

A family of exact eigenstates for a single trapped ion interacting with a laser field

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(Dated: October 13, 2006)

Abstract

We show that, under certain combinations of the parameters governing the interaction of a harmonically trapped ion with a laser beam, it is possible to find one or more exact eigenstates of the Hamiltonian, with no approximations except the optical rotating-wave approximation. These are related via a unitary equivalence to exact eigenstates of the full Jaynes-Cummings model (including counter-rotating terms) supplemented by a static driving term.

PACS numbers: 42.50.-p, 03.65.Bz, 42.50.Dv

I. INTRODUCTION

In recent years, trapped ions interacting with laser beams [1, 2] have become an extremely interesting system for the investigation of fundamental physics. For instance, they have been used to generate nonclassical motional states [3, 4, 5], and have been proposed for applications such as quantum computation [1, 6, 7, 8] and precision spectroscopy [9].

Despite the relative simplicity of this system, the full theoretical treatment of its dynamics is a nontrivial problem, as the laser-ion interaction is highly nonlinear. Even in the simplest case where only a single ion is in the trap, one is usually forced to employ physically motivated approximations in order to find a solution. A well-known example is the *Lamb-Dicke* approximation [11], in which the ion is considered to be confined within a region much smaller than the laser wavelength. Many treatments also assume a *weak coupling* approximation, i.e., a sufficiently weak laser-ion coupling constant. Under these conditions, tuning the laser frequency to integer multiples of the trap frequency results in effective Hamiltonians of the Jaynes-Cummings type [12], in which the centre-of-mass of the trapped ion plays the role of the field mode in cavity QED [13, 14].

Recently, a new approach to this problem has been suggested [10], based on the application of a unitary transformation $\hat{T}$ which linearises the total ion-laser Hamiltonian. Under this transformation the Hamiltonian becomes *exactly* (not effectively!) equivalent to the full Jaynes-Cummings model (JCM), including counter-rotating terms, together with an extra atomic driving term. Remarkably, the ion-trap system is thus formally equivalent to an atom interacting with a single-mode quantised electromagnetic field.

The existence of this correspondence is very useful, since it allows us to map interesting properties of each model onto their counterparts in the other. For example, we can make use of the well-known ‘rotating-wave’ limit where the JCM is analytically, albeit approximately, soluble [12]. Using the map $\hat{T}^\dagger$ to translate this solution back into the ion-trap scenario has led to the identification of a new dynamically interesting regime for that system [10], where phenomena such as “super-revivals” in the ion-laser interaction can occur. The same solution has also led to a scheme for realising substantially faster logic gates for quantum information processing in a linear ion chain [8].

In the present note, we apply this line of reasoning to the problem of obtaining *exact* eigenstates of either system. To our best knowledge, this has never been achieved in the

case of the ion-trap Hamiltonian. In the case of the JCM (with no extra driving), exact eigenstates have been derived for those combinations of system parameters at which level crossings occur in the spectrum [15, 16]. In this paper we use a simple physical ansatz similar to the one in [16] to derive a family of exact eigenstates for the ion-trap system (again, these are valid only under certain combinations of system parameters). In the special case where the ion is driven on resonance, these states correspond, via the mapping $\hat{T}$, to the ones found in refs. [15, 16]. In addition, we study the asymptotic limit of large Lamb-Dicke parameter η in the ion-trap system, which under $\hat{T}$ corresponds to a large coupling constant in the JCM. We obtain in this way a very simple explanation of the asymptotic forms of *all* eigenvalues and eigenvectors in this limit (these have also been derived in [16] using other methods).

II. ION TRAP-JCM CORRESPONDENCE

Let us start then by recalling the equivalence between the ion-trap system and the JCM [10]. The Hamiltonian for the ion-laser dipole interaction, with no approximations (except the optical rotating-wave approximation [9]) can be written as

$$H_{ion} = \nu \hat{n} + \frac{\delta}{2} \sigma_z + \Omega \left(\sigma_+ \hat{D}(i\eta) + \sigma_- \hat{D}^\dagger(i\eta) \right), \quad (1)$$

where $\hat{D}(i\alpha) = e^{i\alpha(a+a^\dagger)}$ is the displacement operator, ν is the harmonic trapping frequency, $\delta = \omega_{atom} - \omega_{laser}$ the laser-ion detuning, Ω the (real) Rabi frequency of the ion-laser coupling, η the Lamb-Dicke parameter, and we have chosen units where $\hbar = 1$. On the other hand, the Jaynes-Cummings Hamiltonian with counter-rotating terms can be written as

$$H_{JCM} = \omega \hat{n} + \frac{\omega_0}{2} \sigma_z + i\lambda (\sigma_+ + \sigma_-) (a - a^\dagger). \quad (2)$$

Although these two models appear to be physically and mathematically quite distinct, they are in fact exactly equivalent. The easiest way to see this is by rewriting eq.(1) in a notation where operators acting on the internal ionic levels are represented explicitly in terms of their matrix elements:

$$H_{ion} = \begin{pmatrix} \nu \hat{n} + \frac{\delta}{2} & \Omega \hat{D}(i\eta) \\ \Omega \hat{D}^\dagger(i\eta) & \nu \hat{n} - \frac{\delta}{2} \end{pmatrix}. \quad (3)$$

Consider now the unitary operator

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} \hat{D}^\dagger(\beta) & \hat{D}(\beta) \\ -\hat{D}^\dagger(\beta) & \hat{D}(\beta) \end{pmatrix} \quad (4)$$

where $\beta = i\eta/2$. It is possible to check after some algebra [10] that:

$$\mathcal{H} \equiv TH_{ion}T^\dagger = \begin{pmatrix} \nu\hat{n} + \Omega + \frac{\nu\eta^2}{4} & \frac{\nu\eta}{2}(a - a^\dagger) + \frac{\delta}{2} \\ \frac{\nu\eta}{2}(a - a^\dagger) + \frac{\delta}{2} & \nu\hat{n} - \Omega + \frac{\nu\eta^2}{4} \end{pmatrix} \quad (5)$$

Returning to the usual notation, we obtain

$$\mathcal{H} = \nu\hat{n} + \Omega\sigma_z + \frac{\nu\eta}{2}(\sigma_+ + \sigma_-)(a - a^\dagger) + \frac{\delta}{2}(\sigma_+ + \sigma_-) + \frac{\nu\eta^2}{4} \quad (6)$$

Comparing with eq. (2) it can be seen that this is precisely the Jaynes-Cummings interaction, supplemented by two additional terms: the first corresponds to an extra static electric field interacting with the atomic dipole, and the second is just a constant energy shift. In particular, a purely Jaynes-Cummings form is recovered when $\delta = 0$, corresponding to a resonant laser-ion interaction in eq. (1). Of course, the various parameters of the Hamiltonian in eq. (6) have different meanings than they do in eq. (1): ν becomes the cavity field frequency ω , 2Ω the atomic transition frequency ω_0 , δ the coupling strength with the static field and η the ratio between the Jaynes-Cummings Rabi frequency 2λ and the cavity frequency ω . In what follows we shall refer to eq. (6) as the ‘Jaynes-Cummings picture’ of the ion-trap Hamiltonian, eq. (1).

As noted above, this correspondence enables one to map interesting properties of each model onto the other. In this paper, we will use it to translate between corresponding eigenstates of each system (it is clear that, if $|\psi\rangle$ is an eigenstate of H_{ion} , then $T|\psi\rangle$ is a corresponding one for $\mathcal{H}$). In this regard it is important to point out that, although H_{ion} and $\mathcal{H}$ both describe systems consisting of a two-level atom interacting with a bosonic mode, one should not identify each of these subsystems with their counterparts after the transformation has been applied. This is due to the fact that T is an *entangling* transformation: separable internal-motional states of the trapped ion can be mapped into entangled atom-cavity states in the corresponding cavity QED system.

III. EIGENSTATES IN THE ASYMPTOTIC LIMIT

In order to demonstrate the convenience of this correspondence, let us consider what happens to the systems above in the ‘asymptotic’ limit where $\eta \gg \Omega/\nu$ (this problem has been studied in [16] in the special case $\delta = 0$, using different methods). In the ion-trap picture, this limit corresponds to having a very large atomic recoil after the absorption of a photon from the laser; in the JCM picture, to having a very large atom-mode coupling constant. From the point of view of current ion-trap and cavity QED experiments, neither of these conditions can be considered realistic. Nevertheless, it may well be possible to engineer such a situation artificially, for example an ‘effective’ JCM with large ‘coupling constant’ can be obtained when a single ion is simultaneously illuminated with weak lasers tuned to the first red and blue sidebands [1].

It is simpler to consider the problem first in the JCM picture. If $\eta \gg \Omega/\nu$, then the term containing Ω in eq. (6) makes a negligible contribution to the total energy. We can therefore expect that the asymptotic eigenvalues and eigenvectors will not be affected if this term is in fact absent. Setting $\Omega = 0$ implies that $\sigma_x \equiv \sigma_+ + \sigma_-$ becomes a constant of the motion, so the asymptotic eigenstates can be chosen to be of the form $|\pm\rangle \otimes |\phi_\pm\rangle$, where $|\pm\rangle = (|g\rangle \pm |e\rangle)/\sqrt{2}$. In order to determine their form explicitly, and also their corresponding eigenvalues, let us switch back to the ion-trap picture. Setting $\Omega = 0$ in eq. (1) reduces the Hamiltonian to that of an *uncoupled* spin-boson system. The asymptotic eigenstates of the ion-trap are therefore simply $|e\ m\rangle, |g\ m\rangle$. Transforming back to the JCM picture using eq. (4), we obtain thus $|\phi_\pm\rangle = |\pm i\eta/2; n\rangle$, where $|\alpha; k\rangle \equiv \hat{D}(\alpha)|k\rangle$ is a displaced number state [18] (note that they are independent of δ). The corresponding eigenvalues are $m\nu \pm \delta/2$ [21]. Note in particular that, when $|\delta| = k\nu$ for integer k , all but k of these energy levels are degenerate. In the special case $\delta = 0$ it can in fact be shown from symmetry considerations alone that the entire asymptotic spectrum must be two-fold degenerate [19]). Conversely, no such asymptotic degeneracies occur for $|\delta| \neq k\nu$.

IV. EIGENSTATES FROM A SIMPLE ANSATZ

Let us return now to the ion-trap Hamiltonian, eq. (1). We will construct an *ansatz* which allows the determination of exact eigenstates of this system, provided certain relations are

satisfied between the parameters Ω, δ, η . Consider the possibility of finding an eigenstate of the form

$$|\psi_{ion}^{m+}\rangle = |g\rangle |\phi\rangle + \frac{\Omega}{\nu} |e\rangle \sum_{n=0}^{m+1} c_n |n\rangle, \quad c_{m+1} \neq 0, \quad (7)$$

Let us see whether the eigenvalue equation

$$H_{ion} |\psi_{ion}^{m+}\rangle = E_m^+ |\psi_{ion}^{m+}\rangle, \quad (8)$$

can be satisfied. Eq. (3) shows that this requires $|\phi\rangle$ to be of the form

$$|\phi\rangle = D^\dagger(i\eta) \sum_{n=0}^{m+1} d_n |n\rangle = \sum_{n=0}^{m+1} d_n |-i\eta; n\rangle. \quad (9)$$

We thus require

$$H_{ion} |\psi\rangle = \sum_{n=0}^{m+1} \Omega \left(\left(n + \frac{\delta}{2\nu} \right) c_n + d_n \right) |e\rangle |n\rangle + \left(\frac{\Omega^2}{\nu} c_n + d_n \left(\hat{n} - \frac{\delta}{2} \right) \right) |g\rangle |-i\eta; n\rangle. \quad (10)$$

Now, using the simple fact that $\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha$ [20], it is easy to show that displaced number states satisfy the recursion relation

$$\hat{n} |\alpha; k\rangle = (|\alpha|^2 + k) |\alpha; k\rangle + \alpha \sqrt{k+1} |\alpha; k+1\rangle + \alpha^* \sqrt{k} |\alpha; k-1\rangle. \quad (11)$$

Substituting then eqs. (7), (10) and (11) into eq. (8) and using the fact that $\{|\alpha; k\rangle\}_{k=0}^\infty$ is an orthonormal basis gives the following eigenstate conditions:

$$E_m^+ = (m+1)\nu + \frac{\delta}{2}, \quad c_n = \begin{cases} \frac{d_n}{m+1-n}; & 0 \leq n \leq m \\ \frac{i\eta\nu^2}{\Omega^2} \sqrt{m+1} d_m; & n = m+1 \end{cases} \quad (12)$$

where the d_n coefficients satisfy $d_{m+1} = 0$ and

$$\begin{bmatrix} \varepsilon_m & -i\eta & & & \\ i\eta & \varepsilon_{m-1} & -i\eta\sqrt{2} & & \\ & i\eta\sqrt{2} & \ddots & \ddots & \\ & & \ddots & \varepsilon_1 & -i\eta\sqrt{m} \\ & & & i\eta\sqrt{m} & \varepsilon_0 \end{bmatrix} \begin{bmatrix} d_0 \\ \vdots \\ d_m \end{bmatrix} = \vec{0} \quad (13)$$

where

$$\varepsilon_j = (j+1 - \eta^2) + \frac{\delta}{\nu} - \frac{\Omega^2}{(j+1)\nu^2}. \quad (14)$$

Using operator $\hat{T}$ we can map this state into a corresponding eigenstate of the generalised JCM model in eq. (6)

$$|\psi_{JCM}^{m+}\rangle = \hat{T} |\psi_{ion}^{m+}\rangle = \sum_{n=0}^{m+1} \left(d_n |+\rangle - \frac{\Omega}{\nu} c_n |-\rangle \right) |-i\eta/2; n\rangle. \quad (15)$$

Once again we verify that the solution for this system can be easily expressed in terms of a displaced number state basis.

Further solutions can also be found if we note that the Hamiltonian H_{ion} is invariant under the combined transformations

$$|e\rangle \leftrightarrow |g\rangle; \quad \delta \leftrightarrow -\delta; \quad \eta \leftrightarrow -\eta. \quad (16)$$

Defining then

$$|\psi_{ion}^{m-}\rangle = \frac{\Omega}{\nu} \sum_{n=0}^{m+1} c'_n |g\rangle |n\rangle + \sum_{n=0}^m d'_n |e\rangle |i\eta, n\rangle, \quad (17)$$

and applying this symmetry transformation to eqs. (10)- (14) we can see that $|\psi_{ion}^{m-}\rangle$ is also an eigenstate of H_{ion} , with eigenvalue $E_m^- = (m+1)\nu - \frac{\delta}{2}$, as long as

$$c'_n = \begin{cases} \frac{d'_n}{m+1-n}; & 0 \leq n \leq m \\ -\frac{i\eta\nu^2}{\Omega^2} \sqrt{m+1} d'_m; & n = m+1 \end{cases} \quad (18)$$

where the d'_n coefficients satisfy an equation analogous to eq.(13) but with

$$\varepsilon_j = (j+1 - \eta^2) - \frac{\delta}{\nu} - \frac{\Omega^2}{(j+1)\nu^2}. \quad (19)$$

Condition (13) means that each of these ansätze succeeds only for certain combinations of Ω, δ, η . This is because the tridiagonal matrix above (which we will refer to as $M_m^\pm$) must have a zero eigenvalue, or equivalently $\det M_m^\pm = 0$. Since $\det M_m^\pm$ is a polynomial of degree $m+1$ in Ω^2, δ or η^2 , fixing two of these quantities determines up to $m+1$ real solutions for the third one. For $m \leq 3$ it is possible to solve this ‘compatibility’ condition algebraically,

resulting in explicit expressions in terms of any of the three parameters. For example, in terms of η , the first few solutions are

$$m = 0 : \quad \eta^2 = 1 - a + sd \quad (20)$$

$$m = 1 : \quad \eta^2 = 2 + sd - \frac{3a}{4} \pm \sqrt{\frac{a^2}{16} - \frac{a}{2} + 2 + sd} \quad (21)$$

where $a = (\Omega/\nu)^2$, $d = \delta/\nu$ and $s = \pm 1$ according to which ansatz we are referring to. For $m > 3$ a numerical solution is necessary.

The physical meaning of these solutions can be clarified by examining the spectrum of H_{ion} , obtained by numerically diagonalising this operator (or, equivalently, $\mathcal{H}$). Let us first examine the case where $\delta = 0$ (previously studied in [15, 16]). In fig. (1a) we plot the first few energy levels as functions of η , for the case where $\Omega = \nu/2$ (corresponding to a resonant JCM with no extra driving in eq.(6)). It can be readily seen that pairs of adjacent energy levels form braids around the lines $E = m\nu$ for integer m , converging to these lines in the limit of large η (as expected from our considerations in section III). The level crossings occur precisely on top of these lines, and there are m such degeneracy points on the line $E = m\nu$ [21]. (In fact, it can be shown [16] that these are the *only* energy values at which degeneracies may occur in this system).

Note now that, when $\delta = 0$ equations (14) and (19) coincide, and thus, $|\psi_{ion}^{m\pm}\rangle$ are simultaneous (in fact, *degenerate*) eigenstates for any combination of Ω, η at which $\det M_m^\pm = 0$. Another way of understanding this fact is to note that the ion-trap Hamiltonian H_{ion} has in this case an extra symmetry: it commutes with the parity-like observable $\sigma_x \exp(i\pi a^\dagger a)$ (in the JCM picture, the corresponding symmetry operator is $\sigma_z \exp(i\pi a^\dagger a)$). It can be easily checked that neither $|\psi_{ion}^{m+}\rangle$ nor $|\psi_{ion}^{m-}\rangle$ have this symmetry, but simple linear combinations of them do. Every solution to the equation $\det M_m^\pm = 0$ must therefore correspond to one of the level crossing points; in other words, the location of these points can be calculated directly from this condition (for $m > 3$ this cannot be done algebraically, of course). In fact, it turns out that, when $\Omega = \nu/2$, this equation always has m real and positive solutions for η [16], and so *every* crossing point in fig. 1a is accounted for in this way. (See [16] for a detailed discussion of what happens as the value of Ω is changed).

For more general values of δ the situation changes as follows (fig 1b,c): whenever $\delta/\nu = k$ where k is a nonzero integer (i.e., the laser is tuned to a sideband transition), it is easy to see that $E_m^+ = E_{m+k}^- = (m+1+k/2)\nu$. In other words $|\psi_{ion}^{m+}\rangle$ and $|\psi_{ion}^{m+k-}\rangle$ are again degenerate

eigenstates, and again the degenerate energies coincide with the asymptotic values for large η (recall sec. III). In fact, it can be verified that $\det M_m^+ = 0 \Rightarrow \det M_{m+k}^- = 0$, so that both eigenstates occur for the same system parameters, thus corresponding once again to line crossings in the spectrum (fig. 1b). (For low values of m, k this can be shown explicitly, for higher values one can again resort to numerical methods). It seems likely that this coincidence may once again be due to an underlying symmetry, however in this case we have not been able to determine it. Finally, when δ is not an integer, the crossings in the spectrum become avoided crossings (fig. 1c), and in fact it is easy to show, using a method analogous to that in ref. [16], that *no* degeneracies can occur in the system. In this case, the solutions obtained above simply mark (some of) the points where the spectrum crosses its asymptotic value (fig. 1c).

V. CONCLUSION

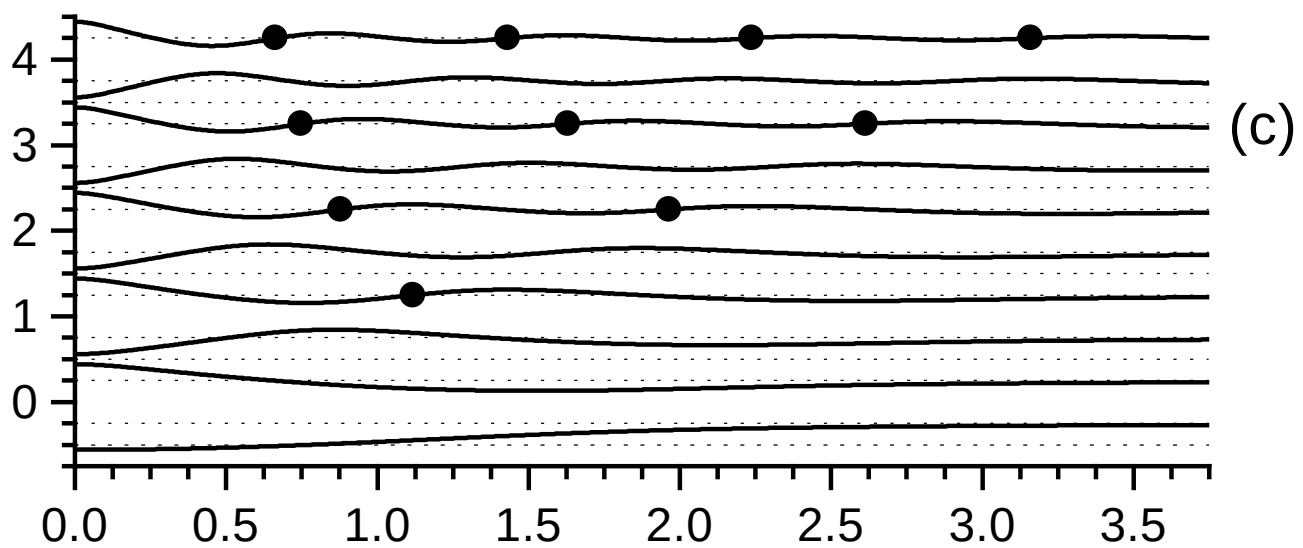
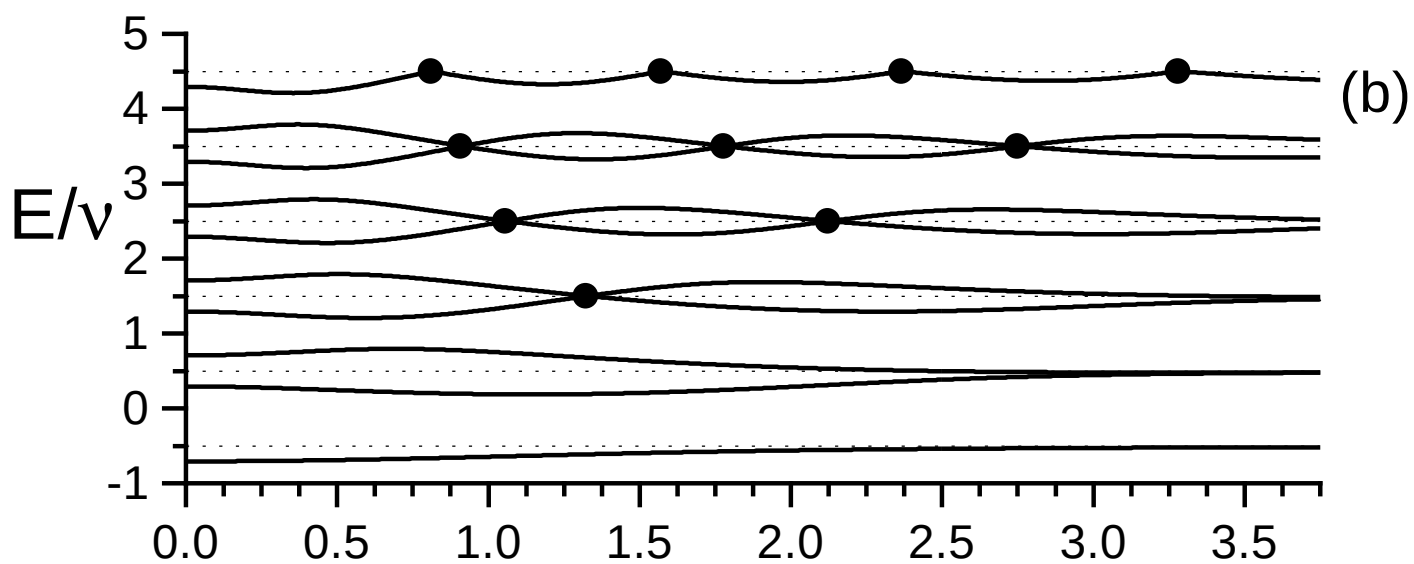
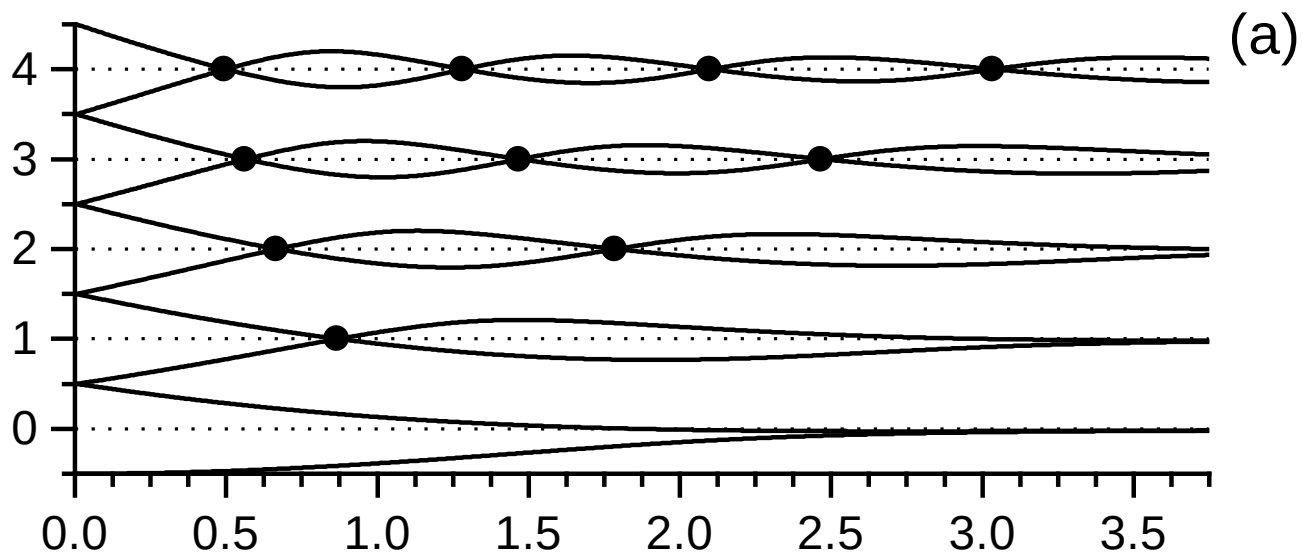
In conclusion, we have shown that in certain circumstances it is possible to obtain exact eigenstates for the Hamiltonian of a single trapped ion. These states are often (but not always) connected with the existence of level crossings in the system's energy spectrum viewed as a function of the Lamb-Dicke parameter. Another property is that they are always expressible as a finite expansion in terms of certain physically well-motivated vibrational states (number states and displaced number states). In this regard, it is worth recalling a long-standing conjecture concerning a possible *complete* solution to the JCM in terms of known functions (valid for any values of the system parameters). It was suggested in [22] that the eigenstates of this system may always be expressible as a finite expansion in terms of a certain basis set, which in Bargmann representation corresponds to a particular class of transcendental functions. As far as we know this proposal has never been analytically verified, although numerically it seems to hold [22, 23]. In the light of our present results, it is natural to speculate that this result should still hold in the presence of an additional static field, and thus should also apply to the single-ion system.

H. M.-C. would like to thank CONACYT (Consejo Nacional de Ciencia y Tecnología) and the Royal Society of London for support, and also Imperial College for hospitality. D.J. was supported in part by the Brazilian Government's Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and by Pembroke College, Cambridge.

FIG. 1: First few energy levels (in units of $\hbar$) of a single trapped ion driven by a laser beam, as a function of the Lamb-Dicke parameter η , for coupling constant $\Omega = \nu/2$. Dark circles denote parameter values for which exact eigenstates of this system can be found using the ansatz described in section IV. (a) When the laser is resonant ($\delta = 0$), a solution exists whenever the energy E is an integer multiple of the trap frequency ν , or in other words whenever the curves intersect the horizontal asymptotes to which all energy levels tend in the limit $\eta \rightarrow \infty$. These are also the level crossing points in the spectrum [16]. (b) A similar behaviour occurs when $\delta/\nu = k$ for integer $k \neq 0$ (i.e., the laser is tuned to one of its sidebands) (here $k = 1$). In this case the level crossings are at points where $E = (m + 1 + 0.5)\nu$, which are again the asymptotic values of E . (c) Finally, when δ is not tuned to a sideband ($\delta = 0.5\nu$ here), level crossings become avoided crossings, and our ansatz finds exact eigenstates only at some of the points where the curves cross the asymptotes $E_m = (m + 1.25)\nu$.

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