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A Geometrically Nonlinear Approach for the Aeroelastic Analysis of Commercial Transport Aircraft

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Declaration

I hereby declare that this thesis is the product of my own research only and no parts of it has been submitted for another degree. References or quotations from other works are fully acknowledged. The copyright of this thesis rests with the author and is made available under a Creative Commons Attribution Non-Commercial No Derivatives licence. Researchers are free to copy, distribute or transmit the thesis on the condition that they attribute it, that they do not use it for commercial purposes and that they do not alter, transform or build upon it. For any reuse or redistribution, researchers must make clear to others the licence terms of this work.

Álvaro Cea Esteban

Supposing truth is a woman –what then? Are there not grounds for the suspicion that all philosophers, insofar as they were dogmatists, have been very inexpert about women? that the gruesome seriousness, the clumsy obtrusiveness with which they have usually approached truth so far have been awkward and very improper methods for winning a woman’s heart? What is certain is that she has not allowed herself to be won: –and today every kind of dogmatism is left standing dispirited and discouraged. If it is left standing at all! For there are scoffers who claim that it has fallen, that all dogmatism lies on the ground, even more, that all dogmatism is dying. Speaking seriously, there are good reasons why all philosophical dogmatizing, however solemn and definitive its airs used to be, may nevertheless have been no more than a noble childishness and tyronism; and perhaps the time is at hand when it will be comprehended again and again what actually was sufficient to furnish the cornerstone for such sublime and unconditional philosophers’ edifices as the dogmatists have built so far –any old popular superstition from time immemorial (like the soul superstition which, in the form of the subject and ego superstition, has not even yet ceased to do mischief), some play on words perhaps, a seduction by grammar, or an audacious generalization of very narrow, very personal, very human, all too human facts.

...that we should finally learn from this Sphinx to ask questions, too? Who is it really that puts questions to us here? What in us really wants “truth”? Indeed we came to a long halt at the question about the cause of this will, until we finally came to a complete stop before a still more basic question. We asked about the value of this will. Suppose we want truth: why not rather untruth? And uncertainty? Even ignorance? The problem of the value of truth came before us –or was it we who came before the problem? Who of us is Oedipus here? Who the Sphinx? It is a rendezvous, it seems, of questions and question marks. And though it scarcely seems credible, it finally also seems to us as if the problem had never even been put so far –as if we were the first to see it, fix it with our eyes, risk it? For it does involve a risk, and perhaps there is none that is greater.

...And only on this now solid, granite foundation of ignorance could knowledge rise so far, the will to knowledge on the foundation of a far more powerful will, the will to no knowledge, to uncertainty, to the untruth! Not as its opposite, but rather –as its refinement! Even if language, here as elsewhere, will not get over its awkwardness, and will continue to talk of opposites where there are only degrees and many subtleties of gradation; even if the inveterate Tartuffery of morals, which now belongs to our unconquerable “flesh and blood”, infects the words even of those of us who know better: here and there we understand it and laugh at the way in which precisely science at its best seeks most to keep us in this simplified, thoroughly artificial, suitably constructed and suitably falsified world, at the way in which, willy-nilly, it loves error, because, being alive –it loves life!

Friedrich Nietzsche, *Beyond Good and Evil*

To my grandmother Emilia.

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The understanding of how much I owe my parents in coming this far belongs in places that the concious mind cannot reach. Therefore I will only mention their support throughout all of my education; the unconditional freedom they granted me when, perhaps, it was not the obvious choice; and their encouragement to always pursue a career that I enjoy –if I was never easy,

hopefully this work backs that you were right. Going back home, and the two summers they spent with me in London, my siblings were always a source of joy, directly proportional to the headaches they give me, so I am very grateful for the former –any peak I conquer, I am sure you three will surpass.

The last shall be first: I have been the luckiest to have Maria as copilot on this ride, this dissertation is hers too. We started the adventure as two enthusiastic youngsters and we come out of it as a family, enduring a global pandemic, many other setbacks that put us on the edge, and two babies that changed our lives on every possible level. I am very happy we made it through –I’m a flame, you are the fire, half & half, let’s keep the dance! Clefs aside, I hope for no better outcome of this work than for it to inspire our son and our daughter a pleasure for Science and a will to rigorous thinking.

Abstract

A new approach is proposed that seamlessly integrates with current industrial aeroelastic load analysis methods and aims to bring together the complexity of computational models in the production environment of aircraft (normally enhanced with experimental data) and the inherent difficulties associated with geometrically nonlinear analysis. Motivation stems from next-generation, ultra-efficient aircraft exhibiting high aspect-ratio wings.

Efficient incorporation of geometrically-nonlinear effects to standard (linear) approaches –based on generic Finite-Element models and Aerodynamic Influence Coefficient matrices– is accomplished through a two-step process: firstly, a reduction of the structure through dynamic condensation techniques on nodes along the main load paths of the vehicle; and secondly, a manipulation of the resulting condensed stiffness and mass matrices, their linear normal modes, and the nodal coordinates provide the nonlinear modal coefficients of the intrinsic beam equations that describe the dynamics of these load paths. The original model is preserved and effectively augmented with geometric stiffening, variations to the inertia and shortening effects, and aerodynamic follower forces (which naturally rotate in a formulation cast in material coordinates). The approach further caters to multibody and trajectory constraints using Lagrange multipliers on the velocity level set.

Structural and aeroelastic static and dynamic solutions are presented, including rigid-body and multibody dynamics, using the resulting nonlinear modal description. Comparison with full FE calculations of a representative wing illustrates both the accuracy and efficiency of the formulation. Excellent approximation of the flutter instability of a clamped wing is obtained under linear assumptions, while limit-cycle oscillations are found due to structural nonlinearities. Studies on a large transport aircraft configuration show the importance of nonlinear effects on the analysis of manoeuvres, trimmed flight, and dynamic gust loads. Moreover, the flutter speed of the airplane decreases as deflections from a steady angle-of-attack increase; and a first exploration into movable wing-tips is carried out within the multibody framework.

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Nomenclature

Structural description

g	metric tensor
$\mathbf{x}$	current configuration points
$\mathbf{v}$	velocity of current configuration points
$\mathbf{X}$	material configuration points
$\mathbf{e}$	Almansi-Eulerian strain tensor
$\mathbf{E}$	Green-Lagrangian strain tensor
$\mathbf{F}_x$	Deformation gradient tensor
$\mathbf{B}$	left Cauchy-Green deformation tensor
$\mathbf{C}$	right Cauchy-Green deformation tensor
$\mathbf{L}$	velocity gradient tensor
$\mathbf{D}$	Green-Lagrangian strain tensor
$\boldsymbol{\sigma}$	Cauchy stress tensor
$\mathbf{P}$	first Piola-Kirchhoff stress tensor
$\mathbf{S}$	second Piola-Kirchhoff stress tensor
$\mathcal{T}$	traction force vector
$\mathbf{n}$	surface normal unit vector
V	solid volume
ρ	density per unit volume
$\mathcal{P}$	linear momentum solid body
$\mathcal{H}$	angular momentum solid body
$\mathbf{b}_f$	external body forces
$\mathbf{A}$	beam stretch tensor
$\mathbf{a}$	reference local strain vector
$\mathbf{f}$	internal forces in material coordinates
$\mathbf{m}$	internal moments in material coordinates
$\mathbf{v}$	translational velocities in material coordinates
$\boldsymbol{\omega}$	angular velocities in material coordinates
$\mathbf{x}_1$	local translational/angular velocity state vector
$\mathbf{x}_2$	internal force/moment state vector
$\mathbf{k}$	curvature vector

k_0	initial curvature
γ	beam sectional strains
κ	beam sectional moment strains
p	sectional linear momentum
h	sectional angular momentum
f_e	applied forces per unit beam length
m_e	applied moments per unit beam length
f_1	applied forces/moments per unit beam length
$\mathcal{M}$	sectional inertia matrix
$\mathcal{C}$	sectional compliance matrix
K	full stiffness matrix
M	full mass matrix
F	external forces/moments in FE formulation
u_n	global displacements in FE formulation
K_a	stiffness matrix after condensation
M_a	mass matrix after condensation
T	Transformation matrix in condensation process
T_{oa}	Transformation matrix from active to omitted set
ϕ_1	continuous velocity modes
ϕ_2	continuous force modes
ϕ_0	continuous displacement modes
ψ_1	continuous momentum modes
ψ_2	continuous strain modes
Φ_1	discrete velocity modes
Φ_2	discrete force modes
Φ_a	discrete displacement modes
Ψ_1	discrete momentum modes
Ψ_2	discrete strain modes
$\Gamma_{1,2}$	nonlinear coupling coefficients in modal coordinates
η	external load in modal coordinates
ω_j	natural frequency of mode j
a_i	unit vectors in reference frame of undeformed configuration
b_i	unit vectors in reference frame of deformed configuration
$\mathbb{E}_i$	unit vectors in reference configuration
r_a	position vector of condensed model in global coordinates
r	position vector of condensed model in local initial reference system
u	displacement vector of condensed model in local initial reference system
θ	rotation vector of condensed model in initial reference system
$\hat{r}_a$	position vector of 3D model in global coordinates
$\hat{r}$	position vector of 3D model in local initial reference system
$\hat{v}$	velocity vector of 3D model in local initial reference system
w	warping field vector
ξ	cross-sectional vector in material coordinates
e_x	unit vector along the deformed reference line
$\mathcal{K}$	Kinetic energy per unit length
$\mathcal{U}$	Potential energy per unit length
$\mathcal{W}_e$	Virtual work of external forces per unit length
A_ξ	Cross-sectional area

ρ_{c0}	Cross-sectional mass density
$\mathbf{q}_1$	intrinsic modal coordinates (velocity component)
$\mathbf{q}_2$	intrinsic modal coordinates (stress resultant component)
$\mathbf{q}_0$	modal coordinates (displacement component)
s	curvilinear coordinate along main load path
t	time

Aerodynamic and aeroelastic formulation

ρ	flow density field
p	flow pressure field
$\mathbf{q}$	flow velocity field
a_s	speed of sound
U_∞	free stream flow velocity
ρ_∞	free stream air density
M_∞	free stream Mach number
a_∞	free stream speed of sound
ϕ	aerodynamic potential
$\hat{\phi}$	linearised aerodynamic potential
ϕ_s	aerodynamic source potential
ϕ_d	aerodynamic doublet potential
$\bar{R}$	contracted radial measure from source
$\hat{\bullet}$	linearised quantity
$\bar{\bullet}$	frequency domain quantity
w	upwash or downwash
h	aerodynamic surface definition
k	reduced frequency
c	reference chord
$\mathbf{u}_k$	normal displacements at aerodynamic grid points
$\mathbf{A}$	kernel DLM matrix
$\mathbf{G}$	interpolation DLM matrices
$\mathbf{S}$	pressure integration DLM matrix
$\mathbf{D}_1, \mathbf{D}_2$	boundary conditions DLM matrices
$\mathbf{Q}_i$	aerodynamic influence coefficient and generalized aerodynamic force matrices
$\mathbf{A}_s, \mathbf{B}_s, \mathbf{D}_s$	Roger's approximation aerodynamic matrices
ζ	Quaternions to parameterised rotations
Ω_j	natural frequency of mode j
λ_p	aerodynamic states
γ_p	aerodynamic poles
$\boldsymbol{\eta}_a$	aerodynamic loads in modal components
$\mathbf{w}_g$	gust upwash
$\mathbf{q}_x$	control states
$\boldsymbol{\alpha}_x$	aerodynamic steady component
$\boldsymbol{\eta}_a$	aerodynamic loads in modal components
$\mathbf{x}$	linearised aeroelastic state vector
$\mathbf{G}_x$	Jacobian matrix of aeroelastic system
$\mathbf{L}_i$	Jacobian matrices of aeroelastic system

Multibody framework

$\mathbf{g}$	constraint equations
$\mathbf{G}$	Jacobian of constraints
λ	Lagrange multipliers
$\mathbf{g}^v$	constraint equations on velocities
$\mathbf{g}^\omega$	constraint equations on angular velocities
$\mathbf{G}_{q_1}^v$	jacobian of $\mathbf{g}^v$ constraints with respect to the modal velocity components in the first body
$\mathbf{G}_{\zeta_1}^v$	jacobian of $\mathbf{g}^v$ constraint with respect to quaternions components in the first body
$\mathbf{G}_{q_2}^v$	jacobian of $\mathbf{g}^v$ constraint with respect to the modal velocity components in the second body
$\mathbf{G}_{\zeta_2}^v$	jacobian of $\mathbf{g}^v$ constraint with respect to the quaternion component in the second body
$\mathbf{G}_{q_1}^\omega$	jacobian of $\mathbf{g}^\omega$ constraints with respect to the modal velocity components in the first body
$\mathbf{G}_{\zeta_1}^\omega$	jacobian of $\mathbf{g}^\omega$ constraint with respect to quaternions components in the first body
$\mathbf{G}_{q_2}^\omega$	jacobian of $\mathbf{g}^\omega$ constraint with respect to the modal velocity components in the second body
$\mathbf{G}_{\zeta_2}^\omega$	jacobian of $\mathbf{g}^\omega$ constraint with respect to the quaternion component in the second body
$(\bullet)^{(1)}$	quantity at the connecting node in the first body
$(\bullet)^{(2)}$	quantity at the connecting node in the first body
$\Phi_{1\theta}$	velocity modes, rotational component
Φ_{1u}	velocity modes, displacement component
$\mathbf{F}_{11}$	right hand side modal velocity equation in the first body
$\mathbf{F}_{12}$	right hand side modal velocity equation in the second body
$\mathbf{F}_{\zeta_1}$	right hand side quaternion equation of the connecting node in the first body
$\mathbf{F}_{\zeta_2}$	right hand side quaternion equation of the connecting node in the second body

Symbols and operators

$(\bullet)^t$	current configuration
$(\bullet)'$	spatial derivative with respect to s
$(\dot{\bullet})$	current configuration
$(\dot{\bullet})$	time derivative
$\frac{D(\bullet)}{Dt}$	material time derivative
$d(\bullet)$	differential operator
$\delta(\bullet)$	variational operator
$\nabla \cdot (\bullet)$	divergence operator
$\nabla(\bullet)$	nabla operator
$\tilde{\bullet}, \times$	cross product operator
$\otimes$	tensor product
$:$	double dot product
$\langle \bullet \rangle$	inner product in the 1D domain

Introduction



Once you have tasted flight, you will forever walk the earth with your eyes turned skyward, for there you have been, and there you will always long to return.

Leonardo da Vinci

Whoever fights monsters should see to it that in the process he does not become a monster. And if you gaze long enough into an abyss, the abyss will gaze back into you.

Friedrich Nietzsche, *Beyond Good and Evil*

Year 2020, the aviation industry has probably taken its biggest hit in history. The COVID-19 pandemic has shaken the entire world with a particular bad economical impact on the tertiary sector. Nonetheless, we should not be pessimistic, the aeronautical industry stands on firm grounds. In the past, the SARS crisis, the first Gulf War or the 9/11 attacks impacted air transport growth for only a few months [1]. In fact, air-traffic has been continuously rising since the advent of commercial air transport. Remarkably, over the last few decades, the cost of travel has fallen by about 60% while the number of travellers has increased by an order of magnitude [2]. This growth has been accompanied lately by the commitment of major industry

players and governments to reduce emissions and work towards zero-emission flights¹. Fortunately, the pandemic does not seem to have significantly undermined these environmental goals. Given people's tendency to grow accustomed to, and take for granted major achievements and landmarks, it is worth recalling that 120 years ago flying was but a mere dream.

Globalisation and free-market economies have allowed unprecedented growth and wealth, helping to shape the world of today. The exchange of goods at a global scale –ultimately the exchange of expertise and the enhancement of competition– has led to technological advances and improvements in the efficiency of value-generation processes. With regards to aviation, freight aircraft, for instance, have directly contributed to commerce on a worldwide basis at speeds never before imagined (cargo ship, the other mode of transatlantic transport, travels at 30-40 km/h, compared to typical commercial aircraft that can surpass the 900 km/h). And, more generally, the continuously increasing demand and a better supply have resulted in affordable flying fares for a vast majority of people in developed countries, which in turn have boosted the intrinsic human nature to visit new horizons, opening people's minds and getting everyone a bit closer. As a double-edged sword, however, the very first principles leading to this progress may also put it in danger: highly contagious local diseases can turn into global pandemics as a consequence of mobility; uncertainty and fast changing dynamics may leave people with outdated skills behind if pertinent bodies do not act swiftly, even more so when labour markets are no longer limited to their own borders; and climate change and the environmental impact of human action can no longer be overlooked and scientists around the world are warning about the potential threat to our way of life if emissions and the destruction of ecosystems keep on the rise, which directly points to aviation as long as fossil fuels are its primary energy resource. As a result, an increasing number of voices, coming from different directions, are calling for isolationism policies, restrictions to travel as an imperative need for the environment, limitations to trade, and the implementation of state-driven economies. It is out of anyone's predictions to forecast what new equilibria are formed and whether technological developments and sensible

¹The Intergovernmental Panel on Climate Change (IPCC) has gathered a lot of facts [3], such as over 1°C increase in the Globe surface temperature since beginning of the twentieth century or the 46% decrease of Arctic sea ice in 2019 compared to 1970, to call for action; the direct impact on ecological systems [4] is not to be overlooked either. These are symptoms that the Earth's equilibrium maybe changing with unpredictable consequences if the situation is not reverted.

management can overcome the aforementioned issues or, on the contrary, if more restricting systems will arise. Advocating for the former –which is not to say that laissez-faire will solve all problems–, the author’s expectation for this work is for it to play an infinitesimal but still non-zero part in that direction.

Reports from the Air Transport Action Group (ATAG) [5] help to gain insight into the weight of aviation worldwide: in total, 4.5 billion passengers were carried in 2019 by around 1,400 airlines with a global fleet in excess of 25,000 aircraft. The growth forecast (pre-COVID) for the next two decades indicated a need for over 32,000 new aircraft for a total value of \$4.9 trillion. The European contribution to this progress is remarkable: from designing and manufacturing 10% of the world’s airline aircraft in the 1960s to more than 50% nowadays [6]. The impact of COVID-19, however, is still to be unveiled, but as of April 2020 there was a 94.4% year on year drop in passengers, with yearly estimations predicting a 60% to 70% decrease in air travellers, which illustrates the dramatic scenario the industry has faced this year and will continue to face, at least in the short term. The International Air Transport Association (IATA) has forecast a loss for airlines of \$84.3 billion in 2020 [7]. This jeopardises the position of 1.4 million people in Europe alone working in the aviation industry, which also indirectly supports between 4.8 and 5.5 million jobs and generates over 100 billion euros of the EU’s GDP each year [8]. In addition, Europe also leads the efforts to accomplish emission-free air travel, which is the next big challenge after this crisis is overcome. The objectives set by the European commission for 2050 [9] are a 75% reduction of CO₂ emissions per passenger, a 90% nitrogen oxide (NO_x) reduction, and a 65% reduction in perceived noise with respect to new aircraft in 2000. To this end, the SESAR and CLEAN SKY European programmes have been put in place to develop a more efficient air traffic management system (SESAR Joint Undertaking) and greener air transport (CLEAN SKY Joint Technology Initiative).

In the quest for performance and the reduction of emissions, three major areas need to be revised and researched: rethinking aircraft design, developing new engine technologies, and improving air traffic management (ATM). A straightforward way of seeing this is by looking at the Breguet equation that describes the range of the aircraft based on cruise flight and simple

assumptions:

$$Range = \frac{U_\infty}{g_0} \times \underbrace{\frac{1}{TSFC}}_{\text{Propulsion system}} \times \underbrace{\frac{L}{D}}_{\text{Aerodynamic design}} \times \underbrace{\ln \left(1 + \frac{W_{fuel}}{W_{landing}} \right)}_{\text{structural design}} \quad (1.1)$$

Improvements in ATM directly decrease the required flight *Range* (m) and costs associated with it. The term g_0 (m/s²) represents the acceleration of gravity. Increasing the flight speed, U_∞ (m/s), is positive, though it of course finds limitations in transonic and supersonic effects. Reducing the Thrust Specific Fuel Consumption (TSFC (s/m)) is part of the propulsion system efficiency (new technologies need also consideration as using sustainable aviation fuels or electric engines may feature smaller efficiencies that are now in a preliminary stage, and still contaminate much less than conventional engines). A lighter design, i.e. less landing weight, $W_{landing}$, will imply a similar reduction in the fuel mass, W_{fuel} , for a given range. Finally, improving the lift to drag ratio, $\frac{L}{D}$, is both the aerodynamicist's task and the result of the aircraft concept itself. This work focuses on analysis methods for next generation commercial aircraft, which are lighter by design and are expected to exhibit higher aspect ratio wings which has the potential of important fuel savings by increasing the $\frac{L}{D}$ ratio. Having underlined the social, environmental, and economical need for affordable, emissions-free, fast travel, we revise next the challenges and prospects of the new technologies that could enable these goals.

1.1 Motivation and challenges

There has always been a compromise between aerodynamic and structural performance in the design of an aircraft. The aspect ratio of the wing has traditionally been taken as the first indicator of aerodynamic efficiency [10]. High-aspect ratio wings do not experience as much loss of lift and increase of drag due to tip effects (vortex induced downwash) as low-aspect-ratio wings of equal area do. On the other hand, the higher the aspect ratio, the heavier the wing needs to be in order to support larger moments. The advancements in new materials have allowed for lighter, high-aspect-ratio wings (HARW) with an inevitable increase in flexibility. Thus HARW aircraft aim at reducing the induced drag which is associated with the shedding

of vorticity along the span and corresponds to around 50 percent of the total drag in a typical flight profile (see [11] for a historical review of induced drag). For the reduction of viscous or parasite drag, aerodynamic optimisation together with active flow control strategies are being studied [12]. Though they are out of the scope of this work, these concepts are to be combined in the next-generation of ultra-efficient airplanes.

In the design process, however, many aspects need to be considered: a major factor of HARWs is manoeuvrability, as the longer the wing is, the less manoeuvrable it becomes, due to the higher moment of inertia and lesser roll-rate; the space inside the wings for a retractile landing gear and for fuel sloshing might be compromised in very long, thinner wings; and finally, for large HARWs aircraft, manoeuvre and taxing around airports can also be an issue. All these considerations must be carefully taken into account and that is precisely the beauty of aircraft design: the final solution is never the optimal aerodynamic design, or the best structure, or the most controllable vehicle, but it is an overall optimum with economical constraints. If that is not complex enough, the characteristics of these new planes may cause some of the linear assumptions on which current analysis is based no longer valid.

1.1.1 Nonlinear analysis –the way of nature

Nature manifests itself as a combination, succession and opposition of nonlinear processes:

“Carelessness about decompositions (or ruptures) and unconcern about scales have been fostered by linear theory. Thus, nonlinear is crucial for contemporary science to regain an understanding of the scale of events that make the connections of elements irreducible; it is a challenge to the fictitious vision of abstracted events... Therefore we should not restrict our discussions to a matter of “dichotomy” of mathematical structures. The relation between linear and nonlinear should be analysed as a problem retroactive to the genesis of science” [13].

And yet, historically, engineers have been reluctant to use nonlinear analysis because of its complexity and the prohibitively expensive cost of simulations (when nonlinear blindly replaces linear in a design process). This has gradually changed due to the feedback combination of improved algorithms and computational power, with the need for higher accuracy in the simulations. During a structural dynamics forum in 2000 [14], leading researchers were invited to

propose ideas with respect to the big challenges in structural dynamics for the next decades. Issues and questions were raised regarding nonlinear analysis, acknowledging the lack of mathematical models for addressing complex, real-life engineering problems. From nuclear reactor structures, seismic activity, and micro-mechanical systems to airplanes, it was generally accepted that inaccuracies in the models had been overcome in the past by introducing excessive safety factors into the development process. Some of the points highlighted at the meeting were the formulation of approaches that account for complex, nonlinear effects, and making the new methods more user-friendly to practical engineers.

In aircraft analysis, nonlinearities may appear in the actuation of the control surfaces, due to structural deformations, on the flight mechanics and in the highly nonlinear nature of fluid dynamics when viscous, separation, or transonic effects are present. This work only deals with the structural-type of nonlinearity which, based on its origin, can be classified as: geometrical nonlinearities, that occur due to the change of the shape of the original model; material nonlinearities, where large strains change the material properties; and boundary nonlinearities, such as contact stresses between different parts [15]. Because aircraft are currently deliberately designed to fly in the linear material region, geometric nonlinearity will be the most relevant structural nonlinearity, which can be included as a feature of future designs without a necessarily detrimental impact on the vehicle's performance. Understanding and capturing it is one of the aims of this project. There are a range of different mechanisms by which geometrical nonlinearities become significant: **large displacements and finite rotations**, as a result of adding up small local deformations to very long structures and leads to **large inertia changes** of the modified geometry; **geometrical stiffening**, due to the change in the shape of the structure which produces a redistribution of internal stresses, especially important in thin structures where the bending stiffness is much smaller than the axial stiffness; and **follower forces**, which are the result of distributed forces on the surface (e.g. pressure) that retain their direction with respect to deformations of the body, and when these are not small, the force direction can change substantially, typically tilting inwards in aircraft wings.

Fig. 1.1 shows four concepts where modeling nonlinear structural dynamics is inherently important: the Airbus future aircraft concept shown in Fig. 1.1(a), displays a common feature of

new configurations even if they are not extreme: the slenderness of their structures, specially the wings. The other example by Airbus in Fig. 1.1(b) is a blended wing body, which seeks to maximise aerodynamic performance. Both examples are intended to be zero-emission commercial aircraft based on hydrogen as primary fuel and could enter service by 2035². Fig. 1.1(c) features the Boeing truss-braced wing aircraft, which doubles the AR of current commercial aircraft and is intended to be ready for operation in 2030-2035³. The solar-powered NASA Helios⁴ in Fig. 1.1(d), with an aspect ratio of nearly 31, is a perfect example where geometrical nonlinearities have to be taken into account, otherwise linear analysis may yield results that are too conservative or not accurate enough [16].



(a) Airbus new HARW concept



(b) Airbus new BWB Concept



(c) Boeing truss-braced wing



(d) NASA Helios solar power aircraft

Figure 1.1: High aspect ratio flying vehicles (Image copyright by Airbus, Boeing and NASA respectively)

In the report of the catastrophic failure of the Helios, it was determined that the lack of adequate analysis tools, mostly constrained to linear methodologies, was one of the main causes of its failure [17]: firstly, large deformations induce geometrically-nonlinear effects that become

²<https://www.airbus.com/innovation/zero-emission/hydrogen/zeroe.html>

³<http://www.boeing.com/features/2019/01/spreading-our-wings-01-19.page>

⁴<https://www.nasa.gov/centers/dryden/news/ResearchUpdate/Helios/>

important in high-aspect-ratio components; secondly, the aerodynamics should accurately represent large motions of the main surfaces; and lastly, if a flying vehicle's natural frequencies are of the same order than the flight dynamics response, the coupling between the flight dynamics, structural response, and aerodynamics has to be taken into account.

A multidisciplinary approach is thus required for the analysis of these configurations such that the nonlinear structural, aerodynamic, and flight mechanics responses are captured. However, certification of a new air vehicle requires analysis on a number of load cases that can exceed the 100,000s [18]. Manoeuvres and gust loads at different velocities and altitudes, and for a range of mass cases and configurations, conform the load envelope. The addition of nonlinear effects to such a large amount of computations makes Reduced Order Models (ROMs) an integral part of the design phase.

Departure from linear assumptions made us realise that an independent analysis of the parts cannot hold as a good approximation of the whole when dealing with complex systems; in this scenario the field of aeroelasticity emerges with its variants -thermo, -servo, etc., which are only a reflection of an interplay between disciplines that cannot be neglected. Therefore, before presenting our approach to geometrically nonlinear aeroelasticity, we review some recent advances and the state-of-the-art of this subject.

1.2 Selected contributions to current aeroelastic theory

Aeroelastic theory has its origins in the assurance of the integrity of the vehicle; classical phenomena such as flutter, divergence, or control reversal are studied to anticipate catastrophic failure. Hence, traditionally, aeroelastic studies were carried out late in the design process, when the configuration had already been defined to a large extent. Traditional paradigms, however, have been revised in new aircraft concepts and aeroelasticity has become an active part of the design from the early stages. The analysis of flexible aircraft is an example of a multidisciplinary field that feeds from disciplines that traditionally were separated from each other. It is therefore difficult to keep precise track of the advancements in each discipline separately, but leading groups have pushed in the last decades for integration towards the development

of computational tools that, encompassing solid, fluid, and flight mechanics, aim to: capture the geometrical nonlinearities in the structural configuration; account for unsteady flow and possibly stall aerodynamics; and incorporate flight dynamics and control methodologies that can benefit from clever mastering of nonlinearities in both structure and flow. This section is intended to give an overview of the current methodologies and the process to their development. The field has come under scrutiny in a series of published works. Friedmann [19] has remarked the renaissance of aeroelasticity for fixed-wing and rotary-wing vehicles with nonlinear computational techniques integrated into the design process. Livne [20] gave guidelines for the future of aeroelasticity with remarks to numerical simulation frontiers, flight testing, management of uncertainty, the importance of reduced order models, the incorporation of active control, challenges in multidisciplinary design optimisation, particular issues faced by industry, and the application to non-conventional configurations. Bhatia has revised the aeroelastic practise from an industrial perspective and advocates for its further integration into the design processes, for which increase of model fidelity and reduction of computational times are deemed necessary [21]; and Afonso et al. [22] have made a review with emphasis in the nonlinear tools available to tackle the challenges encountered by HARW aircraft, which include the appearance of unexpected limit cycle oscillations (LCOs), reduction of flutter speeds with deformations, coupling of rigid-body modes with structural dynamics, the need for multidisciplinary design optimisation and assessment of uncertainty using nonlinear approaches, and the enlargement of experiments and tests looking at these characteristics.

Thus, sound grounds are already established to remark the need for nonlinear aeroelastic analysis and, in the next sections, we outline the efforts made to construct efficient and accurate structural models, predict unsteady aerodynamics, assess flight dynamics effects on flexible aircraft, and finally give some examples of how these disciplines are combined for the aeroelastic design of aircraft.

1.2.1 Structural models for slender structures

In the design process of aircraft structures, as nonlinear effects in their global response become relevant, an issue arises involving the strategies required in order to reduce the computational

effort and accurately solve the problem. A natural way of solving large systems has been reducing the system's DOF and tackling only the dimensions where the main features of the system take place. This is the case with 1D beam models, used as efficient basis for the representation of slender wings, rotor blades, fuselages, etc. Challenges arise to take this modelling approach beyond the conceptual design phase and it is an open question the implementation of model reduction techniques that bridge the gap between 1D and 3D models on structures displaying geometric nonlinearities. Yet, for typical dynamic loads and aeroelastic analysis of high aspect ratio aircraft, the apparent mismatch between a full 3D FEM and a composite beam model maybe overcome due to special characteristics of the models [23]:

- Low frequency vibration modes feature bending and torsional shapes, which are well captured by beams.
- Wing loads are computed as the resultant of forces and moments at different locations along the span (monitoring stations in the main load path).
- Industrial-level models for dynamic loads and aeroelasticity often use lumped masses along the longitudinal axes of wings, fuselage, or tail to model inertia, as shown in Fig. 1.2.

The solution strategy to solve complex slender structures in this case is often divided into three major steps: firstly, transform the full model into a beam-like structure; secondly, solve the beam dynamics of the “skeleton” structure with an appropriate beam model; thirdly, recover the 3D quantities from the 1D solution. The jump to nonlinear problems involves a series of challenges, such as the formulation of finite strains, the parameterisation of finite rotations and the objectivity problem, and the solution of high-order differential equations. All these aspects are described next.

Geometrically-exact beams and shells

The overall structural response of aircraft is modelled using 1D and 2D FE elements, i.e. beams and shells, and only certain components require of independent 3D analysis. An important step

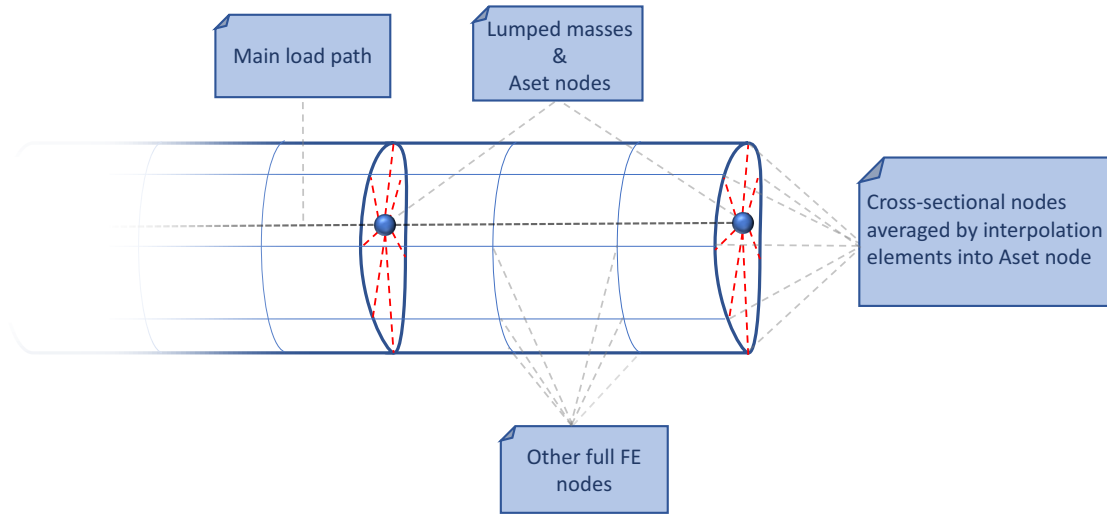


Figure 1.2: Wing idealisation for the aeroelastic analysis

in the solution of nonlinear problem is the development of geometrically-exact theories. They introduce finite strains and rotations, contrary to infinitesimal assumptions. Reissner was one of the initial precursors with studies on beams of arbitrarily large deflections, mainly for static problems [24], [25]. Simo introduced the concept of geometrically-exact beams and gave a dynamic formulation of Reissner's approach [26]. Simo et al. developed the theory further, not only for beams [27] but also for shells [28, 29]. Hodges developed a more general beam theory, based on an intrinsic mixed formulation, i.e. devoid of displacement and rotation variables, in terms of local velocities and forces [30, 31]. He also extended the intrinsic formulation to shell structures [32], though very little work has been carried forward on that front. The two-field formulation doubled the number of variables but reduced the nonlinearities to quadratic terms. Since the first developments of the geometrically-exact beam theory, a range of formulations using different primary variables and ways to parameterise the rotations have appeared. Geradin and Cardona used a displacement formulation with the Cartesian rotation vector and presented the complete theory in the monograph of flexible multibody dynamics [33]. Cesnik [34] introduced an intrinsic theory in terms of strains only. Reviews on the different methodologies and how they perform can be found in Palacios et al. [23, 35]. Far from being an exhausted field, new formulations to geometrically nonlinear beams are still being proposed [36–39].

An extensive number of shell and plate theories have been developed with different features [40, 41], though they inevitably introduce many more DoF than beams and the nonlinear

calculations in this work are based on the 1D description where we put the focus from here onwards.

Cross-sectional analysis methods

Geometrically-exact beam theories assume knowledge of the properties along the beam, i.e. the mass and stiffness per unit length. In complex 3D structures, such as wings, there needs to be a mapping between the known 3D properties and the approximated 1D properties required by beam models. Cross-sectional analysis techniques can be employed in order to accomplish this mapping.

The variational asymptotic method (VAM) is a mathematical methodology used to minimise a functional that depends on small parameters. Berdichevsky [42] proposed this method to resolve the 3D beam problem into two parts: the large dimension of the beam axis; and the smaller dimension, the cross-section. The resulting approach provides a way of building the equations without ad hoc assumption. Cesnik and Hodges [43] applied the VAMs to beams, which gave rise to the Variational Asymptotical Beam Sectional Analysis code (VABS). In the case of beams, the cross-section dimensions are much smaller than the main axis. Perturbations are introduced in the form of a small warping field and the cross-sectional energy is minimised using VAM techniques. After the 1D deformation has been found, the 3D displacements and stresses can be recovered by means of a set of warping coefficients. Cesnik and Palacios extended VABS with new capabilities such as the incorporation of linear electroelastic equations of the cross section for analysis with piezoelectric actuators [44] and the introduction of deformable cross-sectional modes allowing the nonlinear beam model to undergo arbitrarily deformable cross-sections [45].

Another cross-sectional analysis tool based on similar principles is the BEam Cross section Analysis Software (BECAS) [46], from the theoretical work by Giovotto [47]. Alternative methods using homogenisation and unit-cell analysis have been proposed by Palacios and co-workers [48] whereby conservation of strain energy and equivalence of the strain in the large and small scale are used to build the 1D model. Unlike other approaches based on homogenisation [49], this methodology does not require the cross-section to remain constant along the structure,

but it does need periodicity.

Nonlinear modal approaches

Cross-sectional methods rely on approximations such as constant cross-section or periodicity of the structure. Alternatively stick methodologies divide the structure into a set of beams of equivalent bending and torsional stiffness to the 3D structure, obtained from linear static computations [50, 51]. Either it be via cross-sectional analysis or stick modelling approaches, for large structures it has been shown that either approach can lead to significant errors when compared to nonlinear simulations using built-up finite-element models [52, 53]. While it is possible to run nonlinear solutions of full 3D models in the solvers where they are built (Nastran, Abaqus, etc.), they proved to be very expensive, maladroit, or impossible if flight mechanics and other aeroelastic descriptions are to be included.

This has encouraged the development of methods that aim to increase the fidelity of structural computations with geometrical nonlinearities while maintaining the computational cost low. System identification approaches have been proposed [54] that capture the nonlinear response of the structure, approximated by a combination of normal and higher order modes [54, 55], and used to construct ROMs for fast computations in nonlinear dynamic aeroelasticity [53]. The main issue here lies in the prior calculation of a training database made of a large number of static nonlinear computations, which then acts as the model internal physics in the actual simulations. This approach also needs the full model to be suitable for nonlinear simulations, which could be a real constraint, e.g. on linear finite-element models calibrated against GVT. Another modal-based method has recently been proposed for nonlinear static computations from generic FE models that rely on the calculation of curvature mode shapes, along a reference line, and their subsequent integration to obtain a displacement field in a geometric nonlinear fashion [56]. And in a different solution [57], a structure is divided into substructures, attached via a fictitious mass approach, and the deformations of the condensed beam-like structure are represented as the sum of rigid-body displacements plus the modal elastic deformations within the segment. While these two approaches have been positively tested in simple cantilever structures, they have an extensive validation ahead to reach the full aircraft structures treated

in this work under static, dynamic and nonlinear rigid-body dynamics [52, 58]. In the last method, for instance, splitting in substructures a full aircraft configuration as the one in Ch. 7 would be, at best, tedious.

A different approach to nonlinear structural computations is the use of the so-called Nonlinear Normal Modes, which represent an extension of the LNMs. They were first introduced by Rosenberg [59], and have been more recently defined as invariant manifolds in the phase space [60]. Their computation and application are reviewed in [61, 62]. The NNMs are derived in [63] from Hodges' intrinsic beam equations and applied to the solution of beams undergoing large displacements; based on that work, we could extend our approach to obtain the NNMs of the full aircraft configurations treated herein. Actually the NNMs approach has been tested in a full aircraft configuration [64] showing that modes may interact in the nonlinear analysis generating additional NNMs that have no linear counterpart. And yet, there are reasonable doubts about their usefulness in numerical solutions, for which many other bases might be utilised [65]. On the other hand, NNMs might be particularly useful when combined with experimental data, as shown in [66], where a geometrically nonlinear model updating strategy is implemented using NNMs as a correlation metric.

The flourishing of different methodologies indicates important efforts are being put into finding robust approaches that incorporate nonlinear effects to complex 3D FEMs that were built in a linear fashion and which encompass the knowledge and expertise of organisations but now require additional features as linear assumptions may no longer be valid. This thesis contains some of the most advanced examples of nonlinear modal ROMs approaches, including static and dynamic problems on representative wings undergoing large deflections, multibody capabilities and trim, manoeuvre, and gust analyses in a full aircraft configuration. The first step of this approach is the linear reduction of a full FE model into its main loads paths, for which different techniques are outlined next.

Reduced order models in linear structural dynamics

The finite element method has been consolidated in diverse areas of engineering as a principal tool for the analysis of structures. Even under linear assumptions, the rise in the complexity

of the models has led to excessively expensive computations. A significant amount of research has been carried out to produce ROMs from the linear FE models, especially in the field of mechanics, where structures with millions of DoF are studied. A good review of the different reduction techniques in structural mechanics can be found in [67, 68]. The aim of these methods is not restricted to computational efficiency but they have also found application in experimental sensor location [69], signal processing and damage detection [70], vibration testing and model verification [71], and nonlinear analysis [72, 73].

The modal coordinate reduction is based on the transformation from physical coordinates to a set of generalised coordinates, with modal superposition as the most widely used of these methods. It relies on extracting the minimum number of linear normal modes (LNMs) to describe the dynamic behaviour of a structure. The modal displacement method, the mode acceleration method, or the modal truncation augmentation method all fall into this category of modal reduction, and they are reviewed and compared in [74]. Methodologies based on Ritz vectors can be used instead and are faster to compute than the standard eigenvectors. Substructuring techniques such as Component Mode Synthesis (CMS) or the Quasi-Static Modes (QSM) have proved very useful in the industrial environment with the Craig-Bampton method [75]; in [76] these are used to build multibody systems from complex FE models. Other general reduction methods such as the Proper Orthogonal Decomposition (POD) [77] have also been applied to structural problems.

Dynamic condensation techniques are commonly used in the reduction of structural systems. Generally, the type of condensation can be divided into three categories [67]: physical dynamic reduction, where only the stiffness and mass matrices are needed to obtain the condensed matrices; modal-type dynamic condensation, where some of the modes of the full model are required to calculate the condensed matrices; and lastly, in hybrid condensation both the system matrices and its modes are used to produce the condensed matrices. Guyan condensation [78] is an example of physical dynamic condensation and the System Equivalent Reduction Expansion Process (SEREP) approach of modal dynamic condensation (exercised in [69] for optimal node selection). The advantages of modal-type reductions are that the reduced model preserves the modes selected from the full system and the accuracy of the method is independent from the

selected nodes for condensation, as long as the modes are observable at these nodes. The main disadvantage is having to calculate the modes of the full system. A special scheme is also needed to remove the null values of the resulting condensed model prior to doing its eigenvalue analysis. In order to overcome these issues, a hybrid-type condensation combines the Guyan reduction with a modal-type reduction.

1.2.2 Unsteady aerodynamics

The field of unsteady aerodynamics is too broad to give a reasonable account of its main developments. Thus, we shall only indicate some approaches relevant to the aeroelastic analysis. It has been found appropriate to classify the theories according to their level of fidelity or the degree to which they approximate the real physics of the problem, highlighting the main attributes or lack thereof.

Lower fidelity approaches

In the lower fidelity range of approaches, 2D aerodynamics based on strip methods have proved to be useful for capturing the main features of the flow with inexpensive solutions. Furthermore, models for viscous effects and stall dynamics can be added readily. On the other hand, they fail to predict spanwise effects and interference between lifting surfaces. The accuracy of these methods is limited to high frequency dynamics [79] and high aspect ratio wings –and even in those cases significant differences appear when compared to 3D methods [80]. See the work by Peters [81] on the features and developments of inviscid 2D unsteady airfoil theory. His theory on finite state aerodynamics [82] has been broadly used in aeroelastic studies of both fixed-wing and rotor aircraft. The ONERA semi-empirical method for dynamic stall, initially developed by Petot [83], can be combined with the finite state model [84] to have a nonlinear and unsteady model, as utilised in [85] to study post stall flutter phenomena. In a similar manner, the Leishman-Beddoes method is another type of semi-empirical model [86] that aims at modelling the dynamic aerodynamic loads after stall (see [15] for a unified view of these methods).

Medium fidelity formulations

A reduction of the Navier-Stokes equations can be carried out by neglecting viscosity (Euler) and the vorticity of the flow. The resulting equation is the full potential equation, which after some simplifications can be solved via the transonic small disturbance (TSD) approach – accounting for entropy changes and shock waves–; moreover, linearisation of this equation allows a solution by superposition of elementary singularities (over a surface) that make the flow match the boundary conditions of the problem. The associated reduction in computational effort makes these linear methods widely used within industry and academia. The Doublet Lattice Method (DLM) has been the most popular numerical solution among these methods [87, 88]. It is formulated in the frequency domain with doublets singularities. A further reduction of the equations is obtained by neglecting the compressible effects, which leads to the Laplace equation of potential flow. The unsteady vortex lattice method (UVLM) [89] is a time domain numerical solution of this equation based on vortex ring singularities. This method is especially useful for low speed unsteady aerodynamics as it does not require linearisation of the geometry and the free wake is convected by the local flow velocity. The monograph [90] is a standard reference for low speed aerodynamics and the principles of the UVLM are well explained there. Despite being based on potential flow, the UVLM can be modified to incorporate nonlinear effects such as post-stall dynamics [91], and partially compressible effects [92]. It has also been formulated in the frequency domain for fast computations and stability analysis in a similar fashion than the DLM [93]. It is important to note that these methods do not model thickness unless the panels are distributed wrapping the surface of the bodies [94], which implies more costly solutions. 3D higher order panel methods that solve the linearised potential equation are also available [95]. The solution of the compressible flow equations under arbitrary geometry changes (as allowed in the UVLM for incompressible flow) using a panel approach rather than solving the entire 3D fluid domain remains an open problem, if at all solvable. Some indications are given in Sec. 3.4.3 to partially tackle this problem.

Towards higher fidelity analysis

High fidelity aerodynamics are undoubtedly set to substitute many of the past and present methods of low and medium fidelity aerodynamics. Computational Fluid Dynamics (CFD) solve the Navier-Stokes equations, with a range of approaches that also carry different fidelities. Neglecting viscous forces yields the Euler equations, which still can account for compressible effects. If viscosity is accounted for, computational methods of increasing complexity are the Reynolds-averaged Navier-Stokes (RANS), large eddy simulations (LES), and direct numerical simulations (DNS). The last two are still under fundamental research and very far from the intended studies herein. Moreover, for preliminary design phases, the highly multidisciplinary character of aeroelastic analysis makes approaches with CFD very expensive. Aeroelastic and multidisciplinary optimisation studies have been mostly restricted to static analysis, when geometrically nonlinear effects were included, [96, 97] or to linear structural analysis with Euler or RANS aerodynamics [98, 99]. Nevertheless, fully-coupled nonlinear aeroelastic simulation are now available [100, 101] and will be gaining momentum in the coming years. Reduction order models of the aerodynamics, however, can be a good compromise between accuracy and simulation speeds, and are a trending research subject.

Reduced order models and multifidelity approaches

One of the options to palliate the cost of high fidelity analysis while maintaining its accuracy is to build aerodynamic reduced order models. Several methods have been assessed in recent years and they are a popular area of research. A first approximation consists of correcting the aerodynamic matrices from unsteady panel methods with high fidelity computations or wind tunnel data. In [102] they are revised and exercised in an industrial environment. More recently [103] have corrected the aerodynamics from DLM matrices with RANS CFD and used the resulting ROM to study the gust response of a full aircraft configuration.

Another strategy is to build the ROM directly from CFD by computing the response to certain impulses such as Gaussian or unit-step [104], and extract the generalised aerodynamic forces (GAFs) for use in the frequency domain. Along these lines, [105] have developed a system identification approach that improves the construction of the ROMs yielding the GAFs needed

to compute flutter points or assemble the aeroelastic systems with less effort. For highly nonlinear systems, the use of Volterra series, proper orthogonal decomposition (POD) and harmonic balance (HB) to reduce CFD models are reviewed by [106] and used in the solution of different aeroelastic problems. In [107] they have corrected a solution based on indicial functions with RANS simulations, and subsequently applied it to transonic flutter calculations of a truss-braced aircraft. Alternatively, a method based on Dynamic Mode Decomposition (DMD) has been proposed [108] to extract the fluid modes of an oscillating structure and the subsequent derivation of the GAFs.

Alternatively, multifidelity techniques have been proposed that combine multiple methods to fuse the best characteristics of each one of them. While these techniques are relatively new, GAFs from DLM and Euler CFD aerodynamics have been merged in [109] through a co-kriging multifidelity approach. They have carried out flutter calculations and found that difficulties could arise when training the models due to highly nonlinear transonic effects at the Mach number, $M_\infty = 0.775\text{--}0.9$. A nonhierarchical multifidelity formulation is proposed in [110] to fuse data for multidisciplinary design. They have combined vortex lattice aerodynamics with Euler CFD, RANS CFD, and wind tunnel data, and carried out uncertainty propagation via Monte Carlo sampling.

1.2.3 Flight dynamics of flexible aircraft

Traditionally aeroelastic effects and flight dynamics have been decoupled and analysed separately in the design of aircraft. This is justified on the basis of relatively stiff components that undergo small deformations at much higher frequencies than the rigid body responses. In this case, the aerodynamic characteristics for the flight dynamics can be obtained from databases built from experiments, previous empirical relations, flight tests, or computational calculations [111]. Stability and control derivatives can also be produced in this manner. Including elastic effects as a first approximation is attained via elastified derivatives using a linear quasi-static approximation [112, 113].

Changes in the inertia characteristics and stability derivatives in the deformed configuration or the interaction between rigid body and elastic modes might be unavoidable in flexible designs

and new aircraft concepts. Body freedom flutter, for instance, appears as a consequence of coupling the flight dynamics with the elastic deformations [34, 114]. Integration models for flight and aeroelastic dynamics have been proposed with different levels of nonlinearities. The mean-axes approximation is a popular one whereby a floating axes reference system is attached to the instantaneous center of mass and follows the principal axes of inertia (the approximation assumes that the relative linear and angular momenta due to elastic motion vanish). As a result, the rigid-body and elastic equations are decoupled, which on the other hand might not be the best solution, especially in aeroelastic problems where the coupling effects will remain through the aerodynamic forces [115]. Nonetheless, it has been extensively used because it allows a nonlinear description of the rigid body dynamics with linear deformations. Furthermore, different approaches have been proposed to include a flight dynamics description into the DLM aerodynamics via a RFA approach using this approximation [116, 117], or its variant, the practical mean-axes, where the floating system also follows the center of inertia but the axes themselves are fixed in the reference configuration [118].

More general descriptions can be obtained by choosing a material point and orientation-fixed body-axis frame. Even though there is no mathematical decoupling of rigid and elastic components, this allows geometrically nonlinear unified formulations based on velocities [119], strains [120], and displacements variables [23].

1.2.4 Nonlinear aeroelastic tools in very flexible aircraft

Combining medium fidelity structural and aerodynamic models, a range of aeroelastic tools have been developed in order to study very flexible aircraft. Drela was one of the first ones to integrate nonlinear structural methods with flight mechanics and controls into the software ASWING [121], which combines a UVLM methodology with compressible corrections and beams linked with nonlinear connections. It has been used in [114] to study the body freedom flutter behaviour of a flying wing model.

Hodges and Patil have developed NATASHA (Nonlinear Aeroelastic Trim and Stability of HALE Aircraft) software, which combines Hodges intrinsic structural formulation, Peters' in-flow state model, and ONERA stall model [122]. They have shown the flight dynamics of a

HALE configuration [123], the importance of high order nonlinear effects in the computation of LCOs and flutter speeds of very flexible aircraft [124], and the application of nonlinear aeroelasticity to the design of a solar power flying wing studying engine positioning to avoid flutter [125].

Cesnik and collaborators have used an intrinsic structural formulation based on strains [126] and Peters' inflow states aerodynamics to build UM/NAST (University of Michigan's Nonlinear Aeroelastic Simulation Toolbox). It has been employed in structural dynamics calculations [127]; they have studied the response of a HALE aircraft [128] under gust disturbances; performed nonlinear stability studies of a novel configuration [34], showing the occurrence of body freedom flutter due to the coupling of the short-period and bending modes; and also assessed the geometrically nonlinear effects on the manoeuvrability of a HARW transport aircraft configuration [51].

Palacios and co-workers have developed the code SHARPy (Simulation of high aspect ratio planes in Python) [129], which has provided a large amount of studies and results that have directly inspired this work. The nonlinear structural module is based on a geometrically-exact beam formulation in displacements [33]. The main characteristics of the structural methodology together with a linearisation of the dynamic equations are shown in [130]. The UVLM based aerodynamic theory is revised in [89] with stability studies and the open-loop response of a HALE model. They solved the problem of T-tail flutter using UVLM aerodynamics, which would not have been possible with a standard DLM (they then proposed enhanced methods for the standard DLM for this problem and provided experimental validation [131]). Wake-tail interference effects were also assessed [80] and induced drag calculations using UVLM were revised in [132]. Combining the fields of control and design in SHARPy [133], also known as co-design, they studied the optimal control of rolling manoeuvres in very flexible aircraft [134]. More recently they have incorporated atmospheric disturbances from high fidelity into the simulation capabilities [135], and they have explored ROMs of the UVLM that dramatically reduce the computational times [136, 137], while capturing the dynamics over arbitrary deformed geometries.

The challenge for the coming years will be to enhance these descriptions with higher fidelity

approaches and introduce them into the overall design of HARW commercial transport aircraft, that requires evaluation of many other complexities apart from the inherent ones of low speed (e.g. solar-powered) aircraft for which these frameworks were originally developed.

1.2.5 Aeroelasticity and design

Geometrically nonlinear deformations from slender designs, coupling with flight dynamics, and the appearance of radical configurations to enlarge performance boundaries, are pushing aeroelastic analysis to become an active design tool. As a first step, the aeroelastic tools employed in the industrial context for a wide range of problems and applications need to be enhanced: with geometrically nonlinear effects in static loads [138] as well as in gust and manoeuvre loads [58], and concentrated nonlinearities [139]; incorporating control surfaces in the aeroelastic design loop [140], thus also allowing for variations in longitudinal and lateral stability [141] as part of the optimal configuration development; and deploying high fidelity aerodynamics with focus on the aeroelastic problem [142]. The importance of these effects in HARWs has been more than established already [16, 143], although their incorporation into the design process, assessment of their computational overhead, and further consolidation of the benefits of these configurations are far from being resolved.

The second step relates to the integration of the enhanced methods, both from a theoretical and computational perspective. Tools and architectures in Multidisciplinary Design Optimization (MDO) are going to be critical to move forward in this direction [144, 145]. Uncertainty Quantification (UQ) applied to the aeroelastic analysis [146] also holds a great potential in reducing safety margins in the overall process as well as decreasing the number of cases required for certification. Multifidelity analysis [109, 110] may also prove useful to blend solutions and reduce computational times. We describe next the areas where such advancements are of special relevance.

Aeroelastic tailoring

Certainly one of the most promising areas for the application of aeroelasticity to the design of aircraft is aeroelastic tailoring. Making use of the varying properties of composite material gives

a great flexibility to design, but it also brings the challenge of proper cross-discipline interaction management. In general, the problem is laid as an optimisation of local (composite) material properties and structural-sizing variables. A more disruptive strategy has been shown in [147] where a lattice structure is optimised to increase the flutter speed, thereby using the topology rather than the material properties and geometry parameters to attain a superior design. The multiphysics software SU2 [148], which features CFD aerodynamics, nonlinear structural capabilities and their adjoint solvers to calculate sensitivities, has recently shown promising methods to solve fluid-structure optimisation problems under large deformations [149] and obtain topology optimised designs [150]. Scaling these studies from simple models to full aircraft configurations, as it has been done within the suite with aerodynamic-only optimisations [151], will be the next challenge for this novel approach.

In this sense, the work of Martins and co-workers has been pioneering in bringing high-fidelity descriptions to the optimisation of aircraft configurations with great focus on the parallelisation of the methods. They have developed several tools, including TACS (Toolkit for Analysis of Composite Structures)[152], for CSM calculations coupled to a second-order CFD code ADflow [153] via MACH (MDO of Aircraft Configurations with High-Fidelity) [154]. Sensitivities for the different subcomponents are calculated through adjoint solvers. They have performed aeroelastic static optimisation of the NASA Common Research Model (CRM), studying aerostructural tradeoffs [155] and finding important savings by increasing the AR of the wing from 9 to 13.5 [156]. And yet all the structural calculations were based on linear assumptions which may not capture the complete response for these slender wings. So there is room for improvement on these methods by including geometrically nonlinear effects and dynamic loadings.

De Breuker and his team have also made important advances in this area. They have developed the in-house aeroelastic tool PROTEUS, which combines a cross-sectional modeller, a geometrically nonlinear beam, and UVLM aerodynamics [157]. It has been used to perform dynamic aeroelastic tailoring for gust loadings in the CRM [158]. They have also performed aeroelastic optimisation of the Airbus XRF1 model with Nastran solvers and CFD enhancement of the DLM [159], and have continued the analysis to include blended constraints under static and dynamic loadings [160].

In fact a common way of doing tailoring is performing the calculations within the MSC Nastran toolbox: in [161] optimisation of manoeuvres and gust loads has been performed under flutter and control-effectiveness constraints; a matrix perturbation theory is presented in [162] to calculate sensitivities in the aeroelastic analysis and the subsequent fibre angle distribution in a clamped wing; and Bret Stanford has made important contributions using Nastran solvers, such as including control surfaces in the (aeroservoelastic) optimisation [140], incorporating a nested procedure with both surrogate and gradient-based optimisations [163], and highlighting important differences in the optimisation as a consequence of using high fidelity or DLM aerodynamics [164].

These and other research efforts have shown the potential of introducing the aeroelastic analysis into the design process. In most instances, however, geometrical nonlinearities have not been included and limited dynamic analyses were considered. Imposing realistic constraints like aeroelastic instabilities is also a challenge that will strengthen the results of the optimisation. Thus, the road is open to further progress in a wide range of directions.

Flutter and post-flutter prediction and suppression

The problem of flutter has always posed big challenges in the design cycle of aircraft. The first flight test demonstration of active flutter suppression is described in [165]. The flight test was accompanied by experimental work with ground vibration tests (GVTs), theoretical modelling with DLM aerodynamics, and control systems. Since then, strategies have been proposed to build state-space systems and assess gust load reduction and flutter suppression [166].

A thorough review of current techniques is given in [167]: Livne, as in previous articles, strongly encourages for flutter suppression technologies and nonlinear analysis tools to be incorporated early in the design process as a way of accomplishing significant weight savings and performance gains. Another important review paper [168] outlines the current state-of-the-art methods for predicting flutter and nonlinear post-flutter behaviour with emphasis on optimisation problems and how to include flutter as a constraint of the optimisation. In fact, adding flutter constraints into the aeroelastic design optimisation is a problem that is recently gaining attention [164].

The appearance of limit cycle oscillations (LCOs) is a consequence of nonlinear effects in the

structure or the aerodynamics. These have been studied in a number of works, comparing experiments and theory [169], using increased-order modelling [139], a fast medium fidelity method [170], and a high fidelity CFD/CSD approach [100]. Work has also been done for predicting and controlling this phenomenon [171]. Methods to calculate the flutter margins have been established [172], which can also incorporate nonlinear effects such as LCOs. [173]. Harmonic balance and continuation methods have proved specially suitable to study this phenomenon and, more generally, the stability and bifurcation behaviour of nonlinear systems undergoing periodic motions [174, 175].

Regarding reduced models, DMD has been used to construct low-order models for flutter enhanced with high fidelity data [176], and to directly construct the aerodynamic ROMs for flutter calculations [108]. Multifidelity approaches have also been used to study this instability in simple airfoil models [109, 177].

Load alleviation techniques

One important application of bringing the aeroelastic analysis into preliminary stages is the introduction of load alleviation systems that reduce the critical loads thereby allowing for lighter designs. The design of control systems for flutter-suppression and gust load alleviation is not a new subject [166]. Yet the modelling capabilities have dramatically increased up to the point that high fidelity aeroservoelastic analysis is now available in the frequency domain [178]. In order to account for nonlinear effects, however, time domain simulations have to be run as in [179, 180], where dynamic load alleviation studies are undertaken with medium fidelity solutions.

A technology enabler of higher AR wings is the Semi-Aeroelastic-Hinge (SAH) technology, a system devised by Wilson and co-workers [181–183] at Airbus. The wing tip acts as a fixed hinge that can be released in-flight at the notice of an atmospheric disturbance, thereby not transmitting the extra bending moment to the other part of the wing and alleviating the overall loads. As the aspect ratio of the wings keeps increasing for aerodynamic performance, issues may arise for large aircraft that have to fit into standard airport gates. For these very reasons, the wing-span of the Airbus A380 had to be reduced to the detriment of performance. Hence

the second attribute of this device is its ability to fold when taxing. Initial aeroelastic studies were determinant to demonstrate the feasibility of this technology [184]. At the end of the thesis this concept is briefly studied, but further nonlinear analysis will have to be conducted to consolidate such a promising technology.

Non-conventional aircraft configurations

The idea that new aircraft configurations will be needed to meet stricter emission targets and to remain ahead of a competition with an increasing number of players is gaining more acceptance. Less conservative designs and the lack of previous statistical data make paramount the refinement of analysis tools. In the review of non-conventional planes by Livne and Weisshaar [185], they remark the need to study the coupling between flight mechanics and aeroelastic effects, use active control technologies and aeroelastic tailoring that benefits from composite materials. It is also underlined that innovative advancements such as swept-back wing concept, the T-tail, or the fly-by-wire control have become standard today, which might be extrapolated to new configurations in the next century. We briefly review now some studies on these configurations. A widely assessed configuration is the blended wing body (BWB) that aims to take advantage of all wetted surfaces to generate lift, thus achieving great improvements in the aerodynamic performance. The fusion between wing and fuselage brings significant challenges into the structural design and pressurisation of the cabin. A nonlinear aeroelastic analysis is carried out in [34] using beam models and 2D finite state aerodynamics, describing its dynamic response, post-flutter behaviour, and the reduction in the flutter speed with deformations. Aerodynamic optimisation of this concept has been performed in [186] using a high fidelity approach and remarking the close coupling between aerodynamic, trim, and stability states. A multifidelity analysis has been proposed in [110] to do a conceptual design analysis of this configuration with uncertainty quantification. A BWB demonstrator has been built by NASA, named the X-56A Multi-Utility Aeroelastic Demonstrator, designed to study active flutter suppression and provide data to enhance the development of ‘performance aggressive’ aircraft [187]. Time-domain high-fidelity aeroelastic research on the X-56A is presented in [188] and its body-freedom flutter studied.

The joined-wing or box-wing aircraft is a radical configuration that carries a lot of uncertainties but also encouraging benefits: better resistance to bending, due to the wings no longer being a cantilever, so that the aspect ratio of the wings can be increased while achieving lighter structures; and aerodynamically, a reduction of induced drag due to the Prandtl wing system [189] is also claimed but it has not formally been proved in a vehicle-scale concept. This configuration is not new and has been the subject of research for a long time [190]. An extended review of this concept [191] based on previous works [192, 193] has pointed to the development of non-linear aeroelastic tools as a major step to get this configuration from a promising concept to real aircraft. A scale model has been proposed in [194] to replicate the geometrically nonlinear behaviour of a full box-wing concept, acknowledging the need to incorporate these effects into experiments too.

Finally, the truss-braced aircraft is a pure solution to enable higher AR wings. The addition of parasite drag and weight from the truss are expected to be compensated by the lighter wing and the reduction in induced drag from the longer span. Load alleviation on this concept has been assessed in [195] with standard MSC Nastran aeroelastic tools. Its nonlinear transient response is studied in [196] by parametric updating of the state-space matrices. A multifidelity strategy has been shown in [197] to design this type of aircraft combining tabular methods of conceptual design with physics-based analysis including vortex-lattice and CFD aerodynamics, and MDO strategies to optimise the final design.

Other technologies that may be introduced in classical tube-and-wing or more radical configurations will also require a more integrated analysis. Morphing wings, for instance, undoubtedly need special treatment and considerations regarding fluid-structure interactions [198–200].

Progress on experimental methods in nonlinear aeroelasticity

The amount of research that goes into experimentally predicting the nonlinear aeroelastic behaviour of airplanes is not as large as the computational counterpart. This is not a matter of importance but of management of resources and the need for better theoretical understanding before going into the laboratory or the flight tests. Nonetheless, we shall mention some of the advancements on this end and remark that once the theoretical and computational tools are

consolidated, it will be the combination of experiments and theory that will make aforementioned applications a reality.

Some of the most frequent experiments in nonlinear aeroelasticity have been concerned with assessing the LCO behaviour of clamped wings. These have been reported in [169, 201] comparing theory and experiments. Equally transonic effects in the LCOs are evaluated in [202]. Control flutter suppression has been satisfactorily tested in [203] and structural free-play nonlinearities are investigated in [204]. Experiments and the ONERA semi-empirical model for unsteady aerodynamics and stall have been compared in [205].

Scaling methodologies are an important part of experiments in fitting the models into wind tunnels or flying smaller prototypes to gather important data. Scale models have been optimised for vibration eigenpairs, and the linear static response in [194], where the validity of the method is claimed for nonlinear conditions too. A nonlinear aeroelastic scaling methodology based on equivalent static loads has been shown in [206].

Regarding flying tests, vertical and rolling manoeuvres have been simulated in [207] using a DLM+RFA approach and satisfactorily compared to a sail-plane flight test: despite the simplicity of the models, good agreements were shown for the vertical manoeuvres while rolling manoeuvres posed a harder challenge. Similarly, flight test data has been predicted for the dynamic response of the X-56A flying wing configuration in [208]. The AlbatrossONE aircraft is a small scale prototype manufactured to study the characteristics of inflight foldable wing-tips as a passive load alleviation system. The technology aims to enable higher AR wings and details on its development and maiden flight have been described in [209].

Extracting the relevant flight data is a challenge due to the noise it incorporates and the large amount of information involved, even more so if the response is highly nonlinear. [210] have compared methods for flutter prediction from flight test data. In [211], a higher order DMD is used to extract the aircraft natural frequencies and modes from the flight-testing data.

1.3 Current approach and research goals

Current nonlinear structural and aeroelastic problems pose some questions and challenges that this work aims to shed some light on. It has been explained how the incorporation of geometrically nonlinearities coupled with reliable aerodynamic models might be, in more radical configurations, unavoidable; while in more standard concepts, they could bring a range of benefits and make the design less conservative. High fidelity aeroelastic analysis has mostly remained structurally linear due to the computational effort required to run fluid-structure interaction problems in a nonlinear fashion. And despite fully-coupled nonlinear aeroelastic simulation are now available [100], certification of a new air vehicle still requires 100,000s of load case simulations [18], as it considers manoeuvres and gust loads at different velocities and altitudes, and for a range of mass cases and configurations. Furthermore, the design of control strategies also demands fast reduced models, and standard approaches with linear models can fail under large wing deformations [212]. Thus, innovative solutions that combine the accuracy of high fidelity models with the speed of reduced models will have to be proposed.

This work continues the development of an alternative approach [170, 213–215] to study the geometrically nonlinear effects of already existing (linear) industrial-scale aeroelastic models. The method blends the efficiency and accuracy of geometrically-exact 1D descriptions for problems involving slender components, with the precision of condensation techniques in preserving the full 3D characteristics. It seamlessly integrates with standard linear aeroelastic analysis based on finite element solvers and compressible potential aerodynamics. The novelty is that the geometric-nonlinearity is included by considering information about the nodal coordinates, which are not tracked in linear modal analysis. The information for the geometric nonlinearities is introduced from the geometric layout of the structure which results in coupling between modes. This is facilitated by Hodges’ intrinsic equations [216]: by construction, the sectional velocities and internal forces have local influence regardless of large rigid-body rotations, and this does not need updating of the local mass and stiffness properties from the initial configuration. This feature permits the addition of geometrically-nonlinear effects to a predefined complex linear model. In addition, only quadratic terms of the main variables are needed to

capture nonlinear effects. Our concern is a problem of relatively high geometrical complexity for which a (linear) finite-element model (FEM) already exists, not necessarily built with beam elements, but representing a physical domain with a dominant dimension. The goal is to seek a computationally-efficient solution strategy that leverages on the slenderness of the domain to characterise geometrically-nonlinear effects, while utilising the information encapsulated in the original model –which is built, validated and refined during the design cycle of the specific engineering asset under consideration. The solution process is sustained on previous works [214, 217] that showed the capability of the approach to analyse simple 3D structures undergoing large deformations.

As for the modelling of the aerodynamic forces, although 2D airfoil unsteady aerodynamics are still often used for high-aspect-ratio wing aeroelasticity, 3D effects have been shown to play an important role [218] and will be considered here –moreover, it has been shown that the cost of 3D is not significantly larger than 2D after a ROM is applied [137, 219]. The aerodynamic forces are obtained in this work from the Doublet-Lattice Method (DLM) [88, 220], which solves a linearisation of the compressible, inviscid, unsteady flow equations in the 3D domain, and it is commonly used in linear aircraft aeroelastic analysis [221]. Since the structural equations are written in the material frame, the aerodynamic forces naturally appear as follower forces on the structure. It has been shown [170] that key geometrically nonlinear aeroelastic effects (wing inextensionality, geometric stiffening, and follower force effects), can be taken into account despite the linear –i.e. not updating with geometry- aerodynamic matrices. The Unsteady Vortex Lattice Method (UVLM) [80] is another potential method for geometrically-nonlinear aeroelastic simulations, as it fully accounts for large deformations both in the structure and the wake. However, it is substantially more computationally demanding and it is limited to incompressible conditions, and therefore not appropriate for the typical flow regimes of commercial aircraft. As the approach is verified in this dissertation, improvements can be easily introduced by amending the aerodynamic matrices from the DLM with experiments and higher fidelity aerodynamics [102, 103]. Moreover, the current description can directly accommodate higher fidelity aerodynamic models, such as 3D panel methods [18] that include thickness effects and CFD-generated aerodynamic influence coefficient matrices [108, 222].

The resulting solution procedure can be divided into the five stages shown in Fig. 1.3: 1) A linear (arbitrarily complex) model with slender components is the input for the analysis. 2) Model condensation is employed to derive a skeleton-like substructure, along the main load path, containing the main features of the full 3D model. 3) The modes of the reduced structure are evaluated in intrinsic variables (velocities and strains) and used as a basis of a Galerkin-projection of the geometrically-nonlinear intrinsic beam equations. 4) The projected equations are solved in time-domain under given forces: aerodynamic influence coefficient matrices are obtained from DLM and a rational function approximation (RFA) is used to transform to the time domain. 5) The intrinsic modes, the reduced order transformations, and the nonlinear 1D solution are combined to reproduce the full 3D solution. Geometrically-nonlinear behaviour is captured along the principal skeleton and the linear response of the cross-sections (in the form of ribs and fuselage reinforcements) is also represented –if nonlinear deformations also occur in the cross-sections, there is no reliable analysis other than high-fidelity solutions of the full model. The overall procedure has been implemented in what we have named as Nonlinear Modal Reduced Order Model (NMROM).

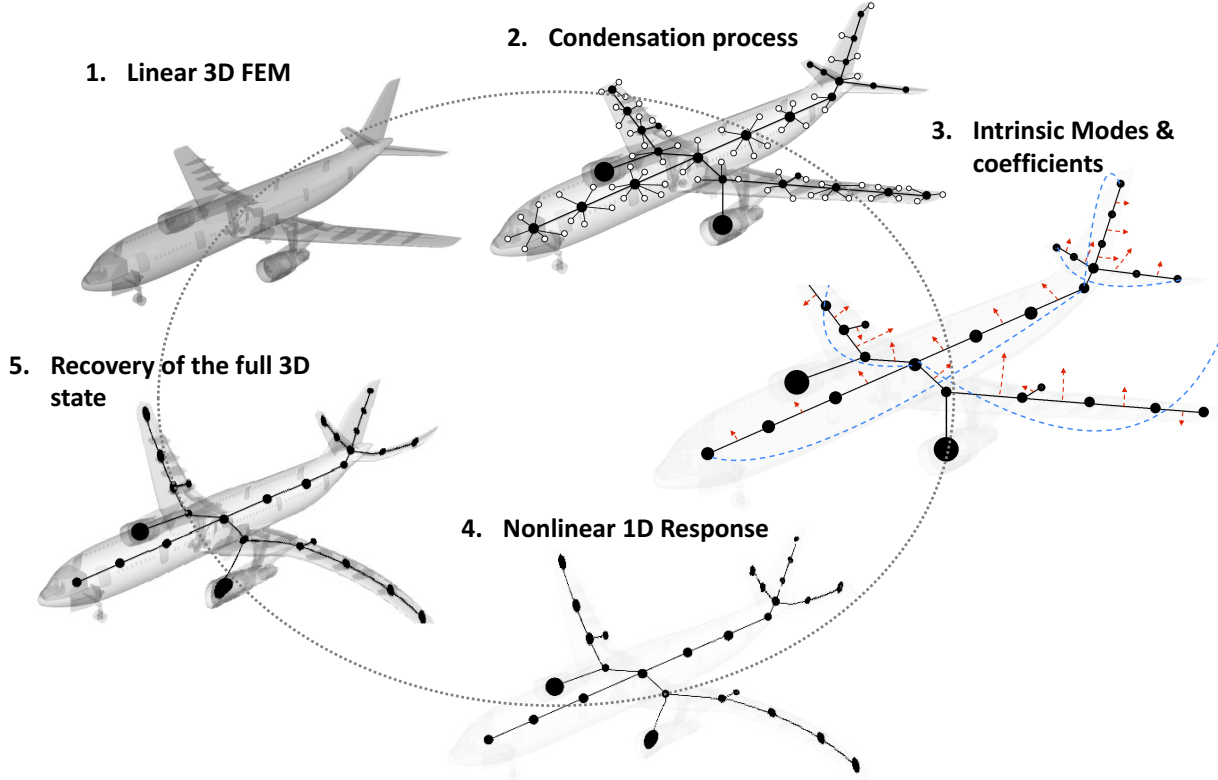


Figure 1.3: Solution process chart using the present approach

1.3.1 Research Objectives

Along this chapter it has been established that new passenger aircraft specifications demand lower emissions and higher fuel efficiency. Furthermore, the competition for performance is pushing into more advanced, lighter materials that reduce the operating weight, and longer wings which can significantly reduce the induced drag of aircraft. Current analysis tools based on linear theory may not be suitable for the new operating conditions and designs. **Thus the goal of this work is to develop efficient computational tools to study geometrically nonlinear effects on flexible aircraft loads.** It will be shown how the proposed methodology, exemplified in Fig. 1.3, can enhance already existing (linear) industrial-scale aeroelastic models with new features from nonlinear theory and is suitable for an industrial loads and aeroelastics environment. Moreover, the modal solution of the equations facilitates a fast computation of an approximated nonlinear response by selection of a reduced set of modes of interest. While our approach could be employed for the nonlinear control of large commercial aircraft or for the nonlinear aeroelastic tailoring of wings, **the focus herein is on the assessment of these nonlinear effects in the aircraft response, including dynamic and static flight loads at a fundamental level, stability analysis and comparison with linear theories.** In order to attain this, a number of milestones have been completed:

- Developing a structural framework that brings together the accuracy of 3D FEMs with the nonlinear capabilities of geometrically exact beam models. In addition to this, it has to be compatible with existing industrial practice for loads and aeroelasticity.
- Exploring fidelity-preserving reduction modelling techniques that allow to: a) condense the full FE model to a beam-like skeleton; b) recover the 3D displacement field from one-dimensional calculations.
- Consolidation of the intrinsic modal theory, previously exercised only on simple models, through further algorithmic developments; and demonstration of its suitability to the aeroelastic analysis of complex models, including multibody connections.

On the road to achieving these goals, the project has been divided into three major parts: the

first is the implementation of a nonlinear structural framework coupled with reduction order techniques. Together they allow the analysis of complex structures by means of detailed finite element models. The second part consists of adding and integrating aerodynamic models to the framework, the doublet lattice method in this case, such that an efficient nonlinear aeroelastic system is built and tested. The final part is the solution of relevant problems with the new toolbox. Following these fundamental modelling techniques, their application to industry-orientated problems conforms the additional set of objectives which were agreed upon with the industrial partner, Airbus:

- Computation of static and dynamic loads, including manoeuvres and gusts, with and without nonlinear effects on complex industrial concepts.
- Assessment of the computational cost associated to nonlinear analysis.
- In addition to these objectives, a collaboration was established during the project to work on the analysis of a foldable wing-tip for load alleviation.

The last point introduced the need to perform multibody computations and gave rise to the multibody theory presented in Ch. 4. The analysis of the flight loads on a commercial transport configuration is presented in Ch. 7 to demonstrate these points.

1.4 Dissertation outline

After this introduction to the motivation, the state-of-the-art, and the present approach, the theories on which this work stands are outlined in the next three chapters. At the beginning of each chapter a historical view of the origins of the treated subject is described as introduction to the chapter and review of its foundations.

Chapter 2 is probably the most dense and difficult of the thesis due to the cross-discipline work presented. The three general theories of solid mechanics, reduction order modelling and geometrically exact nonlinear beams are revised and brought together in a novel description. The link between the principles of elasticity and the intrinsic theory of beams is described; classical

condensation methods in structural dynamics are used to reduce the full models along their main load-paths; the 3D and 1D theories are bridged through approximations in the modal spaces; and expressions are given to recover the full 3D displacement field by mapping the cross-sectional deformations from the 3D LNMs into the 1D solution using the condensation matrices. Chapter 3 begins with a brief description of potential transonic flow, highlights how follower force effects are accounted for and gives some hints into the treatment of other geometric nonlinearities. The integration between aerodynamic forces and the structural modal system is described using a rational function approximation (RFA) of the aerodynamics in the frequency domain; and trim analysis, gust loadings and control inputs are incorporated to finally obtain the full nonlinear aeroelastic system. Since the system is bound to quadratic nonlinearities, linearisation of the equations is given, which provides an efficient framework for aeroelastic studies. Chapter 4 introduces a novel multibody method to couple independent bodies, described with the nonlinear intrinsic formulation, via Lagrange multipliers. An interesting feature is that the constraints are imposed in the velocity level set which might be beneficial for reducing the index of the differential-algebraic-equations (DAEs). But the most significant aspect of the resulting approach is that it is used in combination with the rest of the theory so a geometrically nonlinear multibody system can be constructed out of independent, linear FE models.

From theory to practice, the next three chapters show the numerical exercises carried out in a process that has covered the validation of the main routines, test of the methods in relevant aircraft models, and application to a full airplane configuration. Satisfactory comparisons have been done against MSC Nastran nonlinear and multibody solutions, flutter calculations in the frequency domain, and linear aeroelastic computations. Chapter 5 gives an overview of the software implementation, expanded in more detail in Appendix A. Then a series of exercises involve nonlinear static calculations with simple beam models, dynamic comparisons between models built with shells and their equivalent beams, a double pendulum to test the multibody capabilities, and aeroelastic computations on a HARW including the static equilibrium under angle of attack variations and dynamic perturbations around the equilibrium. Chapter 6 presents a representative simplified aircraft model on which static and dynamic calculations are

carried out, showing the capability of our solvers to accomplish nonlinear solutions in models with discrete mass models. Next, the implemented condensation techniques are tested and the efficiency and accuracy of our NMROMs are remarked when compared to full FE calculations. In the last example of the chapter, a cantilever wing is used to validate the calculation of flutter stabilities when linear analysis is selected, and subsequently the post-flutter nonlinear behaviour is studied showing the appearance of LCOs. In Chapter 7 the computational tools developed are put into practice in a full aircraft configuration built by Airbus for research and collaboration between industry and academia. Static manoeuvres and trimmed flight conditions are computed in linear and nonlinear fashions; dynamic loads are calculated for a range of gust shapes and lengths according to regulations; the open loop response of the aircraft is calculated for a combination of trimmed flight and gust disturbances showing the differences in the wing bending moments between linear and nonlinear approaches; and finally a hinge device on the tip of the wing is released to simulate a foldable wing technology and results regarding the alleviation of wing loads are highlighted.

To conclude the thesis, Chapter 8 summarises the accomplishments of this work and the main research outcomes, and gives recommendations for future work.

A Formulation of Structural Mechanics from an Intrinsic Perspective



If we must have heroes and wars whereinto make them, there is no war so brilliant as a war with the wrong, no hero so fit to be sung, as he who has gained the bloodless VICTORY of truth and mercy.

Isambard K. Brunel

The ignorant man is not free, because what confronts him is an alien world, something outside him and in the offing, on which he depends, without his having made this foreign world for himself and therefore without being at home in it by himself as in something his own. The impulse of curiosity, the pressure for knowledge, from the lowest level up to the highest rung of philosophical insight arises only from the struggle to cancel this situation of unfreedom and to make the world one's own in one's ideas and thought.

Georg W. Friedrich Hegel, Lectures on Aesthetics

The basis for the mechanics of solids is laid in this chapter and represents the foundation for the structural solution that incorporates geometrical nonlinear effects to already existing (linear) models in a non-intrusive manner. Firstly, the physics driving the dynamics of solid bodies are described and will serve to derive, from first principles, the nonlinear beam equations ultimately conforming the aeroelastic methodology in following chapters. A subset of (well-known) structural problems are those of linear elasticity, and reduced order modelling techniques will allow to bridge the gap between them and a nonlinear one-dimensional description through a

mapping of modal information. Therefore the fields of 3D elasticity, reduction order modelling, and geometrically-nonlinear dynamics of slender structures are brought together in a compact formulation: the starting point is an arbitrarily complex FE model for which structural dynamic condensation techniques are employed to reduce the 3D components into nodes along all major load paths (a skeleton-like substructure), in line with current industrial practice. The resulting linear normal modes (LNMs) are evaluated in intrinsic variables (velocities and strains) and used as a basis of a Galerkin-projection of the geometrically-nonlinear intrinsic beam equations in modal space. Thus a second reduction is possible by selecting the size of the modal basis employed in the solution. We generalize previous formulations with a four field description (velocity, momentum, strain, and internal forces) that enables the use of arbitrary reduction techniques and structures with distributed inertia. The projected equations are solved in time domain and the full 3D field can be obtained as a post-processing step.

The workflow of the different theories presented in this chapter is shown in Fig. 2.1, with the starting point as the airframe of an aircraft that we want to approximate mathematically. The fundamentals of solid mechanics are first introduced, which leads to a nonlinear 1D approximation after a set of kinematic assumptions on one hand, and to the equations of linear elasticity on the other; discretisation of the linear equations and applying the appropriate boundary conditions produce the FE models used by organizations in the design process. They are enhanced with experimental tests to account for the missing details in mathematical process. The combination of the 1D description with a condensed version of the FE model along the main load paths allows the construction of a *Nonlinear Modal Reduced Order Model* (NMROM) that combines all the theories in the figure and, as presented in the next chapters, it serves as the core to attain efficient aeroelastic computations. In fact, the construction of an aeroelastic system based on this NMROM is one of the main novelties of this work. In the process, the reduction techniques presented in sec. 2.2 have been explored to capture, as accurate as possible, the characteristics of the full FE model derived from linear elasticity; the solution of the intrinsic equations in modal space, sec. 2.3, and the subsequent derivation of the relation between modal shapes and their couplings in the equations, sec. 2.4, is a key result that allows the seamlessly incorporation of geometrically-nonlinear effects to the (condensed) linear FE models; along the

entire chapter, some light is shed into the direct relation between the 1D description and the main principles of solid mechanics, including the recovery of the 3D displacement field from the reduced nonlinear solution, presented for the first time in sec. 2.4.2. Therefore all subjects in Fig. 2.1 are introduced in this chapter, except for the experimental part of model updating, which could well be part of a future development and validation with real structures.

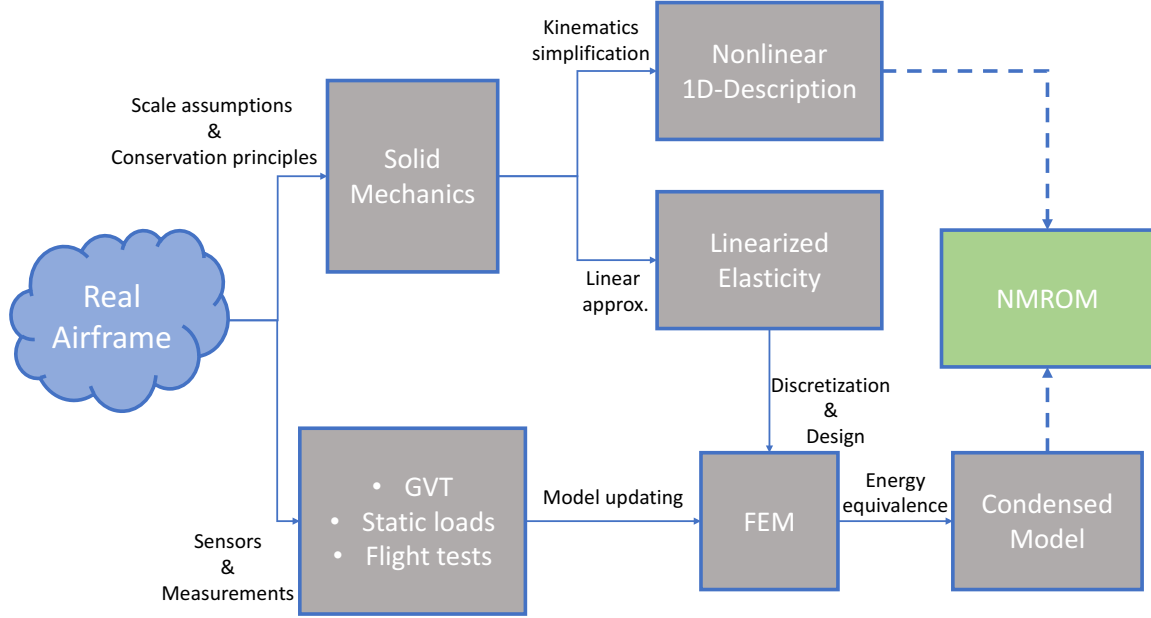


Figure 2.1: Fields and assumptions leading to the construction of a structural NMROM of representative airliners.

2.1 General description of nonlinear solid mechanics

In this section we review the main principles and definitions of solid mechanics and elasticity. They are the backbone of the nonlinear structural theories presented later and, in general, of the mathematical representation of the aircraft structure and its components. Firstly, we show the historical milestones that have led to the development of one of the oldest theories in engineering. Then, we present the main definitions of strains under continuous deformations, their conjugate stresses, the equations of motion from conservation principles, and the discretization of these equations when linear conditions are assumed.

2.1.1 Historical remarks

It may be said that Solid Mechanics has been one of the major branches of physical sciences throughout history, going back to the magnificent constructions of ancient Egyptians, Greeks and later Romans. While they probably used empirical rules for their buildings [223], it is Archimedes (287-212 BC) description of the lever law –with which he claimed he could move the earth given an appropriate support and a sufficiently long bar–, that cast the first stone in the theoretical building of the subject. Most of the knowledge gathered by these civilizations was lost in the Middle-Ages, and it is in The Renaissance that the field gains attention again, thanks to visionaries such as Leonardo da Vinci (1452-1519) who studied the strength of materials experimentally and the equilibrium of structures.

However it is the 17th century that marks the beginning of the theory of elasticity [223] when Galileo Galilei (1564-1642) acknowledged the cantilever beam problem and gave the first accounts of yielding of materials –although he treated the body as inelastic; Robert Hook (1635-1703) announced the linear relation between forces and deformations (Hook’s law) and Edme Mariotte (1620-1684) enunciated the resistance of a beam to flexure originating from extension and compression of its fibres [224].

In the 18th century James Bernoulli (1654-1705) and Leonhard Euler (1707-1783) set the basis for the analysis of beams and vibrations of bars, the latter giving the critical load for columns under buckling and starting the Elastica Theory using his *Calculus of Variations*. Charles-Augustin de Coulomb (1736-1806), through a theory of equilibrium of forces and moments, obtained a correct definition of the *neutral line* in Galilei’s problem, calculated the moment of the elastic forces and was also the first to consider the resistance to torsion and to establish failure criteria [224, 225].

The 19th century was an inflection point with the formulation of the three-dimensional differential equations of elasticity: firstly by Luis M. H. Navier (1785-1836) in 1821 and then by Simeon D. Poisson (1781-1840) and L. Cauchy (1789-1857), who established the theory and gave the definitions of stress and strain known today [225]. It is interesting that, at the time, very different approaches were taken to arrive to the same equations: Navier’s method was based on intermolecular forces and equilibrium of displaced molecules, while Poisson and Cauchy formed

the differential equations by applying the equilibrium of stresses and the relation between the stress and relative displacements [224]. Later George Green (1793-1841) derived the equations from conservation of energy and by application of the principle of Virtual Work, which greatly boosted further developments. Barré de Saint-Venant (1797-1886) accurately linked shear to strain, remarked its importance in the analysis of elastic materials, and gave the definition of principal stresses; he also carried out investigations that led to his principle regarding statically equivalent loads whose effect becomes very small at sufficiently large distances from the loads. George Stokes (1819-1903) conducted a large number of experiments, he observed that Hook's law is a consequence of the experimental fact by which all solids are in a state of isochronous vibration [224], deduced the equations of elasticity from experiments and attempted to move the theory from perfect elastic solids to perfect fluids, passing through plastic solids (thus giving rise to the science of Continuous Mechanics). Following Saint-Venant's theory, Gustav Robert Kirchhoff (1824-1887) sought experiments on the torsion and flexure of steel bars, and established the theory for the analysis of plates under small deformations on two generally-accepted assumptions about the middle-plane of the plate; he then extended the theory to not-small deformations in plates and bars and his results were of great importance in future developments of nonlinear analysis. Unification of free vibrations of solids is due to Alfred Clebsch (1833-1872), who described the oscillations of systems with an infinite number of DoF and formulated general results from particular dynamic problems previously treated by Euler, Poisson, Kirchhoff, and others [223]. William Thomson, Lord Kelvin (1824-1907), brought together the fields of Solid Mechanics and Thermodynamics, conducted investigations onto thermal changes of bodies under deformations and gave the first proof of a strain-energy function; he studied the phenomenon of damped vibrations in elastic solids and their imperfections; all this leading to the development of nonlinear material science.

Most of the developments and theories in structural mechanics were based on linear assumptions until the 20th century when, initially, the focus was on analytical solutions of linear and nonlinear problems; and then the irruption of computers on one hand and techniques such as the finite-element-method on the other allowed for accurate analysis of complex 3D structures with thousands of degrees of freedom. The complexity of nonlinear analysis, however, makes

some of these computations tremendously expensive and it is still today an open research field. Numerous masterminds have led the developments of nonlinear solid mechanics, computational mechanics and the FE Method during this century: Stephen Timoshenko (1878-1972), Richard E. von Mises (1883-1953), Richard Courant (1888-1972), Albert E. Green (1912-1999), Johann H. Argyris (1913-2004), Eric Reissner (1913-1996), Jonathan Turner (1915-1995), Clifford A. Truesdell (1919-2000), Yuan-Cheng Fung (1919-2019), Ray W. Clough (1920-2016), Olgierd Zienkiewicz (1921-2009), John Tinsley Oden (1936-), Thomas J. R. Hughes (1943-) or Juan Carlos Simo (1952-1994), are only a few of the many who have contributed to a subject that branches out into many others, built over the last 400 years, and which will be very briefly described next.

2.1.2 Deformation of bodies and conservation laws

A solid domain Ω represents a body on which deformations occur and are considered as a mapping from an initial to a deformed state. The sort of deformations treated herein are always within the context of continuum mechanics, i.e. any deformation will occur in a continuous manner, from neighbourhood to neighbourhood. Otherwise different fields would have to be introduced, such as fracture and contact mechanics, where new boundary surfaces may appear or disappear, and are out of the scope of this work. Therefore, we shall only deal with smooth deformations, which can be characterised by a continuous mapping, $\hat{\mathbf{x}}$, between the initially undeformed body – denoted as the *reference configuration*, $\mathcal{B}_0$ – and the *current configuration*, $\mathcal{B}_t$, in time, $t \in \mathbb{R}^+$. Points within $\mathcal{B}_0$ are called material points, $\mathbf{X} = (X_1, X_2, X_3) \in \mathcal{B}_0$, and points in the *current configuration* spatial points, $\mathbf{x} = (x_1, x_2, x_3) \in \mathcal{B}_t$, related through $\hat{\mathbf{x}}$ as, $\mathcal{B}_t \subset \mathbb{R}^3 := \hat{\mathbf{x}}(\mathbf{X}, t)$, $\mathbf{x} = \hat{\mathbf{x}}(\mathbf{X}, t) \quad \forall \mathbf{X} \in \mathcal{B}_0$. Fig. 2.2 schematically shows the two configurations and their respective vectors. The metric of the space in which deformations occur is of paramount importance when generalizing the measurement of distortions to any kind of curvilinear reference system. If we take a differential of length dl_0 which becomes dl after deformation, they will be given in the configuration space as $dl_0^2 = d\mathbf{X} \cdot d\mathbf{X} = G_{ij}dX^i dX^j = \left(G_{ij} \frac{\partial X_i}{\partial x_l} \frac{\partial X_j}{\partial x_m} \right) dx^l dx^m$ and $dl^2 = d\mathbf{x} \cdot d\mathbf{x} = g_{ij}dx^i dx^j = \left(g_{ij} \frac{\partial x_i}{\partial X_l} \frac{\partial x_j}{\partial X_m} \right) dX^l dX^m$, with G_{ij} and g_{ij} the covariant metric tensors in the respective coordinates systems, that are contracted by the

contravariant differential elements dX^i and dx^i . The deformation of dl_0 is given as [225],

$$\begin{aligned} dl^2 - dl_0^2 &= \left(g_{ij} \frac{\partial x_i}{\partial X_l} \frac{\partial x_j}{\partial X_m} - G_{lm} \right) dX^l dX^m = 2E_{lm} dX^l dX^m \\ dl^2 - dl_0^2 &= \left(g_{lm} - G_{ij} \frac{\partial X_i}{\partial x_l} \frac{\partial X_j}{\partial x_m} \right) dx^l dx^m = 2e_{lm} dx^l dx^m \end{aligned} \quad (2.1)$$

where E_{ij} is the Green-Lagrangian finite strain tensor, introduced by Green and St. Venant, and e_{ij} is known as the Almansi-Eulerian strain tensor, firstly described by Cauchy for infinitesimal strains, then used by Almansi and Hamel for the general case of finite strains. There are two particularly suitable coordinates choices for solid mechanics: Cartesian coordinates for both initial and deformed configurations, in which case $G_{ij} = g_{ij} = \delta_{ij}$ (δ_{ij} being the Kronecker delta); and a reference frame in the deformed configuration such that the deformed coordinates x_i have the same value as the original coordinates, X_i , in that case $\partial X_i / \partial x_j = \partial x_i / \partial X_j = \delta_{ij}$. These are called convected coordinates, and they allow all the information about strain to be contained in the change of the metric tensor, $E_{ij} = e_{ij} = \frac{1}{2}(g_{ij} - G_{ij})$. For the sake of simplicity we continue the description without distinguishing between covariant and contravariant tensors unless otherwise stated, albeit losing generality.

A description that tracks $\mathbf{X}$ and t as independent variables is called a material or Lagrangian description, whereas the formulation in terms of $\mathbf{x}$ is named spatial or Eulerian description – both are due to Euler though. The former is more commonly used in solid mechanics as it tracks a volume of material when it deforms and moves in space, while the latter is utilised in fluid mechanics descriptions that solve the equations in each differential volume fixed in space. An important difference between the two is how the change in time of a tensor quantity following a material particle is calculated:

$$\dot{\mathcal{F}} = \frac{D\mathcal{F}}{Dt} = \begin{cases} \frac{\partial \mathcal{F}}{\partial t}(\mathbf{X}, t) & \text{[Lagrangian description]} \\ \frac{\partial \mathcal{F}}{\partial t}(\mathbf{x}, t) + v_i \frac{\partial \mathcal{F}}{\partial x_i}(\mathbf{x}, t) & \text{[Eulerian description]} \end{cases} \quad (2.2)$$

referred as the material derivative, $\frac{D(\bullet)}{Dt}$, with $\mathbf{v}$ the velocity of the particle under consideration, it comes from the derivative chain rule and is zero in the material description that naturally follows the particle. It is also possible to transform between reference and current or spatial

representations and vice-versa, achieved by *push-forward* and *pull-back* operations, through $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}^{-1}$ respectively.

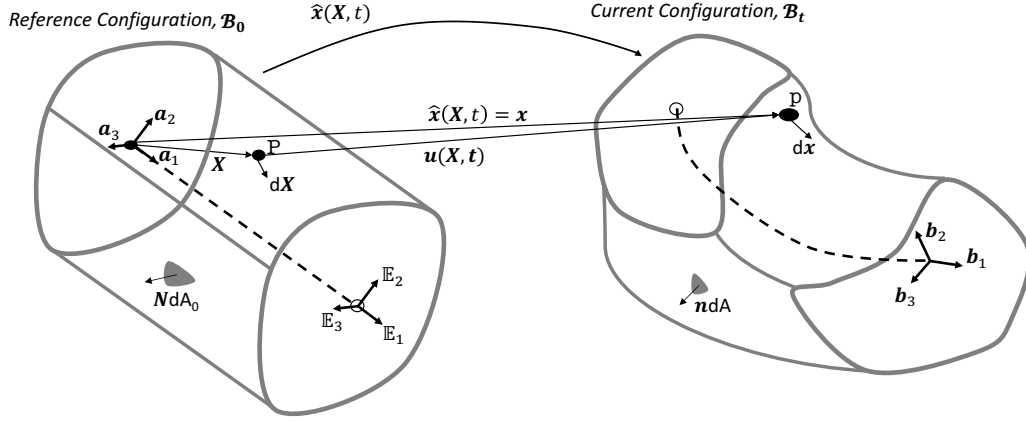


Figure 2.2: Configuration map of deformations

In the analysis of deformations a key quantity is the *deformation gradient tensor*, $\mathbf{F}_x$, which gives the deformation of the current configuration, $\mathcal{B}_t$, relative to the reference configuration $\mathcal{B}_0$, i.e. $\mathbf{F}_x$ at a point P transforms a segment $d\mathbf{X}$ in the reference configuration to the current or deformed configuration, $d\mathbf{x} = \mathbf{F}_x d\mathbf{X}$.

$$\begin{aligned} \mathbf{F}_x &= \frac{\partial \hat{\mathbf{x}}}{\partial X_j} \otimes \mathbf{E}_j = \frac{\partial x_i}{\partial X_j} \mathbf{b}_i \otimes \mathbf{E}_j \\ &= \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{I} + \nabla_X \mathbf{u} \end{aligned} \quad (2.3)$$

and as long as its Jacobian determinant is not singular, $J_F = \det \mathbf{F}_x \neq 0$, the spatial deformation gradient can also be defined, $\mathbf{F}_x^{-1} = \frac{\partial \mathbf{X}}{\partial \mathbf{x}}$; ∇_X is the gradient in material coordinates and $\mathbf{u}$ is the displacement vector in Fig. 2.2, in the material description $\mathbf{u}(\mathbf{X}, t) = \hat{\mathbf{x}}(\mathbf{X}, t) - \mathbf{X}$, and in the spatial description, $\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \hat{\mathbf{x}}^{-1}(\mathbf{x}, t)$. Applying the *Polar Decomposition Theorem* to $\mathbf{F}_x$, it is possible to split the tensor into a proper orthogonal tensor $\mathbf{R}$ and a positive definite tensor,

$$\mathbf{F}_x = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \quad (2.4)$$

$\mathbf{R}$ represents a rigid body rotation, so $\mathbf{U}$ and $\mathbf{V}$ characterize a pure strain in the body and are

called *right* and *left stretch tensors*. Considering again a differential line element,

$$d\mathbf{x} \cdot d\mathbf{x} = (\mathbf{F}_x d\mathbf{X})^\top \mathbf{F}_x d\mathbf{X} = d\mathbf{X}^\top (\mathbf{F}_x^\top \mathbf{F}_x) d\mathbf{X} \quad (2.5a)$$

$$d\mathbf{X} \cdot d\mathbf{X} = (\mathbf{F}_x^{-1} d\mathbf{x})^\top \mathbf{F}_x^{-1} d\mathbf{x} = d\mathbf{x}^\top (\mathbf{F}_x^{-\top} \mathbf{F}_x^{-1}) d\mathbf{x} \quad (2.5b)$$

this allows the definition of the *right Cauchy–Green* deformation tensor $\mathbf{C} = \mathbf{F}_x^\top \mathbf{F}_x = \mathbf{U}^2$ and the *left Cauchy–Green* deformation tensor $\mathbf{B}^{-1} = \mathbf{F}_x^{-\top} \mathbf{F}_x^{-1}$ so that $\mathbf{B} = \mathbf{F}_x \mathbf{F}_x^\top = \mathbf{V}^2$. It follows the relation of these tensors with the *Green–Lagrangian* and *Almansi–Eulerian* tensors, $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$ and $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$. Each of the described second-order tensors $\mathbf{F}_x$, $\mathbf{E}$, $\mathbf{e}$, $\mathbf{C}$, and $\mathbf{B}$, can be used to characterize finite strain deformations, and are useful depending on the problem at hand. Furthermore, the time-rate of these tensors is important in the dynamic description of deformations; the material derivative of the deformation gradient is

$$\dot{\mathbf{F}}_x = \frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{X}} = \frac{\partial \dot{\mathbf{x}}}{\partial \hat{\mathbf{x}}} \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{X}} = \mathbf{L} \cdot \mathbf{F}_x \quad (2.6)$$

where $\mathbf{L}$ is the velocity gradient, $L_{ij} = \frac{\partial v_i}{\partial x_j}$. The symmetric part of $\mathbf{L}$ is the deformation rate tensor $\mathbf{D} = (\mathbf{L} + \mathbf{L}^\top)/2$, and the antisymmetric part of $\mathbf{L}$ the spin tensor, $\mathbf{G} = (\mathbf{L} - \mathbf{L}^\top)/2$. Green's strain rate, for instance, is $\dot{\mathbf{E}} = \mathbf{F}_x^\top \cdot \mathbf{D} \cdot \mathbf{F}_x$, and the rate of other strain measures can be found in textbooks of nonlinear solid mechanics [225]. Having described the relation between line elements in the reference and current configurations, changes in area and volume need to be quantified in order to find out the equations of motion of the deforming body. Let dA and $\mathbf{n}$ be a differential of area and its normal in the current configuration, and dA_0 and $\mathbf{N}$ their counterparts in the reference configuration, as shown in Fig. 2.2. $\mathbf{F}_x$ and its determinant, J_F , also characterises changes of area between configurations, which are found by applying the cross-product on line elements; and similarly for changes in volumes, dV and dV_0 , using the triple product:

$$\mathbf{n} dA = d\mathbf{x}_1 \times d\mathbf{x}_2 = J_F \mathbf{F}_x^{-\top} d\mathbf{X}_1 \times d\mathbf{X}_2 = J_F \mathbf{F}_x^{-\top} \mathbf{N} dA_0 \quad (2.7a)$$

$$dV = d\mathbf{x}_1 \cdot (d\mathbf{x}_2 \times d\mathbf{x}_3) = J_F d\mathbf{X}_1 \cdot (d\mathbf{X}_2 \times d\mathbf{X}_3) = J_F dV_0 \quad (2.7b)$$

In this way we have established the main definitions of finite deformations, with strain tensors in both reference and current configurations, and the correspondent changes in area and volume. Next it is necessary to define the stress tensors, that follow for each particular strain as to comply with fundamental conservation laws. An example of this is the invariance of the internal power, $\mathbb{P}$, computed as the double inner product of the stress conjugate with its correspondent rate of strain,

$$\begin{aligned}\mathbb{P} &= \int_{\mathcal{B}_t} \rho \dot{\mathcal{W}} dV = \int_{\mathcal{B}_t} \boldsymbol{\sigma} : \dot{\mathbf{D}} dV \\ &= \int_{\mathcal{B}_0} \mathbf{P} : \dot{\mathbf{F}}_x dV_0 = \int_{\mathcal{B}_0} (J_F \boldsymbol{\sigma} \mathbf{F}_x^{-T}) : \dot{\mathbf{F}}_x dV_0 \\ &= \int_{\mathcal{B}_0} \mathbf{S} : \dot{\mathbf{E}} dV_0 = \int_{\mathcal{B}_0} (\mathbf{F}_x^{-1} \mathbf{P}) : \dot{\mathbf{E}} dV_0 = \int_{\mathcal{B}_0} (J_F \mathbf{F}_x^{-1} \boldsymbol{\sigma} \mathbf{F}_x^{-T}) : \dot{\mathbf{E}} dV_0\end{aligned}\quad (2.8)$$

$\boldsymbol{\sigma}$ is the Cauchy stress tensor, also known as true stress tensor and conjugate of the deformation rate tensor, $\mathbf{D}$; $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, conjugate of the deformation gradient rate, $\dot{\mathbf{F}}_x$; and $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor, conjugate of the Green-Lagrangian strain rate, $\dot{\mathbf{E}}$. The stress tensors has the significance of yielding the traction force, $\boldsymbol{\mathcal{T}}$, acting on each differential element of area, $d\mathbf{A}$, in either the reference or current configurations. The Cauchy stress tensor is an Eulerian tensor that gives the traction force in the current configuration, $d\boldsymbol{\mathcal{T}}^t$, when multiplied by a differential of area in the current configuration, while the first Piola-Kirchhoff stress tensor is a two-point tensor that also gives the traction force in the current configuration, though when multiplied by a differential of area in the reference configuration:

$$d\boldsymbol{\mathcal{T}}^t = \boldsymbol{\sigma} \cdot \mathbf{n} dA = \mathbf{P} \cdot \mathbf{N} dA_0 \quad (2.9)$$

the same force in the reference configuration is $d\boldsymbol{\mathcal{T}}$, given by the second Piola-Kirchhoff tensor,

$$d\boldsymbol{\mathcal{T}} = \mathbf{F}_x^{-1} d\boldsymbol{\mathcal{T}}^t = \mathbf{S} \cdot \mathbf{N} dA_0 \quad (2.10)$$

thus $\mathbf{S}$ is a Lagrangian tensor that gives the traction force in the reference configuration when multiplied by a differential of area in the reference configuration too.

An important concept for the theory of mechanics is that of *frame indifference* or *objectivity*,

developed by Truesdell and Noll [226], and by which a quantity is objective if it is invariant under changes of frame of reference. For two motions that only differ by a rigid body rotation, $\mathbf{Q}$, and rigid body translation, $\mathbf{c}_0$, such that $\mathbf{x}_q(\mathbf{X}, t) = \mathbf{Q}\mathbf{x}(\mathbf{X}, t) + \mathbf{c}_0(t)$, an objective vector quantity $\mathbf{y}$ or a second-order tensor $\mathbf{Y}$ transforms as,

$$\mathbf{y}_q = \mathbf{Q}\mathbf{y} \quad (2.11a)$$

$$\mathbf{Y}_q = \mathbf{Q}\mathbf{Y}\mathbf{Q}^\top \quad (2.11b)$$

for instance, the material properties of a body must be objective. While a line element $d\mathbf{x}$ is objective, $\mathbf{F}_x$ is not; Eulerian strain tensors are objective but Lagrangian strain tensors are not, and the same is true for the stresses, $\boldsymbol{\sigma}$ is objective but not $\mathbf{S}$ or $\mathbf{P}$.

Equations of motion

Within the principles of Newtonian physics, the equations of motion are derived from the basic conservation principles of mass, linear momentum, and angular momentum (which at the same time rest on the assumption of continuity, valid over length scales of 10^{-6} to 10^3 m and times scales of 10^{-6} to 10^6 s [227], as atoms and molecules are separated by distances of the order of 10^{-10} m. Most engineering applications occur within these scales, though heterogeneous and composite materials, with microstructures that fall within those distances, potentially require modification of the modelling equations). The *Continuity Equation* or conservation of the total mass $\mathfrak{m}$ is written in an Eulerian framework as,

$$\begin{aligned} \dot{\mathfrak{m}} &= \frac{D}{Dt} \int_{\mathcal{B}_t} \rho dV = \int_{\mathcal{B}_t} \frac{\partial \rho}{\partial t} dV + \int_{\partial \mathcal{B}_t} \rho \mathbf{v} \cdot \mathbf{n} da = \int_{\mathcal{B}_t} \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} \right) dV = 0 \\ &\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0 \end{aligned} \quad (2.12)$$

where ρ and $\mathbf{v}$ are the density and velocity fields of the solid respectively. Conservation of linear momentum states that the time rate of linear momentum $\mathcal{P}$ equals the sum of external

body forces, $\mathbf{b}_f$, and traction forces over the body surface, $\mathcal{T}$:

$$\begin{aligned}
\dot{\mathcal{P}} &= \frac{D}{Dt} \int_{\mathcal{B}_t} \rho \mathbf{v} dV = \int_{\mathcal{B}_t} \mathbf{b}_f dV + \int_{\partial \mathcal{B}_t} \mathcal{T} d\mathbf{a} = \int_{\mathcal{B}_t} \mathbf{b}_f dV + \int_{\partial \mathcal{B}_t} \boldsymbol{\sigma} \cdot \mathbf{n} d\mathbf{a} = \int_{\mathcal{B}_t} \left(\mathbf{b}_f + \frac{\partial \sigma_{ij}}{\partial x_j} \right) dV \\
&\Rightarrow \int_{\mathcal{B}_t} \left(\frac{\partial \rho \mathbf{v}}{\partial t} + \frac{\partial \rho \mathbf{v} v_j}{\partial x_j} \right) dV = \int_{\mathcal{B}_t} \left(\rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) + v_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} \right) \right) dV \quad (2.13) \\
&\Rightarrow \rho \left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = b_{fi} + \frac{\partial \sigma_{ij}}{\partial x_j} \Rightarrow \rho \frac{D\mathbf{v}}{Dt} = \mathbf{b}_f + \nabla \cdot \boldsymbol{\sigma}
\end{aligned}$$

The variation of angular momentum equals the moment exerted by the external forces, $\dot{\mathcal{H}} = \int_{\mathcal{B}_t} \mathbf{x} \times \mathbf{b}_f dV + \int_{\partial \mathcal{B}_t} \mathbf{x} \times \mathcal{T} d\mathbf{a}$, which can be proved to be equivalent to the symmetry of the stress tensor [227, Ch. 5], $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\top$. Conservation laws can also be cast in terms of the density and body forces in the reference configuration using the first Piola-Kirchhoff stress tensor, so that equilibrium of linear momentum is formulated as,

$$\rho_0 \ddot{\mathbf{u}} = \mathbf{b}_{f_0} + \nabla_X \cdot \mathbf{P} \quad (2.14)$$

while $\mathbf{P}$ is not symmetric, balance of angular momentum reduces to $\mathbf{F}\mathbf{P}^\top = \mathbf{P}\mathbf{F}^\top$.

Constitutive laws of materials

Four equations have been derived, Eqs. (2.12)-(2.13), for ten unknowns, density, the velocity or displacement field, and the six components of the symmetric stress tensor. Therefore another set of equations have to be added to close the problem. These are the constitutive laws, which describe the mechanical properties of a material relating the stress state to the correspondent internal strains. Principles of thermodynamics are necessary to define the conservation principles governing the state of the material and, in general, for hyperelastic materials a strain energy function, $\mathcal{E}_s$, can be defined such as the stress-strain relationship is,

$$\boldsymbol{\sigma} = \frac{1}{J_F} \frac{\partial \mathcal{E}_s}{\partial \mathbf{e}} \quad ; \quad \mathbf{S} = \frac{\partial \mathcal{E}_s}{\partial \mathbf{E}} \quad (2.15)$$

Fundamentally constitutive equations seek to describe the macroscopic behaviour of solids that results from their internal -and microscopic- constitution. This leads to experiments and phenomenological models employed, rather than first principles, to describe functions such as

$\mathcal{E}_s$.

2.1.3 Linear elasticity

For many applications a good approximation of the equations of solid mechanics is obtained by assuming a linear change of both the mechanical properties and the change between the reference and current configurations. The resulting equations conform the linearised theory of elasticity. The first assumption is that of linear material properties which give rise to the generalization of Hooke's law,

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{D}}\mathbf{e} \quad (2.16)$$

with the coefficients of $\boldsymbol{\mathcal{D}}$ usually obtained from experiments too. It is worth noting that this assumption is valid for most aircraft operating in normal conditions. Next a linearisation is carried out of the strain tensors defined in Eq. (2.1) and written in Cartesian coordinates as follows,

$$E_{ij} = \frac{1}{2} \left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} + \frac{\partial u_k}{\partial X_i} \frac{\partial u_k}{\partial X_j} \right) \approx \frac{1}{2} \left(\frac{\partial u_j}{\partial X_i} + \frac{\partial u_i}{\partial X_j} \right) \quad (2.17)$$

$$e_{ij} = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \approx \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \quad (2.18)$$

In this case $\mathbf{E}$ and $\mathbf{e}$ are equivalent. The velocity $\mathbf{v}$ and its material derivative (the acceleration) are just,

$$\mathbf{v} \approx \frac{\partial \mathbf{u}}{\partial t} = \dot{\mathbf{u}} \quad \frac{D\mathbf{v}}{Dt} \approx \frac{\partial \mathbf{v}}{\partial t} = \ddot{\mathbf{u}} \quad (2.19)$$

and (2.13) becomes

$$\rho \ddot{\mathbf{u}} = \mathbf{b}_f + \nabla \cdot \boldsymbol{\sigma} \quad (2.20)$$

then Eqs. (2.16)-(2.20) are combined which gives a linear set of equations on $\mathbf{u}$. Discretisation of these equations is required to find a solution on realistic geometries, which is briefly described next.

The linear finite-element method

The finite-element-method is a numerical method developed during the the 1950s-1970s to approximate the solution of partial differential equations over complex domains. Although solving nonlinear structural dynamics with the equations described above, and within broader multi-physics problems [149] is among its many applications, in this work it is used as the linear mathematical model that approximates the characteristics of real commercial-aircraft as lifelike and accurate as possible. We then seek to enhance this model with geometrically nonlinear effects employing the methods presented in the next sections. Thus a linear Finite Element Model of an aircraft structure is built combining experimental data from Ground Vibration Tests (GVTs), static load measurements and flight tests, and the linear equations of structural dynamics obtained after linearization of the solid mechanics principles presented above (the entire procedure can be seen in Fig. 2.1). The process of modifying a mathematical model with tests and experiments is often necessary as predictions do not entirely capture the response of the actual structure. Model updating techniques of FE models have been developed [228, 229] with the aim of closing the gap between the mathematical model and observations, and it is an asset of the current methodology being able to take a linear, modified FE model and add to it the nonlinear effects from finite rotations or geometrical stiffening.

The domain in the FE method is divided into sub-regions called *elements* and approximated solutions are found within each element on the nodes of the element. For an element of n_e nodes, shape functions $\mathcal{N}(x, y, z)$ are defined such as the displacement field in an element e is approximated as $\hat{\mathbf{u}}^e = [\mathcal{N}_1^\top; \dots; \mathcal{N}_{n_e}^\top]^\top [\mathbf{u}_1^e; \dots; \mathbf{u}_{n_e}^e] \Rightarrow \hat{\mathbf{u}}^e = \mathcal{N}^e(x, y, z) \mathbf{u}^e(t)$ with $\mathbf{u}^e(t)$ the combined vector of displacements at the nodes. Similarly the strains within the element are defined as $\hat{\mathbf{e}} = \mathcal{N}'^e \mathbf{u}^e$ with $\mathcal{N}'$ the derivative of the shape functions as to comply with linearisation of the strain tensors Eqs. (2.18) The stress in the element is found by using the linear relation in Eq. (2.16), $\hat{\boldsymbol{\sigma}}^e = \mathcal{D} \hat{\mathbf{e}}^e = \mathcal{D} \mathcal{N}'^e \mathbf{u}^e$. The linearized equations of motion, Eqs. (2.20), are written in its weak form using the conservation principle of potential energy, $\mathcal{E}_p = \mathbf{e}^\top \boldsymbol{\sigma}$, kinetic energy, $\mathcal{E}_k = \frac{1}{2} \rho \dot{\mathbf{u}}^2$, and the work of external forces, $\mathcal{W}_f = \mathbf{u}^\top \bar{\mathbf{F}}$, including all body forces, surface tractions and point loads. Applying virtual displacements, a stationary point occurs in the functional $\Pi = \mathcal{E}_p + \mathcal{E}_k - \mathcal{W}_f$ for the actual trajectory of the body, characterized

by a total of N_e elements:

$$\delta\Pi = \sum_{N_e=1}^{N_e} \left(\delta\mathbf{u}^{e\top} \int_{\Omega^e} \mathcal{N}'^\top \mathcal{D} \mathcal{N}' \mathbf{u}^e + \delta\mathbf{u}^{e\top} \int_{\Omega^e} \mathcal{N}^\top \rho \mathcal{N} \ddot{\mathbf{u}}^e - \delta\mathbf{u}^{e\top} \int_{\Omega^e} \mathcal{N}^\top \bar{\mathbf{F}} \right) = \mathbf{0} \quad (2.21)$$

the first integral represents the stiffness terms, the second one the inertia terms, and the last one the equivalent combined forces on each element. When the finite-elements are assembled, the final set of equations are,

$$\mathbf{M} \ddot{\mathbf{u}}_n + \mathbf{K} \mathbf{u}_n = \mathbf{F} \quad (2.22)$$

with $\mathbf{M}$ and $\mathbf{K}$ the global mass and stiffness matrices, respectively; $\mathbf{F}$ the vector of equivalent nodal forces/moments; and $\mathbf{u}_n$ the vector of nodal displacements/linear rotations.

In the next section we introduce dynamic condensation techniques that significantly decrease the size of the problem and allow to reduce the models into the aircraft major load paths on which the geometrically nonlinear 1-dimensional equations will be applied to finally form the NMROM that combines all the theories in Fig. 2.1.

2.2 Dynamic condensation of linear FE models

Model order reduction techniques aim to increase computational efficiency by reducing the size of the problem while maintaining its fundamental characteristics. In the realm of industrial applications, ROMs are very important, even when the majority of the analysis is linear, due to the need for fast simulations and the very large models that can surpass the hundreds of thousands of DoF. Also they have been useful in experimental modal analysis, for measurement devices can only be put in a limited number of places and the complete response has to be reconstructed for validation purposes; structural vibrations and buckling, structural optimization, or damage detection are some of the applications of these techniques [67]. In aircraft loads analysis, trim and stability studies, it is common practice condensing the (linear) full-FE models into the major load paths of the aircraft, to reduce the total number of DoF to a few hundreds. Herein we not only use a condensation approach to reduce the size of the model, but also to represent a structure that can be modelled as a one-dimensional domain –making use of

the slenderness of high AR aircraft, where the span of wetted-surfaces and the fuselage length take predominance over the other dimensions. Therefore, consider again the solid domain Ω in Sec. 2.1.2, defining the 3D structure, and a 1D sub-domain in which the deformable curve Γ defines an internal load path. The points conforming Γ either belong to the 3D structure or are a weighted average of neighboring points. Fig. 2.3 illustrates a typical discretisation of both 3D and 1D domains in a simplified aircraft structure. In the aerodynamic surfaces, the interpolation elements in blue relate the active nodes in the load path to the corresponding cross-sectional nodes of the 3D structure. On the other hand, the fuselage is modelled as a rigid beam, and its nodes are part of both the full and reduced models. The nodes of the condensed model, in red, represent the discretization of Γ , where the geometrically nonlinear dynamics will be described. After a finite-element discretization, eq. (2.22) yields the linear

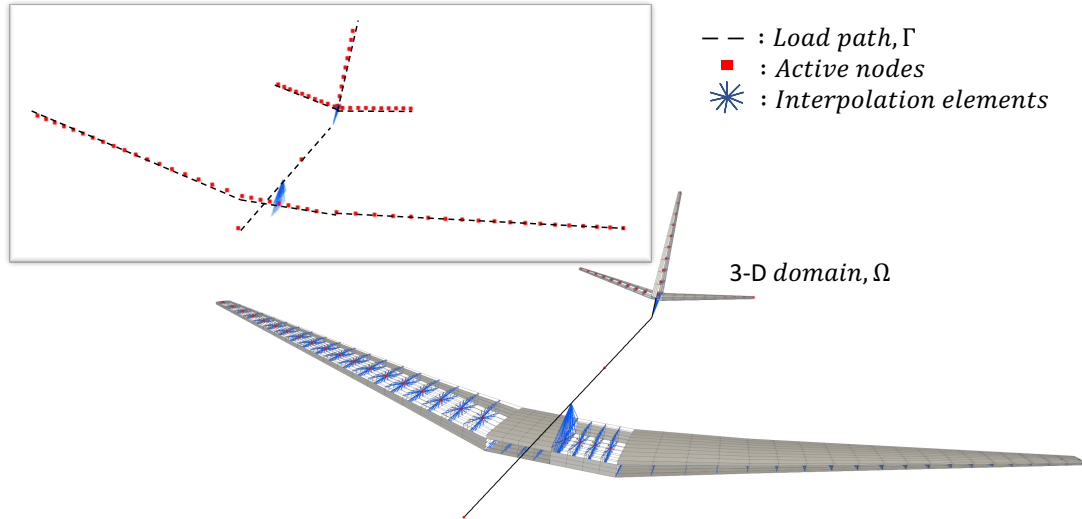


Figure 2.3: Structural condensation, aircraft 3D and 1D domains

structural dynamics equations with respect to a common inertial reference frame. In order to obtain the condensed model, the degrees of freedom of the full model are divided into those on active (or master) nodes, located along the load path Γ , and those of the omitted (or slave) nodes. The displacement vector can therefore be reordered as, $\mathbf{u}_n = (\mathbf{u}_a^\top \mathbf{u}_o^\top)^\top$, and similarly for the LNMs of the unloaded structure $\Phi = [\Phi_a^\top \Phi_o^\top]^\top$, so that the eigenvalue problem of $\mathbf{K}$

and $\mathbf{M}$ is arranged as,

$$\left(\begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ao} \\ \mathbf{K}_{oa} & \mathbf{K}_{oo} \end{bmatrix} - \omega^2 \begin{bmatrix} \mathbf{M}_{aa} & \mathbf{M}_{ao} \\ \mathbf{M}_{oa} & \mathbf{M}_{oo} \end{bmatrix} \right) \begin{pmatrix} \Phi_a \\ \Phi_o \end{pmatrix} = 0 \quad (2.23)$$

A linear dependency is now assumed between the omitted and the active degrees of freedom,

$$\Phi_o = \mathbf{T}_{oa} \Phi_a \quad (2.24)$$

with $\mathbf{T}_{oa} \in \mathbb{R}^{o \times a}$ the transformation matrix between both sets. The quality of the transformation is determined by its accuracy in capturing a subset of interest of the full-system LNMs, which is linked to the spatial sampling that Φ_a introduces on the full domain. In general, the condensation is dependent on the frequencies and forms a nonlinear eigenvalue problem where each LNM, with natural frequency, ω_j , has one transformation matrix,

$$\mathbf{T}_{oa}(\omega_j) = (\mathbf{K}_{oo} - \omega_j^2 \mathbf{M}_{oo})^{-1} (\mathbf{K}_{oa} - \omega_j^2 \mathbf{M}_{oa}) \approx -(\mathbf{K}_{oo}^{-1} + \omega_j^2 \mathbf{K}_{oo}^{-1} \mathbf{M}_{oo} \mathbf{K}_{oo}^{-1}) (\mathbf{K}_{oa} - \omega_j^2 \mathbf{M}_{oa}) \quad (2.25)$$

This is the so-called exact-condensation matrix, where we have also introduced Kidder's mode expansion [67] (right-hand-side expression in (2.25)) to avoid repeated calculation of the inverse in the first term. The first-order approximation of this equation is attained by letting $\omega_j = 0$, thereby removing inertia effects. This results in a static condensation, proposed by Guyan and Irons in 1965 [78], still one of the most popular condensation methods today, also known as Guyan reduction. Note that when the mass model consists only of lumped masses on the active degrees of freedom, $\mathbf{M}_{oo} = \mathbf{M}_{oa} = \mathbf{0}$, Guyan reduction is the exact condensation. It can be proved [67] that this reduction technique is the zeroth order approximation to the power series expansion of the dynamic condensation. A further approximation including the effect of distributed inertia is obtained by selecting a single natural frequency in (2.25) for all LNMs such that the terms $\mathbf{M}_{oo}$ and $\mathbf{M}_{oa}$ appear in the approximation and the inertia effects are included at the selected frequency. Therefore that one frequency is exactly approximated, while accuracy is gradually lost when moving away from it. After calculation of $\mathbf{T}_{oa}$, the transformation from the active set and the full model is defined as $\mathbf{T} = [\mathbf{I}_a \ \mathbf{T}_{oa}^\top]^\top$, with $\mathbf{I}_a$ the identity matrix of

dimension a . The condensed mass and stiffness matrices are obtained by equating the kinetic energy, $\mathcal{E}_k$ and the potential energy, $\mathcal{E}_p$ in the linear reduced and complete systems; if external loads are applied to the omitted nodes, equating virtual work gives the equivalent loads in the condensed model:

$$\begin{aligned}\mathcal{E}_p &= \frac{1}{2} \mathbf{u}_n^\top \mathbf{K} \mathbf{u}_n \cong \frac{1}{2} \mathbf{u}_a^\top \mathbf{T}^\top \mathbf{K} \mathbf{T} \mathbf{u}_a = \frac{1}{2} \mathbf{u}_a^\top \mathbf{K}_a \mathbf{u}_a \\ \mathcal{E}_k &= \frac{1}{2} \dot{\mathbf{u}}_n^\top \mathbf{M} \dot{\mathbf{u}}_n \cong \frac{1}{2} \dot{\mathbf{u}}_a^\top \mathbf{T}^\top \mathbf{M} \mathbf{T} \dot{\mathbf{u}}_a = \frac{1}{2} \dot{\mathbf{u}}_a^\top \mathbf{M}_a \dot{\mathbf{u}}_a \\ \mathcal{W}_f &= \delta \mathbf{u}_n^\top \mathbf{F} \cong \delta \mathbf{u}_a^\top \mathbf{T}^\top \mathbf{F} = \delta \mathbf{u}_a^\top \mathbf{F}_a\end{aligned}\tag{2.26}$$

so that condensed stiffness and mass matrix are obtained as $\mathbf{K}_a = \mathbf{T}^\top \mathbf{K} \mathbf{T}$, $\mathbf{M}_a = \mathbf{T}^\top \mathbf{M} \mathbf{T}$, and the external forces $\mathbf{F}_a = \mathbf{T}^\top \mathbf{F}$. The LNMs in the active set are then $\mathbf{K}_a \Phi_{aa} = \mathbf{M}_a \Lambda_a \Phi_{aa}$, with Λ_a the diagonal matrix of squared natural frequencies (in what follows the approximation between the mode shapes of the full FE at the condensed points, Φ_a , and the shapes directly obtained from the condensed matrices is assumed equivalent, $\Phi_{aa} \approx \Phi_a$, and Φ_{aa} is just referred as Φ_a for simplicity). Let us look into more accurate approaches than Guyan and first order dynamic condensation, such as iterative methods which refine the reduced matrices at every step. In particular, the iterative scheme shown in Ref. [67] based on Kidder expansion is considered here with its computational implementation presented in Alg. (1). The first step is to employ Guyan condensation to obtain $\mathbf{T}_{oa}^{(0)}$, $\mathbf{K}_a^{(0)}$ and $\mathbf{M}_a^{(0)}$. Let $\Phi_a^{(0)}$ be the matrix that includes all the modes on the resulting condensed structure, obtained from the function ϕ that solves the eigenvalue problem of $\mathbf{K}_a$ and $\mathbf{M}_a$. For a prescribed number of iterations, n_i , the process starts with the use of Kidder expansion in Eq. (2.25) to calculate the mode shapes at the omitted nodes, $\Phi_{oj}^{(i)}$, where the subindex j refers to columns in the modal matrix and therefore iterates up to the full number of modes in the reduced system, n_m ; the transformation matrix, $\mathbf{T}_{oa}^{(i)}$, is recalculated next, using the relation in Eq. (2.24); finally $\mathbf{T}^{(i)}$ is assembled and the new $\mathbf{K}_a^{(i)}$, $\mathbf{M}_a^{(i)}$ and $\Phi_a^{(i)}$ are found. The iteration process is repeated until the maximum number of iterations is reached or when the convergence criteria with a prescribed ϵ is met. The level of fidelity preserved in the condensation will depend on how the master nodes are selected and on the ratio between master and omitted nodes.

Algorithm 1: Finite Element Linear Condensation

input : Full FE matrices, $\mathbf{K}$ and $\mathbf{M}$; master points, $\mathbf{p}$;
algorithm to calculate the eigenvalue problem, ϕ ; split function, $\mathbf{S}$
output: Condensed FE matrices, $\mathbf{K}_a$ and $\mathbf{M}_a$
begin

$\mathbf{K}_{aa}, \mathbf{K}_{ao}, \mathbf{K}_{oa}, \mathbf{K}_{oo} \leftarrow \mathbf{S}(\mathbf{K}, \mathbf{p})$
 $\mathbf{M}_{aa}, \mathbf{M}_{ao}, \mathbf{M}_{oa}, \mathbf{M}_{oo} \leftarrow \mathbf{S}(\mathbf{M}, \mathbf{p})$
 $\mathbf{K}_a^{(0)} \leftarrow \mathbf{K}_{aa} - \mathbf{K}_{ao} \mathbf{K}_{oo}^{-1} \mathbf{K}_{oa}$
 $\mathbf{M}_a^{(0)} \leftarrow \mathbf{M}_{aa} + \mathbf{K}_{ao} \mathbf{K}_{oo}^{-1} \mathbf{M}_{oo} \mathbf{K}_{oo}^{-1} \mathbf{K}_{oa}$
 $\omega^{(0)}, \Phi_a^{(0)} \leftarrow \phi \left(\mathbf{K}_a^{(0)}, \mathbf{M}_a^{(0)} \right)$
 $\mathbf{T}_{oa}(\omega) \leftarrow -(\mathbf{K}_{oo}^{-1} + \omega^2 \mathbf{K}_{oo}^{-1} \mathbf{M}_{oo} \mathbf{K}_{oo}^{-1})(\mathbf{K}_{oa} - \omega^2 \mathbf{M}_{oa})$
 $n_m \leftarrow 6 \times \text{length}(\mathbf{p})$
for $i \in [1, \dots, n_i]$ **do**

for $j \in [1, \dots, n_m]$ **do**

$\Phi_{oj}^{(i)} \leftarrow \mathbf{T}_{oa}(\omega_j^{(i-1)}) \Phi_{aj}^{(i-1)}$

 $\mathbf{T}_{oa}^{(i)} \leftarrow \Phi_o^{(i)} \left(\Phi_a^{(i-1)} \right)^{-1}$
 $\mathbf{T}^{(i)} \leftarrow [\mathbf{I}_a \mathbf{T}_{oa}^{(i)\top}]^\top$
 $\mathbf{K}_a^{(i)} \leftarrow \mathbf{T}^{(i)\top} \mathbf{K} \mathbf{T}^{(i)}$
 $\mathbf{M}_a^{(i)} \leftarrow \mathbf{T}^{(i)\top} \mathbf{M} \mathbf{T}^{(i)}$
 $\omega^{(i)}, \Phi_a^{(i)} \leftarrow \phi \left(\mathbf{K}_a^{(i)}, \mathbf{M}_a^{(i)} \right)$
if $\|\Phi_a^{(i)} - \Phi_a^{(i-1)}\| < \epsilon$ **and** $\|\omega^{(i)} - \omega^{(i-1)}\| < \epsilon$ **then**

break

2.3 Geometrically nonlinear mechanics of slender structures

Since the times of Galileo, engineers and physicists have approached problems involving slender structures by constructing one dimensional or beam/rod models. The equations of the 3D structure are reduced to equations in terms of one dimension, the beam axis. A high number of methodologies have been developed over time in order to capture the dynamics of beams with its limitations and assumptions. Traditionally, theories were formulated using ad-hoc

assumptions to define the kinematics of the problem, yielding a particular displacement field. The most popular among classical beam models is the Euler-Bernoulli beam. It has been used extensively and it rests on three assumptions: a) The cross-sectional plane is infinitely rigid. b) The cross-section remains plane after the deformation. c) The cross-section remains normal to the deformed axis of the beam. The theory yields good results for bending problems of isotropic beams. Saint-Venant analysed the problem of torsion and changed the second assumption of the Euler-Bernoulli beam to account for torsion as follows: each section undergoes a rigid body rotation and the cross section warps proportionally to the rate of twist. A more general theory of beams was proposed by Timoshenko, obtained by relaxing the third condition of the Euler-Bernoulli beam, so that the cross-section does not necessarily remain normal to the beam axis and, as consequence, shear deformation appears in the model. On the nonlinear description of beams, Eugene Cosserat (1866-1931) and his brother François, continuing on Kirchhoff's work, proposed the idea of modelling a surface or a rod as points with their own local coordinate systems, a triad of vectors, also called *directors*. When the 1D or 2D bodies undergo a continuous deformation, directors follow each point to the new location. The present work follows Timoshenko and Cosserat assumptions with the novelty that an initial 3D FE model is already available and its characteristics are directly fed into the rod properties.

Therefore an aircraft structure as the one shown in Fig. 2.3 is idealised into a linear finite-element model, then a reduced order model is derived which captures its main characteristics along the aircraft load paths – where aerodynamics and mass forces are to be applied in the aeroelastic solution, and now we set out to describe the nonlinear dynamics of the reduced model using a nonlinear geometrically-exact composite beam. The formulation is based on the works by Hodges [30, 31, 216] and Palacios [52, 63, 230]. The intrinsic beam theory is a fully Lagrangian mixed formulation that uses velocity, strain, sectional forces and their respective rotational counterparts as the main variables. A full description of the theory can be found in the monograph by its initial precursor [231]. Herein the relation between the intrinsic theory and the general principles of elasticity will be presented, seeking to shed some light onto the boundaries of the one-dimensional theory and possibly expand on them. The derivation is as follows: The deformation of slender bodies is firstly defined, followed by a set of kinematic

relations to obtain the appropriate strain and velocity fields; the theory in Sec. 2.1 is then employed to investigate the nature of deformations and derive the virtual work of internal forces from first principles of elasticity; the intrinsic equations of motion are finally written in strong form after application of Hamilton's principle of least action, and solved in modal space using a Galerkin projection, which allows finding a solution to the equations without directly specifying the cross-sectional properties as described in the next section, Sec. 2.4.

2.3.1 Kinematics of aircraft main load-paths

Consider a deformable curve in space, $\Gamma(t)$, conveniently linked to the reduced space defined in Sec. 2.2, that represents the assembled central points of material cross-sections in a slender body. Three variables will unequivocally define the state of such slender body, namely the position of Γ , $\mathbf{r}(s, t) \in \mathbb{R}^3$, its local orientation defined by the rotation tensor, $\mathbf{R} \in \mathbb{R}^3 \otimes \mathbb{R}^3$, and the warping field, $\mathbf{w} \in \mathbb{R}^3$, which acts as the cross-sectional deformation after rotations have been removed. Three reference systems are important in defining those quantities, a Cartesian inertial system, $\mathbf{a}$, a local reference system in the initial configuration, $\mathbf{b}_0$, and the local reference system in the current configuration, $\mathbf{b}$, as illustrated in Fig. 2.4. They are related by the rotational operator as,

$$\mathbf{b}_i = \mathbf{R}^{ab} \mathbf{a}_i \mid \mathbf{R}^{ab} = \delta^{ij} \mathbf{b}_i \otimes \mathbf{a}_j \quad (2.27)$$

and similarly there is a $\mathbf{R}^{b_0b}$ for the local coordinate systems in the undeformed and deformed configurations, from now on simply called $\mathbf{R}^1$. From Euler's theorem, any rotation matrix $\mathbf{R}$ represents a rotation of an angle θ about an axis $\mathbf{n}_r$, so that a vector $\boldsymbol{\theta} = \theta \mathbf{n}_r$ is formed, and the relation between $\mathbf{R}$ and $\boldsymbol{\theta}$ is attained through the exponential mapping:

$$\mathbf{R}|_{\boldsymbol{\theta}} = \exp(\boldsymbol{\theta}) = \mathbf{I} + \tilde{\boldsymbol{\theta}} + \frac{1}{2!} \tilde{\boldsymbol{\theta}}^2 + \dots + \frac{1}{n!} \tilde{\boldsymbol{\theta}}^n \quad (2.28)$$

¹Some of the challenges in tracking finite rotations relate to the parametrization and integration of the rotation matrices $\mathbf{R}$, which are part of the special orthogonal group, $SO(3)$, a non-commutative Lie-group defined as: $SO(3) := \{\mathbf{R} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \mathbf{R}^\top \mathbf{R} = \mathbf{I}, \det \mathbf{R} = +1\}$

where $\tilde{\boldsymbol{\theta}}$ is the skew-symmetric second order tensor or matrix, such that $\tilde{\boldsymbol{\theta}}\mathbf{y} = \boldsymbol{\theta} \times \mathbf{y} \quad \forall \mathbf{y} \in \mathbb{R}^3$. With this we have the main tools to describe deformations in the slender body, such that a point in the structure is defined in the inertial system as,

$$\hat{\mathbf{r}}_a(s, t) = \mathbf{r}_a(s, t) + \mathbf{R}^{ab}(s, t)(\boldsymbol{\xi} + \mathbf{w}(s, t)) \quad (2.29)$$

where $\mathbf{r}_a$ is the position vector of the points along Γ , $\boldsymbol{\xi}$ are the points in the undeformed cross-section, $\boldsymbol{\xi} = [0, X_2, X_3]^\top$, and $\mathbf{w}$ is the warping field representing the cross-section deformations prior to any rigid-body motion. Equally the equation is cast in the initial local coordinate system, noting that $\mathbf{R}(s, t) = \mathbf{R}^{ba}(s, 0)\mathbf{R}^{ab}(s, t)$, the position vector in this system is $\hat{\mathbf{r}} = \mathbf{R}^{ba}(s, 0)\hat{\mathbf{r}}_a$,

$$\hat{\mathbf{r}}(s, t) = \mathbf{r}(s, t) + \mathbf{R}(s, t)(\boldsymbol{\xi} + \mathbf{w}(s, t)) = \mathbf{r}(s, 0) + \mathbf{u}(s, t) + \mathbf{R}(s, t)(\boldsymbol{\xi} + \mathbf{w}(s, t)) \quad (2.30)$$

here the displacement vector, $\mathbf{u}(s, t)$, in the local undeformed system has been introduced. Fig. 2.4 shows definitions of the reference frames and positional vectors of the slender structure. In order to define velocities and strains of the displacement field, it is necessary to look at the variation of the rotation tensor and put together its principal characteristics. Note first the skew-symmetric property of this operator,

$$\frac{d(\mathbf{R}^\top \mathbf{R})}{dt} = \dot{\mathbf{R}}^\top \mathbf{R} + \mathbf{R}^\top \dot{\mathbf{R}} = \mathbf{0} \Rightarrow \mathbf{R}^\top \dot{\mathbf{R}} = -\dot{\mathbf{R}}^\top \mathbf{R} = -\left(\mathbf{R}^\top \dot{\mathbf{R}}\right)^\top \quad (2.31)$$

therefore $\mathbf{R}^\top \dot{\mathbf{R}}$ is a skew tensor from which we can define the material angular velocity. Consider the time derivative of a vector, $\mathbf{p}_b$, sitting in a frame $\mathbf{b}$ that moves with respect to an inertial system $\mathbf{a}$ with angular velocity $\boldsymbol{\omega}$, and $\mathbf{p}^a$ the vector coordinates in the inertial frame,

$$\begin{aligned} \dot{\mathbf{p}}_b &= \frac{\partial \mathbf{p}_b}{\partial t} + \boldsymbol{\omega} \times \mathbf{p}_b = \frac{\partial (\mathbf{R}^{ba} \mathbf{p}_a)}{\partial t} + \tilde{\boldsymbol{\omega}} \mathbf{R}^{ba} \mathbf{p}_a = \dot{\mathbf{R}}^{ba} \mathbf{p}_a + \mathbf{R}^{ba} \dot{\mathbf{p}}_a + \tilde{\boldsymbol{\omega}} \mathbf{R}^{ba} \mathbf{p}_a \\ &\Rightarrow \dot{\mathbf{p}}_b - \mathbf{R}^{ba} \dot{\mathbf{p}}_a = \dot{\mathbf{R}}^{ba} \mathbf{p}_a + \tilde{\boldsymbol{\omega}} \mathbf{R}^{ba} \mathbf{p}_a = \mathbf{0} \Rightarrow \tilde{\boldsymbol{\omega}} = -\dot{\mathbf{R}}^{ba} \mathbf{R}^{ab} = \mathbf{R}^{ba} \dot{\mathbf{R}}^{ab} \end{aligned} \quad (2.32)$$

In the same way the curvature of the cross-sectional centroids, $\mathbf{k}$, is calculated through the spatial derivative of the rotation tensor, $\tilde{\mathbf{k}} = \mathbf{R}^{ba} \mathbf{R}^{ab'}$, where $\bullet' = \frac{\partial(\bullet)}{\partial s}$. The 1D local force

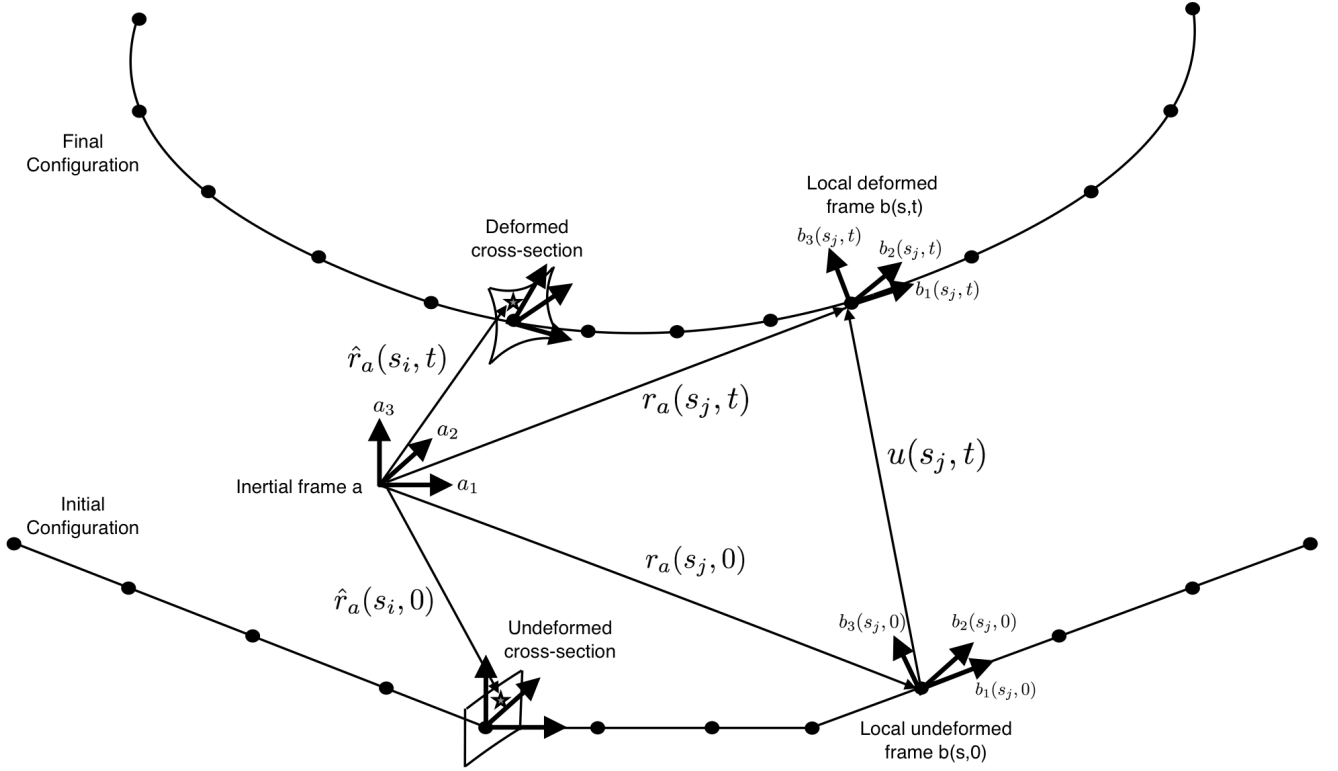


Figure 2.4: Schematic of vector definitions and frames of reference in the mixed intrinsic theory.

strains have been defined by Hodges [31] as the difference between the spatial variation of the reference line in the local reference system; and the moment strains as the difference between current and initial-undeformed curvatures:

$$\boldsymbol{\gamma}(s, t) = \mathbf{R}^{ba}(s, t) \mathbf{r}'_a(s, t) - \mathbf{R}^{ba}(s, 0) \mathbf{r}'_a(s, 0) \quad (2.33)$$

$$\boldsymbol{\kappa}(s, t) = \mathbf{k}(s, t) - \mathbf{k}(s, 0) \quad (2.34)$$

it will be shown in Sec 2.3.2 how these strains link to the 3-dimensional deformation. In the initially undeformed beam, it is always possible to define the reference frame so that $\mathbf{R}^{ba}(s, 0) \mathbf{r}'_a(s, 0) = [1, 0, 0]^\top = \mathbf{e}_x$, by making the first director of $\mathbf{R}^{ab}(s, 0)$ run parallel to $\mathbf{r}'_a(s, 0)$ –as long as the beam is not initially stretched, in which case an initial strain $\boldsymbol{\gamma}_0$ would be carried forward instead. The strain field can be put into a more suitable form using the the

rotation matrices and Eq. (2.30),

$$\begin{aligned}
\boldsymbol{\gamma}(s, t) &= \mathbf{R}^\top(s, t) \mathbf{R}^{ba}(s, 0) \left(\mathbf{R}^{ab'}(s, 0) \mathbf{r}(s, t) + \mathbf{R}^{ab}(s, 0) \mathbf{r}'(s, t) \right) - \mathbf{e}_x \\
&= \mathbf{R}^\top(s, t) \left(\tilde{\mathbf{k}}(s, 0) \mathbf{r}(s, t) + \mathbf{r}'(s, t) \right) - \mathbf{e}_x \\
&= \mathbf{R}^\top(s, t) \left(\tilde{\mathbf{k}}(s, 0) \mathbf{u}(s, t) + \mathbf{u}'(s, t) + \mathbf{e}_x \right) - \mathbf{e}_x
\end{aligned} \tag{2.35}$$

and similarly the curvature is resolved as,

$$\begin{aligned}
\tilde{\mathbf{k}}(s, t) &= \mathbf{R}^\top(s, t) \mathbf{R}^{ba}(s, 0) \left(\mathbf{R}^{ab}(s, 0) \mathbf{R}(s, t) \right)' \\
&= \mathbf{R}^\top(s, t) \mathbf{R}^{ba}(s, 0) \left(\mathbf{R}^{ab}(s, 0) \mathbf{R}'(s, t) + \mathbf{R}^{ab'}(s, 0) \mathbf{R}(s, t) \right) \\
&= \mathbf{R}^\top(s, t) \mathbf{R}'(s, t) + \mathbf{R}^\top(s, t) \tilde{\mathbf{k}}(s, 0) \mathbf{R}(s, t)
\end{aligned} \tag{2.36}$$

The initial curvature, $\mathbf{k}(s, 0)$, has been introduced and it will be regarded as $\mathbf{k}_0$, however, in the computational implementation this term will always be zero: firstly because curvature can be approximated by discretization of linear straight elements, and secondly because, in complex configurations, it would be difficult to find the actual curvature of each element.

The velocity field is built on the assumption of having rigid cross-sections (i.e. neglecting the warping field). In the local reference frame, or in a material description, the velocity of a point in the cross-section is,

$$\begin{aligned}
\hat{\mathbf{v}}(s, \boldsymbol{\xi}, t) &= \mathbf{R}^\top(s, t) \dot{\mathbf{r}}(s, t) = \mathbf{R}^\top(s, t) \frac{d}{dt} (\mathbf{r}(s, 0) + \mathbf{u}(s, t) + \mathbf{R}(s, t) \boldsymbol{\xi}) \\
&= \mathbf{v}(s, t) + \tilde{\boldsymbol{\omega}}(s, t) \boldsymbol{\xi} = \mathbf{R}^\top(s, t) (\dot{\mathbf{r}}(s, 0) + \dot{\mathbf{u}}(s, t) + \tilde{\boldsymbol{\omega}}(s, 0) \mathbf{u}(s, t)) + \tilde{\boldsymbol{\omega}}(s, t) \boldsymbol{\xi}
\end{aligned} \tag{2.37}$$

with $\mathbf{v}$ and $\boldsymbol{\omega}$ the linear and angular velocities respectively of a point in Γ in the local basis. From the definition of $\boldsymbol{\omega}$ in Eq. (2.32), and developing in a similar way than the curvature, Eq. (2.36), the angular velocity can be put as

$$\tilde{\boldsymbol{\omega}}(s, t) = \mathbf{R}^\top(s, t) \dot{\mathbf{R}}(s, t) + \mathbf{R}^\top(s, t) \tilde{\boldsymbol{\omega}}(s, 0) \mathbf{R}(s, t) \tag{2.38}$$

Again the terms $\dot{\mathbf{r}}(s, 0)$ and $\boldsymbol{\omega}(s, 0)$ will be called $\mathbf{v}_0$ and $\boldsymbol{\omega}_0$ for simplicity.

2.3.2 Nonlinear 1D theory from principles of solid mechanics

We now seek to bring together the general equations of solid mechanics laid in Sec. 2.1.2 and the kinematics of a 1D domain, Sec. 2.3.1. Derivations are sustained in the works by Simo [26, 232] and Auricchio et al. [233]. Bringing both descriptions together will shed some light into the limitations of the 1D theory as well as open new solutions to overcome these limitations. It is paramount, as a first step, to realize their difference and equivalence: the nonlinear beam description takes a vector in the current configuration and expresses it with respect to a global reference system, $\mathbf{a}$, or with respect to a moving frame of directors, $\mathbf{b}_i$, such that a vector $\mathbf{y}$ in the current configuration is $\mathbf{y}^t = X_i \mathbf{b}_i = x_i \mathbf{a}_i$; in the solid mechanics framework, quantities are described in either the current configuration or the reference configuration, which entails two different vectors, $\mathbf{y}^t$ and $\mathbf{y}$ respectively; the components of $\mathbf{y}^t$ with respect to $\mathbf{b}_i$ equal those of $\mathbf{y}$ with respect to $\mathbf{E}$, the base of the reference configuration: $\mathbf{y}^t \cdot \mathbf{b}_i = \mathbf{y} \cdot \mathbf{E}_i$. For the sake of simplicity, the following derivations assume an initial straight body with zero initial curvature and coincident frame of reference $\mathbf{a}$, $\mathbf{b}_0$ and $\mathbf{E}_i$ very much like in Fig. 2.2. The relation between the moving frame $\mathbf{b}$ and $\mathbf{E}$ is given by the rotation operator as,

$$\mathbf{b}_i = \mathbf{R} \mathbf{E}_i \quad (2.39)$$

With the 1D kinematics assumed above, changes of objective vectors and tensors between configurations follow the Lie algebra adjoint transformation on $so(3)$ [234], $\tilde{\mathbf{y}}^t = \mathbf{R} \tilde{\mathbf{y}} \mathbf{R}^\top$, and $\mathbf{y} = \mathbf{R}^\top \mathbf{y}^t$, that is $\mathbf{y}$ is obtained by a pull-back of $\mathbf{y}^t$. This is in perfect agreement with the definitions of frame indifference in Eq. (2.11). Despite dealing with spatial and material quantities, they all are parametrized with respect to s in the reference configuration for convenience, $s = X_1$, and the spatial derivative is $\frac{\partial(\bullet)}{\partial X_1} = (\bullet)'$. From Eqs. (2.36), (2.38) and the simplifications made in this section, curvatures and angular velocities in the current and reference configurations are related as,

$$\tilde{\mathbf{k}} = \mathbf{R}^\top \mathbf{R}' \rightarrow \tilde{\mathbf{k}}^t = \mathbf{R} \tilde{\mathbf{k}} \mathbf{R}^\top = \mathbf{R}' \mathbf{R}^\top \quad (2.40)$$

$$\tilde{\boldsymbol{\omega}} = \mathbf{R}^\top \dot{\mathbf{R}} \rightarrow \tilde{\boldsymbol{\omega}}^t = \mathbf{R} \tilde{\boldsymbol{\omega}} \mathbf{R}^\top = \dot{\mathbf{R}} \mathbf{R}^\top \quad (2.41)$$

A starting point in the analysis of deformations is the definition of the gradient deformation tensor based on the 1D kinematics, Eq. (2.30), where it will be assumed that the cross sections of the solid body in the reference configuration remain plane as moving through configurations in time, therefore effectively removing warping, $\mathbf{w} = \mathbf{0}$; presenting the formulation in this way, however, will potentially allow future incorporation of such a warping effect as in [235]. $\mathbf{F}_x$ is thus calculated as follows,

$$\begin{aligned}
\mathbf{F}_x &= \frac{\partial \hat{\mathbf{x}}}{\partial \mathbf{X}} = \frac{\partial \hat{\mathbf{x}}}{\partial X_1} \otimes \mathbf{E}_1 + \frac{\partial \hat{\mathbf{x}}}{\partial X_2} \otimes \mathbf{E}_2 + \frac{\partial \hat{\mathbf{x}}}{\partial X_3} \otimes \mathbf{E}_3 \\
&= \frac{\partial \hat{\mathbf{r}}}{\partial \mathbf{X}} = \frac{\partial (\mathbf{r} + \mathbf{R}\boldsymbol{\xi})}{\partial \mathbf{X}} = \left(\frac{\partial \mathbf{r}}{\partial X_1} + \frac{\partial \mathbf{R}}{\partial X_1} \boldsymbol{\xi} \right) \otimes \mathbf{E}_1 + \mathbf{R} \frac{\partial \boldsymbol{\xi}}{\partial X_2} \otimes \mathbf{E}_2 + \mathbf{R} \frac{\partial \boldsymbol{\xi}}{\partial X_3} \otimes \mathbf{E}_3 \\
&= \left(\frac{\partial \mathbf{r}}{\partial X_1} + X_2 \frac{\partial \mathbf{R}}{\partial X_1} \mathbf{E}_2 + X_3 \frac{\partial \mathbf{R}}{\partial X_1} \mathbf{E}_3 \right) \otimes \mathbf{E}_1 + \mathbf{b}_2 \otimes \mathbf{E}_2 + \mathbf{b}_3 \otimes \mathbf{E}_3 \\
&= \left(\frac{\partial \mathbf{r}}{\partial X_1} + X_2 \tilde{\mathbf{k}}^t \mathbf{b}_2 + X_3 \tilde{\mathbf{k}}^t \mathbf{b}_3 \right) \otimes \mathbf{E}_1 + \mathbf{b}_2 \otimes \mathbf{E}_2 + \mathbf{b}_3 \otimes \mathbf{E}_3
\end{aligned} \tag{2.42}$$

adding and subtracting $\mathbf{b}_1 \otimes \mathbf{E}_1$ from this equation and taking into account from Eq. (2.27) that $\mathbf{b}_1 \otimes \mathbf{E}_1 + \mathbf{b}_2 \otimes \mathbf{E}_2 + \mathbf{b}_3 \otimes \mathbf{E}_3 = \mathbf{R}$,

$$\begin{aligned}
\mathbf{F}_x &= \mathbf{R} + \left((\mathbf{r}' - \mathbf{b}_1) + X_2 \tilde{\mathbf{k}}^t \mathbf{b}_2 + X_3 \tilde{\mathbf{k}}^t \mathbf{b}_3 \right) \otimes \mathbf{E}_1 \\
&= \mathbf{R} \left(\mathbf{I} + \left(\mathbf{R}^\top (\mathbf{r}' - \mathbf{b}_1) + \mathbf{R}^\top (X_2 \tilde{\mathbf{k}}^t \mathbf{b}_2 + X_3 \tilde{\mathbf{k}}^t \mathbf{b}_3) \right) \otimes \mathbf{E}_1 \right) \\
&= \mathbf{R} \left(\mathbf{I} + \left(\mathbf{R}^\top \boldsymbol{\gamma}^t + \mathbf{R}^\top \tilde{\mathbf{k}}^t \boldsymbol{\xi}^t \right) \otimes \mathbf{E}_1 \right) = \mathbf{R} (\mathbf{I} + \mathbf{R}^\top \mathbf{a}^t \otimes \mathbf{E}_1) = \mathbf{R} \mathbf{A}
\end{aligned} \tag{2.43}$$

Here the spatial strain vector in the current configuration has been introduced, $\boldsymbol{\gamma}^t = \mathbf{r}' - \mathbf{b}_1$, and the current local strain vector, $\mathbf{a}^t = \mathbf{r}' - \mathbf{b}_1 + X_2 \tilde{\mathbf{k}}_2^t \mathbf{b}_2 + X_3 \tilde{\mathbf{k}}_3^t \mathbf{b}_3 = \boldsymbol{\gamma}^t + \tilde{\mathbf{k}}^t \boldsymbol{\xi}^t$. The material counterparts $\boldsymbol{\gamma}$ and $\mathbf{a}$ can be calculated by a pull-back operation, but also developing Eq. (2.43) from the right hand side [233],

$$\mathbf{F}_x = (\mathbf{I} + (\mathbf{a}^t \otimes \mathbf{E}_1) \mathbf{R}^\top) \mathbf{R} = \mathbf{A}^t \mathbf{R} \tag{2.44}$$

Note how $\boldsymbol{\gamma} = \mathbf{R}^\top \mathbf{r}' - \mathbf{E}_1$ is similar to the strain defined by Hodges uniquely for a beam in the second equality of Eq. (2.35) when the initial curvature $\mathbf{k}(s, 0)$ has been removed. Effectively global rotations are removed from the strain and curvatures definitions in the material description, so $\boldsymbol{\gamma} = \mathbf{0}$ and $\mathbf{k} = \mathbf{0}$ for rigid body motions, and similarly for the reference local strain

vector, $\mathbf{a} = \boldsymbol{\gamma} + \tilde{\mathbf{k}}\boldsymbol{\xi} = \mathbf{0}$. The stretch tensor $\mathbf{A}$ can be interpreted as the pure stretch part of a decomposition of the deformation gradient, analogously to the deformation gradient of general elasticity, Eq. (2.4), a left extended polar decomposition in Eq. (2.44) and a right extended polar decomposition in Eq. (2.43),

$$\mathbf{A}^t = \mathbf{I} + \mathbf{a}^t \otimes \mathbf{b}_1 \quad (2.45a)$$

$$\mathbf{A} = \mathbf{I} + \mathbf{a} \otimes \mathbf{E}_1 \quad (2.45b)$$

Subsequently to the characterisation of the deformation gradient tensor we can write the strain tensors of finite elasticity: the left Cauchy–Green tensor will be [233],

$$\mathbf{B} = \mathbf{A}^t \mathbf{A}^{t\top} = \mathbf{I} + (\mathbf{a}^t \otimes \mathbf{b}_1 + \mathbf{b}_1 \otimes \mathbf{a}^t) + \mathbf{a}^t \otimes \mathbf{a}^t \quad (2.46)$$

and the Almansi–Eulerian strain tensor, $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$. Equally, the Lagrangian tensors are calculated, the right Cauchy–Green tensor, $\mathbf{C} = \mathbf{F}_x^\top \mathbf{F}_x$, and the Green–Lagrangian tensor, $\mathbf{E}$:

$$\mathbf{C} = \mathbf{A}^\top \mathbf{A} = \mathbf{I} + (\mathbf{a} \otimes \mathbf{E}_1 + \mathbf{E}_1 \otimes \mathbf{a}) + (\mathbf{a} \cdot \mathbf{a})\mathbf{E}_1 \otimes \mathbf{E}_1 \quad (2.47)$$

$$\mathbf{E} = (\mathbf{C} - \mathbf{I})/2 = \frac{1}{2}(\mathbf{a} \otimes \mathbf{E}_1 + \mathbf{E}_1 \otimes \mathbf{a}) + \frac{1}{2}(\mathbf{a} \cdot \mathbf{a})\mathbf{E}_1 \otimes \mathbf{E}_1 \quad (2.48)$$

therefore the strain tensors of 3D elasticity in the restriction to Cosserat’s kinematics have been written as a function of the 1D characterization of total strain, $\mathbf{a} = \boldsymbol{\gamma} + \tilde{\mathbf{k}}\boldsymbol{\xi}$.

Strong form of equations of motion in the spatial description

The strong form of the equations of motion is found in spatial form by defining an internal force, $\mathbf{f}^t$, and internal moment, $\mathbf{m}^t$, in the current coordinates as an integral over the cross sectional area of the distributed traction forces, $\boldsymbol{\mathcal{T}}$, given by the first-Piola tensor as described in Eq. (2.10). Since the cross-sections remain undistorted, distributed internal stresses will act only through the normal, $\mathbf{E}_1$, of the undeformed cross-section. Carrying the integration over

the cross sectional reference area, A_ξ , gives [26]:

$$\mathbf{f}^t = \int_{A_\xi} \mathbf{P}(s, \hat{\mathbf{r}}) \mathbf{E}_1 dA_\xi = \int_{A_\xi} \mathcal{T}_1^t dA_\xi \quad (2.49)$$

$$\mathbf{m}^t = \int_{A_\xi} (\hat{\mathbf{r}} - \mathbf{r}) \times \mathbf{P}(s, \hat{\mathbf{r}}) \mathbf{E}_1 dA_\xi = \int_{A_\xi} (\hat{\mathbf{r}} - \mathbf{r}) \times \mathcal{T}_1^t dA_\xi \quad (2.50)$$

where both $\mathbf{r}$ and $\hat{\mathbf{r}}$ have been defined in the current configuration in Eq. (2.30) (they are written without the $\bullet^t$ for convenience). The restriction of internal stresses acting only in the direction of the body's main axis in the reference configuration would have to be relaxed when warping of the cross-section is taken into account. Eq. (2.14) is employed to get an expression for $\mathcal{T}_1^t$. Firstly the divergence of $\mathbf{P}$ is developed in material coordinates,

$$\nabla_X \cdot \mathbf{P} = \frac{\partial \mathbf{P}}{\partial X_1} \mathbf{E}_1 + \frac{\partial \mathbf{P}}{\partial X_2} \mathbf{E}_2 + \frac{\partial \mathbf{P}}{\partial X_3} \mathbf{E}_3 = \frac{\partial \mathcal{T}_1^t}{\partial X_1} + \frac{\partial \mathcal{T}_2^t}{\partial X_2} + \frac{\partial \mathcal{T}_3^t}{\partial X_3} \quad (2.51)$$

consider now a solid cross-section as in Fig. 2.5 of volume $\Delta\Omega_0$ in the reference configuration, and the increment of the reference axis of length Δs , the integral of the 3D quantities in Eq. (2.14) over this volume divided by Δs will yield the 1D equivalent quantities per unit of reference length (increments Δ are written instead differentials to make clearer the integration of 3D quantities, but it should be understood Δs becomes infinitesimally small and so it does the increment of volume Ω_0). Then, analogously to Eqs. (2.49), (2.53), the linear and angular momentum of the cross-section are defined as

$$\mathbf{p}^t = \frac{1}{\Delta s} \int_{\Delta\Omega_0} \rho_0 \dot{\hat{\mathbf{r}}} \approx \int_{A_\xi} \rho_{c0} \dot{\hat{\mathbf{r}}} dA_\xi \quad (2.52)$$

$$\mathbf{h}^t = \frac{1}{\Delta s} \int_{\Delta\Omega_0} \rho_0 (\hat{\mathbf{r}} - \mathbf{r}) \times \dot{\hat{\mathbf{r}}} \approx \int_{A_\xi} \rho_{c0} (\hat{\mathbf{r}} - \mathbf{r}) \times \dot{\hat{\mathbf{r}}} dA_\xi \quad (2.53)$$

with ρ_{c0} the density of the cross section instead of the volume density. Next we take the variation of $\mathbf{f}^t$ with respect to the beam axis, $\frac{\partial}{\partial s} = \frac{\partial}{\partial X_1}$, apply equilibrium of forces between adjacent cross-sections and the volume they confine, $\Delta\Omega_0$, and using Eq. (2.14), (2.49), (2.52),

(2.51):

$$\frac{\partial}{\partial s} \mathbf{f}^t \approx \frac{1}{\Delta s} \int_{\Delta\Omega_0} \frac{\partial}{\partial s} \mathcal{T}_1^t d\Omega_0 = \frac{1}{\Delta s} \int_{\Delta\Omega_0} \rho_0 \frac{d\dot{\mathbf{r}}}{dt} d\Omega_0 - \frac{1}{\Delta s} \int_{\Delta\Omega_0} \left(\frac{\partial \mathcal{T}_2^t}{\partial X_2} + \frac{\partial \mathcal{T}_3^t}{\partial X_3} + \mathbf{b}_{f0} \right) d\Omega_0 \quad (2.54)$$

the last integral represents the forces $\mathcal{T}_\alpha^t$ (with $\alpha = 2, 3$) at the boundary $\partial\Omega_0$ after application of the divergence theorem – as depicted in Fig. 2.5, which together with the body forces have been combined to form the external forces per unit length, $\mathbf{f}_e^t$. The same exercise is done with

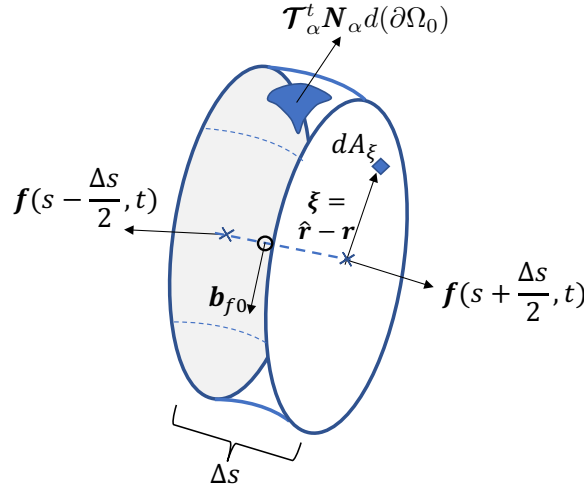


Figure 2.5: Forces acting on the solid cross-section

the internal moments,

$$\begin{aligned} \frac{\partial}{\partial s} \mathbf{m}^t &\approx \frac{1}{\Delta s} \int_{\Delta\Omega_0} \frac{\partial}{\partial s} ((\hat{\mathbf{r}} - \mathbf{r}) \times \mathcal{T}_1^t) d\Omega_0 = -\frac{1}{\Delta s} \int_{\Delta\Omega_0} \frac{\partial}{\partial s} \mathbf{r} \times \mathcal{T}_1^t d\Omega_0 \\ &+ \frac{1}{\Delta s} \int_{\Delta\Omega_0} (\hat{\mathbf{r}} - \mathbf{r}) \times \left(\rho_0 \frac{d\dot{\mathbf{r}}}{dt} \right) d\Omega_0 - \frac{1}{\Delta s} \int_{\Delta\Omega_0} (\hat{\mathbf{r}} - \mathbf{r}) \times \left(\frac{\partial \mathcal{T}_2^t}{\partial X_2} + \frac{\partial \mathcal{T}_3^t}{\partial X_3} + \mathbf{b}_{f0} \right) d\Omega_0 \end{aligned} \quad (2.55)$$

with the integral over $\hat{\mathbf{r}}' \times \mathcal{T}_1^t$ being $\mathbf{0}$ from balance of angular momentum. Similar to the equilibrium of forces, the last integral defines the external moment per unit length, $\mathbf{m}_e^t$, generated by body forces and surface tractions $\mathcal{T}_\alpha$. The term involving the inertial forces can be further developed,

$$\begin{aligned} \frac{1}{\Delta s} \int_{\Delta\Omega_0} (\hat{\mathbf{r}} - \mathbf{r}) \times \left(\rho_0 \frac{d\dot{\mathbf{r}}}{dt} \right) d\Omega_0 &= \frac{1}{\Delta s} \int_{\Delta\Omega_0} \left(\frac{d}{dt} ((\hat{\mathbf{r}} - \mathbf{r}) \times \rho_0 \dot{\mathbf{r}}) - (\dot{\hat{\mathbf{r}}} - \dot{\mathbf{r}}) \times \rho_0 \dot{\mathbf{r}} \right) d\Omega_0 \\ &= \frac{d}{dt} \frac{1}{\Delta s} \int_{\Delta\Omega_0} ((\hat{\mathbf{r}} - \mathbf{r}) \times \rho_0 \dot{\mathbf{r}}) d\Omega_0 + \frac{1}{\Delta s} \int_{\Delta\Omega_0} \dot{\mathbf{r}} \times \rho_0 \dot{\mathbf{r}} d\Omega_0 = \dot{\mathbf{h}}^t + \dot{\mathbf{r}} \times \mathbf{p}^t \end{aligned} \quad (2.56)$$

note that when $\mathbf{r}$ is the centroid of the section, $\dot{\mathbf{r}} \times \mathbf{p}^t$ equals $\mathbf{0}$, as in the analysis by Simo [26], who, neglecting that term, arrived to the final set of equations in spatial form describing the dynamics of the nonlinear rod,

$$\frac{\partial}{\partial s} \mathbf{f}^t + \mathbf{f}_e^t = \frac{d\mathbf{p}^t}{dt} \quad (2.57a)$$

$$\frac{\partial \mathbf{m}^t}{\partial s} + \frac{\partial \mathbf{r}}{\partial s} \times \mathbf{f}^t + \mathbf{m}_e^t = \frac{d\mathbf{h}^t}{dt} + \dot{\mathbf{r}} \times \mathbf{p}^t \quad (2.57b)$$

Internal virtual work

In the previous section the dynamic equations of a slender body have been formulated in spatial form as an equilibrium of forces and momenta. Here we seek to find the relation between these forces and the strains in Eq. (2.45), as well as the expression of the forces in material form. From Eq. (2.8) we can put the variation of internal work $\delta\mathcal{W}$ as a function of the first and second Piola-Kirchhoff stresses, $\delta\mathcal{W}^P$ and $\delta\mathcal{W}^S$ respectively:

$$\delta\mathcal{W}^P = \int_{\Omega_0} \mathbf{P} : \delta\mathbf{F}_x d\Omega_0 \quad (2.58a)$$

$$\delta\mathcal{W}^S = \int_{\Omega_0} \mathbf{S} : \delta\mathbf{E} d\Omega_0 \quad (2.58b)$$

it can be proved that the first equation develops into [233],

$$\delta\mathcal{W}^P = \int_{\Omega_0} \mathbf{P} : \delta\mathbf{F}_x d\Omega_0 = \int_{L_0} (\mathbf{f}^t \cdot \delta_R \boldsymbol{\gamma}^t + \mathbf{m}^t \cdot \delta_R \mathbf{k}^t) ds \quad (2.59)$$

where δ_R is the co-rotational or Lie variation, $\delta_R(\cdot) = \delta(\cdot) + \mathbf{R}\delta(\mathbf{R}^\top(\cdot))$, and L_0 the reference axis length (note that the forces and strains on this integral are given in the current configuration although parametrised by the reference arc length s for convenience [26]). However, it is more interesting for this work to develop the virtual work in terms of $\mathbf{S}$ to find a relation between the forces and strains in material form. The variation of the Green-Lagrangian strain tensor as a function of 1D strains is,

$$\delta\mathbf{E} = \frac{1}{2}(\delta\mathbf{a} \otimes \mathbf{E}_1 + \mathbf{E}_1 \otimes \delta\mathbf{a}) + (\delta\mathbf{a} \cdot \mathbf{a})\mathbf{E}_1 \otimes \mathbf{E}_1 \quad (2.60)$$

and the variation of $\mathbf{a}$,

$$\delta \mathbf{a} = \delta \boldsymbol{\gamma} + \delta \mathbf{k} \times \boldsymbol{\xi} \quad (2.61)$$

The traction in the reference configuration face normal to $\mathbf{E}_1$ in material coordinates is $\boldsymbol{\mathcal{T}}_1 = \mathbf{S} \cdot \mathbf{E}_1$. Using Eqs. (2.58)-(2.61), $\delta \mathcal{W}^S$ is

$$\begin{aligned} \delta \mathcal{W}^S &= \int_{\Omega_0} \mathbf{S} : \delta \mathbf{E} d\Omega_0 = \int_{\Omega_0} \mathbf{S} : (\delta \mathbf{a} \otimes \mathbf{E}_1) d\Omega_0 + \mathbf{S} : (\delta \mathbf{a} \cdot \mathbf{a}) \mathbf{E}_1 \otimes \mathbf{E}_1 d\Omega_0 \\ &= \int_{\Omega_0} \boldsymbol{\mathcal{T}}_1 \cdot \delta \mathbf{a} d\Omega_0 + \boldsymbol{\mathcal{T}}_1 \cdot (\delta \mathbf{a} \cdot \mathbf{a}) \mathbf{E}_1 d\Omega_0 = \int_{\Omega_0} ((\mathbf{I} + \mathbf{a} \otimes \mathbf{E}_1) \boldsymbol{\mathcal{T}}_1) \cdot \delta \mathbf{a} d\Omega_0 \\ &= \int_{\Omega_0} (\mathbf{A} \boldsymbol{\mathcal{T}}_1) \cdot \delta \mathbf{a} d\Omega_0 = \int_{\Omega_0} (\mathbf{A} \boldsymbol{\mathcal{T}}_1) \cdot (\delta \boldsymbol{\gamma} + \delta \mathbf{k} \times (X_2 \mathbf{E}_2 + X_3 \mathbf{E}_3)) d\Omega_0 \\ &= \int_{L_0} \left(\delta \boldsymbol{\gamma} \cdot \int_{A_\xi} (\mathbf{A} \boldsymbol{\mathcal{T}}_1) dA_\xi + \delta \mathbf{k} \cdot \int_{A_\xi} (\mathbf{A} \boldsymbol{\mathcal{T}}_1) \times \boldsymbol{\xi} dA_\xi \right) ds \end{aligned} \quad (2.62)$$

The relation between the traction forces in the current and reference configurations is found as,

$$\mathbf{P} = \mathbf{F}_x \mathbf{S} = (\mathbf{R} \mathbf{A}) \mathbf{S} \Rightarrow \mathbf{P} \mathbf{E}_1 = \mathbf{R} \mathbf{A} (\mathbf{S} \mathbf{E}_1) \Rightarrow \mathbf{R}^\top \boldsymbol{\mathcal{T}}_1^t = \mathbf{A} \boldsymbol{\mathcal{T}}_1 \quad (2.63)$$

so that the relation between the forces and moments in the two configurations is [233],

$$\mathbf{f} = \int_{A_\xi} (\mathbf{A} \boldsymbol{\mathcal{T}}_1) dA_\xi = \mathbf{R}^\top \int_{A_\xi} \boldsymbol{\mathcal{T}}_1^t dA_\xi = \mathbf{R}^\top \mathbf{f}^t \quad (2.64)$$

$$\mathbf{m} = \int_{A_\xi} (\mathbf{A} \boldsymbol{\mathcal{T}}_1) \times \boldsymbol{\xi} dA_\xi = \mathbf{R}^\top \int_{A_\xi} \boldsymbol{\mathcal{T}}_1^t \times \boldsymbol{\xi}^t dA_\xi = \mathbf{R}^\top \mathbf{m}^t \quad (2.65)$$

and it can be proved that $\delta \mathcal{W}^P = \delta \mathcal{W}^S$ by using the invariance of the dot product under orthogonal transformations and Eq. (2.59):

$$\begin{aligned} \delta \mathcal{W}^S &= \int_{L_0} (\mathbf{f} \cdot \delta \boldsymbol{\gamma} + \mathbf{m} \cdot \delta \mathbf{k}) ds = \int_{L_0} ((\mathbf{R} \mathbf{f}) \cdot (\mathbf{R} \delta \boldsymbol{\gamma}) + (\mathbf{R} \mathbf{m}) \cdot (\mathbf{R} \delta \mathbf{k})) ds \\ &\int_{L_0} (\mathbf{f}^t \cdot (\mathbf{R} \delta (\mathbf{R}^\top \boldsymbol{\gamma}^t) + \mathbf{m}^t \cdot (\mathbf{R} \delta (\mathbf{R}^\top \mathbf{k}^t))) ds = \delta \mathcal{W}^P \end{aligned} \quad (2.66)$$

2.3.3 Intrinsic equations of motion

The equations of motion in local coordinates are derived from Hamilton's principle on the 1D domain Γ , representing the load-path of the aircraft structures, from the kinematics presented in Sec. 2.3.1 and the forces and moments defined in Sec. 2.3.2 based on principles of general elasticity. Initially a linearisation of the intrinsic variables needs to be carried out in order to apply the principle of least action, which leads to the dynamics of Γ described in its local material frame. Then the equations of motion are solved in modal space after a Galerkin projection is performed using the modes of the intrinsic linear system. Finally recovery of displacements and rotations along Γ can be obtained as a post-processing step.

Linearisation of intrinsic quantities

An initial step in the calculation of variational quantities is to introduce the variation of the rotation matrix as variation of its three independent rotational degrees of freedom, $\boldsymbol{\theta}$. Making use of the exponential map representation of the rotation, Eq. (2.28), the variation is $\delta \mathbf{R}|_{\theta} = e^{\tilde{\boldsymbol{\theta}}} \delta \tilde{\boldsymbol{\theta}} = \mathbf{R}|_{\theta} \delta \tilde{\boldsymbol{\theta}}$. With this we can define a virtual rotation vector, $\widetilde{\delta \boldsymbol{\theta}}$, the overbar indicates the virtual character of the variable. A virtual displacement vector, $\overline{\delta \mathbf{u}}$, is also defined in the local reference system from the variation of the displacement field given in Eq. (2.30) in the initial base system:

$$\widetilde{\delta \boldsymbol{\theta}} = \mathbf{R}^{\top} \delta \mathbf{R} = -\delta \mathbf{R}^{\top} \mathbf{R} \quad (2.67)$$

$$\overline{\delta \mathbf{u}} = \mathbf{R}^{\top} \delta \mathbf{u} \quad (2.68)$$

We now seek to put the intrinsic variables of velocity and strain as a function of the virtual vectors. From the strain field in the last identity of Eq. (2.35):

$$\begin{aligned} \delta \boldsymbol{\gamma} &= \delta \mathbf{R}^{\top} (\mathbf{u}' + \tilde{\mathbf{k}}_0 \mathbf{u} + \mathbf{e}_x) + \mathbf{R}^{\top} (\delta \mathbf{u}' + \tilde{\mathbf{k}}_0 \delta \mathbf{u}) \\ &= \delta \mathbf{R}^{\top} \mathbf{R} (\boldsymbol{\gamma} + \mathbf{e}_x) + (\overline{\delta \mathbf{u}}' - \mathbf{R}'^{\top} \mathbf{R} \overline{\delta \mathbf{u}}) + \mathbf{R}^{\top} \tilde{\mathbf{k}}_0 \mathbf{R} \overline{\delta \mathbf{u}} \\ &= (\tilde{\boldsymbol{\gamma}} + \tilde{\mathbf{e}}_x) \widetilde{\delta \boldsymbol{\theta}} + \overline{\delta \mathbf{u}}' + \tilde{\mathbf{k}} \overline{\delta \mathbf{u}} \end{aligned} \quad (2.69)$$

The derivative of the rotation matrix with respect to the beam axis and its variation are put as,

$$\mathbf{R}' = \mathbf{R}\tilde{\mathbf{k}} - \tilde{\mathbf{k}}_0\mathbf{R} \Rightarrow \delta\mathbf{R}' = \mathbf{R}'\widetilde{\delta\theta} + \mathbf{R}\widetilde{\delta\theta}' = \left(\mathbf{R}\tilde{\mathbf{k}} - \tilde{\mathbf{k}}_0\mathbf{R}\right)\widetilde{\delta\theta} + \mathbf{R}\widetilde{\delta\theta}' \quad (2.70)$$

Using this and Eq. (2.36), the variation of the moment strain and curvatures is obtained,

$$\begin{aligned} \delta\tilde{\kappa} &= \delta\tilde{\mathbf{k}} = \delta\mathbf{R}^\top\mathbf{R}' + \mathbf{R}^\top\delta\mathbf{R}' + \delta\mathbf{R}^\top\tilde{\mathbf{k}}_0\mathbf{R} + \mathbf{R}^\top\tilde{\mathbf{k}}_0\delta\mathbf{R} \\ &= -\widetilde{\delta\theta}\mathbf{R}^\top\left(\mathbf{R}\tilde{\mathbf{k}} - \tilde{\mathbf{k}}_0\mathbf{R}\right) + \mathbf{R}^\top\left(\left(\mathbf{R}\tilde{\mathbf{k}} - \tilde{\mathbf{k}}_0\mathbf{R}\right)\widetilde{\delta\theta} + \mathbf{R}\widetilde{\delta\theta}'\right) + \\ &\quad -\widetilde{\delta\theta}\mathbf{R}^\top\tilde{\mathbf{k}}_0\mathbf{R} + \mathbf{R}^\top\tilde{\mathbf{k}}_0\mathbf{R}\widetilde{\delta\theta} \\ &= \widetilde{\delta\theta}' + \tilde{\mathbf{k}}\widetilde{\delta\theta} - \widetilde{\delta\theta}\tilde{\mathbf{k}} \end{aligned} \quad (2.71)$$

the tilde operator can be removed applying the Lie algebra property to the last two summands², giving the variation of curvatures as,

$$\delta\kappa = \overline{\delta\theta}' + \tilde{\mathbf{k}}\overline{\delta\theta} \quad (2.72)$$

Next the variation of the velocity field is calculated following Eqs. (2.37) and the definitions of virtual displacement and rotation:

$$\begin{aligned} \delta\mathbf{v} &= \delta\mathbf{R}^\top(\mathbf{v}_0 + \dot{\mathbf{u}} + \tilde{\omega}_0\mathbf{u}) + \mathbf{R}^\top(\delta\dot{\mathbf{u}} + \tilde{\omega}_0\delta\mathbf{u}) \\ &= \delta\mathbf{R}^\top\mathbf{R}\mathbf{v} + (\overline{\delta\dot{\mathbf{u}}} - \dot{\mathbf{R}}^\top\mathbf{R}\overline{\delta\mathbf{u}}) + \mathbf{R}^\top\tilde{\omega}_0\mathbf{R}\overline{\delta\mathbf{u}} \\ &= \tilde{\mathbf{v}}\overline{\delta\theta} + \overline{\delta\dot{\mathbf{u}}} + \tilde{\omega}\overline{\delta\mathbf{u}} \end{aligned} \quad (2.73)$$

and the variation of the angular velocity is calculated from Eq. (2.38) very similar to the variation of moment strains but with derivatives taken with respect to time,

$$\begin{aligned} \delta\tilde{\omega} &= \delta\mathbf{R}^\top\dot{\mathbf{R}} + \mathbf{R}^\top\delta\dot{\mathbf{R}} + \delta\mathbf{R}^\top\tilde{\omega}_0\mathbf{R} + \mathbf{R}^\top\tilde{\omega}_0\delta\mathbf{R} \\ &= \overline{\delta\dot{\theta}} + \tilde{\omega}\widetilde{\delta\theta} - \widetilde{\delta\theta}\tilde{\omega} \\ \delta\omega &= \overline{\delta\dot{\theta}} + \tilde{\omega}\overline{\delta\theta} \end{aligned} \quad (2.74)$$

²If $\mathbf{y}$ and $\mathbf{z}$ are two vectors, $\tilde{\mathbf{y}}\tilde{\mathbf{z}} - \tilde{\mathbf{z}}\tilde{\mathbf{y}} = \widetilde{\mathbf{y}\mathbf{z}}$

Structural dynamics of aircraft load-paths

Densities per unit length of strain energy, $\mathcal{U}$, kinetic energy, $\mathcal{K}$ and virtual work of external forces $\mathcal{W}_e$ are computed in order to apply Hamilton's principle from time t_1 to t_2 , on the load path Γ :

$$\int_{t_1}^{t_2} \left(\int_{\Gamma} \delta(\mathcal{K} - \mathcal{U}) + \mathcal{W}_e ds \right) dt = \overline{\delta A} \quad (2.75)$$

$\overline{\delta A}$ is the action term that introduces the boundary conditions. The strain energy is calculated following the 3D internal energy under 1D assumptions in Eq. (2.66) and replacing the curvatures $\mathbf{k}$ with the moment strains, $\boldsymbol{\kappa}$,

$$\begin{aligned} \delta \mathcal{U} = \delta \boldsymbol{\gamma} \cdot \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}} + \delta \boldsymbol{\kappa} \cdot \frac{\partial \mathcal{U}}{\partial \boldsymbol{\kappa}} = \delta \boldsymbol{\gamma} \cdot \mathbf{f} + \delta \boldsymbol{\kappa} \cdot \mathbf{m} = \left(\overline{\delta \mathbf{u}'} + \tilde{\mathbf{k}} \overline{\delta \mathbf{u}} + (\tilde{\boldsymbol{\gamma}} + \tilde{\mathbf{e}}_x) \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{f} \\ + \left(\overline{\delta \boldsymbol{\theta}'} + \tilde{\mathbf{k}} \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{m} \end{aligned} \quad (2.76)$$

Note that to take warping effects into account, this expression would be extended with the energy from cross-section deformations as in [45]. The expression for the kinetic energy follows from the velocity field within each cross-section, Eq. (2.37),

$$\begin{aligned} \mathcal{K} &= \frac{1}{2} \int_{A_\xi} \rho_{c0} \hat{\mathbf{v}} \cdot \hat{\mathbf{v}} dA_\xi = \frac{1}{2} \int_{A_\xi} \rho_{co} (\mathbf{v} + \tilde{\boldsymbol{\omega}} \boldsymbol{\xi}) \cdot (\mathbf{v} + \tilde{\boldsymbol{\omega}} \boldsymbol{\xi}) dA_\xi \\ &= \frac{1}{2} \mathbf{v} \cdot \mathbf{v} \int_{A_\xi} \rho_{c0} dA_\xi + \mathbf{v}^\top \tilde{\boldsymbol{\omega}} \int_{A_\xi} \rho_{c0} \boldsymbol{\xi} dA_\xi + \frac{1}{2} \boldsymbol{\omega} \otimes \boldsymbol{\omega} : \int_{A_\xi} \rho_{c0} (\boldsymbol{\xi} \cdot \boldsymbol{\xi} \mathbf{I} - \boldsymbol{\xi} \otimes \boldsymbol{\xi}) dA_\xi \end{aligned} \quad (2.77)$$

The first integral represents the mass per unit length, μ ; the second integral is the first moment of inertia of the cross-section, $\boldsymbol{\xi}_\mu$, a vector that will be $\mathbf{0}$ when calculated from the centroid of the cross-section; the last integral is the second moment of inertia of the cross-section, the matrix $\mathbf{I}_\mu$ ³. Linear and angular momenta can be calculated from $\mathcal{K}$ as

$$\mathbf{p} = \frac{\partial \mathcal{K}}{\partial \mathbf{v}} = \mu \mathbf{v} - \boldsymbol{\xi}_\mu \times \boldsymbol{\omega} \quad (2.78)$$

$$\mathbf{h} = \frac{\partial \mathcal{K}}{\partial \boldsymbol{\omega}} = \mathbf{I}_\mu \boldsymbol{\omega} + \boldsymbol{\xi}_\mu \times \mathbf{v} \quad (2.79)$$

³The following property is used in the derivation: $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \Rightarrow (\boldsymbol{\omega} \times \boldsymbol{\xi}) \cdot (\boldsymbol{\omega} \times \boldsymbol{\xi}) = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) - (\boldsymbol{\omega} \cdot \boldsymbol{\xi})^2 = (\boldsymbol{\omega} \otimes \boldsymbol{\omega}) : \mathbf{I}(\boldsymbol{\xi} \cdot \boldsymbol{\xi}) - (\boldsymbol{\omega} \otimes \boldsymbol{\omega}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi})$

We can derive these expressions by pull-back operation of Eqs. (2.52), (2.53), which were derived as approximations of the 3D cross-section. Neglecting warping deformation, using the definition of $\hat{\mathbf{r}}$ in Eq. (2.30), and remarking that both $\mathbf{r}$ and $\hat{\mathbf{r}}$ have been defined in the current configuration, the linear momentum is,

$$\mathbf{p} = \mathbf{R}^\top \mathbf{p}^t = \int_{A_\xi} \rho_{c0} \mathbf{R}^\top \dot{\mathbf{r}} dA_\xi = \int_{A_\xi} \rho_{c0} (\mathbf{R}^\top \dot{\mathbf{r}} + \mathbf{R}^\top \dot{\mathbf{R}} \boldsymbol{\xi}) dA_\xi = \mu \mathbf{v} + \tilde{\boldsymbol{\omega}} \boldsymbol{\xi}_\mu \quad (2.80)$$

as previously mentioned, when $\boldsymbol{\xi}_\mu = \mathbf{0}$, the linear momentum is proportional to $\mathbf{v}$ and $\tilde{\mathbf{v}}\mathbf{p} = \mathbf{0}$.

Similarly the angular momentum is

$$\begin{aligned} \mathbf{h} &= \mathbf{R}^\top \mathbf{h}^t = \int_{A_\xi} \rho_{c0} \mathbf{R}^\top (\hat{\mathbf{r}} - \mathbf{r}) \times \dot{\mathbf{r}} dA_\xi = \int_{A_\xi} \rho_{c0} (\mathbf{R}^\top \mathbf{R} \boldsymbol{\xi}) \times (\mathbf{R}^\top \dot{\mathbf{r}} + \mathbf{R}^\top \dot{\mathbf{R}} \boldsymbol{\xi}) dA_\xi \\ &= \int_{A_\xi} \rho_{c0} (\boldsymbol{\xi} \times \mathbf{v} + \boldsymbol{\xi} \times \boldsymbol{\omega} \times \boldsymbol{\xi}) dA_\xi = \int_{A_\xi} \rho_{c0} (\boldsymbol{\xi} \times \mathbf{v} + (\boldsymbol{\xi} \cdot \boldsymbol{\xi} \mathbf{I} - \boldsymbol{\xi} \otimes \boldsymbol{\xi}) \boldsymbol{\omega}) dA_\xi \\ &= \boldsymbol{\xi}_\mu \times \mathbf{v} + \mathbf{I}_\mu \boldsymbol{\omega} \end{aligned} \quad (2.81)$$

where the properties of the triple cross product have been used⁴. Therefore we have derived equivalent expressions of linear and angular momenta both from energy principles and 3D cross-section integration principles. The variation of the kinetic energy is finally put as,

$$\begin{aligned} \delta \mathcal{K} &= \delta \mathbf{v} \cdot \frac{\partial \mathcal{K}}{\partial \mathbf{v}} + \delta \boldsymbol{\omega} \cdot \frac{\partial \mathcal{K}}{\partial \boldsymbol{\omega}} = \delta \mathbf{v} \cdot \mathbf{p} + \delta \boldsymbol{\omega} \cdot \mathbf{m} = \left(\dot{\overline{\delta \mathbf{u}}} + \tilde{\boldsymbol{\omega}} \overline{\delta \mathbf{u}} + \tilde{\mathbf{v}} \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{p} \\ &\quad + \left(\dot{\overline{\delta \boldsymbol{\theta}}} + \tilde{\boldsymbol{\omega}} \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{m} \end{aligned} \quad (2.82)$$

The external forces and moments on the 3D structure are condensed to the points through the cross-sections conforming Γ , as indicated in Eqs. (2.54), (2.55), and their work for virtual displacements and rotations is simply,

$$\delta \mathcal{W}_e = \overline{\delta \mathbf{u}} \cdot \mathbf{f}_e + \overline{\delta \boldsymbol{\theta}} \cdot \mathbf{m}_e \quad (2.83)$$

⁴ $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \Rightarrow \boldsymbol{\xi} \times \boldsymbol{\omega} \times \boldsymbol{\xi} = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})\boldsymbol{\omega} - (\boldsymbol{\xi} \cdot \boldsymbol{\omega})\boldsymbol{\xi} = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})\boldsymbol{\omega} - (\boldsymbol{\xi} \otimes \boldsymbol{\xi})\boldsymbol{\omega}$

Application of Hamilton's principle, Eq. (2.75), finally yields the equations of motion:

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{\Gamma} \left(- \left(\overline{\delta \mathbf{u}'} + \tilde{\mathbf{k}} \overline{\delta \mathbf{u}} + (\tilde{\gamma} + \tilde{\mathbf{e}}_x) \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{f} - \left(\overline{\delta \boldsymbol{\theta}'} + \tilde{\mathbf{k}} \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{m} \right. \\
& \quad \left. + \left(\tilde{\mathbf{v}} \overline{\delta \boldsymbol{\theta}} + \overline{\delta \dot{\mathbf{u}}} + \tilde{\boldsymbol{\omega}} \overline{\delta \mathbf{u}} \right) \cdot \mathbf{p} + \left(\overline{\delta \dot{\boldsymbol{\theta}}} + \tilde{\boldsymbol{\omega}} \overline{\delta \boldsymbol{\theta}} \right) \cdot \mathbf{h} + \overline{\delta \mathbf{u}} \cdot \mathbf{f}_e + \overline{\delta \boldsymbol{\theta}} \cdot \mathbf{m}_e \right) ds dt \\
& = \int_{\Gamma} \left(\overline{\delta \mathbf{u}} \cdot \mathbf{p}^* + \overline{\delta \boldsymbol{\theta}} \cdot \mathbf{h}^* \right) \Big|_{t_1}^{t_2} ds - \int_{t_1}^{t_2} \left(\overline{\delta \mathbf{u}} \cdot \mathbf{f}^* + \overline{\delta \boldsymbol{\theta}} \cdot \mathbf{m}^* \right) \Big|_{\Gamma} dt
\end{aligned} \tag{2.84}$$

$(\bullet)^*$ are the values of force and moment at the boundaries of Γ , or the values of impulses of linear and angular momenta at the time intervals where the motion takes place. After integration by parts and grouping virtual displacements and rotations,

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{\Gamma} \left(\left(\mathbf{f}' + \tilde{\mathbf{k}} \mathbf{f} - \dot{\mathbf{p}} - \tilde{\boldsymbol{\omega}} \mathbf{p} + \mathbf{f}_e \right) \cdot \overline{\delta \mathbf{u}} \right. \\
& \quad \left. + \left((\tilde{\gamma} + \tilde{\mathbf{e}}_x) \mathbf{f} + \mathbf{m}' + \tilde{\mathbf{k}} \mathbf{m} - \tilde{\mathbf{v}} \mathbf{p} - \dot{\mathbf{h}} - \tilde{\boldsymbol{\omega}} \mathbf{h} + \mathbf{m}_e \right) \cdot \overline{\delta \boldsymbol{\theta}} \right) ds dt \\
& = \int_{\Gamma} \left(\overline{\delta \mathbf{u}} \cdot (\mathbf{p}^* - \mathbf{p}) + \overline{\delta \boldsymbol{\theta}} \cdot (\mathbf{h}^* - \mathbf{h}) \right) \Big|_{t_1}^{t_2} ds - \int_{t_1}^{t_2} \left(\overline{\delta \mathbf{u}} \cdot (\mathbf{f}^* - \mathbf{f}) + \overline{\delta \boldsymbol{\theta}} \cdot (\mathbf{m}^* - \mathbf{m}) \right) \Big|_{\Gamma} dt
\end{aligned} \tag{2.85}$$

Imposing natural boundaries conditions at the open ends, the final equations in strong form are:

$$\mathbf{f}' + (\tilde{\kappa} + \tilde{\mathbf{k}}_0) \mathbf{f} + \mathbf{f}_e = \dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}} \mathbf{p} \tag{2.86a}$$

$$\mathbf{m}' + (\tilde{\kappa} + \tilde{\mathbf{k}}_0) \mathbf{m} + \mathbf{m}_e + (\tilde{\mathbf{e}}_x + \tilde{\gamma}) \mathbf{f} = \dot{\mathbf{h}} + \tilde{\boldsymbol{\omega}} \mathbf{h} + \tilde{\mathbf{v}} \mathbf{p} \tag{2.86b}$$

These equations are Hodges' intrinsic equations [31], but are just the material form of Eqs. (2.57), which were derived in spatial form from 3D elasticity principles – and under simplified assumptions such as $\mathbf{k}_0 = \mathbf{0}$. To prove this, we take Eq. (2.57a), multiply it by $\mathbf{R}^\top$ and using the relation between current and reference quantities, $\mathbf{f}^t = \mathbf{R} \mathbf{f}$:

$$\begin{aligned}
\mathbf{R}^\top \frac{\partial (\mathbf{R} \mathbf{f})}{\partial s} + \mathbf{R}^\top \mathbf{f}_e^t &= \mathbf{R}^\top \left(\mathbf{R} \tilde{\mathbf{k}} \mathbf{f} + \mathbf{R} \mathbf{f}' \right) + \mathbf{f}_e = \mathbf{f}' + \tilde{\mathbf{k}} \mathbf{f} + \mathbf{f}_e \\
&= \mathbf{R}^\top \frac{d \mathbf{p}^t}{dt} = \mathbf{R}^\top \frac{d (\mathbf{R} \mathbf{p})}{dt} = \mathbf{R}^\top \dot{\mathbf{R}} \mathbf{p} + \dot{\mathbf{p}} = \dot{\mathbf{p}} + \tilde{\boldsymbol{\omega}} \mathbf{p}
\end{aligned} \tag{2.87}$$

Analogously for the equilibrium of moments, Eq. (2.57b) is transformed considering $\mathbf{m}^t = \mathbf{R}\mathbf{m}$, and $\mathbf{r}' = \mathbf{R}(\mathbf{e}_x + \boldsymbol{\gamma})$ after Eq. (2.35), giving:

$$\begin{aligned} \mathbf{R}^\top \frac{\partial (\mathbf{R}\mathbf{m})}{\partial s} + \mathbf{R}^\top \left(\frac{\partial \mathbf{r}}{\partial s} \times \mathbf{R}\mathbf{f} \right) + \mathbf{R}^\top \mathbf{m}_e^t &= \mathbf{R}^\top \left(\mathbf{R}\tilde{\mathbf{k}}\mathbf{m} + \mathbf{R}\mathbf{m}' \right) + \mathbf{R}^\top (\mathbf{r}' \times \mathbf{R}\mathbf{f}) \\ + \mathbf{m}_e &= \mathbf{m}' + \tilde{\mathbf{k}}\mathbf{m} + \mathbf{R}^\top (\mathbf{R}(\mathbf{e}_x + \boldsymbol{\gamma}) \times \mathbf{R}\mathbf{f}) + \mathbf{m}_e = \mathbf{m}' + \tilde{\mathbf{k}}\mathbf{m} + (\tilde{\mathbf{e}}_x + \tilde{\boldsymbol{\gamma}})\mathbf{f} + \mathbf{m}_e \\ &= \mathbf{R}^\top \frac{d\mathbf{h}^t}{dt} + \mathbf{R}^\top (\dot{\mathbf{r}} \times \mathbf{p}^t) = \mathbf{R}^\top \frac{d\mathbf{R}\mathbf{h}}{dt} + (\mathbf{R}^\top \dot{\mathbf{r}}) \times (\mathbf{R}^\top \mathbf{p}^t) = \dot{\mathbf{h}} + \tilde{\boldsymbol{\omega}}\mathbf{h} + \tilde{\mathbf{v}}\mathbf{p} \end{aligned} \quad (2.88)$$

where the cross product equivalence under proper rotations, $\mathbf{R}(\mathbf{e}_x + \boldsymbol{\gamma}) \times \mathbf{R}\mathbf{f} = \mathbf{R}((\mathbf{e}_x + \boldsymbol{\gamma}) \times \mathbf{f})$, has been used. Thus we arrived to the equations describing the geometrically nonlinear dynamics of slender structures in spatial and material form, after the kinematic assumptions in Sec. 2.3.1, the general principles of elasticity of Sec. 2.3.2 and the concretion of this section. In order to close the formulation, another set of six equations ought to be added to Eqs. (2.86) because velocities and strains are not independent from each other, the former are the derivative of position and rotation with respect to time, the latter with respect to space. Therefore compatibility equations are derived by taking the time derivatives of strain and moment strain (equivalent to taking the expressions for the variation of strain, $\delta\boldsymbol{\gamma}$ in Eq. (2.69), and moment strain, $\delta\boldsymbol{\kappa}$ in Eq. (2.71), and realizing that $\overline{\delta\boldsymbol{\theta}}/\delta t = \boldsymbol{\omega}$, $\overline{\delta\mathbf{u}}/\delta t = \mathbf{v}$, and $\boldsymbol{\kappa} + \mathbf{k}_0 = \mathbf{k}$):

$$\dot{\boldsymbol{\gamma}} = \mathbf{v}' + \tilde{\boldsymbol{\kappa}}\mathbf{v} + \tilde{\mathbf{k}}_0\mathbf{v} + (\tilde{\mathbf{e}}_x + \tilde{\boldsymbol{\gamma}})\boldsymbol{\omega} \quad (2.89a)$$

$$\dot{\boldsymbol{\kappa}} = \boldsymbol{\omega}' + \tilde{\boldsymbol{\kappa}}\boldsymbol{\omega} + \tilde{\mathbf{k}}_0\boldsymbol{\omega} \quad (2.89b)$$

Combining the intrinsic equations, Eqs. (2.86), and compatibility equations, (2.89), the final intrinsic system is:

$$\begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{h}} \end{bmatrix} - \begin{bmatrix} \dot{\mathbf{f}} \\ \dot{\mathbf{m}} \end{bmatrix}' - \begin{bmatrix} \tilde{\mathbf{k}}_0 & \mathbf{0} \\ \tilde{\mathbf{e}}_x & \tilde{\mathbf{k}}_0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} + \begin{bmatrix} \tilde{\boldsymbol{\omega}} & \mathbf{0} \\ \tilde{\mathbf{v}} & \tilde{\boldsymbol{\omega}} \end{bmatrix} \begin{bmatrix} \mathbf{p} \\ \mathbf{h} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{f}} \\ \tilde{\mathbf{f}} & \tilde{\mathbf{m}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_e \\ \mathbf{m}_e \end{bmatrix} \quad (2.90a)$$

$$\begin{bmatrix} \dot{\boldsymbol{\gamma}} \\ \dot{\boldsymbol{\kappa}} \end{bmatrix} - \begin{bmatrix} \dot{\mathbf{v}} \\ \dot{\boldsymbol{\omega}} \end{bmatrix}' + \begin{bmatrix} -\tilde{\mathbf{k}}_0 & -\tilde{\mathbf{e}}_x \\ \mathbf{0} & -\tilde{\mathbf{k}}_0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix} - \begin{bmatrix} -\tilde{\boldsymbol{\omega}} & -\tilde{\mathbf{v}} \\ \mathbf{0} & -\tilde{\boldsymbol{\omega}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\gamma} \\ \boldsymbol{\kappa} \end{bmatrix} = \mathbf{0} \quad (2.90b)$$

These equations form a system of 8 variables with 3 components each, $\mathbf{p}, \mathbf{h}, \mathbf{f}, \mathbf{m}, \mathbf{v}, \boldsymbol{\omega}, \boldsymbol{\gamma}, \boldsymbol{\kappa}$, i.e. 24 unknowns for a total of 12 equations. The material properties introduce the final set of relations, named the constitutive equations, which under linear assumptions are written as,

$$\begin{bmatrix} \boldsymbol{\gamma}(s, t) \\ \boldsymbol{\kappa}(s, t) \end{bmatrix} = \boldsymbol{\mathcal{C}}(s) \begin{bmatrix} \mathbf{f}(s, t) \\ \mathbf{m}(s, t) \end{bmatrix} \quad ; \quad \begin{bmatrix} \mathbf{p}(s, t) \\ \mathbf{h}(s, t) \end{bmatrix} = \boldsymbol{\mathcal{M}}(s) \begin{bmatrix} \mathbf{v}(s, t) \\ \boldsymbol{\omega}(s, t) \end{bmatrix} \quad (2.91)$$

the compliance matrix $\boldsymbol{\mathcal{C}}$ relates sectional forces and moments to strains and curvatures – in problems such as plasticity, where this relation cannot be assumed linear, the constitutive connection would have to be taken to the variational equation (2.84), though in that case it would not be possible to link the nonlinear equations of the reduced model to a (linear) full-FE model which is the aim of this work. This matrix is difficult to obtain for complex structures with composite materials and usually homogenization or asymptotic methods are utilized to predict it [43, 48]. The mass matrix, $\boldsymbol{\mathcal{M}}$, links velocities and momenta, and is not trivial to obtain for structures with distributed inertia either. In Sec. 2.4 we show how to circumvent having to calculate explicit expressions of $\boldsymbol{\mathcal{C}}$ and $\boldsymbol{\mathcal{M}}$ by solving the equations in modal space; furthermore, they are time-independent, due to the material formulation that considers quantities in the reference configuration, which greatly facilitates the nonlinear computations. In order to attain a more compact formulation, the unknowns in the problem are grouped into the velocity state variable, $\mathbf{x}_1$, the internal force variable, $\mathbf{x}_2$, and the applied forces and moments per unit length, $\mathbf{f}_1$,

$$\mathbf{x}_1 = \begin{bmatrix} \mathbf{v} \\ \boldsymbol{\omega} \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} \mathbf{f} \\ \mathbf{m} \end{bmatrix} \quad \text{and} \quad \mathbf{f}_1 = \begin{bmatrix} \mathbf{f}_e \\ \mathbf{m}_e \end{bmatrix} \quad (2.92)$$

all expressed in terms of their components in the deformed material frame, consequently, $\mathbf{f}_e(s, t)$ and $\mathbf{m}_e(s, t)$ are follower forces and moments per unit span length, respectively. Note also that displacements and rotations do not appear explicitly in equations. Thus, combining Eqs. (2.90), (2.91) and (2.92) a Cosserat rod model is built, where the deformed state on the full domain is approximated by a deformable space curve Γ – identified with the aircraft major load-paths – whose dynamics are described by the intrinsic equations, written here in compact form as in

[236],

$$\mathcal{M}\dot{\mathbf{x}}_1 - \mathbf{x}'_2 - \mathbf{E}\mathbf{x}_2 + \mathcal{L}_1(\mathbf{x}_1)\mathcal{M}\mathbf{x}_1 + \mathcal{L}_2(\mathbf{x}_2)\mathcal{C}\mathbf{x}_2 = \mathbf{f}_1 \quad (2.93a)$$

$$\mathcal{C}\mathbf{x}_2 - \mathbf{x}'_1 + \mathbf{E}^\top \mathbf{x}_1 - \mathcal{L}_1^\top(\mathbf{x}_1)\mathcal{C}\mathbf{x}_2 = \mathbf{0} \quad (2.93b)$$

The linear operators, $\mathcal{L}_1$, $\mathcal{L}_2$, and the matrix $\mathbf{E}$ are defined as,

$$\mathcal{L}_1(\mathbf{x}_1) = \begin{bmatrix} \tilde{\omega} & \mathbf{0} \\ \tilde{\mathbf{v}} & \tilde{\omega} \end{bmatrix} \quad ; \quad \mathcal{L}_2(\mathbf{x}_2) = \begin{bmatrix} \mathbf{0} & \tilde{\mathbf{f}} \\ \tilde{\mathbf{f}} & \tilde{\mathbf{m}} \end{bmatrix} \quad ; \quad \mathbf{E} = \mathcal{L}_1 \left(\begin{bmatrix} \mathbf{e}_x \\ \mathbf{k}_0 \end{bmatrix} \right) \quad (2.94)$$

The above description is geometrically-exact with quadratic nonlinearities only. It easily allows discretisation using finite elements [23] and finite differences [216], although here a global basis will be considered, as defined below. It is worth mentioning structural damping is not included in the equations, for aerodynamic damping has a much larger effect in aeroelastic analysis and modelling the structural damping relies heavily on experiments. It would not be difficult to add to the equations if necessary as shown in [237].

Solution in modal coordinates via Galerkin projection

The LNMs of the structure are used to construct, as a linear combination, a set of intrinsic modes. The intrinsic equations are then solved by projecting the state variables into these modes, namely, the velocity modes, $\phi_1(s) \in \mathbb{R}^6$, the internal force modes, $\phi_2(s) \in \mathbb{R}^6$, the momentum modes, $\psi_1 = \mathcal{M}\phi_1(s) \in \mathbb{R}^6$, and the strain modes, $\psi_2 = \mathcal{C}\phi_2(s) \in \mathbb{R}^6$. Using Einstein's summation convention, the modal projection of the state vectors is then defined as,

$$\begin{aligned} \mathbf{x}_1(s, t) &= \phi_{1j}(s)q_{1j}(t) & ; & & [\mathbf{p}(s, t); \mathbf{h}(s, t)] &= \psi_{1j}(s)q_{1j}(t) \\ \mathbf{x}_2(s, t) &= \phi_{2j}(s)q_{2j}(t) & ; & & [\boldsymbol{\gamma}(s, t); \boldsymbol{\kappa}(s, t)] &= \psi_{2j}(s)q_{2j}(t) \end{aligned} \quad (2.95)$$

where q_{1j} and q_{2j} are the dual intrinsic modal coordinates for the j -th LNM. Intrinsic mode shapes are obtained as the non-trivial solutions to the linear homogeneous equations derived by linearisation of Eqs. (2.93) about a fixed point, here the null state (unloaded reference) is

considered, which gives [236]

$$\mathcal{M}\dot{\mathbf{x}}_1 - \mathbf{x}'_2 - \mathbf{E}\mathbf{x}_2 = \mathbf{f}_1 \quad (2.96a)$$

$$\mathcal{C}\dot{\mathbf{x}}_2 - \mathbf{x}'_1 + \mathbf{E}^\top \mathbf{x}_1 = \mathbf{0} \quad (2.96b)$$

The solution to this linear equation is found by letting $\mathbf{x} = [\mathbf{x}_1, \mathbf{x}_2] = \boldsymbol{\phi}_j q_j$ with $\boldsymbol{\phi} = [\boldsymbol{\phi}_1, \mathbf{i}\boldsymbol{\phi}_2]$ and $q_j = q_{1j} = q_{2j} = e^{i\omega_j t}$ such as the relation between the modes is,

$$\omega_j \boldsymbol{\psi}_{1j}(s) = \boldsymbol{\phi}'_{2j}(s) + \mathbf{E}\boldsymbol{\phi}_{2j}(s) \quad (2.97a)$$

$$\omega_j \boldsymbol{\psi}_{2j}(s) = -\boldsymbol{\phi}'_{1j}(s) + \mathbf{E}^\top \boldsymbol{\phi}_{1j}(s) \quad (2.97b)$$

Analytical expressions of $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$ are given in [63] for a simple cantilever, though they will be approximated here for complex structures built with FE models after application of the reduction techniques defined in section 2.2. Using the intrinsic modes and the projection of the state variables, a Galerkin projection is performed on Eqs. (2.93),

$$\int_{\Gamma} \boldsymbol{\phi}_{1j}^\top [\boldsymbol{\psi}_{1k} \dot{q}_{1k} - (\boldsymbol{\phi}'_{2k} + \mathbf{E}\boldsymbol{\phi}_{2k}) q_{2k} + \mathcal{L}_1(\boldsymbol{\phi}_{1k}) q_{1k} \boldsymbol{\psi}_{1l} q_{1l} + \mathcal{L}_2(\boldsymbol{\phi}_{2k}) q_{2k} \boldsymbol{\psi}_{2l} q_{2l}] ds \quad (2.98a)$$

$$= \int_{\Gamma} \boldsymbol{\phi}_{1j}^\top \mathbf{f}_a ds$$

$$\int_{\Gamma} \boldsymbol{\phi}_{2j}^\top [\boldsymbol{\psi}_{2k} \dot{q}_{2k} - (\boldsymbol{\phi}'_{1k} - \mathbf{E}^\top \boldsymbol{\phi}_{1k}) q_{1k} - \mathcal{L}_1(\boldsymbol{\phi}_{1k}) q_{1k} \boldsymbol{\psi}_{2l} q_{2l}] ds = 0 \quad (2.98b)$$

Considering the inner product in the 1D domain, $\langle \mathbf{u}, \mathbf{v} \rangle = \int_{\Gamma} \mathbf{u}^\top \mathbf{v} ds$, for any $\mathbf{u} \in \mathbb{R}^6$ and $\mathbf{v} \in \mathbb{R}^6$, from Eqs. (2.97) it can be proved that, with the appropriate normalisation, the intrinsic modes satisfy [238]

$$\langle \boldsymbol{\phi}_{1j}, \boldsymbol{\psi}_{1k} \rangle = \langle \boldsymbol{\phi}_{2j}, \boldsymbol{\psi}_{2k} \rangle = \delta_{jk} \quad (2.99a)$$

$$\langle \boldsymbol{\phi}_{1j}, (\boldsymbol{\phi}'_{2k} + \mathbf{E}\boldsymbol{\phi}_{2k}) \rangle = -\langle \boldsymbol{\phi}_{2j}, (\boldsymbol{\phi}'_{1k} - \mathbf{E}^\top \boldsymbol{\phi}_{1k}) \rangle = \omega_j \delta_{jk} \quad (2.99b)$$

where δ^{ij} is the Kronecker delta. Using these relations on the Galerkin projection of the equations, (2.98), the modal amplitudes are finally obtained and written in tensor notation as,

$$\begin{aligned}\dot{q}_{1j} &= \delta^{ji} \omega_i q_{2i} - \Gamma_1^{jik} q_{1i} q_{1k} - \Gamma_2^{jik} q_{2i} q_{2k} + \eta_j \\ \dot{q}_{2j} &= -\delta^{ji} \omega_i q_{1i} + \Gamma_2^{ijk} q_{1i} q_{2k}\end{aligned}\tag{2.100}$$

The coefficients $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are third-order tensors that encapsulate all modal couplings in the response, the former introduces the gyroscopic terms in the dynamics and the latter introduces the strain-force nonlinear relation; $\boldsymbol{\eta}$ is the modal projection of the external forcing terms:

$$\begin{aligned}\Gamma_1^{jkl} &= \langle \boldsymbol{\phi}_{1j}, \mathcal{L}_1(\boldsymbol{\phi}_{1k}) \boldsymbol{\psi}_{1l} \rangle, \\ \Gamma_2^{jkl} &= \langle \boldsymbol{\phi}_{1j}, \mathcal{L}_2(\boldsymbol{\phi}_{2k}) \boldsymbol{\psi}_{2l} \rangle, \\ \eta_j &= \langle \boldsymbol{\phi}_{1j}, \mathbf{f}_1 \rangle\end{aligned}\tag{2.101}$$

In the case of static elastic problems, the equations are solved by setting $\frac{d}{dt} = 0$ and the velocity to zero, $q_1 = 0$. The resulting equations are nonlinear algebraic equations in q_2 :

$$\delta^{ji} \omega_i q_{2i} - \Gamma_2^{jkl} q_{2k} q_{2l} + \eta_{1j} = 0\tag{2.102}$$

We will refer to equation (2.100) as the nonlinear modal reduced-order model (NMROM) of the original 3D structure. It has dimension $2 \times N_m$, where N_m is the number of LNMs retained from the condensed system in section 2.2. In practice, the intrinsic equations of motion can become very stiff if the whole set of modes are retained. Convergence in the time marching algorithm requires the maximum allowable time-step to be of the order of the inverse of the highest frequency, $\Delta t \propto \frac{1}{\omega_m}$, which can be too small. As shown in [214], while high frequency modes are needed to retain the couplings in the quadratic terms for large deformations, the linear part in Eq. (2.100) (multiplied by the frequency) operates at much higher frequencies than the forcing and quadratic terms. A nonlinear residualisation scheme is implemented where a cut-off frequency, ω_c , is selected and sets a division on low and high frequency modes: $\mathbf{q}(t) = [\mathbf{q}_L(t); \mathbf{q}_H(t)]$. If N_m is the total number of modes in the computations, N_L is the number

of modes up to the cut-off frequency. The high-frequency dynamic equations ($N_j > N_L$) are time-averaged which yields $\dot{\mathbf{q}}_H(t) = \mathbf{0}$, and results in a set of differential-algebraic-equations (DAE):

$$\begin{aligned}\dot{q}_{1j} &= \delta^{ji} \omega_i q_{2i} - \Gamma_1^{jik} q_{1i} q_{1k} - \Gamma_2^{jik} q_{2i} q_{2k} + \eta_j \\ \dot{q}_{2j} &= -\delta^{ji} \omega_i q_{1i} + \Gamma_2^{ijk} q_{1i} q_{2k} \\ q_{1s} &= \frac{1}{\omega_s} \Gamma_2^{isk} q_{1i} q_{2k} \\ q_{2s} &= \frac{1}{\omega_s} (\Gamma_1^{sik} q_{1i} q_{1k} + \Gamma_2^{sik} q_{2i} q_{2k} - \eta_{1s})\end{aligned}\tag{2.103}$$

with $i, k \in (1, 2, \dots, N_m)$, $j \in (1, 2, \dots, N_L)$ and $s \in (N_L + 1, \dots, N_m)$. The time step in this system is of the order of ω_c^{-1} . Both Newton-Raphson and fix-point approaches have successfully been shown to solve these algebraic equations (ω_c needs to be relatively large for the fix-point to be stable). The approach leads to important computational savings while maintaining accuracy as demonstrated in Sec. 5.2.2.

Finally in (nonlinear) free vibrations, the total energy of the system is given by the functional $\mathcal{E}_0$:

$$\mathcal{E}_0 = \frac{1}{2} (\langle \mathbf{x}_1, \mathcal{M} \mathbf{x}_1 \rangle + \langle \mathbf{x}_2, \mathcal{C} \mathbf{x}_2 \rangle) = \sum_{j=1}^{\infty} \frac{1}{2} (q_{1j}^2 + q_{2j}^2)\tag{2.104}$$

Given a set of initial conditions, $\mathcal{E}_0$ remains constant for any number of modes N_m within that modal subspace [238]. This will be used as a metric for the validation of complex models and time-integration error checking.

Recovery of displacements and rotations along load-paths

The displacements and rotations at each node are not variables of the intrinsic description but can be recovered from the calculated velocity or strains. This can be attained by time-integrations of local velocities, $\mathbf{x}_1$, as it is done in rigid body dynamics. Following Eqs. (2.37) and (2.38) and assuming $\boldsymbol{\omega}_0 = \mathbf{0}$ for simplicity,

$$\begin{aligned}\dot{\mathbf{R}}^{ab} &= \mathbf{R}^{ab} \tilde{\boldsymbol{\omega}} \\ \dot{\mathbf{r}}_a &= \mathbf{R}^{ab} \mathbf{v}\end{aligned}\tag{2.105}$$

As shown in [63], quaternions $\boldsymbol{\zeta} = [\zeta_0, \zeta_1, \zeta_2, \zeta_3](s, t) = [\zeta_0, \boldsymbol{\zeta}_x](s, t)$ can be used to parameterize the rotation, $\mathbf{R}^{ab}$, so that the first of the equations in (2.105) is replaced by

$$\dot{\boldsymbol{\zeta}} = \begin{bmatrix} \dot{\zeta}_0 \\ \dot{\boldsymbol{\zeta}}_x \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\boldsymbol{\omega}^\top \boldsymbol{\zeta}_x \\ \frac{1}{2}(\zeta_0 \boldsymbol{\omega} - \tilde{\boldsymbol{\omega}} \boldsymbol{\zeta}_x) \end{bmatrix} \quad (2.106)$$

Time integration of $\mathbf{r}_a$ is straightforward after calculation of the rotation matrix $\mathbf{R}^{ab}$, which is built from $\boldsymbol{\zeta}$ as,

$$\begin{aligned} \mathbf{R}^{ab} &= [-\boldsymbol{\zeta}_x \quad \zeta_0 \mathbf{I}_3 + \tilde{\boldsymbol{\zeta}}_x] [-\boldsymbol{\zeta}_x \quad \zeta_0 \mathbf{I}_3 - \tilde{\boldsymbol{\zeta}}_x]^\top \\ &= \begin{bmatrix} \zeta_0^2 + \zeta_1^2 - \zeta_2^2 - \zeta_3^2 & -2\zeta_0\zeta_3 + 2\zeta_1\zeta_2 & 2\zeta_0\zeta_2 + 2\zeta_1\zeta_3 \\ 2\zeta_0\zeta_3 + 2\zeta_1\zeta_2 & \zeta_0^2 - \zeta_1^2 + \zeta_2^2 - \zeta_3^2 & -2\zeta_0\zeta_1 + 2\zeta_2\zeta_3 \\ -2\zeta_0\zeta_2 + 2\zeta_1\zeta_3 & 2\zeta_0\zeta_1 + 2\zeta_2\zeta_3 & \zeta_0^2 - \zeta_1^2 - \zeta_2^2 + \zeta_3^2 \end{bmatrix} \end{aligned} \quad (2.107)$$

In the second instance, the rotation and position in the inertial reference system are calculated by integration of strains along the domain, as in the Frenet-Serret formulas of differential geometry. Following definition of strains and curvatures, Eqs. (2.33) and (2.36),

$$\begin{aligned} \mathbf{R}^{ab'} &= \mathbf{R}^{ab} \tilde{\mathbf{k}} \\ \mathbf{r}'_a &= \mathbf{R}^{ab} (\boldsymbol{\gamma} + \mathbf{e}_x) \end{aligned} \quad (2.108)$$

Analytical solutions to Eq. (2.108) can be obtained when the strain is assumed constant between nodes and a piecewise constant integration is carried out, as is the case in the current implementation. If the beam path is discretized in $n+1$ points, strain and curvatures are defined in the mid-points of the spatial discretization (n in total). γ_n and κ_n are constant within the segment $s_{n-1} \leq s \leq s_n$, and the position and rotation matrix after integration are

$$\begin{aligned} \mathbf{R}^{ab}(s) &= \mathbf{R}^{ab}(s_{n-1}) \mathcal{H}^0(\mathbf{k}, s) \\ \mathbf{r}_a(s) &= \mathbf{r}_a(s_{n-1}) + \mathbf{R}^{ab}(s_{n-1}) \mathcal{H}^1(\mathbf{k}, s) (\mathbf{e}_x + \boldsymbol{\gamma}_n) \end{aligned} \quad (2.109)$$

Where the operators $\mathcal{H}^0(\mathbf{k}, s)$ and $\mathcal{H}^1(\mathbf{k}, s)$ are obtained from integration of the exponential function and defined as in [23],

$$\begin{aligned}\mathcal{H}^0(\mathbf{k}, s) &= e^{\Delta\tilde{\mathbf{k}}} = \mathbf{I} + \frac{\sin(\Delta\phi)}{\Delta\phi}\Delta\tilde{\Psi} + \frac{1 - \cos(\Delta\phi)}{\Delta\phi^2}\Delta\tilde{\Psi}\Delta\tilde{\Psi} \\ \mathcal{H}^1(\mathbf{k}, s) &= \Delta s \left(\mathbf{I} + \frac{1 - \cos(\Delta\phi)}{\Delta\phi^2}\Delta\tilde{\Psi} + \frac{\Delta\phi - \sin(\Delta\phi)}{\Delta\phi^3}\Delta\tilde{\Psi}\Delta\tilde{\Psi} \right)\end{aligned}\tag{2.110}$$

with $\Delta s = s - s_{n-1}$, $\Delta\Psi = \mathbf{k}\Delta s$ and $\Delta\phi = \|\Delta\Psi\|$. Note that when position and rotations are recovered from strain integration, there is still one point that is either clamped or needs to be tracked from integration of its local velocity.

2.4 Bridging full and reduced descriptions through modal spaces

In the previous sections we showed how to reduce a full FEM into a set of nodes conforming a skeleton-like condensed model along the major load paths of the aircraft, Γ ; the nonlinear dynamics of Γ were derived using an intrinsic formulation and linked with the general equations of elasticity. In this section we present a solution to Eqs. (2.93) without specific knowledge of the sectional mass and stiffness properties of the 1D model, $\mathcal{M}$ and $\mathcal{C}$. Rather the FE matrices in Eq. (2.26), $\mathbf{M}_a$, $\mathbf{K}_a$, and their eigenvectors, Φ_a , are employed to approximate the intrinsic modes, Eqs. (2.95), in the solution of the modal equations, (2.100). Importantly, only linear solutions of the 3D built-up FE problem are needed to construct the final NMROM. A method to recover the 3D state is also presented, based on the assumption of linear cross-section deformations, which are obtained from the linear full-FE model, then superimposed to nonlinear solution of Γ .

2.4.1 Approximation of intrinsic modes and nonlinear coefficients

Let Φ_a be the solution of the eigenvalue problem using the condensed matrices, $\mathbf{M}_a$ and $\mathbf{K}_a$, defined in Eq. (2.26). Φ_a includes the full set of modes in the condensed system written as displacement and linear rotations at the nodes along Γ . Those mode shapes also define

velocity and strain distributions, which are computed next. Moreover, standard FE solvers yield results in the global reference frame while the intrinsic modes are defined in the initial local configuration (with the convention of the x -direction running along the local beam). Therefore, a matrix $\Xi_0(s) = [\mathbf{R}^{ba}(s, 0), \mathbf{0}; \mathbf{0}, \mathbf{R}^{ba}(s, 0)]$ is introduced to rotate the 6-component vectors from the global to the local initial frame, $\mathbf{R}^{ba}(s, 0)$ known analytically or calculated from the structural nodes position. In order to construct the functionals that will be integrated to obtain the nonlinear coefficients in Eqs. (2.101) we define a continuous displacement/rotation mode shape in the initial local reference frame, $\phi_0(s) = \Xi_0(s)\mathcal{I}_i(\Phi_a)$, obtained from interpolation in Γ between the discrete values of Φ_a , where $\mathcal{I}_i$ is the interpolation operator of order i . The distribution of nodal velocity associated to the modes shapes is obtained by a differentiation in time of the nodal linear displacement and rotation variables,

$$\Delta \mathbf{x}_1 = \Delta \dot{\mathbf{x}}_0 \Rightarrow \begin{cases} \phi_{1j}(s) = \dot{\phi}_{0j}(s) \\ q_{1j}(t) = \dot{q}_{0j}(t) \end{cases} \quad (2.111)$$

Thus the discrete velocity mode is defined as $\Phi_{1j} = \dot{\Phi}_{0j}$. Initially a linear interpolation is sought for the continuous displacement, $\phi_0(s)$, and velocities modes, $\phi_1(s)$. The linear interpolation between nodes i and $i + 1$, with $s \in [s_i, s_{i+1}]$ and $\Delta s_i = s_{i+1} - s_i$ has to satisfy $\phi_{0j}(s_i) = \Phi_{0j,i}$ and $\phi_{0j}(s_i + \Delta s_i) = \Phi_{0j,i+1}$, and similarly for the velocity modes. The interpolation becomes,

$$\phi_{0j}(s) = \Xi_0(s_i) \left(\Phi_{0j,i} \frac{s_{i+1} - s}{\Delta s_i} + \Phi_{0j,i+1} \frac{s - s_i}{\Delta s_i} \right) \quad (2.112)$$

$$\phi_{1j}(s) = \dot{\phi}_{0j}(s) \quad (2.113)$$

The corresponding distribution of linear and rotational momenta at the master nodes can be obtained using the condensed inertia matrix, $\Psi_{1j} = \mathbf{M}_a \Phi_{1j} = \mathbf{M}_a \dot{\Phi}_{0j}$, expressed in their components in the global frame of reference. The introduction of this momentum mode has allowed calculations to be performed on distributed mass models and the use of any type of condensation technique. Because the mass matrix is already calculated as an integral along the 3D domain and then condensed to a set of master nodes, the continuous momentum mode

shapes, $\boldsymbol{\psi}_1$, are considered lumped and defined as,

$$\boldsymbol{\psi}_{1j}(s) = \boldsymbol{\Xi}_0(s_i) \boldsymbol{\Psi}_{1,j,i} \delta(s - s_i) \quad (2.114)$$

where δ is Dirac's delta function. Each displacement mode also generates a corresponding internal stress state. This defines discrete force/moment modes, $\boldsymbol{\Phi}_2$, which are obtained from the displacement modes and the condensed stiffness matrix using a summation-of-forces approach from aircraft load analysis, [239, Ch. 18]. If N_n is the total number of nodes, the internal node forces are for each mode j computed as $\mathbf{f}_{(j)} = \mathbf{K}_a \boldsymbol{\Phi}_{0j}^{1-3}$, with $\mathbf{f}_{(j)} \in \mathbb{R}^{3N_n}$; and analogously the internal moments as $\mathbf{m}_{\mathbf{f}(j)} = \mathbf{K}_a \boldsymbol{\Phi}_{0j}^{4-6}$. Next, a function $\mathcal{S}(\mathbf{h}, s_i)$ is defined so that it sums the value of $\mathbf{h}$ at each node in the path from a free end to the node s_i in the structure. $\boldsymbol{\Phi}_2$ is calculated as

$$\boldsymbol{\Phi}_{2j,i+\frac{1}{2}} = \begin{bmatrix} \mathcal{S}(\mathbf{f}_{(j)}, s_i) \\ \mathcal{S}(\mathbf{m}_{\mathbf{f}(j)} + (\mathbf{r}_i - \mathbf{r}_{i+\frac{1}{2}}) \times \mathbf{f}_{(j)}, s_i) \end{bmatrix} \quad (2.115)$$

where $\mathbf{r}_i$ is the position vector of the nodes summed by $\mathcal{S}$, and $\mathbf{r}_{i+\frac{1}{2}}$ the mid position between nodes s_i and s_{i+1} . The first term is the sum of forces due to modal displacements and the second one the sum of moments due to modal rotations and the cross product of the position vector and the previous calculated force. This equation represents an equilibrium of internal forces and moments – between node i and $i + 1$. Results are presented in the mid-point $i + \frac{1}{2}$ because more information cannot be extracted in terms of linear stresses from one node to the other. The sum $\mathcal{S}$ is taken as positive in the direction of increasing s , and negative otherwise, so that the addition of internal forces and moments from all nodes in the model equals zero. An algorithm to calculate the optimal path for this sum was designed. The continuous force modes, $\boldsymbol{\phi}_2$, are now interpolated as piece-wise constant between nodes,

$$\boldsymbol{\phi}_{2j}(s) = \boldsymbol{\Xi}_0(s_i) \boldsymbol{\Phi}_{2j,i+\frac{1}{2}} \quad (2.116)$$

with $s \in [s_i, s_{i+1}]$, as before. Finally, the strain modes $\boldsymbol{\psi}_2$ are obtained from spatial derivatives of the displacement modes along Γ , following Eq. (2.97b), and interpolated as piece-wise constant

too,

$$\psi_{2j}(s) = -\frac{\phi_{1j}(s_{i+1}) - \phi_{1j}(s_i)}{\Delta s_i} + \mathbf{E}^\top \frac{\phi_{1j}(s_{i+1}) + \phi_{1j}(s_i)}{2} \quad (2.117)$$

It is worth mentioning that approximation at the midpoint of ψ_2 makes the derivative in the first term of the equation second order accurate. In line with increasing the accuracy of the computations, it is possible to introduce a cubic interpolation for the bending components of ϕ_0 and ϕ_1 , $(\bullet)^{2-3}$, noticing that their respective angular components, $(\bullet)^{5-6}$ are not independent ($(\bullet)^1$ and $(\bullet)^4$ are the axial and torsional components respectively). This increases the convergence rate of the solution significantly, as shown in section 5.2.1 below. To the two conditions for the linear approximation in Eq. (2.112) (repeated here as $\phi_{1j}^{1-6}(s_i) = \Phi_{1j,i}^{1-6}$ and $\phi_{1j}^{1-6}(s_i + \Delta s_i) = \Phi_{1j,i+1}^{1-6}$), the following are added $\frac{\partial \phi_{1j}^{2-3}}{\partial s}(s_i) = \Phi_{1j,i}^{5-6}$ and $\frac{\partial \phi_{1j}^{2-3}}{\partial s}(s_i + \Delta s_i) = \Phi_{1j,i+1}^{5-6}$. And the cubic interpolation for the bending components takes the form of, $\phi_{1j}^{2-3}(s) = a_0 + a_1(s - s_i) + a_2(s - s_i)^2 + a_3(s - s_i)^3$. The coefficients a_k can be calculated using these conditions for the bending components, which form a linear system (for DoF 2 and 5),

$$\begin{bmatrix} \Phi_{1j,i}^2 \\ \Phi_{1j,i}^5 \\ \Phi_{1j,i+1}^2 \\ \Phi_{1j,i+1}^5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & \Delta s_i & \Delta s_i^2 & \Delta s_i^3 \\ 0 & 1 & 2\Delta s_i & 3\Delta s_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (2.118)$$

solved after inverting the system, yielding the a_k coefficients as a function of discrete mode values,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3\Delta s_i^{-2} & -2\Delta s_i^{-1} & 3\Delta s_i^{-2} & \Delta s_i^{-1} \\ 2\Delta s_i^{-3} & \Delta s_i^{-2} & -2\Delta s_i^{-3} & \Delta s_i^{-2} \end{bmatrix} \begin{bmatrix} \Phi_{1j,i}^2 \\ \Phi_{1j,i}^5 \\ \Phi_{1j,i+1}^2 \\ \Phi_{1j,i+1}^5 \end{bmatrix} \quad (2.119)$$

The same analysis is done for DoF 3 and 6. As the continuous form of the intrinsic modes has been built, we can set out to find the integrals along the load paths needed to solve Eqs. (2.100). Firstly the integral of ϕ_{1j} after a linear interpolation is,

$$\int_{s_i}^{s_{i+1}} \phi_{1j}(s) ds = \frac{\Phi_{1j,i} + \Phi_{1j,i+1}}{2} \Delta s_i \quad (2.120)$$

and taking the calculated values of a_k , the integration of the bending components of ϕ_{1j} with the cubic approximation yields an elegant expression,

$$\int_{s_i}^{s_{i+1}} \phi_{1j}^{2-3}(s) ds = \frac{\Phi_{1j,i}^{l2-3} + \Phi_{1j,i+1}^{l2-3}}{2} \Delta s_i + \frac{\Phi_{1j,i}^{l5-6} - \Phi_{1j,i+1}^{l5-6}}{12} \Delta s_i^2 \quad (2.121)$$

It is seen that the previous linear expression is recovered and an additional quadratic term appears as a function of the angular variables. As Δs_i approaches zero the quadratic term is less and less important; however, for already built models that cannot be modified and therefore one has no control over Δs_i , as is the case in industrial models, this improves accuracy. $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$, can be now calculated from Eq. (2.101). Application of Lebesgue integral on Dirac's function makes the integral of $\mathbf{\Gamma}_1$ the discretised set of sums:

$$\Gamma_1^{jkl} = \sum_i \phi_{1j}^\top(s_i) \mathcal{L}_1(\phi_{1k}(s_i)) \psi_{1l}(s_i) \quad (2.122)$$

Piece-wise constant interpolation of ϕ_2 and ψ_2 , makes the integral to approximate $\mathbf{\Gamma}_2$ be proportional to the integral of ϕ_{1j} for each line segment,

$$\Gamma_2^{jkl} = \sum_i \mathcal{F}_2(\phi_{1j}(s_i))^\top \mathcal{L}_2(\phi_{2k}(s_{i+\frac{1}{2}})) \psi_{2l}(s_{i+\frac{1}{2}}) \Delta s_i \quad (2.123)$$

The linear interpolation of ϕ_1 gives $\mathcal{F}_2$ as

$$\mathcal{F}_2^{lin}(\phi_1(s_i)) = \frac{\phi_1(s_i) + \phi_1(s_{i+1})}{2} \quad (2.124)$$

and the cubic interpolation of the bending components gives a quadratic approximation

$$\mathcal{F}_2^{quad}(\phi_1(s_i)) = \mathcal{F}_2^{lin}(\phi_1(s_i)) + \left[0, \left(\frac{\phi_1(s_i) - \phi_1(s_{i+1})}{12} \right) \Big|_{5-6}, \mathbf{0}_{3 \times 1} \right]^\top \Delta s_i \quad (2.125)$$

Lastly, if external forces and moments, $\mathbf{f}_1$, are defined as point-loads, the same principle in the computation of $\mathbf{\Gamma}_1$ applies, and the projection of the forces into the velocity modes is,

$$\eta_j = \sum_i \phi_{1j}(s_i)^\top \mathbf{f}_1(s_i) \quad (2.126)$$

Thus we have arrived to a solution to create reduced order models with geometrically nonlinear capabilities from linear FE models.

2.4.2 Recovery of full 3D state from the intrinsic solution

The recovery of the full 3D model state solution consists in finding the position of the omitted set of DoF left out of the analysis after the reduction process in Sec. 2.2. While this might not be a critical part for the aeroelastic analysis with aerodynamics given as GAFs, however, it adds some important features: 1) the derivation is built on the same assumptions made to construct the beam intrinsic equations, i.e. small cross-sectional dimensions compared to the length of the main components and small cross-sectional deformations compared to the cross-sectional dimensions. Therefore comparing the 3D displacements recovery with full FE computations will also validate the rationale behind beam assumptions; 2) Forces and stresses on the 3D structure can be recovered, which could be useful to do efficient and reliable analysis in other fields, such as aeroelastic tailoring; 3) if higher fidelity aerodynamics such as CFD are to be included, this method can be incorporated inside the solution process. 4) Showing the full state of the aircraft would help picture the different dynamics in simulations.

The process is based on the condensation matrix $\mathbf{T}_{oa}$ that linearly relates displacements of the master nodes conforming the 1D-skeleton, to those of the omitted nodes. The first step is to define a warping field, $\mathbf{w}_c$, that describes the displacement of the omitted nodes in the cross-section relative to a central node. For this, a structure such that showed in Fig. 2.3 has to be assumed, where a group of cross-sectional points in the omitted set (o-set) are linked to

a master point. The resulting warping field is the deformation of a cross-sectional point after deformations and rotations of the central point have been removed. Let $\mathbf{r}^{ik}$ be the position vector of an omitted node i in the cross-section, which is linked to a master node k with position $\mathbf{r}_a^k$; similarly $\mathbf{R}_a^k$ is the rotation matrix from the local to the global reference frame of the master node k ($\mathbf{R}^{ab}$ in Eq. (2.29)). Initially the position vector between i and k is $\mathbf{p}^{ik}$ and the warping field introduces the cross-sectional deformations in this node, $\mathbf{w}_c^{ik}$. This is illustrated in Fig. 2.6. Therefore the position vectors of the cross-sections can be put as,

$$\mathbf{r}^{ik} = \mathbf{r}_a^k + \mathbf{R}_a^k (\mathbf{p}^{ik} + \mathbf{w}_c^{ik}) \quad (2.127)$$

and the deformation field would then be,

$$\mathbf{u}^{ik} = \mathbf{u}_a^k + \mathbf{R}_a^k (\mathbf{p}^{ik} + \mathbf{w}_c^{ik}) - \mathbf{p}^{ik} \quad (2.128)$$

Now consider the (linear) deformation of any natural mode j in the model. The following

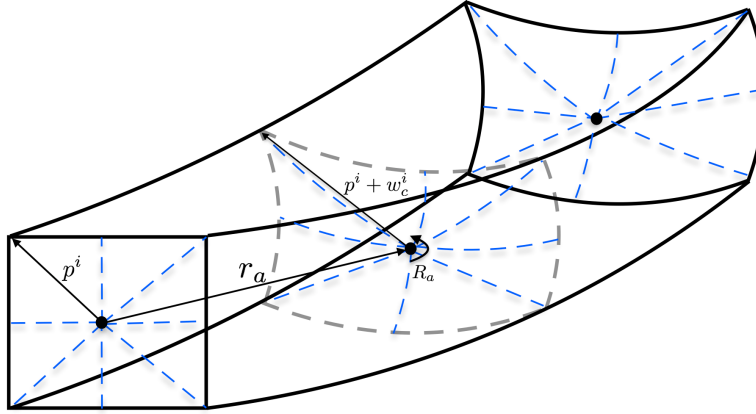


Figure 2.6: 3D description of deformations

approximations are the core of the recovery process:

- (i) The 3D displacement field is obtained for each of the mode j and approximated by the transformation matrix $\mathbf{T}_{oa}$ in Eq. (2.24),

$$\mathbf{u}^{ijk} = \Theta^{ik} \mathbf{T}_{oa} \Phi_0^j \quad (2.129)$$

where Φ_0 are the LNMs at the master nodes after condensation and Θ^{ik} the matrix that yields the displacement components of node i associated to node k :

$$\Theta^{ik} = \begin{bmatrix} \mathbf{0}_3 & \cdots & \mathbf{I}_3 & \cdots & \mathbf{0}_3 \end{bmatrix} \quad (2.130)$$

Note that $\mathbf{u}^{ijk}$ can also be retrieved directly from the 3D eigenvectors, what is important is the linearity condition, which on the other hand comes from the definition.

- (ii) Rotations of each master node k , are very small, i.e. linear, as per definition of the LNMs, so the rotation matrix is written as $\mathbf{R}_{\Omega_a}$ to differentiate it from the NMROM one. It is formed as an increment of the identity matrix and the pull back operation of vector $\mathbf{p}^i$ can be put as,

$$\mathbf{R}_{\Omega_a}^{jk} = \mathbf{I}_3 + d\mathbf{R}_{\Omega_a}^{jk} \rightarrow (\mathbf{R}_{\Omega_a}^{jk} - \mathbf{I}_3) \mathbf{p}^i = d\mathbf{R}_{\Omega_a}^{jk} \mathbf{p}^i = d\Omega_{\Omega_a}^{jk} \times \mathbf{p}^i \quad (2.131)$$

where $d\Omega_a^{jk}$ equals the rotational part of the LNMs, $d\Omega_a^{jk} = \Phi_0^{jk}|_{4-6}$ or a linear combination of those depending on the parametrisation of the rotation matrix used. The linear relation in Eq. (2.131) is consistent with infinitesimal rotations of the rotational group SO(3). Note that such approximation is not valid in our NMROM that accounts for finite rotations, the property highlighted here is only for the LNMs. What we are assuming is that there is a region where significant rotations take place on the master nodes but their effect on cross-sectional displacements remains linear, otherwise at the point where nonlinear deformation of the central nodes also give rise to nonlinear cross-sectional distortions, the type of beam analysis would not be adequate.

- (iii) Because the warping field is small compared to the cross-section, or to $\mathbf{p}$ for that matter, the small rotation on the warping field has second order effects,

$$\mathbf{R}_{\Omega_a}^k \mathbf{w}_c^{ijk} = (\mathbf{I}_3 + d\mathbf{R}_a^{jk}) \mathbf{w}_c^{ijk} \approx \mathbf{I}_3 \mathbf{w}_c^{ijk} \quad (2.132)$$

Based on these relations and Eq. (2.128), the warping field of node i associated to a LNM j is obtained by removing both displacement and rotational components of the reference node k :

$$\mathbf{w}_c^{ijk} = \Theta^{ik} \mathbf{T}_{oa} \Phi_0^j - \mathbf{T}_u^k \Phi_0^j - d\Omega_a^{jk} \times \mathbf{p}^i \quad (2.133)$$

with $\mathbf{T}_u^k$ being a matrix that selects the displacement components at the node k . This is a linear relation that can be developed further considering the rotation operator in Eq. (2.131):

$$\mathbf{w}_c^{ijk} = \Theta^{ik} \mathbf{T}_{oa} \Phi_0^j - \mathbf{T}_u^k \Phi_0^j - \mathbf{T}_p^{ik} \Phi_0^j = \mathbf{T}_w^{ik} \Phi_0^j \quad (2.134)$$

these matrices are then written as,

$$\begin{aligned} \mathbf{T}_u^k &= \begin{bmatrix} \mathbf{0}_3 & \cdots^k & -\mathbf{I}_3 & \cdots \mathbf{0}_3 \end{bmatrix} \\ \mathbf{T}_p^{ik} &= \begin{bmatrix} \mathbf{0}_3 & \cdots^k & -\tilde{\mathbf{p}}^i & \cdots \mathbf{0}_3 \end{bmatrix} \\ \mathbf{T}_w^{ik} &= \Theta^{ik} \mathbf{T}_{oa} + \mathbf{T}_u^k + \mathbf{T}_p^{ik} \end{aligned} \quad (2.135)$$

$\mathbf{T}_w$ is the warping transformation matrix that linearly relates the deformation of the main structure, master nodes, to the linear cross-sectional deformation of the omitted or slaves nodes. Therefore once the matrix $\mathbf{T}_w$ has been assembled, and the nonlinear 1D solution obtained, this expression allows a seamless calculation of the 3D state. At every time step a set of modes are excited with amplitude q_{0j} . The position vector of slave node i linked to the master k is written as,

$$\mathbf{r}^{ik}(t) = \mathbf{r}_a^k(t) + \mathbf{R}_a^k(t) \left(\mathbf{p}^i + \sum_j q_{0j}(t) \mathbf{T}_w^{ik} \Phi_0^j \right) \quad (2.136)$$

note that the $\mathbf{q}_0$ terms, controlling the amplitude of the warping, are obtained from integration of the velocity modal component that are part of the nonlinear equations (3.31). Effects such as Poisson-ratio contraction after extension of the main structure, widening of cross-sections after torsion, or chord-wise bending would appear as they are already incorporated in the LNMs of the (linear) 3D FEM. An alternative mathematical solution to extract the mass and compliance beam cross sectional properties has been implemented into the software VABS

[43, 231]. It relies on asymptotic analysis of slender structures with one dominant dimension such that mathematical reduction of the smaller dimensions can be applied in the approximation of the strain energy. In [44], for instance, a set of cross-sectional modes are included in the asymptotic solution, which would be very similar in shape to the warping functions in Eq. (2.134), only that those would come directly from the LNMs, i.e. the 3D FEM. While that could be an advantage of the present formulation, a limitation must also be acknowledged in that the procedure is posed only as a recovery of 3D quantities where the cross-sectional state do not affect the actual solution. Inspiration might be taken from that work to incorporate the cross-sectional (linear) deformations from the warping functions into the equations of motion, Eq. (2.93), which was also done in [235] with a predefined set of warping functions.

2.5 Brief summary

This chapter has presented a compendium of interrelated theories in structural mechanics that hold a significantly level of complexity independently and even more so when they are put together. Thus it was deemed important to summarise the main achievements of the chapter, distinguishing the author's from the literature's. We began with the main principles and equations in the theory of solid mechanics and, while the discussion followed that of textbooks, a concise exposition helped to merge together the subsequent theories under the very first principles of continuum mechanics.

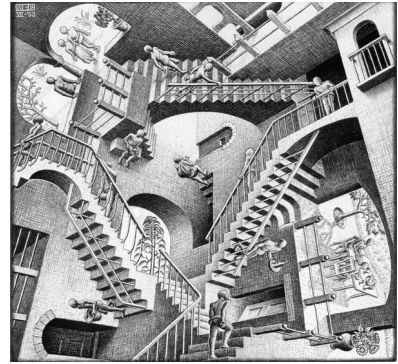
The link between general principles of elasticity and the geometrically exact beam theory is mostly found in Simo's work (in what concerns to the present work as there have been many approaches to build nonlinear beam theories). Some of the definitions in his work have been generalised and made clearer. The intrinsic description for nonlinear beams is due to Hodges. However the complete equivalence of Hodges and Simo equations from a material to a spatial description of the 3D quantities via a pull-back operation is believed to be a novelty of this work.

The project stands on a key contribution by Palacios, that is, the solution of the intrinsic equations in modal space to approximate the intrinsic variables from the LNMs of an arbitrary

structure after a Guyan condensation. The approach has been generalised herein to structures with distributed inertia and to arbitrary reduction techniques that allow a better approximation of the full FE model. Improvement to the continuous modal shapes and the integrals of the modal couplings has been introduced in Eqs. (2.118)-(2.125). Moreover, a method to recover the 3D state of the structural deformations has also been proposed using the condensation mapping and the representation of a warping field based on the LNMs.

Aeroelastic Integration and Flight Dynamics

Description



It is not knowledge, but the act of learning, not possession but the act of getting there, which grants the greatest enjoyment. When I have clarified and exhausted a subject, then I turn away from it, in order to go into darkness again; the never-satisfied man is so strange if he has completed a structure, then it is not in order to dwell in it peacefully, but in order to begin another.

Carl F. Gauss

Science and art sometimes can touch one another, like two pieces of the jigsaw puzzle which is our human life, and that contact may be made across the borderline between the two respective domains.

Maurits C. Escher

In this chapter we couple a (simplified) theory of unsteady aerodynamics with the equations describing the structural domain in Ch. 2 to form a nonlinear aeroelastic system that is efficient and suitable for the assessment of inflight dynamic and static loads. The theory of unsteady aerodynamics is first presented under inviscid assumptions, which allows to solve the fluid dynamics using potential-flow aerodynamics. An important feature of the coupling strategy is that because the structural description has been written in a material form, the aerodynamic

forces seamlessly follow the structure deformations and rotations. Changing boundary conditions in the flow as the structure moves non-linearly, however, is not accounted for in a straight manner in the solution of the potential equations around the reference configuration. Therefore, indications and solutions are also given to partially tackle this issue. The aerodynamic forces are converted from the frequency domain to the time domain via an interpolation strategy and the aeroelastic integration is carried out in the modal space, which allows for effortless construction of reduced order models. Next, a simplified description of the aircraft flight dynamics is introduced, which includes loads due to atmospheric disturbances, a trim strategy for steady level flight and static manoeuvres, and flight controls. All together they conform a powerful aeroelastic description that rests as the backbone of this work.

3.1 Historical remarks

Aeroelastic theory has reached many fields and flourished in a broad range of disciplines, from bridge engineering to wind turbine design; its roots, however, are found in the science of flight and the various incidents concerning airborne vehicles that boosted the need for a better understanding of the iterations between fluid and structural domains. A glance into the origins of human flight unravels how intertwined aviation itself is with the science of aeroelasticity, which might appear sometimes as an abstruse discipline due to the complexity of the mathematical formulations.

We are going to start this review with George Cayley (1773–1857), who bridged the gap between theory, engineering, and the experience of flight with gliders [240]. He explained the flying machine concept as a combination of systems that separately would provide lift, propulsion force, and a mechanism for control. Otto Lilienthal (1848–1896) set the foundations of experimental flight [241]. Through designing, building, and flying several monoplane and biplane gliders, he measured and collected data for wing designs, including the forces acting on cambered wings at various angles of attack. Samuel P. Langley (1834–1906) achieved the first sustained, powered flights in 1896, developing propulsion engines for the planes but he struggled with their control. This progress culminated in the work of Orville Wright (1871–1948)

and Wilbur Wright(1867-1912): they devised the systems for control, including wing warping for roll and elevator actuation for pitch; manufactured combustion engines for propulsion; and designed their own wind tunnels to study airfoils properties. Their efforts led to the first flight of a controlled and powered aircraft in 1903, which set a historical milestone in a discipline that rocketted after that point.

With the aviation world growing and thriving, a subject was emerging in the 1920s from the lessons learnt in the numerous accidents that occurred at the time: Arthur R. Collar (1908-1986) described the field of aeroelasticity as the study of the interaction between elastic, inertia, and aerodynamic forces. In his review of the first fifty years of the subject [242], he showed the importance of bringing these three disciplines together as their interplay had been the cause of many airplane catastrophic failures, including Langley's flying machine weeks prior to the Wright brothers' successful flight. Collar and colleagues Roxbee Cox (1902-1997), Alfred G. Pugsley (1903-1998), and Jolly W. Duncan (1894-1960) were pioneers in producing some of the first works to understand and prevent flutter, aileron reversal, and roll effectiveness, among others. The principles to characterise elastic and inertia forces were laid out in the previous chapter, so to complete Collar's triangle we briefly describe herein the theory of aerodynamics, which began as an application of the principles of fluid dynamics to the science of flight. We shall first highlight some of the outstanding figures that developed the theory in its early stages (see Anderson's monograph [243] for a thorough review of the historical development of the subject).

In the first decade of the twentieth century, Frederick W. Lanchester (1868-1946) introduced the circulation theory to create lift as a vortex action. His work was more qualitative and it was Martin W. Kutta (1867-1944) who described the computation of lift on an infinite-span wing under incompressible flow assumptions. In parallel, Nikolay Y. Joukowski (1847-1921) derived the classical formula $L = \rho_{\infty} U_{\infty} \Gamma_a$, that relates lift per unit span, L , to density, ρ_{∞} , flow velocity, U_{∞} , and circulation strength, Γ_a . These gave rise to the Kutta-Joukowski theory of lift, which established the foundations of aerodynamic theory in the years to come [244].

Continuing with these advancements, the theoretical and experimental work of Ludwig Prandtl (1875- 1953) is a turning point in consolidating the aerodynamic theory into a branch in its

own right. He developed the lifting line theory and approximated the lift and induced drag of finite-span wings; he started the boundary-layer theory and gave practical solutions to those problems; and he also set the basis for compressible aerodynamics analysis. His legacy was expanded by many of his students and co-workers. One of them was Max M. Munk (1890-1986) who developed the thin airfoil theory, still relevant in today's analysis, by which the airfoil can be decoupled into a camber line and a thickness, responsible for lift and parasite drag respectively. Another was Herbert A. Wagner (1900-1982) who introduced the Wagner function, the first derivation of lift on a 2D flat plate subjected to a sudden increase in angle of attack. Walter Birnbaum (1897-1925) studied the 2D flapping wing and, for the first time, described flutter as a structural dynamic stability problem [244]. As airplanes gained flying speed, the study of compressible flow became crucial, a subject in which Theodor Meyer (1882-1972) was a pioneer with the study of shock waves, as well as Adolf Busemann (1901-1986) who promoted the use of swept wings to alleviate compressible adverse effects. Another of Prandtl's students, Theodore von Kármán (1881-1963), went on to become one of the most prominent theoretical aerodynamicists contributing to many areas, such as unsteady flows, stability of laminar flow, transonic and supersonic aerodynamics, and boundary-layers or turbulence [241].

Moving away from Prandtl's orbit, we find in Theodore Theodorsen (1897-1978) a critical figure in the analysis of unsteady aerodynamics and aeroelasticity. He derived a set of complex functions in the frequency domain that connected an airfoil's translation, angle of attack, and aileron rotation, with the resulting unsteady lift, pitching moment and aileron hinge moment, thereby establishing the grounds for mathematical flutter and control analysis. His idealised theory of a flat plate airfoil was the simplest exact theory at the time and has been used to these days in strip theory [245]. Hermann Glauert (1892-1934), following Wagner's methods, studied the angular oscillations of flat plates and gave improved integral expressions with respect to Birnbaum's theory for the numerical evaluation of lift and moment [245]. Hans G. Küssner's (1900-1984) general concept of a lifting surface led to the so-called kernel function, which relates normal velocity (downwash) of the structure to the resulting pressure distribution [246]. William R. Sears (1913-2002), who carried out his thesis on unsteady airfoil motions under von Kármán, showed the importance of the unsteady airfoil theory and made it accessible to

engineers with derivations and results suitable for immediate application to certain flutter and gust problems [241]. Edward Garrick (1910–1981), using the Fourier transform demonstrated that the Theodorsen function was related to the Wagner function describing the circulatory lift under an impulse change in the angle of attack, and to the Küssner function representing the lift response of an airfoil penetrating a vertical gust. Thus, the link was established between the indicial lift functions of Wagner and Kussner and their frequency response counterparts by Theodorsen and Sears [245]. The classical textbook by Bisplinghoff et al on Aeroelasticity [247], published in 1955, has reviewed and expanded all these developments and it is still today a good starting point into the subject.

We move next into describing the unsteady aerodynamics used in this work.

3.2 Unsteady aerodynamics

Description of the physical phenomena of fluids always begins with the Navier-Stokes equations, derived between 1827 and 1845 by Navier, Poisson, Saint-Venant, and Stokes. However, it should be underlined that the theory is a subset of the broader Continuum Mechanics, to which solid mechanics also belongs, and with which it shares the limitation set by the continuity of the material-body: the NS equations are valid for as long as the representative length scale of the system is much larger than the mean free-path of the molecules that conform the fluid, its ratio is the Knudsen number, and the rule-of-thumb suggests it should be less than 0.01 [248, Ch. 4] for assumptions to be accurate¹. The distinction between solid and fluids rests on the impossibility of the fluid to sustain a static equilibrium under external shear stresses. This has led to the constitutive assumption that the applied shear stress is not a function of the strain on the body, as in elastic solids, but rather of the rate of the strain [227, Ch. 6]. The other assumption is that diagonal stresses are all equal within the differential of volume, hence the pressure forces, $\sigma_{ii} = -p$. Therefore, NS equations are similar to the solid equations, i.e. the continuity equation (2.12) and the conservation of linear momentum, Eq. (2.13), with

¹Statistical methods need to be employed for non-small Knudsen numbers, such as Boltzmann equations, from which the NS equations can be derived using asymptotic analysis, in the limit of very small mean free-paths [249]. While this is not the case for the altitudes typical aircraft operate in ($\lesssim 35000$ feet), it is important for re-entry vehicles going through the upper-atmosphere.

the difference that the stress tensor, $\boldsymbol{\sigma}$, is a function of the pressure and the deformation tensor $\mathbf{D}$, defined after Eq. (2.6) as the symmetric part of the velocity gradient in the body, $\mathbf{D} = \frac{1}{2} \left(\frac{\partial \mathbf{q}_i}{\partial x_j} + \frac{\partial \mathbf{q}_j}{\partial x_i} \right)$. Here we have defined the fluid velocity as $\mathbf{q}$ instead of $\mathbf{v}$ to differentiate the fluid domain from the solid domain in problems where they are coupled. $\boldsymbol{\sigma}$ is then written for Newtonian fluids as [227, Ch. 6],

$$\boldsymbol{\sigma} = -p\mathbf{I} + \lambda \text{tr}(\mathbf{D})\mathbf{I} + 2\mu\mathbf{D} \quad (3.1)$$

$\mathbf{D}$ being a symmetric tensor and its trace $\text{tr}(\mathbf{D}) = \nabla \cdot \mathbf{q}$. Moreover, λ and μ are the viscosity material constants, called the dilatation and shear viscosity coefficients, respectively, and generally assumed to be related by *Stokes condition*, $\lambda + \frac{2}{3}\mu = 0$. They are of critical importance in finding quantities such as the parasite drag, however, their effect is mostly limited to the boundary layer of the aircraft surfaces and therefore not essential for the aeroelastic analysis. They are also necessary to predict the flow separation or stall in the aerodynamic surfaces but this is out of the scope of this work and certainly not an operating condition in normal flight. Furthermore, accounting for the viscosity of the flow makes the equations extremely difficult to solve so these terms will be neglected in subsequent analysis, which leads to the Euler flow equations.

3.2.1 Inviscid compressible fluid equations

Commercial aircraft fly at transonic speeds where compressible effects play a key role in the response. Whether a fluid can be considered compressible is given by the Mach number, Ma , defined as the ratio between the fluid velocity and the speed of sound, a_s : in a medium of air, when $\text{Ma} < 0.3$, changes in density are under 5% and designers' rule of thumb indicates that incompressible assumptions may be acceptable, whereas when $\text{Ma} > 0.3$, compressible effects are to be taken into account [250]. Because of the importance of this effect on the aeroelastic analysis, the aerodynamic forces in the present work are obtained from a linearisation of the compressible, inviscid Navier-Stokes equations (the assumption of inviscid flow is common in the aeroelastic analysis for loads which neglects drag forces, and the rationale behind it is that

maximum loads and wing deformations are mostly due to the balance of lift, internal elastic forces, control devices and gravity force. On the contrary, for the aerodynamic performance or the design of the propulsion system, a good comprehension of the drag forces is essential). Thus the equations conforming the aerodynamic system are the continuity equation and the momentum equation with no viscosity, assuming that gravitational or body forces are negligible; that the gas (air) can be treated according to the laws of ideal gases; and that the process is isentropic, i.e. the entropy of the system remains constant as the process is reversible and adiabatic (two reasonable assumptions for the flow outside the boundary layer). With this, Eqs. (2.12), (2.13) are written as,

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{q}) &= 0 \\ \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} + \frac{\nabla p}{\rho} &= \mathbf{0} \\ \frac{p}{\rho^{\gamma_p}} - \frac{p_\infty}{\rho_\infty^{\gamma_p}} &= 0\end{aligned}\tag{3.2}$$

the last equation is obtained from the isentropic conditions with $\gamma_p = 1.4$ the ratio of specific heats for the air, and p_∞ and ρ_∞ the far-field pressure and density respectively. Thus the state variables are pressure, p , air density, ρ , and the flow velocity field, $\mathbf{q}$. The absence of viscosity makes the flow irrotational, $\nabla \times \mathbf{q} = \mathbf{0}$, which allows the introduction of a potential function such that $\mathbf{q} = \nabla \phi$. After manipulation of Eqs. (3.2) the resulting nonlinear potential equation is [251],

$$\nabla^2 \phi - \frac{1}{a_s^2} \left(\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t} (\nabla \phi \cdot \nabla \phi) + \frac{1}{2} \nabla \phi \cdot \nabla (\nabla \phi \cdot \nabla \phi) \right) = 0\tag{3.3}$$

where the speed of sound is $a_s^2 = \frac{dp}{d\rho}$. This is the (nonlinear) unsteady compressible potential flow equation, which holds well for subsonic and supersonic flow conditions but misses some characteristics of transonic flow since compression shock and temperature losses are not accounted for. The nonlinear potential equation has to be complemented with the expression for the speed of sound. This is obtained from the Bernoulli equation, which at the same time is obtained from the integration of the momentum equation in (3.2):

$$\frac{\partial \phi}{\partial t} + \frac{\mathbf{q} \cdot \mathbf{q}}{2} + \int \frac{dp}{\rho} = 0 \Rightarrow \frac{\partial \phi}{\partial t} + \frac{\nabla \phi \cdot \nabla \phi}{2} = \frac{a_s^2 - a_\infty^2}{\gamma_p - 1}\tag{3.4}$$

Solving Eqs. (3.3), (3.4) for complex geometries involves discretization of the entire 3D domain, bringing the solution process to the field of computational fluid dynamics (CFD), which is not ideal for fast computations. On the other hand, for compressible effects to be apparent, the forward speed of the vehicle will be much larger than other velocities due to deformations or rigid-body motions. Therefore a linearization is sought around the uniform flow with velocity U_∞ , such that

$$\mathbf{q} = (U_\infty + \hat{u})\mathbf{i} + \hat{v}\mathbf{j} + \hat{w}\mathbf{k} \quad (3.5a)$$

$$\phi = U_\infty x + \hat{\phi} \quad (3.5b)$$

$$p = p_\infty + \hat{p} \quad (3.5c)$$

$$a_s = a_\infty + \hat{a} \quad (3.5d)$$

$$\rho = \rho_\infty + \hat{\rho} \quad (3.5e)$$

using these approximations Eq. (3.3) simplifies to,

$$(1 - M_\infty^2) \frac{\partial^2 \hat{\phi}}{\partial x^2} + \frac{\partial^2 \hat{\phi}}{\partial y^2} + \frac{\partial^2 \hat{\phi}}{\partial z^2} - \frac{2M_\infty^2}{U_\infty} \frac{\partial^2 \hat{\phi}}{\partial x \partial t} - \frac{M_\infty^2}{U_\infty^2} \frac{\partial^2 \hat{\phi}}{\partial t^2} = 0 \quad (3.6)$$

here the Mach number, $M_\infty = U_\infty/a_\infty$ is also assumed to be time-invariant in the linearization process. Note that when the Mach number is set to zero, the Laplace equation emerges, that is, an inviscid, incompressible description of the fluid. The potential equation is now linear and can then be solved by elementary solutions, which will allow finding the solution around complex geometries via superposition in much faster computational speeds than CFD. In fact the simple change of coordinates $x' = x - U_\infty t$, transforms Eq. (3.6) into the classical acoustic equation [251], which has a stationary point source with $\frac{S(t-r/a_\infty)}{r}$ decay as elementary solution. Based on this, an elementary moving source is obtained and then transformed to a moving frame of reference in the x -direction, so that the source, ϕ_s , is finally calculated satisfying the aerodynamic potential equation, Eq. (3.6), as given in [87]:

$$\phi_s(x, y, z, \xi, \eta, \zeta, t) = \frac{1}{R} S(t - \tau) \quad (3.7)$$

where S is the source intensity, τ is the time delay for the pulse of a source located at (ξ, η, ζ) to travel to the point (x, y, z) ,

$$\tau = \frac{-M_\infty(x - \xi) + \bar{R}}{a\beta^2} \quad (3.8)$$

and $\bar{R}$ the contracted radial measure:

$$\bar{R} = \sqrt{(x - \xi)^2 + \beta^2(y - \eta)^2 + \beta^2(z - \zeta)^2} \quad (3.9)$$

β has been introduced as a compressible effect $\beta = \sqrt{1 - M_\infty^2}$, – it holds a mathematical relation with the Lorentz factor of special relativity², which should deserve more attention in finding solutions to the unsteady compressibility equations under arbitrary deformations, by converting compressible effects to spatial contraction transformations [255].

Due to its symmetric nature, the elementary solution ϕ_s does not produce a pressure difference, hence lift will not be generated. This issue is overcome by bringing together two sources in the $z = 0$ plane with opposite strengths, which remains a solution of the aerodynamic potential equation and do generate a pressure difference [87]. The result is equivalent to taking the derivative with respect to z , giving the doublet source potential, ϕ_d :

$$\phi_d = \frac{\partial}{\partial z} \phi_s \quad (3.10)$$

Next in the process is the introduction of the boundary conditions in order to fully solve Eq. (3.6). The surface of the unsteady structure of the vehicle is described by the function $\mathcal{F}$ as $\mathcal{F}(x, y, z, t) = 0$. Furthermore, $\mathcal{F}$ is decoupled into a mid-surface, $\mathcal{F}_m$, and a thickness envelope, $\mathcal{F}_{th}$, such that $\mathcal{F}(x, y, t) = \mathcal{F}_m(x, y, t) \pm \mathcal{F}_{th}(x, y)$. Usually thickness effects are assumed not to be time-dependent, and their effect is much more important to the aerodynamics engineer in decreasing the overall drag, predicting the stall point, controlling shock waves or delaying the

²Relativistic descriptions of Euler equations with compressible effects have been derived [252], and coupling general relativity with theories of relativistic viscous fluid dynamics is a current subject of research [253]. Derivation of relativity principles and Lorentz transformation from fluid dynamics have led to the postulation of a compressible superfluid ether in the space-time [254]. And even though this is very much out of the scope of this work, it shows the importance of initial premises and assumptions in the development of theories and the strength of cross-discipline analysis in finding new ways for unsolved problems.

critical Mach number; and to the systems engineer in allowing enough room for fuel sloshing and retractable high-lift devices. Therefore, the analysis is carried out only in the mid surfaces and the appropriate boundary conditions translate to non-penetration across these aerodynamic surfaces. In mathematical form, $\mathcal{F}_m(x, y, z, t) = z - h(x, y, t) = 0$, so that its parametrization is $(x, y, h(x, y))$, and the normal to the surface is simply $\mathbf{n} = \nabla \mathcal{F}_m$. The boundary condition reduces to [251],

$$\frac{D\mathcal{F}_m}{Dt} = \frac{\partial \mathcal{F}_m}{\partial t} + \mathbf{q} \cdot \nabla \mathcal{F}_m = 0 \Rightarrow \frac{\partial h}{\partial t} + (U_\infty + \hat{u}) \frac{\partial h}{\partial x} + \hat{v} \frac{\partial h}{\partial y} = \hat{w} \quad (3.11)$$

It is common practice in compressible aerodynamics to linearise this equation around the uniform flow field, as done with the nonlinear aerodynamic potential equation, so the upwash $\hat{w}$ around the $z = 0$ plane is,

$$\hat{w} = \frac{\partial \hat{\phi}}{\partial z}(x, y, 0, t) = \frac{\partial h}{\partial t} + U_\infty \frac{\partial h}{\partial x} \quad (3.12)$$

The solution process thus far described involves computing the aerodynamic potential with the appropriate boundary conditions, then calculating the velocity field, to finally obtain the pressure forces from Eq. (3.4). Linearisation of this equation circumvents this process and allows finding a direct relation between pressure forces and the boundary displacements. The linear pressure field is

$$\hat{p} = -\rho_\infty \left(\frac{\partial \hat{\phi}}{\partial t} + U_\infty \frac{\partial \hat{\phi}}{\partial x} \right) \quad (3.13)$$

Using this equation and differentiating with respect to t and x and manipulating the aerodynamic potential equation yields the pressure potential equation with identical structure to Eq. (3.6) but replacing $\hat{\phi}$ with $\hat{p}$; so an elementary pressure source, ϕ_p , similar to ϕ_s , can be employed to solve the equation. As all quantities have been linearised the problem is better solved in the frequency domain and the function S in Eq. (3.7) is,

$$\phi_p = \frac{A_a}{\bar{R}} e^{i\omega(t-\tau)} \quad (3.14)$$

with A_a the source intensity and $\bar{R}$ and τ as in Eqs. (3.9),(3.8). However, the source ϕ_p encounters the same issue in the generation of lift and a doublet pressure source, ϕ_a , is defined

as,

$$\phi_a = \frac{1}{\rho_\infty} \frac{\partial \phi_p}{\partial z} \quad (3.15)$$

The problem is better solved in the frequency domain, thus restricting the analysis to harmonic motion, such that $\hat{\phi} = \bar{\phi}e^{i\omega t}$ or $\hat{w} = \bar{w}e^{i\omega t}$. For the pressure field, $\Delta p/\rho_\infty = \phi_a$ and $\Delta \hat{p} = \rho_\infty \bar{\phi}_a e^{i\omega t}$. The relation between the acceleration potential and the velocity or aerodynamic potential is found,

$$\phi_a = - \left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \right) \bar{\phi} e^{i\omega t} \quad (3.16)$$

hence the name acceleration potential, since it is the linearised material derivative of the velocity potential. Bringing Eq. (3.14) into Eq.(3.15), and substituting Eq. (3.8) for τ , $\bar{\phi}_a$ is written as,

$$\bar{\phi}_a(x, y, z) = A_a \frac{\partial}{\partial z} \left(\frac{1}{\bar{R}} e^{i\omega \frac{M_\infty(x-\xi)-\bar{R}}{a_\infty \beta^2}} \right) = A_a \beta^2 (z - \zeta) \left(\frac{1}{\bar{R}^2} - \frac{1}{\bar{R}^3} \right) e^{i\omega \frac{M_\infty(x-\xi)-\bar{R}}{a_s \beta^2}} \quad (3.17)$$

Taking the limit in the last identity at $z = 0$ from the left and the right hand side gives the jump $\Delta \bar{\phi}_a$ across a doublet sheet [87] as $\Delta \bar{p} = 4\pi \rho_\infty A_a$. Next the method of characteristics is applied to Eq. (3.13), so the relation between the velocity potential and the pressure is found [87],

$$\hat{\phi} = \frac{1}{\rho_\infty U_\infty} \int_{-\infty}^x \left(p \left(\lambda, y, z, t - \frac{x-\lambda}{U_\infty} \right) - p_\infty \right) d\lambda \quad (3.18)$$

Combining this with Eq. (3.16) for a single oscillating doublet located at $x = \xi$, the relation in the frequency domain between the velocity potential and the acceleration potential is,

$$\bar{\phi}(x, y, z) = -\frac{1}{U_\infty} e^{-i\omega(x-\xi)/U_\infty} \int_{-\infty}^{x-\xi} e^{i\omega\lambda/U_\infty} \bar{\phi}_a(\lambda, y, z) d\lambda \quad (3.19)$$

Consider now a continuous surface of doublets instead of a single one, such that $\Delta \bar{p} = 4\pi \rho_\infty A_a d\xi d\eta$; also from the definition of the velocity potential, $\bar{w} = \frac{\partial \bar{\phi}}{\partial z}$; substituting the first identity of Eq. (3.17) in Eq. (3.19) and integrating over the surface of doublets gives the relation between

boundary velocities and the pressures generated:

$$\begin{aligned} \frac{\bar{w}}{U_\infty} &= -\frac{1}{4\pi\rho U_\infty^2} \int \int_S \Delta \bar{p} e^{-i\omega(x-\xi)/U_\infty} \frac{\partial^2}{\partial z^2} \left(\int_{-\infty}^{x-\xi} e^{i\omega(\lambda/U_\infty + (M_\infty\lambda - \bar{R})/a_\infty\beta^2)} d\lambda \right) d\xi d\eta \\ &= \int \int_S \frac{\bar{p}(\xi, \eta)}{\frac{1}{2}\rho_\infty U_\infty^2} K((x-\xi), (y-\eta), 0) d\xi d\eta \end{aligned} \quad (3.20)$$

$K(x, y)$ is the Kernel function and links the nondimensional upwash at x, y and the pressure at (ξ, η) [251]. Doublet sheet sources can be used to solve the time dependant aerodynamics and other source panel to model the time independent flow over wing, that is the $\mathcal{F}_m$ and $\mathcal{F}_{th}$ surfaces respectively. Although as previously pointed out for aeroelastic problems only the planar solution will be used. In the linearised problem, the wing and the wake are in the $z = 0$ plane and far field boundary conditions are enforced for the flow to be uniform – superposition of doublets automatically satisfies this condition as their influence dies at infinity. For the trailing wake boundary condition, the generation of lift in inviscid flow requires the introduction of the Kutta condition. This is, however, not very clearly characterised for compressible flow. The only condition imposed on the trailing sheet is such that the pressure difference across the sheet is zero. For complex problems these integrals are solved by computational approximation techniques as described next.

3.2.2 The doublet-lattice-method

The doublet-lattice-method (DLM) [88, 220] represents a numerical solution to equation (3.20), where the main surfaces are discretised into boxes and doublet lines placed at the quarter chord of each box or panel, while the downwash, $\hat{w}(x, y, 0)$, is evaluated at the 3/4 chord midspan (note the change in notation from the previous section in the upwash, $\hat{w}_u$, and the downwash, $\hat{w}_d$, related as $\hat{w}_u = -\hat{w}_d$ depending on a reference frame in or out of the structure). One of the main advantages of the method is that $\bar{p} = 0$ off the surfaces, i.e the integrals, only need to be calculated on the surfaces of the main structures and there is no need for special treatments on the wake or the edges. The DLM computes the downwash at the panel j , due to a pressure

in panel i , solving the kernel function in Eq (3.20) and yielding the matrix $\mathbf{A}$:

$$\bar{w}_j = A_{ji} \frac{2\bar{p}_i}{\rho_\infty U_\infty^2} \quad (3.21)$$

The unsteady boundary conditions translate to non-penetration across the aerodynamic surfaces, which after linearisation shown in Eq. (3.12) are written in matrix form as [256],

$$\frac{\hat{\mathbf{w}}}{U_\infty} = \mathbf{D}_1 \mathbf{u}_k + \frac{1}{U_\infty} \mathbf{D}_2 \dot{\mathbf{u}}_k + \frac{\mathbf{w}_g}{U_\infty} \quad (3.22)$$

where $\mathbf{u}_k$ are the normal displacements at the aerodynamic grid points; $\mathbf{w}_g$ the downwash due to an external disturbance such as a gust or the static incidence distribution from an initial angle of attack, camber, or twist; and $k = \frac{\omega c}{2U_\infty}$ is the reduced frequency with c the airplane chord reference. Next, the normal forces f_k at the aerodynamic grid points are obtained from integration of the pressure at each panel, p_j , that is, $\bar{f}_k = S^{kj} \bar{p}_j$. Moreover, coupling the structural and aerodynamic sets is attained through the interpolation matrices, $\mathbf{G}_{kg}^D$ for displacements and $\mathbf{G}_{kg}^F$ for the forces: $\mathbf{u}_k = \mathbf{G}_{kg}^D \mathbf{u}_g$ and $\mathbf{F}_k = \mathbf{G}_{kg}^F \mathbf{F}_g$. For general problems the two matrices are equal and considered here as $\mathbf{G}$. The matrix $\bar{\mathbf{Q}}_{GG}$ is built, which relates displacements and forces at the structural DoF:

$$\bar{\mathbf{Q}}_{GG} = \mathbf{G}^\top \mathbf{S} \mathbf{A}^{-1} \mathbf{D} \mathbf{G} \quad (3.23)$$

Note how the subscript J represents the aerodynamic panels DoF, K the aerodynamic set and G the structural grid points. According to Walt Silva [104, 105] these $\bar{\mathbf{Q}}_{(\bullet)}(k, M_\infty)$ are the aerodynamic influence coefficient matrices (AICs), and when they are projected into the modal space they become the generalised aerodynamic force matrices (GAFs). In this work, however, the terms AIC and GAF are used indistinctly and it will be clear when they refer to actual forces or their modal projection. Thus the aerodynamic forces are projected into the discrete LNMs to obtain a closed-form relation between generalised aerodynamic forces, $\bar{\boldsymbol{\eta}}_a$, and the modal structural displacements, $\bar{\mathbf{q}}_0$, so that for a structural modal base with a total of

H modes the modal GAF $\bar{\mathbf{Q}}_{HH}$ relates these variables:

$$\bar{\eta}_a = \frac{1}{2}\rho_\infty U_\infty^2 \bar{\mathbf{Q}}_{HH} \bar{\mathbf{q}}_0 = \frac{1}{2}\rho_\infty U_\infty^2 (\Phi_0^\top \mathbf{G}^\top \mathbf{S} \mathbf{A}^{-1} (\mathbf{D}_1 + ik\mathbf{D}_2) \mathbf{G} \Phi_0) \bar{\mathbf{q}}_0 \quad (3.24)$$

Similarly for gust and random aerodynamic excitations, $\bar{\mathbf{Q}}_{HJ}$ gives the generalised aerodynamic forces, $\bar{\eta}_{ag}$, due to disturbances at each panel j ,

$$\bar{\eta}_{ag} = \frac{1}{2}\rho_\infty U_\infty^2 \bar{\mathbf{Q}}_{HJ} \bar{\mathbf{w}}_g = \frac{1}{2}\rho_\infty U_\infty^2 (\Phi_0^\top \mathbf{G}^\top \mathbf{S} \mathbf{A}^{-1}) \bar{\mathbf{w}}_g \quad (3.25)$$

In the same fashion, a GAF matrix $\bar{\mathbf{Q}}_{HX}$ links a number X of control points such as aileron or rudder deflections to the airframe modal forces; and the steady-state matrix, $\bar{\mathbf{Q}}_{HC}$, that gives the modal forces, non frequency-dependent, for steady terms such as angle of attack or sideslip angle. The process is summarised as follows: A_{ij} relates downwash and unit pressure at panels i and j respectively; the boundary condition in Eq. (3.12) is given by $\mathbf{D}_1$ and $\mathbf{D}_2$, thus linking deformations and upwash; integration of the pressures into forces is given by the matrix $\mathbf{S}$. Note that these matrices are a function of the reduced frequency and the Mach number. The interpolation matrix $\mathbf{G}$ maps the components of structural grid point deflections, to the deflections of the aerodynamic grid points. Using these matrices, $\bar{\mathbf{Q}}_{GG}$ and $\bar{\mathbf{Q}}_{HH}$ are assembled to output forces and modal forces, from displacements or modal displacements, respectively. MSC Nastran aerodynamic module is utilised herein to obtain these aerodynamic matrices [256]. ZAERO software from Zona Technology is another alternative to model with panel methods the compressible aerodynamics [196, 257].

3.3 Aeroelastic formulation

The full aeroelastic system is formed by coupling the structural NMROM, Eq. (2.100), described in Sec. 2.3, with the modal generalised aerodynamic forces given by the DLM. This was deemed to be a cost effective solution yet superior than other aeroelastic tools based on 2D airfoil aerodynamics. More importantly it allows to validate the methods with current industrial practise for loads and showing discrepancies between linear and nonlinear analysis;

and once the approach is verified, it can be improved by amending the aerodynamic matrices with experiments and higher fidelity aerodynamics [102]. In terms of nonlinear effects, since the structural equations are written in the material frame, the aerodynamic forces naturally appear as follower forces on the structure; on the other hand the change in the aerodynamic panels geometry is not accounted for a priori and possible solutions will be discussed later on in the chapter.

The first step in the construction of an aeroelastic system is the transformation of the aerodynamics forces given by the DLM in frequency domain into the time domain. This is attained by using the method of the rational function approximation. It consists of curve-fitting the AICs calculated for a range of reduced frequencies and fixed Mach number. Depending on the functional used for the fitting different techniques have been developed: Roger's approximation using Padé polynomials [258] and a least-square approximation; Karpel's minimum state method [166]; a combination of the two [259], and Chebyshev polynomials [260]. Despite the benefit of methods such as Karpel's in reducing the number states in the final system, Boeing has reported [261] that methods such as the one proposed by Roger are required in a production engineering environment for its robustness and accuracy in the aeroelastic simulation of loads and gust responses. Therefore, modal aerodynamic forces are obtained in the time domain from Roger's approximation and coupled to the structural description to get the nonlinear aeroelastic system. Subsequently a strategy to trim the aircraft in a longitudinal fashion is introduced in Sec 3.4.1, followed by a method to incorporate gust loads with arbitrary shapes in the time domain simulations, so that a full description of the aircraft under operating conditions and geometrically-nonlinear deformations is achieved.

It should be remarked that the methods described herein are not limited to DLM aerodynamics and can be extended to more advanced aerodynamics. In fact, Kier and co-workers have presented a similar approach with AICs and RFAs to unify manoeuvre and gust loads analysis. They have used several methods to obtain the AICs, starting from the DLM [117, 262], a 3D panel method [18], and a mix of DLM with CFD corrections [222] for multidisciplinary design optimization (MDO). However, they have kept the analysis linear and the aim of this work is to demonstrate how nonlinear effects can also be included. Once the validation and implemen-

tation of the methodology has been carried out, it is rather straight forward to substitute the aerodynamic matrices from the DLM for a higher fidelity method.

3.3.1 Rational-function approximation

A rational-function approximation (RFA) is used to transform the frequency-domain physics into the time-domain, based on the classical method by Roger [258]. Accordingly, the GAFs in Eq. (3.24) are approximated here for a given Mach number as,

$$\mathbf{Q}_{HH}(k) = \mathbf{A}_0 + (ik)\mathbf{A}_1 + (ik)^2\mathbf{A}_2 + \sum_{p=1}^{N_p} \frac{ik}{\gamma_p + ik} \mathbf{A}_{p+2} \quad (3.26)$$

$\mathbf{A}_0$ is just $\bar{\mathbf{Q}}_{HH}(0)$. For a set of reduced frequencies k_n and a fixed Mach number, $\bar{\mathbf{Q}}_{HH}(k_n) \in \mathbb{R}^{N_m \times N_m}$ matrices are computed and the $\mathbf{A}_i$ obtained through a least square fit after the following system is formed,

$$\begin{bmatrix} \bar{\mathbf{Q}}_{HH}(k_1) - \mathbf{A}_0 \\ \bar{\mathbf{Q}}_{HH}(k_2) - \mathbf{A}_0 \\ \vdots \\ \bar{\mathbf{Q}}_{HH}(k_n) - \mathbf{A}_0 \end{bmatrix} = \begin{bmatrix} ik_1 \mathbf{I}_{N_m} & (ik_1)^2 \mathbf{I}_{N_m} & \frac{ik_1}{\gamma_1 + ik_1} \mathbf{I}_{N_m} & \cdots & \frac{ik_1}{\gamma_{N_p} + ik_1} \mathbf{I}_{N_m} \\ ik_2 \mathbf{I}_{N_m} & (ik_2)^2 \mathbf{I}_{N_m} & \frac{ik_2}{\gamma_1 + ik_2} \mathbf{I}_{N_m} & \cdots & \frac{ik_2}{\gamma_{N_p} + ik_2} \mathbf{I}_{N_m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ik_n \mathbf{I}_{N_m} & (ik_n)^2 \mathbf{I}_{N_m} & \frac{ik_n}{\gamma_1 + ik_n} \mathbf{I}_{N_m} & \cdots & \frac{ik_n}{\gamma_{N_p} + ik_n} \mathbf{I}_{N_m} \end{bmatrix} \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \vdots \\ \mathbf{A}_{N_p+2} \end{bmatrix} \quad (3.27)$$

The matrix containing the reduced frequencies is usually not square so its matrix pseudo-inverse is calculated using linear algebra techniques such as singular-value-decomposition or the Moore-Penrose pseudo-inverse. Multiplying this matrix's pseudo-inverse by the $[\bar{\mathbf{Q}}_{HH}(k_i) - \mathbf{A}_0]$ terms yields the rest of the $\mathbf{A}_i$ matrices. In order to complete the RFA the last sum in Eq. (3.26) defines the aerodynamic states, λ_p :

$$\bar{\lambda}_p = \frac{ik}{\gamma_p + ik} \mathbf{A}_{p+2} \bar{\mathbf{q}}_0 \quad (3.28)$$

where γ_p is the aerodynamic lag associated to $\lambda_p \in \mathbb{R}^{N_m}$, for $p = 1, \dots, N_p$ number of lags in the solution. The selection of these lags can greatly affect the quality of the aeroelastic simulations [263] and gradient-based optimization methods are used here to find them. In such

cases, the optimizer needs constraints to prevent the lags from collapsing into a single value, or a multipole approach as shown in [264]. In this work, the optimization is carried out in a two-step process: firstly, a non-gradient discrete optimization is performed on the aerodynamic lags with a selected fixed spacing between them; the results are then the initial values of a gradient optimization. Using this approach, it has been found lags do not collapse and more optimal values are obtained (further details are given in Appendix A.4).

3.3.2 Time-domain integration

The forces in nondimensional time, $t_s = \frac{U_\infty t}{c/2}$, are obtained after applying the inverse Fourier transform to Roger's approximation, Eq. (3.26), and to the definition of the aerodynamic states, Eq. (3.28). The final expression for the aerodynamic modal forces and states in time domain are,

$$\boldsymbol{\eta}_a = \frac{1}{2}\rho U_\infty^2 \left(\boldsymbol{\mathcal{A}}_0 \mathbf{q}_0 + \frac{c}{2U_\infty} \boldsymbol{\mathcal{A}}_1 \mathbf{q}_1 + \left(\frac{c}{2U_\infty} \right)^2 \boldsymbol{\mathcal{A}}_2 \dot{\mathbf{q}}_1 + \sum_{p=1}^{N_p} \boldsymbol{\lambda}_p \right) \quad (3.29)$$

$$\dot{\boldsymbol{\lambda}}_p = -\frac{2U_\infty \gamma_p}{c} \boldsymbol{\lambda}_p + \boldsymbol{\mathcal{A}}_{p+2} \mathbf{q}_1 \quad \text{for } p = 1, \dots, N_p \quad (3.30)$$

The term $\mathbf{q}_0$ is the displacement modal amplitude and $\mathbf{q}_1$ is the velocity modal amplitude, as given by eq. (2.111); note both displacement and velocity amplitudes are used to define the inputs to the aerodynamic subproblem. For convenience in the formulation the aerodynamic states are rescaled with the dynamic pressure such as $\boldsymbol{\lambda}_p \rightarrow \frac{1}{2}\rho U_\infty^2 \boldsymbol{\lambda}_p$. Substituting these expressions into equations (2.100) results in the aeroelastic NMROM equations of motion,

$$\begin{aligned} \mathcal{M}^{ij} \dot{q}_{1j} &= \delta^{ij} \omega_j q_{2j} - \Gamma_1^{ijk} q_{1j} q_{1k} - \Gamma_2^{ijk} q_{2j} q_{2k} \\ &\quad + \frac{\rho U_\infty^2}{2} \mathcal{A}_0^{ij} q_{0j} + \frac{c \rho U_\infty}{4} \mathcal{A}_1^{ij} q_{1j} + \delta^{pp} \lambda_{p,i} + \eta_{fi} \\ \dot{q}_{2i} &= -\delta^{ij} \omega_j q_{1j} + \Gamma_2^{jik} q_{1j} q_{2k} \\ \dot{q}_{0i} &= q_{1i} \\ \dot{\lambda}_{p,i} &= \frac{\rho U_\infty^2}{2} \mathcal{A}_{p+2}^{ij} q_{1j} - \frac{2U_\infty \gamma_p}{c} \lambda_{p,i} \end{aligned} \quad (3.31)$$

for $i, j, k \in \{1, \dots, N_m\}$ and $p \in \{1, \dots, N_p\}$. Here, we have introduced the constant matrix $\mathcal{M}^{ij} = \delta^{ij} - \frac{\rho c^2}{8} \mathcal{A}_2^{ij}$, $\boldsymbol{\eta}_f$ refers to any external loads that do not come from the aerodynamics, after being projected into modal coordinates. The resulting aeroelastic description is therefore built with $2N_m$ nonlinear differential equations that include geometrically-nonlinear modal couplings on the structure, N_m integrators to evaluate the modal displacements and $N_m \times N_p$ linear ODEs for the aerodynamic states (due to our particular choice of RFA).

3.4 Flight dynamics for aircraft loads

In industrial practise of aeroelastic loads, the flight dynamics of the aircraft are greatly simplified in order to be able to analyse the over 100,000s load cases that may occur in flight. The same principle is followed in this work with the aim of adding a geometrically nonlinear solution to the current (linear) environments, without having to modify the already built complex models or finding equivalent ones that might loose some details from the original. Static manoeuvres and the trim state in steady level flight are required in finding the load envelopes that the airframe will have to withstand; they are calculated as the steady equilibrium of a dynamic simulation, which for the trim condition incorporates rigid body modes, gravity, forces from control devices and a linear controller to find the state of nonlinear equilibrium. Furthermore, assessment of dynamic gust responses is one of the most important requirements in complying with regulations; they are computed herein in the time domain which allows treatment of arbitrary gust shapes. In what follows we therefore present the methodology to compute trim level flight and atmospheric disturbances, and Eqs. (3.31) are augmented accordingly to construct the mathematical description that accounts for the different cases encountered in the assessment of loads during flight.

3.4.1 Aircraft longitudinal trim

Finding the trim of an aircraft involves calculating the state of equilibrium between inertial forces; aerodynamic forces, i.e. lift and drag; the thrust coming from the engines; and those forces from actuators. In steady level flight, a first simplification is taken by assuming symmetry

of the aircraft around the longitudinal plane so that gravity and drag forces are cancelled out by the lift and thrust forces respectively, and the pitching moment is balanced by the tail elevator. A second simplification is carried out in this work as drag and thrust forces are neglected by clamping the vehicle in the x -direction. This assumption is common in load analysis and the rationale behind it is that maximum loads and wing deformations are mostly due to the balance of lift, the pitching moment, and gravity forces. Because the trim condition is a static equilibrium, the nonlinear differential equations (3.31) could be transformed into a system of nonlinear algebraic equations, as in Eq. (2.102), and this system solved using a Newton-Raphson or a fix point approach, as demonstrated in [89]. The problem with the approach in (3.31) lies in the difficulty of linking displacement, $\mathbf{q}_0$, and strain, $\mathbf{q}_2$, modal coordinates, which would be needed when setting $\mathbf{q}_1$ to zero (this can be attained by linearisation of strains but the use of velocities was better suited for the dynamic computations carried out in along this work). Therefore the longitudinal trim of the aircraft is achieved through a dynamic simulation where the rigid body modes for pitching and plunging are included in the analysis and a controller drives the tail elevator angle until equilibrium is achieved. As per the model setting, a point $\mathbf{x}_c$ is clamped in all directions but the rotation in y and the vertical position displacements. $\mathbf{x}_c$ is chosen to coincide with the center of gravity of the initial configuration, Cg_0 , or close to it (it does not matter because the Cg position will change as the nonlinear simulation advances). A linear controller aims at keeping $\mathbf{x}_c$ in the same location, so it takes $\mathbf{x}_c$, its time derivative and its integral over time, and outputs the elevator deflection, q_x , such that when $\mathbf{x}_c < 0$ the elevator creates a pitching up moment, the angle of attack and hence the lift increase, and vice-versa when $\mathbf{x}_c > 0$. Reaching the steady state in the dynamic simulation will ensure the zero pitching moment condition and having $\mathbf{x}_c = 0$ makes lift, elevator and gravity forces cancel out. Note that despite the Cg position changing with the nonlinear deformations, the condition of zero pitching moments applies to the whole structure and therefore it is a valid method of finding the nonlinear trim of the aircraft. In addition to this, two external forces are applied proportional to the displacement and the velocity of $\mathbf{x}_c$ – effectively attaching a spring and a damper to the free DoF of the aircraft. These forces will go into the system after the modal projection in the form of $\boldsymbol{\eta}_f$ terms, and they allow for the aircraft to not go to unrealistic positions when

the simulation begins. Fig. 3.1 shows this process schematically: a dynamic simulation runs by advancing in time Eqs. (3.42) which define the aeroelastic NMROM – encapsulating the aircraft structure, mass properties, and aerodynamic surfaces. Other external aerodynamic disturbances will go as $\mathbf{w}_g$, external structural forces as $\boldsymbol{\eta}_f$, and the control state inputs as $\mathbf{q}_x$. Between time steps the position of x_c is recovered and fed to the controller.

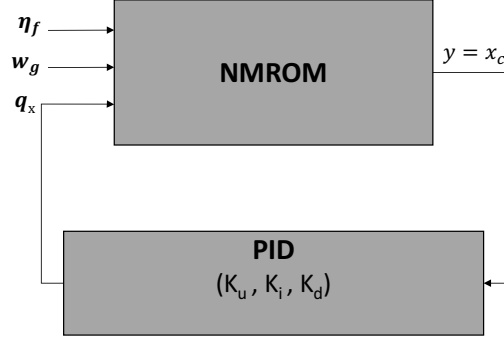


Figure 3.1: Schematic for aircraft nonlinear trim approach

3.4.2 Modelling gust loads

For gust analysis the approach introduced in [117] is followed, so a RFA on $\bar{\mathbf{Q}}_{HJ}$ is carried out allowing the definition of any upwash function and preventing the spiral nature of the gust problem when approximating directly $\bar{\mathbf{Q}}_{HJ}\bar{w}_{gj}$. The force in the frequency domain due to a spanwise homogeneous gust is

$$\bar{\eta}_{gust}(\omega) = \frac{1}{2}\rho_{\infty}U_{\infty}^2\bar{\mathbf{Q}}_{HJ}(\omega)\bar{w}_{gj}(\omega) \quad (3.32)$$

The upwash, $\bar{w}_g$, on a panel j is

$$\bar{w}_{gj}(\omega) = \mathbf{n}_j \cdot \frac{\mathbf{v}_g}{U_{\infty}} e^{-i\omega(\mathbf{x}_j - \mathbf{x}_0)/U_{\infty}} \quad (3.33)$$

where $\mathbf{n}_j$ is the normal to panel j , $\mathbf{v}_g$ is the vector velocity of the gust (usually in the z-direction for normal gust and in y-direction for lateral gust), $\mathbf{x}_j$ is the location of the control point in panel j , and $\mathbf{x}_0$ the position of the gust origin. The normal of each panel could be updated as the structure deforms for a complete nonlinear analysis of the gust. However this would entail

updating a complex panel geometry for a very short period of time and it is deemed a second order effect here, so no updating of $\mathbf{n} \cdot \mathbf{v}_g$ has been implemented; rather a fully vertical gust is assumed as in [181],

$$\bar{w}_{gj}(\omega) = \bar{w}_{ref} \cos \gamma_j e^{-i\omega(x_j - x_0)/U_\infty} \quad (3.34)$$

with γ_j being the dihedral of panel j . After application of the Fourier transform the gust upwash in the time domain is put as,

$$w_{gj} = w_{ref} \frac{b(y_j)}{2U_\infty} \cos \gamma_j \left(1 - \cos \left(\frac{\pi U_\infty}{H_g} \left(t - \frac{x_0 - x_j}{U_\infty} \right) \right) \right) \delta_{tj} \quad (3.35)$$

where $b(y)$ was introduced to change the gust intensity along the spanwise position, and for a standard $1 - \cos$ is set to 1. H_g is the gust gradient, which is half the gust length L_g , and the Kronecker delta is defined as,

$$\delta_{tj} = \begin{cases} 1 & \text{if } \frac{x_0 - x_j}{U_\infty} \leq t \leq \frac{x_0 - x_j}{U_\infty} + \frac{U_\infty}{L_g} \\ 0 & \text{otherwise} \end{cases} \quad (3.36)$$

and w_{ref} is defined according to regulations as [239],

$$w_{ref} = w_{g0} \left(\frac{H}{106.17} \right)^{1/6} \quad (3.37)$$

with H given in meters. w_{g0} is also tabulated as a function of the altitude, although it will be used here as a scaling factor to clearly differentiate the boundaries between linear and nonlinear analysis. Note that $\dot{\mathbf{w}}_g$ and $\ddot{\mathbf{w}}_g$ are easily calculated from Eq. (3.35) and they will go directly to Eq. (3.42) for the gust loads in the full-aeroelastic simulation. Because the aeroelastic system is solved in modal space, the total number of modes utilised in the solution depends on the maximum frequencies involved in the system dynamics. One of them may be set by the minimum gust length: $\omega^{(max)} = \pi U_\infty / L_g^{(min)}$ so that the aerodynamics should be calculated up to at least a reduced frequency of $k^{(max)} = \frac{1}{2} \pi c / L_g^{(min)}$. A typical range of lengths for vertical gusts is 18 m to 214 m [239]. Although for nonlinear computations a larger modal base is generally needed than in linear analysis, which will set higher frequencies than what the

minimum gust length demands.

3.4.3 Full aeroelastic description

We are now in position to describe the full aeroelastic solution extending Eq. (3.31) with gravity forces, $\boldsymbol{\eta}_g$, extra aerodynamic forces from gust disturbances, $\boldsymbol{w}_g$, and control states, $\boldsymbol{q}_x$. For a set of reduced frequencies, k_n , and a given Mach number, the DLM yields the modal forces in the frequency domain: $\bar{\boldsymbol{Q}}_{HH}(k_n, M_\infty)$ gives the forces for a particular set of modal displacements and velocities of the structure; $\bar{\boldsymbol{Q}}_{HJ}(k_n, M_\infty)$ gives the effect of a flow disturbance $\boldsymbol{w}_g$ at each panel j ; $\bar{\boldsymbol{Q}}_{HX}(k_n, M_\infty)$, yields the effect of each control surface in the model; and $\bar{\boldsymbol{Q}}_{HC}(0) = \boldsymbol{C}_0$ is a steady term for angle of attack, side-slip, pitch, roll, or yaw, which are combined in the vector $\boldsymbol{\alpha}_x$. The RFA of $\bar{\boldsymbol{Q}}_{HH}$, $\bar{\boldsymbol{Q}}_{HX}$ and $\bar{\boldsymbol{Q}}_{HJ}$ defines the aerodynamic matrices, $\boldsymbol{A} \in \mathbb{R}^{N_m \times N_m}$, $\boldsymbol{B} \in \mathbb{R}^{N_m \times N_x}$ and $\boldsymbol{D} \in \mathbb{R}^{N_m \times N_j}$ respectively; and the aerodynamic states, $\boldsymbol{\lambda}_p \in \mathbb{R}^{N_m}$, for $p = 1, \dots, N_p$, with the associated aerodynamic lags or poles, γ_p . Thus the total contribution of the aerodynamic forces is:

$$\begin{aligned} \boldsymbol{\eta}_a = \frac{1}{2}\rho_\infty U_\infty^2 & \left(\boldsymbol{A}_0 \boldsymbol{q}_0 + \frac{c}{2U_\infty} \boldsymbol{A}_1 \dot{\boldsymbol{q}}_1 + \left(\frac{c}{2U_\infty} \right)^2 \boldsymbol{A}_2 \ddot{\boldsymbol{q}}_1 + \right. \\ & + \boldsymbol{B}_0 \boldsymbol{q}_x + \frac{c}{2U_\infty} \boldsymbol{B}_1 \dot{\boldsymbol{q}}_x + \left(\frac{c}{2U_\infty} \right)^2 \boldsymbol{B}_2 \ddot{\boldsymbol{q}}_x + \\ & + \boldsymbol{D}_0 \boldsymbol{w}_g + \frac{c}{2U_\infty} \boldsymbol{D}_1 \dot{\boldsymbol{w}}_g + \left(\frac{c}{2U_\infty} \right)^2 \boldsymbol{D}_2 \ddot{\boldsymbol{w}}_g + \\ & \left. + \boldsymbol{C}_0 \boldsymbol{\alpha}_x + \sum_{p=1}^{N_p} \boldsymbol{\lambda}_p \right) \end{aligned} \quad (3.38)$$

$\boldsymbol{\eta}_a$ is accompanied by a system of linear ODEs for the aerodynamic states, which are re-scaled as before, $\boldsymbol{\lambda}_p \rightarrow \frac{1}{2}\rho U_\infty^2 \boldsymbol{\lambda}_p$, so that

$$\dot{\boldsymbol{\lambda}}_p + \frac{2U_\infty \gamma_p}{c} \boldsymbol{\lambda}_p = \frac{1}{2}\rho_\infty U_\infty^2 (\boldsymbol{A}_{2+p} \boldsymbol{q}_1 + \boldsymbol{B}_{2+p} \dot{\boldsymbol{q}}_x + \boldsymbol{D}_{2+p} \dot{\boldsymbol{w}}_g) \quad (3.39)$$

The total set of aerodynamic, inertial and any other external forces are combined in $\boldsymbol{\eta}$ after being projected into the modal base $\boldsymbol{\Phi}_1$: $\boldsymbol{\eta} = \boldsymbol{\eta}_a + \boldsymbol{\eta}_g + \boldsymbol{\eta}_f$. Thus $\boldsymbol{w}_g$ is a vector with length the number of aerodynamic panels, N_j ; $\boldsymbol{q}_x$ the control surfaces in the system, total of N_x ; and

α_x encompasses the components of any steady manoeuvre, of length N_α . For N_m the number of linear modes selected in the system, N_p the number of poles used in the aerodynamics, the final aeroelastic system is built with $2N_m$ nonlinear differential equations; N_m integrators to evaluate the modal displacements; $N_a = N_m \times N_p$ linear ODEs for the aerodynamic states; $4N_n$ (four times the total number of nodes) in order to track the rotations through the quaternions $\zeta = [\zeta_0; \zeta_x]$ in Eq. (2.106), so that gravity forces or other type of dead forces can be rotated to the material frame of reference (it can be the full set of nodes or a reduced set on which the rest are interpolated). Eq. (2.107) describing the relation between the quaternions and the rotation matrix is put here in a compact form:

$$\begin{aligned} \mathbf{R}^{ab} &= \zeta_x \otimes \zeta_x + \zeta_0^2 \mathbf{I}_3 + 2\zeta_0 \tilde{\zeta}_x + (- (\zeta_x \cdot \zeta_x) \mathbf{I}_3 + \zeta_x \otimes \zeta_x) \\ &= (2\zeta_x \otimes \zeta_x + (\zeta_0^2 - \zeta_x \cdot \zeta_x) \mathbf{I}_3) + 2\zeta_0 \tilde{\zeta}_x \end{aligned} \quad (3.40)$$

note that the first parenthesis in the second equality of this equation is the symmetric part of the rotation and the last term the antisymmetric part. Another possibility is to track one node using the quaternions and integrate strains to obtain the rotations for the rest of the nodes as in Eq. (2.108). Either way, once the rotation matrix $\mathbf{R}^{ab}$ is formed for each node l , and written as $\mathbf{R}_l$ for simplification, the gravity forcing term η_g is calculated,

$$\eta_g = \Phi_1^\top \begin{bmatrix} \begin{pmatrix} \mathbf{R}_1^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_1^\top \end{pmatrix} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \dots & \begin{pmatrix} \mathbf{R}_{N_n}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{N_n}^\top \end{pmatrix} \end{bmatrix} \mathbf{M}_a \begin{bmatrix} \begin{pmatrix} \mathbf{g}_0 \\ \mathbf{0} \end{pmatrix} \\ \vdots \\ N_n \\ \vdots \\ \begin{pmatrix} \mathbf{g}_0 \\ \mathbf{0} \end{pmatrix} \end{bmatrix} \quad (3.41)$$

with $\mathbf{M}_a$ the condensed mass matrix in Eqs. (2.26), and $\mathbf{g}_0$ the gravity vector as a function of Earth's radius which can be approximated at sea level as $\mathbf{g}_0 \approx [0 \ 0 \ -9.807]^\top$. Putting the definitions of this and previous chapter together, the full-aeroelastic system is built and written

in tensorial notation as,

$$\begin{cases}
 \dot{q}_{1i} &= \hat{\Omega}^{ij} q_{2j} - \hat{\Gamma}_1^{ijk} q_{1j} q_{1k} - \hat{\Gamma}_2^{ijk} q_{2j} q_{2k} + \hat{\mathcal{A}}_0^{ij} q_{0j} + \hat{\mathcal{A}}_1^{ij} q_{1j} + \hat{\mathcal{B}}_0^{it} q_{xt} + \hat{\mathcal{B}}_1^{it} \dot{q}_{xt} + \hat{\mathcal{B}}_2^{it} \ddot{q}_{xt} \\
 &\quad + \hat{\mathcal{D}}_0^{is} w_{gs} + \hat{\mathcal{D}}_1^{is} \dot{w}_{gs} + \hat{\mathcal{D}}_2^{is} \ddot{w}_{gs} + (\mathcal{M}^{-1})^{ij} \delta^{pp} \lambda_{pj} + \hat{\mathcal{C}}_0^{ir} \alpha_{xr} + \hat{\eta}_{gi} + \hat{\eta}_{fi} \\
 \dot{q}_{2i} &= -\delta^{ij} \Omega_j q_{1j} + \Gamma_2^{jik} q_{1j} q_{2k} \\
 \dot{q}_{0i} &= q_{1i} \\
 \dot{\lambda}_{p,i} &= \hat{\mathcal{A}}_{p+2}^{ij} q_{1j} + \hat{\mathcal{B}}_{p+2}^{it} \dot{q}_{xt} + \hat{\mathcal{D}}_{p+2}^{is} \dot{w}_{gs} - \frac{2U_\infty \gamma_p}{c} \lambda_{p,i} \\
 \dot{\zeta}_{0l} &= -\frac{1}{2} \delta^{lm} (\omega_{l1} \zeta_{1m} + \omega_{l2} \zeta_{2m} + \omega_{l3} \zeta_{3m}) \\
 \dot{\zeta}_{1l} &= \frac{1}{2} (\delta^{lm} \omega_{1l} \zeta_{0m} - \varepsilon^{lmn} \omega_m \zeta_n) \\
 \dot{\zeta}_{2l} &= \frac{1}{2} (\delta^{lm} \omega_{2l} \zeta_{0m} - \varepsilon^{2mn} \omega_m \zeta_n) \\
 \dot{\zeta}_{3l} &= \frac{1}{2} (\delta^{lm} \omega_{3l} \zeta_{0m} - \varepsilon^{3mn} \omega_m \zeta_n)
 \end{cases} \quad (3.42)$$

for $i, j, k \in \{1, \dots, N_m\}$, $p \in \{1, \dots, N_p\}$, $l \in \{1, \dots, N_n\}$, $t \in \{1, \dots, N_x\}$, $s \in \{1, \dots, N_j\}$, $r \in \{1, \dots, N_\alpha\}$. The Levi-Civita symbol, ε_{mno} , is introduced for the cross-product in the quaternion terms. The natural frequencies in the system, ω_j , have been written as Ω_j to distinguish them from the angular velocities, ω , and the following re-scaling of the matrices and tensors has been used in order to simplify the notation:

$$\begin{aligned}
 \mathcal{M}^{ij} &= \delta^{ij} - \frac{\rho_\infty c^2}{8} \mathcal{A}_2^{ij} \quad ; \quad \hat{\Omega}^{ij} = (\mathcal{M}^{-1})^{ik} \delta^{kj} \Omega_j \quad ; \quad [\hat{\Gamma}_1^{ijk} \hat{\Gamma}_2^{ijk}] = (\mathcal{M}^{-1})^{ih} [\Gamma_1^{hjk} \Gamma_2^{hjk}] \\
 [\hat{\mathcal{A}}_0^{ij} \hat{\mathcal{B}}_0^{ij} \hat{\mathcal{C}}_0^{ij} \hat{\mathcal{D}}_0^{ij}] &= \frac{\rho_\infty U_\infty^2}{2} (\mathcal{M}^{-1})^{ik} [\mathcal{A}_0^{kj} \mathcal{B}_0^{kj} \mathcal{C}_0^{kj} \mathcal{D}_0^{kj}] \\
 [\hat{\mathcal{A}}_1^{ij} \hat{\mathcal{B}}_1^{ij} \hat{\mathcal{D}}_1^{ij}] &= \frac{c \rho_\infty U_\infty}{4} (\mathcal{M}^{-1})^{ik} [\mathcal{A}_1^{kj} \mathcal{B}_1^{kj} \mathcal{D}_1^{kj}] \\
 [\hat{\mathcal{B}}_2^{ij} \hat{\mathcal{D}}_2^{ij}] &= \frac{\rho_\infty c^2}{8} (\mathcal{M}^{-1})^{ik} [\mathcal{B}_2^{kj} \mathcal{D}_2^{kj}] \\
 [\hat{\mathcal{A}}_{p+2}^{ij} \hat{\mathcal{B}}_{p+2}^{ij} \hat{\mathcal{D}}_{p+2}^{ij}] &= \frac{c \rho_\infty U_\infty}{4} [\mathcal{A}_{p+2}^{kj} \mathcal{B}_{p+2}^{kj} \mathcal{D}_{p+2}^{kj}] \\
 [\hat{\eta}_{gi} \hat{\eta}_{fi}] &= (\mathcal{M}^{-1})^{ij} [\eta_{gj} \eta_{fj}]
 \end{aligned} \quad (3.43)$$

A residualisation scheme can be implemented for this system in the same way as in Eq. (2.103), so that the frequencies above cut-off are time-averaged and the system becomes a set of DAE.

System linearisation

The aeroelastic state variables are grouped as $\mathbf{x} = [\mathbf{q}_1^\top \mathbf{q}_2^\top \mathbf{q}_0^\top \boldsymbol{\lambda}_p^\top \boldsymbol{\zeta}^\top]^\top$ and Eqs. (3.42) can be written as $\dot{\mathbf{x}} = \mathbf{g}(\mathbf{x})$, where $\mathbf{g}(\mathbf{x})$ is a nonlinear function with second order nonlinearities. This aeroelastic system is then linearised around any reference condition $\mathbf{x}^\circ$, so that $\mathbf{x} = \mathbf{x}^\circ + \mathbf{x}^*$. Typically the reference is a static equilibrium (i.e., $\mathbf{q}_1^\circ = \mathbf{0}$ and $\boldsymbol{\lambda}_p^\circ = \mathbf{0}$ for all p). Thus an efficient approach is generated whereby a steady-state nonlinear solution is firstly obtained, and subsequent analysis can be performed using the linear system. The linear perturbation around the steady state is described as $\dot{\mathbf{x}}^* = \mathbf{G}_x(\mathbf{x}^\circ)\mathbf{x}^* + \boldsymbol{\eta}^*(\mathbf{x}^\circ)$ with $\mathbf{G}_x$ the Jacobian matrix of the aeroelastic system, $\mathbf{G}_x = \frac{\partial \mathbf{g}}{\partial \mathbf{x}}$, a linear function of the steady solution variables, $\mathbf{x}^\circ$; and $\boldsymbol{\eta}^*$ any given perturbation of the external forcing terms.

$$\mathbf{G}_x(\mathbf{x}^\circ) = \begin{bmatrix} \hat{\mathbf{A}}_1 - \mathbf{L}_{q_1}(\mathbf{q}_1^\circ) & \hat{\boldsymbol{\Omega}} - \mathbf{L}_{q_{12}}(\mathbf{q}_2^\circ) & \hat{\mathbf{A}}_0 & \mathcal{M}^{-1}[\mathbf{I}_{N_m \dots N_p} \mathbf{I}_{N_m}] & \mathbf{L}_{q_1 \zeta}(\boldsymbol{\zeta}^\circ) \\ -\boldsymbol{\Omega} + \mathbf{L}_{q_{21}}(\mathbf{q}_2^\circ) & \mathbf{L}_{q_2}(\mathbf{q}_1^\circ) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I}_{N_m} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hat{\mathbf{A}}_P & \mathbf{0} & \mathbf{0} & -\frac{2U_\infty}{c} \boldsymbol{\gamma}_p & \mathbf{0} \\ \mathbf{L}_{\zeta q_1}(\boldsymbol{\zeta}^\circ) & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{L}_\zeta(\mathbf{q}_1^\circ) \end{bmatrix} \quad (3.44)$$

$\boldsymbol{\gamma}_P$ and $\hat{\mathbf{A}}_P$ comprise the lag terms and their corresponding aerodynamic matrices:

$$\hat{\mathbf{A}}_P = \begin{bmatrix} \hat{\mathbf{A}}_3 \\ \vdots \\ \hat{\mathbf{A}}_{2+p} \end{bmatrix} \quad ; \quad \boldsymbol{\gamma}_P = \text{diag}(\gamma_1 \mathbf{I}_{N_m}, \dots, \gamma_{n_p} \mathbf{I}_{N_m}) \quad (3.45)$$

The $\mathbf{L}_q \in \mathbb{R}^{N_m \times N_m}$ matrices contain the derivatives of quadratic nonlinear couplings corresponding to gyroscopic and stiffness terms:

$$\begin{aligned} L_{q_1}^{ij} &= \left(\hat{\Gamma}_1^{ijk} + \hat{\Gamma}_1^{ikj} \right) q_{1k}^\circ \quad ; \quad L_{q_{12}}^{ij} = \left(\hat{\Gamma}_2^{ijk} + \hat{\Gamma}_2^{ikj} \right) q_{2k}^\circ \\ L_{q_{21}}^{ij} &= \hat{\Gamma}_2^{jik} q_{2k}^\circ \quad ; \quad L_{q_2}^{ij} = \hat{\Gamma}_2^{kij} q_{1k}^\circ \end{aligned} \quad (3.46)$$

The $\mathbf{L}_{q_1 \zeta} \in \mathbb{R}^{N_m \times 4N_n}$ matrix is the Jacobian of the gravity forces in Eq. (3.41) combined with any other dead force in the model, $\mathbf{f}^d \in \mathbb{R}^{6N_n}$, that needs to be rotated to the material frame

of reference:

$$\mathbf{L}_{q_1\zeta}^l = \Phi_{1l}^\top \begin{bmatrix} \mathbf{R}_{l,\zeta_0}^\top & \mathbf{0} & \mathbf{R}_{l,\zeta_1}^\top & \mathbf{0} & \mathbf{R}_{l,\zeta_2}^\top & \mathbf{0} & \mathbf{R}_{l,\zeta_3}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{l,\zeta_0}^\top & \mathbf{0} & \mathbf{R}_{l,\zeta_1}^\top & \mathbf{0} & \mathbf{R}_{l,\zeta_2}^\top & \mathbf{0} & \mathbf{R}_{l,\zeta_3}^\top \end{bmatrix} \begin{bmatrix} \mathbf{f}_l^d & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{f}_l^d & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{f}_l^d & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{f}_l^d \end{bmatrix} \quad (3.47)$$

where $\mathbf{L}_{q_1\zeta}^l \in \mathbb{R}^{N_m \times 4}$ and there is no sum in this case over the index l , rather it represents the values of the referred quantity at each particular node. The matrices $\mathbf{R}_{\zeta}$ are found for each node l after differentiating Eq. (3.40):

$$\begin{aligned} \mathbf{R}_{\zeta_0} &= \frac{\partial \mathbf{R}(\zeta^\circ)}{\partial \zeta_0} = 2\zeta_0^\circ \mathbf{I}_3 + 2\tilde{\zeta}_x^\circ \\ \mathbf{R}_{\zeta_k} &= \frac{\partial \mathbf{R}}{\partial \zeta_x^k} = \frac{\partial}{\partial \zeta_x^k} (2\zeta_x^i \zeta_x^j \mathbf{e}_i \otimes \mathbf{e}_j - \delta_{mn} \zeta_x^m \zeta_x^n \delta^{ij} \mathbf{e}_i \otimes \mathbf{e}_j + 2\zeta_0 (\zeta_x^j \mathbf{e}_j \times \mathbf{e}_i) \otimes \mathbf{e}_i) \\ &= \frac{\partial \mathbf{R}(\zeta^\circ)}{\partial \zeta_x^k} = 2\zeta_x^{j\circ} (\mathbf{e}_k \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_k) - 2\zeta_x^{k\circ} \delta^{ij} \mathbf{e}_i \otimes \mathbf{e}_j + 2\zeta_0^\circ (\mathbf{e}_k \times \mathbf{e}_i) \otimes \mathbf{e}_i \quad i, j, k = 1, 2, 3 \end{aligned} \quad (3.48)$$

note that these matrices are linear with ζ° . Similarly $\mathbf{L}_{\zeta q_1} \in \mathbb{R}^{4N_n \times N_m}$ is written for the node l and the mode j ,

$$\mathbf{L}_{\zeta q_1}^{lj} = \begin{bmatrix} -\frac{1}{2}(\Phi_{1\theta}^{lj})^\top \zeta_{xl}^\circ \\ \frac{1}{2}(\zeta_{0l}^\circ \Phi_{1\theta}^{lj} - \tilde{\Phi}_{1\theta}^{lj} \zeta_{xl}^\circ) \end{bmatrix} \quad (3.49)$$

the rest of the entries in the matrix being $\mathbf{0}$ with $\Phi_{1\theta} \in \mathbb{R}^{3N_n \times N_m}$ the angular part of $\Phi_1 \in \mathbb{R}^{6N_n \times N_m}$: $\Phi_{1\theta} = \Phi_1|^{3-6}$. Finally the diagonal matrix $\mathbf{L}_\zeta \in \mathbb{R}^{4N_n \times 4N_n}$ is,

$$\mathbf{L}_\zeta^l = \begin{bmatrix} 0 & -\frac{1}{2}\Phi_{1\theta}^l \mathbf{q}_1^\circ \\ \frac{1}{2}(\Phi_{1\theta}^l \mathbf{q}_1^\circ)^\top & -\frac{1}{2}\widetilde{\Phi_{1\theta}^l \mathbf{q}_1^\circ} \mathbf{I}_3 \end{bmatrix} \quad (3.50)$$

As previously mentioned, if the nonlinear reference condition comes from a static equilibrium, $\mathbf{q}_1^\circ = \mathbf{0}$, and $\mathbf{L}_{q_1} = \mathbf{L}_{q_2} = \mathbf{L}_\zeta = \mathbf{0}$.

Note on aerodynamic nonlinear effects

The path followed to construct a nonlinear aeroelastic system neglects the effect of large displacements in the aerodynamic boundary conditions, i.e. the geometry of aerodynamic panels is not updated in the simulations. Even though we will show in Ch. 6 that accounted effects such as wing stiffening or follower forces are predominant in typical simulations, it is important to bear the limitations of the methodology in mind as well as find possible solutions to them. We may differentiate between strategies that solve the nonlinear unsteady equations, Eqs. (3.3), or their linearised version, Eqs. (3.6):

- To the best of the author's knowledge, there has not been a strategy presented to solve the full-nonlinear Eq. (3.3) with arbitrary boundary conditions in an efficient manner (i.e. without having to resort to CFD solutions where a large volume of the 3D domain needs discretisation, which entails many operations at every step). Closed-form solutions have been given in the 2D domain by employing a hodograph transformation that converts the nonlinear equations into a hodograph plane where they become linear [265]. It would be interesting to extend this approach to the 3D domain so the complete set of shockless transonic effects can be accounted for.
- On the other hand, for the speeds where compressible effects play a role, the assumption by which Eq. (3.3) is linearised around the uniform flow may cover the wide range of operating conditions even in HARW aircraft experiencing significant wing excursions. Three possible approaches are sought to fully or partially consider the effects stemming from the nonlinear boundary conditions:
 - An adequate approach to deal with nonlinear boundary conditions is the unsteady-vortex-lattice-method, which updates the vortex rings elementary solutions as they deform in space and time with the lifting surfaces [89]. This method, however, solves the Laplace equation, i.e. Eq. (3.6) with no compressible effects ($\nabla^2 \hat{\phi} = 0$). It should be further researched whether it is feasible to transform the compressible subsonic equations to incompressible solutions, as done in [255], by effectively contracting

space in a change of coordinates, and subsequently solve these equations with a vortex-lattice approach and untangle the compressible effect at every iteration.

- Another possible solution could be achieved by taking the linear unsteady equations, (3.6), imposing the nonlinear boundary condition, Eq. (3.11), with the velocity field approximated from velocity and doublet elementary solutions, Eqs. (3.7), (3.10), that move with the aerodynamic surfaces. The boundary conditions would have to be updated and imposed at every step in the computations and a tightly or loosely couple approach implemented to exchange information from the aero and the structural solvers - very much like in most fluid-structure solutions-, instead of the fully coupled system shown in Eqs. (3.42).
- Improvement to the standard DLM has been presented [118] whereby the AICs are modified to be given with respect to a mean-axes frame instead of an inertial frame, such that nonlinear rigid body motions are combined with linear structural deformations in the analysis. This solution is not satisfactory in this work as it entails nonlinear deformations. Instead, a strategy is sought to directly update the DLM panels with deformations: the panels do not necessarily have to be in the $z = 0$ plane, but the chord of each panel has to be parallel to the x -direction. This means that if the aerodynamic panels were to be updated from a steady equilibrium geometry with, for instance, an increasing angle of attack, at every new iteration the lift would be zero and simulations would not progress. The proposed solution would be to recompute only the $\mathcal{C}_0$ matrix that yields the steady forces for changes in angle of attack, side-slip etc., and keep the rest of the aerodynamic matrices from the reference configuration. In the analysis of trim the forces from the rigid body modes (pitch and plunge) would be updated in, for instance, an increasing range of gravity acceleration from ϵ to the desired value. This way a completely unsteady nonlinear analysis would not be achieved and the effect of sectional-twist changing due to nonlinear deformations would not be captured. However, for commercial-type of aircraft, where large deformations will come mostly from the steady state, this approximation could provide good insight into the effect of large wing excursions

combined with compressible effects. An example of its application is a hinge on the wing that deflects in-flight to 90° , the aerodynamic forces on the hinge will be zero as the hinge at 90° will remain parallel to the flow with angle of attack variations. Standard DLM, yielding forces from perturbations of the reference configuration (with the hinge at 0°), would not capture the effect of nonlinear folding hinge angles, while this updating approach could provide the main features in the problem. We have started building on this approach although it has been left as a further development for future work.

Main contributions

This chapter has presented a novel aeroelastic approach to solve problems of aircraft loads and stability at industrial level. Direct coupling of the intrinsic formulation with aerodynamics in modal space has permitted the use of compressible 3D panel methods as opposed to previously utilised 2D aerodynamics. Moreover, the resulting approach is easily extendible to higher fidelity aerodynamics and since the equations are cast in a material frame of reference, the forces naturally rotate with the geometry. A new optimisation of the poles in the RFA has been envisaged that significantly improves the process of bringing the aerodynamics from the frequency domain to the time domain, and may reduce computational times by allowing a smaller number of poles in the solution. A strategy to obtain the nonlinear trim state has been proposed, alongside a method to perturb the aircraft with arbitrary gust shapes in the time domain. The aeroelastic system built in Eqs. (3.42) has every ingredient to perform analyses with rigid and elastic geometrically nonlinear deformations: from steady manoeuvres, trimmed flight, gust loads, to dynamic manoeuvres –though for the latter a controller of both lateral and longitudinal dynamics would be needed, which is left for future work. Finally, the linearisation presented in Eqs. (3.44) shows the potential of a formulation which is geometrically exact but reduces to quadratic nonlinearities, as a consequence of velocities and internal forces being the main variables.

Multibody Framework for the Intrinsic Aeroelastic Theory



The world says: “You have needs – satisfy them. You have as much right as the rich and the mighty. Don’t hesitate to satisfy your needs; indeed, expand your needs and demand more.” This is the worldly doctrine of today. And they believe that this is freedom. The result for the rich is isolation and suicide, for the poor, envy and murder.

Fyodor Dostoyevsky, *The Brothers Karamazov*

Freedom, Sancho, is one of the most precious gifts that heaven has bestowed upon men; no treasures that the earth holds buried or the sea conceals can compare with it; for freedom, as for honour, life may and should be ventured.

Miguel de Cervantes, *Don Quixote*

In the last two chapters, a geometrically nonlinear formulation of a complex, slender FEM was introduced together with the aeroelastic description of the one-body system. We show herein how to extend the formulation to an arbitrary number of bodies attached through different joint connections. We begin with a general overview of constrained dynamic systems that provides some insight into the historical developments of the theory and current state-of-the-art solutions; then we formulate the problem in intrinsic modal variables, which will allow a nonlinear

description both in the rigid-body rotations and the elastic deformations, starting from a set of linear FEMs independent of each other. Following the general theory, particular examples of connections are given such as spherical, hinge or cylindrical joints. A linear approximation to the multibody equations is presented for a specific set of problems. The chapter finishes with some considerations about the aeroelastic modelling in these systems and future work to consolidate the methodology.

4.1 Constrained mechanical systems

The general case of a multibody system consists of bodies, which may be rigid or flexible, attached by one or more types of connectors: rigid links that constrain a particular set of DoF between the bodies, also called joints; spring-like and damper-like connections that impose forces based on their relative position or velocities respectively; and actuators, which usually generate a force response following a predefined control law. Fig. 4.1 depicts a general multibody system with these ingredients. Note that the topology of the interlinked bodies is also important for how the formulation is built, with trees, chains or loops as possible topological configurations [266]. The power of systematic approaches and generalisations to solve these

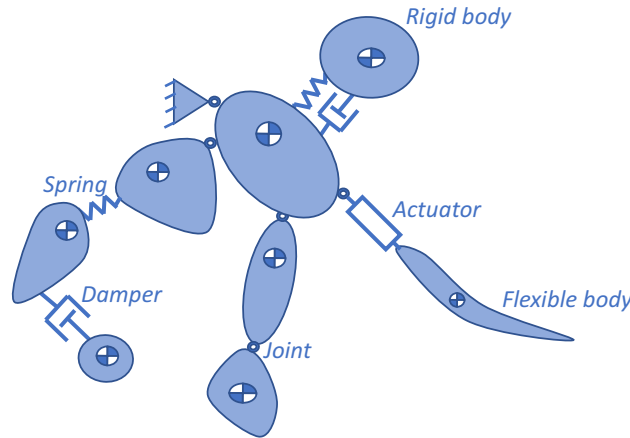


Figure 4.1: General multibody system example

type of systems is sought in the broad spectrum of applications where they are encountered, ranging from cars and trains that require modelling of many interconnected components such as wheels, shock absorbers, breaks or chassis [267]; helicopter rotor systems [268]; biomechanics

of humans with synovial joints linking muscles, tendons, and bones as part of a complex system of actuators and rigid elements [269]; robotic devices which may encompass complex topologies and very large rotations of each element; deployable structures in aircraft [181] and satellites [270]. In what follows we give a brief overview of the fundamental ideas and mathematical techniques to build multibody systems. These methods are applied in the next section to develop constrained connections for flexible bodies described by the modal intrinsic set of equations.

4.1.1 A historical glimpse

Some of most challenging mathematical problems posed throughout history have been related to the interaction between different bodies. Perhaps the eighteenth and nineteenth centuries were the most prolific in establishing the mathematical tools still prevalent in today's engineering practise. During that time, many endeavours were directed towards understanding and describing planetary dynamics; shall we take an instance such as the famous 3-body-problem, we realise that it can be idealised as a multibody system formed of three rigid bodies connected by springs whose force acts according to Newton's gravitational law. Therefore we might say that the realm of multibody systems has its roots in classical mechanics theory and its fundamentals were developed in parallel to this subject: Sir Isaac Newton's (1642-1727) infinitesimal calculus and interpretation of forces and dynamics of particles represents the fabric for all succeeding descriptions of mechanical systems. As the dynamics of different particles can be described using Newton's laws, we can think of a multibody system consisting of infinitesimally small particles that are bound to move at a fixed distance from each other in all possible configurations across time, i.e. a finite rigid body. The constraints for the rigid-body, however, do not need to explicitly appear in the dynamic equations, as derived by Leonhard Euler (1707-1783), and this is the case for many other instances of constrained mechanical systems. The problem lies in finding the (a priori unknown) forces imposed by the constraints, but these can be removed from the analysis if their virtual work vanishes [271], as shown by Jean Le Rond d'Alembert (1717-1783) with the principle that carries his name, which paved the way to solve problems involving holonomic constraints. Furthermore, d'Alembert's principle naturally leads to the principle of virtual work, which is a pillar of the foundation of analytic mechanics, a branch initially de-

veloped by Joseph-Louis Lagrange (1736-1813) that allows putting the dynamic equations as a function of generalised coordinates that are minimal, independent (only for holonomic systems) and implicitly hold the constraints of the system embedded on them. Major theoretical breakthroughs were made by Sir William Rowan Hamilton (1805–1865), whose work on variational principles and canonical transformations was of special relevance in the study of conserved quantities and symmetries in the dynamical systems [271]; the epitome of these advancements is found in Noether’s theorem, by Emmy Noether (1882-1935), which links symmetries in the action to conserved quantities and has impacted not only the science of classical mechanics but also modern physics.

4.1.2 Background theory

Moving on from the historical figures that established the main analytical tools used today, the rise in computational capabilities during the last century boosted the field and allowed bringing the theory to all sort of real life applications –as it occurred with the field of solid mechanics introduced in Ch. 2. Therefore recent developments have focused mainly on the computational aspects of the theory, as reviewed in [266, 272] for instance. In this section we present some of the basic methods to build and solve constrained mechanical systems, which will be applied in the next section to the intrinsic theory, Eqs. (2.93). We start by describing the constraints that will restrict the movement of one or more bodies: the types of constraints are classified according to different criteria, though the more important and recurrent type are the aforementioned holonomic constraints, which can be put into mathematical form as algebraic equations of the generalised coordinates in the system:

$$\mathbf{g}(\mathbf{q}) = \mathbf{0} \tag{4.1}$$

These algebraic equations are added to the differential equations that describe the body dynamics to form a set of DAE. A common strategy to turn the DAE system into an ODE system is to augment the Lagrangian of the system with the constraint equations, which have a null

value, so that Hamilton's principle of least action reads as,

$$\delta \int_{t_1}^{t_2} (\mathcal{K} - \mathcal{U} + \mathcal{W}_e + \mathbf{g}(\mathbf{q})^\top \boldsymbol{\lambda}) dt = 0 \quad (4.2)$$

where $\boldsymbol{\lambda}$ are known as Lagrange multipliers, only dependent on time and not on the generalised coordinates. Solving this expression via calculus of variations results in the Lagrange equations of the first kind. When generalised coordinates represent displacements, the equations take the following form,

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{f}(\dot{\mathbf{q}}, \mathbf{q}, t) - \mathbf{G}(\mathbf{q})^\top \boldsymbol{\lambda} \quad (4.3)$$

with $\mathbf{G}(\mathbf{q})$ the Jacobian of the constraints. In contrast, applying principles of minimal coordinates may eliminate the constraints altogether, thus directly converting the DAEs into a system of ODEs. These are known as the Lagrange equations of the second kind, which have obvious advantages such as avoiding special treatments on the constraints and time-integration. However, they are very problem-specific, difficult to apply in complex systems and the existence of a globally valid set of minimal coordinates is not generally guaranteed [273]. Therefore, as ultimately we are interested in the application to arbitrary aircraft models we continue the analysis with Lagrange first kind equations. Taking the time-derivative of Eq. (4.1) gives the constraints at velocity level, $\mathbf{G}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0}$, and taking a second time-derivative yields the acceleration level constraints, $\dot{\mathbf{G}}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{0}$. One way of calculating the Lagrange multipliers in Eq. (4.3) is substituting the expression for $\ddot{\mathbf{q}}$ in this equation into the acceleration level constraints, then solving for $\boldsymbol{\lambda}$, the so-called Lagrange multiplier method. Despite being straightforward, this approach encounters a number of issues: it is numerically unstable and research efforts have been put into the stabilisation of the solutions [269]; the mass matrix, $\mathbf{M}(\mathbf{q})$, has to be inverted during simulation, which is not efficient, and recursive approaches have been proposed to avoid this [266] –interestingly the intrinsic formulation circumvents these steps because, the way the equations are cast, the mass matrix is constant; the differential index of these equations is three¹ (one differentiation for the Jacobians plus two differentiations of the constraints), which

¹The index of a DAE measures its well-posedness in comparison to an ODE, and it is important because the higher the index, the more numerical difficulties are found in solving the equations[273]. Over time a range of indices have been defined, see [274] for a thorough classification, where it is shown that for regular, solvable systems, the various index concepts are equivalent to the differentiation index, the most frequently used: in

can be an issue for finding a solution, see [275, Ch. 11-12] for alternative procedures in eliminating Lagrange's multipliers, reducing the index of the DAEs and tailored numerical methods. Other approaches apart from the Lagrange multiplier method are the penalty method, which hinders movements that do not satisfy the constraints by prescribing a penalisation function on Eq. (4.2) as $\mathbf{F}_p(\mathbf{g}(\mathbf{q}))$ instead of $\mathbf{g}(\mathbf{q})^\top \boldsymbol{\lambda}$; and the augmented Lagrange method, which is a combination of the other two. A formalised description of these three methods can be found in [276]. We will only apply the Lagrange multiplier method in what follows, though future improvements could include implementation of the augmented Lagrange method and assessment of its advantages and the problems where needed.

Thus far only scleronomic constraints that do not depend explicitly on time have been presented, but the same strategies are applicable to rheonomic constraints, which take the form $\mathbf{g}(\mathbf{q}, t)$. On the other hand, non-holonomic constraints cannot always be put as algebraic equations and are given in differential form as,

$$\mathbf{G}_h(\mathbf{q}, t)d\mathbf{q} + \mathbf{g}_h(\mathbf{q}, t)dt = \mathbf{0} \quad (4.4)$$

where $\mathbf{G}_h$ and $\mathbf{g}_h$ cannot be expressed as the derivative of a vectorial function $\mathbf{g}$ with respect to the generalised coordinates and the time respectively, i.e. they cannot be integrated [275, Ch. 9]. In many practical problems Eqs. (4.4) are divided by dt and non-holonomic constraints are written as $\mathbf{g}(\dot{\mathbf{q}}, \mathbf{q}, t) = \mathbf{0}$. A classical example is a wheel rolling on a surface. Problems involving inequalities, as a body confined to a container, are also non-holonomic. Generalised theories for these constraints are not feasible and strategies are built only for subsets of specific problems.

After presenting some general conceptions of multibody systems, highlighting the various types of constraints and their mathematical modelling, we narrow our interest to flexible multibody systems that will represent the behaviour of slender, interconnected components. They carry some inherent difficulties from the point of numerical analysis [277]: good approximation of large elastic deformations and small system dimensions may prove impossible, and rigid-body motions and elastic deformations can exhibit very different time scales, which can be a challenge

simple terms, it is the number of differentiations needed to transform the DAE into an ODE.

for time integration. A number of computational methods have been proposed to surmount these issues and are reviewed in [272]: the most extended is the floating frame of reference formulation (FFRF), in which linear deformations are superimposed to large rigid-body motions, for it allows working and incorporating multibody features to large models, applying various reduction techniques to decrease computational times [278], and it is not overly complex; for aircraft applications it has been consolidated in the form of mean-axes techniques [117, 118]. In order to account for large body-deformations, two formulations have stood out for multibody applications, geometrically exact approaches and the absolute nodal coordinate (ANC) formulation; the former we have been using along this work, the latter has been developed on the grounds of approximating the displacement spatial field as polynomials [279] (for instance, for 1D elements the basis is linear in the cross-sectional dimensions, $[1, x, y, z, xy, xz, x^2, x^3, \dots, x^{N-5}]$), and directly applying the equations of solid mechanics to the resulting ad-hoc displacement field. The two formulations have been compared in [280], with the outcome that geometrically-exact formulations are more involved in terms of implementation and mathematical modelling, but offer an edge in computational speeds and accuracy; the ANC therefore might be easier to incorporate to already existing FE or multibody codes, and also gives better interfacing with other material codes and facilitates the calculation of 3D quantities, but would be inferior in terms of performance. In the following sections we incorporate multibody features to the already described geometric exact intrinsic theory in modal space, with the possibility that such enhancements as the application of reduction model techniques, use of general FE models, or the recovery of 3D quantities can also be applied in the multibody system.

4.2 Coupling independent aeroelastic bodies

We set out now to build a multibody system out of the geometrically nonlinear intrinsic formulation, whether it be for structural-only problems, Eqs. (2.100), for a simple aeroelastic system, Eqs. (3.31), or for the full aeroelastic description in Eqs. (3.42). They all are included in the computational implementation described in Ch. 5, along with an arbitrary number of bodies and constraints – as long as the topology of the interlinked bodies is a tree or a chain and

not a loop, for which a special treatment would need to be added. In essence, the multi-body problem considered herein is an idealisation of contact mechanics considering the link between bodies rigid and friction-free. We will study the so-called lower kinematic pairs, where the constraints occur along a surface common to both bodies, as opposed to upper kinematic pairs, in which contact takes place through a point or a line. The hinges on a door exemplify the former and a wheel rolling on a surface the latter. Different strategies were initially surveyed in order to cast the constrained problem: the modal equations can accommodate any number of rigid body modes so projection to a base with LNMs from a constrained system was thought a viable solution; while this proved to be a good and efficient approximation, described later on, it only worked within the linear region of displacements. Discontinuities in the PDE beam equations, Eqs. (2.93), made this approach fail under large deformations assumptions. Precisely a more standard solution could be achieved by writing these system of PDEs together with each particular joint and solving the resulting system via an FE discretisation. While this may provide a good solution for the beam problem, we strive to incorporate a multibody capability into arbitrary FE models, which inevitably requires a modal approach such that the properties of the FEM are embedded in the modes that conform the solution, as we did with the single body. A solution is finally attained by adding algebraic equations, as a function of the intrinsic variables, linear and angular local velocities², $\mathbf{g}(\mathbf{v}, \boldsymbol{\omega}, t)$, to the differential equations conforming the NMROM; then employing Lagrange multipliers to convert back the resulting DAE system into an ODE system, as laid out next.

4.2.1 General multi-body approach in intrinsic modal variables

Initially, a reformulation of some variables is needed to denote the connecting nodes of the different bodies. The components of the intrinsic velocity modes, Φ_1 , at the body $B^{(i)}$ and the connecting node j are simply written as $\Phi_1^{(i)} \in \mathbb{R}^{6 \times N_{m_i}}$. N_{m_i} are the number of modes used in the solution of each body i . The corresponding modal coordinates, $\mathbf{q}_1$, are called $\mathbf{q}_1^{(i)}$. In order to simplify the formulation we restrict the description to two bodies, $B^{(1)}$ and $B^{(2)}$, although

²Constrains could also be imposed on strains ($\boldsymbol{\gamma}$ or $\mathbf{q}_2$ in the differential equations), though they would be less useful in defining multi-body problems, and potentially more applicable in structural design or optimal control problems, such that maximum strains are bounded in the constrains.

the computational implementation has been generalised to n bodies with body 1 as the parent body and the rest attached to it. Therefore we have two translational local velocities, $\mathbf{v}$, their projection on the global reference frame, $\mathbf{v}_a$, and the local angular velocities, $\boldsymbol{\omega}$:

$$\begin{aligned} \mathbf{v}^{(1)} &= \Phi_{1u}^{(1)} \mathbf{q}_1^{(1)} & \boldsymbol{\omega}^{(1)} &= \Phi_{1\theta}^{(1)} \mathbf{q}_1^{(1)} & \mathbf{v}_a^{(1)} &= \mathbf{R}^{(1)} \mathbf{v}^{(1)} \\ \mathbf{v}^{(2)} &= \Phi_{1u}^{(2)} \mathbf{q}_1^{(2)} & \boldsymbol{\omega}^{(2)} &= \Phi_{1\theta}^{(2)} \mathbf{q}_1^{(2)} & \mathbf{v}_a^{(2)} &= \mathbf{R}^{(2)} \mathbf{v}^{(2)} \end{aligned} \quad (4.5)$$

where $\Phi_{1u}^{(i)} = \Phi_1^{(i)}|^{1-3}$ and $\Phi_{1\theta}^{(i)} = \Phi_1^{(i)}|^{3-6}$. Note how the angular velocity in the global reference frame, $\boldsymbol{\omega}_a$ is not needed to impose the constraints, as velocities due to rotations in the points of solid bodies are better described from the local reference, i.e. a Lagrangian description. Whenever constraints are given in the global reference frame, the rotation matrix at each connecting node, $\mathbf{R}^{(i)}$, will be needed, as provided in Eqs. (2.107) and equivalently in Eqs. (3.40). Quaternions are employed to parametrise rotations, Eqs. (2.106), and they will be part of the DAE system for the nodes at the joint,

$$\dot{\boldsymbol{\zeta}}^{(i)} = \begin{bmatrix} \dot{\zeta}_0^{(i)} \\ \dot{\boldsymbol{\zeta}}_x^{(i)} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \boldsymbol{\omega}^{(i)} \cdot \boldsymbol{\zeta}_x^{(i)} \\ \frac{1}{2} \left(\zeta_0^{(i)} \boldsymbol{\omega}^{(i)} - \tilde{\boldsymbol{\omega}}^{(i)} \boldsymbol{\zeta}_x^{(i)} \right) \end{bmatrix} \quad (4.6)$$

An important point to remark is that the calculation of the local velocities in Eq. (4.5) is an approximation by truncation of the modal base, i.e. for it to be exact the full set of modes would need to be employed. Previously we discussed the size of the NMROM to be dependent mostly on the $\mathbf{\Gamma}$ terms, Eq. (2.101), to approximate the nonlinear couplings, but also on the integration of the strain or the velocity field to obtain positions and rotations. Here we face the same challenge and if the minimum number of modes in the solution was determined by the amount of modes required to accurately impose the multibody constraints, this approach would be, at the least, suboptimal. We believe this local effect determining the size of the solution over the global response would not be the case in most practical problems. Should we model for instance a second attached body as being rigid and thereby consisting only of the 6 rigid body modes, no effects would be captured regarding the body deformation, however, those modes would suffice to keep the second body attached to the first one at one point. Hence we carry on with the analysis though more numerical problems to verify this assumption are pending.

Up to three algebraic equations, $\mathbf{g}^v$, are added to the intrinsic equations to restrain the movement of the 2 connecting points in their translational DoF. Similarly, for the general multibody case, rotations need to be constrained with up to three algebraic equations, $\mathbf{g}^\omega$, which are added to the positional constraints, $\mathbf{g}^v$. The Lagrange multipliers take the form, $\boldsymbol{\lambda}_{12} = [\boldsymbol{\lambda}_{12}^v \boldsymbol{\lambda}_{12}^\omega]^\top$. We denote with $\mathbf{F}_{11}$ and $\mathbf{F}_{12}$ the right hand side of the modal velocity components for body 1 and 2 respectively, such that for the structure-only description the first equation in Eqs. (2.100) would be written for the $B^{(i)}$ body as $\dot{\mathbf{q}}_1^{(i)} = \mathbf{F}_{1i}(\mathbf{q}_1^{(i)}, \mathbf{q}_2^{(i)})$. Similarly for the quaternions describing the rotations at the connections between bodies, $\dot{\boldsymbol{\zeta}}^{(i)} = \mathbf{F}_{\zeta i}(\mathbf{q}_1^{(i)}, \boldsymbol{\zeta}^{(i)})$. In the general case of the aeroelastic system in Eqs. (3.42), the generalised coordinates comprising all the modal states for both bodies except for the velocity component are collected in the vector $\mathbf{q}_n$ that contains: the force modal coordinates, $\mathbf{q}_2$; the displacement modal components, $\mathbf{q}_0$, and aerodynamic lag terms, $\boldsymbol{\lambda}_p$, if an aeroelastic system is built; and the quaternions, $\boldsymbol{\zeta}$, in the rest of the nodes that do not connect different bodies –if rotations on those nodes need to be tracked, for gravity or dead loadings for instance. The time derivative of $\mathbf{q}_n$ is expressed as a function $\mathbf{F}_{q_n}$. The resulting DAE system to be solved is,

$$\begin{aligned}
\dot{\mathbf{q}}_1^{(1)} &= \mathbf{F}_{11}(\mathbf{q}_1^{(1)}, \mathbf{q}_n) - \mathbf{G}_{q_1}^{v\top} \boldsymbol{\lambda}_{12}^v - \mathbf{G}_{q_1}^{\omega\top} \boldsymbol{\lambda}_{12}^\omega \\
\dot{\mathbf{q}}_1^{(2)} &= \mathbf{F}_{12}(\mathbf{q}_1^{(2)}, \mathbf{q}_n) - \mathbf{G}_{q_2}^{v\top} \boldsymbol{\lambda}_{12}^v - \mathbf{G}_{q_2}^{\omega\top} \boldsymbol{\lambda}_{12}^\omega \\
\dot{\boldsymbol{\zeta}}^{(1)} &= \mathbf{F}_{\zeta 1}(\mathbf{q}_1^{(1)}, \boldsymbol{\zeta}^{(1)}) - \mathbf{G}_{\zeta 1}^{v\top} \boldsymbol{\lambda}^v - \mathbf{G}_{\zeta 1}^{\omega\top} \boldsymbol{\lambda}_{12}^\omega \\
\dot{\boldsymbol{\zeta}}^{(2)} &= \mathbf{F}_{\zeta 2}(\mathbf{q}_1^{(2)}, \boldsymbol{\zeta}^{(2)}) - \mathbf{G}_{\zeta 2}^{v\top} \boldsymbol{\lambda}^v - \mathbf{G}_{\zeta 2}^{\omega\top} \boldsymbol{\lambda}_{12}^\omega \\
\dot{\mathbf{q}}_n &= \mathbf{F}_{q_n}(\mathbf{q}_1^{(1)}, \mathbf{q}_1^{(2)}, \mathbf{q}_n) \\
\mathbf{g}^v(\mathbf{q}_1^{(1)}, \mathbf{q}_1^{(2)}, \boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}) &= \mathbf{0} \\
\mathbf{g}^\omega(\mathbf{q}_1^{(1)}, \mathbf{q}_1^{(2)}, \boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}) &= \mathbf{0}
\end{aligned} \tag{4.7}$$

The $\mathbf{G}^v$ matrices are the Jacobians of the constraint equations $\mathbf{g}^v$, with respect to the modal velocity of the connecting node in the first body, $\mathbf{G}_{q_1}^v = \frac{\partial \mathbf{g}^v}{\partial \mathbf{q}_1^{(1)}} \in \mathbb{R}^{3 \times N_{m1}}$, in the second body, $\mathbf{G}_{q_2}^v = \frac{\partial \mathbf{g}^v}{\partial \mathbf{q}_1^{(2)}} \in \mathbb{R}^{3 \times N_{m2}}$, and with respect to the quaternion variables defining the rotations of the same nodes, $\mathbf{G}_{\zeta 1}^v = \frac{\partial \mathbf{g}^v}{\partial \boldsymbol{\zeta}^{(1)}} \in \mathbb{R}^{3 \times 4}$ and $\mathbf{G}_{\zeta 2}^v = \frac{\partial \mathbf{g}^v}{\partial \boldsymbol{\zeta}^{(2)}} \in \mathbb{R}^{3 \times 4}$ respectively. In the same way the $\mathbf{G}^\omega$ matrices are the Jacobians of the constraint equations $\mathbf{g}^\omega$. The differential terms $\dot{\mathbf{q}}_1^{(i)}$ and

$\dot{\zeta}^{(i)}$ are rearranged as,

$$\begin{aligned} \begin{bmatrix} \dot{q}_1^{(1)} \\ \dot{\zeta}^{(1)} \end{bmatrix} &= \begin{bmatrix} F_{11} \\ F_{\zeta 1} \end{bmatrix} - \begin{bmatrix} G_{q_1}^{v\top} & G_{q_1}^{\omega\top} \\ G_{\zeta 1}^{v\top} & G_{\zeta 1}^{\omega\top} \end{bmatrix} \lambda_{12} \\ \begin{bmatrix} \dot{q}_1^{(2)} \\ \dot{\zeta}^{(2)} \end{bmatrix} &= \begin{bmatrix} F_{12} \\ F_{\zeta 2} \end{bmatrix} - \begin{bmatrix} G_{q_2}^{v\top} & G_{q_2}^{\omega\top} \\ G_{\zeta 2}^{v\top} & G_{\zeta 2}^{\omega\top} \end{bmatrix} \lambda_{12} \end{aligned} \quad (4.8)$$

The Lagrange multipliers λ_{12} associated to the constraints are calculated by differentiating the constraints with respect to time:

$$\begin{aligned} \dot{g}^v &= G_{q_1}^v \dot{q}_1^{(1)} + G_{q_2}^v \dot{q}_1^{(2)} + G_{\zeta 1}^v \dot{\zeta}^{(1)} + G_{\zeta 2}^v \dot{\zeta}^{(2)} + \partial_t g^v \\ \dot{g}^\omega &= G_{q_1}^\omega \dot{q}_1^{(1)} + G_{q_2}^\omega \dot{q}_1^{(2)} + G_{\zeta 1}^\omega \dot{\zeta}^{(1)} + G_{\zeta 2}^\omega \dot{\zeta}^{(2)} + \partial_t g^\omega \end{aligned} \quad (4.9)$$

where $\partial_t g^v$ and $\partial_t g^\omega$ are the partial time derivatives of the constraints if they are of the rheonomic type. Since $\dot{g}^v = \dot{g}^\omega = \mathbf{0}$, Eq. (4.9) can be arranged in matrix form as follows,

$$\begin{bmatrix} G_{q_1}^v & G_{\zeta 1}^v \\ G_{q_1}^\omega & G_{\zeta 1}^\omega \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{\zeta}_1 \end{bmatrix} + \begin{bmatrix} G_{q_2}^v & G_{\zeta 2}^v \\ G_{q_2}^\omega & G_{\zeta 2}^\omega \end{bmatrix} \begin{bmatrix} \dot{q}_2 \\ \dot{\zeta}_2 \end{bmatrix} + \begin{bmatrix} \partial_t g^v \\ \partial_t g^\omega \end{bmatrix} = \mathbf{0} \quad (4.10)$$

Combining this expression with Eqs. (4.8) gives the Lagrange multipliers for the general case,

$$\begin{aligned} &\begin{bmatrix} G_{q_1}^v & G_{\zeta 1}^v \\ G_{q_1}^\omega & G_{\zeta 1}^\omega \end{bmatrix} \begin{bmatrix} F_{11} \\ F_{\zeta 1} \end{bmatrix} + \begin{bmatrix} G_{q_2}^v & G_{\zeta 2}^v \\ G_{q_2}^\omega & G_{\zeta 2}^\omega \end{bmatrix} \begin{bmatrix} F_{12} \\ F_{\zeta 2} \end{bmatrix} + \begin{bmatrix} \partial_t g^v \\ \partial_t g^\omega \end{bmatrix} - \\ &- \left(\begin{bmatrix} G_{q_1}^v & G_{\zeta 1}^v \\ G_{q_1}^\omega & G_{\zeta 1}^\omega \end{bmatrix} \begin{bmatrix} G_{q_1}^{v\top} & G_{q_1}^{\omega\top} \\ G_{\zeta 1}^{v\top} & G_{\zeta 1}^{\omega\top} \end{bmatrix} + \begin{bmatrix} G_{q_2}^v & G_{\zeta 2}^v \\ G_{q_2}^\omega & G_{\zeta 2}^\omega \end{bmatrix} \begin{bmatrix} G_{q_2}^{v\top} & G_{q_2}^{\omega\top} \\ G_{\zeta 2}^{v\top} & G_{\zeta 2}^{\omega\top} \end{bmatrix} \right) \lambda_{12} = \mathbf{0} \end{aligned} \quad (4.11)$$

Defining $G_1 = \begin{bmatrix} G_{q_1}^v & G_{\zeta 1}^v \\ G_{q_1}^\omega & G_{\zeta 1}^\omega \end{bmatrix}$ and $G_2 = \begin{bmatrix} G_{q_2}^v & G_{\zeta 2}^v \\ G_{q_2}^\omega & G_{\zeta 2}^\omega \end{bmatrix}$, Lagrange multipliers are written as

$$\lambda_{12} = (G_1 G_1^\top + G_2 G_2^\top)^{-1} \left(G_1 \begin{bmatrix} F_{11} \\ F_{\zeta 1} \end{bmatrix} + G_2 \begin{bmatrix} F_{12} \\ F_{\zeta 2} \end{bmatrix} + \begin{bmatrix} \partial_t g^v \\ \partial_t g^\omega \end{bmatrix} \right) \quad (4.12)$$

In relation with the inversion of the first parenthesis we note that the matrix inside is positive semidefinite as it is formed from a sum of matrices multiplied by its transpose. Therefore ensuring that each $\mathbf{G}_{(i)}$ is invertible makes the overall expression determinant larger than zero, which might require removal of linearly dependant rows and columns on these matrices as shown below in the specific examples. The process to time-march the multi-body system is to first compute all the $\mathbf{F}_{(\bullet)}$ terms which depend on the current state variables, $\mathbf{q}_1^{(1)}$, $\mathbf{q}_1^{(2)}$ and $\mathbf{q}_n$; afterwards the Jacobians $\mathbf{G}_{(i)}$, problem specific depending on the type of constraint, are obtained from the current state variables to; then λ_{12} can be calculated from Eqs. (4.12); finally all the terms on the right-hand side of Eqs. (4.7) are computed to advance the ODE system to the next time step. Note that because generalised coordinates represent velocities instead of displacements, the differential index of the resulting equations is two (one derivative of $\mathbf{g}^v$ and $\mathbf{g}^\omega$, and substitution of λ_{12} to convert the DAE to ODE), as opposed to classical formulations of flexible bodies which exhibit an index of three. This theory is applied in the next section to the most common set of rigid connections.

4.2.2 Lower kinematic pairs

The previous section have shown all necessary mathematical tools in order to couple, via a point connection, two independent bodies. Displacement and rotational DoF are to be restricted as to build the required kinematic pairs. Fig. 4.2 presents some of the most common types of connections that will be modelled within the nonlinear intrinsic framework. A spherical pair or joint restricts the three positional DoF of the connecting node of the second-body to be equal to those of the first-body, and rotations are independent in the two bodies; therefore out of the twelve DoF in the two nodes, three are constrained in this link. In the hinge or revolute joint, similar to the spherical pair, the second-body is bound to move with the first-body, however, only rotation along a hinge-axis is permitted in the second-body; thus there is only one independent DoF in the second-body. The cylindrical joint allows independent movement and rotation in the second-body only through one axis. The prismatic joint is similar to the cylindrical joint, adding to it a constraint in the free rotational DoF, such that only a linear movement in one direction with respect the first-body is allowed in the second one. Fig. 4.2

shows with arrows the independent DoF of the second-body for these four joints. Other types of joints can be derived as a combination of these ones, such as the screw joint; and even higher kinematic pairs can be put as a combination of these joints, the universal joint is an example of it, consisting of two hinges in orthogonal planes.

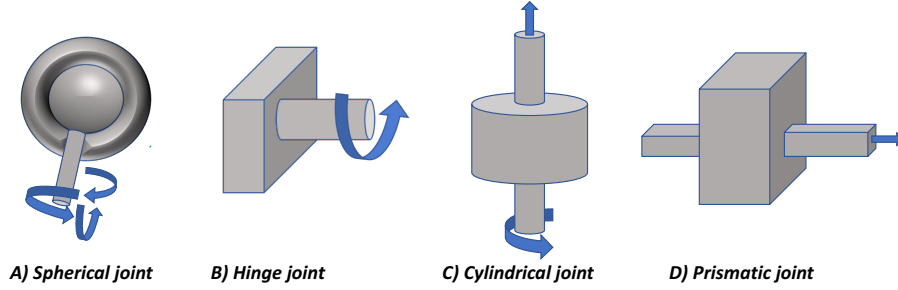


Figure 4.2: Typical examples of lower kinematic pairs

Spherical joint

The three positional constraints in the spherical joint are modelled by equating the global velocities at each connecting node, giving three algebraic equations:

$$\mathbf{g}^v = \dot{\mathbf{r}}_a^{(1)} - \dot{\mathbf{r}}_a^{(2)} = \mathbf{0} \quad (4.13)$$

where $\mathbf{r}_a^{(i)}$ is the position in the global reference frame and $\dot{\mathbf{r}}_a^{(i)} = \mathbf{R}^{(i)}\mathbf{v}^{(i)} = \mathbf{R}^{(i)}\Phi_{1u}^{(i)}\mathbf{q}_1^{(i)}$. The Jacobian of $\dot{\mathbf{r}}_a$ with respect to the modal velocities is given by the function $\mathbf{G}_q^v$,

$$\mathbf{G}_q^v(\zeta) = \frac{\partial \dot{\mathbf{r}}_a}{\partial \mathbf{q}_1} = \mathbf{R}(\zeta)\Phi_{1u} \quad (4.14)$$

similarly the function $\mathbf{G}_\zeta^v$ gives the Jacobian of $\dot{\mathbf{r}}_a$ with respect to the rotational quaternions after Eq. (3.40),

$$\mathbf{G}_\zeta^v(\mathbf{v}, \zeta) = \frac{\partial \dot{\mathbf{r}}_a}{\partial \zeta} = \frac{\partial R^{ij}}{\partial \zeta^k} v_j \mathbf{e}_i \otimes \mathbf{e}_k = \begin{bmatrix} 2\zeta_0 v_1 + 2\zeta_2 v_3 - 2\zeta_3 v_2 & 2\zeta_1 v_1 + 2\zeta_2 v_2 + 2\zeta_3 v_3 \\ 2\zeta_0 v_2 - 2\zeta_1 v_3 + 2\zeta_3 v_1 & -2\zeta_0 v_3 - 2\zeta_1 v_2 + 2\zeta_2 v_1 \\ 2\zeta_0 v_3 + 2\zeta_1 v_2 - 2\zeta_2 v_1 & 2\zeta_0 v_2 - 2\zeta_1 v_3 + 2\zeta_3 v_1 \\ 2\zeta_0 v_3 + 2\zeta_1 v_2 - 2\zeta_2 v_1 & -2\zeta_0 v_2 + 2\zeta_1 v_3 - 2\zeta_3 v_1 \\ 2\zeta_1 v_1 + 2\zeta_2 v_2 + 2\zeta_3 v_3 & 2\zeta_0 v_1 + 2\zeta_2 v_3 - 2\zeta_3 v_2 \\ -2\zeta_0 v_1 - 2\zeta_2 v_3 + 2\zeta_3 v_2 & 2\zeta_1 v_1 + 2\zeta_2 v_2 + 2\zeta_3 v_3 \end{bmatrix} \quad (4.15)$$

with v_j and ζ_k the components of the local velocities and the rotational quaternions respectively. Using Eqs. (4.14) and (4.15), the multi-body system in Eqs. (4.7) is assembled with the Lagrange multipliers in (4.12) and the Jacobians of the system as,

$$\begin{aligned} \mathbf{G}_{q_1}^v &= \mathbf{G}_q^v(\zeta^{(1)}) = \mathbf{R}^{(1)} \Phi_{1u}^{(1)} \quad ; \quad \mathbf{G}_{q_2}^v = -\mathbf{G}_q^v(\zeta^{(2)}) = -\mathbf{R}^{(2)} \Phi_{1u}^{(2)} \\ \mathbf{G}_{\zeta_1}^v &= \mathbf{G}_\zeta^v(\mathbf{v}^{(1)}, \zeta^{(1)}) \quad ; \quad \mathbf{G}_{\zeta_2}^v = -\mathbf{G}_\zeta^v(\mathbf{v}^{(2)}, \zeta^{(2)}) \end{aligned} \quad (4.16)$$

Spherical joints can be found in joystick devices, the steering knuckle part of automotive suspensions or the human hip-joint, making it an important type of joint in the multibody modelling.

Hinged joint

For the revolute or hinge joint three equations, $\mathbf{g}^v$, represent the attachment of the two connecting nodes, exactly as in the spherical joint; and two $\mathbf{g}^\omega$ equations limit the rotation of the second-body with respect to the first-body to an axis of rotation as follows: let $\mathbf{p}_0$ be a unit vector along the hinge axis, the velocity (in the global frame) of the end-point of $\mathbf{p}_0$, $\mathbf{v}_{ap}$, will be equal in both reference systems:

$$\mathbf{v}_{ap} = \dot{\mathbf{r}}_a^{(1)} + \dot{\mathbf{p}}_{01} + \tilde{\boldsymbol{\omega}}^{(1)} \mathbf{p}_{01} = \dot{\mathbf{r}}_a^{(1)} + \tilde{\boldsymbol{\omega}}^{(1)} \mathbf{p}_{01} = \dot{\mathbf{r}}_a^{(2)} + \tilde{\boldsymbol{\omega}}^{(2)} \mathbf{p}_{02} \quad (4.17)$$

where $\mathbf{p}_{0i}$ is the hinge-axis vector $\mathbf{p}_0$ in the initial local reference system of the body i , and its derivative $\dot{\mathbf{p}}_{0i}$ is equal to $\mathbf{0}$ as to model a rigid hinge. The constraints will then be,

$$\begin{aligned}\mathbf{g}^v &= \dot{\mathbf{r}}_a^{(1)} - \dot{\mathbf{r}}_a^{(2)} = \mathbf{0} \\ \mathbf{g}^\omega &= \tilde{\boldsymbol{\omega}}^{(1)} \mathbf{p}_{01} - \tilde{\boldsymbol{\omega}}^{(2)} \mathbf{p}_{02} = \mathbf{0}\end{aligned}\tag{4.18}$$

The positional constraints $\mathbf{g}^v$ are similar to the spherical pair so the Jacobians, $\mathbf{G}_{q_1}^v$, $\mathbf{G}_{q_2}^v$, $\mathbf{G}_{\zeta_1}^v$ and $\mathbf{G}_{\zeta_2}^v$, are equal to those given in Eqs. (4.16). The Jacobians of the rotational constraints and the rest of the terms needed to complete the Lagrange multipliers in Eq. (4.12) are,

$$\begin{aligned}[\mathbf{G}_{q_1}^\omega \ \mathbf{G}_{q_2}^\omega] &= [\tilde{\boldsymbol{\Phi}}_\theta^{(1)} \mathbf{p}_{01} \quad -\tilde{\boldsymbol{\Phi}}_\theta^{(2)} \mathbf{p}_{02}] \\ \mathbf{G}_{\zeta_1}^\omega &= \mathbf{G}_{\zeta_2}^\omega = \mathbf{0} \\ [\partial_t \mathbf{g}^v \ \partial_t \mathbf{g}^\omega] &= \mathbf{0}\end{aligned}\tag{4.19}$$

Note that $\mathbf{G}_q^\omega$ are defined in a subspace which is the plane perpendicular to $\mathbf{p}_0$ and therefore $\mathbf{G}_{q_i}^\omega \mathbf{G}_{q_i}^{\omega\top}$ is not full rank – which is consistent with the five DoF that have to be constrained in this kinematic pair. One of the rows in $\mathbf{G}_q^\omega$ is zero or dependent on the other two and has to be removed, yielding 2 constraints for the rotation, as expected.

Hinge joints appear in all sort of mechanisms, from engine systems to deployable wings.

Cylindrical joint

In the cylindrical pair problem the 2 bodies share an axis through which free rotations and displacements may occur. Constraining rotational DoF is done in a similar fashion to the hinge joint. Despite the reference systems moving differently along the axis of rotation, the assumption by which the velocity of the end point of $\mathbf{p}_0$ is equal in both reference systems remains valid. For the positional constraints, the velocity component perpendicular to the axis of rotation has to be the same in both bodies (note that no distinction is made here for $\mathbf{p}_0$ in the reference systems as they can be initially defined such that $\mathbf{p}_0$ is a unit-normed vector in the x-direction of both bodies). And similarly to the rotational constraints, which are defined in a subspace, here the positional constraints are defined through the subspace of the velocities perpendicular to the axis of rotation. The 3 eqs. are not independent and can be reduced to

2, by removing one row in the Jacobians. These can be calculated in the same way as in the hinge problem. The constraints are,

$$\begin{aligned}\mathbf{g}^v &= \mathbf{R}^{(1)} (\mathbf{v}^{(1)} - (\mathbf{v}^{(1)} \cdot \mathbf{p}_0) \mathbf{p}_0) - \mathbf{R}^{(2)} (\mathbf{v}^{(2)} - (\mathbf{v}^{(2)} \cdot \mathbf{p}_0) \mathbf{p}_0) = \mathbf{0} \\ \mathbf{g}^\omega &= (\tilde{\boldsymbol{\omega}}^{(1)} - \tilde{\boldsymbol{\omega}}^{(2)}) \mathbf{p}_0 = \mathbf{0}\end{aligned}\tag{4.20}$$

and the resulting Jacobians,

$$\begin{aligned}[\mathbf{G}_{q_1}^v \ \mathbf{G}_{q_2}^v] &= [\mathbf{R}^{(1)} (\mathbf{I}_3 - \mathbf{p}_0 \otimes \mathbf{p}_0) \boldsymbol{\Phi}_{1u}^{(1)} \quad - \mathbf{R}^{(2)} (\mathbf{I}_3 - \mathbf{p}_0 \otimes \mathbf{p}_0) \boldsymbol{\Phi}_{1u}^{(2)}] \\ [\mathbf{G}_{\zeta_1}^v \ \mathbf{G}_{\zeta_2}^v] &= [\mathbf{G}_\zeta^v ((\mathbf{I}_3 - \mathbf{p}_0 \otimes \mathbf{p}_0) \mathbf{v}^{(1)}, \boldsymbol{\zeta}^{(1)}) \quad - \mathbf{G}_\zeta^v ((\mathbf{I}_3 - \mathbf{p}_0 \otimes \mathbf{p}_0) \mathbf{v}^{(2)}, \boldsymbol{\zeta}^{(2)})] \\ [\mathbf{G}_{q_1}^\omega \ \mathbf{G}_{q_2}^\omega] &= [\tilde{\boldsymbol{\Phi}}_{1\theta}^{(1)} \mathbf{p}_0 \quad - \tilde{\boldsymbol{\Phi}}_{1\theta}^{(2)} \mathbf{p}_0] \\ [\mathbf{G}_{\zeta_1}^\omega \ \mathbf{G}_{\zeta_2}^\omega] &= [\mathbf{0} \ \mathbf{0}] \\ [\partial_t \mathbf{g}^v \ \partial_t \mathbf{g}^\omega] &= \mathbf{0}\end{aligned}\tag{4.21}$$

with the function $\mathbf{G}_\zeta^v$ given by Eq. (4.15).

An example with this pair that would also required geometrically-nonlinear capabilities is that of a flexible catenary formed by a cable and a mass attached to it through a cylindrical joint to model a zip-line.

Prismatic pair

For this joint, the second body has only one DoF unconstrained to slide through a common axis $\mathbf{p}_0$. If the initial reference frames are chosen to be equal, they will remain so as the three rotational DoF of the second body are constrained. This allows putting the positional constraints in the local frame:

$$\begin{aligned}\mathbf{g}^v &= (\mathbf{v}^{(1)} - \mathbf{v}^{(2)}) \times \mathbf{p}_0 = \mathbf{0} \\ \mathbf{g}^\omega &= \boldsymbol{\omega}^{(1)} - \boldsymbol{\omega}^{(2)} = \mathbf{0}\end{aligned}\tag{4.22}$$

yet again the three $\mathbf{g}^v$ equations are not independent. In fact, if $\mathbf{e}_1$ of the initial reference systems is chosen to be parallel to $\mathbf{p}_0$, the two resulting constraint equations will be just

$\mathbf{v}_2^{(1)} = \mathbf{v}_2^{(1)}$ and $\mathbf{v}_3^{(2)} = \mathbf{v}_3^{(2)}$. The Jacobians are,

$$\begin{aligned} [\mathbf{G}_{q_1}^v \ \mathbf{G}_{q_2}^v] &= [\tilde{\Phi}_{1u}^{(1)} \mathbf{p}_0 \quad -\tilde{\Phi}_{1u}^{(2)} \mathbf{p}_0] \\ [\mathbf{G}_{q_1}^\omega \ \mathbf{G}_{q_2}^\omega] &= [\Phi_{1\theta}^{(1)} \quad -\Phi_{1\theta}^{(2)}] \\ \mathbf{G}_{\zeta_1}^\omega &= \mathbf{G}_{\zeta_2}^\omega = \mathbf{0} \\ [\partial_t \mathbf{g}^v \ \partial_t \mathbf{g}^\omega] &= \mathbf{0} \end{aligned} \tag{4.23}$$

A slider-crank mechanism transforms linear into angular motion, and is formed with three revolute joints and one prismatic joint, which in the modelling of a combustion engine represents the piston motion.

4.2.3 Constrained trajectories

A further generalisation of the theory is explored by considering the restriction of one or more nodes in the system to a particular trajectory, or by restraining its configuration space to a given 1D or 2D subspace. The same strategy followed to obtain the lower kinematic pairs can be employed, which will give rise to other types of constraints such rheonomic and nonholonomic as shown in what follows.

Point trajectory

Let us take the example of a point that is to be defined through the velocity function $\mathbf{f}_v(t)$. If this function is given in the global reference frame, the constraints are written as,

$$\mathbf{g}^v = \dot{\mathbf{r}}_a^{(1)} - \mathbf{f}_v(t) = \mathbf{0} \tag{4.24}$$

and the required Jacobians are computed using Eqs. (4.14) and (4.15),

$$\begin{aligned} \mathbf{G}_{q_1}^v &= \mathbf{G}_q^v(\boldsymbol{\zeta}^{(1)}) = \mathbf{R}^{(1)} \Phi_{1u}^{(1)} \\ \mathbf{G}_{\zeta_1}^v &= \mathbf{G}_\zeta^v(\mathbf{v}^{(1)}, \boldsymbol{\zeta}^{(1)}) \\ \partial_t \mathbf{g}^v &= -\partial_t \mathbf{f}_v(t) \end{aligned} \tag{4.25}$$

On the other hand, if the velocity is defined in the local reference system,

$$\mathbf{g}^v = \mathbf{v}^{(1)} - \mathbf{f}_v(t) = \mathbf{0} \quad (4.26)$$

with the respective Jacobians:

$$\begin{aligned} \mathbf{G}_{q_1}^v &= \Phi_{1u}^{(1)} \\ \mathbf{G}_{\zeta_1}^v &= \mathbf{0} \\ \partial_t \mathbf{g}^v &= -\partial_t \mathbf{f}_v(t) \end{aligned} \quad (4.27)$$

Previous lower kinematic pairs yielded holonomic, scleronomic types of constraints. In defining the trajectory across time of a point in the structure we see an example of a rheonomic constraint, although the solution does not require any special treatment. In a similar manner, rotations around a point could be prescribed. This would permit, for instance, a direct prescription of any aircraft manoeuvre.

Point-plane constraint

When a node is to be restrained to move within a fixed plane the velocity component parallel to the normal of the plane should be zero, only one constraint equation is thus needed:

$$g^v = \dot{\mathbf{r}}_a^{(1)} \cdot \mathbf{p}_0 = \mathbf{v}^{(1)\top} \mathbf{R}^\top \mathbf{p}_0 = 0 \quad (4.28)$$

Defining the the function $\mathbf{G}_{\zeta p_0}^v$ as,

$$\mathbf{G}_{\zeta p_0}^v(\mathbf{p}, \boldsymbol{\zeta}) = \frac{\partial R^{\top ij}}{\partial \zeta^k} p_j \mathbf{e}_i \otimes \mathbf{e}_k \quad (4.29)$$

the Jacobians of the constraints are,

$$\begin{aligned} \mathbf{G}_{q_1}^v &= \Phi_u^{(1)\top} \mathbf{R}^{(1)\top} \mathbf{p}_0 \\ \mathbf{G}_{\zeta_1}^v &= \mathbf{v}^{(1)\top} \mathbf{G}_{\zeta p_0}^v(\mathbf{p}_0, \boldsymbol{\zeta}^{(1)}) \\ \partial_t \mathbf{g}^v &= \mathbf{0} \end{aligned} \quad (4.30)$$

Further considerations with manifolds constraints

Restricting the 3D domain of a point in the structure to an arbitrary curve or to a surface carries the difficulty that knowledge of the actual position within the subdomain is required in the definition of the constraints. But the generalised coordinates in the description represent velocities so the constraints will actually depend on the integration of a function of the coordinates. First we take the case where the movement of a point is to be restricted to the 1D domain defined by a given curve $\mathbf{r}_x$, parametrised by the arc-length s . Two constraints will be required such that the velocity perpendicular to the tangent of the curve is zero:

$$\mathbf{g}^v = \dot{\mathbf{r}}_a^{(1)} \times \mathbf{p}_0(s) = \mathbf{0} \quad (4.31)$$

where $\mathbf{p}_0$ is the tangent given as $\mathbf{p}_0(s) = \mathbf{r}'_x(s)$ and the parameter s can be calculated as $s(t) = \int_0^t \|\dot{\mathbf{r}}_a^{(1)}\| dt$. Another parametrisation of $\mathbf{r}_x$ could be used as a function of $\mathbf{r}_a^{(1)}$, $\mathbf{r}_x(x) = [x, y(x), z(x)]$ with the tangent as $\mathbf{p}_0(\mathbf{r}_a^{(1)}) = [1, y'(\mathbf{r}_a^{(1)} \cdot [1, 0, 0]), z'(\mathbf{r}_a^{(1)} \cdot [1, 0, 0])]$. In both cases the constraint equations would not only depend on the system coordinates but also on their integration.

Similarly in order to constrain the movement of one point to a 2D manifold, the surface is first parametrised as $\mathbf{r}_x = [x, y, z(x, y)]$, and its normal given as $\mathbf{p}_0(x, y) = [1, 0, \frac{\partial z}{\partial x}] \times [0, 1, \frac{\partial z}{\partial y}] = [-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1]$. The velocity normal to this surface should vanish so the one equation constraint is,

$$g^v = \dot{\mathbf{r}}_a^{(1)} \cdot \mathbf{p}_0(\mathbf{r}_a^{(1)} \cdot [1, 0, 0], \mathbf{r}_a^{(1)} \cdot [0, 1, 0]) = 0 \quad (4.32)$$

In this case also the position $\mathbf{r}_a^{(1)}$ is required in enforcing the constraints. It is worth remarking that as opposed to $\mathbf{r}_a^{(1)}$, the rotation matrix $\mathbf{R}^{(1)}$, that needs to be recovered to put the constraints in a global reference system, has been parameterised with quaternions which are part of the DAE variables so its Jacobians can be analytically calculated as in Eq. (4.15).

4.2.4 Aeroelastic modelling and linear multibody approximation

The current description is fully geometrically non-linear as long as the $\mathbf{F}_{(\bullet)}$ terms in (4.7) maintain all the nonlinear couplings in Eqs. (2.100) for structural-only problems, or in Eqs. (3.42) for the full aeroelastic description. When large deformations are negligible and only the relative rotations between bodies show a nonlinear behaviour, the nonlinear coefficients of the intrinsic equations can be removed, thus simplifying and speeding up the computations. Another possible scenario that leads to an elegant solution is found when small rotations and displacements occur between bodies while large deformations on each body remain relevant. Though this might not be very common in pure mechanical systems, effects such as aerodynamic damping or actuators in airborne vehicles may prevent deployable structures to incur in significant deformations between contact bodies, but the high aspect ratio of components still needs a geometrically nonlinear treatment. In this case what is important from the structural point of view is to capture the main dynamics and effects such as the no-transmission of moments through a hinge joint. This can be achieved seamlessly by using the LNMs from a FEM initially built as the intended multibody system. The added steps in the process are as follows,

- In the original 3D FE linear model, the constrained DoF at the coincident nodes of the multibody system appear on the mass and stiffness matrices as rows and columns of zeros and they need to be removed.
- The removed DoF of the dependent node are brought back into the LNMs by copying the information from the coincident (independent) node.
- For each free body, 6 rigid-body modes with null frequency are obtained and used in the intrinsic equations to solve the rigid body dynamics. In the same way each DoF in the connection between bodies that is free will add a rigid-body mode with zero frequency. For example, a free-free multibody system with 2 bodies joint by a spherical pair will have $6 + 3$ rigid body modes.
- In the NMROM, the recovery of the position vector from the local velocities or strains needs a special treatment: at the coincident node between bodies, local velocities are

equal (as the modal information of the constrained DoF on the connected body has been copied). The rotation at these nodes, however, is different, so that the system gets detached at the connections. Constraints have to be imposed on the integration of the dependent body, bringing the positional information from the connecting nodes, $\mathbf{r}_a^{(1)}$ and $\mathbf{r}_a^{(2)}$, into all the points in the second-body $\mathbf{r}_a^{B_2}$. For example, in spherical joints or in hinge joints where the two nodes are located in the same position, the integration of velocities is carried as follows:

$$\begin{aligned}\dot{\mathbf{R}}^{B_1} &= \mathbf{R}^{B_1} \tilde{\boldsymbol{\omega}}_a^{B_1} ; & \dot{\mathbf{r}}_a^{B_1} &= \mathbf{R}^{B_1} \mathbf{v}_a^{B_1} \\ \dot{\mathbf{R}}^{B_2} &= \mathbf{R}^{B_2} \tilde{\boldsymbol{\omega}}_a^{B_2} ; & \dot{\mathbf{r}}_a^{B_2} &= \mathbf{R}^{B_2} \mathbf{v}_a^{B_2} + (\mathbf{R}^{(1)} \mathbf{v}_a^{(1)} - \mathbf{R}_a^{(2)} \mathbf{v}_a^{(2)})\end{aligned}\tag{4.33}$$

For the integration of strains, the connecting point on each body is tracked using the local velocities and integration of strains yields the rest of the nodes' position.

Aerodynamic integration

Having described and implemented different strategies for the structural multi-dynamic problem, we turn our attention into the aerodynamic theory and the coupling between the aeroelastic bodies. The same difficulties underlined in Sec. 3.4.3 are encountered here: the aerodynamic loads in a dynamic analysis do not distinguish between single or multi body approaches, but the boundary conditions of the structural elements, imposed as non-penetrations across the aerodynamic surfaces, need to be met and could vary enormously in multibody problems. This requires an updating of the aerodynamic panels for the full nonlinear problem to be solved. Within methodologies based on AICs matrices, they can be updated around a series of steady solutions as previously described. If the AICs are not updated because their effect is deemed of second order, two approaches have been utilised in the current implementation: when, as presented in this section, the multibody system is built by projecting the nonlinear equations into a subspace of LNMs that already features the desired constraints, the $\mathcal{A}_i$ matrices in Eq. (3.38) are obtained and incorporated in the aeroelastic system in a similar fashion as for a single body, thus the aeroelastic analysis is carried out with no modification other than the points made above and in Eq. (4.33). On the other hand, if a full multibody approach is

pursued using the equations introduced in this chapter, the structural model will be built with no connections between bodies, yielding the corresponding linear stiffness and mass matrices, $\mathbf{K}^{(i)}$ and $\mathbf{M}^{(i)}$. The aerodynamic panels will be built on top of the structural model without a special distinction between bodies, though important care must be taken in the interpolation of aerodynamics and structural elements at the connections (a good study on the interpolation between structural and aerodynamic nodes at the connection of bodies can be found in [184]). For two bodies, the aerodynamic matrices will have the size of the total number of modes of the two bodies combined, $\mathcal{A}_i \in \mathbb{R}^{(N_{m1}+N_{m2}) \times (N_{m1}+N_{m2})}$, and accounts for the aerodynamic coupling, hence entering Eqs. (4.7) by combining the terms $\mathbf{F}_{11} \in \mathbb{R}^{N_{m1}}$ and $\mathbf{F}_{12} \in \mathbb{R}^{N_{m2}}$ as

$$\begin{bmatrix} \mathbf{F}_{11} \\ \mathbf{F}_{12} \end{bmatrix} = \dots + \mathcal{A}_0 \begin{bmatrix} \mathbf{q}_0^{(1)} \\ \mathbf{q}_0^{(2)} \end{bmatrix} + \mathcal{A}_1 \begin{bmatrix} \mathbf{q}_1^{(1)} \\ \mathbf{q}_1^{(2)} \end{bmatrix} + \dots \quad (4.34)$$

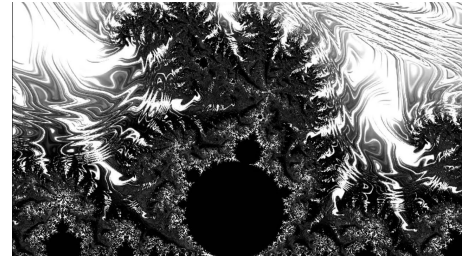
4.3 Methodology features and further work

The derivations presented in this chapter are the building blocks of a systematic approach to solving constrained mechanical problems written in the intrinsic formulation, including multi-body systems and accounting for the coupling between rigid-body and elastic modes and all other geometrically non-linear effects. A range of features have been discovered that may offer valuable solutions over other techniques: a) a major advantage is the possibility of including geometric nonlinearities and multibody effects to independent arbitrary complex linear FEMs (computational codes based on the floating frame of reference formulation [272], for instance, would take into account only linear deformations in each body as mentioned in Sec. 4.1.2); b) the same reduction capabilities shown for a single body are still valid, i.e. firstly condensation of the large FE matrices and then management of the number of modes utilised in the solution (substructuring techniques that reduce the size of the problem also consider linear deformations within each body [33, Ch. 8], and ad-hoc multibody techniques that deal with both nonlinear deformations and rotations are most commonly implemented for smaller problems); c) relatively simple Jacobians are obtained for the coupling between bodies, nonlinearities are

of second order and the mass matrix is constant, which prevents repeated calculations of its inverse as in other methods; d) the versatility of the theory in coupling with the aerodynamic forces has been shown, and the methodology may prove specially suitable to solve aeroelastic problems with morphing components in the modal space.

The dynamics of a double pendulum are tested in Sec. 5.2.3 to validate the theory and the implementation of spherical and hinge joints. However, further work is still required to test the convergence under different scenarios (if a very large base of LNMs is needed for convergence, the method quickly becomes inefficient). Tailored FE methods for multibody problems have become the standard of implementations and proper benchmark against them should be carried out to better establish the merits and disadvantages of the proposed approach. The application to the aeroelastic response of a full aircraft configuration in Sec. 7.5, however, illustrates the potential of the procedure for multidisciplinary problems.

Numerical Implementation and Verification of Aeroelastic Approach



Men's ideas are the most direct emanations of their material state.

Karl Marx

Two things are infinite, the universe and human stupidity, and I am not yet completely sure about the universe.

Albert Einstein

The formulation presented in previous chapters has been implemented into the research code FEM₄INAS (Finite-Element-Models for Intrinsic Nonlinear Aeroelastic Simulations), which is briefly described in this chapter and in more detail in Appendix A. The goal of this overview is to outline the main capabilities and characteristics of the program. As the code grew larger, a test suite was implemented with canonical cases that covered many of the challenges encountered along this work. The numerical results of part of these test cases are introduced next in the chapter as validation of both theory and implementation, as well as providing some insight into the features of geometrically nonlinear modelling.

5.1 Program description

The toolbox FEM₄INAS was developed for applications in nonlinear aeroelasticity. It has a size of around 15-20K lines of code, written and parallelised in the Python interpreted language. Even though it lacks the maturity of a production tool, it was conceived from the beginning to be a general program that could incorporate any type of suitable model with its corresponding input files, rather than a problem-specific solution. Otherwise, it would have been nearly impossible to analyse the industrial-scale models of this work with confidence. It has been tested in a large number of problems including: 1) those shown in this chapter, which are simple constructions but powerful in outlining the nonlinear aspects of the theory; 2) the ones presented in the next chapter, representative of aeronautical structures; 3) the full aircraft FE model in the last numerical chapter; 4) and even a larger, confidential model handled while on-site at Airbus that yielded promising results. It is worth remarking that the software is not a stand-alone tool for analysis as it needs of linear FE models and aerodynamic AIC matrices provided by an external program –MSC Nastran in this work. The framework is thus designed for applications in nonlinear aeroelasticity as a postprocessing tool, with emphasis on nonlinear dynamic responses and inflight load calculations. Appendix A gives a detailed overview into the program and here we just go over the main workflow and capabilities of the approach.

5.1.1 FEM₄INAS workflow

The main workflow of the software is illustrated in Fig. 5.1 and explained in the following steps with links to the equations in previous chapters:

- **Step 1 (Linear 3D analysis):** initially, in the structural process, the full (linear) mass and stiffness matrices, $\mathbf{K}$ and $\mathbf{M}$, from an external FE solver are extracted. Slender components are ideally built with interpolation elements linking the cross sections to the central nodes conforming the main skeleton. This can help on the accuracy of the ROM as selection of master nodes is an important part of reduction order model techniques.
- **Step 2 (Condensation):** The full FE matrices are read and manipulated (inside a

module that makes extensive use of the open-source software PyNastran [281]). These matrices are taken by the condensation module which outputs the condensed mass and stiffness matrices, $\mathbf{K}_a$ and $\mathbf{M}_a$, using Eqs. (2.26) and the selected reduction technique to approximate Eq. (2.25).

- Step 3 (Intrinsic modes and nonlinear couplings):** Initial input files define general variables like the connectivities in the model, the number of modes used in the solution, time discretisation or file locations. The intrinsic modes, ϕ_{1-2} and ψ_{1-2} in Eqs. (2.112)-(2.117), are computed after the reduced model matrices, its LNMs, and the geometry of the main skeleton. These are input into the integrals module that calculates the nonlinear coefficients Γ_{1-2} , Eqs. (2.122)-(2.125), and checks Eqs. (2.99) are satisfied. This part of the process involves calculating third order tensors that scale with the number of modes to the power of 3. Along with the ODE solver, it takes most time of the computation. Parallelisation of this module on an 8 cores computer gets computations to over 5 times faster. It is worth mentioning that calculations up to here only need to be performed once.
- Step 4 (Aerodynamic matrices):** the aerodynamic process reads tabulated AICs (in particular, from DLM) and generates time-domain, state-space models by employing an optimised RFA process with the techniques in Eqs. (3.26)-(3.28). It would also be possible to fully account for the geometric nonlinearities in the aerodynamics by periodically updating the AICs with the geometry as the structure deforms.
- Step 5 (Nonlinear 1D response):** The main solvers are inside a module that builds the nonlinear algebraic equations, the ODEs or the DAEs required to build a NMROM with structural capabilities, Eqs (2.102) and (2.100) for statics and dynamics respectively; aeroelastic features, including nonlinear dynamics, trim analysis and gust response, Eqs. (3.42); and multibody boundary conditions using Lagrange multipliers in the aeroelastic description, Eqs. (4.7). Runge-Kutta or predictor-corrector integrators are used for time marching of the equations. The final solution is assembled using the intrinsic modes and the modal coordinates, Eqs. (2.95), yielding the local velocities and forces, $\mathbf{x}_1$ and

$\mathbf{x}_2$, as the main variables in the formulation. Integration of velocities and strains using Eqs. (2.105)-(2.108) gives the position vector, $\mathbf{r}_a$ and rotation matrices, $\mathbf{R}$, in the main structure.

- **Step 6 (3D recovery):** The full 3D state is recovered if needed by combining the transformation matrix between master and slave nodes, $\mathbf{T}_{oa}$ in Algorithm 1, the intrinsic modes and the 1D nonlinear solution, as described in Sec. 2.4.2.

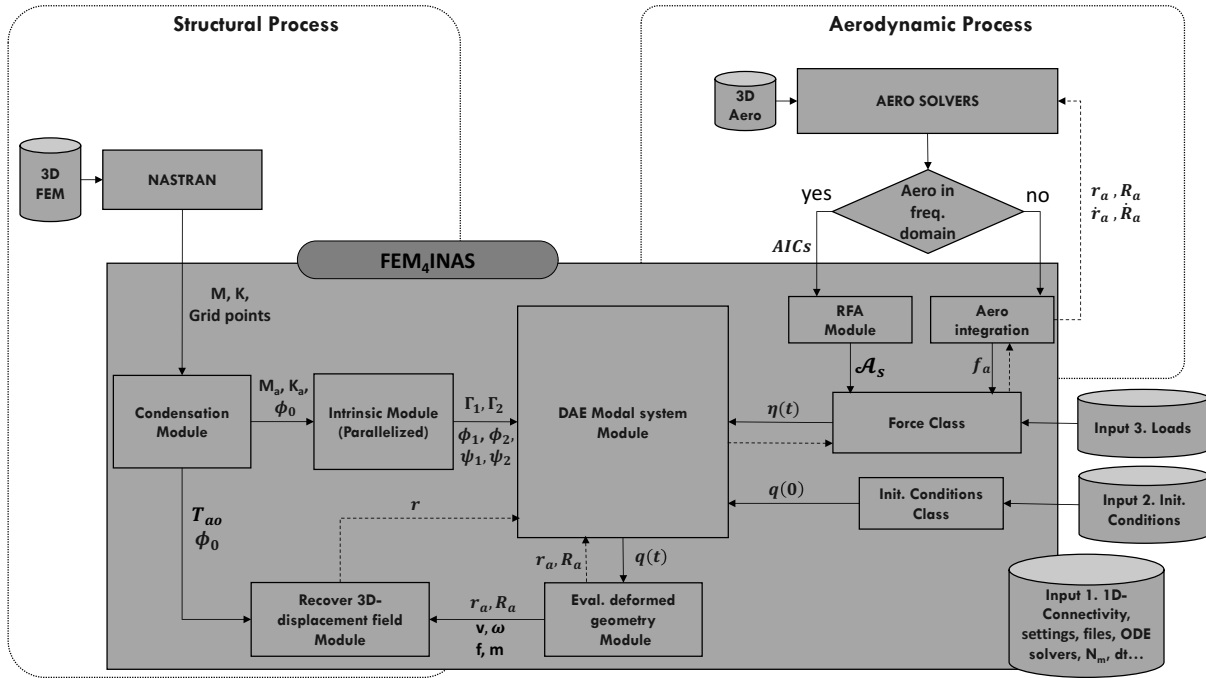


Figure 5.1: FEM₄INAS implementation

5.1.2 Program features

The main file of the program is `feminas_main.py` which manages the flow of the different functions and solutions. Inside the module `./intrinsic` we find the main routines in the solution process, the functions in `./aerodynamics` perform the RFA and other operations related to the aerodynamics, within `./condensation` the reduction of the FE model is carried out, and other modules include tools to support the main process, generate Nastran solutions and manipulate its outputs or post-processing the results. The combination of flags from the input files determines the specific solver called by `feminas_functions.py` that controls the structural or aeroelastic type of analysis, whether gravity or other dead loads are to be included, the sort of linear

approximation, and if the multibody approach is needed. The resulting capabilities and features will be demonstrated in the next numerical sections and chapters and include the following: geometrically exact static and dynamic solutions of structural models; calculation of condensed matrices from full FE models; communication with MSC Nastran and model generators of airplane structures; optimised rational function approximation of AICs matrices; construction of multibody systems from independent, linear bodies; calculation of trimmed flight conditions and the nonlinear response under static manoeuvres; nonlinear dynamic aeroelasticity with gust disturbances of any type, dynamic manoeuvres, and control inputs; assessment of instabilities under nonlinear deformations; and aeroelastic solutions with multibody components.

5.1.3 Test suite

A common problem when developing programs is validating its different parts and maintaining the confidence in previous verified routines as new features are added. This is even more important when codes are being written as a collaboration effort between many people. Despite the fact that only one person has developed the program so far, it was deemed important to start building a test suite for the code. First, because it is expected that the software might grow with more people contributing to it, and, secondly, because as it is a tool intended for industrial use, it would be too risky to not have a testbed to check every part is working properly. The `unittest` tool within Python is being used to prepare the tests which have been organised as follows: `./Test/test_static.py` runs a series of structural static problems and checks the results are within the specified boundaries when compared with results from MSC Nastran or the literature; similarly `./Test/test_dynamic.py` runs structural dynamic problems and `./Test/test_aero.py` does it with aeroelastic problems. Some of the examples in the test suite will be presented in the next section, thereby highlighting the different aspects of the theory and showing a thorough validation before facing more complex exercises in the next two chapters.

5.2 Numerical examples

The theory introduced in previous chapters is put into practise here in order to show its suitability in problems that involve large deformations and rotations. We start by studying examples originally introduced by the precursors of geometrically-nonlinear beam theory, thereby subjecting simple beam models to static loads and demonstrating key aspects of the theory: firstly, we show how sources of geometric nonlinearity may come from the coupling between LNMs or from the integration of internal strains; secondly, we deal with initially-curved configurations and test the application of both follower and dead external loadings; and, finally, we demonstrate the advantage of using a quadratic interpolation of the intrinsic modes over linear ones. Thereafter, equivalent beam and shell structures are built and dynamic computations carried out in order to demonstrate how detailed modelling features are captured by the current methodology. Residualisation of the equations can significantly reduce computational times while maintaining accuracy, which is assessed in the dynamic simulation of a free-free flexible structure.

The ability of the formulation to transform a set of linear, independent bodies into a full multi-body system is partially exemplified in the solution of a double pendulum falling under the action of gravity. Spherical and hinge-type joints are validated and the bounds of the linear approximation are tested using the LNMs from a linear multibody FEM.

The numerical examples end with a flexible clamped wing that is used to evaluate the aeroelastic formulation under large deformations. Computations include the nonlinear static equilibrium of the wing under different angles of attack and dynamic perturbations around the equilibrium state. The results are compared to a state-of-the-art approach in solar power aircraft analysis, obtaining good agreement for moderately large deflections.

5.2.1 Static analysis of canonical cantilevers

Models in this section are created using MSC Nastran two-noded beam elements (CBEAM) and since the linear FEM is already 1D, no condensation needs to be employed. Herein the suitability of the intrinsic modal formulation for problems involving large deformations for both dead and follower forces will be shown. It should be noted that convergence is defined by three different parameters: 1) the spatial discretisation used in the linear FEM analysis (the number of nodes); 2) the truncation on the modal basis used for the projection of the intrinsic equations (the number of modes); and 3) the tolerance on the nonlinear static solver, which for all cases has been set to 1.49×10^{-8} .

Cantilever under tip moment: a linear problem in intrinsic variables

This exercise consists of a clamped cantilever beam with an end moment applied to it, which gives rise to a pure bending problem. It can be found in Bathe and Bolourchi [282] and in Simo [27], and it represents a good test case as there is an analytical solution to the problem. Moreover, because only the bending modes in one plane are excited¹, the nonlinear terms, $\mathbf{\Gamma}_2$, in Eq. (2.102) have no contribution to the solution, leaving the integration of strains, Eq. (2.108) to obtain displacements as the only source of nonlinearity. The deformed shape is an arc of a circle of radius ρ_s and total angle θ_s :

$$\theta_s = \frac{ML}{EI} \quad ; \quad \rho_s = \frac{EI}{M} \quad (5.1)$$

Where M is the moment at the tip; L is the length of the beam, in this case equal to 10 ($L = \rho \cdot \theta = 10$); and EI , the bending stiffness, equals 100. When the moment is 20π , the angle θ is 2π and a complete circle is obtained. Fig. 5.2 shows the deformed configuration with a discretisation of 25 nodes. Incremental moments are applied and the comparison is made using the current methodology with the full modal base, Nastran nonlinear solution (SOL 400) and the analytical solution.

¹Torsion and out-of-plane bending are clearly not excited; pure applied moments do not transmit shear and there is no pull or push force along the beam main-axis so axial modes have no contribution either. See [63] for analytical expression of the LNMs on this problem.

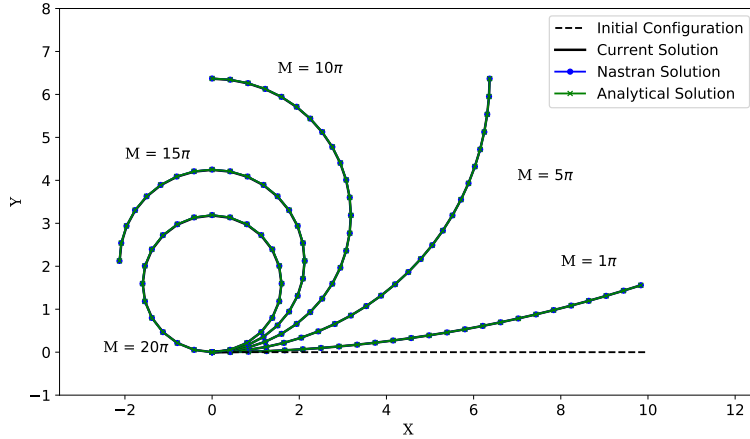


Figure 5.2: Spatial configuration of a cantilever subjected to a concentrated end moment

Excellent agreement is found between the three, however, the problem is worth studying in more depth: as the beam is loaded with a pure bending moment, only the bending modes in one axis are excited. The curvature, κ_s , is constant and it is interesting to see how many modes are needed to attain this constant. This is shown for a beam with 20 nodes and a moment of 20π in Fig. 5.3. Between 15 to 20 bending modes were needed to reproduce the exact solution

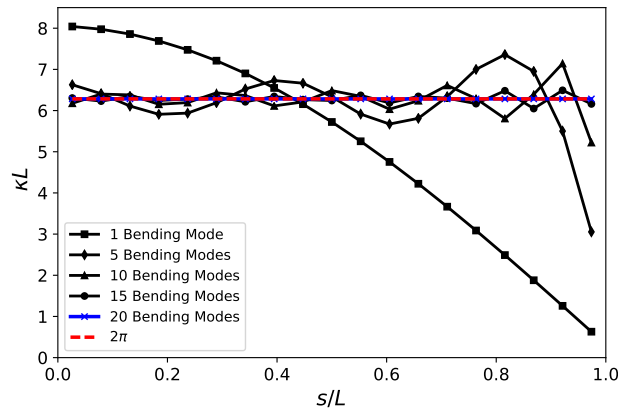


Figure 5.3: Curvature along the beam for $M = 20\pi$, approximation with the number of modes

(a circle), even for moments of the order of 1π that do not close the circumference. Similar results were found with a coarser and a finer discretisation: the problem was solved with 40 and 10 nodes and accurate results were obtained, but in both cases the use of the majority of the XY bending modes was necessary. Fig. 5.4 presents the contribution of each modal component, $\mathbf{q}_2$, to the closed-circle solution.

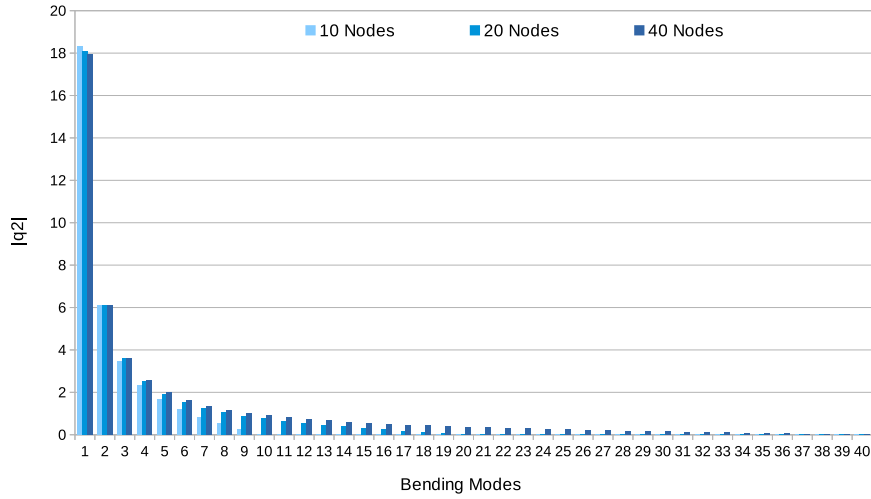


Figure 5.4: Cantilever subjected to a concentrated end moment: XY bending modes contributing to the response

This indicates that the full base of modes was needed not only to find the equilibrium condition but to be able to integrate the strain correctly. And it poses the question as to when is the modal solution or the recovery of displacements what determines the minimum modal base. In most cases it has been found it is the former, but not in this case.

45-Deg bend cantilever beam under follower and dead static loads

A curved cantilever under static and follower loads is studied in this section. The main objective is to show the analysis of initially curved structures and to investigate the convergence of the nonlinear solvers when linear and quadratic approximation of the LNMs are employed. It was first analysed by Bathe and Bolourchi [282] and it has extensively been used to validate nonlinear structural implementations [23, 27, 157]. The geometry consists of a 45-degree bend circle of 100 m radius, 1 m square cross section, Young's modulus $E = 10^7$ Pa, and negligible Poisson ratio. Nastran's LNMs are used to construct a geometrically-nonlinear intrinsic model (NMROM). Fig. 5.5 shows the static deformations under dead and follower tip forces in the z -direction with amplitude increasing from 0 to 3000N. In both problems a discretisation of 20 nodes is used with the full set of modes (120) employed in the solution. The three components of the tip displacement in the same global frame of reference used in Fig. 5.5, for increasing values of the loading, and on the same 20-node discretisation, are shown in Fig. 5.6. Dead and follower force problems are compared to the displacement-based solutions of Simo and Vu-Quoc

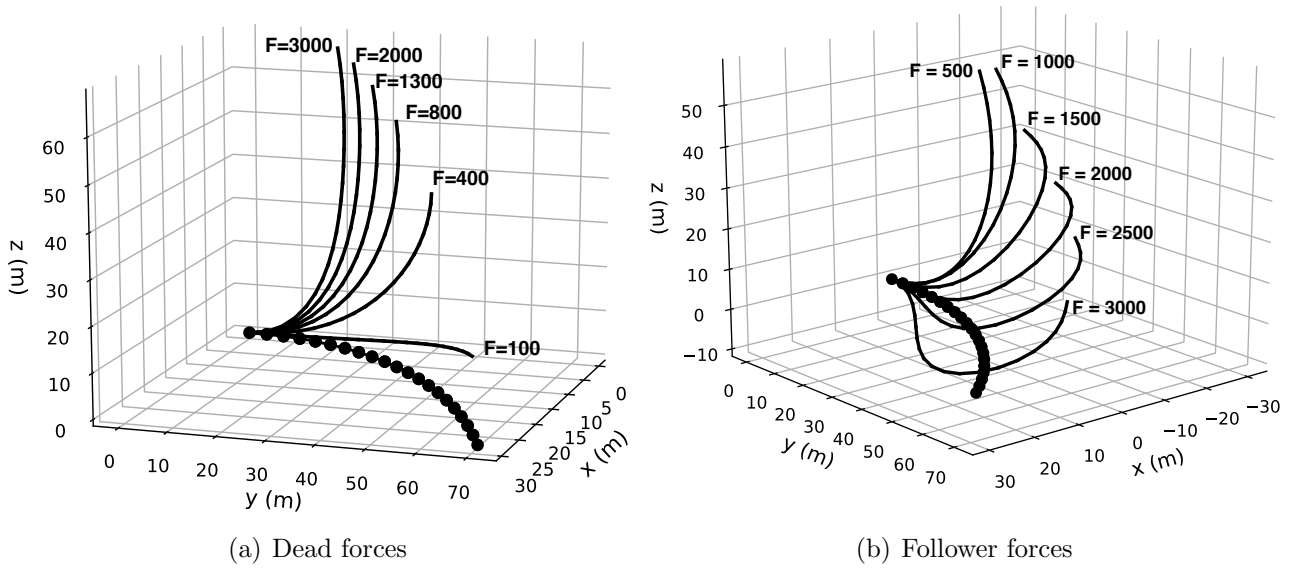


Figure 5.5: 45-deg bend cantilever deformations

[27] and Palacios et al. [23] respectively. In addition to this, details of the mapping between

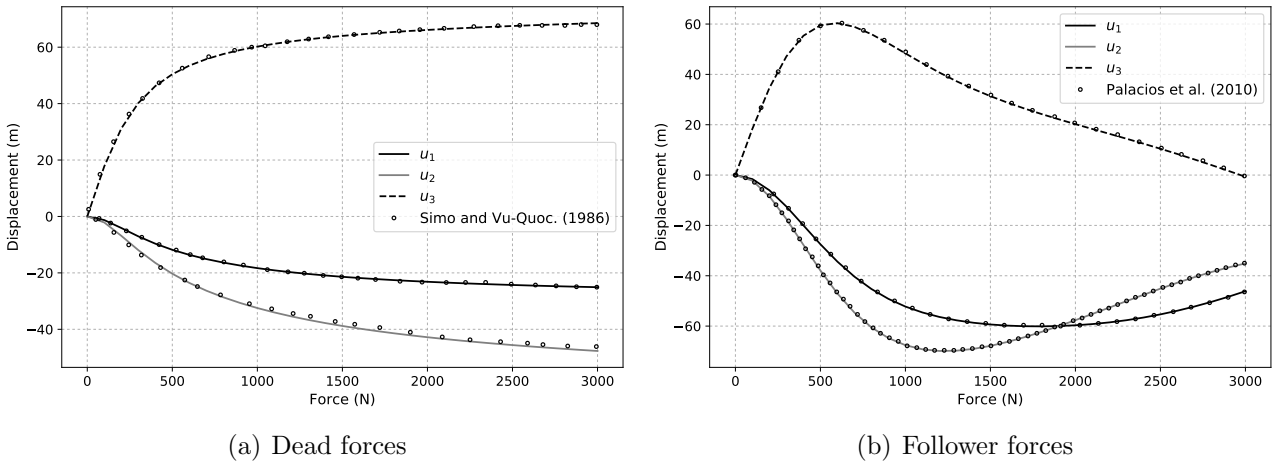


Figure 5.6: Components of the tip displacement of the 45-deg bend cantilever on the global reference frame

the discrete linear normal modes, which are computed at the finite-element nodes, and the continuous modes (along the load path) in intrinsic variables, can accelerate the convergence of the approach. Fig. 5.7 presents a comparison of the error as the spatial discretisation is refined and using the linear and quadratic approximations for $\mathbf{\Gamma}_2$, Eqs. (2.123)-(2.125). The L^2 relative error norm is defined as $\epsilon = \|\mathbf{X} - \mathbf{X}_{ref}\| / \|\mathbf{X}_{ref}\|$, with $\mathbf{X}$ being the computed position vector and $\mathbf{X}_{ref}$ the reference solution. This is averaged for a range of incremental loads (from 0 to 3000 in steps of 100) to form a single error metric. Solutions with linear

approximation of $\mathbf{\Gamma}_2$ and 50 and 100 nodes discretisation are considered as the reference, $\mathbf{X}_{ref}$, in the dead and follower force problems respectively. In both cases the quadratic solution outperforms the linear approximation. This can be especially useful on finite-element models where discretisation cannot be easily refined (e.g., after a model update against experiments or in industry built-up FEMs initially constructed for linear analysis).

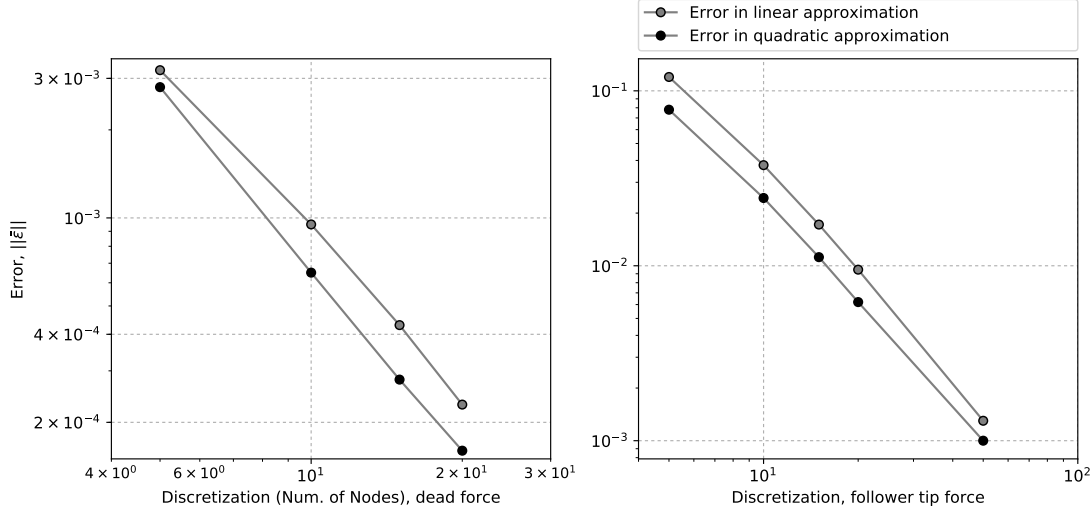


Figure 5.7: Convergence as finite element beam model is refined and comparison of linear versus quadratic approximation of continuous intrinsic modes

5.2.2 Structural dynamics of NMROMs and beam models

Next, we turn our attention to simple shell models for which an equivalent beam can be easily found. Models are built using MSC Nastran 4-noded elements (CQUADs); mass properties are given either as density in the material cards or as discrete mass elements (CONM2s) representing sectional inertia; and interpolation elements (RBE3s) which link the nodes in full and reduced models. Because the main strength of the proposed approach is its ability to add geometrically-nonlinear effects to generic built-up (linear) finite-element models, this section aims to show the differences between the shell and beam models, specially for large deformations, and how those are captured by the current methodology in dynamical problems. The computational gains achieved by residualisation of the equations will also be examined.

Thin-walled straight cantilever with lumped and distributed mass

The nonlinear free vibrations of a simple cantilever box-beam structure are analysed, the dimensions are length $L = 20$, width $w = 1$, height $h = 0.1$ and wall-thickness $t = 0.01$ as shown in Fig. 5.8. It is built with linear elastic material with $E = 10^6$, $\nu = 0.3$ and $\rho = 1$. Three models are built, 1) with beam elements and lumped masses, 2) shell elements with lumped inertia, and 3) shell elements with distributed inertia. This problem has been studied in [214] using modal bases obtained from a shell model with lumped mass, as well as analytical solutions from beam theory. The results there are expanded herein to account for the case of models with distributed inertia, using the new approach proposed in Ch. 2. After condensation on the same nodal set, each model generates slightly different LNMs and we investigate how this affects the geometrically-nonlinear solution.

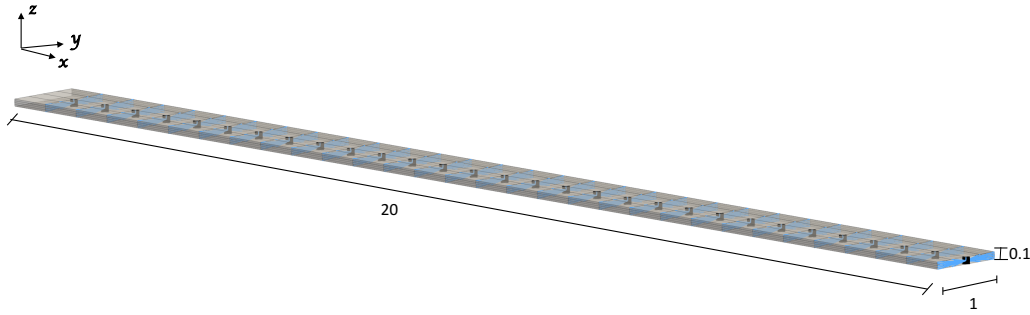


Figure 5.8: 3D cantilever-box model

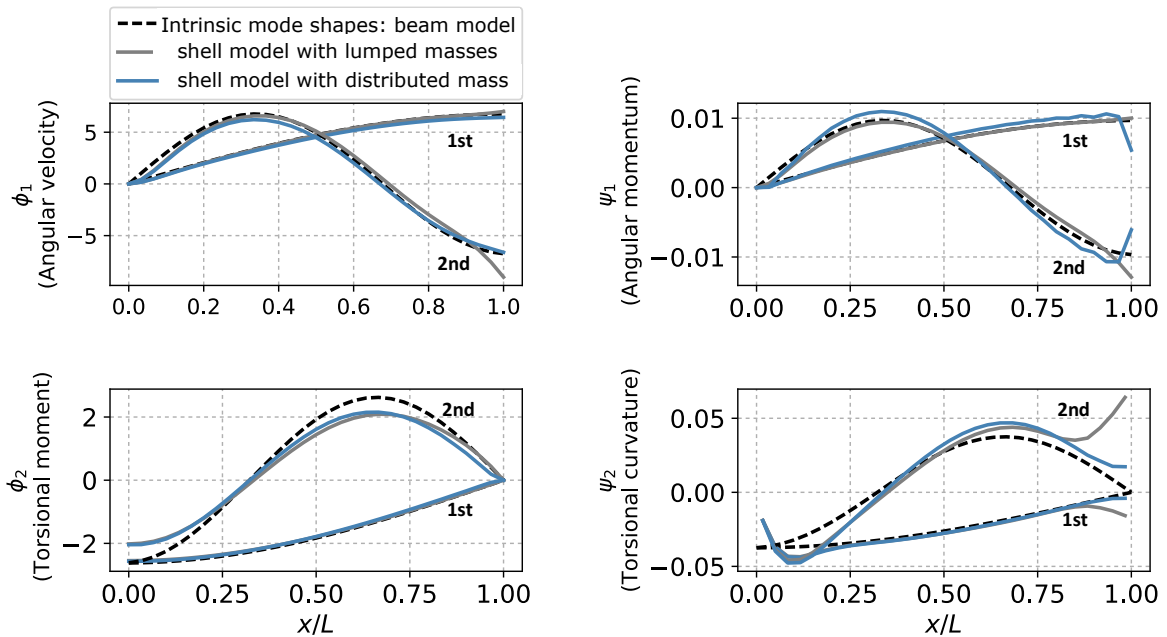
NMROMs are built from three linear models in MSC Nastran and a 30-node spanwise discretisation along the main load path is used for the model condensation using Guyan reduction. This was found to provide converged solution for the nonlinear response studied –which surpassed 30% of displacements with respect to the cantilever length. Both shell models are built using 480 CQUAD elements. Table 5.1 shows the natural frequencies for the first three bending modes in each axis and the first two torsion and axial modes. The frequency of the distributed-mass shell model without condensation, ω_{full} , is also included. The mode number of the corresponding frequency is shown in parentheses.

Bending and axial modes are well captured by all models. It can also be observed that the frequencies of the distributed-mass model are very close to the full model without condensation and the lumped model frequencies are closer to the beam model. Larger discrepancies are seen

Table 5.1: Natural frequencies for the thin-walled cantilever (in parentheses, the mode order number)

Mode Type	ω_{beam}	ω_{lumped}	$\omega_{distributed}$	ω_{full}
1 st x-z bending	0.419 (1)	0.42 (1)	0.426 (1)	0.426 (1)
2 nd x-z bending	2.617 (2)	2.608 (2)	2.65 (2)	2.654 (2)
3 rd x-z bending	7.291 (4)	7.2 (4)	7.34 (4)	7.33 (4)
1 st x-y bending	2.71 (3)	2.71 (3)	2.76 (3)	2.76 (3)
2 nd x-y bending	16.85 (7)	16.87 (7)	17.03 (7)	17.03 (7)
3 rd x-y bending	46.55 (11)	46.64 (13)	46.73 (14)	46.72 (15)
1 st Torsion	13.94 (5)	13.62 (5)	12.96 (5)	12.95 (5)
2 nd Torsion	41.79 (10)	33.84 (10)	31.25 (9)	31.09 (10)
1 st axial	78.52 (16)	78.6 (21)	78.62 (38)	78.61 (64)
2 nd axial	235.35 (33)	235.44 (52)	235.86 (68)	235.59 (146)

in the torsion modes, mostly due to the lack of warping restraint in the beam model. Fig. 5.9 shows the first two torsional modes in terms of intrinsic variables: angular velocity ϕ_1 , torsional moment ϕ_2 , angular momentum ψ_1 , and torsional curvature ψ_2 . The chosen lumped mass model splits the mass equally along all 30 nodes, which nearly doubles the angular momentum at the free end compared to the condensation from a distributed mass. More interestingly, a better matching between angular velocity and curvatures at the free end is observed between the beam and the shell with distributed mass.

**Figure 5.9:** 1st and 2nd torsional modes for the cantilever structure in intrinsic variables

The free-vibrations of the system are investigated by imposing an initial parabolic velocity distribution along the undeformed cantilever, as $\mathbf{x}_1(s, 0) = \mathbf{x}_{10}(s/L)^2$, with $\mathbf{x}_{10} \in \mathbb{R}^6$ being a parameter that will be modified in different computations. An important feature of the proposed approach is that it preserves the linear solution of the condensed FE model, which can be retrieved by simply removing the nonlinear coupling terms in equation 2.100 and subsequent linearisation of the displacement recovery relations. The displacements in y and z directions given by the resulting linear model are shown in Fig. 5.10 after an initial velocity $\mathbf{x}_{10} = (0, 0.3, 0.3, 0, 0, 0)$. Axial displacements are exactly zero. The first 85 modes were used in all cases to match linear Nastran results. This number here is conditioned by the number of modes necessary to replicate the initial parabolic velocity, even though as there is no previous selection of the modes, many of them do not contribute to the solution. And while in some specific cases, such as linear or planar problems, a selection of modes could be carried out, it is however difficult to know a priori which modes are needed in a general three-dimensional nonlinear problem. The lumped shell and beam models show identical response while the distributed mass model is slightly shifted with respect to them. This is a expected behaviour from the natural frequencies of the models, shown in Table 5.1.

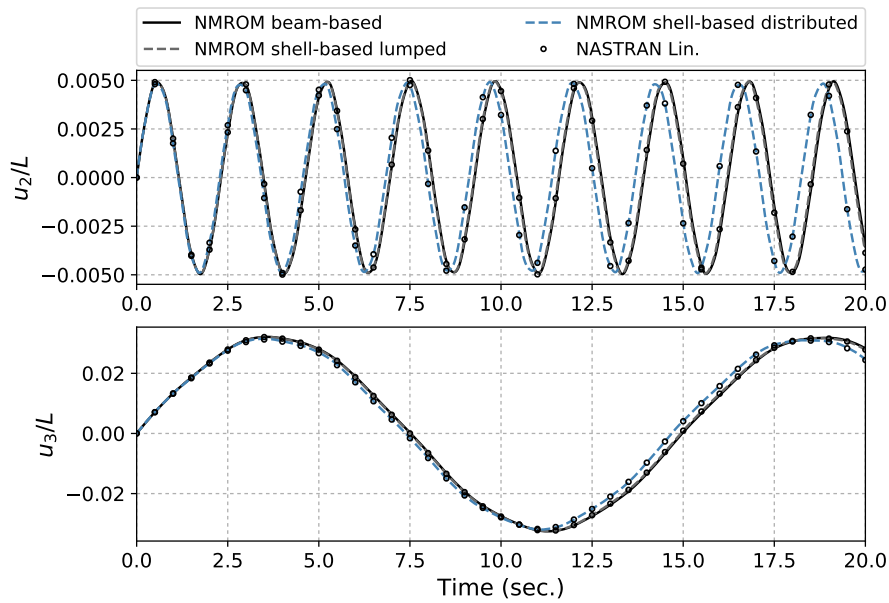


Figure 5.10: Displacements at the free end under small initial parabolic velocity distribution, $\mathbf{x}_{10} = (0, 0.3, 0.3, 0, 0, 0)$

Geometrically-nonlinear effects become relevant as the amplitude of the initial velocity is in-

creased. Displacements over 35% are obtained with $\mathbf{x}_{10} = (0, 3, 3, 0, 0, 0)$, for which the time history of the free-end displacements is shown in Fig. 5.11. The axial component of the dynamics is no longer null and the lumped shell model deviates from the beam model. Converged simulations are obtained with 85 modes and a time step of $\Delta t = 0.002$. NASTRAN nonlinear computations are shown for the beam model with the same simulation time step as the intrinsic computations. The agreement is very good, which further confirms the validity of the proposed approach under dynamic large displacements. Larger discrepancies between the three different NMROMs appear in the displacements along the y-direction, u_2 . Torsional modes are approximated differently and the nonlinear couplings between modes become important under large deformations. The beam and distributed mass models present a closer response, as they have rather similar torsional mode shapes. However, lumped shells and beams, which had nearly identical natural frequencies but different mode shapes, show differences in their nonlinear free vibrations. Therefore, the current methodology, which uses the mode shapes and the coordinates of the condensation points, directly incorporates local information of the initially built linear model into the prediction of the behaviour of the deformed structure in the geometrically-nonlinear simulations.

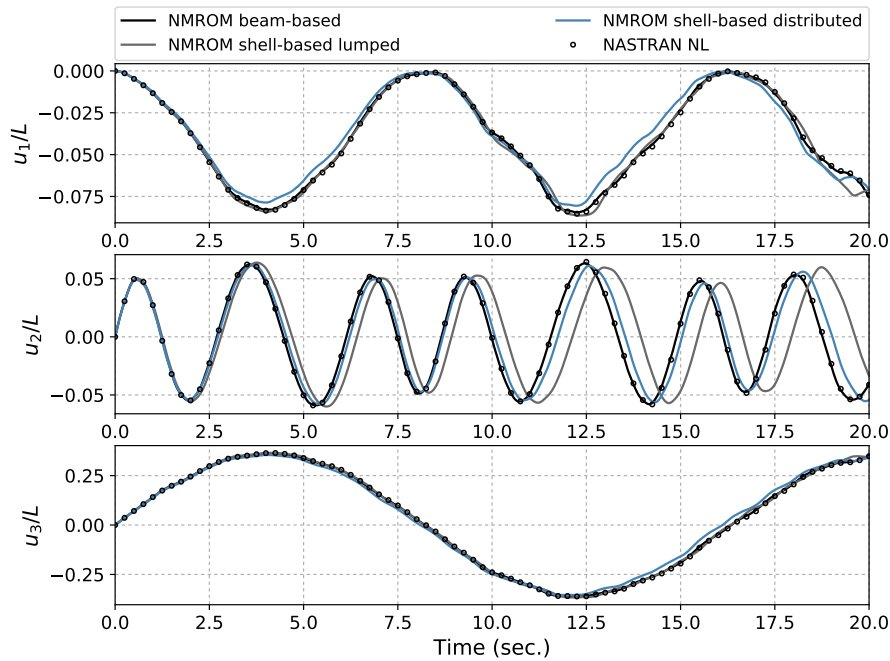


Figure 5.11: Nonlinear displacements at the free end under initial parabolic velocity distribution, $\mathbf{x}_{10} = (0, 3, 3, 0, 0, 0)$

Very flexible unsupported structure with elastic and rigid-body motions

This example shows that nonlinear rigid-body dynamics can also be obtained from a generic linear free-free finite-element model. It also exemplifies the use of residualisation techniques to decrease the computational time while maintaining accuracy. The geometry is defined in Fig. 5.12 and will be referred to as the flexible flying structure (FFS). It was first studied by Simo and Vu-Quoc [232, 283] and has served to verify several implementations of nonlinear beam dynamics with rigid body motions [130, 284]. As before, two linear finite-element models are considered to build the NMROM: a beam model with the properties of Ref. [283] and a shell structure of square cross section (side = $\sqrt{3}$, wall thickness = $\sqrt{3}/10$) with the same sectional mass and stiffness (see Fig. 5.12). Two solutions, with 25 and 50 spanwise condensation nodes are considered, the former consisting of 384 4-noded shell elements, the latter of 784 elements. The inclined straight structure (50 nodes) is shown in Fig. 5.12, together with the material properties and two types of loading: firstly, a dead-force in the x-direction and dead-moment in the y-direction that yield a planar motion in the x-z plane; and secondly, the addition of a moment in the z-direction which produces a three dimensional motion.

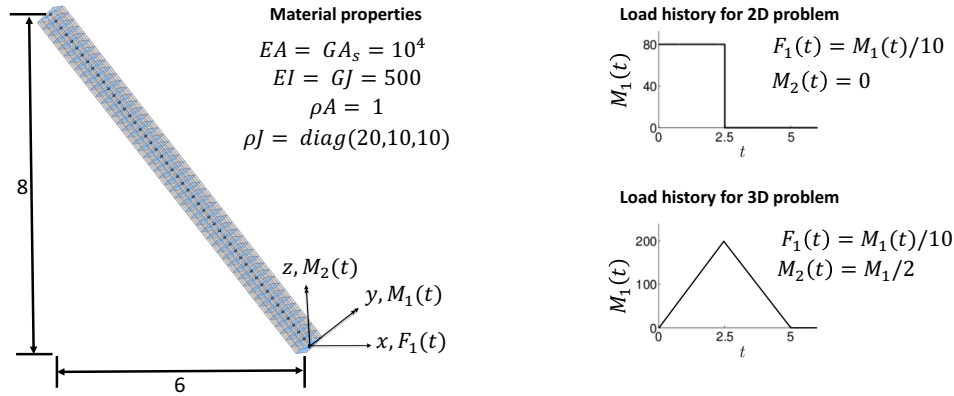


Figure 5.12: FFS geometry, material properties and load cases

Fig. 5.13 shows the planar dynamics of the FFS in a ten-second simulation. The solution with beam elements and 25-node discretisation (in black) is obtained using the full set of modes of the system. The solution of the shell model (in dashed red) is obtained using the first 75% of the modes ($N_m = 112$) and the remaining being residualised, therefore solving the full set of DAE, Eqs. (2.103). A small time-step $\Delta t = 0.0075$ is needed for this simulation because of high frequencies even in the mid-range of the spectrum. The beam and shell solutions show very

good agreement. The larger models with 50-nodes yielded very similar results. Comparisons with [130], which used a 10-node displacement-based nonlinear model and time step $\Delta t = 0.01$, are shown in dashed green. Small differences in the initial model and time integration errors make discrepancies grow larger over time. They are believed to be due to the bigger time step ($\Delta t = 0.1$) and a coarser discretisation used in the reference simulations.

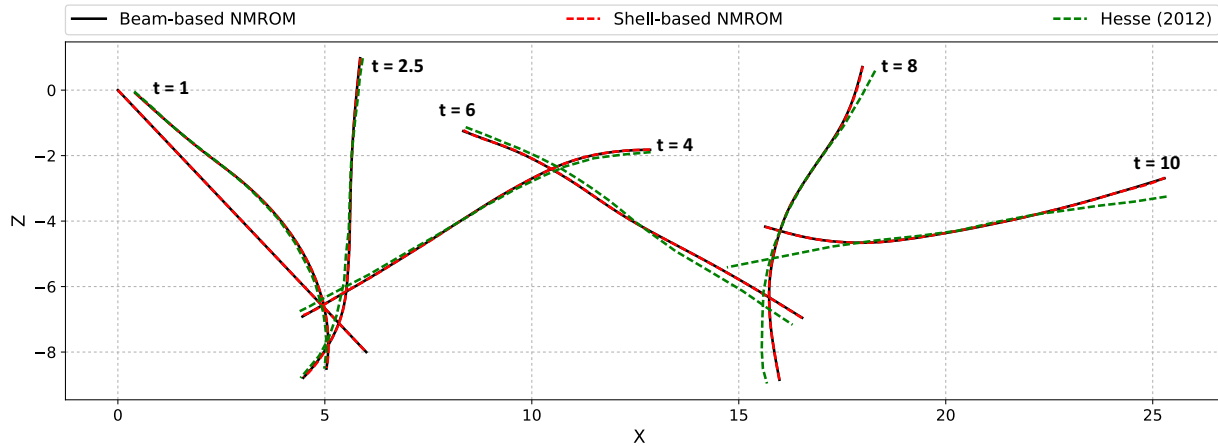


Figure 5.13: Deformed shapes of the FFS moving in the x - z plane (load case (a))

The shell problem on the 3D problem is used to show convergence of the models under finer discretisation, as well as how modal reduction and residualisation of the high frequency dynamics affect the solution. As a triangular dead force is the only external force, the position of the center of mass can be traced analytically. Fig. 5.14(a) shows the root-mean-square (RMS) error of the center of mass (C.M.) between the analytical solution and eight NMROMs: beam and shell based models with 25 and 50 node discretisation are used, and solutions are given with truncated and residualised set of modes. The residualised system consists of N ODEs and $N_{full} - N$ algebraic equations, which have been solved with a Newton-Raphson approach. A significant reduction of the error is obtained from refining the discretisation from 25 to 50 nodes. It is also remarkable that comparing the full-set of modes solution, $N/N_{full} = 1$, to the residualised solution with 75% of the modes, no accuracy is lost, while the computational time is greatly decreased. On the other hand, if the modes are just truncated, there is a notable increase of the error. Shell and beam models show a similar error across the spectrum of modes used in the solution for the residualised system and for all truncated solutions except for the case with 75% modes. This can be explained looking at the natural frequencies of the

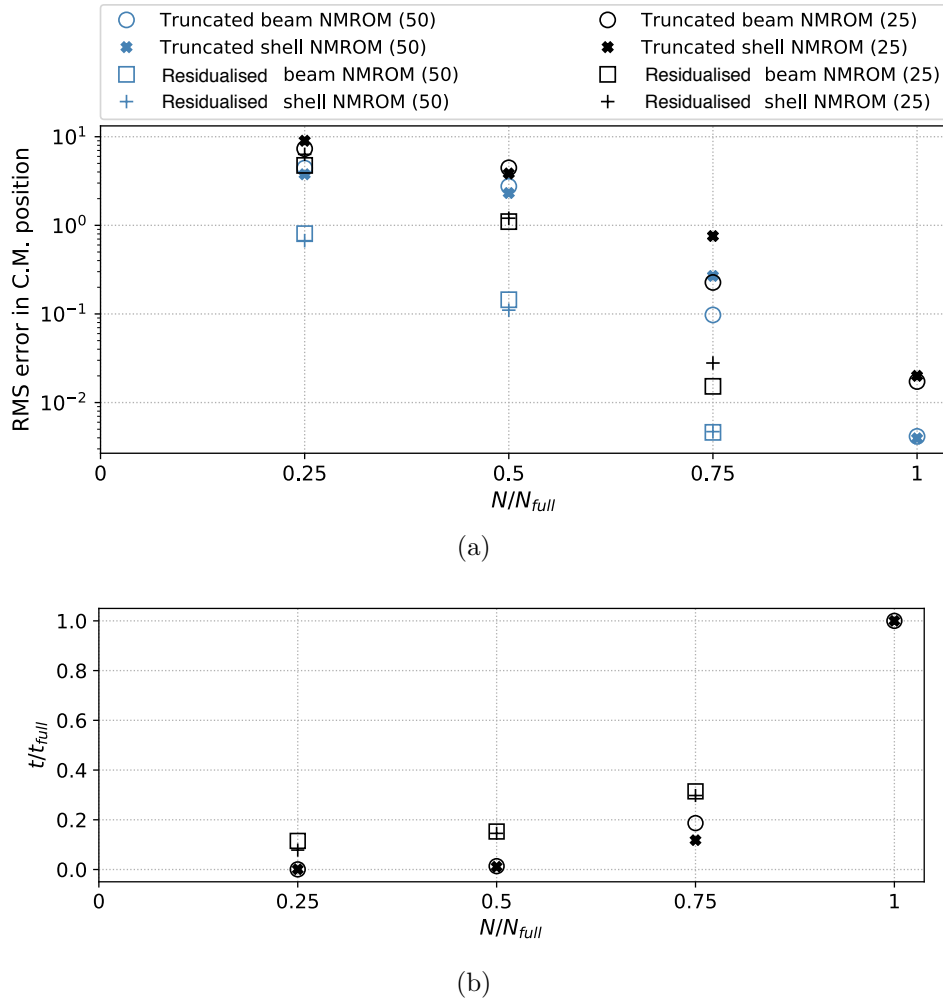


Figure 5.14: Error and normalised computational time of a 10 sec. simulation with respect to the number of modes used in the solution. Load case (b)

structures in the 25-node discretisation: for the beam model, $\omega_{75} = 33.6$ and $\omega_{112} = 236.5$, and for the shell model $\omega_{75} = 30.9$ and $\omega_{112} = 114.7$, which correspond to 50% and 75% of the total number of modes respectively. Therefore the truncation of the last 25% modes in the beam model affects higher frequency modes (which usually have a much smaller contribution) than the shell model. Fig. 5.14(b) illustrates the time taken for each of the simulations on the 25-node models with respect to the simulation time with the full modal basis. While the time can vary from simulation to simulation, the trend shows that not including the highest frequencies in the solution, which determine the largest stable time step, can have a great effect in decreasing the simulation cost. Note that using a fixed point strategy to solve the DAE system achieves faster simulations (at the expense of accuracy if the frequencies residualised are not large enough). Thus, for this problem, a residualised system with around 60-75% of

the modes provides a good compromise between time and accuracy.

5.2.3 Simulation of multibody systems

The multibody approach described in Ch. 4 is validated using two independent bodies that are coupled to build a rigid pendulum and a double-pendulum. Firstly a spherical joint is used to connect the two rigid bars, and the resulting double-pendulum is left to free-fall within the plane formed by gravity and the initial configuration (the XY plane in this case). Then the hinge joint is tested by using it to join the two bodies: setting the hinge-axis perpendicular to the XY plane yields a similar system to the spherical joint because the movement is restricted to the 2D plane; and when the hinge-axis is set parallel to the two bodies in the initial configuration, an equivalent single pendulum is formed, because the independent DoF between the bodies is a torsional one, which is not excited when falling under gravity. Lastly, assessment of the boundaries of the linearised approach, whereby the LNMs of an already built multibody FEM are utilised, is performed on the double-pendulum.

Double rigid pendulum

A spherical joint is utilised to combine two rigid, 2-noded beams of length 60 and 20 respectively, and with a discrete mass of 80 at the tip of each one of them. Note that this dynamical system can become chaotic, a situation after which comparisons are not reliable. In fact, computations have been performed over a period of 30 seconds, when chaotic dynamics begin to appear. The system is released to fall under the force of gravity acting in the y -direction. A Nastran constrained model is built using MPCs entries and solved with the nonlinear solver in Nastran. The NMROM is constructed with the RB modes of a pendulum-like first body and free-free conditions for the second one. The converged time step used in the three simulations is $\Delta t = 0.003$. Comparisons of the three approaches are depicted in Fig. 5.15.

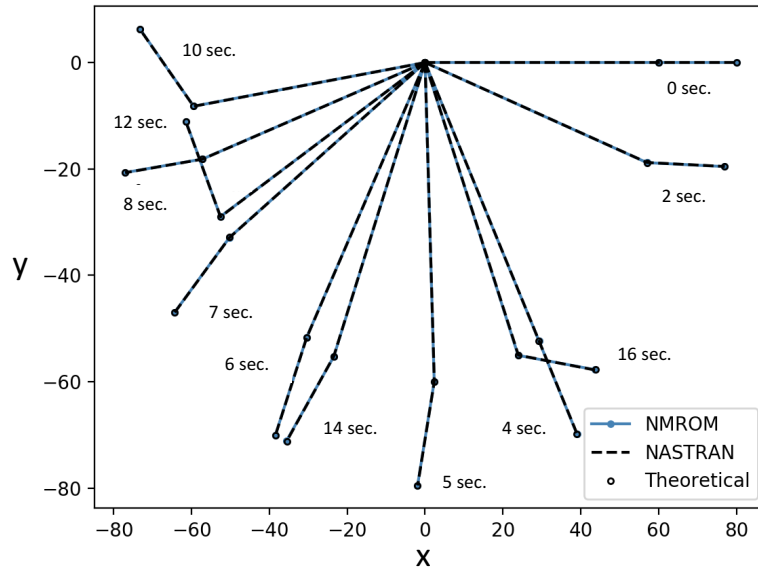


Figure 5.15: Double-pendulum free falling under gravity

Excellent agreement is found between the three solutions. The time evolution of the tip of the two bodies is shown in Fig. 5.16, where disturbances can be appreciated on top of a periodic motion, especially when compared to Fig. 5.17 which shows the same system with a fixed connection, i.e. a single pendulum. These results validate the multibody approach with a spherical type of connection, and also the implementation of gravity forces, which are more involved to include in the dynamical system as dead forces need to be rotated from the global to local reference frame in which the formulation is written (a good compromise given that aerodynamic forces will rotate with the structural deformations).

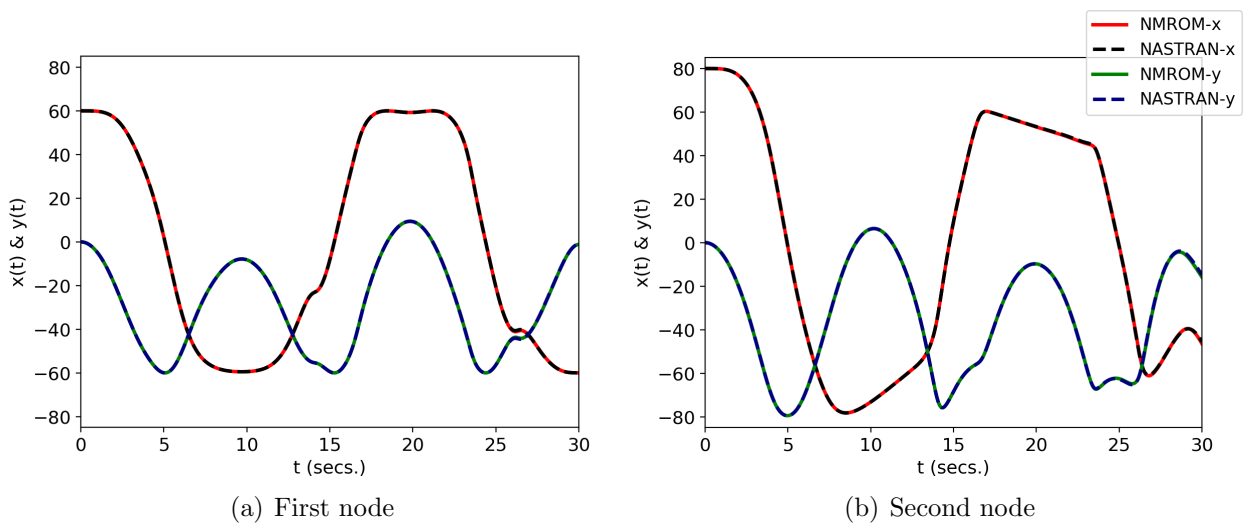


Figure 5.16: Dynamic evolution of double-pendulum nodes

Single rigid pendulum

In order to validate the hinge constraint, we artificially built a single pendulum out of the previous double pendulum by putting a hinge connecting the bodies with its axis running in the x-direction (parallel to the two bodies axis). A rigid pendulum is then constructed in Nastran, and a simulation is run where the pendulum is released under gravity, similar to the double pendulum exercise. A perfectly periodic motion is obtained, as expected in the absence of friction forces. A good match is found between the NMROM with a hinge and the commercial software, as presented in Fig. 5.17, confirming the implementation of hinge constraints.

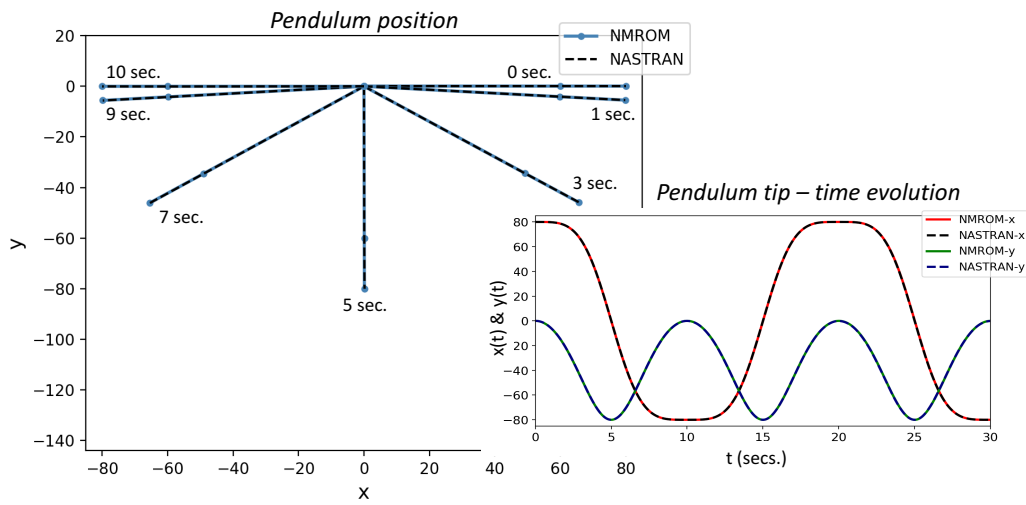


Figure 5.17: Single rigid-pendulum dynamics

Linear approximation of the multibody dynamics

In section 4.2.4 we described a linear approximation that does not require Lagrange multipliers to couple independent bodies. Rather, it solves the intrinsic equations using the LNMs from a FEM built with multibody characteristics. Two problems are solved using the approximation and compared with a nonlinear analysis: the double pendulum released under gravity, as previously studied, and the same system without gravity and imposing a vertical velocity as initial condition on the tip node of each body. This provides some insight as to when this linear approach can and cannot be used. For each body rotation, the approximation is good up to angles of roughly 20 degrees, and for angles over 35 degrees a nonlinear multibody approach ought to be employed.

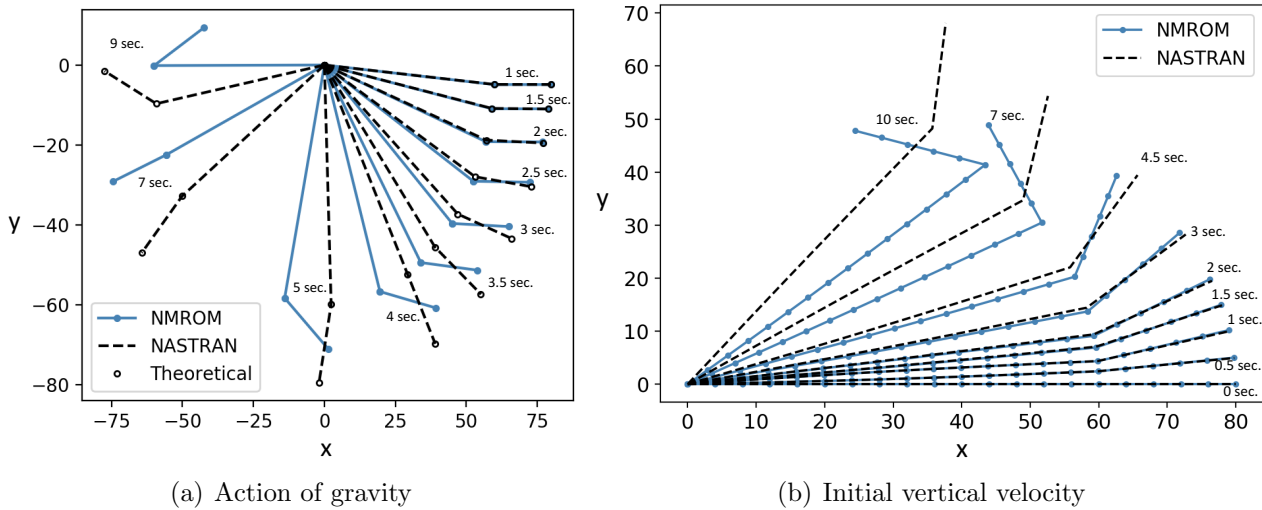


Figure 5.18: Double pendulum linear approximation on a gravity falling example and an initial vertical velocity excitation

Remark on the multibody results

We conclude this section by highlighting the multibody features that were put into practice in the double pendulum problem: we have only utilised the linear stiffness and mass matrices of two independent bars, together with their LNMs, to construct a nonlinear multibody system. If instead a single initial model is used with the constraints already embedded on it, the multibody dynamics are also captured, but its accuracy range is limited. Even though the problems solved only involved rigid bodies, the formulation equally caters for flexible bodies undergoing geometrically nonlinear deformations, as shown in previous sections with a single body. Further work needs to be carried out with flexible structures in order to assess the suitability of the approach in large dimensionality problems. However this work's focus is not on multibody systems but rather on nonlinear aeroelastic problems, which we introduce in the next section.

5.2.4 Static and dynamic aeroelasticity of a highly flexible wing

A very flexible model is built to verify the implementation of the proposed nonlinear aeroelastic solver and assess the validity of the linear aerodynamic model. The wing considered is that of [285], which has been extensively used in the literature as a validation test case for static aeroelasticity with geometrically-nonlinear models. As the original wing is dynamically unstable, the elastic axis has been displaced forward for the results in this paper, which uses the

dynamic equations of motion to compute equilibrium conditions. The wing has a span of 16 m and chord of 1 m, with no sweep or taper. The elastic axis is at 25% of the chord, the in-plane and out-of-plane bending stiffness are $6 \times 10^6 \text{ Nm}^2$ and $2 \times 10^4 \text{ Nm}^2$, respectively, the torsional stiffness is 10^4 Nm^2 , the mass per unit length 0.75 kg/m and the torsional inertia $0.1 \text{ Kg}\cdot\text{m}$. It flies at 25 m/s with air density 0.0029 kg/m^3 . For this test case, the structural model is based on beam elements in MSC Nastran and therefore no condensation is necessary. This simplification allows us to use SHARPy [129] as the reference for comparison, a well-validated solver based on geometrically-nonlinear beams in displacements and unsteady vortex-lattice aerodynamics (UVLM), which is the current state-of-the-art for this nonlinear aeroelastic simulation (although the UVLM allows arbitrary wing excursions, it does not account for compressible effects which are very important for commercial transport aircraft flying in the transonic regime; thus the DLM was deemed a good compromise for moderately large deflections as shown in this exercise). For small deformations, SHARPy and the aeroelastic NMROM will be shown to give similar results, while for large amplitudes the only difference is the effect of change of wing geometry in the aerodynamic model in SHARPy, which is neglected in FEM₄INAS. This assumption, which gives huge computational savings, will be therefore tested below. The wing is clamped at the root and the model has been created in both solvers with 16 beam elements and a 32×5 aerodynamic lattice.

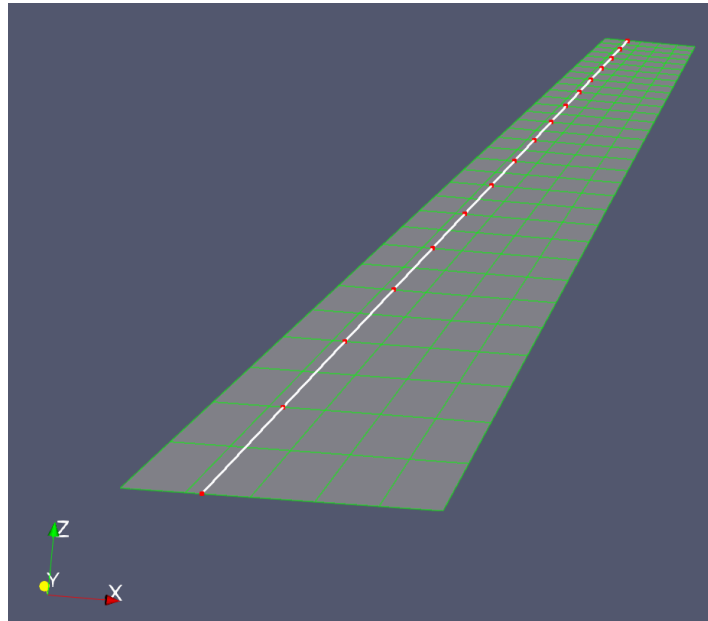
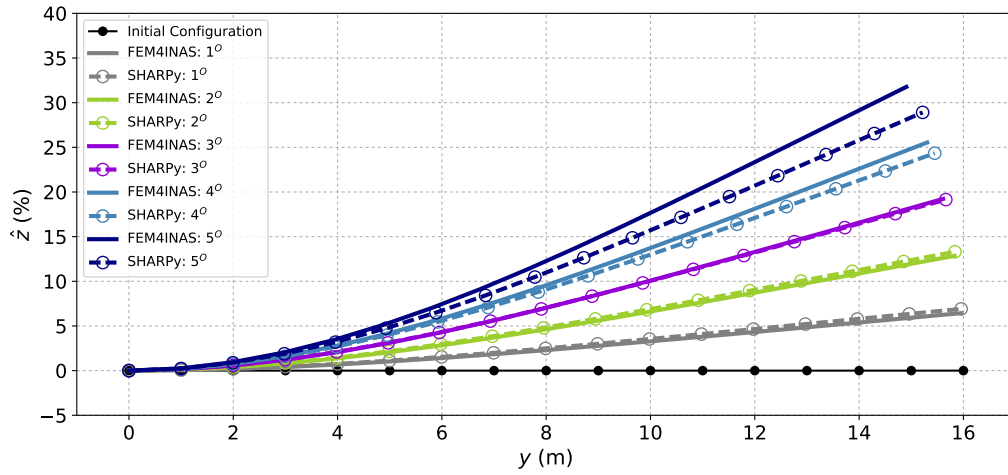


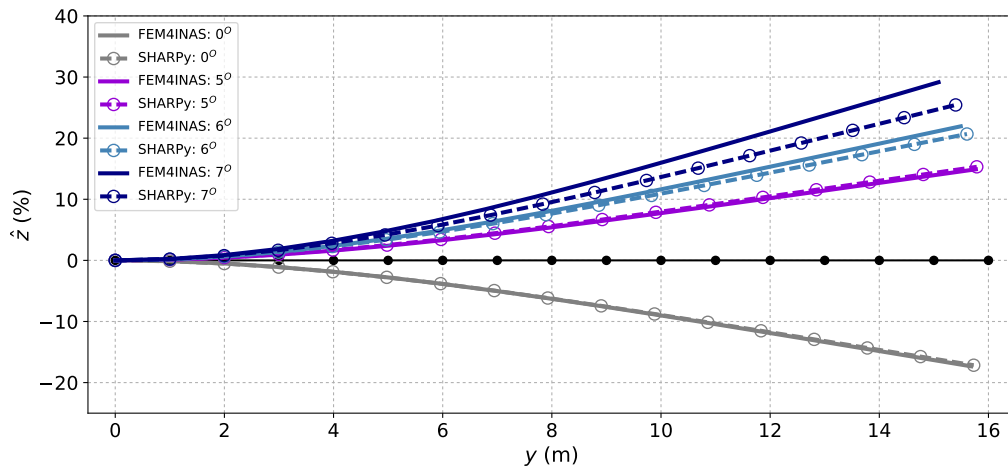
Figure 5.19: Beam discretization and aerodynamic panels to model the wing

Nonlinear aeroelastic equilibrium

Firstly, the aeroelastic equilibrium conditions are determined for different angles of attack at the wing root. Fig. 5.20 presents the comparison between SHARPy and an aeroelastic NMROM simulation converged with 35 modes. Results are presented normalized with the wing semi-span, with and without gravity.



(a) No gravity



(b) Gravity on

Figure 5.20: Deformed shape of the wing under fixed angles of attack with and without gravity

The excellent agreement between models indicates the suitability of the proposed method for moderately large deformations, since discrepancies become significant only with deformations over approximately 25%. For large deformations, the DLM in the initial configuration does not account for the nonlinear effect in the local angles of attack, thus the module of the follower forces does not decrease and deformations are larger. In terms of modelling gravity, figure

5.20(b), it is worth noting that tracking of geometric information allows a very good accuracy in the wing nonlinear deformation under its own weight, even though only the linear initial mass matrix is known to the NMROM, unlike SHARPy, which updates it in every fluid-structure iteration. Although discrepancies in the two methods are more significant when gravity forces are taken into account: it was found that the inclusion of gravity reflected a very sensible model in the aeroelastic equilibrium to the initial parametric quantities, for instance, defining the mass with discrete elements or distributed density. Small differences between approaches in the building and solution strategy, contributed to the disagreements seen in the plot.

A comparison between linear and nonlinear equilibrium shapes of the wing is shown in figure 5.21.

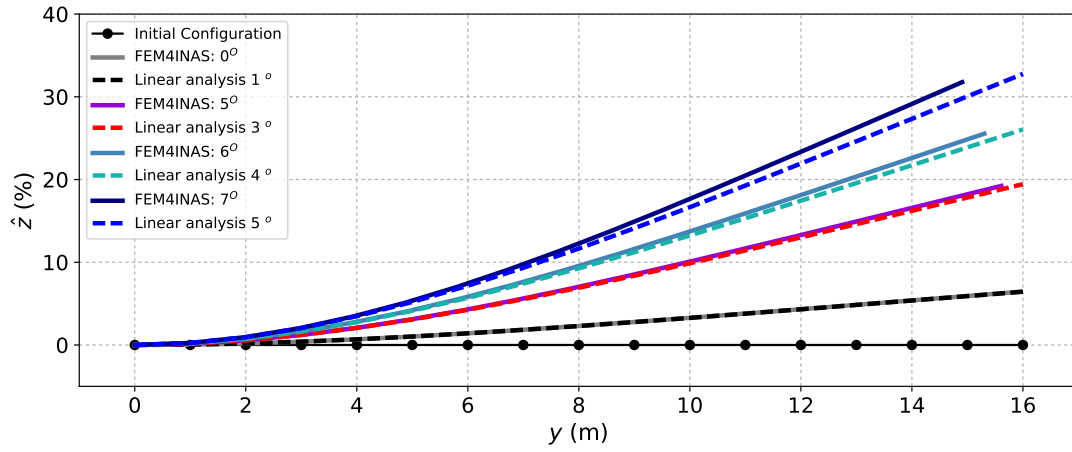


Figure 5.21: Linear VS Nonlinear static equilibrium with no gravity effects

The linear equilibrium is calculated using Eq. (3.44) around the undeformed configuration and retrieving positions from the expansion of the displacement field: modal coordinates, $\mathbf{q}_0$, multiplied by the correspondent mode-shapes, $\boldsymbol{\phi}_0$, instead of the nonlinear integration of strains or velocities. The figure shows clearly the shortening effect from the near-inextensionality in the geometrically-nonlinear model, as well as the small inward rotation resulting from follower-force effect in the aerodynamics.

Nonlinear dynamic response

In order to verify the proposed method for dynamic aeroelasticity, three different forces are applied at the tip of the wing, which is initially in equilibrium at 3° angle of attack and no

gravity. Eq. (5.2) defines the time history of the forces, which are $\mathbf{f}_1 = [0, 0, \pm 25]f(0, 0.5, 0.5)$, $\mathbf{f}_2 = [0, 0, \pm 20]f(0, 1, 1)$, $\mathbf{f}_3 = [0, 0, \pm 15]f(0, 2, 2)$. The minus sign of the load acts at time zero in the simulation and the positive sign after 10 seconds.

$$f(a, b, c) = \begin{cases} a + \frac{1-a}{b}t & \text{if } t \leq b \\ 1 - \frac{1}{b+c}t & \text{if } b < t \leq b + c \\ 0 & \text{otherwise} \end{cases} \quad (5.2)$$

Results are shown in Fig. 5.22, where the response to force $\mathbf{f}_i$ is identified as $-i$ in the legend. They show a close agreement between both models, thus demonstrating that the unsteady aerodynamic effects can be appropriately captured by the (linear) DLM as forcing term to the intrinsic equations. The comparison is slightly better for negative loading because deformations decrease and therefore bring both aerodynamic models closer. A lesser difference is seen for the maximum and minimum points of the disturbance, as these are driven mostly by the structural follower force effects.

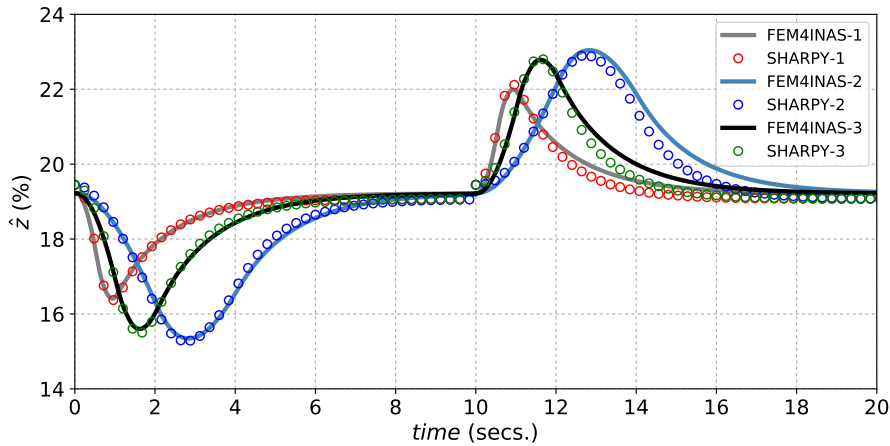
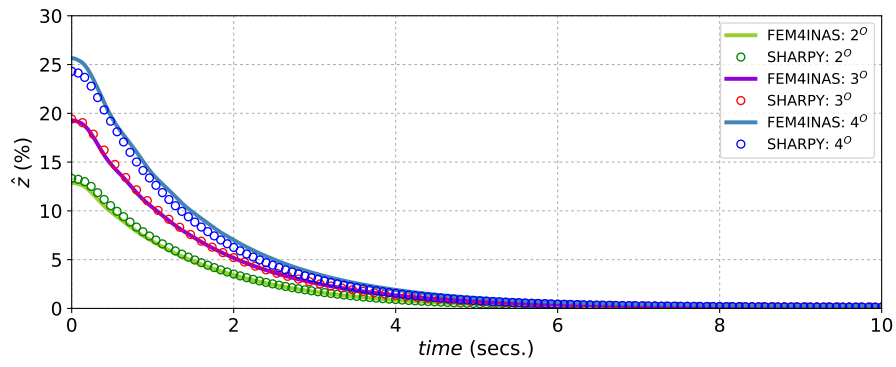
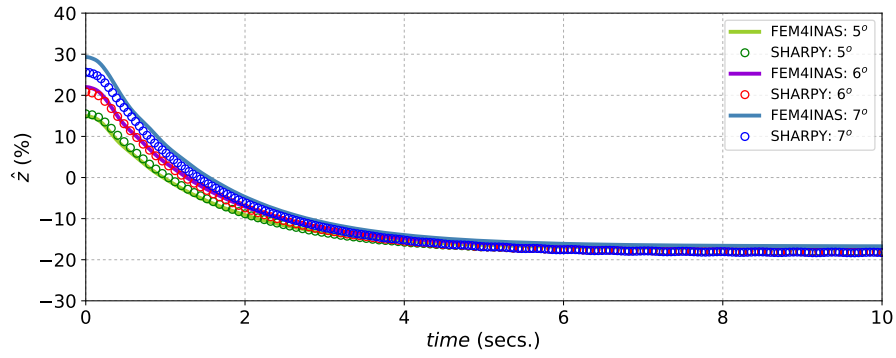


Figure 5.22: Wing tip vertical displacement for three different doublet-type forces from 3° static equilibrium

In the second example, the wing is released from the static equilibrium to a zero angle of attack in a sudden change of loading. Fig. 5.23 illustrates the dynamics of the wing-tip after equilibrium when gravity forces are included and neglected. Note that FEM₄INAS calculates the static equilibrium as part of the dynamic simulation, while SHARPy solves the nonlinear static equations first, then runs the dynamic solvers from the equilibrium position. The aerodynamic



(a) No gravity

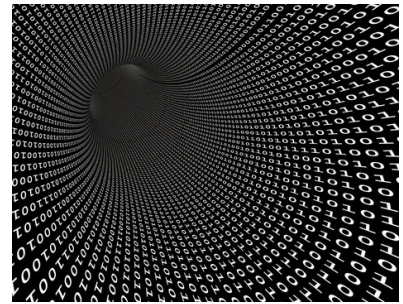


(b) Gravity on

Figure 5.23: Wing release to zero angle of attack from equilibrium position

damping dominates the dynamic response and the overall behaviour is similarly captured by the two approaches, apart from the variations in the static equilibrium when deformations over 25% take place.

Fidelity Preserving Methods for Airplane Computational Models



Any consistent formal system F within which a certain amount of elementary arithmetic can be carried out is incomplete; i.e., there are statements of the language of F which can neither be proved nor disproved in F .

Kurt Gödel, First incompleteness theorem

The limits of my language mean the limits of my world.

Ludwig Wittgenstein, *Tractatus Logico-Philosophicus*

In the previous chapter we have shown the implementation of the present approach and its application to simple problems whereby the nuances and details of geometrically nonlinear theory could be thoroughly delved into. In contrast, along the course of this chapter we increase the complexity of the models to the level of current industrial expertise such that the structural and aeroelastic solutions are verified on those models. Moreover, we present a series of exercises that show the novelty of the methodology in augmenting linear systems with important physics otherwise not captured by standard aeroelastic tools. Initially a simplified aircraft configuration, built by the former Airbus Group Innovations team, serves as a platform to test FEM₄INAS structural solvers and to remark the importance of nonlinear analysis in complex

aircraft models whenever loads are large enough that follower forces, geometrical stiffening and shortening effects along the component's principal dimension cannot be neglected. Structural static and dynamic analyses are carried out and differences between linear and nonlinear approximations are highlighted. Next the wing-box of this aircraft is modified in order to have a model with distributed mass in which to apply the condensation techniques of Sec. 2.2. The benefit of iterative reduction methods in capturing higher natural frequencies of the model is underlined. Furthermore, comparison with nonlinear full-FE computations demonstrates the accuracy and efficiency of the current approach. Finally classical problems in aeroelasticity are shown, including flutter instabilities and limit cycle oscillations. For this, a flexible clamped wing commonly used as a benchmark in the literature have been chosen. The model consists of a wing-box built with shells and beams, and condensed into a single main load path; together with aerodynamic forces given by the DLM, for which optimisation of aerodynamic poles have proved useful.

6.1 A simplified aircraft FEM built to industry standards

A representative FE model of a full aircraft without engines is used to demonstrate the capabilities of the current methodology on large finite-element models where only linear dynamic analysis would be currently possible. The aircraft's main wing is composed of wing surfaces, rear and front spars, wing box and ribs. Flexible tail and rear stabiliser are rigidly attached to the wing. Isotropic and anisotropic materials are utilised, and the inertia is defined by lumped masses with interpolation elements connecting it to the structure. Guyan reduction is employed to condense the model onto the lumped-mass nodes. Fig. 6.1 shows the model built in MSC Nastran as well as the interpolation elements (RBE3s) connecting master nodes (where the inertia is given) and slave nodes (in blue). A total of 3848 shell elements are used in the full model, which is reduced to 78 master nodes along the main load path. For this study only the static and dynamic structural responses are considered. The model is clamped at the node connecting the main wing to the fuselage bar and the corresponding linear normal modes are

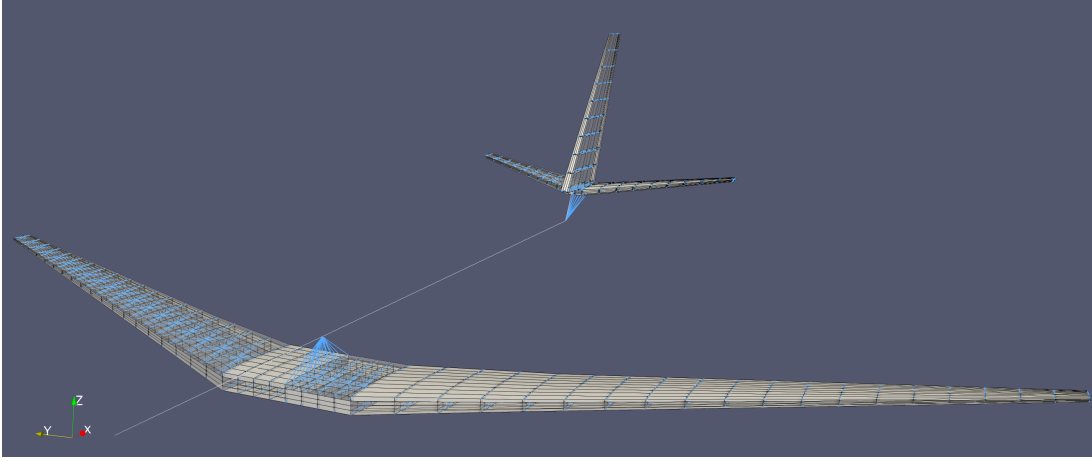


Figure 6.1: Full-vehicle model with shells, beams and interpolation elements

then obtained on the reduced nodal set. As an illustration, three mode shapes of the aircraft are shown in Fig. 6.2 in intrinsic variables. Each mode contains multiple elastic degrees of freedom. For instance, the first symmetrical out-of-plane bending mode has a significant torsional component.

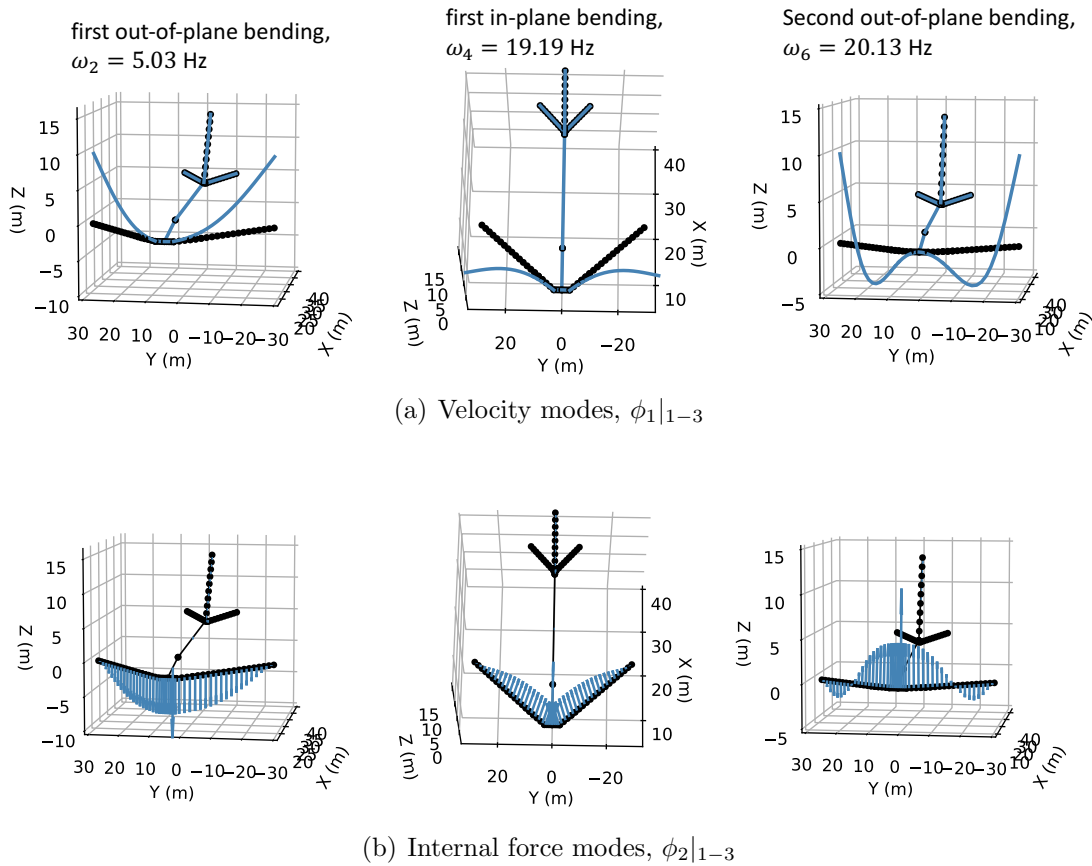


Figure 6.2: Selected aircraft modes in intrinsic coordinates: first out-of-plane and in-plane bending, and second out-of-plane bending

6.1.1 Static deflections under wing-tip loads

The static solution of the aircraft is first studied with a NMROM built with the first 50 modes ($N_m = 50$) and a tolerance of 1.49×10^{-8} for the nonlinear solver. Follower loads normal to the wing are applied at each tip of the main wings. The response under loads of 200, 300, 400, 480 and 530 KN is shown in Fig. 6.3. Nonlinear static simulations on the original full model (before condensation) are also carried out in MSC Nastran and are included in the figure. The interpolation elements in Nastran are used to output the displacements at the condensation nodes for direct comparison with the NMROM results. To quantify the difference between both sets of results, tip displacements, in global coordinates, for the 530 KN load and the full model calculations are $u_x = -0.217$ m $u_y = -1.348$ m, $u_z = 7.236$ m; while calculations from the present methodology yield $u_x = -0.219$ m $u_y = -1.352$ m, $u_z = 7.249$ m. This represents an error of 0.19% for a 25.6% tip deformation of the wing semi-span, $b = 28.8$ m.

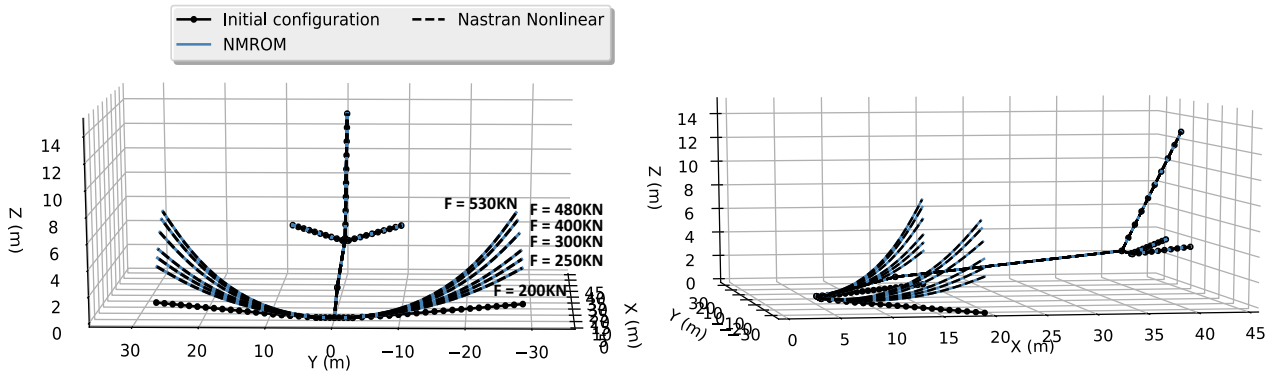
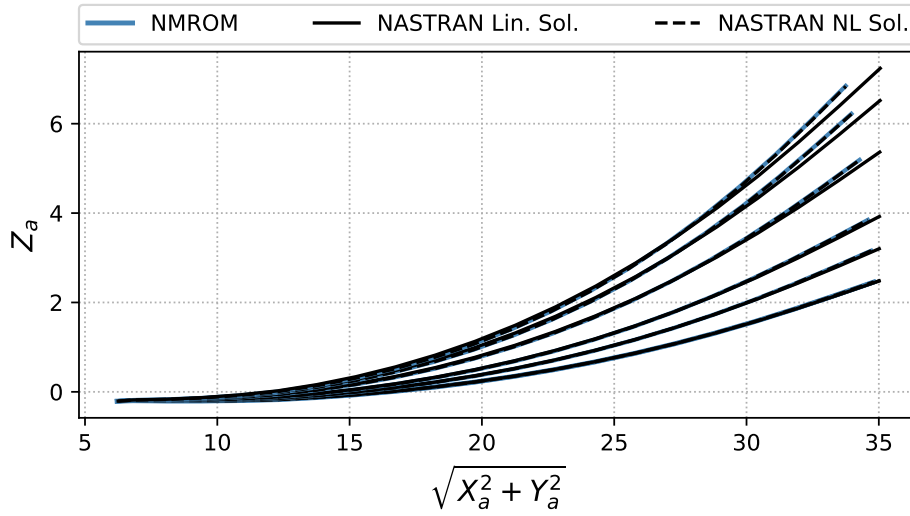


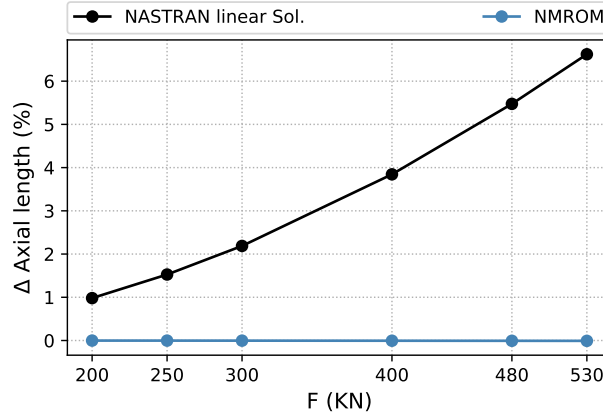
Figure 6.3: Aircraft static response under wing-tip follower loads

A further investigation on how geometrically-nonlinear effects affect the static response is shown in Fig. 6.4. MSC Nastran linear solutions are computed for the set of forces in the z-direction previously described. Deformations in the z-direction versus the metric $\sqrt{x^2 + y^2}$ are shown in Fig. 6.4(a) for solutions with the present NMROM, and linear and nonlinear computations in the full aircraft model. This allows appreciating more clearly the shortening effect in nonlinear computations. On the other hand, the length of the main wing after reduction to the 1D domain is computed before and after deformations ($L_w = \int_{\Gamma_w} ds$). Because the resultant axial stiffness is much higher than bending or torsional stiffness, the structure is nearly inextensible. This effect, however, is not captured by linear approximations. Fig. 6.4(b) shows the percentage

change in the total length of the main wings with the driving set of forces.



(a) Nonlinear shortening effects



(b) Elongation of the main wing under linear and non-linear approaches

Figure 6.4: Static geometrically-nonlinear effects on the aircraft main wing

Linear computations fail to capture axial effects and the total length of the wing increases with the loading. Excellent agreement is obtained between the nonlinear static calculations from MSC Nastran and those of the proposed approach. These results and the dynamic simulations presented next in Sec. 6.2, are in contradiction with the claims in [53] that methodologies based on beam models fail to capture wing responses with displacements over 20%. There, a different strategy is presented to build ROMs based on system identification of build-up FEMs, however, surprisingly enough they only show results with beams. While no claim is made regarding the usability of our method to any structure, the novelty of the approach is that

it directly incorporates the properties of the FEM and, as shown herein, it attains very good approximations in the analysis of slender structures, such as those of aircraft –in this case not even a very high aspect ratio one.

6.1.2 Dynamic nonlinear excitations

For models built with lumped masses and linear interpolation elements, such as this one, no direct nonlinear dynamic analysis is available. However, the current nonlinear condensation strategy provides a solution to this problem. To exemplify this, an initial parabolic velocity as in Sec. 5.2.2 is imposed as a function of the y-direction and the wing semi-span: $\mathbf{x}_1(s, 0) = \mathbf{x}_{10}(y/b)^2$. MSC Nastran linear dynamics can be run and serve to verify the current implementation for small displacements; when larger displacements are considered, comparisons between nonlinear and linear solutions allow to quantify the correction included by the NMROM. An initial small velocity is given with $\mathbf{x}_{10} = (3, 0, 2, 0, 0, 0)$ and the position of the wing tip over a simulation of 5 seconds is shown in Fig. 6.5(a). As the NMROM preserves the linear model, an excellent

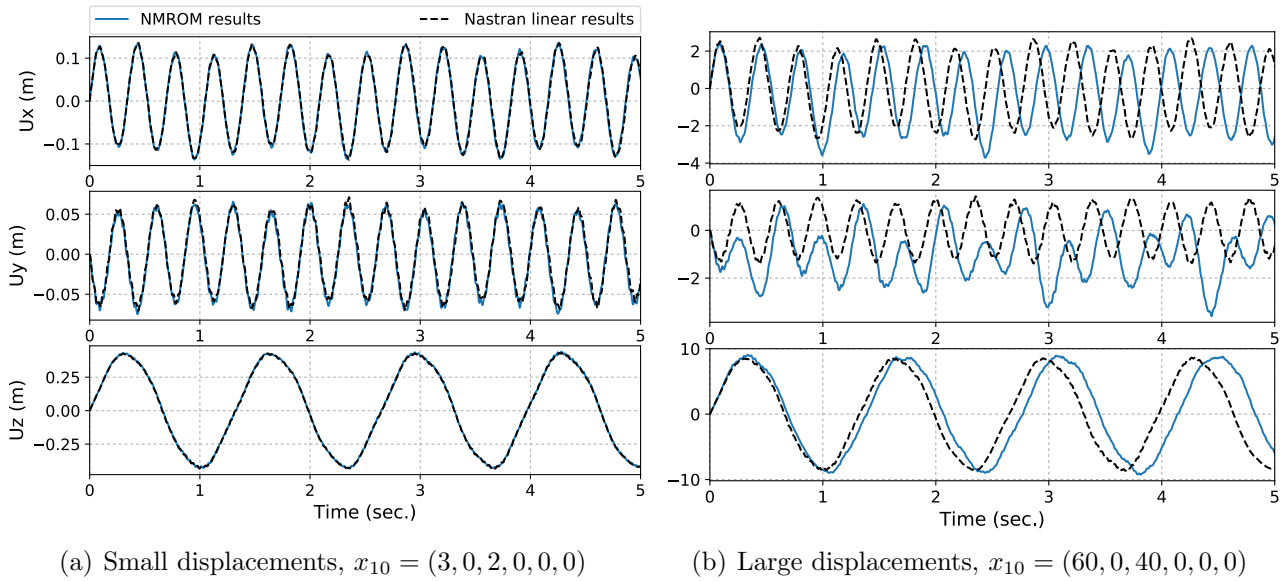
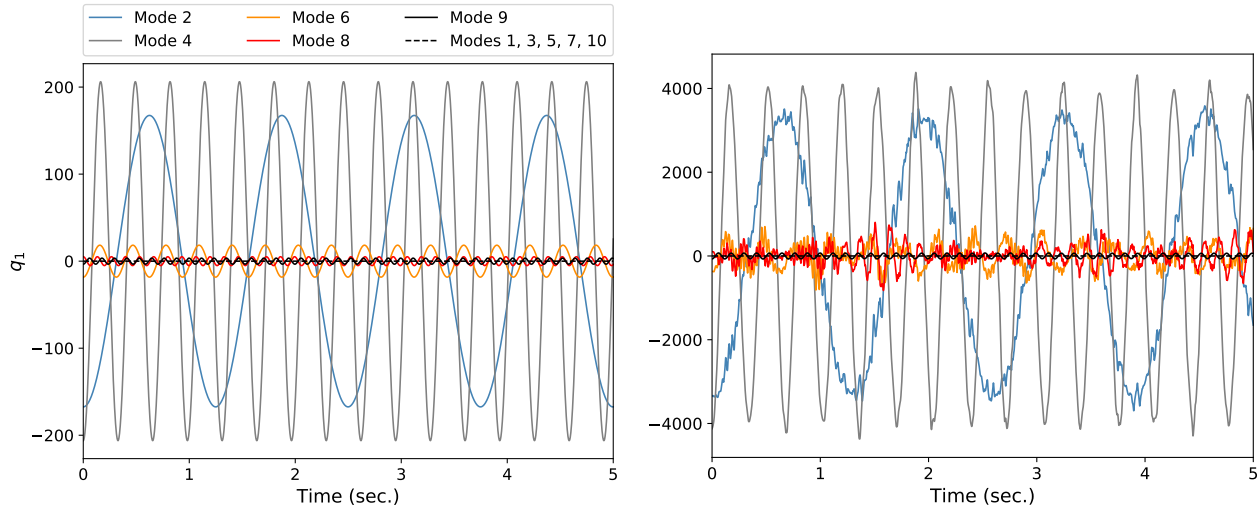


Figure 6.5: Wing tip displacements in free vibrations with small and large amplitudes on full-aircraft model

correlation is found between the linear results on the full model (dashed black) and NMROM calculations (solid blue) with $N_m = 90$ and time step of $\Delta t = 0.001$ s. in the RK4 algorithm used to time march the solution. Note that high-frequency filtering was not needed to converge

to the solution, but a higher number of modes are required in comparison to the static solution. If the initial velocity is increased, $\mathbf{x}_{10} = (60, 0, 40, 0, 0, 0)$, large deformations of up to 35% of the semi-span occur. Fig. 6.5(b) shows that the dynamic response predicted by the NMROM quickly deviates from that of the linear model, particularly in the axial and in-plane bending directions. The effect of nonlinearities can be seen in more detail in the modal coordinates on which the system is solved, Eq. (2.100). Fig. 6.6 shows the time history of the first 10 components of the modal velocity coordinates, $\mathbf{q}_1$. For the small deformations case, harmonic oscillations with the natural frequencies of the system are obtained as seen in Fig. 6.6(a). The same results would be obtained with a linear analysis. Modes 2, 4 and 6 are the first out-of-plane, in-plane and second out-of plane modes shown in Fig. 6.2 respectively. Mode 9 is the first bending mode of the tail and modes 1, 3, 5, 7 and 10 are anti-symmetric modes which do not contribute to the solution of this problem. Some effect of the nonlinear terms can be clearly seen in Fig. 6.6(b): 1) higher frequencies are coupled to lower frequency modes, as in mode 2; 2) there are small changes on the dominant frequency of modes 2 and 4; 3) some additional modes are excited, and 4) some modal amplitudes are proportionally larger (or smaller), as in mode 8 (third symmetrical bending mode).



(a) Small deformations problem, $\mathbf{x}_{10} = (3, 0, 2, 0, 0, 0)$ (b) Large deformations problem, $\mathbf{x}_{10} = (60, 0, 40, 0, 0, 0)$

Figure 6.6: First 10 velocity modal coordinates, $\mathbf{q}_1$

Finally, the total energy of the system is calculated using Eq. (2.104) to verify conservation of fundamental quantities. Results are normalised with the initial energy of the system in Fig. 6.7,

which shows the time history of total, kinetic and potential energy for the free small (linear) and large vibrations of the aircraft. Both problems exhibit a conservation of the total energy. Importantly, the nonlinear system conserves energy even when a truncated basis is used in the solution as proven in [238].

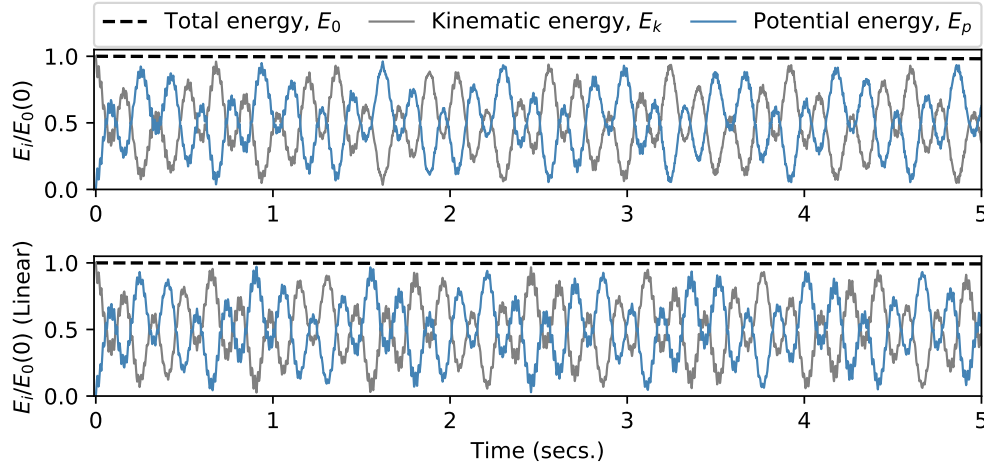


Figure 6.7: Energy distribution in aircraft NMROM linear and nonlinear free-vibrations

6.2 Condensation techniques on a wing-box model

This test case demonstrates the suitability of the condensation methods described in Sec. 2.2 for large FE models, and the accuracy of the NMROM approach for dynamic geometrically-nonlinear calculations. While in the previous example only nonlinear static results could be computed with MSC Nastran, due to the mass model given only as discrete masses at certain nodes, here we consider a distributed mass model so that dynamic nonlinear calculations on the full FE model can be performed and the validation process is completed. The initial design (M_1), shown in Fig. 7.8(b), is a wing-box of 28.8 m. semi-span, built to industrial practice and representative of current aircraft airliners. A total of 1462 elements are employed and the 1255 grid points are condensed to 23 active nodes in the center of the ribs. Composite material properties are used for the shells and mass is represented as continuous density of the structural elements. In order to depart from standard designs and increase its flexibility, a modified design (M_2 , shown in Fig. 7.8(a)) is considered in which the ribs are modified to have both holes and a 2/3 reduction in thickness. In both wings a virtual load path is constructed through the center

of each rib section using interpolation elements and the condensation nodes are shown in Fig. 6.8 in yellow (details of the location for this skeleton are less important than the alignment of the nodes).

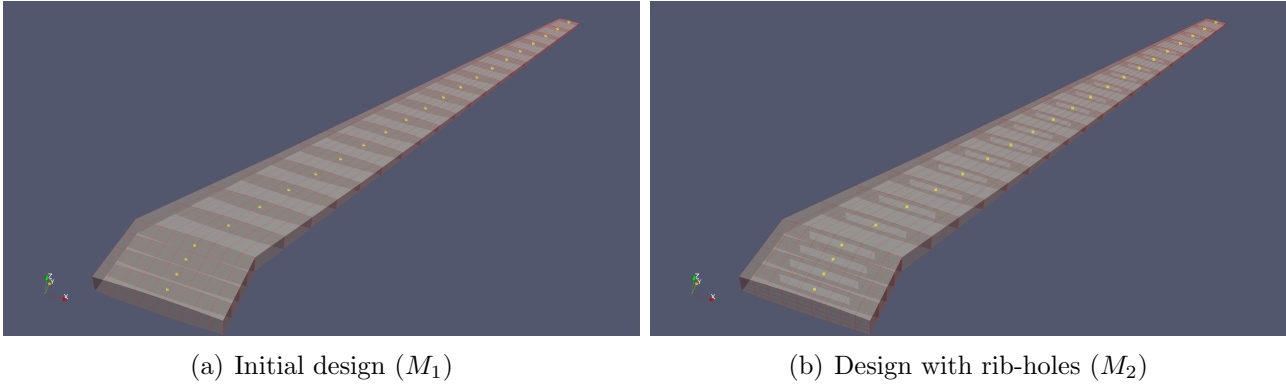


Figure 6.8: Two representative wing models with the chordwise direction along the x-axis and wing-span along the y-axis. Dots indicate the condensation nodes

Figure 6.9 shows, for the first 32 modes on each wing, the relative error with respect to the full model in the natural frequencies predicted by the Guyan and Kidder condensations. For reference, the first natural frequency is 0.355 Hz for M_1 and 0.342 Hz for M_2 , while for the 32nd mode they are 64.84 Hz and 40.90 Hz, respectively. As it can be easily seen, for the wing with the more rigid ribs, the maximum error in the Guyan condensation is just above 0.5% and therefore this approach is sufficient to capture the higher frequencies. For wing M_2 , where more complex cross-sectional behaviour occurs, the iterative method is necessary to capture the higher-order frequencies.

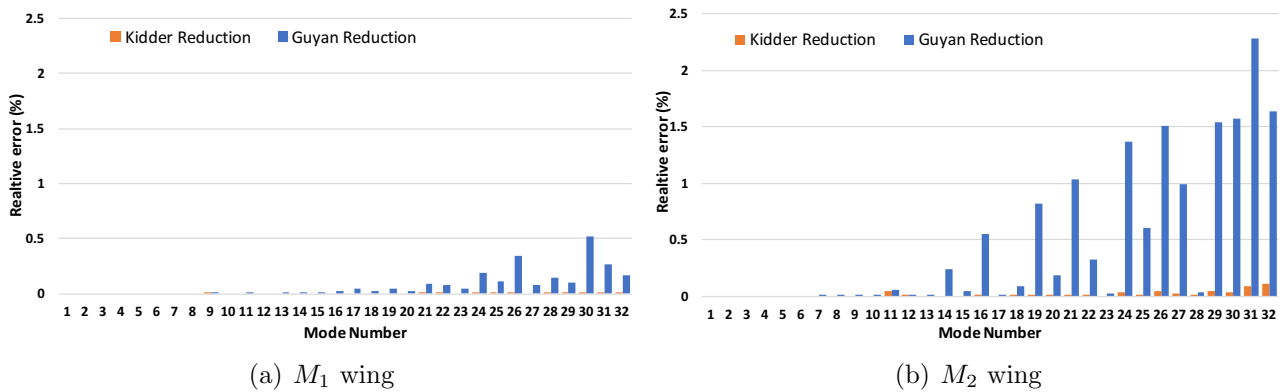


Figure 6.9: Relative errors in the models

The density map of the stiffness matrices of the models gives some insight into the condensation

process. Fig. 6.10 shows the condensed stiffness matrix for both the rigid (M_1) and flexible (M_2) wings, using Kidder condensation.

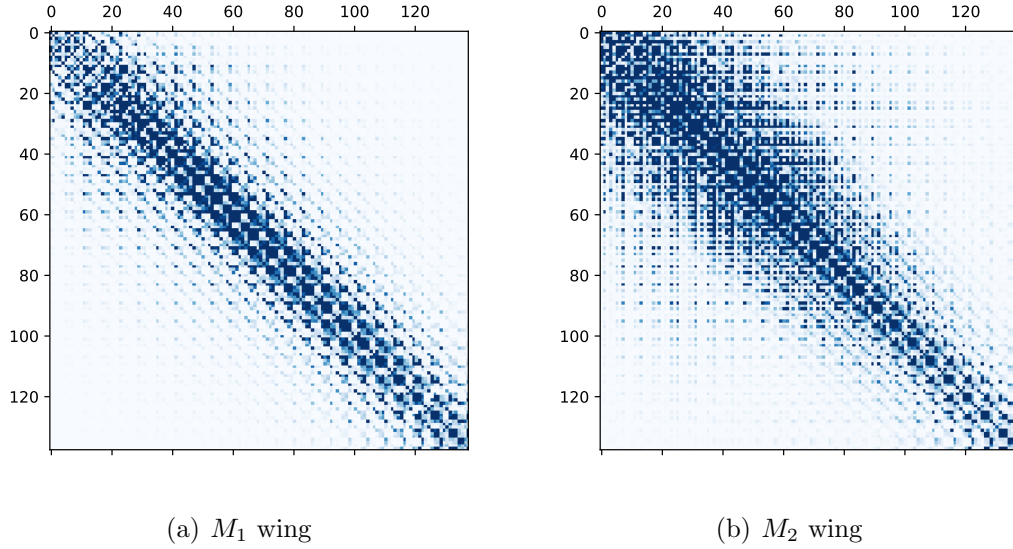


Figure 6.10: Density map of the condensed stiffness matrix in both wing models

For M_1 there are virtually no changes in the updated process of the Kidder reduction and this density map is identical to the one obtained from Guyan reduction. The density map of the wing with flexible ribs, Fig. 6.10(b), shows a much denser condensed matrix, which suggests that the iterative methods pack more information into the matrix to preserve the fidelity of the full model. Dynamic nonlinear simulations are carried out next and compared to MSC Nastran linear and nonlinear analysis (SOL 109 and 400, respectively) on the full FE model. A force is applied at the wing tip with a triangular loading profile given as

$$f(a, b, c) = \begin{cases} a + \frac{1-a}{b}t & \text{if } t \leq b \\ 1 - \frac{1}{b+c}t & \text{if } b < t \leq b + c \\ 0 & \text{otherwise} \end{cases} \quad (6.1)$$

The follower force applied to the tip of M_1 is a ramp, $\mathbf{f}_{tip} = [-2 \times 10^5, 0, 6 \times 10^5]f(0.05, 4, 0.)$ and the dynamic response is presented in Fig. 6.11, where results have been normalised with the wing semi-span. As expected, linear analysis overpredicts vertical displacements and does not capture displacements in the x and y directions. Using Guyan condensation, which is sufficient

for this wing, NMROMs were built with 15 and 50 modes and run with time-steps of 6×10^{-3} and 3×10^{-3} respectively.

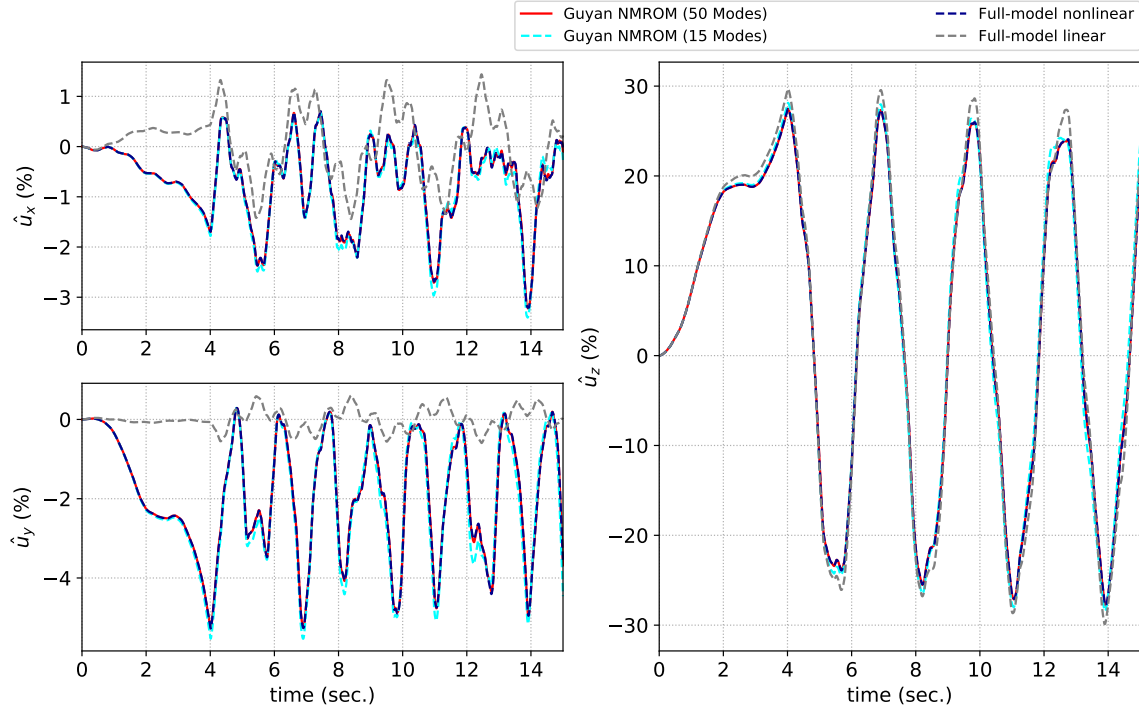


Figure 6.11: Span-normalised tip displacements in the dynamic simulation of M_1

The fully converged 50 modes NMROM and the nonlinear solution from the full FE solution are in complete agreement, even though deformations of +28% of the semi span take place in the simulation. The accuracy of the NMROM with 15 modes is also good, yielding small differences with the full FE analysis and running at nearly real time (less than 30 seconds in an i7-6700 processor running at 3.40GHz), which is over 100 times faster than the time to run the full nonlinear simulation in MSC Nastran (3504 sec. in the same computer with a time step of 1.5×10^{-3} for convergence). The 50 modes NMROM took 2226 s. and, while this is still faster than the full FE, it highlights the relatively rapid growth in computational cost with the number of modes due to both the cost of evaluation of the nonlinear coupling tensors and the need for a smallest time step. For large modal basis, additional savings can be obtained using residualisation techniques, which reduced the computational effort by 75% in [215]. It is also worth remarking that the NMROM results come from a non-optimised Python implementation, while MSC Nastran is written and optimized for compiled language, thus further savings can be expected if a next version of FEM₄INAS was written for performance.

Next, the local angle of attack has also been measured along the wing. It is defined from the difference in vertical position of a central point and the corresponding leading edge. The 3D-plot in Fig. 6.12 shows the angle of attack evolution in time and along the wing-sections for the simulation with design M_1 . In these results, linear analysis overpredicts it by up to 3.6 degrees with respect to the full FE solution and the NMROM. A similar exercise is run on the

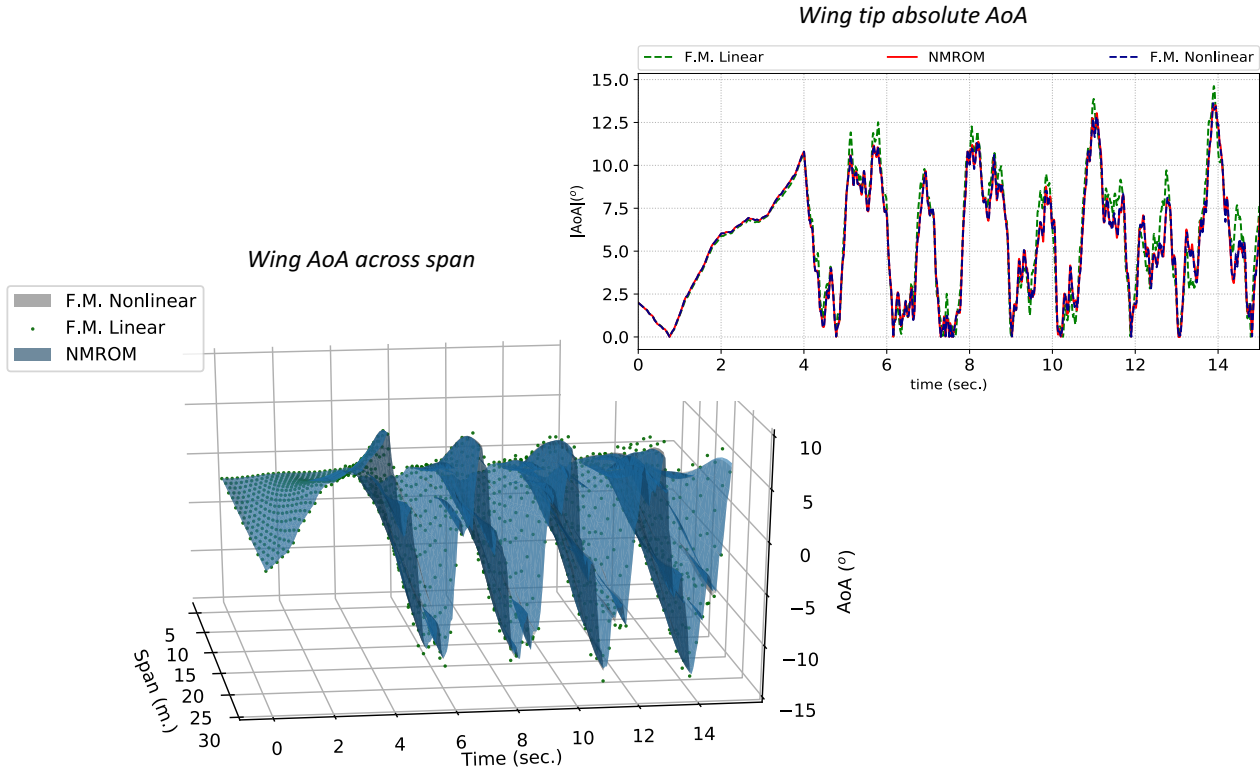


Figure 6.12: Wing-tip angle of attack evolution in time and along span (design M_1)

flexible design, M_2 , now also adding moments in the y-direction to excite higher frequencies. The forcing load at the tip is $f_{tip} = [3 \times 10^5, 0, 1.5 \times 10^5, 0, 6 \times 10^5, 0]f(0.05, 3, 0)$. Fig. 6.13 shows again the time histories of the three components of the wing tip displacements using the full-model, its linearisation, and NMROM with 50 modes using both condensation schemes. Linear analysis shows even larger errors than in the previous example, and both condensation methods show again excellent comparison with the full FE model.

Even though the Kidder method was shown to have a small improvement in the higher modes, the differences did not have a significant impact on the nonlinear response. These results suggest that the overall accuracy in the large-amplitude simulations is still largely driven by the dominant low frequency dynamics.

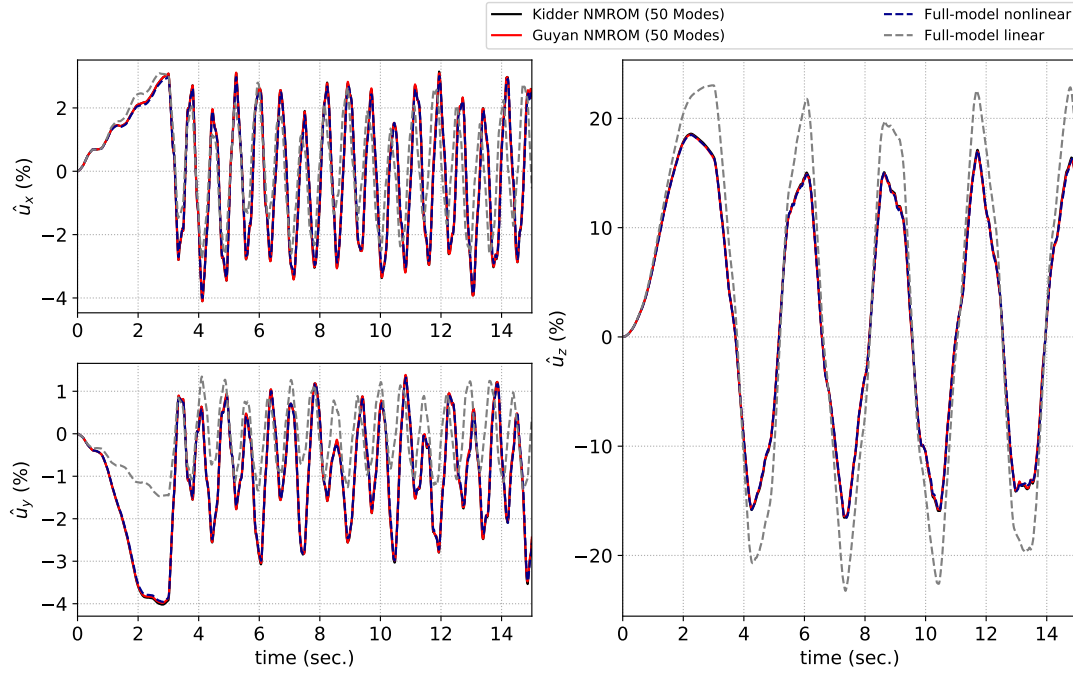


Figure 6.13: Span-normalised tip displacements in the dynamic simulation of M_2

The evolution of the error with the number of modes in the solution is assessed by using a 2-norm of the tip displacements difference between the nonlinear solution on the full FE model and the NMROM averaged over time and it is shown in figure 6.14. An error below 1% is achieved using only 15 LNMs, and above 30 no significant improvements are obtained. For reference, the solution of M_2 took 298 sec. using 30 LNMs. Thus apart from the computational aspects underlined above, it is clear that a good assessment of the number on modes required in the solution makes the biggest difference in attaining both accurate and efficient calculations.

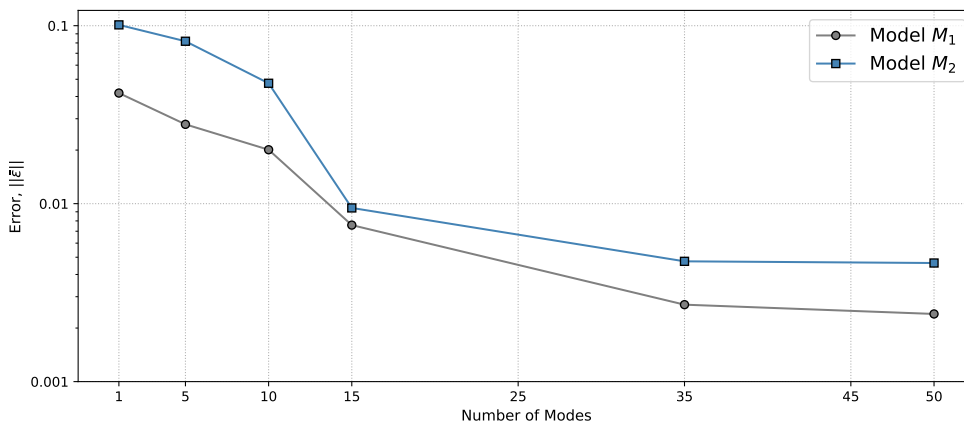


Figure 6.14: Error between NMROM and full model with number of modes in the NMROM solution

6.3 Aeroelastic stability and nonlinear response of a cantilever wing

In this test case, the postflutter behavior of a wing will be considered. The test case is the modified (or *heavy*) Goland wing [286], which has been stretched for these results to seek large deformations, and the analysis is performed without the tip stores. A (linear) MSC Nastran model has been built and it is shown in Fig. 6.15. It is a cantilever wing, initially of 20 ft span, but here increased to 40 ft (for an aspect ratio of 13.3); it has 4 ft structural chord and 6 ft aerodynamic chord. While the geometry is rather simple, it incorporates shell and beam elements to model the skin, ribs and spars, as well as lumped masses. The DLM model has been built with 24 panels in the spanwise direction and 20 panels along the chord. Guyan condensation is employed onto the central points of the ribs.

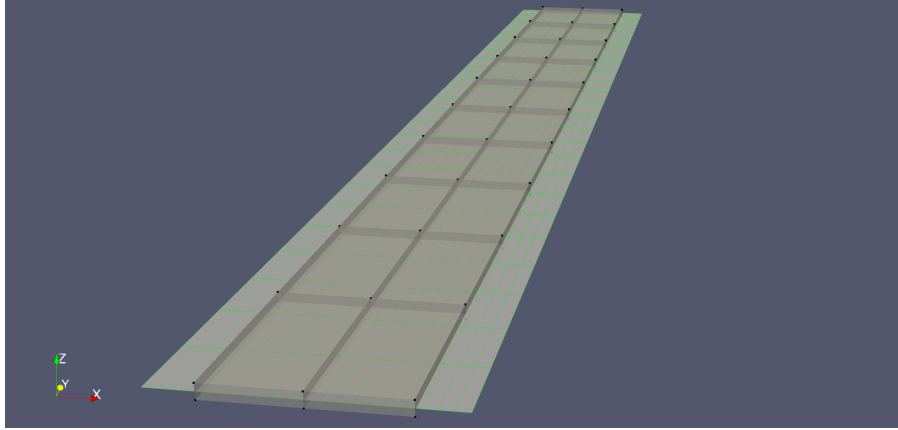
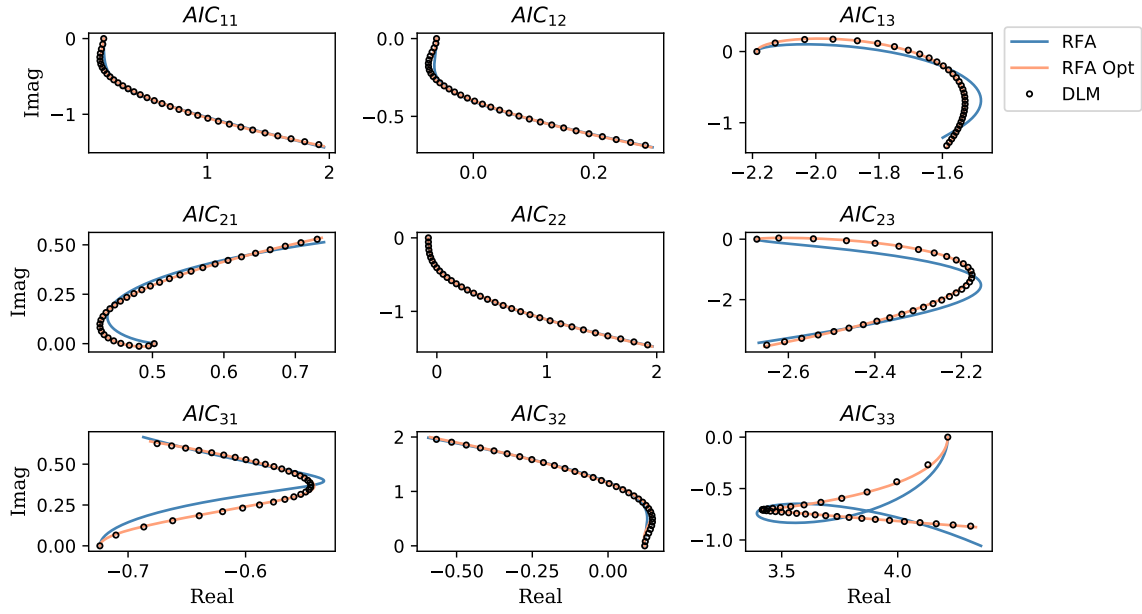


Figure 6.15: Stretched Goland wing structural and aerodynamic model

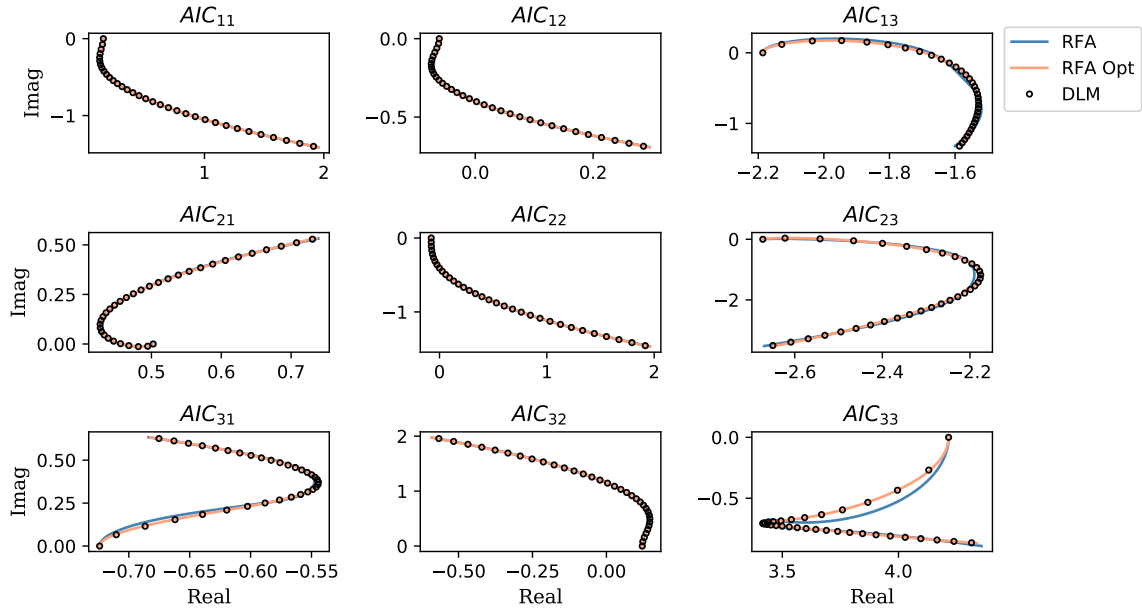
Linear flutter results

The flutter mechanism for the stretched wing is still driven by the aeroelastic coupling between the first bending and first torsional modes. Therefore, only the first three LNMs are retained for the initial linear analysis. The corresponding AICs, $\mathbf{Q}_a(\text{Ma}, k)$, are calculated for a range of reduced frequencies $k = (10^{-9}, 0.01, 0.02, \dots, 1)$ for each Mach number of interest. RFAs with 2 and 4 aerodynamic lags are considered. Figures 6.16 and 6.17 show the imaginary versus real parts of all the components of the AIC, $\mathbf{Q}_a$ for $\text{Ma}=0$ and $\text{Ma}=0.7$, respectively and for the

considered range of reduced frequencies (in increments of $\Delta k = 0.03$).



(a) 2-pole approximation

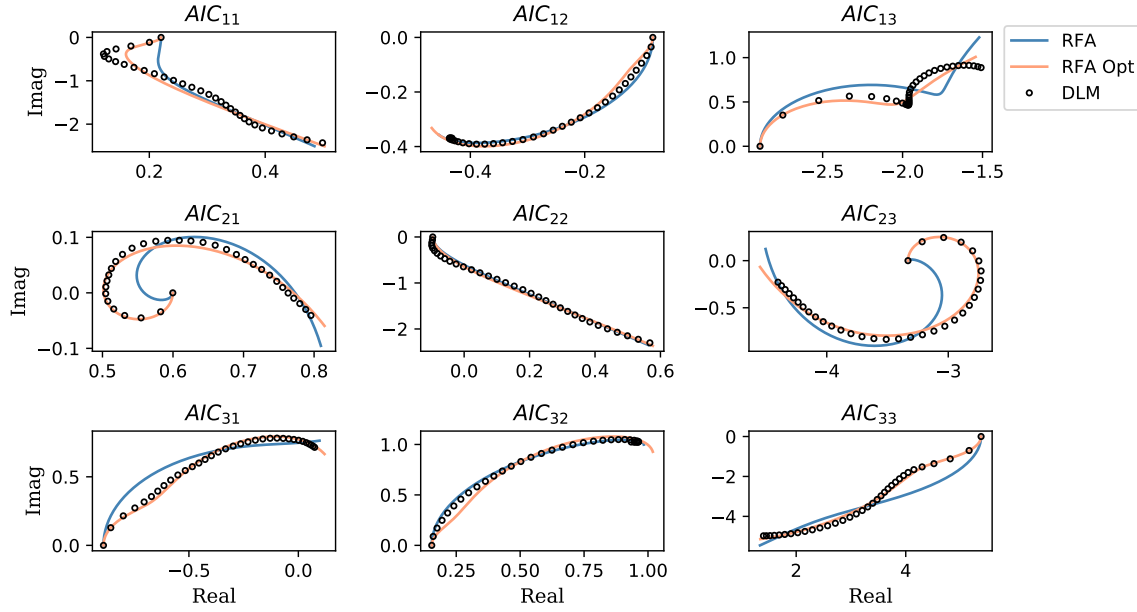


(b) 4-pole approximation

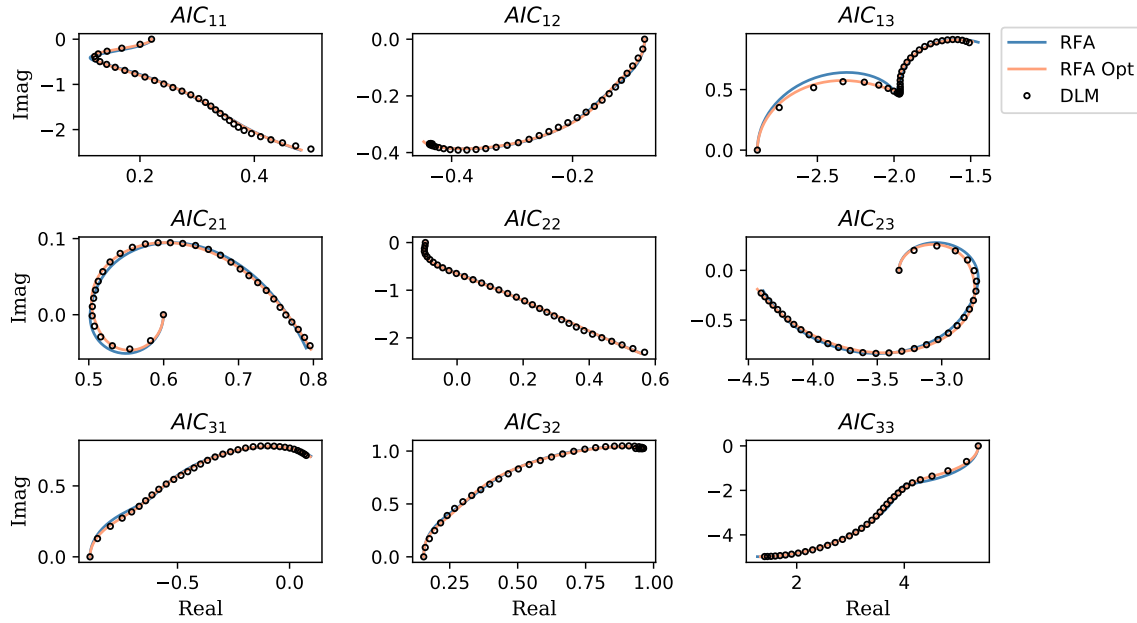
Figure 6.16: Rational function approximation for $Ma=0$, for equally spaced reduced frequencies in $0 \leq k \leq 1$

They compare the discrete calculations obtained from the DLM, a RFA using Rogers' approximation (equation (3.26)) and with the poles chosen as in [181], which is representative of industrial practice, and a second RFA with optimized selection of the aerodynamic lags, as described in section A.4. In [181], the aerodynamic lags are chosen as $\gamma_{0p} = \frac{k_{max}}{p}$, while after

optimization the 2-pole RFA results in $[0.265, 0.077]$ and the 4-pole in $[0.435, 0.335, 0.165, 0.054]$ at $Ma=0$, and they are $[1, 0.067]$ and $[1, 0.9, 0.8, 0.096]$ for $Ma=0.7$.



(a) 2-pole approximation



(b) 4-pole approximation

Figure 6.17: Rational function approximation for $Ma=0.7$, for equally spaced reduced frequencies in $0 \leq k \leq 1$

Optimization of pole selection produces a clear improvement on the accuracy of the RFA, which is more marked at higher Mach numbers, where the AICs present higher variations with reduced frequencies. Although the optimized 2-pole approximation results in a good enough fit, 4-poles

are used in subsequent analysis. The flutter speed is next calculated from the eigenvalues of Eq. (3.44) and compared with the direct frequency-domain results using the pk-method in MSC Nastran. Figure 6.18 shows the flutter speed and frequency for varying Mach numbers, $Ma = 0, 0.3, 0.5, 0.7$ and 0.85 . Both full and reduced model results are employed in the MSC Nastran calculations. Stability conditions defined by the eigenvalue analysis of the linearized system are also included, using the 4-lag RFA defined above. As the NMROM reduces to a description in the original LNs under linear assumptions, its flutter boundary is very close to that obtained by MSC Nastran using either the full or the condensed models, with the small differences being driven by the different manipulation of the same AICs between both solution processes. The preservation of the linear model is a fundamental feature of the proposed approach.

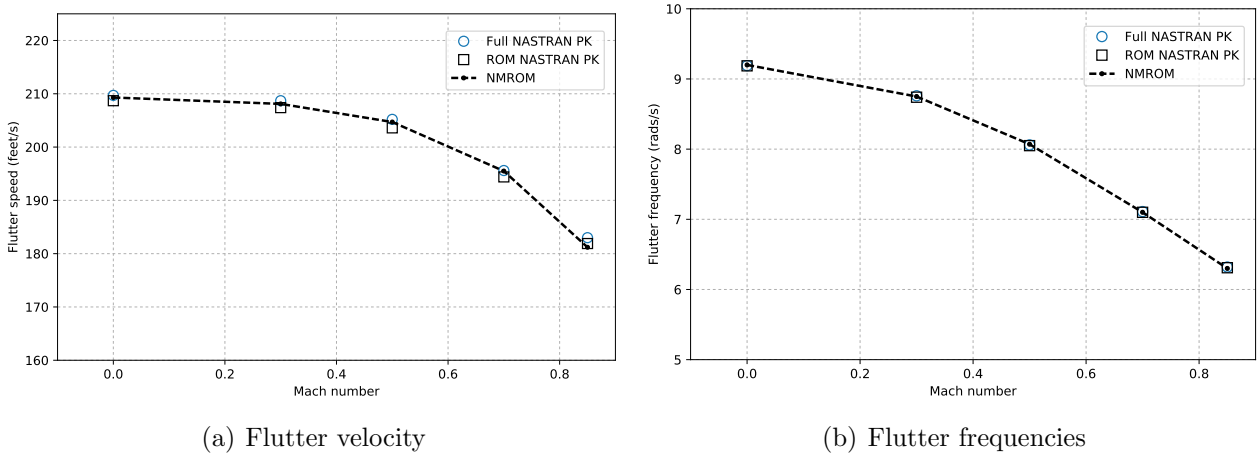


Figure 6.18: Flutter conditions of the Goland wing for changing Mach number

Nonlinear response

The nonlinear system is used to explore the post-flutter behavior of this wing. Twelve FE modes were employed in the solution, $N_m = 12$, which was found to be sufficient to capture the main features on the nonlinear response while keeping computational cost very low. Four aerodynamic lags were again selected, $N_p = 4$, thereby resulting in 48 aerodynamic states. Calculations are carried out at $Ma=0$, where the flutter equivalent airspeed is just below 210 ft/s. An initial small perturbation in the form of a parabolic velocity in the z -direction is given to initiate the wing dynamics. Figure 6.19 shows the time history of the vertical tip-displacements, normalized by the semi-span, for both linear and nonlinear analyses and equivalent airspeed of

210 ft/s, that is, just above the flutter onset. The simulation time is 25 seconds, with time step of $\Delta t = 0.002$ s. While both analysis featured a similar initial response, simulations diverge in the linear approach and, on the other hand, stabilise eventually into a limit cycle oscillation (LCO) in the nonlinear model.

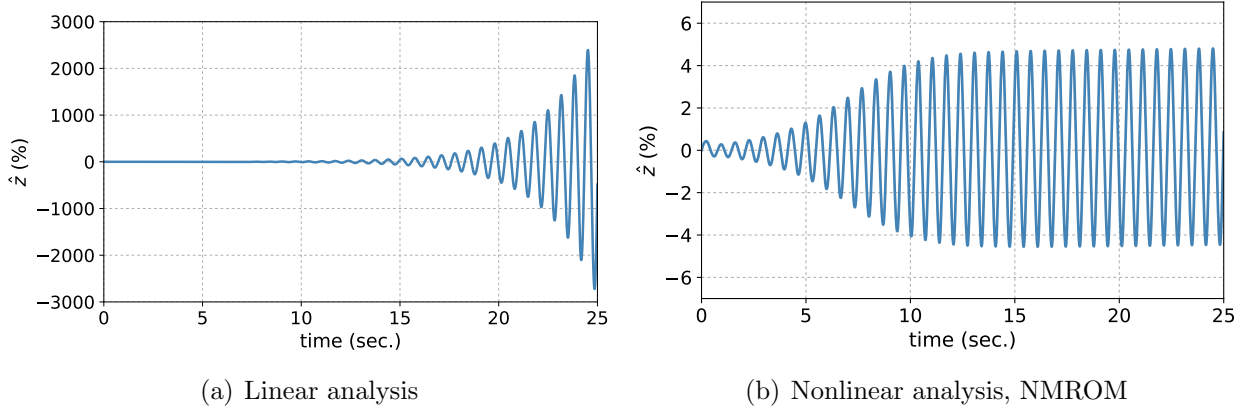


Figure 6.19: Normalized vertical response of the wing-tip at 0 Mach number and $U_\infty = 210$ ft/s

The nonlinear behavior is further explored through the phase plot of the amplitudes of the first two velocity modes, q_{11} and q_{12} . These modes comprise both out-of-plane bending and torsional components. In the linear analysis unbounded growing ellipses are formed, while the nonlinear analysis exhibits a LCO, as shown in Fig. 6.20.

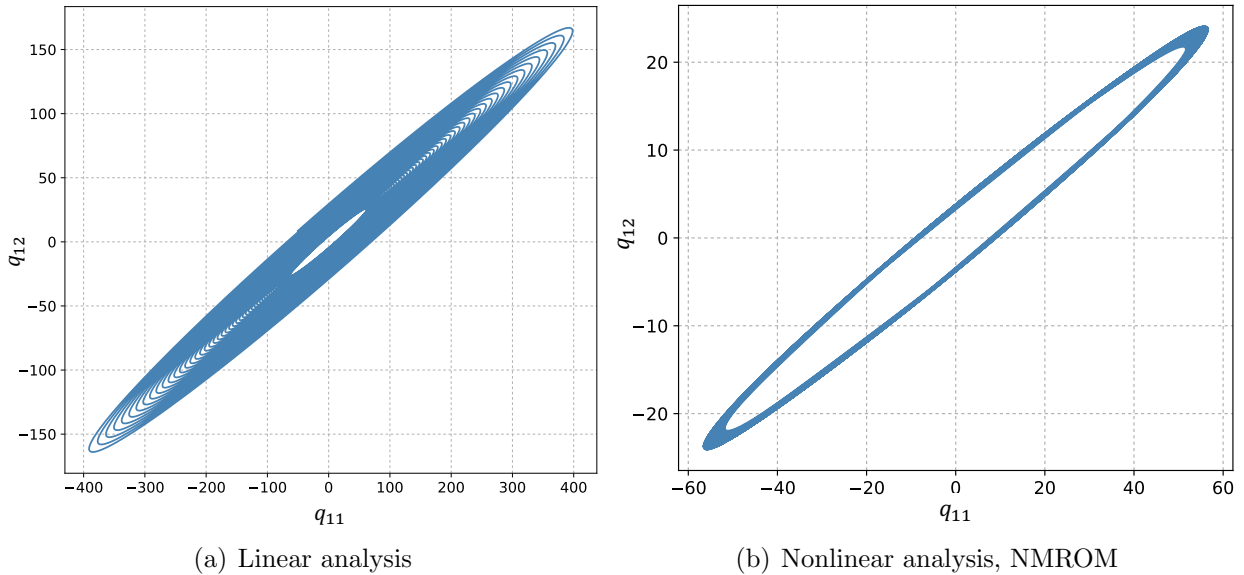


Figure 6.20: Linear VS nonlinear trajectories of first two modal amplitudes at 210 ft/s

As the velocity of the flow is increased from 210 ft/s, the evolution of the LCO grows in

amplitude but also in its nonlinear behavior. It can be seen in Fig. 6.21 how at a speed of 218 ft/s the shape drawn by the LCO is not an ellipse anymore. This highlights the need for nonlinear approaches even for moderate deformations, as also recently shown in [287].

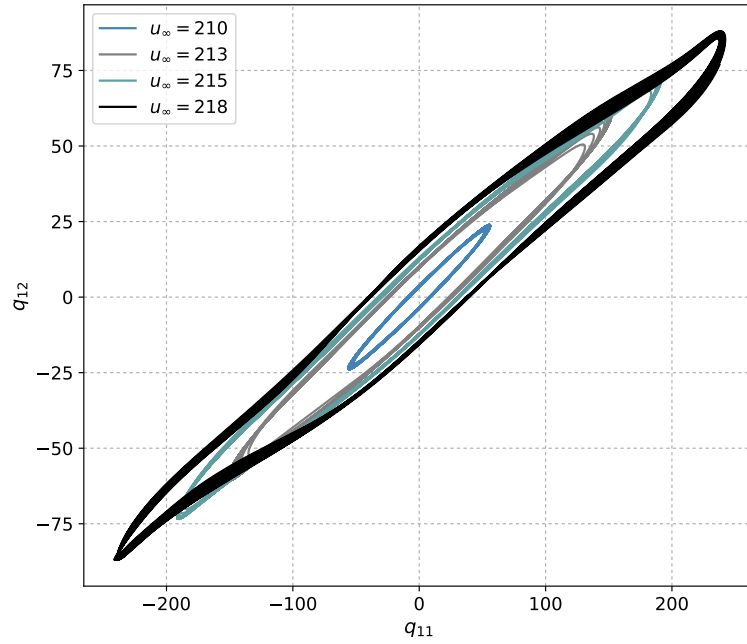


Figure 6.21: Trajectories of first two modal amplitudes with flow speed from 210 ft/s to 218 ft/s

A key effect in the evolution of the LCOs with larger deformations is the increased nonlinear coupling between modes. In-plane modes are excited through those couplings even though only out-of-plane modes were perturbed in the initial excitation.

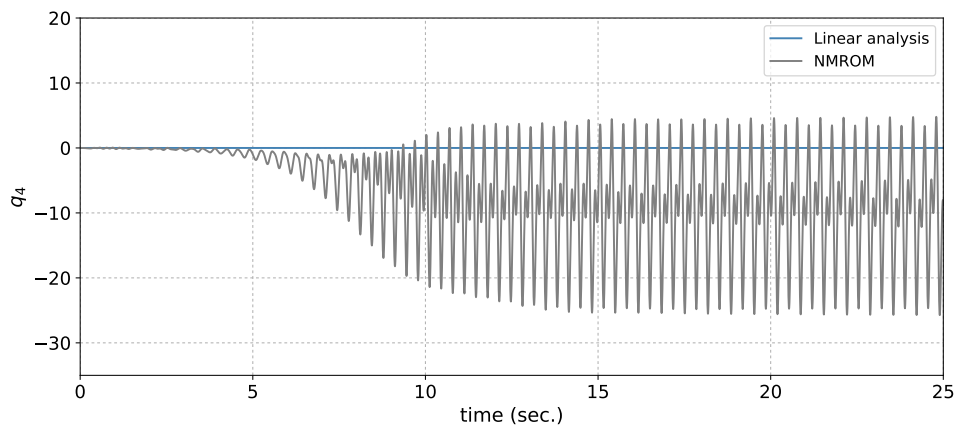


Figure 6.22: Fourth velocity modal coordinate evolution in time for both linear and nonlinear solutions

As an example, Fig. 6.22 shows the time evolution of the amplitude of the fourth velocity mode,

which is a pure in-plane bending mode. Linear and nonlinear results are presented in blue and grey respectively, and it is clear the excitation of this mode only takes place in the nonlinear computations. We note however that the in-plane kinematics of the structure do not translate into aerodynamic forces, as they have no effect in the (linearized) aerodynamics of the DLM. Either higher-order panel methods [18], a linearized UVLM [131] or a linearized CFD would be needed to account for this effect.

Non-Intrusive Study on a Transport Aircraft Configuration



Man is the measure of all things.

Protagoras of Abdera

If a machine is expected to be infallible, it cannot also be intelligent.

Alan Turing

This chapter concludes the numerical exercises with the detailed study of a long range airplane that is a close representation of the configurations currently developed in the industrial environment for linear aeroelasticity. Ultimately, it features the sort of design and characteristics that the presented computational framework was developed for, albeit its conventional AR which would likely be increased in a future concept. Thus, the range of tools and techniques shown in previous chapters are utilised herein to demonstrate the importance of nonlinear analysis when large enough loads act on a flexible design. While it is difficult to verify the findings due to the novelty of the methodology and the dimensionality of the model, the mindset from the beginning of this work was to demonstrate that the physics and fundamentals of simple canonical cases could be well captured. Then, scaling up the complexity of the models would not affect the accuracy of results, as long as the computational implementation itself had been

done properly. To this end we have gathered plenty of evidence above and we also have several examples on this chapter that replicate pure linear results obtained from commercial software when running the solvers under small, linear assumptions.

The first section of the chapter starts with the main characteristics of the aircraft model under analysis, showing that the reduced model that goes into the nonlinear system of equations is actually a highly accurate representation of the original model. Numerical results begin with the static equilibrium for different angles of attack; then the trim solution is tested for a range of load cases, proving a reduction of angle of attack in the equilibrium condition with respect to the linear analysis; the dynamic response for different gust shapes is compared under linear and nonlinear settings, which points towards the idea that geometrically nonlinear analysis may be even more important in dynamic simulations; the nonlinear flutter speed is investigated for rising angles of attack that induce large deformations in the model, and the result is a significant reduction in the flutter speed when compared to the linear solution.

Next, a preliminary study of foldable wing-tips is carried out that shows the relevance of this device in reducing the loads on the wing, thereby allowing designs with more slender wings that improve the overall aerodynamic performance, but in turn may require more advanced numerical tools for predicting the resulting changes in the geometric configurations during flight. It also highlights the capability of the approach to incorporate multibody dynamics into 3D FEMs not necessarily built for multibody analysis.

7.1 Reference aircraft description: XRF1 model

The studies presented in this chapter are based on a reference configuration developed to industry standards by Airbus as part of the eXternal Research Forum (XRF), from which the aircraft takes its name, XRF1. The aircraft represents a long-range wide-body transport airplane and has been used as a research platform for collaboration between the company, universities and research institutions [288]. The original model has been modified for this work and contains a wing tip extension that makes the overall aspect ratio of the wing slightly higher and can be used as an aeroelastic hinge device for load alleviation, although for most of this

work the whole mechanism remains attached and only at the end it is released to carry out a preliminary study of a foldable wings concept. Fig. 7.1 shows the full aeroelastic model split up into the structural, mass and aerodynamic components. The FE model contains a total of around 177400 nodes, which are condensed into 176 active nodes along the reference load axes through interpolation elements (RBE3s with the UM option as described in [289]). A Guyan or static condensation approach is used for the reduction, which is equivalent to a dynamic condensation when the mass model is given as discrete elements at the condensation points, as is the case here – for distributed mass elements other dynamic condensation techniques might give better results as shown above. Among the FE elements conforming the model there are around ~ 500 spring, or 0-dimension elements; $\sim 57,000$ beam elements, $\sim 55,500$ CQUAD4 shell elements, $\sim 3,800$ CTRIA3 shell elements; and ~ 800 rigid elements. As for the mass configurations, the one utilised in this study is the estimated cruise reference mass, which amounts to a total of $\sim 188,500$ kg. The aerodynamic model contains $\sim 1,500$ aerodynamic panels.

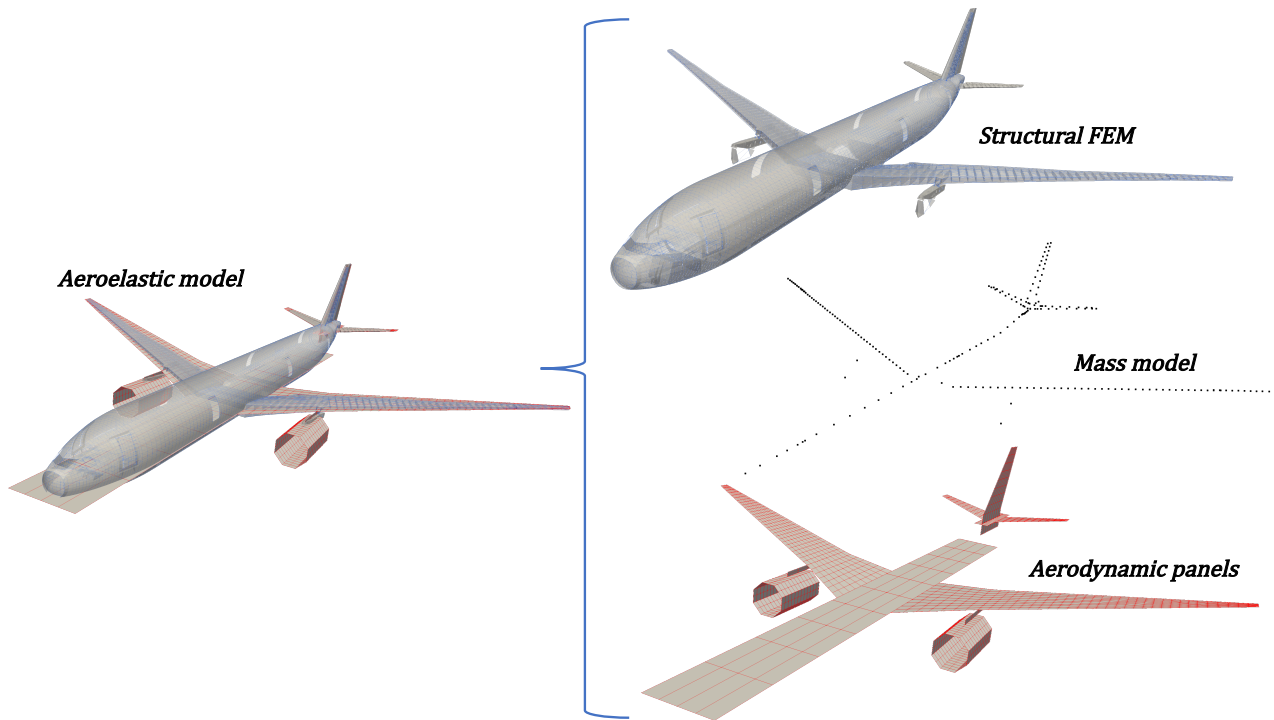


Figure 7.1: Modified XRF1 aeroelastic subcomponents

There are other methods for working with these large models in an efficient manner and incorporating geometrically nonlinear effects, for instance deriving equivalent stick models by

imposing unitary loads [50] in the original model and finding the beams that better match the linear response to these loadings. Another alternative is creating a data base with nonlinear static computations of the full model, which is used for system identification of the properties of a reduced order model of the structure [53]. However, one of the major advantages of the current methodology is that prior nonlinear calculations are not required to find the ROM, nor equivalent designs built that could lose some details of the original model. The only operation is a condensation step where the full aircraft model is reduced into the major load-paths. This is already a common practice in the aeroelastic industrial environment and the reduced model of the XRF1 was also provided. The strength of the approach can be seen by looking at the natural frequencies of the full and reduced models, shown in Table 7.1 normalised with the first natural frequency. An approximate categorisation of the LNMs is also included.

Table 7.1: Normalised natural frequencies of the modified XRF1 clamped model

Mode Type	ω_{full}	$\omega_{condensation}$	Error (%)
First out-of-plane wing bending (1)	1.0	1.0	0.0001
First out-of-plane wing bending (2)	1.04948	1.04949	0.0004
First fuselage bending (3)	1.4285	1.4288	0.025
Fuselage lateral bending + wing out-of-plane bending (4)	1.5093	1.5099	0.038
Wing and pylon lateral roll bending (5)	2.1449	2.1449	0.0007
⋮			
First in-plane wing bending (13)	3.156	3.155	0.03
Second in-plane wing bending (14)	3.25655	3.2566	0.0016
⋮			
First wing torsion (28)	7.675	7.676	0.01
Second wing torsion (29)	7.7368	7.7366	0.002

From the point of view of the aerodynamics, 8 poles are used in the rational function approximation of the frequency domain calculations obtained from the DLM. As shown in Fig. 7.2, an excellent fit is attained with the optimised approximation of the poles. It is worth noting that, while in the cantilever example shown in Sec. 6.3 a good approximation could be found without optimising the poles just by increasing their number, in this more complex case an accurate fit is not obtained in the non-optimised case. In particular, when the number of poles is increased, unwanted oscillations occur due to over-fitting. Thus a good preservation in both the aerodynamics and structural models is attained. The accuracy of the reduced structural

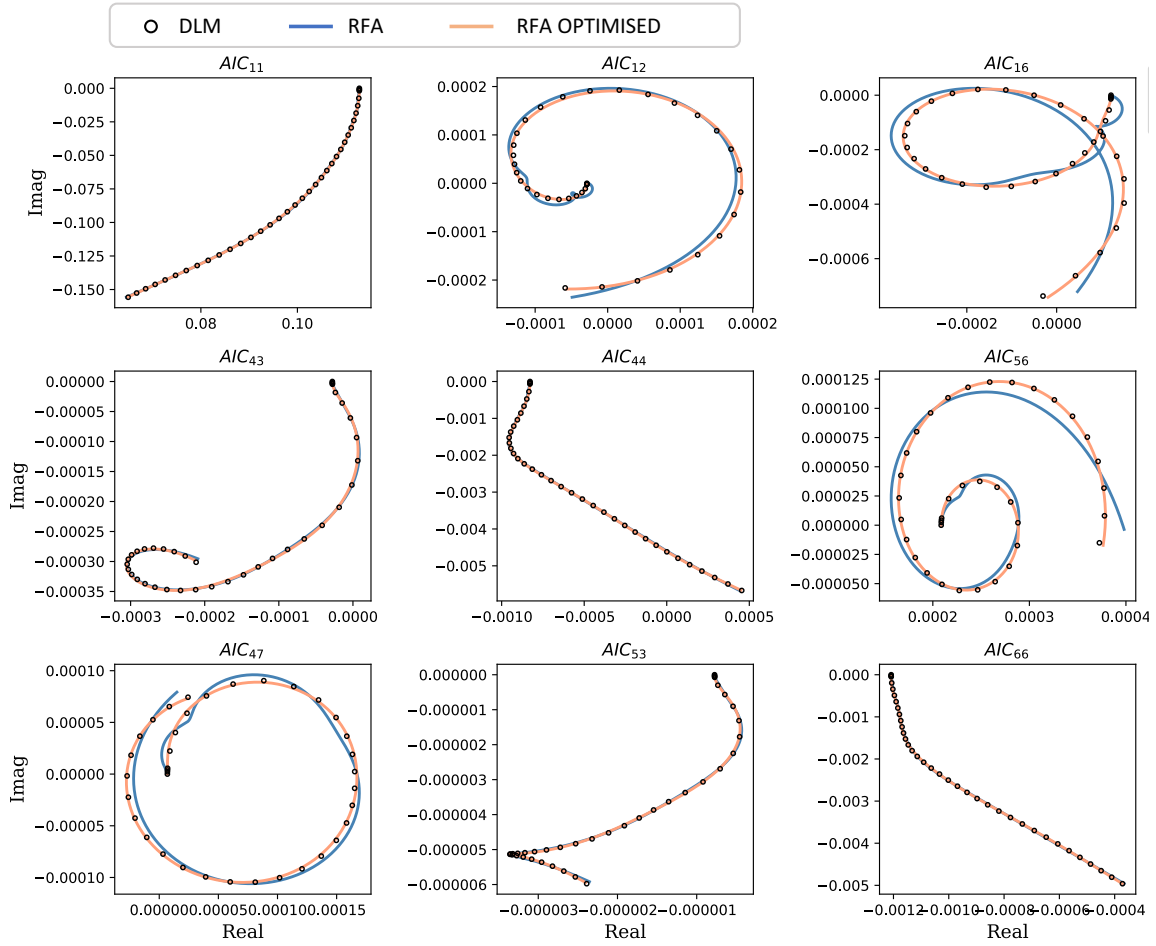


Figure 7.2: Optimised RFA of the XRF1 DLM aeordynamics, $M_\infty = 0$

model is further seen in the natural frequencies of the full and condensed models in Table 7.1: a maximum error of 0.038% in the first 10 modes, 0.35 % in the first 25 and 3.21% in the first 50. In comparison, Riso et al [51] have obtained an equivalent beam model of the XRF1 with a FEM to stick approach which incurs in errors of up to 10.3% only within the first 10 modes; the formulation by Carrera et al [290], which can predict more complex deformations than classical beam models, shows errors of up to 5.6% and 25.5 % in the first 5 and 10 modes respectively when compared to a large model of a part of a fuselage and wing connection built in MSC Nastran; and in the multibody approach proposed by Castellani et al. [291] a stick model was built out of a simple wing box with an error of 5.3% in the first torsional mode, the fifth mode¹.

¹In the same work they have also reported an error of 11.3% using a Guyan condensation approach, for which two things are to be kept in mind: firstly, a Guyan reduction needs some expertise in choosing the condensed points adequately, use of interpolation elements correctly to capture the characteristics of the 3D cross-sections and so on; secondly they have tried to assess different methodologies for reducing a full FE model into a beam equivalent, including an initial version of the approach presented in this work. They, however,

7.2 Aircraft steady state in manoeuvres and trimmed flight

The steady state equilibrium of the aircraft is studied either from a static manoeuvre or a trimmed flight condition. They are fundamental in the calculation of flight loads and the load envelopes built for the design and certification processes.

7.2.1 Static manoeuvres

The aircraft is clamped close to the center of the wing-box. The steady response under aerodynamic loading from angles of attack, $AoA = [1^\circ, 3^\circ, 7^\circ, 11^\circ, 14^\circ]$, is calculated and the configuration of the condensed structure along the load paths shown in Fig. 7.3.

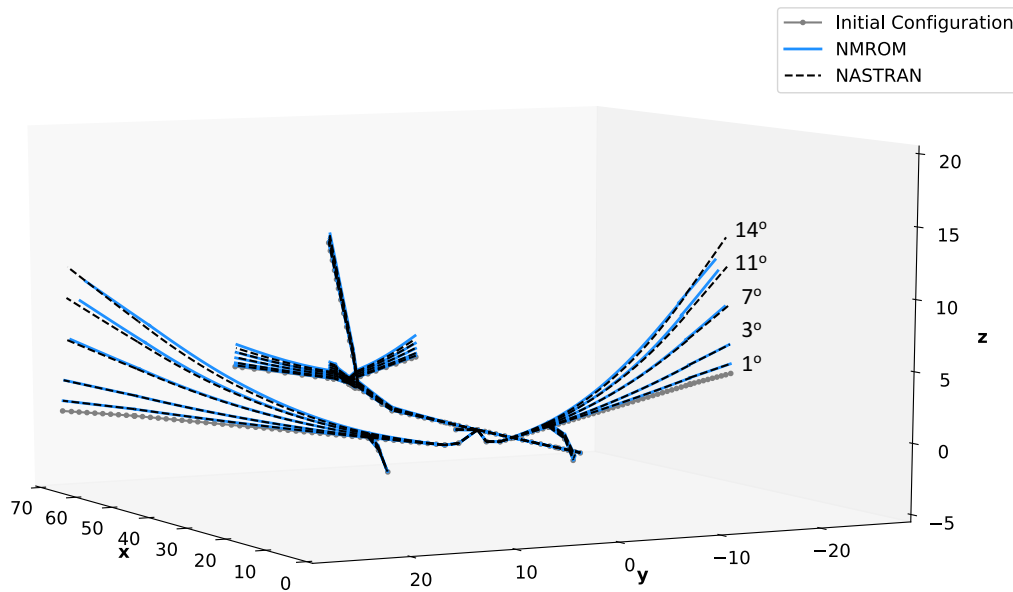


Figure 7.3: Static response of aircraft condensation points for angle of attack variations, $q_\infty = 24500Pa$ and $M_\infty = 0$

The flow conditions are a dynamic pressure of $q_\infty = 24500Pa$ and 0 Mach number. Solutions are presented from MSC Nastran static linear solvers (SOL 144 on the full model, i.e. without

failed in the implementation by building beam elements with their properties given by the 6×6 diagonal terms of the condensed FE matrices –the entries of the full FE matrices in a dynamic system have a direct relation to the topology of the elements conforming the system; this relation is lost when a reduction technique is employed, which is nothing but a mathematical artifact to compress large sparse matrices into much smaller, fully populated ones. Precisely the beauty of the current approach is that the whole information encapsulated in the condensed FE matrices is carried forward in the nonlinear analysis.

Guyan reduction), and solving the dynamic equilibrium of a NMROM built in FEM₄INAS, Eq. (3.42) with α_x containing only the particular angle of attack. A good match is found for small angles and, for larger angles, nonlinear effects such as wing shortening and follower forces are visible.

The NMROM for these results is constructed with 70 structural modes, after a convergence analysis was carried out taking a solution with 100 modes as the reference and using the L^2 relative error norm defined as $\epsilon = \|\mathbf{r}_a - \mathbf{r}_a^{(100)}\| / \|\mathbf{r}_a^{(100)}\|$ on the wing displacements. Fig. 7.4 shows this error between approximations as the angle of attack augments, i.e. for an increasing level of wing deformations. A shaded blue region represents errors where differences are small enough that solutions can be considered good for the purposes of this work (below $3\text{--}4^\circ$, an almost identical response has been found between MSC Nastran linear solution and our solvers). Note that the equilibrium condition itself in the NMROM comes from a dynamic simulation where small, undamped oscillations may affect decimal places unnoticeable in the overall structural state but having small effects on the error convergence.

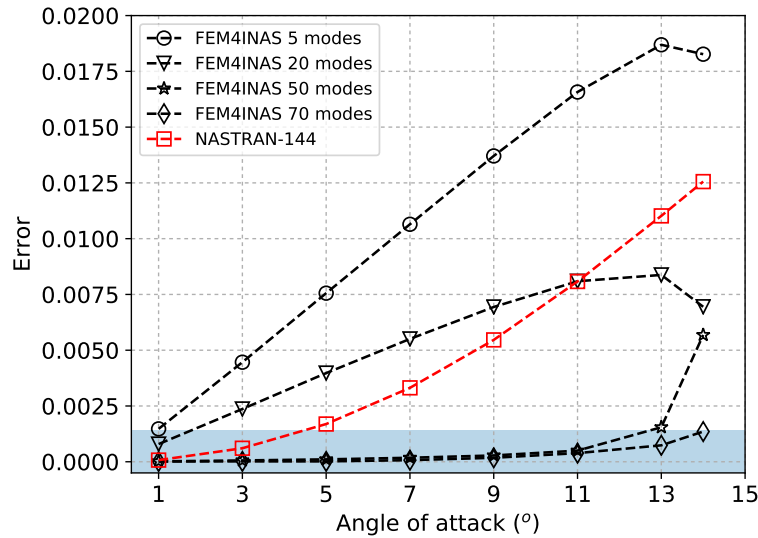


Figure 7.4: Error convergence in the XRF1 static response under AoA (NMROM with 100 structural modes used as the reference)

We have taken a sectional view of the wing condensation points in Fig. 7.5 by showing the normalised wing vertical position versus a metric of x and y components, $\sqrt{x_a^2 + y_a^2}$. It can be seen that up to around 5° , which represents a vertical displacement of about 10% of the wing semi-span, there is a good agreement between MSC Nastran and FEM₄INAS results and, from

there onwards, differences become more prominent.

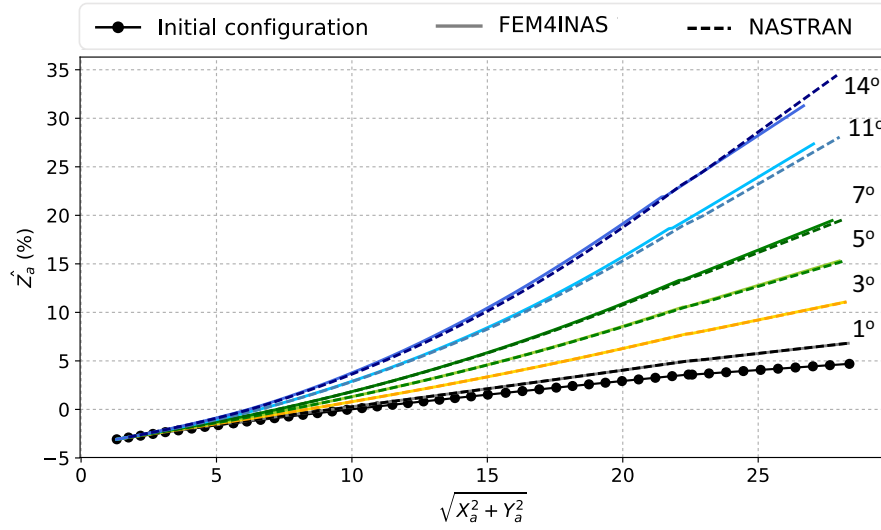


Figure 7.5: Sectional view of the aircraft right-wing

In order to better see the nonlinear effects, the wing tip x, y, z deflections are plot in Fig. 7.6 after normalised ($\hat{\bullet}$) with the wing semi-span. If the linear evolution of MSC Nastran is clear, the NMROM results show larger x and y components, while the z component decreases with respect to the linear results. This is exactly what is expected as the structure deforms and aerodynamic forces are no longer pointing only in the z -direction.

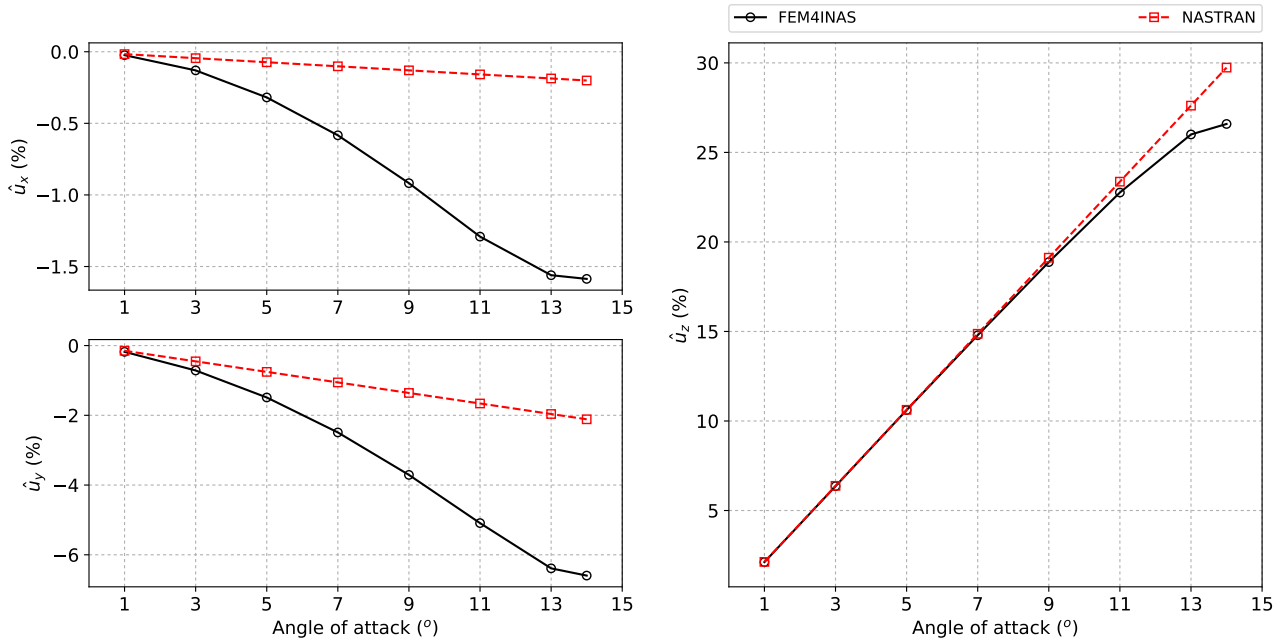


Figure 7.6: Aircraft wing-tip deflections in percentage of span length VS angle of attack

7.2.2 Nonlinear aeroelastic trim

Next in the analysis we consider the trim state of the aircraft for accelerations going from 1g to 3.5g. Even though the upper limit for design is 2.5g, this is a conventional aircraft not conceived to undergo very large deflections so that design limits have been surpassed to display strong nonlinearities, comparable to those expected within the design limits in next generation aircraft with ultra high aspect ratio wings and very flexible structures. Thus the aircraft trimming configurations are shown in Fig. 7.7 for 1g, 2g and 3.5g loadings. The comparison is made on the reduced model between MSC Nastran linear solution on the full model (SOL 144) and a NMROM built with 70 structural modes –based on the convergence exercise above. The flight conditions are $q_\infty = 13313.5Pa$ and $M_\infty = 0$. Note that in this simulation the aircraft is not clamped but includes the rigid body modes corresponding to pitching and plunging, all of which is automatically taken care of in our implementation.

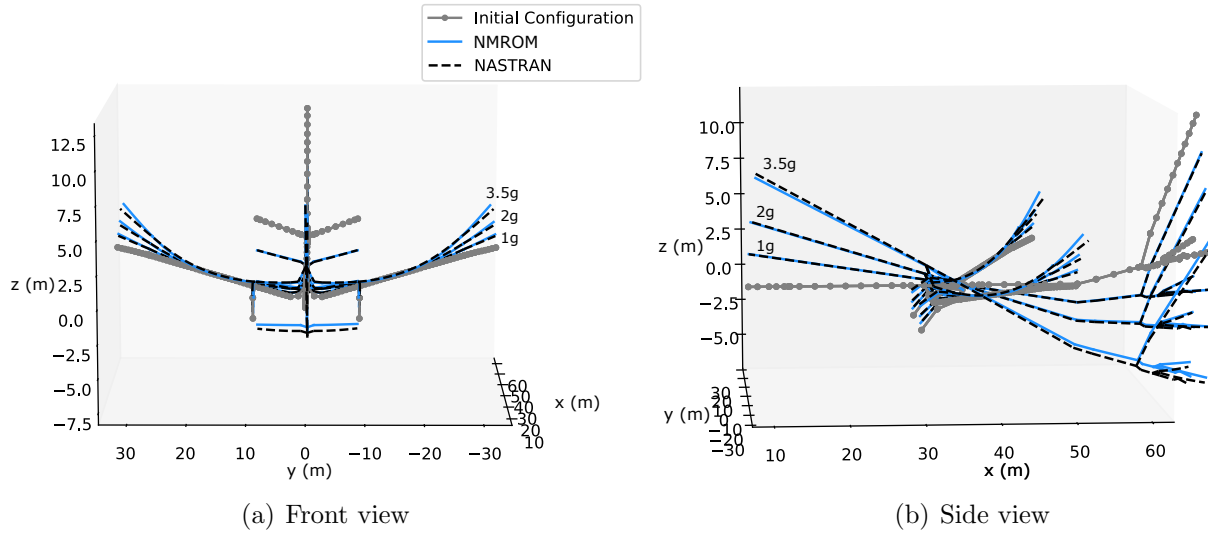


Figure 7.7: 3D trimmed configuration of aircraft under increasing gravity accelerations

Fig. 7.7 shows that nonlinear effects are negligible at 1g and some discrepancies appear at 2g which are intensified at 3.5g. Perhaps the differences between the linear and nonlinear analysis are better illustrated by showing the key figures in controlling the overall lift and moment of the aircraft, i.e. the AoA and elevator angle at the trim condition. These are shown in Fig. 7.8 with the added computation of a new flight condition with $q_\infty = 7076Pa$ and $M_\infty = 0$. We see a very good agreement for small loadings and a slight decrease in the AoA for higher loadings

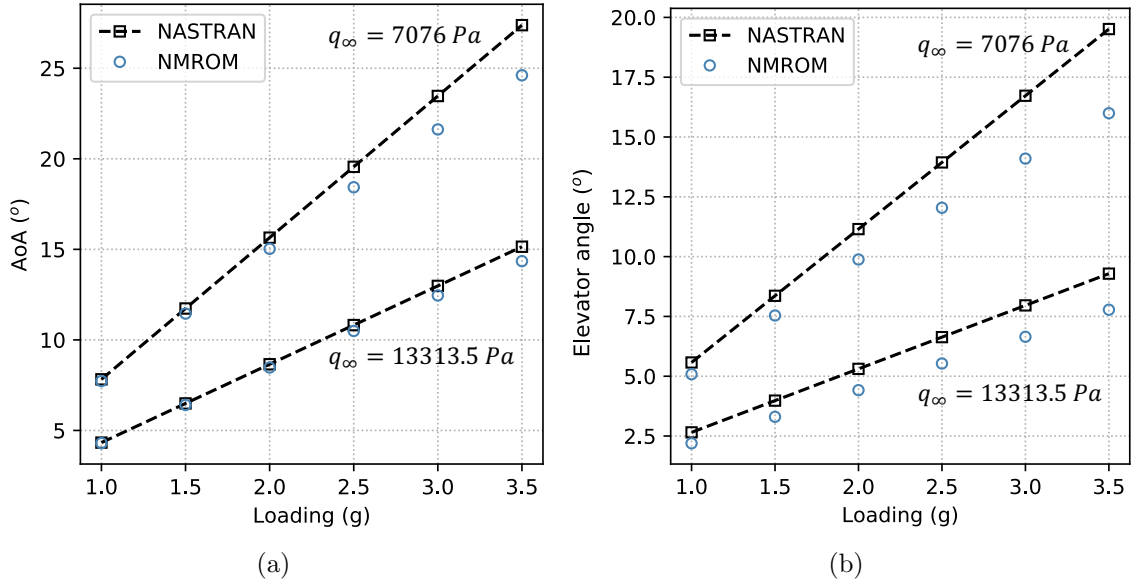


Figure 7.8: Aircraft angle of attack and elevator deflection

with respect to the linear computations. Differences are more significant in the elevator angle, specially for higher AoA . This might be due to the correlation between the decrease in AoA and elevator angle, and also the effect of follower forces in creating the required arm moments. Because the trim condition is achieved through a dynamic simulation that converges to a steady state, we can show some figures of merit in the time-domain simulations such as angle of attack, tail elevator deflection or the z-component of the reference point, x_c , that the linear controller aims to put to zero. These are shown in Fig. 7.9. Moreover, the controller behaves differently

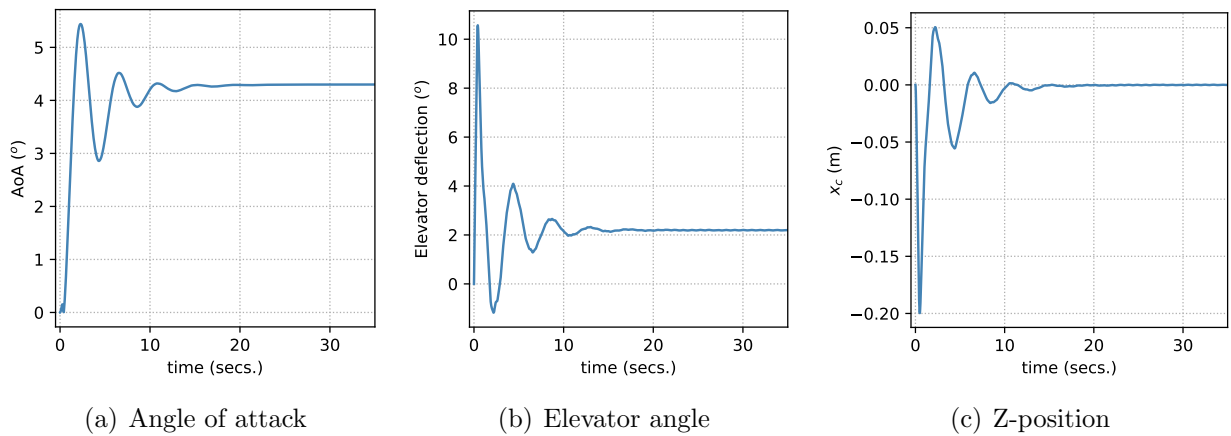


Figure 7.9: Evolution in time of key variables in the 1g flight trim simulation, $q_\infty = 13313.5$, $M_\infty = 0$

depending on the flight conditions but the important metric to consider is that the x_c point

is nearly zero and with no oscillation, i.e. the steady state is converged to zero total forces and moments. For instance, Fig. 7.10 illustrates the resultant of a simulation where the linear controller struggles more to find the equilibrium. We can see the overshoot in both figures and some oscillations until the steady state is attained. With some work on the controller design these responses could be improved so that shorter times are attained to find the equilibrium position. This is not within the scope of this work and, in fact, it could be set as part of a broader future exercise in which a nonlinear controller is explored not only to optimise the trim equilibrium but also to perform optimal load alleviation under large deformations.

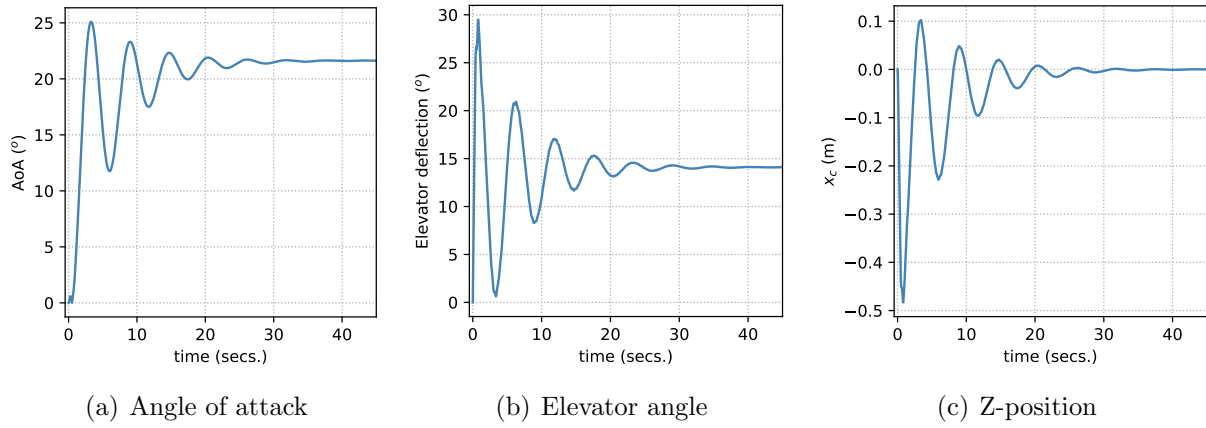


Figure 7.10: Evolution in time of key variables in the 3g flight trim simulation, $q_\infty = 7076 Pa$, $M_\infty = 0$

7.3 Analysis of the aircraft dynamic loads

Characterisation of the overall aircraft dynamics involve an added level of complexity compared to the steady state calculations. We focus here on gust disturbances of relatively simple shapes under clamped and trimmed conditions, but the analysis could also be extended to dynamic manoeuvres and turbulent atmospheric fields.

7.3.1 Gust response

If the previous two exercises were aimed at predicting the aeroelastic static equilibrium of the aircraft, we set out now to study its dynamic response under linear and nonlinear conditions. It is important for certification to analyse the effect of atmospheric disturbances in the aircraft

loads. Control strategies and load alleviation techniques do also rely on having a good description of the aircraft dynamics in flight. A simple representation of atmospheric disturbances is the $1-\cos$ gust type of shape, constant along the wing span and travelling in the x-direction with that particular shape. This type of gust will be used to validate the approach by comparing to the linear results from MSC Nastran, which performs the calculations in the frequency domain and converts them to the time domain via an inverse Fourier transform.

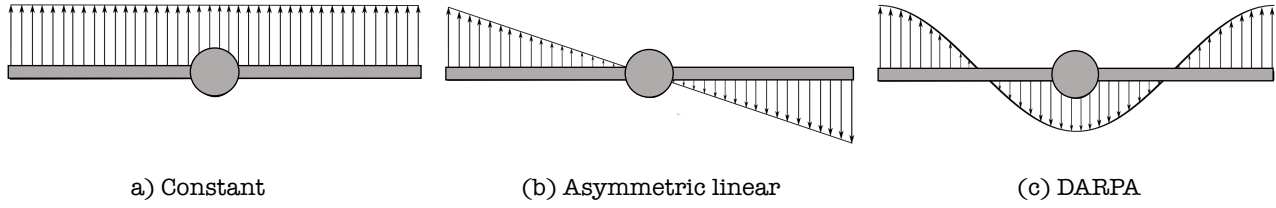


Figure 7.11: Span-wise gust shapes

Arbitrary gust shapes, even random type of signals, can be input within the current framework too, so this will be demonstrated with the span shapes illustrated in Fig. 7.11. The travelling wave in the x-direction remains a cosine shape for convenience, but any other type could have been used. Hence the gust profile in Eq. (3.35) is used with $b(y)$ catering for the shapes across the span just described. The constant shape represents the classical $1-\cos$ disturbance, the antisymmetric linear is good to excite antisymmetric modes and for assessing rolling dynamics, and the DARPA shape gust features a cosine shape that can also excite more complex dynamics. We begin validating the gust input system as well as the the aeroelastic dynamic solvers by comparing the dynamic response of the clamped aircraft for a range of $1-\cos$ gusts with small intensities. As per CS-25 and FAR regulations, gust lengths must be within the range (in meters) of $L_g \in [18, 214]$. Based on this, four types of gusts are selected, with the following gust lengths, $[1 : 18 \text{ m}; 2 : 67 \text{ m}; 3 : 116 \text{ m}; 4 : 214 \text{ m}]$, gust intensity of $w_{g0} = 0.01$ in Eq. (3.37). The flow conditions are air speed $U_\infty = 185 \text{ m/s}$, air density of $\rho_\infty = 0.778 \text{ kg/m}^3$ and the Mach number is set to 0, so no compressible effects are accounted for. A very good match is found considering the differences in the approaches. The wing-tip normalised displacements are shown in Fig. 7.12,

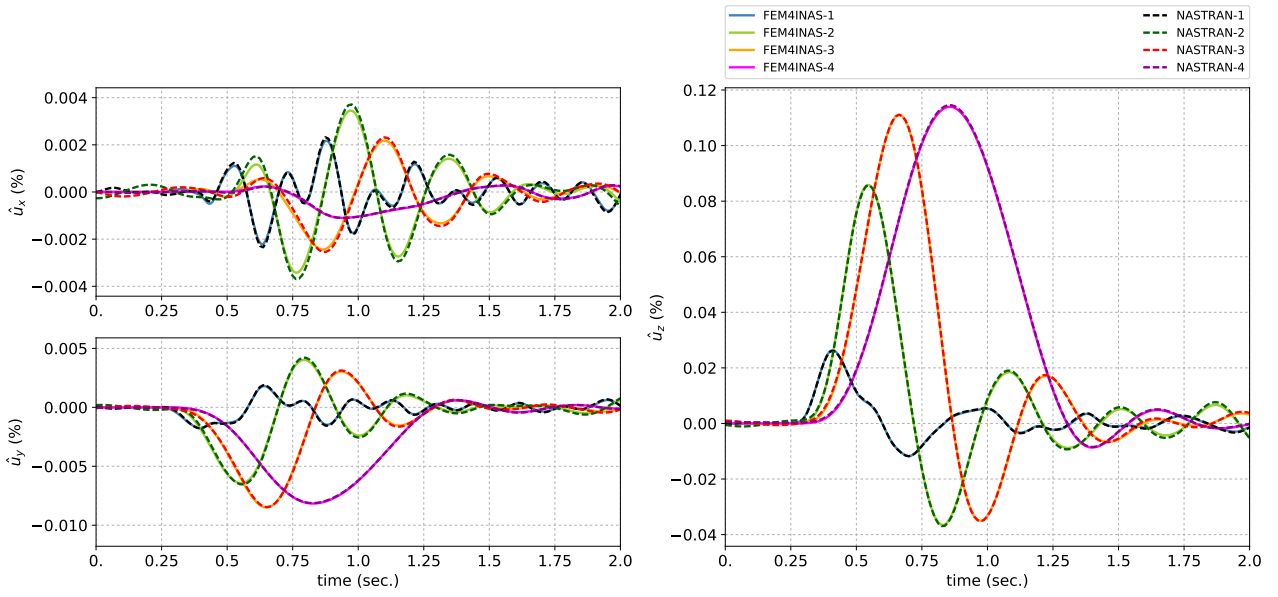


Figure 7.12: Wing-tip dynamic response for low intensity gust at Mach 0

The gust intensity is then increased so that nonlinear effects become relevant. The gust intensity is set to $w_{g0} = 3$ for gust number 2 and to $w_{g0} = 2$ for gust number 4 and results are presented in Fig 7.13.

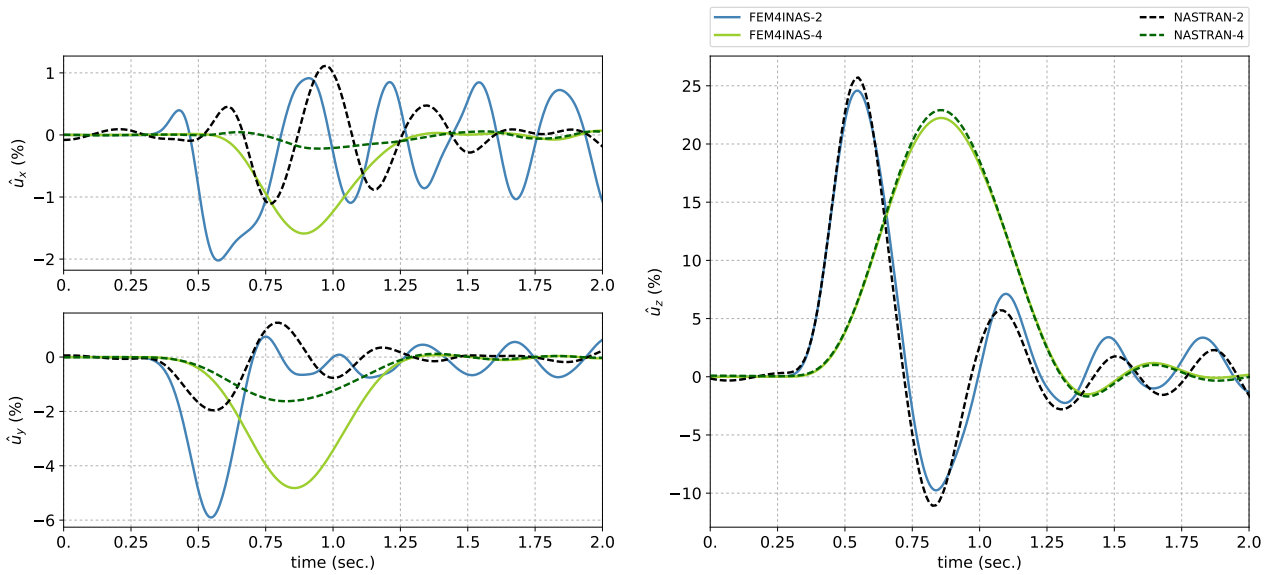


Figure 7.13: Wing-tip dynamic response for high intensity gust at Mach 0

As expected, larger discrepancies are seen in the x and y directions and a slightly smaller component appears in the z direction of the nonlinear computations. It is also apparent a frequency shifting do occur with respect to the linear results specially under gust 2, which has been reported previously in geometrically nonlinear dynamic studies of simpler problems

[52, 53].

The introduction and verification of compressible effects was performed replicating the exercise in Fig. 7.12 with gust number 3 but including results from a simulation at 0.81 Mach number. A very good match is again found and it is shown in Fig. 7.14 that the compressible effects act to increase the overall loading on the wings.

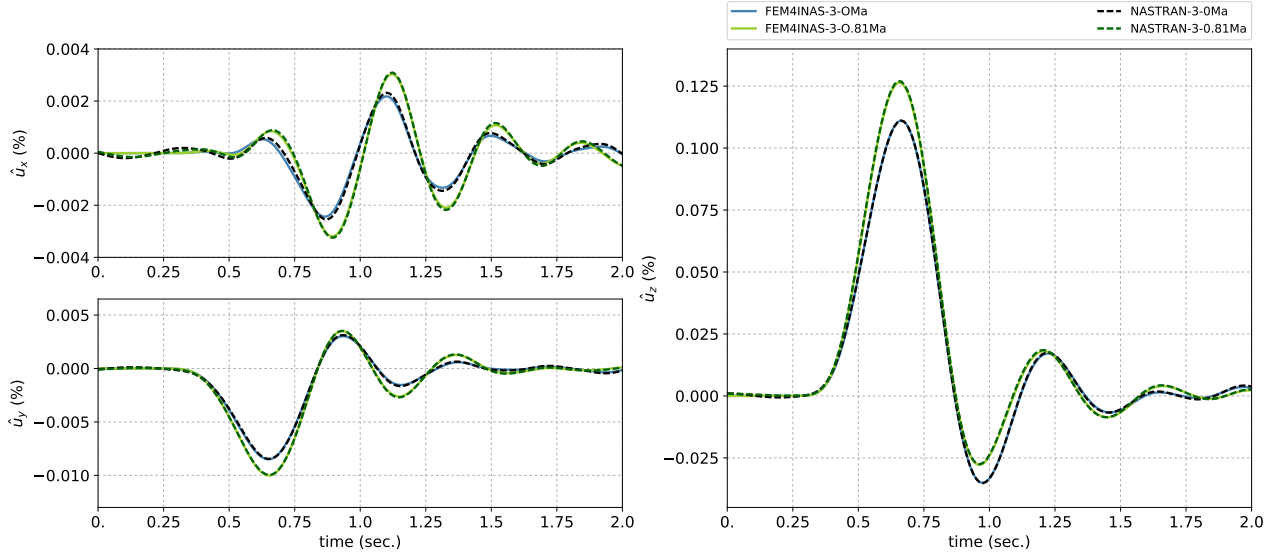


Figure 7.14: Wing-tip dynamic response for low intensity gust at Mach 0-0.81

Similar to the previous exercise, comparisons are presented for large displacements by setting $w_{g0} = 2$ for gust 3 at $M_\infty = 0$ and $M_\infty = 0.81$. If compressible and geometrically nonlinear effects are analysed separately, it may be said that the former has a more important effect in the z -component of the displacements, i.e. in the module of the aerodynamic loading; while the latter changes the response in x and y components more significantly, i.e. in the direction of the loading. It should also be noted that the dynamics excited by these type of disturbances, mostly driven by the first bending modes, are not very complex and very different effects are expected in more involved exercises as it could be the combination of dynamic manoeuvres with random atmospheric disturbances. In this initial phase, however, the primary goal is to demonstrate that linear results from commercial loads packages are replicated for small deflections and that geometric nonlinear effects are seamlessly incorporated when significant deflections take place. Moreover, the approach has to work with the full FE models used in load analysis and take into account compressible effects. These characteristics combined are not met by current state-of-the-art methods.

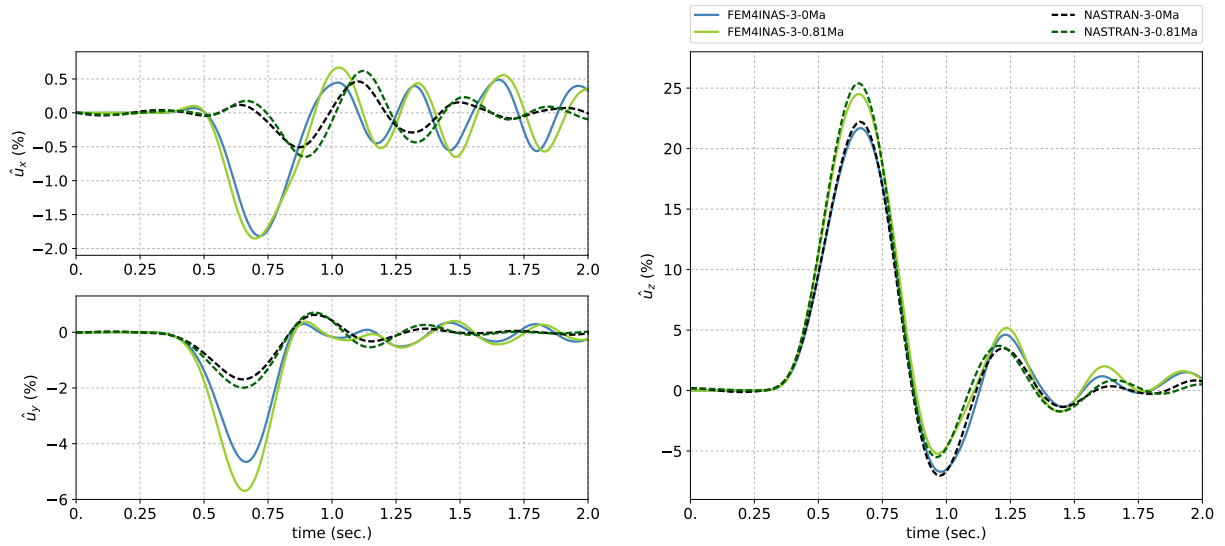


Figure 7.15: Wing-tip dynamic response for high intensity gust at Mach 0-0.81

Next we are carrying out simulations with gust shapes (b) and (c) in Fig. 7.11. The gust length is 67 m., air density of 0.778 kg/m^3 and speed of 185 m/s, and 0 Mach number. The gust intensity is $w_{g0} = 7$ in the asymmetric gust and $w_{g0} = 10$ in the DARPA gust in order to achieve maximum wing tip deformations within 27-33 % in both cases. Instead of plotting those wing displacements, the comparison of moments at the wing root is shown in Fig. 7.16 as a more relevant metric for design. The plot corresponds to the first second of simulation when the gust acts and maximum loads appear in the structure.

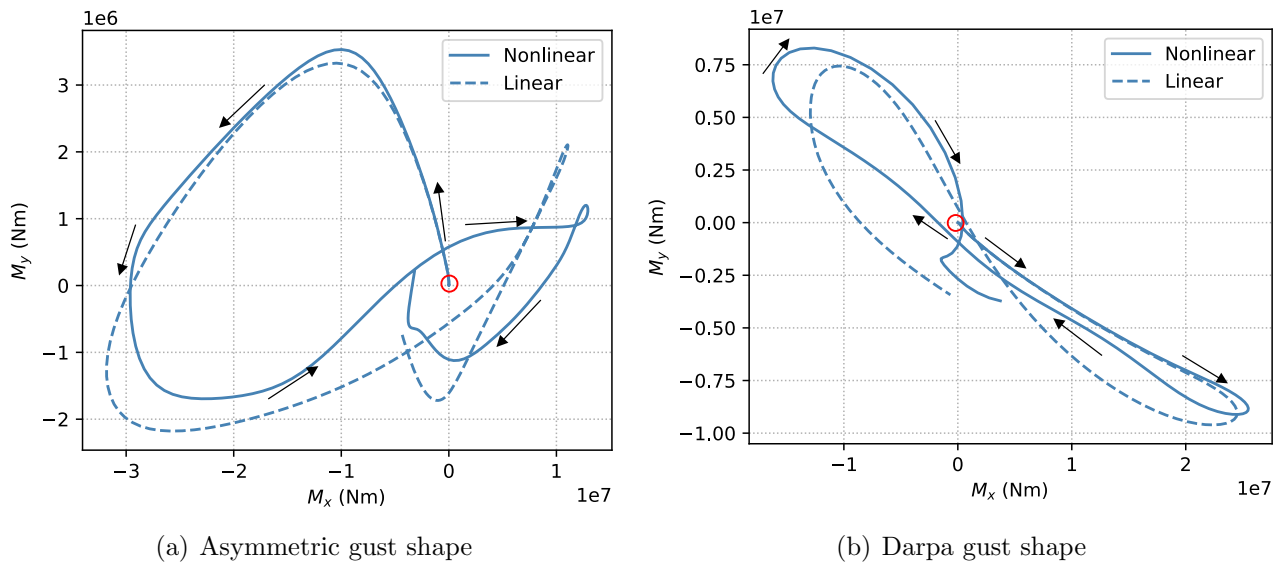


Figure 7.16: Wing root moments evolution due to gust disturbance

Linear and nonlinear comparisons, both carried out within FEM₄INAS framework, show that

linear analysis may produce some non-conservative critical loads, which could lead to failure, or overly conservative in other design points, which could mean heavier than necessary structures. This is a known effect in the literature of high aspect ratio, long endurance aircraft [16], although it has been rarely explored on large transport aircraft. The next step in the analysis would be integrating the approach into an automatic process such that thousands of simulations can be run automatically in parallel, maximum loads extracted and the final load envelopes built comparing the linear versus nonlinear assumptions across all flight conditions. Yet we are showing the methods can deal with industrial models, not trying to solve the entire industrial process.

It is interesting to look at the force/moment modal component, q_2 , for these two excitations, which being (span) asymmetric one and symmetric the other excite different dynamics. Firstly in Fig. 7.17 the q_2 modal evolution of the first 9 modes during the initial 2 seconds is shown for the DARPA gust.

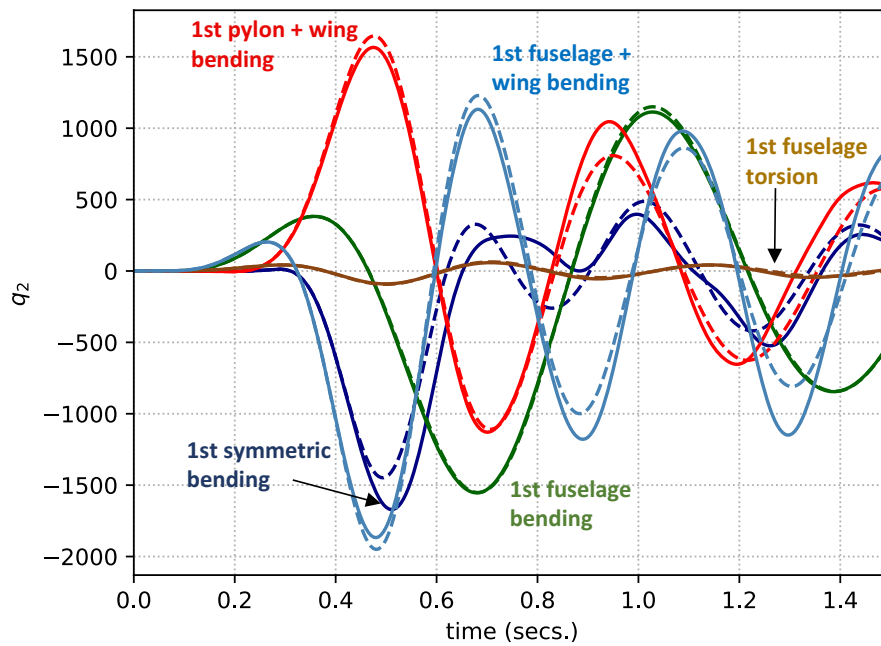


Figure 7.17: q_2 modal components for the DARPA gust excitation. Linear and nonlinear results in dash and solid lines respectively

Modes within the plotting range that have a nearly null contribution are not shown. Differences in the modal amplitudes are clearly visible as well as the frequency shifting built up in time. Some nonlinear frequency superposition is seen in the first symmetric bending mode and a small contribution from a fuselage torsional mode is surprising given the symmetrical nature

of the excitation. The same plot is cast for the antisymmetric gust shape, illustrating a richer modal response. In addition to the previous statements, it can be remarked that two modes are excited in the nonlinear response much more than their linear counterpart. Moreover one of them is the first symmetrical wing bending mode (to a pure asymmetric excitation), which shows that the nonlinear couplings may cause excitation of a mode that was not initially excited by the external forcing term. In fact the response of this mode takes a slightly longer time to develop than the other modes, such as the wing asymmetric modes. It might also be the case that possible asymmetries in the model itself magnify when large excursions take place.

One problematic of nonlinear analysis unveils in these plots: selection of the modes used in the solution a priori from the type of excitation is not easy or even recommendable as couplings between modes or intrinsic features of the model, such as asymmetries, can lead to unexpected dynamics. It will be a matter of expertise and experience of the engineer to be able to cook a solution with the minimum number of selected modes, hence building smaller NMROMs while capturing the pertinent responses. Otherwise a convergence analysis will have to be carried out on every different excitation or perform the calculations with a much larger modal base than necessary (as it is done here).

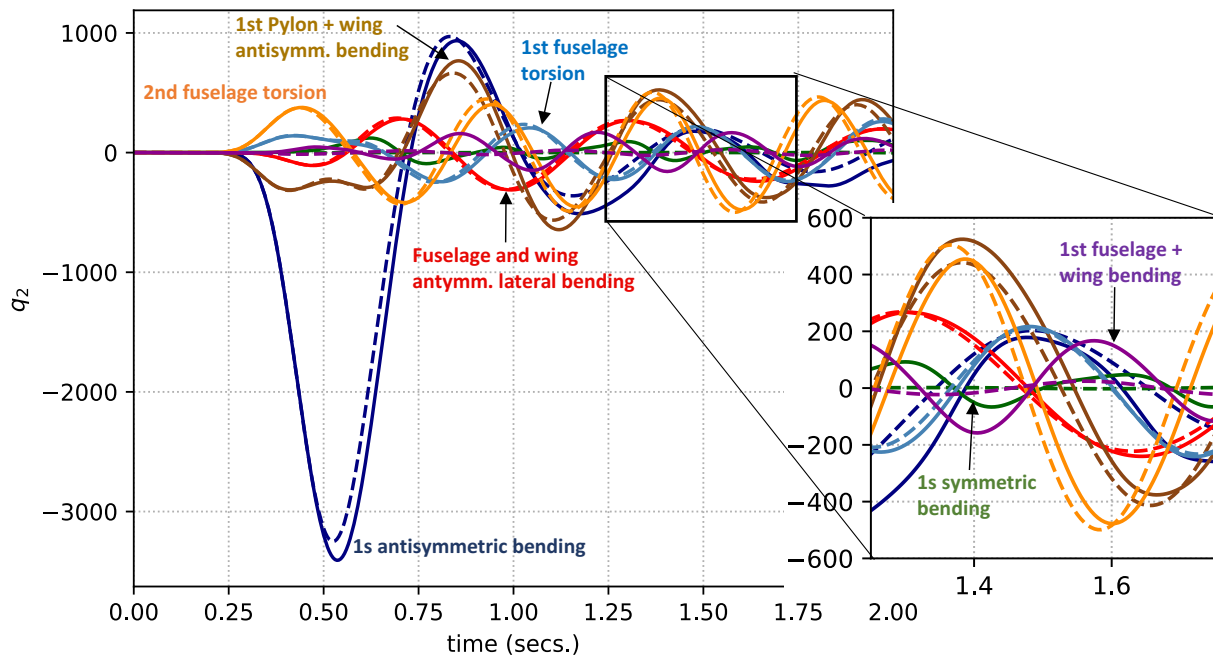


Figure 7.18: q_2 modal components for the antisymmetric gust excitation. Linear and nonlinear results in dash and solid lines respectively

7.3.2 Open-loop response

Previously the gust response of the modified XRF1 clamped configuration was studied. A further step into a more realistic analysis is taken by combining the trim configuration with a gust excitation in an open-loop setting, i.e. without any control input other than the fixed elevator angle to attain trimmed flight. The trim state is calculated for $1g$ level-flight. The flow conditions are air speed $U_\infty = 185$ m/s, air density of $\rho_\infty = 0.778$ kg/m³ and the Mach number is set to 0. The gust length is $L_g = 18$ m and the intensity $w_{g_0} = 7$. We first show in Fig. 7.19 the overall z -component of the wing-tip normalised displacement and after removal of the rigid-body dynamics (represented as the dynamics of a point in the center of the wing-box).

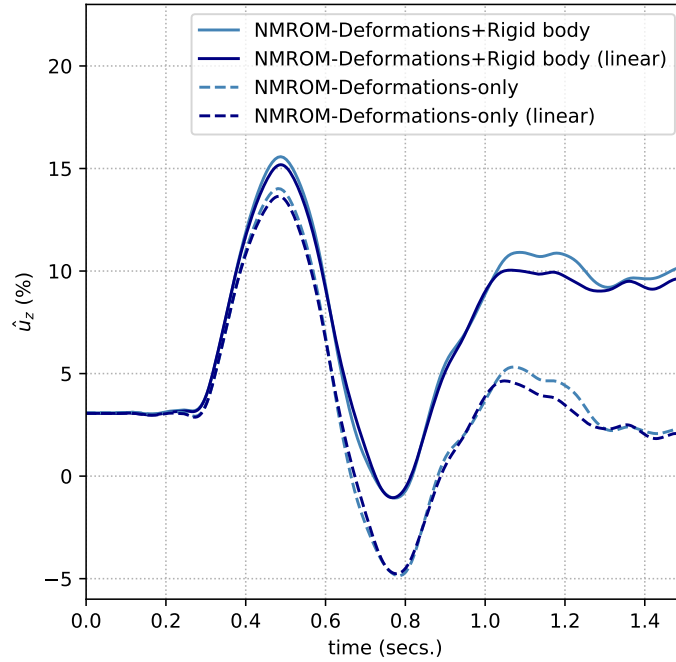


Figure 7.19: Normalised wing tip z -component for the open loop simulation with $U_\infty = 185$ m/s, $\rho_\infty = 0.778$ kg/m³, $M_\infty = 0$

Next we check the loads both at the wing root as a function of time and across the span for the load peak. They are shown in Fig. 7.20, with slightly smaller loads found in the linear analysis. It is also noticeable the jump in the loads at the engine location. We should also remark that the gust loads enter the system as follower forces, i.e. follow the geometry, but their module is not updated with deformations. To fully account for this geometric effect the dot product should be taken on the normal of each panel with the upwash from the gust. While this is

doable within the current framework, it is deemed a second order effect and it will be a matter for future work to further assessed its impact in the overall solution.

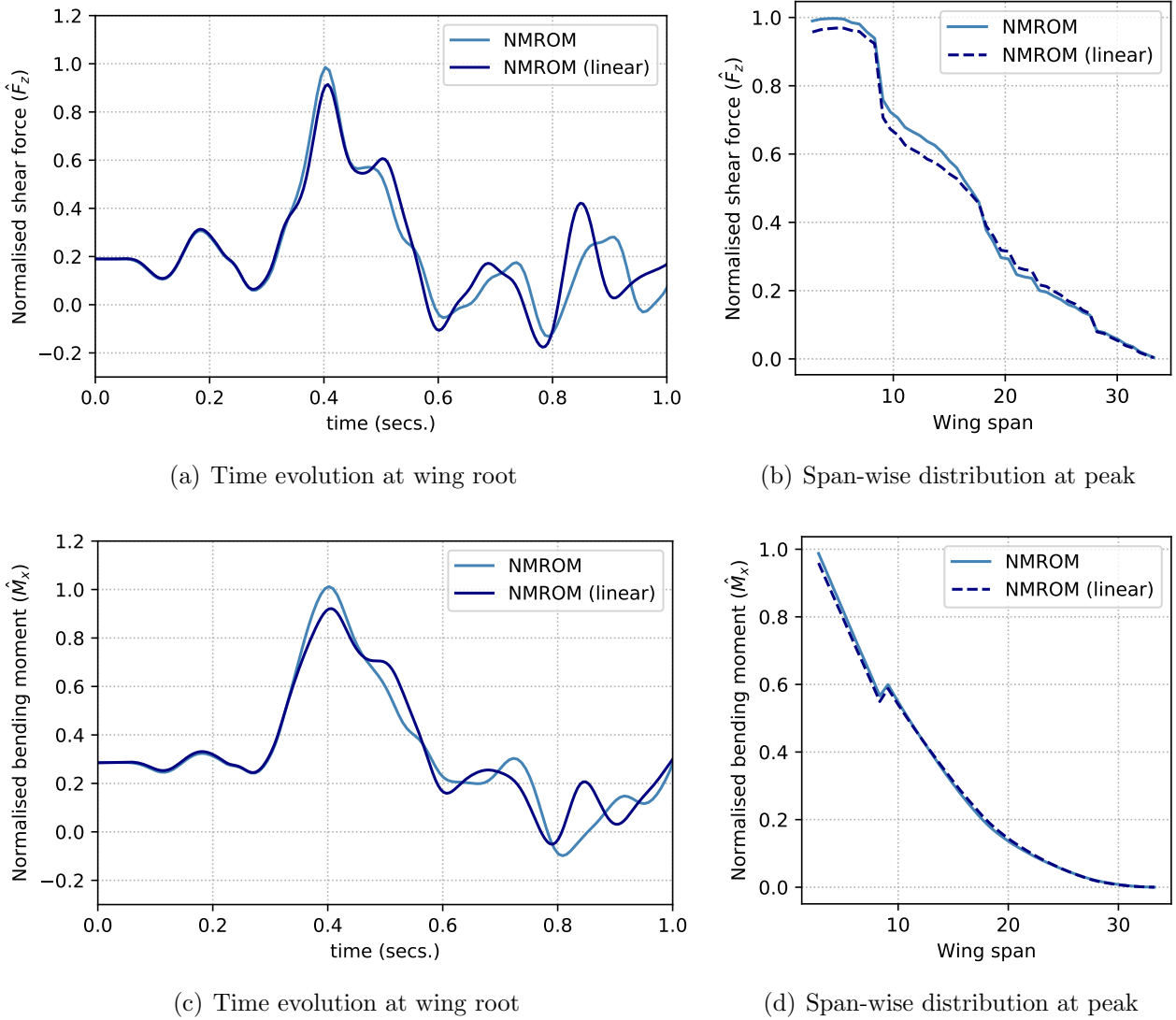
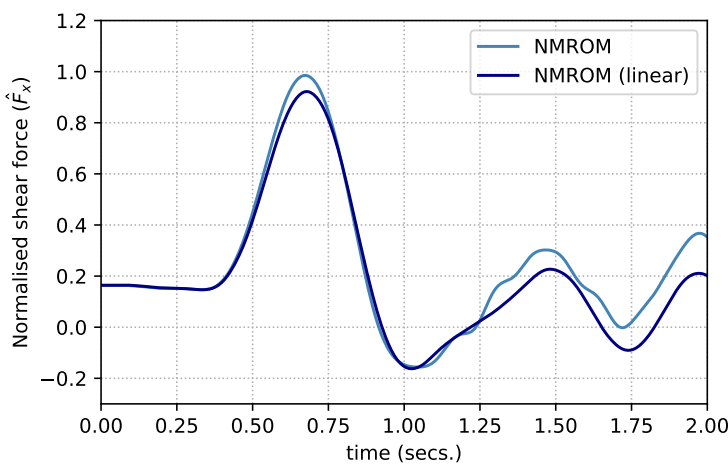


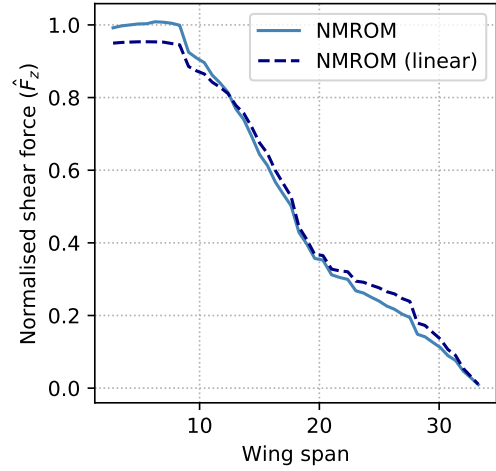
Figure 7.20: Normalised wing bending moment and shear force after gust disturbance on trimmed aircraft, $L_g = 18$ m and $w_{g0} = 7$

The same exercise is run changing the gust length to $L_g = 116$ m and the intensity $w_{g0} = 3$. The overall response is smoother in this case, but apart from that, the behaviour of the loads is very similar. The higher loads in the nonlinear analysis indicate a different response when the rigid-body dynamics are included in the simulations to the case of a clamped aircraft. We see that linear calculations can lead to non-conservative dynamic loadings, for which safety factors are in place, of course, but could be problematic in more flexible designs. Such effects have also been reported in [16], and yet, further investigations need to be carried out on this

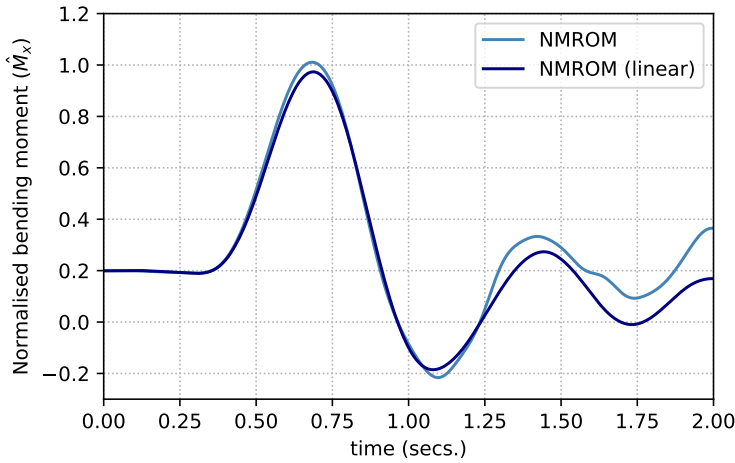
front to properly assess the coupling of trim and dynamic disturbances, and the causes for this increment in the loads from nonlinear analysis.



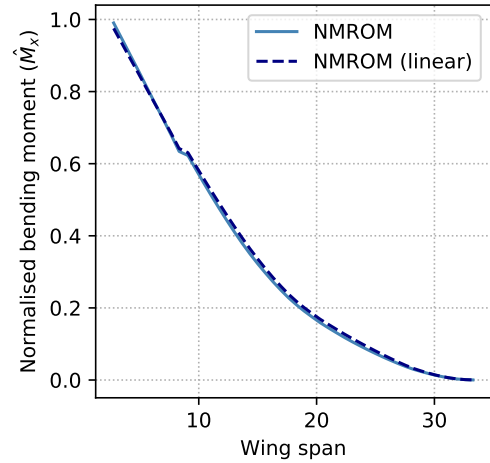
(a) Time evolution at wing root



(b) Span-wise distribution at peak



(c) Time evolution at wing root



(d) Span-wise distribution at peak

Figure 7.21: Normalised wing bending moment and shear force after gust disturbance on trimmed aircraft, $L_g = 116$ m and $w_{g0} = 3$

7.4 Nonlinear flutter

A critical effect in the design of flexible aircraft undergoing large deformations is that of *nonlinear flutter*: this dynamic instability can change its behaviour when the structural configuration changes due to large deformations, making the flutter point to significantly reduce, which could have drastic consequences for the airframe integrity. This has been shown in, for instance, the

work by Su and Cesnik [34], though in a model of less complexity. Similar to the previous studies, unrealistic angles of attack are used to obtain a proper assessment of the nonlinear effects but the same characteristics are expected to appear for smaller angles if a less conventional aircraft is studied with very slender and flexible components. Fig. 7.22 shows the reduction of the flutter speed with AoA and the resultant normalised wing tip deflections just before the instability occurs. The points of study have been $AoA = [0.1, 1, 4, 7, 10, 13]^\circ$, density of air $\rho_\infty = 0.5 \text{ kg/m}^3$ and Mach number $M_\infty = 0.81$. While this is an exercise to demonstrate the capabilities of the proposed tool and the effect of nonlinearities in changing stability boundaries, in order to properly assess the flutter envelope of this particular configuration we would have to run many more simulations considering all other combinations of density and Mach number that may be encountered in flight. The linear flutter speed has also been calculated in MSC Nastran using the PK method on the undeformed geometry, and a very good agreement was obtained with an error of less than 0.3% when compared to the NMRM for very small deformations.

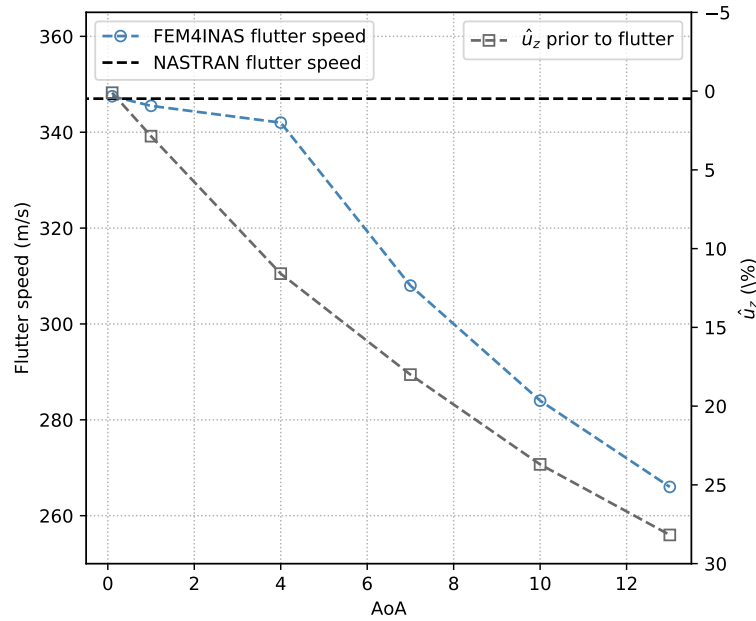


Figure 7.22: XRF1 flutter assesment at one design point: $M_\infty = 0.81$ and $\rho_\infty = 0.5$

The difference between the linear and nonlinear flutter speed at the 13° AoA point is of 23.5%. This could have major implications if it was to occur at an AoA of 5° instead. It is worth remarking that the flutter mechanism is not the same when the equilibrium state is close to

the initial configuration or when large deformations have significantly changed the geometry. It has been found that for $AoA = 0.1^\circ$ oscillations quickly decay or start growing at a rate which is not excessive as shown in Fig. 7.23(a). At $AoA = 10^\circ$, on the contrary, oscillations seem to stabilised into the equilibrium position, and all of a sudden grow to infinity in a very sharp manner after the nonlinear flutter velocity, Fig. 7.23(c), as oppose to Fig. 7.23(b) where equilibrium is reached. It is clearly seen that the post-flutter dynamics are not symmetric

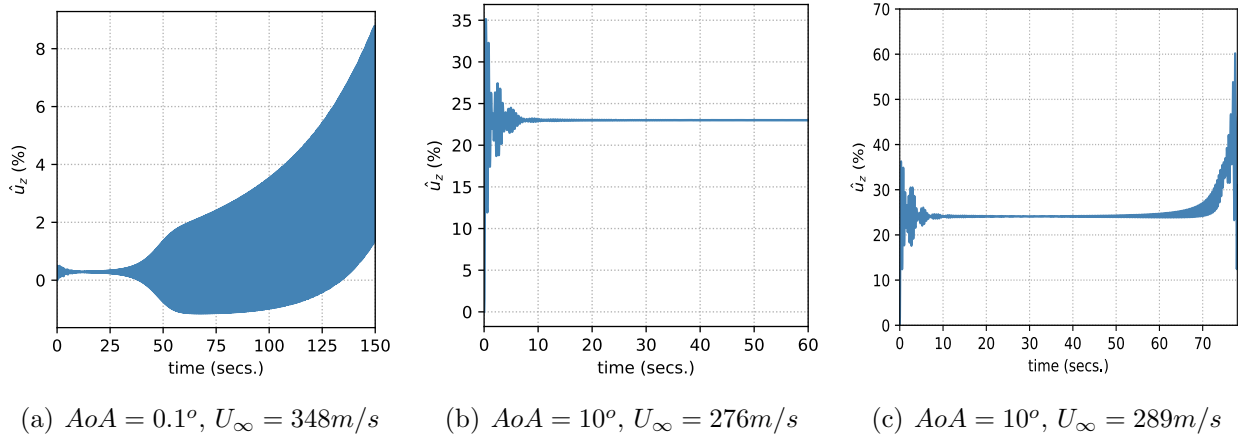


Figure 7.23: Wing tip deformations for pre and post flutter points. Flutter occurs at $U_\infty = 347m/s$ and $U_\infty = 284m/s$ for $AoA = 0.1^\circ$ and $AoA = 10^\circ$ respectively

around a pre-flutter equilibrium state. The first thing to note is that the wing structure is not symmetric, therefore neither it is the stiffness and upward forces do not produce equal displacements as equivalent forces pointing downward. Furthermore, the behaviour of the nonlinear dynamics might be non-intuitive departing from steady conditions, as it has been seen, for instance, in the development of LCOs in [127, 285]. This response certainly deserves more attention and analysis. From the point of view of design, it might also be beneficial to incorporate such effects as to implement strategies to delay the flutter onset.

7.5 Foldable wing-tips for load alleviation

The need to cut emissions and in particular the understanding that increasing the AR of aircraft holds a huge potential in boosting the overall flight efficiency, are pushing for innovative designs that depart from standard configurations. One of the technologies that could enable

higher aspect ratio wings is the Semi Aeroelastic Hinge (SAH) [181], a concept that has taken inspiration from the albatross bird, which can glide over large distances with a mechanism in the shoulder that locks for soaring and unlocks in steering and atmospheric encounters. The concept is translated to aircraft wings through a wing-tip that is attached to the main wing via a hinge mechanism and kept fixed during cruise flight. When a gust or an aggressive manoeuvre is detected, the wing-tip is released, hence acting as a passive load alleviation system. After the triggering event has passed, an actuator brings the wing-tip back to its original position [183]. The concept has been successfully demonstrated on the AlbatrossONE aircraft, a small scale prototype manufactured to study the characteristics of the SAH technology as described in [209]. Fig. 7.24 shows pictures of its flight at Filton's air field, which are available on the Airbus site².

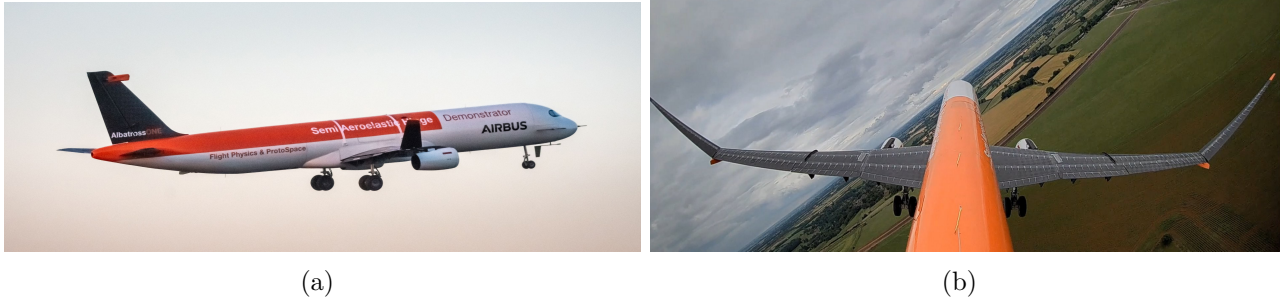


Figure 7.24: AlbatrossONE maiden flight

A preliminary study of this concept was carried out during a secondment at Airbus and time constraints have not allowed a thorough investigation. Fig. 7.25 features the 3D equilibrium of the condensed points in the main structure for AoA between 1° and 10° , flow speed $U_\infty = 120$ m/s and density $\rho_\infty = 1.225$ kg/m³, $M_\infty = 0$.

Differences grow with the AoA and a smaller overall deflection of the main wing becomes visible in the nonlinear computations as well as the shortening effect. Therefore the main goal is to show the potential of the developed multibody approach and give some insight into possible niches where nonlinear analysis would be required. It is important to note that linear aeroelastic analysis has been fundamental in proving the feasibility of this concept and in the design of the flying prototype. Yet this demonstrator was not designed as a HARW and did not exhibit large

²<https://www.airbus.com/newsroom/stories/freely-flapping-wing-tips-took-a-leap-forward.html>

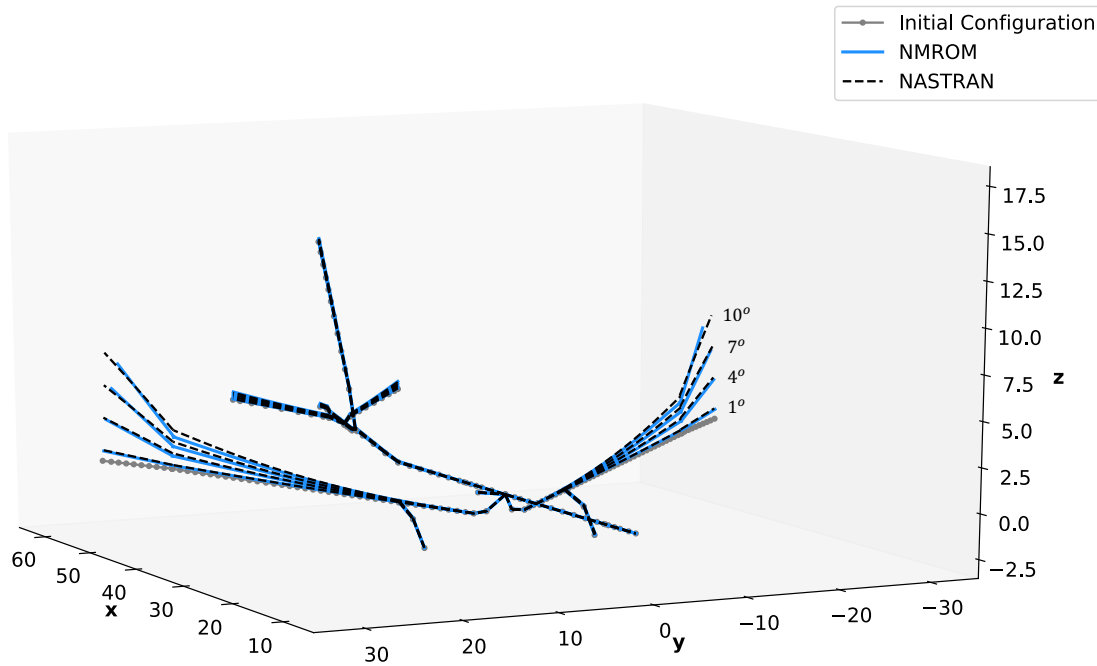


Figure 7.25: Static response of modified XRF1 with SAH technology for AoA variations

wing excursions apart from the wing-tip rotations (enough complexity was already involved in proving the technology alone). Thus a nonlinear strategy that can cater for the coupling between the large rigid-body rotations of the hinge and the large deformations of the main wing could be of paramount importance in further proving the viability of the technology and assessing its benefits.

One interesting aspect has been discovered with the free-hinge model: LCOs appear in the response at high enough speeds. For the fixed-hinge, it was shown in Sec. 7.4 how the flutter speed would decrease with deformations but LCOs were not apparent, the wing response would either damp or grow to infinity. Fig. 7.26 shows such a damped behaviour for $AoA = 1^\circ$ and a highly nonlinear LCO at $AoA = 8^\circ$ with flow speed of $U_\infty = 160$ m/s, density of air $\rho_\infty = 1.225$ kg/m³ and Mach number $M_\infty = 0$. The normalised z -component at the hinge point is plot, which offers a more stable point than the tip but still they show the undamped oscillations. This type of response requires further investigation as it is not captured by linear analysis and could lead to unexpected issues.

A combination of a steady angle of attack and gust loads are employed next to show the potential of this technology as load alleviation system. The angle of attack is set to $AoA = 4^\circ$

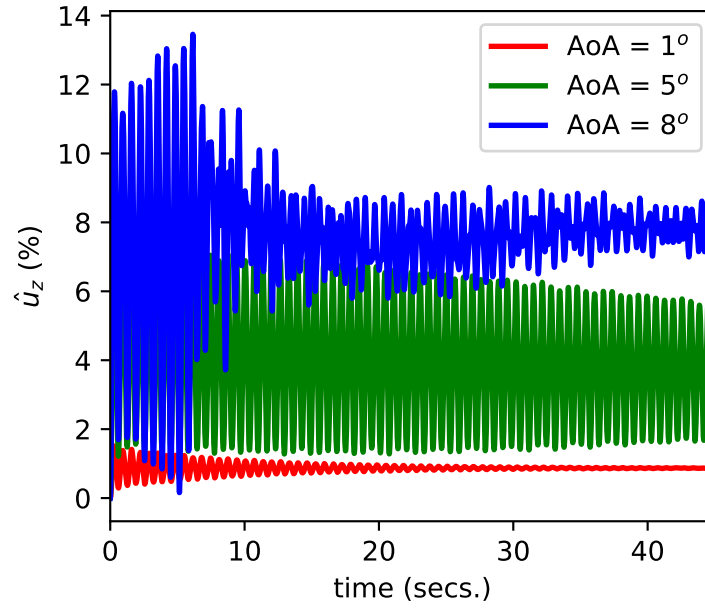


Figure 7.26: *AoA* equilibrium and LCO dynamics of the modified XRF1 with hinge device

and a *1-cos* gust is run with length $L_g = 165$ m and intensity $w_{g0} = 3$. Flow conditions are $U_\infty = 120$ m/s, $\rho_\infty = 1.225$ kg/m³ and $M_\infty = 0$. The wing-root bending moment and the vertical displacement at the hinge point illustrate that along the hinge axis moments are not transferred to the main wing, thereby reducing the overall loads and displacements. It is seen that the steady component (from the *AoA*) has a larger effect on the load reduction. The real behaviour has not been fully modelled as the hinge is intended to be fixed in normal flight and released on immediate contact with the gust. This would require a parameterisation of the terms in the equations such that at the appropriate time there is a switch between them, or a nonlinear hinge as studied in [292]. It is also worth remarking that oscillations with the fixed hinge damp much more quickly than with the free hinge. This is expected as the movable tip may excite higher frequencies and introduce more complex dynamics. Similar to the LCOs shown above, this needs to be studied carefully as it could lead to adverse effects that would not be captured by linear methods.

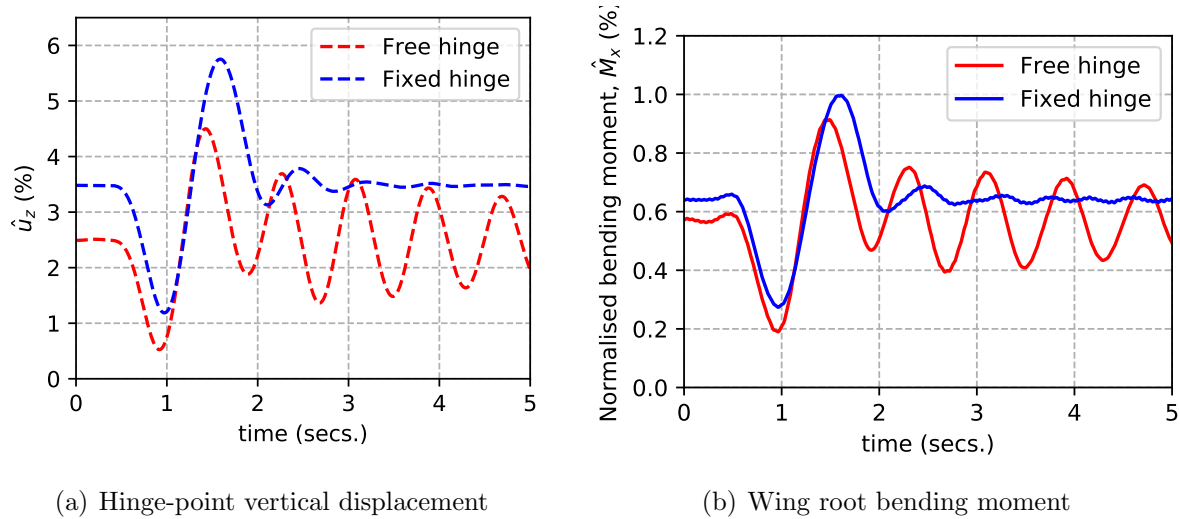


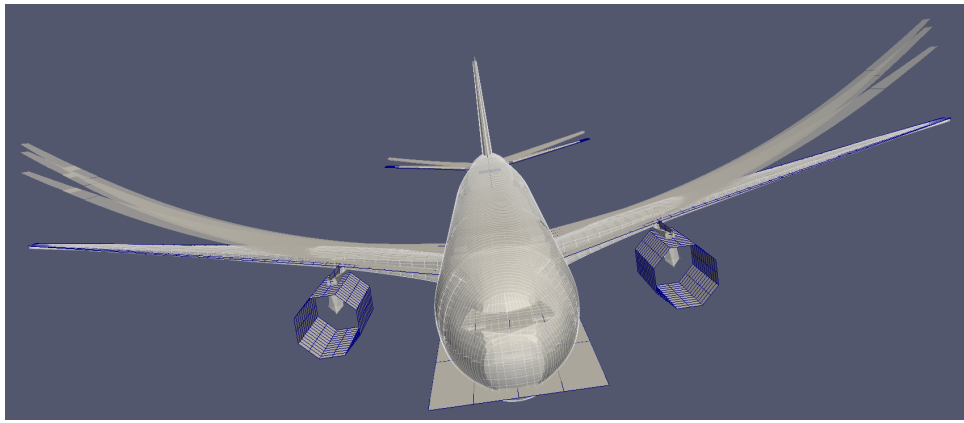
Figure 7.27: Load and displacement comparison on the fixed and free hinge model for $AoA = 4^\circ$ and gust disturbance.

7.6 Chapter summary

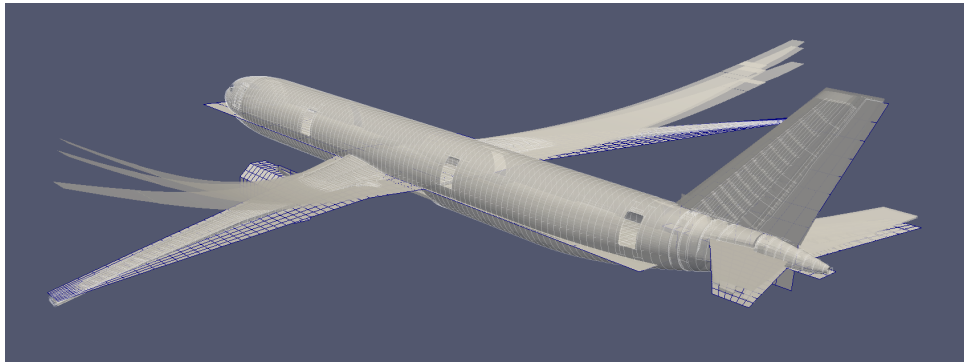
Along this chapter we have exercised the main characteristics of the methodology theoretically described in Ch 2-Ch 4. The XRF1 platform has served to show the suitability of the methods in industrial applications. Static manoeuvres for increasing AoA have been utilised to perform a modal convergence as deformations become large. The result of these simulations are represented in Fig. 7.28 after post-processing the equilibrium of the aerodynamic surfaces (without given specific values due to model confidentiality), and reflects the nonlinear behaviour of deformation³.

Computation of the aircraft trimmed state has revealed a reduction of the AoA in comparison to linear approximations. Gust disturbances have been verified for small deflections with commercial software, and follower force and wing shortening effects shown in dynamic simulations with large displacements. Differences have been found in the wing-root loadings for clamped and free conditions under linear and nonlinear assumptions, which back the need for coupled flight dynamics and aeroelastic simulations under nonlinear architectures. The flutter speed of the aircraft reduces quite significantly with increasing deformations –that were the result of

³Recovery of the 3D structural field could be accomplished using Sec. 2.4.2; and a further mapping into the 3D geometry using interpolation radial functions could lead to CFD type of computations.



(a) Front view



(b) Side view

Figure 7.28: Modified XRF1 fixed wing tip representation under AoA deformations

imposing an initial AoA in the stability problem. This would not be captured by linear tools and therefore having this feature in the analysis could be of paramount importance in flexible HARW aircraft.

Incorporation of multibody capabilities to the NMROMs has been accomplished in a novel implementation that makes use of Lagrange multipliers on the velocity field to impose the selected constraints between bodies. It has been partially tested on the modified XRF1 with a hinge connecting the wing-tip to the main wing, which represents a modelling simplification of the SAH technology. Preliminary results have been shown including the steady state response to AoA variations, the root bending moment reduction when compared to the fixed-hinge and the occurrence of an LCO. Fig. 7.29 presents the full structural model and the aerodynamic surface deformations for a steady case calculation. Further work needs to be carried out to properly assess the effect of geometric nonlinearities on the stability of this concept under large deformations and the benefits/limitations it may encounter when used in combination with a

HARW aircraft configuration.

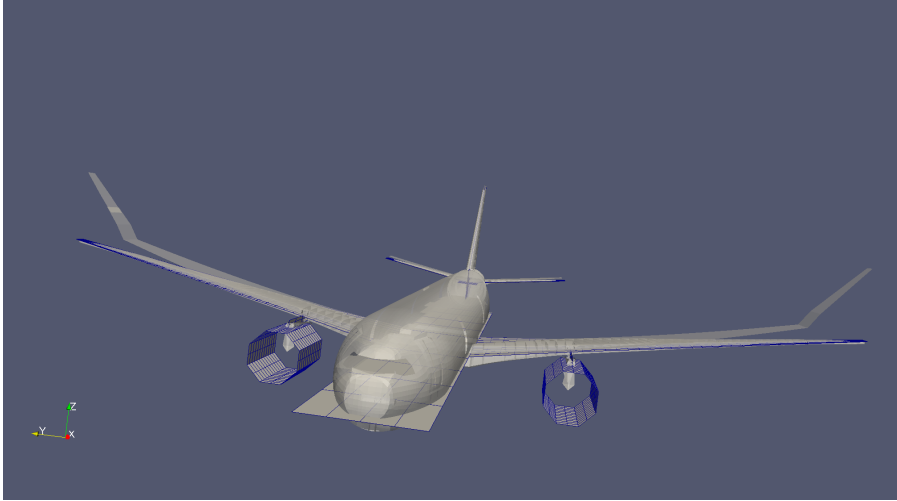


Figure 7.29: Modified XRF1 free wing tip representation under AoA deformation

Conclusions



Woman was always the custodian of human sentiment, morality and honour, and in these respects, man always has yielded woman the palm.

Maria Montessori

Happiness (is) only real when shared.

Christopher McCandless

Approaching the end of this dissertation, making a high level overview of what has been done turns pertinent, so that we do not fail to see the forest for the trees. The chapter begins with a summary of the researched matters and the numerical results that back the theory and show the novelties incorporated to the current methods of aeroelastic practice. We continue by highlighting both the theoretical and implementation achievements. Finally, suggestions are made to carry the present approach forward, since the methods need further consolidation, the computational code should be optimised and, as a result, applications could be broadened into other areas of aircraft design.

8.1 Summary

This dissertation started in Ch. 1 with an introduction to the past, present, and future prospects in computational aeroelasticity. A modal-based description, sustained on previous work by Wang et al. [214, 217, 238], was presented that seamlessly introduces geometrically-nonlinear effects on generic models built for linear dynamic analysis. The proposed approach preserves the linear solution, which is then augmented with the nonlinear effects in beam theory, namely geometric stiffening, follower forces, and changes of inertia properties. To achieve this, firstly, a condensed structure along the main load-path of the full 3D model is obtained using techniques of modal condensation. Secondly, the dynamics of the resulting skeleton-like structure are assumed to be driven by the nonlinear beam intrinsic equations, written in modal space. All the parameters in those equations are obtained through closed-form expressions involving the condensed stiffness and mass matrices, the linear normal modes and, critically, the coordinates of the grid points, which are disregarded in linear theory. This is described in Ch. 2 and linked to the first principles of solid mechanics. Thirdly, influence coefficient matrices, here obtained from the DLM, describe the unsteady aerodynamics in those modal coordinates. This results in a geometrically-nonlinear yet highly computationally efficient aeroelastic system that is presented in Ch. 3 together with the principles of potential compressible aerodynamics and some indications to improve the description. The three distinctive features of the proposed approach are that: 1) only linear analysis methods are used to manipulate the original 3D model; 2) the equivalent beam model is directly built in modal space with no need to explicitly identify sectional stiffness and inertia; and 3) the problem size can be reduced by selecting the number of modes included in the NMROM. Importantly, no model updates are required on either the fluid or the structure (for which only nonlinear quadratic terms are present), which greatly reduces computational time. In addition to these features, a novel multibody framework has been implemented for lower kinematic pairs and constrained trajectories in Ch 4. Subsequent chapters present the numerical results that might prove the present approach a good compromise of fidelities for the aeroelastic analysis of future transport aircraft exhibiting high aspect ratio wings.

In Ch. 5 a brief introduction is given into the software FEM₄INAS, which has been used for the numerical calculations in combination with MSC Nastran. Next, six numerical studies on simple models demonstrate the various aspects of the theory. The problem of a curved beam subjected to very large deformations has been utilised to show validation and improvements to the convergence of the solvers when quadratic interpolation of the intrinsic modes is introduced. A cantilever straight structure modelled independently with beams and shells, and with lumped and distributed inertia serves to show the impact in the level of fidelity of the original model. The proposed approach incorporates additional information, encapsulated in the reduced stiffness and mass matrices of the condensed model, into the nonlinear model. The methodology can also accommodate rigid-body dynamics and large overall motions as illustrated with a free-flying structure. Computational times were shown to be reduced by more than 60 % when the high frequency dynamics were approximated as algebraic equations. The response of a double pendulum falling under gravity provides a preliminary verification of the multibody implementation with spherical and hinge constraints. The results from the linearised approximation are also shown to give an estimation of its range of validity. While the previous five exercises have only dealt with structural dynamics, the last problem in this chapter involves an aeroelastic investigation into a high aspect ratio wing. Despite the linear aerodynamics from the DLM, very good approximations are obtained for moderately large deformations (over 20% of the wing span in our simulations). This is due to geometrically nonlinear effects, such as stiffening and follower forces, naturally being taken into account in the present formulation.

In Ch. 6 the complexity in the models is increased to be representative of aircraft structures. Firstly, a simplified aircraft was statically and dynamically analysed after being condensed into its main load paths via a Guyan condensation. A very good match was found between nonlinear (structural) static computations on the full and reduced models. Comparisons to static linear solutions showed the shortening effect in nonlinear computations as well as the lack of axial restraining in linear approaches. Dynamic calculations involving small displacements were in agreement with commercial software –which can only cope with linear computations on this sort of models with lumped masses. However, as displacements became large, the linear solution was not accurate and the frequency shifting in the dynamics was clearly seen. Secondly,

static and iterative dynamic condensation techniques were performed on a representative wing model. While there was an improvement in the natural frequencies of the full model from the iterative approach to the Guyan condensation, this did not translate to a significant impact on the nonlinear structural response: the wing was subjected to tip loads and the dynamics compared with the full FE linear and nonlinear analysis. While linear analysis failed to capture the response, it was found that a small NMROM sufficed to closely capture the nonlinear behaviour, which in turn led to dramatic improvements in simulation times. And thirdly, the aeroelastic stability and post-flutter nonlinear dynamics were calculated on a cantilever wing box. Flutter speeds were verified using the linear solvers of our implementation and MSC Nastran for Mach numbers between 0 and 0.85. When the nonlinear terms were included in the NMROM, an LCO was discovered and its evolution with the flow velocity described. This is an important attribute of the methodology: turning a ‘linear’ flag variable on or off leads to preservation of the originally linear aeroelastic model or a nonlinear solution with added characteristics respectively. A subset of the results in this chapter and the previous one has been published in [52, 170, 215].

A full aircraft model provided by Airbus and representing a large transport aircraft has been studied in Ch. 7. Numerical examples have shown the accuracy of the formulation in preserving the characteristics of the initial configuration. Linear aeroelastic calculations in MSC Nastran were in very good agreement with the NMROMs for cases where only small displacements took place. On the other hand, geometrically nonlinear effects such as follower forces, wing shortening, and frequency shifting of the dynamics were shown for trimmed flight under high acceleration loads, manoeuvres at large angles of attack, and dynamic gust disturbances. Differences in the wing loadings for clamped and trimmed conditions have led . Furthermore, the nonlinear flutter speed was investigated for increasing variations of angles of attack, which led to a reduction in the flutter speed of up to 23.5% when compared to the linear solution. These results have been partially published in [58] and will be fully available in [293]. To conclude the results of this chapter, we have included a preliminary study onto a load alleviation technology based on foldable wing-tips. The multibody framework in Ch. 4 was devised in order to tackle the problem of a flying aircraft with movable wing tips (attached through a

hinge joint) that are released to alleviate the wing loads in manoeuvres and gust disturbance conditions. The results include the steady state of the aircraft for different AoA , the reduction in the wing-root bending moments for the free-hinge, and the appearance of an LCO behaviour (something that does not show in the fixed-hinge configuration on which flutter speeds decrease with deformations but sustained oscillations were not found). The multibody formulation and a more thorough study of this device should be the subject of a future publication.

8.2 Key contributions

The key contribution presented in this dissertation is a new method to extend current industrial aeroelastic practice based on linear approximations with geometrically nonlinear effects stemming from the slenderness of components in new aircraft configurations. To this end, efforts are divided into three subgroups according to its theoretical and computational nature, and their application to numerical problems. It is believed that a good balance was accomplished between those three.

- **Theoretical advances:** these comprise additional features to a formulation that builds on previous works [213, 214, 217], and allows incorporating geometrical nonlinear effects to (linear) generic FE models.
 - Derivation of the intrinsic **beam theory from first principles of elasticity**.
 - **Generalisation of the formulation** to structures with distributed inertia and arbitrary condensation techniques.
 - Expressions for the **quadratic interpolation** of intrinsic modes that improve convergence.
 - **3D recovery of deformations** based on the full model LNMs to represent cross-sectional deformations.
 - Unified **nonlinear aeroelastic theory** for transport aircraft configurations and linearisation about a state of equilibrium. Methods have been devised for steady

manoeuvres, trim analysis, incorporation of arbitrary gust shapes, and stability analysis.

- **Multibody formulation** for the intrinsic modal solution and integration with linear, compressible aerodynamics.

- **Computational developments:** The program **FEM₄INAS** has been written from scratch to be used in arbitrary aeronautical structures with slender components in combination with a general FE solver, in this work MSC Nastran. Major milestones in its development have included the following:
 - Implementation of **Guyan, classical, and iterative condensation techniques** to reduce 3D FE models into a set of load-axis where nonlinear analysis takes place.
 - **Optimal** calculation of **intrinsic modes** and **parallelisation of nonlinear modal couplings** (cubic tensors obtained as integrals along the reduced domain).
 - **Geometrically nonlinear characterisation:** treatment of **finite rotations**, algebra of **quaternions** and management of rotational matrices, time and spatial **integration of the velocity and strain fields** respectively to obtain the reduced model configuration.
 - **Optimised RFA** to transform frequency-based aerodynamics into the time domain as accurately as possible.
 - Integration of **AICs-based aerodynamics** into the intrinsic formulation to build a **modular aeroelastic code** with nonlinear capabilities that include **follower and dead loadings, stability analysis, calculation of trimmed flight, steady manoeuvres and gust responses in the time domain.**
 - **Multibody framework** based on Lagrange multipliers to constrain the velocity set between bodies and coupling strategy with aerodynamic loadings.

- **Numerical applications:** a large number of problems have been solved in order to verify both the fundamental theory and its implementation. Highlights of the most relevant numerical solutions obtained along this dissertation are outlined next.

- Demonstration of the differences in the **dynamics of 3D FE structures** and their equivalent beams, which are **well captured** by the present method.
- The ability of **condensation methods** to **preserve the fidelity** of full aeronautical structures even for large excursions. **Good agreement** shown with the nonlinear structural response of a **full FE wing-box** while significantly **reducing the computational effort**.
- **Accurate static and dynamic aeroelastic computations** for moderately large deflections ($\approx 20\%$) when compared with a geometrically nonlinear aerodynamic solution.
- Assessment of **geometrically nonlinear effects on the XRF1** transport aircraft configuration: **wing shortening and follower aerodynamic forces in steady manoeuvres; decrease in the AoA to attain trimmed flight; dynamic gust perturbations with a slight shift in the driving frequency and change of the wing loads in the clamped and open-loop response of the aircraft.**
- **Post-flutter analysis** with **LCOs** described in representative wing models. **Non-linear flutter** calculations on the XRF1 model, showing a significant decrease in the speed at which the instability is triggered with large deformations.
- Preliminary **evaluation of foldable wing-tips** with a multibody nonlinear approach.

8.3 Open research questions

From the previous section we see that the research goals outlined in the first chapter of this work, Sec. 1.3.1, albeit ambitious, have been mostly fulfilled, thus highlighting the potential of the current approach. But they only lay the first stone of a much longer effort until the methodology proves useful and makes it into the actual design of new generation aircraft, thereby helping to improve their performance and safety. It is nonetheless a good exercise to highlight the limitations that have arisen alongside the attributes. Consequently, we further challenge the current work with some questions.

The approach has been tested on a significant number of problems but always using a single shot strategy whereby the results of each simulation were checked, *how is it integrated into larger environments that produce thousands of simulations and how is the pertinent data distilled from nonlinear computations?* Moreover, *is it really feasible to run such a large number of cases with nonlinear assumptions, or will error propagation methods and assessment of the most critical cases provide the grounds for a blended type of analysis, and how?* Increasing the accuracy of the aerodynamic models is an important step, *what is the best strategy to incorporate high fidelity aerodynamics into the approach in an efficient manner?* Given that one of the applications is dynamic aeorelastic computations, *how are nonlinear transonic effects, such as the movement of shock waves, accounted for or otherwise what is the impact of neglecting them?* Whether this is the right approach for control design needs to be explored in more detail, as it could be suitable for small systems but not ideal for very large ones, *what is the best strategy for designing an efficient feedback control system for industrial aircraft models undergoing large deformations?* If the applications are to be expanded from classic aeroelastic problems and assessment of loads to aircraft design and aeroelastic tailoring, *how are sensitivities calculated, communication between software implemented, and how does the optimiser wrap the overall solution?*

Based on this questions, we pinpoint more precisely in the next section what problems should be tackled right after this work for further consolidation and development. It must also be acknowledged that, as part of the research endeavour, the possibility exists that the proposed method stays as a more or less elegant solution that does not evolve into a practical application; either because it becomes obsolete and is surpassed by other methods; or because the conditions under which the approach stands out –i.e. computational efficiency, versatility to combine aeroelastic theories in large aircraft models, and accounting of geometric nonlinearities- are no longer a requirement, if computational power soars, for example, or if the current paradigm for building the aircraft mathematical models changes.

8.4 Recommendations for future work

Following the questions challenging this work, future efforts should be directed towards improving both the accuracy of the aerodynamic modelling and the performance of the computational code, as well as exploring new applications beyond aeroelastic load analysis. Therefore we review some possibilities along these lines.

- **Expanding the physics of the approach:** increasing the level of fidelity in the analysis while keeping the cost of calculations low enough for the target applications is the eternal problem of the engineer.
 - **Enhancement of aerodynamic models:** nonlinear effects coming from the change in the aerodynamic grid should be researched more thoroughly. An update of the DLM panels with deformations might be the first approximation for this and also for compressible effects to be accounted for; but certainly it would not be the most efficient nor elegant solution. AICs from CFD or multifidelity methods that are updated according to some strategy of deformations would be an option worth investigating.
 - **Comparison with CFD-CSM:** if CFD features are incorporated into the NMROM, it would be useful to compare with full nonlinear CFD-CSM calculations and assess the trade-off between missing physics and the computational savings of the resulting approach.
 - **Cross-sectional deformations fed into the 1D nonlinear dynamics:** a method to recover the 3D configuration state has been presented and should be numerically verified. There is however no model of the physics relating the effect of the 3D deformations into the 1D dynamics as in [45, 235]. This is not believed to be a relevant effect on aircraft structures, though it would be an exciting mathematical challenge.
 - **Further validation of multibody approach:** the multibody approach needs to be numerically verified for the coupling of large deformations within each body and large rotations between bodies. Especially to determine the efficiency of the

methodology as the problem is mathematically well posed but the solution could require a very large modal base that would hinder its usability.

- **Improvement to the flight dynamics:** the current description of the aircraft flight dynamics has been greatly simplified in order to account for the main characteristics in a dynamic loads environment. Nonetheless, the mathematics behind the formulation allow for a fully nonlinear flight dynamics description. Revision of processes such as the evaluation of the trim state or the introduction of the engine thrust force would be necessary. The approach also offers a good platform to couple with a nonlinear controller and to evaluate load alleviation techniques in extreme flight conditions of aircraft undergoing geometrically nonlinear deformations. This has already been proposed in an intrinsic framework formed of beams and 2D aerodynamics [212], though extension to full aircraft configurations would certainly be a challenge in terms of computational speeds and management of the model complexities.
- **Investigation of the foldable wing-tips technology:** continuing investigating such a promising concept for increasing the AR of wings should be one of the aims of the enhanced tools proposed in this section. These would include the aforementioned updating of aerodynamics and studies about the nonlinear stability behaviour of this concept, which might be critical for its success in real aircraft.
- **Computational efficiency:** Once the physics of the problem are well captured, there are some procedures that could greatly improve the performance of the methodology.
 - **Compiled language for heavy computations:** important computational savings can be accomplished by changing from a high level to a compiled language. The most demanding parts of the approach are the calculation of the nonlinear couplings and the solution of the ODE or DAE system. The former has been parallelised in Python and needs to be computed only once per model. Hence it is suggested that just the ODE system and solvers are written in a compiled language, then wrapped within the program in Python.

- **Time parallelisation:** as the solution of the ODE system is the most expensive computational part, it might be worth exploring parallelisation of the time domain solution with algorithms such as those described in [294].
- **Mode filtering:** the NMROMs have hitherto been built based on a cut-off frequency up to which every nonlinear term was evaluated regardless of the existence of modal couplings. A better strategy could be obtained by first determining the nonlinear coupling terms with a significant weight and discarding the other ones in the calculations.
- **Residualisation scheme extension:** the residualisation scheme by which high frequency dynamic equations are turned into algebraic constraints has proved to be an efficient technique to speed-up the calculations. It needs to be extended to the aeroelastic solvers as it was only implemented and tested in structural problems.
- **Towards aircraft design:** if the enhanced capabilities prove to be highly efficient and accurate even under large deformations and transonic flow conditions, a window of possibilities would open for aeroelastic design of next generation of ultra-efficient aircraft. Introduction of the approach into aeroelastic tailoring processes and MDO architectures could be a good added feature to existing options, incorporating dynamic aeroelastic solutions under ‘aggressive’ conditions, nonlinear stability constraints, control sub-problems and flight mechanics. An immediate application would be the exploration of radical new configurations, which could bring great benefits beyond what conventional designs can offer, at the expense of more complicated and expensive analysis and design phases.

Hence there is plenty of room for improvement in a methodology that hopefully has been shown to hold great potential too. Important efforts to consolidate and expand the approach will be needed, as well as input from industrial players to focus those efforts on adding value to real aircraft development cycles, thus meeting the environmental, economic and social demands highlighted at the beginning of this dissertation. Along the way, challenges that require innovative solutions are surely expected, which in turn will remark the relevance of ‘academic’ contributions to overcome these problems.

Appendices

Software Implementation



From my skin inwards, I am in charge. There begins my exclusive jurisdiction, and I choose whether I should or should not cross that border. I am a sovereign state.

Antonio Escohotado

Without music, life would be a mistake... And those who were seen dancing were thought to be insane by those who could not hear the music.

Friedrich Nietzsche

This appendix expands on the description of FEM₄INAS in Ch. 5 to give a more detailed overview of the numerical implementation in this work as well as guidelines to a potential user. Firstly, an introduction is given to the models built in MSC Nastran and to how the extraction of information from its outputs is carried out; secondly, some guidelines are presented for the initialisation of the main program and the required input files; thirdly, the aerodynamic and condensation modules, that calculate the FE and aerodynamic matrices utilised by the main program, are briefly described; and finally the solution workflow with the main modules is described.

A.1 Model generators and MSC Nastran compatibility

A range of tools have been written to create Nastran models and to read and manipulate the output files of this program. The `.f06` file is the main output file, and the binary `.op2` and text `.pch` files are optional. Within the folder `./utils` there are the necessary modules to read and perform different operations with the data in these files. Nastran internal language DMAP (Direct Matrix Abstraction Program) is also used to output FE and aerodynamic matrices into `.op4` files. Inside `./generators/modelGen.py` there are functions to create arbitrary aircraft geometries with beams, and simple wing-box structures with shell and beam elements. The modelling options include linear static analysis (SOL 101), modal analysis (SOL 103), geometrically nonlinear and multibody dynamics (SOL 400), and static-flutter-dynamic aeroelastic analysis (SOL 144-145-146). The initial intention for this module was to communicate with the PCL (Patran Command Language) and to be able to create general aircraft designs on which sensitivity and parametric analysis could be carried out, but this has proved to be out of scope. Nevertheless these tools have been utilised to build the simple beam and shell models studied in the following sections (the CPACS toolbox for MDO applications [288] and the multifidelity tool for conceptual design SUAVE [295] are more mature frameworks to generate arbitrary aircraft models). In `./utils/run_nastran.sh` there are `shell` functions that facilitate running Nastran by linking solution files to predefined shell options, automatically checking there are no FATAL errors in the `f06` output file –which is a red flag indicating something went wrong inside Nastran routines, and moving all the output files to a selected folder. Even though this is a general purpose FE commercial solver, there are some very distinctive features in the representation of aircraft structures that can be shortly summarised:

- The stiffness elements of aircraft GFEMs mostly comprise beams and shells (CBEAM, CTRIA and CQUAD entries) together with rigid connections via RBEs cards.
- The inertia is usually represented as lumped masses along the main loads reference axis, input with CONM entries. The MAT cards, which allow to input not only the elastic properties of elements but also the mass density to represent distributed inertia, are

another option.

- ASET entries define the master nodes for a Guyan condensation inside MSC Nastran.
- RBE3s (weighted average constraint elements) are used to set the DoF displacements of the reference grid points (placed in the a-set) as the average of the displacements of slave nodes (placed in the m-set) surrounding the master node. A problem encountered is that the reference grid point (REFGRID) of the RBE3 cannot connect to an ASET point or to a downstream super-element (a divided part of a large model that is analysed independently). To get around this, two implementations are proposed: one is to create two coincident nodes connected by zero length, very large stiffness CBUSH; the REFGRID point and the mass element are assigned to one and the other to the ASET. This type of penalty solution might yield ill-conditioned stiffness matrix in some cases. A more robust solution can be attained using the 'UM' option of the RBE3, which reassigns the DoF that will be placed in the m-set. The selection must be careful as the points must not be colinear or have other constraints set on them.
- DLM aerodynamics are created with CAERO cards to define the aerodynamic panels, MKAERO to select Mach number and reduced frequencies points, AERO prescribes the reference aerodynamic parameters in the simulation, and SPLINE entries are used to interpolate aerodynamic and structural DoF.

A.2 Code initialisation

The code is run with the main file `feminas_main.py` which calls `feminas_functions.py` and together control the flow of the program, whose location path should be on the Bash variable `$PATH` such that the main can be run directly. Input files are inside a folder called `Runs`. Two input files are mandatory, one for most of the variables and paths and other for the definition of external loads; and a third file that is not required for structural-only calculations manages the aerodynamic loads, trim calculations etc. These three files will be imported as modules and called inside the program as `V.` for the general variables, `F.` for the loads and `A.` for the

aero. Let us take a model called **X1** as an example: inside `./Runs/X1` there can be an arbitrary number of input files that are combined as required. A configuration file controls the input to be used in each simulation and whether to run the code or load data at the different stages of the process (for example, the intrinsic modes or the nonlinear coefficients need to be calculated only one time, then they can be loaded). This model would be run as `feminas_main.py X1 Config_X1.py`. Alongside the input files, the condensed FE matrices and the aerodynamic matrices need to be loaded into the solution procedure. Lastly, a text file pointed by the variable **Grid** contains the master nodes' locations and their belonging to a particular member frame. Nodes within each member are aligned and share an initial reference frame, $\mathbf{R}^{ab}(0, s)$, which is automatically generated with the x -axis running along the nodes direction. For curved geometries 2-noded frames are selected. The connectivities between frames are manually input in a list with the entries as given in Fig. A.1. This figure shows the main load-paths of the aircraft in Fig. 6.1 with the numbered member frames and its connectivities. One important difference is seen between clamped and free-free structures: the former requires the first beam to contain the clamped point and the latter the first beam to be at a free-end.

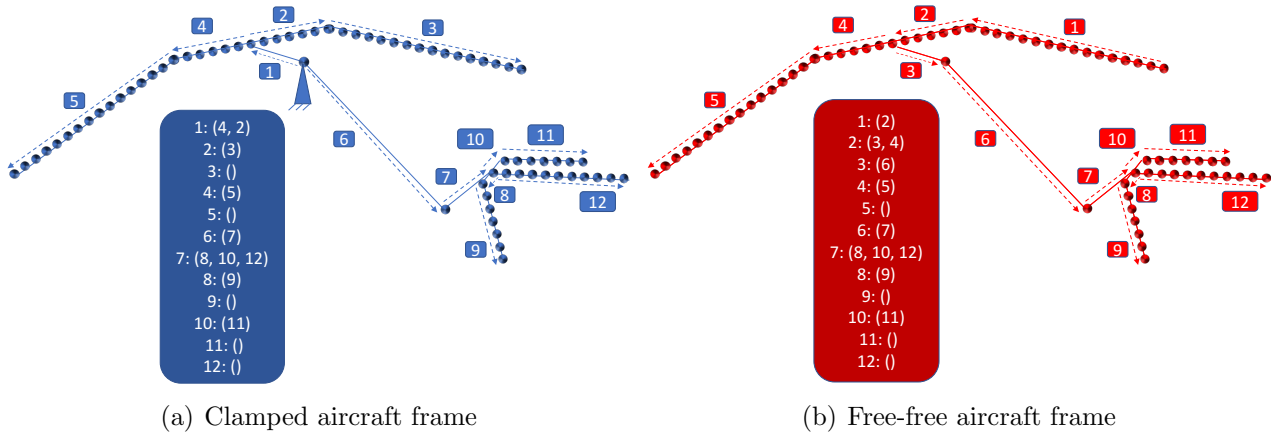


Figure A.1: Example of an aircraft frame structure

A.3 Condensation module

The condensation module takes sparse full FE matrices from a generic FEM and outputs its fully-populated, condensed equivalent. The code is inside a single file, `./condensation.condensation.py`, and it is rather rudimentary: the full stiffness and mass matrices are loaded with one a priori

condition, namely that the nodes where the model is condensed into hold the higher ID in the full FE model. That facilitates the division of the matrices which only requires the selection of the number of condensed nodes, N_a , to know that the last $6N_a \times 6N_a$ entries correspond to the active nodes and the other to the omitted nodes and couplings. The types of reduction techniques implemented are Guyan condensation, classical dynamic condensation upon selection of one frequency of interest, and iterative techniques as described in Sec. 2.2.

A.4 Aerodynamics module

The main aerodynamic calculations are performed by external software that yields the AICs matrices, in this work given by the DLM from MSC Nastran. Two major computations are however carried out in the **aerodynamics** module: the optimisation of the poles in the RFA, and the updating procedure of aerodynamic panels with deformations, which is currently under development and not described in detail.

In this work the optimisation of the poles is performed as follows: given $\bar{\mathbf{Q}}_a(k_j)$ as the AICs in the frequency domain at a frequency k_j , $\mathbf{Q}_a(k_j, \gamma_p)$ its approximation using Roger's technique, Eq. (3.26), the term $\epsilon_a = \frac{1}{n} \sum \epsilon \left(\frac{\bar{\mathbf{Q}}_a(k_j) - \mathbf{Q}_a(k_j, \gamma_p)}{\bar{\mathbf{Q}}_a(k_j)} \right)$ yields an error metric for the optimisation and the functional ϵ can be set as a standard l^2 norm or a weighted sum with higher relevance on the steady terms. The total error to minimise is the average from the whole range of frequencies picked. The variables in the optimisation are the aerodynamic poles, initially selected as in [181], γ_0 . The output are the matrices $\mathbf{A}$ for Roger's approximation and the optimised poles. The schematic of the process can be seen in Fig. A.2.

$$\epsilon_a = \frac{1}{n} \sum_{j=1} \epsilon \left(\frac{\bar{\mathbf{Q}}_a(k_j) - \mathbf{Q}_a(k_j, \gamma_p)}{\bar{\mathbf{Q}}_a(k_j)} \right) \quad (\text{A.1})$$

Two types of optimisation can be selected with the number of poles set fixed. The first one is gradient-based with the solvers available in `scipy.optimize`, however, constraints on the separation of the poles are needed to prevent the poles collapsing into one. The second is a brute discrete optimisation in which one selects the minimum separation between poles and the problem becomes a combinatorial problem: as the number of poles is fixed, they are selected

to be bounded within a frequency range, and their minimum separation is also predefined, the number of possible combinations is not very large. A generator of all possible combinations is evaluated, yielding a vector with the corresponding errors from which the minimum is picked. The two methods can be combined such that the gradient optimisation follows the discrete one, which resulted in a non-collapsing behaviour of the poles. In fact it was found that the gradient optimisation barely moved from the discrete one.

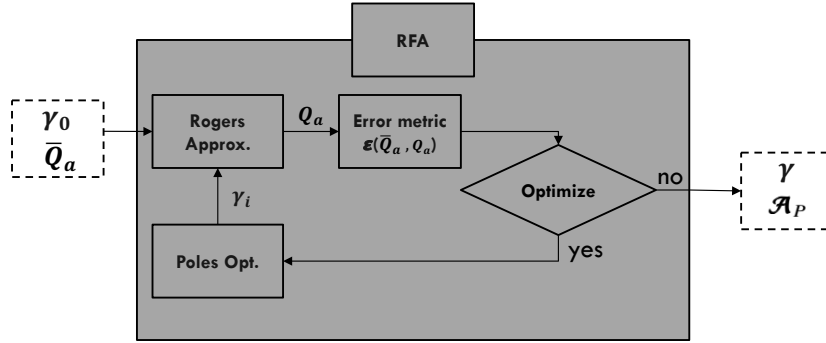


Figure A.2: Schematic of the optimisation of the poles γ

A.5 Solution procedure

The main workflow in the aeroelastic solution is shown in Alg. 2. The 3D finite element model together with the model reduction are input and solved by Nastran (SOL 103). From the FEM process, the structural grid, the stiffness, mass matrices and linear normal modes are obtained. Beam selection, connectivity definition, local reference system selection or node reordering are performed in the `intrinsic.geometry` module. The outcome is a graph representation of the structure. Intrinsic modes and nonlinear couplings are obtained in the modules `intrinsic.modes` and `intrinsic.integrals`. The time-marched solution is managed by the module `qsolvers` with options for ODE and DAE numerical solutions, including Runge-Kutta algorithms and predictor-corrector as Adams-Bashforth. The `dq` module contains classes for static, dynamic and multibody analysis, and inside the functions with the nonlinear solvers, linearised systems, tracking of rotations etc. The `forces` module calculates and incorporates all the external forcing terms into the systems built in `dq`, such as interpolation of input loads, gravity forces, controllers and elevator loads for trim analysis or gust generators. Finally `sol`

assembles velocities and forces and recovers the displacement field and the rotations. Regarding the computational effort of the approach, most of the heavy computations are performed within `intrinsic.integrals` and `intrinsic.dq`. The former computes large third order tensors, though it has been parallelised and it is only calculated one time per set of LNMs so it is not a bottleneck for the solution. The latter, however, is the advancement of the time solution and it would be desirable to speed-up the code by rewriting this module in a compiled language and wrap it inside python.

Algorithm 2: FEM₄INAS main solution

input : $V, F, A, \text{Grid.txt}, K, M$
output: $\phi_{0-2}, \psi_{1-2}, \Gamma_{1-2}, q, x_{1-2}, r_a, R$
begin
 $K_a, M_a \leftarrow \text{condensation.condensation.py} \leftarrow K, M$
 $A, B, D \leftarrow \text{aerodynamics.rfa_main.py} \leftarrow \bar{Q}_a$
 $BeamSeg \leftarrow \text{intrinsic.geometry.py} \leftarrow \text{StructGrid.txt}$
 $\phi_{0-2}, \psi_{1-2} \leftarrow \text{intrinsic.modes.py} \leftarrow BeamSeg$
 $\Gamma_{1-2} \leftarrow \text{intrinsic.integrals.py} \leftarrow \phi_{0-2}, \psi_{1-2}, BeamSeg$
 $q(0) \leftarrow \text{intrinsic.init_cond.py}$
 $q \leftarrow \text{intrinsic.qsolvers.py} \leftarrow \phi_1, A, D, \Gamma_{1-2}$
for $t_i \in V.t_i$ **do**
 $\eta \leftarrow \text{intrinsic.forces.py} \leftarrow \phi_1, R, A, B, D, q(t_{i-1})$
 $\eta_f \leftarrow \text{intrinsic.forces.forceFollower_eta} \leftarrow \phi_1$
 $\eta_a \leftarrow \text{intrinsic.forces.forceAero_eta} \leftarrow A, D, q(t_{i-1})$
 $w_g, \dot{w}_g, \ddot{w}_g \leftarrow \text{intrinsic.forces.gust_interpol}(t_i)$
if $F.gravity_on$ **or** $F.DeadLoads > 0$ **then**
 $R \leftarrow \text{intrinsic.sol.py} \leftarrow \phi_1, q$
 $\eta_d \leftarrow \text{intrinsic.forces.forceDead_eta} \leftarrow \phi_1, R$
 $\eta_g \leftarrow \text{intrinsic.forces.forceGravity_eta} \leftarrow \phi_1, R$
 $\eta_t \leftarrow \text{intrinsic.forces.forceAero_trim} \leftarrow \phi_1, R, B, q(t_{i-1})$
 $q(t_i) \leftarrow \text{intrinsic.dq.py} \leftarrow q(t_{i-1}), \eta, \Gamma_{1-2}$
 $x_1, x_2, r_a, R \leftarrow \text{intrinsic.sol.py} \leftarrow \phi_1, q$

A.5.1 Multibody approach

While some of the aforementioned modules are slightly modified to cater for the multibody description, it is in `dq` where the multibody aeroelastic or structural system is built and most of the features are incorporated. As the majority of the required calculations are already implemented in the single-body class, `DynamicODE`, they are called from the multibody class `MultibodyODE` instead of repeating functions and routines. Thus a validation of the multibody approach under different conditions only entails the correct use and implementation of Lagrange multipliers. The main strategy is shown in Fig. A.3: the vector $\mathbf{q}$ comprising the modal components in all bodies gets to the specific function (depending on the type of analysis) within `MultibodyODE`, $\mathbf{q}$ is then split into the components of each body, which are fed into the appropriate function in `DynamicODE` to obtain the $\mathbf{F}_()$ terms in Eqs. (4.8); Jacobians $\mathbf{G}_()^0$ and Lagrange multipliers λ_{12} are then developed such that the right hand side of Eq. (4.7) is complete, the $d\mathbf{q}$ vector is assembled and the simulation can be advanced to the next time step.

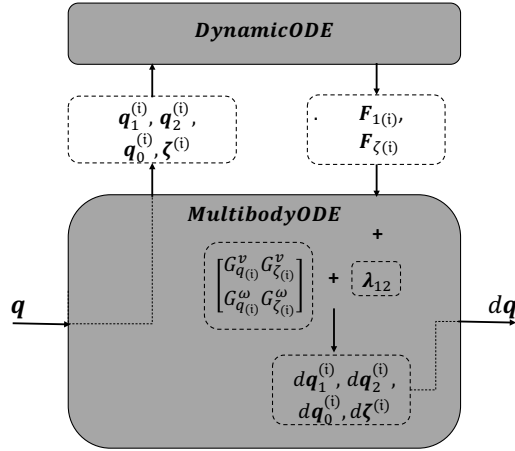


Figure A.3: Main technique for multibody solution

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