements. The overall stiffness-redistribution

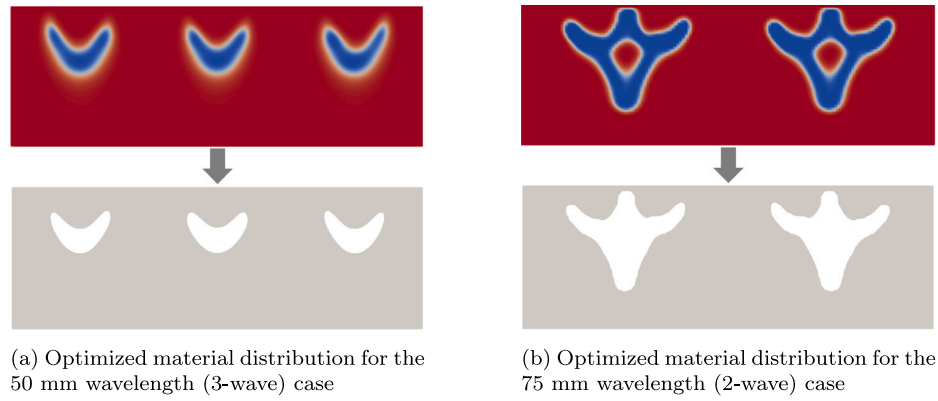


Fig. 8. Optimized material distributions and post-processing via thresholding for both cases.

trend is similar to that in the Winkler foundation example: soft, void-like regions allow and amplify surface buckling, while solid regions restrict deformation. The optimizer automatically discovers complex void geometries: it first places voids in locations where large membrane deflections are needed and then gradually adjusts their boundaries to refine the match to the target profile. The underlying mechanism is that topology optimization modifies the distribution of solid and void regions in the substrate, creating a spatially varying stiffness field that in turn determines where large and small buckling deformations occur on the membrane surface.

In the optimized designs, regions with extremely low density (below 0.01) are classified as voids and have negligible influence on the structural response. These regions, including the isolated solid area identified in the 75 mm case, are removed during post-processing, as illustrated in Fig. 8. To further validate the effectiveness of the optimized structures, buckling simulations are performed on the post-processed designs.

4.2. Nonlinear simulation of optimized structures

Based on the previously established workflow for obtaining buckling modes, eigenvalue analyses are first conducted for both cases. As shown in Fig. 9(a), none of the mode shapes in the 3-wave case match the target pattern; however, multiple modes exhibit repeated occurrences of individual features of the desired shape. This is due to the high degree of symmetry in the optimized structure, where repeated local voids can lead to spatially separated modes with similar eigenvalues—a common phenomenon in structures with repetitive features (Systèmes, 2024). A similar situation occurs in the 2-wave case (Fig. 9(b)), although the target shape appears as the second mode in this case.

To capture more realistic responses at the onset of the buckling together with the post-buckling, nonlinear simulations were then carried out based on the obtained eigenvalues (0.8113 for the 3-wave case and 0.2036 for the 2-wave case). As shown in Figs. 9(c) and 9(e), the evolution of the deformation in both cases exhibits a sudden and pronounced change in the vicinity of the first eigenvalue, corresponding to the sharp transition in the curves and indicating the onset of buckling. Unlike linear eigenmodes, which appear sequentially with distinct symmetry or periodicity, nonlinear modes obtained by loading the structure incrementally emerge simultaneously during compression and provide a more realistic description of the buckling response. Additional mesh-convergence checks on the optimized voided substrates confirmed that the load–displacement response and buckling mode shape are insensitive to further refinement, indicating that the adopted discretization is already in the converged regime.

As shown in Figs. 9(c) and 9(e), the intermediate deformations are closest to the targets. A slight further increase in compression then causes the deformation to deviate rapidly from target profile and

generates crests that bulge upwards, as seen in the final deformations, which are also expected to be observed in the prototype tests. Tracking the discrepancy as $RMSE(u_{actual,y}, u_{target,y})$ between the actual and the target deformation as a function of compression shows that the error first decreases and then increases, reaching a minimum near the eigenvalue, where the deformation most closely matches the target mode, and then growing as the system evolves into the post-buckling regime (as shown in Figs. 9(d) and 9(f)), which is not within the scope of this study.

This behavior is consistent with the optimization objective, which focuses on a narrow load interval in the vicinity of the critical eigenvalue; because the nonlinear evolution does not occur exactly at the linear eigenvalue and an initial imperfection is introduced through the thresholding procedure, achieving the minimum discrepancy near the eigenvalue (onset of buckling) which satisfies the requirements of the present study. Nevertheless, mode selection in terms of wavenumber and wavelength remains correct: in the 3-wave configuration the structure develops three waves with a wavelength of approximately 50 mm, whereas in the 2-wave configuration it develops two waves with a wavelength of approximately 75 mm, in both cases matching the prescribed optimization targets.

5. Physical model validation

The previous section demonstrated the optimization framework on a 2D membrane–substrate system and converted the optimized solution into a manufacturable structure. This section presents prototype tests of these designs. Three silicone samples (the original model, a three wave optimized model, and a two wave optimized model) were fabricated with the same geometry and material properties as in the simulations, and were subjected to compression tests to compare their buckling modes. The specimens were mounted on a wooden base between two lateral compression blocks (Figs. 10(a)–10(c)). The blocks were driven towards each other by a hand-operated screw, so that turning the threaded rod compressed the specimen quasi-statically. With the given screw pitch, the imposed compression increased in steps of about $\Delta u \approx 0.1$ mm per loading step. The target compressive displacements were chosen as 1.1 mm for the original specimen, 0.9 mm for the 3-wave optimized specimen, and 0.3 mm for the 2-wave optimized specimen, reaching slightly beyond the eigenvalues while remaining compatible with the 0.1 mm step size. To approximate the boundary conditions used in the simulations, a thin layer of lubricant was applied to the wooden base, the compression blocks and the specimen surfaces.

The comparisons between the prototype tests and simulations are shown in Figs. 10(d)–10(f) and 10(g)–10(i). In the simulations, the un-optimized specimen consists of 5.5 waves, while the optimized designs consist of three waves and two waves, with corresponding wavelengths of 26.9 mm, 49.4 mm, and 74.8 mm, respectively. The wavelength is

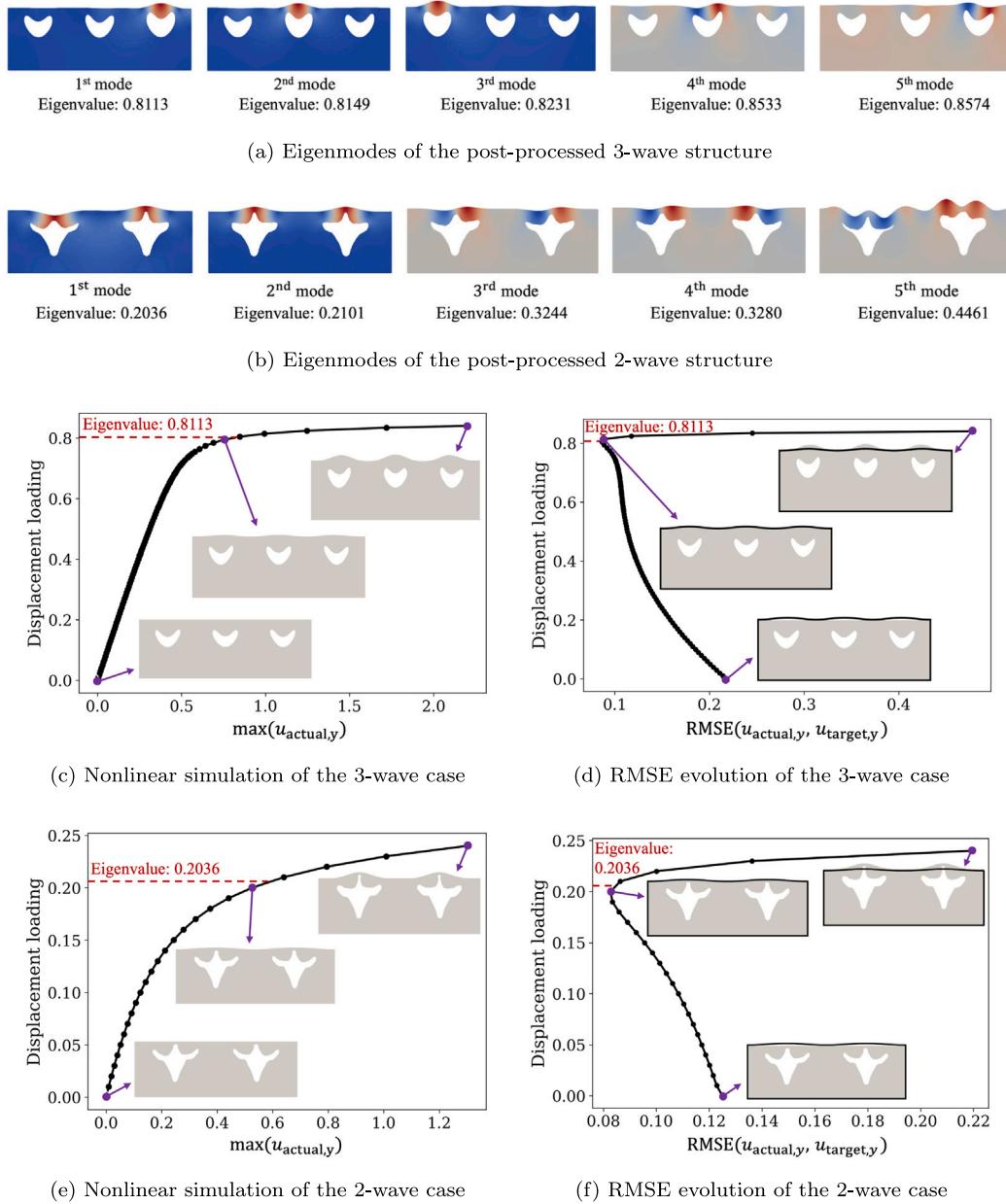


Fig. 9. Eigenvalue analysis and nonlinear simulations of both post-processed structures.

quantified as the horizontal distance between two neighboring minima on the top-surface profile in the test. For the unoptimized specimen (Figs. 10(d) and 10(g)), the top surface exhibits 5.5 waves, but with noticeably non-uniform wavelengths: four representative waves yield an average wavelength of about 19.7 mm, which differs significantly from 26.9 mm, indicating that, without internal voids to guide the deformation, small geometric or boundary-condition deviations can steer the buckling response away from the idealized mode. By contrast, in the 3-wave and 2-wave optimized specimens (Figs. 10(e)–10(f) and 10(h)–10(i)), the internal voids effectively guide the structure towards specific modes, yielding three and two distinct waves, respectively, matching the prescribed wave numbers. The measured wavelengths are 49.6 mm and 74.7 mm for the 3-wave and 2-wave specimens, both within about 0.5% of the corresponding simulations. This level of agreement in wave number and wavelength demonstrates that the optimized substrate topologies robustly and quantitatively realize the targeted buckling modes.

Given that the hand-operated loading imposes displacements in discrete steps of order 0.1 mm, the deformations inevitably lie slightly beyond the ideal onset states in the tests. In this regime, nonlinear simulations (Figs. 9(d) and 9(f)) show that the near-onset sinusoidal modes exist only within a very narrow load window and rapidly evolve into post-buckling patterns once the applied compression slightly exceeds the first eigenvalue in each case. The tests are effectively showing an early post-buckling deformation that is fully consistent with the simulations. Particularly in the 2-wave the profile consists of two locally amplified crests rather than a smooth sinusoid, even though the wavenumber remains correct.

Amplitude differences between the simulations and the physical specimens are also observed and can be attributed to geometric imperfections unavoidably introduced by the transition zone between void and solid when thresholding the optimized design to a manufacturable structure. A sensitivity study with three threshold levels (0.01, 0.008 and 0.006, see Fig. 10(j)) shows that changing the threshold modifies

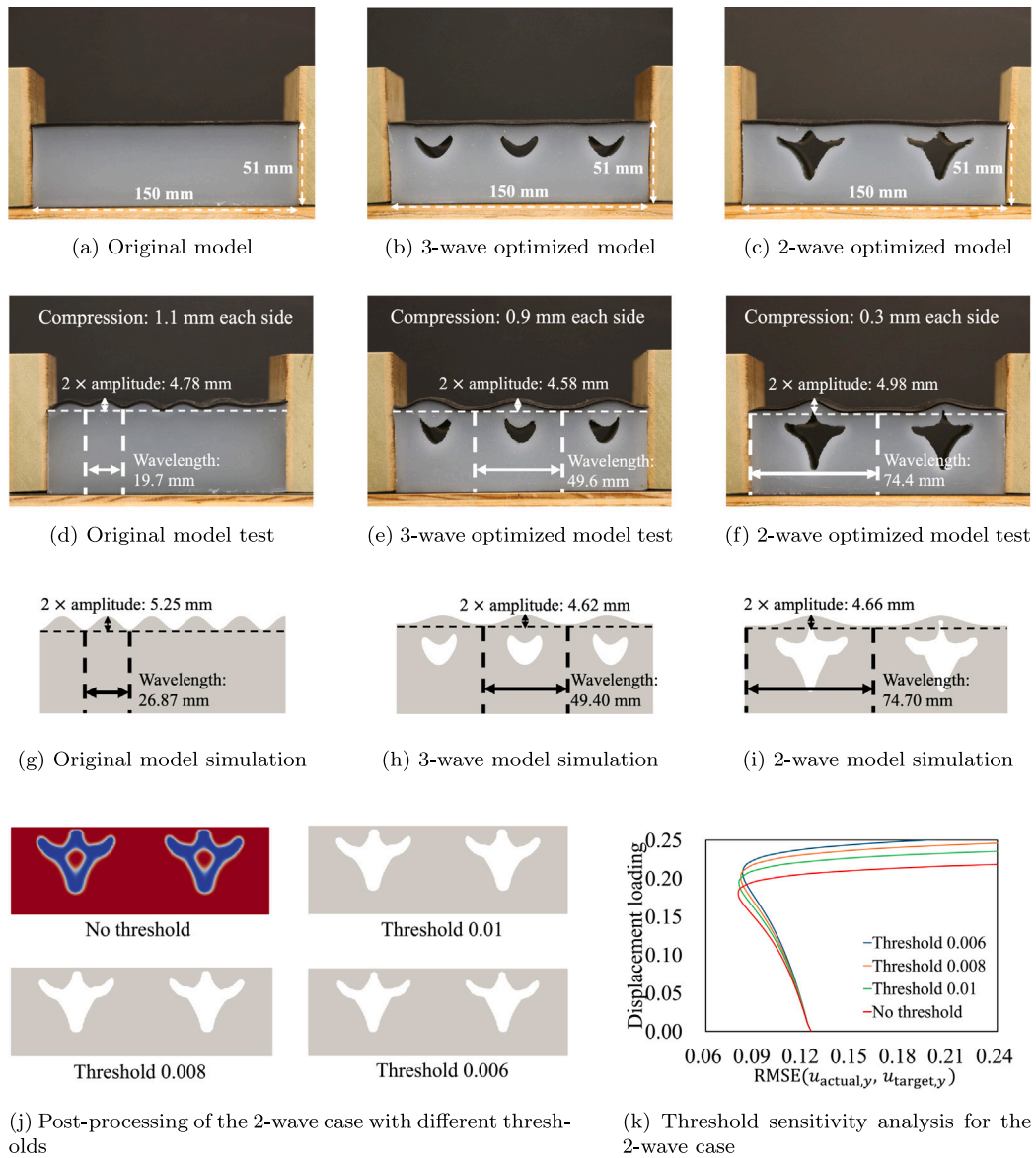


Fig. 10. Comparison between simulations and prototype mechanical tests for the 3-wave and 2-wave cases, including imperfection assessment via threshold sensitivity analysis.

the void area and the character of these imperfections, leading to variations in the amplitude of the local profile (Fig. 10(k)), while leaving the selected wavenumber and wavelength unchanged. In all cases, a slight increase in compression beyond onset quickly produces the same two-crest signature of early post-buckling as observed in the tests. Thus, the remaining discrepancies in amplitude and detailed shape are naturally attributed to post-buckling evolution and threshold-induced imperfections, whereas control of the buckling mode itself is robust.

6. Conclusion

Building on the study of buckling mode optimization in Winkler foundation, this work proposes a novel topology optimization framework that directly controls surface buckling patterns by optimal tailoring of the internal geometry of the substrate. Unlike traditional approaches that rely solely on adjusting material properties or thickness ratios, the method leverages a gradient-based algorithm to optimize the topological layout of a membrane-substrate system, enabling precise

control over its buckling shape. Two distinct target shapes were successfully achieved through the design. The optimized structures, after minimal post-processing, are highly manufacturable and consistently reproduce the desired buckling modes. Compression tests on prototypes show good agreement with the simulations, indicating that the framework is practically applicable and robust.

Mechanically, the optimal density distributions illustrate how topology optimization can be used to control membrane-substrate buckling. By modifying the local geometry of the substrate, the method optimizes the distribution of solid and void regions and thereby creates an effective stiffness field with soft and stiff zones, so that large buckling deformations form in low-stiffness areas, whereas stiff regions strongly restrict bending. In this sense, this optimization framework significantly expands the design space for buckling-driven applications, such as drag-reduction skins, morphing surfaces, and soft robotic systems, where performance depends strongly on controlled surface patterning.

Future work will extend the framework to three dimensional structures, where uniaxial or biaxial compression can induce more intricate, spatially varying buckling patterns. By applying loads sequentially in different directions, path dependent instabilities could be exploited to

obtain programmable and reconfigurable buckling responses, further increasing the versatility of buckling control in functional and adaptive material systems.

CRedit authorship contribution statement

Wei Zhou: Writing – review & editing, Writing – original draft, Visualization, Validation, Methodology, Investigation, Formal analysis. **Robert Hewson:** Writing – review & editing, Supervision, Project administration, Methodology, Conceptualization. **Matthew Santer:** Writing – review & editing, Supervision, Project administration, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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