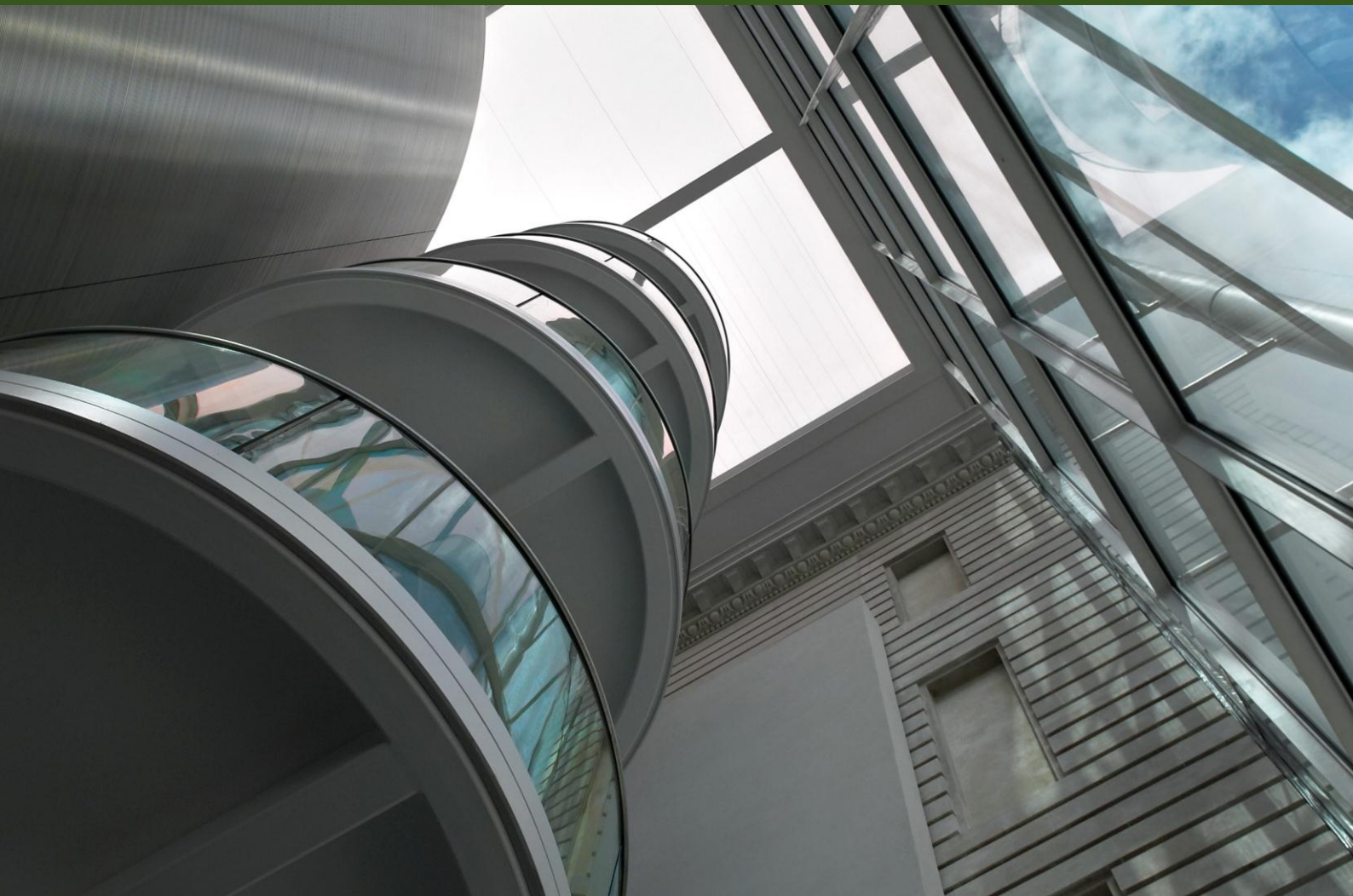


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INNOVATION, ENTREPRENEURSHIP AND KNIGHTIAN UNCERTAINTY

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ABSTRACT. At any given point in time, the collection of assets existing in the economy is observable. Each asset is a function of a set of contingencies. The union, taken over all assets, of these contingencies is what we call the “set of publicly known states”. An innovation is a set of states that are not publicly known along with an asset (in a broad sense) that pays contingent on those states. The creator of an innovation is an entrepreneur. He is represented by a probability measure on the set of new states. All other agents perceive the innovation as “ambiguous”: each of them is represented by a set of probabilities on the new states. The agents in the economy are classified with respect to their attitude toward this Ambiguity: the financiers are (locally) Ambiguity-seeking while the consumers are Ambiguity-averse. An entrepreneur and a financier come together when the former seeks funds to implement his project and the latter seeks new profit opportunities. The resulting contracting problem does not fall within the standard theory due to the presence of Ambiguity (on the financier’s side) and to the heterogeneity in the parties’ beliefs. We prove existence and monotonicity (i.e., truthful revelation) of an optimal contract. We characterize this contract under the additional assumption that the financiers are globally Ambiguity-seeking. Finally, by further strengthening our assumptions, we obtain an explicit form for the optimal innovation contract.

Key Words and Phrases: Innovation, Entrepreneurship, Knightian Uncertainty, Ambiguity.

JEL Classification: C62, D80, D81, D86, L26, P19.

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1. INTRODUCTION

While, undoubtedly, the entrepreneur plays a vital role in the economy of any free-enterprise society and is a key factor of economic growth, there is no universally agreed-upon theoretical approach to the notion of entrepreneurship and to the idea of innovation. Moreover, most theoretical models fail to capture the role of the entrepreneur adequately (see Baumol [4] or Bianchi and Henrekson [5] for a discussion). This has been a major hurdle in the theory of entrepreneurship – especially within neoclassical economic theory – and it is all the more disconcerting, given the importance attributed to the entrepreneur since the early works of Knight [36] and Schumpeter [53]. Arguably, the most celebrated neoclassical models of entrepreneurship are the ones of Lucas [37] and Kihlstrom and Laffont [34]. For Lucas [37], the entrepreneur is the manager of a firm, whose role is that of planning the optimal use of inputs such as labor and capital in order to maximize output. In this model, one of the inputs of the production function is the managerial talent, and it is assumed that while all agents have equal productivity, agents do differ with respect to their managerial talent. Entrepreneurs are then the agents with a managerial talent that exceeds a certain (endogenously determined) benchmark. Uncertainty and risk-attitude play no role in this deterministic model. Kihlstrom and Laffont [34] consider a general equilibrium model of the firm under uncertainty, in which agents are Expected-Utility maximizers. The model of Kihlstrom and Laffont [34] is essentially an occupational choice model where, at the equilibrium, the degree of risk aversion of individual agents determines their becoming an entrepreneur who operates a risky firm and receives an uncertain profit, or a paid employee receiving a risk-free wage. Entrepreneurs are characterized by their lower degree of risk aversion, whereas paid employees have a higher degree of risk aversion. However, the fundamental distinction between risk and Knightian uncertainty as well as the idea of innovation are absent from this model.

In the Austrian tradition, the entrepreneur is at the forefront of the reflection on the nature of economic markets and on the elusive idea of equilibrium. Austrian economics sees the market equilibrium as a process, a systematic *equilibration process*, and the entrepreneur is the fundamental driving force behind this process. Moreover, Austrian theory has long recognized the fundamental role of Knightian uncertainty in economic markets and its intertwining with the notion of entrepreneurship itself (see for instance Kirzner [35]). In Schumpeter’s [53] view, entrepreneurship goes hand in hand with the very idea of innovation and is at the core of economic change. The entrepreneur constantly challenges the *status quo* through a process of “creative destruction” that drives economic development, and he acts in a context of Knightian uncertainty (Ambiguity).

In this paper, we contribute to the literature on innovation and entrepreneurship by formalizing a definition of innovation and of the entrepreneur in the spirit of Schumpeter [53], and by formally examining how the idea of Ambiguity surrounds the actions of the entrepreneur. Our interest in this problem is rooted in a broad project [2, 3, 6, 28, 44, 45] which aims at answering the following questions: *How do capitalist systems generate their dynamism? Why is a capitalist economy inherently different from a centrally planned one?* Our research has been inspired by the fundamental belief that in order to study these issues, we must study the mechanisms of *entrepreneurship* and *innovation* in capitalist economies: the role of entrepreneurs in seeing commercial possibilities for developing and adopting products that exploit new technologies; the role of entrepreneurs in conceiving and developing new products and methods; the role of financiers in identifying entrepreneurs to back and to advise; and the incentives and disincentives for entrepreneurship inside established corporations.

The study of entrepreneurial economies imposes some new challenges, and is radically different from the classical approach. In this paper, we focus on the microfoundations of entrepreneurial economies: we give a formal definition of innovation, of entrepreneur and of a financier, and we study the interaction of entrepreneurs and financiers as micro actors. We refer the interested reader to [2]

for a preliminary inquiry into the properties of entrepreneurial economies, including considerations about equilibria and welfare. In this paper, we propose a model of entrepreneurship and innovation in which the entrepreneur is an innovator, and the introduction of an innovation into an economy by an entrepreneur generates Knightian uncertainty. We argue that at the core of this entrepreneurial economy lies a problem of contracting between an *entrepreneur* and a *financier*. It is a contracting problem involving not only differences in beliefs, but also Ambiguity. We call this problem a problem of *contracting for innovation*, and a large part of this paper focuses on studying this problem of contracting for innovation between an entrepreneur and a financier. Two strands of literature merge in our work: the literature on entrepreneurship and innovation and the literature on contracting under Ambiguity. We contribute to the former by building a theoretical framework where we can answer questions such as What does “innovation” mean? Are “entrepreneur” and “financier” just two labels, or is there something substantial behind these denominations? Why does “contracting for innovation” differ from other contracting problems? We contribute to the literature on contracting under Ambiguity by studying and solving a novel problem of contracting under Ambiguity. We refer the reader to Section 8 for a thorough discussion of our contributions in relation to the existing literature.

The remainder of the paper is organized as follows. We present our ideas on entrepreneurship and innovation in Sections 2 and 3. Section 2 deserves special mention since it contains the formal definition of *Innovation* that we introduce in this paper. The ideas elaborated in these sections lead to the formulation of a certain contracting problem in Section 4. In Section 5, we state our theorem on the existence and monotonicity (truth-telling) of an optimal “innovation contract”. In Section 6, for the case of globally Ambiguity-seeking financiers, we present additional results which allow for an easier determination of the optimal innovation contract. We sharpen these results further in Section 7 where we fully characterize the optimal innovation contract in explicit form, but we do so at the cost of introducing additional assumptions. We believe, nonetheless, that this result may be still useful in more relaxed settings, for instance by providing the basis for a computational procedure. As anticipated, we conclude in Section 8 by discussing our contributions in the context of the existing literature. A number of [Appendices](#) containing some background material as well as the proofs omitted from the main text complete the exposition.

2. INNOVATION

2.1. Objective States vs Subjective States. When dealing with uncertainty, a central concept in classical theories is that of a *state of the world*. Following the Bayesian tradition, a state of the world is a complete specification of all the parameters defining an environment. For instance, a state of the world for the economy would consist of a specification of temperature, humidity, consumers’ tastes, technological possibilities, detailed maps of all possible planets, etc. According to this view, the future is uncertain because it is not known in advance which state will obtain. In principle (but this is clearly beyond human capabilities), one might come up with the full list of all possible states, and classical theories postulate that each and every agent would be described by a probability measure on such a list. While (*ex ante*) different agents might have different views (i.e., different probability distributions), the information conveyed by the market eventually leads them to entertain the same view: at an economy’s equilibrium no two agents are willing to bet against each other about the resolution of uncertainty. Thus, in classical theories, there is nothing uncontroversial about the way one deals with uncertainty.

The contrast between this prediction of the theory and what happens in real life is striking. Actual economic agents usually disagree about the resolution of uncertainty, and even the assumption of a list of contingencies known to all agents as well as the assumption of each agent having a probability

distribution over such contingencies seem hardly tenable. It is an old idea, dating back at least to F. Knight [36], that some – and, perhaps, the most relevant – economic decisions are made in circumstances where the information available is too coarse to make full sense of the surrounding environment, where things look too fuzzy for one to have a probability distribution over a set of relevant contingencies. In such situations, Risk Theory is simply of no use. We fully adhere to this view.

The concept of a state of the world is central to our theory as well. We depart, however, from classical theories in that we do not assume the existence of a list of all possible states which is known to all agents. We do so for several reasons. First, we believe that this assumption is too artificial. Second, a theory built on such an assumption would not be testable, not even in principle. Third, and more importantly, we believe that by making such an assumption we would lose sight of the actual role played by entrepreneurs and financiers in actual economies.

We take a point of view that might be deemed “objective”. We start with the (abstract) notion of an *asset*. In its broadest interpretation, an asset is, by definition, something that pays off depending on the realization of certain contingencies. In other words, in order to define an asset, one must specify a list of contingencies along with the amount that the asset pays as a function of those contingencies. At each point in time, the set of assets existing in the economy is observable. Thus, in principle, the set of contingencies associated with each asset is objectively given. The union, taken over all the assets, of all these contingencies is then objectively given, in the sense that it is derived from observables. We call this set *the set of publicly known states of the world*, and we denote it by SP . We assume that each and every agent in the economy is aware of all the states contained in SP . We stress, however, that what is more important is that this set be knowable rather than be known by every agent. Of course, there is no reason why agents in the economy, individually considered, should be restricted to holding the same view. In other words, while we assume that each agent is aware of the set SP of publicly known states of the world, we are also open to the possibility that each agent might consider states that are not in SP . Formally, we admit that each agent i has a subjective state space S_i of the form $S_i = SP \cup I_i$, where I_i is the list of contingencies in agent i ’s set of states that are not publicly known (i.e., $SP \cap I_i = \emptyset$). An example might clarify. In the ’50s, IBM was investing in the creation of (big) computers. In our terminology, this means that IBM had envisioned states of the world where computers would be produced and sold, where hardware and software for computers would be produced and sold, etc. Since IBM stocks were tradable, these states would be part of the publicly known states according to our definition. Some time between the late ’50s and the early ’60s, Doug Englebart envisioned a world where PCs existed and where software and hardware for PCs would be produced and sold. According to our view, before Doug Englebart began patenting his ideas, these states existed only in his mind (and maybe in those of few others); that is, they were part of Doug Englebart’s subjective state space, but they were not publicly knowable.

We will assume that each agent i is a Bayesian decision maker *with respect to his own subjective state space*. That is, agent i with subjective state space S_i makes his decisions according to a probability distribution P_i on S_i . In the terminology that we will be using in Section 3 below, this means that we assume that each agent believes that he has a good understanding of his own state space. While this assumption could be removed, we believe that it is a good first approximation. Moreover, we believe that it follows quite naturally from the idea of a subjective state space, as we define it.

2.2. Innovation and the Entrepreneur. The idea of “innovation” and the way we model it is central to our theory. Undoubtedly, the ability to “innovate” is one of the most distinguishing features of capitalist economies. Innovations occur in the form of new consumption goods, new technological

processes, new institutions, new forms of organizations in trading activities, etc. We abstract from the differences existing across different types of innovation, and focus on what is common among them. For us, an innovation is defined as follows.

Definition 2.1 (Innovation). *An innovation is a set of states of the world which are not publicly known along with an asset which pays contingent on those states (and, possibly, on the publicly known ones).*

An example will clarify momentarily. For now, we would like to point out that the word “asset” in the definition should be interpreted in a broad sense. That is, by *asset* we mean any activity capable of generating economic value. An innovation will be denoted by a pair (S_j, X_j) , where j is the innovator, S_j is his subjective state space and X_j is the asset that pays contingent on states in S_j . Notice that, as it is encoded in the definition of the subjective state space, (S_j, X_j) pays off also contingent on states in SP .

In order to illustrate the definition, let us imagine an economy where historically only two types of cakes have been consumed: carrot cakes and coconut cakes. Each year, each individual consumer might be of one of two types: she either likes carrot cakes (consumers of type 1) or coconut cakes (consumers of type 2), but not both. The fraction of the population made of consumers of type 1 varies from year to year according to some known stochastic process. In this economy, there are two productive processes: one for producing carrot cakes and one for coconut cakes. There is a continuum of tomorrow’s states, where each state corresponds to the fraction of consumers of type 1. These states are understood by everyone in the economy. That is, $SP = [0, 1]$ and a point x in $[0, 1]$ means that the fraction of type 1 consumers is x . Moreover, there is a given probability distribution on $[0, 1]$, which is known to everyone in the economy. Now, suppose that an especially creative individual, whom we call e , comes into the scene and (a) figures out a new productive process that produces banana cakes; and, (b) believes that each consumer, whether of type 1 or 2, would switch to banana cakes with probability $1/3$ if given the opportunity. What is happening here is that agent e has: (i) imagined a whole set of new states, those in which consumers might like banana cakes (in fact, the subjective state space for agent e is two-dimensional, while SP is one-dimensional); (ii) imagined that a non-negligible probability mass might be allocated to the extra dimension conditional on the consumers being given the chance to consume banana cakes; and, (iii) figured out a device (the productive process) that makes the new states capable of generating economic value.

The definition of innovation given above conveys the essential features which identify any innovation: the new states along with the new activity. We believe that one of the virtues of our definition of innovation is that it makes it clear that the process of innovation is truly associated with the appearance of new and fundamentally different possibilities: from the viewpoint of the innovator, both the state space and the space of production possibilities have higher dimensionality¹.

Definition 2.2 (Entrepreneur). *An agent e who issues an innovation is called an entrepreneur.*

Recall that we have assumed that each agent has a probability distribution on his subjective state space. Thus, an entrepreneur is described by a triple (S_e, X_e, P_e) , where (S_e, X_e) is the innovation and P_e is his subjective probability on the subjective state space S_e .

¹Given two sets, A and B , the Cartesian product $A \times B$ and the disjoint union $A \vee B$ are trivially isomorphic. Thus, in the definition of subjective state space we could have used products (as we have done in the example) rather than disjoint unions. In this paper, we have opted for disjoint unions as this greatly simplifies the wording of certain statements.

3. UNCERTAINTY AND THE CLASSIFICATION OF ECONOMIC AGENTS

In our story, the innovators are the entrepreneurs. But what happens once an entrepreneur comes up with an innovation? In the economy above, how would consumers react if they are told that banana cakes will be available? We follow up here on the idea expressed above that an innovation is associated to a new scenario, i.e., something that the economy as whole has not yet experienced. It is then natural to view such a situation as one of *Knightian uncertainty* (or *Ambiguity*): the information available is (except, possibly, for the entrepreneur) too coarse to form a probability distribution on the relevant contingencies. Notice that Ambiguity enters our model in a rather novel way: its source is not some device (Nature) outside the economic system; rather, it is some of the economic actors – the entrepreneurs – who are the primary source of Ambiguity.

Decision theorists have developed several models to deal with this problem, all of which stipulate that the behavior of agents facing Ambiguity is described not by a single probability but rather by a set of those (See Gilboa and Marinacci [25] for a comprehensive survey). Formally, the problem is as follows. Let e be an entrepreneur, and let i be another agent. Agent i is represented by a pair (S_i, P_i) , where S_i is her subjective state space and P_i a probability on S_i . Suppose that i has never thought of the subjective states of the entrepreneur. Now, suppose that agent i is made aware, directly or otherwise, of the innovation (S_e, X_e) as well as of the probability P_e of the entrepreneur. What is i going to do? Is she going to believe e and adopt his view (i.e., the probability P_e), or is i going to form a different opinion? In fact, is i going to form an opinion at all? Clearly, each of these cases is possible, and there is no real reason to favor one over the other. Thus, we need a way to model all these possibilities simultaneously. We do so as follows. When agent i becomes aware of the subjective states of agent e , the set of states for agent i becomes $S_i \cup S_e$. Thus, agent i 's problem is that of extending her view from S_i to the union $S_i \cup S_e$ as this is necessary for evaluating assets that pay contingent on S_e . We assume that agent i makes this extension by using all the probability distributions on $S_i \cup S_e$ which are compatible with her original view, that is, all those probabilities on $S_i \cup S_e$ whose conditional on S_i is P_i . The exact way in which agent i will evaluate the assets defined on S_e depends, loosely speaking, on the way all of these probabilities are aggregated and, in general, different agents would aggregate them in different ways. To put it in a different terminology, an agent's evaluation of the assets defined on S_e depends on the agent's attitude toward Ambiguity. This observation suggests a natural classification of economic agents: in one category we would put those agents who are going to share, at least partially, the view of at least one entrepreneur, while in the other we would put those who are not going to do so under any circumstance. The former have the potential to become business partners of some entrepreneurs, whereas the latter will never do so. Thus, we will distinguish between *consumers* and *financiers* that are defined as follows.

1. **Consumers:** Their subjective state space coincides with the publicly known set of states. They are Ambiguity-averse, in the sense that they always evaluate their options according to the worst probability (i.e., worst case scenario, or *Maxmin Expected Utility* [26]). Formally, a consumer c is represented by a pair (SP, P_c) , where P_c is a probability measure on SP . When facing an innovation (S_e, X_e) , c evaluates it by using the functional

$$C(X_e) = \min_{P \in \mathcal{C}_c} \int u_c(X_e) dP$$

where $\mathcal{C}_c$ is the set of all probabilities on $SP \cup S_e$ whose conditional on SP is P_c and u_c is the consumer's utility on outcomes.

Notice that this description easily implies that (i) if there exists a bond in the economy, and (ii) if there exists a state in S_e such that the worth of the innovation is below the bond, then the consumer

will not buy that innovation at any positive price. Under these circumstances, these agents will never become business partners of any entrepreneur, which explain why we call them consumers.

2. **Financiers:** Their subjective state space coincides with the publicly known set of states. They are less Ambiguity-averse than the consumers. A financier φ is represented by a pair (SP, P_φ) , where P_φ is a probability measure on SP . When facing an innovation (S_e, X_e) , φ evaluates it by using the functional

$$(3.1) \quad \Phi(X_e) = \alpha(X_e) \min_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ + (1 - \alpha(X_e)) \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ$$

where $\mathcal{C}_\varphi$ is the set of probabilities on $SP \cup S_e$ whose conditional on SP is P_φ and u_φ is the financier's utility on outcomes. For each asset X_e , the coefficient $\alpha(X_e) \in [0, 1]$.

Thus, the functional (3.1) is a combination of aversion toward projects that involve new states (the *min* part of the functional) and favor toward the same projects (the *max* part). Intuitively, the coefficient $\alpha(X_e)$ represents the degree of Ambiguity aversion of the financier (see [21, 22]), and this degree is allowed to vary with the asset (i.e., the entrepreneurial project) to be evaluated. In fact, the coefficient $\alpha(\cdot)$ represents the financier's view: a high $\alpha(\cdot)$ means that the financier does not believe in the project, while a low $\alpha(\cdot)$ means that the financier shares, for the most part, the view of the entrepreneur. We suppose that for at least one asset X_e , $\alpha(X_e) < 1$; that is, there exists at least a project that, once in existence, the financier would consider worth funding at some positive price. A special case obtains when the financier's attitude toward Ambiguity does not depend on the project to be evaluated. In such a case, projects are evaluated by using the functional

$$\Phi(X_e) = \alpha \min_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ + (1 - \alpha) \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ$$

where we suppose that $\alpha < 1$.

We believe that our categorization captures the essential (functional) distinction between the concept of a “consumer” and that of a “financier”: a (pure) consumer is someone who rejects the unknown, and a financier is somebody who is willing to bet on it. The condition in the above definitions that both the consumer's and the financier's state space is SP only means that consumers and financiers are not entrepreneurs. One might argue that this assumption is natural in the case of consumers but it is not so in the case of financiers. This is not problematic as a financier's subjective state space bigger than SP can be easily accommodated in our framework by suitably re-defining the function $\alpha(X_e)$, which represents the financier's Ambiguity aversion.

In sum, we have three types of agents: entrepreneurs, financiers, and consumers. The study of economies populated by these types of agents (in the way we have defined them above) poses entirely new problems. We refer the interested reader to [2] for a preliminary inquiry into the properties of entrepreneurial economies, including equilibria and welfare considerations. In this paper, we focus on the microfoundations underlying an entrepreneurial economy: we study the way in which an entrepreneur and a financier interact, as micro actors. In Section 4 of this paper, we argue that this interaction is a problem of contracting between the entrepreneur and the financier that we call a problem of *contracting for innovation*.

4. CONTRACTING FOR INNOVATION

Our previous discussion leads to the following problem. An entrepreneur comes up with a new idea. Not having enough wealth to implement it, he goes to a financier and describes his project, hoping to obtain the necessary funds. We have seen that the entrepreneur's project – the innovation – is a

pair (S_e, X_e) , where S_e contains the publicly observable states as well as the new states envisioned by the entrepreneur, and $X_e : S_e \rightarrow \mathbb{R}$ expresses the monetary return of the project as a function of the contingencies in S_e . On his end, the entrepreneur has (in his subjective opinion) a clear probabilistic view of the new world that he has envisioned. This is described by a probability measure P_e (see Appendix A for a precise definition of this probability and of the σ -algebra Σ_e on which it is defined). On the other end, the financier, by facing a set of states he had never conceived of, perceives some Ambiguity in the entrepreneur's description. This is described by the fact that the financier evaluates the project by using a functional of the form (3.1) above. Two features place this problem outside the realm of standard contract theory. First, we have *heterogeneity* in the parties' beliefs: their views are different and, in fact, they are formed independently of each other. Second, one of the parties perceives *Ambiguity*, i.e., this party's beliefs are not represented by a probability measure. The remainder of this section will formalize this contracting problem, and Section 5 will discuss its solution.

4.1. Definition of a Contract. The formal definition of an innovation contract is as follows.

Definition 4.1 (Innovation Contract). *An innovation contract (or contract) between an entrepreneur and a financier is a pair (H, Y) , where $H \geq 0$ and $Y \in B(\Sigma_e)$ is such that $Y \leq X_e$.*

The interpretation is that a contract is a scheme according to which the financier pays H (which may be 0) to the entrepreneur, and in exchange receives a claim on part of the amount $X_e(s)$, which obtains when the state of the world $s \in S$ realizes. This claim may consist of all $X_e(s)$ or just a part of it. The amount that the entrepreneur receives when $s \in S$ realizes is denoted by $Y(s)$ (which may be 0). In the definition, we demand that the function $Y(\cdot)$ be an element of $B(\Sigma_e)$, the Banach space of bounded measurable functions on the σ -algebra Σ_e . In turn, this is defined as the coarsest σ -algebra which makes the function X_e measurable. The condition $Y \in B(\Sigma_e)$ encodes the informational constraints that the parties face at the moment of writing a contract. Intuitively, the condition simply states that the parties write the contract on the basis of the information available to them and no more than that (for more on this issue, the reader is referred to Appendix A of this paper). The definition includes as special cases the following types of contracts:

- (a) The financier simply buys the project, and has no further obligation toward the entrepreneur. This obtains for $Y(s) = 0$, for every $s \in S$;
- (b) The financiers acquires ownership of the project. When the state $s \in S$ realizes, he obtains the amount $X_e(s)$ and transfers $Y(s)$ to the entrepreneur;
- (c) The entrepreneur retains ownership of the project, but commits to paying the amount $Z(s) = X_e(s) - Y(s)$ to the financier when $s \in S$ realizes. He does so in exchange for an up front (that is, before the uncertainty resolves) payment of H ;
- (d) The entrepreneur transfers part of the ownership to the financier in exchange for H , and the parties agree to a sharing rule that specifies that when $s \in S$ realizes the amount $Z(s) = X_e(s) - Y(s)$ goes to the financier and the amount $Y(s)$ goes to the entrepreneur.

In a static setting, the distinction between cases (b), (c) and (d) is purely a matter of interpretation because the contract is formally the same. However, case (a) differs from the other three cases in that in case (a) one can actually talk of a transfer of ownership.

Example 4.2 (Publishing). In the case of “Author meets Publisher”, the innovation is a new book, or music, or film, or other intellectual property. In publishing, the up-front payment H is called the

“advance”. The Publisher purchases the residual claim on the work, and contracts to pay the Author a royalty stream based on sales revenue, which corresponds to the function Y .

In the remainder of the paper, the subscripts e will be suppressed (except from the entrepreneur’s utility function) since we are going to consider one entrepreneur only. Consequently, the measurable space (S_e, Σ_e) will be denoted by (S, Σ) .

4.2. The Entrepreneur. As previously mentioned, the entrepreneur has, in his subjective opinion, a clear probabilistic view of the new world S he has envisioned. This view is represented by a (countably additive) probability measure P on (S, Σ) , which he uses to evaluate the possible contracts that he might sign. Formally,

Assumption 4.3. *The entrepreneur evaluates contracts by means of the Subjective Expected Utility (SEU) criterion*

$$\int u_e(Y) dP, \quad Y \in B(\Sigma)$$

where $u_e : \mathbb{R} \rightarrow \mathbb{R}$ is the entrepreneur’s utility for monetary outcomes.

Mainly for reasons of comparison with other parts of the contracting literature, we assume that the uncertainty on S is diffused. Precisely, we assume the following.

Assumption 4.4. *X is a continuous random variable on the probability space (S, Σ, P) . That is, $P \circ X^{-1}$ is nonatomic.*

Finally, we make the following assumption on the entrepreneur’s utility function.

Assumption 4.5. *The entrepreneur’s utility function u_e satisfies the following properties:*

- (1) $u_e(0) = 0$;
- (2) u_e is strictly increasing and strictly concave;
- (3) u_e is continuously differentiable;
- (4) u_e is bounded.

Thus, in particular, we assume that the entrepreneur is risk-averse.

4.3. The Financier. When presented with the innovation $((S, \Sigma), X)$, the financier φ perceives Ambiguity. This is represented by the set $\mathcal{C}_\varphi$ (of probabilities on $SP \cup S_e$ whose conditional on SP is P_φ) which appears in equation (3.1), above. In order to describe the financier’s evaluation of the innovation, we are going to restrict to a sub-class of the functionals of type (3.1): that of *Choquet Expected Utility* (CEU). This class still allows for a wide variety of behavior as these functionals need not be either concave or convex. In the CEU model introduced by Schmeidler [52], the functional (3.1) takes the form of an integral (in the sense of Choquet) with respect to a non-additive, monotone set function (a capacity). While the use of Choquet integrals has become quite common in the applications of decision theory, it is probably still not part of the toolbox of most professionals. We have included a few basic facts about capacities and Choquet integrals in Appendix B.1. In sum, a financier φ is described as follows.

Assumption 4.6. *The financier evaluates contract by means of the functional $\Phi : B(\Sigma) \rightarrow \mathbb{R}$ defined by*

$$\Phi(Y) = \int u_\varphi(Y) dv, \quad Y \in B(\Sigma)$$

where $u_\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is the financier's utility for money, v is a capacity on Σ and the integral is taken in the sense of Choquet (see Appendix B.1).

In line with Assumption 4.4, we also assume the following:

Assumption 4.7. *v is a continuous capacity (see Appendix B.1).*

Finally, we make the following assumption.

Assumption 4.8. *The financier is risk-neutral. We take u_φ to be the identity on $\mathbb{R}$.*

Henceforth, we will assume that the random variable X which describes the profitability of the project is a positive random variable, that is $X \in B^+(\Sigma)$. This is without loss of generality since it can always be obtained by suitably re-normalizing the parties utility functions.

4.4. The Contracting Problem. The problem of finding an optimal innovation contract (H, Y) may be split into two parts: we first determine the optimal Y given H , and then use this to find the optimal H . In line with the description of economic agents of Section 3, we have in mind situations characterized by the fact that, while the entrepreneur is the sole potential provider of the innovation, there is competition among financiers to acquire it. Hence, the problem of finding an optimal contingent payment scheme Y can be formulated as follows

$$(4.1) \quad \begin{aligned} & \sup_{Y \in B(\Sigma)} \int u_e(W_0^e + H - X + Y) dP \\ & \text{s.t.} \quad 0 \leq Y \leq X \\ & \quad \int (X - Y) dv \geq (1 + \rho) H \end{aligned}$$

The argument of the utility u_e in problem (4.1) is the entrepreneur's wealth as a function of the state $s \in S$ that will realize

$$W^e(s) = W_0^e + H - X(s) + Y(s)$$

where W_0^e denotes the entrepreneur's initial wealth, which can be zero. None of our results will be modified if the entrepreneur's initial wealth is assumed to be zero. Clearly, $W^e(\cdot)$ is a measurable function on (S, Σ) . The last constraint is the financier's *participation constraint*. It states that a necessary condition for the financier to offer the contract is that his evaluation of the random variable $X - Y$ (the amount that he receives, as a function of the state, if he signs the contract) be at least as high as the amount H that he has to pay up front to the entrepreneur. In fact, the financier's evaluation of $X - Y$ might have to be strictly higher than H since by funding the entrepreneur the financier might give up other investment opportunities, for instance those present in the standard asset market defined by SP , the publicly known states. This condition is expressed by the factor $(1 + \rho)$, where $\rho \geq 0$. The other constraint $(0 \leq Y \leq X)$ expresses two conditions: (i) the right-hand inequality states that, in each state of the world, the transfer from the financier to the entrepreneur does not exceed the profitability of the project; and, (ii) the left-hand inequality states that a transfer from the financier to the entrepreneur is nonnegative.

Once problem (4.1) is solved and the optimal contingent payment scheme Y_H^* is determined as a function of H , the next step is to determine the optimal H . This is typically an easier problem to handle, and it is solved with standard methods.

4.5. Truthful Revelation of the Profitability of the Project. When studying a problem of contracting in a situation of uncertainty, one typically adds one more constraint to the ones we considered above. This is a monotonicity constraint that, in our case, would stipulate that the payment from the financier to the entrepreneur is an increasing function of X , that is $Y = \Xi \circ X$ for some increasing function $\Xi : \mathbb{R} \rightarrow \mathbb{R}$. This would guarantee that the entrepreneur does not downplay the profitability of the project. For the moment, we ignore this problem altogether. The reason is that, in our main theorem, *we will show that the monotonicity of Y is a feature that appears in all optimal innovation contracts that we determine*. Notice that this feature guarantees that, even in the case where the project profitability depends on (state-contingent) unobserved actions taken by the entrepreneur, there would be neither adverse selection nor moral hazard problems with our optimal contracts.

5. EXISTENCE AND MONOTONICITY OF AN OPTIMAL INNOVATION CONTRACT

In this section, we show that the contracting problem (problem (4.1) of Section 4) between the entrepreneur and the financier admits a solution. Moreover, we show that this solution is increasing in X , thus clearing up the field from concerns of project's misrepresentation on the part of the entrepreneur.

In a Bayesian setting (= no Ambiguity), Milgrom [39] has shown that the key assumption for obtaining monotonicity properties of optimal solutions is the Monotone Likelihood Ratio (MLR) property. In fact, this is precisely how truth-telling obtains in a contracting problem without Ambiguity. The trouble for us is that the MLR property cannot be exported to our setting. The MLR condition is the monotonicity property of the ratio of the two density functions, which represent the parties' beliefs in the standard Bayesian setting. In our setting, the financier is not a Bayesian decision maker, and his beliefs cannot be represented by means of a density function. A way around this problem is provided by Ghossoub [23]. In his study of insurance problems under heterogeneous beliefs, he introduced a novel condition, called *vigilance*, which plays a role similar to that of the MLR property. The vigilance condition displays two remarkable properties. First, it is strictly weaker than the MLR property; that is, whenever the two can be defined simultaneously, the MLR property implies vigilance but the converse is not true. Second, owing to its formulation, the vigilance condition can easily be exported to settings involving any type of non-additive beliefs. Below is the extension that we need.

Definition 5.1 (Vigilance). *Let ν be a capacity on Σ , let P be a measure on the same σ -algebra and let X be a random variable on (S, Σ) . We say that ν is (P, X) -vigilant if for any $Y_1, Y_2 \in B^+(\Sigma)$ such that*

- (i) Y_1 and Y_2 have the same distribution under P ; and
- (ii) Y_2 and X are comonotonic² (Y_2 is a nondecreasing function of X),

the following holds

$$\int (X - Y_2) d\nu \geq \int (X - Y_1) d\nu$$

²For the definition of comonotonic functions, see Appendix B.1.

As the concept of vigilance and its application to non-additive settings are rather new, we remark below that certain classes of capacities that have repeatedly appeared in robust statistics, decision theory and finance (asset pricing and market structure) are in fact (P, X) -vigilant.

Example 5.2 (Wasserman-Kadane Symmetric Capacities). Capacities have a long and rich tradition in robust statistics (see, for instance, Huber and Ronchetti [29]). An important class of capacities are the *symmetric* capacities introduced by Wasserman and Kadane [56].

Definition 5.3. Let (S, Σ, P) be a probability space. A capacity v on (S, Σ) is said to be *Wasserman-Kadane P -symmetric* if for any two random variables Z_1 and Z_2 on (S, Σ, P) that are identically distributed for P , one has

$$\int Z_1 dv = \int Z_2 dv$$

It is easy to verify that Wasserman-Kadane P -symmetric capacities on (S, Σ) are (P, X) -vigilant.

Example 5.4 (Weighted Probability Measures). Another class of capacities that is used in robust statistics is the class of *weakly symmetric* capacities of Wasserman and Kadane [56] and Kadane and Wasserman [32].

Definition 5.5. Let (S, Σ, P) be a probability space. A capacity v on (S, Σ) is said to be *weakly P -Symmetric* if for any $A, B \in \Sigma$, one has:

$$P(A) = P(B) \implies v(A) = v(B)$$

An example of a weakly P -symmetric capacity is a distorted (weighted) probability measure of the form $T \circ P$, where $T : [0, 1] \rightarrow [0, 1]$ is an increasing function with $T(0) = 0$ and $T(1) = 1$. The function T is called a probability weighting (or distortion) function. Distortions of probability measures have played a major role in the theory of choice under uncertainty since the work of Yaari [57], Quiggin [47], and Kahneman and Tversky [33, 55]. It is easy to verify that all capacities that are distortions of the probability measure P are (P, X) -vigilant.

We can now state our main result. Its proof is given in Appendix C.

Theorem 5.6 (Truth-Telling Optimal Innovation Contracts). *If v is (P, X) vigilant, then problem (4.1) admits a solution Y which is comonotonic with X ; that is, there exists an optimal innovation contract which is truth-telling.*

6. AMBIGUITY-LOVING FINANCIERS

In Section 3, we mentioned that a fairly general description of the way financiers deal with Ambiguity is provided by the functionals of the form (see equation (3.1), Section 3)

$$\Phi(X_e) = \alpha(X_e) \min_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ + \left(1 - \alpha(X_e)\right) \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ$$

where the coefficient $\alpha(\cdot)$ is allowed to vary with the project to be evaluated. The variability of the coefficient expresses the financier's preference for certain projects over others, maybe because they are closer to his subjective vision (we pointed out in Section 3 that we can allow for financiers to have a subjective visions by simply re-defining the function $\alpha(\cdot)$). A natural special case of this description obtains when the coefficient $\alpha(\cdot)$ is constant. This would represent the case where the financier is not really concerned about the kind of Ambiguity he faces. Rather, he is only interested

in the fact that there is Ambiguity, and he is willing to bet on its resolution. Since financiers have to be willing to deal with Ambiguity, it suffices to focus on the case $\alpha = 0$ (in fact, the case $\alpha = 1$ identifies the consumers; see Section 3):

$$(6.1) \quad \Phi(X_e) = \max_{Q \in \mathcal{C}_\varphi} \int u_\varphi(X_e) dQ$$

By a result of Schmeidler [51], a subclass of these functionals obtains as a special case of Choquet integrals. Precisely, a Choquet integral can be written in the form (6.1) if and only if the capacity that defines it is submodular (see Appendix B.1). In this case, we can give a characterization of the solution whose existence we proved in Theorem 5.6. Proposition 6.1 below shows that, when the capacity representing the financier is submodular, an optimal solution to the contracting problem (4.1) is the same as the solution of another contracting problem, which involves heterogeneity but no Ambiguity. It is important to stress, as the proof of Proposition 6.1 makes it clear, that this is not a statement about the type of uncertainty involved in this problem (4.1) but only a device which allows us to characterize the solution. The usefulness of the equivalence proved in Proposition 6.1 stems from the fact that the solution can now be characterized by using the methods introduced in Ghossoub [23]. In fact, as we shall see in Section 7, under some mild additional conditions, this solution can even be characterized analytically.

So, let us assume that the capacity v representing the financier in Assumption 4.6 is submodular. Then, the functional Φ takes the form (6.1). The set $\mathcal{C}_\varphi$ is a non-empty, weak*-compact and convex set of probability measures, and it is called the *anticore* of v . For $Q \in \mathcal{C}_\varphi$, consider the following problem

$$(6.2) \quad \begin{aligned} & \sup_{Y \in B(\Sigma)} \int u_e(W_0^e + H - X + Y) dP \\ & s.t. \quad 0 \leq Y \leq X \\ & \quad \int (X - Y) dQ \geq (1 + \rho) H \end{aligned}$$

That is, problem (6.2) is a problem similar to problem (4.1) but (ideally) involves a financier that is an Expected-Utility maximizer, with $Q \in \mathcal{C}_\varphi$ being the probability representing the financier. If Q is (P, X) -vigilant, then by Theorem 5.6, problem (6.2) for $Q \in \mathcal{C}_\varphi$ admits a solution which is comonotonic with X . Let us denote by $Y^*(Q)$ this optimal solution.

Proposition 6.1. *If the capacity v in Assumption 4.6 is submodular, and if every $Q \in \text{anticore}(v)$ is (P, X) -vigilant, then there exists a $Q^* \in \text{anticore}(v)$ such that $Y^*(Q^*)$ solves the contracting problem (4.1).*

It is perhaps superfluous to stress that Proposition 6.1 does *not* say that a financier described by equation (6.1) behaves just like an EU-maximizer with subjective probability measure Q^* : when presented with a contract $Y \neq Y^*$, the financier described by equation (6.1) will evaluate Y using a probability measure $Q \neq Q^*$. Equivalently, Proposition 6.1 is only a statement that a maximum is obtained, but this simple observation buys us a lot of mileage. We refer the reader to Ghossoub [23] for an extensive inquiry into the properties of these solutions. Inspection of the proof of Proposition 6.1 (Appendix D) shows that this result can be extended to general functionals of the form (6.1), that is functionals of the form (6.1) that are not necessarily Choquet integrals.

Corollary 6.2. *Assume that in problem (4.1) the financier is described by a functional of the form*

$$\Phi(X_e) = \max_{Q \in \mathcal{C}} \int u_\varphi(X_e) dQ$$

where $\mathcal{C}$ is a weak-compact set of probability measures on (S, Σ) . If there exists a solution Y^{**} to the contracting problem, and if every $Q \in \mathcal{C}$ is (P, X) -vigilant, then there exists a $Q^* \in \mathcal{C}$ such that $Y^{**} = Y^*(Q^*)$.*

7. THE CASE OF A PROBABILITY DISTORTION

In the previous sections, we have ascertained the existence of an optimal contract Y^* as a function of X , which in turn is a function of the states of the world. We have also established that such a contract would have the truth-telling property. While this is a necessary and fundamental result, it is still quite far from telling us how to write a contract were we financiers meeting with an entrepreneur. We now turn to filling this gap. What we seek is an explicit formula giving us the contract Y^* as a function of the entrepreneur's project X . By obtaining this result, we would achieve a full range of applicability for our theory, since by inputting the entrepreneur's project we would be able to actually write the optimal contract (depending on the financier's view of uncertainty). In this section, we do present this result, but we need some additional assumptions to achieve it. While these assumptions are certainly limiting, we believe that the restricted setting they identify is still very interesting. Moreover, both the result and our methods of achieving it may provide a valuable guide for writing down optimal contracts in more general settings, for instance through approximation procedures.

We have already encountered probability distortions (Example 5.4 above) and mentioned that they have found fruitful applications in decision theory, robust statistics and finance. In this section, we show that when the financier's beliefs are represented by a probability distortion, it is possible to obtain an explicit analytical form of the optimal innovation contract. This full characterization is helpful in practice since it permits to actually compute the optimal innovation contract as a function of the underlying innovation.

When the financier's beliefs are a distortion of a probability measure, it is convenient to replace the assumption of vigilance with two assumptions (Assumption 7.4 and Assumption 7.7 below) that closely mimic the Monotone Likelihood Ratio (MLR) assumption. Similar assumptions have been extensively used in the recent literature, as discussed below.

Thus, let us suppose that the financier's capacity v is of the form $v = T \circ Q$, for some probability measure $Q \neq P$ on (S, Σ) and some function $T : [0, 1] \rightarrow [0, 1]$, which is increasing, concave, and continuous with $T(0) = 0$ and $T(1) = 1$. Then, $T \circ Q$ is a submodular capacity on (S, Σ) . In this setting, Assumption 4.4 and Assumption 4.7 of Section 4 are restated as follows.

Assumption 7.1. *We assume that $v = T \circ Q$, where:*

- (1) Q is a probability measure on (S, Σ) such that $Q \circ X^{-1}$ is nonatomic;
- (2) $T : [0, 1] \rightarrow [0, 1]$ is increasing, concave and continuously differentiable; and,
- (3) $T(0) = 0$, $T(1) = 1$, and $T'(0) < +\infty$.

Kahneman and Tversky [33] interpret the slope of the probability weighting function as a measure of the sensitivity of preferences to very small changes in probabilities. Adopting such a viewpoint, the condition $T'(0) < +\infty$ states that the financier is not overly sensitive to small changes in probabilities of very unlikely events. We will also assume that the lump-sum start-up financing H

that the entrepreneur receives from the financier guarantees a nonnegative wealth process for the entrepreneur. This can be interpreted as a limited liability assumption. Specifically, we shall assume the following.

Assumption 7.2. $X \leq W_0^e + H$, P -a.s.

In order to state the next assumption – the one that replaces vigilance in this setting – we need to introduce a certain amount of notation. The reader who is not interested in the details can skim through this part, and simply record the results contained in Theorem 7.5 and Corollary 7.8. For each $Z \in B^+(\Sigma)$, let $F_Z(t) = Q(\{s \in S : Z(s) \leq t\})$ denote the *distribution function* of Z with respect to the probability measure Q , and let $F_Z^{-1}(t)$ be the left-continuous inverse of the distribution function F_Z (that is, the *quantile function* of Z), defined by

$$(7.1) \quad F_Z^{-1}(t) = \inf \left\{ z \in \mathbb{R}^+ : F_Z(z) \geq t \right\}, \quad \forall t \in [0, 1]$$

Definition 7.3. Denote by $\mathcal{AQuant}$ the collection of all quantile functions f of the form F^{-1} , where F is the distribution function of some $Z \in B^+(\Sigma)$ such that $0 \leq Z \leq X$.

That is, $\mathcal{AQuant}$ is the collection of all quantile functions f that satisfy the following properties:

- (1) $f(z) \leq F_X^{-1}(z)$, for each $0 < z < 1$;
- (2) $f(z) \geq 0$, for each $0 < z < 1$.

Denoting by $Quant = \left\{ f : (0, 1) \rightarrow \mathbb{R} \mid f \text{ is nondecreasing and left-continuous} \right\}$ the collection of all quantile functions, we can then write $\mathcal{AQuant}$ as follows:

$$(7.2) \quad \mathcal{AQuant} = \left\{ f \in Quant : 0 \leq f(z) \leq F_X^{-1}(z), \text{ for each } 0 < z < 1 \right\}$$

By Lebesgue's Decomposition Theorem [1, Th. 10.61] there exists a unique pair (P_{ac}, P_s) of (non-negative) finite measures on (S, Σ) such that:

- $P = P_{ac} + P_s$;
- $P_{ac} \ll Q$ (P_{ac} is absolutely continuous with respect to Q); and,
- $P_s \perp Q$ (P_s and Q are mutually singular).

That is, for all $B \in \Sigma$, $P_{ac}(B) = 0$ whenever $Q(B) = 0$. Moreover, there exists some $A \in \Sigma$ such that $Q(S \setminus A) = P_s(A) = 0$, which then implies that $P_{ac}(S \setminus A) = 0$ and $Q(A) = 1$. Note also that for all $Z \in B(\Sigma)$, $\int Z dP = \int_A Z dP_{ac} + \int_{S \setminus A} Z dP_s$. Furthermore, by the Radon-Nikodým Theorem [15, Th. 4.2.2] there exists a Q -a.s. unique Σ -measurable and Q -integrable function $h : S \rightarrow [0, +\infty)$ such that $P_{ac}(C) = \int_C h dQ$, for all $C \in \Sigma$. Hence, for all $Z \in B(\Sigma)$, $\int Z dP = \int_A Z h dQ + \int_{S \setminus A} Z dP_s$. Also, since $P_{ac}(S \setminus A) = 0$, it follows that $\int_{S \setminus A} Z dP_s = \int_{S \setminus A} Z dP$. Thus, for all $Z \in B(\Sigma)$, $\int Z dP = \int_A Z h dQ + \int_{S \setminus A} Z dP$.

Moreover, since $h : S \rightarrow [0, +\infty)$ is Σ -measurable and Q -integrable, there exists a Borel-measurable and $Q \circ X^{-1}$ -integrable map $\phi : X(S) \rightarrow [0, +\infty)$ such that $h = dP_{ac}/dQ = \phi \circ X$. We will also make the following assumption, which can be interpreted as a kind of monotone likelihood ratio property.

Assumption 7.4. *The Σ -measurable function $h = \phi \circ X = dP_{ac}/dQ$ is anti-comonotonic with X , i.e., ϕ is nonincreasing.*

7.1. A Characterization of An Optimal Innovation Contract. Since $Q \circ X^{-1}$ is nonatomic (by Assumption 7.1), it follows that $F_X(X)$ has a uniform distribution over $(0, 1)$ [20, Lemma A.21], that is, $Q(\{s \in S : F_X(X)(s) \leq t\}) = t$ for each $t \in (0, 1)$. Letting $U := F_X(X)$, it follows that U is a random variable on the probability space (S, Σ, Q) with a uniform distribution on $(0, 1)$. Consider the following quantile problem:

$$(7.3) \quad \begin{aligned} &\text{For a given } \beta \geq (1 + \rho)H, \\ &\sup_f V(f) = \int u_e(W_0^e + H - f(U)) \phi(F_X^{-1}(U)) dQ \\ &\text{s.t. } f \in \mathcal{A}Quant \\ &\int T'(1 - U) f(U) dQ = \beta \end{aligned}$$

The following theorem characterizes the solution of the entrepreneur's problem (problem (4.1) with $v = T \circ Q$) in terms of the solution of the relatively easier quantile problem given in problem (7.3), provided the previous assumptions hold. The proof is given in Appendix E.

Theorem 7.5. *Under the previous assumptions, there exists a parameter $\beta^* \geq (1 + \rho)H$ such that if f^* is optimal for problem (7.3) with parameter β^* , then the function*

$$Y^* = (X - f^*(U))\mathbf{1}_A + X\mathbf{1}_{S \setminus A}$$

is optimal for problem (4.1) (with $v = T \circ Q$).

In particular, $Y^* = X - f^*(U)$, Q -a.s. That is, the set E of states of the world s such that $Y^*(s) \neq (X - f^*(U))(s)$ has probability 0 under the probability measure Q (and hence $v(E) = T \circ Q(E) = 0$). The contract that is optimal for the entrepreneur will be seen by the financier to be almost surely equal to the function $X - f^*(U)$.

Another immediate implication of Theorem 7.5 is that the collection of states of the world in which the optimal innovation contract is a full transfer rule is a set of states to which the financier assigns a zero likelihood. On the set of all other states of the world, the innovation contract deviates from a full transfer rule by the function $f^*(U)$.

Under the following two assumptions, it is possible to fully characterize the shape of an optimal innovation contract. This is done in Corollary 7.8.

Assumption 7.6. *The Σ -measurable function $h = \phi \circ X = dP_{ac}/dQ$ is such that ϕ is left-continuous.*

Assumption 7.7. *the function $t \mapsto \frac{T'(1-t)}{\phi(F_X^{-1}(t))}$, defined on $t \in (0, 1) \setminus \{t : \phi \circ F_X^{-1}(t) = 0\}$, is nondecreasing.*

Assumption 7.7 is also a monotone likelihood ratio type assumption. Similar assumptions have been used in Jin and Zhou [31] in their study of portfolio choice under Prospect Theory, in He and Zhou [27] in their study of a portfolio choice problem under Yaari's [57] Dual Theory of choice,

in Jin and Zhou [30] in their study of greed and leverage within a portfolio choice problem under Prospect Theory, and in Carlier and Dana [12] in their study of the demand for contingent claims under Rank-Dependent Expected Utility [47].

When the previous assumptions hold, we can give an explicit characterization of an optimal contract, as follows.

Corollary 7.8 (The Shape of an Optimal Innovation Contract). *Under the previous assumptions, there exists a parameter $\beta^* \geq (1 + \rho)H$ such that an optimal solution Y^* for problem (4.1) (with $v = T \circ Q$) takes the form*

$$Y^* = \min \left[X, \max \left(0, X - d \right) \right] \mathbf{1}_A + X \mathbf{1}_{S \setminus A},$$

where

$$d = W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1 - U)}{\phi(X)} \right),$$

$U = F_X(X)$, and $\lambda^* \leq 0$ is chosen so that

$$\int T'(1 - U) \max \left[0, \min \left\{ X, W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1 - U)}{\phi(X)} \right) \right\} \right] dQ = \beta^*$$

The proof of Corollary 7.8 is given in Appendix F. Note that if Assumption 7.4 holds, then Assumption 7.6 is a weak assumption³.

8. CONCLUSION

In this paper, we have tackled some of the basic issues associated with the idea of entrepreneurial economy. We have departed from the classical tradition by introducing the notion of *publicly known states of the world*. We argued that this notion not only increases the realism of economic models but that it also allows one to unveil the true role played by entrepreneurs and financiers in actual economies. We have, then, defined an *innovation* as a collection of new states (conceived by the entrepreneur/innovator) together with an asset that pays contingent on these new states as well as on the publicly known ones. We believe that one of the main virtues of our definition is that it makes it clear that the process of innovation changes the fundamentals of the economy as it is associated to the appearance of new possibilities. Due to its nature, an innovation generates Knightian uncertainty (Ambiguity) for agents in the economy, except for the entrepreneur/innovator, and we have classified the agents with respect to the attitude that they display toward this Ambiguity. Those agents that are not strictly averse to this Ambiguity have been called *financiers*. We argued that at the core of this entrepreneurial economy lies a problem of contracting between an entrepreneur and a financier. We called this problem a problem of *contracting for innovation*. It is a contracting problem involving not only belief heterogeneity, but also Ambiguity. We showed existence and monotonicity (truth-telling) of optimal innovation contracts. We characterized these contracts in the case of financiers that are globally ambiguity-seeking, and we found an explicit analytical form for the innovation contract when the financiers' perceived ambiguity is represented by a probability distortion.

The literature on innovation is vast. Spanning from Schumpeter [53] to the works of Reinganum [48], Romer [49], Scotchmer [54] and Boldrin and Levine [7], it contains many more important papers

³Indeed, any monotone function is Borel-measurable and hence “almost continuous”, in view of Lusin’s Theorem [18, Theorem 7.5.2]. Also, any monotone function is almost surely continuous for Lebesgue measure.

than we could reasonably cite here. We refer to Bianchi and Henrekson [5], Endres and Woods [19], and Pollock [46] for an overview of the literature. It is probably fair to say that most of the literature has focused on a particular aspect of innovation or on a particular role played by it, a choice usually dictated by the problem under study (see Bianchi and Henrekson [5] for a discussion). The definition that we have introduced in this paper is an attempt to account simultaneously for all those aspects. In this way, we hope that it will appear as a concept that can be easily exported and particularized to any setting where the intuitive idea of innovation might play a significant role.

Undoubtedly, our construction has a strong Schumpeterian flavor: the entrepreneur is the creator of the innovation⁴, the entrepreneur is a singular actor, our financiers are quite like Schumpeter's bankers, the functional classification of the economic agents, etc. Clearly, there are considerable differences as well. The most notable is in the definition of innovation: ours is a far reaching generalization of Schumpeter's notion, which consists only of a new combination of the inputs in the productive process. Another difference worth stressing is the following. Schumpeter's work, as it is well-known, is regarded as a celebration of the entrepreneur: this is viewed as a privileged individual that, in a condition of severe uncertainty (the newly thought states), has a "vision" (the project/asset) that might change the course of the economy⁵. While this is true in our construction as well, the appearance of this "vision" would be rather inconsequential if it were not coupled with another "vision", that of the financier. In our construction, the vision of the entrepreneur leads to the appearance of Ambiguity. It is only the insight of the financier into this Ambiguity that recognizes the vision of the entrepreneur and makes the change possible. Formally, this insight appears in the form of the coefficient $\alpha(X_e)$ being low enough, which means precisely that the financier believes in the profitability of the entrepreneur's project.

Unlike the literature on innovation, the literature on contracting under heterogeneity and Ambiguity is not vast, but it does enlist several important papers, which tackle a variety of issues (see, for instance, [40, 41, 42]). Important contributions to the problem of existence and monotonicity of an optimal contract in situations of Ambiguity and/or heterogeneity have been made in [8, 9, 10, 11, 14, 16]. Carlier and Dana [9, 10] and Dana [16] show existence and monotonicity in settings characterized by the presence of Ambiguity but where there is no heterogeneity. Carlier and Dana [8] study a setting similar to ours, but impose the additional restriction that the capacity of one party is a distortion of the probability of the other party, thus retaining a certain (weak) form of homogeneity. Chateauneuf, Dana and Tallon [14] allow for capacities (i.e., Ambiguity) on both sides, but they assume that both capacities are submodular distortions and that the state space is finite. Finally, Carlier and Dana [11] also allow for capacities on both sides, but demand that both capacities be distortions of the same measure, and that the heterogeneity be "small" (in a sense made precise in that paper). In relation to this literature, we have contributed an existence and monotonicity result in a setting where, while we have Ambiguity only on one side, we allow for a larger degree of heterogeneity. To this, we have also added the characterization of Section 6 obtained under the additional assumption of a submodular (concave) capacity (but which also covers the general maxmax case) as well as the explicit analytical form of the optimal innovation contract of Section 7.

⁴Schumpeter distinguishes between those who create ideas and those, the entrepreneurs, who turn them into something of economic value. Roughly, in our model this would correspond to distinguishing between those who come up with the new states (inventors) and those who make those states suitable of generating economic value (entrepreneurs) by issuing assets that pay contingent on those states.

⁵In Schumpeter's work, the entrepreneur faces Ambiguity, while in our construction all of his uncertainty is reduced to Risk. This is not a substantial difference as we could allow for the entrepreneur to be described by non-additive criteria. This would result only in a technical complication without changing the essence of the problems we study.

APPENDIX A. INFORMATIONAL CONSTRAINTS AS MEASURABILITY CONDITIONS

This Appendix briefly discusses two aspects of the contracting problem that are seemingly technical. In fact, these aspects play a substantial role not only here but also elsewhere, for instance in the problem of whether or not a central authority is able to replicate the outcomes produced by an economy with innovation. In the present setting, the easiest way to grasp these aspects is also the most intuitive: just think of an entrepreneur and a financier coming together into a room; the former describes his project because he is seeking funding, and the latter has to decide what to do.

The first issue has to do with the measurable structure on the set S_e . In our story, the financier is somebody who not only sees the innovation, i.e., the pair (S_e, X_e) , for the first time in his life but has never conceived of it either. This implies that a contract between the financier and the entrepreneur may only be written on the basis of the information that is revealed in the room. The way to formalize this requirement is by endowing S_e with the coarsest σ -algebra which makes X_e measurable: this expresses precisely the fact that *all the information available is derived from the description of the innovation*. We denote this σ -algebra by Σ_e . Accordingly, the innovation can be written as $((S_e, \Sigma_e), X_e)$, and X_e is a random variable on (S_e, Σ_e) . By Doob's Measurability Theorem [1, Theorem 4.41], any measurable function g on (S_e, Σ_e) has the form $g = \zeta \circ X_e$, where ζ is a Borel-measurable function $\mathbb{R} \rightarrow \mathbb{R}$. The collection of all bounded measurable functions on (S_e, Σ_e) is denoted by $B(\Sigma_e)$, and the set of its positive elements by $B^+(\Sigma_e)$.

The second issue has to do with the probability P_e according to which the entrepreneur evaluates his own innovation. We assume that the entrepreneur declares truthfully this belief P_e , which is thus a common knowledge among the parties. Formally, this probability is just a mathematical representation of certain parts of the entrepreneur's project. Thus, *de facto*, we assume that the entrepreneur reveals truthfully (*ex ante*) some aspects of his project (precisely those that admit a representation in the form of a probabilistic assessment). We believe that this assumption sounds heavier than what it really is, and this is so for at least two reasons. First, when they come in contact with each other, the entrepreneur knows nothing about the financier (formally, this is encoded in the requirement on the σ -algebra). Thus, if he were to lie about those aspects of the project (i.e., declare a probability different from P_e), he would have no reason to think that this might increase his chances to get funded. Second, and perhaps more importantly, the financier's ambiguous beliefs (in the non-additive sense) are formed independently of P_e . That is, the view the financier ends up with after being presented with the innovation would be the same whether P_e or any other probability is declared by the entrepreneur. Formally, what drives the feature that the financier might find the project worthwhile is not the probability P_e but the coefficient of Ambiguity aversion $\alpha(X_e)$, which depends only on the random variable X_e and not on the probability P_e .

We have mentioned that the probability describes certain aspects of the entrepreneur's project. All the other aspects are encoded in the mapping X_e , which expresses the gains/losses that the project allegedly generates as a function of the new states. Needless to say, we do not make any assumption about how truthfully this part is revealed, as this is the very essence of the contracting problem.

APPENDIX B. BACKGROUND MATERIAL

B.1. Capacities and the Choquet Integral. Here, we summarize the basic definitions about capacities, Choquet integrals and Šipoš integrals. The proofs of the statements listed below can be found, for instance, in [38] or [43].

Definition B.1. A (normalized) capacity on a measurable space (S, Σ) is a set function $v : \Sigma \rightarrow [0, 1]$ such that:

- (1) $v(\emptyset) = 0$;
- (2) $v(S) = 1$; and
- (3) $A, B \in \Sigma$ and $A \subset B \implies v(A) \leq v(B)$.

Definition B.2. A capacity v on (S, Σ) is continuous from above (resp. below) if for any sequence $\{A_n\}_{n \geq 1} \subseteq \Sigma$ such that $A_{n+1} \subseteq A_n$ (resp. $A_{n+1} \supseteq A_n$) for each n , it holds that

$$\lim_{n \rightarrow +\infty} v(A_n) = v\left(\bigcap_{n=1}^{+\infty} A_n\right) \quad \left(\text{resp. } \lim_{n \rightarrow +\infty} v(A_n) = v\left(\bigcup_{n=1}^{+\infty} A_n\right)\right)$$

A capacity that is continuous both from above and below is said to be continuous.

Definition B.3. Given a capacity v and a function $\psi \in B(\Sigma)$, the Choquet integral of ψ w.r.t. v is defined by

$$\int \phi \, dv = \int_0^{+\infty} v(\{s \in S : \phi(s) \geq t\}) \, dt + \int_{-\infty}^0 [v(\{s \in S : \phi(s) \geq t\}) - 1] \, dt$$

where the integrals on the RHS are taken in the sense of Riemann.

Unlike the Lebesgue integral, the Choquet integral is not additive. One of its characterizing properties, however, is that it respects additivity on comonotonic functions.

Definition B.4. Two functions $Y_1, Y_2 \in B(\Sigma)$ are comonotonic if for all $s, s' \in S$

$$\left[Y_1(s) - Y_1(s')\right] \left[Y_2(s) - Y_2(s')\right] \geq 0$$

As mentioned above, if $Y_1, Y_2 \in B(\Sigma)$ are comonotonic then

$$\int (Y_1 + Y_2) \, dv = \int Y_1 \, dv + \int Y_2 \, dv$$

Definition B.5. A capacity v on (S, Σ) is submodular (or concave) if for any $A, B \in \Sigma$

$$v(A \cup B) + v(A \cap B) \leq v(A) + v(B)$$

It is supermodular (or convex) if the reverse inequality holds for any $A, B \in \Sigma$.

As a functional on $B(\Sigma)$, the Choquet integral $\int \cdot \, dv$ is concave (resp. convex) if and only if v is submodular (resp. supermodular).

Proposition B.6. Let v be a capacity on (S, Σ) .

- (1) If $Y \in B(\Sigma)$ and $c \in \mathbb{R}$, then $\int (Y + c) \, dv = \int Y \, dv + c$.
- (2) If $A \in \Sigma$ then $\int \mathbf{1}_A \, dv = v(A)$.
- (3) If $Y \in B(\Sigma)$ and $a \geq 0$, then $\int a Y \, dv = a \int Y \, dv$.
- (4) If $Y_1, Y_2 \in B(\Sigma)$ are such that $Y_1 \leq Y_2$, then $\int Y_1 \, dv \leq \int Y_2 \, dv$.
- (5) If v is submodular, then for any $Y_1, Y_2 \in B(\Sigma)$, $\int (Y_1 + Y_2) \, dv \leq \int Y_1 \, dv + \int Y_2 \, dv$.

Definition B.7. *The Šipoš integral, or the symmetric Choquet integral (see [43]), is a functional $\check{S} : B(\Sigma) \rightarrow \mathbb{R}$ defined by*

$$\check{S}(Y) = \int Y^+ dv - \int Y^- dv$$

where the integrals on the RHS are taken in the sense of Choquet and Y^+ (resp. Y^-) denotes the positive (resp. negative) part of $Y \in B(\Sigma)$. Obviously, the Šipoš integral coincides with the Choquet integral for positive functions.

B.2. Nondecreasing Rearrangements. All the definitions and results that appear in this Appendix are taken from Ghossoub [23, 24] and the references therein. We refer the interested reader to Ghossoub [23, 24] for proofs and for additional results.

B.2.1. The Nondecreasing Rearrangement. Let $(S, \mathcal{G}, P)$ be a probability space, and let $X \in B^+(\mathcal{G})$ be a continuous random variable (i.e., $P \circ X^{-1}$ is a nonatomic Borel probability measure) with range $X(S) = [0, M]$. Denote by Σ the σ -algebra generated by X , and let

$$\phi(B) := P\left(\{s \in S : X(s) \in B\}\right) = P \circ X^{-1}(B),$$

for any Borel subset B of $\mathbb{R}$.

For any Borel-measurable map $I : [0, M] \rightarrow \mathbb{R}$, define the *distribution function* of I as the map $\phi_I : \mathbb{R} \rightarrow [0, 1]$ given by $\phi_I(t) := \phi(\{x \in [0, M] : I(x) \leq t\})$. Then ϕ_I is a nondecreasing right-continuous function.

Definition B.8. *Let $I : [0, M] \rightarrow [0, M]$ be any Borel-measurable map, and define the function $\tilde{I} : [0, M] \rightarrow \mathbb{R}$ by*

$$(B.1) \quad \tilde{I}(t) := \inf \left\{ z \in \mathbb{R}^+ : \phi_I(z) \geq \phi([0, t]) \right\}$$

Then $\tilde{I}$ is a nondecreasing and Borel-measurable mapping of $[0, M]$ into $[0, M]$ such that I and $\tilde{I}$ are ϕ -equimeasurable, in the sense that for any $\alpha \in [0, M]$,

$$\phi(\{t \in [0, M] : I(t) \leq \alpha\}) = \phi(\{t \in [0, M] : \tilde{I}(t) \leq \alpha\})$$

Moreover, if $\bar{I} : [0, M] \rightarrow \mathbb{R}^+$ is another nondecreasing, Borel-measurable map which is ϕ -equimeasurable with I , then $\bar{I} = \tilde{I}$, ϕ -a.s. $\tilde{I}$ is called the nondecreasing ϕ -rearrangement of I .

Now, define $Y := I \circ X$ and $\tilde{Y} := \tilde{I} \circ X$. Since both I and $\tilde{I}$ are Borel-measurable mappings of $[0, M]$ into itself, it follows that $Y, \tilde{Y} \in B^+(\Sigma)$. Note also that $\tilde{Y}$ is nondecreasing in X , in the sense that if $s_1, s_2 \in S$ are such that $X(s_1) \leq X(s_2)$ then $\tilde{Y}(s_1) \leq \tilde{Y}(s_2)$, and that Y and $\tilde{Y}$ are P -equimeasurable. That is, for any $\alpha \in [0, M]$, $P(\{s \in S : Y(s) \leq \alpha\}) = P(\{s \in S : \tilde{Y}(s) \leq \alpha\})$.

We will call $\tilde{Y}$ a **nondecreasing P -rearrangement of Y with respect to X** , and we shall denote it by $\tilde{Y}_P$. Note that $\tilde{Y}_P$ is P -a.s. unique. Note also that if Y_1 and Y_2 are P -equimeasurable and if $Y_1 \in L_1(S, \mathcal{G}, P)$, then $Y_2 \in L_1(S, \mathcal{G}, P)$ and $\int \psi(Y_1) dP = \int \psi(Y_2) dP$, for any measurable function ψ such that the integrals exist.

B.2.2. Supermodularity and Hardy-Littlewood Inequalities. A *partially ordered set (poset)* is a pair $(A, \succeq)$, where $\succeq$ is a reflexive, transitive and antisymmetric binary relation on A . For any $x, y \in A$, we denote by $x \vee y$ (resp. $x \wedge y$) the least upper bound (resp. greatest lower bound) of the set $\{x, y\}$. A poset $(A, \succeq)$ is a *lattice* when $x \vee y, x \wedge y \in A$ for every $x, y \in A$. For instance, the Euclidian space $\mathbb{R}^n$ is a lattice for the partial order $\succeq$ defined as follows: for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we write $x \succeq y$ when $x_i \geq y_i$, for each $i = 1, \dots, n$.

Definition B.9. Let $(A, \succeq)$ be a lattice. A function $L : A \rightarrow \mathbb{R}$ is said to be *supermodular* if for each $x, y \in A$,

$$L(x \vee y) + L(x \wedge y) \geq L(x) + L(y)$$

In particular, a function $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ is *supermodular* if for any $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $y_1 \leq y_2$, we have

$$L(x_2, y_2) + L(x_1, y_1) \geq L(x_1, y_2) + L(x_2, y_1)$$

It is then easily seen that supermodularity of a function $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ is equivalent to the function $\eta(y) = L(x + h, y) - L(x, y)$ being nondecreasing for any $x \in \mathbb{R}$ and $h \geq 0$.

Example B.10. The following are useful examples of supermodular functions on $\mathbb{R}^2$:

- (1) If $g : \mathbb{R} \rightarrow \mathbb{R}$ is concave and $a \in \mathbb{R}$, then the function $L_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $L_1(x, y) = g(a - x + y)$ is supermodular;
- (2) The function $L_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $L_2(x, y) = -(y - x)^+$ is supermodular;
- (3) If $\eta : \mathbb{R} \rightarrow \mathbb{R}^+$ is a nonincreasing function, $h : \mathbb{R} \rightarrow \mathbb{R}$ is concave and nondecreasing, and $a \in \mathbb{R}$, then the function $L_3 : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $L_3(x, y) = h(a - y) \eta(x)$ is supermodular.

Lemma B.11. Let $Y \in B^+(\Sigma)$, and denote by $\tilde{Y}_P$ the nondecreasing P -rearrangement of Y with respect to X . Then,

- (1) If L is a supermodular and $P \circ X^{-1}$ -integrable function on the range of X then

$$\int L(X, Y) dP \leq \int L(X, \tilde{Y}_P) dP$$

- (2) If $0 \leq Y \leq X$ then $0 \leq \tilde{Y}_P \leq X$.

APPENDIX C. PROOF OF THEOREM 5.6

Let us denote by $\mathcal{F}_{SB}$ the feasibility set for problem (4.1) (which we assume nonempty to rule out trivial situations):

$$(C.1) \quad \mathcal{F}_{SB} = \left\{ Y \in B(\Sigma) : 0 \leq Y \leq X \text{ and } \int (X - Y) dv \geq (1 + \rho) H = H' \right\}$$

Let $\mathcal{F}_{SB}^\uparrow$ be the set of all the $Y \in \mathcal{F}_{SB}$ which, in addition, are comonotonic with X :

$$\mathcal{F}_{SB}^\uparrow = \left\{ Y = I \circ X \in \mathcal{F}_{SB} : I \text{ is nondecreasing} \right\}$$

Lemma C.1. *If v is (P, X) -vigilant, then for each $Y \in \mathcal{F}_{SB}$ there exists a $\tilde{Y} \in \mathcal{F}_{SB}$ such that:*

- (1) $\tilde{Y}$ is comonotonic with X ,
- (2) $\int u_e(W_0^e + H - X + \tilde{Y}) dP \geq \int u_e(W_0^e + H - X + Y) dP$,
- (3) $\int (X - \tilde{Y}) dv \geq \int (X - Y) dv$

Proof. Choose any $Y = I \circ X \in \mathcal{F}_{SB}$, and let $\tilde{Y}_P$ denote the nondecreasing P -rearrangement of Y with respect to X . Then (i) $\tilde{Y}_P = \tilde{I} \circ X$ where $\tilde{I}$ is nondecreasing, and (ii) $0 \leq \tilde{Y}_P \leq X$, by Lemma B.11. Furthermore, since v is (P, X) -vigilant, it follows that $\int (X - \tilde{Y}_P) dv \geq \int (X - Y) dv$. But $\int (X - Y) dv \geq H'$ since $Y \in \mathcal{F}_{SB}$. Hence, $\tilde{Y}_P \in \mathcal{F}_{SB}^\uparrow$. Moreover, since the utility u_e is concave (Assumption 4.5), the function $\mathcal{U}(x, y) = u_e(W_0^e + H - x + y)$ is supermodular (as in Example B.10 (1)). Then, by Lemma B.11, $\int u_e(W_0^e + H - X + \tilde{Y}) dP \geq \int u_e(W_0^e + H - X + Y) dP$. $\square$

Now, by Lemma C.1, we can choose a maximizing sequence $\{Y_n\}_n$ in $\mathcal{F}_{SB}^\uparrow$ for problem (4.1). That is,

$$\lim_{n \rightarrow +\infty} \int u_e(W_0^e + H - X + Y_n) dP = N \equiv \sup_{Y \in B^+(\Sigma)} \left\{ \int u_e(W_0^e + H - X + Y) dP \right\} < +\infty$$

Since $0 \leq Y_n \leq X \leq M \equiv \|X\|_\infty$, the sequence $\{Y_n\}_n$ is uniformly bounded. Moreover, for each $n \geq 1$ we have $Y_n = I_n \circ X$, with $I_n : [0, M] \rightarrow [0, M]$. Consequently, the sequence $\{I_n\}_n$ is a uniformly bounded sequence of nondecreasing Borel-measurable functions. Thus, by Helly's First Theorem [13, Lemma 13.15] (a.k.a. Helly's Compactness Theorem), there is a nondecreasing function $I^* : [0, M] \rightarrow [0, M]$ and a subsequence $\{I_m\}_m$ of $\{I_n\}_n$ such that $\{I_m\}_m$ converges pointwise on $[0, M]$ to I^* . Hence, I^* is also Borel-measurable, and so $Y^* = I^* \circ X \in B^+(\Sigma)$ is such that $0 \leq Y^* \leq X$. Moreover, the sequence $\{Y_m\}_m$, $Y_m = I_m \circ X$, converges pointwise to Y^* . Thus, the sequence $\{X - Y_m\}_m$ is uniformly bounded and converges pointwise to $(X - Y^*)$. By the assumption that v is continuous (Assumption 4.7), it follows from a Dominated Convergence-type Theorem [43, Theorem 7.16]⁶ that

$$H' \leq \lim_{m \rightarrow +\infty} \int (X - Y_m) dv = \int (X - Y^*) dv$$

and so $Y^* \in \mathcal{F}_{SB}^\uparrow$. Now, by continuity and boundedness of the function u_e , and by Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21], we have

$$\begin{aligned} \int u_e(W_0^e + H - X + Y^*) dP &= \lim_{m \rightarrow +\infty} \int u_e(W_0^e + H - X + Y_m) dP \\ &= \lim_{n \rightarrow +\infty} \int u_e(W_0^e + H - X + Y_n) dP = N \end{aligned}$$

Hence Y^* solves problem (4.1). This concludes the proof of Theorem 5.6. $\square$

⁶The Theorem of Pap [43] is for the *Šipoš integral*, or the *symmetric Choquet integral*. However, the latter coincides with the Choquet integral for nonnegative functions (see Appendix B.1).

APPENDIX D. PROOF OF PROPOSITION 6.1

Let $\mathcal{C}_\varphi$ denote the anticore of v . Since each $Q \in \mathcal{C}_\varphi$ is (P, X) -vigilant, it follows that v is (P, X) -vigilant. Hence, by Theorem 5.6, there exists a solution Y^{**} to problem (4.1). Fix $Q \in \mathcal{C}_\varphi$ arbitrarily, and let $Y^*(Q)$ be an optimal solution of problem (6.2) for this given $Q \in \mathcal{C}_\varphi$. The existence of $Y^*(Q)$ follows from the (P, X) -vigilance of Q , in light of Theorem 5.6. Then, $Y^*(Q)$ satisfies $0 \leq Y^*(Q) \leq X$ and $\int (X - Y^*(Q)) dQ \geq (1 + \rho) H$. Hence,

$$\max_{R \in \mathcal{C}_\varphi} \int (X - Y^*(Q)) dR \geq \int (X - Y^*(Q)) dQ \geq (1 + \rho) H,$$

which shows that $Y^*(Q)$ is feasible for problem (4.1). Since Y^{**} solves problem (4.1), we must have that

$$(D.1) \quad \int u_e(W_0^e + H - X + Y^{**}) dP \geq \int u_e(W_0^e + H - X + Y^*(Q)) dP$$

To conclude the proof, it suffices to find some $Q^{**} \in \mathcal{C}_\varphi$ such that inequality (D.1) holds as an equality. Suppose, by the way of contradiction, that no such Q^{**} exists. Then, for all $Q \in \mathcal{C}_\varphi$ it holds that

$$(D.2) \quad \int u_e(W_0^e + H - X + Y^{**}) dP > \int u_e(W_0^e + H - X + Y^*(Q)) dP$$

Since, by definition, $Y^*(Q)$ solves the problem of type (6.2) defined by Q , inequality (D.2) implies that Y^{**} must not be feasible for any problem of the type (6.2). That is, for all $Q \in \mathcal{C}_\varphi$,

$$\int (X - Y^{**}) dQ < (1 + \rho) H$$

However, by the feasibility of Y^{**} for problem (4.1), we have that for all $Q \in \mathcal{C}_\varphi$,

$$\int (X - Y^{**}) dQ < (1 + \rho) H \leq \max_{R \in \mathcal{C}_\varphi} \int (X - Y^{**}) dR$$

which, since $(X - Y^{**}) \in B(\Sigma)$, contradicts the fact that $\mathcal{C}_\varphi$ is weak*-compact and convex. $\square$

APPENDIX E. PROOF OF THEOREM 7.5

E.1. “Splitting”. Recall that by Lebesgue’s Decomposition Theorem [1, Th. 10.61] there exists a unique pair (P_{ac}, P_s) of (nonnegative) finite measures on (S, Σ) such that $P = P_{ac} + P_s$, $P_{ac} \ll Q$ and $P_s \perp Q$. That is, for all $B \in \Sigma$ with $Q(B) = 0$, we have $P_{ac}(B) = 0$, and there is some $A \in \Sigma$ such that $Q(S \setminus A) = P_s(A) = 0$. It then also follows that $P_{ac}(S \setminus A) = 0$ and $Q(A) = 1$. In the following, the Σ -measurable set A on which Q is concentrated is assumed to be fixed all throughout. Consider now the following two problems:

$$(E.1) \quad \begin{aligned} &\text{For a given } \beta \geq (1 + \rho) H, \\ &\sup_{Y \in B(\Sigma)} \int_A u_e(W_0^e + H - X + Y) dP \\ &s.t. \quad 0 \leq Y \leq X \\ &\int (X - Y) dT \circ Q = \beta \end{aligned}$$

and

$$(E.2) \quad \begin{aligned} & \sup_{Y \in B(\Sigma)} \int_{S \setminus A} u_e(W_0^e + H - X + Y) \, dP \\ \text{s.t. } & 0 \leq Y \mathbf{1}_{S \setminus A} \leq X \mathbf{1}_{S \setminus A} \\ & \int_{S \setminus A} (X - Y) \, dT \circ Q = 0 \end{aligned}$$

Remark E.1. By the boundedness of u_e , the supremum of each of the above two problems is finite when their feasibility sets are nonempty. Now, the function X is feasible for problem (E.2), and so problem (E.2) has a nonempty feasibility set.

Definition E.2. For a given $\beta \geq (1 + \rho)H$, let $\Theta_{A,\beta}$ be the feasibility set of problem (E.1) with parameter β . That is,

$$\Theta_{A,\beta} := \left\{ Y \in B^+(\Sigma) : 0 \leq Y \leq X, \int (X - Y) \, dT \circ Q = \beta \right\}$$

Denote by Γ the collection of all β for which the feasibility set $\Theta_{A,\beta}$ is nonempty:

Definition E.3. Let $\Gamma := \left\{ \beta \geq (1 + \rho)H : \Theta_{A,\beta} \neq \emptyset \right\}$

Lemma E.4. $\Gamma \neq \emptyset$.

Proof. Choose $Y \in \mathcal{F}_{SB}$ arbitrarily, where $\mathcal{F}_{SB}$ is defined by equation (C.1). Then $Y \in B^+(\Sigma)$ is such that $0 \leq Y \leq X$ and $\int (X - Y) \, dT \circ Q \geq (1 + \rho)H$. Let $\beta_Y = \int (X - Y) \, dT \circ Q$. Then, by definition of β_Y , and since $0 \leq Y \leq X$, we have $Y \in \Theta_{A,\beta_Y}$, and so $\Theta_{A,\beta_Y} \neq \emptyset$. Consequently, $\beta_Y \in \Gamma$, and so $\Gamma \neq \emptyset$. $\square$

Lemma E.5. X is optimal for problem (E.2).

Proof. The feasibility of X for problem (E.2) is clear. To show optimality, let Y be any feasible solution for problem (E.2). Then for each $s \in S \setminus A$, $Y(s) \leq X(s)$. Therefore, since u_e is increasing, we have $u_e(W_0^e + H - X(s) + Y(s)) \leq u_e(W_0^e + H - X(s) + X(s)) = u_e(W_0^e + H)$, for each $s \in S \setminus A$. Thus,

$$\int_{S \setminus A} u_e(W_0^e + H - X + Y) \, dP \leq \int_{S \setminus A} u_e(W_0^e + H - X + X) \, dP = u(W_0^e + H)P(S \setminus A)$$

$\square$

Remark E.6. Since $Q(S \setminus A) = 0$ and $T(0) = 0$, it follows that $T \circ Q(S \setminus A) = 0$, and so $\int \mathbf{1}_{S \setminus A} \, dT \circ Q = T \circ Q(S \setminus A) = 0$, by Proposition B.6. Therefore, for any $Z \in B^+(\Sigma)$, it follows from the monotonicity and positive homogeneity of the Choquet integral (Proposition B.6) that

$$0 \leq \int_{S \setminus A} Z \, dT \circ Q = \int Z \mathbf{1}_{S \setminus A} \, dT \circ Q \leq \int \|Z\|_s \mathbf{1}_{S \setminus A} \, dT \circ Q = \|Z\|_s \int \mathbf{1}_{S \setminus A} \, dT \circ Q = 0$$

and so $\int_{S \setminus A} Z \, dT \circ Q = 0$. Consequently, it follows from Proposition B.6 that for any $Z \in B^+(\Sigma)$,

$$\int Z \, dT \circ Q \leq \int Z \mathbf{1}_A \, dT \circ Q = \int_A Z \, dT \circ Q$$

Now, consider the following problem:

$$(E.3) \quad \sup_{\beta \in \Gamma} \left\{ F_A^*(\beta) : F_A^*(\beta) \text{ is the supremum of problem (E.1), for a fixed } \beta \in \Gamma \right\}$$

Lemma E.7. *Under Assumption 7.1, if β^* is optimal for problem (E.3), and if Y_1^* is optimal for problem (E.1) with parameter β^* , then $Y^* := Y_1^* \mathbf{1}_A + X \mathbf{1}_{S \setminus A}$ is optimal for problem (4.1) (with $v = T \circ Q$).*

Proof. By the feasibility of Y_1^* for problem (E.1) with parameter β^* , we have $0 \leq Y_1^* \leq X$ and $\int (X - Y_1^*) \, dT \circ P = \beta^*$. Therefore, $0 \leq Y^* \leq X$, and

$$\begin{aligned} \int (X - Y^*) \, dT \circ Q &= \int [(X - Y_1^*) \mathbf{1}_A + (X - X) \mathbf{1}_{S \setminus A}] \, dT \circ Q \\ &= \int_A (X - Y_1^*) \, dT \circ Q \geq \int (X - Y_1^*) \, dT \circ Q = \beta^* \geq (1 + \rho) H \end{aligned}$$

where the inequality $\int_A (X - Y_1^*) \, dT \circ Q \geq \int (X - Y_1^*) \, dT \circ Q$ follows from the same argument as in Remark E.6. Hence, Y^* is feasible for problem (4.1) (with $v = T \circ Q$). To show optimality of Y^* for problem (4.1) (with $v = T \circ Q$), let $\bar{Y}$ be any other feasible function for problem (4.1) (with $v = T \circ Q$), and define α by $\alpha = \int (X - \bar{Y}) \, dT \circ Q$. Then $\alpha \geq (1 + \rho) H$, and so $\bar{Y}$ is feasible for problem (E.1) with parameter α , and α is feasible for problem (E.3). Hence

$$F_A^*(\alpha) \geq \int_A u_e(W_0^e + H - X + \bar{Y}) \, dP$$

Now, since β^* is optimal for problem (E.3), it follows that $F_A^*(\beta^*) \geq F_A^*(\alpha)$. Moreover, $\bar{Y}$ is feasible for problem (E.2) (since $0 \leq \bar{Y} \leq X$ and so $\int_{S \setminus A} (X - \bar{Y}) \, dT \circ Q = 0$ by Remark E.6). Thus,

$$\begin{aligned} F_A^*(\beta^*) + u_e(W_0^e + H)P(S \setminus A) &\geq F_A^*(\alpha) + u_e(W_0^e + H)P(S \setminus A) \\ &\geq \int_A u_e(W_0^e + H - X + \bar{Y}) \, dP + u_e(W_0^e + H)P(S \setminus A) \\ &\geq \int_A u_e(W_0^e + H - X + \bar{Y}) \, dP + \int_{S \setminus A} u_e(W_0^e + H - X + \bar{Y}) \, dP \\ &= \int u_e(W_0^e + H - X + \bar{Y}) \, dP \end{aligned}$$

However, $F_A^*(\beta^*) = \int_A u_e(W_0^e + H - X + Y_1^*) \, dP$. Therefore,

$$\int u_e(W_0^e + H - X + Y^*) \, dP = F_A^*(\beta^*) + u_e(W_0^e + H)P(S \setminus A) \geq \int u_e(W_0^e + H - X + \bar{Y}) \, dP$$

Hence, Y^* is optimal for problem (4.1) (with $v = T \circ Q$). $\square$

Remark E.8. By Lemma E.7, we can restrict ourselves to solving problem (E.1) with a parameter $\beta \in \Gamma$.

E.2. Solving Problem (E.1). Recall that for all $Z \in B(\Sigma)$, $\int Z dP = \int_A Zh dQ + \int_{S \setminus A} Z dP$, where $h = dP_{ac}/dQ$ is the Radon-Nikodým derivative of P_{ac} with respect to Q . Moreover, by definition of the set $A \in \Sigma$, we have $Q(S \setminus A) = P_s(A) = 0$. Therefore, $\int_A Zh dQ = \int Zh dQ$, for each $Z \in B(\Sigma)$. Hence, we can rewrite problem (E.1) (restricting ourselves to parameters $\beta \in \Gamma$ and recalling that $h = \phi \circ X$) as the following problem:

For a given $\beta \in \Gamma$,

$$(E.4) \quad \begin{aligned} & \sup_{Y \in B(\Sigma)} \int u_e(W_0^e + H - X + Y) \phi(X) dQ \\ & \text{s.t. } 0 \leq Y \leq X \\ & \int (X - Y) dT \circ Q = \beta \end{aligned}$$

Now, consider the following problem:

For a given $\beta \in \Gamma$,

$$(E.5) \quad \begin{aligned} & \sup_{Z \in B(\Sigma)} \int u_e(W_0^e + H - Z) \phi(X) dQ \\ & \text{s.t. } 0 \leq Z \leq X \\ & \int Z dT \circ Q = \beta = \int_0^{+\infty} T(Q(\{s \in S : Z(s) \geq t\})) dt \end{aligned}$$

Lemma E.9. *If Z^* is optimal for problem (E.5) with parameter β , then $Y^* := X - Z^*$ is optimal for problem (E.4) with parameter β .*

Proof. Let $\beta \in \Gamma$ be given, and suppose that Z^* is optimal for problem (E.5) with parameter β . Define $Y^* := X - Z^*$. Then $Y^* \in B(\Sigma)$. Moreover, since $0 \leq Z^* \leq X$, it follows that $0 \leq Y^* \leq X$. Now,

$$\int (X - Y^*) dT \circ Q = \int (X - (X - Z^*)) dT \circ Q = \int Z^* dT \circ Q = \beta$$

and so Y^* is feasible for problem (E.4) with parameter β . To show optimality of Y^* for problem (E.4) with parameter β , suppose, by way of contradiction, that $\bar{Y} \neq Y^*$ is feasible for problem (E.4) with parameter β and

$$\int u_e(W_0^e + H - X + \bar{Y}) h dQ > \int u_e(W_0^e + H - X + Y^*) h dQ$$

that is, with $\bar{Z} := X - \bar{Y}$, we have

$$\int u_e(W_0^e + H - \bar{Z}) h dQ > \int u_e(W_0^e + H - Z^*) h dQ$$

Now, since $0 \leq \bar{Y} \leq X$ and $\int (X - \bar{Y}) dT \circ Q = \beta$, we have that $\bar{Z}$ is feasible for problem (E.5) with parameter β , hence contradicting the optimality of Z^* for problem (E.5) with parameter β . Thus, $Y^* := X - Z^*$ is optimal for problem (E.4) with parameter β . $\square$

Definition E.10. If $Z_1, Z_2 \in B^+(\Sigma)$ are feasible for problem (E.5) with parameter β , we will say that Z_2 is a Pareto improvement of Z_1 (or is Pareto-improving) when the following hold:

- (1) $\int u_e(W_0^e + H - Z_2) h dQ \geq \int u_e(W_0^e + H - Z_1) h dQ$; and,
- (2) $\int Z_2 dT \circ Q \geq \int Z_1 dT \circ Q$.

The next result shows that for any feasible claim for problem (E.5), there is a another feasible claim for problem (E.5), which is comonotonic with X and Pareto-improving.

Lemma E.11. Fix a parameter $\beta \in \Gamma$. If Z is feasible for problem (E.5) with parameter β , then $\tilde{Z}$ is feasible for problem (E.5) with parameter β , comonotonic with X and Pareto-improving, where $\tilde{Z}$ is the nondecreasing Q -rearrangement of Z with respect to X .

Proof. Let Z be feasible for problem (E.5) with parameter β , and note that by Assumption 7.4, the map $\xi(X, Z) := u_e(W_0^e + H - Z) \phi(X)$ is supermodular (see Example B.10). Let $\tilde{Z}$ denote the nondecreasing Q -rearrangement of Z with respect to X . Then by Lemma B.11 (2) and by equimeasurability of Z and $\tilde{Z}$, the function $\tilde{Z}$ is feasible for problem (E.5) with parameter β . Also, by Lemma B.11 (1) and by supermodularity of $\xi(X, Z)$, it follows that $\tilde{Z}$ is Pareto-improving. $\square$

E.3. Quantile reformulation. Fix a parameter $\beta \in \Gamma$, let $Z \in B^+(\Sigma)$ be feasible for problem (E.5) with parameter β , let $F_Z(t) = Q(\{s \in S : Z(s) \leq t\})$ denote the distribution function of Z with respect to the probability measure Q and let $F_X(t) = Q(\{s \in S : X(s) \leq t\})$ denote the distribution function of X with respect to the probability measure Q . Let $F_Z^{-1}(t)$ be the left-continuous inverse of the distribution function F_Z (that is, the quantile function of Z), defined by

$$F_Z^{-1}(t) = \inf \left\{ z \in \mathbb{R}^+ : F_Z(z) \geq t \right\}, \quad \forall t \in [0, 1]$$

Let $\tilde{Z}$ denote the nondecreasing Q -rearrangement of Z with respect to X . Since $Z \in B^+(\Sigma)$, it can be written as $\psi \circ X$ for some nonnegative Borel-measurable and bounded map ψ on $X(S)$. Moreover, since $0 \leq Z \leq X$, ψ is a mapping of $[0, M]$ into $[0, M]$. Let $\zeta := Q \circ X^{-1}$ be the image measure of Q under X . By Assumption 7.1, ζ is nonatomic. We can then define the mapping $\tilde{\psi} : [0, M] \rightarrow [0, M]$ as in Appendix B.2 (see equation (B.1) on p. 21) to be the nondecreasing ζ -rearrangement of ψ , that is,

$$\tilde{\psi}(t) := \inf \left\{ z \in \mathbb{R}^+ : \zeta(\{x \in [0, M] : \psi(x) \leq z\}) \geq \zeta([0, t]) \right\}$$

Then, as in Appendix B.2, $\tilde{Z} = \tilde{\psi} \circ X$. Therefore, for each $s_0 \in S$,

$$\tilde{Z}(s_0) = \tilde{\psi}(X(s_0)) = \inf \left\{ z \in \mathbb{R}^+ : \zeta(\{x \in [0, M] : \psi(x) \leq z\}) \geq \zeta([0, X(s_0)]) \right\}$$

However, for each $s_0 \in S$,

$$\zeta([0, X(s_0)]) = Q \circ X^{-1}([0, X(s_0)]) = F_X(X(s_0)) := F_X(X)(s_0)$$

Moreover,

$$\begin{aligned} \zeta(\{x \in [0, M] : \psi(x) \leq z\}) &= Q \circ X^{-1}(\{x \in [0, M] : \psi(x) \leq z\}) \\ &= Q(\{s \in S : \psi(X(s)) \leq z\}) = F_Z(z) \end{aligned}$$

Consequently, for each $s_0 \in S$,

$$\tilde{Z}(s_0) = \inf \left\{ z \in \mathbb{R}^+ : F_Z(z) \geq F_X(X)(s_0) \right\} = F_Z^{-1}(F_X(X)(s_0)) := F_Z^{-1}(F_X(X))(s_0)$$

That is,

$$(E.6) \quad \tilde{Z} = F_Z^{-1}(F_X(X))$$

where F_Z^{-1} is the left-continuous inverse of F_Z , as defined in equation (7.1).

Hence, by Lemma E.11 and equation (E.6), we can restrict ourselves to finding a solution to problem (E.5) of the form $F^{-1}(F_X(X))$, where F is the distribution function of a function $Z \in B^+(\Sigma)$ such that $0 \leq Z \leq X$ and $\int Z dT \circ Q = \beta$. Moreover, since X is a nondecreasing function of X and Q -equimeasurable with X , it follows from the Q -a.s. uniqueness of the equimeasurable nondecreasing Q -rearrangement (see Appendix B.2) that $X = F_X^{-1}(F_X(X))$, Q -a.s. (see also [20, Lemma A.21]). Thus, for any $Z \in B^+(\Sigma)$,

$$\begin{aligned} \int u_e(W_0^e + H - F_Z^{-1}(F_X(X))) \phi(F_X^{-1}(F_X(X))) dQ &= \int u_e(W_0^e + H - \tilde{Z}) \phi(X) dQ \\ &\geq \int u_e(W_0^e + H - Z) \phi(X) dQ \end{aligned}$$

where the inequality follows from the proof of Lemma E.11. Moreover, since $\zeta = Q \circ X^{-1}$ is nonatomic (by Assumption 7.1), it follows that $F_X(X)$ has a uniform distribution over $(0, 1)$ [20, Lemma A.21], that is, $Q(\{s \in S : F_X(X)(s) \leq t\}) = t$ for each $t \in (0, 1)$. Finally, letting $U := F_X(X)$,

$$\begin{aligned} \int F^{-1}(U) dT \circ Q &= \int_0^{+\infty} T[Q(\{s \in S : F^{-1}(U)(s) \geq t\})] dt \\ &= \int_0^{+\infty} T[Q(\{s \in S : F^{-1}(U)(s) > t\})] dt \\ &= \int_0^{+\infty} T[1 - F(t)] dt \\ &= \int_0^1 T'(1 - t) F^{-1}(t) dt = \int T'(1 - U) F^{-1}(U) dQ \end{aligned}$$

where the third and last equalities above follow from the fact that U has a uniform distribution over $(0, 1)$, and where the second-to-last equality follows from a standard argument⁷.

Now, recall from Definition 7.3 that $\mathcal{A}Quant$ given in equation (7.2) is the collection of all *admissible quantile functions*, that is the collection of all functions f of the form F^{-1} , where F is the distribution function of a function $Z \in B^+(\Sigma)$ such that $0 \leq Z \leq X$, and consider the following problem:

⁷See, e.g. Denneberg [17], Proposition 1.4 on p. 8 and the discussion on pp. 61-62. See also [31, p. 418], [27, p. 210, p. 213], or [8, p. 207].

For a given $\beta \in \Gamma$

$$(E.7) \quad \begin{aligned} \sup_f V(f) &= \int u_e(W_0^e + H - f(U)) \phi(F_X^{-1}(U)) dQ \\ \text{s.t. } f &\in \mathcal{A}Quant \\ \int T'(1-U) f(U) dQ &= \beta \end{aligned}$$

Lemma E.12. *If f^* is optimal for problem (E.7) with parameter $\beta \in \Gamma$, then the function $f^*(U)$ is optimal for problem (E.5) with parameter β , where $U := F_X(X)$. Moreover, $X - f^*(U)$ is optimal for problem (E.4) with parameter β .*

Proof. Fix $\beta \in \Gamma$, suppose that $f^* \in \mathcal{A}Quant$ is optimal for problem (E.7) with parameter β and let $Z^* \in B^+(\Sigma)$ be a corresponding function. That is, f^* is the quantile function of Z^* and $0 \leq Z^* \leq X$. Let $\tilde{Z}^* := f^*(U)$. Then $\tilde{Z}^*$ is the equimeasurable nondecreasing Q -rearrangement of Z^* with respect to X , and so $0 \leq \tilde{Z}^* \leq X$ by Lemma B.11 (2). Moreover,

$$\begin{aligned} \beta &= \int T'(1-U) f^*(U) dQ = \int f^*(U) dT \circ Q \\ &= \int \tilde{Z}^* dT \circ Q = \int_0^{+\infty} T[Q(\{s \in S : \tilde{Z}^*(s) \geq t\})] dt \\ &= \int_0^{+\infty} T[Q(\{s \in S : Z^*(s) \geq t\})] dt = \int \tilde{Z}^* dT \circ Q \end{aligned}$$

where the second-to-last equality follows from the Q -equimeasurability of Z^* and $\tilde{Z}^*$. Therefore, $\tilde{Z}^* = f^*(U)$ is feasible for problem (E.5) with parameter β . To show optimality, let Z be any feasible solution for problem (E.5) with parameter β and let F be the distribution function for Z . Then, by Lemma E.11, the function $\tilde{Z} := F^{-1}(U)$ is feasible for problem (E.5) with parameter β , comonotonic with X and Pareto-improving. Moreover, $\tilde{Z}$ has also F as a distribution function. To show optimality of $\tilde{Z}^* = f^*(U)$ for problem (E.5) with parameter β it remains to show that

$$\int u_e(W_0^e + H - \tilde{Z}^*) \phi(X) dQ \geq \int u_e(W_0^e + H - \tilde{Z}) \phi(X) dQ$$

Now, let $f := F^{-1}$, so that $\tilde{Z} = f(U)$. Since $\tilde{Z}$ is feasible for problem (E.5) with parameter β , we have

$$\begin{aligned} \beta &= \int \tilde{Z} dT \circ Q = \int F^{-1}(U) dT \circ Q \\ &= \int_0^1 T'(1-t) F^{-1}(t) dt = \int T'(1-U) f(U) dQ \end{aligned}$$

Hence, f is feasible for problem (E.7) with parameter β . Since f^* is optimal for problem (E.7) with parameter β we have

$$\int u_e(W_0^e + H - f^*(U)) \phi(F_X^{-1}(U)) dQ \geq \int u_e(W_0^e + H - f(U)) \phi(F_X^{-1}(U)) dQ$$

Finally, since $X = F_X^{-1}(U)$, Q -a.s., we have

$$\int u_e(W_0^e + H - \tilde{Z}^*) \phi(X) dQ \geq \int u_e(W_0^e + H - \tilde{Z}) \phi(X) dQ$$

Therefore, $\tilde{Z}^* = f^*(U)$ is optimal for problem (E.5) with parameter β . Finally, by Lemma E.9, $Y^* := X - \tilde{Z}^* = X - f^*(U)$ is optimal for problem (E.4) with parameter β . $\square$

By Lemmata E.7 and E.12, this completes the proof of Theorem 7.5. $\square$

APPENDIX F. PROOF OF COROLLARY 7.8

Corollary F.1. *Under the assumptions of Section 7, there exists a parameter $\beta^* \geq (1 + \rho)H$ such that an optimal solution Y^* for problem (4.1) (with $v = T \circ Q$) takes the following form:*

$$Y^* = \left(X - \max \left[0, \min \left\{ F_X^{-1}(U), f_{\lambda^*}^*(U) \right\} \right] \right) \mathbf{1}_A + X \mathbf{1}_{S \setminus A}$$

where for each $t \in (0, 1) \setminus \{t : \phi \circ F_X^{-1}(t) = 0\}$,

$$f_{\lambda^*}^*(t) = W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-t)}{\phi(F_X^{-1}(t))} \right)$$

and λ^* is chosen so that

$$\int_0^1 T'(1-t) \max \left[0, \min \left\{ F_X^{-1}(t), f_{\lambda^*}^*(t) \right\} \right] dt = \beta^*$$

Proof. Recall from equation (7.2) that

$$\mathcal{A}Quant = \left\{ f \in Quant : 0 \leq f(z) \leq F_X^{-1}(z), \text{ for each } 0 < z < 1 \right\},$$

where $Quant = \left\{ f : (0, 1) \rightarrow \mathbb{R} \mid f \text{ is nondecreasing and left-continuous} \right\}$. Define the collection $\mathcal{K}$ of functions on $(0, 1)$ as follows:

$$(F.1) \quad \mathcal{K} = \left\{ f : (0, 1) \rightarrow \mathbb{R} \mid 0 \leq f(z) \leq F_X^{-1}(z), \text{ for each } 0 < z < 1 \right\}$$

Then $\mathcal{A}Quant = Quant \cap \mathcal{K}$. Consider the following problem, with parameter $\beta \in \Gamma$:

For a given $\beta \in \Gamma$

$$(F.2) \quad \begin{aligned} \sup_f V(f) &= \int_0^1 u_e(W_0^e + H - f(t)) \phi(F_X^{-1}(t)) dt \\ \text{s.t. } f &\in \mathcal{A}Quant \\ \int_0^1 T'(1-t) f(t) dt &= \beta \end{aligned}$$

Lemma F.2. *For a given $\beta \in \Gamma$, if $f^* \in \mathcal{A}Quant$ satisfies the following:*

$$(1) \int_0^1 T'(1-t) f^*(t) dt = \beta;$$

(2) *There exists $\lambda \leq 0$ such that for all $t \in (0, 1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$,*

$$f^*(t) = \arg \max_{0 \leq y \leq F_X^{-1}(t)} [u_e(W_0^e + H - y) \phi(F_X^{-1}(t)) - \lambda T'(1-t)y]$$

Then f^* solves problem (F.2) with parameter β

Proof. Fix $\beta \in \Gamma$, suppose that $f^* \in \mathcal{AQuant}$ satisfies conditions (1) and (2) above. Then, in particular, f^* is feasible for problem (F.2) with parameter β . To show optimality of f^* for problem (F.2) with parameter β , let f be any other feasible solution for problem (F.2) with parameter β . Then, for all $t \in (0, 1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$,

$$\begin{aligned} u_e(W_0^e + H - f^*(t)) \phi(F_X^{-1}(t)) - \lambda T'(1-t)f^*(t) \\ \geq u_e(W_0^e + H - f(t)) \phi(F_X^{-1}(t)) - \lambda T'(1-t)f(t) \end{aligned}$$

That is, $[u_e(W_0^e + H - f^*(t)) - u_e(W_0^e + H - f(t))] \phi(F_X^{-1}(t)) \geq \lambda T'(1-t)[f^*(t) - f(t)]$. Integrating yields $V(f^*) - V(f) \geq \lambda[\beta - \beta] = 0$, that is $V(f^*) \geq V(f)$, as required. $\square$

Hence, in view of Lemma F.2, in order to find a solution for problem (F.2) with a given parameter $\beta \in \Gamma$ and a given $\lambda \leq 0$, one can start by solving the problem

$$(F.3) \quad \max_{0 \leq f_\lambda(t) \leq F_X^{-1}(t)} [u_e(W_0^e + H - f_\lambda(t)) \phi(F_X^{-1}(t)) - \lambda T'(1-t)f_\lambda(t)]$$

for a fixed $t \in (0, 1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$.

Consider first the following problem:

$$(F.4) \quad \max_{f_\lambda(t)} [u_e(W_0^e + H - f_\lambda(t)) \phi(F_X^{-1}(t)) - \lambda T'(1-t)f_\lambda(t)]$$

for a fixed $t \in (0, 1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$.

By concavity of the utility function u , in order to solve problem (F.4), it suffices to solve for the first-order condition

$$-u'_e(W_0^e + H - f_\lambda^*(t)) \phi(F_X^{-1}(t)) - \lambda T'(1-t) = 0$$

which gives

$$(F.5) \quad f_\lambda^*(t) = W_0^e + H - (u'_e)^{-1} \left(\frac{-\lambda T'(1-t)}{\phi(F_X^{-1}(t))} \right)$$

Then the function $f_\lambda^*(t)$ solve problem (F.4), for a fixed $t \in (0, 1) \setminus \{t : \phi \circ F_x^{-1}(t) = 0\}$.

By Assumption 7.7, the function $t \mapsto \frac{T'(1-t)}{\phi(F_X^{-1}(t))}$ is nondecreasing. By Assumption 4.5, the function u_e is strictly concave and continuously differentiable. Hence, the function u'_e is both continuous and strictly decreasing. This then implies that $(u'_e)^{-1}$ is continuous and strictly decreasing, by the Inverse Function Theorem [50, pp. 221-223]. Therefore, the function $f_\lambda^*(t)$ in equation (F.5) is nondecreasing ($\lambda \leq 0$). Moreover, by Assumption 7.1 and Assumption 7.6, $f_\lambda^*(t)$ is left-continuous.

Define the function f_λ^{**} by

$$(F.6) \quad f_\lambda^{**}(t) = \max \left[0, \min \left\{ F_X^{-1}(t), f_\lambda^*(t) \right\} \right]$$

Then $f_\lambda^{**}(t) \in \mathcal{K}$. Moreover, since both F_X^{-1} and f_λ^* are nondecreasing and left-continuous functions, it follows that f_λ^{**} is nondecreasing and left-continuous. Consequently, $f_\lambda^{**}(t) \in \mathcal{AQuant}$. Finally, it is easily seen that $f_\lambda^{**}(t)$ solves problem (F.3) for the given λ . Now, for a given $\beta_0 \in \Gamma$, if λ^* is chosen so that $\int_0^1 T'(1-t) f_{\lambda^*}^{**}(t) dt = \beta_0$, then by Lemma F.2, $f_{\lambda^*}^{**}$ is optimal for problem (F.2) with parameter β_0 .

Hence, in view of Theorem 7.5, to conclude the proof of Corollary 7.8, it remains to show that for each $\beta_0 \in \Gamma$, there exists a $\lambda^* \leq 0$ such that $\int_0^1 T'(1-t) f_{\lambda^*}^{**}(t) dt = \beta_0$. This is given by Lemma F.3 below.

Lemma F.3. *Let ψ be the function of the parameter $\lambda \leq 0$ defined by $\psi(\lambda) := \int_0^1 T'(1-t) f_\lambda^{**}(t) dt$. Then for each $\beta_0 \in \Gamma$, there exists a $\lambda^* \leq 0$ such that $\psi(\lambda^*) = \beta_0$.*

Proof. First note that ψ is a continuous and nonincreasing function of λ , where continuity of ψ is a consequence of Lebesgue's Dominated Convergence Theorem [1, Theorem 11.21]. Indeed, since X is bounded and since F_X^{-1} is nondecreasing, it follows that for each $t \in [0, 1]$,

$$\min \left\{ F_X^{-1}(t), f_\lambda^*(t) \right\} \leq F_X^{-1}(t) \leq F_X^{-1}(1) \leq M = \|X\|_s < +\infty.$$

Moreover, since T is concave and increasing, T' is nonincreasing and nonnegative, and so for each $t \in [0, 1]$, $0 \leq T'(1-t) \leq T'(0)$. But $T'(0) < +\infty$, by Assumption 7.1. Hence, for each $t \in [0, 1]$,

$$\min \left\{ F_X^{-1}(t), f_\lambda^*(t) \right\} T'(1-t) \leq F_X^{-1}(1) T'(0) \leq \|X\|_s T'(0) < +\infty$$

Moreover, $\psi(0) = 0$ (by Assumption 4.5), and

$$\begin{aligned} \lim_{\lambda \rightarrow -\infty} \psi(\lambda) &= \int_0^1 T'(1-t) \min \left\{ F_X^{-1}(t), W_0^e + H \right\} dt \\ &= \int_0^{F_X(W_0^e + H)} T'(1-t) F_X^{-1}(t) dt + (W_0^e + H) \int_{F_X(W_0^e + H)}^1 T'(1-t) dt \end{aligned}$$

However, by Assumption 7.2, we have $F_X(W_0^e + H) = 1$. This then implies that

$$\lim_{\lambda \rightarrow -\infty} \psi(\lambda) = \int_0^1 T'(1-t) F_X^{-1}(t) dt = \int X dT \circ Q$$

Now, for any $\beta_0 \in \Gamma$, and for any $Y \in B^+(\Sigma)$ which is feasible for problem (E.1) with parameter β_0 , one has:

- (i) $0 \leq Y \leq X$; and,
- (ii) $\int (X - Y) dT \circ Q = \beta_0$.

Hence, $0 \leq X - Y \leq X$, and so, by monotonicity of the Choquet integral (Proposition B.6), it follows that $\beta_0 = \int (X - Y) dT \circ Q \leq \int X dT \circ Q$. Consequently, for any $\beta_0 \in \Gamma$,

$$0 = \psi(0) \leq \beta_0 \leq \int X dT \circ Q = \lim_{\lambda \rightarrow -\infty} \psi(\lambda)$$

Hence, by the Intermediate Value Theorem [50, Theorem 4.23], for each $\beta_0 \in \Gamma$ there exists some $\lambda^* \leq 0$ such that $\psi(\lambda^*) = \beta_0$. $\square$

By Lemmata F.2 and F.3, this concludes the proof of Corollary F.1. $\square$

Now, since the function $f_{\lambda}^{**}(t) = \max \left[0, \min \left\{ F_X^{-1}(t), f_{\lambda^*}^*(t) \right\} \right]$ is optimal for problem (F.2) with parameter β^* , it is optimal for problem (E.7) with parameter β^* . Therefore, by Lemma E.12, the function

$$f_{\lambda}^{**}(U) = \max \left[0, \min \left\{ F_X^{-1}(U), f_{\lambda^*}^*(U) \right\} \right]$$

is optimal for problem (E.5) with parameter β^* . However, since $Q \circ X^{-1}$ is nonatomic (by Assumption 7.1), it follows that $F_X(X)$ has a uniform distribution over $(0, 1)$ [20, Lemma A.21], and that $X = F_X^{-1}(F_X(X))$, Q -a.s. Therefore,

$$\begin{aligned} f_{\lambda}^{**}(U) &= \max \left[0, \min \left\{ F_X^{-1}(U), f_{\lambda^*}^*(U) \right\} \right] \\ &= \max \left[0, \min \left\{ F_X^{-1}(U), W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(F_X^{-1}(U))} \right) \right\} \right] \\ &= \max \left[0, \min \left\{ X, W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(X)} \right) \right\} \right], \quad Q\text{-a.s.} \end{aligned}$$

This Q -a.s. equality implies that

$$\max \left[0, \min \left\{ X, W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(X)} \right) \right\} \right]$$

is also optimal for problem (E.5) with parameter β^* . Lemma E.9 then implies that

$$X - \max \left[0, \min \left\{ X, W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(X)} \right) \right\} \right]$$

is optimal for problem (E.4) (and hence for problem (E.1)) with parameter β^* . Finally, it can easily be verified that

$$\begin{aligned} X - \max \left[0, \min \left\{ X, W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(X)} \right) \right\} \right] \\ = \min \left[X, \max \left(0, X - \left[W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(X)} \right) \right] \right) \right] \end{aligned}$$

The rest follows from Lemma E.7 and Corollary F.1, noting that since $X = F_X^{-1}(U)$, Q -a.s., one has:

$$\begin{aligned} \beta^* &= \int_0^1 T'(1-t) \max \left[0, \min \left\{ F_X^{-1}(t), f_{\lambda^*}^*(t) \right\} \right] dt \\ &= \int T'(1-U) \max \left[0, \min \left\{ F_X^{-1}(U), f_{\lambda^*}^*(U) \right\} \right] dQ \\ &= \int T'(1-U) \max \left[0, \min \left\{ X, W_0^e + H - (u_e')^{-1} \left(\frac{-\lambda^* T'(1-U)}{\phi(X)} \right) \right\} \right] dQ \end{aligned}$$

This concludes the proof of Corollary 7.8. □

REFERENCES

- [1] C.D. ALIPRANTIS and K.C. BORDER. *Infinite Dimensional Analysis - 3rd edition*. Springer-Verlag, 2006.
- [2] M. AMARANTE, O. ÖZGÜR, and E.S. PHELPS. The Entrepreneurial Economy II: Equilibria and Welfare. mimeo (2011).
- [3] M. AMARANTE and E.S. PHELPS. The Entrepreneurial Economy. Kauffman Foundation Report, 2005.
- [4] W.J. BAUMOL. Entrepreneurship in Economic Theory. *The American Economic Review*, 58(2):64–71, 1968.
- [5] M. BIANCHI and M. HENREKSON. Is Neoclassical Economics Still Entrepreneurless? *Kyklos*, 58(3):353–377, 2005.
- [6] R. BOJILOV and E.S. PHELPS. Dynamism and Economic Performance: A Preliminary Report. Columbia University, The Center on Capitalism and Society, Working Paper No. 60 (July 2010).
- [7] M. BOLDRIN and D.K. LEVINE. Perfectly Competitive Innovation. *Journal of Monetary Economics*, 55(3):435–453, 2008.
- [8] G. CARLIER and R.A. DANA. Core of Convex Distortions of a Probability. *Journal of Economic Theory*, 113(2):199–222, 2003.
- [9] G. CARLIER and R.A. DANA. Pareto Efficient Insurance Contracts when the Insurer’s Cost Function is Discontinuous. *Economic Theory*, 21(4):871–893, 2003.
- [10] G. CARLIER and R.A. DANA. Rearrangement Inequalities in Non-convex Insurance Models. *Journal of Mathematical Economics*, 41(4-5):483–503, 2005.
- [11] G. CARLIER and R.A. DANA. Two-persons Efficient Risk-sharing and Equilibria for Concave Law-invariant Utilities. *Economic Theory*, 36(2):189–223, 2008.
- [12] G. CARLIER and R.A. DANA. Optimal Demand for Contingent Claims when Agents Have Law Invariant Utilities. *Mathematical Finance*, 21(2):169–201, 2011.
- [13] N.L. CAROTHERS. *Real Analysis*. Cambridge University Press, 2000.
- [14] A. CHATEAUNEUF, R.A. DANA, and J.M. TALLON. Optimal Risk-sharing Rules and Equilibria with Choquet-expected-utility. *Journal of Mathematical Economics*, 34(2):191–214, 2000.
- [15] D.L. COHN. *Measure Theory*. Birkhauser, 1980.
- [16] R.A. DANA. On Equilibria when Agents have Multiple Priors. *Annals of Operations Research*, 114(1):105–115, 2002.
- [17] D. DENNEBERG. *Non-Additive Measure and Integral*. Kluwer Academic Publishers, 1994.
- [18] R.M. DUDLEY. *Real Analysis and Probability*. Cambridge University Press, 2002.
- [19] A.M. ENDRES and C.R. WOODS. Modern Theories of Entrepreneurial Behavior: A Comparison and Appraisal. *Small Business Economics*, 26(2):189–202, 2006.
- [20] H. FÖLLMER and A. SCHIED. *Stochastic Finance: An Introduction in Discrete Time - 3rd ed.* Walter de Gruyter, 2011.
- [21] P. GHIRARDATO, F. MACCHERONI, and M. MARINACCI. Differentiating Ambiguity and Ambiguity Attitude. *Journal of Economic Theory*, 118(2):133–173, 2004.
- [22] P. GHIRARDATO and M. SINISCALCHI. A More Robust Definition of Multiple Priors. mimeo (2010).
- [23] M. GHOSSOUB. Belief Heterogeneity in the Arrow-Borch-Raviv Insurance Model. mimeo (2011).
- [24] M. GHOSSOUB. Monotone Equimeasurable Rarrangements with Non-Additive Probabilities. mimeo (2011).
- [25] I. GILBOA and M. MARINACCI. Ambiguity and the Bayesian Paradigm. In *D. Acemoglu, M. Arellano, and E. Dekel (eds.), Advances in Economics and Econometrics: Theory and Applications, Tenth World Congress of the Econometric Society*. Cambridge University Press, 2012.
- [26] I. GILBOA and D. SCHMEIDLER. Maxmin Expected Utility with a Non-Unique Prior. *Journal of Mathematical Economics*, 18(2):141–153, 1989.
- [27] X.D. HE and X.Y. ZHOU. Portfolio Choice via Quantiles. *Mathematical Finance*, 21(2):203–231, 2011.
- [28] H.T. HOON and E.S. PHELPS. Innovation and Employment. Columbia University, The Center on Capitalism and Society, Working Paper No. 69 (April 2011).
- [29] P.J. HUBER and E.M. RONCHETTI. *Robust Statistics - Second Edition*. John Wiley and Sons, 2011.
- [30] H. JIN and X.Y. ZHOU. Greed, Leverage, and Potential Losses: A Prospect Theory Perspective. *Mathematical Finance*, Forthcoming.
- [31] H. JIN and X.Y. ZHOU. Behavioral Portfolio Selection in Continous Time. *Mathematical Finance*, 18(3):385–426, 2008.
- [32] J.B. KADANE and L. WASSERMAN. Symmetric, Coherent, Choquet Capacities. *The Annals of Statistics*, 24(3):1250–1264, 1996.
- [33] D. KAHNEMAN and A. TVERSKY. Prospect Theory: An Analysis of Decision Under Risk. *Econometrica*, 47(2):263–291, 1979.
- [34] R.E. KIHLSSTROM and J.J. LAFFONT. A General Equilibrium Entrepreneurial Theory of Firm Formation Based on Risk Aversion. *Journal of Political Economy*, 87(4):719–748, 1979.

- [35] I.M. KIRZNER. Entrepreneurial Discovery and the Competitive Market Process: An Austrian Approach. *Journal of Economic Literature*, 35(1):60–85, 1997.
- [36] F.H. KNIGHT. *Risk, Uncertainty, and Profit*. Boston/New York: Houghton Mifflin, 1921.
- [37] R.E. LUCAS. On the Size Distribution of Business Firms. *The Bell Journal of Economics*, 9(2):508–523, 1978.
- [38] M. MARINACCI and L. MONTRUCCHIO. Introduction to the Mathematics of Ambiguity. In *I. Gilboa (ed.), Uncertainty in Economic Theory: Essays in Honor of David Schmeidlers 65th Birthday*. Routledge, London, 2004.
- [39] P. MILGROM. Good News and Bad News: Representation Theorems and Applications. *The Bell Journal of Economics*, 12(2):380–391, 1981.
- [40] S. MUKERJI. Ambiguity Aversion and Incompleteness of Contractual Form. *American Economic Review*, 88(5):1207–1231, 1998.
- [41] S. MUKERJI. A Survey of Some Applications of the Idea of Ambiguity Aversion in Economics. *International Journal of Approximate Reasoning*, 24(2):221–234, 2000.
- [42] S. MUKERJI and J.M. TALLON. An Overview of Economic Applications of David Schmeidlers Models of Decision Making under Uncertainty. In *I. Gilboa (ed.), Uncertainty in Economic Theory: Essays in Honor of David Schmeidlers 65th Birthday*. Routledge, London, 2004.
- [43] E. PAP. *Null-Additive Set Functions*. Kluwer Academic Publishers, 1995.
- [44] E.S. PHELPS. The Economic Prosperity of Nations: Prosperity Depends on Dynamism, Dynamism on Institutions. Columbia University, The Center on Capitalism and Society, Working Paper No. 2 (January 2005).
- [45] E.S. PHELPS. Toward a Model of Innovation and Performance Along the Lines of Knight, Keynes, Hayek and m. Polanyi. Columbia University, The Center on Capitalism and Society, Working Paper No. 19 (November 2006).
- [46] R. POLLOCK. Economics Bibliography: Primarily Innovation and IP Oriented. <http://rufuspollock.org/economics/biblio/>.
- [47] J. QUIGGIN. A Theory of Anticipated Utility. *Journal of Economic Behavior*, 3(4):323–343, 1982.
- [48] J. REINGANUM. The Timing of Innovation: Research, Development, and Diffusion. In *R. Schmalensee & R. Willig (ed.), Handbook of Industrial Organization, vol. 1, ch. 14, 849-908*. Elsevier, 1989.
- [49] P.M. ROMER. Increasing Returns and Long-Run Growth. *The Journal of Political Economy*, 94(5):1002–1037, 1986.
- [50] W. RUDIN. *Principles of Mathematical Analysis – 3rd ed.* New York: McGraw-Hill Book Company, 1976.
- [51] D. SCHMEIDLER. Integral Representation without Additivity. *Proceedings of the American Mathematical Society*, 97(2):255–261, 1986.
- [52] D. SCHMEIDLER. Subjective Probability and Expected Utility without Additivity. *Econometrica*, 57(3):571–587, 1989.
- [53] J.A. SCHUMPETER. *The Theory of Economic Development*. Cambridge: Harvard University Press [English Translation 1936], 1911.
- [54] S. SCOTCHMER. *Innovation and Incentives*. The MIT Press, 2004.
- [55] A. TVERSKY and D. KAHNEMAN. Advances in Prospect Theory: Cumulative Representation of Uncertainty. *The Journal of Risk and Uncertainty*, 5(4):297–323, 1992.
- [56] L. WASSERMAN and J.B. KADANE. Symmetric Upper Probabilities. *The Annals of Statistics*, 20(4):1720–1736, 1992.
- [57] M. YAARI. The Dual Theory of Choice under Risk. *Econometrica*, 55(1):95–115, 1987.

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