MONOPOLES IN NON-ABELIAN GAUGE THEORIES

by

DAVID MICHAEL SCOTT

A thesis presented for
the Degree of Doctor of Philosophy of the University of London
and the Diploma of Membership of Imperial College.

Department of Physics
Blackett Laboratory
Imperial College
LONDON SW7 2BZ

September 1979

Monopoles in Non-Abelian Gauge Theories

D.M. Scott

ABSTRACT

The existence of 'spherically' or '$\underline{J} + \underline{I}$' symmetric monopoles is considered. A complete set of basis functions for '$\underline{J} + \underline{I}$' symmetric field configurations is constructed for an arbitrary compact, non-abelian, Lie group. The basis has convient algebraic properties. The relationship between this basis and the formalism of Wilkinson and Goldhaber is examined, then a hybrid of the two approaches is used to show that monopole solutions exist in a class of gauge theories which includes all of those theories for which solutions are known in the Prasad-Sommerfield limit, provided that a suitable potential can be constructed. The proof does not require that this limit be taken, hence it guarantees solutions of the full theories including their respective interactions between the Higgs fields. Careful attention is paid to showing that the solutions are well behaved at the origin of the coordinates about which the '$\underline{J} + \underline{I}$' symmetry is defined.

The asymptotic behaviour of monopole solutions is discussed briefly. It is shown that in certain theories the asymptotic behaviour of the gauge fields is calculable.

A new model which illustrates the preceding general treatment is described. This model has the smallest possible non-abelian symmetry group after symmetry breakdown has taken place. The gauge group is SU(3) which is broken spontaneously to SO(3) by a Higgs multiplet which transforms according to the (2,0) representation of SU(3). The theory is unusual in that $\pi_2(SU(3)/SO(3)) = \pi_1(SO(3)) = Z_2$ rather than Z. The asymptotic behaviour of the one spherically symmetric monopole the theory possesses is found.

A new potential which leads to the symmetry breaking scheme of the SU(5) model of Georgi and Glashow is presented. It is much simpler than previously suggested potentials. The vector boson masses and the masses of the monopoles of this model are shown to satisfy

$$m^2 \geqslant V^2 \left(\frac{y^2}{4\cos^2\theta_W} + \cos^2\theta_W g^2 \right)$$

θ_W = Weinberg angle, y = hypercharge, g = magnetic charge, and V = the length of the Higgs field (the 24-plet) in the vacuum.

PREFACE

The work presented in this thesis was carried out in the Blackett Laboratory, Imperial College, beteen October 1976 and September 1979 under the supervision of Dr. D.I. Olive. The material presented is the original work of the author except where otherwise stated, and has not been previously submitted for a degree of this or any other university.

I should like to thank Dr. D.I. Olive for his guidance, help, and encouragement throughout the course of the work. I have also benifited from conversations with S.J. Avis and P.M. Stevenson, and from correspondence with A.S. Goldhaber.

I wish to thank the S.R.C. for financial support.

There was a young theorist called Scott,
Who found ten monopoles on the trot,
But the first nine they say,
Can be transformed away,
And the tenth is a sign he forgot.

Various students in the theory group 1977-78.

To my parents

CONTENTS

	Page
Abstract	2
Preface	3
Notation	7
CHAPTER ONE: INTRODUCTION	9
1.1 Introduction to the introduction	10
1.2 Strings and the possibility of their removal	12
1.3 Topological quantum numbers and non-singular monopoles	15
1.4 A criticism of the preceding arguments and assertions	18
1.5 Things to come	21
CHAPTER TWO: FORMALISM FOR '$\underline{J}$ + $\underline{T}$' SYMMETRIC FIELD CONFIGURATIONS	23
2.1 Introduction	24
2.2 Definition of '$\underline{J}$ + $\underline{T}$' symmetry	25
2.3 The covariant derivative, field tensor, and equations of motion for '$\underline{J}$ + $\underline{T}$' invariant fields	26
2.4 Connection with the formalism of Goldhaber and Wilkinson	30
2.5 The construction of the Higgs fields and gauge potentials using tensors	32
CHAPTER THREE: THE EXISTENCE OF MONOPOLE SOLUTIONS	36
3.1 Introduction	37

3.2 The form of the energy integral for '$\underline{J}$ + $\underline{T}$' symmetric field configurations 42

3.3 Existence of solutions of the radial equations on $(0,\infty)$ 44

3.4 Differentiability at the origin 50

3.5 The group which transforms '$\underline{J}$ + $\underline{T}$' symmetric field configurations into '$\underline{J}$ + $\underline{T}$' symmetric field configurations 54

3.6 Asymptotic behaviour 57

CHAPTER FOUR: AN SU(3) EXAMPLE 61

4.1 Introduction 62

4.2 The model 62

4.3 '$\underline{J}$ + $\underline{T}$' symmetric monopoles 66

CHAPTER FIVE: MONOPOLES IN A GRAND UNIFIED THEORY BASED ON SU(5) 74

5.1 Introduction 75

5.2 The symmetry breaking potential 77

5.3 Particle masses 80

5.4 Other stationary points of $V(\Phi,H)$ 82

5.5 Connection with the Weinberg-Salam model 82

5.6 A modification of the Bogomol'nyi bound 84

5.7 The magnetic charges of '$\underline{J}$ + $\underline{T}$' symmetric monopoles 88

APPENDIX A 91

APPENDIX B 92

APPENDIX C 95

REFERENCES 96

NOTATION

$\mathbb{R}$ = the set of real numbers.

$\mathbb{C}$ = the set of complex numbers.

$\mathbb{R}^n(\mathbb{C}^n)$ = real(complex) Euclidean n-space.

A function is said to be C^r if and only if all of its r^{th} (partial) derivatives exist and are continuous.

sup = supremum = least upper bound = l.u.b.

inf = infimum = greatest lower bound = g.l.b.

$:=$ means 'is defined by'.

$\forall$ = for all.

ϵ^{abc} $(a,b,c = 1,2,3)$ is the totally antisymmetric tensor defined by $\epsilon^{123} = +1$.

S^n = the n dimensional unit sphere in $\mathbb{R}^{n+1}$.

Z = the group of integers with addition as the group operation.

Z_2 = the group based on the set $\{0,1\}$ with addition modulo two as the group operation.

G denotes a compact Lie group on which some gauge theory is based.

$\mathcal{L}(G)$ denotes the Lie algebra of G.

$S^2(\infty)$ denotes the "sphere at infinity".

$\underline{\Phi}$ denotes a multiplet of Higgs fields transforming according to some linear, finite-dimensional, unitary representation of a Lie group.

$V(\underline{\Phi})$ denotes a continuous, non-negative function of $\underline{\Phi}$. It is called a Higgs potential. Further restrictions are placed on $V(\underline{\Phi})$ as they are required.

W = the vector space in which $\underline{\Phi}$ takes its values.

$\mathcal{M}_0 = \{w \in W \mid V(w) = 0\}$

$\mathcal{H}(r)$ is defined by $E = 4\pi \int_0^\infty dr\, \mathcal{H}(r)$, where E is the energy of a '$\underline{J}$ + $\underline{I}$' invariant field configuration.

Units in which $\epsilon_0 = \mu_0 = \hbar = c = 1$ are used throughout.

$W_\mu = W_\mu^a L^a$ where $\{L^a\}$ is a basis of $\mathcal{L}(G)$ corresponding to the components W_μ^a.

CHAPTER 1

INTRODUCTION

1.1 Introduction to the Introduction

In 1931 Dirac [12] showed that a necessary condition for the existence of monopoles in a consistent quantum mechanical treatment of electromagnetism which places no arbitrary restrictions on unobservable phases is the quantisation of electric charge. The quantisation condition is

$$\frac{qq}{\frac{\hbar c}{4\pi}} = \frac{n}{2} \;, \quad n \in \mathbb{Z}$$

This provides a good reason for studying monopoles as the experimentally observed quantisation of charge has intrigued physicists for many years. Also, electromagnetism with both electric and magnetic charges is dual symmetric (as it is with no charges), whereas with electric charges alone it is not. This dual symmetry is a kind of rotational symmetry beteen electric and magnetic quantities.

$$\begin{pmatrix} \underline{E}' \\ \underline{B}' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \underline{E} \\ \underline{B} \end{pmatrix}, \quad \begin{pmatrix} \rho_e' \\ \rho_m' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \rho_e \\ \rho_m \end{pmatrix}, \quad \text{etc.}$$

ρ_e = electric charge density, ρ_m = magnetic charge density.
This is encouraging as symmetry principles play an important role in current thinking.

In recent years it has come to light that there is almost certainly a hitherto unstudied, indeed unsuspected, component of the particle spectrum of some quantum field theories. What is more, these recently discovered features arise in non-abelian gauge theories, and such theories are generally accepted as being of the type most likely to be useful in describing nature. The objects to which I refer are also called magnetic monopoles because of their close connection with such objects in electromagnetic theory. This connection will be explained later.

Above I said that there are almost certainly some quantum field theories in which monopoles exist. The reason for the qualification of the statement is that, as yet, there is no complete quantum field theoretic treatment of magnetic monopoles in non-abelian gauge theories; most of the results which have been obtained so far deal with classical field theories. If monopoles can be shown to persist after quantisation, and if there masses can be calculated, then they will provide new possible tests of these theories as soon as sufficiently high energies are available for experimental purposes. The disturbing thing is that no magnetic monopoles have yet been experimentally observed. This may not prove too serious an objection, however, as all of the indications are that they should be very massive and also should interact with each other very strongly (the quantisation condition implies a reciprocity beteen the magnitudes of the smallest units of electric and magnetic charge).

Owing to the difficulties of constructing a quantum field theory of monopoles a lot of effort has gone into finding specific examples of, and general properties of, monopoles in classical field theories. This is not a pointless exercise as one may hope that this information will prove useful in developing a functional integral approach to the problem of quantisation [3]. Also, these solutions may be used in calculations to determine the quantum mass corrections to the "classical masses" of the monopoles [11].

The contents of the rest of this chapter are as follows:

In section 1.2 it is explained why the 'Dirac string' [13] associated with a monopole in ordinary electrodynamics cannot be gauged away. To do this use is made of an argument involving homotopy

theory. This line of argument is suitable when considering groups other than the U(1) group of ordinary electrodynamics and leads immediately to a generalisation of the Dirac monopole. More importantly, other objects appear which resemble Dirac monopoles but which, in contrast to the direct generalisations of the Dirac monopole, can be written without a string if a suitable gauge is chosen.

In section 1.3 Higgs fields are introduced into the discussion. Rather than considering a general model the Georgi-Glashow model [17] is described and the usual arguments and assertions relating to 'topological quantum numbers' and finite energy non-singular monopole solutions in that model are gone through. The section concludes with a brief description of the asymptotic behaviour of the gauge fields and generalisations to other groups.

Section 1.4 contains a systematic criticism of the arguments given in section 1.3. Often the arguments are shown to be too loose, and examples are given which contradict some of the plausible statements of section 1.3.

Section 1.5 outlines the contents of the succeeding chapters.

1.2 Strings and the Possibility of Their Removal†

The first significant work on monopoles, for my purposes at least, was done by Dirac [13]. He considered monopoles in ordinary electromagnetism. When monopoles are included, electrodynamics plus quantum mechanics implies charge quantisation, and hence that the

---

† I am using Dirac's original definition of a string [13], see appendix C.

theory can be thought of as a U(1) gauge theory (charge quantisation is necessary to ensure compactness).

If magnetic charges exist $\nabla.\underline{B} \neq 0$ and we can no longer write $\underline{B}=\nabla\times\underline{A}$ with $\underline{A}$ a twice continuously differentiable function. Consequently, if we are to use a vector potential (as we do in a Lagrangian formulation of electrodynamics) the introduction of monopoles necessitates singularities of $\underline{A}$: these are the familiar Dirac strings. I shall write

$$\nabla.\underline{B}=\rho_m \quad , \rho_m = \text{magnetic charge density,}$$

and $$\nabla.\underline{E}=\rho_e \quad , \rho_e = \text{electric charge density,}$$

for the microscopic Maxwell equations. The magnetic field of a point magnetic monopole of strength g is then

$$\underline{B}=\frac{g\hat{\underline{x}}}{4\pi r^2} \quad ; \quad \hat{\underline{x}} = \text{unit radial vector.} \quad r=\left(\sum_{i=1}^{3} x_i^2\right)^{\frac{1}{2}}. \qquad (1.1)$$

One vector potential which leads to this magnetic field is

$$\underline{A}_D=\frac{g}{4\pi r}\cdot\frac{(1-\cos\theta)}{\sin\theta}\,\hat{\underline{\phi}} \qquad (1.2)$$

which has a string along the negative z axis. The string can be moved by a gauge transformation but it cannot be eliminated. I shall now describe one way of seeing this: it is not the simplest way but it is the best for generalising the argument to non-abelian groups.

Under a gauge transformation $U=\exp[-ie\Lambda(r,\theta,\phi)]$

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \Lambda(r,\theta,\phi) .$$

Suppose the string can be removed by this gauge transformation. For the string to be removed one must have

$$\Lambda(r,\theta,\phi) = \frac{q\phi}{2\pi} + f(r,\theta,\phi)$$

in the region of $\theta=\pi$ with $f(r>0,\theta,\phi)$ differentiable. The gauge transformation $U(r,\theta,\phi) = \exp(-ie\Lambda)$ must be differentiable everywhere except on the negative z axis if $\underline{A}$ is to have no singularities after the gauge transformation. The map $U_\pi: S^1 \to S^1$ defined by $U_\pi(\phi) = U(1,\pi,\phi)$ is homotopic to the identity map of S^1 onto itself.† (S^1 has been parametrized by the angle ϕ, $0 \le \phi < 2\pi$. The structure of U(1) as a differentiable manifold is S^1.) The map $U_c: S^1 \to S^1$ defined by $U_c(\phi) = U(1,0,\phi)$ is a constant map as U is well defined on the positive z axis. A homotopy beteen these maps can be constructed pictorially by envisaging a closed loop moving over the unit sphere. The loop starts off very small, enclosing the South Pole of the sphere, and moves upwards over the sphere until it shrinks to enclose the North Pole (see figure 1 below).

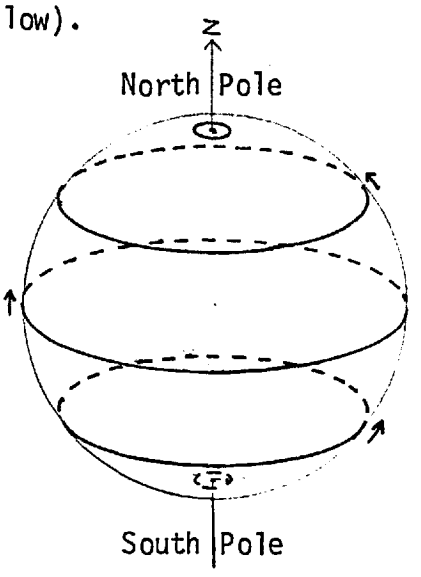


Figure 1

$U(1,\theta,\phi)$ restricted to the moving loop provides the homotopy. So, if such a gauge transformation exists, the identity map of S^1 onto S^1 is homotopic to a constant map. But there is no such homotopy, therefore such a gauge transformation does not exist, and the string cannot be

---

† $U_\pi(\phi)$ describes a closed curve as ϕ ranges from 0 to 2π because of the Dirac quantisation condition $\frac{eq}{2\pi}$ = integer. The integer is taken to be 1.

gauged away.

The next major contribution came from 't Hooft [21] . The gauge potentials in a theory based on the group G can be thought of as taking their values in the Lie algebra of G, $\mathcal{L}(G)$. The immediate generalisation of equation (1.2) is

$$\underline{W} = \underline{A}_{\mathcal{D}} Q \tag{1.3}$$

where Q is a matrix in $\mathcal{L}(G)$. The obvious question to ask is: 'Can the string be gauged away?'. For the group U(1) the answer, as has just been shown, is 'No', and there are direct generalisations of the Dirac monopole in other groups too. However, for some groups and specific choices of Q the answer is 'Yes'. The group space is larger and the loop in the space corresponding to encircling the South Pole may be capable of being shrunk to a point. 't Hooft illustrated this point by finding the large r behaviour of a string-free monopole in the Georgi-Glashow model.

The energy associated with the field in equation (1.3) is infinite. It is a property of certain gauge theories*that they have solutions the gauge field part of which is for large r like the string-free point monopoles described above, but which are regular at the origin and have finite energy. It is these theories which are the objects of study in this thesis.

1.3 Topological Quantum Numbers and Non-Singular Monopoles

The Georgi-Glashow model [17] is a gauge theory based on the group O(3). In addition to the gauge potentials it contains a vector multiplet of Higgs scalars. The Lagrangian is

---

* These theories contain Higgs fields in addition to the gauge fields.

$$\int_{\mathbb{R}^3} d^3x \left[-\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2} (D_\mu \phi)^a (D^\mu \phi)^a - \lambda (\phi^a \phi^a - v^2)^2 \right].$$

A repeated index implies summation.

$a = 1,2,3$ is a group index.

$\mu, \nu = 0,1,2,3$ are Lorentz indices.

The metric is $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$.

$$G^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e \epsilon^{abc} W^b_\mu W^c_\nu .$$

$$(D_\mu \phi)^a = \partial_\mu \phi^a + e \epsilon^{abc} W^b_\mu \phi^c .$$

The energy is given by

$$E = \int_{\mathbb{R}^3} d^3x \left[\frac{1}{2} G^a_{0i} G^a_{0i} + \frac{1}{4} G^a_{ij} G^a_{ij} + \frac{1}{2} (D_0 \phi)^a (D_0 \phi)^a + \frac{1}{2} (D_i \phi)^a (D_i \phi)^a + \lambda (\phi^a \phi^a - v^2)^2 \right]$$

$$i, j = 1,2,3 .$$

The usual arguments concerning the existence and form of monopole solutions will now be given.

Consider a sphere centred on the origin expanding outwards in space towards infinity. The values of ϕ^a ($a = 1,2,3$) on this sphere must tend to values which make $(\phi^a \phi^a - v^2)^2$ zero if the field configuration has finite energy for if they were to approach some limit which did not make $(\phi^a \phi^a - v^2)^2$ zero then the energy, which involves integrating this expression, would be infinite. Consequently, ϕ restricted to the 'sphere at infinity' is a map from S^2 into $\mathbb{R}^3$ with the image of S^2 contained in the 2-sphere of radius v. It therefore induces a map of S^2 into S^2. Continuous maps of S^2 into S^2 can be split into equivalence classes of maps: two maps are equivalent if they can be "continuously deformed" into each other (with the image of S^2 under the map always confined to S^2). The equivalence classes

can be labelled by the integers. A field configuration with a boundary value corresponding to one equivalence class cannot evolve continuously into a field configuration with a boundary value corresponding to a different equivalence class because at some stage during the transition $V(\underline{\Phi})$ would become non-zero in an infinite region of space and the energy would become infinite. That is there is an infinite potential barrier between field configurations corresponding to different equivalence classes. Thus the continuous evolution of the solution in time does not change the integer associated with the boundary value, and it is called a 'topological quantum number'.

Corresponding to each topological quantum number there is a solution of the equations of motion with that number. An example of a non-trivial asymptotic behaviour is given by

$$\phi^a = \phi(r)\hat{x}_a \quad \text{with} \quad \lim_{r\to\infty}\phi(r) = v.$$

If solutions of the form

$$\phi^a(\underline{x},t) = \phi(r)\hat{x}_a$$

$$W_\mu^a(\underline{x},t) = \epsilon_{\mu ab}\hat{x}_b W(r)\,, \quad \mu = 1,2,3; \quad W_0(\underline{x},t) \equiv 0$$

are sought, then $W_\mu^a(\underline{x},t) = -\epsilon_{\mu ab}\dfrac{\hat{x}_b}{er}$ and $\phi^a(\underline{x},t) = v\hat{x}_a$ solve the equations of motion and are well defined everywhere except at the origin; although the energy is infinite because of the behaviour of the solution near the origin. Corresponding to this point solution there is a regular finite energy solution with the same large r behaviour.

The singular gauge transformation which takes $v\hat{x}_a$ to $v\delta_{a3}$ produces

the transformation

$$-\frac{\epsilon_{\mu ab}\hat{x}_b}{er} \to -\frac{1}{e} A_D \delta_{a3} .$$

This should be compared with equation (1.3). The asymptotic forms of finite energy solutions are related by gauge transformations to those generalisations of the Dirac monopole which can be written without strings.

In a more general context, finiteness of energy implies that $\mathfrak{D}_\mu \Phi \to 0$ as infinity is approached. Here Φ transforms according to any finite dimensional, unitary representation of the group G on which the theory is based, and $\mathfrak{D}_\mu$ represents the appropriate covariant derivative. If a small region enclosing the origin is excluded a gauge can be chosen in which $\hat{x}_i W_i = 0$. Then, as for large r Φ becomes independent of r,

$$\lim_{r\to\infty} \mathfrak{D}_i \Phi = 0 \Rightarrow \lim_{r\to\infty} \frac{\hat{\theta}}{r}\frac{\partial \Phi}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial \Phi}{\partial \phi} = \lim_{r\to\infty} ie(W_\theta \hat{\theta} + W_\phi \hat{\phi})\Phi \quad (1.4)$$

and so W_i falls off like $\frac{1}{r}$, and when G_{ij} is calculated it falls off like $\frac{1}{r^2}$ (see Coleman [8,p.26]).

The existence of a topological Quantum number is important because it guarantees that the solution is non-dissipative (Coleman [8,p.2] defines a solution to be dissipative if

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^3} \theta_{00}(x,t) = 0$$

where θ_{00} is the energy density of the solution).

1.4 A Criticism of the Preceding Arguments and Assertions

The arguments and assertions in the preceding section are typical of those found in the literature. However, I do not wish to imply

that they are accepted by everyone - particularly as I am about to attack them.

The first error in the preceding section is the statement that

$$\int_{\mathbb{R}^3} d^3x\, V(\Phi) < \infty \tag{1.5}$$

($V(\Phi)$ is the Higgs potential) implies that Φ must approach a zero of $V(\Phi)$ as $r\to\infty$ in any given direction. It is true that if $\lim_{r\to\infty} \Phi(r,\theta,\phi)$ exists then it must be a zero of $V(\Phi)$, but the limit need not exist if one considers only those restrictions imposed by (1.5). This is easily proved. First note that there exists a function defined on $[0,\infty)$ with the following properties:

(i) f is infinitely differentiable everywhere.

(ii) $f \geq 0 \quad \forall r \in [0,\infty)$.

(iii) $f(r) \not\to 0$ as $r \to \infty$.

(iv) $\int_0^\infty r^2 f^n(r)\, dr < \infty \quad ; \quad n = 1, 2, 3, 4$.

Details of this function are given in appendix A, but the function can be described qualitatively in the following manner. The function is zero except in small regions surrounding each integer. In each of these regions there is a spike. The height of the spike increases with the size of the integer, but its width decreases, and in such a manner that the stated integrals are finite. In the Georgi-Glashow model both $\phi^a(\underline{x}) = v\hat{x}_a$ and $\phi^a(\underline{x}) = (v + f(r))\hat{x}_a$ are C^∞ functions on $\mathbb{R}^3 - \{0\}$ which give finite values when substituted into $\int_{\mathbb{R}^3} d^3x\, V(\Phi)$. The second function does not have a well defined topological quantum number as it does not have a well defined limit as $r\to\infty$. If '$\underline{J} + \underline{I}$' symmetry is assumed (it is defined in chapter 2) it can be shown that a well defined limit exists, but the proof involves the kinetic term which

provides information about the derivatives of $\underline{\Phi}$. The proof of this statement is given in chapter 3, along with an explanation of the difficulties encountered when trying to extend the method to include arbitrary field configurations. I know of no proof that finiteness of energy guarantees a well defined topological quantum number for an arbitrary field configuration. The preceding discussion also raises the question as to whether or not a topological quantum number is conserved in time.

The next point I wish to make is that it has never been rigorously demonstrated that there is always a connection between point solutions and the asymptotic behaviour of regular solutions, although it is commonly assumed that one does exist. I am not saying that it is not true, in fact the known regular solutions support this conjecture [2, 26, 35], I simply wish to draw attention to the fact that as yet such a connection is unproven in general.

It is wrong to say that finiteness of energy implies $\lim_{r\to\infty} \mathcal{D}_i\underline{\Phi} = 0$. The limit may not exist as was demonstrated above in the case of $V(\underline{\Phi})$. The argument that $W_i \sim \frac{1}{r}$ for large r therefore breaks down. It would have done so in any case as equation (1.4) does not imply that $W_i \sim \frac{1}{r}$; it gives no information about those components of W_i which annihilate $\underline{\Phi}$ as $r\to\infty$.

Finally, to my knowledge, no general proof exists in the literature showing that a solution with a non-zero topological quantum number is non-dissipative. Coleman [8] deals separately with each model he discusses. If one considers a sphere contracting in from infinity towards the origin then in an obvious way this provides (via the Higgs field at any given instant) a homotopy between the boundary condition and the constant map of S^2 into the vector space in which $\underline{\Phi}$ takes its values given by $S^2 \to \underline{\Phi}(\infty)$. If the topological quantum

number is non-zero there must be regions of space where $V(\Phi) \neq 0$. The difficulty is that as the argument stands it does not say how different $V(\Phi)$ must be from zero in those regions, and so it does not preclude the possibility that the amount by which it differs from zero may become arbitrarily small as time progresses. Some information about $V(\Phi)$ is needed before one can show that a solution is non-dissipative. A proof based on the two assumptions that the topological charge is conserved in time, and that the constant fields that make $V(\Phi)$ zero form a quotient space is given in appendix B.

1.5 Things to Come

In view of the above discussion it can be seen that there is a need to establish rigorous results concerning the existence and properties of monopole solutions. This thesis contains a partial solution of the problem, and some illustrative examples.

In chapter 2 I introduce the formalism, and a number of equations which are used in subsequent chapters.

Chapter 3 contains a proof of the existence of '$\underline{J} + \underline{I}$' symmetric monopoles in a class of theories which includes almost all of those theories which are known to have solutions in the Prasad-Sommerfield limit and others too* The proof does not involve taking that limit. The asymptotic behaviour of finite energy solutions is considered briefly.

Chapter 4 illustrates the general discussion of chapter 3 with a simple example with some unusual properties. The group which classifies the possible "monopoles" is Z_2 rather than Z, and the Higgs

---

* There is a slight restriction, see page 41.

fields do not transform according to the adjoint representation of the group on which the theory is based.

Chapter 5 describes a possible unified model of strong, weak, and electromagnetic interactions. At the classical level the monopoles of the theory are shown to be very massive if some of the vector bosons are. This result follows from an interesting modification of the Bogomol'nyi bound which involves the Weinberg angle.

CHAPTER 2

FORMALISM FOR '$\underline{J}$ + $\underline{T}$' SYMMETRIC FIELD CONFIGURATIONS

2.1 Introduction

In this chapter all indices, whether greek* or latin, run through the values 1, 2, and 3. A repeated index implies summation in all but section 2.5. A prime denotes differentiation with respect to r.

In this chapter the formal apparatus required in subsequent chapters for dealing with '$\underline{J}$ + $\underline{T}$' symmetric fields is developed. Imposing this symmetry greatly simplifies many calculations, and the expressions for various quantities. At first an algebraic approach is used, then a tensor interpretation is given which leads to some further useful results.

In section 2.2 '$\underline{J}$ + $\underline{T}$' symmetry is defined. In section 2.3 simplfied expressions for $\mathfrak{D}_\mu\Phi$ and $G_{\mu\nu}$ are found for field configurations which possess this symmetry and also have $W_c(x)\equiv 0$. The equations of motion are found for such fields: they have a particularly simple form when the radial component of W_μ is zero (i.e. $\hat{x}_\mu W_\mu = 0$). This gauge can be chosen everywhere in space except for an arbitrarily small neighbourhood of the origin. This approach is linked to the formalism of Wilkinson and Goldhaber [36] in section 2.4.

In section 2.5 the theory of '$\underline{J}$ + $\underline{T}$' symmetric field configurations is developed in terms of tensors. The most general possible expressions for both the Higgs fields and the gauge potentials are given. It is shown that the radial part of W_μ cannot be gauged away at the origin.

Unless otherwise stated only static fields with $W_c(x)\equiv 0$ are considered.

---

* Unless otherwise stated.

2.2 Definition of '$\underline{J}$ + $\underline{T}$' Symmetry

The matrix gauge group, G, is supposed non-abelian and compact, and to act on the Higgs fields, $\bar{\Phi}$, via a linear, finite-dimensional, unitary representation. The Lie algebra of G, $\mathcal{L}(G)$, contains an SU(2) subalgebra. Indeed $\mathcal{L}(G)$ may contain more than one inequivalent SU(2) subalgebra. A particular SU(2) algebra is specified and its generators in the fundamental representation of $\mathcal{L}(G)$ are denoted by T_i (i = 1,2,3). The T_i's are assumed to be hermitean and to satisfy the commutation relations

$$[T_i, T_j] = i\epsilon_{ijk}T_k . \tag{2.1}$$

Firstly 'rotation' in the 'internal' space is defined via the T_i's. That is, the angles θ_i (i = 1,2,3) define the 'rotation' $\exp(-i\theta_i T_i)$. '$\underline{J}$ + $\underline{T}$' symmetry is then defined as symmetry under simultaneous rotations in external and internal space by the same amount; the parameters being position independent. In other words invariance is demanded under transformations of the form $\exp(-i\theta_i[J_i+T_i])$, where the J_i's generate ordinary rotations (and include the appropriate spin matrices). Note that inequivalent SU(2) subalgebras lead to different notions of '$\underline{J}$ + $\underline{T}$' symmetry. Also '$\underline{J}$ + $\underline{T}$' symmetry is not a gauge invariant property. In fact those gauge transformations which preserve the symmetry will play an important part later on when it is shown that energy integral can sometimes be simplified.

Gauge invariant quantities calculated from '$\underline{J}$ + $\underline{T}$' symmetric quantities are rotationally invariant.

2.3 The Covariant Derivative, Field Tensor, and Equations of Motion for '$\underline{J}$ + $\underline{T}$' Invariant Fields

An element F of a vector space carrying a linear, unitary representation of G is '$\underline{J}$ + $\underline{T}$' invariant if

$$\mathcal{D}(\underline{J}+\underline{T})F = 0 \tag{2.2}$$

Where $\mathcal{D}(\underline{J}+\underline{T})$ denotes the appropriate representation of $\underline{J}+\underline{T}$. The Higgs field $\Phi(x)$ is an SO(3) (rotation group) scalar and so equation (2.2), for $F=\Phi(x)$, becomes

$$\mathcal{D}(\underline{L}+\underline{I})\Phi = 0 \ , \quad \underline{L} = -i\underline{x}\times\nabla \ .$$

Taking the cross product of this equation with $\frac{\underline{x}}{r}$ gives

$$\mathcal{D}(-i\hat{x}_j\frac{\partial}{\partial r} + i\partial_j + \frac{1}{r}\epsilon_{jkl}\hat{x}_k T_l)\Phi = 0$$

which can be rewritten as

$$\partial_j\Phi - i\mathcal{D}(i[T,\partial_j T])\Phi = \hat{x}_j\frac{\partial\Phi}{\partial r} \tag{2.3}$$

Here $T=\hat{x}_i T_i$. This, and other equations, can be shortened by introducing the object $\mathcal{D}_{Tj}$ which is defined to be the "covariant derivative" with $eW_\mu = i[T,\partial_\mu T]$. Using this notation equation (2.3) becomes

$$\mathcal{D}_{Tj}\Phi = \hat{x}_j\frac{\partial\Phi}{\partial r} \ . \tag{2.4}$$

If F is an SO(3) vector, V_k, equation (2.2) reads

$$\mathcal{D}(L_j + T_j)V_k = i\epsilon_{jkl}V_l \ ,$$

and so $$\mathcal{D}_{Tj}V_k = \hat{x}_j\frac{\partial V_k}{\partial r} + \delta_{jk}\frac{\hat{x}_l}{r}V_l - \frac{\hat{x}_k}{r}V_j \ . \tag{2.5}$$

When F is a 2-tensor, G_{ij}, equation (2.2) becomes

$$\mathcal{D}(L_j+T_j)G_{kl} = i\epsilon_{jkp}G_{pl} + i\epsilon_{jlp}G_{kp}$$

which implies

$$\mathcal{D}_{Tj}G_{kl} = \hat{x}_j\frac{\partial G_{kl}}{\partial r} + \delta_{jk}\frac{\hat{x}_p}{r}G_{pl} - \frac{\hat{x}_k}{r}G_{jl} + \delta_{jl}\frac{\hat{x}_p}{r}G_{kp} - \frac{\hat{x}_l}{r}G_{kj}.$$

Hence $\mathcal{D}_{Tj}G_{jl} = \frac{\partial}{\partial r}(\hat{x}_jG_{jl}) + 2\frac{\hat{x}_j}{r}G_{jl} + \frac{\hat{x}_j}{r}G_{lj} - \hat{x}_lG_{jj}.$

This simplifies to $\mathcal{D}_{Tj}G_{jl} = \frac{\partial}{\partial r}(\hat{x}_jG_{jl}) + \frac{\hat{x}_j}{r}G_{jl}$ (2.6)

if G_{ij} is antisymmetric. If we now identify the G_{ij} in equation (2.6) as the field tensor, then

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - ie[W_\mu, W_\nu].$$

Let $eW_\mu = X_\mu + i[T, \partial_\mu T]$, then

$$\begin{aligned}
eG_{\mu\nu} &= \partial_\mu(eW_\nu) - \partial_\nu(eW_\mu) - i[X_\mu, X_\nu] - i[i[T,\partial_\mu T], i[T,\partial_\nu T]] \\
&\quad - i[X_\mu, i[T,\partial_\nu T]] - i[i[T,\partial_\mu T], X_\nu] \\
&= \partial_\mu(eW_\nu) - \partial_\nu(eW_\mu) - i[X_\mu, X_\nu] + i[i[T,\partial_\mu T], i[T,\partial_\nu T]] \\
&\quad - i[eW_\mu, i[T,\partial_\nu T]] - i[i[T,\partial_\mu T], eW_\nu] \\
&= [\mathcal{D}_{T\mu}, eW_\nu] - [\mathcal{D}_{T\nu}, eW_\mu] - i[X_\mu, X_\nu] + i[[T,\partial_\mu T],[T,\partial_\nu T]] \\
&= \hat{x}_\mu\frac{\partial}{\partial r}(eW_\nu) - \frac{\hat{x}_\nu}{r}eW_\mu - \hat{x}_\nu\frac{\partial}{\partial r}(eW_\mu) + \frac{\hat{x}_\mu}{r}eW_\nu \\
&\quad - i[X_\mu, X_\nu] - \frac{1}{r^2}\epsilon_{\mu\nu j}\hat{x}_jT
\end{aligned}$$

using equations (2.1) and (2.5).

So $eG_{\mu\nu} = \frac{\hat{x}}{r}_\mu (reW_\nu) - \frac{\hat{x}}{r}_\nu (reW_\mu) - i[X_\mu, X_\nu] - \frac{1}{r^2}\epsilon_{\mu\nu j}\hat{x}_j T$ (2.7)

Let $X_\mu = \frac{V_\mu}{r}$, then $\frac{\partial}{\partial r}(reW_\nu) = \frac{\partial V_\nu}{\partial r}$ as $\frac{\partial}{\partial r}(r[T, \partial_\nu T]) = 0$ and equation (2.7) becomes

$$eG_{\mu\nu} = \frac{\hat{x}}{r}_\mu \frac{\partial V_\nu}{\partial r} - \frac{\hat{x}}{r}_\nu \frac{\partial V_\mu}{\partial r} - \frac{i}{r^2}[V_\mu, V_\nu] - \frac{1}{r^2}\epsilon_{\mu\nu j}\hat{x}_j T. \qquad (2.8)$$

Substituting equation (2.8) into equation (2.6) we obtain

$$[\mathcal{D}_{rj}, eG_{jl}] = \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial V_l}{\partial r} - \frac{\hat{x}_l}{r}\frac{\partial}{\partial r}(\hat{x}_j V_j) - \frac{i}{r^2}[\hat{x}_j V_j, V_l]\right\}$$
$$+ \frac{1}{r}\left\{\frac{1}{r}\frac{\partial V_l}{\partial r} - \frac{\hat{x}_l}{r}\frac{\partial}{\partial r}(\hat{x}_j V_j) - \frac{i}{r^2}[\hat{x}_j V_j, V_l]\right\}.$$

A commutator has appeared because G_{jl} has been written as a matrix belonging to $\mathcal{L}(G)$. For any given $\delta > 0$ one may choose the gauge $\hat{x}_j V_j = 0$ for $r \geq \delta$ without disturbing the '$\underline{J} + \underline{T}$' symmetry [36]. Then

$$[\mathcal{D}_{rj}, eG_{jl}] = \frac{1}{r}\frac{\partial^2 V_l}{\partial r^2}$$

therefore $[\mathcal{D}_j, eG_{jl}] = \frac{1}{r}\frac{\partial^2 V_l}{\partial r^2} - \frac{i}{r}[V_j, G_{jl}]$

$$= \frac{1}{r}\frac{\partial^2 V_l}{\partial r^2} - \frac{i}{r}\left[V_j, \frac{\hat{x}_j}{r}\frac{\partial V_l}{\partial r} - \frac{\hat{x}_l}{r}\frac{\partial V_j}{\partial r} - \frac{i}{r^2}[V_j, V_l] - \frac{1}{r^2}\epsilon_{jlk}\hat{x}_k T\right]$$

$$= \frac{1}{r}\frac{\partial^2 V_l}{\partial r^2} + \frac{V_l}{r^3} + \frac{i}{r^3}[V_j, i[V_j, V_l]] + \frac{i}{r^2}\hat{x}_l[V_j, \frac{\partial V_j}{\partial r}] \qquad (2.9)$$

using $\hat{x}_j V_j = 0$, and $i[V_j, \epsilon_{jlk}\hat{x}_k T] = V_l$.

I shall now calculate $\mathcal{D}_\rho \mathcal{D}_\rho \overline{\Phi}$ as it is required in the equations of motion.

$$\mathcal{D}_\rho \mathcal{D}_\rho \overline{\Phi} = \mathcal{D}_\rho\left(\mathcal{D}_{r\rho}\overline{\Phi} - \frac{i}{r}D(V_\rho)\overline{\Phi}\right)$$

$$= \mathcal{D}_\rho\left(\hat{x}_\rho \frac{\partial \overline{\Phi}}{\partial r} - \frac{i}{r}D(V_\rho)\overline{\Phi}\right) \qquad \text{using equation (2.4)}$$

$$= \mathcal{D}_{r\rho}\left(\hat{x}_\rho \frac{\partial \overline{\Phi}}{\partial r} - \frac{i}{r}D(V_\rho)\overline{\Phi}\right) - \frac{1}{r^2}D(V_\rho)D(V_\rho)\overline{\Phi}$$

$$= \hat{x}_\rho \frac{\partial}{\partial r}\left(\hat{x}_\rho \frac{\partial \Phi}{\partial r} - iD(V_\rho)\Phi\right) + 2\frac{\hat{x}_\rho}{r}\left(\hat{x}_\rho \frac{\partial \Phi}{\partial r} - \frac{i}{r}D(V_\rho)\Phi\right) - \frac{1}{r^2}D(V_\rho)D(V_\rho)\Phi$$

using equation (2.5)

$$= \frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} - \frac{1}{r^2}D(V_\rho)D(V_\rho)\Phi$$

So $$\mathfrak{D}_\rho \mathfrak{D}_\rho \Phi = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) - D(V_\rho)D(V_\rho)\Phi\right]. \qquad (2.10)$$

We are now ready to write down the equations of motion. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4} G_{\mu\nu}{}^a G^{\mu\nu a} - (\mathfrak{D}_\mu \Phi)^\dagger \mathfrak{D}^\mu \Phi - V(\Phi),$$

which leads to the equations of motion

$$(\mathfrak{D}_\mu \mathfrak{D}^\mu \Phi)_b = \frac{\partial V}{\partial \Phi^{*b}}, \qquad (2.11)$$

and $$(\mathfrak{D}^\nu G_{\mu\nu})^a = -ie\,\Phi^\dagger D(L^a)\mathfrak{D}_\nu \Phi + ie(\mathfrak{D}_\nu \Phi)^\dagger D(L^a)\Phi. \qquad (2.12)$$

The indices a and b are associated with the group representations and do not necessarily run through the values 1, 2, and 3. For '$\underline{J} + \underline{I}$' symmetric fields which are static and have $W_0(x) \equiv 0$ these equations reduce to

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = D(V_\rho)D(V_\rho) + r^2 \frac{\partial V}{\partial \Phi^\dagger}$$

(from equations (2.11) and (2.10)), and

$$\frac{\partial^2 V_l}{\partial r^2} = -\frac{V_l}{r^2} + \frac{1}{r^2}\left[V_j, [V_j, V_l]\right] - i\frac{\hat{x}_l}{r}\left[V_j, \frac{\partial V_j}{\partial r}\right]$$
$$-\frac{e^2}{r}\left[\Phi^\dagger D(L^a)D(V_l)\Phi + \Phi^\dagger D(V_l)D(L^a)\Phi\right]L^a$$
$$-ie^2 \hat{x}_l \left[\Phi^\dagger D(L^a)\frac{\partial \Phi}{\partial r} - \frac{\partial \Phi^\dagger}{\partial r} D(L^a)\Phi\right]L^a.$$

(from equations (2.12) and (2.9)).

2.4 Connection with the Formalism of Goldhaber and Wilkinson

Wilkinson and Goldhaber [36] write eW_μ in the form

$$eW_\mu = \frac{1}{r}\epsilon_{\mu ij}(\mathcal{M}_i - T_i)\hat{x}_j \tag{2.13}$$

where $[T, \mathcal{M}_k] = i\epsilon_{jkl}\hat{x}_j\mathcal{M}_l$ so that

$$V_\mu = \epsilon_{\mu ij}\mathcal{M}_i\hat{x}_j . \tag{2.14}$$

Note that $\mathcal{M}_i$ is not completely defined by equation (2.14) as $\mathcal{M}_i$ and $\mathcal{M}_i + \hat{x}_i X$ lead to the same V_μ for any SO(3) and '$\underline{J} + \underline{T}$' scalar X. I am therefore free to fix the radial component as I wish. I shall do that by first imposing

$$\mathcal{M}_i\hat{x}_i = 0 , \tag{2.15}$$

and by then defining $\mathcal{I}_i$ by

$$\mathcal{I}_i = \mathcal{M}_i - \frac{i}{2}\hat{x}_i\hat{x}_l\epsilon_{ljk}[\mathcal{M}_j, \mathcal{M}_k] .$$

From equations (2.8) and (2.14)

$$eB_\sigma = \frac{1}{2}\epsilon_{\sigma\mu\nu}eG_{\mu\nu}$$

$$= \frac{1}{2}\epsilon_{\sigma\mu\nu}\left(\frac{\hat{x}_\mu}{r}\epsilon_{\nu ij}\mathcal{M}_i'\hat{x}_j - \frac{\hat{x}_\nu}{r}\epsilon_{\mu ij}\mathcal{M}_i'\hat{x}_j\right)$$

$$-\frac{i}{2r^2}\epsilon_{\sigma\mu\nu}\epsilon_{\mu ij}\hat{x}_j\epsilon_{\nu kl}\hat{x}_l[\mathcal{M}_i, \mathcal{M}_k] - \frac{1}{2r^2}\epsilon_{\sigma\mu\nu}\epsilon_{\mu\nu j}\hat{x}_j T$$

So $$eB_\sigma = \frac{1}{r}\mathcal{M}_\sigma' + \frac{\hat{x}_\sigma}{r^2}\left\{-\frac{i}{2}\hat{x}_l\epsilon_{ljk}[\mathcal{M}_j, \mathcal{M}_k] - T\right\} \tag{2.16}$$

using $\mathcal{M}_\sigma'\hat{x}_\sigma = 0$, or in terms of $\mathcal{I}_i$

$$eB_\sigma = \frac{1}{r}(\mathcal{I}_\sigma' - \hat{x}_\sigma\mathcal{I}') + \frac{\hat{x}_\sigma}{r^2}(\mathcal{I} - T)$$

where $\mathcal{I} = \mathcal{I}_j\hat{x}_j$ and $eW_\mu = \frac{1}{r}\epsilon_{\mu ij}(\mathcal{I}_i - T_i)\hat{x}_j$.

$$[\mathcal{D}_j, eG_{jl}] = [\mathcal{D}_j, \epsilon_{jlm} eB_m]$$

$$= [\mathcal{D}_{Tj}, \epsilon_{jlm} eB_m] - \frac{i}{r}[\epsilon_{jpq} I_p \hat{x}_q, \epsilon_{jlm} eB_m]$$

$$= \epsilon_{jlm}\left(\hat{x}_j \frac{\partial}{\partial r}(eB_m) + \delta_{jm}\hat{x}_p eB_m - \frac{\hat{x}_m}{r} eB_j\right) - \frac{i}{r}[I_l \hat{x}_m - I_m \hat{x}_l, eB_m]$$

using equation (2.5)

$$= \epsilon_{jlm}\left(\frac{\hat{x}_j}{r} I''_m - \frac{\hat{x}_j}{r^2} I'_m - \frac{\hat{x}_m}{r^2} I'_j\right) - \frac{i}{r^3}[I_l, I-T] + i\frac{\hat{x}_l}{r}[I_m, I'_m]$$

$$= \epsilon_{jlm}\frac{\hat{x}_j}{r} I''_m + i\frac{\hat{x}_l}{r}[I_m, I'_m] - \frac{i}{r^3}[I_l, I-T].$$

If $[\mathcal{D}_j, G_{jl}] = 0$ and the derivatives of I_m with respect to r vanish - as is true for point solutions and presumably also for regular solutions, to a good approximation, for large r - this last equation implies

$$[I_l, I-T] = 0.$$

As I_l is a vector with respect to '$\underline{J}$ + $\underline{I}$' symmetry this yields

$$[I_j, I_k] = i\epsilon_{jkl} I_l .$$

The I_l's form an SU(2) algebra. This is a direct proof, using the equations of motion, of a result first derived by Wilkinson and Goldhaber [36] for point solutions using angular momentum conservation relations.

Nothing really new has been derived in this and the preceding section. However this treatment of the topics covered has not appeared in print.

2.5 The constuction of the Higgs Fields and Gauge Potentials Using Tensors

Under the action of the group generated by the SU(2) subalgebra the space of Higgs fields decomposes into a direct sum of invariant subspaces. Each subspace is spanned by the components of a position independent SU(2) spinor when position dependent coefficients are allowed. In each subspace of odd dimension the spinor components can be replaced by those of a totally symmetric, traceless, SO(3) tensor (a scalar in a one dimensional space). For an odd dimensional space let the conponents be

$$\phi^{\alpha}_{(ijk\cdots q)}\,, \qquad \sum_{i=1}^{3} \phi^{\alpha}_{(iik\cdots q)} = 0\,.$$

The latin indices run through the values 1, 2, and 3. The label α distinguishes between different subspaces of the same dimension. The round brackets denote symmetrization.

No '$\underline{J}$ + $\underline{I}$' invariants can be constructed in even dimensional subspaces because the spatial dependence of a field may be written in terms of SO(3) tensors (which are related to spherical harmonics).

In each odd dimensional subspace one simply forms

$$\phi^{\alpha}(l) = \sum_{i,j,\ldots,q} \phi^{\alpha}_{(ij\cdots q)}\, \hat{x}_i \hat{x}_j \cdots \hat{x}_q\,.$$

l = the number of indices, $2l+1$ = the dimension of the space. The most general '$\underline{J}$ + $\underline{I}$' invariant expression for the Higgs field is then

$$\Phi(x) = \sum_{\alpha, l} c^{\alpha}_{l}(r)\phi^{\alpha}(l)\,, \quad c^{\alpha}_{l}(0) = 0 \;\forall_{\alpha} \text{ when } l \neq 0\,. \qquad (2.17)$$

I shall use a matrix notation for the gauge potentials so that they become functions from space-time into $\mathcal{L}(G)$ (viewed as a matrix algebra rather than an abstract algebra), i.e. $W_{\mu} = W^{a}_{\mu} L^{a}$.

The construction of '$\underline{J}$ + $\underline{T}$' invariant basis functions for the gauge potentials is similar to that for the Higgs fields, but now a space index must be added. This index runs over the same values as the internal SO(3) indices and can either be symmetrized with them and then split into a part with trace and a traceless part, or it can be anti-symmetrized with them. Thus three linearly independent invariants are formed in each space of odd dimension greater than one. One invariant is formed in each one dimensional subspace.

Let the odd dimensional subspaces in $\mathcal{L}(G)$ be spanned by $\{L^{\alpha}_{(ij\cdots q)}\}$. The most general '$\underline{J}$ + $\underline{T}$' invariant expression for the gauge potentials is

$$eW_{\mu}(\underline{x}) = \sum_{\alpha,\ell,i} \frac{\ell}{r}\left[a^{\alpha}_{\ell}(r)(\delta_{\mu i} - \hat{x}_{\mu}\hat{x}_i) + b^{\alpha}_{\ell}(r)\sum_j \epsilon_{\mu ij}\hat{x}_j\right] L^{\alpha}(\ell)_i + \sum_{\alpha,\ell}\frac{e^{\alpha}_{\ell}(r)}{r}\hat{x}_{\mu}L^{\alpha}(\ell) \qquad (2.18)$$

$$= \sum_{\alpha,\ell}\left\{a^{\alpha}_{\ell}\partial_{\mu}L^{\alpha}(\ell) - ib^{\alpha}_{\ell}[T,\partial_{\mu}L^{\alpha}(\ell)] + \frac{e^{\alpha}_{\ell}}{r}\hat{x}_{\mu}L^{\alpha}(\ell)\right\}$$

$$= \sum_{\alpha,\ell}\left\{A^{\alpha}_{\ell}\partial_{\mu}L^{\alpha}(\ell) - iB^{\alpha}_{\ell}[T,\partial_{\mu}L^{\alpha}(\ell)] + \frac{e^{\alpha}_{\ell}}{r}\hat{x}_{\mu}L^{\alpha}(\ell)\right\} + i[T,\partial_{\mu}T] \qquad (2.19)$$

by definition.

$$L^{\alpha}(\ell)_i = \sum_{j,k,\ldots,q} L^{\alpha}_{(ijk\cdots q)}\hat{x}_j\hat{x}_k\cdots\hat{x}_q\,, \quad L^{\alpha}(\ell) = \sum_i L^{\alpha}(\ell)_i\hat{x}_i\,, \text{ and } T = \sum_i T_i\hat{x}_i\,.$$

μ runs through the values 1, 2, and 3. W_{μ} has been split up into three parts; one radial and two orthogonal transverse parts.

$W_{\mu}(\underline{x})$ must be well defined at $r=0$, and so $\lim_{\underline{x}\to 0} W_{\mu}(\underline{x})$ must be independent of the direction from which the origin is approached. Let $\underline{x}\to 0$ along the z axis, then

$$\lim_{\underline{x}\to 0} eW_3(\underline{x}) = \lim_{r\to 0}\sum_{\alpha,\ell}\frac{e^{\alpha}_{\ell}(r)}{r}L^{\alpha}_{(33\cdots 3)}\,.$$

However, if $\underline{x}\to 0$ along the x axis

$$\lim_{\underline{x}\to 0} eW_3(\underline{x}) = \lim_{r\to 0}\sum_{\alpha,\ell}\frac{\ell}{r}\left\{a^{\alpha}_{\ell}(r)L^{\alpha}_{(311\cdots 1)} - b^{\alpha}_{\ell}(r)L^{\alpha}_{(211\cdots 1)}\right\}.$$

For W_μ to be well defined we must have

$$\lim_{r\to 0} \sum_{\alpha,l} \left\{ \frac{e^\alpha_l(r)}{r} L^\alpha_{(33\cdots 3)} + \frac{l}{r}\left[b^\alpha_l(r) L^\alpha_{(211\cdots 1)} - a^\alpha_l(r) L^\alpha_{(311\cdots 1)} \right] \right\} = 0 .$$

This equation must be satisfied individually for each α and l.

$l=0$ yields $\lim_{r\to 0} \frac{e^\alpha_0(r)}{r} = 0 \ , \ \forall \alpha ;$

$l=1$ yields $\lim_{r\to 0} \frac{1}{r}\left(e^\alpha_1(r) - a^\alpha_1(r)\right) = \lim_{r\to 0} \frac{b^\alpha_1(r)}{r} = 0 , \ \forall \alpha ;$

$l \geq 2$ yields $\lim_{r\to 0} \frac{e^\alpha_l(r)}{r} = \lim_{r\to 0} \frac{a^\alpha_l(r)}{r} = \lim_{r\to 0} \frac{b^\alpha_l(r)}{r} = 0 , \ \forall \alpha ,$

as $L^\alpha_{(33\cdots 3)}$, $L^\alpha_{(211\cdots 1)}$, and $L^\alpha_{(311\cdots 1)}$ are all orthogonal if they have two or more indices. To see this note that the constraints

$$\sum_i L^\alpha_{(ii k\cdots q)} = 0$$

relate only components which differ from each other by precisely two indices. $\frac{e^\alpha_1}{r}$ and $\frac{a^\alpha_1}{r}$ need not vanish as $r\to 0$. This is because if

$$\lim_{r\to 0} \frac{e^\alpha_1(r)}{r} = \lim_{r\to 0} \frac{a^\alpha_1(r)}{r}$$

then $\lim_{r\to 0} \left[\frac{a^\alpha_1}{r}(\delta_{\mu i} - \hat{x}_\mu \hat{x}_i) L^\alpha(1)_i + \frac{e^\alpha_1}{r} L^\alpha(1) \hat{x}_\mu \right]$

$$= \lim_{r\to 0} \frac{a^\alpha_1(r)}{r} L^\alpha(1)_\mu + \lim_{r\to 0} \frac{1}{r}\left[a^\alpha_1(r) - e^\alpha_1(r) \right] \hat{x}_\mu L^\alpha(1)$$

$$= \lim_{r\to 0} \frac{a^\alpha_1(r)}{r} L^\alpha(1)_\mu .$$

Now $L^\alpha(1)_\mu$ has no angular dependence so the limit is well defined.

The radial part of eW_μ is $\sum_{\alpha,l} \frac{e^\alpha_l(r)}{r} \hat{x}_\mu L^\alpha(l)$ and it is in general not well defined because of the $l=1$ part. It cannot, therefore, be gauged away without the gauge potentials becoming ill defined. The expression

$$\sum_{\substack{\alpha,l \\ l\neq 1}} \frac{e^\alpha_l(r)}{r} \hat{x}_\mu L^\alpha(l)$$

vanishes at the origin and is well defined, hence it may be gauged away.

The basis given above is a generalisation of that used by Corrigan, Olive, Fairlie and Nuyts [10], and Brihaye and Nuyts [5]. A closely related basis has been written down by Leznov and Saveliev [22].

If $W_0(x)$ is included it has no space index and the most general form for it is identical to that for a Higgs multiplet transforming according to the adjoint representation. In all that follows it could be treated as an extra massless Higgs multiplet for

$$G_{ic} = \partial_i W_c - ie[W_i, W_c] = \mathcal{D}_i W_c \quad \text{as} \quad \partial_0 W_i = 0,$$

and $\mathcal{D}_c \Phi = -ie\mathcal{D}(W_c)\Phi$ as $\partial_0 \Phi = 0$.

Nothing would be gained, however, in the existence proof of chapter 3 as $W_0(x) \equiv 0$ satisfies the W_c equation of motion identically. The significance of this remark will be explained in chapter 3.

A tensor expression for $\mathcal{M}_i$ can be given. From equations (2.14), (2.15), and (2.19)

$$\mathcal{M}_i = \sum_{\alpha,\ell} \ell \left[B_\ell^\alpha (\delta_{ij} - \hat{x}_i \hat{x}_j) + A_\ell^\alpha \epsilon_{ikj} \hat{x}_k \right] L^\alpha(U)_j . \qquad (2.20)$$

CHAPTER 3

THE EXISTENCE OF MONOPOLE SOLUTIONS

3.1 Introduction

This chapter deals with the existence of classical monopole solutions in non-abelian gauge theories, and with their properties.

't Hooft [21] and Polyakov [25] independently found point monopole solutions of the equations of motion of the Georgi-Glashow model [17], and asserted that a corresponding finite energy solution existed. 't Hooft supported this assertion with a numerical calculation. However that does not constitute a proof of the existence of a regular solution of the desired form. When other theories were shown to have point solutions it was assumed, understandably, by many people that in those non-abelian gauge theories there would exist regular solutions corresponding to many of the point solutions. In general this was and still is an assumption, albeit a very plausible one, but it is shown below that in certain circumstances a regular solution can be proved to exist.

Neither the existence of a topological charge, nor the existence of a point solution of the equations of motion guarantees that the equations of motion have a static solution valid for all values of the radial coordinate; at least not in any obvious way. The topological quantum numbers which occur depend only on $V(\Phi)$. On the other hand, many things other than $V(\Phi)$ affect the equations of motion, and there are many time independent field configurations which have well defined topological quantum numbers which are not static solutions of the equations of motion. Also it is by no means obvious that the known point solutions can be modified to give a true solution regular in all of space. If one thinks of the Laplace equation it has the singular solution $\frac{1}{r}$ which cannot be smoothed out to give a regular solution with $\frac{1}{r}$ behaviour asymptotically.

One way of proving the existence of a monopole solution is, of course, to exhibit that solution. This also gives the most infomation. However, the equations of motion are non-linear and difficult to solve. Explicit solutions are known only for certain theories [2, 26, 35] in which the Higgs fields transform according to the adjoint representation of a special unitary group. Moreover, even these theories have been solved only in the Prasad-Sommerfield limit [26] of vanishing Higgs potential, and even then only for certain asymptotic limits. In the Prasad-Sommerfield limit the equations of motion of these theories can be reduced to the Bogomol'nyi equations [4] which are first order; this is not the case for other representations.

Another, more abstract, approach is to attempt to show that a stationary "point" of the action (energy integral for static solutions) exists. Tyupkin, Fateev, and Schvarts [33] started a proof along these lines which was completed by Rawnsley [27, 28]. They examined certain SU(2) and SU(3) gauge theories. Rawnsley wrote down an ansatz for a general group, but it only included one function to describe the gauge potential which is insufficient to ensure that variation of the energy integral <u>after</u> the ansatz has been inserted leads to the same equations of motion as varying the energy integral and then inserting the ansatz.

Nearly all attempts at finding or proving the existence of static solutions have dealt with '<u>J</u> + <u>I</u>' symmetric field cofigurations, and the work reported below is no exception. It is reasonable to demand this symmetry as it is, roughly speaking, the largest symmetry which can be required without excluding solutions with a non-zero topological charge. Also one would expect any solution describing an elementary particle to possess a high degree of symmetry. Imposing this symmetry has the technical advantage of

reducing the equations of motion to a set of coupled, non-linear, ordinary differential equations in the radial coordinate, r. I shall call these equations the radial equations.

In section 3.2 the form the energy integral takes for 'J + T' symmetric field configurations is considered. It turns out that to evaluate this integral it is necessary to know the values of those quantities appearing in the energy integral on only, let us say, the z axis. As the most general possible form for a 'J + T' symmetric field configuration is used, varying this integral leads to the correct equations of motion (see ref. 8 appendix 4). An extra assumption is made which simplify the energy further whilst retaining this property. In section 3.5 examples which satisfy this assumption are given.

In sections 3.3 and 3.4 I prove the existence of static monopole solutions in a class of theories. The points (a)→(g) listed below establish some sufficient conditions for a theory to belong to that class. The proof does not involve taking the Prasad-Sommerfield limit. I consider static 'J + T' symm. solutions with $W_0(x) \equiv 0$; thus excludes 'dyon' solutions. In section 3.3 it is shown that if the energy integral can be taken to have the simplified form given in section 3.2 then a solution of the radial equations exists on $(0,\infty)$. In section 3.4 this solution is shown to correspond to a solution of the full equations of motion in all of space. To be precise consider a non-abelian gauge theory with the symmetry broken spontaneously by the Higgs potential $V(\Phi)$. Let the most general 'J + T' invariant ansatz corresponding to a particular embedding of the SU(2) algebra (i.e. identification of the T_i's) be

$$\Phi(x) = \sum_{\alpha,\ell} c^{\alpha}_{i\ell}(r)\,\phi^{\alpha}(\ell).$$

If the following conditions are satisfied:

(i) $V(c_\ell^\alpha) := V(\sum_{\alpha,\ell} c_\ell^\alpha \phi^\alpha(1)) \geq 0 \quad \forall\ c_\ell^\alpha$,

(ii) $V(c_\ell^\alpha)$ is a quartic with no linear term,

(iii) $V(c_\ell^\alpha)=0$ for some constant values $\{\eta_\ell^\alpha\}$ of the c_ℓ^α's,

(iv) there exists a set of functions (any set including the a_ℓ^α's and e_ℓ^α's of equation (2.18) will do) with the property that setting all of the functions in the set equal to zero satisfies the equations of motion of those functions identically (this ensures that the simplified form of the energy integral may be used);

then there is a '$\underline{J} + \underline{T}$' symmetric solution of the Euler-Lagrange equations with $W_0(x)\equiv 0$, and $\lim_{r\to\infty} c_\ell^\alpha(r)\phi^\alpha(\ell) = \eta_\ell^\alpha \phi^\alpha(\ell)$.

Note:

(a) There is no linear term in $V(\Phi)$ if Φ transforms according to a representation with no trivial (i.e. one dimensional) irreducible part.

(b) It will be shown that (iv) is satisfied if the following two conditions are satisfied:

(i) The $\ell\neq 0$ part of the group which transforms '$\underline{J} + \underline{T}$' symmetric fields into '$\underline{J} + \underline{T}$' symmetric fields is an abelian subgroup of the group containing all ℓ values (ℓ is a sort of angular momentum defined with respect to $\{T_i\}$).

(ii) The $\ell=1$ part of the radial component of an arbitrary '$\underline{J} + \underline{T}$' symmetric $W_i(x)$ annihilates an arbitrary '$\underline{J} + \underline{T}$' symmetric $\Phi(x)$.

(c) If the group mentioned in (b) (including the $\ell=0$ part) is abelian and the Higgs fields transform according to the adjoint representation, then both conditions in (b) are satisfied. This is explained in section 3.5.

(d) The group is abelian if T_3 has no repeated eigenvalues.

(e) Both conditions in (b) are satisfied by the 'maximal' and 'minimal' embeddings of the SU(2) algebra in the Lie algebra of SU(n). More details and further examples can be found in section 3.5.
(f) It has not been assumed that a topological quantum number exists.
(g) The representation according to which $\underline{\Phi}$ transforms has not been assumed to be irreducible, so the treatment includes cases where there is more than one multiplet of Higgs fields. This is relevant to the example of chapter 5.

These observations show that there is quite a large class of theories which contain monopole solutions. All of the known exact solutions [2, 26, 35] (exact in the Prasad-Sommerfield limit that is) are solutions of SU(n) theories. The fields of these solutions transform according to the adjoint representation, and the SU(2) embedding is the maximal embedding defined in section 3.5. The theorem ensures that a solution of the full theory (i.e. with a non-zero potential) exists in each of these cases , provided only that a suitable potential can be constructed (the quoted authors do not give a potential except for the case of the Georgi-Glashow model)**.

In section 3.6 the asymptotic behaviour of monopole solutions is briefly considered.

Section 3.3 contains a proof of the fact that any finite energy '$\underline{J} + \underline{I}$' symmetric field configuration has a well defined topological charge.

** A suitable potential can be constructed if $\underline{\Phi}$,asymptotically, has precisely two eigenvalues when written as a Hermitean matrix. This is the case for the solutions of refs. 2 and 26, and the explicit examples given in reference 35.

3.2 The Form of the Energy Integral for 'J + T' Symmetric Field Configurations

The form of the energy integral for a static, 'J + T' symmetric field configuration with $W_0(\underline{x})\equiv 0$ is derived below, then an assumption is made which further simplfies the expression.

The field configurations considered are 'J + T' invariant so if $\mathcal{R}$ is a rotation which satisfies $\mathcal{R}\hat{\underline{z}} = \hat{\underline{x}}$

$\hat{\underline{z}}$ = unit vector along z axis

$\hat{\underline{x}}$ = unit radial vector

then $$\Phi(\underline{x}) = \mathcal{D}(U(\mathcal{R})^{-1})\Phi(r\hat{\underline{z}}) \tag{3.1a}$$

$$W_\mu(\underline{x}) = U^{-1}(\mathcal{R})W_\rho(r\hat{\underline{z}})U(\mathcal{R})\mathcal{R}_{\mu\rho} \quad (\mu,\rho = 1,2,3) \tag{3.1b}$$

$$(\mathfrak{D}_\mu\Phi)(\underline{x}) = \mathcal{D}(U^{-1}(\mathcal{R}))(\mathfrak{D}_\rho\Phi)(r\hat{\underline{z}})\mathcal{R}_{\mu\rho} \tag{3.1c}$$

$$G_{\mu\nu}(\underline{x}) = U^{-1}(\mathcal{R})G_{\rho\sigma}(r\hat{\underline{z}})U(\mathcal{R})\mathcal{R}_{\mu\rho}\mathcal{R}_{\nu\sigma} \tag{3.1d}$$

Now the energy integral is gauge and rotationally invariant, it therefore follows from (3.1a)→(3.1d) that the energy density can be expressed solely in terms of quantities defined on the z axis. So the energy of a static 'J + T' symmetric field configuration is given by

$$E = 4\pi\int_0^\infty dr\, r^2\left[\tfrac{1}{2}B_i^a(r\hat{\underline{z}})B_i^a(r\hat{\underline{z}}) + (\mathfrak{D}_i\Phi)^+(r\hat{\underline{z}})(\mathfrak{D}_i\Phi)(r\hat{\underline{z}}) + V(\Phi(r\hat{\underline{z}}))\right].$$

Given some indexed object X_i (i = 1, 2, 3) $X_\pm$ are defined by $X_\pm = X_1 \pm iX_2$. $\underline{M}(r)$ is defined to be $\underline{\mathcal{M}}(r\hat{\underline{z}})$, $\Phi_0(r)$ to be $\Phi(r\hat{\underline{z}})$, and $N(r)$ to be $\mathcal{N}(r\hat{\underline{z}})$. From equation (2.16)

$$eB_\pm(r\hat{\underline{z}}) = \frac{M'_\pm}{r} + \frac{i}{r^2}[M_\pm, N] \tag{3.2}$$

and $$eB_3(r\hat{\underline{z}}) = \frac{1}{r^2}\left(\tfrac{1}{2}[M_+, M_-] - T_3\right). \tag{3.3}$$

From equations (2.4) and (2.13)

$$(\mathfrak{D}_{\pm}\Phi)(r\hat{z}) = \mp D(M_{\pm})\Phi_0$$

and $$(\mathfrak{D}_3\Phi)(r\hat{z}) = \Phi_0' - \frac{i}{r}D(N)\Phi_0 .$$

Therefore

$$E = 4\pi\int_0^\infty dr\, r^2\Big[\frac{1}{e^2K^2}Tr\Big\{\Big(\frac{M_+'}{r}+\frac{i}{r^2}[M_+,N]\Big)\Big(\frac{M_-'}{r}+\frac{i}{r^2}[M_-,N]\Big)+\frac{1}{r^4}\Big(\frac{1}{2}[M_+,M_-]-T_3\Big)^2\Big\}$$
$$+\Big(\Phi_0'-\frac{i}{r}D(N)\Phi_0\Big)^{\dagger}\Big(\Phi_0'-\frac{i}{r}D(N)\Phi_0\Big)+\frac{1}{2r^2}\Phi_0^{\dagger}\{D(M_+)D(M_-)\}\Phi_0+V(\Phi_0)\Big]. \quad (3.4)$$

The generators are normalized so that $Tr(L^aL^b)=K^2\delta^{ab}$. This expression is still too complex for my purposes. Using equations (2.17), (2.18), (2.19), and (2.20) E can be expressed in terms of a_ℓ^α, b_m^β, e_n^γ, and c_p^τ. If setting a subset of these functions equal to zero satisfies the equations of motion associated with the elements of that subset identically then those functions may be set equal to zero in the energy integral <u>before</u> variation and the correct equations of motion will still result from varying the new expression. The resulting equations will be the original equations with that set of functions set equal to zero in them. I shall assume that such a subset exists and that it includes all of the a_ℓ^α's and e_n^γ's. This assumption is required because it ensures that the derivatives of the remaining b_m^β's and c_p^τ's separate off from the terms which do not involve derivatives. The form which the integrand of the energy integral takes is important for the existence proof. With this assumption the most general form that the energy integral can assume is

$$E = 4\pi \int_0^\infty dr \sum_{\gamma,n} \frac{\Delta^2}{e^2 K^2} \left\{ n(n+1) b_n^{\gamma\,2} + \frac{1}{r^2} \left[n(n+1) b_n^\gamma + \sum_{\alpha,\ell;\beta,m} F^{\alpha\beta\gamma}_{\ell m n} b_\ell^\alpha b_m^\beta \right]^2 \right\}$$
$$+ 4\pi \int_0^\infty dr \sum_{\tau,\rho} \Gamma^2 \left\{ r^2 c_\rho^{\tau\,2} + \rho(\rho+1) \left[c_\rho^\tau + \sum_{\alpha,\ell;\gamma,n} J^{\alpha\gamma\tau}_{\ell n \rho} b_\ell^\alpha c_n^\gamma \right]^2 + \rho(\rho+1) \left[\sum_{\alpha,\ell;\gamma,n} H^{\alpha\gamma\tau}_{\ell n \rho} b_\ell^\alpha c_n^\gamma \right]^2 \right\}$$
$$+ 4\pi \int_0^\infty dr\, r^2 V(c_n^\gamma). \qquad (3.5)$$

The indices on the c's do not necessarily run through the same values as those on the b's. Δ is the common norm of the basis functions $L^\alpha(\ell)$, i.e. $\Delta^2 = \mathrm{Tr}\,(L^\alpha(\ell)^2); \forall \alpha, \ell$. Similarly $\Gamma^2 = \phi^\tau(\rho)^\dagger \phi^\tau(\rho); \forall \tau, \rho$. Sufficient conditions for such a subset to exist are given in section 3.5.

The $\ell(\ell+1)$ factors in equation (3.5) arise from the relations

$$\partial_\mu(-i[T, \partial_\nu L^\alpha(\ell)]) - \partial_\nu(-i[T, \partial_\mu L^\alpha(\ell)]) = \frac{\ell(\ell+1)}{r^2} \epsilon_{\mu\nu\sigma} \hat{x}_\sigma L^\alpha(\ell),$$

$$\sum_{\mu,\nu} \mathrm{Tr}\left\{ (\hat{x}_\mu [T, \partial_\nu L^\alpha(\ell)] - \hat{x}_\nu [T, \partial_\mu L^\alpha(\ell)])^2 \right\} = -\frac{\ell(\ell+1)}{r^2} \mathrm{Tr}\,(L^\alpha(\ell)^2)$$

and $$\sum_\mu \partial_\mu \phi^\alpha(\ell)^\dagger \partial_\mu \phi^\alpha(\ell) = \frac{\ell(\ell+1)}{r^2} \phi^\alpha(\ell)^\dagger \phi^\alpha(\ell).$$

3.3 Existence of Solutions of the Radial Equations on $(0,\infty)$

In this section two things are proved.

(i) Any '$\underline{J}$ + $\underline{T}$' symmetric field configuration of finite energy, whether time independent or not, which is C^1 in $\mathbb{R}^3$ has a well defined topological quantum number.

(ii) The equations of motion obtained by varying the integral of equation (3.5) have a solution defined on $(0,\infty)$ for each possible boundary condition.

At the end of the proof of the first assertion I explain how one might try to extend the proof to arbitrary field configurations, and then point out the difficulties which arise when one tries to do so. The proofs are generalisations and slight modifications of the methods

of Tyupkin, Fateev and Schvarts [33] , and Rawnsley [27, 28] .

Proof of (i)

Outside any small open ball, B, enclosing the origin the radial part of W_μ ($\mu = 1, 2, 3$) may be gauged away. The most general form for a '$\underline{J} + \underline{I}$' invariant W_μ is given by equation (2.18). The most general '$\underline{J} + \underline{I}$' invariant expression for $W_0(x)$ is

$$W_0(x) = \sum_{\alpha,l} f_l^\alpha(r)\, L^\alpha(l).$$

It follows that the energy integral for a '$\underline{J} + \underline{I}$' symmetric field configuration with $W_\mu x_\mu = 0$ outside B has the form (in the region outside B)

$$\int dr \left[\sum_{\alpha,l} \left(a_l^{\alpha\,2} + b_l^{\alpha\,2} + r^2 f_l^{\alpha\,2} \right) + \sum_{\gamma,n} r^2 c_n^{\gamma\,2} \right] \qquad (3.6)$$

+ a positive term independent of a_l^α, b_l^α, f_l^α, and c_n^γ.

Constants have been omitted in the sums. The terms involving derivatives with respect to r of the radial functions separate off because they are the only radial components in $\mathfrak{B}_i^a$ and $\mathfrak{D}_i\Phi$.

For a finite energy field configuration with energy E equation (3.6) implies

$$\int_a^\infty dr\, r^2 c_n^{\gamma\,2} \leq E \,, \qquad a \geq \text{radius of B}$$

$$\text{So} \quad |c_n^\gamma(b) - c_n^\gamma(a)| = \left| \int_a^b c_n^{\gamma\,\prime} dr \right| \,, \quad b \geq a$$

$$\leq \left(\int_a^b r^2 c_n^{\gamma\,2} dr \right)^{\frac{1}{2}} \left(\int_a^b \frac{1}{r^2} dr \right)^{\frac{1}{2}},$$

$$\Rightarrow |c_n^\gamma(b) - c_n^\gamma(a)|^2 \leq E \left(\frac{1}{a} - \frac{1}{b} \right) \leq \frac{E}{a},$$

and it follows immediately that the limit $\lim_{r\to\infty} c_n^\gamma(r) := c_n^\gamma(\infty)$ exists,

and hence the limit $\lim_{r\to\infty} \sum_{\beta,n} c_n^\gamma(r)\phi^\gamma(n) = \sum_{\beta,n} c_n^\gamma(\infty)\phi^\gamma(n)$ exists as the sum is a finite sum. Finiteness of energy, combined with the fact that the limit is now known to exist, ensures that a well defined topological quantum number exists. The proof is simple.

$$\int r^2 V(c_n^\gamma)\,dr \leq \frac{E}{4\pi} \tag{3.7}$$

Now suppose $V(c_n^\gamma(\infty)) = V_0 \neq 0$, then, as V is a continuous function of the c's, for any given $\epsilon > 0$ there is an $r_0 > 0$ such that for $r \geq r_0$ $|V(c_n^\gamma(r)) - V_0| < \epsilon$. Take $\epsilon < \frac{V_0}{2}$ then

$$\int_{r_0}^{b} r^2 V(c_n^\gamma(r))\,dr \geq \int_{r_0}^{b} r^2 \frac{V_0}{2}\,dr \quad , \quad b \geq r_0 . \tag{3.8}$$

But $\int_{r_0}^{b} r^2 \frac{V_0}{2}\,dr$ diverges as $r \to \infty$ so equation (3.8) contradicts equation (3.7) and it follows that $V(c_n^\gamma(\infty)) = 0$.

One might think of expanding a general field configuration in terms of spherical harmonics:

$$\phi^a(x) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}^a(r)\, Y_{\ell m}(\theta,\phi)$$

One can then show, just as for the c's above, that $\lim_{r\to\infty} f_{\ell m}^a(r)$ exists. However the sum is an infinite sum and it no longer follows immediately that $\lim_{r\to\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m}^a(r)\, Y_{\ell m}(\theta,\phi)$ exists, and even if it does it may be discontinuous. Consequently I have not yet been able to prove that a general, continuously differentiable, Higgs field configuration of finite energy has a well defined topological quantum number associated with it.

An Outline of the Proof of (ii)

A sufficient condition for a field configuration that is C^1 (i.e. has continuous first derivatives) to be a solution of the Euler-Lagrange equations derived from expression (3.5) is that it be a

stationary "point" of the integral under C^∞ variations of compact support. I shall concentrate, therefore, on showing that the integral has a stationary point.

The energy is bounded below and the greatest lower bound (or infimum) of E taken over a given space of functions, X, is well defined and finite if there is a function in X which makes E finite. By the definition of infimum there is a sequence, $\{g_n\}^\dagger$, of functions ($g_n \in X$) such that the energies $E_n = E(g_n)$ approach the infimum as $n \to \infty$. If the sequence has a limit function $g \in X$, then

$$E(g) \geq \inf_{\tilde{g} \in X} E(\tilde{g}) \quad (= E_0 \text{ say}) \text{ by definition.}$$

So if it can be shown that $E(g) \leq E_0$ then $E(g) = E_0$ and g is a stationary point of E.

The space of functions I shall consider will not consist of just C^1 functions, and because of this the Euler-Lagrange equations will not follow immediately from the fact that there is a stationary point. In fact the functions I shall consider will have weak first derivatives, and the Euler-Lagrange equations will be satisfied weakly. However, the equations are elliptic and from this it follows that they are strongly satisfied, i.e. the solution will be a solution in the sense of the classical theory of differential equations.

The radial coordinate takes values in $[0,\infty)$. The restriction in the title of this section to $(0,\infty)$ arises from the fact that the integrand in equation (3.5) viewed as a function of $\{b^\alpha_\ell, c^\tau_n\}$ and r is not continuous in r at $r = 0$. The origin will be dealt with in the next section.

---

† g_n is a shorthand notation for a set of functions $\{b^\alpha_\ell, c^\tau_p\}_n$

Proof of (ii)

(a) The existence of a limit function for each boundary condition. First of all note that for each set of numbers $\{\eta_n^\gamma\}$ which make $V(c_n^\gamma)$ zero there exist C^1 functions defined on $(0,\infty)$ with the following properties:

$$\left.\begin{aligned} &(\alpha)\ \lim_{r\to\infty} r(c_n^\gamma(r)-\eta_n^\gamma)^2 = \lim_{r\to 0} r(c_n^\gamma(r)-\eta_n^\gamma)^2 = 0\,,\ \forall \gamma,n. \\ &\quad\ \lim_{r\to\infty} \frac{b_\ell^{\alpha 2}(r)}{r} = \lim_{r\to 0} \frac{b_\ell^{\alpha 2}(r)}{r} = 0\ ,\ \forall \alpha,\ell\,. \end{aligned}\right\} \qquad (3.9)$$

(β) $E(b,c)$ is finite.

Let $\dfrac{b(r)}{\sqrt{r}} = (B\circ\log)(r)$, $\sqrt{r}(c(r)-\eta) = (C\circ\log)(r)$,

and $y=\log r$ - I have dropped the indices - then

$$\int_0^\infty b'^2\,dr = \int_{-\infty}^\infty (B'^2+\tfrac{1}{4}B^2)\,dy\,,\ \text{and}\ \int_0^\infty r^2c'^2\,dr = \int_{-\infty}^\infty (C'^2+\tfrac{1}{4}C^2)\,dy\,.^*$$

B and C are thus elements of the Sobolev space $W^{1,2}(-\infty,\infty)$ [1]. The space of sets of functions over which variation is allowed is taken to be the completion of the space of sets of C^1 functions satisfying (3.9) in the norm topology given by the norm

$$\sum_{\alpha,\ell}\int_0^\infty b_\ell^{\alpha\prime 2}\,dr + \sum_{\gamma,n}\int_0^\infty r^2c_n^{\gamma\prime 2}\,dr = \sum_{\alpha,\ell}\int_{-\infty}^\infty (B_\ell^{\alpha\prime 2}+\tfrac{1}{4}B_\ell^{\alpha 2})\,dy + \sum_{\gamma,n}\int_{-\infty}^\infty (C_n^{\gamma\prime 2}+\tfrac{1}{4}C_n^{\gamma 2})\,dy.$$

Now consider the sequence $\{b_{\ell,n}^\alpha, c_{\rho,n}^\gamma\} := \{\{b_\ell^\alpha, c_\rho^\gamma\}_n\}$ with $E_n := E(\{b_\ell^\alpha, c_\rho^\gamma\}_n) \to E_0$. Since $\{E_n\}$ is a bounded sequence the sequences $\int (B_{\ell,n}^{\alpha\prime 2}+\frac{1}{4}B_{\ell,n}^{\alpha 2})\,dy$ and $\int (C_{\rho,n}^{\gamma\prime 2}+\frac{1}{4}C_{\rho,n}^{\gamma 2})\,dy$ are bounded. Every bounded sequence in a Hilbert space has a weakly convergent subsequence, and a Sobolev space is a Hilbert space. Also every bounded sequence of numbers has a limit point. Therefore, without loss of generality, it can be assumed that there are numbers N_ℓ^α , M_ρ^γ

---

* $B' := \dfrac{dB}{dy}$, $C' := \dfrac{dC}{dy}$.

and functions B_ℓ^α and C_p^τ such that $\int_{-\infty}^{\infty}(B_{\ell,n}^{\alpha/2}+\frac{1}{4}B_{\ell,n}^{\alpha 2})dy \to N_\ell^\alpha$ in $\mathbb{R}$ as $n\to\infty$

$$\int_{-\infty}^{\infty}(C_{p,n}^{\tau/2}+\frac{1}{4}C_{p,n}^{\tau 2})dy \to M_p^\tau \text{ in } \mathbb{R} \text{ as } n\to\infty$$

$$B_{\ell,n}^\alpha \xrightarrow{\text{weakly}} B_\ell^\alpha \;,\; C_{p,n}^\tau \xrightarrow{\text{weakly}} C_p^\tau \text{ in } W^{1,2}(-\infty,\infty) \text{ as } n\to\infty.$$

The Sobolev Embedding Theorem [1] asserts that the embedding $W^{1,2}(-\infty,\infty) \to C_B^0(-\infty,\infty)$ exists, therefore B_ℓ^α and C_p^τ are continuous, and we have the result that $b_{\ell,n}^\alpha \to b_\ell^\alpha$ and $c_{p,n}^\tau \to c_p^\tau$ pointwise on $(0,\infty)$, with b_ℓ^α and c_p^τ being in the original set of functions over which the infimum was taken.

(b) The set of limit functions is a stationary point.

One can write $E_n = \int_0^\infty F_n\,dr + \int_0^\infty G_n\,dr$ where F_n contains only derivatives of the radial functions and G_n contains no derivatives. It is a property of a Hilbert space, H, with norm $\|\cdot\|_H$ that if $h_n \xrightarrow{\text{wk.}} h$ in H then $\lim_{n\to\infty}\|h_n\|_H \geq \|h\|_H$. Consequently

$$\lim_{n\to\infty}\int_0^\infty F_n\,dr \geq \int_0^\infty F\,dr \tag{3.10}$$

F and G are the functions obtained from F_n and G_n respectively by replacing the set $\{b_{\ell,n}^\alpha, c_{p,n}^\tau\}$ by $\{b_\ell^\alpha, c_p^\tau\}$. Now $G_n \to G$ pointwise on $(0,\infty)$. Thus

$$0 \leq \int_a^b G(r)\,dr = \int_a^b (G(r)-G_n(r))\,dr + \int_a^b G_n(r)\,dr \;,\; b>a>0$$

$$\Rightarrow\; 0 \leq \int_a^b G(r)\,dr \leq \int_a^b (G(r)-G_n(r))\,dr + \int_0^\infty G_n(r)\,dr$$

$$\text{as } G(r), G_n(r) \geq 0 \;,\; 0<r<\infty$$

$$\Rightarrow\; 0 \leq \int_a^b G(r)\,dr \leq \lim_{n\to\infty}\int_0^\infty G_n(r)\,dr \;,\; \forall\, b<\infty$$

$$\Rightarrow\; 0 \leq \int_0^\infty G(r)\,dr \leq \lim_{n\to\infty}\int_0^\infty G_n(r)\,dr. \tag{3.11}$$

Note that $\int_0^\infty G(r)\,dr$ exists as $G(r) \geq 0$ and $\int_0^b G(r)\,dr$ is bounded for all values of b. Equations (3.10) and (3.11) show that

$$E(b_\ell^\alpha, c_p^\gamma) \leq \lim_{n\to\infty}\left\{\int_0^\infty F_n\,dr + \int_0^\infty G_n\,dr\right\} = E_0$$

and so the limit functions are a stationary point of E.

(c) Solutions of the radial equations.

The radial equations (which are Euler-Lagrange equations) have a weak solution because the integral has a stationary point. The adjective 'weak' is required because the Sobolev space used contains functions which have only weak derivatives. However the radial equations are ordinary differential equations and hence elliptic. This property can be used to show that the weak solution is in fact a strong solution [14].

3.4 Differentiability at the Origin

In this section it is shown that each of the solutions of the radial equations shown to exist in the preceding section generates a solution of the full equations. To do this it must be shown that when the radial functions are inserted in the expressions (2.17) and (2.18) they lead to fields which are C^1 at the origin. This is not trivially true as the radial functions have not yet been shown to be differentiable at the origin, and the basis functions are not even well defined at the origin.

Let $f_q^\beta(r) = r c_q^\beta(r)$. Omitting indices and constants in the sums the equations of motion which derive from the energy integral (3.5) are

$$b_\ell^{\alpha\prime\prime} = \frac{1}{r^2}\,l(l+1)\,b_\ell^\alpha + \frac{1}{r^2}\sum(b^2+b^3) + \frac{1}{r^2}\sum(f^2+bf^2), \qquad (3.12)$$

and $$f^{\xi\,\prime\prime}_{q} = \frac{1}{r^2} q(q+1) f^{\xi}_{q} + \frac{1}{r^2}\sum(bf + b^2 f) + \frac{1}{r^2}\sum(r^2 f + r f^2 + f^3) \quad (3.13)$$

The fact that V is a quartic with no linear term implies that its first derivatives are cubics with no constant term - which explains the form of the last sum in equation (3.13). If $q \neq 0$ then the right hand side of equation (3.13) contains no functions $f^{\gamma}_{n}(r)$ with $n=0$.

For small and large r the b's and f's are $o(1/r)$, and finiteness of energy guarantees that they are bounded by N/r for some constant N for all r. Changing notation $\{b^{\beta}_{m}, f^{\gamma}_{n}\} \to \{d_i\}$ (the 'angular momentum' associated with d_i is denoted t_i) it follows from equations (3.12) and (3.13) that

$$|d_i''| \leq \frac{\text{const.}}{r^{3/2}}$$

and $$\frac{d}{dr}\left[r^{2t_i+2}\frac{d}{dr}\left(\frac{d_i}{r^{t_i+1}}\right)\right] = \sum_j d_j \times o\left(r^{t_i-\frac{1}{2}}\right), \; r \text{ small.} \quad (3.14)$$

To derive this last equation the result

$$\frac{d}{dr}\left[r^{2t_i+2}\frac{d}{dr}\left(\frac{d_i}{r^{t_i+1}}\right)\right] = r^{t_i+1}\left[d_i'' - t_i(t_i+1)\frac{d_i}{r^2}\right]$$

is required. Proceeding in a similar manner to Rawnsley [27, 28] one can show from equation (3.14) that

$$\left.\begin{aligned} d_i(r) &\leq \text{constant} \times r \;, && t_i = 0 \\ d_i(r) &\leq \text{constant} \times r^2 \;, && t_i = 1 \\ d_i(r) &= o(r^2) \;, && t_i > 1 \end{aligned}\right\} \quad (3.15)$$

for small r.

Let $$\frac{d}{dr}\left[r^{2t_i+2}\frac{d}{dr}\left(\frac{d_i}{r^{t_i+1}}\right)\right] = r^{t_i+1} G_i(r) \;, \; r \neq 0. \quad (3.16)$$

$G_i(r)$ is the non-linear part of whichever of the equations (3.12) or

(3.13) is relevant; $G_i(r) \leqslant$ constant$\times r$. From equation (3.16)

$$r^{2t_i+2}\frac{d}{dr}\left[\frac{d_i}{r^{t_i+1}}\right] = \int_0^r u^{t_i+1}G_i(u)\,du = r^{t_i+2}\int_0^1 u^{t_i+1}G_i(ur)\,du, \quad r\neq 0; \tag{3.17}$$

which implies

$$\frac{d_i'(r)}{r} - (t_i+1)\frac{d_i(r)}{r^2} = \int_0^1 u^{t_i+1}G_i(ur)\,du, \quad r\neq 0. \tag{3.18}$$

If $t_i = 1$ equation (3.17) implies

$$\frac{d}{dr}\left[\frac{d_i}{r^2}\right] = \frac{1}{r}\int_0^1 du\, u^2 G_i(ur)$$

and so $$\frac{d_i(y)}{y^2} - \frac{d_i(x)}{x^2} = \int_x^y dr\int_0^1 du\,\frac{u^2}{r}G_i(ur). \tag{3.19}$$

Now the limit as $x \to 0$ of the right hand side exists, therefore $\lim_{x\to 0}\frac{d_i(x)}{x^2}$ exists. It can now be shown that $W_\mu(\underline{x})$ is C^1 at $\underline{x} = \underline{0}$.

If $a_\ell^\alpha(r) \equiv e_\ell^\alpha(r) \equiv 0$, $eW_\mu(\underline{x}) = \sum_{\alpha,\ell,i,j}\frac{\ell}{r}b_\ell^\alpha(r)\epsilon_{\mu ij}\hat{x}_j L^\alpha(\ell)_i$, and

$$e\partial_\nu W_\mu(\underline{x}) = \sum_{\alpha,\ell,i}\ell\left[\left(\frac{b_\ell^{\alpha\prime}}{r} - (\ell+1)\frac{b_\ell^\alpha}{r^2}\right)\hat{x}_\nu\sum_j\epsilon_{\mu ij}\hat{x}_j + \frac{b_\ell^\alpha}{r^2}\epsilon_{\mu i\nu}\right]L^\alpha(\ell)_i + \sum_{\alpha,\ell,i,j}\ell(\ell-1)\frac{b_\ell^\alpha}{r^2}\epsilon_{\mu ij}\hat{x}_j L^\alpha(\ell)_{\nu i}$$
$$\underline{x}\neq\underline{0}.$$

Now $\lim_{r\to 0}\left(\frac{b_\ell^{\alpha\prime}}{r} - (\ell+1)\frac{b_\ell^\alpha}{r^2}\right) = 0$, and $\lim_{r\to 0}\ell(\ell-1)\frac{b_\ell^\alpha}{r^2} = 0$ follow, for all α and ℓ, from equations (3.15) and (3.18). Also $\lim_{r\to 0}\frac{\ell b_\ell^\alpha}{r^2} = 0$ if $\ell \neq 1$, but it is equal to λ_1^α say, a constant, if $\ell = 1$. So

$$\lim_{r\to 0} e\partial_\nu W_\mu(\underline{x}) = \sum_{\alpha,i}\lambda_1^\alpha\epsilon_{\mu i\nu}L_i^\alpha(1).$$

As only the $\ell = 1$ part contributes there is no angular dependence.

Also

$$e\partial_\nu W_\mu(\underline{x}=\underline{0}) = \lim_{r\to 0}\sum_{\alpha,\ell,i}\frac{\ell b_\ell^\alpha(r)}{r^2}\epsilon_{\mu i\nu}L^\alpha_{(i\nu\nu\ldots\nu)} = \sum_{\alpha,i}\lambda_1^\alpha\epsilon_{\mu i\nu}L^\alpha(1).$$

This proves that $W_\mu(\underline{x})$ is C^1 in all of space.

As for $\underline{\Phi}$:

$$\partial_\mu \underline{\Phi}(\underline{x}=\underline{0}) = \lim_{r\to 0} \sum_{\alpha,l} \frac{1}{r}\left(c_l^\alpha(r) - c_l^\alpha(0)\right)\phi^\alpha_{(\mu\mu\cdots\mu)}$$

$$= \sum_{\alpha,l} c_l^{\alpha\prime}(0)\,\phi^\alpha_{(\mu\mu\cdots\mu)} \quad \text{if } c_l^{\alpha\prime}(0) \text{ exists.} \qquad (3.20)$$

For $\underline{x} \neq \underline{0}$, $\partial_\mu \underline{\Phi}(\underline{x}) = \sum_{\alpha,l}\left[\hat{x}_\mu\left(c_l^{\alpha\prime} - \frac{l c_l^\alpha}{r}\right)\phi^\alpha(l) + \frac{l c_l^\alpha}{r}\phi^\alpha(l)_\mu\right]$. (3.21)

Now $\lim_{r\to 0}\left[c_l^{\alpha\prime} - \frac{l c_l^\alpha}{r}\right] = \lim_{r\to 0}\left[\frac{f_l^{\alpha\prime}}{r} - (l+1)\frac{f_l^\alpha}{r^2}\right] = 0$ for all α and l by equation (3.18). Also $\lim_{r\to 0}\frac{l c_l^\alpha}{r} = 0$ unless $l=1$ as $c_l^\alpha(r)$ is $o(r)$ for $l>1$, for $l=1$ the limit is $c_1^{\alpha\prime}(0)$ if this derivative exists. If $b_i = 0$ equation (3.17) implies that

$$\frac{d_i(y)}{y} - \frac{d_i(x)}{x} = \int_x^y dr \int_0^1 du\, u\, G_i(ur).$$

This shows that $\lim_{x\to 0}\frac{d_i(x)}{x}$ exists and it is equal to $c_0^\xi(0)$ for some ξ. So

$$c_0^\xi(y) - c_0^\xi(0) = \int_0^y dr \int_0^1 du\, u\, G_i(ur) = y\int_0^1 dv \int_0^1 du\, u\, G_i(uvy)$$

and $c_0^{\xi\prime}(0) = \lim_{y\to 0}\frac{1}{y}\left(c_0^\xi(y) - c_0^\xi(0)\right) = 0.$

If $l=1$, $c_l^{\alpha\prime}(0)$ exists because of equation (3.19). If $l>1$, $c_l^{\alpha\prime}(0) = 0$ as $c_l^\alpha(y)$ is $o(y)$. Using all of these results in equations (3.20) and (3.21) one finds

$$\lim_{\underline{x}\to\underline{0}} \partial_\mu \underline{\Phi}(\underline{x}) = \partial_\mu \underline{\Phi}(\underline{x}=\underline{0}) = \sum_\alpha c_1^{\alpha\prime}(0)\,\phi^\alpha(1)_\mu$$

and it is well defined as only $l=1$ terms contribute, therefore $\underline{\Phi}$ is C^1.

The energy density is C^1 as a function of W_μ, $\underline{\Phi}$, $\partial_\nu W_\mu$ and $\partial_\nu \underline{\Phi}$, therefore the fact that both W_μ and $\underline{\Phi}$ are C^1, combined with the fact that they are a stationary point of the energy integral, guarantees

that they satisfy the Euler-Lagrange equations even though they are second order equations [7].

3.5 The Group Which Transforms '$\underline{J}$ + $\underline{T}$' Symmetric Field Configurations into '$\underline{J}$ + $\underline{T}$' Symmetric Field Configurations

In this section the question of the existence of theories for which the energy integral can be reduced to the form (3.5) before variation is considered. For such theories it has been shown that corresponding to each set of numbers $\{\eta^{\gamma}_{n}\}$ with $V(\eta^{\gamma}_{n}) = 0$ there is a solution. However the method of proof is such that it cannot give any indication of whether or not there is more than one solution for a given boundary value of the Higgs field - despite the fact that there may be more than one point solution. Another property of the proof is that if a certain subset of the set of radial functions can be set equal to zero before variation without adversely affecting the equations of motion then nothing is gained by retaining them. The reason is that the method of proof cannot demonstrate the existence of a solution for which they are non-zero. This is why there is no point in trying to show that dyons exist; $W_o(\underline{x}) \equiv 0$ satisfies the $W_c(\underline{x})$ equation identically.

I shall now show that there are theories in which some of the radial functions can be set equal to zero.

The generators of the group which transforms '$\underline{J}$ + $\underline{T}$' invariant field configurations into '$\underline{J}$ + $\underline{T}$' invariant field configurations are $\{ L^{\alpha}(l) \mid \text{all } \alpha \text{ and } l \}$ or, on the z axis, $\{ L^{\alpha}_{(33\ldots3)} \}$. If the $l \neq 0$ generators (i.e. those with indices) of this group commute then all of them may be simultaneously diagonalised to real matrices. Also, when T_3 is diagonal, T_+ can be chosen real with non-zero entries in only

the diagonal above the leading diagonal. So for an abelian $\ell \neq 0$ subalgebra $[T_+, L^\alpha_{(33\ldots3)}]$ can be chosen real with non-zero entries in only the diagonal above the leading diagonal. Given this form for $[T_+, L^\alpha_{(33\ldots3)}]$ it follows from equations (3.2), (3.3) and the following equation (obtained from equations (2.19) and (2.20))

$$M_+ = \sum_{\alpha,\ell} \ell(B^\alpha_\ell + iA^\alpha_\ell) L^\alpha(\ell)_+ = -\sum_{\alpha,\ell} (B^\alpha_\ell + iA^\alpha_\ell)[T_+, L^\alpha_{(33\ldots3)}]$$

that $a^\alpha_\ell \equiv e^\alpha_\ell \equiv 0$ satisfies the equations of the a's and e's obtained by varying only the $\mathrm{Tr}(G_{\mu\nu}G^{\mu\nu})$ term identically. The covariant derivative term is more troublesome.

$$(\mathfrak{D}_\mu \Phi)^+(\mathfrak{D}_\mu \Phi)(r\hat{z}) = \left(\Phi_0' - \frac{i}{r}\mathcal{D}(N)\Phi_0\right)^+\left(\Phi_0' - \frac{i}{r}\mathcal{D}(N)\Phi_0\right) + \frac{1}{2r^2}\Phi_0^+\{\mathcal{D}(M_+),\mathcal{D}(M_-)\}\Phi_0 .$$

The second term gives no trouble as $\{\mathcal{D}(M_+),\mathcal{D}(M_-)\}$ contains no cross terms between the A's and the B's. The first term is

$$\Phi_0'^+\Phi_0' + \frac{1}{r^2}\Phi_0^+\mathcal{D}(N)^2\Phi_0 + \frac{i}{r}\left[\Phi_0^+\mathcal{D}(N)\Phi_0' - \Phi_0'^+\mathcal{D}(N)\Phi_0\right] .$$

The part in the square brackets is the problem. It has been shown in chapter 2 that all but the $\ell=1$ part of N maybe gauged away. In this gauge the term in square brackets is zero if $\mathcal{D}(L^\alpha(1))\Phi = 0$ for all α ($\ell=1$) for an arbitrary '$\underline{J} + \underline{I}$' symmetric Φ. This is the case if there is only one basis function $L^\alpha(1)$, for this basis function must then be T and $\mathcal{D}(T)\Phi=0$ is a consequence of '$\underline{J} + \underline{I}$' symmetry. If Φ transforms according to the adjoint representation and the $\ell=1$ generators of this algebra commute with each of the other generators of the algebra which transforms '$\underline{J} + \underline{I}$' invariant fields into '$\underline{J} + \underline{I}$' fields then $\mathcal{D}(L^\alpha(1))\Phi = 0$.

In SU(n) theories ($n>2$) there are two embeddings of the SU(2) algebra which have $\hat{x}_\mu T$ as the only $\ell=1$ basis function for the radial

part of ϵW_μ, these are:

(i) The "maximal" embedding for which the fundamental representation of SU(n) gives an irreducible representation of SU(2). The n dimensional vector space decomposes in the manner $n \to n$, so that the adjoint representation which is $(n^2 - 1)$ dimensional decomposes in the manner

$(n^2 - 1) \to (2n-1) \oplus (2n-3) \oplus \ldots \oplus 3$. E.g. $15 \to 7 \oplus 5 \oplus 3$ for SU(4).

(ii) The "minimal" embedding for which the fundamental representation of SU(n) decomposes in the fashion

$n \to 2 \oplus 1 \oplus 1 \oplus \ldots \oplus 1$ with (n-2) ones.

The adjoint representation decomposes as follows:

$(n^2 - 1) \to 3 \oplus 1 \oplus 1 \oplus \ldots \oplus 1$ with (n^2-4) ones.

E.g. SU(4) $4 \to 2 \oplus 1 \oplus 1$

and $15 \to 3 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 1$.

For these embeddings the generators $\{L^\alpha_{(33\ldots3)} \mid \ell \geq 1\}$ commute with all of the generators of the algebra. For (i) this follows from the fact that T_3 has no repeated eigenvalues, and for (ii) from the fact that T_3 is the only element of the set $\{L^\alpha_{(33\ldots3)} \mid \ell \geq 1\}$. In these cases the integral (3.5) can be used no matter what representation is chosen for $\bar{\Phi}$.

For n odd and $n \geq 5$ there are still more embeddings of $\mathcal{L}(SU(2))$ in $\mathcal{L}(SU(n))$ for which the elements of $\{L^\alpha_{(33\ldots3)} \mid \ell \geq 1\}$ commute with all of the generators. These embeddings correspond to partitioning the fundamental representation into two irreducible representations of SU(2); one of even dimension and one of odd dimension.

E.g. SU(5) (a) $5 \to 3 \oplus 2$

$$24 \to 5 \oplus 4 \oplus 4 \oplus 3 \oplus 3 \oplus 2 \oplus 2 \oplus 1$$

(b) $5 \to 4 \oplus 1$

$$24 \to 7 \oplus 5 \oplus 4 \oplus 4 \oplus 3 \oplus 1$$

The generators $\{ L^{\alpha}_{(33\ldots3)} \mid \text{all } \ell \text{ including } \ell = 0 \}$ commute because T_3 has no repeated eigenvalues. These embeddings do not necessarily satisfy $\mathcal{D}(L^{\alpha}(1))\underline{\Phi} = 0$ for all α with $\underline{\Phi}$ transforming to an arbitrary representation, but they do if $\underline{\Phi}$ transforms according to the adjoint representation.

3.6 Asymptotic Behaviour

In this section some minor results concerning the asymptotic behaviour of monopole solutions will be derived. Further results can be found in ref. [30].

The $(\text{mass})^2$ matrix in a spontaneously broken gauge theory is proportional to $v^{\dagger}\{\mathcal{D}(G^{\alpha}),\mathcal{D}(G^{\beta})\}v$, where v is the asymptotic limt of $\underline{\Phi}$ in some direction and $G^{\alpha}, G^{\beta} \in \mathcal{L}(G)$. When considering '$\underline{J} + \underline{I}$' symmetric monopoles a term $\underline{\Phi}^{+}_{0}(\infty)\{\mathcal{D}(M_+),\mathcal{D}(M_-)\}\underline{\Phi}_{0}(\infty)$ appears. This is like a mass term. It corresponds to a $(\text{mass})^2$ matrix of the form

$$\underline{\Phi}^{+}_{0}(\infty)\left[\mathcal{D}(L^{\alpha}(1)_+)\mathcal{D}(L^{\beta}(m)_-) + \mathcal{D}(L^{\alpha}(1)_-)\mathcal{D}(L^{\beta}(m)_+)\right]\underline{\Phi}_{0}(\infty)$$

However the eigenvalues of this matrix will only correspond to the true particle masses of the theory if the full $(\text{mass})^2$ matrix can be written in block diagonal form with this matrix appearing as one of the blocks. (in writing these expressions I am ignoring an overall factor). I shall now show that it does indeed form such a block.

Consider the algebra $\mathcal{L}(G)$. If complex coefficients are allowed

when forming linear combinations of vectors in this space the complexification of the algebra is generated. In the complexification of $\mathcal{L}(G)$ a basis may be chosen each of the elements of which is an eigenvector of T_3. Let the basis elements associated with the eigenvalue λ be denoted by $V^a(\lambda)$. $\Phi_0(\infty)$ is an eigenvector of $D(T_3)$ with eigenvalue 0. $D(V^a(\lambda))\Phi_0(\infty)$ is another with eigenvalue λ. Hence

$$[D(V^a(\lambda_1)\Phi_0(\infty)]^+ D(V^b(\lambda_2))\Phi_0(\infty) = 0$$

unless $\lambda_1 = \lambda_2$. Thus the $(\text{mass})^2$ matrix splits into blocks corresponding to different eigenvalues of T_3. The $\lambda=+1$ space is spanned by $\{L^\alpha(l)_+\}$ and the $\lambda=-1$ space by $\{L^\alpha(l)_-\}$. In the direct sum of these two subspaces (which gives rise to a block on the diagonal) the $(\text{mass})^2$ matrix is proportional to

$$[D(L^\alpha(l)_+ + L^\alpha(l)_-)\Phi_0(\infty)]^+ D(L^\beta(m)_+ + L^\beta(m)_-)\Phi_0(\infty)$$

$$= \Phi_0(\infty)^+ [D(L^\alpha(l)_-)D(L^\beta(m)_+) + D(L^\alpha(l)_+)D(L^\beta(m)_-)]\Phi_0(\infty)$$

which proves the assertion.

A similar statement is <u>not</u> true for the Higgs fields; the eigenvalues of $\frac{\partial^2 V(c^\gamma_\ell)}{\partial c^\gamma_p \partial c^\gamma_n}$ are not, in general, the masses of the Higgs particles of the theory (see chapter 4 for an example of this).

I shall now show that if the Higgs multiplet transforms according to the adjoint representation, or if there are a number of multiplets which do so, and

$$\Phi_0(\infty)^+ \{D(M_+), D(M_-)\}\Phi_0(\infty) = 0 \Rightarrow M_+ = 0$$

and furthermore the group which transforms '$\underline{J}$ + $\underline{I}$' symmetric fields into '$\underline{J}$ + $\underline{I}$' symmetric fields is abelian, then for the solutions which

have been shown to exist the gauge fields approach the form

$$eB_i = -\frac{\hat{x}_i}{r^2}T$$

exponentially quickly for $r \gg \frac{1}{m}$ where m is the mass of the lightest vector boson present in the $\ell = \pm 1$ block of the (mass)^2 matrix. For simplicity I shall consider only the case the case when one multiplet is present. The extension to any finite number of multiplets is trivial.

When $T_3 = diag(\lambda_1, \lambda_2, \ldots, \lambda_{n+1})$ M_+ can be chosen to have the form

$$\begin{bmatrix} 0 & h_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & h_2 & 0 & \cdots & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & h_n \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

M_+ is an $(n+1)\times(n+1)$ matrix. The h's are real functions of r; some may be zero. One can take Φ_0 to be

$$\Phi_0 = diag(g_1, g_2, \ldots, g_{n+1}).$$

$$[M_+, \Phi_0] = diag[h_1(g_2-g_1), h_2(g_3-g_2), \ldots, h_n(g_{n+1}-g_n)],$$

and $$[M_+, M_-] = diag(h_1^2, h_2^2-h_1^2, h_3^2-h_2^2, \ldots, -h_n^2).$$

Hence $$e^2 Tr(B_i B_i) = \sum_{i=1}^{n} \left[h_i'^2 + \frac{1}{4r^2}(h_i^2 - h_{i-1}^2 - 2\lambda_i)^2 \right],$$

and $$Tr\left([M_+, \Phi_0][\Phi_0, M_-]\right) = \sum_{i=1}^{n} h_i^2 (g_{i+1}-g_i)^2.$$

The h equations are therefore

$$h_i'' = \frac{h_i}{2r^2}\left[2h_i^2 - h_{i+1}^2 - h_{i-1}^2 - 2\lambda_i + 2\lambda_{i+1}\right] + e^2 K^2 h_i (g_{i+1}-g_i)^2. \ *$$

---

* Similar equations are satisfied if further multiplets of the same sort are added.

If $h_i \not\equiv 0$ then by hypothesis $\lim_{r\to\infty} e^2k^2(g_{i+1}-g_i)^2 = m_i^2$, say, with $m_i > 0$. Now $h_i(r) = o(\sqrt{r})$ for large r, so for large r

$$h_i'' = h_i\left(m_i^2 + o\left(\frac{1}{\sqrt{r}}\right)\right) \quad **$$

and so $h_i = \text{constant} \times \exp(-m_i r)$ for large r.

A little more information, concerning the behaviour of the Higgs fields may be found in ref. [30].

** Similar equations are satisfied if further multiplets of the same sort are added.

CHAPTER 4

AN SU(3) EXAMPLE

4.1 Introduction

Below I describe a model which illustrates many of the features of the general arguments given in the preceding chapters. This model is of interest not because it is useful for describing the physical world, but because it has the smallest possible non-abelian residual symmetry group after spontaneous symmetry breakdown has occured (i.e. the symmetry group of the particle states is SO(3)), and because the Higgs fields transform according to a representation other than the adjoint representation - a circumstance which has not been extensively investigated. Owing to the model's simplicity it is possible to determine the large r behaviour of the one '$\underline{J} + \underline{I}$' symmetric monopole that the model possesses and this is done. This solution has a topological charge but it is not possible to define a magnetic charge in the usual way because the Higgs fields do not transform according to the adjoint representation. It also has the peculiar feature that the group which classifies the possible asymptotic limits of the Higgs fields up to homotopic equivalence is Z_2. A physical theory would require the group of integers, Z.

4.2 The Model

The model is that of an SU(3) gauge theory with the Higgs fields transforming according to the (2,0) representation of SU(3). The potential is chosen so that the symmetry is broken spontaneously to SO(3). The Higgs fields can be arranged so as to form a 3×3, complex, symmetric matrix. The action of SU(3) on the Higgs fields is given by

$$\Phi \to U\Phi U^{T} \quad , \quad U \in SU(3).$$

The potential is

$$V(\Phi) = \lambda_1 \mathrm{Tr}[(\Phi^\dagger - \mathrm{adj}\Phi)(\Phi - \mathrm{adj}\Phi^\dagger)] + \lambda_2 \mathrm{Tr}[(I - \Phi^\dagger\Phi)^2]$$

$$= \lambda_1 (\Phi^*_{ij} - \tfrac{1}{2}\epsilon_{ilm}\epsilon_{jpq}\Phi^{lp}\Phi^{mq})(\Phi^{ij} - \tfrac{1}{2}\epsilon^{irs}\epsilon^{jtu}\Phi^*_{rt}\Phi^*_{su})$$
$$+ \lambda_2 (\delta^i_i - \Phi^*_{ip}\Phi^{pj})(\delta^i_j - \Phi^*_{jq}\Phi^{qi})$$

$$= \lambda_1 [\mathrm{Tr}(\Phi^\dagger\Phi) - 3\det\Phi - 3\det\Phi^\dagger + \tfrac{1}{2}\{\mathrm{Tr}(\Phi^\dagger\Phi)\}^2 - \tfrac{1}{2}\mathrm{Tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi)]$$
$$+ \lambda_2 [3 - 2\mathrm{Tr}(\Phi^\dagger\Phi) + \mathrm{Tr}(\Phi^\dagger\Phi\Phi^\dagger\Phi)].$$

$\mathrm{adj}\Phi$ is the adjugate matrix.

I shall now show that the potential has the property that I have claimed for it.

$V(\Phi)=0$ implies $\quad I = \Phi^\dagger\Phi, \qquad (4.1)$

and $\quad \Phi^\dagger = \mathrm{adj}\,\Phi. \qquad (4.2)$

(4.1) implies that Φ is unitary, and as it is symmetric it can be written in the form

$$\Phi = K U U^T$$

with $K \in \mathbb{C}$, $|K|=1$, and $U \in SU(3)$, $U = U^T$, as any SU(3) matrix can be written in the form $U^2 (U \in SU(3))$ with U symmetric if the original matrix is symmetric. Equation (4.2) now implies that $K^* = K^2$, i.e. $K^3 = 1$. Hence $KI = Z$ where Z is an element of the centre of SU(3). Now $Z = Z'^2 = Z'Z'^T$ for some other element Z' of the centre, so

$$\Phi = KUU^T = (Z'U)(Z'U)^T.$$

That is any Φ which minimises $V(\Phi)$ may be written in the form UU^T with $U \in SU(3)$. Now $\Phi = I$ minimises $V(\Phi)$ and the element U of SU(3) takes $\Phi = I$ to $\Phi = UU^T$. Combining this with the previous result we have that Φ minimises V if and only if Φ lies on the orbit of the identity matrix under the given SU(3) action. The little group of

the identity matrix is obviously SO(3). The space of Higgs fields which minimise V is thus homeomorphic to the quotient space SU(3)/SO(3). SU(3) is simply connected, therefore

$$\pi_2(SU(3)/SO(3)) = \pi_1(SO(3)) = Z_2$$

SU(3) has eight generators whilst SO(3) has three generators so there are five generators corresponding to broken symmetries, the Higgs potential will therefore give rise to five massless particles (ignoring the coupling to gauge potentials). The SO(3) subgroup referred to is the actual matrix group SO(3), so the fundamental representation of SU(3) decomposes in the manner $3 \to 3$ under its action. It follows that the adjoint, (1,1), representation decomposes in the manner $8 \to 5 \oplus 3$, and the (2,0) in the fashion $6 \to 5 \oplus 1$, or $12 \to 5 \oplus 5 \oplus 1 \oplus 1$ in terms of real fields. It is easy to give an elementary explanation of the last decomposition. As the SO(3) matrices are real the real and imaginary parts of Φ transform separately under their action. Each of these six dimensional invariant subspaces decomposes further for both the real and imaginary parts can be written as a multiple of the identity matrix plus a traceless symmetric matrix. Now

$$Tr(OXO^T) = Tr(O^TOX) = Tr(X) \quad \text{for} \quad O \in SO(3)$$

and $OIO^T = I$,

so the decomposition is invariant under the action of the SO(3) subgroup, and the decomposition predicted above has been explicitly given.

I shall now find the masses of the Higgs particles. Under the variation

$$\Phi \to \Phi + \delta\Phi \quad , \text{ with } \Phi = I \text{ initially,}$$

(i) $\delta[\mathrm{Tr}(\Phi^+\Phi)] = \mathrm{Tr}(\delta\Phi^+ + \delta\Phi) + \mathrm{Tr}(\delta\Phi^+\delta\Phi)$,

(ii) $\delta\{[\mathrm{Tr}(\Phi^+\Phi)]^2\} = [\mathrm{Tr}(\Phi^+\Phi + \delta\Phi^+\Phi + \Phi^+\delta\Phi + \delta\Phi^+\delta\Phi)]^2\big|_{\Phi=I} - 9$

$= 6\,\mathrm{Tr}(\delta\Phi^+ + \delta\Phi) + 6\,\mathrm{Tr}(\delta\Phi^+\delta\Phi) + [\mathrm{Tr}(\delta\Phi^+ + \delta\Phi)]^2$

to second order,

(iii) $\delta(\mathrm{Tr}(\Phi^+\Phi\Phi^+\Phi)) = \mathrm{Tr}[(I+\delta\Phi^+)(I+\delta\Phi)(I+\delta\Phi^+)(I+\delta\Phi)] - 3$

$= 2\,\mathrm{Tr}(\delta\Phi^+ + \delta\Phi) + \mathrm{Tr}(\delta\Phi\delta\Phi + \delta\Phi^+\delta\Phi^+ + 4\,\delta\Phi^+\delta\Phi)$

to second order,

(iv) $\det\Phi = \prod_{i=1}^{3}\lambda_i$, where λ_1, λ_2, and λ_3 are the eigenvalues of Φ,

$\delta(\det\Phi)\big|_{\Phi=I} = \sum_i \delta\lambda_i + \frac{1}{2}\sum_{i,j,k} |\epsilon_{ijk}|\,\delta\lambda_j\,\delta\lambda_k$ to second order

$= \sum_i \delta\lambda_i + \frac{1}{2}\left(\sum_i \delta\lambda_i\right)^2 - \frac{1}{2}\sum_i (\delta\lambda_i)^2$

$= \mathrm{Tr}(\delta\Phi) + \frac{1}{2}[\mathrm{Tr}(\delta\Phi)]^2 - \frac{1}{2}\mathrm{Tr}(\delta\Phi^2)$.

So $\delta V(\Phi=I) = \lambda_1\left[\mathrm{Tr}\{(\delta\Phi^+ + \delta\Phi)^2\} - \{\mathrm{Tr}(\delta\Phi^+)\}^2 - \{\mathrm{Tr}(\delta\Phi)\}^2 + \mathrm{Tr}(\delta\Phi^+)\mathrm{Tr}(\delta\Phi)\right]$

$+ \lambda_2\,\mathrm{Tr}[(\delta\Phi^+ + \delta\Phi)^2]$

Let $\delta\Phi = \eta + i\xi$, $\eta = \eta_0 + \eta_1$, and $\xi = \xi_0 + \xi_1$, where η_0, η_1, ξ_0 and ξ_1 are real, η_0 and ξ_0 are proportional to the identity matrix, and η_1 and ξ_1 are traceless and symmetric. Then

$\delta V(\Phi=I) = \lambda_1\left[\mathrm{Tr}\{(2\eta)^2\} - \{\mathrm{Tr}(\eta - i\xi)\}^2 - \{\mathrm{Tr}(\eta + i\xi)\}^2 + \mathrm{Tr}(\eta - i\xi)\,\mathrm{Tr}(\eta + i\xi)\right]$

$+ \lambda_2\,\mathrm{Tr}[(2\eta)^2]$.

$$\delta V(\Phi=I) = \lambda_1\left[4\,\mathrm{Tr}(\eta_0^2+\eta_1^2) - 2(\mathrm{Tr}\,\eta_0)^2 + 2(\mathrm{Tr}\,\xi_0)^2 + (\mathrm{Tr}(\eta_0))^2 + (\mathrm{Tr}(\xi_0))^2\right]$$

$$+ 4\lambda_2\,\mathrm{Tr}\left[\eta_0^2+\eta_1^2\right]$$

$$= \lambda_1\left[4\,\mathrm{Tr}(\eta_0^2+\eta_1^2) - 3\,\mathrm{Tr}(\eta_0^2) + 9\,\mathrm{Tr}(\xi_0^2)\right] + 4\lambda_2\,\mathrm{Tr}(\eta_0^2+\eta_1^2)$$

as $(\mathrm{Tr}\,\eta_0)^2 = 3\,\mathrm{Tr}(\eta_0^2)$ and $(\mathrm{Tr}\,\xi_0)^2 = 3\,\mathrm{Tr}(\xi_0^2)$.

$$\therefore\ \delta V(\Phi=I) = 9\lambda_1\,\mathrm{Tr}(\xi_0^2) + (\lambda_1+4\lambda_2)\,\mathrm{Tr}(\eta_0^2) + 4\lambda_1\,\mathrm{Tr}(\eta_1^2)$$

to second order.

The $(\text{mass})^2$ matrix of the vector bosons is given by

$$M^2_{jk} = \frac{e^2}{4}\mathrm{Tr}\left[\Phi^+\lambda_j\lambda_k\Phi + \Phi^+\lambda_j\Phi\tilde{\lambda}_k + \tilde{\lambda}_j\Phi^+\lambda_k\Phi + \tilde{\lambda}_j\Phi^+\Phi\tilde{\lambda}_k\right]$$

with $\Phi = I$.

$$M^2_{jk} = 2e^2(1-\delta_{2j}-\delta_{5j}-\delta_{7j})\,\delta_{jk} \qquad \text{- j not summed.}$$

We see that the Higgs-Kibble mechanism gives rise to three massless vector bosons, five massive vector bosons of mass $\sqrt{2}e$, one scalar boson of mass $\sqrt{18\lambda_1}$, one boson of mass $\sqrt{2\lambda_1+8\lambda_2}$, and five bosons of mass $\sqrt{8\lambda_1}$. The definition

$$\mathcal{L} = -\tfrac{1}{2}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right) - \tfrac{1}{2}\mathrm{Tr}\left[(\mathcal{D}_\mu\Phi)^+(\mathcal{D}^\mu\Phi)\right] - V(\Phi)$$

has been used with $G_{\mu\nu} = G^a_{\mu\nu}\frac{\lambda_a}{2}$, the λ's being the usual Gell-Mann matrices.

4.3 'J + T' Symmetric Monopoles

There are two inequivalent embeddings of the algebra of SU(2) into the algebra of SU(3). They correspond to choosing either the natural SO(3) subgroup or the subgroup of matrices of the form

$$\begin{pmatrix} SU(2) & 0 \\ 0 & 1 \end{pmatrix}.$$

In the first case, as noted above, the (1,1) representation decomposes in the following way:

$$8 \to 5 \oplus 3$$

and the (2,0) representation decomposes in the manner

$$6 \to 5 \oplus 1$$

under the action of the subgroup. For the second choice of subgroup

$$8 \to 3 \oplus 2 \oplus 2 \oplus 1$$

and $6 \to 3 \oplus 2 \oplus 1$.

Demanding '$\underline{J} + \underline{T}$' symmetry rules out the possibility of a topological charge for the first embedding (if reasonable assumptions are made about the asymptotic behaviour of monopole solutions), but not for the second [20] . I shall therefore consider only the second embedding from now onwards.

For the second, SU(2), embedding one has to construct a $\Phi_0(r)$ such that

$$\lambda_3 \Phi_0 + \Phi_0 \lambda_3^T = 0.$$

This implies that has the form

$$\begin{pmatrix} 0 & \phi_1 & 0 \\ \phi_1 & 0 & 0 \\ 0 & 0 & \phi_2 \end{pmatrix}$$

To find the values of off the z axis one can either calculate

$$U(R)^\dagger \Phi_0 U^*(R)$$

or solve the following equations to obtain basis functions:

$$\lambda_i K_j + K_j \lambda_i^T = \sum_k 2i \epsilon_{ijk} K_k \; ; \quad i,j,k = 1,2,3. \tag{4.3}$$

and $$\lambda_i S + S\lambda_i^T = 0. \tag{4.4}$$

Solutions of equations (4.3) and (4.4) are

$$\underline{K} = (K_1, K_2, K_3) = \left[\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right]$$

$$= (i\lambda_3, \tfrac{2}{3}I + \tfrac{1}{\sqrt{3}}\lambda_8, -i\lambda_1)$$

and $$S = \tfrac{1}{3}(I - \sqrt{3}\lambda_8) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $\psi_3 = \sum_{i=1}^{3} \frac{1}{\sqrt{2}} \hat{x}_i K_i$, and $\psi_4 = S$, then $\Phi(x) = a(r)\psi_3 + b(r)\psi_4$ is the most general expression for a '$\underline{J} + \underline{I}$' symmetric Φ. a and b are complex functions of r.

The most general '$\underline{J} + \underline{I}$' symmetric expression for the gauge potentials is [10]

$$eW_\mu(x) = \gamma(r)\partial_\mu \psi_1 + i(1-\delta(r))[\psi_1, \partial_\mu \psi_1] + \zeta(r)\frac{x_\mu}{r}\psi_1 + \eta(r)\frac{x_\mu}{r}\psi_2.$$

$$\psi_1 = \sum_{i=1}^{3} \tfrac{1}{2} \hat{x}_i \lambda_i \quad \text{and} \quad \psi_2 = \tfrac{1}{2}\lambda_8$$

The quantities which appear in the energy integral can now be calculated.

$$M_+ = \begin{bmatrix} 0 & \delta + i\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad N = \frac{1}{2}\begin{bmatrix} \zeta + \frac{\eta}{2\sqrt{3}} & 0 & 0 \\ 0 & -\zeta + \frac{\eta}{2\sqrt{3}} & 0 \\ 0 & 0 & -\frac{\eta}{\sqrt{3}} \end{bmatrix}$$

$$\Phi_0 = \begin{bmatrix} 0 & \frac{-ia}{\sqrt{2}} & 0 \\ \frac{-ia}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & b \end{bmatrix}.$$

$$(\mathcal{D}_3\Phi)(r\hat{z}) = \Phi_0' - \frac{i}{r}(N\Phi_0 + \Phi_0 N^T) = \begin{bmatrix} 0 & -\frac{i}{\sqrt{2}}(a' - \frac{i\eta a}{2\sqrt{3}r}) & 0 \\ -\frac{i}{\sqrt{2}}(a' - \frac{i\eta a}{2\sqrt{3}r}) & 0 & 0 \\ 0 & 0 & b' + \frac{i\eta b}{\sqrt{3}r} \end{bmatrix}$$

$$(\mathcal{D}_+\Phi)(r\hat{z}) = -\frac{1}{r}(M_+\Phi_0 + \Phi_0 M_+^T) = \begin{bmatrix} -i\sqrt{2}(\delta + i\gamma)a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$eB_3(r\hat{z}) = \frac{1}{r^2}\left(\frac{1}{2}[M_+, M_-] - T_3\right) = \frac{1}{2r^2}\begin{bmatrix} \delta^2+\gamma^2-1 & 0 & 0 \\ 0 & 1-\delta^2-\gamma^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$eB_+(r\hat{z}) = \frac{M_+'}{r} + \frac{1}{r^2}[M_+, N] = \frac{1}{r}\begin{bmatrix} 0 & \delta' + i\gamma' - \frac{i}{r}(\delta + i\gamma) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Phi_0^+ - \mathrm{adj}\,\Phi_0 = \begin{bmatrix} 0 & \frac{i}{\sqrt{2}}(a^* - ab) & 0 \\ \frac{i}{\sqrt{2}}(a^* - ab) & 0 & 0 \\ 0 & 0 & b^* - \frac{a^2}{2} \end{bmatrix}$$

$$I - \Phi_0^+\Phi_0 = \begin{bmatrix} 1 - \frac{|a|^2}{2} & 0 & 0 \\ 0 & 1 - \frac{|a|^2}{2} & 0 \\ 0 & 0 & 1 - |b|^2 \end{bmatrix}.$$

So

$$\mathcal{H}(r) = \frac{1}{e^2}\left[\left(\delta' + \frac{\gamma\delta}{r}\right)^2 + \left(\gamma' - \frac{\delta\delta}{r}\right)^2 + \frac{1}{2r^2}(\delta^2+\gamma^2-1)^2\right]$$

$$+ \frac{1}{2}\left[\left|a' - \frac{i\eta a}{2\sqrt{3}r}\right|^2 + \left|b' + \frac{i\eta b}{\sqrt{3}r}\right|^2 + 2(\delta^2+\gamma^2)|a|^2\right]$$

$$+ r^2\left[\lambda_1\left(|a^* - ab|^2 + |b^* - \tfrac{a^2}{2}|^2\right) + \lambda_2\left(2(1 - \tfrac{|a|^2}{2})^2 + (1-|b|^2)^2\right)\right].$$

The generators of the group which transforms '$\underline{J} + \underline{T}$' symmetric field configurations into '$\underline{J} + \underline{T}$' symmetric field configurations are (the appropriate representations of) ψ_1 and ψ_2, or on the z axis $\frac{1}{2}\lambda_3$, and $\frac{1}{2}\lambda_8$. The first has no effect on the Higgs fields by the definition of '$\underline{J} + \underline{T}$' symmetry, but the second does have an effect. Under an infinitessimal transformation involving ψ_2, $e^{i\delta\theta\psi_2}$ ($\delta\theta$ infinitessimal)

$$\Phi \rightarrow \Phi + i\delta\theta(\psi_2\Phi + \Phi\psi_2^T) = \Phi + i\delta\theta\{\psi_2, \Phi\}.$$

Now $\Phi = a\psi_3 + b\psi_4$

$$\{\psi_2, \psi_3\} = \frac{1}{\sqrt{3}}\psi_3 \quad \left(\{\tfrac{1}{2}\lambda_8, \tfrac{1}{2}\lambda_3\} = \frac{1}{2\sqrt{3}}\lambda_3\right),$$

and $\{\psi_2, \psi_4\} = -\frac{2}{\sqrt{3}}\psi_4$.

So under a finite transformation corresponding to $e^{i\theta\psi_2}$

$$\Phi = a\psi_3 + b\psi_4 \rightarrow \Phi' = a\exp\left(\frac{i\theta}{\sqrt{3}}\right)\psi_3 + b\exp\left(-\frac{2i\theta}{\sqrt{3}}\right)\psi_4.$$

$V(\Phi) = 0$ implies $a^* - ab = 0$, (4.5)

$$|b|^2 = 1, \tag{4.6}$$

and $$|a|^2 = 2. \tag{4.7}$$

Let $a = |a|e^{i\tau}$ and $b = |b|e^{i\phi}$, then equation (4.5) implies $\tau = -\frac{1}{2}\phi$. This is precisely the relationship between the changes in the arguments of a and b under a ψ_2 gauge transformation. It follows that a gauge can be chosen in which $a = |a|$ and $b = |b|$ asymptotically, and so $b = 1$ and $a = \sqrt{2}$ at infinity using equations (4.6) and (4.7)

Let $a = a_1 + ia_2$, and $b = b_1 + ib_2$; a_1, a_2, b_1, and b_2 being real functions of r. Then

$$\mathcal{H}(r)=\frac{1}{e^2}\left[\left(\delta'+\frac{\gamma\varepsilon}{r}\right)^2+\left(\gamma'-\frac{\delta\varepsilon}{r}\right)^2+\frac{1}{2r^2}(\delta^2+\gamma^2-1)^2\right]$$

$$+\frac{1}{2}\left(a_1'+\frac{\eta a_2}{2\sqrt{3}r}\right)^2+\frac{1}{2}\left(a_2'-\frac{\eta a_1}{2\sqrt{3}r}\right)^2+\frac{1}{2}\left(b_1'-\frac{\eta b_2}{\sqrt{3}r}\right)^2+\frac{1}{2}\left(b_2'+\frac{\eta b_1}{\sqrt{3}r}\right)^2+(\delta^2+\gamma^2)(a_1^2+a_2^2)$$

$$+r^2\Big[\lambda_1\{(a_1-a_1b_1+a_2b_2)^2+(a_2+a_1b_2+a_2b_1)^2+(b_1-\tfrac{1}{2}a_1^2+\tfrac{1}{2}a_2^2)^2$$

$$+(b_2+a_1a_2)^2\}+\lambda_2\{\tfrac{1}{2}(a_1^2+a_2^2-2)^2+(b_1^2+b_2^2-1)^2\}\Big].$$

Setting $a_2=b_2=\gamma=\eta=\varsigma=0$ satisfies the equations of motion of these functions identically, so they may be set equal to zero in the energy density. Doing that yields

$$\mathcal{H}(r)=\frac{1}{e^2}\left[\delta'^2+\frac{1}{2r^2}(\delta^2-1)^2\right]+\frac{r^2}{2}a_1'^2+\frac{r^2}{2}b_1'^2+\delta^2a_1^2$$

$$+r^2\lambda_1\left[a_1^2(1-b_1)^2+(b_1-\tfrac{1}{2}a_1^2)^2\right]+r^2\lambda_2\left[\tfrac{1}{2}(a_1^2-2)^2+(b_1^2-1)^2\right].$$

This is the form covered by the theorem proved earlier and so a monopole exists.

The equations of motion are:

$$\delta''=\frac{\delta}{r^2}(\delta^2-1)+e^2a_1^2\delta \tag{4.8}$$

$$\frac{d}{dr}(r^2a_1')=2\delta^2a_1+2r^2\lambda_1a_1\left[(1-b_1)^2+(\tfrac{1}{2}a_1^2-b_1)\right]+2r^2\lambda_2a_1(a_1^2-2) \tag{4.9}$$

and
$$\frac{d}{dr}(r^2b_1')=2r^2\lambda_1b_1\left[(b_1-1)a_1^2+(b_1-\tfrac{1}{2}a_1^2)\right]+4r^2\lambda_2b_1(b_1^2-1). \tag{4.10}$$

For large r equation(4.8) becomes

$$\delta''\approx 2e^2\delta \qquad \delta=o(\sqrt{r}) \qquad a_1^2\approx 2,$$

$$\Rightarrow \delta\approx \text{constant}\times\exp(-\sqrt{2}\,er).$$

Let $a_1(r)=\sqrt{2}+\xi(r)$, and $b_1(r)=1+\chi(r)$, then

$$\frac{d}{dr}(r^2\xi')=r^2\xi(2\lambda_1+8\lambda_2)+r^2\chi(-2\sqrt{2}\lambda_1)+2^{nd}\text{ order terms.}$$

and $\frac{d}{dr}(r^2\chi') = r^2\xi(-2\sqrt{2}\lambda_1) + r^2\chi(4\lambda_1+8\lambda_2) + 2^{nd}$ order terms.

follow from equations (4.9) and (4.10). The $(\text{mass})^2$ matrix is

$$\begin{pmatrix} 2\lambda_1+8\lambda_2 & -2\sqrt{2}\lambda_1 \\ -2\sqrt{2}\lambda_1 & 4\lambda_1+8\lambda_2 \end{pmatrix}$$

Choosing $\lambda_1=\lambda_2=\frac{1}{2}$ and $e=\sqrt{2}$ the eigenvalues of this matrix are 7 and 4. On the other hand the masses of the Higgs particles are 3, $\sqrt{5}$, and 2, so the masses of the Higgs particles do not govern the exponential decay of the Higgs fields in any simple manner. The $(\text{mass})^2$ of the gauge particle is 4 , consequently the inequalities of reference 30 are satisfied and the assumption of exponential decay is consistent with the equations of motion. A solution of the equations of motion for large r is

$$\Phi(x) \approx \left(\text{const.} \times \frac{e^{-2r}}{r} + \sqrt{2}\right)\psi_3 + \left(\text{const.} \times \frac{e^{-2r}}{r} + 1\right)\psi_4,$$

$$W_\mu(x) \approx \frac{i}{\sqrt{2}}\left(1 - \text{const.} \times e^{-2r}\right)[\psi_1, \partial_\mu \psi_1]$$

$$= \frac{1}{\sqrt{2}r}\left(1 - \text{const.} \times e^{-2r}\right)\epsilon_{\mu ij}\hat{x}_i \frac{\lambda_j}{2}.$$

For large r $B_i(x) = \frac{-\hat{x}_i}{\sqrt{2}r^2}\psi_1 +$ exponentially small terms.

On the y axis $\Phi(\infty)=I$ so its little group is the natural SO(3) subgroup of SU(3), and $\lim_{r\to\infty} er^2 B(r\hat{y}) = -\frac{\lambda_2}{2}$. The gauge field satsifies the conditions necessary for the application of the results of Goddard and Olive [20] which means that the topological charge of the solution is given by the homotopy class of the closed path

$$h(s) = \exp(-2\pi i \lambda_2 s) \quad , \quad 0 \le s \le 1$$

in SO(3). As this path is homotopically non-trivial in SO(3) the solution has a topological charge.

CHAPTER 5

MONOPOLES IN A GRAND UNIFIED THEORY

BASED ON SU(5)

5.1 Introduction *

In recent years interest in unifying weak, electromagnetic, and strong interactions has been growing. One of the proposed unifying schemes, due to Georgi and Glashow [18], is based on the group SU(5). This theory has been studied by a number of people [6, 15, 16, 19]. Buras et al. [6] and Frampton et al. [16] have written down Higgs potentials for this model. Each of these potentials involves a multiplet of fields transforming according to the adjoint representation of SU(5), but the former has in addition a Higgs multiplet transforming according to the 5 representation, whilst the latter has one transforming according to the 45. Both of these potentials have very complex relationships between the parameters they involve, which makes them hard to analyse.

Below, in section 5.2, a new potential involving fields transforming according to the 24 and 5 dimensional representations is written down. This potential is easy to analyse and it can be shown, quite simply, that the absolute minima of the potential give the required vacuum structure. The potential leads to simple algebraic expressions for the masses of the particles at the classical level. The parameters of the potential are subject to only very simple constraints - typically that they be real and greater than zero. The breaking of the SU(2)⊗U(1) symmetry is produced solely by the $I = \frac{1}{2}$ part of the 5 dimensional multiplet, so the Weinberg-Salam model [29, 34] with $\sin^2\theta_W = \frac{3}{8}$ is embedded in this theory in a direct way. This in contrast to refs. 6 and 16 . In those papers the 24 dimensional representation is also involved in this breaking, and a cancelation between terms in the mass expressions arising from

---

* The content of this chapter is essentially that of ref. 31.

the two different multiplets is needed if a gauge hierarchy is to exist at the classical level. The expressions for the masses in the new version of the model are given in section 5.3.

Section 5.4 contains some observations on the stationary points of the potential. I do not know whether the model can sustain a gauge hierarchy as a quantum theory.

In section 5.5 expressions for the gauge coupling constants of the Weinberg-Salam model in terms of the parameters of the SU(5) theory are derived. The useful expression

$$\frac{Q}{\|Q\|} = \cos\theta_W \frac{Y}{\|Y\|} + \sin\theta_W \frac{I_3}{\|I_3\|}$$

is also derived.

The model is of a type which has been proved to contain '$\underline{J} + \underline{I}$' symmetric monopoles of the 't Hooft-Polyakov variety [21, 25] for at least some choices of $\underline{I}$ [30]. This is in contrast to the Weinberg-Salam model which does not contain such objects. Monopoles are likely to be a feature of any unified theory using a Higgs field transforming according to the adjoint representation. It would be a major event if magnetic monopoles were discovered, however the monopoles of this theory are very heavy, and there is as yet no real chance of seeing them.

Section 5.6 contains a derivation of the inequality

$$m^2 \geq v^2 \left(\frac{4^2}{4\cos^2\theta_W} + \cos^2\theta_W g^2 \right)$$

for the masses of the vector bosons and the monopoles of the theory. It is shown that the quantisation condition

$$q_e y_g = \frac{4\pi}{3} \times \text{integer} \qquad (1.1)$$

must be satisfied. q_e is the charge on the electron. From these results, in a suitable approximation, it follows that

$$\frac{\text{the mass of a monopole}}{\text{the mass of a superheavy vector boson}} = \frac{3}{20} \cdot \frac{1}{\alpha} .$$

α is the fine structure constant.

In section 7 it is explicitly demonstrated that the quantisation condition (1.1) is satisfied by '$\underline{J} + \underline{T}$' symmetric monopoles if the gauge fields obey an inverse square law for large r. The results obtained by algebraic means are related to the topological aspects of the model.

Units in which $\hbar = c = \mu_0 = 1$ are used throughout.

5.2 The Symmetry Breaking Potential

The SU(5) gauge symmetry is broken spontaneously to U(3) by two Higgs multiplets. One of them transforms according to the 24 (real) dimensional adjoint representation of SU(5). It is written as a 5x5 hermitean, traceless matrix, $\underline{\Phi}$. The other transforms according to the 5 (complex) dimensional fundamental representation, and is denoted by H, a column vector. A possible form for the pair of fields $\{\underline{\Phi}_v, H_v\}$ which gives the required symmetry breaking is

$$\underline{\Phi}_v = \text{diag}\left(a, a, a, -\frac{3a}{2}, -\frac{3a}{2}\right); \quad H_v = \begin{vmatrix} 0 \\ 0 \\ 0 \\ h \\ 0 \end{vmatrix} , \; h \text{ real.}$$

The little group of Φ_V is $(SU(3)\otimes SU(2)\otimes U(1))/Z_6$, whilst that of the pair $\{\Phi_V, H_V\}$ is U(3). In each case the "special unitary part" of the little group contains matrices of the form

$$\begin{pmatrix} SU(3) & 0 \\ 0 & I_2 \end{pmatrix}$$

I shall call the group of matrices of this form $SU(3)_C$. The "U(1) parts" are generated by

$$Y = \text{diag}_C\left(-\tfrac{2}{3}, -\tfrac{2}{3}, -\tfrac{2}{3}, 1, 1\right)$$

and

$$\tilde{Q} = \text{diag}\left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 0, 1\right)$$

respectively. Y is the weak hypercharge matrix for the fundamental representation, and $\tilde{Q}$ is the charge matrix in units of the proton charge. The global group structure has been given in each case as the global as well as the local structure of the groups is relevant for monopoles.

A Higgs potential which gives rise to $\{\Phi_V, H_V\}$ is

$$V(\Phi, H) = \lambda_1 \text{Tr}\left[(\Phi - a)^2(\Phi + \tfrac{3a}{2})^2\right] + \lambda_2\left[\text{Tr}(\Phi^2) - \tfrac{15a^2}{2}\right]^2 + \lambda_3\left[H^\dagger H - h^2\right]^2 + \lambda_4 H^\dagger(\Phi + \tfrac{3a}{2})^2 H. \quad \lambda_1, \lambda_2, \lambda_3, \lambda_4 > 0.$$

This potential is greater than or equal to zero for any $\{\Phi, H\}$, and is equal to zero if and only if $\{\Phi, H\}$ is related to $\{\Phi_V, H_V\}$ by an SU(5) transformation. To see this note that Φ can be diagonalized to a real matrix by an SU(5) transformation. The condition that the first term be zero then states that each eigenvalue satisfies a

quadratic equation which has two roots a and $-\frac{3a}{2}$. The tracelessness condition fixes the number of times each root occurs as an eigenvalue, and they can be brought into the given order by an SU(5) transformation. $Tr(\Phi^2) = Tr(\Phi_V^2) = \frac{15a^2}{2}$ so the second term is zero. The third term is zero if and only if

$$\sum_{i=1}^{5} |h_i|^2 = h^2. \tag{5.1}$$

With $\Phi = \Phi_V$ the fourth term is zero if and only if

$$\sum_{i=1}^{3} |h_i|^2 = 0. \tag{5.2}$$

(5.1) and (5.2) together imply

$$h_1 = h_2 = h_3 = 0, \quad |h_4|^2 + |h_5|^2 = h^2.$$

An SU(5) matrix of the form

$$\begin{pmatrix} I_3 & 0 \\ 0 & SU(2) \end{pmatrix}$$

produces no change when it acts on Φ_V. However

$$\left(\begin{array}{ccc|cc} & I_3 & & 0 & \\ \hline & 0 & & \alpha & -\beta \\ & & & \beta^* & \alpha \end{array}\right) \begin{pmatrix} 0 \\ 0 \\ 0 \\ h \\ 0 \end{pmatrix} = h \begin{pmatrix} 0 \\ 0 \\ 0 \\ \alpha \\ \beta^* \end{pmatrix} \qquad (|\alpha|^2 + |\beta|^2 = 1)$$

which is the most general form for $H = \begin{pmatrix} 0 \\ 0 \\ 0 \\ h_4 \\ h_5 \end{pmatrix}$ with $H^\dagger H = h^2$. So any pair $\{\Phi, H\}$ which minimises $V(\Phi, H)$ is related to $\{\Phi_V, H_V\}$ by an SU(5)

transformation. That is SU(5) acts transitively on the vacuum manifold.

5.3 Particle Masses

The vector boson matrix is (the vector index has been suppressed)

$$\begin{pmatrix} V + \left(\frac{Z^0}{\sqrt{20}} - \frac{A}{\sqrt{12}}\right) I_3 & X^- & Y^- \\ X^+ & -\sqrt{\frac{4}{5}}\, Z^0 & W^- \\ Y^+ & W^+ & \frac{Z^0}{\sqrt{20}} + \sqrt{\frac{3}{4}}\, A \end{pmatrix}$$

V is a 3x3 matrix representating the gluons. $W^{\pm}$, Z^0, and A are the bosons familiar from the Weinberg-Salam model, and X^- and Y^- are new colour triplets of vector bosons which have charge $-\frac{1}{3}$ and $-\frac{4}{3}$ respectively. $X^- = \frac{1}{\sqrt{2}}(X_1 - iX_2)$ with X_1 and X_2 real. Note that the A component is a multiple of the charge matrix Q. X^- and Y^- have hypercharge $-\frac{5}{3}$.

$$m^2_{X^\pm} = e^2\left[\left(\frac{5a}{2}\right)^2 + h^2\right] \quad , \quad m^2_{Y^\pm} = \left(\frac{5ae}{2}\right)^2 ,$$

$$m^2_{Z^0} = \frac{8}{5} e^2 h^2 \quad , \quad m^2_{W^\pm} = e^2 h^2$$

Note $\dfrac{m^2_{W^\pm}}{m^2_{Z^0}} = \dfrac{5}{8} = \cos^2\theta_W$ as promised.

It is convenient to write the Higgs boson matrix as

$$\underline{\Phi} = \left(\begin{array}{c|c|c} \phi - \sqrt{\frac{2}{15}}E I_3 & M^- & N^- \\ \hline M^+ & \sqrt{\frac{3}{10}}E + \frac{F}{\sqrt{2}} & D^- \\ \hline N^+ & D^+ & \sqrt{\frac{3}{10}}E - \frac{F}{\sqrt{2}} \end{array}\right) + \underline{\Phi}_v$$

and $H = \sqrt{2}\begin{pmatrix} K^- \\ \tilde{h}_4 \\ \tilde{h}_5 \end{pmatrix} + H_v \quad , \quad \tilde{h}_4 = \tilde{h}_{4R} + i\tilde{h}_{4I}$.

M^-, N^-, and K^- are colour triplets with charges $-\frac{1}{3}$, $-\frac{4}{3}$, and $-\frac{1}{3}$ respectively. M^- and K^- mix to produce mass eienstates J^- and J'^-, and E and F mix to produce mass eigenstates B and C.

$$m^2_{\tilde{h}_{4I}} = m^2_{\tilde{h}_{5R}} = m^2_{\tilde{h}_{5I}} = m^2_{J'^\pm} = m^2_{N^\pm} = 0$$

$$m^2_\phi = \frac{(5a)^2}{2}\lambda_1 \quad , \quad m^2_{J^\pm} = \left[\frac{(5a)^2}{2} + h^2\right]\lambda_4$$

$$m^2_{D^\pm} = \frac{(5a)^2}{2}\lambda_1 + h^2\lambda_4 \quad , \quad m^2_{\tilde{h}_{4R}} = (2h)^2\lambda_3$$

$$m^2_{B,C} = \frac{(5a)^2}{2}\lambda_1 + 30a^2\lambda_2 + \frac{4}{5}h^2\lambda_4 \pm \left[\left(30a^2\lambda_2 - \frac{h^2}{5}\lambda_4\right)^2 + \frac{3}{5}h^4\lambda_4\right]^{\frac{1}{2}}$$

5.4 Other Stationary Points of $V(\Phi,H)$

$V(\Phi,H)$ has at least three orbits of stationary points besides the orbit of absolute minima. As the pair $\{\Phi_1, H_1\}$ given by

$$\Phi_1 = \mathrm{diag}(b, b, b, -\tfrac{3b}{2}, -\tfrac{3b}{2}) \quad , \quad H_1 = 0$$

is the most general form of an $(SU(3)\otimes SU(2)\otimes U(1))/Z_6$ invariant, position independent, Higgs field a stationary point of $V(\Phi,H)$ with respect to variations of b is a stationary point with respect to arbitrary variations of Φ and H [8]. $\frac{\partial V}{\partial \Phi}(\Phi=\Phi_1, H=0) = 0$ is a cubic in b and has three roots. $\Phi=\Phi_1$ is one. $\Phi=0$, $H=0$ is the most general SU(5) invariant field configuration and hence is another stationary point. As these two solutions are real, and the coefficients of the cubic are real, the third solution is also real.

If one considers perturbations around the pair $\{\Phi=\Phi_1, H=0\}$ (which is a stationary point of the potential) and one throws away all of the particles whose classical masses depend on the parameter a then one obtains the Weinberg-Salam model (with $g' = \sqrt{\frac{3}{5}}\, g$) at its unstable stationary point, so perhaps this potential can sustain a gauge hierarchy.

5.5 Connection with the Weinberg-Salam Model

Let $W_\mu = W_\mu^a L^a$ with L^a an element of the Lie algebra of SU(5), and let $\mathrm{Tr}(L^a L^b) = \|L\|^2 \delta^{ab}$.* A_μ, B_μ, and $A_{\mu I_3}$ are defined by

$$A_\mu = \frac{\mathrm{Tr}(W_\mu Q)}{\|L\|\cdot\|Q\|}, \quad B_\mu = \frac{\mathrm{Tr}(W_\mu Y)}{\|L\|\cdot\|Y\|}, \quad A_{\mu I_3} = \frac{\mathrm{Tr}(W_\mu I_3)}{\|L\|\cdot\|I_3\|}. \qquad (5.1)$$

This definition is manifestly independent of $\|L\|$, and leads to the standard normalization of the kinetic terms of the gauge fields.

---

* The norm of a matrix X is defined by $\|X\|^2 = \mathrm{Tr}(X^\dagger X)$.

$$W_\mu = \frac{\|L\|}{\|Q\|} A_\mu Q + S_\mu = \frac{\|L\|}{\|Y\|} B_\mu Y + \frac{\|L\|}{\|I_3\|} A_{\mu I_3} I_3 + Z_\mu$$

for some S_μ and Z_μ satisfying $Tr(QS_\mu) = Tr(YZ_\mu) = Tr(I_3 Z_\mu) = 0$.

The covariant derivative of the H multiplet is

$$\partial_\mu H + ieW_\mu H = \partial_\mu H + ie\left(\frac{\|L\|}{\|Q\|} A_\mu Q + S_\mu\right) H .$$

Comparing this with the minimal electromagnetic coupling

$$\partial_\mu H + i A_\mu Q H$$

one finds

$$e = \frac{\|Q\|}{\|L\|} \tag{5.2}$$

and

$$eW_\mu = \frac{\|Q\|}{\|Y\|} B_\mu Y + \frac{\|Q\|}{\|I_3\|} A_{\mu I_3} I_3 + eZ_\mu$$

The coupling constants in the Weinberg-Salam model are therefore

$$\frac{g'}{2} = \frac{\|Q\|}{\|Y\|}$$

and

$$g = \frac{\|Q\|}{\|I_3\|}$$

which implies

$$\frac{g'}{g} = \sqrt{\frac{3}{5}}$$

Let $\tilde{Q} = \frac{1}{|q_e|} Q$ with q_e the charge on the electron

$$\tilde{Q} = \frac{1}{2} Y + I_3 \tag{5.3}$$

From equations (5.1) and (5.3)

$$A_\mu = \frac{\mathrm{Tr}(W_\mu Q)}{\|L\|.\|Q\|} = \frac{\mathrm{Tr}\left[W_\mu\left(\frac{1}{2}Y+I_3\right)\right]}{\|L\|.\|\tilde{Q}\|} = \frac{1}{2}\frac{\|Y\|}{\|\tilde{Q}\|}B_\mu + \frac{\|I_3\|}{\|\tilde{Q}\|}A_{\mu I_3}$$

$$= \cos\theta_W B_\mu + \sin\theta_W A_{\mu}. \quad \text{by definition.}$$

$$\left.\begin{aligned} &\text{Therefore} \quad \cos\theta_W = \frac{1}{2}\frac{\|Y\|}{\|\tilde{Q}\|} \\ &\text{and} \qquad \sin\theta_W = \frac{\|I_3\|}{\|\tilde{Q}\|} \end{aligned}\right\} \tag{5.4}$$

which combined with equation (5.3) yields

$$\frac{Q}{\|Q\|} = \frac{\tilde{Q}}{\|\tilde{Q}\|} = \cos\theta_W \frac{Y}{\|Y\|} + \sin\theta_W \frac{I_3}{\|I_3\|} \tag{5.5}$$

an equation which could be used as a convenient definition of θ_W.

5.6 A Modification of the Bogomol'nyi Bound

Suppose $\lim_{r\to\infty}\{\Phi(r,\theta,\phi), H(r,\theta,\phi)\}$ exists for each θ and ϕ, and that it is a zero of $V(\Phi,H)$, then one can associate a charge matrix, $Q(\theta,\phi)$, with each θ and ϕ. $Q(\theta,\phi)$ and $\{\Phi(\infty,\theta,\phi), H(\infty,\theta,\phi)\}$ are unitarily related to Q and $\{\Phi_v, H_v\}$ respectively. Owing to this their angular dependence is irrelevant in gauge invariant expressions, and

it will be ignored henceforth.

In this SU(5) model the multiplet of fields transforming according to the adjoint representation is associated not with charge, but with hypercharge. Consequently, for the vector bosons and monopoles of the theory, one would expect a mass inequality something like

$$m^2 \geq v^2(y^2 + g_Y^2)$$

v is the length of the Higgs field $\underline{\Phi}$ at infinity, y is the hypercharge of the particle, and g_Y is the "magnetic hypercharge" of the particle. However this is not quite right. In fact an extra factor appears in the y^2 term as $\|Y\| \neq \|Q\|$. If χ is defined by $\chi^{ab} = i f^{acb} \phi^c(\infty)$, then the (mass)^2 matrix of the vector bosons is given by $M^2 = e^2\chi^2$, ignoring the contribution from the multiplet H. Let Y_{24} be the hypercharge matrix in the adjoint representation, then (as $Y \propto \underline{\Phi}(\infty)$)

$$M^2 = e^2 \frac{\|\Phi\|^2}{\|Y\|^2} Y_{24}^2 = v^2 \frac{\|Q\|^2}{\|Y\|^2} Y_{24}^2$$

using equation (5.2). Thus the mass inequality is really

$$m^2 \geq v^2 \left(\frac{\|Q\|^2}{\|Y\|^2} y^2 + g_Y^2 \right). \tag{6.1}$$

Let $B_i = \frac{1}{2} \epsilon_{ijk} \left(\partial_j W_k - \partial_k W_j + ie[W_j, W_k] \right)$,*

then the definition of g_Y is

$$g_Y = \int_{S^2(\infty)} dS_i \frac{\mathrm{Tr}(B_i \underline{\Phi})}{\|L\| . \|\underline{\Phi}\|} .$$

A relationship exists between g_Y and the magnetic charge g.

$$g = \int_{S^2(\infty)} dS_i \frac{\mathrm{Tr}(B_i Q)}{\|L\| . \|Q\|} .$$

---

* The sign of e has been changed to make comparison with ref. 20 easier.

If mild assumptions are made about the asymptotic behaviour of monopole solutions one may write [20]:

$$\Phi(x) = U(x)\Phi_V U^\dagger(x) \quad \text{for some } U(x) \in SU(5) \text{ for large } r, \text{ and}$$

$$B_i(x) = \frac{\hat{x}_i}{r^2} U(x) B_V U(x) \text{ for some } B_V \text{ satisfying } [B_V, \Phi_V] = 0.$$

If this is so, then

$$g_Y = 4\pi \frac{\mathrm{Tr}(B_V \Phi_V)}{\|L\| . \|\Phi_V\|} = 4\pi \frac{\mathrm{Tr}(B_V Y)}{\|L\| . \|Y\|}$$

Similarly $$g = 4\pi \frac{\mathrm{Tr}(B_V Q)}{\|L\| . \|Q\|}$$

Now $[B_V, \Phi_V]=0$ implies $B_V = \beta Q + k$, where $\beta \in \mathbb{R}$, Q is the charge matrix corresponding to $\{\Phi_V, H_V\}$, and $k \in \mathcal{L}(SU(3)_C)$. Therefore

$$g_Y = 4\pi\beta \frac{\mathrm{Tr}(QY)}{\|L\| . \|Y\|} \quad \text{and} \quad g = 4\pi\beta \frac{\|Q\|}{\|L\|}$$

So $\frac{g_Y}{g} = \frac{\mathrm{Tr}(QY)}{\|Q\| . \|Y\|}$. Combining this with equations (5.4) and (5.5) one finds

$$g_Y = \cos\theta_W g = \frac{1}{2} \frac{\|Y\|}{\|\tilde{Q}\|} g \tag{6.2}$$

Equation (6.1) becomes

$$m^2 \geq v^2 \left(\frac{\|\tilde{Q}\|^2}{\|Y\|^2} q_e^2 y^2 + \frac{1}{4} \frac{\|Y\|^2}{\|\tilde{Q}\|^2} g^2 \right)$$

$$= v^2 \left(\frac{q_e^2 y^2}{4\cos^2\theta_W} + \cos^2\theta_W g^2 \right) . \tag{6.3}$$

To calculate the ratio of the mass of a monopole to that of a vector boson the quantisation rule involving y and g is needed. To find this rule it is helpful to consider a model obtained from the original model by disposing of the H multiplet. Given an acceptable boundary condition for the full model, denoted $\{\Phi(\infty), H(\infty)\}$, an

acceptable boundary condition for the model obtained by setting $\lambda_3 = \lambda_4 = 0$ and throwing away the H multiplet is $\Phi(\infty)$. In this reduced model Φ is the only multiplet, and there is a standard definition for the charge matrix in such a case [20]. It is usual to write $Q_Y = \frac{e\Phi(\infty)}{v}$. The subscript Y indicates that this matrix is proportional to the hypercharge matrix in the full model. The magnetic charge of this new model is g_Y. The quantisation condition

$$q_Y g_Y = \frac{2\pi}{3} \times \text{integer}$$

is satisfied [9]. Now

$$Y = \frac{\|Y\|}{\|\Phi_v\|}\Phi_v = \frac{\|Y\|}{e\|L\|}Q_Y = \frac{\|Y\|}{\|Q\|}Q_Y$$

using $Q_Y = \frac{e\Phi_v}{v}$ and equation (5.2). Therefore

$$y g_Y = \frac{\|Y\|}{\|Q\|} \cdot \frac{2\pi}{3} \cdot \text{integer},$$

but $g_Y = \frac{1}{2}\frac{\|Y\|}{\|Q\|} g$ (equation (6.2)) so

$$q_e y g = \frac{4\pi}{3} \cdot \text{integer}. \qquad (6.4)$$

It follows from equations (6.3) and (6.4) that

$$\frac{\text{the mass of a monopole}}{\text{the mass of a superheavy vector boson}} = 2\cos^2\theta_w \left|\frac{g}{q_e y}\right|$$

$$= \cos^2\theta_w \cdot \frac{1}{\alpha} \cdot \frac{1}{y^2} \cdot \left|\frac{q_e y g}{2\pi}\right| = \cos^2\theta_w \cdot \frac{1}{\alpha y^2} \cdot \left|\frac{2}{3} \times \text{integer}\right|$$

provided that the Prasad-Sommerfield limit [26] is taken, and the contribution from H is ignored. The superheavy vector bosons have $|y| = \frac{5}{3}$. so taking the integer to be 1:

$$\frac{\text{the mass of a monopole}}{\text{the mass of a superheavy vector boson}} = \frac{\cos^2\theta_w}{\alpha} \times \frac{6}{25} = \frac{3}{20} \cdot \frac{1}{\alpha}.$$

5.7 The Magnetic Charges of '$\underline{J}$ + $\underline{I}$' Symmetric Monopoles

Assuming an inverse square law behaviour for the gauge fields at large distances from the origin, the magnetic charge of a '$\underline{J}$ + $\underline{I}$' symmetric monopole defined with respect to the generators $\{T_i\}$ of an SU(2) subalgebra of the gauge group algebra is [20, 24]

$$g = \frac{4\pi \mathrm{Tr}(T_3 Q)}{\|Q\|^2} \tag{7.1}$$

Here Q is the charge matrix corresponding to the asymptotic values of $\underline{\Phi}$ and H on the z axis. The generator $T = T_i \hat{x}_i$ is an exact symmetry generator for the monopole and so $T_3 \in \mathcal{L}(U(3))$

$$[T_3, \underline{\Phi}(r\hat{\underline{z}})] = 0 \quad \text{and} \quad T_3 H(r\hat{\underline{z}}) = 0. \tag{7.2}$$

All possible magnetic charges corresponding to '$\underline{J}$ + $\underline{I}$' symmetric monopoles may be calculated [20, 36] by taking

$$Q = \mathrm{diag}\left(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{1}{3}, 0, 1\right),$$

$$\lim_{r\to\infty} \underline{\Phi}(r\hat{\underline{z}}) = \mathrm{diag}\left(a, a, a, -\tfrac{3a}{2}, -\tfrac{3a}{2}\right), \quad \lim_{r\to\infty} H(r\hat{\underline{z}}) = \begin{vmatrix} 0 \\ 0 \\ 0 \\ h \\ 0 \end{vmatrix}$$

then finding all diagonal T_3's which satisfy equation (7.2), and finally using these T_3's and Q in equation (7.1). The fourth eigenvalue of each T_3 is necessarily zero, and each T_3 is traceless, so one can write

$$T_3 = \mathrm{diag}(a, b, c, 0, d) \quad \text{with} \quad -(a+b+c) = d.$$

Then $$g = \frac{4\pi\left[-\frac{1}{2}(a+b+c)+d\right]|q_e|}{4q_e^2/3} = \frac{4\pi d}{|q_e|}.$$

The possible non-zero values are

$$\frac{gq_e}{4\pi} = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2.$$

$\frac{gq_e}{4\pi}$ is an integer multiple of $\frac{1}{2}$, and any charge is an integer multiple of $\frac{1}{3}q_e$, therefore

$$q_g = \frac{2\pi}{3} \times \text{integer}$$

and $\exp(igQ) \in$ centre of $SU(3)_c$. Thus the generalised Dirac quantisation condition [9] is satisfied.

One can see in a more abstract way why there is a relationship between the "magnetic hypercharge" and the magnetic charge of this theory. Suppose one sets $\lambda_3 = \lambda_4 = 0$ in the potential $V(\Phi, H)$, then one obtains a potential $V'(\Phi)$ which depends on Φ only. The absolute minima of this potential form precisely the orbit of Φ_v, hence the vacuum manifold of $V'(\Phi)$ is

$$\mathcal{M}_0' = SU(5)/[SU(3) \otimes SU(2) \otimes U(1)/Z_6]$$

for $(SU(3) \otimes SU(2) \otimes U(1))/Z_6$ is the little group of Φ_v. The relevant second homotopy groups can be calculated using the homotopy sequence of a bundle [32].

$$\pi_2(\mathcal{M}_0') = \pi_1(SU(3) \otimes SU(2) \otimes U(1)/Z_6) \text{ as } \pi_1(SU(5)) = 0$$

$$= Z$$

The vacuum manifold of the potential $V(\Phi, H)$ is $\mathcal{M}_c = SU(5)/U(3)$ because the little group of the pair $\{\Phi_v, H_v\}$ is $U(3)$.

$$\pi_2(\mathcal{M}_c) = \pi_1(U(3)) = Z$$

just as for $\mathcal{M}_0'$. This is because any closed path in $(SU(3)\otimes SU(2)\otimes U(1))/Z_6$ can be continuously deformed into a closed path in the U(3) little group of $\{\Phi_v, H_v\}$. Alternatively, one can look directly at $\mathcal{M}_0$ and $\mathcal{M}_0'$.

$$V(\Phi_v, H) = 0 \Rightarrow h_1 = h_2 = h_3 = 0 \text{ and } |h_4|^2 + |h_5|^2 = h^2$$

Adding the H multiplet (to convert $V'(\Phi)$ to $V(\Phi, H)$) enlarges the vacuum manifold. $\mathcal{M}_0$ is obtained from $\mathcal{M}_0'$ by "glueing" a three sphere S^3 onto each point of $\mathcal{M}_0'$. As $\Pi_2(S^3) = 0$ this process does not affect the second homotopy group of the space; the exact details of the glueing process are irrelevant.

Appendix A

An example of a function, f, satisfying the following conditions:

(i) f is infinitely differentiable on $[0,\infty)$,

(ii) $f(x) \not\to 0$ as $x \to \infty$;

(iii) $\int_0^\infty x^2 f^m(x)dx < \infty$, $m = 1,2,3,4$;

(iv) $f(x) \geq 0 \quad \forall x \in [0,\infty)$.

Let $$g(x;\alpha,\epsilon) = \begin{cases} \exp\left[-1/((x-\alpha)^2-\epsilon^2)^2\right] & , \alpha-\epsilon<x<\alpha+\epsilon \\ 0 & , \text{otherwise.} \end{cases}$$

Let $f_n(x) = g\left(x; n, \frac{1}{n^2(2^{n+2})^5}\right)$,

and $f(x) = \sum_{n=1}^{\infty} f_n(x)\, 2^{n+2} \exp\left(n^8(2^{n+2})^{20}\right)$.

Now

(i) is satisfied;

(ii) at $x = n$, $f(n) = 2^{n+2}$, i.e. $f(n) \to \infty$ as $n \to \infty$;

(iii) $f(x) \leq 2^{n+2}$ in $\left[n - \frac{1}{n^2(2^{n+2})^5}, n + \frac{1}{n^2(2^{n+2})^5}\right]$

$= 0$ otherwise, therefore (as $2^{n+2} > 1$, and $\frac{1}{n^2(2^{n+2})^5} < \frac{1}{2}$ for $n \geq 1$)

$$\int_0^\infty x^2 f^m(x)dx \leq \sum_{n=1}^{\infty} \frac{2\cdot(n+\frac{1}{2})^2(2^{n+2})^4}{n^2(2^{n+2})^5} \quad , \quad m = 1,2,3,4;$$

$$< 8 \sum_{n=1}^{\infty} \frac{1}{2^{n+2}} = 2;$$

(iv) is trivially satisfied.

APPENDIX B

The results of this appendix were obtained in conjunction with S.J. Avis.

As has been shown in section 1.4 the existence of a non-zero topological quantum number at a given time ensures that there is a region of space where $V(\underline{\Phi}) \neq 0$. To show that a solution is non-dissipative when there is a conserved non-zero topological charge one must show that the amount that $V(\underline{\Phi})$ differs from zero in this region (the region itself may vary with time) cannot become arbitrarily small as time progresses.

Definition: A solution is non-dissipative if and only if there exists some $\delta > 0$ such that $\sup \mathcal{E}(\underline{x},t) \geq \delta$ for all t.

Here (i) $\mathcal{E}(\underline{x},t)$ = the energy density at the point $\underline{x}$ at time t,

(ii) $\delta \in \mathbb{R}$.

This is a stronger definition than that given by Coleman [8].

The following theorem is needed [23]:

Theorem (Local retraction theorem): Let M be a manifold without boundary, and $f: M \to \mathbb{R}^n$ be a C^r imbedding, then there is a neighbourhood U of f(M) and a C^r retraction $r: U \to f(M)$. (Recall that this means that $r(y) = y$ for y in f(M).)

Note that a manifold which is a homogeneous space (such as a quotient space) has no boundary.

Theorem: If a solution of the equations of motion of a classical gauge theory (with gauge group G) in a flat space time (with any number of spatial dimensions) has a conserved topological quantum number which arises from a continuous G-invariant Higgs potential $V(\underline{\Phi})$, and if $V(\underline{\Phi}) \geq 0$ with the set of constant $\underline{\Phi}$'s which make V zero non-empty and forming a quotient space, then the solution is non-dissipative.

Outline of the proof

Let the vector space in which $\underline{\Phi}$ takes its values be W (W is $\mathbb{C}^n$ or $\mathbb{R}^n$ for some n), and $Y_t = \{w \in W \mid w = \underline{\Phi}(\underline{x},t) \text{ for some } \underline{x}\}$. Define the scatter of Y_t to be $S_t = \sup_{\substack{y \in Y_t \\ m \in \mathcal{M}_o}} \|y - m\|$. This is, roughly speaking, the maximum distance between any point of Y_t and $\mathcal{M}_o$. Now $V(\underline{\Phi})$ is a continuous function of $\underline{\Phi}$, therefore if $S_t \to 0$ as $t \to \infty$ the potential term in the energy density will approach zero as $t \to \infty$. It will be shown that $S_t \not\to 0$. To do this it is shown that there is an open set $U \supset \mathcal{M}_o$ such that for each t $Y_t \not\subset U$. This shows that for each t there is a point of Y_t greater than some fixed distance from $\mathcal{M}_o$ determined by U. It will be shown that for those points $\tilde{y} \in Y_t$ held at a sufficient distance from $\mathcal{M}_o$ that $V(\tilde{y})$ is greater than or equal to some fixed number δ.

Proof: The set $\mathcal{M}_o = \{w \in W \mid V(w) = 0\}$ is bounded. By the local retraction theorem there is a bounded open set U, $\mathcal{M}_o \subset U \subset W$, with U retractible to $\mathcal{M}_o$. W-U is closed. Let B be a closed ball, $\mathcal{M}_o \subset U \subset B$, then $(W-U) \cap B = B-U$ is closed and bounded and hence compact. It is also non-empty. Let $\inf_{w \in B-U} V(w) = \delta$, then there is a $w' \in B-U$ such that $V(w') = \delta$ as B-U is compact. However, $(B-U) \cap \mathcal{M}_o = \phi$ so $\delta > 0$. Now Y_t is path connected (this follows from its definition and the fact that the space-time is path connected) so $Y_t \cap (B-U) \neq \phi$ because if the intersection were empty it would imply $Y_t \subset U$. $Y_t \not\subset U$ for if it were $\underline{\Phi}(\underline{x},t)$ would have a trivial boundary condition in U (i.e. the boundary condition would be homotopic to a constant map when continuous deformations in U are allowed), and $(r \circ \underline{\Phi})(\underline{x},t)$ would have a trivial boundary condition in $\mathcal{M}_o$. This cannot be so because at spatial

infinity $\overline{\Phi}(\underline{x},t)$ lies in $\mathfrak{M}_0$, and $r \circ \overline{\Phi} = \overline{\Phi}$ at spatial infinity, which implies that the boundary condition is trivial in $\mathfrak{M}_0$. This contradicts the assumption of a non-zero topological quantum number. So $Y_t \cap (B-U) \neq \emptyset$ which implies

$$\sup_{y \in Y_t} V(y) \geqslant \sup_{y \in Y_t \cap (B-U)} V(y) \geqslant \inf_{y \in Y_t \cap (B-U)} V(y) \geqslant \inf_{y \in (B-U)} V(y) = \delta.$$

APPENDIX C

Dirac's definition of a string [13] is given in the two following quotations.

"The idea which makes this generalization possible consists in supposing each pole to be at the end of an unobservable string, which is the line along which the electromagnetic poyentials are singular,.. ..."

"If we consider one instant of time, Eq. (12) or (4) requires that the total magnetic flux crossing any closed surface at this time shall be zero. This is not true if there is a magnetic monopole inside the closed surface. Equation (12) must then fail somewhere on the surface, and we may suppose that it fails at only one point. Equation (12) will fail at one point on every closed surface surrounding the pole, so that it will fail on a line of points, which we shall call a string, extending outward from the pole."

The equations refered to are:

$$\partial_\nu (F^+)_{\mu\nu} = 0 \qquad (4)$$

and

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \qquad (12)$$

REFERENCES

1. Adams,R.A.: Sobolev Spaces (Academic Press, New York, 1975).
2. Bais,F.A., Weldon,H.A.: Phys Rev. Letts. 41 (1978) 601.
3. Bardakci,K., Samuel,S.: Phys. Rev. D18 (1978) 2849.
4. Bogomol'nyi,E.B.: Sov. J. Nucl. Phys. 24 (1976) 449.
5. Brihaye,Y., Nuyts,J.: J.M.P. 18 (1977) 2177.
6. Buras,A.J., Ellis,J., Gaillard,M.K., Nanopoulos,D.V.: Nucl. Phys. B135 (1978) 66.
7. Cartan,H.: Differential Forms (Kershaw Publishing Co., London, 1971).
8. Coleman,S.: New Phenomena in Subnuclear Physics - Proc. 1975 Int. School of Physics 'Ettore Majorana' (ed. A. Zichichi) (Plenum, New York) pp.297-421.
9. Corrigan,E., Olive,D.I.: Nucl. Phys. B110 (1976) 237.
10. Corrigan,E., Olive,D.I., Fairlie,D.B., Nuyts,J.: Nucl. Phys. B106 (1976) 475.
11. D'Adda,A., Horsley,R., Di Vecchia,P.: Supersymmetric Magnetic Monopoles and Dyons, Nordita preprint March 1978.
12. Dirac,P.A.M.: Proc. Roy. Soc. A133 (1931) 60.
13. Dirac,P.A.M.: Phys. Rev. 74 (1948) 817.
14. Dunford,N., Schwarts,J.T.: Linear Operators Part II (John Wiley and Sons, New York, 1963) 1704.
15. Fradkin,E.S., Kalashnikov,O.K.: Phys. Letts. 64B (1976) 177.
16. Frampton,P.H., Nandi,S., Scanio,J.J.G.: Estimate of flavour number from SU(5) grand unification, Ohio State University preprint COO-1542-2.
17. Georgi,H., Glashow,S.L.: Phys. Rev. D6 (1972) 2977.
18. Georgi,H., Glashow,S.L.: Phys. Rev. Letts. 32 (1974) 438.

19. Georgi,H., Quinn,H.R., Weinberg,S.: Phys.Rev. Letts. 33 (1974) 451.

20. Goddard,P., Olive,D.I.: Rep. Prog. Phys. 41 (1978) 1357.

21. 't Hooft,G.: Nucl.Phys. B79 (1974) 276.

22. Leznov,A.N., Saveliev,M.V.:Phys. Letts. 79B (1978) 294.

23. Munkres,J.R.: Elementary Differential Topology, Annals of Mathematics Studies 54 (Princeton University Press, Princeton, 1966) 51.

24. Olive,D.I.: Nucl. Phys. B113 (1976) 413.

25. Polyakov,A.M.: JETP Letts. 20 (1974) 194.

26. Prasad,M.K., Sommerfield,C.M.: Phys. Rev. Letts. 35 (1975) 760.

27. Rawnsley,J.H.: Existence and regularity of solutions of gauge field equations, Dublin Institute for Advanced Studies preprint DIAS-TP-77-09.

28. Rawnsley,J.H.: J. Phys. A10 (1977) L139.

29. Salam,A.: Proc. 8^{th} Nobel Symposium, Stockholm (ed. N. Svartholm) (Almqvist and Wiksells, Stockholm, 1968) 367.

30. Scott,D.M.: On the existence and smoothness of spherically symmetric monopole solutions in non-abelian gauge theories (revised), Imperial College preprint ICTP/78-79/5A.

31. Scott,D.M.: Monopoles in a grand unified theory based on SU(5), Imperial College preprint ICTP/78-79/32.

32. Steenrod,N.: The Topology of Fibre Bundles (Princeton University Press, Princeton, 1951) 91.

33. Tyupkin,Yu.S., Fateev,V.A., Schvarts,A.S.: Theor. Math. Phys. 26 (1976) 270.

34. Weinberg,S.: Phys.Rev. Letts. 19 (1967) 1264.

35. Wilkinson,D., Bais,F.A.: Phys. Rev. D19 (1979) 2410.

36. Wilkinson,D., Goldhaber,A.S.: Phys. Rev. D16 (1977) 1221.