

Cosmological perturbations from non-equilibrium phenomena in the very early universe

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Abstract

I predict the cosmological perturbations in temperature and galaxy distribution observable today if the non-equilibrium phenomena of preheating or vacuum transition occurred in the very early universe.

Preheating involves the rapid production of daughter particles after the end of inflation. I study the case where an inflaton field decays to a single light scalar spectator field during preheating. A separate universe approach combined with non-perturbative delta N formalism is required to calculate its observable effects on the non-Gaussianity of primordial curvature perturbations. By extending the non-perturbative formalism to include scale dependence of the spectator field, I find that cosmic variance plays a key role in determining the non-Gaussianity. Using lattice field theory simulations, I perform a full numerical calculation of the non-Gaussianity parameter f_{NL} . This calculation is illustrated for an observationally-viable model of preheating motivated by non-minimal coupling to gravity. Furthermore, I also study the dependence of f_{NL} on the parameters in this model.

Vacuum or first order phase transitions are ubiquitous in theories containing multiple fields when the potential has several minima. If such a transition takes place during inflation in the presence of a light spectator field, I show that it can generate a large, potentially detectable non-Gaussianity. To do so I calculate the instanton solution that describes quantum tunnelling between vacuum states on a de Sitter background, obtain its dependence on the spectator field and thereby its modulation of the expansion of space at different locations. I apply this procedure to a toy inflationary model using the thin-wall approximation in Minkowski space, which makes analytical calculation feasible. Then I construct a realistic model based on Higgs inflation and study the parameter dependence of f_{NL} in this scenario as well.

Statement of originality

This thesis is the author's own work, all else has been appropriately referenced. A major portion of this work appears in the following two collaborative articles written by the author:

- [1] P. S. Ghoderao and A. Rajantie, *Curvature perturbations from preheating with scale dependence*, JCAP 05 (2024) 106, DOI:10.1088/1475-7516/2024/05/106.
eprint: [arXiv:2311.02173](#) [[astro-ph.CO](#)]

- [2] B. Garbrecht, P. S. Ghoderao and A. Rajantie, *Curvature perturbations from vacuum transition during inflation*, JCAP 05 (2025) 049, DOI:10.1088/1475-7516/2025/05/049.
eprint: [arXiv:2502.01632](#) [[astro-ph.CO](#)]

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Conventions, abbreviations and symbols used

Unless otherwise specified, the following are used throughout the thesis:

1. t = time coordinate
2. η = conformal time
3. x = three spatial coordinates represented as a vector, the vector sign has been dropped for simplicity
4. k = momentum
5. $g_{\mu\nu}$ = spacetime metric
6. metric signature = “mostly plus”, i.e. $(-+++)$
7. $\mathcal{R}$ = Ricci scalar
8. G = Newton’s gravitational constant
9. units are such that $\hbar = c = 1$
10. M_P = reduced Planck mass
11. k_B = Boltzmann constant
12. ξ = non-minimal coupling parameter
13. T = temperature
14. a = scale factor
15. z = cosmological redshift
16. H = Hubble rate
17. R = radius of bubble
18. Γ = decay rate per unit volume
19. subscripts FV and TV denote quantities in the false and true vacuum respectively, for example, σ_{FV} denotes transition field value in its true vacuum
20. $\mathcal{D}$ = functional derivative
21. p = percolation parameter
22. $\mathcal{P}$ = probability

- 23. P = power spectrum
- 24. $\mathcal{P}$ = reduced power spectrum
- 25. P = Pressure
- 26. $\mathcal{N}$ = lapse function in the Arnowitt-Deser-Misner (ADM) formalism of gravity
- 27. $\mathcal{N}$ = number of efolds during inflation, to distinguish them from number of efolds during preheating denoted by N , while discussing preheating throughout Chapter 3
- 28. Fourier Transform:

$$g(x) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} g(k) \quad \text{and} \quad g(k) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} g(x)$$

- 29. Delta function: $\int e^{i\vec{k}\cdot\vec{x}} d^3x = (2\pi)^3 \delta^3(k)$
- 30. $\iota = \sqrt{-1}$
- 31. QFT = Quantum Field Theory
- 32. SM = Standard Model of particles
- 33. Λ CDM = Lambda Cold Dark Matter also known as the Concordance/Standard model of cosmology
- 34. BBN = Big Bang Nucleosynthesis
- 35. CMB = Cosmic Microwave Background
- 36. LSS = Large Scale Structure
- 37. GW = Gravitational Waves
- 38. HM = Hawking-Moss (also used as a subscript to indicate quantities associated with the Hawking-Moss transition, for example p_{HM} indicates the percolation parameter for Hawking-Moss decay rate)
- 39. CDL = Coleman-De Luccia
- 40. ADM = Arnowitt-Deser-Misner

Chapter 1

Introduction

“it is all about predicting things, making us glorious fortune tellers of matter, space and time”

1.1 General introduction

1.1.1 The very early universe

Let us begin by drawing a distinction between the early and very early eras in our universe’s evolution. It has been confirmed through numerous observational tests that our universe expanded from a very small size. Assuming the size was zero at the start, around 1 second later the universe was a hot soup made up of all the particles that we know and see around us today such as the electrons, photons etc. After this time, we know how the universe evolved based on well-established laws of physics. Therefore from around 1 second onwards we say the early universe begins and before that is what we then call the very early universe.

One of the most well-motivated and widely accepted hypothesis for what happened in the very early universe is known as cosmic inflation. It is an era of extremely rapid expansion. To visualise the evolution during this era, it is helpful to compare it to a marble rolling-down a flat hill as shown in Figure 1.1. The marble represents inflaton particle that is responsible for inflation. When it is on the nearly flat part of the hill, the universe expands very rapidly. One of the simplifications usually made is to assume that this hill is exactly flat, denoted by the dashed line in the Figure 1.1. However, if it is exactly flat then the marble will always remain at one place on the hill and the universe will continue to expand rapidly even until today. But we know that the universe is not expanding rapidly today, and hence the hill must be slightly sloped initially and eventually the marble must settle down at the bottom of the valley and end the era of rapid expansion.

One of the highlights of this work is that we will take into account this slight slope during inflation.

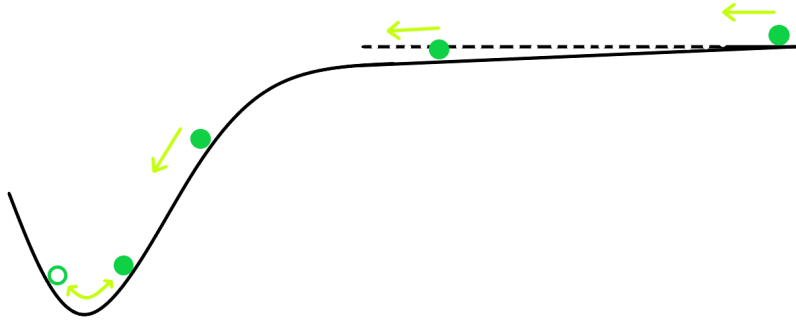


Figure 1.1: A marble (inflaton particle) slowly rolling down a nearly flat hill before entering and oscillating around the bottom of a valley.

1.1.2 Non-equilibrium phenomena

As the marble leaves the nearly flat portion of the hill and enters the valley, it will gain speed and instead of stopping at the bottom of the valley, will overshoot it and undergo oscillations before eventually settling down.

Non-equilibrium phenomena can be understood as a process occurring on a different time scale than the rest of the system. When the marble is rolling down the nearly flat hill, its position is changing slowly. After it has settled down at the bottom of the valley, there is no further change at all. But when the marble is undergoing oscillations, it is doing so quickly and thus displaying non-equilibrium behaviour.

Another way to understand non-equilibrium phenomena is through the energy of a system. The hill and valley of Figure 1.1 actually represent how much energy is stored in the marble or inflaton particle, also known as its potential. The greater the height of the marble from its stable minimum at the bottom of the valley, the greater is its potential energy. So the universe starts out with a lot of potential energy during inflation. As it settles down to the bottom of the valley it loses all its energy through oscillations by transferring it to the everyday particles around us such as the electron, photon etc. Thus a change in the system's energy is another way to understand non-equilibrium phenomena.

In the context of the universe's evolution, temperature is essentially a measure of how much particles are hitting each other. As the universe expands, the particles have more space to move about and so the universe cools. During inflation, when the expansion is rapid, the universe cools by a lot. But we saw that at the start of the early universe, it was a hot soup of the everyday particles electrons, photons etc. So there must occur an intervening era that reheats the universe back into a hot soup. Therefore this era is named reheating.

We have already seen what the reheating era is, it is when the inflaton particle undergoes oscillations around the bottom of its valley and transfers its potential energy to reheating the

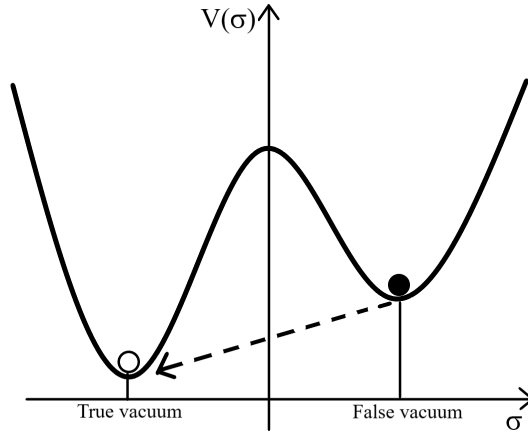


Figure 1.2: Schematic potential of a particle (field) that undergoes vacuum decay by quantum tunnelling from its unstable minimum to a stable minimum.

universe. Thus the reheating era is responsible for giving a fiery birth to all the everyday particles around us such as the electrons, photons etc. starting from just the one type of inflaton particle.

Reheating is one of the two non-equilibrium phenomena that are the focus of this work. The other non-equilibrium phenomenon studied in this work is known as vacuum transition. To understand this phenomenon we can go back to the analogy of the marble. But this time, the marble is not the inflaton particle, but another particle that might be present alongside it during inflation. While trying to incorporate the forces of gravity in the tiny size of the very early universe, we can get such particles that are different from the inflaton particle and have a potential energy that looks like Figure 1.2. Thus instead of a flat hill this particle has a potential that looks like two valleys separated by a peak.

If the marble is in the slightly higher valley's bottom, then based on our everyday classical experience, we know that the marble will remain stuck there. But at tiny sizes, the correct description of particles is through quantum mechanics. In quantum mechanics, the particle is allowed to tunnel through the hill from a higher position until it reaches the lowest possible potential energy.

The bottom of the valley which is at a slightly higher position is known as the false vacuum or unstable minimum because the particle can tunnel from it. While the bottom of the valley which is at the lowest height is known as the true vacuum or stable minimum because the particle cannot go any lower in its potential.

Vacuum transition is the process of tunnelling from the false to the true vacuum. The potential energy of the system changes abruptly during vacuum transition and thus it is an example of non-equilibrium phenomena.

1.1.3 Cosmological perturbations

The universe is uniform to a large degree. This is evident from the experimental observations of galaxy and temperature distributions today. However, if it was exactly uniform then matter could not have clumped together to form galaxies, stars, planets and ultimately us. These slight non-uniformities are called cosmological perturbations.

Inflation provides a mechanism for these slight non-uniformities. At tiny sizes, quantum theory dictates that fluctuations would arise, i.e. particles would be concentrated more in some locations and less in others. Due to the rapid expansion during inflation, these fluctuations at tiny sizes get enlarged to large, universal sizes. It is like taking a snapshot of these fluctuations and zooming the photo to huge sizes in a very small amount of time.

When we say that the early universe was like a soup-like fluid, what we mean is that it is made up of many different particles, just like water is made up of many individual H_2O molecules, but collectively it appears like a homogenous fluid.

Now suppose there are two different kinds of particles present in the soup-like fluid during inflation, one is the inflaton that is responsible for the rapid expansion and cooling of the temperature and the other is a particle that undergoes vacuum transition. Then this system is analogous to air plus water. As the air temperature cools, water starts to freeze by forming bubbles of ice (aka ice crystals) that grow and meet, thus forming a solid block of ice.

Now if we add milk into the mix, then the milk particles (aka lactose molecules) are concentrated more in some places and less in others. So when the air temperature drops, the water freezes at slightly different times at different locations. This is what happens in ice-cream!

Just like ice-cream has three main components: air, water and milk (plus sugar and flavouring to make it tasty), in the very early universe too there can be three kinds of fluids: a first kind made up of the inflaton particles, a second kind made up of particles that undergo vacuum transition and a third kind made up of particles that play the role of milk particles.

The third type of particles are just there to spectate, but importantly since the size of the universe is tiny, they can have quantum fluctuations that through the rapid expansion would show up in the non-uniformity of galaxy and temperature distributions observed today. This is one of the main results of this work.

Through this work we will argue that non-uniformity in galaxy and temperature distributions in our universe today can be generated by quantum fluctuations in a particle that is present along with the inflaton particle in the very early universe if the universe undergoes some kind of non-equilibrium phenomenon.

We saw how this can occur for the non-equilibrium phenomena of vacuum transition, and a similar thing can occur for reheating too. This is the second main result of this work.

In the rest of this thesis, I make the above ideas more precise and express them in a rigorous mathematical framework.

1.2 The standard cosmological model

Sitting on Earth at present times, we develop an understanding of the universe by making observations of the sky. On length scales larger than about a hundred megaparsecs, we find that the universe largely looks the same in all directions, making it isotropic; and at all points, making it homogeneous. Then assuming the Copernican Principle that our particular position on Earth is not a special location in the universe, we use the homogeneity and isotropicity to be the starting point of cosmology. This is known as the Cosmological Principle.

The combination of homogeneity and isotropicity place geometrical constraints on the metric of spacetime, restricting it to a Friedmann-Robertson-Walker (FRW) form

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right) \quad (1.1)$$

in spherical coordinates $\{r, \theta, \phi\}$ and with time coordinate t . The parameter K captures spatial curvature and can take the values $K = 0, \pm 1$. $a(t)$ is known as the scale factor. It captures the expansion of space.

In the general theory of relativity, the metric $g_{\mu\nu}$ of spacetime is related to the energy-momentum tensor $T_{\mu\nu}$ of matter through the Einstein equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = 8\pi G T_{\mu\nu}, \quad (1.2)$$

where $\mathcal{R}_{\mu\nu}$, $\mathcal{R}$ denote the Ricci tensor and Ricci scalar respectively and G is the Newton constant of gravitation. In terms of field theory, this equation can be obtained from the Einstein-Hilbert action with minimal coupling,

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{8\pi G} \mathcal{R} + \mathcal{L}_{\text{matter}} \right) \quad (1.3)$$

where g denotes the determinant of the metric and $\mathcal{L}_{\text{matter}}$ is the matter Lagrangian related to the energy-momentum tensor by a variational derivative with respect to the metric,

$$T_{\mu\nu} = g_{\mu\nu}\mathcal{L}_{\text{matter}} - 2\frac{\delta\mathcal{L}_{\text{matter}}}{\delta g^{\mu\nu}}. \quad (1.4)$$

For a perfect fluid with energy density ρ , pressure P and four velocity u^μ , its energy momentum tensor is given by

$$T_{\mu\nu} = Pg_{\mu\nu} + (\rho + P)u_\mu u_\nu. \quad (1.5)$$

Assuming matter behaves as a perfect fluid in the FRW metric one can construct a model, known as the Lambda Cold Dark Matter (Λ CDM) model, that describes the evolution of our universe in terms of the scale factor. The Λ CDM model of cosmology has been very successful in describing a number of observed phenomena such as galaxy formation, baryon acoustic oscillations and the

cosmic microwave background radiation.

Corresponding to the equation of state $P = w\rho$, the components of the fluid in Λ CDM are comprised of: radiation with $w = 1/3$ denoted by subscript r ; matter (ordinary + dark) with $w = 0$ denoted by subscript m ; dark energy with $w = -1$ denoted by subscript Λ . The fractional energy density Ω of the individual component can be defined with respect to the critical energy density today ρ_{crit} as

$$\Omega_{\Lambda} = \frac{\rho_{\Lambda}}{\rho_{\text{crit}}}, \quad \Omega_m = \frac{\rho_m}{\rho_{\text{crit}}}, \quad \Omega_r = \frac{\rho_r}{\rho_{\text{crit}}}. \quad (1.6)$$

The 00-component of the Einstein equation yields the evolution of the metric (1.1) in terms of the energy density ρ . This relation is known as the Friedmann equation and is given by

$$H^2 + \frac{K^2}{a^2} = \frac{\rho}{3M_P^2}. \quad (1.7)$$

Here H denotes the Hubble rate defined as

$$H(t) = \frac{1}{a(t)} \frac{da(t)}{dt} \quad (1.8)$$

and M_P denotes the reduced Planck Mass given by $M_P^2 = 1/8\pi G$.

One can also include the spatial curvature as a form of energy density by defining

$$\Omega_K = -\frac{K^2}{a_0^2 H_0^2} \quad (1.9)$$

where subscript 0 is used to indicate the value of the corresponding quantities today, for example a_0 , H_0 used above respectively denote the scale factor and Hubble rate today. With this the Friedmann equation for Λ CDM model can be written as

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{\Lambda} + \Omega_K \left(\frac{a_0}{a}\right)^2 + \Omega_m \left(\frac{a_0}{a}\right)^3 + \Omega_r \left(\frac{a_0}{a}\right)^4, \quad (1.10)$$

such that

$$\rho_{\text{crit}} = \frac{3H_0^2}{8\pi G}. \quad (1.11)$$

Today, the addition of all the fractional energy densities should equal to unity, $\Omega_{\Lambda} + \Omega_K + \Omega_m + \Omega_r = 1$. From cosmological experiments [6] we roughly have $\Omega_{\Lambda} = 0.68$, $\Omega_m = 0.31$ of which baryon (ordinary) matter denoted by subscript b is $\Omega_b = 0.05$ and cold dark matter (CDM) denoted by subscript c is $\Omega_c = 0.26$ and $\Omega_r \sim 10^{-4}$. The spatial curvature Ω_K is consistent with being zero.

Thus the universe is measured to be a spatially flat FRW spacetime,

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \quad (1.12)$$

in Cartesian coordinates $\{x, y, z\}$. Hereafter in this work, the usage ‘FRW spacetime’ should be understood to mean flat FRW spacetime.

Using the Friedmann equation for Λ CDM (1.10), we can divide the universe’s evolution into different eras based on the component of the cosmic fluid that dominates the energy density in that era. Starting from a negligibly small scale factor, the first component that dominates is radiation because it evolves as $\rho_r \sim a^{-4}$. For radiation, the energy density ρ_r and temperature T are related as [7]

$$\rho_r = \frac{\pi^2}{30} g_{\text{eff}} T^4, \quad (1.13)$$

where g_{eff} are the effective radiation (i.e. massless) degrees of freedom in the system. Therefore during radiation domination we have

$$a \sim \frac{1}{T}. \quad (1.14)$$

As the scale factor increases, eventually the universe reaches radiation-matter equality when energy density of radiation ρ_r becomes equal to that of matter ρ_m . In terms of the redshift z defined by

$$z + 1 = \frac{a_0}{a}, \quad (1.15)$$

this occurs at $z_{\text{eq}} \sim 3500$ [6] corresponding to a temperature of $T_{\text{eq}} \sim 10^5 K$. Thereafter the matter domination era with $\rho_m \sim a^{-3}$ begins.

After $z \sim 0.1$ corresponding to matter-dark energy equality, the universe starts to be dominated by dark energy $\rho_\Lambda = \text{const.}$ The universe is dark energy dominated at present times and would remain dark energy dominated hereafter according to the Λ CDM model.

A fundamental description of dark matter and dark energy in the Λ CDM model in terms of particle physics or a modification of general relativity is currently missing. In recent years, there is a lot of theoretical and observational effort underway to address this issue. Candidates for dark matter [8] include weakly interacting massive particles (WIMPs), axions, sterile neutrinos, primordial black holes, dark sectors eg. dark photon, etc. Candidate theories for dark energy [8, 9] are the cosmological constant, quintessence models, galileon or massive gravity etc.

One of the most striking success of the Λ CDM model is the precise prediction of element abundances observed today by following the thermal history of the universe. The era in the universe’s evolution that births these elements is termed as Big Bang Nucleosynthesis (BBN), and is a coming together of disparate physical theories, namely the amplitude calculations performed using Quantum Field Theory, non-equilibrium Thermodynamics and Cosmology. Thus establishing a connection of the smallest scales to the largest scales known in nature. The starting conditions required by BBN are a radiation dominated universe in local thermal equilibrium at temperature $T_{\text{nuc}} \sim 10^{10} K$ in the presence of all the Standard Model particles. Thereafter Λ CDM follows the evolution of the

universe from the radiation dominated era, to a matter dominated era and finally the current dark energy dominated era.

1.3 Cosmic inflation

Inflation is a postulated era before radiation domination during which the universe undergoes an accelerated expansion, $\ddot{a} > 0$. It was originally postulated to address some shortcomings of the Λ CDM model, namely the horizon, flatness and monopole problems.

Horizon problem:

Measurements of the cosmic microwave background radiation reveal that the universe has a uniform thermal distribution with a temperature of approximately $2.725K$ everywhere. However, if one follows backwards the evolution of two patches of the sky which are separated by more than 2° , the Λ CDM model predicts they were never in causal contact with each other. If this is the case, they could not have interacted with each other to settle on a uniform distribution in temperature. This inconsistency is known as the horizon problem. The most widely accepted solution to this problem is the theory of inflation. By adding an additional stage to the evolution before Λ CDM, in which the universe expanded at an accelerated rate, almost exponentially, the entire observable sky could be taken to have evolved from a single causally connected patch.

Flatness problem:

The spatial curvature fraction of the universe at some general time t can be written as

$$|\Omega_K|_t = \frac{|K|}{a(t)^2 H(t)^2}. \quad (1.16)$$

Ignoring the recent dark energy domination, the quantity aH always decreases during the matter and radiation dominated eras (see Figure 1.5) where the scale factor goes as $a \sim t^{2/3}$ and $a \sim t^{1/2}$ respectively. In matter domination, $aH \sim a^{-1/2}$ so $|\Omega_K|_{\text{eq}} \approx |\Omega_K|_0/z_{\text{eq}}$ at radiation matter equality. In radiation domination, $aH \sim a^{-1} \sim T$, so at nucleosynthesis temperatures $T_{\text{nuc}} \sim 10^{10}K$ the curvature fraction $|\Omega_K|_{\text{nuc}}$ in terms of the curvature fraction today $|\Omega_K|_0$ is

$$|\Omega_K|_{\text{nuc}} \approx \frac{|\Omega_K|_0}{z_{\text{eq}}} \frac{T_{\text{eq}}^2}{T_{\text{nuc}}^2} \sim 10^{-14} |\Omega_K|_0, \quad (1.17)$$

where the temperature at radiation matter equality $T_{\text{eq}} = 10^5 K$. As $|\Omega_K|_0 < 1$ today, during nucleosynthesis it would be unusually small. Thus there is a naturalness puzzle to explain such a negligible value of the spatial curvature in the early stages of the Λ CDM model. This is known as the flatness problem. By postulating an inflationary era of near exponential expansion, where aH increases significantly (see Figure 1.5), even a highly spatially curved

initial state of the universe can be made nearly flat today by compensating for the increase in $|\Omega_K|$ in the radiation and matter eras.

Monopole problem:

The Standard Model (SM) of particle physics has enjoyed remarkable experimental success. However, in its nascent form, with neutrinos assumed to be massless, it cannot explain the phenomenon of neutrino oscillations. It is possible to add neutrino masses in an ad-hoc fashion by introducing additional parameters, but it makes the notion of unification, i.e. a description of physical theories in terms of as few parameters as possible, which had been a guiding element in the construction of the Standard Model itself, worse. As a continuation of the idea of unification, there have been proposals to describe the SM gauge group of $SU(3)_c \times SU(2)_L \times U(1)_Y$ as an effective field theory emerging from more unified ones such as $SU(5)$ or $SO(10)$. Such proposals fall under the banner of Grand Unified Theories (GUTs).

At a more fundamental level, the SM does not describe gravity. The unification of gravity with the other SM forces is a major research endeavour of current theoretical physics, not least because it can help explain what occurs at or near the singularity of black holes. One of the leading candidates for a theory of quantum gravity is string theory.

Since we only observe the SM gauge group at terrestrial energies, many of the beyond Standard Model scenarios involving GUTs and string theory involve symmetry breaking from the unified gauge group to the SM one. The generation of magnetic monopoles is a consequence of such symmetry breaking for a wide range of gauge groups [10, 11]. The monopoles thus produced have masses corresponding to the typical scale of the symmetry breaking, which is $\gtrsim 10^{15} \text{ GeV}$.

In the context of early universe, and the Λ CDM model in particular, the GUT symmetry gets restored due to high temperatures as one moves closer to the big bang singularity [12]. Thus the symmetry breaking occurs naturally in the course of the evolution of the universe. Then using the Kibble mechanism [13], assuming one monopole per Hubble volume H^{-3} one can put a lower limit on the number density of the monopoles n_{mono} at formation [14]

$$n_{\text{mono}} \gtrsim H^3 \sim \left(\frac{g_{\text{eff}}^{1/2} T^2}{M_P} \right)^3, \quad (1.18)$$

where g_{eff} are the effective massless degrees of freedom at the GUT scale. In typical models $g_{\text{eff}} \sim 10$ and $T \sim 10^{15} \text{ GeV}$ as remarked before, therefore the monopole density written as a dimensionless quantity is [15]

$$\frac{n_{\text{mono}}}{T^3} \gtrsim 10^{-10}. \quad (1.19)$$

Due to the high mass of the produced monopole, it quickly drops out (freezes out) of the cosmological expansion and hence there is no interaction between monopole pairs that can lead to their annihilation. Therefore if there is no non-adiabatic process affecting the monopoles,

their density remains $\sim 10^{-10}$ until today [15]. But this is equal to the baryon number density today, implying that there are as many monopoles as baryons, which is clearly not the case as no monopoles have been detected yet [8]. Moreover, if their mass is around the GUT scale, then such a high number density would result in the energy density of monopoles being much larger than the critical energy density ρ_{crit} today implying that the universe could never have achieved its size today. This is known as the monopole problem.

The monopole problem is only a problem in as much as the unified gauge group fails to account for the non-abundance of monopoles today. However, a sufficiently large period of accelerated expansion in the early universe can help to dilute the number density of monopoles to acceptable levels today irrespective of the unified model. In this way inflation renders the monopole problem moot.

It is useful to follow the history of the theory of inflation as it originally developed in order to provide context for the discussion of vacuum transition in the very early universe in Chapter 4. Ref. [16] introduced the notion of inflation that arose out of a strongly supercooled phase transition in the very early universe. A supercooled transition is a first order phase transition with a very high potential barrier separating the stable minima from the unstable one (see Figure 1.2).

When the field is trapped in its unstable minima or false vacuum, the matter source in the Einstein equation is just $T_{\mu\nu} = \rho_{FV} g_{\mu\nu}$, where ρ_{FV} is a constant energy density of the false vacuum. This resembles a cosmological constant and the resulting metric is known to be a de Sitter spacetime. To see this, let us look at the Friedmann equation (1.7) in flat FRW spacetime for a scalar field.

A scalar field remains invariant under general coordinate transformations, i.e. under a coordinate transform $x \rightarrow \tilde{x}$, a scalar field $\phi(x)$ transforms as $\tilde{\phi}(\tilde{x}) = \phi(x)$. Similarly one can define a general tensor field of rank (m, n) $\Phi^{\mu_1 \mu_2 \dots \mu_m}_{\nu_1 \nu_2 \dots \nu_n}(x)$ through its behaviour under a general coordinate transform $x \rightarrow \tilde{x}$,

$$\tilde{\Phi}^{\mu_1 \mu_2 \dots \mu_m}_{\nu_1 \nu_2 \dots \nu_n}(\tilde{x}) = \left(\frac{\partial \tilde{x}^{\mu_1}}{\partial x^{\rho_1}} \frac{\partial \tilde{x}^{\mu_2}}{\partial x^{\rho_2}} \dots \frac{\partial \tilde{x}^{\mu_m}}{\partial x^{\rho_m}} \right) \left(\frac{\partial x^{\sigma_1}}{\partial \tilde{x}^{\nu_1}} \frac{\partial x^{\sigma_2}}{\partial \tilde{x}^{\nu_2}} \dots \frac{\partial x^{\sigma_n}}{\partial \tilde{x}^{\nu_n}} \right) \Phi^{\rho_1 \rho_2 \dots \rho_m}_{\sigma_1 \sigma_2 \dots \sigma_n}(x). \quad (1.20)$$

In this work we will mostly be interested in the dynamics of scalar $(m, n) = (0, 0)$ fields in the very early universe. Vector $(m, n) = (1, 0)$ or $(0, 1)$, and rank-2 tensor fields $(m, n) = (0, 2)$ or $(2, 0)$ also do appear. For example, we have already made use of the metric $g_{\mu\nu}$ which is a covariant rank-2 tensor field.

For a single scalar field σ in a potential $V(\sigma)$, the energy density is given by

$$\rho = \frac{\dot{\sigma}^2}{2} + V(\sigma) \quad (1.21)$$

where a dot denotes derivative with respect to time t . In the case of the transition potential Figure 1.2 the field is trapped in the false vacuum σ_{FV} , so we have $\dot{\sigma} = 0$ and are left with $\rho_{FV} = V(\sigma_{FV})$ where σ_{FV} denotes the field in the false vacuum. Then the Friedmann equation

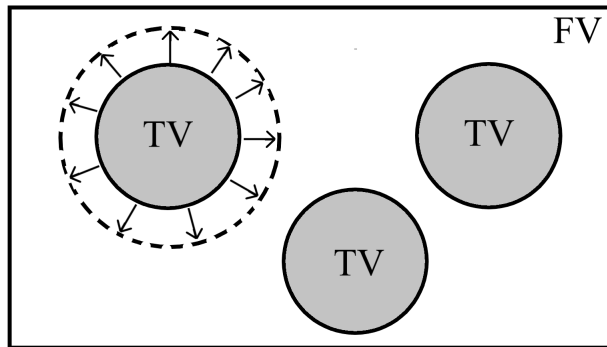


Figure 1.3: Bubbles of true vacuum (TV) are nucleated and grow in the false vacuum (FV) background in a first order phase transition.

gives

$$a(t) = a(t = 0) \exp \left(\sqrt{\frac{V(\sigma_{FV})}{3}} \frac{t}{M_P} \right) \quad (1.22)$$

where $a(t = 0)$ is the scale factor at some initial time. Thus we see that the universe expands exponentially. We also see that the expansion is accelerated $\ddot{a} > 0$. A particle theory explanation for such a vacuum transition potential to arise within the early universe is provided by Grand Unified Theories (GUTs) which can also possess a very high barrier height such that the decay rate is very low, the field remains trapped for a long time and the resulting exponential expansion continues until at least all of the currently observable universe can arise from the same causal volume, thus solving the horizon problem.

First order phase transition proceeds via the formation of bubbles of true vacuum in the false vacuum background (see Figure 1.3). However, assuming inflation ends when the first order phase transition is completed, the theory has a ‘graceful exit’ problem whereby the space between the bubbles grows faster than they are nucleated and hence they can never meet and percolate to complete the transition.

A possible solution to the ‘graceful exit’ problem was provided in ref. [17] which suggested introducing a second field so that the decay rate became field-dependent and increased towards the end of inflation, allowing the transition to complete. We will return to this field-dependent decay rate in Chapter 4. However, in the meantime, refs. [18–20] showed it was possible to account for an accelerated expansion by the dynamics of a single field, even without having a first order phase transition altogether [21]. Such a scenario came to be known as slow-roll inflation because it resembled a scalar field in a sufficiently flat potential classically rolling under the influence of friction term towards its minimum. Slow-roll inflation has become the most widely accepted solution, not least because it also accounts for an almost scale-invariant spectrum for primordial perturbations, which is observed in cosmological experiments today.

Following the standard treatment [14, 22, 23] for slow-roll inflation, we consider a potential $V(\phi)$ where scalar field ϕ is called the inflaton field. This treatment is also valid for multiple scalar fields which have a single field behaving as the inflaton that slowly-rolls. As before the Friedmann equation is given by

$$H^2 = \frac{1}{3M_P^2} \left(\frac{\dot{\phi}^2}{2} + V(\phi) \right). \quad (1.23)$$

While the equation of motion for the inflaton is given by

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0, \quad (1.24)$$

where dot denotes derivative with respect to time and the notation comma in subscript denotes derivative with respect to the corresponding quantity, here the field ϕ . For example, the double derivative of V with respect to the field ϕ would be denoted as $V_{,\phi\phi}$. Slow-roll is defined by $\frac{|\dot{H}|}{H^2} \ll 1$ or alternatively by the slow-roll parameters

$$\epsilon = \frac{M_P^2}{2} \left(\frac{V_{,\phi}}{V} \right)^2, \quad (1.25)$$

$$\eta = M_P^2 \frac{V_{,\phi\phi}}{V} \quad (1.26)$$

being negligible, i.e. $\epsilon, \eta \ll 1$. Note that the slow-roll parameter η is not to be confused with conformal time. In this case, we can neglect the terms in $\dot{\phi}^2$ and $\ddot{\phi}$ to obtain the slow-roll Friedmann equation

$$H^2 = \frac{V}{3M_P^2} \quad (1.27)$$

and the slow-roll equation of motion

$$3H\dot{\phi} = -V_{,\phi}. \quad (1.28)$$

In the Λ CDM model, Big Bang Nucleosynthesis (BBN) puts stringent constraints on any new physics in terms of the element abundances observed in the universe today. This implies that inflation must end at some point before the radiation dominated era begins. In the slow-roll scenario, this can be achieved when the slow-roll parameter ϵ becomes unity. Throughout this work, we shall take $\epsilon = 1$ to signify the end of inflation.

A schematic illustration of the situation at the end of inflation is given by Figure 1.4. The stage in the universe's evolution between inflation and Big Bang Nucleosynthesis is called the Reheating era. During this era, the high potential energy stored in the inflaton field is transferred to birthing the Standard Model particles as well as towards reheating the universe, conditions that are necessary for BBN to start. Thus by postulating the existence of the inflationary and reheating eras, the

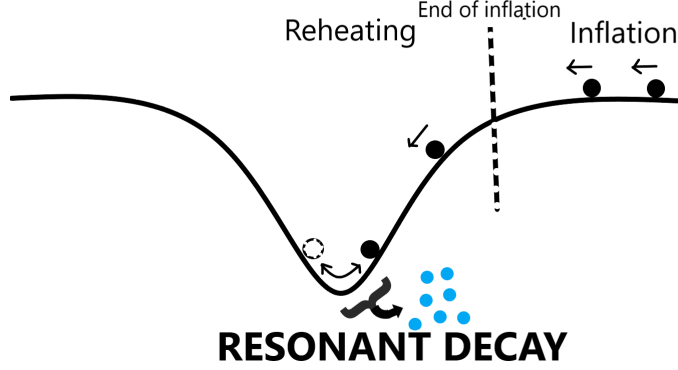


Figure 1.4: Schematic description of an inflaton field (depicted by a black dot) slowly-rolling during inflation, picking up speed after the end of inflation and undergoing oscillations at the bottom of its potential. Also shown is the generation of daughter particles (depicted by cyan dots) by the resonant decay of the inflaton. The era in the universe’s evolution after the end of inflation and before the onset of Big Bang Nucleosynthesis is termed as Reheating.

history of the universe, between some as yet unknown fundamental origins beyond the Standard Model and the present times, is fully described.

The success of the theory of inflation is not just limited to providing a solution to the horizon problem. It also provides a natural framework whereby quantum fluctuations of a scalar field give rise to primordial perturbations in the metric that eventually seed the formation of structure in our universe. We shall describe curvature perturbations in detail in Chapter 2. Here, we examine the origin of long distance field correlations by applying quantum field theory in curved spacetime during inflation. In broad terms, how the accelerated expansion of spacetime connects the quantum to the cosmic, the smallest scales to the largest.

1.3.1 Super-horizon fluctuations

The gravitationally minimally coupled action of a scalar field ϕ in a potential V is given by

$$S = \int \sqrt{-g} \left(\frac{M_P^2}{2} \mathcal{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) \quad (1.29)$$

where $g_{\mu\nu}$ is the metric and $\mathcal{R}$ its Ricci scalar. We shall solve for the correlator of a scalar field with the above action on a flat FRW background. The canonical momentum for this action is

$$\Pi_\phi(t, x) = \frac{\partial S}{\partial \dot{\phi}} = a^3(t) \dot{\phi}(t, x). \quad (1.30)$$

For purposes of second quantisation, we promote the field and canonical momentum to corresponding operators and divide them into background part indicated by an overbar and first order perturbation

indicated by delta as

$$\hat{\phi}(x, t) = \bar{\phi}(t) + \delta\hat{\phi}(t, x), \quad (1.31)$$

$$\hat{\Pi}_\phi(x, t) = a^3(t)\dot{\bar{\phi}}(t) + \delta\hat{\Pi}_\phi(t, x). \quad (1.32)$$

These field operators should satisfy the equal time commutation relations

$$[\hat{\phi}(t, x), \hat{\phi}(t, y)] = 0 ; [\hat{\Pi}_\phi(t, x), \hat{\Pi}_\phi(t, y)] = 0 ; [\hat{\phi}(t, x), \hat{\Pi}_\phi(t, y)] = i\delta^3(x - y). \quad (1.33)$$

This can be achieved by Fourier expanding the field in terms of its comoving momentum k classical mode functions $\delta\phi_k$ and the creation and annihilation operators $\hat{a}_k, \hat{a}_k^\dagger$ as

$$\delta\hat{\phi}(t, x) = \int d\vec{k} \left(\delta\phi_k(t) e^{i\vec{k}\cdot\vec{x}} \hat{a}_k + \delta\phi_k(t)^* e^{-i\vec{k}\cdot\vec{x}} \hat{a}_k^\dagger \right), \quad (1.34)$$

where the mode functions satisfy the mode equations

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \left(\frac{k^2}{a^2} + V_{,\phi\phi}(\bar{\phi}) \right) \delta\phi_k = 0. \quad (1.35)$$

We know that in reality both the Hubble rate H and effective mass $m_{\text{eff}}^2 \sim V_{,\phi\phi}(\bar{\phi})$ are not constant. But to illustrate the long-range fluctuation of a quantum field during inflation, as a first approximation, here let us take $H \approx \text{const}$ and $V_{,\phi\phi}(\bar{\phi}) \approx 0$. We shall go beyond this approximation in the later parts of this work.

With the above approximation, we obtain a simplified equation

$$\delta\ddot{\phi}_k + 3H\delta\dot{\phi}_k + \frac{k^2}{a^2} \delta\phi_k = 0 \quad (1.36)$$

which has a general solution

$$\delta\phi_k = \frac{c_k}{a} e^{\frac{ik}{aH}} \left(1 + \frac{iaH}{k} \right), \quad (1.37)$$

where c_k is an integration constant that is fixed (assuming it is real) by the equal time commutation relations to be

$$c_k = \frac{1}{\sqrt{2k(2\pi)^3}}. \quad (1.38)$$

With a Bunch-Davies vacuum state $|0\rangle$ [24, 25], the two point field correlator is given by

$$\langle 0 | \delta\hat{\phi}(t, x) \delta\hat{\phi}(t, y) | 0 \rangle = \int d\vec{k} |\delta\phi_k|^2 e^{i\vec{k}\cdot(\vec{x}-\vec{y})} = \frac{1}{(2\pi)^3} \int d\vec{k} \frac{1}{2ka^2} \left(1 + \frac{a^2 H^2}{k^2} \right) e^{i\vec{k}\cdot(\vec{x}-\vec{y})}. \quad (1.39)$$

For modes which are sub-horizon, i.e. having $k \gg aH$, the two point field correlator falls off

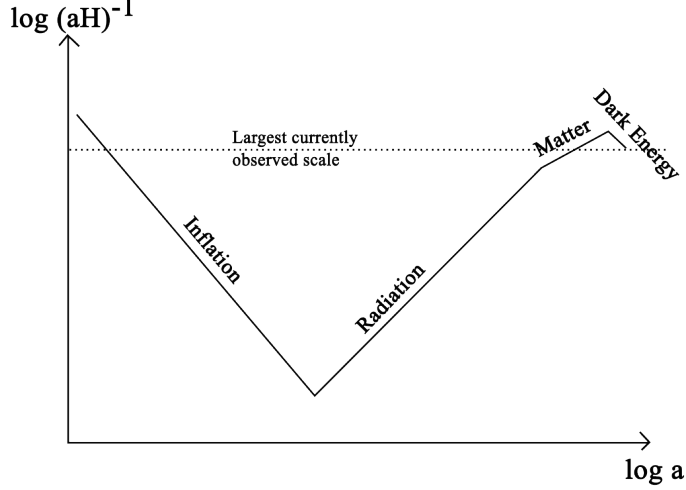


Figure 1.5: Evolution of the comoving Hubble length as a function of the scale factor assuming instantaneous reheating. The dashed line indicates the largest currently observed scale k_* .

polynomially with the distance squared

$$\langle 0 | \hat{\delta}\phi(t, x) \hat{\delta}\phi(t, y) | 0 \rangle \sim \frac{1}{a^2 |x - y|^2}. \quad (1.40)$$

However, for modes which are super-horizon, i.e. having $k \ll aH$, the two point correlator goes as the logarithm of the distance

$$\langle 0 | \hat{\delta}\phi(t, x) \hat{\delta}\phi(t, y) | 0 \rangle \sim H^2 \ln |x - y|. \quad (1.41)$$

Thus, quantum fluctuations are correlated on large scales during inflation.

Figure 1.5 describes the evolution of scales in the universe. The comoving Hubble length is given by $(aH)^{-1}$. During inflation the comoving Hubble length decreases on a logarithmic plot. Whereas the radiation dominated $a \sim \sqrt{t}$ and matter dominated $a \sim t^{2/3}$ eras appear with the appropriate positive slopes. Any particular comoving scale k does not evolve with time and is represented as a horizontal line. It first becomes equal to the comoving Hubble length, $k = aH$, during the inflationary epoch. At this point, the scale ‘exits (or leaves) the horizon’. A scale which exited the horizon during inflation eventually ‘reenters the horizon’ when it again becomes equal to the comoving Hubble length during the radiation, matter or dark energy dominated epochs. Thus the field correlations that are established at quantum sizes during inflation, become independent on time on super-horizon scales (1.41) and get ‘frozen-in’. They are then carried forward until horizon reentry when the size of the universe has grown exponentially.

This exponential growth is characterised by the number of efolds before the end of inflation N

which is defined as

$$N = -\ln\left(\frac{a}{a_{\text{end}}}\right) \quad (1.42)$$

where a_{end} is the scale factor at the end of inflation. The Hubble rate is related to the number of efolds by the relation

$$N = -\int dt \, H. \quad (1.43)$$

The number of efolds N is useful to characterise the time evolution during inflation in a model-independent way. Typically its value lies between $N \sim 30$ to 70 in order to satisfy various requirements like solving the horizon problem and providing correct initial conditions for Big Bang Nucleosynthesis. Throughout this work we will assume the largest currently observed scale left the horizon $N_* = 55$ efolds before the end of inflation [26].

1.3.2 Inflation and non-minimal coupling

In the early universe context, especially that of inflation, a non-minimal coupling to gravity must necessarily be included in the action due to QFT in curved spacetime considerations [27]. The non-minimal coupling to a scalar field ϕ takes the form $\xi\phi^2\mathcal{R}$, where $\mathcal{R}$ is the Ricci scalar and the dimensionless parameter ξ is known as the non-minimal coupling parameter. A value $\xi = 1/6$ results in the action being conformally invariant for a massless, free field.

Curved spacetime QFT calculations show [28] that the non-minimal coupling parameter ξ possesses no fixed point under renormalisation group running, and therefore it has no preferred value theoretically. Its value also remains essentially unconstrained empirically [29].

There are a multitude of inflationary models with or without the inclusion of non-minimal coupling that each have certain desirable theoretical motivations [30]. In this work, we would be dealing with two such models that both involve non-minimal coupling to gravity:

1. α -attractor [31]

These models arise from supergravity considerations and define a universal class of inflationary attractor solutions [32]. The salient feature of these models is that their potential for large field values becomes exponentially flat, resulting in a generic prediction for observational parameters such as the spectral tilt and tensor-to-scalar ratio that lie in the sweet spot of values allowed by cosmological experiments [33].

The models are classified by the form of their potential. In terms of the parameter α , so called E-models have a potential of the form

$$V(\phi) \sim \left(1 - e^{-\sqrt{\frac{2}{3\alpha}}\phi}\right)^n, \quad (1.44)$$

whereas T-models have a potential of the form

$$V(\phi) \sim \tanh^n \left(\frac{\phi}{\sqrt{6\alpha}} \right). \quad (1.45)$$

Here n is an integer greater than one.

In section 3.3, we shall see how the T-model form also arises when converting a polynomial potential with small non-minimal coupling ($\xi \ll 1/6$) from Jordan frame to the Einstein frame.

2. Higgs inflation [34]

There is a natural candidate for the inflaton field present in the Standard Model, it is the Higgs boson. For sufficiently high value of non-minimal coupling $\xi \gg 1/6$, it can be shown that the Higgs potential in the Einstein frame satisfies the observational constraints on curvature power spectrum and tensor-to-scalar ratio placed by cosmological experiments. We shall see this calculation in section 4.5.

1.4 Einstein and Jordan frames

Consider a scalar tensor theory of gravity with some matter action S_{matter}

$$S = \int d^4x \sqrt{-g} \left(\frac{M_P^2}{2} A(\phi) \mathcal{R} - \frac{1}{2} B(\phi) (\nabla\phi)^2 - V(\phi) \right) + S_{\text{matter}}(e^{2\alpha(\phi)} g_{\mu\nu}, \psi_m).$$

Note that S_{matter} depends non-trivially on ϕ through the modified metric, $\hat{g} = e^{2\alpha\phi} g$. As we can see, the total independent parameter functions for a scalar tensor theory are $A(\phi), B(\phi), \alpha(\phi)$ and $V(\phi)$. However, the form of the above action is preserved under a conformal transformation [35]:

$$g_{\mu\nu} = e^{2\gamma(\bar{\phi})} \bar{g}_{\mu\nu}, \quad (1.46)$$

$$\phi = f(\bar{\phi}). \quad (1.47)$$

These transformations allow us to make a conformal rescaling of the metric followed by a redefinition of the scalar field. With these two transformations, the four independent parameter functions in the scalar tensor action can be reduced to two, bringing the action into forms defined by

Jordan frame: Defined by $\alpha = 0$ and $B = 1$. Parameter functions are A and V .

Einstein frame: Defined by $A = 1$ and $B = 1$. Parameter functions are α and V .

Based on the physicality of changing coordinates, we see a potential problem with regards to the Einstein and Jordan frames. This is because General Relativity is known to be non-renormalisable as a quantum field theory. Thus we cannot be certain what a complete theory of quantum gravity looks like. In that case the Einstein and Jordan frames might describe physically inequivalent theories. However, classically there is no physical difference between the two.

If renormalisability is assumed to be paramount, then Jordan frame is physically intuitive because it keeps the canonical forms of the matter action. The matter fields then evolve under geodesics of a matter field-independent metric.

In terms of a more general relativistic picture, in the Einstein frame, the gravity side retains the canonical behaviour of the Einstein equation.

Thus we have the dichotomy, Jordan frame better describes particle side and Einstein frame better describes gravity side. We shall see this come into play while discussing vacuum decay during inflation, where the Einstein frame is convenient to perform the inflationary background and non-Gaussianity calculations whereas the Jordan frame is helpful to calculate vacuum decay dynamics.

For the details of converting from Jordan to Einstein frame, see appendix [A](#).

1.5 Non-perturbative introduction

Calculations in physics can usually be condensed to a set of integro-differential equations whose solutions are not known or difficult to obtain. In most cases, perturbation theory helps to start from the known solution of a simpler system and iteratively build up the solution to the desired system by expanding in some small parameter. For example, in the path integral formalism of Quantum Field Theory (QFT) the Lagrangian is split into the free field part $\mathcal{L}_0$ and the interacting part $\mathcal{L}_1$ as $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$. The path integral of a scalar field φ for the free part is given by

$$Z_0(J) = \int \mathcal{D}\varphi e^{\frac{i}{\hbar} \int d^4x (\mathcal{L}_0 + J\varphi)}, \quad (1.48)$$

where $\mathcal{D}$ indicates a functional derivative. Then the total path integral is evaluated by taking repeated derivatives of the free path integral $Z_0(J)$ [\[36\]](#) as

$$\begin{aligned} Z(J) &= \int \mathcal{D}\varphi e^{\frac{i}{\hbar} \int d^4x (\mathcal{L}_0 + \mathcal{L}_1 + J\varphi)} \\ &= e^{\frac{i}{\hbar} \int d^4x \mathcal{L}_1(\frac{1}{i} \frac{\delta}{\delta J(x)})} \int \mathcal{D}\varphi e^{\frac{i}{\hbar} \int d^4x (\mathcal{L}_0 + J\varphi)} \\ &= e^{\frac{i}{\hbar} \int d^4x \mathcal{L}_1(\frac{1}{i} \frac{\delta}{\delta J(x)})} Z_0(J). \end{aligned} \quad (1.49)$$

Taking the quartic scalar theory as a particular example,

$$S = \int d^4x \mathcal{L} = \int d^4x \left(\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 - \lambda \varphi^4 \right), \quad (1.50)$$

the free field part is taken to be $\mathcal{L}_0 = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2$ while the interacting part is then $\mathcal{L}_1 = -\lambda \varphi^4$. The path integral can be expanded in powers of the coupling constant λ as

$$Z(J) \sim \sum_{n=0}^{\infty} \frac{1}{n!} \left(-i \frac{\lambda}{\hbar} \int d^4x \left(\frac{1}{i} \frac{\delta}{\delta J(x)} \right)^4 \right)^n e^{\frac{i}{2\hbar} \int d^4y d^4z J(y) \Delta(y-z) J(z)}. \quad (1.51)$$

Thus, if λ is small, the above expression provides an accurate estimate of the path integral upto a given order in λ by iteratively solving it with $\Delta(y - z)$ which is the known free field propagator.

Now if one rescales the field in the quartic action (1.50) above to $\varphi' = \sqrt{\lambda}\varphi$ then the path integral contains $e^{\iota \frac{S(\varphi')}{\hbar\lambda}}$. Thus an expansion in small λ is equivalent to an expansion in small $\hbar$ [37]. And hence taking the semi-classical limit $\hbar \rightarrow 0$ is equivalent to taking the weak coupling limit $\lambda \rightarrow 0$.

Consider now the phenomenon of quantum tunnelling through a potential barrier. By solving the Schrödinger's equation, the transmission coefficient through the barrier is found to be

$$T(E) = e^{-\int dx \frac{1}{\hbar} \sqrt{2m(V-E)}}. \quad (1.52)$$

This expression is of Order($1/\hbar$) and is therefore not captured by perturbation theory in the limit $\hbar \rightarrow 0$. Thus first order phase transitions that proceed through quantum tunnelling through a barrier are an example of non-perturbative phenomena. In the preceding section, we saw the original theory of inflation based on first order phase transition in the very early universe. Another viewpoint on its non-perturbative nature is obtained by considering the schematic potential Figure 1.2 of vacuum decay from its false to the true vacuum. Perturbation theory is essentially a local Taylor expansion near one of the two vacuum expectation values (vev), and is blind to the other vev that is a global feature of the potential. In Chapter 4, we study the non-perturbative dynamics of vacuum decay in more detail, especially when it occurs during inflation. Another example of first order phase transition that can take place in the very early universe is the electroweak phase transition [38, 39] in beyond Standard Model theories. It is often postulated to be the source of the observed baryon matter antimatter asymmetry in our universe and is one of the important candidates for gravitational wave searches with the Laser Interferometer Space Antenna (LISA) [40].

Second order phase transitions, also known as continuous phase transitions, can also occur during the evolution of the universe. Just like first order phase transitions create bubbles, second order phase transitions in the early universe can give rise to topological defects [13] such as monopoles, cosmic strings, domain walls and textures [41]. These topological defects or solitons are not continuously connected to the vacuum field configuration and hence are akin to global features that are not visible to perturbation theory. Phase transitions in the early universe, both first and second order, are generically predicted to arise via symmetry breaking within a plethora of beyond the Standard Model theories. However, in this work we do not study the observational signatures from second order phase transitions in the early universe.

Perturbation theory can break down for certain class of non-linear ordinary differential equations (ODEs) with periodic coefficients. In the presence of some small parameter, the non-linear system can be broken down into a linear system plus non-linear corrections in higher orders of the small parameter.

Floquet theory describes solutions of the system of linear ODEs of the form $\dot{x} = A(t)x$ where $A(t)$ is some periodic matrix with period T such that $A(t+T) = A(t)$. The solutions take the form $x(t) = x_0(t)e^{\mu t}$ where μ is called the Floquet or characteristic exponent of the system. The Floquet

exponent is a complex number dependent on the parameters of the system. Therefore there can exist some parameters for which the Floquet exponent is real and positive, giving rise to exponentially growing solutions also called an ‘instability’ and this phenomenon is known as parametric resonance. The simplest example of parametric resonance is a driven harmonic oscillator system.

If there is a parametric resonance, the solution to the linear system grows exponentially and hence even though a perturbative expansion in some small parameter might be possible, the non-linear terms at higher order in the small parameter can quickly grow to comparable magnitudes to the linear term, thus breaking the perturbative description.

Such non-perturbative effects can arise in the very early universe during the reheating era. As described in the preceding section, reheating generically involves oscillation of the inflaton field at the bottom of its potential. The equations of motion for the inflaton and its daughter fields then are non-linear ODEs in a periodic background. We shall study these non-perturbative effects of reheating in detail in [Chapter 3](#).

Chapter 2

Curvature perturbations

“Beyond the horizon set off the light,
with dreams of flight shimmering,
then simmered troubles, oscillations and bubbles,
that sent the heavens a-shivering!”

Homogeneity and isotropicity are the two keystones of the Λ CDM description of the universe. However, we know that the universe is not exactly homogeneous and isotropic, especially on short distances where we encounter voids, galaxies and other cosmic structures. Precise measurements of the Cosmic Microwave Background show that there are temperature fluctuations on the order of $10^{-5} - 10^{-6} K$ [4, 42] arising even from the early epoch of recombination. The smallness of departures from homogeneity and isotropicity allow their treatment in the form of perturbations. These perturbations can be described by the over-densities and under-densities around an unperturbed background.

2.1 Separate Universe Approximation

In the Arnowitt-Deser-Misner (ADM) formalism of gravity [43], the metric is foliated into spatial hypersurfaces and temporal threadings through them,

$$ds^2 = -\mathcal{N}^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \quad (2.1)$$

where $\mathcal{N}$ is the lapse function, β^i the shift vector and γ_{ij} is the metric on spatial slices. In terms of this 1 + 3 split the Einstein Hilbert action (1.3) without including matter part becomes

$$S = \int d^4x \mathcal{N} \sqrt{\gamma} \frac{M_P^2}{2} \left({}^{(3)}\mathcal{R} + K_{ij}K^{ij} - K^2 \right) \quad (2.2)$$

where the extrinsic curvature K_{ij} is given by

$$K_{ij} = \frac{1}{2\mathcal{N}} (-\partial_t \gamma_{ij} + D_i \beta_j + D_j \beta_i), \quad (2.3)$$

and $K = K^i_i$. Including a source term $T_{\mu\nu}$, the Einstein equation (1.2) can be written as [44]

$$(\partial_t - \mathcal{L}_\beta) \gamma_{ij} = -2\mathcal{N} K_{ij} \quad (2.4)$$

$$(\partial_t - \mathcal{L}_\beta) K_{ij} = -D_i D_j \mathcal{N} + \mathcal{N} \left({}^{(3)}\mathcal{R}_{ij} + K K_{ij} - 2K_{ik} K^k_j + 4\pi ((S - E)\gamma_{ij} - 2S_{ij}) \right) \quad (2.5)$$

$${}^{(3)}\mathcal{R} + K^2 - K_{ij} K^{ij} = 16\pi E \quad (2.6)$$

$$D_j K^j_i - D_i K = 8\pi p_i. \quad (2.7)$$

where $\mathcal{L}_\beta$ denotes the Lie derivative with respect to the shift vector β^i and $D_i, {}^{(3)}\mathcal{R}_{ij}, {}^{(3)}\mathcal{R}$ denote covariant derivative, Ricci tensor and Ricci scalar with respect to the metric γ_{ij} respectively. Source quantities are defined using the normal vector $n_\mu = (-\mathcal{N}, 0, 0, 0)$ and $n^\mu = (1/\mathcal{N}, -\beta_i/\mathcal{N})$ and orthogonal projector $\gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu$ so that energy density $E = n^\mu n^\nu T_{\mu\nu}$, momentum density $p_\alpha = -T_{\mu\nu} n^\mu \gamma^\nu_\alpha$ and $S_{\alpha\sigma} = \gamma^\mu_\alpha \gamma^\nu_\sigma T_{\mu\nu}$, $S = S^\alpha_\alpha$.

In general it is difficult to solve these equations analytically, and numerical methods [45, 46] become useful. For cosmological applications, it is possible to treat the equations non-perturbatively using the gradient expansion method [47]. Smoothing involves replacing quantities within a certain distance by the average value of the quantity over that distance. We saw in Chapter 1 that smoothing was involved even in the definition of the Cosmological Principle when we say “on scales larger than a particular one (~ 100 Mpc)” the universe appears homogeneous and isotropic. Motivated by super-horizon correlations during inflation, we assume the universe is smoothed at some scale $k \ll aH$. Then introducing a small parameter $\varepsilon = k/aH$, we can expand the Einstein’s equations in this parameter to obtain the gradient expansion.

We shall closely follow the calculation in ref. [48]. At zeroth order in the expansion, the Einstein tensor $G_{\mu\nu} = \mathcal{R}_{\mu\nu} - (1/2)g_{\mu\nu}\mathcal{R}$ in terms of the 1 + 3 split can be written as

$$G^0_0 = -\frac{1}{3}K^2 + \text{Order}(\varepsilon^2) \quad (2.8)$$

$$G^0_i = 0 + \text{Order}(\varepsilon) \quad (2.9)$$

$$G^i_j = \frac{1}{3} \left(\frac{2}{\mathcal{N}} \partial_t K - K^2 \right) \delta^i_j + \text{Order}(\varepsilon^2). \quad (2.10)$$

For the source we take an energy momentum tensor corresponding to a fluid. At zeroth order this is just given by the perfect fluid form $T_{\mu\nu} = (\rho + P)u_\mu u_\nu + g_{\mu\nu}P$ with 4-velocity $u_\mu = (u_0, v_i)$ written in the 1 + 3 split as

$$u^0 = \left(\mathcal{N}^2 + (\beta_k + v_k)(\beta^k + v^k) \right)^{-1/2}, \quad u^i = u^0 v^i. \quad (2.11)$$

In terms of this split, the energy momentum tensor for a perfect fluid is

$$T^0_0 = -(u^0)^2(\rho + P)(\mathcal{N}^2 + \beta^k(v_k + \beta_k)) + P \quad (2.12)$$

$$T^0_i = (u^0)^2(\rho + P)(v_i + \beta_i) \quad (2.13)$$

$$T^i_j = (u^0)^2(\rho + P)u^i(v_j + \beta_j) + \delta^i_j P. \quad (2.14)$$

Now using the coordinate freedom to choose coordinates that comove with the fluid so that $v_i = 0$, then the $0i$ -component of Einstein's equation implies that $\beta_i = \text{Order}(\varepsilon)$. Hence the 00 -component at zeroth order becomes

$$T^0_0 = -\rho + \text{Order}(\varepsilon^2) = -\frac{1}{3}K^2 = G^0_0. \quad (2.15)$$

If we identify $K = -3H$ with H being the local Hubble rate, then the above equation is just the Friedmann equation for a locally homogeneous and isotropic FRW universe. Hence each volume within the smoothing scale $k \ll aH$ can be treated as a separate universe. This is known as the separate universe approximation.

Thus the separate universe approximation arises from the zeroth order terms in the gradient expansion of the ADM equations. Alternatively, one can frame the separate universe approximation in terms of causality [49]. If two volumes in the early universe are separated by a distance greater than $(aH)^{-1}$ then light cannot travel between them and they are not in causal contact. Hence their evolution can be treated independently of each other. This is not an approximation, but a straightforward consequence of causality. Now the main assumption or approximation is that each of these Hubble volumes evolves as an FRW spacetime i.e. as a separate universe of its own. This is the separate universe approximation.

The zeroth order Einstein equations depend on the quantity $K = K^i_i$. Therefore separating the extrinsic curvature into its trace K and traceless $\bar{K}_{ij}$ parts we have

$$K_{ij} = \bar{K}_{ij} + \frac{1}{3}K\gamma_{ij}. \quad (2.16)$$

If the shift vector $\beta_i = 0$, which as we saw is true at the zeroth order in gradient expansion then eq. (2.5) simplifies to [47]

$$\frac{\partial \bar{K}^i_k}{\mathcal{N} \partial t} = K \bar{K}^i_k = -\frac{\partial \ln(\sqrt{\gamma})}{\mathcal{N} \partial t} \bar{K}^i_k \quad (2.17)$$

where we have used the definition of the extrinsic curvature eq. (2.3), $K_{ij} = -\partial_t \gamma_{ij}/2\mathcal{N}$ and contracted it with γ^{ij} to substitute $K = -\gamma^{ij} \partial_t \gamma_{ij}/2\mathcal{N}$ in the second equality. This equation has a solution $\bar{K}^i_k \sim \gamma^{-1/2}$ with $\gamma = \det(\gamma_{ij})$. Now defining a local scale factor a from the ADM metric

(2.1) in the case where $\beta = \text{Order}(\varepsilon)$ can be ignored, from

$$K = -\frac{\partial \ln(\sqrt{\gamma})}{\mathcal{N} \partial t} = -\frac{1}{2\mathcal{N}\gamma} \frac{d\gamma}{dt} = -3H = -3\frac{1}{a} \frac{da}{\mathcal{N} dt} \quad (2.18)$$

we have $\gamma = a^6$. Therefore the traceless part of the extrinsic curvature then becomes proportional to $\bar{K}_k^i \sim \gamma^{-1/2} \sim a^{-3}$. Assuming that $1/a^3$ is negligible during inflation, we conclude that the traceless part of the extrinsic curvature is negligible. The most general form for the three metric with vanishing $\bar{K}_j^i$ is given by [47]

$$\gamma_{ij}(t, x) = a(t, x)^2 \tilde{\gamma}_{ij}(x) \quad (2.19)$$

where $\det(\tilde{\gamma}_{ij}) = 1$.

The global or background scale factor $\bar{a}(t)$ can be separated out from the locally defined scale factor $a(t, x)$ by defining

$$a(t, x) = \bar{a}(t) e^{\psi(t, x)} \quad (2.20)$$

where $\psi(t, x)$ is a perturbation. Note that one can always make ψ small by defining a over an appropriate vicinity. In a similar fashion, the $\tilde{\gamma}_{ij}(x)$ part can be factored as

$$\tilde{\gamma}_{ij}(x) = \mathbf{I} e^{h_{ij}(x)} \quad (2.21)$$

where h_{ij} is called the tensor perturbation. Because $\det(\tilde{\gamma}_{ij}) = 1$, h_{ij} is traceless.

The local number of efolds corresponding to the locally defined scale factor between times t_1 and t_2 is given by

$$\begin{aligned} N(t_1 \rightarrow t_2; x) &= \int_{t_1}^{t_2} \mathcal{N} dt H = \int_{t_1}^{t_2} \mathcal{N} dt \frac{1}{a} \frac{da}{\mathcal{N} dt} \\ &= \int_{t_1}^{t_2} dt \left(\frac{\dot{\bar{a}}}{\bar{a}} + \dot{\psi} \right) = \ln \left(\frac{\bar{a}(t_2)}{\bar{a}(t_1)} \right) + \psi(t_2, x) - \psi(t_1, x). \end{aligned} \quad (2.22)$$

Identifying $\ln(\bar{a}(t_2)/\bar{a}(t_1))$ as the background number of efolds $\bar{N}$ and writing the difference between the local and background number of efolds $N - \bar{N} = \delta N$, we have

$$\delta N(t_1 \rightarrow t_2, x) = \psi(t_2, x) - \psi(t_1, x). \quad (2.23)$$

We shall refer to this equation as the delta N formula.

The conservation of energy momentum $\nabla^\mu T_{\mu\nu} = 0$ implies the local continuity equation at zeroth order in ε is

$$\dot{\rho}(t, x) + 3H(\rho(t, x) + P(t, x)) = 0. \quad (2.24)$$

We now consider uniform energy density slices such that $\rho(t, x) = \rho(t)$. Renaming ψ on uniform energy density slices by ζ we have

$$\dot{\zeta}(t, x) + \frac{\dot{\bar{a}}(t)}{\bar{a}(t)} = -\frac{1}{3} \frac{\dot{\rho}(t)}{\rho(t) + P(t, x)}. \quad (2.25)$$

Now if the adiabatic condition $P = P(\rho)$ is also satisfied, then all quantities on the right hand side of the above equation become independent of the spatial position x . Therefore we can always choose to incorporate the time dependence into the background scale factor such that it satisfies

$$\frac{\dot{\bar{a}}(t)}{\bar{a}(t)} = -\frac{1}{3} \frac{\dot{\rho}(t)}{\rho(t) + P(\rho(t))} \quad (2.26)$$

leaving

$$\dot{\zeta}(t, x) = 0. \quad (2.27)$$

Thus the perturbation ζ on uniform energy density slices is conserved on large, super-horizon scales and is called the curvature perturbation. It is the seed from inflation that sets up initial conditions for the subsequent growth of structure and the CMB anisotropy (see section 2.3).

In the case where the pressure is non-adiabatic, one cannot express $P(t, x)$ as a function of ρ . In that case eq. (2.25) clearly shows that the curvature perturbation ζ can evolve on super-horizon scales. This would be the scenario we would be most interested in as part of this work. To perform calculations in this scenario it is useful to express ζ in terms of δN .

Consider two time threadings, threading A that starts from a uniform energy density slice, $\psi_A(t_1, x) = \zeta(t_1, x)$ and ends on a uniform energy density slice, $\psi_A(t_2, x) = \zeta(t_2, x)$. And threading B that starts from a flat slice, $\psi_B(t_1, x) = 0$ and ends on a flat slice, $\psi_B(t_2, x) = 0$. To connect these two threadings, let them be coincident at the initial time, i.e. $\psi_A(t_1, x) = \psi_B(t_1, x) = 0 = \zeta(t_1, x)$. Therefore from the delta N formula we have

$$\zeta(x, t_2) = \delta N(x, t_2). \quad (2.28)$$

This is also sometimes referred to as the delta N formula.

2.2 Statistical description

2.2.1 Statistical homogeneity and isotropy

The action and its associated equations of motion describe the evolution from one spatial slice of the universe to the next, in steps along the time thread. However, the spatial description of fluctuations within one slice at a fixed time is random. Since we assume that the fluctuations are ultimately sourced by quantum fields, thinking of it as collection of harmonic oscillators at each point in space, the energy state of each of these harmonic oscillators is random and only admits a probabilistic

description.

Let us assume we have a random field $g(x)$ which takes a random value at each point x . This field can be the inflaton field $\phi(x)$, any spectator scalar field $\chi(x)$ or the curvature perturbation $\zeta(x)$.

If $g(x)$ is real, then $g^*(x) = g(x)$. In Fourier space, it corresponds to $g^*(\vec{k}) = g(-\vec{k})$. This is sometimes referred to as the reality condition for the modes.

Spatial correlators are the expectation values of fields at different spatial points,

$$\Sigma(x, y) = \langle g(x)g(y) \rangle = \int dg \mathcal{P}(g)g(x)g(y) \quad (2.29)$$

where $\mathcal{P}(g)$ denotes the probability distribution function of the random field g .

Statistical homogeneity

Consider a translation operator $\hat{T}$ such that $\hat{T}_a g(x) = g(x-a)$. Statistical homogeneity implies $\mathcal{P}(g) = \mathcal{P}(\hat{T}_a g)$. In terms of the correlator, this means $\langle g(x)g(y) \rangle = \langle g(x-a)g(y-a) \rangle$ for every a , implying the correlator $\Sigma(x, y)$ should only depend on the difference $x - y$, i.e. $\Sigma(x, y) = \Sigma(x - y)$.

In Fourier space, the translation operator acts as $\hat{T}_a g(\vec{k}) = g(\vec{k})e^{-i\vec{k}\cdot\vec{a}}$. Translation invariance then implies the presence of a delta function in the momentum correlator, $\langle g(\vec{k})g^*(\vec{k}') \rangle = G(\vec{k})\delta(\vec{k} - \vec{k}')$, where $G(\vec{k})$ is some function in momentum space.

Statistical isotropy

Isotropy implies invariance under spatial rotations. The rotation operator $\hat{R}$ is such that $\hat{R}g(x) = g(R^{-1}x)$ where R is the rotation matrix. Statistical isotropy implies the correlator $\Sigma(x, y) = \Sigma(R^{-1}x, R^{-1}y)$. By translational symmetry $\Sigma(x, y) = \Sigma(x - y)$, so we have $\Sigma(R^{-1}x, R^{-1}y) = \Sigma(R^{-1}(x - y))$. This is only equal to $\Sigma(x - y)$ for all R if $x - y$ does not transform under rotation. Therefore $\Sigma(x, y) = \Sigma(|x - y|)$.

In Fourier space, the rotation operator acts as $\hat{R}f(\vec{k}) = f(R^{-1}\vec{k})$. Rotational invariance then implies the momentum correlator also depends only on the magnitude of $\vec{k}$, $\langle g(\vec{k})g^*(\vec{k}') \rangle = G(k)\delta(\vec{k} - \vec{k}')$.

2.2.2 Power spectrum

The power spectrum P_g associated with a random field g is defined by

$$\langle g(\vec{k})g(\vec{k}') \rangle = (2\pi)^3 \delta^3(\vec{k} + \vec{k}') P_g(k). \quad (2.30)$$

Note that we have simply replaced $G(k)$ in the preceding section with a definition involving the power spectrum P . One can also define a reduced power spectrum $\mathcal{P}$ as

$$\mathcal{P}_g(k) = \frac{k^3}{2\pi^2} P_g(k) \quad (2.31)$$

to help connect with experiments. For example, CMB measurements fix the value of the reduced power spectrum for curvature perturbations to be $\mathcal{P}_\zeta = 2.1 \times 10^{-9}$ at the scale $k_{\text{phys}} = 0.05 \text{Mpc}^{-1}$ [4]. In this work, we use the terms ‘power spectrum’ and ‘reduced power spectrum’ interchangeably, their meaning should be derived from the context.

Transforming to real space correlator,

$$\langle g(x)g(y) \rangle = \int \frac{d^3k}{4\pi k^3} \mathcal{P}_g(k) e^{i\vec{k} \cdot (\vec{x} - \vec{y})} = \int \frac{dk}{k} \mathcal{P}_g(k) \frac{\sin(k|x-y|)}{k|x-y|} \quad (2.32)$$

where we have performed the angular integration in the last step. As we can see, the position space correlator is only dependent on the magnitude of the difference between the points as expected from statistical homogeneity and isotropy.

The variance Σ is defined as $\Sigma = \lim_{x \rightarrow y} \Sigma(x, y)$ which yields

$$\Sigma = \int \frac{dk}{k} \mathcal{P}_f(k). \quad (2.33)$$

2.2.3 Gaussian perturbation

A Gaussian perturbation is defined as one whose Fourier coefficients have no correlation with each other. In the continuum limit this leads to the one-point and two-point Fourier space correlators

$$\langle g(\vec{k}) \rangle = 0, \quad (2.34)$$

$$\langle g(\vec{k})g^*(\vec{k}') \rangle = P(k)\delta(\vec{k} - \vec{k}'). \quad (2.35)$$

All the higher point correlators can be obtained using these two correlators for a Gaussian perturbation. For example, the three-point correlator $\langle g(\vec{k}_1)g(\vec{k}_2)g(\vec{k}_3) \rangle = \langle g(\vec{k}_1)g(\vec{k}_2) \rangle \langle g(\vec{k}_3) \rangle + \text{perms}$. Where “perms” indicates permutations of the indices $\{1, 2, 3\}$. Each of these terms contain a factor of $\langle g(\vec{k}_i) \rangle = 0$, hence the three-point correlator is zero. Similarly, one can argue that all of the odd-point correlators are zero. The even point correlators on the other hand are generically non-zero. For example, the four point correlator $\langle g(\vec{k}_1)g(\vec{k}_2)g(\vec{k}_3)g(\vec{k}_4) \rangle = \langle g(\vec{k}_1)g(\vec{k}_2) \rangle \langle g(\vec{k}_3)g(\vec{k}_4) \rangle + \text{perms} = \delta(\vec{k}_1 + \vec{k}_2)\delta(\vec{k}_3 + \vec{k}_4)P(k_1)P(k_2) + \text{perms}$.

Taking the probability distribution to be

$$\mathcal{P}(g) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{g^2}{2\sigma^2}}, \quad (2.36)$$

we can verify that all the moments $\langle g(x)^{2n} \rangle = \int_{-\infty}^{\infty} \mathcal{P}(g) g^{2n} dg$ for the Gaussian perturbation above are same as those obtained from this distribution. For example, the moment for the four-point correlator above is $\langle g(x)^4 \rangle = 3\langle g(x)^2 \rangle^2 = 3\sigma^4$, where standard deviation $\sigma = \sqrt{\langle g(x)^2 \rangle}$. In this way, we can motivate that the probability of Gaussian perturbation looks like the Gaussian distribution function using just the definition of the two point correlator.

2.2.4 Non-gaussian perturbation

In general, cosmological perturbations need not be Gaussian. However, statistical homogeneity and isotropy restrict the forms of n -point correlators that can be constructed. In particular, as we saw earlier, homogeneity implies delta function factors while isotropy implies only magnitude dependence. Let us investigate the correlators one by one,

One-point correlator

$\langle g(\vec{k}) \rangle \propto \delta(\vec{k})$ is the only possible choice. In position space this corresponds to $\langle g(x) \rangle = \frac{1}{(2\pi)^3}$, which seems reasonable as the probability of a realisation is one divided by the volume of phase space. But this constant value can be absorbed into the background and hence it may not necessarily be classed as a perturbation.

Two-point correlator

There are two possibilities: either $\langle g(\vec{k})g(\vec{k}') \rangle \propto \delta(\vec{k})\delta(\vec{k}')$ or $\langle g(\vec{k})g(\vec{k}') \rangle \propto \delta(\vec{k} + \vec{k}')P(k)$. So a general correlator can be the linear combination of both, $\langle g(\vec{k})g(\vec{k}') \rangle \sim \delta(\vec{k})\delta(\vec{k}') + \delta(\vec{k} + \vec{k}')P(k)$. This illustrates a general property of the higher point correlators, they contain terms which are made out of lower point correlators and one term which contains the delta function involving all the momenta. In the case of two-point correlator the first term is again just a number, which can be absorbed into the background.

Three-point correlator

Dropping the constant term proportional to the one-point correlator $\langle g(\vec{k}_1) \rangle \langle g(\vec{k}_2) \rangle \langle g(\vec{k}_3) \rangle$, the three-point correlator can be made up of

$$\langle g(\vec{k}_1)g(\vec{k}_2)g(\vec{k}_3) \rangle \sim (\langle g(\vec{k}_1) \rangle \langle g(\vec{k}_2)g(\vec{k}_3) \rangle + \text{perms}) + \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)B_g(k_1, k_2, k_3). \quad (2.37)$$

The first terms each have a factor $\langle g(\vec{k}) \rangle \propto \delta(\vec{k})$ so they vanish in case the momentum is non-zero. If one of the momenta is zero, then it behaves like a two point correlator and can be absorbed into the two-point correlator.

The second term contains a delta function involving all three momenta, so it is non-vanishing only if $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = 0$. This means that the momenta form a closed triangle. See Figure 2.1.

The Bispectrum $B_g(k_1, k_2, k_3)$ should have arguments that only depend on magnitudes, hence it is only dependent on the length of the sides of the momenta triangle. For the Gaussian case considered earlier, the bispectrum is zero. Thus a non-zero bispectrum is the lowest point correlator that is able to capture the deviation from Gaussianity.

In the case of the curvature perturbation, its bispectrum can be used to define a dimensionless quantity known as the non-Gaussianity parameter f_{NL} [50],

$$B_\zeta(k_1, k_2, k_3) = f_{\text{NL}}F(k_1, k_2, k_3), \quad (2.38)$$

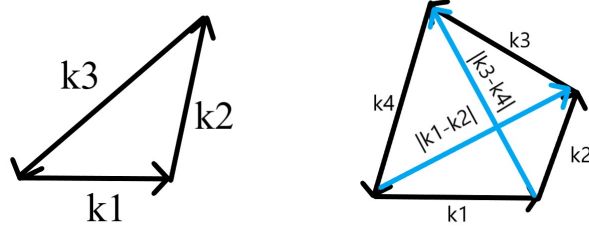


Figure 2.1: Momenta configuration for the three point correlator (left) and four point correlator (right)

where the momentum dependence is isolated into a template function $F(k_1, k_2, k_3)$. Different choices for the momenta $\{k_1, k_2, k_3\}$ yield a momenta triangle with a different shape. For example, $k_1 \approx k_2 \approx k_3$ corresponds to an ‘equilateral’ shape, $k_1 \ll k_2 \approx k_3$ corresponds to a ‘local’ shape and the ‘orthogonal’ shape is defined to be orthogonal to both the ‘equilateral’ and ‘local’ shapes.

Different physical mechanisms for the generation of curvature perturbations correspond to a bispectrum that peaks for a specific shape of the momenta triangle [50]. For example, if the action contains non-canonical kinetic terms then the generated bispectrum typically peaks for an ‘equilateral’ shape. The template $F(k_1, k_2, k_3)$ can be chosen to reflect this behaviour and thus help identify the underlying mechanism for generation of the non-Gaussianity.

The ‘local’ template $F(k_1, k_2, k_3) \sim P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)$ is generally used to characterise non-Gaussianity arising from super-horizon evolution [51]. In this work, we shall define f_{NL} analogously to such a template [52]

$$f_{\text{NL}}(\vec{k}_1, \vec{k}_2, \vec{k}_3) = -\frac{5}{6} \left(\frac{B_\zeta(k_1, k_2, k_3)}{P_\zeta(k_1)P_\zeta(k_2) + P_\zeta(k_1)P_\zeta(k_3) + P_\zeta(k_2)P_\zeta(k_3)} \right). \quad (2.39)$$

However, for simplicity the actual calculation of f_{NL} shall be carried out in the limit $k_1 = k_2 = k_3 = k_*$, where k_* is the scale at which measurements take place. In this limit the ‘local’, ‘equilateral’ and ‘orthogonal’ templates all coincide.

Four-point correlator

Barring terms containing the one-point correlator, which can be absorbed into the definitions of lower point correlators or the background, for the four-point correlator we can have

$$\begin{aligned} \langle g(\vec{k}_1)g(\vec{k}_2)g(\vec{k}_3)g(\vec{k}_4) \rangle &\sim (\langle g(\vec{k}_1)g(\vec{k}_2) \rangle \langle g(\vec{k}_3)g(\vec{k}_4) \rangle + \text{perms}) \\ &+ \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) T_g(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) \end{aligned} \quad (2.40)$$

Following the general trend, the four-point correlator gets contributions from the two-point

correlator and a term which is proportional to the delta function involving all four momenta. Similar to the three point correlator case, the delta function implies that the momenta form a closed quadrilateral.

We have deliberately chosen to highlight the vector nature of momenta in the trispectrum $T_g(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$. This is because even though the trispectrum must depend on scalar quantities due to rotational invariance, here there are six such quantities. Namely, the four sides of the quadrilateral plus the two diagonals as shown in Figure 2.1. Therefore in order to define the reduced trispectrum $\mathcal{T}_g$, we need to include all six scalars,

$$T_g(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \mathcal{T}_g(k_1, k_2, k_3, k_4)(P_g(k_1)P_g(k_2)P_g(|\vec{k}_1 - \vec{k}_4|) + 23 \text{ perms}). \quad (2.41)$$

The number of terms is equal to the possible ways to choose three arguments out of the available six, ${}^6C_3 = 24$. Analogous to f_{NL} , in case of the curvature perturbation the reduced trispectrum can be used to define another non-Gaussianity parameter g_{NL} [53] that corresponds to the four-point curvature correlator. This parameter is useful to study inflationary theories that exhibit parity violation [54].

2.3 Observations

2.3.1 Cosmological perturbation theory

For cosmological applications, it is possible to treat the equations of motion perturbatively in the metric and matter fields using Cosmological Perturbation Theory [55–57].

At a given point in time, the background FRW metric is invariant under rotations and translations in three dimensions. This implies the perturbation in the metric as well as in the energy-momentum tensor can be decomposed into its scalar, vector and rank 2 tensor representations [23]. The perturbed metric around an FRW background in conformal time η can be written as

$$g_{\mu\nu} = a(\eta)^2 \left(-(1 + \Phi)d\eta^2 + (B_{,i} + S_i)d\eta dx^i + ((1 + \psi)\delta_{ij} + 2E_{,ij} + F_{i,j} + F_{j,i} + h_{ij})dx^i dx^j \right), \quad (2.42)$$

in terms of the scalar functions Φ, B, ψ, E ; the vector functions S_i, F_i ; and the tensor functions h_{ij} . If the energy momentum part of the Einstein's equation consists of an irrotational fluid, which is the case for scalar fields in the early universe, then the vector functions can be set to zero.

Due to diffeomorphism invariance, there is ambiguity in defining perturbations that correspond to physical inhomogeneities [23, 58]. By performing a general coordinate transformation, $x_\mu \rightarrow \tilde{x}_\mu = x_\mu + \alpha_\mu$ where α_μ are infinitesimal functions of spacetime, one finds two of the four scalar functions are redundant. This redundancy can be eliminated by fixing a particular gauge. In the cosmological context, there exist many different gauge choices that have been used in the literature: longitudinal, spatially flat, synchronous, comoving orthogonal, total matter and uniform density gauge to name a few [59]. For example in the longitudinal or conformal Newtonian gauge, the perturbed metric is

written as [60]

$$g_{\mu\nu} = a(\eta)^2 \left(-(1 + \Phi) d\eta^2 + ((1 + \psi)\delta_{ij} + h_{ij}) dx^i dx^j \right). \quad (2.43)$$

At first order in perturbation theory, the Einstein equations for the scalar, vector and tensor representations decouple. Thus we can concentrate on the scalar part of the perturbed metric separately. Comparing with the perturbation on a generic slice (2.20) at first order, we can use the delta N formula to write

$$\psi - \zeta = \delta N, \quad (2.44)$$

where ζ is the perturbation on a uniform density slice as before. Under a temporal gauge transformation $t \rightarrow \tilde{t} = t + \delta t(t, x)$, we have $\delta N = -H\delta t$ whereas at first order the scalar field fluctuation $\delta\phi$ transforms as [61]

$$\delta\phi \rightarrow \delta\phi - \dot{\bar{\phi}}\delta t, \quad (2.45)$$

where $\bar{\phi}$ indicates the background value of ϕ . Therefore

$$\zeta = \psi - H \frac{\delta\phi}{\dot{\bar{\phi}}}. \quad (2.46)$$

We can now calculate the power spectrum for curvature perturbation ζ at horizon exit during inflation. Starting from a flat slicing $\psi = 0$,

$$\langle \zeta_1 \zeta_2 \rangle = \frac{H^2}{\dot{\bar{\phi}}^2} \langle \delta\phi_1 \delta\phi_2 \rangle. \quad (2.47)$$

Let the scale at which the effective mass $m_{\text{eff}}^2 = V_{,\phi\phi}$ for potential $V(\phi)$ becomes equal to the the Hubble rate $H(\phi)$ be denoted as k_{crit} . For a mode k that is exiting the horizon before k_{crit} , i.e. $k < k_{\text{crit}}$ we can effectively assume that the field remains massless. Then the approximation in eq. (1.36) remains valid. Therefore we can use the mode function solution $\delta\phi_k$ in eq. (1.37) to obtain the reduced power spectrum a few Hubble times after horizon exit, $k = aH$, when the mode has gone super-horizon,

$$\mathcal{P}_\phi = \frac{k^3}{2\pi^2} \Sigma = \frac{k^3}{2\pi^2} (2\pi)^3 |\delta\phi_k|^2 = \frac{H^2}{4\pi^2}. \quad (2.48)$$

Hence the inflaton contribution to the curvature perturbation in momentum space is given by

$$\mathcal{P}_\zeta^\phi = \frac{1}{4\pi^2} \frac{H^4}{\dot{\bar{\phi}}^2}. \quad (2.49)$$

In slow roll, using $H^2 = V/3M_P^2$ and $3H\dot{\phi} = -V_{,\phi}$ and in terms of the slow-roll parameter ϵ (1.25),

$$\mathcal{P}_\zeta^\phi = \frac{1}{24\pi^2 M_P^4} \frac{V}{\epsilon}. \quad (2.50)$$

The tensor-to-scalar ratio r is defined as the ratio between power spectra for the tensor modes and scalar modes, i.e.

$$r = \frac{\mathcal{P}_h}{\mathcal{P}_\zeta}. \quad (2.51)$$

By using a similar treatment as for the scalar field fluctuations during inflation, the power spectrum for the tensor h_{ij} perturbations can be calculated to be [61]

$$\mathcal{P}_h = \frac{8}{M_P^2} \frac{H^2}{4\pi^2}, \quad (2.52)$$

where the additional $8/M_P^2$ factor arises when accounting for the two polarisations that the tensor field can possess. Assuming all the scalar curvature perturbations arise from the inflaton, using eq. (2.50) above, the tensor-to-scalar ratio becomes

$$r = 16\epsilon. \quad (2.53)$$

2.3.2 CMB anisotropy

The Cosmic Microwave Background originates from the recombination epoch at redshift $z \approx 1100$. After which CMB photons propagate freely in the Λ CDM background. Observations reveal their distribution $f(q, T)$ with photon momentum denoted by q , corresponds to a black body spectrum

$$f(q, T) = \frac{1}{\exp\left(\frac{q}{k_B T}\right) - 1} \quad (2.54)$$

today at temperature $T = 2.725K$ with fluctuations $\delta T/T \sim 10^{-5}$. Therefore one can estimate their distribution from $z \approx 1100$ to $z \sim 10$ by following the Boltzmann equation in a Cosmological Perturbation Theory background eq. (2.43). After $z \sim 10$ one needs to also take into account the re-scattering by electrons in galaxy clusters known as the Sunyaev-Zeldovich effect.

The relation between the temperature fluctuations observed today $\delta T/T$ and the curvature perturbation at recombination ζ_{recomb} is known as the Sachs-Wolfe effect and is given by

$$\frac{\delta T}{T} \approx \zeta_{\text{recomb}}. \quad (2.55)$$

For a derivation of this effect, one can refer to any of the standard cosmology textbooks [14, 22, 23, 61]. Thus by measuring the temperature anisotropy today, we are able to gain information on primordial perturbations originating from the very early universe.

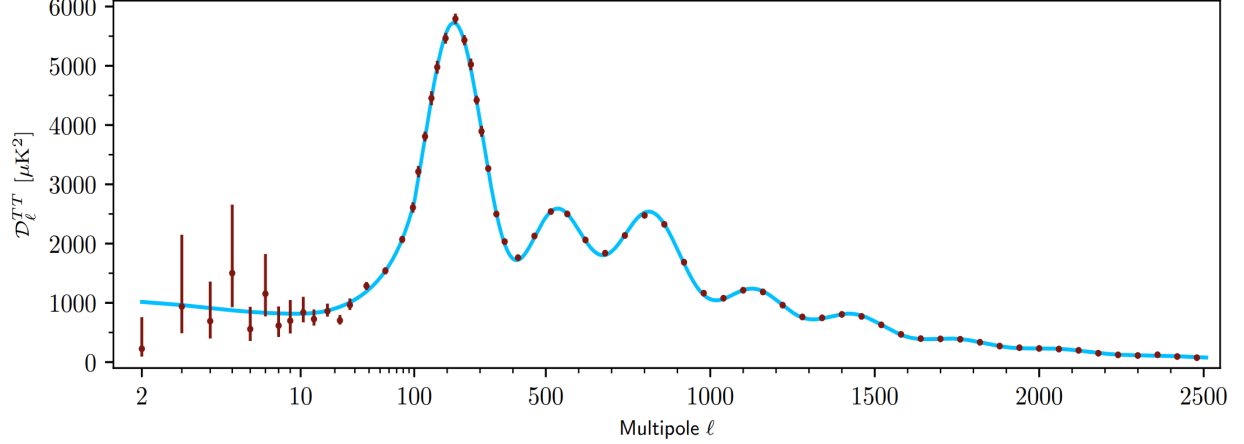


Figure 2.2: Planck CMB power spectrum reproduced from ref. [4]. The superscript TT in $\mathcal{D}_\ell^{TT}$ here denotes that this plot corresponds to the intensity (as opposed to polarisation) spectrum. The continuous light blue curve corresponds to the best-fit Planck 2018 model.

To characterise the temperature fluctuations observed in the sky, it is helpful to expand them in spherical harmonics Y_{lm} with coefficients

$$a_{lm} = \int d\hat{n} Y_{lm}^*(\hat{n}) \frac{\delta T_{\text{today}}}{T_{\text{today}}}(\hat{n}), \quad (2.56)$$

where $\hat{n}$ denotes the direction in the sky. Then assuming statistical isotropy, the power spectrum can be written as

$$\langle a_{lm} a_{l'm'}^* \rangle = \delta_{ll'} \delta_{mm'} C_l. \quad (2.57)$$

In terms of the quantity

$$\mathcal{D}_l = \frac{l(l+1)}{2\pi} C_l \quad (2.58)$$

the observed power spectrum from the Planck satellite [4] is reproduced in Figure 2.2. The plateau at $2 < l \lesssim 20$ is known as the Sachs-Wolfe plateau and is a direct measure of the primordial curvature perturbations. It gives confidence to the near scale-invariant prediction for the curvature perturbations arising from an inflationary era.

The intermediate ranges $20 \lesssim l \lesssim 1000$ show peaks corresponding to the acoustic oscillations of modes that entered the horizon at different times during radiation domination. Whereas the damping at $1000 \lesssim l \lesssim 2500$ is due to the finite width of the last scattering surface, owing to the fact that the transition from the universe being opaque to photons free streaming does not occur instantaneously.

The Planck power spectrum allows to put tight constraints on the parameters of the Λ CDM model as well as to derive constraints on inflation, such as the power spectrum for primordial

perturbations, its scale dependence, tensor-to-scalar ratio and non-Gaussianity.

2.3.3 Features

Let us now characterise some key features of the observable primordial perturbations i.e. those coming from scales $10^{-4}\text{Mpc}^{-1} < k < 10^{-1}\text{Mpc}^{-1}$,

- *Adiabatic condition:* The observed primordial perturbations are adiabatic, i.e. the ratio of number density of the particle species, eg. cold dark matter, baryons and neutrinos, to the number density of photons remains constant. For inflationary models that combine adiabatic and isocurvature perturbations such as the curvaton scenario [62], Planck measurements constrain the non-adiabatic contribution to be less than 2% [63].
- *Scale dependence:* Observations indicate that the curvature power spectrum

$$\mathcal{P}_\zeta(k) = \mathcal{P}_* \left(\frac{k}{k_*} \right)^{n_s-1} \quad (2.59)$$

is almost independent on the scale chosen. With

$$\mathcal{P}_* = 2.1 \times 10^{-9}. \quad (2.60)$$

The spectral index n_s characterises small departures from scale invariance. Current constraints place $n_s = 0.9665 \pm 0.0038$ [63]. A scale-invariant spectrum $n_s = 1$ has been ruled out.

- *Tensor to scalar ratio:* Current constraint on the tensor to scalar ratio (2.51) is [64, 65]

$$r < 0.038 \quad \text{at } k_{\text{phys}} = 0.05\text{Mpc}^{-1}. \quad (2.61)$$

- *Gaussianity:* The observed primordial perturbations are Gaussian to a high degree. Current constraints [53] on the ‘local’ bispectrum are

$$f_{\text{NL}}^{\text{local}} = -0.9 \pm 5.1 \quad (2.62)$$

while for the equilateral and orthogonal templates they are

$$f_{\text{NL}}^{\text{equi}} = -26 \pm 47, \quad f_{\text{NL}}^{\text{ortho}} = -38 \pm 24. \quad (2.63)$$

These constraints are provided by Planck satellite CMB measurements. However, Large Scale Structure (LSS) measurements can also provide a competitive bound on f_{NL} [66] and may even improve the error to below unity through current and upcoming experiments, for example $\Delta f_{\text{NL}}^{\text{local}} \lesssim 0.5$ from the SPHEREx mission [67].

One mechanism for the significant generation of curvature perturbations on large scales in this scenario is when:

- *The field develops significant fluctuations over the separate Hubble volumes*

If we only had a single field, the inflaton, then it is possible to choose a cosmological gauge such that all fluctuations at scales larger than N_{ini} can be made to disappear. Therefore a second scalar field in addition to the inflaton field is needed, we refer to it as the spectator field χ . Furthermore, this spectator field must be light compared to the background Hubble rate, $m_\chi < H$, so that its fluctuations are not damped on large scales.

- *The field fluctuations over the separate Hubble volumes modulate the local expansion of space*

If a non-adiabatic phenomenon occurs at N_{ini} , each Hubble volume expands slightly differently and generates curvature perturbations in accordance with the delta N formula (2.28). There are several possible phenomena that can generate this:

- The curvaton mechanism [68–70]: in which the spectator field comes to dominate the energy density towards the end of inflation
- Preheating [1, 71–78]: in which there is a non-linear, non-perturbative transfer of energy from the inflaton to the spectator field after the end of inflation. We shall perform a detailed study of this mechanism by including scale dependence in Chapter 3.
- Oscillon formation [79]: in which non-linear solitons come to dominate the evolution post-inflation.
- Vacuum transition [80]: in which there is a non-perturbative change in the energy density during inflation due to a transition in the vacuum. We shall develop a formalism to study this new mechanism for generating curvature perturbations in Chapter 4.

Thus if there is a scalar field χ that has a different local value $\chi(x)$ in each Hubble volume, it can affect the amount of expansion, generating a position-dependent curvature perturbation [47]

$$\zeta_\chi(x) = \delta N(\chi(x)). \quad (2.65)$$

In this work, we will assume that there is the inflaton field ϕ which dominates the energy density during inflation and satisfies the slow roll conditions. The spectator field χ , which is light compared to the Hubble rate during inflation so that it has significant superhorizon correlations, and which interacts sufficiently weakly with the inflaton so that its effect can be neglected in the early stages of inflation. We also assume that self-interactions of the spectator field χ are weak so that it can be assumed to be Gaussian. And hence its probability distribution can be written as (2.36)

$$\mathcal{P}_G(\chi) = \frac{1}{\sqrt{2\pi\langle\delta\chi^2\rangle}} \exp\left(-\frac{(\chi - \bar{\chi})^2}{2\langle\delta\chi^2\rangle}\right) \quad (2.66)$$

in terms of the field variance $\langle\delta\chi^2\rangle$, where $\delta\chi = \chi - \bar{\chi}$ and $\bar{\chi}$ is mean value of χ in the observable universe.

The total curvature perturbation $\zeta = \zeta_\phi + \zeta_\chi$ also includes a contribution from the inflaton field ϕ . This additive form is valid if during inflation, the inflaton field dominates the expansion and is slowly rolling and that the back-reaction from χ is negligible, all things we will assume. Eq. (2.65) shows that to obtain the statistics of the curvature perturbation generated by the field χ , one needs to determine $\delta N(\chi)$ which is a function of the local value of the field χ , and the statistics of the field χ .

Assuming statistical homogeneity and isotropy, the statistics of χ are fully described by the two-point correlation function

$$\langle \delta\chi(x)\delta\chi(x') \rangle = \Sigma(|x - x'|), \quad (2.67)$$

or its Fourier transform $\Sigma(k)$ which satisfies

$$\langle \delta\chi(\vec{k})\delta\chi(\vec{k}') \rangle = \Sigma(k)(2\pi)^3\delta(\vec{k} + \vec{k}'). \quad (2.68)$$

In analogy with eq. (2.31) we also define

$$\mathcal{P}_\chi(k) = \frac{k^3}{2\pi^2}\Sigma(k). \quad (2.69)$$

If δN is not a linear function of χ , then it gives rise to a non-Gaussian contribution to the curvature perturbation. In many simple scenarios, δN can be well approximated by a quadratic Taylor expansion,

$$\delta N(\chi) = \delta N(\bar{\chi}) + \delta N_{,\chi}(\bar{\chi})\delta\chi + \frac{1}{2}\delta N_{,\chi\chi}(\bar{\chi})\delta\chi^2. \quad (2.70)$$

Assuming that the dominant contribution to the curvature perturbation is given by ζ_ϕ and that it is Gaussian, the leading term in the bispectrum is [81]

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = (\delta N_{,\chi})^2 \delta N_{,\chi\chi} (\Sigma(k_1)\Sigma(k_2) + \Sigma(k_1)\Sigma(k_3) + \Sigma(k_2)\Sigma(k_3)). \quad (2.71)$$

Considering an equilateral configuration, $k_1 = k_2 = k_3 = k$, eq. (2.39) then gives

$$f_{\text{NL}} = -\frac{5}{6}(\delta N_{,\chi})^2 \delta N_{,\chi\chi} \frac{\Sigma(k)^2}{P_\zeta(k)^2} = -\frac{5}{6}(\delta N_{,\chi})^2 \delta N_{,\chi\chi} \frac{\mathcal{P}_\chi(k)^2}{\mathcal{P}_\zeta(k)^2}. \quad (2.72)$$

However, if $\delta N_{,\chi}$ is sufficiently small, which happens if the theory has symmetry under the sign change $\chi \rightarrow -\chi$ and $\bar{\chi}$ is small, the dominant term is given by

$$\begin{aligned} \langle \zeta(\vec{x}_1)\zeta(\vec{x}_2)\zeta(\vec{x}_3) \rangle &= \frac{1}{8}(\delta N_{,\chi\chi})^3 \langle \delta\chi(\vec{x}_1)^2\delta\chi(\vec{x}_2)^2\delta\chi(\vec{x}_3)^2 \rangle \\ &= (\delta N_{,\chi\chi})^3 \Sigma(|\vec{x}_1 - \vec{x}_2|)\Sigma(|\vec{x}_1 - \vec{x}_3|)\Sigma(|\vec{x}_2 - \vec{x}_3|) + \dots, \end{aligned} \quad (2.73)$$

whose Fourier transform gives

$$B_\zeta(\vec{k}_1, \vec{k}_2, \vec{k}_3) = (\delta N_{,\chi\chi})^3 \int \frac{d^3q}{(2\pi)^3} \Sigma(q)\Sigma(|\vec{q} + \vec{k}_1|)\Sigma(|\vec{q} + \vec{k}_1 + \vec{k}_2|). \quad (2.74)$$

In this case f_{NL} is therefore given by [81]

$$f_{\text{NL}} = -\frac{5}{6} \frac{(\delta N_{,\chi\chi})^3}{P_\zeta(k)^2} \int \frac{d^3 q}{(2\pi)^3} \Sigma(q) \Sigma(|\vec{q} + \vec{k}_1|) \Sigma(|\vec{q} + \vec{k}_1 + \vec{k}_2|). \quad (2.75)$$

These expressions for the non-Gaussianity parameter f_{NL} (2.72)–(2.75) are valid when the Taylor expansion (2.70) for δN is valid. Therefore we shall refer to these expressions as arising in the perturbative delta N formalism.

As we shall see in the next chapter, for preheating the Taylor expansion (2.70) is no longer valid, and hence we will need to go beyond it using a non-perturbative delta N formalism.

Chapter 3

Preheating with scale dependence

“1 2 all”

3.1 Reheating

Reheating is the name given to the era in the universe’s evolution between inflation and Big Bang Nucleosynthesis. During inflation, the universe expands almost exponentially in an adiabatic manner causing any matter present to dilute and cool down. However, the precondition to Big Bang Nucleosynthesis (BBN) is Standard Model matter such as the quarks, electrons and photons in local thermal equilibrium at temperature about 100MeV. Thus the intervening era between inflation and BBN must siphon energy to the SM particles and increase the temperature to $\sim 100\text{MeV}$. Hence this era is referred to as reheating. In principle, we also expect beyond Standard Model physics such as baryogenesis and the electroweak phase transition to occur between inflation and BBN at energies spanning $10^2 - 10^{10}\text{GeV}$. Figure 3.1 shows the broad stages in which reheating can be divided.

In a wide range of models the initial stage of reheating, which begins just after the end of inflation, involves the rapid generation of SM particles and is known as preheating. Due to an oscillatory background provided by the inflaton field towards the end of inflation, Floquet theory

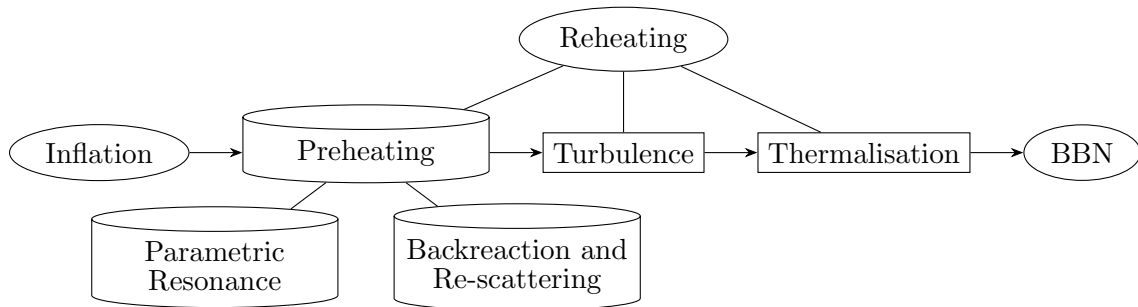


Figure 3.1: Chart illustrating the stages of universe’s history between the end of inflation and Big Bang Nucleosynthesis

implies an instability, i.e. an exponential growth of field fluctuations, can be generated especially in the low momentum (low k) modes. This is known as parametric resonance. However, the produced particles can back-react [82] on the $k = 0$ mode and re-scatter energy to higher k modes, leading to a complicated non-linear behaviour that ultimately causes the growth to stop at some point. This later part of preheating is labelled backreaction and rescattering.

Due to the high occupancy number of the lowest k -modes, their evolution can be treated in a semi-classical instead of quantum fashion [83]. Furthermore, as the coupling constant in inflationary theories is small, this exponential growth persists for a long time such that the semi-classical equations can be treated as classical equations with appropriate initial conditions that can be solved using lattice simulations [84].

After preheating, the number density is replicated to higher k during turbulence [85]. However, the resulting particle spectra does not resemble thermal distribution and further thermalisation needs to take place before a plasma with local thermal equilibrium is achieved [86, 87].

Just like inflation, the reheating era is not directly accessible through current cosmological measurements of cosmic microwave background or the large scale structure. However, as we shall see in the following sections, the non-linear dynamics occurring during preheating can affect the super-horizon evolution of modes that exited during inflation and thus leave an imprint on these observables.

The duration of reheating and its equation of state remain uncertain. Detection of primordial gravitational waves can help break the degeneracy between these two quantities. The uncertainty enters into the calculation of number of efolds during inflation. For example, in the number of efolds when the observed scale $k_{\text{phys}} = 0.05 \text{Mpc}^{-1}$, at which Planck satellite measurements take place, leaves the horizon. Denoting by subscripts ‘phys’, ‘end’, ‘reh’ and ‘th’ the corresponding quantities at the physical scale k_{phys} , the end of inflation, the end of reheating and the start of thermal equilibrium respectively, then in terms of the comoving Hubble length today [26, 88],

$$\frac{k_{\text{phys}}}{a_0 H_0} = \frac{a_{\text{phys}} H_{\text{phys}}}{a_0 H_0} = \frac{a_{\text{phys}}}{a_{\text{end}}} \frac{a_{\text{end}}}{a_{\text{reh}}} \frac{a_{\text{reh}}}{a_0} \frac{\rho_{\text{reh}}^{1/4}}{H_0} \frac{\rho_{\text{end}}^{1/4}}{\rho_{\text{reh}}^{1/4}} \frac{H_{\text{phys}}}{\rho_{\text{end}}^{1/4}}. \quad (3.1)$$

We can then use the definitions $N_{\text{phys}} = \ln(a_{\text{end}}/a_{\text{phys}})$, $N_{\text{reh}} = \ln(a_{\text{reh}}/a_{\text{end}})$ and $H_{\text{phys}}^2 = V_{\text{phys}}/3M_P^2$ in slow-roll to write

$$N_{\text{phys}} = \ln \left(\frac{a_{\text{reh}} \rho_{\text{reh}}^{1/4}}{\sqrt{3} a_0 H_0} \right) + \frac{1}{4} \ln \left(\frac{V_{\text{phys}}^2}{\rho_{\text{end}} M_P^4} \right) - \ln \left(\frac{k_{\text{phys}}}{a_0 H_0} \right) - N_{\text{reh}} + \frac{1}{4} \ln \left(\frac{\rho_{\text{reh}}}{\rho_{\text{end}}} \right). \quad (3.2)$$

Thermal equilibrium is reached when $a_{\text{th}} \rho_{\text{th}}^{1/4} = a_{\text{reh}} \rho_{\text{reh}}^{1/4}$. Assuming thermal equilibrium corresponds to the start of BBN, the first term in the expression for N_{phys} above evaluates to [89]

$$\ln \left(\frac{a_{\text{reh}} \rho_{\text{reh}}^{1/4}}{\sqrt{3} a_0 H_0} \right) = 66.89 - \frac{1}{12} g_{\text{eff}}, \quad (3.3)$$

where g_{eff} denotes the effective massless degrees of freedom during reheating. The dependence on inflationary evolution is isolated in the second term. The third term can be calculated by plugging in the value of the scale k_{phys} . The last two terms are the duration of and energy density during reheating, which are the ones that remain undetermined through observations and depend upon the details of the specific reheating model.

3.2 Preheating

To study the effects of preheating, as a first step it is useful to consider the decay of inflaton to a single scalar field instead of the full particle content of the SM. Massless preheating [90] is the toy model with simplest polynomial conformal potential

$$V(\phi, \chi) = \frac{\lambda}{4}\phi^4 + \frac{g^2}{2}\phi^2\chi^2 \quad (3.4)$$

first used to study preheating dynamics. Here, the field ϕ is the inflaton and χ is called the spectator or daughter field. Both inflaton and spectator are taken to be massless fields.

Let us examine the dynamics in this model through a first order perturbative QFT analysis. We shall closely follow the discussion in ref. [91]. The equations of motion in FRW are

$$\ddot{\phi} + 3H\dot{\phi} + \lambda\phi^3 + g^2\chi^2\phi = 0, \quad (3.5)$$

$$\ddot{\chi} + 3H\dot{\chi} + g^2\phi^2\chi = 0. \quad (3.6)$$

where H is given by the Friedmann equation

$$H^2 = \frac{1}{3M_P^2} \left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\dot{\chi}^2 + \frac{\lambda}{4}\phi^4 + \frac{g^2}{2}\phi^2\chi^2 \right). \quad (3.7)$$

Dividing the fields into their mean and fluctuating parts

$$\phi = \bar{\phi} + \delta\phi, \quad \chi = \bar{\chi} + \delta\chi \quad (3.8)$$

and assuming the background spectator field value is negligible as compared to the inflaton value, we just have

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \lambda\bar{\phi}^3 = 0. \quad (3.9)$$

Conformalising the field and time by the scale factor: $\bar{\phi} = \bar{\varphi}/a, dt = ad\eta$ gives

$$\bar{\varphi}'' + \lambda\bar{\varphi}^3 - \frac{a''}{a}\bar{\varphi} = 0 \quad (3.10)$$

where prime denotes derivative with respect to conformal time. Taking the average over several

oscillations, $a''(\eta) = 0$ [92]. So the equation simplifies to

$$\bar{\varphi}'' + \lambda \bar{\varphi}^3 = 0 \quad (3.11)$$

which has the solution $\bar{\varphi} = \bar{\varphi}_{\text{ini}} \text{cn}(\sqrt{\lambda} \bar{\varphi}_{\text{ini}}(\eta - \eta_{\text{ini}}), 1/\sqrt{2})$ in terms of the Jacobi cosine function, where $\bar{\varphi}_{\text{ini}}$ and η_{ini} are the initial conformal field and time values. Now the quantum fluctuations can be Fourier expanded in terms of their mode functions $\{\phi_k, \chi_k\}$ and creation and annihilation operators $\{a_k^\dagger, a_k\}$ as

$$\delta \hat{\phi}^i(t, x) = \frac{1}{(2\pi)^{3/2}} \int d^3k \left(\hat{a}_k \phi_k^i(t) e^{-i\vec{k} \cdot \vec{x}} + \hat{a}_k^\dagger \phi_k^{*i}(t) e^{+i\vec{k} \cdot \vec{x}} \right) \quad (3.12)$$

where $i = \{\phi, \chi\}$ is just an index labelling the two fields. In terms of the mode functions and ignoring any metric perturbations

$$\ddot{\phi}_k + 3H\dot{\phi}_k + \left(\frac{k^2}{a^2} + 3\lambda \bar{\varphi}^2 \right) \phi_k = 0, \quad (3.13)$$

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left(\frac{k^2}{a^2} + g^2 \bar{\varphi}^2 \right) \chi_k = 0. \quad (3.14)$$

The above mode equations are identical to each other if we replace g^2 with 3λ . In terms of the rescaled variables $\tau = \sqrt{\lambda} \bar{\varphi}_{\text{ini}} \eta$, $\chi_k = X_k/a$ and the oscillating background $\bar{\varphi}(\eta)$,

$$\frac{d^2 X_k}{d\tau^2} + \left(\kappa^2 + \frac{g^2}{\lambda} \text{cn}(\tau; \frac{1}{\sqrt{2}}) \right) X_k = 0 \quad (3.15)$$

where $\kappa^2 = k^2/\lambda \bar{\varphi}_{\text{ini}}^2$. This is in the form of a Lamé equation. Its solution is given by

$$X_k = e^{\mu_k \tau} f(\tau) + e^{-\mu_k \tau} g(\tau) \quad (3.16)$$

where $f(\tau)$ and $g(\tau)$ are functions with the same period as the coefficients of the Lamé equation. The Floquet index μ_k depends on the parameters κ^2 and g^2/λ . Figure 3.2 shows the Floquet index is non-zero, i.e. the equation has an instability for a range of parameter values $g^2/\lambda \in [1, 3] \cup [6, 10]$. One can also find the maximum value for the Floquet index for $k = 0$ lies at $g^2/\lambda = 1.875$ for the massless preheating model. This exponential amplification is known as parametric resonance. The growing amplitude means that energy is transferred rapidly to the field χ and daughter particles are produced.

In reality, the exponential growth cannot persist indefinitely and will stop when back-reaction effects begin to play a role. There have been attempts to study back-reaction effects in an analytical [93] or semi-analytical [94] fashion, for example using the Hartree approximation [82]. However one can account for these effects fully by performing classical lattice simulations.

To motivate the applicability of classical simulations for studying preheating dynamics, it is useful to separately account for the behaviour of the low and high k modes.

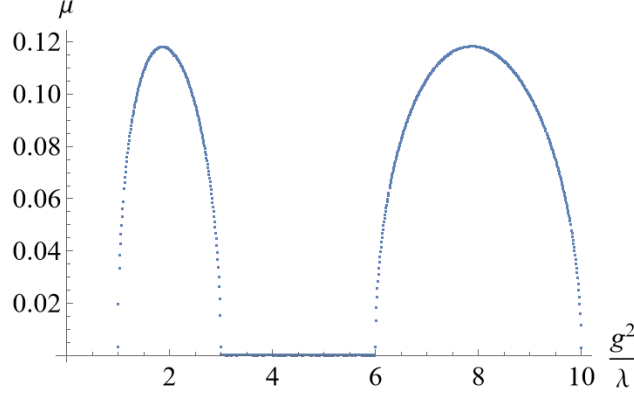


Figure 3.2: Floquet index for the zero momentum ($k = 0$) mode as a function of the parameter g^2/λ for massless preheating model.

The low k modes exhibit exponential growth due to parametric resonance. Consider the canonical commutation relation between the field χ and its conjugate momentum, $[\hat{\chi}(t, x), \hat{\Pi}_\chi(t, y)] = i\delta^3(x - y)$. In terms of the mode functions $\chi_k(\eta)$ [83],

$$i(\chi'_k(\eta)\chi_k^*(\eta) - \chi_k'^*(\eta)\chi_k(\eta)) = 1, \quad (3.17)$$

i.e. $\text{Im}(\chi_k'^*(\eta)\chi_k(\eta)) = 1$. We know that the low k modes grow exponentially, so the real part $\text{Re}(\chi_k'^*(\eta)\chi_k(\eta))$ grows exponentially but the imaginary part must always stay unity. Therefore upto exponentially small corrections, $\chi_k'^*(\eta)\chi_k(\eta)$ can be taken to be real and hence the contribution from the quantum commutator to be negligible [83].

For the behaviour of high k modes, let us go back to the mode equations (3.13)–(3.14) for the fluctuations $k > 0$ in terms of conformalised fields and rescaled time τ ,

$$\frac{d^2\varphi_k^i}{d\tau^2} + \omega_k^2(\tau)\varphi_k^i = 0. \quad (3.18)$$

Here $\omega_k^2(\tau) = \kappa^2 + 3\bar{\varphi}^2(\tau)$ for the inflaton and $\omega_k^2(\tau) = \kappa^2 + (g^2/\lambda)\bar{\varphi}^2(\tau)$ for the daughter field. The initial conditions for the mode equations above are specified as

$$\varphi_k^i(0) = \sqrt{\frac{\lambda}{2\omega_k(0)}} \text{ and } \frac{d\varphi_k^i(0)}{d\tau} = (-i\omega_k(0) + h(0))\varphi_k^i(0) \quad (3.19)$$

where $h(0)$ is the conformalised Hubble rate $h(\tau) = H(\tau)/\sqrt{\lambda\bar{\varphi}_{\text{ini}}}$. It is useful to quantify the comoving number density n_k of these $k > 0$ modes. It can be defined in terms of harmonic oscillator energy density using

$$\left(n_k + \frac{1}{2}\right)\omega_k = \frac{1}{2}\left|\frac{d\varphi}{d\tau}\right|^2 + \frac{1}{2}\lambda\omega_k^2|\varphi|^2. \quad (3.20)$$

If the number density is low $n_k \ll \lambda^{-1}$, the quantum dynamics is well approximated by the linear theory. In the linear theory of quantum to classical transitions [83], the quantum evolution of the fields is equivalent to evolving an ensemble of their modes classically with initial conditions drawn from a statistical distribution given by [84]

$$\mathcal{P}[\varphi_k(0)] \sim \exp\left(-\frac{2}{\lambda} \int d^3k \, \omega_k(0) |\varphi_k(0)|^2\right). \quad (3.21)$$

Once a value for $\varphi_{k>0}(0)$ is drawn from the above Gaussian distribution, its time derivative is calculated using eq. (3.19). Then with the addition of the zero mode value at the end of inflation obtained from the inflationary evolution, one has the mode functions for all k . These can then be inverse Fourier transformed to populate the initial fluctuations $\delta\phi^i$ and their momenta $\delta\dot{\phi}^i$ on a real space lattice.

Thus if the coupling constant λ is small enough, both the low and high k modes can be described classically during preheating.

There exist many lattice simulation programs [95–97] that have been used to study preheating, especially for the massless preheating model [74, 76, 98–100]. I have made use of the program HLattice [96] for performing numerical simulations of the non-minimally coupled (NMC) preheating model discussed later.

The massless preheating model is ruled out by current observational data [64, 65] because it predicts a very high tensor to scalar ratio. To see this, let us investigate the inflationary dynamics for the potential $\lambda\phi^4/4$ assuming the spectator field is negligible during inflation. For this potential, the end of inflation occurs at $\phi_{\text{end}} = 2.83M_P$ and hence the number of efolds can be calculated as $N = \phi^2/8M_P^2 - 1$. The currently observed scale corresponding to $N_* = 55$ efolds left the horizon at $\phi_* = 21.17M_P$. Therefore the tensor to scalar ratio (2.53) $r = 16\epsilon_* = 0.286$, which is much higher than the current conservative limit of $r < 0.1$ [63]. One way to lower the tensor to scalar ratio is by introducing non-minimal coupling to gravity [101]. We shall construct such an observationally-viable non-minimally coupled (NMC) preheating model in the next section.

3.3 NMC preheating

3.3.1 Potential

It has been shown that introduction of non-minimal coupling of inflaton to gravity is able to satisfy observational bounds on spectral index and tensor to scalar ratio [63, 102]. Thus we study massless preheating with non-minimal coupling to gravity, referring to it as NMC preheating, as the simplest observationally-viable model of preheating.

For the preheating portion of this work, we will assume that the non-minimal coupling parameter has a small positive value $0 < \xi \ll 1/6$.

The Jordan frame action for massless preheating with only the inflaton field coupled non-

minimally to gravity is taken to be

$$S = \int d^4x \sqrt{-g} \left(f(\phi) \mathcal{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - V(\phi, \chi) \right), \quad (3.22)$$

$$\text{where } f(\phi) = \frac{M_P^2}{2} + \frac{1}{2} \xi \phi^2 \quad \text{and} \quad V(\phi, \chi) = \frac{\lambda}{4} \phi^4 + \frac{1}{2} g^2 \phi^2 \chi^2. \quad (3.23)$$

As described in section 1.4, this Jordan frame action can be converted to the Einstein frame by conformally rescaling the metric to remove the non-minimal coupling and then redefining the fields to bring their kinetic terms back to the canonical form.

In the present case, using a conformal rescaling $\tilde{g}_{\mu\nu} = (1 + \xi \frac{\phi^2}{M_P^2}) g_{\mu\nu}$ yields [103]

$$S = \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{\mathcal{R}} - \frac{1}{2} \tilde{g}^{\mu\nu} \mathcal{G}_{ij} \partial_\mu \phi^i \partial_\nu \phi^j - \tilde{V}(\phi, \chi) \right), \quad (3.24)$$

where

$$\tilde{V}(\phi, \chi) = \frac{M_P^4}{4f^2(\phi)} V(\phi, \chi), \quad (3.25)$$

and $\mathcal{G}_{ij}$ is the metric in field space with components

$$\mathcal{G}_{\phi\phi} = \frac{1}{1 + \xi \frac{\phi^2}{M_P^2}} + \frac{6\xi^2 \frac{\phi^2}{M_P^2}}{(1 + \xi \frac{\phi^2}{M_P^2})^2} \quad \text{and} \quad \mathcal{G}_{\chi\chi} = \frac{1}{1 + \xi \frac{\phi^2}{M_P^2}}. \quad (3.26)$$

We can check that the Ricci scalar of this field-space metric is not zero at order ξ . Hence even with only the inflaton non-minimally coupled to gravity it is not possible to bring back canonical kinetic terms for both fields as the field-space metric is not conformally flat. Furthermore it has been shown [103] that bringing back canonical kinetic terms is generically not possible if more than one field is non-minimally coupled.

However if only a single field ϕ was present in the Jordan frame action, then the field space metric has only one component $\mathcal{G}_{\phi\phi}$ and a redefined field variable $\tilde{\phi}$ which satisfies the differential equation

$$\frac{d\tilde{\phi}}{d\phi} = \frac{\sqrt{\xi(1 + 6\xi)\phi^2/M_P^2 + 1}}{\xi\phi^2/M_P^2 + 1} \approx \frac{1}{\sqrt{\xi\phi^2/M_P^2 + 1}}, \quad (3.27)$$

where the final form is valid for $\xi \ll 1/6$, brings back the canonical kinetic term. This gives the relation

$$\phi = \frac{M_P}{\sqrt{\xi}} \sinh \left(\frac{\sqrt{\xi}}{M_P} \tilde{\phi} \right), \quad (3.28)$$

and the Einstein frame potential

$$\tilde{V}(\tilde{\phi}) = \frac{\lambda}{4} \left(\frac{M_P}{\sqrt{\xi}} \tanh\left(\frac{\sqrt{\xi}\tilde{\phi}}{M_P}\right) \right)^4. \quad (3.29)$$

Such a potential is also motivated by α -attractor T models of inflation [31].

Therefore, in order to simplify numerical simulations by not having to include non-canonical kinetic terms, we choose a potential for the two-field case as

$$\tilde{V}(\tilde{\phi}, \tilde{\chi}) = \frac{\lambda}{4} \left(\frac{M_P}{\sqrt{\xi}} \tanh\left(\frac{\sqrt{\xi}\tilde{\phi}}{M_P}\right) \right)^4 + \frac{g^2}{2} \tilde{\chi}^2 \left(\frac{M_P}{\sqrt{\xi}} \tanh\left(\frac{\sqrt{\xi}\tilde{\phi}}{M_P}\right) \right)^2. \quad (3.30)$$

A similar approach to preheating simulations in α -attractor motivated potentials has been used in ref. [104]. Here, we keep the identification $\phi \rightarrow (M_P/\sqrt{\xi}) \tanh(\sqrt{\xi}\phi/M_P)$ in the interaction between the two fields to preserve the flatness of the potential at large inflaton values, which maintains the desirable features of α -attractor models. For the remainder of the discussion on preheating, we shall drop the tilde on the fields for simplicity, bearing in mind that they belong to the Einstein frame.

3.3.2 Parameter values

The non-minimally coupled (NMC) preheating model has three parameters: λ, g^2 , and ξ . Of these, we shall now see that λ is constrained by the power spectrum observation and ξ is constrained by tensor-to-scalar ratio observation.

During inflation we assume the field χ is negligible, so there is effectively a single-field inflationary model with the inflaton potential

$$V(\phi) = \frac{\lambda}{4} \left(\frac{M_P}{\sqrt{\xi}} \tanh\left(\frac{\sqrt{\xi}\phi}{M_P}\right) \right)^4. \quad (3.31)$$

The first slow-roll parameter is

$$\epsilon = \frac{M_P^2}{2} \left(\frac{V'}{V} \right)^2 = \frac{8\xi}{\cosh^2\left(\sqrt{\xi}\frac{\phi}{M_P}\right) \sinh^2\left(\sqrt{\xi}\frac{\phi}{M_P}\right)}. \quad (3.32)$$

Setting $\epsilon = 1$, we find that inflation ends when ϕ reaches the value ϕ_{end} given by

$$\cosh\left(\frac{2\sqrt{\xi}}{M_P}\phi_{\text{end}}\right) = \sqrt{1 + 32\xi}. \quad (3.33)$$

We normalise the scale factor a to be unity at this time, $a_{\text{end}} = 1$. Furthermore, to draw a clear distinction between the number of e-folds during inflation and the number of e-folds during preheating, in this chapter, we shall denote the former by $\mathcal{N}$ and the later by just N .

Using the slow-roll field and Friedmann equations, we can then relate the number of e-foldings

until the end of inflation $\mathcal{N} = -\ln a$ to the inflaton field value,

$$\mathcal{N} = \frac{1}{16\xi} \cosh \left(2 \frac{\sqrt{\xi}}{M_P} \phi \right) - \frac{\sqrt{1+32\xi}}{16\xi}, \quad (3.34)$$

and the Hubble rate to $\mathcal{N}$ as

$$H = \sqrt{\frac{\lambda}{12}} \frac{M_P}{\xi} \tanh^2 \left(\frac{\sqrt{\xi}}{M_P} \phi \right) = \sqrt{\frac{\lambda}{12}} \frac{M_P}{\xi} \frac{16\xi\mathcal{N} + \sqrt{1+32\xi} - 1}{16\xi\mathcal{N} + \sqrt{1+32\xi} + 1}. \quad (3.35)$$

The curvature power spectrum due to the inflaton field (2.50) is therefore [61]

$$\mathcal{P}_* = \frac{1}{8\pi^2} \frac{H(\phi_*)^2}{M_P^2 \epsilon(\phi_*)} = \frac{1}{768\pi^2} \frac{\lambda}{\xi^3} \frac{\sinh^6 \left(\sqrt{\xi} \frac{\phi_*}{M_P} \right)}{\cosh^2 \left(\sqrt{\xi} \frac{\phi_*}{M_P} \right)}, \quad (3.36)$$

where we have used eqs. (3.32) and (3.35), and ϕ_* is the field value $\mathcal{N}_*$ e-foldings before the end of inflation, when the observational scale k_* exited the horizon,

$$\phi_* = \frac{M_P}{2\sqrt{\xi}} \cosh^{-1}(16\mathcal{N}_*\xi + \sqrt{1+32\xi}). \quad (3.37)$$

The precise value of $\mathcal{N}_*$ depends on post-inflationary evolution of the universe, but to be specific we will assume $\mathcal{N}_* = 55$. Setting $\mathcal{P}_*$ equal to its observed value, $\mathcal{P}_* = 2.1 \times 10^{-9}$, we obtain

$$\lambda = 6.4 \times 10^{-5} \xi^3 \frac{(16\mathcal{N}_*\xi + \sqrt{32\xi+1} + 1)}{(16\mathcal{N}_*\xi + \sqrt{32\xi+1} - 1)^3}. \quad (3.38)$$

As can be seen, λ is fully determined once we fix ξ .

Parameter ξ is constrained from below by the tensor to scalar ratio observation [63]. At the observational scale $\mathcal{N}_*$, the NMC preheating model gives (2.53) [61]

$$r = 16\epsilon_* = \frac{512\xi}{\sinh^2 \left(2\sqrt{\xi} \frac{\phi_*}{M_P} \right)} = \frac{16}{8\mathcal{N}_*^2\xi + \mathcal{N}_*\sqrt{32\xi+1} + 1}. \quad (3.39)$$

Requiring the conservative bound $r < 0.1$ to be satisfied for $\mathcal{N}_* = 55$ yields the lower bound on the non-minimal coupling in NMC preheating model as

$$\xi > 0.004. \quad (3.40)$$

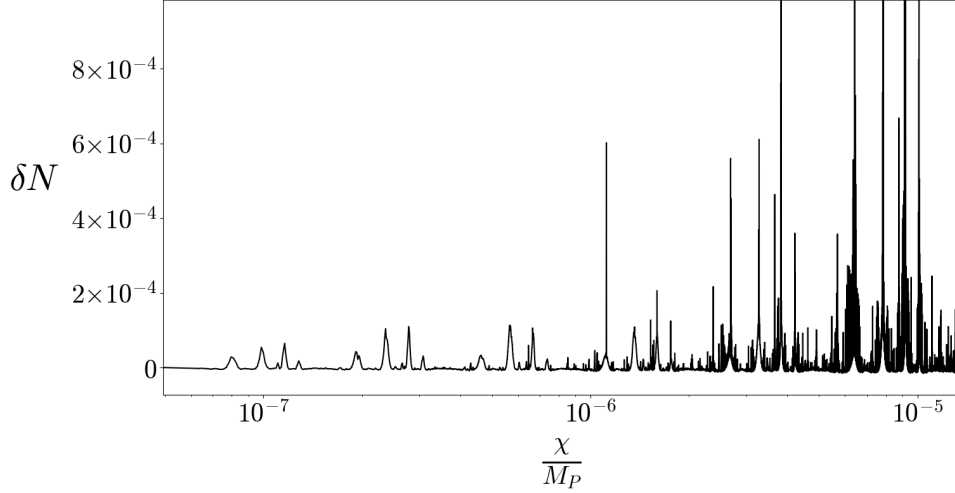


Figure 3.3: delta N for a range of initial χ values for the massless preheating model with parameters $g^2/\lambda = 2$. The delta N for each χ are picked at a uniform energy density slice corresponding to a fixed H_* given by $\ln(M_P/H_*) = 28$. The initial ϕ value is $3.5M_P$.

3.4 Non-perturbative delta N formalism

The perturbative delta N formalism for light spectator field introduced in section 2.4 involves expanding N in terms of the perturbations in the field,

$$\delta N(\chi) = N_{,\chi} \delta\chi + \frac{1}{2!} N_{,\chi\chi} (\delta\chi)^2 + \dots \quad (3.41)$$

However, there exist scenarios where the perturbation in fields is significant. Such cases arise for example in the massless preheating model.

Defining an initial flat slice with a uniform scale factor a_{ini} , the function $\delta N = N(\chi) - \bar{N}$ can be simulated using

$$N(\chi) = \zeta(\chi) = \ln \left(\frac{a(\rho_{\text{ref}}, \chi)}{a_{\text{ini}}} \right) \quad (3.42)$$

where for different field values of χ we evaluate the value of a at a slicing where ρ , the energy density is constant $= \rho_{\text{ref}}$. Recall that the delta N formalism yields the curvature perturbation going from a flat slicing to a slicing where the energy density is uniform. In particular, the final slice is based on energy density and not time. Hence a uniform energy density slice is achieved at different scale factors corresponding to the value of spectator field χ .

For the massless preheating model (3.4), I obtain a plot of the function $N(\chi)$ using lattice simulations as shown in Figure 3.3. The details of the lattice simulations will be covered later in section 3.6. The important feature to note here is that the plot matches ref. [76], which was the first to find the spikes in delta N arising from preheating simulations.

From the perturbative delta N formalism we expect the first term to dominate, $N = N_{,\chi} \delta\chi$.

However as can be seen from the simulation results for massless preheating case in Figure 3.3, δN constitutes a spiky pattern. Therefore the perturbative delta N formalism clearly fails. In general, the perturbative method will fail if the function $N(\chi)$ is highly non-linear because higher order terms in Taylor expansion also need to be included. This is the case for the massless preheating model and would remain the case also for the NMC preheating model (3.30) presented in this work.

An improvement to the perturbative delta N formalism is the *non-perturbative delta N formalism* [78]. It places the curvature correlator as the centrepiece instead of the function N , because observables such as the non-Gaussianity parameter f_{NL} depend on the former. The curvature correlator is defined through the joint probability distribution of $\chi(x)$ as

$$\langle \zeta_1 \zeta_2 \dots \rangle = \int d\chi_1 d\chi_2 \dots \times (N_1 - \bar{N})(N_2 - \bar{N}) \dots \times \mathcal{P}(\chi_1, \chi_2, \dots) \quad (3.43)$$

where the subscripts denote different spatial points $x_1, x_2, \dots$. We know that the distribution of χ fields which are present in the early universe is almost Gaussian. Using this fact, the joint probability distribution of the fields can be expanded around a multivariate Gaussian distribution [105]. Such an expansion is called a Gauss-Hermite expansion. Upto third order it is given by [106]:

$$\mathcal{P}(\nu) = \left(1 + \frac{1}{3!} S_3 \sigma H_3(\nu) + \frac{1}{4!} S_4 \sigma^2 H_4(\nu) + \frac{10}{6!} S_3^2 \sigma^2 H_6(\nu) \right) \mathcal{P}_G(\nu) \quad (3.44)$$

where $S_\ell \equiv M_\ell / \sigma^{2\ell-2}$ and $M_2 = \mu_2 = \sigma^2$, $M_3 = \mu_3$, $M_4 = \mu_4 - 3\sigma^4$, $M_5 = \mu_5 - 10\sigma^2 M_3$, $M_6 = \mu_6 - 15\sigma^2 M_4 - 10M_3^2 - 15\sigma^6$. Here, H_n are the nth order Hermite polynomials given by the Rodriguez formula. μ_n are the nth order moments. In case of the spectator field χ , the 3rd and 4th moments denote its intrinsic bispectrum and trispectrum respectively. We are assuming that the spectator field is Gaussian, therefore these intrinsic bispectrum and trispectrum can be set to zero and one is left with just the Gaussian joint probability distribution function

$$\mathcal{P}_G(\vec{\chi}) = \frac{1}{\sqrt{\det(2\pi\Sigma_{ij})}} \exp\left(-\frac{\delta\chi_i \Sigma_{ij}^{-1} \delta\chi_j}{2}\right) \quad (3.45)$$

where Σ_{ij} is the covariance matrix

$$\Sigma_{ij} = \langle \delta\chi_i \delta\chi_j \rangle. \quad (3.46)$$

The next step in the non-perturbative delta N formalism is to further expand the joint probability distribution in terms of the field covariance [78]. See Appendix B.1 for details of this calculation.

Then in terms of the non-perturbative coefficients defined as

$$\begin{aligned}
\tilde{N}_\chi &= \frac{1}{\langle \delta\chi^2 \rangle} \int d\chi \mathcal{P}_G(\chi) \delta\chi \delta N(\chi), \\
\tilde{N}_{\chi\chi} &= \frac{1}{\langle \delta\chi^2 \rangle^2} \int d\chi \mathcal{P}_G(\chi) \delta\chi^2 \delta N(\chi), \\
\tilde{N}_{\chi\chi\chi} &= \frac{1}{\langle \delta\chi^2 \rangle^3} \int d\chi \mathcal{P}_G(\chi) \delta\chi^3 \delta N(\chi), \\
\tilde{N}_{\chi\chi\chi\chi} &= \frac{1}{\langle \delta\chi^2 \rangle^4} \int d\chi \mathcal{P}_G(\chi) \delta\chi^4 \delta N(\chi), \quad \text{etc.}
\end{aligned} \tag{3.47}$$

where $\mathcal{P}_G(\chi)$ is the Gaussian one-point probability distribution (2.66) which is determined by its mean $\bar{\chi}$ and variance $\langle \delta\chi^2 \rangle$, the two point curvature correlator for the spectator field can be expanded as

$$\langle \zeta_\chi(\vec{x}_1) \zeta_\chi(\vec{x}_2) \rangle = \tilde{N}_\chi^2 \Sigma_{12} + \frac{1}{2} \tilde{N}_{\chi\chi}^2 \Sigma_{12}^2 + \frac{1}{4} \tilde{N}_{\chi\chi\chi}^2 \Sigma_{12}^3 + \frac{1}{8} \tilde{N}_{\chi\chi\chi\chi}^2 \Sigma_{12}^4 + \text{Order}(\Sigma_{12}^5), \tag{3.48}$$

while the three point curvature correlator for the spectator field can be expanded as

$$\begin{aligned}
\langle \zeta_\chi(\vec{x}_1) \zeta_\chi(\vec{x}_2) \zeta_\chi(\vec{x}_3) \rangle = & \\
& \tilde{N}_\chi \tilde{N}_{\chi\chi} \tilde{N}_\chi (\Sigma_{12} \Sigma_{23} + \text{perms}) + \\
& \frac{1}{2} \tilde{N}_{\chi\chi} \tilde{N}_{\chi\chi\chi} \tilde{N}_\chi (\Sigma_{12}^2 \Sigma_{23} + \text{perms}) + \tilde{N}_{\chi\chi} \tilde{N}_{\chi\chi} \tilde{N}_{\chi\chi} (\Sigma_{12} \Sigma_{23} \Sigma_{31}) + \\
& \frac{1}{4} \tilde{N}_{\chi\chi\chi} \tilde{N}_{\chi\chi\chi\chi} \tilde{N}_\chi (\Sigma_{12}^3 \Sigma_{23} + \text{perms}) + \frac{1}{2} \tilde{N}_{\chi\chi\chi} \tilde{N}_{\chi\chi\chi} \tilde{N}_{\chi\chi} (\Sigma_{12}^2 \Sigma_{23} \Sigma_{31} + \text{perms}) + \\
& \frac{1}{4} \tilde{N}_{\chi\chi} \tilde{N}_{\chi\chi\chi\chi} \tilde{N}_{\chi\chi} (\Sigma_{12}^2 \Sigma_{23}^2 + \text{perms}) + \text{Order}(\Sigma_{ij}^5).
\end{aligned} \tag{3.49}$$

This non-perturbative expansion is valid if $\Sigma_{ij} \ll \langle \delta\chi^2 \rangle$.

In ref. [78] the non-perturbative delta N expansion (3.48) and (3.49) were truncated at first order in covariance to yield power spectrum

$$P_\zeta^X(k) = \tilde{N}_\chi^2 \Sigma(k), \tag{3.50}$$

and the bispectrum

$$B_\zeta^X(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \tilde{N}_\chi^2 \tilde{N}_{\chi\chi} (\Sigma(k_1) \Sigma(k_2) + \Sigma(k_1) \Sigma(k_3) + \Sigma(k_2) \Sigma(k_3)), \tag{3.51}$$

which can be seen to be analogous to eq. (2.71), with the substitutions $\delta N_{,\chi} \rightarrow \tilde{N}_\chi$ and $\delta N_{,\chi\chi} \rightarrow \tilde{N}_{\chi\chi}$. Therefore the value of the non-Gaussianity parameter f_{NL} also has the same form,

$$f_{\text{NL}} = -\frac{5}{6} \tilde{N}_\chi^2 \tilde{N}_{\chi\chi} \frac{\Sigma(k)^2}{P_\zeta(k)^2}. \tag{3.52}$$

However, as in the perturbative case, this is not necessarily the dominant term, especially if $\tilde{N}_\chi$ is

small. Therefore we go to the next order of the expansion in powers of $\Sigma(k)$,

$$P_\zeta^\chi(k_1) = \tilde{N}_\chi^2 \Sigma(k_1) + \tilde{N}_{\chi\chi}^2 \int d\vec{q} \Sigma(q) \Sigma(|\vec{k}_1 - \vec{q}|) \quad (3.53)$$

$$\begin{aligned} B_\zeta^\chi(\vec{k}_1, \vec{k}_2, \vec{k}_3) &= (\Sigma(k_1)\Sigma(k_2) + \text{perms}) \left(\tilde{N}_\chi \tilde{N}_{\chi\chi} \tilde{N}_\chi \right) \\ &+ \left(\int d\vec{q} \Sigma(|\vec{k}_1 - \vec{q}|) \Sigma(q) \Sigma(k_2) + \text{perms} \right) \left(\frac{1}{2} \tilde{N}_{\chi\chi\chi} \tilde{N}_\chi \tilde{N}_{\chi\chi} - 2\langle\delta\chi^2\rangle^{-1} \tilde{N}_{\chi\chi} \tilde{N}_\chi^2 \right) \\ &+ \left(\int d\vec{q} \Sigma(|\vec{q} - \vec{k}_1|) \Sigma(q) \Sigma(|\vec{k}_3 + \vec{q}|) d\vec{q} \right) \left(\tilde{N}_{\chi\chi}^3 - 12\langle\delta\chi^2\rangle^{-1} \tilde{N}_\chi^2 \tilde{N}_{\chi\chi} \right), \end{aligned} \quad (3.54)$$

where we have appropriately introduced the convolution integrals while performing the Fourier transform. Considering the case $\tilde{N}_\chi \rightarrow 0$, we therefore obtain for the equilateral limit $|\vec{k}_1| = |\vec{k}_2| = |\vec{k}_3| = k$ the result

$$f_{\text{NL}} = -\frac{5}{6} \frac{\tilde{N}_{\chi\chi}^3}{P_\zeta(k)^2} \int \frac{d^3q}{(2\pi)^3} \Sigma(q) \Sigma(|\vec{q} - \vec{k}_1|) \Sigma(|\vec{q} + \vec{k}_3|), \quad (3.55)$$

which is again analogous to the earlier perturbative result eq. (2.75) but is valid even when the Taylor expansion (2.70) is not.

3.5 Initial conditions including scale dependence

In order to apply the non-perturbative delta N formalism, we need to know the statistics of the χ field at the initial time slice a_{ini} defined in eq. (3.42). Because the field is weakly coupled, its evolution during inflation can be described using linear theory in momentum space.

To summarise the separate universe picture for the light spectator field introduced in section 2.4, here it is convenient to divide the comoving momenta k in three separate ranges:

Modes with $k < a_0 H_0$, where a_0 and H_0 are the scale factor and Hubble rate today, have wavelength longer than the size of the currently observable universe. Therefore from the perspective of any observation, they appear as a uniform, non-zero mean value, which we denote as $\bar{\chi}$. Because of this random origin, the precise value of $\bar{\chi}$ cannot be computed, but if we know the whole inflationary history, we can compute its variance $\langle\bar{\chi}^2\rangle$, which is often referred to as the “cosmic variance”.

Modes with $a_0 H_0 < k < a_{\text{ini}} H_{\text{ini}}$, where H_{ini} is the Hubble rate at the initial slice, have a wavelength that is shorter than the size of the observable universe today but longer than the size of a single “separate universe”. Therefore these modes give rise to a different initial value χ_{ini} in each separate universe and are responsible for the produced curvature perturbation. More precisely, in the context of the non-perturbative delta N calculation, they determine the field correlation function $\Sigma(x)$ and the one-point probability distribution (2.66), which depends on the mean $\bar{\chi}$ and the variance $\langle\delta\chi^2\rangle$.

Modes with $k > a_{\text{ini}} H_{\text{ini}}$, where H_{ini} is the Hubble rate at the initial slice, have a wavelength shorter than the size of a single “separate universe”. Therefore they are statistically homogeneous

on the observational scales and do not contribute directly to the curvature perturbation. However, they play an important role in the lattice calculation and are discussed in the next section 3.6.

At linear order, a mode χ_k with comoving momentum k satisfies the mode equation

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \frac{k^2}{a^2}\chi_k + m_\chi^2(\phi)\chi_k = 0, \quad (3.56)$$

where the effective ϕ -dependent mass of the χ field is

$$m_\chi^2(\phi) \equiv g^2 \left(\frac{M_P}{\sqrt{\xi}} \tanh \left(\frac{\sqrt{\xi}}{M_P} \phi \right) \right)^2. \quad (3.57)$$

Because

$$\frac{m_\chi^2}{H^2} = 12\xi \frac{g^2}{\lambda} \tanh \left(\frac{\sqrt{\xi}}{M_P} \phi \right)^{-2} = 12 \frac{g^2 \xi}{\lambda} \left(1 + \frac{2}{16\xi \mathcal{N} + \sqrt{1 + 32\xi} - 1} \right), \quad (3.58)$$

one can see that assuming $g^2 \xi / \lambda < 1$, the χ field is light early on during inflation, and its mass increases gradually relative to the Hubble rate during inflation.

For convenience, we choose the initial time slice a_{ini} in (3.42) to correspond to the moment when

$$\frac{m_\chi^2(\phi_{\text{ini}})}{H^2(\phi_{\text{ini}})} = \frac{9}{4}, \quad (3.59)$$

which gives

$$\mathcal{N}_{\text{ini}} = \frac{1}{16\xi} \left(\frac{3 + 16g^2 \xi / \lambda}{3 - 16g^2 \xi / \lambda} - \sqrt{1 + 32\xi} \right). \quad (3.60)$$

Because the field is light, one can approximate that when the comoving mode χ_k exists the horizon at time t_k , its amplitude is

$$\mathcal{P}_\chi(k; t_k) = \frac{H(t_k)^2}{4\pi^2}. \quad (3.61)$$

Afterwards, its evolution is overdamped and its amplitude decays with rate

$$\gamma(t) = \frac{3H(t)}{2} \left[1 - \sqrt{1 - \frac{4m_\chi^2(t)}{9H(t)^2}} \right]. \quad (3.62)$$

At the time slice t_{ini} , the amplitude is therefore

$$\mathcal{P}_\chi(k; t_{\text{ini}}) = \frac{H(\mathcal{N}_k)^2}{4\pi^2} e^{-3F(\mathcal{N}_k, \mathcal{N}_{\text{ini}})}, \quad (3.63)$$

where $\mathcal{N}_k$ and $\mathcal{N}_{\text{ini}}$ denote the number of e-foldings at times t_k and t_{ini} , respectively, and

$$F(\mathcal{N}_k, \mathcal{N}) = \frac{2}{3} \int_{\mathcal{N}}^{\mathcal{N}_k} \frac{\gamma}{H} d\mathcal{N}' = \int_{\mathcal{N}}^{\mathcal{N}_k} \left(\sqrt{1 - \frac{16g^2}{3\lambda} \xi \left(\frac{16\xi\mathcal{N}' + \sqrt{1+32\xi} + 1}{16\xi\mathcal{N}' + \sqrt{1+32\xi} - 1} \right)} - 1 \right) d\mathcal{N}', \quad (3.64)$$

which for $\mathcal{N}_{\text{ini}}$ in eq. (3.60) is given by

$$\begin{aligned} F(\mathcal{N}_k, \mathcal{N}_{\text{ini}}) = & \mathcal{N}_k - \frac{1}{16\xi} \left(\frac{3 + 16g^2\xi/\lambda}{3 - 16g^2\xi/\lambda} - \sqrt{1 + 32\xi} \right) \\ & - \frac{16\xi\mathcal{N}_k + \sqrt{1 + 32\xi} - 1}{48\xi} \sqrt{9 - 48\frac{g^2}{\lambda} \xi \left(\frac{16\xi\mathcal{N}_k + \sqrt{1 + 32\xi} + 1}{16\xi\mathcal{N}_k + \sqrt{1 + 32\xi} - 1} \right)} \\ & + \frac{2g^2/\lambda}{\sqrt{9 - 48g^2\xi/\lambda}} \tanh^{-1} \left(\sqrt{\frac{3\lambda - 16g^2\xi \left(\frac{16\xi\mathcal{N}_k + \sqrt{1 + 32\xi} + 1}{16\xi\mathcal{N}_k + \sqrt{1 + 32\xi} - 1} \right)}{3\lambda - 16g^2\xi}} \right). \end{aligned} \quad (3.65)$$

This implies the power spectrum (3.63) is scale dependent through the function $F(\mathcal{N}_k)$. Therefore we need to go beyond the scale-invariant approximation used in other works refs. [81, 100].

In order to compute f_{NL} , we need the two-point correlator $\Sigma(k)$, which is given by eq. (2.69), as well as the one-point probability distribution $p(\chi)$, which is given by eq. (2.66) and depends on the variance $\langle \delta\chi^2 \rangle$ and the mean value $\bar{\chi}$. As discussed earlier, the latter is a random variable whose variance $\langle \bar{\chi}^2 \rangle$ can be computed.

The two variances are both given by a similar integral (2.33), with a different integration range,

$$\langle \bar{\chi}^2 \rangle = \int_0^{a_0 H_0} \mathcal{P}_\chi(k; t_{\text{ini}}) \frac{dk}{k} = \int_{\mathcal{N}_*}^{\infty} \frac{H(\mathcal{N}_k)^2}{4\pi^2} e^{-3F(\mathcal{N}_k, \mathcal{N}_{\text{ini}})} \left(\frac{1}{H(\mathcal{N}_k)} \frac{dH(\mathcal{N}_k)}{d\mathcal{N}_k} - 1 \right) d\mathcal{N}_k, \quad (3.66)$$

$$\langle \delta\chi^2 \rangle = \int_{a_0 H_0}^{a_{\text{ini}} H_{\text{ini}}} \mathcal{P}_\chi(k; t_{\text{ini}}) \frac{dk}{k} = \int_{\mathcal{N}_{\text{ini}}}^{\mathcal{N}_*} \frac{H(\mathcal{N}_k)^2}{4\pi^2} e^{-3F(\mathcal{N}_k, \mathcal{N}_{\text{ini}})} \left(\frac{1}{H(\mathcal{N}_k)} \frac{dH(\mathcal{N}_k)}{d\mathcal{N}_k} - 1 \right) d\mathcal{N}_k, \quad (3.67)$$

where the latter forms are obtained by changing the integration variable to $\mathcal{N}_k$, the value of $\mathcal{N}$ at the time when mode k left the horizon,

$$\frac{dk}{k} = \left(\frac{1}{H(\mathcal{N}_k)} \frac{dH(\mathcal{N}_k)}{d\mathcal{N}_k} - 1 \right) d\mathcal{N}_k, \quad (3.68)$$

with $H(\mathcal{N}_k)$ given by eq. (3.35).

Throughout this work we consistently choose the number of efolds at which the currently observable scale left the horizon to be $\mathcal{N}_* = 55$.

The cosmic variance integral eq. (3.66) at large $\mathcal{N}_k$ goes as

$$\langle \bar{\chi}^2 \rangle \sim \int^\infty \exp \left(- \left(1 - \sqrt{1 - \frac{16}{3} \xi \frac{g^2}{\lambda}} \right) \mathcal{N}_k \right) d\mathcal{N}_k. \quad (3.69)$$

From which it can be deduced that for $\xi = 0$ the integral has no exponential suppression and hence

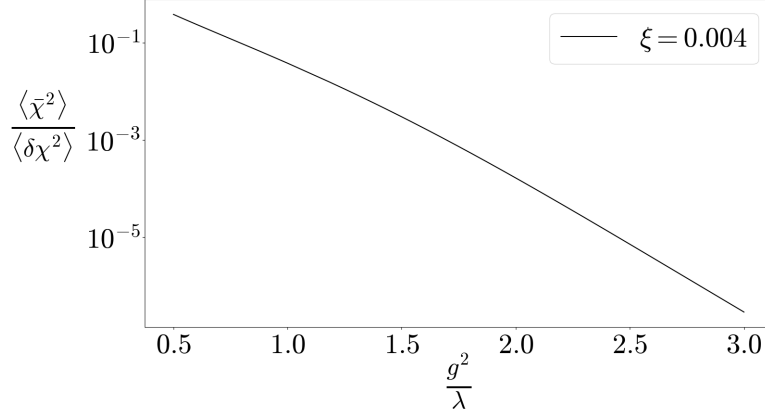


Figure 3.4: Ratio of the cosmic variance to the initial variance for $g^2/\lambda \in [0.5, 3]$ at fixed $\xi = 0.004$. Cosmic variance is two orders of magnitude less than the variance of the initial χ field in the range $1 < g^2/\lambda < 3$ which is relevant for parametric resonance [5]. For $g^2/\lambda > 3$, it is more than four orders less. It only becomes comparable to the initial variance at $g^2/\lambda < 0.5$ which is well outside the parametric resonance band. Figure reproduced from ref. [1].

cosmic variance for the minimal coupling case is infinite [75]. However, the introduction of even a small positive non-minimal coupling $\xi > 0$ gives rise to an exponential suppression and hence we expect cosmic variance to be negligible for the non-minimal coupling case. This is indeed what we find by numerically integrating for the cosmic and initial variances in eqs. (3.66) and (3.67), as summarised in Figure 3.4.

Whereas an infinite cosmic variance would imply that the mean $\bar{\chi}$ that appears in the probability density $p(\chi)$ can have any value and can therefore be considered to be a free parameter, a negligible cosmic variance forces us to choose $\bar{\chi} \approx 0$. Symmetry of the potential around zero dictates $\tilde{N}_\chi$ will be small which, as described in section 3.4, makes the dominant term in the non-perturbative delta N formalism to be eq. (3.55). Thus we illustrate the important role played by cosmic variance in calculating the non-Gaussianity from preheating. If cosmic variance is not negligible, as happens for the massless preheating toy model, then one needs to use eq. (3.52) as done in ref. [100]. However if it is negligible, as happens in the NMC preheating model, then one needs to use eq. (3.55) to calculate f_{NL} . The only qualification is for the potential to be symmetric in χ , which is true in most preheating scenarios.

3.6 Lattice simulations

Lattice field theory simulations involve solving the full equations of motion for the fields and Friedmann equation by discretising space on a grid and evolving the equations in discrete time-steps. Lattice simulations of preheating have shown that an observable level of non-Gaussianity can be generated, especially in the massless preheating model [74–76, 100]. However, this model is incompatible with tensor-to-scalar ratio measurements from Planck satellite [63]. In this work, we will address this issue by considering an observationally viable model of preheating with the inflaton ϕ

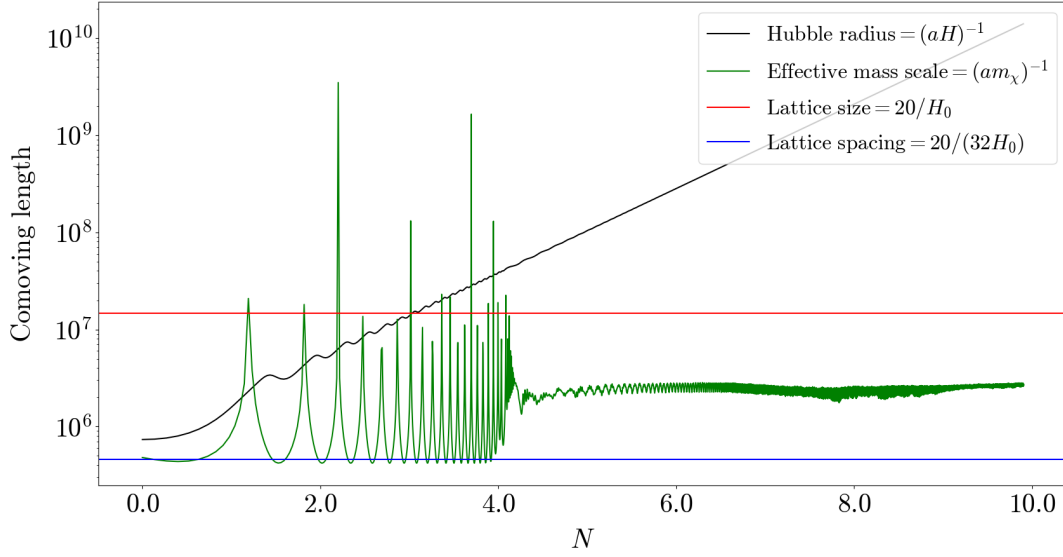


Figure 3.5: Evolution of the typical scales involved in lattice simulation with the number of e-folds during preheating

non-minimally coupled to gravity that decays into a massless spectator field χ , the NMC preheating model described in section 3.3.

I use the program HLattice [96] to perform fully non-linear simulations of preheating dynamics for the NMC preheating model (3.30). See Appendix B.2 for a brief description of the program and modifications to the publicly available code I make for the particular use case of NMC preheating model.

The fields are evolved in a linear fashion till the condition that the potential energy is equal to the kinetic energy is met. Thereafter non-linear simulations begin. Using the principles first outlined in ref. [84], we can study the semi-classical dynamics occurring during preheating by classically simulating an equivalent non-equilibrium, non-linear problem. In HLattice, this is achieved through capturing the quantum nature by initialising the fields with random Gaussian fluctuations for inhomogeneous modes.

The lattice size has to be chosen as a compromise between two factors: Ideally, one would desire the lattice to be large enough so that light cannot travel across the lattice during the nonlinear non-equilibrium stage of the evolution, but if it is larger than the comoving Hubble length, then one would need to include metric perturbations and the use of the FRW metric would not be justified. Therefore, I choose a comoving lattice with 32^3 grid points and length of a side equal to $20/H$, where H is the Hubble rate at the start of the non-linear evolution.

For this lattice size, Figure 3.5 shows that the comoving Hubble length is initially somewhat smaller than the lattice size but always lies above the lattice spacing. I choose the discretisation scheme LATTICEEASY for evolution in HLattice as any metric perturbations are not included.

I have checked that changing the lattice spacing from 32^3 to 16^3 grid points while keeping lattice size $20/H$ fixed as well as changing the lattice spacing from 32^3 to 64^3 grid points while changing

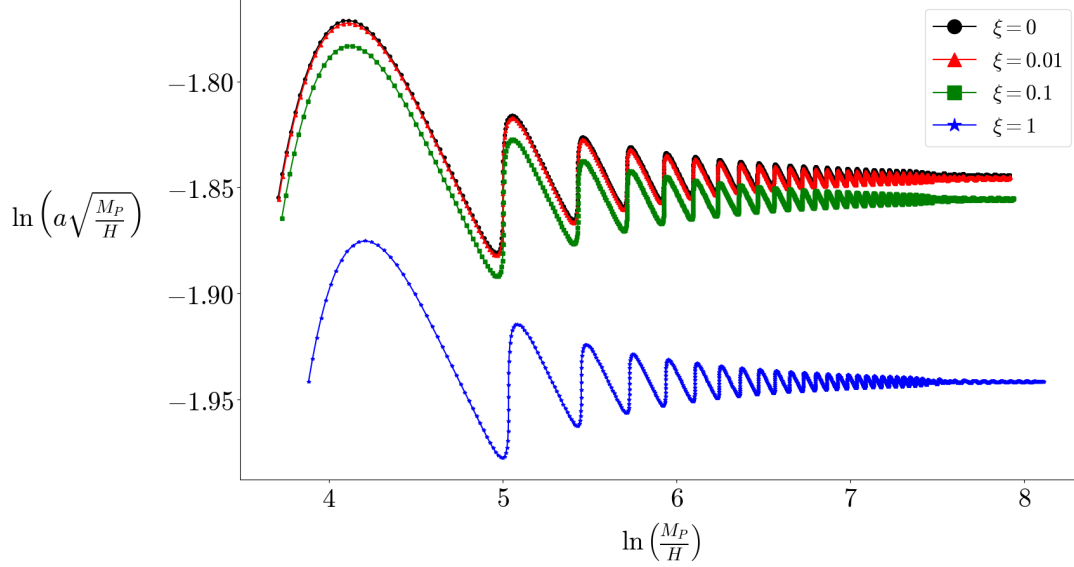


Figure 3.6: The deviation from radiation domination $a \propto \sqrt{H}$ for different values of ξ in the NMC preheating model. All the curves become flat around $\ln(M_P/H_*) = 7.5$ indicating the onset of radiation domination. We have chosen initial values $\phi_{\text{ini}} = 3.5M_P$ and $\chi_{\text{ini}} = 0$. Simulation begins at end of inflation when $\dot{\phi}^2 + \dot{\chi}^2 > V(\phi, \chi)$.

lattice size from $20/H$ to $40/H$ does not alter the simulation results significantly. Furthermore, the comoving mass scale $(am_\chi)^{-1}$ where $m_\chi^2 = g^2 \langle \tanh^2 \sqrt{\xi} \phi \rangle / \xi$ that is a representative of the field dynamics, lies between the lattice spacing and lattice size throughout the non-linear evolution. This can be seen from Figure 3.5.

Plotting the raw a versus H data obtained from HLattice in the form of deviation from radiation domination, Figure 3.6 shows that all curves become flat at late times, indicating onset of radiation domination. It also shows the change in asymptote as the non-minimal coupling ξ is changed indicating that the simulation can in principle be used to capture the NMC preheating model for a large range of non-minimal coupling values.

Similarly, plotting the deviation from radiation domination, Figure 3.7 shows the change in asymptote for two different values of χ_{ini} . Using Gaussian filtering in conformal time with standard deviation of 100 time steps, we can precisely pick the value of $N = \ln(a)$ at $H_{\text{ref}} = 5 \times 10^{-15} M_P$ which would correspond to a constant energy density ρ_{ref} to yield the N function (3.42). The two curves deviate due to field excursion in transverse direction [76], giving rise to a spike in the delta N function at $\chi_{\text{ini}} = 1.581 \times 10^{-6} M_P$ as compared to $\chi_{\text{ini}} = 0$.

For a specific value of g^2/λ and ξ in the NMC preheating model, I draw $N_{\text{runs}} = 10,000$ initial field values χ_i , $i \in \{1, \dots, N_{\text{runs}}\}$, from the probability distribution (2.66) with zero mean, i.e., $\bar{\chi} = 0$, and with $\langle \delta\chi^2 \rangle$ given by eq. (3.67). A lattice simulation is run and the value of δN is extracted for each of the N_{runs} values separately. Figure 3.8 compares the δN functions obtained for three different values of the non-minimal coupling ξ .

Particularly relevant are the values $\xi = 0$ which corresponds to the minimal coupling case

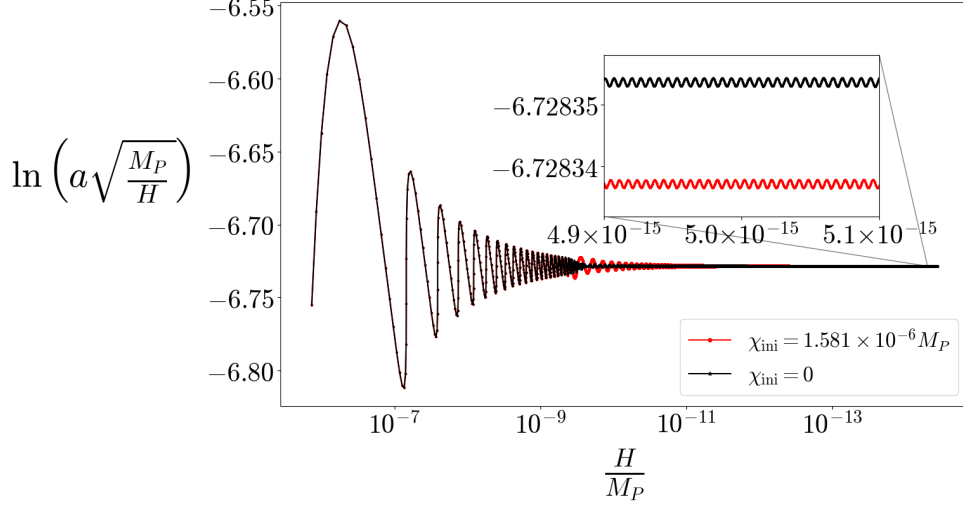


Figure 3.7: The deviation from radiation domination $a \propto 1/\sqrt{H}$ at two χ_{ini} values for $g^2/\lambda = 2$ and $\xi = 0.004$. $\chi_{\text{ini}} = 1.581 \times 10^{-6} M_P$ corresponds to spike in the delta N function. Inset shows zoomed in data around $H_{\text{ref}} = 5 \times 10^{-15} M_P$ where we pick the value of $N = \ln(a)$ after filtering. Figure reproduced from ref. [1].

ξ	λ	ϕ_{ini}	Initial Variance	Cosmic Variance
0	1.769×10^{-13}	$3.266 M_P$	$1.044 \times 10^{-12} M_P^2$	∞
0.004	4.946×10^{-13}	$3.314 M_P$	$4.176 \times 10^{-13} M_P^2$	$2.205 \times 10^{-15} M_P^2$
0.007	7.334×10^{-13}	$3.351 M_P$	$3.481 \times 10^{-13} M_P^2$	$5.757 \times 10^{-17} M_P^2$

Table 3.1: Numerical values of quantities to be specified in simulations for different ξ but all with the same $g^2/\lambda = 2$. Table reproduced from ref. [1].

and $\xi = 0.004$ which is the lower bound from observations eq. (3.40). The same inhomogeneous modes are used to initialise simulation for all χ_{ini} . We expect the uncertainty arising from different random realisations of the inhomogeneous modes to be largely uncorrelated between different Hubble volumes, making its effect subdominant to the statistical error in sampling of χ_{ini} . As a quantitative check I carry out 32 runs with different random realisations of the inhomogeneous modes for $g^2/\lambda = 2$ and $\xi = 0.004$ at $\chi_{\text{ini}} \approx 1.581 \times 10^{-6} M_P$, $1.203 \times 10^{-6} M_P$ and $0 M_P$. I find the standard deviation in δN due to this effect is approximately 40% of the spike heights in Figure 3.8. Therefore the spikes are a genuine feature of the simulations and this effect is subdominant.

Drawing initial χ values from (2.66) amounts to importance sampling the probability distribution, and therefore the non-perturbative coefficients $\tilde{N}_\chi$ and $\tilde{N}_{\chi\chi}$ defined in eqs. (3.47) can be obtained as averages over the runs,

$$\begin{aligned}
\tilde{N}_\chi &= \frac{1}{N_{\text{runs}}} \sum_i (\chi_i - \bar{\chi}) (N(\chi_i) - \bar{N}), \\
\tilde{N}_{\chi\chi} &= \frac{1}{N_{\text{runs}}} \sum_i (\chi_i - \bar{\chi})^2 (N(\chi_i) - \bar{N}).
\end{aligned} \tag{3.70}$$

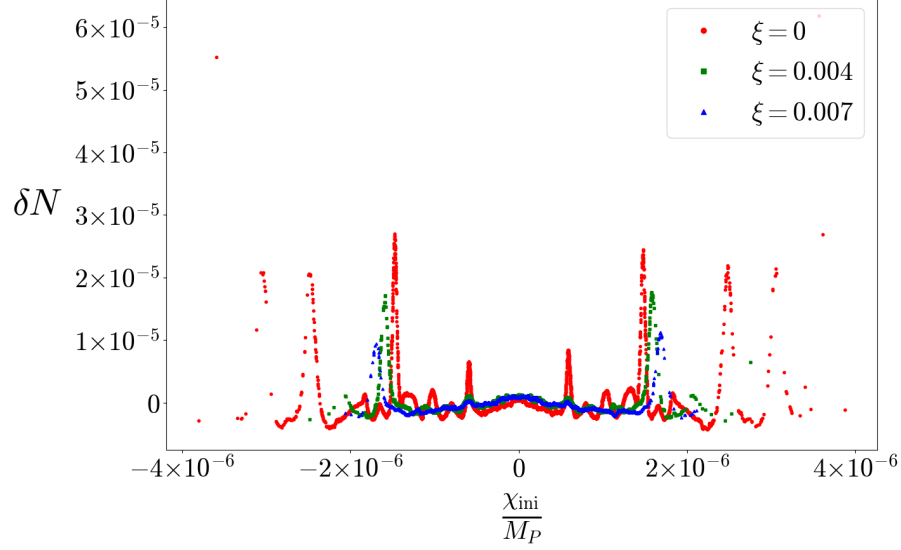


Figure 3.8: δN at $H_{\text{ref}} = 5 \times 10^{-15} M_P$ for $\xi = 0, 0.004, 0.007$ and $g^2/\lambda = 2$. The relevant initial quantities used to plot δN are summarised in Table 3.1. The plot is not exactly symmetric under $\chi \rightarrow -\chi$ due to same random realisation of inhomogeneous modes being used for each run, which breaks the symmetry. Figure reproduced from ref. [1].

Even though the cosmic variance $\langle \bar{\chi}^2 \rangle$ is small, it is not exactly zero, and therefore it is important to consider the full range of possible values of $\bar{\chi}$. Because the relevant values are small, we can achieve this without carrying out new simulations by re-weighting the data: For a general small value of $\bar{\chi}$, the coefficients (3.47) are given by

$$\begin{aligned} \tilde{N}_\chi &= \frac{1}{N_{\text{runs}}} \sum_i \exp \left(-\frac{\bar{\chi}^2 - 2\bar{\chi}\chi_i}{2\langle \delta\chi^2 \rangle} \right) (\chi_i - \bar{\chi}) (N(\chi_i) - \bar{N}), \\ \tilde{N}_{\chi\chi} &= \frac{1}{N_{\text{runs}}} \sum_i \exp \left(-\frac{\bar{\chi}^2 - 2\bar{\chi}\chi_i}{2\langle \delta\chi^2 \rangle} \right) (\chi_i - \bar{\chi})^2 (N(\chi_i) - \bar{N}). \end{aligned} \quad (3.71)$$

Figure 3.9 shows the non-perturbative coefficient $\tilde{N}_{\chi\chi}$ over the range of $\bar{\chi}$ allowed by the cosmic variance. The value of $\tilde{N}_{\chi\chi}$ and error bar for each $\bar{\chi}$ is obtained by using the bootstrap (resampling with replacement) method. For a brief summary of this method, see appendix B.3. We use the symmetry $\chi \rightarrow -\chi$ to double the number of sample points $\delta N(-\chi)$ used in calculating $\tilde{N}_{\chi\chi}$.

For the final value of $\tilde{N}_{\chi\chi}$ and its standard deviation that goes into the f_{NL} calculation (3.55), we take an average over the values and error bars for each $\bar{\chi}$. Figure 3.10 shows that the Monte Carlo treatment we have used gives convergent estimates as the sample size is increased. This provides a further check of the robustness of the calculation.

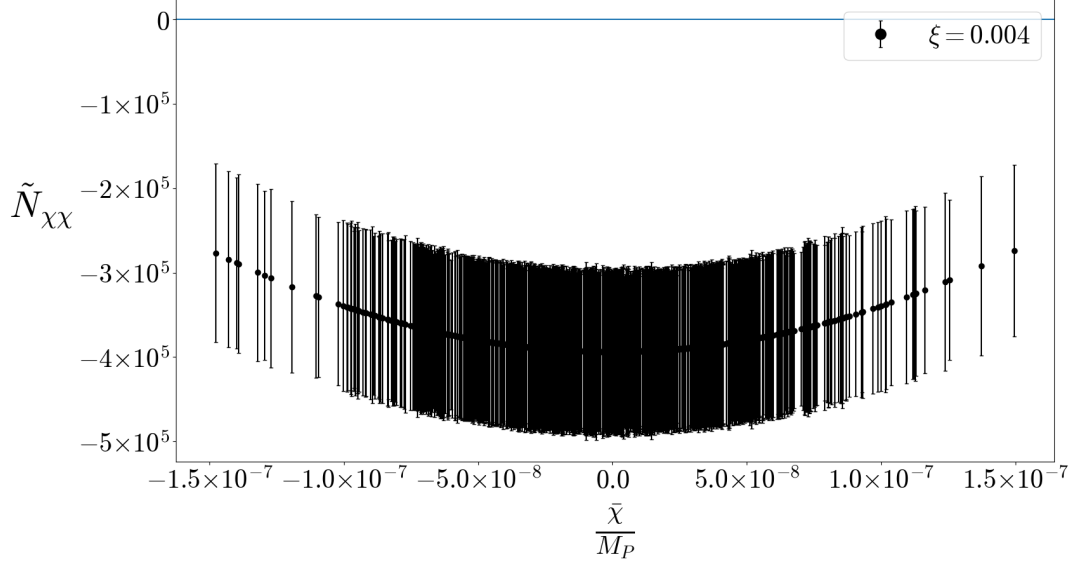


Figure 3.9: $\tilde{N}_{\chi\chi}$ for 1000 $\bar{\chi}$ drawn from a Gaussian distribution with mean equal to zero and variance equal to cosmic variance $\langle \bar{\chi}^2 \rangle$. The model parameters used are $g^2/\lambda = 2$ and $\xi = 0.004$. The final value and error bar for each $\bar{\chi}$ are obtained by bootstrapping 100 resamples. Figure reproduced from ref. [1].

3.7 Non-Gaussianity including scale dependence

In ref. [81], the authors assume a scale-invariant power spectrum for the field χ ,

$$\Sigma(k) = \frac{2\pi^2}{k^3} \frac{H_*^2}{4\pi^2}, \quad (3.72)$$

where H_* is the Hubble rate at the time when the current observational scale left the horizon. With that assumption, the non-Gaussianity parameter f_{NL} given by eq. (3.55) at the observational scale k_* becomes approximately

$$f_{\text{NL}} \approx -\frac{5}{6} \frac{\tilde{N}_{\chi\chi}^3 \mathcal{P}_\chi^3}{\mathcal{P}_*^2} \int_{L^{-1}}^{k_*} \frac{dq}{q} = -\frac{5}{6} \frac{\tilde{N}_{\chi\chi}^3 \mathcal{P}_\chi^3}{\mathcal{P}_*^2} \ln(k_* L), \quad (3.73)$$

with an infrared cut-off L^{-1} corresponding to the size of the observable universe. Therefore its k -dependence appears logarithmic.

However, in practice the field χ will not be exactly scale invariant because of its mass and the time-dependence of the Hubble rate during inflation. To capture the effect of the scale dependence, we approximate the field variance with a power law,

$$\Sigma(k) \approx 2\pi^2 \frac{\mathcal{A}_\chi}{k^{3-n_\chi}}, \quad (3.74)$$

where the spectral index $n_\chi (> 0)$ and the amplitude $\mathcal{A}_\chi$, which has energy dimensions $2-n_\chi$, depend on the parameters of the specific inflation model. This form implies the real space correlator behaves

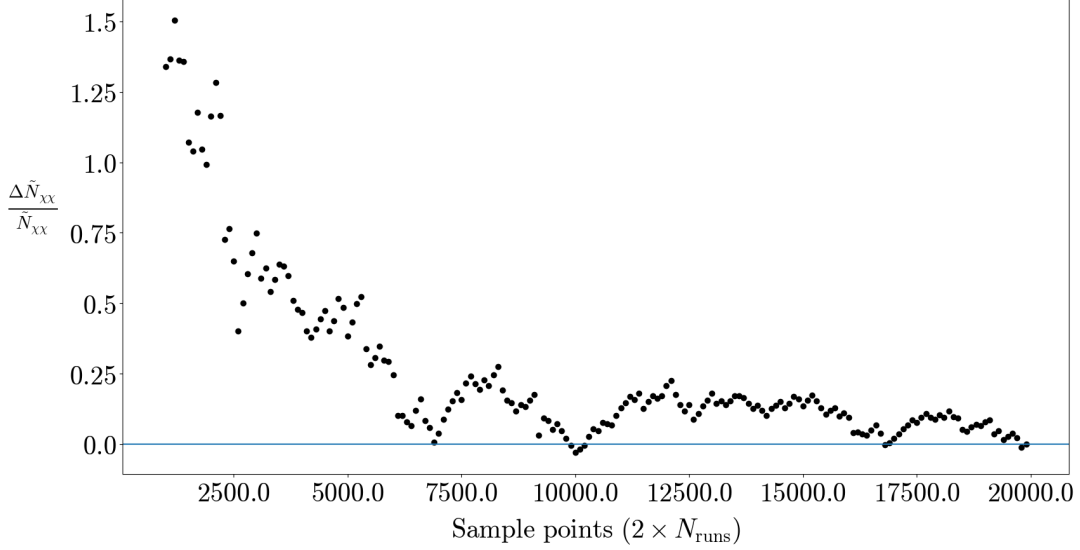


Figure 3.10: Relative deviation in $\tilde{N}_{\chi\chi}$ from its final value used for f_{NL} calculation as a function of the number of sample points. By making use of the symmetry $\chi \rightarrow -\chi$, the number of sample points are double the number of simulation runs.

on asymptotically long distances as $\Sigma_{ij} \sim |\vec{x}_i - \vec{x}_j|^{-n_\chi}$ and is therefore small on observable scales, justifying the non-perturbative delta N expansion (3.49). Then the integral in eq. (3.55) gives for the equilateral case,

$$f_{\text{NL}} \approx -\frac{5}{6} \frac{\tilde{N}_{\chi\chi}^3 \mathcal{A}_\chi^3 k_*^{2n_\chi}}{\mathcal{P}_*^2} \int_0^{k_*} \frac{dq}{q^{1-n_\chi}} = -\frac{5}{6} \frac{\tilde{N}_{\chi\chi}^3 \mathcal{A}_\chi^3 k_*^{3n_\chi}}{n_\chi \mathcal{P}_*^2}. \quad (3.75)$$

This shows that f_{NL} has power-law dependence on the observational scale k_* . Because cosmological observations are carried out on comoving scales that are many orders of magnitude larger than the Hubble scale during inflation, this leads to a significant suppression of f_{NL} if $n_\chi > 0$.

I shall now illustrate this scale dependence of f_{NL} using the NMC preheating model (3.30).

In a scale invariant treatment, f_{NL} is given by eq. (3.73). Substituting in typical values from NMC preheating model: $\xi = 0.004$, $\mathcal{P}_\chi \sim 10^{-11} M_P^2$. $\mathcal{P}_* = 2.1 \times 10^{-9}$ is known from Planck observations [4]. Taking L to be the length of the observable universe, one has $\ln(k_* L) \sim 1$. From lattice simulations, $\tilde{N}_{\chi\chi} \sim 10^6 M_P^{-2}$ for $g^2/\lambda = 2$ and $\xi = 0.004$. Putting these values together gives

$$f_{\text{NL}}^{\text{scale inv.}} \sim 1, \quad (3.76)$$

in the scale invariant approximation.

However, the correct f_{NL} shall be obtained by taking a scale-dependent estimate of the field variance. For NMC preheating, this essentially implies including the over-damped oscillator contribution $F(\mathcal{N})$ from eq. (3.63) in the variance. To obtain $\mathcal{N}$ in terms of k , we need to invert the

horizon crossing relation $k = a(\mathcal{N})H(\mathcal{N})$,

$$k = e^{-\mathcal{N}} H(\mathcal{N}) = e^{-\mathcal{N}} \left(\frac{M_P}{\xi} \sqrt{\frac{\lambda}{12}} \left(\frac{16\xi\mathcal{N} + \sqrt{1+32\xi} - 1}{16\xi\mathcal{N} + \sqrt{1+32\xi} + 1} \right) \right). \quad (3.77)$$

This is difficult to invert analytically. However, for large $\mathcal{N}$ the expression is dominated by the exponential factor. Therefore in order to have an analytical estimate, we use the approximation $H = \text{const}$ that allows us to capture the large $\mathcal{N}$ behaviour. Fixing H from eq. (3.35) to H_*

$$\sqrt{\frac{\lambda}{12}} \frac{M_P}{\xi} \frac{16\xi\mathcal{N}_k + \sqrt{1+32\xi} - 1}{16\xi\mathcal{N}_k + \sqrt{1+32\xi} + 1} = H_* \quad (3.78)$$

and identifying it in the scale-dependent power spectrum at initial time slice (3.63)

$$\mathcal{P}_\chi(k; t_{\text{ini}}) \approx \frac{H_*^2}{4\pi^2} e^{-3F(\mathcal{N}_k, \mathcal{N}_{\text{ini}})} \quad (3.79)$$

with the exponential damping coefficient (3.65) becoming

$$\begin{aligned} F(\mathcal{N}_k, \mathcal{N}_{\text{ini}}) \approx & \left(1 - \frac{1}{3} \sqrt{9 - 48 \frac{g^2}{\lambda} \sqrt{\frac{\lambda}{12}} \frac{M_P}{H_*}} \right) \mathcal{N}_k \\ & - \mathcal{N}_{\text{ini}} - \frac{\sqrt{32\xi+1}-1}{48\xi} \sqrt{9 - 48 \frac{g^2}{\lambda} \sqrt{\frac{\lambda}{12}} \frac{M_P}{H_*}} \\ & + \frac{2g^2/\lambda}{\sqrt{9 - 48g^2\xi/\lambda}} \tanh^{-1} \left(\sqrt{\frac{3\lambda - 16g^2 \sqrt{\frac{\lambda}{12}} \frac{M_P}{H_*}}{3\lambda - 16g^2\xi}} \right), \end{aligned} \quad (3.80)$$

the correlator takes the form

$$\Sigma(q) \approx 2\pi^2 \frac{\mathcal{A}_\chi}{q^{3-n_\chi}}, \quad (3.81)$$

where

$$n_\chi = 3 - \sqrt{9 - 48 \frac{g^2}{\lambda} \sqrt{\frac{\lambda}{12}} \frac{M_P}{H_*}}, \quad (3.82)$$

$$\mathcal{A}_\chi = \frac{H_*^{2-n_\chi}}{2} \exp \left(3\mathcal{N}_{\text{ini}} + \frac{\sqrt{32\xi+1}-1}{16\xi} (3 - n_\chi) - \frac{6g^2/\lambda}{\sqrt{9 - 48 \frac{g^2}{\lambda} \xi}} \tanh^{-1} \frac{3 - n_\chi}{\sqrt{9 - 48 \frac{g^2}{\lambda} \xi}} \right). \quad (3.83)$$

This form separates the momentum q dependence explicitly and is precisely the form used in eq. (3.74).

For typical value of parameters $g^2/\lambda = 2$ and $\xi = 0.004$ in the NMC preheating model, $n_\chi \sim 0.1$, $\mathcal{A}_\chi \sim 10^{-12} M_P^{2-n_\chi}$. $\tilde{N}_{\chi\chi} \sim 10^6 M_P^{-2}$ for $g^2/\lambda = 2$ and $\xi = 0.004$ from numerical simulations

remains the same. However there is a small subtlety in specifying the value of comoving scale k_* . We have fixed scale factor to be one at the end of inflation, $a_{\text{end}} = 1$. However the physical scale $k_{\text{phys}} = 0.05 \text{Mpc}^{-1}$ at which Planck satellite measurements take place assumes scale factor is one at present times, $a_0 = 1$. Therefore $k_* = (a_0/a_{\text{end}})k_{\text{phys}}$. Assuming instantaneous reheating [107],

$$\ln \left(\frac{a_0}{a_{\text{end}}} \right) = \frac{1}{4} \ln \left(\frac{30}{g_{\text{eff}} \pi^2} \right) + \frac{1}{3} \ln \left(\frac{11}{43} g_{\text{eff}} \right) + \ln \left(\frac{\rho_{\text{end}}^{1/4}}{T_0} \right) \approx 65, \quad (3.84)$$

with $g_{\text{eff}} \sim 100$, $T_0 = 2.725K$ and

$$\rho_{\text{end}} = \frac{3\lambda}{8} \left(\frac{M_P}{\sqrt{\xi}} \tanh \left(\frac{\sqrt{\xi}}{M_P} \phi_{\text{end}} \right) \right)^4. \quad (3.85)$$

Therefore $k_* \sim 2.225 \times 10^{-30} M_P$. Putting all the above values together, the calculation gives

$$f_{\text{NL}} \sim 10^{-9}. \quad (3.86)$$

This order of magnitude estimate for f_{NL} can be made more precise by calculating the momentum integrals involved in eq. (3.55) numerically. We split the domain of integration for the momentum q into $q < k_*$ and $q > k_*$. For $q < k_*$, the constant H_* approximation is valid and momentum integral can be obtained analytically as in eq. (3.75). For $q > k_*$, we invert $q(N)$ numerically to give $\Sigma^{\text{num}}(q)$ and momentum integral as

$$\int_{k_*}^{q(\mathcal{N}_{\text{ini}})} d\vec{q} \Sigma(q) \Sigma(|\vec{q} - \vec{k}_1|) \Sigma(|\vec{q} + \vec{k}_3|) \approx 4\pi \int_{k_*}^{q(\mathcal{N}_{\text{ini}})} q^2 dq (\Sigma^{\text{num}}(q))^3. \quad (3.87)$$

Combining the integrals for the above two domains yields the full integral from $q = 0$ to $q = q(\mathcal{N}_{\text{ini}})$. The remaining ingredient to be used in eq. (3.55) is the non-perturbative coefficient $\tilde{N}_{\chi\chi}$. From the lattice simulation results at different $\bar{\chi}$, Figure 3.9, we take the average $\tilde{N}_{\chi\chi}$ and average of the error bars to estimate f_{NL} and its error through standard propagation. For $g^2/\lambda = 2$, $\xi = 0.004$ I find

$$f_{\text{NL}} = (2.3 \pm 1.7) \times 10^{-9}, \quad (3.88)$$

which matches well the order of magnitude estimate above.

One can see that there is a significant decrease in the non-Gaussianity estimate if we perform a complete linearised calculation, rather than using the scale invariant approximation (3.72). Essentially, this occurs because the momentum integral in eq. (3.55) is power law suppressed in $k_* \sim 10^{-30} M_P$ which results in a very low overall factor in f_{NL} . Thus I illustrate that for non-Gaussianity calculation in any model of preheating, it is important to include scale-dependence during inflation to yield an accurate result.

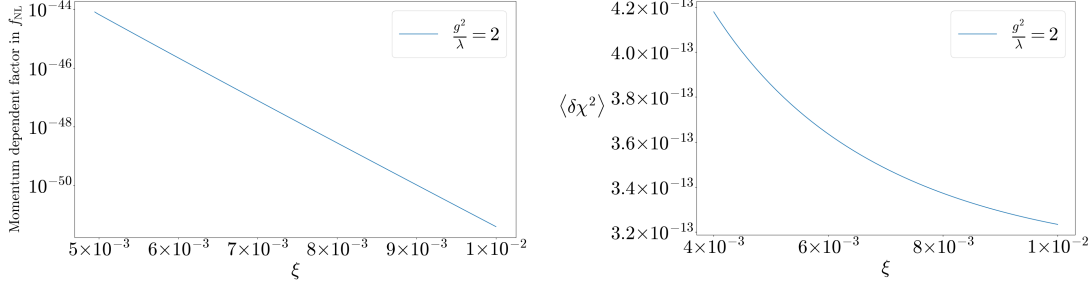


Figure 3.11: ξ dependence of the momentum integral (left) and initial variance (right). Figure reproduced from ref. [1].

3.8 Parameter dependence of f_{NL}

In this section, we study the change in f_{NL} with parameters ξ and g^2/λ in the NMC preheating model (3.30).

The ξ dependence of momentum integral occurs through the factor $\mathcal{A}_\chi^3 k_*^{3n_\chi}/n_\chi$ in eq. (3.75). Figure 3.11 shows that this factor drops down almost exponentially as ξ is increased from the lowest value of $\xi = 0.004$ allowed in the model. The non-perturbative coefficient $\tilde{N}_{\chi\chi}$ (3.47) captures the departure from a constant δN within a spread of χ_{ini} governed by the initial variance. Now as can be seen from Figure 3.8, the height of spikes reduces as ξ increases. Furthermore, the number of spikes decreases as the spread in χ_{ini} shrinks. Figure 3.11 also shows that the initial variance does not change significantly with ξ when compared to the change in momentum integral. Combining both the factors, we surmise f_{NL} will be maximum for lowest observational bound on ξ , $\xi = 0.004$ for a given value of g^2/λ .

We can therefore search for the highest value of f_{NL} in the model by keeping $\xi = 0.004$ fixed and scanning over different g^2/λ . Figure 3.12 shows that the maximum value of f_{NL} occurs around $g^2/\lambda = 1.625$. At exactly $g^2/\lambda = 1.625$ I find it to be

$$f_{\text{NL}} = -(6.3 \pm 0.7) \times 10^{-4}. \quad (3.89)$$

This value still lies well-within the current observation bound of $f_{\text{NL}}^{\text{local}} = -0.9 \pm 5.1$ [53], but it is several orders of magnitude larger than what we obtain for generic g^2/λ , e.g., eq. (3.88).

Based on a linearised analysis of the parametric resonance for the massless preheating case, one might expect the maximum value of f_{NL} to lie at $g^2/\lambda = 1.875$ [75]. However, our simulations indicate this is shifted to $g^2/\lambda \approx 1.625$.

To investigate this shift we compare the growth of zero mode with root mean square of the inhomogeneous fluctuations in Figure 3.13. We see that the zero mode for $g^2/\lambda = 1.875$ grows with a slightly higher slope than $g^2/\lambda = 1.625$ in accordance with parametric resonance considerations. However the major difference in evolution for the two coupling strengths arises from the non-linear behaviour of fluctuations. Figure 3.13 shows that after the evolution becomes non-linear, the fluctuations catch up significantly slower for the $g^2/\lambda = 1.625$ value, thus allowing for exponential

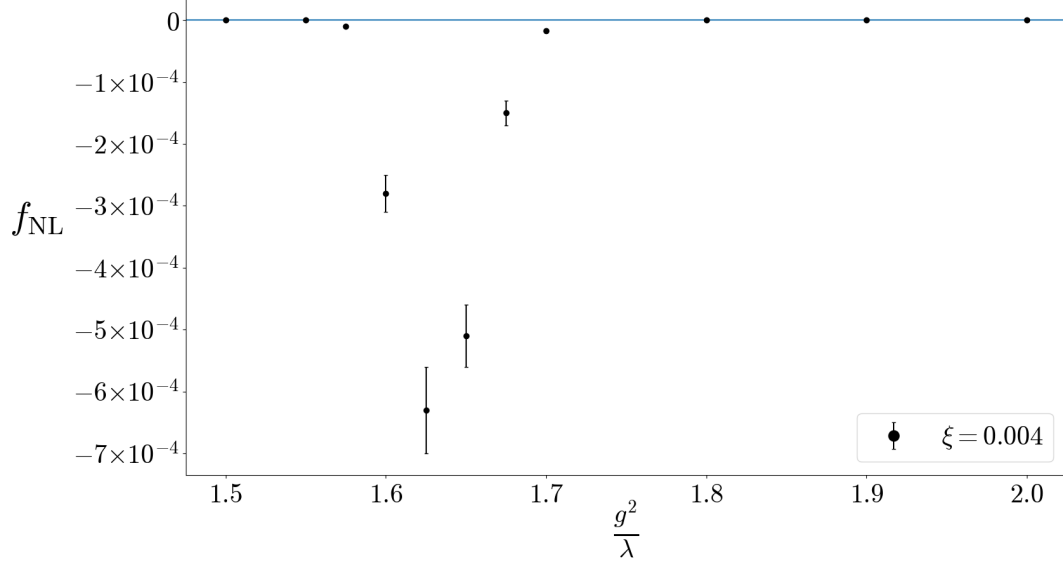


Figure 3.12: Non-Gaussianity parameter f_{NL} at different coupling strengths g^2/λ for a fixed $\xi = 0.004$. Figure reproduced from ref. [1].

growth of zero mode to persist for a longer time and hence increasing the $\delta N(\chi)$ that increases f_{NL} as compared to $g^2/\lambda = 1.875$ value. This effect can only be captured by lattice simulations. Thus by finding many orders of magnitude change in f_{NL} with the coupling strength, we have also demonstrated the need for full lattice simulations to study parameter dependence of preheating models.

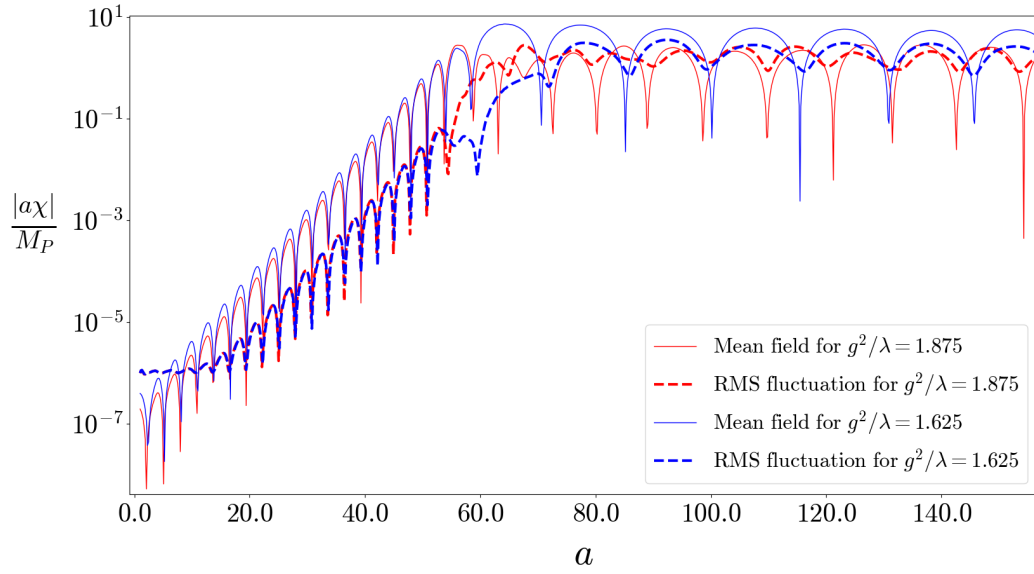


Figure 3.13: Mean field over all lattice sites $\langle\chi\rangle$ and root mean squared (RMS) field fluctuation $\sqrt{\langle\chi^2\rangle - \langle\chi\rangle^2}$ values as a function of scale factor for $g^2/\lambda = 1.875$ and $g^2/\lambda = 1.625$. $\xi = 0.004$ in both cases. Figure reproduced from ref. [1].

Chapter 4

Vacuum transition during inflation

“the universe as an ice-cream”

4.1 Vacuum decay in field theory

A central quantity that characterises vacuum decay is the decay rate per unit volume Γ . There are various approaches to calculate the decay rate in terms of the action in quantum field theory. These include, the original path integral approach [37], the ‘direct’ approach [108] and the real time approach [109] among others. The decay rate per unit volume is usually parametrised in the form

$$\Gamma = Ae^{-B}, \quad (4.1)$$

where A is known as the pre-factor and B is the decay coefficient.

Consider the action S of a scalar field σ in a potential $U(\sigma)$ that has an unstable minimum known as the false vacuum denoted by σ_{FV} and a stable minimum known as the true vacuum denoted by σ_{TV} , separated by a barrier (see Figure 1.2). To calculate the decay rate of false vacuum in such a potential using field theory, it is useful to go to the Euclideanised form of the action, S_E .

Euclideanisation implies analytically continuing the time coordinate in the action to $\tau = it$. This effectively results in the potential being inverted as shown in Figure 4.1.

The decay coefficient B is obtained through the classical solutions of the Euclideanised action S_E [110],

$$B = S_E(\sigma_b) - S_E(\sigma_{FV}) \quad (4.2)$$

where σ_b is known as the bounce solution that necessarily passes through the classical turning point σ_{turn} and σ_{FV} is the constant false vacuum solution.

The pre-factor A is obtained through the one-loop fluctuations around the classical solutions of

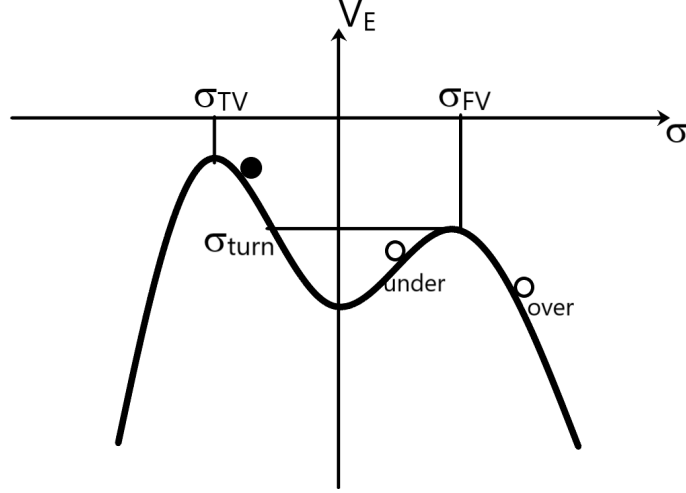


Figure 4.1: Schematic Euclideanised decay potential, $V_E = -V$ in Figure 1.2. σ_{turn} labels the classical turning point. If the initial condition lies somewhere between σ_{TV} and σ_{turn} , then following the Euclidean equation of motion, the solution can undershoot or overshoot the false vacuum. By continuity, there is a solution that ends at σ_{FV} which is known as the bounce solution σ_b .

the Euclideanised action [111],

$$A = \left(\frac{B}{2\pi} \right)^2 \sqrt{\frac{\det(S''_E(\sigma_{FV}))}{\det'(S''_E(\sigma_b))}} \quad (4.3)$$

where the prime on the determinant indicates omission of the four zero modes or collective coordinates [112] and $S''_E(\sigma) = \delta^2 S_E / \delta \sigma^2$.

4.1.1 Decay rate in Minkowski space

In Euclideanised Minkowski space it can be proven that the solution with lowest action is $O(4)$ symmetric [37]. If r is taken to be the Euclidean radial coordinate then the spherically symmetric equation of motion for a vacuum decaying field σ is

$$\frac{d^2 \sigma}{dr^2} + \frac{3}{r} \frac{d\sigma}{dr} - U_{,\sigma} = 0. \quad (4.4)$$

The bounce solution is the one with boundary conditions

$$\sigma(r \rightarrow \infty) = \sigma_{FV}, \quad (4.5)$$

$$\left. \frac{d\sigma}{dr} \right|_{r=0} = 0 \quad (4.6)$$

and can be found by using the shooting method of solving boundary value problems. For the calculation of the functional determinant one can use the Gel'fand Yaglom theorem [113] or the

resolvent method [114].

Considering a generic quartic potential of the form

$$U(\sigma) = \frac{\lambda'}{4!}\sigma^4 + \frac{g'}{3!}\sigma^3 - \frac{\mu^2}{2!}\sigma^2 + U_0 \quad (4.7)$$

where U_0 is a constant independent of σ , it is possible to calculate the decay rate analytically under the thin-wall approximation [114].

The thin-wall approximation entails that the difference between the minima is negligible as compared to height of the barrier, i.e.

$$\frac{U(\sigma_{FV}) - U(\sigma_{TV})}{U(\sigma_{\text{bar}}) - U(\sigma_{TV})} \ll 1 \quad (4.8)$$

where σ_{bar} is the field value corresponding the top of the potential barrier. For the quartic potential (4.7) this translates to

$$\mu^2 \gg \frac{3g'^2}{8\lambda'}, \quad (4.9)$$

as the extrema are given by

$$\sigma_{FV/TV} = -\frac{3g'}{2\lambda'} \left(-1 \pm \sqrt{1 + \frac{8}{3} \frac{\lambda' \mu^2}{g'^2}} \right), \quad \sigma_{\text{bar}} = 0. \quad (4.10)$$

Treating g' to be negligible, the equation of motion

$$\frac{d^2\sigma}{dr^2} + \frac{3}{r} \frac{d\sigma}{dr} - \frac{\lambda'}{3!}\sigma^3 + \mu^2\sigma \approx 0 \quad (4.11)$$

admits a solution

$$\sigma_b \approx \sqrt{\frac{6\mu^2}{\lambda'}} \tanh \left(\frac{\mu}{\sqrt{2}}(r - R) \right) \quad (4.12)$$

which is also known as a ‘kink’ solution. Here R is just an integration parameter for now. Notice that $\sigma_b(r \gg R) - \sigma_b(r \ll R) = \sigma_{FV} - \sigma_{TV}$ and qualitatively one can say that the bounce solution interpolates between the false and true vacuum within a narrow radius around $r = R$. It resembles a bubble $O(4)$ in Euclidean space where inside the bubble of radius R the field is in its true vacuum while outside it sits in the false vacuum.

Connecting back from Euclidean to Minkowski space at some fixed time x_0 we identify $\sigma(x_0, \vec{x}) = \sigma_b(\vec{x}, x_4 = x_0)$ which implies the solution is also a bubble in Minkowski space at its nucleation time. Furthermore we identify a boundary condition for the derivative, $\partial_0\sigma(x_0, \vec{x}) = 0$. The equation of motion in Minkowski space is $(-\partial_0^2 + \nabla^2)\sigma = U_{,\sigma}$. Its solution with these boundary conditions is

just

$$\sigma(x_0, \vec{x}) = \sigma_b \left(r = \sqrt{-x_0^2 + |\vec{x}|^2} \right). \quad (4.13)$$

For the bubble wall at $r = R$, this solution implies evolution on a hyperboloid $-x_0^2 + |\vec{x}|^2 = R^2$. Therefore qualitatively the vacuum decay looks like a bubble nucleating with a (usually microscopic) radius R and then its wall expanding at nearly the speed of light.

The decay coefficient B can be calculated from the bounce action

$$S_E(\sigma_b) = \int d^4x \left(\frac{1}{2} \partial_\mu \sigma_b \partial^\mu \sigma_b + U(\sigma_b) \right) = 2\pi^2 \int_0^\infty dr \, r^3 \left(\frac{1}{2} \left(\frac{d\sigma_b}{dr} \right)^2 + U(\sigma_b) \right) \quad (4.14)$$

which can be separated into three regions: $r < R$, $r > R$ and $r \approx R$. In the region $r > R$, the bounce solution σ_b remains approximately constant at σ_{FV} and we can choose U_0 to make $U(\sigma_{FV}) = 0$ so that the action is zero,

$$S_E(\sigma_b) \Big|_{r>R} = 0. \quad (4.15)$$

In the region $r < R$, the bounce solution σ_b remains approximately constant at σ_{TV} so that the action is

$$S_E(\sigma_b) \Big|_{r<R} = \frac{\pi^2}{2} R^4 U(\sigma_{TV}) = -\sqrt{6}\pi^2 g' \frac{\mu^3}{\lambda^{3/2}} R^4. \quad (4.16)$$

In the remaining region close to $r = R$, using $d\sigma_b/dr = \sqrt{2U(\sigma_b)}$ we have

$$\begin{aligned} S_E(\sigma_b) \Big|_{r \approx R} &= 2\pi^2 R^3 \int_{R-\Delta r}^{R+\Delta r} dr \left(\frac{d\sigma_b}{dr} \right)^2 = 2\pi^2 R^3 \int_{-\sqrt{6\mu^2/\lambda'}}^{\sqrt{6\mu^2/\lambda'}} d\sigma_b \frac{d\sigma_b}{dr} \\ &= 2\pi^2 R^3 \int_{-\sqrt{6\mu^2/\lambda'}}^{\sqrt{6\mu^2/\lambda'}} d\sigma_b \sqrt{2U(\sigma_b)} = 8\sqrt{2}\pi^2 \frac{\mu^3}{\lambda'} R^3. \end{aligned} \quad (4.17)$$

Combining the bounce action in all three regions,

$$S_E(\sigma_b) = -\sqrt{6}\pi^2 g' \frac{\mu^3}{\lambda^{3/2}} R^4 + 8\sqrt{2}\pi^2 \frac{\mu^3}{\lambda'} R^3 \quad (4.18)$$

and extremising it with respect to R , we find that minima occurs at bubble radius

$$R = \frac{2\sqrt{3\lambda'}}{g'} \quad (4.19)$$

and the minimum bounce action is equal to

$$B = 48\sqrt{6\lambda'} \frac{\mu^3}{g'^3}. \quad (4.20)$$

The calculation of the pre-factor A is more involved and here we shall only point out the broad elements that go into it. Ref. [114] provides a complete treatment. The fluctuation determinant around bounce action is equal to the product of eigenvalues of the Green's function defined by

$$(-\Delta^{(4)} + U''(\sigma(x)))G(x, x') = \delta^{(4)}(x - x') \quad (4.21)$$

where $\Delta^{(4)}$ is the four dimensional Laplacian. In spherical coordinates, it can be separated as

$$G(x, x') = \frac{1}{2\pi^2} \sum_{j=0}^{\infty} (j+1) G_j(r, r') U_j(\cos \theta) \quad (4.22)$$

where U_j are Chebyshev polynomials and

$$\left(-\frac{d^2}{dr^2} - \frac{3}{r} \frac{d}{dr} + \frac{j(j+2)}{r^2} + U''(\sigma(r)) \right) G_j(r, r') = \frac{\delta(r - r')}{r'^3}. \quad (4.23)$$

In the thin-wall approximation, one can neglect the gradient term while substituting $r \sim R$ in the remaining terms as

$$\left(-\frac{d^2}{dr^2} + \frac{j(j+2)}{R^2} + U''(\sigma(r)) \right) G_j(r, r') \approx \frac{\delta(r - r')}{R^3}. \quad (4.24)$$

The $j = 0$ case contains a negative eigenvalue equal to $-3/R^2$ whereas the $j = 1$ case contains four zero eigenvalues. Since the fluctuation determinant is obtained using the method of steepest descent these five eigenvalues need to be removed. Omitting them one obtains

$$\frac{\det' G_{j=0}}{\det G_{FV, j=0}} \left(\frac{\det' G_{j=1}}{\det G_{FV, j=1}} \right)^4 = (24\mu^2)^{-5}, \quad (4.25)$$

a result which can also be directly obtained by the application of the Gel'fand Yaglom theorem. For $j > 1$, upto one-loop renormalisation

$$\ln \left(\sqrt{\frac{\det' G_{FV, j>1}}{\det G_{j>1}}} \right) = \frac{\pi}{4\sqrt{6}} R^3 \mu^3. \quad (4.26)$$

Combining all the factors,

$$A = \left(\frac{B}{2\pi} \right)^2 \sqrt{\frac{R^2}{3}} (24\mu^2)^{5/2} \exp \left(\frac{\pi}{4\sqrt{6}} R^3 \mu^3 \right) \quad (4.27)$$

where the factor $\sqrt{B/2\pi}$ appears due to omission of each zero mode and $\sqrt{R^2/3}$ due to omission of the negative mode.

Hereafter, we include the exponential factor in the definition of B making

$$A = 24^5 \sqrt{6} \pi^2 \frac{\lambda'^{3/2} \mu^{11}}{g'^7}, \quad (4.28)$$

$$B = 48 \sqrt{6} \pi^2 \left(1 - \frac{\lambda'}{16 \sqrt{3} \pi} \right) \frac{\sqrt{\lambda'} \mu^3}{g'^3}. \quad (4.29)$$

Together these coefficients are used to calculate the decay rate in Minkowski spacetime under the thin-wall approximation.

4.1.2 Decay rate in de Sitter space

In Euclideanised curved spacetime, there does not exist a proof that the solution with lowest action is $O(4)$ symmetric. In this case there is an interplay between the scalar field and metric which can in principle lead to non-trivial geometries for the solution depending on the curvature of background spacetime and whether it is $O(4)$ symmetric. If the background is slightly curved, then the decay can be approximated to occur on a Minkowski spacetime and $O(4)$ symmetry is a reasonable assumption. This would not necessarily be the case if the background is highly curved, as in the vicinity of a black hole or in the very early universe. However, owing to the assumption of homogeneity and isotropy, the cosmological de Sitter background that we would be concerned with in this work is still $O(4)$ symmetric. And hence the lowest action solution can be reasonably assumed to also be $O(4)$ symmetric.

The $O(4)$ symmetric metric in Euclidean space can be expressed as [115]

$$ds^2 = dr^2 + \beta(r)^2 d\Omega_3^2 \quad (4.30)$$

where r is the Euclidean radius and Ω_3 denotes the area element of a three sphere. The scalar field equation of motion with the above metric and a potential $U(\sigma)$ is

$$\frac{d^2 \sigma}{dr^2} + 3 \frac{1}{\beta} \frac{d\beta}{dr} \frac{d\sigma}{dr} = \frac{\partial U}{\partial \sigma}. \quad (4.31)$$

While the Einstein's equation yields

$$\left(\frac{d\beta}{dr} \right)^2 = 1 + \frac{\beta^2}{3M_P^2} \left(\frac{1}{2} \left(\frac{d\sigma}{dr} \right)^2 - U(\sigma) \right). \quad (4.32)$$

Since $\beta(r)$ quantifies the radius of three sphere in Euclidean space at distance r from origin, at the origin $r = 0$ we have the initial condition to be $\beta(r) = 0$. It can be shown that $\beta(r)$ admits at least one other zero at some finite $r_{\max}$ when $U(\sigma_{FV}) > 0$ [116, 117]. Therefore to ensure the equation of motion for the field (4.31) does not possess any singularities at $r = 0$ and $r = r_{\max}$ we

must ensure the boundary conditions

$$\left. \frac{d\sigma}{dr} \right|_{r=0} = \left. \frac{d\sigma}{dr} \right|_{r=r_{\max}} = 0 \quad (4.33)$$

are satisfied. Thus the solution for the transition field σ is necessarily compact if $U(\sigma_{FV}) > 0$.

It is helpful to think in terms of the strength of the spacetime background. In the case of de Sitter space, since it admits a causal horizon, this can also be quantified through a Gibbons Hawking temperature [118], $T_{\text{Gibbons-Hawking}} = H/2\pi$.

- If the “thermal” Gibbons Hawking background temperature is greater than the height of the barrier, quantified lets say by its effective mass μ_{eff} , i.e. $H \gtrsim \mu_{\text{eff}}$ then the favoured evolution would be a classical lifting of the field upward along the potential without any tunnelling. Thus we can expect a constant solution to the equations of motion in this case where the field always remains at the top of the barrier. This is known as the Hawking Moss (HM) solution [119] or the Hawking Moss instanton.
- If the background is weaker than the height of the barrier $H \lesssim \mu_{\text{eff}}$, then the favoured evolution would in general be a combination of classical lifting and tunnelling. Therefore we would not expect the solution to approach the false vacuum or the true vacuum, it would first be classically lifted from the false vacuum due to the “thermal” Gibbons Hawking background, tunnel and then classically fall down to the true vacuum. This is known as the Coleman De Luccia (CDL) solution [115] or the Coleman De Luccia (CDL) instanton.
- Finally, if the strength of the background is negligible, we approach a Minkowski spacetime where the favoured evolution is a pure tunnelling. So we would expect the solution to approach both the false and true vacuums at infinity as we saw in the previous section.

With this physical picture in mind, the Hawking-Moss (HM) solution can be readily quantified. The field classically lifts to the top of the barrier, which we shall denote as σ_{bar} . Thus the equation of motion for the field (4.31) is trivially satisfied for a constant $\sigma = \sigma_{\text{bar}}$. Whereas the background Einstein’s equation (4.32) has a solution,

$$\beta(r) = \frac{\sin(H_{\text{HM}} r)}{H_{\text{HM}}} \quad (4.34)$$

where H_{HM} is defined analogously to the slow-roll Hubble rate

$$H_{\text{HM}}^2 = \frac{U(\sigma_{\text{bar}})}{3M_P^2} \quad (4.35)$$

for Hawking-Moss transition. Then the decay exponent (4.2), $B_{\text{HM}} = S_E[\sigma_{\text{bar}}] - S_E[\sigma_{FV}]$ is simply obtained to be

$$B_{\text{HM}} = 24\pi^2 M_P^4 \left(\frac{1}{U(\sigma_{FV})} - \frac{1}{U(\sigma_{\text{bar}})} \right). \quad (4.36)$$

The prefactor for Hawking-Moss transition is given by [120]

$$A_{\text{HM}} = \frac{1}{6\pi} \frac{\sqrt{U''(\sigma_{FV})|U''(\sigma_{\text{bar}})|}}{H^2} \quad (4.37)$$

where prime denotes derivative with respect to the field σ and H is the background Hubble rate not necessarily at the top of the barrier. This can be obtained for example using the stochastic formalism [121, 122].

Rather than field value being fixed to the top of the barrier for the Hawking Moss instanton, the Coleman De Luccia (CDL) instanton is the monotonic solution of the Euclidean field equations (4.31)–(4.32) that interpolates between field values on either side of the barrier.

To obtain the CDL solution let us fix the background spacetime to be de Sitter with Hubble rate

$$H_{FV}^2 = \frac{U(\sigma_{FV})}{3M_P^2}, \quad (4.38)$$

owing to the fact that the field remains stuck in its false vacuum before the transition takes place. Then the Einstein equation (4.32) yields

$$\beta(r) = \frac{\sin(H_{FV}r)}{H_{FV}}, \quad (4.39)$$

so that r_{max} in eq.(4.33) can be taken to be $r_{\text{max}} = \pi/H_{FV}$. The equation of motion for the field need not be trivially satisfied, but simplifies to

$$\frac{d^2\sigma}{dr^2} + 3 \cot(H_{FV}r) \frac{d\sigma}{dr} = \frac{\partial U}{\partial \sigma} \quad (4.40)$$

with the boundary conditions

$$\left. \frac{d\sigma}{dr} \right|_{r=0} = \left. \frac{d\sigma}{dr} \right|_{r=\pi/H_{FV}} = 0. \quad (4.41)$$

In the flat space limit $H_{FV} \rightarrow 0$, we recover the equation of motion for the tunneling field in Minkowski spacetime (4.4) and the second boundary condition at $r_{\text{max}} = \pi/H_{FV} \rightarrow \infty$ is no longer necessary. Instead we need to fix $\sigma(r \rightarrow \infty) = \sigma_{FV}$ to obtain the bounce (4.5). Thus the Minkowski bounce solution can be thought of as a special case of the CDL bounce.

The ordinary differential equation above that involves Neumann boundary conditions can be solved numerically using a shooting algorithm [37]. First we bring the equation into program variables, clearly separating the scales involved by rescaling the Euclidean radial coordinate $r \rightarrow rH$ so that the equation of motion and boundary conditions take the form [123]:

$$\frac{d^2\sigma}{dr^2} + 3 \cot(r) \frac{d\sigma}{dr} = \frac{1}{H^2} \frac{dU}{d\sigma} \quad (4.42)$$

and

$$\left. \frac{d\sigma}{dr} \right|_{r=0} = \left. \frac{d\sigma}{dr} \right|_{r=\pi} = 0 \quad (4.43)$$

with the bounce action now given by

$$S_E = 2\pi^2 \int_0^\pi dr \frac{\sin(r)^3}{H^2} \left(\frac{1}{2} \left(\frac{d\sigma}{dr} \right)^2 + \frac{U}{H^2} \right). \quad (4.44)$$

Next we proceed to convert the boundary value problem eq. (4.42) between the interval $r \in [\varepsilon, \pi - \varepsilon]$ into an initial value problem with initial field value, $\sigma_{\text{guess}} = (\sigma_{TV} + \sigma_{FV})/2$ and derivative $\frac{d\sigma}{dr}(r = \varepsilon) = 0$. The small shift in the domain by ε ensures that numerical program avoids the singularities at $r = 0$ and $r = r_{\text{max}} = \pi$. If the solution encounters a minimum or maximum for the field, then it is classified as an ‘undershoot’, integration is stopped and the guess is updated towards the true vacuum i.e. $\sigma_{\text{new guess}} = (\sigma_{\text{guess}} + \sigma_{\text{last overshoot}})/2$. Whereas if the solution does not encounter any extrema, then it is classified as an ‘overshoot’, integration is stopped if the field reaches a very high value, for example $(M \times \sigma_{FV})$ where $M \gg 1$, and the guess is updated towards the false vacuum i.e. $\sigma_{\text{new guess}} = (\sigma_{\text{guess}} + \sigma_{\text{last undershoot}})/2$. This process of binary searching is carried on iteratively until the second boundary condition lies close to zero within some tolerance, for example $|\frac{d\sigma}{dr}(r = \pi - \varepsilon)| < \frac{\sigma_{FV} - \sigma_{TV}}{M}$.

4.2 Characteristics of transition during inflation

4.2.1 Completion of decay

The volume fraction v occupied by bubbles is a measure of the progress of vacuum transition during inflation. See Figure 4.2 for illustration. The volume fraction depends on three time epochs, the time at which one wishes to calculate the volume fraction arising from bubbles that nucleate between some initial and final times. However, to simplify the treatment we denote the volume fraction at the final time, expressed in number of efolds as N_f , from bubbles that formed between some initial time N_i to the final time N_f by $v(N_i, N_f)$. This is given by integrating the volume fraction occupied by a single bubble $v_1(N, N_f)$ multiplied by the average number of bubbles nucleated in our past light cone $n(N)$ as

$$v(N_i, N_f) = \int_{N_i}^{N_f} dn(N) v_1(N, N_f). \quad (4.45)$$

In terms of the conformal time η , the volume fraction occupied by a single bubble is given by

$$v_1(N, N_f) = \frac{\frac{4\pi}{3}(\eta(N_f) - \eta)^3}{\frac{4\pi}{3}(\eta_0 - \eta(N_f))^3}. \quad (4.46)$$

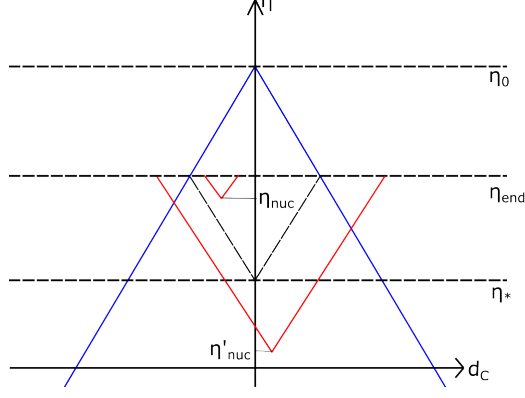


Figure 4.2: Spacetime diagram illustrating bubble nucleation in our past lightcone (in blue). The vertical axis denotes conformal time while the horizontal axis corresponds to comoving distance. η_0 is the conformal time today. For a bubble that nucleates at η_{nuc} and expands at the speed of light, the forward light cone (in red) denotes the fraction of our observable universe covered by it at the end of inflation surface labelled by η_{end} . Similarly a bubble that nucleates at η'_{nuc} , before the largest currently observed scale left the horizon at η_* , covers the whole observable universe at η_{end} . Figure reproduced from ref. [2].

Whereas the average number of bubbles nucleated in our past lightcone is given by [124]

$$n = \int d^4x \sqrt{-g} \Gamma = \int d\eta \frac{4\pi}{3} (\eta_0 - \eta)^3 a(\eta)^4 \Gamma(\eta) \quad (4.47)$$

where η_0 denotes conformal time today. Thus the volume fraction (4.45) becomes

$$v(N_i, N_f) = \int_{\eta(N_i)}^{\eta(N_f)} d\eta \frac{4\pi}{3} (\eta_0 - \eta)^3 a(\eta)^4 \Gamma(\eta) \times \left(\frac{\eta(N_f) - \eta}{\eta_0 - \eta(N_f)} \right)^3, \quad (4.48)$$

where we have assumed $N_i > N_f$.

It is useful to explicitly make the connection between conformal time and number of e-folds during inflation in the expressions for the average number of bubbles (4.47) and volume fraction (4.48). Using the definition of the number of e-folds $dN = -H dt = -H a d\eta$ and the scale factor during inflation $a(N) = a_{\text{end}} e^{-N}$, we find that N e-foldings before the end of inflation, the conformal time has the value

$$\eta(N) = \eta_{\text{end}} - \frac{1}{a_{\text{end}}} \int_0^N \frac{e^{N'}}{H(N')} dN' \quad (4.49)$$

where η_{end} is the conformal time at the end of inflation. Substituting this back into the expression for the average number of bubbles gives

$$n = \int dN \frac{4\pi}{3} \left(\eta_0 - \eta_{\text{end}} + \frac{1}{a_{\text{end}}} \int_0^N dN' \frac{e^{N'}}{H(N')} \right)^3 \frac{a_{\text{end}}^3 e^{-3N}}{H(N)} \Gamma(N). \quad (4.50)$$

For the standard Λ CDM universe (1.10),

$$\eta_0 - \eta_{\text{end}} = \frac{1}{H_0} \int_0^{a_0} \frac{da}{a^2 \sqrt{(\Omega_\Lambda + \Omega_m(a_0/a)^3 + \Omega_r(a_0/a)^4)}} \approx 3.21(a_0 H_0)^{-1}. \quad (4.51)$$

Then, in terms of the observed number of efolds

$$\frac{a_{\text{end}} H_{\text{end}}}{a_0 H_0} = e^{N_*} \quad (4.52)$$

the average number of bubbles becomes

$$n = \int dN \frac{4\pi}{3} \left(3.21 + H_{\text{end}} \int_0^N dN' \frac{e^{N'-N_*}}{H(N')} \right)^3 e^{-3(N-N_*)} \frac{\Gamma(N)}{H(N) H_{\text{end}}^3}. \quad (4.53)$$

Hence the volume fraction (4.48) in terms of the number of efolds becomes

$$\begin{aligned} v(N_i, N_f) = \int_{N_f}^{N_i} dN \frac{4\pi}{3} \left(3.21 + \int_0^N dN' \frac{H_{\text{end}} e^{N'-N_*}}{H(N')} \right)^3 e^{-3(N-N_*)} \frac{\Gamma(N)}{H(N) H_{\text{end}}^3} \\ \times \left(\frac{\int_0^N dN' \frac{e^{N'}}{H(N')} - \int_0^{N_f} dN' \frac{e^{N'}}{H(N')}}{\int_0^{N_*} dN' \frac{e^{N'}}{H(N')} + \int_0^{N_f} dN' \frac{e^{N'}}{H(N')}} \right)^3. \end{aligned} \quad (4.54)$$

Note the appearance of the quantity

$$p(N) = \frac{\Gamma(N)}{H(N)^4} \quad (4.55)$$

in the above expression. In the context of vacuum transition during inflation this is referred to as the percolation parameter [16, 125, 126]. In particular, focusing on one single efold during inflation we have $N_i = N_f + 1$ in which case it can be shown that

$$v(N_f + 1, N_f) \sim p(N_f). \quad (4.56)$$

Thus the percolation parameter is an approximate measure of the instantaneous volume fraction during inflation.

From percolation theory considerations [127], the critical threshold for completion of a transition in a three dimensional continuum with spheres is about $v_c \approx 0.3$. Thus one can approximately take the transition to complete when the percolation parameter becomes unity and we denote it by a subscript T as N_T or ϕ_T , i.e.

$$p(N_T) = 1. \quad (4.57)$$

4.2.2 Hawking-Moss transition never completes during inflation

In this work, we are interested in vacuum transition that occurs during inflation. Considering slow-roll inflation, the background spacetime is nearly de Sitter. Therefore, to simplify the treatment of vacuum decay, we assume that the background is exactly de Sitter. This assumption is justified if the inflaton domination approximation

$$U(\sigma_{FV}) - U(\sigma_{TV}) < H^2 M_P^2 \quad (4.58)$$

holds. Note that the inflaton dominated approximation, which involves the difference between the potential at the false and true vacuum, is independent of the barrier height. That is, the false and true vacuum values can lie very close to each other but the barrier separating them can be arbitrarily high. The barrier height characterises the type of transition, Coleman De Luccia or Hawking Moss that would be favoured [128]. But the de Sitter background assumption relies on the potential difference between the vacua. Thus for a transition occurring during inflation, both the HM and CDL solutions are possible.

There exists an interplay between the Hawking Moss and Coleman De Luccia transitions in the context of inflation. To see this it is useful to collate certain results pertaining to the eigenvalues of the fluctuations about the bounce solution.

- For Minkowski spacetime, it can be proven that the bounce solution with lowest action has only one negative eigenvalue [129]. This is because if more than one negative eigenvalue solution exists then there is a linear combination of these eigenmodes that yields a tunnelling path with lower decay rate exponent B . But a bounce solution must already minimise B in the quantum field theory path integral. Thus, the exponent B for such a bounce solution with more than one negative eigenvalue must be a saddle point of the path integral and not a minimum. Hence it does not describe tunneling and must be discarded.
- The above result can be extended to curved spacetime [130, 131], so that bounce solutions with more than one negative eigenvalues must be discarded. This is the reason why oscillating bounce solutions [132] that cross the barrier more than once can be ignored for decay rate calculation.
- As the Hawking-Moss solution is analytically tractable, it is possible to determine the condition for which it possesses only one negative eigenvalue. This is given by [133, 134]

$$H > \frac{\sqrt{|U''(\sigma_{\text{bar}})|}}{2} . \quad (4.59)$$

Apart from this limit, the Hawking Moss solution does not describe tunnelling.

- It can be shown that the Coleman De Luccia solution always exists [135] apart from the above

limit, i.e. when

$$H < \frac{\sqrt{|U''(\sigma_{\text{bar}})|}}{2} . \quad (4.60)$$

Thus the picture in the context of inflation starts to become clearer. At very early times when the Hubble rate is high, the HM transition is the dominant one. Then as the Hubble rate decreases to beyond the threshold defined by the above limit, the CDL transition becomes the dominant one [124, 128, 136]. Alternatively, assuming the transition occurs towards the end of inflation, the parameters of the decay potential determine which of the HM or CDL transitions dominates.

I shall now show that the percolation parameter (4.55) for a Hawking–Moss transition is always less than unity for a quartic potential. We denote the decay rate as $\Gamma_{\text{HM}} = A_{\text{HM}} \exp(-B_{\text{HM}})$ and de Sitter Hubble rate as H . Note that for this argument it is not necessary for the background Hubble rate to be fixed at the top of the barrier (4.35), any other background Hubble rate also works. The percolation parameter p_{HM} is bounded by

$$p_{\text{HM}} = \frac{A_{\text{HM}}}{H^4} \exp(-B_{\text{HM}}) < \frac{A_{\text{HM}}}{H^4} \quad (4.61)$$

due to the exponent $B_{\text{HM}} > 1$. The prefactor for Hawking–Moss decay rate is given by eq. (4.37), so the bound on p becomes

$$p_{\text{HM}} < \frac{1}{6\pi} \frac{\sqrt{U''(\sigma_{\text{FV}})|U''(\sigma_{\text{bar}})|}}{H^2} . \quad (4.62)$$

Now we use the fact that for a quartic transition potential (4.7), the maximum value of $\sqrt{U''(\sigma_{\text{FV}})}$ can be $2\sqrt{|U''(\sigma_{\text{bar}})|}$ which occurs when the coefficient of cubic term in σ , $g' \rightarrow 0$. Thus the bound on p changes as,

$$p_{\text{HM}} < \frac{1}{3\pi} \frac{|U''(\sigma_{\text{bar}})|}{H^2} . \quad (4.63)$$

Then using the relation (4.59) for the HM transition to exist we finally see that

$$p_{\text{HM}} < \frac{4}{3\pi} < 1. \quad (4.64)$$

Hence we conclude that a Hawking–Moss transition never completes during inflation (cf. eq. (4.57)) for an effectively quartic transition potential.

4.3 Non-Gaussianity formalism

4.3.1 Setup

We will consider the scenario in which transition occurs during the last N_* e-folds, i.e. after the largest currently observable scale left the horizon. Even if a single bubble had nucleated before N_* ,

it has the potential to fill up the entire observable universe, in which case its effects would not be observable. See Figure 4.2 for illustration. This means that no bubbles can have nucleated in our past lightcone prior to the instance given by N_* . The probability of this is given by $\exp(-n(N \geq N_*))$. Therefore the average number of bubbles nucleated in our past lightcone prior to the instance given by N_* must be less than unity,

$$n(N \geq N_*) \lesssim 1. \quad (4.65)$$

In this work, we only consider transitions that occur towards the end of inflation, $N_T \sim O(1)$, such that the transition is completed by bubbles whose sizes are several orders of magnitude smaller than the scale at which CMB measurements take place. Thus their effect on the CMB non-Gaussianity is only through the modulation of background inflationary expansion. Transitions that occur within the observable N_* efolds but not towards the end of inflation can lead to interesting features in the post-inflationary evolution, but these are not the focus of the present work.

For added simplicity we consider that the transition is completed predominantly by bubbles nucleated at N_T , i.e. the percolation parameter rises sharply at N_T ,

$$\left| \frac{dp}{dN} \right|_{N=N_T} \gg 1. \quad (4.66)$$

Physically this translates to multiple bubbles forming inside one Hubble volume and coalescing within an inflationary time scale H^{-1} . In that case we can treat this transition as a sudden change in the transition field from its false to its true vacuum at the level of the background inflationary evolution.

In order to study the different parts of the mechanism generating non-Gaussianity from vacuum transition during inflation, we will isolate the inflation, vacuum transition and spectator sectors into three separate fields while constructing the general formalism. However, it is important to note that this is just for illustration purposes. In reality one expects this mechanism to play a role in generating curvature perturbations for a wide range of beyond the Standard Model (BSM) scenarios that exhibit vacuum transition during inflation. Ultimately, by connecting such theoretically motivated BSM theories with the observations of non-Gaussianity in the CMB, we wish to test the suitability and delineate the parameter spaces of such theories to describe our universe.

Therefore we consider a potential $W(\phi, \sigma, \chi)$ comprised of three fields: the inflaton ϕ , vacuum transition field σ and a light spectator field χ . On constant ϕ and χ slices, the potential in σ admits two minima. The false vacuum denoted by $\sigma_{FV}(\phi, \chi)$. And the true vacuum denoted by $\sigma_{TV}(\phi, \chi)$. Henceforth we shall not indicate the dependence upon ϕ and χ explicitly and refer to the false and true vacuums simply as σ_{FV} and σ_{TV} respectively.

If the transition field remains light throughout inflation until the transition point N_T , then we do not require a separate χ field to play the role of the spectator and generate significant fluctuations over the Hubble volumes in a separate universe picture. However, if the transition field is heavy then its fluctuations get damped at the scale N_T and it does not contribute significantly to the

generation of curvature perturbations.

For our scenario, we assume initial conditions such that the transition field σ is already trapped in its false vacuum. The inflaton ϕ evolves in a slow-roll fashion, and the transition occurs at some point ϕ_T during its evolution. We shall also assume that the inflaton dynamics dominates both the transition and spectator dynamics, and as such there is no back-reaction of the transition or spectator sectors on the inflationary background. This can be captured by

$$\frac{W(\sigma_{FV}) - W(\sigma_{TV})}{M_P^2} < H_T^2, \quad (4.67)$$

where H_T denotes the Hubble rate at transition ϕ_T with the transition field in its false vacuum σ_{FV} . In this case, the inflaton field will continue to evolve in a slow-roll fashion after the transition has occurred. This assumption also allows us to characterise the time evolution by the inflaton field value and to write the number of efolds in slow-roll in terms of the field evolution from ϕ_* , which corresponds to the instance when the currently observed scale left the horizon to ϕ_{end} , which corresponds to the end of inflation as

$$N(\chi) = \int_{\phi_{\text{end}}}^{\phi_*} \frac{W}{W_{,\phi}} d\phi. \quad (4.68)$$

With the sudden change in σ at transition ϕ_T (4.66), the number of efolds get split into

$$\delta N = \int_{\phi_{\text{end}}}^{\phi_T} d\phi \frac{W}{W_{,\phi}} \Big|_{\sigma=\sigma_{TV}} + \int_{\phi_T}^{\phi_*} d\phi \frac{W}{W_{,\phi}} \Big|_{\sigma=\sigma_{FV}}. \quad (4.69)$$

By splitting the integral into two parts at ϕ_T we are also implicitly ignoring the background generated by bubbles nucleated at $\phi > \phi_T$, which is again justified under eq. (4.66).

4.3.2 Non-Gaussianity

Our goal is to calculate the non-Gaussianity arising from the spectator field χ . Assuming that the spectator field only negligibly contributes to the energy density during inflation, any direct dependence of the number of efolds (4.69) on χ through the potential can be ignored. Therefore it only indirectly depends on χ through $\phi_T(\chi)$, the time at which transition occurs in each Hubble volume,

$$\delta N(\chi) = \int_{\phi_{\text{end}}}^{\phi_T(\chi)} d\phi \frac{W}{W_{,\phi}} \Big|_{\sigma=\sigma_{TV}} + \int_{\phi_T(\chi)}^{\phi_*} d\phi \frac{W}{W_{,\phi}} \Big|_{\sigma=\sigma_{FV}}. \quad (4.70)$$

Because the spectator field value is different in different Hubble volumes, each Hubble volume undergoes a different amount of expansion before the transition in it takes place. It is this difference in expansion between the Hubble volumes that can generate sizable non-Gaussianity.

To calculate f_{NL} (2.72)–(2.75), we need only the first and second derivatives of $\delta N(\chi)$ at $\chi = \bar{\chi}$.

But these can be immediately calculated without having to perform the integral (4.70),

$$\delta N_{,\chi} = \left(\left. \frac{W}{W_{,\phi}} \right|_{\sigma=\sigma_{TV}} - \left. \frac{W}{W_{,\phi}} \right|_{\sigma=\sigma_{FV}} \right)_{\phi=\phi_T} \frac{d\phi_T}{d\chi} = \Delta \left(\frac{W}{W_{,\phi}} \right) \Big|_{\phi=\phi_T} \frac{d\phi_T}{d\chi}, \quad (4.71)$$

where we have introduced the notation Δ to indicate the difference between the true and false vacuum values. Similarly,

$$\delta N_{,\chi\chi} = \Delta \left(\frac{\partial}{\partial\phi} \left(\frac{W}{W_{,\phi}} \right) \right) \Big|_{\phi=\phi_T} \left(\frac{d\phi_T}{d\chi} \right)^2 + \Delta \left(\frac{W}{W_{,\phi}} \right) \Big|_{\phi=\phi_T} \frac{d^2\phi_T}{d\chi^2}. \quad (4.72)$$

The remaining ingredients to fully determine the δN derivatives are the derivatives of ϕ_T with respect to χ , i.e. the change in time of transition across different Hubble volumes. These can be determined by numerically solving for the bounce solution with a different initial value for the spectator field corresponding to each separate Hubble volume. However, we can simplify the treatment without having to perform a separate bounce calculation for each Hubble volume for some specific types of potentials. For example, if we choose to capture the vacuum transition with a quartic potential and the coupling to the spectator field via a quadratic term,

$$U(\sigma, \chi) = \frac{\lambda'}{4!} \sigma^4 + \frac{g'}{3!} \sigma^3 + \frac{\mu^2}{2!} \sigma^2 + \frac{g_S^2}{2} \sigma^2 \chi^2. \quad (4.73)$$

Then we can define an “effective” mass

$$\mu_{\text{eff}}^2(\phi, \chi) = \frac{\partial^2 U}{\partial \sigma^2} \Big|_{\sigma=0} \quad (4.74)$$

which is the only parameter in the quartic transition potential that changes as one moves from one Hubble volume to the next. Note that the effective mass can also depend on the inflationary sector through a direct or indirect coupling between the inflaton and transition fields. For example, as I demonstrate in section 4.5 for a Higgs inflation model, the inflaton and transition sectors can be coupled indirectly via a non-minimal coupling to gravity.

Now to calculate the derivatives of ϕ_T with respect to χ , we know that $\phi_T(\chi)$ is a solution of the percolation condition eq. (4.57),

$$A(\mu_{\text{eff}}) e^{-B(\mu_{\text{eff}})} = H^4. \quad (4.75)$$

To proceed further without resort to numerical solutions, we can ignore the ϕ_T dependence of H . Note that while A and B depend on all parameters in the potential for the transition field σ , here we are assuming that the dependence on the fields ϕ and χ enters only through the parameter μ_{eff} .

Then differentiating the percolation condition with respect to χ one has

$$\frac{d}{d\chi}\mu_{\text{eff}}(\phi_T, \chi) = 0 \quad (4.76)$$

which implies

$$\frac{d\phi_T}{d\chi} = -\frac{\mu_{\text{eff},\chi}}{\mu_{\text{eff},\phi_T}}, \quad (4.77)$$

$$\frac{d^2\phi_T}{d\chi^2} = -\frac{d}{d\chi} \left(\frac{\mu_{\text{eff},\chi}}{\mu_{\text{eff},\phi_T}} \right). \quad (4.78)$$

We still need to obtain the transition point ϕ_T by finding out where the percolation parameter becomes unity i.e. by solving eq. (4.57). But we need to do so for only one value of χ at $\bar{\chi}$ and then use eqs. (4.77)–(4.78) above at that $\phi_T(\bar{\chi})$ and $\bar{\chi}$ as part of the non-Gaussianity calculation combining eqs. (2.72)–(2.75) into

$$f_{\text{NL}} = -\frac{5}{6} \frac{\mathcal{P}_\chi^2}{\mathcal{P}_*^2} (\delta N_{,\chi})^2 \delta N_{,\chi\chi} \bigg|_{\chi=\bar{\chi}} - \frac{5}{6} \frac{(\delta N_{,\chi\chi})^3}{\mathcal{P}_*^2} \int \frac{d\vec{q}}{4\pi} \Sigma(q) \Sigma(|\vec{k}_* + \vec{q}|) \Sigma(|\vec{k}_* - \vec{q}|) \bigg|_{\chi=\bar{\chi}}. \quad (4.79)$$

4.4 Illustrative analytical model

4.4.1 Quartic transition in the thin-wall approximation

Consider a potential comprised of the inflaton ϕ part $V(\phi)$ and the vacuum transition field σ part $U(\sigma)$, which are coupled through a dimensionless coupling of the form $g_I^2 \phi^2 \sigma^2 / 2$. The transition is also modulated by a spectator field χ through another dimensionless coupling of the form $g_S^2 \sigma^2 \chi^2 / 2$. The total potential can be written as

$$W(\phi, \sigma, \chi) = V(\phi) + U(\sigma) - \frac{g_I^2}{2} \phi^2 \sigma^2 + \frac{g_S^2}{2} \chi^2 \sigma^2 \quad (4.80)$$

where $U(\sigma)$ takes a generic quartic form (4.7).

Note that we have negative sign for the coupling between inflaton and transition field in order that the effective mass of the transition field

$$\mu_{\text{eff}} = \sqrt{\mu^2 + g_I^2 \phi^2 - g_S^2 \chi^2} \quad (4.81)$$

decreases as the inflaton field slowly rolls down its potential, which makes transition more favourable towards the end of inflation when the field ϕ takes lower values. This can be readily seen through the expression for the decay rate per unit volume in Minkowski space (4.28). The thin-wall condition, which is essential for the validity of the analytical decay rate is given by

$$\mu_{\text{eff}} = x \sqrt{\frac{3}{8} \frac{g'}{\sqrt{\lambda'}}} \quad \text{with } x > 1, \quad (4.82)$$

where I choose to quantify the validity of the thin-wall approximation by a dimensionless number x which has to be at least greater than one.

We can continue to use the analytical form for the decay rate (4.28), which is calculated assuming a Minkowski background, in the curved spacetime scenario corresponding to inflation provided that the radius of bubble at formation R lies well within the Hubble radius at that time,

$$R \ll H_T^{-1}. \quad (4.83)$$

For the quartic potential being used (4.7), the bubble radius at formation in the thin-wall approximation is given by eq. (4.19).

4.4.2 Parameter values and scale of inflation

The simplifying approximations of inflaton dominated evolution (4.67), thin-wall (4.82), Minkowski spacetime decay (4.83) along with the percolation condition (4.57) translate into constraints on the parameters of the vacuum transition potential (4.7): λ' , g' and μ_{eff} .

With the substitution for μ_{eff} from eq. (4.82), the decay rate parameters (4.28) become

$$A = 8.74 \times 10^5 \left(\frac{g'}{\lambda'} \right)^4 x^{11}, \quad (4.84)$$

$$B = 266 \frac{1}{\lambda'} \left(1 - \frac{\lambda'}{16\sqrt{3}\pi} \right) x^3 \quad (4.85)$$

where $x > 1$ for the thin-wall approximation to be valid. Then the Minkowski spacetime decay condition (4.83) implies

$$\frac{g'}{H_T} > 2\sqrt{3\lambda'}, \quad (4.86)$$

whereas the percolation condition implies

$$\frac{g'}{H_T} > \frac{\lambda'}{30.58} \exp \left(\frac{66.62}{\lambda'} \left(1 - \frac{\lambda'}{16\sqrt{3}\pi} \right) \right). \quad (4.87)$$

On the other hand, the inflaton domination condition (4.67) gives

$$\frac{2\sqrt{6}g'\mu_{\text{eff}}^3}{3\lambda'^{3/2}M_P^2} \frac{8\pi}{3} < H_T^2 \quad (4.88)$$

which implies

$$\frac{g'}{\sqrt{H_T}M_P} < \left(\frac{\lambda'^3}{\pi} \right)^{1/4}. \quad (4.89)$$

At a heuristic level, one of the Minkowski spacetime decay or percolation conditions would be

stronger, rendering the other one redundant. For efficient decay, g'/H_T should be as large as possible. So if one chooses $1 < \lambda' < 16\sqrt{3}\pi$ in order to maintain the validity of the one-loop expansion, the inflaton domination condition provides an approximate maximum value of $g' \sim \sqrt{H_T M_P}$. For concreteness, we can choose $\lambda' = \pi^2$ in which case the Minkowski spacetime decay condition gives $g'/H_T > 10.8$ whereas the percolation condition yields a limit $g'/H_T > 41$ making it the stronger condition. Then choosing $g' = \sqrt{H_T M_P}$ implies $H_T < 6 \times 10^{-3} M_P$. Thus one must choose an appropriate inflation model that satisfies the constraint placed on the Hubble rate.

After choosing an inflation model, one can solve the percolation condition to obtain x and thus fix the remaining parameter μ_{eff} . For example, in $\lambda\phi^4$ inflation we can have lowest possible Hubble rate at the end of inflation as $H_{\text{end}} = 2 \times 10^{-6} M_P = H_T$ that implies $x \leq 1.15$.

At this stage, we can also see why the transition field cannot play the role of the spectator to generate significant curvature perturbations. The Minkowski spacetime decay condition implies $g' > H_T$ whereas the thin wall condition requires $\mu_{\text{eff}} > g'$. Therefore $\mu_{\text{eff}} > H_T$ and hence the transition field is not light as compared to Hubble rate and would not generate significant non-Gaussianity. Therefore we need to include a third independent spectator field that is light during inflation in order to generate significant non-Gaussianity. In the current case we have chosen a massless spectator field χ as our third field in the total potential (4.80).

4.4.3 Analytical f_{NL} calculation

In the previous section we saw that there exist parameter values in a quartic decay potential for which the thin-wall approximation is satisfied at the border of the allowed range for $x > 1$, enabling the use of analytic expressions for the decay rate (4.84) to obtain the non-Gaussianity.

With the potential (4.80), the number of efolds are given by

$$N(\chi) = \int_{\phi_{\text{end}}}^{\phi_*} \frac{W}{W_{,\phi}} d\phi = \int_{\phi_{\text{end}}}^{\phi_*} d\phi \left(\frac{V(\phi) + U(\sigma) - g_I^2 \phi^2 \sigma^2 / 2 + g_S^2 \chi^2 \sigma^2 / 2}{V_{,\phi} - g_I^2 \sigma^2 \phi} \right). \quad (4.90)$$

Assuming a sudden change in σ due to vacuum transition at ϕ_T the number of efolds get split into

$$\begin{aligned} \delta N = & \int_{\phi_{\text{end}}}^{\phi_T} d\phi \left(\frac{V(\phi) + U(\sigma_{TV}) - g_I^2 \phi^2 \sigma_{TV}^2 / 2 + g_S^2 \chi^2 \sigma_{TV}^2 / 2}{V_{,\phi} - g_I^2 \sigma_{TV}^2 \phi} \right) \\ & + \int_{\phi_T}^{\phi_*} d\phi \left(\frac{V(\phi) + U(\sigma_{FV}) - g_I^2 \phi^2 \sigma_{FV}^2 / 2 + g_S^2 \chi^2 \sigma_{FV}^2 / 2}{V_{,\phi} - g_I^2 \sigma_{FV}^2 \phi} \right) \\ & - \int_{\phi_{\text{end}}}^{\phi_*} d\phi \frac{V}{V_{,\phi}} \end{aligned} \quad (4.91)$$

where I have subtracted the contribution of evolution from ϕ_* to ϕ_{end} to only capture change in N due to the vacuum transition.

Further assuming that the spectator field is negligible during inflation, the δN function becomes

$$\begin{aligned}\delta N &= \int_{\phi_{\text{end}}}^{\phi_T} d\phi \left(\frac{V(\phi) + U(\sigma_{TV}) - g_I^2 \phi^2 \sigma_{TV}^2/2}{V_{,\phi} - g_I^2 \sigma_{TV}^2 \phi} \right) \\ &+ \int_{\phi_T}^{\phi_*} d\phi \left(\frac{V(\phi) + U(\sigma_{FV}) - g_I^2 \phi^2 \sigma_{FV}^2/2}{V_{,\phi} - g_I^2 \sigma_{FV}^2 \phi} \right) \\ &- \int_{\phi_{\text{end}}}^{\phi_*} d\phi \frac{V}{V_{,\phi}}.\end{aligned}\quad (4.92)$$

The first and second derivatives of $\delta N(\chi)$ at $\chi = \bar{\chi}$ are

$$\frac{\partial(\delta N)}{\partial \chi} = \left(\frac{V(\phi_T) + U(\sigma_{TV}) - g_I^2 \phi_T^2 \sigma_{TV}^2/2}{V_{,\phi_T} - g_I^2 \sigma_{TV}^2 \phi_T} - \frac{V(\phi_T) + U(\sigma_{FV}) - g_I^2 \phi_T^2 \sigma_{FV}^2/2}{V_{,\phi_T} - g_I^2 \sigma_{FV}^2 \phi_T} \right) \frac{d\phi_T}{d\chi}, \quad (4.93)$$

$$\begin{aligned}\frac{\partial^2(\delta N)}{\partial \chi^2} &= \left(\frac{V(\phi_T) + U(\sigma_{TV}) - g_I^2 \phi_T^2 \sigma_{TV}^2/2}{V_{,\phi_T} - g_I^2 \sigma_{TV}^2 \phi_T} - \{\sigma_{TV} \rightarrow \sigma_{FV}\} \right) \frac{d^2 \phi_T}{d\chi^2} \\ &+ \left(\frac{(V(\phi_T) + U(\sigma_{FV}) - g_I^2 \phi_T^2 \sigma_{FV}^2/2)(V_{,\phi_T} \phi_T - g_I^2 \sigma_{FV}^2)}{(V_{,\phi_T} - g_I^2 \sigma_{FV}^2 \phi_T)^2} - \{\sigma_{FV} \rightarrow \sigma_{TV}\} \right) \left(\frac{d\phi_T}{d\chi} \right)^2\end{aligned}\quad (4.94)$$

where the notation $\{\sigma_{TV} \rightarrow \sigma_{FV}\}$ indicates replace σ_{TV} with σ_{FV} in the previous term.

The remaining ingredient that goes into the f_{NL} formula (4.79) is $\phi_T(\chi)$. We know that it is a solution of the percolation condition (cf. eq. (4.84))

$$a\mu_{\text{eff}}^{11} e^{-b\mu_{\text{eff}}^3} = H^4 \quad (4.95)$$

where $a = 8.74 \times 10^5 \lambda'^{3/2}/g'^7$ and $b = 266\sqrt{\lambda'}(1 - \lambda'/16\sqrt{3}\pi)/g'^3$. Approximating that H varies slowly with ϕ_T and differentiating the percolation condition with respect to χ we have

$$\frac{d\mu_{\text{eff}}}{d\chi} = 0 \quad \text{OR} \quad 3\mu_{\text{eff}}^3 b = 11. \quad (4.96)$$

Substituting the second equality back into percolation condition just gives one particular value of g' dependent on λ' . However, since the parameter g' in the potential is arbitrary, then the first equality is the one that must be satisfied and hence

$$\frac{d}{d\chi} \sqrt{\mu^2 + g_I^2 \sigma_{FV}^2 \phi_T^2 - g_S^2 \sigma_{FV}^2 \chi^2} = 0 \quad (4.97)$$

$$\implies \phi_T^2 = \phi_T^2(0) + \frac{g_S^2}{g_I^2} \chi^2 \quad (4.98)$$

where we have once again used the approximation that H or σ_{FV} does not depend on ϕ_T . $\phi_T(0)$ is the solution to the percolation condition when $\chi = 0$. Thus the problem of finding the non-Gaussianity has been reduced to solving the percolation condition for $\phi_T(0)$.

We can use the $\lambda\phi^4$ inflationary model to illustrate how the full calculation proceeds analytically.

However, this model is not observationally viable as it exhibits a high tensor-to-scalar ratio. We shall apply our method to an observationally-viable model in the section 4.5, but by obtaining the decay rate numerically.

In order to satisfy the power spectrum observation of $\mathcal{P}_* = 2.1 \times 10^{-9}$, $\lambda = 1.8 \times 10^{-13}$ in $V(\phi) = \lambda\phi^4/4$. Supposing the decay occurs close to the end of inflation so that it does not generate spurious consequences that affect the successful features of the post-inflationary stage. For $\lambda\phi^4$, the first slow-roll parameter $\epsilon = (M_P^2/2)(V'/V)^2$ becomes unity around $\phi_{\text{end}} \approx 3M_P$. Therefore we choose $\phi_T(0) = 5M_P$. Then $H_T \approx 3 \times 10^{-6}M_P$.

Fixing $\lambda' = \pi^2$ as before we then fix the maximum allowed $g' \sim \sqrt{H_T M_P} = 1.73 \times 10^{-3}M_P$ to obtain $x = 1.09$ from the percolation condition (4.95). Thus the thin-wall condition is just barely satisfied and there is a tension between the thin-wall and percolation conditions.

Assuming scale-invariant spectrum for the spectator field, its variance over our observable universe is

$$\langle \delta\chi^2 \rangle = \int \mathcal{P}(k) \frac{dk}{k} \approx \frac{H_T^2}{4\pi^2} \int \frac{dk}{k} \approx \frac{H_T^2}{4\pi^2} N_* \quad (4.99)$$

where k labels the momentum of a mode. With $H_T = 3 \times 10^{-6}M_P$, we find $\langle \delta\chi^2 \rangle \sim 10^{-10}M_P^2$. From the massless preheating model, it is known that the cosmic variance is infinite [75], making $\bar{\chi}$ a truly free parameter. We choose $\bar{\chi} = \sqrt{\langle \delta\chi^2 \rangle} \sim 10^{-5}M_P$. Then we fix $g_S = 0.01$ in order that the contribution to the inflationary evolution from the spectator field always remains sub-dominant.

Finally we are left with one free parameter g_I which can be varied to obtain the f_{NL} dependence in Figure 4.3. As a sample point, $f_{\text{NL}} = 2.32$ for $g_I = 1.5 \times 10^{-7}M_P$.

In summary, for $V(\phi) = 1.8 \times 10^{-13}\phi^4/4$ and the following sample parameter values in the total potential (4.80):

$$\lambda' = \pi^2, g' = 1.73 \times 10^{-3}M_P, \mu = 3.67 \times 10^{-4}M_P, g_I = 1.5 \times 10^{-7}M_P, g_S = 0.01, \bar{\chi} = 10^{-5}M_P \quad (4.100)$$

one can obtain a large and observable $f_{\text{NL}} \sim 1$. We can also verify that the simplifications made for the formalism in section 4.3 to hold are satisfied

inflaton dominated evolution (4.67): $(W(\sigma_{FV}) - W(\sigma_{TV}))/H_T^2 M_P^2 = 0.0036 \ll 1$

no bubble before N_* efolds (4.65): $n(N \geq N_*) \sim 10^{-468} \ll 1$

sudden change in σ at transition (4.66): $|dp/dN|_{N=N_T} = 1.28 > 1$

Thus we have seen that a vacuum transition during inflation can give rise to observable non-Gaussianity in principle. However, to maintain an analytical handle over the calculation, we have used Minkowski decay rate (4.28) even though the background spacetime during inflation is curved.

In the next section 4.5, we shall improve upon the present calculation by considering the background spacetime to be de Sitter. But as we saw in section 4.2, a Hawking-Moss transition never

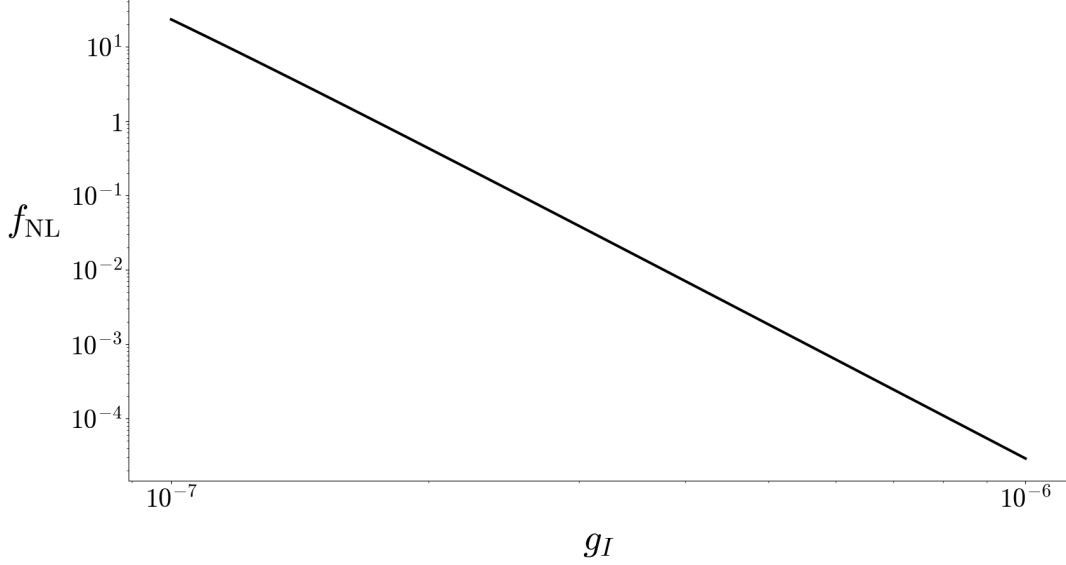


Figure 4.3: Non-Gaussianity parameter f_{NL} as a function of the coupling g_I between inflaton and transition field, for $\lambda\phi^4$ inflation with a quartic decay potential. Other parameters are fixed to the values quoted in eq. (4.100). Over the range shown here, $f_{\text{NL}} \propto g_I^{-6}$ approximately.

completes during inflation, precluding the use of an analytical expression for the decay rate. Therefore we calculate the de Sitter decay rate numerically for a Coleman De Luccia type transition. By doing so we can also alleviate the tension between the thin-wall approximation and percolation condition present in the illustrative treatment of this section. Furthermore, to make the model observationally viable, we consider the inflationary sector to be Higgs inflation instead of the simple $\lambda\phi^4$ quartic inflation that has been ruled out by Planck satellite measurements [63].

4.5 Higgs inflation model

4.5.1 Non-minimal coupling to gravity

We consider the inflationary sector to be Higgs inflation [34], so the inflaton ϕ is the Higgs field. In Higgs inflation, ϕ is non-minimally coupled to gravity with the non-minimal coupling parameter ξ_ϕ being large $\xi_\phi \gg 1/6$. A non-minimal coupling to gravity must necessarily be included in the action due to QFT in curved spacetime considerations [27]. Therefore we can also include a non-minimal coupling for the transition and spectator fields through parameters ξ_σ and ξ_χ respectively.

For simplicity, we will assume that the non-minimal coupling for the transition field σ has the same magnitude but opposite sign to the Higgs field, i.e., $\xi = \xi_\phi = -\xi_\sigma \gg 1/6$. Then a coupling between the Higgs field and the vacuum transition field arises naturally through the non-minimal term and no direct coupling between the fields is necessary. On the other hand, we assume that the spectator field χ is minimally coupled, i.e., $\xi_\chi = 0$.

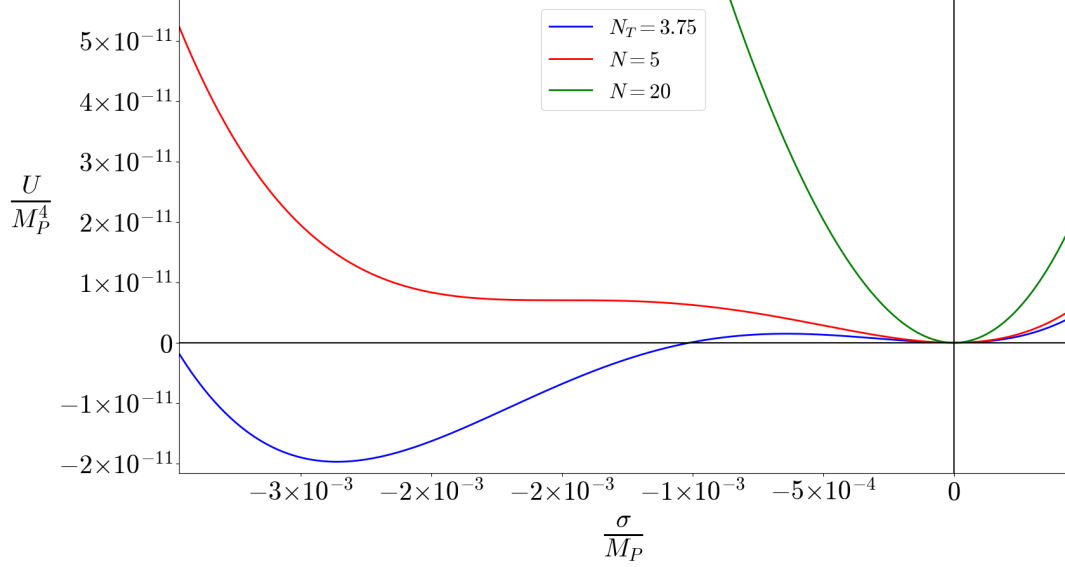


Figure 4.4: $U(\phi, \sigma, \chi = 0)$ (4.104) at different times during inflation, labelled by the number of e-folds N . This potential is chosen in such a way that when $\mathcal{R}$ is sufficiently high, it has a local minimum, a false vacuum, at $\sigma_{FV} = 0$. The transition field parameters are $\lambda' = 100, g' = 10^{-1} M_P$ and $\mu = 5 \times 10^{-3} M_P$. Figure reproduced from ref. [2].

Let us therefore consider the action in Jordan frame

$$S = \int d^4x \sqrt{-g} \left(f(\phi, \sigma) \mathcal{R} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - W(\phi, \sigma, \chi) \right), \quad (4.101)$$

where

$$f(\phi, \sigma) = \frac{M_P^2}{2} + \frac{1}{2} \xi \phi^2 - \frac{1}{2} \xi \sigma^2 \quad (4.102)$$

and

$$W(\phi, \sigma, \chi) = \frac{\lambda \phi^4}{4} + \lambda' \frac{\sigma^4}{4!} + g' \frac{\sigma^3}{3!} - \mu^2 \frac{\sigma^2}{2!} + \frac{1}{2} g_S^2 \chi^2 \sigma^2. \quad (4.103)$$

Here $\mathcal{R}$ is the Ricci scalar. The transition field part of the potential is

$$U(\phi, \sigma, \chi) = \lambda' \frac{\sigma^4}{4!} + g' \frac{\sigma^3}{3!} + \mu_{\text{eff}}^2 \frac{\sigma^2}{2!}, \quad (4.104)$$

where the effective mass of the transition field is given by

$$\mu_{\text{eff}}^2 = -\mu^2 + \xi \mathcal{R} + g_S^2 \chi^2. \quad (4.105)$$

Its evolution during inflation is presented in Figure 4.4.

Let us look at the action from the perspective of inflationary evolution of the background. For a majority of the evolution, the decaying field remains stuck in its false vacuum until the transition occurs at some value of ϕ_T towards the end of inflation. Therefore, we can make an approximation $\sigma \approx \sigma_{FV}$. Similarly, within one Hubble volume we assume that the spectator field χ is homogeneous and denote its value by χ_i . Therefore the kinetic terms for σ and χ vanish. Then the action becomes

$$S \approx \int d^4x \sqrt{-g} \left(f(\phi, \sigma_{FV}) \mathcal{R} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - W(\phi, \sigma_{FV}, \chi_i) \right). \quad (4.106)$$

The inflaton dynamics is best described in the Einstein frame where its slow roll equations of motion possess a simplified form. Making a conformal transformation of the metric

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = \frac{2f(\phi, \sigma_{FV})}{M_P^2} g_{\mu\nu}, \quad (4.107)$$

the field develops a non-canonical kinetic term [103]

$$S \approx \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{\mathcal{R}} - \left(\frac{1}{1 + \xi \frac{\phi^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2}} + \frac{6\xi^2 \frac{\phi^2}{M_P^2}}{(1 + \xi \frac{\phi^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2})^2} \right) \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right. \\ \left. - \frac{1}{(1 + \xi \frac{\phi^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2})^2} W(\phi, \sigma_{FV}, \chi_i) \right). \quad (4.108)$$

We can convert the non-canonical kinetic term into canonical form by redefining the field $\phi \rightarrow \tilde{\phi}$ with

$$\frac{d\tilde{\phi}}{d\phi} = \sqrt{\frac{1}{1 + \xi \frac{\phi^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2}} + \frac{6\xi^2 \phi^2}{\left(1 + \xi \frac{\phi^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2}\right)^2}}. \quad (4.109)$$

This expression, along with the potential in the Einstein frame,

$$\tilde{W} \approx \frac{W(\phi, \sigma_{FV}, \chi_i)}{(1 + \xi \frac{\phi^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2})^2} \quad (4.110)$$

is sufficient to calculate the inflationary evolution numerically in the Einstein frame.

Owing to the inflaton dominated assumption, we neglect the false vacuum term $f(\phi, \sigma_{FV}) \approx f(\phi) \equiv (M_P^2 + \xi \phi^2)/2$ as a further simplification. Then the inflationary sector is identical to that of Higgs inflation [34]. Therefore we fix the self coupling to its measured value for the Higgs field $\lambda \simeq 0.1$ [8] and calculate the inflationary power spectrum and tensor to scalar ratio. Inflation ends when the slow-roll parameter becomes large,

$$\epsilon = \frac{M_P^2}{2} \left(\frac{\tilde{W}_\phi}{\tilde{W}} \frac{d\phi}{d\tilde{\phi}} \right)^2 \sim 1, \quad (4.111)$$

which occurs at

$$\phi_{\text{end}} = \sqrt{\frac{1}{2\xi} \left(\frac{\sqrt{192\xi^2 + 32\xi + 1}}{6\xi + 1} - \frac{1}{6\xi + 1} \right)}. \quad (4.112)$$

The number of efolds are counted as

$$\begin{aligned} N &= \int_{\phi_{\text{end}}}^{\phi} \frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 d\phi \\ &= \left(\frac{\phi^2(6\xi + 1)}{8} + \frac{1 - \sqrt{(8\xi + 1)(24\xi + 1)}}{16\xi} - \frac{3}{4} \log \left(\frac{2(6\xi + 1)(\phi^2\xi + 1)}{12\xi + \sqrt{(8\xi + 1)(24\xi + 1)} + 1} \right) \right) \end{aligned} \quad (4.113)$$

from which we obtain ϕ_* for when the currently observed scale left the horizon $N_* = 55$. Then the curvature power spectrum (2.50) [61]

$$\mathcal{P}_* = \frac{1}{24\pi^2} \frac{\tilde{W}}{\epsilon} \Big|_{\phi=\phi_*}, \quad (4.114)$$

and tensor to scalar ratio (2.53) [61]

$$r = 16\epsilon \Big|_{\phi=\phi_*} \quad (4.115)$$

can be determined. The observed inflationary power spectrum $\mathcal{P}_* = 2.1 \times 10^{-9}$ [4] is obtained for $\xi = 15000$. For this value of the non-minimal coupling we have $\phi_{\text{end}} = 0.00877M_P$, $\phi_* = 0.0722M_P$ and the tensor to scalar ratio $r = 0.0035$. This value lies well within the current observational bound $r < 0.03$ [64, 65]. Thus the inflationary sector corresponds to Higgs inflation and is observationally viable.

4.5.2 Spectator field fluctuations

To calculate the non-Gaussianity, we need to examine the dynamics of the spectator field. Its variance over separate Hubble volumes is calculated in the Einstein frame. Keeping the field χ as dynamical, we have the action

$$S \approx \int d^4x \sqrt{-\tilde{g}} \left(\frac{M_P^2}{2} \tilde{\mathcal{R}} - \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \tilde{\phi} \partial_\nu \tilde{\phi} - \kappa(\tilde{\phi}) \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \chi \partial_\nu \chi - \tilde{W}(\tilde{\phi}, \chi) \right),$$

where the χ field now has a non-minimal kinetic term

$$\kappa(\tilde{\phi}) = \left(1 + \xi \frac{\phi(\tilde{\phi})^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2} \right)^{-1}, \quad (4.116)$$

and the potential in the Einstein frame is

$$\tilde{W}(\tilde{\phi}, \chi) = \frac{W(\phi(\tilde{\phi}), \sigma_{FV}, \chi)}{(1 + \xi \frac{\phi(\tilde{\phi})^2}{M_P^2} - \xi \frac{\sigma_{FV}^2}{M_P^2})^2}. \quad (4.117)$$

Then the equations of motion for the fields are

$$\ddot{\tilde{\phi}} + 3\tilde{H}\dot{\tilde{\phi}} - \frac{\kappa_{,\tilde{\phi}}}{2}\dot{\chi}^2 + \tilde{W}_{,\tilde{\phi}} = 0 \quad (4.118)$$

$$\ddot{\chi} + 3\tilde{H}\dot{\chi} - \frac{\kappa_{,\tilde{\phi}}}{\kappa}\dot{\tilde{\phi}}\dot{\chi} + \frac{\tilde{W}_{,\chi}}{\kappa} = 0. \quad (4.119)$$

Now assuming the background spectator field and its derivative are negligible during inflation, the super-horizon mode equation for the first-order fluctuation of the spectator is

$$\delta\ddot{\chi}_k + 3\tilde{H}\delta\dot{\chi}_k + \frac{\kappa_{,\tilde{\phi}}}{\kappa}\dot{\tilde{\phi}}\delta\dot{\chi}_k + \frac{1}{\kappa}\tilde{W}_{,\chi\chi}\delta\chi_k = 0. \quad (4.120)$$

We use the background slow-roll equations

$$3\tilde{H}\dot{\tilde{\phi}} = -\tilde{W}_{,\tilde{\phi}} \quad \text{and} \quad \tilde{H}^2 = \frac{\tilde{W}}{3M_P^2} \quad (4.121)$$

to obtain

$$\delta\ddot{\chi}_k + 3\tilde{H} \left(1 - M_P^2 \frac{\tilde{W}_{,\tilde{\phi}}}{3\tilde{W}} \frac{\kappa_{,\tilde{\phi}}}{\kappa} \right) \delta\dot{\chi}_k + \frac{1}{\kappa} \tilde{W}_{,\chi\chi} \delta\chi_k = 0. \quad (4.122)$$

Before a mode exits the horizon, we take its evolution to be scale invariant and hence the power spectrum at horizon crossing as $\mathcal{P}(t_k) = \tilde{H}_k^2/4\pi^2$. Thereafter its evolution is described by the first order mode equation above and it develops an overdamped envelope $e^{-F(\phi_k, \phi_T)}$ from horizon crossing at ϕ_k to the transition at ϕ_T . The overdamping coefficient is obtained as

$$\begin{aligned} F(\phi_k, \phi_T) = & \frac{3}{2} \int_{\phi_T}^{\phi_k} d\phi \left(1 - M_P^2 \frac{\tilde{W}_{,\tilde{\phi}}}{3\tilde{W}} \frac{\kappa_{,\tilde{\phi}}}{\kappa} \left(\frac{d\phi}{d\tilde{\phi}} \right)^2 \right) \\ & \times \left(1 - \sqrt{1 - \frac{4}{3} M_P^2 \frac{\tilde{W}_{,\chi\chi}}{\kappa\tilde{W}} \left(1 - M_P^2 \frac{\tilde{W}_{,\tilde{\phi}}}{3\tilde{W}} \frac{\kappa_{,\tilde{\phi}}}{\kappa} \left(\frac{d\phi}{d\tilde{\phi}} \right)^2 \right)^{-2}} \right) \frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2. \end{aligned} \quad (4.123)$$

Hence the variance (2.33) is given by

$$\langle \delta\chi^2 \rangle = \int \frac{dk}{k} \mathcal{P}(k) = \int \frac{dk}{k} \frac{\tilde{H}_k^2}{4\pi^2} |e^{-F(\phi_k, \phi_T)}|^2. \quad (4.124)$$

Using the slow-roll Friedmann equation and the horizon crossing relation $k = a_k \tilde{H}_k = a_{\text{end}} e^{-N_k} \tilde{H}_k$

we can rewrite the integral in terms of the Jordan frame field ϕ as

$$\langle \delta\chi^2 \rangle = \int d\phi \left(\frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 - \frac{1}{2} \frac{\tilde{W}_{,\phi}}{\tilde{W}} \right) \frac{\tilde{W}}{12\pi^2 M_P^2} |e^{-F(\phi, \phi_T)}|^2 \quad (4.125)$$

where the scale factor is assumed to be unity at the end of inflation, $a_{\text{end}} = 1$.

The variance at transition ϕ_T within our observable universe is calculated by choosing the appropriate integration limits

$$\langle \delta\chi_T^2 \rangle = \int_{\phi_T}^{\phi_*} d\phi \left(\frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 - \frac{1}{2} \frac{\tilde{W}_{,\phi}}{\tilde{W}} \right) \frac{\tilde{W}}{12\pi^2 M_P^2} |e^{-F(\phi, \phi_T)}|^2. \quad (4.126)$$

where ϕ_* corresponds to when the currently observed scale left the horizon. Similarly the cosmic variance, the variance of modes that lie outside our observable universe at transition (i.e. modes which left the horizon before N_* efolds) is given by

$$\langle \bar{\chi}^2 \rangle = \int_{\phi_*}^{\phi_{\text{begin}}} d\phi \left(\frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 - \frac{1}{2} \frac{\tilde{W}_{,\phi}}{\tilde{W}} \right) \frac{\tilde{W}}{12\pi^2 M_P^2} |e^{-F(\phi, \phi_T)}|^2, \quad (4.127)$$

assuming inflation begins at some $\phi = \phi_{\text{begin}}$.

After populating the separate universes with a Gaussian distribution (2.66) with mean $\bar{\chi}$ and variance $\langle \delta\chi_T^2 \rangle$ at transition, the non-Gaussianity parameter can be calculated using the δN -formalism (4.79). However, the δN -calculation depends on time evolution and must take place in the Einstein frame. Therefore we need to include the full Einstein frame potential (4.110) to calculate the derivatives of δN (4.71)–(4.72) in the Einstein frame

$$\frac{\partial(\delta N)}{\partial\chi} = \Delta \left(\frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 \right) \bigg|_{\phi=\phi_T} \frac{d\phi_T}{d\chi} \quad (4.128)$$

$$\frac{\partial^2(\delta N)}{\partial\chi^2} = \Delta \left(\frac{\partial}{\partial\phi} \left(\frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 \right) \right) \bigg|_{\phi=\phi_T} \left(\frac{d\phi_T}{d\chi} \right)^2 + \Delta \left(\frac{\tilde{W}}{\tilde{W}_{,\phi}} \left(\frac{d\tilde{\phi}}{d\phi} \right)^2 \right) \bigg|_{\phi=\phi_T} \frac{d^2\phi_T}{d\chi^2}. \quad (4.129)$$

4.5.3 Vacuum transition in de Sitter space

A nearly scale-invariant assumption for inflation implies the transition can be considered to occur in a fixed de Sitter background, $H \approx \text{const}$. The vacuum transition dynamics is best studied in the Jordan frame where the effective mass for the transition field is directly given by eq. (4.105) where $\mathcal{R}$ can be obtained through the background action (4.106) whose equations of motion for the

inflaton field and metric are [137]

$$\nabla_\alpha \nabla^\alpha \phi = W_{,\phi} - \xi \phi \mathcal{R}, \quad (4.130)$$

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathcal{R} = \frac{\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g_{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi - g_{\mu\nu} W + (\nabla_\mu \nabla_\nu - g_{\mu\nu} \nabla_\alpha \nabla^\alpha)(M_P^2 + \xi \phi^2 - \xi \sigma_{FV}^2)}{M_P^2 + \xi \phi^2 - \xi \sigma_{FV}^2}. \quad (4.131)$$

Contracting the metric equation with $g^{\mu\nu}$ and making use of the inflaton field equation we find

$$\mathcal{R} = \frac{(1 + 6\xi) \partial_\mu \phi \partial^\mu \phi + 4W + 6\xi \phi W_{,\phi}}{M_P^2 + \xi(1 + 6\xi)\phi^2 - \xi \sigma_{FV}^2}. \quad (4.132)$$

Derivatives of the field can be ignored under the slow roll assumption, making

$$\mathcal{R} \approx \frac{(1 + 6\xi) \lambda \phi^4 + 4 \left(\lambda' \frac{\sigma_{FV}^4}{4!} + g' \frac{\sigma_{FV}^3}{3!} - \mu^2 \frac{\sigma_{FV}^2}{2!} + g_S^2 \chi_i^2 \frac{\sigma_{FV}^2}{2} \right)}{M_P^2 + \xi(1 + 6\xi)\phi^2 - \xi \sigma_{FV}^2}. \quad (4.133)$$

Owing to the high non-minimal coupling value $\xi \gg 1$ and the inflaton dominated approximation, this can be further simplified to

$$\mathcal{R} \approx \frac{\lambda \phi^2}{\xi}. \quad (4.134)$$

Finally the effective mass (4.105) becomes

$$\mu_{\text{eff}}^2 \approx -\mu^2 + \lambda \phi^2 + g_S^2 \chi^2, \quad (4.135)$$

which decreases towards the end of inflation as required for the vacuum transition to have observable effects.

The fixed de Sitter background in the current case has

$$H_T = \sqrt{\frac{\mathcal{R}(\phi_T)}{12}}, \quad (4.136)$$

which implies ϕ can be fixed to its value at transition ϕ_T for the purposes of the decay rate calculation.

On a fixed de Sitter background, the transition field has an equation of motion [115, 124]

$$\frac{d^2 \sigma}{dr^2} + 3H \cot(Hr) \frac{d\sigma}{dr} = \frac{dU}{d\sigma} \quad (4.137)$$

in Euclideanised space with radius r and with boundary conditions

$$\left. \frac{d\sigma}{dr} \right|_{r=0} = \left. \frac{d\sigma}{dr} \right|_{r=\pi/H} = 0. \quad (4.138)$$

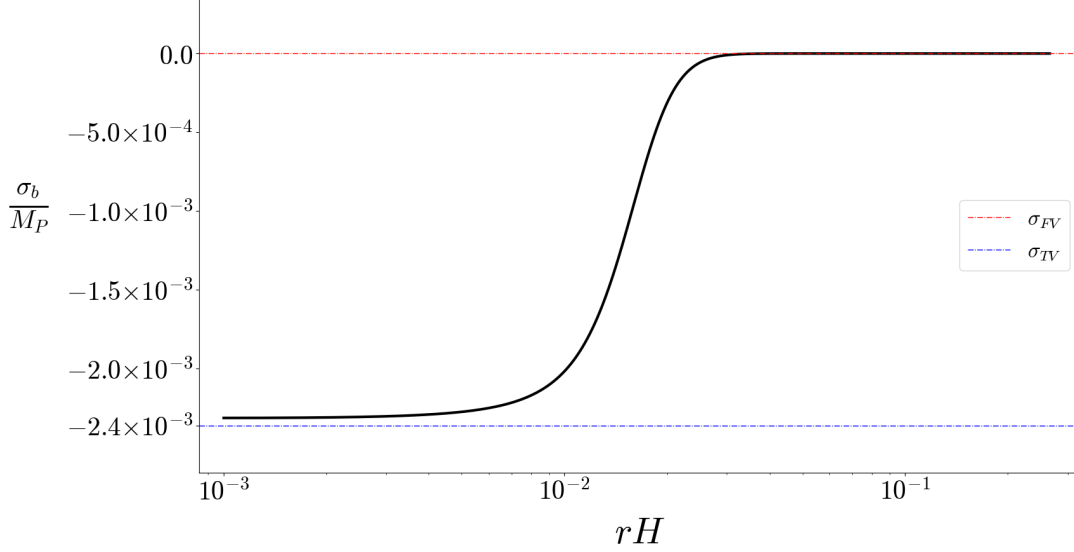


Figure 4.5: Numerical de Sitter bounce solution σ_b at $N_T = 3.75$ efolds for the Higgs inflation model with parameters in eq. (4.143)

I calculate the decay rate numerically using Mathematica by employing a shooting algorithm on a domain $rH \in [10^{-3}, \pi - 10^{-3}]$, with working precision of 70 digits. Once the bounce solution σ_b is obtained as in Figure 4.5, its action is calculated by numerically integrating eq. (4.44). The exponential coefficient in the decay rate B is obtained by subtracting the constant part at false vacuum, $B = S_E[\sigma_b] - S_E[\sigma_{FV}]$. Unless otherwise specified, I always take the pre-factor in the decay rate to be $A = \mu_{\text{eff}}^4$ in numerical calculations.

4.5.4 Parameter dependence of f_{NL}

Due to Hawking-Moss transition remaining incomplete during inflation (4.64), the parameter space of interest in our setup is reduced to

$$\mu_{\text{eff}} \gtrsim H_T, \quad (4.139)$$

where the Coleman–De Luccia transition dominates.

The effective mass for the Higgs inflation scenario is given by eq. (4.135). Because we are always assuming the dynamics of the χ field is subdominant to both the inflationary and decay sectors, the last term in this expression contributes negligibly to μ_{eff} . The middle term, $\lambda\phi^2$ however, gets fixed by the Higgs inflation parameter $\lambda \sim 10^{-1}$ and typical value of ϕ over the last N_* efolds $\phi \sim 10^{-2}M_P$ giving a contribution to $\mu_{\text{eff}} \propto 10^{-3}M_P$. At higher values of ϕ , the potential admits only one vacuum at $\sigma = 0$. To maintain the existence of the false vacuum towards the end of inflation when transition takes place, $\lambda\phi_T^2 - \mu^2 < (3/8)g'^2/\lambda'$. Therefore μ should lie close to $\lambda\phi_T^2$ and we choose $\mu = 5 \times 10^{-3}M_P$.

For an effective mass around $10^{-3}M_P$, the transition completes close to the end of inflation if

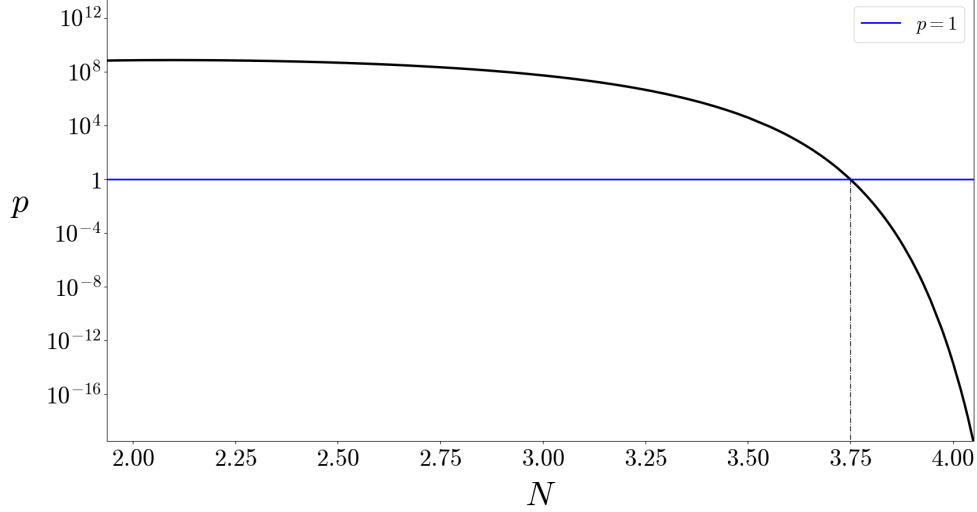


Figure 4.6: Percolation parameter during Higgs inflation as a function of the number of efolds and with transition field parameters $\lambda' = 100, g' = 10^{-1}M_P$ and $\mu = 5 \times 10^{-3}M_P$. Figure reproduced from ref. [2].

the value of g' is around $10^{-1}M_P$. But for such a high value of g' , in order to maintain the inflaton dominated simplification, one has to take $\lambda' \sim 100$. This value is not in the perturbative range and should be seen—just as g' —as a phenomenological parameter describing the tunnelling dynamics. One may, therefore, either assume that the latter occurs in a regime of non-perturbative couplings or construct a more sophisticated but perturbative sector describing the tunnelling field. One may also consider modifying the potential for ϕ beyond minimal Higgs inflation.

Following the above discussion, let us fix the parameter values for transition sector to be

$$\lambda' = 100, g' = 10^{-1}M_P, \mu = 5 \times 10^{-3}M_P. \quad (4.140)$$

The plot of the percolation parameter as a function of number of efolds for these parameters is given by Figure 4.6. One can see the transition completes at $N_T \approx 3.75$. We can also check that all the simplifications for our calculation to hold are satisfied. First, eq. (4.67) is satisfied, $(\tilde{W}(\sigma_{FV}) - \tilde{W}(\sigma_{TV}))/\tilde{H}_T^2 M_P^2 = 0.0093 \ll 1$, and therefore the evolution is inflaton dominated. Second, $\langle n_{\text{bub}} \rangle = 0$ as the potential admits only one vacuum above $N > 5$ efolds, and therefore eq. (4.47) is satisfied implying no bubbles can have nucleated before the largest currently observed scale left the horizon. Third, eq. (4.66) is satisfied, $|dp/dN|_{N=N_T} = 64.46 \gg 1$, and therefore the transition happens within one e-folding.

Our scenario can generate significant non-Gaussianity if the effective mass of the spectator field is light, i.e. $m_\chi = g_S \sigma_{FV} < H_T$. For the action (4.101) with above transition field parameters, the effective mass of spectator field $m_\chi = g_S \sigma_{FV}$ is zero because $\sigma_{FV} = 0$. This makes the cosmic variance (4.127) proportional to how long inflation lasted beyond $N > N_*$ efolds, and as such it depends on the specific inflationary scenario being considered. As the cosmic mean is drawn from

a Gaussian distribution with variance equal to the cosmic variance, it further makes the cosmic mean an independent parameter. We shall only examine the cosmic mean over an arbitrary range $\bar{\chi} \in [0, 4\sqrt{\langle\delta\chi_T^2\rangle}]$. For the transition field parameters fixed above, the variance at transition (4.126) is found to be

$$\langle\delta\chi_T^2\rangle = 4.5 \times 10^{-11} M_P^2. \quad (4.141)$$

The remaining parameter is the coupling between spectator and transition sectors g_S . I check that over a range $g_S \in [0.1, 10]$ and $\bar{\chi} \in [0, 4\sqrt{\langle\delta\chi_T^2\rangle}]$ as before, the transition at $N_T = 3.75$ e-folds is not affected. This is in keeping with the spectator dynamics being assumed sub-dominant to the transition dynamics. However, this allows us to obtain the variation in f_{NL} as a function of the spectator sector parameters g_S and $\bar{\chi}$ in the above range analytically directly using the non-Gaussianity calculation method expressed through eq. (4.79).

To simplify the calculation for the sub-leading term in f_{NL} (4.79), we divide the domain of integration into two: when $0 < q < k_*$, we approximate $\{\Sigma(|\vec{k}_* + \vec{q}|), \Sigma(|\vec{k}_* - \vec{q}|)\} \approx \Sigma(k_*)$ and when $k_* < q < q(N_T)$, we approximate $\{\Sigma(|\vec{k}_* + \vec{q}|), \Sigma(|\vec{k}_* - \vec{q}|)\} \approx \Sigma(q)$. Hence the total momentum integral is approximated as

$$\int \frac{d\vec{q}}{4\pi} \Sigma(q) \Sigma(|\vec{k}_* + \vec{q}|) \Sigma(|\vec{k}_* - \vec{q}|) \approx \Sigma(k_*)^2 \int_{L^{-1}}^{k_*} q^2 dq \Sigma(q) + \int_{k_*}^{q(N_T)} q^2 dq \Sigma(q)^3, \quad (4.142)$$

and calculated numerically. Note that we have also introduced an infrared cut-off L^{-1} corresponding to the size of the observable universe. In reality the cut-off, which arises from the δN formalism, is not sharp. Therefore it is not possible to assign a precise value to L . But we know that the size of the observable universe as compared to the scale at which non-Gaussianity observations take place must satisfy $\ln(k_* L) \gtrsim 1$. The non-Gaussianity parameter f_{NL} (4.79) increases with $\ln(k_* L)$, therefore to be on the conservative side I take $\ln(k_* L) = 1$ in our calculations to clearly demonstrate that non-Gaussianity owing to vacuum transition during inflation can be large.

In Figure 4.7, I plot f_{NL} as a function of the spectator sector parameters g_S and $\bar{\chi}$. The dependence on g_S reveals an approximately power law scaling which I find to be $f_{\text{NL}} \propto g_S^6$. On the other hand, the dependence on $\bar{\chi}$ is slight.

Finally I summarise by providing a sample set of parameter values for the Higgs inflation scenario:

$$\lambda' = 100, g' = 10^{-1} M_P, \mu = 5 \times 10^{-3} M_P, \bar{\chi} = 0, g_S = 0.5. \quad (4.143)$$

For these sample values the non-Gaussianity parameter is calculated to be

$$f_{\text{NL}} = -25.96. \quad (4.144)$$

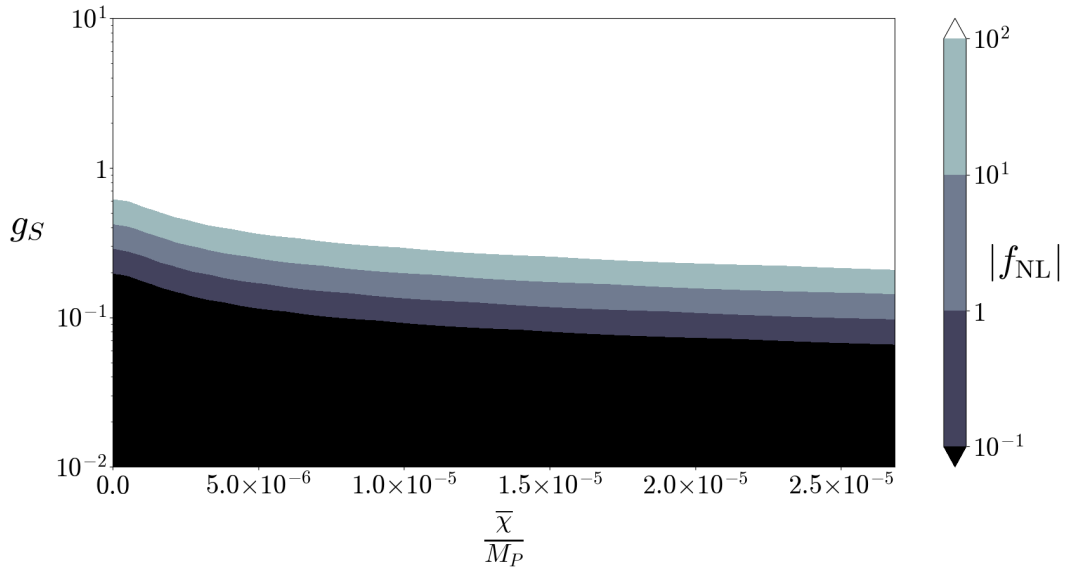


Figure 4.7: Non-Gaussianity parameter f_{NL} as a function of spectator coupling g_S and cosmic mean $\bar{\chi}$. We highlight non-Gaussianity parameter values in the range $f_{\text{NL}} \in [0.1, 100]$ which might be accessible by current and future experiments. Other parameters in the Higgs inflation scenario are fixed to their sample values: $\lambda' = 100$, $g' = 10^{-1} M_P$ and $\mu = 5 \times 10^{-3} M_P$. Figure reproduced from ref. [2].

Chapter 5

Conclusion

5.1 Summary

Curvature perturbations from the very early universe are accessible through cosmological observations of the cosmic microwave background and large scale structure. In the absence of any non-adiabatic pressure the curvature perturbations on these large, observable scales remain conserved from the time they leave the horizon during inflation until horizon re-entry during radiation domination. However, through this work, we saw how non-equilibrium phenomena such as vacuum transition or preheating taking place even after the observable scale has left the horizon can still leave imprints upon the curvature perturbations that would show up in the observations, specifically in terms of their non-Gaussian statistics.

The crucial ingredient to have a significant amount of non-Gaussianity, such that it can be detected by current experiments and those in the near future, is the presence of a light scalar field.

For both preheating and vacuum transition I have tried to present a complete treatment: extending or developing a new model-agnostic formalism to connect a particular non-equilibrium phenomena to the observable of non-Gaussianity, constructing a realistic model, illustrating the formalism by applying it to this model, and finally studying the parameter dependence of non-Gaussianity in that model.

In particular, for preheating we extended the non-perturbative delta N formalism [74, 78, 100] used for non-Gaussianity calculation and illustrated it by applying to an observationally viable model of massless preheating motivated by non-minimal coupling to gravity.

Interestingly, we find that calculating non-Gaussianity from preheating at currently observed scale, which occurs after the end of inflation, involves taking into account the evolution of modes at very early times, well-before they left the horizon.

In a separate universe picture, the evolution of each individual causal volume of our currently observable universe depends sensitively on initial conditions [76]. However, the overall probability distribution of initial conditions is Gaussian and depends on the initial variance as well as the mean over our currently observable universe. This mean itself depends on the cosmic variance over the

entire universe, of which our observable universe is just a small part.

If the spectrum of the spectator field is sufficiently blue-tilted during inflation, then the cosmic variance becomes negligible and hence the dependence on modes before the currently observed scale left the horizon disappears. As an example we have seen how this occurs in our preheating model. We are able to arrive at this finding precisely because of considering the full time-dependence of Hubble rate during inflation.

When time-dependence of Hubble rate is included, it translates to the spectator field having a scale-dependent spectrum. Thus we may no longer use a simple scale-invariant approximation to calculate non-Gaussianity [81]. We have also seen how considering scale dependence yields a f_{NL} value that is many orders of magnitude different from just using a scale-invariant approximation.

We have scanned for f_{NL} over a parameter range $g^2/\lambda = 1$ to 3 in our model, which is relevant from parametric resonance considerations [5]. We find that f_{NL} changes by many orders of magnitude within this range. Therefore both detection or non-detection of non-Gaussianity can be used to constrain the coupling strength in preheating models. The maximal $f_{\text{NL}} \sim 10^{-4}$ occurs around $g^2/\lambda = 1.625$ with $\xi = 0.004$ fixed.

For vacuum transition during inflation in the presence of a light scalar field, I have outlined a general method to calculate the non-Gaussianity of primordial curvature perturbations. By using this method, one can obtain an analytical estimate of the non-Gaussianity parameter (4.79) for a generic potential through the difference in its derivatives before and after the transition (4.71)–(4.72), and the derivatives of its effective mass for the transition field (4.77)–(4.78).

By applying this method to the Higgs inflation model, we have conclusively seen that vacuum transition occurring during inflation in the presence of a light spectator field can be the source of large non-Gaussianity. We find that f_{NL} takes values in the current observationally allowed range of $f_{\text{NL}}^{\text{local}} = -0.9 \pm 5.1$ [53] for a wide range of cosmic mean $\bar{\chi}$ values and a narrower range of spectator field couplings g_S . Higgs inflation model parameters which yield a f_{NL} value larger than the observationally allowed range can already be ruled out. Cosmological experiments in the near future [67] will further restrict this allowed range and can, therefore, more tightly constrain the parameter space of such models involving vacuum transition during inflation.

Finally, we note the requirement of a separate light spectator field in addition to the inflaton and vacuum transition fields in our model. For the transition field to be light, the top of the barrier must be flatter. This implies $|U''_{\text{bar}}| \approx |U''_{FV}|$ for a quartic decay potential, and hence the Hawking Moss transition, which is the dominant one for small effective mass, can never percolate as shown in Section 4.5.4. Thus if the transition field is light then decay never completes and hence a separate light spectator field is required to generate the fluctuations that modulate the vacuum transition rate.

5.2 Outlook

In this work we were only interested in large, CMB scales. However, interesting physical processes such as Primordial Black Hole formation can take place on small scales. Recently, there has been work that extends the gradient expansion of the ADM equations to first order in $\varepsilon = k/aH$ [138]. Such a treatment becomes necessary if one wants to examine the dynamics near horizon crossing $k \sim aH$ which is useful in estimating the production of Primordial Black Holes (PBHs) [139] for example.

In the separate universe picture, we use a Gaussian probability distribution function for the light scalar field to populate the separate Hubble volumes. This is because the light scalar field did not have any self-interactions, for example, a quartic term $\lambda_\chi \chi^4$ in the potential. The stochastic formalism [140] can be used to obtain probability distribution of a light interacting scalar field on super-horizon scales [121].

As alluded to in section 2.4, the non-perturbative phenomena of oscillon formation and evolution post inflation can also be the source of significant curvature perturbations in the presence of a light scalar field. But this has not been covered in the present work and can be the subject of further investigation.

To simplify the preheating simulations, we drop the effects of non-canonical kinetic terms arising from conversion of Jordan to Einstein frame in the two-field case. Simulations for with non-canonical kinetic terms have been performed [141, 142] for non-minimal coupling parameter $\xi > 1$. Such simulations, which incorporate the curvature in field space, are able to capture the full effects of non-minimal coupling and can be used to calculate non-Gaussianity using the formalism outlined in this work.

Lattice simulations of preheating with non-minimal coupling can also be performed directly in the Jordan frame [143]. It would be interesting to perform these simulations for the non-minimally coupled massless preheating model we have considered and compare our results.

For preheating, the value of the non-Gaussianity parameter we found $f_{\text{NL}} \sim 10^{-4}$ is consistent with current observational bounds [53]. It is too small to be detected by cosmological experiments in the near future, but perhaps is large enough that a different measure of non-Gaussianity, for example using wavelet transform [144], can be used to detect it. Finding such a measure that captures the inherently non-linear nature of the fields during preheating is a possible direction for future work.

Nonetheless, the full calculation of f_{NL} in our model enables a complete illustration of our formalism to obtain non-Gaussianity from preheating. This method can very well be applied to other interesting models of inflation and reheating, possibly yielding high, detectable non-Gaussianity.

In the case of vacuum transition during inflation, the role of the light field can be played by the transition field [80] itself or by another scalar field. In this work, we have considered the latter scenario, which can be more natural in theories with multiple scalar fields because it does not require

the transition field to be light.

Scenarios with multiple fields and vacua are ubiquitous in fundamental particle physics theories such as Grand Unified Theories (GUTs) [8] and string theory [145–147]. It is possible that transition between these vacua takes place in the very early universe, during the inflationary stage, when fields take trans-Planckian values.

We have argued that the decay for a light transition field never completes during inflation for a quartic decay potential (4.64). However, if the decay potential contains operators higher than dimension four, then it could be possible to simultaneously satisfy the light effective mass and percolation conditions [80]. When the interactions and particle content required to have a vacuum transition during inflation scenario arise from integrating out sectors beyond the Standard Model, such higher dimensional operators may be generated. Finding well-motivated particle physics models where the transition field remains light can be a direction for further work.

The Higgs inflation model that we have used to study vacuum transition during inflation is attractive because the Higgs particle is present within the Standard Model itself. With similar motivation, the running of the Higgs coupling within the SM can lead to a transition potential in the very early universe that is beyond the simple quartic potential. For example, in the critical Higgs scenario [148, 149]. This could be a scenario where the transition field can remain light and can be investigated further.

It may be possible to relax some of the approximations used to obtain the analytic form for f_{NL} from vacuum transition during inflation. Such as that the evolution is inflaton dominated, the decay occurs in the form of a sudden transition, there is no bubble background at transition, the number of efolds have no extrinsic dependence on the spectator field χ or that the Hubble rate at transition H_T is independent of χ . However, this would require us to extend the treatment and possibly perform the non-Gaussianity calculation numerically. It could be a direction for future work.

First order phase transitions occurring during inflation can also be the source for Gravitational Waves (GWs) [150–154]. Along with the non-Gaussianity analysis presented here, a complementary analysis of theories exhibiting multiple vacua during inflation in terms of GWs can pin down their allowed parameter range even better as upcoming GW experiments [155] generate more data.

5.3 Future work

Currently, the Starobinsky inflation model [156] is used as a benchmark with which to compare the observable predictions of all other inflationary models [63]. The Starobinsky model minimally extends the Einstein-Hilbert action with an $\mathcal{R}^2$ term in the Jordan frame. Introducing such a term results in an inflationary era in the very early universe [18]. In the Einstein frame, the $\mathcal{R}^2$ term can be thought of as Einstein gravity plus a scalar [157, 158]. This scalar is called the scalaron and has a potential $V(\phi) \sim (1 - e^{-\sqrt{2/3}\phi/M_P})^2$.

In this work we have considered the Higgs inflation model with the motivation that the in-

flaton candidate is the Higgs. Its properties have already been or are being measured in collider experiments. However, the large non-minimal coupling parameter value in this model can result in strong coupling behaviour before the Planck scale is reached. Thus perturbative calculations with couplings observed at low energies might become unreliable at inflationary energy scales. It can be shown that the addition of an $\mathcal{R}^2$ term to Higgs inflation ensures that the strong coupling regime is not encountered until the Planck scale [159].

However, the addition of an $\mathcal{R}^2$ term introduces the scalaron field which then becomes a natural candidate for the scalar spectator field during reheating. It has also been shown that there is parametric resonance in the Higgs scalaron model [160]. Thus, this scenario would be a well motivated one for studying preheating and just like the Starobinsky model does for inflation, can be used to set a benchmark for preheating models.

The scale-dependent preheating formalism outlined in this work would be helpful to make predictions of the power spectrum and non-Gaussianity in the Higgs scalaron scenario. However, it would also require some additional theoretical work. Precisely because the non-minimal coupling value is high, the use of lattice simulations to capture the high k modes has to be examined. If the number density n_k lies in the intermediate range, $n_k \lesssim \lambda$, where $\lambda \approx 0.1$ for Higgs inflation, then the initial conditions for lattice simulations need to be modified. This could turn out to be a significant theoretical challenge.

For a vacuum transition occurring during inflation, the percolation condition is never met for a large part of the transition field parameter space. We have proven this explicitly for the Hawking-Moss parameter range (4.139). However, bubbles of true vacuum still form and grow even for these parameters. As inflation is not exactly de Sitter, it must eventually end, and these bubbles that have nucleated during inflation would eventually collide with each other after inflation ends.

The collision of vacuum bubbles can lead to the production of gravitational waves [161]. However, the collisions usually studied in the literature [162–164] occur between bubbles that have nucleated during radiation domination. In these studies, typically the bubbles nucleate and collide well within one Hubble time during radiation domination and hence remain sub-Hubble sized at collision. Thus calculating the GW spectrum arising from bubble collisions assuming a Minkowski background is a valid simplification to make in this case.

But from our work, we know that bubbles that have nucleated during inflation can collide after the end of inflation, during radiation domination. As a result these bubbles have sizes that are much greater than the Hubble radius during radiation domination.

Therefore studying the GW spectrum arising from incomplete vacuum transition during inflation would involve going beyond the treatment of bubble collisions on a Minkowski background to a curved background.

One of the major fundamental physics aim of upcoming GW experiments such as LISA [165] is to detect the gravitational waves arising from bubble collisions in the early universe. Thus, calculating the characteristic signal expected from collisions between bubbles nucleated during inflation would

help distinguishing between these scenarios and provide a window to parameter spaces of BSM theories during inflation.

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Appendix A

Converting from Jordan to Einstein frame

A.1 Single field case

The formulation for conformal transformations of the metric is well known [166]. This section closely follows the calculation in ref. [103].

We assume metric with signature $g_{\mu\mu}$: $(- + + +)$ with usual definitions for the Christoffel symbols and Ricci tensor,

$$\Gamma_{\mu\nu}^{\sigma} = \frac{1}{2}g^{\sigma\alpha}(\partial_{\mu}g_{\alpha\nu} + \partial_{\nu}g_{\mu\alpha} - \partial_{\alpha}g_{\mu\nu}), \quad (\text{A.1})$$

$$\mathcal{R}_{\mu\nu\rho}^{\sigma} = \partial_{\nu}\Gamma_{\mu\rho}^{\sigma} - \partial_{\rho}\Gamma_{\mu\nu}^{\sigma} + \Gamma_{\mu\rho}^{\alpha}\Gamma_{\alpha\nu}^{\sigma} - \Gamma_{\mu\nu}^{\alpha}\Gamma_{\alpha\rho}^{\sigma}, \quad (\text{A.2})$$

$$\mathcal{R}_{\mu\nu} = \mathcal{R}_{\mu\sigma\nu}^{\sigma}, \quad (\text{A.3})$$

$$\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}. \quad (\text{A.4})$$

Under conformal transformation of the metric

$$\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} \quad (\text{A.5})$$

the gravitational quantities change as

$$\tilde{g}^{\mu\nu} = \frac{1}{\Omega^2(x)}g^{\mu\nu}, \quad (\text{A.6})$$

$$\sqrt{-\tilde{g}} = \Omega^D \sqrt{-g}, \quad (\text{A.7})$$

$$\tilde{\Gamma}_{\beta\gamma}^{\alpha} = \Gamma_{\beta\gamma}^{\alpha} + \frac{1}{\Omega}(\delta_{\beta}^{\alpha}\nabla_{\gamma}\Omega + \delta_{\gamma}^{\alpha}\nabla_{\beta}\Omega - g_{\beta\gamma}\nabla^{\alpha}\Omega) \quad (\text{A.8})$$

where D is the dimension of spacetime. The Ricci scalar becomes

$$\tilde{\mathcal{R}} = \frac{1}{\Omega^2} \left(\mathcal{R} - \frac{2(D-1)}{\Omega} \square \Omega - (D-1)(D-4) \frac{1}{\Omega^2} g^{\mu\nu} \nabla_\mu \Omega \nabla_\nu \Omega \right) \quad (\text{A.9})$$

where

$$\square \Omega = g^{\mu\nu} \nabla_\mu \nabla_\nu \Omega = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \Omega). \quad (\text{A.10})$$

Let the Jordan frame action for a scalar field be

$$S = \int d^D x \sqrt{-g} \left(f(\phi) \mathcal{R} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right). \quad (\text{A.11})$$

After the conformal rescaling of metric the gravity part of the above action becomes

$$\int d^D x \sqrt{-g} f(\phi) \mathcal{R} = \int d^D x \frac{\sqrt{-\tilde{g}}}{\Omega^D} f(\phi) \left(\Omega^2 \tilde{\mathcal{R}} + \frac{(2D-1)}{\Omega} \square \Omega + \frac{(D-1)(D-4)}{\Omega^2} g^{\mu\nu} \nabla_\mu \Omega \nabla_\nu \Omega \right). \quad (\text{A.12})$$

To bring back the canonical Einstein-Hilbert term, $\frac{M_P^{(D-2)}}{2} \mathcal{R}$ we must choose Ω as

$$\Omega^{D-2}(x) = \frac{2}{M_P^{D-2}} f(\phi(x)) \quad (\text{A.13})$$

which gives

$$\int d^D x \sqrt{-g} f(\phi) \mathcal{R} = \int d^D x \sqrt{-\tilde{g}} \frac{M_P^{D-2}}{2} \left(\tilde{\mathcal{R}} - (D-1)(D-2) \frac{1}{\Omega^2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \Omega \tilde{\nabla}_\nu \Omega \right). \quad (\text{A.14})$$

The scalar part of action under conformal rescaling of metric looks like,

$$\int d^D x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right) = \int d^D x \sqrt{-\tilde{g}} \left(-\frac{1}{2} \frac{1}{\Omega^{D-2}} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - \tilde{V}(\phi) \right) \quad (\text{A.15})$$

where

$$\tilde{V} = \frac{V}{\Omega^D}. \quad (\text{A.16})$$

In terms of f we can rewrite this as

$$S = \int d^D x \sqrt{-\tilde{g}} \left(-\frac{1}{2} \frac{(D-1)}{(D-2)} M_P^{D-2} \frac{1}{f^2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu f \tilde{\nabla}_\nu f - \frac{1}{4f} M^{D-2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu \phi \tilde{\nabla}_\nu \phi - \tilde{V}(\phi) \right). \quad (\text{A.17})$$

To bring back the canonical kinetic term for scalar, we must have

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} = -\frac{M_P^{D-2}}{4f}\tilde{g}^{\mu\nu}\left(\tilde{\nabla}_\mu\phi\tilde{\nabla}_\nu\phi + \frac{2(D-1)}{(D-2)}\frac{1}{f}\tilde{\nabla}_\mu f\tilde{\nabla}_\nu f\right) \quad (\text{A.18})$$

that gives a field redefinition,

$$\frac{d\tilde{\phi}}{d\phi} = F(\phi) = \sqrt{\frac{M_P^{D-2}}{2f^2(\phi)}}\sqrt{f(\phi) + \frac{2(D-1)}{(D-2)}(f'(\phi))^2}. \quad (\text{A.19})$$

Thus we have seen how a conformal rescaling of the metric and a field redefinition allows us to move from the Jordan to the Einstein frame in the single field case.

A.2 Multi-field case

The recipe for converting from Jordan to Einstein frame, which was a conformal rescaling of metric followed by a field redefinition, in general need not work for the multi-field case. Let us assume a coupling between scalar fields ϕ^i and the Ricci scalar given by a function $f(\phi^i) = f(\phi^1, \phi^2, \phi^3, \dots)$,

$$S = \int d^D x \sqrt{-g} \left(f(\phi^i) \mathcal{R} - \frac{1}{2} \delta_{ij} g^{\mu\nu} \nabla_\mu \phi^i \nabla_\nu \phi^j - V(\phi^i) \right). \quad (\text{A.20})$$

Just like for the single field case, we can try and see what a conformal rescaling of the metric generates. Following the calculation from previous section, the gravity part becomes

$$\int d^D x \sqrt{-g} f(\phi^i) \mathcal{R} = \int d^D x \sqrt{-\tilde{g}} \frac{M_P^{D-2}}{2} \left(\tilde{\mathcal{R}} - \frac{(D-1)}{(D-2)} \frac{1}{f^2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu f \tilde{\nabla}_\nu f \right) \quad (\text{A.21})$$

while the matter part is

$$\int d^D x \sqrt{-\tilde{g}} \left(-\frac{1}{2} \delta_{ij} g^{\mu\nu} \nabla_\mu \phi^i \nabla_\nu \phi^j - V(\phi^i) \right) = \int d^D x \sqrt{-\tilde{g}} \left(-\frac{1}{4f} M_P^{D-2} \tilde{g}^{\mu\nu} \delta_{ij} \tilde{\nabla}_\mu \phi^i \tilde{\nabla}_\nu \phi^j - \tilde{V}(\phi^i) \right). \quad (\text{A.22})$$

Combining the above two parts, the total action is

$$S = \int d^D x \sqrt{-\tilde{g}} \left(\frac{M_P^{D-2}}{2} \tilde{\mathcal{R}} - \frac{1}{2} \frac{(D-1)}{(D-2)} \frac{M_P^{D-2}}{f^2} \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu f \tilde{\nabla}_\nu f - \frac{M_P^{D-2}}{4f} \tilde{g}^{\mu\nu} \delta_{ij} \tilde{\nabla}_\mu \phi^i \tilde{\nabla}_\nu \phi^j - \tilde{V}(\phi^i) \right). \quad (\text{A.23})$$

Now $\tilde{\nabla}_\mu f(\phi^i) = \frac{\partial f}{\partial \phi^i} \tilde{\nabla}_\mu \phi^i$ and so

$$S = \int d^D x \sqrt{-\tilde{g}} \left(\frac{M_P^{D-2}}{2} \tilde{\mathcal{R}} - \frac{1}{2} \tilde{g}^{\mu\nu} \mathcal{G}_{ij} \tilde{\nabla}_\mu \phi^i \tilde{\nabla}_\nu \phi^j - \tilde{V}(\phi^i) \right) \quad (\text{A.24})$$

where

$$\mathcal{G}_{ij} = \frac{M_P^{D-2}}{2f} \delta_{ij} + \frac{(D-1)}{(D-2)} \frac{M_P^{D-2}}{f^2} \frac{\partial f}{\partial \phi^i} \frac{\partial f}{\partial \phi^j}. \quad (\text{A.25})$$

This is like a metric on the target space (ϕ -space/field space). We need $\mathcal{G}_{ij} \rightarrow \tilde{\mathcal{G}}_{ij} = \delta_{ij}$ to bring back the canonical kinetic term for ϕ fields. Now we are aware that for constant Minkowski-like metrics the Riemann tensor must be zero, $\tilde{\mathcal{R}}_{jkl}^i = 0 \implies \tilde{\mathcal{R}} = 0$. We can simplify the form for $\mathcal{G}$ by performing a conformal rescaling in just the ϕ -space

$$\mathcal{G}_{ij} \rightarrow \hat{\mathcal{G}}_{ij} = \frac{2f}{M_P^{D-2}} \mathcal{G}_{ij} = \delta_{ij} + 2 \frac{(D-1)}{(D-2)} \frac{1}{f} f_{,i} f_{,j}. \quad (\text{A.26})$$

This separates the expression into a constant metric and f -dependent terms. If f can be perturbatively expanded, the second term can be neglected upto appropriate order.

From ref. [103]

$$\hat{\mathcal{R}} = \frac{2(D-1)}{L(\phi^i)} \left((D-2) A^{ijkl} (f f_{,ij} f_{,kl} - f_{,i} f_{,j} f_{,kl}) + 2(D-1) B^{ijklmn} (f_{,i} f_{,j} f_{,kl} f_{,mn}) \right), \quad (\text{A.27})$$

where a comma indicates partial derivative with respect to ϕ and

$$L(\phi^i) = \left((D-2)f + 2(D-1) \sum_i f_{,i}^2 \right)^2, \quad (\text{A.28})$$

$$A^{ijkl} = (\delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl}), \quad (\text{A.29})$$

$$B^{ijklmn} = (\delta^{ij} A^{klmn} + 2\delta^{ik} A^{jlmn}). \quad (\text{A.30})$$

Although not a sufficient condition, we necessarily want $\hat{\mathcal{R}}$ to be zero to bring back canonical kinetic term for fields. As can be seen, it is generally dependent on f and its derivatives. So in the general case, we cannot simultaneously bring back canonical kinetic terms for all fields.

We shall see this explicitly with two fields ϕ, χ in four dimensions where $f(\phi)$ depends only on a single field. The action to be considered is

$$S = \int d^4x \sqrt{-g} \left(f(\phi) \mathcal{R} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g^{\mu\nu} \nabla_\mu \chi \nabla_\nu \chi - V(\phi, \chi) \right). \quad (\text{A.31})$$

Converting to Einstein frame we have the field space metric

$$\mathcal{G}_{\phi\phi} = \frac{M_P^2}{2f(\phi)} + \frac{3}{2} \frac{M_P^2}{f^2(\phi)} \left(\frac{\partial f}{\partial \phi} \right)^2, \quad (\text{A.32})$$

$$\mathcal{G}_{\chi\chi} = \frac{M_P^2}{2f(\phi)}. \quad (\text{A.33})$$

Bringing back canonical kinetic terms for the ϕ field,

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi} = -\frac{1}{2}\tilde{g}^{\mu\nu}\left(\frac{M_P^2}{2f(\phi)} + \frac{3}{2}\frac{M_P^2}{f^2(\phi)}\left(\frac{\partial f}{\partial\phi}\right)^2\right)\nabla_\mu\phi\nabla_\nu\phi \quad (\text{A.34})$$

$$\Rightarrow \frac{d\tilde{\phi}}{d\phi} = \sqrt{\left(\frac{M_P^2}{2f(\phi)} + \frac{3}{2}\frac{M_P^2}{f^2(\phi)}\left(\frac{\partial f}{\partial\phi}\right)^2\right)} \quad (\text{A.35})$$

This is exactly the transformation for single field inflaton case we saw in the previous section. However for the second field χ we need

$$-\frac{1}{2}\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\chi}\tilde{\nabla}_\nu\tilde{\chi} = -\frac{1}{2}\tilde{g}^{\mu\nu}\left(\frac{M_P^2}{2f(\phi)}\right)\nabla_\mu\chi\nabla_\nu\chi, \quad (\text{A.36})$$

which is not possible because $\tilde{\chi}$ has to be a function of ϕ which will generate cross terms $\nabla\tilde{\phi}\nabla\tilde{\chi}$. One can also check this by finding the Ricci scalar for the field space metric as generically non-vanishing, whereas it should yield zero for the metric to be conformally flat.

Appendix B

Preheating calculation details

B.1 Correlator expansion in the non-perturbative delta N formalism

I shall make repeated use of the identity

$$\frac{\partial \mathcal{P}_G}{\partial \Sigma_{\alpha\beta}} = \frac{1}{2} \mathcal{P}_G \delta y_\gamma \delta y_\delta \Sigma_{\gamma\alpha}^{-1} \Sigma_{\delta\beta}^{-1} \quad (\text{B.1})$$

to expand $\mathcal{P}_G(\chi)$ to the desired order of Σ_{ij} . For example, at Order(Σ_{ij}^3) for the three point correlator we have two possible terms: 1. ($\Sigma_{12}^2 \Sigma_{23} + \text{perms}$) and 2. $\Sigma_{12} \Sigma_{23} \Sigma_{13}$. Their coefficients can be calculated as

- **Case 1:** Coefficient of ($\Sigma_{12}^2 \Sigma_{23} + \text{perms}$)

$$\frac{\partial \mathcal{P}_G}{\partial \Sigma_{12}} = \frac{1}{2} \mathcal{P}_G(\delta\chi_1)(\delta\chi_2)\Sigma^{-2} \quad (\text{B.2})$$

$$\frac{\partial^2 \mathcal{P}_G}{\partial \Sigma_{12} \partial \Sigma_{12}} = \frac{1}{4} \mathcal{P}_G(\delta\chi_1)^2(\delta\chi_2)^2\Sigma^{-4} - \mathcal{P}_G(\delta\chi_1)(\delta\chi_2)\Sigma^{-3} \quad (\text{B.3})$$

$$\begin{aligned} \frac{\partial^3 \mathcal{P}_G}{\partial \Sigma_{12} \partial \Sigma_{12} \partial \Sigma_{13}} &= \frac{1}{8} \mathcal{P}_G(\delta\chi_1)^3(\delta\chi_2)^2(\delta\chi_3)\Sigma^{-6} - \mathcal{P}_G(\delta\chi_1)^2(\delta\chi_2)^2\Sigma^{-5} \\ &\quad - \frac{1}{2} \mathcal{P}_G(\delta\chi_1)^2(\delta\chi_2)(\delta\chi_3)\Sigma^{-5} + 3\mathcal{P}_G(\delta\chi_1)(\delta\chi_2)\Sigma^{-4} \end{aligned} \quad (\text{B.4})$$

- **Case 2:** Coefficient of $\Sigma_{12} \Sigma_{23} \Sigma_{13}$

$$\frac{\partial \mathcal{P}_G}{\partial \Sigma_{12}} = \frac{1}{2} \mathcal{P}_G(\delta\chi_1)(\delta\chi_2)\Sigma^{-2} \quad (\text{B.5})$$

$$\frac{\partial^2 \mathcal{P}_G}{\partial \Sigma_{23} \partial \Sigma_{12}} = \frac{1}{4} \mathcal{P}_G(\delta\chi_1)(\delta\chi_2)^2(\delta\chi_3)\Sigma^{-4} - \mathcal{P}_G(\delta\chi_1)(\delta\chi_2)\Sigma^{-3} \quad (\text{B.6})$$

$$\begin{aligned} \frac{\partial^3 \mathcal{P}_G}{\partial \Sigma_{13} \partial \Sigma_{23} \partial \Sigma_{12}} &= \frac{1}{8} \mathcal{P}_G(\delta\chi_1)^2(\delta\chi_2)^2(\delta\chi_3)^2\Sigma^{-6} - \mathcal{P}_G(\delta\chi_1)(\delta\chi_2)^2(\delta\chi_3)\Sigma^{-5} \\ &\quad - \frac{1}{2} \mathcal{P}_G(\delta\chi_1)^2(\delta\chi_2)(\delta\chi_3)\Sigma^{-5} + 3\mathcal{P}_G(\delta\chi_1)(\delta\chi_2)\Sigma^{-4} \end{aligned} \quad (\text{B.7})$$

Here and throughout, the notation ‘perms’ indicates permutation of the spatial indices.

If a coefficient does not contain all possible spatial indices ($\{1, 2, 3\}$ for the three point correlator), as happens for example for the last term in (B.7) then in the curvature correlator the resulting term will be

$$\langle \zeta_1 \zeta_2 \zeta_3 \rangle \supset \Sigma_{12} \Sigma_{23} \Sigma_{13} 3\Sigma^{-4} \int d\chi_1 d\chi_2 d\chi_3 \mathcal{P}_G(\chi_1, \chi_2, \chi_3)(\delta\chi_1)(\delta\chi_2)(N_1 - \bar{N})(N_2 - \bar{N})(N_3 - \bar{N}). \quad (\text{B.8})$$

Using the separation property of Gaussian joint probability distribution with equal variance $\Sigma = \langle \delta\chi^2 \rangle$ that $\mathcal{P}_G(\chi_1, \chi_2, \dots) = \mathcal{P}_G(\chi_1)\mathcal{P}_G(\chi_2)\dots$, this term contains an integral over the free spatial index

$$\int d\chi_3 \mathcal{P}_G(\chi_3)(N_3 - \bar{N}) = 0. \quad (\text{B.9})$$

Hence, if a coefficient in the Σ_{ij} expansion does not contain all possible spatial indices, then its contribution to the curvature correlator drops out. This helps simplify the calculation.

Now consider the first term in (B.7) as an illustration, the resulting term in the curvature correlator would be

$$\begin{aligned} \langle \zeta_1 \zeta_2 \zeta_3 \rangle \supset \Sigma_{12} \Sigma_{23} \Sigma_{13} \frac{1}{8} &\left(\Sigma^{-2} \int d\chi_1 \mathcal{P}_G(\chi_1)(\delta\chi_1)^2(N_1 - \bar{N}) \right) \\ &\times \left(\Sigma^{-2} \int d\chi_2 \mathcal{P}_G(\chi_2)(\delta\chi_2)^2(N_2 - \bar{N}) \right) \left(\Sigma^{-2} \int d\chi_3 \mathcal{P}_G(\chi_3)(\delta\chi_3)^2(N_3 - \bar{N}) \right). \end{aligned} \quad (\text{B.10})$$

By defining a non-perturbative coefficient

$$\tilde{N}_{\chi\chi} = \left(\Sigma^{-2} \int d\chi \mathcal{P}_G(\chi)(\delta\chi)^2(N - \bar{N}) \right) \quad (\text{B.11})$$

this term can be written succinctly as

$$\langle \zeta_1 \zeta_2 \zeta_3 \rangle \supset \Sigma_{12} \Sigma_{23} \Sigma_{13} \frac{1}{8} \tilde{N}_{\chi\chi}^3. \quad (\text{B.12})$$

Similarly other terms in the curvature correlator can be expressed in terms of the non-perturbative coefficients $\tilde{N}_\chi, \tilde{N}_{\chi\chi}$, and so on.

B.2 HLattice

HLattice is a freely available program that simulates scalar fields and gravity during inflation and preheating in FORTRAN language. It can be downloaded from: <https://www.cita.utoronto.ca/~zqhuang/hlat/>. For exact details of the numerical algorithm used, see ref. [96]. Here we will provide a brief summary of its working relevant to our simulations.

The variable that is evolved is $\beta_{ij} = \ln(g_{ij})$, where 3×3 metric $g_{ij} = a(t)^2(\delta_{ij} + h_{ij})$ in synchronous gauge. From this the scale factor a can be obtained at each step using

$$a(t) = \frac{1}{L^3} \left(\int \sqrt{g} d^3x \right)^{1/3}. \quad (\text{B.13})$$

The equations of motion of fields have a time and spatial gradient. Spatial gradients are calculated using a special discretisation scheme defined in ref. [96]. Time discretisation is not performed, instead a hybrid of 6-th order symplectic integrator and 4-th order Runge-Kutta integrator is used to integrate the equations of motion to find β_{ij} at each time step.

Modifications to original code

To facilitate our simulations, we have performed a few modifications to the code:

1. Introduce a new subroutine `model_input()` in `model.f90` file. This function reads initial parameter values for ξ, ϕ and χ from a file `model_para.txt`. The `driver.f90` file has to be modified to allow this new subroutine to run.
2. File `io.f90` has been modified to write 15 decimal points in a vs H data to file. And to not write this data to screen.
3. Removed maximum steps criteria to stop simulation from `evolve.f90` file. Simulation stops only when a maximum scale factor is reached.

Simulation parameters

HLattice has two files, `configure.h` and `model.f90` which contain the most relevant parameters to be specified.

In `model.f90`, we specify typical model parameters:

$$\lambda = 10^{-14}$$

$$g^2/\lambda = 2$$

Potential, its first and second derivatives with respect to both fields. Depending on the model, massless preheating or non-minimal coupling, different potentials are specified.

Initial field values from `model_para.txt`.

Initial momenta for both fields = calculated at linear order during inflation internally by the program.

Initial fields are evolved linearly during inflation, non-linear simulations start when DEFROST starting point: Potential Energy < Kinetic Energy is reached.

Initial random Gaussian perturbations with specified Power spectrum are put in for the modes $k > 0$ before non-linear evolution starts.

In `configure.h`, we specify typical simulation parameters:

Size of simulation box = $20/H$

Number of points on each edge of box (Simulation resolution) = 32

Maximum scale factor = 20,000

Feedback interval for writing to file = 7

Only write a vs H data to file.

B.3 Bootstrap method

“Pull yourself from your own bootstraps, and you can fly.”

I use the bootstrap method [167] to find the error in the non-perturbative coefficient $\tilde{N}_{\chi\chi}$ (3.71). This method allows one to calculate the statistical error without knowing the actual probability distribution of the parameter. One simply need a large enough sample of observations.

As an illustrative example, consider the height of persons observed from a sample of $n = 100$ individuals. We want to estimate the mean height and standard error of all humanity from this one sample. Bootstrapping is a well-defined statistical method that allows us to do so.

Steps involved in bootstrapping:

1. Write each observation on a separate slip of paper and put them in a box. In the above example one has $n = 100$ slips of papers.
2. Draw one slip from the box and record the observation on that slip. Then put the slip back into the box. This is known as ‘resampling with replacement’.
3. Repeat previous step until the same number of observations as in original sample are obtained. In this way one has another sample (re-sample) of the same size as the original sample.
4. Calculate desired sample statistic from the re-sample and record it.
5. Repeat steps 2 through 4 multiple times, lets say b times. This way instead of going out and measuring the height of a billion humans in a single sample, we have re-samples of 100 containing a billion observations. Now we have b values for the mean, median etc. from each re-sample.

6. Calculate the standard deviation of the b different values of desired sample statistic. This standard deviation gives the bootstrapped standard error. Typically $b \sim 100$ resamples are sufficient to yield the correct estimate of the error.

In the case of $\tilde{N}_{\chi\chi}$ (Figure 3.9), I have $n = 20,000$ original sample points for χ_{ini} . By resampling with replacement I obtain $b = 100$ resamples and calculate the $\tilde{N}_{\chi\chi}$ for each. Then I calculate the standard deviation of these $b = 100$ $\tilde{N}_{\chi\chi}$ values. Using eq. (3.71), I shift the cosmic mean $\bar{\chi}$ to another value drawn from a Gaussian distribution with zero mean and variance equal to the cosmic variance. Then repeat the bootstrapping procedure for 1000 different $\bar{\chi}$ values. Lastly I take the average of the $\tilde{N}_{\chi\chi}$ and standard deviation values obtained for each of the $\bar{\chi}$ as the final $\tilde{N}_{\chi\chi}$ and its error used in calculating f_{NL} .