

Exact solutions for laminar pressure-driven flow through a pipe with a concentric internal grating

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This paper presents an analytical model for laminar pressure-driven axial flow along a circular pipe containing an internal concentric circular grating of no-slip fins. Explicit expressions for the velocity field are derived using complex analysis techniques and conformal mappings. From these solutions, formulas for the Fanning friction factor associated with axial flow in such finned pipes are derived and, using these, the drag properties of the pipes are examined. The results offer valuable insights for optimizing finned duct designs in applications involving heat transfer, fluid flow and biofilm cultivation.

Keywords: laminar duct flow; pipe flow; friction factor.

1. Introduction

The study of laminar flow in ducts is central to many engineering applications, including the study of forced convective heat transfer. In view of its importance, a monograph, or ‘source book’, by [Shah & London \(2014\)](#) provides a valuable comprehensive collection of analytical results (those known at the time of its publication) for pressure-driven laminar flow and forced convective heat transfer in various duct geometries, i.e. in pipes with differing cross-sections. Those authors consider fully developed, steady-state, pressure-driven flow subject to a no-slip boundary condition and they surveyed the effect of the fluid properties and the duct geometry on the flow patterns. They document analytical solutions for the velocity fields and use this to calculate the friction factor, a quantifier of the degree of drag relative to the dynamic pressure of the flow depending on the geometric configuration. It is of great interest to be able to predict and control quantities such as pressure drop, heat transfer and other characteristics.

Analytical models are essential for this, as they provide valuable insights that are crucial for optimizing these systems.

Scientific interest in pressure-driven flow in pipes is no less important in modern technologies, although the concept of pipe flow has broadened to include the notion of a superhydrophobic pipe wherein portions of the boundary are made up of free surfaces, or menisci, rather than solid no-slip surfaces. Such pipes have the advantage that fluid can slip over the free surface portions of the boundaries meaning that such pipes can have beneficial drag properties and lead to greater flow rates for a given pressure drop. In an important study, Philip (1972) used complex analysis to find analytical solutions for pressure-driven flow in a pipe with a superhydrophobic boundary, i.e. a boundary made up of an alternating array of no-slip and no-shear portions with N -fold rotational symmetry where $N \geq 1$ is an integer. More recently, Crowdy (2021) has generalized this solution of Philip to find analytical solutions for laminar flow in an annular pipe with two concentric walls where one of them is superhydrophobic and the other is a no-slip surface. Another recent extension is to find generalizations of Philip's superhydrophobic pipe flow for general configurations of no-shear slots not necessarily possessing any rotational symmetry (Zimmermann *et al.*, 2025). Being a relatively modern concept, superhydrophobic pipes are not a focus of Shah & London (2014) but recent work by Rodriguez-Broadbent & Crowdy (2022) has shown that certain, but not all, superhydrophobic pipes allow for an enhancement of the heat transfer properties. This result was unexpected since earlier studies had failed to find any such enhancement (Kirk *et al.*, 2017) in other flow situations such as channel flows over superhydrophobic surfaces.

The present paper is not concerned with superhydrophobic pipes, *per se*, although the results here—which involve a more traditional form of laminar duct flow—were derived while the authors were investigating them. Indeed, the mathematical machinery used here to find analytical solutions—the theory of the prime function associated with multiply connected domains (Crowdy, 2020)—is the same as that used in Crowdy's study of a superhydrophobic annular pipe (Crowdy, 2021). That same mathematical machinery has also recently been used by Miyoshi *et al.* (2024) to find analytical solutions for flow in a classical heat transfer problem: pressure-driven flow along a so-called shrouded fin array (Sparrow *et al.*, 1979). There the fins are introduced to increase the surface area available for transfer of heat into the liquid, which then carries it away in a direction aligned with the fins. The recent contribution by Miyoshi *et al.* (2024) is close in spirit to that of the present paper.

It is shown here that the problem of pressure-driven flow along a pipe with a no-slip outer wall and containing an interior concentric circular grating with N rotationally symmetric no-slip fins admits a closed-form analytical solution for the associated flow field. Figure 1 shows a cross-section of such a pipe. The existence of an analytical result is surprising given that the flow cross-section is intrinsically multiply connected; analytical progress in such cases is known to be much harder to achieve. It is notable that only very simple instances of ducts with multiply connected cross-sections are featured in the monograph by Shah & London (2014). The new solutions here add to that source book of analytical results for laminar duct flows.

Finned pipes are of interest in the study of convective heat transfer due to their ability to provide additional fluid-solid contact area and are therefore often studied for optimizing heat exchangers. Usually, pipes with radial slits emanating from the exterior boundary are considered for this purpose (Zeitoun & Hegazy, 2004; Nobari & Gharali, 2006; Qiuwang *et al.*, 2008; Duplain & Baliga, 2009). Their overall efficiency depends on geometrical factors including the number of fins, their individual length, the distance between each fin and their distance from the centre of the tube. More generally, the problem of optimizing heat exchanger efficiency becomes one of maximizing the overall surface area while staying within the acceptable limits of energy loss due to increasing wall shear stress. The pipes considered here, with an interior concentric circular grating of fins that are not directly connected to the outer boundary, furnish a different class of finned pipe that may also find use in heat transfer or related applications.

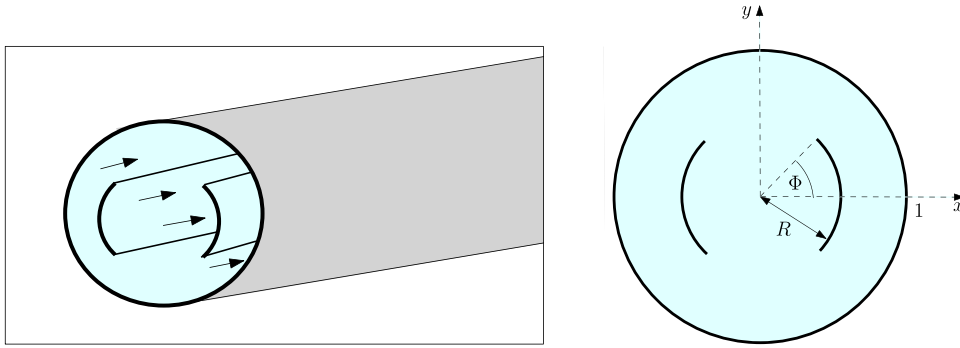


FIG. 1. A circular pipe with an interior concentric circular grating with $N = 2$ no-slip fins (left). The nondimensionalized geometry: a cross section of the pipe with $N = 2$ no-slip fins at radius R and each of arc-length 2Φ (right).

Furthermore, the cultivation of productive biofilms presents a challenge potentially related to finned pipes. Biofilms are often used for efficient industrial production of valuable products, such as citric acid (Anderson *et al.*, 1980), gluconic acid (Heinrich & Rehm, 1982) and many more (Schmeckebier *et al.*, 2022). Finned pipes are a useful design option for such bioreactors because they significantly increase the overall wall surface, enhancing microorganism deposition and enabling biofilm growth (Al-Amshawee *et al.*, 2021). Furthermore, although biofilms provide some protection to cells and enhance resistance to environmental factors, it is essential to establish isolated regions of low wall shear stress to prevent biofilm collapse caused by abrasion. In other cases, however, biofilm accumulation in pipes is highly undesirable, as their deposition can lead to drinking water pollution, pipe corrosion and increased drag (Firouzi & Hashemabadi, 2009; Cowle *et al.*, 2014). Other potential applications for finned duct geometries may include catalyst reactions (Okayama *et al.*, 1999; Fukuhara & Igarashi, 2004), direct flow filtration (Wang *et al.*, 2017; Razak *et al.*, 2020) and more.

The main result of this paper is to show that a (suitably nondimensionalized) solution for the axial flow $\mathbf{u} = (0, 0, w(x, y))$ in a pipe such as that shown in Fig. 1, is given, in terms of a parametric complex variable ζ , by

$$\begin{aligned} w &= \frac{1 - |\mathcal{Z}(\zeta)|^2}{4} - \frac{1 - R^2}{4} \frac{\log |\zeta|}{\log \rho}, \\ z = \mathcal{Z}(\zeta) &= \left(\alpha \frac{P(\zeta/\alpha, \rho)}{P(\zeta\alpha, \rho)} \right)^{1/N}, \end{aligned} \quad (1.1)$$

where $z = x + iy$ and R is the radius of the circular grating as indicated for the two-finned pipe shown in Fig. 1. The parametric variable ζ takes values in a concentric annulus $D_\zeta = \{\zeta | \rho < |\zeta| < 1\}$ and α is an interior point in this domain. The function $P(\cdot, \cdot)$ is related to the so-called prime function (Crowdy, 2020) associated with the annulus D_ζ and can be defined for any $\zeta \neq 0$ by the convergent infinite product

$$P(\zeta, \rho) = (1 - \zeta) \prod_{n=1}^{\infty} (1 - \rho^{2n}\zeta)(1 - \rho^{2n}/\zeta), \quad 0 \leq \rho < 1. \quad (1.2)$$

As will become clear in the derivation to follow, the function $\mathcal{Z}(\zeta)$ is a conformal mapping that transplants D_ζ to a typical sector of the finned pipe.

This paper is structured as follows: Section 2 introduces the underlying boundary value problem governing the axial flow in the class of finned pipes shown in Fig. 1. It is shown that this boundary value problem can be reduced to the task of determining an analytic function $h(z)$ where $z = x + iy$. An explicit parametric representation of this analytic function is determined in Section 3 using conformal mapping from a canonical annulus in a complex parametric ζ plane. With the solutions at hand, some representative axial velocity contours are plotted in Section 4. Section 5 calculates and analyses the friction factor/Reynolds number product fRe (defined later) from the analytical expressions for the velocity field.

2. Boundary value problem for the axial flow

Following the standard formulations of laminar duct flow as treated, for example, in Shah & London (2014), the laminar unidirectional flow $\mathbf{u} = (0, 0, w(x, y))$, with respect to a Cartesian (x, y, Z) plane, where $w(x, y)$ is the axial velocity, is driven by a constant pressure gradient $\partial p / \partial Z = -s$ along the Z axis of a circular pipe of radius $\mathcal{R}$. Preventing the flow from being a simple Poiseuille flow is a concentric circular grating inside the pipe featuring N rotationally symmetric no-slip fins as illustrated in Fig. 1. Nondimensionalizing lengths with respect to $\mathcal{R}$ and the velocity w using $s\mathcal{R}^2/\mu$ where μ is the dynamic viscosity, such a flow is governed by Poisson's equation

$$\nabla^2 w = -1. \quad (2.1)$$

The no-slip condition requires

$$w = 0 \quad (2.2)$$

on the outer boundary, now of unit radius, and on all the N fins of the interior grating located at distance R from the centre of the pipe as shown for a two-finned pipe $N = 2$ in Fig. 1. The nondimensional parameter R is the ratio of the radius of the circular grating to the radius of the outer pipe boundary.

Choosing standard Poiseuille pipe flow, denoted by w_{ps} , as a convenient particular solution, the general solution of the linear boundary value problem (2.1)–(2.2) can be written, as a function of the complex variable $z = x + iy$, as

$$w = w_{ps} + w_h, \quad (2.3)$$

where

$$w_{ps} = \frac{1 - |z|^2}{4}, \quad w_h = \text{Im}[h(z)]. \quad (2.4)$$

An analytic function, or complex potential, $h(z)$ has been introduced of which the harmonic correction to w_{ps} , denoted by w_h is the imaginary part. The challenge now is to determine $h(z)$.

3. Analytical solution using complex variable methods

To solve the boundary value problem stated in §2, a parametric concentric annulus

$$\rho < |\zeta| < 1 \quad (3.1)$$

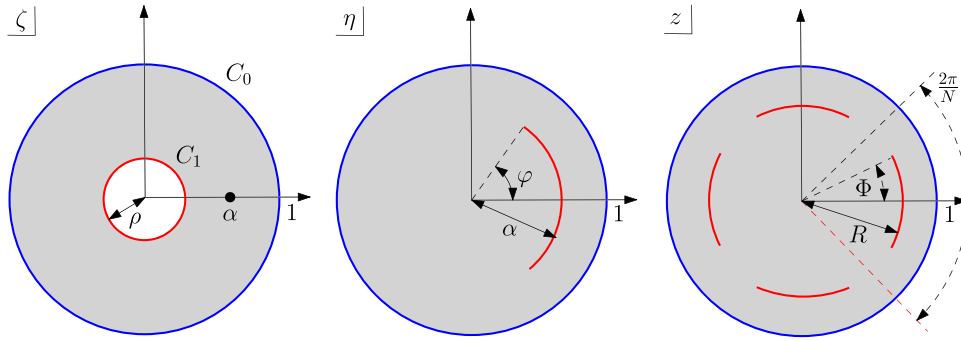


FIG. 2. Composition of conformal mappings. The circular slit map (3.2) transplants the concentric annulus D_ζ to a disc in the η plane with a single circular slit of arc-length 2φ . This is then transplanted, by (3.5), to a representative period window of opening angle $2\pi/N$ in the physical z plane.

in a complex ζ plane, to be denoted by D_ζ , is introduced. The boundary circles, $|\zeta| = 1, \rho$, will be called C_0 and C_1 , respectively. A conformal mapping from D_ζ to another parametric η plane is defined by

$$\eta = f(\zeta) = \alpha \frac{P(\zeta/\alpha, \rho)}{P(\zeta\alpha, \rho)}, \quad (3.2)$$

where $\alpha \in \mathbb{R}$ is inside D_ζ and $P(.,.)$ is defined in (1.2). The latter function, which is related to the prime function of D_ζ , satisfies the two functional relations (Crowdy, 2020)

$$P(1/\zeta, \rho) = -\zeta^{-1}P(\zeta, \rho), \quad P(\rho^2\zeta, \rho) = -\zeta^{-1}P(\zeta, \rho). \quad (3.3)$$

This is readily verified on use of the infinite product definition (1.2). The function (3.2), built as a ratio of prime functions, is an example of a bounded circular slit map as discussed in a recent monograph on prime functions by one of the authors (Crowdy, 2020); it can be shown to transplant D_ζ to the unit disc in the complex η plane with C_0 being transplanted to $|\eta| = 1$ while the image of C_1 is an interior circular arc centred at $\eta = 0$, as depicted in Fig. 2. Appendix A shows how the functional relations (3.3) can be used to establish some of the claimed properties of the bounded circular slit map (3.2), although full details are given in Crowdy (2020). An important observation is that the radius of the circular slit in the η plane is α . To furthermore account for the desired N -fold rotational symmetry in the z plane, η is connected to z by

$$z = g(\eta) \equiv \eta^{1/N} \quad (3.4)$$

On composition of the two mappings (3.2) and (3.4), the required mapping from D_ζ to a representative sector of the z plane is given by

$$z = \mathcal{Z}(\zeta) \equiv g(f(\zeta)) = f(\zeta)^{1/N} = \left(\alpha \frac{P(\zeta/\alpha, \rho)}{P(\zeta\alpha, \rho)} \right)^{1/N}, \quad \alpha = R^N, \quad (3.5)$$

where R is the the radius of the circular grating.

Given this mapping, the next step is to introduce the composed function

$$h(z) = h(\mathcal{Z}(\zeta)) \equiv H(\zeta). \quad (3.6)$$

Relation (2.4) can be rewritten as

$$w = \frac{1 - |\mathcal{Z}(\zeta)|^2}{4} + \operatorname{Im}[H(\zeta)], \quad (3.7)$$

where it is now $H(\zeta)$ that is to be determined. The no-slip boundary conditions imply that

$$\begin{aligned} \operatorname{Im}[H(\zeta)] &= 0, & \zeta \in C_0, \\ \operatorname{Im}[H(\zeta)] &= -\left(\frac{1-R^2}{4}\right), & \zeta \in C_1. \end{aligned} \quad (3.8)$$

Consider the choice

$$H(\zeta) = -i \left(\frac{1-R^2}{4} \right) \frac{\log \zeta}{\log \rho}. \quad (3.9)$$

It is easy to verify, using basic properties of the complex logarithm, that this function is analytic in D_ζ , and also satisfies the boundary conditions (3.8). The solution for the unidirectional flow $w(x, y)$ through the pipe is therefore given, as a function of ζ , by

$$w = \frac{1 - |\mathcal{Z}(\zeta)|^2}{4} - \frac{1-R^2}{4} \frac{\log |\zeta|}{\log \rho} \quad (3.10)$$

with $\mathcal{Z}(\zeta)$ defined explicitly in (3.5). This leads to the parametric solution (1.1) with both w and z given as explicit functions of the parameter ζ .

In any practical setting, the grating will be characterized by the two parameters R and the solid fraction $\varphi/\pi = N\Phi/\pi$ where the angles φ and Φ are shown in Fig. 2; this is the fraction of the circular grating consisting of no-slip fins (rather than gaps). For a given grating radius R , the parametric solution (1.1) also depends on the single real parameter ρ . This mathematical parameter is in correspondence with the solid fraction. To make use of the parametric solution (1.1) it only remains to find ρ given the two physical characteristics R and $\varphi/\pi = N\Phi/\pi$.

To find ρ , first note from Fig. 2 that the edges of the fin in the η plane are at $\alpha e^{\pm i\varphi}$. Let $\zeta_\pm$ be the preimages of these points, i.e.

$$f(\zeta_\pm) = \alpha e^{\pm i\varphi}, \quad \text{or} \quad \pm \varphi = \arg[f(\zeta_\pm)], \quad (3.11)$$

where $\arg[\cdot]$ refers to the argument of the complex quantity in square brackets. The preimages $\zeta_\pm$ are not known *a priori*, but they lie on C_1 so $\zeta_\pm = \rho e^{i\chi_\pm}$ where, by the reflectional symmetry with respect to the real axis in both preimage and target domains, it is expected that $\chi_- = -\chi_+$. It is therefore enough

to consider the single real equation

$$\varphi = \arg \left[\frac{P\left(\frac{\rho e^{i\chi_+}}{\alpha}, \rho\right)}{P(\rho e^{i\chi_+} \alpha, \rho)} \right], \quad (3.12)$$

where $\alpha = R^N$. The points $\zeta_{\pm}$ are the preimages of the two edges of a slit where the derivative of the conformal mapping function $f(\zeta)$ vanishes, i.e.

$$\frac{df}{d\zeta}(\zeta_{\pm}) = 0. \quad (3.13)$$

These conditions can also be stated as the two real equations

$$\left. \frac{d\tilde{\varphi}}{d\theta} \right|_{\theta=\chi_{\pm}} = 0, \quad \eta = \alpha e^{i\tilde{\varphi}}, \quad \zeta = \rho e^{i\theta}, \quad (3.14)$$

only one of which is needed because of the reflectional symmetry. Using (3.2), a second real equation relating ρ and χ_+ resulting from (3.14) is

$$\operatorname{Re} \left[K\left(\frac{\rho e^{i\chi_+}}{\alpha}, \rho\right) - K(\rho e^{i\chi_+} \alpha, \rho) \right] = 0, \quad (3.15)$$

with

$$K(\zeta, \rho) \equiv \frac{\zeta}{P(\zeta, \rho)} \frac{\partial P(\zeta, \rho)}{\partial \zeta} = -\frac{\zeta}{1-\zeta} - \sum_{k=1}^{\infty} \frac{\rho^{2k} \zeta}{1-\rho^{2k} \zeta} + \sum_{k=1}^{\infty} \frac{\rho^{2k} / \zeta}{1-\rho^{2k} / \zeta}, \quad (3.16)$$

where (1.2) has been used. The two real equations (3.12) and (3.15) must therefore be solved simultaneously for the two real parameters χ_+ and ρ for given values of R and $\varphi = N\Phi$. This is easily done using Newton's method, and more will be said about the choice of initial conditions for the Newton iteration in Section 5 where a calculation of volume fluxes provides additional insights.

Table 1 exhibits some sample data obtained from solving this parameter problem for a range of grating geometries. Although χ_+ is solved for, it is only the value of ρ that is needed to complete the specification of the parametric solution (1.1). While this simple numerical step is needed to determine the solution of the parameter problem for a given pipe geometry, once the relevant value of ρ is found, the parametric solution (1.1) gives z and w as explicit functions of ζ .

4. Velocity contours

Figure 3(a–d) shows typical velocity contours of the axial flow for different values of N , φ and R , based on (1.1).

Figure 3(e) compares the contour lines of the analytically calculated velocity field of (3.10) to those obtained from a numerical simulation ($R = 1/2$ and $\varphi = \pi/2$), using the finite-element solver COMSOL Multiphysics. For that, the 2D Poisson equation was solved numerically, subject to a Dirichlet boundary

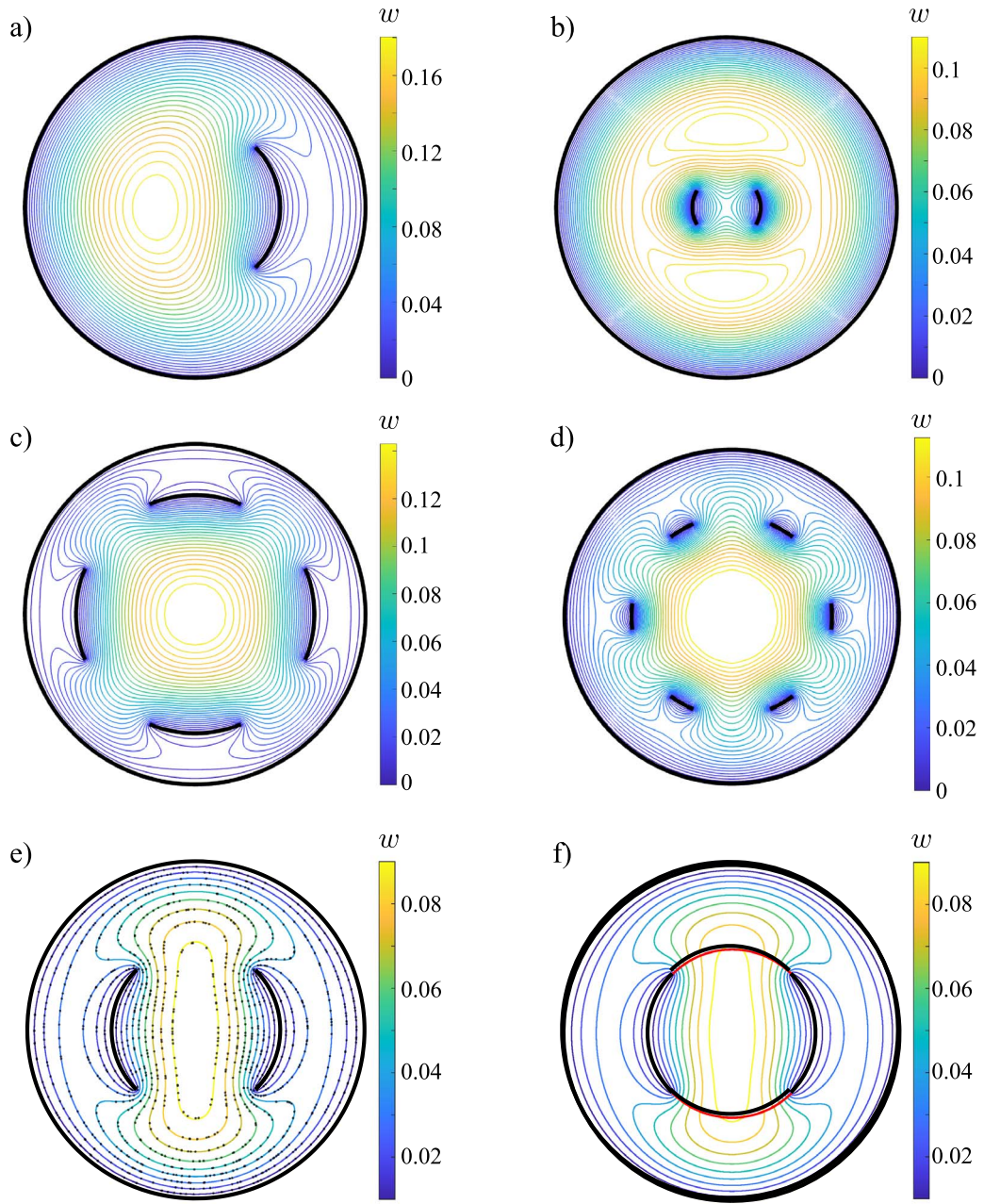


FIG. 3. Constant axial velocity contour lines for laminar flow through pipes with concentric circular gratings. (a) $R = 0.5, N = 1, \phi = \pi/4$; (b) $R = 0.2, N = 2, \phi = \pi/3$; (c) $R = 0.7, N = 4, \phi = \pi/2$; (d) $R = 0.6, N = 6, \phi = \pi/4$; (e) Comparison of the analytical solution (3.10) and a numerical simulation (black dots) for $R = 0.5, N = 2, \phi = \pi/2$; (f) Coupled flow of a pipe-in-pipe configuration as derived in Zimmermann & Schönecker (2024), assuming both regions to be occupied by the same fluid and driven by the same pressure gradient, for $R = 0.5, N = 2, \phi = \pi/2$.

TABLE 1 Sample data from solving (3.12) and (3.15)

N	Radius R	Angle φ	ρ	χ_+
1	0.2	$\pi/4$	0.0788	-1.1535
1	0.4	$\pi/4$	0.1726	-1.0748
1	0.6	$\pi/4$	0.3042	-0.9241
2	0.5	$\pi/4$	0.1002	-1.1394
2	0.5	$\pi/2$	0.1796	-0.7536
2	0.5	$3\pi/4$	0.2313	-0.3855
4	0.5	$3\pi/4$	0.0577	-0.3923

condition $w = 0$ on all walls. These numerically calculated contour lines are represented by black dots and show excellent agreement with the analytical result.

The solution (1.1) can be considered a special case of the analytical models recently described by Zimmermann & Schönecker (2024). These models expand on previous analytical solutions that describe laminar pressure-driven axial flow through pipes (Philip, 1972) and annular tubes (Crowdy, 2021) subject to mixed no-slip and no-shear boundary conditions. They incorporate patterns of finite-shear slits in otherwise no-slip walls. Although the original purpose was to model superhydrophobic and liquid-infused (annular) pipes, both the inner and outer tubes can be coupled along these finite-shear boundary parts to generate a pipe-in-pipe geometry. The velocity field of that set-up is fully determined and described by a system of two equations. Figure 3(f) shows contour lines for $R = 1/2$ and $\varphi = \pi/2$, however, using the solutions from Zimmermann & Schönecker (2024), which are originally given in a different form, namely as a system of equations. The expressions derived and presented here therefore serve to further confirm the solutions from Zimmermann & Schönecker (2024). We assume that both regions are occupied by the same fluid, driven by the same pressure gradient s , and are coupled along the finite-shear boundary parts indicated as red arcs at $|z| = R$. The results demonstrate excellent agreement with the calculations presented in Fig. 3(e). The key difference is that the present analysis has produced a more straightforward solution for generating these velocity fields.

5. The friction factor

Returning temporarily to dimensional quantities, the Fanning friction factor f appears in the standard engineering equation for the pressure drop ΔP over distance L due to viscous losses in a steady laminar pipe flow with cross-section P , namely,

$$\frac{\Delta P}{L} = 4f \left(\frac{\rho w_{\text{avg}}^2}{2} \right) D_H, \quad (5.1)$$

where w_{avg} is an average velocity defined by

$$w_{\text{avg}} = \frac{Q}{\mathcal{A}}, \quad Q = \int_P \int_P w dA, \quad (5.2)$$

where Q is the volumetric flow rate and $\mathcal{A}$ is the cross-sectional area of P while

$$D_H = \frac{4\mathcal{A}}{\mathcal{P}} \quad (5.3)$$

is the hydraulic diameter with $\mathcal{P}$ being the total perimeter of P wetted by the flow. Defining the Reynolds number to be

$$\text{Re} = \frac{\tilde{\rho} w_{\text{avg}} D_H}{\mu}, \quad (5.4)$$

where $\tilde{\rho}$ is the fluid density then, on combining (5.1) and (5.4), the friction-factor-Reynolds number product (henceforth referred to as ‘the friction factor’ for brevity) is

$$f\text{Re} = \frac{1}{2} \left(\frac{\Delta P}{\mu L} \right) \frac{D_H^2}{w_{\text{avg}}}. \quad (5.5)$$

Expressed in terms of the nondimensional quantities introduced earlier, i.e.

$$f\text{Re} = \frac{8\mathcal{A}^3}{Q\mathcal{P}^2}. \quad (5.6)$$

For a circular pipe without an internal grating, $\mathcal{P} = 2\pi$, $\mathcal{A} = \pi$ and with a standard normalized Poiseuille flow in such a pipe, with volumetric flow rate denoted by Q_{ps} , given by

$$Q_{\text{ps}} = \int_0^1 \int_0^{2\pi} \frac{1-r^2}{4} r \, dr \, d\theta = \frac{\pi}{8}, \quad (5.7)$$

where (r, θ) are cylindrical polar coordinates, this becomes

$$f\text{Re} = 16. \quad (5.8)$$

For the more general class of pipes of interest in this paper, $\mathcal{P} = 2\pi + 4\Phi RN = 2\pi + 2\pi R\varphi$, $\mathcal{A} = \pi$, so that the result is

$$f\text{Re} = \frac{2\pi}{Q(1 + 2R(\varphi/\pi))^2}, \quad (5.9)$$

where it remains to calculate the volumetric flow rate Q in such pipes. To do so, it is convenient to split it into components given by

$$Q = Q_{\text{ps}} + Q_{\text{h}}, \quad (5.10)$$

where the second term is

$$Q_h = \iint_P w_h \, dA = - \left(\frac{1-R^2}{4} \right) \frac{1}{\log \rho} \operatorname{Re} \left[\iint_P \log \zeta \, dA \right]. \quad (5.11)$$

The complex form of Stokes theorem can be applied to (5.11) but care is needed because the integrand is not single-valued in P . It can be argued that, on transferring the integral to the ζ plane,

$$Q_h = - \left(\frac{1-R^2}{8} \right) \frac{1}{\log \rho} \operatorname{Re} \left[\int_{\partial A} \frac{\log \zeta \, |\mathcal{Z}'(\zeta)|^2}{i \zeta} \mathcal{F}(\zeta) \, d\zeta \right], \quad (5.12)$$

where ∂A is a special contour introduced in Appendix C and $dz = \mathcal{Z}'(\zeta)d\zeta$ with

$$\mathcal{Z}'(\zeta) = \frac{\mathcal{Z}(\zeta)}{N\zeta} \mathcal{F}(\zeta), \quad \mathcal{F}(\zeta) = \frac{\zeta f'(\zeta)}{f(\zeta)} = K(\zeta/\alpha, \rho) - K(\zeta\alpha, \rho) \quad (5.13)$$

with $K(\zeta, \rho)$ defined in (3.16); Appendix B gives more details of the derivation of (5.13). Note that a factor of $1/N$ has disappeared because the integral around ∂A only gives the contribution to Q_h from one of the N sectors of the pipe. Appendix C also shows how, from (5.12), it can be deduced that,

$$Q_h = \frac{N\pi(1-R^2)^2}{8 \log \rho}. \quad (5.14)$$

Together, (5.10), (5.7) and (5.14) lead to the concise expression

$$Q = \frac{\pi}{8} \left(1 + \frac{N(1-R^2)^2}{\log \rho} \right). \quad (5.15)$$

On combining (5.9) and (5.15),

$$f\operatorname{Re} = \frac{16}{(1 + 2R(\varphi/\pi))^2 (1 + N(1-R^2)^2/\log \rho)}, \quad (5.16)$$

where φ/π is the solid fraction of the grating. At first sight, this formula appears to be explicit, but recall that $\log \rho$ is only known once the nonlinear equations (3.12) and (3.15) have been solved for ρ (and χ_+) given R, N and φ .

It is worth noting that the functional form of the volumetric flow rate Q given in (5.15) is not dissimilar to the expression for the sum of the volumetric flow rate due to pressure-driven flow in a circular pipe of radius R with that, caused by the same constant pressure gradient, in a concentric annulus $R < |z| < 1$ which, after a straightforward calculation, is

$$\frac{\pi}{8} \left(1 + \frac{(1-R^2)^2}{\log R} \right). \quad (5.17)$$

Indeed, when the solid fraction φ/π is close to unity, a comparison of (5.15) and (5.17) suggests the scaling $\rho \sim R^N$ because, as $\varphi/\pi \rightarrow 1$, the flux in the pipe-with-circular-grating as given by (5.15) is

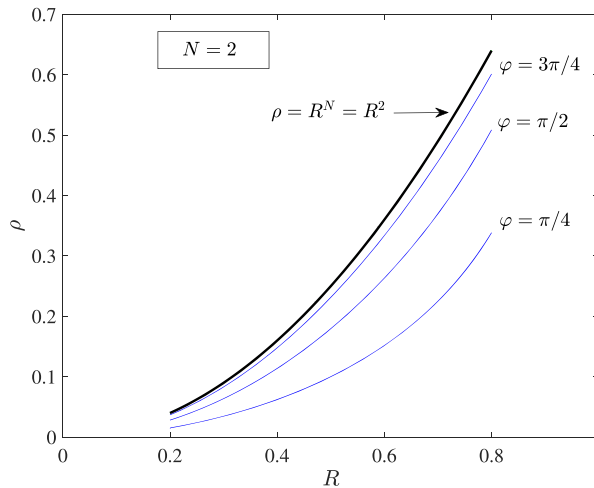


FIG. 4. Graphs of the values of ρ solving (3.12) and (3.15) for $0.2 \leq R \leq 0.8$ with fixed $\varphi = \pi/4, \pi/2$ and $3\pi/4$ and $N = 2$. As the solid fraction $\varphi/\pi \rightarrow 1$ the solution for ρ tends to the graph $\rho = R^N = R^2$ (the solid black line) as suggested by a comparison of the flux expressions (5.15) and (5.17).

expected to tend to (5.17). This was confirmed to be true, as shown in Fig. 4 for the case $N = 2$. It shows graphs of the values of ρ found by solving (3.12) and (3.15) over the range $0.2 \leq R \leq 0.8$ for increasing $\varphi = \pi/4, \pi/2$ and $3\pi/4$: the graphs approach the curve $\rho = R^2$ (shown as a solid black line) as φ increases. This insight is valuable: it motivates taking the initial guess $\rho = R^N$ when using Newton's method to solve (3.12) and (3.15) and this turns out to be a robust strategy for most values of R, N and φ with only the initial guess for χ_+ requiring occasional alteration to ensure convergence to the solution.

5.1 Effect of varying pipe geometry on fRe

Figure 5 features two graphs: that on the left shows fRe as a function of grating radius R for a single-finned grating $N = 1$ when the solid fraction is fixed so that $\varphi = \pi/4, \pi/2$ and $3\pi/4$; that on the right shows similar graphs for a two-finned grating $N = 2$. The graphs for $N = 1$ and $N = 2$ are qualitatively similar, a prominent feature being the presence of a local maximum of the friction factor. This means that for gratings with a given number of fins N and a given solid fraction, there is a critical radius of the grating at which the friction factor is maximal. A similar qualitative feature was observed for the slip lengths associated with superhydrophobic annular pipes (Crowdy, 2021).

Figure 6 shows fRe as a function of R , for $0.2 \leq R \leq 0.8$, as the number of fins increases from $N = 1$ to $N = 7$ for gratings with a fixed solid fraction $\varphi = \pi/4$ meaning that, as N increases, the length of each fin in the grating decreases in such a way that the total solid fraction at a given radius R is unchanged. An interesting feature of this graph is that, for each fixed R shown, and fixed solid fraction, the value of fRe increases monotonically with N . Intuitively, increasing the number of fin edges present appears to increase the friction factor even if the solid fraction and radius R remain unchanged. Again, a similar qualitative feature was observed for superhydrophobic annular pipes (Crowdy, 2021).

It might be expected that by adding a grating, no matter what its geometry, the friction factor should exceed the value 16 for a circular pipe with no grating at all. This is because, by adding fins, more no-slip regions are being added to the circular pipe thereby increasing the resistance to flow. An intriguing

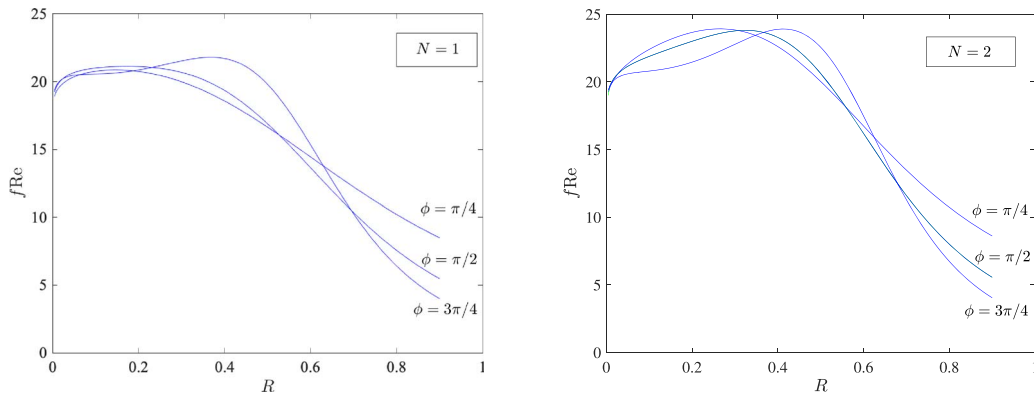


FIG. 5. The effect of changing R and ϕ for a fixed number of fins: graphs are shown of fRe as a function of R for $\phi = \pi/4, \pi/2$ and $3\pi/4$ for $N = 1$ (left) and $N = 2$ (right). All curves tend to the classical value $fRe = 16$ for a circular pipe as $R \rightarrow 0$ since the effect of the grating disappears in this limit.

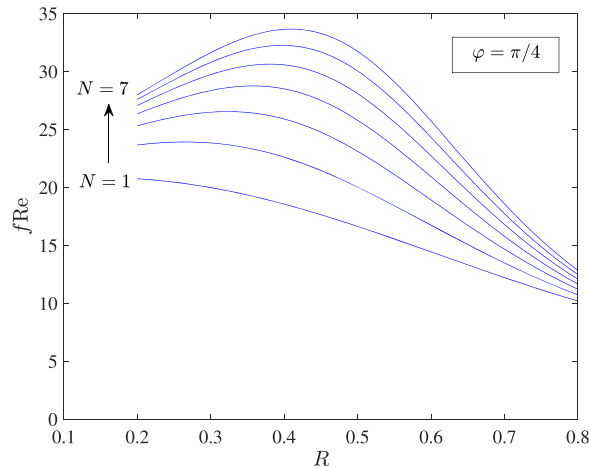


FIG. 6. The effect of increasing the number of fins in the grating for a fixed solid fraction $\phi = \pi/4$. The graphs show fRe as a function of R as N increases from $N = 1$ to $N = 7$.

feature of the graphs shown so far is that while the friction factor fRe does indeed increase for small R , as R gets closer to unity the value of fRe can descend below the value 16. To gain insight into this counter-intuitive feature, Fig. 7 shows graphs of the volumetric flux rate Q as a function of R for fixed $\phi = \pi/4, \pi/2$ and $3\pi/4$ for $N = 1$; graphs for other values of N are similar. In all cases, $Q < \pi/8$ which is the volumetric flow rate for flow in a circular pipe with no grating. Thus, as might be expected given the increased resistance to flow, the volumetric flow rate always decreases when a grating is added, no matter what its geometry. Formula (5.9) shows, however, that the friction factor fRe is not only inversely dependent on Q —and would therefore, all other things being equal, increase if Q decreases from the circular pipe value—but also depends inversely on the wetted perimeter, which clearly increases if the fins of an internal grating are added. The latter circumstance means that, all other things being equal, fRe would decrease if the wetted perimeter is increased. It is evident from Figs 5 and 6, where fRe can

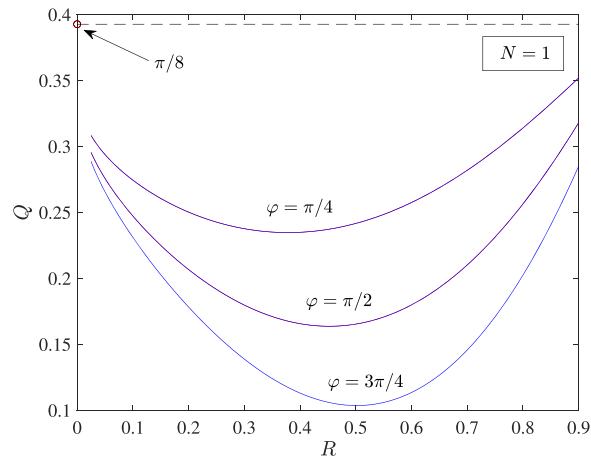


FIG. 7. Graph of volumetric flow rate Q as a function of R for fixed $\varphi = \pi/4, \pi/2$ and $3\pi/4$ for $N = 1$. As expected, no matter what the geometry of the grating, Q is always less than the value $\pi/8$ relevant for flow in a circular pipe without a grating.

decrease below 16 when R is sufficiently large, that, for such gratings, the decrease of fRe due to the increased wetted perimeter ‘wins’ over the increase due to the reduced Q leading to a net decrease in the friction factor.

Given that the pipes considered here always have a circular outer boundary, another choice of length scale in the definition (5.4) of the Reynolds number is the geometrical diameter, which remains constant even as fins are added. It was found that this gives qualitatively similar results to those reported here based on the more common choice of using the hydraulic diameter D_H given in (5.3).

6. Discussion

This paper has derived an analytical expression for the velocity field of a laminar pressure-driven flow in a pipe containing a concentric grating comprised of N rotationally symmetric no-slip interior fins. The solutions for the flow are explicit once a parameter ρ has been found by solving a parameter problem. From this analytical solution, it is an easy matter to evaluate and investigate the friction-factor/Reynolds number product, or fRe , associated with the flow along such a pipe. These friction characteristics have been examined in detail.

The longitudinal velocity for unimpeded pressure-driven longitudinal flow in a circular pipe is maximal at the centre of the pipe. It is reasonable to propose that adding a circular grating of small radius will therefore affect the friction factor most significantly since the flow near the centre of the pipe will be forced to be zero on the fins of the grating. However, since the grating radius in that case is small, the area of flow affected by the velocity reduction is not large and, as a result, the friction factor of the global flow is not affected as strongly as might be imagined. A general observation from Fig. 6 is that for a grating with a fixed no-slip fraction and a fixed number of fins, there is a critical radius at which the friction factor associated with such a grating is increased most. Another lesson from this study is that, at least for some within this class of hybrid pipes, the expected increase in friction due to the adding of a circular grating to a circular pipe flow is not reflected by the friction factor quantifier fRe .

This work constitutes a new addition to the analytical results for laminar pressure-driven flow collected in the source book of [Shah & London \(2014\)](#). It is also worth mentioning that boundary value problems for Poisson's equation with a constant right hand side appear in a wide range of physical and mathematical contexts. [Bazant \(2016\)](#) has reviewed no less than 17 other application areas where the same class of boundary value problems governing pressure-driven pipe flows also provide a model of the system. The new exact solutions identified in this paper are likely to find relevance in many of those other applications making the results here of rather broad significance.

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Conflict of interest

None of the authors report any conflict of interest.

Data availability

This paper reports analytical formulas from which all data can be readily reproduced.

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Appendix A. The bounded circular slit map (3.2)

This appendix sketches out some properties of the circular slit map (3.2). On taking a complex conjugate of (3.2) for $\zeta \in C_0$, it follows that

$$\overline{\eta(\zeta)} = \alpha \frac{P(\bar{\zeta}/\alpha, \rho)}{P(\bar{\zeta}\alpha, \rho)} = \alpha \frac{P((\zeta\alpha)^{-1}, \rho)}{P(\alpha/\zeta, \rho)}, \quad \zeta \in C_0, \quad (\text{A.1})$$

where we have used the fact that $\bar{P}(\zeta, q) \equiv \overline{P(\bar{\zeta}, \bar{q})} = P(\zeta, q)$ and $\bar{\zeta} = 1/\zeta$ on C_0 . On use of (3.3),

$$\overline{\eta(\zeta)} = \alpha \frac{-\frac{1}{\zeta\alpha} P(\zeta\alpha, \rho)}{-\frac{\alpha}{\zeta} P(\zeta/\alpha, \rho)} = \frac{1}{\alpha} \frac{P(\zeta\alpha, \rho)}{P(\zeta/\alpha, \rho)} = \frac{1}{\eta(\zeta)}, \quad \zeta \in C_0. \quad (\text{A.2})$$

Therefore, the image of C_0 lies on the unit circle $|\eta| = 1$. It can be shown, using similar manipulations and the properties (3.3), that the image of points on C_1 satisfy $\bar{\eta} = \alpha^2/\eta$ and therefore lie on a circle of radius α in the η plane, as shown in Fig. 2. Readers should refer to Crowdy (2020) for an explanation of why the image of C_0 is the whole unit circle $|\eta| = 1$ while the image of C_1 is a circular slit.

Appendix B. The derivative $d\mathcal{Z}/d\zeta$

This appendix shows how to derive a formula for $d\mathcal{Z}/d\zeta$ where $\mathcal{Z}(\zeta)$ is defined in (3.5). Using the chain rule

$$\begin{aligned}\frac{d\mathcal{Z}}{d\zeta} &= \frac{1}{N}\eta^{1/N-1} \left(\alpha \left(\frac{1}{\alpha} \frac{P'(\zeta/\alpha, \rho)}{P(\zeta\alpha, \rho)} - \alpha \frac{P(\zeta/\alpha, \rho)P'(\zeta\alpha, \rho)}{P(\zeta\alpha, \rho)^2} \right) \right), \\ &= \frac{1}{N}\eta^{1/N-1} \left(\alpha \frac{P(\zeta/\alpha, \rho)}{P(\zeta\alpha, \rho)} \left(\frac{1}{\alpha} \frac{P'(\zeta/\alpha, \rho)}{P(\zeta/\alpha, \rho)} - \alpha \frac{P'(\zeta\alpha, \rho)}{P(\zeta\alpha, \rho)} \right) \right).\end{aligned}\quad (\text{B.1})$$

But this can be written as

$$\frac{d\mathcal{Z}}{d\zeta} = \frac{1}{N}\eta^{1/N-1} \left(\frac{\eta}{\zeta} \left(\frac{\zeta}{\alpha} \frac{P'(\zeta/\alpha, \rho)}{P(\zeta/\alpha, \rho)} - \zeta\alpha \frac{P'(\zeta\alpha, \rho)}{P(\zeta\alpha, \rho)} \right) \right), \quad (\text{B.2})$$

which, on use of the definition of $K(\zeta, q)$ in (3.16), is

$$\frac{d\mathcal{Z}}{d\zeta} = \frac{1}{N\zeta}\eta^{1/N} (K(\zeta/\alpha, \rho) - K(\zeta\alpha, \rho)), \quad (\text{B.3})$$

from which (5.13) follows.

Appendix C. Evaluating the volume flux correction Q_h

This appendix shows how to calculate

$$Q_h = -\frac{1-R^2}{8} \frac{1}{\log \rho} \operatorname{Re} \left[\int_{\partial A} \frac{\log \zeta |\mathcal{Z}(\zeta)|^2}{i\zeta} \mathcal{F}(\zeta) d\zeta \right], \quad (\text{C.1})$$

as discussed in Section 5. A difficulty arises because the integrand in (C.1) is multi-valued due to the complex logarithm whose branch cut is chosen to run along the negative real axis. It is natural to decompose the term in (C.1) involving the integral into three separate contributions:

$$\operatorname{Re} \left[\int_{\partial A} \frac{\log \zeta |\mathcal{Z}(\zeta)|^2}{i\zeta} \mathcal{F}(\zeta) d\zeta \right] = I_1 + I_2 + I_3, \quad (\text{C.2})$$

where $\{I_j | j = 1, 2, 3\}$ denote contributions from the contour decomposition of ∂A shown in Fig. C8.

The three contributions are now considered separately. The contour in I_1 runs counter-clockwise around C_0 :

$$I_1 = \operatorname{Re} \left[\int_{C_0} \frac{\log \zeta |\mathcal{Z}(\zeta)|^2}{i\zeta} \mathcal{F}(\zeta) d\zeta \right], \quad \text{with } \zeta = e^{i\chi}. \quad (\text{C.3})$$

Since $|\mathcal{Z}(\zeta)| = 1$ on C_0 and $\frac{d\zeta}{i\zeta} = d\chi$ this can be written as

$$I_1 = \operatorname{Re} \left[\int_{-\pi}^{\pi} i\chi \mathcal{F}(e^{i\chi}) d\chi \right] = \operatorname{Re} \left[\int_{-\pi}^{\pi} \chi \frac{d}{d\chi} \log(f(e^{i\chi})) d\chi \right] = 0 \quad (\text{C.4})$$

because $|f(e^{i\chi})| = 1$ so $(d/d\chi) \log |f(e^{i\chi})| = 0$.

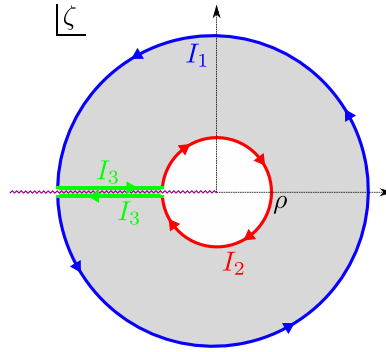


FIG. C8. The contour ∂A : the integral for Q_h is decomposed into the parts I_1, I_2, I_3 for evaluating the volume flux based on equation C.1 in the ζ plane. The branch cut runs along the negative real axis and is depicted as a dashed line.

Turning to I_2 whose contour is C_1 traversed counter-clockwise then with $\zeta = \rho e^{i\chi}$ on C_1 , and on use of the fact that $|\mathcal{Z}(\zeta)|^2 = R^2$ for ζ on C_1 ,

$$I_2 = -\operatorname{Re} \left[\int_{C_1} \frac{\log \zeta |\mathcal{Z}(\zeta)|^2}{i \zeta} \mathcal{F}(\zeta) d\zeta \right] = -R^2 \operatorname{Re} \left[\int_{-\pi}^{\pi} (\log \rho + i\chi) \mathcal{F}(\rho e^{i\chi}) d\chi \right]. \quad (\text{C.5})$$

By similar steps to those just used for I_1 ,

$$-R^2 \operatorname{Re} \left[\int_{-\pi}^{\pi} i\chi \mathcal{F}(\rho e^{i\chi}) d\chi \right] = -R^2 \operatorname{Re} \left[\int_{-\pi}^{\pi} \chi \frac{d}{d\chi} \log f(\rho e^{i\chi}) d\chi \right] = 0 \quad (\text{C.6})$$

because $|f(\rho e^{i\chi})| = \alpha$ so $(d/d\chi) \log |f(\rho e^{i\chi})| = 0$. Thus, I_2 reduces to

$$\begin{aligned} I_2 &= -R^2 \log \rho \operatorname{Re} \left[\int_{C_1} \mathcal{F}(\zeta) \frac{d\zeta}{i\zeta} \right] = -R^2 \log \rho \operatorname{Re} \left[\int_{C_1} \frac{\zeta f'(\zeta)}{f(\zeta)} \frac{d\zeta}{i\zeta} \right] \\ &= -R^2 \log \rho \operatorname{Im} [\log f(\zeta)]_{C_1} = -R^2 \log \rho [\arg f(\zeta)]_{C_1}, \end{aligned} \quad (\text{C.7})$$

where $[\cdot]_{C_1}$ denotes the change in the contained quantity on making a single anticlockwise traversal of C_1 . It follows that

$$I_2 = 0 \quad (\text{C.8})$$

because the image of C_1 under $f(\zeta)$ is a circular slit so there is zero change in the argument of $f(\zeta)$ on traversing C_1 .

It only remains to examine I_3 . On use of the fact that $\mathcal{Z}(\zeta)$ is real and single-valued for ζ on the negative real axis, and on evaluating $\log \zeta$ on the top and bottom of the branch cut along this axis, this can be written

$$I_3 = \int_{-\rho}^{-1} [(\log |\zeta| - i\pi) - (\log |\zeta| + i\pi)] \frac{\mathcal{Z}(\zeta)^2 \mathcal{F}(\zeta)}{i\zeta} d\zeta = -2\pi \int_{-\rho}^{-1} \frac{\mathcal{Z}(\zeta)^2 \mathcal{F}(\zeta)}{\zeta} d\zeta. \quad (\text{C.9})$$

But on use of (5.13) and (3.5) to write the integrand in terms of $f(\zeta)$ and $f'(\zeta)$ this becomes

$$I_3 = -2\pi \int_{-\rho}^{-1} f(\zeta)^{2/N} \frac{\zeta f'(\zeta)}{f(\zeta)} \frac{d\zeta}{\zeta} = -2\pi \int_{f(-\rho)}^{f(-1)} f^{2/N-1} df = -\pi N (f(-1)^{2/N} - f(-\rho)^{2/N}). \quad (\text{C.10})$$

Now

$$f(-1) = \alpha \frac{P(-1/\alpha, \rho)}{P(-\alpha, \rho)} = 1, \quad f(-\rho) = \alpha \frac{P(-\rho/\alpha, \rho)}{P(-\rho\alpha, \rho)} = \alpha \quad (\text{C.11})$$

where, in the last equality, the fact that

$$P(-\rho\alpha, \rho) = P(-\rho^2(\alpha/\rho), \rho) = P(-\rho/\alpha, \rho) \quad (\text{C.12})$$

has been used; this last relation follows from (3.3). The results in (C.11) can also be deduced from studying the geometrical properties of the mapping function $\eta = f(\zeta)$ as shown in Fig. 2. Thus,

$$I_3 = -\pi N (f(-1)^{2/N} - f(-\rho)^{2/N}) = -\pi N (1 - \alpha^{2/N}) = -\pi N (1 - R^2) \quad (\text{C.13})$$

since $\alpha = R^N$.

Finally, on combining (C.1), (C.2), (C.4), (C.8) and (C.13), the result (5.14) follows.