

# **A Structural Approach to Pricing Credit Derivatives with Counterparty Adjustments**

A thesis presented for the degree of  
Doctor of Philosophy of Imperial College  
by

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Ioana-Alexandra Savescu

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# Abstract

This thesis studies the problem of computing adjustments for bilateral counterparty risk for a standard Credit Default Swap (CDS) in a three-factor first passage time default risk model.

Extending the existing literature that gives analytical expression for the transition probability density function for two-dimensional Brownian motions absorbed at the boundaries in the positive quadrant, a method to obtain a semi-analytical expression for the transition probability density function of a three-dimensional Brownian motion absorbed at first exit time from the positive octant is developed. This is done by separating the problem into a radial and an angular part, of which the latter depends only on the correlation matrix. The solution to the angular part is obtained through the finite element method.

These mathematical results are then used to provide semi-analytical expressions for bilateral Credit Value Adjustment (CVA) and Debit Value Adjustments (DVA) of a credit default swap. An example of market data is analysed in detail and it is shown that these value adjustments can be non-negligible. An approximation commonly used by practitioners for the computation of bilateral value adjustments is to use the sum of the unilateral ones as a proxy. The framework developed here allows for an analysis on the precision of this approximation to be performed in the case of the credit default swap, when wrong-way risk is present. Finally, the problem of valuing partially collateralised CDSs is studied and analytical or semi-analytical solutions are developed for computing the potential shortfall caused by the risky nature of the collateral.

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# Chapter 1

## Introduction

Counterparty risk has become increasingly important over the last years, primarily as a consequence of the financial crisis and the realisation that no counterparty can be considered risk free. Counterparty risk is defined as the risk that a party to a financial contract fails to fulfil its contractual obligations. There is a lot of focus currently both in the industry and in the research area about how to take into account this risk when pricing derivative products and more importantly how to manage this risk.

This chapter focuses on defining in detail counterparty risk and other related concepts in Section 1.1, along with presenting the different possible approaches to mitigating this risk. Then, Section 1.2 gives the definition of the main product studied throughout this thesis: the single-name credit default swap. Section 1.3 gives a detailed plan of the thesis that can be used as a reading guide, and finally Section 1.4 summarises the main results.

## 1.1 Counterparty risk – definition and related concepts

Counterparty credit risk (CCR) can be defined as the risk that a party to a financial contract defaults prior to the contract's expiration and does not fulfil all of its obligations. All contracts that are negotiated over-the-counter (OTC derivatives) are subject to this risk.

The recent financial crisis has highlighted the importance of accurately measuring and accounting for counterparty risk, as the default of major participants in the financial derivative transactions have shown that no counterparty can be safely assumed to be risk free (Lehman Brothers being the prime example). This is in consequence a main topic of focus for the regulators in the current environment.

One of the possible ways of mitigating counterparty risk that regulators have considered is to standardise as many of the bespoke products available on the market as possible and impose trading through a centralised counterparty (CCP). This solution relies on the fact that products that are traded through a CCP are generally considered not to have counterparty risk exposure, as the central counterparty guarantees the cashflows for both counterparties to the contract. While this achieves the goal of reducing interconnections in the financial system, many authors have argued that this may lead to a concentration of default risk that can become dangerous (see for example Duffie and Zhu [2011]). The debate is still ongoing though, with a recent study concluding that central counterparties do have a positive effect by reducing inter-dealer exposures (see Cont and Kokholm [2012]). Furthermore, considering central counterparties risk-free is debatable as there have been cases of default events in past history (see Kress [2011] for a detailed discussion).

Although central counterparties could potentially reduce counterparty risk, it is far from a perfect solution and it is unlikely that the size of the bilateral OTC market<sup>1</sup> will reduce so significantly as to make the question of counterparty risk irrelevant. This is because the firm-specific requirements of financial and non-financial firms for tailor-made derivatives suitable to their needs but not to others are not conducive to trading on exchanges or to clearing through central counterparties. As stated in the Second Consultative Document on margin requirements that was recently issued by



the Basel Committee on Banking Supervision (BCBS) and the Board of Governors of the International Organizations of Securities Commissions, a substantial fraction of OTC derivatives will not be able to be centrally cleared. An earlier IMF study, estimates that 25% of the interest rate derivatives market, 33% of the credit default swaps market, and significant percentages of other types of OTC derivatives will remain non-cleared (see ISDA [March 2013] for details).

Another way to attempt to mitigate counterparty risk is using collateral or margining the trades. The idea behind collateral is similar to that used to mitigate lending risk (when the house is collateral for a mortgage for example). In the case of counterparty risk the problem is more complex though, as the exposure is not known upfront and it can change direction through the lifetime of the contract. A few authors have tried to address this issue and estimate that while there are cases where using collateralisation can reduce counterparty risk, it does not always work as there can still be significant gap risk due to sudden changes in the market value. Factors that can have the effect of increasing this gap risk are deterioration of the value of the collateral, re-hypotecation of the collateral, large changes in mark-to-market since the last margining date (see for example Brigo et al. [2011] for a detailed discussion).

A third option to reduce counterparty risk exposure is through netting agreements, and is actively used in the market. “A *netting agreement* is a legally binding contract between two counterparties that, in the event of default, allows aggregation of transactions between two counterparties i.e., transactions with negative value can be used to offset the ones with positive value and only the net positive value represents credit exposure at the time of default.” (Zhu and Pykhtin [2007]). However, there still are trades that are not covered in netting agreements, so not everything can be treated at counterparty level rather than contract level.

Although several possible ways of mitigating counterparty risk have been proposed and are being implemented in practice, it is highly unlikely that this risk will disappear completely. It has recently been the focus of regulators, practitioners and academics alike. Several authors have focused on counterparty risk related issues, and a general

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<sup>1</sup>The most recent BIS data (as of June 30, 2012) puts the notional amount of OTC derivatives outstanding at \$638.9 trillion – see ISDA [March 2013].

overview can be found for example in Zhu and Pykhtin [2007], Gregory [2011], Brigo [2011], Brigo et al. [2013].

There are two general approaches in the market aimed at dealing with counterparty credit risk. The first one is proposed in the Basel agreements and is aimed at measuring the economic and regulatory capital that banks need to set aside to account for potential future losses due to counterparty risk. This is discussed briefly in Section 1.1.1. The second approach aims at marking to market counterparty risk for accounting purposes and to allow hedging. This is the focus of the applications discussed in this thesis and the relevant notions are described in more detail in Section 1.1.2.

### 1.1.1 Risk managing counterparty credit risk

From a risk management point of view, counterparty credit risk is measured through VaR type measures (Value-at-Risk). These calculations require the estimation of potential future losses at a given date with a high degree of statistical confidence. For credit risk though, the computation of these measures is much more involved as historical scenarios are not relevant to today's portfolios and counterparties, hence the estimation is usually performed through a Monte-Carlo simulation method. There are three main components to calculations of this type (see Zhu and Pykhtin [2007]):

- **Scenario Generation:** Future market scenarios are simulated for a fixed set of simulation dates using evolution models for the relevant risk factors. This simulation is usually done under the historical probability measure.
- **Instrument Valuation:** At each simulation date, all instruments in the portfolio are valued under the realisation of each scenario for the market evolution. This is a pricing step, so the valuation is done under the risk-neutral measure, taking as starting point the market data simulated in the first step for a given scenario/simulation date.
- **Aggregation:** For each simulation date and each realisation of the market risk factors, trades are added together within a given netting-set.

Potential Future Exposure (PFE) is one measure of credit risk and is used by banks to monitor when the credit limits with counterparties are breached. For a given date, it is the maximum exposure at that date with a high degree of statistical confidence, so it is estimated by building the whole distribution and estimating a percentile in the far tail (typically 95%).

Other measures linked to credit risk are Exposure At Default (EAD), which is defined as the exposure at the (random) default time of the counterparty, or Credit VaR which can be obtained as a percentile on the loss distribution associated with the position held by the bank, over a given time horizon, when applied to default risk.

The regulatory framework regarding counterparty credit risk is in continuous change as the Basel III agreement looks to improve on the previous ones and ensure that the financial institutions are sufficiently well capitalised to withstand future crisis.

### 1.1.2 Market value of counterparty credit risk

For years, the standard practice in the industry was to mark to market derivatives portfolios without taking into account the credit quality of the counterparty. However, the true market value must include adjustments due to the potential losses that can arise from counterparty default.

Having a choice between trading with a risk-free counterparty and a risky counterparty, one would naturally choose to trade with the risk-free one. Consequently, the mark to market of any derivatives contract traded with a risky counterparty needs to be adjusted to account for the extra risk taken on. Credit value adjustment (CVA) is by definition the difference between the value of a portfolio that considers counterparties risk-free and the value of the portfolio taking into account the possibility of losses due to counterparty default.

Considering a contract between a bank (regarded as risk-free for the time being) and a risky counterparty, the following scenarios may arise on default of the latter:

- If the value of the contract is positive from the point of view of the bank at the

time of default of the counterparty, then the bank will recover only a portion of this positive value, as the counterparty is no longer able to fully fulfil its contractual obligations

- If at the time of default of the counterparty the value of the contract is negative from the bank's point of view, it still needs to pay in full that amount to the counterparty

In view of these potential cashflows, since the bank can incur a loss in some of the scenarios where the counterparty defaults, the valuation of the contract at inception should take into account this risk and should be appropriately discounted. The value of this discount is known as the unilateral credit value adjustment (UCVA).

If the counterparty acknowledges that the bank is risk free, when valuing the contract on its side it will price in its own default probability. The adjustment thus obtained compared to the standard valuation (when neither counterparty is risky) is known as unilateral debit valuation adjustment (UDVA).

If the bank has a non-zero probability of default though, the same contract viewed from the counterparty's point of view can give rise to losses, in scenarios where the bank defaults. Hence the counterparty is also entitled to a discount from its point of view. Thus, when both counterparties are subject to credit risk, the counterparty risk becomes bilateral in nature. This observations is also summarised in the following excerpt from the Basel II regulations (Annexe IV):

“Unlike a firm's exposure to credit risk through a loan, where the exposure to credit risk is unilateral and only the lending bank faces the risk of loss, counterparty credit risk creates a bilateral risk of loss: the market value of the transaction can be positive or negative to either counterparty to the transaction.”

If each counterparty to a contract considers only the other party as risky when pricing, both will end up asking for a discount compared to the standard price (when both counterparties are risk free). Hence, the agreement on the fair price of the contract is possible only if both parties consider both themselves and the counterparty as risky when valuing the contract. On the previous example, the bank computes the price of the contract and then applies two adjustments:

- An adjustment to account for the extra risk it takes on as the counterparty is risky, known as bilateral credit value adjustment (BCVA) – negative in value to the bank
- An adjustment to account for its own default probability and the potential losses the counterparty can incur in that case, known as bilateral debit value adjustment (BDVA) – positive in value to the bank

The sign on the total adjustment depends on the relative credit quality of both counterparties as well as the nature of the contract and the possible losses. Using this bilateral setup makes the problem symmetrical and the two counterparties now agree on the price of the contract.

The question of whether one should account for its own default has given rise to a lot of debate recently. The adjustment coming from this, the DVA term, is positive and hence gives rise to situations where the riskier a counterparty becomes, the higher its gains from this adjustment will be. This has been seen in recent years in the financial industry, as banks have reported higher earnings due to increases in their default probabilities and decrease in value of their debt (see Brigo et al. [2013] for several examples). Regulators have thus questioned the use of DVA and argued that it may give the wrong incentives.

Another important question when discussing DVA is whether it can be hedged or not. Large companies that have significant bond issuances could potentially hedge this by buying-back debt when their credit quality decreases. This is however not always possible. As selling protection on yourself is not allowed, some banks have resorted to proxy-hedging: selling protection on a basket of entities with which their correlation is very high. While this approach hedges the moves in the mark-to-market of DVA, it could potentially increase systemic risk: if one of the companies in the proxy basket actually defaults, not only is the credit quality of the company using this basket as a proxy decreasing as it is highly correlated to the defaulted one, but it also needs to pay up on the protection it has sold, and this ends up intensifying contagion effects.

While the debate is still open on many issues linked to DVA, counterparty risk

is bilateral in nature, and accounting for DVA in pricing ensures that counterparties can agree on the fair price of a contract.

A particular feature of counterparty risk is that the size of the exposure to a counterparty is not known in advance, as it depends on the market environment at the default time of the counterparty. This may lead to what is known as “*Wrong-way risk*”, when the size of the exposure is positively correlated to the likelihood of default of the counterparty. This risk is particularly relevant for credit derivatives, which usually involve selling protection on one or more issuers, as will be discussed in the next section.

Another important issue to consider when discussing counterparty risk is the closeout conventions. There are two main conventions that both have advantages and disadvantages alike. The first one is known as “risk-free” closeout and refers to the case where on default of one of the counterparties the cashflow depends on the value of the derivative valued with risk free counterparties. The second considers that the closeout amount should depend on the value of the residual derivative when taking into account the riskiness of the counterparty still alive. For a detailed discussion on the implication of each of these options see Brigo and Morini [2010].

There are a multitude of issues linked to counterparty risk still under discussion and debate and as such the literature on this subject is rich.

## 1.2 Credit derivative products

The most liquidly traded single name credit product is the credit default swap. This section gives the definition of this product and highlights its particularities when counterparty risk is considered.

A Credit Default Swap (CDS) is a contract in which the protection buyer (PB) agrees to pay a periodic coupon  $c$  to a protection seller (PS) in exchange for a potential cashflow in the event of a default of the reference name (RN) of the swap before the maturity of the contract  $T$ .

The cashflows in a standard CDS can be described as follows:

- At inception: there are two possible options

- Option 1: The coupon that the protection buyer needs to pay periodically to the protection seller is calculated such that the value of the contract at inception is 0. In this case, there is no cashflow at the start of the trade, and the periodical coupon is known as the “*break-even coupon*”.
  - Option 2: The coupon has a fixed value and there an initial payment is made, called “upfront”, that represents the price at time 0 of the contract, given the specified coupon value. Depending on the sign of this price, it can be either counterparty that makes the payment.
- Throughout the life of the contract, the protection buyer pays the periodic coupon (agreed upon at inception) to the protection seller, as long as the reference name has not defaulted and the protection insurance is still in place.
  - In case the reference name defaults before the maturity  $T$  of the contract, the protection seller pays to the protection buyer a fraction of the notional of the contract, the periodic coupon payments stop and the contract is terminated.
  - There is no final cashflow in case the contract reaches maturity without a default of the reference name.

The value of the credit default swap can be naturally decomposed into a coupon leg (also known as premium leg) and a default leg (or protection leg). The coupon leg refers to the periodic payments made by the protection buyer to the protection seller, while the default leg refers to the potential cashflow paid in case of the default of the reference name.

“Wrong-way risk” is of particular interest when studying counterparty risk for credit products. In the case of the CDS, if the party selling protection is positively correlated to the issuer referenced by the contract, the scenarios in which it has to pay the investors for losses are exactly the scenarios where its credit quality deteriorates as well. As a consequence, it might not be able to fulfil its obligation and the investor is faced with a shortfall. Other examples of derivatives that exhibit “wrong-way risk” are commodities derivatives.

As mentioned before, one way of potentially mitigating counterparty risk is to collateralise the trade. Even in this case the risk does not completely vanish, as the credit quality of the collateral comes into play. This case is studied in more detail in chapter 7 by looking at the value of a CDS that is collateralised by a risky bond. The bond can deteriorate in time, and even though there is no counterparty per se, the risk of not getting the full payout in case the reference name defaults is non-negligible, especially when the bond issuer and the reference name of the CDS are highly correlated.

### 1.3 Outline of the thesis

The thesis is organised as follows: chapters 2 to 4 develop the modelling framework and the mathematical results, while chapters 5 to 7 present several applications where the previous results are used.

Chapter 2 presents the modelling framework used throughout the thesis. It discusses briefly the first-passage time model along with the main references in the literature that have addressed default modelling. With the framework in place, analytical results for pricing single-name credit default swaps (CDSs) are presented. These are based on expressions obtained for the transition probability density function of a Brownian motion on the positive semi-axis with an absorbing barrier at 0.

In chapter 3, the extension to the two-dimensional case is considered, and an analytical expression for the transition probability density function for a planar Brownian motion in the positive quartile with absorbing barriers at 0 is presented.

The *simultaneous* and *consistent* calculation of the CVA and DVA for a CDS requires the consideration of three-dimensional first-passage time models and studying the joint evolution of the assets of the reference name, the protection seller and the protection buyer. This task is complex both conceptually and technically and, to the best of the author's knowledge, has not been undertaken before. To this purpose, the well-known results discussed in chapters 2 and 3 are extended in chapter 4, where a new method to compute the transition probability density function for a *three-dimensional* Brownian motion in the positive octant with absorbing barriers at 0 is



developed. A semi-analytical expression is obtained by combining the eigenfunction expansion technique with the finite element method. It is worth noting that the proposed construction of the transition probability density function contributes *both* to mathematical finance and to probability theory.

Chapter 5 applies the results obtained in chapters 3 and 4 to the problem of computing joint survival probabilities for two and three issuers respectively. While the analytical formula for the two-dimensional case has been studied in the literature previously, the semi-analytic formula for the three-dimensional case is new.

Chapter 6 discusses in detail the problem of computing credit and debit value adjustments for a single name credit default swap. Section 6.1 contains a detailed literature review of existing approaches to computing CVA/DVA. The mathematical results obtained in chapters 3 and 4 are then applied to this problem in subsequent sections. The results in Lipton and Sepp [2009] for computing unilateral CVA/DVA are extended to the problem of computing bilateral CVA for a CDS. By using the results developed in chapter 4, the CVA and DVA corrections for a CDS can be computed in a *consistent manner* and new semi-analytical formulas for the computation of bilateral CVA/DVA are obtained. The applications of the proposed technique to real market cases are also discussed, with some realistic examples of pricing CDSs sold by risky sellers to risky buyers.

Several practitioners have considered approximating bilateral CVA/DVA corrections by the sum of the the unilateral CVA/DVA. Some studies have been performed in the literature to investigate how good this approximation is by looking at specific underlying products. For CDSs though this analysis has not been undertaken before, and this is investigated in Section 6.4.

As mentioned previously, the focus has gradually shifted after the credit crisis from more complex structures and products to simpler ones where the risks can be better modelled and understood. Credit linked notes or CDSs with risky collateral are some examples of products that have gained in popularity. Chapter 7 discusses in detail an example and develops analytical and semi-analytical formulas for the valuation of such products. The effect of impaired collateral is taken into account in this framework and the influence of the most important pricing parameters is analysed.

## 1.4 Summary of main results

This thesis contributes to the detailed study of some single-name credit derivatives in an environment where the counterparties can no longer be considered risk-free.

The first step is to develop the necessary mathematical tools for the analysis. A three-dimensional extension of the first-passage time framework, where the joint dynamics of the firms' values are driven by correlated Brownian motions is proposed. A semi-analytic solution for the transition probability density function for a three-dimensional Brownian motion combining the eigenfunction expansion technique with the finite element method is developed and tested (see chapter 4, Proposition 4.1). It is important to note that the expression obtained can be applied to a variety of applications, not restricted to the ones presented in this thesis.

Several applications of the mathematical results are then presented, starting with the computation of joint survival probabilities for three issuers in the first-passage time framework (see chapter 5, Proposition 5.2). The computation of bilateral credit and debit value adjustments for a standard CDS is then discussed in detail. Semi-analytical solutions are proposed (see chapter 6, Propositions 6.3 and 6.4) and the results obtained are analysed, showing that taking into account the riskiness of both counterparties is important and has a non-negligible effect. Although fully analytical solutions are not possible for calculation of the bilateral adjustments, given a triplet of pairwise correlations, the eigenvalues/eigenfunctions can be precomputed, which allows efficient computations across a range of initial points, volatilities or other trade-related data (coupons, recoveries etc.), without repeating the most numerically expensive part.

Using the analytic and semi-analytic expressions developed for the unilateral and bilateral value adjustments, the precision of an approximation commonly used by practitioners is investigated for the particular case of the CDS, so when wrong-way risk is present. The analysis shows that the errors obtained when approximating the bilateral adjustments by the unilateral ones can be significant (see chapter 6, Section 6.4).

Finally, the case where a CDS is partially collateralised is studied, with particular

attention to how the riskiness of the collateral can affect the valuation of the product (see chapter 7). Using the mathematical results in chapter 4, analytical and semi-analytical solutions for the potential shortfall caused by collateral impairment or default are given in propositions 7.2 (risk-free protection buyer) and Proposition 7.3 (risky protection buyer) respectively. The effect of different parameters on the size of the shortfall is then analysed in detail.

There is no shortage of examples of more or less complex products where the risks are not fully understood and for which there is no universally agreed model to be used for their pricing. The aim here has been to delve deeper and provide a better understanding of the different aspects linked to the credit default swap.

A selection of the results presented in this thesis have been published in Lipton and Savescu [April, 2013] and Lipton and Savescu [2013].

## Chapter 2

# Modelling Framework

Modelling firms' defaults is an important issue, especially in the view of the recent financial crisis that has led to a significant number of defaults among companies all over the world. The pricing models for single name credit derivatives can be divided into two large classes: reduced-form models and structural default models. This chapter briefly discusses both classes, with a more detailed presentation of the structural framework which is used throughout this thesis.

Section 2.1 contains an overview of the existing literature on structural default and reduced form models in the context of pricing single name credit products. The next section –2.2– presents the framework in detail, while sections 2.3 and 2.4 show how to obtain the transition probability density function and the survival probability respectively in this context. The last section focuses on the pricing of the most liquidly traded single-name credit product, the Credit Default Swap (CDS).

## 2.1 Literature review

The models used in the existing literature to price credit products can be divided into two broad classes: reduced-form models and structural default models.

*Reduced-form models* (also called intensity models) describe default by means of an exogenous jump process: the default time is the first jump time of a Poisson process with a deterministic or stochastic intensity. Hence, the firm's default is unpredictable: it is not triggered by basic market observables but has an exogenous component that is independent from all default-free market information. The literature regarding this class of models is very rich (Jarrow and Turnbull [1995], Duffie and Singleton [1997], Duffie and Singleton [1999], Lando [1998], Brigo and Alfonsi [2005] to mention just a few) and a comprehensive review of this approach can be found in chapter 8 of Bielecki and Rutkowski [2004].

The reduced-form modelling approach is very flexible and allows for a good calibration of the entire term structure of market quoted credit spreads. It has therefore been used extensively in the pricing of credit derivatives by both academics and practitioners alike. One disadvantage of this approach is that there is no economical interpretation of the default and no obvious way in which to model in a consistent framework both credit and equity products.

The second class of models used for pricing products that are subject to credit risk is the *structural default models*. Merton developed the original version of this model (Merton [1974]). He postulated that the firm's value  $a_t$  is driven by a log-normal diffusion. The firm, which borrowed a zero-coupon bond with face value  $N$  and maturity  $T$ , defaults at time  $T$  if its value  $a_T$  is less than the bond's face value  $N$ . Following this pioneering insight, many authors proposed various extensions of the basic model, see, e.g., Black and Cox [1976], Kim et al. [1993], Longstaff and Schwartz [1995], Leland and Toft [1996], Brigo and Tarenghi [2004], Brigo and Tarenghi [2005] and Albanese and Chen [2005] among many others. They considered more complicated forms of debt and assumed that the default event may be triggered continuously up to the debt maturity (these type of models are also known as *first-passage time models*).

In this approach, the default time is a predictable stopping time with respect to the filtration modelling the information flow available to the traders. This means that the random time of default is announced by an increasing sequence of stopping times. As a consequence of this feature, the implied short-term credit spreads are zero, which is undesirable as it is not inline with market observations. In order to avoid this problem and obtain reasonable short-time spreads several solutions have been proposed in the literature. It has been shown that this can be achieved either by making default barriers curvilinear (Hyer et al. [1999], Avellaneda and Zhu [2001], Hull and White [2001]), or by making default barriers stochastic Finger et al. [2002], or by incorporating jumps into the firm's value dynamics (Zhou [2001b], Hilberink and Rogers [2002], Lipton [2002], Sepp [2004], Sepp [2006], Cariboni and Schoutens [2007], Feng and Linetsky [2008]), or by making the starting value of the asset value process random (Davis and Pistorius [2010], Davis and Pistorius [2013]).

Structural default models are mainly used in credit portfolio and credit risk managing applications as well as to price equity-credit hybrid products. Due to the fact that it is possible to have an economic interpretation of the model, studying counterparty risk on credit products in this framework is an interesting topic as it helps to build-up intuition about several important problems linked to this topic. The next section details the structural default modelling framework which is used for all the developments presented in this thesis.

## 2.2 First-passage time model framework

In this section the structural default framework for single names and in particular the first-passage time framework is discussed. Suppose that the underlying financial market is arbitrage free, such that there exists a risk-neutral probability measure  $\mathbb{Q}$ . This means that the price process of any tradeable security, which pays no coupons or dividends, is a martingale under  $\mathbb{Q}$  when discounted by the *savings account*  $B$  given as  $B_t = \exp\left(\int_0^t \varrho_s ds\right)$ . The short term interest rate  $\varrho_t$  is assumed to be deterministic for the rest of the developments.

Let  $a_t$  be the firm's asset value per share. It is assumed that  $a_t$  is driven by the

following diffusion dynamics under  $\mathbb{Q}$ :

$$da_t = (\varrho_t - \zeta_t) a_t dt + \sigma_t a_t dW_t, \quad (2.1)$$

where  $\zeta_t$  is the dividend rate,  $W_t$  is a standard Brownian motion and  $\sigma_t$  is the deterministic volatility.

Further, it is assumed that the firm defaults when its value per share becomes less than a fraction of its debt per share. In this approach, which is similar to that of Finger et al. [2002] and Lipton [2002], the default barrier of the firm is a deterministic function of time given by:

$$l_t = l_0 E_t, \quad (2.2)$$

where

$$E_t = \exp \left( \int_0^t \left( r_u - \zeta_u - \frac{1}{2} \sigma_u^2 \right) du \right),$$

and  $l_0 = RL_0$ . Here  $R$  is the average recovery of the firm's liabilities (that can be estimated from the prices of its bonds and CDS quotes) and  $L_0$  is its total debt per share (from the balance sheet as the ratio of the firm's total liabilities and the number of common shares outstanding).

As pointed out in Black and Cox [1976], the fact that the time-dependence of the barrier is exponential makes intuitive sense as the expected debt value takes this form. The fact that the growth rate of the barrier is taken to be equal to the drift of the asset value process implies that the leverage is constant in time (see also Zhou [2001a] and Haworth et al. [2008]). A more general case, where the barrier grows at a different rate is studied in Zhou [2001a] for example, with the author concluding that the impact is relatively small and hence the simplification of considering the growth rate of the barrier equal to that of the asset value is attractive.

Following Stamicar and Finger [2006], the following approximation of the firm's equity price per share  $s_t$  is used:

$$s_t = \begin{cases} a_t - l_t, & t < \tau \\ 0, & t \geq \tau \end{cases}, \quad (2.3)$$

where  $\tau$  is the default time. At time  $t = 0$ ,  $s_0$  is specified by the market price of the equity share. Accordingly, the initial asset value is given by  $a_0 = s_0 + l_0$ .

For simplicity we assume going forward that the volatility is constant in time. The solution of the stochastic differential equation (2.1) can be written as a product of a deterministic part and a stochastic exponent:

$$a_t = l_0 E_t e^{\sigma W_t},$$

where  $W_0 = \frac{1}{\sigma} \ln \left( \frac{a_0}{l_0} \right) > 0$ . In the current formulation, the default event occurs at the first time  $\tau$  when  $W_\tau$  becomes negative:

$$\tau = \inf\{t \geq 0, W_t \leq 0\},$$

so it is the first passage time of the process below the fixed barrier at 0. We denote by  $x_t$  the absorbed process at zero, which is therefore driven by the following dynamics under  $\mathbb{Q}$ :

$$dx_t = dW_t, \quad x_0 = \frac{1}{\sigma} \ln \left( \frac{a_0}{l_0} \right), \quad (2.4)$$

with  $x_0$  representing the “relative distance” of the asset value from the default barrier. The default event is determined only by the dynamics of the stochastic driver  $x_t$ .

As was emphasized by Zhou [2001b] and Lipton [2002], introducing jumps in the dynamics of the asset value allows one to calibrate to CDS market spreads even for short maturities. In the framework without jumps, it is well known that the default time is predictable, so that the survival probability is hyper-exponentially flat for very short maturities, and good calibration of distressed names in the market is impossible.

The case without jumps however, allows for analytical solutions in some cases which are useful for the understanding of the problem, and provide good benchmark for the more general case with jumps. Besides, CDSs with medium and long maturities can be adequately dealt with in the case without jumps. Accordingly, this thesis is focused on the simplified case without jumps.

To summarise, the key assumptions used going forward are:

- first-passage time framework, where the default can occur at any point prior to



the maturity of the debt

- deterministic interest rates
- the firm's value is driven by a diffusion process with constant volatility
- the default barrier is a deterministic function of time, taken such that it is flat for the logarithm of the firm's value

### 2.3 Green's function

An important concept that is used throughout this thesis is the transition probability density function associated with a Markov process.<sup>1</sup> This section gives briefly the main definitions and the analytical expressions obtained in the one dimensional structural default framework considered here.

Consider a Markov process  $X_t$  for which the dynamics are given by the SDE:

$$dX_t = \mu(t, X_t) dt + \sigma(t, X_t) dW_t, \quad (2.5)$$

where  $W_t$  is a standard Brownian motion and the coefficients satisfy certain regularity conditions<sup>2</sup>. Let  $p(t', x'|t, \tilde{x})$  denote the transition probability density function from state  $\tilde{x}$  and time  $t$  to state  $x'$  at time  $t'$ , where  $0 \leq t \leq t'$  such that:

$$\mathbb{E}[f(X_{t'}) | X_t = \tilde{x}] = \int_{\mathbb{R}} f(x') p(t', x'|t, \tilde{x}) dx'. \quad (2.6)$$

The transition probability density function satisfies the forward Kolmogorov equation (also known as the Fokker-Planck equation).

**Theorem 2.1.** (*Forward Kolmogorov equation*)

*Let the process  $X_t$  follow the dynamics given in equation (2.5). Then the transition probability density function of  $X_t$  assumed to be at least  $C^{1,2}(\mathbb{R}_+, \mathbb{R})$  function of*

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<sup>1</sup>For a detailed discussion of the concepts presented briefly here see for example Karatzas and Shreve [1991].

<sup>2</sup>For details on the regularity conditions see Karatzas and Shreve [1991] page 364.

“forward” arguments  $t'$  and  $x'$  solves the following forward Kolmogorov equation:

$$\frac{\partial p(t', x'|t, \tilde{x})}{\partial t'} - \frac{1}{2} \frac{\partial^2}{\partial x'^2} (\sigma^2(t', x') p) + \frac{\partial}{\partial x'} (\mu(t', x') p) = 0, \quad (2.7)$$

with the initial condition  $p(t, x'|t, \tilde{x}) = \delta(x' - \tilde{x})$ .

Let  $\mathcal{O}$  denote the second order differential operator associated with the process  $X_t$ . The operator is given by (see Karatzas and Shreve [1991], page 312):

$$\mathcal{O}f(t, x) = \frac{\partial f(t, x)}{\partial t} + \frac{1}{2} \frac{\partial^2}{\partial x^2} (\sigma^2(t, x) f) + \frac{\partial}{\partial x} (\mu(t, x) f); f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}).$$

Let the operator  $\mathcal{O}^*$  define its adjoint:

$$\mathcal{O}^*f(t', x') = \frac{\partial f(t', x')}{\partial t'} - \frac{1}{2} \frac{\partial^2}{\partial x'^2} (\sigma^2(t', x') f) + \frac{\partial}{\partial x'} (\mu(t', x') f); f \in C^{1,2}(\mathbb{R}_+, \mathbb{R}) \quad (2.8)$$

The solution to the Forward Kolmogorov equation is also known under the name of Green's function for the operator  $\mathcal{O}^*$  associated with the process  $X_t$  and with a Dirac-delta initial condition. The term Green's function is usually used in physics and is a generic term that denotes the fundamental solution<sup>3</sup> to a PDE problem. Since the techniques used throughout this thesis are typically PDE techniques, this will be the preferred term used going forward. Note that numerically the transition probability density for process  $X_t$  and the Green's function associated with the operator  $\mathcal{O}^*$  and with a Dirac-delta initial condition are equal.

For homogeneous Markov processes, with stationary transition probabilities, the transition probability density function depends only on the time interval  $\zeta = t' - t$  rather than the moments  $t'$  and  $t$ , and hence the simplified notation  $p(t, x'|t, \tilde{x}) \equiv p(\zeta, x'|\tilde{x})$  is used.

The operator associated to the Brownian motion process is denoted by  $\mathcal{O}_{1D}^* = \frac{\partial}{\partial \zeta} - \frac{1}{2} \frac{\partial^2}{\partial x'^2}$ . The Green's function for this operator satisfies the forward Kolmogorov equation which simplifies to:

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<sup>3</sup>The solution to the PDE augmented with a  $\delta$ -type initial condition.

$$G_\zeta(\zeta, x'|\tilde{x}) - \frac{1}{2}G_{x'x'}(\zeta, x'|\tilde{x}) = 0,$$

with the initial condition  $G(0, x'|\tilde{x}) = \delta(x' - \tilde{x})$ , where  $\tilde{x}$  denotes the starting point of the Brownian motion. The solution for this equation considered on the entire real axis is given by the Gaussian heat kernel:  $H(\zeta, x'|\tilde{x}) = \frac{1}{\sqrt{2\pi\zeta}}e^{-\frac{(x'-\tilde{x})^2}{2\zeta}}$ .

In the structural default framework considered in Section 2.2, the process stops as soon as a default occurs. This means that the process of interest here is  $x_t$  as defined in (2.4), a stopped version of the Brownian motion process, where the random stopping time is the first hitting time of the 0 barrier (default barrier). The problem that the Green's function for operator  $\mathcal{O}_{1D}^*$  with  $\delta$ -type initial condition solves needs thus to be augmented with an absorbing barrier at 0:

$$G_\zeta(\zeta, x'|\tilde{x}) - \frac{1}{2}G_{x'x'}(\zeta, x'|\tilde{x}) = 0, \quad (2.9)$$

$$G(0, x'|\tilde{x}) = \delta(x' - \tilde{x}), \quad (2.10)$$

$$G(\zeta, 0|\tilde{x}) = 0. \quad (2.11)$$

This is augmented with the usual regularity conditions at infinity ( $G(\zeta, x'|\tilde{x})$  and all of its derivatives vanish when  $x' \rightarrow \infty$ ). The domain is the positive semi-axis,  $\mathbb{R}^+$ .

The solution for this problem is well known and can be obtained through two alternative methods. Using the method of images, the solution for the PDE with absorbing boundary at 0 can be constructed starting from the solution for the same PDE given on the whole real axis (for details see for example Lipton [2001] page 72 or Sepp [2007] page 123):

$$G(\zeta, x'|\tilde{x}) = H(\zeta, x'|\tilde{x}) - H(\zeta, x'|- \tilde{x}), \quad \forall x' \in \mathbb{R}^+$$

and the solution for the problem (2.9) - (2.11) is:

$$G(\zeta, x'|\tilde{x}) = \frac{1}{\sqrt{2\pi\zeta}} \left( e^{-\frac{(x'-\tilde{x})^2}{2\zeta}} - e^{-\frac{(x'+\tilde{x})^2}{2\zeta}} \right). \quad (2.12)$$

Alternatively, using Fourier transforms, the solution for problem (2.9) - (2.11) can

also be written using an integral representation.

**Proposition 2.1.** *(Fourier transform for Green's function for operator  $\mathcal{O}_{1D}^*$  on  $\mathbb{R}_+$  with an absorbing barrier at 0 and  $\delta$ -type initial condition)*

*The Green's function that solves problem (2.9) - (2.11) on the positive semi-axis  $\mathbb{R}^+$  has the following integral representation:*

$$G(\zeta, x'|\tilde{x}) = \frac{2}{\pi} \int_0^\infty e^{-\frac{k^2\zeta}{2}} \sin k\tilde{x} \sin kx' dk.$$

*Proof.* To solve the problem, the Fourier sine transform can be used (this is appropriate for PDE problems defined on semi-infinite domains). For a given function  $f(x)$ , the sine Fourier transform of  $f$ ,  $\mathcal{F}_s[f(x)]$ , is defined as:

$$\mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(kx) dx$$

Let  $\hat{G}(\zeta, k)$  denote the Fourier sine transform for Green's function that solves problem (2.9) - (2.11):

$$\hat{G}(\zeta, k) = \mathcal{F}_s[G(\zeta, x'|\tilde{x})].$$

The PDE (2.9) can be simplified by applying the Fourier sine transform to both terms:

$$\begin{aligned} \mathcal{F}_s[G_\zeta(\zeta, x'|\tilde{x})] &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial G(\zeta, x'|\tilde{x})}{\partial \zeta} \sin(kx') dx' \\ &= \frac{\partial}{\partial \zeta} \hat{G}(\zeta, k), \\ \mathcal{F}_s[G_{x'x'}(\zeta, x'|\tilde{x})] &= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 G(\zeta, x'|\tilde{x})}{\partial x'^2} \sin(kx') dx' \\ &= \sqrt{\frac{2}{\pi}} \left[ \frac{\partial G(\zeta, x'|\tilde{x})}{\partial x'} \sin(kx') \Big|_0^\infty - k \int_0^\infty \frac{\partial G(\zeta, x'|\tilde{x})}{\partial x'} \cos(kx') dx' \right] \\ &= -k \sqrt{\frac{2}{\pi}} \left[ G(\zeta, x'|\tilde{x}) \cos(kx') \Big|_0^\infty + k \int_0^\infty G(\zeta, x'|\tilde{x}) \sin(kx') dx' \right] \\ &= -k^2 \hat{G}(\zeta, k), \end{aligned}$$

where the boundary condition  $G(\zeta, 0|\tilde{x}) = 0$  along with the regularity conditions

when  $x' \rightarrow \infty$  were used to cancel the terms that arise from the integration by parts. The PDE (2.9) is now reduced to a simple ODE in  $\zeta$ :

$$\frac{\partial}{\partial \zeta} \hat{G}(\zeta, k) = -\frac{1}{2} k^2 \hat{G}(\zeta, k).$$

The generic solution for this ODE is given by:

$$\hat{G}(\zeta, k) = \hat{G}(0, k) e^{-\frac{1}{2} k^2 \zeta}.$$

To obtain the final form of the solution, the particular form of the initial condition given in equation (2.10) is used.  $\hat{G}(0, k)$  is the Fourier sine transform of this initial condition:

$$\begin{aligned} \hat{G}(0, k) &= \mathcal{F}_s[G(0, x'|\tilde{x})] \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty \delta(x' - \tilde{x}) \sin(kx') dx' \\ &= \sqrt{\frac{2}{\pi}} \sin(k\tilde{x}) \end{aligned}$$

and hence the final form of the solution for the ODE is:

$$\hat{G}(\zeta, k) = \sqrt{\frac{2}{\pi}} \sin(k\tilde{x}) e^{-\frac{1}{2} k^2 \zeta}.$$

The last step needed is the application of the inverse Fourier sine transform. The final solution for Green's function given below is an alternative representation to that obtained through the method of images (see formula (2.12)).

$$G(\zeta, x'|\tilde{x}) = \frac{2}{\pi} \int_0^\infty \sin(k\tilde{x}) \sin(kx') e^{-\frac{1}{2} k^2 \zeta} dk \quad (2.13)$$

□

The solution to the problem (2.9), augmented with the initial condition (2.10) and boundary conditions (2.11) is unique. Suppose that two functions  $G_1$  and  $G_2$  would satisfy the problem. Then, by the linearity of the operator,  $G_1 - G_2$  would also satisfy

the PDE (2.9), would have 0 boundary conditions and null initial condition. But using for example Theorem 3.6, pg 138 in Friedman [2006], yields that this problem has a unique solution (as the operator is uniformly parabolic and the boundary and initial condition are continuous). Since the identically null function satisfies this problem, this yields  $G_1 - G_2 = 0$ , and hence Green's function for the operator  $\mathcal{O}_{1D}^*$  on the positive semi-axis with an absorbing barrier at 0 and  $\delta$ -type initial condition is unique.

Figure 2.1 shows that the expressions obtained through the two different formulations ((2.12) and (2.13)) coincide up to numerical accuracy.

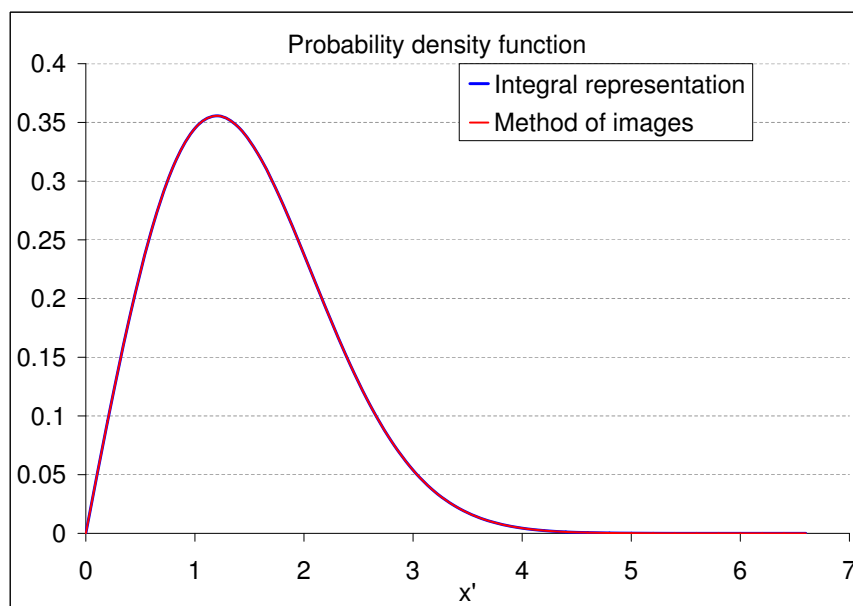


Figure 2.1: Probability density function function ( $\zeta = 1$  year,  $\tilde{x} = 1$ ).

## 2.4 Survival probability

Consider an issuer  $X$  and let  $x_t$  be the process for the stochastic factor associated to the firm asset value of this issuer. In the structural default framework presented in Section 2.2, this process is a standard Brownian motion issued from the initial positive point  $\tilde{x}$ . The issuer is said to default if the value of its firm's assets drops below a specified threshold, which, as discussed previously, is equivalent to the process

$x_t$  dropping below 0.

Let  $\tau$  denote the default time of issuer  $X$ ,

$$\tau = \inf\{u \geq t, x_u \leq 0\}$$

and  $Q(t, T, \tilde{x})$  denote the survival probability at time  $t$  of the reference issuer, up to maturity  $T$ :

$$Q(t, T, \tilde{x}) = \mathbb{E}[\mathbb{1}_{\tau > T} | x_t = \tilde{x}],$$

with  $\tilde{x} > 0$ .

The barrier at 0 is an absorbing barrier for the process  $x_t$ , as once the name has defaulted it cannot come back to life. Hence, the probability of the issuer  $X$  surviving past the maturity  $T$  can also be written as  $Q(t, T, \tilde{x}) = \mathbb{E}[\mathbb{1}_{x_T > 0} | x_t = \tilde{x}]$ . The previous section has shown how to compute the transition probability density function for a Brownian motion process stopped at the first hitting time of 0. The expectation can thus be computed using Green's function for operator  $\mathcal{O}_{1D}^*$  with  $\delta$ -type initial condition and absorbing boundary at 0 as follows (see equation (2.6)):

$$\begin{aligned} Q(t, T, \tilde{x}) &= \mathbb{E}[\mathbb{1}_{x_T > 0} | x_t = \tilde{x}] \\ &= \int_0^\infty G(T - t, x' | \tilde{x}) dx'. \end{aligned}$$

Using the analytical expression obtained in equation (2.12) yields the following expression for the survival probability

$$Q(t, T, \tilde{x}) = 2\Phi\left(\frac{\tilde{x}}{\sqrt{T-t}}\right) - 1, \quad (2.14)$$

where  $\Phi$  denotes the cumulative standard normal distribution:

$$\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

## 2.5 Pricing of a standard CDS

The most liquidly traded single name credit product is the credit default swap. This section shows how to price it in the first-passage time framework presented above. For more details see for example Bielecki and Rutkowski [2004].

A Credit Default Swap (CDS) is a contract in which the protection buyer (PB) agrees to pay a periodic coupon  $c$  to a protection seller (PS) in exchange for a potential cashflow in the event of a default of the reference name (RN) of the swap before the maturity of the contract  $T$ .

For the standard CDS, the protection buyer and protection seller are considered non-risky, and hence the only entity whose evolution needs to be modelled is the reference name. The process  $x_t$  measures the relative distance from the default barrier in time and it has the following dynamics:  $dx_t = \sigma_x dW_t^x$ , with the starting point  $\tilde{x} > 0$  (as discussed in Section 2.2).

Let  $\tau^{RN}$  denote the default time of the reference name given its survival up to time  $t$ :

$$\tau^{RN} = \inf\{u \geq t, x_u \leq 0\},$$

and  $R_{RN}$  its recovery. Then the cashflows (from the protection buyer's point of view) can be written in a mathematical way as follows:

$$V^{CL}(t, T, \tilde{x}) = -\mathbb{E} \left[ \sum_{T_i} cD(t, T_i) \mathbb{1}_{\{T_i \leq \tau^{RN}\}} \Delta T_i \mid \mathcal{F}_t \right], \quad (2.15)$$

$$V^{DL}(t, T, \tilde{x}) = \mathbb{E} \left[ (1 - R_{RN}) D(t, \tau^{RN}) \mathbb{1}_{\{\tau^{RN} < T\}} \mid \mathcal{F}_t \right], \quad (2.16)$$

where  $(\mathcal{F}_t)_{t \geq 0}$  is the natural filtration associated to the Brownian motion process  $W_t$  with the usual properties,  $T_i$ ,  $i \geq 0$  are the coupon payment dates (with  $T_0 = t$ ) and  $D(t, T)$  is the price of a zero-coupon bond with maturity  $T$ :  $D(t, T) = \exp \left( - \int_t^T \varrho_u du \right)$ . (This can be also rewritten in terms of the *savings account*  $B$ :  $D(t, T) = B_t B_T^{-1}$ .)

The risk-neutral valuation formulas (2.15) and (2.16) give the arbitrage-free price of the coupon and default leg respectively as the expected value of all discounted future cashflows. This relies on the assumption that the defaultable claim considered



is hedgeable, and hence that the value of the firm is tradeable. While this is a typical assumption in structural models, it does need some justification. The value of the firm is not directly observable (or tradeable), but it is typically assumed that it is equal to the sum of the equity and the debt of the firm (see equation (2.3)). Since both of these can be considered tradeable, the value of the firm is thus a tradeable security as well.

**Proposition 2.2.** *(Coupon leg valuation for a standard CDS)*

*In the structural default framework presented in Section 2.2, the value of the coupon leg of a standard credit default swap with maturity  $T$ , under the simplifying assumption that the coupons are paid continuously, is given by:*

$$V^{CL}(t, T, \tilde{x}) = -\frac{c}{\varrho} \left[ 1 - e^{-\varrho(T-t)} Q(t, T, \tilde{x}) - e^{\tilde{x}\sqrt{2\varrho}} \Phi \left( -\frac{\tilde{x}}{\sqrt{T-t}} - \sqrt{2\varrho(T-t)} \right) - e^{-\tilde{x}\sqrt{2\varrho}} \Phi \left( -\frac{\tilde{x}}{\sqrt{T-t}} + \sqrt{2\varrho(T-t)} \right) \right], \quad (2.17)$$

where  $c$  is the coupon and  $\tilde{x}$  is the value at time  $t$  of the stochastic process that measures the distance from the default boundary for the reference name.

*Proof.* The value of the premium leg, under the assumptions discussed above, is the expectation of all discounted future cashflows, as given in equation (2.15). For deterministic and constant interest rate  $\varrho$ , this can be written using the expression computed in equation (2.14) for the survival probability as:

$$\begin{aligned} V^{CL}(t, T, \tilde{x}) &= -c \sum_{T_i} D(t, T_i) \mathbb{E} [\mathbb{1}_{\{T_i \leq \tau^{RN}\}} | x_t = \tilde{x}] \Delta T_i \\ &= -c \sum_{T_i} D(t, T_i) Q(t, T_i, \tilde{x}) \Delta T_i \\ &= -c A(t, T, \tilde{x}), \end{aligned}$$

where  $A(t, T, \tilde{x})$  is the risky annuity of the CDS defined as:

$$A(t, T, \tilde{x}) = \sum_{T_i} D(t, T_i) \mathbb{E} [\mathbb{1}_{\{T_i \leq \tau^{RN}\}} | x_t = \tilde{x}] \Delta T_i.$$

Under the simplified assumption that the coupon is paid continuously, the value of the annuity can be written as

$$A(t, T, \tilde{x}) = \int_t^T D(t, t') Q(t, t', \tilde{x}) dt',$$

and the integral has a fully analytical solution.

$$\begin{aligned} A(t, T, \tilde{x}) &= \int_t^T D(t, t') Q(t, t', \tilde{x}) dt' \\ &= \int_t^T e^{-\varrho(t'-t)} \left[ 2\Phi\left(\frac{\tilde{x}}{\sqrt{t'-t}}\right) - 1 \right] dt' \\ &= 2 \int_t^T e^{-\varrho(t'-t)} \Phi\left(\frac{\tilde{x}}{\sqrt{t'-t}}\right) dt' + \frac{1}{\varrho} e^{-\varrho(T-t)} - \frac{1}{\varrho} \end{aligned}$$

Using integration-by-parts and then the change of variable  $\frac{1}{\sqrt{t'-t}} = u$ ,  $dt' = -\frac{2}{u^3} du$ :

$$\begin{aligned} \int_t^T e^{-\varrho(t'-t)} \Phi\left(\frac{\tilde{x}}{\sqrt{t'-t}}\right) dt' &= -\frac{1}{\varrho} e^{-\varrho(t'-t)} \Phi\left(\frac{\tilde{x}}{\sqrt{t'-t}}\right) \Big|_t^T \\ &\quad - \frac{1}{\varrho} \int_t^T e^{-\varrho(t'-t)} \varphi\left(\frac{\tilde{x}}{\sqrt{t'-t}}\right) \frac{\tilde{x}}{2\sqrt{(t'-t)^3}} dt' \\ &= \frac{1}{\varrho} - \frac{1}{\varrho} e^{-\varrho(T-t)} \Phi\left(\frac{\tilde{x}}{\sqrt{T-t}}\right) \\ &\quad - \frac{\tilde{x}}{\varrho\sqrt{2\pi}} \int_{\frac{1}{\sqrt{T-t}}}^{\infty} e^{-\frac{1}{2}\tilde{x}^2 u^2 - \frac{\varrho}{u^2}} du \end{aligned}$$

To get the final expression for the annuity leg, the following indefinite integral (7.4.33 in Abramowitz and Stegun [1964]) is used:

$$\int e^{-a^2 x^2 - \frac{b^2}{x^2}} dx = \frac{\sqrt{\pi}}{2a} \left[ e^{2ab} \Phi\left(\sqrt{2}ax + \frac{\sqrt{2}b}{x}\right) + e^{-2ab} \Phi\left(\sqrt{2}ax - \frac{\sqrt{2}b}{x}\right) \right] + C, \quad (2.18)$$

where  $a \neq 0$ , as well as the well-known identity  $\Phi(x) + \Phi(-x) = 1$ . The analytical

expression for the annuity leg is given by:

$$A(t, T, \tilde{x}) = \frac{1}{\varrho} \left[ 1 - e^{-\varrho(T-t)} Q(t, T, \tilde{x}) - e^{\tilde{x}\sqrt{2\varrho}} \Phi \left( -\frac{\tilde{x}}{\sqrt{T-t}} - \sqrt{2\varrho(T-t)} \right) - e^{-\tilde{x}\sqrt{2\varrho}} \Phi \left( -\frac{\tilde{x}}{\sqrt{T-t}} + \sqrt{2\varrho(T-t)} \right) \right], \quad (2.19)$$

which yields the desired expression for the coupon leg in equation (2.17).  $\square$

**Remark 2.1.** *The expression in equation (2.19) for the annuity of a standard CDS is valid for non-zero interest rates ( $\varrho \neq 0$ ). An analytic expression can be obtained for the case where interest rates are 0 as well, and this can be useful for testing purposes:*

$$A(t, T, \tilde{x}) = (T-t) Q(t, T, \tilde{x}) + 2\tilde{x}\sqrt{T-t} \varphi \left( \frac{\tilde{x}}{\sqrt{T-t}} \right) - 2\tilde{x}^2 \Phi \left( -\frac{\tilde{x}}{\sqrt{T-t}} \right), \quad (2.20)$$

where  $\varphi(\cdot)$  denotes the standard normal probability density function.

**Proposition 2.3.** *(Default leg valuation for a standard CDS)*

*In the structural default framework presented in Section 2.2, the value of the default leg of a standard credit default swap with maturity  $T$ , written on a reference name with recovery  $R_{RN}$  is given by:*

$$V^{DL}(t, T, \tilde{x}) = (1 - R_{RN}) \left[ 1 - e^{-\varrho(T-t)} Q(t, T, \tilde{x}) - \varrho A(t, T, \tilde{x}) \right]. \quad (2.21)$$

*Proof.* As discussed previously, the fair value of the protection leg of the CDS can be written as the discounted value of the future cashflows as shown in equation (2.16). For deterministic and constant interest rates, denoted by  $\varrho$ , this can be written as:

$$V^{DL}(t, T, \tilde{x}) = \mathbb{E} \left[ (1 - R_{RN}) e^{-\varrho(\tau^{RN}-t)} \mathbb{1}_{\{\tau^{RN} < T\}} \middle| x_t = \tilde{x} \right],$$

which is a Feynman-Kac representation of a final value problem, with Dirichlet boundary conditions. As such, by defining appropriately the domain  $D$  and the functions  $f$ ,  $g$  and  $k$ , Theorem A.3 can be used to reduce the calculation of this expectation to

finding the solution to the following PDE problem:

$$V_t^{DL} + \frac{1}{2}V_{xx}^{DL} - \varrho V^{DL} = 0, \quad (2.22)$$

where the domain is the positive semi-axis  $D = \mathbb{R}^+$ , the discounting function  $k(t, x)$  is constant and equal to the short term rate  $k(t, x) = \varrho$ , the final value function is null  $f(x) = 0$  and the boundary condition is given by  $g(t, 0) = 1 - R_{RN}$ .

Consider now function  $U(t, T, \tilde{x}) = e^{\varrho(T-t)}V^{DL}(t, T, \tilde{x})$ . This function solves the standard heat equation with appropriate final and boundary conditions:

$$\begin{aligned} U_t + \frac{1}{2}U_{xx} &= 0, \\ U(T, T, x) &= 0, \\ U(t, T, 0) &= (1 - R_{RN}) e^{\varrho(T-t)}. \end{aligned}$$

In view of the PDE that function  $U(t, T, x)$  solves, the following identity holds:

$$\int_t^T \int_0^\infty \left( U_{t'}(t', x') + \frac{1}{2}U_{xx}(t', x') \right) G(t' - t, x' | \tilde{x}) dx' dt' = 0,$$

where  $G(t' - t, x' | \tilde{x})$  is the Green's function for the one-dimensional operator  $\mathcal{O}_{1D}^*$  with  $\delta$ -type initial condition, solving the forward Kolmogorov problem (2.9) - (2.11) with homogeneous boundary conditions. To find the solution for this problem, the integration-by-parts technique is used with the purpose of switching the partial derivatives from the function  $U$  to Green's function  $G$ . After performing one integration by parts in the time dimension and two in  $x'$  and using the final and boundary conditions for  $U$  and the initial and boundary conditions for  $G$ , the following expression is obtained for  $U$ :

$$\begin{aligned} U(t, T, \tilde{x}) &= - \int_t^T \int_0^\infty U(t', x') \left[ G_{t'}(t' - t, x' | \tilde{x}) - \frac{1}{2}G_{x'x'}(t' - t, x' | \tilde{x}) \right] dx' dt' \\ &\quad + \frac{1}{2} \int_t^T U(t', 0) \left. \frac{\partial G(t' - t, x' | \tilde{x})}{\partial x'} \right|_{x'=0} dt'. \end{aligned}$$

Since  $G$  solves the forward Kolmogorov equation this simplifies to

$$U(t, T, \tilde{x}) = \frac{1}{2} \int_t^T U(t', 0) \left. \frac{\partial G(t' - t, x' | \tilde{x})}{\partial x'} \right|_{x'=0} dt',$$

and going back to the initial function:

$$V^{DL}(t, T, \tilde{x}) = \frac{1}{2} \int_t^T e^{-\varrho(t'-t)} V^{DL}(t', 0) \left. \frac{\partial G(t' - t, x' | \tilde{x})}{\partial x'} \right|_{x'=0} dt'.$$

Using (2.12) for the expression of Green's function for the operator  $\mathcal{O}_{1D}^*$  with  $\delta$ -type initial condition and absorbing barrier at 0:

$$\left. \frac{\partial G(t' - t, x' | \tilde{x})}{\partial x'} \right|_{x'=0} = \frac{2\tilde{x}}{\sqrt{2\pi} (t' - t)^3} e^{-\frac{\tilde{x}^2}{2(t'-t)}},$$

yields for the value of the default leg:

$$\begin{aligned} V^{DL}(t, T, \tilde{x}) &= \frac{1}{2} (1 - R_{RN}) \frac{2\tilde{x}}{\sqrt{2\pi}} \int_t^T e^{-\varrho(t'-t)} e^{-\frac{\tilde{x}^2}{2(t'-t)}} \frac{1}{\sqrt{(t' - t)^3}} dt' \\ &= (1 - R_{RN}) \frac{2}{\sqrt{2\pi}} \int_{\frac{\tilde{x}}{\sqrt{T-t}}}^{\infty} e^{-\frac{u^2}{2} - \frac{\varrho \tilde{x}^2}{u^2}} du. \end{aligned}$$

As in the case of the annuity calculation, the indefinite integral in equation (2.18) is used to obtain the final form for the fair value of the default leg of a standard CDS:

$$V^{DL}(t, T, \tilde{x}) = (1 - R_{RN}) \left[ 1 - e^{-\varrho(T-t)} Q(t, T, \tilde{x}) - \varrho A(t, T, \tilde{x}) \right].$$

□

In view of the expressions for the coupon leg and default leg given in propositions 2.2 and 2.3 respectively, the price of a single-name CDS where both counterparties

are considered non-risky is:

$$\begin{aligned}
 V^{CDS}(t, T, \tilde{x}) &= V^{CL}(t, T, \tilde{x}) + V^{DL}(t, T, \tilde{x}) \\
 &= - (c + \varrho (1 - R_{RN})) A(t, T, \tilde{x}) \\
 &\quad + (1 - R_{RN}) [1 - e^{-\varrho(T-t)} Q(t, T, \tilde{x})] .
 \end{aligned} \tag{2.23}$$

## Chapter 3

# Transition Probability Density Function for a Planar Brownian Motion in a Wedge

In the first-passage time model framework, under certain assumptions about the form of the debt barrier and the dynamics of the firm's asset value (as discussed in Section 2.2), the study of the default event, the first hitting time of the time-dependent barrier by the firm's asset value process, can be simplified to the study of the first hitting time of a fixed barrier at 0 by a Brownian motion. Hence, in the previous chapter, the transition probability density function for a Brownian motion on the positive semi-axis with an absorbing boundary at 0 proved to be the main tool needed to price default contingent claims.

Chapters 3 and 4 study the extension of this problem to two and three dimensions respectively. The generalisation of this simplified problem for the two-dimensional case is straightforward: the quantity of interest is the transition probability density function for a planar Brownian motion in a wedge. In this framework, analytical solutions are well known and have been studied previously. This chapter gives a brief outline of the methodology, as it is instructive and a starting point for the new methodology developed in chapter 4 for the three dimensional case.

The chapter is structured as follows: Section 3.1 gives an overview of the main

references in literature that have studied the problem; the rest of the chapter details the solution and the methodology used to obtain the transition probability density function for this particular case using the eigenfunction expansion method. The expression obtained is then used in sections 5.1 and 6.3.1 to derive analytic solutions for the joint survival probability of two issuers and unilateral counterparty adjustments respectively.

### 3.1 Literature review

Extensions of the one-dimensional transition probability density function for a Brownian motion stopped at the first hitting time of 0 to the two-dimensional case for a planar Brownian motion in a wedge have been studied extensively in the literature. Analytical formulas have been derived and used subsequently in different applications. Two possible methods can be applied in order to obtain the solution in the two dimensional case: the eigenfunction expansion method and the method of images. The solution using the first method is well known and has been introduced in Iyengar [1985], He et al. [1998], Lipton [2001], Zhou [2001a], Metzler [2010]. A novel solution, through the method of images, was announced by Lipton in 2008 at a SIAM meeting, briefly discussed in Lipton and Sepp [2009] and detailed further in Lipton and Savescu [2013].

The analytic expression obtained for the transition probability density function of a planar Brownian motion in a wedge has been applied to a variety of problems: in He et al. [1998] the authors apply it to price double look-back options, in Lipton [2001] for pricing FX products, in Cont and De Larrard [2011] to study the dynamics of order books or in Lipton and Sepp [2009], Blanchet-Scaillet and Patras [2011] for the estimation of unilateral CVA for CDSs. The wide range of applications underline the importance and usefulness of the two-dimensional analytical formula discussed in this chapter.



### 3.2 Green's function

The quantity of interest in the two-dimensional case is the transition probability density function for a two-dimensional correlated Brownian motion in the positive quadrant. Let processes  $x_t$  and  $y_t$  be the two processes considered:

$$\begin{bmatrix} dx_t \\ dy_t \end{bmatrix} = \Sigma \begin{bmatrix} dW_t^1 \\ dW_t^2 \end{bmatrix},$$

where  $(W_t^1, W_t^2)$  is a standard two-dimensional Brownian motion and

$$\Sigma \Sigma^t = \begin{pmatrix} 1 & \rho_{xy} \\ \rho_{xy} & 1 \end{pmatrix},$$

with  $|\rho_{xy}| < 1$ . The case where the processes are perfectly correlated is excluded as in this case both issuers have the same evolution and the two-dimensional problem reduces to the one-dimensional case studied in the previous chapter. The starting points for both processes need to be positive, to ensure the barrier has not yet been touched. In the structural default framework, processes  $x_t$  and  $y_t$  can be interpreted as the relative distance from the default barrier in time for each of the entities considered. Let  $\tau^X = \inf\{u \geq t, x_u \leq 0\}$ ,  $\tau^Y = \inf\{u \geq t, y_u \leq 0\}$  and  $\tau = \min\{\tau^X, \tau^Y\}$   $\tau$  is thus the first exit time of the two-dimensional process from the positive quadrant.

Denote the transition probability density function for the two-dimensional case by  $p(\zeta, x', y' | \tilde{x}, \tilde{y})$ , where  $\tilde{x}$  and  $\tilde{y}$  are the starting points for our processes respectively ( $\tilde{x} > 0$  and  $\tilde{y} > 0$ ) and by  $x'$  and  $y'$  the forward coordinates. The operator associated with the planar Brownian motion is denoted by  $\mathcal{O}_{2D}^*$ :

$$\mathcal{O}_{2D}^* = \frac{\partial}{\partial \zeta} - \frac{1}{2} \frac{\partial^2}{\partial x'^2} - \frac{1}{2} \frac{\partial^2}{\partial y'^2} - \rho_{xy} \frac{\partial^2}{\partial x' \partial y'}.$$

As in the one-dimensional case, the Green's function for the operator  $\mathcal{O}_{2D}^*$ , with the initial condition depending on the starting point of the two-dimensional process and

0 boundary conditions is the solution to the forward Kolmogorov equation (see A.1):

$$G_\zeta - \frac{1}{2}G_{x'x'} - \frac{1}{2}G_{y'y'} - \rho_{xy} G_{x'y'} = 0, \quad (3.1)$$

$$G(0, x', y' | \tilde{x}, \tilde{y}) = \delta(x' - \tilde{x}) \delta(y' - \tilde{y}), \quad (3.2)$$

$$G(\zeta, 0, y' | \tilde{x}, \tilde{y}) = 0, \quad G(\zeta, x', 0 | \tilde{x}, \tilde{y}) = 0. \quad (3.3)$$

The function is also subject to the usual conditions of vanishing at infinity (along with its derivatives). The domain in the  $(x', y')$  plane is given by:

$$\mathcal{D} = \{(x', y') \mid 0 \leq x' < \infty, \quad 0 \leq y' < \infty\}. \quad (3.4)$$

By using, for example, Theorem 3.6, pg 138 in Friedman [2006], the difference of any two solutions for problem (3.1), augmented with the initial condition (3.2) and boundary conditions (3.3) is identically null (the operator is uniformly parabolic and the boundary and initial condition for the associated homogeneous problem are continuous). Hence the solution to the above problem is unique and this property is inherited by all the transformed problems below.

The first step in solving this problem is to apply a change of variables that eliminates the cross derivative:

$$\begin{cases} \alpha'(x', y') = x' \\ \beta'(x', y') = -\frac{1}{\bar{\rho}_{xy}} (\rho_{xy} x' - y'), \end{cases} \quad (3.5)$$

where  $\bar{\rho}_{xy} = \sqrt{1 - \rho_{xy}^2}$ . Along with the change of variables, the domain the problem has to be solved in has changed from the positive quadrant to a wedge (see Figure 3.1). The angle is characterized by  $\cos(\varpi) = -\rho_{xy}$ , so if  $\rho_{xy} > 0$ , the angle is obtuse.

The above change of variables leads to the following simplified version of the

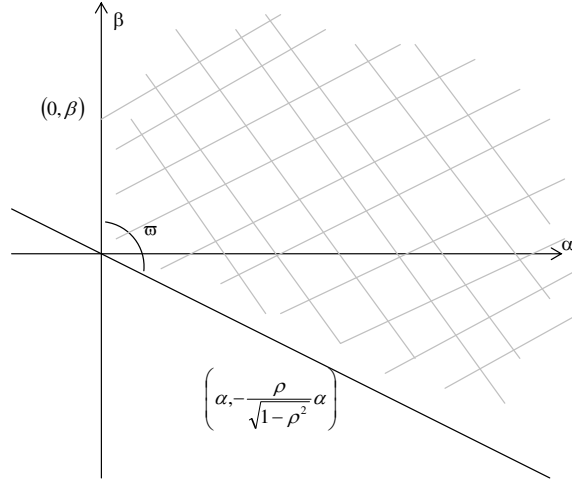


Figure 3.1: The new domain the problem has to be solved in after the change of variables.

problem in (3.1) -(3.3):

$$\begin{aligned}
 G_\zeta - \frac{1}{2}G_{\alpha'\alpha'} - \frac{1}{2}G_{\beta'\beta'} &= 0, \\
 G\left(0, \alpha', \beta' \mid \tilde{\alpha}, \tilde{\beta}\right) &= \delta(\alpha' - \tilde{\alpha}) \delta(\beta' - \tilde{\beta}), \\
 G\left(\zeta, 0, \beta' \mid \tilde{\alpha}, \tilde{\beta}\right) &= 0, \quad G\left(\zeta, \alpha', -\frac{\rho_{xy}}{\bar{\rho}_{xy}}\alpha' \mid \tilde{\alpha}, \tilde{\beta}\right) = 0,
 \end{aligned} \tag{3.6}$$

where  $\tilde{\alpha} = \tilde{x}$  and  $\tilde{\beta} = -\frac{1}{\bar{\rho}_{xy}}(\rho_{xy}\tilde{x} - \tilde{y})$ .

In order to take advantage of the symmetry of the domain, a second change of variables is applied to convert to polar coordinates:

$$\begin{cases} \alpha = -r \sin(\varphi - \varpi) \\ \beta = r \cos(\varphi - \varpi) \end{cases} \longleftrightarrow \begin{cases} r = \sqrt{\alpha^2 + \beta^2} \\ \varphi = \varpi + \arctan\left(-\frac{\alpha}{\beta}\right). \end{cases} \tag{3.7}$$

The problem in equation (3.6) is thus transformed to the following form:

$$G_\zeta - \frac{1}{2} \left( G_{r'r'} + \frac{1}{r'} G_{r'} + \frac{1}{r'^2} G_{\varphi'\varphi'} \right) = 0, \quad (3.8)$$

$$G(0, r', \varphi' | \tilde{r}, \tilde{\varphi}) = \frac{1}{\tilde{r}} \delta(r' - \tilde{r}) \delta(\varphi' - \tilde{\varphi}), \quad (3.9)$$

$$G(\zeta, r', 0) = 0, \quad G(\zeta, r', \varpi) = 0, \quad G(\zeta, 0, \varphi') = 0, \quad (3.10)$$

with the usual conditions that  $G$  and its derivatives vanish when  $r' \rightarrow \infty$  and where the polar coordinates  $(\tilde{r}, \tilde{\varphi})$  of the source are given by:

$$\tilde{r} = \frac{\sqrt{\tilde{x}^2 - 2\rho_{xy}\tilde{x}\tilde{y} + \tilde{y}^2}}{\bar{\rho}_{xy}}, \quad (3.11)$$

$$\tilde{\varphi} = \arccos(-\rho_{xy}) - \arctan\left(\frac{\bar{\rho}_{xy}\tilde{x}}{\tilde{y} - \rho_{xy}\tilde{x}}\right). \quad (3.12)$$

The domain this has to be solved in is a wedge, as shown in Figure 3.1:

$$\tilde{\mathcal{D}} = \{(r', \varphi'), \quad 0 \leq r' < \infty, \quad 0 \leq \varphi' \leq \varpi\}. \quad (3.13)$$

The most common way to solve this problem is through the eigenfunction expansion technique and the solution is given in Proposition 3.1.

**Proposition 3.1.** *The solution for the PDE (3.8) inside the domain  $\tilde{\mathcal{D}}$  given in equation (3.13), augmented with the initial condition (3.9) and the boundary conditions (3.10) has the following analytical expression:*

$$G(\zeta, r', \varphi' | \tilde{r}, \tilde{\varphi}) = \frac{2e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\varpi\zeta} \sum_{n=1}^{\infty} I_{\nu_n} \left( \frac{r'\tilde{r}}{\zeta} \right) \sin(\nu_n \varphi') \sin(\nu_n \tilde{\varphi}), \quad (3.14)$$

where  $\varpi = \arccos(-\rho_{xy})$ ,  $\nu_n = \frac{n\pi}{\varpi}$ .

*Proof.* The first step in solving the problem is to apply the separation of variables technique. Solutions of the following form are proposed:

$$G(\zeta, r', \varphi' | \tilde{r}, \tilde{\varphi}) = g(\zeta, r') f(\varphi').$$

By substituting back in equation (3.8), this can be rewritten such that the left hand side depends only on  $\zeta$  and  $r'$  and the right hand side depends only on  $\varphi'$ . Hence, both sides are equal to some constant value  $C$  and the PDEs that the functions  $g$  and  $f$  satisfy are:

$$g_{\zeta} = \frac{1}{2} \left( g_{r'r'} + \frac{1}{r'} g_{r'} + \frac{C}{r'^2} g \right), \quad (3.15)$$

$$f_{\varphi'\varphi'} = Cf. \quad (3.16)$$

The boundary conditions for Green's function at  $\varphi' = 0$  and  $\varphi' = \varpi$  (see equation (3.10)) yield the following conditions:

$$g(\zeta, r')f(0) = 0, \quad g(\zeta, r')f(\varpi) = 0, \quad \forall (\zeta, r') \in \mathbb{R}^+ \times \mathbb{R}^+.$$

As  $g$  cannot be identically null (as this would imply that Green's function is identically null), the function  $f$  must satisfy the boundary conditions  $f(0) = 0$  and  $f(\varpi) = 0$ , along with the PDE given in (3.16).

It can be easily shown that positive values for the constant  $C$  cannot occur. Suppose that  $C > 0$ ; then there exist real numbers  $A$  and  $B$  such that

$$f(\varphi') = Ae^{\sqrt{C}\varphi'} + Be^{-\sqrt{C}\varphi'}.$$

The boundary conditions for  $f$  imply that both  $A$  and  $B$  are null which would result in Green's function  $G$  being identically null. Supposing that  $C = 0$ , there exist real numbers  $A$  and  $B$  such that

$$f(\varphi) = A\varphi + B.$$

The boundary conditions again would imply that Green's function is identically 0. Consequently, the constant  $C$  has to be negative. Let  $C = -\Lambda^2$ . In this case, there exist real numbers  $A$  and  $B$  such that:

$$f(\varphi) = A \sin(\Lambda\varphi') + B \cos(\Lambda\varphi')$$

The boundary condition  $f(0) = 0$  yields  $B = 0$  and  $f(\varpi) = 0$  yields  $\Lambda = \frac{n\pi}{\varpi}$  for some positive integer  $n$  and the solution is given by  $f(\varphi') = A \sin\left(\frac{n\pi\varphi'}{\varpi}\right)$ . The overall solution for Green's function can thus be written as an infinite sum where  $f$  has the specific form given above.

Function  $g(\zeta, r')$  solves the PDE given in equation (3.15), where  $C = -\Lambda^2$ . From the boundary conditions for Green's function (equation (3.10)) and to avoid obtaining a null solution everywhere, it results that  $g$  should also satisfy the boundary condition  $g(\zeta, 0) = 0$  and it should vanish when  $r' \rightarrow \infty$ :  $g(\zeta, r') \xrightarrow{r' \rightarrow \infty} 0$ .

It can be shown that the solution is given by:

$$g(\zeta, r') = \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta} I_{\Lambda}\left(\frac{r'\tilde{r}}{\zeta}\right), \quad (3.17)$$

where  $I_{\Lambda}(\xi)$  is the modified Bessel function of the first kind and satisfies the following equation:

$$\xi^2 \frac{d^2 I}{d\xi^2} + \xi \frac{dI}{d\xi} - (\xi^2 + \Lambda^2) = 0. \quad (3.18)$$

It can be verified that this solves the PDE in (3.15) by computing the relevant derivatives of  $g$ .

$$\begin{aligned} \frac{\partial g}{\partial \zeta}(\zeta, r') &= \left(\frac{r'^2 + \tilde{r}^2}{2\zeta^2} - \frac{1}{\zeta}\right) g(\zeta, r') - r'\tilde{r} \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta^3} \frac{d}{d\xi} I_{\Lambda}\left(\frac{r'\tilde{r}}{\zeta}\right) \\ \frac{\partial g}{\partial r'}(\zeta, r') &= -\frac{r'}{\zeta} g(\zeta, r') + \tilde{r} \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta^2} \frac{d}{d\xi} I_{\Lambda}\left(\frac{r'\tilde{r}}{\zeta}\right) \\ \frac{\partial^2 g}{\partial r'^2}(\zeta, r') &= \left(-\frac{1}{\zeta} + \frac{r'^2}{\zeta^2}\right) g(\zeta, r') - \frac{2e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta^3} r'\tilde{r} \frac{d}{d\xi} I_{\Lambda}\left(\frac{r'\tilde{r}}{\zeta}\right) + \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta^3} \tilde{r}^2 \frac{d^2}{d\xi^2} I_{\Lambda}\left(\frac{r'\tilde{r}}{\zeta}\right) \end{aligned}$$

Let  $\mathcal{L}$  be the operator applied to  $g$ :  $\mathcal{L}g = g_{\zeta} - \frac{1}{2}\left(g_{r'r'} + \frac{1}{r'g_r} - \frac{\Lambda^2}{r'^2}g\right)$ . Using the previously obtained expressions for the derivatives, the following expression is obtained

for the operator  $\mathcal{L}$  applied to  $g$ :

$$\begin{aligned}\mathcal{L}g(\zeta, r') &= \left( \frac{r'^2 + \tilde{r}^2}{2\zeta^2} - \frac{1}{\zeta} + \frac{\Lambda^2}{2r'^2} + \frac{1}{2\zeta} + \frac{1}{2\zeta} - \frac{r'^2}{2\zeta^2} \right) \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta} I_\Lambda \left( \frac{r'\tilde{r}}{\zeta} \right) \\ &\quad + \left( -\frac{r'\tilde{r}}{\zeta} - \frac{\tilde{r}}{2r'} + \frac{r'\tilde{r}}{\zeta} \right) \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta^2} \frac{d}{d\xi} I_\Lambda \left( \frac{r'\tilde{r}}{\zeta} \right) - \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{2\zeta^3} \tilde{r}^2 \frac{d^2}{d\xi^2} I_\Lambda \left( \frac{r'\tilde{r}}{\zeta} \right) \\ &= \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{2\zeta r'^2} \left\{ \left[ \left( \frac{r'\tilde{r}}{\zeta} \right)^2 + \Lambda^2 \right] I_\Lambda \left( \frac{r'\tilde{r}}{\zeta} \right) - \frac{r'\tilde{r}}{\zeta} \frac{d}{d\xi} I_\Lambda \left( \frac{r'\tilde{r}}{\zeta} \right) - \left( \frac{r'\tilde{r}}{\zeta} \right)^2 \frac{d^2}{d\xi^2} I_\Lambda \left( \frac{r'\tilde{r}}{\zeta} \right) \right\},\end{aligned}$$

which is identically 0 using equation (3.18). To conclude that  $g$  is indeed given by the expression in equation (3.17), the boundary conditions need to be verified. When  $r' \rightarrow \infty$ , the argument of the modified Bessel function goes to infinity, and the asymptotic expansion for large arguments given in Abramowitz and Stegun [1964] (page 377, 9.7.1) can be used:

$$I_\Lambda(\xi) \approx \frac{e^\xi}{\sqrt{2\pi\xi}} \left[ 1 - \frac{4\Lambda^2 - 1}{8\xi} + \frac{(4\Lambda^2 - 1)(4\Lambda^2 - 9)}{2!(8\xi)^2} - \dots \right], \quad \xi \rightarrow \infty.$$

For large values of  $\xi$ , the terms in brackets go to 1, and hence the following approximation can be used for the modified Bessel function:  $I_\Lambda(\xi) \approx \frac{e^\xi}{\sqrt{2\pi\xi}}$ . This yields for function  $g$ :

$$\begin{aligned}g(\zeta, r') &\approx \frac{e^{-\frac{\tilde{r}^2 + r'^2}{2\zeta}}}{\zeta} \frac{\sqrt{\zeta} e^{\frac{\tilde{r}r'}{\zeta}}}{\sqrt{2\pi\tilde{r}r'}} \\ &= \frac{e^{-\frac{(\tilde{r}-r')^2}{2\zeta}}}{\sqrt{2\pi\zeta\tilde{r}r'}} \xrightarrow{r' \rightarrow \infty} 0.\end{aligned}$$

For the limit at  $r' = 0$ , it suffices to remark that  $I_\Lambda(\xi) \xrightarrow{\xi \rightarrow 0} 0$  when  $\Lambda > 0$ , and thus the expression for function  $g$  given in equation (3.17) satisfies all the necessary conditions.

The solution for the Green's function can thus be written as an infinite sum:

$$G(\zeta, r', \varphi' | \tilde{r}, \tilde{\varphi}) = \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta} \sum_{n=1}^{\infty} C_n I_{\frac{n\pi}{\varpi}} \left( \frac{r'\tilde{r}}{\zeta} \right) \sin \left( \frac{n\pi\varphi'}{\varpi} \right).$$

It is important to note that the above sum is absolutely summable by properties of the modified Bessel function (see for example Watson [1995]). To simplify the equations, the following notation is introduced:  $\nu_n = \frac{n\pi}{\varpi}$ . The coefficients  $C_n$  can be computed by imposing the initial condition for Green's function given in equation (3.9). When  $\zeta \rightarrow 0$ , the argument for the modified Bessel function goes to infinity, and the asymptotic expansion for large arguments can again be used to obtain an approximation for function  $g$ :

$$g(\zeta, r') \approx \frac{e^{-\frac{(\tilde{r}-r')^2}{2\zeta}}}{\sqrt{2\pi\zeta\tilde{r}r'}},$$

and the limit when  $\zeta \rightarrow 0$  is:

$$g(\zeta, r') \xrightarrow{\zeta \rightarrow 0} \frac{1}{\tilde{r}} \delta(r' - \tilde{r}).$$

Thus, in order for the initial condition for Green's function to be satisfied, the coefficients  $C_n$  need to verify:

$$\sum_{n=1}^{\infty} C_n \sin(\nu_n \varphi') = \delta(\varphi' - \tilde{\varphi}).$$

After multiplication by  $\sin(\nu_m \varphi')$  and integration from 0 to  $\varpi$ , the coefficients are given by:  $C_n = \frac{2}{\varpi} \sin(\nu_n \tilde{\varphi})$ . The final formula for Green's function in the domain shown in Figure 3.1 is:

$$G(\zeta, r', \varphi' | \tilde{r}, \tilde{\varphi}) = \frac{2e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\varpi\zeta} \sum_{n=1}^{\infty} I_{\nu_n} \left( \frac{r'\tilde{r}}{\zeta} \right) \sin(\nu_n \varphi') \sin(\nu_n \tilde{\varphi}). \quad (3.19)$$

As the expression above verifies the PDE (3.8) inside the domain  $\tilde{\mathcal{D}}$  given in equation (3.13), augmented with the initial condition (3.9) and the boundary conditions (3.10), using the unicity of the solution to this problem discussed previously yields that this is the desired solution.  $\square$

**Remark 3.1.** For the Green's function given in equation (3.14), the following can be



shown:

$$r'^n \frac{\partial^k G(\zeta, r', \varphi' | \tilde{r}, \tilde{\varphi})}{\partial r'^k} \xrightarrow{r' \rightarrow \infty} 0, \quad \forall n \geq 0, \quad k = 0, 1. \quad (3.20)$$

This can be done by using the approximation of the Bessel function for large values of the argument, in the same way as for the proof above of the limit at  $r' \rightarrow \infty$  of Green's function. For  $k = 1$ , the formula (9.6.26) in Abramowitz and Stegun [1964] can be used to obtain the result. Similarly, for the limit at 0:

$$r'^n \frac{\partial^k G(\zeta, r', \varphi' | \tilde{r}, \tilde{\varphi})}{\partial r'^k} \xrightarrow{r' \rightarrow 0} 0, \quad \forall n \geq 0, \quad k = 0, 1. \quad (3.21)$$

These are true also for the derivatives with respect to  $\tilde{r}$ , as the expression is symmetric in  $r'$  and  $\tilde{r}$ . These results will be useful later on.

Figure 3.2 shows the two dimensional Green's function for operator  $\mathcal{O}_{2D}^*$  with absorbing boundaries, for sample values for the input parameters (in the original coordinates  $x$  and  $y$ ). The peak of the distribution is around the initial point, as expected.

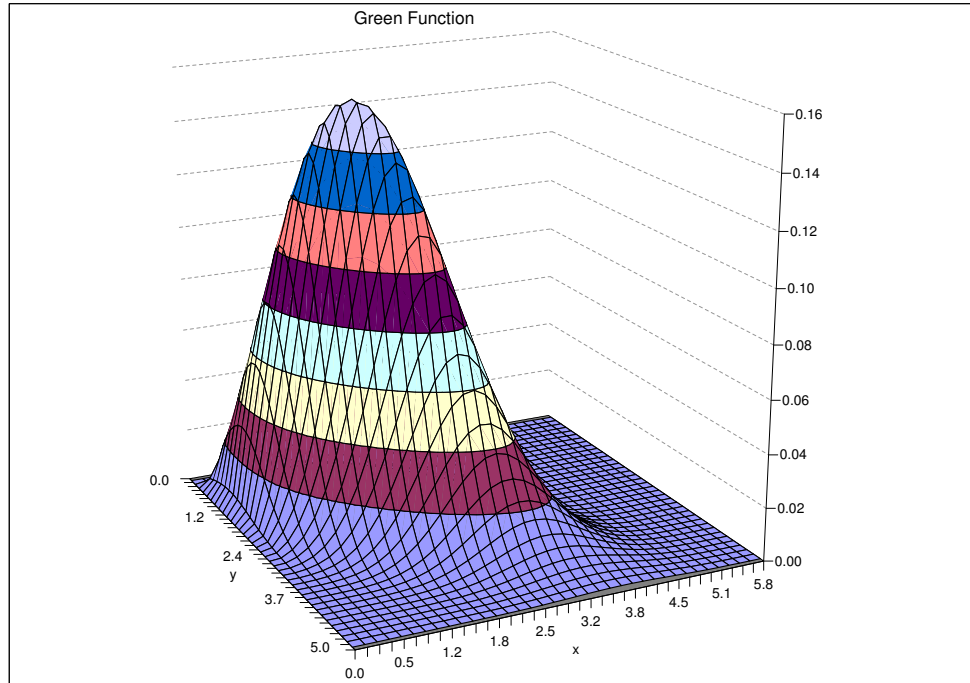


Figure 3.2: Green's function for operator  $\mathcal{O}_{2D}^*$  with 0 boundary conditions and  $\delta$ -type initial condition:  $\tilde{x} = 1.47$ ,  $\tilde{y} = 1.9$ ,  $\rho_{xy} = 70\%$ ,  $\zeta = 1$  year.

## Chapter 4

# Transition Probability Density Function for a Three-dimensional Brownian Motion in a Cone

Chapter 2 discussed how to obtain the transition probability density function for a standard Brownian motion stopped at the first hitting time of a fixed barrier at 0. This was subsequently extended in chapter 3 to a two-dimensional problem, where the methodology used to obtain an analytical expression for the transition probability density function for a planar Brownian motion in a wedge through the eigenfunction expansion method was shown.

The present chapter studies the extension to the three dimensional case: the quantity of interest is the transition probability function for a three-dimensional Brownian motion in a cone, with absorbing boundaries. The eigenfunction expansion method is used to obtain a semi-analytical expression (Section 4.1). While the eigenvalues and eigenvectors are not known analytically as in the two-dimensional case for the planar Brownian motion in a wedge, these can be computed numerically using the finite element method as discussed in Section 4.2.

The expression obtained for the transition probability function for a three-dimensional Brownian motion in a cone is then used in sections 5.2 and 6.3.2 to derive analytic solutions for the joint survival probability for three issuers in the first-passage-time

framework and bilateral counterparty adjustments for a standard CDS respectively.

## 4.1 Green's function

The quantity of interest in the three-dimensional case is the transition probability density function for a three-dimensional correlated Brownian motion in the positive octant. Let processes  $x_t$ ,  $y_t$  and  $z_t$  be the three processes considered:

$$\begin{bmatrix} dx_t \\ dy_t \\ dz_t \end{bmatrix} = \Sigma \begin{bmatrix} dW_t^1 \\ dW_t^2 \\ dW_t^3 \end{bmatrix},$$

where  $(W_t^1, W_t^2, W_t^3)$  is a standard three-dimensional Brownian motion and

$$\Sigma \Sigma^t = \begin{pmatrix} 1 & \rho_{xy} & \rho_{xz} \\ \rho_{xy} & 1 & \rho_{yz} \\ \rho_{xz} & \rho_{yz} & 1 \end{pmatrix}.$$

In order for the problem to be well defined, the correlation matrix needs to be valid, so needs to be positive definite, which yields the following conditions:

$$\begin{aligned} 1 - \rho_{xy}^2 &> 0, \\ 1 - \rho_{xy}^2 - \rho_{xz}^2 - \rho_{yz}^2 + 2\rho_{xy}\rho_{xz}\rho_{yz} &> 0. \end{aligned}$$

The inequalities are strict to eliminate the cases where the three-dimensional problem reduces to two or even one dimensional problems.

Let

$$\tau^X = \inf\{u \geq t, x_u \leq 0\},$$

$$\tau^Y = \inf\{u \geq t, y_u \leq 0\},$$

$$\tau^Z = \inf\{u \geq t, z_u \leq 0\},$$

and  $\tau = \min\{\tau^X, \tau^Y, \tau^Z\}$ , the first exit time of the three-dimensional process from

the positive octant.

Denote the transition probability density function for the three-dimensional case by  $p(\zeta, x', y', z' | \tilde{x}, \tilde{y}, \tilde{z})$ , where  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  are the starting points for our processes respectively ( $\tilde{x} > 0$ ,  $\tilde{y} > 0$  and  $\tilde{z} > 0$ ) and by  $x'$ ,  $y'$  and  $z'$  the forward coordinates. The operator associated with the three-dimensional Brownian motion is denoted by  $\mathcal{O}_{3D}^*$ :

$$\mathcal{O}_{3D}^* = \frac{\partial}{\partial \zeta} - \frac{1}{2} \frac{\partial^2}{\partial x'^2} - \frac{1}{2} \frac{\partial^2}{\partial y'^2} - \frac{1}{2} \frac{\partial^2}{\partial z'^2} - \rho_{xy} \frac{\partial^2}{\partial x' \partial y'} - \rho_{xz} \frac{\partial^2}{\partial x' \partial z'} - \rho_{yz} \frac{\partial^2}{\partial y' \partial z'}.$$

As in the one and two-dimensional cases, the Green's function for the operator  $\mathcal{O}_{3D}^*$ , with the initial condition depending on the starting point of the three-dimensional process and 0 boundary conditions is the solution to the forward Kolmogorov equation (see A.1):

$$G_\zeta - \frac{1}{2} G_{x'x'} - \frac{1}{2} G_{y'y'} - \frac{1}{2} G_{z'z'} - \rho_{xy} G_{x'y'} - \rho_{xz} G_{x'z'} - \rho_{yz} G_{y'z'} = 0, \quad (4.1)$$

$$G(0, x', y', z' | \tilde{x}, \tilde{y}, \tilde{z}) = \delta(x' - \tilde{x}) \delta(y' - \tilde{y}) \delta(z' - \tilde{z}), \quad (4.2)$$

$$G(\zeta, 0, y', z' | \tilde{x}, \tilde{y}, \tilde{z}) = 0, \quad G(\zeta, x', 0, z' | \tilde{x}, \tilde{y}, \tilde{z}) = 0, \quad G(\zeta, x', y', 0 | \tilde{x}, \tilde{y}, \tilde{z}) = 0. \quad (4.3)$$

The function is also subject to the usual conditions of vanishing at infinity (along with its derivatives). The domain in the  $(x', y', z')$  plane is the positive octant:

$$\mathcal{D} = \{(x', y', z'), \quad 0 \leq x' < \infty, \quad 0 \leq y' < \infty, \quad 0 \leq z' < \infty\}. \quad (4.4)$$

The unicity for Green's function for operator  $\mathcal{O}_{3D}^*$  in the positive octant and with absorbing boundaries is obtained using a similar argument as in the one dimensional case (the operator is uniformly parabolic and the problem satisfied by the difference of any two solutions of (4.1)-(4.3) has continuous initial and boundary conditions and hence it is identically null). This property is inherited by all the transformed problems below.

As in the two-dimensional case, the first step in solving this problem is to apply a change of variables that eliminates the cross derivatives from the PDE, bringing a

change in the domain:

$$\begin{cases} \alpha'(x', y', z') = x' \\ \beta'(x', y', z') = \frac{1}{\bar{\rho}_{xy}} (-\rho_{xy}x' + y') \\ \gamma'(x', y', z') = \frac{1}{\bar{\rho}_{xy}\chi} [(\rho_{xy}\rho_{yz} - \rho_{xz})x' + (\rho_{xy}\rho_{xz} - \rho_{yz})y' + \bar{\rho}_{xy}^2 z'] \end{cases} \quad (4.5)$$

where the notations  $\chi = \sqrt{1 - \rho_{xy}^2 - \rho_{xz}^2 - \rho_{yz}^2 + 2\rho_{xy}\rho_{xz}\rho_{yz}}$  and  $\bar{\rho} = \sqrt{1 - \rho^2}$  are used. The conditions necessary for a valid correlation matrix ensure that the change of variables is well defined.

The problem for Green's function for the operator  $\mathcal{O}_{3D}^*$  with  $\delta$ -type initial condition and 0 boundary conditions in the new coordinate system becomes:

$$\begin{aligned} G_\zeta - \frac{1}{2}G_{\alpha'\alpha'} - \frac{1}{2}G_{\beta'\beta'} - \frac{1}{2}G_{\gamma'\gamma'} &= 0, \\ G(0, \alpha', \beta', \gamma' | \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) &= \delta(\alpha' - \tilde{\alpha}) \delta(\beta' - \tilde{\beta}) \delta(\gamma' - \tilde{\gamma}), \\ G(\zeta, 0, \beta', \gamma' | \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) &= 0, \quad G\left(\zeta, \alpha', -\frac{\rho_{xy}}{\bar{\rho}_{xy}}\alpha', \gamma' \middle| \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\right) = 0, \\ G\left(\zeta, \alpha', \beta', -\frac{\rho_{xz}\bar{\rho}_{xy}}{\chi}\alpha' + \frac{\rho_{xy}\rho_{xz} - \rho_{yz}}{\chi}\beta' \middle| \tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\right) &= 0, \end{aligned} \quad (4.6)$$

where  $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma})$  gives the position of the source point in the new coordinate system and can be written directly in terms of  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$ .

With the change of variables, the domain in which the problem needs to be solved has changed. The original domain was the volume bounded by the planes  $x' = 0$ ,  $y' = 0$  and  $z' = 0$ . This now changes to the volume bounded by the planes:  $\alpha' = 0$ ,  $\beta' = -\frac{\rho_{xy}}{\bar{\rho}_{xy}}\alpha'$  and  $\gamma' = \frac{\bar{\rho}_{xy}}{\chi}(-\rho_{xz}\alpha' + \frac{\rho_{xy}\rho_{xz} - \rho_{yz}}{\bar{\rho}_{xy}}\beta')$ , denoted by  $\Pi_1$ ,  $\Pi_2$  and  $\Pi_3$  respectively. Let  $\vec{e}_3$  be the versor corresponding to the edge  $\Pi_1 \cap \Pi_2$ ,  $\vec{e}_2$  the versor

corresponding to the edge  $\Pi_1 \cap \Pi_3$ , and  $\vec{e}_1$  the versor corresponding to the edge  $\Pi_2 \cap \Pi_3$ :

$$\begin{aligned}\vec{e}_3 &= (0, 0, 1), \\ \vec{e}_2 &= \left(0, \frac{\chi}{\bar{\rho}_{xy}\bar{\rho}_{xz}}, -\frac{\rho_{yz} - \rho_{xz}\rho_{xy}}{\bar{\rho}_{xy}\bar{\rho}_{xz}}\right), \\ \vec{e}_1 &= \left(\frac{\chi}{\bar{\rho}_{yz}}, -\frac{\rho_{xy}\chi}{\bar{\rho}_{xy}\bar{\rho}_{yz}}, -\frac{\rho_{xz} - \rho_{yz}\rho_{xy}}{\bar{\rho}_{xy}\bar{\rho}_{yz}}\right).\end{aligned}$$

Then the domain of interest is the hull spanned by these vectors:

$$\tilde{\mathcal{D}} = \{\vec{v} \mid \vec{v} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2 + \omega_3 \vec{e}_3, \quad \omega_i \geq 0\}.$$

In order to take advantage of the symmetry of the problem, a second change of variables to spherical coordinates<sup>1</sup> is applied: the axis  $\alpha' = 0$  and  $\beta' = 0$  is given by  $\theta' = 0$ ; the axis  $\alpha' = 0$  and  $\gamma' = 0$  is given by  $\varphi' = 0$  and  $\theta' = \pi/2$ .

$$\begin{cases} \alpha' = r' \sin \theta' \sin \varphi' \\ \beta' = r' \sin \theta' \cos \varphi' \\ \gamma' = r' \cos \theta' \end{cases} \longleftrightarrow \begin{cases} r' = \sqrt{\alpha'^2 + \beta'^2 + \gamma'^2} \\ \theta' = \arccos\left(\frac{\gamma'}{r'}\right) \\ \varphi' = \arctan\left(\frac{\alpha'}{\beta'}\right) \end{cases} \quad (4.7)$$

In order to obtain the range of possible values for  $\varphi$  for the domain of interest, the versors  $\vec{e}_1$  and  $\vec{e}_2$  are projected onto the  $(\alpha', \beta')$  plane to obtain the following normalized vectors:

$$\begin{aligned}\vec{H}_2 &= (0, 1), \\ \vec{H}_1 &= (\bar{\rho}_{xy}, -\rho_{xy}).\end{aligned}$$

The range of values for  $\varphi'$  is therefore given by:  $0 \leq \varphi' \leq \varpi$ , where  $\varpi = \arccos(-\rho_{xy})$ . It is important to note that  $\varpi < \pi$ , as  $\alpha' \geq 0$  inside of

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<sup>1</sup>Notice that the change to spherical coordinates is not the classical one since  $\varphi' = 0$  and  $\theta' = \pi/2$  denote the  $\beta'$  axis rather than the  $\alpha'$  one. This is done for convenience such that the range of possible values for  $\varphi'$  is between 0 and a maximum value.

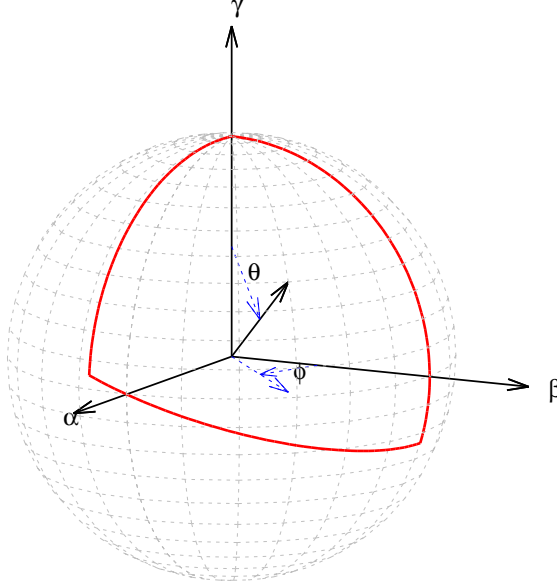


Figure 4.1: Domain after the change in coordinates ( $\rho_{xy} = 20\%$ ,  $\rho_{xz} = 0\%$ ,  $\rho_{yz} = 30\%$ )

the domain. As can be observed in Figure 4.1, the possible range of values for  $\theta'$  depends on  $\varphi'$ :  $0 \leq \theta' \leq \Theta(\varphi')$ .

In order to calculate  $\Theta(\varphi')$ , first consider a vector on the boundary of the domain (in the  $\Pi_3$  plane):

$$\vec{X} = \omega_1 \vec{e}_1 + \omega_2 \vec{e}_2,$$

where  $\omega_1 \geq 0$  and  $\omega_2 \geq 0$ . Using the formulas for  $\vec{e}_1$  and  $\vec{e}_2$  yields:

$$\vec{X} = \left( \omega_1 \frac{\chi}{\bar{\rho}_{yz}}, -\omega_1 \frac{\rho_{xy}\chi}{\bar{\rho}_{xy}\bar{\rho}_{yz}} + \omega_2 \frac{\chi}{\bar{\rho}_{xy}\bar{\rho}_{xz}}, -\omega_1 \frac{\rho_{xz} - \rho_{yz}\rho_{xy}}{\bar{\rho}_{xy}\bar{\rho}_{yz}} - \omega_2 \frac{\rho_{yz} - \rho_{xz}\rho_{xy}}{\bar{\rho}_{xy}\bar{\rho}_{xz}} \right).$$

The projection of this vector onto the  $(\alpha', \beta')$  plane is the following vector:

$$\vec{X}_{\alpha\beta} = \left( \omega_1 \frac{\chi}{\bar{\rho}_{yz}}, -\omega_1 \frac{\rho_{xy}\chi}{\bar{\rho}_{xy}\bar{\rho}_{yz}} + \omega_2 \frac{\chi}{\bar{\rho}_{xy}\bar{\rho}_{xz}} \right),$$

which leads to the following form for the angle  $\varphi'$  as a function of  $\omega_1$  and  $\omega_2$ :

$$\varphi'(\omega_1, \omega_2) = \arccos \left( \frac{-\omega_1 \rho_{xy} \bar{\rho}_{xz} + \omega_2 \bar{\rho}_{yz}}{\sqrt{\omega_1^2 \bar{\rho}_{xz}^2 - 2\omega_1 \omega_2 \rho_{xy} \bar{\rho}_{xz} \bar{\rho}_{yz} + \omega_2^2 \bar{\rho}_{yz}^2}} \right).$$



This expression can be simplified to obtain a parametric form of the angle  $\varphi'$  that depends on a single parameter  $\omega$ , where  $\omega = \frac{\omega_1 \bar{\rho}_{xz}}{\omega_2 \bar{\rho}_{yz}}$ :

$$\varphi'(\omega) = \arccos \left( \frac{1 - \rho_{xy}\omega}{\sqrt{1 - 2\rho_{xy}\omega + \omega^2}} \right). \quad (4.8)$$

It is easy to verify that this parametric form for  $\varphi'$  has the right bounds:  $\varphi'(0) = 0$  and  $\varphi'(\omega) \xrightarrow{\omega \rightarrow \infty} \varpi$ . Similarly, a parametric form can be obtained for  $\Theta$ :

$$\Theta(\omega) = \arccos \left( -\frac{\rho_{yz} - \rho_{xz}\rho_{xy} + \omega(\rho_{xz} - \rho_{yz}\rho_{xy})}{\sqrt{\bar{\rho}_{xy}^2 (\bar{\rho}_{xz}^2 - 2\omega(\rho_{xy} - \rho_{xz}\rho_{yz}) + \omega^2 \bar{\rho}_{yz}^2)}} \right). \quad (4.9)$$

In particular, the values for  $\omega = 0$  and  $\omega \rightarrow \infty$  are given by:

$$\begin{aligned} \Theta(0) &= \arccos \left( -\frac{\rho_{yz} - \rho_{xz}\rho_{xy}}{\bar{\rho}_{xy}\bar{\rho}_{xz}} \right), \\ \Theta(\omega) &\xrightarrow{\omega \rightarrow \infty} \arccos \left( -\frac{\rho_{xz} - \rho_{yz}\rho_{xy}}{\bar{\rho}_{xy}\bar{\rho}_{yz}} \right). \end{aligned}$$

Formulas (4.8) and (4.9) give a parametric characterization of the boundary of the domain which will prove very useful going forward. It is also possible to invert equation (4.8) to obtain  $\omega(\varphi') = \frac{\sin \varphi'}{\sin(\varpi - \varphi')}$ , and then substituting in equation (4.9), obtain directly an expression for  $\Theta(\varphi')$ :

$$\Theta(\varphi') = \arccos \left( -\frac{\rho_{yz} - \rho_{xz}\rho_{xy} + \frac{\sin \varphi'}{\sin(\varpi - \varphi')} (\rho_{xz} - \rho_{yz}\rho_{xy})}{\sqrt{\bar{\rho}_{xy}^2 \left( \bar{\rho}_{xz}^2 - 2\frac{\sin \varphi'}{\sin(\varpi - \varphi')} (\rho_{xy} - \rho_{xz}\rho_{yz}) + \left( \frac{\sin \varphi'}{\sin(\varpi - \varphi')} \right)^2 \bar{\rho}_{yz}^2 \right)}} \right). \quad (4.10)$$

After the two changes of variables, the problem for Green's function for the operator  $\mathcal{O}_{3D}^*$  with  $\delta$ -type initial condition and 0 boundary conditions in the new coordi-

nates has the following form:

$$G_\zeta - \frac{1}{2} \left[ \frac{1}{r'} \frac{\partial^2}{\partial r'^2} (r' G) + \frac{1}{r'^2} \left( \frac{1}{\sin^2 \theta'} G_{\varphi' \varphi'} + \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} (\sin \theta' G_{\theta'}) \right) \right] = 0, \quad (4.11)$$

$$G(0, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = \frac{1}{\tilde{r}^2 \sin \tilde{\theta}} \delta(r' - \tilde{r}) \delta(\varphi' - \tilde{\varphi}) \delta(\theta' - \tilde{\theta}), \quad (4.12)$$

$$\begin{aligned} G(\zeta, 0, \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) &= 0, \quad G(\zeta, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) \xrightarrow{r' \rightarrow \infty} 0. \\ G(\zeta, r', 0, \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) &= 0, \quad G(\zeta, r', \varpi, \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = 0 \\ G(\zeta, r', \varphi', 0 | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) &= 0, \quad G(\zeta, r', \varphi', \Theta(\varphi') | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = 0, \end{aligned} \quad (4.13)$$

where the coordinates of the source  $(\tilde{r}, \tilde{\varphi}, \tilde{\theta})$  can be obtained by applying successively the changes of variables (4.5) and (4.7) to the original coordinates  $(\tilde{x}, \tilde{y}, \tilde{z})$ . The domain this needs to be solved in is:

$$\tilde{D} = \{ (r', \varphi', \theta') \mid r' \geq 0, \quad 0 \leq \varphi' \leq \varpi, \quad 0 \leq \theta' \leq \Theta(\varphi') \}, \quad (4.14)$$

which is an infinite cone, extending from the origin and for which the intersection with the three-dimensional unit sphere is a spherical triangle, as shown in Figure 4.1.

**Proposition 4.1.** *The solution for Green's function for the operator  $\mathcal{O}_{3D}^*$  that solves equation (4.11) inside the domain  $\tilde{D}$ , augmented with initial condition (4.12) and zero boundary conditions (4.13) has the following semi-analytical expression:*

$$G(\zeta, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta \sqrt{r' \tilde{r}}} \sum_{n=1}^{\infty} I_{\sqrt{\Lambda_n + \frac{1}{4}}} \left( \frac{r' \tilde{r}}{\zeta} \right) \Psi_n(\tilde{\varphi}, \tilde{\theta}) \Psi_n(\varphi', \theta'), \quad (4.15)$$

where  $\Lambda_n$  and  $\Psi_n$  are the eigenvalues and eigenfunctions respectively of the following angular problem:

$$\begin{aligned} -\frac{1}{\sin^2 \theta'} \frac{\partial^2 \Psi}{\partial \varphi'^2} - \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left( \sin \theta' \frac{\partial \Psi}{\partial \theta'} \right) &= \Lambda \Psi, \\ \Psi(0, \theta') &= 0, \quad \Psi(\varpi, \theta') = 0, \quad \Psi(\varphi', 0) = 0, \quad \Psi(\varphi', \Theta(\varphi')) = 0. \end{aligned}$$

*Proof.* The solution is built through the eigenfunction expansion method, by extend-

ing the methodology used in the two dimensional case. The first step is to apply the separation of variables technique. Solutions of the following form are proposed:

$$G(\zeta, r', \varphi', \theta') = h(\zeta, r')\Psi(\varphi', \theta'). \quad (4.16)$$

By substituting (4.16) into (4.11), we obtain an equation where the left hand side depends only on  $\zeta$  and  $r'$  and the right hand side depends only on  $\varphi'$  and  $\theta'$ , and hence both sides are equal to some constant value  $\Lambda$ , which yields the following PDEs for functions  $h$  and  $\Psi$ :

$$h_\zeta = \frac{1}{2} \left( \frac{1}{r'} \frac{\partial^2}{\partial r'^2} (r' h) - \frac{\Lambda}{r'^2} h \right), \quad (4.17)$$

$$- \frac{1}{\sin^2 \theta'} \Psi_{\varphi' \varphi'} - \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} (\sin \theta' \Psi_{\theta'}) = \Lambda \Psi. \quad (4.18)$$

The domain for the PDE in equation (4.18) is the spherical triangle shown in Figure 4.1:

$$\Omega = \{(\varphi', \theta') \mid 0 \leq \varphi' \leq \varpi, 0 \leq \theta' \leq \Theta(\varphi')\} \quad (4.19)$$

The zero boundary conditions at  $\varphi' = 0$ ,  $\varphi' = \varpi$ ,  $\theta' = 0$  and for  $\theta' = \Theta(\varphi')$  for Green's function, yield zero boundary conditions for function  $\Psi$ . Indeed, if  $\Psi(0, \theta') \neq 0$ , to satisfy the boundary condition for Green's function and taking into account equation (4.16) implies that  $h(\zeta, r') = 0$ , so an identically null Green's function. With a similar argument for the other boundaries and to have a non-null Green's function results in the following boundary conditions for function  $\Psi$ :

$$\Psi(0, \theta') = 0, \quad \Psi(\varpi, \theta') = 0, \quad \Psi(\varphi', 0) = 0, \quad \Psi(\varphi', \Theta(\varphi')) = 0. \quad (4.20)$$

Consider the standard Riemannian metric on the sphere, denoted by  $g$ :

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 \theta' \end{pmatrix}. \quad (4.21)$$

In a given metric, the expression of the Laplacian is given by:

$$\Delta_g \Psi = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j} \partial_i \left[ g^{i,j} \sqrt{\det(g)} \partial_j \Psi \right],$$

where  $\det(g)$  is the determinant of the metric tensor and  $g^{ij}$  are elements of the inverse matrix  $g^{-1}$  (see for example Bérard and Besson [1986] page 46). For the metric associated with the sphere surface defined in (4.21), the determinant is  $\det(g) = \sin^2 \theta'$ , and the expression for the Laplacian is:

$$\begin{aligned} \Delta_g \Psi &= -\frac{1}{\sin \theta'} \left[ \frac{\partial}{\partial \theta'} \left( \sin \theta' \frac{\partial \Psi}{\partial \theta'} \right) + \frac{\partial}{\partial \varphi'} \left( \frac{1}{\sin^2 \theta'} \sin \theta' \frac{\partial \Psi}{\partial \varphi'} \right) \right] \\ &= -\frac{1}{\sin^2 \theta'} \frac{\partial^2 \Psi}{\partial \varphi'^2} - \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left( \sin \theta' \frac{\partial \Psi}{\partial \theta'} \right). \end{aligned}$$

Thus the problem to solve for function  $\Psi$ , given by the PDE (4.18) on the domain described in (4.19) and with boundary conditions given in (4.20), is an eigenvalue Dirichlet problem for the Laplace operator in the metric  $g$ , also known as the Laplace-Beltrami operator:

$$\begin{aligned} \Delta_g \Psi &= \Lambda \Psi, \quad \text{in } \Omega \\ \Psi|_{\partial\Omega} &= 0, \end{aligned} \tag{4.22}$$

where  $\Omega$  is the domain given in (4.19) and  $\partial\Omega$  denotes its boundary.

The eigenvalue problem given in equation (4.18) but considered on the surface of the whole sphere is a well known problem. It has been shown, in Courant and Hilbert [2008] for example, that this problem has a countably infinite sequence of positive eigenvalues, as well as a corresponding sequence of orthogonal eigenfunctions. The solutions are obtained using the separation of variables technique and are known as the spherical harmonics.

For this specific case however, the domain of the eigenvalue problem (4.18) is not the entire surface of a sphere, but a spherical triangle. The separation of variable technique cannot be applied in this case due to the specific shape of the domain, and

therefore fully analytical solutions are not available. However, the Dirichlet eigenvalue problem in (4.22) also has a countably infinite sequence of positive eigenvalues  $0 < \Lambda_1 \leq \Lambda_2 \leq \dots$  (see for example Theorem 18 in Bérard and Besson [1986] page 53). The same theorem also states that the corresponding eigenfunctions (referred to as spherical surface harmonics in Courant and Hilbert [2008], page 318) are orthogonal with respect to the scalar product associated with the given metric. In this case this means:

$$\int_{\Omega} \Psi_i \Psi_j \sin \theta' d\varphi' d\theta' = \delta_{ij}, \quad (4.23)$$

where  $\Psi_i$  and  $\Psi_j$  are two eigenfunctions of the problem (4.22). Even though the eigenfunctions are not known analytically, these can be computed numerically as will be discussed in Section 4.2.

Since the eigenvalues are positive, this yields the following PDE for function  $h$ :

$$h_{\zeta} = \frac{1}{2} \left( \frac{1}{r'} \frac{\partial^2}{\partial r'^2} (r' h) - \frac{\Lambda}{r'^2} h \right). \quad (4.24)$$

The boundary conditions for Green's function at  $r' = 0$  and  $r' \rightarrow \infty$  transfer to function  $h$ . Indeed, suppose for example that  $h(\zeta, 0) \neq 0$ ; since Green's function has zero boundary conditions (see equation (4.13)), equation (4.16) implies a null  $\Psi$  function everywhere and thus an identically null solution for Green's function. With similar argument for the other boundary, for a non-null Green's function,  $h$  has to satisfy the following boundary conditions:

$$h(\zeta, 0) = 0, \quad h(\zeta, r') \xrightarrow{r' \rightarrow \infty} 0.$$

Let function  $\tilde{h}$  defined on  $\mathbb{R}^+ \times \mathbb{R}^+$  be given by  $\tilde{h}(\zeta, r') = \sqrt{r'} h(\zeta, r')$ . The PDE that this function satisfies can be obtained from equation (4.24), by rewriting the

derivatives of  $h$  in terms of function  $\tilde{h}$  and its derivatives:

$$\begin{aligned}\frac{\partial h}{\partial \zeta} &= \frac{1}{\sqrt{r'}} \frac{\partial \tilde{h}}{\partial \zeta} \\ \frac{\partial}{\partial r'} (r' h) &= \frac{1}{2\sqrt{r'}} \tilde{h} + \sqrt{r'} \frac{\partial \tilde{h}}{\partial r'} \\ \frac{\partial^2}{\partial r'^2} (r' h) &= \frac{\partial}{\partial r'} \left( \frac{1}{2\sqrt{r'}} \tilde{h} + \sqrt{r'} \frac{\partial \tilde{h}}{\partial r'} \right) \\ &= -\frac{1}{4r' \sqrt{r'}} \tilde{h} + \frac{1}{\sqrt{r'}} \frac{\partial \tilde{h}}{\partial r'} + \sqrt{r'} \frac{\partial^2 \tilde{h}}{\partial r'^2}\end{aligned}$$

This yields the following equation for function  $\tilde{h}$ :

$$\tilde{h}_\zeta = \frac{1}{2} \left( \tilde{h}_{r'r'} + \frac{1}{r'} \tilde{h}_{r'} - \frac{\Lambda + \frac{1}{4}}{r'^2} \tilde{h} \right), \quad (4.25)$$

which is similar to equation (3.15), solved for the two dimensional case. Proceeding in a similar manner as in that case it can be shown that the solution for this equation is given by

$$\tilde{h}(\zeta, r') = \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta \sqrt{\tilde{r}}} I_{\sqrt{\Lambda+1/4}} \left( \frac{r' \tilde{r}}{\zeta} \right),$$

where  $I_\Lambda(\xi)$  is the modified Bessel function of the first kind and satisfies equation (3.18). This yields the following solution for function  $h$ :

$$h(\zeta, r') = \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta \sqrt{r' \tilde{r}}} I_{\sqrt{\Lambda+1/4}} \left( \frac{r' \tilde{r}}{\zeta} \right).$$

As in the two dimensional case, the boundary conditions need to be verified by this expression. When  $r' \rightarrow \infty$ , the argument of the modified Bessel function goes to infinity, and the asymptotic expansion for large arguments given in Abramowitz and Stegun [1964] (page 377, 9.7.1) can again be used. This yields the following

approximation for function  $h$  when  $r' \rightarrow \infty$ :

$$\begin{aligned} h(\zeta, r') &\approx \frac{e^{-\frac{\tilde{r}^2 + r'^2}{2\zeta}} \sqrt{\zeta} e^{\frac{\tilde{r}r'}{\zeta}}}{\zeta \sqrt{r'\tilde{r}} \sqrt{2\pi\tilde{r}r'}} \\ &= \frac{e^{-\frac{(\tilde{r}-r')^2}{2\zeta}}}{\sqrt{2\pi\zeta\tilde{r}r'}} \xrightarrow{r' \rightarrow \infty} 0, \end{aligned}$$

and hence the right limit is recovered at  $r' \rightarrow \infty$ . Since  $I_\Lambda(\xi) \xrightarrow[\xi \rightarrow 0]{} 0$  when  $\Lambda > 0$ , function  $h$  vanishes when  $r' \rightarrow 0$  and both boundary conditions are satisfied.

The solution for Green's function for the operator  $\mathcal{O}_{3D}^*$  that solves equation (4.11) inside the domain  $\tilde{\mathcal{D}}$ , augmented with initial condition (4.12) and zero boundary conditions, can thus be written as a superposition of solutions:

$$\begin{aligned} G\left(\zeta, r', \varphi', \theta' \middle| \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right) &= \sum_{n=1}^{\infty} C_n h_n(\zeta, r') \Psi_n(\varphi', \theta') \\ &= \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta \sqrt{r'\tilde{r}}} \sum_{n=1}^{\infty} C_n I_{\sqrt{\Lambda_n + \frac{1}{4}}} \left( \frac{r'\tilde{r}}{\zeta} \right) \Psi_n(\varphi', \theta'), \end{aligned}$$

where  $\Psi_n$  denotes the  $n$ -th eigenfunction of problem (4.22), and  $\Lambda_n$  the corresponding eigenvalue. It is important to note that the above sum is absolutely summable by properties of the modified Bessel functions (see for example Watson [1995]).

The coefficients  $C_n$  can be computed by imposing the initial condition for Green's function:

$$G\left(0, r', \varphi', \theta' \middle| \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right) = \frac{1}{\tilde{r}^2 \sin \tilde{\theta}} \delta(r' - \tilde{r}) \delta(\varphi' - \tilde{\varphi}) \delta(\theta' - \tilde{\theta}).$$

When  $\zeta \rightarrow 0$ , the argument of the modified Bessel function goes to infinity and hence the asymptotic expansion for large arguments can be used to obtain an approximation for function  $g$  and the corresponding limit at  $\zeta \rightarrow 0$ :

$$h(\zeta, r') \approx \frac{e^{-\frac{(\tilde{r}-r')^2}{2\zeta}}}{\sqrt{2\pi\zeta\tilde{r}r'}} \xrightarrow{\zeta \rightarrow 0} \frac{1}{\tilde{r}^2} \delta(r' - \tilde{r}).$$

Thus, in order for the initial condition in equation (4.12) to be satisfied, the coefficients  $C_n$  need to verify:

$$\sum_{n=1}^{\infty} C_n \Psi_n(\varphi', \theta') = \frac{1}{\sin \tilde{\theta}} \delta(\varphi' - \tilde{\varphi}) \delta(\theta' - \tilde{\theta}). \quad (4.26)$$

To compute the coefficients  $C_n$ , the orthogonality condition for the eigenfunctions of problem (4.22), given in equation (4.23) can be used. Equation (4.26) is multiplied by  $\Psi_m(\varphi', \theta') \sin \theta'$  and then integrated over the whole domain:

$$C_m = \iint_{\Omega} \frac{1}{\sin \tilde{\theta}} \Psi_m(\varphi', \theta') \sin \theta' \delta(\varphi' - \tilde{\varphi}) \delta(\theta' - \tilde{\theta}) d\varphi' d\theta',$$

and hence  $C_m = \Psi_m(\tilde{\varphi}, \tilde{\theta})$ . The final formula for Green's function for operator  $\mathcal{O}_{3D}^*$  with  $\delta$ -type initial condition and absorbing boundaries is:

$$G\left(\zeta, r', \varphi', \theta' \mid \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right) = \frac{e^{-\frac{r'^2 + \tilde{r}^2}{2\zeta}}}{\zeta \sqrt{r' \tilde{r}}} \sum_{n=1}^{\infty} I_{\sqrt{\Lambda_n + \frac{1}{4}}} \left(\frac{r' \tilde{r}}{\zeta}\right) \Psi_n(\tilde{\varphi}, \tilde{\theta}) \Psi_n(\varphi', \theta'). \quad (4.27)$$

As the expression above verifies the desired PDE, as well as the initial condition and the boundary conditions, the unicity of the solution to this problem discussed previously yields that this is the desired solution.  $\square$

**Remark 4.1.** *Similarly to the two-dimensional case, for the Green's function given in equation (4.15), the following can be shown:*

$$r'^n \frac{\partial^k G\left(\zeta, r', \varphi', \theta' \mid \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right)}{\partial r'^k} \xrightarrow{r' \rightarrow \infty} 0, \quad \forall n \geq 0, \quad k = 0, 1. \quad (4.28)$$

*This can be done by using the approximation of the Bessel function for large values of the argument, in the same way as for the proof above of the limit at  $r' \rightarrow \infty$  of Green's function. For  $k = 1$ , the formula (9.6.26) in Abramowitz and Stegun [1964] can be used to obtain the result. Similarly, for the limit at 0:*

$$r'^n \frac{\partial^k G\left(\zeta, r', \varphi', \theta' \mid \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right)}{\partial r'^k} \xrightarrow{r' \rightarrow 0} 0, \quad \forall n \geq 0, \quad k = 0, 1. \quad (4.29)$$



*These are true also for the derivatives with respect to  $\tilde{r}$ , as the expression is symmetric with respect to  $r'$  and  $\tilde{r}$ . These results will be useful later on.*

## 4.2 Computation of the eigenvalues and eigenfunctions of the angular problem

This section describes how the eigenvalues and eigenvectors of the angular problem are computed numerically. As discussed in the previous section, this is a Dirichlet eigenvalue problem on the spherical triangle shown in Figure 4.1 and denoted by  $\Omega$ :

$$\Omega = \{(\varphi', \theta') \mid 0 \leq \varphi' \leq \varpi, 0 \leq \theta' \leq \Theta(\varphi')\}$$

where the operator is the Laplace-Beltrami operator (the laplacian associated with the standard Riemannian metric on the sphere given in (4.21)):

$$-\frac{1}{\sin^2 \theta'} \frac{\partial^2 \Psi}{\partial \varphi'^2} - \frac{1}{\sin \theta'} \frac{\partial}{\partial \theta'} \left( \sin \theta' \frac{\partial \Psi}{\partial \theta'} \right) = \Lambda \Psi, \quad \text{in } \Omega, \quad (4.30)$$

$$\Psi(0, \theta') = 0, \quad \Psi(\varpi, \theta') = 0, \quad \Psi(\varphi', 0) = 0, \quad \Psi(\varphi', \Theta(\varphi')) = 0.$$

In Walden and Kellogg [1977] the authors focus on determining numerically, using a finite difference approach, the smallest eigenvalue for the above problem. However, for the purposes of writing an eigenfunction expansion for Green's function for the operator  $\mathcal{O}_{3D}^*$  with  $\delta$ -type initial condition and 0 boundary conditions, the estimation of several eigenvalues and eigenvectors is needed here.

The shape of the domain for the eigenvalue problem is another important factor in the choice of the numerical method to use. The two dimensional spherical surface inside the red line shown in Figure 4.1 can be mapped directly onto the  $(\varphi', \theta')$  plane. This is done in a similar way to the method used by cartographers to map the Earth's surface using Mercator's projection. The southern boundary of the domain is mapped into a continuous curve parametrised by equations (4.8) and (4.9). The boundary at  $\theta' = 0$  is degenerate as it corresponds to the north pole on the sphere.

Figure 4.2 shows the domain projected onto the  $(\varphi', \theta')$  plan when all correlation

values are set to 0. Figures 4.3, 4.4 and 4.5 show the oriented domain for sample positive correlation values, while Figures 4.6 and 4.7 show the domain when some of the correlations have negative values.

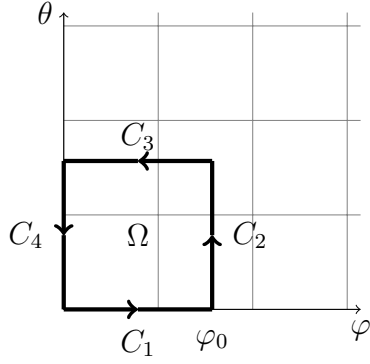


Figure 4.2: 3D spherical triangle projected in 2D:  $\rho_{xy} = 0$ ,  $\rho_{xz} = 0$ ,  $\rho_{yz} = 0$

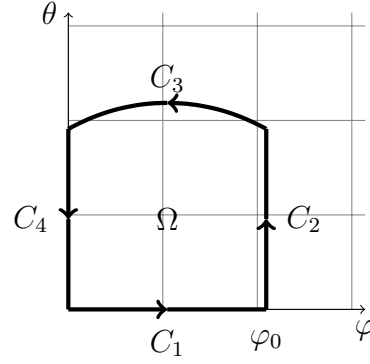


Figure 4.3: 3D spherical triangle projected in 2D:  $\rho_{xy} = 0.5$ ,  $\rho_{xz} = 0.5$ ,  $\rho_{yz} = 0.5$

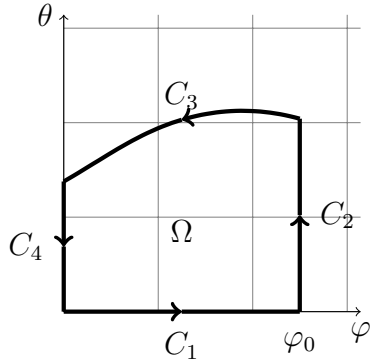


Figure 4.4: 3D spherical triangle projected in 2D:  $\rho_{xy} = 0.8$ ,  $\rho_{xz} = 0.5$ ,  $\rho_{yz} = 0.3$

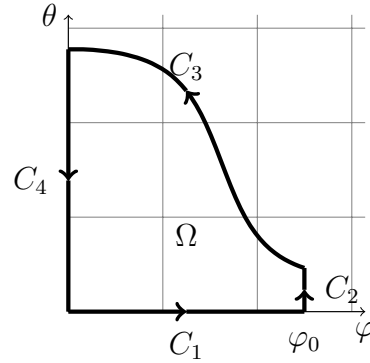


Figure 4.5: 3D spherical triangle projected in 2D:  $\rho_{xy} = 0.8$ ,  $\rho_{xz} = 0.05$ ,  $\rho_{yz} = 0.6$

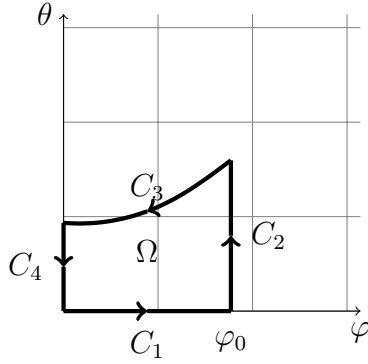


Figure 4.6: 3D spherical triangle projected in 2D:  $\rho_{xy} = 0.2$ ,  $\rho_{xz} = -0.1$ ,  $\rho_{yz} = -0.6$

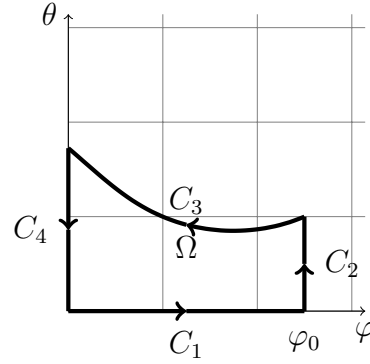


Figure 4.7: 3D spherical triangle projected in 2D:  $\rho_{xy} = 0.8$ ,  $\rho_{xz} = -0.65$ ,  $\rho_{yz} = -0.45$

Taking into account both the fact that several eigenvalues and eigenfunctions are needed as well as the various possible shapes for the domain, with curved boundary, the finite element method (FEM) is an appropriate numerical solution for this problem. In order to apply the FEM, the first step is to write the weak or variational formulation for the Dirichlet eigenvalue problem.

Let  $L^2(\Omega, g)$  be the space of measurable functions  $f$  on  $\Omega$  such that

$$\int_{\Omega} |f|^2 \sin \theta' d\varphi' d\theta' < \infty,$$

where the norm was defined to take into account the standard Riemannian metric  $g$  on the surface of the sphere. The natural spaces that allow for the resolution of the variational formulations of partial differential equations are the Sobolev spaces. Let  $\mathcal{H}^1(\Omega, g)$  denote the Sobolev space which consists of functions  $f$  that have bounded norm  $\|\cdot\|_1$ , where the norm is defined as:

$$\|f\|_1^2 = \int_{\Omega} |f|^2 \sin \theta' d\varphi' d\theta' + \int_{\Omega} |\nabla_g f|^2 \sin \theta' d\varphi' d\theta'.$$

For more details on the definitions of the Sobolev spaces, see for example Bérard and Besson [1986]. Let  $H_0^1(\Omega, g)$  be defined as:

$$\mathcal{H}_0^1(\Omega, g) = \{f \in \mathcal{H}^1(\Omega, g) \mid f = 0 \text{ on } \partial\Omega\}.$$

This is the space in which the solutions for the variational formulation of the problem will be searched for. To obtain the variational formulation, consider a test function  $\hat{\Psi}$ , with  $\hat{\Psi} \in \mathcal{H}_0^1(\Omega, g)$ . Since the test function belongs to the same space as  $\Psi$ , it is also null on the boundary of the domain. To obtain the variational formulation (or weak formulation) of the spectral problem, the PDE in equation (4.30) is multiplied by the test function and an integration over the whole domain is performed. Using Green's formula (see for example Bérard and Besson [1986] pages 48-49), as well as the null boundary conditions for  $\Psi$  and  $\hat{\Psi}$  the following holds:

$$\int_{\Omega} \Delta_g \Psi \hat{\Psi} \sin \theta' d\varphi' d\theta' = \int_{\Omega} \nabla_g \Psi \nabla_g \hat{\Psi} \sin \theta' d\varphi' d\theta',$$

and the weak (or variational) formulation of the spectral problem is:

Find  $(\Lambda, \Psi) \in \mathbb{R} \times \mathcal{H}_0^1(\Omega, g)$  such that:

$$\int_{\Omega} \nabla_g \Psi \nabla_g \hat{\Psi} \sin \theta' d\varphi' d\theta' = \Lambda \int_{\Omega} \Psi \hat{\Psi} \sin \theta' d\varphi' d\theta', \quad \forall \hat{\Psi} \in \mathcal{H}_0^1(\Omega, g). \quad (4.31)$$

For the FEM method, a discretisation is required, so the Hilbert space  $\mathcal{H}_0^1(\Omega, g)$  is replaced by a sub-space of finite dimension (denoted by  $V$ ). Let  $n$  be the dimension of this space and  $(\Phi_i)_{1 \leq i \leq n}$  a basis. The solution for the problem in this finite-dimensional space can be associated with a vector in  $\mathbb{R}^n$ ,  $\Psi_n = (\psi_1, \dots, \psi_n)$  such that:  $\Psi(\varphi', \theta') = \sum_{i=1}^n \Phi_i(\varphi', \theta') \psi_i$ . The weak (or variational) formulation given in (4.31) is then approximated by:

$$\text{Find } (\Lambda, \Psi_n) \in \mathbb{R} \times \mathbb{R}^n \text{ such that : } K\Psi_n = \Lambda M\Psi_n, \quad (4.32)$$

where  $K = (K_{ij})_{1 \leq i, j \leq n}$  is the stiffness matrix, and  $M = (M_{ij})_{1 \leq i, j \leq n}$  the mass matrix, with elements given by:

$$K_{ij} = \int_{\Omega} \nabla_g \Phi_j \nabla_g \Phi_i \sin \theta' d\varphi' d\theta',$$

$$M_{ij} = \int_{\Omega} \Phi_i \Phi_j \sin \theta' d\varphi' d\theta'.$$

For the convergence of the solution of the FEM problem in (4.32) to the solution of the variational formulation in (4.31) see for example Dziuk [1988], Apel and Pester [2005] or Pester [2005], where the authors look specifically at the Laplace-Beltrami operator on spherical surfaces.

Note that both the stiffness and the mass matrices are symmetric. To solve the generalised eigenvalue system, the first step is to do a Cholesky decomposition of matrix  $M$  (see for example Press et al. [1992] page 462):  $M = \mathcal{M}\mathcal{M}^T$  and the eigenvalue system becomes:

$$\mathcal{M}^{-1}K\Psi = \Lambda^2\mathcal{M}^T\Psi.$$

Let  $C$  be a matrix defined as  $C = \mathcal{M}^{-1}K(\mathcal{M}^{-1})^T$ , which is also symmetric, and the eigenvalue system can be rewritten as:

$$C(\mathcal{M}^T\Psi) = \Lambda^2(\mathcal{M}^T\Psi). \quad (4.33)$$

The eigenvalues and eigenvectors for this problem can be computed (using for example a QR algorithm – see Press et al. [1992] page 476), and the eigenvectors for the original problem can be computed as  $(\mathcal{M}^T)^{-1}E$  where  $E$  is an eigenvector of the modified problem. Sample results are discussed in Section 4.2.2.

#### 4.2.1 Constructing the mesh

The discretisation of the domain needed for the finite element method is obtained by constructing a triangular mesh. This section gives a brief description of the methodology used to construct triangular meshes on the domain of interest. The dimension of the sub-space of finite dimension in which the solution is looked for is given by the number of free points in the mesh, denoted by  $n$  (the number of vertices of all triangles in the mesh excluding those that are on the boundary of the domain). The finer the mesh is, the higher the dimension of this space, and the better the approximation of the solution is.

The algorithm used is iterative. The nodes of the mesh are adjusted at each

iteration based on the current element sizes according to the ideas presented in Per-Olof [2005]. The Delaunay triangulation algorithm is then used to adjust the topology (decide the edges) at each iteration. For the Delaunay triangulation, a divide and conquer algorithm is applied, along with the quad-edge data structure described in detail in Guibas and Stolfi [1985].

Figures 4.8 and 4.9 show the uniform meshes obtained with this method for two sample sets of correlations.

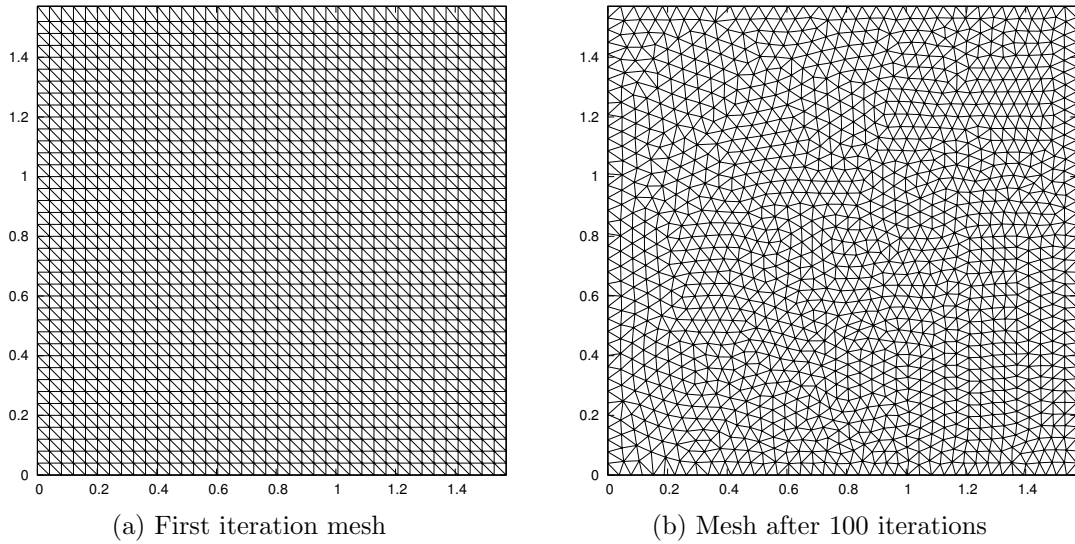


Figure 4.8: Uniform mesh for the domain obtained for  $\rho_{xy} = 0\%$ ,  $\rho_{xz} = 0\%$ ,  $\rho_{yz} = 0\%$ . The mesh is constructed using 1500 points.

There are cases though where it is advantageous to have different sized elements in different regions: where the geometry is more complex or the problem requires more accuracy (for example close to a singularity such that the global accuracy of the solution is good). In order to create adaptive meshes for the domain, the desired edge length distribution over the domain can be specified (this does not have to equal the actual size, but it rather gives the relative distribution over the domain).

Algorithm 1 gives a brief description of the method used to build adaptive triangular meshes. The fixed number of iterations can be replaced by a condition on the largest move of a point in the mesh in the previous iteration.

To obtain the uniform meshes shown in Figures 4.8 and 4.9, the element size

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**Algorithm 1** Algorithm for constructive an adaptive mesh

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**Require:**  $X_1, X_2, Y_1, Y_2$  – bounding box of the domain

**Require:**  $s(x, y)$  – element size function (gives the relative element size distribution over the domain)

- 1: Build a mesh with equally spaced points for the bounding box of the domain
  - 2: Remove points outside the domain
  - 3: Rejection method for adaptive meshes: keep points inside the domain with probabilities proportional to  $1/s(x, y)^2$
  - 4: **while**  $i < \text{MAXITER}$  **do**
  - 5:   Delaunay triangulation using the divide and conquer algorithm described in detail in Guibas and Stolfi [1985]
  - 6:   Assemble triangles obtained through the Delaunay procedure
  - 7:   **for** each triangle **do**
  - 8:     Compute centroid
  - 9:     If centroid outside the domain: remove triangle from list
  - 10:   **end for**
  - 11:   Move mesh points based on current edge lengths using ideas described in Per-Olof [2005]
  - 12:   Bring points that have moved outside of the domain back to the boundary
  - 13:    $i = i + 1$
  - 14: **end while**
-

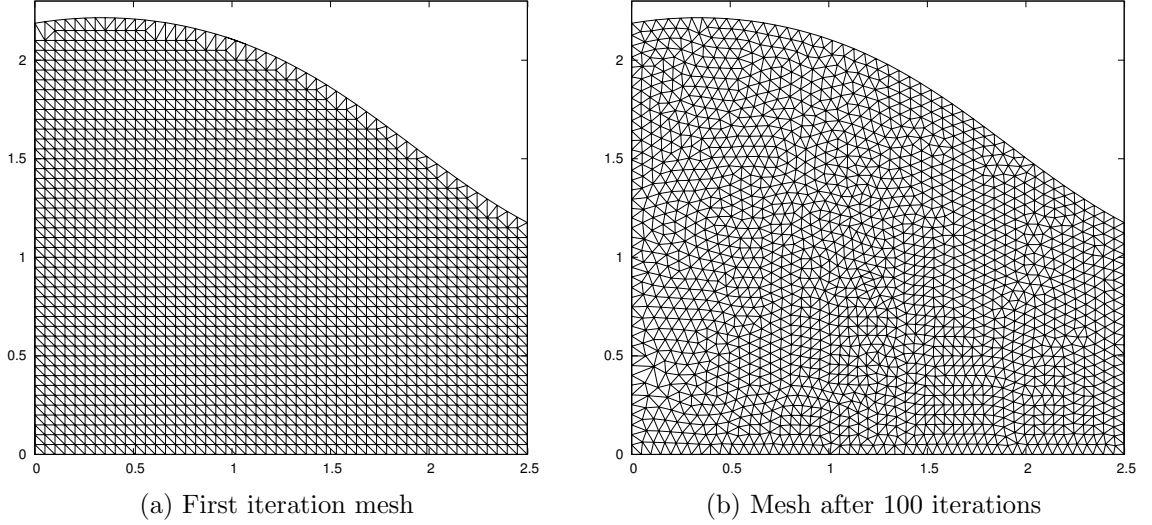


Figure 4.9: Uniform mesh for the domain obtained for  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 20\%$ ,  $\rho_{yz} = 50\%$ . The mesh is constructed using 1800 points.

function is constant over the domain,  $s(\varphi', \theta') = 1$ . This means that step 3 in algorithm 1 does not reject any points.

Figures 4.11, 4.12 and 4.13 show examples of meshes obtained for different values of the correlations and a non-uniform element size function. The meshes are finer close to some or all four of the boundaries. In each case a mesh similar to the ones used as starting point for the uniform case is constructed first (by performing steps 1 and 2 in algorithm 1). Then the rejection method eliminates points in the regions where less precision is required.

To obtain the adaptive mesh in Figure 4.11 for example, the non-uniform size function  $s$  depends on the distance of each point to the closest boundary:

$$s(\varphi', \theta') = 1 + \nu \cdot \min_{C_i} \{\text{dist}((\varphi', \theta'), C_i)\}. \quad (4.34)$$

If the point  $(\varphi', \theta')$  is on the boundary, then  $s(\varphi', \theta') = 1$ , and the probability of rejecting points close to the boundary is almost null. If the point is towards the centre of the domain, the probability of rejecting points is higher as function  $s$  in that point will be larger than 1. Figure 4.10 shows how the probability of keeping



points varies over the domain when using the size function  $s$  in equation (4.34) with different values for the constant  $\nu$  (correlation values are  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$  and  $\rho_{yz} = 30\%$ ). The higher  $\nu$  is, the bigger the size difference between the elements close to the boundaries and the ones in the centre is. A value of  $\nu = 4$  was used to obtain the adaptive grid in Figure 4.11.

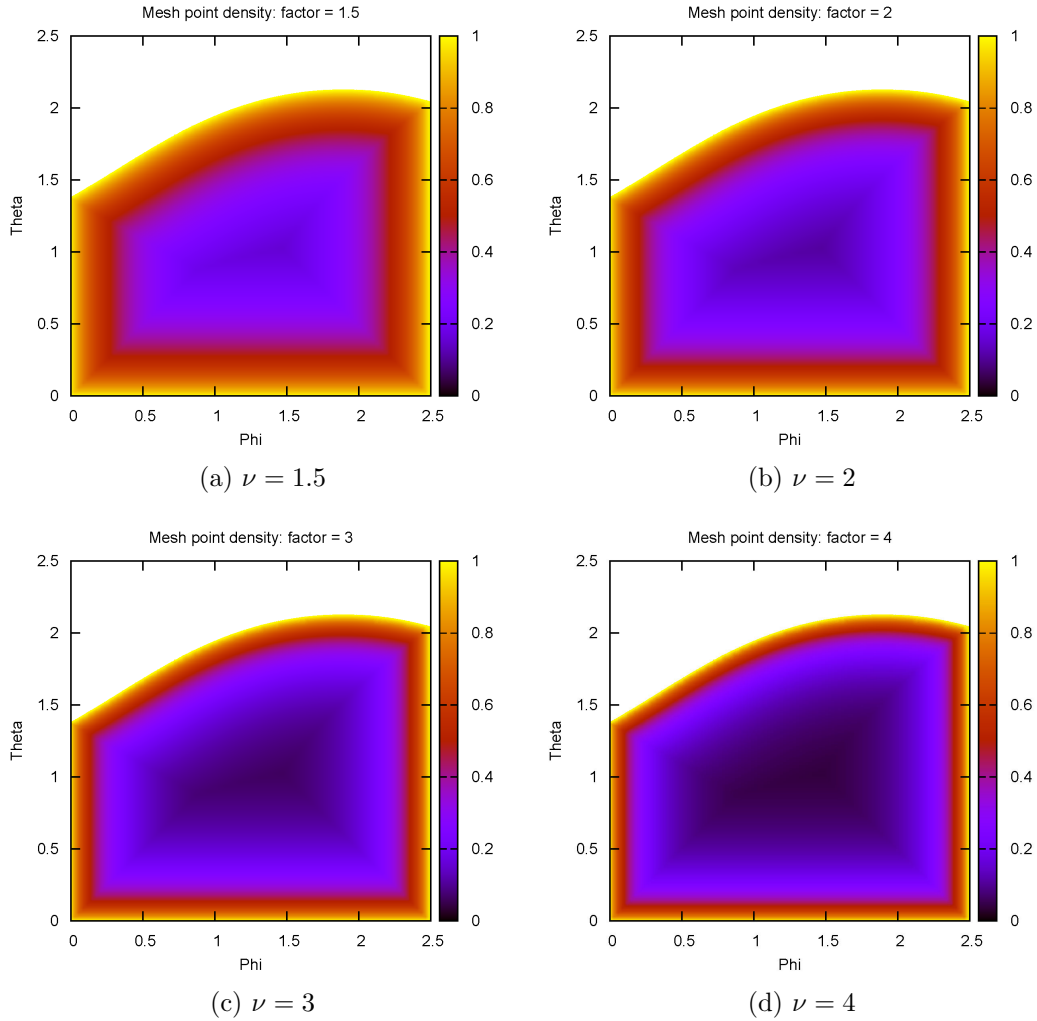


Figure 4.10: The probability of keeping points inside the domain for correlations:  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$ ,  $\rho_{yz} = 30\%$  (relevant for step 3 in algorithm 1).

Once the rejection step is performed, the Delaunay triangulation of the remaining points is used as the starting mesh for the iterative process (steps 4-14), and is denoted in the graphs as the “first iteration” mesh. The figures also show the final

mesh, obtained after 100 iterations. Figure 4.14 shows a similar example, when two of the pairwise correlations are negative.

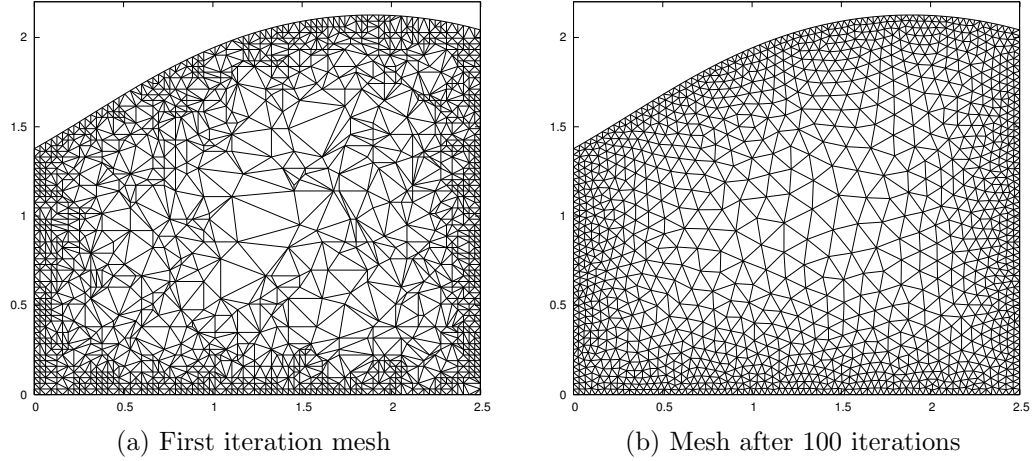


Figure 4.11: Adaptive mesh for the domain obtained for  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$ ,  $\rho_{yz} = 30\%$ . The mesh is constructed using 1500 points and is finer as we get closer to the boundaries.

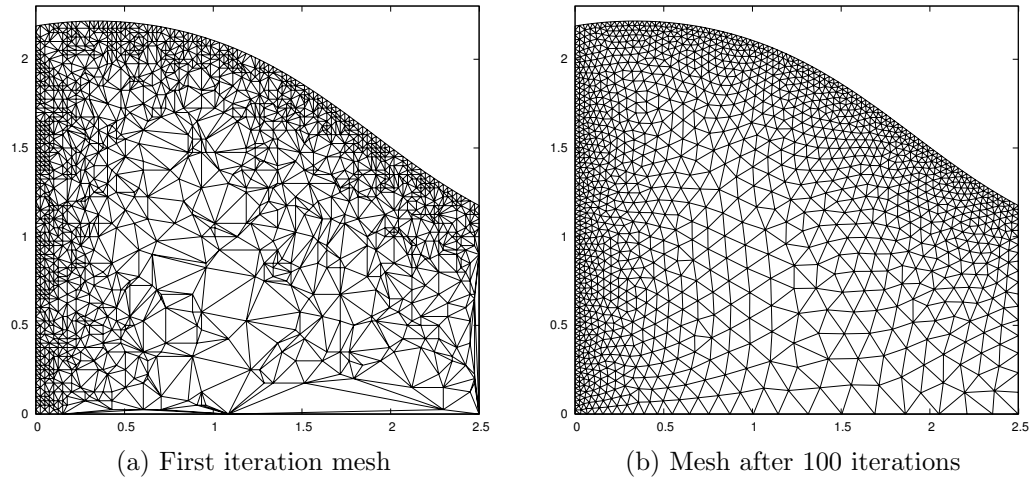


Figure 4.12: Adaptive mesh for the domain obtained for  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$ ,  $\rho_{yz} = 50\%$ . The mesh is constructed using 1500 points. The mesh is finer near two of the boundaries.

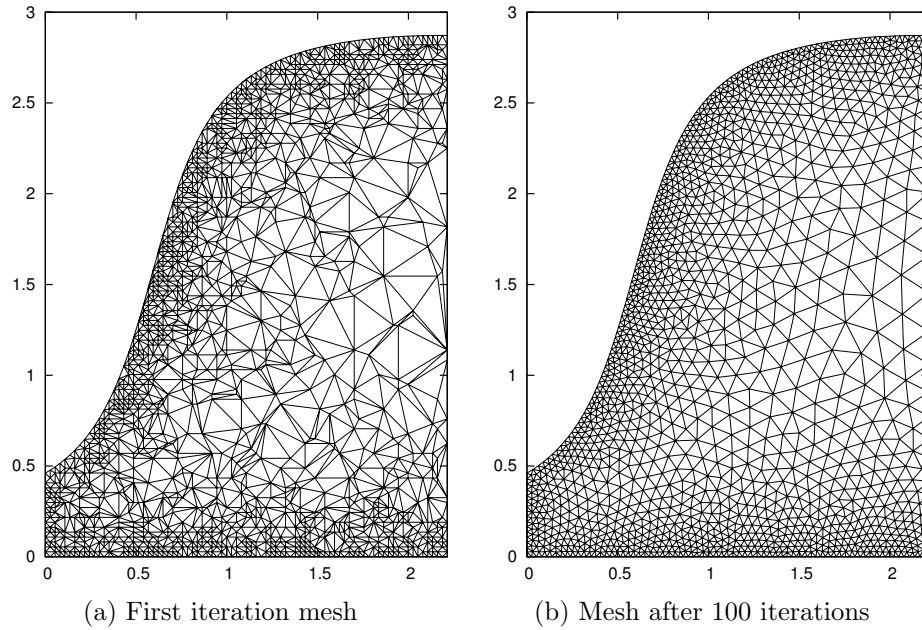


Figure 4.13: Adaptive mesh for the domain obtained for  $\rho_{xy} = 60\%$ ,  $\rho_{xz} = 80\%$ ,  $\rho_{yz} = 5\%$ . The mesh is constructed using 1500 points. The mesh is finer near three of the boundaries.

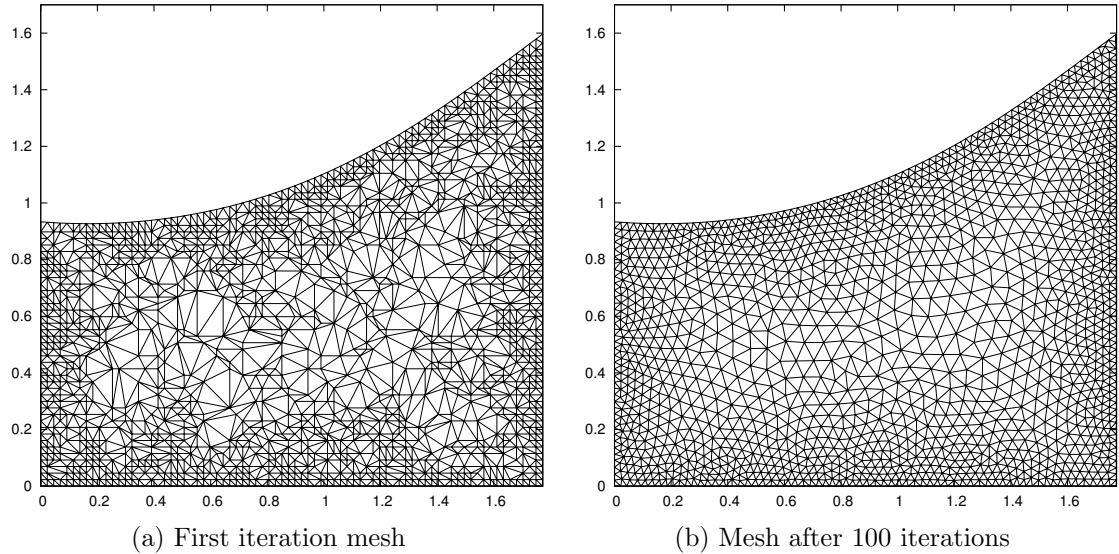


Figure 4.14: Adaptive mesh for the domain obtained for  $\rho_{xy} = 20\%$ ,  $\rho_{xz} = -10\%$ ,  $\rho_{yz} = -60\%$ . The mesh is constructed using 1600 points and is finer as we get closer to the boundaries.

### 4.2.2 Eigenvectors

Once the mesh is constructed, the eigenvalue problem in matrix form given in equation (4.32) is solved to obtain the eigenvalues and corresponding eigenvectors.

Linear basis functions on each triangle are considered. Given any triangle  $T$  of the mesh, there are only three basis functions that are non-zero on  $T$ . Let  $(\varphi'_1, \theta'_1)$ ,  $(\varphi'_2, \theta'_2)$  and  $(\varphi'_3, \theta'_3)$  denote the vertices of triangle  $T$  and  $\Phi_1$ ,  $\Phi_2$  and  $\Phi_3$  the corresponding non-zero basis functions. These are defined by:

$$\Phi_i(\varphi'_j, \theta'_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases},$$

and can be represented by  $\Phi_i(\varphi', \theta') = a_i + b_i\varphi' + c_i\theta'$ ,  $(\varphi', \theta') \in T$ ,  $i = 1, 2, 3$ . The coefficients  $a_i$ ,  $b_i$  and  $c_i$  can be found by solving a system of  $3 \times 3$  equations. Each of the integrals involved in the computation of the elements of matrices  $K$  and  $M$  can be rewritten as a sum of integrals over the triangles where the basis functions are non-zero.

Figure 4.15 shows the case where all correlations are 0. Figures 4.16 and 4.17 shows sample eigenvectors for a domain where all three correlations are positive, while Figure 4.18 shows a case where two of the correlations are negative. Even though the shape of the domain varies significantly between the different examples, similar patterns can be observed, with an increasing number of modes for higher order eigenvectors. Note also that for the first eigenvectors the modes are better defined than for the higher order ones.

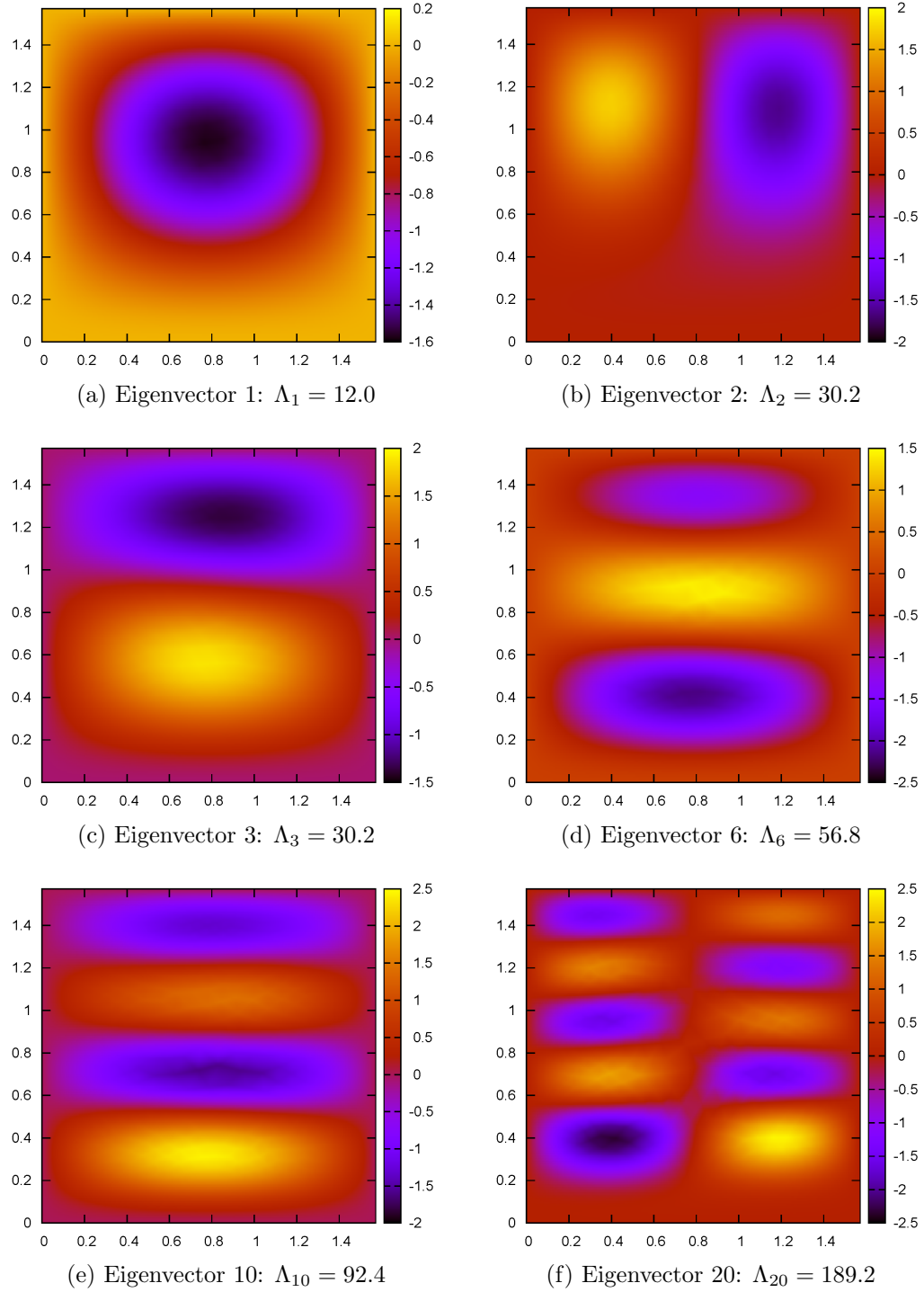


Figure 4.15: Eigenvectors and corresponding eigenvalues for the domain obtained for  $\rho_{xy} = 0\%$ ,  $\rho_{xz} = 0\%$ ,  $\rho_{yz} = 0\%$ . The mesh is constructed using 1500 points. The mesh is finer as we get closer to the boundaries.

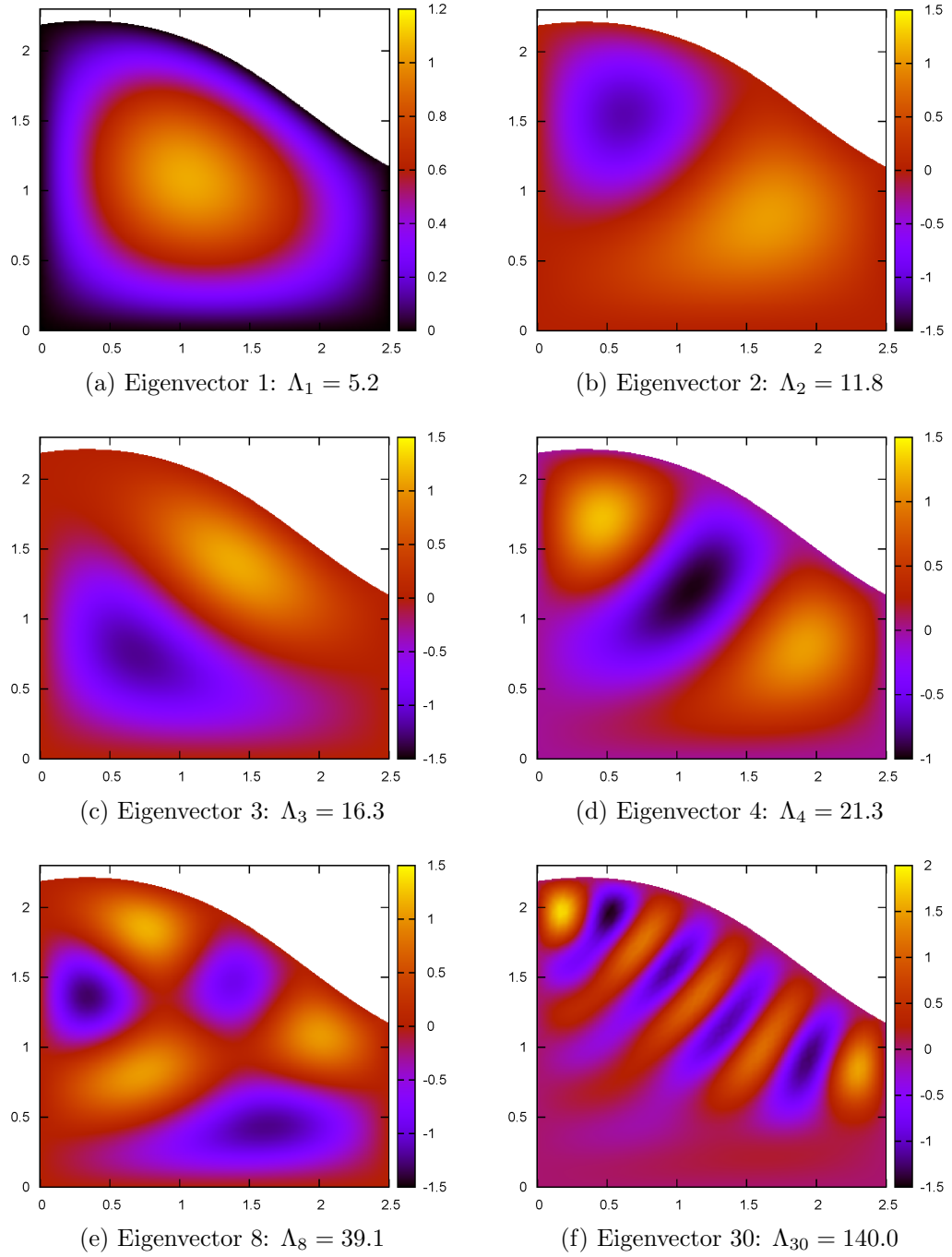


Figure 4.16: Eigenvectors and corresponding eigenvalues for the domain obtained for  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 20\%$ ,  $\rho_{yz} = 50\%$ . The mesh is constructed using 1800 points and is shown in Figure 4.9.

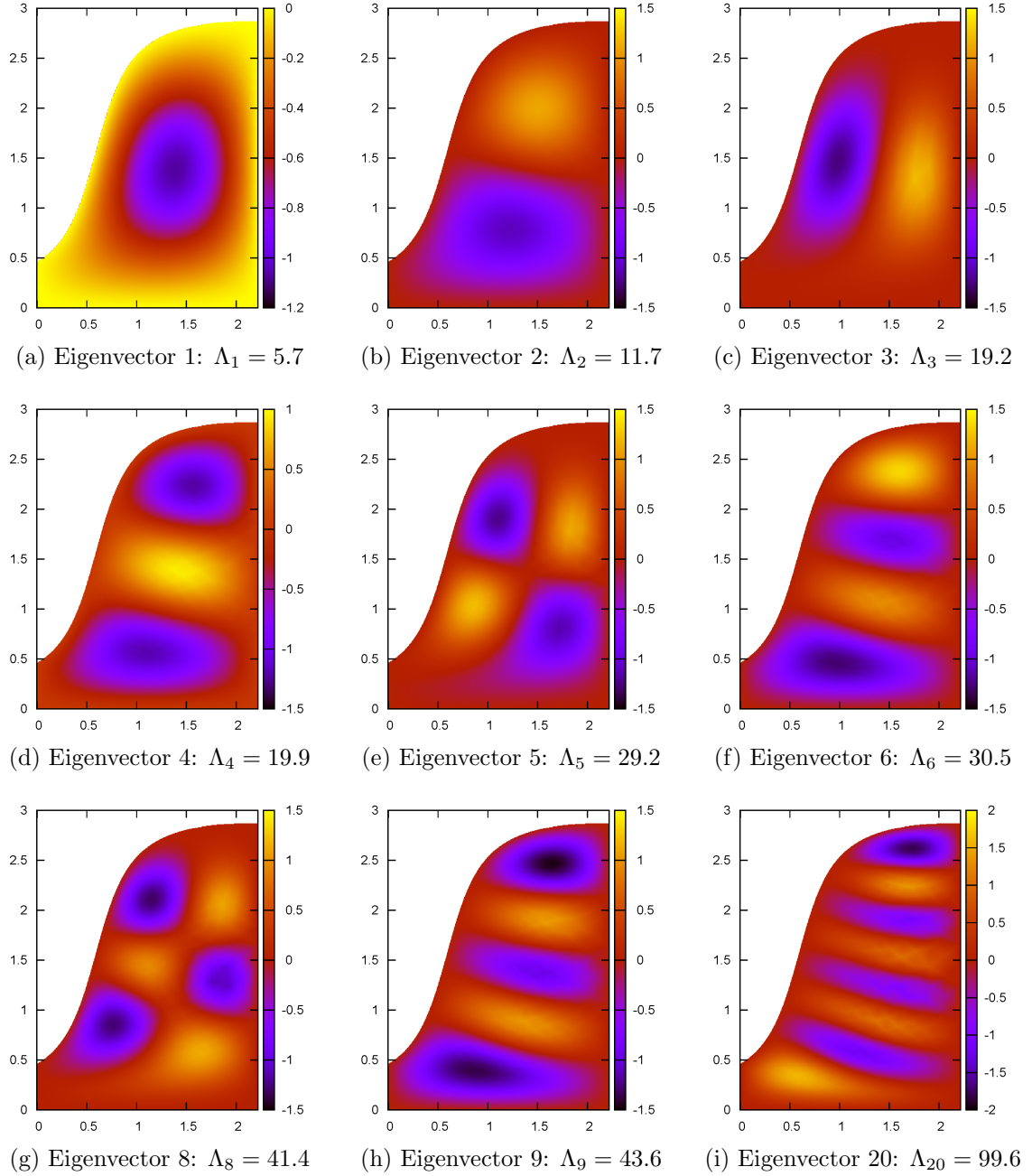


Figure 4.17: Eigenvectors and corresponding eigenvalues for the domain obtained for  $\rho_{xy} = 60\%$ ,  $\rho_{xz} = 80\%$ ,  $\rho_{yz} = 5\%$ . The mesh is constructed using 1500 points and is shown in Figure 4.13.

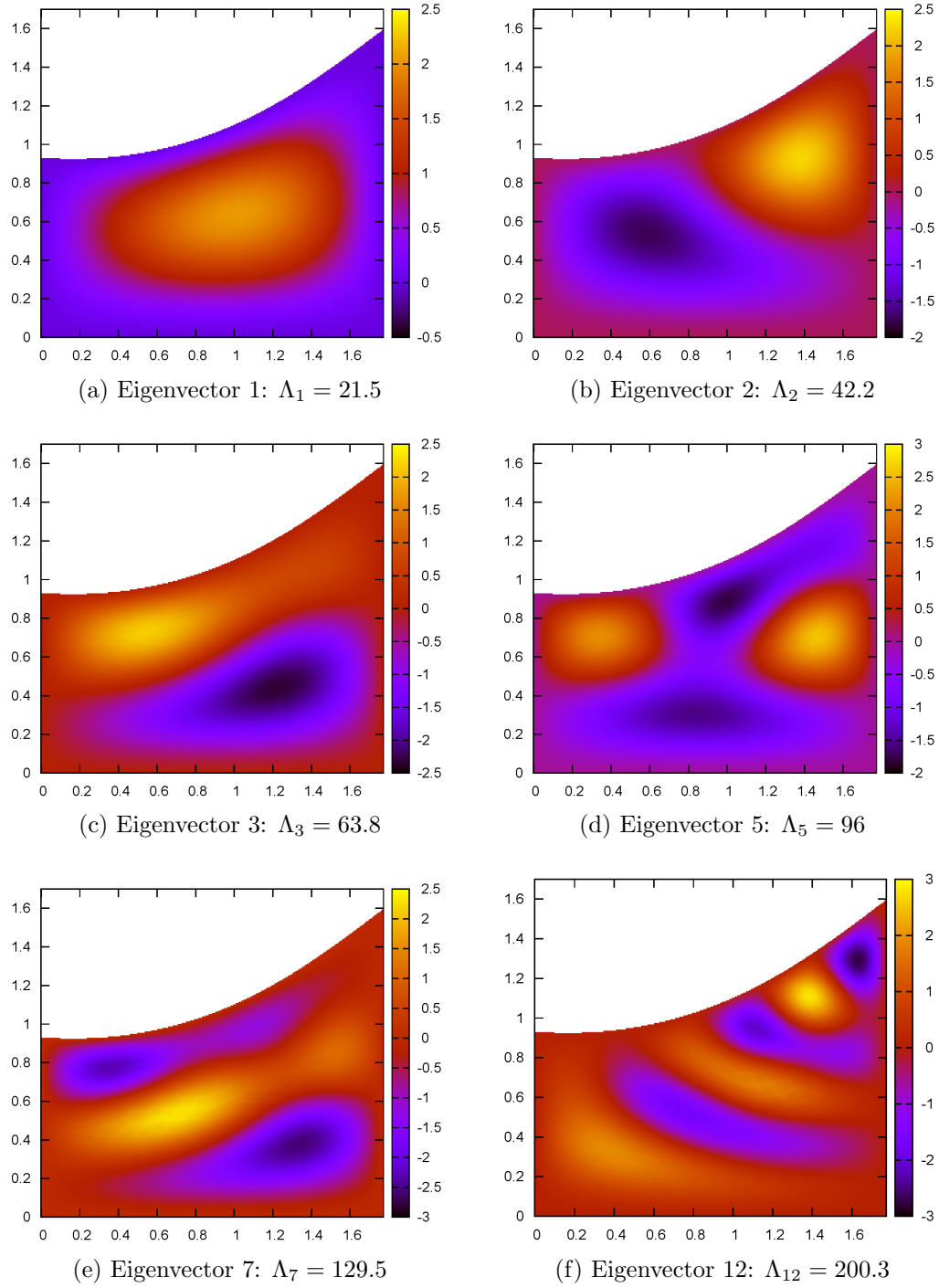


Figure 4.18: Eigenvectors and corresponding eigenvalues for the domain obtained for  $\rho_{xy} = 20\%$ ,  $\rho_{xz} = -10\%$ ,  $\rho_{yz} = -60\%$ . The mesh is constructed using 1600 points and is shown in Figure 4.14.



### 4.3 Efficiency of the proposed method

This section analyses the efficiency of the proposed solution for computing Green's function for the operator  $\mathcal{O}_{3D}^*$  with  $\delta$ -type initial condition and 0 boundary conditions.

In the two dimensional case, the eigenfunction expansion used for Green's function of a two-dimensional operator  $\mathcal{O}_{2D}^*$  in the positive quadrant with absorbing boundaries at 0 and  $\delta$ -type initial condition has proven very popular in a wide range of applications, mainly due to the fact that the formula is fully analytical (the eigenvectors and eigenvalues are well known). For the three-dimensional case though, the formula obtained in equation (4.15) is only semi-analytical, as the eigenvalues and eigenvectors of the angular problem (4.30) with 0 boundary conditions in the domain given in (4.2) are computed numerically as discussed in Section 4.2. So this section aims at showing how the proposed method compares in terms of efficiency with a brute force approach on the full three dimensional problem.

#### 4.3.1 Proposed method

In the method proposed in Section 4.1, the solution is obtained through an eigenfunction expansion. After the separation of variables, the part that is left to solve numerically is a two-dimensional eigenvalue problem, that using a finite element approach requires solving the generalised eigensystem given in equation (4.32). The steps performed to obtain all the eigenvalues and eigenvectors for this problem, along with the algorithm complexities of each step are discussed below.

Denote by  $n$  the size of the symmetric square matrices  $K$  and  $M$  (stiffness and mass matrices respectively). This is given by the number of free points in the mesh constructed for the domain.

- Perform the Cholesky decomposition of matrix  $M$ :  $M = \mathcal{M}\mathcal{M}^T$ . Operation count for this step:  $\frac{1}{6}n^3$  (see Press et al. [1992] page 96).
- Invert matrix  $\mathcal{M}$ :  $O(n^{2.8})$  (using the Strassen algorithm, see Press et al. [1992] page 102).

- Obtain the standard eigensystem in equation (4.33) by computing matrix  $C = \mathcal{M}^{-1}K(\mathcal{M}^{-1})^T$ ; two matrix multiplications:  $O(n^{2.8})$
- Tridiagonalisation of the symmetric matrix  $C$ :  $\frac{4}{3}n^3$  (see Press et al. [1992] page 469)
- Compute eigenvalues and eigenvectors of a tridiagonal matrix: using QR / QL algorithm the complexity is  $3n^3 + 30n^2$  (Press et al. [1992] page 480)
- To obtain the eigenvectors of the original problem (4.32), a final matrix multiplication needs to be performed,  $O(n^{2.8})$

The total complexity of the algorithm for computing eigenvalues and eigenvectors is  $\frac{9}{2}n^3 + O(n^{2.8})$ , where  $n$  is the number of free points in the mesh. Given the mesh is two-dimensional and considering  $k$  points in each direction, a rough approximation for  $n$  is  $n \sim k^2$ .

#### 4.3.2 Brute force 3D method

For comparison, this section looks at the complexity of obtaining a “brute force” solution directly for the problem (4.1). The FEM method in this case is applied to a three-dimensional domain, so a three-dimensional mesh needs first to be constructed on the infinite domain given in (4.4), the positive octant. Let  $N$  be the number of free points in this three-dimensional mesh. The problem reduces to solving the following linear system:

$$A\dot{\alpha} + B\alpha = 0,$$

where  $A$  and  $B$  are the mass and stiffness matrices associated with the FEM discretisation respectively,  $\alpha$  is the desired solution and  $\dot{\alpha}$  denotes its derivative with respect to time (see Dziuk and Elliott [2008] for how to obtain this formulation). An implicit scheme can be applied for the time dimension, which leads to the following system:

$$(A + B\Delta t)\alpha^{m+1} = A\alpha^m,$$

where  $t_m = m\Delta t$ . The following steps need to be performed in order to obtain the solution  $\alpha$ :

- Perform an LU decomposition of matrix  $A + B\Delta t$ : operations count for this step  $\frac{1}{3}N^3$
- For each time step (going from  $m$  to  $m + 1$ ):
  - Compute the right hand-side  $A\alpha^m$ , with complexity  $O(N^2)$
  - Solve a first system through forward-substitution, with complexity  $O(N^2)$
  - Solve a second system through backward-substitution to obtain  $\alpha^{m+1}$ , with complexity  $O(N^2)$

The total complexity using the brute force method is thus given by  $\frac{1}{3}N^3 + 3N_t N^2 + O(N^2)$ , where  $N_t$  is the number of time steps used for the time discretisation. Here  $N$  is the number of free points in the mesh used for the three-dimensional domain, which if  $k$  points are considered in each direction, is roughly  $N \sim k^3$ , so significantly larger than in the previous method.

For a numerical example, let  $k$  be 25, the number of discretisation points in each dimension for both solutions. For a roughly rectangular domain this leads to approximately  $k^2 - 4k$  free points in the two-dimensional mesh, so  $n \sim 500$ . An approximation of the total number of operations leads to  $\sim 4.5 \cdot 500^3 + 4 \cdot 500^{2.8}$ , so  $\sim 600,000,000$  operations. For the three-dimensional mesh, the number of free points is approximately  $k^3 - 6k^2$ , so  $N \sim 12000$ . Let the  $N_t$  be 25, which leads to a total approximate number of operations in the brute force method of  $\sim 0.33 \cdot 12000^3 + 3 \cdot 25 \cdot 12000^2$ , so  $\sim 586,800,000,000$ .

This back-of-the-envelope calculation shows that the method proposed in this chapter for computing Green's function for the operator  $\mathcal{O}_{3D}^*$  in the positive octant with  $\delta$ -type initial condition and 0 boundary conditions is significantly faster than the brute force method (requires fewer operations by a factor of  $\sim 1000$ ).

Another important factor to mention in terms of efficiency is that the numerical part of the calculation in the proposed method does not depend on the coordinates

of the source point, but only on the three pairwise correlations. So for a given set of pairwise correlations the eigenvalues and eigenvectors need to be computed only once, and then daily changes in the riskiness of the names considered, which lead to changes in the coordinates of the source point, are obtained for virtually no extra cost. In the brute force method this does not hold though, as the implicit finite difference scheme in the time dimension needs to be redone when the coordinates of the source are changed, and thus the cost in terms of operations would be  $3N_tN^2 + O(N^2)$ .

Finally, the proposed method can be compared to a finite difference implementation of the computation of the joint survival probability in three dimensions starting from the PDE formulation. Table 4.1 shows a comparison of the timings using the two different methods. For the same precision, the proposed method is significantly faster (more than 10 times faster). Moreover, for the same fixed cost of computing the eigenvalues/eigenvectors through the FEM method, the calculations for different payoffs can then be obtained directly (joint survival probabilities, CVA/DVA, shortfalls etc.). For the finite difference method the numerical part needs to be repeated for every different payoff considered.

| Time steps | Space steps | Maturity | FEM relative error | FD relative error | FEM time | FD time |
|------------|-------------|----------|--------------------|-------------------|----------|---------|
| 50         | 200         | 5Y       | 0.4%               | 1.8%              | 76s      | 196s    |
| 100        | 200         | 5Y       | 0.4%               | 0.4%              | 76s      | 480s    |
| 100        | 200         | 1Y       | 0.1%               | 2.1%              | 76s      | 467s    |
| 100        | 250         | 1Y       | 0.1%               | 1.4%              | 76s      | 895s    |

Table 4.1: Performance comparison for the calculation of joint survival probabilities using the proposed method compared to the finite difference brute force method. The processor used is an Intel Xeon E5-2620 2Ghz. For the computation with FEM of the eigenvalues/eigenvectors is included in the timings for every case, even though once is enough.

## Chapter 5

# Joint Survival Probabilities

This chapter contains details about the calculation of the joint survival probabilities of two or three correlated issuers in the first-passage time framework described in Section 2.2. The solutions obtained for the transition probability density functions in chapters 3 and 4 are used to obtain analytical (for the two issuer case in Section 5.1) or semi-analytical (for the three issuer case in Section 5.2) solutions for this problem.

For the two dimensional case, the analytical formulas for the joint survival probabilities are well known and have been studied extensively in the literature. See for example Iyengar [1985], Zhou [2001a], Haworth [2006], Lipton and Sepp [2009] or Metzler [2010] to mention just a few.

For the three dimensional case though, the semi-analytical formula obtained in Section 5.2 for the joint survival probability is new to the best of the author's knowledge. This formula is then validated in Section 5.3 by performing different sanity checks and comparing with known limit cases. An empirical study of the convergence is also carried out, focusing on both the number of points in the meshes used for the finite element method as well as the number of terms considered in the infinite sum used for the eigenfunction expansion (see equation (5.4)).

## 5.1 Two issuers

Consider two issuers  $X$  and  $Y$  and let  $x_t$  and  $y_t$  respectively be the processes for the stochastic factors associated with the firms' asset values of these issuers (as defined in Section 2.2, equation (2.4)). These processes are Brownian motions issued from the initial positive points  $\tilde{x}$  and  $\tilde{y}$  respectively. The Brownian motions are correlated as described in Section 3.2.

The issuers are said to default if the value of its firm's assets drops below a specified threshold, which as discussed previously is equivalent to the process  $x_t$  or  $y_t$  dropping below 0. Let  $\tau_X$  denote the default time of issuer  $X$ :

$$\tau^X = \inf\{u \geq t, x_u \leq 0\},$$

$\tau_Y$  the default time of issuer  $Y$ :

$$\tau^Y = \inf\{u \geq t, y_u \leq 0\},$$

and  $\tau$  the minimum of the two:  $\tau = \min\{\tau_X, \tau_Y\}$ . Then

$$Q(t, T, \tilde{x}, \tilde{y}) = \mathbb{E}[\mathbb{1}_{\tau > T} | x_t = \tilde{x}, y_t = \tilde{y}]$$

denotes the joint survival probability at time  $t$  of the two issuers, up to maturity  $T$ .

**Proposition 5.1.** *The joint survival probability of the two issuers considered is given by:*

$$Q(t, T, \tilde{r}, \tilde{\varphi}) = \frac{2\tilde{r}e^{-\frac{\tilde{r}^2}{4(T-t)}}}{\sqrt{2\pi(T-t)}} \sum_{k=0}^{\infty} \frac{\sin(\nu_{2k+1}\tilde{\varphi})}{2k+1} \left[ I_{\frac{\nu_{2k+1}-1}{2}}\left(\frac{\tilde{r}^2}{4(T-t)}\right) + I_{\frac{\nu_{2k+1}+1}{2}}\left(\frac{\tilde{r}^2}{4(T-t)}\right) \right]. \quad (5.1)$$

where  $\nu_k = \frac{k\pi}{\varpi}$ ,  $\varpi = \arccos(-\rho_{xy})$  and  $\tilde{r}$  and  $\tilde{\varphi}$  are given by:

$$\tilde{r} = \frac{\sqrt{\tilde{x}^2 - 2\rho_{xy}\tilde{x}\tilde{y} + \tilde{y}^2}}{\bar{\rho}_{xy}},$$

$$\tilde{\varphi} = \varpi - \arctan\left(\frac{\bar{\rho}_{xy}\tilde{x}}{\tilde{y} - \rho_{xy}\tilde{x}}\right).$$

*Proof.* The barrier at 0 is an absorbing barrier, as once the name has defaulted it cannot come back to life. Hence, the probability of both issuers surviving past the maturity  $T$  can also be written as  $Q(t, T, \tilde{x}, \tilde{y}) = \mathbb{E}[\mathbb{1}_{x_T > 0} \mathbb{1}_{y_T > 0} | x_t = \tilde{x}, y_t = \tilde{y}]$ . The problem of computing the joint survival probability in expectation form can be transformed to a PDE problem with terminal condition and Dirichlet boundary conditions through a Feynman-Kac type representation (see Theorem A.3 for example):

$$Q_t + \frac{1}{2}Q_{xx} + \frac{1}{2}Q_{yy} + \rho_{xy}Q_{xy} = 0,$$

$$Q(T, T, x, y) = 1,$$

$$Q(t, T, x, 0) = 0, \quad Q(t, T, 0, y) = 0.$$

The domain is the positive quadrant (see equation (3.4)). After applying the changes of variables (3.5) and (3.7) described in Section 3.2, the problem for the joint survival probability is transformed to:

$$Q_t + \frac{1}{2} \left( Q_{rr} + \frac{1}{r}Q_r + \frac{1}{r^2}Q_{\varphi\varphi} \right) = 0,$$

$$Q(T, T, r, \varphi) = 1,$$

$$Q(t, T, r, 0) = 0, \quad Q(t, T, r, \varpi) = 0,$$

and needs to be solved in a wedge (see equation (3.13)). The solution for this problem can be obtained using the transition probability density function for a two-dimensional process in a wedge, that solves the corresponding forward Kolmogorov equation, as discussed in Section 3.2. This yields for the joint survival probability (using for

example formula (7.26) page 369 in Karatzas and Shreve [1991]):

$$Q(t, T, \tilde{r}, \tilde{\varphi}) = \int_0^\infty \int_0^\varpi G(T - t, r', \varphi' | \tilde{r}, \tilde{\varphi}) r' d\varphi' dr',$$

where  $G(T - t, r', \varphi' | \tilde{r}, \tilde{\varphi})$  is given in equation (3.14).

$$Q(t, T, \tilde{r}, \tilde{\varphi}) = \int_0^\infty \int_0^\varpi \frac{2r' e^{-\frac{r'^2 + \tilde{r}^2}{2(T-t)}}}{\varpi(T-t)} \sum_{n=1}^\infty I_{\nu_n}\left(\frac{r'\tilde{r}}{T-t}\right) \sin(\nu_n \varphi') \sin(\nu_n \tilde{\varphi}) d\varphi' dr'$$

The sum and the integral in  $\varphi'$  can be interchanged and since  $\nu_n = \frac{n\pi}{\varpi}$ , the terms in the sum for even values of  $n$  are null, which yields the following expression for the joint survival probability:

$$Q(t, T, \tilde{r}, \tilde{\varphi}) = \sum_{k=0}^\infty \frac{4}{(2k+1)\pi(T-t)} \sin(\nu_{2k+1}\tilde{\varphi}) \int_0^\infty r' e^{-\frac{r'^2 + \tilde{r}^2}{2(T-t)}} I_{\nu_{2k+1}}\left(\frac{\tilde{r}r'}{T-t}\right) dr' \quad (5.2)$$

The integration by parts method can be then applied to the integral in  $r'$  which is denoted by  $J$ :

$$\begin{aligned} J &= \int_0^\infty r' e^{-\frac{r'^2 + \tilde{r}^2}{2(T-t)}} I_{\nu_{2k+1}}\left(\frac{\tilde{r}r'}{T-t}\right) dr' \\ &= -(T-t) e^{-\frac{\tilde{r}^2}{2(T-t)}} \int_0^\infty \left( e^{-\frac{r'^2}{2(T-t)}} \right)' I_{\nu_{2k+1}}\left(\frac{\tilde{r}r'}{T-t}\right) dr' \\ &= -(T-t) e^{-\frac{\tilde{r}^2}{2(T-t)}} \left[ e^{-\frac{r'^2}{2(T-t)}} I_{\nu_{2k+1}}\left(\frac{\tilde{r}r'}{T-t}\right) \Big|_0^\infty - \frac{\tilde{r}}{T-t} \int_0^\infty e^{-\frac{r'^2}{2(T-t)}} I'_{\nu_{2k+1}}\left(\frac{\tilde{r}r'}{T-t}\right) dr' \right] \end{aligned}$$

Since  $I_\Lambda(\xi) \xrightarrow[\xi \rightarrow 0]{} 0$  when  $\Lambda > 0$ , the limit at  $r' \rightarrow 0$  of the first term is 0. For the limit when  $r' \rightarrow \infty$ , the asymptotic expansion of the modified Bessel function for large arguments given in Abramowitz and Stegun [1964], formula (9.7.1) is used, and the limit is also 0. Hence the expression for integral  $J$  becomes:

$$J = \tilde{r} e^{-\frac{\tilde{r}^2}{2(T-t)}} \int_0^\infty e^{-\frac{r'^2}{2(T-t)}} I'_{\nu_{2k+1}}\left(\frac{\tilde{r}r'}{T-t}\right) dr'$$



The next step is to use formula (9.6.26) in Abramowitz and Stegun [1964]:

$$I'_\nu(z) = \frac{1}{2} [I_{\nu-1}(z) + I_{\nu+1}(z)],$$

to replace the derivative of the modified Bessel function, followed by formula (11.4.31) in Abramowitz and Stegun [1964]:

$$\int_0^\infty e^{-\xi r^2} I_\nu(\eta r) dr = \frac{1}{2} \sqrt{\frac{\pi}{\xi}} e^{\frac{\eta^2}{8\xi}} I_{\nu/2} \left( \frac{\eta^2}{8\xi} \right),$$

for the integrals involving modified Bessel function. This yields the following result for integral  $J$ :

$$J = \frac{\sqrt{2\pi(T-t)}}{4} \tilde{r} e^{-\frac{\tilde{r}^2}{4(T-t)}} \left[ I_{\frac{\nu_{2k+1}-1}{2}} \left( \frac{\tilde{r}^2}{4(T-t)} \right) + I_{\frac{\nu_{2k+1}+1}{2}} \left( \frac{\tilde{r}^2}{4(T-t)} \right) \right].$$

Replacing this into equation (5.2) yields the final formula (5.1).  $\square$

**Remark 5.1.** *The joint survival probability of the two issuers considered can be written in the alternative form:*

$$Q(t, T, \tilde{r}, \tilde{\varphi}) = \sum_{k=0}^{\infty} \frac{4 \sin(\nu_{2k+1} \tilde{\varphi})}{(2k+1) \pi} \left( \frac{\tilde{r}^2}{2(T-t)} \right)^{\frac{\nu_{2k+1}}{2}} \frac{\Gamma(1 + \frac{\nu_{2k+1}}{2})}{\Gamma(1 + \nu_{2k+1})} {}_1F_1 \left( \frac{\nu_{2k+1}}{2}, 1 + \nu_{2k+1}, -\frac{\tilde{r}^2}{2(T-t)} \right), \quad (5.3)$$

where  ${}_1F_1$  denotes the confluent hypergeometric function and  $\Gamma$  denotes the gamma function. While this expression is more complex than the one in equation (5.1), it allows for a generalization to the three dimensional case, which is discussed in Section 5.2.

Figure 5.1 shows the joint survival probability for two issuers for a range of starting point values and two sample correlations.

As the initial values are closer to the boundaries, the joint survival probabilities are lower. The impact of the correlation between the two issuers is most noticeable when the names have roughly the same riskiness (the diagonal of the graphs). For high correlation, the joint survival probability becomes non-negligible even when both

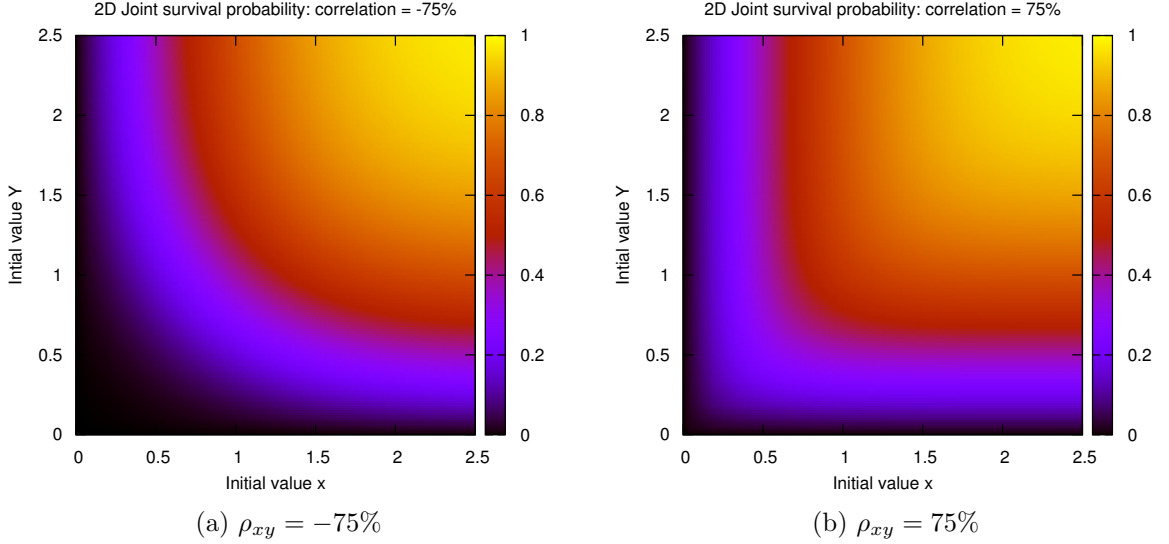


Figure 5.1: Joint survival probability for sample correlations and starting point values (time horizon  $T = 1$  year).

issuers are fairly risky (starting points close to 0), while for high anti-correlation, the joint survival probability has a slower increase with the distance from the default boundary (intuitively, if both names are equally risky and one does not default, the second is likely to default as they are anti-correlated and hence the joint survival probability is lower close to the boundaries).

## 5.2 Three issuers

Consider three issuers  $X$ ,  $Y$  and  $Z$  and let  $x_t$ ,  $y_t$  and  $z_t$  respectively be the processes for the stochastic factors associated to the firms' asset values of these issuers (as defined in Section 2.2, equation (2.4)). These processes are Brownian motions issued from the initial positive points  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  respectively. The Brownian motions are correlated, as described in Section 4.1, with the correlations satisfying the same conditions in order to ensure a positive definite correlation matrix.

The issuers are said to default if the corresponding value of the firms' assets drops below a specified threshold, which, as discussed previously, is equivalent to the process

$x_t$ ,  $y_t$  or  $z_t$  dropping below 0. Let  $\tau_X$  denote the default time of issuer  $X$ :

$$\tau^X = \inf\{u \geq t, x_u \leq 0\},$$

$\tau_Y$  the default time of issuer  $Y$ :

$$\tau^Y = \inf\{u \geq t, y_u \leq 0\},$$

$\tau_Z$  for issuer  $Z$ :

$$\tau^Z = \inf\{u \geq t, z_u \leq 0\},$$

and  $\tau$  the minimum of the three:  $\tau = \min\{\tau_X, \tau_Y, \tau_Z\}$ .  $\tau$  is thus the first exit time of the three-dimensional process from the positive octant. Then  $Q(t, T, \tilde{x}, \tilde{y}, \tilde{z}) = \mathbb{E}[\mathbb{1}_{\tau > T} | x_t = \tilde{x}, y_t = \tilde{y}, z_t = \tilde{z}]$  denotes the joint survival probability at time  $t$  of the three issuers, up to maturity  $T$ .

The next proposition gives a semi-analytical formula for computing the joint survival probability of three correlated issuers in the setup described above, that generalizes the expression obtained in the two dimensional case in equation (5.3).

**Proposition 5.2.** *The joint survival probability of the three issuers considered is given by:*

$$\begin{aligned} Q(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) &= \sum_{n=1}^{\infty} \left( \frac{\tilde{r}^2}{2(T-t)} \right)^{\frac{\nu_n}{2} - \frac{1}{4}} \frac{\Gamma\left(\frac{\nu_n}{2} + \frac{5}{4}\right)}{\Gamma(\nu_n + 1)} {}_1F_1\left(\frac{2\nu_n - 1}{4}, \nu_n + 1, -\frac{\tilde{r}^2}{2(T-t)}\right) \\ &\quad \times \Psi_n(\tilde{\varphi}, \tilde{\theta}) \iint_{\Omega} \Psi_n(\varphi', \theta') \sin \theta' d\varphi' d\theta', \end{aligned} \quad (5.4)$$

where  $\tilde{r}$ ,  $\tilde{\varphi}$  and  $\tilde{\theta}$  are obtained from the initial positions of the three processes  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  through the change of variables given in (4.5) and (4.7),  $\nu_n = \sqrt{\Lambda_n + \frac{1}{4}}$  and  $\Lambda_n$  and  $\Psi_n$  are the eigenvalues and eigenfunctions respectively of the spectral problem (4.30).

*Proof.* The barrier at 0 is an absorbing barrier, as once the name has defaulted it cannot come back to life. Hence, the probability of all three issuers surviving past

the maturity  $T$  can also be written as

$$Q(t, T, \tilde{x}, \tilde{y}, \tilde{z}) = \mathbb{E} [\mathbb{1}_{x_T > 0} \mathbb{1}_{y_T > 0} \mathbb{1}_{z_T > 0} | x_t = \tilde{x}, y_t = \tilde{y}, z_t = \tilde{z}].$$

The problem of computing the joint survival probability in expectation form can be transformed to a PDE problem with terminal condition and Dirichlet boundary conditions through a Feynman-Kac type representation (see Theorem A.3 for example):

$$Q_t + \frac{1}{2}Q_{xx} + \frac{1}{2}Q_{yy} + \frac{1}{2}Q_{zz} + \rho_{xy}Q_{xy} + \rho_{xz}Q_{xz} + \rho_{yz}Q_{yz} = 0,$$

$$Q(T, T, x, y, z) = 1,$$

$$Q(t, T, 0, y, z) = 0, \quad Q(t, T, x, 0, z) = 0, \quad Q(t, T, x, y, 0) = 0.$$

The domain for this problem is the positive octant (see equation (4.4)). Following a similar procedure as in Section 4.1, successive changes of variables (4.5) and (4.7) are applied to obtain the transformed problem:

$$Q_t + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (rQ) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} Q_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta Q_\theta) \right) \right] = 0,$$

$$Q(T, T, r, \varphi, \theta) = 1,$$

$$Q(t, T, r, 0, \theta) = Q(t, T, r, \varpi, \theta) = Q(t, T, r, \varphi, 0) = Q(t, T, r, \varphi, \Theta(\varphi)) = 0.$$

The domain for this transformed problem is given in equation (4.14). The solution for this problem can be obtained, similar to the two dimensional case, using the transition probability density function for a three-dimensional process, that solves the corresponding forward Kolmogorov equation as discussed in Section 4.1. This yields for the joint survival probability:

$$Q(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} G\left(T - t, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right) r'^2 \sin \theta' d\theta' d\varphi' dr'$$

where  $G\left(T - t, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta}\right)$  is given in (4.15). By switching the sum and the

integrals this leads to the following expression:

$$Q(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = \frac{e^{-\frac{\tilde{r}^2}{2(T-t)}}}{(T-t)\sqrt{\tilde{r}}} \sum_{n=1}^{\infty} \Psi_n(\tilde{\varphi}, \tilde{\theta}) \left[ \iint_{\Omega} \Psi_n(\varphi', \theta') \sin \theta' d\varphi' d\theta' \right] \int_0^{\infty} r'^{\frac{3}{2}} e^{-\frac{r'^2}{2(T-t)}} I_{\nu_n} \left( \frac{\tilde{r}r'}{T-t} \right) dr',$$

The modified Bessel function  $I_{\nu}(\cdot)$  can be rewritten in terms of the Bessel function of the first kind  $J_{\nu}$  using the following relationship:  $I_{\nu}(x) = i^{-\nu} J_{\nu}(ix)$ :

$$Q(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = \frac{e^{-\frac{\tilde{r}^2}{2(T-t)}}}{(T-t)\sqrt{\tilde{r}}} \sum_{n=1}^{\infty} \Psi_n(\tilde{\varphi}, \tilde{\theta}) \left[ \iint_{\Omega} \Psi_n(\varphi', \theta') \sin \theta' d\varphi' d\theta' \right] \int_0^{\infty} r'^{\frac{3}{2}} e^{-\frac{r'^2}{2(T-t)}} i^{-\nu_n} J_{\nu_n} \left( i \frac{\tilde{r}r'}{T-t} \right) dr'.$$

To compute the integral in  $r'$ , formula (11.4.28) in Abramowitz and Stegun [1964] can be used:

$$\int_0^{\infty} e^{-a^2 x^2} x^{\mu-1} J_{\nu}(bx) dx = \frac{\Gamma\left(\frac{\mu+\nu}{2}\right)}{\Gamma(\nu+1)} \frac{\left(\frac{b}{2a}\right)^{\nu}}{2a^{\mu}} {}_1F_1\left(\frac{\mu+\nu}{2}, \nu+1, -\frac{b^2}{4a^2}\right),$$

for  $Re(\nu+\mu) > 0$  and  $Re(a^2) > 0$  and where  ${}_1F_1$  denotes the confluent hypergeometric function.

$$Q(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = e^{-\frac{\tilde{r}^2}{2(T-t)}} \sum_{n=1}^{\infty} \left( \frac{\tilde{r}^2}{2(T-t)} \right)^{\frac{\nu_n}{2} - \frac{1}{4}} \frac{\Gamma\left(\frac{\nu_n}{2} + \frac{5}{4}\right)}{\Gamma(\nu_n+1)} {}_1F_1\left(\frac{2\nu_n+5}{4}, \nu_n+1, \frac{\tilde{r}^2}{2(T-t)}\right) \\ \times \Psi_n(\tilde{\varphi}, \tilde{\theta}) \iint_{\Omega} \Psi_n(\varphi', \theta') \sin \theta' d\varphi' d\theta',$$

Finally, the formula in (5.4) is obtained by using (13.1.27) in Abramowitz and Stegun [1964]:

$${}_1F_1(a, b, z) = e^z {}_1F_1(b-a, b, -z).$$

□

### 5.3 Limit cases and sanity checks

This section discusses the calculation of the joint survival probability of three issuers, as described in Section 5.2. Unlike the calculation of individual survival probabilities

or joint probabilities for two issuers, for which fully analytic solutions are available, the three dimensional case has a semi-analytic solution. In formula (5.4), the eigenvalues and eigenvectors are obtained through the finite element method applied to solve the spectral angular problem (4.18).

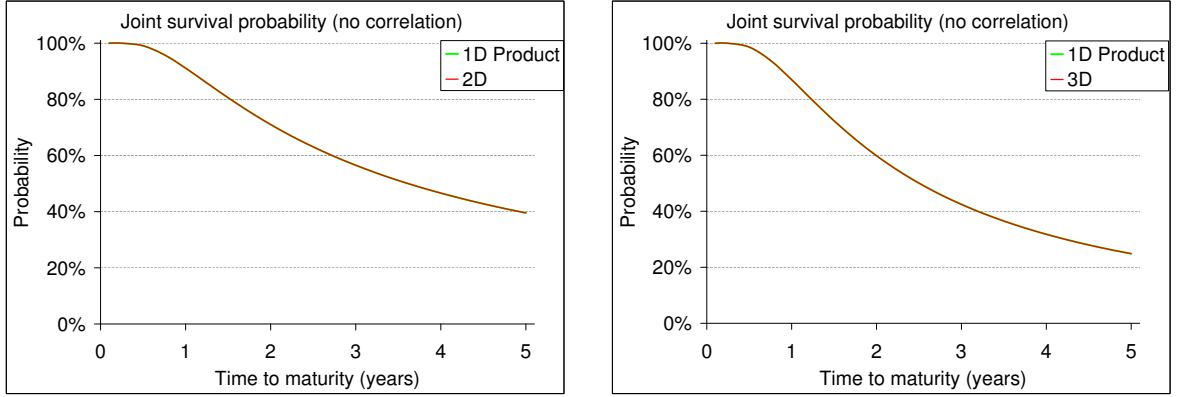
To validate the formula obtained in Proposition 5.2, several sanity checks and limit cases are discussed here: the simple case where the issuers are independent (Section 5.3.1), when one issuer is independent from the other two (Section 5.3.2), when one issuer tends towards being risk-free (Section 5.3.3) or when the pairwise correlations are permuted (Section 5.3.4).

For all examples discussed below all three issuers considered are identical unless otherwise stated, with equal starting values  $\tilde{x} = \tilde{y} = \tilde{z}$ , implying a survival probability of 62.9% at five years for each of the issuers taken independently.

### 5.3.1 Independent issuers

The most straightforward case to check is the one where the issuers are independent, so all pairwise correlations are zero. In this simplified case, the joint survival probabilities obtained using formula (5.1) for two issuers and (5.4) for three issuers need to simply match the product of the individual survival probabilities. Figure 5.2 shows this comparison for a sample set of input parameters. In both cases the agreement is very good (relative differences are below 0.2% for all maturities).

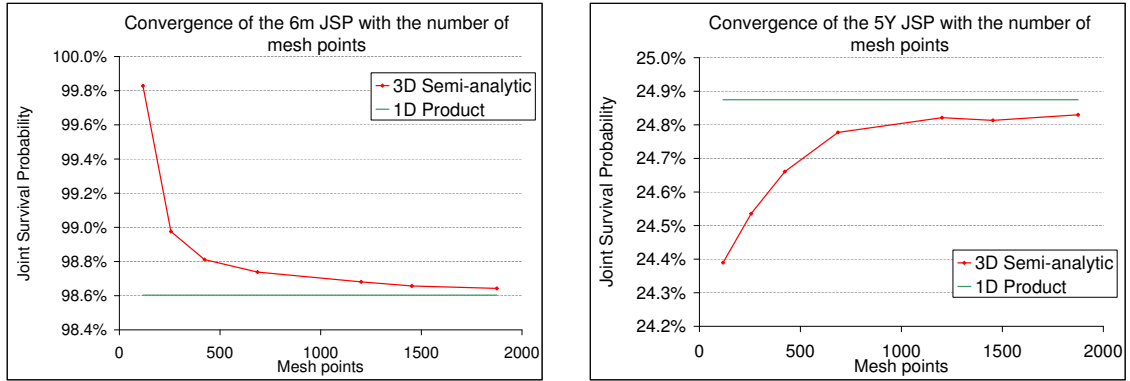
For this simple example, where the joint survival probabilities are known, an empirical study of the convergence of the calculation in the three dimensional case can be performed. Since the calculation uses the finite element method, it is natural to consider the convergence with respect to the number of points in the mesh. Figure 5.3 shows this convergence for two sample maturities. For this analysis, the number of terms considered in the infinite sum is capped at 1000. For the cases where there are less than 1000 points in the mesh, the sum has the same number of terms as the number of eigenvalues/eigenfunctions that are computed. When there are more than 1000 points in the grid, even though the number of terms in the sum remains constant, the accuracy still improves as the precision on all eigenvalues and eigenvectors is



(a) JSP for two issuers using formula (5.1) compared to the product of the one dimensional survival probabilities

(b) JSP for three issuers using formula (5.4) compared to the product of the one dimensional survival probabilities

Figure 5.2: Sanity check for the joint survival probability of multiple issuers. All three issuers are identical and considered to be independent.



(a) JSP for two issuers using formula (5.1) compared to the product of the one dimensional survival probabilities

(b) JSP for two issuers using formula (5.4) compared to the product of the one dimensional survival probabilities

Figure 5.3: Convergence of the joint survival probability for three issuers as a function of the number of points used in the mesh. All three issuers have the same input parameters and are considered to be independent.

increased.

Figure 5.4 shows the convergence of the relative error of the joint survival probability of three issuers with the number of points in the mesh, for several maturities. For each maturity, the relative error is computed with respect to the product of the individual survival probabilities. As the number of points in the mesh increases, the

relative error decreases.

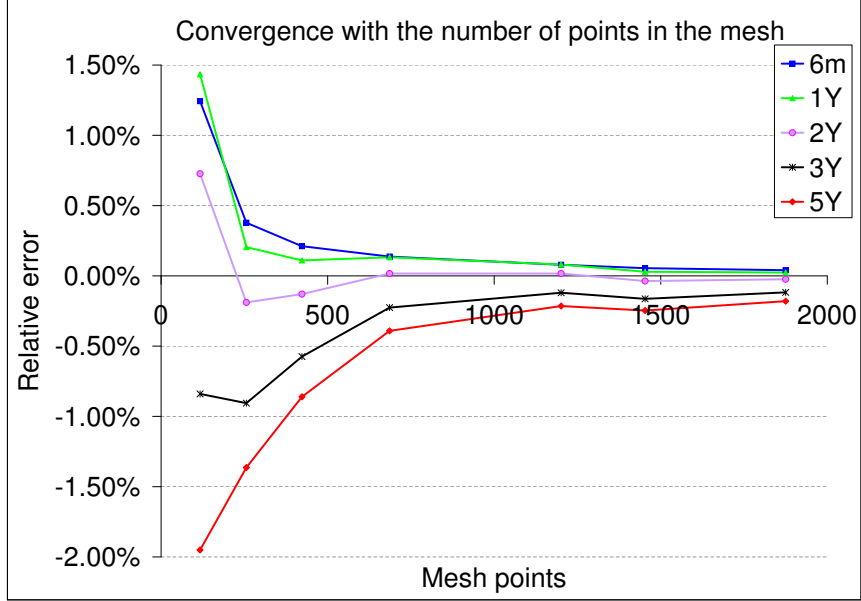


Figure 5.4: Convergence of the relative error of the joint survival probability for three issuers as a function of the number of points used in the mesh. All three issuers have the same input parameters and are considered to be independent. The relative error is defined as  $(JSP_{3D} - SP_1 \cdot SP_2 \cdot SP_3) / (SP_1 \cdot SP_2 \cdot SP_3)$ , where  $SP_i$  is the survival probability of issuer  $i$  and  $JSP_{3D}$  is the joint survival probability using the three dimensional formulation.

When the issuers are independent, the domain used for the computation reduces to a square. The meshes used for the study are shown in Figure 5.5. These are successively finer uniform meshes, with the number of points increasing each time.

The accuracy of finite element methods improves not only with the number of points in the mesh used, but also with the order of the basis functions used in the approximation. The current implementation uses linear basis functions only.

Another source of potential inaccuracies in the calculation is the truncation of the infinite sum. The number of terms considered in this sum impacts both the performance of the calculation and the precision, and hence an analysis of the appropriate cut-off point is relevant.

The analysis is performed using a mesh with approximately 1900 points for the finite element method, so the maximum number of eigenvalues available for the sum



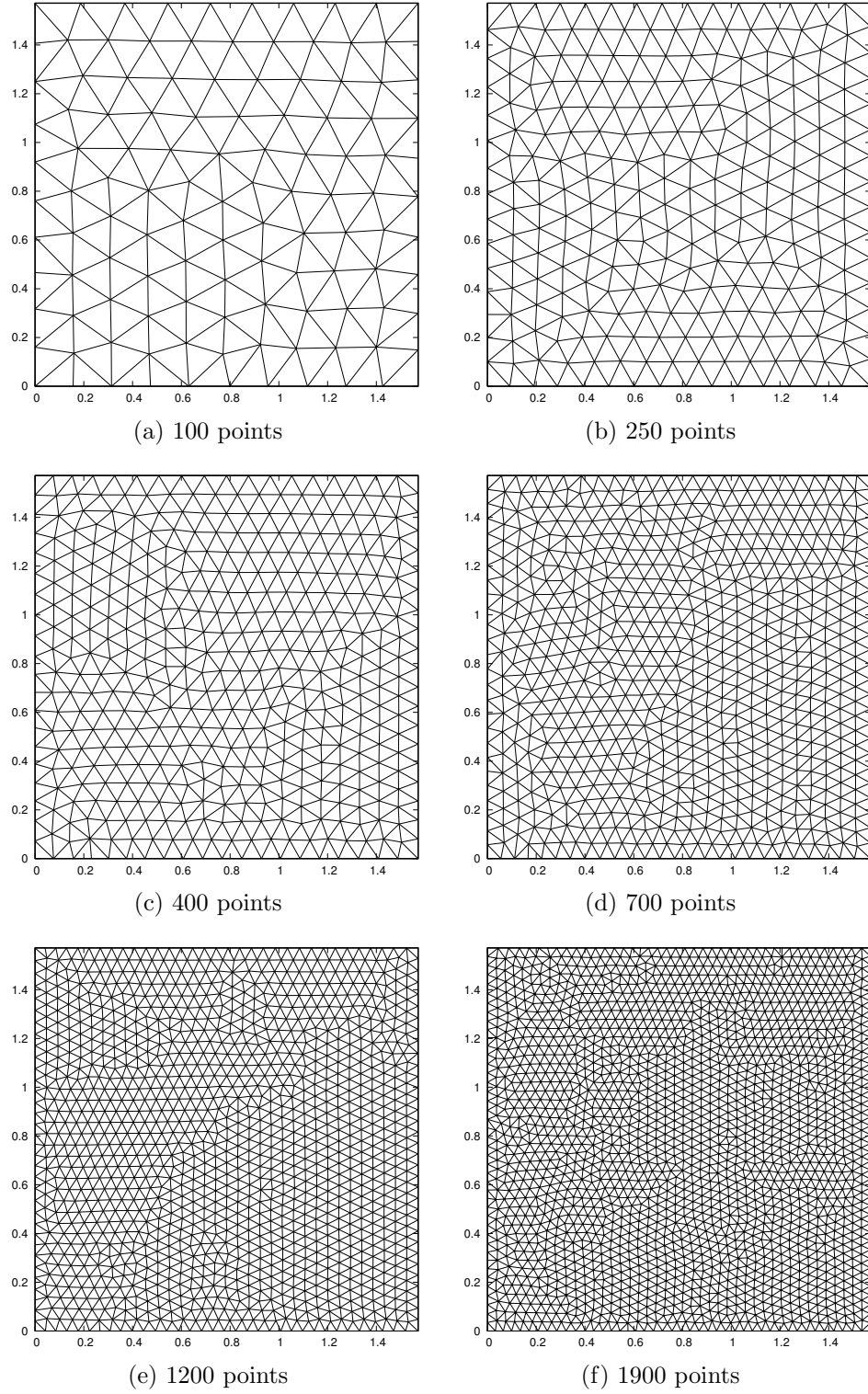


Figure 5.5: Successively finer uniform meshes used for the finite element method in the three dimensional case ( $\rho_{xy} = 0\%$ ,  $\rho_{xz} = 0\%$ ,  $\rho_{yz} = 0\%$ ).

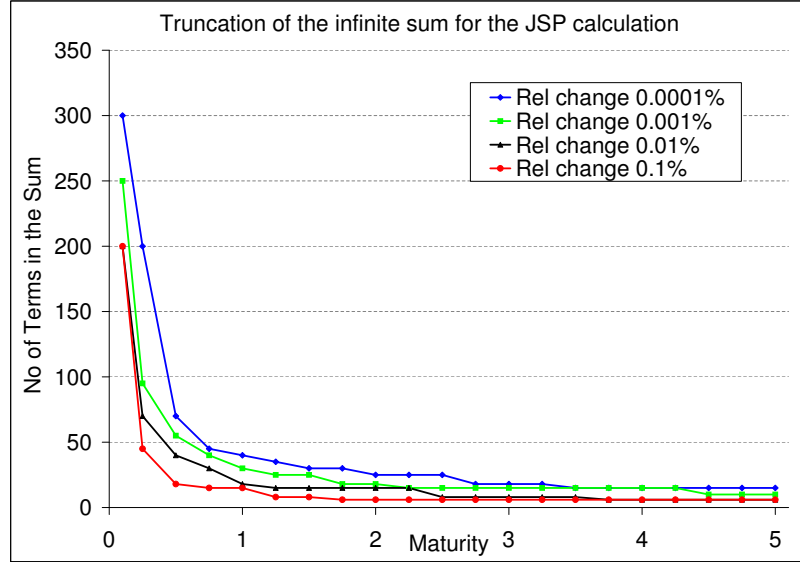
is high enough, as to not limit the analysis. The joint survival probabilities are then computed for several maturities, taking into account additional terms in the sum at each step. As the number of terms in the sum increases, the relative change with respect to the previously computed survival probability is calculated. For a given relative change tolerance, for example 0.1%, the maximum number of terms in the sum after which the relative change stays below the limit is backed out.

Figure 5.6 shows the results for several maturities. For relatively long maturities, of more than three years, the number of terms needed in the sum is fairly low (at most 20), even for very low tolerances. This means that the calculation of the survival probability is fast once the eigenvalues and eigenfunctions are computed, as the sum converges very quickly. However, for shorter maturities, especially for those less than one year, the number of terms needed in the sum increases rapidly, making the computation more time consuming.

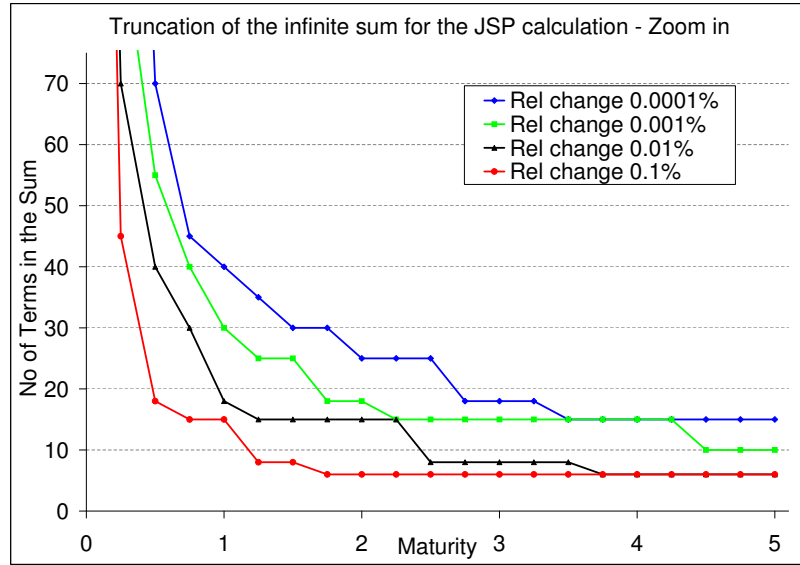
### 5.3.2 One issuer is independent from the other two

For this example one of the pairwise correlations is set to a non-zero value:  $\rho_{xy} = 50\%$ . When issuer  $Z$  is independent from the other two, the corresponding pairwise correlations  $\rho_{xz}$  and  $\rho_{yz}$  are zero, which means that the joint survival probability of the three issuers can simply be computed by multiplying the survival probability of issuer  $Z$  to the joint survival probability of issuers  $X$  and  $Y$  obtained using formula (5.1).

Figure 5.7 shows that for all maturities, when the common value for  $\rho_{xz}$  and  $\rho_{yz}$  decreases from 80% towards 0%, the joint survival probability computed using formula (5.4) tends towards the known value for  $\rho_{xz} = \rho_{yz} = 0\%$ . Figure 5.8 focuses on the five year maturity joint survival probability and plots this as a function of the common correlation value for  $\rho_{xz}$  and  $\rho_{yz}$ . The line obtained through the product of the one and two dimensional probabilities is crossed exactly at 0% correlation, which is further validation of Proposition 5.2.



(a) Number of terms needed in the sum before all subsequent relative changes remain below a given tolerance



(b) Zoom-in (useful for the number of terms needed for longer maturities)

Figure 5.6: Analysis of the truncation of the infinite sum for the joint survival probability calculation. All three issuers have the same input parameters and are considered to be independent. The relative change is defined as  $(JSP_{3D}^N - JSP_{3D}^{N-1}) / (SP_1 \cdot SP_2 \cdot SP_3)$ , where  $SP_i$  is the survival probability of issuer  $i$  and  $JSP_{3D}^N$  is the joint survival probability using the three dimensional formulation and  $N$  terms in the infinite sum.

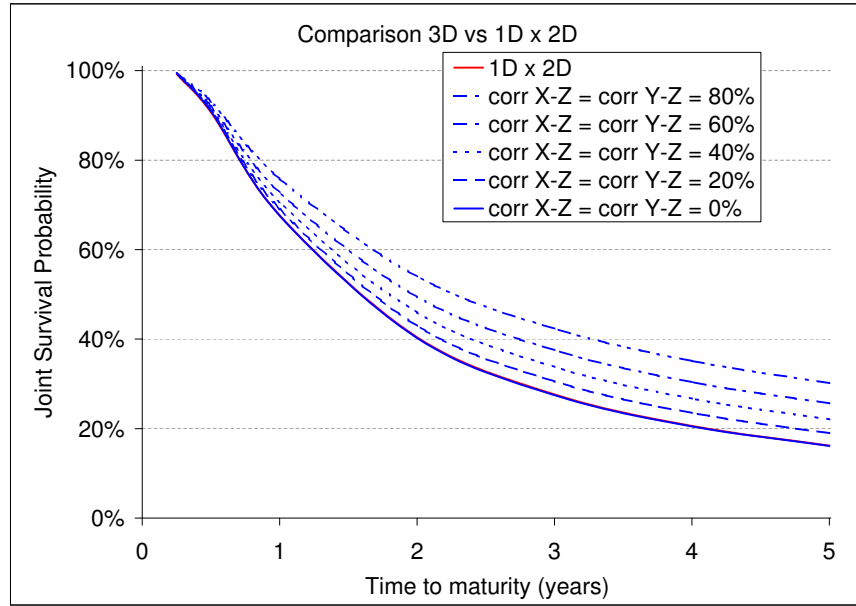


Figure 5.7: Convergence of the joint survival probability curve computed with the 3D formulation to the product of the single name survival probability (1D) and that of two issuers computed with the 2D formulation, as the third issuer becomes uncorrelated to the first two.

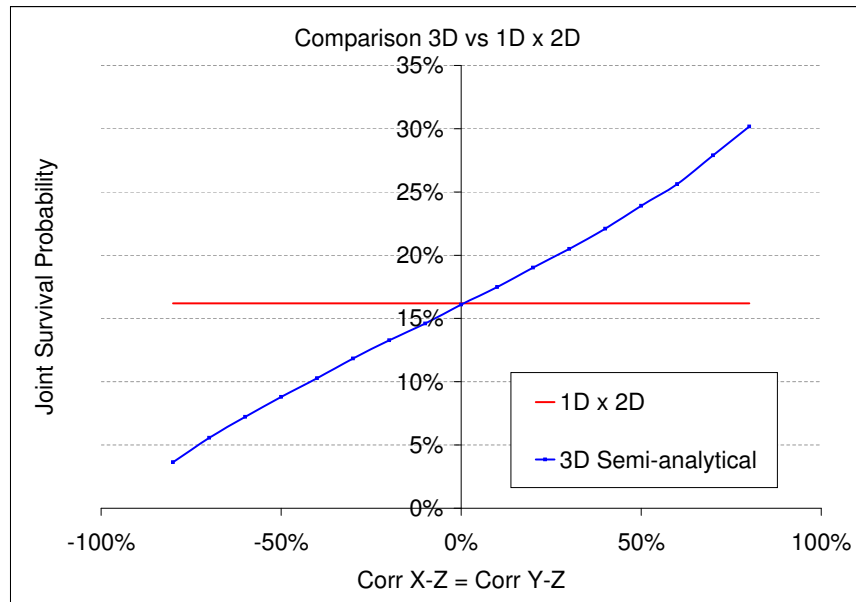


Figure 5.8: Joint survival probability at 5Y maturity for three issuers as a function of the common correlation value  $\rho_{xz} = \rho_{yz}$  (the third pairwise correlation is non-zero:  $\rho_{xy} = 50\%$ ).

### 5.3.3 One virtually risk free issuer

The next step is to consider the case where all pairwise correlations are non-zero. The study of a limit case is first conducted in order to validate the implementation. As the starting point value for one of the issuers is increased, the probability of default for this issuer decreases as it is further and further away from the barrier. As the default probability tends to zero, the joint survival probability for the three issuers should tend to the joint survival probability of the two remaining risky issuers, which can be computed using the two dimensional formulation.

For this example the pairwise correlations are  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 50\%$  and  $\rho_{yz} = 30\%$ . The grid for this triplet of correlations is built and the eigenvalues and eigenvectors are computed using the finite element method. Once this is done, the joint survival probability of the three issuers can be efficiently computed for different starting points for issuer Z. Figures 5.9 and 5.10 show that as issuer Z becomes virtually riskfree, the 3D joint survival probability tends to the 2D one, as expected.

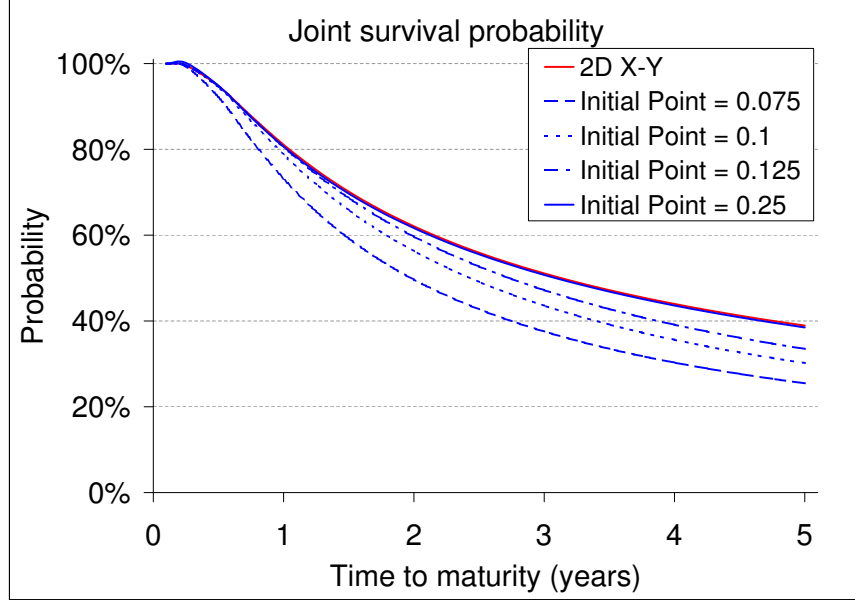


Figure 5.9: Convergence of the joint survival probability curve computed with the 3D formulation to that of two issuers computed with the 2D formulation, as the third issuer becomes less risky.

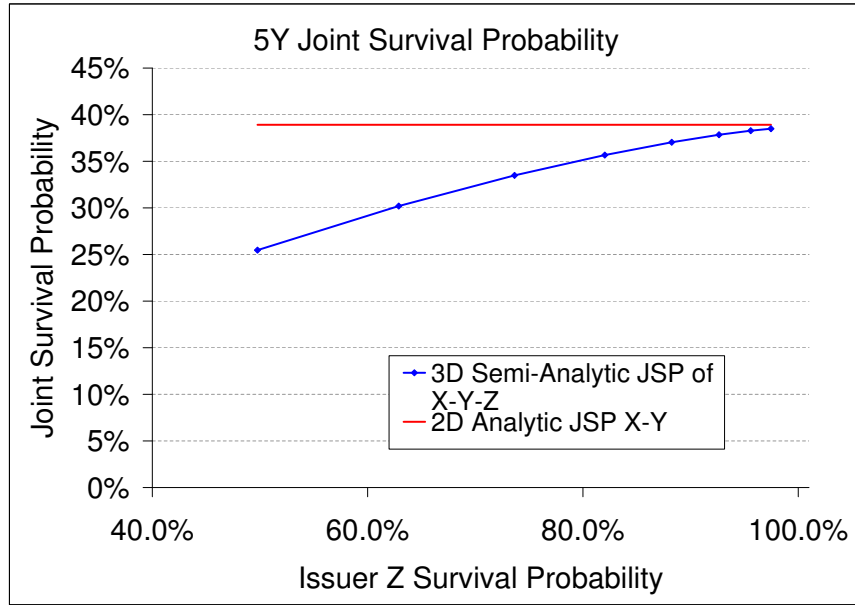


Figure 5.10: Convergence of the 5Y joint survival probability computed with the 3D formulation to the joint survival probability of two issuers computed with the 2D formulation, as the third issuer becomes less risky.

#### 5.3.4 Identical issuers, same correlations triplet

In this example three identical issuers are considered, and all three pairwise correlations are non-zero. As the order of the correlations changes (there are six possible permutations given three distinct correlation values), the shape of the domain on which the finite element method is applied becomes significantly different. Even so, since the three issuers are identical, the joint survival probability should be the same in all cases and hence this is a good test for the robustness of the calculation.

The correlation values used for the numerical example below are 70%,  $-25\%$  and  $-50\%$ . The meshes in Figure 5.12 vary significantly for the different permutations, as  $\varpi$ ,  $\Theta(0)$  and  $\Theta(\varpi)$  as well as the convexity of the curved boundary change from one case to the other.

The joint survival probability curves for the six different cases are shown in Figure 5.11. The agreement between the curves is very good, which shows that the calculation is robust.

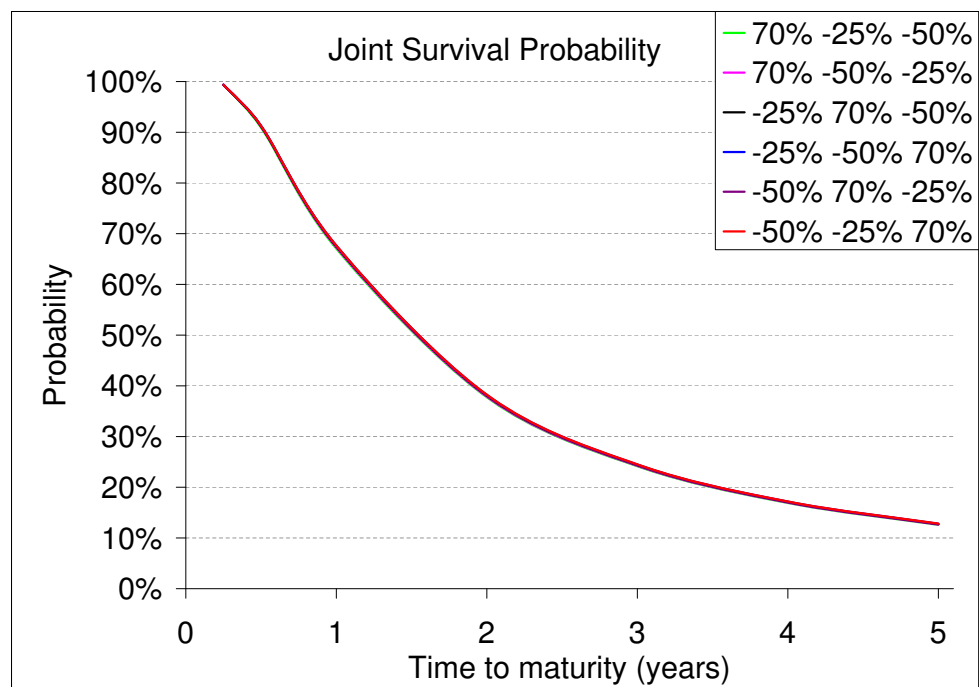


Figure 5.11: Joint survival probability curves for the six different permutations of the pairwise correlations 70%, -25% and -50% (the three issuers are considered identical).

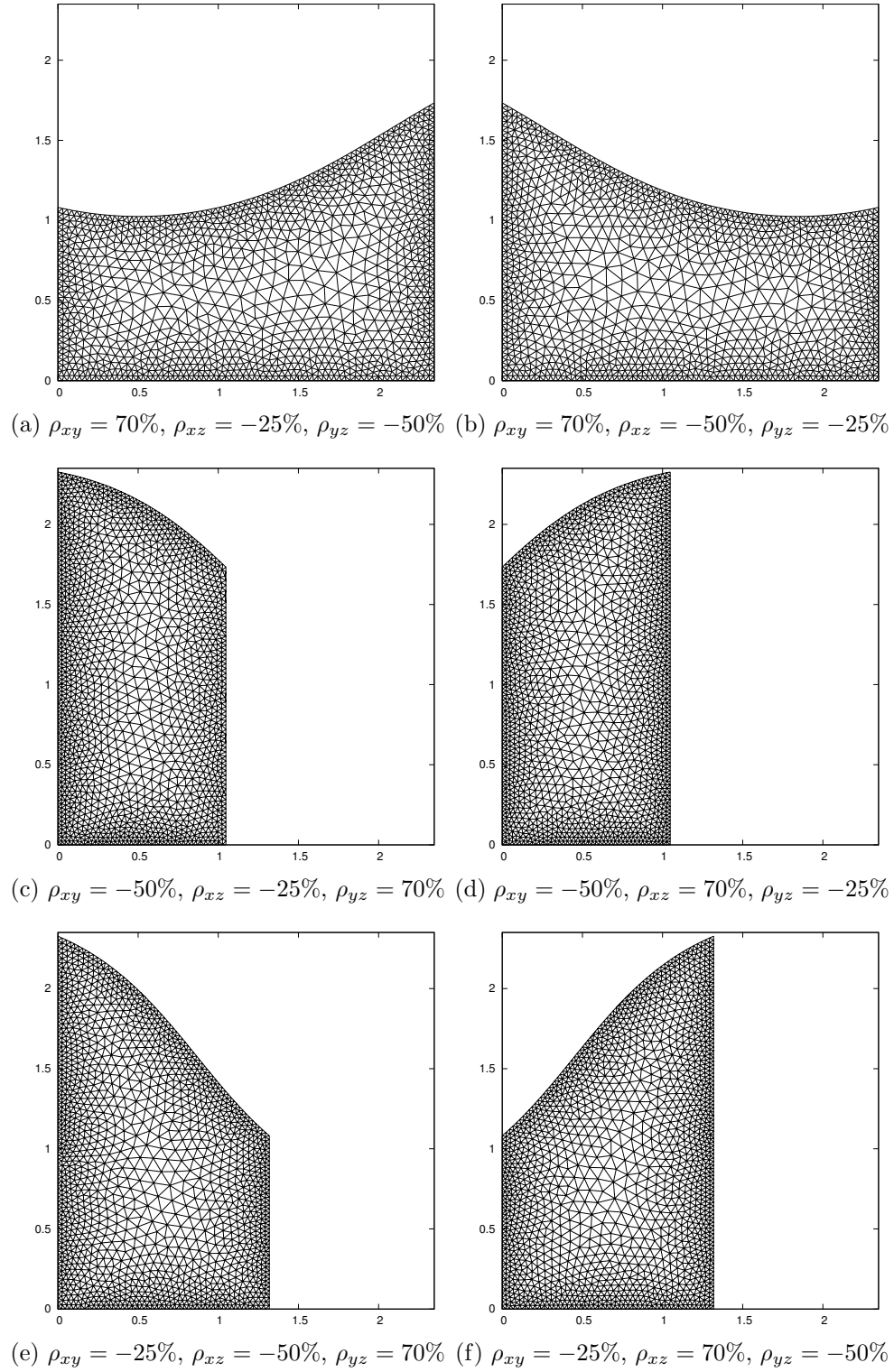


Figure 5.12: Meshes obtained for the six different permutations of the pairwise correlations 70%, -25% and -50%.



## Chapter 6

# CVA and DVA Calculations for a CDS

The present chapter contains details about the calculation of the credit and debit value adjustments for a single name credit default swaps in the first-passage time framework described in Section 2.2.

Section 6.1 discusses the various models proposed in the literature to evaluate counterparty risk adjustments, particularly in the context of credit derivatives, and places the contribution of the applications discussed thereafter in this thesis in the current context. The mathematical results obtained in chapters 3 and 4 are then used to obtain analytical solutions for counterparty adjustments in Section 6.3. The formulas for unilateral adjustments for counterparty risk of a CDS have already been discussed in the literature (see Lipton and Sepp [2009]) and are presented in Section 6.3.1. The semi-analytical formulas for the computation of bilateral CVA/DVA are new and discussed in detail in Section 6.3.2.

Several practitioners have considered approximating bilateral CVA/DVA corrections by the sum of the unilateral CVA/DVA. Some studies have been performed in the literature to investigate how good this approximation is by looking at specific underlying products. For CDSs though, this analysis has not been undertaken before, and this is investigated in Section 6.4.

Finally, Section 6.5 looks at the applications of the proposed technique to real

market cases, with some realistic examples of pricing CDSs sold by risky sellers to risky buyers.

## 6.1 Literature review

This section presents the different models that have been proposed in the literature for pricing counterparty risk, with a particular focus on those that consider counterparty adjustments for credit default swaps valuation. Similarly to the models used for valuing standard single name CDSs with risk free counterparties, the models proposed for the valuation of CDSs with risky counterparties fall into two main categories: reduced-form models (discussed in Section 6.1.1) and structural or first passage time models (discussed in 6.1.2).

### 6.1.1 Reduced-form model approach

The existing reduced-form models that have been considered for CDS counterparty risk valuation can be divided into two broad groups: models with interacting intensities and copula models.

In the class of models with interacting intensities (also referred to as contagion models), the impact of defaults on the default intensities of surviving firms is exogenously specified; the joint distribution of default times is then endogenously derived.

Contagion models were first introduced in Davis and Lo [2001]. The authors introduce a contagion model to account for concentration risk in large portfolios, providing an alternative to the Moody's diversity score methodology of accounting for default clustering. The ideas are then extended in Jarrow and Yu [2002]. Here the authors introduce the concept of "primary" and "secondary" firms: the firms for which the default probabilities are influenced only by macroeconomic conditions are "primary" firms, while the firms that can have their default probabilities influenced by the default of other firms are "secondary" firms. The authors then study counterparty risk on a CDS in the case where the reference name is considered to be a primary firm, while the protection seller is a secondary firm. The default intensities are deterministic constants, with default indicators of the other names as feeds for the secondary firms.

This results in an increase in the default intensity of one party when the default of another party occurs. In this simplified scenario the authors investigate the influence of taking into account these contagion effects on the fair spread of the CDS and conclude that a CDS may be significantly overpriced if this correlation is ignored.

In Leung and Kwok [2005], the authors extend the previous framework in Jarow and Yu [2002] to allow for an arbitrary dependence structure, relaxing the need for primary and secondary firms, and analyse the impact of correlated risks between the three CDS parties. Instead of following Yu's approach, they employ the change of measure introduced by Collin-Dufresne et al. [2004] in the valuation process. If the protection seller defaults prior to the reference entity, then the protection buyer renews the CDS with a new counterparty. Supposing that the default risk of the protection seller and reference entity are positively correlated, they estimate the expected replacement cost due to an increase in the swap rate in the new CDS. They extend the counterparty risk framework to consider the bilateral case as well, by including the possibility of default of the protection buyer.

In the class of copula models, the copula and hence the dependence structure of the default times is exogenously specified; the default intensities and the reaction of default intensities to defaults of other firms in the portfolio (in other words the default contagion) are then endogenously derived.

In Brigo and Chourdakis [2009], the authors deal with the counterparty risk problem in this setup. They focus on unilateral and asymmetric counterparty risk (the party that computes the CVA is considered to be risk free). The model proposed in this paper models explicitly credit spread volatility, besides the default correlation on which previous papers focused. The default intensities are driven by CIR processes, with independent Brownian motions, as typically spread correlation has a much lower impact on dependence of default times than default correlation. The default correlation between the names is defined through a dependence structure on the exponential random variables characterizing the default times, and is modelled using a copula function. Brigo and Capponi [2010] generalise the work in Brigo and Chourdakis [2009] to the case of bilateral risk, and the two parties will now agree on the credit valuation adjustment. Both papers provide numerical examples through a

Monte-Carlo implementation of the framework.

Other approaches, based on reduced form modelling, have also been proposed in the literature for this purpose: Chen and Filipovic [2003], Leung and Kwok [2005], Leung and Kwok [2009], Crépey et al. [2009], Brigo et al. [2012b] and Lipton and Shelton [2011] to mention just a few.

### 6.1.2 Structural default models

As in the case of the reduced-form approach, several papers have addressed the problem of counterparty risk valuation for CDSs in the structural default framework.

In order to compute counterparty adjustments, several authors have considered the straightforward extension to the two dimensional case of the standard first passage time model, where the correlation is introduced through the correlation of the Brownian motions that drive the evolution of the firms' asset values. Considering exponential default boundaries, closed form expressions for the unilateral counterparty adjustments are available and have been given in Blanchet-Scaillet and Patras [2011], Haworth [2006]. These papers build on previous work by Lipton [2001], Zhou [2001a], Patras [2006], where authors study a similar case but apply it to different contexts.

Extension to this framework in several directions have been proposed and studied in the existing literature. The above approach allows for analytic solutions and is therefore a very useful and efficient way of qualitatively studying the impact of unilateral counterparty adjustments. However, one of its main drawbacks is that implied short-term credit spreads are zero given that the default time is predictable, and hence calibration to the entire CDS spread term structure is seldom possible. To avoid this problem and obtain reasonable short-term spreads, several solutions have been proposed in the literature (see for example Hyer et al. [1999], Avellaneda and Zhu [2001], Hull and White [2001] and Overbeck and Schmidt [2005]). In Hull and White [2001] the authors propose to do this by making the default barrier time-dependent. The solution proposed in Overbeck and Schmidt [2005] allows us to maintain the analytical tractability for the joint default probability while calibrating to given marginal default distribution; this is achieved by suitably transforming the time scales of the

Brownian motions. In Lipton and Sepp [2009] the authors propose adding jumps to the dynamics of the asset firms' evolution, which makes the short-term spreads non-zero as the defaults are no longer predictable. They discuss in this framework the calculation of unilateral adjustments on CDSs.

A different direction in which the framework can be extended is to incorporate default contagion in the standard first passage time model. In Haworth et al. [2008], the authors introduce a simple mechanism for default contagion. They adopt a similar mechanism as the one proposed in the reduced-form framework by Jarrow and Yu [2002] in which primary companies impact secondary companies but not vice-versa: they assume that company one defaults on its outstanding debt the first time that the value of either company reaches its default barrier (the model is asymmetric with respect to default risk). A more realistic contagion structure is proposed in Haworth [2006]: the default of one company causes a jump in the volatility of other firms. While this is more realistic in terms of describing the way a default ripples through the market, analytical solutions are no longer possible. The author solves the problem using finite difference methods and extends it to the three dimensional case as well. A further extension where contagion takes the form of a decaying spike in spreads or volatility is also discussed.

Another interesting extension to Zhou's first-passage-time model is presented in Kim et al. [2008]. The authors start from the empirical observations that the volatilities of asset returns vary according to market environments (e.g. business cycle) and that default correlations as well as default probability are higher conditional on the economy starting in a recession than an expansion. In order to reflect the effects of such market cycles, they propose a regime-switching first-passage-time model: they assume that market condition has two states, bull market (or economic expansion) and bear market (or economic recession), and the volatilities of asset returns as well as risk-free interest rate change accordingly. These regimes are determined by a two-state Markov chain evolving in continuous time: a regime change occurs at the next jump time of a Poisson process. In this framework they discuss the single name pricing of CDSs with and without counterparty risk. To obtain the individual and joint survival probabilities they develop PDEs (a system of PDEs) which are then

solved through the finite difference method. As in Zhou's model, the default boundary is exponential. A possible further extension would be to consider that the default boundary is dependent on the business cycle.

In this thesis, the first-passage time framework considered in chapter 2 is adopted to study counterparty risk related problems on CDSs. The mathematical results in chapters 3 and 4 are used to obtain analytical or semi-analytical results for both unilateral and bilateral value adjustments on CDSs. In the other papers in the literature that have studied counterparty risk issues in this framework, the valuation is done using Monte-Carlo simulations or finite difference schemes, which are highly numerically intensive. Unlike these papers, the new semi-analytical solutions studied in this thesis for the bilateral case are significantly faster (see discussion in Section 4.3) and thus provide a good benchmark and the possibility of studying different linked problems. One of these is the precision of the commonly used in practice approximation of the bilateral adjustments by the sum of the unilateral ones, studied in Section 6.4. Chapter 7 then looks into the case where the CDS is partially collateralised.

## 6.2 Arbitrage-free valuation of counterparty risk adjustments

In Section 2.5, the definition of the cashflows for both the coupon leg and the default leg of a standard CDS were given. In order to simplify the formulas we denote by  $CF(t, T)$  the sum of all discounted contractual cashflows between  $t$  and the maturity  $T$  (both coupon leg and default leg), and write the value  $V_t$  of the CDS as:  $V_t = \mathbb{E}[CF(t, T) | \mathcal{F}_t]$ .

Let  $\tau^{RN}$  denote the default time of the reference name of the CDS,  $\tau^{PS}$  and  $\tau^{PB}$  the default times of the protection seller (PS) and protection buyer (PB) respectively, while  $R_{PS}$  and  $R_{PB}$  their recoveries.

## 6.2.1 Risk-free protection buyer

Suppose first that the protection seller can default but consider the protection buyer risk free, and denote by  $\tilde{V}_t$  the value of the derivative in this case:

$$\begin{aligned}\tilde{V}_t = & \mathbb{E} [CF(t, T) \mathbb{1}_{\tau^{PS} > \min\{T, \tau^{RN}\}} | \mathcal{F}_t] \\ & + \mathbb{E} [\mathbb{1}_{\tau^{PS} < \min\{T, \tau^{RN}\}} [CF(t, \tau^{PS}) + D(t, \tau^{PS}) (R_{PS} V_{\tau^{PS}}^+ + V_{\tau^{PS}}^-)] | \mathcal{F}_t].\end{aligned}$$

As usual,  $V^\pm = \pm \max(0, \pm V)$ , and it is assumed that if the position is negative in value (to the PB) at the time of default of the PS, the PB will still be obligated to pay in full, while if the position is positive in value they will recover a fraction  $R_{PS}$  of the value of the position. Using  $V^+ + V^- = V$  and  $V_{\tau^{PS}} = \mathbb{E} [CF(\tau^{PS}, T) | \mathcal{F}_{\tau^{PS}}]$  yields the following expression for the price of the derivative:

$$\begin{aligned}\tilde{V}_t = & \mathbb{E} [CF(t, T) \mathbb{1}_{\tau^{PS} > \min\{T, \tau^{RN}\}} \\ & + \mathbb{1}_{\tau^{PS} < \min\{T, \tau^{RN}\}} [CF(t, \tau^{PS}) + D(t, \tau^{PS}) \mathbb{E} [CF(\tau^{PS}, T) | \mathcal{F}_{\tau^{PS}}] \\ & - D(t, \tau^{PS}) (1 - R_{PS}) V_{\tau^{PS}}^+] | \mathcal{F}_t].\end{aligned}$$

Since both  $D(t, \tau^{PS})$  and  $CF(t, \tau^{PS})$  are  $\mathcal{F}_{\tau^{PS}}$ -measurable, this can be rewritten as

$$CF(t, \tau^{PS}) + D(t, \tau^{PS}) \mathbb{E} [CF(\tau^{PS}, T) | \mathcal{F}_{\tau^{PS}}] = \mathbb{E} [CF(t, T) | \mathcal{F}_{\tau^{PS}}].$$

For  $t < \tau^{PS}$ , the tower law for expectations  $\mathbb{E} [\mathbb{E} [\cdot | \mathcal{F}_{\tau^{PS}}] | \mathcal{F}_t] = \mathbb{E} [\cdot | \mathcal{F}_t]$  leads to:

$$\begin{aligned}\tilde{V}_t = & \mathbb{E} [CF(t, T) | \mathcal{F}_t] \\ & + \mathbb{E} [\mathbb{1}_{\tau^{PS} > \min\{T, \tau^{RN}\}} (CF(t, T) - \mathbb{E} [CF(t, T) | \mathcal{F}_{\tau^{PS}}]) | \mathcal{F}_t] \\ & - \mathbb{E} [\mathbb{1}_{\tau^{PS} < \min\{T, \tau^{RN}\}} D(t, \tau^{PS}) (1 - R_{PS}) V_{\tau^{PS}}^+ | \mathcal{F}_t].\end{aligned}$$

To evaluate the second term on the right-hand-side, there are two possible cases:

- $\min\{T, \tau^{RN}\} = T$ : then  $\tau^{PS} > T$ , and hence  $CF(t, T)$  is  $\mathcal{F}_{\tau^{PS}}$ -measurable, and  $\mathbb{1}_{\tau^{PS} > \min\{T, \tau^{RN}\}} (CF(t, T) - \mathbb{E} [CF(t, T) | \mathcal{F}_{\tau^{PS}}]) = 0$ .

- $\min\{T, \tau^{RN}\} = \tau^{RN}$ : if the reference name defaults before the maturity of the CDS, the total cashflows for the CDS,  $CF(t, T)$  are  $\mathcal{F}_{\tau^{RN}}$ -measurable as the coupon payments stop and the protection payment happens at  $\tau^{RN}$ . Since  $\tau^{PS} > \min\{T, \tau^{RN}\}$ , the total cashflows are also  $\mathcal{F}_{\tau^{PS}}$ -measurable, and hence  $\mathbb{1}_{\tau^{PS} > \min\{T, \tau^{RN}\}} (CF(t, T) - \mathbb{E}[CF(t, T) | \mathcal{F}_{\tau^{PS}}]) = 0$ .

This yields:

$$\tilde{V}_t = V_t - \mathbb{E} \left[ \mathbb{1}_{\tau^{PS} < \min\{T, \tau^{RN}\}} D(t, \tau^{PS}) (1 - R_{PS}) V_{\tau^{PS}}^+ \middle| \mathcal{F}_t \right].$$

The term Credit Value Adjustment (CVA) refers to the adjustment to account for the possibility of the counterparty's default and is defined as  $V^{UCVA} = V_t - \tilde{V}_t$  (UCVA refers to the unilateral nature of the adjustment, as the protection buyer is considered non risky – the trade is seen from the protection buyer's point of view):

$$V^{UCVA} = (1 - R_{PS}) \mathbb{E} \left[ \mathbb{1}_{\tau^{PS} < \min\{T, \tau^{RN}\}} D(t, \tau^{PS}) V_{\tau^{PS}}^+ \middle| \mathcal{F}_t \right], \quad (6.1)$$

where the recovery is considered constant.

Similarly, the case where the protection buyer is risky but the protection seller is risk free can be considered. The term Debt Valuation Adjustment (DVA) represents the additional cost to account for one's own default (again UDVA refers to the unilateral nature of the adjustment):

$$V^{UDVA} = (1 - R_{PB}) \mathbb{E} \left[ \mathbb{1}_{\{\tau^{PB} < \min\{T, \tau^{RN}\}\}} D(t, \tau^{PB}) V_{\tau^{PB}}^- \middle| \mathcal{F}_t \right]. \quad (6.2)$$

### 6.2.2 Risky protection buyer

Given recent events, it can no longer be supposed that one of the counterparties is risk free. The Basel II documentation also makes reference to a bilateral counterparty risk, in which both counterparties involved in the derivative contract are subject to default risk.

One of the advantages of considering the bilateral CVA is the symmetry it introduces in pricing: the two counterparties will now agree on the price of the derivative



(for a detailed discussion on this see for example Brigo and Capponi [2010]). If  $\tau$  denotes the minimum of the two default times:  $\tau = \min\{\tau^{PS}, \tau^{PB}\}$ , then the price of the CDS in this case is given by:

$$\begin{aligned} \tilde{V}_t = & \mathbb{E} [CF(t, T) \mathbb{1}_{\tau > T} + \\ & + \mathbb{1}_{\tau = \tau^{PS} < T} (CF(t, \tau^{PS}) + D(t, \tau^{PS}) R_{PS} V_{\tau^{PS}}^+ + D(t, \tau^{PS}) V_{\tau^{PS}}^-) + \\ & + \mathbb{1}_{\tau = \tau^{PB} < T} (CF(t, \tau^{PB}) + D(t, \tau^{PB}) R_{PB} V_{\tau^{PB}}^- + D(t, \tau^{PB}) V_{\tau^{PB}}^+) ] . \end{aligned}$$

Similar ideas as in the case where only one of the counterparties is risky, the value of the derivative in this case can be simplified to:

$$\tilde{V}_t = V_t - V^{BCVA} - V^{BDVA},$$

where BCVA and BDVA define the bilateral credit and debit value adjustments and are given by:

$$V^{BCVA} = (1 - R_{PS}) \mathbb{E} [\mathbb{1}_{\tau^{PS} < \min\{\tau^{PB}, \tau^{RN}, T\}} D(t, \tau^{PS}) V_{\tau^{PS}}^+ | \mathcal{F}_t], \quad (6.3)$$

$$V^{BDVA} = (1 - R_{PB}) \mathbb{E} [\mathbb{1}_{\tau^{PB} < \min\{\tau^{PS}, \tau^{RN}, T\}} D(t, \tau^{PB}) V_{\tau^{PB}}^- | \mathcal{F}_t]. \quad (6.4)$$

It is important to emphasize that expressions (6.1), (6.2) and (6.3), (6.4) are not identical. The bilateral expressions for CVA/DVA depend on the first to default time between the protection seller and protection buyer, so in order to compute these in a consistent way for CDSs, one needs to consider three-dimensional models.

### 6.3 Counterparty risk adjustments valuation in a first-passage time model

Consider three issuers  $X$ ,  $Y$  and  $Z$  and let  $x_t$ ,  $y_t$  and  $z_t$  respectively be the processes for the stochastic factors associated to the firms' asset value of these issuers (as defined in Section 2.2, equation (2.4)). These processes are standard Brownian motions issued from the initial positive points  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  respectively.

Issuer  $X$  is the issuer associated with the protection seller, issuer  $Y$  is the reference name of the CDS and  $Z$  is associated to the protection buyer. As previously, the CDS is seen from the protection buyer's point of view. In a first passage time framework, the default times are defined as follows:

$$\begin{aligned}\tau^{PS} &= \inf\{u \geq t, x_u \leq 0\}, \\ \tau^{RN} &= \inf\{u \geq t, y_u \leq 0\}, \\ \tau^{PB} &= \inf\{u \geq t, z_u \leq 0\}.\end{aligned}$$

### 6.3.1 Unilateral CVA and DVA

The valuation of the unilateral CVA and DVA, for which the expressions are given in equations (6.1) and (6.2) respectively, in the first-passage time framework is discussed in this section. For the unilateral computations, one of the three issuers involved in the contract is considered risk-free, and hence the problem becomes two-dimensional. As such, the results in chapter 3 can be applied. The Brownian motions associated with the stochastic factors' processes are correlated as described in Section 3.2.

Consider first a risk-free protection buyer. Proposition 6.1 gives an analytic expression for computing the unilateral credit value adjustment for a CDS in the first-passage time default framework.

**Proposition 6.1.** (*Unilateral CVA*)

*The value for the unilateral CVA in the first-passage time model is given by:*

$$V^{UCVA}(t, T, \tilde{r}, \tilde{\varphi}) = -\frac{1 - R_{PS}}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, \varpi) V^{CDS}(t', T, \bar{\rho}_{xy} r)^+ \frac{1}{r} dr dt', \quad (6.5)$$

where  $G(t' - t, r', \varphi' | \tilde{r}, \tilde{\varphi})$  denotes the Green's function for a two-dimensional operator  $\mathcal{O}_{2D}^*$  in a wedge with absorbing boundaries and is given by the expression in Proposition 3.1, and  $V^{CDS}$  denotes the value of a standard CDS with non-risky counterparts.

*Proof.* The expression for computing the unilateral CVA given in expectation form

in equation (6.1) can be transformed to a PDE problem with terminal condition and Dirichlet boundary conditions through a Feynman-Kac type representation (see Theorem A.3 for example). The terminal value for the PDE problem is null, as there is no payment to be made at maturity. In the first-passage time default risk framework considered here, the names default once the associated processes pass below 0, so the domain for the PDE problem is the positive quadrant:

$$\mathcal{D} = \{(x, y), \quad 0 \leq x < \infty, \quad 0 \leq y < \infty\}. \quad (6.6)$$

As can be observed in equation (6.1), the boundary condition is non-null only in the case where the protection seller defaults first (boundary at  $x = 0$ ), as it will no longer be able to honour its payments and hence the shortfall for the protection buyer will be a fraction of the outstanding present value of the single name swap. The problem to solve thus becomes:

$$\begin{aligned} V_t^{\text{UCVA}} + \frac{1}{2}V_{xx}^{\text{UCVA}} + \frac{1}{2}V_{yy}^{\text{UCVA}} + \rho_{xy} V_{xy}^{\text{UCVA}} - \varrho V^{\text{UCVA}} &= 0, \\ V^{\text{UCVA}}(T, T, x, y) &= 0 \\ V^{\text{UCVA}}(t, T, 0, y) &= (1 - R_{PS}) V^{\text{CDS}}(t, T, y)^+ \\ V^{\text{UCVA}}(t, T, x, 0) &= 0 \\ V^{\text{UCVA}}(t, T, \infty, y) &= 0, \quad V^{\text{UCVA}}(t, T, x, \infty) = 0. \end{aligned}$$

By using, for example, Theorem 3.6, pg 138 in Friedman [2006], the difference of any two solutions for the above PDE problem with final condition and boundary conditions is identically null (the operator is uniformly parabolic and the boundary and final conditions for the associated homogeneous problem are continuous). Hence the solution to the above problem is unique and this property is inherited by all the transformed problems below.

Consider the following function  $U^{\text{UCVA}}(t, T, x, y) = e^{\varrho(T-t)} V^{\text{UCVA}}(t, T, x, y)$ . After applying the change of variables that eliminates the cross derivative given in equation

(4.5), the pricing problem becomes:

$$\begin{aligned}
U_t^{\text{UCVA}} + \frac{1}{2}U_{\alpha\alpha}^{\text{UCVA}} + \frac{1}{2}U_{\beta\beta}^{\text{UCVA}} &= 0, \\
U^{\text{UCVA}}(T, T, \alpha, \beta) &= 0, \\
U^{\text{UCVA}}(t, T, 0, \beta) &= e^{\varrho(T-t)} (1 - R_{PS}) V^{CDS}(t, T, \bar{\rho}_{xy}\beta)^+, \\
U^{\text{UCVA}}\left(t, T, \alpha, -\frac{\rho_{xy}}{\bar{\rho}_{xy}}\alpha\right) &= 0, \\
U^{\text{UCVA}}(t, T, \infty, \beta) &= 0, \quad U^{\text{UCVA}}(t, T, \alpha, \infty) = 0.
\end{aligned}$$

The PDE pricing problem now needs to be solved in a wedge. Applying next the second change of variables given in equation (3.7), leads to the following form for the pricing problem:

$$U_t^{\text{UCVA}} + \frac{1}{2}\left(U_{rr}^{\text{UCVA}} + \frac{1}{r}U_r^{\text{UCVA}} + \frac{1}{r^2}U_{\varphi\varphi}^{\text{UCVA}}\right) = 0, \quad (6.7)$$

$$U^{\text{UCVA}}(T, T, r, \varphi) = 0, \quad (6.8)$$

$$U^{\text{UCVA}}(t, T, r, \varpi) = e^{\varrho(T-t)} (1 - R_{PS}) V^{CDS}(t, T, \bar{\rho}_{xy}r)^+, \quad (6.9)$$

$$U^{\text{UCVA}}(t, T, r, 0) = 0, \quad (6.10)$$

$$U^{\text{UCVA}}(t, T, 0, \varphi) = 0, \quad U^{\text{UCVA}}(t, T, \infty, \varphi) = 0, \quad (6.11)$$

where the domain to solve this in is:

$$\tilde{\mathcal{D}} = \{(r, \varphi), \quad 0 \leq r < \infty, \quad 0 \leq \varphi < \varpi\}.$$

Lemma 6.1 below shows how to obtain the solution for the above problem but with generic boundary conditions. Particularising to the specific boundary conditions for the unilateral CVA case, the following fully analytic solution is obtained:

$$V^{\text{CVA}}(t, T, \tilde{r}, \tilde{\varphi}) = -\frac{1 - R_{PS}}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, \varpi) V^{\text{CDS}}(t', T, \bar{\rho}_{xy}r)^+ \frac{1}{r} dr dt', \quad (6.12)$$

where the starting point coordinates  $(\tilde{r}, \tilde{\varphi})$  are given by equations (3.11) and (3.12).

□

Lemma 6.1 gives the solution to the PDE in equation (6.13) with zero final condition (6.14) and generic boundary conditions (given in equations (6.15)–(6.17)). This result is applied to the specific problems of computing UCVA and UDVA.

**Lemma 6.1.** *(2D PDE problem with generic boundary conditions)*

Let  $U$  be a  $\mathbb{C}^{1,2}(\mathbb{R}_+, \tilde{D})$  bounded function and  $u_0$  and  $u_\varpi$  continuous and bounded functions. The solution for the following problem with the corresponding final and boundary conditions:

$$U_t + \frac{1}{2} \left( U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} \right) = 0, \quad (6.13)$$

$$U(T, T, r, \varphi) = 0, \text{ (final condition)} \quad (6.14)$$

$$U(t, T, 0, \varphi) = 0, \quad U(t, T, \infty, \varphi) = 0, \quad (6.15)$$

$$U(t, T, r, 0) = u_0(t, T, r), \quad (6.16)$$

$$U(t, T, r, \varpi) = u_\varpi(t, T, r), \quad (6.17)$$

considered in the angular domain described by

$$\tilde{D} = \{(r, \varphi), \quad 0 \leq r < \infty, \quad 0 \leq \varphi < \varpi\}$$

and shown in Figure 3.1 is given by:

$$U(t, T, \tilde{r}, \tilde{\varphi}) = \frac{1}{2} \int_t^T \int_0^\infty [G_\varphi(t' - t, r, 0) u_0(t', T, r) - G_\varphi(t' - t, r, \varpi) u_\varpi(t', T, r)] \frac{1}{r} dr dt',$$

where  $G(t' - t, r', \varphi' | \tilde{r}, \tilde{\varphi})$  denotes the Green's function for the two-dimensional operator  $\mathcal{O}_{2D}^*$  in a wedge with absorbing boundaries and is given by the expression in Proposition 3.1.

*Proof.* The fundamental solution for the above problem with zero boundary conditions is given by the Green's function for the two-dimensional operator  $\mathcal{O}_{2D}^*$  in the

angular domain, as discussed in Section 3.2. For this problem, Proposition 3.1 gives a solution using the eigenfunction expansion method.

The solution for the above problem with non-zero boundary conditions is supposed to be bounded and with bounded first derivatives. To obtain this solution the following identity is used:

$$\int_t^T \int_0^\infty \int_0^\varpi \left[ U_t + \frac{1}{2} \left( U_{rr} + \frac{1}{r} U_r + \frac{1}{r^2} U_{\varphi\varphi} \right) \right] G(t' - t, r, \varphi) r d\varphi dr dt' = 0,$$

which holds as  $U$  solves equation (6.7). A series of integration by parts are then performed with the purposes of “transferring” the partial derivatives from function  $U$  to Green’s function  $G$ .

$$\begin{aligned} 0 &= \int_0^\varpi \int_0^\infty [U(T, T, r, \varphi) G(T - t, r, \varphi) - U(t, T, r, \varphi) G(0, r, \varphi)] r dr d\varphi \\ &\quad - \int_t^T \int_0^\varpi \int_0^\infty U(t', T, r, \varphi) G_{t'}(t' - t, r, \varphi) r dr d\varphi dt' \\ &\quad + \frac{1}{2} \int_t^T \int_0^\varpi [r U_r(t', T, r, \varphi) G(t' - t, r, \varphi)]_{r=0}^\infty d\varphi dt' \\ &\quad - \frac{1}{2} \int_t^T \int_0^\varpi \int_0^\infty U_r(t', T, r, \varphi) \frac{\partial}{\partial r} (r G(t' - t, r, \varphi)) dr d\varphi dt' \\ &\quad + \frac{1}{2} \int_t^T \int_0^\varpi [U(t', T, r, \varphi) G(t' - t, r, \varphi)]_{r=0}^\infty d\varphi dt' \\ &\quad - \frac{1}{2} \int_t^T \int_0^\varpi \int_0^\infty U(t', T, r, \varphi) G_r(t' - t, r, \varphi) dr d\varphi dt' \\ &\quad + \frac{1}{2} \int_t^T \int_0^\varpi [U_\varphi(t', T, r, \varphi) G(t' - t, r, \varphi)]_{\varphi=0}^{\varphi=\varpi} \frac{1}{r} dr dt' \\ &\quad - \frac{1}{2} \int_t^T \int_0^\varpi \int_0^\infty U_\varphi(t', T, r, \varphi) G_\varphi(t' - t, r, \varphi) \frac{1}{r} dr d\varphi dt' \end{aligned}$$

Using the final condition for  $U$ :  $U(T, T, r, \varphi) = 0$  given in equation (6.14) and the initial condition for Green’s function  $G(0, r', \varphi') = \frac{1}{\tilde{r}} \delta(r' - \tilde{r}) \delta(\varphi' - \tilde{\varphi})$ , the first integral becomes:

$$\int_0^\varpi \int_0^\infty [U(T, T, r, \varphi) G(T - t, r, \varphi) - U(t, T, r, \varphi) G(0, r, \varphi)] r dr d\varphi = -U(t, T, \tilde{r}, \tilde{\varphi}).$$

For the third and fifth integral, the assumption that  $U$  and  $U_r$  are bounded and that  $\lim_{r \rightarrow \infty} rG(t, r, \varphi) = 0$  and  $\lim_{r \rightarrow \infty} G(t, r, \varphi) = 0$  are used (see equation (3.20)). Hence these integrals are 0. For the seventh integral, the boundary conditions for Green's function:  $G(t, r, 0) = 0$ ,  $G(t, r, \varpi) = 0$  come into play, to yield:

$$\begin{aligned}
U(t, T, \tilde{r}, \tilde{\varphi}) &= - \int_t^T \int_0^\varpi \int_0^\infty U(t', T, r, \varphi) \left[ G_t(t' - t, r, \varphi) + \frac{1}{2r} G_r(t' - t, r, \varphi) \right] r dr d\varphi dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\varpi [U(t', T, r, \varphi) (G + rG_r)]_{r=0}^\infty d\varphi dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\varpi \int_0^\infty U(t', T, r, \varphi) \frac{\partial}{\partial r} (G + rG_r) dr d\varphi dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\varpi [U(t', T, r, \varphi) G_\varphi(t' - t, r, \varphi)]_{\varphi=0}^{\varphi=\varpi} \frac{1}{r} dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\varpi \int_0^\infty U(t', T, r, \varphi) G_{\varphi\varphi}(t' - t, r, \varphi) \frac{1}{r} dr d\varphi dt' \\
&= - \int_t^T \int_0^\varpi \int_0^\infty U(t', T, r, \varphi) \left[ G_t - \frac{1}{2} \left( G_{rr} + \frac{1}{r} G_r + \frac{1}{r^2} G_{\varphi\varphi} \right) \right] r dr d\varphi dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\varpi [U(t', T, r, \varphi) G_\varphi(t' - t, r, \varphi)]_{\varphi=0}^{\varphi=\varpi} \frac{1}{r} dr dt'
\end{aligned}$$

where the following limit has been used:  $\lim_{r \rightarrow \infty} rG_r(t, r, \varphi) = 0$ . Since Green's function for the operator  $\mathcal{O}_{3D}^*$  solves the forward equation (3.8), the final solution for the problem is:

$$U(t, T, \tilde{r}, \tilde{\varphi}) = \frac{1}{2} \int_t^T \int_0^\infty [G_\varphi(t' - t, r, 0) u_0(t', T, r) - G_\varphi(t' - t, r, \varpi) u_\varpi(t', T, r)] \frac{1}{r} dr dt'.$$

Given the unicity of the solution to the starting PDE problem, the only thing left to show is that the obtained solution is bounded along with its first derivatives. This is immediately true using the limits for Green's function (see equations (3.20) and (3.21)) along with the fact that the functions  $u_0$  and  $u_\varpi$  are bounded.  $\square$

Using similar techniques and applying the result in Lemma 6.1 to the case of unilateral DVA, a fully analytic expression can be obtained for its valuation in the first-passage time default risk framework:

**Proposition 6.2.** (*Unilateral DVA*)

The value for the unilateral DVA in the first-passage time model is given by:

$$V^{DVA}(t, T, \tilde{r}, \tilde{\varphi}) = -\frac{1 - R_{PB}}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, \overline{\varphi}) V^{CDS}(t', T, \bar{\rho}_{yz} r)^- \frac{1}{r} dr dt', \quad (6.18)$$

where  $V^{CDS}$  denotes the value of a standard CDS with non-risky counterparts and  $G(t' - t, r', \varphi' | \tilde{r}, \tilde{\varphi})$  denotes the Green's function for the two-dimensional operator  $\mathcal{O}_{2D}^*$  in a wedge with absorbing boundaries and is given by the expression in Proposition 3.1; the angle is given by  $\overline{\varphi} = \arccos(-\rho_{yz})$  and the source coordinates by:

$$\tilde{r} = \frac{\sqrt{\tilde{y}^2 - 2\tilde{y}\tilde{z}\bar{\rho}_{yz} + \tilde{z}^2}}{\bar{\rho}_{yz}}$$

$$\tilde{\varphi} = \overline{\varphi} - \arctan\left(\frac{\bar{\rho}_{yz}\tilde{z}}{\tilde{y} - \rho_{yz}\tilde{z}}\right).$$

Propositions 6.1 and 6.2 give analytical expressions for computing unilateral credit and debit value adjustments for a CDS in the first-passage time framework. Figure 6.1 shows the impact of these adjustments on the break-even coupon of a CDS, as a function of the correlation to the reference name. Consider three names that are equally risky ( $\tilde{x} = \tilde{y} = \tilde{z}$  such that the individual survival probability at five year time horizon is  $\approx 50\%$ ).

- When considering the protection buyer as risk-free, the unilateral CVA can be computed using the result in Proposition 6.1; this adjustment is reflected in the break-even coupon of the CDS by lowering it. This makes sense intuitively, as the protection buyer takes on the extra risk of the protection seller not being able to fulfil its contractual obligations in case the reference name defaults. Figure 6.1 shows that the adjustment is highly dependent on the correlation between the reference name and the protection seller: the higher the correlation, the higher the adjustment, which exemplifies the notion of “wrong-way risk”.
- When considering the protection buyer risky but a risk-free protection seller, the unilateral DVA adjustment can be computed using the result in Proposition 6.2;



this adjustment is reflected in the break-even coupon of the CDS by increasing it. Here the correlation has the opposite effect: the adjustment is higher when the protection buyer and the reference name are anti-correlated.

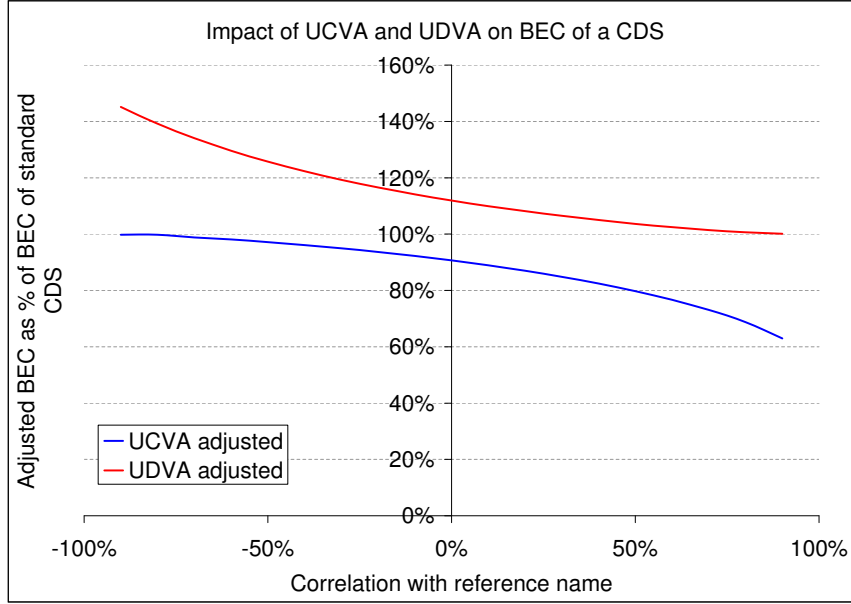


Figure 6.1: Impact of the UCVA and UDVA on the BEC of a CDS as a function of the correlation of the risky party with the reference name. The BECs are computed for a 5Y maturity and  $R_{PB} = R_{PS} = 0\%$ ,  $R_{RN} = 40\%$ .

It is worth noting that in the extreme cases, the adjustments can be quite significant at  $\approx \pm 40\%$  of the original break-even coupon of the standard CDS.

Figure 6.2 shows the impact of the riskiness of the protection seller (protection buyer) on the unilateral credit (debt) adjustment. The CDS considered is at par at the starting date, which means that the coupon leg and default leg are equal. The adjustments are shown as percentage of this common value of the coupon/default leg. The correlations are taken as  $\rho_{xy} = 75\%$  and  $\rho_{yz} = -75\%$ , which as observed previously show the cases where the UCVA/UDVA respectively are non-negligible.

Note that as the default probability of the party in question (protection seller/buyer) goes to 0, the adjustments go to 0 as well, so the limit behaviour expected is correctly replicated. At their maximum, the absolute value of the adjustments can represent up to 30 – 40% of the default leg value, which is significant.

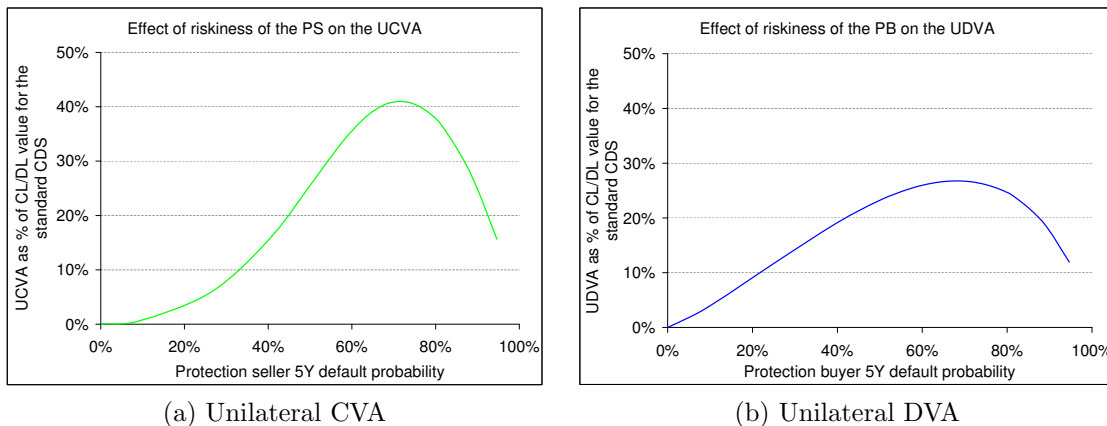


Figure 6.2: Dependence on the riskiness of the PS/PB of the absolute value of the unilateral CVA/DVA adjustments. The adjustments are given as percentage of the absolute common value of the default leg and coupon leg of the standard 5Y CDS (the original CDS is considered at par). The survival probability of the reference name is  $\approx 50\%$  at 5Y and  $R_{PB} = R_{PS} = 0\%$ ,  $R_{RN} = 40\%$ .

Another important feature is that at very high values for the default probability of the protection seller/buyer, the adjustments decrease. To better understand this behaviour, consider the limit case when the protection seller defaults almost immediately after the start date of the trade. By analysing equation (6.1), in this extreme case the unilateral CVA will tend to the positive value of the CDS today, and since a par CDS has been considered in the example in Figure 6.2a, this limit is 0. Figure 6.3 shows the case of a non par CDS (the coupon is 75% of the coupon considered in the previous case), where the limit when the default probability of the protection seller tends to 1 is non-zero.

### 6.3.2 Bilateral CVA and DVA

This section discusses the valuation in the first-passage time framework of the bilateral CVA and DVA, for which the arbitrage-free expressions are given in equations (6.3) and (6.4) respectively. All three issuers are considered risky to compute bilateral adjustments in a consistent manner and as such, the results in chapter 4 can be applied. The Brownian motions associated with the stochastic factors' processes are

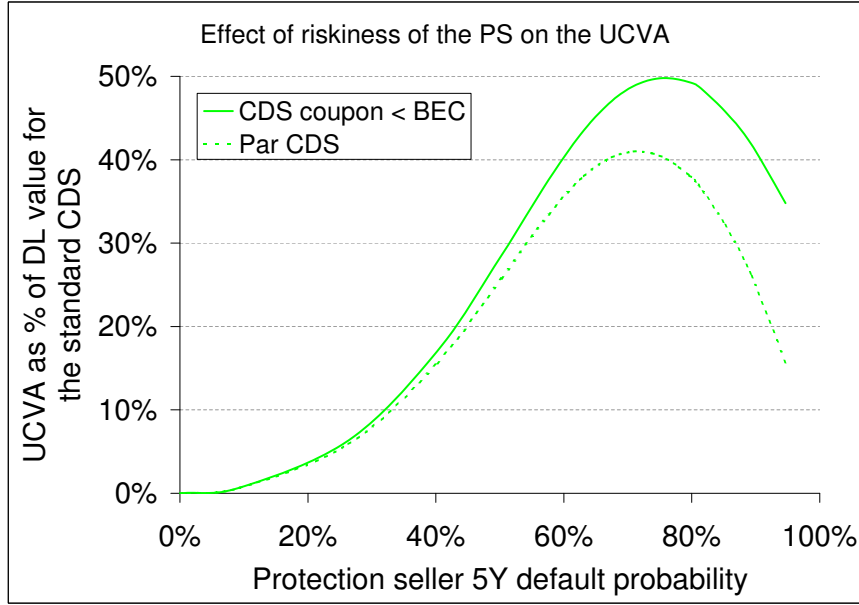


Figure 6.3: Same example as in Figure 6.2, but with the coupon on the CDS at 75% of the BEC coupon, such that the CDS is not at par originally.

correlated as described in Section 4.1.

Lemma 6.2 gives a helpful result that can then be applied to obtain both bilateral CVA and DVA expressions and is used in the proofs of Propositions 6.3 and 6.4.

**Lemma 6.2.** (*3D PDE problem with generic boundary conditions*)

Let  $U$  be a  $\mathbb{C}^{1,2}(\mathbb{R}_+, \tilde{D})$  bounded function and  $u_0$ ,  $u_\varpi$  and  $u_\Theta$  continuous and bounded functions. The solution for the following problem with the corresponding final and boundary conditions:

$$U_t + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (rU) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} U_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right) \right] = 0, \quad (6.19)$$

$$U(T, T, r, \varphi, \theta) = 0, \text{ (final condition)} \quad (6.20)$$

$$U(t, T, 0, \varphi, \theta) = 0, \quad U(t, T, \infty, \varphi, \theta) = 0, \quad (6.21)$$

$$U(t, T, r, 0, \theta) = u_0(t, T, r, \theta), \quad U(t, T, r, \varpi, \theta) = u_\varpi(t, T, r, \theta) \quad (6.22)$$

$$U(t, T, r, \varphi, 0) = 0, \quad U(t, T, r, \varphi, \Theta(\varphi)) = u_\Theta(t, T, r, \varphi) \quad (6.23)$$

considered in the domain shown in Figure 3.1 which is an infinite cone, extending

from the origin and for which the intersection with the three-dimensional unit sphere is a spherical triangle:

$$\tilde{\mathcal{D}} = \{ (r, \varphi, \theta) \mid r \geq 0, \quad 0 \leq \varphi \leq \varpi, \quad 0 \leq \theta \leq \Theta(\varphi) \},$$

is given by:

$$\begin{aligned} U(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = & -\frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \sin \Theta(\varphi) U(t', T, r, \varphi, \Theta(\varphi)) G_\theta(t' - t, r, \varphi, \Theta(\varphi)) d\varphi dr dt' \\ & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\infty \frac{U(t', T, r, \varphi(\omega), \Theta(\omega)) G_\varphi(t' - t, r, \varphi(\omega), \Theta(\omega))}{\sin \Theta(\omega)} \Theta_\omega(\omega) d\omega dr dt' \\ & - \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(\varpi)} \frac{U(t', T, r, \varpi, \theta) G_\varphi(t' - t, r, \varpi, \theta)}{\sin \theta} d\theta dr dt' \\ & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(0)} \frac{U(t', T, r, 0, \theta) G_\varphi(t' - t, r, 0, \theta)}{\sin \theta} d\theta dr dt'. \end{aligned} \quad (6.24)$$

Here  $G(t' - t, r, \varphi, \theta \mid \tilde{r}, \tilde{\varphi}, \tilde{\theta})$  denotes the Green's function for the three-dimensional operator  $\mathcal{O}_{3D}^*$  in the domain  $\tilde{\mathcal{D}}$ , and is given by Proposition 4.1. The coordinates of the source  $(\tilde{r}, \tilde{\varphi}, \tilde{\theta})$  can be obtained by applying successively the changes of variables (4.5) and (4.7) to the original coordinates  $(\tilde{x}, \tilde{y}, \tilde{z})$ .

*Proof.* Let  $\mathcal{L}$  denote the Laplace operator in spherical coordinates:

$$\mathcal{L}U = \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (rU) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} U_{\varphi\varphi} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right) \right].$$

The solution for the above problem with non-zero boundary conditions is supposed to be bounded and with bounded first derivatives. To obtain it, the following identity is used:

$$\int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} (U_t + \mathcal{L}U) G(t' - t, r, \varphi, \theta) r^2 \sin \theta d\theta d\varphi dr dt' = 0, \quad (6.25)$$

which holds as  $U$  solves the PDE in equation (6.19). A series of integrations by parts are then performed with the purpose of “transferring” the partial derivatives from

function  $U$  to Green's function  $G$ :

$$\begin{aligned}
0 &= \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} U(t', T, r, \varphi, \theta) G(t', r, \varphi, \theta) r^2 \sin \theta d\theta d\varphi dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} \frac{\partial^2}{\partial r^2} (rU) G(t', r, \varphi, \theta) r \sin \theta d\theta d\varphi dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ \frac{1}{\sin \theta} U_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right] G(t', r, \varphi, \theta) d\theta d\varphi dr dt' \\
&= \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} [U(t', T, r, \varphi, \theta) G(t', r, \varphi, \theta)]_{t'=t}^{t'=T} r^2 \sin \theta d\theta d\varphi dr \\
&\quad - \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} U(t', T, r, \varphi, \theta) G_t(t', r, \varphi, \theta) r^2 \sin \theta d\theta d\varphi dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ \frac{\partial}{\partial r} (rU) G(t', r, \varphi, \theta) r \right]_{r=0}^\infty \sin \theta d\theta d\varphi dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} \frac{\partial}{\partial r} (rU) \frac{\partial}{\partial r} (rG) \sin \theta d\theta d\varphi dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ \frac{1}{\sin \theta} U_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right] G(t', r, \varphi, \theta) d\theta d\varphi dr dt'
\end{aligned}$$

Using the final condition for  $U$ :  $U(T, T, r, \varphi, \theta) = 0$  and the initial condition for Green's function given in equation (4.12), the first integral becomes:

$$\int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} [U(t', T, r, \varphi, \theta) G(t', r, \varphi, \theta)]_{t'=t}^{t'=T} r^2 \sin \theta d\theta d\varphi dr = -U(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta})$$

As  $U$  and  $U_r$  are assumed bounded and  $\lim_{r \rightarrow \infty} rG(t, r, \varphi, \theta) = 0$  and  $\lim_{r \rightarrow \infty} r^2 G(t, r, \varphi, \theta) = 0$ , the third integral is 0. For the fourth integral, another

integration by parts is applied to obtain:

$$\begin{aligned}
U(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = & - \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} U(t', T, r, \varphi, \theta) G_t(t', r, \varphi, \theta) r^2 \sin \theta d\theta d\varphi dr dt' \\
& - \frac{1}{2} \int_t^T \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ r U \frac{\partial}{\partial r} (rG) \right]_{r=0}^\infty \sin \theta d\theta d\varphi dt' \\
& + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} r U \frac{\partial^2}{\partial r^2} (rG) \sin \theta d\theta d\varphi dr dt' \\
& + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ \frac{1}{\sin \theta} U_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right] G d\theta d\varphi dr dt'.
\end{aligned}$$

The regularity conditions on Green's function and its derivatives when  $r \rightarrow \infty$  yield that  $\lim_{r \rightarrow \infty} r^2 G_r(t, r, \varphi, \theta) = 0$  and thus:

$$\begin{aligned}
U(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = & - \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} U \left[ G_t - \frac{1}{2r} \frac{\partial^2}{\partial r^2} (rG) \right] r^2 \sin \theta d\theta d\varphi dr dt' \\
& + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ \frac{1}{\sin \theta} U_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right] G d\theta d\varphi dr dt'.
\end{aligned} \tag{6.26}$$

The last integral is computed separately. The following notation is used in order to simplify the formulae:

$$I = \int_0^\varpi \int_0^{\Theta(\varphi)} \left[ \frac{1}{\sin \theta} U_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_\theta) \right] G d\theta d\varphi$$

The domain over which the integral is performed is shown in Figure 6.4 (for sample values of the correlations). Let  $\Omega$  denote the domain  $\partial\Omega$  its boundary:  $\partial\Omega = C_1 \cup C_2 \cup C_3 \cup C_4$ .

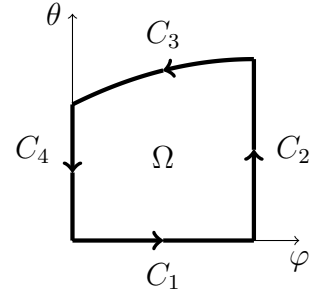


Figure 6.4: Integration domain

$$\begin{aligned}
I &= \iint_{\Omega} \left[ \frac{1}{\sin \theta} U_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_{\theta}) \right] G(t', r, \varphi, \theta) d\varphi d\theta \\
&= \iint_{\Omega} \left[ \frac{\partial}{\partial \varphi} \left( \frac{1}{\sin \theta} U_{\varphi} G \right) - \frac{1}{\sin \theta} U_{\varphi} G_{\varphi} + \frac{\partial}{\partial \theta} (\sin \theta U_{\theta} G) - \sin \theta U_{\theta} G_{\theta} \right] d\varphi d\theta
\end{aligned}$$

The Green-Riemann theorem can be applied here to obtain integrals on the border of the domain:

$$\iint_{\Omega} \left[ \frac{\partial}{\partial \varphi} \left( \frac{1}{\sin \theta} U_{\varphi} G \right) + \frac{\partial}{\partial \theta} (\sin \theta U_{\theta} G) \right] d\varphi d\theta = \int_{\partial\Omega} \frac{1}{\sin \theta} U_{\varphi} G d\theta - \int_{\partial\Omega} \sin \theta U_{\theta} G d\varphi$$

which are 0 as Green's function is null on the boundary. So the integral of interest  $I$  has the following form:

$$\begin{aligned}
I &= - \iint_{\Omega} \left[ \frac{1}{\sin \theta} U_{\varphi} G_{\varphi} + \sin \theta U_{\theta} G_{\theta} \right] d\varphi d\theta \\
&= - \iint_{\Omega} \left[ \frac{\partial}{\partial \varphi} \left( \frac{1}{\sin \theta} U G_{\varphi} \right) - \frac{U G_{\varphi\varphi}}{\sin \theta} + \frac{\partial}{\partial \theta} (U \sin \theta G_{\theta}) - U \frac{\partial}{\partial \theta} (\sin \theta G_{\theta}) \right] d\varphi d\theta \\
&= - \int_{\partial\Omega} \frac{1}{\sin \theta} U G_{\varphi} d\theta + \int_{\partial\Omega} \sin \theta U G_{\theta} d\varphi \\
&\quad + \iint_{\Omega} \left[ \frac{1}{\sin \theta} G_{\varphi\varphi} + \frac{\partial}{\partial \theta} (\sin \theta G_{\theta}) \right] U d\varphi d\theta,
\end{aligned}$$

where the Green-Riemann theorem has been applied once more. In order to compute the integrals on the boundary of the domain, this is split into the four different sections as shown in Figure 6.4. For the  $C_3$  section of the curve, the parametric representation of the boundary given in equations (4.8) and (4.9) is used.

$$\begin{aligned}
\int_{\partial\Omega} U G_{\theta} \sin \theta d\varphi &= \int_{C_1} U G_{\theta} \sin \theta d\varphi + \int_{C_3} U G_{\theta} \sin \theta d\varphi \\
&= \int_0^{\varpi} [U G_{\theta} \sin \theta]_{\theta=0} d\varphi + \int_{C_3} U G_{\theta} \sin \theta d\varphi \\
&= - \int_0^{\varpi} U(t', r, \varphi, \theta(\varphi)) G_{\theta}(t', r, \varphi, \theta(\varphi)) \sin \theta(\varphi) d\varphi
\end{aligned}$$

$$\begin{aligned}
\int_{\partial\Omega} \frac{1}{\sin\theta} U G_\varphi d\theta &= \int_{C_2} \frac{1}{\sin\theta} U G_\varphi d\theta + \int_{C_3} \frac{1}{\sin\theta} U G_\varphi d\theta + \int_{C_4} \frac{1}{\sin\theta} U G_\varphi d\theta \\
&= \int_0^{\theta(\varpi)} \frac{1}{\sin\theta} U(t', r, \varpi, \theta) G_\varphi(t', r, \varpi, \theta) d\theta \\
&\quad - \int_0^\infty \frac{1}{\sin\theta(\omega)} U(t', r, \varphi(\omega), \theta(\omega)) G_\varphi(t', r, \varphi(\omega), \theta(\omega)) \theta'(\omega) d\omega \\
&\quad - \int_0^{\theta(0)} \frac{1}{\sin\theta} U(t', r, 0, \theta) G_\varphi(t', r, 0, \theta) d\theta
\end{aligned}$$

The final expression for the integral considered  $I$  is therefore:

$$\begin{aligned}
I &= \iiint_{\Omega} \left[ \frac{1}{\sin\theta} G_{\varphi\varphi} + \frac{\partial}{\partial\theta} (\sin\theta G_\theta) \right] U d\varphi d\theta \\
&\quad - \int_0^\varpi \sin\theta(\varphi) U(t', r, \varphi, \theta(\varphi)) G_\theta(t', r, \varphi, \theta(\varphi)) d\varphi \\
&\quad + \int_0^\infty \frac{1}{\sin\Theta(\omega)} U(t', r, \varphi(\omega), \theta(\omega)) G_\varphi(t', r, \varphi(\omega), \theta(\omega)) \theta'(\omega) d\omega \\
&\quad - \int_0^{\Theta(\varpi)} \frac{1}{\sin\theta} U(t', r, \varpi, \theta) G_\varphi(t', r, \varpi, \theta) d\theta + \int_0^{\Theta(0)} \frac{1}{\sin\theta} U(t', r, 0, \theta) G_\varphi(t', r, 0, \theta) d\theta
\end{aligned}$$

As in the two dimensional case, the boundary conditions in equations (6.21) to (6.23), the initial condition for Green's function and final condition for  $U$  (6.20), along with the fact that  $G_t - \mathcal{L}G = 0$  (the Green's function solves the forward PDE), yield the final pricing formula for  $U$ :

$$\begin{aligned}
U(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) &= -\frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi \sin\Theta(\varphi) U(t', T, r, \varphi, \Theta(\varphi)) G_\theta(t' - t, r, \varphi, \Theta(\varphi)) d\varphi dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\infty \frac{U(t', T, r, \varphi(\omega), \Theta(\omega)) G_\varphi(t' - t, r, \varphi(\omega), \Theta(\omega))}{\sin\Theta(\omega)} \Theta_\omega(\omega) d\omega dr dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(\varpi)} \frac{U(t', T, r, \varpi, \theta) G_\varphi(t' - t, r, \varpi, \theta)}{\sin\theta} d\theta dr dt' \\
&\quad + \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(0)} \frac{U(t', T, r, 0, \theta) G_\varphi(t' - t, r, 0, \theta)}{\sin\theta} d\theta dr dt'. \tag{6.27}
\end{aligned}$$

We note that for one of the integrals the parametric representation of the boundary



of our domain given by formulas (4.8) and (4.9) has been used. Similarly to the two-dimensional case, using the regularity conditions on Green's functions, it can be shown that the solution  $U$  has the required regularity conditions, and since the is unique this is the desired solution.  $\square$

Proposition 6.3 gives an analytic expression for computing the bilateral credit value adjustment for a CDS in the first passage-time default framework.

**Proposition 6.3.** (*Bilateral CVA*)

*The value for the bilateral CVA in the first-passage time model is given by:*

$$V^{BCVA}(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = -\frac{1 - R_{PS}}{2} \int_t^T \int_0^\infty \int_0^{\Theta(0)} D(t, t') G_\varphi(t' - t, r, 0, \theta) V^{CDS}(t', T, \bar{\rho}_{xy} r \sin \theta)^+ \frac{d\theta}{\sin \theta} dr dt', \quad (6.28)$$

where  $G(t' - t, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta})$  denotes the transition probability density function for a three-dimensional Brownian motion in a cone with absorbing boundaries and is given by the expression in Proposition 4.1, and  $V^{CDS}$  denotes the value of a standard CDS with non-risky counterparts.

*Proof.* The expression for computing the bilateral CVA given in expectation form in equation (6.3) can be transformed to a PDE problem with terminal condition and Dirichlet boundary conditions through a Feynman-Kac type representation (see Theorem A.3 for example). The terminal value for the PDE problem is null, as there is no payment to be made at maturity. The domain for the PDE problem is the positive octant:

$$\mathcal{D} = \{(x, y, z), \quad 0 \leq x < \infty, \quad 0 \leq y < \infty, \quad 0 \leq z < \infty\}. \quad (6.29)$$

As can be observed in equation (6.3), the boundary condition is non-null only in the case where the protection seller defaults first, as it will no longer be able to honour its payments and hence the shortfall for the protection buyer will be a fraction of

the outstanding present value of the single name swap. The problem to solve thus becomes:

$$\begin{aligned}
V_t^{\text{BCVA}} + \frac{1}{2}V_{xx}^{\text{BCVA}} + \frac{1}{2}V_{yy}^{\text{BCVA}} + \frac{1}{2}V_{zz}^{\text{BCVA}} + \rho_{xy}V_{xy}^{\text{BCVA}} + \rho_{xz}V_{xz}^{\text{BCVA}} + \rho_{yz}V_{yz}^{\text{BCVA}} - \varrho V^{\text{BCVA}} &= 0, \\
V^{\text{BCVA}}(T, T, x, y, z) &= 0 \\
V^{\text{BCVA}}(t, T, 0, y, z) &= (1 - R_{PS}) V^{\text{CDS}}(t, T, y)^+ \\
V^{\text{BCVA}}(t, T, x, 0, z) &= 0, \quad V^{\text{BCVA}}(t, T, x, y, 0) = 0
\end{aligned}$$

Similarly to the calculation of the unilateral CVA, consider the following function  $U^{\text{BCVA}}(t, x, y, z) = e^{\varrho(T-t)} V^{\text{BCVA}}(t, x, y, z)$ . After applying the change of variables that eliminates the cross derivatives (see equation (4.5)), the pricing problem becomes:

$$\begin{aligned}
U_t^{\text{BCVA}} + \frac{1}{2}U_{\alpha\alpha}^{\text{BCVA}} + \frac{1}{2}U_{\beta\beta}^{\text{BCVA}} + \frac{1}{2}U_{\gamma\gamma}^{\text{BCVA}} &= 0, \\
U^{\text{BCVA}}(T, T, \alpha, \beta, \gamma) &= 0 \\
U^{\text{BCVA}}(t, T, 0, \beta, \gamma) &= e^{\varrho(T-t)} (1 - R_{PS}) V^{\text{CDS}}(t, T, \bar{\rho}_{xy}\beta)^+, \\
U^{\text{BCVA}}\left(t, T, \alpha, -\frac{\rho_{xy}}{\bar{\rho}_{xy}}\alpha, \gamma\right) &= 0 \\
U^{\text{BCVA}}\left(t, T, \alpha, \beta, -\frac{\rho_{xz}\bar{\rho}_{xy}}{\chi}\alpha + \frac{\rho_{xy}\rho_{xz} - \rho_{yz}}{\chi}\beta\right) &= 0.
\end{aligned}$$

The domain for this PDE is an infinite cone as described in detail in Section 4.1. The second change of variable is applied (see equation (4.7)) and the modified pricing problem is:

$$\begin{aligned}
U_t^{\text{BCVA}} + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (r U^{\text{BCVA}}) + \frac{1}{r^2} \left( \frac{1}{\sin^2 \theta} U_{\varphi\varphi}^{\text{BCVA}} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta U_{\theta}^{\text{BCVA}}) \right) \right] &= 0, \\
U^{\text{BCVA}}(T, T, r, \varphi, \theta) &= 0 \\
U^{\text{BCVA}}(t, T, r, 0, \theta) &= e^{\varrho(T-t)} (1 - R_{PS}) V_{\text{CDS}}(t, T, \bar{\rho}_{xy} r \sin \theta)^+ \\
U^{\text{BCVA}}(t, T, r, \varpi, \theta) &= 0 \\
U^{\text{BCVA}}(t, T, r, \varphi, 0) &= 0 \\
U^{\text{BCVA}}(T, T, r, \varphi, \Theta(\varphi)) &= 0
\end{aligned} \tag{6.30}$$

The domain for this transformed problem is the one given in equation (4.14). The result in Lemma 6.2 below can now be used, with the boundary conditions particularised to the bilateral CVA case, to obtain the solution:

$$U^{\text{BCVA}}(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(0)} \frac{U^{\text{BCVA}}(t', T, r, 0, \theta) G_\varphi(t' - t, r, 0, \theta)}{\sin \theta} d\theta dr dt'. \quad (6.31)$$

Using the boundary condition in equation (6.31), and then transforming back to the initial function gives the expression in equation (6.28).  $\square$

Similarly to the bilateral CVA, the bilateral DVA can be computed using the result in Lemma 6.2.

**Proposition 6.4.** (*Bilateral DVA*)

*The value for the bilateral DVA in the first-passage time model is given by:*

$$\begin{aligned} V^{BDVA}(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = & \\ & - \frac{1}{2} \int_t^T \int_0^\infty \int_0^\varpi D(t, t') V^{BDVA}(t', T, r, \varphi, \Theta(\varphi)) G_\theta(t' - t, r, \varphi, \Theta(\varphi)) \sin \Theta(\varphi) d\varphi dr dt' \\ & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^\infty D(t, t') \frac{V^{BDVA}(t', T, r, \varphi(\omega), \Theta(\omega)) G_\varphi(t' - t, r, \varphi(\omega), \Theta(\omega))}{\sin \Theta(\omega)} \Theta_\omega(\omega) d\omega dr dt', \end{aligned} \quad (6.32)$$

where

$$V^{BDVA}(t, T, r, \varphi, \Theta(\varphi)) = (1 - R_{PB}) V^{CDS}(t, T, (\rho_{xy} \sin \varphi + \bar{\rho}_{xy} \cos \varphi) r \sin \Theta(\varphi))^- ,$$

and  $G(t' - t, r', \varphi', \theta' | \tilde{r}, \tilde{\varphi}, \tilde{\theta})$  denotes the Green's function for the three-dimensional operator  $\mathcal{O}_{3D}^*$  in a cone with absorbing boundaries and is given by the expression in Proposition 4.1, and  $V^{CDS}$  denotes the value of a standard CDS with risk-free counterparts.

These original formulas provide a new way of consistently computing the bilateral

CVA and DVA. Figures 6.5 and 6.6 show the impact of the different correlations on the bilateral CVA and DVA respectively. A concrete example is considered, where all three issuers are equally risky ( $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  are equal and chosen such that the individual five year default probability is  $\approx 50\%$ ). The size of the adjustments is shown as percentage of the common value of the protection and premium legs (these are equal for a par CDS). The graphs show the adjustments when one of the pairwise correlations is set to 0% and the other two are varied.

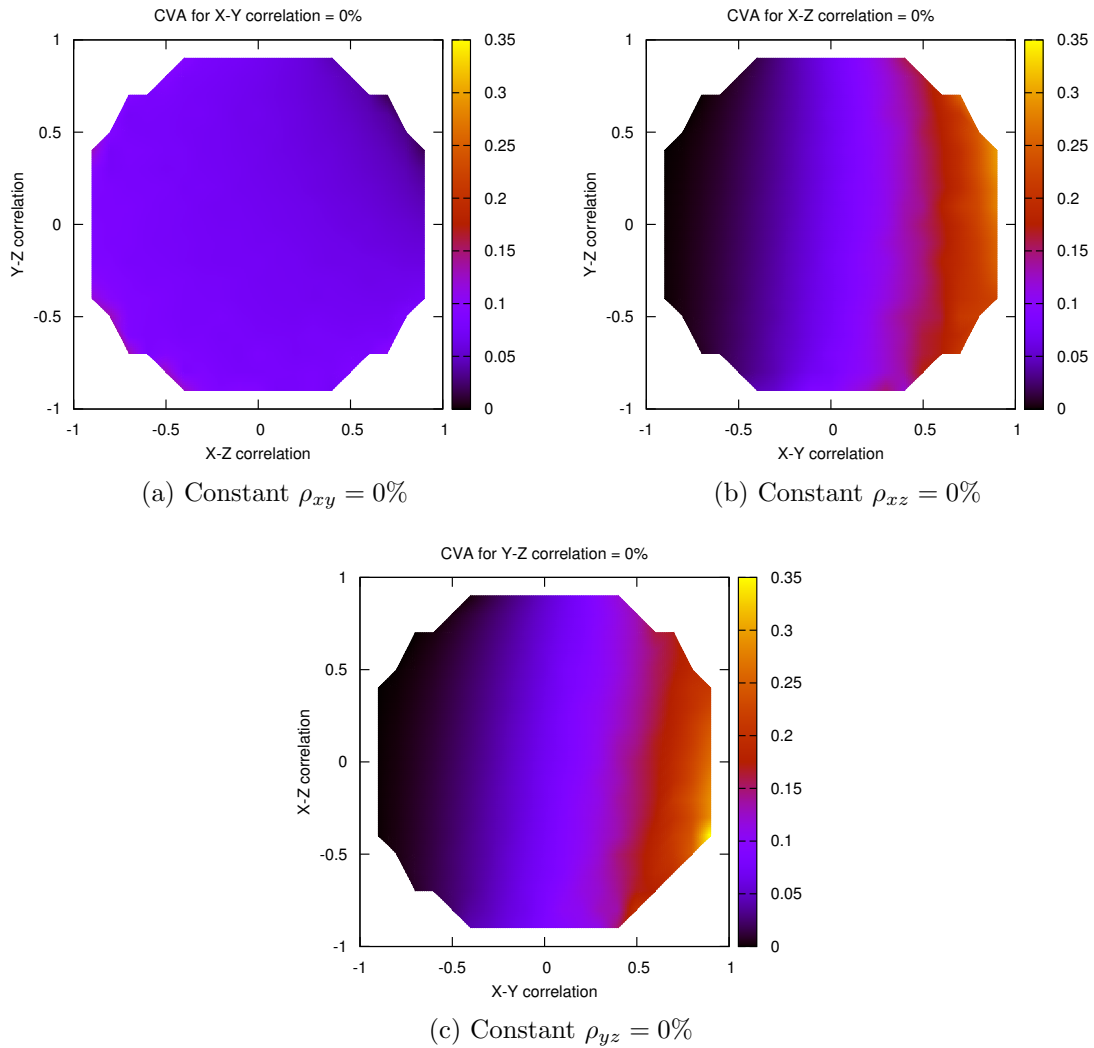


Figure 6.5: Impact of the correlations on the size of the bilateral credit value adjustments.

For the bilateral CVA the protection seller - reference name correlation (X-Y correlation) has the biggest impact. This can be observed first in Figure 6.5a, where the size of the adjustment stays roughly constant when  $\rho_{xy}$  is constant. Furthermore, when  $\rho_{xy}$  is varied in Figures 6.5b and 6.5c, the size of the adjustment varies from roughly 0% to close to 30% of the common absolute value of the premium/protection leg. This is quite significant, and the adjustment is highest when the correlation between the reference name and the protection seller is positive and high.

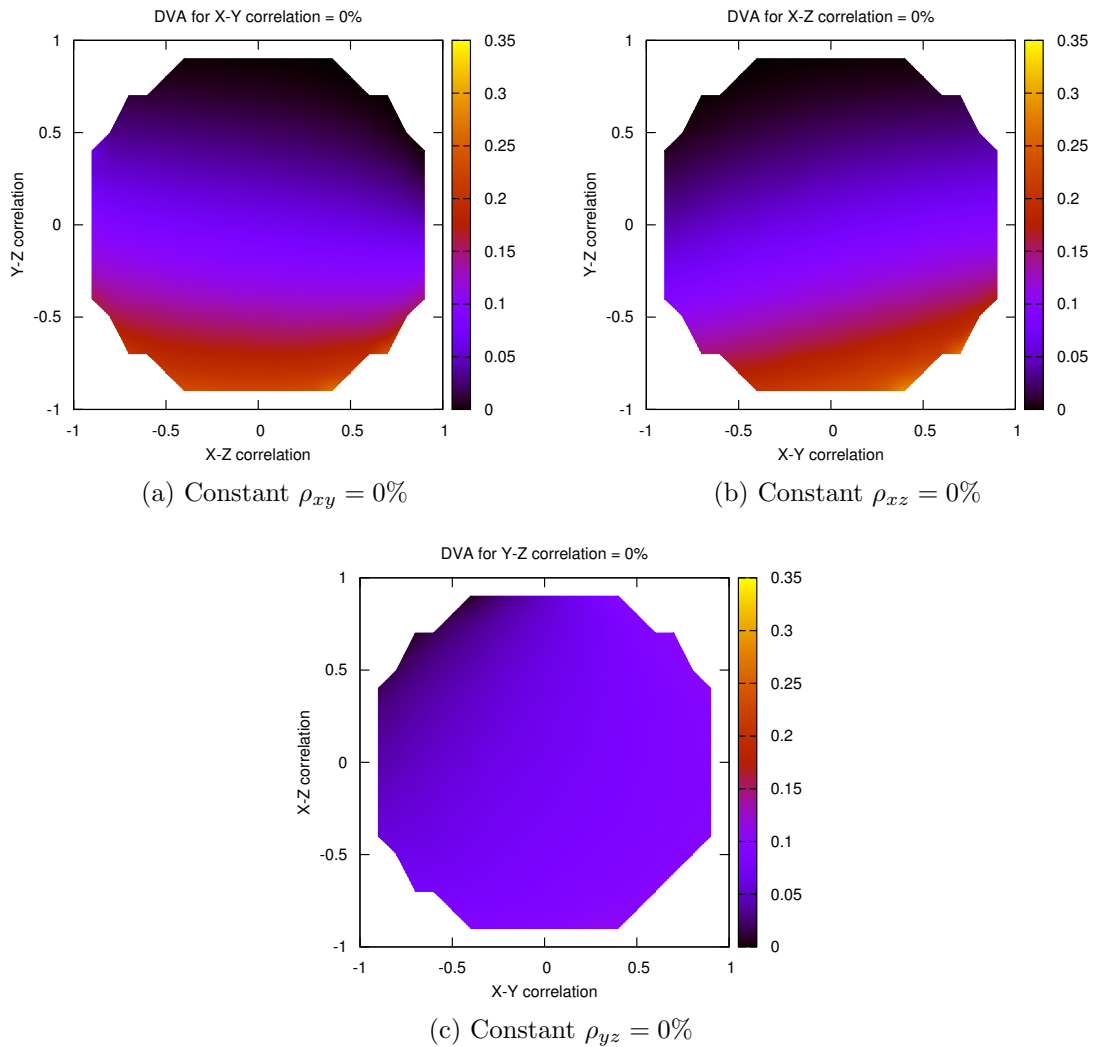


Figure 6.6: Impact of the correlations on the size of the bilateral debit value adjustments.

Figure 6.6 shows the impact of the different correlations on the bilateral debit value adjustment. Similarly to the BCVA, there is one correlation that has a significant impact on the size of the adjustment, namely the Y-Z correlation (reference name to protection buyer correlation). This makes sense intuitively, as the debit value adjustment is a reflection of the protection buyer's riskiness. The higher the anti-correlation between the protection buyer and the reference name is, the larger the size of the adjustment is.

## 6.4 Unilateral versus bilateral adjustments

Both unilateral and bilateral value adjustments for a standard CDS have been computed and discussed in this chapter. Given that for the bilateral adjustments only semi-analytical formulas are available, and as such their computation is more time consuming, it is interesting to investigate if the sum of the unilateral adjustments is a good approximation for the full bilateral calculation. This section therefore looks at whether  $V^{UCVA} + V^{UDVA} \approx V^{BCVA} + V^{BDVA}$ . The approximation considers the counterparties risky one at a time, and therefore ignores the first-to-default time between the protection seller and the protection buyer, and hence their dependence.

This approximation is often used in the industry (see for example Picoult [2005]) and its accuracy has been investigated previously in Brigo et al. [2012a] for a couple of simple products: a zero coupon bond, where the exposure is unidirectional, and an equity forward contract where exposure can go both ways. In Brigo et al. [2012a], the authors found that the difference can reach significant values (5 – 7% of the notional) even when wrong-way risk is not present. The case of a standard CDS has not been investigated previously.

The framework discussed so far is a convenient setup for this analysis, as both unilateral and bilateral adjustments can be computed efficiently, without needing a full Monte-Carlo simulation. For this analysis, three equally risky issuers are considered ( $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$  are equal and chosen such that the five year default probability is  $\approx 50\%$ ) and the unilateral and bilateral credit and debit value adjustments to a CDS are computed. The pairwise correlation values are varied between  $-90\%$  and  $+90\%$

to see what the impact is on the difference between the approximation and the fully bilateral adjustments. The case of the CDS is particularly interesting as it shows an example where wrong-way risk is present.

Figures 6.7, 6.8 and 6.9 show this difference ( $V^{BCVA} + V^{BDVA} - V^{UCVA} - V^{UDVA}$ ) as a percentage of the common value of the default and premium legs of the standard CDS (a par CDS has been considered). As all three pairwise correlations are varied, this yields a three-dimensional data set, which is subsequently split into a series of two-dimensional ones for the purpose of this analysis, by fixing one correlation at a time.

Figure 6.7 shows the case where the X-Y correlation is fixed at a given value and the other two pairwise correlations are varied roughly between  $-1$  and  $1$ , while Figures 6.8 and 6.9 show the cases where the X-Z and Y-Z correlations are fixed respectively. In the case of the CDS, the difference can go up to  $\pm 15\%$  of the default leg value as can be observed in the graphs.

The X-Z correlation, which is the correlation between the protection seller and the protection buyer, is an input missing in the unilateral approximation, as it is precisely this dependence that is being ignored. Its effect is non-negligible, especially at high positive values. For  $\rho_{xz} = 90\%$ , the approximation is significantly different with both positive and negative errors depending on the values of the other two correlations (see for example 6.8i).

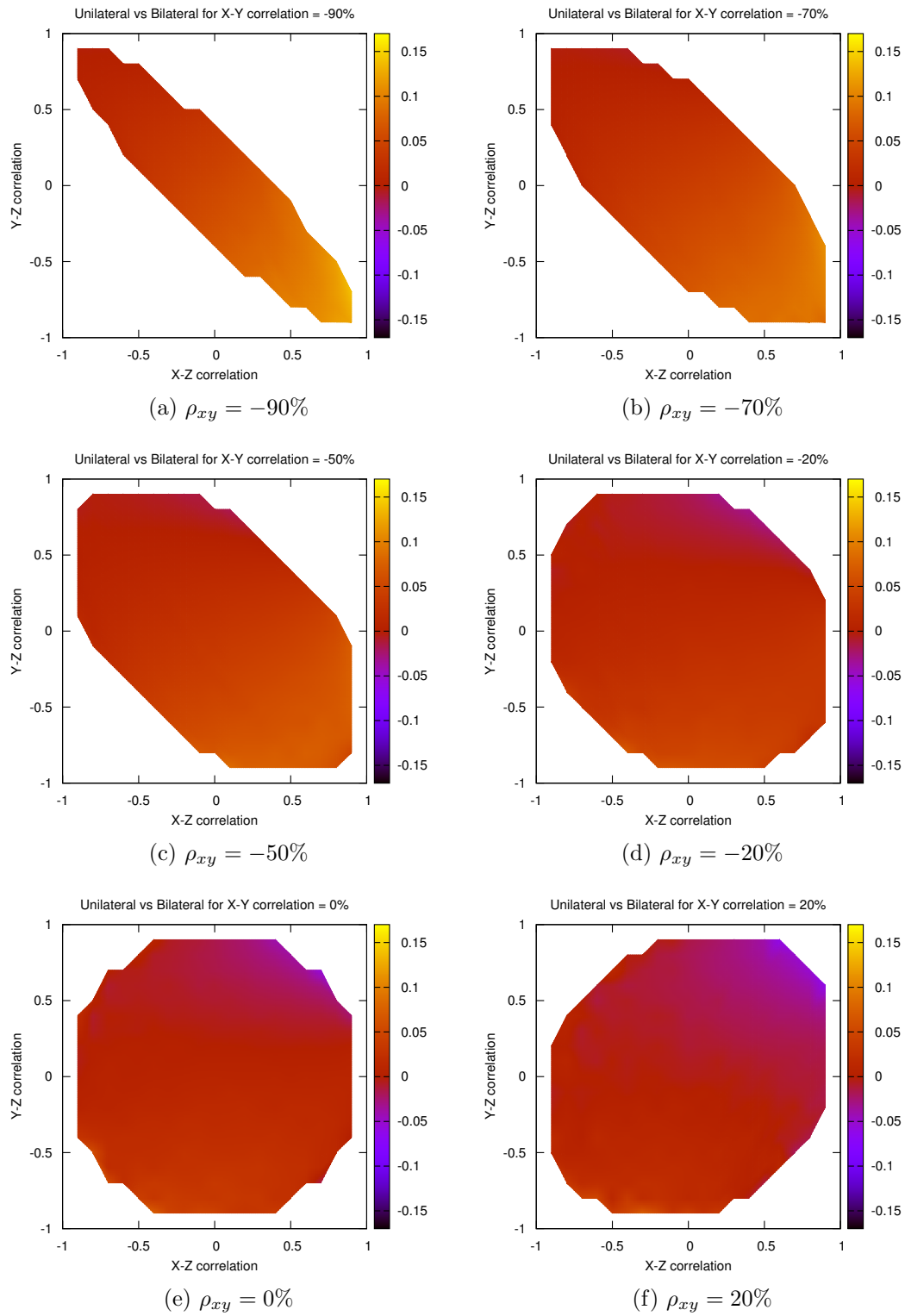
For high positive values for the correlations between the reference name and both the protection buyer and protection seller ( $\rho_{xy}$  and  $\rho_{yz}$  close to  $100\%$ ), the credit value adjustment is high whereas the debit value adjustment is low (see for example Figures 6.5 and 6.6). The significant difference between the approximation and the fully bilateral computation comes therefore from the credit value adjustment. The comparison of equations (6.1) and (6.3) shows that the main difference comes from the indicator functions ( $\mathbb{1}_{\tau^{PS} < \min\{\tau^{PB}, \tau^{RN}, T\}}$  versus  $\mathbb{1}_{\tau^{PS} < \min\{\tau^{RN}, T\}}$  where the first to default between the protection seller and the protection buyer is ignored for the unilateral case). The sum of the bilateral adjustments is therefore lower than that of the unilateral ones, which gives rise to significant negative errors.

For the case where the reference name is highly anti-correlated to both the pro-

tection buyer and the protection seller ( $\rho_{xy}$  and  $\rho_{yz}$  close to  $-100\%$ ), the credit value adjustment is low while the debit value adjustment is high. Similarly to the case above, the errors on the approximation come mainly from the DVA calculation and lead to significant positive errors (see Figure 6.8i).

As the protection seller and protection buyer are often financial institutions, which as the recent crisis has shown are highly correlated, the case with high positive values for the X-Z correlation is very likely. The significant possible errors for the approximation in this case are therefore important and show that consideration of fully bilateral adjustments is important.





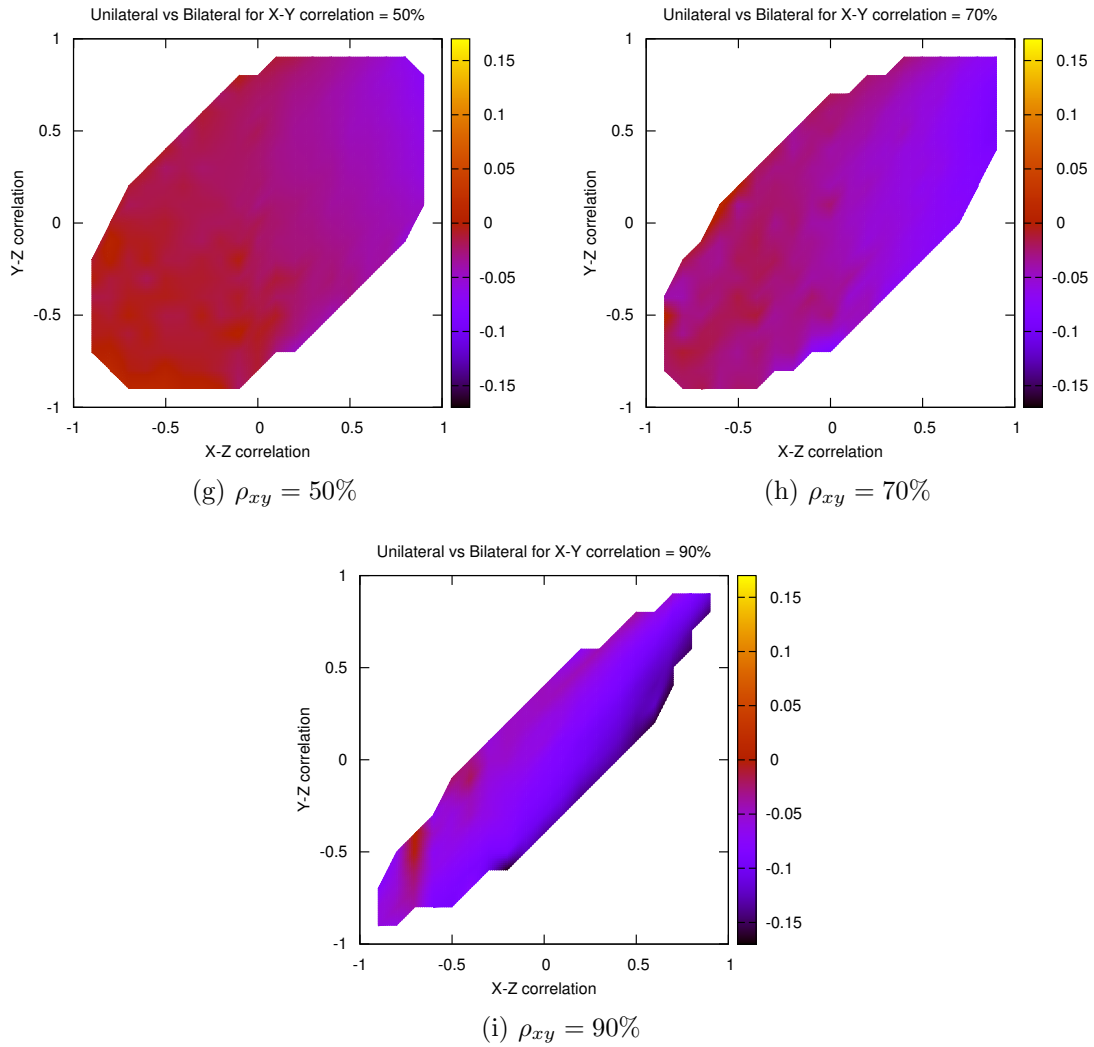
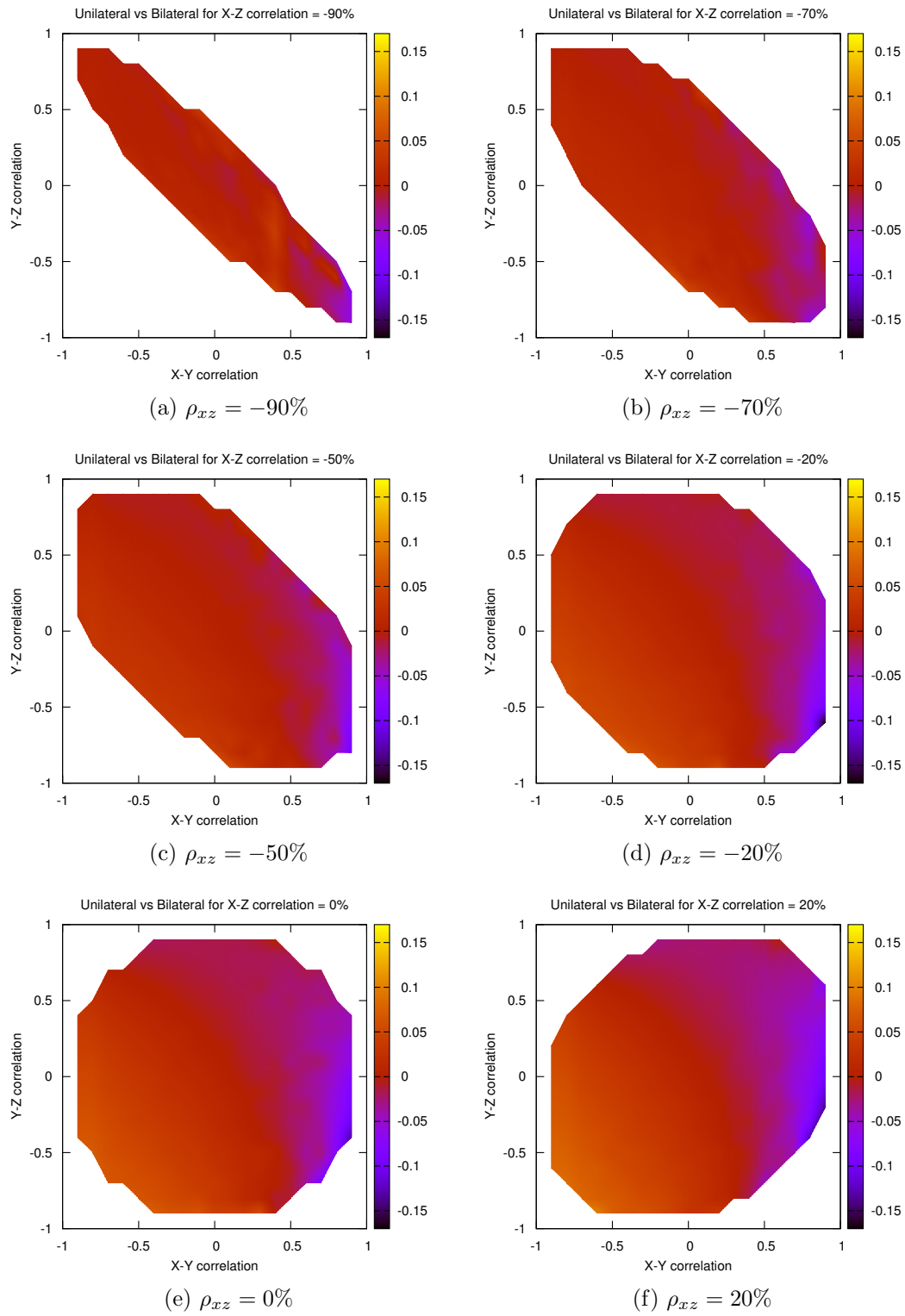


Figure 6.7: Difference between unilateral and bilateral adjustments as a function of the pairwise correlations(  $V^{BCVA} + V^{BDVA} - V^{UCVA} - V^{UDVA}$ ). Each graph shows what the difference is when the X-Y correlation is fixed at a given level and the other two are varied.



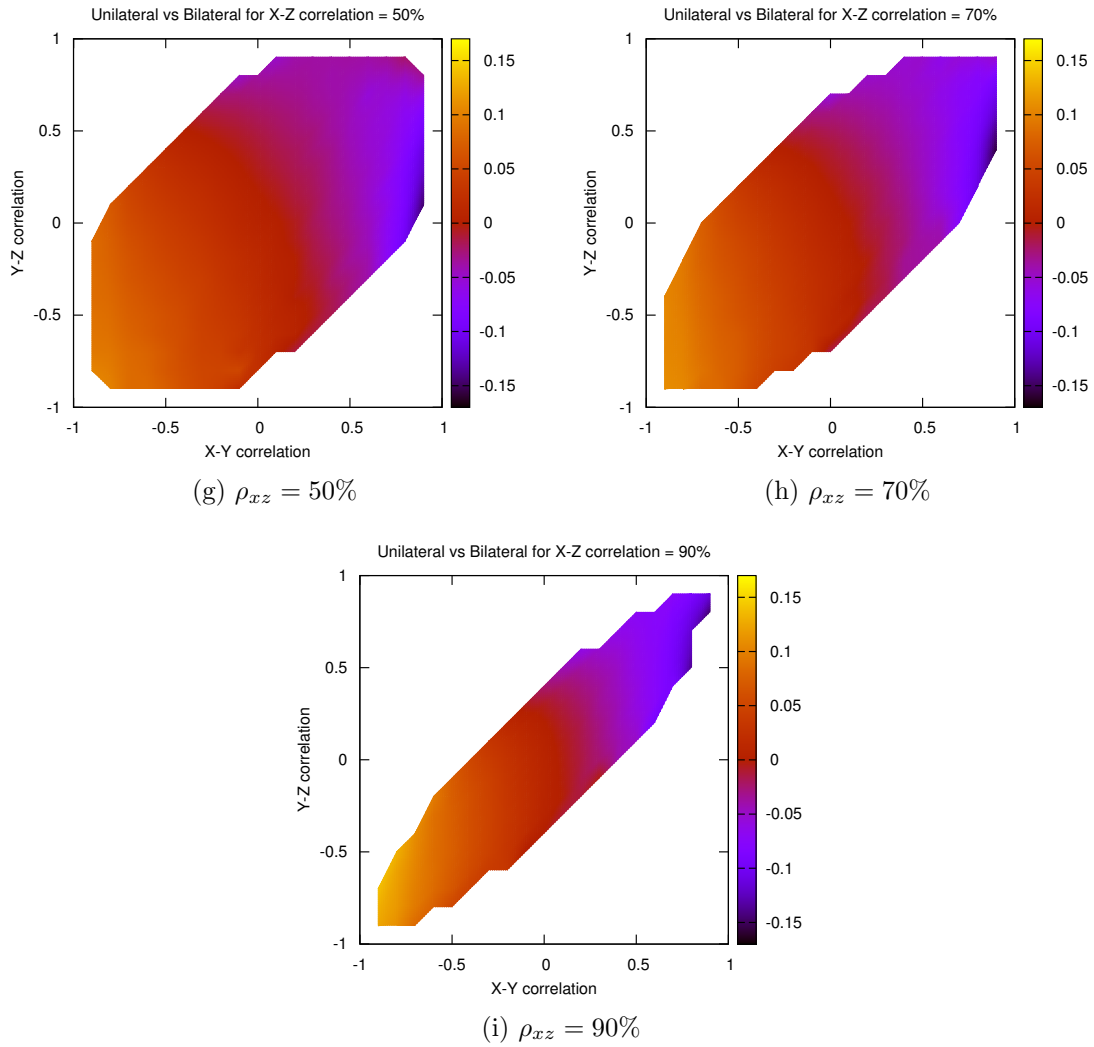
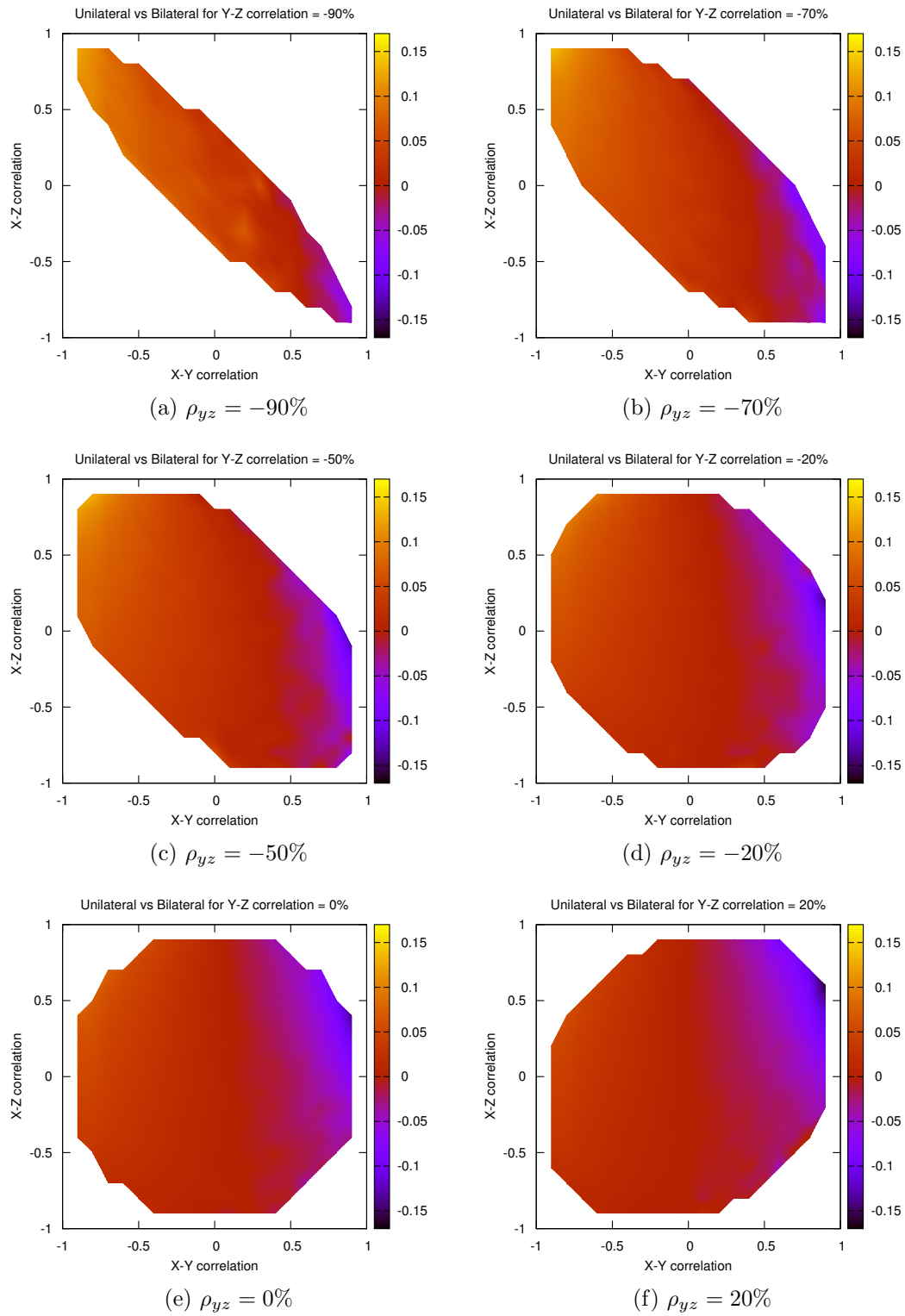


Figure 6.8: Difference between unilateral and bilateral adjustments as a function of the pairwise correlations ( $V^{BCVA} + V^{BDVA} - V^{UCVA} - V^{UDVA}$ ). Each graph shows what the difference is when the X-Z correlation is fixed at a given level and the other two are varied.



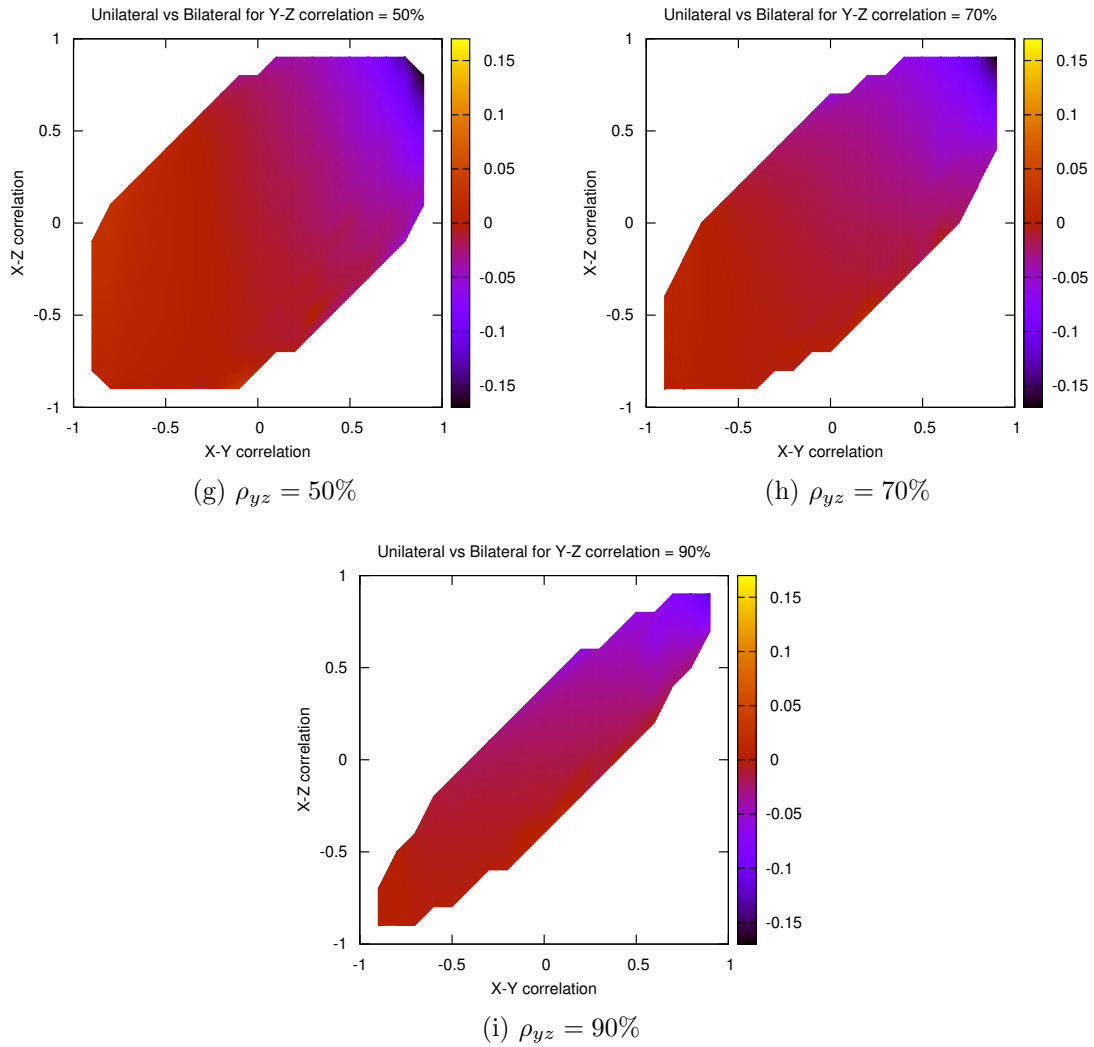


Figure 6.9: Difference between unilateral and bilateral adjustments as a function of the pairwise correlations ( $V^{BCVA} + V^{BDVA} - V^{UCVA} - V^{UDVA}$ ). Each graph shows what the difference is when the Y-Z correlation is fixed at a given level and the other two are varied.

## 6.5 Application to market data

This section shows an example using real market data of the impact of the CVA and DVA adjustments on the value of a standard CDS. The breakeven coupon obtained for a standard CDS with risk-free counterparties is compared to the ones obtained when either the protection buyer or the protection seller are risky (using the two-dimensional formulation and results), as well as when both are risky (using the three-dimensional formulation and results). When using the 2D formulation and considering that either the protection seller or the protection buyer are risky, the two parties will not agree on the breakeven coupon of the CDS. This problem disappears when using the full three dimensional framework, where both are risky, and the problem becomes symmetrical.

### 6.5.1 Input parameters and calibration to real market data

Consider three issuers: **AIG** as a protection seller, **GE** as the reference name of the CDS and **UNICREDIT** the protection buyer. The protection seller and protection buyer entities have been chosen to be relatively risky such that the effect of the CVA and DVA adjustments on the breakeven coupon are non negligible. The inputs to the model are calibrated to market data from the 15th of December 2011 (see table 6.1).

| Inputs              | AIG    | GE    | UNICREDIT |
|---------------------|--------|-------|-----------|
| 5Y CDS Spread (bps) | 713    | 253   | 600       |
| Recovery            | 50%    | 40%   | 40%       |
| Stock Price         | 23.22  | 16.79 | 0.7       |
| Debt/share          | 1270.3 | 118.8 | 13.7      |

Table 6.1: Snapshot of market data used for the model calibration (15th December 2011).

The market data is then used to calibrate the inputs to our model (see table 6.2). The initial value is a measure of the relative distance to default. This has been obtained using the share price on that date, together with the outstanding number of shares and total liabilities for that company (see Lipton and Sepp [2009] for a detailed description of the calibration). The volatility  $\sigma$  has been calibrated such

that the 5Y single name CDS spread is matched to the market spread (the 5Y point has been chosen as it is usually the most liquidly traded contract). Since the 5Y CDS spreads used here are “cleared” CDS spreads (dealt through a clearing house and hence not subject to counterparty risk), the calibration of the volatility is done by pricing a standard CDS with no counterparty risk and matching the price obtained using the market spread. The initial value and the volatility are then used to obtain the starting points for the three processes associated with the considered issuers,  $\tilde{x}$ ,  $\tilde{y}$  and  $\tilde{z}$ .

| Inputs        | AIG    | GE     | UNICREDIT |
|---------------|--------|--------|-----------|
| Initial value | 0.0359 | 0.3035 | 0.1199    |
| $\sigma$      | 2.44%  | 10.45% | 6.3%      |
| Source points | 1.47   | 2.9    | 1.9       |

Table 6.2: Input parameters calibrated to market data (15th December 2011).

Having calibrated the required single-name data, an analysis can be performed on how much the “cleared” spread for the reference name would change if the trade were an inter-dealer trade and counterparty risk had to be taken into account. This effect is investigated hereafter.

For the two and three dimensional cases, the correlations between the different issuers are also required as inputs to the model. These can be calibrated from the prices of first to default swap contracts if such contracts including the relevant names are available on the market. Alternatively, these can be proxied by considering the sector/region each issuer belongs to and then using the historically estimated correlations.<sup>1</sup>

In this section however, the aim is to show the impact of CVA and DVA on the breakeven spread of a CDS, and hence different sets of pairwise correlations are used for the same group of issuers in order to illustrate a variety of cases.

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<sup>1</sup>In regulatory capital charge models, one needs to estimate sector-to-sector and region-to-region correlations. This can be done for example by constructing proxy-portfolios for each sector using all the issuers that belong to it and averaging their CDS spreads and then computing the correlations of the increments of the time series thus obtained.



### 6.5.2 Impact of CVA/DVA adjustments

Figure 6.10 shows the CDS breakeven coupon in different cases. The spreads are hyper-exponentially flat at 0, which is a known problem of models without jumps. However, for longer maturities market prices can be well matched which allows for an analysis of the effect of considering the protection seller or the protection buyer or both as being risky. If the protection seller is risky, the probability of it non paying the full amount due in the case of the default of the reference name is non zero, and hence the protection buyer pays a lower coupon as it takes on that risk as well. If the protection buyer is risky, the breakeven coupon moves in the opposite direction and the two counterparties no longer agree on the coupon. The three dimensional case, where both are considered risky, solves this problem as it becomes symmetrical.

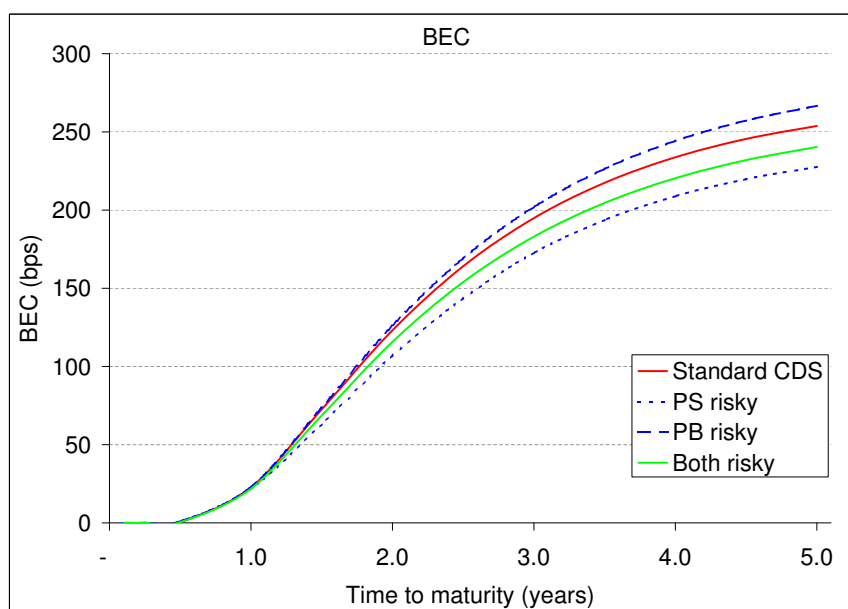
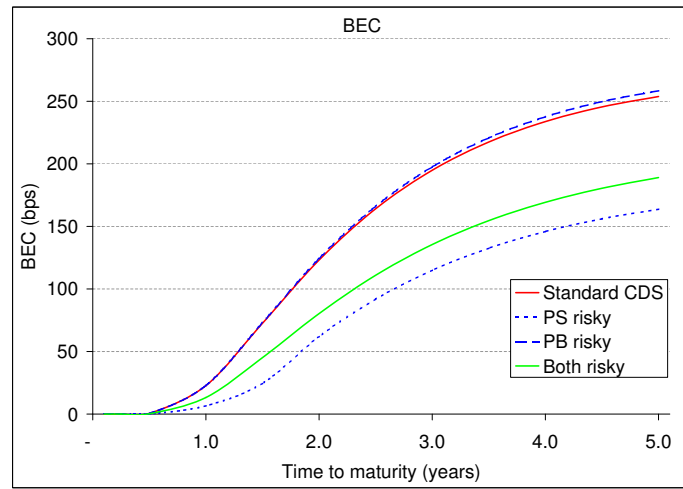
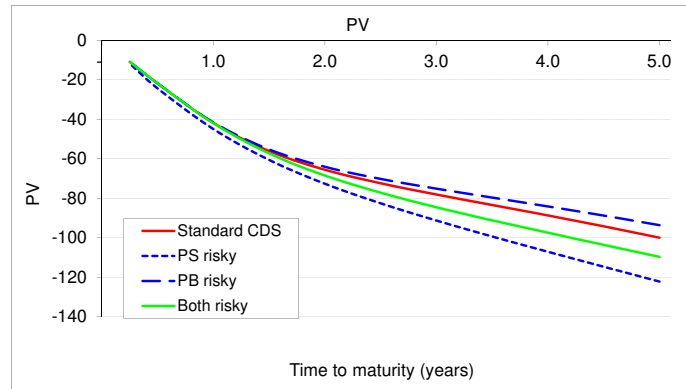


Figure 6.10: Impact of counterparty adjustments on the break-even coupon (BEC) of a CDS:  $\rho_{xy} = 0\%$ ,  $\rho_{xz} = 0\%$ ,  $\rho_{yz} = 0\%$ .

Figure 6.11 shows the case where the protection seller is highly correlated to the reference name. In the case of a default of the reference name, the protection seller is likely to default as well, and hence the shortfall between the contractual payout and what will actually get paid can be significant. The break-even coupon will get adjusted accordingly and will be lower than on a standard fully collateralised CDS as the expectation of the payout is lower from the protection buyer's point of view. The figure also shows the impact on the PV, where the standard CDS PV has been normalised to  $-100$  for the five year maturity (the trade is seen from the protection buyer's point of view).



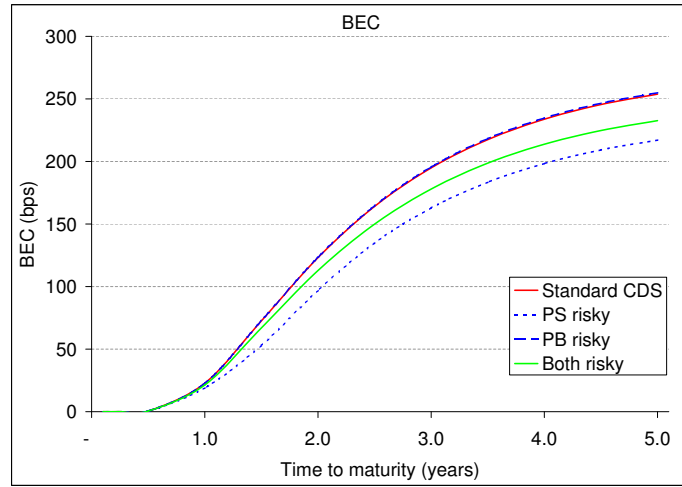
(a) Break-even coupon



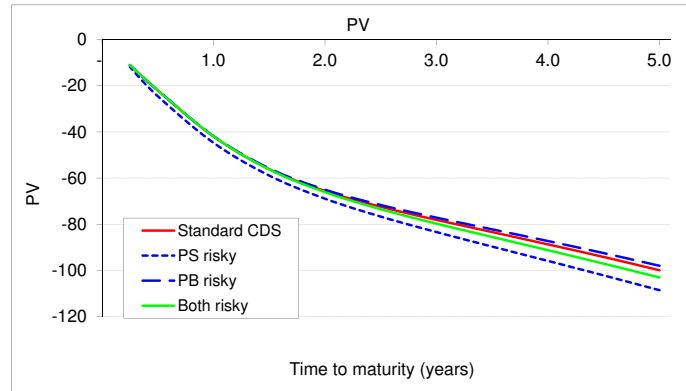
(b) PV impact

Figure 6.11: Impact of counterparty adjustments on the break-even coupon (BEC) and PV of a CDS:  $\rho_{xy} = 80\%$ ,  $\rho_{xz} = 20\%$ ,  $\rho_{yz} = 50\%$ .

Figure 6.12 shows the case where the protection buyer is highly correlated to the reference name. Since on the default of the reference name the coupon payments stop regardless of what happens to the protection buyer, the impact of considering the protection buyer as risky in this case is not significant.



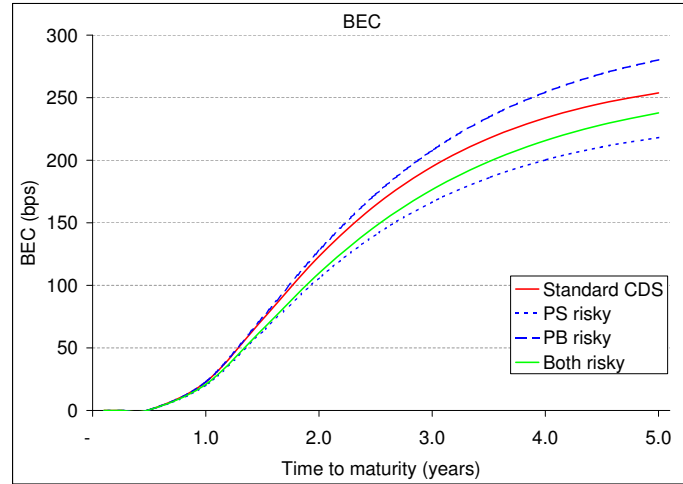
(a) Break-even coupon



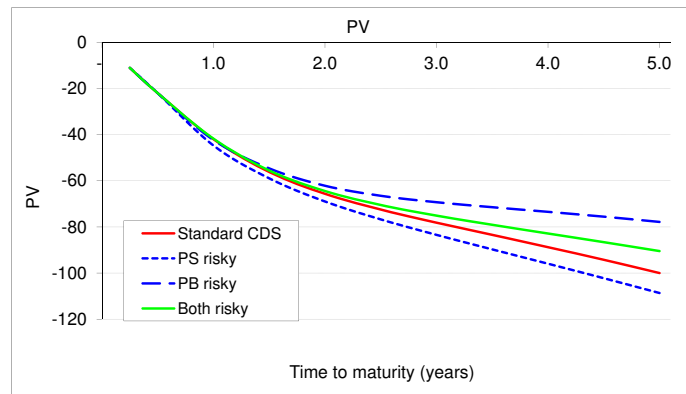
(b) PV impact

Figure 6.12: Impact of counterparty adjustments on the break-even coupon (BEC) and PV of a CDS:  $\rho_{xy} = 20\%$ ,  $\rho_{xz} = 30\%$ ,  $\rho_{yz} = 80\%$ .

Figure 6.13 shows the case where the protection buyer is highly anti-correlated to the reference name. This is intuitively the case where the DVA is largest as it is in the cases where the reference name does not default that the protection buyer is more likely to default on its coupon paying obligation. This leaves the protection seller with a potential shortfall.



(a) Break-even coupon



(b) PV impact

Figure 6.13: Impact of counterparty adjustments on the break-even coupon (BEC) and PV of a CDS:  $\rho_{xy} = 20\%$ ,  $\rho_{xz} = -10\%$ ,  $\rho_{yz} = -60\%$ .

## Chapter 7

# Partially Collateralised CDS

Credit linked notes or CDSs with risky collateral are some examples of products that have gained in popularity after the credit crisis and this chapter focuses on the valuation of such products. While the problem bares similarities with the valuation of the credit and debit value adjustments discussed in chapter 6, it is interesting to analyse the impact of impaired collateral and compare it to the CVA adjustments.

Section 7.1 describes briefly the problem, while Section 7.2 gives the arbitrage-free valuation of the possible shortfall due to the effect of impaired collateral.

The mathematical results in chapters 3 and 4 are then applied in Section 7.3 to obtain analytical and semi-analytical solutions for valuing the potential shortfall in the first-passage time framework considered here. Finally, Section 7.4 contains examples and analyses the results.

## 7.1 Problem description

A *Credit Linked Note* (CLN) is a note paying an enhanced coupon to investors for bearing the credit risk of a reference entity. The buyer of the note (referred to hereafter as the Investor) funds the credit protection that the issuer of the note may thus sell to a third party. In exchange the note pays a higher-than-normal yield. CLNs are primarily issued by special purpose vehicles (SPVs). These transactions are tailored to satisfy specific needs of the investor and hence are not subject to clearing, thus making the question of counterparty risk particularly relevant. To mitigate some of this risk, trades are typically collateralised by collateral held within the SPV.

An example of a CLN, with the participants and the different cashflows involved is shown in Figure 7.1. The investor buys the CLN for an upfront fee that is paid to the issuer of the note (the SPV). Up to the maturity of the note or the default of the reference name, the investor receives fixed coupon payments. If the reference name does not default, at maturity the investor receives back the full notional amount.

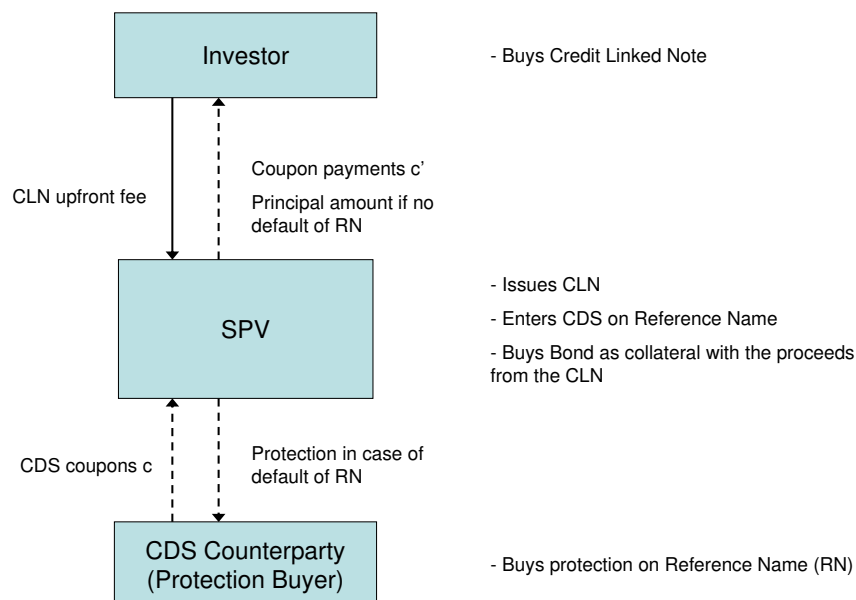


Figure 7.1: Credit Linked Note: participants, cashflows

The SPV sells the protection bought as part of the CLN by entering into a Credit Default Swap on the reference name with a third party. With the proceeds from the note, the SPV buys a bond that is held within the SPV as collateral for the CDS. While this is designed to minimise counterparty exposure, there are several scenarios in which losses can still arise. Depending on the correlations between the issuer of the bond and the reference name of the CDS, the collateral can be significantly impaired in case of default of the reference name, thus leading to “wrong-way” risk scenarios and losses for the protection buyer.

The correct valuation of the CDS involved in this CLN trade needs to take into account the risk of default of the reference name, the issuer of the bond as well as the protection buyer and carefully analyse the cashflows in each scenario. In Shelton [2010], the author looks into the legal aspects of deciding the cashflows for this type of structure, as well as proposes a way of pricing the potential shortfall in an intensity-based model. The approach proposed combines and extends two concepts: the survival measure (Schönbucher [August,2003], Collin-Dufresne et al. [2004]) and the recipe of combining individual single-name dynamics with a given copula function for default times (Schönbucher and Schubert [2001]).

In this chapter a solution for pricing the potential shortfall due to the riskiness of the collateral in the first passage-time framework is proposed. Both the case where the CDS protection buyer is considered non-risky and when it is risky are discussed and analysed.

## 7.2 Arbitrage-free valuation of potential shortfall

Consider the case where the CDS is collateralised by a risky zero-coupon bond. Let  $\tau^B$  denote the default time of the issuer of the bond, and as previously,  $\tau^{RN}$  that of the reference name of the CDS;  $R_B$  and  $R_{RN}$  denote their recoveries respectively.

### 7.2.1 Risk-free protection buyer

The protection buyer is considered risk-free for now, and  $T$  denotes the maturity of both the CDS and the bond.

- $\tau^B < \min\{\tau^{RN}, T\}$ : if the value of the CDS at the time of default of the bond issuer is negative to the protection buyer nothing happens, if the value is positive, the most that can be recovered is the recovery  $R_B$  of the bond:  $\min\{V_{\tau^B}^{CDS}, R_B\}$
- $\tau^{RN} < \min\{\tau^B, T\}$ : the protection buyer can get at most the value of the collateral at that time:  $\min\{1 - R_{RN}, V_{\tau^{RN}}^{Bond}\}$ .

Let  $\tilde{V}_t$  denote the value of the CDS collateralised by a bond:

$$\begin{aligned}\tilde{V}_t &= \mathbb{E} \left[ \mathbb{1}_{T < \min\{\tau^{RN}, \tau^B\}} CF(t, T) \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} [CF(t, \tau^B) + D(t, \tau^B) \min\{V_{\tau^B}^{CDS}, R_B\}] \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B\}} [CF(t, \tau^{RN}) + D(t, \tau^{RN}) \min\{1 - R_{RN}, V_{\tau^{RN}}^{Bond}\}] \middle| \mathcal{F}_t \right],\end{aligned}$$

The cashflows in the case where the collateral default first can be rewritten in the following way:

$$\begin{aligned}\min\{V_{\tau^B}^{CDS}, R_B\} &= V_{\tau^B}^{CDS} + \min\{0, R_B - V_{\tau^B}^{CDS}\} \\ &= V_{\tau^B}^{CDS} - (V_{\tau^B}^{CDS} - R_B)^+, \end{aligned} \tag{7.1}$$

and similarly for the case where the reference name of the CDS default first:

$$\min\{1 - R_{RN}, V_{\tau^{RN}}^{Bond}\} = 1 - R_{RN} - (1 - R_{RN} - V_{\tau^{RN}}^{Bond})^+. \tag{7.2}$$

This yields the following expression for the value of the partially collateralised CDS:

$$\begin{aligned}\tilde{V}_t &= \mathbb{E} \left[ \mathbb{1}_{T < \min\{\tau^{RN}, \tau^B\}} CF(t, T) \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} [CF(t, \tau^B) + D(t, \tau^B) V_{\tau^B}^{CDS}] \middle| \mathcal{F}_t \right] \\ &- \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} D(t, \tau^B) (V_{\tau^B}^{CDS} - R_B)^+ \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B\}} [CF(t, \tau^{RN}) + D(t, \tau^{RN}) (1 - R_{RN})] \middle| \mathcal{F}_t \right] \\ &- \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B\}} D(t, \tau^{RN}) (1 - R_{RN} - V_{\tau^{RN}}^{Bond})^+ \middle| \mathcal{F}_t \right]. \end{aligned} \tag{7.3}$$



The value of the CDS at the default time of the bond issuer can be written as the expected value of all remaining cashflows up to maturity:  $V_{\tau^B}^{\text{CDS}} = \mathbb{E} [CF(\tau^B, T) | \mathcal{F}_{\tau^B}]$ . Since the discount factor  $D(t, \tau^B)$ , the cashflows up to  $\tau^B$  and  $\mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}}$  are  $\mathcal{F}_{\tau^B}$ -measurable:

$$\mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} [CF(t, \tau^B) + D(t, \tau^B) V_{\tau^B}^{\text{CDS}}] = \mathbb{E} [\mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} CF(t, T) | \mathcal{F}_{\tau^B}].$$

Since  $\tau^B > t$ , the tower law for expectations  $\mathbb{E} [\mathbb{E} [\cdot | \mathcal{F}_{\tau^B}] | \mathcal{F}_t] = \mathbb{E} [\cdot | \mathcal{F}_t]$  can be used to obtain for the second term on the right hand side of equation (7.3) the following expression:  $\mathbb{E} [\mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} CF(t, T) | \mathcal{F}_t]$ .

When the reference name of the CDS defaults first, there are no cashflows after the protection payment has been made, so  $CF(t, \tau^{RN}) + D(t, \tau^{RN}) (1 - R_{RN}) = CF(t, T)$ . Terms one, two and four on the right hand side of equation (7.3) thus yield the value of the standard fully collateralised CDS. For the partially collateralised one, the value is given by:

$$\tilde{V}_t = V_t - V_t^{SH},$$

where  $V_t^{SH}$  denotes the present value of the potential shortfall due to the riskiness of the collateral and is given by:

$$\begin{aligned} V_t^{SH} = & \mathbb{E} [\mathbb{1}_{\tau^B < \min\{T, \tau^{RN}\}} D(t, \tau^B) (V_{\tau^B}^{\text{CDS}} - R_B)^+ | \mathcal{F}_t] \\ & + \mathbb{E} [\mathbb{1}_{\tau^{RN} < \min\{T, \tau^B\}} D(t, \tau^{RN}) (1 - R_{RN} - V_{\tau^{RN}}^{\text{Bond}})^+ | \mathcal{F}_t]. \end{aligned} \quad (7.4)$$

Compared to the CVA calculation of a standard CDS where the protection seller is risky, the shortfall in this case has two potential sources:

- when the bond issuer default first, the maximum that can be recovered is limited by the recovery of the bond; this is a similar contribution to the unilateral CVA for a standard CDS
- a shortfall can arise even when the issuer of the bond is still alive, but the value of the bond has fallen below the protection payment due when the reference name defaults

## 7.2.2 Risky protection buyer

Consider now a risky protection buyer. This case will bear similarities with the valuation of the bilateral adjustments discussed in chapter 6.

- $\tau^B < \min\{\tau^{RN}, \tau^{PB}, T\}$ : if the value of the CDS at the time of default of the bond issuer is negative to the protection buyer nothing happens, if the value is positive, the most that can be recovered is the recovery  $R_B$  of the bond:  $\min\{V_{\tau^B}^{CDS}, R_B\}$
- reference name defaults first, and before the maturity of the CDS: the protection buyer can get at most the value of the collateral at that time:  $\min\{1 - R_{RN}, V_{\tau^{RN}}^{Bond}\}$ .
- protection buyer default first, and before the maturity of the CDS

Let  $\tilde{V}_t$  denote the value of the CDS collateralised by a bond. Taking into account the previous cases this can be written as:

$$\begin{aligned}\tilde{V}_t &= \mathbb{E} \left[ \mathbb{1}_{T < \min\{\tau^{RN}, \tau^B, \tau^{PB}\}} CF(t, T) \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}, \tau^{PB}\}} [CF(t, \tau^B) + D(t, \tau^B) \min\{V_{\tau^B}^{CDS}, R_B\}] \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B, \tau^{PB}\}} [CF(t, \tau^{RN}) + D(t, \tau^{RN}) \min\{1 - R_{RN}, V_{\tau^{RN}}^{Bond}\}] \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^{PB} < \min\{T, \tau^B, \tau^{RN}\}} [CF(t, \tau^{PB}) + D(t, \tau^{PB}) (R_{PB} V_{\tau^{PB}}^{CDS^-} + V_{\tau^{PB}}^{CDS^+})] \middle| \mathcal{F}_t \right].\end{aligned}$$

Using equation (7.1) and (7.2) yields the following expression for the value of the partially collateralised CDS with risky protection buyer:

$$\begin{aligned}\tilde{V}_t &= \mathbb{E} \left[ \mathbb{1}_{T < \min\{\tau^{RN}, \tau^B, \tau^{PB}\}} CF(t, T) \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}, \tau^{PB}\}} [CF(t, \tau^B) + D(t, \tau^B) V_{\tau^B}^{CDS}] \middle| \mathcal{F}_t \right] \\ &- \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}, \tau^{PB}\}} D(t, \tau^B) (V_{\tau^B}^{CDS} - R_B)^+ \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B, \tau^{PB}\}} [CF(t, \tau^{RN}) + D(t, \tau^{RN}) (1 - R_{RN})] \middle| \mathcal{F}_t \right] \\ &- \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B, \tau^{PB}\}} D(t, \tau^{RN}) (1 - R_{RN} - V_{\tau^{RN}}^{Bond})^+ \middle| \mathcal{F}_t \right] \\ &+ \mathbb{E} \left[ \mathbb{1}_{\tau^{PB} < \min\{T, \tau^B, \tau^{RN}\}} [CF(t, \tau^{PB}) + D(t, \tau^{PB}) V_{\tau^{PB}}^{CDS}] \middle| \mathcal{F}_t \right] \\ &- \mathbb{E} \left[ \mathbb{1}_{\tau^{PB} < \min\{T, \tau^B, \tau^{RN}\}} (1 - R_{PB}) V_{\tau^{PB}}^{CDS^-} \middle| \mathcal{F}_t \right],\end{aligned}\tag{7.5}$$

Using similar techniques as in the case where the protection buyer is non-risky, it can be shown that terms one, two, four and six on the right hand side of equation (7.5) sum up to the value of the standard CDS, while the rest of the terms represent the adjustments due to the riskiness of the protection buyer and of the collateral:

$$\tilde{V}_t = V_t^{\text{CDS}} - V_t^{SH} - V_t^{BDVA}$$

where  $V^{SH}$  is the value of the shortfall due to the riskiness of the collateral and is given by

$$\begin{aligned} V_t^{SH} = & \mathbb{E} \left[ \mathbb{1}_{\tau^B < \min\{T, \tau^{RN}, \tau^{PB}\}} D(t, \tau^B) (V_{\tau^B}^{\text{CDS}} - R_B)^+ \middle| \mathcal{F}_t \right] \\ & + \mathbb{E} \left[ \mathbb{1}_{\tau^{RN} < \min\{T, \tau^B, \tau^{PB}\}} D(t, \tau^{RN}) (1 - R_{RN} - V_{\tau^{RN}}^{\text{Bond}})^+ \middle| \mathcal{F}_t \right] \end{aligned} \quad (7.6)$$

and  $V^{BDVA}$  is the bilateral DVA adjustment given by (see 6.2):

$$V_t^{BDVA} = \mathbb{E} \left[ \mathbb{1}_{\tau^{PB} < \min\{T, \tau^B, \tau^{RN}\}} (1 - R_{PB}) V_{\tau^{PB}}^{\text{CDS}-} \middle| \mathcal{F}_t \right].$$

The shortfall correction, negative in value, is similar to the ones obtained in the case where the protection buyer is non-risky, but not identical as it depends on the joint distribution of default of all three issuers now: the bond issuer, the reference name of the CDS and the protection buyer. The DVA adjustment is positive and represents the compensation needed for the protection seller to trade with a risky counterparty as opposed to a non-risky one.

### 7.3 Shortfall valuation in first-passage time model

As seen in equations (7.4) and (7.6), the valuation of one of the components of the shortfall requires the valuation of a single name bond in the first-passage time framework considered in Section 2.2. The next proposition gives the value of the bond in this framework:

**Proposition 7.1** (*Zero-coupon Bond Value*). *The value of a zero-coupon bond in the first passage time framework is given by:*

$$V^{Bond}(t, T, \tilde{x}) = D(t, T)Q(t, T, \tilde{x}) + R_B [1 - e^{-\varrho(T-t)}Q(t, T, \tilde{x}) - \varrho A(t, T, \tilde{x})], \quad (7.7)$$

where  $Q(t, T, \tilde{x})$  is given by (2.14) and  $A(t, T, \tilde{x})$  by equation (2.19).

*Proof.* Consider issuer  $X$  as the bond issuer, with the stochastic factors associated to the firm's asset value denoted by  $x_t$  (as defined in Section 2.2, equation (2.4)). The first-passage time framework is again considered, with the issuer defaulting whenever the value of the stochastic factor  $x_t$  drops below 0. The cashflows for the bond holder are as follows:

- If the bond issuer does not default before the maturity  $T$ , the bond holder receives the whole notional amount at maturity (normalised at 1)
- If the bond issuer defaults (denote by  $\tau^B$  the default time), then the bond holder recovers only a fraction of the notional,  $R_B$

Under similar assumptions as in Section 2.5 – the defaultable claim considered is hedgeable, and hence that the value of the firm is tradeable – the arbitrage-free price of the bond can be written as the discounted value of the future cashflows as follows:

$$V^{Bond}(t, T, \tilde{x}) = \mathbb{E} [D(t, T) \mathbb{1}_{\tau^B > T} + \mathbb{1}_{\tau^B < T} R_B D(t, \tau^B) | \mathcal{F}_t]$$

Using a Feynman-Kac representation, the pricing problem for the zero-coupon bond can be transformed from expectation format to a PDE problem with appropriate final value and boundary conditions (see theorem A.3):

$$\begin{aligned} V_t^{Bond} + \frac{1}{2} V_{xx}^{Bond} - \varrho V^{Bond} &= 0, \\ V^{Bond}(T, T, x) &= 1, \\ V^{Bond}(t, T, 0) &= R_B, \end{aligned}$$

where the domain is the positive semi-axis  $\mathbb{R}^+$ .

Consider function  $U(t, T, x) = e^{\varrho(T-t)} V^{Bond}(t, T, x)$ , which solves the modified pricing problem:

$$\begin{aligned} U_t + \frac{1}{2} U_{xx} &= 0, \\ U(T, T, x) &= 1, \\ U(t, T, 0) &= R_B e^{\varrho(T-t)}, \end{aligned}$$

in the same domain as before ( $\mathbb{R}^+$ ). To solve this problem, a similar technique as for the default leg calculation of the standard CDS is used (see Proposition 2.3). The solution is:

$$\begin{aligned} U(t, T, \tilde{x}) &= \int_0^\infty U(T, T, x') G(T-t, x'|\tilde{x}) dx' \\ &\quad + \frac{1}{2} \int_t^T U(t', T, 0) \left. \frac{\partial G(t'-t, x'|\tilde{x})}{\partial x'} \right|_{x'=0} dt' \end{aligned}$$

Notice how this is not exactly the same as in the case of the default leg, as the terminal condition in the case of the bond is non-zero. Going back to the value of the bond yields:

$$V^{Bond}(t, T, \tilde{x}) = D(t, T) \int_0^\infty G(T-t, x'|\tilde{x}) dx' + \frac{1}{2} \int_t^T R_B D(t, t') \left. \frac{\partial G(t'-t, x'|\tilde{x})}{\partial x'} \right|_{x'=0} dt',$$

which using similar techniques as for the default leg calculation:

$$V^{Bond}(t, T, \tilde{x}) = D(t, T) Q(t, T, \tilde{x}) + R_B [1 - e^{-\varrho(T-t)} Q(t, T, \tilde{x}) - \varrho A(t, T, \tilde{x})], \quad (7.8)$$

where  $Q(t, T, \tilde{x})$  is given by (2.14) and  $A(t, T, \tilde{x})$  by equation (2.19).

□

### 7.3.1 Risk-free protection buyer

Consider two issuers  $X$  and  $Y$  and let  $x_t$  and  $y_t$  respectively be the processes for the stochastic factors associated to the firms' asset value of these issuers (as defined in

Section 2.2, equation (2.4)). These processes are standard Brownian motions issued from the initial positive points  $\tilde{x}$  and  $\tilde{y}$  respectively. The Brownian motions are correlated, as described in Section 3.2.

Issuer  $X$  is the issuer of the bond held as collateral, while issuer  $Y$  is the reference name of the CDS. As with the unilateral CVA problem, an analytical solution for the potential shortfall due to the riskiness of the collateral can be obtained in the first passage time framework.

**Proposition 7.2.** *In the first-passage time framework, the value of the potential shortfall for the protection buyer of the CDS caused by the riskiness of the collateral is given by:*

$$\begin{aligned} V^{SH}(t, T, \tilde{r}, \tilde{\varphi}) &= \frac{1}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, 0) \left[ 1 - R_{RN} - V^{Bond}(t', T, r \sin \varpi) \right]^+ \frac{dr}{r} dt' \\ &\quad - \frac{1}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, \varpi) \left[ V^{CDS}(t', T, r \sqrt{1 - \rho_{xy}^2}) - R_B \right]^+ \frac{dr}{r} dt', \quad (7.9) \end{aligned}$$

where  $\varpi = \arccos(-\rho_{xy})$ .

*Proof.* The problem of evaluating the potential shortfall given in expectation form in equation (7.4) can be transformed to a PDE problem with terminal condition and Dirichlet boundary conditions through a Feynman-Kac type representation (see theorem A.3 for example):

$$\begin{aligned} V_t^{SH} + \frac{1}{2} V_{xx}^{SH} + \frac{1}{2} V_{yy}^{SH} + \rho_{xy} V_{xy}^{SH} - \varrho V^{SH} &= 0, \\ V^{SH}(T, T, x, y) &= 0, \\ V^{SH}(t, T, 0, y) &= (V^{CDS}(t, T, y) - R_B)^+, \\ V^{SH}(t, T, x, 0) &= (1 - R_{RN} - V^{Bond}(t, T, x))^+, \\ V^{SH}(t, T, \infty, y) &= 0, \quad V^{SH}(t, T, x, \infty) = 0. \end{aligned}$$

Consider function  $U^{SH}(t, T, x, y) = e^{\varrho(T-t)} V^{SH}(t, T, x, y)$ . After applying the changes of variables (3.5) and (3.7) described in Section 3.2, the problem for function

$U^{SH}$  in the angular domain is given by:

$$\begin{aligned}
U_t^{SH} + \frac{1}{2}U_{xx}^{SH} + \frac{1}{2}U_{yy}^{SH} + \rho_{xy}U_{xy}^{SH} &= 0, \\
U^{SH}(T, T, r, \varphi) &= 0, \\
U^{SH}(t, T, r, \varpi) &= e^{\varrho(T-t)} \left( V^{\text{CDS}}(t, T, r\sqrt{1 - \rho_{xy}^2}) - R_B \right)^+, \\
U^{SH}(t, T, r, 0) &= e^{\varrho(T-t)} (1 - R_{RN} - V^{\text{Bond}}(t, T, r \sin \varpi))^+, \\
U^{SH}(t, T, 0, \varphi) &= 0, \quad U^{SH}(t, T, \infty, \varphi) = 0.
\end{aligned}$$

The analytical solution for the above problem can be obtained in a similar manner with the expressions for the unilateral CVA/DVA by using Lemma 6.1 and particularising for the specific boundary conditions.

$$\begin{aligned}
U^{SH}(t, T, \tilde{r}, \tilde{\varphi}) &= \frac{1}{2} \int_t^T \int_0^\infty G_\varphi(t' - t, r, 0) U^{SH}(t', T, r, 0) \frac{1}{r} dr dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\infty G_\varphi(t' - t, r, \varpi) U^{SH}(t', T, r, \varpi) \frac{1}{r} dr dt'.
\end{aligned}$$

The final expression for the potential shortfall is:

$$\begin{aligned}
V^{SH}(t, T, \tilde{r}, \tilde{\varphi}) &= \frac{1}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, 0) [1 - R_{RN} - V^{\text{Bond}}(t', T, r \sin \varpi)]^+ \frac{dr}{r} dt' \\
&\quad - \frac{1}{2} \int_t^T \int_0^\infty D(t, t') G_\varphi(t' - t, r, \varpi) \left[ V^{\text{CDS}}(t', T, r\sqrt{1 - \rho_{xy}^2}) - R_B \right]^+ \frac{dr}{r} dt' \quad (7.10)
\end{aligned}$$

□

**Remark 7.1.** When the recovery of the bond is  $R_B = 0\%$ , the contribution of the case where the bond issuer defaults first to the shortfall (second term in (7.9)) is the same as the unilateral CVA expression obtained in (6.5) when the recovery of the protection

seller is 0%.

### 7.3.2 Risky protection buyer

When the protection buyer of the CDS is considered risky, the adjustments on the value of the fully-collateralised price of the CDS can be split into two parts:

- the adjustment due to the risky nature of the collateral (defined in the previous case as the shortfall)
- the adjustment due to the riskiness of the protection buyer

While the shortfall is similar in nature to that obtained in the previous case, the valuation depends on the joint dynamics of three issuers (the bond issuer, the CDS reference name and the protection buyer), and hence a three dimensional formulation needs to be used for the valuation. This is given in Proposition 7.3. The adjustment due to the riskiness of the protection buyer is the same as the bilateral DVA adjustment studied in Section 6.3.2 and given in formula (6.32).

**Proposition 7.3.** *In the first-passage time framework, the value of the potential shortfall due to the riskiness of the collateral (in the case where the protection buyer is risky as well) is given by:*

$$\begin{aligned}
 V^{SH}(t, T, \tilde{r}, \tilde{\varphi}, \tilde{\theta}) = & \\
 & - \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(\varpi)} \frac{D(t, t') [1 - R_{RN} - V^{Bond}(t', T, r \sin \theta \bar{\rho}_{xy})]^+ G_\varphi(t' - t, r, \varpi, \theta)}{\sin \theta} d\theta dr dt' \\
 & + \frac{1}{2} \int_t^T \int_0^\infty \int_0^{\Theta(0)} \frac{D(t, t') [V^{CDS}(t', T, r \sin \theta \bar{\rho}_{xy}) - R_B]^+ G_\varphi(t' - t, r, 0, \theta)}{\sin \theta} d\theta dr dt'. \quad (7.11)
 \end{aligned}$$

*Proof.* The problem of evaluating the potential shortfall given in expectation form in equation (7.6) can be transformed to a PDE problem with terminal condition and Dirichlet boundary conditions through a Feynman-Kac type representation (see



theorem A.3 for example):

$$\begin{aligned}
V_t^{SH} + \frac{1}{2}V_{xx}^{SH} + \frac{1}{2}V_{yy}^{SH} + \frac{1}{2}V_{zz}^{SH} + \rho_{xy}V_{xy}^{SH} + \rho_{xz}V_{xz}^{SH} + \rho_{yz}V_{yz}^{SH} - \varrho V^{SH} &= 0, \\
V^{SH}(T, T, x, y, z) &= 0, \\
V^{SH}(t, T, 0, y, z) &= (V^{\text{CDS}}(t, T, y) - R_B)^+, \\
V^{SH}(t, T, x, 0, z) &= (1 - R_{RN} - V^{\text{Bond}}(t, T, x))^+, \\
V^{SH}(t, T, x, y, 0) &= 0, \\
V^{SH}(t, T, \infty, y, z) &= 0, \quad V^{SH}(t, T, x, \infty, z) = 0, \quad V^{SH}(t, T, x, y, \infty) = 0.
\end{aligned}$$

The analytical solution for the above problem can be obtained in a similar manner with the expressions for the bilateral CVA/DVA by using the changes of variables in chapter 4 and the result in Lemma 6.2 and particularizing for the specific boundary conditions. This leads to the expression in equation (7.11).  $\square$

## 7.4 Examples and results analysis

This section focuses on a few examples and analyses the difference between the unilateral/bilateral valuation adjustments due to the riskiness of the counterparties and the valuation adjustment due to the riskiness of the collateral computed through the estimation of the shortfall as described above.

### 7.4.1 Risk-free protection buyer

Figure 7.2 shows the effect of the pairwise correlation between the issuers ( $\rho_{xy}$ ) on the PV of the CDS. Several quantities are shown for comparison:

- PV of the fully collateralised CDS (blue)
- PV when taking into account the riskiness of the protection seller (this is equal to the fully collateralised CDS PV minus the unilateral CVA adjustment computed as in equation (6.5))

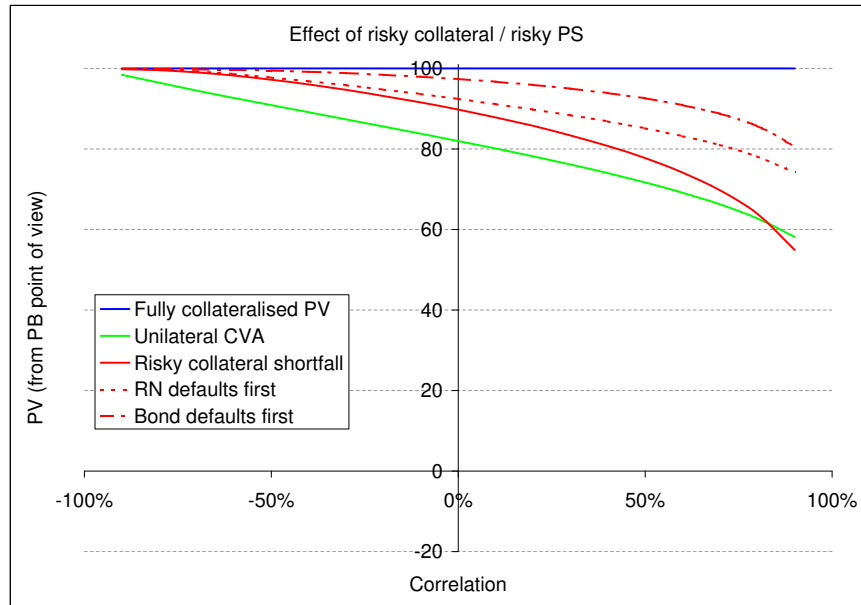
- PV of the CDS collateralised by the risky bond (equal to the fully collateralised PV minus the shortfall given in Proposition 7.2). In this case, as discussed previously, the shortfall may arise from two different sources: either as a consequence of the default of the bond issuer – similar case to the unilateral CVA, although payoff is different, or as a consequence of the impairment of the bond value at the default of the reference name. In order to better understand these effects, the contributions are shown both separately and together.

Figures 7.2a shows the case where both the recovery on the bond and the recovery of the protection seller are set to 30%. The first important observation is the presence of “wrong-way” risk; as the pairwise correlation increases, the value of the shortfall as well as the unilateral CVA adjustment increase, to make the contract less valuable for the protection buyer. Both contributions to the shortfall illustrate the effect of “wrong-way” risk:

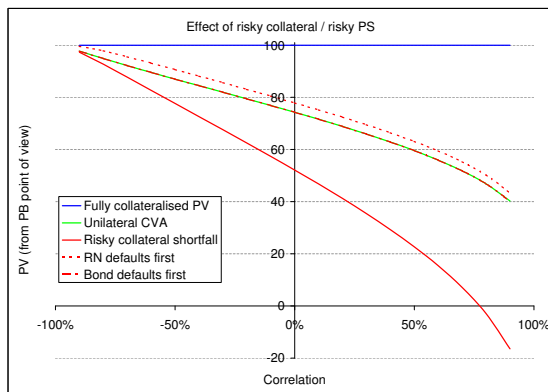
- when the reference name defaults first, as the two issuers are positively correlated, the bond issuer becomes riskier and hence the value of the bond decreases, thus leading to a higher shortfall for the protection buyer
- when the bond issuer defaults first, again as the correlation is positive and high, the reference name is riskier and hence the value of the fully collateralised CDS from the protection buyer’s point of view increases, yielding a larger shortfall

Figures 7.2b and 7.2c show the same effect for two different values of the bond recovery (equal to the protection seller recovery for the unilateral CVA calculation) – 0% and 50% respectively. The value of the shortfall is highly dependent on the recovery assumptions. When the recovery is taken to be 0%, the expected shortfall in the case where the bond issuer defaults first is equal to the unilateral CVA adjustment. The overall shortfall is significantly higher than in the previous case, where the recovery of the bond was 30%, and the “wrong-way” risk effect is again very evident.

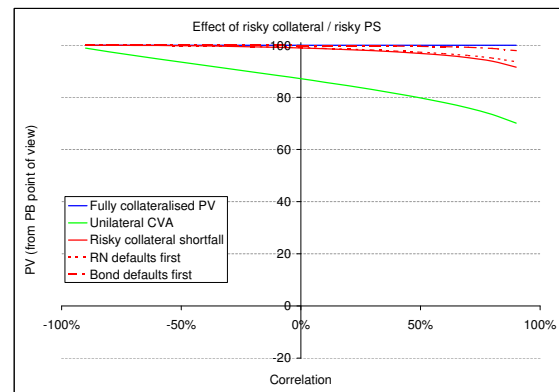
When the recovery of the bond is high enough, the shortfall is almost negligible, even at high correlation values. The advantages of collateralisation can be seen when comparing to the unilateral CVA adjustment, which is still significant even when the recovery of the protection seller is 50%.



(a) Bond recovery and PS recovery are 30%



(b) Bond recovery and PS recovery are 0%



(c) Bond recovery and PS recovery are 50%

Figure 7.2: Effect of the pairwise correlation for the two risky issuers case – PV of the CDS from the protection buyer's point of view. The PV of the fully collateralised CDS is shown in blue (and has been normalised to 100). The effect of a risky PS (so of the unilateral CVA) is shown in green. The shortfall for the CDS with a risky collateral is shown in red, with the two different contributions shown separately as well.

In Figure 7.3, the effect of the value of the interest rates on the size of the shortfall is analysed. While the component coming from the case where the bond defaults first is comparable in the two cases compared (low interest rate scenario in 7.3a and high interest rates in 7.3b), the second component of the shortfall is strongly impacted. When the reference name defaults first, the value of the shortfall depends on the difference between the protection payout and the value of the bond at the time of default. When interest rates are high, the value of the bond is lower as the future cashflows are discounted heavily, and hence the shortfall is higher. The effect of interest rates on the value of the shortfall is more pronounced than on the unilateral CVA adjustment.

Figure 7.4 analyses the behaviour of the shortfall and of the unilateral CVA adjustment when varying the riskiness of the bond issuer or protection seller respectively. Two cases are shown, one when the survival probability at a five year time horizon for the reference name is  $\approx 50\%$  (Figure 7.4a), and one when the reference name is riskier with a survival probability at 5Y of  $\approx 21\%$  (Figure 7.4b). It is interesting to observe that in both cases, when the riskiness of the bond issuer / protection seller increases, the unilateral CVA adjustment monotonically increases, while the shortfall for the bond collateralised CDS stabilises or even decreases.

#### 7.4.2 Risky protection buyer

Figure 7.5 shows the impact of the shortfall on the price of the CDS when the protection buyer is considered risky. The shortfall is valued using the results in Proposition 7.3. The three names are considered in this example equally risky, with the individual survival probabilities at 5Y of  $\approx 50\%$ . The values have been normalised such that the default leg of the fully collateralised CDS is 100.

The analysis of the unilateral versus bilateral approximation, showed that the most significant differences appear when the protection seller and the protection buyer are highly correlated, as the dependence between the counterparties is ignored in the unilateral case. To better observe the differences on the shortfall between the case when the protection buyer is considered risk free and the case where the protection

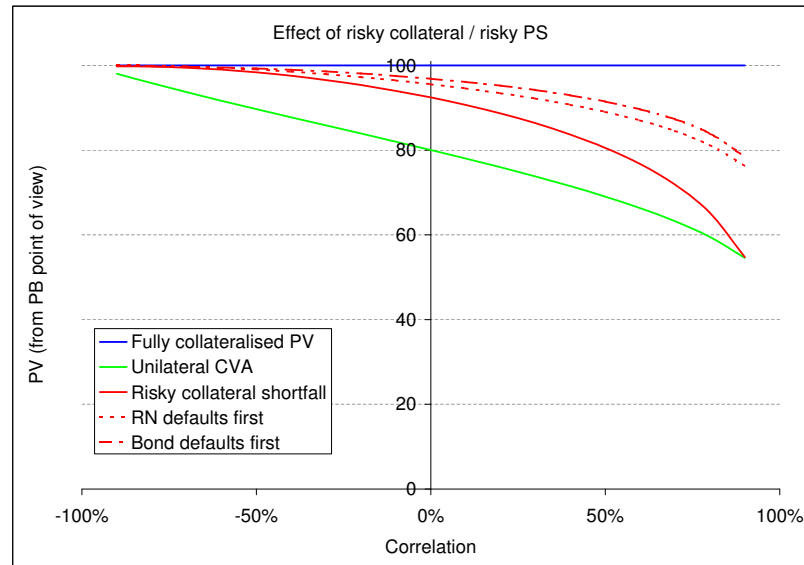
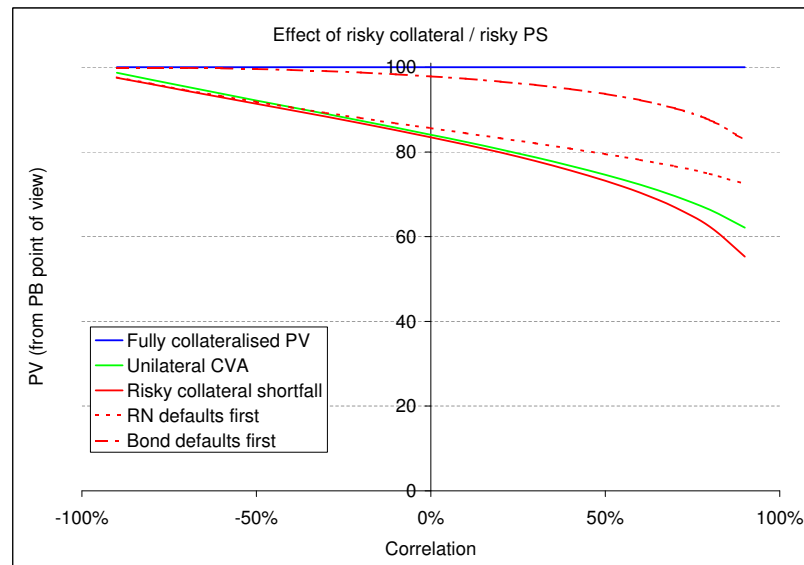
(a) Low interest rate:  $\rho = 1\%$ (b) High interest rate:  $\rho = 10\%$ 

Figure 7.3: Effect of the interest rates – PV of the CDS from the protection buyer's point of view. The PV of the fully collateralised CDS is shown in blue (normalised to 100). The effect of a risky PS (so of the unilateral CVA) is shown in green. The shortfall for the CDS with a risky collateral is shown in red, with the two different contributions shown separately as well.

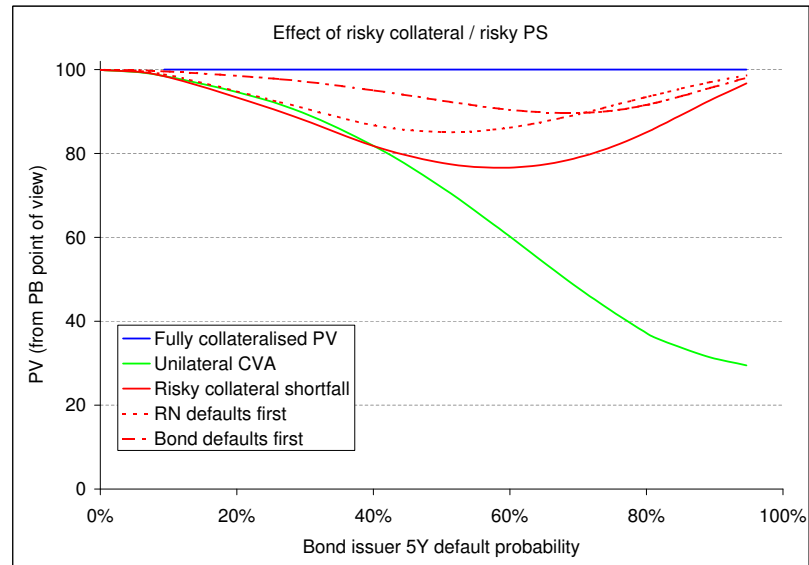
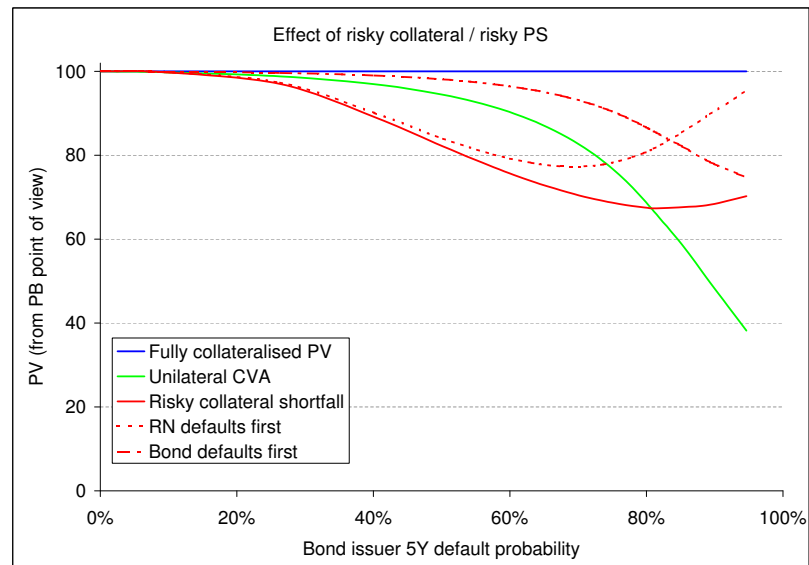
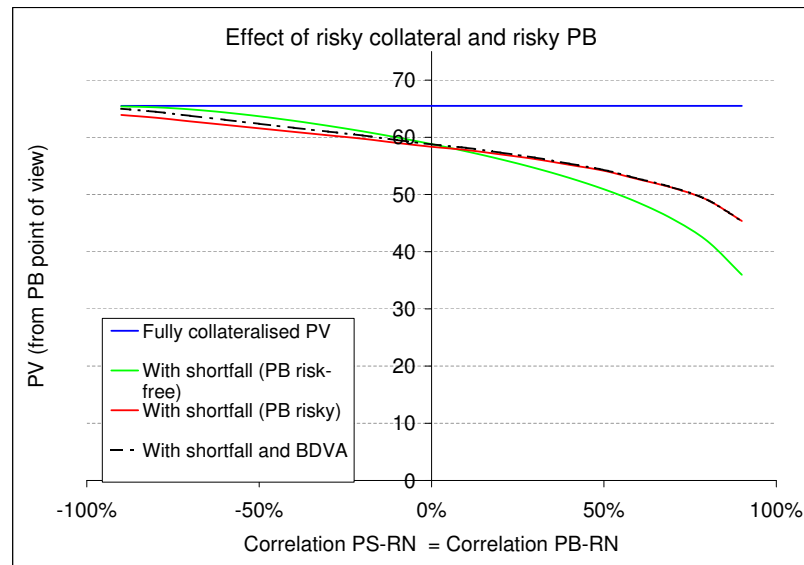
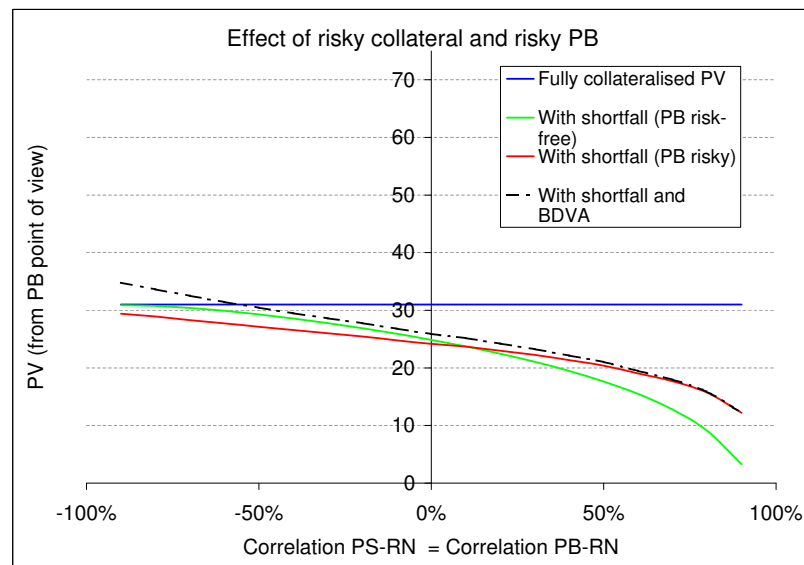
(a) RN survival probability at 5Y  $\approx 50\%$ (b) RN survival probability at 5Y  $\approx 21\%$ 

Figure 7.4: Effect of riskiness of the collateral issuer (protection seller) – PV of the CDS from the protection buyer's point of view. The PV of the fully collateralised CDS is shown in blue (normalised to 100). The effect of a risky PS (so of the unilateral CVA) is shown in green. The shortfall for the CDS with a risky collateral is shown in red, with the two different contributions shown separately as well.



(a) Low coupon (35% of the BEC)



(b) Higher coupon (70% of the BEC)

Figure 7.5: Effect of the common correlation PB-RN and RN-Bond issuer (PB - Bond issuer correlation is fixed at 90%) – PV of the CDS from the protection buyer's point of view. The PV of the fully collateralised CDS is shown in blue (the default leg is normalised to 100). The PV adjusted by the shortfall when the PB is risk-free is shown in green. The PV adjusted by the shortfall when the PB is risky is shown in red, while the dashed violet line shows the PV adjusted by both the shortfall and the DVA component.

buyer is risky, in Figure 7.5 the correlation protection buyer – bond issuer is set to 90%. The other two pairwise correlations are equal and varied between  $-100\%$  and  $100\%$ .

Figure 7.5a shows a case when the coupon of the CDS is relatively low (35% of the break-even coupon), while Figure 7.5b shows the case with a higher coupon. The DVA adjustment is, as expected, higher in the case where the coupon is higher. The graphs also show that the difference in the valuation of the shortfall when considering the protection buyer risk free or not is non-negligible, so as in the case of the CVA/DVA adjustments it is important to take into account the riskiness of both counterparties.



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# Appendix A

## Link between SDEs and PDEs

Consider a Markov process  $X_s$  for which the dynamics are given by the SDE:

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad s \geq 0 \quad (\text{A.1})$$

with  $X_0 = x$ ,  $W_t$  a standard  $d$ -dimensional Brownian motion and where the coefficients satisfy some regularity conditions:

- (i)  $\sigma(t, x)$  are continuous and bounded on  $[0, \infty) \times \mathbb{R}^d$
- (ii)  $\sigma(t, x)$  are Lipschitz-continuous in  $x$  with a Lipschitz constant that does not depend on  $x$
- (iii)  $\mu(t, x)$  is continuous and bounded uniformly in  $t$  and Lipschitz-continuous in  $x$

Let  $\mathcal{A}$  denote the second order differential operator associated with the drift vector  $\mu$  and the diffusion matrix  $a = \{a_{ij}\}_{1 \leq i, j \leq d}$  with the elements defines as:

$$(a_{ij}(t, x))_{1 \leq i, j \leq d} = \sigma(t, x) \sigma^t(t, x).$$

The operator is given by (see Karatzas and Shreve [1991], page 312):

$$\mathcal{A}f(x) = \frac{1}{2} \sum_{i=1}^d \sum_{j=1}^d a_{ij}(t, x) \frac{\partial^2 f(x)}{\partial x_i \partial x_j} + \sum_{i=1}^d \mu_i(t, x) \frac{\partial f(x)}{\partial x_i}; f \in C^2(\mathbb{R}^d),$$

and  $\mathcal{A}^*$  denotes its adjoint operator.

Similar to the one-dimensional case discussed in 2.3, the transition probability density function for this diffusion process (or Green's function associated to the above operator  $\mathcal{A}$ ) satisfies the forward Kolmogorov equation, under the additional conditions:

(iv) Uniform ellipticity: there exists a positive constant  $\delta$  such that:

$$\sum_{i=1}^d \sum_{j=1}^d a_{ij}(t, x) \xi_i \xi_j \geq \delta \|\xi\|^2,$$

holds for every  $\xi \in \mathbb{R}^d$  and  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ .

(v) the functions  $\frac{\partial b_i}{\partial x_i}$ ,  $\frac{\partial a_{ij}}{\partial x_i}$  and  $\frac{\partial^2 a_{ij}}{\partial x_i^2}$  are bounded and Holder-continuous.

**Theorem A.1** (Forward Kolmogorov equation - multi-dimensional). *The Green function associated with the operator  $\mathcal{A}^*$  is at least  $C^{1,2}(\mathbb{R}_+, \mathbb{R}^d)$  function of “forward” arguments  $t'$  and  $x'$  solves the following forward Kolmogorov equation:*

$$\frac{\partial G(t', x'|t, x)}{\partial t'} - \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial x'_i \partial x'_j} [a_{ij}(t', x') G] + \sum_{i=1}^d \frac{\partial}{\partial x'_i} (\mu(t', x'_i) G) = 0, \quad (\text{A.2})$$

with the initial condition  $G(t, x'|t, x) = \delta(x' - x)$ .

For a proof, see for example Karatzas and Shreve [1991], page 368. The two and three dimensional versions of this theorem are used in chapters 3 and 4.

The main tools used throughout this thesis for pricing different claims are partial differential equations specific tools. Thus, it is useful to have a link between the expectation-based pricing formulas obtained through the arbitrage-free assumptions and the corresponding PDE formulations of the pricing problem. This link is provided by Feynman-Kac representations of PDE problems. The main cases that are used throughout this thesis are briefly discussed below, with a particular focus on final value problems with Dirichlet boundary conditions.

Consider a Markov process  $\tilde{X}_s$ <sup>1</sup> for which the dynamics are given by the SDE:

$$d\tilde{X}_s = \tilde{\mu}(s, \tilde{X}_s) ds + \tilde{\sigma}(s, \tilde{X}_s) dW_s, \quad s \geq 0 \quad (\text{A.3})$$

with  $\tilde{X}_0 = x$  and where  $W_t$  is a standard  $d$ -dimensional Brownian motion and the coefficients satisfy the regularity conditions discussed above. The following theorem gives the Feynman-Kac representation of an initial value problem (IVP) with Dirichlet boundary conditions.

**Theorem A.2.** (*Feynman-Kac representation for IVP, with Dirichlet boundary conditions*)

Let  $(\tilde{X}_s)_{s \geq 0}$  be a  $d$ -dimensional diffusion process that follows the dynamics given in equation (A.3). Let  $D$  be a domain in  $\mathbb{R}^d$ , for which the boundary is denoted by  $\partial D$ . Denote by  $\tilde{\tau}$  the first time the process  $\tilde{X}_s$  hits the boundary of the domain  $D$ :  $\tilde{\tau} = \inf \{s \geq 0, \tilde{X}_s \in \partial D\}$ . Assume that function  $u(t, x)$  is bounded and at least  $C^{1,2}(\mathbb{R}_+, D)$ <sup>2</sup> and is a solution to the following problem:

$$\frac{\partial u}{\partial t} = \frac{1}{2} \sum_{i,j=1}^d \tilde{a}_{ij}(t, x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^d \tilde{\mu}_i(t, x) \frac{\partial u}{\partial x_i} + c(t, x)u(t, x),$$

with initial condition  $u(0, x) = f(x)$  and boundary conditions

$$u(t, x) = g(t, x), \quad x \in \partial D.$$

The functions  $c$ ,  $f$  and  $g$  are continuous and bounded in their domain of definition ( $\mathbb{R}_+ \times D$ ,  $D$  and  $\mathbb{R}_+ \times \partial D$  for  $c$ ,  $f$  and  $g$  respectively) and such that  $g(0, x) = f(x)$ ,  $\forall x \in \partial D$ . Then the following representation holds:

$$u(t, x) = \mathbb{E} \left[ f(\tilde{X}_t) e^{\int_0^t c(t-s, \tilde{X}_s) ds} \mathbb{1}_{\tilde{\tau} > t} + g(\tilde{\tau}, \tilde{X}_{\tilde{\tau}}) e^{\int_0^{\tilde{\tau}} c(t-s, \tilde{X}_s) ds} \mathbb{1}_{\tilde{\tau} < t} \middle| \tilde{X}_0 = x \right].$$

---

<sup>1</sup>The notation with  $\tilde{\cdot}$  is used for the initial value problem and the one without  $\tilde{\cdot}$  for the final value problem, which is the one predominant throughout this thesis.

For the proof see Freidlin [1985], Theorem 2.3 page 133. In the case of financial applications, it is interesting to have a Feynman-Kac representation for a final value problem – FVP – (potentially with Dirichlet boundary conditions), as the price of different products is usually defined as the expectation of future cashflows discounted to today (assuming an arbitrage-free market and a tradeable asset). Theorem A.2 can be used to derive a similar result for the case where there is a final condition.

Let  $T$  denote the terminal maturity and consider the function  $v(t, x)$  defined as  $v(t, x) = u(T - t, x)$  (time reversal). With this definition,  $v$  has a terminal condition given by:

$$v(T, x) = u(0, x) = f(x)$$

and satisfies the following PDE:

$$-\frac{\partial v}{\partial t} + k(t, x)v(t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^d \mu_i(t, x_i) \frac{\partial v}{\partial x_i},$$

where  $a(t, x) = \tilde{a}(T - t, x)$ ,  $\mu(t, x) = \tilde{\mu}(T - t, x)$  and  $k(t, x) = -c(T - t, x)$ .

Fix  $t < T$ , and consider the process  $(X_s)_{s \geq T-t}$  with the following dynamics:

$$dX_s = \mu(s, X_s) ds + \sigma(s, X_s) dW_s, \quad X_{T-t} = x, \quad (\text{A.4})$$

where  $\mu(s, x) = \tilde{\mu}(t - s, x)$  and  $\sigma(s, x) = \tilde{\sigma}(t - s, x)$ . It can be shown that this process is related to the one given in equation (A.3):  $X_s = \tilde{X}_{s-T+t}$ ,  $\forall s \geq T - t$ . This implies that  $X_T = \tilde{X}_t$  and that  $\tilde{X}_\theta = X_{\theta+T-t}$ ,  $\forall \theta \geq 0$ . The representation of  $u(t, x)$

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<sup>2</sup> $C^{1,2}(\mathbb{R}_+, D)$  denotes the space of function  $u(t, x)$  defined on  $\mathbb{R}_+ \times D$  which have continuous first derivatives in the first variable and continuous second derivatives in the second one.

as an expectation can thus be rewritten as:

$$\begin{aligned}
 u(t, x) &= \mathbb{E} \left[ f(X_T) e^{\int_0^t c(t-s, X_{s+T-t}) ds} \mathbb{1}_{\tilde{\tau} > t} \right. \\
 &\quad \left. + g(\tilde{\tau}, X_{\tilde{\tau}+T-t}) e^{\int_0^{\tilde{\tau}} c(t-s, X_{s+T-t}) ds} \mathbb{1}_{\tilde{\tau} < t} \middle| X_{T-t} = x \right] \\
 &= \mathbb{E} \left[ f(X_T) e^{\int_0^t c(s', X_{T-s'}) ds'} \mathbb{1}_{\tilde{\tau} > t} \right. \\
 &\quad \left. + g(\tilde{\tau}, X_{\tilde{\tau}+T-t}) e^{\int_{t-\tilde{\tau}}^t c(s', X_{T-s'}) ds'} \mathbb{1}_{\tilde{\tau} < t} \middle| X_{T-t} = x \right] \\
 &= \mathbb{E} \left[ f(X_T) e^{-\int_0^t k(T-s', X_{T-s'}) ds'} \mathbb{1}_{\tilde{\tau} > t} \right. \\
 &\quad \left. + g(\tilde{\tau}, X_{\tilde{\tau}+T-t}) e^{-\int_{t-\tilde{\tau}}^t k(T-s', X_{T-s'}) ds'} \mathbb{1}_{\tilde{\tau} < t} \middle| X_{T-t} = x \right] \\
 &= \mathbb{E} \left[ f(X_T) e^{-\int_{T-t}^T k(s, X_s) ds} \mathbb{1}_{\tilde{\tau} > t} \right. \\
 &\quad \left. + g(\tilde{\tau}, X_{\tilde{\tau}+T-t}) e^{-\int_{T-t}^{T-t+\tilde{\tau}} k(s, X_s) ds} \mathbb{1}_{\tilde{\tau} < t} \middle| X_{T-t} = x \right],
 \end{aligned}$$

which yields the following expression for the function  $v$ :

$$\begin{aligned}
 v(t, x) &= u(T-t, x) \\
 &= \mathbb{E} \left[ f(X_T) e^{-\int_t^T k(s, X_s) ds} \mathbb{1}_{\tilde{\tau} > T-t} \right. \\
 &\quad \left. + g(\tilde{\tau}, X_{\tilde{\tau}+t}) e^{-\int_t^{t+\tilde{\tau}} k(s, X_s) ds} \mathbb{1}_{\tilde{\tau} < T-t} \middle| X_t = x \right]
 \end{aligned}$$

The Feynman-Kac representation of the final value problem with Dirichlet boundary conditions is given in Theorem A.3.

**Theorem A.3.** (*Feynman-Kac representation for FVP, with Dirichlet boundary conditions*)

Let  $(\tilde{X}_s)_{s \geq 0}$  be a  $d$ -dimensional diffusion process that follows the dynamics given in equation (A.3). Let  $D$  be a domain in  $\mathbb{R}^d$ , for which the boundary is denoted by  $\partial D$ . Denote by  $\tilde{\tau}$  the first time the process  $\tilde{X}_s$  hits the boundary of the domain  $D$ :  $\tilde{\tau} = \inf \{s \geq 0, \tilde{X}_s \in \partial D\}$ . Assume that function  $u(t, x)$  is bounded and at least

$C^{1,2}(\mathbb{R}_+, D)$  and is a solution to the following problem:

$$-\frac{\partial v}{\partial t} + k(t, x)v(t, x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^d \mu_i(t, x_i) \frac{\partial v}{\partial x_i},$$

with final condition  $v(T, x) = f(x)$  and boundary conditions

$$v(t, x) = g(t, x), \quad x \in \partial D.$$

The functions  $c$ ,  $f$  and  $g$  are continuous and bounded in their domain of definition ( $\mathbb{R}_+ \times D$ ,  $D$  and  $\mathbb{R}_+ \times \partial D$  for  $c$ ,  $f$  and  $g$  respectively) and such that  $g(T, x) = f(x)$ ,  $\forall x \in \partial D$ . Then the following representation holds:

$$v(t, x) = \mathbb{E} \left[ f(X_T) e^{-\int_t^T k(s, X_s) ds} \mathbb{1}_{\tau > T} + g(\tau, X_\tau) e^{-\int_t^\tau k(s, X_s) ds} \mathbb{1}_{\tau < T} \middle| X_t = x \right].$$

In Freidlin [1985], the authors remark that theorems A.2 and A.3 hold even without assuming  $g(0, x) = f(x)$  or  $g(T, x) = f(x)$  respectively on the boundary of the domain  $\partial D$ , provided that the time-homogeneous case is considered (the coefficients  $\mu$ ,  $\sigma$  do not depend on time). This is the case that is used extensively throughout this thesis. The typical example, cited in Freidlin [1985] as well, is the case of the survival probability, when the final condition is  $f(x) = 1$  and boundary condition  $g(t, x) = 0$ . This type of representation allows for the valuation of mean values of various functionals of the Markov process by solving the corresponding differential equations.