THE MEAN-SQUARE DICHOTOMY SPECTRUM AND A BIFURCATION TO A MEAN-SQUARE ATTRACTOR

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Abstract. The dichotomy spectrum is introduced for linear mean-square random dynamical systems, and it is shown that for finite-dimensional mean-field stochastic differential equations, the dichotomy spectrum consists of finitely many compact intervals. It is then demonstrated that a change in the sign of the dichotomy spectrum is associated with a bifurcation from a trivial to a non-trivial mean-square random attractor.

1. **Introduction.** Mean-square properties are of traditional interest in the investigation of stochastic systems in engineering and physics. This is quite natural since the Ito stochastic calculus is a mean-square calculus. At first sight, it is thus somewhat surprising that the classical theory of random dynamical systems and their spectra is a pathwise theory, although this can be justified by Doss–Sussman-like transformations between stochastic differential equations and path-wise random ordinary differential equations [1]. Such transformations, however, do not apply to mean-field stochastic differential equations, which include expectations of the solution in their coefficient functions [6].

Mean-square random dynamical systems based on deterministic two-parameter semi-groups from the theory of nonautonomous dynamical systems acting on a state space of random variables or random sets with the mean-square topology were introduced in [7]. These act like deterministic systems with the stochasticity built into the state spaces of mean-square random variables. A mean-square random attractor was defined as a nonautonomous pullback attractor for such systems from the theory of nonautonomous dynamical systems [9]. The main difficulty in applying

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the theory is the lack of useful characterisations of compact sets of such spaces of mean-square random variables.

In this paper, a theory of mean-square exponential dichotomies is presented for linear mean-field stochastic differential equations. (It also applies to classical linear stochastic differential equations). Although the corresponding mean-square random dynamical systems are essentially infinite-dimensional their dichotomy spectrum is given by the union of finitely many intervals. This is applied to analyse a nonlinear mean-field stochastic differential equation, for which it is shown that the trivial solution undergoes a mean-square bifurcation leading to a nontrivial mean-square attractor.

The paper is structured as follows. Section 2 contains the definition of a mean-square random dynamical system, and the notions of mean-square exponential dichotomy and mean-square dichotomy spectrum are introduced. Section 3 explains under which conditions, a mean-field stochastic differential equation generates a mean-square random dynamical system. In Section 4, the spectral theorem is established, which says that the mean-square spectrum of a linear mean-field stochastic differential equation consists of finitely many compact intervals. Finally, in the last section, it is shown that for a one-dimensional mean-field SDE of pitchfork-type, a stability change in the mean-square spectrum is associated with a bifurcation from a trivial to a non-trivial mean-square random attractor.

2. Mean-square random dynamical systems.

Consider the time set $\mathbb{R}$, and define $\mathbb{R}^2_{\geq} := \big\{(t,s) \in \mathbb{R}^2 : t \geq s\big\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ be a complete filtered probability space satisfying the usual hypothesis, i.e., $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ is an increasing and right-continuous family of sub-σ-algebras of $\mathcal{F}$, which contain all $\mathbb{P}$-null sets. Essentially, $\mathcal{F}_t$ represents the information about the randomness at time $t \in \mathbb{R}$. Finally, define

$$\mathfrak{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d) \quad \text{and} \quad \mathfrak{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d) \qquad \text{for } t \in \mathbb{R}$$

with the norm $\|X\|_{\mathrm{ms}} := \sqrt{\mathbb{E}|X|^2}$, where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$.

Definition 1. A *mean-square random dynamical system* (MS-RDS for short) φ on the underlying phase space $\mathbb{R}^d$ with the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$ is a family of mappings

$$\varphi(t,s,\cdot) : \mathfrak{X}_s \to \mathfrak{X}_t, \qquad \text{for } (t,s) \in \mathbb{R}^2_{\geq},$$

which satisfies:

(1) *Initial value condition.* $\varphi(s,s,X_s) = X_s$ for all $X_s \in \mathfrak{X}_s$ and $s \in \mathbb{R}$.
(2) *Two-parameter semigroup property.* For all $X \in \mathfrak{X}_w$ and all $(t,s),(s,w) \in \mathbb{R}^2_{\geq}$

$$\varphi(t,w,X) = \varphi(t,s,\varphi(s,w,X)).$$

(3) *Continuity.* φ is continuous.

Mean-square random dynamical systems are essentially deterministic with the stochasticity built into or hidden in the time-dependent state spaces.

A MS-RDS φ is called *linear* if for each $(t,s) \in \mathbb{R}^2_{\geq}$, the map $\varphi_{t,s}(\cdot) := \varphi(t,s,\cdot)$ is a bounded linear operator. It will be denoted by $\Phi_{t,s}$, and $\Phi_{t,s}(X)$ will conventionally be written $\Phi_{t,s}X$. A spectral theory for linear mean-square random dynamical systems can be established based on exponential dichotomies.

Definition 2 (Mean-square exponential dichotomy)**.** Let $\gamma \in \mathbb{R}$. A linear mean-square random dynamical system $\Phi_{t,s} : \mathfrak{X}_s \to \mathfrak{X}_t$ is said to admit an *exponential dichotomy* with *growth rate* γ if there exist positive constants K, α and a time-dependent decomposition

$$\mathfrak{X}_t = U_\gamma(t) \oplus S_\gamma(t) \qquad \text{for } t \in \mathbb{R}$$

such that

$$\|\Phi_{t,s} X_s\|_{\mathrm{ms}} \le K e^{(\gamma-\alpha)(t-s)} \|X_s\|_{\mathrm{ms}} \qquad \text{for } X_s \in S_\gamma(s) \text{ and } t \ge s\,,$$
$$\|\Phi_{t,s} X_s\|_{\mathrm{ms}} \ge \frac{1}{K} e^{(\gamma+\alpha)(t-s)} \|X_s\|_{\mathrm{ms}} \qquad \text{for } X_s \in U_\gamma(s) \text{ and } t \ge s\,.$$

A special case of exponential dichotomy, when the growth rate is equal to zero and the space of initial condition consists of the deterministic vectors in $\mathbb{R}^d$, is also investigated in [2, 12], where a Perron-type condition for existence of this exponential dichotomy is established.

Definition 3 (Mean-square dichotomy spectrum)**.** The *mean-square dichotomy spectrum* for a linear MS-RDS Φ is defined as

$$\Sigma := \Big\{ \gamma \in \mathbb{R} : \Phi \text{ has no exponential dichotomy with growth rate } \gamma \Big\}.$$

The set $\rho := \mathbb{R} \setminus \Sigma$ is called the *resolvent set* of Φ.

The dichotomy spectrum was first introduced in [11] for nonautonomous differential equations. Dichotomy spectra for random dynamical systems have been discussed recently in [3, 4, 13].

3. **Mean-field stochastic differential equations.** Mean-field stochastic differential equations of the form

$$\mathrm{d}X_t = f(t, X_t, \mathbb{E}X_t)\,\mathrm{d}t + g(t, X_t, \mathbb{E}X_t)\,\mathrm{d}W_t \tag{1}$$

were introduced in [6]. Here $\{W_t\}_{t\in\mathbb{R}}$ is a two-sided scalar Wiener process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $F := (f, g) : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d \times \mathbb{R}^d$.

Let $\mathrm{Lip}(\mathbb{R}^d)$ denote the set of Lipschitz continuous functions $f : \mathbb{R}^d \to \mathbb{R}^d$, and for each $f \in \mathrm{Lip}(\mathbb{R}^d)$, set

$$\|f(\cdot)\|_{\mathrm{lg}} := \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{1+|x|}\,.$$

Suppose that

(A1) $\Gamma := \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d} \Big\{ \mathrm{Lip} F(t,x,\cdot) + \|F(t,x,\cdot)\|_{\mathrm{lg}} \Big\} < \infty$.

(A2) For each $R > 0$, there exist a constant L_R and a modulus continuity ω_R such that

$$\|F(t_1,x_1,\cdot) - F(t_2,x_2,\cdot)\|_{\mathrm{lg}}^2 \le L_R |x_1 - x_2|^2 + \omega_R(|t_1 - t_2|)$$

for all $(t_k, x_k) \in \mathbb{R} \times \mathbb{R}^d$ with $t_k + |x_k|^2 \le R$, $k \in \{1, 2\}$.

Let $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$ be the natural filtration generated by $\{W_t\}_{t\in\mathbb{R}}$, and define

$$\mathfrak{X} := L^2(\Omega, \mathcal{F}; \mathbb{R}^d), \quad \mathfrak{X}_t := L^2(\Omega, \mathcal{F}_t; \mathbb{R}^d) \quad \text{for } t \in \mathbb{R}.$$

Given any initial condition $X_s \in \mathfrak{X}_s$, $s \in \mathbb{R}$, a solution of (1) is a stochastic process $\{X_t\}_{t\ge s}$ with $X_t \in \mathfrak{X}_t$ for $t \ge s$, satisfying the stochastic integral equation

$$X_t = X_s + \int_s^t f(u, X_u, \mathbb{E}X_u)\,\mathrm{d}u + \int_s^t g(u, X_u, \mathbb{E}X_u)\,\mathrm{d}W_u\,.$$

It was shown in [6] that the SDE (1) has a unique solution and generates a MS-RDS $\{\varphi_{t,s}\}_{t\geq s}$ on the underlying phase space $\mathbb{R}^d$ with a probability set-up $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}}, \mathbb{P})$, defined by $\varphi_{t,s} : \mathfrak{X}_s \to \mathfrak{X}_t$ with

$$\varphi_{t,s}(X_s) = X_t \qquad \text{for } X_s \in \mathfrak{X}_s\,.$$

4. **Mean-square dichotomy spectrum for linear mean-field stochastic differential equations.** Consider a linear mean-field stochastic differential equation

$$\mathrm{d}X_t = \big(A(t)X_t + B(t)\mathbb{E}X_t\big)\,\mathrm{d}t + \big(C(t)X_t + D(t)\mathbb{E}X_t\big)\,\mathrm{d}W_t\,, \tag{2}$$

where $A, B, C, D : \mathbb{R} \to \mathbb{R}^{d\times d}$ are continuous bounded functions, which generates a linear mean-square random dynamical system $\Phi_{t,s}$.

Proposition 4 (Equations for the first and second moments)**.** Let $X_s \in \mathfrak{X}_s$, and define $X_t := \Phi_{t,s}X_s$ for $t \geq s$. Then $\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}X_t = \big(A(t) + B(t)\big)\mathbb{E}(X_t)$ and for all $i, j \in \{1, \dots, d\}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}X_t^iX_t^j &= \sum_{k=1}^{d}\Big(a_{ik}(t)\mathbb{E}X_t^kX_t^j + a_{jk}(t)\mathbb{E}X_t^kX_t^i\Big)\\ &\quad + \sum_{m,n=1}^{d} c_{im}(t)c_{jn}(t)\mathbb{E}X_t^mX_t^n + \sum_{k=1}^{d}\Big(b_{ik}(t)\mathbb{E}X_t^k\mathbb{E}X_t^j + b_{jk}(t)\mathbb{E}X_t^k\mathbb{E}X_t^i\Big)\\ &\quad + \sum_{m,n=1}^{d}\left(c_{im}(t)d_{jn}(t) + c_{jn}(t)d_{im}(t) + d_{im}(t)d_{jn}(t)\right)\mathbb{E}X_t^m\mathbb{E}X_t^n.\end{aligned}$$

Proof. From (2),

$$\mathrm{d}X_t^i = \sum_{k=1}^{d}\left(a_{ik}(t)X_t^k + b_{ik}(t)\mathbb{E}X_t^k\right)\,\mathrm{d}t + \sum_{k=1}^{d}\left(c_{ik}(t)X_t^k + d_{ik}(t)\mathbb{E}X_t^k\right)\,\mathrm{d}W_t$$

holds for $t \in \mathbb{R}$. Taking the expectation of variables in both sides gives

$$\mathbb{E}X_t = \mathbb{E}X_s + \int_s^t \big(A(u) + B(u)\big)\mathbb{E}X_u\,\mathrm{d}u \quad \text{for } t \geq s,$$

which proves the first statement. Ito's product formula [8, Example 3.4.1] and the expectation then yield the second statement. □

Corollary 5. Let $(U_{i,j}(t,s), V_{i,j}(t,s))$ be the evolution operator of the linear nonautonomous differential equation in $\mathbb{R}^{d(d+1)}$

$$\begin{aligned}\dot{u}_{i,j} &= \sum_{k=1}^{d}\big(a_{ik}(t) + b_{ik}(t)\big)u_{k,j} + \big(a_{jk}(t) + b_{jk}(t)\big)u_{k,i}\\ \dot{v}_{i,j} &= \sum_{k=1}^{d}\left(a_{ik}(t)v_{k,j} + a_{jk}(t)v_{k,i}\right) + \sum_{m,n=1}^{d} c_{im}(t)c_{jn}(t)v_{m,n}\\ &\quad + \sum_{k=1}^{d}\left(b_{ik}(t)u_{k,j} + b_{jk}(t)u_{k,i}\right)\\ &\quad + \sum_{m,n=1}^{d}\left(c_{im}(t)d_{jn}(t) + c_{jn}(t)d_{im}(t) + d_{im}(t)d_{jn}(t)\right)u_{m,n},\end{aligned} \tag{3}$$

where $1 \leq i \leq j \leq d$. Then for any $X_s \in \mathfrak{X}_s$,

$$\|\Phi_{t,s}X_s\|_{\mathrm{ms}} = \left(\sum_{i=1}^{d} V_{i,i}(t,s)(\pi_s X_s)\right)^{\frac{1}{2}}, \tag{4}$$

where the map $\pi_s = \pi_s^1 \times \pi_s^2 : \mathfrak{X}_s \to \mathbb{R}^{\frac{d(d+1)}{2}} \times \mathbb{R}^{\frac{d(d+1)}{2}}$ is defined by

$$(\pi_s^1 X_s)_{i,j} = \mathbb{E}X_s^i \mathbb{E}X_s^j \quad \text{and} \quad (\pi_s^2 X_s)_{i,j} = \mathbb{E}X_s^i X_s^j \,.$$

Remark 6. It is interesting to compare the above equations with the ordinary differential equations for the first moment and second moments of linear stochastic differential equations, see e.g. [5, Section 6.2].

The proofs of the following preparatory results are straightforward.

Lemma 7. Let $\gamma \in \mathbb{R}$ be such that that the linear MS-RDS $\Phi_{t,s} : \mathfrak{X}_s \to \mathfrak{X}_t$ generated by (1) admits an exponential dichotomy with the growth rate γ and a decomposition

$$\mathfrak{X}_t = U_\gamma(t) \oplus S_\gamma(t).$$

Then the subspace $S_\gamma(t)$ is uniquely determined, i.e., if the linear MS-RDS $\Phi_{t,s}$ also admits an exponential dichotomy with the growth rate γ and another decomposition

$$\mathfrak{X}_t = \widehat{U}_\gamma(t) \oplus \widehat{S}_\gamma(t),$$

then $S_\gamma(t) = \widehat{S}_\gamma(t)$ for all $t \in \mathbb{R}$.

The subspaces $S_\gamma(t)$ of an exponential dichotomy with the growth rate γ are its stable subspaces. The following lemma provides an inclusion relation between these stable subspaces. Its proof follows directly from the definition of an exponential dichotomy.

Lemma 8. Let $\gamma_1 < \gamma_2$ be such that the linear MS-RDS admits an exponential dichotomy with the growth rates γ_1 and γ_2. Then $S_{\gamma_1}(t) \subset S_{\gamma_2}(t)$ for all $t \in \mathbb{R}$.

One of the main results of this paper is the following characterisation of the dichotomy spectrum.

Theorem 9 (Spectral Theorem)**.** Suppose that the coefficient functions in the linear mean-field stochastic differential equation (2) satisfy

$$\max\left\{|a_{ij}(t)|, |b_{ij}(t)|, |c_{ij}(t)|, |d_{ij}(t)|\right\} \leq m \qquad \text{for } i,j \in \{1,\dots,d\} \text{ and } t \in \mathbb{R} \tag{5}$$

with some $m > 0$. Then the dichotomy spectrum Σ is the disjoint union of at most $d(d+1)$ compact intervals $[a_1, b_1], \dots, [a_n, b_n]$ with $a_1 \leq b_1 < a_2 \leq b_2 \leq \cdots < a_n \leq b_n$. Furthermore, for each $s \in \mathbb{R}$, there exists a filtration of subspaces

$$\{0\} \subsetneq V_1(s) \subsetneq V_2(s) \subsetneq \cdots \subsetneq V_n(s) = \mathfrak{X}_s$$

which satisfies that for any $i \in \{1,\dots,n\}$, a random variable $X_s \in V_i(s)$ if and only if for any $\varepsilon > 0$, there exists $K(\varepsilon) > 0$ such that

$$\|\Phi_{t,s}X_s\|_{\mathrm{ms}} \leq K(\varepsilon)e^{(b_i+\varepsilon)(t-s)} \quad \text{for } t \geq s\,.$$

Proof. The proof is divided into several steps.
Step 1. First it will be shown that $(-\infty, -\Gamma) \subset \rho$ and $(\Gamma, \infty) \subset \rho$, where $\Gamma := 2dm + 2d^2m^2$. Let $X_s \in \mathfrak{X}_s$ be arbitrary, and define

$$\alpha(t) := \max_{i,j\in\{1,\dots,d\}} \left\{\mathbb{E}X_t^i X_t^j, \mathbb{E}X_t^i \mathbb{E}X_t^j\right\} \quad \text{for } t \geq s.$$

By the inequalities $\mathbb{E}XY \le \sqrt{\mathbb{E}X^2\mathbb{E}Y^2}$ and $(\mathbb{E}X)^2 \le \mathbb{E}X^2$, it follows that

$$\alpha(t) = \max_{1\le i\le d}(\mathbb{E}X_t^i)^2 \le \|\Phi(t,s)X_s\|_{\mathrm{ms}}^2.$$

Then by Corollary 5,

$$\alpha(t) \le \alpha(s) + 2\Gamma\int_s^t \alpha(u)\,\mathrm{d}u\,,$$

and Gronwall's inequality then yields

$$\|\Phi(t,s)X_s\|_{\mathrm{ms}}^2 \le d\alpha(t) \le de^{2\Gamma(t-s)}\alpha(s) \le de^{2\Gamma(t-s)}\|X_s\|_{\mathrm{ms}}^2.$$

This proves that $(\Gamma,\infty)\subset\rho$. Time reversal of the equations in Corollary 5 leads to

$$\|\Phi(t,s)X_s\|_{\mathrm{ms}}^2 \ge \frac{1}{d}e^{-2\Gamma(t-s)}\|X_s\|_{\mathrm{ms}}^2 \quad \text{for } (t,s)\in\mathbb{R}^2_{\ge}\,,$$

which proves $(-\infty,\Gamma)\subset\rho$.

Step 2. It will be shown that for any $t\in\mathbb{R}$, the set

$$\big\{S_\gamma(t) : \gamma\in\rho\cap(-\Gamma-1,\Gamma+1)\big\}$$

consists of at most $d(d+1)+1$ elements. Suppose the contrary, i.e., there exist $n+1$ numbers $\gamma_0<\gamma_1<\cdots<\gamma_n$ in $\rho\cap(-\Gamma-1,\Gamma+1)$, where $n>d(d+1)$, such that

$$S_{\gamma_i}(t)\ne S_{\gamma_j}(t) \qquad \text{for } i\ne j.$$

Then by Lemma 8,

$$S_{\gamma_0}(t)\subsetneq S_{\gamma_1}(t)\subsetneq\cdots\subsetneq S_{\gamma_n}(t).$$

Thus, there exist $X_t^1,\dots,X_t^n$ such that

$$X_t^i\in S_{\gamma_i}(t), \quad X_t^i\notin S_{\gamma_{i-1}}(t) \qquad \text{for } i\in\{1,\dots,n\}\,.$$

By definition of the γ_i, there exist $K,\alpha>0$ and complementary subspaces $U_{\gamma_i}(t)$ such that $\mathfrak{X}_t = U_{\gamma_i}(t)\oplus S_{\gamma_i}(t)$ and

$$\|\Phi_{\bar t,t}X_t\|_{\mathrm{ms}} \le Ke^{(\gamma_i-\alpha)(\bar t-t)}\|X_t\|_{\mathrm{ms}} \qquad \text{for } X_t\in S_{\gamma_i}(t) \text{ and } \bar t\ge t \tag{6}$$

and

$$\|\Phi_{\bar t,t}X_t\|_{\mathrm{ms}} \ge \frac{1}{K}e^{(\gamma_i+\alpha)(\bar t-t)}\|X_t\|_{\mathrm{ms}} \qquad \text{for } X_t\in U_{\gamma_i}(t) \text{ and } t\ge t\,. \tag{7}$$

Since $\mathbb{R}^{\frac{d(d+1)}{2}}\times\mathbb{R}^{\frac{d(d+1)}{2}}$ is $d(d+1)$-dimensional, it follows that there exist $k\le n$ and $\alpha_1,\dots,\alpha_{k-1}$ with $\alpha_1^2+\cdots+\alpha_{k-1}^2\ne 0$ and

$$\pi_t X_t^k = \alpha_1\pi_t X_t^1+\cdots+\alpha_{k-1}\pi_t X_t^{k-1}. \tag{8}$$

Consequently, by Corollary 5,

$$\begin{aligned}
\|\Phi_{\bar t,t}X_t^k\|_{\mathrm{ms}}^2 &= \sum_{i=1}^d V_{i,i}(\bar t,t)(\pi_t X_t^k) = \sum_{i=1}^d\sum_{j=1}^{k-1}\alpha_j V_{i,i}(\bar t,t)(\pi_t X_t^j)\\
&\le \sum_{j=1}^{k-1}|\alpha_j|\sum_{i=1}^d\sum_{j=1}^{k-1}V_{i,i}(\bar t,t)(\pi_t X_t^j)\\
&= \sum_{j=1}^{k-1}|\alpha_j|\left(\|\Phi_{\bar t,t}X_t^1\|_{\mathrm{ms}}^2+\cdots+\|\Phi_{\bar t,t}X_t^{k-1}\|_{\mathrm{ms}}^2\right).
\end{aligned}$$

By definition of γ_i and (6)

$$\|\Phi_{\bar{t},t}X_t^k\|_{\mathrm{ms}}^2 \leq (k-1)K\left(\sum_{j=1}^{k-1}|\alpha_j|\right)\left(\sum_{j=1}^{k-1}\|X_t^j\|_{\mathrm{ms}}^2\right)e^{(\gamma_{k-1}-\alpha)(\bar{t}-t)}.$$

Hence, it follows by (6) and (7) that $X_t^k \in S_{\gamma_{k-1}}(t)$, which leads to a contradiction. *Step 3.* As proved in Step 2, for $t \in \mathbb{R}$, let $S_0(t) \subsetneq S_1(t) \subsetneq \cdots \subsetneq S_n(t)$ with $n \leq d(d+1)$ satisfy

$$\big\{S_\gamma(t) : \gamma \in \rho \cap (-\Gamma-1, \Gamma+1)\big\} = \{S_0(t), S_1(t), \ldots, S_n(t)\big\}.$$

By Step 1, it follows that $S_0(t) = \{0\}$ and $S_n(t) = \mathfrak{X}_t$. For each $i \in \{0, \ldots, n\}$, define

$$\mathcal{I}_i := \big\{\gamma \in \rho \cap (-\Gamma-1, \Gamma+1) : S_\gamma(t) = S_i(t)\big\}.$$

It will be shown that $\mathcal{I}_i = (b_i, a_{i+1})$, where

$$b_i = \inf\{\gamma : \gamma \in \mathcal{I}_i\} \quad \text{and} \quad a_{i+1} = \sup\{\gamma : \gamma \in \mathcal{I}_i\} \qquad \text{for } i \in \{0, \ldots, n\}.$$

First, let $\gamma \in \mathcal{I}_i$ be arbitrary. By the definition of $\mathcal{I}_i$, there exist $K, \alpha > 0$ and a decomposition $\mathfrak{X}_t = U(t) \oplus S_i(t)$ and

$$\|\Phi_{\bar{t},t}X_t\|_{\mathrm{ms}} \leq Ke^{(\gamma-\alpha)(\bar{t}-t)}\|X_t\|_{\mathrm{ms}} \qquad \text{for } X_t \in S_i(t) \text{ and } \bar{t} \geq t$$

and

$$\|\Phi_{\bar{t},t}X_t\|_{\mathrm{ms}} \geq \frac{1}{K}e^{(\gamma+\alpha)(\bar{t}-t)}\|X_t\|_{\mathrm{ms}} \qquad \text{for } X_t \in U(t) \text{ and } \bar{t} \geq t\,.$$

This implies that $(\gamma-\alpha, \gamma+\alpha) \subset \mathcal{I}_i$, so $\mathcal{I}_i$ is open. It can be shown similarly that $\mathcal{I}_i$ is connected. Hence $\mathcal{I}_i = (a_i, b_i)$. Combining this result and Step 1 gives

$$\rho = (-\infty, a_1) \cup (b_1, a_2) \cup \cdots \cup (b_{n-1}, a_n) \cup (b_n, \infty),$$

which implies that

$$\Sigma = [a_1, b_1] \cup \cdots \cup [a_n, b_n].$$

To conclude the proof, the filtration corresponding to the spectral intervals is constructed as follows: for $t \in \mathbb{R}$, $V_0(t) := \{0\}, V_n(t) := \mathfrak{X}_t$, and

$$V_i(t) := S_\gamma(t), \quad \text{where } \gamma \in (b_i, a_{i+1}), \quad i \in \{1, \ldots, n-1\}.$$

Due to Lemma 7, the definition of V_i is independent of $\gamma \in (b_i, a_{i+1})$ for $i \in \{1, \ldots, n-1\}$. The strict inclusion $V_i \subsetneq V_{i+1}$ for $i \in \{0, \ldots, n-1\}$ follows from the construction of the open interval (b_i, a_{i+1}) above. Finally, the dynamical characterisation of V_i follows from the definition of (b_i, a_{i+1}) and the definition of exponential dichotomy. This completes the proof. □

5. **Bifurcation of a mean-square random attractor.** A mean-square random attractor was defined in [7] as the pullback attractor of the nonautonomous dynamical system formulated as a mean-square random dynamical system.

Specifically, a family $\mathcal{A} = \{A_t\}_{t\in\mathbb{R}}$ of nonempty compact subsets of $\mathfrak{X}$ with $A_t \subset \mathfrak{X}_t$ for each $t \in \mathbb{R}$ is called a *pullback attractor* if it pullback attracts all uniformly bounded families $\mathcal{D} = \{D_t\}_{t\in\mathbb{R}}$ of subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, i.e.,

$$\lim_{s\to-\infty} \operatorname{dist}(\varphi(t, s, D_s), A_t) = 0.$$

Uniformly bounded here means that there is an $R > 0$ such that $\|X\|_{\mathrm{ms}} \leq R$ for all $X \in D_t$ and $t \in \mathbb{R}$.

The existence of pullback attractors follows from that of an absorbing family. A uniformly bounded family $\mathcal{B} = \{B_t\}_{t\in\mathbb{R}}$ of nonempty closed subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$

is called a *pullback absorbing family* for a MS-RDS φ if for each $t \in \mathbb{R}$ and every uniformly bounded family $\mathcal{D} = \{D_t\}_{t\in\mathbb{R}}$ of nonempty subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, there exists some $T = T(t, \mathcal{D}) \in \mathbb{R}^+$ such that

$$\varphi(t, s, D_s) \subseteq B_t \quad \text{for } s \in \mathbb{R} \text{ with } s \le t - T.$$

Theorem 10. Suppose that a MS-RDS φ has a positively invariant pullback absorbing uniformly bounded family $\mathcal{B} = \{\mathcal{B}_t\}_{t\in\mathbb{R}}$ of nonempty closed subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ and that the mappings $\varphi(t, s, \cdot) : \mathfrak{X}_s \to \mathfrak{X}_t$ are pullback compact (respectively, eventually or asymptotically compact) for all $(t, s) \in \mathbb{R}^2_{\ge}$. Then, φ has a unique global pullback attractor $\mathcal{A} = \{\mathcal{A}_t\}_{t\in\mathbb{R}}$ with its component sets determined by

$$A_t = \bigcap_{s\le t} \varphi(t, s, B_s) \qquad \text{for } t \in \mathbb{R}.$$

Consider the nonlinear mean-field SDE

$$\mathrm{d}X_t = \left(\alpha X_t + \beta \mathbb{E}X_t - X_t \mathbb{E}X_t^2\right) \mathrm{d}t + X_t \,\mathrm{d}W_t \tag{9}$$

with real-valued parameters α, β. Note that the theory in Section 3 can be easily extended to include the second moment of the solution in the equation.

This SDE has the steady state solution $\bar{X}(t) \equiv 0$. Linearising along this solution gives the bi-linear mean-field SDE

$$\mathrm{d}Z_t = (\alpha Z_t + \beta \mathbb{E}Z_t) \,\mathrm{d}t + Z_t \,\mathrm{d}W_t \,. \tag{10}$$

Theorem 11. The dichotomy spectrum of the linear MS-RDS Φ generated by (10) is given by

$$\Sigma = \begin{cases} \{\alpha + 1/2\} \cup \{\alpha + \beta\} & \text{if } \beta > 1/2 \,, \\ \{\alpha + 1/2\} & \text{if } \beta \le 1/2 \,. \end{cases}$$

Proof. Taking the expectation of two sides of (10) yields that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}Z_t = (\alpha + \beta)\mathbb{E}Z_t,$$

which implies that

$$\mathbb{E}\Phi(t, s)Z_s = e^{(\alpha+\beta)(t-s)}\mathbb{E}Z_s \qquad \text{for } (t, s) \in \mathbb{R}^2_{\ge} \text{ and } Z_s \in \mathfrak{X}_s \,. \tag{11}$$

Ito's formula for the function $U(x) = x^2$ then gives

$$\mathrm{d}Z_t^2 = \left[(2\alpha + 1)Z_t^2 + 2\beta Z_t \mathbb{E}Z_t\right] \mathrm{d}t + 2Z_t^2 \,\mathrm{d}W_t \,.$$

Consequently,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}Z_t^2 = (2\alpha + 1)\mathbb{E}Z_t^2 + 2\beta(\mathbb{E}Z_t)^2.$$

Thus, using (11) for $(t, s) \in \mathbb{R}^2_{\ge}$ and $Z_s \in \mathfrak{X}_s$, it follows that

$$\begin{aligned} \|\Phi(t, s)Z_s\|^2_{\mathrm{ms}} &= e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} + 2\beta \int_s^t e^{(2\alpha+1)(t-u)}(\mathbb{E}Z_u)^2 \,\mathrm{d}u \\ &= e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} + 2\beta(\mathbb{E}Z_s)^2 \int_s^t e^{(2\alpha+1)(t-u)} e^{(2\alpha+2\beta)(u-s)} \,\mathrm{d}u \\ &= e^{(2\alpha+1)(t-s)} \left(\|Z_s\|^2_{\mathrm{ms}} + 2\beta(\mathbb{E}Z_s)^2 \int_s^t e^{(2\beta-1)(u-s)} \,\mathrm{d}u \right) . \end{aligned} \tag{12}$$

The assertions of the lemma will be shown for the three cases $\beta < 1/2$, $\beta = 1/2$ and $\beta > 1/2$.

Case 1 ($\beta < 1/2$). By (12),

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} = e^{(2\alpha+1)(t-s)}\left(\|Z_s\|^2_{\mathrm{ms}} + \frac{2\beta}{1-2\beta}\left(1-e^{(2\beta-1)(t-s)}\right)(\mathbb{E}Z_s)^2\right),$$

which together with the inequality $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$ yields

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} \leq e^{(2\alpha+1)(t-s)}\left(1+\frac{2|\beta|}{(1-2\beta)}\right)\|Z_s\|^2,$$

and

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} \geq \begin{cases} e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} & \text{if } \beta \geq 0, \\ \frac{1}{1-2\beta}e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} & \text{if } \beta < 0. \end{cases}$$

This implies that $\Sigma = \{\alpha + 1/2\}$.

Case 2 ($\beta = 1/2$). By (12),

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} = e^{(2\alpha+1)(t-s)}\left(\|Z_s\|^2_{\mathrm{ms}} + (t-s)(\mathbb{E}Z_s)^2\right),$$

which implies that

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} \geq e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} \quad \text{for } (t,s) \in \mathbb{R}^2_{\geq}.$$

Let $\varepsilon > 0$ be arbitrary. Since $(t-s) \leq \frac{1}{\varepsilon}e^{\varepsilon(t-s)}$ for $t \geq s$ and $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$, it follows that

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} \leq \left(1+\tfrac{1}{\varepsilon}\right)e^{(2\alpha+1+\varepsilon)(t-s)}\|Z_s\|^2_{\mathrm{ms}}.$$

Consequently, $\Sigma \subset \left[\alpha+\frac{1}{2}, \alpha+\frac{1}{2}+\varepsilon\right]$. The limit $\varepsilon \to 0$ leads to $\Sigma = \{\alpha+1/2\}$.

Case 3 ($\beta > 1/2$). By (12),

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} = e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} + \frac{2\beta}{2\beta-1}\left(e^{(2\alpha+2\beta)(t-s)} - e^{(2\alpha+1)(t-s)}\right)(\mathbb{E}Z_s)^2. \tag{13}$$

Together with the inequality $(\mathbb{E}X)^2 \leq \mathbb{E}X^2$, this implies that

$$e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}} \leq \|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} \leq \left(1+\frac{2\beta}{2\beta-1}\right)e^{(2\alpha+2\beta)(t-s)}\|Z_s\|^2_{\mathrm{ms}}.$$

Consequently, $\Sigma \subset \left[\alpha+\frac{1}{2}, \alpha+\beta\right]$. Let $\gamma \in \left(\alpha+\frac{1}{2}, \alpha+\beta\right)$ be arbitrary. Choose and fix $\varepsilon > 0$ such that $(\gamma-\varepsilon, \gamma+\varepsilon) \subset \left(\alpha+\frac{1}{2}, \alpha+\beta\right)$. The aim is to show that Φ admits an exponential dichotomy with the growth rate γ for the decomposition $\mathfrak{X}_s = U_s \oplus S_s$, where

$$S_s := \{f \in \mathfrak{X}_s : \mathbb{E}f = 0\},$$
$$U_s := \{f \in \mathfrak{X}_s : f \text{ is independent of noise}\}.$$

Obviously, any $X \in \mathfrak{X}_s$ can be written as $X = (X - \mathbb{E}X) + \mathbb{E}X$, with $X - \mathbb{E}X \in S_s$ and $\mathbb{E}X \in U_s$. By (13), for any $Z_s \in S_s$,

$$\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} = e^{(2\alpha+1)(t-s)}\|Z_s\|^2_{\mathrm{ms}}.$$

Now $(\mathbb{E}Z_s)^2 = \mathbb{E}Z_s^2$ for any $Z_s \in U_s$, so by (13), for $t - s \geq 1$,

$$\begin{aligned}\|\Phi(t,s)Z_s\|^2_{\mathrm{ms}} &\geq \frac{2\beta}{2\beta-1}\left(e^{(2\gamma+2\varepsilon)(t-s)} - e^{(2\gamma-2\varepsilon)(t-s)}\right)\|Z_s\|^2_{\mathrm{ms}} \\ &\geq \frac{4\varepsilon\beta}{2\beta-1}e^{2\gamma(t-s)}\|Z_s\|^2_{\mathrm{ms}}.\end{aligned}$$

Here the inequality $e^x \geq 1+x$ for $x \geq 0$ has been used. Thus, Φ admits an exponential dichotomy with the growth rate γ, which means that $\Sigma \subset \{\alpha+1/2\} \cup \{\alpha+\beta\}$. Considering the decomposition $S_s \oplus U_s$, it follows that $\alpha+\frac{1}{2}, \alpha+\beta \in \Sigma$. Thus, $\Sigma = \{\alpha+1/2\} \cup \{\alpha+\beta\}$. This completes the proof. □

A globally bifurcation of pullback attractor of (9) with $\beta = 1$, i.e.,

$$\mathrm{d}X_t = \left(\alpha X_t + \mathbb{E}X_t - X_t \mathbb{E}X_t^2\right) \mathrm{d}t + X_t \,\mathrm{d}W_t\,, \tag{14}$$

as α varies, will be investigated in a series of theorems.

The first and second moment equations of the mean-field SDE (14) are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}X_t = (\alpha+1)\mathbb{E}X_t - \mathbb{E}X_t\mathbb{E}X_t^2\,, \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}X_t^2 = (2\alpha+1)\mathbb{E}X_t^2 + 2(\mathbb{E}X_t)^2 - 2(\mathbb{E}X_t^2)^2\,, \tag{16}$$

where Ito's formula with $y = x^2$ was used to derive (16). These can be rewritten as the system of ODEs

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(\alpha+1-y) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = (2\alpha+1)y + 2x^2 - 2y^2\,, \quad \text{where } x^2 \leq y\,,$$

which has a steady state solution $\bar{x} = \bar{y} = 0$ for all α corresponding to the zero solution $X_t \equiv 0$ of the mean-field SDE (14). There also exist valid (i.e., with $y \geq 0$) steady state solutions $\bar{x} = \pm\sqrt{(\alpha+1)/2}$, $\bar{y} = \alpha+1$ for $\alpha > -1$ and $\bar{x} = 0$, $\bar{y} = \alpha+\frac{1}{2}$ for $\alpha \geq -\frac{1}{2}$. It needs to be shown if there are solutions of the SDE (14) with these moments.

Theorem 12. The MS-RDS φ generated by (14) has a uniformly bounded positively invariant pullback absorbing family.

Proof. Let α be arbitrary and define

$$B_t = \left\{X \in \mathfrak{X}_t : \|X\|_{\mathrm{ms}} \leq \sqrt{|\alpha|+2}\right\} \quad \text{for } t \in \mathbb{R}\,.$$

Using

$$(2\alpha+1)\mathbb{E}X_t^2 + 2(\mathbb{E}X_t)^2 - 2(\mathbb{E}X_t^2)^2 \leq (2\alpha+3)\mathbb{E}X_t^2 - 2(\mathbb{E}X_t^2)^2\,,$$

which holds since $(\mathbb{E}X_t)^2 \leq \mathbb{E}X_t^2$, the second moment equation (16) gives the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}X_t^2 \leq (2\alpha+3)\mathbb{E}X_t^2 - 2(\mathbb{E}X_t^2)^2 \tag{17}$$

Let $\mathcal{D} = \{D_t\}_{t\in\mathbb{T}}$ be a uniformly bounded family of nonempty subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$, i.e., $D_t \subset \mathfrak{X}_t$ and there exists $R > 0$ such that $\|X\|_{\mathrm{ms}} \leq R$ for all $X \in D_t$. Specifically, it will be shown that $\varphi(t,s,D_s) \subset B_t$ for $t-s \geq T$, where T is defined by

$$T := \log\left(\frac{R^2}{|\alpha|+2}\right). \tag{18}$$

Pick $X_s \in D_s$ arbitrarily and $(t,s) \in \mathbb{R}^2_{\geq}$ with $t-s \geq T$. Motivated by the differential inequality (17), consider the scalar system

$$\dot{y} = (2\alpha+3)y - 2y^2, \qquad \text{where } y(s) \leq R^2\,. \tag{19}$$

A direct computation yields

$$y(t) = y(s)\exp\left(\int_s^t (2\alpha+3) - 2y(u)\,\mathrm{d}u\right) \leq R^2 \exp\left(\int_s^t (2\alpha+3) - 2y(u)\,\mathrm{d}u\right).$$

From the definition of T in (18), it follows that $\min_{s\le u\le t} y(u) \le |\alpha| + 2$. Furthermore, $y = 0$ and $y = \alpha + \frac{3}{2}$ are stationary points of the ODE (19). For this reason, $\min_{s\le u\le t} y(u) \le |\alpha| + 2$ implies that $y(t) \le |\alpha| + 2$. Then from (17), it follows that $y(t) \ge \|\varphi(t,s)X_s\|_{\mathrm{ms}}^2$. This means that

$$\|\varphi_{t,s}X_s\|_{\mathrm{ms}}^2 \le y(t) \le |\alpha| + 2\,, \tag{20}$$

i.e., $\varphi_{t,s}X_s \in B_t$ for $t - s \ge T$. Hence, $\{B_t\}_{t\in\mathbb{R}}$ is a pullback absorbing family for the MS-RDS φ. It is clear that this family is uniformly bounded and positively invariant for the MS-RDS φ. □

Theorem 13. The MS-RDS φ generated by (14) has a pullback attractor with component sets $\{0\}$ when $\alpha < -1$.

Proof. Let $\alpha < -1$ be arbitrary. Let $\mathcal{D} = \{D_t\}_{t\in\mathbb{R}}$ be a uniformly bounded family of nonempty subsets of $\{\mathfrak{X}_t\}_{t\in\mathbb{R}}$ with $\|X\|_{\mathrm{ms}} \le R$ for all $X \in D_t$ where $R > 0$. Let $(X_s)_{s\in\mathbb{R}}$ be an arbitrary sequence with $X_s \in D_s$. The moment equations (15)–(16) can be written as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\varphi(t,s)X_s] &= \mathbb{E}[\varphi(t,s)X_s]\left(\alpha + 1 - \mathbb{E}[\varphi(t,s)X_s]^2\right)\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[\varphi(t,s)X_s]^2 &= (2\alpha+1)\mathbb{E}[\varphi(t,s)X_s]^2 + 2(\mathbb{E}[\varphi(t,s)X_s])^2 - 2(\mathbb{E}[\varphi(t,s)X_s]^2)^2.
\end{aligned}$$

Then

$$\mathbb{E}[\varphi(t,s)X_s] = \mathbb{E}X_s \exp\left(\int_s^t \alpha + 1 - \mathbb{E}X_u^2\,\mathrm{d}u\right),$$

which implies that

$$(\mathbb{E}[\varphi(t,s)X_s])^2 \le e^{(\alpha+1)(t-s)}R^2\,.$$

Moreover, by the variation of constants formula,

$$\begin{aligned}
&\mathbb{E}[\varphi(t,s)X_s]^2\\
&= e^{(2\alpha+1)(t-s)}\mathbb{E}X_s^2 + 2\int_s^t e^{(2\alpha+1)(t-u)}\left(\mathbb{E}[\varphi(u,s)X_s]^2 - (\mathbb{E}[\varphi(u,s)X_s]^2)^2\right)\mathrm{d}u\\
&\le e^{(2\alpha+1)(t-s)}R^2 + 2R^2\int_s^t e^{(2\alpha+1)(t-u)}e^{(\alpha+1)(u-s)}\,\mathrm{d}u\\
&\le e^{(2\alpha+1)(t-s)}R^2 + 2e^{(\alpha+1)(t-s)}(t-s)R^2\,,
\end{aligned}$$

which implies that $\lim_{s\to-\infty}\|\varphi(t,s,X_s)\|_{\mathrm{ms}} = 0$. Thus $\{0\}$ is the pullback attractor of (14) in this case. □

Theorem 14. The MS-RDS φ generated by (14) has a nontrivial pullback attractor when $-1 < \alpha < -\frac{1}{2}$.

Remark 15. The idea for the proof is taken from [7, Subsection 4.1], but their result cannot be applied directly, since the Lipschitz constant of the nonlinear terms is at least 1, and α is not less than -4. Instead the uniform equicontinuity of the mapping $t \mapsto \mathbb{E}X_t$, and then the positivity of the second moment to obtain a better estimate are used.

Proof. In order to apply Theorem 10, it needs to be shown that φ is pullback asymptotically compact, i.e., given a uniformly bounded family $\mathcal{D} = \{D_t\}_{t\in\mathbb{R}}$ of nonempty subsets of $\mathfrak{X}_t$ and sequences $\{t_k\}_{k\in\mathbb{N}}$ in $(-\infty, t)$ with $t_k \to -\infty$ as $k \to \infty$ and $\{X_k\}_{k\in\mathbb{N}}$ with $X_k \in D_{t_k} \subset \mathfrak{X}_{t_k}$ for each $k \in \mathbb{N}$, then the subset

$\{\varphi(t,t_k,X_k)\}_{k\in\mathbb{N}} \subset \mathfrak{X}_t$ is relatively compact. For this purpose, let $\varepsilon > 0$ be arbitrary. A finite cover of $\{\varphi(t,t_k,X_k)\}_{k\in\mathbb{N}}$ with diameter less than ε will be constructed. Choose and fix $s \in \mathbb{R}$ with $s < t$ such that

$$4e^{(2\alpha+1)(t-s)}(|\alpha|+2)^2 < \frac{\varepsilon^2}{2}\,, \tag{21}$$

and define $Y_s^k := \varphi(s,t_k,X_k)$. Using (20) it can be assumed without loss of generality that

$$\mathbb{E}(Y_s^k)^2 \le |\alpha| + 2 \qquad \text{for } k \in \mathbb{N}. \tag{22}$$

For $u \in [s,t]$, let $Y_u^k := \varphi(u,s)Y_s^k$ and consider a family of functions $f^k : [s,t] \to \mathbb{R}$ defined by $f^k(u) := \mathbb{E}Y_u^k$. By Ito's formula,

$$Y_t^k = e^{\alpha(t-s)}Y_s^k + \int_s^t e^{\alpha(t-u)}\left(\mathbb{E}Y_u^k - Y_u^k\mathbb{E}(Y_u^k)^2\right)\,\mathrm{d}u + \int_s^t e^{\alpha(t-u)}Y_u^k\,\mathrm{d}W_u,$$

from which it can be shown that that $(f^k)_{k\in\mathbb{N}}$ is a uniformly equicontinuous sequence of functions. By the Arzelà–Ascoli theorem, for $\delta := \varepsilon\sqrt{\frac{\alpha}{2}+\frac{1}{4}}$, there exists a finite index set $J(\delta) \subset \mathbb{N}$ such that for any $k \in \mathbb{N}$ there exists $n_k \in J(\delta)$ for which

$$\sup_{u\in[s,t]} |\mathbb{E}Y_u^k - \mathbb{E}Y_u^{n_k}| < \delta\,. \tag{23}$$

To conclude the proof of asymptotic compactness, it is sufficient to show the inequality $\|Y_t^k - Y_t^{n_k}\|_{\mathrm{ms}} \le \varepsilon$. Indeed, for $u \in [s,t]$, a direct computation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}(Y_u^k - Y_u^{n_k})^2 = (1+2\alpha)\mathbb{E}(Y_u^k - Y_u^{n_k})^2 + 2(\mathbb{E}Y_u^k - \mathbb{E}Y_u^{n_k})^2 - 2(\mathbb{E}(Y_u^k)^2)^2 \\ - 2(\mathbb{E}(Y_u^{n_k})^2)^2 + 2\mathbb{E}Y_u^kY_u^{n_k}(\mathbb{E}(Y_u^k)^2 + \mathbb{E}(Y_u^{n_k})^2)\,.\end{aligned}$$

Note that

$$2(\mathbb{E}(Y_u^k)^2)^2 + 2(\mathbb{E}(Y_u^{n_k})^2)^2 - 2\mathbb{E}Y_u^kY_u^{n_k}(\mathbb{E}(Y_u^k)^2 + \mathbb{E}(Y_u^{n_k})^2) \ge 0\,.$$

Hence, by the variation of constant formula,

$$\|Y_t^k - Y_t^{n_k}\|_{\mathrm{ms}}^2 \le e^{(2\alpha+1)(t-s)}\|Y_s^k - Y_s^{n_k}\|_{\mathrm{ms}}^2 + 2\int_s^t e^{(2\alpha+1)(t-u)}(\mathbb{E}Y_u^k - \mathbb{E}Y_u^{n_k})^2\,\mathrm{d}u\,,$$

which together with (22) and (23) implies that

$$\|Y_t^k - Y_t^{n_k}\|_{\mathrm{ms}}^2 \le 4e^{(2\alpha+1)(t-s)}(|\alpha|+2)^2 + \frac{2\delta^2}{2\alpha+1} < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2\,.$$

Here (21) was used to obtain the preceding inequality. Thus the set $\{\varphi(t,t_k,X_k)\}_{k\in\mathbb{N}}$ is covered by the finite union of open balls with radius ε centered at $\varphi(t,t_k,X_k)$, where $k \in J(\lambda)$, i.e., is totally bounded.

Hence the MS-RDS is asymptotically compact and by Theorem 10 has a pullback attractor with component subsets $\{A_t\}_{t\in\mathbb{R}}$ that contain the zero solution. It remains to show that the sets A_t also contain other points. Consider a uniformly bounded family of bounded sets defined by

$$D_t := \left\{X \in \mathfrak{X}_t : \mathbb{E}X = \sqrt{(\alpha+1)/2}\,,\ \mathbb{E}X^2 = \alpha+1\right\}.$$

Recall that these values are a steady state solution of the moment equations (15)–(16). Hence $\varphi(t,s,X_s) \in D_t$ for all $t > s$ when $X_s \in D_s$. Then by pullback attraction

$$\operatorname{dist}(\varphi(t,s,X_s),A_t) \le \operatorname{dist}(\varphi(t,s,D_s),A_t) \to 0 \quad \text{as} \quad s \to -\infty\,.$$

The convergence here is mean-square convergence, and $\mathbb{E}\varphi(t,s,X_s)^2 \equiv \alpha + 1$ for all $t > s$. Thus, there exists a random variable $X_t^* \in A_t \cap D_t$ for each $t \in \mathbb{R}$, i.e., the pullback attractor is nontrivial. Note that by arguments in [10] (see also [9]), it follows that there is in fact an entire solution $\bar{X}_t \in A_t \cap D_t$ for all $t \in \mathbb{R}$. □

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