

CONVEX PROGRAMMING FOR OPTIMAL CONTROL:  
ALGORITHMS AND CONVERGENCE RATES

by

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ABSTRACT

Many control problems reduce to the minimization of a scalar valued function  $f$  over a convex and compact subset  $Q$  of a Hilbert space.

The thesis is concerned with algorithms for solving such minimization problems for cases ( of practical importance ) for which some structural properties of  $f$  (e.g. bounds on the Hessian of  $f$  ) are known.

An existing feasible direction algorithm obtains search directions by minimizing exactly lower approximations  $f^S$  to  $f$  on  $Q$ . Global convergence results exist for it. However sometimes the minimization of  $f^S$  on  $Q$  cannot be done exactly. The effects on convergence rate of approximately minimizing  $f^S$  on  $Q$  are investigated and new results derived. These results show that iterating on the subproblem  $\min \{ f^S(z) : z \in Q \}$  improves the bounds for iteration on the main problem  $\min \{ f(z) : z \in Q \}$ . When  $f^S$  is strictly convex the convergence results involve the ratio between second and first order effects for  $f^S$ . Properties of the ratio are stated and proved and a mixed stopping criterion for the overall algorithm derived.

Applications to Classical Optimal Control Problems are considered.

Then another type of minimization algorithm is considered for the case where  $Q$  is a subset of  $R^n$ . At each iteration, it minimizes  $f$  on a subset of  $Q$ , actually a convex polytope approximating  $Q$ , and then it extends the subset. Application to a Non-Classical Optimal Control Problem is considered.

Algorithms where  $f^S$  is either an upper approximation to  $f$  or a lower approximation derived from an upper approximation are investigated and convergence rates obtained.

The problem of minimizing a convex function over an unbounded cone is considered. Algorithms to solve this problem are studied. A convergence rate depending on the ratio between the smallest and the largest eigenvalues ( or appropriate bounds to them ) of the Hessian of  $f$ , is derived.

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To Rosa and Marco

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## CHAPTER 1

### INTRODUCTION AND OUTLINE OF CONTRIBUTIONS

#### 1.1 - INTRODUCTION

A non-linear programming problem can basically be described by:

$$(1.1) \quad \min\{ v(x) : x \in Q \}$$

where  $v$  is a non-linear functional and  $Q$  is the set of points which satisfy all the constraints. In general, the optimum of a non-linear programming problem will not be at an extreme point of the constraint set (unlike for the linear programming case) and may not even be on the boundary. Also, the problem may have local minima distinct from the global minimum. These properties are direct consequences of the non-linearity. Nevertheless there is an important type of potentially non-linear problem for which all local minima are global minima: the convex programming problem. This thesis is about solving such problems.

Convex programming problems are characterized by an objective function  $v$  which is convex and by a constraint

set  $Q$  that is also convex (and compact in order to ensure existence of solutions).

(1.2) DEFINITION[1]: A set of points is said to be convex if, given any two points in the set, the line segment joining them is also in the set.

(1.3) DEFINITION[1]: A function  $v$  is said to be convex if the line segment drawn between any two points in the epigraph of the function never lies below the epigraph.

An important result in convex programming is the following.

(1.4) THEOREM[2]: Any local minimizer of a convex programming problem is a global minimizer.

This is important because it means one can stop optimizing once any local minimizer has been found, as it is automatically a global minimizer.

Note that the problem may have a number of points  $x \in Q$  at which the globally minimal value of  $v$  is achieved. The set of all such points is actually convex.

Although convexity is desirable, many real world problems turn out to be non-convex. In addition, there is no simple way to test a non-linear function for convexity. However the importance of studying convex programming is not diminished since:

- (i) many significant problems of practical interest are actually convex.
- (ii) many algorithms for non-convex problems require the solution of a convex sub-problem at each iteration.
- (iii) theoretical analysis is relatively easy for convex problems, and yields deep understanding of what might be possible for non-convex problems, perhaps in a weakened form. Convexity thus plays a role much the same as that of linearity in the study of dynamical systems. Many results derived from linear theory are used in the design of, for example, non-linear control systems.

Many, if not most, existing methods of non-linear programming fall roughly into three main categories[3]:

- Methods of centers
- Methods of feasible directions
- Penalty function methods

In this thesis, methods of feasible directions will be of fundamental relevance. These methods embody the same philosophy as techniques for unconstrained minimization but are constructed to deal with inequality constraints. Briefly, the idea is to pick a starting point satisfying the constraints and to find a direction such that

- (a) a small move in that direction violates no constraint,
- (b) the objective function improves in that direction.

One then moves some distance in this direction obtaining a new and better estimate and repeats the

procedure until a point is obtained such that no direction satisfying both (a) and (b) can be found.

There are many ways of choosing such directions, thus there exist many methods of feasible directions, e.g. the method of Zoutendijk, the gradient projection method of Rosen etc.

This thesis is concerned mostly with methods for which the search direction for each iteration is obtained by solving a subproblem defined by:

$$(1.5) \quad \min \{ v^S(x_j, x) : x \in Q \}$$

where  $v^S$  is some lower or upper approximation to  $v$ . Such a  $v^S$  is called here a lower-approximation for  $v$  if  $v^S(x_j, x) \leq v(x)$ ,  $\forall x \in Q$ , and an upper-approximation if  $v^S(x_j, x) \geq v(x)$ ,  $\forall x \in Q$ . Ideally a  $v^S$  should be used so that  $v^S$  is a good approximation to  $v$  and so that problem (1.5) is easy to solve.<sup>1</sup>

In Chapter 2 an existing feasible direction algorithm [4] that obtains search directions by minimizing exactly lower approximations  $v^S$  to  $v$  on  $Q$  is considered. Global convergence results exist for it. However sometimes the minimization of  $v^S$  on  $Q$  cannot be done exactly. The effects on convergence rate of approximate minimization of  $v^S$  are investigated and new results derived. These results show that improving the accuracy with which subproblem (1.5) is solved improves the bounds for the convergence of

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1

The study of such methods was motivated by an optimization problem

the iteration for solving the main problem (1.1). When  $v^S$  is strictly convex the convergence results involve the ratio between second and first order effects when minimizing  $v^S$ . Properties of the ratio are stated and proved and a mixed stopping criterion for the overall algorithm derived. Applications to classical optimal control problems are considered.

Then another type of minimization algorithm is considered, in Chapter 3, for the case where  $Q$  is a subset of  $R^n$ . At each iteration, it minimizes  $v$  on a subset of  $Q$ , actually a convex polytope approximating  $Q$ , and then it extends the subset. This is done in such a way that the resulting sequence  $x_j$  converges to a global minimizer of  $v$  on  $Q$ . The convex polytope is defined by its vertices, which are found from past values of  $x_j$ . A mechanism is incorporated to test for redundant vertices.

The case where  $v(x) = ||x||_2^2$  is considered in detail as that arises in some important new optimal control problems (e.g. for dominant output feedback).

In general terms it can be said that while in feasible direction algorithms one seeks to define search directions along which minimization of  $v$  is carried out, for the algorithms considered in this Chapter the aim is to define a search set (a convex polytope) over which minimization of  $v$  is considered. With this perspective,

the feasible direction algorithms of Chapter 2 can be seen as particular cases of these set-approximation algorithms.

In Chapter 4 and for  $v$  a general quadratic, several algorithms are presented differing basically only in the way the search direction is found. In fact, the approximation  $v^S$  to  $v$  is now either an upper approximation or a lower approximation which is derived from an upper approximation function. Several approaches are considered and convergence rates derived. One of the results obtained shows that a similar rate is obtainable independently of whether  $v^S$  is a lower or an upper approximation.

Also methods to derive lower approximation functions  $v$  from available upper approximations are presented.

A new approach is used leading to a non sharp bound for the "distance"  $v(x_j) - v(\hat{x})$ , where  $\hat{x}$  is a global minimizer of  $v$  on  $Q$ . A rate of convergence for the case when  $x_j$  is far away from  $\hat{x}$  is obtained.

Finally Chapter 5 is about a special type of convex programming problem for which  $Q$  is the closed and unbounded set:

$$(1.6) \quad \text{cone}[\Gamma] \triangleq \{ \alpha x : \alpha \in \mathbb{R}^+, x \in \Gamma \}$$

where  $\Gamma$  is a compact convex set and  $0 \notin \Gamma$ . Allwright [5] shows that, for the case where  $v$  is a norm function, an equivalent problem, with the same solution, can be derived wherein minimization of  $v$  is now carried out over a new,

compact convex and bounded, set  $S$ , actually a suitably truncated version of  $\text{cone}[\Gamma]$ . A generalization of Allwright's results to the case where  $v$  is a general quadratic is presented and a convergence rate is derived which depends on the ratio between the smallest and the largest eigenvalues of the Hessian of  $v$ . For the case where  $v$  is a general convex function, whose Hessian is upper and lower bounded, it is shown that a similar equivalent problem can also be formulated. An algorithm to solve the equivalent problem is stated and a convergence rate, depending on both lower and upper bounds of the Hessian, is derived.

Chapter 6 contains some concluding comments on the thesis.

## 1.2 - MAIN CONTRIBUTIONS OF THE THESIS

We list below the main contributions and we refer to the Lemmas and Theorems which are related to them:

### Chapter 2

- Derivation of an upper-bound on the " distance " between the approximate optimum of  $f^S$  and the exact optimum of  $f$ : Lemma (2.20)..
- A stopping condition for the main algorithm : Theorem (2.24).

- A stopping condition for the iteration on the subproblem (SP) : Lemma (2.26) , Theorem (2.32).
- A property of the new search direction  $h_j = \tilde{y}(z_j) - z_j$  Lemma (2.34).
- Convergence rate information for the case when  $S \geq 0$  Theorem (2.37).
- Convergence rate information for the case  $S \geq nI$  ,  $n > 0$  revealing dependence on the ratio  $\psi_j$  between second and first order effects on  $f^S$  : Theorem (2.54).
- Properties of the ratio function  $\psi_j$  : Theorems (2.66) , (2.67) , (2.68) , (2.69) and (2.70).
- An application example : Example (ii).
- Discussion of the connections between Gilbert's results and Allwright's.

### Chapter 3

- Discussion of the connections between a feasible direction algorithm and a set-approximation type of algorithm : algorithm 1a.
- Methods to generate inner approximations and resulting algorithms: algorithms 1b and 1c , Lemma (3.22), Theorem (3.24).

- Methods to find the closest face of a convex polytope : Lemma (3.29), Theorems (3.31) and (3.32).

#### Chapter 4

- Convergence rate information regarding the unconstrained minimization of a convex function  $v$  using a search direction obtained by minimization of a local upper approximation to  $v$  : algorithm (4.9) , Theorem (4.14).
- Convergence rate information regarding the constrained minimization of a quadratic  $v$  using a search direction obtained by minimization of :
  - an upper approximation  $v^U$  to  $v$ : Theorem(4.28).
  - a lower approximation  $\bar{v}$  to  $v$  constructed from an upper bound: Theorem (4.46).
  - a lower approximation  $v^{\alpha U}$  to  $v$ :Theorem (4.63).
- Properties of an algorithm which finds  $h_j$  by minimization of an upper approximation  $v^U$  to  $v$ : Lemmas(4.76),(4.79),(4.80),(4.81),(4.82),(4.83) and Theorem (4.84).
- Derivation of upper and lower bounds on respectively the  $\lambda_{\max}$  and  $\lambda_{\min}$  of positive definite matrices  $C=C'$ : Theorem (4.99).

## Chapter 5

-Minimizing a quadratic over a cone generated by a convex, compact set :

-A relation between the diameter of a set  $S$  and the geometry of  $\Gamma$  : Lemma (5.23).

-Formulation of an equivalent problem: Theorem(5.28).

-A convergence rate :Theorem (5.29).

-Generalization to a convex function:

-Formulation of an equivalent problem: Theorem(5.46).

-A convergence rate :Theorem (5.50).

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Continuation of footnote 1 of page 4

arising in optimal output feedback, outlined in Section 3.6 , for which the feasible set  $Q$  is the convex hull of the image, under a nonlinear function, of the unit sphere. That feasible set does not seem to be describable in the conventional form  $Q = \{x \in \mathbb{R}^n: g(x) \leq 0, h(x)=0\}$ , so most of the existing optimization algorithms are not applicable. One cannot even determine whether a given point is in  $Q$ , one can only compute points known to be in  $Q$  by finding contact points for supporting hyperplanes. Hence methods which do not maintain feasibility also do not seem to be applicable. Computation of contact points corresponds to minimization of  $v^S(x_j, y)$  with respect to  $y$  from  $Q$ , with  $S = 0$  - a special case of  $S \geq 0$ . That is the reason why, for  $S \geq 0$ , most of this thesis is devoted to feasible direction algorithms based on minimization of  $v^S(x_j, y)$  with respect to  $y$  from a set  $Q$  which is not necessarily described by inequalities  $g(x) \leq 0$  and/or equalities  $h(x) = 0$

Other cases for which minimization of  $v^S(x_j, y)$  with respect to  $y$  from  $Q$  is relatively easy include  $S \geq 0$  and either  $Q$  a polytope [generally solved by linear programming when  $S = 0$  and generally solved by quadratic programming when  $S \geq 0$  (but also see Example (i) of page (46)] or  $Q$  the convex hull of a hyper-ellipse (e.g. Example (ii) of page (48)).

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## CHAPTER 2

### FEASIBLE DIRECTION ALGORITHMS FOR CONVEX PROGRAMMING - GLOBAL CONVERGENCE RESULTS

#### 2.1 MOTIVATION

Many practical problems in optimal control are either convex (i.e. have a convex objective function and a convex feasible set) or can be approximated closely by another convex problem.

Consider the constrained optimization problem (P) of minimizing a convex function  $f : H \rightarrow \mathbb{R}$  on a closed and bounded convex subset  $Q$  of a Hilbert-Space  $H$ , where  $f$  is twice continuously differentiable on  $Q$  with second derivative  $\nabla^2 f$  satisfying:

$$(2.1) \quad S \leq \nabla^2 f \leq S + MI \quad \text{for all} \quad z \in Q$$

for some known  $S = S^* \geq 0$  and some  $M < \infty$ , where  $S^*$  denotes the Hilbert-Space adjoint of the operator  $S$ .

For solving problems of this type there exists a group of well known algorithms including [1] methods of centers, feasible direction algorithms such as those of Zoutendijk and Pironneau and Polak as well as the Goldstein-Levitin-Polyak gradient projection method [2]. However the convergence results available for these algorithms are either asymptotic or give a decreased rate as the number of constraints increases.

Allwright [3] has studied the convergence properties of a feasible direction algorithm which employs  $Q$  directly for choosing feasible descent directions. The feasible descent direction is obtained by solving exactly (at each iteration) a subproblem with quadratic or linear objective function. The deduced convergence rate is either independent of the geometry of the constraint set (when the objective function of the subproblem is a strictly convex quadratic) or simply depends on its diameter (when the objective function of the subproblem is linear or is a convex, but not strictly convex, quadratic).

However the exact solution of the subproblem, eventhough possible for some cases of practical importance, turns out to be not possible in other cases regardless of their simple structure.

Then it is important to investigate what happens when only an approximate solution of the subproblem is available. That is studied in this Chapter.

## 2.2 ALGORITHMS: CONVERGENCE RESULTS

### 2.2.1 - A FEASIBLE DIRECTION ALGORITHM

To solve problem (P) Allwright [3] describes a feasible direction algorithm where the feasible descent direction is determined by solving exactly the subproblem (SP) defined at iteration  $j$  as:

(SP) $^j$ :

$$(2.2) \quad \min \{ f^S(z_j, y) : y \in Q \}$$

with

$$(2.3) \quad f^S(z_j, y) \triangleq f(z_j) + \langle \nabla f(z_j), (y - z_j) \rangle + \frac{1}{2} \langle (y - z_j), S(y - z_j) \rangle$$

and where

$$(2.4) \quad \hat{y}(z_j) \in \operatorname{argmin} \{ f^S(z_j, y) : y \in Q \}$$

The algorithm described in [3] will be denoted Algorithm-1 and is stated next:

#### Algorithm - 1

step (i) Choose  $z_0 \in Q$  and set  $j=0$

step (ii) Solve exactly (SP) $^j$

step (iii) Define the search direction  $h_j$  by

$$(2.5) \quad h_j = \hat{y}(z_j) - z_j$$

step (iv) Minimize approximately with respect to  $\lambda$  the function  $f(z_j + \lambda h_j)$  by finding the smallest integer  $l \geq 0$  such that:

$$(2.6) \quad f(z_j + \beta^l h_j) \leq f(z_j) + \frac{1}{2} \beta^l \langle \nabla f(z_j), h_j \rangle$$

and then setting  $\lambda_j = \beta^l$ ,  $\beta \in (0, 1)$

(an Armijo type rule)

step (v) Set  $z_{j+1} = z_j + \lambda_j h_j \in Q$   
 $(z_{j+1} \in Q \text{ because } \lambda \in [0, 1] \text{ and } z_j + \hat{y}(z_j) \in Q)$   
 If  $z_{j+1}$  is optimal STOP, otherwise set  
 $j = j+1$  and GO TO (ii).

(2.7) REMARKS: (i) Algorithm-1 is based on the intuitive idea that if the quadratic approximation  $f^S$  is a reasonably good approximation to  $f$  then  $\hat{y}(z)$  should be a fairly good approximation to  $\hat{z}$ .

(ii) If  $f$  is actually quadratic then  $\lambda_j$  can obviously be found analytically and the minimization involved in step (iv) can be carried out exactly.

(iii) The exact solution of (SP) in step (ii) is only possible for some cases.

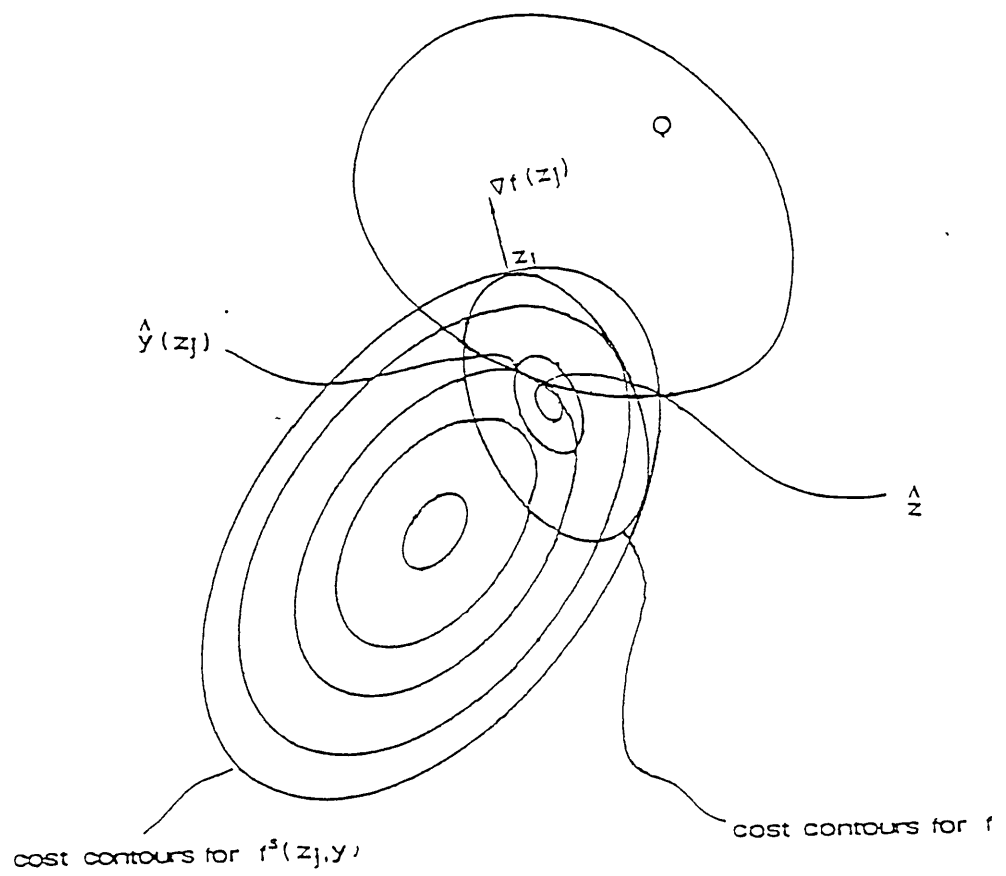


figure 2.1

### 2.2.2 CONVERGENCE RESULTS

In [3] is given a convergence rate, for the sequence  $\{f(z_j)\}$  generated by Algorithm-1 which is either independent of the size and geometry of  $Q$  (when  $f^S$  is quadratic with  $S > 0$ ) or only dependent of the diameter of  $Q$  (when  $f^S$  is linear, i.e.  $S=0$ , or when  $S \geq 0$  but  $S \neq 0$ ).

(2.8) THEOREM [3]: Suppose that  $0 \leq S \leq NI; ||x-y|| \leq \gamma < \infty; \forall y, x \in Q$

Then  $\{f(z_j)\} \rightarrow f(\hat{z})$  for the sequence generated by algorithm-1 and convergence rate information is provided by:

(a) If

$$(2.9) \quad [f(z_j) - f(\hat{z})] \geq (M + \frac{1}{2} N) \gamma^2 = \bar{M} \gamma^2$$

then

$$(2.10) \quad [f(z_{j+1}) - f(\hat{z})] \leq \frac{1}{2} M \gamma^2 \leq \frac{1}{2} [f(z_j) - f(\hat{z})]$$

(b) If

$$(2.11) \quad [f(z_j) - f(\hat{z})] < \bar{M} \gamma^2$$

then

$$(2.12) \quad [f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \{1 - \frac{1}{2} \beta \frac{[f(z_j) - f(\hat{z})]}{(M + N) \gamma^2}\}.$$

where  $\beta$  is the Armijo parameter

(2.13) REMARKS: The result of (a) reveals that the "distance" from  $f(\hat{z})$  is at least halved in the first iteration if (2.9) holds and is small if  $M$ , a measure of the error between  $S$  and  $\nabla^2 f$ , is small. Indeed if  $M=0$  then  $f(z_1) = f(\hat{z})$  which is plausible since in that case the quadratic approximation  $f^S(z, y)$  to  $f(y)$  is exact.

(2.14) THEOREM [3]: Suppose that  $S \geq nI$  for  $n > 0$ .

Then for algorithm-1:

$$(2.15) [f(z_j) - f(\hat{z})] \leq \{1 - [\beta(1 - \frac{1}{2}\beta)/(1 + \frac{M}{n})]\}^j [f(z_0) - f(\hat{z})].$$

(2.16) REMARKS: (i) The convergence rate result of theorem (2.14) is completely independent of the geometry of  $Q$ .

(ii) If the minimization of  $f(z_j + \lambda h_j)$  with respect to  $\lambda$  is exact then theorems (2.8) and (2.14) apply with  $\beta=1$ .

## 2.3 RESULTS WHEN (SP) IS SOLVED ONLY APPROXIMATELY

### 2.3.1 AN ALGORITHM

As was said before and will be illustrated later, there are cases of interest for which exact minimization of  $f^S(z, y)$  on  $Q$  can not be achieved in practice.

Consider then  $\tilde{y}(z_j)$  to be an approximate solution for  $(SP)^j$  - approximate in that  $\tilde{y}(z_j)$  satisfies:

$$(2.17) \quad 0 \leq [f^S(z_j, \tilde{y}(z_j)) - f^S(z_j, \hat{y}(z_j))] \leq \epsilon [f(z_j) - f^S(z_j, \hat{y}(z_j))]$$

for a pre-specified  $\epsilon \in [0, 1)$ .

The corresponding algorithm is denoted algorithm-2, which follows:

Algorithm - 2

step (i) Set  $z_0 \in Q$  ,  $j=0$

step (ii) Solve  $(SP)^j$  approximately by finding a  $\tilde{y}(z_j)$  which satisfies (2.17).

step (iii) Define the search direction  $\tilde{h}_j$

$$(2.18) \quad \tilde{h}_j = \tilde{y}(z_j) - z_j$$

step (iv) Minimize approximately  $f(z_j + \lambda \tilde{h}_j)$  as in step (iv) of algorithm-1.

step (v) Set  $z_{j+1} = z_j + \lambda_j \tilde{h}_j$ .

If  $z_{j+1}$  satisfies a stopping condition STOP, otherwise set  $j=j+1$  and go to (ii).

### 2.3.2. STOPPING CONDITIONS

A stopping condition is needed for the approximate solution of  $(SP)^j$ , so that one can tell when a  $\tilde{y}(z_j)$  satisfying (2.17) has been found and another stopping condition is needed for the solution of (P).

#### (i) STOPPING CRITERION FOR (P)

(2.19) LEMMA [3]

$$f^S(z, \hat{y}(z)) \leq f(\hat{z}), \quad \forall z \in Q$$

Proof: The Taylor formula for second-order expansions gives:

$$f(y) = f(z) + \langle \nabla f(z), (y-z) \rangle + \int_0^1 (1-t) \langle (y-z), \nabla^2 f(z+t[y-z]) \rangle (y-z) dt$$

and as by (2.1)  $\nabla^2 f \geq S$ ,

$$\begin{aligned} f(y) &\geq f(z) + \langle \nabla f(z), (y-z) \rangle + \int_0^1 (1-t) \langle (y-z), S(y-z) \rangle dt \\ &= f(z) + \langle \nabla f(z), (y-z) \rangle + \frac{1}{2} \langle (y-z), S(y-z) \rangle \\ &= f^S(z, y) . \end{aligned}$$

Hence  $f^S(z, y) \leq f(y) \quad \forall y, z \in Q$  ;

consequently,

$$f^S(z, \hat{y}(z)) = \min_{y \in Q} f^S(z, y) \leq \min_{y \in Q} f(y) = f(\hat{z})$$

Q.E.D.

(2.20) LEMMA For  $\tilde{y}(z_j)$  satisfying (2.17)

$$(2.21) \quad f(z_j) - f^S(z_j, \tilde{y}(z_j)) \geq (1-\epsilon) [f(z_j) - f^S(z_j, \hat{y}(z_j))]$$

Proof: Consider

$$\begin{aligned} [f(z_j) - f^S(z_j, \hat{y}(z_j))] &= [f(z_j) - f^S(z_j, \tilde{y}(z_j))] \\ &\quad + [f^S(z_j, \tilde{y}(z_j)) - f^S(z_j, \hat{y}(z_j))] \\ &\leq [f(z_j) - f^S(z_j, \tilde{y}(z_j))] \\ &\quad + \epsilon [f(z_j) - f^S(z_j, \hat{y}(z_j))] \\ &\hspace{15em} \text{by (2.17)} \end{aligned}$$

Then

$$(1-\epsilon) [f(z_j) - f^S(z_j, \hat{y}(z_j))] \leq f(z_j) - f^S(z_j, \tilde{y}(z_j))$$

Q.E.D.

(2.22) REMARK : LEMMA (2.20) provides a lower bound on the "distance" between the approximate optimum of  $f^S$  and  $f(z_j)$ .

In order to keep the iterative process finite, it might be acceptable to cease iterating once an  $\delta$ -approximation has been achieved, for a pre specified small positive  $\delta$ , in that

$$(2.23) \quad f(z_j) - f(\hat{z}) \leq \delta$$

However  $f(\hat{z})$  is not known apriori and so some indirect test has to be developed. The following theorem gives the basis for such test:

(2.24) THEOREM - For  $\delta > 0$  and  $\epsilon \in [0, 1)$ :

$$\text{If} \quad f(z_j) - f^S(z_j, \tilde{y}(z_j)) \leq \delta (1 - \epsilon)$$

$$\text{then} \quad f(z_j) - f(\hat{z}) \leq \delta$$

Proof: By LEMMA (2.20)

$$f(z_j) - f^S(z_j, \hat{y}(z_j)) \leq \frac{1}{(1 - \epsilon)} [f(z_j) - f^S(z_j, \tilde{y}(z_j))].$$

Now by assumption

$$f(z_j) - f^S(z_j, \tilde{y}(z_j)) \leq (1 - \epsilon) \delta$$

Consequently

$$f(z_j) - f^S(z_j, \hat{y}(z_j)) \leq \delta$$

implying by LEMMA (2.19)

$$f(z_j) - f(\hat{z}) \leq f(z_j) - f^S(z_j, \hat{y}(z_j)) \leq \delta$$

Q.E.D.

(2.25) REMARK: Theorem (2.24) reveals that for a smaller  $\epsilon$  (corresponding to a better estimate of the subproblem being obtained) the number of iterations  $j$  needed by algorithm-2 to satisfy the stopping condition <sup>(2.23)</sup> decreases - since a smaller  $\epsilon$  corresponds to a bigger  $\delta(1-\epsilon)$  so that the condition

$$f(z_j) - f^S(z_j, \tilde{y}(z_j)) \leq \delta(1-\epsilon)$$

is less stringent and might be satisfied for a smaller  $j$ .

(ii) A STOPPING CRITERION FOR (SP)<sup>j</sup>

It has been assumed that iteration on the subproblem stops when an  $\epsilon$ -approximation  $\tilde{y}(z_j)$  to  $\hat{y}(z_j)$  is achieved, in that

$$[f^S(z_j, \tilde{y}(z_j)) - f^S(z_j, \hat{y}(z_j))] \leq \epsilon [f(z_j) - f^S(z_j, \hat{y}(z_j))]$$

However the fact that  $f^S(z_j, \hat{y}(z_j))$  is not known doesn't allow the direct use of the expression above to test the quality of the estimate  $\tilde{y}(z_j)$ . Some sort of indirect test has to be used. The following Lemma and Theorem give the basis for such a test.

(2.26) LEMMA: For given  $y_k \in Q$ , let

$$(2.27) \quad \Theta(y, y_k) = f^S(z_j, y_k) + \langle \nabla_y f^S(z_j, y_k), (y - y_k) \rangle \quad \forall y \in Q$$

and

$$(2.28) \quad \bar{y}(y_k) \in \operatorname{argmin} \{ \Theta(y, y_k) : y \in Q \}.$$

Then

$$(2.29) \quad \Theta(\bar{y}(y_k), y_k) \leq f^S(z_j, \hat{y}(z_j))$$

Proof: By the definition of  $\bar{y}(y_k)$

$$\Theta(\bar{y}(y_k), y_k) \leq \Theta(y, y_k) \quad \forall y \in Q$$

and hence from the definition of  $\Theta(.,.)$

$$(2.30) \quad \langle \nabla_y f^S(z_j, y_k), (\bar{y}(y_k) - y_k) \rangle \leq \langle \nabla_y f^S(z_j, y_k), (y - y_k) \rangle \quad \forall y \in Q$$

Now since  $\nabla_y f^S(z_j, y_k) = \nabla_z f(z_j) + S(y_k - z_j)$  and by replacing  $y$  by  $\hat{y}(z_j)$  in (2.30),

$$(2.31) \quad \begin{aligned} & \langle [\nabla_z f(z_j) + S(y_k - z_j)], (\bar{y}(y_k) - y_k) \rangle \\ & \leq \langle [\nabla_z f(z_j) + S(y_k - z_j)], (\hat{y}(z_j) - y_k) \rangle \end{aligned}$$

Consider now the Taylor expansion of  $f^S(z_j, \hat{y}(z_j))$  around  $y_k$ :

$$\begin{aligned} f^S(z_j, \hat{y}(z_j)) &= f^S(z_j, y_k) + \langle [\nabla_z f(z_j) + S(y_k - z_j)], (\hat{y}(z_j) - y_k) \rangle \\ &\quad + \frac{1}{2} \langle (\hat{y}(z_j) - y_k), S(\hat{y}(z_j) - y_k) \rangle \end{aligned}$$

Now since  $S \geq 0$  and by (2.31)

$$\begin{aligned} f^S(z_j, \hat{y}(z_j)) &\geq f^S(z_j, y_k) + \langle [\nabla_z f(z_j) + S(y_k - z_j)], (\hat{y}(z_j) - y_k) \rangle \\ &\geq f^S(z_j, y_k) + \langle [\nabla_z f(z_j) + S(y_k - z_j)], (\bar{y}(y_k) - y_k) \rangle \end{aligned}$$

and hence  $f^S(z_j, \hat{y}(z_j)) \geq \Theta(\bar{y}(y_k), y_k)$  Q.E.D.

(2.32) THEOREM:

If

$$(2.32a) \quad [f^S(z_j, y_k) - \Theta(\bar{y}(y_k), y_k)] \leq \epsilon [f(z_j) - f^S(z_j, y_k)]$$

then  $[f^S(z_j, y_k) - f^S(z_j, \hat{y}(z_j))] \leq \epsilon [f(z_j) - f^S(z_j, \hat{y}(z_j))]$

Proof: Now  $f^S(z_j, y_k) - \Theta(\bar{y}(y_k), y_k) \leq \epsilon [f(z_j) - f^S(z_j, y_k)]$

implies by Lemma (2.26), that

$$[f^S(z_j, y_k) - f^S(z_j, \hat{y}(z_j))] \leq \epsilon [f(z_j) - f^S(z_j, y_k)]$$

and so since  $f^S(z_j, \hat{y}(z_j)) \leq f^S(z_j, y_k)$ ,

$$[f^S(z_j, y_k) - f^S(z_j, \hat{y}(z_j))] \leq \epsilon [f(z_j) - f^S(z_j, \hat{y}(z_j))] \quad \text{Q.E.D.}$$

Continuation of Footnote 1 of page 26.

of Algorithm 2 of page 26. For, by Theorem 2.8, for the function  $f^S(z_j, y)$

$$(2.32c) \quad f^S(z_j, y_k) \rightarrow f^S(z_j, \hat{y}).$$

Also, by [3, page 12], for  $\alpha_k$  of Algorithm 2, step 5:

if  $\tilde{\alpha}_k < 1$  then

$$(2.32d) \quad f^S(z_j, y_{k+1}) \leq f^S(z_j, y_k) - 1/2 \langle \nabla_y f^S(z_j, y_k), (\bar{y}(y_k) - y_k) \rangle^2 / [\gamma^2 \lambda_{\max}[S]]$$

else

$$(2.32e) \quad f^S(z_j, y_{k+1}) \leq f^S(z_j, y_k) + 1/2 \langle \nabla_y f^S(z_j, y_k), (\bar{y}(y_k) - y_k) \rangle.$$

From (2.32d-e), it can be seen that (2.32c) holds only if

$\langle \nabla_y f^S(z_j, y_k), (\bar{y}(y_k) - y_k) \rangle$  converges to zero

i.e. if  $f^S(z_j, y_k) - \Theta(\bar{y}(y_k), y_k) = -\langle \nabla_y f^S(z_j, y_k), (\bar{y}(y_k) - y_k) \rangle$  converges to zero.

Hence (2.32a) must hold after a sufficient number of iterations  $k$ .

Algorithm-2 is restated with the stopping mechanisms included:

Algorithm-2

Step (i) Set  $z_0 \in Q$ ,  $j=0$  and choose  $\delta > 0$  and small

Step (ii) Solve  $(SP)^j$  approximately by:

step 1 - Choose  $\epsilon > 0$  and small

set  $y_0 = z_j$ ,  $k=0$

step 2 - Find  $\bar{y}(y_k)$  such that

$$\bar{y}(y_k) \in \operatorname{argmin} \{ \Theta(y, y_k) : y \in Q \}$$

step 3 - Stopping test

If  $\bar{y}(y_k)$  satisfies (2.32a)<sup>1</sup> set

$\tilde{y}(z_j) = y_k$  and GO TO step (iii),

otherwise CONTINUE.

step 4 - Define a search direction  $d_k$

$$d_k = \bar{y}(y_k) - y_k$$

step 5 - Minimize approximately with respect to

$\alpha_k \in [0, 1]$  the function  $f^S(z_j, [y_k + \alpha_k d_k])$

Let  $\tilde{\alpha}_k$  be the approximate solution

step 6 - Set  $y_{k+1} = y_k + \tilde{\alpha}_k d_k$

Set  $k=k+1$  and GO TO step 2

---

<sup>1</sup>

Condition (2.32a) will eventually be satisfied by a  $y_k$  generated by step (ii)

Step (iii) Stopping test

If  $[f(z_j) - f^S(z_j, \tilde{y}(z_j))] \leq \delta(1-\epsilon)$  STOP,  
otherwise CONTINUE.

Step (iv) Define the search direction  $\tilde{h}_j$

$$\tilde{h}_j = \tilde{y}(z_j) - z_j$$

Step (v) Minimize approximately  $f(z_j + \lambda \tilde{h}_j)$   
as in step (iv) of algorithm-1

Step (vi) Set  $z_{j+1} = z_j + \lambda_j \tilde{h}_j$  ,  $j=j+1$   
and GO TO step (ii)

### 2.3.3 NEW CONVERGENCE RESULTS

The fact that the subproblem (SP) is solved approximately affects the convergence results obtained by Allwright [3].

In order to facilitate the derivation of the new convergence results, the following result is helpful .

(2.34) LEMMA:

For  $\epsilon \in [0,1)$  and the  $\epsilon$ - approximation  $\tilde{y}(z_j)$  of (2.17):

$$(2.35) \quad \langle \nabla f(z_j), \tilde{h}_j \rangle \leq (1-\epsilon)[f(\hat{z}) - f(z_j)] \leq 0$$

where  $\tilde{h}_j = \tilde{y}(z_j) - z_j$

Proof: By the definition of  $f^S(z_j, y)$  and the fact that

$$\tilde{y}(z_j) = z_j + \tilde{h}_j$$

$$f^S(z_j, \tilde{y}(z_j)) = f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle + \frac{1}{2} \langle \tilde{h}_j, S \tilde{h}_j \rangle$$

Using LEMMA (2.20) and the fact that  $S \geq 0$

$$f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle \leq f(\hat{z}) + \epsilon [f(z_j) - f(\hat{z})]$$

$$\text{Hence } \langle \nabla f(z_j), \tilde{h}_j \rangle \leq (1-\epsilon) [f(\hat{z}) - f(z_j)] \leq 0 \quad \forall z_j \in Q$$

Q.E.D.

(2.36) REMARK: The Lemma (2.34) shows that  $\tilde{h}_j$  is a descent direction and establishes a negative upper bound on  $\langle \nabla f(z_j), \tilde{h}_j \rangle$ .

The following Theorem gives global convergence rate information for algorithm-2 when the approximation function is convex.

(2.37) THEOREM: Suppose  $0 \leq S \leq NI$ ,  $\bar{M} = \frac{1}{2}N + M$

$$\|x - y\| \leq \gamma < \infty \quad \forall x, y \in Q$$

Then  $\{f(z_j)\} \rightarrow f(\hat{z})$  and

$$(2.38) \text{ (a) If } [f(z_j) - f(\hat{z})] \geq \frac{1}{1-\epsilon} (M + \frac{1}{2}N) \gamma^2 = \frac{1}{1-\epsilon} \bar{M} \gamma^2$$

$$(2.39) \text{ then } [f(z_{j+1}) - f(\hat{z})] \leq \epsilon [f(z_j) - f(\hat{z})] + \frac{1}{2} M \gamma^2$$

$$\leq \left(\frac{\epsilon+1}{2}\right) [f(z_j) - f(\hat{z})]$$

$$(2.40) \quad (b) \quad \text{If} \quad [f(z_j) - f(\hat{z})] < \frac{1}{1-\epsilon} \bar{M}_\gamma^2$$

then

$$(2.41) \quad [f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \{1 - \frac{1}{2} \beta (1-\epsilon)^2 \frac{[f(z_j) - f(\hat{z})]}{(M+N)\gamma^2}\},$$

(2.42) REMARKS: (i) With  $\epsilon=0$  the results provided by theorem (2.37) are consistent with those of theorem (2.8), as would have been expected because the algorithms are identical.

(ii) The results of theorem (2.37) show that using a smaller  $\epsilon$  for the subproblem improves the convergence rate bounds for the main algorithm.

(iii) Theorem (2.37) reveals that eventually after at most  $k$  iterations  $[f(z_j) - f(\hat{z})] < \frac{1}{1-\epsilon} \bar{M}_\gamma^2$  where  $k$  is the smallest integer greater than or equal to:

$$\frac{\log \frac{\bar{M}_\gamma^2}{(1-\epsilon)[f(z_0) - f(\hat{z})]}}{\log \frac{\epsilon+1}{2}}$$

#### Proof of Theorem (2.37)

The Taylor formula for second order expansions gives:

$$\begin{aligned} f(z_j + \lambda \tilde{h}_j) &= f(z_j) + \lambda \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &+ \lambda^2 \int_0^1 (1-t) \langle \tilde{h}_j, \nabla^2 f(z_j + t \lambda \tilde{h}_j) \tilde{h}_j \rangle dt \end{aligned}$$

and hence by (2.1)

$$(2.43) f(z_j + \lambda \tilde{h}_j) \leq f(z_j) + \lambda \langle \nabla f(z_j), \tilde{h}_j \rangle + \frac{1}{2} \lambda^2 \langle \tilde{h}_j, \tilde{S} \tilde{h}_j \rangle + \frac{1}{2} \lambda^2 M \gamma^2$$

Consequently

$$(2.44) \quad f(z_j + \lambda \tilde{h}_j) \leq f(z_j) + \frac{1}{2} \lambda \langle \nabla f(z_j), \tilde{h}_j \rangle$$

$$(2.45) \quad \text{if} \quad 0 \leq \lambda \leq \frac{-\langle \nabla f(z_j), \tilde{h}_j \rangle}{\langle \tilde{h}_j, \tilde{S} \tilde{h}_j \rangle + M \gamma^2} \triangleq \hat{\lambda}_j$$

a) If  $\hat{\lambda}_j \geq 1$  then (2.44) is satisfied for all  $\lambda \in [0, 1]$  and the value of  $\lambda_j$  chosen (according to the Armijo type rule) will be  $\lambda_j = 1$ .

So, if  $\hat{\lambda}_j \geq 1$ , (2.44) gives:

$$\begin{aligned} f(z_{j+1}) &= f(z_j + \tilde{h}_j) \\ &\leq f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle + \frac{1}{2} \langle \tilde{h}_j, \tilde{S} \tilde{h}_j \rangle + \frac{1}{2} M \gamma^2 \\ &= f^S(z_j, \tilde{y}(z_j)) + \frac{1}{2} M \gamma^2. \end{aligned}$$

Consequently, from Lemma (2.20)

$$f(z_{j+1}) \leq f(\hat{z}) + \epsilon [f(z_j) - f(\hat{z})] + \frac{1}{2} M \gamma^2.$$

Hence

$$(2.46) \quad [f(z_{j+1}) - f(\hat{z})] \leq \epsilon [f(z_j) - f(\hat{z})] + \frac{1}{2} M \gamma^2$$

Now from (2.45)  $\hat{\lambda}_j \geq 1$  iff

$$-\langle \nabla f(z_j), \tilde{h}_j \rangle \geq \langle \tilde{h}_j, \tilde{S}h_j \rangle + M\gamma^2$$

or, equivalently, iff

$$(2.47) \quad -\langle \nabla f(z_j), \tilde{h}_j \rangle - \frac{1}{2} \langle \tilde{h}_j, \tilde{S}h_j \rangle \geq \frac{1}{2} \langle \tilde{h}_j, \tilde{S}h_j \rangle + M\gamma^2$$

But as  $0 \leq S \leq NI$  (2.47) will hold if

$$(2.48) \quad -\langle \nabla f(z_j), \tilde{h}_j \rangle - \frac{1}{2} \langle \tilde{h}_j, \tilde{S}h_j \rangle \geq \frac{1}{2} N\gamma^2 + M\gamma^2 = \bar{M}\gamma^2$$

with  $\bar{M} = \frac{1}{2}N + M$ .

Now by Lemma (2.20)

$$\begin{aligned} f^S(z_j, \tilde{y}(z_j)) &= f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle + \frac{1}{2} \langle \tilde{h}_j, \tilde{S}h_j \rangle \\ &\leq f(\hat{z}) + \epsilon [f(z_j) - f(\hat{z})] \end{aligned}$$

i.e.

$$(2.49) \quad -\langle \nabla f(z_j), \tilde{h}_j \rangle - \frac{1}{2} \langle \tilde{h}_j, \tilde{S}h_j \rangle \geq (1-\epsilon)[f(z_j) - f(\hat{z})]$$

Hence from (2.48) and (2.49) it follows that

$$(2.50) \quad \text{if } (1-\epsilon)[f(z_j) - f(\hat{z})] \geq \bar{M}\gamma^2 \text{ then } \hat{\lambda}_j \geq 1$$

Now (2.50) together with (2.46) give result (a) of theorem (2.37), i.e.

$$\text{if} \quad [f(z_j) - f(\hat{z})] \geq \frac{1}{1-\epsilon} \bar{M}_\gamma^2$$

then

$$\begin{aligned} [f(z_{j+1}) - f(\hat{z})] &\leq \epsilon [f(z_j) - f(\hat{z})] + \frac{1}{2} M_\gamma^2 \\ &\leq \epsilon [f(z_j) - f(\hat{z})] + \frac{1}{2} \bar{M}_\gamma^2 \\ &\leq \epsilon [f(z_j) - f(\hat{z})] + \frac{1}{2} (1-\epsilon) [f(z_j) - f(\hat{z})] \\ &= \left(\frac{\epsilon+1}{2}\right) [f(z_j) - f(\hat{z})] \end{aligned}$$

The converse of the implication in Theorem (2.37) (a) is not true, as

$$(2.51) \quad \text{if} \quad [f(z_j) - f(\hat{z})] < \frac{1}{1-\epsilon} \bar{M}_\gamma^2,$$

then  $\hat{\lambda}_j$  can be either  $\geq 1$  or  $< 1$ .

(b) If  $\hat{\lambda}_j < 1$  the algorithm will choose  $\lambda_j$  satisfying

$$\lambda_j > \beta \hat{\lambda}_j \geq 0$$

and since  $\langle \nabla f(z_j), \tilde{h}_j \rangle \leq 0$ , (2.44) yields

$$\begin{aligned} f(z_j + \lambda_j \tilde{h}_j) &\leq f(z_j) + \frac{1}{2} \lambda_j \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &\leq f(z_j) + \frac{1}{2} \beta \hat{\lambda}_j \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &= f(z_j) - \frac{1}{2} \beta \frac{\langle \nabla f(z_j), \tilde{h}_j \rangle^2}{\langle \tilde{h}_j, S \tilde{h}_j \rangle + M_\gamma^2} \\ &\leq f(z_j) - \frac{1}{2} \beta \frac{\langle \nabla f(z_j), \tilde{h}_j \rangle^2}{(M+N)_\gamma^2} \end{aligned}$$

But from Lemma (2.34)

$$(2.52) \quad \langle \nabla f(z_j), \tilde{h}_j \rangle^2 \geq (1-\epsilon)^2 [f(z_j) - f(\hat{z})]^2 \geq 0$$

Hence

$$f(z_{j+1}) = f(z_j + \lambda_j \tilde{h}_j) \leq f(z_j) - \frac{1}{2} \beta (1-\epsilon)^2 \frac{[f(z_j) - f(\hat{z})]^2}{(M+N)\gamma^2}$$

i.e.

$$(2.53) \quad [f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \left\{ 1 - \frac{1}{2} \beta (1-\epsilon)^2 \frac{[f(z_j) - f(\hat{z})]}{(M+N)\gamma^2} \right\}$$

So the result of part (b) has been proved for the case when  $\hat{\lambda}_j < 1$ . It remains to be shown that (2.53) will hold even if  $\hat{\lambda}_j \geq 1$  in which case the algorithm will choose  $\lambda_j = 1$  and then:

$$\begin{aligned} f(z_{j+1}) &\leq f(z_j) + \frac{1}{2} \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &\leq f(z_j) - \frac{1}{2} (1-\epsilon) [f(z_j) - f(\hat{z})] \quad (\text{by Lemma (2.34)}) \\ &\leq f(z_j) - \frac{1}{2} \beta (1-\epsilon) [f(z_j) - f(\hat{z})] \quad (\text{since } \beta \in [0, 1]) \\ &\leq f(z_j) - \frac{1}{2} \beta (1-\epsilon) [f(z_j) - f(\hat{z})] \frac{(1-\epsilon) [f(z_j) - f(\hat{z})]}{(M+N)\gamma^2} \end{aligned}$$

since by (2.51)

$$[f(z_j) - f(\hat{z})] < \frac{1}{1-\epsilon} \bar{M} \gamma^2 \leq \frac{1}{1-\epsilon} (M+N) \gamma^2 \quad \text{so that}$$

$$\frac{(1-\epsilon) [f(z_j) - f(\hat{z})]}{(M+N)\gamma^2} \leq 1$$

This concludes the proof of theorem (2.37).

When the approximation function  $f^S(.,.)$  is strictly convex, global convergence results for algorithm-2 are provided by the following theorem:

(2.54) THEOREM: Suppose  $S \geq nI$  for  $n > 0$ .

Then whenever:

(2.55)  $z_j \notin \operatorname{argmin} \{ f(z) : z \in Q \}$

(2.56)  $[f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \{ 1 - (1-\epsilon) \frac{\beta(1 - \frac{1}{2}\beta)}{\psi_j(1 + \frac{M}{n})} \}$

where

(2.57)  $\psi_j \triangleq \psi(z_j, \tilde{y}(z_j)) = \frac{\langle \tilde{h}_j, S\tilde{h}_j \rangle}{-\langle \nabla f(z_j), \tilde{h}_j \rangle}$

and  $\tilde{h}_j = \tilde{y}(z_j) - z_j$

(2.58) REMARKS: (i) The fact that under assumption (2.55) the scalar valued ratio function  $\psi_j$  only takes values in  $(0, 2]$  will be shown later when properties of  $\psi_j$  are discussed.

(ii) The result of theorem (2.54) shows that improvement on the global problem bounds will be achieved by decreasing the pre-specified  $\epsilon$  and by iterating on the subproblem until  $\psi_j$  lies in the interval  $(0, 1+\mu]$  with  $\mu$  small, which can always be done (see theorem (2.70)).

Proof of theorem (2.54)

Since  $S^* = S \geq nI$  for  $n > 0$ ,  $S$  has a symmetric self-adjoint square root  $S^{1/2} > 0$ .

Applying the second order Taylor formula yields:

$$\begin{aligned} f(z_j + \lambda \tilde{h}_j) &= f(z_j) + \lambda \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &\quad + \lambda^2 \int_0^1 (1-t) \langle S^{1/2} \tilde{h}_j, S^{-1/2} \nabla^2 f(z_j + t\lambda \tilde{h}_j) S^{-1/2} S^{1/2} \tilde{h}_j \rangle dt \end{aligned}$$

Hence, since  $S \geq nI$   $n > 0$  and also

$$0 \leq \nabla^2 f - S \leq MI$$

$$\begin{aligned} (2.59) \quad f(z_j + \lambda \tilde{h}_j) &\leq f(z_j) + \lambda \langle \nabla f(z_j), \tilde{h}_j \rangle + \frac{1}{2} \lambda^2 \langle \tilde{h}_j, S \tilde{h}_j \rangle (1 + \frac{M}{n}) \\ &= f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle [\lambda - \frac{1}{2} \lambda^2 \psi_j (1 + \frac{M}{n})] \triangleq \phi_j(\lambda) \end{aligned}$$

Therefore

$$\begin{aligned} f(z_j + \lambda \tilde{h}_j) &\leq f(z_j) + \frac{1}{2} \lambda \langle \nabla f(z_j), \tilde{h}_j \rangle \\ (2.60) \quad \text{if } 0 \leq \lambda &\leq \frac{1}{\psi_j (1 + \frac{M}{n})} = \tilde{\lambda}_j \end{aligned}$$

As  $(1 + \frac{M}{n}) \geq 1$  and  $\psi_j \in (0, 2]$  (shown later), one of the following three conditions must always be true:

$$(2.61) \quad (i) \quad 1/(1+M/n) < \psi_j \leq 1$$

$$(2.62) \quad (ii) \quad \psi_j \leq 1/(1+M/n) \leq 1$$

$$(2.63) \quad (iii) \quad 1/(1+M/n) < 1 < \psi_j$$

Consider first case (i). Now  $\tilde{\lambda}_j$  is the exact minimizer of  $\phi_j(\lambda)$ .

If  $\tilde{\lambda}_j \leq 1$  it is easy to see that the algorithm will choose  $\lambda_j \in (\beta\tilde{\lambda}_j, \tilde{\lambda}_j)$  and hence, from (2.59)

$$f(z_j + \lambda_j \tilde{h}_j) \leq \phi_j\left(\beta \frac{1}{\psi_j(1+M/n)}\right) \leq \phi_j\left(\beta \frac{1}{1+M/n}\right)$$

since  $\frac{\beta}{1+M/n} \leq \frac{\beta}{\psi_j(1+M/n)} \leq \tilde{\lambda}_j$

and  $\phi_j$  becomes more negative as  $\lambda$  increases when  $\lambda \in [0, \tilde{\lambda}_j]$ .

Therefore:

$$(2.64) \quad f(z_j + \lambda_j \tilde{h}_j) = f(z_{j+1}) \leq \phi_j\left(\frac{\beta}{1+M/n}\right), \quad \text{i.e.}$$

$$\begin{aligned} f(z_{j+1}) &\leq f(z_j) + \frac{\beta}{(1+M/n)} \langle \nabla f(z_j), \tilde{h}_j \rangle - \frac{1}{2} \frac{\beta^2}{(1+M/n)} \psi_j \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &= f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle \left( \beta - \frac{1}{2} \beta^2 \psi_j \right) \frac{1}{(1+M/n)} \\ &\leq f(z_j) + \langle \nabla f(z_j), \tilde{h}_j \rangle \beta \left( 1 - \frac{1}{2} \beta \right) \frac{1}{(1+M/n)} \end{aligned}$$

since  $\psi_j \leq 1$  and  $\langle \nabla f(z_j), \tilde{h}_j \rangle \leq 0$ .

Next from a result of Lemma (2.34)

$$(2.65) \quad [f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \{1 - (1-\epsilon) \frac{\beta(1 - \frac{1}{2}\beta)}{(1+M/n)}\}$$

Consider now case (ii):

Since  $0 < \psi_j \leq \frac{1}{(1+M/n)} \leq 1$ , it follows from (2.60) that  $\tilde{\lambda}_j \geq 1$ .

Therefore the algorithm will choose  $\lambda_j = 1$  and so, from (2.59)

$$f(z_j + \lambda_j \tilde{h}_j) = f(z_{j+1}) \leq \varphi_j(1) \leq \varphi_j\left(\frac{1}{1+M/n}\right) \leq \varphi_j\left(\frac{\beta}{1+M/n}\right)$$

so that the bound of (2.64) established for case (i) will be valid for case (ii) as well. Consequently (2.65) is valid for case (ii).

Finally consider case (iii):

$$\frac{1}{1+M/n} < 1 < \psi_j \leq 2$$

In this case the algorithm will choose  $\lambda$  s.t.

$$f(z_{j+1}) \leq \varphi_j\left(\frac{\beta}{\psi_j(1+M/n)}\right) \text{ and then, from (2.59)}$$

$$\begin{aligned} f(z_{j+1}) &\leq f(z_j) + \frac{\beta(1 - \frac{1}{2}\beta)}{\psi_j(1+M/n)} \langle \nabla f(z_j), \tilde{h}_j \rangle \\ &\leq f(z_j) - (1-\epsilon) \frac{\beta(1 - \frac{1}{2}\beta)}{\psi_j(1+M/n)} [f(z_j) - f(\hat{z})] . \end{aligned}$$

Thus a general bound valid for the three cases is:

$$[f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \{1 - (1-\epsilon) \frac{\beta(1 - \frac{1}{2}\beta)}{\psi_j(1+M/n)}\}$$

with  $0 < \psi_j \leq 2$  .

#### 2.3.4 PROPERTIES OF THE RATIO FUNCTION $\Psi_j$

$$\text{Now } \Psi_j \triangleq \Psi(z_j, \tilde{y}(z_j)) = \frac{\langle (\tilde{y}(z_j) - z_j), S(\tilde{y}(z_j) - z_j) \rangle}{-\langle \nabla f(z_j), (\tilde{y}(z_j) - z_j) \rangle}$$

where  $z_j, \tilde{y}(z_j) \in Q$

Recall that  $\hat{y}(z_j)$  is the constrained global minimizer of  $f^S(z_j, y)$ , on  $Q$ , i.e

$$\hat{y}(z_j) \in \operatorname{argmin} \{ f^S(z_j, y) : y \in Q \} = \{ \hat{y}(z_j) \}.$$

The last equality occurs owing to the convexity of  $Q$  and the positive definiteness of  $S$ .

Let  $y^*(z_j)$  be the unconstrained global minimizer of  $f^S(z_j, y)$  on  $H$ , i.e. let

$$y^*(z_j) \in \operatorname{argmin} \{ f^S(z_j, y) : y \in H \} = \{ y^*(z_j) \}$$

(2.66) THEOREM: Assume  $S > 0$ . Then

$$\tilde{h}_j = 0 \quad \text{iff} \quad z_j \in \operatorname{argmin} \{ f(z) : z \in Q \}$$

Proof:

(i) Proof that:  $[z_j \in \operatorname{argmin} \{ f(z) : z \in Q \}] \Rightarrow [\tilde{h}_j = 0]$

Now

$$z_j \in \operatorname{argmin} \{ f(z) : z \in Q \} \Rightarrow f(z_j) = f(\hat{z})$$

and from Lemma (2.20) with  $z_j = \hat{z}$

$$f^S(\hat{z}, \tilde{y}(\hat{z})) \leq f(\hat{z}).$$

But by the definition of  $\hat{y}(z)$

$$f^S(\hat{z}, \hat{y}(\hat{z})) \leq f^S(\hat{z}, \tilde{y}(\hat{z}))$$

and by [3]

$$f^S(\hat{z}, \hat{y}(\hat{z})) = f(\hat{z})$$

So:

$$f(\hat{z}) = f^S(\hat{z}, \hat{y}(\hat{z})) \leq f^S(\hat{z}, \tilde{y}(\hat{z})) \leq f(\hat{z})$$

therefore

$$f^S(\hat{z}, \hat{y}(\hat{z})) = f^S(\hat{z}, \tilde{y}(\hat{z})) = f(\hat{z})$$

and so

$$\langle \nabla f(\hat{z}), (\tilde{y}(\hat{z}) - \hat{z}) \rangle + \frac{1}{2} \langle (\tilde{y}(\hat{z}) - \hat{z}), S(\tilde{y}(\hat{z}) - \hat{z}) \rangle = 0$$

Now if the quadratic term is greater than zero the linear term has to be smaller than zero which contradicts optimality of  $\hat{z}$ .

Therefore the quadratic term has to be zero which implies (since  $S > 0$ )

$$\tilde{y}(\hat{z}) - \hat{z} = 0 \quad \text{i.e.} \quad \tilde{h}_j = 0$$

(ii) Proof that:  $[\tilde{h}_j = 0] \implies [z_j \in \operatorname{argmin} \{ f(z) : z \in Q \}]$

Suppose  $\tilde{h}_j = \tilde{y}(z_j) - z_j = 0$ . Then  $f^S(z_j, \tilde{y}(z_j)) = f(z_j)$  since

$$\begin{aligned} f^S(z_j, \tilde{y}(z_j)) &= f(z_j) + \langle \nabla f(z_j), (\tilde{y}(z_j) - z_j) \rangle + \\ &\quad + \frac{1}{2} \langle (\tilde{y}(z_j) - z_j), S(\tilde{y}(z_j) - z_j) \rangle. \end{aligned}$$

But by Lemma (2.20)

$$f^S(z_j, \tilde{y}(z_j)) \leq f(\hat{z}) + \epsilon [f(z_j) - f(\hat{z})]$$

So

$$f^S(z_j, \tilde{y}(z_j)) = f(z_j) \leq f(\hat{z}) + \epsilon [f(z_j) - f(\hat{z})]$$

$$\text{i.e. } [f(z_j) - f(\hat{z})] \leq \epsilon [f(z_j) - f(\hat{z})] .$$

But for  $\epsilon \neq 0$  the previous inequality only holds when

$$f(z_j) = f(\hat{z}) .$$

Hence

$$z_j \in \operatorname{argmin} \{ f(z) : z \in Q \} .$$

(2.67) THEOREM If  $z_j \notin \operatorname{argmin} \{ f(z) : z \in Q \}$  and  $S > 0$  then

$$\psi_j \triangleq \frac{\langle \tilde{h}_j, S \tilde{h}_j \rangle}{- \langle \nabla f(z_j), \tilde{h}_j \rangle} \in (0, 2]$$

Proof: The fact that  $\psi_j \leq 2$  can be established by just noticing that, from (2.17)

$$\begin{aligned} f^S(z_j, \tilde{y}(z_j)) &\leq f^S(z_j, \hat{y}(z_j)) + \epsilon [f(z_j) - f^S(z_j, \hat{y}(z_j))] \\ &= \epsilon f(z_j) + (1-\epsilon) f^S(z_j, \hat{y}(z_j)) \\ &\leq \epsilon f(z_j) + (1-\epsilon) f(z_j) = f(z_j) \end{aligned}$$

and hence, from the definition of  $f^S(.,.)$ ,

$$\langle (\tilde{y}(z_j) - z_j), S(\tilde{y}(z_j) - z_j) \rangle \leq -2 \langle \nabla f(z_j), (\tilde{y}(z_j) - z_j) \rangle .$$

Hence, since from Lemma (2.34) and assumption on  $z_j$

$$\langle \nabla f(z_j), (\tilde{y}(z_j) - z_j) \rangle < 0 ,$$

this yields  $\psi_j \leq 2$

To show that  $\psi_j > 0$ , note that from theorem (2.65)  $\tilde{h}_j \neq 0$  since here it is assumed that  $z_j \notin \operatorname{argmin} \{ f(z) : z \in Q \}$ .

So  $\langle \tilde{h}_j, S\tilde{h}_j \rangle > 0$  and as above  $\langle \nabla f(z_j), \tilde{h}_j \rangle < 0$ .

Hence  $\psi_j > 0$ .

Consequently  $\psi_j \in (0, 2]$ .

Q.E.D.

(2.68) THEOREM: If  $y^*(z_j) \in Q$  and  $z_j \notin \operatorname{argmin} \{ f(z) : z \in Q \}$ ,

then  $\psi(z_j, \hat{y}(z_j)) = \psi(z_j, y^*(z_j)) = 1$

for  $y^*(z_j)$  of page 38

Proof: Since  $f^S(.,.)$  is convex, a necessary and sufficient condition for  $y^*(z_j)$  to be an unconstrained minimizer of  $f^S(z_j, y)$  is:

$$\left. \frac{\partial}{\partial y} f^S(z_j, y) \right|_{y=y^*} = 0$$

Equivalently

$$\nabla f(z_j) + S(y^*(z_j) - z_j) = 0 .$$

and hence

$$\langle [\nabla f(z_j) + S(y^*(z_j) - z_j)], (y^*(z_j) - z_j) \rangle = 0$$

implying

$$-\langle \nabla f(z_j), (y^*(z_j) - z_j) \rangle = \langle (y^*(z_j) - z_j), S(y^*(z_j) - z_j) \rangle$$

i.e.  $\psi(z_j, y^*(z_j)) = 1$

Since  $S > 0$  and  $Q$  is convex, if  $y^*(z_j) \in Q$  then

$$\hat{y}(z_j) = y^*(z_j) .$$

Consequently

$$\psi(z_j, \hat{y}(z_j)) = \psi(z_j, y^*(z_j)) = 1 .$$

Q.E.D.

(2.69) THEOREM: If  $y^*(z_j) \notin Q$  and  $z_j \in \operatorname{argmin}\{f(z) : z \in Q\}$ ,

$$\text{then } \psi(z_j, \hat{y}(z_j)) \leq 1$$

Proof: From the definition of  $\hat{y}(z_j)$

$$\left\langle \frac{\partial}{\partial y} f^S(z_j, y) \right|_{y=\hat{y}(z_j)}, (\hat{y}(z_j) - y) \rangle \leq 0 \quad \forall y \in Q$$

$$\text{But } \left. \frac{\partial}{\partial y} f^S(z_j, y) \right|_{y=\hat{y}(z_j)} = \nabla f(z_j) + S(\hat{y}(z_j) - z_j)$$

$$\text{so } \langle [\nabla f(z_j) + S(\hat{y}(z_j) - z_j)], (\hat{y}(z_j) - y) \rangle \leq 0, \quad \forall y \in Q .$$

For  $y = z_j$ , this gives

$$\langle (\hat{y}(z_j) - z_j), S(\hat{y}(z_j) - z_j) \rangle \leq -\langle \nabla f(z_j), (\hat{y}(z_j) - z_j) \rangle$$

consequently by the assumption  $z_j \in \operatorname{argmin}\{f(z) : z \in Q\}$ ,

$$\langle \nabla f(z_j), (\hat{y}(z_j) - z_j) \rangle < 0 .$$

Hence  $\psi(z_j, \hat{y}(z_j)) \leq 1$  .

Q.E.D.

(2.70) THEOREM:

If  $[y_k(z_j) \rightarrow \hat{y}(z_j)]$  and  $[z_j \notin \operatorname{argmin}\{f(z) : z \in Q\}]$   
 then for any  $\mu > 0$ , there exists  $N(\mu) < \infty$  such that  
 (2.71)  $\psi(z_j, y_k(z_j)) \leq 1 + \mu$  for all  $k > N(\mu)$

Proof: If  $y_k(z_j) \rightarrow \hat{y}(z_j)$  then  $\psi(z_j, y_k) \rightarrow \psi(z_j, \hat{y})$   
 i.e. for all  $\mu > 0$  there exists  $N(\mu)$  such that for  $k > N(\mu)$

$$|\psi(z_j, y_k) - \psi(z_j, \hat{y}(z_j))| < \mu$$

i.e.

$$\psi(z_j, y_k) < \psi(z_j, \hat{y}(z_j)) + \mu$$

So, since by Theorem (2.69)  $\psi(z_j, \hat{y}(z_j)) \leq 1$ ,

$$\psi(z_j, y_k) \leq 1 + \mu$$

Q.E.D.

(2.72) REMARK: Consider algorithm-2 with the stopping conditions for the subproblem being that both conditions (2.17) with a pre-specified  $\epsilon$  and condition (2.71) for a pre-specified  $\mu$  are satisfied. Then from theorems (2.54) and (2.70):

$$[f(z_{j+1}) - f(\hat{z})] \leq [f(z_j) - f(\hat{z})] \left\{ 1 - \frac{(1-\epsilon)\beta(1-\frac{1}{2}\beta)}{(1+\mu)(1+\frac{M}{n})} \right\}$$

## 2.4 APPLICATION TO AN OPTIMAL CONTROL PROBLEM

Application of Algorithms 1 and 2 to a basic open loop control problem is considered here.

Consider the control constrained linear quadratic problem of choosing a control  $u \in Q$  to minimize

$$f(u) \triangleq \frac{1}{2} \int_T \{ \langle x^u(t), Kx^u(t) \rangle + \langle u(t), Ru(t) \rangle \} dt$$

for a finite-dimensional linear dynamic system,

$$\dot{x}^u(t) = Ax^u(t) + Bu(t) \quad : \quad x^u(t_s) = x_s$$

where

$$Q = \{ \text{square integrable } u : T \rightarrow \Omega \subset \mathbb{R}^m \}$$

$\Omega$  is a convex subset of  $\mathbb{R}^m$

$$K=K' \geq 0 \quad R=R' > 0$$

$$T=[t_0, t_1] \quad t_1 - t_0 < \infty$$

This problem can be considered as a particular case of the problem stated in 2.1. If  $\nabla^2 f$  could be shown to be bounded on a similar way and an approximation  $f^S(u, v)$  to  $f(v)$  were available then either algorithm 1 or 2 could be used. In fact Mayne, using a strong variation approach, finds exact and approximate expressions for

$$(2.73) \quad \Delta f = f(v) - f(u)$$

where neither  $v$  nor  $u$  have to be optimal. An estimate of  $\Delta f$  is known [4] to be:

$$(2.74) \quad \hat{\Delta}f = \int_T [H(x^u(t), v(t), \lambda^u(t), t) - \\ H(x^u(t), u(t), \lambda^u(t), t)] dt$$

where  $H(x, u, \lambda, t)$  is the Hamiltonian and  $\lambda^u$  is the solution to the usual adjoint differential equation evaluated at  $x^u$ .

Now by choosing an appropriate Hilbert Space (in fact  $L_2[t_0, t_1]$  with an induced inner product) the system response  $x^u$  can be written:

$$x^u = x^s + Wu$$

where  $x^s$  is the initial condition contribution and  $W$  is the input-output map (a convolution operator).

The cost function can now be expressed as:

$$(2.75) \quad f(u) = \frac{1}{2} \langle x^u, \mathcal{K} x^u \rangle + \frac{1}{2} \langle u, \mathcal{R} u \rangle \\ = \frac{1}{2} \langle x^s, \mathcal{K} x^s \rangle + \langle u, W^* \mathcal{K} x^s \rangle + \frac{1}{2} \langle u, [\mathcal{R} + W^* \mathcal{K} W] u \rangle$$

$$\text{where} \quad [\mathcal{K} x^u](t) \triangleq K x^u(t) \quad \text{a.e. } t \in T \\ [\mathcal{R} u](t) \triangleq R u(t) \quad \text{a.e. } t \in T$$

$$\mathcal{K} = \mathcal{K}^* \geq 0 \quad \mathcal{R} = \mathcal{R}^* > 0$$

$$\text{and} \quad \langle u, \mathcal{R} u \rangle \triangleq \int_T \langle u(t), R u(t) \rangle dt$$

where  $u$  is a function belonging to  $Q$  and  $u(t)$  is a vector in  $\mathbb{R}^m$  belonging to  $\Omega$ .

From (2.75)

$$\nabla^2 f(u) = \mathcal{R} + W^* \mathcal{K} W.$$

Now since the gradient is known to be given by:

$$[\nabla f(u)](t) = H[x^u(t), v, \lambda^u(t), t]_{v|v=u(t)}$$

and the Hamiltonian is defined as:

$$H[x, u, \lambda, t] \triangleq \frac{1}{2} \langle x, Kx \rangle + \frac{1}{2} \langle u, Ru \rangle + \langle \lambda, Ax + Bu \rangle$$

we get from (2.73) and (2.74) an estimate of  $f(v)$  given by:

$$\hat{f}(u, v) = f(u) + \langle \nabla f(u), (v-u) \rangle + \frac{1}{2} \langle (v-u), \mathcal{R}(v-u) \rangle$$

and then, for  $S \triangleq \mathcal{R}$

$$\hat{f}(u, v) = f^S(u, v) .$$

For this choice of  $S$ ,  $M$  can be chosen to be:

$$M = \| \nabla^2 f(u) - \mathcal{R} \| = \| W^* \mathcal{K} W \|$$

and then the control problem can be viewed as a special case of the general problem of 2.1.

In certain particular conditions the approximation function  $f^S(u, v)$  can be minimized exactly and then Algorithm-1 can be used. This is the case for the application example considered by Allwright [3], which will be reviewed next. Another example will be considered later for which exact minimization of  $f^S$  is not possible.

#### Example (i) [3]

$$\text{Let } \Omega = \{ u \in \mathbb{R}^m : |u_i| \leq 1 \}$$

$$R = \text{diag}\{r_{11}, \dots, r_{mm}\} > 0$$

In this case it is easy to minimize  $f^S(u,v)$  with respect to  $v \in Q$  as:

$$f^S(u,v) = f(u) + \int_T \{ \langle [\nabla f(u)](t), [v(t)-u(t)] \rangle + \frac{1}{2} \langle [v(t)-u(t)], R[v(t)-u(t)] \rangle \} dt$$

and  $\{\hat{v}(u)\} = \operatorname{argmin} \{ f^S(u,v) : v \in Q \}$   
i.e.

$$\{[\hat{v}(u)](t)\} = \operatorname{argmin} \{ \langle [\nabla f(u)](t), [v-u(t)] \rangle + \frac{1}{2} \langle [v-u(t)], R[v-u(t)] \rangle ; v(t) \in \Omega \}$$

Due to the diagonality assumption for  $R$ , the problem of finding  $[\hat{v}(u)](t) \in \Omega \subset \mathbb{R}^m$  splits into the  $m$  simpler problems of finding

$$[\hat{v}_i(u)](t) \in \operatorname{argmin} \{ \langle [\frac{\partial f(u)}{\partial u_i}](t), [v_i - u_i(t)] \rangle + \frac{1}{2} r_{ii} [v_i - u_i(t)]^2 : |v_i| \leq 1 \}$$

The solution for each value of  $i$  involves minimizing a quadratic on a line segment. That is easy to do. The result is that

$$[\hat{v}_i(u)](t) = \begin{cases} [v_i^*(u)](t) & \text{if } |v_i^*| \leq 1 \\ 1 & \text{if } v_i^* > 1 \\ -1 & \text{if } v_i^* < -1 \end{cases}$$

where

$$[v_i^*(u)](t) = u_i(t) - \frac{1}{r_{ii}} \left[ -\frac{\partial f}{\partial u_i} \right](t)$$

Then  $[\hat{v}_i(u)](t)$  can be expressed in closed form as:

$$[\hat{v}_i(u)](t) = \text{sat}\{ u_i(t) - \frac{1}{r_{ii}} \left[ \frac{\partial f(u)}{\partial u_i} \right](t) \}$$

so that finally

$$[\hat{v}(u)](t) = \text{sat}\{ u(t) - R^{-1} [\nabla f(u)](t) \}$$

where sat is defined by

$$\text{Sat}\{u\}_i = \begin{cases} u_i & \text{if } |u_i| \leq 1 \quad \forall i \\ 1 & \text{if } u_i > 1 \quad \text{for any } i \\ -1 & \text{if } u_i < -1 \quad \text{for any } i \end{cases}$$

Eventhough this exact closed form solution has been obtained under severe assumptions on  $R$  and  $\Omega$ , they happen to be of practical interest.

#### Example (ii)

$$\text{Let } \Omega = \{ u \in \mathbb{R}^m : ||u|| \leq 1 \}$$

$$R = \text{diag}\{r_{11}, \dots, r_{mm}\} > 0$$

Now again the problem consists of finding  $[\hat{v}(u)](t)$

so

$$[\hat{v}(u)](t) \in \underset{v(t) \in \Omega}{\text{argmin}} \{ \langle [\nabla f(u)](t), [v - u(t)] \rangle + \frac{1}{2} \langle [v - u(t)], R[v - u(t)] \rangle : v(t) \in \Omega \}$$

or consequently:

$$\begin{aligned} [\hat{v}(u)](t) \in \operatorname{argmin} \{ \langle ([\nabla f(u)](t) - Ru(t)), v \rangle \\ + \frac{1}{2} \langle v, Rv \rangle : \|v\| \leq 1 \} \end{aligned}$$

Again the unconstrained solution will be given by:

$$[v^*(u)](t) = -R^{-1} \{ [\nabla f(u)](t) - Ru(t) \}$$

and then

$$[\hat{v}(u)](t) = \begin{cases} [v^*(u)](t) & \text{if } \|v^*\| \leq 1 \\ \hat{v} \text{ with } \|\hat{v}\|=1 & \text{if } \|v^*\| > 1 \end{cases}$$

and

$$\left. \frac{\partial f^S(u, v)}{\partial v} \right|_{v=\hat{v}} = [\nabla f(u)](t) - Ru(t) + R\hat{v} = -\alpha \hat{v} \text{ for some } \alpha \geq 0$$

Hence

$$\begin{aligned} [\hat{v}(u)](t) = -[R + \alpha I]^{-1} \{ [\nabla f(u)](t) - Ru(t) \} \text{ for some } \alpha > 0 \\ \text{such that: } \|\hat{v}\| = 1 \end{aligned}$$

The condition  $\|\hat{v}\|=1$  is equivalent to

$$(2.76) \quad \langle g, [R + \alpha I]^{-2} g \rangle = 1$$

where  $g = [\nabla f(u)](t) - Ru(t)$

As  $R$  is diagonal (2.76) is equivalent to:

$$(2.77) \quad \sum_i \frac{g_i^2}{(r_{ii} + \alpha)^2} = 1$$

where  $g_i = \left[ \frac{\partial f(u)}{\partial u_i} \right](t) - r_{ii} u_i(t)$

For  $i=2$  the equation (2.77) is of  $4^{\text{th}}$  degree and eventhough an exact closed form solution is available it's complexity is such that an approximate solution is preferable in most cases. For  $i>2$  only an approximate solution  $\tilde{\alpha}$  can be found. Then we will get the situation of (2.17).

REMARK: The assumption on the diagonality of  $R$  can be relaxed since (i) inner products (and hence norms) are invariant under orthogonal transformations and (ii) because  $R$  is positive definite, an orthogonal change of basis ensuring diagonalization of  $R$  is possible.

## 2.5 MINIMIZING A NORM OVER A CONVEX SET

### GILBERT CONVERGENCE RESULTS

The particular case with the function  $f(\cdot)$  having a simple and known structure is discussed here. If  $f(\cdot)$  is a quadratic, some of the results, derived in [3] and in the present chapter, will apply if the appropriate identifications are made.

Let  $f(\cdot)$  be the quadratic

$$(2.79) \quad f(z) \triangleq \langle z, Qz \rangle$$

where  $Q' = Q \geq 0$

and consider the following identifications:

$$S=0, \quad N=0, \quad M=2 \lambda_{\max}[Q]$$

and

$$f^S(z_j, y) = f^0(z_j, y) = f(z_j) + \langle \nabla f(z_j), (y - z_j) \rangle$$

The results of theorem (2.8) are valid for this case and for the sequence generated by Algorithm-1 global convergence rate information is given by:

(a) If  $[f(z_j) - f(\hat{z})] \geq 2\gamma^2 \lambda_{\max}[Q]$ , then

$$(2.80) \quad [f(z_{j+1}) - f(\hat{z})] \leq \gamma^2 \lambda_{\max}[Q] \leq \frac{1}{2} [f(z_j) - f(\hat{z})]$$

(b) If  $[f(z_j) - f(\hat{z})] < 2\gamma^2 \lambda_{\max}[Q]$

$$(2.81) \quad f(z_{j+1}) - f(\hat{z}) \leq [f(z_j) - f(\hat{z})] \left\{ 1 - \frac{1}{4\lambda_{\max}[Q]} \gamma^2 [f(z_j) - f(\hat{z})] \right\}$$

Consider the case  $Q=I$ . This corresponds to the function  $f(z)$  being the usual 2-norm of  $z$ :

$$(2.82) \quad f(z) = \langle z, Iz \rangle = ||z||^2$$

Then the above results (a) (b) will assume the form

(a) If  $||z_j||^2 - ||\hat{z}||^2 \geq 2\gamma^2$ , then

$$(2.83) \quad ||z_{j+1}||^2 - ||\hat{z}||^2 \leq \gamma^2 \leq \frac{1}{2} (||z_j||^2 - ||\hat{z}||^2)$$

(b) If  $||z_j||^2 - ||\hat{z}||^2 < 2\gamma^2$ , then

$$(2.84) \quad ||z_{j+1}||^2 - ||\hat{z}||^2 \leq (||z_j||^2 - ||\hat{z}||^2) \left\{ 1 - \frac{1}{4} \gamma^{-2} (||z_j||^2 - ||\hat{z}||^2) \right\}$$

(2.85) REMARK:  $\beta$  has been taken equal to 1 since minimization of a quadratic on a line segment can be done exactly.

### Gilbert's Results

Working on the extension by Fancher [5] of an idea by Ho [6], Gilbert [7] proved convergence of an iterative procedure for computing the minimum of a quadratic form on a convex set. The proof is based on an algorithm which is essentially algorithm-1 but where the objective function is a squared norm as in (2.82)

The basic problem (B.P.) stated by Gilbert [3] is essentially:

(B.P.) Given  $K$ , a compact and convex set in  $E^n$ , find a point

$$\hat{z} \in K \text{ such that } \|\hat{z}\|^2 = \min \{ \|z\|^2 : z \in K \}$$

In order to define the iterative procedure to compute the solution of (B.P.), Gilbert introduces the function  $s(\cdot)$  as a specific contact function of  $K$  and defines the function  $\beta(\cdot)$ :

$$(2.86) \quad \beta(z) = \begin{cases} \|z - s(-z)\|^2 \langle z, (z - s(-z)) \rangle & \text{if } z - s(-z) \neq 0 \\ 0 & \text{if } z - s(-z) = 0 \end{cases}$$

whose geometric significance is as follows:

$x = z + \beta(z)(s(-z) - z)$  is the point on the line  $L(z; s(-z))$  with minimum euclidean norm.

The iterative procedure considered by Gilbert defines a sequence of vectors  $\{z_k\}$  by:

$$(2.87) \quad z_{k+1} = z_k + \alpha_k (s(-z_k) - z_k) \quad k=0,1,\dots$$

where  $z_0$  is an arbitrary point in  $K$  and the scalars  $\alpha_k$  are selected arbitrarily from the closed interval  $I(z)$ ,

$$(2.88) \quad I(z) = [\min\{\delta\beta(z), 1\}, \min\{(2-\delta)\beta(z), 1\}]$$

where  $0 < \delta = \text{fixed number} \leq 1$

For the case under consideration it can easily be shown how similar Gilbert's and Allwright's procedures are. In fact let  $s(-z_j)$  be, as in [7], the solution of:

$$(2.89) \quad \max\{ \langle -z_j, y \rangle : y \in K \}$$

or equivalently of:

$$(2.90) \quad \min\{ \langle z_j, y \rangle : y \in K \}$$

If it is remembered that for the particular case under study  $\nabla f(z_j) = Zz_j$  and that  $s(wy) = s(y)$  for any  $w \in \mathbb{R}^+$  (a property of the contact function) then  $s(-z_j)$  can be said to be the solution of:

$$\min\{ \langle \nabla f(z_j), y \rangle : y \in K \}$$

or finally of

$$(2.91) \quad \min\{ f(z_j) + \langle \nabla f(z_j), (y - z_j) \rangle = f^0(z_j, y) : y \in K \}$$

and so the equivalence between  $s(-z_j)$  in [7] and  $\hat{y}(z_j)$  in [3] is obvious.

Again in order to emphasize the connections between the two approaches, an outline of Gilbert's proof of convergence for the algorithm is presented and comparative comments made.

Consider the functions  $\Gamma(z)$  and  $\Delta(z; \alpha)$  defined by [7]:

$$(2.92) \quad \Gamma(z) \triangleq ||z||^2 - ||\hat{z}||^2$$

and

$$(2.93) \quad \Delta(z; \alpha) \triangleq \Gamma(z) - \Gamma(z + \alpha[s(-z) - z])$$

where  $\alpha \in I(z)$

Now  $\Delta(z; \alpha)$  can be expressed as

$$(2.94) \quad \Delta(z; \alpha) = 2\alpha[||z||^2 - \langle s(-z), z \rangle] - \alpha^2 ||z - s(-z)||^2$$

Because the coefficient of  $\alpha^2$  is not positive,  $\min\{\Delta(z; \alpha) : \alpha \in I(z)\}$  is attained at one of the end points of  $I(z)$ .

Also it is readily shown that

$$\Delta(z; \delta\beta(z)) = \Delta(z; (2-\delta)\beta(z))$$

Thus from the definition of  $I(z)$

$$(2.95) \quad \min\{\Delta(z; \alpha) : \alpha \in I(z)\} = \begin{cases} \Delta(z; \delta\beta(z)) & \text{if } \delta\beta(z) \leq 1 \\ \Delta(z, 1) & \text{if } \delta\beta(z) \geq 1 \end{cases}$$

This result is used by Gilbert to derive a lower bound on  $\Delta(z; \alpha)$ ,  $\alpha \in I(z)$ , as follows. From (2.86) it follows that

$$(2.96) \quad \begin{aligned} \Delta(z; \delta\beta(z)) &= ||z - s(-z)||^{-2} [\langle z, (z - s(-z)) \rangle]^2 (2\delta - \delta^2) \\ &\geq \gamma^{-2} \delta [\langle z, (z - s(-z)) \rangle]^2 \end{aligned}$$

since  $2\delta - \delta^2 > \delta$ , where  $\gamma$  is the diameter of  $K$  defined in the usual way.

Now, by definition of  $\Gamma(z)$

$$\begin{aligned}\Gamma(z) &= ||z||^2 - ||\hat{z}||^2 \\ &= ||z - \hat{z}||^2 + 2\langle \hat{z}, (z - \hat{z}) \rangle \\ &\leq 2||z - \hat{z}||^2 + 2\langle \hat{z}, (z - \hat{z}) \rangle \\ &= 2\langle z, (z - \hat{z}) \rangle \\ &\leq 2\langle z, (z - s(-z)) \rangle \text{ by the definition of } s(-z)\end{aligned}$$

and then

$$(2.97) \quad \langle z, (z - s(-z)) \rangle \geq \frac{1}{2} \Gamma(z)$$

Using this result in (2.96) yields

$$(2.98) \quad \Delta(z; \delta\beta(z)) \geq \frac{1}{4} \gamma^{-2} \delta \Gamma^2(z)$$

Now consider  $\Delta(z; 1)$ .

If  $\beta(z) \geq 1$  then, from (2.86),  $\langle z, (z - s(-z)) \rangle \geq ||z - s(-z)||^2$

and consequently from (2.94) it follows that

$$\begin{aligned}(2.99) \quad \Delta(z; 1) &= 2\langle z, (z - s(-z)) \rangle - ||z - s(-z)||^2 \\ &\geq \langle z, (z - s(-z)) \rangle \\ &\geq \frac{1}{2} \Gamma(z) \quad \text{by (2.97)}\end{aligned}$$

Using this result and (2.98) in (2.95) yields:

$$\Delta(z; \alpha) \Big|_{\alpha \in I(z)} \geq \min\left\{\frac{1}{4} \gamma^{-2} \delta \Gamma^2(z), \frac{1}{2} \Gamma(z)\right\}$$

and finally by setting  $z=z_k$  and using (2.87) it can be seen that

$$(2.100) \quad \Delta(z_k, \alpha_k) = \Gamma(z_k) - \Gamma(z_{k+1}) \\ \geq \min\left\{\frac{1}{4} \gamma^{-2} \delta \Gamma^2(z_k), \frac{1}{2} \Gamma(z_k)\right\} \geq 0$$

This is the lower bound on  $\Delta$  derived by Gilbert. Clearly the sequence  $\{\Gamma(z_k)\}$  is decreasing and since it is bounded from below by zero, has a limit point. Passing to the limit on the left side of (2.100) gives zero and therefore from the right side  $\Gamma(z_k) \rightarrow 0$ .

#### Connections between Gilbert's results and Allwright's

Now since  $\delta$  expresses the effect of not minimizing exactly  $||z||^2$  along the line  $L(z, s(-z))$  and as this can be done exactly, take  $\delta=1$ .

Suppose that  $\Gamma(z_k) \geq 2\gamma^2$ .

Then  $\Gamma^2(z_k) \geq 2\gamma^2 \Gamma(z_k)$

and so  $\frac{1}{4} \gamma^{-2} \Gamma^2(z_k) \geq \frac{1}{2} \Gamma(z_k)$

Hence, from (2.100) it follows

$$\Gamma(z_{k+1}) \leq \frac{1}{2} \Gamma(z_k)$$

i.e. from (2.92)  $[||z_{k+1}||^2 - ||\hat{z}||^2] \leq \frac{1}{2} [||z_k||^2 - ||\hat{z}||^2]$

which is exactly the same result as the one, (2.83) obtained by Allwright under the same assumption.

Considering now that

$$\Gamma(z_k) \leq 2\gamma^2$$

Then  $\frac{1}{4} \gamma^{-2} \Gamma^2(z_k) \leq \frac{1}{2} \Gamma(z_k)$

and from (2.100) it follows that

$$(2.101) \quad \Gamma(z_{k+1}) \leq \Gamma(z_k) \{1 - \frac{1}{4} \gamma^{-2} \Gamma(z_k)\}$$

again the same as (2.84) obtained by Allwright under the same assumption.

However by considering the upper bound to the rate given by:

$$[1 - \frac{1}{4} \gamma^{-2} \Gamma(z_k)] \leq [1 + \frac{1}{4} \gamma^{-2} \Gamma(z_k)]^{-1}$$

Gilbert presents a convergence result in closed form:

**THEOREM** [7] - Let  $\theta_k = \theta_0 (1 + \frac{1}{4} \gamma^{-2} \theta_0 k)^{-1}$ ,  $\theta_0 = \|z_0\|^2 - \|\hat{z}\|^2$

and assume that  $\|z_0\|^2 \leq \|\hat{z}\|^2 + 2\gamma^2$ .

Then for the sequence  $\{z_k\}$  generated by the algorithm the following inequalities hold for  $k \geq 0$ :

$$\|z_k\| \leq \sqrt{\theta_k + \|\hat{z}\|^2}$$

$$\|z_k - \hat{z}\| \leq \sqrt{\theta_k}$$

It is now clear that for the particular case under consideration both procedures, i.e. Gilbert's [7] and Allwright's [3], have similar features and the convergence results, eventhough obtained using different approaches, are the same. However Allwright's results apply to a larger class of problems (i.e. for objective functions which are more general convex functions).

## 2.6 REFERENCES

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CHAPTER 3  
INNER APPROXIMATION ALGORITHMS

3.1 METHODS INVOLVING APPROXIMATION  
OF THE FEASIBLE SET

Consider the constrained optimization problem:

$$(3.1) \quad \min \{ f(z) : z \in Q \}$$

Suppose that  $Q, Q_n$  belong to some overall set. In many methods for constrained optimization, the feasible set  $Q$  is approximated by a sequence of sets  $Q_n$  and the following sequence of problems is solved

$$(3.2) \quad \min \{ f(z) : z \in Q_n \} \quad n=0,1,\dots$$

Under certain assumptions on  $f(\cdot)$ ,  $Q$  and the sequence  $Q_n$ , problems (3.1) and (3.2) are equivalent in the sense that the limit of the sequence of minimal values for (3.2) is the minimal value for (3.1). In detail:

(3.3) THEOREM [1]

Let  $Q$  be any set in Banach space  $E$  and let  $f(z)$  be a functional satisfying a Lipschitz condition on  $\bigcup_{n=1}^{\infty} Q_n$  and bounded from below on  $Q$  in that

$$(3.4) \quad \inf_{z \in Q} f(z) = f^* > -\infty$$

Suppose  $x^n$  minimizes  $f$  approximately on  $Q_n$  in that

$$(3.5) \quad f(x^n) \leq \inf_{x \in Q_n} f(x) + \epsilon_n$$

where  $\lim_{n \rightarrow \infty} \epsilon_n = 0$

and where  $Q$  and  $Q_n$  have the properties that

$$(3.6) \quad \lim_{n \rightarrow \infty} \inf_{y \in Q_n} \|x - y\| = 0 \quad x \in Q$$

$$(3.7) \quad \lim_{n \rightarrow \infty} \inf_{x \in Q} \|x^n - x\| = 0$$

$$(3.8) \quad \text{then} \quad \lim_{n \rightarrow \infty} f(x^n) = f^* .$$

(3.9) REMARKS: (i) (3.6) means that by taking  $n$  big enough one can get arbitrarily near any  $x \in Q$  with a  $y$  in  $Q_n$ .

(ii) (3.7) means that by taking  $n$  big enough one can get arbitrarily near  $x^n \in Q_n$  with an  $x$  from  $Q$ .

(iii) If the approximation sets  $Q_n$  are "outer sets" (i.e.  $Q_n \supset Q \quad \forall n$ ) then 3.6 is obviously satisfied. If the

approximation sets are "inner" ( $Q_n \subset Q \quad \forall n$ ) then (3.7) is satisfied automatically.

(iv) the set sequence  $Q_n$  can be chosen in advance (as in the Ritz method) or the next  $Q_n$  can be determined from the results of previous problems (as in the cut-off methods).

The set approximation approach consists essentially of a restatement of a given problem in an alternative form that is equivalent but hopefully more amenable to solution. Associated with this type of approach there are usually some specific aims like:

- (i) To isolate familiar special structures embedded in a given problem.
- (ii) To induce linearity in a non-linear problem via judicious approximation.
- (iii) To induce separation.

Amongst the most popular devices used in pursuit of these objectives are Dualization, Projection and Inner (outer) Linearization. The last two are particular cases of set approximation methods. Inner linearization will be the main subject of interest in the following.

### 3.2 INNER LINEARIZATION

Rockafellar [2] distinguishes between two representations of a closed convex set  $Q$ , which are dual to each other.  $Q$  is said to be externally (outerly) represented if it is defined as the intersection of all half-spaces that contain  $Q$ .  $Q$  is said to be internally (innerly) represented if it is defined as the union of all  $d$ -dimensional simplices whose vertices belong to  $Q$ , where  $d = \dim Q$ .

For convex functions, equivalent representations exist since convex functions on  $\mathbb{R}^n$  can be identified with certain convex sets of  $\mathbb{R}^{n+1}$  (their epigraphs). A function approximation will be "inner" ("outer") if the epigraph of the approximation function lies entirely within (outside) the epigraph of the approximated function.

A convex set  $Q$  is said to have been linearized if the approximating set is a convex polytope and a convex function  $f$  is said to be linearized if the epigraph of the approximating function is a convex polytope. If the convex polytope is defined as the convex hull of a finite set of points (called a base) belonging to  $Q$  (or  $\text{epi } f$ ), then the set  $Q$  (the function  $f$ ) is said to be "inner" linearized. Inner linearization is then an approximation to convex functions and sets which is conservative in the sense that

it does not underestimate the value of a convex function or include any points outside the approximated convex set.

The accuracy of the approximation can be made as great as desired by making the density of the base sufficiently large. However from theorem (3.3) it is known that the approximating sets only have to be accurate in a neighbourhood of the optimum, this suggests the desirability of a procedure to generate base points as actually needed in the course of computationally solving the inner linearized problem.

### 3.3 A FEASIBLE DIRECTION TYPE OF ALGORITHM VERSUS AN SET-APPROXIMATION TYPE OF ALGORITHM

Consider the conceptual algorithm that results from using the iterative method presented in section 2.2.1. to solve problem (P), when the optimal steplength is used along each search direction. This conceptual algorithm can be given a different interpretation with emphasis on its decomposable nature - Polak has already pointed out the decomposable nature of methods of feasible directions since they effectively replace the original problem by a sequence of simplex problems [3] pp. 173.

In fact the algorithm solves (P) by decomposing it into two complementary problems which are solved alternately through successive rounds. These two

subproblems associated with algorithm-1 will be denoted  $(P_1)$  and  $(P_2)$  and at iteration  $j$  are defined by :

—  $(P_1)$  Find  $\hat{y}(z_j)$  such that

$$(3.10) \quad \hat{y}(z_j) \in \operatorname{argmin} \{ f^S(z_j, y) : y \in Q \}$$

where

$$(3.11) \quad f^S(z_j, y) \triangleq f(z_j) + \langle \nabla f(z_j), (y - z_j) \rangle + \frac{1}{2} \langle (y - z_j), S(y - z_j) \rangle$$

—  $(P_2)$  Find  $z_{j+1}$  such that

$$(3.12) \quad z_{j+1} \in \operatorname{argmin} \{ f(z) : z \in Q_{j+1} \}$$

where

$$(3.13) \quad Q_{j+1} = \operatorname{conv} \{ \{z_j\} \cup \{\hat{y}(z_j)\} \}.$$

(3.14) REMARKS:(i) There exists a conceptual connection between  $(P_1)$  and  $(P_2)$ . In fact for  $(P_1)$  the feasible set is kept exactly as defined for  $(P)$  but the objective function is replaced by a quadratic (or linear) outer approximation. For  $(P_2)$  the objective function is exactly the same as in  $(P)$  but the feasible set is now a linear inner approximation to the original feasible set.

(ii)  $(P_2)$  can be viewed as a partial version of  $(P)$ , i.e., the original problem solved over a linearized version of  $Q$ .

(iii)  $Q_j$  is an "inner-approximation" (since  $Q$  is convex and both solutions to  $(P_1)$  and  $(P_2)$  are feasible) and is defined to be the convex hull of the union of the last solution to  $(P_2)$  and the latest solution to  $(P_1)$ .

In the light of this new interpretation, the conceptual algorithm of Section 2.2.1 can now be restated as follows:

Algorithm-1a

Initialization: let  $z_0 \in Q$ ,  $Q_0 \triangleq \{z_0\}$ ,

set  $j=0$ .

(i) Solve  $(P_1)$ , thereby finding  $\hat{y}(z_j)$ . If solution is optimal for  $(P)$ , STOP, otherwise CONTINUE.

(ii) Define the next "inner" approximation  $Q_{j+1}$ ,

$$Q_{j+1} = \text{conv}\{ \{z_j\} \cup \{\hat{y}(z_j)\} \}.$$

(iii) Solve  $(P_2)$  by finding a  $z_{j+1}$  which minimizes  $f$  on  $Q_{j+1}$ .

(iv) Set  $j=j+1$  and GO TO (i).

(3.15) REMARK: It is now obvious that the iterative procedure of 2.2.1. is a member of a class of set approximation methods.

### 3.4 A NATURAL EXPANSION

Eventhough  $(P_1)$  and  $(P_2)$  are problems which are simpler to solve than  $(P)$  - because the feasible set in  $(P_2)$  has a linear structure and the objective function in  $(P_1)$  has a quadratic (or linear) structure - solving  $(P_1)$  remains computationally expensive (even in simple cases e.g.  $f^S(\cdot, \cdot)$  linear and  $Q$  stricly convex). The number of times  $(P_1)$  has to be solved depends on how much cost reduction the solution of  $(P_2)$  gives. Intuitively this will tend to become larger if  $Q_j$  is a better approximation to  $Q$ .

Consider then replacing  $Q_j$  (as defined previously) by  $S_j$  (a convex subset of  $Q$ ) such that

$$(3.16) \quad Q_j \subseteq S_j \subset Q$$

Obviously this will imply

$$(3.17) \quad f(z_j^{S_j}) \leq f(z_j^{Q_j})$$

where

$$(3.18) \quad z_j^{S_j} \in \operatorname{argmin}\{f(z) : z \in S_j\}$$

$$(3.19) \quad z_j^{Q_j} \in \operatorname{argmin}\{f(z) : z \in Q_j\}.$$

### 3.4.1 Generating an Inner Approximation $S_j$

In (3.13) the inner approximation  $Q_j$  is defined as the convex hull of the last solution to  $(P_2)$  and the latest point generated by  $(P_1)$ :

$$Q_j = \text{conv} \{ \{z_{j-1}\} \cup \{\hat{y}(z_{j-1})\} \}$$

where  $\{z_{j-1}\}$  is a singleton subset of  $Q_{j-1}$ .

A natural way of generating an  $S_j$  satisfying (3.16) is to set

$$S_j = \text{conv} \{ S_{j-1} \cup \{\hat{y}(\bar{z}_{j-1})\} \}, \quad S_0 \triangleq Q_0$$

where  $\bar{z}_{j-1} \in \text{argmin} \{ f(z) : z \in S_{j-1} \}$ .

In fact, if  $z_0 = \bar{z}_0$ , implying

$$\{z_0\} \triangleq Q_0 = S_0 \triangleq \{\bar{z}_0\},$$

then:

$$Q_1 \triangleq \text{conv} \{ \{z_0\} \cup \{\hat{y}(z_0)\} \} = \text{conv} \{ \{\bar{z}_0\} \cup \{\hat{y}(\bar{z}_0)\} \} \triangleq S_1$$

$$Q_2 \triangleq \text{conv} \{ \{\bar{z}_1\} \cup \{\hat{y}(\bar{z}_1)\} \} \subseteq \text{conv} \{ S_1 \cup \{\hat{y}(\bar{z}_1)\} \} \triangleq S_2.$$

Carrying further this argument, on a step by step basis, it is obvious that

$$S_j \supseteq Q_j \quad \forall j.$$

An extended version of the conceptual algorithm-1a can now be stated:

Algorithm-1b

Initialization:      let  $z_0 \in Q$  ,  $S_0 \triangleq \{z_0\}$ ,  
                         set  $j=0$

(i)    Solve  $(P_1)$  by minimizing  $f^S(z_j, y)$  with respect to  
          $y \in Q$ . If the solution is optimal for  $(P)$ , STOP  
         otherwise CONTINUE.

(ii)   Define an "inner" approximation  $S_{j+1}$

$$S_{j+1} = \text{conv} \{ S_j \cup \{\hat{y}(z_j)\} \} .$$

(iii) Solve  $(P_2)$  by finding a  $z_{j+1}$  which minimizes  $f$  on  
          $S_{j+1}$  .

(iv)   Set  $j=j+1$  and GO TO (i) .

(3.20) REMARKS:

(i)    In algorithm-1a,  $\dim \text{Aff}[Q_j]=1$  for all  $j$ . In  
         algorithm-1b,  $\dim \text{Aff}[S_j]$  can increase to  $n$ .

(ii)   In algorithm 1b it is not necessary to store the  
         convex hull but merely the base points.

- (iii) From remark (i), it can be seen that with algorithm 1a the maximum number of base points to be stored at each interaction is two, while with algorithm-1b it can be  $j$ .
- (iv) Remark (iii) reveals the desirability of a method (based on some kind of redundancy) which allows the dropping of points which are not needed for the solution (i.e. a method for base reduction).

### 3.4.2 A First Method

From theorem (3.3), the approximating sets only have to be a good approximation to  $Q$  in a neighbourhood of the optimum. The idea is then, before starting iterating on  $(P_2)$ , to assess which subset of the original set  $Q$  is worthwhile approximating.

Let  $z_j$  be the last available solution to  $(P_2)$ , i.e.

$$z_j \in \operatorname{argmin}\{f(z) : z \in S_j\}$$

where  $S_j = \operatorname{conv} b(S_j)$

and  $b(S_j) = \{y_1, \dots, y_m\}$ .

Then, assuming  $z_j$  is not optimal for  $(P)$ , we require a  $z_{j+1}$  satisfying:

$$f(z_{j+1}) \leq f(z_j) .$$

A necessary condition for a certain  $z$  to be a candidate for  $z_{j+1}$  is:

$$\langle \nabla f(z_j), (z - z_j) \rangle \leq 0 .$$

This means that the candidates for  $z_{j+1}$  are the elements of the subset  $B_j \subset Q$  defined by:

$$B_j = \{ z \in Q : \langle \nabla f(z_j), (z - z_j) \rangle \leq 0 \} .$$

Only this set is worthwhile approximating.

One possible criterion is to keep from the previous inner approximation only those points belonging to  $B_j$ , yielding the set  $\bar{F}_j$  defined by:

$$\bar{F}_j \triangleq S_j \cap B_j = \{ z \in S_j : \langle \nabla f(z_j), (z - z_j) \rangle \leq 0 \}$$

or

$$\bar{F}_j = \{ z \in S_j : \langle \nabla f(z_j), (z - z_j) \rangle = 0 \} \cup \{ z \in S_j : \langle \nabla f(z_j), (z - z_j) \rangle < 0 \} .$$

However since  $z_j$  is the exact solution of:

$$\min \{ f(z) : z \in S_j \}$$

the set  $\{ z \in S_j : \langle \nabla f(z_j), (z - z_j) \rangle < 0 \}$  is empty and hence:

$$(3.21) \quad \bar{F}_j = \{ z \in S_j : \langle \nabla f(z_j), (z - z_j) \rangle = 0 \} \triangleq F_j$$

the set  $F_j$ , denoted the "optimal face" of  $S_j$ , together with the latest solution to  $(P_1)$  - which certainly belongs to  $B_j$  - defines the next inner approximation  $S_{j+1}$ :

$$S_{j+1} = \text{conv} \{ F_j \cup \{\hat{y}(z_j)\} \} .$$

It will be shown next that the vertices of  $F_j$  are those of  $S_j$  which are in  $B_j$ .

(3.22) LEMMA: Let  $S_j = \text{conv } b(S_j)$  where  $b(S_j) = \{y_1, \dots, y_m\}$ .

Then  $F_j = \text{conv } b(F_j)$  where

$$(3.22a) \quad b(F_j) = \{ y \in b(S_j) : \langle \nabla f(z_j), (y - z_j) \rangle = 0 \} .$$

Proof: Since  $S_j = \text{conv}\{y_1, \dots, y_m\}$ , any  $x \in S_j$  can be written

$$\text{as} \quad x = \lambda_1 y_1 + \dots + \lambda_m y_m \quad \text{where} \quad 0 \leq \lambda_i \leq 1$$

$$\text{and} \quad \lambda_1 + \dots + \lambda_m = 1.$$

Hence, by the definition of  $F_j$ , if  $x \in F_j$  then

$$x = \lambda_1 y_1 + \dots + \lambda_m y_m$$

$$\text{and} \quad \langle \nabla f(z_j), (x - z_j) \rangle = 0 .$$

$$\text{But} \quad x - z_j = \lambda_1 (y_1 - z_j) + \dots + \lambda_m (y_m - z_j) \text{ and so}$$

$$(3.23) \quad 0 = \langle \nabla f(z_j), (x - z_j) \rangle$$

$$= \lambda_1 \langle \nabla f(z_j), (y_1 - z_j) \rangle + \dots + \lambda_m \langle \nabla f(z_j), (y_m - z_j) \rangle .$$

Since  $z_j$  minimizes  $f$  on  $S_j$  and  $y_1, \dots, y_m \in S_j$

$$\langle \nabla f(z_j), (y_i - z_j) \rangle \geq 0 \quad \forall i .$$

So, from (3.23) if for any  $i$

$$\langle \nabla f(z_j), (y_i - z_j) \rangle > 0$$

then  $\lambda_i = 0, \forall x \in F_j$ . Hence every  $x \in F_j$  can be written as  $\sum \lambda_i y_i$

where  $\lambda_i \geq 0, \forall i, \sum \lambda_i = 1$  and  $\lambda_i = 0$  when  $\langle \nabla f(z_j), (y_i - z_j) \rangle > 0$ .

Consequently  $F_j \subset \text{conv } b(F_j)$ .

Now from (3.21) and (3.22a) it is easy to see that  $b(F_j) \subset F_j$  and since  $F_j$  is a convex set  $\text{conv } b(F_j) \subset F_j$ .

Hence  $F_j = \text{conv } b(F_j)$ , i.e.

$$F_j = \text{conv } \{ y \in b(S_j) : \langle \nabla f(z_j), (y - z_j) \rangle = 0 \}$$

and the base for  $F_j$  is then obtained by rejecting all the elements of the base of  $S_j$  that do not satisfy:

$$\langle \nabla f(z_j), (y - z_j) \rangle = 0$$

This will be called a base reduction.

By incorporating this base reduction philosophy in algorithm-1b <sup>there</sup> results the conceptual algorithm-1c:

Algorithm-1c

Initialization: Let  $z_0 \in Q$ ,  $F_0 \triangleq \{z_0\}$ ,  $S_0 \triangleq \{z_0\}$

$$b(F_0) = \{z_0\}, \quad b(S_0) = \{z_0\}, \quad j=0.$$

i) Solve  $(P_1)$

If solution is optimal for  $(P)$  STOP, otherwise CONTINUE.

(ii) Define  $b(S_{j+1}) = \{ b(F_j) \cup \{\hat{y}(z_j)\} \}$

and  $S_{j+1} = \text{conv } \{ b(S_{j+1}) \}$ .

(iii) Solve  $(P_2)$ .

(iv) Find  $b(F_{j+1})$  by base reduction.

(v) Set  $j=j+1$  and GO TO (i).

### 3.4.3 Convergence Results

Due to the initial step of algorithm-1c and to the way in which  $F_j$  is constructed

$$z_j \in F_j \quad \forall j \geq 0$$

and so for each  $j$  :

$$S_{j+1} \triangleq \text{conv}\{F_j \cup \{\hat{y}(z_j)\}\} \supseteq \text{conv}\{z_j \cup \{\hat{y}(z_j)\}\} \triangleq Q_{j+1}.$$

These facts imply that the conceptual algorithm-1c will perform in the worst case as well as conceptual algorithm -1a and that it will hopefully perform better than algorithm -1a in some cases. This means that the convergence rates obtained by Allwright [4] and in Chapter 2 of this thesis will apply with  $\beta=1$  ( $\beta=1$  because only the case when subproblem  $(P_2)$  is solved exactly has been considered so far).

### 3.4.4 A Second Method

The first storage reduction method described previously is very much based on the assumption that it is possible to solve exactly subproblem  $(P_2)$ . When this is not so, the way in which the "optimal" face is defined may leads us to drop points which are important or to keep in points which are irrelevant. To drop points which are important will lead to wrong results or to non

converging sequences while keeping in irrelevant points may lead to the storage of an unnecessarily large number of points.

In method one it was shown that points  $p$  satisfying at iteration  $j$  the condition  $\langle \nabla f(z_j), (p - z_j) \rangle > 0$  could be removed where  $z_j$  was the exact solution of  $(P_2)$ . However if what is known is  $\tilde{z}_j$ , as an approximation to  $z_j$ , the same procedure cannot be used.

In the present situation, two sorts of problems should be considered: (i) How to avoid removing important points (ii) How to avoid keeping irrelevant points.

Consider problem (i).

It will be shown that, at least in certain cases, sufficient conditions can be established to ensure that important points are not rejected.

(3.24) THEOREM: Let  $f(x)$  be a convex function with

$$nI \leq \nabla^2 f(x) \leq \bar{M}.I, \quad n > 0$$

Suppose that  $\tilde{z}$  is an approximation to the unknown  $z_j$  which minimizes  $f$  on  $S_j$ .

Then  $\langle (p - z_j), \nabla f(z_j) \rangle > 0$ , and  $p$  can be discarded, from  $b(S_j)$

if  $\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle > 0$  and

$$(3.25) \quad \|z_j - \tilde{z}\| < \frac{\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle}{\bar{M} \|p - \tilde{z}\|}.$$

Proof: If

$$||z_j - \tilde{z}|| < \frac{\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle}{\bar{M} ||p - \tilde{z}||}$$

then

$$\bar{M} ||z_j - \tilde{z}|| < \frac{\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle}{||p - \tilde{z}||}$$

so

$$||\nabla f(z_j) - \nabla f(\tilde{z})|| \leq \bar{M} ||z_j - \tilde{z}||$$

implies

$$||\nabla f(z_j) - \nabla f(\tilde{z})|| < \frac{\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle}{||p - \tilde{z}||}$$

and then

$$\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle > ||p - \tilde{z}|| ||\nabla f(z_j) - \nabla f(\tilde{z})||$$

or

$$(3.24a) \quad \langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle > -\langle (p - \tilde{z}), [\nabla f(z_j) - \nabla f(\tilde{z})] \rangle.$$

Now since  $\langle (\tilde{z} - z_j), \nabla f(z_j) \rangle \geq 0$  ( as  $z_j$  minimizes  $f$  on  $S_j$  )

(3.24a) yields

$$\langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle + \langle (\tilde{z} - z_j), \nabla f(z_j) \rangle > -\langle (p - \tilde{z}), [\nabla f(z_j) - \nabla f(\tilde{z})] \rangle$$

$$\text{i.e. } \langle (p - \tilde{z}), \nabla f(\tilde{z}) \rangle + \langle (p - \tilde{z}), [\nabla f(z_j) - \nabla f(\tilde{z})] \rangle + \langle (\tilde{z} - z_j), \nabla f(z_j) \rangle > 0$$

$$\text{i.e. } \langle (p - \tilde{z}), \nabla f(z_j) \rangle + \langle (\tilde{z} - z_j), \nabla f(z_j) \rangle > 0.$$

Hence

$$\langle (p - z_j), \nabla f(z_j) \rangle > 0 \quad \text{and } p \text{ can be discarded from } b(S_j).$$

Q.E.D.

### 3.5 MINIMIZING A NORM OVER A CONVEX SET

Consider again the case where the objective function is

$$f(z) = \|z\|_2^2$$

and consider minimizing  $f(z)$  over  $Q$  where  $Q$  is a convex and compact subset of  $\mathbb{R}^n$  which does not include the origin (the solution would be trivial in that case).

In the light of the approach considered in this chapter, the problem of minimizing  $f(z)$  over  $Q$  will be replaced by a sequence of simpler problems to be solved alternately. They are defined by:

$$— (P_1) \quad \min_{y \in Q} f^0(z, y)$$

where  $f^0(z, y) = f(z) + \langle \nabla f(z), (y - z) \rangle$

(let  $\hat{y}(z)$  denote a global minimizer of  $f^0(z, y)$  with respect to  $y \in Q$ ).

$$— (P_2) \quad \min_{z \in \Omega_j} f(z)$$

where  $\Omega_j$  is an inner linearised version of  $Q$  (a polytope) defined as the convex hull of a finite number of points.

Algorithm-2a

Initialization:      Let     $z_0 \in Q$

Set  $j=0$  and  $\Omega_0 \triangleq \{z_0\}$  .

(i) Solve  $(P_1)$  over  $Q$ , i.e. determine a global minimizer

$\hat{y}(z_j)$  for the problem

$$\min_{y \in Q} f^0(z_j, y) .$$

(ii) Set  $\Omega_{j+1} = \text{conv}\{\Omega_j \cup \{\hat{y}(z_j)\}\}$  .

(iii) Solve  $(P_2)$  over  $\Omega_{j+1}$  i.e. find a  $z_{j+1}$  which minimizes  $f(z)$  over  $\Omega_{j+1}$  .

(iv) Set  $j=j+1$  and go to (i) .

Algorithm-2a here may involve the storage of a large number of points since at each of its iterations a new point generated at (i) will be added by (ii) to the inner approximation .

In order to help reduce the number of points to be stored  $\Omega_j$  could be updated by replacing it by its closest face  $CF_j$  where  $CF_j$  is defined as the intersection between  $\Omega_j$  and the support hyperplane tangent to  $\Omega_j$  at  $z_j$  and with outward normal  $z_j/\|z_j\|$  ( see figure 3.1 ).

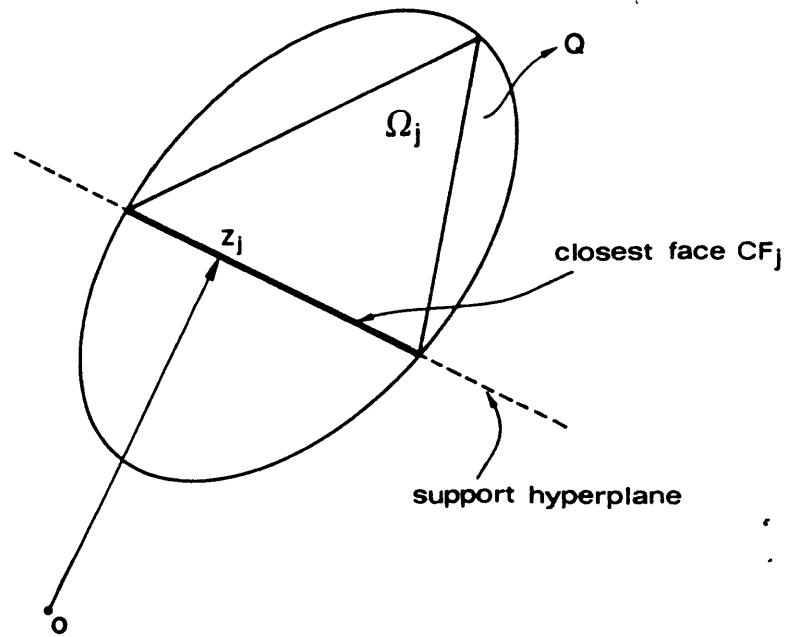


figure 3.1

### 3.5.1. Finding the closest face $CF_j$

As was mentioned previously, if  $z_j$  is the exact solution of  $(P_2)$ , finding the closest face is easy as it only requires the removal from  $\Omega_j$  of all points  $p$  satisfying

$$\langle \nabla f(z_j), (p - z_j) \rangle > 0 .$$

The remaining points will satisfy

$$\langle \nabla f(z_j), (p - z_j) \rangle = 0$$

and their convex hull will define the closest face.

However the exact solution  $z_j$  of  $(P_2)$  may not be available and an estimate  $\tilde{z}_j$  must then be used instead.

In that case a different method for finding the closest face is needed, and will be developed next.

Let  $\{z_j^i\}$  be a sequence in  $\Omega_j$  of approximations to  $z_j$  which converges to  $z_j$ .

Introduce the sets  $S_j^i$  and  $B_j^i$  defined as follows

$$(3.26) \quad S_j^i \triangleq \{ z \in \mathbb{R}^n : \|z\|^2 \leq \|z_j^i\|^2 \text{ and } \langle z, z_j^i \rangle \geq \langle y_j^i, z_j^i \rangle \}$$

where

$$(3.27) \quad y_j^i \in \operatorname{argmin} \{ \langle y, z_j^i \rangle : y \in \Omega_j \}$$

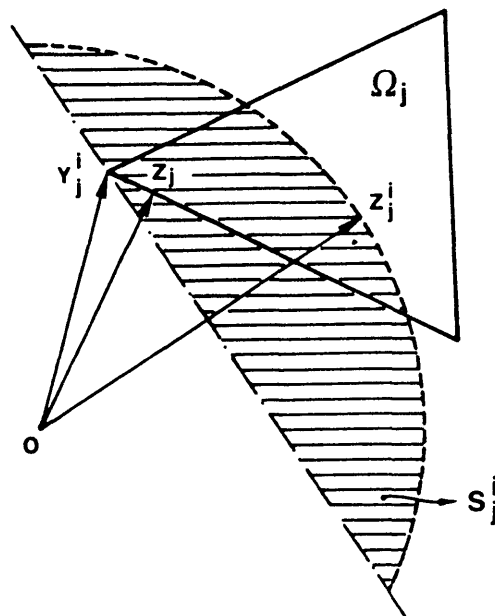


figure 3.2

and

$$(3.28) \quad B_j^i = \{z \in \mathbb{R}^n : \|z - z_j^i\|^2 \leq 2\|z_j^i\|^2 - 2\langle z_j^i, y_j^i \rangle \triangleq r^2(B_j^i)\}$$

where  $B_j^i$  is a closed sphere centered in  $z_j^i$  and whose radius  $r(B_j^i)$  goes to zero when  $z_j^i$  converges to  $z_j$ .

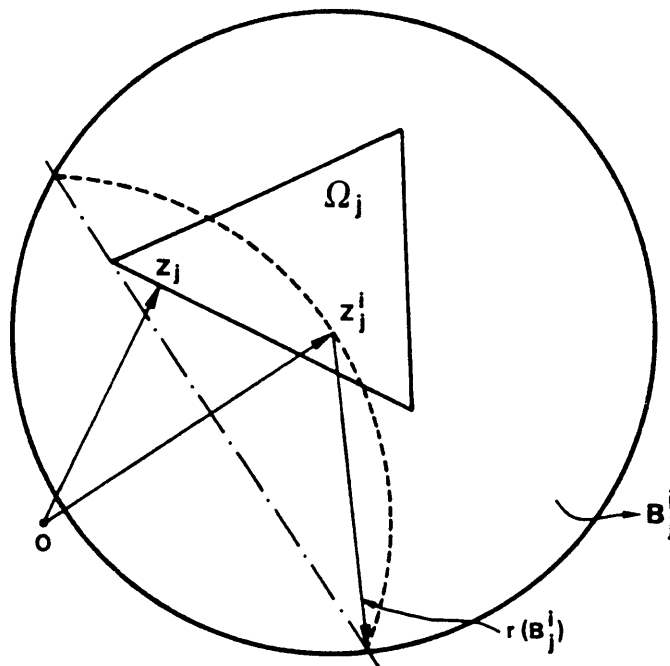


figure 3.3

$$(3.29) \quad \text{LEMMA: (i) } z_j \in B_j^i$$

$$(ii) \quad z_j \in S_j^i$$

Proof:

$$\begin{aligned}
 (i) \quad ||z_j - z_j^i||^2 &= ||z_j||^2 + ||z_j^i||^2 - 2 \langle z_j^i, z_j \rangle \\
 &\leq 2||z_j^i||^2 - 2\langle z_j^i, z_j \rangle \quad \text{since } ||z_j||^2 \leq ||z_j^i||^2 \\
 &\leq 2||z_j^i||^2 - 2\langle z_j^i, y_j^i \rangle \quad \text{since, from (3.27),} \\
 &\quad \langle y_j^i, z_j^i \rangle \leq \langle z, z_j^i \rangle \quad \forall z \in \Omega_j.
 \end{aligned}$$

So, from (3.28):  $z_j \in B_j^i$

$$(ii) \quad ||z_j||^2 \leq ||z_j^i||^2 \text{ by definition of } z_j.$$

Also

$$\langle z_j, z_j^i \rangle \geq \langle y_j^i, z_j^i \rangle \text{ by (3.27) and so, by (3.26) } z_j \in S_j^i$$

Q.E.D.

(3.30) REMARKS: (i) The result just proved for  $z_j$  extends to any  $z_j^k$  with  $k \geq i+1$ .

(ii) When  $z_j^i \rightarrow z_j$  obviously  $z - z_j^i \rightarrow 0$  and  $B_j^i \rightarrow \{z_j\}$ .

(3.31) THEOREM: Suppose  $\Omega_j = \text{conv}\{p_1, \dots, p_r\}$ .

If  $\bar{p} \in \{p_1, \dots, p_r\}$  and

$$\langle b, \bar{p} \rangle > \langle b, p_k \rangle \quad , \quad k=1, \dots, r$$

for all  $b \in B_j^i$ , then  $\bar{p}$  does not belong to the closest face,  $CF_j$ , of  $\Omega_j$  and can be eliminated.

Proof: By Lemma (3.29)  $z_j \in B_j^i$  and hence if  $\bar{p}$  maximizes (over all  $p_k$ ) the function  $\langle b, p_k \rangle$  for all  $b \in B_j^i$  then

$$\langle z_j, \bar{p} \rangle > \langle z_j, p_k \rangle \quad \forall p_k \neq \bar{p}.$$

Since  $z_j$  is the closest point to 0 in  $\Omega_j$  and since each  $p_i \in \Omega_j$

$$\langle z_j, p_i \rangle \geq ||z_j||^2.$$

Hence

$$\langle z_j, \bar{p} \rangle > ||z_j||^2 = f(z_j).$$

Since the closest face,  $CF_j$ , is defined by

$$CF_j \triangleq \{ z \in \Omega_j : \langle z, z_j \rangle = ||z_j||^2 \},$$

we see that  $\bar{p} \notin CF_j$ .

Q.E.D.

(3.32) THEOREM: Suppose

$$\alpha_j(z_j^i) = \langle z_j^i, \bar{p} \rangle - \max_{p_k \neq \bar{p}} \langle z_j^i, p_k \rangle > 0$$

Then  $\langle z, \bar{p} \rangle - \max_{p_i \neq \bar{p}} \langle z, p_i \rangle > 0$

for all  $z \in S \triangleq \{ z : ||z - z_j^i|| < \frac{\alpha_j(z_j^i)}{||\bar{p}|| + \max_{p_k \neq \bar{p}} ||p_k||} \triangleq r_s(z_j^i) \}.$

Proof: Suppose

$$\langle z_j^i, \bar{p} \rangle - \max_{p_k \neq \bar{p}} \langle z_j^i, p_k \rangle = \alpha_j(z_j^i) > 0 .$$

Consider now a perturbation  $\delta z$  from  $z_j^i$ . Then

$$\begin{aligned} & \langle (z_j^i + \delta z), \bar{p} \rangle - \max_{p_k \neq \bar{p}} \langle (z_j^i + \delta z), p_k \rangle \\ & \geq \langle z_j^i, \bar{p} \rangle + \langle \delta z, \bar{p} \rangle - \max_{p_k \neq \bar{p}} \langle z_j^i, p_k \rangle - \max_{p_k \neq \bar{p}} \langle \delta z, p_k \rangle \\ & = \alpha_j(z_j^i) + \langle \delta z, \bar{p} \rangle - \max_{p_k \neq \bar{p}} \langle \delta z, p_k \rangle \\ & \geq \alpha_j(z_j^i) - \|\delta z\| \|\bar{p}\| - \|\delta z\| \max_{p_k \neq \bar{p}} \|p_k\| \end{aligned}$$

$$\text{and hence if } \|\delta z\| \triangleq \|z - z_j^i\| < \frac{\alpha_j(z_j^i)}{\|\bar{p}\| + \max_{p_k \neq \bar{p}} \|p_k\|} ,$$

i.e. if  $z \in S$ , then

$$\langle z, \bar{p} \rangle - \max_{p \neq \bar{p}} \langle z, p \rangle > 0 .$$

Q.E.D.

(3.33) REMARK:  $S$  is an open sphere centered at  $z_j^i$  whose radius converges to

$$r_s(z_j) \triangleq \frac{\alpha_j(z_j)}{\|\bar{p}\| + \max_{p_k \neq \bar{p}} \|p_k\|} \neq 0 \quad \text{as } z_j^i \rightarrow z_j .$$

A procedure can now be considered to test the vertices of  $\Omega_j$

(i) Iterate until

$$\langle z_j^i, \bar{p} \rangle > \langle z_j^i, p \rangle \quad \forall p \neq \bar{p} .$$

(ii) Evaluate the radius  $r(B_j^i)$  of  $B_j^i$  .

(iii) Evaluate the radius  $r_S(z_j^i)$  of  $S$  .

If  $r(B_j^i) < r_S(z_j^i)$  and since  $z_j \in B_j^i$  it will imply  $z_j \in S$  and then

$$\langle z_j, \bar{p} \rangle - \max_{p \neq \bar{p}} \langle z_j, p \rangle > 0$$

$$\text{i.e.} \quad \langle z_j, \bar{p} \rangle > \max_{p \neq \bar{p}} \langle z_j, p \rangle \geq ||z_j||^2$$

and then  $\bar{p}$  can be discarded.

If  $r_S(z_j^i) < r(B_j^i)$  either  $\bar{p}$  is not discardable or iteration <sup>on i</sup> has to proceed <sup>further</sup> before discardability of  $\bar{p}$  is revealed.

Incorporating this procedure into algorithm-2a it results algorithm-2b:

Algorithm-2b

Initialization: Let  $z_0 \in Q$ , set  $j=0$  and  $\Omega_0 = \{z_0\} = CF_0$

(i) Solve  $(P_1)$  over  $Q$

$$\hat{y}(z_j) \in \operatorname{argmin} \{ f^0(z_j, y) : y \in Q \} .$$

(ii) Set  $\Omega_{j+1} = \operatorname{conv} \{ CF_j \cup \{ \hat{y}(z_j) \} \}$  .

(iii) Solve  $(P_2)$  over  $\Omega_{j+1}$

$$z_{j+1} \in \operatorname{arg min} \{ f(z) : z \in \Omega_{j+1} \} .$$

(iv) Find  $CF_{j+1}$  .

(v) Set  $j=j+1$  and go to (i) .

(3.33a) REMARK: It should be emphasized that even if an upper limit on the number of points to be stored is fixed, the algorithm must work at least as well as the Allwright - feasible direction algorithm when function evaluations are expensive.

### 3.6 - APPLICATION TO A NON-CLASSICAL OPTIMAL CONTROL PROBLEM

#### 3.6.1 - A Linear Quadratic Problem

Consider the linear time invariant system described by the model

$$\dot{x}(t) = A x(t) + B u(t) \quad : \quad x(0) = x_0$$

$$y(t) = C x(t)$$

where  $x(t) \in \mathbb{R}^n$ ,  $y(t) \in \mathbb{R}^r$  and  $A$  is asymptotically stable, with the additional constraint on the control

$$u(t) = F y(t) \quad \text{where } u(t) \in \mathbb{R}^m.$$

Let  $J$  be the quadratic cost defined by:

$$J = \int_0^{\infty} \{x'(t) Q x(t) + u(t)' R u(t)\} dt$$

with  $Q = Q' \geq 0$        $R = R' > 0$

The closed-loop system is described by the model

$$\dot{x}(t) = [A + BFC] x(t) \quad : \quad x(0) = x_0$$

so

$$x(t) = \exp[A + BFC]t \cdot x_0$$

The cost can now be written as:

$$(3.34) J[x_0, F] = x'(0) \left[ \int_0^{\infty} e^{[A+BFC]t} [Q + C'F'RFC] e^{[A+BFC]t} dt \right] x(0)$$

$$(3.35) \quad = x'(0) K(F) x(0)$$

where  $K(F) = \int_0^\infty e^{[A+BFC]t} [Q+C'F'RFC] e^{[A+BFC]t} dt$

satisfies the Lyapunov equation:

$$[A+BFC]'K(F) + K(F)[A+BFC] = -Q - C'F'RFC.$$

(3.36) REMARK: The expression (3.35) above emphasizes the dependence of  $J$  on both  $F$  and  $x(0)$ .

### 3.6.2 - The Levine Athans Approach

Obviously, for any particular  $x_0$ , one could attempt to find an  $F$  which minimizes  $J[x_0, F]$ , but an  $F$  which is good for many initial conditions is really needed. Levine and Athans [5] took the view that the expected value of  $J$  should be minimized, assuming that the initial state  $x(0)$  is a random variable distributed on the surface of the  $n$ -dimensional unit sphere, i.e. assuming that

$$E\{x_0\} = 0 \quad E\{x(0)x'(0)\} = X_0 > 0.$$

The expected cost is then given by:

$$E\{J\} = E\{x'(0)K(F)x(0)\} = \text{trace}[K(F)X_0]$$

this approach however only guarantees that

$$\text{trace}[K(F^*)X_0] \leq \text{trace}[K(0)X_0],$$

where  $F^*$  denotes an  $F$  which minimizes  $\text{trace}[K(F)X_0]$  and  $K(0)X_0$  is for the open loop ( $F=0$ ) case.

It doesn't guarantee that

$$J^* = x_0' K(F^*) x_0 < x_0' K(0) x_0, \quad \forall x_0$$

i.e. doesn't guarantee that the closed loop cost for  $F^*$  will not be greater than the open loop cost (for  $F=0$ ) for some initial conditions.

### 3.6.3 - Allwright's Dominant Output Feedback Approach[6]

(3.37) DEFINITION:

$F$  is said to dominate  $\bar{F}$  iff  $K(F) \leq K(\bar{F})$

where  $K(F) \leq K(\bar{F})$  denotes

$$x' K(F) x \leq x' K(\bar{F}) x \quad \forall x \in \mathbb{R}^n$$

with  $x' K(F) x < x' K(\bar{F}) x$  for some  $x$ .

The approach considered involves generating a sequence  $\{F_k\}$  with the property that

$$K(F_k) \leq K(F_{k-1}) \quad k = 1, 2, \dots$$

The limit  $F^*$  of the sequence  $\{F_k\}$  (if it exists) will then satisfy

$$x_0' K(F^*) x_0 \leq x_0' K(F_0) x_0 \quad \forall x_0$$

i.e. the optimal feedback  $F^*$  dominates the initial feedback (usually  $F_0=0$ , for the open-loop system).

(3.38) REMARK: Obviously if

$$x'_0 K(F)x_0 \leq x'_0 K(F_0)x_0 \quad \forall x_0$$

then  $\text{trace } K(F^*) \leq \text{trace } K(F)$ , but the converse is not true.

This has relevance to the expected cost when  $X_0=I$ .

To generate the solution, Allwright applies a steepest descent type of algorithm which at each iteration determines a search direction by solving a subsidiary problem (SP) with non-differentiable convex cost and non convex constraint set:

$$(SP) \quad \min_{\mu \in B_k} \lambda_{\max} \left[ \sum_{i=1}^k \mu_i H_i \right]$$

where  $B_k = \{ \mu \in \mathbb{R}^k : ||\mu||=1 \}$

$$H_i = H'_i \in \mathbb{R}^{n \times n} \text{ and}$$

$\lambda_{\max}[D]$  denotes the most positive eigenvalue of  $D=D'$ .

Allwright has shown that in certain conditions the solution of (SP) reduces to the solution of a simpler problem (SP<sub>1</sub>) where the cost is convex and the constraint set is convex and compact:

$$(SP_1) \quad \min_{z \in Q} ||z||_2 \quad \text{with } 0 \notin Q$$

The constraint set  $Q$  is defined by  $Q = \text{conv}\{q(B_n)\}$  where  $q(z) = [\langle z, H_1 z \rangle, \dots, \langle z, H_k z \rangle]^T \in \mathbb{R}^k$ .

Hence it seems that  $Q$  cannot be expressed in terms of a set of equalities and/or inequalities and so conventional algorithms are not applicable.

Also, for this particular set  $Q$  it turns out that a global minimizer for  $\min\{\langle z, q \rangle : q \in Q\}$  is relatively easy to find and is given by  $\hat{q} = q[v]$  where  $v$  is a normalized eigenvector associated with the largest eigenvalue of  $\sum z_k H_k$ .

These facts about  $Q$  suggest the use of the algorithm of Y.C. Ho [7], as presented by Gilbert [8], for solving (SP).

Since each iteration of the Ho-Gilbert algorithm involves some expensive computations, it would be useful to apply a potentially improved algorithm - such as algorithm-2b. This is one area in which the work of this Chapter might be useful.

### 3.7 - REFERENCES

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CHAPTER 4  
PROPERTIES OF SOME ALGORITHMS  
WHICH FIND SEARCH DIRECTIONS BY MINIMIZING UPPER BOUNDS  
AND FUNCTIONS DERIVED FROM THEM

4.1 - INTRODUCTION

In order to determine search directions, it seems necessary to have a local approximation to the cost  $v$ . The Allwright algorithms depend on a local approximation which is everywhere lower than  $v$  (i.e. one which we may think of as an outer approximation). Unless  $v$  has an especially nice structure, finding such a lower bound will be hard (or impractical). However, finding an upper-bound (i.e. which we may think of as an inner approximation) is much easier.

This can be clarified by considering the simple case of a quadratic  $v$ :

$$(4.1) \quad v(x) = a + \langle b, x \rangle + \frac{1}{2} \langle x, Cx \rangle \quad : \quad C' = C > 0 .$$

To obtain a non-linear lower-bound for  $v$  seems to require knowledge of  $\lambda_{\min}[C]$  or a lower-bound for it - which are relatively hard to find. However, an upper-bound for  $v$  can be found relatively easily. For example, an upper bound for  $\lambda_{\max}[C]$  is the Frobenius norm

$$(4.2) \quad \|C\|_F = \left( \sum_i \sum_j c_{ij}^2 \right)^{1/2}$$

which is easy to evaluate.

As another example, consider again the optimal constrained problem introduced in Chapter 2.

$$(4.4) \quad \text{Let } C = R + W^* K W \quad (R \neq J)$$

with the usual optimal control interpretations of  $R$ ,  $W$  and  $K$ .

Then we could define  $U$  by

$$(4.5) \quad U = R + \|W^* K W\| I > C$$

This  $U$  could be a good approximation to  $C$  and is easy to invert as

$$(U^{-1}z)(t) = [R + \|W^* K W\| I]^{-1} z(t) \quad \forall t.$$

This fact alone justifies considering methods to find search directions other than the ones based on the existence of a lower bound for the second order derivative matrix.

In Section 4.2, the use of upper bounds for choosing search directions for unconstrained problems is considered, when  $\nabla^2 v(x)$  has the following properties:

$$(4.6) \quad aI \leq \nabla^2 v(x) \leq bI, \quad \forall x \in H \quad \text{with } a > 0$$

and also

$$(4.7) \quad \nabla^2 v(x) < U, \quad \forall x \in H \quad \text{yielding}$$

$$(4.7a) \quad \alpha I \leq U^{-1/2} \nabla^2 v(x) U^{-1/2} \leq \beta I, \quad \forall x \in H$$

for suitable  $0 < \alpha \leq \beta$ .

Then in Sections 4.3 and onward, constrained problems are considered and search directions found by optimizing various local approximations to  $v(x)$ .

#### 4.2 - AN ALGORITHM TO SOLVE AN UNCONSTRAINED PROBLEM

Consider minimizing  $v(x)$  on a Hilbert-Space  $H$ , where  $v(x)$  is a general twice continuously differentiable strictly convex function satisfying (4.6) and (4.7), using a search direction obtained by minimization of a local approximation to  $v$  defined by:

$$(4.8) \quad v^u(x, y) = v(x) + \langle \nabla v(x), (y-x) \rangle + \frac{1}{2} \langle (y-x), U(y-x) \rangle$$

with  $U^* = U > 0$

(4.9) Algorithm: The algorithm is described by:

$$x_{j+1} = x_j + \lambda_j h_j : x_0 \in H$$

where

$$(4.10) \quad h_j = y^*(x_j) - x_j \text{ is the search direction, with}$$

$$(4.11) \quad \{y^*(x_j)\} = \operatorname{argmin} \{v^u(x_j, y) : y \in H\}$$

and  $\lambda_j$  is chosen using the Armijo scheme:

Choose  $\mu > 1$  and  $\delta \in (0, 1)$ .

Increase  $p$ , from 0, in unit steps until

$$(4.12) \quad v(x_j + \mu^p h_j) \leq v(x_j) + \frac{1}{2} \mu^p \langle \nabla v(x_j), h_j \rangle .$$

Then increase  $q$ , from 0, in unit steps until

$$(4.13) \quad v(x_j + \delta^q \mu^p h_j) \leq v(x_j) + \frac{1}{2} \delta^q \mu^p \langle \nabla v(x_j), h_j \rangle$$

and set  $\lambda_j = \delta^q \mu^p$ .

The following theorem gives convergence rate information:

(4.14) THEOREM:

For the sequence generated by algorithm (4.9):

$$(4.15) \quad [v(x_{j+1}) - v(x^*)] \leq [v(x_j) - v(x^*)] \{1 - \delta \frac{\alpha}{\beta}\}$$

where  $\alpha$  and  $\beta$  are as in (4.7a) and

$$(4.16) \quad \{x^*\} = \operatorname{argmin} \{ v(x) : x \in H \} .$$

Proof: The Taylor formula for second order expansions gives:

$$v(x_j + \lambda h_j) = v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \lambda^2 \int_0^1 (1-t) \langle h_j, \nabla^2 v(x_j + t h_j) h_j \rangle dt .$$

Since  $U^* = U > 0$ ,  $U$  has a symmetric self-adjoint square root  $U^{1/2}$  and

$$\begin{aligned} (4.17) \quad v(x_j + \lambda h_j) &= v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle \\ &\quad + \lambda^2 \int_0^1 (1-t) \langle U^{1/2} h_j, U^{-1/2} \nabla^2 v(x_j + t h_j) U^{-1/2} U^{1/2} h_j \rangle dt \\ &\leq v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \lambda^2 \beta \langle h_j, U h_j \rangle \quad \text{by (4.7a)} . \end{aligned}$$

Now from (4.8), (4.10) and (4.11) it follows that

$$(4.18) \quad h_j = -U^{-1} \nabla v(x_j)$$

and so (4.17) becomes:

$$(4.19) \quad v(x_j + \lambda h_j) \leq v(x_j) - \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle [\lambda - \frac{1}{2} \lambda^2 \beta] = g(\lambda) .$$

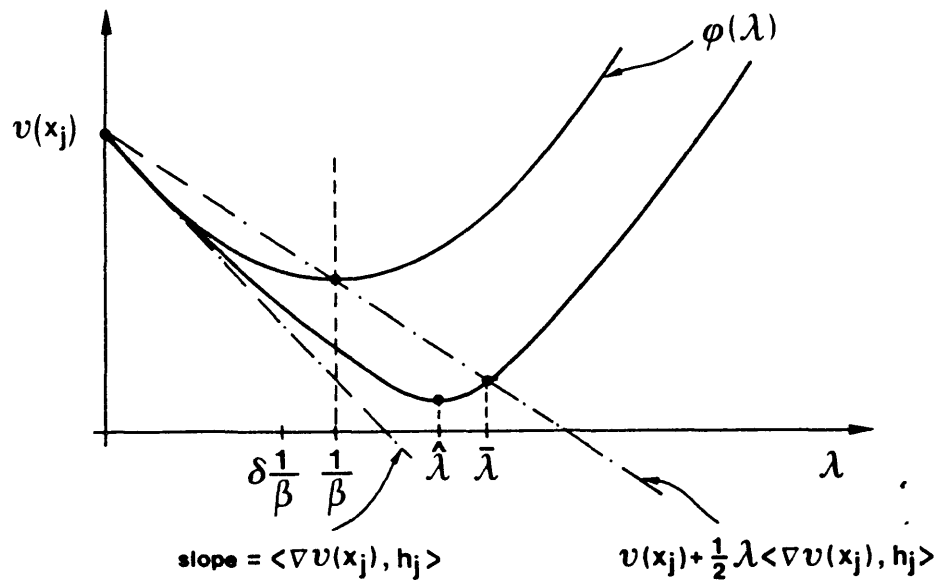


figure 4.1

Choosing  $\lambda$  according to Armijo scheme we get  $\lambda_j \geq \delta(\frac{1}{\beta})$ ,  
 where  $1/\beta$  is the unconstrained minimizer of  $\phi(\lambda)$ .

So

$$(4.20) \quad v(x_{j+1}) = v(x_j + \lambda_j h_j)$$

$$\leq v(x_j) + \frac{1}{2} \lambda_j \langle \nabla v(x_j), h_j \rangle$$

$$(4.20a) \quad \leq v(x_j) - \frac{\delta}{2\beta} \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \quad [\text{from (4.18)}]$$

and since  $\lambda_j \geq \delta(1/\beta)$ .

Now, use of the Taylor formula for second order expansion about  $x_j$ , yields:

$$\begin{aligned} v(x) &= v(x_j) + \langle \nabla v(x_j), (x - x_j) \rangle \\ &\quad + \int_0^1 (1-t) \langle (x - x_j), \nabla^2 v(x_j + t[x - x_j]) (x - x_j) \rangle dt \\ &\geq v(x_j) + \langle \nabla v(x_j), (x - x_j) \rangle + \frac{1}{2} \alpha \langle (x - x_j), U(x - x_j) \rangle \quad x \in H \end{aligned}$$

by (4.7a) .

Then

$$\begin{aligned} v(x^*) &= \min \{ v(x) : x \in H \} \\ &\geq \min \{ v(x_j) + \langle \nabla v(x_j), (x - x_j) \rangle + \frac{1}{2} \alpha \langle (x - x_j), U(x - x_j) \rangle : x \in H \} \\ &= v(x_j) - \frac{1}{2\alpha} \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \end{aligned}$$

and hence

$$(4.21) \quad \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \geq 2\alpha [v(x_j) - v(x^*)] .$$

Now, use of (4.21) in (4.20a) yields

$$v(x_{j+1}) \leq v(x_j) - \frac{\delta\alpha}{\beta} [v(x_j) - v(x^*)]$$

and consequently

$$(4.22) \quad [v(x_{j+1}) - v(x^*)] \leq [v(x_j) - v(x^*)] \{1 - \frac{\delta\alpha}{\beta}\}$$

Q.E.D.

(4.23) REMARK: (i) If  $U$  is chosen to be a scalar multiple of  $I$ , say  $U = MI$ , then algorithm (4.9) becomes a steepest descent algorithm. Then  $\alpha$  and  $\beta$  of (4.7a) can be taken to be  $a$  and  $b$  of (4.6), and typically  $b \gg a$ .

(ii) If  $\bar{x}_j$  is the sequence associated with the steepest descent version of algorithm (4.9) then:

$$(4.23a) \quad [v(\bar{x}_{j+1}) - v(\bar{x}^*)] \leq [v(\bar{x}_j) - v(\bar{x}^*)] \{1 - \delta \frac{a}{b}\}.$$

(4.24) REMARK : If an  $U \neq MI$  can be found easily and such that  $U - \nabla^2 v(x)$  is small, then  $\alpha$  and  $\beta$  of (4.7a) will be approximately 1.

Hence, (4.22) becomes

$$(4.24a) \quad [v(x_{j+1}) - v(x^*)] \leq [v(x_j) - v(x^*)] \{1 - \delta\}.$$

Since, typically,  $(1 - \delta \frac{a}{b}) > (1 - \delta)$ , this means that algorithm (4.9) with  $U \neq MI$  may be preferable to steepest descent in some cases.

#### 4.3 - A CONSTRAINED PROBLEM

Consider now the problem

$$(4.25) \quad \min\{v(x) : x \in Q\}$$

where  $v(x)$  is the quadratic defined in (4.1) and  $Q$  is a compact convex subset of  $\mathbb{R}^n$ , with diameter  $\gamma$ .

(4.26) Algorithm: The algorithm is described by:

$$x_{j+1} = x_j + \lambda_j h_j \quad : \quad x_0 \in Q$$

where  $\lambda_j \in \operatorname{argmin}\{v(x_j + \lambda h_j) : \lambda \in [0, 1]\}$

$h_j$  is the search direction.

The main objective of this chapter is to study properties of algorithm (4.26) when the search direction  $h_j$  is found by optimizing different local approximations to  $v(x)$ .

#### 4.4 - DEFINITION OF $h_j$ BY MINIMIZATION OF AN UPPER APPROXIMATION TO $v(x)$

Suppose  $U' = U > C = C' > 0$

$$h_j = \hat{y}(x_j) - x_j$$

$$(4.27) \quad \{\hat{y}(x_j)\} = \operatorname{argmin}\{v^U(x_j, y) : y \in Q\}$$

where

$$(4.27a) \quad v^U(x_j, y) \triangleq v(x_j) + \langle \nabla v(x_j), (y - x_j) \rangle \\ + \frac{1}{2} \langle (y - x_j), U(y - x_j) \rangle .$$

(4.28) THEOREM:

For  $h_j = \hat{y}(x_j) - x_j$  and  $\hat{y}(x_j)$  defined as in (4.27), algorithm (4.26) has the following convergence properties:

$$(a) \quad \text{If } v(x_j) - v(\hat{x}) \geq \gamma^2 \lambda_{\max}[U] \quad \text{then} \\ (4.29) \quad v(x_{j+1}) - v(\hat{x}) \leq \frac{1}{2} [v(x_j) - v(\hat{x})] \\ (b) \quad \text{If } v(x_j) - v(\hat{x}) < \gamma^2 \lambda_{\max}[U] \quad \text{then}$$

$$(4.30) \quad v(x_{j+1}) - v(\hat{x}) \leq [v(x_j) - v(\hat{x})] \left\{ 1 - \frac{1}{2} \frac{[v(x_j) - v(\hat{x})]}{\gamma^2 \lambda_{\max}[U]} \right\}$$

Proof: Now

$$v(\hat{x}) = v(x_j) + \langle \nabla v(x_j), (\hat{x} - x_j) \rangle + \frac{1}{2} \langle (\hat{x} - x_j), C(\hat{x} - x_j) \rangle$$

so

$$v(\hat{x}) \geq v(x_j) + \langle \nabla v(x_j), (\hat{x} - x_j) \rangle \quad \text{and hence}$$

$$(4.31) \quad 0 \geq v(\hat{x}) - v(x_j) \geq \langle \nabla v(x_j), (\hat{x} - x_j) \rangle.$$

Also

$$\begin{aligned} v(x_{j+1}) &\leq v^U(x_j, \hat{y}(x_j)) \\ &\leq \min \{ v^U(x_j + \alpha(\hat{x} - x_j)) : \alpha \in [0, 1] \} \\ (4.31a) \quad &= \min \{ v(x_j) + \alpha \langle \nabla v(x_j), (\hat{x} - x_j) \rangle \\ &\quad + \frac{1}{2} \alpha^2 \langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle : \alpha \in [0, 1] \}. \end{aligned}$$

Let  $\alpha^* \in \operatorname{argmin} \{ v^U(x_j + \alpha(\hat{x} - x_j)) : \alpha \in \mathbb{R}^+ \}$ .

Then from (4.27)

$$(4.32) \quad \alpha^* = - \frac{\langle \nabla v(x_j), (\hat{x} - x_j) \rangle}{\langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle}.$$

Now  $\alpha^*$  is either  $< 1$  or  $\geq 1$ .

(i) If  $\alpha^* \geq 1$  the  $\hat{\alpha} \in [0, 1]$  that minimizes  $v^U[x_j + \alpha(\hat{x} - x_j)]$

is  $\hat{\alpha} = 1$ , so from (4.31a)

$$\begin{aligned} (4.33) \quad v(x_{j+1}) &\leq v(x_j) + \langle \nabla v(x_j), (\hat{x} - x_j) \rangle + \frac{1}{2} \langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle \\ &\leq v(x_j) + \frac{1}{2} \langle \nabla v(x_j), (\hat{x} - x_j) \rangle \quad \text{from (4.32), since } \alpha^* \geq 1 \\ &\leq v(x_j) - \frac{1}{2} [v(x_j) - v(\hat{x})] \quad \text{by (4.31)} \end{aligned}$$

and hence

$$(4.33a) \quad v(x_{j+1}) - v(\hat{x}) \leq \frac{1}{2} [v(x_j) - v(\hat{x})] .$$

ii) If  $\alpha^* < 1$  then the  $\hat{\alpha} \in [0, 1]$  which minimizes

$v^U[x_j + \alpha(\hat{x} - x_j)]$  is  $\hat{\alpha} = \alpha^*$  and consequently

$$\begin{aligned} (4.34) \quad v(x_{j+1}) &\leq v(x_j) - \frac{1}{2} \frac{\langle \nabla v(x_j), (\hat{x} - x_j) \rangle^2}{\langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle} \quad \text{from (4.31a)} \\ &\leq v(x_j) - \frac{1}{2} \frac{[v(x_j) - v(\hat{x})]^2}{\|\hat{x} - x_j\|^2 \lambda_{\max}[U]} \quad \text{from (4.31)} \\ &\leq v(x_j) - \frac{1}{2} \frac{[v(x_j) - v(\hat{x})]^2}{\gamma^2 \lambda_{\max}[U]} \end{aligned}$$

and hence

$$(4.34a) \quad v(x_{j+1}) - v(\hat{x}) \leq [v(x_j) - v(\hat{x})] \left\{ 1 - \frac{1}{2} \frac{[v(x_j) - v(\hat{x})]}{\gamma^2 \lambda_{\max}[U]} \right\}$$

It will now be shown that if

$$v(x_j) - v(\hat{x}) \geq \gamma^2 \lambda_{\max}[U] \quad \text{then } \alpha^* \geq 1.$$

Now, since  $\gamma$  is the diameter of  $Q$

$$(4.35) \quad v(x_j) - v(\hat{x}) \geq \gamma^2 \lambda_{\max}[U] \geq \langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle.$$

$$\text{Also } v(\hat{x}) = v(x_j) + \langle \nabla v(x_j), (\hat{x} - x_j) \rangle + \frac{1}{2} \langle (\hat{x} - x_j), C(\hat{x} - x_j) \rangle$$

so

$$(4.36) \quad v(x_j) - v(\hat{x}) = - \langle \nabla v(x_j), (\hat{x} - x_j) \rangle - \frac{1}{2} \langle (\hat{x} - x_j), C(\hat{x} - x_j) \rangle$$

Hence, from (4.35)

$$- \langle \nabla v(x_j), (\hat{x} - x_j) \rangle - \frac{1}{2} \langle (\hat{x} - x_j), C(\hat{x} - x_j) \rangle \geq \langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle.$$

Rearranging gives

$$-\langle \nabla v(x_j), (\hat{x} - x_j) \rangle \geq \langle (\hat{x} - x_j), U(\hat{x} - x_j) \rangle + \frac{1}{2} \langle (\hat{x} - x_j), C(\hat{x} - x_j) \rangle \geq 0$$

Therefore, from (4.32),  $\alpha^* \geq 1$ . Since  $\alpha^* \geq 1$ , (4.33a) is valid and (4.29) holds.

Next, if

$$(4.37) \quad v(x_j) - v(\hat{x}) < \gamma^2 \lambda_{\max} [U]$$

then  $\alpha^*$  is either  $< 1$  or  $\geq 1$

If  $\alpha^* < 1$  then (4.34a), and hence (4.30), holds

If  $\alpha^* \geq 1$  then from (4.33)

$$\begin{aligned} v(x_{j+1}) &\leq v(x_j) - \frac{1}{2} [v(x_j) - v(\hat{x})] \\ &\leq v(x_j) - \frac{1}{2} \frac{[v(x_j) - v(\hat{x})]^2}{\gamma^2 \lambda_{\max} [U]} \quad \text{by (4.37)} \end{aligned}$$

and then (4.30) holds as well. Hence (4.30) is valid whether  $\alpha^* \geq 1$  or  $< 1$

Q.E.D.

(4.38) REMARKS: (i) The convergence rate information given by theorem (4.28) is similar to that obtained by Allwright [1] when using a lower approximation to  $v(x)$  i.e. an  $L < C$  instead of an  $U > C$ . This meaning that in all cases where to obtain  $U$  is easier than to obtain  $L$  algorithm (4.26) is preferable .

(ii) Theorem (4.28) does not claim convergence of  $v(x_j) \rightarrow v(\hat{x})$ . However, the proof is much the same as in [3] (Section 1.3.3).

#### 4.5 - DEFINITION OF $h_j$ BY MINIMIZATION OF A LOWER

#### BOUND $\bar{v}(x)$ TO $v(x)$ CONSTRUCTED FROM AN UPPER-BOUND

Suppose  $U$  and  $L$  are known with  $U \geq C \geq L, > 0$ . Then  $U=C+E$  for some  $E \geq 0$ . Suppose also that an  $F$  is known such that  $F \geq E$ .  $L, U, F$ , are all self-adjoint.

$$(4.39) \quad \text{Let } v^L(x,y) = v^O(x,y) + \frac{1}{2} \langle (y-x), L(y-x) \rangle$$

where  $v^O(x,y) = v(x) + \langle \nabla v(x)(y-x) \rangle$  is a lower first order approximation to  $v(y)$ .

Now let

$$(4.40) \quad \bar{v}(x,y) = v^U(x,y) - \frac{1}{2} \langle (y-x), F(y-x) \rangle$$

and consider

$$(4.41) \quad v(y) - v^L(x,y) = \frac{1}{2} \langle (y-x), [C-L](y-x) \rangle$$

and

$$(4.42) \quad v(y) - \bar{v}(x,y) = \frac{1}{2} \langle (y-x), [C-U+F](y-x) \rangle$$

then  $\bar{v}(x,y)$  will be a better approximation to  $v(y)$  than  $v^L(x,y)$  if

$$[C-U+F] < [C-L]$$

or equivalently if

$$(4.43) \quad U > F+L$$

which will hold if, for example

$$(4.44) \quad \lambda_{\min}[U] \geq ||F||+||L|| \quad .$$

Now convergence rate information for algorithm (4.26) when  $\hat{y}(x_j)$  is defined by:

$$(4.45) \quad \hat{y}(x_j) \in \operatorname{argmin}\{ \bar{v}(x_j, y) : y \in Q \}$$

is given by the following theorem:

(4.46) THEOREM: For  $h_j = \hat{y}(x_j) - x_j$  and  $\hat{y}(x_j)$  defined as in (4.45), algorithm (4.26) generates a sequence  $x_j$  satisfying

$$v(x_{j+1}) - v(\hat{x}) \leq [v(x_j) - v(\hat{x})] \left\{ \frac{\lambda_{\max}[F]}{\lambda_{\min}[U]} \right\} \quad .$$

Proof: Let

$$\bar{v}(x_j, y) \triangleq v(x_j) + \langle \nabla v(x_j), (y - x_j) \rangle + \frac{1}{2} \langle (y - x_j) [U - F] (y - x_j) \rangle$$

$$\{\hat{y}(x_j)\} = \operatorname{argmin}\{ \bar{v}(x_j, y) : y \in Q \}$$

then

$$\left\langle \frac{\partial \bar{v}(x_j, y)}{\partial y} \Big|_{y=\hat{y}(x_j)}, (x - \hat{y}(x_j)) \right\rangle \geq 0 \quad \forall x \in Q$$

i.e.

$$\langle [\nabla v(x_j) + (U - F)(\hat{y}(x_j) - x_j)], (x - \hat{y}(x_j)) \rangle \geq 0 \quad \forall x \in Q$$

so for  $x=x_j$  and  $h_j = \hat{y}(x_j) - x_j$  <sup>this</sup> yields:

$$(4.47) \quad \langle \nabla v(x_j), h_j \rangle \leq -\langle h_j, [U-F]h_j \rangle.$$

Now

$$(4.48) \quad v(x_j + \lambda h_j) = v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, Ch_j \rangle$$

$$(4.49) \quad \leq v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, Uh_j \rangle$$

$$(4.50) \quad \leq v(x_j) - \lambda \langle h_j, [U-F]h_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, Uh_j \rangle$$

by (4.47).

So the  $\hat{\lambda}$  which minimizes the right hand side of (4.50) and the  $\tilde{\lambda}$  which minimizes the right hand side of (4.49) are given by

$$(4.51) \quad \hat{\lambda} = \frac{\langle h_j, [U-F]h_j \rangle}{\langle h_j, Uh_j \rangle} < 1$$

$$(4.52) \quad \tilde{\lambda} = - \frac{\langle \nabla v(x_j), h_j \rangle}{\langle h_j, Uh_j \rangle} \geq \hat{\lambda} \quad \text{by (4.47).}$$

Since  $\tilde{\lambda} \geq \hat{\lambda}$ , the right hand side of (4.49) at  $\tilde{\lambda}$  will be less than or equal to its value at  $\hat{\lambda}$ .

Hence, from (4.49)

$$\begin{aligned} (4.53) \quad v(x_{j+1}) &\leq v(x_j) + \frac{\langle h_j, [U-F]h_j \rangle}{\langle h_j, Uh_j \rangle} \cdot \langle \nabla v(x_j), h_j \rangle \\ &\quad + \frac{1}{2} \frac{[\langle h_j, [U-F]h_j \rangle]^2}{\langle h_j, Uh_j \rangle^2} \cdot \langle h_j, Uh_j \rangle \\ &= v(x_j) + \frac{\langle h_j, [U-F]h_j \rangle}{\langle h_j, Uh_j \rangle} [\langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \langle h_j, [U-F]h_j \rangle] \end{aligned}$$

On the other hand

$$(4.54) \quad v(y) \geq \bar{v}(x_j, y) \quad \forall x_j, y \in Q.$$

So, by minimizing each side of (4.54) with respect to  $y$

$$\begin{aligned} v(\hat{x}) &\geq \bar{v}(x_j, \hat{y}(x_j)) \\ &= v(x_j) + \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \langle h_j, [U-F]h_j \rangle \end{aligned}$$

and then

$$(4.55) \quad v(\hat{x}) - v(x_j) \geq \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \langle h_j, [U-F]h_j \rangle.$$

By using (4.55) in (4.53)

$$v(x_{j+1}) \leq v(x_j) - \frac{\langle h_j, [U-F]h_j \rangle}{\langle h_j, Uh_j \rangle} [v(x_j) - v(\hat{x})].$$

$$\text{i.e.} \quad v(x_{j+1}) - v(\hat{x}) \leq [v(x_j) - v(\hat{x})] \left\{ 1 - \frac{\langle h_j, [U-F]h_j \rangle}{\langle h_j, Uh_j \rangle} \right\}.$$

Now

$$\frac{\langle h_j, [U-F]h_j \rangle}{\langle h_j, Uh_j \rangle} = 1 - \frac{\langle h_j, Fh_j \rangle}{\langle h_j, Uh_j \rangle} \geq 1 - \frac{\lambda_{\max}[F]}{\lambda_{\min}[U]}$$

and so finally

$$[v(x_{j+1}) - v(\hat{x})] \leq [v(x_j) - v(\hat{x})] \left\{ \frac{\lambda_{\max}[F]}{\lambda_{\min}[U]} \right\}.$$

Q.E.D.

(4.56) REMARK: If  $F$  is such that  $\|F\|_2$  is small, then

convergence to  $v(\hat{x})$  will be fast.

#### 4.6 - DEFINITION OF $h_j$ BY MINIMIZATION

OF A LOWER BOUND  $v^{\alpha U}(x,y)$  FOR  $v(x)$ .

Suppose an  $U > C$  is available. It will be shown that under certain assumptions it is possible to find an

$\alpha \in (0,1)$  such that  $\alpha U < C$ .

Let  $U = C+E$ . Then  $U > E$ , since  $C > 0$ , and consequently  $\|U\| > \|E\|$ .

Now consider  $\alpha$  such that

$$(4.57) \quad \alpha U \leq C \quad \text{i.e.} \quad \alpha U \leq U-E$$

Now since  $U-E \geq U-\|E\|$ ,

(4.57) will hold if

$$(4.58) \quad \alpha U \leq U-\|E\|$$

i.e. if

$$(4.59) \quad (1-\alpha)U \geq \|E\|$$

But (4.59) will hold if

$$(1-\alpha)\lambda_{\min}[U] \geq \|E\|$$

i.e. if

$$(4.60) \quad \alpha \leq 1 - \frac{\|E\|}{\lambda_{\min}[U]} < 1.$$

For  $E$  small enough  $\alpha$  can be positive, i.e. for  $U$  near  $C$ ,  $\alpha \in (0,1)$ .

Suppose now that  $\alpha U \leq C$  for some  $\alpha \in (0,1)$  and let

$$(4.61) \quad v^{\alpha U}(x_j, y) = v(x_j) + \langle \nabla v(x_j), (y-x_j) \rangle + \frac{1}{2} \alpha \langle (y-x_j), U(y-x_j) \rangle$$

$$(4.62) \quad \{\hat{y}(x_j)\} = \operatorname{argminf} \{ v^{\alpha U}(x_j, y) : y \in Q \}.$$

Then convergence rate information is given by the following theorem:

(4.63) THEOREM: For  $\alpha \in (0,1)$  giving  $\alpha U \leq C$ ,  $h_j = \hat{y}(x_j) - x_j$  and  $\hat{y}(x_j)$  defined as in (4.62), algorithm (4.26) generates a sequence  $x_j$ , satisfying

$$[v(x_{j+1}) - v(\hat{x})] \leq (1-\alpha)[v(x_j) - v(\hat{x})] .$$

Proof: By definition of  $\hat{y}(x_j)$  in (4.62)

$$\begin{aligned} & - \langle [\nabla v(x_j) + \alpha U(\hat{y}(x_j) - x_j)], h_j \rangle \\ & = \left\langle \frac{\partial v^{\alpha U}(x_j, y)}{\partial y} \Big|_{y=\hat{y}(x_j)}, (x_j - \hat{y}(x_j)) \right\rangle \geq 0 \end{aligned}$$

and so

$$(4.64) \quad \langle \nabla v(x_j), h_j \rangle \leq -\alpha \langle h_j, U h_j \rangle$$

$$\text{Also } v(y) \geq v^{\alpha U}(x_j, y) \quad \forall y \in Q$$

since  $\alpha U \leq C$  and consequently, by minimizing each side with respect to  $y$

$$\begin{aligned} v(\hat{x}) & \geq v^{\alpha U}(x_j, \hat{y}(x_j)) \\ & = v(x_j) + \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \alpha \langle h_j, U h_j \rangle . \end{aligned}$$

Hence

$$(4.65) \quad v(\hat{x}) - v(x_j) \geq \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \alpha \langle h_j, U h_j \rangle .$$

Now

$$(4.66) \quad v(x_j + \lambda h_j) = v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, Ch_j \rangle$$

$$(4.67) \quad \leq v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, Uh_j \rangle$$

$$(4.68) \quad \leq v(x_j) - \lambda \alpha \langle h_j, Uh_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, Uh_j \rangle = \phi(\lambda).$$

Since  $\alpha \in (0,1)$  and since  $\lambda_j$  minimizes  $v(x_j + \lambda h_j)$  with respect to  $\lambda \in [0,1]$

$$v(x_{j+1}) = v(x_j + \lambda_j h_j) \leq v(x_j + \alpha h_j).$$

Hence, from (4.66-68)

$$\begin{aligned} v(x_{j+1}) &\leq v(x_j) + \alpha \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \alpha^2 \langle h_j, Uh_j \rangle \\ &= v(x_j) + \alpha [\langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \alpha \langle h_j, Uh_j \rangle] \\ &\leq v(x_j) + \alpha [v(\hat{x}) - v(x_j)] \quad \text{by (4.65)} \end{aligned}$$

and hence:

$$[v(x_{j+1}) - v(\hat{x})] \leq (1-\alpha)[v(x_j) - v(\hat{x})]$$

Q.E.D.

4.7-SOME PROPERTIES OF AN ALGORITHM WHICH FINDS  $h_j$  BY

MINIMIZATION OF AN UPPER APPROXIMATION  $v^U(x,y)$  to  $v(y)$

Consider now  $U > C > 0$  and

$$v^U(x_j, y) \triangleq v(x_j) + \langle \nabla v(x_j), (y-x_j) \rangle + \frac{1}{2} \langle (y-x_j), U(y-x_j) \rangle$$

$$\{\hat{y}_1(x_j)\} = \operatorname{argmin} \{ v^U(x_j, y) : y \in Q \}$$

$$(4.69) \quad h_j = \hat{y}_1(x_j) - x_j.$$

Then

$$v(x_j + \lambda h_j) \leq v(x_j) + \lambda \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \lambda^2 \langle h_j, U h_j \rangle$$

$$\hat{\lambda} = \frac{-\langle \nabla v(x_j), h_j \rangle}{\langle h_j, U h_j \rangle} \geq 1$$

$$\text{So } \lambda_j = 1 \text{ and } v(x_{j+1}) \leq v(x_j) + \frac{1}{2} \langle \nabla v(x_j), h_j \rangle.$$

In order to find a convergence rate, one needs to obtain a lower bound for  $v(\hat{x})$ . To find such a lower bound from  $v^U(x,y)$  a new approach is considered.

Let  $v^{\alpha U}(x,y)$  be as defined in (4.61) where  $\alpha$  is chosen so  $\alpha U < C$  and let

$$(4.70) \quad \hat{y}_\alpha(x_j) \in \operatorname{argmin} \{ v^{\alpha U}(x_j, y) : y \in Q \}$$

$$(4.71) \quad y_\alpha^*(x_j) \in \operatorname{argmin} \{ v^{\alpha U}(x_j, y) : y \in \mathbb{R}^n \}$$

$$(4.72) \quad \bar{y} = \hat{y}_1(x_j) + \left( \frac{\alpha-1}{\alpha} \right) U^{-1} \nabla v(x_j)$$

$$(4.73) \quad y_1^*(x_j) = x_j - U^{-1} \nabla v(x_j) \in \operatorname{argmin} \{ v^U(x_j, y) : y \in \mathbb{R}^n \}.$$

Figure 4.2 shows the cost contours for  $v^U$  and  $v^{\alpha U}$  together with the vectors defined above. The cost contours for  $v$  were not drawn in order to make the picture more clear

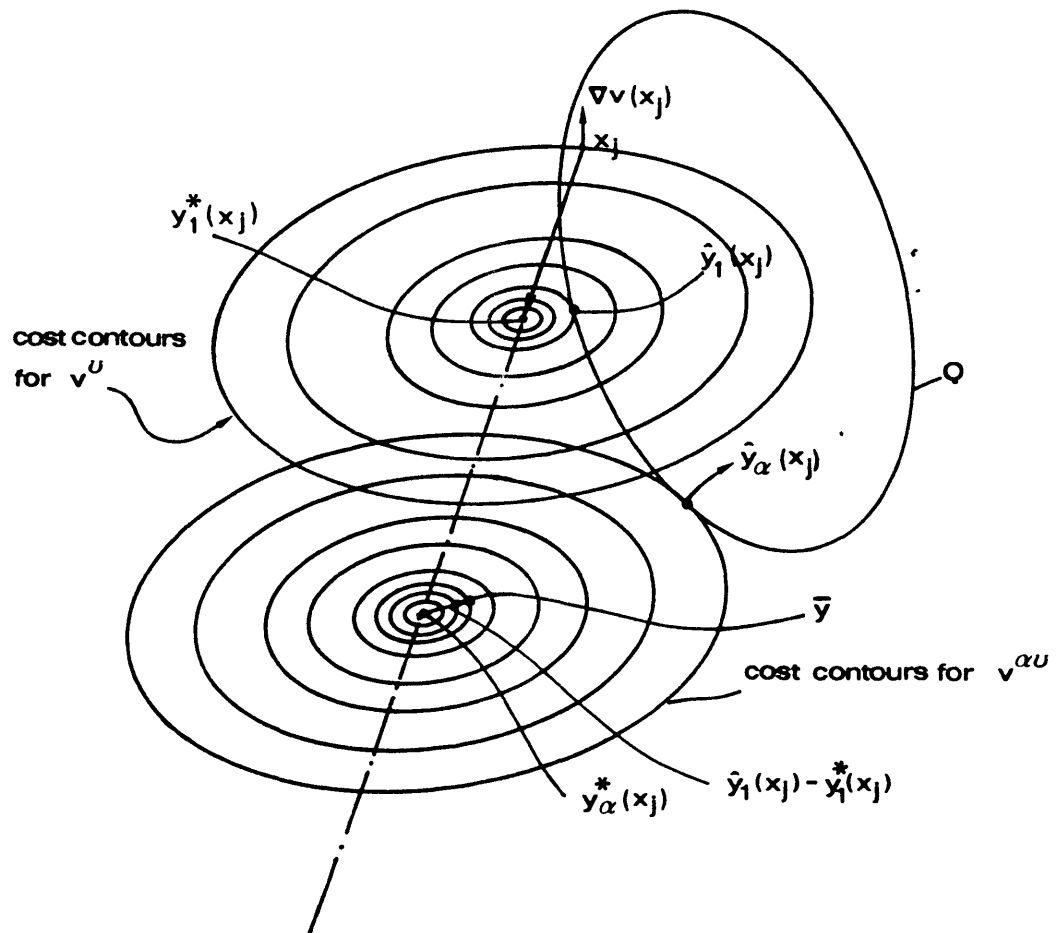


figure 4.2

Since  $\alpha U < C$

$$v(y) \geq v^{\alpha U}(x_j, y) \quad \forall y \in Q$$

and

$$(4.74) \quad v(\hat{x}) = \min_{y \in Q} v(y) \geq \min_{y \in Q} v^{\alpha U}(x_j, y)$$

Consider now  $v^{\alpha U}(x_j, y)$ , regarded as

$$v^{\alpha U}(x_j, [y_{\alpha}^*(x_j) + (y - y_{\alpha}^*(x_j))])$$

Since  $y_{\alpha}^*(x_j)$  is the unconstrained global minimizer of

$v^{\alpha U}(x_j, y)$  with respect to  $y$

$$v^{\alpha U}(x_j, y) = v^{\alpha U}(x_j, y_{\alpha}^*(x_j)) + \frac{1}{2} \|y - y_{\alpha}^*(x_j)\|_{\alpha U}^2$$

The cost contours for  $v^{\alpha U}(x_j, y)$  are obviously centered on  $y_{\alpha}^*(x_j)$  - as shown in figure 4.2 .

From (4.74) and (4.71)

$$(4.75) \quad v(\hat{x}) \geq \min_{y \in Q} \{v^{\alpha U}(x_j, y_{\alpha}^*(x_j)) + \frac{1}{2} \|y - y_{\alpha}^*(x_j)\|_{\alpha U}^2\} \\ = v^{\alpha U}(x_j, y_{\alpha}^*(x_j)) + \frac{1}{2} \|\hat{y}_{\alpha}(x_j) - y_{\alpha}^*(x_j)\|_{\alpha U}^2$$

In order to express this lower bound in useful terms, i.e. in terms of  $h_j$ , a number of Lemmas and a theorem will be stated and proved.

(4.76) LEMMA

$$0 \leq \langle [\hat{y}_1(x_j) - y_1^*(x_j)], U[\hat{y}_1(x_j) - y_1^*(x_j)] \rangle \\ \leq \langle \nabla v(x_j), (\hat{y}_1(x_j) - y_1^*(x_j)) \rangle$$

Proof: Now

$$\begin{aligned}
 (4.77) \quad & \langle [\hat{y}_1(x_j) - y_1^*(x_j)], U[\hat{y}_1(x_j) - y_1^*(x_j)] \rangle \\
 &= \langle [(\hat{y}_1(x_j) - x_j) - (y_1^*(x_j) - x_j)], U[(\hat{y}_1(x_j) - x_j) - (y_1^*(x_j) - x_j)] \rangle \\
 &= \langle (\hat{y}_1(x_j) - x_j), U(\hat{y}_1(x_j) - x_j) \rangle - 2 \langle (\hat{y}_1(x_j) - x_j), U(y_1^*(x_j) - x_j) \rangle \\
 &\quad + \langle (y_1^*(x_j) - x_j), U(y_1^*(x_j) - x_j) \rangle .
 \end{aligned}$$

On the other hand, by the definition of  $\hat{y}_1(x_j)$

$$\langle \nabla_y v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)}, (y - \hat{y}_1(x_j)) \rangle \geq 0 \quad \forall y \in Q$$

and so, for  $y = x_j$

$$\langle [\nabla v(x_j) + U(\hat{y}_1(x_j) - x_j)], (x_j - \hat{y}_1(x_j)) \rangle \geq 0 .$$

Hence

$$(4.78) \quad \langle [\hat{y}_1(x_j) - x_j], U[\hat{y}_1(x_j) - x_j] \rangle \leq -\langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle .$$

Use of (4.78) in (4.77) yields

$$\begin{aligned}
 & \langle [\hat{y}_1(x_j) - y_1^*(x_j)], U[\hat{y}_1(x_j) - y_1^*(x_j)] \rangle \\
 & \leq -\langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle - 2 \langle (\hat{y}_1(x_j) - x_j), U(y_1^*(x_j) - x_j) \rangle \\
 & \quad + \langle (y_1^*(x_j) - x_j), U(y_1^*(x_j) - x_j) \rangle \\
 & = -\langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle + 2 \langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle \\
 & \quad + \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \\
 & = \langle \nabla v(x_j), [\hat{y}_1(x_j) - x_j + U^{-1} \nabla v(x_j)] \rangle \\
 & = \langle \nabla v(x_j), [\hat{y}_1(x_j) - y_1^*(x_j)] \rangle
 \end{aligned}$$

Q.E.D.

The next Lemma states that the  $\alpha$  factor moves the center  $y_\alpha^*(x_j)$  further from  $\hat{y}_1(x_j)$ .

(4.79) LEMMA

$$||\hat{y}_1(x_j) - y_\alpha^*(x_j)||_{\alpha U}^2 \leq ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 .$$

Proof:

$$\text{Clearly } \hat{y}_1(x_j) - y_\alpha^*(x_j) = (\hat{y}_1(x_j) - y_1^*(x_j)) + (y_1^*(x_j) - y_\alpha^*(x_j)) .$$

$$\text{Now } y_1^*(x_j) = x_j - U^{-1} \nabla v(x_j)$$

$$y_\alpha^*(x_j) = x_j - \frac{1}{\alpha} U^{-1} \nabla v(x_j)$$

so

$$\hat{y}_1(x_j) - y_\alpha^*(x_j) = (\hat{y}_1(x_j) - y_1^*(x_j)) + \left(\frac{1-\alpha}{\alpha}\right) U^{-1} \nabla v(x_j) .$$

Hence:

$$\begin{aligned} & ||\hat{y}_1(x_j) - y_\alpha^*(x_j)||_{\alpha U}^2 = ||(\hat{y}_1(x_j) - y_1^*(x_j)) + \left(\frac{1-\alpha}{\alpha}\right) U^{-1} \nabla v(x_j)||_{\alpha U}^2 \\ & = ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 + 2(1-\alpha) \langle \nabla v(x_j), [\hat{y}_1(x_j) - y_1^*(x_j)] \rangle \\ & \quad + ||\left(\frac{1-\alpha}{\alpha}\right) U^{-1} \nabla v(x_j)||_{\alpha U}^2 \\ & \geq ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 + 2(1-\alpha) \langle \nabla v(x_j), [\hat{y}_1(x_j) - y_1^*(x_j)] \rangle \\ & \geq ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 \quad \text{by Lemma (4.76) and the fact that} \\ & \quad \alpha < 1. \end{aligned}$$

Q.E.D.

The facts that the cost contours for  $v^U$  and  $v^{\alpha U}$  have the same shape, and that  $\hat{y}_1(x_j)$  is located relatively to the center  $y_1^*(x_j)$  in the same position as  $\bar{y}$  relatively to  $y_\alpha^*$ , explain the following result.

(4.80) LEMMA

$$\nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)} = \frac{1}{\alpha} \nabla v^{\alpha U}(x_j, y) \Big|_{y=\bar{y}} .$$

Proof:

$$\begin{aligned} \nabla v^{\alpha U}(x_j, y) \Big|_{y=\bar{y}} &= \nabla v(x_j) + \alpha U(\bar{y} - x_j) \\ &= \nabla v(x_j) + \alpha U[\hat{y}_1(x_j) - x_j] + \left(\frac{\alpha-1}{\alpha}\right) U^{-1} \nabla v(x_j) \\ &= \nabla v(x_j) + \alpha U[\hat{y}_1(x_j) - x_j] + (\alpha-1) \nabla v(x_j) \\ &= \alpha \nabla v(x_j) + \alpha U[\hat{y}_1(x_j) - x_j] \\ &= \alpha [\nabla v(x_j) + U[\hat{y}_1(x_j) - x_j]] \\ &= \alpha \nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)} . \end{aligned}$$

Q.E.D.

(4.81) LEMMA

$$\langle \nabla v(x_j), U^{-1} \nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)} \rangle \geq 0 .$$

**Proof:** Now

$$\nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)} = \nabla v(x_j) + U(\hat{y}_1(x_j) - x_j)$$

so

$$\begin{aligned} \langle \nabla v(x_j), U^{-1} \nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)} \rangle &= \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \\ &\quad + \langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle \\ &= \langle \nabla v(x_j), [U^{-1} \nabla v(x_j) + \hat{y}_1(x_j) - x_j] \rangle \\ &= \langle \nabla v(x_j), (\hat{y}_1(x_j) - y_1^*(x_j)) \rangle \geq 0 \text{ by (4.73) and Lemma (4.76).} \end{aligned}$$

Q.E.D.

(4.82) **LEMMA**

$$\langle \nabla v^{\alpha U}(x_j, y) \Big|_{y=\bar{y}}, (y - \bar{y}) \rangle = \langle [\nabla v(x_j) + \alpha U(\bar{y} - x_j)], (y - \bar{y}) \rangle \geq 0 \quad \forall y \in Q$$

**Proof:** Since  $y - \bar{y} = (y - \hat{y}_1(x_j)) + (\frac{1-\alpha}{\alpha}) U^{-1} \nabla v(x_j)$ , and by Lemma (4.80), we have that

$$\begin{aligned} \langle \nabla v^{\alpha U}(x_j, y) \Big|_{y=\bar{y}}, (y - \bar{y}) \rangle &= \alpha \langle \nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)}, (y - \bar{y}) \rangle \\ &= \alpha \langle [\nabla v(x_j) + U(\hat{y}_1(x_j) - x_j)], (y - \bar{y}) \rangle \\ &= \alpha \langle [\nabla v(x_j) + U(\hat{y}_1(x_j) - x_j)], [(y - \hat{y}_1(x_j)) + (\frac{1-\alpha}{\alpha}) U^{-1} \nabla v(x_j)] \rangle \\ &= \alpha \langle [\nabla v(x_j) + U(\hat{y}_1(x_j) - x_j)], (y - \hat{y}_1(x_j)) \rangle \\ &\quad + (1-\alpha) \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle + (1-\alpha) \langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle \end{aligned}$$

$$\begin{aligned}
 &\geq (1-\alpha) \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle + (1-\alpha) \langle \nabla v(x_j), (\hat{y}_1(x_j) - x_j) \rangle \\
 &= (1-\alpha) \langle \nabla v(x_j), U^{-1} [\nabla v(x_j) + U(\hat{y}_1(x_j) - x_j)] \rangle \\
 &= (1-\alpha) \langle \nabla v(x_j), U^{-1} \nabla v^U(x_j, y) \Big|_{y=\hat{y}_1(x_j)} \rangle \geq 0 \text{ by Lemma (4.81)}.
 \end{aligned}$$

Q.E.D.

(4.83) LEMMA

$$||y - y_\alpha^*(x_j)||_{\alpha U}^2 \geq ||\bar{y} - y_\alpha^*||_{\alpha U}^2 = ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 \quad \forall y \in Q.$$

Proof:

$$\begin{aligned}
 &||y - y_\alpha^*(x_j)||_{\alpha U}^2 = ||(y - \bar{y}) + (\bar{y} - y_\alpha^*(x_j))||_{\alpha U}^2 \\
 &= ||y - \bar{y}||_{\alpha U}^2 + 2\alpha \langle (y - \bar{y}), U(\bar{y} - y_\alpha^*(x_j)) \rangle + ||\bar{y} - y_\alpha^*(x_j)||_{\alpha U}^2 \\
 &= ||y - \bar{y}||_{\alpha U}^2 + 2\alpha \langle (y - \bar{y}), U(\bar{y} - x_j + \frac{1}{\alpha} U^{-1} \nabla v(x_j)) \rangle + ||\bar{y} - y_\alpha^*||_{\alpha U}^2 \\
 &= ||y - \bar{y}||_{\alpha U}^2 + 2\alpha \langle (y - \bar{y}), [\nabla v(x_j) + \alpha U(\bar{y} - x_j)] \rangle + ||\bar{y} - y_\alpha^*||_{\alpha U}^2 \\
 &= ||y - \bar{y}||_{\alpha U}^2 + 2\alpha \langle \nabla v^{\alpha U}(x_j, y) \Big|_{y=\bar{y}}, (y - \bar{y}) \rangle + ||\bar{y} - y_\alpha^*||_{\alpha U}^2 \\
 &\geq ||\bar{y} - y_\alpha^*||_{\alpha U}^2 \text{ by Lemma (4.82)}.
 \end{aligned}$$

It remains to be shown that

$$||\bar{y} - y_\alpha^*||_{\alpha U}^2 = ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2.$$

Now

$$\begin{aligned}
 ||\bar{y} - y_\alpha^*||_{\alpha U}^2 &= ||(\hat{y}_1(x_j) + (\frac{\alpha-1}{\alpha})U^{-1}\nabla v(x_j)) - (x_j - \frac{1}{\alpha}U^{-1}\nabla v(x_j))||_{\alpha U}^2 \\
 &= ||\hat{y}_1(x_j) + U^{-1}\nabla v(x_j) - \frac{1}{\alpha}U^{-1}\nabla v(x_j) - x_j + \frac{1}{\alpha}U^{-1}\nabla v(x_j)||_{\alpha U}^2 \\
 &= ||\hat{y}_1(x_j) - x_j + U^{-1}\nabla v(x_j)||_{\alpha U}^2 \\
 &= ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 \quad \text{Q.E.D..}
 \end{aligned}$$

(4.84) THEOREM: For  $h_j$  as defined in (4.69) and  $\alpha$  such that

$$\alpha U < C,$$

$$v(\hat{x}) - v(x_j) \geq \alpha \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \alpha \langle h_j, U h_j \rangle + \frac{1}{2} (\alpha - \frac{1}{\alpha}) \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle$$

Proof: From (4.75)

$$v(\hat{x}) \geq v^{\alpha U}(x_j, y_\alpha^*(x_j)) + \frac{1}{2} ||\hat{y}_\alpha(x_j) - y_\alpha^*(x_j)||_{\alpha U}^2$$

and use of the result of Lemma (4.83) with  $y = \hat{y}_\alpha(x_j)$  yields

$$\begin{aligned}
 v(\hat{x}) &\geq v^{\alpha U}(x_j, y_\alpha^*(x_j)) + \frac{1}{2} ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 \\
 &= v(x_j) - \frac{1}{2\alpha} \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle + \frac{1}{2} ||\hat{y}_1(x_j) - y_1^*(x_j)||_{\alpha U}^2 \\
 &= v(x_j) - \frac{1}{2\alpha} \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle + \frac{1}{2} ||\hat{y}_1(x_j) - x_j||_{\alpha U}^2 \\
 &\quad + \alpha \langle (\hat{y}_1(x_j) - x_j), \nabla v(x_j) \rangle + \frac{1}{2} \alpha \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \\
 &= v(x_j) + \frac{1}{2} \alpha \langle h_j, U h_j \rangle + \alpha \langle \nabla v(x_j), h_j \rangle \\
 &\quad + \frac{1}{2} (\alpha - \frac{1}{\alpha}) \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle
 \end{aligned}$$

i.e.

$$\begin{aligned}
 (4.85) \quad v(\hat{x}) - v(x_j) &\geq \alpha \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \alpha \langle h_j, U h_j \rangle \\
 &\quad + \frac{1}{2} (\alpha - \frac{1}{\alpha}) \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle \quad \text{Q.E.D.}
 \end{aligned}$$

Now since  $v(x_{j+1}) \leq v^U(x_j, \hat{y}(x_j))$

it follows that

$$(4.86) \quad v(x_{j+1}) - v(\hat{x}) \leq v^U(x_j, \hat{y}(x_j)) - v(\hat{x}) .$$

In order to find a convergence rate, one would seek a  $\theta \in [0,1]$  such that.

$$(4.87) \quad v^U(x_j, \hat{y}(x_j)) - v(\hat{x}) \leq \theta [v(x_j) - v(\hat{x})] .$$

However, as the bound (4.85) is non-sharp (i.e. when  $x_j \rightarrow \hat{x}$  the bound doesn't go to zero) there is not such a  $\theta$  near  $\hat{x}$ . The alternative is then to see if such a  $\theta$  exists when one is a long way from the optimum.

#### 4.7.1 - Rate of Convergence far from the Optimum

Now

$$(4.88) \quad v^U(x_j, \hat{y}(x_j)) - v(\hat{x}) = [v(x_j) - v(\hat{x})] + \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \langle h_j, U h_j \rangle$$

From theorem (4.84) it follows that:

$$(4.89) \quad \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \langle h_j, U h_j \rangle \leq \frac{1}{\alpha} [v(\hat{x}) - v(x_j)]$$

$$- \frac{1}{2} \left( \frac{\alpha^2 - 1}{\alpha^2} \right) \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle$$

and since

$$(4.90) \quad v^U(x_j, y^*(x_j)) - v(x_j) = - \frac{1}{2} \langle \nabla v(x_j), U^{-1} \nabla v(x_j) \rangle$$

(4.89) becomes:

$$(4.91) \quad \langle \nabla v(x_j), h_j \rangle + \frac{1}{2} \langle h_j, U h_j \rangle \leq \frac{1}{\alpha} [v(\hat{x}) - v(x_j)]$$

$$+ \left( \frac{1 - \alpha^2}{\alpha^2} \right) [v(x_j) - v^U(x_j, y^*(x_j))] .$$

Now, using (4.91) in (4.88), yields

$$(4.92) \quad v(x_j, \hat{y}(x_j)) - v(\hat{x}) \leq [v(x_j) - v(\hat{x})] + \frac{1}{\alpha} [v(\hat{x}) - v(x_j)] \\ + \left(\frac{1-\alpha^2}{\alpha^2}\right) [v(x_j) - v^U(x_j, \hat{y}(x_j))] .$$

It can, however, be argued that if  $\|x_j - \hat{x}\|$  is big (i.e. away from the optimum) then

$$(4.93) \quad v^U(x_j, \hat{y}(x_j)) \geq v(\hat{x})$$

and hence:

$$(4.94) \quad v(x_j) - v^U(x_j, \hat{y}(x_j)) \leq v(x_j) - v(\hat{x}) .$$

Then, (4.94) together with (4.92) and (4.86), give

$$(4.95) \quad v(x_{j+1}) - v(\hat{x}) \leq \left[\frac{1}{\alpha^2} - \frac{1}{\alpha}\right] [v(x_j) - v(\hat{x})] \\ = \theta [v(x_j) - v(\hat{x})]$$

where  $\theta > 0$  since  $\frac{1}{\alpha^2} > \frac{1}{\alpha}$  for  $\alpha < 1$

and  $\theta < 1$  for  $\alpha > -\frac{1}{2} + \frac{\sqrt{5}}{2} = 0.618$  .

The significance is that if there is an  $\alpha > 0.618$  for which  $\alpha U < C$  then, under assumption (4.93), the algorithm will satisfy (4.95).

(4.96) REMARK: It has not been possible to obtain very strong results, because the bound of (4.85) turns out to not be sharp, but nevertheless some new information regarding the behaviour of the algorithm has been found.

4.8 - COMMENTS ON BOUNDS FOR THE EIGENVALUES OF  
POSITIVE DEFINITE MATRICES  $C=C'$ .

In the Introduction to this Chapter it was referred<sup>r</sup> to the practical difficulties associated with the determination of lower bounds for  $\lambda_{\min}[C]$ . However in the context of matrices  $C=C'$ , a lot can be said about  $\lambda_{\min}[C]$  and  $\lambda_{\max}[C]$  by considering the diagonal entries of  $C$ .

(4.97) THEOREM ( Gershgorin's theorem) [2]

The eigenvalues of an  $n \times n$  complex matrix  $C$  lie in the union of the discs

$$|s - c_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}|, \quad i=1, 2, \dots, n$$

and also in the union of the discs

$$|s - c_{ii}| \leq \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ji}|, \quad i=1, 2, \dots, n.$$

(4.98) REMARK: Gershgorin's theorem says that there is an eigenvalue  $\lambda$  near the entry  $c_{ii}$ .

From theorem (4.97) the following result was derived:

(4.99) THEOREM: For  $C=C'>0$  ,  $C \in \mathbb{R}^{n \times n}$

$$\lambda_{\max}[C] \leq \max_i \{ c_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \} = M(C)$$

$$\lambda_{\min}[C] \geq \max_i \{ \min [ c_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| ] , 0 \} = N(C) .$$

Proof: By Gershgorin's Theorem

and since  $c_{ji}=c_{ij} \in \mathbb{R}$  ,  $c_{ii}>0$  , it yields

$$\begin{aligned} & U_i \left[ \left( c_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \right), \left( c_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \right) \right] \\ &= \left[ \min_i \left( c_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \right), \max_i \left( c_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \right) \right] \end{aligned}$$

and since  $\lambda > 0$ , it follows immediately:

$$\lambda_{\max}[C] \leq \max_i \{ c_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \} = M(C)$$

and

$$\lambda_{\min}[C] \geq \max_i \{ \min_i \left( c_{ii} - \sum_{\substack{j=1 \\ j \neq i}}^n |c_{ij}| \right) , 0 \} = N(C)$$

Q.E.D.

(4.100) REMARK : At first sight, these results seem to contradict what has been said about the difficulties in obtaining lower bounds for  $\lambda_{\min}[C]$ . However, the fact is that the lower bound for  $\lambda_{\min}[C]$  given by theorem (4.99) will often be zero which is not telling us anything new.

4.9 - REFERENCES

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CHAPTER 5  
ALGORITHMS FOR OPTIMIZING A FUNCTION OVER A CONE

5.1 - STATEMENT OF THE PROBLEM

Consider the constrained optimization problem defined by:

$$(5.1) \quad \min \{ v(x) : x \in \text{cone}[\Gamma] \}$$

where  $v(x)$  is a convex function,  $\Gamma$  is a convex, compact subset of  $\mathbb{R}^n$  with  $0 \notin \Gamma$  and where  $\text{cone}[\Gamma]$  is the conical hull

$$\text{cone}[\Gamma] = \text{conv} \{ \alpha \gamma : \alpha \in \mathbb{R}^+, \gamma \in \Gamma \}$$

where  $\text{conv} \{A\}$  denotes the convex hull of a set  $A \neq \emptyset$ . This type of problem was introduced in [1].

It will be assumed that  $x^* \notin \text{cone}[\Gamma]$  where  $x^*$  is the unconstrained minimizer of  $v(x)$ , i.e.

$$x^* \in \text{argmin} \{ v(x) : x \in \mathbb{R}^n \}.$$

If  $x^* \in \text{cone}[\Gamma]$  the solution is trivial.

Also it will be assumed that:

(i) real numbers  $\Pi_{\min}$  and  $\Pi_{\max}$  exist and are known, so that:

$$(5.2) \quad 0 < \pi_{\min} \leq \min \{ \|x\| : x \in \Gamma \}$$

$$(5.3) \quad \pi_{\max} \geq \max \{ \|x\| : x \in \Gamma \},$$

(ii) for each  $s \in \mathbb{R}^n$ , the minimization subproblem:

$$(5.4) \quad \min \{ \langle s, \gamma \rangle : \gamma \in \Gamma \}$$

has an easy and exact solution.

It is also assumed that (5.1) has a solution.

## 5.2 - MINIMIZING A NORM OVER A CONE

Allwright [1] has presented an algorithm and corresponding rate of convergence, for approximating the solution  $\hat{x}$  of (5.1) when  $v(x)$  is the norm function:

$$(5.5) \quad v(x) = \|z - x\|^2$$

That algorithm for approximating  $\hat{x}$  on the unbounded set  $\text{cone}[\Gamma]$  is based on the fact that  $\hat{x}$  is also the solution of an equivalent problem:

$$(5.6) \quad \min \{ v(x) : x \in S \}$$

where  $S$  is a bounded set, actually a suitable truncated version of  $\text{cone}[\Gamma]$ .

When  $v(x)$  is as in (5.5) the fundamental result that justifies the equivalence between (5.1) and (5.6) is stated next.

(5.7) THEOREM: [1]

$$\text{Let } \eta = \frac{||z||}{\Pi_{\min}} \quad ; \quad S = \{ \alpha \gamma : \alpha \in [0, \eta], \gamma \in \Gamma \}$$

Then S is a convex set and  $\hat{x} \in S$ , where

$$\hat{x} \in \operatorname{argmin} \{ v(x) : x \in \operatorname{cone}[\Gamma] \}.$$

Proof

(i) The convexity of S may be verified easily.

(ii) Since  $S \subset \operatorname{cone}[\Gamma]$ ,  $\hat{x}$  will minimize  $||z-x||$  on S provided  $\hat{x} \in S$ . Clearly  $||\hat{x}|| \leq ||z||$ . Because  $\hat{x} \in \operatorname{cone}[\Gamma]$ ,  $\hat{x}$  may be written as  $\hat{x} = \theta \hat{\gamma}$  for some  $\theta \geq 0$  and some  $\hat{\gamma} \in \Gamma$ .

Due to assumption (5.2) it follows that

$$||\hat{\gamma}|| \geq \Pi_{\min}$$

and hence

$$(5.8) \quad \theta \leq \frac{||\hat{x}||}{\Pi_{\min}} \leq \frac{||z||}{\Pi_{\min}} = \eta$$

Consequently  $\hat{x} \in S$ .

Q.E.D.

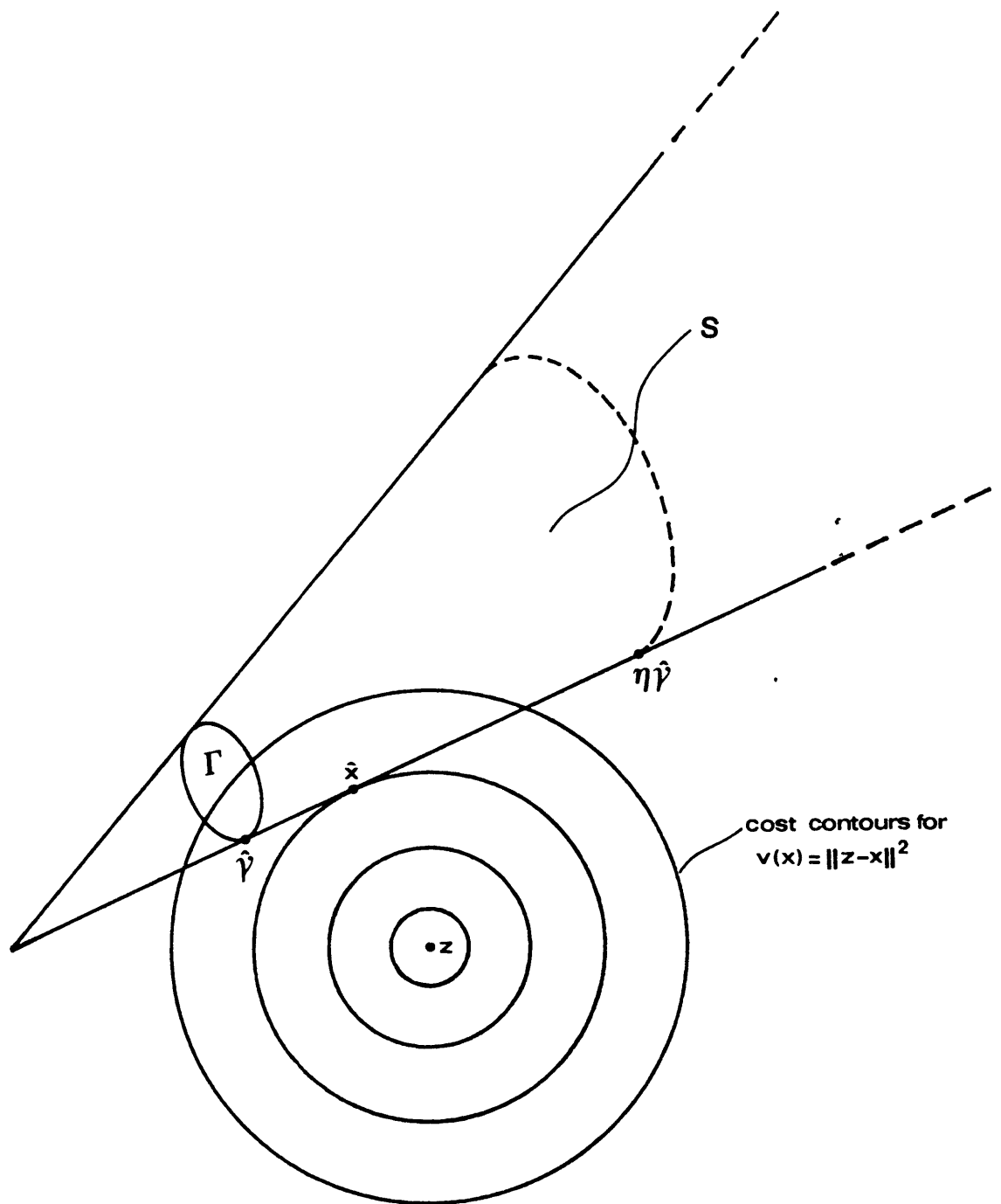


figure 5.1

The algorithm, due to Allwright, for finding an acceptably-accurate approximation  $\tilde{x}$  to  $\hat{x}$  by minimizing  $v$  on  $S$  is stated next:

(5.9) Algorithm [1]

(i) Initial step

Set  $x_0=0$  ,  $i=0$  .

(ii) Find a  $y_i \in \operatorname{argmin} \{ \langle \nabla v(x_i), (y-x_i) \rangle : y \in S \}$

by finding a

$$y_i \in \operatorname{argmin} \{ \langle \nabla v(x_i), y \rangle : y \in \Gamma \}$$

and setting

$$(5.10) \quad y_i = \begin{cases} x_i & \text{if } \langle \nabla v(x_i), y_i \rangle \geq 0 \\ \eta y_i & \text{otherwise} \end{cases}$$

(iii) Compute a lower bound for  $v(\hat{x})$

$$l_i = v(x_i) + \langle \nabla v(x_i), (y_i - x_i) \rangle .$$

Compute  $\hat{l}_i$ , the best lower bound for  $v(\hat{x})$  found so far

$$\hat{l}_i = \max \{ \hat{l}_{i-1}, l_i \} .$$

If  $v(x_i) - \hat{l}_i$  satisfies a stopping condition set  $\tilde{x} = x_i$

and STOP, else CONTINUE.

(iv) Find  $x_{i+1}$  using

$$\{x_{i+1}\} = \operatorname{argmin} \{ v(x) : x \in \operatorname{cone} [\operatorname{line} \{x_i, y_i\}] \} .$$

Set  $i=i+1$  and GO TO (ii) .

(5.11) REMARK:  $x_{i+1}$  may be found simply and exactly due to the simple nature of the set  $\text{cone}[\text{line } \{x_i, y_i\}]$ . In fact, suppose

$$x_{i+1}^* \in \text{argmin } \{v(x) : x \in \text{linear subspace spanned by } x_i, y_i\}.$$

Then  $x_{i+1}^*$  is easy to find.

If  $x_{i+1}^* \in \text{cone}[\text{line } \{x_i, y_i\}]$ , then the problem is solved. Otherwise, the best of the best points in the rays through  $x_i$  and  $y_i$  must then be the global solution.

(5.12) REMARK: It should be noticed that in algorithm (5.9) the search direction is found by optimizing the linear function  $\langle \nabla v(x_i), \gamma \rangle$  over  $\Gamma$  instead of over  $S$ .

Consider the determination of  $y_i$  in step (ii) of algorithm (5.9). Step (iv) ensures that, for  $i > 0$ ,  $x_i$  minimizes  $v$  on the ray through  $x_i$ . So, since  $x_0 = 0$ ,  $\langle \nabla v(x_i), x_i \rangle = 0$ ,  $\forall i \geq 0$ . Also, as any  $y \in S$  may be expressed as  $y = \theta \gamma$  for some  $\theta \in [0, \eta]$  and some  $\gamma \in \Gamma$ ,

$$\begin{aligned} \min_{y \in S} \langle \nabla v(x_i), (y - x_i) \rangle &= \min_{\theta \in [0, \eta]} \min_{\gamma \in \Gamma} \theta \langle \nabla v(x_i), \gamma \rangle \\ &= \begin{cases} \eta \langle \nabla v(x_i), \gamma_i \rangle & \text{if } \langle \nabla v(x_i), \gamma_i \rangle < 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

The above minimal value for  $\langle \nabla v(x_i), (y - x_i) \rangle$  is achieved by  $y_i$  of (5.10) which proves that  $y_i$  minimizes  $\langle \nabla v(x_i), (y - x_i) \rangle$  on  $S$ .

Convergence rate information for algorithm (5.9) was determined by Allwright [1] and is given by:

$$(5.13) \quad [v(x_{i+1}) - v(\hat{x})] \leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{[v(x_i) - v(\hat{x})]}{4D} \right\}$$

where

$$(5.14) \quad D = \|\nabla v(\hat{x})\|d + d^2$$

and  $d$  is the diameter of  $S$ , i.e.

$$\|x - y\| \leq d, \quad \forall x, y \in S.$$

### 5.3 - MINIMIZING A GENERAL QUADRATIC OVER A CONE

To solve the problem  $\min \{v(x) : x \in S\}$ , with  $v(x)$  a general quadratic, an algorithm with the same structure of algorithm (5.9) may be used.

A rate of convergence could then be obtained using (for this particular case) the results obtained by Allwright either in [1] or [2].

Let

$$(5.15) \quad v(x) = a + \langle b, x \rangle + \frac{1}{2} \langle x, Cx \rangle; \quad C = C' \succ 0$$

and

$$(5.16) \quad v^{s=0}(x_i, y) = v(x_i) + \langle \nabla v(x_i), (y - x_i) \rangle$$

then

$$(5.17) \quad 0 < \nabla^2 v(x) = C \leq \lambda_{\max}[C] I$$

In these circumstances convergence information would be given by [2]:

$$a) \quad \text{If } [v(x_i) - v(\hat{x})] \geq d^2 \lambda_{\max}[C]$$

then

$$(5.18) \quad [v(x_{i+1}) - v(\hat{x})] \leq \frac{1}{2} [v(x_i) - v(\hat{x})] .$$

$$b) \quad \text{If } [v(x_i) - v(\hat{x})] < d^2 \lambda_{\max}[C]$$

then

$$(5.19) \quad [v(x_{i+1}) - v(\hat{x})] \leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{1}{2} \frac{[v(x_i) - v(\hat{x})]}{d^2 \lambda_{\max}[C]} \right\} .$$

Another way of finding the rate would be to consider an approach similar to that of Allwright in [1].

First consider

$$\begin{aligned} v(x) &= v(x_i) + \langle \nabla v(x_i), (x - x_i) \rangle + \frac{1}{2} \langle (x - x_i), C(x - x_i) \rangle \\ &\geq v(x_i) + \langle \nabla v(x_i), (x - x_i) \rangle \quad [\text{from (5.15)}] \end{aligned}$$

so

$$\begin{aligned} v(\hat{x}) &= \min_{x \in S} v(x) \geq \min_{x \in S} \{ v(x_i) + \langle \nabla v(x_i), (x - x_i) \rangle \} \\ &= v(x_i) + \langle \nabla v(x_i), (y_i - x_i) \rangle \end{aligned}$$

and hence

$$(5.19a) \quad v(\hat{x}) - v(x_i) \geq \langle \nabla v(x_i), (y_i - x_i) \rangle .$$

Consider now  $v(x)$  expanded about the constrained minimizer  $\hat{x}$ , i.e.

$$v(x_i) = v(\hat{x}) + \langle \nabla v(\hat{x}), (x_i - \hat{x}) \rangle + \frac{1}{2} \langle (x_i - \hat{x}), C(x_i - \hat{x}) \rangle$$

$$(5.19b) \quad \leq v(\hat{x}) + D \quad \text{where}$$

$$(5.20) \quad D = \|\nabla v(\hat{x})\| d + \frac{1}{2} d^2 \lambda_{\max}[C] .$$

Also, let

$$v(x_{i+1}) = \min \{ v(x) : x \in \text{cone}[\text{line } \{x_i, y_i\}] \}$$

$$\leq \min \{ v(x_i + \alpha[y_i - x_i]) : \alpha \in [0, 1] \} .$$

The last inequality results from the fact that

$$\text{line } \{x_i, y_i\} \subset \text{cone}[\text{line } \{x_i, y_i\}]$$

Let  $\alpha_i$  be the value of  $\alpha$  which minimizes  $v(x_i + \alpha[y_i - x_i])$

with respect to  $\alpha \in [0, 1]$  and  $\alpha_i^*$  be the unconstrained global minimizer:

$$(5.21) \quad \alpha_i^* = \frac{-\langle \nabla v(x_i), (y_i - x_i) \rangle}{\langle (y_i - x_i), C(y_i - x_i) \rangle} .$$

(1) if  $\alpha_i^* \in [0, 1]$  then  $\alpha_i = \alpha_i^*$  and

$$\begin{aligned} v(x_{i+1}) &\leq v(x_i) - \frac{1}{2} \frac{[\langle \nabla v(x_i), (y_i - x_i) \rangle]^2}{\langle (y_i - x_i), C(y_i - x_i) \rangle} \\ &\leq v(x_i) - \frac{1}{2} \frac{[v(x_i) - v(\hat{x})]^2}{\|y_i - x_i\|^2 \lambda_{\max}[C]} \quad [\text{from (5.19a)}] \\ &\leq v(x_i) - \frac{1}{2} \frac{[v(x_i) - v(\hat{x})]^2}{d^2 \lambda_{\max}[C]} \\ &\leq v(x_i) - \frac{1}{2} \frac{[v(x_i) - v(\hat{x})]^2}{2D} \end{aligned}$$

and hence

$$(5.22) \quad [v(x_{i+1}) - v(\hat{x})] \leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{[v(x_i) - v(\hat{x})]}{4D} \right\}$$

(ii) If  $\alpha_i^* > 1$  then  $\alpha_i = 1$  and

$$\begin{aligned} v(x_{i+1}) &\leq v(x_i + 1[y_i - x_i]) \\ &= v(x_i) + \langle \nabla v(x_i), (y_i - x_i) \rangle + \frac{1}{2} \langle (y_i - x_i), C(y_i - x_i) \rangle \\ &\leq v(x_i) + \frac{1}{2} \langle \nabla v(x_i), (y_i - x_i) \rangle \quad [\text{since } \alpha_i^* > 1] \\ &\leq v(x_i) - \frac{1}{2} [v(x_i) - v(\hat{x})] \quad [\text{from (5.19a)}] \\ &\leq v(x_i) - \frac{[v(x_i) - v(\hat{x})]^2}{4D} \quad [\text{from (5.19b)}] \end{aligned}$$

and so (5.22) holds again.

Neither (5.19) nor (5.22) reveal dependence of convergence rate information on the conditioning of  $C$ .

In the next section using a different approach, a rate depending on both  $\lambda_{\min}[C]$  and  $\lambda_{\max}[C]$  is derived.

### 5.3.1 - A New Rate of Convergence

Before considering the determination of the new rate of convergence for the case where algorithm (5.9) is used to minimize a general quadratic, an intermediate result revealing the relation between the diameter of  $S$  and the geometry of  $\Gamma$  will be stated and proved.

(5.23) LEMMA:

Let  $d = \max \{ \|x - y\| : x, y \in S \}$  denote the diameter of the set  $S$ , then

$$(5.24) \quad d \leq 2 \|x^*\| \frac{\Pi_{\max}}{\Pi_{\min}}$$

where  $\Pi_{\min}$  and  $\Pi_{\max}$  are as defined in (5.2) and (5.3).

Proof:

$$\begin{aligned} d &= \max \{ \|\alpha \gamma_x - \beta \gamma_y\| : \gamma_x, \gamma_y \in \Gamma; \alpha, \beta \in [0, \eta] \} \\ &\leq \max \{ \|\alpha \gamma_x\| + \|\beta \gamma_y\| : \gamma_x, \gamma_y \in \Gamma; \alpha, \beta \in [0, \eta] \} \\ &\leq \max \{ \|\alpha \gamma_x\| : \gamma_x \in \Gamma, \alpha \in [0, \eta] \} + \\ &\quad \max \{ \|\beta \gamma_y\| : \gamma_y \in \Gamma, \beta \in [0, \eta] \} \\ &= \max \{ \{ \max \|\alpha \gamma_x\| : \alpha \in [0, \eta] \} : \gamma_x \in \Gamma \} \\ &\quad + \max \{ \{ \max \|\beta \gamma_y\| : \beta \in [0, \eta] \} : \gamma_y \in \Gamma \} \\ &= \max \{ \eta \|\gamma_x\| : \gamma_x \in \Gamma \} + \\ &\quad \max \{ \eta \|\gamma_y\| : \gamma_y \in \Gamma \} \\ (5.24a) &= 2 \eta \Pi_{\max} \\ &= 2 \|x^*\| \frac{\Pi_{\max}}{\Pi_{\min}} \quad [ \text{from (5.8)} ] . \end{aligned}$$

Q.E.D.

(5.25) REMARK: The Lemma (5.23) establishes an upper bound on the diameter of  $S$  that depends on  $\Gamma$ , through  $\Pi_{\min}$  and  $\Pi_{\max}$ .

Consider then the problem of minimizing  $\bar{v}(z)$  on  $\text{cone}[\bar{\Gamma}]$  where

$$(5.26) \quad \bar{v}(z) = \langle (z^* - z), C(z^* - z) \rangle = \|z^* - z\|_C^2$$

$$(5.27) \quad C = C' > 0 \quad \text{and} \quad z^* \notin \text{cone}[\bar{\Gamma}] .$$

The following theorem establishes that by a suitable change of basis the problem of minimizing a general quadratic over a cone transforms into the problem of minimizing a norm over a new related cone. The type of transformation of  $\Gamma$  used next was introduced in [1].

(5.28) THEOREM: For  $\bar{v}(z)$  as defined on (5.26) and under assumptions (5.27),

$$\min \{ \bar{v}(z) : z \in \text{cone}[\bar{\Gamma}] \} = \min \{ v(x) : x \in \text{cone}[\Gamma] \}$$

where  $v(x) = \|x^* - x\|^2$ ,  $x^* = C^{1/2} z^*$ ,  $x = C^{1/2} z$  and

$$\Gamma = C^{1/2} \bar{\Gamma} = \{ C^{1/2} z : z \in \bar{\Gamma} \} .$$

Proof: Since  $C = C' > 0$ ,  $C$  has a symmetric self adjoint square root  $C^{1/2}$ .

$$\begin{aligned} \text{Then } \bar{v}(z) &= \|z^* - z\|_C^2 = \langle (z^* - z), C(z^* - z) \rangle \\ &= \langle C^{1/2}(z^* - z), C^{1/2}(z^* - z) \rangle \\ &= \|C^{1/2} z^* - C^{1/2} z\|^2 \\ &= \|x^* - x\|^2 = v(x) . \end{aligned}$$

So  $\min \{ \bar{v}(z) : z \in \text{cone}[\bar{\Gamma}] \} = \min \{ v(x) : x \in \text{cone}[\Gamma] \}$

Q.E.D.

Now the next theorem gives convergence rate information for the minimization of  $\bar{v}(z) = \|z^* - z\|_C^2$ .

(5.29) THEOREM:

Consider minimizing  $\bar{v}(z)$  by generating a sequence  $z_i$  defined by  $z_i = C^{-1/2} x_i$  where  $x_i$  is the sequence generated by applying algorithm (5.9) to  $v(x)$  [ of theorem (5.28) ].

Convergence rate information is given by:

$$a) \quad \text{If } \bar{v}(z_i) - \bar{v}(\hat{z}) \geq 2\bar{d}^2 \lambda_{\max}[C]$$

then

$$(5.30) \quad [\bar{v}(z_{i+1}) - \bar{v}(\hat{z})] \leq \frac{1}{2} [\bar{v}(z_i) - \bar{v}(\hat{z})].$$

$$b) \quad \text{If } \bar{v}(z_i) - \bar{v}(\hat{z}) < 2\bar{d}^2 \lambda_{\max}[C]$$

then

$$(5.31)$$

$$[\bar{v}(z_{i+1}) - \bar{v}(\hat{z})] \leq [\bar{v}(z_i) - \bar{v}(\hat{z})] \left\{ 1 - \frac{[\bar{v}(z_i) - \bar{v}(\hat{z})]}{16 \bar{\eta}^2 \bar{\Pi}_{\max}^2 \lambda_{\max}[C]} \left( \frac{\lambda_{\min}[C]}{\lambda_{\max}[C]} \right) \right\}.$$

Proof: By Theorem (5.28)

$$\min\{ \bar{v}(z) : z \in \text{cone}[\bar{\Gamma}] \} = \min\{ v(x) : x \in \text{cone}[\Gamma] \}.$$

Now since  $v(x) = \|x^* - x\|^2$ , by theorem (5.7) it is known that

$$\{\hat{x}\} = \text{argmin} \{ v(x) : x \in \text{cone}[\Gamma] \} = \text{argmin} \{ v(x) : x \in S \}$$

$$\text{where } S = \{ \alpha \gamma : \alpha \in [0, \eta] , \gamma \in \Gamma \} \quad \text{and} \quad \eta = \frac{\|x^*\|}{\Pi_{\min}}$$

Consider applying algorithm (5.9) to  $v(x)$ .

First consider  $v(x)$  expanded about  $x_i$

$$\begin{aligned} v(x) &= v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle + ||x-x_i||^2 \\ &\geq v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle . \end{aligned}$$

Then

$$v(\hat{x}) = \min\{ v(x) : x \in S \} \geq \min\{ v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle : x \in S \}$$

$$\text{Let } y_i \in \operatorname{argmin} \{ v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle : x \in S \} .$$

Hence

$$(5.32) \quad \langle \nabla v(x_i), (y_i-x_i) \rangle \leq v(\hat{x}) - v(x_i)$$

giving a lower bound for  $v(\hat{x}) - v(x_i)$  .

Now

$$v(x_{i+1}) = \min \{ v(x) : x \in \operatorname{cone}[\operatorname{line} \{y_i, x_i\}] \}$$

$$\leq \min \{ v(x_i + \alpha[y_i - x_i]) : \alpha \in [0,1] \}$$

$$(5.32a) = \min \{ v(x_i) + \alpha \langle \nabla v(x_i), (y_i - x_i) \rangle + \alpha^2 ||y_i - x_i||^2 : \alpha \in [0,1] \} .$$

Let  $\alpha^*$  be the unconstrained minimizer of  $v(x_i + \alpha[y_i - x_i])$ :

$$(5.33) \quad \alpha^* = \frac{-\langle \nabla v(x_i), (y_i - x_i) \rangle}{2||y_i - x_i||^2} .$$

Now either  $\alpha^* < 1$  or  $\alpha^* \geq 1$  .

(i) It will be shown next that:

(5.33a) If  $\alpha^* < 1$ , then

(5.33b)

$$[v(x_{i+1}) - v(\hat{x})] \leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{1}{16} \frac{[v(x_i) - v(\hat{x})]}{\bar{\eta}^2 \bar{\Pi}_{\max}^2 \lambda_{\max}[C]} \left( \frac{\lambda_{\min}[C]}{\lambda_{\max}[C]} \right) \right\}$$

Suppose  $\alpha^* < 1$ . Then  $\hat{\alpha} = \alpha^*$  where  $\hat{\alpha}$  denotes the value of  $\alpha \in [0, 1]$  that minimizes  $v(x_i + \alpha [y_i - x_i])$ .

Also

$$(5.34) \quad v(x_{i+1}) \leq v(x_i) - \frac{1}{4} \frac{\langle \nabla v(x_i), (y_i - x_i) \rangle^2}{\|y_i - x_i\|^2}$$

$$(5.35) \quad \leq v(x_i) - \frac{1}{4} \frac{[v(x_i) - v(\hat{x})]^2}{d^2} \text{ by (5.32),}$$

where  $d$  denotes the diameter of  $S$ .

From Lemma (5.23)

$$d \leq 2 \|x^*\| \frac{\Pi_{\max}}{\Pi_{\min}}$$

and hence

$$(5.36) \quad v(x_{i+1}) \leq v(x_i) - \frac{1}{16} \frac{[v(x_i) - v(\hat{x})]^2}{\|x^*\|^2 \Pi_{\max}^2} \cdot \Pi_{\min}^2$$

Now, from (5.28)

$$(5.37) \quad \|x^*\|^2 = \|C^{1/2} z^*\|^2 \leq \|z^*\|^2 \lambda_{\max}[C]$$

and from (5.3)

$$\begin{aligned}
 \Pi_{\max} &= \max \{ ||x|| : x \in \Gamma \} \\
 &= \max \{ ||C^{1/2}z|| : z \in \bar{\Gamma} \} \\
 (5.38) \quad &\leq \bar{\Pi}_{\max} \lambda_{\max}^{1/2}[C]
 \end{aligned}$$

where

$$\bar{\Pi}_{\max} = \max \{ ||z|| : z \in \bar{\Gamma} \}$$

Also from (5.2)

$$\begin{aligned}
 \Pi_{\min} &= \min \{ ||x|| : x \in \Gamma \} \\
 &= \min \{ ||C^{1/2}z|| : z \in \bar{\Gamma} \} \\
 (5.39) \quad &\geq \bar{\Pi}_{\min} \lambda_{\min}^{1/2}[C]
 \end{aligned}$$

Now use of (5.37), (5.38) and (5.39) in (5.36) yields

$$\begin{aligned}
 v(x_{i+1}) &\leq v(x_i) - \frac{1}{16} \frac{[v(x_i) - v(\hat{x})]^2}{||z^*||^2 \bar{\Pi}_{\max}^2 \lambda_{\max}^2[C]} (\bar{\Pi}_{\min}^2 \lambda_{\min}[C]) \\
 &= v(x_i) - \frac{1}{16} \frac{[v(x_i) - v(\hat{x})]^2}{\eta^2 \bar{\Pi}_{\max}^2 \lambda_{\max}[C]} \left( \frac{\lambda_{\min}[C]}{\lambda_{\max}[C]} \right)
 \end{aligned}$$

which proves that (5.33b) holds.

(ii) It will be shown next that:

If  $\alpha^* \geq 1$  then

$$(5.40) \quad [v(x_{i+1}) - v(\hat{x})] \leq \frac{1}{2} [v(x_i) - v(\hat{x})] .$$

Suppose  $\alpha^* \geq 1$ . Then  $\hat{\alpha}=1$  and from (5.32a)

$$v(x_{i+1}) \leq v(x_i) + \langle \nabla v(x_i), (y_i - x_i) \rangle + \|y_i - x_i\|^2$$

But since  $\alpha^* \geq 1$  it follows from (5.33) that

$$\|y_i - x_i\|^2 \leq -\frac{1}{2} \langle \nabla v(x_i), (y_i - x_i) \rangle$$

$$\text{and so } v(x_{i+1}) \leq v(x_i) + \frac{1}{2} \langle \nabla v(x_i), (y_i - x_i) \rangle$$

$$\leq v(x_i) - \frac{1}{2} [v(x_i) - v(\hat{x})] \quad [\text{from (5.32)}]$$

and then

$$(5.41) \quad [v(x_{i+1}) - v(\hat{x})] \leq \frac{1}{2} [v(x_i) - v(\hat{x})]$$

which proves that (5.40) holds.

(iii) It will be shown next that:

$$(5.42) \quad \text{if } v(x_i) - v(\hat{x}) \geq 2 \bar{d}^2 \lambda_{\max}[C], \text{ then } \alpha^* \geq 1.$$

From (5.32) it is known that:

$$v(x_i) - v(\hat{x}) \leq -\langle \nabla v(x_i), (y_i - x_i) \rangle$$

Now if  $v(x_i) - v(\hat{x}) \geq 2 \bar{d}^2 \lambda_{\max}[C]$  then

$$-\langle \nabla v(x_i), (y_i - x_i) \rangle \geq 2 \bar{d}^2 \lambda_{\max}[C] \geq 2 \bar{d}^2 \geq 2 \|x_i - y_i\|^2$$

where the central inequality occurs because

$$\bar{d}^2 = \max \{ \|x - \bar{x}\|^2 : x, \bar{x} \in S \}$$

$$\leq \max \{ \|z - \bar{z}\|^2 \lambda_{\max}[C] : z, \bar{z} \in \bar{S} \}$$

$$= \bar{d}^2 \lambda_{\max}[C]$$

Then, from (5.33),  $\alpha^* \geq 1$ , which proves (5.42).

(iv) It will be shown next that:

(5.43) if  $v(x_i) - v(\hat{x}) < 2 \bar{d}^2 \lambda_{\max}[C]$  then (5.33b) holds.

Suppose

$$(5.44) \quad v(x_i) - v(\hat{x}) < 2 \bar{d}^2 \lambda_{\max}[C].$$

Clearly either  $\alpha^* < 1$  or  $\alpha^* \geq 1$ .

if  $\alpha^* < 1$ , then (5.33b) holds, from (5.33a).

Next, suppose  $\alpha^* \geq 1$ . Then

$$\begin{aligned} v(x_{i+1}) &\leq v(x_i) - \frac{1}{2} [v(x_i) - v(\hat{x})] \quad [\text{from (5.40)}] \\ &\leq v(x_i) - \frac{1}{2} \frac{[v(x_i) - v(\hat{x})]^2}{4 \bar{d}^2 \lambda_{\max}[C]} \quad [\text{from (5.44)}] \\ &\leq v(x_i) - \frac{1}{16} \frac{[v(x_i) - v(\hat{x})]^2}{\bar{\eta}^2 \bar{\Pi}_{\max}^2 \lambda_{\max}[C]} \quad [\text{from (5.24a)}] \\ &\leq v(x_i) - \frac{1}{16} \frac{[v(x_i) - v(\hat{x})]^2}{\bar{\eta}^2 \bar{\Pi}_{\max}^2 \lambda_{\max}[C]} \left( \frac{\lambda_{\min}[C]}{\lambda_{\max}[C]} \right) \end{aligned}$$

showing that (5.33b) holds.

Hence, whether  $\alpha^* \geq 1$  or  $\alpha^* < 1$ , (5.33b) holds, which proves (5.43).

(v) In order to conclude the proof of the theorem it will be shown that (5.33b) and (5.40) imply (5.31) and (5.30) respectively.

This will be done by showing that  $\bar{v}(z_i) = v(x_i) \quad \forall i$ .

Now, by assumption  $z_i = C^{-1/2} x_i \quad \forall i$ , then

$$\begin{aligned} \bar{v}(z_i) &= \|z^* - z_i\|_C^2 \\ &= \langle C^{1/2}(z^* - z_i), C^{1/2}(z^* - z_i) \rangle \\ &= \|C^{1/2}z^* - C^{1/2}z_i\| \\ &= \|x^* - x_i\| = v(x_i) \quad \forall i. \end{aligned}$$

Q.E.D.

#### 5.4 - GENERALIZATION TO A CONVEX FUNCTION $v(x)$

Now consider the problem

$$\min \{ v(x) : x \in \text{cone}[\Gamma] \}$$

where  $\Gamma$  is as before but  $v(x)$  is a convex function with second derivative matrix satisfying:

$$(5.45) \quad 0 < nI \leq C=C' \leq \nabla^2 v(x) \leq MI$$

Here again the iterative procedure to approximate the minimizer  $\hat{x}$  of  $v$  on the unbounded set  $\text{cone}[\Gamma]$  is based on the fact that  $\hat{x}$  is also the minimizer of  $v$  on a bounded set  $S$  -as before a suitably truncated version of  $\text{cone}[\Gamma]$ .

In order to define the new set  $S$ , when  $v(x)$  is a general convex function, the following theorem is of importance.

(5.46) THEOREM: For any  $\tilde{x} \in \text{cone}[\Gamma]$ ,

$$\text{let } \eta = \frac{\|\tilde{x}^*\|}{\Pi_{\min}} + \frac{a(\tilde{x}, x^*)}{\Pi_{\min}} \quad \text{where}$$

$$(5.47) \quad a(\tilde{x}, x^*) = \left[ \frac{2}{n} (v(\tilde{x}) - v(x^*)) \right]^{1/2},$$

which is a computable quantity.

Suppose  $S = \{\alpha \gamma : \alpha \in [0, \eta], \gamma \in \Gamma\}$

Then  $S$  is convex and  $\hat{x} \in S$ , where  $\hat{x}$  is the global minimizer of  $v$  on  $\text{cone}[\Gamma]$

Proof: (i) Convexity of  $S$

Consider arbitrary  $s_1, s_2 \in S$

then

$$s_1 = \alpha_1 \gamma_1, s_2 = \alpha_2 \gamma_2 \quad \text{for } \alpha_1, \alpha_2 \in [0, \eta] \text{ and for some } \gamma_1, \gamma_2 \in \Gamma.$$

Now let  $\bar{s} \triangleq \beta s_1 + (1-\beta) s_2$  where  $\beta \in [0, 1]$

then if  $\bar{s} = \beta \alpha_1 \gamma_1 + (1-\beta) \alpha_2 \gamma_2 \in S$ ,  $S$  is convex.

Now by the convexity of  $\Gamma$

$$\text{implying } \bar{\beta} \gamma_1 + (1-\bar{\beta}) \gamma_2 \in \Gamma \quad \text{where } \bar{\beta} \in [0, 1]$$

$$(5.47a) \quad \alpha [\bar{\beta} \gamma_1 + (1-\bar{\beta}) \gamma_2] \in \alpha \Gamma \subset S \quad \text{for some } \alpha \in [0, \eta]$$

$$\text{Choose } \bar{\beta} = \frac{\alpha_1 \beta}{\alpha_2(1-\beta) + \alpha_1 \beta} \in [0, 1] \quad \text{and}$$

$$\alpha = \alpha_2(1-\beta) + \alpha_1 \beta \in [0, \eta]$$

Then from (5.47a)

$$\gamma_2 \alpha_2(1-\beta) + \alpha_1 \beta \gamma_1 = \bar{s} \in S$$

Hence  $S$  is convex .

(ii) Proof that  $\hat{x} \in S$

Take any  $\tilde{x} \in \text{cone}[\Gamma]$

then

$$(5.48) \quad v(\hat{x}) \leq v(\tilde{x})$$

Now use of the Taylor formula for a second order expansion about the unconstrained global minimizer  $x^*$  of  $v$ , yields:

$$v(\hat{x}) = v(x^*) + \int_0^1 (1-t) \langle \hat{x} - x^*, \nabla^2 v(x^* + t(\hat{x} - x^*)) (\hat{x} - x^*) \rangle dt$$

$$\geq v(x^*) + \frac{1}{2} n \|\hat{x} - x^*\|^2 \quad [\text{from (5.45)}]$$

and hence

$$\begin{aligned} \|\hat{x} - x^*\|^2 &\leq \frac{2}{n} [v(\hat{x}) - v(x^*)] \\ &\leq \frac{2}{n} [v(\tilde{x}) - v(x^*)] \quad [\text{from (5.48)}] \end{aligned}$$

Also, since  $\|\hat{x}\| = \|x^* + \hat{x} - x^*\|$ ,

$$\|\hat{x}\| \leq \|x^*\| + \|\hat{x} - x^*\|$$

Therefore, finally

$$\|\hat{x}\| \leq \|x^*\| + \left[ \frac{2}{n} [v(\tilde{x}) - v(x^*)] \right]^{1/2}$$

i.e.

$$\|\hat{x}\| \leq \|x^*\| + a(\tilde{x}, x^*)$$

Now since  $\hat{x} \in \text{cone}[\Gamma]$ , it follows that  $\hat{x} = \theta \hat{\gamma}$  for some  $\theta \in \mathbb{R}^+$ ,  $\hat{\gamma} \in \Gamma$ .

Then from (5.2),  $\|\hat{x}\| = \theta \|\hat{y}\| \geq \theta \Pi_{\min}$  and consequently

$$\theta \leq \frac{\|\hat{x}\|}{\Pi_{\min}} \leq \frac{\|x^*\|}{\Pi_{\min}} + \frac{a(\tilde{x}, x^*)}{\Pi_{\min}} = \eta$$

Hence  $\hat{x} = \theta \hat{y} \in S$ .

Q.E.D.

Consider now a new version of algorithm (5.9)

(5.49) Algorithm

(i) Initial step

Set  $x_0 = 0$ ,  $i = 0$ .

Compute  $\eta = \frac{\|x^*\|}{\Pi_{\min}} + \frac{a(x_0, x^*)}{\Pi_{\min}}$ .

(ii) Find a  $y_i \in \operatorname{argmin} \{ \langle \nabla v(x_i), y - x_i \rangle : y \in S \}$

by finding a

$$y_i \in \operatorname{argmin} \{ \langle \nabla v(x_i), y \rangle : y \in \Gamma \}$$

and setting

$$y_i = \begin{cases} x_i & \text{if } \langle \nabla v(x_i), y_i \rangle \geq 0 \\ \eta y_i & \text{otherwise} \end{cases}$$

(iii) Compute a lower bound for  $v(\hat{x})$  and STOP if  $v(x_i)$

satisfies a stopping condition. Otherwise, CONTINUE.

(iv) Compute an approximate solution  $x_{i+1}$  to

$$\min \{ v(x) : x \in \text{cone}[\text{line} \{x_i, y_i\}] \}$$

by solving  $\min \{ v(x_i + \lambda(y_i - x_i)) : \lambda \in [0, 1] \}$

using an ARMIJO TYPE rule, where the  $\lambda_i \in [0, 1]$  that minimizes approximately  $v(x_i + \lambda(y_i - x_i))$  will be chosen to be  $\beta^l$  for the smallest integer  $l \geq 0$  such that:

$$v(x_i + \beta^l(y_i - x_i)) \leq v(x_i) + \frac{1}{2} \beta^l \langle \nabla v(x_i), (y_i - x_i) \rangle$$

where  $\beta \in (0, 1)$  and is independent of  $i$ .

(v) Set  $i=i+1$  and GO TO (ii).

Convergence rate information regarding algorithm (5.49) is given by the following theorem:

(5.50) THEOREM: For algorithm (5.49), global convergence rate information is given by:

(a) If  $v(x_i) - v(\hat{x}) \geq \frac{1}{n} M^2 d^2$

Then  $[v(x_{i+1}) - v(\hat{x})] \leq \frac{1}{2} [v(x_i) - v(\hat{x})]$

(b) If  $v(x_i) - v(\hat{x}) < \frac{1}{n} M^2 d^2$

then

$$[v(x_{i+1}) - v(\hat{x})] \leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{1}{8} \beta \frac{[v(x_i) - v(\hat{x})]}{M \eta^2 \Pi_{\max}^2} \left( \frac{n}{M} \right) \right\}$$

Proof: The Taylor formula for second order expansions gives

$$\begin{aligned} v(x) &= v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle \\ &\quad + \int_0^1 (1-t) \langle (x-x_i), \nabla^2 v(x_i + t[x-x_i]) (x-x_i) \rangle dt \\ &\geq v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle + \frac{1}{2} \langle (x-x_i), C(x-x_i) \rangle \text{ [by (5.45)]} \\ &\geq v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle \quad \text{since } C \geq nI > 0. \end{aligned}$$

Hence

$$\min\{v(x) : x \in S\} \geq \min\{v(x_i) + \langle \nabla v(x_i), (x-x_i) \rangle : x \in S\}$$

$$\text{and so } v(\hat{x}) \geq v(x_i) + \langle \nabla v(x_i), (y_i - x_i) \rangle$$

i.e.

$$(5.51) \quad v(\hat{x}) - v(x_i) \geq \langle \nabla v(x_i), (y_i - x_i) \rangle$$

Also

$$(5.52) \quad v(x_{i+1}) \leq \min\{v(x_i + \lambda(y_i - x_i)) : \lambda \in [0, 1]\}.$$

Minimization of  $v(x_i + \lambda(y_i - x_i))$  with respect to  $\lambda \in [0, 1]$  is done approximately.

Then the  $\lambda_i \in [0, 1]$  that minimizes approximately  $v(x_i + \lambda(y_i - x_i))$  will be chosen to be  $\beta^l$  for the smallest integer  $l \geq 0$  such that:

$$(5.53) \quad v(x_i + \beta^l(y_i - x_i)) \leq v(x_i) + \frac{1}{2} \beta^l \langle \nabla v(x_i), (y_i - x_i) \rangle$$

Now, using the Taylor formula

$$\begin{aligned} v(x_i + \lambda[y_i - x_i]) &= v(x_i) + \lambda \langle \nabla v(x_i), (y_i - x_i) \rangle \\ &\quad + \lambda^2 \int_0^1 (1-t) \langle (y_i - x_i), \nabla^2 v(x_i + t\lambda[y_i - x_i]) (y_i - x_i) \rangle dt \end{aligned}$$

Since  $C=C'>0$ , by (5.45),  $C^{1/2}>0$  exists, so

$$\begin{aligned} v(x_i + \lambda[y_i - x_i]) &= v(x_i) + \lambda \langle \nabla v(x_i), (y_i - x_i) \rangle \\ &\quad + \lambda^2 \int_0^1 (1-t) \langle C^h(y_i - x_i), C^{-h} \nabla^2 v(x_i + t\lambda[y_i - x_i]) C^{-h} C^h(y_i - x_i) \rangle dt \\ &\leq v(x_i) + \lambda \langle \nabla v(x_i), (y_i - x_i) \rangle + \frac{1}{2} \lambda^2 \frac{M}{n} \langle (y_i - x_i), C(y_i - x_i) \rangle \\ &= \phi_i(\lambda) \quad . \end{aligned}$$

Consequently

$$(5.54) \quad v(x_i + \lambda[y_i - x_i]) \leq v(x_i) + \frac{1}{2} \lambda \langle \nabla v(x_i), (y_i - x_i) \rangle$$

if

$$\begin{aligned} &v(x_i) + \lambda \langle \nabla v(x_i), (y_i - x_i) \rangle + \frac{1}{2} \lambda^2 \frac{M}{n} \langle (y_i - x_i), C(y_i - x_i) \rangle \\ &\leq v(x_i) + \frac{1}{2} \lambda \langle \nabla v(x_i), (y_i - x_i) \rangle \end{aligned}$$

i.e. if

$$(5.54a) \quad 0 \leq \lambda \leq \frac{-\langle \nabla v(x_i), (y_i - x_i) \rangle}{\langle (y_i - x_i), C(y_i - x_i) \rangle} \cdot \frac{n}{M} = \lambda_i^*$$

where  $\lambda_i^*$  is the unconstrained minimizer of  $\phi_i(\lambda)$ .

(i) Now if  $\lambda_i^* \geq 1$  then (5.54) holds for all  $\lambda \in [0, 1]$  and

the ARMIJO rule will choose  $\lambda=1$  and from (5.52) and (5.54)

$$\begin{aligned} v(x_{i+1}) &\leq v(x_i) + \frac{1}{2} \langle \nabla v(x_i), (y_i - x_i) \rangle \\ &\leq v(x_i) - \frac{1}{2} [v(x_i) - v(\hat{x})] \quad [\text{from (5.54)}] \end{aligned}$$

and hence

$$(5.55) \quad [v(x_{i+1}) - v(\hat{x})] \leq \frac{1}{2} [v(x_i) - v(\hat{x})]$$

(ii) If  $\lambda_i^* < 1$ , the value chosen by the ARMIJO TYPE rule will satisfy  $\lambda_i > \beta \lambda_i^*$ . In fact (5.54) will be satisfied for all  $\lambda \in [0, \lambda_i^*]$  and as  $\lambda$  will be chosen as the largest number of the form  $\beta^l$  ( $l \geq 0$ ) it cannot be smaller than  $\beta \lambda_i^*$ .

Again from (5.52) and (5.54)

$$\begin{aligned} v(x_{i+1}) &\leq v(x_i) + \frac{1}{2} \lambda_i \langle \nabla v(x_i), (y_i - x_i) \rangle \\ &\leq v(x_i) + \frac{1}{2} \beta \lambda_i^* \langle \nabla v(x_i), (y_i - x_i) \rangle \\ &\leq v(x_i) - \frac{1}{2} \beta \frac{\langle \nabla v(x_i), (y_i - x_i) \rangle^2}{\langle (y_i - x_i), C(y_i - x_i) \rangle} \left(\frac{n}{M}\right) \\ &\leq v(x_i) - \frac{1}{2} \beta \frac{[v(x_i) - v(\hat{x})]^2}{M d^2} \left(\frac{n}{M}\right) \end{aligned}$$

where  $d$  is the diameter of  $S$ .

Hence

$$\begin{aligned} (5.56) \quad [v(x_{i+1}) - v(\hat{x})] &\leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{1}{2} \beta \frac{[v(x_i) - v(\hat{x})]}{M d^2} \left(\frac{n}{M}\right) \right\} \\ &\leq [v(x_i) - v(\hat{x})] \left\{ 1 - \frac{1}{8} \beta \frac{[v(x_i) - v(\hat{x})]}{M \eta^2 \Pi_{\max}^2} \left(\frac{n}{M}\right) \right\} \\ &\hspace{15em} \text{by (5.24a)} \end{aligned}$$

where  $\eta$  is as in theorem (5.46).

It will now be proved that if

$$(5.57) \quad v(x_i) - v(\hat{x}) \geq \frac{M^2 d^2}{n}$$

then  $\lambda_i^* \geq 1$ .

From (5.51)

$$v(x_i) - v(\hat{x}) \leq - \langle \nabla v(x_i), (y_i - x_i) \rangle$$

and by (5.57) and (5.45)

$$\begin{aligned} - \langle \nabla v(x_i), (y_i - x_i) \rangle &\geq \frac{M^2 d^2}{n} \geq \left(\frac{M}{n}\right) M \|y_i - x_i\|^2 \\ &\geq \frac{M}{n} \langle (y_i - x_i), C(y_i - x_i) \rangle \end{aligned}$$

and so, from (5.54a)  $\lambda_i^* \geq 1$

Now, if

$$(5.58) \quad v(x_i) - v(\hat{x}) < \frac{M^2 d^2}{n}$$

$\lambda_i^*$  is either  $\geq 1$  or  $< 1$

If  $\lambda_i^* < 1$  then (5.56) holds.

If  $\lambda_i^* \geq 1$  (5.56) holds again since as  $\lambda$  will be chosen equal to 1:

$$\begin{aligned} v(x_{i+1}) &\leq v(x_i) + \frac{1}{2} \langle \nabla v(x_i), (y_i - x_i) \rangle \\ &\leq v(x_i) - \frac{1}{2} [v(x_i) - v(\hat{x})] \\ &\leq v(x_i) - \frac{1}{2} \beta \frac{[v(x_i) - v(\hat{x})]^2}{M d^2} \left(\frac{n}{M}\right) \end{aligned}$$

as  $\beta \in (0, 1)$  and  $\frac{[v(x_i) - v(\hat{x})]}{M d^2} \left(\frac{n}{M}\right) < 1$  [from (5.58)] .

Q.E.D.

5.5 - REFERENCES

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## CHAPTER 6

### CONCLUSIONS

6.1 - Here some general comments about the thesis are given and some topics for future research are mentioned.

As was emphasized earlier, many optimal control problems can either be formulated as convex programming problems (e.g. the classical open-loop optimal control problem introduced in [1] analysed in Chapter 2) or their solution involves the solution of a convex programming problem (e.g. the non-classical dominant output feedback problem [2] considered in Chapter 3).

In this thesis, feasible direction algorithms or extended versions of feasible direction algorithms (e.g. set approximation algorithms) are studied for convex programming problems. Each iteration of a feasible direction method can be decomposed into the solution of two subproblems,  $(P_1)$  and  $(P_2)$ . The solution of one of these minor problems  $(P_1)$  gives the search direction that generates the feasible set over which the other minor problem  $(P_2)$  is going to be solved. Finding search directions is a fundamental step of a feasible direction

algorithm. Very often search directions  $h_j$  are found by optimizing over the constraint set  $Q$  an approximation  $f^S$  to the objective function  $f$ , e.g.

$$(6.1) \quad h_j = \hat{y}(z_j) - z_j \quad \text{where}$$

$$(6.2) \quad \hat{y}(z_j) \in \operatorname{argmin} \{ f^S(z_j, y) : y \in Q \}$$

Algorithms based on this sort of approach were considered by Allwright [1] and rate of convergence information derived for the case where  $\hat{y}(x_j)$  is the exact solution of

$$(6.3) \quad \min \{ f^S(z_j, y) : y \in Q \}.$$

In Chapter 2 is investigated what happens when exact solution of (6.3) is not possible and only an  $\epsilon$ -approximation  $\tilde{y}(z_j)$  to  $\hat{y}(z_j)$  is available. The effects on the convergence behaviour of the algorithm were studied in detail and new results presented. It was shown that improvement of the global problem convergence rate bounds will be achieved by iterating further on the subproblem. For the case where  $S > 0$  it was revealed that the convergence rate depends on the ratio between second and first order effects on  $f^S$  and properties of this ratio were studied. An optimal control problem where the results obtained in Chapter-2, seem to have full applications was outlined.

As mentioned before feasible direction methods involve the solution of two subproblems,  $(P_1)$  and  $(P_2)$ :

$(P_1)$  - The optimization, of an approximation  $f^S$  to  $f$ , over the feasible set.

$(P_2)$  - The optimization of the objective function over the search line obtained from the solution of  $(P_1)$ .

Intuitively, if  $(P_2)$  were solved over a larger set, containing the search line, a bigger cost reduction could be expected and consequently a faster convergence. Based on this intuitive idea a connection between feasible direction algorithms and set-approximation methods was introduced on Chapter 3. Methods to generate inner sets (search sets) as convex hulls of sets of base points were envisaged. At each step of the algorithm, some new points are added and others dropped, updating the basis and hence the inner sets over which the optimization is going to be carried out. Particular cases of interest were studied. So far it has been assumed that  $f^S$  is lower than  $f$  everywhere. However, not always is such an approximation available or easy to derive. Chapter 4 is then dedicated to the study of methods of feasible direction where the search direction is found by optimization of an  $f^S$  that is either an upper approximation to  $f$  everywhere or is a

lower approximation derived from an available upper approximation. Rate of convergence information for the related algorithms were derived. It was shown that when  $f^S$  is an upper approximation the rate of convergence is of the same type as that obtained by Allwright in [1] for an  $f^S$  which was a lower approximation. In fact the result tells us that close to the optimum the two approaches are equivalent. By introducing a lower bound  $f^{\alpha U}$  derived from an upper bound  $f^U$  a non-sharp bound for the "distance" from the optimum was derived and convergence rate information obtained for the case where one is too far away from the optimum.

Eventhough this new approach didn't give very strong results it seems to be interesting.

Chapter 5 deals with a special type of convex programming problem where the feasible set is a convex, closed but unbounded cone generated by a convex and compact set . The results obtained by Allwright [3] for an objective function which was a norm were extended first to the case of a general quadratic and then to a general strictly convex function. The solution considered in [3] involves the formulation of an equivalent problem where the unbounded feasible set is replaced by another bounded set actually a truncated version of the original one. It was shown that for the more general case an equivalent problem can also be formulated. When the objective

function is a quadratic, the convergence rate obtained, using a coordinate transformation, shows dependence on the ratio between the smallest and the largest eigenvalues of the second-derivative matrix. This is interesting since the dependence on  $\lambda_{\min}$  is not apparent when one uses the results of Allwright [1], [3].

In this thesis it has been assumed that the objective functions to be optimized were either twice differentiable and convex (with second derivative satisfying some properties) or had a particular quadratic nature.

One possible extension of the work contained in this thesis regards consideration of objective functions of a different nature [4], perhaps:

$$f(u) = h(\underline{v}(u))$$

where  $\underline{v}$  is a vector of convex functions and  $h$  is monotone (e.g.  $f(u) = \exp v_1(u) + v_2(u)$ ).

This is relevant in a control context because control problems may have several scalar valued performance measures  $v_i(u)$  which are conflicting, describing the performance in such a way that the objective is best achieved when  $v_i(u)$  is minimal ( $\forall i$ ). If no single control exists which satisfies that aim, it is reasonable to construct a scalar valued objective  $f$  from the  $v_i$  using a monotone  $h$ , as then  $f$  decreases whenever any  $v_i(u)$  decrease and the others do not increase.

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