Coercive Inequalities and Nonlinear Dynamics on Infinite Spin Systems

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Declaration

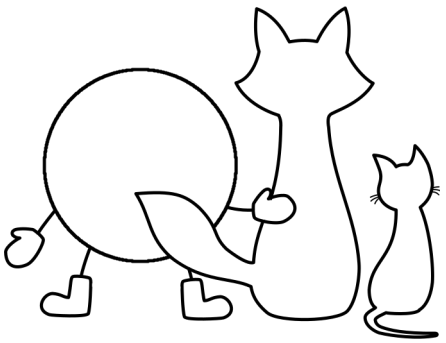
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To my wife, to Wickey, and to little friends



Abstract

This thesis concerns the study of spin systems on an infinite lattice with the Langevin diffusion dynamics. The projects are divided into two main topics.

In the first topic we construct Gibbs measures in infinite dimensions which are reversible with respect to the Markov semigroup associated to the Langevin dynamics. We then find sufficient conditions for the Gibbs measures to satisfy spectral gap or logarithmic Sobolev inequalities, so that one can obtain ergodicity or contractivity of the semigroup. We pay particular attention to the cases where the single-state space is a Carnot group, and the Gibbs measure is defined by homogeneous norms with vanishing or blowing up sub-gradient.

For the second topic, we study the quantum dynamics of the system by considering the Schrödinger equation in infinite dimensions with a logarithmic nonlinearity. Namely, we obtain estimates that hold in arbitrary dimensions, and prove the existence of infinite-dimensional weak solutions for potentials with bounded multi-particle interactions.

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Symbols and Notations

$\mathbb{N}$	Set of natural numbers, $\{0, 1, 2, ...\}$
$\mathbb{R}^m$	m-dimensional Euclidean space, m is a positive integer
$\mathbb{Z}^D$	D-dimensional integer lattice, D is a positive integer
$\mathbb{H}$	3-dimensional Heisenberg group, defined in Example 2.4.2
$\mathbb{G}$	Homogeneous Carnot group, defined in Definition 2.4.1
$\nabla_{\mathbb{G}}$	(Canonical) sub-gradient on $\mathbb{G}$, defined in Definition 2.4.3
$\Delta_{\mathbb{G}}$	(Canonical) sub-Laplacian on $\mathbb{G}$, defined in Definition 2.4.3
N	Kaplan norm on $\mathbb{H}$, defined in Example 2.4.9
d	Control distance on $\mathbb{G}$, defined in Definition 2.4.13
Ω	Configuration space (e.g. $\Omega = \mathbb{R}^{\mathbb{Z}}$)
ω	Boundary condition in Ω, defined in (2.1.4)
$\mathbb{E}_\Lambda^\omega$	Conditional expectation (or finite-volume Gibbs measure), defined in (2.1.5)
ν	Infinite-volume Gibbs measure associated to $\mathbb{E}_\Lambda$, defined in (2.1.8)
$\mathcal{I}$	Indicator function, $\mathcal{I}_A(a) = 1$ if $a \in A$ and 0 if $a \notin A$
$\|\cdot\|_{l^p}$	$\|x\|_{l^p} = (\sum_i \|x_i\|^p)^{1/p}$ if $p \in [1, \infty)$ and $\|x\|_{l^\infty} = \sup_i \|x_i\|$, for any vector $x = (x_i)$
$j \sim k$	$j, k \in \mathbb{Z}^D$ are neighbours
$\{\sim j\}$	The neighbouring set of $j \in \mathbb{Z}^D$, i.e. $\{k \in \mathbb{Z}^D : j \sim k\}$
$A \Subset B$	A is a compact subset of B
$M \succcurlyeq \rho$	$\langle Mx, x\rangle \geq \rho\langle x, x\rangle$ for matrix $M \in \mathbb{R}^{m\times m}$, vector $x \in \mathbb{R}^m$ and constant $\rho \in \mathbb{R}$
z^*	Complex conjugate of $z \in \mathbb{C}$
$\mathrm{Dom}(f)$	Domain of a function or operator f
$f^{(\alpha)}$	The partial derivative $\partial^{\|\alpha\|} f/(\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m})$ for any $f : \mathbb{R}^m \to \mathbb{R}$ or $\mathbb{C}$ and the multi-index $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ with $\|\alpha\| = \alpha_1 + \cdots + \alpha_n$
$C^k(\Omega, S)$	Set of functions $f : \Omega \to S$ such that $f^{(\alpha)}$ is continuous for all $\|\alpha\| \leq k$, where $k \in \mathbb{N}$. For simplicity, denote by $C^k(\Omega) = C^k(\Omega, \mathbb{R})$ and $C = C^0$
$\mu(f)$ or μf	The integral $\int_\Omega f(x)\mu(dx)$ for any measure μ and measurable function f on Ω
$\|\cdot\|_{\mathbb{L}^p(\mu)}$ or $\|\cdot\|_p$	$\|f\|_{\mathbb{L}^p(\mu)} = (\mu\|f\|^p)^{1/p}$ if $p \in [1, \infty)$
$\|\cdot\|_{\mathbb{L}^\infty(\Omega)}$ or $\|\cdot\|_\infty$	$\|f\|_{\mathbb{L}^\infty(\Omega)} = \sup_{x\in\Omega} \|f(x)\|$
$\mathbb{L}^p(\mu, \Omega)$ or $\mathbb{L}^p(\mu)$	Set of functions $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$ such that $\|f\|_{\mathbb{L}^p(\mu)} < \infty$, where $p \in [1, \infty)$
$\mathbb{L}^\infty(\Omega)$	Set of functions $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$ such that $\|f\|_{\mathbb{L}^\infty(\Omega)} < \infty$
$W^{k,p}(\mu, \Omega)$	Sobolev space, set of functions $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$ such that $f^{(\alpha)}$ exists in weak sense and is in $\mathbb{L}^p(\mu, \Omega)$ for all $\|\alpha\| \leq k$, where $k \in \mathbb{N}$ and $p \in [1, \infty)$
$H^k(\mu, \Omega)$	Sobolev space $W^{k,2}(\mu, \Omega)$ for any $k \in \mathbb{N}$
Hölder conjugates	p, q are Hölder conjugates if $p, q \in (0, \infty)$ such that $1/p + 1/q = 1$
$\mathbf{\Gamma}_1$, $\mathbf{\Gamma}_2$	Garré du champ operator and $\mathbf{\Gamma}_2$ operator, defined in (2.3.8) and (2.3.9)

Chapter 1

Introduction

This thesis concerns the study of large interacting systems in statistical mechanics, a branch of mathematical physics that takes its origin from Maxwell, Boltzmann and Gibbs in the 19th century [1]. The central idea is to deduce macroscopic behaviours of a system by combining the microscopic information with a probabilistic approach.

Given the stochastic nature of microscopic systems, one may consider each microscopic state as a random variable X_j, the evolution of the entire system is then described by a stochastic equation of $\boldsymbol{X} = (X_1, X_2, ..., X_N)$. If the equilibrium is reached, the probability distribution of $\boldsymbol{X}$ will be concentrated on the most probable configuration that satisfies the laws of thermodynamics (e.g. the maximum entropy principle) and the macroscopic constraints (e.g. total volume, energy). Such distribution in general takes the form of a Gibbs measure, which is the central object of equilibrium statistical mechanics.

As what follows, one can either have an equilibrium description by Gibbs measures or a non-equilibrium description by stochastic dynamics, it is therefore interesting to develop frameworks that can link both of them. In a finite-dimensional system, we see such a link by constructing a Markov diffusion process for which the Gibbs measure is reversible. In case some coercive inequalities are satisfied (e.g. spectral gap or logarithmic Sobolev

inequalities), one can obtain ergodicity or contractivity of the Markov semigroup. However, delicate issues may arise at the thermodynamic limit since a Markov semigroup or Gibbs measure is not necessarily well-defined at the limit and it is difficult to extend coercive inequalities to infinite dimensions when there exist unbounded multi-particle interactions.

These problems have been tackled in infinite-dimensional analysis since the last century, while many important results were obtained for systems with compact state space and bounded interactions [2, 3], which were generalised to unbounded spaces with perturbed quadratic interactions [4]. This PhD project serves as an exploration along this path, with the aim to investigate coercive inequalities and nonlinear PDEs defined on infinite spin systems, in which the interactions are of higher order growth and arbitrary range. In this thesis, we present the results of the study on these topics and the structure is as follows.

In Chapter 2, we set up the preliminaries and review the literature surrounding the subject. Namely, we will define the configuration space as $\Omega = \mathbf{M}^{\mathbb{Z}^D}$ for some single-spin space $\mathbf{M}$, covered by some finite-range interactions such that the potential energy on any compact $\Lambda \subset \mathbb{Z}^D$ is given by U_Λ. In each finite-volume Λ, we consider the Langevin dynamics

$$dX_\Lambda(t) = -\nabla_\Lambda U_\Lambda^\omega(X_\Lambda)dt + \sqrt{2}dB_\Lambda(t),$$

which is a stochastic gradient flow with the standard Brownian motion B_Λ on $\mathbf{M}$. Let $\mathcal{P}_t^\Lambda$ be the semigroup of the above dynamics in Λ, our aim is to investigate for which potential energy U_Λ the infinite-volume semigroup $\mathcal{P}_t = \lim_{\Lambda\uparrow\mathbb{Z}^D} \mathcal{P}_t^\Lambda$ exists and is ergodic or contractive. To this end, we will construct the reversible measure for $\mathcal{P}_t$, that is, the (infinite-volume) Gibbs measure μ on Ω associated to the potential U_Λ. In case μ satisfies a spectral gap (or log-Sobolev) inequality, one immediately obtains the ergodicity (or hypercontractivity) of the semigroup $\mathcal{P}_t$. Therefore, the key is to find sufficient conditions on U_Λ such that μ satisfies the spectral gap or log-Sobolev inequality.

This topic has been investigated in many works, while various results have been established firstly for compact space $\mathbf{M}$, and then for unbounded $\mathbf{M}$ (e.g. $\mathbb{R}^m$). The main strategy is the use of the Bakry-Emery condition. In case $\mathbf{M}$ is given by a nilpotent Lie group $\mathbb{G}$, the Bakry-Emery condition fails and one needs an alternative approach to obtain the coercive inequalities. This line of work was firstly developed in [5] and then generalised to various situations in [6, 7, 8]. In these results, the potential energy are all defined by a homogeneous norm whose sub-gradient are bounded above and below almost everywhere by some positive constant, in which case one obtains the log-Sobolev inequality. However, when the potential is given by a homogeneous norm with vanishing sub-gradient, the log-Sobolev inequality can no longer hold and one has to consider the spectral gap inequality. This topic is not yet well studied in infinite dimensions due to the dedicate properties of vanishing sub-gradient. In our project, we specifically deal with this problem and provide a detailed classification of potentials, defined by norms with vanishing sub-gradient, for which the spectral gap inequality is satisfied. The results are presented in Chapter 3. Instead of the vanishing sub-gradient, we also propose a systematic approach to construct homogeneous norms with unbounded sub-gradient, and establish log-Sobolev inequalities for examples of such norms. The corresponding results will be shown in Chapter 4.

In Chapter 5, we shall explore the quantum counterpart of the Langevin dynamics and study the Schrödinger equation in infinite dimensions with logarithmic nonlinearity. We shall construct an infinite-volume Gibbs measure through ground state representations and show that it satisfies the log-Sobolev inequality. We then establish several properties of the solution that hold in arbitrary dimensions, including the mass and energy conservation, the gradient and entropy bound, and the finite-speed propagation of information. Using these properties and the log-Sobolev inequality, we can establish the existence of weak solutions in infinite dimensions for potentials with bounded multi-spin interactions.

Chapter 2

Theoretical Background

In this chapter, we set up the theoretical framework for the infinite-dimensional spin system. We start by defining the configuration space with finite-range potentials, by which we construct the finite and infinite-volume Gibbs measures. We then introduce Langevin dynamics on the system with the associated Markov semigroup, and show that the Gibbs measures are reversible with respect to the semigroup. We also consider extending the dynamics to the quantum setting, introducing the logarithmic Schrödinger equation derived from the Langevin dynamics.

To study the Markov semigroup, we introduce the coercive inequalities and explain how they are related to the ergodicity and contractivity of the semigroup. In particular, we are interested in developing the inequalities when the underlying space is given by a homogeneous Carnot group $\mathbb{G}$ and the potential is defined by homogeneous norms on $\mathbb{G}$. To this end, we will properly set up the Carnot group with the non-commutative setting. In addition, we introduce a systematic strategy, called U-bound, to obtain coercive inequalities on the Carnot groups. We then close this chapter by discussing how the results of this thesis enters in the context of the literature.

2.1 Configuration space

We consider the physical space modelled by a D-dimensional integer lattice $\mathbb{Z}^D$, endowed with a norm $\|\cdot\|$[1]. A single vertex $j \in \mathbb{Z}^D$ is called a *site* and we assign each site a state (usually called *spin*) with values in some Polish space[2] $\mathbf{M}$. Denote by $x_j \in \mathbf{M}$ the spin at site j, the resulting configuration space is therefore $\Omega := \mathbf{M}^{\mathbb{Z}^D}$. Intuitively, one may regard Ω as a particle system where each particle is located at a site $j \in \mathbb{Z}^D$, with spin $x_j \in \mathbf{M}$. The entire configuration $x \in \Omega$ can thus be treated as an infinite-dimensional vector $x = (x_j)_{j\in\mathbb{Z}^D}$ with index set $\mathbb{Z}^D$ and underlying space $\mathbf{M}$. Similarly for any sub-lattice $\Lambda \subset \mathbb{Z}^D$, denote by $\Omega_\Lambda := \mathbf{M}^\Lambda$ the sub-system and $x_\Lambda = (x_j)_{j\in\Lambda}$ the spins inside Λ. We now define the notion of measurability on Ω.

2.1.1 Measurable properties.

The σ-algebra $\mathcal{F}$ on the space Ω is the smallest σ-algebra generated by cylinder sets $\mathrm{Cyl}(\omega_\Lambda) := \{x \in \Omega : x_\Lambda = \omega_\Lambda\}$ for all $\omega_\Lambda \in \Omega_\Lambda$ and all compact $\Lambda \subset \mathbb{Z}^D$, i.e.

$$\mathcal{F} = \sigma\left(\left\{\mathrm{Cyl}(\omega_\Lambda) \;\middle|\; \omega_\Lambda \in \Omega_\Lambda,\, \Lambda \Subset \mathbb{Z}^D\right\}\right), \tag{2.1.1}$$

where $\Lambda \Subset \mathbb{Z}^D$ means Λ is a compact subset of $\mathbb{Z}^D$. For any function $f : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}')$, we say f is *$\mathcal{F}$-measurable* if $f^{-1}(A) \in \mathcal{F}$ for any $A \in \mathcal{F}'$. Similarly for any $K \subset \mathbb{Z}^D$, the σ-algebra $\mathcal{F}_K$ on Ω_K is given by (2.1.1) with $\mathbb{Z}^D$ replaced by K. Thus, if f is $\mathcal{F}_K$-measurable it is also $\mathcal{F}$-measurable and only depends on the spins in Ω_K. We shall say f is *localised*

[1] Unless stated otherwise, $\|\cdot\|$ will denote the norm on $\mathbb{Z}^D$ throughout this thesis. In chapter 3 we consider $\|\cdot\| = \|\cdot\|_{l^1(\mathbb{Z}^D)}$ and in Chapter 5 we consider $\|\cdot\| = \|\cdot\|_{l^\infty(\mathbb{Z}^D)}$.

[2] A Polish space is a separable completely metrizable topological space, in this thesis we mainly consider the case where $\mathbf{M}$ is $\mathbb{R}^m$ or a homogeneous Carnot group $\mathbb{G}$.

in $K \subset \mathbb{Z}^D$ if f is $\mathcal{F}_K$-measurable, and we call f a *local function* if f is localised in some compact $K \Subset \mathbb{Z}^D$ with Λ_f denoting the smallest such K. With local functions we can now define the potential energy as follows.

2.1.2 Finite-range potential.

The potential on Ω is given by a family of functions $\mathcal{J} \coloneqq \{J_K\}_{K \Subset \mathbb{Z}^D}$ such that for each $K \Subset \mathbb{Z}^D$ the potential $J_K : \Omega \to \mathbb{R}$ is $\mathcal{F}_K$-measurable. For any finite volume $\Lambda \Subset \mathbb{Z}^D$, the potential energy inside Λ is given by

$$U_\Lambda(x) = \sum_{\Lambda \cap K \neq \varnothing} J_K(x). \tag{2.1.2}$$

We shall call $\mathcal{J}$ a potential of finite range R if $J_K \equiv 0$ for all K such that $\operatorname{diam}(K) > R$, where $\operatorname{diam}(K) \coloneqq \max_{j,k \in K} \|j - k\|$ is the set diameter of K. In the following we give a few examples of such finite-range potentials.

Example 2.1.1. Below we present some well-known lattice models in statistical mechanics. One can find rich literature on this topic, for example, see [9] and the references therein.

(1) **The Ising model ([10]).** Consider the system $\Omega = \{-1, 1\}^{\mathbb{Z}^D}$ with the potential

$$J_K = \begin{cases} \alpha_j x_j & \text{if } K = \{j\} \\ \beta_{jk} x_j x_k & \text{if } K = \{j, k\} \\ 0 & \text{otherwise} \end{cases}$$

where α_j, β_{jk} are real constants such that $\beta_{jk} = 0$ if $j \not\sim k$ (i.e. j, k are not neighbours in $\mathbb{Z}^D$). By (2.1.2), the potential energy in finite volume $\Lambda \Subset \mathbb{Z}^D$ is given by

$$U_\Lambda(x) = -\sum_{j \in \Lambda} \alpha_j x_j - \sum_{\{j,k\} \cap \Lambda \neq \varnothing} \beta_{jk} x_j x_k.$$

This model was initially introduced to describe magnets, assuming the particles are arranged on a lattice structure $\mathbb{Z}^D$ with magnetic moments pointing either north $(+1)$ or south (-1). α_j is the strength of an external magnetic field and β_{jk} is the interaction strength between neighbours, such that the magnet is ferromagnetic if $\beta_{jk} > 0$ and antiferromagnetic if $\beta_{jk} < 0$. The Ising model is widely adapted in both natural and social sciences (e.g. a voting system), while many results concerning thermodynamic limits and phase transitions were obtained (see [9, Chapter 3]).

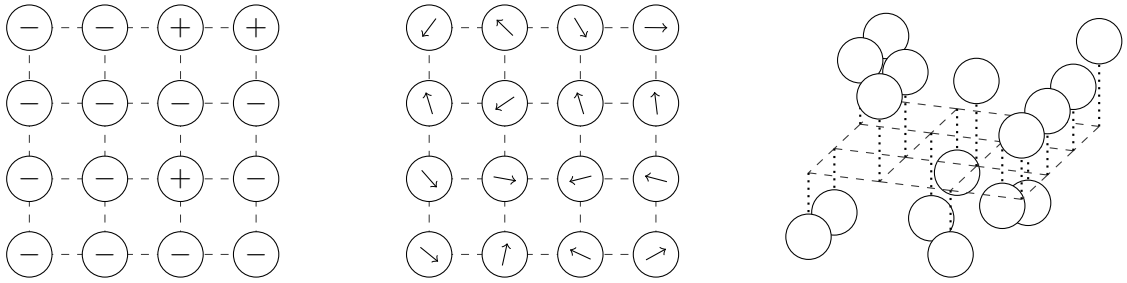
(2) **The classical Heisenberg model ([11]).** Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$. Consider $\Omega = (\mathbb{S}^2)^{\mathbb{Z}^D}$ and the potential

$$J_K = \begin{cases} \langle \alpha_j, x_j \rangle & \text{if } K = \{j\} \\ \beta_{jk} \langle x_j, x_k \rangle & \text{if } K = \{j, k\} \\ 0 & \text{otherwise} \end{cases}$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product, $\alpha_j \in \mathbb{R}^3$ is an external field and $\beta_{jk} \in \mathbb{R}$ again satisfies $\beta_{jk} = 0$ if $j \not\sim k$. The potential energy in any $\Lambda \Subset \mathbb{Z}^D$ is given by

$$U_\Lambda(x) = -\sum_{j \in \Lambda} \langle \alpha_j, x_j \rangle - \sum_{\{j,k\} \cap \Lambda \neq \varnothing} \beta_{jk} \langle x_j, x_k \rangle,$$

This is a generalisation of the Ising model to continuous state spaces, where the magnetic spins can take arbitrary directions on the unit sphere. In general, one can consider any smooth compact finite dimensional Riemannian manifold as the single-site state space.



(a) Ising model on $\mathbb{Z}^2$ (b) Heisenberg model on $\mathbb{Z}^2$ (c) Gaussian model on $\mathbb{Z}^2$

Figure 2.1: Lattice models in Example 2.1.1

(3) **The Gaussian model ([12]).** Consider the space $\Omega = \mathbb{R}^{\mathbb{Z}^D}$ and the potential

$$J_K = \begin{cases} \alpha_j x_j^2 & \text{if } K = \{j\} \\ \beta_{jk}(x_j - x_k)^2 & \text{if } K = \{j,k\} \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha_j, \beta_{jk} \geq 0$ and $\beta_{jk} = 0$ if $j \not\sim k$. The potential energy in $\Lambda \Subset \mathbb{Z}^D$ reads

$$U_\Lambda(x) = \sum_{j\in\Lambda} \alpha_i x_j^2 + \sum_{\substack{\{j,k\}\cap\Lambda\neq\varnothing \\ j\sim k}} \beta_{jk}(x_j - x_k)^2$$

This is an adaptation of the Ising model to non-compact spaces and it lies in the study of Gaussian random fields. The thermodynamic limit for the massive case ($\alpha_j > 0$) and massless case ($\alpha_j = 0$) are discussed in [9, Chapter 8].

Inspired by the above examples, we shall hereafter only consider potential energy in the following form

$$U_\Lambda(x) = \sum_{j\in\Lambda} \alpha_j \phi(x_j) + \sum_{\{j,k\}\cap\Lambda\neq\varnothing} \beta_{jk} V(x_j, x_k) \tag{2.1.3}$$

for some external field $\phi \in C^2(\mathbf{M}, \mathbb{R})$ and pairwise interaction $V \in C^2(\mathbf{M} \times \mathbf{M}, \mathbb{R})$, with

interaction strength $\beta_{ij} \in \mathbb{R}$ such that $\beta_{ij} = 0$ if $\|j - k\| > R$ for some finite range $R > 0$. This is loosely justified since many interacting systems only involve pairwise interactions of the same type. Moreover, it is a much cleaner setup as we only need to deal with two functions instead of the whole family $\mathcal{J}$ appeared in definition (2.1.2).

For a potential energy given by (2.1.3), we notice that U_Λ may have dependence outside Λ due to the interaction V. It will turn out to be useful to interpret such external dependence as boundary conditions fixed by some constant $\omega \in \Omega$, in the sense that

$$U_\Lambda^\omega(x) \coloneqq U_\Lambda(x_\Lambda \bullet \omega_{\Lambda^\mathsf{c}}) \tag{2.1.4}$$

where $\Lambda^\mathsf{c} \coloneqq \mathbb{Z}^D \setminus \Lambda$ is the complement of Λ, and

$$(x_\Lambda \bullet \omega_{\Lambda^\mathsf{c}})_j \coloneqq \begin{cases} x_j, & \text{for } j \in \Lambda \\ \omega_j, & \text{for } j \in \Lambda^\mathsf{c}. \end{cases}$$

With this notation, we are treating the spins inside Λ as variables x_Λ, while the spins outside Λ are fixed by boundary conditions $x_{\Lambda^\mathsf{c}} = \omega_{\Lambda^\mathsf{c}}$. In the next subsection, we will use this boundary condition setup to define a family of local (or finite-volume) Gibbs measures and the associated global (or infinite-volume) Gibbs measure.

2.1.3 Local and global Gibbs measures

Suppose the macroscopic system Ω is at equilibrium with a constant temperature, we consider local Gibbs measure as the probability distribution of the spins in finite volumes, and global Gibbs measure as the associated infinite-volume distribution. We shall hereinafter restrict to the continuous unbounded case $\Omega = (\mathbb{R}^m)^{\mathbb{Z}^D}$ to take advantage of the simplicity of Lebesgue measure, but the setup can be easily generalised to discrete spaces or other smooth Riemannian manifolds.

Local Gibbs measures.

For finite volume $\Lambda \Subset \mathbb{Z}^D$, the local Gibbs measure in Λ is given by the following definition.

Definition 2.1.2 (Local Gibbs Measure)**.** Let $\Omega_\Lambda = (\mathbb{R}^m)^\Lambda$ be the subsystem in any finite volume $\Lambda \Subset \mathbb{Z}^D$, with boundary conditions $\omega \in \Omega$, and the potential energy U^ω_Λ defined in (2.1.4). Let $\mathbb{E}^\omega_\Lambda$ be a probability measure on Ω_Λ with the density

$$d\mathbb{E}^\omega_\Lambda = \delta_{\omega_{\Lambda^c}} \otimes \frac{1}{Z^\omega_\Lambda} e^{-U^\omega_\Lambda} dx_\Lambda,$$

where $\delta_{\omega_{\Lambda^c}}$ denotes the point measure (Direc measure) on Ω_{Λ^c} concentrated at ω_{Λ^c}, and

$$Z^\omega_\Lambda := \int_{\Omega_\Lambda} e^{-U^\omega_\Lambda(x_\Lambda)} dx_\Lambda$$

is the normalisation constant with Lebesgue measure dx_Λ on $(\mathbb{R}^m)^\Lambda$. The measure $\mathbb{E}^\omega_\Lambda$ restricted to the sigma algebra of Lebesgue measurable subsets of Ω_Λ is called a *local Gibbs measure*[3] (in the finite volume Λ, with external boundary conditions $\omega \in \Omega$).

To see why the above definition provides an interesting and reasonable probability measure in finite volumes, one may refer to the arguments in [9, Section 1.2.2]. We also note that in case U_Λ is given by (2.1.3), one needs to choose ϕ and V such that $Z^\omega_\Lambda < \infty$ for all $\omega \in \Omega$, since otherwise $\mathbb{E}^\omega_\Lambda$ may not be well-defined.

With the probability measure $\mathbb{E}^\omega_\Lambda$, we can now define the conditional expectation of random variables on Ω. That is, for any function $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$, the expectation of f in

[3] Also called finite-volume or finite-dimensional Gibbs measure in this thesis.

finite volume Λ (with boundary conditions $\omega \in \Omega$) is given by

$$\mathbb{E}^{\omega}_{\Lambda}(f) \coloneqq \int_{\Omega_\Lambda} f d\mathbb{E}^{\omega}_{\Lambda} = \frac{1}{Z^{\omega}_{\Lambda}} \int_{\Omega_\Lambda} f(x_\Lambda \bullet \omega_{\Lambda^c}) e^{-U^{\omega}_{\Lambda}(x_\Lambda)} dx_\Lambda. \tag{2.1.5}$$

With this notation $\mathbb{E}^{\omega}_{\Lambda}$ can be regarded as a linear operator acting on measurable functions, in the sense that $\mathbb{E}_\Lambda f(\omega) = \mathbb{E}^{\omega}_{\Lambda}(f)$. This way we can drop ω in the notation when its role is insignificant and only focus on the properties of $\mathbb{E}_\Lambda$. The family of operators $\mathcal{E} = \{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ shall be called a *local specification*, since it specifies the distribution in all local system Ω_Λ. It is easy to verify the following result for $\mathbb{E}_\Lambda$.

Proposition 2.1.3. *The local Gibbs measure $\mathbb{E}_\Lambda$ defined in (2.1.5) satisfies the following properties:*

(S1) *Locality: $\mathbb{E}^{\omega}_{\Lambda}(f) = f(\omega)$ for all $\omega \in \Omega$, $\Lambda \Subset \mathbb{Z}^D$ and f that is $\mathcal{F}_{\Lambda^c}$-measurable.*

(S2) *Compatibility: $\mathbb{E}^{\omega}_{\Lambda_1}\big(\mathbb{E}^{\cdot}_{\Lambda_2}(f)\big) = \mathbb{E}^{\omega}_{\Lambda_2}\big(\mathbb{E}^{\cdot}_{\Lambda_1}(f)\big) = \mathbb{E}^{\omega}_{\Lambda_1}(f)$ for all $\omega \in \Omega$, $\Lambda_2 \subset \Lambda_1 \Subset \mathbb{Z}^D$ and measurable f.*

Here we present a simple example of the local Gibbs measures.

Example 2.1.4. Let $\Omega = \mathbb{R}^{\mathbb{Z}}$. For any $\Lambda \Subset \mathbb{Z}$ and boundary conditions $\omega \in \mathbb{R}^{\mathbb{Z}}$, consider the Gaussian model in $\mathbb{R}^\Lambda$ (see Example 2.1.1.(3)) with the following potential energy

$$U^{\omega}_{\Lambda}(x) = \sum_{i,j \in \Lambda} C_{ij} x_i x_j + \sum_{i \in \Lambda, j \notin \Lambda} C_{ij} x_i \omega_j, \tag{2.1.6}$$

where $C_{ij} \in \mathbb{R}$ are such that $C_{ii} > 0$ and $C_{ij} = 0$ if $\|i - j\| > 1$. The corresponding local Gibbs measure is therefore a Gaussian measure

$$\mathbb{E}^{\omega}_{\Lambda}(f) = \frac{1}{Z^{\omega}_{\Lambda}} \int f e^{-\left(\sum_{i,j \in \Lambda} C_{ij} x_i x_j + \sum_{i \in \Lambda, j \notin \Lambda} C_{ij} x_i \omega_j\right)} dx_\Lambda. \tag{2.1.7}$$

Note that for the integral to converge, we may need some conditions on f and C_{ij} (e.g. f is continuous and bounded while C_{ij} is symmetric and positive definite).

Infinite-volume Gibbs measure.

We now find the probability distribution ν of spins on the entire Ω, which should obey the microscopic behaviours described by the local specification $\mathcal{E}$. The main question is in what sense should ν be consistent with $\mathcal{E}$? The answer was provided by Dobrushin, Lanford and Ruelle [13, 14] who introduced the following DLR consistency.

Definition 2.1.5 (Global Gibbs measure and DLR equation)**.** Let $\mathcal{E} = \{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ be a local specification on $(\Omega, \mathcal{F})$. A probability measure ν on Ω is called a global Gibbs measure[4] for $\mathcal{E}$ if ν satisfies

$$\nu\mathbb{E}_\Lambda = \nu \qquad \forall\, \Lambda \Subset \mathbb{Z}^D, \tag{2.1.8}$$

where $\nu\mathbb{E}_\Lambda$ is the probability measure given by $\nu\mathbb{E}_\Lambda(f) \coloneqq \nu\big(\mathbb{E}_\Lambda(f)\big)$ for all measurable f. Condition (2.1.8) is called the DLR equation and a global Gibbs measure ν is also called a *DLR measure*.

Remark 2.1.6. To avoid confusion, we shall hereafter use ν to denote the global Gibbs measure, while using μ for the measure in a general probability space.

To see why the DLR equation defines a reasonable consistency, let $A_{\omega,\Lambda} \coloneqq \{x \in \Omega \mid x_{\Lambda^c} = \omega_{\Lambda^c}\}$ be a cylinder set for any $\Lambda \Subset \mathbb{Z}^D$, then (2.1.8) is equivalent to

$$\nu(f \mid A_{\omega,\Lambda}) \coloneqq \frac{\nu(\mathcal{I}_{A_{\omega,\Lambda}} f)}{\nu(\mathcal{I}_{A_{\omega,\Lambda}})} = \mathbb{E}_\Lambda(f, \omega)$$

[4] Also called infinite-volume or infinite-dimensional Gibbs measure in this thesis.

where $\mathcal{I}_A(a) = 1$ if $a \in A$ and 0 otherwise. This shows that the macroscopic expectation ν coincides with the microscopic expectation $\mathbb{E}_\Lambda$ once the boundary condition $x_{\Lambda^c} = \omega_{\Lambda^c}$ is fixed. Note that this property holds true for all finite system Ω_Λ and therefore a macro-microscopic consistency is achieved.

Let $\mathcal{G}(\mathcal{E})$ be the set of global Gibbs measures for $\mathcal{E} = \{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$. For any $\mu_1, \mu_2 \in \mathcal{G}(\mathcal{E})$, it follows from (2.1.8) that $t\mu_1 + (1-t)\mu_2 \in \mathcal{G}(\mathcal{E})$ for any $t \in [0,1]$, i.e. $\mathcal{G}(\mathcal{E})$ is convex. We therefore have $|\mathcal{G}(\mathcal{E})| = 0, 1$ or ∞, where $|\mathcal{G}(\mathcal{E})|$ is number of elements in $\mathcal{G}(\mathcal{E})$. The case $|\mathcal{G}(\mathcal{E})| = \infty$ corresponds to phase transitions and one may refer to [15, 16, 17] for more details. Our focus is on the case $|\mathcal{G}(\mathcal{E})| = 1$ and in Chapter 3 we will formulate sufficient conditions for this to hold.

Remark 2.1.7. Compared to the DLR consistency one may be more familiar with Kolmogorov's consistency [18, Section 15.6], which is the standard approach to construct probability measures on infinite product spaces. The issue of Kolmogorov's consistency in our setup is that it always produces a unique global measure, making it impossible to model phase transitions. A detailed explanation on this matter can be found in [9, page 251].

So far we have seen how the system at equilibrium is described by Gibbs measures, in the next section we will introduce Langevin dynamics on the lattice and study how the system evolves when out of equilibrium.

2.2 Langevin dynamics

In this section, we assume the time evolution during non-equilibrium can be modelled as a Markov process with Langevin dynamics. We will first restrict the dynamics inside a finite volume $\Lambda \Subset \mathbb{Z}^D$ and fix the external spins by some boundary conditions $\omega \in \Omega$. Later we send $\Lambda \to \mathbb{Z}^D$ and investigate the infinite-volume dynamics.

We shall again focus on the space $\Omega = (\mathbb{R}^m)^{\mathbb{Z}^D}$ and clarify the following notations. For any $j \in \mathbb{Z}^D$ and $\Lambda \subset \mathbb{Z}^D$, denote by $\nabla_j \coloneqq \nabla_{x_j}$ the $\mathbb{R}^m$-gradient at site j and $\nabla_\Lambda \coloneqq (\nabla_j)_{j\in\Lambda}$ the vector composed by single-site gradients in Λ. A dot product is given by

$$\nabla_\Lambda f \cdot \nabla_\Lambda g \coloneqq \sum_{j\in\Lambda} \langle \nabla_j f, \nabla_j g \rangle_{\mathbb{R}^m}$$

with $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$ denoting the Euclidean inner product, and therefore the length of gradient is

$$|\nabla_\Lambda f| \coloneqq (\sum_{j\in\Lambda} \|\nabla_j f\|_{l^2}^2)^{\frac{1}{2}}.$$

Similarly define the Laplacian $\Delta_j \coloneqq \Delta_{x_j}$ and $\Delta_\Lambda \coloneqq \sum_{j\in\Lambda} \Delta_j$.

Denote by $X_j(t)$ the random variable representing the spin at site j and time t. Fix any finite volume $\Lambda \Subset \mathbb{Z}^D$ and boundary conditions $\omega \in \Omega$, denote by $X_\Lambda(t) = (X_j(t))_{j\in\Lambda}$ the vector of spins in Λ and let U_Λ^ω be the potential defined in (2.1.3) and (2.1.4). We consider the following stochastic Langevin dynamics

$$dX_\Lambda(t) = -\nabla_\Lambda U_\Lambda^\omega(X_\Lambda)dt + \sqrt{2}dB_\Lambda(t), \tag{2.2.1}$$

where $B_\Lambda(t) \coloneqq (B_j(t))_{j\in\Lambda}$ with $B_j(t)$ being the $\mathbb{R}^m$-standard Brownian motion at site j. Intuitively, the equation describes a system in which the spins are subject to a drag force $-\nabla_\Lambda U_\Lambda$ and a Gaussian noise B_Λ. In the following, we will list some basic properties of this dynamics and discuss its connections with Schrödinger equation in the quantum setting.

Example 2.2.1. Consider the Gaussian model defined in Example 2.1.4. The Langevin dynamics for this model is simply

$$dX_\Lambda(t) = -\Big(C_{ii}X_i(t) + \textstyle\sum_{j:|i-j|=1} C_{ij}\omega_j\Big)_{i\in\Lambda} dt + \sqrt{2}dB_\Lambda(t) \tag{2.2.2}$$

where $\omega = (\omega_j)_{j\in\Omega}$ are the fixed boundary conditions.

2.2.1 Markov semigroups and generators.

Let $f : \Omega_\Lambda \to \mathbb{R}$ be an observable of the system described by (2.2.1). To study the stochastic evolution of $f(X_\Lambda(t))$ it is sometimes easier to focus on the expectation $\mathbf{E}(f(X_\Lambda(t)))$, which is deterministic as long as an initial condition $X_\Lambda(0) = x_\Lambda$ is fixed. More formally, we need to deduce the evolution of the quantity $u_\Lambda(x_\Lambda, t) \coloneqq \mathbf{E}(f(X_\Lambda(t)) \mid X_\Lambda(0) = x_\Lambda)$. Using Itô's lemma, it can be shown that if X_Λ solves (2.2.1) then u_Λ is a solution of the following Cauchy problem

$$\begin{cases} \partial_t u_\Lambda = \mathcal{L}_\Lambda^\omega u_\Lambda \\ u_\Lambda(x_\Lambda, t = 0) = f(x_\Lambda \bullet \omega_{\Lambda^c}) \end{cases} \tag{2.2.3}$$

with the diffusion operator $\mathcal{L}_\Lambda^\omega \coloneqq \Delta_\Lambda - \nabla_\Lambda U_\Lambda^\omega \cdot \nabla_\Lambda$.

If there exists a unique solution to (2.2.3) we can write it informally as $u_\Lambda = e^{t\mathcal{L}_\Lambda^\omega} f$, where the linear operator $\mathcal{P}_t^{\Lambda,\omega} \coloneqq e^{t\mathcal{L}_\Lambda^\omega}$ satisfies a semigroup property $\mathcal{P}_{t+s} = \mathcal{P}_t \circ \mathcal{P}_s$. Motivated by this property, we can define the notion of Markov semigroups in the following sense [19].

Definition 2.2.2 (Markov semigroup and generator)**.** For any Banach space $(\mathfrak{B}, \|\cdot\|_\mathfrak{B})$ of real-valued functions, let $L(\mathfrak{B})$ be the space of linear operators on $\mathfrak{B}$. A Markov semigroup on $\mathfrak{B}$ is a mapping $\mathcal{P}_t : [0, \infty) \to L(\mathfrak{B})$ such that

(i) $\mathcal{P}_0 = I$, the identity operator on $\mathfrak{B}$;

(ii) $\mathcal{P}_{t+s} = \mathcal{P}_t \circ \mathcal{P}_s$ for all $s, t \geq 0$;

(iii) $\|\mathcal{P}_t f - f\|_\mathfrak{B} \to 0$ as $t \downarrow 0$ for all $f \in \mathfrak{B}$;

(iv) If $\mathbb{1} \in \mathfrak{B}$ then $\mathcal{P}_t \mathbb{1} = \mathbb{1}$ for all $t \geq 0$, where $\mathbb{1}$ is the function constantly equal to 1;

(v) $\mathcal{P}_t f \geq 0$ for all $t \geq 0$ and non-negative $f \in \mathfrak{B}$.

The infinitesimal generator $\mathcal{L}$ of a Markov semigroup $\mathcal{P}_t$ is defined by the limit

$$\mathcal{L}f = \lim_{t \downarrow 0} \frac{1}{t}(\mathcal{P}_t - I)f$$

and it shall be called a Markov generator. The domain $\mathrm{Dom}(\mathcal{L})$ of $\mathcal{L}$ contains all $f \in \mathfrak{B}$ such that the above limit is well-defined.

Remark 2.2.3. Property (iv) and (v) in Definition 2.2.2 imply that a Markov semigroup is contractive, i.e. $\|\mathcal{P}_t f\|_{\mathfrak{B}} \leq \|f\|_{\mathfrak{B}}$ for all $t \geq 0$ and $f \in \mathfrak{B}$, which then implies, with property (ii) and (iii), that $\mathcal{P}_t$ is continuous at all time t.

Example 2.2.4. For the same model as in Example 2.1.4, the infinitesimal generator in the finite volume Λ is given by $\mathcal{L}_\Lambda^\omega := \Delta_\Lambda - \nabla_\Lambda U_\Lambda^\omega \cdot \nabla_\Lambda$. The Markov semigroup in Λ is therefore $\mathcal{P}_t^{\Lambda,\omega} = e^{t\mathcal{L}_\Lambda^\omega}$.

In the following we introduce a necessary and sufficient set of conditions for a linear operator $\mathcal{L}$ to be a valid Markov generator.

Theorem 2.2.5 (Hille-Yoshida theorem [20])**.** *A linear operator $\mathcal{L}$ is the generator of a Markov semigroup $\mathcal{P}_t$ on $\mathfrak{B}$ if and only if the following hold:*

(1) $\mathrm{Dom}(\mathcal{L})$ *is dense in* $\mathfrak{B}$.

(2) $\mathbb{1} \in \mathrm{Dom}(\mathcal{L})$ *and* $\mathcal{L}\mathbb{1} = 0$.

(3) $\mathcal{L}$ *is closed in the sense that* $\lim_{n\to\infty} \mathcal{L}f_n = \mathcal{L}f$ *for any* $f_n \to f$ *in* $(\|\cdot\|_{\mathfrak{B}}, \mathrm{Dom}(\mathcal{L}))$.

(4) *For all* $\lambda > 0$, λ *belongs to the resolvent set of* $\mathcal{L}$ *and* $(\lambda I - \mathcal{L})^{-1}$ *is bounded, positivity preserving, and satisfies*

$$\sup_{\|f\| \leq 1} \|(\lambda I - \mathcal{L})^{-1} f\|_{\mathfrak{B}} \leq \frac{1}{\lambda}.$$

2.2.2 Semigroup in infinite dimensions.

Now that we have introduced Markov semigroups $\mathcal{P}_t^\Lambda$ in finite volume Λ, the next step is to send $\Lambda \to \mathbb{Z}^D$ to obtain the infinite-volume semigroup. However, as pointed out in the introduction, the limit $\lim_{\Lambda\to\mathbb{Z}^D} \mathcal{P}_t^\Lambda$ may not be well-defined for an arbitrary potential U_Λ and configuration space Ω, it is essential to find sufficient conditions for the limit to exist. In Guionnet and Zegarliński's work [19, Theorem 8.2], the following result was developed in response to this problem.

Theorem 2.2.6. *Let* $\mathbf{M}$ *be a smooth compact connected Riemannian manifold,* $\Omega = \mathbf{M}^{\mathbb{Z}^D}$ *be the configuration space, and* $\mathfrak{C}(\Omega)$ *be the space of real-valued bounded continuous functions on* Ω*. Let* U_Λ *be a finite-range potential as defined in* (2.1.2) *and* $\mathcal{P}_t^{\Lambda,\omega} = e^{t(\Delta_\Lambda - \nabla_\Lambda U_\Lambda^\omega \cdot \nabla_\Lambda)}$ *be the corresponding semigroup. Then for any local functions* $f \in \mathfrak{C}(\Omega)$ *and boundary conditions* $\omega \in \Omega$*, the following limit*

$$\mathcal{P}_t f \coloneqq \lim_{\Lambda\to\mathbb{Z}^D} \mathcal{P}_t^{\Lambda,\omega} f$$

exists in the uniform norm $\|\cdot\|_\infty \coloneqq \|\cdot\|_{\mathbb{L}^\infty(\Omega)}$ *and defines a Markov semigroup on* $\mathfrak{C}(\Omega)$*. Furthermore, for any* $\varepsilon > 0$ *there exists* $\delta \geq 1$ *such that for any finite volume* Λ *and time* t *satisfying* $\operatorname{dist}(\Lambda_f, \Lambda^{\mathsf{c}}) \geq \delta t$ *the following error bound holds*

$$\left\|\mathcal{P}_t f - \mathcal{P}_t^{\Lambda,\omega} f\right\|_\infty \leq e^{-\varepsilon t} \sum_{i\in\mathbb{Z}^D} \|\nabla_i f\|_\infty.$$

To prove the theorem, the strategy is to consider two volumes $\Lambda_1 \subset \Lambda_2$ and deduce the interpolation bound $\|\mathcal{P}_t^{\Lambda_2} f - \mathcal{P}_t^{\Lambda_1} f\|_\infty \leq c\int_0^t ds \sum_{i\in\Lambda_2\setminus\Lambda_1} \|\nabla_i \mathcal{P}_s^{\Lambda_2} f\|_\infty$ as well as the gradient bound $\|\nabla_i \mathcal{P}_s^{\Lambda_2} f\|_\infty \leq c\int_0^t \sum_j C_{ij} \|\nabla_j \mathcal{P}_t^{\Lambda_2} f\|_\infty$ for some $c > 0$ independent of f, where $C_{ij} = 0$ if $\|i - j\|$ exceeds the finite range of potential. Then we use the gradient

bound inductively to get $\|\nabla_i \mathcal{P}_s^{\Lambda_2} f\|_\infty \leq e^{-\delta t} \sum_i \|\nabla_i f\|_\infty$ where δ increases linearly with the gap between Λ_1 and Λ_2. Plugging this δ-bound into the first interpolation bound, we can show that the sequence $\{\mathcal{P}_t^\Lambda f\}_{\Lambda \in \mathbb{Z}^D}$ is Cauchy in $\mathbb{L}^\infty(\Omega)$ and therefore convergent.

Remark 2.2.7. The above theorem was shown in [19] for compact state spaces, while in our setup the space is unbounded (e.g. $\Omega = (\mathbb{R}^m)^{\mathbb{Z}^D}$ or $\mathbb{G}^{\mathbb{Z}^D}$ for a homogeneous Carnot group $\mathbb{G}$). With the same interpolating strategy, similar results can be obtained for unbounded spaces. The reader may refer to [19] for more discussions on this matter.

2.2.3 Invariant and reversible measure.

Given a semigroup $\mathcal{P}_t$, it is interesting to investigate which distribution is preserved or reversible under the evolution of $\mathcal{P}_t$ in the sense of the following definition.

Definition 2.2.8. Let Ω be a Polish space and $\mathfrak{B}$ be a Banach space of real-valued measurable functions on Ω. Let $\mathcal{P}_t$ be a Markov semigroup on $\mathfrak{B}$ with generator $\mathcal{L}$. A probability measure ν on Ω is *invariant* with respect to $\mathcal{P}_t$ (or $\mathcal{L}$) if

$$\nu(\mathcal{P}_t f) = \nu(f) \text{ for } f \in \mathfrak{B},\, t \geq 0 \quad \text{or equivalently} \quad \nu(\mathcal{L} f) = 0 \text{ for } f \in \mathrm{Dom}(\mathcal{L})$$

for all $t \geq 0$, and ν is *reversible* with respect to $\mathcal{P}_t$ (or $\mathcal{L}$) if

$$\nu(f \mathcal{P}_t g) = \nu(g \mathcal{P}_t f) \text{ for } f, g \in \mathfrak{B},\, t \geq 0$$
$$\text{or equivalently} \quad \nu(f \mathcal{L} g) = \nu(g \mathcal{L} f) \text{ for } f, g \in \mathrm{Dom}(\mathcal{L}).$$

Note that reversibility implies invariance by letting $g = \mathbb{1}$.

For the case $\Omega = (\mathbb{R}^m)^{\mathbb{Z}^D}$, it turns out that, under mild conditions, the local Gibbs measure $\mathbb{E}_\Lambda^\omega$ and the global Gibbs measure ν are invariant and reversible with respect to

the semigroup $\mathcal{P}_t^{\Lambda,\omega}$ generated by $\mathcal{L}_\Lambda^\omega = \Delta_\Lambda - \nabla_\Lambda U_\Lambda^\omega \cdot \nabla_\Lambda$. A formal statement is provided in the following lemma.

Lemma 2.2.9. *For any $\Lambda \Subset \mathbb{Z}^D$ and boundary conditions $\omega \in \Omega$, denote by $H^k(\mathbb{E}_\Lambda^\omega, \Omega_\Lambda)$ the (Sobolev) space of functions $f : \Omega_\Lambda \to \mathbb{C}$ such that the weak derivatives of f up to order k are $\mathbb{L}_2(\mathbb{E}_\Lambda^\omega)$-integrable. Then for any $U_\Lambda^\omega \in H^1(\mathbb{E}_\Lambda^\omega, \Omega_\Lambda)$, the Gibbs measures $\mathbb{E}_\Lambda^\omega$ defined in (2.1.5) and ν given in (2.1.8) are invariant and reversible with respect to $\mathcal{L}_\Lambda^\omega$ on $H^2(\mathbb{E}_\Lambda^\omega, \Omega_\Lambda)$, i.e.*

$$\nu(\mathcal{L}_\Lambda^\omega f) = \mathbb{E}_\Lambda^\omega(\mathcal{L}_\Lambda^\omega f) = 0$$

and

$$\nu(f\mathcal{L}_\Lambda^\omega g) = \nu(g\mathcal{L}_\Lambda^\omega f), \quad \mathbb{E}_\Lambda^\omega(f\mathcal{L}_\Lambda^\omega g) = \mathbb{E}_\Lambda^\omega(g\mathcal{L}_\Lambda^\omega f)$$

for all $f, g \in H^2(\mathbb{E}_\Lambda^\omega, \Omega_\Lambda)$.

Proof. Let $j \in \Lambda$ and for invariance we have

$$\mathbb{E}_\Lambda^\omega(\Delta_j f) = \frac{1}{Z_\Lambda^\omega}\int (\Delta_j f) e^{-U_\Lambda^\omega} dx_\Lambda = -\frac{1}{Z_\Lambda^\omega}\int \nabla_j f \cdot \nabla_j\big(e^{-U_\Lambda^\omega}\big) dx_\Lambda = \mathbb{E}_\Lambda^\omega(\nabla_j U_\Lambda^\omega \cdot \nabla_j f)$$

due to integration by parts. Since $U_\Lambda \in H^1$ and $f \in H^2$, all integrals are finite and thus $\mathbb{E}_\Lambda^\omega(\mathcal{L}_\Lambda^\omega f) = 0$. Similarly for reversibility, we have

$$\begin{aligned}\int (f\Delta_j g - g\Delta_j f) e^{-U_\Lambda} dx_\Lambda &= \int \nabla_j (f\nabla_j g - g\nabla_j f) e^{-U_\Lambda} dx_\Lambda \\ &= -\int (f\nabla_j g - g\nabla_j f)\nabla_j\big(e^{-U_\Lambda}\big) dx_\Lambda\end{aligned}$$

which gives $\mathbb{E}_\Lambda^\omega(f\mathcal{L}_\Lambda^\omega g) = \mathbb{E}_\Lambda^\omega(g\mathcal{L}_\Lambda^\omega f)$. The DLR equation leads to the same invariance and reversibility of ν. □

2.2.4 Connection with the Schrödinger equation.

In this section we also consider dynamics in the quantum setting, making it necessary to establish connections between the Langevin dynamics and the Schrödinger equation $i\partial_t\psi = -\Delta\psi + V\psi$. The starting point is to rewrite the wave function into polar form $\psi = \sqrt{\rho}e^{i\xi}$ with the probability density $\rho = |\psi|^2$ and the phase ξ. Substituting this into the Schrödinger equation and splitting the imaginary and real part, it yields

$$\text{Imaginary:} \quad \partial_t\rho = -\nabla\cdot(2\rho\nabla\xi) \tag{2.2.4}$$

$$\text{Real:} \quad \partial_t\xi = -\big(|\nabla\xi|^2 + V + V_B\big) \tag{2.2.5}$$

where $V_B \coloneqq -(\Delta\sqrt{\rho})/\sqrt{\rho}$ is the Bohm quantum potential. This system is known as the Madelung equations [21], an equivalent formulation of the Schrödinger equation, consisting of a continuity equation (2.2.4) and a modified Hamilton-Jacobi equation (2.2.5).

To link with the Langevin dynamics in our setup, we shall use the Fokker-Planck equation. That is, let $X(t)$ be a stochastic process on $\mathbb{R}^m$ that satisfies the equation

$$dX(t) = a(X(t),t)dt + b(X(t),t)dB(t)$$

with a drift vector a in $\mathbb{R}^m$, a diffusion matrix b in $\mathbb{R}^{m\times m}$, and a standard Brownian motion $B(t)$ on $\mathbb{R}^m$. Let $p(x,t)$ be the probability density of $X(t)$, then $p(x,t)$ satisfies the differential equation

$$\partial_t p(x,t) = -\sum_{j=1}^{m}\partial_{x_j}(p(x,t)a_j(x,t)) + \sum_{j=1}^{m}\sum_{k=1}^{m}\partial_{x_j}\partial_{x_k}\big(p(x,t)B_{jk}(x,t)\big)$$

where $B = bb^{\mathsf{T}}$. This is known as the Fokker-Plank equation, introduced in [22, 23].

Applying this equation for the Langevin dynamics (2.2.1), we have

$$\partial_t p_\Lambda = \nabla_\Lambda\big(p_\Lambda(\nabla_\Lambda U_\Lambda)\big) + \Delta_\Lambda p_\Lambda \tag{2.2.6}$$

where $p_\Lambda(x,t)$ is the probability density of the process $X_\Lambda(t)$ that satisfies (2.2.1). Taking $\rho = p_\Lambda$ and $\xi = (U_\Lambda + \log p_\Lambda)/2$, we immediately establish the equivalence between (2.2.4) and (2.2.6). Namely, the Langevin dynamics can be adapted into the quantum setting with probability density p_Λ and phase $(U_\Lambda + \log p_\Lambda)/2$. To utilise the second Madelung equation, take the same substitution into (2.2.5) and we find

$$V = \frac{1}{2}\Delta_\Lambda U_\Lambda - \frac{1}{4}|\nabla_\Lambda U_\Lambda|^2 - V_B, \tag{2.2.7}$$

which connects the lattice potential U_Λ and the quantum potential V. With this relation, we find, after a transformation $\psi = e^{-U_\Lambda/2}\phi$, that the Schrödinger equation takes the form

$$i\partial_t\phi = -(\mathcal{L}_\Lambda + V_B)\phi$$

where $\mathcal{L}_\Lambda = \Delta_\Lambda - \nabla_\Lambda U_\Lambda \cdot \nabla_\Lambda$ is the diffusion operator defined in (2.2.3).

If we discard the Bohm potential V_B (which is the case in classical mechanics), then one has the linear equation $i\partial_t\phi = -\mathcal{L}_\Lambda\phi$ that enjoys several nice properties in the $\mathbb{L}^2(\nu)$ function space, where ν is the global Gibbs measure. In Chapter 5, we will study this equation with a logarithmic nonlinearity, that is, $i\partial_t\phi = -\mathcal{L}_\Lambda\phi + \lambda\phi\log\frac{|\phi|^2}{\nu|\phi|^2}$ for some $\lambda \in \mathbb{R}$. We shall find various estimates that are uniform in the size $|\Lambda|$, and establish the existence of weak solutions to the equation when $\Lambda \uparrow \mathbb{Z}^D$.

2.3 Coercive inequalities and semigroup

After setting up a Markov semigroup $\mathcal{P}_t$ in the previous section, we now study its behavior over time. In particular, we are interested in finding out when $\mathcal{P}_t$ is ergodic or contractive. For this purpose, a well-established strategy is to use coercive inequalities, as once they are satisfied for a semigroup we immediately get the desired ergodicity or contractivity. Below we present three well-known coercive inequalities.

Definition 2.3.1. Let $\mathcal{P}_t$ be a Markov semigroup with generator $\mathcal{L}$, and we denote by $\mathcal{E}_\mu(f,g) = \mu(f(-\mathcal{L})g)$ the Dirichlet form. For any probability measure μ that is reversible with respect to $\mathcal{P}_t$, we say

(1) μ satisfies a *spectral gap inequality* (SGI) if for some $C_{SG} \geq 0$

$$\mu(f-\mu f)^2 \leq C_{SG}\mathcal{E}_\mu(f,f), \tag{2.3.1}$$

(2) μ satisfies a *classical Sobolev inequality* (CSI) if for some $C_{CS}, D_{CS} \geq 0$ and $p > 2$

$$\big(\mu|f|^p\big)^{2/p} \leq C_{CS}\mathcal{E}_\mu(f,f) + D_{CS}\mu f^2, \tag{2.3.2}$$

(3) μ satisfies a *logarithmic Sobolev inequality* (LSI) if for some $C_{LS}, D_{LS} \geq 0$

$$\mu\left(f^2 \log \frac{f^2}{\mu f^2}\right) \leq C_{LS}\mathcal{E}_\mu(f,f) + D_{LS}\mu f^2, \tag{2.3.3}$$

where $f \in \mathbb{L}^2(\mu) \cap \mathrm{Dom}(\mathcal{E}_\mu)$ and all constants are independent of f.

Before discussing how these inequalities are related to semigroups, we first explain why they are regarded as 'coercive'. One reason lies in the definition of coercive bilinear

forms: a real-valued bilinear form A on a normed space $\mathcal{B}$ with norm $\|\cdot\|_{\mathcal{B}}$ is coercive if $\|x\|_{\mathcal{B}}^2 \leq cA(x,x)$ for some $c > 0$, i.e. the norm $\|\cdot\|_{\mathcal{B}}$ is coerced by A. In the above inequalities, the left-hand sides are all (semi-)norms, coerced by the right-hand sides which are all bilinear forms, it is therefore natural to regard the inequalities as being coercive.

Remark 2.3.2. While Definition 2.3.1 provides an abstract formulation for the coercive inequalities by using generators and reversible measures, we remark that for the setup of this thesis the Dirichlet form $\mathcal{E}_\mu(f,f)$ can be re-written into a more explicit form. Namely, substituting the space $\Omega = (\mathbb{R}^m)^{\mathbb{Z}^D}$, the generator $\mathcal{L} = \mathcal{L}_\Lambda = \Delta_\Lambda - \nabla_\Lambda U_\Lambda \cdot \nabla_\Lambda$, and its reversible measure $\mu = \mathbb{E}_\Lambda$ in Definition 2.3.1, one can use integration by parts to show that the Dirichlet form follows $\mathcal{E}_{\mathbb{E}_\Lambda}(f,f) = \mathbb{E}_\Lambda|\nabla_\Lambda f|^2$.

Now we illustrate how the coercive inequalities are applied in the theory of Markov semigroups. For any semigroup $\mathcal{P}_t$ and $p, q > 0$ define the operator norm

$$\|\mathcal{P}_t\|_{p,q} := \sup\left\{\|\mathcal{P}_t f\|_{\mathbb{L}^q(\mu)} : \|f\|_{\mathbb{L}^p(\mu)} = 1\right\}.$$

In the following theorem (a collection of results in [19], [24], [25]), we establish the equivalence between the above inequalities and the ergodicity/contractivity of a semigroup.

Theorem 2.3.3. *For the same setup in Definition 2.3.1, the following statements hold:*

(1) *if ν satisfies SGI with coefficient $C_{SG} \geq 0$, then $\mathcal{P}_t$ is exponentially ergodic in the following sense:*

$$\mu(\mathcal{P}_t f - \mu f)^2 \leq e^{-2t/C_{SG}}\mu(f - \mu f)^2 \qquad \forall\, t > 0, \tag{2.3.4}$$

(2) *if ν satisfies CSI with coefficient $C_{CS}, D_{CS} \geq 0$, then $\mathcal{P}_t$ is ultra-contractive:*

$$\|\mathcal{P}_t\|_{2,\infty} < \infty \qquad \forall\, t > 0 \text{ sufficiently large}, \tag{2.3.5}$$

(3) *if ν satisfies LSI with coefficient $C_{LS} \geq 0, D_{LS} = 0$, then $\mathcal{P}_t$ is hyper-contractive:*

$$\|\mathcal{P}_t\|_{2,p} \leq 1 \qquad \forall\, p > 2 \text{ and } \forall\, t > 0 \text{ sufficiently large}. \tag{2.3.6}$$

As shown in the theorem, one can develop the above inequalities to obtain the corresponding properties of the Markov semigroup, this makes coercive inequalities an important candidate in the study of Markov process and stochastic differential equations.

Remark 2.3.4. We notice that the hypercontractivity in (2.3.6) requires $D_{LS} = 0$ in the LSI, otherwise the right hand side will grow with time (see [19] for more details). We shall call the LSI non-defective if $D_{LS} = 0$ (defective if $D_{LS} > 0$). According to the convention, we shall hereafter refer to a non-defective LSI simply as LSI since many interesting results only hold for non-defective LSIs.

2.3.1 Properties of SGI and LSI.

In this thesis we are interested in developing inequalities on infinite-dimensional space, and for this purpose we shall hereafter only focus on SGI and LSI, as CSI cannot be tensorised to infinite dimensions and thus loses its usefulness (see Remark 2.3.6 below for more details). In the following, we list a set of important properties of SGI and LSI, starting with the tensorisation properties adapted from [19, Theorem 2.5, Theorem 4.4].

Theorem 2.3.5. *For $i = 1, 2$ let μ_i be a reversible probability measure for generator $\mathcal{L}_i$ on (Ω_i, Σ_i), assume that μ_i satisfies an SGI with coefficient C_i. Let $\Omega = \Omega_1 \times \Omega_2$ be the product*

space with the product σ-algebra. For any function f on Ω, denote by $f_1^\omega(\cdot) = f(\cdot, \omega)$ and $f_2^\omega(\cdot) = f(\omega, \cdot)$, we extend the generator $\mathcal{L}_i$ to Ω in the sense that

$$\tilde{\mathcal{L}}_1 f(x_1, x_2) = \mathcal{L}_1 f_1^{x_2}(x_1), \quad \tilde{\mathcal{L}}_2 f(x_1, x_2) = \mathcal{L}_2 f_2^{x_1}(x_2).$$

Then $\mathcal{L} = \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2$ is a well-defined generator on Ω such that the product measure $\mu_1 \otimes \mu_2$ is reversible for $\mathcal{L}$ and satisfies an SGI with coefficient $C = \max\{C_1, C_2\}$. The same result holds for LSI (but not CSI).

The above property can be shown with a telescoping technique. For example, in the case of SGI, let μ_1, μ_2 be two measures as in Theorem 2.3.5, with $\mu_1\mu_2 \equiv \mu_1 \otimes \mu_2$ we have

$$\begin{aligned}
\mu_1\mu_2(f - \mu_1\mu_2(f))^2 &= \mu_1\mu_2(f^2) - (\mu_1\mu_2(f))^2 \\
&= \Big(\mu_1\mu_2(f^2) - \mu_1\big((\mu_2 f)^2\big)\Big) + \Big(\mu_1\big((\mu_2 f)^2\big) - (\mu_1\mu_2(f))^2\Big) \\
&= \mu_1\mu_2(f - \mu_2 f)^2 + \mu_1(\mu_2 f - \mu_1\mu_2 f)^2.
\end{aligned}$$

Applying SGI for μ_2 in the first term and SGI for μ_1 in the second term, we obtain the SGI for the product measure $\mu_1\mu_2$. It is not hard to adapt the same strategy for LSI. In fact, if an inequality takes the form $\mu\Phi(f) - \Phi(\mu(f)) \le C\mathcal{E}_\mu(f, f)$ for some $\Phi : \mathbb{R} \to \mathbb{R}$, then it satisfies the tensorisation property due to the same telescoping technique (note that $\Phi(x) = x^2$ gives SGI and $\Phi(x) = x \log x$ gives LSI).

By this theorem, one can extend the SGI and LSI to infinite-dimensional product spaces, i.e. $\Omega = \prod_{n=1}^\infty \Omega_n$ with measure $\mu = \otimes_{n=1}^\infty \mu_n$. However, we note that due to the multi-spin interactions in our setup, the measure on the infinite-dimensional product space is not the product of the measure on each individual space. Therefore, one cannot directly use the tensorisation property to extend SGI/LSI to infinite dimensions. This is one of the main challenges that we shall overcome in this thesis.

Remark 2.3.6. The CSI does not satisfy the tensorisation property. To see this, let μ be a probability measure on $\mathbb{R}$ that satisfies the CSI $\big(\mu|f|^p\big)^{\frac{2}{p}} \le a\mu f^2 + b\mu|\nabla_{\mathbb{R}} f|^2$ for some $p > 2$, $a \ge 0$, $b \ge 0$. Note that $a < 1$ is impossible since $f \equiv 1$ would be a counterexample; moreover, $(a, b) = (1, 0)$ is impossible since $\mu f^2 = \mu(|f|^p{}^{\frac{2}{p}}) < \big(\mu|f|^p\big)^{\frac{2}{p}}$ for non-constant f by Jensen's inequality. This implies that $(a, b) \in [1, \infty) \times [0, \infty) \setminus \{(1, 0)\}$, hence there exist $f : \mathbb{R} \to \mathbb{R}$ and constants $\tilde{a}, \tilde{b}$ such that

$$\big(\mu|f|^p\big)^{\frac{2}{p}} > \tilde{a}\mu f^2 + \tilde{b}\mu|\nabla_{\mathbb{R}} f|^2 \quad \text{where} \quad \begin{cases} \tilde{a} \in (1, a],\ \tilde{b} = 0 \text{ if } a > 1 \\ \tilde{a} = 1,\ \tilde{b} \in (0, b] \text{ if } b > 0. \end{cases} \tag{2.3.7}$$

Now define $g : \mathbb{R}^n \to \mathbb{R}$ by $g(x) = f(x_1)f(x_2)\cdots f(x_n)$, and let μ^n be the product measure on $\mathbb{R}^n$. It then follows

$$\begin{aligned} \big(\mu^n|g|^p\big)^{\frac{2}{p}} = \big(\mu|f|^p\big)^{\frac{2n}{p}} &> \Big(\tilde{a}\mu f^2 + \tilde{b}\mu|\nabla_{\mathbb{R}} f|^2\Big)^n \\ &\ge \tilde{a}^n\big(\mu f^2\big)^n + n\tilde{a}^{n-1}\tilde{b}\mu|\nabla_{\mathbb{R}} f|^2\big(\mu f^2\big)^{n-1} \\ &= \tilde{a}^n\mu^n(g^2) + n\tilde{a}^{n-1}\tilde{b}\mu^n|\nabla_{\mathbb{R}^n} g|^2. \end{aligned}$$

Sending $n \to \infty$, either $\tilde{a}^n$ or $n\tilde{a}^{n-1}\tilde{b}$ blows up due to the conditions on $(\tilde{a}, \tilde{b})$ in (2.3.7). Thus, if μ^n satisfies the CSI $\big(\mu^n|g|^p\big)^{\frac{2}{p}} \le c_1\mu^n(g^2) + c_2\mu^n|\nabla_{\mathbb{R}^n} g|^2$, at least one of c_1 and c_2 blows up on infinite-dimensional product spaces. This shows that CSI cannot be tensorised to infinite dimensions as SGI or LSI.

On top of the tensorisation property, we also have the following relation between the coefficients of the SGI and LSI.

Theorem 2.3.7. *If a probability measure μ satisfies the LSI with coefficient C_{LS}, then μ satisfies the SGI with coefficient C_{SG} such that*

$$C_{SG} \leq \frac{1}{2} C_{LS}.$$

The above property and proof is due to Rothaus [26]. Another idea (due to S. G. Bokov and F. Götze [27]) is to use the fact that the right-hand side of LSI is invariant under a shift by a constant and so that one has

$$\frac{1}{2}\mu|f - \mu f|^2 \leq \sup_{a\in\mathbb{R}} \mu\left(|f|^2 \log \frac{|f|^2}{\mu|f|^2}\right).$$

For an interesting discussion when $C_{SG} = \frac{1}{2}C_{LS}$ is or is not true, one may see [28].

Later we will explore the same setup in the quantum setting where the variables and functions are in the complex domain. To this end, we shall extend the inequalities to complex functions, which is discussed in the following remark.

Remark 2.3.8. Note that LSI for real functions implies similar inequality for complex functions. To see that, let $f_R \coloneqq \mathrm{Re}(f)$, $f_I \coloneqq \mathrm{Im}(f)$ and assume that $\mu|f|^2 = \mu|f_R|^2 + \mu|f_I|^2 = 1$ by the homogeneity of the LSI. Then by the convexity of the entropy function $\mathrm{Ent}(x) \coloneqq x \log x$ we have

$$\begin{aligned}
\mu\mathrm{Ent}(|f|^2) &= \mu\mathrm{Ent}\left(\mu|f_R|^2 \frac{|f_R|^2}{\mu|f_R|^2} + \mu|f_I|^2 \frac{|f_I|^2}{\mu|f_I|^2}\right) \\
&\leq \mu\mathrm{Ent}\left(\frac{|f_R|^2}{\mu|f_R|^2}\right)\mu|f_R|^2 + \mu\mathrm{Ent}\left(\frac{|f_I|^2}{\mu|f_I|^2}\right)\mu|f_I|^2.
\end{aligned}$$

Hence applying LSI with the real functions f_R and f_I, we get desired LSI for complex functions f with the same coefficient C_{LS}.

2.3.2 Bakry-Emery condition and extension to infinite dimensions

To obtain the SGI or LSI, many useful tools have ever since been developed. One of them includes the famous *Bakry and Emery condition* [29], for which we first need to define the so-called *carré du champ operator*

$$\mathbf{\Gamma}_1(f,g) = \frac{1}{2}(\mathcal{L}(fg) - f\mathcal{L}g - g\mathcal{L}f) \tag{2.3.8}$$

and the $\mathbf{\Gamma}_2$ operator

$$\mathbf{\Gamma}_2(f,g) = \frac{1}{2}(\mathcal{L}\mathbf{\Gamma}_1(f,g) - \mathbf{\Gamma}_1(f,\mathcal{L}g) - \mathbf{\Gamma}_1(\mathcal{L}f,g)) \tag{2.3.9}$$

for a diffusion operator[5] $\mathcal{L}$ with a reversible probability measure μ. The condition states that if the following inequality holds

$$\mathbf{\Gamma}_2(f,f) \geq \rho\mathbf{\Gamma}_1(f,f) \tag{2.3.10}$$

for some $\rho > 0$, then μ satisfies the LSI with coefficient $2/\rho$ and thus the SGI with coefficient $1/\rho$ by Theorem 2.3.7.

In this thesis, we shall only consider the situation of a smooth Riemannian manifold $\mathbf{M}$ and the generator $\mathcal{L} = \Delta - \nabla U \cdot \nabla$ for a C^2 potential $U : \mathbf{M} \to \mathbb{R}$, with the gradient ∇ and Laplace-Beltrami operator Δ. The reversible measure is given by $d\mu = e^{-U}dx/Z$ with the Riemannian volume element dx and normalisation constant Z. In this setup, the

[5] An operator $\mathcal{L}$ on a Banach space $\mathcal{B}$ is a diffusion if $\mathcal{L}F(f) = F'\mathcal{L}f + F''(f)\mathbf{\Gamma}_1(f,f)$ for all smooth function f on $\mathcal{B}$ and F on $\mathbb{R}$.

$\mathbf{\Gamma}_1, \mathbf{\Gamma}_2$ operators take the following form

$$\begin{aligned} \mathbf{\Gamma}_1(f,f) &= |\nabla f|^2, \\ \mathbf{\Gamma}_2(f,f) &= \mathrm{Ric}(\nabla f, \nabla f) + |\mathrm{Hess}(f)|^2 + \langle \mathrm{Hess}(U)\nabla f, \nabla f\rangle, \end{aligned} \tag{2.3.11}$$

where $\mathrm{Ric}(\cdot,\cdot)$ is the Ricci curvature and $\mathrm{Hess}(U)$ is the Hessian matrix of U. One may see [29] for a derivation of (2.3.11) by direct calculations with Bochner's formula.

Although $\mathbf{M}$ can be kept as a general Riemannian manifold, we shall only focus on the case $\mathbf{M} = \mathbb{R}^m$ or a Carnot group $\mathbb{G}$ in this thesis, and for this purpose we will not provide a detailed setup of the Riemannian manifold and its related topics but refer the reader to [30] for more details on this matter.

Remark 2.3.9. In the special case where $\mathbf{M} = \mathbb{R}^m$, it is sufficient to have $\mathrm{Hess}(U) \succcurlyeq \rho$, i.e. $\langle \mathrm{Hess}(U)x, x\rangle \geq \rho\langle x, x\rangle$ for all $x \in \mathbb{R}^m$ and some $\rho > 0$ independent of x, where $\mathrm{Hess}(U)$ is the Hessian matrix with entries $[\mathrm{Hess}(U)]_{jk} = \partial_{x_j}\partial_{x_k}U$. By the Bakry-Emery condition, the probability measure $e^{-U}dx/Z$ satisfies an LSI with coefficient $2/\rho$ if $\mathrm{Hess}(U) \succcurlyeq \rho$.

In the study of LSIs for global Gibbs measures, the Bakry-Emery condition provides the possibility for extending LSIs from finite to infinite dimensions. For the example of $\Omega = \mathbb{R}^{\mathbb{Z}^D}$, the idea is to construct a potential U_Λ such that $\mathrm{Hess}_\Lambda(U_\Lambda) \succcurlyeq \rho$ for all $\Lambda \Subset \mathbb{Z}^D$ and $\rho > 0$ independent of Λ, where $\mathrm{Hess}_\Lambda(U_\Lambda)$ is the matrix $A \in \mathbb{R}^{|\Lambda|\times|\Lambda|}$ with entries $A_{jk} = \partial_{x_j}\partial_{x_k}U_\Lambda$ for $j,k \in \Lambda$. Then by the above remark, the finite-dimensional Gibbs measure $d\mathbb{E}_\Lambda = e^{-U_\Lambda}dx_\Lambda/Z_\Lambda$ satisfies LSI with coefficient $2/\rho$ independent of Λ. Using this dimension-uniform LSI, one can deduce several interpolation estimates which lead to the LSI for the infinite-dimensional Gibbs measure ν associated to $\mathbb{E}_\Lambda$. For a rigorous proof of LSI using this idea, see Chapter 5 and in particular Section 5.3. In the following we provide a simple example of such potential.

Example 2.3.10. Consider the Gaussian model introduced in Example 2.1.4, with $\Omega = \mathbb{R}^{\mathbb{Z}}$

and $U_\Lambda = \sum_{j\in\Lambda, k\in\mathbb{Z}} C_{jk}x_j x_k$. If $C_{jj} \geq 3$, $|C_{jk}| \leq 1$ for $j \neq k$ and $C_{jk} = 0$ if $\|j-k\| > 1$, then it is easy to verify $\mathrm{Hess}_\Lambda(U_\Lambda) \succcurlyeq 1$ by using the AM-GM inequality $x_j x_k \geq -(x_j^2 + x_k^2)/2$. This implies the finite-dimensional Gaussian measure $d\mathbb{E}_\Lambda = e^{-U_\Lambda} dx_\Lambda / Z_\Lambda$ satisfies LSI with coefficient 2 and SGI with coefficient 1 for all $\Lambda \Subset \mathbb{Z}$.

Now that we have discussed the case $\mathbf{M} = \mathbb{R}^m$, we shall focus on the other case $\mathbf{M} = \mathbb{G}$ for a homogeneous Carnot group $\mathbb{G}$. Due to the geometry of $\mathbb{G}$, the Ricci curvature is no longer bounded from below and therefore the Bakry-Emery condition in general fails to work [5, 30]. This requires new approaches to develop SGI or LSI in manifolds like $\mathbb{G}$. In the next section, we will set up the Carnot group with the non-commutative setting, and discuss some of these new approaches.

2.4 Carnot groups and non-commutative setting

So far the configurations in our setup are mainly restricted in $\mathbb{R}^m$, which enjoys many nice properties that fail to hold on a general metric space. One example is the space of Nilpotent Lie groups $\mathbb{G}$ where we lose the commutativity and the simple Euclidean gradient structure. Furthermore, since the Ricci curvature of $\mathbb{G}$ is unbounded from below, the Bakry-Emery condition no longer holds and one needs an alternative strategy to establish an LSI [5, 30]. All these challenging yet interesting properties of $\mathbb{G}$ have been motivating researchers to develop more tools for coercive inequalities in such abstract spaces. In this thesis, we shall focus on one particular family of Lie groups, namely the *homogeneous Carnot groups*.

Definition 2.4.1 (Homogeneous Carnot group [31])**.** Let $\mathbb{G} = (\mathbb{R}^m, \circ)$ be a Lie group on $\mathbb{R}^m$, consider the following properties:

(CG1) For some stratification $\mathbb{R}^m = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ with positive integer r, the function

$\mathcal{D}_\lambda : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\mathcal{D}_\lambda(x) = \mathcal{D}_\lambda(x_1, ..., x_n) \coloneqq (\lambda^{s_1} x_1, \lambda^{s_2} x_2, ..., \lambda^{s_m} x_m) \tag{2.4.1}$$

is an automorphism of $\mathbb{G}$ for all $\lambda > 0$, where $0 < s_j \leq s_k$ for all j, k.

(CG2) For m_1 defined as above, let $X^{(1)}, ..., X^{(m_1)}$ be the left invariant vector fields[6] on $\mathbb{G}$ with $X^{(k)}(0) = \frac{\partial}{\partial x_k}|_{x_k=0}$ for $k = 1, ..., m_1$, then

$$\operatorname{rank}(\operatorname{Lie}\{X^{(1)}, ..., X^{(m_1)}\}(x)) = m \quad \forall\, x \in \mathbb{R}^m \tag{2.4.2}$$

where $\operatorname{Lie}\{\tilde{X}\}$ is the least sub-algebra containing $\tilde{X}$.

If the above properties hold, we say $\mathbb{G} = (\mathbb{R}^m, \circ, \mathcal{D}_\lambda)$ is a step-r (homogeneous) Carnot group with dilation $\mathcal{D}_\lambda$ and (Jacobian) generators $X^{(1)}, ..., X^{(m_1)}$.

Below we introduce a simple example of Carnot groups.

Example 2.4.2 (Heisenberg group)**.** Consider the group $\mathbb{H} = (\mathbb{R}^3, \circ)$ with the group law

$$(x_1, x_2, x_3) \circ (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3 + (x_1 y_2 - x_2 y_1)/2).$$

Then $\mathbb{H}$ is a step-2 Carnot group with dilation $\mathcal{D}_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^2 x_3)$ and generators

$$X^{(1)} = \partial_{x_1} - \frac{1}{2} x_2 \partial_{x_3}, \quad X^{(2)} = \partial_{x_2} + \frac{1}{2} x_1 \partial_{x_3}.$$

[6] A vector field X on $\mathbb{R}^m$ is left-invariant if $X(\varphi \circ \tau_a) = (X\varphi) \circ \tau_a$ for all $a \in \mathbb{G}$ and smooth $\varphi : \mathbb{R}^m \to \mathbb{R}$, where $\tau_a(h) = a \circ h$ is the left-translation by a.

2.4.1 Sub-gradient and sub-Laplacian on Carnot groups

The sub-gradient and sub-Laplacian in Carnot groups play a role similar to that in Euclidean spaces, but are adapted to the non-commutative nature of the Carnot group's Lie algebra. They allow the generalization of concepts like gradients and directional derivatives to this non-Euclidean setting. Below we give the definition of sub-gradient and sub-Laplacian.

Definition 2.4.3. Let $\mathbb{G}$ be a Carnot group with generators $X^{(1)}, \cdots, X^{(m_1)}$, the vector

$$\nabla_{\mathbb{G}} \coloneqq (X^{(1)}, \cdots, X^{(m_1)})$$

is called the (canonical) sub-gradient and the operator

$$\Delta_{\mathbb{G}} \coloneqq \nabla_{\mathbb{G}} \cdot \nabla_{\mathbb{G}} = \sum_{k=1}^{m_1} \left(X^{(k)}\right)^2$$

is called the (canonical) sub-Laplacian. In general, for any basis $Y^{(1)}, ..., Y^{(m_1)}$ of the space $\text{span}\{X^{(1)}, \cdots, X^{(m_1)}\}$ the operator $\sum_{k=1}^{m_1} \left(Y^{(k)}\right)^2$ is called a sub-Laplacian.

Remark 2.4.4. On a Carnot group $\mathbb{G}$, the carré du champ operator Γ_1 defined in (2.3.8) takes the form $\Gamma_1(f, f) = |\nabla_{\mathbb{G}} f|^2$.

Example 2.4.5. Consider the Heisenberg group $\mathbb{H}$, the sub-gradient on $\mathbb{H}$ is given by

$$\nabla_{\mathbb{H}} = (X^{(1)}, X^{(2)}) = \left(\partial_{x_1} - \frac{1}{2} x_2 \partial_{x_3},\ \partial_{x_2} + \frac{1}{2} x_1 \partial_{x_3}\right)$$

with the sub-Laplacian

$$\Delta_{\mathbb{H}} = \left(X^{(1)}\right)^2 + \left(X^{(2)}\right)^2 = \partial_{x_1}^2 + \partial_{x_2}^2 + \frac{1}{4}\Big(\left(x_1^2 + x_2^2\right)\partial_{x_3} + x_2 \partial_{x_1} - x_1 \partial_{x_2}\Big)\partial_{x_3}.$$

2.4.2 Homogeneous norms on Carnot groups

In Carnot groups, one can also introduce the notion of distance, which takes into account the non-commutative nature and therefore differs from the Euclidean metric. Such distance on a Carnot group $\mathbb{G}$ is given by *homogeneous norms*, which are defined as below.

Definition 2.4.6 ([31, Definition 5.1.1])**.** The homogeneous norm on a Carnot group $\mathbb{G}$ is any continuous functions $\rho : \mathbb{G} \to [0, \infty)$ that satisfies

(1) $\rho(\mathcal{D}_\lambda x) = \lambda\rho(x)$ for any $\lambda > 0$ and $x \in \mathbb{G}$ (i.e. ρ is homogeneous of degree 1),

(2) $\rho(x) > 0$ if and only if $x \neq 0$,

where $\mathcal{D}_\lambda(x)$ is the dilation defined in (2.4.1). Moreover, we say ρ is symmetric if

(3) $\rho(x) = \rho(x^{-1})$ for all $x \in \mathbb{G}$, where x^{-1} is the group inverse of x.

With this definition, it follows that all homogeneous norms are equivalent.

Proposition 2.4.7 ([31, Proposition 5.1.4])**.** *Let ρ_1, ρ_2 be two homogeneous norms on $\mathbb{G}$, then there exists a constant $c \geq 1$ such that*

$$c^{-1}\rho_1(x) \leq \rho_2(x) \leq c\rho_1(x)$$

for all $x \in \mathbb{G}$, while c is bounded above by

$$c \leq \max\Big\{ \sup\{\rho_2(x) : \rho_1(x) = 1\}, 1/\inf\{\rho_2(x) : \rho_1(x) = 1\}\Big\}.$$

We remark that in some contexts a homogeneous norm may also be defined with the extra condition of triangle inequality $\rho(x \circ y) \leq \rho(x) + \rho(y)$. In this thesis, we shall keep the above general definition as introduced in [31, Definition 5.1.1], so that a homogeneous

norm may not satisfy a triangle inequality in the usual sense. Nevertheless, one can still show it satisfies a generalised version of triangle inequality.

Proposition 2.4.8 ([31, Proposition 5.1.7])**.** *Let ρ be a homogeneous norm on a Carnot group $\mathbb{G}$, then ρ satisfies the following generalised triangle inequality*

$$\rho(x \circ y) \leq c(\rho(x) + \rho(y))$$

where $c := \max\{\rho(x \circ y) : \rho(x) + \rho(y) = 1\}$. If $c = 1$, we shall call the above inequality a strict triangle inequality.

In the sequel we give two important examples of homogeneous norms on Carnot groups. The first one is the Kaplan norm N, which is associated to the fundamental solution of the sub-Laplacian $\Delta_{\mathbb{G}}$; while the second is the control distance (or Carnot-Carathéodory distance) d, which has a geodesic property on the group. We will give definitions and introduce some important properties of both norms. We shall also discuss the main differences between the two norms, and explain how the differences are related to the coercive inequalities that we develop in this thesis. As for a broad introduction of homogeneous norms on Carnot groups, one may refer to [31, Chapter 5].

Kaplan norm N

Below we provide the definition of the Kaplan norm N on Heisenberg group $\mathbb{H}$, although we remark that N can be defined in a similar manner on more general Carnot groups (such as a H-type group [30, Section 3.2]).

Example 2.4.9 (Kaplan norm)**.** Consider the Heisenberg group $\mathbb{H}$ and the function

$$N(x_1, x_2, x_3) = ((x_1^2 + x_2^2)^2 + 16x_3^2)^{\frac{1}{4}}, \tag{2.4.3}$$

one can verify that N is a homogeneous norm on $\mathbb{H}$ which is smooth on $\mathbb{H} \setminus \{0\}$. We shall call N the *Kaplan norm*[7] and it satisfies

$$|\nabla_{\mathbb{G}} N(x)| = \frac{\|x\|_E}{N(x)} \quad \text{and} \quad \Delta_{\mathbb{G}} N(x) = \frac{3\|x\|_E^2}{N^3(x)} \tag{2.4.4}$$

where $\|x\|_E \coloneqq \sqrt{x_1^2 + x_2^2}$ is the Euclidean norm of the step-1 coordinates. For a derivation of (2.4.4), one may refer to [30, Proposition 3.2.16 and 3.2.17]. In addition, N satisfies the strict triangle inequality (see [33]) with $N(x) = N(x^{-1})$.

Remark 2.4.10. On the Heisenberg group $\mathbb{H}$, the function N^{-2} is the fundamental solution of the sub-Laplacian $\Delta_{\mathbb{H}}$, in the sense that N^{-2} is smooth on $\mathbb{H} \setminus \{0\}$ and

$$\Delta_{\mathbb{H}} N^{-2}(h) = 0$$

for all $h \in \mathbb{H} \setminus \{0\}$. For a proof of this result and a detailed introduction of fundamental solutions for a sub-Laplacian, we refer the reader to [31, Chapter 5].

Control distance d

To properly define the control distance on a Carnot group $\mathbb{G}$, we shall first introduce the length of an (admissible) curve in $\mathbb{G}$.

Definition 2.4.11. For a Carnot group $\mathbb{G}$ with generators $\{X^{(1)}, ..., X^{(m_1)}\}$, let $\gamma : [0, 1] \to \mathbb{G}$ be an admissible curve, i.e. γ is piecewise differentiable on $(0, 1)$ and

$$\dot{\gamma}(t) = \sum_{k=1}^{m_1} \alpha_k X^{(k)}(\gamma(t))$$

[7] N may have different names (such as $\mathcal{L}$-gauge in [31]), in this thesis we adopt the name 'Kaplan norm' after Kaplan's work [32]

almost everywhere for some measurable $\alpha_k : [0,1] \to \mathbb{R}$. The length of γ is defined to be

$$|\gamma| \coloneqq \int_0^1 \left(\sum_{k=1}^{m} \alpha_k^2(t) \right) dt.$$

Remark 2.4.12. For the existence of such admissible curve between any pair of points in $\mathbb{G}$, one needs the Carathéodory–Chow–Rashevsky connectivity theorem in the case of stratified vector fields, which was proved in [31, Theorem 19.1.3]. In addition, with this notion of length, one can now define the geodesic between $a, b \in \mathbb{G}$ as the admissible curve γ joining a, b with the shortest length $|\gamma|$, and the existence of geodesics on Carnot groups is proved in [31, 5.15 Appendix C].

We are now ready to define the control distance d as follows.

Definition 2.4.13. For any Carnot group $\mathbb{G}$, the control distance (or Carnot-Carathéodory distance) d between $x, y \in \mathbb{G}$ is defined by

$$d(x,y) \coloneqq \inf\{|\gamma| : \gamma : [0,1] \to \mathbb{G} \text{ is an admissible curve with } \gamma(0) = x,\ \gamma(1) = y\}.$$

This definition guarantees d to be the length of geodesics in $\mathbb{G}$ and it associates to the sub-gradient via the identity

$$|\nabla_{\mathbb{G}} f| = \limsup_{d(x,y)\to 0} \frac{|f(x) - f(y)|}{d(x,y)} \tag{2.4.5}$$

for any differentiable function f (see [30, (3.15)] for a simple derivation). By further defining $d(x) \coloneqq d(x, 0)$ and thus $d(x,y) = d(x \circ y^{-1}) = d(y \circ x^{-1})$, one can show that d is a (in general non-smooth) homogeneous norm on $\mathbb{G}$ with the following properties.

Theorem 2.4.14 ([31, Section 5.2])**.** *On a Carnot group $\mathbb{G}$ of step r, let $d(x) \coloneqq d(x, 0)$ for the control distance d defined in Definition 2.4.13. Then d is a homogeneous norm,*

only smooth on $S = \{(x_1, x_2, ..., x_r) \in \mathbb{G} : x_1 \neq 0\}$, *that satisfies*

$$|\nabla_{\mathbb{G}} d(x)| = 1$$

for all $x \in S$, *and there exists* $\varepsilon > 0$ *such that*

$$\Delta_{\mathbb{G}} d \leq \frac{\varepsilon}{d}$$

in the sense of distributions. Also, d *satisfies strict triangle inequality with* $d(x) = d(x^{-1})$.

Comparing the above properties of d with the properties of Kaplan norm N in (2.4.4), we notice that the sub-gradient and sub-Laplacian of d are bounded away from zero outside the unit ball whereas N has vanishing sub-gradient and sub-Laplacian on $\|x\|_E$. This property of d turns out to be crucial for constructing an LSI, as one can see from the following theorem.

Theorem 2.4.15 ([5]). *Let* d *be the control distance and* N *be the Kaplan norm on the Heisenberg group* $\mathbb{H}$. *Then the probability measure* $e^{-d^p} dx/Z$ *satisfies an LSI for all* $p \geq 2$, *but the measure* $e^{-N^p} dx/Z$ *does not satisfies LSI for any* $p \in \mathbb{R}$. *In fact, any smooth homogeneous norm* ρ *on* $\mathbb{H}$ *has a vanishing sub-gradient similar to* N *and therefore the measure* $e^{-\rho^p} dx/Z$ *cannot satisfy an LSI for* $p \geq 2$.

2.4.3 Coercive inequalities on Carnot groups.

Due to the special properties of a Carnot group $\mathbb{G}$, obtaining an SGI or LSI on $\mathbb{G}$ turns out to be difficult. This problem has been tackled in the works of Hebisch and Zegarliński [5], where a systematic strategy was developed to obtain coercive inequalities on general metric spaces. The key is the use of a functional inequality called U-bound that takes the

following form

$$\nu(Uf^2) \leq C\nu|\nabla_{\mathbb{G}} f|^2 + D\nu|f|^2$$

for some $U : \mathbb{G} \to \mathbb{R}$ called the *U-bound function* and some probability measure ν on $\mathbb{G}$. In our setup, U will mostly be the finite-range potential or some norm-like functions that can bound the potential. Below we illustrate how U-bounds can lead to coercive inequalities.

Theorem 2.4.16 (U-bound to SGI [5])**.** *Let $\mathbb{G}$ be a Carnot group and μ a probability measure on $\mathbb{G}$ which is absolutely continuous with respect to Lebesgue measure dx. Suppose the U-bound*

$$\mu(\eta f^2) \leq C\mu|\nabla_{\mathbb{G}} f|^2 + D\mu|f|^2 \tag{2.4.6}$$

holds for some $\eta : \mathbb{G} \to [0, \infty)$ and constants $C, D > 0$ independent of f. If for all $\varepsilon > 0$, there exists a finite ball containing the set $\{x : \eta(x) \leq \varepsilon\}$ and there exists $a_\varepsilon > 0$ such that $d\mu/dx \in (1/a_\varepsilon, a_\varepsilon)$ on $\{\eta < \varepsilon\}$, then the SGI holds

$$\mu(f - \mu f)^2 \leq C_{SG}\mu|\nabla_{\mathbb{G}} f|^2.$$

Theorem 2.4.17 (U-bound to LSI [5])**.** *Let $\mathbb{G}$ be a Carnot group and $d\mu = e^{-U}dx/Z$ a probability measure on $\mathbb{G}$ for some continuously differentiable $U : \mathbb{G} \to \mathbb{R}$. Suppose the U-bound*

$$\mu((|\nabla_{\mathbb{G}} U|^2 + U)f^2) \leq C\mu|\nabla_{\mathbb{G}} f|^2 + D\mu|f|^2 \tag{2.4.7}$$

holds and the Sobolev inequality $(\int |f|^{2+\varepsilon}dx)^{\frac{2}{2+\varepsilon}} \leq a\int |\nabla_{\mathbb{G}} f|^2 dx + b\int |f|^2 dx$ holds for some $\varepsilon > 0$, then the defective LSI holds

$$\mu\left(f^2 \log \frac{f^2}{\mu f^2}\right) \leq C_{LS}\mu|\nabla_{\mathbb{G}} f|^2 + D_{LS}\mu(f^2).$$

If in addition μ satisfies an SGI, then $D_{LS} = 0$ (i.e. one has a non-defective LSI).

Remark 2.4.18. In [5], the above two theorems are proved with power 2 replaced by any $q \in (1, 2]$, in which case the SGI and LSI become q-SGI and q-LSI. This is a stronger result since for $1 < q < q'$ one can show that q-LSI implies q'-LSI (same for q-SGI), and q-LSI leads to a stronger 'decay of tail' estimate than q'-LSI. For demonstration of these results and a general introduction to q-SGI/q-LSI, one may see [34] and the references therein.

Using the above theorems, one can now construct a U-bound (2.4.6) or (2.4.7) to deduce the corresponding SGI or LSI. However, it is not obvious how this can be useful, indeed, we are only shifting our aim from SGI/LSI to another type of functional inequality. In the remark below, we list a few reasons for considering U-bounds.

Remark 2.4.19. To understand the advantages of U-bounds, it is important to first grasp the main ideas for constructing a U-bound. We will sketch the ideas for the measure $d\mu = e^{-\rho^2/2}dx/Z$ given by a quadratic potential $\rho^2/2$ and some homogeneous norm ρ on $\mathbb{G}$. We start by calculating the integral

$$-\mu((\Delta_{\mathbb{G}}\rho)f) = \frac{1}{Z}\int \langle \nabla_{\mathbb{G}}\rho, \nabla_{\mathbb{G}}\big(e^{-\rho^2/2}f\big)\rangle = -\mu\big(|\nabla_{\mathbb{G}}\rho|^2\rho f\big) + \mu(\langle \nabla_{\mathbb{G}}\rho, \nabla_{\mathbb{G}}f\rangle), \quad (2.4.8)$$

where the first step is due to integration by parts and the second is by Leibniz rule on the term $\nabla\big(e^{-\rho^2}f\big)$. In case ρ satisfies the following property

$$\exists\, a \geq 1 \text{ such that } 1/a \leq |\nabla_{\mathbb{G}}\rho| \leq a \text{ and } \Delta_{\mathbb{G}}\rho \leq a \text{ outside the } \rho\text{-unit ball}, \quad (2.4.9)$$

equation (2.4.8) can be rearranged into the inequality

$$\mu(\rho f) \leq a^2\mu|\nabla_{\mathbb{G}}f| + a^2\mu(f) \quad (2.4.10)$$

for all $f \geq 0$ supported outside the ρ-unit ball. For a general f, replace f by $|f|$ and cut it

off on the unit ball by a smooth mollifier, then (2.4.10) becomes $\mu(\rho|f|) \leq b\mu|\nabla f| + b\mu|f|$ for some $b > 0$ independent of f, where we used the metric inequality $|\nabla f| \leq |\nabla|f||$ that holds almost everywhere. Finally, substituting f with f^2 and using Jensen's inequality $2\mu|f\nabla_{\mathbb{G}} f| \leq \mu(f^2) + \mu|\nabla_{\mathbb{G}} f|^2$, we get the U-bound

$$\mu(\rho f^2) \leq b\mu|\nabla_{\mathbb{G}} f|^2 + 2b\mu(f^2).$$

Based on the sketched proof, we now discuss a few reasons for considering U-bounds.

(1) U-bounds are relatively easy to obtain: The steps above are straightforward to follow and they require only basic lemmas from measure integration theorem.

(2) U-bounds are versatile: By changing the properties of U-bound functions, the U-bound may lead to a variety of functional inequalities in addition to SGI and LSI, e.g. modified LSI [35], isoperimetric inequalities [34] and concentration inequalities of Gibbs measures [36].

(3) U-bounds provide a link from potential to functional inequalities: By observing the sketched proof, we notice that the U-bound function in general takes the form of the potential and its gradient. On the other hand, the U-bound function links with SGI/LSI by Theorem 2.4.16 and 2.4.17. This makes the U-bound a bridge to connect the SGI/LSI and the potential in Gibbs measures, allowing us to easily check which properties of the potentials are needed in order to obtain an SGI/LSI.

Based on the U-bound strategy, many results concerning coercive inequalities on Carnot groups have been published, including some recent works by Inglis, Konstantopoulos, and Papageorgiou [6, 7, 8, 30, 37], where they obtained LSI for quadratic or higher-order interactions defined by control distance d on the Heisenberg group. The main strategy is to deduce a single-site U-bound and LSI, and extend them to the whole lattice by iterations.

In our work, we will follow the same U-bound strategy to obtain infinite-dimensional SGI/LSIs, but we shall replace d by a norm ρ with vanishing or blowing up sub-gradient along some directions. This makes a significant difference for the following reasons:

(1) The measure given by ρ cannot satisfy an LSI in general (as discussed in Theorem 2.4.15). Therefore, we shall instead aim for infinite-dimensional SGIs for measures constructed by homogeneous norms with vanishing sub-gradient in some directions, which are not yet covered in the literature.

(2) When deducing single-site U-bounds like (2.4.6) or (2.4.7), the approach sketched in Remark 2.4.19 fails to work as property (2.4.9) are not satisfied. For a workaround, we consider U-bounds and SGI/LSI in a weak form, in the sense that the inequality at site j depends on all sites k that are within the interaction range of j. A similar concept has been used in [7] but not for the same purpose, and here we shall generalise this method systematically and study some variations of the weak forms.

(3) In addition, we shall consider ρ with blowing up sub-gradients in some directions, which are rarely discussed in the literature. We will find a systematic way to construct norms with unbounded (and vanishing) sub-gradients, and test whether an SGI/LSI can be obtained in each case.

The main results addressing these key points are presented in chapter 3 and 4.

Chapter 3

Spectral gap inequalities on infinite product of Carnot groups

In this chapter we present the results of the joint research project with E. Bou Dagher and Y. Qiu, on the topic of constructing spectral gap inequalities (SGI) where the configuration space is $\Omega = \mathbb{G}^{\mathbb{Z}^D}$, with a homogeneous Carnot group $\mathbb{G}$ and an integer lattice $\mathbb{Z}^D$.

In particular, we consider potential energy that takes the following form

$$U_\Lambda^\omega(x) := \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda} \beta_{ij} V(x_i, x_j) + \sum_{i\in\Lambda,\, j\notin\Lambda} \beta_{ij} V(x_i, \omega_j)$$

for a phase $\phi \in C^2(\mathbb{G}, \mathbb{R})$ and an interaction $V \in C^2(\mathbb{G}\times\mathbb{G}, \mathbb{R})$, where $\beta_{ij} = 0$ if $\|i-j\|_{l^1} > R$ for some range $R > 0$. Let $\mathbb{E}_\Lambda^\omega$ be the local Gibbs measure given by U_Λ^ω. Our aim is to find sufficient conditions on ϕ, V, β_{ij} so that there exists a unique global Gibbs measure ν associated to $\{\mathbb{E}_\Lambda\}_{\Lambda\Subset\mathbb{Z}^D}$ and ν satisfies the spectral gap inequality. Below we present an informal statement of the main result (see Theorem 3.3.1 for the full statement).

Theorem 3.0.1. *Let U_Λ and $\mathbb{E}_\Lambda$ be defined as above. Suppose the following two conditions*

hold on ϕ and V for some $q \in (1, 2]$:

(1) *There exist a function $\eta \in C(\mathbb{G}, [0, \infty))$ diverging to infinity in all directions and a constant $B > 0$ such that*

$$\int |g(h)|^q \eta(h) e^{-\phi(h)} dh \leq B \int \left(|\nabla_h g(h)|^q + |g(h)|^q \right) e^{-\phi(h)} dh$$

for all $g \in C(\mathbb{G})$ for which the left-hand side is finite, and

(2) *the interaction V satisfies the gradient bound*

$$\left|\nabla_{h_1} V(h_1, h_2)\right|^q + \left|\nabla_{h_2} V(h_1, h_2)\right|^q \leq B(1 + \eta(h_1) + \eta(h_2)).$$

If $\sup_{i,j} |\beta_{ij}|$ is finite and sufficiently small, then there exists a unique global Gibbs measure ν associated to $\{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ and ν satisfies the q-SGI

$$\nu|f - \nu f|^q \leq C_{SG} \nu |\nabla f|^q$$

for all $f \in W^{1,q}(\nu)$ such that $\sup_{i \in \mathbb{Z}^D} \nu(\eta(x_i)|f|^q) < \infty$ with some constant $C_{SG} > 0$.

To formally introduce the above result, we need to set up the background and framework. That is, in the first section, we will give a brief introduction to SGI, explain where our work enters into the literature, and recall some important notations from Chapter 2. We then present the results in the remaining sections and discuss some examples and applications of the main theorem.

3.1 Introduction

The main object under study is the spectral gap inequality (SGI), defined by

$$\int_\Omega \left| f - \int_\Omega f d\mu \right|^2 d\mu \leq C_{SG} \int_\Omega |\nabla f|^2 d\mu \tag{3.1.1}$$

on a probability space (Ω, Σ, μ) with a smooth (sub-)gradient ∇, where $C_{SG} > 0$ is a constant independent of f for which the right-hand side is finite. This inequality plays an important role in the theory of Markov semigroups and stochastic differential equations as discussed in Section 2.3 and Theorem 2.3.3.

We note that there exist several generalisations of (3.1.1) in literature. For instance, [38] studied the case where $C_{SG} > 0$ can be made arbitrarily small by introducing a possibly arbitrarily large $\mathbb{L}^1$-norm on the right-hand side, and [39] studied the opposite of this in which C_{SG} can be made large and a 2-homogeneous functional of f is made small. We also remark, since it will be of main focus of this project, that [34] explored the generalisation of (3.1.1) in which 2 is replaced with any $q \in (1, 2]$.

As for SGIs on unbounded spin systems, the literature is wide. A good starting point may be the works of [3, 4, 40, 41, 42, 43] and the references within. Our work can be regarded as the intersection of [43, 44, 45] which studied the extension of finite-dimensional SGI to infinite dimensions, and the paper [5] which studied the problem of constructing finite-dimensional SGI on Carnot groups, with a new strategy called U-bound as introduced in Section 2.4.3.

In the work of [6, 46], the two approaches are combined to obtain a log-Sobolev inequality on $\Omega = \mathbb{H}^{\mathbb{Z}^D}$, for a global Gibbs measure defined by the control distance d (as in Definition 2.4.13). Our work is partly inspired by this and has a similar layout: we start from the U-bounds on a single-spin space to obtain a finite-dimensional SGI, and then find

conditions under which it can be extended to infinite dimensions. The difference lies in the fact that we consider Gibbs measures defined by norms that are smooth outside the origin (e.g. the Kaplan norm N in Example 2.4.9), in which case the log-Sobolev inequality fails. More details on this matter are provided in Theorem 2.4.15.

Organisation of this chapter is as follows. In the end of this section, we introduce the infinite-dimensional setup for $\mathbb{G}^{\mathbb{Z}^D}$. In Section 3.2 we present a general framework to extend single-site SGI to infinite-dimensional SGI by introducing a weak form of U-bound. In Section 3.3 we find sufficient conditions on the potential energy to prove the weak U-bound. This condition also allows us to easily generalise Dobrushin's compactness Theorem to our situation, which guarantees the existence of a global Gibbs measure. In Section 3.4 we present examples of potentials such that the SGI holds on infinite product of Heisenberg groups, and in particular the potentials that depend on the Kaplan norm N.

3.1.1 The infinite-dimensional setting

The infinite-dimensional setup is the same as introduced in Chapter 2, but for the convenience of the reader, we briefly recall some important notations. The configuration space is given by $\Omega = \mathbb{G}^{\mathbb{Z}^D}$ for a Carnot group $\mathbb{G}$ defined in Definition 2.4.1. For finite volume $\Lambda \Subset \mathbb{Z}^D$ and boundary conditions $\omega \in \Omega$, we consider the finite-range potential

$$U_\Lambda^\omega(x_\Lambda) := U_\Lambda(x_\Lambda \bullet \omega_{\Lambda^c}) = \sum_{i\in\Lambda} \phi(x_i) + \sum_{i,j\in\Lambda} \beta_{ij} V(x_i, x_j) + \sum_{i\in\Lambda,\, j\notin\Lambda} \beta_{ij} V(x_i, \omega_j) \tag{3.1.2}$$

for a phase $\phi \in C^2(\mathbb{G}, \mathbb{R})$ and an interaction $V \in C^2(\mathbb{G}\times\mathbb{G}, \mathbb{R})$, where $|\beta_{ij}| < \beta$ for some $\beta > 0$ and $\beta_{ij} = 0$ if $\|i-j\| := \|i-j\|_{l^1} > R$ for some range $R > 0$. We also require ϕ and V to satisfy the condition

$$Z_\Lambda^\omega := \int_{\mathbb{G}^\Lambda} e^{-U_\Lambda^\omega(x_\Lambda)} dx_\Lambda < \infty$$

for all $\Lambda \Subset \mathbb{Z}^D$ and $\omega \in \Omega$, where dx_Λ is the Lebesgue measure on $\mathbb{G}^\Lambda$. Then the local Gibbs measure $\mathbb{E}_\Lambda$ is well-defined and given by

$$\mathbb{E}^\omega_\Lambda f = \int_{\mathbb{G}^\Lambda} f d\mathbb{E}^\omega_\Lambda = \frac{1}{Z^\omega_\Lambda} \int_{\mathbb{G}^\Lambda} f(x_\Lambda \bullet \omega_{\Lambda^c}) e^{-U^\omega_\Lambda(x_\Lambda)} dx_\Lambda. \tag{3.1.3}$$

By identifying $\mathbb{E}_\Lambda f(\omega) = \mathbb{E}^\omega_\Lambda(f)$ as a linear operator, we call the family of operators $\mathcal{E} = \{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ a local specification. The global Gibbs measure ν associated to $\mathcal{E}$ is then given by the DLR equation

$$\nu \mathbb{E}_\Lambda(f) = \nu(f)$$

for all $\Lambda \Subset \mathbb{Z}^D$ and all measurable function f.

In the sequel, we shall reserve the notation $p = q/(q-1)$ for its Hölder conjugate. Moreover, for $j \in \mathbb{Z}^D$ denote by ∇_j the sub-gradient $\nabla_{\mathbb{G}}$ on the single-space $\mathbb{G}^{\{j\}}$, and similarly $\nabla_\Lambda = (\nabla_j)_{j\in\Lambda}$ for $\Lambda \subset \mathbb{Z}^D$ and $\nabla = (\nabla_j)_{j\in\mathbb{Z}^D}$. The length of gradient is therefore given by

$$|\nabla_\Lambda f|^q = \sum_{j\in\Lambda} \|\nabla_j f\|^q_{l^2}$$

and

$$|\nabla f|^q = \sum_{j\in\mathbb{Z}^D} \|\nabla_j f\|^q_{l^2}.$$

Unless otherwise stated, ∇ always denotes the global sub-gradient on $\mathbb{G}^{\mathbb{Z}^D}$ and not the Euclidean or sub-gradient on one variable. In addition, for convenience we adopt the shorthand notation $\nu(f) = \int_\Omega f d\nu$ throughout this chapter.

The aim of this project is to find sufficient conditions on U_Λ (and thus on β_{ij}, ϕ and V) so that the global Gibbs measure ν exists, is unique, and satisfies the following q-SGI

$$\nu|f - \nu f|^q \le C_{SG} \nu |\nabla f|^q$$

for fixed constants $q \in (1,2]$ and C_{SG} independent of f. To keep the proof easy to read, we choose $R = 1$ for the finite range of interaction, meaning $\beta_{ij} = 0$ whenever $\|i-j\| := \|i-j\|_{l^\infty(\mathbb{Z}^D)} > 1$. The general case $1 < R < \infty$ is discussed briefly at the end of the following section.

3.2 Passage from single-site to global inequalities

In this section, we show that the global q-SGI on Ω can be obtained by a weak U-bound and a weak q-SGI on single-spin space. The strategy of U-bounds were first introduced in the setting of general metric spaces in [5] where they established conditions under which a q-SGI and other coercive inequalities hold for Gibbs measures on a finite-dimensional space. As introduced in Section 2.4.3, the standard (or strong) U-bound on a single-site space Ω_j only has terms of f and $\nabla_j f$ on the right-hand side. In comparison, the one we presented here is weak in the sense that the right-hand side contains extra terms of $\nabla_k f$ for all k in the interaction range of j. It turns out that this is actually sufficient, provided that the extra terms of $\nabla_k f$ decay sufficiently fast as k moves further away from the given site $j \in \mathbb{Z}^D$. Similarly, the q-SGI here is weak in the sense that the right-hand side contains similar extra terms than just $\nu|\nabla_i f|^q$ in (3.1.1). In the following we introduce the main theorem for this section.

Theorem 3.2.1. *Let $q \in (1,2]$, and let $\{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ be the local specification for a potential given in (3.1.2) with interaction range $R = 1$. Suppose there exists a global Gibbs measure ν associated to $\{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ such that the following two conditions hold for all $f \in W^{1,q}(\nu)$ and all $\beta > 0$ sufficiently small:*

(A1) *for any* $i \in \mathbb{Z}^D$ *and* $j \in \mathbb{Z}^D$ *with* $\|j - i\| = 1$*, one has the weak single-site* U*-bound*

$$\nu\left(|f|^q |\nabla_j V(x_i, x_j)|^q\right) \leq A\nu|f|^q + A\sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q\,,$$

(A2) *and, for any* $i \in \mathbb{Z}^D$*, one has the weak single-site* q*-SGI*

$$\nu\mathbb{E}_i^\omega|f - \mathbb{E}_i^\omega f|^q \leq A\sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q$$

for some constant $A > 0$ *and some symmetric matrix* $(M_{ij} \geq 0)_{i,j\in\mathbb{Z}^D}$ *such that*

(B1) *The constants in* (**A1**) *and* (**A2**) *satisfies that* $A = \mathcal{O}(\beta^{-1/2})$ *as* $\beta \to 0^+$*, and*

(B2) $M_{ij} \geq 0$ *for all* $i, j \in \mathbb{Z}^D$ *and satisfies* $\sup_{i\in\mathbb{Z}^D}\sum_{j\in\mathbb{Z}^D} M_{ij} < \infty$*.*

Then for all $\beta > 0$ *sufficiently small, the global Gibbs measure* ν *is unique and satisfies the following* q*-SGI*

$$\nu|f - \nu f|^q \leq C_{SG}\nu|\nabla f|^q$$

with some constant $C_{SG} > 0$ *possibly depending on* β*, for all* $f \in W^{1,q}(\nu)$*.*

Before presenting the proof, we first give a few remarks about the above theorem.

Remark 3.2.2. The $l^1(\mathbb{Z}^D)$-summability of $(M_{i,j})_{j\in\mathbb{Z}^D}$ of **(B2)** implies $\left\|(M_{ij})\right\|_{l^\infty} < \infty$. Without loss of generality we may assume $\left\|(M_{ij})\right\|_{l^\infty} \leq 1$ by re-scaling M and absorbing the constant into A.

Remark 3.2.3. To avoid confusion, we also remark that the Landau notations $\mathcal{O}$ and o are defined with respect to small β. That is, since β needs to be made sufficiently small, we may as well restrict β in the range $(0, \bar{\beta})$ for some $\bar{\beta} > 0$. Then for any positive constants $A = A(\beta)$ and $B = B(\beta)$ that depend on β, the notations $\mathcal{O}$ and o have the following meaning:

(1) $A = \mathcal{O}(B)$ if $A(\beta) \leq cB(\beta)$ for all $\beta \in (0, \bar{\beta})$ and some $c > 0$ independent of β;

(2) $A = \mathcal{O}(1)$ if $A(\beta)$ is bounded above uniformly for all $\beta \in (0, \bar{\beta})$;

(3) $A = o(1)$ if $A \to 0$ as $\beta \to 0^+$.

As what follows, we will use $\mathcal{O}(1)$ to show boundedness, and use $A = \mathcal{O}(B)$ to emphasize the asymptotic behaviour of A and B. If we only need A to vanish with β without concerning the convergence rate, then $o(1)$ shall be used.

The basic idea is that the summability condition **(B2)** implies that if the interaction matrix M_{ij} decays sufficiently fast at infinity (in particular it suffices that it decays exponentially fast), then the defective terms can be dispensed of. The generic case is that M_{ij} is a matrix which decays like $\varepsilon^{\|i-j\|}$ where $0 < \varepsilon < 1$ but which may be bounded away from zero in a finite neighbourhood of i.

To prove this theorem, we first prove some lemmata. The first is a sweeping out inequality - these were introduced in [19] in order to prove infinite-dimensional logarithmic Sobolev inequalities. Throughout the remainder of this chapter we fix $q \in (1, 2]$ and assume the conditions of Theorem 3.2.1 hold. Moreover, we use the notation $j \sim i$ to mean j is a neighbour of site i, i.e. j is adjacent to i in the sense $\|j - i\| = 1$, and we use the notation $\{\sim i\}$ to denote the set of neighbours of i, that is the $2D$ points $j \in \mathbb{Z}^D$ with $\|i - j\| = 1$. (Thus $j \sim i$ if and only if $j \in \{\sim i\}$.) Lastly, since one needs $\beta \to 0^+$ to make the arguments work; we will need to track the decay (or growth) of some constants and we will typically reserve uppercase letters such as K for large constants that are $\mathcal{O}(1)$ and lowercase letters such as c for small constants that are $o(1)$. In addition, we will drop the notation of boundary conditions ω for simplicity whenever its role is insignificant.

Lemma 3.2.4. *For all $i, j \in \mathbb{Z}^D$, the following single-site sweeping out inequality holds*

$$\nu\big|\nabla_j(\mathbb{E}_i^\omega f)\big|^q \le K_1\nu\big|\nabla_j f\big|^q + c_1\sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q \tag{3.2.1}$$

with some constants $K_1 = \mathcal{O}(1)$ and $c_1 = o(1)$ independent of i, j and f.

Proof. Let us recall that $\mathbb{E}_i^\omega f$ does not depend on x_i; in general it depends on every other coordinate x_j for $j \neq i$, but in the specific case of interactions of unit range, it depends on neighbouring spins x_j for $j \sim i$. Thus if $j \notin \{\sim i\}$ then the probability kernel of the measure $d\mathbb{E}_i^\omega$ does not depend on j, and only f may depend on j. It follows that one simply has $\nu\big|\nabla_j(\mathbb{E}_i f)\big|^q = \nu\big|\mathbb{E}_i(\nabla_j f)\big|^q \le \nu\big|\nabla_j f\big|^q$ by interchanging differentiation and integration in the first equality and by Jensen's inequality together with the DLR equation in the second inequality, so we shall focus on the case $j \sim i$, i.e. $\|i - j\| = 1$. Note also that to emphasise the dependence of the function $\mathbb{E}_i f(\omega) = \mathbb{E}_i^\omega f$ on ω, we will write $\mathbb{E}_i^\omega f$ with the understanding it replaces $\mathbb{E}_i f$ even though $\mathbb{E}_i^\omega f$ is a real number and $\mathbb{E}_i f$ is a real-valued function. Let $X^{(l)}$ for $l = 1, \cdots, m_1$ and $m_1 \in \mathbb{N}$ be the vector fields generating the Lie algebra $\mathcal{G}$ of a nilpotent Lie group $\mathbb{G}$. For $j \in \mathbb{Z}^D$, let $\nabla_j = (X_j^{(l)})_{l=1,..,m_1}$ denote the sub-gradient on the copy $\mathbb{G}^{\{j\}}$ of $\mathbb{G}$ at site j.

Thus for a bounded differentiable function with bounded derivative

$$\nabla_j\mathbb{E}_i^\omega f = (X_j^{(l)}(\mathbb{E}_i^\omega f))_{l=1,\cdots,m_1}.$$

If $\rho_i := d\mathbb{E}_i^\omega/\lambda(dx_i)$ is the density of the measure $\mathbb{E}_i^\omega$ with respect to the Lebesgue measure $\lambda(dx_i)$ on $\mathbb{G}^{\{i\}}$, then

$$X_j^{(l)}(\mathbb{E}_i^\omega f) = \int\Big(\rho_i(X_j^{(l)}f) + (X_j^{(l)}\rho_i)f\Big)\lambda(dx_i) = \mathbb{E}_i^\omega(X_j^{(l)}f) + \int(X_j^{(l)}\rho_i)f\lambda(dx_i). \tag{3.2.2}$$

A direct calculation yields

$$X_j^{(l)}\rho_i = X_j^{(l)}\left(\frac{e^{-U_i^\omega}}{\int e^{-U_i^\omega}\lambda(dx_i)}\right) = \frac{(-X_j^{(l)}U_i^\omega)e^{-U_i^\omega}}{\int e^{-U_i^\omega}\lambda(dx_i)} - \frac{e^{-U_i^\omega}\int(-X_j^{(l)}U_i^\omega)e^{-U_i^\omega}\lambda(dx_i)}{(\int e^{-U_i^\omega}\lambda(dx_i))^2}$$
$$= -\rho_i(X_j^{(l)}U_i^\omega) + \rho_i\mathbb{E}_i^\omega(X_j^{(l)}U_i^\omega)$$

and therefore

$$\begin{aligned}\int(X_j^{(l)}\rho_i)f\lambda(dx_i) &= -\mathbb{E}_i^\omega((X_j^{(l)}U_i^\omega)f) + (\mathbb{E}_i^\omega f)(\mathbb{E}_i^\omega(X_j^{(l)}U_i^\omega)) \\ &= -\mathbb{E}_i^\omega((f-\mathbb{E}_i^\omega f)(X_j^{(k)}U_i^\omega))\end{aligned} \tag{3.2.3}$$

Inserting (3.2.3) into (3.2.2), we obtain

$$\begin{aligned}\nabla_j(\mathbb{E}_i^\omega f) &= (\mathbb{E}_i^\omega\nabla_j f) - \mathbb{E}_i^\omega((f-\mathbb{E}_i^\omega f)(\nabla_j U_i^\omega)) \\ &= (\mathbb{E}_i^\omega\nabla_j f) - \beta_{ij}\mathbb{E}_i^\omega((f-\mathbb{E}_i^\omega f)(\nabla_j V(x_i,\omega_j))),\end{aligned}$$

which implies by Jensen's inequality

$$\left|\nabla_j\mathbb{E}_i^\omega f\right|^q \le 2\mathbb{E}_i^\omega\left|\nabla_j f\right|^q + 2\beta^q\mathbb{E}_i^\omega\left(|f-\mathbb{E}_i^\omega f|^q\left|\nabla_j V(x_i,\omega_j)\right|^q\right)$$

Integrating with respect to ν and applying the DLR equation $\nu\mathbb{E}_i = \nu$, we find

$$\begin{aligned}\nu\left|\nabla_j\mathbb{E}_i^\omega f\right|^q &\le 2\nu\left|\nabla_j f\right|^q + 2\beta^q\nu\left(|f-\mathbb{E}_i^\omega f|^q\left|\nabla_j V(x_i,\omega_j)\right|^q\right) \\ &\le 2\nu\left|\nabla_j f\right|^q + 2A\beta^q\left(\nu|f-\mathbb{E}_i^\omega f|^q + \sum_{k\in\mathbb{Z}^D} M_{ki}\nu\left|\nabla_k(f-\mathbb{E}_i^\omega f)\right|^q\right)\end{aligned} \tag{3.2.4}$$

where the second inequality is by **(A1)**. For the first term in the bracket, it holds

$$\nu|f - \mathbb{E}_i^\omega f|^q \leq A \sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q \tag{3.2.5}$$

by **(A2)**, and for the second term, it holds

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^D} M_{ki}\nu\left|\nabla_k(f - \mathbb{E}_i^\omega f)\right|^q \\
&\quad\leq 2\sum_{k\in\mathbb{Z}^D} M_{ki}\left(\nu|\nabla_k f|^q + \nu|\nabla_k \mathbb{E}_i^\omega f|^q\right) \\
&\quad\leq 4\sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q + 2\sum_{k\sim i, k\neq j} M_{ki}\nu|\nabla_k \mathbb{E}_i^\omega f|^q + 2\nu\left|\nabla_j \mathbb{E}_i^\omega f\right|^q
\end{aligned} \tag{3.2.6}$$

where in the third inequality we have simplified $\nu\left|\nabla_k \mathbb{E}_i^\omega f\right|^q$ by considering the two cases $k \sim i$ and $k \not\sim i$ separately. In the first case, $\left\|(M_{ij})\right\|_{l^\infty} \leq 1$ by Remark 3.2.2 so

$$\begin{aligned}
2\sum_{k\sim i} M_{ki}\nu|\nabla_k \mathbb{E}_i^\omega f|^q &= 2\sum_{k\sim i, k\neq j} M_{ki}\nu|\nabla_k \mathbb{E}_i^\omega f|^q + 2M_{ji}\nu\left|\nabla_j \mathbb{E}_i^\omega f\right|^q \\
&\leq 2\sum_{k\sim i, k\neq j} M_{ki}\nu|\nabla_k \mathbb{E}_i^\omega f|^q + 2\nu\left|\nabla_j \mathbb{E}_i^\omega f\right|^q,
\end{aligned}$$

and in the case where $k \not\sim i$ we have $\nu\left|\nabla_k \mathbb{E}_i^\omega f\right|^q \leq \nu|\nabla_k f|^q$ and thus it can be absorbed into the first sum $2\sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q$. Inserting (3.2.5) and (3.2.6) back into (3.2.4), we find

$$\nu\left|\nabla_j \mathbb{E}_i^\omega f\right|^q \leq a_1\nu\left|\nabla_j f\right|^q + a_2\sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q + a_3\sum_{k\sim i, k\neq j} M_{ki}\nu|\nabla_k \mathbb{E}_i^\omega f|^q \tag{3.2.7}$$

with

$$a_1 := \frac{2}{1 - 4A\beta^q}, \quad a_2 := \frac{2A\beta^q(A+4)}{1 - 4A\beta^q}, \quad a_3 := \frac{4A\beta^q}{1 - 4A\beta^q}.$$

By the estimate on A of **(B1)**, we may assume $a_1 < 4$ and $a_3 < a_2 < \beta^{(q-1)/2} < 1$, for β small enough. The difference between (3.2.7) with (3.2.1) is the appearance of the summation $a_3 \sum_{k\sim i, k\neq j} M_{ki}\nu\left|\nabla_k \mathbb{E}_i^\omega f\right|^q$ which we claim can be eliminated by the following induction procedure.

For notational convenience, let us denote by

$$W_k := \nu|\nabla_k \mathbb{E}_i^\omega f|^q\,, \quad Y_k := \nu|\nabla_k f|^q\,, \quad S := \sum_{k\in\mathbb{Z}^D} M_{ki} Y_k,$$

in which case (3.2.7) reads, for $\beta > 0$ small enough and $a_2 > a_3$ replacing a_3 for simplicity,

$$W_j \leq a_1 Y_j + a_2 S + a_2 \sum_{k\sim i, k\neq j} M_{ki} W_k. \tag{3.2.8}$$

We can apply (3.2.8) on each W_k appearing in the sum; one iteration yields

$$\begin{aligned} a_2 \sum_{k\sim i,k\neq j} M_{ki} W_k &\leq a_2 \sum_{k\sim i,k\neq j} M_{ki} \left(a_1 Y_k + a_2 S + a_2 \sum_{\ell\sim i,\ell\neq k} M_{\ell i} W_\ell \right) \\ &\leq a_1 a_2 S + 2D a_2^2 S + 2D a_2^2 \sum_{\ell\sim i,\ell\neq k} M_{\ell i} W_\ell \\ &\leq a_1 a_2 S + a_2 (2D a_2) S + a_2 (2D a_2) \sum_{\ell\sim i,\ell\neq k} M_{\ell i} W_\ell \end{aligned}$$

since $\sum_{k\sim i,k\neq j} M_{\ell i} Y_k \leq S$ and $\left\|(M_{ij})\right\|_{l^\infty} \leq 1$ by Remark 3.2.2, and each site has exactly $2D$ neighbours. With $\varepsilon = 2Da_2 = o(1)$, we obtain after a second iteration

$$\begin{aligned} W_j &\leq a_1 Y_j + a_2 S + a_1 a_2 S + \varepsilon a_2 S + \varepsilon a_2 \sum_{\ell\neq j,\ell\sim i} M_{\ell i} W_\ell \\ &\leq a_1 Y_j + a_2 S + a_1 a_2 S + \varepsilon a_2 S + \varepsilon a_1 a_2 S + \varepsilon^2 a_2 S + \varepsilon^2 a_2 \sum_{m\neq j, m\sim i} M_{mi} W_m \end{aligned}$$

and so on and so forth; iterating this procedure gives in the limit

$$W_j \leq a_1 Y_j + \frac{a_2 + a_1 a_2}{1 - \varepsilon} S$$

by taking $\varepsilon = \mathcal{O}(a_2) = o(1)$ small enough. In particular if β is small enough then since $a_1 < 4 = \mathcal{O}(1)$ and $a_2 < \beta^{(q-1)/2}$ one obtains the desired inequality with $K_1 = 4$ and $c_1 = \beta^{(q-1)/4}$. □

Remark 3.2.5. The estimates given here are somewhat wasteful but can easily be improved. For instance $a_2 = \mathcal{O}(\beta^{q-1})$ so c_1 can be taken as $\mathcal{O}(\beta^{q-1})$ as well, but the choice of estimate is intentional and eliminates the need to write down the controlling constant.

Remark 3.2.6. For the limit of the inductive procedure to be well-defined, we need to show $\sum_{k\sim i, k\neq j} M_{ki} W_k \leq \sum_{k\sim i} M_{ki} W_k = \nu \left| \nabla_{\{\sim i\}} \mathbb{E}_i^{\omega} f \right|^q \leq \nu |\nabla f|^q$ is bounded. Adding $a_2 M_{ji} W_j$ to the right-hand side of (3.2.8) for simplicity and summing over $j \sim i$ gives

$$\begin{aligned} \sum_{j\sim i} W_j &\leq a_1 \sum_{j\sim i} Y_j + 2Da_2 S + 2Da_2 \sum_{k\sim i} M_{ki} W_k \\ &\leq (a_1 + 2Da_2)\nu |\nabla f|^q + 2Da_2 \sum_{k\sim i} W_k. \end{aligned}$$

Taking the sum $2Da_2 \sum_{k\sim i} W_k$ to the left hand side we obtain $(1 - 2Da_2) \sum_{j\sim i} W_j \leq (a_1 + 2Da_2)\nu |\nabla f|^q$; since $a_2 = o(1)$, if β is small enough then

$$\sum_{j\sim i} W_j \leq 4a_1 \nu |\nabla f|^q < 16\nu |\nabla f|^q < \infty$$

by assumption of $f \in W^{1,q}(\nu)$.

Using the single-site sweeping out inequality (3.2.1), we will prove a global sweeping out inequality on a carefully chosen infinite volume subsystem. The first step is to form a

finite partition of $\mathbb{Z}^D$ whose components are infinite volume subsystems $(\Gamma_n)_{n=0}^{2^D-1}$ with the property that no interaction exists within each Γ_n which, in our case, translates to each pair of points $i, j \in \Gamma_n$ satisfying $\|i-j\| > 1$ whenever $i \neq j$. The partition chosen here will be consist of 2^D translations of the sub-lattice $\Gamma_0 = (2\mathbb{Z})^D$ translating 0 to another point in $[0,1]^D \subset \mathbb{Z}^D$, although we remark that this is not the only partition. The choice of partition enters the proof through the controlling constants so the partitions are equivalent as far as obtaining the q-SGI (with no regard for the constant) is concerned.

In any case we fix in the sequel our choice of partition $(\Gamma_n)_{n=0}^{2^D-1}$ as above, and we begin by proving global sweeping out inequalities for Γ_n; the self-similarity of the Γ_n will mean the proof follows for all if it follows for one.

Lemma 3.2.7. *For any $\Lambda \subset \mathbb{Z}^D$ and $n \in \{0, 1, \cdots, N-1\}$, the sub-lattice sweeping out inequality*

$$\nu\big|\nabla_\Lambda(\mathbb{E}_{\Gamma_n} f)\big|^q \leq K_2 \nu|\nabla_\Lambda f|^q + c_2 \nu|\nabla_{\Lambda^c} f|^q \tag{3.2.9}$$

holds for some constants $K_2 = \mathcal{O}(1)$ and $c_2 = o(1)$ independently of n and Λ.

Remark 3.2.8. We emphasise the reason that the sub-gradient is taken over a general subset $\Lambda \subset \mathbb{Z}^D$ of the lattice here instead of at a single site $j \in \mathbb{Z}^D$ as in Lemma 3.2.4 is because in the latter case the case $j \not\sim i$ is trivial and thus the sweeping out inequality is only meaningful for $\Lambda = \{\sim i\}$.

Proof. As mentioned earlier, it suffices to prove the case $n = 0$. First, rewrite the gradient into its single-site form

$$\nu\big|\nabla_\Lambda(\mathbb{E}_{\Gamma_0} f)\big|^q = \sum_{j \in \Lambda} \nu\big|\nabla_j(\mathbb{E}_{\Gamma_0} f)\big|^q . \tag{3.2.10}$$

Let $i \in \Gamma_0$ and $j \in \Lambda$. Since Γ_0 has no self-interaction, the measure $\mathbb{E}_{\Gamma_0}$ can be realised

as the product of the single-site measures $\mathbb{E}_{\Gamma_0} = \mathbb{E}_{i_1}\mathbb{E}_{i_2}\cdots$ where $(i_m)_{m\geq 1}$ enumerates Γ_0. This reduces the problem of studying $\nabla_j \mathbb{E}_{\Gamma_0}$ to the problem of studying $\nabla_j \mathbb{E}_i$ for $i \in \Gamma_0$.

Note if $\|i_m - j\| > 1$, that is if site j does not interact with site i_m, then $\nabla_j \mathbb{E}^{\omega}_{i_m} f = \mathbb{E}^{\omega}_{i_m} \nabla_j f$ together with Jensen's inequality allows us to pull $\mathbb{E}^{\omega}_{i_m}$ out, which is then eliminated by ν with the DLR equation, so the only issue are sites $i_m \in \Gamma_0$ such that $i_m \sim j$. Thinking of Γ_0 as partitioning $\mathbb{Z}^D$ into D-dimensional cubes of edge length 2, each containing exactly 2^D points in Γ_0, it follows there is at least one cube containing j. (It might be the case j falls on the boundary of two cubes; we may without loss of generality choose the cube whose centre is closest to the origin.) Denote by Q_j this cube. By construction,

$$\Gamma_0 \cap \{\sim j\} \subset \Gamma_0 \cap Q_j := \{i^j_1, i^j_2, \cdots, i^j_{2^D}\} \subset \Gamma_0$$

where $(i^j_m)_{m=1,2,\cdots,2^D}$ enumerates $\Gamma_0 \cap Q_j$. Thus we can reduce the sites in (3.2.10) to

$$\nu\left|\nabla_\Lambda(\mathbb{E}_{\Gamma_0} f)\right|^q \leq \sum_{j\in\Lambda} \nu\left|\nabla_j(\mathbb{E}_{Q_j} f)\right|^q = \sum_{j\in\Lambda} \nu\left|\nabla_j(\mathbb{E}_{i^j_{2^D}} \cdots \mathbb{E}_{i^j_2}\mathbb{E}_{i^j_1} f)\right|^q \tag{3.2.11}$$

Let us fix j and drop the superscript, that is write i_m for i^j_m. Denote by $f_0 := f$ and $f_m := \mathbb{E}_{i_m}\cdots\mathbb{E}_{i_2}\mathbb{E}_{i_1} f$ for $m = 1, 2, \cdots, 2^D$, so that $\nu\left|\nabla_j(\mathbb{E}_{\Gamma_0} f)\right|^q \leq \nu\left|\nabla_j f_{2^D}\right|^q$. By Lemma 3.2.4, we have

$$\nu\left|\nabla_j f_m\right|^q \leq K_1 \nu\left|\nabla_j f_{m-1}\right|^q + c_1 \sum_{k\in\mathbb{Z}^D} M_{k,i_m} \nu|\nabla_k f_{m-1}|^q\,. \tag{3.2.12}$$

Note that if k is sufficiently far from Q_j then site k does not interact with any of the N sites in Q_j. We can make this observation more precise: if R_j is the smallest cube strictly

containing Q_j, then for $k \notin R_j$ one has $\nu|\nabla_k f_m|^q \leq \nu|\nabla_k f|^q$ by Jensen's inequality, and

$$M_{k,i_m} \leq M_{k,Q_j} := \sup_{i \in Q_j} M_{k,i}$$

since $i_m \in Q_j$. Moreover, $M_{k,i_m} \leq 1$ by Remark 3.2.2, so

$$\nu\big|\nabla_j f_m\big|^q \leq K_1 \nu\big|\nabla_j f_{m-1}\big|^q + c_1 \sum_{k \in R_j} \nu|\nabla_k f_{m-1}|^q + c_1 \sum_{k \notin R_j} M_{k,Q_j} \nu|\nabla_k f|^q \,. \tag{3.2.13}$$

In fact, the only dependence on j in this equation comes from the fact a choice of j gives a choice of Q_j and therefore a choice of sites i_m for $m = 1, 2, \cdots, N$ with respect to which we take a conditional expectation. This means we may replace ∇_j with ∇_k for $k \in \mathbb{Z}^D$ without replacing Q_j, and thus $(i_m) = (i_m^j)$, with Q_k and (i_m^k) respectively. We can then apply Lemma 3.2.4 as before, and, again, analyse the summation through the decomposition $\mathbb{Z}^D = R_j \sqcup (R_j)^{\mathsf{c}}$ and not $\mathbb{Z}^D = R_k \sqcup (R_k)^{\mathsf{c}}$. Thus we arrive at

$$\nu|\nabla_k f_m|^q \leq K_1 \nu|\nabla_k f_{m-1}|^q + c_1 \sum_{\ell \in R_j} \nu|\nabla_\ell f_{m-1}|^q + c_1 \sum_{\ell \notin R_j} M_{\ell,Q_j} \nu|\nabla_\ell f|^q \,. \tag{3.2.14}$$

In fact we only need this for $k \in R_j$ but the statement holds nonetheless unconditionally.

We now follow an inductive procedure in the spirit of the proof of Lemma 3.2.4. Denote by

$$W_m^j := \nu\big|\nabla_j f_m\big|^q \,, \quad Y_m^j := \sum_{k \in R_j} \nu|\nabla_k f_m|^q = \sum_{k \in R_j} W_m^k, \quad S^{(j)} := \sum_{k \notin R_j} M_{k,Q_j} \nu|\nabla_k f|^q$$

so that (3.2.14) now reads

$$W_m^k \leq K_1 W_{m-1}^k + c_1 Y_{m-1}^j + c_1 S^{(j)}. \tag{3.2.15}$$

Noting $\left|R_j\right| = 5^D$, we can sum the left hand side over $k \in R_j$ to conclude

$$\begin{aligned} Y_m^j \leq \sum_{k \in R_j} (K_1 W_{m-1}^k + c_1 Y_{m-1}^j + c_1 S^{(j)}) &\leq (K_1 + 5^D c_1) Y_{m-1}^j + 5^D c_1 S^{(j)} \\ &\leq 5 Y_{m-1}^j + S^{(j)} \end{aligned} \tag{3.2.16}$$

for β small enough, since $K_1 = 4$ and $c_1 = o(1)$. Note if we use (3.2.15) inductively we find

$$\begin{aligned} W_{m-1}^j &\leq K_1 W_{m-2}^j + c_1 Y_{m-2}^j + c_1 S^{(j)} \\ &\leq c_1 Y_{m-2}^j + c_1 S^{(j)} + K_1^2 W_{m-3}^j + K_1 c_1 Y_{m-3}^j + K_1 c_1 S^{(j)} \\ &\vdots \\ &\leq K_1^{m-1} W_0^j + c_1 \sum_{i=2}^{m} K_1^{i-2} Y_{m-i}^j + c_1 \sum_{i=2}^{m} K_1^{i-2} S^{(j)} \end{aligned}$$

so that $K_1 W_{m-1}^j \leq K_1^m W_0^j + c_1 \sum_{i=1}^{m-1} K_1^i Y_{m-1-i}^j + c_1 \sum_{i=1}^{m} K_1^i S^{(j)}$. Similarly, iterating (3.2.16) gives

$$Y_m^j \leq 5^m (Y_0^j + S^{(j)}) \leq 5^N (Y_0^j + S^{(j)})$$

so that

$$\begin{aligned} \nu \left|\nabla_j f_{2^D}\right|^q = W_{2^D}^j &\leq K_1^{2^D} W_0^j + c_1 \sum_{i=0}^{2^D} K_1^i Y_{2^D-1-i}^j + c_1 \sum_{i=0}^{2^D} K_1^i S^{(j)} \\ &\leq L_1 W_0^j + c_1 L_2 Y_0 + c_1 L_2 S^{(j)} \end{aligned} \tag{3.2.17}$$

for $L_1 = K_1^{2^D}$ and $L_2 = (2^D + 1)(5K_1)^{2^D}$. If we take this back into the summation (3.2.11),

we find

$$\nu\big|\nabla_\Lambda(\mathbb{E}_{\Gamma_0}f)\big|^q \le \sum_{j\in\Lambda}\left(L_1\nu\big|\nabla_j f\big|^q + c_1L_2\sum_{k\in R_j}\nu|\nabla_k f|^q + c_1L_2\sum_{k\notin R_j}M_{k,Q_j}\nu|\nabla_k f|^q\right)$$

where $M_{k,Q_j} := \sup_{i\in Q_j} M_{k,i}$. Let us denote by I_1, I_2, I_3 respectively the three terms.

I_1, by definition, is $L_1\nu|\nabla_\Lambda f|^q$, while estimating I_2 boils down to a counting exercise: if $O_k = \{j \in \mathbb{Z}^D \mid k \in R_j\}$ then $|S_k|$ is finite and bounded above uniformly by say L_3. Indeed, if j is sufficiently far from k then R_j is far from k also, that is there is a finite neighbourhood of k which contains O_k. It follows

$$I_2 = c_1L_2\sum_{j\in\Lambda}\sum_{k\in R_j}\nu|\nabla_k f|^q \le c_1L_2\sum_{k\in\mathbb{Z}^D}\sum_{j\in O_k}\nu|\nabla_k f|^q \le c_1L_2L_3\nu|\nabla f|^q\,. \tag{3.2.18}$$

To estimate I_3, we expand the summation from $j \in \Lambda$ and $k \in R_j$ to $j, k \in \mathbb{Z}^D$, and following the arguments for I_2, we conclude

$$\sum_{j\in\mathbb{Z}^D} M_{k,Q_j} \le \sum_{\substack{j\in\mathbb{Z}^D\\ \ell\in Q_j}} M_{k,\ell} \le \sum_{\substack{j\in\mathbb{Z}^D\\ \ell\in R_j}} M_{k,\ell} = \sum_{\substack{\ell\in\mathbb{Z}^D\\ j\in O_\ell}} M_{k,\ell} \le L_3\sum_{\ell\in\mathbb{Z}^D} M_{k,\ell} \le L_3L_0 =: L_4$$

for $L_0 = \sup_{i\in\mathbb{Z}^D}\sum_{j\in\mathbb{Z}^D} M_{ij}$ the supremum of **(B2)**. It follows

$$\begin{aligned} I_3 &\le c_1L_2\sum_{j\in\mathbb{Z}^D}\sum_{k\in\mathbb{Z}^D} M_{k,Q_j}\nu|\nabla_k f|^q \\ &\le c_1L_2\sum_{k\in\mathbb{Z}^D}\nu|\nabla_k f|^q\sum_{j\in\mathbb{Z}^D} M_{k,Q_j} \le c_1L_2L_4\nu|\nabla f|^q\,. \end{aligned} \tag{3.2.19}$$

Since $|\nabla f|^q = |\nabla_\Lambda f|^q + |\nabla_{\Lambda^c} f|^q$, by collecting the estimates $I_1 = L_1\nu|\nabla_\Lambda f|^q$ and (3.2.18), (3.2.19), this proves the lemma with $K_2 = L_1 + c_1L_2(L_3 + L_4)$ and $c_2 = c_1L_2(L_3 + L_4)$. More concretely for β small enough we may take $K_2 = 4L_1 = 4^{2^D+1}$ and $c_2 = \beta^{(q-1)/8}$

since $c_2 = \mathcal{O}(c_1) = \mathcal{O}(\beta^{(q-1)/4})$. □

The next problem is to use the sweeping out inequality on the sub-lattice to transfer the weak single-site q-SGI to a weak sub-lattice q-SGI; in particular we want to show the weak q-SGI holds for each of the 2^D sub-lattices Γ_n.

Lemma 3.2.9. *For any $n \in \{0, 1, \cdots, 2^D - 1\}$, the following weak sub-lattice q-SGI*

$$\nu \mathbb{E}^{\omega}_{\Gamma_n} \left| f - \mathbb{E}^{\omega}_{\Gamma_n} f \right|^q \leq C_0 \nu |\nabla f|^q \tag{3.2.20}$$

holds for some constant $C_0 > 0$ independently of n such that $C_0 = \mathcal{O}(\beta^{-1/2})$.

Note that this is still a weak q-SGI as the right-hand side contains $|\nabla f|^q$ instead of $|\nabla_{\Gamma_n} f|^q$; but unlike the weak q-SGI of **(A2)** the interaction matrix M_{ki} is conspicuously absent. In a sense this reflects the fact the sub-lattices infiltrate the lattice so that the distance between an arbitrary point and the sub-lattice is bounded above.

Proof. We again consider only the case $i = 0$, the other cases being similar. Let $(i_m)_{m \geq 1}$ enumerate Γ_0, and denote as before $f_0 := f$ and $f_m = \mathbb{E}_{i_m} \cdots \mathbb{E}_{i_2} \mathbb{E}_{i_1} f$, so that $\mathbb{E}_{\Gamma_0} f = f_\infty$. By the DLR equation and a simple telescoping argument it can be shown that the following interpolation identity holds:

$$\nu \mathbb{E}_{\Gamma_0} \left| f - \mathbb{E}_{\Gamma_0} f \right|^q = \sum_{m=0}^{\infty} \nu \left(\mathbb{E}^{\omega}_{i_{m+1}} \left| f_m - \mathbb{E}_{\Gamma_0} f \right|^q - \left| \mathbb{E}^{\omega}_{i_{m+1}} (f_m - \mathbb{E}_{\Gamma_0} f) \right|^q \right).$$

(In particular the negative part of the summand at m cancels with the positive part of the summand at $m+1$ so that as $m \to \infty$ only the $\nu \mathbb{E}^{\omega}_{i_1} \left| f_0 - \mathbb{E}_{\Gamma_0} f \right|^q = \nu \mathbb{E}_{\Gamma_0} \left| f - \mathbb{E}_{\Gamma_0} f \right|^q$ part survives.) To estimate the difference, we use the following sub-quadratic inequality

$$|a|^q - |b|^q \leq \operatorname{sgn}(b) q |b|^{q-1} (a - b) + 2|a - b|^q \tag{3.2.21}$$

which holds for all $a, b \in \mathbb{R}$ and $q \in (1, 2]$ (see Lemma A.1.1). We can take $a = f$ and $b = \mathbb{E}_i^\omega f$, and integrate with respect to $\mathbb{E}_i^\omega$ and obtain

$$\mathbb{E}_i^\omega |f|^q - |\mathbb{E}_i^\omega f|^q \leq 2\mathbb{E}_i^\omega |f - \mathbb{E}_i^\omega f|^q \,. \tag{3.2.22}$$

(The first term vanishes since $\mathbb{E}_i^\omega f$ is a constant, and so $\mathbb{E}_i^\omega (f - \mathbb{E}_i^\omega f) = 0$.) Using this bound, we obtain

$$\nu \big| f - \mathbb{E}_{\Gamma_0} f \big|^q \leq 2 \sum_{m=0}^{\infty} \nu \mathbb{E}_{i_{m+1}}^\omega \Big| f_m - \mathbb{E}_{i_{m+1}}^\omega f_m \Big|^q \,. \tag{3.2.23}$$

Applying **(A2)** to each summand, we find

$$\begin{aligned} \nu \mathbb{E}_{i_{m+1}}^\omega \Big| f_m - \mathbb{E}_{i_{m+1}}^\omega f_m \Big|^q &\leq A \sum_{j \in \mathbb{Z}^D} M_{j, i_{m+1}} \nu \big| \nabla_j f_m \big|^q \\ &\leq A \sum_{j \in \Gamma_0} M_{j, i_{m+1}} \nu \big| \nabla_j f \big|^q + A \sum_{j \in (\Gamma_0)^{\mathsf{c}}} M_{j, i_{m+1}} \nu \big| \nabla_j f_m \big|^q \,, \end{aligned} \tag{3.2.24}$$

since there is no self-interaction in Γ_0 and f_m depends only on coordinates in $(\Gamma_0)^{\mathsf{c}}$ so that $\nabla_j f_m = (\nabla_j f)_m$ whenever $j \in \Gamma_0$. (Jensen's inequality then provides the final simplification.) For the summation over $(\Gamma_0)^{\mathsf{c}}$, we apply the recursion (3.2.17) on each $\nu \big| \nabla_j f_m \big|$ which gives

$$\nu \big| \nabla_j f_m \big|^q \leq L_1 \nu \big| \nabla_j f \big|^q + c_1 \sum_{k \in R_j} \nu |\nabla_k f|^q + c_1 L_2 \sum_{k \notin R_j} M_{k, Q_j} \nu |\nabla_k f|^q \tag{3.2.25}$$

(Note $m \to \infty$ here but recall there is a cube Q_j on which $\nu |\nabla_j (\mathbb{E}_{\Gamma_0} f)|^q \leq \nu |\nabla_j (\mathbb{E}_{Q_j} f)|^q$.)

We can insert (3.2.25) into (3.2.24) and again into (3.2.23) to obtain

$$\begin{aligned}\frac{1}{2A}\nu\big|f-\mathbb{E}_{\Gamma_0 f}\big|^q \le L_1\sum_{m=0}^{\infty}\sum_{j\in\mathbb{Z}^D} M_{j,i_{m+1}}\nu\big|\nabla_j f\big|^q \\ + c_1L_2\sum_{m=0}^{\infty}\sum_{j\in(\Gamma_0)^{\mathsf{c}}} M_{j,i_{m+1}}\sum_{k\in R_j}\nu|\nabla_k f|^q \\ + c_1L_2\sum_{m=0}^{\infty}\sum_{j\in(\Gamma_0)^{\mathsf{c}}} M_{j,i_{m+1}}\sum_{k\notin R_j} M_{k,Q_j}\nu|\nabla_k f|^q\end{aligned}$$

since $L_1 > 1$.

Let us denote by I_1, I_2, I_3 respectively the three terms. We note as in the previous proof that $\sum_{m=0}^{\infty} M_{j,i_{m+1}} \le \sum_{j\in\mathbb{Z}^D} M_{i,j} \le L_0$ uniformly in j by **(B2)** since (M_{ij}) is symmetric so column sums coincide with row sums, so I_1 is estimated by $I_1 \le L_1L_0\nu|\nabla f|^q$. I_2 and I_3 can be estimated with the arguments in the previous proof; the latter is entirely analogous to (3.2.18) and (3.2.19) but with the extra appearance of $\sum_{m=0}^{\infty} M_{j,i_{m+1}} \le L_0$ so

$$\begin{aligned}I_2 &\le c_1L_0L_2L_3\nu|\nabla f|^q\,,\\ I_3 &\le c_1L_0L_2L_4\nu|\nabla f|^q\,.\end{aligned}$$

Collecting everything, we arrive at the weak q-SGI

$$\nu\mathbb{E}_{\Gamma_0}\big|f-\mathbb{E}_{\Gamma_0 f}\big|^q \le C_0|\nabla f|^q$$

with $C_0 = 2A(L_1 + c_1L_0L_2(L_3+L_4)) = \mathcal{O}(A) = \mathcal{O}(\beta^{-1/2})$. □

Having obtained a weak q-SGI for the sub-lattices, we give a final argument to show this implies the (strong) q-SGI on the entire lattice.

Lemma 3.2.10. *Let* $\Pi := \mathbb{E}_{\Gamma_{N-1}} \cdots \mathbb{E}_{\Gamma_0} \mathbb{E}_{\Gamma_0}$. *Then* Π *satisfies the following four conditions:*

(1) Π *satisfies the DLR equation*

$$\nu f = \nu(\Pi f).$$

(2) *There exists a constant* C_1 *such that*

$$\nu|f|^q - \nu|\Pi f|^q \leq C_1 \nu|\nabla f|^q \,,$$

(3) *There exists* $\xi \in (0,1)$ *such that for any* $m \in \mathbb{Z}_{\geq 0}$ *one has*

$$\nu\left|\nabla(\Pi^m f)\right|^q \leq \xi^m \nu|\nabla f|^q \,.$$

(4) *The following limit*

$$\lim_{m\to\infty} \Pi^m f = \nu f$$

holds ν*-almost surely.*

Proof. **(1)** follows immediately from the DLR equation. To prove **(2)**, denote by $f_{-1} = f$ and $f_n = \mathbb{E}_{\Gamma_n} \cdots \mathbb{E}_{\Gamma_1} \mathbb{E}_{\Gamma_0} f$, for $n = 0, 1, \cdots, 2^D - 1$. (This is a departure from the usual notation since the index of the sub-lattices starts at zero to distinguish the index from their points, which starts at one.) By applying the sub-quadratic inequality (3.2.21) and the weak sub-lattice q-SGI (3.2.20) we obtain

$$\begin{aligned}\nu|f|^q - \nu|\Pi f|^q &= \sum_{n=0}^{2^D-1} \nu\mathbb{E}_{\Gamma_n}\left(|f_{n-1}|^q - |\mathbb{E}_{\Gamma_n} f_{n-1}|^q\right) \leq 2\sum_{n=0}^{2^D-1} \nu\mathbb{E}_{\Gamma_n}|f_{n-1} - \mathbb{E}_{\Gamma_n} f_{n-1}|^q \\ &\leq 2C_0 \sum_{n=0}^{2^D-1} \nu|\nabla f_{n-1}|^q = 2C_0\nu|\nabla f|^q + 2C_0 \sum_{n=0}^{2^D-2} \nu\left|\nabla_{(\Gamma_n)^{\mathsf{c}}}(\mathbb{E}_{\Gamma_n} f_{n-1})\right|^q\end{aligned}$$

since $f_n = \mathbb{E}_{\Gamma_n} f_{n-1}$ depends only on coordinates in $(\Gamma_n)^{\mathsf{c}}$. The first term requires no

further estimation, whilst we can apply Lemma 3.2.7 inductively on the second term to obtain

$$\nu\left|\nabla_{(\Gamma_n)^c}(\mathbb{E}_{\Gamma_n} f_{n-1})\right|^q \leq K_2^{n+1}\nu|\nabla f|^q$$

which gives

$$\nu|f|^q - \nu|\Pi f|^q \leq C_1\nu|\nabla f|^q$$

which is exactly **(2)** with $C_1 = 2C_0\sum_{n=0}^{2^D-1}(K_2)^n = 2C_0\frac{(K_2)^{2^D}-1}{K_2-1} \leq 2C_0(K_2)^{2^D}$ since $K_2 = 4^{2^D+1} > 2$.

To prove **(3)**, we define $\Gamma^{(m)} = \cup_{n=0}^m \Gamma_n$ and apply Lemma 3.2.7 again to obtain

$$\nu\left|\nabla_{\Gamma^{(m)}} f_m\right|^q = \nu\left|\nabla_{\Gamma^{(m-1)}} f_m\right|^q \leq K_2\nu\left|\nabla_{\Gamma^{(m-1)}} f_{m-1}\right|^q + c_2\nu|\nabla f_{m-1}|^q\,.$$

Applying Lemma 3.2.7 as in the proof for **(2)**, we obtain

$$\nu\left|\nabla_{\Gamma^{(m)}} f_m\right|^q \leq K_2\nu\left|\nabla_{\Gamma^{(m-1)}} f_{m-1}\right|^q + c_2K_2^m\nu|\nabla f|^q\,. \tag{3.2.26}$$

This equation can be inductively applied on the first term, that is applying (3.2.26) on $\nu\left|\nabla_{\Gamma^{(m-1)}} f_{m-1}\right|^q$ and starting from $m = 2^D - 1$ we find

$$\nu\left|\nabla(\Pi f)\right|^q \leq (K_2)^{2^D}\nu\left|\nabla_{\Gamma_0} f_0\right|^q + c_2(K_2)^{2^D}\nu|\nabla f|^q = c_2(K_2)^{2^D}\nu|\nabla f|^q$$

since $\nabla_{\Gamma_0} f_0 = \nabla_{\Gamma_0}\mathbb{E}_{\Gamma_0} f = 0$. Since $K_2 = \mathcal{O}(1)$ and $c_2 = o(1)$, we obtain **(3)** at $m = 1$ with $\xi = c_2(K_2)^{2^D}$ with β small enough; the general case for $m \in \mathbb{Z}_{\geq 1}$ follows by replacing f by Πf and proceeding by straightforward induction.

To prove the final statement **(4)**, following the arguments of **(2)** we first bound the

q-variance with respect to Π by

$$\nu|f-\Pi f|^q \leq 2^{D(q-1)} \sum_{n=0}^{2^D-1} \nu \mathbb{E}_{\Gamma_n} |f_{n-1} - \mathbb{E}_{\Gamma_n} f_{n-1}|^q \leq 2^{D(q-1)} C_1 \nu |\nabla f|^q$$

using the Hölder inequality $(\sum_{i=1}^{2^D} |x_i|)^q \leq 2^{D(q-1)} \sum_{i=1}^{2^D} |x_i|^q$ for non-negative (x_i) on the telescoping sum $f - \Pi f = \sum_{n=0}^{2^D-1} f_{n-1} - f_n$. For $m \in \mathbb{Z}_{\geq 1}$, replacing f by $\Pi^m f$ in the above estimate we obtain

$$\nu \left| \Pi^m f - \Pi^{m+1} f \right|^q \leq 2^{D(q-1)} C_1 \nu \left| \nabla(\Pi^m f) \right|^q \leq 2^{D(q-1)} C_1 \xi^m \nu |\nabla f|^q \,.$$

Since $\xi < 1$, the Borel-Cantelli lemma implies Π^m converges ν-almost surely to some function g, and similarly $\left|\nabla(\Pi^m f)\right|$ converges ν-almost surely to zero by **(3)**, so that g is constant ν-almost everywhere. Note the decay of C_1 with respect to β does not matter since β is sufficiently small, but fixed. Assuming that f is bounded, by dominated convergence we have

$$\lim_{m\to\infty} \Pi^m f = g = \nu(g) = \lim_{m\to\infty} \nu(\Pi^m f) = \nu(f)$$

with the last equality obtained by **(1)**. The general case of possibly unbounded $f \in W^{1,q}(\nu)$ is obtained by density. □

We can now prove the main theorem of this section.

Proof of Theorem 3.2.1. Denote by $f_m = \Pi^m f$ for $m \in \mathbb{N}$, we have

$$\begin{aligned} \nu|f-\nu f|^q &= \sum_{n=0}^{\mathcal{N}-1} \left(\nu|f_n - \nu f|^q - \nu \left| \Pi(f_n - \nu f) \right|^q \right) + \nu|f_{\mathcal{N}} - \nu f|^q \\ &\leq C_1 \sum_{n=0}^{\mathcal{N}-1} \nu|\nabla f_n|^q + \nu|f_{\mathcal{N}} - \nu f|^q \end{aligned}$$

by **(1)** and **(2)** of Lemma 3.2.10. Applying **(3)** for the term $\nu|\nabla f_n|^q$, it follows

$$\nu|f-\nu f|^q \leq C_1\left(\sum_{n=0}^{\mathcal{N}-1}\xi^n\right)\nu|\nabla f|^q+\nu|f_{\mathcal{N}}-\nu f|^q \leq \frac{C_1}{1-\xi}\nu|\nabla f|^q+\nu|f_{\mathcal{N}}-\nu f|^q\,.$$

Taking $\mathcal{N}\to\infty$, the remainder term satisfies $\nu|f_{\mathcal{N}}-\nu f|^q\to 0$ by **(4)** which gives the global q-SGI

$$\nu|f-\nu f|^q \leq C_{SG}\nu|\nabla f|^q$$

with $C_{SG}=C_1/(1-\xi)$. As for the uniqueness of ν, it follows from (4) in Lemma 3.2.10. □

3.2.1 Generalisations to interactions of finite range R

The analysis can be generalised to the case where the interaction has finite range $R>1$, meaning $\beta_{ij}\equiv 0$ for $\|i-j\|>R$. No modification to the main theorem in this case is necessary: one modifies the definition $j\sim i$ to mean $1\leq\|j-i\|\leq R$, i.e. j is a neighbour when j is allowed to interact with i, and much of the analysis is left unchanged modulo changes to $\mathcal{O}(1)$ estimates. In Lemma 3.2.4, the only dependence on the range of the interaction (beyond the fact the definition $j\sim i$ depends on the range) appears in the definition of ε. When $R=1$, the number of neighbours of $i\in\mathbb{Z}^D$ is exactly $2D$; when $R>1$ the number of neighbours grows like R^D but for fixed R is finite, so that we can replace $\varepsilon=2Da_2$ with $\varepsilon=N_Ra_2$ where $N_R=\big|\{\sim i\}\big|$ is the number of neighbours of i.

A generalisation of the sub-lattices $(\Gamma_n)_{n=0}^{N-1}$ is also straightforward: it is easy to see the base cube $\Gamma_0=(R+1)\mathbb{Z}^D$ has no self-interaction and that there are $N_1=(R+1)^D$ translations of Γ_0 shifting the base point 0 to another point in $[0,R]^D\subset\mathbb{Z}^D$ partition $\mathbb{Z}^D$. Note one needs to make a distinction between the number N_1 of sub-latties in our partition and the number $N_2=2^D$ of vertices in a D-dimensional cube, which coincide for $R=1$. In any case, only N_2 has a role to play in the proof of Lemmas 3.2.7 and 3.2.9, where the

sub-lattices are fixed (and so the number of sub-lattices is not relevant) so we only need to change the estimate on $\left|R_j\right| = (2R+3)^D = \mathcal{O}(1)$, which makes no difference to the validity of the proof because its influence is only felt in the $\mathcal{O}(1)$ constants L_2, L_3, L_4. Similarly the N appearing in the proof of **(3)** and **(4)** in Lemma 3.2.10 is actually N_1 and not N_2, but again its influence is $\mathcal{O}(1)$ and can be dealt with by taking $c_2 = o(1)$ sufficiently small.

3.3 Sufficient conditions on the phase and interaction

Having now given general conditions - the existence of a weak single-site U-bound and a weak single-site q-SGI - which guarantee the existence of a global q-SGI, we now consider conditions on the phase $\phi \in C^2(\mathbb{G})$ and interaction $V \in C^2(\mathbb{G} \times \mathbb{G})$ appearing in (3.1.2) such that the conditions of Theorem 3.2.1 hold. The first will be a (strong) single-site U-bound without interaction in terms of a general function η, and the second will place conditions on the interaction V in terms of η.

Theorem 3.3.1. *Let $q \in (1, 2]$, and let $\{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ be the local specification for a potential given in (3.1.2) with interaction range $R = 1$. Suppose the following two conditions hold on ϕ and V for sufficiently small $\beta > 0$:*

(C1) *There exist a bicompact*[1] *function $\eta \in C(\mathbb{G}, [0, \infty))$ and a constant $B > 0$ such that*

$$\int |g|^q \eta(h) e^{-\phi(h)} dh \leq B \int \left(|\nabla g|^q + |g|^q \right) e^{-\phi(h)} dh$$

for all $g \in C(\mathbb{G})$ for which the left-hand side is finite, and

[1] A function $f : \Omega \to [0, \infty)$ is bicompact if the set $\{x \in \Omega : f(x) \leq c\}$ is compact for any $c \in [0, \infty)$

(C2) *the interaction V satisfies the gradient bound*

$$\left|\nabla_{h_1} V(h_1, h_2)\right|^q + \left|\nabla_{h_2} V(h_1, h_2)\right|^q \leq B(1 + \eta(h_1) + \eta(h_2)).$$

Then for all sufficiently small $\beta > 0$, there exists a unique global Gibbs measure ν associated to $\{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ that satisfies the q-SGI

$$\nu|f - \nu f|^q \leq C_{SG} \nu |\nabla f|^q$$

for all $f \in W^{1,q}(\nu)$ such that $\sup_{i \in \mathbb{Z}^D} \nu(\eta(x_i)|f|^q) < \infty$ with some constant $C_{SG} > 0$.

Note that although this setup allows for more general U-bounds, the generic case is when the phase, the interaction, and the U-bound are all powers of a homogeneous norm. For instance, in case of the control distance d, [6] proves the q-logarithmic Sobolev inequality with $\phi = d^p$, $V(x, y) = (d(x) \pm d(y))^2$, and $\eta = d^p$, for all $p \geq 2$. This setup allows for more general U-bound functions η which may exist outside the generic case and asserts the only requirement of η is that it controls the sub-gradient of the interaction in both coordinates up to a constant.

The main problem in the proof is to show that **(C1)** and **(C2)**, which we assume hold in the sequel, imply the existence of a weak q-SGI. Since $\left|\nabla_j V\right|$ is controlled by η, the weak U-bound of **(A1)** follows immediately, and the difficulty lies with **(A2)**. In [6, Lemma 4.3] it was shown one can pass from a strong single-site U-bound to the strong q-SGI. Here we show the statement remains true with "weak" replacing "strong".

The first step in establishing the weak U-bound for η is to prove a weak U-bound for η_i that is defective in the sense the right-hand side contains copies of the weak U-bound for neighbouring η_j. We will then approach in the spirit of the proofs in Section 3.2 whereby we prove that inductively inserting the U-bound into these defective copies "converges"

appropriately. It will also be noted as a corollary that this weak U-bound will lead to the existence of the global Gibbs measure ν by a criterion of Dobrushin. As for the uniqueness of ν, it follows from both the weak U-bound and q-SGI as a result of Theorem 3.2.1. In the sequel, we write for notational convenience $\eta_i \coloneqq \eta(x_i)$ the U-bound function η in the x_i-coordinate, and similarly $\phi_i \coloneqq \phi(x_i)$, and $V_{ij} \coloneqq V(x_i, x_j)$ with the understanding $x_j = \omega_j$ whenever x_j is left unspecified.

Lemma 3.3.2. *Assume for $q \in [1, \infty)$ we have*

$$\beta^q \equiv \sup_i \sum_{j \neq i} \beta_{ij}^q < \infty \quad \textit{and} \quad 2^{q-1} B^2 \beta^q < 1.$$

Then for all $i \in \mathbb{Z}^D$, it follows

$$\mathbb{E}_i^\omega(\eta_i |f|^q) \leq K_1 \big(\mathbb{E}_i^\omega |\nabla_i f|^q + \mathbb{E}_i^\omega |f|^q \big) + c_1 \sum_{j \neq i} \beta_{ij}^q \, \mathbb{E}_i^\omega(\eta_j |f|^q) \tag{3.3.1}$$

with

$$K_1 = \frac{\max\big\{2^{q-1}B,\, B(1 + 2^{q-1} B \beta^q)\big\}}{(1 - 2^{q-1} B^2 \beta^q)} \quad \textit{and} \quad c_1 = \frac{2^{q-1} B^2}{(1 - 2^{q-1} B^2 \beta^q)}.$$

Hence for $f \equiv 1$, we have

$$\mathbb{E}_i^\omega(\eta_i) \leq K_1 + c_1 \sum_{j \neq i} \beta_{ij}^q \, \eta_j(\omega). \tag{3.3.2}$$

Proof. Let f be compactly supported and differentiable. Since $V \in C^2(\mathbb{G} \times \mathbb{G})$, it follows that V_{ij}, $\nabla_i V_{ij}$ and $\nabla_j V_{ij}$ are almost everywhere bounded on compact sets, then we can

replace g with $fe^{-\frac{1}{q}\sum_{j\neq i}\beta_{ij}V_{ij}}$ in **(C1)** to obtain

$$\mathbb{E}_i^\omega(\eta_i|f|^q) \leq B\int\left|\nabla_i f - \frac{1}{q}f\sum_{j\neq i}\beta_{ij}\nabla_i V_{ij}\right|^q e^{-U_i^\omega}dx_i + B\int|f|^q\,e^{-U_i^\omega}dx_i$$

$$\leq 2^{q-1}B\int|\nabla_i f|^q\,e^{-U_i^\omega}dx_i + 2^{q-1}B\sum_{j\neq i}\beta_{ij}^q\int|f|^q|\nabla_i V_{ij}|^q\,e^{-U_i^\omega}dx_i + B\int|f|^q\,e^{-U_i^\omega}dx_i,$$

that is

$$\mathbb{E}_i^\omega(\eta_i|f|^q) \leq 2^{q-1}B\,\mathbb{E}_i^\omega|\nabla_i f|^q + 2^{q-1}B\sum_{j\neq i}\beta_{ij}^q\mathbb{E}_i^\omega(|f|^q|\nabla_i V_{ij}|^q) + B\,\mathbb{E}_i^\omega|f|^q\,.$$

Applying **(C2)** to the second term, we have

$$\mathbb{E}_i^\omega(\eta_i|f|^q) \leq 2^{q-1}B\,\mathbb{E}_i^\omega|\nabla_i f|^q + 2^{q-1}B^2\sum_{j\neq i}\beta_{ij}^q\,\mathbb{E}_i^\omega(|f|^q\,(1+\eta_i+\eta_j)) + B\,\mathbb{E}_i^\omega|f|^q\,.$$

and after rearranging, with $\beta^q = \sup_i\sum_{j\neq i}\beta_{ij}^q$, we get

$$(1-2^{q-1}B^2\beta^q)\mathbb{E}_i^\omega(\eta_i|f|^q) \leq 2^{q-1}B\,\mathbb{E}_i^\omega|\nabla_i f|^q + 2^{q-1}B^2\sum_{j\neq i}\beta_{ij}^q\,\mathbb{E}_i^\omega(\eta_j|f|^q)$$
$$+ B(1+2^{q-1}B\beta^q)\,\mathbb{E}_i^\omega|f|^q\,.$$

Now this relation can be generalised to functions which are not compactly supported. In this way the relation (3.3.1) follows. The relation (3.3.2) is the special case for $f\equiv 1$.

□

Although the U-bounds were introduced in [5] to obtain coercive inequalities on general metric spaces, it turns out the U-bounds also play a crucial role in Dobrushin's existence theorem (for the existence of global Gibbs measures). In the following, we introduce an adapted version of this theorem to fit in our setting.

Theorem 3.3.3 ([47, Theorem 1]). *Let $\{\mathbb{E}_\Lambda\}_{\Lambda\Subset\mathbb{Z}^D}$ be a local specification that satisfies the following two conditions:*

(1) *There exist a bicompact*[2] *function $h:\mathbb{G}\to[0,\infty)$, and constants $C\geq 0$, $c_{ij}\geq 0$ such that*

$$\mathbb{E}_i^\omega(h(x_i))\leq C+\sum_{j\in\mathbb{Z}^D}c_{ij}h(\omega_j)\quad \text{and}\quad \sum_{j\in\mathbb{Z}^D}c_{ij}<1$$

for all $i\in\mathbb{Z}^D$ and $\omega\in\mathbb{G}^{\mathbb{Z}^D}$.

(2) *For any $i\in\mathbb{Z}^D$, there exist constants $D_n\geq 0$, $d_{ij}^{(n)}\geq 0$ with $n=1,2,\ldots$ such that $D_n\to 0$ as $n\to\infty$ and*

$$\sum_{j\in\mathbb{Z}^D}d_{ij}^{(n)}\leq D_n;$$

moreover, there exists a sequence of finite volumes $\Lambda_i^1\subset\Lambda_i^2\subset\cdots$ whose union is $\mathbb{Z}^D$, such that for any function $g\in C(\mathbb{G},[-1,1])$ there exists a sequence functions $f_n\in C\big(\mathbb{G}^{\Lambda_i^n},\mathbb{R}\big)$ such that

$$\left|\mathbb{E}_i^\omega(g(x_i))-f_n(\omega_j,j\in\Lambda_i^n)\right|\leq D_n+\sum_{j\in\mathbb{Z}^D}d_{ij}^{(n)}h(x_j).$$

Then there exists a global Gibbs measure ν associated to the local specification $\{\mathbb{E}_\Lambda\}_{\Lambda\Subset\mathbb{Z}^D}$.

Corollary 3.3.4. *Let $\{\mathbb{E}_\Lambda\}_{\Lambda\Subset\mathbb{Z}^D}$ be a local specification that satisfies the conditions in Theorem 3.3.1, then there exists a global Gibbs measure ν associated to $\{\mathbb{E}_\Lambda\}_{\Lambda\Subset\mathbb{Z}^D}$.*

Proof. It is sufficient to verify the two conditions in Theorem 3.3.3. Condition (1) is simply the weak U-bound (3.3.2) with $C=K_1$ and $c_{ij}=c_1\beta_{ij}^q$. Since $\beta_{ij}\equiv 0$ for $\|i-j\|>1$ and $c_1\to 0$ as $\beta\to 0$, the condition $\sum_j c_{ij}<1$ can be satisfied for sufficiently small

[2] See footnote 1

$\beta = \sup_{i,j} |\beta_{i,j}$, which is one of the conditions in Theorem 3.3.1. For condition (2), we can choose $\Lambda_i^n = \{j \in \mathbb{Z}^D : \|i-j\| \le n\}$ and $f_n(\omega_j, j \in \Lambda_i^n) = \mathbb{E}_i^{\omega_0}(g)$ for all $n = 1, 2, ...$, where the boundary conditions ω_0 is given by $(\omega_0)_j = \omega_j$ for $\|i-j\| \le 1$ and 0 otherwise. Then it follows $\mathbb{E}_i^{\omega}(g) - f_n = \mathbb{E}_i^{\omega}(g) - \mathbb{E}_i^{\omega_0}(g) = 0$ for all ω since the boundary conditions ω and ω_0 coincide on $\{j \in \mathbb{Z}^D : \|i-j\| \le 1\}$. This means that the left-hand side of the inequality in condition (2) is 0, hence the inequality simply holds by choosing $D_n = d_{ij}^{(n)} \equiv 0$. □

Remark 3.3.5. Although Theorem 3.3.3 was introduced for systems with infinite-range interactions, it can be greatly simplified in our setting. Indeed, for a finite-range potential given by (3.1.2), condition (2) always holds by the same argument in the above proof, and in condition (1) the sum $\sum_j c_{ij}$ is finite and can be made smaller than 1 with small β. Hence the key is whether one has a weak U-bound for the single-site measure $\mathbb{E}_i$. This relates U-bound with the existence of global Gibbs measures.

We now continue with the proof of Theorem 3.3.1. As mentioned earlier, the next step uses the inductive arguments of Section 3.2 to show the defective term $c_1 \sum_{j:j\sim i} \nu(\eta_j |f|^q)$ can be controlled.

Lemma 3.3.6. *For all $i \in \mathbb{Z}^D$, the following weak single-site U-bound*

$$\nu(\eta_i |f|^q) \le K_2 \left(\nu|f|^q + \sum_{k \in \mathbb{Z}^D} c_2^{\|k-i\|} \nu|\nabla_k f|^q \right) \tag{3.3.3}$$

holds for some constants $K_2 = \mathcal{O}(1)$ and $c_2 = o(1)$.

Proof. For notational convenience, denote by

$$W_i := \nu(\eta_i |f|^q), \quad Y_i := \nu|\nabla_i f|^q, \quad Z = \nu|f|^q$$

so that (3.3.1) reads

$$W_i \leq K_1 Y_i + K_1 Z + c_1 \sum_{k \sim i} W_k \tag{3.3.4}$$

whilst (3.3.3) reads $W_i \leq K_2 Z + K_2 \sum_{k \in \mathbb{Z}^D} c^{\|k-i\|} Y_k$. The difference between the induction argument here and that which was given in Section 3.2 is that in the previous case the left-hand side depended on j while the defective on the right-hand side was a summation over a set which depended on i. This is in contrast to the current case where the left-hand side *also* depends on i. This means that in the first case only terms depending on the neighbours of i appear whereas when we apply (3.3.4) to each W_k in the sum we expect terms depending on neighbours of neighbours of i to appear, and iterating this procedure we expect the neighbours of their neighbours, and so on so forth - eventually each term $\nu|\nabla_k f|^q$ should appear.

Indeed, let us look at

$$\sum_{k \sim i} W_k \leq 2DK_1 Z + K_1 \sum_{k \sim i} Y_k + c \sum_{k \sim i} \sum_{\ell \sim k} W_\ell, \tag{3.3.5}$$

and in general if the nested summation is denoted by

$$\mathcal{S}_M(W_i) \coloneqq \sum_{k_1 \sim i} \sum_{k_2 \sim k_1} \cdots \sum_{k_M \sim k_{M-1}} W_{k_M},$$

then

$$\mathcal{S}_M(W_i) \leq K_1 \mathcal{S}_M(Z) + K_1 \mathcal{S}_M(Y_i) + c\mathcal{S}_{M+1}(W_i). \tag{3.3.6}$$

where $\mathcal{S}_M(Z)$ and $\mathcal{S}_M(Y_i)$ are defined similarly. The first term $\mathcal{S}_M(Z)$ is easily handled, since Z has no dependence on any indices so

$$\mathcal{S}_M(Z) = \sum_{k_1 \sim i} \sum_{k_2 \sim k_1} \cdots \sum_{k_M \sim k_{M-1}} Z = (2D)^M Z.$$

The second term $\mathcal{S}_M(Y_i)$ requires some counting: the summation contains, for instance, terms Y_k for which $\|k-i\| = M$, but these terms can only occur in the innermost summation, of which there are $2D$ terms, so a crude estimate is that the contribution to $\mathcal{S}_M(Y_i)$ of those Y_k for which $\|k-i\| = M$ is controlled by $2D\sum_{\|k-i\|=M} Y_k$. Similarly, terms Y_k for which $\|k-i\| = M-1$ can only occur in the two innermost summations, of which there are $(2D)^2$ terms, and so on so forth; continuing in this way we find

$$\mathcal{S}_M(Y_i) \leq \sum_{n=0}^{M} \sum_{\|k-i\|=n} (2D)^{M-n} Y_k \leq (2D)^M \sum_{n=0}^{M} \sum_{\|k-i\|=n} Y_k. \tag{3.3.7}$$

The term $\mathcal{S}_{M+1}(W_i)$ is handled by the assumption $\sup_{i\in\mathbb{Z}^D} \nu(\eta_i|f|^q) = \sup_{i\in\mathbb{Z}^D} W_i < \infty$, implying each $W_k \leq L$ for some constant $L > 0$ uniformly in k. We can then crudely estimate the number of terms by $(2D)^{M+1}$ so that $\mathcal{S}_{M+1}(W_i) \leq (2D)^{M+1}L$.

With $\mathcal{S}_0(X) = X$ where X is one of W_i, Y_i, Z, we find

$$\begin{aligned}
W_i &\leq K_1\mathcal{S}_0(Y_i) + K_1\mathcal{S}_0(Z) + c_1\mathcal{S}_1(W_i) \\
&\leq K_1\mathcal{S}_0(Y_i) + K_1\mathcal{S}_0(Z) + c_1K_1\mathcal{S}_1(Z) + c_1K_1\mathcal{S}_1(Y_i) + c_1^2\mathcal{S}_2(W_i) \\
&\vdots \\
&\leq K_1\sum_{r=0}^{M} c_1^r\mathcal{S}_r(Y_i) + K_1\sum_{r=0}^{M} c_1^r\mathcal{S}_r(Z) + c_1^{M+1}\mathcal{S}_{M+1}(W_i) \\
&\leq K_1\sum_{r=0}^{M} c_1^r\mathcal{S}_r(Y_i) + K_1\sum_{r=0}^{M}(2dc_1)^r Z + (2dc_1)^{M+1}L
\end{aligned}$$

so that $\varepsilon \coloneqq 2Dc_1 < 1/2$ for small enough β, giving $K_1\sum_{r=0}^{M}(2Dc_1)^r Z \leq \frac{K_1}{1-\varepsilon}Z \leq 2K_1Z$

and $(2Dc_1)^{M+1}L \leq \varepsilon^{M+1}L$. For the first term, our previous estimate (3.3.7) gives

$$\begin{aligned}\sum_{r=0}^{M} c_1^r \mathcal{S}_r(Y_i) &\leq \sum_{r=0}^{M} c_1^r (2D)^r \sum_{n=0}^{r} \sum_{\|k-i\|=n} Y_k = \sum_{n=0}^{M}\sum_{r=n}^{M} (2Dc_1)^r \sum_{\|k-i\|=n} Y_k \\ &\leq 2\sum_{n=0}^{M} (2Dc_1)^n \sum_{\|k-i\|=n} Y_k\end{aligned}$$

Thus we arrive at

$$W_i \leq 2\sum_{n=0}^{M} (2Dc_1)^n \sum_{\|k-i\|=n} Y_k + 2K_1 Z + (2Dc_1)^{M+1} L$$

for all $M \in \mathbb{Z}_{\geq 0}$ and β small enough. Taking $M \to \infty$ we conclude (3.3.3) holds for $K_2 = \max(2, 2K_1) = \mathcal{O}(1)$ and $c_2 = 2Dc_1 = \mathcal{O}(c_1) = o(1)$. □

We can now verify that the conditions of Theorem 3.2.1 hold and prove the q-SGI. The key ingredient is the previous lemma - it is straightforward to see that the weak U-bound **(A1)** is immediate, and more difficult to see that the weak U-bound implies the weak q-SGI of **(A2)**, but we show that the strategy for 'strong U-bound ⇒ strong q-SGI' as in Theorem 2.4.16 can be adapted to the weak case. The basic idea is that one can truncate the left-hand side of **(A2)** into two parts, one where the boundary conditions are bounded and the other where they are not. We present these ideas in the following proof of Theorem 3.2.1.

Proof of Theorem 3.3.1. By applying Lemma 3.3.6 and **(C2)**, we obtain

$$\begin{aligned}\nu\big(|f|^q\big|\nabla_j V_{ij}\big|^q\big) &\leq B\nu|f|^q + B\nu(\eta_i|f|^q) + B\nu(\eta_j|f|^q)\\ &\leq B(1+2K_2)\nu|f|^q + BK_2\sum_{k\in\mathbb{Z}^D}\left(c_2^{\|k-i\|}+c_2^{\|k-j\|}\right)\nu|\nabla_k f|^q\\ &\leq B(1+2K_2)\left(\nu|f|^q + \sum_{k\in\mathbb{Z}^D}\left(c_2^{\|k-i\|}+c_2^{\|k-j\|}\right)\nu|\nabla_k f|^q\right) \qquad (3.3.8)\end{aligned}$$

This is **(A1)** with $A = B(1+2K_2) = \mathcal{O}(1)$ but with an incorrect interaction matrix M_{ki} since $c_2^{\|k-i\|} + c_2^{\|k-j\|}$ has dependence on j. For any $i, k \in \mathbb{Z}^D$ define the distance

$$\operatorname{dist}(k,i) \coloneqq \inf_{j\in\{i\}\cup\{\sim i\}} \|k-j\|, \qquad (3.3.9)$$

we see that $c_2^{\|k-i\|} + c_2^{\|k-j\|} \leq 2c_2^{\operatorname{dist}(k,i)}$ since $j \sim i$ by assumption. Thus the correct interaction matrix is $M_{ki} = c_2^{\operatorname{dist}(k,i)}$ modulo a constant. Note $M_{ij} \geq 0$ for all $i, j \in \mathbb{Z}^D$, and $\sum_{j\in\mathbb{Z}^D} M_{ij}$ is clearly independent of i and finite for $c_2 < 1$ since the number of sites with $\|i-j\| = n$ grows polynomially like n^d while M_{ij} for such sites decays as c_2^n, meaning $\sum_{j\in\mathbb{Z}^D} M_{ij}$ is bounded above by a multiple of $\sum_{n\geq 0} c_2^n n^D < \infty$. It follows **(B2)** holds, but we hold off judgment on **(B1)** since we may need to enlarge the constant $B(1+2K_2)$ depending on the constant we find for **(A2)**.

To obtain **(A2)**, fix a parameter $L > 0$ and denote by $\mathbb{B}_r$ the metric ball of radius r in Carnot group $\mathbb{G}$ associated to the control distance d. By the conditions on η, we have $\eta(h) \to +\infty$ as $d(h) \to \infty$ and thus there exists $r > 0$ such that

$$\{\eta_i \leq L\} \subset \mathbb{B}_r.$$

Now we study the left-hand side of **(A2)** by applying the cutoff

$$\nu|f-\mathbb{E}_i^\omega f|^q = \nu|f-\mathbb{E}_i^\omega f|^q \mathcal{I}_{\{\eta_i+\bar{\eta}_i\le L\}} + \nu|f-\mathbb{E}_i^\omega f|^q \mathcal{I}_{\{\eta_i+\bar{\eta}_i> L\}} \tag{3.3.10}$$

where $\mathcal{I}$ is the indicator function[3] and $\bar{\eta}_i \coloneqq \sum_{j:j\sim i}\eta_j$, so that $\eta_i+\bar{\eta}_i$ is a function of x_i and x_j for $j\sim i$. Let $F \coloneqq f-\mathbb{E}_i^\omega f$ by dropping ω for simplicity, we first study the cutoff $I_1 \coloneqq |F|^q \mathcal{I}_{\{\eta_i+\bar{\eta}_i\le L\}}$.

Fix any $h\in\mathbb{G}$ with $d(h)=2r$. Since η is non-negative, one has the bound $\{\eta_i+\bar{\eta}_i\le L\}\subset \mathbb{B}_r\cup S_i$ where $S_i\coloneqq\{\bar{\eta}_i\le L\}$, we can then bound νI_1 by

$$\nu I_1 \le 2\nu \underbrace{\left(|F(x_i)-F(x_i\circ h)|^q \mathcal{I}_{\mathbb{B}_r}\mathcal{I}_{S_i}\right)}_{=:I_2} + 2\nu \underbrace{\left(|F(x_i\circ h)|^q \mathcal{I}_{\mathbb{B}_r}\mathcal{I}_{S_i}\right)}_{=:I_3}. \tag{3.3.11}$$

For the term I_2, let $\gamma:[0,2r]\to\mathbb{G}$ be a geodesic from 0 to h such that $|\dot\gamma(t)|\le 1$ for all t, then

$$\begin{aligned} \big|F(x_i)-F(x_i\circ h)\big|^q &= \left|\int_0^{2r} \frac{d}{dt}F(x_i\circ\gamma(t))\,dt\right|^q \\ &= \left|\int_0^{2r} \nabla_i F(x_i\circ\gamma(t))\cdot\dot\gamma(t)\,dt\right|^q \\ &\le 2r\int_0^{2r}\big|\nabla_i f(x_i\circ\gamma(t))\big|^q\,dt \end{aligned} \tag{3.3.12}$$

where the last step is by Hölder's inequality and the fact $\nabla_i\mathbb{E}_i^\omega f(x_i\circ\gamma(t))=0$. Since ϕ and V are continuous, the potential can be bounded by

$$|U_i^\omega|\,\mathcal{I}_{\mathbb{B}_r}\mathcal{I}_{S_i} \le \sup_{h\in\mathbb{B}_{3r}}|\phi(h)| + 2D\sup_{h_1,h_2\in\mathbb{B}_{3r}}|V(h_1,h_2)| =: T$$

[3] $\mathcal{I}_X(x)=1$ if $x\in X$ and 0 otherwise for any set X.

provided that $\beta \leq 1$, which easily holds by making β sufficiently small. Then T is independent of β and only dependent on L. With this bound and (3.3.12), it follows

$$\begin{aligned}\mathbb{E}_i^\omega I_2 &\leq 2r \iint_0^{2r} |\nabla_i f(x_i \circ \gamma(t))|^q \frac{e^{-U_i^\omega}}{Z_i^\omega} \mathcal{I}_{\mathbb{B}_r} dt dx_i \, \mathcal{I}_{S_i} \\ &\leq 2r \frac{e^T}{Z_i^\omega} \iint_0^{2r} |\nabla_i f(x_i \circ \gamma(t))|^q \mathcal{I}_{\mathbb{B}_r} dt dx_i \, \mathcal{I}_{S_i} \\ &\leq 4r^2 \frac{e^T}{Z_i^\omega} \int |\nabla_i f(x_i)|^q \mathcal{I}_{\mathbb{B}_{3r}} dx_i \, \mathcal{I}_{S_i}\end{aligned}$$

where the last step is by the change of variable $x_i \circ \gamma(t)$ to x_i and the strict triangle inequality (see Theorem 2.4.14)

$$d(x_i \circ \gamma(t)) \leq d(x_i) + d(\gamma(t)) \leq 3r$$

for $x_i \in \mathbb{B}_r$ and $\gamma(t) \in \mathbb{B}_{2r}$ since $d(h) = 2r$. Now we can reinsert $e^{-U_i^\omega}$ back into the integral to obtain

$$\nu I_2 = \nu \mathbb{E}_i^\omega I_2 \leq 4r^2 e^{2T} \nu \int |\nabla_i f(x_i)|^q \frac{e^{-U_i^\omega(x_i)}}{Z_i^\omega} dx_i \, \mathcal{I}_{S_i} \leq 4r^2 e^{2T} \nu \mathbb{E}_i^\omega |\nabla_i f|^q. \tag{3.3.13}$$

This concludes the estimate of νI_2 in (3.3.11).

As for νI_3 in (3.3.11), note that when $x \in \mathbb{B}_r$ one has the strict triangle inequality

$$d(x_i \circ h) \geq d(h) - d(x_i^{-1}) \geq r$$

since $d(x_i^{-1}) = d(x_i)$ by Theorem 2.4.14. With this and a change of variable $x_i \circ h$ to x_i, it follows

$$\nu I_3 \leq \nu\Big(|F(x_i)|^q \mathcal{I}_{\{d(x_i) \geq r\}} \mathcal{I}_{S_i}\Big) \leq \nu\Big(|F(x_i)|^q \mathcal{I}_{\{\eta_i \geq L\}} \mathcal{I}_{S_i}\Big) \leq \frac{1}{L} \nu(|F(x_i)|^q \eta_i),$$

where the second step is due to the fact $\{d(x_i) \geq r\} \subset \{\eta_i \geq L\}$. Now taking this and the estimate (3.3.13) back into (3.3.11) and (3.3.10), we arrive at

$$\nu|F|^q \leq 8r^2 e^{2T} \nu|\nabla_i f|^q + \frac{3}{L}\nu\big(|F|^q(\eta_i + \bar{\eta}_i)\big). \tag{3.3.14}$$

As for the second term, one can simply apply the weak U-bound in Lemma 3.3.6, which gives

$$\begin{aligned}\nu\left(|F|^q(\eta_i + \bar{\eta}_i)\right) &\leq (2D+1)K_2 \left(\nu|F|^q + \sum_{k\in\mathbb{Z}^D}\sum_{\|j-i\|\leq 1} c_2^{\|k-j\|}\nu|\nabla_k F|^q\right) \\ &\leq 4(2D+1)K_2 \left(\nu|F|^q + \sum_{j:j\sim i} c_2\nu|\nabla_j \mathbb{E}_i^\omega f|^q + \sum_{k\in\mathbb{Z}^D}\sum_{\|j-i\|\leq 1} c_2^{\|k-j\|}\nu|\nabla_k f|^q\right)\end{aligned}$$

since $\nabla_i \mathbb{E}_i^\omega f = 0$ and $\nu|\nabla_k \mathbb{E}_i^\omega f|^q = \nu|\mathbb{E}_i^\omega(\nabla_k f)|^q \leq \nu|\nabla_k f|^q$ for $\|k-i\| > 1$ by Jensen's inequality. Using the $\mathrm{dist}(\cdot,\cdot)$ function defined in (3.3.9), we can simplify the bound as

$$\nu\left(|F|^q(\eta_i + \bar{\eta}_i)\right) \leq 4(2D+1)^2 K_2 \left(\nu|F|^q + \sum_{j:j\sim i} c_2\nu|\nabla_j \mathbb{E}_i^\omega f|^q + \sum_{k\in\mathbb{Z}^D} c_2^{\mathrm{dist}(k,i)}\nu|\nabla_k f|^q\right).$$

Choose L large enough such that $12(2D+1)^2 K_2/L < 1/2$, then substitute the above bound in (3.3.14) and collect $\nu|F|^q$ on the left-hand side, it follows

$$\nu|f - \mathbb{E}_i^\omega f|^q \leq 16r^2 e^{2T}\nu|\nabla_i f|^q + \sum_{j:j\sim i} c_2\nu|\nabla_j \mathbb{E}_i^\omega f|^q + \sum_{k\in\mathbb{Z}^D} M_{ki}\nu|\nabla_k f|^q \tag{3.3.15}$$

with $M_{ki} = c_2^{\mathrm{dist}(k,i)}$, which coincides with the situation for **(A1)**. The only work left is to bound the term $\nu|\nabla_j \mathbb{E}_i^\omega f|^q$ for $j \sim i$.

Note that we already proved condition **(A1)**, thus one can follow the same steps as in the proof of Lemma 3.2.4, only to skip (3.2.5) since we do not have **(A2)** yet. This way the q-variance $\nu|f - \mathbb{E}_i^\omega f|^q$ in (3.2.4) is kept all the way to the induction step (3.2.7), which

now has an extra term $a_2\nu|f-\mathbb{E}_i^\omega f|^q$ where $a_2=\frac{2A\beta^q(A+4)}{1-4A\beta^q}=\mathcal{O}(\beta^q)$ is the same as the one defined in (3.2.7). Note that this only works when $A\beta^q$ is sufficiently small, which can be made possible since $A=B(1+2K_2)=\mathcal{O}(1)$ as in (3.3.8). Now we can start the induction step in (3.2.8) with the extra term $a_2\nu|f-\mathbb{E}_i^\omega f|^q$, and after exactly the same procedure one eventually finds that

$$\nu|\nabla_j\mathbb{E}_i^\omega f|^q\leq a_1\nu|\nabla_j f|^q+\frac{2a_1a_2}{1-2Da_2}\left(\nu|f-\mathbb{E}_i^\omega f|^q+\sum_{k\in\mathbb{Z}^D}M_{ki}\nu|\nabla_k f|^q\right)$$

for $a_1=\frac{2}{1-4A\beta^q}=\mathcal{O}(1)$. Taking this back in (3.3.15) while making $a_2,c_2=o(1)$ small enough, one can collect $\nu|f-\mathbb{E}_i^\omega f|^q$ on the left-hand side and obtain **(A2)**. The condition **(B1)** is easily satisfied since all constants appearing in the bounds are of $\mathcal{O}(1)$. □

3.4 Examples

In this section, we construct example potentials satisfying conditions **(C1)** and **(C2)**. The usual strategy is to find a phase ϕ and a function η satisfying the U-bound in **(C1)**, and then equipping this with an interaction V satisfying **(C2)**. Below we demonstrate a few such constructions using the control distance d and Kaplan norm N.

3.4.1 The control distance d

The problem of finding ϕ and η satisfying **(C1)** on the Heisenberg group was studied in the literature. As mentioned earlier, one already has the q-logarithmic Sobolev inequality courtesy of [6, Theorem 5.2], which made use of the following U-bound for the control

distance d first appearing in [5, Theorem 2.4]:

$$\int |g|^q \, d^p e^{-\alpha d^p} dh \leq C \int \left(|\nabla g|^q + |g|^q \right) e^{-\alpha d^p} dh \tag{3.4.1}$$

for $\alpha > 0$, $p \geq 2$ and some $C > 0$. In other words, for q Hölder conjugate to p, one obtains a U-bound for the phase $\phi(h) = \alpha d^p(h)$ and function $\eta(h) = d^p(h)$.

If we consider the interactions V for which $\left|\nabla_i V_{ij}(h_i, h_j)\right|^q + \left|\nabla_j V_{ij}(h_i, h_j)\right|^q \leq B(1 + d^p(h_i) + d^p(h_j))$, one starting point is $V_{ij} = P(d_i, d_j)$ for a polynomial P and where $V_{ij} \coloneqq V(h_i, h_j)$, $d_i \coloneqq d(h_i)$, and $d_j \coloneqq d(h_j)$, in analogue with the shorthand for η_i and η_j in the previous section. The sub-gradient is

$$\left|\nabla_i V_{ij}\right| = |\nabla_i d_i| \left|\partial_1 P(d_i, d_j)\right| \leq \left|\partial_1 P(d_i, d_j)\right|$$

by Theorem 2.4.14, and assuming P has degree r its sub-gradient is controlled by a polynomial $\partial_1 P$ of degree at most r. The same goes for $\left|\nabla_j V_{ij}\right|$, which means $\left|\nabla_i V_{ij}(h_i, h_j)\right|^q + \left|\nabla_j V_{ij}(h_i, h_j)\right|^q$ is controlled by $1 + d_i^{qr} + d_j^{qr}$ modulo a constant. Since $q = p/(p-1)$ it follows $qr \leq p$ amounts to $r \leq p - 1$, i.e. **(C2)** holds for $r \leq p - 1$.

If p is an integer, then in fact the extremal case $r = p$ can be handled too as derivatives of mixed terms $d_i^k d_j^{p-k}$ for $1 \leq k \leq p - 1$ are strictly of degree $p - 1$ and thus their contribution to $\left|\nabla_i V_{ij}\right|^q + \left|\nabla_j V_{ij}\right|^q$ can be controlled by $d_i^{q(p-1)} + d_j^{q(p-1)} = d_i^p + d_j^p$, and control over the pure terms d_i^p and d_j^p is automatic. In other words, our discussion has lead to the following.

Theorem 3.4.1. *Let $\alpha > 0$ and $p \geq 2$. Define a local specification on $\mathbb{H}^{\mathbb{Z}^D}$ by the following potential:*

$$U_\Lambda^\omega(x_\Lambda) = \alpha \sum_{i \in \Lambda} d^p(x_i) + \sum_{i,j \in \Lambda} \beta_{ij} P(d(x_i), d(x_j)) + \sum_{i \in \Lambda,\, j \notin \Lambda} \beta_{ij} P(d(x_i), d(\omega_j))$$

with $|\beta_{i,j}| < \beta$ *for all* $i, j \in \mathbb{Z}^D$*, and for* P *a polynomial of degree at most* p*. Then for* q *Hölder conjugate to* p *and for all sufficiently small* $\beta > 0$*, there exists a unique Gibbs measure* ν *which satisfies the* q*-spectral gap inequality*

$$\nu|f - \nu f|^q \leq C_{SG}\nu|\nabla f|^q \,.$$

In particular this also proves that when αd^p satisfies a q-SGI then αd^p plus lower degree terms also satisfies a q-SGI as a special case when P is a polynomial solely in the first variable. Note the arguments can be generalised to the case where V is not a polynomial, but a sum of monomials of the form $d_i^m d_j^n$, since the proof does not need the powers to be integers. We note it is required that $m \in [1, \infty)$ since if $0 < m < 1$ then $\nabla_i d_i^m = m\nabla_i d \cdot d_i^{m-1}$ has negative exponent $m - 1$ which blows up at zero whilst the right-hand side of **(C2)** is finite. More remarkable however is that the theorem also allows the interaction to depend on not just the control distance, but also the Kaplan norm. If $V_{ij} = P(N_i, N_j)$ where, for N denoting the Kaplan norm, we set $N_i \coloneqq N(x_i)$ and $N_j \coloneqq N(x_j)$, then since $|\nabla N| \leq 1$ (see e.g. [30, Proposition 3.2.16]), can obtain the same bound because all homogeneous norms are equivalent. In fact, one can go further since

$$\big|\nabla(d^j N^k)\big| \leq \big|\nabla d^j\big|N^k + d^j\big|\nabla N^k\big| \leq j d^{j-1} N^k + k d^j N^{k-1}$$

by similar reasoning, meaning we can generalise the previous theorem to the following.

Consider the polynomial

$$Q(x_i, x_j) = \sum_{k=1}^{n} a(r_k^{(1)}, s_k^{(1)}, r_k^{(2)}, s_k^{(2)}) d(x_i)^{r_k^{(1)}} d(x_j)^{s_k^{(1)}} N(x_i)^{r_k^{(2)}} N(x_j)^{s_k^{(2)}},$$

which is a finite sum of $n \in \mathbb{N}$ monomials in $d(x_i), d(x_j), N(x_i), N(x_j)$ of degree at most p

with non-negative exponents $r_k^{(1)}, s_k^{(1)}, r_k^{(2)}, s_k^{(2)} \in \mathbb{Z}_{>0}$ for $k = 1, 2, \cdots, n$.

Theorem 3.4.2. *Let $\alpha > 0$ and $p \geq 2$. Define a local specification on $\mathbb{H}^{\mathbb{Z}^D}$ by the following potential:*

$$U_\Lambda^\omega(x_\Lambda) = \alpha \sum_{i \in \Lambda} d^p(x_i) + \sum_{i,j \in \Lambda} \beta_{ij} Q(x_i, x_j) + \sum_{i \in \Lambda,\, j \notin \Lambda} \beta_{ij} Q(x_i, \omega_j)$$

with $\left|\beta_{i,j}\right| < \beta$ for all $i, j \in \mathbb{Z}^D$. Then for q Hölder conjugate to p and for all sufficiently small $\beta > 0$, there exists a unique Gibbs measure ν which satisfies the q-spectral gap inequality

$$\nu|f - \nu f|^q \leq C_{SG} \nu |\nabla f|^q .$$

In fact for the control distance one also has a U-bound in the sub-quadratic case $p < 2$, from [5, Theorem 2.6]:

$$\int |g|^2 \, d^{2(p-1)} e^{-\alpha d^p} dh \leq C \int \left(|\nabla g|^2 + |g|^2 \right) e^{-\alpha d^p} dh$$

for $\alpha > 0$, $1 \leq p < 2$ and some $C > 0$. Note $2(p-1) < p$, meaning this is a strictly worse U-bound than that given in the quadratic $p = 2$ and super-quadratic $p > 2$ cases. We note the U-bound here is not useful if $p = 1$ however because the bound is trivial since $d^{2(p-1)} \equiv 1$. Moreover, by analogue with the previous argument the interaction is trivial for $p < \frac{3}{2}$ since allowed monomials have exponents that are either zero or at least one, and that are bounded above by $2(p-1)$. Thus the arguments can be extended only in the case where $\frac{3}{2} < p < 2$ and we find the following.

Theorem 3.4.3. *Let $\alpha > 0$ and $1 < p < 2$. Define a local specification on $\mathbb{H}^{\mathbb{Z}^D}$ by the*

following potential:

$$U_\Lambda^\omega(x_\Lambda) = \alpha \sum_{i\in\Lambda} d^p(x_i) + \sum_{i,j\in\Lambda} \beta_{ij} Q(x_i, x_j) + \sum_{i\in\Lambda,\, j\notin\Lambda} \beta_{ij} Q(x_i, \omega_j)$$

with $|\beta_{i,j}| < \beta$ *for all* $i, j \in \mathbb{Z}^D$*, and for*

$$Q(x_i, x_j) = \sum_{k=1}^{N} d(x_i)^{r_k^1} d(x_j)^{s_k^1} N(x_i)^{r_k^2} N(x_j)^{s_k^2}$$

a finite sum of $N \in \mathbb{Z}_{\geq 1}$ *monomials in* $d(x_i), d(x_j), N(x_i), N(x_j)$ *of degree at most* $2(p-1)$ *with non-negative exponents* $r_k^1, s_k^1, r_k^2, s_k^2$ *for* $k = 1, 2, \cdots, N$ *belonging to* $\{0\} \cup [1, \infty)$*. Then for all sufficiently small* $\beta > 0$*, there exists a unique Gibbs measure* ν *which satisfies the* 2*-spectral gap inequality*

$$\nu|f - \nu f|^2 \leq C_{SG} \nu|\nabla f|^2 .$$

3.4.2 The Kaplan norm N

One serious deficiency in the case of the Kaplan norm is that at the time the q-SGI was proven for the measure $Z^{-1}e^{-\alpha N^p}$ in [30, Theorem 4.5.5] for $\alpha > 0$ and $p \geq 2$, the arguments used a degenerate U-bound $\eta = N^{p-2}\|h\|_E^2$ where $\|h\|_E = \|(w, z)\|_E = \|w\|_{l_2}$ is the (Euclidean) norm of the Euclidean part of $h = (w, z)$, in the sense η fails to diverge to infinity in all directions; clearly along the z-axis where $w = 0$ one has $\eta \equiv 0$. (This property is needed to ensure everything is nicely bounded when η is bounded above.) Recently however a new U-bound for the Kaplan norm was derived which is more suitable and satisfies the divergence condition, which we state.

Theorem 3.4.4 ([48, Theorem 1])**.** *The measure* $e^{-\alpha N^p}/Z$ *for* $\alpha > 0$ *and* $p \geq 2$ *satisfies*

the U-bound

$$\int |g|^q \, N^{p-3} e^{-\alpha N^p} dh \leq C \int \left(|\nabla g|^q + |g|^q \right) e^{-\alpha N^p} dh$$

for any $q \geq 2$ and g supported outside the unit ball $\{N < 1\}$, with positive constant C independent of g.

Remark 3.4.5. By perturbation theory of [5] the result can be extended to any semi-bounded polynomial of order $p \geq 2$.

If $p > 3$, then $\eta = N^{p-3}$ diverges to infinity in all directions as required, and the U-bound can be extended to all of $\mathbb{H}$ as $N^{p-3} \leq 1$ in $\{N < 1\}$, meaning we obtain the U-bound

$$\int |g|^q \, N^{p-3} e^{-\alpha N^p} dh \leq C \int \left(|\nabla g|^q + |g|^q \right) e^{-\alpha N^p} dh \tag{3.4.2}$$

for $\alpha > 0$ and $p > 3$. Since our methods work when $q \in (1, 2]$, it follows we can transfer the U-bound for $\phi = \alpha N^p$ and $\eta = N^{p-3}$ into the 2-SGI by applying the same arguments as for the control distance. This is our main result - although it was shown in [30, Theorem 4.5.5] the q-SGI for the Kaplan norm in finite dimension holds for $p \geq 2$ and with $q < 2$ Hölder conjugate to p, at the moment our techniques allow us to extend this to the infinite-dimensional case for $p > 3$ and $q \equiv 2$.

Theorem 3.4.6. *Let $\alpha > 0$ and $p > 3$. Define a local specification on $\mathbb{H}^{\mathbb{Z}^D}$ by the following potential:*

$$U_\Lambda^\omega(x_\Lambda) = \alpha \sum_{i \in \Lambda} N^p(x_i) + \sum_{i,j \in \Lambda} \beta_{ij} Q(x_i, x_j) + \sum_{i \in \Lambda,\, j \notin \Lambda} \beta_{ij} Q(x_i, \omega_j)$$

with $\left|\beta_{i,j}\right| < \beta$ for all $i, j \in \mathbb{Z}^D$, and for

$$Q(x_i, x_j) = \sum_{k=1}^{N} d(x_i)^{r_k^1} d(x_j)^{s_k^1} N(x_i)^{r_k^2} N(x_j)^{s_k^2}$$

a finite sum of $N \in \mathbb{Z}_{\geq 1}$ *monomials in* $d(x_i), d(x_j), N(x_i), N(x_j)$ *of degree at most* $\max(1, p-3)$ *with non-negative exponents* $r_k^1, s_k^1, r_k^2, s_k^2$ *for* $k = 1, 2, \cdots, N$ *belonging to* $\{0\} \cup [1, \infty)$. *Then for all sufficiently small* $\beta > 0$, *there exists a unique Gibbs measure* ν *which satisfies the* 2*-spectral gap inequality*

$$\nu|f - \nu f|^2 \leq C_{SG}\nu|\nabla f|^2 .$$

In fact in both examples one does not need to restrict our attention to just the control distance d and the Kaplan norm N - as far as finding valid interactions is concerned we may work with any monomial which is composed of homogeneous norms ϕ in either x_i or x_j raised to appropriate exponents, provided $|\nabla\phi|$ is bounded. Even if $|\nabla\phi|$ is not bounded we may be able to obtain q-SGIs, provided $|\nabla\phi|$ does not grow too fast. The matter of replacing the *phase* with a different homogeneous norm is more delicate as it requires the existence of a U-bound as in **(C1)**, but nonetheless the results here can be applied if and when these U-bounds are found.

Remark 3.4.7. We remark that in the above examples, one can replace d and N in the multi-spin interaction V by a set of different homogeneous norms $\{\rho_n\}$, provided that each $|\nabla_{\mathbb{G}}\rho_n|$ is uniformly bounded and their powers are small enough for **(C2)** to hold. An example of such ρ would be the arithmetic or geometric mean of d and N. Moreover, instead of constructing V as a function on $\rho(x_i), \rho(x_j)$, one may consider V as a function of $\rho(x_j^{-1} \circ x_i)$. The motivation comes from the fact that a potential in Euclidean space is often given by a function on $|x_i - x_j|$, and therefore the choice of $\rho(x_j^{-1} \circ x_i)$ is more natural in the setting of a Carnot group. The above theorems still hold since if $|\nabla_{x_i}\rho(x_i)|$ is bounded then $|\nabla_{x_i}\rho(x_j^{-1} \circ x_i)|$ is also bounded due to the left-invariant property[4] of the

[4]See footnote 6

vector fields (note that $|\nabla_{x_i}\rho(x_i \circ x_j^{-1})|$ may not be bounded since the vector fields are not right-invariant).

Chapter 4

Logarithmic Sobolev Inequality for Unbounded Sub-gradient

In this chapter, we construct homogeneous norms with unbounded sub-gradient or sub-Laplacian and explore the possibility of developing logarithmic Sobolev inequality (LSI) for Gibbs measures defined by such norms.

For simplicity, we will construct homogeneous norm ρ on the Heisenberg group $\mathbb{H}$, with the form $\rho(h) = F(h)N(h)$ where $h \in \mathbb{H}$, N is the Kaplan norm (defined in Example 2.4.9), and $F : \mathbb{H} \to [0, \infty)$ is some scaling function that makes $|\nabla_{\mathbb{H}}\rho|$ blow up in some directions. We will give some interesting examples of such ρ, and in particular, we focus on the case $\rho = N - \frac{1}{2}\sqrt{N|w|}$ where $|w|$ is the l_2-norm of the Euclidean part (i.e. $w = (x, y)$ and $|w| = \sqrt{x^2 + y^2}$ for $(x, y, z) \in \mathbb{H}$). We show later that $|\nabla_{\mathbb{H}}\rho|$ and $\Delta_{\mathbb{H}}\rho$ blow up on $\{w = 0\}$, which is a reversed property as opposed to N since $|\nabla_{\mathbb{H}}N|$ and $\Delta_{\mathbb{H}}N$ vanish on $\{w = 0\}$ (see (2.4.4)). This is an interesting property that are not well explored in literature.

In addition, we show that, despite the unbounded sub-gradient of ρ, the Gibbs measure

$d\mu = e^{-\rho^p} dh / (\int e^{-\rho^p} dh)$ can still satisfy the q-LSI for any $p \geq 2$ with $1/p + 1/q = 1$, i.e.

$$\mu \left(|f|^q \log \frac{|f|^q}{\mu |f|^q} \right) \leq C \mu |\nabla_{\mathbb{H}} f|^q$$

for some $C > 0$ independent of f. The full statement is provided in Theorem 4.3.1.

4.1 Introduction

The homogeneous norms considered in most works all have bounded sub-gradients, the two examples considered so far (Kaplan norm N and control distance d) both fall into this category. In fact, $|\nabla_{\mathbb{G}} \rho|$ is bounded for any symmetric and differentiable homogeneous norm ρ with a strict triangle inequality. To see this, simply use the identity (2.4.5) to obtain

$$|\nabla_{\mathbb{G}} \rho| = \limsup_{d(x \circ y^{-1}) \to 0} \frac{|\rho(x) - \rho(y)|}{\rho(x \circ y^{-1})} \cdot \frac{\rho(x \circ y^{-1})}{d(x \circ y^{-1})},$$

where the first ratio is bounded due to the reverse triangle inequality $|\rho(x) - \rho(y)| \leq \rho(x \circ y^{-1})$ while the second is bounded due to the equivalence of homogeneous norms (see Proposition 2.4.7). Note that this resembles the fact Lipschitz continuity implies bounded gradient on $\mathbb{R}^m$, since strict triangle inequality provides a Lipschitz property.

In our setup, we consider the general definition of homogeneous norms (Definition 2.4.6) without requiring the strict triangle inequality, and this opens up the possibility for defining norms with unbounded sub-gradient. In the sequel, we shall develop strategies to construct such norms and, in particular, investigate if the LSI still holds for the Gibbs measures in case of the unbounded sub-gradient.

4.2 Constructing norms with unbounded sub-gradient

In this section, we will work in the Heisenberg group $\mathbb{H}$ for simplicity, while noting that the same results can be generalised to similar spaces (e.g. H-type groups or in general step-2 Carnot groups) with the cost of possibly large computational complexity. The strategy is to use the Kaplan norm N as a base, and obtain new norms by adding suitable modifications to N. The aim is to explore which modification can provide interesting properties such as the unbounded sub-gradient or sub-Laplacian.

To be specific, we will modify N by reshaping the unit ball $B_N = \{N \leq 1\}$ to define new norms. This is due to the observation that a homogeneous norm is completely defined by its unit ball[1], so that one can use B_N as the base and scale B_N along different polar angles, then the scaled unit ball defines a new homogeneous norm whose properties are based on the scaling.

Following this idea, it will be convenient to introduce the spherical coordinates for N as below. Let $\mathbb{H} = (\mathbb{R}^3, \circ)$ be the Heisenberg group and $N(x, y, z) = ((x^2+y^2)^2+z^2)^{1/4}$ be the Kaplan norm as in Example 2.4.9. Consider the Kaplan-polar coordinate transformation $T : (x, y, z) \to (N, \varphi, \theta)$ given by

$$\begin{aligned} N &= \left((x^2+y^2)^2+z^2\right)^{1/4} \in [0,\infty), \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \in [0, 2\pi), \\ \theta &= \tan^{-1}\left(\frac{x^2+y^2}{z}\right) \in [0,\pi]. \end{aligned} \tag{4.2.1}$$

[1] This can been seen by considering the dilation $\mathcal{D}_\lambda$ (see Definition 2.4.6) on the unit ball.

The azimuth φ and inclination θ are homogeneous[2] of degree 0 and therefore invariant under dilation. The inverse transformation $T^{-1} : (N, \varphi, \theta) \to (x, y, z)$ is given by

$$x = N \cos\varphi \sqrt{\sin\theta}, \quad y = N \sin\varphi \sqrt{\sin\theta}, \quad z = N^2 \cos\theta.$$

Here are some basic properties of the Kaplan-polar coordinate:

$$\begin{aligned} dV &= dx\, dy\, dz = N^3\, dN d\varphi\, d\theta, \\ |\nabla_{\mathbb{H}} f|^2 &= \frac{\sin(\theta)}{N^2}\Big(\big(2\partial_\theta f + \partial_\varphi f\big)^2 + \big(N\partial_N f - \cot(\theta)\partial_\varphi f\big)^2 \Big). \end{aligned} \tag{4.2.2}$$

An interesting application of the Kaplan-polar coordinate is to construct homogeneous norms by scaling the Kaplan norm. That is, we can construct a new norm $\rho = F(\varphi, \theta)N$ where F is the scaling of N along the direction of (φ, θ). We present such construction in the following proposition.

Proposition 4.2.1. *Let $F : [0, 2\pi) \times [0, \pi] \to \mathbb{R}$ be any continuous function such that the range of F is contained in (a, b) for some $a, b > 0$. Then the following function*

$$\rho = F(\varphi, \theta)N$$

defines a homogeneous norm on $\mathbb{H}$. Moreover, ρ satisfies the generalised triangle inequality

$$\rho(x \circ y) \leq \frac{b}{a}\big(\rho(x) + \rho(y)\big).$$

Proof. To show ρ is a homogeneous norm, we verify all the conditions in Definition 2.4.6. Indeed, ρ is continuous because F is continuous and φ, θ are continuous in the spherical

[2] A function $f : \mathbb{G} \to \mathbb{R}$ is homogeneous of degree $n \in \mathbb{N}$ if $f(\mathcal{D}_\lambda x) = \lambda^n f(x)$ for all λ.

coordinate transformation (4.2.1). Moreover, since F is bounded above and below by positive constants, it holds $\rho \geq 0$ and $\rho(h) = 0$ only when $h = 0$. The homogeneity of ρ follows from the homogeneity of N and the fact φ, θ are homogeneous of degree 0. As for the triangle inequality, one simply use the fact $F \in (a, b)$ and the strict triangle inequality for N as in Example 2.4.9. □

Remark 4.2.2. Although F depends on both polar angles, it will turn out that the dependence on θ alone is enough to produce interesting example norms. For this reason, we first set up some identities for later calculations. If $\rho = F(\theta)N$ is independent of φ, then one can use (2.4.4) to show that

$$|\nabla_{\mathbb{H}}\rho|^2 = (4|F'|^2 + F^2)\sin\theta, \quad \Delta_{\mathbb{H}}\rho = \frac{\sin\theta}{N}\left(4F'' + 4(\cot\theta)F' + 3F\right).$$

Example 4.2.3. By proposition 4.2.1 we can now construct any homogeneous norm by scaling the Kaplan norm. Below we give some example norms whose sub-gradient has interesting properties:

(1) Consider the following homogeneous norm

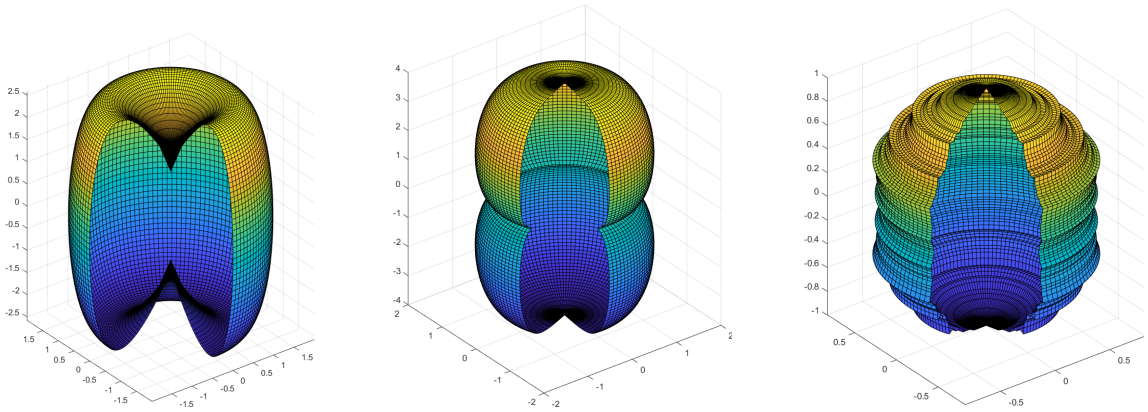
$$\rho_1 = \left(1 - (1/2)\sqrt[4]{\sin\theta}\right) N,$$

then ρ_1 is smooth on $\mathbb{H}\setminus\{z = 0\}, |\nabla_{\mathbb{H}}\rho_1|$ never vanishes and blows up on $\{(x, y) = 0\}$.

(2) Consider the following homogeneous norm

$$\rho_2 = \left(1 - (1/2)\sqrt[4]{|\cos\theta|}\right) N,$$

ρ_2 is smooth on $\mathbb{H}\setminus\{z = 0\}, |\nabla_{\mathbb{H}}\rho_2|$ vanishes on $\{(x, y) = 0\}$ and blows up on $\{z = 0\}$.



(a) ρ_1: Blow up (b) ρ_2: Blow up / vanish (c) ρ_3: Non-differentiable

Figure 4.1: Unit ball of ρ_1, ρ_2, ρ_3 in Example 4.2.3

(3) Let $W : [0, \pi] \to [-1, 1]$ be a Weierstrass function such that $W(0) = W(\pi) = 0$, then

$$\rho_3 = \big(1 - (1/2)W(\theta)\big)\, N$$

is a homogeneous norm such that $\nabla_{\mathbb{H}}\rho_3$ does not exist everywhere.

Remark 4.2.4. We note that ρ_1 and ρ_2 can be simplified in Cartesian form:

$$\rho_1 = N - \frac{1}{2}\sqrt{|w|N}, \quad \rho_2 = N - \frac{1}{2}\sqrt{|z|^{1/2}N}$$

where $w = (x, y) \in \mathbb{R}^2$ is the horizontal variable. Moreover, all three norms satisfy the generalised triangle inequality with a constant 2 by Proposition 4.2.1.

In the next section, we show that, in spite of the unboundedness of the sub-gradient, the Gibbs measure defined by ρ_1 can still satisfy the LSI under some growth conditions of the potential.

4.3 LSI for unbounded sub-gradient

Let $\mathbb{H} = (\mathbb{R}^{2+1}, \circ)$ be the Heisenberg group and N be the Kaplan norm $N = (|w|^4 + z^2)^{1/4}$ where $w = (x, y) \in \mathbb{R}^2$. Consider the homogeneous norm

$$\rho := N - \frac{1}{2}\sqrt{N|w|}$$

as in Remark 4.2.4. It is smooth outside $\{|w| = 0\}$, and, with the aid of Remark 4.2.2, one can obtain the sub-gradient and sub-Laplacian

$$|\nabla_{\mathbb{H}}\rho|^2 = \frac{z^2}{16N^3|w|} + R_1(|w|/N), \quad \Delta_{\mathbb{H}}\rho = -\frac{z^2}{8|w|^{\frac{3}{2}}N^{\frac{7}{2}}} + \frac{1}{N}R_2(|w|/N), \tag{4.3.1}$$

where $R_1(a) = a^3/4 - a^{5/2} + a^2$ and $R_2(a) = -a^{5/2} + 3a^2$. It then follows $|\nabla_{\mathbb{H}}\rho| \geq 1/3$ and $|\nabla_{\mathbb{H}}\rho| \to \infty$ as $|w| \to 0$. Therefore, ρ is an almost everywhere smooth homogeneous norm that has a sub-gradient unbounded above but bounded below by a positive constant. In the following, we show that the single-spin Gibbs measure defined by ρ satisfies the q-LSI.

Theorem 4.3.1. *Let $\phi = \rho^p$ for some $p \geq 2$ and let $q = p/(p-1)$ be the Hölder conjugate of p. The probability measure $d\mu = \left(\int_{\mathbb{H}} e^{-\phi(h)}dh\right)^{-1} e^{-\phi(h)}dh$ satisfies the q-LSI*

$$\mu\left(|f|^q \log \frac{|f|^q}{\mu|f|^q}\right) \leq C_{LS}\mu|\nabla_{\mathbb{H}}f|^q$$

for some constant $C_{LS} > 0$.

By Theorem 4.1 in Hebisch and Zegarliński's work [5], i.e. an application of Theorem

2.4.17 with power 2 replaced by $q \in (1,2]$, one has the following implication

$$\left.\begin{aligned} \text{Sobolev inequality:}\quad & \big(\textstyle\int_{\mathbb{H}} |f|^{2+\varepsilon} dh\big)^{\frac{2}{2+\varepsilon}} \le a_1 \int_{\mathbb{H}} |\nabla_{\mathbb{H}} f|^2 dh + b_1 \int_{\mathbb{H}} |f|^2 dh \\ q\text{-SGI:}\quad & \mu|f - \mu f|^q \le a_2 \mu |\nabla_{\mathbb{H}} f|^q + b_2 \mu |f|^q \\ \text{U-bound:}\quad & \mu\big((\phi + |\nabla_{\mathbb{H}}\phi|^q)|f|^q\big) \le a_3 \mu |\nabla_{\mathbb{H}} f|^q + b_3 \mu |f|^q \end{aligned}\right\} \Longrightarrow q\text{-LSI} \tag{4.3.2}$$

where $\varepsilon > 0$ and $a_i \ge 0, b_i \ge 0, i = 1,2,3$. We will verify that all three inequalities on the left-hand side indeed hold. The Sobolev inequality already exists in literature, e.g. see [49, Theorem IV.7.2]. For the q-SGI, note that

$$\phi + |\nabla_{\mathbb{H}}\phi|^q = \rho^p + p^q \rho^{q(p-1)} |\nabla_{\mathbb{H}}\rho|^q \ge (1 + (p/3)^q)\rho^p$$

since $q(p-1) = p$ and $|\nabla_{\mathbb{H}}\rho| \ge 1/3$ by (4.3.1). If the U-bound in (4.3.2) holds, one can then obtain the q-SGI by Theorem 2.4.16 with $\eta = \rho^p$. Therefore, the key is to obtain the U-bound in (4.3.2), for which we prove the following lemma.

Lemma 4.3.2. *Let $\phi = \rho^p$ for some $p \ge 2$ and q be the Hölder conjugate of p, there exists a constant $C > 0$ such that the measure $e^{-\phi}$ satisfies the U-bound*

$$\int_{\mathbb{H}} \big(1 + \rho^p\big) |\nabla_{\mathbb{H}}\rho|^q \, |f|^q e^{-\phi} dh \le C \int_{\mathbb{H}} \big(|\nabla_{\mathbb{H}} f|^q + |f|^q\big) e^{-\phi} dh. \tag{4.3.3}$$

Proof. For simplicity, we shall denote by $\nabla \equiv \nabla_{\mathbb{H}}$ in this proof. Following the idea in [5], assume $f \ge 0$ is smooth and we start with the Leibniz rule

$$(\nabla\phi) f e^{-\phi} = (\nabla f) e^{-\phi} - \nabla\big(f e^{-\phi}\big).$$

Applying the functional $\int \rho \langle \nabla\rho/|\nabla\rho|^\alpha, \cdot\rangle$ with $\alpha = (p-2)/(p-1) \in [0,1)$ on both sides,

we have

$$p\int \rho^p|\nabla\rho|^q\, fe^{-\phi} = \int \rho\left\langle \frac{\nabla\rho}{|\nabla\rho|^\alpha}, \nabla f\right\rangle e^{-\phi} - \int \left\langle \frac{\rho\nabla\rho}{|\nabla\rho|^\alpha}, \nabla\left(fe^{-\phi}\right)\right\rangle$$
$$\leq \int \rho|\nabla\rho|^{\frac{q}{p}}|\nabla f|\, e^{-\phi} + \int \nabla\left(\frac{\rho\nabla\rho}{|\nabla\rho|^\alpha}\right) fe^{-\phi}. \tag{4.3.4}$$

For the last term, we use chain rule to get

$$\nabla\left(\frac{\rho\nabla\rho}{|\nabla\rho|^\alpha}\right) = \frac{1}{|\nabla\rho|^\alpha}\left(|\nabla\rho|^2 + \rho\Delta\rho\right) + \rho\left\langle \nabla|\nabla\rho|^{-\alpha}, \nabla\rho\right\rangle. \tag{4.3.5}$$

By the calculations in (4.3.1) and the equivalence $N/2 \leq \rho \leq 3N/2$, the sub-Laplacian term satisfies

$$|\nabla\rho|^2 + \rho\Delta\rho \leq 4 + \left(1 - \sqrt{\tfrac{N}{|w|}}\right)\frac{z^2}{16|w|\, N^3} =: 4 + I, \tag{4.3.6}$$

where the constant 4 is due to $\sup_{|a|\leq 1}(R_1(a) + 3R_2(a)/2) \leq 4$. Next, we show that $1 - I \geq |\nabla\rho|^2$. To this end, we first calculate the following term

$$\frac{16|w|N^3}{z^2}(1/2 - I) = \frac{8|w|N^3}{z^2} + \sqrt{\tfrac{N}{|w|}} - 1 = \frac{8a}{1-a^4} + \frac{1}{\sqrt{a}} - 1$$

where $a \coloneqq |w|/N$. Since $a \in [0,1]$ one can show that $8a/(1-a^4) + 1/\sqrt{a} \geq 3$, then by the above equality we have $1/2 - I \geq 2z^2/(16|w|N^3)$ and it follows

$$1 - I \geq \frac{z^2}{16|w|N^3} + \frac{1}{2} \geq \frac{z^2}{16|w|N^3} + R_1(|w|/N) = |\nabla\rho|^2.$$

Taking this into (4.3.6), the first term in (4.3.5) is then bounded by

$$\frac{1}{|\nabla\rho|^\alpha}\left(|\nabla\rho|^2 + \rho\Delta\rho\right) \leq \frac{5}{|\nabla\rho|^\alpha} - |\nabla\rho|^{2-\alpha} \leq 15 - |\nabla\rho|^q \tag{4.3.7}$$

where the second step is by $|\nabla\rho| \geq 1/3$ and the relation $2 - \alpha = q$. For the last term in

(4.3.5), direct calculation gives

$$\rho\langle\nabla|\nabla\rho^{-\alpha}|,\nabla\rho\rangle = 4^{2\alpha-1}\alpha\rho(I_1 - z^4)/I_2$$

where the term I_1 and I_2 are given by

$$\begin{aligned} I_1 &:= 8z^2|w|^3\left(-|w| + 5\sqrt{|w|N} - 4N\right), \\ I_2 &:= |w|^{\frac{3-\alpha}{2}} N^{\frac{7-3\alpha}{2}}\left(z^2 + 4|w|^3\left(\sqrt{|w|} - 2\sqrt{N}\right)^2\right)^{1+\alpha/2}. \end{aligned}$$

Since $I_2 \geq 0$, the term $-z^4/I_2$ is non-positive and can be simply dropped. The other term $\rho I_1/I_2$ is bounded because it is homogeneous of degree 0 and smooth on the unit ball $\{N = 1\}$. This implies $\rho\langle\nabla\left|\nabla\rho^{-\alpha}\right|,\nabla\rho\rangle \leq c$ for some $c > 0$. Collecting this and (4.3.4)-(4.3.7), it follows

$$\int (1 + p\rho^p)|\nabla\rho|^q f e^{-\phi} \leq \int \rho|\nabla\rho|^{\frac{q}{p}}|\nabla f|\, e^{-\phi} + (15 + c)\int f e^{-\phi}.$$

Replacing f by f^q and using Young's inequality for the term $\nabla f^q = f^{q/p}\nabla f$, it follows

$$\rho|\nabla\rho|^{\frac{q}{p}} f^{q/p}|\nabla f| \leq \frac{1}{p}\rho^p|\nabla\rho|^q f^q + \frac{1}{q}|\nabla f|^q.$$

Moving the f^q term to the right hand side and we conclude on the U-bound

$$\int (1 + \rho^p)|\nabla\rho|^q f^q e^{-\phi} \leq \frac{1}{q}\int |\nabla f|^q e^{-\phi} + (15 + c)\int f^q e^{-\phi}.$$

For $f < 0$, the above U-bound holds by replacing f with $|f|$ and using the metric inequality $\left|\nabla|f|\right| \leq |\nabla f|$ that holds almost everywhere. □

Remark 4.3.3. Since $\phi + |\nabla_{\mathbb{H}}\phi|^q \leq c(1 + \rho^p)|\nabla_{\mathbb{H}}\rho|^q$ for some $c > 0$, the U-bound (4.3.3)

holds for $\phi + |\nabla_{\mathbb{H}}\phi|^q$. In fact, the U-bound holds for any semi-bounded polynomial ϕ in the form

$$\phi = a\rho^p + \sum_{n=0}^{M} b_n \rho^{r_n}$$

where $a > 0$, $M \in \mathbb{N}$, $b_n \in \mathbb{R}$ and $r_n \in [0, p)$.

For the LSI on infinite lattice $\mathbb{Z}^D$, if the interaction $V = V(x_i, x_j)$ satisfies the growth conditions $V \leq B\big(1 + \phi(x_i) + \phi(x_j)\big)$ and

$$|\nabla_i V|^q + |\nabla_j V|^q \leq B\big(1 + |\nabla_i \phi(x_i)|^q + |\nabla_j \phi(x_j)|^q\big)$$

for some $B > 0$, then we can use the results in [5] to show that the global Gibbs measure ν associated with the potential $U_i = \phi(x_i) + \sum_j \beta_{ij} V(x_i, x_j)$ satisfies a q-LSI for sufficiently small $|\beta_{ij}|$. However, we note that the polynomial interaction cannot satisfy the above growth condition, for example, consider $V = \rho(x_i)\rho(x_j)$ then $|\nabla_i V|^q = \rho^q(x_j)|\nabla\rho(x_i)|^q$ which cannot be bounded by $|\nabla\rho|^q$. For a workaround, we may consider $V = f(\rho(x_i), \rho(x_j))$ for some f with bounded derivatives, for example, $V = \sqrt{\rho^2(x_i) + \rho^2(x_j)}$.

Remark 4.3.4. Although we only obtained the LSI for one specific example of homogeneous norms with unbounded sub-gradient, it is easy to generalise the same idea for a class of similar examples. In general, it is possible to expect an LSI for a norm ρ such that $|\nabla_{\mathbb{H}}\rho|$ is bounded below by a positive constant but may blow up in some directions.

Chapter 5

Logarithmic Schrödinger Equations in Infinite Dimensions

In this chapter, we present the results of the joint research project with Larry Read, on logarithmic Schrödinger equations (LSE) in infinite-dimensional configuration space. Namely, for the system $\Omega = \mathbb{R}^{\mathbb{Z}^D}$ and any finite volume $\Lambda \Subset \mathbb{Z}^D$, we consider the nonlinear Cauchy problem

$$\begin{aligned} i\partial_t \phi &= -\mathcal{L}_\Lambda \phi + \lambda \phi \log \frac{|\phi|^2}{\nu |\phi|^2} \\ \phi(x, t=0) &= f(x) \end{aligned} \tag{LSE$_\Lambda$}$$

where $x \in \Omega$, $f : \Omega \to \mathbb{C}$ is localised[1] in Λ, $\mathcal{L}_\Lambda = \Delta_\Lambda - \nabla_\Lambda U_\Lambda \cdot \nabla_\Lambda$ with a finite-range potential U_Λ, and ν is the global Gibbs measure associated to U_Λ. All the notations and terminologies are the same as defined in Chapter 2.

Our aim is to find various estimates that are uniform in the size $|\Lambda|$, and use them to

[1] f is localised in $\Lambda \subset \mathbb{Z}^D$ if f only depends on x_j for $j \in \Lambda$ (see Chapter 2, Section 2.1.1).

establish the existence of weak solutions when $\Lambda \uparrow \mathbb{Z}^D$. To this end, we will first prove in Section 5.3 the following log-Sobolev inequality (LSI) for ν to control the nonlinear logarithmic term in the LSE.

Theorem 5.0.1. *The global Gibbs measure ν associated to U_Λ satisfies a log-Sobolev inequality on $\Omega = \mathbb{R}^{\mathbb{Z}^d}$ (under suitable growth conditions on U_Λ), that is,*

$$\mathrm{Ent}_\nu\left(|\phi|^2\right) \equiv \nu\left(|\phi|^2 \log \frac{|\phi|^2}{\nu|\phi|^2}\right) \leq c\|\nabla\phi\|^2_{L^2(\nu)} \tag{5.0.1}$$

for all ϕ such that the right-hand side is well-defined and finite, and some constant $c \geq 0$ independent of the function ϕ.

The conditions on U_Λ are provided in Theorem 5.3.1 (the full form of Theorem 5.0.1) and the remarks/examples following that. Using this inequality, we develop several estimates for the solutions of LSE in Section 5.4. We summarise these in the statement below.

Theorem 5.0.2. *Assume that U_Λ satisfies the same conditions in Theorem 5.0.1. For any $\Lambda \Subset \mathbb{Z}^D$, let ϕ be the solution to (LSE_Λ) with initial data $f \in H^1(\nu)$ localised in $\Lambda_f \subset \Lambda$. Then, for $\lambda > 0$, it follows that*

$$\|\nabla\phi(\cdot,t)\|^2_{L^2(\nu)} \leq e^{2\lambda t}\|\nabla f\|^2_{L^2(\nu)} \quad \text{and} \quad \mathrm{Ent}_\nu\left(|\phi(\cdot,t)|^2\right) \leq (\lambda^{-1}+c)\|\nabla f\|^2_{L^2(\nu)}$$

for all $t > 0$, where c is the constant in (5.0.1). Furthermore, for any $j \in \mathbb{Z}^d$

$$\left\|\nabla_{\{x_j\}}\phi(\cdot,t)\right\|^2_{L^2(\nu)} \leq e^{-\mathcal{N}_j}\|\nabla f\|^2_{L^2(\nu)}$$

holds for all $t \leq \varepsilon\mathcal{N}_j$, where $\varepsilon = \varepsilon(\lambda, U_\Lambda) > 0$ and $\mathcal{N}_j$ is proportional to the distance between j and Λ_f on the lattice.

Note that the above bounds are independent of the size $|\Lambda|$ once the initial condition

is fixed. Therefore, we can construct a sequence $\Lambda_1 \subset \Lambda_2 \subset \cdots$ invading the whole lattice $\mathbb{Z}^D$ and use the above bounds to show that the solutions to $(\mathrm{LSE}_{\Lambda_n})$ converge to a weak solution to the infinite-dimensional LSE. The result is provided in Section 5.5 with the following informal statement of the main theorem.

Theorem 5.0.3. *Suppose the lattice is 1-dimensional* $(\Omega = \mathbb{R}^{\mathbb{Z}})$*, and the potential satisfies the same conditions in Theorem 5.0.1 and has bounded multi-spin interactions. Then for any localised initial data* $f \in H^1(\nu)$ *there exists a weak solution to the infinite-dimensional LSE problem, i.e. there exists* $\phi \in H^1(\nu)$ *such that*

$$\begin{aligned} \nu\left(gi\partial_t\phi\right) &= \nu\left(\nabla_\Omega(g)\cdot\nabla_\Omega(\phi) + g\lambda\phi\log\frac{|\phi|^2}{\nu|\phi|^2}\right) \\ \phi(x,t=0) &= f(x) \end{aligned} \tag{5.0.2}$$

for all smooth, compactly supported, and localised $g : \mathbb{R}^{\mathbb{Z}} \to \mathbb{C}$*, where* $\nabla_\Omega := \left(\nabla_{\{x_j\}}\right)_{j\in\mathbb{Z}}$*.*

We will give an introduction of LSE in the first section and set up the framework in the next one. We then formally state and prove the above theorems in the remaining sections.

5.1 Introduction

The logarithmic Schrödinger equation (LSE) was first introduced in [50], given by

$$i\partial_t\psi = H\psi \equiv -\Delta\psi + V\psi + \lambda\psi\log\frac{|\psi|^2}{\int|\psi|^2 d\mathbf{x}} \tag{5.1.1}$$

with $\mathbf{x} \in \mathbb{R}^m$, $\lambda \in \mathbb{R}$ and Δ denoting the Laplacian in $\mathbb{R}^m$. For non-interacting systems, the LSE enjoys a statistical independence property in the following sense. Let $\{\mathbf{x}_1\} = \mathbb{R}^{m_1}$

and $\{\mathbf{x}_2\} = \mathbb{R}^{m_2}$ be two non-interacting spaces. Suppose for $j = 1, 2$ the LSE

$$i\partial_t \psi_j = H_j \psi_j \equiv -\Delta_j \psi_j + V_j \psi_j + \lambda \psi_j \log \frac{|\psi_j|^2}{\int |\psi_j|^2 d\mathbf{x}_j}$$

is satisfied on $\{\mathbf{x}_j\}$, with wavefunction ψ_j, potential V_j, initial condition $\psi_j(\mathbf{x}_j, 0) = f_j(\mathbf{x}_j)$, and $\Delta_j \equiv \Delta_{\{\mathbf{x}_j\}}$. Then, the following LSE

$$i\partial_t \psi = H\psi \equiv -\Delta\psi + V\psi + \lambda\psi \log \frac{|\psi|^2}{\int |\psi|^2 d\mathbf{x}} = (\psi_2 H_1 \psi_1 + \psi_1 H_2 \psi_2)$$

is satisfied on the composite system $\{\mathbf{x}\} = \{(\mathbf{x}_1, \mathbf{x}_2)\} = \mathbb{R}^{m_1+m_2}$, with wavefunction $\psi(\mathbf{x}, t) = \psi_1(\mathbf{x}_1, t)\psi_2(\mathbf{x}_2, t)$, potential energy $V(\mathbf{x}) = V_1(\mathbf{x}_1) + V_2(\mathbf{x}_2)$, initial condition $\psi(\mathbf{x}, 0) = f_1(\mathbf{x}_1) f_2(\mathbf{x}_2)$, and $\Delta \equiv \Delta_{\{\mathbf{x}\}}$.

Thus the probability density $|\psi|^2$ of a composite system is given by the product of the densities $|\psi_j|^2, j = 1, 2$ of the subsystems, provided that they are initially separate. This shows that non-interacting systems remain statistically independent. Moreover, the normalisation factor $\int |\psi|^2 d\mathbf{x} = 1$ is invariant in time and we have the following additivity of energy property

$$\langle \psi, H\psi \rangle = \sum_j \langle \psi_j, H_j \psi_j \rangle.$$

To study the asymptotic behaviour of large interacting systems we shall use the idea of ground state representation. That is, we consider the wavefunction $\psi \equiv \phi e^{-\frac{1}{2}U}$ with some $U : \mathbb{R}^m \to \mathbb{R}$ independent of time. If U and V satisfy the following relation

$$-\frac{1}{2}\Delta U + \frac{1}{4}|\nabla U|^2 + \lambda U = V, \tag{5.1.2}$$

the LSE (5.1.1) has the following transformation

$$i\partial_t\phi = -\mathcal{L}\phi + \lambda\phi \log \frac{|\phi|^2}{\mu|\phi|^2} \tag{5.1.3}$$

for the linear diffusion operator $\mathcal{L} = \Delta - \nabla U \cdot \nabla$ and the measure $\mu(\phi) = \int \phi e^{-U} d\mathbf{x}$.

In the sequel, we will study (5.1.3) in finite and infinite dimensions where the Lebesgue measure is no longer well-defined. Therefore, it is essential to define the Gibbs measure $d\mu = e^{-U} d\mathbf{x}$ as a reference probability measure in the system. For this purpose, we need to introduce some conditions on the potentials. First we require V to have an additive structure, with a multi-particle interaction and an external potential for single particle. However, due to the nonlinear relation (5.1.2) between U and V, this is not guaranteed even if U has a similar additive structure. To avoid the issue, one may in addition require U to have a finite range of interactions. This set of conditions provides an interesting family of potentials that we will introduce later in the setup.

The LSE plays an important role in the study of nonlinear Schrödinger equations. This is due to the ubiquity of logarithmic nonlinearity in mathematical physics, for instance, in quantum mechanics [50, 51], nuclear physics [52], information theory [53, 54], quantum optics [55], quantum gravity [56], modelling magma transport [57], transport and diffusion phenomena [58], theory of Bose–Einstein condensation [59] and others. For results on existence and regularity of solutions to different variations of LSE, the literature is wide and one may refer to [60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73].

In the above cited research works, the LSE is studied on finite-dimensional configuration space, which rules out the possibility for systems with infinitely many particles. This project aims to fill in this gap by introducing an infinite-dimensional setting for the LSE in the context of statistical mechanics. Namely, we consider the configuration space $\Omega = \mathbb{R}^{\mathbb{Z}^D}$ with a finite-range potential U. We construct the associated global Gibbs measure ν and

the operator $\mathcal{L}$ corresponding to the Dirichlet form in $\mathbb{L}^2(\nu)$.

As the first result, we show that under some growth conditions of U, the global measure ν satisfies the logarithmic Sobolev inequality

$$\mathrm{Ent}_\nu\left(|\phi|^2\right) := \nu\left(|\phi|^2 \log \frac{|\phi|^2}{\nu|\phi|^2}\right) \leq C\|\nabla\phi\|^2_{\mathbb{L}^2(\nu)} \tag{5.1.4}$$

for some constant $C \in (0,\infty)$ independent of ϕ. This inequality implies, under a proper choice of λ, that the right hand side of the LSE (5.1.3) is well defined in arbitrary dimensions. The proof follows the general idea in [2, 4, 19, 74, 75, 76], while we need some extra attention on the non-compactness of the space $\Omega = \mathbb{R}^{\mathbb{Z}^D}$ and the unboundedness of the potential U. The main theorem is presented in section 5.3.

With the aid of this inequality, we will find various estimates for the solutions to the LSE. These include gradient and entropy bounds in the following form

$$\|\nabla\phi(\cdot,t)\|^2_{\mathbb{L}^2(\nu)} \leq e^{2\lambda t}\|\nabla f\|^2_{\mathbb{L}^2(\nu)} \quad \text{and} \quad \mathrm{Ent}_\nu\left(|\phi(\cdot,t)|^2\right) \leq (\lambda^{-1}+C)\|\nabla f\|^2_{\mathbb{L}^2(\nu)}$$

where $t>0$, f is the initial data and $C>0$ is the constant in (5.1.4). Moreover, we prove a finite-speed propagation of information property of the solution, in the sense that

$$\left\|\nabla_{\{x_j\}}\phi(\cdot,t)\right\|^2_{\mathbb{L}^2(\nu)} \leq e^{-\mathcal{N}_j}\|\nabla f\|^2_{\mathbb{L}^2(\nu)}$$

holds for all $t \leq \varepsilon\mathcal{N}_j$, where $\varepsilon = \varepsilon(\lambda, U) > 0$ and $\mathcal{N}_j$ is proportional to the distance between j and the dependence of f on the lattice $\mathbb{Z}^D$. This bound shows that, if the initial data depends on finitely many variables, the LSE can only enlarge the initial dependence with finite speed. This property resembles the one in the case of semigroups (see e.g. [4, 74]) and is crucial to prove the existence of solutions to the infinite-dimensional LSE. The results concerning these bounds are presented in Section 5.4.

Using the finite-dimensional estimates, we establish in Section 5.5 the existence of weak solutions to the following infinite-dimensional LSE

$$\begin{aligned} \nu\left(gi\partial_t\phi\right) &= \nu\left(\nabla g\cdot\nabla\phi + g\lambda\phi\log\frac{|\phi|^2}{\nu|\phi|^2}\right) \\ \phi(x,t=0) &= f(x) \end{aligned} \tag{5.1.5}$$

for all smooth compactly supported function $g:\mathbb{R}^{\mathbb{Z}}\to\mathbb{C}$ that depends on finitely many variables. We then conclude by discussing the construction of solitary solutions to the LSE in finite and infinite dimensions.

5.2 Infinite-dimensional setting

The setup is almost the same as in Chapter 2, for the convenience of the readers we briefly recap some important notations and definitions in this section. Consider the particles being located on a lattice $\mathbb{Z}^D$, endowed with the l^∞ distance

$$\|i\| \coloneqq \max_{1\le n\le D}|i_n|$$

for $i=(i_1,...,i_D)\in\mathbb{Z}^D$. We also define the set distance by

$$\operatorname{dist}(\Lambda,\Lambda') \equiv \inf_{i\in\Lambda, j\in\Lambda'}\|i-j\|, \quad \operatorname{diam}(\Lambda)\equiv\sup_{i,j\in\Lambda}\|i-j\| \quad \forall\,\Lambda,\Lambda'\subset\mathbb{Z}^D.$$

For each particle at $j\in\mathbb{Z}^D$, we assign a random variable x_j called spin with values in $\mathbb{R}$. The configuration of the entire system is therefore $x=(x_j)_{j\in\mathbb{Z}^D}$, living in the infinite-dimensional space $\Omega=\mathbb{R}^{\mathbb{Z}^D}$.

For any function f on Ω, we say f is localised in some $\Lambda\subset\Omega$ if f only depends on spins inside $\mathbb{R}^\Lambda$. Let $\Lambda_f\subset\mathbb{Z}^D$ be the smallest set in which f is localised, we call f a local

or cylinder function if Λ_f is compact. The potential on Ω is given by a family of functions $\mathcal{J} = \{J_K\}_{K \Subset \mathbb{Z}^D}$ where for each $K \Subset \mathbb{Z}^D$, $J_K \in C^2(\Omega, \mathbb{R})$ is localised in K and $J_K \equiv 0$ if $\mathrm{diam}(K) > R$ for some range $R \in \mathbb{N}$. The potential energy U_Λ in any finite volume $\Lambda \Subset \mathbb{Z}^D$ is well-defined by

$$U_\Lambda(x) = \sum_{\Lambda \cap X \neq \varnothing} J_K(x).$$

Example 5.2.1 (Billinear Potential)**.** Consider the bilinear potential

$$J_K(x) = \begin{cases} C_{ii}x_i^2 & \text{if } K = \{i\} \\ C_{ij}x_i x_j & \text{if } K = \{i,j\} \\ 0 & \text{otherwise} \end{cases}$$

where $C_{ij} \in \mathbb{R}$ is uniformly bounded and $C_{ij} = 0$ if $\|i-j\| > R$ for some finite range $R > 0$. The potential energy inside Λ is then given by

$$U_\Lambda(x) = \sum_{\{i,j\} \cap \Lambda \neq \varnothing} C_{ij}x_i x_j.$$

With the same boundary condition setup in Chapter 2, we define the local Gibbs measure $\mathbb{E}_\Lambda$ by

$$\mathbb{E}_\Lambda^\omega(f) \coloneqq \int_{\Omega_\Lambda} f d\mathbb{E}_\Lambda^\omega = \frac{1}{Z_\Lambda^\omega} \int_{\Omega_\Lambda} f(x_\Lambda \bullet \omega_{\Lambda^c}) e^{-U_\Lambda^\omega(x_\Lambda)} dx_\Lambda$$

for any $\Lambda \Subset \mathbb{Z}^D$ and boundary conditions $\omega \in \Omega$. For $\mathbb{E}_\Lambda^\omega$ to be well-defined in all finite volumes, we shall hereafter only consider U_Λ that satisfies

$$0 < \int_{\mathbb{R}^\Lambda} e^{-U_\Lambda^\omega} \, dx_\Lambda < \infty \quad \forall \, \Lambda \Subset \mathbb{Z}^D \quad \forall \, \omega \in \Omega. \tag{5.2.1}$$

The family $\{\mathbb{E}_\Lambda\}_{\Lambda \Subset \mathbb{Z}^D}$ is called a local specification, with which one can associate a global

Gibbs measure ν by the DLR equation

$$\nu \mathbb{E}_\Lambda f = \nu f \tag{5.2.2}$$

for all $\Lambda \Subset \mathbb{Z}^D$ and measurable function f on Ω.

Here we adopt the same definitions for ∇ and Δ as in Chapter 2. That is, define the derivatives $\nabla_j := \partial/\partial x_j$ and $\Delta_j := (\partial/\partial x_j)^2$ on $\mathbb{R}^{\{j\}}$, the gradient $\nabla_\Lambda f := \big(\nabla_j f\big)_{j\in\Lambda}$ and $\nabla f := \big(\nabla_j f\big)_{j\in\mathbb{Z}^D}$ for $\Lambda \subset \mathbb{Z}^D$, and the Laplacian $\Delta_\Lambda := \sum_{j\in\Lambda}\Delta_j$ and $\Delta := \sum_{j\in\mathbb{Z}^D}\Delta_j$. The dot product is given by

$$\nabla f \cdot \nabla g := \sum_{j\in\mathbb{Z}^D} (\nabla_j f)(\nabla_j g), \quad \nabla_\Lambda f \cdot \nabla_\Lambda g := \sum_{j\in\Lambda} (\nabla_j f)(\nabla_j g),$$

with the l^2 length $|\nabla f|^2 := \nabla f \cdot \nabla f$ and $|\nabla_\Lambda f|^2 := \nabla_\Lambda f \cdot \nabla_\Lambda f$. With these notations, we define the following function space

$$\mathfrak{F}(\Omega) := \left\{ f : \Omega \to \mathbb{C} \;\middle|\; \begin{array}{l} f \text{ is local, bounded and twice differentiable} \\ f \in H^2(\mathbb{E}^\omega_\Lambda, \mathbb{C}) \text{ for all } \Lambda \Subset \mathbb{Z}^D \text{ and } \omega \in \Omega \end{array} \right\}, \tag{5.2.3}$$

where $H^2(\mathbb{E}^\omega_\Lambda)$ is understood as $\mathbb{E}^\omega_\Lambda|f|^2 + \mathbb{E}^\omega_\Lambda|\nabla_\Lambda f|^2 + \mathbb{E}^\omega_\Lambda|\Delta_\Lambda f|^2 < \infty$. Note that this implies $f \in H^2(\nu)$ by the DLR equation. Then for any $f \in \mathfrak{F}(\Omega)$ the linear diffusion operator

$$\mathcal{L}^\omega_\Lambda f = \Delta_\Lambda f - \nabla_\Lambda U^\omega_\Lambda \cdot \nabla_\Lambda f$$

is well-defined for all $\Lambda \Subset \mathbb{Z}^D$ and $\omega \in \Omega$. Next, we recall some coercive inequalities for $\mathbb{E}^\omega_\Lambda$.

Definition 5.2.2. Let f^* be the complex conjugate of f. We say $\mathbb{E}_\Lambda$ satisfies the spectral

gap inequality (SGI) with coefficient $C_{SG} \in (0, \infty)$ if

$$\mathbb{E}_\Lambda^\omega |f - \mathbb{E}_\Lambda^\omega f|^2 \leq \mathbb{E}_\Lambda^\omega (f^*(-\mathcal{L}_\Lambda) f) = C_{SG} \mathbb{E}_\Lambda^\omega |\nabla_\Lambda f|^2$$

and $\mathbb{E}_\Lambda$ satisfies the logarithmic Sobolev inequality (LSI) with coefficient $C_{LS} \in (0, \infty)$ if

$$\mathrm{Ent}_\Lambda(|f|^2) := \mathbb{E}_\Lambda^\omega \left(|f|^2 \log \frac{|f|^2}{\mathbb{E}_\Lambda^\omega |f|^2} \right) \leq C_{LS} \mathbb{E}_\Lambda^\omega |\nabla_\Lambda f|^2$$

for all boundary conditions $\omega \in \Omega$.

5.3 Log Sobolev inequalities for the Gibbs measure

In this section, we present sufficient conditions on the potential energy U_Λ such that the associated global Gibbs measure ν satisfies a logarithmic Sobolev inequality (LSI). We will hereafter assume the existence of ν and give more discussion in Remark 5.3.6 below. We also drop the notation of ω for simplicity when its role is not important.

Theorem 5.3.1. *Let U_Λ^ω be a twice differentiable potential of some finite range $R > 0$ for all $\Lambda \Subset \mathbb{Z}^D$ and $\omega \in \Omega$. Suppose U_Λ^ω satisfies the condition (5.2.1) and $\|\nabla_i \nabla_j U_\Lambda^\omega\|_\infty \leq A$ for all $\Lambda \Subset \mathbb{Z}^D$, $\omega \in \Omega$, and $i, j \in \mathbb{Z}^D$ with $i \neq j$ where $A > 0$ is independent of i, j, Λ, ω. If the finite-volume log-Sobolev inequality*

$$\mathbb{E}_\Lambda^\omega \left(|f|^2 \log \frac{|f|^2}{\mathbb{E}_\Lambda^\omega |f|^2} \right) \leq \tilde{c}\, \mathbb{E}_\Lambda^\omega |\nabla_\Lambda f|^2 \tag{5.3.1}$$

holds for all $\Lambda \Subset \mathbb{Z}^D$, $\omega \in \Omega$ and some $\tilde{c} > 0$ independent of f, Λ and ω, then the global

Gibbs measure ν associated to $(\mathbb{E}_\Lambda)_{\Lambda\Subset\mathbb{Z}}$ is unique and satisfies the log-Sobolev inequality

$$\nu\left(|f|^2\log\frac{|f|^2}{\nu|f|^2}\right)\leq C_{LS}\nu|\nabla f|^2 \tag{5.3.2}$$

for all $f\in\mathfrak{F}(\Omega)$ and some constant $C_{LS}>0$ independent of f.

Remark 5.3.2. A sufficient condition for the volume-uniform LSI (5.3.1) is that

$$\sum_{i,j\in\Lambda}\big(\nabla_i\nabla_j U_\Lambda^\omega(x_\Lambda)\big)x_ix_j\geq\rho\sum_{j\in\Lambda}x_j^2$$

for all $\Lambda\Subset\mathbb{Z}^D$, $x_\Lambda\in\mathbb{R}^\Lambda$, $\omega\in\Omega$ and some $\rho>0$ independent of Λ,x,ω. This is a consequence of the Bakry-Emery condition as discussed in Remark 2.3.9.

Below we give some example potentials that satisfy the conditions in the above theorem.

Example 5.3.3. Consider the bilinear potential on $\mathbb{Z}^D$ given by

$$U_\Lambda(x)=\sum_{\{i,j\}\cap\Lambda\neq\varnothing}C_{ij}x_ix_j,$$

where $|C_{ij}|\leq 1$ for $i\neq j$, $C_{ij}=0$ if $\|i-j\|>R$ for some finite range $R>0$, and $(2R+1)^D\leq C_{ii}\leq\overline{C}$ for some $\overline{C}\geq(2R+1)^D$. Then U_Λ satisfies all conditions in Theorem 5.3.1 by Remark 5.3.2. This can be seen by the AM-GM inequality $2x_ix_j\geq -x_i^2-x_j^2$.

Example 5.3.4. Consider the same bilinear potential but with bounded perturbations

$$U_\Lambda(x)=\sum_{\{i,j\}\cap\Lambda\neq\varnothing}C_{ij}x_ix_j+\varepsilon\sum_{\substack{K\Subset\mathbb{Z}^D\\K\cap\Lambda\neq\varnothing}}W_K(x_K),$$

where $\varepsilon\in\mathbb{R}$, $W_K\in C^2(\mathbb{R}^K)$ has derivatives bounded uniformly in K of order $n=0,1,2$ and $W_K\equiv 0$ if $\operatorname{diam}(K)>R$. Suppose C_{ij} satisfies the same conditions as in the above

example, then for sufficiently small $|\varepsilon|$ all conditions in Theorem 5.3.1 are satisfied.

As another example we introduce the restricted setup with no boundary conditions. In Section 5.5 we will use this setup extensively to establish the existence results for the LSE in infinite dimensions.

Example 5.3.5. For any $\Lambda \Subset \mathbb{Z}^D$, consider the following potential

$$U^\circ_\Lambda(x) \coloneqq \sum_{K \subseteq \Lambda} J_K(x) \tag{5.3.3}$$

where J_K satisfies the same conditions as in Section 5.2 but now is strictly localised inside $K \subset \Lambda$. This means the potential has no boundary conditions. Then we denote by $\mathbb{E}^\circ_\Lambda$ the local Gibbs measure corresponding to U°_Λ, i.e. $d\mathbb{E}^\circ_\Lambda \coloneqq (e^{-U^\circ_\Lambda} dx_\Lambda)(\int e^{-U^\circ_\Lambda} dx_\Lambda)^{-1}$, and by ν° the global Gibbs measure associated to $(\mathbb{E}^\circ_\Lambda)_{\Lambda \Subset \mathbb{Z}^D}$ through the DLR equation. Then Theorem 5.3.1 holds true with $(U_\Lambda, \mathbb{E}_\Lambda, \nu)$ replaced by $(U^\circ_\Lambda, \mathbb{E}^\circ_\Lambda, \nu^\circ)$.

Remark 5.3.6 (Existence)**.** Concerning the existence of ν in our setting, one can restrict the configuration space Ω to the following space of slowly increasing sequences

$$\mathfrak{S}_\Omega \coloneqq \big\{x \in \Omega : \exists n > 0 \text{ such that } \sup_j |x_j|/|j|^n < \infty\big\}, \tag{5.3.4}$$

and introduce suitable growth conditions for the multi-particle interaction (e.g. the bi-linear interaction in the examples). Then by the compactness arguments in [77], there exists a global Gibbs measure ν associated to the potential on $\mathfrak{S}_\Omega$. The restriction to the subspace $\mathfrak{S}_\Omega$ can be justified by the fact that $\mathfrak{S}_\Omega$ is dense in Ω, i.e. $\nu(\mathfrak{S}_\Omega) = 1$ (see [77, Proposition A.1]), and that one can obtain compactness argument on such space.

5.3.1 Strategy to prove Theorem 5.3.1

The proof of Theorem 5.3.1 draws from the concepts in [2, 4, 19, 74, 75, 76], with modifications to accommodate the non-compact nature of the space $\Omega = \mathbb{R}^{\mathbb{Z}^D}$ and the unboundedness of the potential U. The main idea is to construct a probability measure (called *transition matrix*) Π on Ω that satisfies the LSI and $\Pi^n \to \nu$ as $n \to \infty$. Then one can derive the ν-LSI by iteratively applying the Π-LSI in a telescoping series of entropy terms given by the sequence $\{\Pi^n\}_{n\in\mathbb{N}}$.

Herein we demonstrate a construction of the transition matrix Π. The strategy is to define a set of equally sized cubes $\{Q_k\}$ that partition the lattice $\mathbb{Z}^D$, then define Π as the infinite product of local Gibbs measures $\mathbb{E}_{Q_k}$ for each Q_k. To this end, fix a parameter $L \in \mathbb{N}$ and the interaction range $R \in \mathbb{N}$, we define the cubes

$$Q_0 \coloneqq [0, 2(L+R)]^D \cap \mathbb{Z}^D \quad \text{and} \quad Q_k \coloneqq k + Q_0,\ k \in \mathbb{Z}^D \tag{5.3.5}$$

in the sense that Q_0 is the base cube and Q_k is the translation of Q_0 by a vector $k \in \mathbb{Z}^D$. For any $s \in \mathbb{N}$ let $v_s = (v_s^{(1)}, ..., v_s^{(D)}) \in \{0,1\}^D$ be the binary representation of s, i.e. $s = \sum_{n=1}^{D} v_s^{(n)} 2^{n-1}$. We define the following set of cubes

$$\Gamma_s \coloneqq \bigcup_{k\in T_s} Q_k \tag{5.3.6}$$

for $s = 0, 1, ..., 2^D - 1$ and the translation set $T_s \coloneqq \left\{k \in \mathbb{Z}^D : k \in 2(L+2R)\mathbb{Z}^D + (L+2R)v_s\right\}$. See Figure 5.1 for a demonstration in the 2-dimensional lattice $\mathbb{Z}^2$.

By such construction, each Γ_s contains disjoint cubes of the same shape as Q_0, equally distributed in $\mathbb{Z}^D$, and separated from each other by a distance greater than $2R$. Moreover, as s goes over $\{0, 1, ..., 2^D - 1\}$ the vector v_s covers all directions in $\{0,1\}^D$, thus the union

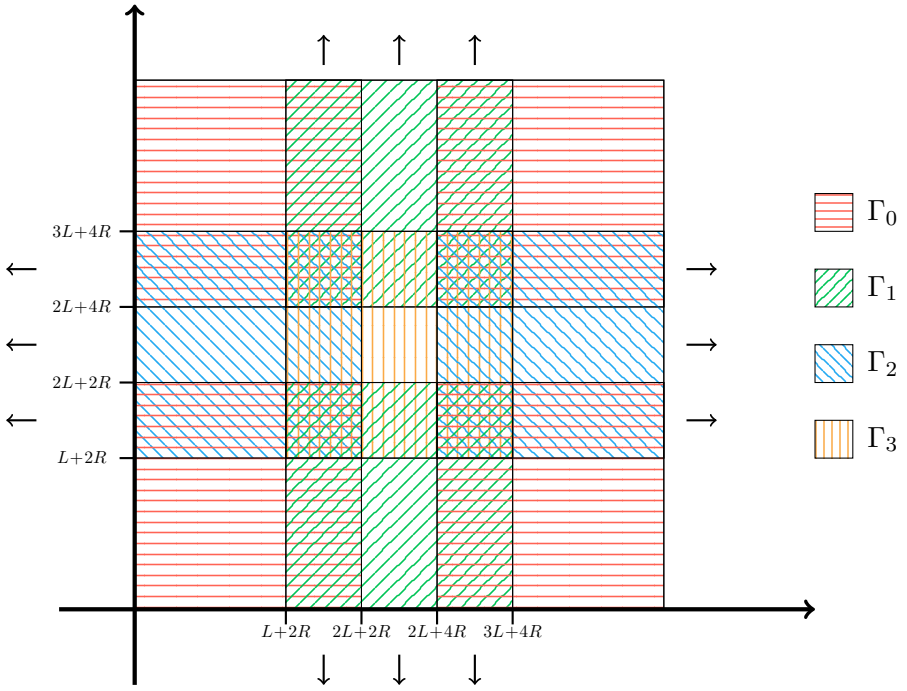


Figure 5.1: Construction of Q_k and Γ_s in $\mathbb{Z}^2$

$\bigcup_{s=0,1,\ldots,2^D-1} \Gamma_s = \mathbb{Z}^D$ covers the entire lattice. With $\mathbb{E}_{Q_k}$ denoting the local Gibbs measure on cube Q_k, we define the product probability measure $\mathbb{E}_s^\omega := \bigotimes_{Q_k \subset \Gamma_s} \mathbb{E}_{Q_k}^\omega$. Then one can construct the transition matrix Π by

$$\Pi^\omega := \mathbb{E}_{2^D-1}^\omega \mathbb{E}_{2^D-2} \cdots \mathbb{E}_1 \mathbb{E}_0$$

with boundary conditions $\omega \in \Omega$, in the sense that $x_{(\Gamma_{2^D-1})^c} = \omega_{(\Gamma_{2^D-1})^c}$.

One may notice that this construction is very similar to the one in Chapter 3, Lemma 3.2.10 for the ν-SGI. The difference lies in that here Π is the product of measures on cubes, whose size will be changed later as a parameter to aid the proof. In comparison, the Π in Chapter 3 is the product of measures on single sites, which are fixed and does not act as parameters as the cube Q_k. In the following, we show that, despite the said difference, the

measure Π here satisfies similar properties as those given in Lemma 3.2.10.

Proposition 5.3.7. *Let Π be defined as above by the cubes, then the following hold.*

(i) *The DLR equation holds:*

$$\nu\Pi f = \nu f.$$

(ii) *There exists $\bar{c} > 0$ such that*

$$\nu\Big(\Pi\big(|f|^2 \log |f|^2\big) - \Pi|f|^2 \log \Pi|f|^2\Big) \leq \bar{c}\, \nu|\nabla f|^2.$$

(iii) *There exists $\gamma \in (0,1)$ such that*

$$\nu\big|\nabla(\Pi|f|^2)^{\frac{1}{2}}\big|^2 \leq \gamma\nu|\nabla f|^2.$$

(iv) *The sequence Π^n converges to ν almost surely, i.e.*

$$\lim_{n\to\infty} \Pi^n(f) = \nu(f) \qquad \nu\text{-}a.s,$$

and the limit defines a unique global Gibbs measure ν associated to $(\mathbb{E}_\Lambda)_{\Lambda\in\mathbb{Z}^D}$.

Before proving Proposition 5.3.7, we first demonstrate how it leads to the ν-LSI (5.3.2).

Proof of Theorem 5.3.1. Let $f_0 = f$ and $f_{n+1} = \Pi f_n$ for $n \in \mathbb{N}$. By (i) we have

$$\nu\left(|f|^2 \log \frac{|f|^2}{\nu|f|^2}\right) = \sum_{n=0}^{\mathcal{N}-1} \nu\Big(\Pi\big(|f_n|^2 \log |f_n|^2\big) - \Pi|f_n|^2 \log \Pi|f_n|^2\Big) + \nu\left(|f_{\mathcal{N}}|^2 \log \frac{|f_{\mathcal{N}}|^2}{\nu|f|^2}\right)$$

then by (ii), it follows

$$\nu\left(|f|^2\log\frac{|f|^2}{\nu|f|^2}\right)\leq\bar{c}\sum_{n=0}^{\mathcal{N}-1}\nu|\nabla f_n|^2+\nu\left(|f_{\mathcal{N}}|^2\log\frac{|f_{\mathcal{N}}|^2}{\nu|f|^2}\right).$$

Since $\gamma\in(0,1)$, applying (iii) inductively gives

$$\nu\left(|f|^2\log\frac{|f|^2}{\nu|f|^2}\right)\leq\frac{\bar{c}}{1-\gamma}\nu|\nabla f|^2+\nu\left(|f_{\mathcal{N}}|^2\log\frac{|f_{\mathcal{N}}|^2}{\nu|f|^2}\right).$$

Then by (iv) and the monotone convergence theorem, the last term satisfies

$$\nu\left(|f_{\mathcal{N}}|^2\log\frac{|f_{\mathcal{N}}|^2}{\nu|f|^2}\right)\to 0$$

as $\mathcal{N}\to\infty$, which yields the LSI for ν with coefficient $C_{LS}=\bar{c}/(1-\gamma)$. □

Remark 5.3.8. We remark that the construction of Π is not unique. In fact, the base set Q_0 is not necessarily a cube; rather, it can take any shape as long as it is optimised under certain conditions. Our preference for a homogeneous cube partition is simply to provide an easy read for the proof.

Proof of Proposition 5.3.7

Here we demonstrate the proof of conditions (i) to (iv). Condition (i) simply follows from the DLR equation (5.2.2) of $\mathbb{E}_{Q_k}$ for each cube Q_k. To verify conditions (ii) to (iv), we use the following lemma.

Lemma 5.3.9. *Suppose all the conditions in Theorem 5.3.1 are satisfied, and let Q_k be*

the cubes as constructed above. Then for any Q_k and $j \notin Q_k$ the 'sweeping out' inequalities

$$\nu\Big|\nabla_j(\mathbb{E}_{Q_k} f)\Big|^2 \leq \sum_{i\in\mathbb{Z}^D} \alpha_{ji}\nu|\nabla_i f|^2 \tag{5.3.7}$$

and

$$\nu\Big|\nabla_j\big(\mathbb{E}_{Q_k}|f|^2\big)^{\frac{1}{2}}\Big|^2 \leq \sum_{i\in\mathbb{Z}^D} \alpha_{ji}\nu|\nabla_i f|^2 \tag{5.3.8}$$

hold with $0 \leq \alpha_{ji} \leq \tilde{M}|Q_0|e^{-M\|j-i\|}$ for some constants $M, \tilde{M} > 0$ independent of j and Q_k. Here $|Q_0|$ denotes the number of sites contained in the base cube Q_0.

Assuming Lemma 5.3.9 is true, we now verify condition (ii). Denote by $f_{-1} \coloneqq f$ and $f_s \coloneqq (\mathbb{E}_s\mathbb{E}_{s-1}\cdots\mathbb{E}_0|f|^2)^{1/2}$, we obtain the telescoping series

$$\nu\Big(\Pi\big(|f|^2\log|f|^2\big) - \Pi|f|^2\log\Pi|f|^2\Big) = \sum_{s=0}^{2^D-1} \nu\mathbb{E}_s\left(f_{s-1}^2\log\frac{f_{s-1}^2}{\mathbb{E}_s f_{s-1}^2}\right)$$

by the DLR equation. For any s, each local measure $\mathbb{E}_{Q_k}$ with $Q_k \subset \Gamma_s$ satisfies the LSI with coefficient $\tilde{c}$ by the condition (5.3.1), their product $\mathbb{E}_s$ therefore satisfies the same LSI by Theorem 2.3.5, which implies

$$\nu\Big(\Pi\big(|f|^2\log|f|^2\big) - \Pi|f|^2\log\Pi|f|^2\Big) \leq \tilde{c}\sum_{s=0}^{2^D-1}\nu|\nabla_{\Gamma_s} f_{s-1}|^2. \tag{5.3.9}$$

We now bound the quantity $\nabla_{\Gamma_s} f_{s-1}$. For any $j \in \Gamma_s$, if $j \in \Gamma_{s-1}$ then $\nabla_j f_{s-1} = 0$. If $j \notin \Gamma_{s-1}$, there exists only one cube $Q_{k(j)} \subset \Gamma_{s-1}$ such that $\mathrm{dist}(j, Q_{k(j)}) \leq R$. In the latter case, we have

$$\nabla_j f_{s-1} = \nabla_j(\mathbb{E}_{\Gamma_{s-1}\setminus Q_{k(j)}}\mathbb{E}_{Q_{k(j)}} f_{s-2}^2)^{\frac{1}{2}} \leq \mathbb{E}_{\Gamma_{s-1}\setminus Q_{k(j)}}\Big|\nabla_j(\mathbb{E}_{Q_{k(j)}} f_{s-2}^2)^{\frac{1}{2}}\Big|^2$$

by the Cauchy-Schwartz inequality. Integrating this with ν and using the sweeping out

inequality (5.3.8) for the cube $Q_{k(j)}$, it follows

$$\nu|\nabla_j f_{s-1}|^2 \leq \sum_{i\in\mathbb{Z}^D\setminus\Gamma_{s-2}} \alpha_{ji}\nu|\nabla_i f_{s-2}|^2\,, \tag{5.3.10}$$

where $i \in \mathbb{Z}^D \setminus \Gamma_{s-2}$ since $f_{s-2} = (\mathbb{E}_{\Gamma_{s-2}}|f_{s-3}|^2)^{1/2}$ has no dependence in Γ_{s-2}. Now that $\nabla_j f_{s-1}$ is bounded by $\nabla_i f_{s-2}$, one can use the same estimate iteratively to reach the gradients on f, i.e.

$$\nu|\nabla_j f_{s-1}|^2 \leq \sum_{i\in\mathbb{Z}^D} \eta_{ji}^{(s)}\nu|\nabla_i f|^2\,, \tag{5.3.11}$$

where

$$\eta_{ji}^{(s)} = \sum_{i_1\in\mathbb{Z}^D\setminus\Gamma_{s-2}} \sum_{i_2\in\mathbb{Z}^D\setminus\Gamma_{s-3}} \cdots \sum_{i_{s-1}\in\mathbb{Z}^D\setminus\Gamma_0} \big(\alpha_{ji_1}\alpha_{i_1i_2}\cdots\alpha_{i_{s-1}i}\big). \tag{5.3.12}$$

Summing (5.3.11) over all $j \in \Gamma_s$, the constant before $\nu|\nabla_i f|^2$ is given by

$$\sum_{j\in\Gamma_s} \eta_{ji}^{(s)} \leq \bigg(\sup_{i\in\mathbb{Z}^D} \sum_{k\in\mathbb{Z}^D} \alpha_{ki}\bigg)^s \leq (\tilde{M}|Q_0|)^s \bigg(\sum_{k\in\mathbb{Z}^D} e^{-M\|k\|}\bigg)^s \leq (\tilde{M}|Q_0|C_D)^s \tag{5.3.13}$$

where we used the condition $\alpha_{ji} \leq \tilde{M}|Q_0|e^{-M\|j-i\|}$ in Lemma 5.3.7, and $C_D = \sum_k e^{-M\|k\|}$ is a finite constant that only depends on M and the dimension D of the lattice. Taking this bound in (5.3.9), one gets

$$\nu\Big(\Pi\big(|f|^2\log|f|^2\big) - \Pi|f|^2\log\Pi|f|^2\Big) \leq \bar{c}\nu|\nabla f|^2 \tag{5.3.14}$$

with $\bar{c} = \tilde{c}\big(1 + (\tilde{M}|Q_0|C_D) + ... + (\tilde{M}|Q_0|C_D)^{2^D}\big)$, and this concludes the proof of (ii).

To verify (iii), we use the same iterative strategy as in (5.3.10) and (5.3.11) to get

$$\begin{aligned}\nu\big|\nabla(\Pi|f|^2)^{\frac{1}{2}}\big|^2 &= \sum_{i_1\in\mathbb{Z}^D\setminus\Gamma_{2^D-1}} \nu|\nabla_{i_1} f_{2^D-1}|^2 \\ &\le \sum_{i_1\in\mathbb{Z}^D\setminus\Gamma_{2^D-1}} \sum_{i_2\in\mathbb{Z}^D\setminus\Gamma_{2^D-2}} \alpha_{i_1 i_2}\nu|\nabla_{i_2} f_{2^D-2}|^2 \\ &\le \sum_{i_1\in\mathbb{Z}^D\setminus\Gamma_{2^D-1}} \sum_{i_2\in\mathbb{Z}^D\setminus\Gamma_{2^D-2}} \sum_{i_3\in\mathbb{Z}^D\setminus\Gamma_{2^D-3}} \alpha_{i_1 i_2}\alpha_{i_2 i_3}\nu|\nabla_{i_3} f_{2^D-3}|^2 \\ &\le \cdots \le \sum_{i\in\mathbb{Z}^D} \tilde{\eta}_i \nu|\nabla_i f|^2, \end{aligned} \tag{5.3.15}$$

with the constant

$$\tilde{\eta}_i = \sum_{i_1\in\mathbb{Z}^D\setminus\Gamma_{2^D-1}} \sum_{i_2\in\mathbb{Z}^D\setminus\Gamma_{2^D-2}} \cdots \sum_{i_{2^D}\in\mathbb{Z}^D\setminus\Gamma_0} \big(\alpha_{i_1 i_2}\alpha_{i_2 i_3}\cdots\alpha_{i_{2^D} i}\big). \tag{5.3.16}$$

For simplicity, instead of the above 2^D summations over each i_n, we shall combine them into one summation over each path in $P \coloneqq \big\{(i_1, i_2, ..., i_{2^D}) : i_n \in \mathbb{Z}^D \setminus \Gamma_{2^D-n}, n = 1, ..., 2^D\big\}$. It then follows

$$\tilde{\eta}_i \le (\tilde{M}|Q_0|)^{2^D} \sum_{(i_1,...,i_n)\in P} e^{-M\left(\|i_1-i_2\|+\|i_2-i_3\|+\cdots+\|i_{2^D}-i\|\right)}$$

due to the condition $\alpha_{ji} \le \tilde{M}|Q_0|e^{-M\|j-i\|}$.

Now we claim that on each path $(i_1, ..., i_{2^D}) \in P$, there exists at least one pair (i_n, i_{n+1}) such that $\|i_n - i_{n+1}\| \ge L/2^{D+1}$. Indeed, by the construction of the cubes Q_k in (5.3.5) and the set Γ_s in (5.3.6), the intersection between any two overlapping cubes Q_k and $Q_{\tilde{k}}$ is a (hyper)rectangle whose shortest edge has a length of L. Since each path has $2^D - 1$ segments, there is at least one segment with length $\frac{L}{2(2^D-1)} \ge \frac{L}{2^{D+1}}$. This property allows us to partition P in the following way. For $n = 1, ..., 2^D - 1$, define $P_n \coloneqq \big\{(i_1, ..., i_{2^D}) \in$

$P : \|i_n - i_{n+1}\| \geq L/2^{D+1}\}$ then $P = \bigcup_{n=1}^{2^D-1} P_n$. For the sum over P_1, we have

$$\begin{aligned}
&\sum_{(i_1,\dots,i_{2^D})\in P_1} e^{-M\left(\|i_1-i_2\|+\|i_2-i_3\|+\cdots+\|i_{2^D}-i\|\right)} \\
&\leq \sum_{i_{2^D}\in\mathbb{Z}^D} \cdots \sum_{i_2\in\mathbb{Z}^D} \sum_{\substack{i_1\in\mathbb{Z}^D \\ \|i_1-i_2\|\geq L/2^{D+1}}} e^{-M\left(\|i_1-i_2\|+\cdots+\|i_{2^D}-i\|\right)} \\
&\leq L^D e^{-ML/2^{D+1}} \sum_{i_{2^D}\in\mathbb{Z}^D} \cdots \sum_{i_2\in\mathbb{Z}^D} \sum_{i_1\in\mathbb{Z}^D} e^{-M\left(\|i_1-i_2\|+\cdots+\|i_{2^D}-i\|\right)} \\
&= L^D e^{-ML/2^{D+1}} (C_D)^{2^D},
\end{aligned}$$

where $C_D = \sum_j e^{-M\|j\|}$ as in (5.3.13). Using the same idea for sums on all other P_n, it follows $\tilde{\eta}_i \leq (2L)^D (\tilde{M}|Q_0|C_D)^{2^D} e^{-ML/2^{D+1}}$ and therefore by (5.3.15) we have

$$\nu\left|\nabla(\Pi|f|^2)^{\frac{1}{2}}\right|^2 \leq (2L)^D (\tilde{M}|Q_0|C_D)^{2^D} e^{-ML/2^{D+1}} \nu|\nabla f|^2. \tag{5.3.17}$$

Since Q_0 has edge length $2(L+R)$, the size $|Q_0|$ grows polynomially in L. One can therefore make L large enough so that $\gamma = (2L)^D (\tilde{M}|Q_0|C_D)^{2^D} e^{-ML/2^{D+1}} < 1$, which concludes (iii).

To verify (iv), we redefine $f_{-1} \coloneqq f$ and $f_s \coloneqq \mathbb{E}_s \mathbb{E}_{s-1} \cdots \mathbb{E}_0(f)$. By the argument for (5.3.9), each $\mathbb{E}_s$ satisfies the LSI and thus the SGI (by Theorem 2.3.7) with coefficient $\tilde{c}$. Therefore, one has the following estimate for the Π-variance

$$\nu|f - \Pi f|^2 \leq 2^{2^D} \sum_{s=0}^{2^D-1} \nu\mathbb{E}_s|f_{s-1} - \mathbb{E}_s f_{s-1}|^2 \leq 2^{2^D} \tilde{c} \sum_{s=0}^{2^D-1} \nu|\nabla_{\Gamma_s} f_{s-1}|^2.$$

Note that the right-hand side is the same as in (5.3.9) without the factor 2^{2^D}, thus one can

follow the same steps from (5.3.9) until (5.3.14) to conclude that

$$\nu\Pi|f - \Pi f|^2 \leq 2^{2^D}\bar{c}\nu|\nabla f|^2$$

for the same $\bar{c}$ in (5.3.14). Similarly, the argument for (5.3.15) and (5.3.17) leads to the gradient bound

$$\nu|\nabla(\Pi f)|^2 \leq \beta\nu|\nabla f|^2$$

for some $\beta \in (0,1)$. Applying the above two estimates with f replaced by $\Pi^n f$, if follows

$$\nu|\Pi^n f - \Pi^{n+1} f|^2 \leq 2^{2^D}\bar{c}\nu|\nabla(\Pi^n f)|^2 \leq 2^{2^D}\beta^n\bar{c}\nu|\nabla f|^2.$$

Given the fact $\beta \in (0,1)$ and $\nu|\nabla f|^2 < \infty$ for $f \in H^1(\nu)$, we can use the Borel-Cantelli lemma to show that $\Pi^n f \to F_{\text{lim}}$ almost surely in ν for some function $F_{\text{lim}} : \Omega \to \mathbb{C}$ (for a proof see [30, Lemma 5.4.6]). Similarly, one has $|\nabla(\Pi^n f)| \to 0$ almost surely and therefore the limit F_{lim} must be a constant function. This observation leads to

$$\lim_{n\to\infty} \Pi^n f = F_{\text{lim}} = \nu(F_{\text{lim}}) = \lim_{n\to\infty} \nu(\Pi^n f) = \nu(f) \qquad \nu\text{-almost surely}$$

where the limit and ν can be interchanged by the dominated convergence theorem (since f is bounded in $\mathfrak{F}(\Omega)$), and the last equality follows from the DLR equation (i).

Proof of Lemma 5.3.9

We will only prove (5.3.8) since the situation for (5.3.7) is similar. Let Λ be any cube Q_k in the setup (hence $\text{diam}(\Lambda) > 2R$). Fix any $j \notin \Lambda$ and $i \in \Lambda$ such that $\|j - i\| > R$, we denote by $F \coloneqq (\mathbb{E}_{\Lambda\setminus i}|f|^2)^{1/2}$ where $\Lambda \setminus i \coloneqq \Lambda \setminus \{i\}$ for simplicity. By the compatibility

property $\mathbb{E}_\Lambda = \mathbb{E}_\Lambda \mathbb{E}_{\Lambda\setminus i}$ in Proposition 2.1.3, we have

$$\left|\nabla_j \big(\mathbb{E}_\Lambda |f|^2\big)^{\frac{1}{2}}\right|^2 = \left|\nabla_j \big(\mathbb{E}_\Lambda F^2\big)^{\frac{1}{2}}\right|^2 = \frac{|\nabla_j \big(\mathbb{E}_\Lambda F^2\big)|^2}{4\mathbb{E}_\Lambda F^2}.$$

To bound the numerator, we notice that for any function g it follows

$$\begin{aligned}\nabla_j(\mathbb{E}_\Lambda g) &= \int \nabla_j \left(\frac{g e^{-U_\Lambda}}{\int e^{-U_\Lambda} dx_\Lambda}\right) dx_\Lambda \\ &= \mathbb{E}_\Lambda(\nabla_j g) - \mathbb{E}_\Lambda(g \nabla_j U_\Lambda) + \mathbb{E}_\Lambda(g)\mathbb{E}_\Lambda(\nabla_j U_\Lambda) \\ &= \mathbb{E}_\Lambda(\nabla_j g) - \mathbb{E}_\Lambda(g; -\nabla_j U_\Lambda)\end{aligned} \tag{5.3.18}$$

with the covariance $\mathbb{E}_\Lambda(f;g) \coloneqq \mathbb{E}_\Lambda(fg) - (\mathbb{E}_\Lambda f)(\mathbb{E}_\Lambda g)$. Using the above calculation, we have

$$\left|\nabla_j \big(\mathbb{E}_\Lambda |f|^2\big)^{\frac{1}{2}}\right|^2 \le \frac{\left|\mathbb{E}_\Lambda(\nabla_j F^2)\right|^2}{4\mathbb{E}_\Lambda F^2} + \frac{\left|\mathbb{E}_\Lambda(F^2; -\nabla_j U_\Lambda)\right|^2}{4\mathbb{E}_\Lambda F^2}. \tag{5.3.19}$$

Let I_1 and I_2 be the first and second term on the right-hand side of (5.3.19). For I_1 it simply follows $I_1 \le \mathbb{E}_\Lambda |\nabla_j F|^2$ by Cauchy-Schwartz inequality. For I_2, note that F has no dependence in $\Lambda \setminus i$ due to the integral $\mathbb{E}_{\Lambda\setminus i}$, hence by substituting $\mathbb{E}_\Lambda = \mathbb{E}_\Lambda \mathbb{E}_{\Lambda\setminus i}$ the covariance becomes $\mathbb{E}_\Lambda(F^2; -\nabla_j U_\Lambda) = \mathbb{E}_\Lambda(F^2; \mathcal{U})$ with $\mathcal{U} \coloneqq \mathbb{E}_{\Lambda\setminus i}(-\nabla_j U_\Lambda)$. To proceed, we first define $\tilde{\mathbb{E}}^\omega_\Lambda \coloneqq \mathbb{E}^\omega_\Lambda(d\tilde{x}_\Lambda)$ as an isomorphic copy of $\mathbb{E}^\omega_\Lambda(dx_\Lambda)$ on a variable $\tilde{x}_\Lambda$ independent of x_Λ, while $x_{\Lambda^c} = \tilde{x}_{\Lambda^c} = \omega_{\Lambda^c}$ for the same boundary conditions $\omega \in \Omega$. Similarly define $\tilde{F} \coloneqq F(\tilde{x}_\Lambda)$, $\tilde{\mathcal{U}} \coloneqq \mathcal{U}(\tilde{x}_\Lambda)$, and one can bound the covariance by

$$\begin{aligned}\left|\mathbb{E}_\Lambda(F^2;\mathcal{U})\right|^2 &= \frac{1}{4}\left|\mathbb{E}_\Lambda \tilde{\mathbb{E}}_\Lambda \Big((F^2 - \tilde{F}^2)(\mathcal{U} - \tilde{\mathcal{U}})\Big)\right|^2 \\ &\le \frac{1}{4}\Big(\mathbb{E}_\Lambda \tilde{\mathbb{E}}_\Lambda (F + \tilde{F})^2\Big)\mathbb{E}_\Lambda \tilde{\mathbb{E}}_\Lambda \Big((F - \tilde{F})^2(\mathcal{U} - \tilde{\mathcal{U}})^2\Big) \\ &\le \Big(\mathbb{E}_\Lambda F^2\Big)\mathbb{E}_\Lambda \tilde{\mathbb{E}}_\Lambda \Big((F - \tilde{F})^2(\mathcal{U} - \tilde{\mathcal{U}})^2\Big).\end{aligned} \tag{5.3.20}$$

Combining I_1 and I_2, we obtain the estimate

$$\left|\nabla_j\big(\mathbb{E}_\Lambda|f|^2\big)^{\frac{1}{2}}\right|^2 \le \mathbb{E}_\Lambda|\nabla_j F|^2 + \frac{1}{4}\mathbb{E}_\Lambda\tilde{\mathbb{E}}_\Lambda\Big((F-\tilde{F})^2(\mathcal{U}-\tilde{\mathcal{U}})^2\Big). \tag{5.3.21}$$

Note that $\mathcal{U}$ only contains one free variable on x_i since all other variables are integrated by $\mathbb{E}_{\Lambda\setminus i}$ in $\Lambda\setminus i$ and fixed by ω outside Λ. Therefore, we can write $\mathcal{U}(x_\Lambda)=\mathcal{U}(x_i)$ and bound the difference

$$|\mathcal{U}(x_i)-\mathcal{U}(\tilde{x}_i)| = \left|\int_{\tilde{x}_i}^{x_i}\nabla_i\mathcal{U}(\hat{x}_i)\,d\hat{x}_i\right| \le |x_i-\tilde{x}_i|\|\nabla_i\mathcal{U}\|_\infty.$$

To bound the term $\|\nabla_i\mathcal{U}\|_\infty$, we use the same calculation as in (5.3.18) to get

$$\nabla_i\mathcal{U} = \nabla_i\big(\mathbb{E}_{\Lambda\setminus i}(-\nabla_j U_\Lambda)\big) = \mathbb{E}_{\Lambda\setminus i}(\nabla_i\nabla_j U_\Lambda) - \mathbb{E}_{\Lambda\setminus i}(\nabla_j U_\Lambda;\nabla_i U_{\Lambda\setminus i}).$$

Since $\|i-j\|>R$, it follows $\nabla_i\nabla_j U_\Lambda=0$ and $\nabla_j U_\Lambda=\nabla_j U_{\Lambda\setminus i}$ due to the finite range of potential, which implies

$$|\mathcal{U}(x_i)-\mathcal{U}(\tilde{x}_i)| \le |x_i-\tilde{x}_i|\big\|\mathbb{E}_{\Lambda\setminus i}(\nabla_j U_{\Lambda\setminus i};\nabla_i U_{\Lambda\setminus i})\big\|_\infty.$$

To deal with the above covariance, we use the following result adapted from [78, Theorem 2.4 and Remark 2.6].

Theorem 5.3.10. *If $\|\nabla_i\nabla_j U_\Lambda\|_\infty \le \tilde{A}e^{-\|j-i\|}$ for some $\tilde{A}>0$ independent of i,j,Λ, then the volume-uniform LSI (5.3.1) is equivalent to the following mixing condition*

$$|\mathbb{E}_\Lambda(f;g)|^2 \le C|\Lambda_f||\Lambda_g|e^{-M\mathrm{dist}(\Lambda_f,\Lambda_g)}\Big(\mathbb{E}_\Lambda|\nabla_\Lambda f|^2\Big)\Big(\mathbb{E}_\Lambda|\nabla_\Lambda g|^2\Big) \tag{5.3.22}$$

for all $f,g\in(\Omega)$ localised in Λ_f,Λ_g respectively and $C,M>0$ independent of Λ,f,g.

By the condition $\|\nabla_i \nabla_j U_\Lambda\|_\infty \le A$ in Theorem 5.3.1 and the finite range of the interaction (i.e. $\nabla_i \nabla_j U_\Lambda = 0$ for $\|i-j\| > R$), the condition $\|\nabla_i \nabla_j U_\Lambda\|_\infty \le \tilde{A} e^{-\|j-i\|}$ in Theorem 5.3.10 is satisfied with $\tilde{A} = Ae^R$. Therefore, the above theorem holds in the current setup and one can apply (5.3.22) to obtain the following covariance bound

$$\left\| \mathbb{E}_{\Lambda\setminus i}(\nabla_j U_{\Lambda\setminus i}; \nabla_i U_{\Lambda\setminus i}) \right\|_\infty^2 \le C\tilde{A}^4 e^{2MR}(2R+1)^{4D} e^{-M\|j-i\|}.$$

Collecting every term in (5.3.20), one can estimate I_2 by

$$I_2 \le \frac{1}{4} C\tilde{A}^4 e^{2MR}(2R+1)^{4D} e^{-M\|j-i\|} \mathbb{E}_\Lambda \tilde{\mathbb{E}}_\Lambda \Big((F-\tilde{F})^2 (x_i - \tilde{x}_i)^2 \Big).$$

Denote by $\hat{F} \coloneqq F - \tilde{F}$, $\hat{x}_i \coloneqq x_i - \tilde{x}_i$ and $\hat{\mathbb{E}}_\Lambda \coloneqq \mathbb{E}_\Lambda \tilde{\mathbb{E}}_\Lambda$, we then have the following estimate derived from the relative entropy inequality (see Lemma A.2.1 and Remark A.2.2)

$$\hat{\mathbb{E}}_\Lambda \Big(\hat{F}^2 \hat{x}_i^2 \Big) \le \frac{1}{\varepsilon} \hat{\mathbb{E}}_\Lambda \left(\hat{F}^2 \log \frac{\hat{F}^2}{\hat{\mathbb{E}}_\Lambda \hat{F}^2} \right) + \frac{1}{\varepsilon} \big(\hat{\mathbb{E}}_\Lambda \hat{F}^2 \big) \log \hat{\mathbb{E}}_\Lambda \Big(e^{\varepsilon \hat{x}_i^2} \Big) \tag{5.3.23}$$

for any $\varepsilon > 0$. By the uniform LSI (5.3.1) and Theorem 2.3.5, the double measure $\hat{\mathbb{E}}_\Lambda$ satisfies the LSI with coefficient $\tilde{c}$ and thus the first term on the right-hand side is bounded by $\tilde{c}\hat{\mathbb{E}}_\Lambda(|\nabla_{x_\Lambda}\hat{F}|^2 + |\nabla_{\tilde{x}_\Lambda}\hat{F}|^2) = 2\tilde{c}\,\mathbb{E}_\Lambda |\nabla_i F|^2$, where only ∇_i remains since F has no dependence in $\Lambda \setminus i$. By Theorem 2.3.7 the measure $\hat{\mathbb{E}}_\Lambda$ also satisfies the SGI with coefficient $\tilde{c}/2$, hence

$$\hat{\mathbb{E}}_\Lambda \hat{F}^2 = 2\mathbb{E}_\Lambda (F - \mathbb{E}_\Lambda F)^2 \le \tilde{c}\,\mathbb{E}_\Lambda |\nabla_\Lambda F|^2 = \tilde{c}\,\mathbb{E}_\Lambda |\nabla_i F|^2.$$

For the logarithm on $\varepsilon \hat{\mathbb{E}}_\Lambda \hat{x}_i^2$, we use the following lemma adapted from [5, Theorem 4.5].

Lemma 5.3.11. *Let ν be a probability measure on $\mathbb{R}^m$ that satisfies a LSI with coefficient $\tilde{c} > 0$. Let $g : \mathbb{R}^m \to \mathbb{R}$ be any differentiable function such that $|\nabla_{\mathbb{R}^m} g|^2 \le ag$ for some*

$a > 0$, then for all $\varepsilon \in (0, 1/a\tilde{c})$ the following bound holds

$$\log \nu\big(e^{\varepsilon g}\big) \leq 2\varepsilon\nu(g).$$

Since $|\nabla_{x_i}\hat{x}_i^2|^2 + |\nabla_{\tilde{x}_i}\hat{x}_i^2|^2 = 8\hat{x}_i^2$, we can let $\varepsilon < \frac{1}{8\tilde{c}}$ in Lemma 5.3.11 to show that

$$\log \hat{\mathbb{E}}_\Lambda\Big(e^{\varepsilon\hat{x}_i^2}\Big) \leq 2\varepsilon\hat{\mathbb{E}}_\Lambda\hat{x}_i^2 = 4\varepsilon\mathbb{E}_\Lambda(x_i - \mathbb{E}_\Lambda(x_i))^2 \leq 2\varepsilon\tilde{c}\,\mathbb{E}_\Lambda|\nabla_\Lambda x_i|^2 = 2\varepsilon\tilde{c}.$$

Taking this bound to (5.3.23), (5.3.20), (5.3.19) and integrating with ν, one gets

$$\nu\big|\nabla_j\big(\mathbb{E}_\Lambda|f|^2\big)^{\frac{1}{2}}\big|^2 \leq \nu|\nabla_j F|^2 + C_R e^{-M\|j-i\|}\nu|\nabla_i F|^2 \tag{5.3.24}$$

with $C_R = \max\big\{1,\, \tilde{c}C\tilde{A}^4 e^{2MR}(2R+1)^{4D}(\varepsilon^{-1} + \tilde{c})\big\}$ independent of i, j, Λ.

Based on $F = (\mathbb{E}_{\Lambda\setminus i}|f|^2)^{1/2}$, we can define $F_2 \coloneqq (\mathbb{E}_{\Lambda\setminus\{i,i_2\}}|f|^2)^{1/2}$ with $\|j - i_2\| > R$ and use (5.3.24) again to bound $\nabla_j F$ by $\nabla_j F_2$ and $\nabla_{i_2} F_2$. With this idea, one may construct a finite sequence $\{i_n\}$ in Λ such that $\|j - i_n\|$ is decreasing and the sequence terminates at $i_{\overline{n}}$ when $\|j - i_{\overline{n}+1}\| \leq R$. Then we apply (5.3.24) inductively to bound each $\nabla_j F_n$ where $F_n \coloneqq (\mathbb{E}_{\Lambda\setminus\{i_1,i_2,\ldots,i_n\}}|f|^2)^{1/2}$. For the second term $\nabla_{i_n} F_n$ during each iteration, we will use the following estimate

$$\nu\big|\nabla_i\big(\mathbb{E}_{\Lambda'}|f|^2\big)^{\frac{1}{2}}\big|^2 \leq \nu|\nabla_i f|^2 + \tilde{C}_R\nu|\nabla_{\Lambda'} f|^2, \tag{5.3.25}$$

which holds for all $\Lambda' \subseteq \Lambda$, $i \notin \Lambda'$ and some $\tilde{C}_R \geq 1$ depending only on $\tilde{c}, A, R, D$. The bound (5.3.25) can be derived by a similar strategy as for (5.3.24) while keeping f instead of F in the beginning. Although we lose the exponential factor $e^{-M\|j-i\|}$ in this bound, one can use (5.3.25) to stop the iteration by getting the gradient bound directly on f.

With (5.3.24) and (5.3.25), we now perform the iteration as follows. At each step n, two

gradient terms $\nabla_j F_n$ and $\nabla_{i_n} F_n$ are generated by (5.3.24). For the $\nabla_j F_n$ term, we continue the iteration by using (5.3.24) again which produces $\nabla_j F_{n+1}$ and $\nabla_{i_{n+1}} F_{n+1}$. For the $\nabla_{i_n} F_n$ term, we stop the iteration by applying (5.3.25) which produces $\nabla_{\Lambda\setminus\{i_1,i_2,\dots,i_{n-1}\}} f$. Once the iteration reaches the last step $\overline{n}$ we simply apply (5.3.25) to bound both $\nabla_j F_N$ and $\nabla_{i_N} F_N$, which terminates the iteration with all gradient terms being on f. Therefore, the final output of this procedure is given by

$$\nu\Big|\nabla_j\big(\mathbb{E}_\Lambda|f|^2\big)^{\frac{1}{2}}\Big|^2 \le \nu|\nabla_j f|^2 + C_R\tilde{C}_R\beta_{\overline{n}}\nu|\nabla_{\Lambda\setminus\{i_1,\dots,i_{\overline{n}}\}} f|^2 + C_R\tilde{C}_R\sum_{n=1}^{\overline{n}}\beta_n\nu|\nabla_{i_n} f|^2,$$

where $\beta_n = \sum_{m=1}^{n} e^{-M\|j-i_m\|} \le |\Lambda| e^{-M\|j-i_n\|}$ since $\|j-i_n\|$ is decreasing along the sequence. Rearranging the terms and grouping the constants, we finally obtain the sweeping out inequality (5.3.8)

$$\nu\Big|\nabla_j\big(\mathbb{E}_\Lambda|f|^2\big)^{\frac{1}{2}}\Big|^2 \le \sum_{i\in\mathbb{Z}^D}\alpha_{ji}\nu|\nabla_i f|^2$$

with $0 < \alpha_{ji} \le e^{MR} C_R\tilde{C}_R|\Lambda| e^{-M\|j-i\|}$.

The idea for (5.3.7) is essentially the same. Let $F \coloneqq \mathbb{E}_{\Lambda\setminus i} f$ in this case and we get

$$|\nabla_j(\mathbb{E}_\Lambda f)|^2 = |\nabla_j(\mathbb{E}_\Lambda F)|^2 \le 2\mathbb{E}|\nabla_j F|^2 + 2|\mathbb{E}_\Lambda(F;\nabla_j U_\Lambda)|^2. \tag{5.3.26}$$

Denote by $\mathcal{U} \coloneqq \mathbb{E}_{\Lambda\setminus i}(-\nabla_j U_\Lambda)$ and the covariance follows

$$|\mathbb{E}_\Lambda(F;\nabla_j U_\Lambda)|^2 = \frac{1}{4}\big|\mathbb{E}_\Lambda\tilde{\mathbb{E}}_\Lambda\big((F-\tilde{F})(\mathcal{U}-\tilde{\mathcal{U}})\big)\big|^2 \le \frac{1}{4}\mathbb{E}_\Lambda\tilde{\mathbb{E}}_\Lambda\Big((F-\tilde{F})^2(\mathcal{U}-\tilde{\mathcal{U}})^2\Big)$$

similarly as (5.3.20). Taking this in (5.3.26), the right-hand side becomes the same as that in (5.3.21), up to multiplication by a constant. Therefore, one can apply the same procedure as above and conclude (5.3.7).

5.4 Estimates in arbitrary dimensions

After developing the log-Sobolev inequality for ν, we now return to the log-Schrödinger equation (LSE) in finite volume and find various estimates that are uniform in dimension.

To this end, recall that the LSE in finite volume $\Lambda \Subset \mathbb{Z}^D$ is given by

$$\begin{aligned} i\partial_t \phi &= -\mathcal{L}_\Lambda \phi + \lambda \phi \log \frac{|\phi|^2}{\nu |\phi|^2} \\ \phi(x, t=0) &= f(x) \end{aligned} \tag{LSE$_\Lambda$}$$

where ν is the infinite-volume Gibbs measure, and $\mathcal{L}_\Lambda \coloneqq \Delta_\Lambda - \nabla_\Lambda U_\Lambda \cdot \nabla_\Lambda$ for a finite-range potential U_Λ. We consider initial data $f \in \mathfrak{F}(\Omega)$ localised in some $\Lambda_f \subset \Lambda$ to ensure that all the initial particles are covered by the potential in Λ. Due to the finite-range interaction, the solution ϕ is localised in $\{j : \operatorname{dist}(j, \Lambda) \leq R\}$ for some range $R \in \mathbb{N}$, hence the Cauchy problem (LSE$_\Lambda$) remains in finite-dimensional space and is well-defined.

In this section we find estimates for the solutions of (LSE$_\Lambda$) which hold uniformly for all $\Lambda \Subset \mathbb{Z}^D$. In the following section we will use these estimates to prove the existence of solutions to the infinite-volume LSE as $\Lambda \to \mathbb{Z}^D$. To begin with, denote by $\|\cdot\|_\nu \coloneqq \sqrt{\nu|\cdot|^2}$ the $\mathbb{L}^2(\nu)$ norm, we show that (LSE$_\Lambda$) satisfies the conservation of mass and energy.

Theorem 5.4.1. *For any $\Lambda \Subset \mathbb{Z}^D$, let $\phi(\cdot, t)$ be the solution of (LSE$_\Lambda$) with initial data $f \in \mathfrak{F}(\Omega)$. Then for all time $t \geq 0$ the following conservation of mass property holds*

$$\|\phi(\cdot, t)\|_\nu^2 = \|f\|_\nu^2.$$

If U_Λ is twice differentiable, the following conservation of energy property holds

$$\|\nabla_\Lambda \phi(\cdot, t)\|_\nu^2 + \lambda \mathrm{Ent}_\nu |\phi(\cdot, t)|^2 = \|\nabla_\Lambda f\|_\nu^2 + \lambda \mathrm{Ent}_\nu |f|^2 \tag{5.4.1}$$

where $\mathrm{Ent}_\nu F \coloneqq F\log(F/\nu F)$ *denotes the* ν*-entropy.*

Proof. For the mass conservation, we first find the time evolution of $\nu|\phi|^2$ by

$$\partial_t \nu|\phi|^2 = 2\mathrm{Re}(\nu(\phi^*\partial_t\phi)) = -2\mathrm{Im}\left(\nu\left(\phi^*\mathcal{L}_\Lambda\phi - \lambda|\phi|^2\log\frac{|\phi|^2}{\nu|\phi|^2}\right)\right) = -2\mathrm{Im}\big(\nu(\phi^*\mathcal{L}_\Lambda\phi)\big),$$

where the logarithmic term is real and therefore eliminated by $\mathrm{Im}(\cdot)$. Since ν is reversible for $\mathcal{L}_\Lambda$ (by Lemma 2.2.9), it follows

$$\mathrm{Im}\big(\nu(\phi^*\mathcal{L}_\Lambda\phi)\big) = \mathrm{Im}\big(\nu(\phi\mathcal{L}_\Lambda\phi^*)\big) = -\mathrm{Im}\big(\nu(\phi^*\mathcal{L}_\Lambda\phi)\big) = 0.$$

This shows $\partial_t\nu|\phi|^2 = 0$ and thus the mass conservation. For the energy conservation, we will similarly show that

$$\partial_t\big(\nu|\nabla_\Lambda\phi|^2 + \lambda\mathrm{Ent}_\nu|\phi|^2\big) = 0. \tag{5.4.2}$$

Hereinafter we consider regularised solution $e^{-\varepsilon\mathcal{L}_\Lambda}\phi$ of (LSE_Λ) and then send ε to 0 at the end, but for simplicity, we shall only keep this in mind and demonstrate the proof without explicit reference to ε.

For the gradient term in (5.4.2), we first analyse the single-site gradient by

$$\partial_t|\nabla_j\phi|^2 = 2\mathrm{Im}\bigg(-\nabla_j(\mathcal{L}_\Lambda\phi)\cdot\nabla_j\phi^* + \lambda\nabla_j\bigg(\phi\log\frac{|\phi|^2}{\nu|\phi|^2}\bigg)\cdot\nabla_j\phi^*\bigg). \tag{5.4.3}$$

The first product on the right-hand side follows

$$\nabla_j(\mathcal{L}_\Lambda\phi)\cdot\nabla_j\phi^* = [\nabla_j,\mathcal{L}_\Lambda]\phi\cdot\nabla_j\phi^* + (\nabla_j\phi^*)\mathcal{L}_\Lambda(\nabla_j\phi)$$

where the commutator satisfies $[\nabla_j,\mathcal{L}_\Lambda] = -\sum_{k\in\Lambda}\nabla_{jk}U_\Lambda\cdot\nabla_k$ with $\nabla_{jk}\coloneqq\nabla_j\nabla_k$. Taking

this in (5.4.3) and integrating with ν, one gets

$$\partial_t \nu |\nabla_j \phi|^2 = \mathrm{Im}\Big(2\nu \sum_{k\in\Lambda} \nabla_{jk} U_\Lambda \cdot \nabla_k \phi \cdot \nabla_j \phi^*\Big) - \mathrm{Im}\Big(2\nu\Big((\nabla_j \phi^*)\mathcal{L}_\Lambda(\nabla_j \phi)\Big)\Big) \tag{5.4.4}$$

$$+ \mathrm{Im}\Big(2\lambda\nu\Big(\nabla_j\Big(\phi \log \frac{|\phi|^2}{\nu|\phi|^2}\Big) \cdot \nabla_j \phi^*\Big)\Big). \tag{5.4.5}$$

After a summation over $j \in \Lambda$, the first term in (5.4.4) vanishes due to cancellation in Λ and the second term also vanishes by the reversibility of ν for $\mathcal{L}_\Lambda$. Hence only the last term survives after the summation, leading to

$$\partial_t \nu |\nabla_\Lambda \phi|^2 = \mathrm{Im}\Big(2\lambda\nu\big(\nabla_\Lambda(\phi \log |\phi|^2) \cdot \nabla_\Lambda \phi^*\big)\Big). \tag{5.4.6}$$

As for the entropy term in (5.4.2), we use mollification argument and the $\mathbb{L}^2$-invariance property to obtain

$$\frac{\partial}{\partial t}\mathrm{Ent}_\nu |\phi|^2 = -\mathrm{Im}\Big(2\nu\big(\phi^* \log |\phi|^2 \mathcal{L}_\Lambda \phi\big)\Big) - \mathrm{Im}\Big(2\nu\Big(\phi^* \mathcal{L}_\Lambda \phi + \lambda(1 + \log |\phi|^2)\mathrm{Ent}_\nu |\phi|^2\Big)\Big).$$

The second term vanishes again due to the reversibility of ν and thus it follows

$$\frac{\partial}{\partial t}\mathrm{Ent}_\nu |\phi|^2 = -\mathrm{Im}\Big(2\nu\big(\phi^* \log |\phi|^2 \mathcal{L}_\Lambda \phi\big)\Big) = \mathrm{Im}\Big(2\nu\big(\nabla_\Lambda(\phi^* \log |\phi|^2) \cdot \nabla_\Lambda \phi\big)\Big) \tag{5.4.7}$$

using integration by parts. Since the measure in (5.4.7) is simply the complex conjugate of that in (5.4.6), one easily verifies (5.4.2) and hence the energy conversation property. □

Remark 5.4.2. The mass conservation property holds true for any measure μ_Λ and the operator $\mathcal{L}_\Lambda$ associated to the Dirichlet form $\mu_\Lambda(f\mathcal{L}_\Lambda g) = -\mu_\Lambda(\nabla_\Lambda f \cdot \nabla_\Lambda g)$.

With similar steps one can obtain time-dependent estimates for the gradient and the entropy. In particular, we show that these estimates can be independent of time if ν

satisfies the log-Sobolev inequality (i.e. if the conditions in Theorem 5.3.2 are satisfied in this setup).

Theorem 5.4.3. *For any $\Lambda \Subset \mathbb{Z}^D$ let ϕ be the solution of* (LSE_Λ) *with localised initial data $f \in \mathfrak{F}(\Omega)$. If U_Λ is twice differentiable, then for all $t \geq 0$ one has the gradient bound*

$$\|\nabla_\Lambda \phi(\cdot, t)\|_\nu^2 \leq e^{2|\lambda|t} \|\nabla f\|_\nu^2 \tag{5.4.8}$$

and the entropy bound

$$\mathrm{Ent}_\nu |\phi(\cdot, t)|^2 \leq \mathrm{Ent}_\nu |f|^2 + \frac{1}{|\lambda|}(e^{2|\lambda|t} - 1)\|\nabla f\|_\nu^2. \tag{5.4.9}$$

If $\lambda > 0$ and ν satisfies a log-Sobolev inequality with coefficient $c > 0$, one has the time-uniform bounds

$$\|\nabla_\Lambda \phi(\cdot, t)\|_\nu^2 \leq (1 + \lambda c)\|\nabla f\|_\nu^2 \quad \textit{and} \quad \mathrm{Ent}_\nu |\phi(\cdot, t)|^2 \leq (1/\lambda + c)\|\nabla f\|_\nu^2. \tag{5.4.10}$$

Proof. Following the proof of Theorem (5.4.1), the estimate (5.4.6) implies

$$\partial_t \nu |\nabla_\Lambda \phi|^2 \leq 2|\lambda| \sum_{j \in \Lambda} \nu \Big| \mathrm{Im}\big(\phi \nabla_j \phi^*\big)\big(\nabla_j \log |\phi|^2\big) \Big| \leq 2|\lambda| \sum_{j \in \Lambda} \nu |\nabla_j \phi|^2.$$

One then obtains the gradient bound (5.4.8) with Grönwall's lemma . For the entropy bound, continuing the calculation for (5.4.7) leads to

$$\frac{\partial}{\partial t} \mathrm{Ent}_\nu |\phi|^2 = -2 \sum_{j \in \Lambda} \mathrm{Im}\left(\nu \left(\frac{\phi^*}{\phi} (\nabla_j \phi)^2 \right) \right) \leq 2\nu |\nabla_\Lambda \phi|^2 \leq 2e^{2|\lambda|t} \nu |\nabla f|^2,$$

where the last step is by the gradient bound (5.4.8) that we just derived. Integrating with respect to t and one obtains the entropy bound (5.4.9). The uniform bounds (5.4.10)

easily follow from the energy conservation property (5.4.1) and the log-Sobolev inequality of ν. □

With the gradient bound of $\nabla_\Lambda \phi$ inside Λ, we now study $\nabla_j \phi$ for j outside Λ. Due to the finite range of interaction, it can be shown that the norm $\|\nabla_j \phi\|_\nu$ decays exponentially as j travels away from the initial region covered by Λ_f. This means the information in (LSE_Λ) can only be propagated with finite speed, which is presented in the following theorem.

Theorem 5.4.4. *For any $\Lambda \Subset \mathbb{Z}^D$ let ϕ be the solution of (LSE_Λ) with initial data $f \in \mathfrak{F}(\Omega)$ localised in some $\Lambda_f \Subset \mathbb{Z}^D$. Suppose the potential U_Λ satisfies $\|\nabla_j \nabla_k U_\Lambda\|_\infty \leq A$ for all $\Lambda \Subset \mathbb{Z}^D$, $j, k \in \mathbb{Z}^D$ with $j \neq k$ and some $A > 0$ independent of i, j, Λ. Then the following estimate holds*

$$\|\nabla_j \phi(\cdot, t)\|_\nu^2 \leq e^{-\mathcal{N}_j} \|\nabla f\|_\nu^2$$

for any $j \in \mathbb{Z}^D$ and all time $t \leq \varepsilon \mathcal{N}_j$ where $\varepsilon^{-1} = 9(2|\lambda| + A)(2R+1)^{2D}$ is the propagation speed and $\mathcal{N}_j$ is the distance $\mathcal{N}_j = \lfloor \mathrm{dist}(j, \Lambda_f) R^{-1} \rfloor$.

Proof. Using the results in (5.4.4), (5.4.5) and (5.4.6), the estimate holds

$$\partial_t \big(\nu |\nabla_j \phi|^2\big) \leq 2|\lambda|\, \nu |\nabla_j \phi|^2 + \sum_{k \in \Lambda \setminus \{j\}} \|\nabla_{jk} U_\Lambda\|_\infty \big(\nu |\nabla_k \phi|^2 + \nu |\nabla_j \phi|^2\big) \tag{5.4.11}$$

for any $j \in \mathbb{Z}^D$. With the condition $\|\nabla_{jk} U_\Lambda\|_\infty \leq A$, we obtain the differential inequality

$$\partial_t \big(\nu |\nabla_j \phi|^2\big) \leq \sum_{k \in \mathbb{Z}^D} A_{jk}\, \nu |\nabla_k \phi|^2 \tag{5.4.12}$$

where $A_{jj} = 2|\lambda| + A(2R+1)^D$, $A_{jk} = A$ if $0 < \|j - k\| \leq R$ and $A_{jk} = 0$ otherwise. In the following we shall use this inequality to get an improved estimate on $\|\nabla_j \phi\|_\nu$. First we

remark that as a consequence of (5.4.8) and (5.4.11) it follows

$$\partial_t \nu|\nabla_j \phi|^2 \leq A_{jj}\nu|\nabla_j\phi|^2 + A\nu|\nabla_\Lambda \phi|^2 \leq A_{jj}\nu|\nabla_j\phi|^2 + Ae^{2|\lambda|t}\nu|\nabla f|^2.$$

By [79, Theorem A], the above inequality implies $\nu|\nabla_j\phi(\cdot,t)|^2 \leq ae^{\gamma t}\nu|\nabla f|^2$ for the constants $a = 1 + A(2|\lambda|)^{-1}$ and $\gamma = A_{jj} + 2|\lambda|$ independent of j. Therefore, one has the following estimate

$$\bar{\phi}_t \coloneqq \sup_{k\in\mathbb{Z}^D}\sup_{s\in[0,t]} \nu|\nabla_k\phi(\cdot,s)|^2 \leq ae^{\gamma t}\nu|\nabla f|^2 \tag{5.4.13}$$

in any finite time $t \geq 0$. Suppose the initial data $f \in \mathfrak{F}(\Omega)$ is localised in some finite volume $\Lambda_f \Subset \mathbb{Z}^D$. With the finite range $R \in \mathbb{N}$ of the interaction, we will show that in a finite time period $\|\nabla_j\phi\|_\nu$ decays exponentially with $\mathrm{dist}(j,\Lambda_f)$. To this end, let $\mathcal{N}_j = \lfloor \mathrm{dist}(j,\Lambda_f)R^{-1}\rfloor$ and consider the integral inequality

$$\nu|\nabla_j\phi(\cdot,t)|^2 \leq \nu|\nabla_j f|^2 + \sum_{k\in\mathbb{Z}^D} A_{jk}\int_0^t dt_1\, \nu|\nabla_k\phi(\cdot,t_1)|^2 \tag{5.4.14}$$

deduced from (5.4.12). Note that $\nabla_j f$ is zero if $j \notin \Lambda_f$, thus we can iterate (5.4.14) for $M > \mathcal{N}_j$ times until we get a nonzero derivative of f. This yields

$$\begin{aligned}\nu|\nabla_j\phi(\cdot,t)|^2 \leq \sum_{n=\mathcal{N}_j}^{M}\frac{t^n}{n!}\sum_{k_1,\dots,k_n\in\mathbb{Z}^D}(A_{jk_1}\dots A_{k_{n-1}k_n})\nu|\nabla_{k_n}f|^2 \\ + \sum_{k_1,\dots,k_{M+1}\in\mathbb{Z}^D}(A_{jk_1}\dots A_{k_Mk_{M+1}})\int_0^t\cdots\int_0^{t_M} dt_{M+1}\,\nu\left|\nabla_{k_{M+1}}\phi(\cdot,t_{M+1})\right|^2.\end{aligned}$$

Let $A_* = A_{jj}(2R+1)^D$, the right-hand side in the above inequality is bounded above by

$$\left(\sum_{n=\mathcal{N}_j}^{M}\frac{(A_*t)^n}{n!}\nu|\nabla f|^2\right) + \frac{(A_*t)^M}{M!}\bar{\phi}_t$$

with $\bar{\phi}_t$ given in (5.4.13). Since $\bar{\phi}_t < \infty$ for all finite time t, sending $M \to \infty$ leads to

$$\nu|\nabla_j \phi(\cdot,t)|^2 \le \sum_{n=\mathcal{N}_j}^{\infty} \frac{(A_* t)^n}{n!} \nu|\nabla f|^2 \le \frac{(A_* t)^{\mathcal{N}_j}}{\mathcal{N}_j!} e^{A_* t} \nu|\nabla f|^2,$$

where by Stirling's formula $n! \ge n^n e^{-n}$ one has

$$\frac{(A_* t)^{\mathcal{N}_j}}{\mathcal{N}_j!} e^{A_* t} \le \exp\left(\mathcal{N}_j \left(\log(A_*) + 1 - \log \frac{\mathcal{N}_j}{t} + A_* \frac{t}{\mathcal{N}_j} \right)\right).$$

For all $t \le \frac{\mathcal{N}_j}{9A_*}$ it easily follows $\log(A_*) + 1 - \log(\mathcal{N}_j/t) + A_*(t/\mathcal{N}_j) \le -1$, which concludes the proof of Theorem 5.4.4. □

5.4.1 Example potentials for the estimates

In the following we examine whether the above estimates hold true for the examples introduced in Section 5.3, that is, the (perturbed) bilinear potential in Example 5.3.3, 5.3.4 and the Λ-restricted potential in Example 5.3.5.

In case U_Λ is given by the (perturbed) bilinear potentials, it is easy to check that $\|\nabla_j \nabla_k U_\Lambda\|_\infty$ is bounded uniformly for all j, k, Λ. Moreover, the global measure ν associated to U_Λ satisfies the LSI by the argument in Example 5.3.3 and 5.3.4. Therefore, all the estimates in Theorem 5.4.1, 5.4.3, 5.4.4 will hold for the system (LSE_Λ) given by the potential U_Λ. We also remark that U_Λ is the potential in the ground state representation of the original solution $\psi \equiv \phi e^{-\frac{1}{2}U_\Lambda}$, with the original potential V_Λ. For consistency, we relate U_Λ back to V_Λ in the following remark by the differential equation (5.1.2), namely $V_\Lambda = -\frac{1}{2}\Delta_\Lambda U_\Lambda + \frac{1}{4}|\nabla_\Lambda U_\Lambda|^2 + \lambda U_\Lambda$.

Remark 5.4.5. Consider the bilinear example of U_Λ with conditions given in Example

5.3.3. Assuming $C_{jk} = C_{kj}$ for simplicity, direct calculation gives

$$V_\Lambda(x) = \tilde{C}_\Lambda + \sum_{j,k\in\Lambda_R} \tilde{C}_{jk} x_j x_k, \tag{5.4.15}$$

with the range-R set $\Lambda_R \coloneqq \{j : \mathrm{dist}(j,\Lambda) \le R\}$ and the coefficients

$$\tilde{C}_\Lambda = -\sum_{j\in\Lambda} C_{jj}, \quad \tilde{C}_{jk} = \lambda C_{jk}\mathcal{I}_{(j,k\in\Lambda)} + \sum_{l\in\Lambda} C_{jl}C_{kl},$$

where $\mathcal{I}_{(j,k\in\Lambda)} = 1$ when $j,k \in \Lambda$ and 0 otherwise. By the conditions on C_{jk} in Example 5.3.3, it follows that $|\tilde{C}_{jk}|$ is uniformly bounded and of finite range $2R$. In case $\lambda > 0$, the quadratic coefficient $\tilde{C}_{jj}$ is uniformly bounded below by some positive constant; if $\lambda < 0$ and $|\lambda|$ is large (e.g. $\lambda < -\sup_j C_{jj}$ when $C_{jk} = 0$ for $j \neq k$) the quadratic terms become negative, in which case one may lose the log-Sobolev inequality and thus the uniqueness and strict positivity of the ground state. We remark that the latter case where LSE is defined by negative bilinear V was studied in [64] recently, as it provides an interesting situation where the positive stationary solutions could be non-unique, with each one generating a continuous family of solitary waves.

Now we discuss the Λ-restricted setting in Example 5.3.5. That is, the system in any $\Lambda \Subset \mathbb{Z}^D$ with removed boundary conditions, where $U^\circ_\Lambda = \sum_{K\subseteq\Lambda} J_K$ is the potential strictly inside Λ and $\mathbb{E}^\circ_\Lambda$ is the local Gibbs measure corresponding to U°_Λ. In this case, the LSE is defined strictly in Λ with no boundary conditions, and is given by

$$\begin{aligned} i\partial_t \phi_\Lambda &= -\mathcal{L}^\circ_\Lambda \phi_\Lambda + \lambda \phi_\Lambda \log \frac{|\phi_\Lambda|^2}{\mathbb{E}^\circ_\Lambda |\phi_\Lambda|^2} \\ \phi_\Lambda|_{t=0} &= f \end{aligned} \tag{LSE$^\circ_\Lambda$}$$

with $\mathcal{L}^\circ_\Lambda \coloneqq \Delta_\Lambda - \nabla_\Lambda U^\circ_\Lambda \cdot \nabla_\Lambda$ and initial condition f localised in $\Lambda_f \subset \Lambda$. For this setup,

we have the following results.

Corollary 5.4.6. *If the ν-estimates in Theorem 5.4.1, 5.4.3, 5.4.4 hold for* (LSE_Λ)*, they remain true for* ($\mathrm{LSE}^\circ_\Lambda$) *with ν replaced by $\mathbb{E}^\circ_\Lambda$. That is, for the same conditions on U°_Λ, one has the mass and energy conservation property*

$$\mathbb{E}^\circ_\Lambda|\phi_\Lambda|^2 = \mathbb{E}^\circ_\Lambda|f|^2 \quad \text{and} \quad \mathbb{E}^\circ_\Lambda|\nabla\phi_\Lambda|^2 + \lambda\mathrm{Ent}_{\mathbb{E}^\circ_\Lambda}|\phi_\Lambda|^2 = \mathbb{E}^\circ_\Lambda|\nabla f|^2 + \lambda\mathrm{Ent}_{\mathbb{E}^\circ_\Lambda}|f|^2,$$

the gradient and entropy bounds

$$\mathbb{E}^\circ_\Lambda|\nabla\phi_\Lambda|^2 \le e^{2|\lambda|t}\mathbb{E}^\circ_\Lambda|\nabla f|^2_\nu \quad \text{and} \quad \mathrm{Ent}_{\mathbb{E}^\circ_\Lambda}|\phi_\Lambda|^2 \le \mathrm{Ent}_{\mathbb{E}^\circ_\Lambda}|f|^2 + |\lambda|^{-1}(e^{2|\lambda|t}-1)\mathbb{E}^\circ_\Lambda|\nabla f|^2,$$

and the finite-speed propagation of information property

$$\mathbb{E}^\circ_\Lambda|\nabla_j\phi_\Lambda|^2 \le e^{-\mathcal{N}_j}\mathbb{E}^\circ_\Lambda|\nabla f|^2 \tag{5.4.16}$$

for all $j \in \mathbb{Z}^D$ and all time $t \le \varepsilon\mathcal{N}_j$, where ε and $\mathcal{N}_j$ are the same as in Theorem 5.4.4.

Note that in the above corollary we have uniform bounds for the total gradient $\nabla\phi_\Lambda$ instead of $\nabla_\Lambda\phi_\Lambda$ since the solution ϕ_Λ of ($\mathrm{LSE}^\circ_\Lambda$) is strictly localised in Λ due to the removed boundary conditions. In the next section, we will use this ($\mathrm{LSE}^\circ_\Lambda$) setup with the estimates in Corollary 5.4.6 to construct solutions to the LSE in infinite dimensions.

5.5 Existence of weak solutions in infinite dimensions

In this section we establish the existence of weak solutions to the infinite-volume LSE by finite-dimensional approximations. The strategy is to define a sequence of finite volumes $\{\Lambda_n\}$, increasing in size, such that the sequence of solutions $\{\phi_{\Lambda_n}\}$ to the problem ($\mathrm{LSE}^\circ_{\Lambda_n}$)

is compact in $H^1(\nu)$. Then we choose a convergent subsequence whose limit defines a weak solution to the infinite-volume LSE.

Before presenting the main result, we briefly explain the reason to choose ($\mathrm{LSE}^\circ_\Lambda$) instead of ($\mathrm{LSE}_\Lambda$) for the finite-dimensional approximations. The advantage of ($\mathrm{LSE}^\circ_\Lambda$) is that it lives entirely inside $\mathbb{R}^\Lambda$, for which there already exist many results concerning existence and uniqueness of the solutions. In addition, the setup is technically simpler without the boundary conditions. As for the uniqueness of the infinite-volume solution, it will also be convenient to compare the solution obtained from ($\mathrm{LSE}^\circ_\Lambda$) with the one from other settings such as (LSE_Λ).

For simplicity, we shall present the proof for the space $\Omega = \mathbb{R}^{\mathbb{Z}}$ and assume the multi-particle interaction is bounded, i.e. the potential J_K have uniformly (in K) bounded derivatives of order $n = 0, 1, 2$ for all $|K| \geq 2$. For a discussion of higher dimensional lattice and unbounded interaction, see Remark 5.5.3. Moreover, we assume that the measures $\mathbb{E}_\Lambda$ and $\mathbb{E}^\circ_\Lambda$ associated to $\{J_K\}$ satisfy a log-Sobolev inequality with coefficient uniform in Λ. With these conditions, we now to introduce the existence theorem.

Theorem 5.5.1. *Let $\Omega = \mathbb{R}^{\mathbb{Z}}$ and suppose $\{J_K\}$ satisfies the above conditions. For any $\lambda \in \mathbb{R}$ and $f \in \mathfrak{F}(\Omega)$, if the Cauchy problem ($\mathrm{LSE}^\circ_\Lambda$) with initial data f and coupling constant λ admits at least one solution for all $\Lambda \Subset \mathbb{Z}^D$, then there exists a weak solution to the infinite-volume LSE problem with initial data f and coupling constant λ, i.e. there exists $\phi \in H^1(\nu)$ such that*

$$\begin{aligned} \nu\left(gi\partial_t\phi\right) &= \nu\left(\nabla g \cdot \nabla\phi + g\lambda\phi \log \frac{|\phi|^2}{\nu|\phi|^2}\right) \\ \phi(t=0) &= f \end{aligned} \tag{LSE_∞}$$

for all smooth, local and compactly supported functions $g : \Omega \to \mathbb{C}$.

Proof of Theorem 5.5.1. Let $f \in (\Omega)$ be localised in $\Lambda_f \subset [-L, L]$ for some $L \geq R$, where R is the range of interaction. Define the sequence $\Lambda_n \coloneqq [-8nL, 8nL] \cap \mathbb{Z}$ and suppose (LSE$^\circ_\Lambda$) with initial condition f admits at least one solution $\phi_n \coloneqq \phi_{\Lambda_n}$ for all $n \in \mathbb{N}$. Our aim is to prove the following convergence of the norms

$$\lim_{n\to\infty} \nu|\phi_n|^2 = \nu|f|^2. \tag{5.5.1}$$

Denote by $\rho_n \coloneqq d\mathbb{E}_{\Lambda_n}/d\mathbb{E}^\circ_{\Lambda_n}$ the density ratio, it follows $\nu|\phi_n|^2 = \nu\mathbb{E}_\Lambda|\phi_n|^2 = \nu\mathbb{E}^\circ_\Lambda(|\phi_n|^2\rho_n)$. With the notation $\nu_n \coloneqq \nu\mathbb{E}^\circ_{\Lambda_n}$ and the ν-covariance $\nu(g;h) \coloneqq \nu(gh) - \nu(g)\nu(h)$, we have

$$\nu|\phi_n|^2 = \nu_n(|\phi_n|^2;\rho_n) + \big(\nu_n|\phi_n|^2\big)\big(\nu_n(\rho_n)\big) = \nu_n(|\phi_n|^2;\rho_n) + \nu_n|f|^2 \tag{5.5.2}$$

since $\mathbb{E}^\circ_{\Lambda_n}(\rho_n) = 1$ and $\mathbb{E}^\circ_{\Lambda_n}|\phi_n|^2 = \mathbb{E}^\circ_{\Lambda_n}|f|^2$ by the mass conservation property in Corollary 5.4.6. Using (5.5.2) for the norms of ϕ_n and ϕ_{n+1}, one has the following estimate

$$\Big|\nu|\phi_{n+1}|^2 - \nu|\phi_n|^2\Big| \leq \Big|\nu_n\big(|\phi_n|^2;\rho_n\big)\Big| + \Big|\nu_{n+1}\big(|\phi_{n+1}|^2;\rho_{n+1}\big)\Big| + \Big|\nu_{n+1}|f|^2 - \nu_n|f|^2\Big|. \tag{5.5.3}$$

To bound the first covariance, let $\Gamma = ([-12nL, -4nL]\cup[4nL, 12nL])\cap\mathbb{Z}$ and we rewrite the covariance by

$$\nu_n(|\phi_n|^2;\rho_n) = \nu_n\big(|\phi_n|^2 - \mathbb{E}_\Gamma|\phi_n|^2;\rho_n\big) + \nu_n(\mathbb{E}_\Gamma|\phi_n|^2;\rho_n). \tag{5.5.4}$$

By conditions of the theorem, the lattice is one-dimensional and J_K is uniformly bounded for $|K| \geq 2$, the ratio ρ_n is thus bounded uniformly in n by $1/B_1 \leq \rho_n \leq B_1$, where

$$B_1 = \exp\left(4\sup_{k\in\mathbb{Z}}\left(\sum_{K\ni k,\,|K|\geq 2}\|J_K\|_\infty\right)\right) \tag{5.5.5}$$

is finite since the interaction is of finite range. Hence, for any function $g \geq 0$ one has $\nu(g) \leq B_1 \nu_n(g)$ and $\nu_n(g) \leq B_1 \nu(g)$. With this density ratio bound, the first covariance in (5.5.4) can be estimated by

$$\big|\nu_n\big(|\phi_n|^2 - \mathbb{E}_\Gamma|\phi_n|^2; \rho_n\big)\big| \leq 2B_1\nu\big||\phi_n|^2 - \mathbb{E}_\Gamma|\phi_n|^2\big| \leq 2B_1\nu\mathbb{E}_\Gamma\tilde{\mathbb{E}}_\Gamma\big||\phi_n|^2 - |\tilde{\phi}_n|^2\big|. \tag{5.5.6}$$

For the double measure $\mathbb{E}_\Gamma\tilde{\mathbb{E}}_\Gamma$, we apply the same strategy as in (5.3.20) to get

$$(5.5.6) \leq 4B_1\nu\Big(\big(\mathbb{E}_\Gamma|\phi_n|^2\big)^{1/2}\big(\mathbb{E}_\Gamma|\phi_n - \mathbb{E}_\Gamma\phi_n|^2\big)^{1/2}\Big) \leq 4(B_1)^2\big(\mathbb{E}^\circ_\Gamma|f|^2\big)^{1/2}\nu\big(\mathbb{E}_\Gamma|\nabla_\Gamma\phi_n|^2\big)^{1/2},$$

where we used the mass conservation for $\mathbb{E}^\circ_\Gamma$ and the spectral gap inequality for $\mathbb{E}_\Gamma$ (since $\mathbb{E}_\Gamma$ satisfies the LSI by conditions of the theorem). Using the finite-speed propagation property (5.4.16) for $\mathbb{E}^\circ_\Gamma|\nabla_\Gamma\phi_n|^2$, one has the estimate

$$(5.5.6) \leq D_1|\Lambda_n|e^{-\lfloor \mathrm{dist}(\Gamma,\Lambda_f)/R\rfloor}(\nu|f|^2)^{1/2}(\nu|\nabla f|^2)^{1/2}$$

with $D_1 = 4(B_1)^5$. By the construction of Γ and Λ_n, we have $\mathrm{dist}(\Gamma, \Lambda_f) \geq \frac{1}{8}\mathrm{diam}(\Lambda_n)$ and therefore there exists $\varepsilon_1 > 0$ independent of n such that

$$\big|\nu_n\big(|\phi_n|^2 - \mathbb{E}_\Gamma|\phi_n|^2; \rho_n\big)\big| \leq D_1|\Lambda_n|e^{-\varepsilon_1\mathrm{diam}(\Lambda_n)}\big(\nu|f|^2 + \nu|\nabla f|^2\big). \tag{5.5.7}$$

Here we note that due to the finite-speed propagation property (5.4.16), the bound (5.5.7) only holds in finite time range $t \leq C\mathrm{diam}(\Lambda_n)$ for some $C > 0$ independent of n. As $n \to \infty$ in the sequence, this time range will extend to $[0, \infty)$ and the above estimate becomes uniform in time.

To bound the second covariance in (5.5.4), we use the mixing condition (5.3.22) for the measure $\mathbb{E}^\circ_{\Lambda_n}$, which holds true since $\mathbb{E}^\circ_{\Lambda_n}$ satisfies a uniform LSI and $\nabla_{ij}U^\circ_\Lambda$ is bounded due

to the bounded multi-particle interaction. Therefore, there exist $C_{\mathrm{mix}}, M > 0$ independent of n such that

$$\left|\mathbb{E}^{\circ}_{\Lambda_n}(\mathbb{E}_\Gamma|\phi_n|^2;\rho_n)\right| \leq C_{\mathrm{mix}}|\Lambda_\Gamma||\Lambda_{\rho_n}|e^{-M\mathrm{dist}(\Gamma^{\mathsf{c}},\Lambda_{\rho_n})}\left(\mathbb{E}^{\circ}_{\Lambda_n}\left|\nabla_{\Lambda_n}\mathbb{E}_\Gamma|\phi_n|^2\right|^2\right)^{\frac{1}{2}}\left(\mathbb{E}^{\circ}_{\Lambda_n}|\nabla_{\Lambda_n}\rho_n|^2\right)^{\frac{1}{2}}.$$

Here Λ_Γ, Λ_{ρ_n} are the smallest sets in which $\mathbb{E}_\Gamma|\phi_n|^2$ and ρ_n are localised, respectively. First, it is easy to see that both $|\Lambda_\Gamma|$ and $|\Lambda_{\rho_n}|$ are smaller than $|\Lambda_n|$. Moreover, since $\mathrm{dist}(\Gamma^{\mathsf{c}},\Lambda_{\rho_n}) \geq \frac{1}{8}\mathrm{diam}(\Lambda_n)$ there exists $\varepsilon_2 > 0$ independent of n such that the exponential term is bounded by $e^{-\varepsilon_2\mathrm{diam}(\Lambda_n)}$. As for the $\nabla_{\Lambda_n}\rho_n$ term, we use the following bound

$$\|\nabla_{\Lambda_n}\rho_n\|_\infty \leq 2B_1 \sup_{k\in\mathbb{Z}}\left(\sum_{K\ni k,\,|K|\geq 2}\|\nabla J_K\|_\infty\right) = B_2$$

where B_2 is finite since J_K is of finite range and has uniformly bounded derivatives for $|K| \geq 2$. To estimate the gradient $\nabla_{\Lambda_n}\mathbb{E}_\Gamma|\phi_n|^2$, direct calculation gives

$$\nabla_j\mathbb{E}_\Gamma|\phi_n|^2 = \mathbb{E}_\Gamma\big(\nabla_j|\phi_n|^2\big) - \mathbb{E}_\Gamma\big(|\phi_n|^2;\nabla_j U_\Gamma\big)$$

for any $j \notin \Gamma$. The first term on the right is bounded above by $\mathbb{E}_\Gamma|\phi_n|^2 + \mathbb{E}_\Gamma|\nabla_j\phi_n|^2$ due to Young's inequality. For the covariance, an application of Lemma 5.3.11 and the relative entropy inequality (similar as (5.3.23)) gives

$$\left|\mathbb{E}_\Gamma\big(|\phi_n|^2;\nabla_j U_\Gamma\big)\right| \leq C_{RE}|\Lambda_n|\big(\mathbb{E}_\Gamma|\phi_n|^2 + \mathbb{E}_\Gamma|\nabla_j\phi_n|^2\big)$$

for some $C_{RE} > 0$ independent of n. Collecting all the bounds, we have

$$\left|\nu\mathbb{E}^{\circ}_{\Lambda_n}(\mathbb{E}_\Gamma|\phi_n|^2;\rho_n)\right| \leq B_2C_{\mathrm{mix}}(1+C_{RE})|\Lambda_n|^4e^{-\varepsilon_2\mathrm{diam}(\Lambda_n)}\big(\nu|\phi_n|^2 + \nu|\nabla\phi_n|^2\big).$$

Using the mass conservation for $\mathbb{E}^\circ_{\Lambda_n}|\phi_n|^2$ and the gradient bound for $\mathbb{E}^\circ_{\Lambda_n}|\nabla\phi_n|^2$, it follows

$$\left|\nu\mathbb{E}^\circ_{\Lambda_n}(\mathbb{E}_\Gamma|\phi_n|^2;\rho_n)\right| \le D_2|\Lambda_n|^4 e^{-\varepsilon_2 \mathrm{diam}(\Lambda_n)}\left(\nu|f|^2 + e^{2|\lambda|t}\nu|\nabla f|^2\right) \tag{5.5.8}$$

with $D_2 = B_1^2 B_2 C_{\mathrm{mix}}(1+C_{RE})$. Taking (5.5.7) and (5.5.8) into (5.5.4), one obtains

$$\left|\nu_n(|\phi_n|^2;\rho_n)\right| \le D_{12}|\Lambda_n|^4 e^{-\varepsilon_{12}\mathrm{diam}(\Lambda_n)}\left(\nu|f|^2 + e^{2|\lambda|t}\nu|\nabla f|^2\right) \tag{5.5.9}$$

with $D_{12} = \max\{D_1, D_2\}$ and $\varepsilon_{12} = \min\{\varepsilon_1, \varepsilon_2\}$. This concludes the estimate of the first term in (5.5.3). The second term in (5.5.3) follows similarly and we will finish the proof of (5.5.1) by bounding the third term in (5.5.3) as below.

For any $\theta \in [0,1]$ we define the interpolating potential

$$U^\circ_\theta := U^\circ_{\Lambda_n} + \sum_{k\in\Lambda_{n+1}\setminus\Lambda_n} J_{\{k\}} + \sum_{\substack{K\subseteq\Lambda_{n+1},|K|\ge 2\\ K\cap(\Lambda_{n+1}\setminus\Lambda_n)\neq\varnothing}} \theta J_K$$

and the corresponding local Gibbs measure $d\mathbb{E}^\circ_\theta := \left(\int e^{-U^\circ_\theta}dx_{\Lambda_{n+1}}\right)^{-1} e^{-U^\circ_\theta}dx_{\Lambda_{n+1}}$. Then one can rewrite the following difference into an integral

$$\mathbb{E}^\circ_{\Lambda_{n+1}}|f|^2 - \mathbb{E}^\circ_{\Lambda_n}|f|^2 = \int_0^1 d\theta \frac{d}{d\theta}\mathbb{E}^\circ_\theta|f|^2,$$

where the θ-derivative follows $\frac{d}{d\theta}\mathbb{E}^\circ_\theta|f|^2 = -\mathbb{E}^\circ_\theta\left(|f|^2;\frac{d}{d\theta}U^\circ_\theta\right)$. Since the lattice is one-dimensional, the size $|\Lambda_{n+1}\setminus\Lambda_n|$ is independent of n and thus the density $d\mathbb{E}^\circ_{\Lambda_{n+1}}/d\mathbb{E}^\circ_\theta$ is bounded above uniformly in n and θ. Then by the argument in [80, Lemma 5.1], if $\mathbb{E}^\circ_{\Lambda_{n+1}}$ satisfies the volume-uniform LSI the measure $\mathbb{E}^\circ_\theta$ also satisfies the LSI uniformly in n and θ. Thus, Theorem 5.3.10 holds for $\mathbb{E}^\circ_\theta$ and the mixing condition

$$\left|\mathbb{E}^\circ_\theta\left(|f|^2;\tfrac{d}{d\theta}U^\circ_\theta\right)\right| \le C'_{\mathrm{mix}}|\Lambda_\theta||\Lambda_f|e^{-M'\mathrm{dist}(\Lambda_\theta,\Lambda_f)}\left(\mathbb{E}^\circ_\theta|\nabla_{\Lambda_{n+1}}|f|^2|^2\right)^{\frac{1}{2}}\left(\mathbb{E}^\circ_\theta|\nabla_{\Lambda_{n+1}}\left(\tfrac{d}{d\theta}U^\circ_\theta\right)|^2\right)^{\frac{1}{2}}$$

holds for some $C'_{\text{mix}}, M' > 0$ independent of n and θ, where Λ_θ is the smallest set where $\frac{d}{d\theta}U^\circ_\theta$ is localised in. By the construction of Λ_n, it follows $|\Lambda_\theta||\Lambda_f| \leq |\Lambda_n|^2$ and $\operatorname{dist}(\Lambda_\theta, \Lambda_f) \geq \frac{1}{4}\operatorname{diam}(\Lambda_n)$. In addition, the gradient on $\frac{d}{d\theta}U^\circ_\theta$ can be bounded by

$$\left\|\nabla_{\Lambda_{n+1}}\left(\tfrac{d}{d\theta}U^\circ_\theta\right)\right\|_\infty \leq \sup_{n\in\mathbb{N}}\left(\sum_{\substack{K\cap(\Lambda_{n+1}\setminus\Lambda_n)\neq\varnothing\\|K|\geq 2}}\|\nabla J_K\|_\infty\right) = B_3.$$

Then by noticing that $\mathbb{E}^\circ_\theta(g) \leq B_1\nu(g)$ for all $\theta \in [0,1]$ and $g \geq 0$, we get the estimate

$$\left|\nu_{n+1}|f|^2 - \nu_n|f|^2\right| \leq D_3|\Lambda_n|^2 e^{-\varepsilon_3 \operatorname{dist}(\Lambda_n)}\left(\nu|f|^2 + \nu|\nabla f|^2\right) \tag{5.5.10}$$

with $D_3 = B_1B_3C'_{\text{mix}}$ and $\varepsilon_3 = \frac{1}{4}M'$. Collecting (5.5.9) and (5.5.10) in (5.5.3), we finally arrive at

$$\left|\nu|\phi_{n+1}|^2 - \nu|\phi_n|^2\right| \leq D|\Lambda_n|^4 e^{-\varepsilon\operatorname{diam}(\Lambda_n)}\left(\nu|f|^2 + e^{2|\lambda|t}\nu|\nabla f|^2\right)$$

for some $D, \varepsilon > 0$ independent of n. Since the series $\sum_n |\Lambda_n|^4 e^{-\varepsilon\operatorname{diam}(\Lambda_n)}$ converges, the real sequence $\nu|\phi_n|^2$ is Cauchy and thus convergent to some $C_{\text{lim}} \geq 0$. Using the same idea with triangle inequality, it can be shown that $\left|\nu|\phi_n(t_1)|^2 - \nu|\phi_n(t_2)|^2\right|$ tends to zero for any time t_1, t_2. Therefore, the limit C_{lim} is independent of time and identical to $\nu|f|^2$, which concludes the proof of (5.5.1).

Using the finite-dimensional estimates from Corollary 5.4.6, in particular the energy conservation and entropy bound, we can obtain a compactness of the set of solutions ϕ_n to (LSE°_Λ). When the entropy term is bounded the sequence $\{\phi_n\}$ is weakly compact, together with the convergence of the $\mathbb{L}^2(\nu)$ norms $\nu|\phi_n|^2$ by (5.5.1), this implies that the sequence of solutions converges strongly in $\mathbb{L}^2(\nu)$. Hence there exists a subsequence ϕ_{n_m} with $m \in \mathbb{N}$ that converges ν-almost surely to some $\phi \in \mathbb{L}^2(\nu)$. This concludes that ϕ is a solution to the infinite-volume LSE in the ultra weak sense, i.e. for any smooth, local and

compactly supported function $g : \Omega \to \mathbb{C}$ we have

$$\nu\left(gi\partial_t\phi\right) = \nu\left((-\mathcal{L}g)\phi + g\lambda\phi\log\frac{|\phi|^2}{\nu|\phi|^2}\right)$$

where the operator $\mathcal{L} = \lim_{\Lambda\to\mathbb{Z}}\mathcal{L}_\Lambda$ is well-defined for all such g.

In addition, due to the finite-dimensional uniform bounds in $H^1(\nu)$ from Section 5.4, one can choose a suitable weakly convergent subsequence (for which the nonlinear term also converges by the uniform bound for entropy), such that the limit ϕ of the subsequence solves the weak infinite-volume LSE

$$\nu\left(gi\partial_t\phi\right) = \nu\left(\nabla g\cdot\nabla\phi + g\lambda\phi\log\frac{|\phi|^2}{\nu|\phi|^2}\right).$$

This concludes the proof of Theorem 5.5.1. □

In the following, we give an example potential that satisfies the conditions for the existence theorem.

Example 5.5.2. For any $K \Subset \mathbb{Z}$, consider the perturbed quadratic potential

$$J_K(x) = \begin{cases} C_j x_j^2 & \text{if } K = \{j\} \\ \varepsilon W_K(x) & \text{if } |K| \geq 2 \end{cases}.$$

where $\varepsilon \in \mathbb{R}$, $W_K \in C^2(\mathbb{R}^K)$ has uniformly in K bounded derivatives of order $n = 0, 1, 2$ and $W_K \equiv 0$ if $\operatorname{diam}(K) > R$. Suppose C_j is uniformly bounded above and below by positive constants, then for sufficiently small $|\varepsilon|$ the corresponding measures $\mathbb{E}_\Lambda$ and $\mathbb{E}_\Lambda^\circ$ both satisfy the volume-uniform LSI by Remark 5.3.2. Moreover, by [81, Proposition 1.3], the Cauchy problem ($\mathrm{LSE}_\Lambda^\circ$) defined by the potential $\{J_K\}_{K\subset\Lambda}$ admits a unique solution for all $\Lambda \Subset \mathbb{Z}$, $\lambda \in \mathbb{R}$ and initial condition $f \in \mathfrak{F}(\Omega)$. This shows that all conditions in Theorem 5.5.1 are satisfied and there exists a weak solution of (LSE_∞) associated to the

perturbed quadratic potential on $\mathbb{R}^{\mathbb{Z}}$.

Remark 5.5.3. In the proof of Theorem 5.5.1, the condition $D = 1$ on the lattice dimension is required to ensure $B_1 < \infty$ in (5.5.5). To see this, note that B_1 depends on the boundary size $|\partial\Lambda|$ of the cube $\Lambda \Subset \mathbb{Z}^D$. When $D = 1$, $|\partial\Lambda| \equiv 2$ is independent of $|\Lambda|$. In higher dimensions $D \geq 2$, $|\partial\Lambda|$ grows indefinitely with $|\Lambda|$ and thus make B_2 blow up. However we remark that this could be offset with further assumptions on the potentials.

5.6 Solitary solutions for LSE

The solitons were originally introduced in [82] for the LSE in finite dimensions, and were later studied in a number of works (e.g. see [58, 60, 61, 83, 55, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98]). In $\mathbb{R}^m$ one can find general solutions to the following LSE

$$i\partial_t\psi = -\Delta\psi + V\psi + \lambda\psi \log|\psi|^2$$

by performing separation of variables $\psi(t,x) = e^{-iEt}\phi(x)$, where $\phi : \mathbb{R}^m \to \mathbb{R}$ is the stationary state satisfying the eigen-equation

$$-\Delta\psi + V\psi + \lambda\psi \log|\psi|^2 = E\psi \tag{5.6.1}$$

with energy eigenvalue $E \in \mathbb{R}$. For simplicity, we shall assume $V = 0$ and try to solve

$$-\Delta\psi + \lambda\psi \log|\psi|^2 = E\psi. \tag{5.6.2}$$

Inspired by the works in [50], we insert the soliton-like ansatz $\psi = \exp(-\frac{a}{2}|x|^2)$ into (5.6.2) and find that $a = -\lambda$ provides a solution with $E = -\lambda n$. By setting $\psi_0 = \exp(-\frac{\lambda}{2}|x|^2)$, it follows that $\psi_b \coloneqq b\psi_0$ is also a solution to (5.6.2) for any nonzero $b \in \mathbb{R}$, with eigenvalue

$E_b = -\lambda(n - \log b^2)$. To normalise the wavefunction ψ, we will need $\lambda < 0$ and the normalised solution is $\psi = (-2\pi/\lambda)^{\frac{n}{2}} \exp(-\frac{\lambda}{2}|x|^2)$ with eigenvalue $E = -\lambda n(1 - \log(-2\pi/\lambda))$. Next, we investigate (5.6.1) with a quadratic potential $V = \sum_{j=1}^n a_j x_j^2$ where $a_j > 0$, and this time we use a more general ansatz $\psi = \exp(-U(x))$ to obtain

$$\Delta U - |\nabla U|^2 - 2\lambda U = E - \sum_{j=1}^n a_j x_j^2.$$

With $U = \sum_{j=1}^n \frac{1}{2} c_j x_j^2$ and $c_j = \frac{1}{2}(\sqrt{4a_j + \lambda^2} - \lambda) > 0$, one gets a normalised solution $\psi = \sqrt{(2\pi)^n / \Pi_j c_j} \exp(-\frac{1}{2}\sum_j c_j x_j^2)$ with eigenvalue $E = \lambda n \log(2\pi) + \sum_j (c_j - \lambda \log c_j)$.

We remark that the use of Galilean covariance could provide more soliton type solutions [83], but one will need suitable adaptation for more general potentials. As for LSE in infinite-dimensional systems, the situation is more complicated for nontrivial interactions. In case log-Sobolev inequality is satisfied, one has the following analog of spectral gap property

$$\nu\left(-\phi^* \mathcal{L}\phi + \lambda|\phi|^2 \log \frac{|\phi|^2}{\nu|\phi|^2}\right) \geq \frac{2(1-\varepsilon)}{c}\left(\frac{1}{c} + \lambda\right)\nu|\phi|^2,$$

provided that $\lambda > -\varepsilon/c$ for some $\varepsilon \in [0, 1)$. However, we note that the structure of spectrum is not discrete in general, and even in the best cases (e.g. in statistical mechanics [99] or quantum field theory [100] and the references therein) one has separated windows of continuous spectrum. Therefore the classical concept of solitons may not make sense in this case. Nevertheless, it would be possible to expect states with solitary behaviour on large timescale, and it would be interesting to develop such theory (with potential applications e.g. to quantum computing).

Chapter 6

Summary and Outlook

To summarise, the central object under study is the interacting system on an infinite-dimensional lattice. We constructed a finite-range potential and the associated Gibbs measure in finite and infinite volume. We then introduced the Langevin dynamics on the system with the corresponding Markov semigroups, for which the Gibbs measures are invariant and reversible. To study the ergodicity/contractivity of the semigroups, we make use of the coercive inequalities, in particular, the spectral gap and log-Sobolev inequalities. We then generalised the said inequalities to the homogeneous Carnot group to investigate the non-commutative settings. In addition, the nonlinear quantum dynamics are also considered on the system, with results concerning the finite-dimensional estimates and the existence of infinite-dimensional weak solutions to the logarithmic Schrödinger equation.

Based on these results, we discuss a few possibilities that may lead to potentially interesting topics. First of all, on top of SGI and LSI we could consider other coercive inequalities interpolating between these two, for example, the modified log-Sobolev inequality or the super-Poincaré inequality (see [30, 101] and the references therein for more of such inequalities). With a similar strategy presented in Chapter 3, it could be possible to

extend these inequalities to infinite dimensional spaces. However, we note that our strategy relies on iterating finite-dimensional inequalities and delicate issues may arise when trying to control the constants during the iterations.

Along this path, we could consider an even more abstract setup, where the left-hand side of the inequality is given by $\mu(\Phi(f)) - \Phi(\mu(f))$ for some smooth convex function $\Phi : \mathbb{R} \to \mathbb{R}$ (note that $\Phi = x^2$ gives SGI and $\Phi = x \log x$ gives LSI). One could therefore try to adapt the current strategy into this general setting, and explore which properties must be set on Φ in order to extend the inequality to infintie dimensions.

As for the results in Chapter 4, we only showed that the LSI is possible for homogeneous norms with unbounded sub-gradients, but left the window open for norms with both blowing up and vanishing sub-gradients. In this case, the LSI should not hold due to the argument in Theorem 2.4.15. However, an SGI is still possible since one may obtain a U-bound by a similar strategy as for the Kaplan norm N (see [48]). Then the rest follows from [5] and a similar strategy as in the end of Chapter 3.

On the topic of the Schrödinger equation, a natural generalisation is to consider the LSE on Heisenberg groups $\mathbb{H}^{\mathbb{Z}}$, with the gradient and Laplacian replaced by $\nabla_{\mathbb{H}}$ and $\Delta_{\mathbb{H}}$. Then we could utilise a similar approach as in Chapter 5 with the coercive inequalities developed in Chapter 3 and 4 to establish similar results as in Section 5.4 and 5.5. Concerning the existence theorem 5.5.1, it is possible that with similar ideas in [102] we could develop a strategy for higher order estimates which would lead to a stronger convergence of the approximating sequence, corresponding to a reasonable potential (e.g. the quadratic potential perturbed by smooth, bounded and short-range multi-particle interactions). Moreover, with these high-order estimates and further assumptions on the initial data, it is possible to obtain the existence results for potentials with unbounded multi-particle interactions and on lattice $\mathbb{Z}^D$ with $D \geq 2$.

Appendix A

A.1 A sub-quadratic inequality

Lemma A.1.1. *Let $a, b \in \mathbb{R}$, and $q \in (1, 2]$. Then*

$$|a|^q - |b|^q \leq \operatorname{sgn}(b) q |b|^{q-1} (a - b) + 2|a - b|^q . \tag{A.1.1}$$

Proof. If $b = 0$ then (A.1.1) follows immediately, so assume $b \neq 0$. If we divide both sides of (A.1.1) by $|b|^q$ it reads

$$|x|^q - 1 \leq q(x - 1) + 2|x - 1|^q \tag{A.1.2}$$

with $x \coloneqq a/b$. It then suffices to prove (A.1.2) for all $x \in \mathbb{R}$ and $q \in (1, 2]$. Define the difference function $\varphi(x) = 2|x - 1|^q + q(x - 1) - |x|^q + 1$ and its dervative

$$\psi(x) \coloneqq \frac{1}{q}\varphi'(x) = 2\operatorname{sgn}(x - 1)|x - 1|^{q-1} + 1 - \operatorname{sgn}(x)|x|^{q-1} . \tag{A.1.3}$$

The proof is a straightforward exercise in verifying the following four possible cases.

1. If $x > 1$, then $\psi(x) > 0$.

2. If $x = 1$, then $\psi(x) = 0$.

3. If $0 \leq x < 1$, then $\psi(x) = 1 - x^{q-1} - 2(1-x)^{q-1}$ and since

$$x^{q-1} + (1-x)^{q-1} > (x + (1-x))^{q-1} = 1$$

it follows $\psi(x) < 0$.

4. If $x < 0$, then $\psi(x) = 1 + (-x)^{q-1} - 2(1-x)^{q-1}$ and since

$$1 + u^m - 2(1+u)^m = (u^m - (1+u)^m) - (1 - (1+u)^m) < 0$$

for any $u, m > 0$ it follows $\psi(x) < 0$.

We conclude the derivative $\psi = \varphi'/q$ is negative on $\{x < 1\}$, positive on $\{x > 1\}$, and zero at $x = 1$, which means φ is minimised at $\varphi(1) = 0$. We conclude (A.1.1) holds with equality if and only if $a = b$. □

A.2 Relative entropy inequality

Lemma A.2.1. *Let $\Omega = \mathbb{R}^m$ and μ be a probability measure on Ω. For any measurable non-negative f, g on Ω and any $\varepsilon > 0$, we have the relative entropy inequality*

$$\mu\left(f \log \frac{g}{\mu g}\right) \leq \mu\left(f \log \frac{f}{\mu f}\right).$$

Proof. We first prove that $xy \leq x \log x - x + e^y$ for any $x \geq 0$, $y \geq 0$. To see this, note that $e^{y-u} \geq 1 + (y-u)$ for any $u \in \mathbb{R}$, which implies $e^u - e^y \leq e^u u - e^u y$. Replacing u with $\log x$ gives the desired inequality for $x > 0$, $y \geq 0$. As for $x = 0$, the equality holds since $\lim_{x \downarrow 0}(x \log x) = 0$. Substituting $x = f/\mu f$, $y = \log(g/\mu g)$ in $xy \leq x \log x - x + e^y$ and integrating with μ, one obtains the relative entropy inequality. □

Remark A.2.2. For any $\varepsilon > 0$ and non-negative f, g, it follows

$$\mu(fg) = \frac{1}{\varepsilon}\mu\left(f \log\left(e^{\varepsilon g}\right)\right) = \frac{1}{\varepsilon}\mu\left(f \log \frac{e^{\varepsilon g}}{\mu\left(e^{\varepsilon g}\right)}\right) + \frac{1}{\varepsilon}\mu(f)\mu\left(e^{\varepsilon g}\right).$$

Applying the relative entropy inequality for the first term on the right-hand side, it follows

$$\mu(fg) \leq \frac{1}{\varepsilon}\mu\left(f \log \frac{f}{\mu f}\right) + \frac{1}{\varepsilon}\mu(f) \log\left(\mu(e^{\varepsilon g})\right).$$

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