

A SELF-SUSTAINING MECHANISM FOR PLANE COUETTE FLOW WITH A FLEXIBLE LOWER BOUNDARY

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Summary

The stability of plane Couette flow to travelling-wave disturbances is studied asymptotically at high Reynolds numbers Re when the lower boundary possesses a degree of flexibility modelled by a spring-backed plate. First, it is shown that a three-dimensional (3D) linear instability exists, with streamwise and spanwise wavelengths comparable with the channel width. Building on this, nonlinear effects from the self-interaction of the wave are introduced, leading to a self-sustaining interaction between a roll/streak flow and the 3D wave. Governing nonlinear vortex-wave interaction (VWI) equations are derived and a perturbation analysis is carried out to guide a numerical investigation of the equations. The co-existence of two families of finite-amplitude solutions, each with different flow structures, is found. Numerical solutions of the VWI equations in each case show that a small wave amplitude of $O(Re^{-1}(\log Re)^{-1/2})$ is all that is necessary to provoke an $O(1)$ change to the basic Couette flow.

1. Introduction

The important question of how and why a laminar flow undergoes a transition to a turbulent state is still exercising the minds of the scientific community over a century and a half after Reynolds pioneering experiment (1) concerning the susceptibility of pipe flow to naturally occurring perturbations in the flow-field. This experiment and more recent studies show very clearly that the route to transition depends crucially on the disturbance environment, and indeed for many flows the process is highly nonlinear from an early stage. In order to elucidate the relevant nonlinear mechanisms, it is instructive to consider experimentally a fully turbulent flow. In such a flow, it often proves possible to discern coherent streak-like structures despite the apparent randomness of the overall motion (e.g. see (2)). These streaks appear to often be accompanied by a longitudinal vortex structure in the cross-stream plane (3).

In recent years, with the substantial increase in computing power available to researchers, it has proved possible to identify numerically the individual building blocks of a turbulent flow by starting from a fully turbulent simulation and reducing the size of the computational domain and the Reynolds number Re of the flow until much of the randomness is eliminated (4). One is then witness to the interaction between three distinct structures: a flow in the cross-stream plane independent of downstream distance, known as a roll flow, a streamwise component with cross-stream variation (a

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streak), and a travelling wave which propagates in the streamwise direction but again possesses a complicated spanwise dependence.

At finite Reynolds number, this so-called theory of self-sustaining processes or SSP (initially formulated by Waleffe in an unpublished paper in 1990 and verified in (4)) has successfully formed the foundation of a numerical approach to generating three-dimensional (3D) solutions of the Navier–Stokes equations and has been applied to many bounded flows including plane Couette flow (5), plane Poiseuille flow (6), and pipes of circular, rectangular and elliptical cross-sections (7 to 10). At high Reynolds number, the theory was developed independently in a series of papers (11 to 14) and is referred to as ‘vortex-wave interaction’ (VWI) since the first application was to a flow in a curved channel where Görtler vortices are the primary instability mechanism. The wave involved in a VWI can be governed primarily by inviscid or viscous dynamics and the majority of the nonlinear solutions referred to above essentially correspond to the former scenario. In this article, however, we will concentrate on a VWI in which the latter situation holds, and the wave more resembles the Tollmien–Schlichting (TS) instability found in boundary layers. One such VWI, mentioned briefly in (14), and in more detail in (15), is the ‘square vortex interaction’, so-called because both the streamwise and spanwise scales are comparable with the channel width, in contrast with a classical TS wave which typically possesses a much longer streamwise length scale. The square vortex has received little attention due to the fact that in the case of rigid boundaries and spanwise periodicity, it has been proved that no neutral solutions exist (16). However, Hall (17) recently revisited this interaction for both bounded and unbounded flows subject to a prescribed forcing at the wall and showed that neutral solutions are possible in this case. In this article we will demonstrate that, for plane Couette flow with a compliant lower surface in which boundary motion is induced by pressure differences within the fluid, it is possible for the square vortex interaction to operate and to produce a self-sustaining process at high Reynolds number in which the Couette base state is completely engulfed by the roll/streak motion.

Over the years, there have been many investigations of the effects of wall flexibility on the stability of shear flows. Motivated by the experiments of Krämer (18), an early investigation on the effect of a flexible wall on the stability of boundary layer flow was carried out in (19). It was found that the TS instability was stabilised by wall compliance but destabilised by the dissipation in the wall. Carpenter and Garrad (20, 21) considered two types of instabilities for a passive surface: those which could not exist without viscosity (termed TS type instabilities), and those that depend on the surface flexibility (termed flow-induced surface instabilities). They remark that this distinction may be artificial, particularly as the instabilities arise from the coupled fluid–solid system; indeed the instability which we find here, in which the wave is viscous but cannot operate without surface flexibility, appears to be a combination of the two types.

In their studies, Carpenter and Garrad modelled a Krämer-type compliant surface as a spring-backed plate. An elastic plate is supported by an array of springs that rests on a rigid surface. A viscous fluid substrate also backs the plate. This model has been used extensively in various forms (22–27). In particular Gajjar and Sibanda (28) use a spring-backed plate model to investigate the stability of plane Poiseuille flow between one rigid wall and one compliant wall at large Reynolds numbers. They find an increase in wall damping to be destabilising on the Tollmien–Schlichting instability. Although this model is obviously very simplistic, and viscoelastic continuum models for flexibility are an alternative option (e.g. (29)), we will use a spring-backed plate model in this analysis as a first step in establishing how wall flexibility can self-sustain a roll/streak/wave interaction in plane Couette flow at high Reynolds number: an instability that would not be operational in the case of rigid boundaries. The relative simplicity of this part of the model allows

us to identify the critical wave amplitude required for the interaction to take place, and to explore the effect of the wall properties on the flow-field in a systematic fashion. Moreover, small amplitude solutions to the governing equations can be found in terms of elementary functions and are useful in their own right, and as a partial check on the full numerical simulations we present.

The set-up of the remainder of the paper is as follows. In section 2, we will introduce the governing equations, including the specifics of the compliant wall model we use. In section 3, we demonstrate that, for a range of properties of the compliant wall, a linear neutral instability exists at high Reynolds number in which the wave has a streamwise wavelength comparable with the channel width and propagates at close to the speed of the lower wall. In section 4, we generalize this asymptotic framework and demonstrate that a self-sustaining process in the form of the square wave VWI mentioned above is possible provided the wave perturbation in the bulk of the flow is of the critical size $\text{Re}^{-1}(\log \text{Re})^{-1/2} \Delta$ with Δ an $O(1)$ parameter, and corresponds to a wall displacement of order $\text{Re}^{-2/3}(\log \text{Re})^{-1/2}$. This tiny disturbance is sufficient to completely alter the basic Couette profile throughout the flow-field for moderate values of Δ , as our numerical simulations of the interaction equations will demonstrate. For a given set of wall properties, we found two distinct families of solutions with very different spanwise distributions. The first of these is essentially a nonlinear continuation in Δ of the linear instability outlined in section 3, while the second only exists above a finite value of that parameter. Conclusions and proposals for future work are outlined in section 5.

2. The governing equations

Consider the flow of an incompressible fluid through a region bounded by two infinitely long walls. The upper wall, which is taken to be rigid, moves to the right with speed U^* , while the lower wall moves to the left with the same constant speed. This lower plate is flexible and responds to the fluid flow in a way to be outlined below. The space between the plates is filled with an incompressible fluid of constant density ρ^* and kinematic viscosity ν^* . The flow is readily described by a Cartesian coordinate system with (x^*, y^*, z^*) representing distances in the streamwise, normal and spanwise directions, respectively. The upper wall is located at $y^* = h^*$, while the position of the lower wall y^* is given by

$$y_w^* = -h^* + \eta^*(x^*, z^*, t^*), \quad (2.1)$$

with t^* denoting dimensional time. Here, $y^* = -h^*$ is the position of the lower wall when unflexed, while η^* is the displacement of the wall due to its flexibility. It is convenient to non-dimensionalize by writing

$$(x^*, y^*, z^*, y_w^*, \eta^*) = h^*(x, y, z, y_w, \eta), \quad t^* = (h^*/U^*)t, \quad (2.2a)$$

$$(u^*, v^*, w^*) = U^*(u, v, w), \quad p^* = \rho^*(U^*)^2 p, \quad (2.2b)$$

so that the governing unsteady 3D Navier–Stokes equations may be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (2.3a)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} + \text{Re}^{-1} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right), \quad (2.3b)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \text{Re}^{-1} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right), \quad (2.3c)$$

$$\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \text{Re}^{-1} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right), \quad (2.3d)$$

where the Reynolds number Re is given by

$$\text{Re} = \frac{U^* h^*}{\nu^*}. \quad (2.4)$$

Since we are assuming the upper wall to be rigid, the no-slip condition of viscous flow applies there, and in non-dimensional form we have

$$u = 1, \quad v = w = 0 \quad \text{on } y = 1. \quad (2.5)$$

We now formulate the boundary conditions on the lower wall, which has the non-dimensional position

$$y_w = -1 + \eta(x, z, t), \quad (2.6)$$

as seen from (2.1) and (2.2a). Since the wall flexes only in the normal direction, the streamwise and spanwise conditions are simply

$$u = -1, \quad w = 0 \quad \text{on } y = y_w. \quad (2.7)$$

Applying a kinematic condition to relate the motion of the fluid to that of the wall, we have

$$v = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} + w \frac{\partial \eta}{\partial z} \quad \text{on } y = y_w. \quad (2.8)$$

Finally, we apply a dynamic condition by modelling the response of the lower wall to the fluid using a spring-backed plate model (20, 21):

$$m \frac{D^2 \eta}{Dt^2} = -p - \text{Re}^{-2} K \eta, \quad (2.9)$$

where m and K are the non-dimensional mass and spring constant per unit area. Here, we have made use of the non-dimensionalization

$$m^* = \rho^* h^* m, \quad K^* = (\rho^* \nu^{*2} / h^{*3}) K. \quad (2.10)$$

A viscous stress term could also be added to the right-hand side of (2.9), but this is a higher order effect at the large Reynolds numbers we will consider here. We have also neglected other effects such as flexural rigidity and damping in order to keep the wall model as simple as possible as a first step in understanding how wall compliance can affect self-sustaining processes.

In the absence of any wall displacement, the steady parallel solution to (2.3) is simply undisturbed plane Couette flow, i.e.

$$u = y, \quad v = w = 0, \quad p = 0, \quad (-1 \leq y \leq 1). \quad (2.11)$$

In what follows, we suppose that the wall displacement is small and demonstrate that the basic Couette flow supports a linear instability in the form of an oblique neutral wave propagating at close to the speed of the lower wall. This sets up an asymptotic framework which we can then modify to allow for nonlinear effects in section 4. There we will identify the critical (but still small) size of the wall displacement in terms of the Reynolds number that leads to an $O(1)$ change in the basic flow by setting up a roll/streak/wave self-sustaining process.

3. Linear stability properties at high Reynolds number

We perturb the basic Couette flow by seeking a solution in normal-mode form:

$$(u, v, w, p) = (y, 0, 0, 0) + \varepsilon(\tilde{u}(y), \tilde{v}(y), \tilde{w}(y), \tilde{p}(y)) \exp[i\alpha(x - ct) + i\beta z] + c.c.. \quad (3.1)$$

Here, the wave amplitude $\varepsilon \ll 1$, $c.c.$ denotes complex conjugate, α, β are respectively the streamwise and spanwise wavenumbers of the disturbance, while c is the component of the phase speed in the streamwise direction. We represent the wall displacement in the form

$$y_w = -1 + \delta \tilde{\eta} \exp[i\alpha(x - ct) + i\beta z] + c.c., \quad (3.2)$$

where δ is another small parameter to be related to ε in due course. We will prescribe a real value of the spanwise wavenumber β and seek neutral modes so that α and c are real quantities to be determined as part of the solution. Substitution of (3.1) into the Navier–Stokes equations (2.3) leads, at $O(\varepsilon)$, to the perturbation equations

$$i\alpha \tilde{u} + \tilde{v}' + i\beta \tilde{w} = 0, \quad (3.3a)$$

$$i\alpha(y - c)\tilde{u} + \tilde{v} = -i\alpha \tilde{p} + \text{Re}^{-1} (\tilde{u}'' - \gamma^2 \tilde{u}), \quad (3.3b)$$

$$i\alpha(y - c)\tilde{v} = -\tilde{p}' + \text{Re}^{-1} (\tilde{v}'' - \gamma^2 \tilde{v}), \quad (3.3c)$$

$$i\alpha(y - c)\tilde{w} = -i\beta \tilde{p} + \text{Re}^{-1} (\tilde{w}'' - \gamma^2 \tilde{w}), \quad (3.3d)$$

where

$$\gamma^2 = \alpha^2 + \beta^2. \quad (3.4)$$

The boundary conditions on the rigid upper surface are, from (2.5) and (3.1):

$$\tilde{u}(1) = \tilde{v}(1) = \tilde{w}(1) = 0. \quad (3.5)$$

Linearizing the kinematic condition (2.8), we obtain

$$i\alpha(1 + c)\tilde{u}(-1) - \tilde{v}(-1) = 0, \quad (3.6)$$

while the dynamic condition (2.9) now reads

$$\tilde{p}(-1) = (\text{Re}^{-2}K - \alpha^2(1+c)^2m)\tilde{u}(-1), \quad (3.7)$$

and, from linearization of (2.7), the condition on the spanwise velocity is simply

$$\tilde{w}(-1) = 0. \quad (3.8)$$

In what follows, we analyse (3.3)–(3.8) in the high Reynolds number limit

$$\text{Re} \gg 1, \quad (3.9)$$

and show that a neutral mode solution exists in which the streamwise wavenumber is $O(1)$ and the phase speed component c is close to the speed of the lower flexible wall. We start by writing

$$\alpha = \alpha_0 + \dots, \quad c = -1 + \text{Re}^{-1/3}c_1 + \dots, \quad (3.10)$$

with α_0 and c_1 to be determined. The $-1/3$ Reynolds number scaling in (3.10) is to ensure a viscous-inertial balance within a lower wall layer of thickness $O(\text{Re}^{-1/3})$, through the balancing of the terms $i\alpha(y-c)$ and $\text{Re}^{-1}\partial^2/\partial y^2$. We start by considering the solution in the main bulk of the flow, referred to here as the ‘core’.

3.1 Core solution

Here, the fluid-dynamic quantities expand in the form

$$(\tilde{u}, \tilde{v}, \tilde{w}, \tilde{p}) = (u_0, v_0, w_0, p_0) + \dots, \quad (3.11)$$

with all the subscripted quantities dependent on the $O(1)$ normal variable y . From substitution of (3.11) into the governing perturbation equations (3.3), along with the expansions (3.10) for the wavenumber and phase speed, we find:

$$i\alpha_0 u_0 + v'_0 + i\beta w_0 = 0, \quad (3.12a)$$

$$i\alpha_0(y+1)u_0 + v_0 = -i\alpha_0 p_0, \quad (3.12b)$$

$$i\alpha_0(y+1)v_0 = -p'_0, \quad (3.12c)$$

$$i\alpha_0(y+1)w_0 = -i\beta p_0. \quad (3.12d)$$

The velocity components can be eliminated, leaving the pressure satisfying the Rayleigh equation

$$(y+1)(p''_0 - \gamma_0^2 p_0) = 2p'_0, \quad (3.13)$$

where

$$\gamma_0^2 = \alpha_0^2 + \beta^2. \quad (3.14)$$

Applying the condition $p'_0(1) = 0$, which ensures, from (3.12c), zero flow through the upper wall, we have

$$p_0 = (y+1) \cosh[\gamma_0(1-y)] + \gamma_0^{-1} \sinh[\gamma_0(1-y)], \quad (3.15)$$

where the constant of proportionality has been taken to be unity without loss of generality as a normalization condition on the eigenvalue problem. The corresponding solutions for (u_0, v_0, w_0) follow from (3.12) as

$$u_0 = -\cosh[\gamma_0(1-y)] + \beta^2 \gamma_0^{-1} \alpha_0^{-2} (y+1)^{-1} \sinh[\gamma_0(1-y)], \quad (3.16a)$$

$$v_0 = -i\gamma_0\alpha_0^{-1} \sinh[\gamma_0(1-y)], \quad (3.16b)$$

$$w_0 = -\beta\alpha_0^{-1} \left(\cosh[\gamma_0(1-y)] + \gamma_0^{-1}(y+1)^{-1} \sinh[\gamma_0(1-y)] \right), \quad (3.16c)$$

so that as the upper and lower walls are approached:

$$(u_0, v_0, w_0, p_0) \rightarrow (-1, -i\gamma_0^2\alpha_0^{-1}(1-y), -\beta\alpha_0^{-1}, 2) \text{ as } y \rightarrow 1, \quad (3.17a)$$

$$(u_0, v_0, w_0, p_0) \sim \Lambda(\beta^2\alpha_0^{-2}(y+1)^{-1}, -i\gamma_0^2\alpha_0^{-1}, -\beta\alpha_0^{-1}(y+1)^{-1}, 1) \text{ as } y \rightarrow -1, \quad (3.17b)$$

where we have defined

$$\Lambda = \gamma_0^{-1} \sinh 2\gamma_0. \quad (3.18)$$

Clearly, the boundary conditions (3.5), (3.8) are not satisfied by the solutions in the core, implying the existence of thin wall layers where viscosity must be re-introduced. We will start by considering the upper layer, which possesses a classical structure, before moving to the lower layer where the inclusion of wall flexibility affects the flow dynamics.

3.2 Upper wall layer

Equation (3.17a) implies the existence of a tangential slip velocity in the streamwise and spanwise directions which must be reduced to zero at the wall. Using the usual boundary-layer arguments, the appropriate expansion here is

$$u = u_+(Y_+) + \dots, \quad v = -\text{Re}^{-1/2}v_+(Y_+) + \dots, \quad w = w_+(Y_+) + \dots, \quad p = p_+(Y_+) + \dots, \quad (3.19)$$

where $Y_+ = \text{Re}^{1/2}(1-y)$ is the $O(1)$ normal variable in the upper wall layer. Substitution of (3.19) into (3.3) yields

$$i\alpha_0 u_+ + v'_+ + i\beta w_+ = 0, \quad (3.20a)$$

$$2i\alpha_0 u_+ = -i\alpha_0 p_+ + u''_+, \quad (3.20b)$$

$$0 = -p'_+, \quad (3.20c)$$

$$2i\alpha_0 w_+ = -i\beta p_+ + w''_+, \quad (3.20d)$$

subject to the matching conditions to the core from (3.17a):

$$u_+ \rightarrow -1, \quad w_+ \rightarrow -\beta\alpha_0^{-1} \text{ as } Y_+ \rightarrow \infty, \quad (3.21)$$

and the no-slip conditions (3.5), which become

$$u_+ = v_+ = w_+ = 0 \text{ on } Y_+ = 0.$$

It is easy to see from (3.20c) that the pressure remains constant to leading order across the layer and therefore assumes the form

$$p_+ = 2 \text{ for all } Y_+,$$

in view of (3.17a), while the solutions for the velocity perturbations are found to be

$$u_+ = -(1 - \exp[-\mu Y_+]), \quad v_+ = i\gamma_0^2 \alpha_0^{-1} (Y_+ - \mu^{-1} + \mu^{-1} \exp[-\mu Y_+]), \quad w_+ = \beta \alpha_0^{-1} u_+, \quad (3.22)$$

with

$$\mu \equiv \alpha_0^{1/2} (1 + i). \quad (3.23)$$

We now turn to the lower wall layer, where we will be able to determine an eigenrelation for the neutral wave.

3.3 Lower wall layer analysis and the linear eigenrelation

The near-wall form (3.17b) implies that as the lower wall is approached, the streamwise and spanwise velocities become singular. This singularity is smoothed out by the inclusion of viscosity in the lower wall layer which is essentially a near-wall critical layer. Motivated by the asymptotic form of the core solution as the wall is approached, we pose the expansion:

$$\tilde{u} = \text{Re}^{1/3} u_-(Y_-) + \dots, \quad \tilde{v} = v_-(Y_-) + \dots, \quad \tilde{w} = \text{Re}^{1/3} w_-(Y_-) + \dots, \quad \tilde{p} = p_-(Y_-) + \dots, \quad (3.24)$$

with $y = -1 + \text{Re}^{-1/3} Y_-$ and Y_- of $O(1)$. In terms of this scaled normal coordinate, the position of the lower wall is given by

$$Y_-^w = \text{Re}^{1/3} \delta \tilde{\eta} \exp[i\alpha(x - ct) + i\beta z] + c.c.,$$

with $\text{Re}^{1/3} \delta \ll 1$ assumed (see (3.31), (3.32) below). The leading-order governing equations are, from substitution of (3.24) into (3.3):

$$i\alpha_0 u_- + v'_- + i\beta w_- = 0, \quad (3.25a)$$

$$i\alpha_0(Y_- - c_1)u_- + v_- = -i\alpha_0 p_- + u''_-, \quad (3.25b)$$

$$0 = -p'_-, \quad (3.25c)$$

$$i\alpha_0(Y_- - c_1)w_- = -i\beta p_- + w''_-, \quad (3.25d)$$

with matching to the core behaviour (3.17b):

$$u_- \sim \beta^2 \alpha_0^{-2} \Lambda / Y_-, \quad v_- \rightarrow -i\gamma_0^2 \alpha_0^{-1} \Lambda, \quad w_- \sim -\beta \alpha_0^{-1} \Lambda / Y_-, \quad \bar{P}_0 \rightarrow \Lambda \text{ as } Y_- \rightarrow \infty, \quad (3.26)$$

and subject to the kinematic and no-slip conditions at the wall, which become:

$$i\alpha_0 c_1 u_-(0) - v_-(0) = 0, \quad w_-(0) = 0. \quad (3.27)$$

We see from (3.25c) that, as in the upper wall layer, the leading-order pressure remains constant and therefore takes on the value

$$p_- = \Lambda, \quad (3.28)$$

throughout the lower wall layer. However, the solution here must also obey the dynamic condition (3.7) at the wall, which in terms of the wall layer variables is

$$\Lambda = (K_0 - \alpha_0^2 c_1^2 m_0) u_-(0). \quad (3.29)$$

Here, in order to ensure that the wall flexibility is a leading-order effect we have scaled the spring stiffness and mass per unit area with Reynolds number as

$$K = \text{Re}^{5/3} K_0, \quad m = \text{Re}^{1/3} m_0, \quad (3.30)$$

with K_0, m_0 regarded as $O(1)$ quantities. Thus the instability is relevant to walls consisting of material which is relatively heavy and stiff. For example, at a Reynolds number $\text{Re} = 10^5$, the values of K, m used in this article are close to those possessed by the Kr  mer surface considered in (20). It is worth noting this scaling implies, via the wall equation (2.9), that the wall displacement δ (introduced in (3.2)) is related to the core wave amplitude ε by

$$\delta = \text{Re}^{1/3} \varepsilon. \quad (3.31)$$

Since the perturbation to the wall position is required to be asymptotically smaller than the thickness of the wall layer, we therefore require the inequality

$$\varepsilon \ll O(\text{Re}^{-2/3}) \quad (3.32)$$

as a bound on the wave amplitude. In fact, we will see later in section 4 that it is possible for ε to be comfortably smaller than this and still completely alter the Couette flow in the core.

The solution of the system (3.25)–(3.29), which leads to the desired linear eigenrelation, proceeds as follows. First, the solution of (3.25d) subject to (3.26) and (3.27) can be represented in the form

$$w_- = \frac{i\beta\Lambda}{(i\alpha_0)^{2/3}} \left(\mathcal{L}(\xi) - \frac{\mathcal{L}(\xi_0)}{Ai(\xi_0)} Ai(\xi) \right), \quad \xi \equiv (i\alpha_0)^{1/3} (Y_- - c_1), \quad \xi_0 \equiv -(i\alpha_0)^{1/3} c_1. \quad (3.33)$$

Here, the function $\mathcal{L}$ is the particular solution of the forced Airy equation $\mathcal{L}'' - \xi\mathcal{L} = 1$, while Ai is the usual Airy function. Next, we differentiate the streamwise momentum equation (3.25b) with respect to Y_- , and use the continuity equation (3.25a). This yields the shear equation

$$\frac{d^2\tau}{d\xi^2} - \xi\tau = -i\beta(i\alpha_0)^{-2/3} w_-, \quad \tau \equiv u'_-. \quad (3.34)$$

Substituting for w_- from (3.33), and using the properties of $\mathcal{L}$ and Ai , we find that

$$\tau = C Ai(\xi) + \frac{\beta^2\Lambda}{(i\alpha_0)^{4/3}} \left(\mathcal{L}'(\xi) - \frac{\mathcal{L}'(\xi_0)}{Ai(\xi_0)} Ai'(\xi) \right), \quad (3.35)$$

where C is an unknown constant. Integrating between $\xi = \xi_0$ and $\xi = \infty$, and applying conditions in the far-field (3.26) and at the wall (3.29) we obtain

$$-\frac{\Lambda}{K_0 - \alpha_0^2 c_1^2 m_0} = (i\alpha_0)^{-1/3} C \kappa(\xi_0), \quad \kappa(\xi_0) \equiv \int_{\xi_0}^{\infty} Ai(s) ds, \quad (3.36)$$

which determines C . Finally we return to the streamwise momentum equation (3.25b), set $Y_- = 0$ (i.e. $\xi = \xi_0$), and apply the kinematic condition in (3.27). This yields

$$u''_-(0) = i\alpha_0\Lambda. \quad (3.37)$$

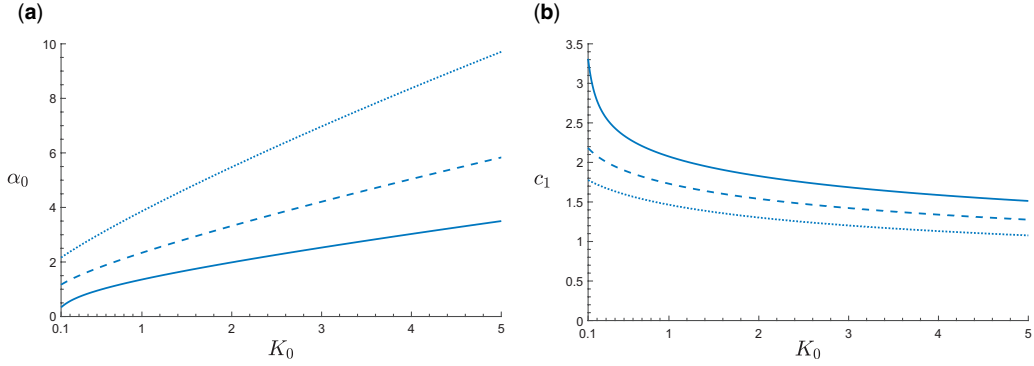


Fig. 1 Results from the linear eigenrelation (3.41). (a) Streamwise wavenumber α_0 versus stiffness parameter K_0 , (b) phase speed correction c_1 versus K_0 . Three different values of the mass parameter m_0 are shown. Dotted: $m_0 = 0.05$; dashed: $m_0 = 0.1$; solid: $m_0 = 0.2$.

Calculating the left-hand side of (3.37) using (3.35), and substituting for C from (3.36), this becomes

$$\frac{\alpha_0^{5/3} \mathcal{G}(\xi_0)}{K_0 - \alpha_0^2 c_1^2 m_0} = \gamma_0^2, \quad \mathcal{G}(\xi_0) \equiv i^{5/3} \frac{Ai'(\xi_0)}{\kappa(\xi_0)}. \quad (3.38)$$

It is well-known (e.g. see (30, 31)) that the function $\mathcal{G}(\xi_0)$ is only real if

$$\xi_0 = \xi_N = -i^{1/3} s_N, \quad s_N \simeq 2.2972, \quad (3.39)$$

with

$$\mathcal{G}(\xi_N) \simeq -1.0007. \quad (3.40)$$

Bearing in mind the definition of ξ_0 in (3.33), this implies that a neutral solution is possible, provided α_0, c_1 are such that

$$\frac{\alpha_0^{5/3} \mathcal{G}(\xi_N)}{K_0 - \alpha_0^2 c_1^2 m_0} = \alpha_0^2 + \beta^2, \quad c_1 = s_N \alpha_0^{-1/3}. \quad (3.41)$$

The first of equations (3.41) can be written as a 10th order polynomial for $\alpha_0^{1/3}$ and readily solved using a polynomial root solver. It can be shown that a unique neutral solution for α_0 always exists. Sample results are shown in Fig. 1 for the case $\beta = 1$ and $m_0 = (0.05, 0.1, 0.2)$ where we plot the leading-order wavenumber α_0 and the phase speed correction c_1 versus the stiffness parameter K_0 . We will return to this figure in the next section when we discuss the effects of nonlinearity.

Now that we have established that a linear instability exists, in the next section we will identify the critical size of the wall displacement ε which allows the Couette base state to be completely altered; we will also derive the governing equations for this flexible roll/streak/wave interaction. Following this we will present numerical solutions of the full nonlinear problem, confirming that the base state can indeed be altered in this way.

4. Nonlinear roll/streak/wave interaction at high Reynolds number

Having established that a linear neutral wave can be supported by Couette flow with a flexible lower boundary, we aim to build on the asymptotic framework presented in the previous section and introduce nonlinear effects from the wave in the lower wall layer in such a way that they provide a leading-order forcing on the core. This sets up a roll/streak/wave interaction which potentially can completely alter the base state throughout the channel.

We take the streamwise wavenumber α to be $O(1)$ as before, comparable with the channel width, and again seek travelling-wave solutions propagating close to the speed of the lower wall, with the phase speed expansion given in (3.10). We also scale the spring stiffness and mass with Reynolds number in accordance with (3.30). The governing equations are the nonlinear Navier–Stokes equations (2.3) subject to the boundary conditions (2.5), (2.7)–(2.9). The lower wall is again perturbed by an $O(\delta)$ amount as in (3.2) but we now allow for a more general spanwise dependence by writing the equation for the lower wall as

$$y_w = -1 + \varepsilon \text{Re}^{1/3} \tilde{\eta}(z) E + c.c., \quad (4.1)$$

where we have used the value of δ in (3.31) and adopted the notation

$$E \equiv \exp[i\alpha_0(x - ct)]. \quad (4.2)$$

An expansion of the streamwise wavenumber in terms of the Reynolds number is also strictly necessary, but we only require the leading-order term in our analysis.

4.1 The core flow

We now pose the expansion

$$u = U_S(y, z) + \varepsilon u_0(y, z) E + c.c. + \dots, \quad (4.3a)$$

$$v = \text{Re}^{-1} V_R(y, z) + \varepsilon v_0(y, z) E + c.c. + \dots, \quad (4.3b)$$

$$w = \text{Re}^{-1} W_R(y, z) + \varepsilon w_0(y, z) E + c.c. + \dots, \quad (4.3c)$$

$$p = \text{Re}^{-2} P_R(y, z) + \varepsilon p_0(y, z) E + c.c. + \dots. \quad (4.3d)$$

In (4.3), the subscript S denotes the streak flow, while R refers to the roll. Our aim is to determine the wave disturbance size ε such that the flow in the core becomes nonlinear. Substitution of (4.3) into the Navier–Stokes equations (2.3) leads to the roll/streak equations:

$$\frac{\partial V_R}{\partial y} + \frac{\partial W_R}{\partial z} = 0, \quad (4.4a)$$

$$V_R \frac{\partial U_S}{\partial y} + W_R \frac{\partial U_S}{\partial z} = \frac{\partial^2 U_S}{\partial y^2} + \frac{\partial^2 U_S}{\partial z^2}, \quad (4.4b)$$

$$V_R \frac{\partial V_R}{\partial y} + W_R \frac{\partial V_R}{\partial z} = -\frac{\partial P_R}{\partial y} + \frac{\partial^2 V_R}{\partial y^2} + \frac{\partial^2 V_R}{\partial z^2}, \quad (4.4c)$$

$$V_R \frac{\partial W_R}{\partial y} + W_R \frac{\partial W_R}{\partial z} = -\frac{\partial P_R}{\partial z} + \frac{\partial^2 W_R}{\partial y^2} + \frac{\partial^2 W_R}{\partial z^2}. \quad (4.4d)$$

Note that these equations are independent of Re and that the roll and streak flows uncouple so that the roll flow can be determined first. It is clear that one possible solution of this system, in which the roll flow is identically zero, is simply plane Couette flow. Energy arguments (e.g. see (32)) easily establish that if this roll/streak flow is subject to rigid boundaries with no-slip conditions, then the basic plane Couette flow is in fact the only solution of this steady system. Thus, in order to generate alternative solutions, we will need to force equations (4.4) in some way. In fact this forcing will manifest itself in the form of a boundary condition arising from the nonlinear self-interaction of the wave components in the lower wall layer as we shall see presently. As regards the wave in the core, we have, from substitution of (4.3) into (2.3):

$$i\alpha_0 u_0 + \frac{\partial v_0}{\partial y} + \frac{\partial w_0}{\partial z} = 0, \quad (4.5a)$$

$$i\alpha_0 (U_S + 1)u_0 + \frac{\partial U_S}{\partial y} v_0 + \frac{\partial U_S}{\partial z} w_0 = -i\alpha_0 p_0, \quad (4.5b)$$

$$i\alpha_0 (U_S + 1)v_0 = -\frac{\partial p_0}{\partial y}, \quad (4.5c)$$

$$i\alpha_0 (U_S + 1)w_0 = -\frac{\partial p_0}{\partial z}, \quad (4.5d)$$

so that in this region the wave is simply driven by the streak flow. These equations reduce to (3.12) in the case where $U_S = y$ and $\partial/\partial z = i\beta$. As in section 3, the velocity components can be eliminated from these equations, leaving

$$(U_S + 1) \left(\frac{\partial^2 p_0}{\partial y^2} + \frac{\partial^2 p_0}{\partial z^2} - \alpha_0^2 p_0 \right) = 2 \frac{\partial U_S}{\partial y} \frac{\partial p_0}{\partial y} + 2 \frac{\partial U_S}{\partial z} \frac{\partial p_0}{\partial z}, \quad (4.6)$$

which is a generalized version of the Rayleigh equation (3.13). Thus, in the core, the roll/streak flow incorporates nonlinearity and viscosity, while the wave appears to be a linear inviscid instability supported by the streak.

Thus far, we have not discussed suitable boundary conditions. Considering the roll/streak flow first, it seems reasonable to impose the usual no-slip conditions on the upper boundary, namely

$$U_S = 1, V_R = W_R = 0 \text{ on } y = 1. \quad (4.7)$$

On the lower boundary, we set

$$U_S = -1, V_R = 0 \text{ on } y = -1, \quad (4.8)$$

but we allow for a possible spanwise slip velocity by writing

$$W_R \rightarrow Q(z) \text{ as } y \rightarrow -1. \quad (4.9)$$

We will see that this slip velocity is produced by the nonlinear self-interaction of the wave in a lower wall layer to be discussed presently.

We now turn to the boundary conditions on the wave components. At the upper and lower walls, we impose the condition of zero normal flow which, in terms of the pressure can be expressed as

$$\frac{\partial p_0}{\partial y} = 0 \quad \text{on } y = \pm 1. \quad (4.10)$$

Unlike the linear instability problem (3.12), the Rayleigh equation (4.6) subject to (4.10) does not have an analytic solution: nevertheless, it suffices for our purposes to calculate the asymptotic behaviour of the wave components as the walls are approached. We find as $y \rightarrow 1$:

$$p_0 \sim p_+(z) - \frac{1}{2}(p_+''(z) - \alpha_0^2 p_+(z))(y-1)^2, \quad (4.11a)$$

$$u_0 \rightarrow -\frac{1}{2}p_+(z), \quad (4.11b)$$

$$v_0 \sim -\frac{i}{2\alpha_0}(p_+''(z) - \alpha_0^2 p_+(z))(y-1), \quad (4.11c)$$

$$w_0 \rightarrow \frac{i}{2\alpha_0}p_+'(z), \quad (4.11d)$$

while as $y \rightarrow -1$:

$$p_0 \sim p_-(z) + \frac{1}{2}(p_-''(z) - \frac{2\lambda_-'(z)}{\lambda_-(z)}p_-'(z) - \alpha_0^2 p_-(z))(y+1)^2, \quad (4.12a)$$

$$u_0 \sim -\frac{1}{\alpha_0^2 \lambda_-(z)^2}(\lambda_-(z)p_-''(z) - \lambda_-'(z)p_-'(z))(y+1)^{-1}, \quad (4.12b)$$

$$v_0 \rightarrow \frac{i}{\alpha_0 \lambda_-(z)}(p_-''(z) - \frac{2\lambda_-'(z)}{\lambda_-(z)}p_-'(z) - \alpha_0^2 p_-(z)), \quad (4.12c)$$

$$w_0 \sim \frac{i}{\alpha_0 \lambda_-(z)}p_-'(z)(y+1)^{-1}, \quad (4.12d)$$

so that we obtain the same algebraic growth of the streamwise and spanwise components as seen in the linear case (3.17b) but now with spanwise-dependent coefficients. Here, we have defined the wall wave pressures

$$p_{\pm}(z) = p_0(\pm 1, z), \quad (4.13)$$

and the unknown streak shear stress at the lower wall:

$$\lambda_-(z) = \frac{\partial U_S}{\partial y}(-1, z). \quad (4.14)$$

This latter quantity turns out to play a crucial role in the interaction, representing the effect of the roll/streak flow on the wave in the lower wall layer. The quantities defined in (4.13), (4.14) are referred to as ‘wall quantities’ because, in view of (4.1), $y_w = -1$ to leading order. Before looking more closely into the effect of the streak shear stress, we briefly discuss the details of the upper wall layer.

4.2 The upper wall layer

The details here are almost identical to that of the linear case presented in section 3.2. Since the roll/streak flow in the core already satisfies the no-slip conditions on the upper wall, the role of this wall layer is purely to reduce the streamwise and spanwise wave slip velocities evident in (4.11b), (4.11d) to zero. As in the linear case, the layer has the classical thickness $\text{Re}^{-1/2}$ and we write

$$Y_+ = \text{Re}^{1/2}(1 - y) \quad (4.15)$$

to represent the $O(1)$ normal coordinate. The relevant flow expansions are, by analogy with those in (3.19):

$$u = 1 - \text{Re}^{-1/2}\lambda_+(z)Y_+ + \cdots + \varepsilon u_+(Y_+, z)E + c.c., \quad (4.16a)$$

$$v = -\varepsilon \text{Re}^{-1/2}v_-(Y_+, z)E + c.c. + \cdots, \quad (4.16b)$$

$$w = \varepsilon w_+(Y_+, z)E + c.c. + \cdots, \quad (4.16c)$$

$$p = \varepsilon p_+(z)E + c.c. + \cdots. \quad (4.16d)$$

Here, $\lambda_+(z)$ represents the streak shear stress at the upper wall: this quantity does not play a role in the leading-order dynamics. We have also anticipated that, as in the linear case, the leading-order pressure is independent of Y_+ , and therefore must be equal to its value at the edge of the core region. Substitution of (4.16) into (2.3) yields

$$i\alpha_0 u_+ + \frac{\partial v_+}{\partial Y_+} + \frac{\partial w_+}{\partial z} = 0, \quad (4.17a)$$

$$2i\alpha_0 u_+ = -i\alpha_0 p_+ + \frac{\partial^2 u_+}{\partial Y_+^2}, \quad (4.17b)$$

$$2i\alpha_0 w_+ = -p'_+(z) + \frac{\partial^2 w_+}{\partial Y_+^2}, \quad (4.17c)$$

subject to the matching conditions to the core (4.11a):

$$u_+ \rightarrow -\frac{1}{2}p_+(z), \quad w_+ \rightarrow \frac{i}{2\alpha_0}p'_+(z) \text{ as } Y_+ \rightarrow \infty, \quad (4.18)$$

and the no-slip conditions (2.5), which become

$$u_+ = v_+ = w_+ = 0 \text{ on } Y_+ = 0. \quad (4.19)$$

Equations (4.17) are simply a generalization of (3.20), with the solutions for the velocity perturbations given by

$$u_+ = -\frac{1}{2}p_+(z)(1 - \exp[-\mu Y_+]), \quad (4.20a)$$

$$v_+ = -\frac{i}{2\alpha_0}(p''_+(z) - \alpha_0^2 p_+(z))(Y_+ - \mu^{-1} + \mu^{-1} \exp[-\mu Y_+]), \quad (4.20b)$$

$$w_+ = \frac{i}{2\alpha_0} p'_+(z)(1 - \exp[-\mu Y_+]), \quad (4.20c)$$

with μ defined in (3.23). Solutions (4.20) should be compared with their counterparts (3.22) in the linear instability structure. We now discuss the details of the lower wall layer and determine the critical size of the wave amplitude ε that ensures a strong feedback from the wall layer into the core.

4.3 The lower wall layer

As the lower wall is approached, we see from (4.12b), (4.12d) that the streamwise and spanwise wave velocity components become singular. Thus, the wave dominates over the roll/streak flow in this layer, where we write

$$u = -1 + \text{Re}^{-1/3} \lambda_-(z) Y_- + \cdots + \varepsilon \text{Re}^{1/3} u_-(Y_-, z) E + c.c. + \cdots, \quad (4.21a)$$

$$v = \varepsilon v_-(Y_-, z) E + \cdots + \varepsilon^2 \text{Re}^{2/3} \bar{V}_R(Y_-, z) + \cdots, \quad (4.21b)$$

$$w = \varepsilon \text{Re}^{1/3} w_-(Y_-, z) E + c.c. + \cdots + \varepsilon^2 \text{Re} \bar{W}_R(Y_-, z) + \cdots, \quad (4.21c)$$

$$p = \varepsilon p_-(z) E + c.c. + \cdots + \varepsilon^2 \bar{P}_R(Y_-, z) + \cdots, \quad (4.21d)$$

with $y = -1 + \text{Re}^{-1/3} Y_-$, so that in terms of the wall layer variable, the position of the lower wall corresponds to $Y_- = \text{Re}^{2/3} \varepsilon \tilde{\eta} E + c.c.$, and $\varepsilon \ll \text{Re}^{-2/3}$. Here, the first set of dots in the expansion for u represents the higher-order terms in the Taylor expansion of the streak flow about $y = -1$, and we have again anticipated that the leading-order wave pressure is independent of the normal coordinate. The scaling for the roll contributions (with subscript R) is chosen by requiring the roll to be forced by nonlinear effects from the wave. Substitution of expansions (4.21) into (2.3) shows that the roll flow ($\bar{V}_R, \bar{W}_R, \bar{P}_R$) satisfies

$$\frac{\partial \bar{V}_R}{\partial Y_-} + \frac{\partial \bar{W}_R}{\partial z} = 0, \quad (4.22a)$$

$$i\alpha_0 u_-^* v_- + v_- \frac{\partial v_-^*}{\partial Y_-} + w_- \frac{\partial v_-^*}{\partial z} + c.c. = -\frac{\partial \bar{P}_R}{\partial Y_-} + \frac{\partial^2 \bar{V}_R}{\partial Y_-^2}, \quad (4.22b)$$

$$i\alpha_0 u_-^* w_- + v_- \frac{\partial w_-^*}{\partial Y_-} + w_- \frac{\partial w_-^*}{\partial z} + c.c. = \frac{\partial^2 \bar{W}_R}{\partial Y_-^2}, \quad (4.22c)$$

with an asterisk denoting complex conjugate. Here, we can see in (4.22) the explicit nonlinear forcing by the wave components referred to above. It is also noteworthy that spanwise pressure gradient effects are absent from (4.22c) and are a higher-order effect. Also, from substitution of (4.21) into (2.3), and equating the terms proportional to E , we find the wave components (u_-, v_-, w_-, p_-) to be governed by

$$i\alpha_0 u_- + \frac{\partial v_-}{\partial Y_-} + \frac{\partial w_-}{\partial z} = 0, \quad (4.23a)$$

$$i\alpha_0 (\lambda_-(z) Y_- - c_1) u_- + \lambda_-(z) v_- + \lambda'_-(z) Y_- w_- = -i\alpha_0 p_- + \frac{\partial^2 u_-}{\partial Y_-^2}, \quad (4.23b)$$

$$i\alpha_0(\lambda_-(z)Y_- - c_1)w_- = -p'_-(z) + \frac{\partial^2 w_-}{\partial Y_-^2}. \quad (4.23c)$$

Note how the wave dynamics are driven by the streak shear $\lambda_-(z)$. Matching to the near-wall core solution behaviour (4.12a) requires

$$u_- = O(1/Y_-), \quad w_- = O(1/Y_-) \quad \text{as } Y_- \rightarrow \infty. \quad (4.24)$$

The conditions on the flexible surface are slightly modified from those of the linear study in section 3, now incorporating the spanwise-dependent streak shear:

$$i\alpha_0 c_1 u_-(0, z) - \lambda_-(z) v_-(0, z) = 0, \quad w_-(0, z) = 0, \quad p_-(z) = (K_0 - \alpha_0^2 c_1^2 m_0) \frac{u_-(0, z)}{\lambda_-(z)}. \quad (4.25)$$

Equations (4.23), together with the boundary and matching conditions (4.25), (4.24), are a generalization of the system (3.25)–(3.29) and can be solved by a similar procedure to that presented in section 3.2, although considerably more algebra is involved. For full details see Chotai (33), who adapted the method of Smith (34). We find that the wave velocity components can be eliminated, leaving the wave pressure satisfying

$$p_-''(z) - \frac{\lambda'_-(z)}{\lambda_-(z)} \mathcal{F}(\xi_0(z)) p'_-(z) - \alpha_0^2 p_- = -\frac{(\alpha_0 \lambda_-(z))^{5/3} \mathcal{G}(\xi_0(z))}{K_0 - \alpha_0^2 c_1^2 m_0} p_-. \quad (4.26)$$

Here, the function $\mathcal{G}$ is as defined in (3.38), but now the argument ξ_0 is a function of z , and is given by

$$\xi_0 = -(i\alpha_0)^{1/3} c_1 / (\lambda_-(z))^{2/3}, \quad (4.27)$$

which reduces to the previous definition given in (3.33) when $\lambda_-(z) \equiv 1$. A new feature arising is the middle term on the left-hand side of (4.26), where the function $\mathcal{F}$ is given by

$$\mathcal{F}(\xi_0) = \frac{3}{2} + \frac{\xi_0}{2Ai(\xi_0)} (\xi_0 \kappa(\xi_0) + Ai'(\xi_0)). \quad (4.28)$$

Since (4.26) is linear in p_- , we can write

$$p_- = A\tilde{p}_-(z), \quad (4.29)$$

where we can take $\tilde{p}_-(0) = 1$ without loss of generality, while A is a complex constant. In order to complete the analysis, we have to study the effect of the wave forcing in (4.22) on the roll flow and examine how it feeds back into the core. In order to do this, it is necessary only to study the solution at the edge of the wall layer and for this purpose we can just infer the far-field behaviour of the wave velocity components in (4.22) from their behaviour (4.12a) at the edge of the core, so that

$$u_- \sim -\frac{A}{\alpha_0^2 \lambda_-(z)^2} (\lambda_-(z) \tilde{p}_-''(z) - \lambda'_-(z) \tilde{p}_-'(z)) Y_-^{-1}, \quad (4.30a)$$

$$v_- \rightarrow \frac{iA}{\alpha_0 \lambda_-(z)} (\tilde{p}_-''(z) - \frac{2\lambda'_-(z)}{\lambda_-(z)} \tilde{p}_-'(z) - \alpha_0^2 \tilde{p}_-(z)), \quad (4.30b)$$

$$w_- \sim \frac{iA}{\alpha_0 \lambda_-(z)} \tilde{p}'_-(z) Y_-^{-1}, \quad (4.30c)$$

as $Y_- \rightarrow \infty$. Substituting these asymptotic forms into the spanwise roll equation (4.22c), we find that

$$\overline{W}_R \sim -q(z) \log Y_- \text{ as } Y_- \rightarrow \infty, \quad (4.31)$$

where

$$q(z) = \frac{|A|^2}{\alpha_0^2 \lambda_-(z)^2} \left[\alpha_0^2 (\tilde{p}_-(z) \tilde{p}_-^*(z))' + (\tilde{p}'_-(z) \tilde{p}_-^*(z))' \right]. \quad (4.32)$$

It is evident that the parameter $|A|^2$ is a measure of the nonlinearity of the system. Equation (4.31) implies that a non-zero spanwise roll field is generated at the edge of the wall layer; we wish this to exert a leading-order effect on the core flow. Taking into account the spanwise roll scaling in (4.21), we see that in the core, where $Y_- = O(\text{Re}^{1/3})$, the behaviour (4.31) leads to a spanwise roll component $W_R = O(\varepsilon^2 \text{Re} \log \text{Re})$. For this to be of the same order as the spanwise roll component in the core expansion (4.3), we therefore require the balance

$$\text{Re}^{-1} \sim \varepsilon^2 \text{Re} \log \text{Re}, \quad (4.33)$$

implying that the wave amplitude necessary to potentially provoke an $O(1)$ change to the Couette flow base state is

$$\varepsilon = \text{Re}^{-1} (\log \text{Re})^{-1/2}, \quad (4.34)$$

corresponding to a wall displacement $\delta = O(\text{Re}^{1/3} \varepsilon) = O(\text{Re}^{-2/3} (\log \text{Re})^{-1/2})$. Note that this confirms that $\varepsilon \ll O(\text{Re}^{-2/3})$ as assumed earlier in our treatment of the lower wall layer and is the same scaling as that obtained in (17). The far-field behaviour (4.31) is consistent with the spanwise slip condition (4.9) imposed on (4.4), provided

$$Q(z) = -q(z)/3. \quad (4.35)$$

4.4 The nonlinear interaction equations

It is worth stating the full roll/streak/wave interaction problem again, removing annotations for the sake of clarity. The roll/streak flow equations in the core are

$$\frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0, \quad (4.36a)$$

$$V \frac{\partial U}{\partial y} + W \frac{\partial U}{\partial z} = \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}, \quad (4.36b)$$

$$V \frac{\partial V}{\partial y} + W \frac{\partial V}{\partial z} = -\frac{\partial P}{\partial y} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}, \quad (4.36c)$$

$$V \frac{\partial W}{\partial y} + W \frac{\partial W}{\partial z} = -\frac{\partial P}{\partial z} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2}, \quad (4.36d)$$

subject to the boundary conditions

$$U = 1, \quad V = W = 0 \quad \text{on } y = 1, \quad (4.37a)$$

$$U = -1, \quad V = 0, \quad W = -q(z)/3 \quad \text{on } y = -1, \quad (4.37b)$$

where the spanwise slip velocity $q(z)$ is given in terms of the wave pressure on the lower wall by

$$q(z) = \frac{\Delta^2}{\alpha^2 \lambda(z)^2} \left[\alpha^2 (p(z)p^*(z))' + (p'(z)p'^*(z))' \right], \quad (4.38)$$

with $\lambda(z)$ the lower wall streak shear

$$\lambda(z) = \left. \frac{\partial U}{\partial y} \right|_{y=-1}, \quad (4.39)$$

and $\Delta = |A|$ representing the wave amplitude, while the wave pressure satisfies the eigenvalue problem

$$p''(z) - \frac{\lambda'(z)}{\lambda(z)} \mathcal{F}(\xi_0(z)) p'(z) - \alpha^2 p(z) = -\frac{(\alpha \lambda(z))^{5/3} \mathcal{G}(\xi_0(z))}{K_0 - \alpha^2 c_1^2 m_0} p(z), \quad (4.40)$$

subject to periodicity conditions

$$p(0) = p(2\pi/\beta) = 1, \quad p'(0) = p'(2\pi/\beta), \quad (4.41)$$

where β is a prescribed spanwise wavenumber. The coefficients in (4.40) are

$$\mathcal{F}(\xi_0) = \frac{3}{2} + \frac{\xi_0}{2Ai(\xi_0)} (\xi_0 \kappa(\xi_0) + Ai'(\xi_0)), \quad \mathcal{G}(\xi_0) = i^{5/3} \frac{Ai'(\xi_0)}{\kappa(\xi_0)}, \quad \xi_0 = -\frac{(i\alpha)^{1/3} c_1}{(\lambda(z))^{2/3}}. \quad (4.42)$$

We can see how the nonlinear interaction of the wave pressure forces the roll (and hence the streak) via the boundary condition (4.38), while the streak shear feeds back into the wave equation (4.40), both explicitly and via the function ξ_0 . The pressure equation (4.40) subject to the periodic boundary conditions (4.41) can be considered to be an eigenvalue problem to be solved for the real values of the streamwise wavenumber α_0 and the phase speed correction c_1 .

4.5 Perturbation analysis

Before turning to the numerical solution of the interaction problem at $O(1)$ wave amplitudes Δ , we can make some analytical progress by assuming that the wave pressure is small, and seeking a leading-order solution of the form

$$p(z) = \cos \beta z. \quad (4.43)$$

Since Δ is small we anticipate that the streak will simply be a small perturbation to plane Couette flow and hence we can take the shear term $\lambda(z)$ to be unity at leading order. Writing

$$\alpha = \alpha_0 + \dots, \quad c_1 = c_{11} + \dots, \quad \lambda(z) = 1 + \dots, \quad \xi_0 = \xi_{00} + \dots, \quad (4.44)$$

an order-of-magnitude analysis of the wave pressure equation (4.40) results in the leading-order eigenrelation

$$\frac{\mathcal{G}(\xi_{00})\alpha_0^{5/3}}{K_0 - \alpha_0^2 c_{11}^2 m_0} = \alpha_0^2 + \beta^2, \quad c_{11} = -i^{-1/3} \xi_{00} \alpha_0^{-1/3}, \quad (4.45)$$

which is exactly that obtained in section 3.2 in the linear analysis: see (3.38). Here, $\xi_{00} = -i^{1/3} s_N$ with s_N given in (3.39), and $G(\xi_{00}) \simeq -1.0007$. The solutions for α_0 and c_{11} (c_1 in section 3) were plotted versus K_0 in Fig. 1 for $m_0 = (0.05, 0.1, 0.2)$, $\beta = 1$. We see that the disturbances become shorter in streamwise wavelength as the lower plate becomes more rigid and propagate at a speed ever closer to that of the lower wall. It can be established that a neutral solution exists for all positive values of the parameters K_0 , m_0 and β .

We now wish to study the behaviour at next order in the expansions (4.44), where we expect a roll/streak flow to be generated by the wave. In the remainder of this section, we will concentrate on a spanwise wavenumber

$$\beta = 1, \quad (4.46)$$

although the analysis can readily be generalized to non-unit β . To leading order, the form for p in (4.43) yields a spanwise forcing of the form

$$q(z) = \Delta^2 (\alpha_0^{-2} - 1) \sin 2z + \dots, \quad (4.47)$$

upon recollection of the definition of q in (4.38). We now investigate how this forcing drives the roll/streak solution in the core. In light of the boundary condition (4.37), the form for $q(z)$ suggests that the spanwise roll $W(z) = O(\Delta^2)$, with the continuity equation (4.36a) indicating the normal roll component V will also be of this order. Together, this roll-field will feed into (4.36b) where it will generate a streak with $U = y + O(\Delta^2)$. Finally, the normal momentum equation (4.36c) suggests that $P = O(\Delta^2)$. A roll/streak expansion is therefore anticipated in which

$$(U - y, V, W, P) = \Delta^2 (U_1(y, z), V_1(y, z), W_1(y, z), P_1(y, z)) + \dots. \quad (4.48)$$

Substituting (4.48) into the roll/streak equations (4.36), we obtain

$$\frac{\partial V_1}{\partial y} + \frac{\partial W_1}{\partial z} = 0, \quad (4.49a)$$

$$V_1 = \frac{\partial^2 U_1}{\partial y^2} + \frac{\partial^2 U_1}{\partial z^2}, \quad (4.49b)$$

$$0 = -\frac{\partial P_1}{\partial y} + \frac{\partial^2 V_1}{\partial y^2} + \frac{\partial^2 V_1}{\partial z^2}, \quad (4.49c)$$

$$0 = -\frac{\partial P_1}{\partial z} + \frac{\partial^2 W_1}{\partial y^2} + \frac{\partial^2 W_1}{\partial z^2}. \quad (4.49d)$$

These linearized equations are subject to the matching conditions (4.37), giving

$$U_1 = V_1 = W_1 = 0 \quad \text{on } y = 1, \quad (4.50a)$$

$$U_1 = V_1 = 0, \quad W = -\frac{1}{3}(\alpha_0^{-2} - 1) \sin 2z \quad \text{on } y = -1. \quad (4.50b)$$

Guided by the form of the forcing on the lower wall, we seek a solution for the roll flow of the form

$$V_1 = \frac{2}{3}(\alpha_0^{-2} - 1)F(y) \cos 2z, \quad W_1 = -\frac{1}{3}(\alpha_0^{-2} - 1)F'(y) \sin 2z. \quad (4.51)$$

After some manipulation of equations (4.49), and application of the boundary conditions (4.50), the function F can be determined as

$$F(y) = C_1 e^{2y} + C_2 e^{-2y} + C_3 y e^{2y} + C_4 y e^{-2y}, \quad (4.52)$$

with coefficients given by

$$C_1 = -\frac{9 - e^{-8}}{e^6 - 66e^{-2} + e^{-10}}, \quad C_2 = \frac{1 + 7e^{-8}}{e^2 - 66e^{-6} + e^{-14}}, \quad (4.53a)$$

$$C_3 = \frac{7 + e^{-8}}{e^6 - 66e^{-2} + e^{-10}}, \quad C_4 = \frac{1 - 9e^{-8}}{e^2 - 66e^{-6} + e^{-14}}. \quad (4.53b)$$

Introducing a perturbation streamfunction ϕ_1 such that

$$V_1 = \frac{\partial \phi_1}{\partial z}, \quad W_1 = -\frac{\partial \phi_1}{\partial y}, \quad (4.54)$$

we may write

$$\phi_1 = \frac{1}{3}(\alpha_0^{-2} - 1)F(y) \sin 2z \quad (4.55)$$

up to a constant.

The roll flow forces the streak in accordance with equation (4.49b), and leads to the solution

$$U_1 = -\frac{1}{3}(\alpha_0^{-2} - 1)R(y) \cos 2z, \quad (4.56)$$

where

$$R(y) = C_5 e^{2y} + C_6 e^{-2y} + \left[-\frac{C_3}{4} y^2 + \left(\frac{C_3}{8} - \frac{C_1}{2} \right) y \right] e^{2y} + \left[\frac{C_4}{4} y^2 + \left(\frac{C_4}{8} + \frac{C_2}{2} \right) y \right] e^{-2y}, \quad (4.57)$$

with

$$C_5 = -\frac{39 + 90e^{-8} - e^{-16}}{8(e^6 - 67e^{-2} + 67e^{-10} - e^{-18})}, \quad C_6 = \frac{3 + 130e^{-8} - 5e^{-16}}{8(e^2 - 67e^{-6} + 67e^{-14} - e^{-22})}. \quad (4.58)$$

Figure 2 depicts the streamfunction and streak flow corresponding to a lower wall with properties $K_0 = 1$ and $m_0 = 0.2$. We observe that spanwise variations are more intense in the lower half of the channel, with the streamfunction being more concentrated towards the lower wall than the streak flow.

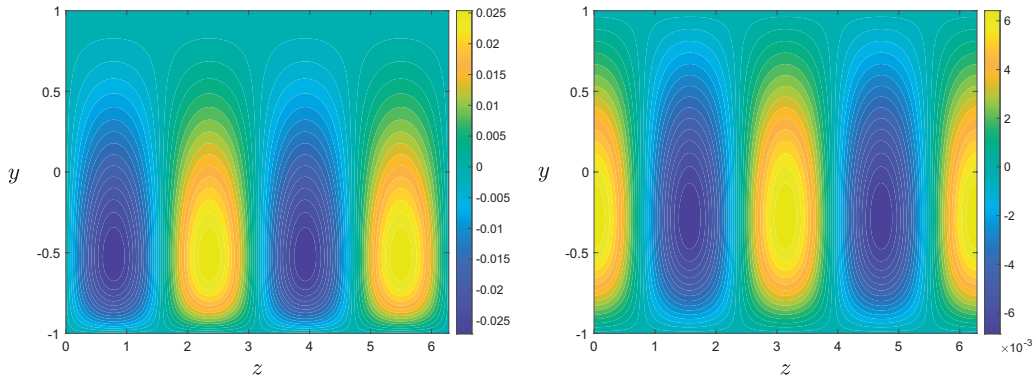


Fig. 2 Depiction of the streamfunction ϕ_1 (left) and streak flow U_1 (right) for $K_0 = 1$ and $m_0 = 0.2$.

We are now able to determine the behaviour of the streak shear as

$$\lambda(z) = 1 + \Delta^2 \lambda_1(z) + \dots, \quad (4.59)$$

with

$$\lambda_1(z) = \left. \frac{\partial U_1}{\partial y} \right|_{y=-1} = \lambda_{11} \cos 2z, \quad (4.60)$$

and

$$\lambda_{11} = -\frac{1}{3} (\alpha_0^{-2} - 1) \left(\frac{1}{8} - \frac{24 + 40e^{-8}}{e^8 - 67 + 67e^{-8} - e^{-6}} \right), \quad (4.61)$$

from differentiation of (4.56). In order to determine the corrections to the streamwise wavenumber and phase speed, we need to investigate the effect of this shear term on the pressure equation (4.40). A ‘maximal-interaction approach’ suggests $O(\Delta^2)$ corrections to the wavenumber, phase speed and pressure, leading to the expansions

$$\alpha = \alpha_0 + \Delta^2 \alpha_1 + \dots, \quad c_1 = c_{11} + \Delta^2 c_{12} + \dots, \quad p = \cos z + \Delta^2 p_1(z) + \dots. \quad (4.62)$$

Upon substitution of (4.59) and (4.62), (4.40) yields at $O(\Delta^2)$ an equation for p_1 of the form

$$p_1''(z) + p_1 = L_1 \sin 2z \sin z + L_2 \cos z + L_3 \cos 2z \cos z, \quad (4.63)$$

where L_1, L_2, L_3 are complex-valued coefficients dependent on the unknown quantities α_1 and c_{12} . In order that p_1 satisfies the periodic boundary conditions (4.41), there must be no $\cos z$ component on the right-hand side of (4.63). This leads us to the solvability condition

$$L_1 + 2L_2 + L_3 = 0. \quad (4.64)$$

Since the L_i are complex-valued, (4.64) can be written as two real equations to determine α_1, c_{12} for given values of the wall properties K_0 and m_0 . These solutions are depicted in Fig. 3 for various

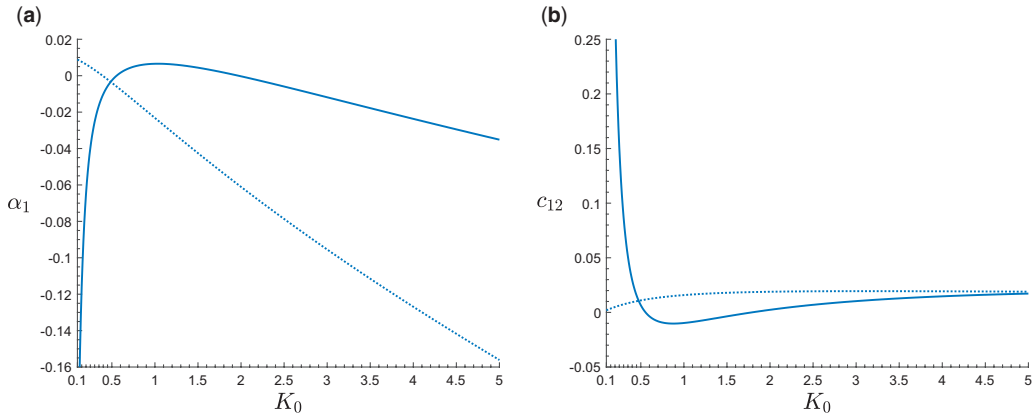


Fig. 3 Results from the solvability condition (4.64). (a) Streamwise wavenumber α_1 versus stiffness parameter K_0 , (b) phase speed correction c_{12} versus K_0 . Two different values of the mass parameter m_0 are shown. Dotted: $m_0 = 0.05$; solid: $m_0 = 0.2$.

K_0 and two choices of m_0 . We can see that for a spring stiffness $K_0 = 1$, we have $\alpha_1 < 0$ for $m_0 = 0.05$, indicating a subcritical bifurcation from the linear state, while we find $\alpha_1 > 0$ for $m_0 = 0.2$, corresponding to a supercritical bifurcation. In the full numerical results that follow, we will concentrate on these choices of parameters to examine whether there are any qualitative differences in the full nonlinear states which emerge due to this alteration in the nature of the bifurcation.

It is possible to continue this perturbation process to higher-order, determining next the $O(\Delta^4)$ contribution to the spanwise forcing and calculating the corresponding correction to the roll/streak flow and so on, although the algebra becomes progressively more complicated. The process indicates that if we start with a solution for the wave pressure which is even, as in (4.43), then the solution will retain this symmetry at higher order.

4.6 Numerical investigations

At $O(1)$ wave amplitudes Δ , the solution of the roll/streak/wave problem posed in section 4.4 is a formidable numerical task. We describe the numerical methods we used first and then discuss the results for various values of the wall properties.

4.6.1 Numerical method for solving (4.36)–(4.42). We include brief details of the numerical methods employed here with the reader directed to (33) for a comprehensive account. We will concentrate on the case of unit spanwise wavenumber $\beta = 1$, although the method readily extends to more general β . Our procedure involves, at a prescribed wave amplitude Δ , iterating over the spanwise roll forcing $q(z)$ for a given spring stiffness K_0 and mass m_0 . For small values of Δ , the initial guess used for q is provided by the perturbation analysis of the previous section. For larger Δ , a more appropriate initial guess is the converged $q(z)$ from the solution for a slightly smaller value of Δ .

To solve the roll equations (4.36a), (4.36c), (4.36d), the roll pressure is eliminated by cross-differentiation, followed by a Fourier decomposition in z of the roll velocity components. This

results in an infinite set of nonlinear ordinary differential equations for the Fourier coefficients which we truncate at a suitably large value N , say. A value of $N = 28$ proved sufficient for all our calculations. The truncated system was discretized using a second-order finite-differencing scheme and the nonlinear terms treated as a forcing, so that a fixed point scheme could be applied iteratively to obtain a converged solution. Once the roll flow is obtained, we can construct a roll streamfunction ϕ (up to a constant) defined by $V = \partial\phi/\partial z$, $W = -\partial\phi/\partial y$.

The roll flow forms the coefficients in the streak equation (4.36b). Again, this equation was Fourier-decomposed in z , truncated, and solved using finite differences. The linear nature of equation (4.36b) meant that in this case no iterative process was required. Once the streak is obtained, we can calculate the shear $\lambda(z)$ using (4.39).

We now need to solve the pressure equation (4.40) with this distribution of $\lambda(z)$ in order to determine α and c_1 such that $p(z)$ is periodic in z . Guided by the perturbation analysis of section 4.5, we will assume that if we start the interaction process with a pressure distribution at small Δ which is even in z , then it will retain this symmetry for all Δ . We therefore seek an even periodic solution of (4.40) for which we can impose $p(0) = 1$, $p'(0) = 0$, and $p'(\pi) = 0$. Firstly, the coefficients $\mathcal{F}$ and $\mathcal{G}$ in this equation are calculated using the $\lambda(z)$ distribution arising from the roll/streak equations together with suitable guesses for α_0 , c_1 . Equation (4.40) was then solved by marching from $z = 0$ using a Runge–Kutta scheme and then performing a Newton iteration on the values of α_0 , c_1 until $p'(\pi) = 0$ is achieved to some acceptable tolerance. The solution for $p(z)$ thus obtained is then fed into the formula (4.38) to obtain $q(z)$, and this newly–updated forcing is then re-applied to the roll/streak problem (4.36).

The whole procedure is repeated until the forcing $q(z)$ converges. Once a converged solution is obtained, it is used as an initial guess for the solution to the system at a slightly increased value of the wave amplitude Δ . In this way, we can construct solutions at progressively larger values of Δ .

4.6.2 Numerical results. We found two families of solutions, each with different characteristics. We are able to categorize the solutions as emerging from either zero amplitude ($\Delta = 0$), or finite amplitude ($\Delta = \Delta_c$). We will refer to these as Type I and Type II solutions, respectively.

Consider a lower wall with the properties $(K_0, m_0) = (1, 0.05)$. In Fig. 4, we compare Type I solutions for the streamwise wavenumber α_0 and phase speed correction c_1 from our numerical computations with the results of our perturbation analysis of section 4.5 as the wave amplitude Δ is increased. We see good agreement between the two approaches for $\Delta \lesssim 1$. We expect that including more terms in the perturbation expansion would maintain the close agreement to larger values of Δ . With increasing Δ , both the streak shear $\lambda(z)$ and spanwise roll forcing $q(z)$ broadly appear to retain the shapes they possess at small Δ (Fig. 5), though a slight skewing is noticeable in $q(z)$. The magnitudes of the maximum and minimum values of $\lambda(z)$ and $q(z)$ increase considerably as Δ becomes larger.

A similar trend is observed in the streamfunction and streak flow (Figs. 6, 7): the magnitudes of the flow becomes greatly amplified with increasing Δ , as would be expected. The spanwise spacing, however, is relatively unaltered up to $\Delta \simeq 1.9$. We see that the streamfunction is concentrated towards the lower wall, perhaps not unsurprisingly since it is only forced by the wave on the lower boundary. At $\Delta = 1.9$, it is clear that the streak flow is no longer a small perturbation to plane Couette flow and the base flow has therefore been completely altered by this self-sustained process. We are unable to continue this solution to higher values of wave amplitude due to the failure of the numerical method to converge; we do, however, find the existence of a Type II solution for these

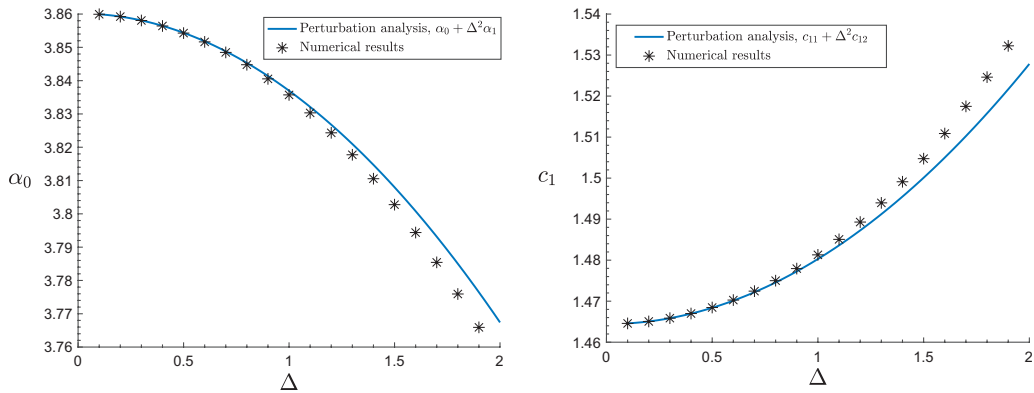


Fig. 4 Comparison of numerically obtained Type I solutions (asterisks) with analytically obtained solutions from our perturbation analysis (solid line). Solutions α_0 (left) and c_1 (right) are shown as Δ varies. The wall properties are $K_0 = 1$, $m_0 = 0.05$.

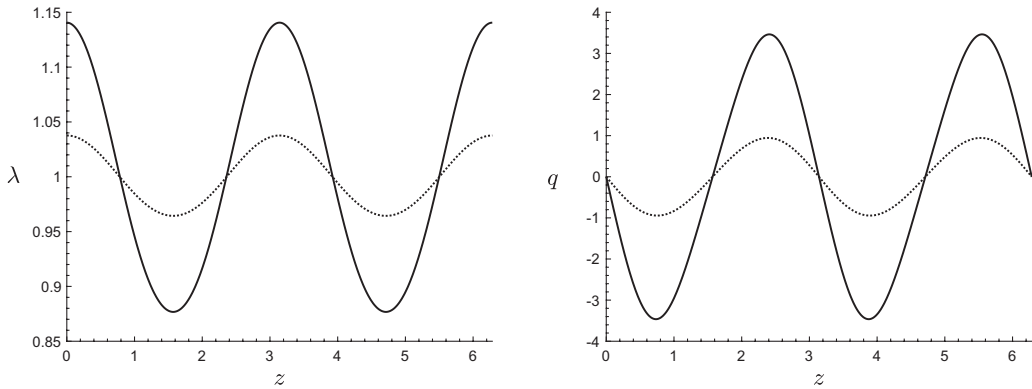


Fig. 5 Type I solutions. Plots of the numerically obtained shear $\lambda(z)$ (left) and $q(z)$ (right) for $\Delta = 1$ (dotted) and $\Delta = 1.9$ (solid). The wall properties are $K_0 = 1$, $m_0 = 0.05$.

same wall properties. We will see shortly that these Type I and II solutions co-exist for certain values of Δ .

Before discussing the Type II solution further, we examine the effect on the Type I solution of increasing m_0 . The Type I solutions for $K_0 = 1$, $m_0 = 0.2$ are shown in Figs. 8 to 11; again α_0 and c_1 show good agreement with results obtained using the perturbation method of section 4.5. Unlike the case $m_0 = 0.05$, we see that c_1 now decreases with increasing Δ , so that the wave is propagating ever closer to the speed of the lower wall. This is consistent with the change in the nature of the bifurcation which we noted in section 4.5.

As Δ increases in Fig. 9, we see that the shape of $q(z)$ deviates from that predicted by the perturbation expansion, with this being particularly noticeable around $z = \pi$. In Fig. 10, the rolls

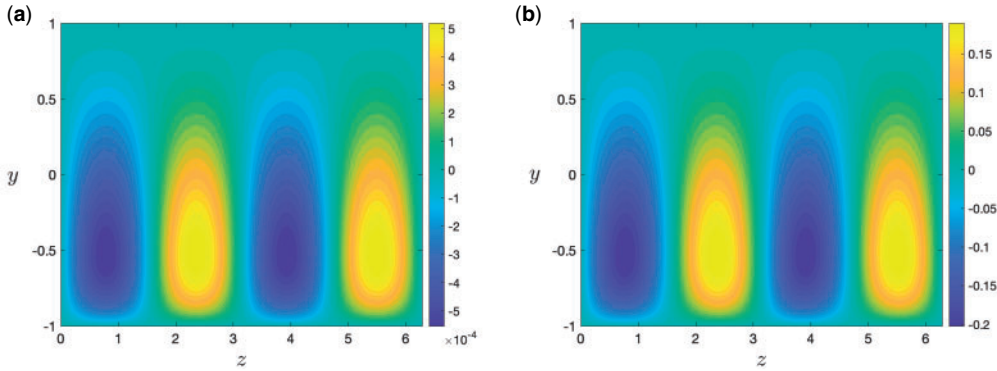


Fig. 6 Type I solution. Plots of the numerically obtained streamfunction for (a) $\Delta = 0.1$ and (b) $\Delta = 1.9$. The wall properties are $K_0 = 1$, $m_0 = 0.05$.

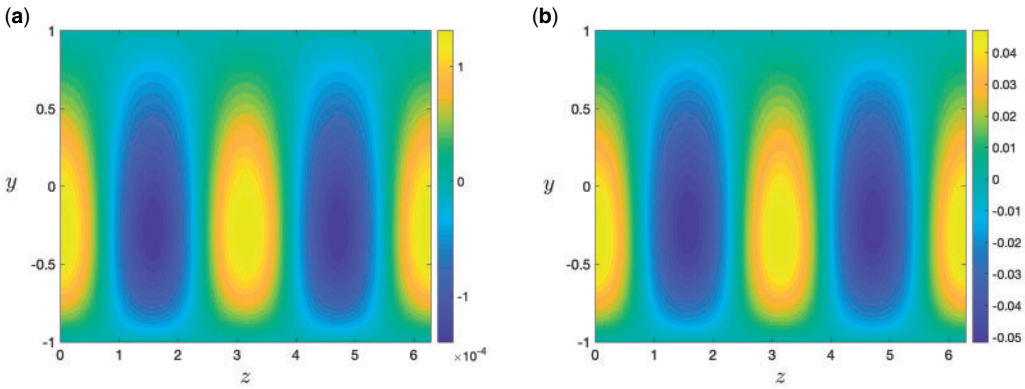


Fig. 7 Type I solution. Plots of the numerically obtained streak ($U - y$) for (a) $\Delta = 0.1$ and (b) $\Delta = 1.9$. The wall properties are $K_0 = 1$, $m_0 = 0.05$.

appear to change shape as Δ is increased, with each roll losing its spanwise symmetry. At $\Delta = 3.2$, the streamfunction solution has become highly disturbed. The streak flow (Fig. 11) increases in intensity with Δ , eventually becoming comparable in magnitude with the original Couette flow, particularly near the channel centreline. By $\Delta = 3.2$, the positive contributions to the streak pattern have spread over a larger spanwise extent near the lower wall.

Before continuing with our description of the numerical results, it is worth remarking that if we perform a perturbation analysis along the lines of section 4.5, starting with a uniform shear as in (4.44) but, rather than (4.43), we begin with a spanwise-independent pressure $p = 1$ ($0 \leq z \leq 2\pi$), then at leading order the wave equation (4.40) generates a solution for α_0 and c_1 obtained from (4.45) with $\beta = 0$. However, from (4.38), this solution leads to zero spanwise forcing, i.e. $q(z) \equiv 0$, and hence does not generate a roll/streak flow.

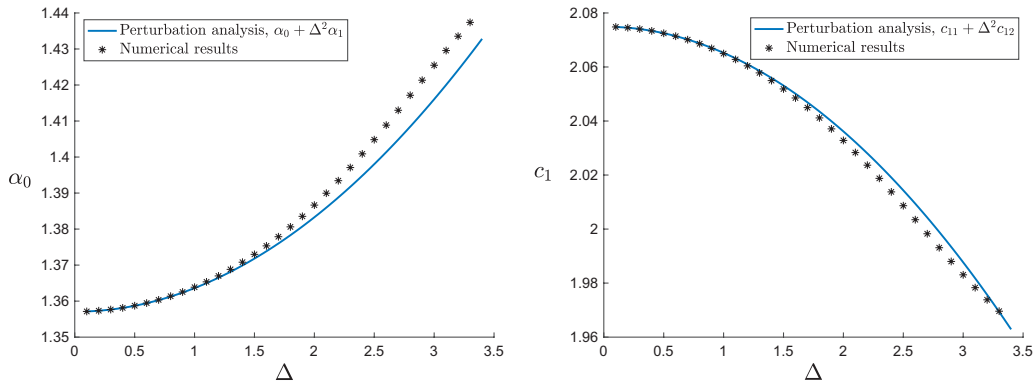


Fig. 8 Comparison of numerically obtained Type I solutions (asterisks) with analytically obtained solutions from our perturbation analysis (solid line). Solutions α_0 (left) and c_1 (right) are shown as Δ varies. The wall properties are $K_0 = 1$, $m_0 = 0.2$.

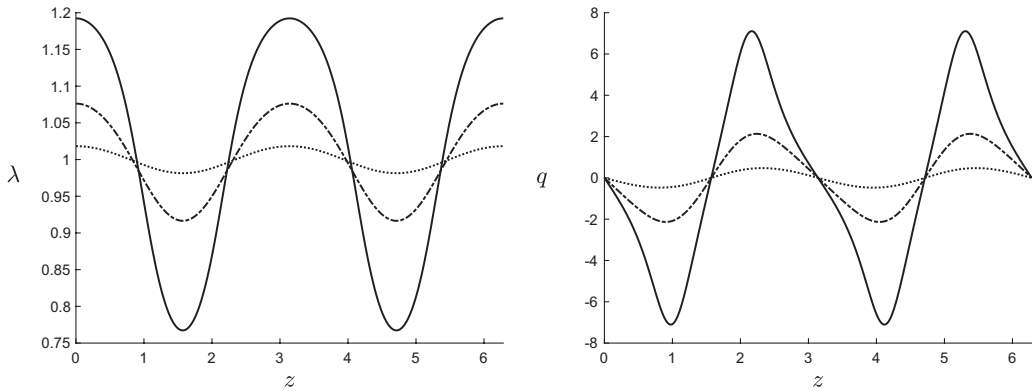


Fig. 9 Type I solutions. Plots of the numerically obtained shear $\lambda(z)$ (left) and forcing $q(z)$ (right) for $\Delta = 1$ (dotted), $\Delta = 2$ (dot-dashed) and $\Delta = 3.2$ (solid). The wall properties are $K_0 = 1$, $m_0 = 0.2$.

The Type II solution found for the wall properties $(K_0, m_0) = (1, 0.05)$ appears to bifurcate from the constant pressure solution mentioned above at a value $\Delta_c \simeq 1.2$ as shown in Fig. 12. The maximum magnitudes of the streak shear $\lambda(z)$ and spanwise forcing $q(z)$ increase significantly as Δ is increased (Fig. 13). The solution is quite different in character to its Type I counterpart: there is no even or odd symmetry about $z = \pi/2$ or $z = 3\pi/2$, and the peak value of the shear $\lambda(z)$ never exceeds its undisturbed value of unity. The forcing is almost zero over a large part of the spanwise domain, becoming localized near two spanwise locations that vary with Δ . We observe that the roll flow is concentrated towards the lower wall more so than the streak, as seen in Figs. 14, 15. We also notice fewer rolls than in the Type I streamfunction solution for the same wall properties. The rolls increase in intensity with increasing Δ and their spacing becomes non-uniform. In Fig. 15, where

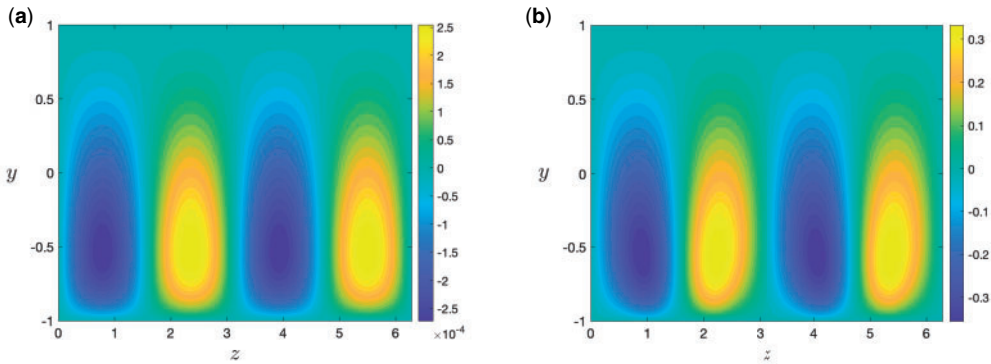


Fig. 10 Type I solution. Plots of the numerically obtained streamfunction for (a) $\Delta = 0.1$ and (b) $\Delta = 3.2$. The wall properties are $K_0 = 1$, $m_0 = 0.2$.

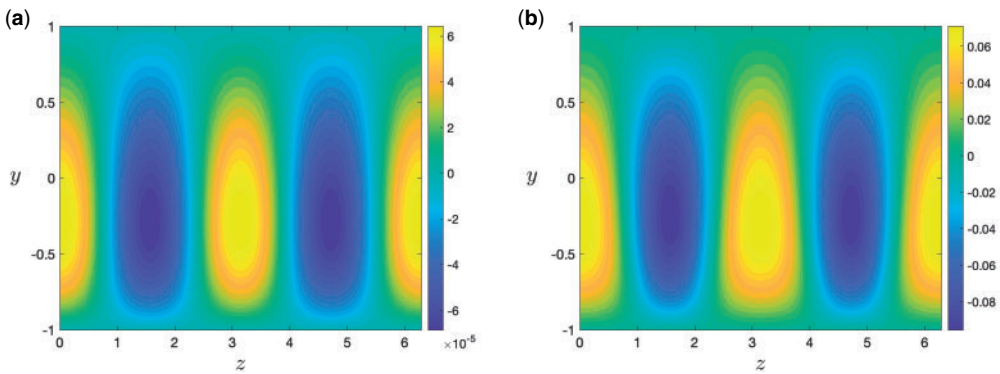


Fig. 11 Type I solution. Plots of the numerically obtained streak ($U - y$) for (a) $\Delta = 0.1$ and (b) $\Delta = 3.2$. The wall properties are $K_0 = 1$, $m_0 = 0.2$.

we plot contours of the streak, we see two distinct minimum regions developing and the spanwise variations in the streak becoming more complicated with increasing Δ . By $\Delta = 6.5$, the streak flow has become an $O(1)$ perturbation to plane Couette flow over the entire flow-field.

We now consider a lower wall with $K_0 = 1$ as before, but with an increased mass of $m_0 = 0.2$. A Type II solution again emerges at finite amplitude, in this case at a critical wave amplitude $\Delta_c \simeq 3$ (see the right-hand panel of Fig. 12). Figure 16 shows that the spanwise distributions of $\lambda(z)$ and $q(z)$ are quite different to the Type II solution for $m_0 = 0.05$: the shear term $\lambda(z)$ now has only one trough and no peaks, while the forcing $q(z)$ has a steepening gradient near $z = \pi$ as Δ is increased. These differences feed into the streamfunction and streak flow, which are illustrated in Figs. 17, 18. As the wave amplitude is increased to $\Delta = 5$ when $m_0 = 0.2$, the rolls become more ‘pointed’ towards $z = \pi$ near the lower wall. This is in contrast to the Type II streamfunction for $m_0 = 0.05$ (Fig. 14). The effect of increasing Δ on the shape of the streak is also pronounced for $m_0 = 0.2$;

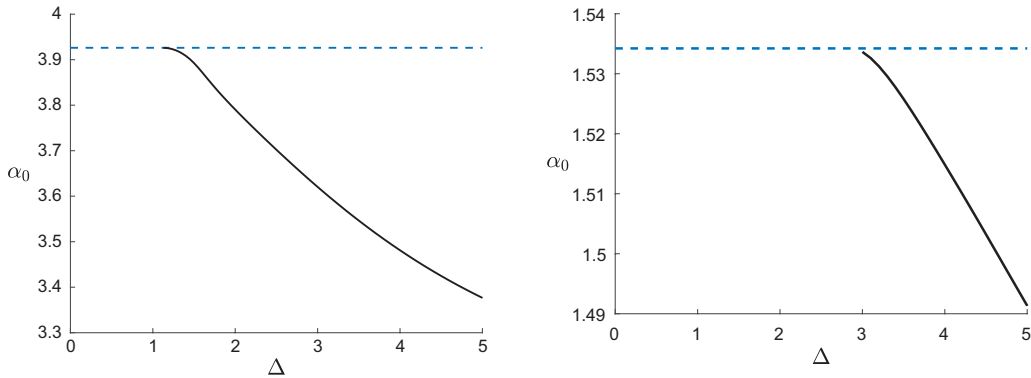


Fig. 12 Type II solutions. Solid curves depict solutions α_0 to the vortex-wave interaction equations when the wall properties are $K_0 = 1$, $m_0 = 0.05$ (left) and $K_0 = 1$, $m_0 = 0.2$ (right). The dashed line represents the constant pressure wave solution with α_0 calculated from the linear eigenrelation (4.45) with $\beta = 0$.

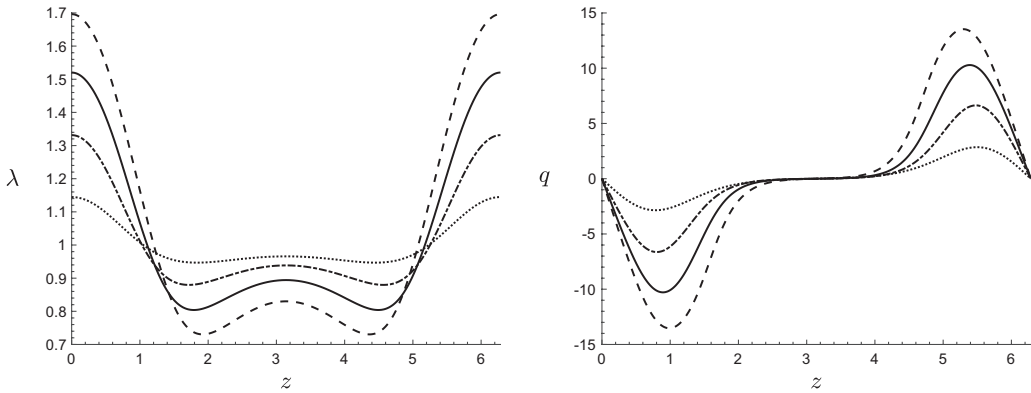


Fig. 13 Type II solutions. Plots of the numerically obtained shear $\lambda(z)$ (left) and $q(z)$ (right) for $\Delta = 2$ (dotted), $\Delta = 3$ (dot-dashed), $\Delta = 4$ (solid) and $\Delta = 5$ (dashed). The wall properties are $K_0 = 1$, $m_0 = 0.05$.

the negative region has a trend towards concentration near $z = \pi$, while the positive regions have a greater spanwise extent.

5. Conclusions

We investigate the instability of plane Couette flow through a channel with an upper rigid wall, and a lower flexible wall governed by a simple spring-backed plate model with parameters controlling the mass and stiffness per unit area of the plate. The plate responds to pressure fluctuations in the fluid through a dynamic boundary condition which supplements the usual no-slip and kinematic conditions.

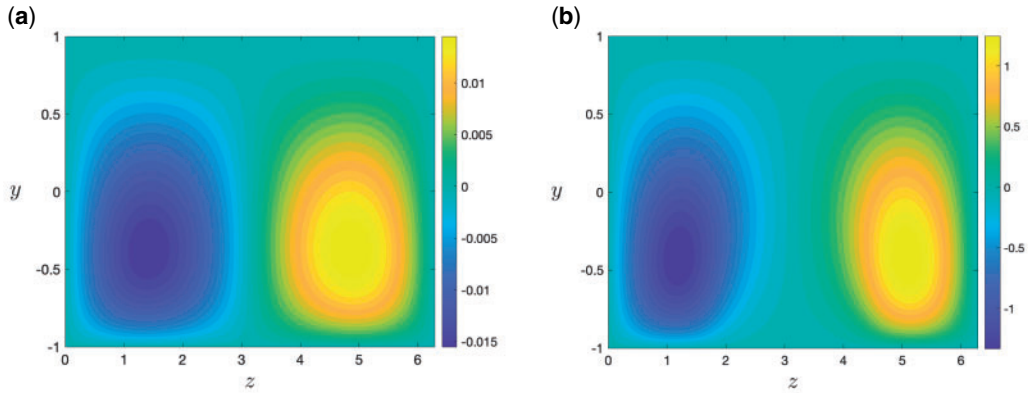


Fig. 14 Type II solutions. Plots of the numerically obtained streamfunction for (a) $\Delta = 1.2$ and (b) $\Delta = 6.5$. The wall properties are $K_0 = 1$, $m_0 = 0.05$.

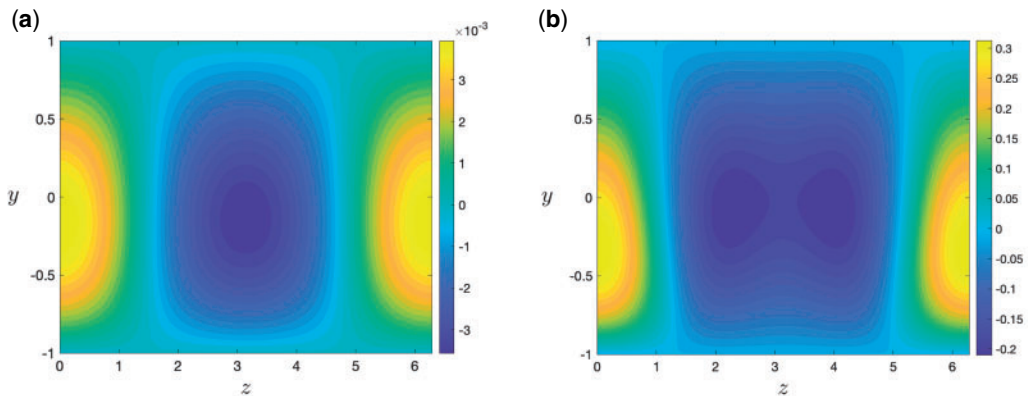


Fig. 15 Type II solutions. Plots of the numerically obtained streak $U - y$ for (a) $\Delta = 1.2$ and (b) $\Delta = 6.5$. The wall properties are $K_0 = 1$, $m_0 = 0.05$.

We first establish that, at high Reynolds number, for a wall of relatively high density and stiffness, the plane Couette base flow supports oblique linear neutral wave disturbances of streamwise wavelength comparable with the channel width, which propagate at close to the speed of the lower wall. The neutral modes possess an asymptotic structure in which the predominantly inviscid core flow is supplemented by a Stokes layer near the upper wall, and a critical layer near the lower flexible wall.

We then extend the analysis to the nonlinear regime, where we establish that when the amplitude of the waves in the core reaches a (still tiny) $O(\text{Re}^{-1}(\log \text{Re})^{-1/2})$ size, the lower layer becomes nonlinear, generating a spanwise slip velocity that induces a roll/streak flow in the core. A perturbation analysis demonstrates analytically how the spanwise dependence of the roll/streak/wave flow becomes more complicated as the wave amplitude Δ is increased.

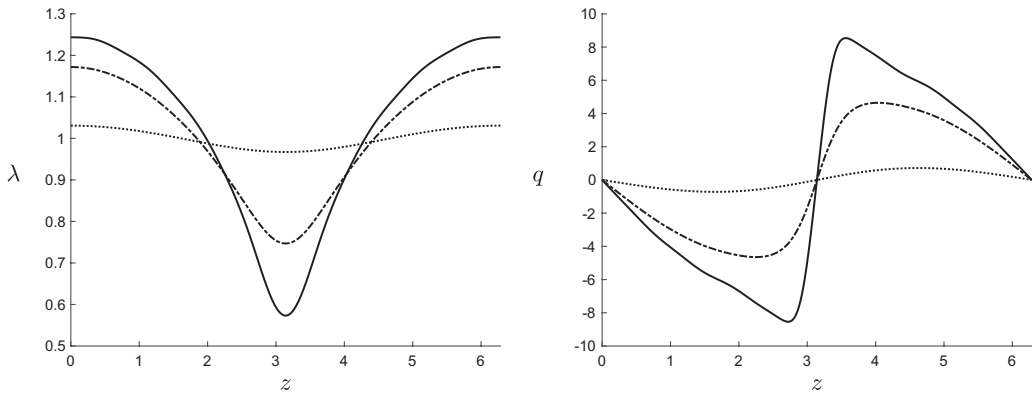


Fig. 16 Type II solutions. Plots of the numerically obtained shear $\lambda(z)$ (left) and $q(z)$ (right) for $\Delta = 3$ (dotted), $\Delta = 4$ (dot-dashed) and $\Delta = 4.8$ (solid). The wall properties are $K_0 = 1$, $m_0 = 0.2$.

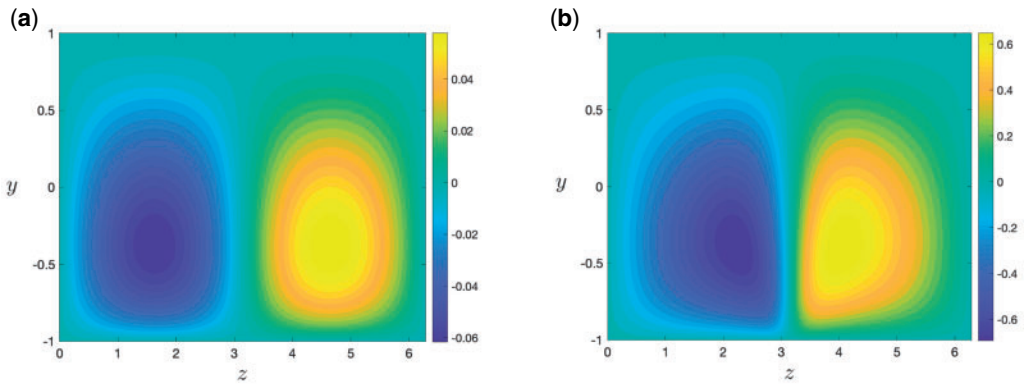


Fig. 17 Type II solutions. Plots of the numerically obtained streamfunction for (a) $\Delta = 3$ and (b) $\Delta = 5$. The wall properties are $K_0 = 1$, $m_0 = 0.2$.

Full numerical simulations of the nonlinear interaction equations show that the roll flow is typically concentrated towards the lower wall, and is greatly affected in magnitude and shape by an increase in the wave amplitude. The streak flow induced by the $O(\text{Re}^{-1})$ roll field is an $O(1)$ perturbation to the underlying Couette flow, and its intensity is fairly evenly distributed across the channel despite the fact that the forcing arises purely from nonlinear interactions near the flexible surface. We have computed two families of solutions for which the spanwise distributions of the roll/streak flows are quite different. Type I solutions exist for all non-zero amplitudes and their small amplitude behaviour is consistent with the perturbation analysis mentioned above. Type II solutions only exist above a critical amplitude, the value of which is dependent on the properties of the flexible wall. Both Type I and Type II roll/streak flows become more intense as the wave amplitude is increased.

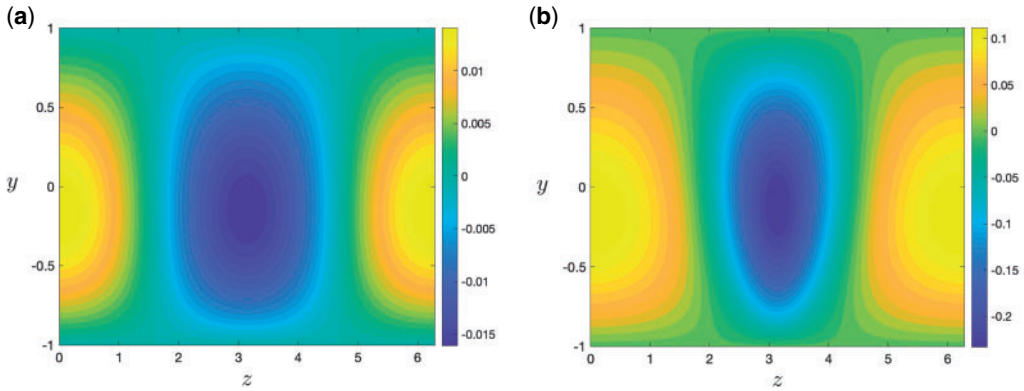


Fig. 18 Type II solutions. Plots of the numerically obtained streak $U - y$ for (a) $\Delta = 3$ and (b) $\Delta = 5$. The wall properties are $K_0 = 1$, $m_0 = 0.2$.

An increase in the mass m_0 of the wall for a fixed stiffness K_0 appears to change the behaviour of the Type I streamwise wavelength α_0 and phase speed c_1 as the wave amplitude Δ increases. A plate mass of $m_0 = 0.05$ is associated with a wave of increasing wavelength and phase speed as Δ is increased. On the other hand, a larger mass of $m_0 = 0.2$ corresponds to a wave with a shortening wavelength that travels more closely to the speed of the lower wall as Δ is increased. This is consistent with the subcritical and supercritical bifurcations observed in Fig. 3 for $K_0 = 1$ and the two choices of mass.

The effect of mass is more distinguished among the Type II solutions: the wave amplitude at which the Type II solution emerges increases with the plate mass. Though the streamfunction and streak flow for both $m_0 = 0.05$ and $m_0 = 0.2$ share similar flow characteristics for Δ near their respective emergence amplitudes Δ_c , the flow structures at larger Δ differ greatly. While plates of the smaller mass $m_0 = 0.2$ lead to a ‘splitting’ in the streak flow as Δ increases, plates with a larger mass of $m_0 = 0.5$ are associated with a streak flow that localizes in the spanwise direction with increasing Δ . In addition, at a given Δ (for which solutions exist), a larger mass is associated with a forcing $q(z)$ that has a much stronger spanwise gradient near $z = \pi$.

However, we have only explored the (K_0, m_0) parameter space in a very limited way due to the expensive and time-consuming nature of the calculations, and further investigation of the effects of the wall properties on the roll/streak flow is certainly merited, along with studying the effects of varying the spanwise wavenumber β . It would also be interesting to see how introducing compliance to the upper surface alters the flow-field. The present compliant wall model is extremely unsophisticated, and including more physical effects into the equation governing the behaviour of the wall is a natural next step. Nevertheless, the current research demonstrates that tiny perturbations to a very simple compliant surface are capable of completely altering the underlying base flow.

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