

UNIVERSITY OF LONDON

ABELIAN THREEFOLDS OF
GEOMETRIC GENUS ONE

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ABSTRACT

An Abelian variety of rank r can always be mapped birationally on the sets of an involution of order r on a Picard variety. One method of generating such an involution is by means of a finite linear group operating on the independent variables of the underlying Abelian function field. In particular, if the involution is not compounded of a Picard involution (an assumption which involves no loss of generality) and if some plurigenus of the Abelian variety is non-zero, the corresponding involution can always be generated in this fashion.

This thesis investigates the Abelian threefolds of geometric genus one in terms of these linear groups. Since the geometric genus is one, the homogeneous part of each substitution must be modular. This fact, together with the existence of a sextic characteristic equation for any substitution, determines the possible forms for the homogeneous groups obtained by omitting the constant vectors. The Riemann matrices (each with an associated function field and Picard threefold) which admit these groups are next determined as admitting appropriate complex multiplications. Finally, the matrices determine the forms of the

constant vectors in the non-homogeneous groups.

It is found that there are 64 possible homogeneous groups. For a selection of these, which covers all the types, the sets of matrices and non-homogeneous linear groups are found in full. In all other cases it is established by example that matrices admitting the group exist.

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CHAPTER 1. ABELIAN FUNCTIONS AND ABELIAN THREEFOLDS

1. Scope of the chapter. The first part of this chapter describes, without proofs, the general theorems of the theory of Abelian functions and Riemann matrices of genus three. The outline follows the development in the treatise of Conforto (4), to which reference may be made for full details. Conforto in fact deals with functions of arbitrary genus p , but since we shall not require p to exceed three the results are given here for $p=3$. This theory leads to the introduction of Abelian threefolds in the second part of the chapter.

Abelian functions of genus three

2. Definition. An Abelian function of genus three is a meromorphic function of three essential independent complex variables possessing six linearly independent periods.

3. Fundamental periods. Let the variables u_1, u_2, u_3 form a column vector $\underline{u}$ so that $\tilde{\underline{u}} = (u_1, u_2, u_3)$, the $\sim$ denoting transposition, and let $f(\underline{u})$ be an Abelian function^{*}. Then there are six linearly independent vectors $\underline{\omega}_i$, with $\tilde{\underline{\omega}}_i = (\omega_{1i}, \omega_{2i}, \omega_{3i})$, such that

$$f(\underline{u} + \underline{\omega}_i) = f(\underline{u}), \quad i = 1, 2, \dots, 6.$$

Any sum of integral multiples of the $\underline{\omega}_i$ is also a period.

The vector $\underline{u}$ may be represented by the point in a real

* In the general theory of the first two chapters we use the variable $\underline{u}$, but for the work of later chapters it is convenient to go over to variables x, y, z .

Euclidean space of six dimensions, S_6 , whose coordinates are the real and imaginary parts of the components of $\underline{u}$. The periods $\underline{\omega}$ are represented in S_6 by a discrete set of points; for otherwise the set would admit a finite limit point and this may be shown to imply infinitesimal periods and further that the variables are not "essential", contrary to the definition.

We can choose, in an infinity of ways, a set of six periods such that every other period is obtained as an integral combination of these periods; such a set is termed a fundamental or primitive set of periods. The periods of a primitive set are represented by linearly independent points of S_6 which, together with the origin, determine a fundamental cell of non-zero volume; the points of the cell are all those whose coordinates are of the form

$$\sum \lambda_i \omega_i, \quad 0 \leq \lambda_i < 1, \quad i = 1, 2, \dots, 6.$$

As a corollary it is seen that $f(\underline{u})$ cannot admit more than six independent periods.

4. Riemann matrices. The period vectors of a primitive set may be arranged to form the columns of a 3×6 matrix

$$\underline{\Omega} = (\underline{\omega}_1, \underline{\omega}_2, \underline{\omega}_3, \underline{\omega}_4, \underline{\omega}_5, \underline{\omega}_6) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} & \omega_{16} \\ \omega_{21} & \omega_{22} & \omega_{23} & \omega_{24} & \omega_{25} & \omega_{26} \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & \omega_{35} & \omega_{36} \end{pmatrix}.$$

This is a Riemann matrix of genus three. A Riemann matrix is any matrix whose columns form a primitive set of periods for some Abelian function. The matrix $\underline{\Omega}$ is a Riemann matrix for $f(\underline{u})$, and $f(\underline{u})$ is attached to $\underline{\Omega}$.

If $\underline{\Omega}'$ is any other Riemann matrix for $f(\underline{u})$, the columns of $\underline{\Omega}'$ are expressible integrally in terms of those of $\underline{\Omega}$ and vice versa. Thus

$$\underline{\Omega}' = \underline{\Omega} \underline{M}$$

where $\underline{M}$ is a 6x6 unimodular matrix, i.e. a matrix with integral elements and determinant ± 1 .

5. Change of variable. If a new variable vector $\underline{u}'$ be taken, given by the equation

$$\underline{u}' = \underline{A} \underline{u}$$

where $\underline{A}$ is a non-singular 3x3 matrix with complex elements, then $f(\underline{u})$ becomes $f'(\underline{u}')$ and a Riemann matrix for $f'(\underline{u}')$ is given by $\underline{A} \underline{\Omega}$. Any other Riemann matrices for $f'(\underline{u}')$, $f(\underline{u})$ are related by an equation

$$\underline{\Omega}' = \underline{A} \underline{\Omega} \underline{M} \tag{1}$$

where the 6x6 matrix $\underline{M}$ is unimodular.

Two Riemann matrices $\underline{\Omega}$, $\underline{\Omega}'$ satisfying an equation of the form (1) where $\underline{A}$ is a non-singular complex 3x3 matrix and $\underline{M}$ is a 6x6 unimodular matrix are said to be equivalent. The relationship is reflexive, symmetric and transitive, and divides the Riemann matrices of genus three into equivalence classes. Any two equivalent matrices are period matrices for two Abelian functions related by a non-singular linear transformation of variable.

6. Intermediary functions. Cousin's theorem states that: Every meromorphic function $f(u)$ may be expressed as the quotient of two entire functions $\phi(u)$, $\psi(u)$ which have no

common divisor at any point and which vanish simultaneously only at the points of essential indeterminacy of $f(u)$.

The functions $\phi(u)$ and $\psi(u)$ are determined only to within a factor which is a non-vanishing entire function, i.e. $\phi(u)$ and $\psi(u)$ may both be multiplied by a factor of the form

$$e^{G(u)}$$

where $G(u)$ is an entire function.

Applying this theorem to an Abelian function $f(u)$ with Riemann matrix Ω , it follows that

$$f(u) = f(u + \omega_i) = \frac{\phi(u + \omega_i)}{\psi(u + \omega_i)} = \frac{e^{g_i(u)} \phi(u)}{e^{g_i(u)} \psi(u)}, \quad i = 1, 2, \dots, 6,$$

where the $g_i(u)$ are entire functions. It can be shown that for a suitable choice of $G(u)$, the $g_i(u)$ all reduce to non-homogeneous linear functions of u , and we write

$$g_i(u) = 2\pi i(\tilde{\lambda}_i u + \gamma_i), \quad i = 1, 2, \dots, 6.$$

The six complex 3-vectors λ_i may be arranged as a matrix

$$\underline{\Lambda} = (\lambda_i) = (\lambda_{hi}), \quad \begin{matrix} h = 1, 2, 3; \\ i = 1, 2, \dots, 6, \end{matrix}$$

which is the second period matrix of $\phi(u), \psi(u)$ - Ω being their first period matrix - and the γ_i give a complex 6-vector $\underline{\gamma}$, with $\tilde{\gamma} = (\gamma_i)$, which is the parameter of $\phi(u), \psi(u)$. Such functions $\phi(u), \psi(u)$, for which the $g_i(u)$ are linear, are called intermediary functions. Two intermediary functions with the same $\Omega, \underline{\Lambda}$ and $\underline{\gamma}$ are said to be "of the same type".

7. The Existence theorem for Abelian functions. By evaluating $\phi((u + \omega_i) + \omega_j)$ and $\phi((u + \omega_j) + \omega_i)$ and equating, it is found that Ω and $\underline{\Lambda}$ must be so related that

$$\underline{N} = \underline{\tilde{Q}} \underline{A} - \underline{\tilde{A}} \underline{Q}$$

is an integral skew-symmetric 6x6 matrix. The matrix $\underline{N}$ is uniquely determined by $f(\underline{u})$ and $\underline{Q}$ and is a characteristic matrix of $f(\underline{u})$; it is found to satisfy the condition

$$\underline{Q} \underline{N}^{-1} \underline{\tilde{Q}} = 0. \quad (2)$$

Further, for $f(\underline{u})$ to exist as a non-degenerate function it is necessary that

$$1 \underline{\tilde{u}} \underline{Q} \underline{N}^{-1} \underline{\tilde{Q}} \underline{u} > 0, \text{ for all non-zero } \underline{u}. \quad (3)$$

The bar - denotes a complex conjugate. The matrix $\underline{L} = 1 \underline{Q} \underline{N}^{-1} \underline{\tilde{Q}}$ is Hermitian, i.e. $\underline{\tilde{L}} = \underline{L}$, and (3) asserts that the corresponding Hermitian form or, more briefly, the matrix itself is positive definite; we write simply

$$1 \underline{Q} \underline{N}^{-1} \underline{\tilde{Q}} > 0. \quad (4)$$

The conditions (2) and (4) are also sufficient for $\underline{Q}$ to be a Riemann matrix. Since $\underline{P} = \underline{N}^{-1}$ is a rational matrix, there is no loss of generality in multiplying by some integer to give an integral matrix. We can then state the following Existence theorem for Abelian functions of genus three:

The necessary and sufficient condition for a 3x6 matrix $\underline{Q}$ to be the period matrix of some Abelian function of genus three, i.e. to be a Riemann matrix of genus three, is that there exists an integral skew-symmetric 6x6 matrix $\underline{P}$ such that

$$(1) \quad \underline{Q} \underline{P} \underline{\tilde{Q}} = 0;$$

$$(11) \quad 1 \underline{Q} \underline{P} \underline{\tilde{Q}} > 0.$$

The relations (1) and (11) are the Riemann equality and Riemann inequality respectively. The matrix $\underline{P}$ is a principal matrix of $\underline{Q}$. Thus $\underline{Q}$ is a Riemann matrix if and only

if it possesses a principal matrix. There is no loss of generality in assuming that the elements of $\underline{P}$ are relatively prime. A given principal matrix $\underline{P}$ gives rise to an infinite sequence of characteristic matrices $s\underline{N}$ ($s = 1, 2, \dots$) where $\underline{N} = k\underline{P}^{-1}$ with k the positive rational number such that $\underline{N}$ is integral with relatively prime elements.

8. Isomorphic Riemann matrices. A 3×6 matrix $\underline{\Omega}'$ is said to be isomorphic with a Riemann matrix $\underline{\Omega}$ of genus 3 when there is a relation

$$\underline{\Omega}' = \underline{A} \underline{\Omega} \underline{R}$$

where $\underline{A}$ is a non-singular complex 3×3 matrix and $\underline{R}$ is a non-singular rational 6×6 matrix.

If $\underline{P}$ is a principal matrix for $\underline{\Omega}$ it may be verified that $\underline{P}' = \underline{R}' \underline{P} \underline{\widetilde{R}}$ is a principal matrix for $\underline{\Omega}'$. Thus $\underline{\Omega}'$ is also a Riemann matrix. The relationship of isomorphism is reflexive, symmetric and transitive. In particular, for our purposes, we note that to ensure that a given matrix satisfies the Riemann conditions, i.e. admits a principal matrix, it suffices to verify this fact for some isomorphic matrix.

9. Compound Riemann matrices. A matrix of the form

$$\underline{\Omega} = \begin{pmatrix} \underline{\Omega}_1 & \underline{0} & . & . & \underline{0} \\ \underline{0} & \underline{\Omega}_2 & . & . & \underline{0} \\ . & . & . & . & . \\ \underline{0} & \underline{0} & . & . & \underline{\Omega}_r \end{pmatrix}$$

where the $\underline{\Omega}_i$ are Riemann matrices is said to be compound,

and is itself a Riemann matrix. If $\underline{P}_i$ are principal matrices for the $\underline{Q}_i$, then a principal matrix for $\underline{Q}$ is given by

$$\underline{P} = \begin{pmatrix} \underline{P}_1 & \underline{0} & . & . & \underline{0} \\ \underline{0} & \underline{P}_2 & . & . & \underline{0} \\ . & . & . & . & . \\ \underline{0} & \underline{0} & . & . & \underline{P}_r \end{pmatrix}.$$

We shall be concerned later with matrices isomorphic with compound matrices and the Riemann conditions will be known to be satisfied by virtue of the fact that the component matrices are known to be Riemannian.

10. Canonical forms for $\underline{Q}$, $\underline{A}$ and $\underline{y}$. On transforming from $\underline{Q}$ to an equivalent Riemann matrix $\underline{Q}' = \underline{A} \underline{Q} \underline{M}$, we find

$$\underline{A}' = \underline{\tilde{A}'} \underline{A} \underline{M}, \quad \underline{N}' = \underline{\tilde{M}} \underline{N} \underline{M}, \quad \underline{P}' = \underline{\tilde{M}'} \underline{P} \underline{\tilde{M}'}.$$

The matrices $\underline{N}$, $\underline{N}'$ and also $\underline{P}$, $\underline{P}'$ are unimodular congruent.

From a theorem of Frobenius it follows that $\underline{M}$ may be so chosen that (dropping the dash)

$$\underline{P} = \begin{pmatrix} \underline{0} & \underline{B} \\ -\underline{B} & \underline{0} \end{pmatrix},$$

where

$$\underline{B} = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{pmatrix},$$

and $1 = b_1 | b_2 | b_3$, the latent roots of $\underline{P}$ being equal in pairs to b_1 , b_2 , b_3 .

The general characteristic matrix $s\underline{N}$ of the sequence becomes

$$s \begin{pmatrix} \underline{0} & -\underline{B}^{-1} \\ \underline{B}^{-1} & \underline{0} \end{pmatrix},$$

where $n = sb_1$. We shall denote this general matrix by $\underline{N}$ simply.

The choice of variable is still at our disposal, and permits the reduction of $\underline{Q}$ to canonical form. When the above value for $\underline{N}$ is substituted in the Riemann inequality, it is seen that the first 3×3 matrix in $\underline{Q}$ must be non-singular. Hence by a change of variable this 3×3 matrix can be transformed to any other desired non-singular matrix, in particular to $\pi i \underline{B}^{-1}$. The Riemann equality now shows that the final 3×3 matrix in $\underline{Q}$ must be symmetric; and finally, the Riemann inequality shows that the real part of this matrix must be negative definite:

$$\underline{Q} = (\pi i \underline{B}^{-1}, \underline{T})$$

with $\tilde{T} = \underline{T} = \underline{T}' + i \underline{T}''$, $\underline{T}' < 0$.

If now we multiply the intermediary functions $\phi(\underline{u})$, $\psi(\underline{u})$ by an exponential whose exponent is a homogeneous quadratic in $\underline{u}$, and consider

$$\phi'(\underline{u}) = e^{\pi i \tilde{u} \underline{S} u} \phi(\underline{u}), \quad \psi'(\underline{u}) = e^{\pi i \tilde{u} \underline{S} u} \psi(\underline{u}),$$

where $\underline{S}$ is a complex 3×3 matrix, then $\phi'(\underline{u})$, $\psi'(\underline{u})$ are again intermediary functions with principal matrix $\underline{P}$ and the whole of the above reduction is unaffected. By suitable choice of $\underline{S}$, the second period matrix may be reduced to the form

$$\underline{\Lambda} = (\underline{Q}, -\frac{n}{\pi i} \underline{E}),$$

where $\underline{E}$ is the unit 3×3 matrix.

It remains to consider the parameter $\underline{y}$. The functions

$$\phi''(\underline{u}) = e^{2\pi i \tilde{u} \underline{u}} \phi'(\underline{u} + \underline{u}), \quad \psi''(\underline{u}) = e^{2\pi i \tilde{u} \underline{u}} \psi'(\underline{u} + \underline{u})$$

are intermediary with the same $\underline{Q}$, $\underline{\Lambda}$, $\underline{P}$ and $\underline{N}$ as ϕ' , ψ' , but

by suitable choice of the 3-vectors $\underline{\alpha}$, $\underline{c}$, their parameter may be reduced to any desired form. We choose the form

$$\underline{y} = \begin{pmatrix} \underline{0} \\ -\frac{n}{2\pi i} \text{Sp } \underline{T} \end{pmatrix},$$

where $\text{Sp } \underline{T}$ is the spur vector of $\underline{T}$.

With $\underline{\Omega}$, $\underline{\Lambda}$ and $\underline{y}$ in the above canonical forms, an expression may be obtained for any corresponding intermediary function, but before proceeding to this we note the existence of a certain invariant of an intermediary function.

11. The determinant of an intermediary function. From the first and second period matrices $\underline{\Omega}$, $\underline{\Lambda}$ of an intermediary function we form the determinant

$$\delta = \begin{vmatrix} \underline{\Omega} \\ \underline{\Lambda} \end{vmatrix}.$$

The absolute value $|\delta|$ is the determinant of the intermediary function, and is invariant on transforming to an equivalent matrix $\underline{\Omega}'$. Since

$$\underline{N} = \underline{\tilde{\Omega}} \underline{\Lambda} - \underline{\tilde{\Lambda}} \underline{\Omega} = (\underline{\tilde{\Omega}}, -\underline{\tilde{\Lambda}}) \begin{pmatrix} \underline{\Lambda} \\ \underline{\Omega} \end{pmatrix},$$

it follows that $|\delta|$ is the absolute value of the pfaffian of $\underline{N}$. It is a non-zero integer and, in the above notation, is given by

$$|\delta| = \frac{n^3}{b_1 b_2 b_3} = s^3 \cdot \frac{b_3}{b_1} \cdot \frac{b_3}{b_2} \cdot 1.$$

12. Theta functions. The canonical forms found for the matrices associated with a given intermediary function lead us to seek a function $\phi(\underline{u})$ satisfying the equations

$$\phi(\underline{u} + \frac{\pi i}{b_h}) = \phi(\underline{u}), \quad h = 1, 2, 3,$$

$$\phi(\underline{u} + T\underline{e}_h) = e^{-2n\tilde{g}_h(\underline{u} + \frac{1}{2}SpT)} \phi(\underline{u}), \quad h = 1, 2, 3,$$

($\underline{e}_h$ being the column vector whose h -th component is 1, the others being zero).

It is found that $\phi(\underline{u})$ must be expressible linearly in terms of the $|\delta|$ functions

$$\Theta_n[g](\underline{u}) = \sum_m e^{(\tilde{m} + \frac{1}{n}\tilde{g}B)n[T(\underline{m} + \frac{1}{n}B\underline{g}) + 2\underline{u}]},$$

where the sum is over all 3-vectors $\underline{m}$,

$$\tilde{m} = (m_1, m_2, m_3), \quad m = 0, \pm 1, \pm 2, \dots,$$

and

$$\tilde{g} = (g_1, g_2, g_3) \quad \text{where} \quad 0 \leq g_h < \frac{n}{b_h} = \frac{sb_2}{b_h}, \quad h = 1, 2, 3.$$

The function $\Theta_n[g](\underline{u})$ is a theta function of order n .

13. Abelian functions. It may now be proved, using the above representation of an intermediary function, that if $\phi(\underline{u})$ is a suitable intermediary function of non-zero determinant attached to a Riemann matrix $\underline{\Omega}$, then the vector $\underline{c}$ may be chosen such that

$$f(\underline{u}) = \frac{\phi(\underline{u} + \underline{c}) \phi(\underline{u} - \underline{c})}{\phi^2(\underline{u})}$$

is an Abelian function attached to $\underline{\Omega}$.

14. Fields of Abelian functions. Let $\underline{\Omega}$ be a Riemann matrix and let $f_1(\underline{u})$, $f_2(\underline{u})$ be any two meromorphic functions admitting the columns of $\underline{\Omega}$ as periods. Then the sum, difference, product and quotient of $f_1(\underline{u})$, $f_2(\underline{u})$ ($\neq 0$) are again meromorphic functions admitting these same periods. Thus the totality of such functions constitutes a field which is termed the Abelian Function Field attached to $\underline{\Omega}$.

Since $\underline{\Omega}$ is a Riemann matrix there do exist functions of the field for which the columns of $\underline{\Omega}$ form a primitive set of periods. But besides these functions the field includes other Abelian functions for which the columns of $\underline{\Omega}$ are not primitive periods, and includes also degenerate functions which depend effectively on fewer than three variables and which therefore remain constant as u varies along some of the period vectors $\underline{\omega}$ in S_3 ; in particular, the function field contains the field of complex constants. It is necessary to admit these degenerate Abelian functions in order to obtain the field.

The essential properties of the function field are independent of the particular coordinate system chosen; and within that system the period matrix may be subjected to a unimodular substitution. Thus one field of Abelian functions is associated with an equivalence class of Riemann matrices.

Abelian threefolds

15. The algebraic relation between five intermediary functions of the same type. The product of m intermediary functions all admitting the first period matrix $\underline{\Omega}$ but with second period matrices $\underline{\Lambda}_r$ and parameters $\underline{y}_r$ ($r = 1, 2, \dots, m$) is an intermediary function admitting the matrix $\underline{\Omega}$ and with second period matrix $\sum_{r=1}^m \underline{\Lambda}_r$ and parameter $\sum_{r=1}^m \underline{y}_r$. In particular, the product of m intermediary functions of the same type with period matrices $\underline{\Omega}$, $\underline{\Lambda}$, parameter $\underline{y}$, characteristic matrix $\underline{N}$,

and determinant $|\delta| = \sqrt{|\underline{N}|}$ ($\neq 0$) is an intermediary function admitting the periods of $\underline{\Omega}$, with second period matrix $m\underline{A}$, parameter $m\underline{y}$, characteristic matrix $m\underline{N}$, and determinant $\sqrt{|m\underline{N}|} = m^2\sqrt{|\underline{N}|} = m^3|\delta|$.

Now let

$$x_0 = \phi_0(\underline{u}), \quad x_1 = \phi_1(\underline{u}), \quad \dots, \quad x_4 = \phi_4(\underline{u}),$$

be five intermediary functions of the same type with determinant $|\delta| \neq 0$. The number of products of degree m in $x_0, \dots, x_4$ is $\binom{m+4}{4}$. These products are all intermediary functions of the same type with determinant $m^3|\delta|$, and so belong to a linear system determined by $m^3|\delta|$ linearly independent functions. For m sufficiently large

$$\binom{m+4}{4} > m^3|\delta|,$$

since the left-hand side contains a positive term in m^4 .

It follows that for such a value of m there is at least one form of degree m

$$\Phi(x_0, \dots, x_4)$$

which vanishes identically in $\underline{u}$ on substituting $\phi_i(\underline{u})$ for x_i .

If Φ is reducible, then at least one of its factors must vanish identically as a function of $\underline{u}$, and so we may take Φ to be irreducible as a function of the x_i .

The same result holds for degenerate intermediary functions of the same type; for multiplying all by a suitable non-degenerate function we obtain non-degenerate functions for which the result holds and some power of the multiplier

may then be taken out as a factor. Thus

Given any 5 intermediary functions $\phi_1(u), \dots, \phi_5(u)$ of genus 3 and of the same type, there is an irreducible form $\Phi(x_1, \dots, x_5)$ which, on substituting $\phi_i(u)$ for x_i , vanishes identically as a function of u .

If $f_1(u), \dots, f_4(u)$ are Abelian functions of a field $\mathcal{F}$, they may be represented as quotients of intermediary functions of the same type all with the same denominator (on taking a suitable common denominator), and the form Φ for these intermediary functions gives rise to a polynomial in 4 variables which, on substituting the $f_i(u)$ for these variables, vanishes identically as a function of u . Thus

Any 4 Abelian functions of genus 3 of the same field satisfy an algebraic equation.

16. Abelian threefolds. Consider, in an affine space $S_4(y_1, \dots, y_4)$ or in the corresponding projective space with coordinates $(x_1, x_2, \dots, x_4)$, the variety W represented parametrically by

$$y_i = f_i(u)$$

in terms of Abelian functions and by

$$x_i = \phi_i(u)$$

in terms of intermediary functions. The variety W is called an Abelian variety. Congruent points u give rise to the same point of the variety so that only the points of a fundamental cell need be considered.

If the $f_i(u)$ are chosen such that some 3 of them are

algebraically independent then W together with its limit points gives all the points of an algebraic threefold - an Abelian threefold - which is otherwise defined by the algebraic equations satisfied by the $f_i(\underline{u})$. In future W will be understood to include these limit points. A general point of the threefold W arises from a finite set of points of the fundamental cell, i.e. from a finite set of incongruent points. If there are r points in this set then the threefold is said to be of rank r . We shall not have occasion to consider the situation where a general point of W arises from an infinite set of points of the fundamental cell and W is of dimension less than 3.

17. Picard threefolds. In the particular case when a general point of the threefold W arises from a single point of the fundamental cell, W is called a Picard threefold; we denote this by V .

If the parametric equations of a Picard threefold in terms of Abelian functions are

$$y_i = f_i(\underline{u}), \quad i = 1, 2, \dots, t,$$

then some three of the $f_i(\underline{u})$ are algebraically independent and hence every other Abelian function $f(\underline{u})$ of the field is an algebraic function of these and so is an algebraic function of the y_i . Further, since $f(\underline{u})$ is a one-valued function of $\underline{u}$ and so of the y_i , it must be a rational function of the y_i ; and conversely, a rational function of the y_i is an Abelian function of the field. Thus

Functions of an Abelian function field are rational functions on a Picard threefold associated with the field, and conversely.

In particular it follows that

Any two Picard threefolds associated with an Abelian function field are birationally equivalent.

We can thus refer to the Picard threefold of an Abelian function field. This abstract threefold may be represented by any one of an infinity of birationally equivalent models. It may be shown that

A model of the Picard threefold may be chosen which is non-singular and whose points are in unexceptional (1,1) correspondence with the points of the fundamental cell.

The Picard threefold is, of course, independent of the choice of fundamental cell, which does not enter into the definition. In fact, if $\underline{\Omega}$ and $\underline{\Omega}' = \underline{\Omega} \underline{M}$ ($\underline{M}$ unimodular) are equivalent matrices, then to each point of the fundamental cell determined by one corresponds one congruent point of the cell determined by the other - the basic property of a fundamental cell.

Again, if $\underline{\Omega}$ and $\underline{\Omega}' = \underline{A} \underline{\Omega}$ ($|\underline{A}| \neq 0$) are equivalent matrices under the change of variable $\underline{u}' = \underline{A} \underline{u}$, then the Picard threefold V' for the field associated with $\underline{\Omega}'$ is represented parametrically by Abelian functions of $\underline{u}'$ and hence by Abelian functions of $\underline{u}$, and so is a rational transform of the Picard threefold V of the field associated with $\underline{\Omega}$; and vice versa.

Hence V and V' are birationally equivalent.

Thus the Picard threefold is independent of the coordinate system and particular fundamental cell chosen. This is in accord with the earlier remarks that the properties of an Abelian function field are independent of the coordinate system chosen and that one field corresponds to all the Riemann matrices of an equivalence class.

18. The superficial irregularity of a Picard threefold.

Consider a model of the Picard threefold associated with a field $\mathcal{F}$ and represented parametrically in an affine space S_4 by

$$y_i = f_i(\underline{u}), \quad i = 1, 2, 3, 4. \quad (5)$$

We can assume that the first three $f_i(\underline{u})$ are algebraically independent, whence for these $f_i(\underline{u})$

$$J = \left| \frac{\partial f_i(\underline{u})}{\partial u_j} \right| \neq 0.$$

Then

$$dy_i = \sum_{j=1}^3 \frac{\partial f_i(\underline{u})}{\partial u_j} du_j, \quad i = 1, 2, 3; \quad (6)$$

and, since $J \neq 0$, these equations may be inverted to give the du_j . The $\frac{\partial f_i(\underline{u})}{\partial u_j}$ are functions of the field $\mathcal{F}$ and so are rational functions of the four y_i , and thus we obtain

$$du_j = \sum_{i=1}^3 R_{ij}(y_1, y_2, y_3, y_4) dy_i, \quad j = 1, 2, 3,$$

where the R_{ij} are rational functions of the y_i . Integrating, we obtain the u_j as everywhere finite integrals on the Picard threefold, i.e. as Picard integrals of the first kind. Since each point $P(y_i)$ of the Picard threefold arises from a set

of congruent points $\underline{u}$, the values of one of these integrals taken from a point P_0 to P along different paths may differ by a component of a period generated by $\underline{\omega}$, and only by such a component.

The three Picard integrals are linearly independent, i.e., no linear combination with constant coefficients is equal to a constant. There are no further linearly independent Picard integrals of the first kind. For, using (5) and (6), such an integral may be expressed in the form

$$v = \int_{\underline{u}=\underline{u}_0}^{\underline{u}} \sum_{j=1}^3 g_j(\underline{u}) du_j$$

where the $g_j(\underline{u})$ are functions of the Abelian field; and since the integral is to be everywhere finite, the meromorphic functions $g_j(\underline{u})$ must be entire functions and so constants (the only entire functions belonging to an Abelian field). Thus v is linearly dependent on the three u_j . Hence

A Picard threefold possesses precisely three linearly independent Picard integrals of the first kind.

The Picard threefold is thus of superficial irregularity 3.

19. The geometric genus of a Picard threefold. The geometric genus of a threefold is defined transcendently as the number of linearly independent everywhere finite triple integrals on the variety. If in such an integral on the Picard threefold

$$\iiint R(y_1, y_2, y_3, y_4) dy_1 dy_2 dy_3,$$

where R is rational, we substitute for the y_i in terms of $\underline{u}$, we obtain an integral

$$\iiint g(\underline{u}) \, du_1 du_2 du_3$$

where $g(\underline{u})$ is an Abelian function and, as in the last section, the integral is everywhere finite if and only if $g(\underline{u})$ is constant. Thus on the Picard threefold every such integral is a multiple of

$$\iiint du_1 du_2 du_3,$$

and the Picard threefold is of geometric genus 1.

20. Improperly Abelian and pseudo-Abelian threefolds.

It is convenient to define here two other types of threefold which appear in our work. The study of both is due to Roth.

An Abelian threefold is said to be improperly Abelian if it can be represented, simply or multiply, on a product of Abelian varieties of genera less than 3. It consequently admits a parametric representation by means of algebraic functions of Abelian functions of genera less than 3 (15, 16; see also 17).

A pseudo-Abelian threefold of type q ($q = 1, 2$) is one birationally equivalent to a non-singular threefold possessing a continuous commutative ∞^q group of automorphisms whose trajectories are elliptic curves when $q = 1$ and hyperelliptic surfaces when $q = 2$. The pseudo-Abelian threefold is termed elliptic or hyperelliptic accordingly (13, 14; see also 17).

An improperly Abelian threefold may, in particular, be elliptic or hyperelliptic.

21. Ordinary automorphisms of a Picard threefold. Let $\mathcal{F}$ be an Abelian function field and let V be a non-singular,

unexceptional model of its Picard threefold given by parametric equations

$$y_i = f_i(\underline{u}), \quad i = 1, 2, \dots, t.$$

Any function of the field is a rational function of the $f_i(\underline{u})$.

Consider the linear transformation of $\underline{u}$ determined by

$$\underline{u}' = \underline{u} + \underline{c}$$

where $\underline{c}$ is a constant vector. Congruent points of the $\underline{u}$ space transform into congruent points, and so we may also write

$$\underline{u}' = \underline{u} + \underline{c} \pmod{\underline{Q}},$$

giving a (1,1) transformation of the fundamental cell into itself. Correspondingly, we have an automorphism of V , since

$$y_i' = f_i(\underline{u}') = f_i(\underline{u} + \underline{c})$$

is a rational function of the y_i ; and vice versa y_i is a rational function of the y_i' .

The product of the transformations corresponding to vectors $\underline{c}_1, \underline{c}_2$ is the transformation corresponding to the vector $\underline{c}_1 + \underline{c}_2$. As $\underline{c}$ varies $\pmod{\underline{Q}}$ we obtain a continuous ∞^3 commutative group of automorphisms of V . These transformations are said to be of the first kind (Roth, 17, 106). There is precisely one automorphism of the group transforming any given point P into any given point P' : there are no exceptional points, and the group is said to be absolutely transitive.

We obtain a second ∞^3 system of automorphisms of V from the linear transformations

$$\underline{u}' = -\underline{u} + \underline{c};$$

these are of the second kind. They do not form a group alone,

but do so in conjunction with the automorphisms of the first kind.

The possession of the group of automorphisms of the first kind is characteristic of a (non-singular) Picard threefold, i.e.

A non-singular algebraic threefold which possesses an ∞^3 continuous commutative group of automorphisms which is absolutely transitive is a Picard threefold.

The automorphisms of the first and second kinds are termed ordinary.

22. Singular automorphisms of a Picard threefold.

Every Picard threefold admits the above ordinary automorphisms which impose no condition on the Riemann matrix $\underline{\Omega}$. Any other automorphism of a Picard threefold will be represented by a linear substitution on $\underline{u}$

$$\underline{u}' = \underline{A} \underline{u} + \underline{a} \quad (7)$$

where $\underline{A} (\neq \pm \underline{E})$ is a non-singular 3×3 matrix. For (7) to represent an automorphism we must first ensure that congruent points $\underline{u}$ transform into congruent points $\underline{u}'$. This implies

$$\underline{A} \underline{\Omega} = \underline{\Omega} \underline{R} \quad (8)$$

where $\underline{R}$ is a non-singular integral 6×6 matrix. When condition (8) is satisfied one point $\underline{u} \pmod{\underline{\Omega}}$ gives rise to one point $\underline{u}' \pmod{\underline{\Omega}}$. To ensure further that the converse is true we require that

$$\underline{A}' \underline{\Omega} = \underline{\Omega} \underline{R}' \quad (9)$$

with $\underline{R}'$ an integral 6×6 matrix. From (8) and (9) it follows

that $\underline{R}' = \underline{R}^{-1}$, whence $\underline{R}$ must be unimodular. If this is so then one point $\underline{u}' \pmod{\underline{Q}}$ corresponds to only one point $\underline{u} \pmod{\underline{Q}}$. Thus (7) represents an automorphism of the Picard threefold if and only if (8) is satisfied with $\underline{R}$ unimodular.

The relation (8) is a particular instance of a Hurwitz relation. More generally, such a relation holds for an arbitrary algebraic correspondence* but $\underline{A}$ and $\underline{R}$ need not then be non-singular. The cases when $\underline{A} = \underline{0}$, $\underline{R} = \underline{0}$ and $\underline{A} = \rho \underline{E}$, $\underline{R} = \rho \underline{E}$, are termed trivial. We require a particular type of non-trivial relation to be satisfied. A general Riemann matrix $\underline{Q}$ does not admit any non-trivial Hurwitz relation. When the matrix does admit such a relation, the corresponding function field is said to admit complex multiplication by $\underline{A}$; if $f(\underline{u})$ belongs to the field so also does $f(\underline{A} \underline{u})$. The terminology derives from the case of one variable when $\underline{A}$ is then necessarily a complex multiplier.

* Possibly between two different threefolds, in which case, on the right, the matrix $\underline{Q}$ is replaced by some other Riemann matrix $\underline{Q}'$.

CHAPTER 2. INVOLUTIONS ON A PICARD THREEFOLD

1. The generation of involutions on a Picard threefold.

Let V be the Picard threefold associated with an Abelian function field $\mathcal{F}$ and let W be an Abelian threefold of rank r associated with this field. Then W is a rational transform of V and is represented on V by an involution I_r of order r . One method of generating an involution on V is by means of a finite group of automorphisms. It is fundamental to our work that, under certain conditions, it can be asserted that the involution I_r may in fact be generated in this manner. The result is arrived at in the following series of three theorems.

Theorem 1. If I_r possesses ∞^2 coincidences then the geometric genus and plurigenera of W are all zero.

The proof is due to Roth (16). By hypothesis, there is a coincidence surface on V and hence a branch surface B on W . Consider, then, a general linear system $|C|$ of surfaces on W ; this maps an irreducible system $|C_r|$ on V , which belongs to I_r and whose adjoint system $|C'|$ is equivalent to the sum of the transform of $|C|$ and the coincidence surface. Now since V has a canonical surface of order zero, we have $C' \equiv C$, whence it follows that $C \equiv C' + B$; that is to say, W possesses an anticanonical system (12). Hence, on W , the process of successive adjunction always terminates (12), so that W must have geometric genus and plurigenera zero.

Theorem 2. Any Abelian threefold W of superficial irregularity 3 is a Picard threefold and the involution mapping it on another Picard threefold V can be generated by automorphisms of the first kind.

As above, let the Abelian threefold W be mapped on the involution I_7 on the Picard threefold V possessing integrals of the first kind u_i ($i = 1, 2, 3$) and associated with the Riemann matrix Ω . Let W lie in an S_4 (y_1, y_2, y_3, y_4) and let three linearly independent simple integrals of the first kind on W be

$$\int (R_{i1} dy_1 + R_{i2} dy_2 + R_{i3} dy_3), \quad i = 1, 2, 3,$$

where the R_{ij} are rational functions of y_1, y_2, y_3, y_4 .

Substituting for the y_i their values in terms of the u_i , we obtain three linearly independent simple integrals of the first kind on V and we may take these to be the u_i themselves so that

$$R_{i1} dy_1 + R_{i2} dy_2 + R_{i3} dy_3 = du_i, \quad i = 1, 2, 3. \quad (1)$$

If $\underline{u}$ and $\underline{u}'$ are points of a set of the involution I_7 , the left-hand side of (1) is unaltered when we replace $\underline{u}$ by $\underline{u}'$, and therefore

$$du'_i = du_i, \quad i = 1, 2, 3,$$

whence

$$u'_i = u_i + c_i, \quad i = 1, 2, 3,$$

or

$$\underline{u}' = \underline{u} + \underline{c}.$$

Thus the involution on V is generated by a finite group of automorphisms of the first kind. The corresponding group, Γ , of linear substitutions on $\underline{u}$ contains all those generated

by the columns of the Riemann matrix $\underline{\Omega}$. Now the Abelian functions in the parametric representation of W are invariant under all the substitutions of Γ whence it follows that Γ may be generated by 6 substitutions, the constant vectors of which give the columns of a Riemann matrix $\underline{\Omega}'$. A general point of W arises from one point $u \pmod{\underline{\Omega}'}$, and so W is a model of the Picard threefold associated with $\underline{\Omega}'$. This completes the proof of Theorem 2.

Theorem 3. If the Abelian threefold W has some pluri-genus greater than zero and the involution I_r mapping W on a Picard threefold V is not compounded of a Picard involution^{*}, then I_r can be generated by a finite group of automorphisms of V .

This theorem for varieties of arbitrary dimension is due to Andreotti (1). The proof in the case of surfaces was given by De Franchis (5), thus removing a restriction[‡] which was earlier found necessary in Bagnera-De Franchis (2).

In proving the theorem for threefolds (the procedure for any dimension is the same) the first step is to show that no point of V is fundamental for the involution I_r , i.e. no point corresponds to a curve or surface in the involution. This follows from a consideration of the tensorial form

* That is, an involution whose model is a Picard threefold.

‡ It was found necessary to assume the non-existence of fundamental curves for the involution; their exclusion in (5) was by the method given here in the text.

shown by Kähler (8) to be associated with the non-zero plurigenus, p_n say, of W . Under the rational transformation between W and V this form becomes the tensorial form corresponding to the n -th plurigenus of V , and this form may be written

$$\left(\frac{\partial(u_1, u_2, u_3)}{\partial(s_1, s_2, s_3)} \right)^n, \quad (2)$$

the u_i being the simple integrals of the first kind on V and the s_i auxiliary variables.

The expression (2) must thus be invariant on passing from the neighbourhood of a point $\underline{u}$ to the neighbourhood of a point $\underline{u}'$ conjugate to $\underline{u}$ in the involution, and hence the Jacobian of $\underline{u}$ with respect to $\underline{u}'$ must equal an n -th root of unity and so must be non-zero. This excludes the possibility of fundamental points.

From this result and Theorem 1 it follows that the groups of the involution I , which contain united points are ∞' at most and that the corresponding locus in the real S_4 of the $\underline{u}$ is at most a real surface, D say.

Now suppose V to lie in an S_4 with non-homogeneous coordinates $\underline{x} = (x_1, \dots, x_4)$. Let the parametric equations of V be

$$x_i = f_i(\underline{u}), \quad i = 1, 2, 3, 4,$$

where the f_i belong to the matrix $\underline{\Omega}$.

Let $\underline{x}^\circ$ and $\underline{x}'^\circ$ with respective parameters $\underline{u}^\circ$ and $\underline{u}'^\circ$ be two points conjugate in the involution. Then in some neighbourhood of $\underline{x}'^\circ$, the x'_i are analytic functions of the x_i

in some neighbourhood of $\underline{x}^0$; and hence of $\underline{u}$. These functions can be extended analytically to every point of the S_6 except points of the real surface D corresponding to coincidences in the involution. Further, every closed path in the S_6 may be reduced to zero keeping one point fixed, without meeting the surface D , whence the functions are meromorphic.

If, then, $\underline{u}$ describes a closed cycle in the S_6 not meeting D , the x_i^r return to their initial values. On the other hand, if the point $\underline{x}$ describes a closed cycle on V , this corresponds in general to increasing the $\underline{u}$ by some period dependent on $\underline{\Omega}$ and not to a closed cycle in the $\underline{u}$ space, and the effect in general will be to permute the points conjugate to $\underline{x}$ in the involution. However, since the order of this permutation must be a factor of $r!$, if the cycle on V is described $r!$ times the identical substitution on the points of the set must result. In other words, if $\underline{u}$ is increased by any period from the matrix $r!\underline{\Omega}$, the x_i^r are unchanged. Thus the x_i^r are Abelian functions of $\underline{u}$ for which $r!\underline{\Omega}$ is a period matrix; though not necessarily a fundamental period matrix. This matrix is fundamental for the functions $f_i(\frac{\underline{u}}{r!})$, so that if $\bar{V}$ is the variety, birationally identical to V , which is defined by the equations

$$\bar{x}_i = f_i\left(\frac{\underline{u}}{r!}\right), \quad i = 1, 2, 3, 4,$$

it follows that the $x_i^r(\underline{u})$ are rational functions of the $\bar{x}_i$, and hence that the integrals u_i^r are expressible linearly in

terms of the $\frac{u_i}{r!}$, i.e. in terms of the u_i .

We have therefore

$$\underline{u}' = \underline{\lambda} \underline{u} + \underline{c} \pmod{r!\underline{\Omega}}, \quad (3)$$

where $|\underline{\lambda}| \neq 0$.

Consider the set of all linear substitutions by which we pass from the parameters of one point of the group to those of another point of the group. These form a group G containing all the above substitutions (3) and also the substitutions which augment $\underline{u}$ by any period dependent on $\underline{\Omega}$ since these latter leave all the points of the group invariant.

In G the set of all substitutions of the form

$$\underline{u}' = \underline{u} + \underline{c} \quad (4)$$

form a subgroup Γ . Since the coordinates of a point on W , as functions of $\underline{u}$, are invariant for G and so in particular for Γ , this latter group admits a set of 6 generating substitutions, say

$$\underline{u}' = \underline{u} + \underline{\omega}_h, \quad h = 1, 2, \dots, 6, \quad (5)$$

where $\underline{\Omega}' = (\underline{\omega}_h)$ is a Riemann matrix. If Z is the Picard variety attached to the matrix $\underline{\Omega}'$, then

(i) the coordinates of a point on W , as functions of $\underline{u}$, admit $\underline{\Omega}'$ and so are rational functions of the coordinates $z_i (i = 1, \dots, 4)$ of a model of Z in S_4 ;

(ii) the coordinates z_i of a point on Z , as functions of $\underline{u}$, admit the matrix $\underline{\Omega}$, since the substitutions (4), as noted, include all those dependent on $\underline{\Omega}$. Thus the z_i are rational functions of the x_i .

By hypothesis, the involution I , on V is not compounded of an Abelian involution of rank 1, whence the transformation from V to Z must be birational, and

$$\underline{Q}' = \underline{Q}.$$

If, therefore, in any substitution of G we increase the $\underline{u}$ by any period dependent on $\underline{Q}$, the $\underline{u}'$ must also increase by a period dependent on $\underline{Q}$, for otherwise (5) would not represent the generating substitutions of Γ . Thus the equations for the substitutions of G are satisfied mod $\underline{Q}$ and so represent birational transformations of V . This is the required theorem*.

In a discussion of threefolds W representable on involutions on Picard threefolds V , the condition required in the proof of the last theorem that the involution should not be compounded of a Picard involution can always be met by replacing V by Z when necessary. Thus, for our purposes, there is no loss of generality in supposing this condition to be satisfied and this we shall do in future.

That the theorem is not universally true without this condition is demonstrated by a counter-example in (1).

2. The linear group on $\underline{u}$. We have seen that two transformations of the group G which have the same $\underline{\lambda}$ have $\underline{Q}$ which differ only by a period generated by $\underline{Q}$. Therefore if we consider the finite group $G \pmod{\underline{Q}}$, i.e. the factor group G/Γ , this group is isomorphic with the homogeneous linear group consisting of substitutions of the form

$$\underline{u}' = \underline{\lambda} \underline{u}.$$

* The orders of the group of automorphisms and of the group of substitutions $G \pmod{\underline{Q}}$ equal the order of the involution.

We investigate the possible forms of this group, and to this end consider the various conditions which λ satisfies. The homogeneous group will also be denoted by G .

3. The geometric genus of W . The geometric genus is determined by the following theorem:

The threefold W has geometric genus 1 if and only if each substitution of G is modular; when G contains a non-modular substitution the geometric genus is zero.

The geometric genus is given by the number of linearly independent everywhere finite triple integrals on W . When the parametric expressions for the coordinates are substituted, each such integral gives rise to a similar integral on V , and linear independence is preserved. Thus the geometric genus of W cannot exceed that of V and so can have only the values 0, 1.

Suppose first that W possesses one everywhere finite triple integral

$$\iiint R(y_1, y_2, y_3, y_4) dy_1 dy_2 dy_3.$$

On substituting for the y_i in terms of u we obtain the unique everywhere finite triple integral on V which is

$$\iiint du_1 du_2 du_3$$

apart from an arbitrary constant factor. Hence, dividing R by the appropriate constant, we can write

$$R(y_i) \frac{D(y_1, y_2, y_3)}{D(u_1, u_2, u_3)} = 1$$

for all u , where the second factor on the left is the Jacobian of the y_i with respect to u . Writing u' for u

where the two points belong to a set of the involution does not alter $R(y_i)$. Hence we must have

$$\frac{D(y_1, y_2, y_3)}{D(u'_1, u'_2, u'_3)} = \frac{D(y_1, y_2, y_3)}{D(u_1, u_2, u_3)} = \frac{D(y_1, y_2, y_3)}{D(u'_1, u'_2, u'_3)} \cdot \frac{D(u'_1, u'_2, u'_3)}{D(u_1, u_2, u_3)}; (6)$$

from which it follows that

$$\frac{D(u'_1, u'_2, u'_3)}{D(u_1, u_2, u_3)} = |\underline{\lambda}| = 1,$$

and the substitution transforming $\underline{u}$ into $\underline{u}'$ is modular. This must be so for every substitution of the group G .

Conversely, if each substitution of G is modular, then (6) shows that the Jacobian

$$\frac{D(y_1, y_2, y_3)}{D(u_1, u_2, u_3)},$$

which is an Abelian function admitting the periods of $\underline{\Omega}$, takes the same value at all the points of a group of I_γ . It is therefore a rational function of y_1, y_2, y_3, y_4 , and so we can write

$$\frac{D(u_1, u_2, u_3)}{D(y_1, y_2, y_3)} = R(y_1, y_2, y_3, y_4),$$

and $du_1 du_2 du_3 = R(y_1, y_2, y_3, y_4) dy_1 dy_2 dy_3$.

Thus the right-hand side of this equation gives an everywhere finite triple integral on W , and the geometric genus of W is 1.

In all other cases the geometric genus of W must be zero.

4. The superficial irregularity of W . This is given by the number of linearly independent everywhere finite simple integrals on W . Since each of these gives rise to a similar integral on V , linear independence being preserved, the

superficial irregularity of W cannot exceed that of V , namely 3. We have already eliminated the case where this irregularity equals 3 and W is consequently a Picard threefold.

When the superficial irregularity of W is 2 we can choose u such that the integrals on V corresponding to the chosen ones on W are u_1, u_2 . If u, u' belong to the same set of the involution I_r on V , it then follows that

$$du'_1 = du_1, \quad du'_2 = du_2,$$

and so $u'_1 = u_1 + c_1, \quad u'_2 = u_2 + c_2$.

In this case every substitution of G may be put in the form

$$u'_1 = u_1 + c_1,$$

$$u'_2 = u_2 + c_2,$$

$$u'_3 = \alpha u_1 + \beta u_2 + \gamma u_3 + c_3.$$

Similarly, when W has superficial irregularity 1, the expression for u' in every substitution may be put in the form

$$u'_1 = u_1 + c_1,$$

if u is suitably chosen.

Conversely, when the group can be thrown into one or other of the above forms, the threefold W is superficially irregular.

5. Reduction of one substitution of G to a standard form.

Any particular substitution of G generates a finite cyclic group (mod $\mathfrak{g}$), isomorphic with the homogeneous linear group obtained by ignoring the constant terms, and hence homomorphic with a finite cyclic plane projective group. Now any such

group possesses at least one set of three non-collinear invariant points and, referred to such a set as vertices of the triangle of reference, the projective group takes the form of a multiplication on the variables. Further, starting from any one or two invariant points we can find the remainder to make up a triad.

Thus, using the reductions of the last section, we see that when W is of superficial irregularity 2, one substitution of G may be put in the form

$$u'_1 = u_1 + c_1,$$

$$u'_2 = u_2 + c_2,$$

$$u'_3 = \lambda_3 u_3 + c_3,$$

with $\lambda_3 \neq 1$. Replacing u_3 by $u_3 + \frac{c_3}{1-\lambda_3}$, the last equation becomes

$$u'_3 = \lambda_3 u_3.$$

Similarly, when W is of superficial irregularity 1, one substitution of G may be reduced to the form

$$u'_1 = u_1 + c_1,$$

$$u'_2 = \lambda_2 u_2,$$

$$u'_3 = \lambda_3 u_3,$$

with λ_2, λ_3 not both equal to 1.

When W is superficially regular, one substitution of G may be reduced to the form

$$u'_1 = \lambda_1 u_1,$$

$$u'_2 = \lambda_2 u_2,$$

$$u'_3 = \lambda_3 u_3,$$

with $\lambda_1, \lambda_2, \lambda_3$ not all equal to 1. If, of course, this cyclic group constitutes the whole of the group G , then none of $\lambda_1, \lambda_2, \lambda_3$ can equal 1.

6. Threefolds W of geometric genus 1. In proving Theorem 3 above it was necessary to assume that some plurigenus of W was greater than zero. From now onwards we shall restrict our attention to those threefolds W which are of geometric genus one, and so this condition will be satisfied.

In the linear group G every substitution must now be modular, and when the substitution is reduced to a standard form we must have

$$\lambda_1 \lambda_2 \lambda_3 = 1. \quad (7)$$

It is now seen that the variety W of geometric genus 1 cannot be of irregularity 2. For otherwise we could make $\lambda_1 = \lambda_2 = 1$, and then by (7) we should also have $\lambda_3 = 1$, and G would contain a substitution of the first kind, a situation we have shown can be excluded.

7. The characteristic equation of a substitution of the group G . Consider a substitution of G reduced, as is always possible, to the form

$$\begin{aligned} u'_1 &= \lambda_1 u_1 + c_1, \\ u'_2 &= \lambda_2 u_2, \\ u'_3 &= \lambda_3 u_3, \end{aligned} \quad (8)$$

where $\lambda_2, \lambda_3 \neq 1$ and λ_1 may or may not equal unity. The λ_i are termed the multipliers of the substitution. Since G does not contain any substitutions of the first kind it

follows that if $\underline{u}$ is increased by a period from $\underline{Q}$ then also $\underline{u}'$ must increase by a period dependent on $\underline{Q}$. This leads to the equation

$$\underline{\lambda} \underline{Q} = \underline{Q} \underline{R} \quad (9)$$

where

$$\underline{\lambda} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

and $\underline{R}$ is a 6x6 integral matrix; (9) is the Hurwitz relation of the automorphism.

The complex conjugate of (9) gives

$$\bar{\underline{\lambda}} \bar{\underline{Q}} = \bar{\underline{Q}} \underline{R},$$

$\underline{R}$ being real. Hence

$$\begin{pmatrix} \underline{\lambda} & \underline{Q} \\ \underline{Q} & \bar{\underline{\lambda}} \end{pmatrix} \begin{pmatrix} \underline{Q} \\ \bar{\underline{Q}} \end{pmatrix} = \begin{pmatrix} \underline{Q} \\ \bar{\underline{Q}} \end{pmatrix} \underline{R},$$

from which it follows that

$$\begin{pmatrix} \underline{\lambda} & \underline{Q} \\ \underline{Q} & \bar{\underline{\lambda}} \end{pmatrix}$$

is the transform of $\underline{R}$ by

$$\begin{pmatrix} \underline{Q} \\ \bar{\underline{Q}} \end{pmatrix}^{-1}.$$

This implies that

$$|\underline{\lambda} - x\underline{E}_3| \cdot |\bar{\underline{\lambda}} - x\underline{E}_3| = |\underline{R} - x\underline{E}_6|, \quad (10)$$

$\underline{E}_r$ being the unit matrix of order r as before.

The sextic equation

$$|\underline{R} - x\underline{E}_6| = 0,$$

which has integral coefficients, is the characteristic

equation of the substitution (8). From (10) it is seen that the roots of the characteristic equation are

$$\lambda_1, \lambda_2, \lambda_3, \bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3.$$

Further, since the characteristic matrix $(\underline{R} - x\underline{E}_6)$ is the transform of

$$\begin{pmatrix} \underline{\lambda} - x\underline{E}_3 & \underline{0} \\ \underline{0} & \bar{\underline{\lambda}} - x\underline{E}_3 \end{pmatrix}$$

by a non-singular matrix, the ranks of these two matrices are equal when any particular value is substituted for x . Since, when any root of the characteristic equation is substituted in the latter matrix the rank is reduced by the multiplicity of the root, i.e. by the maximum possible amount, this also occurs for the characteristic matrix.

We have therefore the following three facts:

- (i) in addition to the multipliers, their complex conjugates are also roots of the characteristic equation;
- (ii) any real root of the characteristic equation must be of even multiplicity;
- (iii) if two or all three of the roots are equal their value, if complex, is a double or triple root of the characteristic equation and, if real, is a quadruple or sextuple root; and when substituted in the matrix $(\underline{R} - x\underline{E}_6)$ the rank of this matrix is reduced to 4, 3, 2 or 0 respectively.

The n -th power of the substitution (8) has multipliers $\lambda_1^n, \lambda_2^n, \lambda_3^n$ and, if the substitution is of period n , these must all be equal to 1. In addition, if $\lambda_1 = 1$ so that

possibly $c, \neq 0$, we must have

$$(nc, 0, 0) \equiv 0 \pmod{Q}.$$

8. The primitive m -th roots of unity. In applying the above considerations to determine possible sets of multipliers, the following observation on the roots of unity is needed.

Let ω_m be a primitive m -th root of unity^{*}, and consider the polynomial $x^m - 1$. When $x^m - 1$ is expressed as a product of factors irreducible over the field of rationals $\mathcal{R}$, ω_m is a root of the equation obtained by equating one of these factors, $f(x)$ say of degree s , to zero; $f(x)$ is the characteristic polynomial of ω_m . The field $\mathcal{R}(\omega_m)$ obtained by adjoining ω_m to $\mathcal{R}$ is of degree s and is isomorphic with the field of polynomials in x over $\mathcal{R}$, modulo $f(x)$, (6, 103). Every primitive m -th root of unity belongs to this field, and each such root determines the same field. Thus all such roots are roots of the equation $f(x) = 0$. Further, $f(x)$ has no other zeros; for if one such zero were a primitive m' -th root of unity where m' divides m , it would follow that $f(x)$ was a factor of $x^{m'} - 1$, and ω_m could not be a primitive m -th root, contrary to hypothesis.

Let $\phi(m)$ be the Gauss-Euler function of m , i.e. the number of integers not greater than m and prime to m . Then

* We shall generally use ω_m in this sense; but primitive cube and fourth roots will, as customary, be denoted by ϵ and i respectively.

there are $\phi(m)$ primitive m -th roots of unity, and the degree s of $f(x)$ equals $\phi(m)$.

9. Scope of the following chapters. The next chapter combines the theory developed above with that of ternary linear groups to determine all modular homogeneous groups G which can give rise to involutions of the required type on Picard threefolds. Chapter 4 considers improperly Abelian threefolds and sets out the associated preliminary results concerning elliptic curves and hyperelliptic surfaces which are required in the discussion of these and other threefolds. Chapters 5-8 investigate the Riemann matrices for the Picard threefolds on which involutions can be generated by the various types of homogeneous group. Some cases are worked out in full to illustrate the techniques involved, while for the remainder one matrix is given in each case to establish that involutions can be obtained from all the groups. Chapter 9 is devoted to the problem of finding for the matrices which admit any given homogeneous group all the non-homogeneous groups admitted by each of these matrices. Again, although the principles are general, their application in a particular instance can be lengthy and require special arguments, and so they are illustrated by reference to the cyclic groups and a selection of other groups. The chapter is followed by a list of the 64 homogeneous groups which give rise to Abelian threefolds of geometric genus one.

CHAPTER 3. THE HOMOGENEOUS LINEAR GROUPS

1. Finite modular ternary linear groups. The modular ternary linear groups are in (1, 1) or (3, 1) correspondence with plane projective groups according as they do not or do contain the substitution with multipliers $\varepsilon, \varepsilon, \varepsilon$. The finite plane projective groups were investigated incompletely by Jordan (7) and Valentiner (19) and completely by Blichfeldt (3; or 11). A brief summary of the results is contained in Van der Waerden (20). It is convenient for our present purpose to distinguish five types of finite plane projective group; the authors mentioned were concerned only with the last of these - the primitive - since the general structure of the other types is either easily arrived at or else reduces to that of binary groups.

A plane projective group is said to be reducible when it leaves invariant a point or line; it then also leaves invariant in addition a line or point respectively, in each case the point not lying on the line. Thus in all reducible plane cases we have a non-incident invariant point and line. In some cases the group is further reducible and the line contains two points each of which is invariant, giving three invariant points. In the other reducible cases we have a binary group on the line, and the possibilities here are well-known (9, 11). In this work we recognise three types of reducible group:

(A) Cyclic groups, concerning whose nature nothing

further need be said;

(B) Multiplicative groups. This term is used to denote those groups, other than cyclic groups, leaving three points invariant. They are obtained from two or more cyclic groups leaving these three points invariant;

(C) Reducible ternary groups of the regular solids. In all reducible groups other than those covered by (A) and (B), the group determined on the invariant line is the binary group corresponding to one of the groups of the regular solids: dihedral, tetrahedral, octahedral, icosahedral.

A projective group is said to be imprimitive when the space on which it operates can be expressed as a sum of subspaces which are permuted by the group. Such a group is either reducible or else the subspaces are all of the same dimension, and when this dimension is zero the group is said to be monomial. In the case of plane groups the only irreducible imprimitive groups are monomial ones, and these constitute our fourth type.

(D) Monomial groups. These leave invariant a triad of points but no individual point of the triad.

All remaining groups fall into the fifth class:

(E) Primitive groups. These do not leave invariant any point or triad of points.

We now obtain in turn the linear groups G belonging to the above five categories and satisfying the conditions implied by the existence of a characteristic equation for each substitution.

2. Notation. We shall use G_r to denote any group of order r . The specific groups found below will be denoted by Roman numerals, and so we shall write, for example, $G_6(\text{IV})$, or simply $G(\text{IV})$ if we are not concerned with the order. It will be convenient from now onwards to replace the variables u_1, u_2, u_3 by x, y, z . The substitution

$$x' = \lambda_1 x, \quad y' = \lambda_2 y, \quad z' = \lambda_3 z$$

will be denoted by

$$S(\lambda_1 x, \lambda_2 y, \lambda_3 z),$$

and further cyclic substitutions by $S_1, S_2, \dots$

(A) Cyclic groups

3. Conditions imposed on the multipliers. From the results in the last chapter, the multipliers λ_i of a generating substitution are roots of a sextic characteristic equation, and if λ_i is a primitive m -th root of unity

$$\phi(m) \leq 6.$$

The values of m for which this is so, together with the corresponding values of $\phi(m)$ are as follows

m	2	3	4	5	6	7	8	9	10	12	14	18
$\phi(m)$	1	2	2	4	2	6	4	6	4	4	6	6

Assume $\lambda_1, \lambda_2, \lambda_3$ to be ordered so that

$$m_1 \leq m_2 \leq m_3.$$

Let the least common multiple of m_1, m_2 be m . Then since

$$\lambda_3 = \frac{1}{\lambda_1 \lambda_2}$$

it follows that $m_3 = m$; and the substitution is of period m . Thus in the case of a ternary modular substitution, the period

is given by the index of the multiplier of highest index. Possible orders for the group are therefore the values of m in the table above.

The following four facts are constantly needed.

- (i) $\lambda_1 \lambda_2 \lambda_3 = 1.$
- (ii) If one λ_i takes the value $+1$ or -1 , this is a double root of the characteristic equation and the remaining two multipliers are at most quartic irrationalities.
- (iii) If two λ_i are equal, their characteristic polynomial, being a repeated factor of the characteristic equation, is at most a cubic, so that $\lambda_i = \omega_m$ where $m = 2, 3, 4$, or 6 .
- (iv) If two of the λ_i are m -th roots of unity where $\phi(m) > 3$, i.e.
 $m = 5, 7, 8, 9, 10, 12, 14, 18,$

they cannot be conjugate roots. For both must be simple roots of the characteristic equation and so, if conjugate, would lead on substitution in equation (9) of the last chapter to the (determinate) ratios of the elements of one row of $\underline{\Omega}$ being the complex conjugates of the corresponding ratios for another row. With a suitable change of variable we should obtain a matrix one of whose rows consisted entirely of real elements, and the representative points in the S_6 of Chapter 1.3 would all lie in a prime, contrary to the fact that they must be linearly independent.

In listing possible triads of multipliers, the order in a triad is of no significance, and further we give only one generating triad for each cyclic group.

4. Enumeration of cyclic groups. If $\lambda_1 = \lambda_2 = \lambda_3$, there is one possibility.

$$G_3(I) \quad S(\epsilon x, \epsilon y, \epsilon z).$$

If two λ_i are equal, say $\lambda_2 = \lambda_3$, there are three possibilities.

$$G_2(II) \quad S(x, -y, -z);$$

$$G_4(III) \quad S(-x, iy, iz);$$

$$G_6(IV) \quad S(\epsilon x, -\epsilon y, -\epsilon z).$$

That there are no further possibilities follows from (iii).

If all three λ_i are different we find the following cases.

(a) One multiplier, say λ_1 , equals 1. The other two multipliers must be complex conjugates and so by (iv) there are - excluding G(II) above -

$$G_3(V) \quad S(x, \epsilon y, \epsilon^2 z);$$

$$G_4(VI) \quad S(x, iy, -iz);$$

$$G_6(VII) \quad S(x, -\epsilon y, -\epsilon^2 z).$$

(b) The above seven groups are of orders 2, 3, 4 or 6. We show now that there is only one further group with one of these orders. The multipliers of such a group must be chosen from

$$\pm 1, \pm \epsilon, \pm \epsilon^2, \pm i.$$

If one multiplier is ± 1 , one of the other two must be $-1, \pm \epsilon$ or i , giving four cases all enumerated above. If no multiplier is ± 1 or -1 , then all three must be chosen from

$\pm \epsilon, \pm \epsilon^2$, and we obtain I or IV. If one multiplier is -1

then, apart from II and III, we have one further group, namely

$G_6(\text{VIII}) \quad S(-x, \varepsilon y, -\varepsilon^2 z).$

(c) The remaining groups are discussed according to order.

For a G_5 , noting (a) above, we see that each multiplier must be a primitive fifth root of unity. But then two of these multipliers must be complex conjugates, and this is ruled out by (iv). Thus there are no modular groups of order 5 of the required type; and hence, also, no modular groups of order 10.

For a G_7 , the multipliers must be distinct primitive seventh roots of unity. Calling any particular multiplier ω , the other two must be ω^2 , ω^4 , and we have one possibility.

$G_7(\text{IX}) \quad S(\omega, x, \omega^2 y, \omega^4 z).$

For a G_8 , one multiplier must be a primitive eighth root of unity, say ω_8 , where we can assume $\omega_8^2 = 1$. The other two multipliers are powers of ω_8 , other than ω_8 , ω_8^7 , 1, the sum of whose exponents is 7. There are two possibilities.

$G_8(\text{X}) \quad S(-x, \omega_8 y, i\omega_8 z);$

$G_8(\text{XI}) \quad S(ix, \omega_8 y, -\omega_8 z).$

For a G_9 , one multiplier must be a primitive ninth root of unity and so, since $\phi(9) = 6$, the multipliers must be three distinct primitive ninth roots, no two of which are conjugate, and the sum of whose exponents is 9. These primitive roots being of the form

$$\omega_9, \quad \omega_9^2, \quad \omega_9^4, \quad \omega_9^5, \quad \omega_9^7, \quad \omega_9^8,$$

it is seen that the conditions cannot be fulfilled and so there are no modular cyclic groups of order 9 of the required

type; and also no such groups of order 18.

The next possibility is a G_{12} . One multiplier must be a primitive twelfth root of unity, say ω_{12} where we can take $\omega_{12} = i\epsilon$. By (iii) and (iv) no two multipliers can be equal or complex conjugates. Two possibilities are

$$G_{12}(\text{XII}) \quad S(-x, i\epsilon y, i\epsilon^2 z);$$

$$G_{12}(\text{XIII}) \quad S(\epsilon x, i\epsilon y, -i\epsilon z).$$

Two other apparent possibilities are the groups generated by

$$S(\epsilon^2 x, -iy, i\epsilon z) \quad \text{and} \quad S(-\epsilon^2 x, iy, i\epsilon z),$$

but these may be dismissed; for in each case the multipliers can only be obtained from an octavic characteristic equation and not from a sextic.

Finally, for a G_{14} , since $\phi(14) = 6$, each multiplier is a primitive fourteenth root of unity. The six primitive roots are of the form

$$\omega_{14}, \quad \omega_{14}^3, \quad \omega_{14}^5, \quad \omega_{14}^9, \quad \omega_{14}^{11}, \quad \omega_{14}^{13}.$$

As each exponent is odd it is impossible to choose three such that the sum of the exponents is unity. Thus there are no modular cyclic groups of order 14 of the required type.

The above thirteen groups are all the cyclic groups which can possibly lead to Abelian threefolds of geometric genus one.

(B) Multiplicative groups

5. Representation as a direct product. These groups are arrived at by considering in turn the possible values for the period of the substitution of highest order belonging to the group. Since all multiplicative groups are Abelian they can be represented as direct products of cyclic groups, i.e. for each multiplicative group there exists a set of cyclic groups such that any substitution of the multiplicative group can be written uniquely as the product of a set of substitutions, one from each of the cyclic groups (10, 140 or 11, 87). The order of the multiplicative group is the product of the orders of the cyclic groups. We shall give a set of generating substitutions and state what values the multipliers can assume in each case.

(a) Substitutions of period 2. There is a single group.
 $G_4(\text{XIV}) \quad S(-x, y, -z), \quad S(x, -y, -z); \quad \lambda_1, \lambda_2 = \pm 1.$

(b) Substitutions of period 3. Every substitution of the group, other than identity, must be of period 3, for otherwise the group would contain substitutions of period exceeding 3. In all there are 9 possible modular substitutions and, bearing in mind the expression as a direct product, we see that the only possibility is the entire group of order 9.
 $G_9(\text{XV}) \quad S(\epsilon x, \epsilon y, \epsilon z), \quad S(x, \epsilon y, \epsilon z); \quad \lambda_1, \lambda_2 = 1, \epsilon, \epsilon^2.$

(c) Substitutions of period 4. Every substitution of the group is of period 4, 2 or 1. In all there are 16 such modular substitutions, and possible orders for the group

are 8 and 16. To obtain a G_8 we must multiply the G_4 (III) or G_4 (VI) by a G_2 of type II - with the multipliers in any relative order. This leads to a single G_8 .

G_8 (XVI) $S(-x, iy, iz), S_1(-x, y, -z); \lambda_1 = \pm 1, \lambda_2 = \pm 1, \pm i$.

There is also the group of order 16.

G_{16} (XVII) $S(-x, iy, iz), S_1(ix, -y, iz); \lambda_1, \lambda_2 = \pm 1, \pm i$.

(d) Substitutions of period 6. Every substitution of the group is of period 6, 3, 2, or 1. There are 36 such modular substitutions, and possible orders for the group are 12, 18, and 36.

Multiplying G_6 (VII) or G_6 (VIII) by the G_2 with multipliers -1, 1, -1 (or -1, -1, 1) we obtain a G_{12} .

G_{12} (XVIII) $S(-x, \varepsilon y, -\varepsilon^2 z), S_1(-x, -y, z); \lambda_1 = \pm 1, \lambda_2 = \pm 1, \pm \varepsilon, \pm \varepsilon^2$.

If, on the other hand, we multiply G_6 (IV) by either of these G_2 s we obtain a different G_{12} .

G_{12} (XIX) $S(\varepsilon x, -\varepsilon y, -\varepsilon z), S_1(-x, y, -z); \lambda_1 = \pm 1, \pm \varepsilon, \pm \varepsilon^2, \lambda_2 = \pm \lambda_1$.

Consider next the effect of multiplying a G_6 by a G_3 .

Multiplying G_6 (VII) or G_6 (VIII) by G_3 (I) we obtain a G_{18} .

G_{18} (XX) $S(\varepsilon x, -\varepsilon y, -\varepsilon z), S_1(x, \varepsilon y, \varepsilon^2 z); \lambda_1 = 1, \varepsilon, \varepsilon^2, \lambda_2 = \pm 1, \pm \varepsilon, \pm \varepsilon^2$.

If either of these two G_6 s is multiplied by the G_3 with multipliers $\varepsilon, 1, \varepsilon^2$ or $\varepsilon, \varepsilon^2, 1$, we obtain again this same group; and likewise if we multiply G_6 (IV) by either of these G_3 s or by G_3 (V).

The only other group is that of order 36.

G_{36} (XXI) $S(\varepsilon x, -\varepsilon y, -\varepsilon z), S_1(-x, \varepsilon y, -\varepsilon^2 z); \lambda_1, \lambda_2 = \pm 1, \pm \varepsilon, \pm \varepsilon^2$.

(e) Substitutions of period 8. Every substitution of the group is of period 8, 4, 2 or 1. There are 64 such modular substitutions, and possible orders for multiplicative groups are 16, 32 and 64. Groups may be formed by multiplying $G_8(X)$ or $G_8(XI)$ by $G_2(II)$, $G_4(III)$ or $G_4(VI)$, in each case with the multipliers taken in any order, or by so multiplying $G(X)$ by $G(XI)^*$. In all cases where the cyclic group is extended to a multiplicative group it is found that the resulting group contains the substitution with multipliers -1, 1, -1. But this implies that the groups resulting from $G(X)$ and $G(XI)$ contain respectively the substitutions with multipliers 1, ω_8 , $-\omega_8$ and -1, ω_8 , ω_8 ; and as the cyclic groups generated by these sets of multipliers are not included in the cyclic groups I-XIII found earlier, we conclude that no multiplicative groups containing substitutions of period 8 are admissible.

(f) Substitutions of period 12. Possible periods for substitutions are 12, 6, 4, 3, 2, 1. There are 144 such substitutions. Groups can arise from $G_{12}(XII)$ and $G_{12}(XIII)$. As in the last paragraph, any extension containing the substitution with multipliers -1, 1, -1 (or -1, -1, 1) is excluded since this implies also the presence of the substitutions with multipliers 1, ϵ , $-\epsilon^2$ and $-\epsilon$, ϵ , ϵ

* In some of these cases the two generating groups have some substitutions in common so that the resulting group is not obtained explicitly as a direct product. This is irrelevant here. The same remark applies when considering substitutions of period 12.

respectively, and these do not appear in any of the cyclic groups.

The only remaining possibility arising from $G_{12}(XII)$ is obtained on multiplying by $G_3(I)$ (or by $G_6(IV)$ or by the groups generated by the substitutions with multipliers $\epsilon, 1, \epsilon^2$ or $\epsilon, -1, -\epsilon^2$); but this group contains the substitution with multipliers $-\epsilon, i\epsilon^2, 1$ and so is ruled out.

Similarly the only possible group arising from $G_{12}(XIII)$ is obtained on multiplying by $G_3(V)$ (or equally by $G_6(VII)$ or by the groups generated by the substitutions with multipliers $\epsilon, 1, \epsilon^2$ or $\epsilon, -1, -\epsilon^2$); this group contains the substitution with multipliers $\epsilon, i\epsilon^2, -1$, and so it also is ruled out.

Thus no multiplicative groups containing substitutions of period 12 are admissible, and this concludes the discussion of all the periods found for the cyclic substitutions.

(C) Reducible ternary groups of the regular solids

6. The icosahedral group. It was shown in the discussion of cyclic groups that no substitution of period 5 is admissible. Since the icosahedral group contains such substitutions it is thereby excluded - both in its reducible and irreducible ternary forms. The remaining reducible groups are the dihedral, tetrahedral, and octahedral, and these are considered in turn.

7. Dihedral groups. These groups are obtained on extending a cyclic or multiplicative group by the adjunction

of a substitution interchanging two of the united points of the group. There is no need to consider the effect of adjoining more than one transposing substitution. For the set of all multiplicative substitutions in the resulting group would constitute one of the groups I-XXI; and the product of two transposing substitutions would belong to this subgroup.

When the multipliers of the groups I-XXI are taken in the orders given above, it is necessary to consider successively the interchange of the three possible pairs of united points. For the moment, however, a substitution transposing y and z will be taken, and the question of reducing this to one of a set of standard forms investigated.

Consider then a transposing substitution

$$x' = -\frac{1}{\alpha\beta}x, \quad y' = \alpha z, \quad z' = \beta y. \quad (1)$$

Keeping the same triangle of reference so as not to alter the forms of the cyclic groups, choose new variables X, Y, Z , given by

$$x = X, \quad y = \sqrt{-\alpha}Y, \quad z = \sqrt{\beta}Z.$$

The substitution (1) now assumes the form

$$X' = \frac{1}{\beta^2}X, \quad Y' = -pZ, \quad Z' = pY,$$

where $p = \sqrt{-\alpha\beta}$. Equally we can choose variables such that $p = -\sqrt{-\alpha\beta}$.

Alternatively, if we choose X, Y, Z given by

$$x = X, \quad y = \sqrt{\alpha}Y, \quad z = \sqrt{\beta}Z,$$

the substitution (1) takes the form

$$X' = \frac{1}{p^2}X, \quad Y' = ipZ, \quad Z' = ipY,$$

where again $p = \sqrt{-\alpha\beta}$; and equally we can make $p = -\sqrt{-\alpha\beta}$.

Thus the substitution depends on the one parameter p and, reverting to the old variables, may be taken in either of the equivalent forms

$$x' = \frac{1}{p^2}x, \quad y' = -pz, \quad z' = py; \quad (2)$$

$$x' = \frac{1}{p^2}x, \quad y' = ipz, \quad z' = ipy. \quad (3)$$

To determine the possible values for p , observe that the square of (2) or (3) is the substitution

$$S(\frac{1}{p^4}x, -p^2y, -p^2z). \quad (4)$$

Possible values for p^2 are thus

$$1, -1, \epsilon, -\epsilon, \epsilon^2, -\epsilon^2, i, -i.$$

For p we need only consider one of the square roots in each case, and we choose the values

$$1, -1, \epsilon^2, i\epsilon^2, \epsilon, i\epsilon, -\omega, -i\omega.$$

In view of the presence of the substitution (4) in the group, the possibilities which need to be considered may be reduced. Thus when $p = \epsilon^2$, we see that the product of the transposing substitution and the corresponding substitution (4), i.e. the cube of the transposing substitution, is that transposing substitution for which $p = -1$. In this way the above values for p may be reduced to one or other of

$$1, -1, -i\omega.$$

The first of these we shall use with the transposing substitution in the form (2) and the other two with the substitution in the form (3). Thus we have three transposing substitutions

which may be denoted by

$$T(x, -z, y), \quad T(-x, z, y), \quad T(ix, \omega, z, \omega, y)$$

respectively. As remarked earlier, in each case the two corresponding substitutions involving a multiplication on y or z must also be considered. We examine the groups arising from the three types of substitution in turn. Each group can, of course, be generated in more than one way and may contain a number of the substitutions T - as will become apparent in the discussion.

Note that if in a cyclic group the multipliers of two variables are always equal, a substitution transposing these variables need not be considered since by a change of variable any resulting group can be reduced to a multiplicative form which must have been determined already.

8. Dihedral groups whose generators include $T(x, -z, y)$ or $T(z, y, -x)$ or $T(-y, x, z)$. The substitution T^2 is of period 2, whence groups of odd order, i.e. groups I, V, IX and XV, need not be considered.

$G_2(\text{II})$ gives rise to one group, containing also $G_4(\text{XIV})$.
 $G_8(\text{XXII}) \quad S(x, -y, -z), \quad T(-y, x, z).$

$G_4(\text{III})$ gives rise to one group, containing also $G_{16}(\text{XVII})$.
 $G_{32}(\text{XXIII}) \quad S(-x, iy, iz), \quad T(-y, x, z).$

$G_6(\text{IV})$ gives rise to one group, containing also $G_{12}(\text{XIX})$.
 $G_{24}(\text{XXIV}) \quad S(\epsilon x, -\epsilon y, -\epsilon z), \quad T(-y, x, z).$

$G_4(\text{VI})$ gives rise to one group
 $G_8(\text{XXV}) \quad S(x, iy, -iz), \quad T(x, -z, y).$

A second possibility is the group generated by

$$S(x, iy, -iz), \quad T(-y, x, z);$$

but this coincides with $G_{32}(XXIII)$.

$G_6(VII)$ gives rise to two groups.

$$G_{12}(XXVI) \quad S(x, -\varepsilon^2 y, -\varepsilon z), \quad T(x, -z, y);$$

$$G_{72}(XXVII) \quad S(x, -\varepsilon^2 y, -\varepsilon z), \quad T(-y, x, z).$$

This last group contains $G_{36}(XXI)$.

$G_6(VIII)$ gives rise to two groups.

$$G_{24}(XXVIII) \quad S(-x, \varepsilon y, -\varepsilon^2 z), \quad T(x, -z, y);$$

$$G_{36}(XXIX) \quad S(-x, \varepsilon y, -\varepsilon^2 z), \quad T(z, y, -x).$$

The first contains $G_{12}(XVIII)$; and the second $G_{18}(XX)$ - with x, y interchanged.

A third possibility is the group generated by

$$S(-x, \varepsilon y, -\varepsilon^2 z), \quad T(-y, x, z);$$

but this coincides with $G_{72}(XXVII)$.

$G_8(X)$ and $G_8(XI)$ each give rise to one group.

$$G_{16}(XXX) \quad S(-x, \omega_8 y, i\omega_8 z), \quad T(x, -z, y);$$

$$G_{16}(XXXI) \quad S(ix, \omega_8 y, -\omega_8 z), \quad T(x, -z, y).$$

In each case the group obtained by adjoining instead $T(-y, x, z)$ contains a substitution of period 16 and is ruled out.

$G_{12}(XII)$ and $G_{12}(XIII)$ each give rise to one group.

$$G_{24}(XXXII) \quad S(-x, i\varepsilon y, i\varepsilon^2 z), \quad T(x, -z, y);$$

$$G_{24}(XXXIII) \quad S(\varepsilon x, i\varepsilon y, -i\varepsilon z), \quad T(x, -z, y).$$

In each case the group obtained by adjoining instead $T(-y, x, z)$ contains a substitution of period 24 and is

ruled out.

We have noted in passing the most ample multiplicative group contained in each dihedral group found. Consideration of the effect of adjoining one of the three T's at present under discussion to the multiplicative groups shows that in fact only one new group arises, and this from $G_6(XVI)$.

$G_{16}(XXXIV) \quad S(-x, iy, iz), \quad S(-x, y, -z), \quad T(x, -z, y).$

In all other cases earlier dihedral groups are obtained, possibly with the variables permuted.

9. Dihedral groups whose generators include $T(-x, z, y)$ or $T(z, -y, x)$ or $T(y, x, -z)$. Here all the groups I-XXI must be considered, including those of odd order.

The first group to give a new dihedral group is $G_6(V)$, from which we obtain two cases, the second containing $G_6(XV)$.

$G_6(XXXV) \quad S(x, \epsilon y, \epsilon^2 z), \quad T(-x, z, y);$

$G_{18}(XXXVI) \quad S(x, \epsilon y, \epsilon^2 z), \quad T(y, x, -z).$

$G_4(VI)$, $G_6(VII)$ and $G_6(VIII)$ each give rise to one group.

$G_6(XXXVII) \quad S(x, iy, -iz), \quad T(-x, z, y);$

$G_{12}(XXXVIII) \quad S(x, -\epsilon^2 y, -\epsilon z), \quad T(-x, z, y);$

$G_{36}(XXXIX) \quad S(-x, \epsilon y, -\epsilon^2 z), \quad T(z, -y, x).$

This last group contains $G_{18}(XX)$ - with x, y permuted - but differs from $G_{36}(XXIX)$.

Adjoining any substitution T to $G_7(IX)$ leads to substitutions of period 14 which are inadmissible.

$G_8(X)$ gives rise to one group.

$G_{16}(XL) \quad S(-x, \omega_8 y, i\omega_8 z), \quad T(-x, z, y).$

Adjoining instead $T(y, x, -z)$ leads to substitutions of period 16 and so the resulting group is ruled out.

Adjoining $T(-x, z, y)$ to $G_8(XI)$ and $G_{12}(XII)$ we obtain a G_{16} and G_{24} in which S^2T is given by

$$(x, iz, iy) \text{ and } (x, -iz, -iy)$$

respectively, whence these groups coincide with $G_{16}(XXXI)$ and $G_{24}(XXXII)$, the second generators being taken in the alternative form. Adjoining instead $T(y, x, -z)$ leads to substitutions of periods 16 and 24 respectively and so these groups are inadmissible.

$G_{12}(XIII)$ gives rise to one group.

$$G_{24}(XLI) \quad S(ex, icy, -iez), \quad T(-x, z, y).$$

Adjoining instead $T(y, x, -z)$ leads to substitutions of period 24 and so the group is inadmissible.

The result of adjoining $T(-x, z, y)$ to $G_4(XIV)$ is the same as that of adjoining $T(x, -z, y)$, which has already been considered; and similarly for $T(z, -y, x)$ and $T(y, x, -z)$. Thus nothing new is obtained from $G_4(XIV)$ or from any group of which it is a subgroup. This leaves $G_8(XV)$ and $G_{16}(XX)$. From $G_8(XV)$ we obtain only $G_{16}(XXXVI)$. From $G_{16}(XX)$ we obtain $G_{32}(XXXIX)$ - with x and y permuted - and $G_{72}(XXVII)$.

10. Dihedral groups whose generators include

$T(ix, \omega, z, \omega, y)$ or $T(\omega, z, iy, \omega, x)$ or $T(\omega, y, \omega, x, iz)$. The substitutions T^2 are respectively

$$(-x, iy, iz), \quad (ix, -y, iz), \quad (ix, iy, -z),$$

and we need only consider a given T in association with the groups from I-XXI which contain the corresponding T^2 . In this way all possibilities will be met. Thus we consider groups III, XI, XII, XVI and XVII.

The possibility arising from G_4 (III), namely the group generated by

$$S(-x, iy, iz), \quad T(ix, \omega_z z, \omega_y y)$$

is cyclic since $S = T^2$.

From G_8 (XI) we have the group generated by

$$S(ix, \omega_y y, -\omega_z z), \quad T(ix, \omega_z z, \omega_y y),$$

which is G_{16} (XXXI).

G_{12} (XII) gives rise to one group.

$$G_{24}$$
(XLII) $S(-x, 1ey, 1e^2 z), \quad T(ix, \omega_z z, \omega_y y).$

From G_8 (XVI) we have the group generated by

$$S(-x, iy, iz), \quad S(-x, y, -z), \quad T(ix, \omega_z z, \omega_y y).$$

By a change of variable we can interchange the roles of the last two generators while leaving the first generator unaltered. Thus taking

$$X = x, \quad Y = y - z, \quad Z = y + z,$$

we have the group generated by

$$S(-X, iY, iZ), \quad T(-X, Z, Y), \quad S(iX, -\omega_y Y, \omega_z Z),$$

and this is G_{16} (XXXI) with the generator T in the alternative form.

G_{16} (XVII) gives rise to one group.

$$G_{32}$$
(XLI) $S(-x, iy, iz), \quad T(\omega_y y, \omega_x x, iz).$

The same group is obtained on adjoining instead either of

the other two substitutions T. The second generator of G_{16} (XVII) is not needed, and the group is in fact obtainable from G_4 (III) directly as the above generators show, but was not derived earlier because T^2 is not contained in G_4 (III) so that consideration of this case was deferred.

This completes the enumeration of the possible homogeneous modular ternary dihedral groups.

11. Reducible tetrahedral groups. In terms of a parameter θ on a line, the tetrahedral group may be generated by the two substitutions

$$\theta' = -\theta \quad \text{and} \quad \theta' = \frac{1(\theta + 1)}{1 - \theta}.$$

Hence for our purpose we may take the modular generators of the ternary group to be

$$S(-\frac{1}{\alpha^2}x, \alpha y, -\alpha z),$$

$$U(\frac{1}{\beta^2}x, \frac{1}{2}(1+1)\beta(y+z), -\frac{1}{2}(1-1)\beta(y-z)).$$

We shall need to know the resulting forms for some other substitutions of the group. Thus

$$U^3(\frac{1}{\beta^6}x, -\beta^3y, -\beta^3z),$$

$$U^5(\frac{1}{\beta^{10}}x, \frac{1}{2}(1-1)\beta^5y - \frac{1}{2}(1+1)\beta^5z, \frac{1}{2}(1-1)\beta^5y + \frac{1}{2}(1+1)\beta^5z),$$

$$U^6(\frac{1}{\beta^{12}}x, \beta^6y, \beta^6z),$$

$$US(-\frac{1}{(\alpha\beta)^2}x, \frac{1}{2}(1+1)\alpha\beta(y-z), -\frac{1}{2}(1-1)\alpha\beta(y+z)),$$

$$(US)^3(-\frac{1}{(\alpha\beta)^6}x, 1(\alpha\beta)^3y, 1(\alpha\beta)^3z).$$

In reducing the possible forms for U we shall have occasion to interchange the variables y and z; hence in order to cover all cases we must not also interchange these variables in the reduction of S. We must consider then

the following seven possibilities for α

$$1, -1, i, \omega, \varepsilon, -\varepsilon, i\varepsilon,$$

any other permissible value for α giving the same cyclic group as one of these.

From the form of the substitution U^3 we see that possible values for β^3 are

$$\pm 1, \pm i, \pm \varepsilon, \pm \varepsilon^2.$$

However, the last four values imply that U is of period 9 or 18 and so these are ruled out. The first four values for β^3 lead to the following twelve values for β

$$\pm 1, \pm \varepsilon, \pm \varepsilon^2, \pm i, \pm i\varepsilon, \pm i\varepsilon^2.$$

The last four values for β give substitutions U of period 12 in which the multiplier of x is $-\varepsilon$ or $-\varepsilon^2$. Carrying out a change of variable on y and z to reduce U to the standard cyclic form does not alter the multiplier of x . Since there is no permissible cyclic substitution of period 12 one of whose multipliers is $-\varepsilon$ or $-\varepsilon^2$, it follows that these four values for β are ruled out. From the form for U^6 , and hence for U^7 , we see that $\beta = i, -i$ lead to the same group, so that $\beta = i$ may be omitted.

If we make the change of variable

$$X = x, \quad Y = iz, \quad Z = y,$$

in U^5 we find that the substitution takes the form

$$\left(\frac{1}{\beta^5}X, \frac{1}{2}(1+i)\beta^5(Y+Z), -\frac{1}{2}(1-i)\beta^5(Y-Z)\right).$$

This shows that $\beta = \varepsilon, \varepsilon^2$ give the same group; and so also do $\beta = -\varepsilon, -\varepsilon^2$.

We are thus left with five possibilities for β

$$1, -1, \epsilon, -\epsilon, -i,$$

the periods of the corresponding substitutions U being

$$6, 3, 6, 3, 12.$$

Taking now the value ω_8 for α with any of these five values for β we see that in $(US)^3$ the multipliers of y and z are both equal to ω_8 or $-\omega_8$ or $-i\omega_8$; none of these is permissible and so $\alpha = \omega_8$ is ruled out. This leaves six values for α , namely

$$1, -1, i, \epsilon, -\epsilon, i\epsilon,$$

the periods of the corresponding substitutions S being

$$2, 2, 4, 6, 6, 12.$$

These must be considered in conjunction with the above five values for β .

A number of combinations are eliminated by considering the substitution US . Thus, for example, if $\alpha = \epsilon$, $\beta = 1$, US is of period 12 and the multiplier of x is $-\epsilon$; which is inadmissible. Again when $\beta = 1$ or ϵ , the form of U^3 shows that $\alpha = 1, -1$ give rise to the same group, as also do $\alpha = \epsilon, -\epsilon$; and the same result follows from U^6 when $\beta = -1$. Using these facts we find that the combinations remaining for consideration are

$$\beta = 1, \alpha = 1, i, i\epsilon; \quad \beta = -1, \alpha = 1, -1, i, i\epsilon;$$

$$\beta = \epsilon, \alpha = 1, i\epsilon; \quad \beta = -\epsilon, \alpha = 1, i\epsilon;$$

$$\beta = -i, \alpha = 1, i, \epsilon.$$

The orders of groups arising may be found by observing how

many substitutions give rise to the identical substitution on the line YZ when the group is regarded as a projective one, and multiplying by 12, the order of the projective tetrahedral group.

The value $\beta = 1$ gives rise to three groups, which we take in the order corresponding to $\alpha = 1, 1, \epsilon$ since the first is a subgroup of the other two.

$$G_{24}(\text{XLIV}) \quad S(x, iy, -iz), \quad U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z));$$

$$G_{48}(\text{XLV}) \quad S(-x, y, -z), \quad U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z));$$

$$G_{72}(\text{XLVI}) \quad S(\epsilon x, i\epsilon y, -i\epsilon z), \quad U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z)).$$

When $\beta = -1$ and $\alpha = 1, -1$ or $i, i\epsilon$, the substitutions $(US)^6$ or $(US)^3$ respectively are given by $(x, -y, -z)$ so that these groups coincide with those considered under $\beta = 1$.

The values $\beta = \epsilon, \alpha = 1$ give rise to a new group.

$$G_{24}(\text{XLVII}) \quad S(x, iy, -iz), \quad U(\epsilon x, \frac{1}{2}(1+i)\epsilon(y+z), -\frac{1}{2}(1-i)\epsilon(y-z)).$$

When $\beta = \epsilon, \alpha = i\epsilon, S^8$ is given by $(\epsilon^2 x, \epsilon^2 y, \epsilon^2 z)$, and the group coincides with that given by $\beta = 1, \alpha = i\epsilon$.

When $\beta = -\epsilon$ and $\alpha = 1, i\epsilon$, the substitutions S^2, S^6 respectively are given by $(x, -y, -z)$ so that the resulting groups coincide with those considered under $\beta = \epsilon$.

When $\beta = -1, U^3$ is given by $(-x, -iy, -iz)$ and it follows that the groups given by $\beta = -1, \alpha = 1$ and $\beta = -1, \alpha = i$ both coincide with that given by $\beta = \alpha = 1$. When $\beta = -1, \alpha = \epsilon$, so that S^4 is given by $(\epsilon x, \epsilon y, \epsilon z)$, the group contains the substitution for which $\beta = -i\epsilon$, and this has been seen to be inadmissible.

In all, therefore, there are four reducible tetrahedral groups to be considered further, each of which is generated by two substitutions.

It remains to see whether groups can be obtained requiring three or more generators but still determining a tetrahedral group on the line YZ of the projective plane. Such a group must contain substitutions of the forms of S and U above and hence must contain one of the four groups found already. If to one of these groups we adjoin a substitution, U, say, similar to U^r then $(U,)^{-1}U^r$ is a substitution in the standard cyclic form. Multiplying this by the substitution S already in the group we obtain a new substitution S_1 ; and from S_1 and substitutions of the original group we can, conversely, obtain U. Hence we need only consider the adjunction of substitutions of the form of the first generator S. This implies that these substitutions must, in fact, be the substitutions S already determined, so that a group of the type sought must arise from combining two or more of the groups G(XLIV) - G(XLVII).

Now G(XLIV) is a subgroup of both G(XLV) and G(XLVI), so that it gives nothing new when combined with either of them. Again, a group containing both G(XLV) and G(XLVI) would contain the substitution with multipliers $-\epsilon$, ϵ , ϵ , and so is inadmissible.

The only remaining possibility is to combine G(XLVII) with one of the first three groups. Now G(XLVII) is a subgroup

of G(XLVI) so that these two together give nothing new. Taken with G(XLIV), G(XLVII) gives G(XLVI); and taken with G(XLV), we obtain again the inadmissible substitution with multipliers $-\epsilon$, ϵ , ϵ .

This exhausts the possibilities and we conclude that there are no further reducible tetrahedral groups calling for consideration.

12. Reducible octahedral groups. In terms of a parameter θ on a line, the octahedral group may be generated by the two substitutions

$$\theta' = 1\theta \quad \text{and} \quad \theta' = \frac{1(\theta + 1)}{1 - \theta}.$$

The square of the first generator here gives the first generator used earlier for the tetrahedral group. In modular ternary form, the first generator may be written

$$S\left(\frac{1}{iy^2}x, 1\gamma y, \gamma z\right).$$

It is convenient to regard the octahedral group as obtained by adjoining S to one of the four tetrahedral groups.

Perusal of the cyclic groups shows that the values for γ which must be considered are

$$1, -1, i, -i, \omega, \epsilon, -\epsilon.$$

Since each of the tetrahedral groups contains the substitution $(x, -y, -z)$, the second and third values for γ may be omitted. Again, either of the last two values for γ taken with any of the tetrahedral groups implies a multiplicative group containing substitutions of period 12, and no such group is

admissible. We are left with the three values for γ

$$1, -1, \omega_8.$$

The groups generated by the substitutions S corresponding to $\gamma = 1, -1$ differ only in the interchange of the variables y, z . Now in the reduction of the tetrahedral group it was found that a suitable multiplication on the variables followed by an interchange of y and z led to U^5 assuming the same form as U but with β replaced by β^5 . In all cases the cyclic subgroup is symmetrical in y and z and is unaltered; for the first three cases, the form of U is also unaltered, but for the final case the effect is to replace ϵ by ϵ^2 . Therefore, with the sole proviso that in $G(\text{XLVII})$ we must consider both $\beta = \epsilon, \epsilon^2$, we may carry out the change of variables and so reduce the values of γ which must be considered to

$$1, \omega_8.$$

$G_{24}(\text{XLIV})$ gives rise to two groups.

$$G_{96}(\text{XLVIII}) \quad S(-ix, iy, z), \quad U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z)).$$

The first generator of $G_{24}(\text{XLIV})$ is not needed. The group also contains $G_{48}(\text{XLV})$. There are four substitutions giving the identical transformation on the projective line YZ , whence the group is of order $4 \times 24 = 96$.

$$G_{48}(\text{XLIX}) \quad S(-x, i\omega_8 y, \omega_8 z), \quad U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z)).$$

Again the first generator of $G_{24}(\text{XLIV})$ is not needed.

Adjoining $S(-ix, iy, z)$ to $G_{48}(\text{XLV})$ gives $G_{96}(\text{XLVIII})$ as noted above; adjoining instead $S(-x, i\omega_8 y, \omega_8 z)$ gives

a group containing the substitution $S_2(x, i\omega, y, -\omega, z)$, which is inadmissible.

Adjoining $S(-ix, iy, z)$ to G_{72} (XLVI) gives a group containing $S_2(-\epsilon x, -i\epsilon y, -i\epsilon z)$; adjoining $S(-x, i\omega, y, \omega, z)$ the substitution S_U is of period 24; and so both are inadmissible.

Adjoining $S(-ix, iy, z)$ to G_{24} (XLVII) gives a group containing $S_2(-x, y, -z)$ and implies an inadmissible tetrahedral group. Adjoining $S(-x, i\omega, y, \omega, z)$, $(US,)^4$ is $S_2(\epsilon x, \epsilon y, \epsilon z)$ and the group coincides with the second considered under G_{72} (XLVI). The same conclusions hold when the β of G_{24} (XLVII) takes the value ϵ^2 instead of ϵ .

Both octahedral groups found may be obtained from G_{24} (XLIV). Any further octahedral group must thus be obtainable by adjoining more than one substitution S_i to G_{24} (XLIV) and so must in fact arise as the combination of the two groups already found. But the resulting group contains $S_2(x, i\omega, y, -\omega, z)$ and so is ruled out. Hence there are no further admissible reducible octahedral groups.

(D) Monomial groups

13. Preliminary results. The group leaves invariant a triad of points, which we take as vertices of the triangle of reference, and is easily seen to contain a substitution permuting cyclically the three points of the triad. Consider, then, a substitution in the cyclic form

$$S(\alpha x, \beta y, \frac{1}{\alpha\beta} z)$$

and a permuting substitution which we may take in the form

$$P(Ay, Bz, \frac{1}{AB}x).$$

We find

$$P^{-1}SP(\frac{1}{\alpha\beta}x, \alpha y, \beta z),$$

$$PSP^{-1}(\beta x, \frac{1}{\alpha\beta}y, \alpha z).$$

Thus if the group contains the substitution with multipliers $\lambda_1, \lambda_2, \lambda_3$, it also contains those with multipliers $\lambda_1, \lambda_1, \lambda_2$ and $\lambda_2, \lambda_3, \lambda_1$. Among groups I-XXI we need only consider further I, IX, XIV, XV, XVII, XIX and XXI, and among the dihedral groups we need only consider those whose maximal multiplicative (or cyclic) subgroup is one of these seven; these dihedral groups are XXII, XXIII, XXIV, XXVII, XXXVI, and XLIII.

No further groups will be obtained by adjoining a second permuting substitution P_1 , for $P_1^{-1}P$ is of the cyclic form, whence the resulting group can be obtained by adjoining only P to a suitable, possibly more ample, group.

14. Groups arising from cyclic and multiplicative groups. In the case of a cyclic or multiplicative group we can at once reduce the substitution P to a simple form. Choosing new variables X, Y, Z given by

$$X = Hx, \quad Y = Ky, \quad Z = Lz,$$

we find that P now takes the form

$$P(\frac{H}{K}AY, \frac{K}{L}BZ, \frac{L}{HAB}X),$$

and taking H, K, L such that

$$H : K : L = 1 : A : AB,$$

the substitution P becomes

(Y, Z, X).

This change of variable does not alter the form of the multiplicative group. We take the seven cyclic and multiplicative groups in turn.

The cyclic group $G_3(I)$ gives nothing new since the group obtained by adjoining P can be reduced to the multiplicative form.

The other six groups lead respectively to the following.

$$G_{21}(I) \quad S(\omega, x, \omega^2 y, \omega^4 z), \quad P(y, z, x);$$

$$G_{12}(II) \quad S(x, -y, -z), \quad P(y, z, x);$$

$$G_{27}(LII) \quad S(x, \varepsilon y, \varepsilon^2 z), \quad P(y, z, x);$$

$$G_{48}(LIII) \quad S(x, iy, -iz), \quad P(y, z, x);$$

$$G_{36}(LIV) \quad S(\varepsilon x, -\varepsilon y, -\varepsilon z), \quad P(y, z, x);$$

$$G_{108}(LV) \quad S(-x, \varepsilon y, -\varepsilon^2 z), \quad P(y, z, x).$$

No further generator is needed in any of the last five cases, as easily follows from the earlier remark that the substitutions obtained by permuting the multipliers in S necessarily belong to the group.

15. Groups arising from dihedral groups. The above reduction of P cannot be applied when dealing with dihedral groups since the change of variable would destroy the form of the transposing generator. For the first four of the six relevant dihedral groups the transposing substitution is

$$T(-y, x, z).$$

Thus the permuting substitution P may be replaced by

$$P^{-1}TP(x, \frac{B}{A}z, \frac{A}{B}y),$$

since, as is easily verified, the substitution $S_1 = (PTP^{-1}TP)$ is of the standard cyclic form and so belongs to the original group, and $S_1(P^{-1}TP)T^{-1}$ is the substitution

$$(Ay, -Bz, -\frac{1}{AB}x)$$

which, on multiplication by $(x, -y, -z)$ - which belongs to the dihedral group - gives the substitution P . Thus adjoining either P or $P^{-1}TP$ to the dihedral group leads to the same monomial group.

On choosing new variables X, Y, Z given by

$$X = x, \quad Y = y, \quad Z = \frac{B}{A}z,$$

the form of the dihedral group is unaltered, but $P^{-1}TP$ assumes the form

$$(X, -Z, Y).$$

We consequently obtain from $G(\text{XXII})$, $G(\text{XXIII})$, $G(\text{XXIV})$ and $G(\text{XXVII})$ the following four monomial groups, where T is now used as the general sign for a transposing substitution.

$$G_{24}(\text{LVI}) \quad T(-y, x, z), \quad T(x, -z, y).$$

This group contains $G_4(\text{XIV})$ as a subgroup. The first generator S of $G(\text{XXII})$ is not needed.

$$G_{96}(\text{LVII}) \quad S(-x, iy, iz), \quad T(-y, x, z), \quad T(x, -z, y);$$

$$G_{72}(\text{LVIII}) \quad S(\epsilon x, \epsilon y, \epsilon z), \quad T(-y, x, z), \quad T(x, -z, y);$$

$$G_{216}(\text{LIX}) \quad S(x, \epsilon y, \epsilon^2 z), \quad T(-y, x, z), \quad T(x, -z, y).$$

The last three groups contain $G_{16}(\text{XVII})$, $G_{12}(\text{XIX})$ and $G_{36}(\text{XXI})$ respectively as subgroups.

The four groups contain, in addition to the subgroups

noted, the G_6 rearranging the variables in all possible orders, whence the orders of the groups are as written.

In the dihedral group G_{18} (XXXVI), the transposing generator is $T(y, x, -z)$. We find that $P^{-1}TP$ is given by

$$T(-x, \frac{B}{A}z, \frac{A}{B}y)$$

and $S_1 = PTT_1$ is given by

$$(-Ax, -Ay, \frac{1}{A^2}z),$$

whence

$$S_1 T_1 T = P.$$

Since S_1, T_1 already belong to the dihedral group, we may thus again replace P by T_1 . Choosing new variables X, Y, Z as in the last instance, T_1 takes the form

$$(-X, Z, Y).$$

Therefore from G_{18} (XXXVI) we obtain the following group.

$$G_{54}(LX) \quad S(x, \epsilon y, \epsilon^2 z), \quad T(y, x, -z), \quad T_1(-x, z, y).$$

This group contains G_9 (XV) as a subgroup.

Finally we have the dihedral group G_{32} (XLIII). By an argument similar to the last, it is seen that the permuting substitution P may be replaced by the transposing substitution

$$T_1(ix, \omega_2 z, \omega_2 y),$$

and the following group arises.

$$G_{96}(LXI) \quad T(\omega_2 y, \omega_2 x, iz), \quad T_1(ix, \omega_2 z, \omega_2 y).$$

This group contains G_{16} (XVII) as a subgroup. The generator $S(-x, iy, iz)$ of G_{32} (XLIII) is not needed here, being obtained as the square of T_1 .

(E) Primitive groups

16. Admissible modular linear representations.

References to the authors who have determined the primitive groups were made at the beginning of this chapter. The work of Blichfeldt (3, 11) shows that there are six primitive ternary projective groups. When represented by modular linear groups, the representations for four of these six necessarily contain the substitution $(\varepsilon x, \varepsilon y, \varepsilon z)$ and so the correspondence is (3,1); the other two can be represented by modular linear groups with which they are in (1, 1) correspondence, and a second representation in (3, 1) correspondence is obtained by adjoining the substitution $(\varepsilon x, \varepsilon y, \varepsilon z)$. The groups are as follows.

$$G_{108} \text{ (LXII) } S(x, \varepsilon y, \varepsilon^2 z), T(-y, -x, -z), \\ V\left(\frac{1}{\varepsilon - \varepsilon^2} (x + y + z), \frac{1}{\varepsilon - \varepsilon^2} (x + \varepsilon y + \varepsilon^2 z), \frac{1}{\varepsilon - \varepsilon^2} (x + \varepsilon^2 y + \varepsilon z)\right).$$

Of order 36 as a projective group.

$$G_{216} \text{ (LXIII) } S, T, V \text{ as above,} \\ W\left(\frac{1}{\varepsilon - \varepsilon^2} (x + y + \varepsilon^2 z), \frac{1}{\varepsilon - \varepsilon^2} (x + \varepsilon y + \varepsilon z), \frac{1}{\varepsilon - \varepsilon^2} (\varepsilon x + y + \varepsilon z)\right).$$

Of order 72 as a projective group.

$$G_{648} \text{ } S, T, V \text{ as above, } U(\omega, x, \omega, y, \omega, z), \omega, \omega^2 = 1.$$

Of order 216 as a projective group - the Hessian group. The presence of the substitution U of period 9 shows that this group is inadmissible for generating an involution of genus one on a Picard threefold.

The groups G_{108} and G_{216} are normal divisors of the Hessian

group, and the G_{54} generated by

$$S, T \text{ as above, } T(-x, -z, -y)$$

is a normal divisor of all three. The remaining types of primitive group are simple.

G_{60} . Of order 60 also as a projective group - the icosahedral group. This group is inadmissible for our purpose since it contains substitutions of period 5.

G_{180} . The group obtained by adjoining the substitution $(\epsilon x, \epsilon y, \epsilon z)$ to the $(1, 1)$ representation of the icosahedral group is likewise inadmissible.

G_{1080} . Of order 360 as a projective group - Valentiner's group. This is also obtained by adjoining a further generator to the icosahedral group G_{60} , and so is inadmissible.

G_{168} (LXIV) $S(\omega x, \omega^2 y, \omega^4 z), P(y, z, x),$

$$U: x' = h((\omega^4 - \omega^3)x + (\omega^2 - \omega^5)y + (\omega - \omega^6)z),$$

$$y' = h((\omega^2 - \omega^5)x + (\omega - \omega^6)y + (\omega^4 - \omega^3)z),$$

$$z' = h((\omega - \omega^6)x + (\omega^4 - \omega^3)y + (\omega^2 - \omega^5)z),$$

$$h = \frac{1}{7}(\omega + \omega^2 + \omega^4 - \omega^6 - \omega^5 - \omega^3) = \frac{1}{\sqrt{-7}},$$

$$\omega^7 = 1.$$

Of order 168 as a projective group - Klein's group.

G_{504} . The group obtained by adjoining the substitution $(\epsilon x, \epsilon y, \epsilon z)$ to the above representation of Klein's group. This group is ruled out as it contains substitutions of period 21.

CHAPTER 4. IMPROPERLY ABELIAN THREEFOLDS AND HYPERELLIPTIC
SURFACES

1. Abelian threefolds W of superficial irregularity 1.

In the cases of four of the cyclic groups and of $G(\text{XXV})$, $G(\text{XXVI})$ and $G(\text{XLIV})$ one particular multiplier of every substitution of the group is equal to unity. Most of the remaining groups possess one of these seven groups as a subgroup. We consider now the implications of this unit multiplier for the threefolds W and the matrices associated with the groups. The results obtained also provide starting points for the other groups referred to.

In Chapter 2 it was seen that when the group can be put in a form in which one multiplier is always unity, the threefold W is of superficial irregularity 1; this is therefore true of the present groups. It may further be shown that W is elliptic. For every substitution on, say, the first variable can be put in the form

$$x' = x + c,$$

and it immediately follows that W is invariant under the ∞^1 group of substitutions of the first kind obtained by allowing c to vary continuously. The trajectories are curves on each of which the group acts in a completely transitive manner, whence they are elliptic curves and W is

an elliptic threefold.

Also W is improperly Abelian. For the Picard threefold V carrying the involution must also contain a congruence of elliptic curves, namely those giving rise to the trajectories on W ; it is said to be special of type 1. It consequently (17, 107) contains also a pencil of hyperelliptic surfaces (and is equally special of type 2) and this gives rise to a pencil of surfaces on W . Now both the congruence of curves and the pencil of surfaces on W are Abelian, whence W may be represented multiply on the product variety of an Abelian surface and curve (17, 113), and admits a parametric representation by means of algebraic functions of Abelian functions of genera 2 and 1, i.e. W is improperly Abelian.

Therefore: the Abelian threefolds of geometric genus 1 and superficial irregularity 1 are elliptic improperly Abelian.

An Abelian threefold of geometric genus 1 cannot be hyperelliptic, for this would imply its being of superficial irregularity 2 and we have observed that this is impossible.

We return now to the Picard threefold V carrying the involution giving rise to an elliptic threefold W and which, as we have seen, must be special. The following theorem holds.

When a Picard threefold V carries an involution which is mapped by an elliptic threefold of geometric genus one, the Riemann matrix of V is equivalent to a matrix of the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & & & & \\ & & (\tau_{ij}) & & & \\ \omega_{31} & \omega_{32} & & & & \end{pmatrix}, \begin{matrix} i = 2, 3, \\ j = 3, 4, 5, 6, \end{matrix} \quad (1)$$

where (τ_{ij}) is a Riemann matrix of genus 2.

For the involution is generated by one of the four cyclic groups above or by $G(\text{XXV})$, $G(\text{XXVI})$, $G(\text{XLIV})$, and we consider one substitution in the standard form with multipliers $\lambda_1 = 1, \lambda_2, \lambda_3$. In each case +1 is precisely a double root of the characteristic equation; and when substituted in the characteristic matrix must reduce the rank to 4 (Chapter 2). This last fact may be seen directly: for otherwise the ratios of the elements of the first row of the Riemann matrix would be (determinate and) real, which is impossible. Thus, corresponding to the multiplier +1 we have precisely four linearly independent real integral linear equations connecting the elements of the first row of the Riemann matrix. By virtue of these equations the elements of the row may be expressed integrally in terms of submultiples of two of them. If these two independent periods, say ω_{11}, ω_{12} , are chosen such that the representative points in the plane x of a real S_6 are at the two smallest distances from the origin, with $|\omega_{11}| \leq |\omega_{12}|$, then the representation of the remaining elements must be integral in ω_{11}, ω_{12} . For otherwise we could make

$$\omega_{13} = \alpha \omega_{11} + \beta \omega_{12}$$

with

$$0 < |\alpha|, |\beta| \leq \frac{1}{2},$$

whence

$$|\omega_{13}| < \frac{1}{2}|\omega_{11}| + \frac{1}{2}|\omega_{12}| < |\omega_{12}|,$$

with strict inequality since the points representing ω_{11}, ω_{12} are not collinear with the origin; and this contradicts the definition of ω_{12} .

Subtracting suitable multiples of the first two columns of the matrix from the remaining four, we can now make

$$\omega_{1i} = 0, \quad i = 3, 4, 5, 6,$$

and replacing x by $\omega_{11}x$ we obtain the form (1) with $\tau = \omega_{12}/\omega_{11}$.

The 2×4 matrix (τ_{ij}) is a Riemann matrix for the Abelian functions when regarded as functions of y, z only. This concludes the proof of the theorem stated above. The reasoning holds equally if the 3×6 matrix admits a substitution in which precisely one multiplier equals -1 , as in $G_+(III)$.

In cases where the matrix admits a group containing one of the above types as a subgroup, the reduction again holds, but here the 2×4 matrix (τ_{ij}) admits a larger group. In general this group for (τ_{ij}) will not be modular. It follows that we shall need to know the Riemann matrices of genus 2 which admit a finite group, modular or not. We need also the corresponding results for matrices of genus 1.

2. Riemann matrices of genus 1 which admit finite groups.

There are two types of Riemann matrix of genus 1 admitting finite groups (other than that generated by $x' = -x$, which imposes no condition on the matrix). These matrices, which are well known from the theory of elliptic curves, are equivalent to the following

- (1, ϵ) admitting the group generated by $x' = \epsilon x$,
and also that generated by $x' = -\epsilon x$;

- (1, 1) admitting the group generated by $x' = ix$.

3. Riemann matrices of genus 2 which admit finite groups.

For matrices of genus 2 it is necessary in some cases to note more than one form for each matrix, these forms being progressively simpler but calling for changes of variable which we shall not always find it advantageous to carry out. It is convenient to adopt slight variants of the forms found by Bagnara-De Franchis in their study of such matrices (2); our forms were also obtained by Scorza (18). The forms assumed by the Riemann conditions for the matrices are also given where these are not trivial.

$$(1) \quad x' = \epsilon x, \quad y' = \epsilon y,$$

$$\begin{pmatrix} \omega_{11} & \epsilon \omega_{11} & \omega_{13} & \epsilon \omega_{13} \\ \omega_{21} & \epsilon \omega_{21} & \omega_{23} & \epsilon \omega_{23} \end{pmatrix},$$

with $\omega_{11}\omega_{23} - \omega_{13}\omega_{21} \neq 0$.

Choosing new variables X, Y given by

$$x = \omega_{11}X, \quad y = \omega_{21}Y,$$

the matrix assumes the form

$$\begin{pmatrix} 1 & \epsilon & \tau & \epsilon\tau \\ 1 & \epsilon & \tau' & \epsilon\tau' \end{pmatrix};$$

alternatively, choosing new variables given by

$$x = \omega_{11}X + \omega_{13}Y, \quad y = \omega_{21}X + \omega_{23}Y,$$

the matrix assumes the form

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 \\ 0 & 0 & 1 & \epsilon \end{pmatrix}.$$

$$(11) \quad \begin{aligned} x' &= ix, \quad y' = iy, \\ \begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{21} & i\omega_{21} & \omega_{23} & i\omega_{23} \end{pmatrix}, \end{aligned}$$

where $\omega_{11}\omega_{23} - \omega_{13}\omega_{21} \neq 0$.

Choosing new variables as in (i), the matrix assumes respectively the forms

$$(111) \quad \begin{pmatrix} 1 & i & \tau & i\tau \\ 1 & i & \tau' & i\tau' \end{pmatrix}, \quad \begin{pmatrix} 1 & i & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix}.$$

$$(111) \quad x' = \varepsilon x, \quad y' = \varepsilon^2 y,$$

$$\text{and also} \quad x' = -\varepsilon x, \quad y' = -\varepsilon^2 y,$$

$$\begin{pmatrix} \omega_{11} & \varepsilon\omega_{11} & \omega_{13} & \varepsilon\omega_{13} \\ \omega_{21} & \varepsilon^2\omega_{21} & \omega_{23} & \varepsilon^2\omega_{23} \end{pmatrix}.$$

Choosing new variables X, Y given by $x = \omega_{11}X$, $y = \omega_{21}Y$, the matrix becomes

$$\begin{pmatrix} 1 & \varepsilon & \tau & \varepsilon\tau \\ 1 & \varepsilon^2 & \tau' & \varepsilon^2\tau' \end{pmatrix}.$$

The Riemann conditions are that for some integral α, β, p, q ,

$$\beta\tau\tau' - (p + \varepsilon^2q)\tau - (p + \varepsilon q)\tau' - \alpha = 0,$$

$$p^2 + q^2 - pq + \alpha\beta > 0, \quad \tau' \neq \bar{\tau}.$$

$$(iv) \quad x' = ix, \quad y' = -iy,$$

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{21} & -i\omega_{21} & \omega_{23} & -i\omega_{23} \end{pmatrix}.$$

Choosing new variables as in (iii), the matrix becomes

$$\begin{pmatrix} 1 & i & \tau & i\tau \\ 1 & -i & \tau' & -i\tau' \end{pmatrix}.$$

The Riemann conditions are that for some integral α, β, p, q ,

$$\beta \tau \tau' - (p - iq)\tau - (p + iq)\tau' - \alpha = 0,$$

$$p^2 + q^2 + \alpha\beta > 0, \quad \tau' \neq \bar{\tau}.$$

$$(v) \quad x' = ix, \quad y' = -iy; \quad x' = y, \quad y' = -x.$$

First

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{13} & -i\omega_{13} & -\omega_{11} & i\omega_{11} \end{pmatrix},$$

or, dividing each variable by ω_{11} ,

$$\begin{pmatrix} 1 & i & \tau & i\tau \\ \tau & -i\tau & -1 & i \end{pmatrix}.$$

The Riemann conditions imply that for some integral p, q, s ,

$$(p - iq)\tau^2 + 2s\tau - (p + iq) = 0.$$

A second matrix admitting the group is

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{21} & -i\omega_{21} & \omega_{23} & -i\omega_{23} \end{pmatrix}$$

where $\omega_{21} = \omega_{11} - (1 + i)\omega_{13}$, $\omega_{23} = (1 + i)\omega_{11} - i\omega_{13}$.

Dividing each variable by ω_{11} , writing $\omega_{13}/\omega_{11} = \tau$, and making a substitution on the columns of the matrix leads to

$$\begin{pmatrix} 1 + i - i\tau & \tau & 1 + \tau & \tau - 1 \\ -\tau & 1 + i - i\tau & 1 - \tau & 1 + \tau \end{pmatrix}.$$

The Riemann conditions imply that for some integral a, b, s ,

$$(s - ib)\tau^2 + [(a + b - s) + i(a + b + s)]\tau - (a + is) = 0.$$

$$(vi) \quad x' = \varepsilon x, \quad y' = \varepsilon^2 y; \quad x' = y, \quad y' = -x,$$

$$\begin{pmatrix} \omega_{11} & \varepsilon\omega_{11} & \omega_{13} & \varepsilon\omega_{13} \\ \omega_{13} & \varepsilon^2\omega_{13} & -\omega_{11} & -\varepsilon^2\omega_{11} \end{pmatrix},$$

or, dividing each variable by ω_{11} ,

$$\begin{pmatrix} 1 & \varepsilon & \tau & \varepsilon\tau \\ \tau & \varepsilon^2\tau & -1 & -\varepsilon^2 \end{pmatrix}.$$

For some integral p, q, s , we must have

$$(p + \epsilon^2 q)\tau^2 + 2s\tau - (p + \epsilon q) = 0.$$

$$(vii) \quad x' = ix, \quad y' = -iy;$$

$$x' = \frac{1}{2}(1+i)(x+y), \quad y' = -\frac{1}{2}(1-i)(x-y),$$

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{21} & -i\omega_{21} & \omega_{23} & -i\omega_{23} \end{pmatrix},$$

$$\text{where } \omega_{21} = \omega_{11} - (1+i)\omega_{13}, \quad \omega_{23} = (1+i)\omega_{11} - i\omega_{13}.$$

This reduces as in (v).

$$(viii) \quad x' = x, \quad y' = \epsilon y.$$

We have

$$\begin{pmatrix} \omega_{11} & 0 & \omega_{13} & 0 \\ 0 & \omega_{22} & 0 & \epsilon\omega_{22} \end{pmatrix}, \quad \begin{pmatrix} \omega_{11} & 0 & \omega_{13} & 0 \\ 0 & \omega_{22} & \frac{1}{3}(1-\epsilon)\omega_{22} & \epsilon\omega_{22} \end{pmatrix},$$

reducing respectively to

$$\begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & 0 & \epsilon \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 1 & \frac{1}{3}(1-\epsilon) & \epsilon \end{pmatrix}.$$

$$(ix) \quad x' = \omega_8 x, \quad y' = i\omega_8 y, \quad \omega_8^2 = 1 \quad (\omega_8 = \frac{1}{\sqrt{2}}(1+i)).$$

Bagnara-De Franchis did not find it necessary for their purpose to seek a complete set of non-equivalent matrices admitting this group, since from one matrix (to which all the others were isomorphic) it was possible to show that the surfaces mapping the involutions were all rational. Here we must investigate the matrices more fully.

The square of the substitution has multipliers 1, -1, and so from (iv), on dividing the variables by suitable constants, the matrix takes the form

$$\begin{pmatrix} 1 & 1 & \tau & i\tau \\ 1 & -1 & \tau' & -i\tau' \end{pmatrix}. \quad (2)$$

Expressing the fact that the first column admits the multipliers ω_8 , $i\omega_8$, and solving the resulting equations, we find

$$\tau = \frac{\omega_8 + h + ik}{1 + im}, \quad \tau' = \frac{i\omega_8 + h - ik}{1 - im}, \quad (3)$$

with h, k, l, m integral. These values for τ, τ' also ensure that the second column admits the multipliers $\omega_8, i\omega_8$.

For the third, and so also the last, column to admit these multipliers we must have

$$\begin{aligned} \omega_8 \tau &= r_{31} + ir_{32} + r_{33} \tau + ir_{34} \tau, \\ i\omega_8 \tau' &= r_{31} - ir_{32} + r_{33} \tau' - ir_{34} \tau'. \end{aligned}$$

Substituting for τ in the first of these equations and equating the real and imaginary rational and irrational parts, we obtain the following four equations which must be satisfied by h, k, l, m :

$$\begin{aligned} lr_{31} - mr_{32} &= kr_{34} - hr_{33}, \\ mr_{31} + lr_{32} &= 1 - hr_{34} - kr_{33}, \\ r_{33} &= h, \\ r_{34} &= k. \end{aligned}$$

The last two equations serve to determine r_{33}, r_{34} . Substituting these values in the first two equations and writing $r_{31} = a$, $r_{32} = b$, we have that h, k, l, m are not arbitrary integers but must satisfy the two equations

$$al - bm = k^2 - h^2, \quad am + bl = 1 - 2hk, \quad (4)$$

for some integral a, b . The equation satisfied by τ' leads to these same two conditions.

To investigate the Riemann conditions we observe that the matrix (2) in which τ, τ' have the values given by (3) is isomorphic with the matrix

$$\begin{pmatrix} 1 & 1 & \omega_s & i\omega_s \\ 1 & -1 & i\omega_s & \omega_s \end{pmatrix}.$$

The conditions are those found in (iv) above, and we find on substituting $\tau = \omega_s, \tau' = i\omega_s,$

$$\alpha = -\beta, \quad p = q,$$

and then the inequality is satisfied on choosing p, α such that

$$2p^2 - \alpha^2 > 0.$$

Thus the Riemann conditions are always satisfied.

In conclusion, the matrices admitting the group are of the form (2) where τ, τ' are given by (3) subject to the conditions (4). It may be noted that the above work does not involve any further change of variable to modify the matrix (2), but only an evaluation of τ, τ' .

$$(x) \quad x' = \omega_s x, \quad y' = -\omega_s y.$$

The discussion of this group is similar to that just given. The involution on the Picard surface is rational, and so possible forms for the matrix were not investigated by Bagnera-De Franchis. Arguing as in the last case, we find that the matrix must be equivalent to one of the form

$$\begin{pmatrix} 1 & 1 & \tau & i\tau \\ 1 & 1 & \tau' & i\tau' \end{pmatrix},$$

where

$$\tau = \frac{\omega_s + h + ik}{1 + im}, \quad \tau' = \frac{-\omega_s + h + ik}{1 + im},$$

h, k, l, m being integers satisfying the equations

$$al - bm = k^2 - h^2, \quad am + bl = 1 - 2hk,$$

for some integral a, b . Here, again, these results follow simply from an evaluation of τ, τ' without any change of variable after the matrix has been thrown into the above form.

$$(xi) \quad x' = ix, \quad y' = i\epsilon^2 y.$$

The remarks made for the last group apply. The matrix is found to be equivalent to one of the form

$$\begin{pmatrix} 1 & i & \tau & i\tau \\ 1 & i & \tau' & i\tau' \end{pmatrix},$$

where $\tau = \frac{\epsilon + h + ik}{1 + im}, \quad \tau' = \frac{\epsilon^2 + h + ik}{1 + im},$

h, k, l, m being integers satisfying the equations

$$al - bm = k^2 - h^2 + h - 1, \quad am + bl = k - 2hk,$$

for some integral a, b .

$$(xii) \quad x' = ix, \quad y' = -i\epsilon y.$$

Again this group is similar to (ix). The matrix is equivalent to one of the form

$$\begin{pmatrix} 1 & i & \tau & i\tau \\ 1 & -i & \tau' & -i\tau' \end{pmatrix},$$

where $\tau = \frac{\epsilon + h + ik}{1 + im}, \quad \tau' = \frac{\epsilon + h - ik}{1 - im},$

h, k, l, m being integers satisfying the equations

$$al - bm = k^2 - h^2 + h - 1, \quad am + bl = k - 2hk,$$

for some integral a, b .

$$(xiii) \quad x' = \omega_8 x, \quad y' = i\omega_8 y; \quad x' = -y, \quad y' = x.$$

The groups (v) and (ix) are subgroups of the present group.

We require to find for what values of τ the two matrices found in (v) will admit the group (ix). Taking these matrices in the forms

$$\begin{pmatrix} 1 & i & \tau & i\tau \\ \tau & -i\tau & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & i & \tau & i\tau \\ \omega_{21} & -i\omega_{21} & \omega_{23} & -i\omega_{23} \end{pmatrix},$$

$$\text{where} \quad \omega_{21} = 1 - (1 + i)\tau, \quad \omega_{23} = (1 + i) - i\tau,$$

the reasoning of (ix) may be applied with the single change that the second row is now multiplied through by a constant, i.e. the τ' of (ix) is replaced by $-1/\tau$ in the first matrix and by ω_{23}/ω_{21} in the second matrix, and the bottom rows are then multiplied by τ , ω_{21} respectively. Thus τ in the present matrices must assume the form found in (ix) and $-1/\tau$ and ω_{23}/ω_{21} must take the form found there for τ' .

For the first matrix this implies

$$\frac{i\omega_8 + h - ik}{1 - im} = \frac{-(1 + im)}{\omega_8 + h + ik},$$

which leads to

$$h = k, \quad l^2 + m^2 = 1 - 2h^2. \quad (5)$$

The conditions (4) imposed on h, k, l, m in (ix) now become

$$al - bm = 0, \quad am + bl = 1 - 2h^2,$$

and are satisfied by taking

$$a = m, \quad b = l,$$

when conditions (5) are satisfied.

The solutions of (5) are

$$h = k = 0 \quad \text{and either} \quad l = \pm 1, \quad m = 0 \quad \text{or} \quad l = 0, \quad m = \pm 1,$$

and the corresponding values for τ are

$$\pm\omega_s, \mp i\omega_s.$$

The matrices obtained from $\tau = -\omega_s, -i\omega_s$ differ from those obtained from $\tau = \omega_s, i\omega_s$ only in the interchange of the rows (and the signs of two columns). Thus, since x, y can be interchanged in the equations of the group without changing the form, we need only take the two matrices

$$\begin{pmatrix} 1 & 1 & \omega_s & i\omega_s \\ \omega_s & -i\omega_s & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \omega_s & i\omega_s \\ i\omega_s & \omega_s & -1 & 1 \end{pmatrix}.$$

For the second of the two matrices written earlier we have

$$\tau' = \frac{\omega_{23}}{\omega_{21}} = \frac{(1+i) - i\tau}{1 - (1+i)\tau},$$

or $(1+i)\tau\tau' - \tau' - i\tau + (1+i) = 0.$

Substituting for τ, τ' their values from (ix), we find that

h, k, l, m must satisfy the equations

$$h - k - l = 0, \quad h^2 + k^2 + l^2 + m^2 - lh + lk - mh - mk - 1 = 0.$$

Substituting for h in the second equation we obtain

$$2k^2 + 2k(1-m) + (l^2 + m^2 - lm - 1) = 0.$$

For real k we must therefore have

$$(1-m)^2 \geq 2(l^2 + m^2 - lm - 1),$$

or $l^2 + m^2 - 2 \leq 0.$

The solution $l = m = 0$ may be eliminated since, in (ix), τ must be finite; alternatively, we note that these values lead to a non-integral value for k . There remain 12 sets of values for h, k, l, m , namely

$$\begin{array}{cccccccccccc}
 h & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 \\
 k & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \\
 l & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
 m & -1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1
 \end{array}$$

The corresponding values for τ are

$$\pm i\omega_s, \pm i\omega_s + (1 - i), \pm i\omega_s + 1, \pm i\omega_s - i, \frac{\pm i\omega_s + 1}{1 + i}, \frac{\pm i\omega_s + 1}{1 + i},$$

Thus there are two matrices of the first type and twelve matrices of the second type.

$$(xiv) \quad x' = ix, \quad y' = -iy; \quad x' = y, \quad y' = x.$$

From (iv), the matrix can be put in the form

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{21} & -i\omega_{21} & \omega_{23} & -i\omega_{23} \end{pmatrix}.$$

Now $(\omega_{21}, \omega_{11})$ etc. are periods. Hence, following methods used by Bagnara-De Franchis in discussing group (v), we observe that we must have

$$\omega_{21} = p\omega_{11} + q\omega_{13}, \quad \omega_{23} = r\omega_{11} + s\omega_{13}, \quad (6)$$

$$\text{and} \quad \omega_{11} = \bar{p}\omega_{21} + \bar{q}\omega_{23}, \quad \omega_{13} = \bar{r}\omega_{21} + \bar{s}\omega_{23}, \quad (7)$$

where p, q, r, s are integers in the ring $[1, i]$ and the bar denotes the conjugate..

The determinant $D = ps - qr$ cannot be zero, for otherwise we should have a relation between the elements of the second row from (6), and the same relation between the elements of the first row from (7). For the same reason, on substituting from (6) in (7) we must obtain identities. For otherwise on substituting from (6) in (7) we should obtain a relation

between the elements of the first row, and on substituting from (7) in (6) we should obtain the same relation between the elements of the second row.

It follows that on solving (6) for ω_{11} , ω_{13} we must obtain (7): the solution is

$$\omega_{11} = \frac{s}{D}\omega_{21} - \frac{q}{D}\omega_{23}, \quad \omega_{13} = -\frac{r}{D}\omega_{21} + \frac{p}{D}\omega_{23}. \quad (8)$$

Hence

$$\frac{s}{D} = \bar{p}, \quad -\frac{q}{D} = \bar{q}, \quad -\frac{r}{D} = \bar{r}, \quad \frac{p}{D} = \bar{s}. \quad (9)$$

It follows that $|D| = 1$, and since D is an integer in the ring $[1, i]$ we must have $D = \pm 1, \pm i$.

If we put

$$\omega_{11} = i\Omega_{11}, \quad \omega_{13} = \Omega_{13}, \quad \omega_{21} = -i\Omega_{21}, \quad \omega_{23} = \Omega_{23},$$

we find the effect is to change the sign of D . So we need only consider $D = 1, i$.

(a) $D = 1$. From (9)

$$p = a + ib, \quad s = a - ib, \quad q = ic, \quad r = id,$$

and

$$D = ps - qr = a^2 + b^2 + cd = 1.$$

$$\text{So } \omega_{21} = (a + ib)\omega_{11} + ic\omega_{13}, \quad \omega_{23} = id\omega_{11} + (a - ib)\omega_{13}, \quad (10)$$

$$\text{with } a^2 + b^2 + cd = 1. \quad (11)$$

Putting

$$\begin{aligned} \omega_{11} &= \Omega_{11} + (h + ik)\Omega_{13}, & \omega_{13} &= \Omega_{13}, \\ \omega_{21} &= \Omega_{21} + (h - ik)\Omega_{13}, & \omega_{23} &= \Omega_{23}, \end{aligned} \quad (12)$$

we find that the relations between the Ω_{ij} are

$$\begin{aligned} \Omega_{21} &= [a - kd + i(b-hd)]\Omega_{11} + [ic + 2i(ak+bh) - (h^2+k^2)id]\Omega_{13}, \\ \Omega_{23} &= id\Omega_{11} + [a - kd - i(b-hd)]\Omega_{13}. \end{aligned} \quad (13)$$

If instead we put

$$\begin{aligned}\omega_{11} &= \Omega_{11}, & \omega_{13} &= (h + ik)\Omega_{11} + \Omega_{13}, \\ \omega_{21} &= \Omega_{21}, & \omega_{23} &= (h - ik)\Omega_{21} + \Omega_{23},\end{aligned}\tag{14}$$

then we find

$$\begin{aligned}\Omega_{21} &= [a - kc + i(b+hc)]\Omega_{11} + ic\Omega_{13} \\ \Omega_{23} &= [id + 2i(ak-bh) - i(h^2+k^2)c]\Omega_{11} + [a - kc - i(b+hc)]\Omega_{13}.\end{aligned}\tag{15}$$

First, say that neither c nor d is zero. Then, in view of (13) and (15), we can reduce a, b by multiples of c, d ; in this process either one of c, d becomes zero and we then proceed as later, or else we can make

$$|a|, |b| \leq \frac{1}{2}|c|, \frac{1}{2}|d|.$$

Now equation (11) may be written

$$a^2 + b^2 - 1 = -cd.$$

Say $|c| \leq |d|$ (otherwise we work in terms of $|d|$). Then if a or b is non-zero

$$|a^2 + b^2 - 1| = a^2 + b^2 - 1 \leq \frac{1}{2}|c|^2 - 1,$$

$$\text{and} \quad |-cd| \geq |c|^2,$$

$$\text{whence} \quad |c|^2 \leq \frac{1}{2}|c|^2 - 1;$$

which is impossible. Hence

$$a = b = 0$$

$$\text{and so} \quad c = d = 1 \quad \text{or} \quad c = d = -1.$$

The second possibility reduces to the first on taking periods Ω_{ij} where

$$\omega_{11} = \Omega_{11}, \quad \omega_{13} = -\Omega_{13}; \quad \omega_{21} = \Omega_{21}, \quad \omega_{23} = -\Omega_{23}.$$

We thus obtain here a first matrix, namely

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ i\omega_{13} & \omega_{13} & i\omega_{11} & \omega_{11} \end{pmatrix}.$$

Next we must consider the possibility (still with $D = 1$) that one or other of c, d , say c , is zero. Then from (11)

$$a = \pm 1, \quad b = 0 \quad \text{or} \quad a = 0, \quad b = \pm 1.$$

If we put

$$\omega_{11} = i\Omega_{11}, \quad \omega_{13} = i\Omega_{13}; \quad \omega_{21} = -i\Omega_{21}, \quad \omega_{23} = -i\Omega_{23}, \quad (16)$$

the effect is to change the signs of a, b, c, d in (10).

Hence we need only consider

$$a = 1, \quad b = 0; \quad a = 0, \quad b = 1.$$

After this we see from equations (15) that we can make

$$d = 0, 1.$$

In the same way, starting from $d = 0$ we obtain the same two sets of possible values for a, b and then, by (13), we can make

$$c = 0, 1.$$

The four matrices in which one or other of c, d is non-zero reduce to the matrix found earlier. Thus if $a = c = 1, b = d = 0$, putting $h = 0, k = 1$ in (15) gives the required result; and similarly for the other three cases.

We are thus left with two of the present matrices, namely

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{11} & -i\omega_{11} & \omega_{13} & -i\omega_{13} \end{pmatrix}, \quad \begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ i\omega_{11} & \omega_{11} & -i\omega_{13} & -\omega_{13} \end{pmatrix},$$

making three in all arising from the value $D = 1$.

In the last of these matrices we take $i\omega_{13}$ as a new ω_{13} ; this changes the sign of D and reduces the matrix to the form

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ i\omega_{11} & \omega_{11} & i\omega_{13} & \omega_{13} \end{pmatrix}.$$

(b) D = 1. From (9)

$p = a + ib$, $s = b + ia$, $q = c(1 - i)$, $r = d(1 - i)$,
with $D = ps - qr = 1(a + ib)(a - ib) - cd(1 - i)^2 = 1$.

So

$$\begin{aligned}\omega_{21} &= (a + ib)\omega_{11} + c(1 - i)\omega_{13}, \\ \omega_{23} &= d(1 - i)\omega_{11} + (b + ia)\omega_{13},\end{aligned}\tag{17}$$

with

$$a^2 + b^2 + 2cd = 1.\tag{18}$$

Making the substitution (12) we find

$$\begin{aligned}\Omega_{21} &= [a - d(h-k) + i(b + d(h+k))]\Omega_{11} \\ &\quad + [c + a(h-k) - b(h+k) - d(h^2+k^2)](1-i)\Omega_{13},\end{aligned}\tag{19}$$

$$\Omega_{23} = d(1-i)\Omega_{11} + [b + d(h+k) + i(a - d(h-k))]\Omega_{13},$$

and making instead the substitution (14) we find

$$\begin{aligned}\Omega_{21} &= [a + c(h+k) + i(b - c(h-k))]\Omega_{11} + c(1-i)\Omega_{13}, \\ \Omega_{23} &= [d - a(h+k) + b(h-k) - c(h^2+k^2)](1-i)\Omega_{11} \\ &\quad + [b - c(h-k) + i(a + c(h+k))]\Omega_{13}.\end{aligned}\tag{20}$$

If d is non-zero, then putting $h-k = 1$ in (19) we see that we can replace a , b by

$$a - id, \quad b + (1 + 2k)d,$$

and so we can make

$$|a| \leq \frac{1}{2}|d|, \quad |b| \leq |d|.\tag{21}$$

If now c is non-zero, then using (20) we can make

$$|a| \leq \frac{1}{2}|c|, \quad |b| \leq |c|.\tag{22}$$

Proceeding thus we find either that one of c , d becomes zero or else we can satisfy both (21) and (22) simultaneously. In the latter case say $|c| \leq |d|$ (if not we work in terms of $|d|$); then from (18) we have

$$a^2 + b^2 - 1 = -2cd,$$

and so a, b cannot both be zero, whence

$$\begin{aligned} |a^2 + b^2 - 1| &\leq \frac{1}{4}|c|^2 + |c|^2 - 1 \\ &= \frac{5}{4}|c|^2 - 1. \end{aligned}$$

Also $|-2cd| = 2|c| \cdot |d| \geq 2|c|^2.$

Hence $2|c|^2 \leq \frac{5}{4}|c|^2 - 1,$

which is impossible.

It follows that at least one of c, d must be zero.

Then from (18)

$$a = \pm 1, \quad b = 0 \quad \text{or} \quad a = 0, \quad b = \pm 1,$$

and again using the substitution (16) these reduce to

$$a = 1, \quad b = 0; \quad a = 0, \quad b = 1.$$

Following this, we see from (19) and (20) that we can reduce the remaining one of c, d to zero. We have left two matrices, the second of which reduces to the first on interchanging ω_{11} and ω_{13} , and so from $D = 1$ we obtain finally the single matrix

$$\begin{pmatrix} \omega_{11} & i\omega_{11} & \omega_{13} & i\omega_{13} \\ \omega_{11} & -i\omega_{11} & i\omega_{13} & \omega_{13} \end{pmatrix}.$$

Choosing new variables X, Y given by $x = \omega_{11}X, y = \omega_{11}Y$, and writing $\omega_{13}/\omega_{11} = \tau$, the four matrices found become

$$\begin{pmatrix} 1 & 1 & \tau & i\tau \\ i\tau & \tau & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \tau & i\tau \\ 1 & -1 & \tau & -i\tau \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & \tau & i\tau \\ 1 & 1 & i\tau & \tau \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & \tau & i\tau \\ 1 & -1 & i\tau & \tau \end{pmatrix}.$$

The Riemann conditions stated in (iv) may be applied to

the present matrices if (as is seen by multiplying the second row by a suitable constant), we take respectively

$$\tau' = \frac{1}{\tau}, \tau, \tau, 1\tau.$$

It is found that the required integral α, β, p, q always exist, the only condition effective being the last, $\tau' \neq \bar{\tau}$, which implies for the first matrix that $|\tau| \neq 1$, for the second and third that τ is not real, and for the fourth that τ is not a real multiple of $(1 - 1)$. This concludes the discussion of Riemann matrices of genus 2.

4. Some general considerations relating to compound Riemann matrices of genus three. The results of this chapter permit us in many cases to write a matrix in the form (1), with (τ_{ij}) a known matrix of genus 2 and τ either arbitrary or equal to 1 or ϵ . The problem of specifying the permissible matrices then reduces to that of determining the possible forms for $\omega_{21}, \omega_{22}, \omega_{31}, \omega_{32}$. We shall find that the matrix is isomorphic to the compound matrix

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & & & & \\ 0 & 0 & & (\tau_{ij}) & & \\ 0 & 0 & & & & \end{pmatrix},$$

whence, by the result proved for compound matrices in Chapter 1, the Riemann conditions for the matrix of genus 3 are satisfied when the conditions are satisfied by the component matrices. For (τ_{ij}) the conditions are as stated in the present chapter and for the matrix of genus 1, $(1, \tau)$, the condition is simply that τ is not real. Hence for the

matrices of this type the Riemann conditions need not be considered further.

The determination of the values of the four ω_j and the reduction to a minimum set of non-equivalent matrices depends on simple arithmetic and algebraic arguments which must, however, to a large extent be developed specially for each group. It does not appear necessary or desirable to set out these arguments in full; detailed consideration and/or enunciation of results will be restricted to representative cases, points of particular interest, and to those groups whose matrices are required as a starting point for later groups. For each of the remaining groups it will be established by example that matrices do exist which admit the group; and application of the methods illustrated will yield the complete set of non-equivalent matrices.

It may be noted here that among the matrices found in the following chapters, some have fixed uniquely determined elements while others contain parameters which can vary continuously subject to the Riemann conditions. In stating results no distinction will be drawn between the two cases.

5. Coincidences of the involutions. Some remarks may be made concerning the nature of the loci of coincidences of the involutions on the Picard threefolds V , and their relation to the threefold W . Justification and illustrations are to be found in the general results already given and in the explicit groups later shown to exist. We have seen that

the Abelian threefold W may be elliptic or superficially regular; Picard threefolds have been eliminated and hyperelliptic threefolds cannot arise. It follows, incidentally, from the proof of Theorem 1 of Chapter 2.1 that the pure canonical surface of an Abelian threefold of geometric genus 1 is of order zero, since it is mapped on the Picard threefold by such a surface. Hence the arithmetic genus, P_a , of W equals 1, and so the tridimensional irregularity is zero. The superficially regular threefolds W are therefore completely regular.

Consider first an involution generated by a cyclic group. The generating substitution may be put in the form (8) of Chapter 2.7. Coincidences are obtained when the two sides of (8) are congruent, mod $\underline{q}$. When the threefold W is elliptic, $\lambda_1 = 1$, and the involution has no coincidences unless $c_1 = 0$, in which case the coincidence loci are curves obtained by varying the first parameter while the last two parameters take values for a coincidence in the substitution group determined on these. Such coincidences always exist, and thus a set of curves belonging to the congruence of elliptic curves on V is obtained. If W is completely regular, $\lambda_1 \neq 1$ and $c_1 = 0$. A finite set of coincidence points is always obtained.

Next consider a threefold W arising from a group other than a cyclic group. When W is elliptic, the substitutions on the first variable can all be reduced simultaneously to

the form of (8) with $\lambda_1 = 1$, and for any particular cyclic subgroup the last two substitutions can also be put in the form of (8). Coincidences are obtained only if for some substitution $c_1 = 0$, and then, as before, the loci are curves of the congruence of elliptic curves on V which gives rise to the congruence on W . Now, however, an involution giving rise to a completely regular threefold W may contain elliptic subgroups. It follows that on V there may be no coincidences or a finite number of coincidences or a set of coincidence curves (belonging to one or more congruences) or there may be both curves and isolated points. An example of a completely regular threefold represented by an involution without coincidences may be constructed from group $G_+(XIV)$. Thus the matrix

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \tau' & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \tau'' \end{pmatrix}$$

admits the non-homogeneous group, G_+ , generated by

$$S(-x, y + \frac{1}{2}, -z), \quad S(x + \frac{1}{2}, -y, -z + \frac{1}{2});$$

and it is easily verified that the resulting involution, I_+ , on the Picard threefold V has no coincidences. (On the other hand, when the homogeneous group is taken, the coincidence loci are curves from the three congruences on V .) The only general conclusion is that if there are any isolated coincidences the threefold W must be completely regular.

Coincidences will give rise to multiple points and

curves on a projective model of W unless particular steps are taken to resolve these. To investigate the possibilities would call for detailed examination of the parametric representation of W .

CHAPTER 5. MATRICES ADMITTING CYCLIC GROUPS

1. $G_3(I)$. $S(\epsilon x, \epsilon y, \epsilon z)$. Here ϵ is a triple root of the characteristic equation

$$|\underline{R} - \epsilon \underline{E}_6| = 0$$

and the matrix $(\underline{R} - \epsilon \underline{E}_6)$ is of rank 4 (2.7). We show that the matrix $\underline{R}$ may be reduced to the form

$$\begin{pmatrix} 0 & 1 & & & & \\ & -1 & -1 & & & \\ & & & 0 & 1 & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{matrix} \quad (1)$$

The method is that of Bagnera-De Franchis (2, 293 et seq.).

Choose new periods given by the modular substitution

$$T_{\alpha\beta}^{\sigma} : \quad \underline{\omega}'_{\alpha} = \underline{\omega}_{\alpha} - \sigma \underline{\omega}_{\beta}, \quad \underline{\omega}'_i = \underline{\omega}_i, \quad i \neq \alpha.$$

Then in the new matrix $\underline{R}'$ only the elements of the α -th row and β -th column differ from those of $\underline{R}$: in particular, for $\gamma \neq \beta$, $r_{\alpha\gamma}$ is replaced by $r_{\alpha\gamma} - \sigma r_{\beta\gamma}$. Hence, by the use of $T_{\alpha\beta}^{\sigma}$, we can arrange that

$$0 < r_{26} \leq |r_{16}| ;$$

and then by the use of T_{12}^{σ} we can reduce the absolute value of r_{16} . By repetition we make $r_{16} = 0$. In the same way, using in succession pairs of substitutions $T_{\alpha\beta}^{\sigma}$ we make

$$r_{26} = r_{36} = r_{46} = 0 ; \quad r_{15} = r_{25} = r_{35} = 0 ;$$

$$r_{14} = r_{24} = 0 ; \quad r_{13} = 0.$$

The matrix $\underline{R}$ is now of the form

$$\begin{pmatrix} r_{11} & r_{12} & 0 & 0 & 0 & 0 \\ r_{21} & r_{22} & r_{23} & 0 & 0 & 0 \\ r_{31} & r_{32} & r_{33} & r_{34} & 0 & 0 \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & 0 \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} \\ r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66} \end{pmatrix}.$$

Now $r_{34} \neq 0$, for otherwise the characteristic equation would resolve into two instead of three real factors; and also $r_{12} \neq 0$, $r_{56} \neq 0$, for otherwise r_{11} or r_{66} would be a real root of the characteristic equation. Consider then the determinant obtained from $|\underline{R} - \epsilon \underline{E}_6|$ by omitting the fourth and sixth rows and the first and fifth columns; this must vanish, and so

$$r_{12} \cdot r_{23} \cdot r_{34} \cdot r_{56} = 0,$$

whence

$$r_{23} = 0.$$

Similarly

$$r_{45} = 0.$$

The characteristic equation is now the product of three real quadratic factors obtained from the second order determinants along the leading diagonal. The first of these gives

$$\begin{vmatrix} r_{11} - x & r_{12} \\ r_{21} & r_{22} - x \end{vmatrix} = x^2 - (r_{11} + r_{22})x + r_{11}r_{22} - r_{12}r_{21} \\ = x^2 + x + 1,$$

since the zeros must be ϵ , ϵ^2 . Hence

$$r_{11} + r_{22} = -1, \quad (2)$$

$$r_{11}r_{22} - r_{12}r_{21} = 1. \quad (3)$$

Using $T_{12}^\sigma, T_{21}^\sigma$ we can make r_{11} such that

$$|r_{11}| \leq \frac{1}{2}|r_{12}|, \frac{1}{2}|r_{21}|,$$

whence $4r_{11}^2 \leq |r_{12}| \cdot |r_{21}|.$

From (2) and (3) we have

$$r_{11}^2 + r_{11} + r_{12}r_{21} + 1 = 0,$$

so that $4r_{11}^2 \leq |r_{12}| \cdot |r_{21}| \leq r_{11}^2 + |r_{11}| + 1.$

It follows that $r_{11} = 0,$

and then from (2) $r_{22} = -1.$

From (3) $r_{12}r_{21} = -1,$

so that either

$$r_{12} = -r_{21} = 1 \quad \text{or} \quad r_{12} = -r_{21} = -1.$$

In the first case we have

$$\varepsilon\omega_1 = \omega_2, \quad \varepsilon\omega_2 = -\omega_1 - \varepsilon\omega_2;$$

and in the second case

$$\varepsilon\omega_1 = -\omega_2, \quad \varepsilon\omega_2 = \omega_1 - \varepsilon\omega_2;$$

and on replacing ω_2 by $-\omega_2$ these latter equations reduce to the former. We can therefore take

$$r_{11} = 0, \quad r_{12} = -r_{21} = -r_{22} = 1.$$

Precisely the same reduction may be carried out on the other two second order matrices on the leading diagonal.

Next consider the determinant obtained from $|\underline{R} - \varepsilon\underline{E}_6|$ by suppressing the second and third rows and the first and sixth columns. On expansion, since the value is zero, we find

$$r_{54} = r_{53} + r_{63} = s \quad \text{say,} \quad r_{64} = -r_{53} = t \quad \text{say.}$$

Using T_{63}^t we make $r_{64} = r_{53} = 0$, and then using T_{53}^s we make

$r_{54} = r_{63} = 0$. Finally, using $T_{61}^\sigma, T_{51}^\sigma$ and then $T_{41}^\sigma, T_{31}^\sigma$ we make

the last four elements of the first and second columns vanish. The matrix $\underline{R}$ is now in the desired form (1).

The linear relations connecting the periods take the form

$$\varepsilon \underline{\omega}_1 = \underline{\omega}_2, \quad \varepsilon \underline{\omega}_3 = \underline{\omega}_4, \quad \varepsilon \underline{\omega}_5 = \underline{\omega}_6.$$

We must have

$$|\underline{\omega}_1, \underline{\omega}_3, \underline{\omega}_5| \neq 0,$$

for otherwise the Riemann matrix would be of rank less than 3.

Choosing new variables X, Y, Z given by

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = (\underline{\omega}_1, \underline{\omega}_3, \underline{\omega}_5) \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

the Riemann matrix reduces to the form

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \varepsilon \end{pmatrix}.$$

The Picard variety attached to this matrix is the product of three equianharmonic elliptic curves. The method of reduction applies quite generally when a Picard variety is invariant under a substitution all of whose multipliers are equal to ε ; these varieties have been termed equianharmonic by Scorza (18, 336).

In conclusion: all matrices admitting the group $G_3(I)$ $S(\varepsilon x, \varepsilon y, \varepsilon z)$ are equivalent to the matrix

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \varepsilon \end{pmatrix}.$$

2. G₂(II) S(x, -y, -z). Two roots of the characteristic equation are equal to +1, and the matrix may be put in the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & 0 & \tau' & \tau'' \\ \omega_{31} & \omega_{32} & 0 & \frac{1}{s} & \tau'' & \tau''' \end{pmatrix},$$

where the bottom right 2x4 matrix is a general Riemann matrix and the variables y, z have been chosen to reduce this to the canonical form of Bagnera-De Franchis.

Consideration of the first row of the above matrix shows that in the matrix R we must have

$$r_{11} = r_{22} = 1, \quad r_{12} = r_{21} = 0.$$

Hence

$$-\omega_{21} = \omega_{21} + 0 + r_{13} + 0 + r_{15}\tau' + r_{16}\tau'',$$

$$-\omega_{31} = \omega_{31} + 0 + 0 + r_{14}\frac{1}{s} + r_{15}\tau'' + r_{16}\tau''';$$

with similar equations for ω_{22} , ω_{32} . It follows that, by adding multiples of the last four columns to the first two, we can reduce the matrix to the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}(\alpha + \beta\tau' + \gamma\tau'') & \frac{1}{2}(\alpha' + \beta'\tau' + \gamma'\tau'') & 1 & 0 & \tau' & \tau'' \\ \frac{1}{2}(\frac{\alpha_1}{s} + \beta\tau'' + \gamma\tau''') & \frac{1}{2}(\frac{\alpha'_1}{s} + \beta'\tau'' + \gamma'\tau''') & 0 & \frac{1}{s} & \tau'' & \tau''' \end{pmatrix},$$

where $\alpha, \beta, \gamma, \alpha', \beta', \gamma', \alpha_1, \alpha'_1 = 0, 1$.

Now we can always make $\gamma' = 0$ by subtracting column 1 (col. 1) from col. 2 or interchanging these columns if necessary. Also a $1 + \tau'$ in col. 2 may always be reduced to (a new) τ' by adding col. 3 to col. 5. After this there

remain six possibilities for col.2, namely

$$\begin{array}{cccccc} \tau & \tau & \tau & \tau & \tau & \tau \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2}\tau' & \frac{1}{2}\tau' \\ 0 & 0 & \frac{1}{2}\frac{1}{s} & \frac{1}{2}\frac{1}{s} & \frac{1}{2}\tau'' & \frac{1}{2}(\frac{1}{s} + \tau''). \end{array}$$

For each of these forms for the second column we must consider how far the first column may be reduced. In this reduction we may subtract the second column from the first or, in some cases, interchange the two, reduce $\frac{1}{s} + \tau'''$ to τ''' , and reduce $1 + \tau''$ to τ'' where this has not already been done to simplify the second column. It is found that the above forms for the second column yield respectively 8, 5, 3, 3, 2 and 2 forms for the first column, omitting repetitions. With suitable ordering of the first two columns we can say:

there are 23 matrices admitting the group

$G_2(\text{II})$ $S(x, -y, -z)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\tau'\epsilon_2 + \frac{1}{2}\tau''\epsilon_3 & 1 & 0 & \tau' & \tau'' \\ \frac{1}{2}\frac{1}{s}\epsilon_1 & \frac{1}{2}\tau''\epsilon_2 + \frac{1}{2}\tau'''\epsilon_3 & 0 & \frac{1}{s} & \tau'' & \tau''' \end{pmatrix}, \epsilon, \epsilon_1, \epsilon_2, \epsilon_3 = 0, 1,$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\tau'' + \frac{1}{2}\epsilon & \frac{1}{2}\tau' & 1 & 0 & \tau' & \tau'' \\ \frac{1}{2}\tau''' & \frac{1}{2}\tau'' + \frac{1}{2}\frac{1}{s}\epsilon_1 & 0 & \frac{1}{s} & \tau'' & \tau''' \end{pmatrix}, \epsilon, \epsilon_1 = 0, 1,$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\tau'\epsilon & \frac{1}{2}\epsilon_1 & 1 & 0 & \tau' & \tau'' \\ \frac{1}{2}\frac{1}{s} + \frac{1}{2}\tau''\epsilon & 0 & 0 & \frac{1}{s} & \tau'' & \tau''' \end{pmatrix}, \epsilon, \epsilon_1 = 0, 1, \epsilon\epsilon_1 \neq 0.$$

3. $G_2(\text{III})$ $S(-x, iy, iz)$. From the theorem of 4.1 and the form of the matrix of genus 2 given in group (ii) of 4.3, it follows that with a suitable choice of variables y, z , any matrix for the present group can be put in the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & 1 & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 & 0 & 1 & 1 \end{pmatrix}.$$

From the first row, we see that in the matrix $\underline{R}$

$$r_{11} = r_{22} = -1, \quad r_{12} = r_{21} = 0.$$

Thus the ω satisfy equations

$$i\omega_{21} = -\omega_{21} + r_{13} + ir_{14},$$

$$i\omega_{22} = -\omega_{22} + r_{23} + ir_{24},$$

$$i\omega_{31} = -\omega_{31} + r_{15} + ir_{16},$$

$$i\omega_{32} = -\omega_{32} + r_{25} + ir_{26}.$$

Solving for ω_{21} , we see that if r_{13}, r_{14} are both odd or both even, then ω_{21} may be reduced to zero on adding suitable multiples of cols. 3 and 4, and that otherwise ω_{21} may be reduced to $\frac{1}{2}(1+i)$; and similarly for the other ω_{ij} . This gives the possibilities

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\alpha(1+i) & \frac{1}{2}\beta(1+i) & 1 & 1 & 0 & 0 \\ \frac{1}{2}\gamma(1+i) & \frac{1}{2}\delta(1+i) & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \alpha, \beta, \gamma, \delta = 0, 1.$$

If α, β are not both zero, we can make $\beta = 0$. Then if $\delta = 1$, we can make $\gamma = 0$. Since the second and third rows, as also the first and second columns, may be interchanged, this leaves four cases:

$$\begin{aligned} \alpha = \beta = \gamma = \delta = 0; \quad \alpha = 1, \quad \beta = \gamma = \delta = 0; \\ \alpha = \delta = 1, \quad \beta = \gamma = 0; \quad \alpha = \gamma = 1, \quad \beta = \delta = 0. \end{aligned}$$

The second case is obtained from the last by subtracting the second row from the third and then adding cols.5 and 6 to 3 and 4 respectively. Hence

there are 3 matrices admitting the group $G_4(\text{III})$ $S(-x, iy, iz)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & 0 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2}\epsilon\epsilon(1+i) & 0 & 0 & 1 & 1 \end{pmatrix}, \epsilon, \epsilon_1 = 0, 1.$$

4. $G_4(\text{IV})$ $S(\epsilon x, -\epsilon y, -\epsilon z)$. The cube of this substitution has multipliers 1, -1, -1, the first row admits the multiplier ϵ , and last two rows the multipliers ϵ, ϵ , whence, by a suitable multiplication on the variable x and choice of variables y, z , the matrix can be put in the form

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & \epsilon & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 & 0 & 1 & \epsilon \end{pmatrix}.$$

Since the matrix also admits the substitution with multipliers $\epsilon, \epsilon, \epsilon$, we see that we can take

$$\omega_{22} = \epsilon\omega_{21}, \quad \omega_{32} = \epsilon\omega_{31}.$$

Next, since the matrix admits the substitution with multipliers 1, -1, -1, we have (from col.1)

$$-\omega_{21} = \omega_{21} + a + b\epsilon,$$

whence

$$\omega_{21} = \frac{1}{2}(p + q\epsilon),$$

and using cols.3 and 4 we can make $p, q = 0, 1$.

In the same way we can take

$$\omega_{31} = \frac{1}{2}(p' + q'\epsilon), \quad p', q' = 0, 1.$$

If, now, $q = 1$ it can be reduced to zero either by interchanging the second and third rows (if $q' = 0$) or by subtracting row 3 from row 2 (if $q' = 1$); after this latter operation the last four columns can again be reduced to the form already given. Thus we can make $q = 0$.

If $p = 0$, we have four possibilities for ω_{31}, ω_{32} , namely

$$0, 0; \quad \frac{1}{2}, \frac{1}{2}\epsilon; \quad \frac{1}{2}\epsilon, \frac{1}{2}\epsilon^2; \quad \frac{1}{2}(1+\epsilon), \frac{1}{2}(\epsilon+\epsilon^2).$$

The third possibility reduces to the second on adding col.4 to col.3, col.6 to col.5, dividing rows 2 and 3 by ϵ , and finally re-ordering the last four columns. Similarly, the last possibility, which is in fact $-\frac{1}{2}\epsilon^2, -\frac{1}{2}$, also reduces to the second. Thus when

$$\omega_{21} = \omega_{22} = 0$$

there are two possibilities for ω_{31}, ω_{32} , namely

$$0, 0; \quad \frac{1}{2}, \frac{1}{2}\epsilon.$$

If $p = 1$, we can always make $p' = 0$. If also $q' = 0$, we obtain the second of the cases just found on interchanging rows 2 and 3. If $q' = 1$, we add col.6 to col.5 and divide row 3 by ϵ . This gives us the case with $p' = 1, q' = 0$, which in turn reduces to $p' = q' = 0$.

No further reduction is possible without altering the form of the generating substitution. Thus

there are 2 matrices admitting the group $G_4(\text{IV})$ $S(\epsilon x, -\epsilon y, -\epsilon z)$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon & 0 & 0 & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1.$$

5. $G_2(V)$ $S(x, \epsilon y, \epsilon^2 z)$. In view of Chapter 4.3.(iii), the matrix may be written

$$\begin{pmatrix} 1 & r & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & \epsilon & r' & \epsilon r' \\ \omega_{31} & \omega_{32} & 1 & \epsilon^2 & r'' & \epsilon^2 r'' \end{pmatrix}.$$

Considering the transforms of cols.1 and 2 under S , we find

$$\begin{aligned} \epsilon \omega_{21} &= \omega_{21} + r_{13} + \epsilon r_{14} + r_{15} r' + \epsilon r_{16} r', \\ \epsilon^2 \omega_{31} &= \omega_{31} + r_{13} + \epsilon^2 r_{14} + r_{15} r'' + \epsilon^2 r_{16} r'', \end{aligned}$$

with similar equations for ω_{22} , ω_{32} .

Solving for ω_{ij} and writing

$$r_{13} + r_{14} = p, \quad r_{15} + r_{16} = q, \quad r_{13} = r, \quad r_{15} = s,$$

with similar p' , q' , r' , s' , we find

$$\begin{aligned} \omega_{21} &= \frac{1}{3}[p(1-\epsilon) - 3r + r'(q(1-\epsilon) - 3s)], \\ \omega_{31} &= \frac{1}{3}[p(1-\epsilon^2) - 3r + r''(q(1-\epsilon^2) - 3s)], \\ \omega_{22} &= \frac{1}{3}[p'(1-\epsilon) - 3r' + r'(q'(1-\epsilon) - 3s')], \\ \omega_{32} &= \frac{1}{3}[p'(1-\epsilon^2) - 3r' + r''(q'(1-\epsilon^2) - 3s')]. \end{aligned}$$

Now r , r' , s , s' reduce to zero on adding suitable multiples of cols.3 and 5; and similarly p , p' , q , q' are made to assume the values 0, 1 or 2. This gives the matrices

$$\begin{pmatrix} 1 & r & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon)(p + qr') & \frac{1}{3}(1-\epsilon)(p' + q'r') & 1 & \epsilon & r' & \epsilon r' \\ \frac{1}{3}(1-\epsilon^2)(p + qr'') & \frac{1}{3}(1-\epsilon^2)(p' + q'r'') & 1 & \epsilon^2 & r'' & \epsilon^2 r'' \end{pmatrix},$$

$p, p', q, q' = 0, 1, 2.$

If both p, p' are non-zero, then either they are equal or their sum is 3. Thus in every case we can make $p' = 0$. Then either $q' = 0$ or, by changing the sign of the second column and adding (col.5 - col.6) if necessary, q' can be made equal to 1. If $q' = 1$ we can make $q = 0$; and then either $p = 0$ or we can make $p = 1$.

There remain the cases for which $p' = q' = 0$. Here we have $p, q = 0, 1, 2$, giving nine cases. The cases for which $p = 2$ reduce to those for which $p = 1$ on changing the sign of the first column. The case $p = 0, q = 2$ similarly reduces to $p = 0, q = 1$; and $p = 0, q = 1$ ($q' = 0$) becomes $p = q = 0, q' = 1$ on interchanging the first two columns.

So far, without any change to the last four columns, we have reduced the possibilities to six, namely (p' being always zero)

$p = q = q' = 0$; $p = 1, q = q' = 0$; $p = q = 0, q' = 1$;
 $p = q' = 1, q = 0$; $p = q = 1, q' = 0$; $p = 1, q = 2, q' = 0$.

On changing the signs of the last two columns, the last set can be reduced to the fifth, and on adding cols.3 and 4 to cols.5 and 6 respectively, the fifth reduces to the third. Finally, on dividing rows 2 and 3 by τ', τ'' respectively, the third reduces to the second set. This leaves three sets

$p = q = q' = 0$; $p = 1, q = q' = 0$; $p = q' = 1, q = 0$.

Thus: there are 3 matrices admitting the group $G_3(V)$ $S(x, \epsilon y, \epsilon^2 z)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon)\tau' & 1 & \epsilon & \tau' & \epsilon\tau' \\ \frac{1}{3}\epsilon(1-\epsilon^2) & \frac{1}{3}\epsilon\epsilon(1-\epsilon^2)\tau'' & 1 & \epsilon^2 & \tau'' & \epsilon^2\tau'' \end{pmatrix},$$

(i) $\epsilon = \epsilon_1 = 0$; (ii) $\epsilon = 1, \epsilon_1 = 0$; (iii) $\epsilon = \epsilon_1 = 1$.

6. $G_4(\text{VI}) S(x, iy, -iz)$. Discussion of this group is on precisely the same lines as that of the previous one, and we conclude that

there are 3 matrices admitting the group $G_4(\text{VI}) S(x, iy, -iz)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon(1+i)\tau' & 1 & i & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1-i) & \frac{1}{2}\epsilon\epsilon(1-i)\tau'' & 1 & -i & \tau'' & -i\tau'' \end{pmatrix},$$

(i) $\epsilon = \epsilon_1 = 0$; (ii) $\epsilon = 1, \epsilon_1 = 0$; (iii) $\epsilon = \epsilon_1 = 1$.

7. $G_4(\text{VII}) S(x, -\epsilon y, -\epsilon^2 z)$. The fourth power of this substitution has multipliers 1, ϵ , ϵ^2 , and the discussion of $G_3(\text{V})$ applies. The first of the three matrices found there clearly admits the present group. It is easily verified that neither of the other two admits the substitution with multipliers 1, -1, -1, and so both are excluded here. Hence

there is only one matrix admitting the group $G_4(\text{VII}) S(x, -\epsilon y, -\epsilon^2 z)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & \tau' & \epsilon\tau' \\ 0 & 0 & 1 & \epsilon^2 & \tau'' & \epsilon^2\tau'' \end{pmatrix}.$$

8. $G_4(VIII) S(-x, \epsilon y, -\epsilon^2 z)$. It is convenient to interchange the first two variables and consider the substitution $S(\epsilon x, -y, -\epsilon^2 z)$, reverting eventually to the original variables. The cube of the substitution now has multipliers 1, -1, -1, and since also the first variable admits the multiplier ϵ , the first row of the Riemann matrix may be reduced to $(1 \ \epsilon \ 0 \ 0 \ 0 \ 0)$.

The bottom right 2×4 matrix of the Riemann matrix admits the multipliers -1, $-\epsilon^2$, and so also 1, ϵ . By Chapter 4.3.(viii) it is therefore of one or other of two forms, and we have the two possible types of matrix

$$(a) \quad \begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & 0 & \tau' & 0 \\ \omega_{31} & \omega_{32} & 0 & 1 & 0 & \epsilon \end{pmatrix},$$

$$(b) \quad \begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & 0 & \tau' & 0 \\ \omega_{31} & \omega_{32} & 0 & 1 & \frac{1}{3}(1-\epsilon) & \epsilon \end{pmatrix}.$$

Type (a). Expressing that the first two columns admit the multipliers ϵ , -1, $-\epsilon^2$, we find that ω_{21} , ω_{22} are expressible integrally in terms of 1, τ' , and hence may be reduced to zero; and ω_{31} , ω_{32} can be put in the forms

$$\omega_{31} = \frac{1}{2}(p + q\epsilon), \quad \omega_{32} = \frac{1}{2}(p + q + p\epsilon), \quad p, q = 0, 1.$$

The four pairs of values are

$$0, 0; \quad \frac{1}{2}, -\frac{1}{2}\epsilon^2; \quad \frac{1}{2}\epsilon, \frac{1}{2}; \quad -\frac{1}{2}\epsilon^2, \frac{1}{2}\epsilon;$$

the last being a reduced form of that arising from $p = q = 1$.

Using the fourth and sixth columns, the minus signs may be

eliminated, and then multiplying the third row by ϵ , ϵ^2 respectively, the second and fourth cases may be reduced to the third; cols. 4 and 6 being rearranged to take their original forms.

Hence, taking the variables in their original order, we have two matrices

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ 0 & 0 & \frac{1}{2}\epsilon\epsilon & \frac{1}{2}\epsilon & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1.$$

Type (b). The reduction follows that of type (a). First, ω_{21} and ω_{22} reduce to zero. It then follows that in the equations satisfied by ω_{31} , ω_{32} , the coefficient of $\frac{1}{3}(1-\epsilon)$ is zero, and this leads to the four pairs of values for ω_{31} , ω_{32} again. These reduce to two pairs as before, it being necessary only to observe that the $\frac{1}{3}(1-\epsilon)$ can in each instance be retained.

Taking the variables in their original order, we obtain the two matrices found above with the single change that

$$\omega_{32} = \frac{1}{3}(1-\epsilon).$$

Hence: there are 4 matrices admitting the group G_4 (VIII) $S(-x, \epsilon y, -\epsilon^2 z)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ 0 & \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{2}\epsilon\epsilon & \frac{1}{2}\epsilon & 1 & \epsilon \end{pmatrix}, \quad \epsilon, \epsilon^2 = 0, 1.$$

9. $G_7(\text{IX})$ $S(\omega_7 x, \omega_7^2 y, \omega_7^4 z)$. Let $(\omega_{11}, \omega_{21}, \omega_{31})$ be any period. Then $(\omega_7 \omega_{11}, \omega_7^2 \omega_{21}, \omega_7^4 \omega_{31}), \dots, (\omega_7^5 \omega_{11}, \omega_7^3 \omega_{21}, \omega_7^6 \omega_{31})$, obtained by successive multiplications of the components by $\omega_7, \omega_7^2, \omega_7^4$ respectively, are also periods, and the six periods are linearly independent. It follows that no component can be zero, and, on choosing $\omega_{11} x, \omega_{21} y, \omega_{31} z$ as new variables, the periods may be combined to give the period matrix as written at the end of this section. Any fundamental period matrix is isomorphic with this one. That the matrix is in fact a Riemann matrix, i.e. that Abelian functions admitting the substitution S exist, is assured by the fact that Klein's quartic curve gives rise to such functions (18', 4; 10', 471: these papers also show, it may be observed, that the matrix is equivalent to a compound one).

It is an interesting question whether, for a suitable choice of the initial period, the matrix as written is always fundamental. This is well known to be so for the similar matrix of genus 1 (which is equianharmonic), and can be shown to be so in the case of genus 2 (where the multipliers of the group are complex fifth roots of unity). At present, however, the question remains open for the matrix of genus 3. Thus we state

one matrix which admits the group

$$G_7(\text{IX}) \quad S(\omega_7 x, \omega_7^2 y, \omega_7^4 z) \text{ is}$$

$$\begin{pmatrix} 1 & \omega_7 & \omega_7^2 & \omega_7^3 & \omega_7^4 & \omega_7^5 \\ 1 & \omega_7^2 & \omega_7^4 & \omega_7^6 & \omega_7 & \omega_7^3 \\ 1 & \omega_7^4 & \omega_7 & \omega_7^5 & \omega_7^2 & \omega_7^6 \end{pmatrix}.$$

10. $G_2(X) S(-x, \omega_2 y, i\omega_2 z)$. The square of this substitution has multipliers 1, 1, -1. Hence, from G(VI) and Chapter 4.3.(ix), the Riemann matrix must be of one or other of the three forms

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon_1(1+i)\tau' & 1 & i & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1-i) & \frac{1}{2}\epsilon\epsilon_1(1-i)\tau'' & 1 & -i & \tau'' & -i\tau'' \end{pmatrix}, \epsilon, \epsilon_1 = 0, 1,$$

where $\tau' = \frac{\omega_2 + h + ik}{1 + im}, \quad \tau'' = \frac{i\omega_2 + h - ik}{1 - im}$

with $al - bm = k^2 - h^2, \quad am + bl = 1 - 2hk, \quad (4)$
 h, k, l, m, a, b being integral.

When $\epsilon = \epsilon_1 = 0$, the matrix is compound and clearly admits the group.

When $\epsilon = 1, \epsilon_1 = 0$, we must consider the effect of the multipliers -1, $\omega_2, i\omega_2$ on the first column. We require

$$\omega_2 \frac{1}{2}(1+i) = -\frac{1}{2}(1+i) + 0 + r_{13} + ir_{14} + \frac{(\omega_2 + h + ik)}{1 + im}(r_{15} + ir_{16});$$

and a corresponding equation for the first element of the third row which, however, yields nothing further. The equation written is equivalent to the four integral equations

$$\begin{aligned} 1 &= r_{15} + r_{16}, & 0 &= -\frac{1}{2}(1-m) + r_{13}l - r_{14}m + r_{15}h - r_{16}k, \\ -m &= r_{15} - r_{16}, & 0 &= -\frac{1}{2}(1+m) + r_{13}m + r_{14}l + r_{15}k + r_{16}h. \end{aligned}$$

Solving for the r_{1i}

$$r_{15} = \frac{1}{2}(1-m), \quad r_{16} = \frac{1}{2}(1+m), \quad (5)$$

$$r_{13} = \frac{1}{2}(1-h+k), \quad r_{14} = \frac{1}{2}(1-h-k). \quad (6)$$

From equations (5), 1 and m are either both even or both odd,

and from the second of the equations (4) it follows that they must both be odd. Equations (6) show that of h, k , one is even and the other odd. Thus for this matrix the following further conditions are imposed on τ', τ''

$$l, m, h+k \text{ odd.}$$

Now observing that

$$al - bm \equiv am + bl + a(1-m) - b(1+m),$$

we see that, if l and m are both odd, $al - bm$ and $am + bl$ are of the same parity; and since the second is odd, by (4), so also is the first. Hence $h+k$ is necessarily odd if l, m are, and the third condition above may be dropped.

Consider finally the matrix for which $\epsilon = \epsilon_1 = 1$. The conditions just found must be satisfied. In addition, applying similar considerations to the second column, and using equations (4), we find that $a+b$ must be even. From the second of equations (4), a and b must then both be odd. But now a, b, l, m are all odd and so the left-hand side of this equation is even, which is inadmissible. Hence this matrix does not admit the group.

That the previous matrix, for which $\epsilon = 1, \epsilon_1 = 0$, can effectively exist is seen by taking

$$h = l = m = 1, \quad k = 0.$$

In conclusion

there are 2 matrices admitting the group

$$G_2(X) \quad S(-x, \omega, y, i\omega, z), \text{ namely}$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & 0 & 1 & i & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1-i) & 0 & 1 & -i & \tau'' & -i\tau'' \end{pmatrix}, \quad \epsilon = 0, 1,$$

where $\tau' = \frac{\omega_s + h + ik}{1 + im}, \quad \tau'' = \frac{i\omega_s + h - ik}{1 - im},$

with $al - bm = k^2 - h^2, \quad am + bl = 1 - 2hk,$
 a, b, h, k, l, m being integral. In addition when $\epsilon = 1,$
 l, m are both odd.

11. $G_s(XI) S(ix, \omega_s, y, -\omega_s, z)$. Using Chapter 4.3.(x),
 and in view of the first multiplier of S , the Riemann matrix
 may be reduced to the form

$$\begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & i & \tau' & i\tau' \\ \omega_{31} & \omega_{32} & 1 & i & \tau'' & i\tau'' \end{pmatrix},$$

where $\tau' = \frac{\omega_s + h + ik}{1 + im}, \quad \tau'' = \frac{-\omega_s + h + ik}{1 + im},$

with $al - bm = k^2 - h^2, \quad am + bl = 1 - 2hk,$
 a, b, h, k, l, m being integral.

Applying the multipliers $-1, 1, 1$ to the first two
 columns we find that, after reduction, there are four forms
 which must be considered, namely

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon(1+i) & \frac{1}{2}(1+i) & \frac{1}{2}(1+i)\tau' \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon(1+i) & \frac{1}{2}(1+i) & \frac{1}{2}(1+i)\tau'' \end{array}, \quad \epsilon, \epsilon = 0, 1.$$

These cannot be reduced further without altering the
 elements of the first row.

When $\epsilon = \epsilon_1 = 0$, the matrix is compound and admits the group.

When $\epsilon = 1, \epsilon_1 = 0$, we find, arguing as for the last group, that we must have l, m odd and $h+k$ even. As pointed out there, this is not possible, and so this matrix is ruled out.

When $\epsilon = \epsilon_1 = 1$, we find we must have

l, m both odd.

Finally, for the fourth matrix we find

$l+m$ odd, $h+k$ even

(which imply also $a+b$ odd). To verify that this is possible, we observe that one solution is given by

$$h = k = l = -b = 1, \quad a = m = 0.$$

In conclusion

there are 3 matrices admitting the group

$G_8(XI) \ S(ix, \omega, y, -\omega, z)$, namely

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon(1+i) & 1 & 1 & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon(1+i) & 1 & 1 & \tau'' & i\tau'' \end{pmatrix},$$

$$(i) \ \epsilon = 0; \quad (ii) \ \epsilon = 1;$$

$$(iii) \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+i) & \frac{1}{2}(1+i)\tau' & 1 & 1 & \tau' & i\tau' \\ \frac{1}{2}(1+i) & \frac{1}{2}(1+i)\tau'' & 1 & 1 & \tau'' & i\tau'' \end{pmatrix},$$

where $\tau' = \frac{\omega_8 + h + ik}{1 + im}, \quad \tau'' = \frac{-\omega_8 + h + ik}{1 + im},$

with $al - bm = k^2 - h^2, \quad am + bl = 1 - 2hk,$

a, b, h, k, l, m being integral. In addition, we have

for matrix (ii) $l, m, h+k$ odd;

for matrix (iii) $l+m$ odd, $h+k$ even.

12. $G_{12}(XII)$ $S(-x, 1\epsilon y, 1\epsilon^2 z)$. Using Chapter 4.3.(xi) and reducing $\omega_{21}, \dots, \omega_{32}$ as in the last group (with the simplification that ω_{12} has no longer to be maintained equal to 1), we find that any matrix must be equivalent to one of the three

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon_1(1+i)\tau' & 1 & 1 & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon_1(1+i)\tau'' & 1 & 1 & \tau'' & i\tau'' \end{pmatrix}, \quad \epsilon, \epsilon_1 = 0, 1,$$

where τ', τ'' are as specified at the end of this section.

When $\epsilon = \epsilon_1 = 0$, the matrix is compound and admits the group.

For the other two matrices, consider the effect of multiplying the first column by the multipliers $1, \epsilon, \epsilon^2$. For the resulting expression to be a period we must have

$$l+m \text{ even, } h+k \text{ odd.}$$

Addition of the two conditions already imposed on h, k, l, m in specifying τ', τ'' , leads to

$$a(1+m) + b(1-m) = k(k+1) - h(h-1) - 2hk - 1.$$

Since the right-hand side is odd, this shows that $l+m$ also must be odd, and so the two matrices are ruled out. Thus

there is only one matrix admitting the group

$G_{12}(XII)$ $S(-x, 1\epsilon y, 1\epsilon^2 z)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & \tau' & i\tau' \\ 0 & 0 & 1 & i & \tau'' & i\tau'' \end{pmatrix}$$

where $\tau' = \frac{\varepsilon+h+ik}{1+im}$, $\tau'' = \frac{\varepsilon^2+h+ik}{1+im}$,

with $am + bl = k - 2hk$, $al - bm = k^2 - h^2 + h - 1$,
 a, b, h, k, l, m being integral.

13. $G_{12}(XIII)$ $S(\varepsilon x, i\varepsilon y, -i\varepsilon z)$. Starting from Chapter 4.3.(xii) and reasoning as in the last three cases we find that

there are 2 matrices admitting the group $G_{12}(XIII)$, namely

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ \frac{1}{2}\varepsilon(1+i) & \frac{1}{2}\varepsilon(1+i)\tau' & 1 & i & \tau' & i\tau' \\ \frac{1}{2}\varepsilon(1-i) & \frac{1}{2}\varepsilon(1-i)\tau'' & 1 & -i & \tau'' & -i\tau'' \end{pmatrix}, \quad \varepsilon = 0, 1,$$

where $\tau' = \frac{\varepsilon+h+ik}{1+im}$, $\tau'' = \frac{\varepsilon+h-ik}{1-im}$,

with $am + bl = k - 2hk$, $al - bm = k^2 - h^2 + h - 1$,
 a, b, h, k, l, m being integral. In addition, when $\varepsilon = 1$, we have

$h+k$ even.

CHAPTER 6. MATRICES ADMITTING MULTIPLICATIVE GROUPS

1. $G_+(XIV)$ $S(-x, y, -z), S(x, -y, -z); \lambda_1, \lambda_2 = \pm 1.$

In view of the multipliers 1, the elements of each row of the Riemann matrix satisfy four integral linear relations, and the matrix may be put in the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & \tau' & 0 & 0 \\ \omega_{31} & \omega_{32} & \omega_{33} & \omega_{34} & 1 & \tau'' \end{pmatrix}.$$

Consideration of the transforms of the first two columns shows that ω_{21}, ω_{22} may be reduced to one or other of the three pairs

$$0, 0; \quad \frac{1}{2}, 0; \quad \frac{1}{2}, \frac{1}{2}\tau'.$$

For each of these cases the possible values for ω_{33}, ω_{34} and then ω_{31}, ω_{32} may be determined; they are all found to be reducible to 0, $\frac{1}{2}$ or $\frac{1}{2}\tau''$. It is not proposed to list all the resulting matrices of which there are 18.

2. $G_+(XV)$ $S(\varepsilon x, \varepsilon y, \varepsilon z), S(x, \varepsilon y, \varepsilon^2 z); \lambda_1, \lambda_2 = 1, \varepsilon, \varepsilon^2.$

From the first generator and the fact that there is a substitution in which any particular multiplier equals 1 (and the other two take the values $\varepsilon, \varepsilon^2$), we see that the matrix can be put in the form

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ \tau & \varepsilon\tau & 1 & \varepsilon & 0 & 0 \\ \tau' & \varepsilon\tau' & \tau'' & \varepsilon\tau'' & 1 & \varepsilon \end{pmatrix}.$$

Applying the multipliers $\varepsilon^2, 1, \varepsilon$ to the third column,

we must have

$$\epsilon \tau'' = \tau'' + a + b\epsilon,$$

whence

$$\tau'' = \frac{1}{3}(b - 2a - (a + b)\epsilon),$$

or, using the last two columns,

$$\tau'' = \frac{1}{3}(b - 2a)(1-\epsilon).$$

Values of $b - 2a$ need only be considered modulo 3, so that we have the three values 0, 1, -1. For the last of these we change the signs of the third and fourth columns and that of the second row, thus converting the -1 to +1. Hence

$$\tau'' = 0, \frac{1}{3}(1-\epsilon).$$

The same reduction applies also to τ .

The possible values for τ' are now determined by applying the multipliers 1, ϵ^2 , ϵ to the first column. After allowing for the fact that some of the resulting matrices (namely those in which just one of τ , τ' , τ'' is non-zero) differ only by an interchange of rows and columns, we arrive at the following eight matrices.

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}\epsilon\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon\epsilon(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix}, \quad \epsilon, \epsilon^2 = 0, 1;$$

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 0 & 0 & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 1, -1;$$

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \tau' & \epsilon\tau' & \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix},$$

$$\tau' = \frac{1}{3}, \frac{1}{3}\epsilon, \frac{1}{3}\epsilon^2.$$

The fifth of these reduces to the fourth on changing the sign of the third row and last two columns. The seventh and eighth reduce to the sixth on choosing $\epsilon^2 z$, ϵz respectively as a new third variable and adjusting the last four columns. Thus

there are 5 matrices admitting the group

$G_4(XV) \ S(\epsilon x, \epsilon y, \epsilon z), \ S_1(x, \epsilon y, \epsilon^2 z); \ \lambda_1, \lambda_2 = 1, \epsilon, \epsilon^2$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}\epsilon\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon\epsilon(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix},$$

(i) $\epsilon = \epsilon_1 = 0$; (ii) $\epsilon = 1, \epsilon_1 = 0$; (iii) $\epsilon = \epsilon_1 = 1$;

$$(iv) \begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 1 & \epsilon \end{pmatrix},$$

$$(v) \begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}\epsilon & \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix}.$$

The matrices have been numbered (i)-(v) for later reference.

3. $G_2(XVI) \ S(-x, iy, iz), \ S(-x, y, -z); \ \lambda_1 = \pm 1,$

$\lambda_2 = \pm 1, \pm i$. The matrix can first be put in the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & i & 0 & 0 \\ \omega_{31} & \omega_{32} & \tau'' & i\tau'' & 1 & 1 \end{pmatrix}.$$

Applying the first generator to the first two columns in turn and using cols. 3 and 4 we find that we can make

$$\omega_{21}, \omega_{22} = 0, \frac{1}{2}(1+i).$$

The cases where $\omega_{22} = \frac{1}{2}(1+i)$ are eliminated by interchanging the first two columns or else subtracting col.1 from col.2.

Applying the second generator to the third column we find

$$\tau'' = 0, \frac{1}{2}, \frac{1}{2}i, \frac{1}{2}(1+i);$$

and the third of these reduces to the second on multiplying the third row by -1 (by choosing iz as a new variable).

To determine ω_{31} , again apply the first generator to the first column. We find

$$\text{when } \omega_{21} = 0, \quad \omega_{31} = 0, \frac{1}{2}(1+i);$$

$$\text{when } \omega_{21} = \frac{1}{2}(1+i), \quad \omega_{31} = \frac{1}{2}(1+i)\tau'', \frac{1}{2}(1+i)(1 + \tau'').$$

The second generator applied to the first column shows that when $\omega_{21} = \frac{1}{2}(1+i)$ we cannot have $\tau'' = \frac{1}{2}$.

As we have seen, we can make $\omega_{22} = 0$; then, as for ω_{31} , we find that

$$\omega_{32} = 0, \frac{1}{2}(1+i).$$

It is convenient to consider the above possibilities in three groups. First, when $\omega_{21} = \omega_{22} = 0$, $\tau'' = 0, \frac{1}{2}, \frac{1}{2}(1+i)$, we can make

$$\omega_{31} = 0, \frac{1}{2}(1+i), \quad \omega_{32} = 0,$$

giving six matrices.

When $\omega_{21} = \frac{1}{2}(1+i)$, $\omega_{22} = \tau'' = 0$, then if $\omega_{31} = \omega_{32} = 0$, we have one of the last matrices with rows 2 and 3 interchanged, and the other three possibilities reduce to the two where

$$\omega_{31}, \omega_{32} = \frac{1}{2}(1+i), 0; \quad 0, \frac{1}{2}(1+i).$$

Finally, when $\omega_{21} = \frac{1}{2}(1+i)$, $\omega_{22} = 0$, $r'' = \frac{1}{2}(1+i)$, we again reduce the possibilities to two

$$\omega_{31} = \frac{1}{2}, \quad \omega_{32} = 0, \quad \frac{1}{2}(1+i).$$

Interchanging the first two columns of the two matrices of the first set for which $\omega_{31} = \frac{1}{2}(1+i)$, $r'' = 0$, $\frac{1}{2}(1+i)$, we can say

there are 10 matrices admitting the group $G, (XVI) \ S(-x, iy, iz), \ S(-x, y, -z); \lambda_1 = \pm 1, \lambda_2 = \pm 1, \pm i$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & 0 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2}\epsilon(1+i) & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$(i) \ \epsilon = \epsilon_1 = 0; \quad (ii) \ \epsilon = 1, \ \epsilon_1 = 0; \quad (iii) \ \epsilon = \epsilon_1 = 1;$$

$$(iv) \ \begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+i) & 0 & 1 & 1 & 0 & 0 \\ \frac{1}{2}(1+i) & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & 0 & \frac{1}{2} & \frac{1}{2}i & 1 & 1 \end{pmatrix},$$

$$(v) \ \epsilon = 0; \quad (vi) \ \epsilon = 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & 0 & 1 & 1 & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon(1+i) & \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 1 & 1 \end{pmatrix},$$

$$(vii) \ \epsilon = \epsilon_1 = 0; \quad (viii) \ \epsilon = 1, \ \epsilon_1 = 0;$$

$$(ix) \ \epsilon = 0, \ \epsilon_1 = 1; \quad (x) \ \epsilon = \epsilon_1 = 1.$$

4. $G_{16}(XVII) \ S(-x, iy, iz), S(ix, -y, iz); \lambda_1, \lambda_2 = \pm 1, \pm i.$

The matrix first reduces to the form

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & i & 0 & 0 \\ \omega_{31} & \omega_{32} & r'' & ir'' & 1 & i \end{pmatrix}.$$

Applying the multipliers 1, i, -1 to the first column shows that we can take

$$\omega_{22} = i\omega_{21}, \quad \omega_{32} = -\omega_{31}.$$

Again, applying -1, i, i and i, -1, i to the first and third columns we find we can make

$$\omega_{21}, r'' = 0, \frac{1}{2}(1+i);$$

and further, when $\omega_{21}, r'' = 0$ we can make

$$\omega_{31} = 0, \frac{1}{2}(1+i),$$

and when $\omega_{21}, r'' = \frac{1}{2}(1+i)$

$$\omega_{31} = \frac{1}{2}, \frac{1}{2}i.$$

When $\omega_{31} = \frac{1}{2}(1+i)$, we can change ω_{32} from $-\omega_{31}$ to $i\omega_{31}$, and this shows that the three cases when two of $\omega_{21}, \omega_{31}, r''$ are zero and the remaining one equals $\frac{1}{2}(1+i)$ are equivalent, differing only by a permutation of rows and columns.

The case $\omega_{31} = \frac{1}{2}i$ becomes $\omega_{31} = \frac{1}{2}$ on multiplying the last row by -1 and adjusting the last four columns; and we can then make $\omega_{32} = \frac{1}{2}$. Hence

there are 5 matrices admitting the group

$$G_{16}(XVII) \ S(-x, iy, iz), S(ix, -y, iz); \lambda_1, \lambda_2 = \pm 1, \pm i, \text{ namely}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon(-1+i) & \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon(-1+i) & 1 & i \end{pmatrix}, \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+1) & \frac{1}{2}(-1+1) & 1 & 1 & 0 & 0 \\ \frac{1}{2}\epsilon(1+1) & \frac{1}{2}\epsilon(-1+1) & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+1) & \frac{1}{2}(-1+1) & 1 & 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2}(1+1) & \frac{1}{2}(-1+1) & 1 & 1 \end{pmatrix}.$$

In these matrices, as in some for the previous group, some elements have been written $\frac{1}{2}(-1+1)$ to display one column as derived from the preceding one in a particular manner; these elements can easily be transformed to $\frac{1}{2}(1+1)$ with, except in the case of the last matrix, no change to the other elements of the column.

5. $G_{12}(\text{XVIII}) \ S(-x, \epsilon y, -\epsilon^2 z), \ S_1(-x, -y, z); \lambda_1 = \pm 1,$
 $\lambda_2 = \pm 1, \pm \epsilon, \pm \epsilon^2$. The matrix first reduces to the form

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & \epsilon & 0 & 0 \\ \omega_{31} & \omega_{32} & \tau'' & \epsilon^2 \tau'' & 1 & \epsilon^2 \end{pmatrix},$$

and we then find that the ω_{ij} may be reduced to zero.

Applying the multipliers -1, -1, 1 to the third column, we find

$$\tau'' = 0, \frac{1}{2}, \frac{1}{2}\epsilon^2, -\frac{1}{2}\epsilon,$$

and the last two reduce to the second on dividing the third row by ϵ^2 , ϵ respectively and using cols. 5 and 6. Hence

there are 2 matrices admitting the group

$G_{12}(\text{XVIII}) \ S(-x, \epsilon y, -\epsilon^2 z), \ S_1(-x, -y, z); \lambda_1 = \pm 1,$

$\lambda_2 = \pm 1, \pm \epsilon, \pm \epsilon^2$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ 0 & 0 & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon^2 & 1 & \epsilon^2 \end{pmatrix}, \quad \epsilon = 0, 1.$$

6. $G_{12}(XIX) \quad S(\epsilon x, -\epsilon y, -\epsilon z), S(-x, y, -z); \lambda_1 = \pm 1, \pm \epsilon, \pm \epsilon^2$.

$\lambda_2 = \pm \lambda_1$. The discussion follows the lines of the preceding ones and leads to the conclusion

there are 4 matrices admitting the group $G_{12}(XIX)$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon & 1 & \epsilon & 0 & 0 \\ \frac{1}{2}\epsilon\epsilon & \frac{1}{2}\epsilon\epsilon\epsilon & 0 & 0 & 1 & \epsilon \end{pmatrix}, \quad \epsilon, \epsilon^2 = 0, 1;$$

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\epsilon & \frac{1}{2} & \frac{1}{2}\epsilon & 1 & \epsilon \end{pmatrix}.$$

7. $G_{12}(XX) \quad S(\epsilon x, -\epsilon y, -\epsilon z), S(x, \epsilon y, \epsilon^2 z); \lambda_1 = 1, \epsilon, \epsilon^2$.

$\lambda_2 = \pm 1, \pm \epsilon, \pm \epsilon^2$. We find that

there are 2 matrices admitting the group $G_{12}(XX)$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ 0 & 0 & \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1.$$

8. $G_{36}(XXI) \quad S(\epsilon x, -\epsilon y, -\epsilon z), S(-x, \epsilon y, -\epsilon^2 z);$

$\lambda_1, \lambda_2 = \pm 1, \pm \epsilon, \pm \epsilon^2$. Of the two matrices admitting the last group, that for which $\epsilon = 1$ does not admit the present group, whence

there is only one matrix admitting the group $G_{36}(XXI)$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \epsilon \end{pmatrix}.$$

CHAPTER 7. MATRICES ADMITTING REDUCIBLE TERNARY GROUPS
OF THE REGULAR SOLIDS

(a) Dihedral groups

1. Groups to be considered in detail. There are 22 dihedral groups but we do not propose to consider all these at length. We shall restrict ourselves to a full examination of 7 of them; these will provide results necessary for later work and will also illustrate the methods required in dealing with the remainder. For these remaining groups we quote in each instance one matrix which establishes that such groups are possible on a Picard variety. The groups dealt with fully are XXIV, XXV, XXIX, XXX, XXXIV, XXXVI, XXXVII.

2. $G_{32}(\text{XXII})$ $S(x, -y, -z)$, $T(-y, x, z)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \tau & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \tau'' \end{pmatrix} \quad (1)$$

3. $G_{32}(\text{XXIII})$ $S(-x, iy, iz)$, $T(-y, x, z)$. The matrix (1) with $\tau = \tau'' = 1$ admits this group.

4. $G_{24}(\text{XXIV})$ $S(\epsilon x, -\epsilon y, -\epsilon z)$, $T(-y, x, z)$. This group contains $G(\text{XIX})$ as a subgroup. We note that in considering matrices for the present group, we cannot interchange either of the first two rows and the third row since this would alter the form of the substitutions of the group. Also we cannot divide both the first and second rows by arbitrary

constants for the same reason, so that only one of these rows can be taken initially in its simplest form. Thus instead of the 4 matrices found for G(XIX), we must here consider the 5 matrices,

$$\begin{pmatrix} \alpha & \varepsilon\alpha & 0 & 0 & 0 & 0 \\ \frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon\varepsilon & 1 & \varepsilon & 0 & 0 \\ \frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon\varepsilon & 0 & 0 & 1 & \varepsilon \end{pmatrix}, \quad \varepsilon, \varepsilon = 0, 1,$$

$$\begin{pmatrix} \alpha & \varepsilon\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\varepsilon & \frac{1}{2} & \frac{1}{2}\varepsilon & 1 & \varepsilon \end{pmatrix}.$$

For the three matrices in which $\omega_{21} = \omega_{22} = 0$, we have that $\alpha, \varepsilon\alpha$ are expressible integrally in terms of $1, \varepsilon$, and conversely. Hence

$$1 = h\alpha + k\varepsilon\alpha,$$

$$\text{and so} \quad \varepsilon = h\varepsilon\alpha + k\varepsilon^2\alpha = -k\alpha + (h - k)\varepsilon\alpha,$$

$$\text{so that} \quad \begin{vmatrix} h & k \\ -k & h - k \end{vmatrix} = h^2 - hk + k^2 = 1,$$

$$\text{since} \quad h^2 - hk + k^2 \geq (|h| - |k|)^2 \geq 0,$$

and the transformation is unimodular.

The possible values for h, k are

$$\pm 1, 0; \quad 0, \pm 1; \quad 1, 1; \quad -1, -1;$$

leading to the values for α

$$\pm 1, \pm \varepsilon, \pm \varepsilon^2.$$

Two of the corresponding matrices, namely

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ \frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon\varepsilon & \frac{1}{2}\varepsilon & \frac{1}{2}\varepsilon\varepsilon & 1 & \varepsilon \end{pmatrix}, \quad \varepsilon = 0, 1,$$

do admit the group; each of the remaining matrices is either reducible to one or other of these or else can be seen to be inadmissible by consideration of the transform of the first column by the second generating substitution.

For the two matrices in which $\omega_{21} = \frac{1}{2}$, $\omega_{22} = \frac{1}{2}\epsilon$, we find, arguing as earlier, that α can take the values

$$\pm\frac{1}{2}, \pm\frac{1}{2}\epsilon, \pm\frac{1}{2}\epsilon^2.$$

Again two of these matrices, namely

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\epsilon & 1 & \epsilon & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon & 0 & 0 & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1,$$

admit the group, and each of the remainder may be ruled out or is equivalent to one of these. Hence

there are 4 matrices admitting the group

G_{24} (XXIV) $S(\epsilon x, -\epsilon y, -\epsilon z)$, $T(-y, x, z)$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\epsilon & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\epsilon & 1 & \epsilon & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon\epsilon & 0 & 0 & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1.$$

5. G_2 (XXV) $S(x, iy, -iz)$, $T(x, -z, y)$. By Chapter

4.3.(v), the matrix is equivalent to one or other of

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & i & \tau' & i\tau' \\ \omega_{31} & \omega_{32} & \tau' & -i\tau' & -1 & i \end{pmatrix},$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 + i - i\tau' & \tau' & 1 + \tau' & \tau' - 1 \\ \omega_{31} & \omega_{32} & -\tau' & 1 + i - i\tau' & 1 - \tau' & 1 + \tau' \end{pmatrix},$$

and the problem is to determine the ω_{ij} .

For the first matrix the initial reduction is similar to that of $G(V)$, and we find that, by virtue of the matrix admitting the substitution S , we can make

$$\omega_{21} = \frac{1}{2}(1+i)(p + q\tau'), \quad \omega_{22} = \frac{1}{2}(1+i)q'\tau',$$

$$\omega_{31} = \frac{1}{2}(1-i)(p\tau' - q), \quad \omega_{32} = -\frac{1}{2}(1-i)q',$$

with six sets of values for p, q, q' ,

$$\begin{array}{ccccccc} p & 0 & 1 & 1 & 0 & 1 & 0 \\ q & 0 & 0 & 0 & 1 & 1 & 0 \\ q' & 0 & 0 & 1 & 0 & 0 & 1. \end{array}$$

The last set becomes the fourth on interchanging the first two columns of the matrix, so we need only consider the first five sets.

The matrix corresponding to the first set is compound and clearly admits the group. The matrix obtained from the second set does not admit the group. For the transform of the first column by T is

$$(1, -\frac{1}{2}(1-i)\tau', \frac{1}{2}(1+i)),$$

and so we require

$$-\frac{1}{2}(1-i)\tau' = \frac{1}{2}(1+i) + (h+ik) + (l+im)\tau',$$

$$\frac{1}{2}(1+i) = \frac{1}{2}(1-i)\tau' + (h-ik)\tau' - (l-im),$$

with h, k, l, m integral. Eliminating τ' leads to

$$(2h+1)^2 + (2k+1)^2 + (2l+1)^2 + (2m-1)^2 = 0,$$

which is impossible. Thus the second set of parameters, and likewise the third set, is ruled out. The fourth set leads to this same equation with the signs before the units in the first and last brackets changed, and so these parameters are also ruled out.

The two linear equations in τ' implied by the fifth set of parameters are satisfied identically on taking

$$h = m = 0, \quad k = l = -1.$$

Hence we obtain two matrices from the first of the two matrices written at the beginning.

Consider now the second matrix at the beginning. It will be recalled that the bottom right 2×4 matrix is obtained from

$$\begin{pmatrix} 1 & 1 & \tau' & 1\tau' \\ 1-(1+1)\tau' & -1[1-(1+1)\tau'] & (1+1)-1\tau' & -1[(1+1)-1\tau'] \end{pmatrix}$$

by a transformation of the columns. Using this form for the matrix, the values for ω_{ij} are found to be

$$\begin{aligned} \omega_{21} &= \frac{1}{2}p(1+1) + \frac{1}{2}q(1+1)\tau' & , & \quad \omega_{22} = \frac{1}{2}q'(1+1)\tau' & , \\ \omega_{31} &= \frac{1}{2}p[1-1-2\tau'] + \frac{1}{2}q[2-(1+1)\tau'] & , & \quad \omega_{32} = \frac{1}{2}q'[2-(1+1)\tau'] & , \end{aligned}$$

where the values of p , q , q' are

$$\begin{array}{rcccccc} p & 0 & 1 & 1 & 0 & 1 \\ q & 0 & 0 & 0 & 0 & 1 \\ q' & 0 & 0 & 1 & 1 & 0. \end{array}$$

The above sets of values ensure that the matrix admits the substitution S . It may be verified that the transforms of the first two columns under the substitution T are also

expressible in terms of the columns of the matrix, so that all five sets are permissible. Hence

there are 7 matrices admitting the group G_6 (XXV) $S(x, iy, -iz)$, $T(x, -z, y)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i)(1+\tau') & 0 & 1 & i & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1-i)(\tau'-1) & 0 & \tau' & -i\tau' & -1 & 1 \end{pmatrix},$$

$$(i) \epsilon = 0; \quad (ii) \epsilon = 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon(1+i)\tau' & 1+i-i\tau' & \tau' & i+\tau' & \tau'-1 \\ \frac{1}{2}\epsilon[1-i-2\tau'] & \frac{1}{2}\epsilon[2-(1+i)\tau'] & -\tau' & 1+i-i\tau' & 1-\tau' & i+\tau' \end{pmatrix},$$

$$(iii) \epsilon = \epsilon_1 = 0; \quad (iv) \epsilon = 1, \epsilon_1 = 0;$$

$$(v) \epsilon = 0, \epsilon_1 = 1; \quad (vi) \epsilon = \epsilon_1 = 1;$$

$$(vii) \begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+i)(1+\tau') & 0 & 1+i-i\tau' & \tau' & i+\tau' & \tau'-1 \\ \frac{1}{2}[3-i-(3+i)\tau'] & 0 & -\tau' & 1+i-i\tau' & 1-\tau' & i+\tau' \end{pmatrix}.$$

6. G_{12} (XXVI) $S(x, -\epsilon^2 y, -\epsilon z)$, $T(x, -z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & \tau' & \epsilon\tau' \\ 0 & 0 & \tau' & \epsilon^2\tau' & -1 & -\epsilon^2 \end{pmatrix}.$$

7. G_{12} (XXVII) $S(x, -\epsilon^2 y, -\epsilon z)$, $T(-y, x, z)$. The equianharmonic matrix admits this group.

8. G_{24} (XXVIII) $S(-x, \epsilon y, -\epsilon^2 z)$, $T(x, -z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \epsilon \end{pmatrix}.$$

9. $G_{36}(\text{XXIX})$ $S(-x, \epsilon y, -\epsilon^2 z), T(z, y, -x)$. This group contains $G(\text{XX})$ as a subgroup, the only change being that the first two variables are permuted. Hence we have to consider the matrices

$$\begin{pmatrix} \alpha & \epsilon\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 0 & 0 & 1 & \epsilon \end{pmatrix}, \quad \epsilon = 0, 1.$$

When $\epsilon = 0$ we see, arguing as in $G(\text{XXIV})$, that we can take $\alpha = 1$, and the matrix admits the group.

When $\epsilon = 1$, we note that $\alpha, \epsilon\alpha$ can be expressed integrally in terms of $\frac{1}{3}(1-\epsilon), \frac{1}{3}\epsilon(1-\epsilon), 1, \epsilon$ and hence in terms of the first two, and conversely. This leads to six values for α , which may be written

$$\pm\frac{1}{3}(2+\epsilon), \quad \pm\frac{1}{3}(1-\epsilon), \quad \pm\frac{1}{3}(1+2\epsilon).$$

Consider the first of these, $\alpha = \frac{1}{3}(2+\epsilon)$. Then

$$\epsilon\alpha = \frac{1}{3}(2\epsilon+\epsilon^2) = -\frac{1}{3}(1-\epsilon).$$

We have that $(\frac{1}{3}(1-\epsilon), 0, -\frac{1}{3}(2+\epsilon))$, obtained as the transform of the first column by T , must be a period, whence

$$\begin{aligned} -\frac{1}{3}(2+\epsilon) &= -\frac{1}{3}\epsilon(1-\epsilon) + p + q\epsilon, \\ &= -\frac{1}{3}(1+2\epsilon) + p + q\epsilon, \end{aligned}$$

which is impossible with integral p, q . In a similar way the other five possible values for α may be eliminated. Thus

there is only one matrix admitting the group $G_{36}(\text{XXIX})$ $S(-x, \epsilon y, -\epsilon^2 z), T(z, y, -x)$, namely the equianharmonic matrix.

10. $G_{16}(XXX) S(-X, \omega, Y, i\omega, Z), T(X, -Z, Y)$. This group contains $G(XXV)$ as a subgroup so that in the first place the matrix is of one or other of the two forms quoted at the beginning of section 5. Further, the bottom right 2×4 matrix now admits the group of Chapter 4.3.(xiii), and so the possible values for r' are the 2 and 12 values found there for the two matrices respectively. After this, the first two columns of the matrices of genus 3 reduce to the 2 and 5 possibilities found in $G(XXV)$ for the two types of matrix. It remains to investigate whether these matrices now admit the substitution S , i.e. $G(X)$; and the only questionable columns are the first two.

The first matrix ($\epsilon = 0$) and third matrix ($\epsilon = \epsilon, = 0$) are compound and admit the group for all relevant values of r' .

In the cases of the remaining matrices, since we are here concerned with a cyclic group, we can multiply the last two variables by constants thus reducing the matrices to the forms considered in $G(X)$ (at the cost of destroying the particular form of the last four columns which is irrelevant for the moment). Further, we need in fact only consider the elements of the second row since in $G(X)$ the third row imposes no extra conditions.

For the second matrix, therefore, we add cols.3 and 4 to cols.5 and 6 respectively and divide the second row by $1 + r'$. For clarity let us denote the r' of $G(X)$ by $\bar{r}'$ to distinguish it from the r' of the present group. Then when

$\tau' = \omega_0$, we have

$$\overline{\tau'} = \frac{1}{1 + \omega_0} = \frac{\omega_0 - 1}{-1 + 1},$$

and this $\overline{\tau'}$ satisfies the three conditions specified at the end of G(X), and hence the matrix admits G(X) and so also the present group. The same conclusion is reached when $\tau' = i\omega_0$.

The third matrix having already been seen to be admissible, we come to matrix (iv) of G(XXV). Here $\overline{\tau'} = \tau'$. The conditions that $h+k, l, m$ be all odd are satisfied by the last four of the twelve values for τ' , but not by the first eight.

For matrix (v), $\overline{\tau'}$ is the reciprocal of τ' . Hence

$$\overline{\tau'} = \frac{1 + im}{\omega_0 + h + ik} = \frac{\omega_0 + H + iK}{L + iM},$$

where $H = -h, \quad L = \frac{1}{1^2 + m^2} [1(k^2 - h^2) + m(1 - 2hk)],$
 $K = -k, \quad M = \frac{1}{1^2 + m^2} [-m(k^2 - h^2) + 1(1 - 2hk)].$

Here $H+K, L, M$ are odd for the middle four sets of values of τ' only.

For matrix (vi), we have $\overline{\tau'} = \tau'$. Hence, as in G(X), none of these matrices is admissible.

For matrix (vii)

$$\overline{\tau'} = \frac{1}{1 + \tau'} = \frac{1 + im}{\omega_0 + (h+1) + i(k+m)} = \frac{\omega_0 + H + iK}{L + iM},$$

where

$$H = -(h+1), \quad L = \frac{1[(k+m)^2 - (h+1)^2] + m[1 - 2(h+1)(k+m)]}{1^2 + m^2},$$

$$K = -(k+m), \quad M = \frac{m[-(k+m)^2 + (h+1)^2] + 1[1 - 2(h+1)(k+m)]}{1^2 + m^2}.$$

Here $H+K$, L , M are odd for the first four values of τ' only.

Thus in all we obtain $2+2+12+4+4+4 = 28$ matrices, and

there are 28 matrices admitting the group

$G_{16}(\text{XXX})$ $S(-x, \omega, y, i\omega, z)$, $T(x, -z, y)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i)(1+\tau') & 0 & 1 & i & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1-i)(\tau'-1) & 0 & \tau' & -i\tau' & -1 & i \end{pmatrix},$$

$$\tau' = \omega, i\omega; \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & 0 & 1+i-i\tau' & \tau' & i+\tau' & \tau'-1 \\ \frac{1}{2}\epsilon(1-i-2\tau') & 0 & -\tau' & 1+i-i\tau' & 1-\tau' & i+\tau' \end{pmatrix},$$

$$\tau' = \frac{\pm\omega+1}{1+i}, \frac{\pm i\omega+1}{1+i}; \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}\epsilon(1+i)\tau' & 1+i-i\tau' & \tau' & i+\tau' & \tau'-1 \\ 0 & \frac{1}{2}\epsilon[2-(1+i)\tau'] & -\tau' & 1+i-i\tau' & 1-\tau' & i+\tau' \end{pmatrix},$$

$$\tau' = \pm\omega+1, \pm\omega-1; \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i)(1+\tau') & 0 & 1+i-i\tau' & \tau' & i+\tau' & \tau'-1 \\ \frac{1}{2}\epsilon[3-i-(3+i)\tau'] & 0 & -\tau' & 1+i-i\tau' & 1-\tau' & i+\tau' \end{pmatrix},$$

$$\tau' = \pm i\omega, \pm i\omega+(1-i); \quad \epsilon = 0, 1.$$

11. $G_{16}(\text{XXXI})$ $S(ix, \omega, y, -\omega, z), T(x, -z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \omega & i\omega \\ 0 & 0 & -1 & 1 & i\omega & -\omega \end{pmatrix}.$$

12. $G_{24}(\text{XXXII})$ $S(-x, i\epsilon y, i\epsilon^2 z), T(x, -z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 1 & i\epsilon \\ 0 & 0 & 1 & i\epsilon^2 & -1 & -\epsilon^2 \end{pmatrix}.$$

13. $G_{24}(\text{XXXIII})$ $S(\epsilon x, i\epsilon y, -i\epsilon z), T(x, -z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i-i\epsilon & -\epsilon & 1-\epsilon & -\epsilon-1 \\ 0 & 0 & \epsilon & 1+i-i\epsilon & 1+\epsilon & 1-\epsilon \end{pmatrix}.$$

14. $G_{16}(\text{XXXIV})$ $S(-x, iy, iz), S(-x, y, -z), T(x, -z, y)$.

This group contains $G(\text{XVI})$ as a subgroup and so we must consider the 10 matrices found there, but with the second or third row multiplied through by a constant. Since y and z appear on an equal footing in the present group, the interchange of the second and third rows made at one point in the reduction of $G(\text{XVI})$ is still valid and does not call for any further matrix to be considered.

For the first four matrices, in which $\omega_{33} = \omega_{34} = 0$, we see, as in $G(\text{XXIX})$, that we can take the multiplier to

be unity, and the matrices as written in G(XVI) admit the group.

For the next two matrices of G(XVI), we insert a multiplier α in the third row and find that $\frac{1}{2}\alpha$, $\frac{1}{2}1\alpha$ are expressible integrally in terms of 1, i , and conversely; whence we obtain

$$\alpha = \pm 2, \pm 2i.$$

The matrices arising on choosing the negative signs are equivalent to those arising from the positive, whence we need only take the latter.

For the last four matrices of G(XVI), we multiply the second row through by α ; and then find we must have

$$\alpha = \pm \frac{1}{2}(1+i), \pm \frac{1}{2}(-1+i).$$

The corresponding values for ω_{21} are

$$\pm \frac{1}{2}\epsilon i, \mp \frac{1}{2}\epsilon, \quad \epsilon = 0, 1.$$

For all four values of α the last four columns can be rearranged to make the bottom right 2×4 matrix

$$\begin{pmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 0 & 0 \\ \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 1 & 1 \end{pmatrix};$$

and then the values for ω_{21} reduce to only

$$\frac{1}{2}\epsilon i, \frac{1}{2}\epsilon.$$

This gives 8 matrices all of which admit the group. Hence there are 14 matrices admitting the group

$G_{16}(\text{XXXIV})$ $S(-x, iy, iz)$, $S(-x, y, -z)$, $T(x, -z, y)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon\epsilon(1+i) & 0 & 1 & i & 0 & 0 \\ 0 & \frac{1}{2}\epsilon(1+i) & 0 & 0 & 1 & 1 \end{pmatrix}, \epsilon, \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+i) & 0 & 1 & 1 & 0 & 0 \\ \frac{1}{2}(1+i) & 0 & 0 & 0 & 1 & 1 \end{pmatrix};$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \epsilon(1+i) & 0 & 1 & 1 & 2 & 2i \end{pmatrix}, \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ \epsilon(1+i) & 0 & 1 & 1 & 2 & 2i \end{pmatrix}, \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon & 0 & \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 0 & 0 \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon(1+i) & \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 1 & 1 \end{pmatrix}, \quad \epsilon, \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}i & 0 & \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 0 & 0 \\ \frac{1}{2} & \frac{1}{2}\epsilon(1+i) & \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 1 & 1 \end{pmatrix}, \quad \epsilon = 0, 1.$$

15. $G_6(\text{XXXV})$ $S(x, \epsilon y, \epsilon^2 z), T(-x, z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & \tau' & \epsilon\tau' \\ 0 & 0 & 1 & \epsilon^2 & \tau' & \epsilon^2\tau' \end{pmatrix}.$$

16. $G_{10}(\text{XXXVI})$ $S(x, \epsilon y, \epsilon^2 z), T(y, x, -z)$. As in $G(\text{XV})$, the matrix may be put in the form

$$\begin{pmatrix} \alpha & \epsilon\alpha & 0 & 0 & 0 & 0 \\ \tau & \epsilon\tau & 1 & \epsilon & 0 & 0 \\ \tau' & \epsilon\tau' & \tau'' & \epsilon\tau'' & 1 & \epsilon \end{pmatrix},$$

and, changing if necessary the signs of the first two rows and columns, we can ensure that

$$\tau, \tau^n = 0, \frac{1}{3}(1-\epsilon).$$

When $\tau = 0$, we must have

$$\alpha = \pm 1, \pm \epsilon, \pm \epsilon^2,$$

and these values all reduce to $\alpha = 1$, with possibly a new value for τ' .

When $\tau = \frac{1}{3}(1-\epsilon)$, then, by an argument similar to that of G(XXIX), we have the possibilities

$$\alpha = \pm \frac{1}{3}(1-\epsilon), \pm \frac{1}{3}(1+2\epsilon), \pm \frac{1}{3}(2+\epsilon),$$

$$\text{i.e.} \quad = \pm \frac{1}{3}(1-\epsilon), \pm \frac{1}{3}\epsilon(1-\epsilon), \pm \frac{1}{3}\epsilon^2(1-\epsilon).$$

When α takes one of the middle two values, we multiply all three rows of the matrix by ϵ^2 . This makes $\alpha = \pm \frac{1}{3}(1-\epsilon)$. The last two columns at once reduce to the original form. Interchanging cols. 3 and 4, adding the new col. 3 to col. 4, and then changing the sign of col. 4 reduces cols. 3 and 4 to the original form. Finally, adding the new cols. 3 and 4 to cols. 1 and 2 respectively, restores the form of ω_{21}, ω_{22} , i.e. brings them to $\frac{1}{3}(1-\epsilon), \frac{1}{3}\epsilon(1-\epsilon)$. Corresponding operations apply if α has one or other of the last two values. Thus we need only consider

$$\alpha = \pm \frac{1}{3}(1-\epsilon).$$

In all the above, the last two elements of cols. 1 and 2 are left in the form $\tau', \epsilon\tau'$. Consider the possibilities for τ' . When $\tau = 0$ (so that $\alpha = 1$), we can have

$$\tau' = 0, \pm \frac{1}{3}(1-\epsilon).$$

Applying the substitution T, we see that

$$\tau'' = -\tau',$$

and thus there are two matrices for which $\tau = 0$, namely the equianharmonic matrix and

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3}\varepsilon(1-\varepsilon) & \frac{1}{3}(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & 1 & \varepsilon \end{pmatrix}.$$

When $\tau = \frac{1}{3}(1-\varepsilon)$, then as indicated in considering $G(XV)$, which is a subgroup of the present group, we find values for τ' depending on the value of τ'' as follows

$$\text{if } \tau'' = 0, \quad \tau' = 0, \pm \frac{1}{3}(1-\varepsilon);$$

$$\text{if } \tau'' = \frac{1}{3}(1-\varepsilon), \quad \tau' = \frac{1}{3}, \frac{1}{3}\varepsilon, \frac{1}{3}\varepsilon^2.$$

The case $\tau'' = 0$, $\tau' = -\frac{1}{3}(1-\varepsilon)$ reduces to $\tau'' = 0$, $\tau' = \frac{1}{3}(1-\varepsilon)$ on changing the signs of the third row and of the last two columns. When $\tau'' = \frac{1}{3}(1-\varepsilon)$, $\tau' = \frac{1}{3}\varepsilon$, we multiply the third row by ε^2 . This makes $\omega_{31} = \frac{1}{3}$, $\omega_{32} = \frac{1}{3}\varepsilon$. Cols.5 and 6 at once reduce to their original form; and cols.3 and 4 are so reduced using cols.5 and 6. A similar argument holds when $\tau' = \frac{1}{3}\varepsilon^2$. Thus when $\tau = \frac{1}{3}(1-\varepsilon)$ we have at the moment the six possibilities

$$\alpha = \pm \frac{1}{3}(1-\varepsilon) \text{ with } \tau' = \tau'' = 0; \text{ or } \tau' = \frac{1}{3}(1-\varepsilon), \tau'' = 0; \\ \text{or } \tau' = \frac{1}{3}, \tau'' = \frac{1}{3}(1-\varepsilon).$$

These matrices admit the substitutions of $G(XV)$; it remains to find whether they admit the generator T of the present group. It may be verified that the two matrices corresponding to

$\alpha = \tau' = \frac{1}{3}(1-\epsilon)$, $\tau'' = 0$; $\alpha = \frac{1}{3}(1-\epsilon)$, $\tau' = \frac{1}{3}$, $\tau'' = \frac{1}{3}(1-\epsilon)$, do not admit the group since the transform of the first columns under T are not expressible in terms of the columns of the matrix. The other four matrices do admit the group. Hence

there are 6 matrices admitting the group

$G_{18}(\text{XXXVI})$ $S(x, \epsilon y, \epsilon^2 z)$, $T(y, x, -z)$, namely

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & 0 & 0 \\ -\frac{1}{3}\epsilon(1-\epsilon) & -\frac{1}{3}\epsilon\epsilon(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix},$$

$$(i) \epsilon = 0; \quad (ii) \epsilon = 1;$$

$$\begin{pmatrix} \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \epsilon \end{pmatrix},$$

$$(iii) \epsilon = 1; \quad (iv) \epsilon = -1;$$

$$(v) \begin{pmatrix} -\frac{1}{3}(1-\epsilon) & -\frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 1 & \epsilon \end{pmatrix};$$

$$(vi) \begin{pmatrix} -\frac{1}{3}(1-\epsilon) & -\frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3} & \frac{1}{3}\epsilon & \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix}.$$

17. $G_{18}(\text{XXXVII})$ $S(x, iy, -iz)$, $T(-x, z, y)$. The last four elements of the first row of the Riemann matrix may be reduced to zero and then, by virtue of Chapter 4.3.(xiv), there are four possible forms for the bottom right 2×4 matrix.

We have thus to consider matrices

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \omega_{21} & \omega_{22} & 1 & i & \tau' & i\tau' \\ \omega_{31} & \omega_{32} & \omega_{33} & -i\omega_{33} & \omega_{35} & -i\omega_{35} \end{pmatrix},$$

where the four possible pairs of values for ω_{33} , ω_{35} are

$$i\tau', i; 1, \tau'; i, i\tau'; 1, i\tau'.$$

Applying to the last two elements of each of the first two columns considerations similar to those in $G(V)$, we find we can make, by virtue of the substitution S ,

$$\omega_{21} = \frac{1}{2}(1+i)(p + q\tau'), \quad \omega_{22} = \frac{1}{2}(1+i)q'\tau',$$

$$\omega_{31} = \frac{1}{2}(1-i)(p\omega_{33} + q\omega_{35}), \quad \omega_{32} = \frac{1}{2}(1-i)q'\omega_{35},$$

$p, q, q' = 0, 1$. Taking the transform of the first column under the substitution T ,

$$(-1, \frac{1}{2}(1-i)(p\omega_{33} + q\omega_{35}), \frac{1}{2}(1+i)(p + q\tau'))$$

must be a period.

When $\omega_{33} = i\tau'$, $\omega_{35} = i$, we have therefore

$$\frac{1}{2}(1+i)(p\tau' + q) = -\frac{1}{2}(1+i)(p + q\tau') + (h+ik) + (1+im)\tau',$$

$$\text{or} \quad \frac{1}{2}(1+i)(p + q)(1 + \tau') = (h+ik) + (1+im)\tau'$$

$$= (ih+k)\tau' + (1l+m),$$

the last expression being obtained from the elements of the third row. These equations must be satisfied identically, for otherwise the last four columns of the Riemann matrix would be linearly dependent. Hence we must have

$$p = q = 0 \text{ or } 1.$$

Similar considerations applied to the second column show that $q' = 0$.

For the second and third pairs of values for ω_{33} , ω_{35} , we find that the substitution T imposes no condition on the values of p, q, q'. Then as in G(VI), these values for p, q, q' may be reduced to the three sets

$$p = q = q' = 0; \quad p = 1, q = q' = 0; \quad p = q' = 1, q = 0.$$

In making this reduction it is necessary in some cases to divide the second and third rows by τ' , thus making $\frac{1}{\tau'}$ a new τ' ; or, again, to add cols.5 and 6 to cols.3 and 4 respectively and then divide the last two rows by $1 + \tau'$.

Again for the fourth matrix the substitution T imposes no further condition. Here, however, the bottom right 2x4 matrix is not symmetrical in the 1 and τ' and the reductions indicated for the last two matrices cannot be carried out. The only reduction possible is to interchange the first two columns when $p = q' = 0, q = 1$, thus obtaining $p = q = 0, q' = 1$; there are then 5 matrices remaining. In all we have $2+3+3+5 = 13$ matrices, and

there are 13 matrices admitting the group $G_4(\text{XXXVII})$ $S(x, 1y, -1z)$, $T(-x, z, y)$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i)(1+\tau') & 0 & 1 & 1 & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1+i)(1+\tau') & 0 & i\tau' & \tau' & 1 & 1 \end{pmatrix}, \quad \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+i) & \frac{1}{2}\epsilon\epsilon(1+i)\tau' & 1 & 1 & \tau' & i\tau' \\ \frac{1}{2}\epsilon(1-i) & \frac{1}{2}\epsilon\epsilon(1-i)\tau' & 1 & -1 & \tau' & -i\tau' \end{pmatrix}, \quad \epsilon, \epsilon, = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+1) & \frac{1}{2}\epsilon\epsilon(1+1)\tau' & 1 & 1 & \tau' & 1\tau' \\ \frac{1}{2}\epsilon(1+1) & \frac{1}{2}\epsilon\epsilon(1+1)\tau' & 1 & 1 & 1\tau' & \tau' \end{pmatrix}, \quad \epsilon, \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}\epsilon(1+1) & \frac{1}{2}\epsilon(1+1)\tau' & 1 & 1 & \tau' & 1\tau' \\ \frac{1}{2}\epsilon(1-1) & \frac{1}{2}\epsilon(1+1)\tau' & 1 & -1 & 1\tau' & \tau' \end{pmatrix}, \quad \epsilon, \epsilon = 0, 1;$$

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ \frac{1}{2}(1+1)(1+\tau') & 0 & 1 & 1 & \tau' & 1\tau' \\ \frac{1}{2}(1-1) + \frac{1}{2}(1+1)\tau' & 0 & 1 & -1 & 1\tau' & \tau' \end{pmatrix}.$$

18. $G_{12}(\text{XXXVIII})$ $S(x, -\epsilon^2 y, -\epsilon z), T(-x, z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \epsilon & \tau' & \epsilon\tau' \\ 0 & 0 & 1 & \epsilon^2 & \tau' & \epsilon^2\tau' \end{pmatrix}.$$

19. $G_{36}(\text{XXXIX})$ $S(-x, \epsilon y, -\epsilon^2 z), T(z, -y, x)$. The equianharmonic matrix admits this group.

20. $G_{16}(\text{XL})$ $S(-x, \omega y, i\omega z), T(-x, z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \tau' & 1\tau' \\ 0 & 0 & 1\tau' & \tau' & 1 & 1 \end{pmatrix}$$

where $\tau' = \frac{\omega_2 + h(1+i)}{1+im}$ with $2h^2 - 1 = l^2 + m^2$,

h, l, m being integral.

21. G_{24} (XLI) $S(\epsilon x, i\epsilon y, -i\epsilon z), T(-x, z, y)$. The following matrix admits this group.

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \tau' & i\tau' \\ 0 & 0 & 1 & -1 & \tau' & -i\tau' \end{pmatrix}$$

where $\tau' = \frac{\epsilon+h}{1}$ with $a1 = -h^2 + h - 1,$

$a, h, 1$ being integral.

22. G_{24} (XLII) $S(-x, i\epsilon y, i\epsilon^2 z), T(ix, \omega, z, \omega, y)$.

The following matrix admits this group.

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \tau' & i\tau' \\ 0 & 0 & i\omega_s \tau' & -\omega_s \tau' & -\omega_s & -i\omega_s \end{pmatrix}$$

where $\tau' = \frac{\epsilon+h+ik}{1+im}$

with $h^2 - k^2 - h + 1 + 2lm = 0, \quad k - 2hk + l^2 - m^2 = 0,$
 h, k, l, m being integral.

23. G_{24} (XLIII) $S(-x, iy, iz), T(\omega, y, \omega, x, iz)$. The following matrix admits this group.

$$\begin{pmatrix} \omega_s & i\omega_s & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

This concludes the discussion of the dihedral groups and we turn to the reducible tetrahedral groups, the second of our three types of reducible ternary group.

(b) Reducible tetrahedral groups

24. $G_{24}(XLIV) S(x, iy, -iz), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$.

This group and the next, but not the final two of this type, will be considered in detail.

The present group contains $G(XXV)$ as a subgroup and hence we have to consider the 7 matrices found there. Further, the bottom right 2×4 matrix must admit the tetrahedral group of Chapter 4.3.(vii), whence only the last 5 matrices of $G(XXV)$ are possible. We therefore investigate whether the first two columns of these matrices admit the present group.

The first of these matrices, (iii), clearly does admit the group. In the case of matrix (iv), applying the substitution U to the first column, $(1, \frac{1}{2}(1+i)(1-\tau'), -\frac{1}{2}(1-i)(1+\tau'))$ must be a period, whence

$$\begin{aligned} \frac{1}{2}(1+i)(1-\tau') &= \frac{1}{2}(1+i) + a(1+i-1\tau') + b\tau' + c(1+\tau') + d(\tau'-1) \\ -\frac{1}{2}(1-i)(1+\tau') &= \frac{1}{2}(1-i-2\tau') - a\tau' + b(1+i-1\tau') + c(1-\tau') + d(1+\tau'). \end{aligned}$$

These equations are satisfied on taking

$$a = -b = -c = d = \frac{1}{2}.$$

It follows that the equations cannot also be satisfied by integral values of a, b, c, d , for otherwise the columns of the bottom right 2×4 matrix would be linearly dependent. Hence matrix (iv) is ruled out.

Similar reasoning eliminates the remaining matrices, and so

there is only one matrix admitting the group

G_{24} (XLIV) $S(x, iy, -iz)$, $U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$,
namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i-i\tau' & \tau' & 1+\tau' & \tau'-1 \\ 0 & 0 & -\tau' & 1+i-i\tau' & 1-\tau' & 1+\tau' \end{pmatrix}.$$

25. G_{24} (XLV) $S(-x, y, -z)$, $U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$.

This group contains G (XXXIV) as a subgroup and so we must examine the 14 matrices found there to see if they admit the generator U .

Applying U to the third column of the first four matrices we have that $(0, \frac{1}{2}(1+i), -\frac{1}{2}(1-i))$ should be a period; this is not so and hence these matrices are excluded. Similarly, the next two pairs of matrices are excluded. On the other hand, we verify that the bottom right 2×4 matrix common to the last six matrices does admit U . This 2×4 matrix is

$$\begin{pmatrix} \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 0 & 0 \\ \frac{1}{2}(1+i) & \frac{1}{2}(-1+i) & 1 & 1 \end{pmatrix}$$

which, on multiplying both rows by $1+i$ and carrying out a substitution on the columns, becomes

$$\begin{pmatrix} 0 & 1-i & 1 & -1 \\ -(1-i) & 0 & 1 & 1 \end{pmatrix}.$$

In this form, the matrix is obtained from that found for the previous tetrahedral group, G (XLIV), which is a subgroup of the present group, by putting $\tau' = 1-i$. It follows that the remaining elements of the second and third rows of the

3x6 matrix must be zero, and so

there is only one matrix admitting the group

$G_{24}(XLV) S(-x, y, -z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$, namely

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-i & 1 & -1 \\ 0 & 0 & -(1-i) & 0 & 1 & 1 \end{pmatrix}.$$

26. $G_{24}(XLVI) S(\epsilon x, i\epsilon y, -i\epsilon z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$.

Matrices admitting this group must also admit $G(XLIV)$ and $G(XXXIII)$ which are subgroups. In particular, it may be verified that the following matrix admits the group.

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i+i\epsilon & -\epsilon & 1-\epsilon & -\epsilon-1 \\ 0 & 0 & \epsilon & 1+i+i\epsilon & 1+\epsilon & 1-\epsilon \end{pmatrix}.$$

27. $G_{24}(XLVII) S(x, iy, -iz), U(\epsilon x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$.

The following matrix admits this group.

$$\begin{pmatrix} 1 & \epsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & \epsilon+i\epsilon^2 & -\epsilon^2+i\epsilon \\ 0 & 0 & \epsilon+i\epsilon^2 & \epsilon^2-i\epsilon & -1 & 1 \end{pmatrix}.$$

(c) Reducible octahedral groups

28. $G_{24}(XLVIII) S(-ix, iy, z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$

This group contains $G(XLV)$ as a subgroup and so there is only one type of matrix to be considered. This matrix must now admit the substitution S above, and hence in the matrix

we can make $\tau = 1$; it is then at once verified that the resulting matrix admits the group. This matrix is

$$\begin{pmatrix} 1 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-i & 1 & -1 \\ 0 & 0 & -(1-i) & 0 & 1 & 1 \end{pmatrix}.$$

29. $G_{22}(\text{XLIX})$ $S(-x, i\omega, y, \omega, z)$, $U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$

The discussion of this group, which will not be pursued, follows on from the existence of the subgroups $G(\text{XXX})$ and $G(\text{XLIV})$. In particular, the following matrix admits the group.

$$\begin{pmatrix} 1 & \tau & 0 & 0 & 0 & 0 \\ 0 & 0 & 1+i+\omega, & i\omega, & i(1+\omega,) & i\omega, -1 \\ 0 & 0 & -i\omega, & 1+i+\omega, & 1-i\omega, & i(1+\omega,) \end{pmatrix}.$$

CHAPTER 8. MATRICES ADMITTING MONOMIAL AND PRIMITIVE GROUPS

(a) Monomial groups

1. Group to be considered in detail. There are twelve monomial groups, $G(L)$ - $G(LXI)$. We shall consider in detail $G(LX)$ which illustrates the type and which is also required for the discussion of two primitive groups. We quote for each of the other groups one matrix which admits the group.

2. $G_{21}(L)$ $S(\omega_7 x, \omega_7^2 y, \omega_7^4 z), P(y, z, x)$. The following matrix admits this group.

$$\begin{pmatrix} \omega_7 & \omega_7^2 & \omega_7^3 & \omega_7^4 & \omega_7^5 & \omega_7^6 \\ \omega_7^2 & \omega_7^4 & \omega_7^6 & \omega_7 & \omega_7^3 & \omega_7^5 \\ \omega_7^4 & \omega_7 & \omega_7^5 & \omega_7^2 & \omega_7^6 & \omega_7^3 \end{pmatrix}.$$

3. Groups $G(LI)$ - $G(LIX)$. These are:

- $G_{12}(LI) \quad S(x, -y, -z), \quad P(y, z, x);$
 $G_{27}(LII) \quad S(x, \varepsilon y, \varepsilon^2 z), \quad P(y, z, x);$
 $G_{48}(LIII) \quad S(x, iy, -iz), \quad P(y, z, x);$
 $G_{36}(LIV) \quad S(\varepsilon x, -\varepsilon y, -\varepsilon z), \quad P(y, z, x);$
 $G_{108}(LV) \quad S(-x, \varepsilon y, -\varepsilon^2 z), \quad P(y, z, x);$
 $G_{24}(LVI) \quad T(-y, x, z), \quad T(x, -z, y);$
 $G_{96}(LVII) \quad S(-x, iy, iz), \quad T(-y, x, z), \quad T(x, -z, y);$
 $G_{72}(LVIII) \quad S(\varepsilon x, \varepsilon y, \varepsilon z), \quad T(-y, x, z), \quad T(x, -z, y);$
 $G_{216}(LIX) \quad S(x, \varepsilon y, \varepsilon^2 z), \quad T(-y, x, z), \quad T(x, -z, y).$

Each of these groups is admitted by one of the matrices

$$\begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \varepsilon \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

$$4. \quad G_{54}(LX) \quad S(x, \varepsilon y, \varepsilon^2 z), T(y, x, -z), T(-x, z, y).$$

The first two substitutions generate the group $G(XXXVI)$, and so we must consider the six matrices found there. In view of the existence of the third generator for the present group, either the last row or else the first two rows of the matrices must be multiplied through by a constant. Thus the first two matrices we take in the forms

$$\begin{pmatrix} \tau & \varepsilon\tau & 0 & 0 & 0 & 0 \\ 0 & 0 & \tau & \varepsilon\tau & 0 & 0 \\ -\frac{1}{3}\varepsilon(1-\varepsilon) & -\frac{1}{3}\varepsilon\varepsilon(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & \frac{1}{3}\varepsilon\varepsilon(1-\varepsilon) & 1 & \varepsilon \end{pmatrix}, \varepsilon = 0, 1.$$

When $\varepsilon = 0$ then, as in $G(XXIV)$, we find the possible values

$$\tau = \pm 1, \pm \varepsilon, \pm \varepsilon^2,$$

and all these cases reduce to $\tau = 1$.

When $\varepsilon = 1$ then, as in $G(XXIX)$, we find the possible values

$$\tau = \pm \frac{1}{3}(1-\varepsilon), \pm \frac{1}{3}(1+2\varepsilon), \pm \frac{1}{3}(2+\varepsilon),$$

$$\text{i.e.} \quad \tau = \pm \frac{1}{3}(1-\varepsilon), \pm \frac{1}{3}\varepsilon(1-\varepsilon), \pm \frac{1}{3}\varepsilon^2(1-\varepsilon).$$

For the third and fourth values of τ we multiply each row of the matrix by ε^2 , i.e. each variable by ε . The matrices are then reducible to those for the first two values of τ ; and similarly for the last two values of τ . Consider then $\tau = \pm \frac{1}{3}(1-\varepsilon)$. Under the third generator, the first column transforms to $(-\tau, -\frac{1}{3}(1-\varepsilon), 0)$. So we must have

$$-\frac{1}{3}(1-\varepsilon) = a\tau + b\varepsilon\tau,$$

$$0 = \frac{1}{3}(1-\varepsilon) + a\frac{1}{3}(1-\varepsilon) + b\frac{1}{3}\varepsilon(1-\varepsilon) + c + d\varepsilon.$$

These equations can be satisfied when $\tau = \frac{1}{3}(1-\epsilon)$, but not when $\tau = -\frac{1}{3}(1-\epsilon)$, and it is easily verified that the matrix arising from the first value of τ does admit the group.

We turn now to the second two matrices of G(XXXVI), which we take in the form

$$\begin{pmatrix} \frac{1}{3}\epsilon(1-\epsilon) & \frac{1}{3}\epsilon\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & \tau & \epsilon\tau \end{pmatrix}, \quad \epsilon = \pm 1.$$

These must admit the substitution $(-z, -y, -x)$, which is the transform of the second generator by the third. Hence, arguing as for the first matrix of this group, we can take

$$\tau = \frac{1}{3}(1-\epsilon).$$

But then $(\frac{1}{3}(1-\epsilon), 0, 0)$ must be a period; and since this cannot be expressed integrally in terms of the columns of the matrix, these two matrices are excluded.

The next matrix of G(XXXVI) we take in the form

$$\begin{pmatrix} -\frac{1}{3}(1-\epsilon) & -\frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}\tau(1-\epsilon) & \frac{1}{3}\epsilon\tau(1-\epsilon) & 0 & 0 & \tau & \epsilon\tau \end{pmatrix}.$$

Possible values for τ may be reduced to ± 1 , and of these only $\tau = -1$ is admissible.

The final matrix of G(XXXVI) we consider in the form

$$\begin{pmatrix} -\frac{1}{3}(1-\epsilon) & -\frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}\tau & \frac{1}{3}\epsilon\tau & \frac{1}{3}\tau(1-\epsilon) & \frac{1}{3}\epsilon\tau(1-\epsilon) & \tau & \epsilon\tau \end{pmatrix}.$$

Admissible values for τ are found to be

$$\tau = -(1-\varepsilon), -\varepsilon(1-\varepsilon), -\varepsilon^2(1-\varepsilon),$$

and for each of these values the final four columns of the matrix can be reduced to the form shown in (iv)-(vi) below.

Hence

there are 6 matrices admitting the group

$G_{54}(LX)$ $S(x, \varepsilon y, \varepsilon^2 z)$, $T(y, x, -z)$, $T(-x, z, y)$, namely

$$(i) \begin{pmatrix} 1 & \varepsilon & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \varepsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \varepsilon \end{pmatrix},$$

$$(ii) \begin{pmatrix} \frac{1}{3}(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & 0 & 0 \\ -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3}\varepsilon(1-\varepsilon) & \frac{1}{3}(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & 1 & \varepsilon \end{pmatrix},$$

$$(iii) \begin{pmatrix} -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3}\varepsilon(1-\varepsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & 1 & \varepsilon & 0 & 0 \\ -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3}\varepsilon(1-\varepsilon) & 0 & 0 & 1 & \varepsilon \end{pmatrix},$$

$$\begin{pmatrix} -\frac{1}{3}(1-\varepsilon) & -\frac{1}{3}\varepsilon(1-\varepsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\varepsilon) & \frac{1}{3}\varepsilon(1-\varepsilon) & 1 & \varepsilon & 0 & 0 \\ -\frac{1}{3}\varepsilon(1-\varepsilon) & -\frac{1}{3}\varepsilon^2(1-\varepsilon) & 1 & \varepsilon & 1-\varepsilon & \varepsilon(1-\varepsilon) \end{pmatrix},$$

$$(iv) \tau = 1; \quad (v) \tau = \varepsilon; \quad (vi) \tau = \varepsilon^2.$$

5. $G_{54}(LXI)$ $T(\omega_2 y, \omega_2 x, iz)$, $T(ix, \omega_2 z, \omega_2 y)$. The

following matrix admits this group.

$$\begin{pmatrix} \omega_2 & i\omega_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega_2 & i\omega_2 \end{pmatrix}.$$

(b) Primitive groups

$$6. \quad G_{102}(\text{LXII}) \quad S(x, \epsilon y, \epsilon^2 z), T(-y, -x, -z), \\ V\left(\frac{1}{\epsilon - \epsilon^2}(x + y + z), \frac{1}{\epsilon - \epsilon^2}(x + \epsilon y + \epsilon^2 z), \frac{1}{\epsilon - \epsilon^2}(x + \epsilon^2 y + \epsilon z)\right).$$

The G_{54} generated by S , T and $T(-x, -z, -y)$ is, as was noted in Chapter 3.16, a subgroup of the present group. This subgroup differs from $G_{54}(\text{LX})$ only in choice of variable, and the latter assumes the form of the present G_{54} on choosing $-y$ as a new second variable. For the G_{102} , therefore, we must consider the 6 matrices found for $G(\text{LX})$, but the signs of the elements of the second row must be changed.

In finding if these matrices admit the substitution V it is convenient to write the coefficient $\frac{1}{\epsilon - \epsilon^2}$ in the alternative form $\frac{1}{3}\epsilon(\epsilon - 1)$. Applying V to the first column of the first matrix we see that for this matrix to admit the group

$$\left(\frac{1}{3}\epsilon(\epsilon - 1), \frac{1}{3}\epsilon(\epsilon - 1), \frac{1}{3}\epsilon(\epsilon - 1)\right)$$

must be a period; and as this is not so the matrix is ruled out.

For the second and third matrices it may be verified that the transform of each column is again a period, and both these matrices admit the group.

For each of the three remaining matrices, consideration of the transforms of the columns (the first and fifth columns suffice) shows that the matrix is inadmissible.

Hence, (after changing the signs of some columns), we have

there are 2 matrices admitting the group

$$G_{108}(\text{LXII}) \quad S(x, \epsilon y, \epsilon^2 z), \quad T(-y, -x, -z),$$

$$V\left(\frac{1}{\epsilon - \epsilon^2}(x + y + z), \frac{1}{\epsilon - \epsilon^2}(x + \epsilon y + \epsilon^2 z), \frac{1}{\epsilon - \epsilon^2}(x + \epsilon^2 y + \epsilon z)\right),$$

namely

$$(i) \quad \begin{pmatrix} \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 \\ -\frac{1}{3}(1-\epsilon) & -\frac{1}{3}\epsilon(1-\epsilon) & -\frac{1}{3}(1-\epsilon) & -\frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon \end{pmatrix},$$

$$(ii) \quad \begin{pmatrix} \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 1 & \epsilon & 0 & 0 \\ \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 1 & \epsilon \end{pmatrix}.$$

7. $G_{216}(\text{LXIII})$ S, T, V as in the previous group.

$$W\left(\frac{1}{\epsilon - \epsilon^2}(x + y + \epsilon^2 z), \frac{1}{\epsilon - \epsilon^2}(x + \epsilon y + \epsilon z), \frac{1}{\epsilon - \epsilon^2}(\epsilon x + y + \epsilon z)\right).$$

We must consider whether the two matrices found for the last group admit the further generator W . It is easily verified that they do, and hence

the 2 matrices found for $G(\text{LXII})$ admit $G_{216}(\text{LXIII})$.

8. $G_{168}(\text{LXIV})$ $S(\omega x, \omega^2 y, \omega^4 z), P(y, z, x),$

$$U: \quad x' = h((\omega^4 - \omega^3)x + (\omega^2 - \omega^5)y + (\omega - \omega^6)z),$$

$$y' = h((\omega^2 - \omega^5)x + (\omega - \omega^6)y + (\omega^4 - \omega^3)z),$$

$$z' = h((\omega - \omega^6)x + (\omega^4 - \omega^3)y + (\omega^2 - \omega^5)z),$$

$$h = \frac{1}{7}(\omega + \omega^2 + \omega^4 - \omega^6 - \omega^5 - \omega^3) = \frac{1}{\sqrt{-7}},$$

$$\omega^7 = 1.$$

The matrix given for $G_{21}(L)$ in Section 2 of this Chapter admits $G_{168}(\text{LXIV})$.

CHAPTER 9. THE NON-HOMOGENEOUS LINEAR GROUPS

1. The problem. In general the equations of the linear group which generates the sets of some involution on a Picard threefold will be non-homogeneous. In some cases the constant terms can all be made to vanish by a suitable choice of variables; in particular, this can always be achieved for a cyclic group in which each multiplier of a generating substitution differs from unity. Apart from this case, it is necessary, in order to determine the properties of the involutions, to investigate what forms the constant vectors can assume. The key fact is that whenever a product yields a substitution whose homogeneous part is the identical substitution, the constant vector must be zero, modulo periods. Additionally, there may be some latitude in choosing variables which do not alter the forms of the homogeneous group and matrices already determined, but which permit simplification of the constant vectors.

The cyclic groups calling for consideration (other than $G_2(II)$ which will be omitted) are $G(V)$, $G(VI)$ and $G(VII)$. In addition, we shall examine the multiplicative group $G(XV)$, the dihedral group $G(XXXVI)$, the monomial group $G(LX)$, and the primitive groups $G(LXII)$, $G(LXIII)$, which form a sequence with each a subgroup of the next.

2. The non-homogeneous $G_3(V)$. The generating substitution must be taken in the form $S(x+c, \epsilon y, \epsilon^2 z)$. The form of S^3 shows that $(3c, 0, 0)$ must be a period, whence

$$c = \frac{1}{3}(a + b\tau).$$

Using the period matrix we ensure that

$$a, b = 0, 1, 2,$$

and a translation of the variables y, z maintains the forms of the substitutions on these variables.

Considering the 3 matrices found for $G(V)$ in Chapter 5.5, we now see that

$$(0, a\frac{1}{3}\epsilon(1-\epsilon) + b\frac{1}{3}\epsilon\epsilon(1-\epsilon)\tau, a\frac{1}{3}\epsilon(1-\epsilon^2) + b\frac{1}{3}\epsilon\epsilon(1-\epsilon^2)\tau)$$

must be a period. Since this must be expressible in terms of the last four fundamental periods, it must be the null period. Hence

$$a\epsilon = b\epsilon\epsilon, = 0,$$

and so $\epsilon (= \epsilon_1) = 0$ or $a = 0$ with $b = 0$ or $\epsilon_1 = 0$. Except in the case of the homogeneous group ($a = b = 0$), $\epsilon_1 = 0$, and the value of c may always be reduced to $\frac{1}{3}\tau$.

We thus obtain two groups from $G_3(V)$, (including the homogeneous one which will, of course, always arise), namely those given by

$$c = 0 \quad \text{admitted by all 3 matrices;}$$

$$c = \frac{1}{3}\tau \quad \text{admitted by the matrices (i), (ii).}$$

3. The non-homogeneous $G_4(VI)$. The generating substitution is of the form $S(x+c, iy, -iz)$, and so $(4c, 0, 0)$ is a period. The discussion is similar to that of the previous

group, $G_3(V)$, and we find the following values for c

$c = 0, \frac{1}{2}\tau$ admitted by all 3 matrices;

$c = \frac{1}{4}\tau$ admitted by matrices (1), (11);

$c = \frac{1}{2}, \frac{1}{2} + \frac{1}{4}\tau$ admitted by matrix (11).

4. The non-homogeneous $G_3(VII)$. The generating substitution is of the form $S(x+c, -\epsilon y, -\epsilon^2 z)$. There is only one matrix and the constant c may take the values

$$0, \frac{1}{6}\tau, \frac{1}{3}\tau, \frac{1}{2}\tau, \frac{1}{2} + \frac{1}{3}\tau.$$

5. The non-homogeneous $G_3(XV)$. The generators of this group must be taken in the form

$$S(\epsilon x, \epsilon y, \epsilon z), \quad S(x+c_1, \epsilon y + c_2, \epsilon^2 z + c_3).$$

From the form of S_1^3 we see that

$$(3c_1, 0, 0) \equiv 0, \text{ mod } 9. \quad (1)$$

Again, since SS_1 and S_1S have the same multipliers, their constant terms must be congruent. The condition is

$$((1-\epsilon)c_1, (1-\epsilon)c_2, (1-\epsilon)c_3) \equiv 0, \text{ mod } 9. \quad (2)$$

Using this condition we can express any product of S and S_1 in the form $S^m S_1^n$ and then, using (1), we can ensure that $m, n < 3$. Thus there are no further independent conditions to be satisfied by the c_i .

Taking the value of c_1 , modulo periods, (1) gives the possibilities, considering only this first element,

$$c_1 = \frac{1}{3}(a + b\epsilon), \quad a, b = 0, 1, -1;$$

$$\text{and then } c_1(1-\epsilon) = \frac{1}{3}(a + b - (a - 2b)\epsilon),$$

$$\text{and using (2) } a + b \equiv 0, \text{ mod } 3.$$

$$\text{Hence } c_1 = 0, \frac{1}{3}(1-\epsilon), -\frac{1}{3}(1-\epsilon).$$

Changing the signs of all three variables, the third value for c_1 reduces to the second. We now verify that for these two values, $(3\epsilon, 0, 0)$ is a period with the exception that $c_1 = \frac{1}{3}(1-\epsilon)$ is not permissible with matrix (v).

Substituting in (2) the two values found for c_1 leads to the values of c_2 ; and then use of (2) again leads to the values of c_3 . In finding the vectors (c_1, c_2, c_3) we desire to maintain the forms of the equations of the group and of the matrix under consideration and to retain only one of a set of equivalent vectors. Depending on the matrix, we can multiply the variables by some of $-1, \pm\epsilon, \pm\epsilon^2$; and for matrices symmetrical in two rows we can eliminate some vectors on inspection by interchanging two variables. In this way we arrive at the following table of vectors and corresponding matrices. Further ways of eliminating redundant vectors are discussed below.

Table 1

Vector			Matrices
c_1	c_2	c_3	
0	0 $\frac{1}{3}(1-\epsilon)$	$0, \frac{1}{3}(1-\epsilon)$	(1)-(v)
		$\frac{1}{3}(1-\epsilon)$	(1), (1v)
		$-\frac{1}{3}(1-\epsilon)$	(1v)
		$-\frac{1}{3}$	(11), (111), (v)
		$-\frac{1}{3}\epsilon, -\frac{1}{3}\epsilon^2$	(v)
$\frac{1}{3}(1-\epsilon)$	0 $\frac{1}{3}(1-\epsilon)$	$0, \frac{1}{3}(1-\epsilon)$	(11)
		$\frac{1}{3}(1-\epsilon)$	(1)
		$-\frac{1}{3}$	(11)

Table 1 cont.

$\frac{1}{3}(1-\epsilon)$	$\frac{1}{3}(1-\epsilon)$	$\frac{1}{3}$	(iii)
	$-\frac{1}{3}(1-\epsilon)$	$0, \frac{1}{3}(1-\epsilon), -\frac{1}{3}(1-\epsilon)$	(iii)
	$-\frac{1}{3}$	$-\frac{1}{3}$	(iv)

Next consider the effect of a translation of variables.

Take new variables X, Y, Z where

$$X = x + k_1, \quad Y = y + k_2, \quad Z = z + k_3.$$

Then $S(\epsilon X + k_1(1-\epsilon), \epsilon Y + k_2(1-\epsilon), \epsilon Z + k_3(1-\epsilon))$,

and $S_1(X + c_1, \epsilon Y + c_2 + k_2(1-\epsilon), \epsilon^2 Z + c_3 + k_3(1-\epsilon^2))$.

We therefore require

$$(k_1(1-\epsilon), k_2(1-\epsilon), k_3(1-\epsilon)) \equiv \underline{0}, \text{ mod } \underline{Q}, \quad (3)$$

and then the vector (c_1, c_2, c_3) is replaced by

$$(c_1, c_2 + k_2(1-\epsilon), c_3 + k_3(1-\epsilon^2)). \quad (4)$$

For matrices (ii) and (iii), taking

$$k_1 = 0, \quad k_2 = -\frac{1}{3}(1-\epsilon^2), \quad k_3 = -\frac{1}{3},$$

we find that where $c_3 = \frac{1}{3}(1-\epsilon)$ it may be reduced to zero;

and changing the signs of the k_i , $c_3 = -\frac{1}{3}(1-\epsilon)$ is similarly reduced to zero.

For matrix (iv), taking

$$k_1 = -\frac{1}{3}(1-\epsilon^2), \quad k_2 = k_3 = -\frac{1}{3},$$

we find that $(0, \frac{1}{3}(1-\epsilon), -\frac{1}{3}(1-\epsilon))$ and $(0, \frac{1}{3}(1-\epsilon), \frac{1}{3}(1-\epsilon))$

may be reduced to zero and $(0, 0, \frac{1}{3}(1-\epsilon))$ respectively.

For matrix (v), taking

$$k_1 = -\frac{1}{3}(1-\epsilon^2), \quad k_2 = -\frac{1}{3}, \quad k_3 = -\frac{1}{3}(1-\epsilon^2),$$

we find $(0, \frac{1}{3}(1-\epsilon), -\frac{1}{3}\epsilon^2)$ reduces to zero: taking

$$k_1 = 0, \quad k_2 = \frac{1}{3}(1-\epsilon^2), \quad k_3 = \frac{1}{3},$$

the same reduction follows when $c_3 = -\frac{1}{3}$, $-\frac{1}{3}\epsilon$ instead of $-\frac{1}{3}\epsilon^2$.

Table 1 may now be replaced by Table 2.

Table 2

Vector			Matrices
c_1	c_2	c_3	
0	0	0	(1)-(v)
		$\frac{1}{3}(1-\epsilon)$	(1), (1v), (v)
		$\frac{1}{3}(1-\epsilon)$	(1)
	$\frac{1}{3}(1-\epsilon)$	$-\frac{1}{3}$	(11), (111)
		0	(11)
		$\frac{1}{3}(1-\epsilon)$	(1)
$\frac{1}{3}(1-\epsilon)$	0	$-\frac{1}{3}$	(11)
		$\frac{1}{3}$	(111)
		0	(111)
	$-\frac{1}{3}(1-\epsilon)$	0	(111)
		$-\frac{1}{3}$	(1v)
		$-\frac{1}{3}$	(1v)

Finally, we recall that the homogeneous group $G_3(XV)$ is symmetrical in the three variables and that in the matrices (1)-(v) we chose to make zero four elements of the first row and two elements of the second row. From symmetry, the rows on which the zeros are imposed can be chosen in any way*, and the generator S_3 may be replaced

* An exception occurs in the case of matrix (11), where in the reduction we had occasion to interchange rows. Correspondingly although, as can be seen, we may transpose the second and third rows we cannot transpose any other pair.

by one in which the multipliers $1, \epsilon, \epsilon^2$ appear in the corresponding order. In achieving this permutation of the roles of the variables in the second generator, these variables may need to be multiplied by constants, and possibly subjected to a translation. The result of such a permutation is that the vector (c_1, c_2, c_3) is replaced by a new vector which must itself be reducible to one of the entries in Table 2. But this may be a different entry, showing that some vector may be discarded.

The change of variable needed to bring about any given permutation can be quickly determined by trial. For the present group each new vector obtained is found to be equivalent to the one from which it arose except in the case of matrix (iii), and the examination of this matrix will illustrate the procedure.

For matrix (iii), then, we propose to interchange the roles of the variables x and z . By inspection, we decide to multiply the variables by $\frac{1}{3}(1-\epsilon), -1, 1-\epsilon^2$ respectively. The period matrix now takes the form

$$\begin{pmatrix} \frac{1}{3}(1-\epsilon) & \frac{1}{3}\epsilon(1-\epsilon) & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -\epsilon & 0 & 0 \\ 1 & \epsilon & 1 & \epsilon & 1-\epsilon^2 & \epsilon-1 \end{pmatrix}.$$

Subtract cols.1 and 2 from cols.3 and 4 respectively.

Subtract $(2 \times \text{col.1} + \text{col.2})$ from col.5 and $(\text{col.2} - \text{col.1})$ from col.6. Change the signs of the last four columns.

The matrix is now in the required form.

The substitution S is unaltered but S_1 becomes

$$x' = x + \frac{1}{3}(1-\varepsilon)c_1, \quad y' = \varepsilon y - c_2, \quad z' = \varepsilon^2 z + (1-\varepsilon^2)c_3,$$

and $S^2 S_1^2$ becomes

$$x' = \varepsilon^2 x - \frac{2}{3}(1-\varepsilon^2)c_1, \quad y' = \varepsilon y + \varepsilon c_2, \quad z' = z - (1-\varepsilon^2)c_3.$$

Here the multipliers are in the order $\varepsilon^2, \varepsilon, 1$, and so this substitution is to be taken as the new second generator.

The vector (c_1, c_2, c_3) is thus replaced by

$$(-(1-\varepsilon^2)c_1, \varepsilon c_2, -\frac{2}{3}(1-\varepsilon^2)c_3).$$

The particular vector $(\frac{1}{3}(1-\varepsilon), -\frac{1}{3}(1-\varepsilon), 0)$ is replaced by $(0, -\frac{1}{3}\varepsilon(1-\varepsilon), -\frac{2}{3})$. Changing the signs of all 3 variables and subtracting the period $(0, \varepsilon, \frac{1}{3}\varepsilon(1-\varepsilon))$, we get $(0, \frac{1}{3}(1-\varepsilon), \frac{1}{3}(1-2\varepsilon))$. Adding the period $(0, 0, \varepsilon)$, we have $(0, \frac{1}{3}(1-\varepsilon), -\frac{1}{3}\varepsilon^2)$, and, on choosing ε times the third variable as a new variable, we have $(0, \frac{1}{3}(1-\varepsilon), -\frac{1}{3})$. This is the second vector found for matrix (iii), and so the fourth vector $(\frac{1}{3}(1-\varepsilon), -\frac{1}{3}(1-\varepsilon), 0)$ may be discarded.

As remarked earlier, no other vectors are found in this way to be equivalent. Thus the non-homogeneous groups arising from $G_4(XV)$ are as determined by Table 2, omitting the penultimate case which is shown in brackets.

6. The non-homogeneous $G_{12}(XXXVI)$. The generators must be considered in the forms

$$S(x+c_1, \varepsilon y, \varepsilon^2 z), \quad T(y+c_2, x+c_3, -z+c_4).$$

We remark first that in the reductions made below it is important for later work to restrict ourselves wherever possible to changes of variable which consist of translations

or of multiplying all three variables by the same constant, and to note where other substitutions prove necessary.

If we choose $x+k_1$, $y+k_2$, $z+k_3$ as new variables, the equations of S are unaltered, mod $\underline{Q}$, provided

$$(0, k_1(1-\epsilon), k_3(1-\epsilon^2)) \equiv \underline{0}, \text{ mod } \underline{Q}; \quad (5)$$

and (c_1, c_2, c_3) is replaced by

$$(c_1 + (k_1 - k_2), c_2 - (k_1 - k_2), c_3 + 2k_3).$$

Taking $k_1 = -c_1$, $k_2 = k_3 = 0$, we see that we can always make $c_1 = 0$; and this is retained if afterwards we always take $k_1 = k_2$, the form of the vector now being

$$(0, c_2, c_3 + 2k_3). \quad (6)$$

We shall take $c_1 = 0$ in future.

$$\text{Since} \quad S^3 = T^2 = I, \quad (7)$$

$$\text{we have} \quad (3c, 0, 0) \equiv \underline{0}, \text{ mod } \underline{Q}, \quad (8)$$

$$\text{and} \quad (c_2, c_2, 0) \equiv \underline{0}, \text{ mod } \underline{Q}. \quad (9)$$

Again, from a consideration of the homogeneous parts,

$$(TS)^2 = (ST)^2, \quad (10)$$

and so the difference in their constant vectors is congruent to zero:

$$(c(1-\epsilon), -(c + c_2)(1-\epsilon), 3\epsilon^2 c_3) \equiv \underline{0}, \text{ mod } \underline{Q}. \quad (11)$$

We can now show that there are no further conditions imposed on the constants. For if (8), (9) and (11) are satisfied then (7) and (10) hold, and by considering products of S and T , taken r at a time where r is successively 1, 2, ... we find that every product can be reduced

to one of the following eighteen:

I S S² (ST)² S(ST)² TST (STS)² TS²T STS²T
T ST S²T STS S²TS TS S²TS² TS² STS²

These form the entire group, every product of the generators being equivalent to one of them: consequently no further conditions can arise.

Using the three conditions found and making only changes of variable which maintain the forms of the generators and of the matrix under consideration, the following possibilities are obtained for c, c_2, c_3 ($c_1 = 0$ always).

Table 3

c	c_2	c_3	Matrices
0	0	0	(i)-(vi)
		$\frac{1}{3}$	(i), (iii)-(v)
	$\frac{1}{3}(1-\epsilon)$	$0, \frac{1}{3}$	(iii)
$\frac{1}{3}(1-\epsilon)$	0	$0, \frac{1}{3}$	(i)
		$-\frac{1}{9}(1-\epsilon)$	(ii)
$\frac{1}{3}$	0	$0, \frac{1}{3}$	(iv)
		$-\frac{1}{9}(1-\epsilon)$	} (v)
		$-\frac{1}{3} - \frac{1}{9}(1-\epsilon)$	
		$\frac{1}{3} - \frac{1}{9}(1-\epsilon)$	

In reducing the possibilities to those of Table 3, the only change of variable needed which is not of the types mentioned at the beginning of the section is a choice of $-z$.

as a new variable with x, y unaltered. This is needed to change $c_3 = -\frac{1}{3}$ to $+\frac{1}{3}$ in the three cases where $c = \frac{1}{3}(1-\epsilon)$ for matrix (i), $c_2 = \frac{1}{3}(1-\epsilon)$ for matrix (iii), and $c = \frac{1}{3}$ for matrix (iv). Of these only the first arises in the discussion of the next group, $G(LX)$.

Finally, since x, y appear symmetrically in the homogeneous group, it is possible to interchange the first two rows of any matrix and the roles of x, y . For all matrices this may, in fact, be effected after taking $-z$ as a new variable. The generators S and T must be replaced by the new forms of TST and T , namely

$$TST(\epsilon x + \epsilon c_2, y + c + c_2, \epsilon^2 z - (1-\epsilon^2)c_3),$$

$$T(y, x + c_2, -z - c_3).$$

Since $(\epsilon c_2, c_2, 0)$ is a period, the ϵc_2 and c_2 may be removed from the equations for TST . After this, taking $y - c_2$ and $z + c_3$ as new variables, the new generators become

$$TST(\epsilon x, y + c, \epsilon^2 z), \quad T(y + c_2, x, -z + c_3),$$

which are the original forms with the roles of x and y interchanged. Thus in all cases the set of constants c, c_2, c_3 is equivalent to itself, and the table given does not reduce further.

In conclusion, the non-homogeneous groups arising from $G(XXXVI)$ are as determined by Table 3.

7. The non-homogeneous $G_{54}(LX)$. The first two generators of the group generate $G_{18}(XXXVI)$. The six matrices of the present group arise from the six for $G(XXXVI)$ as follows: (1), (11), (111) from (1), (11), (v) respectively, (1v)-(v1) from (vi); the relevant factors for multiplying the rows are to be found in the discussion of the homogeneous group.

The work done for $G(XXXVI)$ may be applied at once to the present group, except that in discussing matrix (1) we do not use the change of sign of the single variable z since we wish to preserve the form of the third generator. Thus the generators may be taken in the forms $S(x+c, \epsilon y, \epsilon^2 z)$, $T(y, x, -z+c_3)$, $T_1(-x+d_1, z+d_2, y+d_3)$. The table of constants c, c_3 becomes the following.

Table 4

c	c_3	Matrices
0	0	(1)-(v1)
	$\frac{1}{3}$	(1), (111)
$\frac{1}{3}(1-\epsilon)$	$0, \pm \frac{1}{3}$	(1)
$\frac{1}{3}$	$\frac{1}{9}(1-\epsilon)$	(11), (111)
	$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	} (111)
	$-\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	

The only pair whose form calls for explanation is the pair $\frac{1}{3}, \frac{1}{9}(1-\epsilon)$ for matrix (11). Here the constants are first obtained as $-\frac{1}{3}\epsilon, -\frac{1}{9}(1-\epsilon)$. Subtracting col.1 of matrix (11)

from the right-hand side of S, we have

$$S(x - \frac{1}{3}, \epsilon y, \epsilon^2 z + \frac{1}{3}(1 - \epsilon)).$$

Now choosing $x + \frac{1}{3}$, $y + \frac{1}{3}$ as new variables, we have

$$S(x - \frac{1}{3}, \epsilon y + \frac{1}{3}(1 - \epsilon), \epsilon^2 z + \frac{1}{3}(1 - \epsilon)),$$

and, from (6) of the previous section, the form of T is unaltered. Using col.4 of matrix (11), the constant vector in S reduces to $(-\frac{1}{3}, 0, 0)$, and changing the sign of each variable we have the form quoted for the constants.

To investigate the possible forms for the constants d_1, d_2, d_3 , we require the generating relations satisfied by S, T, T_1 . We have

$$I = S^3 = T^2 = (TS)^2 (TS^2)^2,$$

these being the relations found in the last section (with the third in an equivalent form); and also

$$I = T_1^2 = (ST_1)^2 = (TT_1)^3 = (STT_1)^3. \quad (12)$$

Using these relations, it may be verified that every product of 7 (or more) generators can be expressed as a product of fewer generators, and in fact that every product may be reduced to one of the 54 which follow.

These give the 54 substitutions of the group and so there can be no further relations to be satisfied. The substitutions are:

I	S	S ²	(ST) ²	(ST) ² S	TST	(STS) ²	TS ² T	STS ² T
T	ST	S ² T	STS	S ² TS	TS	S ² TS ²	TS ²	STS ²
T ₁	ST ₁	TS	(ST) ² T ₁	TTST	TSTT ₁	TS ² TT ₁ S	TS ² TT ₁	TT ₁ S ² T
TT ₁	STTT ₁	TSTT ₁	STSTT ₁	TTST ₁	TSTT ₁	STTT ₁ S	TTST ₁	TTT ₁ S
TT ₁	STT ₁	S ² TT ₁	STST ₁	TTSTT ₁	TST ₁	S ² TT ₁ S	TT ₁ S	STT ₁ S
TT ₁	STT ₁	TST ₁	TSTS	TT ₁ S	STT ₁ S	STT ₁ S ²	TTSTT ₁	TT ₁ S ²

The equations (12) imply the following congruences.

$$(0, d_2 + d_3, d_2 + d_3) \equiv 0, \text{ mod } \underline{Q}, \quad (13)$$

$$(0, \varepsilon d_2 + d_3, d_2 + \varepsilon^2 d_3) \equiv 0, \text{ mod } \underline{Q}, \quad (14)$$

$$(c_3 - d_1 + d_2 - d_3, -(c_3 - d_1 + d_2 - d_3), c_3 - d_1 + d_2 - d_3) \equiv 0, \text{ mod } \underline{Q}, \quad (15)$$

$$(c + \varepsilon^2 c_3 - d_1 + d_2 - \varepsilon^2 d_3, -\varepsilon(c + \varepsilon^2 c_3 - d_1 + d_2 - \varepsilon^2 d_3), \\ c + \varepsilon^2 c_3 - d_1 + d_2 - \varepsilon^2 d_3) \equiv 0, \text{ mod } \underline{Q}, \quad (16)$$

The congruences (13) and (14) serve to determine d_2 and d_3 ; and (15) then gives d_1 . The values so found must satisfy (16) or, what is often more convenient to use, the congruence obtained by subtracting (15) from (16). In this way, and simplifying d_2 , d_3 and then d_1 as far as possible without altering c , c_3 , we arrive at Table 5, given on the next page.

Consider now the effect of a translation of variables.

If when $d_1 = -d_2 = d_3 = \frac{1}{3}(1-\varepsilon)$ for matrix (i) we take $z = \frac{1}{3}(1-\varepsilon)$ as a new variable, S is unaltered, in T we obtain $c_3 = -\frac{2}{3}(1-\varepsilon)$ (which is congruent to $\frac{1}{3}(1-\varepsilon)$), and in T₁ we now have $d_2 = d_3 = 0$. Again, when $d_2 = -d_3 = \frac{1}{3}$ for matrix (ii), we take $x + \frac{1}{3}$, $y + \frac{1}{3}$, $z + \frac{2}{3}$ as new variables. Then S is unaltered, in T, c_3 is increased by $\frac{1}{3}$, and in T₁,

Table 5

c	c_3	d_1	d_2	d_3	Matrices
0	0	0	0	0	(1)-(v1)
		$\frac{1}{3}(1-\epsilon)$	$-\frac{1}{3}(1-\epsilon)$	$\frac{1}{3}(1-\epsilon)$	(1)
			0	0	(11), (111)
		$-\frac{1}{3}, -\frac{1}{3}\epsilon, -\frac{1}{3}\epsilon^2$	$\frac{1}{3}$	$-\frac{1}{3}$	(11)
		1	0	0	(1v)-(v1)
	$\frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}\epsilon, \frac{1}{3}\epsilon^2$	0	0	(111)
$\frac{1}{3}(1-\epsilon)$	$-\frac{1}{3}$	$-\frac{1}{3}$	0	0	(1)
$\frac{1}{3}$	$\frac{1}{9}(1-\epsilon)$	$\frac{1}{9}(1-\epsilon)$	0	0	
		$-\frac{2}{9}(1-\epsilon)$			(11), (111)
		$\frac{4}{9}(1-\epsilon)$			
		$-\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$\frac{1}{3}$	$-\frac{1}{3}$	
		$-\frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$			
		$-\frac{1}{3}\epsilon^2 + \frac{1}{9}(1-\epsilon)$			
		$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$-\frac{1}{3}$	$\frac{1}{3}$	(11)
		$\frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$			
		$\frac{1}{3}\epsilon^2 + \frac{1}{9}(1-\epsilon)$			
	$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	0	0	
		$\frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$			
		$\frac{1}{3}\epsilon^2 + \frac{1}{9}(1-\epsilon)$			
	$-\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$-\frac{1}{3} + \frac{1}{9}(1-\epsilon)$			(111)
		$-\frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$			
		$-\frac{1}{3}\epsilon^2 + \frac{1}{9}(1-\epsilon)$			

d_1 is increased by $\frac{2}{3}$ and d_2, d_3 become zero. Similar results can be obtained when $d_2 = -d_3 = -\frac{1}{3}$ for matrix (ii). It follows that we can always take

$$d_2 = d_3 = 0,$$

and the possible constants are given below.

Table 6

c	c_3	d_1	Matrices
0	0	0	(i)-(vi)
		$\frac{1}{3}(1-\epsilon)$	(ii), (iii)
		1	(iv)-(vi)
	$\frac{1}{3}(1-\epsilon)$	$\frac{1}{3}(1-\epsilon)$	(i)
	$\frac{1}{3}$	$\frac{1}{3}, \frac{1}{3}\epsilon, \frac{1}{3}\epsilon^2$	(ii), (iii)
$\frac{1}{3}(1-\epsilon)$	$-\frac{1}{3}$	$-\frac{1}{3}$	(i)
$\frac{1}{3}$	$\frac{1}{9}(1-\epsilon)$	$\frac{1}{9}(1-\epsilon)$	(ii), (iii)
		$-\frac{2}{9}(1-\epsilon)$	
		$\frac{4}{9}(1-\epsilon)$	
	$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	
		$\frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$	
		$\frac{1}{3}\epsilon^2 + \frac{1}{9}(1-\epsilon)$	
	$-\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$-\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	
		$-\frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$	
		$-\frac{1}{3}\epsilon^2 + \frac{1}{9}(1-\epsilon)$	

Finally, the present group, in the homogeneous form, is symmetrical in x and z . It may be verified that in the case of each matrix the forms of the first and third rows

may be interchanged by manipulation of columns. We therefore require to investigate the constant vectors obtained in the non-homogeneous forms of the group when the roles of x and z are interchanged. The roles of the generators T and T_1 are interchanged and S must be replaced by

$$STS^2T(\epsilon^2x + c, \epsilon y + 2\epsilon c, z - (1-\epsilon^2)c_3).$$

Choosing

$$x - \frac{1}{3}(1-\epsilon)c, \quad y - \frac{1}{3}(1-\epsilon)c, \quad z - d_3 - \frac{1}{3}(1-\epsilon)c,$$

as new variables, we find, on using the fact that $(0, 3\epsilon c, 0)$ is a period,

$$STS^2T(\epsilon^2x, \epsilon y, z - (1-\epsilon^2)c_3),$$

$$T(y, x, -z + c_3 - 2d_3 - \frac{2}{3}(1-\epsilon)c),$$

$$T_1(-x + d_1 - \frac{2}{3}(1-\epsilon)c, z + d_2 + d_3, y).$$

Hence, corresponding to the original constants c, c_3, d_1 , we have, since $d_2 = d_3 = 0$,

$$-(1-\epsilon^2)c, \quad d_1 - \frac{2}{3}(1-\epsilon)c, \quad c_3 - \frac{2}{3}(1-\epsilon)c.$$

It may now be verified that in each case the new constants reduce to those from which they arose, provided suitable multiplications and translations of variables are carried out. Thus Table 6 is not reduced further.

In conclusion: the non-homogeneous groups arising from $G_{54}(LX)$ and the matrices admitting them are as shown in Table 6.

8. The non-homogeneous $G_{108}(LXII)$. The group $G_{54}(LX)$ becomes a subgroup of the present group on choosing $-y$ as a new variable. It follows that the generators for the

non-homogeneous G_{108} (LXII) may be written

$$S(x+c, \varepsilon y, \varepsilon^2 z), \quad T(-y, -x, -z+c_3),$$

$$V: x' = \frac{1}{\varepsilon-\varepsilon^2}(x+y+z) + e_1,$$

$$y' = \frac{1}{\varepsilon-\varepsilon^2}(x+\varepsilon y+\varepsilon^2 z) + e_2,$$

$$z' = \frac{1}{\varepsilon-\varepsilon^2}(x+\varepsilon^2 y+\varepsilon z) + e_3.$$

We have

$$V^2 = T_1,$$

where $T_1(-x, -z, -y)$ is the third generator of $G(LX)$ with $-y$ as a new variable. The two possible matrices (i), (ii) arise from (ii), (iii) of $G(LX)$, and so from Table 6 there are 14 possible triads to be considered for c, c_3, d , (where d is expressible in terms of the e_i).

We now require the set of conditions satisfied by the constants here. The subgroup G_{54} being of index 2, any substitution of the present group may be written

$$VA \quad (17)$$

where A is a product of S, T, T_1 . If we have expressions whereby the products

$$SV \quad \text{and} \quad TV$$

can be written with the V on the left, then any product can be reduced to a power of V postmultiplied by a substitution of G_{54} and so, using the relation $V^2 = T_1$, the expression assumes the form (17). The conditions imposed on the constants are therefore those found under $G(LX)$ (which are already satisfied), together with those implied by the expressions for SV and TV and by the relation $V^2 = T_1$. There are thus three new vector conditions to be satisfied.

By evaluating the homogeneous parts we verify that

$$SV = VTT, \quad \text{and} \quad TV = VTS.$$

The conditions imposed in making the constant vectors equivalent are respectively

$$\begin{pmatrix} \frac{1}{3}\varepsilon(1-\varepsilon)(c_3 - d_1) + c \\ -e_2(1-\varepsilon) + \frac{1}{3}\varepsilon(1-\varepsilon)(\varepsilon^2 c_3 - \varepsilon d_1) \\ -e_3(1-\varepsilon^2) + \frac{1}{3}\varepsilon(1-\varepsilon)(\varepsilon c_3 - \varepsilon^2 d_1) \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}, \quad (18)$$

$$\text{and} \begin{pmatrix} e_1 + e_2 + \frac{1}{3}\varepsilon(1-\varepsilon)(c - d_1) \\ e_1 + e_2 + \frac{1}{3}\varepsilon(1-\varepsilon)(c - d_1) \\ 2e_3 + \frac{1}{3}\varepsilon(1-\varepsilon)(c - d_1) - c_3 \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}. \quad (19)$$

The condition imposed by the relation $V^2 = T$, is

$$\begin{pmatrix} \frac{1}{3}\varepsilon(1-\varepsilon)(e_1 + e_2 + e_3) - e_1 + d_1 \\ \frac{1}{3}\varepsilon(1-\varepsilon)(e_1 + \varepsilon e_2 + \varepsilon^2 e_3) - e_2 \\ \frac{1}{3}\varepsilon(1-\varepsilon)(e_1 + \varepsilon^2 e_2 + \varepsilon e_3) - e_3 \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}. \quad (20)$$

A further condition results from expressing the fact that $V^4 = I$. This is found to be

$$\begin{pmatrix} 0 \\ 2(e_2 - e_3) \\ -2(e_1 - e_3) \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}. \quad (21)$$

Condition (21) is not, of course, independent of the previous ones, being a consequence of (20) and the earlier $T^2 = I$, but its simplicity makes it more convenient than the stronger condition (20) in those cases where it proves adequate.

The relation (18) serves to determine e_2 and e_3 which must then satisfy (21). Following this, (19) determines

e_1 , and finally (20) must be satisfied. It is found that most of the 14 triads c, c_3, d_1 are ruled out by the form of the first element of (18) or of the third element of (19) or by (21). The only group, other than the homogeneous one, which is permissible arises from $c = \frac{1}{3}$, $c_3 = \frac{1}{3} + \frac{1}{9}(1-\epsilon)$, $d_1 = \frac{1}{3}\epsilon + \frac{1}{9}(1-\epsilon)$, and then in the case of each of the two matrices we can have

$$e_1 = \frac{1}{9}\epsilon, \quad e_2 = e_3 = \frac{4}{9}.$$

Thus for the non-homogeneous G_{108} (LXII) we have the two groups determined by the following table of constants, each group being admitted by both matrices (i) and (ii).

Table 7

c	c_3	e_1	e_2	e_3
0	0	0	0	0
$\frac{1}{3}$	$\frac{1}{3} + \frac{1}{9}(1-\epsilon)$	$\frac{1}{9}\epsilon$	$\frac{4}{9}$	$\frac{4}{9}$

9. The non-homogeneous G_{216} (LXIII). The generators are S, T, V as in the last group together with

$$\begin{aligned} W: x' &= \frac{1}{\epsilon - \epsilon^2}(x + y + \epsilon^2 z) + f_1, \\ y' &= \frac{1}{\epsilon - \epsilon^2}(x + \epsilon y + \epsilon z) + f_2, \\ z' &= \frac{1}{\epsilon - \epsilon^2}(\epsilon x + y + \epsilon z) + f_3. \end{aligned}$$

The conditions which must be satisfied by the constant vectors, beyond those for G (LXII), are the ones involving the f_i . We find

$$W^2 = T, S \quad \text{and} \quad W^4 = I, \quad (22)$$

where T_1 is as in the last section.

Again, the subgroup G_{108} (LXII) being of index 2, any substitution of the present group may be written

$$WA$$

where now A belongs to G (LXII). As in the last section, the remaining conditions are to be found by obtaining expressions for

$$SW, \quad TW, \quad VW$$

in which the W appears on the left. We find, from consideration of the homogeneous parts, that

$$SW = WTT, S, \quad TW = WST, \quad VW = WVST, TT. \quad (23)$$

The congruences imposed by (22) are

$$\begin{pmatrix} -c + d_1 + \frac{1}{3}\epsilon(1-\epsilon)(f_1 + f_2 + \epsilon^2 f_3) - f_1 \\ \frac{1}{3}\epsilon(1-\epsilon)(f_1 + \epsilon f_2 + \epsilon f_3) - f_2 \\ \frac{1}{3}\epsilon(1-\epsilon)(\epsilon f_1 + f_2 + \epsilon f_3) - f_3 \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}, \quad (24)$$

$$\begin{pmatrix} 0 \\ -2(f_2 - \epsilon^2 f_3) \\ 2(\epsilon f_2 - f_3) \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}, \quad (25)$$

(25) being a consequence of (24) and earlier conditions.

The congruences imposed by (23) are

$$\begin{pmatrix} \frac{1}{3}\epsilon(1-\epsilon)(c - d_1 + \epsilon^2 c_3) - c \\ -f_2(1-\epsilon) + \frac{1}{3}\epsilon(1-\epsilon)(\epsilon c - \epsilon d_1 + \epsilon c_3) \\ -f_3(1-\epsilon) + \frac{1}{3}\epsilon(1-\epsilon)(c - d_1 + \epsilon c_3) \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}, \quad (26)$$

$$\begin{pmatrix} -(f_1 + f_2) + \frac{1}{3}\epsilon(1-\epsilon)(c + d_1) \\ -(f_1 + f_2) + \frac{1}{3}\epsilon(1-\epsilon)(c + d_1) \\ -2f_3 + \frac{1}{3}\epsilon^2(1-\epsilon)(c + d_1) + c_3 \end{pmatrix} \equiv \underline{0}, \text{ mod } \underline{\Omega}, \quad (27)$$

$$\begin{pmatrix} \frac{1}{3}(2+\varepsilon)(\varepsilon(c+d_1)-c_3+d_1)-\frac{1}{3}\varepsilon(1-\varepsilon)(e_1+e_2+\varepsilon^2 e_3-f_1-f_2-f_3)-e_1+f_1 \\ \frac{1}{3}(2+\varepsilon)(\varepsilon^2(c+d_1)-\varepsilon^2 c_3+d_1)-\frac{1}{3}\varepsilon(1-\varepsilon)(e_1+\varepsilon e_2+\varepsilon e_3-f_1-\varepsilon f_2-\varepsilon^2 f_3)-e_2+f_2 \\ \frac{1}{3}(2+\varepsilon)(\varepsilon^2(c+d_1)-c_3+\varepsilon^2 d_1)-\frac{1}{3}\varepsilon(1-\varepsilon)(\varepsilon e_1+e_2+\varepsilon e_3-f_1-\varepsilon^2 f_2-\varepsilon f_3)-e_3+f_3 \end{pmatrix} \\ \equiv \underline{0}, \text{ mod } \underline{Q}. \quad (28)$$

We must consider the two sets of values found for c, c_3, e_1, e_2, e_3 in the last section. When these are all zero, conditions (25), (26) and (27) lead to the conclusion that in the case of each of the two matrices we must have

$$f_1 = f_2 = f_3 = 0;$$

and these clearly satisfy the remaining conditions. For the second possible set of constants $c, \dots, c_3$, condition (26) leads to 3 pairs of values for f_2, f_3 in the case of each matrix, but none of these pairs satisfies (25).

Thus, in the case of G_{216} (LXIII), all the constant vectors must be zero and the only group is the homogeneous one.

LIST OF HOMOGENEOUS GROUPS GENERATING INVOLUTIONS OF
GEOMETRIC GENUS ONE ON PICARD THREEFOLDS

(A) Cyclic groups

$G_3(I)$	$S(\varepsilon x, \varepsilon y, \varepsilon z)$	$G_6(VIII)$	$S(-x, \varepsilon y, -\varepsilon^2 z)$
$G_2(II)$	$S(x, -y, -z)$	$G_7(IX)$	$S(\omega, x, \omega^2 y, \omega^4 z)$
$G_4(III)$	$S(-x, iy, iz)$	$G_8(X)$	$S(-x, \omega, y, i\omega, z)$
$G_6(IV)$	$S(\varepsilon x, -\varepsilon y, -\varepsilon z)$	$G_9(XI)$	$S(ix, \omega, y, -\omega, z)$
$G_3(V)$	$S(x, \varepsilon y, \varepsilon^2 z)$	$G_{12}(XII)$	$S(-x, i\varepsilon y, i\varepsilon^2 z)$
$G_4(VI)$	$S(x, iy, -iz)$	$G_{12}(XIII)$	$S(\varepsilon x, i\varepsilon y, -i\varepsilon z)$
$G_6(VII)$	$S(x, -\varepsilon y, -\varepsilon^2 z)$		

(B) Multiplicative groups

$G_4(XIV)$	$S(-x, y, -z), S_1(x, -y, -z), \lambda_1, \lambda_2 = \pm 1$
$G_9(XV)$	$S(\varepsilon x, \varepsilon y, \varepsilon z), S_1(x, \varepsilon y, \varepsilon^2 z), \lambda_1, \lambda_2 = 1, \varepsilon, \varepsilon^2$
$G_8(XVI)$	$S(-x, iy, iz), S_1(-x, y, -z), \lambda_1 = \pm 1, \lambda_2 = \pm 1, \pm i$
$G_{16}(XVII)$	$S(-x, iy, iz), S_1(ix, -y, iz), \lambda_1, \lambda_2 = \pm 1, \pm i$
$G_{12}(XVIII)$	$S(-x, \varepsilon y, -\varepsilon^2 z), S_1(-x, -y, z), \lambda_1 = \pm 1, \lambda_2 = \pm 1, \pm \varepsilon, \pm \varepsilon^2$
$G_{12}(XIX)$	$S(\varepsilon x, -\varepsilon y, -\varepsilon z), S_1(-x, y, -z), \lambda_1 = \pm 1, \pm \varepsilon, \pm \varepsilon^2, \lambda_2 = \pm \lambda_1$
$G_{18}(XX)$	$S(\varepsilon x, -\varepsilon y, -\varepsilon z), S_1(x, \varepsilon y, \varepsilon^2 z), \lambda_1 = 1, \varepsilon, \varepsilon^2, \lambda_2 = \pm 1, \pm \varepsilon, \pm \varepsilon^2$
$G_{36}(XXI)$	$S(\varepsilon x, -\varepsilon y, -\varepsilon z), S_1(-x, \varepsilon y, -\varepsilon^2 z), \lambda_1, \lambda_2 = \pm 1, \pm \varepsilon, \pm \varepsilon^2$

(C) Reducible ternary groups of the regular solids

(a) Dihedral groups

$G_8(XXII)$	$S(x, -y, -z), T(-y, x, z)$
$G_{32}(XXIII)$	$S(-x, iy, iz), T(-y, x, z)$

$G_{24}(\text{XXIV})$	$S(\varepsilon x, -\varepsilon y, -\varepsilon z), T(-y, x, z)$
$G_8(\text{XXV})$	$S(x, iy, -iz), T(x, -z, y)$
$G_{12}(\text{XXVI})$	$S(x, -\varepsilon^2 y, -\varepsilon z), T(x, -z, y)$
$G_{72}(\text{XXVII})$	$S(x, -\varepsilon^2 y, -\varepsilon z), T(-y, x, z)$
$G_{24}(\text{XXVIII})$	$S(-x, \varepsilon y, -\varepsilon^2 z), T(x, -z, y)$
$G_{36}(\text{XXIX})$	$S(-x, \varepsilon y, -\varepsilon^2 z), T(z, y, -x)$
$G_{16}(\text{XXX})$	$S(-x, \omega_8 y, i\omega_8 z), T(x, -z, y)$
$G_{16}(\text{XXXI})$	$S(ix, \omega_8 y, -\omega_8 z), T(x, -z, y)$
$G_{24}(\text{XXXII})$	$S(-x, i\varepsilon y, i\varepsilon^2 z), T(x, -z, y)$
$G_{24}(\text{XXXIII})$	$S(\varepsilon x, i\varepsilon y, -i\varepsilon z), T(x, -z, y)$
$G_{16}(\text{XXXIV})$	$S(-x, iy, iz), S(-x, y, -z), T(x, -z, y)$
$G_6(\text{XXXV})$	$S(x, \varepsilon y, \varepsilon^2 z), T(-x, z, y)$
$G_{18}(\text{XXXVI})$	$S(x, \varepsilon y, \varepsilon^2 z), T(y, x, -z)$
$G_8(\text{XXXVII})$	$S(x, iy, -iz), T(-x, z, y)$
$G_{12}(\text{XXXVIII})$	$S(x, -\varepsilon^2 y, -\varepsilon z), T(-x, z, y)$
$G_{36}(\text{XXXIX})$	$S(-x, \varepsilon y, -\varepsilon^2 z), T(z, -y, x)$
$G_{16}(\text{XL})$	$S(-x, \omega_8 y, i\omega_8 z), T(-x, z, y)$
$G_{24}(\text{XLI})$	$S(\varepsilon x, i\varepsilon y, -i\varepsilon z), T(-x, z, y)$
$G_{24}(\text{XLII})$	$S(-x, i\varepsilon y, i\varepsilon^2 z), T(ix, \omega_8 z, \omega_8 y)$
$G_{32}(\text{XLIII})$	$S(-x, iy, iz), T(\omega_8 y, \omega_8 x, iz)$

(b) Reducible tetrahedral groups

$G_{24}(\text{XLIV})$	$S(x, iy, -iz), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$
$G_{48}(\text{XLV})$	$S(-x, y, -z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$
$G_{72}(\text{XLVI})$	$S(\varepsilon x, i\varepsilon y, -i\varepsilon z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$
$G_{24}(\text{XLVII})$	$S(x, iy, -iz), U(\varepsilon x, \frac{1}{2}(1+i)\varepsilon(y+z), -\frac{1}{2}(1-i)\varepsilon(y-z))$

(c) Reducible octahedral groups

$$G_9(\text{XLVIII}) \quad S(-ix, iy, z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$$

$$G_{48}(\text{XLIX}) \quad S(-x, i\omega_8 y, \omega_8 z), U(x, \frac{1}{2}(1+i)(y+z), -\frac{1}{2}(1-i)(y-z))$$

(D) Monomial groups

$$G_{21}(\text{L}) \quad S(\omega_7 x, \omega_7^2 y, \omega_7^4 z), P(y, z, x)$$

$$G_{12}(\text{LI}) \quad S(x, -y, -z), P(y, z, x)$$

$$G_{27}(\text{LII}) \quad S(x, \epsilon y, \epsilon^2 z), P(y, z, x)$$

$$G_{48}(\text{LIII}) \quad S(x, iy, -iz), P(y, z, x)$$

$$G_{36}(\text{LIV}) \quad S(\epsilon x, -\epsilon y, -\epsilon z), P(y, z, x)$$

$$G_{108}(\text{LV}) \quad S(-x, \epsilon y, -\epsilon^2 z), P(y, z, x)$$

$$G_{24}(\text{LVI}) \quad T(-y, x, z), T(x, -z, y)$$

$$G_{96}(\text{LVII}) \quad S(-x, iy, iz), T(-y, x, z), T(x, -z, y)$$

$$G_{72}(\text{LVIII}) \quad S(\epsilon x, \epsilon y, \epsilon z), T(-y, x, z), T(x, -z, y)$$

$$G_{216}(\text{LIX}) \quad S(x, \epsilon y, \epsilon^2 z), T(-y, x, z), T(x, -z, y)$$

$$G_{54}(\text{LX}) \quad S(x, \epsilon y, \epsilon^2 z), T(y, x, -z), T(-x, z, y)$$

$$G_{96}(\text{LXI}) \quad T(\omega_8 y, \omega_8 x, iz), T(ix, \omega_8 z, \omega_8 y)$$

(E) Primitive groups

$$G_{108}(\text{LXII}) \quad S(x, \epsilon y, \epsilon^2 z), T(-y, -x, -z), \\ V\left(\frac{1}{\epsilon - \epsilon^2}(x+y+z), \frac{1}{\epsilon - \epsilon^2}(x+\epsilon y+\epsilon^2 z), \frac{1}{\epsilon - \epsilon^2}(x+\epsilon^2 y+\epsilon z)\right)$$

$$G_{216}(\text{LXIII}) \quad S, T, V \text{ as above,} \\ W\left(\frac{1}{\epsilon - \epsilon^2}(x+y+\epsilon^2 z), \frac{1}{\epsilon - \epsilon^2}(x+\epsilon y+\epsilon z), \frac{1}{\epsilon - \epsilon^2}(\epsilon x+y+\epsilon z)\right)$$

$$G_{168}(\text{LXIV}) \quad S(\omega x, \omega^2 y, \omega^4 z), P(y, z, x), \\ U: \begin{aligned} x' &= h((\omega^4 - \omega^3)x + (\omega^2 - \omega^5)y + (\omega - \omega^6)z) \\ y' &= h((\omega^2 - \omega^5)x + (\omega - \omega^6)y + (\omega^4 - \omega^3)z) \\ z' &= h((\omega - \omega^6)x + (\omega^4 - \omega^3)y + (\omega^2 - \omega^5)z), \\ h &= \frac{1}{7}(\omega + \omega^2 + \omega^4 - \omega^6 - \omega^5 - \omega^3) = \frac{1}{\sqrt{-7}}, \omega^7 = 1. \end{aligned}$$

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Some types of irregular threefold

by RAYMOND HALL (a Londra)

Few examples of irregular threefolds have so far been given: Severi, in his fundamental paper ⁽¹⁾ on threefolds, discusses two types, one being the topological product of a curve and a surface, at least one of which is irrational, and the other the topological product of triads of points of an irrational curve; and Maxwell ⁽²⁾ has discussed the irregular primals of S_4 of order $k\lambda$ having a k ple point at each of the λ^4 points of intersection of four primals of order λ .

In this paper we give further examples of irregular threefolds, the projective models of which are primals of S_4 with suitable multiple elements.

The arithmetic genus P_a of each such threefold is the virtual number of linearly independent canonical surfaces calculated for the model considered by using postulation formulae. The geometric genus P_g is the actual number of linearly independent canonical surfaces and must be found in each case by *ad hoc* methods. Here we use methods based on the extended form of Noether's theorem due to Torelli and Giambelli ⁽³⁾, and it is the justification of these methods which constitutes a considerable part of the work. The difference $P_g - P_a$ is the *tridimensional irregularity* g_1 . Again, the irregularity of a generic ⁽⁴⁾ surface on the threefold is a constant for

⁽¹⁾ F. SEVERI, Rend. Circ. Mat. Palermo, 28 (1909), 33-87: see also L. ROTH, Rend. di Mat. (5) 10 (1951), 297-346.

⁽²⁾ E. A. MAXWELL, Proc. Cambridge Phil. Soc. 32 (1936), 185-93.

⁽³⁾ R. TORRELLI, Atti Acc. Torino 41 (1905), 224-34; G. Z. GIAMBELLI *ibid.* 235-58.

⁽⁴⁾ The surface must belong at least to a net whose free characteristic curve is irreducible; cfr. SEVERI, *loc. cit.*

the threefold; this is the *superficial irregularity* q_2 , and may be calculated by considering a generic prime section.

1. *Primals possessing a multiple curve.* — Consider, in S_4 , a primal V^n , of order n , whose only singularity is an r -ple irreducible non singular curve C^r , of order r and genus ϱ . The canonical system (if any) is cut on V^n by primals of order $n - 5$ passing $r - 2$ times through C^r . The virtual postulation of C^r for these primals is given by the formula (5)

$$(1) \quad \binom{r}{r-3} r(n-5) - 5 \binom{r}{4} - \left\{ \binom{r}{4} + \binom{r+1}{4} \right\} (\varrho - 1).$$

Subtracting this postulation from the total number of linearly independent primals of S_4 of order $n - 5$, which is $\binom{n-1}{4}$, we have the arithmetic genus P_a .

We take as the curve C the complete intersection of three general primals, each of order m , with equations $A_1 = 0, A_2 = 0, A_3 = 0$; the curve is of order m^3 and genus $p = \frac{3}{2} m^3(m-1) - (m^3 - 1)$. Equating to zero a general form of degree r in A_1, A_2, A_3 , with constant coefficients, we have the equation of an irreducible primal V^{rm} , of order rm , on which C is r -ple. The canonical system of V^{rm} is cut by primals V^{rm-5} passing $r - 2$ times through C . By (1), the arithmetic genus is given by

$$P_a = \binom{rm-1}{4} - \binom{r}{r-3} m^3(rm-5) + 5 m^3 \binom{r}{4} + \frac{1}{2} m^3(3m-5) \left\{ \binom{r}{4} + \binom{r+1}{4} \right\}.$$

To find the geometric genus we use the method developed by Maxwell⁽⁶⁾. The argument applies here at once on making the obvious modifications. First, by the theorem of Torrelli and Giambelli,

(5) B. SEGRE, Proc. Cambridge Phil. Soc. 38 (1942), 368-77

(6) E. A. MAXWELL, *op. cit.*

the equation of any one of the required adjoints may be written in the form

$$\Sigma M_{\alpha\beta\gamma} A_1^\alpha A_2^\beta A_3^\gamma = 0,$$

where $\alpha + \beta + \gamma = r - 2$ and $M_{\alpha\beta\gamma}$ is of degree $2m - 5$. To obtain the linearly independent members of this system we must, following Maxwell, write $M_{\alpha\beta\gamma}$ in the form

$$M'_{\alpha\beta\gamma} + \Theta A_1 + \Phi A_2 + \Psi A_3,$$

where Θ, Φ, Ψ are general forms of degree $m - 5$ and $M'_{\alpha\beta\gamma}$ consists of the remaining linearly independent terms of $M_{\alpha\beta\gamma}$. Thus the number of terms in $M'_{\alpha\beta\gamma}$ is $\binom{2m-1}{4} - 3\binom{m-1}{4}$. The equation of an adjoint primal, expressed in terms of the linearly independent members (7), may now be written in the form

$$(2) \quad \Sigma M'_{\alpha\beta\gamma} A_1^\alpha A_2^\beta A_3^\gamma + \Sigma \Theta_{stu} A_1^s A_2^t A_3^u = 0, \quad \begin{cases} \alpha + \beta + \gamma = r - 2 \\ s + t + u = r - 1 \end{cases}$$

where $M'_{\alpha\beta\gamma}$ is defined as above and Θ_{stu} is a general form of degree $m - 5$. Since there are $\binom{r}{2}$ polynomials $M'_{\alpha\beta\gamma}$ and $\binom{r+1}{2}$ polynomials Θ_{stu} , we have that

$$P_g = \binom{r}{2} \left\{ \binom{2m-1}{4} - 3\binom{m-1}{4} \right\} + \binom{r+1}{2} \binom{m-1}{4}.$$

The tridimensional irregularity is given by

$$q_1 = P_g - P_a = \frac{(r-1)(r-2)}{2}.$$

Taking a generic prime section of the threefold we have a surface of the first type considered by Maxwell, whence the superficial irregularity is given by

$$q_2 = \frac{(r-1)(r-2)}{2}.$$

(7) That the members displayed in this form are linearly independent is proved below.

It remains to be proved that the members exhibited in (2) are linearly independent; actually we shall prove a more general result than that required here.

Consider, in S_n , a set of $l (\leq n)$ general primals $P_1, P_2, \dots, P_l$ of the same order m and a primal of order $(r+k)m + \delta$ ($0 \leq \delta < m$) passing r -ply through their intersection. If Θ_s denotes a general polynomial of degree $s m + \delta$, the equation of this primal is of the form

$$\sum \Theta_k (P)_r = 0,$$

where $(P)_r$ denotes a homogeneous product of total degree r in the P_i s. To ensure that no two members of this are identical, we write the equation in the form

$$(3) \sum \Theta'_k (P)_r + \sum \Theta'_{k-1} (P)_{r+1} + \dots + \sum \Theta'_1 (P)_{r+k-1} + \sum \Theta_0 (P)_{r+k} = 0,$$

where Θ'_s denotes the terms which arise when Θ_s is expressed in the form

$$\Theta'_s + \sum \Theta_{s-1} (P)_1.$$

We now prove that the members of (3) are linearly independent.

Suppose that all the terms involving $\Theta_0, \Theta'_1, \dots, \Theta'_{p-1}$ are linearly independent but that, when we introduce Θ'_p , we have a linear identity: then this is of the form

$$(4) \quad \sum \Theta'_p (P)_{r+k-p} + \sum \Psi_{p-1} (P)_{r+k-p+1} \equiv 0,$$

where the second sum consists of terms involving $\Theta'_{p-1}, \dots, \Theta_0$. We first cancel out any common factor P_i which appears throughout (4). Now take any P_i , say P_1 , which appears in a Θ'_p term in (4); collecting all the terms independent of P_1 on the left, we have

$$\begin{aligned} & \sum \Theta'_p (P_2, \dots, P_l)_{r+k-p} + \sum \Psi_{p-1} (P_2, \dots, P_l)_{r+k-p+1} \\ & \equiv P_1^a \{ \sum \Theta'_p (P)_{r+k-p-a} + \sum \Psi_{p-1} (P)_{r+k-p+1-a} \}, \end{aligned}$$

where $a > 0$ and where there are terms Θ'_p on the right. Now the left-hand side, equated to zero represents a primal passing $r+k-p$ times through $P_2 \cdot P_3 \dots P_l$ and hence so does the bracket on the right. We have therefore

$$\sum \Theta'_p (P)_{r+k-p-a} + \sum \Psi_{p-1} (P)_{r+k-p+1-a} \equiv \sum \Psi_{p-a} (P)_{r+k-p},$$

and, absorbing the terms on the right into the second sum on the left, we have another identity of the form (4) with r replaced by $r - a (< r)$. We can therefore repeat the process. Proceeding in this way we arrive either at an identity involving only one term Θ'_p or else at a value of r such that $r + k - p$ is less than the number of P_i 's remaining in the Θ'_p terms of the identity. In the latter case we choose one of the P_i 's which appears in some but not all of the Θ'_p terms, and so reduce the number of Θ'_p terms in the next identity obtained. With these processes of reducing r , eliminating Θ'_p terms and cancelling any P_i which appears throughout (4), we can always obtain eventually an identity

$$\Theta'_p (P)_{r+k-p} + \Sigma \Psi_{p-1} (P)_{r+k-p+1} \equiv 0,$$

involving only one Θ'_p term, by choosing at the last stage a P_i which appears in only one Θ'_p term.

We can now, by the above method, eliminate each P_i from the Θ'_p term of this identity and obtain finally

$$\Theta'_p + \Sigma \Psi_{p-1} (P)_1 \equiv 0.$$

Since this contradicts the definition of Θ'_p we conclude that the members of (3) must be linearly independent.

2. *Primals with a multiple surface.* — We consider a type of threefold with a double surface, the surface consisting of two irreducible components. The threefolds have equations of the type

$$(5) \quad P^2 (U^2 + V^2) + Q^2 (U^2 - V^2) = 0,$$

where P, Q are general forms of degree m , and U, V are general forms of degree n . This type of equation has been used by PEDOE⁽⁸⁾ to obtain irregular surfaces in S_3 . On the threefold the surfaces F^{m^2} , given by $P = Q = 0$, and F^{n^2} , given by $U = V = 0$, are double. Canonical surfaces are cut by primals of order $2(m + n) - 5$ passing simply through F^{m^2} and F^{n^2} . We show that any such primal has an equation of the form

$$(6) \quad U(A_1 P + A_2 Q) + V(A_3 P + A_4 Q) = 0,$$

where A_i is of degree $m + n - 5$.

(8) D. PEDOE, Proc. Cambridge Phil. Soc. 31 (1935), 48-9.

By the extension of Noether's theorem, the equation of any adjoint primal (which must contain F^{m^2}), is of the form

$$A U + B V = 0,$$

where A, B are of degree $2m + n - 5$. Write this, with an obvious notation, in the form

$$(A' + A_1 P + A_2 Q) U + (B' + A_3 P + A_4 Q) V = 0.$$

To obtain (6) we require to show that any member of the system

$$(7) \quad A' U + B' V = 0$$

which contains F^{m^2} is included in (6). For any member of this system which contains F^{m^2} we have an identity

$$A' U + B' V \equiv L P + M Q,$$

or

$$A' U \equiv L P + M Q - B' V,$$

whence $A' U = 0$ passes through the curve $P = Q = V = 0$. Since U does not contain this curve it follows that A' must contain it and so, remembering the definition of A' , we have an identity of the form

$$A' \equiv H V; \text{ and similarly } B' \equiv K U.$$

With these values in (7) the equation becomes

$$(H + K) U V = 0;$$

and if the primal so represented contains F^{m^2} , we must have

$$H + K \equiv X P + Y Q;$$

hence this primal is included in (6), which therefore gives the entire system.

We may note that the proof applies, with trivial modification, to the corresponding problem in three or any higher number of dimensions.

To find P_g we require the number of linearly independent terms in (6). There is no loss of generality in assuming $n \geq m$; let $n = cm + d$ where $0 \leq d < m$, let Θ_s denote a polynomial of degree $sm + d - 5$ and let $(P)_s, (U)_s$ denote products of degree s in P, Q and in U, V respectively. We write (6) in the form

$$\Sigma A'(P)_1(U)_1 + \Sigma \Theta_c(P)_2(U)_1 + \Sigma \Phi(P)_1(U)_2 = 0,$$

where Φ is of degree $m - 5$, and then in the form

$$(8) \quad \Sigma \Phi(P)_1(U)_2 + \{ \Sigma A'(P)_1(U)_1 + \Sigma \Theta'_c(P)_2(U)_1 + \\ + \Sigma \Theta'_{c-1}(P)_3(U)_1 + \dots + \Sigma \Theta'_1(P)_{c+1}(U)_1 + \Sigma \Theta_0(P)_{c+2}(U)_1 \} = 0,$$

where Θ_0 is zero if $d < 5$.

To show that the terms exhibited in (8) are linearly independent, we absorb the Φ terms in the A' terms and write the equation as

$$(9) \quad \Sigma A^*(P)_1(U)_1 + \Sigma \Theta'_c(P)_2(U)_1 + \dots + \Sigma \Theta'_1(P)_{c+1}(U)_1 + \\ + \Sigma \Theta_0(P)_{c+2}(U)_1 = 0.$$

Written in this form, the set of terms having U as factor are linearly independent, by the result obtained at the end of § 1 and, similarly, so are the terms having V as a factor. Suppose now that a combination of terms of the former set, which we can write as

$$(10) \quad U \{ \Sigma A^*(P)_1 + \Sigma \Theta_c(P)_2 \},$$

is identical with a combination of terms from the latter, which will possess the factor V . We must then have an identity of the form

$$\Sigma A^*(P)_1 + \Sigma \Theta_c(P)_2 \equiv V \Sigma \Phi(P)_1,$$

which may be rearranged as

$$(11) \quad P \{ A^* + \Theta_c P + V \Phi \} \equiv Q \{ A^* + \Sigma \Theta_c(P)_1 + V \Phi_1 \};$$

this shows that the bracket on the left has Q as a factor, and so we have an identity of the form

$$A^* + \Sigma \Theta_c(P)_1 + V \Phi \equiv 0.$$

From the definition of A^* , such an identity is only possible if each Θ_c in it is zero. This proves that there is no term in P^2 in (10). Similarly there is no term in Q^2 . To see that there is no term in PQ we deal as above with the bracket on the right of (11). A similar argument holds for the set of terms having V as a factor; and as a result we have established that any linear dependence in (9) involves only the A^* terms. To express these in terms of linearly independent members it is easily seen, by an argument of the above type, that we must write them as in (8); and this proves the result.

To find the number of terms in (8), let $N(\theta')$ denote the number of terms in θ' , and so on. The required number is then

$$(12) \quad 6N(\Phi) + 2\{2N(A') + 3N(\theta'_c) + 4N(\theta'_{c-1}) + \dots \\ + (c+2)N(\theta'_1) + (c+3)N(\theta_0)\},$$

where

$$N(\theta_c) = N(\theta'_c) + 2N(\theta'_{c-1}) + 3N(\theta'_{c-2}) + \dots$$

$$+ cN(\theta'_1) + (c+1)N(\theta_0)$$

and

$$N(A) = N(A') + 2N(\Phi) + 2N(\theta'_c) + 3N(\theta'_{c-1}) + \dots$$

$$+ (c+1)N(\theta'_1) + (c+2)N(\theta_0).$$

Substituting for $N(A')$ in (12) and using the expression for $N(\theta_c)$, we find that (12) reduces to

$$4N(A) - 2N(\theta_c) - 2N(\Phi).$$

Thus the geometric genus is given by

$$P_g = 4 \binom{m+n-1}{4} - 2 \binom{n-1}{4} - 2 \binom{m-1}{4}.$$

To find the arithmetic genus we require the virtual postulation for $V^{2(m+n)-5}$ of a surface F^{m^2} , of arithmetic genus equal to the number of linearly independent primals of order $2m-5$ i. e. $\binom{2m-1}{4}$, and sectional genus $p = m^3 - 2m^2 + 1$, together with

a surface P^{n^2} having corresponding characters, the two surfaces meeting in $m^2 n^2$ points. Using the postulation formulae we have

$$P_a = \binom{2m + 2n - 1}{4} - \left\{ (m^2 + n^2) \binom{2m + 2n - 4}{2} - (2m + 2n - 5)(m^3 + n^3 - 2m^2 - 2n^2) + \binom{2m - 1}{4} + \binom{2n - 1}{4} + 2 \right\} + m^2 n^2.$$

The tridimensional irregularity is given by

$$q_1 = P_g - P_a = \frac{1}{12} \{ m^4 + n^4 - 10(m^3 + n^3) + 35(m^2 + n^2) - 50(m + n) + 36 \}.$$

Taking a general prime section of the threefold, we have a surface whose equation may be written in the form (5); the equations of the adjoint surfaces are of the form (6), where now the A_i are of degree $m + n - 4$. By an analysis similar to the above we find that the geometric genus of this surface section is given by

$$p_g = 4 \binom{m + n - 1}{3} - 2 \binom{n - 1}{3} - 2 \binom{m - 1}{3},$$

and from the postulation formulae the arithmetic genus is given by

$$p_a = 3 + 2 \binom{n - 1}{3} + 2 \binom{m - 1}{3} + 2mn(m + n - 4).$$

Thus the superficial irregularity of the primal is given by

$$q_2 = p_g - p_a = 1.$$

One interesting result may be singled out. The sextic threefold primal possessing a double plane and double Segre quartic surface (obtained when P, Q are linear and U, V quadratic) has characters $P_a = 1, P_g = 0$, and may be obtained from a sextic threefold possessing a double Segre surface, for which $P_a = P_g = 0$, by making this contain further a double plane. The imposition of this additional double locus thus *increases* the arithmetic genus by unity.

THREEFOLDS POSSESSING AN INFINITE GROUP OF
BIRATIONAL SELF-TRANSFORMATIONS

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The algebraic curves and surfaces invariant under an infinite group of birational transformations have been investigated in great detail; some threefolds with this property have been well known for some time and

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others have been considered recently by Roth ([10], [11], [12]). In this paper we survey the possible types of such threefold and examine the question of their existence.

The set of all (birational) transformations leaving a threefold invariant forms a group which may be, or contain, a finite continuous group, or contain continuous series of transformations, or else be (strictly) discontinuous. These possibilities are considered in turn.

1. *Finite continuous groups.* We first remark that, if an algebraic variety is invariant for a finite continuous group of transformations, it is necessarily invariant for a group in which the parameters figure *algebraically*. Further, it is always possible to find a sub-group of this group which is *commutative* (and in which the parameters still appear algebraically). These two results are given by Painlevé ([6], 257, 266) with particular reference to surfaces, but his method is quite general. Thus we need only consider algebraic groups; in §4 we use the fact of the existence of commutative sub-groups.

2. *One-parameter groups.* In this case the trajectories of the group are the ∞^2 rational or elliptic curves of a congruence*. We show that if the curves are rational the threefold is *scrollar*, i.e. birationally equivalent to a line-ruled threefold. For this purpose we require to determine a surface unisecant to the curves of the congruence; the transformation to a line-ruled threefold is then classical.

Now the ∞^1 group of transformations determined on a general curve of the congruence consists of all the projectivities with two fixed united points or else all the elations with one fixed united point. In the latter case, the locus of this single united point gives the unisecant surface desired. When the united points are in general distinct, their locus is a bisecant surface. We show that this surface must degenerate into two unisecant surfaces. To see this, we apply the result of Painlevé ([6], 270 *et seq.*) that if a surface is invariant for a rational ∞^1 group of transformations the locus of united points (assumed distinct) consists of two curves each unisecant to the (rational) trajectories.

Let the congruence of rational curves on the threefold, V_3 , be mapped on the points of a surface F . To a curve on F corresponds a surface on V_3 compounded of the congruence and, by Painlevé's result, the locus of united points on this surface degenerates into two curves, these lying on the surface, Φ say, the locus of united points on V_3 . Now let $|C|$ be an ∞^r ($r > 1$) linear system of irreducible curves of positive grade on F . To $|C|$ corresponds a rational system of surfaces on V_3 . Further,

* We intend always a congruence of index one.

r general points of V_3 determine r curves of the congruence and so r points of F ; and these points determine a unique curve of $|C|$ to which corresponds a unique surface of the rational system on V_3 passing through the r chosen points, *i.e.* the system of surfaces is of index one, and so is linear. This linear system of surfaces will meet the surface Φ in a linear system of curves, each curve of which degenerates into two variable components. This system is not compounded of a pencil since it has positive grade. By Bertini's theorem, this situation is impossible on an irreducible surface. We conclude that Φ degenerates into two unisecant surfaces, and hence that V_3 is scrollar.

When the trajectories of the group are elliptic curves there is a birational correspondence between the points of any one of these curves and the operations of the group, or the points of a curve mapping the group. Thus all the curves have the same modulus, and any surface compounded of the congruence is an elliptic surface. A finite or infinite number of the curves, considered as members of the congruence, may be reducible; each of these will be a multiple curve as is seen by considering one of the elliptic surfaces through it. Such threefolds have been termed *elliptic* by Roth [11].

So, finally,

A threefold invariant under an ∞^1 algebraic group of birational transformations is scrollar or elliptic.

3. *Two-parameter groups.* If a threefold is invariant for an ∞^2 algebraic group, the trajectories of a point may be rational curves, or the rational, hyperelliptic or elliptic* scroller surfaces of a pencil.

In the first case, the group leaving a general point invariant is ∞^1 and so, as above, the threefold is scrollar.

If the trajectories are rational surfaces, we show that the threefold is *planar*, *i.e.* birationally equivalent to a plane-ruled threefold. We first determine an ∞^1 rational sub-group of the given group. To this end we consider the ∞^2 group as operating on any particular one of the rational trajectories, say A .

Since A is a regular surface, we can determine a birational transform $\bar{A}$ of A such that the corresponding transformations of $\bar{A}$ are projective. For on A consider any simple linear, or more generally continuous, system of curves. The transforms of these curves under the ∞^2 group form a continuous system which must be contained in a complete (simple) linear system, as the surface is regular. This linear system is invariant for the ∞^2 group, and its projective model gives the desired surface $\bar{A}$. This technique is due to Fano (*cf.* [5]).

* If the transforms of a point lie on a general scrollar surface, the trajectories must form the pencil of rational curves on the surface, and so we have, in fact, a congruence of rational curves as trajectories; the same remark applies in §4.

Now given any ∞^2 group, we can determine many ∞^1 sub-groups; and further, if the group is both algebraic and projective, a sub-group may be found which is also algebraic. This follows from the fact that any ∞^2 group is integrable (since a non-integrable group must contain an ∞^3 simple sub-group) and the integrable projective groups have all been classified*; when algebraic, such a group contains a descending chain of invariant *algebraic* sub-groups. (It may be noted that a group which is algebraic but not projective does not necessarily contain ∞^1 algebraic sub-groups; thus the group for the general Picard surface cannot do so, for otherwise the surface would contain a pencil of rational or elliptic curves.)

The ∞^1 rational group which we have just found operates on the particular trajectory A chosen, when we transfer back from $\bar{A}$ to A . If each transformation of A arises from one transformation of the original group for the entire threefold†, we have at once the ∞^1 rational sub-group desired. It may, however, happen that a given transformation of A arises from a finite number $n > 1$ of transformations of the threefold; this will occur if there is some transformation, other than identity, leaving A totally invariant. In this case, the transformations of the original group giving rise to the ∞^1 group on A form an algebraic group, which may be reducible. In any case, it consists of m ($1 \leq m \leq n$) components; each component is of dimension 1; and one (at least) must contain the identity and will be a continuous algebraic group. This theorem for a reducible algebraic group is proved by Enriques and Chisini (*Teoria geometrica* ..., Vol. III, 250-1); the fact that the transformations there are homographies is incidental and does not enter into the proof. Since the trajectories on A of this continuous group are rational curves, the group must itself be rational ([2], 243), and so in every case we have the required sub-group.

The trajectories of the ∞^1 group are rational curves, ∞^1 of which lie on each rational surface. By means of this group, the threefold may be transformed into a line-ruled threefold, say a cone, and the pencil of rational surfaces transforms into a pencil of cones. The generic surface section of the threefold cone thus contains a pencil of rational curves and so is scrollar. Transforming this surface into a scroll, we have the original threefold transformed into a cone generated by ∞^1 planes. Algebraically, assuming that the threefold V_3 lies in $S_4(x_0, x_1, \dots, x_4)$, the first transformation gives an equation which can be put in the form

$$V'(x_0', \dots, x_3') = 0,$$

* See the account of ∞^2 groups of projective transformations given in §§ 9, 10 of Enriques-Fano, *Annali di Mat.* (2), 26 (1897), 59-99.

† This always happens when the trajectories are elliptic curves or hyperelliptic surfaces (cf. [10], 400); any other situation is only possible with rational trajectories,

and the second transformation which is of the form

$$x_i'' = f_i(x_0', \dots, x_3') \quad (i = 0, \dots, 3),$$

$$x_4'' = x_4'^n,$$

where n is the degree of the forms f_i , leads to an equation

$$V''(x_0'', x_1'', x_2'') = 0,$$

representing a plane cone, whence

Any threefold which is invariant under an ∞^2 group whose trajectories are rational surfaces is birationally equivalent to a plane-ruled threefold.

When the trajectories of the ∞^2 group are hyperelliptic surfaces (of rank 1) there is one transformation under which a point P of a general surface of the pencil transforms into a given point P' of the same surface. There is thus a birational correspondence between any surface of the pencil and a surface mapping the transformations of the group, and so a correspondence between any two surfaces of the pencil. A finite number of the surfaces of the pencil may be multiple, consisting of an irreducible surface counted a certain number of times. These threefolds have been studied by Roth [10], who terms them *hyperelliptic*.

The remaining possibility is that the trajectories of the ∞^2 group are elliptic scrollar surfaces. In this case the transformations leaving the rational curves invariant form an ∞^1 sub-group and the threefold is a particular case of a scrollar threefold. So

A threefold invariant under an ∞^2 algebraic group of birational transformations is scrollar, planar or hyperelliptic.

There are, of course, particular cases, *e.g.* birational threefolds.

4. *Three-parameter groups.* The trajectories of an ∞^3 algebraic group may be rational curves, rational or elliptic scrollar surfaces, or the group may operate transitively on the threefold.

In any of the *intransitive* cases, earlier methods at once show that *the threefold is scrollar or planar*.

When the group is transitive but not commutative, we can, as pointed out in §1, obtain an ∞^1 or ∞^2 commutative sub-group; the threefold may then be treated by the methods of §§2, 3.

There remains the case of an ∞^3 transitive commutative group. This type of threefold has been considered in detail by Severi ([13], 166), who

shows that it is either an Abelian threefold of rank 1 or else is birationally equivalent to the product of an Abelian variety and a flat space, i.e. a hyperelliptic surface and line or an elliptic curve and plane or a (point and) solid.

A particular case of an Abelian threefold is that in which the parametric representation is by functions of genus less than three; the threefold is then termed *improperly* Abelian ([12], 406). If the functions are of genera two and one, the threefold contains a pencil of hyperelliptic surfaces and also a congruence of elliptic curves; it is thus both hyperelliptic and elliptic. If the functions are all of genus one, the threefold contains three congruences of elliptic curves and so may be regarded as elliptic in three different ways.

5. The above results may be summarized thus:

A threefold invariant under a finite continuous group of birational transformations is birational, planar, scollar, Abelian, hyperelliptic or elliptic.

6. We next consider *threefolds invariant for a continuous series of transformations but not for any finite continuous group*. There are no algebraic curves or surfaces with this property.

The possibilities are that the transforms of a general point lie on a curve or surface or are not restricted to any algebraic sub-variety. When the transforms lie on a curve, this must be rational or elliptic; we prove that the former case can arise, but not the latter.

The threefolds of the required type are those containing a congruence of rational curves which cannot be transformed into lines, provided always that a finite continuous group does not arise in some way unconnected with the congruence. For the curves of the congruence may be reduced to conics on a suitable transform*, and then a prime of the space determines a pair of points on a general conic of the system and so an involutory transformation having these for double points; these involutions and their products give the required series. One such threefold is the "Enriques primal" V_3^6 of S_4 ([9], 267). This possesses six double planes, the intersections two by two of four primes, four triple lines and a quadruple point; the generic prime section is a regular Enriques surface with geometric genus zero and bigenus one. If we fix on a pair of "opposite" double planes (i.e. planes meeting only in the quadruple point), then, through any simple point of the primal, there passes a unique plane meeting these planes in lines and so meeting the primal residually in a conic, in general irreducible. These conics form a rational congruence (of index one); each of the three pairs of opposite double planes gives rise to a different congruence. Now

* See [3], 268 for a discussion of this reduction.

the primal is unirational but not birational and so the rational congruence of conics cannot be transformed into a congruence of lines, for if this were possible it would establish the birationality of the primal. Thus this primal illustrates the type in question. In this case the curves of the congruence are already conics on the model considered.

To see that the transforms of a point cannot lie on an elliptic curve we remark that any surface compounded of such a congruence would be an elliptic surface. On this surface, a transformation of the second species, given on one of the curves, determines a unique transformation, and hence a unique transformation on each of the curves of the pencil. On the threefold, then, choose a general curve C of the congruence and a pencil of (elliptic) surfaces compounded of the congruence and each containing C . Every transformation of the elliptic group on C determines a unique transformation of the other curves of the congruence, whence the threefold is invariant under this one-parameter group, contrary to hypothesis. Thus the curves of the congruence cannot be elliptic.

7. If the transforms of a point under a continuous series of transformations are restricted to a surface, this surface must be rational, scrollar, elliptic or hyperelliptic. The scrollar and elliptic surfaces give congruences of invariant curves, which have been considered in the previous section; the elliptic scrollar surface is no exception since those transformations leaving the rational curves invariant must still form a continuous series. There remain the rational and hyperelliptic surfaces. The existence of threefolds of the required type containing pencils of such surfaces is suggested, but not definitely established, by the following considerations.

In an attempt to obtain a threefold on which the transforms of a point are *necessarily* restricted to a rational surface (as opposed to the example given at the end of §8 below), we may make use of Baldassarri's result ([1], 397) that, given a V_3^m of S_4 with an $(m-3)$ -ple plane, there exists a system of curves unisecant to the cubic surfaces of the pencil determined by primes through the plane. Each curve determines an involutory transformation, by projection from the points where it meets each surface, and so we have a continuous series of such transformations. Unless every such threefold is planar, and so birational, we thus obtain models not invariant under a finite continuous group.

That no example of this type can be constructed containing a pencil of quadric surfaces, follows from Conforto's result ([3], 279) that a threefold with a pencil of quadric surfaces is necessarily planar.

In the case of a threefold containing a pencil of hyperelliptic surfaces, the obvious type to look for is that in which the surfaces are of varying moduli and have a continuous system of unisecant curves; the two points determined on a surface by any two curves taken in one order or the other would give a unique transformation of the second species. More generally

the curves might be multisequant, giving sets of points the sums of whose parameters are not equivalent (*cf.* [4], 669). It would be desirable to construct the parametric equations of such a variety, if one exists.

8. Turning to types for which the transforms of a point invade the whole variety, we have Fano's result [5] that a threefold which is unirational but not birational cannot admit finite continuous groups of transformations. The general non-singular cubic primal in S_4 certainly admits continuous series of transformations, for example those obtained as products of projections from the points of the primal, and to show that there are no finite continuous groups would be equivalent to proving the irrationality of the primal; this has been pointed out by Snyder ([14], 344), who has constructed various series of transformations. By taking only the products of projections of the cubic primal from the points of a plane section, we have a continuous series of transformations leaving invariant the surfaces of the pencil through this plane.

9. *Threefolds with an infinite discontinuous group of transformations* (but no continuous series). Here, as before, the transforms of a point may lie on a curve or surface, or not be restricted to any algebraic sub-variety.

If a threefold contains a congruence of rational curves, we have seen that it possesses continuous series of transformations. On the other hand, a variety with a congruence of elliptic curves may give rise to a discontinuous group. Examples of such varieties are the quartic primal with two nodes, the V_3^8 and V_3^{18} given by Roth ([8], 127), and the product of a curve of genus greater than unity and any surface with a pencil of elliptic curves invariant for a discontinuous group; for references to the literature on such surfaces see [15].

10. When the threefold contains a pencil of invariant surfaces not compounded of a congruence of elliptic curves, these surfaces may be rational, hyperelliptic or surfaces (with all their genera equal to unity) of the type invariant for a discontinuous group ([4], 665).

An example of the last kind is immediately provided by the product of such a surface and a curve of genus greater than 1.

There are no known examples of threefolds of this type which contain a pencil of rational or hyperelliptic surfaces; one possible construction for the latter is similar to that indicated at the end of §7, except that now only a finite set of secant curves is called for. By analogy with Enriques's construction ([4], 669) for surfaces with a pencil of elliptic curves, we might specify such a threefold by imposing on hyperelliptic surfaces the conditions of meeting given curves in specified numbers of points so as to

reduce the freedom to one; but it would be preferable to have a parametric representation.

11. Finally we have the case of a discontinuous group where the transforms of a point are not restricted. An example is provided by a quartic primal with a finite number (≥ 4) of nodes, not all lying in the same plane. The transformations are obtained as products of the projections from the nodes. Other examples may be obtained as generalizations of the surfaces given by Painlevé [7]. Thus the threefold given (in non-homogeneous coordinates in S_5) by

$$x_1 = \wp(u_1), \quad x_2 = \wp(u_2), \quad x_3 = \wp(u_3), \quad x_4 = \frac{\wp'(u_1)}{\wp'(u_2)}, \quad x_5 = \frac{\wp'(u_2)}{\wp'(u_3)},$$

each point of which corresponds to two non-congruent sets of parameters of the form (u_1, u_2, u_3) , $(-u_1, -u_2, -u_3)$, is invariant under the transformations

$$u_i' = a_{ij} u_j,$$

where the a_{ij} are integers such that $|a_{ij}| = \pm 1$.

To see this, we observe that the threefold is invariant for the substitutions permuting the parameters, changing signs, and also for

$$u_1' = u_1 + u_2, \quad u_2' = u_2, \quad u_3' = u_3,$$

and five similar substitutions, *i.e.* those adding any row of the column-vector of the u_i (and so any integral multiple of a row) to any other row. We have to show how to build up the above matrix (a_{ij}) from these substitutions or, what is equivalent (since the inverse of one of these substitutions is of the same kind, or a product of substitutions of the same kind), how to obtain the unit matrix from (a_{ij}) by means of these substitutions.

Since $|a_{ij}| = \pm 1$, the elements of the third column cannot have any common factor and so, by suitable operations, we can make $a_{33} = 1$; and then make $a_{13} = a_{23} = 0$. In the new matrix, a_{12} , a_{22} must be co-prime and so we can make $a_{22} = 1$; and then $a_{12} = 0$. It now follows that $a_{11} = \pm 1$, and so can be made $+1$. The matrix now has the form

$$\begin{pmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{pmatrix};$$

we make $a_{21} = 0$ and finally make $a_{31} = a_{32} = 0$. This proves that the threefold is invariant for the group stated; it will also be invariant for other transformations involving half-periods.

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ON ALGEBRAIC VARIETIES WHICH POSSESS FINITE
CONTINUOUS COMMUTATIVE GROUPS OF BIRATIONAL
SELF-TRANSFORMATIONS

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An algebraic variety V_n , of dimension n , invariant under a finite continuous ∞^n commutative transitive group of birational transformations is either a Picard variety (when the group is absolutely transitive) or else is birationally equivalent to the product of a Picard variety and a linear space† (when the group is only generally transitive). This theorem was proved by Severi ([7], 166) subject, apart from particular cases, to a certain

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† A variety invariant for a generally transitive group is termed *quasi-Abelian* by Severi; in particular, the variety may be birational.

hypothesis, which he has since shown to be unnecessary [8]. The object of the present note, which continues a previous one [2] in this Journal, is to deduce from Severi's theorem that

an algebraic variety V_n , invariant for a continuous ∞^r commutative group whose trajectories are of dimension $r' (< n)$, is birationally equivalent to the product of a linear space S_d ($0 \leq d \leq r'$) and a variety V_{n-d} which, when $d < r'$, is pseudo-Abelian of type $r' - d^*$.

The assumption that the group is commutative is made in Severi's work in order to invoke a result of Picard ([4], 246) which establishes the existence of a certain system of linearly independent simple integrals when this condition is satisfied; the inversion of the system then leads to the parametric representation of the variety. In the present work, the commutative property ensures further that any subgroup which arises is *invariant* in the original group. This result is necessary for the step by step reduction of §4.

There remain for consideration the continuous non-commutative groups. Now any continuous group leaving an algebraic variety invariant is contained in an algebraic group with the same property ([3], 257); so there is no loss of generality in restricting attention to algebraic groups. Moreover, any such group necessarily contains continuous algebraic commutative subgroups; but it seems that the exploitation of these gives only partial results. For further progress it may be necessary to consider the structure of the groups in detail. In the following work, however, all groups are assumed to be commutative.

1. In the first place we observe that Severi's theorem holds equally when the transitive group for V_n is of dimension $n' > n$; for the simple integrals of Picard's work are obtained *a fortiori* in this case ([4], 248). When $n' > n$, the classification according to absolute or general transitivity fails; V_n must be equivalent to a product variety possessing a linear component of dimension greater than zero, while the group may or may not be absolutely transitive.

2. *Algebraic varieties V_n invariant for an ∞^r commutative algebraic group.* We know that the trajectories of the group must be either Picard varieties or else quasi-Abelian (in particular, birational) varieties of some dimension $r' \leq r$; and, in the case requiring investigation, $r' < n$. When they are

* A variety W_p is said to be *pseudo-Abelian of type q* ($1 \leq q \leq p-1$) when it contains an ∞^{p-q} system of index one of Picard varieties V_q which are the trajectories of an ∞^q commutative group of birational transformations. The group acts in an absolutely transitive manner on any irreducible trajectory, and any two such trajectories are each equivalent to the group (regarded as a variety) and so to each other ([5], 363). See also [6].

Picard varieties, $r' = r$; the group then operates in a simply transitive manner on the general trajectory, and V_n is a pseudo-Abelian variety of type r ([5], 363).

When the trajectories are quasi-Abelian we begin by fixing our attention on any simple irreducible trajectory A . The group determined on A is of dimension r'' ($r' \leq r'' \leq r$). Consider first the case when $r'' = r$, so that each transformation of A arises from a finite number of transformations of V_n . By Severi's theorem, A is birationally equivalent to the product, say $\bar{A}$, of a Picard variety V_p and a space $S_{r'-p}$. Regarding the group as acting on $\bar{A}$, we see that it contains a subgroup corresponding to those transformations which determine the identical transformation on V_p . This subgroup thus acts on $S_{r'-p}$, and so is equivalent to a projective group, whence it contains a rational ∞^1 subgroup ([1], 67).

Now the corresponding transformations of A arise from an ∞^1 algebraic group for V_n . If this is reducible, it consists of a finite number of components, each of dimension 1, and of these one, at least, is a finite continuous group, determining a congruence of rational trajectories on V_n ([2], 422). In order to transform V_n into a line-ruled variety whose generators correspond to these trajectories, we require to construct a variety unisecant to these curves.

3. *Varieties invariant for an ∞^1 rational group.* To obtain a unisecant to the rational trajectories, we proceed by induction from Painlevé's theorem ([3], 270) that, in the case of a surface, the locus of the two united points, if distinct, on the trajectories, degenerates into two unisecant curves; our proof generalizes that given in [2] for threefolds.

Assume the corresponding theorem to hold for a V_{n-1} , and consider a V_n invariant for an ∞^1 rational group. If the generic trajectory contains only one united point, we have at once the desired unisecant. When there are two united points, the locus is a bisecant variety, and there is no loss of generality in assuming that this is a pure variety of $n-1$ dimensions, say U_{n-1} , for it will become so on transforming by all the primals of sufficiently high order passing through it. By considering the trajectories as points of a variety, it is seen that we can construct on V_n a linear system of V_{n-1} , compounded of the congruence of trajectories, whose generic member is irreducible and such that $n-1$ members have a free intersection consisting of curves of the congruence.

Each variety V_{n-1} meets U_{n-1} in a variety U_{n-2} which, by hypothesis, degenerates into two components. Therefore the linear system $|V_{n-1}|$ meets U_{n-1} in a linear system $|U_{n-2}|$ each variety of which degenerates into two variable components; and the free curves of intersection of the V_{n-1} give rise to free points of intersection of the U_{n-2} . Now by Bertini's theorem, a reducible linear system without fixed components on an irreducible variety must be compounded of a pencil, and so cannot have

positive grade. We conclude that U_{n-1} must be reducible, and so consists of two varieties unisecant to the curves of the congruence. This completes the inductive proof since, for $n=2$, the theorem is that of Painlevé. Therefore, in every case V_n may be transformed into a line-ruled variety, in particular into a cone.

4. Since, by hypothesis, the original group for V_n is commutative, every subgroup, and in particular the above ∞^1 subgroup, is invariant. This implies that the congruence of generators of the cone is invariant for the entire group, any transformation of which determines a birational transformation acting on the curves as elements. Therefore the general prime section of the cone is invariant for a group induced by that operating on the generators, and the trajectories of the former will have dimension $r'-1$.

The considerations of §§2, 3 may be applied in turn to this prime section. We continue to reason thus for r' stages in succession unless, at some intermediate stage, on taking a further section, we obtain a pseudo-Abelian variety. Suppose we proceed for d stages ($d \leq r'$) before obtaining, as section, a pseudo-Abelian or (as happens when $d=r'$) a variety on which no infinite group is induced. We then have V_n expressed as the product of d lines and a variety V_{n-d} ; and the product of d lines is itself equivalent to a linear space S_d . When $d < r'$, the variety V_{n-d} is pseudo-Abelian of type $r'-d$; when $d=r'$, nothing can be said concerning V_{n-d} , which is quite general. We have thus established our theorem in the case when the dimension r'' of the group on a trajectory is equal to the dimension r of the group for V_n itself.

5. In the case when $r'' < r$, an ∞^1 group on a trajectory no longer arises from an ∞^1 group for V_n , and the argument at the end of §2 fails. In this case the identical transformation on A arises from $\infty^{r-r''}$ transformations of V_n ; these form an algebraic group which either is or contains a continuous algebraic group (of dimension $r-r''$). The trajectories of this group constitute a congruence of which the original trajectories are compounded; in particular, the two sets of trajectories may be the same.

These new trajectories are quasi-Abelian, and not Picard, varieties since those which lie on A are totally invariant for the group. If the group determined on one of these trajectories has dimension $r-r''$, we can, by §2, find an ∞^1 subgroup for V_n , and this will be an invariant subgroup of the *entire* group of dimension r .

On the other hand, if the group on a trajectory has dimension less than $r-r''$, we can apply again the method of this section and obtain a continuous group of still smaller dimension. After a finite number of steps either we arrive at a group for V_n such that the group determined on its (quasi-Abelian) trajectories is of the same dimension as the former, and

then proceed as in §2, or we obtain as trajectories a congruence of curves invariant for a group of dimension greater than unity operating on V_n . In this latter case the curves must therefore be rational. If, now, we choose those transformations leaving a general point of one of these curves invariant, we impose one further algebraic condition, and obtain a subgroup containing a continuous group of dimension one less than the previous group. Hence, by imposing successive invariant points, we obtain, as before, an ∞^1 rational group.

Thus in every case we have an ∞^1 rational group by means of which V_n is transformable into a line-ruled variety, and our theorem is established in all generality.

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