

**Unsteady Aerodynamics, Reduced-Order
Modelling, and Predictive Control in
Linear and Nonlinear Aeroelasticity
with Arbitrary Kinematics**

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Declaration

I hereby declare that this thesis, and the research to which it refers, are the product of my own work, and has not been submitted for any other reward. Ideas and quotations from the work of other people, published or otherwise, are fully acknowledged.

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Robert J. S. Simpson

*To my Mum and Dad, Jenny and Euan,
for cultivating a curious mind*

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Abstract

This thesis concerns the development of computational methods for efficient flexible-aircraft flight dynamics analyses. An argument is made for a hierarchy of methods that provide predictive capability for loads and stability analyses, and the ability to create low-order dynamic models for control system synthesis. The proposed aeroelastic models are formulated using three-dimensional unsteady aerodynamics in the form of an unsteady vortex-lattice method developed to model the relatively complex kinematics inherent in flexible-aircraft dynamics, and in particular the unsteady induced drag. No assumptions are made relating to the kinematics of the fluid-structure interface (inputs) and use of the three-dimensional Joukowski relation naturally resolves all components of the unsteady aerodynamic forcing (outputs). A consistent linearization of this method about an arbitrary reference state yields nondimensional (independent of free-stream dynamic pressure) discrete-time state-space models that resolve frequencies up to a spatio-temporal Nyquist limit defined by the wake discretization, and have a convenient form for coupling with structural dynamics models. Aircraft structural components are modelled using a geometrically-exact composite beam formulation, and, additionally, in the case of linear dynamics, a generic modal description. The latter allows the linear aerodynamics to be expressed in a reduced set of inputs and outputs, thus obtaining a time-domain alternative to the classical frequency-domain-based doublet-lattice method. The models modified for these modal degrees-of-freedom are shown to be amenable to balanced realization and truncation, and are verified in flutter analyses where only 10-100 balanced states are required (compared to 1000-10,000 physical states) for converged results. Finally, predictive controllers and linear-quadratic regulators are synthesized using reduced-order aeroelastic models, and are applied in nonlinear simulations for gust-load alleviation.

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Nomenclature

Symbols

A	vortex ring (panel) area
$\mathbf{A}_1$	matrix of variations of CRV derivatives
$\mathcal{A}$	aerodynamic influence coefficient matrix
C	aerodynamic coefficient
$C(k)$	Theodorsen's function, $C(k) = F(k) + iG(k)$
$\mathcal{C}$	constant matrix, state-space matrix, coordinate transformation
$\mathbf{C}$	damping matrix
c	chord, m
E	Young's modulus
f	frequency
$\mathbf{f}$	force
G	shear modulus
g	acceleration due to gravity
H	control horizon, or Heaviside function
$\mathbf{H}$	segment-lattice interpolation matrix
h	plunge displacement
I	second moment of area
$\mathbf{I}, \mathcal{I}$	Identity matrix
J	polar moment of area
K	number of vortex rings
$\mathbf{K}$	curvature
k	reduced frequency, or vortex ring index
$\mathbf{l}$	vortex segment vector
M	number of chordwise vortex rings
	Mach number (Section 1.1)
$\mathcal{M}$	mass matrix
$\mathbf{m}$	moment
n	system size
N	number of spanwise vortex rings

$\mathbf{N}$	matrix of normal vectors
$\mathbf{n}$	panel normal vector
P	number of panels
	orthogonal projection operator
$\mathbf{Q}$	column matrix of discrete/generalized forces
q	dynamic pressure
$\mathbf{q}$	modal coordinates
$\mathbf{R}$	beam nodal displacement
r	reduced model-order
$\mathbf{r}$	vortex segment kernel vector
	displacement of body-fixed frame (Chapter 3)
$\mathbf{S}$	stiffness matrix
s	vortex segment index, or beam arc-length
$\mathbf{T}$	similarity transformation
$\mathbf{T}$	tangential operator
$\mathbf{T}_{as}$	structural to aerodynamic DoF transformation
t	time
Δt	time step
$\mathbf{u}$	induced velocity field
$\mathbf{u}$	system inputs
V	velocity
$\mathbf{V}$	sectional inertial velocity
$\mathbf{v}$	fluid-lattice relative velocity, or velocity of body-fixed frame
$\mathbf{W}$	downwash projection matrix
w	downwash due to gust
$\mathbf{x}$	system states
$\mathbf{x}$	spatial field
$\mathbf{Y}$	aerodynamic state-space submatrix
$\mathbf{y}$	system outputs

Greek

α	angle of attack
β	flap angle
	Prandtl-Glauert factor (Section 1.1.1)
β	velocities of body-fixed frame
Γ	vortex circulation strength
$\mathbf{\Gamma}$	vector of panel circulation strengths
γ	force strains
ϵ_1, ϵ_2	convergence criteria
$\delta(\cdot)$	variational operator
$\boldsymbol{\eta}$	column matrix of structural degrees of freedom
$\boldsymbol{\Theta}$	Euler angles
ζ	aerodynamic lattice vertices
ζ	column matrix of aerodynamic grid nodes
κ	moment strains
λ	continuous-time eigenvalue, or wake wavelength
$\boldsymbol{\nu}$	vector of atmospheric fluid velocities
Ξ	lattice interpolation matrix
ξ	cross-sectional coordinates
ρ	air density
σ	Hankel Singular Value
ϕ	velocity potential
Φ	infinitesimal rotation, or matrix of modes
χ	quaternion
Ψ	Cartesian rotation vector
ω	angular velocity
$\boldsymbol{\omega}$	angular velocity of body-fixed frame
	vector-valued vorticity (Section 1.1.1)
Ω	angular velocity of material frame, or matrix of natural frequencies

Subscript

a	pertaining to aerodynamic system
ext	external
ea	elastic axis
f	flutter
fb	pertaining to flexible-body dynamics system
fh	flap hinge
g	gust
gyr	gyroscopic
ref	reference condition
s	pertaining to structural system
$stiff$	stiffness
w	pertaining to the wake
∞	free-stream condition

Superscript

c	at collocation point
m	at segment midpoint
ref	reference trajectory
$(\dot{\cdot})$	time derivative
$(\hat{\cdot})$	unit vector, or dimensional aerodynamic state
$(\tilde{\cdot})$	pertaining to a balanced system
$(\check{\cdot})$	skew-symmetric matrix
$(\cdot)'$	nondimensional time derivative
	derivative with respect to arc-length (Chapter 3)
$(\bar{\cdot})$	skew-symmetric matrix dual

Others

$\frac{D(\cdot)}{Dt}$	substantial derivate
∇	gradient operator

$\mathcal{O}$ of the order of

$\partial(\cdot)$ partial derivative

Abbreviations

ACARE	Advisory Council for Aviation Research and Innovation in Europe
AIC	Aerodynamic Influence Coefficient
BFF	Body-Freedom-Flutter
CFD	Computational Fluid Dynamics
CRV	Cartesian Rotation Vector
DLM	Doublet-Lattice Method
DoF	Degree of Freedom
FAFD	Flexible Aircraft Flight Dynamics
FMM	Fast Multipole Method
FoR	Frame of Reference
FSI	Fluid-Structure Interface
GECEB	Geometrically-Exact Composite Beam
HALE	High-Altitude Long-Endurance
HSV	Hankel Singular Value
LCO	Limit-Cycle Oscillation
LQR	Linear-Quadratic Regulator
MPC	Model Predictive Control
NASA	National Aeronautics and Space Administration (formerly NACA)
ONERA	Le Office National d'Etudes et de Recherches Aerospatiales
RFA	Rational-Function Approximation
ROM	Reduced-Order Model
SFW	Swept Flying-Wing
SHARPy	Simulation of High-Aspect-Ratio Planes in Python
UAV	Unmanned Aerial Vehicle
UVLM	Unsteady Vortex-Lattice Method
VFA	Very Flexible Aircraft
VLM	Vortex-Lattice Method

Chapter 1

Introduction

Over a hundred years of evolution in aircraft design has shown a relentless drive for ever more efficient vehicles. Long-term objectives in reduction of fuel consumption, noise, and emissions, while increasing range, payload, and comfort, are periodically set up by the aviation industry. In Europe, ACARE has recently establish broad objectives for air transport over the next few decades in their Flightpath 2050 report [34], while NASA periodically updates its technology roadmap over 20-year horizons [99]. Environmental concerns now bring additional pressure to the long-standing commercial interests, and have resulted in very ambitious targets in fuel-burn reduction. This has brought a demand for technologies that enable next-generation ultra-efficient engines, but also very lightweight airframes. In parallel, the rapid development of unmanned air systems is pushing for platforms with extremely high range and endurance. The confluence of both technological demands is forcing a paradigm shift in the design of next-generation aircraft: Ever-increasing structural flexibility caused by low-weight, high-aspect-ratio designs has brought about a new class of very flexible aircraft (VFA), that is, large vehicles with substantial geometry changes under loading and much more complex dynamical behaviour. In particular, the vibration characteristics of a VFA are of such low frequency that they interact with the classical flight dynamics response in a way that can only be described by a unified aeroelastic analysis incorporating rigid-body, elastic, and aerodynamics models simultaneously [143], and the large geometry changes bring the need for geometrically-nonlinear models of the aircraft primary structures for an accurate characterization.

Examples of very flexible aircraft exhibiting this behaviour are: future generation Unmanned Aerial Vehicles (UAVs) for military intelligence, surveillance and reconnaissance [142]; High-Altitude Long-Endurance (HALE) aircraft developed for many applications including

communication in civilian applications [8]; aircraft designed for high-altitude atmospheric and/or persistent environmental research [128]; and high-performance gliders. All of them are characterized by very tight weight constraints and very high requirements on aerodynamic efficiency, which result in very high aspect-ratio as well as very light wings. Further comment on the appropriate mathematical modelling of these vehicles will be presented later, however, at this stage, it is important to note that high-fidelity computational tools (i.e., Navier-Stokes Computational Fluid Dynamics (CFD) and detailed finite-element modelling for Structural Dynamics) are too costly, and often impractical, for flight dynamics simulations of VFA. The cost of such techniques is further compounded by the requirement that the analysis be in the time-domain, and that many iterations are required in both preliminary aircraft sizing and control law design. If CFD-based aeroelastic analysis is still considered too costly on current-generation aircraft, which can be analyzed using linear, frequency-domain-based methods, the argument is even stronger for VFA. Consequently, this work will focus on appropriate mid-fidelity analysis methods and tools to support the development of VFA. Other applications of this analysis, which are beyond the scope of this PhD, include large wind turbines, helicopter rotors, and flexible/articulated flapping aircraft.

As epitomized by NASA's X-56 experimental vehicle [129], the successful development of VFA is tightly linked to control systems that are designed to provide stability augmentation, to prevent flutter instances within the flight envelope, as well as gust load alleviation, to guarantee the structural integrity in adverse conditions. Crucially, from a control perspective where system dynamics are often described in state-space form, the relatively-large static and dynamic deformations of VFA introduce state-dependent (and therefore temporal) variations of the aircraft dynamics. Thus, the assumption of a linear, time-invariant (LTI) system description, which is commonly found in the literature of flexible aircraft dynamics [70, 11], may not be appropriate when attempting to formulate suitable control strategies. Fortunately, however, the dynamics of VFA are quite slow, and suitable for real-time optimization-based control methods, such as model predictive control (MPC). MPC is a nonlinear control strategy developed originally for the process industries, that is capable of handling multiple-input multiple-output systems naturally; enforcing constraints on system inputs, input rates, and states; accounting for unmodelled plant dynamics and nonlinearities; and, dealing with uncertainties, for example disturbances to the system input and/or output, in a robust way [81]. It therefore offers an attractive solution to the challenging control problems inherent

in flexible aircraft design and operation, which are perhaps most strongly articulated in the Helios mishap report [100].

The Helios mishap is the foremost example of a failure of previous uncoupled, linear methodologies, and the subsequent NASA report [100] made several key points, including:

- “The aircraft represents a **nonlinear** stability and control problem involving complex interactions among the flexible structure, unsteady aerodynamics, flight control system, propulsion system, the environmental conditions, and vehicle flight dynamics.”
- “[Scientists and Engineers should] develop more advanced, multidisciplinary (structures, aeroelastic, aerodynamics, atmospheric, materials, propulsion, controls, etc) **time-domain** analysis methods appropriate to highly flexible, morphing vehicles.”
- “[...and] develop **multidisciplinary** (structures, aerodynamic, controls, etc) models, which can describe the nonlinear dynamic behavior of aircraft modifications...”

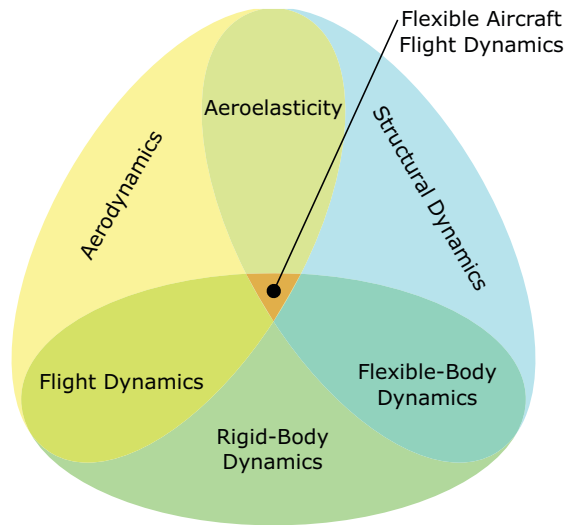


Figure 1.1: The interaction between rigid-body dynamics, structural dynamics and aerodynamics required for VFA dynamic analysis. This is an adapted form of Collar’s triangle [24].

From the above observations and recommendations it is clear that a new approach to aircraft design is necessary for VFA because their nonlinear, coupled aeroservoelastic (the synthesis of aerodynamics, structural dynamics, flight dynamics and control analysis) behaviour is not captured by current uncoupled, linear methodologies. The disciplinary synthesis required is shown in the Venn diagram of Figure 1.1. As will be discussed later in

this chapter, there has already been a substantial amount of effort in this direction in recent years. However, the focus has been in understanding key issues in VFA performance using low-fidelity formulations. This work aims to improve the state-of-the-art in computational methods and control strategies for the analysis of VFA using industrial-quality (or higher) levels of modelling fidelity. We will be particularly concerned with modelling strategies for the development of solar-powered aircraft of the type currently being considered by Google (codename *Solara 50*), Facebook (*Aquila*) and Airbus Defense & Space (*Zephyr*) as high-altitude platforms for communication and observation. Although it is not reported here, the author has benefited from direct involvement in the aeroelastic analysis of two of these platforms during this doctoral work.

1.1 Background

The focus of this thesis will be on unsteady aerodynamic modelling of VFA and the integration of resulting models into full-vehicle nonlinear aeroservoelastic analysis methods. Therefore, the state of the art in unsteady aerodynamic modelling will be discussed first in Section 1.1.1. The challenge is the development of reliable unsteady aerodynamics modelling of a complete aircraft configuration, including complex kinematics and large dynamic deflections of the aerodynamic surfaces. This is then supplemented in Section 1.1.2 with a discussion on the Unsteady Vortex Lattice Method (UVLM), which will be developed further in this work as it is capable of capturing many nonlinearities that other methods cannot, at a reasonable cost. Next, the available modelling approaches to investigate the dynamics of flexible air vehicles with coupled aeroelastic/flight dynamics response are discussed in Section 1.1.3. Finally, Section 1.1.4 will provide an overview of the state of the art in active aeroelastic control methods.

1.1.1 Full-vehicle unsteady aerodynamic modelling

Although there has been some attempts in aeroelastic modelling of highly-flexible wings using high-fidelity solutions [103, 36, 52, 88], the high computational cost has so far restricted their applicability to static equilibrium analyses. Much of the aerodynamic analysis of flexible aircraft flight dynamics is therefore based on potential flow methods, in which the assumption is that the aerodynamics are governed by Laplace’s equation for the scalar-valued velocity

potential. In many cases, especially for large-scale HALE-type aircraft, the simple description of the flow physics provided by potential flow can be justified as the aircraft operate exclusively within the low subsonic flow regime, the Reynolds number is large enough for viscous effects to be confined to infinitesimally-thin shear-layers, and the flow remains attached throughout the large but slow wing excursions. Laplace's equation is elliptic and the instantaneous flow velocity is completely defined by the boundary conditions at each instant, which is an excellent approximation (with huge computational advantages) to compute velocity fields in an incompressible inviscid fluid where the speed of sound (and therefore the speed at which information travels in the flow) is very large with respect to the convective time scales in the fluid. With this approach the velocity field can then be built by superposition of elementary solutions used to model aerodynamic surfaces and their wakes, the position and strength of which will generally vary in time.

Potential flow theory

Before the following aerodynamic methods are described it is useful to discuss the assumptions made in potential flow theory. A physical starting point for the derivation of the potential flow equations is the non-conservative form of the continuity equation [5],

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{U} = 0, \quad (1.1)$$

where $\frac{D(\cdot)}{Dt}$ is the substantial derivative, ρ is the fluid density, and $\mathbf{U}(\mathbf{x}, t)$ is the vector field of fluid velocity. This is the mathematical expression of conservation of mass in a frame moving with the fluid. It is assumed that the flow is locally incompressible everywhere and that therefore the density, ρ , can be assumed constant leading to,

$$\nabla \cdot \mathbf{U} = 0. \quad (1.2)$$

In addition to incompressibility (1.2) the assumption of flow irrotationality is also assumed. Vorticity, $\boldsymbol{\omega}$, is a metric of local flow rotation (it is equal to twice the angular velocity of an infinitesimal fluid element) defined as the *curl* of the velocity field, $\boldsymbol{\omega} = \nabla \times \mathbf{U}$. In potential flow it is assumed that the vorticity is zero everywhere except within infinitesimally-thin regions around around the aerodynamic surfaces (and their wakes). Hence, the flow in the

majority of the domain is assumed to be irrotational and,

$$\boldsymbol{\omega} = \nabla \times \mathbf{U} = 0. \quad (1.3)$$

The *curl* of the *grad* of a scalar function is identically zero, i.e $\nabla \times (\nabla \phi) = 0$ [74]. Therefore, the velocity field can be represented by the gradient of a scalar field, i.e $\mathbf{U} = \nabla \phi$. Substituting this result into Eq. (1.2) gives

$$\nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0, \quad (1.4)$$

which is a second-order, linear, elliptic, partial-differential equation known as Laplace’s equation. As discussed above, the linearity of the equation allows for the principle of *superposition* to be used to build-up a complex global solution from the summation of simple elementary solutions. Examples of such elementary solutions are the uniform flow, source, sink, doublet and vortex as described in many texts on the fundamentals of aerodynamics; see, for example, [71], or [5]. It is the three-dimensional vortex filament that is used to build a solution to Laplace’s equation in the UVLM.

Two-dimensional methods

Theodorsen used the potential flow model to formulate a closed-form solution for the pressure distribution on a thin aerofoil in 2-D undergoing oscillations in plunge, pitch and flap angle [141]. The resulting lift and moments were used to calculate the air velocity at which flutter will occur. The reduced frequency, $k = \omega c / 2V_\infty$,¹ was introduced in addition to Theodorsen’s lift deficiency function, which is a linear frequency response function between effective angle-of-attack and the normalized lift. The unsteady effects described by Theodorsen’s function are captured by including the effect of convection of vorticity in a flat (linear) wake. The key concepts introduced by Theodorsen have been used extensively in unsteady aerodynamics since their inception.

A review of potential-flow-based, oscillatory, thin-aerofoil motion with a focus on resulting lift and pitching-moments can be found in the work of [84] and, more recently, [114]. The focus is often on incompressible solutions, but the theory extends to compressible subsonic unsteady aerodynamics [10]. However, even in those cases simple compressibility corrections, such as the well-known Prandtl-Glauert correction factor, are suggested as effective strate-

¹Here ω is the angular rate of oscillations, c is the aerofoil chord, and V_∞ is the free-stream fluid velocity.

gies [84]. The pressure, lift and pitching-moment scale inversely with the Prandtl-Glauert factor, $\beta = \sqrt{1 - M_\infty^2}$,² where the reduced frequency parameter is also scaled such that $k_\beta = k/\beta^2$. Compressibility corrections using the Prandtl-Glauert factor have also been applied to three-dimensional potential flow methods such as the steady vortex lattice method [86] and unsteady-lifting-line method [32]. For thin aerofoil motions it is stated that the out-of-phase component in Theodorsen's lift deficiency function, is more sensitive to Mach number effects. The tenuous validity of the Kutta-condition for unsteady flows is stressed, and limit of $k \approx 1$ is suggested as beyond this the flow does not leave the trailing-edge smoothly.

Garrick, building on the work of Theodorsen, formulated a closed-form solution for the propulsive force experienced by an aerofoil undergoing oscillations in plunge, pitch and flap angle [37]. The major contribution of this work was the inclusion of leading-edge suction effects in the unsteady formulation; the aerofoil propulsive force being the sum of the leading-edge suction force and any component of the pressure force orientated towards the free-stream. Garrick's solution uses the thin-aerofoil approximation which leads to a formulation with infinite vorticity at an infinitesimally-small, rounded leading-edge, resulting in a finite force contribution [147].

Since unsteady aerodynamic methods formulated using Theodorsen's approach are based on functions evaluated at discrete frequencies they are useful for aeroelastic stability analyses based on iterative frequency matching such as the p and $p-k$ methods [65], however there is no direct way to obtain time-domain solutions. The development of two-dimensional incompressible unsteady aerofoil theory, including methods capable of time-marching, is chronicled in [114]. It is noted that the indicial lift response of an aerofoil undergoing a transient step-change in angle-of-attack was originally presented by Wagner [78] and was later shown by Garrick to be related to Theodorsen's lift deficiency function by an inverse Fourier transform [38]. Time-marching became possible as early as 1938 when R. T. Jones developed a two-state approximation to Wagner's indicial response ([68], originally cited in [114]) and since then finite-state approximations to unsteady aerofoil aerodynamics have blossomed.

Peters' finite-state airloads theory [116] and finite-state inflow theory [117] form an approximation to Laplace's equation for potential flow in which the effect of shed vorticity is captured by a finite number of state variables. The effect of control surface motion and displacements can be captured in this model as a specific case of general aerofoil chord line

²Here M_∞ is the free-stream Mach number.

deformations. Also, the inflow theory can be coupled with stall models such as the ONERA dynamic stall model (see [110] for an application to flexible aircraft dynamics). These finite-state models form a set of ordinary-differential equations that can be written in matrix form which allows them to be easily coupled with structural models for aeroelastic calculations in the frequency or time-domain [114]. A similar state-space formulation was presented in [77] for the unsteady lift of a flapped aerofoil which was found to perform better at low-frequencies for a given number of states [104].

Unsteady two-dimensional panel methods directly solve Laplace's equation for the scalar potential numerically and allow inclusion of geometric nonlinearities that are missing from the aforementioned analytical models [71]. Firstly, the non-penetration boundary condition is enforced on the instantaneous location of the surfaces, as opposed to a fixed reference position. Secondly, a force-free wake can be allowed to develop in accordance with Helmholtz's theorem that vortices are convected with the local flow velocity, as opposed to a linear wake fixed at a reference position. This free-wake nonlinearity was found to account for significant differences in periodically-averaged-thrust and efficiency ($\approx 10\%$) predicted by a two-dimensional panel code applied to plunging oscillations of a NACA0012 section [67], albeit at a high reduced-frequency of $k = 4$.

Three-dimensional methods

An adequate approximation of the spanwise lift distribution, even for high-aspect-ratio wings, requires 3-D modelling tools. Aerodynamic and structural nonlinearities in the context of the aeroelastic characterization of VFA wings are discussed in [104] and [106]. Since the underlying fluid mechanics are linear (potential flow) the aerodynamic nonlinearities present in the analysis are all geometrical in nature. The authors note that linear structural analysis coupled with three-dimensional geometrically-exact unsteady aerodynamics (the UVLM) was previously shown to significantly affect the aeroelastic response, suggesting aerodynamic geometric nonlinearities are important. Essentially, using a three-dimensional geometrically-exact aerodynamic theory amounts to solving Laplace's equation for the scalar potential by enforcing the non-penetration boundary condition on the entire, instantaneous, deformed aircraft geometry. Therefore, finite-wing effects, large-deformations, and cross-influence of multiple surfaces and their wakes can be captured.

Unsteady-lifting-line [1, 32] is an extension of the well-known lifting-line method to capture

in first-approximation spanwise variations in potential-flow due to finite-wing effects. In the unsteady extension a linear wake is shed instantaneously from each lifting-line in the direction of the free-stream. Only the quasi-steady three-dimensional influence of the wake is retained in the formulation of [32], while unsteady effects are approximated by a two-dimensional empirical lag term. Finite-wing effects, large-deformations, and cross-influence of multiple surfaces can be captured with this method, but it is only suitable in the limit of very-large-aspect-ratio surfaces with very slow motions. Numerical investigations have shown that the two-dimensional approximation of unsteady effects are valid for simulations featuring very high aspect-ratios ($AR \approx 200$), where flutter speeds were shown to compare well to those used by Theodorsen's theory. However, no verification has been shown for a three-dimensional case, such as the Goland wing [44].

The large majority of aircraft aeroelastic analyses are carried out using lifting-surface methods for unsteady flows. The most popular one is without doubt the Doublet-Lattice Method (DLM) [2], although alternatives exist, such as the Kernel Function Method [146] and the Unsteady Vortex Lattice Method (UVLM) [71]. These formulations all find a solution to the potential flow equations for the scalar potential (either velocity or acceleration potential) by placing an arrangement of quadrilateral elements over the mean-lifting-surface of a geometry. The DLM has found major application in the aeronautics industry for the prediction of aeroelastic instabilities of complete, relatively stiff, aircraft configurations. In the DLM, chordwise and free-stream directions are assumed to be parallel, and the non-penetration boundary condition is enforced on the undeformed geometry [14]. It is perhaps not suitable, therefore, for the analysis of flexible aircraft, which are typically subject to large deflections in both torsion and bending.

The (steady) Vortex Lattice Method (VLM) [71] is closely related to the DLM because a constant strength doublet panel is equivalent to a vortex ring of the same strength. Indeed, the DLM at $k = 0$ reduces to the VLM [123], but only for small angles-of-attack as the DLM contains linear assumptions about the geometry of the lifting-surfaces and their wakes [2]. As will be seen in the next section, there is a much greater distinction between the DLM and the Unsteady Vortex Lattice Method, however.

DLM-based aerodynamics are typically used to produce frequency-domain unsteady loads in modal bases. In this approach a transfer function is sought that relates the downwash and pressure distributions on a set of trapezoidal doublet elements that discretize infinitesimally-

thin approximate mean-surfaces that represent an aircraft's wings and other primary structures. In doing so only the incident flow and resulting forces normal to the free-stream are resolved, and any contribution to the unsteady aerodynamics from the steady-state loading distribution is lost [72]. As a result, it is not suitable for the analysis of T-tails, which are subject to in-plane motions on the horizontal tail [95]. In addition to that, a flat infinite wake that is parallel to the free-stream flow is also assumed, which stops the method from being applied in problems where this is not a reasonable assumption, for example in wind-turbine aeroelasticity [54]. It should be noted that the DLM is formulated in the frequency domain for small-amplitude simple-harmonic-motions, which is very appropriate for aeroelastic stability analysis. When time-domain models are required, rational-function approximations (RFA) of the unsteady aerodynamics are obtained using, in most cases, Roger's method [124, 16, 35, 29], which is based on Padé polynomials augmented by terms that represent the equivalent mass, damping and stiffness of the aerodynamic system. Following this an inverse Laplace transform yields a time-domain model, which is typically used for gust response and manoeuvre load analysis.

1.1.2 The unsteady vortex-lattice method (UVLM)

The UVLM offers alternative approach to DLM-based aerodynamics for incompressible flows. It is a three-dimensional, “geometrically-exact” (that is, capable to follow the lifting surfaces in their large excursions), lifting-surface method used to solve potential flows around non-steady aerodynamic bodies. As it is a lifting-surface method the aerodynamic bodies are represented by their mean aerodynamic surfaces, which are discretized in this case by a lattice of rectilinear vortex rings. Perhaps the most comprehensive modern description of the UVLM is given in [71], although discussion of induced drag calculation and efficient numerical implementation is very limited. In its standard formulation, the UVLM is limited to thin wings (which are modelled as lifting surfaces) in attached flow conditions and does not include the effect of non-lifting bodies.

The UVLM finds a potential flow solution for the entire domain by solving an integral equation on a thin lifting-surface immersed in the flow. The surface is discretized by vortex rings, or “panels”, each with circulation strength Γ . Solution to the integral equation is found at each time step by solving a system of linear algebraic equations of the form $\mathbf{A}\mathbf{\Gamma} = \mathbf{b}$, where the matrix $\mathbf{A}$ is known as the aerodynamic influence coefficient (AIC) matrix, $\mathbf{\Gamma}$ is the

vector of panel circulation strengths, and $\mathbf{b}$ is the right-hand-side vector formed by imposing the non-penetration boundary condition. Further details of the solution methodology will be given in Chapter 2 of this thesis.

As opposed to the DLM, the UVLM is formulated in the time-domain and naturally results in a discrete-time system of equations. A continuous-time formulation has been presented, but only in two dimensions [157]. Another distinction from the DLM is the geometrically-exact representation, where the non-penetration boundary condition is satisfied on the deformed aircraft geometry at every time step. The wake modelling also distinguishes the two models; the DLM uses a flat wake, whereas the UVLM has a wake which is shed from the deformed trailing-edge at each time step (referred to as a prescribed wake). In the UVLM the wake may also be allowed to roll up over time (a free wake) in accordance with Helmholtz’s laws of vortex motion [130].

Geometrical nonlinearities due to a free-wake may be of importance even in the aerodynamic analysis of simple geometries. Steady results with a fully-developed free-wake show that there is only a marginal difference in lift-distribution and overall lift compared to results obtained using a prescribed wake [140], however the effect on drag was not considered. The effect of free-wake nonlinearity in unsteady motion was also not considered.

Wake interference effects have been studied using the UVLM in the context of HALE aircraft. Murua and his co-workers at Imperial College London studied the wake-tail interference of a flexible HALE configuration for prescribed motions [96]. Under a prescribed phugoid-type motion, where aircraft pitch and plunge was combined to produce a constant angle of attack, the inclusion of a prescribed wake was shown to induce sinusoidal variations in lift from the tail. Pure plunging and pitching motions also highlighted interference effects, and such effects were also found to influence the dynamics of the aircraft open-loop response in [94]. The analysis in both cases was limited due the linear increase in the number of wake panels, P_w , with every time step because the wake roll-up routine has an algorithmic time-complexity of $\mathcal{O}(P_w(P_w + P_b))$, where P_b is the number of bound panels (those used to discretize the lifting-surfaces). Therefore, relatively coarse spatial and temporal discretizations were used, however they were adequate to illustrate that wake roll-up has a clear effect on tail lift in prescribed plunging motions of a flexible HALE aircraft. Wake roll-up was also included in the work of [33] where the UVLM was shown to be good candidate for, at the least, preliminary studies of interference effects. The UVLM is the only aerodynamic model discussed so

far that is capable of capturing wake interference effects that include free-wake nonlinearity.

Wake modelling and acceleration methods

Vortex methods are characterized by approximations of the vorticity in the boundaries of the domain, which, critically, includes the wake. When this discretization confines vector-valued vorticity $\boldsymbol{\omega}_p$ to infinitesimal points in space, such an approach is called a vortex particle method. A practical advantage of this approach is that connectivity of the vorticity distribution can be completely discarded [148], which can also simplify the creation of new vorticity shed from solid boundaries [79]. Also, this formulation is particularly amenable to fast solution using acceleration methods that will be discussed later in this section. However, particle methods may not satisfy the condition of Kelvin’s circulation theorem, $\frac{D\Gamma_{total}}{Dt} = 0$, where

$$\Gamma_{total} = \int \boldsymbol{\omega}_p(\mathbf{x}, t) d\mathbf{x} \quad (1.5)$$

is the total circulation. Special treatment of the solution by, for example, using the transpose scheme introduced in [26], must be sought.

Conservation of circulation is only one of many invariants of incompressible, inviscid flows; others include linear impulse, angular impulse, kinetic energy and helicity [26]. Also, the vorticity field should always be divergence-free due to its definition, $\boldsymbol{\omega} = \nabla \times \mathbf{U}$, and the vector calculus identity that the *divergence* of the *curl* of a vector field is zero in three dimensions. A vorticity field discretized by vortex particles will only be approximately divergence-free [79], and the divergence of vorticity will in general be amplified by convection and stretching [26]. The effect of violating the divergence-free criterion in UVLM implementations that use vortex particle methods is rarely quantified.

A vortex filament on the other hand is a concentration of vorticity supported on a curve in space that extends to infinity or forms a closed-loop [26]. Each infinitesimal segment of the filament is convected by the local flow velocity, and the whole filament has a constant scalar circulation strength, Γ . Hence, the total circulation is automatically conserved. The vorticity field is also divergence-free irrespective of the accuracy of the time-integration scheme or filament discretization [73]. Also, vortex filament methods conserve linear impulse, angular impulse, and kinetic energy [26], and the re-orientation and stretching of vorticity is automatically handled by the convection of points along the vortex filament. The free-wake

calculations of [120] and [71] use piecewise linear vortex filaments.

A piecewise linear vortex filament is very similar to a series of vortex particles, however vortex filament methods use the connectivity to enable substantial computational savings (stretching and re-orientation is captured by convection of the endpoints of each segment) and benefit from the vorticity field being divergence-free. As discussed in [26] an interesting hybrid of the two methods is presented in [73] that makes use of the advantages of filament methods.

Thus far it has been shown that the UVLM is the favoured method for capturing the numerous geometric nonlinearities required for appropriate aerodynamic analysis of flexible aircraft. Although this method is very fast compared to high-fidelity aerodynamic analysis it has the potential to be much faster. The UVLM has three bottlenecks which are all similar to that of the n -body problem: The first is the roll-up of the free-wake; the second is construction of the Aerodynamic Influence Coefficient (AIC) matrix and solution of the subsequent linear system; and the third is the post-processing of the solution for aerodynamic forces.

The roll-up of a free-wake in the UVLM is almost a typical n -body problem³, except that the influence of vorticity bound to the aerodynamic surfaces must be considered also. This results in a problem of complexity $\mathcal{O}(P_w(P_w + P_b))$ where P_w is the number of wake elements, and P_b is the number of bound elements. The number of wake elements grows linearly over time and therefore this bottleneck is most prohibitive when small time steps, or simulations over long times, are required. This has proved to set the current computational limit, for example, in the nonlinear open-loop and closed-loop response of flexible aircraft to gusts [98].

Many fast solutions to the n -body problem rely on the influence of one body on another decaying rapidly with distance. This allows the effect of neighbouring elements, clustered far away from an evaluation point, to be approximated in a single evaluation. This concept was leveraged in [6] and [12] in the context of astrophysics to quickly calculate the mutual interaction of point masses in space. In this work, relatively-closely grouped point masses (solar-systems, galaxies, etc) are represented by a simple monopole approximation [6], or a multipole expansion [12] in which the moments of mass distribution within a group are included.

In order to determine whether grouping the effect of a set of particles is appropriate, the size of the group and the distance to the evaluation point must be considered. This is

³The problem of predicting the individual motions of n interacting objects, which typically has $\mathcal{O}(n^2)$ complexity.

achieved in the so-called tree-codes of [6] and [12] by using a hierarchical decomposition of space in which the solution domain is recursively bisected along every dimension into ever finer sub-cells. This leads to binary tree data structure in one dimension, a quadtree in two dimensions, or an octree in three dimensions. At each level of refinement the separation of a group in one cell from evaluation points in another can be inferred from their relative positions in the tree-structure; for example, by insisting that immediately adjacent cells are not *well-separated*. In the case where particles are not well-separated their mutual influence must be calculated directly. Such tree-codes reduce the complexity of the n -body problem from $\mathcal{O}(P^2)$ to $\mathcal{O}(P \log P)$.

The tree-code concept was taken a step further by Greengard and Rokhlin [46, 47] when they presented the Fast Multipole Method (FMM), which allows the n -body problem to be solved with $\mathcal{O}(P)$ complexity. They introduced three main mathematical tools for manipulating multipole expansions which are necessary for the implementation of the FMM. Firstly, a formula for shifting the origin of a multipole expansion is provided; this allows multipole expansions calculated in sub-cells to be combined to find the total multipole expansion of a parent cell in a very efficient manner. A method for converting a multipole expansion into a remote local Taylor expansion is also presented, which allows efficient evaluation of the effect of a multipole anywhere within a well-separated region. Finally, a mechanism is provided for shifting the centre of a local Taylor expansion. Another major contribution of this work is the introduction of rigorous relative error bounds that can be set *a priori* by choosing the order of the multipole expansion, p . In order to fix a precision relative to the sum of quantities being expanded, ϵ , one must choose an order of expansion $p \approx \log_2(\epsilon)$ [47]. A tree data structure is used to infer the well-separatedness criterion, which is defined as $|\mathbf{x}_1 - \mathbf{x}_2| > 3r$, where r is a radius large enough to contain the particles of each group, and $\mathbf{x}_1$ and $\mathbf{x}_2$ are the centroids of each group.

The FMM has been applied to many problems across a wide range of physical applications including Astrophysics, Electrodynamics, Molecular Dynamics and Fluid Dynamics. There are many variants of the FMM, the development of which are chronicled in the introduction of [155]. Recently there has been an emphasis on implementing FMM-type algorithms efficiently on parallel computer architectures. For example, in [131], a parallel tree-code for two- and three- dimensional problems in gravitation and fluid dynamics is presented. An open-source fast-multipole library was made available by the authors of [28] which is reliant on the PetSc

library¹ (a Parallel extensible toolkit for Scientific applications). Finally, another open-source FMM library, ExaFMM², which has no such external dependencies and is written in C++, is under development at the University of Boston [155]. ExaFMM is targeted towards very large (exascale) physical systems, and has been developed to dynamically allocate computational effort for maximum efficiency on heterogeneous parallel architectures and even CUDA-capable graphics cards.³

The solution acceleration methods reviewed thus far are categorised as particle-particle (PP) methods [63]. There exists another class of solution acceleration methods for the n -body problem known as particle-mesh (PM) methods, or Particle-In-Cell (PIC) methods. In these methods the particle attributes are interpolated onto a regular mesh, and a fast Poisson solver is used to solve the potential on the mesh points. The effect of the potential on the grid points (e.g. resulting fluid velocity, or gravitational force) is then interpolated back onto the particle positions. Such an approach has a complexity $\mathcal{O}(P + Q \log Q)$, where Q is the number of mesh points [46]. Only those field variations which have a wavelength longer than the mesh spacing can be resolved by the PM method [63], and highly-non-uniform particle distributions cause a significant loss of accuracy [46]. Given the tendency of vortex sheets to roll-up into non-uniform distributions, PM methods could be regarded as inappropriate in the context of Vortex Methods.

An extension of the PM method, in which nearby particle interactions are computed directly, is called the particle-particle particle-mesh (P³M) method [63]. In these methods the long-range forces are computed using the PM technique and the relatively few PP interactions required at close-range are computed directly. When high accuracy is required and the spacing of particles is nonuniform the CPU cost of P³M methods tend to be excessive [46]. The pre-corrected-FFT method, originally developed for electrostatics analysis [118], and later applied to solve the surface panel strengths of rigid aircraft [153], is an example of a P³M method applied to a BEM. The authors of [153] also use a hybrid multipole-tree-code to evaluate wake-wake, wake-body, and body-wake interactions. This is because PP methods are more versatile, time-efficient, and memory-efficient than P³M methods if the particle distribution is non-uniform and changing significantly with time.

The notion of using multipoles to represent distant groups of elements has also been used

¹mcs.anl.gov/petsc/

²www.exafmm.org

³www.developer.nvidia.com/what-cuda

to solve boundary element problems efficiently. In the n -body problem the element strengths are known and the influence of the each element must be evaluated; conversely, in the boundary element problem, element strengths must be determined by enforcing a set of boundary conditions. In panel methods this amounts to constructing the AIC matrix and solving the linear algebraic system for panel source, doublet or circulation strengths, depending on the particular method. A panel method accelerated with multipole expansions and a quadtree data-structure is described in [145] that solves panels strengths with $\mathcal{O}(P \log P)$ complexity, where P in this context is the number of panels. Considering that direct inversion of the AIC matrix has complexity $\mathcal{O}(P^3)$, and formation of the matrix is $\mathcal{O}(P^2)$, then the benefits are clear. This approach is advanced in [15] where a method with reported complexity $\mathcal{O}(P)$ is presented that uses multipole and Taylor expansions (as in [47]). For very detailed configurations it may be necessary to incorporate a fast panel solver into the UVLM; for example, if detailed panelling of control surfaces was required.

Linearized solutions

Linearizations of vortex methods for aeroelastic analysis were pioneered by Kenneth Hall (in [51]), who wrote a linearized form of the unsteady vortex-lattice equations in state-space form, thus directly obtaining time-domain models. Eigenanalysis of his equations, and aeroelastic equations that used the same fluid model, revealed the structure of eigenvalues underlying the unsteady aerodynamics model and their interaction with structural dynamics modes [55, 56]. Later, Joseba Murua developed a more general approach in which induced-drag, in-plane motions, and the effects of steady-state aerodynamic loading were captured, using a linearized UVLM, in an aeroelastic formulation of flexible aircraft flight dynamics equations [93]. The force calculations in this method, which was the first linearized model to include induced-drag terms, have their origins in the approximations suggested in [71] and [138]. This formulation has proven to be extremely useful for multi-disciplinary stability analysis as it allows seamless integration with geometrically-nonlinear structural dynamics models, including rigid-body degrees of freedom, and has allowed linearizations to be obtained about the trimmed aircraft configuration, which may include the large static deformations anticipated to occur in HALE aircraft. Applications of the method are discussed extensively in [97] where it is shown to successfully predict the characteristics of T-tail flutter, something which requires bespoke corrections to DLM-based aeroelasticity [144]. In the UVLM-based approach, free-wake effects

around the trim condition are neglected, resulting in a “frozen” wake geometry. The different wake geometries that can be found in flexible aircraft dynamics modelling are presented in Table. 1.1 for linearised and nonlinear formulations, where such formulations exist.

	Unsteady lifting line	DLM	UVLM
Linearised	wind-aligned	flat	frozen
Nonlinear	wind-aligned	N/A	prescribed / free

Table 1.1: Wake geometry specifications in 3D potential flow methods.

1.1.3 Aeroelasticity and flexible aircraft dynamics modelling

As mentioned in the introduction to this chapter, the need to model flexible aircraft flight dynamics (FAFD) using coupled, nonlinear analysis tools was highlighted by the Helios mishap [100] in 2003. However, a relative few researchers had already identified this need and published work in the field from the late 1990s. Since then there have been numerous advancements in modelling methodologies and the identification of significant results, which are discussed next.

Perhaps the first authors to present a description of the nonlinear dynamics of a full aircraft were Patil, Hodges and Cesnik [110]. Their starting point was a geometrically-nonlinear structural dynamics model based on the mixed-variational formulation of the exact intrinsic equations of moving beams of Hodges [64]. The extended model included motion in the global frame as a variable, gravitational potential energies, and control surface dynamics. The unsteady aerodynamics were introduced using Peters’ finite-state airloads theory [116, 117]. As discussed above, this is an unsteady, two-dimensional, thin-aerofoil solution to Laplace’s equation for potential flow in which the effect of shed vorticity is captured by a finite number of state variables. The effect of control surface motions/displacements can be captured in this model as a specific case of general aerofoil chord line deformations. This allowed the authors to formulate a nonlinear aeroelastic model of the entire aircraft which included the coupling of the rigid-body dynamics, structural dynamics and unsteady aerodynamics including the effect of control inputs. They also included further aerodynamic nonlinearities by integrating the semi-empirical ONERA dynamic stall model [83] into their aerodynamic model.

The model developed in [110] allowed the authors to perform stability analysis about a nonlinear steady-state, and also time-march the nonlinear system. The model was used to

perform flutter analysis on the Goland Wing [44], a relatively stiff, uniform, cantilevered wing that is used as a benchmark for flutter analysis. The inclusion of structural nonlinearities was found to increase the flutter speed due to geometric stiffening. Nonlinear effects introduced by stall models were also assessed and it was found that a simple static stall model also generally increased the flutter speed, however inclusion of the dynamic stall model dramatically reduced the flutter speed at high angles of attack. This was attributed to coupling of the stall states and the structural states. They extended their analysis to a complete aircraft in order to assess the effect of rigid-body dynamics on the problem, however the relative stiffness of the aircraft led to little coupling between the aircraft flight dynamics and structural dynamics. The authors noted that a highly-flexible aircraft's structural modes would be of lower frequency and would therefore interact with the flight dynamics modes more.

Further results obtained using the model of [110] are presented in [108] in which Limit-Cycle-Oscillations (LCOs) of a high-aspect-ratio wing were captured by time-marching the nonlinear model. By time-marching the nonlinear model the effect of large disturbances could be evaluated, as opposed to evaluating only small (linear) disturbances about a nonlinear trim condition. It was shown that their model may experience a flutter LCO at speeds well below the flutter speed depending on the amplitude of an initial disturbance. In this case the initial disturbance was applied by applying extra forces and moments to the wing-tip at the initial condition. This model was also used to design a feedback controller for gust-load alleviation and flutter suppression on a highly-flexible flying wing [105].

A HALE aircraft configuration was proposed in [109] which has the large aspect-ratio and relatively high flexibility expected of such aircraft. Using the analysis originally presented in [110] they investigated wing flutter under cantilever boundary conditions. They referred to "Nonlinear Flutter" to characterize the flutter analysis about a deformed trim condition (calculated using nonlinear analysis), as opposed to flutter analysis about the undeformed configuration. They showed that flutter speed and frequency calculated about the deformed configuration was generally lower than that calculated about the undeformed configuration, and that using the deformed configuration results in flutter speeds that vary significantly with root angle of attack. This was ultimately attributed to a coupling of torsion and edgewise-bending modes facilitated by the large steady-state bending deformation of the wing.

The stability analysis of a complete HALE configuration was also performed in [109] where root loci of a flexible aircraft were compared to those of a representative stiff aircraft.

Flexibility was shown to change the stability characteristics of the aircraft significantly. Coupling of the low-frequency main-wing structural modes and the flight dynamics modes was shown, while the high-frequency structural modes were relatively unaffected. The interaction of flight-dynamic and structural-dynamic modes is therefore of prime concern when dealing with flexible aircraft, which have structural natural frequencies low enough such that they might couple with the aircraft flight-dynamics. Indeed, this was emphasised in work on the aeroelastic characteristics of a highly flexible aircraft designed for human-powered flight [143] although the analysis of dynamics in this work is strictly linear, albeit about a nonlinear trim condition.

Also in the late nineties, Mark Drela developed a tool for the preliminary aerodynamic, structural, and control-law design of aircraft [32]. In his methodology, referred to as ASWING, the main structural elements, including the fuselage, in a general high-aspect-ratio aircraft are represented by nonlinear isotropic beams, and individual beams can be joined directly or by rigid pylons. The aerodynamics of the lifting surfaces are represented by Unsteady-Lifting-Line theory which is a three-dimensional method based on potential flow theory; spanwise variations in aerodynamic loading due to finite-wing effects are captured. The nonlinear time-domain response of a sailplane flying through an intense gust was shown in addition to root-locus analysis. The short-period mode and first-bending mode of the sailplane main wing were shown to have similar frequencies which the author noted may complicate control law design due to coupling.

Once the basic analysis methodologies were established, more complex configurations, such as those of the joined-wing “sensorcraft” [142] were analysed in [22]. They used a strain-based formulation of the nonlinear beam dynamics [20] and Peters’ finite-state aerodynamics. The single-wing Sensorcraft configuration presented in that work showed little change in roll response with the addition of fuselage and/or tail flexibility, although a configuration with a rigid fuselage and a very flexible tail section (0.25 stiffness of main wing) was found to experience tail flutter at a flight velocity below that at which the main wing would flutter. A later extension of this methodology, presented in [23], includes the aerodynamic forces from the fuselage although the influence of the fuselage on other aerodynamic surfaces is not yet considered. Importantly, the validity of analysing complex three-dimensional aircraft geometries with two-dimensional aerodynamics was not addressed.

The modelling effort of [22] was extended in [134] in which integration of the coupled

equations of motion was performed using a modified Newmark method, namely, a Modified Generalized- α method [133]. A spanwise lift distribution function was also used to correct the two-dimensional finite-state aerodynamics for tip loss effects on a surface-by-surface basis; no cross-influence between aerodynamic surfaces and their wakes were captured. They used their formulation to compare the response of a representative twin-tailed HALE aircraft using three simulation types: A flight dynamics simulation based on the deformed aircraft configuration at trim with no elastic degrees of freedom; a flight dynamics simulation including linearised structural dynamics about the trim condition; and a fully coupled nonlinear solution of the complete aeroservoelastic model. It was shown that full nonlinear analysis was required for antisymmetric manoeuvres of highly-loaded configurations in order to properly capture the aircraft response. For the aircraft presented the authors concluded that linearised structural dynamics about the deformed configuration was adequate for symmetric manoeuvres.

The unsteady vortex lattice method (UVLM) was first used as the aerodynamic model in an aeroelastic description of HALE wings in [149]. The nonlinear-beam/UVLM analysis was applied to the Goland wing and results were in very good agreement to those obtained with a higher-order DLM (in the commercial code ZAERO⁴). This is expected as the deformations are small (within the linear regime). As already discussed, for large wing deformations the UVLM offers an advantage over the DLM as the boundary conditions are geometrically nonlinear, i.e., the non-penetration boundary condition is enforced on the deformed geometry [94].

A review of the aforementioned structural and aerodynamics models was carried out in [104] in the context of flight dynamics of very flexible aircraft. Displacement-based, strain-based, and intrinsic finite-element models for nonlinear beams were shown to be consistently obtainable from a single set of equations. The intrinsic description, in which velocities and internal forces are the structural states, was found to offer efficiency advantages over traditional displacement-based formulations. Two finite-state approximations of two-dimensional unsteady aerodynamics were compared: Peters' finite-state inflow model [117], and Leishman's indicial response model [77]. The latter was found to perform better at low frequencies, i.e. those expected of VFA, for a given number of states. This model was then compared to a three-dimensional UVLM, and it was found that the UVLM is required to capture the effect of large-amplitude wing dynamics, especially at low-frequencies where the flow is highly three-dimensional (at high frequencies it was found that 3D analysis was not so necessary).

⁴www.zonatech.com/ZAERO.htm

Finally, for cases in which the aircraft structure has very large deformations under static (trimmed) equilibrium, but the excitation in the dynamic response is bounded, a perturbation analysis can be performed on the flexible multibody dynamics equations [60]. The resulting formulation retains the nonlinear couplings in the rigid-body degrees of freedom, but only retains first order terms in the structural degrees of freedom, which can then be projected onto the linear normal modes of the free structure on the reference configuration. This captures the effect of gyroscopic loads on the structure, which are neglected in earlier linearized formulations based on the mean axis approach [150].

1.1.4 Active load control with aeroelastic models

The first successful attempt at aeroelastic stability augmentation (which is typically referred to as *flutter suppression*) was on a prototype of the B-52 bomber in the early 1970s [125], although that was never implemented in a production aircraft. Active aeroelastic technology further evolved to include disturbance rejection systems for gust load alleviation [69, 89] in the 1980s, and culminated with the Active Aeroelastic Wing (AAW) project on a modified F/A-18 in the 1990s [112]. Later studies on the AAW configuration [156] have shown that feedforward control methods provide substantial improvements for gust load alleviation. Most large transport aircraft are currently fitted with relatively simple manoeuvre/gust load alleviation systems, and this is being considered for next-generation regional aircraft as well [121]. In addition to this, a major test program has been started recently at NASA with the Lockheed-developed X-56 [129] that specifically targets the phenomenon of aeroelastic/rigid-body coupling with the intention of developing active aeroelastic control systems for stability augmentation and disturbance rejection. Importantly, this has created a focal point for public domain research into flexible aircraft flight dynamics (FAFD) and control [129, 19, 62].

A major assumption in the above studies is that the airframe is still relatively stiff and the wings can be analyzed using linear structural dynamics. However, as mentioned above, the system dynamics of very flexible vehicles must include the relevant nonlinear effects due to large geometry changes. Due to the complexity of designing nonlinear control systems on large dynamical systems, most attempts to design active gust alleviation mechanisms on VFA have so far been restricted to the use of linear controllers obtained from the linearization of the vehicle dynamics around a reference condition. The work of [25] and [30] applied $\mathcal{H}_\infty$ and LQG controllers, respectively, to full aircraft models and investigated the response to gust

and continuous disturbances. These studies have highlighted the difficulty of achieving robust control with linear control schemes on very flexible vehicles as the disturbances increase in size.

A potential strategy is to develop on-line optimization-based control strategies in which the control is synthesized based on the current state of the system and information about the predicted plant dynamics. This, in particular, is the basis for model-predictive control (MPC) strategies. Recently, an MPC scheme for gust load alleviation was formulated using a linear, aeroelastic, reduced-order model (ROM) of a commercial airliner [42]. The proposed solution also included preview information of the future gust environment, provided by light detection and ranging sensors. When the controller was applied to the linear, full-order model it showed superior performance to that of a linear reference controller. Although this application still assumes a linear system response, extensions of MPC are available for nonlinear systems [48] which could provide new strategies for load alleviation valid in more complex scenarios.

The work of [50] has also shown that linear MPC, with a prediction enhancement based on output disturbance feedback,⁵ was more effective than a linear quadratic regulator (LQR) when applied to a linearized flexible aircraft model. The robustness of the MPC scheme was emphasized by successfully applying the same controller to a plant whose control effectiveness was reduced by 50% (to simulate damage to the aircraft control system).

One potential drawback of using vortex-based methods to directly formulate linear time-domain models is the large number of degrees of freedom required to discretize the wake. To date these methods have been based on the assumption that vorticity in the discrete wake, which is frozen in physical space, passes downstream one element per time step [51, 93]. Therefore the streamwise discretization of the wake sets the temporal discretization of the dynamics in the problem. This is in contrast to frequency domain methods such as the DLM in which dynamics at discrete frequencies are calculated directly, as the time-domain models resolve a frequency range starting at zero with an upper bound set by double the element spacing in the wake – this is effectively a Nyquist limit on the sampling of the discrete aerodynamics. A small wake spacing will therefore resolve a large frequency range, and vice-versa, with the drawback of adding many more states to the model.

Hall overcame the limits introduced by using his relatively-high order models by projecting their dynamics onto the first few eigenvalues of the system [51]. Soon after, the application

⁵Model predictive control with constant output disturbance feedback is the essence of one of the original proprietary predictive control products, and is known as dynamic matrix control (DMC) [81].

of balanced transformations to unsteady aerodynamics was first proposed in [9]. The work of [127] subsequently compared balanced realizations with direct projection on the system eigenvalues and showed a much better performance for a given system size. The same balancing techniques have also been proposed to reduce problem size in doublet-lattice-based rational function approximations obtained by Roger’s method [53, 91]. Low-Reynolds unsteady aerodynamics were investigated in [18] who used balanced realizations of Theodorsen’s function to obtain low-order, parameterized models of flat-plate pitching and plunging dynamics.

Since eigenvalues describe the internal dynamics of a system, but not the dependency with any specific inputs and outputs, they typically provide a less than ideal basis for model reduction. Also, since the computational effort of finding balanced realizations goes as $\mathcal{O}(n^3)$, which is similar to an eigenvalue analysis, both methods become impractical for systems of size $n > 10^4$. For larger systems, balanced realizations can be combined with Proper Orthogonal Decomposition, which is used initially to reduce the system size. This was first demonstrated in [152] where reduced-order models of transonic flows over moving airfoils were obtained.

1.2 Research objectives

A good characterization of the unsteady aerodynamics is clearly essential for coupled dynamic aeroelastic and flight dynamic analyses, and, it has been seen, the basis of many computational tools in either field is a potential-flow fluid mechanics model. In traditional linear aeroelasticity, both in research and industrial settings, the frequency-domain-based doublet-lattice method (DLM) [2] has been the *de facto* approach for aerodynamic analysis. Dynamic aeroelastic stability analysis (flutter prediction) and gust and manoeuvre loads analyses are provided by a frequency-domain solution of the subsonic, three-dimensional, unsteady aerodynamics using a Green’s function method whereby doublet elements are placed over the mean-surface of aerodynamic bodies. The development of VFA, which are subject to large deformations in trim, dynamic effects due to non-zero loading in the reference condition, and for which one may need to calculate unsteady induced-drag effects for flight dynamics predictions, asks for a more general formulation than the one offered by the DLM, however. An additional complexity of DLM, which we will aim to overcome even for linear problems, is that, as it is based on frequency domain descriptions, requires rational-function approximations to obtain time-domain models. Those are time-consuming and introduce additional complexity

and a lack of robustness in the modelling process [35].

Linear state-space models of 3-D potential-flow-based unsteady aerodynamics were proposed by Hall [51]. The equivalent nonlinear method would be a full unsteady vortex-lattice method (UVLM) that includes unsteady induced drag, the effects of large dynamic deflections and the dynamics of a force-free wake [97]. A first linearization of that more general approach, with inputs defined by the degrees-of-freedom of a geometrically-exact beam dynamics model, was also developed by Murua [93] for aeroelastic stability analysis of flexible aircraft. However, a general solution still needs to be developed such that it can be used with, for example, mode shapes obtained from a general-purpose finite-element solver.

Unsteady induced-drag (which comes particularly from the presence of leading suction in the moving wing) would be therefore of increased relevance to flight dynamics applications, but it is relatively unexplored in the literature. Most references ignore induced-drag effects altogether, or at least do not articulate their approach in a clear way. Murua [97] and Stanford and Beran [138] were among the first to present a method to calculate unsteady induced-drag forces without restrictive assumptions on the kinematics of the aerodynamic surface. Their approaches are based on augmenting the method presented by Katz and Plotkin [71] to compute the forces on the panels. However, an alternative approach based on Joukowski's three-dimensional relationship to resolve all components of the forcing on each vortex segment in the lattice could provide much accurate estimates for a given discretization [113]. This will be investigated in this thesis, where we will aim to develop a general nonlinear formulation of the UVLM for complex kinematics.

Further to additional development of a general aerodynamic theory, there is a need for computationally-efficient solutions for full-aircraft analysis. This is particularly relevant if the models were to be used to drive an MPC-based load alleviation strategy suitable for VFA. New strategies for model reduction of the vehicle dynamic equations will be necessary which ensure a sufficient level of fidelity with very good computational performance. To address the objectives, this work will include nonlinear plant dynamics in the closed-loop, which are critical for adequately describing the dynamics of VFA, and develop optimization-based gust alleviation and trajectory control strategies. Importantly, hard limits on control inputs and input rates will be enforced. Also, since model predictive control relies on faster than real-time computation of the discrete dynamics, special attention will be paid to both the size of the model used for control synthesis and the timing characteristics of the resulting

implementation, with real-application feasibility in mind.

1.3 Dissertation outline

In the chapter ahead a general nonlinear formulation of the UVLM is set out first, in Section 2.1, where the nonlinear problem is formulated for arbitrary motions considering the aforementioned boundary conditions. Details of two alternative force calculation methods, building on the aforementioned work, are articulated for arbitrary kinematics in Section 2.2. A critical evaluation of these methods is carried out later, in the results of Section 5.1.1.

This will allow, in the next part of that chapter the development of a discrete-time, linear, time-invariant, state-space description of vortex-lattice-based aerodynamics, which generalizes those proposed by [97] and [51]. In the new approach a consistent linearization is obtained around an arbitrary reference configuration, which allows the possibility of including the effect of non-zero loading and deformed geometries in the linearized system dynamics. Moreover, a nondimensional expression of the equations gives rise to models that are independent of free-stream dynamic pressure, which is key to building aeroelastic models for efficient stability analyses. These new linearized models, described in Section 2.4, resolve frequencies up to a spatio-temporal Nyquist limit defined by the wake discretization, and have a convenient form for coupling with structural dynamics models. No assumptions will be made relating to the kinematics of the fluid-structure interface (inputs) and use of the three-dimensional Joukowski relation naturally resolves all components of the unsteady aerodynamic forcing (outputs).

Chapter 3 will introduce the finite-element formulation for geometrically-nonlinear beams that defines the structural model, which is based on the work of [58]. The resulting aeroelastic simulation framework, including details of the implementation will then be discussed in Chapter 4. Solutions for static and dynamic, as well as linear and nonlinear analysis are introduced there and will be then exercised in Chapter 5, which focuses on numerical investigations on the performance of the proposed formulation on relevant full-state problems. Results include several verification cases to test key features of the implementation, including classical frequency-domain aerodynamics of aerofoils in Section 5.1.2, and flutter of cantilever wings. This chapter concludes with an exercise on body-freedom flutter characterization of swept-flying wings.

Balanced realizations of the state-space aerodynamic equations are finally explored in the

context of aeroelastic stability analyses in Chapter 6. This will lead into an exercise in gust load alleviation on a flexible cantilever wing with flap saturation using predictive control. This thesis concludes with a summary of its main contributions and some recommendations for further work.

1.4 Publications based on this work

Maraniello S., Simpson R.J.S., Palacios R., “Optimal Manoeuvres with Very Flexible Wings.” In *Proceedings of the 57th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, January 2016, San Diego, Florida, USA

Simpson R.J.S., Palacios R., “Integrated Flight Dynamics and Aeroelasticity of Flexible Aircraft with Application to Swept Flying Wings.” In *Proceedings of the 56th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, 5-9 January 2015, Kissimmee, Florida, USA

Simpson R.J.S., Palacios R., Hesse H., Goulart P.J., “Predictive Control for Alleviation of Gust Loads on Very Flexible Aircraft.” In *Proceedings of the 55th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, 13-17 Jan 2014, National Harbor, Maryland, USA

Simpson R.J.S., Palacios R., “Numerical Aspects of Nonlinear Flexible Aircraft Flight Dynamics Modeling.” In *Proceedings of the 54th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics and Materials Conference*, 8-11 April 2013, Boston, Massachusetts, USA

Simpson R.J.S., Palacios R., Murua J., “Induced-Drag Calculations in the Unsteady Vortex-Lattice Method.” *AIAA Journal*, Vol. 61, No. 7, pp. 1775-1779, July 2013

Chapter 2

Unsteady Vortex-Lattice-Based Aerodynamics with Arbitrary Kinematics

In recent times the nonlinear form of the unsteady vortex-lattice model (UVLM) has experienced a resurgence for flexible-aircraft flight dynamics (FAFD) modelling. Unsteady induced-drag, which is of increased relevance to flight dynamics applications, is relatively unexplored in the literature however, with some authors ignoring induced-drag effects altogether. Murua [97], building on the work of Stanford and Beran [138], was among the first to present a method to calculate unsteady induced-drag forces without restrictive assumptions on the kinematics of the aerodynamic surface. Their approaches, which are based on augmenting the method presented by Katz and Plotkin [71], are not the only option, however. An alternative approach is to use Joukowski's three-dimensional relationship to resolve all components of the forcing on each vortex segment in the lattice, while using Bernoulli's equation across the surface to capture added-mass contributions [113]. This approach allows expressions for all components of the aerodynamic forcing, under arbitrary kinematics, to be written in a far more succinct and clear way. In the chapter ahead a general nonlinear formulation of the UVLM is set out first, in Section 2.1, where the nonlinear problem is formulated for arbitrary motions considering the aforementioned boundary conditions. Details of two alternative force calculation methods, building on the aforementioned work, are articulated for arbitrary kinematics in Section 2.2. A critical evaluation of these methods is carried out later, in the results

of Section 5.1.1.

In the next part of the chapter a novel, discrete-time, linear, time-invariant, state-space description of vortex-lattice-based aerodynamics is developed, inspired by the work of Murua in FAFD modelling [97] and the seminal work of Hall [51]. In the new approach a consistent linearization is obtained around an arbitrary reference configuration, which allows the possibility of including the effect of non-zero loading and deformed geometries in the linearized system dynamics – something that the classical doublet-lattice method does not achieve. Moreover, a nondimensional expression of the equations gives rise to models that are independent of free-stream dynamic pressure, which is key to building aeroelastic models for efficient stability analyses. These new linearized models, described in Section 2.4, resolve frequencies up to a spatio-temporal Nyquist limit defined by the wake discretization, and have a convenient form for coupling with structural dynamics models. No assumptions are made relating to the kinematics of the fluid-structure interface (inputs) and use of the three-dimensional Joukowski relation naturally resolves all components of the unsteady aerodynamic forcing (outputs). The models are subsequently used to predict the classical frequency-domain aerodynamics of aerofoils in Section 5.1.2, and flutter in the three-dimensional problems of Section 5.3. Balanced realizations of these equations are then explored in the context of aeroelastic stability analyses in Chapter 6.

2.1 Nonlinear discrete-time equations

The unsteady vortex-lattice method (UVLM) is a three-dimensional, geometrically-nonlinear, potential-flow-based method that is used to calculate the unsteady loads induced on a body as it moves through a fluid medium. 3-D geometries are idealized as infinitesimally-thin surfaces that support a time-varying vorticity distribution discretized by a lattice of rectilinear vortex ring elements. The wake of each surface, which starts at the trailing-edge and trails a finite length downstream, is also idealized as a vortex sheet and is discretized in the same way. The fluid velocity field is then calculated by superimposing the velocities induced by all vortex rings whilst enforcing the non-penetration boundary condition on the moving surface [71]. Unsteady effects are included by modelling the convection of vortex rings in the wake, which may be allowed to form into a so-called ‘free wake’ under Helmholtz laws of vortex motion, whilst satisfying the Kutta condition and Joukowski’s hypothesis [92]. In the following

section a general, geometrically-nonlinear formulation of the UVLM is described: details of the vorticity discretization, calculation of the induced velocity field, and satisfaction of the problem boundary conditions are treated here. Post-processing of the vorticity distribution to obtain the aerodynamic forces is described in the the next section, Section 2.2.

The surface lattice is discretized using a regular grid with M rings in the chordwise direction and N rings in the spanwise direction, with $K = MN$ in total. The corresponding wake lattice has $K_w = M_w N$ rings with $M_w \gg M$ being the relatively large number of chordwise rings in the wake. Hence, the geometry is described by $K_\zeta = (M + 1)(N + 1)$ lattice vertices on the surface with $K_{\zeta_w} = (M_w + 1)(N + 1)$ in the wake. Each of the vortex rings has a scalar circulation strength, and these are collected in vectors $\mathbf{\Gamma} \in \mathbb{R}^K$ and $\mathbf{\Gamma}_w \in \mathbb{R}^{K_w}$ that correspond to the surface and wake vorticity distributions, respectively. Using this description the instantaneous induced velocity field can be expressed, at the point $\mathbf{x}$, by the linear relationship

$$\mathbf{u}(\mathbf{x}, t) = \mathcal{A}^3(\mathbf{x}, \boldsymbol{\zeta}(t)) \mathbf{\Gamma}(t) + \mathcal{A}_w^3(\mathbf{x}, \boldsymbol{\zeta}_w(t)) \mathbf{\Gamma}_w(t) \in \mathbb{R}^3, \quad (2.1)$$

where $\mathcal{A}^3 \in \mathbb{R}^{3 \times K}$ and $\mathcal{A}_w^3 \in \mathbb{R}^{3 \times K_w}$ are *Aerodynamic Influence Coefficient* (AIC) matrices corresponding to the surface and wake lattices, respectively, which are calculated using the Biot-Savart law. The vector-valued terms $\boldsymbol{\zeta} \in \mathbb{R}^{3K_\zeta}$ and $\boldsymbol{\zeta}_w \in \mathbb{R}^{3K_{\zeta_w}}$ contain the lattice vertices used to describe the surface and wake geometries, respectively, which are illustrated in Figure 2.1.

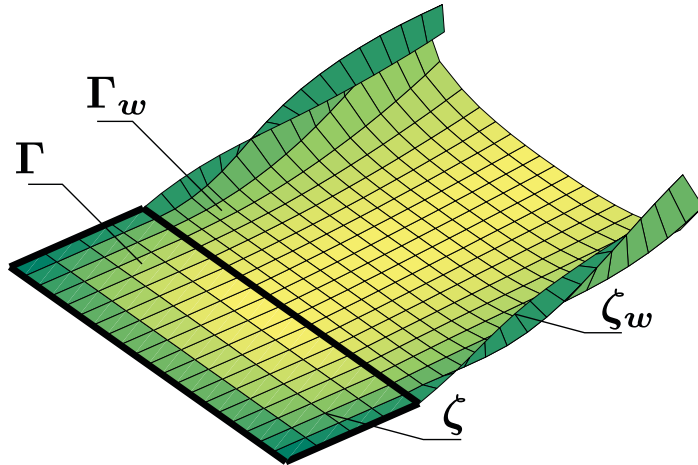


Figure 2.1: Vortex lattice geometry and circulation distribution.

Each column of the matrix $\mathcal{A}^3$, denoted $\mathcal{A}_k^3 \in \mathbb{R}^3$ where $k = 1, \dots, K$, is the velocity induced by the k -th vortex ring at the target point $\mathbf{x}$, assuming unit circulation, $\Gamma_k = 1$.

This is calculated by splitting the unit-strength vortex ring into its four constituent linear segments and summing their contributions, i.e. [71]

$$\mathcal{A}_k^3 = \frac{1}{4\pi} \sum_{l=1}^4 \mathbf{r}_{0l}^\top \left(\frac{\mathbf{r}_{1l}}{\|\mathbf{r}_{1l}\|} - \frac{\mathbf{r}_{2l}}{\|\mathbf{r}_{2l}\|} \right) \frac{\widetilde{\mathbf{r}_{1l}\mathbf{r}_{2l}}}{\|\widetilde{\mathbf{r}_{1l}\mathbf{r}_{2l}}\|^2}, \quad (2.2)$$

where the definition of segment vectors $\mathbf{r}_0, \mathbf{r}_1$ and $\mathbf{r}_2$ depends on the geometry of the k -th vortex ring and location of the target point $\mathbf{x}$ as illustrated in Figure 2.2, and l is the segment counter on the k -th vortex ring. The symbol $\widetilde{(\cdot)}$ denotes the skew-symmetric matrix associated with the vector cross product $(\cdot) \times (\cdot)$, and the parentheses $\|\cdot\|$ denote l^2 -norms.

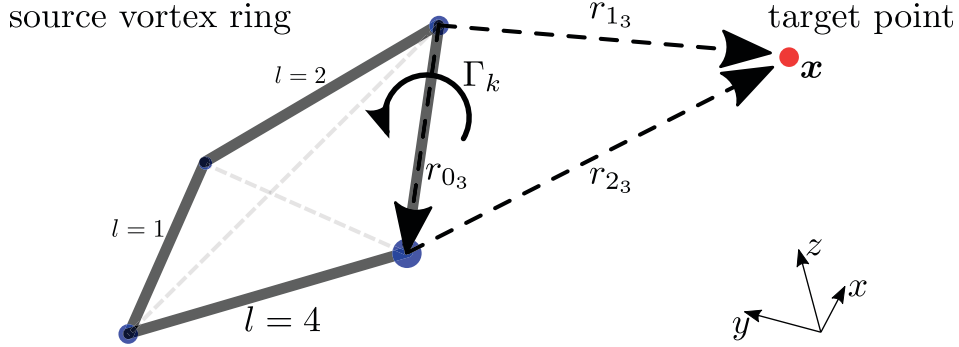


Figure 2.2: Source vortex ring and target point in application of the Biot-Savart law. Vectors corresponding to the third segment, $l = 3$, are indicated with dashed arrows.

To enforce the non-penetration boundary condition, the induced velocity field (2.1) must be projected along the surface normal vectors at control points on the surface; these are chosen to be at the centre of each vortex ring, and are referred to as collocation points. If ζ_k^c are the $k = 1, \dots, K$ collocation points then the matrix $\mathcal{A}^* = [\mathcal{A}^3(\zeta_1^c); \mathcal{A}^3(\zeta_2^c); \dots; \mathcal{A}^3(\zeta_K^c)] \in \mathbb{R}^{3K \times K}$ is the linear map from the surface circulation distribution to the velocities at those points. Defining $\mathbf{N} \in \mathbb{R}^{K \times 3K}$ as the block-diagonal matrix of the normal vectors components at the collocation points, the AIC matrix $\mathcal{A} = \mathbf{N}\mathcal{A}^* \in \mathbb{R}^{K \times K}$ describes the instantaneous linear map between the vorticity distribution and the *normalwash* at the collocation points. Analogously, the wake influence matrix is defined as $\mathcal{A}_w = \mathbf{N}\mathcal{A}_w^* \in \mathbb{R}^{K \times K_w}$, where $\mathcal{A}_w^* = [\mathcal{A}_w^3(\zeta_1^c); \mathcal{A}_w^3(\zeta_2^c); \dots; \mathcal{A}_w^3(\zeta_K^c)] \in \mathbb{R}^{3K \times K_w}$. Consequently, the non-penetration condition

can be expressed, at the discrete time step $n + 1$, as

$$\mathcal{A}\Gamma^{n+1} + \mathcal{A}_w\Gamma_w^{n+1} + \mathbf{W}(\boldsymbol{\nu}^{n+1} - \dot{\boldsymbol{\zeta}}^{n+1}) = 0, \quad (2.3)$$

where $\boldsymbol{\nu} \in \mathbb{R}^{3K_\zeta}$ and $\dot{\boldsymbol{\zeta}} \in \mathbb{R}^{3K_\zeta}$ are, respectively, the atmospheric fluid velocities (which can be spatially and temporally varying in general) and surface inertial velocities at the lattice vertices. The matrix $\mathbf{W} = \mathbf{N}\boldsymbol{\Xi} \in \mathbb{R}^{K \times 3K_\zeta}$ projects the kinematic and atmospheric velocities along the normal vectors at the collocation points, and the constant matrix $\boldsymbol{\Xi} \in \mathbb{R}^{3K \times 3K_\zeta}$ represents a bilinear interpolation of velocities from the lattice vertices to the collocation points, as detailed in Appendix A.

Unsteady effects are included through convection of the wake states $\boldsymbol{\zeta}_w$ and Γ_w . The convection of the wake lattice is given by

$$\boldsymbol{\zeta}_w^{n+1} + \mathbf{C}_\zeta \boldsymbol{\zeta}_w^{n+1} = \mathbf{C}_{\zeta_w} \boldsymbol{\zeta}_w^n + \Delta t (\mathcal{A}_1 \Gamma^n + \mathcal{A}_2 \Gamma_w^n + \boldsymbol{\nu}_w^n), \quad (2.4)$$

where $\mathbf{C}_\zeta \in \mathbb{N}^{3K_{\zeta_w} \times 3K_\zeta}$ is a sparse, constant matrix that joins the wake lattice with the trailing-edge points at the current time step, while $\mathbf{C}_{\zeta_w} \in \mathbb{N}^{3K_{\zeta_w} \times 3K_{\zeta_w}}$ shifts the wake lattice downstream. $\boldsymbol{\nu}_w^n \in \mathbb{R}^{3K_{\zeta_w}}$ are the instantaneous atmospheric fluid velocities at the wake lattice vertices. The matrices $\mathcal{A}_1 \in \mathbb{R}^{3K_{\zeta_w} \times K}$ and $\mathcal{A}_2 \in \mathbb{R}^{3K_{\zeta_w} \times K_w}$ account for the fluid velocities induced by the surface and wake vortex rings at the wake lattice vertices, respectively. Including these velocities allows the UVLM to capture wake roll-up.

The convection of circulation strengths can be written as

$$\Gamma_w^{n+1} = \mathbf{C}_\Gamma \Gamma^n + \mathbf{C}_{\Gamma_w} \Gamma_w^n, \quad (2.5)$$

in which $\mathbf{C}_{\Gamma_w} \in \mathbb{N}^{K_w \times K_w}$ is a sparse constant matrix that shifts the wake circulation downstream by one chordwise element. Similarly, $\mathbf{C}_\Gamma \in \mathbb{N}^{K_w \times K}$ propagates the circulation strength from the trailing-edge at the previous time step into the first row of wake vortex rings.

As mentioned earlier, (2.3) gives an instantaneous solution to Laplace's equation at each time step. Unsteadiness, i.e. time-dependence, is introduced to the system through the boundary conditions and the convection of vorticity in the wake. The Joukowski hypothesis, which requires that flow leaves the trailing-edge smoothly, i.e. trailing-edge vortex segments have zero circulation, is approximately satisfied by setting the newly-shed panel strengths

equal to the corresponding trailing-edge panel strength at the previous time step, (2.4) & (2.5). This also approximately satisfies the Kutta condition, which is the requirement for no pressure jump to exist at the trailing-edge [92]. It is worthwhile noting that neither of these conditions are met if the kinematic velocity at the trailing-edge is large (specifically, as large as the characteristic fluid velocity [71]). In any case, the particular form of (2.4) corresponds to a first-order, explicit, time-stepping scheme, which is widely used for the UVLM [71, 149, 93]. Classical vortex methods are unconditionally stable (Cottet *et al.* [26], pg. 43 - 44) in that their time-stepping schemes are not restricted by a Courant-Friedricks-Levy (CFL) condition, however, there are Kelvin-Helmholtz instabilities in the free wake. This is a physical phenomenon that can be observed experimentally and is captured by the UVLM which is characterized by the wake rolling up and stretching into increasingly-tight bundles [7]. However, since the velocity induced by the Biot-Savart law decreases rapidly with distance, and the wake is convected downstream, the effect on satisfying (2.3) is likely to be small. Together, equations (2.3) - (2.5) describe a discrete-time method to obtain the unsteady flowfield around an infinitesimally-thin surface moving through an incompressible fluid.

2.2 Force calculations: Induced-drag and leading-edge suction

Once the instantaneous circulation distribution on the surface is obtained through (2.3) the resulting aerodynamic loads can be calculated. While traditional linear aeroelasticity is focussed on predicting lift forces, accurate prediction of the unsteady induced drag is of great importance in flexible aircraft flight dynamics, and flapping flight applications [138], in which all components of the aerodynamic forces play an important role. The main difficulty associated with this in lifting-surface methods, such as the UVLM and the DLM, is the inclusion of leading-edge suction effects.

The simplest example of a solution to the leading-edge suction problem comes from thin-aerofoil theory. In the steady case the component of pressure force in the free-stream direction is cancelled by the leading-edge suction force; the drag is therefore zero – D’Alembert’s paradox. However, in the unsteady case there may be drag, or thrust, depending on the details of the unsteady motion. Garrick [37] introduced a closed-form solution for the propulsion of a sinusoidally oscillating aerofoil by considering the suction force due to the contribution of

infinite, but integrable, vorticity at an infinitely-small, rounded leading-edge [147].

An equivalent approach in three dimensions is found in solutions built by superposition of horseshoe vortices, such as the Unsteady Lifting-Line Method [32]. In this approach the unsteady force contribution from an incremental segment of vorticity, described by vector $\partial \mathbf{l}$ orientated in the spanwise wing direction along the quarter-chord, is calculated using the unsteady vector form of the Joukowski theorem, resulting in the force increment $\partial \mathbf{F} = \partial \mathbf{F}_{st} + \partial \mathbf{F}_{unst}$ where

$$\partial \mathbf{F}_{st} = \rho_{\infty} \Gamma (\mathbf{v} \times \partial \mathbf{l}), \quad (2.6)$$

and

$$\partial \mathbf{F}_{unst} = \rho_{\infty} \frac{\partial \Gamma}{\partial t} c (\hat{\mathbf{v}} \times \partial \mathbf{l}). \quad (2.7)$$

In the above equations Γ is the circulation around an incremental vortex segment of length $\partial \mathbf{l}$, and $\mathbf{v}$ is the local fluid-segment relative velocity at the location of $\partial \mathbf{l}$. The unit vector $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$ describes the direction of the local fluid-segment relative velocity. This approach has been extended to the UVLM in two ways [113, 71] although the resulting methods have not been described fully, or compared, as they are here. The first approach, which will be referred to as the *Joukowski method*, involves direct application of (2.6) to the segments of the surface vortex lattice [113], as described below in Section 2.2.1. The second approach is a version of Katz & Plotkin's method [71] generalized for complex kinematics. In this case only the spanwise vortex segments contribute to a local drag term, which is calculated using a modified induced-velocity field. This method, referred to as the *Katz & Plotkin method*, is described in Section 2.2.2.

2.2.1 The Joukowski method

The basis of what we will call the Joukowski method is that the majority of the forcing is obtained through direct application of (2.6) to the linear vortex segments that make up the surface lattice, while added mass terms, calculated by applying Bernoulli's equation across the surface lattice, are added separately. For each of the $s = 1, \dots, 4K$ segments the quasi-steady force contribution acting at the midpoint of each segment can be written as

$$\mathbf{f}_s = \rho_{\infty} \tilde{\mathbf{v}}_s l_s \Gamma_k \quad \text{with} \quad k = \left\lfloor \frac{s-1}{4} \right\rfloor + 1, \quad (2.8)$$

where $\mathbf{l}_s$ is the line segment describing the vortex geometry, and $\mathbf{v}_s$ is the fluid-segment relative velocity, defined as

$$\mathbf{v}_s = \mathbf{u}(\boldsymbol{\zeta}_s^m) + \mathbf{H}_s^\top (\boldsymbol{\nu} - \dot{\boldsymbol{\zeta}}), \quad (2.9)$$

where $\boldsymbol{\zeta}_s^m$ are the midpoint coordinates of the s -th segment, and $\mathbf{H}_s^\top$ linearly interpolates the velocities defined in (2.3) from the segment vertices to the same midpoint. An illustration of the vector quantities in (2.8) is given in Figure 2.3.

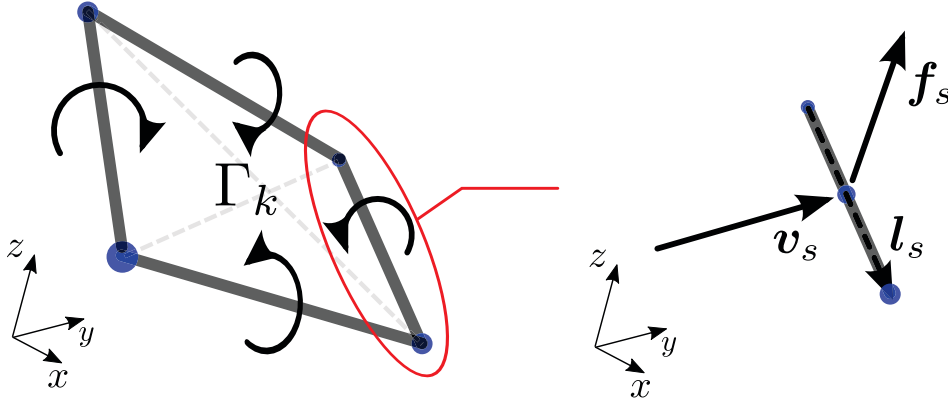


Figure 2.3: The geometry of a linear vortex segment on the k -th vortex ring, and the corresponding fluid-segment relative velocity and quasi-steady force at the segment midpoint.

When calculating $\mathbf{u}(\boldsymbol{\zeta}_s^m)$, defined by (2.1), it is often the case that $\boldsymbol{\zeta}_s^m$ exists on a line collinear with the source vortex segment, and from inspection of (2.2) and Figure 2.2 it appears that this results in infinite induced velocities. However, writing the kernel in terms of small perturbations from the collinear line reveals that induced velocities are identically zero there and have an analytically obtainable gradient. The velocities induced by a segment on itself are singular, however, and are avoided by cutting-off the Biot-Savart kernel, i.e. setting the velocity to zero.

After finding the force distribution on the segments, (2.8), all that remains is to include the added mass terms which are calculated by applying Bernoulli's equation across the surface lattice. These contributions are expressed at the k -th collocation point as

$$\mathbf{f}_{u_k} = \rho_\infty A_k \mathbf{n}_k \dot{\Gamma}_k, \quad (2.10)$$

where the product of vortex ring area A_k and normal vector $\mathbf{n}_k$ is a quadratic function of the surface geometry, and $\dot{\Gamma}_k$ is the derivative of the circulation strength with respect to

time, calculated using backward differencing (in-keeping with the first order approximation of dynamics in the method).

Therefore the vector of aerodynamic force components, $\mathbf{f}_a \in \mathbb{R}^{3K_\zeta}$, defined at the lattice vertices has the form

$$\mathbf{f}_a^n = \mathbf{H} \mathbf{f}^n + \mathbf{\Xi}^\top \mathbf{f}_u^n, \quad (2.11)$$

in which quasi-steady force contributions, $\mathbf{f} \in \mathbb{R}^{12K}$, act at the midpoint of each vortex segment and the unsteady contributions, $\mathbf{f}_u \in \mathbb{R}^{3K}$, act at the panel collocation points. The sparse matrix $\mathbf{H} \in \mathbb{R}^{3K_\zeta \times 12K}$, detailed in Appendix A, is used to distribute force contributions from the vortex segments to the lattice vertices, and is based on the same bilinear relationships as $\mathbf{\Xi} \in \mathbb{R}^{3K \times 3K_\zeta}$.

2.2.2 Generalized Katz and Plotkin method

An approximation to the solution above is given by Katz & Plotkin [71]. In this case the unsteady Bernoulli equation is applied to the top and bottom surfaces of each panel to find the pressure jump, and therefore the corresponding normal forces. Leading-edge suction effects are then included by considering the induced drag forces arising from local downwash, calculated using a modified induced-velocity field, acting on spanwise vortex elements. This is a more cumbersome formulation which may have come about because of two potential advantages: first, it avoids the calculation of induced velocities at points where Katz & Plotkin's three-dimensional kernel appears singular, and second, many of the terms required for the force calculations can be efficiently computed alongside the matrices required for (2.3).

Borrowing the nomenclature from Katz & Plotkin's work, which will be used in this section (but dropped for the remainder of the thesis) the local lift contribution at the ij -th collocation point can be expressed as

$$\begin{aligned} L_{ij}^{local} = \rho_\infty & \left((\mathbf{U}_{ij}^u + \mathbf{U}_{ij}^w) \cdot \hat{\boldsymbol{\tau}}_{ij}^c \frac{\Gamma_{ij} - \Gamma_{i-1,j}}{\Delta c_{ij}} \right. \\ & \left. + (\mathbf{U}_{ij}^u + \mathbf{U}_{ij}^w) \cdot \hat{\boldsymbol{\tau}}_{ij}^s \frac{\Gamma_{ij} - \Gamma_{i,j-1}}{\Delta b_{ij}} + \frac{\partial \Gamma_{ij}}{\partial t} \right) A_{ij} \cos \alpha_{ij}, \end{aligned} \quad (2.12)$$

where $\mathbf{U}_{ij}^u$ is the fluid-lattice relative velocity at each collocation point due to aerodynamic system inputs, i.e $\mathbf{U}_{ij}^u = \mathbf{U}_{ij}^{ext} - \dot{\boldsymbol{\zeta}}_{ij}^{col}$, where $\mathbf{U}_{ij}^{ext}$ and $\dot{\boldsymbol{\zeta}}_{ij}^{col}$ are external fluid velocities (such as free-stream or gust velocities) and the motion of the collocation points, respectively. $\mathbf{U}_{ij}^w$ are the

velocity contributions from wake vorticity, evaluated at the collocation points. The contribution of bound vorticity is approximated by two components arising from segment circulations perpendicular to the panel tangential vectors $\hat{\boldsymbol{\tau}}_{ij}^c$ and $\hat{\boldsymbol{\tau}}_{ij}^s$, in the chordwise and spanwise directions, respectively. Δc_{ij} and Δb_{ij} are the chordwise and spanwise extent of the ij -th vortex ring, respectively, and the local angle of attack is defined as $\alpha_{ij} = \tan^{-1} \left(\mathbf{U}_{ij}^u \cdot \hat{\mathbf{n}}_{ij} / \mathbf{U}_{ij}^u \cdot \hat{\boldsymbol{\tau}}_{ij}^c \right)$ [138].

The induced drag is then calculated using the component of downwash that acts along the local lift vector. The local lift vector is found by projecting the panel normal vector onto the plane perpendicular to the relative inertial velocity. In this work this is achieved using an orthogonal projection operator [40], $P_{\hat{\mathbf{U}}_{ij}^u} = \mathbf{I}_3 - \hat{\mathbf{U}}_{ij}^u \hat{\mathbf{U}}_{ij}^{u\top}$, where $\mathbf{I}_3$ is a 3×3 identity matrix. The local drag is then

$$D_{ij}^{local} = \rho_\infty \left[- \left(\mathbf{U}_{ij}^{bc} + \mathbf{U}_{ij}^w \right) \cdot \left(P_{\hat{\mathbf{U}}_{ij}^u} \hat{\mathbf{n}}_{ij} \right) (\Gamma_{ij} - \Gamma_{i-1,j}) \Delta b_{ij} + \frac{\partial \Gamma_{ij}}{\partial t} A_{ij} \sin \alpha_{ij} \right], \quad (2.13)$$

where the superscript bc indicates the induced velocity calculated considering chordwise-orientated vorticity on the surface only. At the leading-edge $\Gamma_{i-1,j}$ is set to zero. The total force contribution from each panel is then, $\mathbf{f}_{ij} = D_{ij}^{local} \hat{\mathbf{U}}_{ij}^u + L_{ij}^{local} P_{\hat{\mathbf{U}}_{ij}^u} \hat{\mathbf{n}}_{ij}$, which are collected in the vector $\mathbf{f}_{katz} \in \mathbb{R}^{3K}$. Therefore the aerodynamic force components at the lattice vertices can be expressed as

$$\mathbf{f}_a = \boldsymbol{\Xi}^\top \mathbf{f}_{katz}. \quad (2.14)$$

where the definition of $\boldsymbol{\Xi} \in \mathbb{R}^{3K \times 3K_\zeta}$ is the same as in (2.11), and in the definition of $\mathbf{W}$ in (2.3).

Either this method, encompassing (2.12) - (2.14), or the one of the previous section, can be used to obtain the aerodynamic forcing, $\mathbf{f}_a \in \mathbb{R}^{3K_\zeta}$, defined at the lattice vertices. The methods are compared against each other in convergence exercises and in comparison with analytical methods in Section 5.1.1.

2.3 Nonlinear state-space formulation

The methods discussed so far can be written in the form of a general, nonlinear, discrete-time state-space equation with the form

$$\mathbf{p}(\hat{\mathbf{x}}_a^{n+1}, \hat{\mathbf{u}}_a^{n+1}) = \mathbf{q}(\hat{\mathbf{x}}_a^n, \hat{\mathbf{u}}_a^n), \quad (2.15)$$

$$\mathbf{f}_a^n = \mathbf{r}(\hat{\mathbf{x}}_a^n, \hat{\mathbf{u}}_a^n), \quad (2.16)$$

where (2.15) is a difference equation in the states of the model

$$\hat{\mathbf{x}}_a := \left[\mathbf{\Gamma}; \mathbf{\Gamma}_w; \dot{\mathbf{\Gamma}}; \boldsymbol{\zeta}_w \right] \quad (2.17)$$

defined by the equations (2.3) - (2.5) and subject to the inputs

$$\hat{\mathbf{u}}_a := \left[\dot{\boldsymbol{\zeta}}; \boldsymbol{\zeta}; \boldsymbol{\nu}; \boldsymbol{\nu}_w \right], \quad (2.18)$$

and (2.16) is the output equation that gives the force distribution on the aerodynamic lattice using either the *Joukowski* (Section 2.2.1) or *Katz & Plotkin* (Section 2.2.2) method.

2.4 Linearized formulation

A linearization of the aerodynamics model described by equations (2.15) and (2.16) is desirable as it allows the formulation of linear aeroelastic models with several advantages over a DLM-based approach, as partially demonstrated by Murua [93]. In the following section a novel linearization is developed that improves-upon that work in three ways:

- A consistent linearization of equations (2.15) and (2.16) is obtained with respect to all of the aerodynamic model states and inputs (apart from the wake geometry, which is assumed to be frozen with no loss of generality).¹ This allows the linear aerodynamics to be coupled with *any* structural model through the definition of a linear map between its degrees of freedom and the kinematics of the aerodynamic lattice.
- The Joukowski method is used as a basis for the linearized forces and the equations

¹In Murua's work [93] the linearization is obtained with respect to the degrees-of-freedom of a geometrically-exact beam structural dynamics model. Additionally, a frozen surface geometry is assumed, and variations in downwash due to the linear lattice displacements are limited to those due to rotational degrees-of-freedom in the beam.

contain no assumptions about the kinematics of the surface lattice or the resulting force components (apart from they are small variations on an arbitrary reference configuration). The description is also far more concise than the existing methods, which are based on the work of *Katz & Plotkin*.

- The equations are nondimensionalized so that they are independent of free-stream dynamic pressure which is critical for efficient stability analysis and common in implementations of the DLM.

Aside from those inherent in any linearization the only assumption made in the development of the linear model is that the wake geometry is frozen, but can take any shape, for example a nonplanar prescribed wake or that obtained from nonlinear time-marching of the aerodynamic equations (2.3) - (2.5). Therefore all the terms relating to the dynamics of ζ_w are removed from the description; the effect of such terms on the system dynamics is at least quadratic anyway, so cannot be captured in any linear model.

The approach is therefore very general: discrete-time state-space models are obtained directly; they include the effects of a nonzero reference condition in all the states and inputs of the model, and; can predict linear induced drag effects. Additionally, the description is independent of any structural model, and yields models that are independent of the free-stream dynamic pressure.

As a starting point the aerodynamic states and inputs are redefined in nondimensional form.² The aerodynamic states are defined as

$$\mathbf{\Gamma} := \frac{\hat{\mathbf{\Gamma}}}{cV_{\text{ref}}}, \quad \mathbf{\Gamma}_w := \frac{\hat{\mathbf{\Gamma}}_w}{cV_{\text{ref}}}, \quad \text{and} \quad \mathbf{\Gamma}' := \frac{\dot{\hat{\mathbf{\Gamma}}}}{2V_{\text{ref}}^2}, \quad (2.19)$$

where c is the wing chord and V_{ref} is the mean fluid-lattice relative velocity at the reference condition. The superscript $(\cdot)'$ denotes a derivative with respect to nondimensional time which, in-keeping with the aeroelastic definition of reduced frequency, is defined as

$$\Delta t := \frac{V_{\text{ref}}\Delta\hat{t}}{c/2}. \quad (2.20)$$

Unless stated otherwise the nondimensional time step (2.20) is set such that the vortex rings in the wake have the same chordwise extent as those on the surface, i.e. $\Delta t = 2/M$. The

²In the following definitions the superscript $(\cdot)$ is used to denote the dimensional form of the variables introduced in Section 2.1.

geometric and kinematic degrees-of-freedom are defined as

$$\zeta := \frac{\hat{\zeta}}{c}, \quad \zeta_w := \frac{\hat{\zeta}_w}{c}, \quad \text{and} \quad \zeta' := \frac{\dot{\hat{\zeta}}}{2V_{\text{ref}}}, \quad (2.21)$$

and the atmospheric fluid velocity is defined as $\boldsymbol{\nu} := \hat{\boldsymbol{\nu}}/V_{\text{ref}}$. Substituting these definitions into the equations (2.3), (2.5) and (2.8) - (2.11) gives a nondimensional form of the nonlinear aerodynamic equations, which are subsequently linearized by assuming small perturbations in all the remaining degrees-of-freedom of the model, about an arbitrary reference configuration.

The circulation states are decomposed as

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_0 + \partial\boldsymbol{\Gamma}, \quad \boldsymbol{\Gamma}_w = \boldsymbol{\Gamma}_{w_0} + \partial\boldsymbol{\Gamma}_w, \quad \text{and} \quad \boldsymbol{\Gamma}' = \boldsymbol{\Gamma}'_0 + \partial\boldsymbol{\Gamma}', \quad (2.22)$$

along with the geometric and kinematic variables, which take the form

$$\zeta = \zeta_0 + \partial\zeta, \quad \zeta' = \zeta'_0 + \partial\zeta', \quad \text{and} \quad \zeta_w = \zeta_{w_0}, \quad (2.23)$$

where the $\partial(\cdot)$ operator indicates small perturbations around arbitrary reference values $(\cdot)_0$. Additionally, the reference atmospheric fluid velocity field is set to zero, and so $\boldsymbol{\nu} = \partial\boldsymbol{\nu}$. This presents no real limitation however, since the kinematic velocities of the lattice can be modified to represent the fluid-lattice relative motion; the matrices required for a linearized model about a nonzero atmospheric velocity are the same as those developed for the lattice velocities (but of opposite sign). Linearizing the nondimensionalized form of (2.3) using the definitions of (2.22) and (2.23) leads to a general expression for the linearized normalwash, which can be written as

$$\begin{aligned} & \frac{\partial(\mathcal{A}\boldsymbol{\Gamma}_0)}{\partial\zeta} \partial\zeta^{n+1} + \mathcal{A}_0 \partial\boldsymbol{\Gamma}^{n+1} + \frac{\partial(\mathcal{A}_w\boldsymbol{\Gamma}_{w_0})}{\partial\zeta} \partial\zeta^{n+1} + \mathcal{A}_{w_0} \partial\boldsymbol{\Gamma}_w^{n+1} \dots \\ & + \mathbf{W}_0 (\partial\boldsymbol{\nu}^{n+1} - 2\partial\zeta'^{n+1}) - 2\frac{\partial(\mathbf{W}\zeta'_0)}{\partial\zeta} \partial\zeta^{n+1} = 0, \end{aligned} \quad (2.24)$$

where all the matrices are functions of the nondimensional reference geometry, and terms involving variations of AIC matrices are the matrix-valued functions

$$\frac{\partial(\mathcal{A}\boldsymbol{\Gamma}_0)}{\partial\zeta} \in \mathbb{R}^{K \times 3K\zeta}, \quad (2.25)$$

and

$$\frac{\partial(\mathcal{A}_w \Gamma_{w_0})}{\partial \zeta} \in \mathbb{R}^{K \times 3K_\zeta}, \quad (2.26)$$

whose elements are obtained by linearizing the matrix kernels analytically, as described in Appendix B. A similar description can be found there for the matrix-valued function

$$\frac{\partial(\mathbf{W} \zeta'_0)}{\partial \zeta} \in \mathbb{R}^{K \times 3K_\zeta}, \quad (2.27)$$

that gives the variations in normalwash due to variations of the lattice geometry, and in particular any changes in orientation of the surface normal vectors. Note that the costly evaluation of third-order tensors in (2.25) - (2.27) is avoided by analytically differentiating the corresponding matrix-vector products directly.

The second equation required for the linear model describes the propagation of circulation through the wake, and is given by the linearization of the nondimensional form of (2.5) as

$$\partial \Gamma_w^{n+1} = \mathbf{C}_\Gamma \partial \Gamma^n + \mathbf{C}_{\Gamma_w} \partial \Gamma_w^n, \quad (2.28)$$

where the matrices are identical to those of (2.5). Finally, variations of the rate-of-change of circulation strengths on the body are calculated using the backwards difference formula

$$\Delta t \partial \Gamma'^{n+1} = \partial \Gamma^{n+1} - \partial \Gamma^n. \quad (2.29)$$

Defining the linear aerodynamic state vector as

$$\mathbf{x}_a := \begin{bmatrix} \partial \Gamma; & \partial \Gamma_w; & \partial \Gamma' \end{bmatrix}, \quad (2.30)$$

and the linear inputs as

$$\mathbf{u}_a := \begin{bmatrix} \partial \zeta'; & \partial \zeta; & \partial \nu \end{bmatrix}, \quad (2.31)$$

the linearized equations (2.24), (2.28) and (2.29) can be written in the state-space descriptor form

$$\mathbf{E}_a \mathbf{x}_a^{n+1} = \mathbf{F}_a \mathbf{x}_a^n + \mathbf{G}_a \mathbf{u}_a^{n+1}, \quad (2.32)$$

where the matrices $\mathbf{E}_a, \mathbf{F}_a \in \mathbb{R}^{(2K+K_w) \times (2K+K_w)}$ have the form

$$\mathbf{E}_a = \begin{bmatrix} \mathcal{A}_0 & \mathcal{A}_{w_0} & 0 \\ 0 & \mathcal{I}_{K_w} & 0 \\ -\mathcal{I}_K & 0 & \Delta t \mathcal{I}_K \end{bmatrix}, \quad (2.33)$$

and

$$\mathbf{F}_a = \begin{bmatrix} 0 & 0 & 0 \\ \mathcal{C}_\Gamma & \mathcal{C}_{\Gamma_w} & 0 \\ -\mathcal{I}_K & 0 & 0 \end{bmatrix}, \quad (2.34)$$

and the input matrix for the state equation, $\mathbf{G}_a \in \mathbb{R}^{(2K+K_w) \times 9K_\zeta}$, has the form

$$\mathbf{G}_a = \begin{bmatrix} 2\mathbf{W}_0 & \left(-\frac{\partial(\mathcal{A}\Gamma_0)}{\partial\zeta} - \frac{\partial(\mathcal{A}_w\Gamma_{w_0})}{\partial\zeta} + 2\frac{\partial(\mathbf{W}\zeta'_0)}{\partial\zeta} \right) & -\mathbf{W}_0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (2.35)$$

Note that the inputs on the right-hand-side of (2.32) are, atypically, taken at time $n + 1$. This is a consequence of (2.3), and hence (2.24), being elliptic in nature – i.e. the effect of a fluid disturbance is felt everywhere instantaneously. Hence, all of the unsteadiness in the fluid response arises from the time-history of instantaneous solutions to (2.24), typically with time-varying boundary conditions, through the propagation equation (2.28). It may seem counter-intuitive to use the (future) system inputs at the same time for which one makes a prediction, however, it is consistent with staggered fluid-structure interaction algorithms: the old fluid states, and new information about the kinematics of the fluid-structure interface, are combined to find the new fluid states. A consequence of this is that these discrete-time equations fit naturally into a weakly-coupled (explicit) fluid-structure-interaction time-stepping procedure in which the structural solution is advanced first, and therefore this approach will be followed here (see Chapter 4). Also, since $\mathbf{E}_a$ is non-singular, the state-space form $\mathbf{x}_a^{n+1} = \mathbf{A}_a \mathbf{x}_a^n + \mathbf{B}_a \mathbf{u}_a^{n+1}$ can be obtained, if necessary, where $\mathbf{A}_a = \mathbf{E}_a^{-1} \mathbf{F}_a$ and $\mathbf{B}_a = \mathbf{E}_a^{-1} \mathbf{G}_a$. The descriptor form is obtained for numerical efficiency however (since $\mathbf{E}_a^{-1}$ need rarely be evaluated explicitly).

The outputs, $\mathbf{y}_a \in \mathbb{R}^{3K_\zeta}$, are defined to be the perturbations in aerodynamic forces at the lattice vertices and are obtained through linearization of the Joukowski method, described by

(2.11), giving

$$\mathbf{y}_a^n = \partial \mathbf{f}_a^n = \mathbf{H} \partial \mathbf{f}^n + \mathbf{\Xi}^\top \partial \mathbf{f}_u^n, \quad (2.36)$$

where the variation in force on the s -th segment is

$$\partial \mathbf{f}_s = \widetilde{\mathbf{v}}_{0_s}^m \mathbf{l}_{0_s} \partial \Gamma_k + \Gamma_{0_k} \widetilde{\mathbf{v}}_{0_s}^m \frac{\partial \mathbf{l}_s}{\partial \zeta_q} \partial \zeta_q - \Gamma_{0_k} \widetilde{\mathbf{l}}_{0_s} \partial \mathbf{v}_s^m, \quad (2.37)$$

where $q = 1, \dots, K_\zeta$ is an index through the lattice vertices, and the variations in velocities at the segment midpoints are expressed as

$$\begin{aligned} \partial \mathbf{v}^m &= \mathcal{A}_0^m \partial \Gamma + \frac{\partial(\mathcal{A}_0^m \Gamma_0)}{\partial \zeta} \partial \zeta + \mathcal{A}_{w_0}^m \partial \Gamma_w + \frac{\partial(\mathcal{A}_w^m \Gamma_{w_0})}{\partial \zeta} \partial \zeta \dots \\ &+ \mathbf{H}^\top (\partial \nu - 2 \partial \zeta'), \end{aligned} \quad (2.38)$$

in which the AIC matrices $\mathcal{A}_0^m \in \mathbb{R}^{12K \times K}$ and $\mathcal{A}_{w_0}^m \in \mathbb{R}^{12K \times K_w}$ map the surface and wake circulation distributions to induced velocities at the segment midpoints. The added mass force contribution at each collocation point, from (2.10), has the nondimensional linearized form

$$\partial \mathbf{f}_{u_k} = 2A_{0_k} \mathbf{n}_{k_0} \partial \Gamma'_k + 2\Gamma'_{0_k} \frac{\partial(A_k \mathbf{n}_k)}{\partial \zeta_q} \partial \zeta_q, \quad (2.39)$$

where the product of the panel area, A_k , and corresponding normal vector, $\mathbf{n}_k$, is a quadratic function of the dimensionless lattice geometry.

Gathering the equations (2.36) - (2.39) allows the state-space matrices for the output equation, which has the form

$$\mathbf{y}_a^n = \mathbf{C}_a \mathbf{x}_a^n + \mathbf{D}_a \mathbf{u}_a^n, \quad (2.40)$$

where $\mathbf{C}_a \in \mathbb{R}^{3K_\zeta \times (2K + K_w)}$ and $\mathbf{D}_a \in \mathbb{R}^{3K_\zeta \times 9K_\zeta}$, to be written as

$$\mathbf{C}_a = \begin{bmatrix} \mathbf{H} (\mathbf{Y}_1 - \mathbf{Y}_3 \mathcal{A}_0^m), & -\mathbf{H} \mathbf{Y}_3 \mathcal{A}_{w_0}^m, & \mathbf{\Xi}^\top \mathbf{Y}_4 \end{bmatrix}, \quad (2.41)$$

and

$$\mathbf{D}_a = \begin{bmatrix} 2\mathbf{H} \mathbf{Y}_3 \mathbf{H}^\top, & \mathbf{H} \mathbf{Y}_2 - \mathbf{H} \mathbf{Y}_3 \left(\frac{\partial(\mathcal{A}_0^m \Gamma_0)}{\partial \zeta} + \frac{\partial(\mathcal{A}_w^m \Gamma_{w_0})}{\partial \zeta} \right) + \mathbf{\Xi}^\top \mathbf{Y}_5, & -\mathbf{H} \mathbf{Y}_3 \mathbf{H}^\top \end{bmatrix}, \quad (2.42)$$

where $\mathbf{Y}_{(1-5)}$ are either block-diagonal or sparse matrices, which are relatively simple to compute and are defined in Appendix B. This description is based on linearizations of the

three-dimensional form of Joukowski's relation (2.6). Consequently, no component of the problem kinematics or force distribution is neglected. This means that the effects of arbitrary (small) kinematics – including in-plane motions – are all captured, and that the unsteady induced drag is resolved naturally as a component of the resulting force distribution.

The nondimensional linear aerodynamics model described by (2.32) & (2.40) are subsequently used to predict the classical frequency-domain aerodynamics of aerofoils in Section 5.1.2, and flutter in the three-dimensional problems of Section 5.3 where the model is easily modified for input-output in a structural dynamics modal basis. Balanced realizations of these equations are then explored in the context of aeroelastic stability analyses in Chapter 6.

2.4.1 Resolution and physical timescales

It is necessary to achieve a 'discretization-independent' (converged, spatially and temporally) representation of the aerodynamics through mesh convergence exercises. Further to this, the nondimensional formulation above, in Section 2.4, opens up the possibility of having a single converged model that is valid accross all incompressible dynamic pressures of interest (which is, in a restrictive sense, the flight envelope; changes in geometry due to different trim states would require new linear models). This subsection provides guidelines to determine such a model, unique for a given aerodynamic body (or thin-wing approximation thereof), based on two parameters specific to the problem of interest:

- the maximum physical frequency that must be captured in the model, f_{lim} . This is usually set in relation to the pertinent natural mode frequencies of an underlying structural model.
- the minimum physical true airspeed of interest, V_{lim} . In aeroelastic stability analysis this may be very low so as to give continuity with the *in vacuo* structural modes.

The Nyquist frequency, which in this context refers to the maximum reduced frequency that can be adequately sampled in the model, is determined by the spatial discretization of the wake. It can be expressed as

$$k_{\text{nyq}} = \frac{\pi}{2\Delta\zeta_w}, \quad (2.43)$$

where $\Delta\zeta_w$ is the chordwise length of the wake vortex rings in the nondimensional model. To ensure that all relevant physical frequencies are captured, the chordwise wake discretization

can be set as

$$\Delta\zeta_w = \frac{V_{\text{lim}}}{2f_{\text{lim}}c}, \quad (2.44)$$

which corresponds to a nondimensional time step of $\Delta t = 2\Delta\zeta_w$. Since the physical frequencies captured by the model increase linearly with free-stream velocity, as $f = kV_\infty/\pi c$, then an adequate sampling of the unsteady aerodynamics is guaranteed throughout the flight envelope.

If analysis at very low velocities is desired, a prohibitively large number of wake elements may be necessary. This is because the total number of wake vortex rings, and hence the total length of the wake used in the model (ideally infinite), must be large enough to show convergence. In such cases it may be possible to further limit the reduced frequency range at V_{lim} and obtain smaller, yet adequate, models.

If coupling with a structural model is desired, the maximum frequency required in the structural analysis may force a very small time step into the aerodynamic model. In such cases the maximum reduced frequency at which potential flow aerodynamics are valid for the typical modes of oscillation in the problem must be considered; general guidelines are given in [71] ($k_{\text{max}} = 1$). If the structural frequencies are beyond this then high-fidelity aerodynamic models or leading-edge separation approximations would be required. Such cases are outside the scope of this work.

Convergence of the aerodynamic response of aerofoils in the (reduced) frequency domain, calculated using the linear model described in this chapter, is investigated in Section 5.1.2. Additionally, the convergence of three-dimensional models is investigated in an aeroelastic context in Section 5.3.

Chapter 3

Geometrically-Exact Models of Beam Dynamics

Assuming geometric slenderness, the primary structures of aircraft are modelled using a *geometrically-exact* composite beam description, formulated in a moving frame of reference. A displacement-based approach is used [135, 136, 40] in which the position of the beam reference line and orientation of the associated cross-sections are expressed in a body-attached frame, which is free to translate and rotate in all six *rigid-body* degrees-of-freedom. Following closely the work of Hesse [59], a nonlinear model is described that captures the elastic, inertial and gyroscopic effects of structures experiencing arbitrarily-large rigid-body (unconstrained) motions and dynamic deflections. A consistent linearization of this description, which retains all of the coupling between rigid-body and elastic degrees-of-freedom, forms the basis of Jacobians, useful for time-marching of the nonlinear equations, and linear structural dynamics models obtained at reference states with arbitrary deflections.

The nonlinear beam equations are developed first in Section 3.1, which opens with a description of the beam kinematics in terms of displacement and rotational degrees-of-freedom, in Section 3.1.1. Hamilton's principle is then used to derive the dynamic equations of motion for an unrestrained, initially-curved, composite beam expressed in these coordinates, in Section 3.1.2. Finally, the resulting partial-differential-equations are discretized using finite elements and cast as a system of discrete, second-order ordinary-differential-equations which are subsequently time-marched using a modified Newmark method, as described in Section 3.1.3 and Section 3.1.4, respectively. Linearized models of the beam dynamics, which

will be used for aeroelastic stability analysis, are discussed in Section 3.2. These include a consistent linearization of the nonlinear model mentioned above (Section 3.2.1) state-space models (Section 3.2.2), and a canonical modal form of the equations (Section 3.2.3) written in the same reduced-time basis as the linearized aerodynamics model of Section 2.4.

3.1 Geometrically-nonlinear dynamics of composite beams in a moving reference frame

A multi-beam configuration comprised of curvilinear composite (anisotropic) beam segments that may be initially curved and twisted, and are capable of large deformations and global rotations, is used to represent the primary structures of aircraft. Beam reference line coordinates, defined as $\mathbf{R}(s, t)$ where s is the beam arc-length ordinate, are expressed in relation to a moving body-fixed reference frame, the A -frame, depicted in Figure 3.1, which is free to translate with velocity $\mathbf{v}(t)$ and rotate with angular velocity $\boldsymbol{\omega}(t)$. The position of the A -frame relative to an inertial (ground) frame, G , is described by the vector $\mathbf{r}$.

The rotations of cross-sections along the beam reference line are described by the orientation of the local material frame-of-reference (FoR), B , which is parameterized using the Cartesian rotation vector (CRV) [40]. This is a minimal representation of the possibly finite (large) rotations between material (B) and body-fixed (A) frames.

For the remainder of this section the notation used is similar to that of Hesse [59] and Hodges [64] with whom this formulation originates. Where necessary the projection of vector components into a specific FoR is indicated by a subscript, for example the velocities $\mathbf{v}_A$ and $\boldsymbol{\omega}_A$ have their components expressed in the body-fixed frame, A .

3.1.1 Geometrically-exact beam kinematics

In this section the kinematics of aircraft primary structures are described in a Lagrangian manner by the linear and angular motions of a beam reference line under the assumption that the cross sections of the beam remain rigid. Defining $\boldsymbol{\xi}_B$ as the coordinates of a point within a beam cross-section expressed in the local material FoR, then the displacement of that point relative to the inertial FoR, and expressed in that same frame, can be written as

$$\mathbf{X}_G = \mathbf{r}_G + \mathbf{C}^{GA} \mathbf{R}_A + \mathbf{C}^{GA} \mathbf{C}^\top \boldsymbol{\xi}_B, \quad (3.1)$$

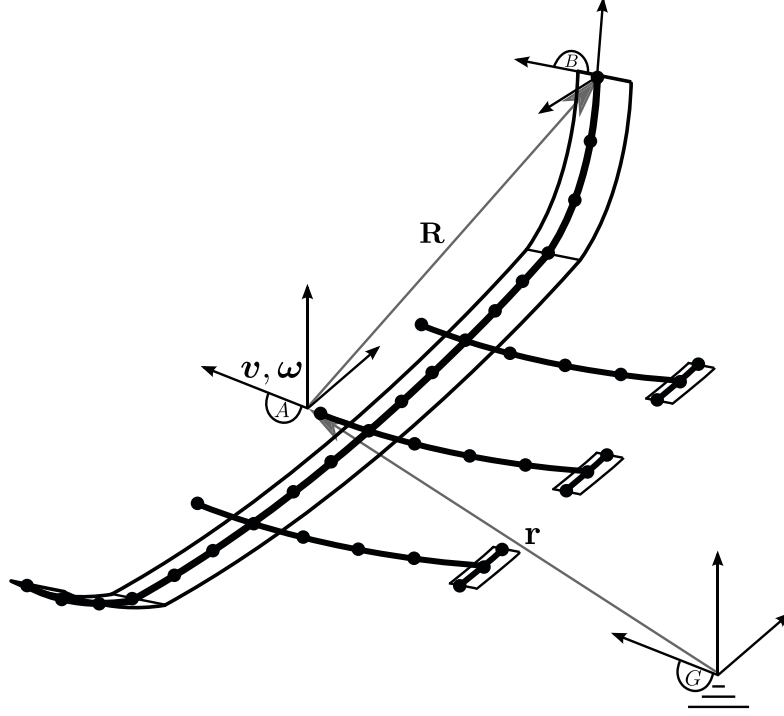


Figure 3.1: HALE aircraft represented by aerodynamic surfaces and beams. The frames of reference used in the definition of beam quantities are shown.

where $\mathbf{C}^{GA}$ is the linear transformation of vector components from the A to the G frame, and $\mathbf{C} = \mathbf{C}^{BA}$ is a similar transformation from A to B frames. The former is parameterized by quaternions, $\chi = [\chi_0; \chi_v]$, and has the definition

$$\mathbf{C}^{GA} = (2\chi_0^2 - 1) \mathbf{I} + 2 \left(\chi_v \chi_v^\top + \chi_0 \tilde{\chi}_v \right). \quad (3.2)$$

where $\mathbf{I}$ is a 3×3 identity matrix and $\tilde{\chi}_v$ is the dual skew-symmetric matrix of χ_v associated with the vector cross-product. This form is convenient as it is singularity-free and facilitates expression of finite three-dimensional rotations in terms of algebraic relations [40]. Also, rotations of the A -frame in inertial space can be tracked, assuming a flat Earth, using the attitude propagation equations [139]

$$\begin{Bmatrix} \dot{\chi}_0 \\ \dot{\chi}_v \end{Bmatrix} = -\frac{1}{2} \begin{bmatrix} 0 & \omega_A^\top \\ \omega_A & \tilde{\omega}_A \end{bmatrix} \begin{Bmatrix} \chi_0 \\ \chi_v \end{Bmatrix}, \quad (3.3)$$

where the superscript $(\dot{\cdot})$ denotes a derivative with respect to time. Note that despite the assumption that ξ_B is constant above, higher-order effects can be included in the beam for-

mulation that follows since shear strains are accounted for. However, this requires appropriate reductions of the cross-sectional geometry, mass and stiffness distributions (homogenisation) [31, 102] that will not be described here.

A Lagrangian description of the beam reference line curvature is facilitated by using the CRV, denoted $\Psi(s, t)$, to parameterize the finite rotations of cross-sections along the beam, allowing for arbitrarily-large curvatures in the deformed and reference configurations. This parameterization, also known as an *axis-angle* representation is a direct representation of Euler's rotation theorem, and therefore has the geometrical interpretation $\Psi = \phi \mathbf{n}$, in which ϕ is a scalar rotation about an axis defined by the unit vector $\mathbf{n}$. Using this notation the transformation matrix $\mathbf{C}$ can be defined as [40]

$$\mathbf{C}(\Psi) = \mathbf{I} + \frac{\sin \phi}{\phi} \tilde{\Psi} + \frac{1 - \cos \phi}{\phi^2} \tilde{\Psi}^2 \quad (3.4)$$

where $\mathbf{I}$ is the unit matrix and $\tilde{\Psi}$ is the dual skew-symmetric matrix of Ψ . Alternatively, the transformation matrix can be calculated using the matrix exponential as $\mathbf{C}(\Psi) = e^{\tilde{\Psi}} = \sum_{k=0}^{\infty} \frac{1}{k!} \tilde{\Psi}^k$, in which the infinite sum is truncated. The inverse transformation that maps vector components expressed in the B frame to the A frame is given by the transpose of (3.4), i.e. $\mathbf{C}^\top$.

Despite its name the CRV is not actually a member of the standard (Euclidean) vector space, $\mathbb{R}^3$, due to the non-commutative nature of finite rotations, and belongs to the special orthogonal Lie group, $so(3)$, which defines the necessary algebraic relationships between minimum parameter representations of finite rotations (spherical motion) [40]. The main implication of this is that in a finite-element context, which typically assumes interpolated elements are members of Euclidean space, special treatment of rotational degrees of freedom may be required, as discussed in Section 3.1.3.

Spatial and temporal variations of (3.4) are required as they define the curvature and angular rate on the beam reference line, respectively, and in turn allow the definition of strain and rotational kinetic energies in terms of $\Psi(s, t)$. General variations are described by a skew-symmetric matrix of infinitesimal rotations, defined in the local material frame as [58]

$$\delta \tilde{\Phi}_B = \mathbf{C} \delta \mathbf{C}^\top = \widetilde{\mathbf{T} \delta \Psi}, \quad (3.5)$$

where the tangential rotation operator, $\mathbf{T}(\Psi)$, is [40]

$$\mathbf{T}(\Psi) = \mathbf{I} + \frac{\cos \phi - 1}{\phi^2} \tilde{\Psi} + \left(1 - \frac{\sin \phi}{\phi}\right) \frac{\tilde{\Psi}^2}{\phi^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} \tilde{\Psi}^k. \quad (3.6)$$

In the definitions of (3.4) and (3.6) the apparent singularity at $\phi = 0$ is handled by truncating the series expansion, and noticing that in the limit of $\phi \rightarrow 0$ both expressions approach the identity matrix [40]. From (3.5) the local reference line curvature and cross-sectional angular velocity can be expressed, respectively, as

$$\widetilde{\mathbf{K}}_B = \mathbf{C} \left(\mathbf{C}^\top \right)' = \mathbf{T}(\Psi) \Psi', \quad (3.7)$$

where $(\cdot)'$ denotes a derivative with respect to arc-length, and

$$\widetilde{\Omega}_B^\Psi = \mathbf{C}(\dot{\mathbf{C}}^\top) = \widetilde{\mathbf{T}(\Psi) \dot{\Psi}}, \quad (3.8)$$

where Ω^Ψ is the angular velocity due to the time-derivative of the reference line CRV.

The force and moment strains, which describe the deformation of the reference line from the underformed state (at $t = 0$) to the current state (at time t), are therefore written in terms of the CRV as [64]

$$\begin{aligned} \gamma(s, t) &= \mathbf{C}(\Psi(s, t)) \mathbf{R}'_A(s, t) - \mathbf{C}(\Psi(s, 0)) \mathbf{R}'_A(s, 0), \\ \kappa(s, t) &= \mathbf{T}(\Psi(s, t)) \Psi'(s, t) - \mathbf{T}(\Psi(s, 0)) \Psi'(s, 0). \end{aligned} \quad (3.9)$$

The force strain γ , which has three components, is the difference in the spatial derivative of the reference line displacements between reference and deformed configurations written in the material frame, and includes the effects of extension and shear in both directions. The moment strain, κ , includes the torsional and bending strains and is the difference in local curvature, $\mathbf{K}_B = \mathbf{T}(\Psi) \Psi'$, from the reference to deformed configuration.

Similarly, the total (inertial) rate-of-change of the reference line displacements and orientation is given, in the material frame, as

$$\begin{aligned} \mathbf{V}_B(s, t) &= \mathbf{C}(\Psi(s, t)) \dot{\mathbf{R}}_A(s, t) + \mathbf{C}(\Psi(s, t)) [\mathbf{v}_A(t) + \tilde{\omega}_A(t) \mathbf{R}_A(s, t)], \\ \Omega_B(s, t) &= \mathbf{T}(\Psi(s, t)) \dot{\Psi}(s, t) + \mathbf{C}(\Psi(s, t)) \omega_A(t), \end{aligned} \quad (3.10)$$

where $\mathbf{V}_B$ is the inertial velocity, and $\mathbf{\Omega}_B$ is the inertial angular velocity found by summing the contributions of $\boldsymbol{\omega}$ and $\mathbf{\Omega}^\Psi$, which is defined in (3.8).

Infinitesimal Beam Kinematics

In order to formulate expressions for the virtual potential (elastic) and kinetic energies required in the application of Hamilton's principle, variations of the strain and velocity measures developed above are required. Starting with the force and moment strains, (3.9), their infinitesimal changes can be written as

$$\begin{aligned}\delta\boldsymbol{\gamma} &= \mathbf{C}\delta\mathbf{R}'_A + \mathbf{C}\widetilde{\mathbf{R}}'_A\mathbf{C}^\top\delta\boldsymbol{\Phi}_B, \\ \delta\boldsymbol{\kappa} &= \delta\boldsymbol{\Phi}'_B + \widetilde{\mathbf{K}}_B\delta\boldsymbol{\Phi}_B,\end{aligned}\tag{3.11}$$

where $\delta\boldsymbol{\Phi}_B$ has been defined in terms of the CRV in (3.5). When infinitesimal changes in the derivatives of finite rotations are required, such as $\delta\boldsymbol{\Phi}'_B$ above, the following matrix, written in terms of a general 3-component column matrix $\mathbf{h}$, is used [40]:

$$\begin{aligned}\mathbf{A}_1(\boldsymbol{\Psi}, \mathbf{h}) &= \frac{\delta(T(\boldsymbol{\Psi})\mathbf{h})}{\delta\boldsymbol{\Psi}} = \phi^{-2}(1 - \cos\phi)\widetilde{\mathbf{h}} + \phi^{-1}[\sin\phi - 2\phi^{-1}(1 - \cos\phi)]\widetilde{\mathbf{h}}\mathbf{n}\mathbf{n}^\top + \\ &\quad \phi^{-1}(2 + \cos\phi - 3\phi^{-1}\sin\phi)\widetilde{\mathbf{n}}\widetilde{\mathbf{h}}\mathbf{n}\mathbf{n}^\top + \\ &\quad \phi^{-1}(1 - \phi^{-1}\sin\phi)(\widetilde{\mathbf{h}}\widetilde{\mathbf{n}} - 2\widetilde{\mathbf{n}}\widetilde{\mathbf{h}}),\end{aligned}\tag{3.12}$$

where the CRV is interpreted in its geometric form $\boldsymbol{\Psi} = \phi\mathbf{n}$.

Using the same definitions the infinitesimal beam velocities can be written as

$$\begin{aligned}\delta\mathbf{V}_B &= \mathbf{C}\left(\delta\dot{\mathbf{R}}_A + \widetilde{\boldsymbol{\omega}}_A\delta\mathbf{R}_A - \widetilde{\mathbf{R}}_A\delta\boldsymbol{\omega}_A + \delta\mathbf{v}_A\right) + \widetilde{\mathbf{V}}_B\delta\boldsymbol{\Phi}_B, \\ \delta\mathbf{\Omega}_B &= \delta\dot{\boldsymbol{\Phi}}_B + \widetilde{\mathbf{\Omega}}_B\delta\boldsymbol{\Phi}_B + \mathbf{C}\delta\boldsymbol{\omega}_A.\end{aligned}\tag{3.13}$$

where infinitesimal changes in the linear and angular velocities of the A -frame can be written as

$$\begin{aligned}\delta\mathbf{v}_A &= \delta\dot{\mathbf{r}}_A + \widetilde{\boldsymbol{\omega}}_A\delta\mathbf{r}_A, \\ \delta\boldsymbol{\omega}_A &= \delta\dot{\boldsymbol{\varphi}}_A + \widetilde{\boldsymbol{\omega}}_A\delta\boldsymbol{\varphi}_A,\end{aligned}\tag{3.14}$$

in which infinitesimal rotations of the A -frame are formulated as $\delta\boldsymbol{\varphi}_A = \mathbf{C}^{AG}\delta\mathbf{C}^{GA}$ in analogy

with (3.5).

3.1.2 Equation of motion of an arbitrary reference line

Having established geometrically-exact expressions for the beam kinematics, the structural dynamics equation of motion in the interval $[t_1, t_2]$ is formulated using Hamilton's principle, which states

$$\int_{t_1}^{t_2} \int_L [\delta \mathcal{T} - \delta \mathcal{U} + \delta \mathcal{W}] ds dt = 0, \quad (3.15)$$

where the spatial dimension of integration L denotes all of the reference lines in a multi-beam configuration. The terms $\mathcal{T}$ and $\mathcal{U}$ denote the kinetic and internal potential (elastic) energy densities per unit length, respectively, and $\delta \mathcal{W}$ is the virtual work of applied loads per unit length. These energies are now expressed in terms of the strain and velocity measures developed above.

Internal and Kinetic Energy Densities

The internal potential energy is expressed in terms of the geometrically-exact strain measures developed above; no matter how large the displacements and rotations of the beam, the strain relations are still valid, assuming that the deformations remain within the linear elastic regime. Thus the virtual elastic potential energy is expressed as [64]

$$\delta \mathcal{U} = \begin{bmatrix} \delta \boldsymbol{\gamma}^\top & \delta \boldsymbol{\kappa}^\top \end{bmatrix} \boldsymbol{\mathcal{S}}_{cs} \begin{bmatrix} \boldsymbol{\gamma}^\top & \boldsymbol{\kappa}^\top \end{bmatrix}^\top = \begin{bmatrix} \delta \boldsymbol{\gamma}^\top & \delta \boldsymbol{\kappa}^\top \end{bmatrix} \begin{bmatrix} \mathbb{F}_B^\top & \mathbb{M}_B^\top \end{bmatrix}^\top, \quad (3.16)$$

where $\boldsymbol{\mathcal{S}}_{cs}$ is a 6×6 cross-sectional stiffness matrix which includes constitutive relations for local extension, shear, bending and torsion degrees-of-freedom, and any linear coupling between them. The corresponding internal forces and moments have been defined as $\mathbb{F}_B$ and $\mathbb{M}_B$, respectively.

The virtual kinetic energies are written as [64]

$$\delta \mathcal{T} = \begin{bmatrix} \delta \mathbf{V}_B^\top & \delta \boldsymbol{\Omega}_B^\top \end{bmatrix} \boldsymbol{\mathcal{M}}_{cs} \begin{bmatrix} \mathbf{V}_B^\top & \boldsymbol{\Omega}_B^\top \end{bmatrix}^\top = \begin{bmatrix} \delta \mathbf{V}_B^\top & \delta \boldsymbol{\Omega}_B^\top \end{bmatrix} \begin{bmatrix} \mathbf{P}_B^\top & \mathbf{H}_B^\top \end{bmatrix}^\top, \quad (3.17)$$

where the linear and angular cross-sectional momenta have been defined as $\mathbf{P}_B$ and $\mathbf{H}_B$,

respectively, and the cross-sectional mass matrix is

$$\mathcal{M}_{cs} = \begin{bmatrix} m\mathbf{I} & -m\tilde{\boldsymbol{\xi}}_B^{cg} \\ m\tilde{\boldsymbol{\xi}}_B^{cg} & \mathbf{J} \end{bmatrix}, \quad (3.18)$$

in which m is the mass per unit length, $\mathbf{I}$ is a 3×3 identity matrix, $\boldsymbol{\xi}_B^{cg}$ is the coordinates of the cross-sectional mass centroid expressed in the material frame of reference, and $\mathbf{J}$ is the sectional inertia tensor at the origin of the local material frame. Both $\mathcal{M}_{cs}$ and $\mathcal{S}_{cs}$ can be obtained through appropriate cross-sectional analysis [102].

Virtual Work of External Forces

The virtual work per unit length done on a beam cross-section is defined as $\delta\mathcal{W} = \langle \delta \mathbf{X}_G^\top \boldsymbol{\mu} \rangle$ where $\mathbf{X}_G$ is a material point in the cross-section, as defined in (3.1), at which the external force per-unit-length $\boldsymbol{\mu}$ is applied, and the brackets $\langle \cdot \rangle$ denote integration over the cross-section. Defining the total applied forces and moments at the reference line as

$$\mathbf{F}_B := \langle \boldsymbol{\mu}_B \rangle, \quad \text{and} \quad \mathbf{M}_B := \langle \tilde{\boldsymbol{\xi}}_B \boldsymbol{\mu}_B \rangle, \quad (3.19)$$

respectively, which typically arise due to thrust, gravity and aerodynamic loads, the cross-sectional work per-unit-length is written as

$$\delta\mathcal{W} = \left[\delta \mathbf{r}_G^\top \mathbf{C}^{GA} + \delta \mathbf{R}_A^\top + \delta \boldsymbol{\varphi}_A^\top \left(\tilde{\mathbf{R}}_A + \mathbf{C}^\top \tilde{\boldsymbol{\xi}}_B \mathbf{C} \right) \right] \mathbf{C}^\top \mathbf{F}_B + \delta \boldsymbol{\Phi}_B^\top \mathbf{M}_B. \quad (3.20)$$

where $\delta \boldsymbol{\Phi}_B$ and $\delta \boldsymbol{\varphi}_A$ are the cross-section and A -frame *virtual rotations*, respectively, with the former defined in (3.5), and the latter in an analogous manner noting that $\delta \tilde{\boldsymbol{\varphi}}_A = \mathbf{C}^{AG} \delta \mathbf{C}^{GA}$.

Equation of Motion of an Arbitrary Reference Line

Finally, substituting the expressions of virtual measures from (3.16), (3.17) and (3.20) into Hamilton's principle, (3.15), and integrating by parts in time, gives the weak form of the

equations of motion as [104]

$$\begin{aligned}
 & \int_{t_1}^{t_2} \left\{ \int_L \left\{ \delta \mathbf{R}_A^\top \mathbf{C}^\top \left(\dot{\mathbf{P}}_B + \tilde{\boldsymbol{\Omega}}_B \mathbf{P}_B - \mathbf{F}_B \right) + \delta \mathbf{R}'_A{}^\top \mathbf{C}^\top \mathbb{F}_B \right. \right. \\
 & \quad \left. \left. + \delta \Phi_B^\top \left(\dot{\mathbf{H}}_B + \tilde{\boldsymbol{\Omega}}_B \mathbf{H}_B + \tilde{\mathbf{V}}_B \mathbf{P}_B - \tilde{\mathbf{K}}_B \mathbb{M}_B - \mathbf{C} \tilde{\mathbf{R}}'_A \mathbf{C}^\top \mathbb{F}_B - \mathbf{M}_B \right) + \delta \Phi'_B{}^\top \mathbb{M}_B \right\} ds \right. \\
 & \quad \left. + \delta \mathbf{r}_G^\top \mathbf{C}^{GA} \left(\dot{\mathbf{P}}^R + \tilde{\boldsymbol{\omega}}_A \mathbf{P}_A^R - \mathbf{F}_A^R \right) + \delta \varphi_A^\top \left(\dot{\mathbf{H}}_A^R + \tilde{\boldsymbol{\omega}}_A \mathbf{H}_A^R - \mathbf{M}_A^R \right) \right\} dt = \\
 & = \int_L \left[\delta \mathbf{R}_A^\top \mathbf{C}^\top \mathbf{P}_B + \delta \Phi_B^\top \mathbf{H}_B \right]_{t_1}^{t_2} ds + \left[\delta \mathbf{r}_G^\top \mathbf{C}^{GA} \mathbf{P}_A^R + \delta \varphi_A^\top \mathbf{H}_A^R \right]_{t_1}^{t_2}
 \end{aligned} \tag{3.21}$$

where the total momenta and external forces and moments are isolated as

$$\begin{aligned}
 \mathbf{P}_A^R &= \int_L \mathbf{C}^\top \mathbf{P}_B ds, & \mathbf{H}_A^R &= \int_L \left(\tilde{\mathbf{R}}_A \mathbf{C}^\top \mathbf{P}_B + \mathbf{C}^\top \mathbf{H}_B \right) ds, \\
 \mathbf{F}_A^R &= \int_L \mathbf{C}^\top \mathbf{F}_B ds, & \mathbf{M}_A^R &= \int_L \left(\tilde{\mathbf{R}}_A \mathbf{C}^\top \mathbf{F}_B + \mathbf{C}^\top \mathbf{M}_B \right) ds.
 \end{aligned} \tag{3.22}$$

and represent the integral through all beam reference lines of the respective momenta, forces and moments expressed in the A -frame. Consequently, in (3.21), the Lagrangian of the unconstrained beam is now expressed in terms of the independent degrees-of-freedom. Integration along the beam reference lines, L , is facilitated using a finite-element approach, as discussed next.

3.1.3 Discrete equations of motion

The beam equation of motion, (3.21), is solved by approximating the reference line displacements, $\mathbf{R}_a(s)$, and rotations, $\boldsymbol{\Psi}(s)$, using a finite-element discretization with elemental shape functions $N_i(s)$ as

$$\begin{aligned}
 \mathbf{R}_A(s) &\cong \sum_{i=1}^n N_i(s) \mathbf{R}_{A_i}, \\
 \boldsymbol{\Psi}(s) &\cong \sum_{i=1}^n N_i(s) \boldsymbol{\Psi}_i,
 \end{aligned} \tag{3.23}$$

where $\mathbf{R}_{A_i}$ and $\boldsymbol{\Psi}_i$ are nodal values of the reference line degrees-of-freedom, and N_i are the weights at the i -th node. Both linear and quadratic interpolation has been implemented,

which corresponds to 2-noded ($n = 2$) and 3-noded ($n = 3$) elements. The *objectivity* of strain relations, which in this context refers to their invariance under global finite rotations [66, 126], is not guaranteed by the approximation of (3.23), however. The effect of this is slow convergence of the beam discretization using linear elements, however quadratic elements effectively reduce the associated errors to negligible levels even in problems featuring large out-of-plane static deflections, as shown later in Section 5.2.1.

Letting $\boldsymbol{\eta}_i := [\mathbf{R}_{A_i}; \boldsymbol{\Psi}_i]$ define the blocks in a column-matrix of all the nodal degrees of freedom, and $\boldsymbol{\beta} := [\mathbf{v}_A; \boldsymbol{\omega}_A]$ be the rigid-body degrees-of-freedom, the discrete form of (3.21) can be written in compact form as [59]

$$\mathcal{M}(\boldsymbol{\eta}) \begin{Bmatrix} \ddot{\boldsymbol{\eta}} \\ \dot{\boldsymbol{\beta}} \end{Bmatrix} + \begin{Bmatrix} \mathbf{Q}_{gyr}^S \\ \mathbf{Q}_{gyr}^R \end{Bmatrix} + \begin{Bmatrix} \mathbf{Q}_{stif}^S \\ 0 \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q}_{ext}^S \\ \mathbf{Q}_{ext}^R \end{Bmatrix}, \quad (3.24)$$

where discrete gyroscopic, elastic and external forces are respectively denoted as $\mathbf{Q}_{gyr}$, $\mathbf{Q}_{stif}$ and $\mathbf{Q}_{ext}$, and their structural and rigid-body components are indicated using the superscripts S and R , respectively. These discrete forces are in general nonlinear functions of the beam degrees-of-freedom, and in the case of $\mathbf{Q}_{ext}$ any other form of external forcing, while the *tangent* mass matrix $\mathcal{M}$ is a function of the current structural deformations, $\boldsymbol{\eta}$. No structural damping is assumed in the equations of motion, which is a conservative approximation from an aeroelastic stability point of view.

The set of nonlinear second-order ordinary-differential-equations of (3.24) describes the dynamics of unconstrained composite beams that can be initially-curved and twisted with no restrictions in the underlying kinematic description. All coupling between large dynamic deflections of the beam structure and the rigid-body motion of the body-fixed reference frame is captured. Aerodynamic effects, calculated using the models developed in Chapter 2, are included as discrete external forces as will be discussed in Chapter 4.

3.1.4 Temporal integration of the discrete equations

Defining the time-rate-of-change of flexible-body dynamics states as $\dot{\mathbf{x}}_{fb} := [\dot{\boldsymbol{\eta}}; \dot{\boldsymbol{\beta}}; \dot{\boldsymbol{\chi}}]$ to include the structural, rigid-body and attitude degrees-of-freedom, the integration of the discrete system (3.24) (and also in general (3.21)) in time can be described as an initial-value

problem of the form

$$\begin{cases} \dot{\mathbf{x}}_{\text{fb}}(t) \\ \ddot{\mathbf{x}}_{\text{fb}}(t) \end{cases} = \mathbf{f}(\mathbf{x}_{\text{fb}}, \dot{\mathbf{x}}_{\text{fb}}, t), \quad \text{and} \quad \mathbf{f}_0 = \mathbf{f}(\mathbf{x}_{\text{fb}}(t_0), \dot{\mathbf{x}}_{\text{fb}}(t_0), t_0), \quad (3.25)$$

where the subscript $(\cdot)_0$ denotes the initial conditions.

As noted in [58] the coupled equations of motion are *numerically stiff*, meaning that there is a wide range of frequencies present in the characteristic response of the system. Numerically this is diagnosed by a large condition number, owing to the large spread in singular values between the relatively-high-frequency structural modes and the low-frequency rigid-body modes [134]. Consequently, explicit time-integration schemes are impractical because a very small time step is required to avoid the propagation of high-frequency errors. An unconditionally-stable implicit method is therefore chosen to solve (3.25), based on modifying an implicit Newmark- β scheme [41, 40], in which the prediction of flexible-body-dynamics states at the next time step, $n + 1$, is written as

$$\begin{aligned} \mathbf{x}_{\text{fb}}^{n+1} &= \mathbf{x}_{\text{fb}}^n + \Delta t \dot{\mathbf{x}}_{\text{fb}}^n + \left(\frac{1}{2} - \vartheta_2\right) \Delta t^2 \ddot{\mathbf{x}}_{\text{fb}}^n + \vartheta_2 \Delta t^2 \ddot{\mathbf{x}}_{\text{fb}}^{n+1}, \\ \dot{\mathbf{x}}_{\text{fb}}^{n+1} &= \dot{\mathbf{x}}_{\text{fb}}^n + (1 - \vartheta_1) \Delta t \ddot{\mathbf{x}}_{\text{fb}}^n + \vartheta_1 \Delta t \ddot{\mathbf{x}}_{\text{fb}}^{n+1}, \end{aligned} \quad (3.26)$$

where Δt is a constant time step, and the constants ϑ_1 and ϑ_2 define the accuracy and stability properties of the scheme. Following [58] these are chosen as

$$\vartheta_1 = \frac{1}{2} + \alpha \quad \text{and} \quad \vartheta_2 = \frac{1}{4} \left(\vartheta_1 + \frac{1}{2} \right)^2, \quad (3.27)$$

where $\alpha > 0$, which introduces a small positive algorithmic damping. To ensure that (3.24) is satisfied in the presence of large (nonlinear) deformations and rigid-body motion a correction method is employed in which successive linearizations of (3.24), defined next in Section 3.2.1, are used to converge the solution during Newton-Raphson sub-iterations.

3.2 Linearized beam dynamics

A consistent linearization of the geometrically-exact beam model described by the nonlinear equations of (3.24) is considered first. The resulting linear equations, which include all the possible couplings of the rigid-body and structural degrees-of-freedom, allow natural vibration

modes to be computed, as described in Section 3.2.1, and facilitate the formulation of a linear time-invariant state-space model written in physical degrees-of-freedom, as described in Section 3.2.2. Finally, a canonical modal form of the structural dynamics equations are written in the same reduced-time basis as the nondimensional aerodynamics of Section 2.4, and is also cast into a discrete-time state-space form.

3.2.1 Consistently-linearized unconstrained GECB equations

Linearizations of the equations of motion (3.24) representing the geometrically-exact composite beam (GECB) dynamics are obtained through a perturbation approach. It is assumed that in the reference condition, denoted $(\dot{\eta}_0, \beta_0, \eta_0, \chi_0)$, accelerations are zero [40], and therefore

$$\mathbf{Q}_{gyr}(\eta_0, \dot{\eta}_0, \beta_0) + \mathbf{Q}_{stif}(\eta_0) = \mathbf{Q}_{ext}(\eta_0, \dot{\eta}_0, \beta_0, \chi_0). \quad (3.28)$$

In general this reference condition can include: large elastic deformations and any associated instantaneous velocities; gyroscopic effects arising from the rigid-body motion of the system, for example during an aircraft manoeuvre; and any external forcing encompassed by $\mathbf{Q}_{ext}$. Assuming small perturbations around this point, denoted $(\partial\ddot{\eta}, \partial\dot{\beta}, \partial\dot{\eta}, \partial\beta, \partial\eta, \partial\chi)$, the beam dynamics degrees-of-freedom are approximated as

$$\begin{aligned} \ddot{\eta} &= \partial\ddot{\eta} \quad \text{and} \quad \dot{\beta} = \partial\dot{\beta}, \quad \text{while,} \\ \dot{\eta} &= \dot{\eta}_0 + \partial\dot{\eta}, \\ \beta &= \beta_0 + \partial\beta, \\ \eta &= \eta_0 + \partial\eta, \\ \chi &= \chi_0 + \partial\chi. \end{aligned} \quad (3.29)$$

Substituting the variables (3.29) into the nonlinear equations reveals linearized dynamics of the form derived in [58]

$$\mathcal{M}(\eta_0) \begin{Bmatrix} \partial\ddot{\eta} \\ \partial\dot{\beta} \end{Bmatrix} + \mathcal{C}(\eta_0, \dot{\eta}_0, \beta_0) \begin{Bmatrix} \partial\dot{\eta} \\ \partial\beta \end{Bmatrix} + \mathcal{K}(\eta_0, \dot{\eta}_0, \beta_0) \begin{Bmatrix} \partial\eta \\ 0 \end{Bmatrix} = \partial\mathbf{Q}_{ext}, \quad (3.30)$$

where the tangent mass matrix, $\mathcal{M}$, is now a function of the elastic DoFs (the beam's shape) at the reference condition, and the tangent damping and stiffness matrices, $\mathcal{C}$ and $\mathcal{K}$, respectively,

are obtained by directly linearizing the discrete elastic and gyroscopic forces of (3.24), which are defined in detail in [58]. The resulting matrices have the structure

$$\begin{aligned}\mathcal{C}(\boldsymbol{\eta}_0, \dot{\boldsymbol{\eta}}_0, \boldsymbol{\beta}_0) &= \begin{bmatrix} (\partial \mathbf{Q}_{gyr}^S / \partial \dot{\boldsymbol{\eta}}) & (\partial \mathbf{Q}_{gyr}^S / \partial \boldsymbol{\beta}) \\ (\partial \mathbf{Q}_{gyr}^R / \partial \dot{\boldsymbol{\eta}}) & (\partial \mathbf{Q}_{gyr}^R / \partial \boldsymbol{\beta}) \end{bmatrix}, \\ \mathcal{K}(\boldsymbol{\eta}_0, \dot{\boldsymbol{\eta}}_0) &= \begin{bmatrix} (\partial \mathbf{Q}_{gyr}^S / \partial \boldsymbol{\eta}) + (\partial \mathbf{Q}_{stif}^S / \partial \boldsymbol{\eta}) & 0 \\ (\partial \mathbf{Q}_{gyr}^S / \partial \boldsymbol{\eta}) & 0 \end{bmatrix},\end{aligned}\tag{3.31}$$

which shows there is both apparent damping and stiffness due to gyroscopic effects. This linearized (incremental) form of (3.24) is used to build state-space dynamics models, as discussed next in Section 3.2.2, and Jacobian matrices that aid convergence in nonlinear dynamic analyses (as mentioned above, in Section 3.1.4).

To obtain the unconstrained natural vibration modes of a beam configuration, which are often used as coordinate bases in structural dynamics and aeroelastic analysis, the dynamic equations of motion (3.24) are further simplified by assuming a static equilibrium. Specifically this means that, in addition to the assumption of zero accelerations in all the DoFs, there is also zero relative velocity ($\dot{\boldsymbol{\eta}}_0 = 0$) in the elastic DoFs. The resulting form of the unforced linear equations of motion is simply

$$\mathcal{M}(\boldsymbol{\eta}_0) \begin{Bmatrix} \partial \ddot{\boldsymbol{\eta}} \\ \partial \dot{\boldsymbol{\beta}} \end{Bmatrix} + \mathcal{K}(\boldsymbol{\eta}_0, \boldsymbol{\beta}_0) \begin{Bmatrix} \partial \boldsymbol{\eta} \\ 0 \end{Bmatrix} = \mathbf{0}.\tag{3.32}$$

Defining states $\mathbf{x}_b := [\boldsymbol{\eta}; \int \boldsymbol{\beta} dt]$ as the unconstrained beam degrees-of-freedom and applying a Laplace transform to (3.32), the nontrivial solutions are found using eigenanalysis and have the form

$$(\mathcal{K} - \boldsymbol{\Omega}^2 \mathcal{M}) \boldsymbol{\Phi}_b = \mathbf{0},\tag{3.33}$$

in which $\boldsymbol{\Omega} = \text{diag}\{\omega_1, \omega_2, \dots, \omega_n\}$ are the natural frequencies associated to the columns of the matrix of natural modes $\boldsymbol{\Phi}_b$.

If the body-fixed frame has all six DoFs then the first six columns of $\boldsymbol{\Phi}_b$ are zero-frequency modes corresponding to the rigid-body translation and rotation (roll, pitch and yaw) of the reference axes. The remaining modes are the unconstrained modes of vibration of the structure, which may be truncated by removing the last $n - n_m$ columns from $\boldsymbol{\Phi}_b$, preserving the n_m lowest frequency modes.

3.2.2 State-space description

Following the description by Murua [93] the linearized beam equations (3.30) are cast in a discrete-time state-space form. The Newmark- β method is used again, leading to implicit predictions for the linear perturbations of the form

$$\begin{aligned} \partial \boldsymbol{\eta}^{n+1} &= \partial \boldsymbol{\eta}^n + \Delta t \partial \dot{\boldsymbol{\eta}}^n + \left(\frac{1}{2} - \vartheta_2 \right) \partial \Delta t^2 \ddot{\boldsymbol{\eta}}^n + \vartheta_2 \Delta t^2 \ddot{\boldsymbol{\eta}}^{n+1}, \\ \partial \dot{\boldsymbol{\eta}}^{n+1} &= \partial \dot{\boldsymbol{\eta}}^n + (1 - \vartheta_1) \Delta t \partial \ddot{\boldsymbol{\eta}}^n + \vartheta_1 \Delta t \partial \ddot{\boldsymbol{\eta}}^{n+1}, \\ \partial \boldsymbol{\beta}^{n+1} &= \partial \boldsymbol{\beta}^n + (1 - \vartheta_1) \Delta t \partial \dot{\boldsymbol{\beta}}^n + \vartheta_1 \Delta t \partial \dot{\boldsymbol{\beta}}^{n+1}, \\ \partial \boldsymbol{\Theta}^{n+1} &= \partial \boldsymbol{\Theta}^n + \frac{1}{2} \Delta t \partial \dot{\boldsymbol{\Theta}}^n + \frac{1}{2} \Delta t \partial \dot{\boldsymbol{\Theta}}^{n+1}, \end{aligned} \tag{3.34}$$

in which Euler angles, $\boldsymbol{\Theta} = \boldsymbol{\Theta}_0 + \partial \boldsymbol{\Theta}$, are used to track the linear changes in orientation of the body-fixed frame and are preferred to quaternions in the linearized dynamics equations [25]. A midpoint integration is used for the Euler angles since it admits integration in terms of their first derivatives only, which is convenient since in a linear setting $\partial \dot{\boldsymbol{\Theta}} = \partial \boldsymbol{\omega}_a$. The accelerations in (3.34) are obtained as a function of the beam states, and general inputs $\mathbf{u}_s$, as [93]

$$\begin{aligned} \begin{Bmatrix} \partial \ddot{\boldsymbol{\eta}}^{n+\epsilon} \\ \partial \ddot{\boldsymbol{\beta}}^{n+\epsilon} \end{Bmatrix} &= -\mathcal{M}^{-1}(\boldsymbol{\eta}_0) \left[(\mathcal{C} + \mathcal{C}_{ext}) \begin{Bmatrix} \partial \dot{\boldsymbol{\eta}}^{n+\epsilon} \\ \partial \dot{\boldsymbol{\beta}}^{n+\epsilon} \end{Bmatrix} + (\mathcal{K} + \mathcal{K}_{ext}) \begin{Bmatrix} \partial \boldsymbol{\eta}^{n+\epsilon} \\ 0 \end{Bmatrix} \dots \right. \\ &\quad \left. \dots + \frac{\partial \mathcal{Q}_{ext}}{\partial \boldsymbol{\Theta}} \partial \boldsymbol{\Theta}^{n+\epsilon} + \frac{\partial \mathcal{Q}_{ext}}{\partial \mathbf{u}_s} \partial \mathbf{u}_s^{n+\epsilon} \right] \end{aligned} \tag{3.35}$$

in which

$$\begin{aligned} \mathcal{K}_{ext} &= \begin{bmatrix} \frac{\partial \mathcal{Q}_{ext}}{\partial \boldsymbol{\eta}} & \mathbf{0} \end{bmatrix}, \\ \mathcal{C}_{ext} &= \begin{bmatrix} \frac{\partial \mathcal{Q}_{ext}}{\partial \dot{\boldsymbol{\eta}}} & \frac{\partial \mathcal{Q}_{ext}}{\partial \dot{\boldsymbol{\beta}}} \end{bmatrix} \end{aligned} \tag{3.36}$$

are the apparent stiffness and damping due to external effects, for example aerodynamic forcing.

Defining the linearized GECB states at the discrete time step n as

$$\mathbf{x}_s^n := [\partial \dot{\boldsymbol{\eta}}; \partial \boldsymbol{\eta}; \partial \boldsymbol{\beta}; \partial \boldsymbol{\Theta}]^n, \tag{3.37}$$

the linear dynamics of equation (3.30) can be expressed in the implicit linear time-invariant discrete-time state-space form

$$\begin{aligned} \mathbf{E}_s \mathbf{x}_s^{n+1} + \mathbf{F}_s \mathbf{u}_s^{n+1} &= \mathbf{A}_s \mathbf{x}_s^n + \mathbf{B}_s \mathbf{u}_s^n \\ \mathbf{y}_s^n &= \mathbf{x}_s^n \end{aligned} \quad (3.38)$$

where the outputs $\mathbf{y}_s$ are chosen to be the unconstrained beam states, which define the displacements and velocities required to construct a fluid-structure interface for the aerodynamics model, as will be discussed in Section 4.1.1. The inputs $\mathbf{u}_s$ are the external forces and moments on the beam nodes which arise from aerodynamic, thrust and gravity forces.

3.2.3 Canonical modal form

Here the linear dynamics of beams are described using modal coordinates, and time-derivatives are taken with respect to the nondimensional time defined in Section 2.4 and are once again indicated by the *prime* superscript $(\cdot)'$. A set of mass-orthonormalized modes are obtained from (3.33) such that $\mathbf{x}_b(s, t) = \mathbf{\Phi}(s) \mathbf{q}(t)$, where $\mathbf{q}$ are the modal amplitudes. Thus, using the definition of nondimensional time introduced in (2.20), the modal form of the beam dynamics equations [39] can be expressed as

$$\frac{4V_\infty^2}{c_{\text{ref}}^2} \mathbf{q}'' + \mathbf{\Omega}^2 \mathbf{q} = \rho_\infty V_\infty^2 c_{\text{ref}}^2 \mathbf{Q}(t), \quad (3.39)$$

where $\mathbf{Q}(t) = \mathbf{\Phi}^\top \mathbf{Q}_{\text{ext}}$ are the generalized forces nondimensionalised with free-stream density ρ_∞ , free-stream velocity V_∞ , and a reference chord length c_{ref} , in accordance with the scheme used in Section 2.4. Since a state-space realization of the aeroelastic equations is desired (3.39) is written in first-order form using the definitions $\mathbf{x}_m := [\mathbf{q}'; \mathbf{q}]$ and $\bar{\mathbf{\Omega}} := \frac{c_{\text{ref}}}{2V_\infty} \mathbf{\Omega} = \text{diag}\{k_1, k_2, \dots, k_n\}$, resulting in the continuous-time, linear, time-invariant equations $\mathbf{x}_m' = \mathbf{A}_c \mathbf{x}_m + \mathbf{B}_c \mathbf{Q}$ where

$$\mathbf{A}_c = \begin{bmatrix} 0 & -\bar{\mathbf{\Omega}}^2 \\ \mathbf{I} & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{B}_c = \begin{bmatrix} \frac{\rho_\infty c_{\text{ref}}^4}{4} \mathbf{I} \\ 0 \end{bmatrix}.$$

A discrete-time equivalent of this system is then found – with the same sampling period as (2.20) – by assuming zero-order-hold on the structural inputs, which gives the discrete-time

modal form of the structural dynamics equations as

$$\begin{aligned}\mathbf{x}_m^{n+1} &= \mathbf{A}_m \mathbf{x}_m^n + \mathbf{B}_m \mathbf{Q}^n, \\ \mathbf{y}_m^n &= \mathbf{x}_m^n,\end{aligned}\tag{3.40}$$

where $\mathbf{A}_m = e^{\Delta t \mathbf{A}_c}$ and $\mathbf{B}_m = \mathbf{A}_c^{-1} (e^{\Delta t \mathbf{A}_c} - \mathbf{I}) \mathbf{B}_c$ [85]. Outputs, $\mathbf{y}_m$, are chosen to be the modal coordinates and their rates-of-change, and the aerodynamic model will be augmented to accept these as inputs in the aeroelastic formulation discussed in Section 5.3. It will be necessary when doing so to recover the physical beam degrees of freedom from these modal amplitudes.

Chapter 4

Multidisciplinary Integration

In this chapter the flexible-body dynamics models introduced in Chapter 3, and the nonlinear and linearized unsteady aerodynamics models of Chapter 2, are brought together to form a family of nonlinear and linearized models for the aeroelastic and flight dynamic analysis of flexible aircraft. While there have been similar efforts by other groups in recent years [32, 107, 134, 149], the fidelity of the underlying models gives advantages over alternative coupled methods:

- The nonlinear equations of motion of unconstrained composite beam dynamics include gyroscopic effects which are maintained, if present, in the consistent linearization described in Section 3.2 (which is based on the work of Hesse [58]).
- The nonlinear discrete-time equations of vortex-lattice-based aerodynamics introduced in Chapter 2 include the effects of three-dimensionality, a force-free wake, and are suitable for, at least, preliminary analysis of wake interference effects [33, 98]. Additionally, the boundary conditions governing the flowfield are formulated on the instantaneous deforming aerodynamic geometry, and unsteady induced-drag forces are resolved [97].

Further to this, the novel linearization of discrete-time aerodynamics in this work (Section 2.4) is formulated without assumptions on the kinematics of the fluid-structure interface, and facilitates a complete separation of linearized expressions which describe the aerodynamics and structural dynamics. This is achieved by formulating the aerodynamics as a state-space model in which the inputs are as general as possible, i.e. in terms of the instantaneous displacements and velocities of the underlying lattice geometry. Consequently, the linearized aerodynamics can be coupled easily with any structural model (whether linear or nonlinear).

With this in mind, the fluid-structure interface between the vortex-lattice-based aerodynamics model and the flexible-body dynamics modelled by beams is developed first in Section 4.1. Mapping the kinematics of the beam description to the aerodynamic lattice is covered first, in Section 4.1.1 (nonlinear) and Section 4.1.2 (linearized), followed by the converse procedure whereby the forces calculated using the aerodynamic lattice are transformed back onto the beam degrees-of-freedom, in Section 4.1.3. The aeroelastic models formed by coupling the aforementioned methods, including nonlinear static and dynamic models and consistent linearizations thereof, are introduced in Section 4.2, followed by a discussion of aeroelastic stability analysis, which is carried out in this work by direct eigenanalysis of the coupled linear dynamics, in Section 4.2.3. Finally, the software framework developed to carry out the research in this thesis, namely *Simulation of High-Aspect-Ratio Planes in Python* (SHARPy), is introduced in Section 4.3.

4.1 The fluid-structure interface

In this work the dynamics of aircraft structures are represented by the kinematics of a reference line (curve) while the aerodynamics are formulated on an infinitesimally-thin lattice (surface). Thus, a mapping is required between the beam degrees-of-freedom and those of the aerodynamics which will be referred to as the fluid-structure interface. In this section the kinematics of the beam are mapped on to the lattice under the assumption that cross-sections of the aerodynamic surfaces remain rigid, and forces calculated on the lattice are mapped back to the beam under the same assumption. A coincident spanwise discretization is also assumed, meaning that a lattice cross-section is defined for each node in the discrete description of the beam reference line described in Section 3.1.3.

4.1.1 Mapping beam kinematics to the aerodynamic model

The reference line of a multibeam configuration is discretized by nodes at arc-lengths s_j along the beam, where j is the node index. Depending on the aircraft configuration there may be an aerodynamic cross-section at the node, which is then defined, in the local material frame, as

$$\boldsymbol{\xi}_{B_i}(s_j) = \begin{bmatrix} \xi_{1_{ij}} & \xi_{2_{ij}} & \xi_{3_{ij}} \end{bmatrix}^T \quad \text{for } i = 1, \dots, M(s_j) + 1, \quad (4.1)$$

where $M(s_j)$ is the number of vortex rings used to discretize the aerofoil camber line at that location along the beam.¹ Subsequently, the aerodynamic lattice, which is defined in the inertial frame (G in Figure 3.1) by the discrete points $\zeta_{G_{ij}}$, is written using the expressions introduced in Section 3.1 as

$$\zeta_{G_{ij}} = \mathbf{r}_G(t) + \mathbf{C}^{GA}(t) \left[\mathbf{R}_A(s_j, t) + \mathbf{C}^\top(s_j, t) \boldsymbol{\xi}_{B_i}(s_j) \right], \quad (4.2)$$

where $\mathbf{r}_G(t)$ is the origin of the body-fixed frame, $\mathbf{C}^{GA}(t)$ is the linear transformation from the body-fixed frame to the inertial frame, $\mathbf{R}_A(s_j, t)$ is the deformed position of the j -th structural node, and $\mathbf{C}^\top(s_j, t)$ is the linear transformation from the material B -frame to the A -frame based on the CRV at the j -th structural node, $\boldsymbol{\Psi}(s_j, t)$. Using the subscript $(\cdot)_j$ to denote quantities at the arc-lengths s_j the lattice velocities, which are obtained through differentiation of (4.2), can be expressed in shorthand notation as

$$\dot{\zeta}_{G_{ij}} = \mathbf{C}^{GA} \left[\mathbf{v}_A + \dot{\mathbf{R}}_{A_j} - \mathbf{C}_j^\top \tilde{\boldsymbol{\xi}}_{B_{ij}} \mathbf{T}_j \dot{\boldsymbol{\Psi}}_j + \tilde{\boldsymbol{\omega}}_A \left[\mathbf{R}_{A_j} + \mathbf{C}_j^\top \boldsymbol{\xi}_{B_{ij}} \right] \right] \quad (4.3)$$

where $\mathbf{v}_A$ and $\boldsymbol{\omega}_A$ are the translational and angular velocities of the body-fixed frame, $\mathbf{T}_j$ is the *tangential operator* at the j -th node, defined in (3.6), and $\dot{\boldsymbol{\Psi}}_j$ is the time-rate-of-change of the CRV at the j -th node, all of which are generally function of time.

4.1.2 Linearized mapping

The linearized unsteady aerodynamic formulation introduced in Section 2.4 is defined as a stand-alone module and as such can be integrated easily with any structural dynamics model. Only two matrices are required to define the linearized fluid-structure interface (FSI): Firstly, transformation from the physical coordinates of the structural model, $\boldsymbol{\eta}_s$, and their rates of change, $\boldsymbol{\eta}'_s$,² to the displacements and velocities the aerodynamic grid, $\boldsymbol{\zeta}$ and $\boldsymbol{\zeta}'$, denoted $\mathbf{T}_{as}$; and secondly, transformation of the force distribution on the aerodynamic grid, $\mathbf{f}^a$, to the boundary of the structure, $\mathbf{f}^s$, denoted $\mathbf{T}_{sa}$. It is also helpful to note that, providing the structural force distribution is defined at the physical coordinates of the model, the transformation of the structural displacement/rotation field to the aerodynamic grid displacements is simply $\mathbf{T}_{sa}^\top$ [17]. Figure 4.1 illustrates the coupling of aerodynamic and structural models,

¹To allow for the lattice structure used in the aerodynamics model adjacent aerodynamic cross-sections must have the same number of points in the chordwise direction.

²These may include rigid-body degrees-of-freedom.

where the latter is expressed in terms of any suitable generalized coordinates, $\mathbf{q}$, that are related to the physical coordinates and forces through the matrix Φ . Generally, any structural model – whether linear or nonlinear, finite-element, finite-difference, or modal – may be integrated with the aerodynamic model of Section 2.4. In the following three subsections linearized interfaces of increasing complexity will be introduced, starting with the one required for two-dimensional aeroelastic models of aerofoils. For consistency with the notation used in previous chapters the $\partial(\cdot)$ symbol is used to denote linear perturbations.

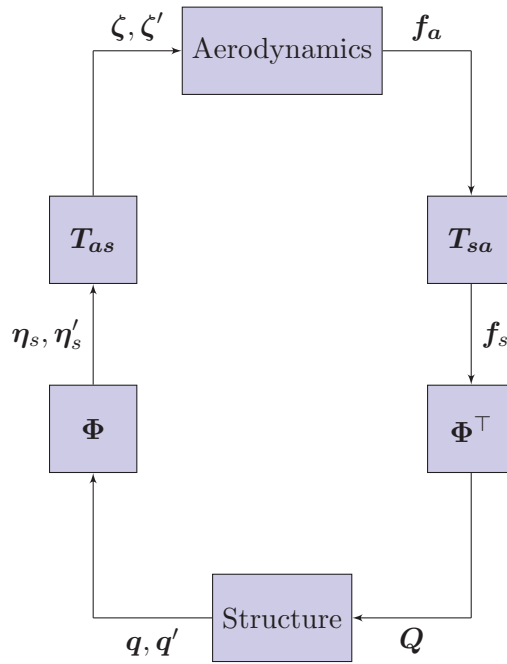


Figure 4.1: Coupling between aerodynamic and structural models.

Aerofoil in 2-D

A rigid airfoil model with linear pitch, plunge and flap degrees of freedom is considered first. The aerofoil boundary is idealized as an infinitesimally-thin *flat plate* and is placed in a parallel free-stream flow — deformations of the surface perpendicular to the free-stream flow leads to an effective downwash arising from both the velocity and displacement of the surface. Rotations are assumed about a point x_{ea} from the leading edge, and at the hinge of a trailing-edge flap at x_{fh} . If α is the angle of incidence, h is the plunge degree-of-freedom (positive down), and β is the flap angle, the structural inputs relevant to the aerodynamic

model can be defined as

$$\mathbf{x}_s^{(1)} := \begin{bmatrix} \alpha & \alpha' & \frac{h'}{c} & \beta & \beta' \end{bmatrix}^\top, \quad (4.4)$$

where $(\cdot)'$ indicates derivatives with respect to the nondimensional time of (2.20). Corresponding perturbations in the vertical velocities, z' , and geometry, z , of the aerofoil boundary can be expressed, respectively, as

$$z' = -(x - x_{ea}) \alpha' - b(x - x_{fh}) \beta' - \frac{h'}{c_{\text{ref}}}, \quad (4.5)$$

$$z = -(x - x_{ea}) \alpha - b(x - x_{fh}) \beta, \quad (4.6)$$

where x and z denote chordwise and vertical coordinates, respectively, and $b = H(x - x_{fh})$. After discretizing the chord the relationships (4.5) and (4.6) define the elements of $\mathbf{T}_{as}^{(1)} \in \mathbb{R}^{12(M+1) \times 5}$ in the structural to aerodynamic transformation

$$\begin{bmatrix} \partial \zeta' \\ \partial \zeta \end{bmatrix} = \mathbf{T}_{as}^{(1)} \mathbf{x}_s^{(1)}, \quad (4.7)$$

where the perturbations of lattice velocities on the left-hand-side are those required for the linearized aerodynamic model defined in Section 2.4. The mapping above is used to investigate the frequency-domain response of classical 3-DoF aerofoils, modelled using linearized aerodynamics, in Section 5.1.2.

Planar Wings

The linear deformations of finite-span planar wings, including planforms with sweep and taper, can be represented in terms of the linear degrees-of-freedom of a cantilever beam model as

$$\partial \zeta_{ij} = \frac{1}{c_{\text{ref}}} \left(\partial \mathbf{R}_j - \mathbf{C}_j^\top \tilde{\boldsymbol{\xi}}_{B_{ij}} \mathbf{T}_j \partial \boldsymbol{\Psi}_j \right) \quad (4.8)$$

using the same shorthands introduced for (4.3) and assuming the inertial and body-fixed frames are coincident. Similarly, the velocity perturbations can be expressed as

$$\partial \zeta'_{ij} = \frac{1}{c_{\text{ref}}} \left(\partial \mathbf{R}'_j - \mathbf{C}_j^\top \tilde{\boldsymbol{\xi}}_{B_{ij}} \mathbf{T}_j \partial \boldsymbol{\Psi}'_j + \mathbf{C}_j^\top \left(\left(\widetilde{\tilde{\boldsymbol{\xi}}_{B_{ij}} \mathbf{T}_j \boldsymbol{\Psi}'_j} \right) \mathbf{T}_j - \tilde{\boldsymbol{\xi}}_{B_{ij}} \mathbf{A}_{1j} \right) \partial \boldsymbol{\Psi}_j \right) \quad (4.9)$$

where $\mathbf{A}_{1_j} = \mathbf{A}_1(\Psi_j, \Psi'_j)$ which is a matrix-valued function defined in Section 3.1.1. Subsequently the mapping $\mathbf{T}_{as}^{(2)} \in \mathbb{R}^{6K_\zeta \times 12N_s}$ can be defined such that

$$\begin{bmatrix} \partial \zeta' \\ \partial \zeta \end{bmatrix} = \mathbf{T}_{as}^{(2)} \begin{bmatrix} \partial \eta' \\ \partial \eta \end{bmatrix}, \quad (4.10)$$

which transforms perturbations in the beam elastic degrees-of-freedom, defined at N_s discrete nodes, to the displacements and velocities of the aerodynamic grid points, of which there are K_ζ . This transformation is used to define nondimensional aerodynamic models of cantilever wings in terms of the modes of an underlying beam dynamics model in Section 5.3.

General Deformed Configurations

An unconstrained beam configuration is considered now, linearized about an arbitrary reference condition, with the linear states of (3.37). In this case the linearized displacements of the aerodynamic lattice can be written as

$$\partial \zeta_{G_{ij}} = \frac{1}{c_{\text{ref}}} \left(\partial \mathbf{r}_G + \mathbf{C}^{GA} \left(\partial \mathbf{R}_{A_j} - \mathbf{C}^{GA} \widetilde{\mathbf{P}}_{ijA} \mathbf{T}^A \partial \boldsymbol{\Theta} - \mathbf{C}_j^\top \widetilde{\boldsymbol{\xi}}_{B_i} \mathbf{T}_j \partial \Psi_j \right) \right) \quad (4.11)$$

where $\mathbf{P}_{ijA} = \mathbf{R}_{A_j} + \mathbf{C}_j^\top \boldsymbol{\xi}_{B_i}$, and $\mathbf{T}^A = \mathbf{T}(\boldsymbol{\Theta})$ is a tangential operator where $\boldsymbol{\Theta}(t)$ are the Euler angles parameterizing the orientation of the body-fixed frame with respect to the inertial frame. Taking perturbations of the lattice velocities, (4.3), results in their linear expression as

$$\begin{aligned} \partial \zeta'_{G_{ij}} = \frac{1}{2V_{\text{ref}}} \mathbf{C}^{GA} \left(\partial \mathbf{v}_A + \widetilde{\mathbf{P}}_{ijA} \partial \boldsymbol{\omega}_A - \widetilde{\dot{\zeta}}_{ijA} \mathbf{T}^A \partial \boldsymbol{\Theta} + \partial \dot{\mathbf{R}}_{A_j} + \mathbf{C}_j^\top \widetilde{\boldsymbol{\xi}}_{B_{ij}} \mathbf{T}_j \partial \dot{\Psi}_j \right. \\ \left. + \widetilde{\boldsymbol{\omega}}_A \partial \mathbf{R}_{A_j} + \left(\mathbf{C}_j^\top \widetilde{\boldsymbol{\xi}}_{B_i} \mathbf{A}_{1_j} - \widetilde{\boldsymbol{\omega}}_A \mathbf{C}_j^\top \widetilde{\boldsymbol{\xi}}_{B_i} \mathbf{T}_j \right) \partial \Psi_j \right), \end{aligned} \quad (4.12)$$

where $\mathbf{v}_A$ and $\boldsymbol{\omega}_A$ are the translational and angular velocities of the body-fixed frame defined in Section 3.1.1, $\widetilde{\dot{\zeta}}_{ijA} = \mathbf{C}^{AG} \widetilde{\dot{\zeta}}_{G_{ij}} \mathbf{C}^{GA}$ is the transformation of the underlying skew-symmetric matrix from the G - to the A frame, and $\mathbf{A}_{1_j} = \mathbf{A}_1(\Psi_j, \dot{\Psi}_j)$. Subsequently, on gathering (4.11)

and (4.12), the transformation $\mathbf{T}_{as}^{(3)} \in \mathbb{R}^{6K_\zeta \times 12N_s + 9}$ is defined such that

$$\begin{bmatrix} \partial \zeta' \\ \partial \zeta \end{bmatrix} = \mathbf{T}_{as}^{(3)} \begin{bmatrix} \partial \dot{\eta} \\ \partial \eta \\ \partial \beta \\ \partial \Theta \end{bmatrix}, \quad (4.13)$$

and is used to formulate coupled flexible aircraft models in Chapter 5.

4.1.3 Mapping aerodynamic forces to the structural model

After calculating the (dimensional) forces on the vertices of the deformed aerodynamic grid, $\mathbf{f}_{G_{ij}}^a$, they are mapped back onto the beam reference line resulting in forces, $\mathbf{f}_B^s(s_j, t)$, and moments $\mathbf{m}_B^s(s_j, t)$, defined in the local material frame-of-reference. Since there is one cross-section for each beam node, the nodal forces and moments are

$$\mathbf{f}_B^s(s_j, t) = \mathbf{C}(s_j, t) \mathbf{C}^{AG}(t) \sum_{i=1}^{M+1} \mathbf{f}_{G_{ij}}^a, \quad (4.14)$$

which is the sum of forces acting on the the j -th cross-section, and

$$\mathbf{m}_B^s(s_j, t) = \sum_{i=1}^{M+1} \widetilde{\boldsymbol{\xi}_{B_i}(s_j)} \mathbf{C}(s_j, t) \mathbf{C}^{AG}(t) \mathbf{f}_{G_{ij}}^a, \quad (4.15)$$

which is the sum of moments with moment arms $\boldsymbol{\xi}_{B_i}(s_j)$. In a linear FSI the transformation from aerodynamic forces to forces and moments on the beam is simply the transpose of one of the linearized displacement mappings of (4.6), (4.8) or (4.11) (depending on the complexity of the model) as alluded to in Section 4.1.2.

4.2 Aeroelastic models

Coupling the methods developed for unsteady aerodynamics and flexible-body dynamics, which are found in Chapter 2 and Chapter 3, respectively, with the fluid-structure interface model presented above (Section 4.1) allows a hierarchy of aeroelastic models to be formulated. The models presented in this section include nonlinear and linearized static aeroelastic models of clamped structures, nonlinear discrete-time aeroelastic models of the dynamics of wings

and aircraft, and linearized state-space (monolithic) formulations of unsteady aeroelastic behaviour that are convenient for stability analysis and control law synthesis.

4.2.1 Static aeroelastic analysis

An equilibrium condition for clamped structures can be defined between the elastic, aerodynamic and gravitational forces and can be expressed in the nonlinear setting as $\mathbf{Q}_{gyr} + \mathbf{Q}_{stif} = \mathbf{Q}_{ext}$ from (3.24) under static assumptions (no time-derivatives). These nonlinear static equilibria are computed in this work by iterating the aerodynamic and structural solvers until convergence, as shown in the flow diagram of Figure 4.2. In this case a steady form of the vortex-lattice-based aerodynamics of Section 2.1 is used in which time-derivatives are zero and a very long (but finite) wake is assumed that trails downstream from the deformed trailing-edge, approximating horse-shoe vortices [71].

Convergence of the coupled system is monitored by tracking the difference in discrete forces and structural displacements between successive iterations, with the criteria

$$\max (\|\Delta(\mathbf{Q}_{stif} - \mathbf{Q}_{ext})\|_{\infty}, \|\Delta\boldsymbol{\eta}\|_{\infty}) < \epsilon_1 \quad (4.16)$$

where the Δ indicates the difference between successive iterations, $\|\cdot\|_{\infty}$ is the infinity-norm. Linearized forms of either model can be substituted into the analysis framework depicted in Figure 4.2 as long as the deflections in such cases are sufficiently small – something that can be determined by comparing the nonlinear and linear analyses.

In cases where the static response approach aeroelastic divergence (where the problem is singular), the divergence dynamic pressure is obtained using extrapolation from results calculated at sub-critical dynamic pressures using a Divergence Index (DI) projection [57].

4.2.2 Nonlinear dynamic analysis

Integration of the coupled system, described by the discrete nonlinear beam equations (3.24) and the aerodynamics of (2.15) and (2.16), in time is done using a partitioned procedure [119] whereby separate integration schemes are used to advance the flexible-body and aerodynamic states. The structural equations are discretized in time using the Newmark- β scheme [41] described in Section 3.1.4, which for each time step, n , uses Newton-Raphson sub-iterations, k , to achieve a converged solution to (3.24). In the loosely-coupled (explicit) coupled time-

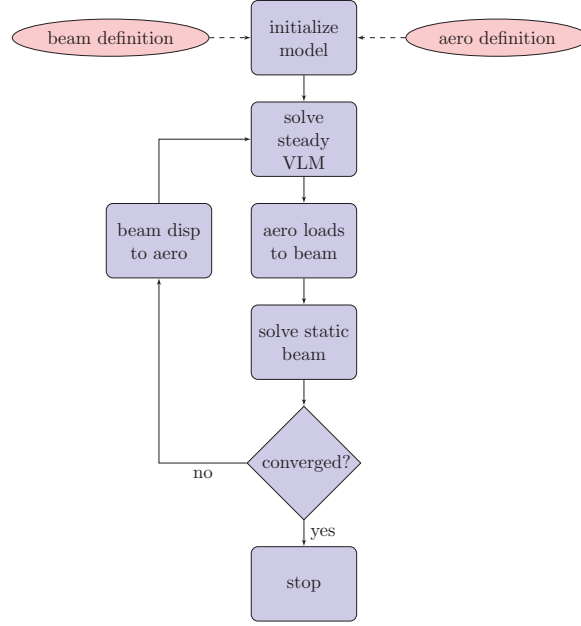


Figure 4.2: Information flow in the static aeroelastic solver.

integration scheme the coupled system residual, $\mathbf{r}$, has the form,

$$\mathbf{r}_{n+1}^k = \mathcal{M}(\boldsymbol{\eta}_{n+1}^k) \begin{Bmatrix} \ddot{\boldsymbol{\eta}}_{n+1}^k \\ \dot{\boldsymbol{\beta}}_{n+1}^k \end{Bmatrix} + \mathbf{Q}_{gyr_{n+1}}^k + \mathbf{Q}_{stiff_{n+1}}^k - \mathbf{Q}_{ext_n}, \quad (4.17)$$

in which the external forces are calculated once at the start of each time step, as indicated in the flow diagram of Figure 4.3. Therefore, (3.24) is not strictly satisfied at time $n + 1$ and the solution of structural and aerodynamic states are staggered. The convergence criteria is defined as $\|\mathbf{r}_{n+1}^k\|_\infty / \|\mathbf{r}_{n+1}^0\|_\infty < \epsilon_2$, and unless otherwise stated $\epsilon_2 = 10^{-5}$ is used.

4.2.3 Linearized models for stability analysis

The linearized aeroelastic models used in this work are formed by coupling the linear beam equations (3.38) with the linearized aerodynamics of (2.32) and (2.40) using the interface described above in Section 4.1.2. The dynamics of this coupled system, in the absence of control inputs and disturbances, can be expressed by the system of equations

$$\mathbf{E}_{\text{sys}} \mathbf{x}_{\text{sys}}^{n+1} = \mathbf{A}_{\text{sys}} \mathbf{x}_{\text{sys}}^n, \quad (4.18)$$

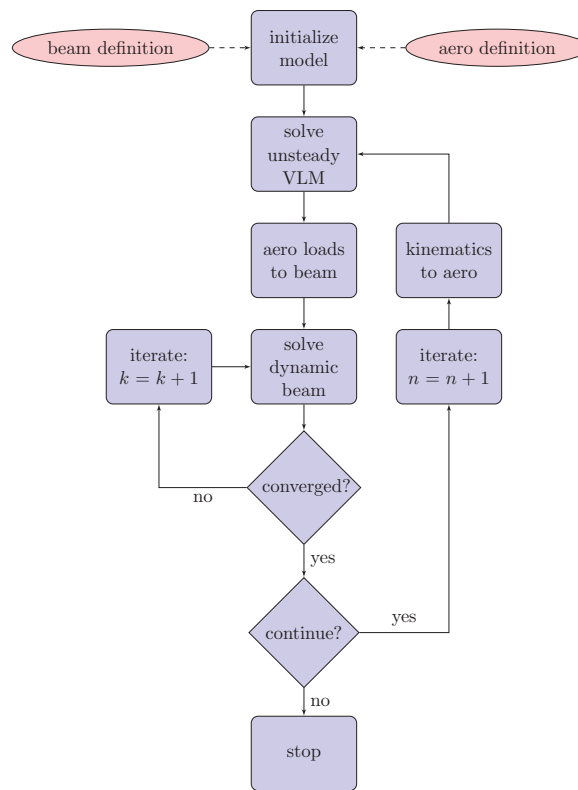


Figure 4.3: Information flow in the dynamic aeroelastic solver.

where the states of the full coupled system are defined as

$$\mathbf{x}_{\text{sys}} := [\mathbf{x}_a; \mathbf{x}_s] = \left[\partial\mathbf{\Gamma}; \partial\mathbf{\Gamma}_w; \partial\dot{\mathbf{\Gamma}}; \partial\dot{\boldsymbol{\eta}}; \partial\boldsymbol{\eta}; \partial\boldsymbol{\beta}; \partial\boldsymbol{\Theta} \right] \in \mathbb{R}^{n_{\text{sys}}}, \quad (4.19)$$

in which $n_{\text{sys}} = 2K + K_w + 12(N_s - 1) + 9$ where K is number of vortex rings on the body of the configuration, K_w is the number in the wake, and N_s is the total number of finite-element nodes used to discretize the beam model. Note that the aerodynamic states $\left[\partial\mathbf{\Gamma}; \partial\mathbf{\Gamma}_w; \partial\dot{\mathbf{\Gamma}} \right]$ are expressed in their dimensional form, and the matrices $\mathbf{E}_{\text{sys}}$ and $\mathbf{A}_{\text{sys}}$ are functions of the reference condition at which the linearization is obtained.

Stability of the aeroelastic system (4.18) is determined by solving the generalized eigenproblem

$$\mathbf{E}_{\text{sys}}(\rho_\infty, V_\infty) \mathbf{x}_{\text{sys}_i} = z_i \mathbf{A}_{\text{sys}}(\rho_\infty, V_\infty) \mathbf{x}_{\text{sys}_i} \quad (4.20)$$

where z_i is the i -th discrete-time eigenvalue (generally complex) and $\mathbf{x}_{\text{sys}_i} \in \mathbb{C}^{n_{\text{sys}}}$ the corresponding eigenvector. The free-stream density ρ_∞ and velocity V_∞ affect the aeroelastic coupling through a scaling on the aerodynamic forces (with the dynamic pressure) and through the (true) airspeed, which introduces an additional effect due to changes in the effective reduced frequency at different altitudes. If the deformed aircraft configuration, obtained through a trimming procedure, is used as the reference condition during linearization of the coupled equations, then finding the unstable conditions of (4.20) may be referred to as *matched flutter* analysis in the context of flexible aircraft flight dynamics [93].

4.3 Implementation framework

The methods described in Chapters 2, 3 and 4 are implemented in a Python-based modular framework called *Simulation of High-Aspect-Ratio Planes in Python* (SHARPy). High-level scripting (defining/running simulations, post-processing results...) is performed using Python 3.2 (www.python.org) while computationally-intensive tasks are executed by pre-compiled libraries written in Fortran90, in the case of beam finite-element calculations, or C/C++ in the case of the aerodynamics methods. Thus the relatively-fast development afforded by a scripting language is complemented by the performance advantages offered by compiled code. Maintaining control over the large multi-language code-base is facilitated by

the Git³ version control system (www.git-scm.com), documentation is produced automatically directly from the source code using Doxygen (www.doxygen.org), and automated testing is performed using Python's native `unittest` module.

The SHARPy framework, depicted in Figure 4.4, is comprised of Python modules that are combined to provide aeroelastic analysis. The aerodynamic and structural routines are defined independently and can be easily loaded into a new development effort using Python's `import` statement. The PyAero module contains the general nonlinear aerodynamic method and routines required to create the matrices for the linear aerodynamics model of Chapter 2, while the PyBeam module contains linear/nonlinear static and dynamic solvers and routines for the calculation of beam finite-element matrices. Coupling method for these models are defined in the PyFSI module which facilitates static and dynamic aeroelastic analysis routines defined in the PyCoupled module. Moreover, each of these modules have `unittests` associated to their constituent methods which aid debugging of the decoupled, and coupled, implementations.

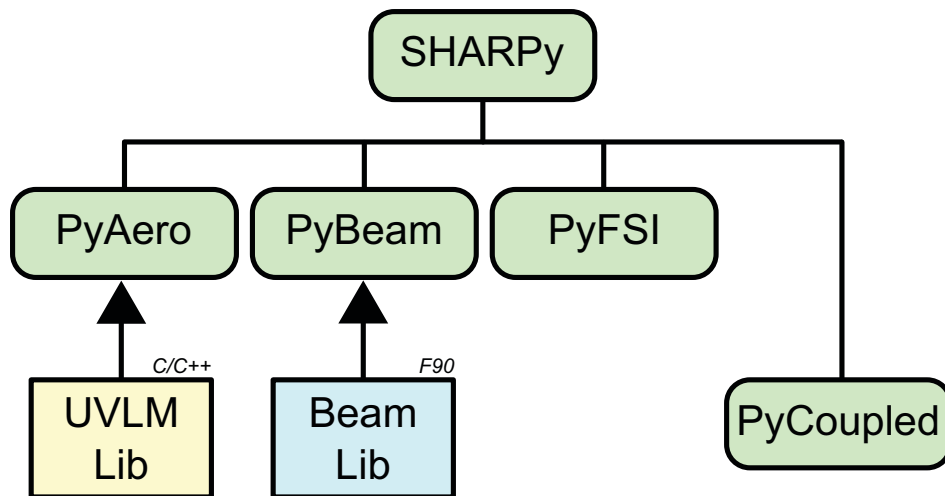


Figure 4.4: Python modules in the Simulation of High-Aspect-Ratio Planes (SHARPy) framework.

³The code is also hosted on GitHub at <https://github.com/aeroelastics/SHARPy.git>.

Chapter 5

Numerical Investigations with Full-Order Systems

In this chapter the analysis methods presented so far, which are implemented in the **SHARPy** framework, are applied to problems in aerodynamics, aeroelasticity and flexible aircraft dynamics. Verification and comparison is sought with other numerical approaches, analytical solutions, and experimental data where possible. In doing so, the vortex-lattice-based aerodynamics and the fluid-structure interface bear much of the focus, since the beam models have been validated extensively in the work of Palacios [104] and Hesse [59, 58]. Extensions beyond the cases which can be verified are made when their validity can be reasonably induced, culminating in the aeroelastic analysis of a swept flying-wing configuration at the end of the chapter.

The aerodynamic models of Chapter 2 are treated in isolation first, in Section 5.1. A verification of the general nonlinear formulation, including both the *Joukowski* and *Katz & Plotkin* force calculation methods, is sought for finite wings in steady-state conditions. Following this, the unsteady induced-drag of aerofoils oscillating in pitch and plunge degrees-of-freedom is investigated and comparison is made with the closed-form solution of Garrick [37] before extending the study to finite wings. The spatial and temporal convergence of linearized aerodynamic models of aerofoils with pitch, plunge and trailing-edge-flap degrees-of-freedom is then investigated. Comparison with the linear solutions of Theodorsen [141] (for aerofoil motions) and Sears [132] (for vertical gusts) is made, and the convergence of the vortex-based method assessed in the frequency domain.

The static aeroelastic response of high-aspect-ratio wings is then investigated using the coupled methods of the **SHARPy** framework in Section 5.2. Initially a convergence exercise on the beam finite-elements is carried out to assess the error due to non-objectivity in the spatial approximation of the finite (large) three-dimensional rotations expected in the response. Following this a comparison is made with high-fidelity computations of a representative high-altitude long-endurance (HALE) wing [106, 137] under aerodynamic loading to verify the coupled aeroelastic model for static cases.

Flutter of the Goland wing [44, 45] is then investigated by coupling the linearized aerodynamics equations, in which the input/output behaviour is redefined in terms of the modal degrees-of-freedom of a beam model, with the state-space description of structural dynamics in Section 3.2.3. Efficient stability analysis is facilitated by the nondimensional form of the aerodynamic model, which is also written directly in discrete-time form and without the approximations made in frequency-domain based approaches, such as the doublet-lattice method [2, 35]. Convergence of the aerodynamics method in modal coordinates is investigated, and comparison of the flutter characteristics are then made with analytical and numerical predictions by other authors to verify the implementation in dynamic cases, in Section 5.3.

Finally, the aeroelasticity of swept flying-wings is investigated in Section 5.4 in which the dynamic phenomenon of body-freedom flutter is predicted. To build confidence in the coupled model, the static and dynamic aeroelastic response of swept, isotropic cantilever wings calculated using **SHARPy** are verified against experimental results and numerical predictions from other authors [122]. Following this, a representative swept flying-wing model is defined and the unrestrained natural modes and body-freedom flutter characteristics are investigated under variations of the aircraft mass distribution.

5.1 Aerodynamic test cases

In this section the implementation of aerodynamic methods, described in Chapter 2 and implemented in the **PyAero** module of the **SHARPy** framework, are tested on problems of finite wings and aerofoils in steady and unsteady conditions. In the cases where aerofoil (2-D) models are required the vortex-lattice is defined such that there is a single spanwise panel with a very large spanwise dimension (10^5 chord-lengths). Confidence in the methods is built by comparing the lift, drag and moments calculated in different kinematic conditions with

closed-form analytical solutions, other numerical approaches and experimental results.

5.1.1 Verification of the nonlinear method

The nonlinear form of vortex-lattice-based aerodynamics, described in Sections 2.1 and 2.2 and summarised by the general, nonlinear, discrete-time equations defined in (2.15)-(2.18), is investigated here. Throughout the comparison with alternative results, both of the force calculation methods — namely the *Joukowski* and *Katz & Plotkin* methods — are used, firstly, for cross-checking purposes, and secondly, with a view to ascertaining their convergence properties in static and dynamic problems.

Steady-State Loads

Lifting characteristics of planar wings are considered first to validate the modelling approach in problems featuring the three-dimensional effects of finite-span and wing sweep. The effect of varying the wing aspect-ratio on lift coefficient slope is shown for two wing planforms in Figure 5.1; results from a simple rectangular planform are shown, alongside those from a wing with 45 degrees sweep-back. It can be observed that increasing the aspect-ratio, and increasing sweep angle, reduces the lift coefficient slope. This effect is expected since the tip vortices induce downwash over the outboard region of the wing, which tilts the resulting force vector rearwards, reducing lift, and increasing induced drag. The comparison between experimental results, [71]’s vortex-lattice method (VLM), and the current VLM implementation, made in Figure 5.1, is very good. Also, both force calculation methods in the current implementation predict an almost identical lift-curve slope for both wing configurations and all aspect-ratios.

Steady drag calculations are considered next. In both this steady analysis, and the unsteady analysis presented in the next section, it is the induced drag, or drag-due-to-lift [75], that is presented. The total drag includes contributions from the induced drag, also referred to as pressure drag, and the viscous drag – no attempt has been made here to include viscous drag. For validation of steady induced drag, the VLM implemented in **SHARPy** is compared to Prandtl’s lifting-line method (see, for example, [5]).

The Lifting Line method models a lifting surface as an array of horseshoe vortices, and is only suitable for wings of moderate or high-aspect-ratio since there is no resolution of the surface’s chordwise extent. This is evident above in figure 5.2(a) where the lifting line drag prediction for low aspect-ratios is higher than both the VLM drag prediction methods,

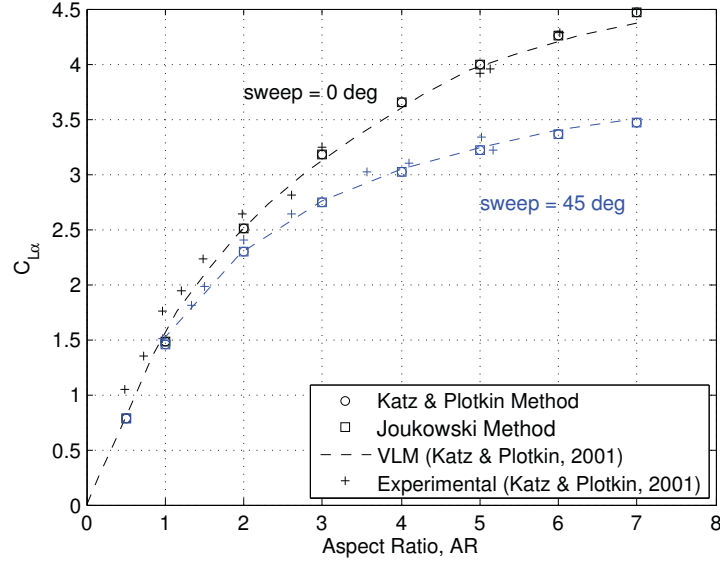
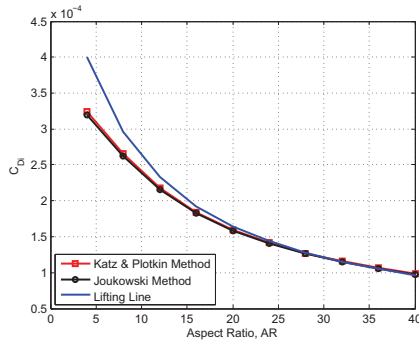
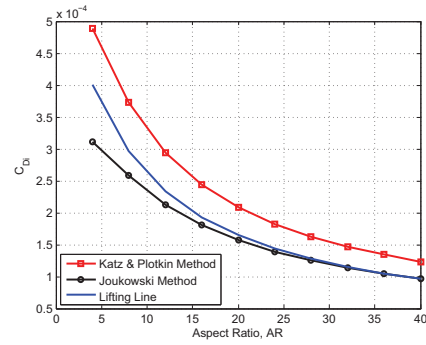


Figure 5.1: Effect of aspect-ratio and sweep on the lift coefficient slope of planar wings. VLM and experimental results from [71]. Zero sweep results are shown in black, while 45 deg sweep-back results are shown in blue.



(a) converged with respect to discretisation



(b) 1 chordwise panel

Figure 5.2: Steady drag coefficients for flat plates of varying aspect ratio at 1 deg AoA.

which include ample chordwise resolution. For lifting surfaces with an aspect ratio of approximately 20 or above the drag predictions have coalesced as any chordwise dimension is small in comparison to the span of the wing.

The drag predictions from a limiting case using only 1 chordwise panel are shown in figure 5.2(b), which reveals the subtle differences in the solution methodologies. In the method of Katz and Plotkin [71], the downwash used to calculate induced drag is calculated at the panel 3/4 chord, however the spanwise bound vorticity is located at the panel 1/4 chord line; this

causes an over-prediction in drag that vanishes with increasing numbers of chordwise panels. The Joukowski method is similar to lifting-line in that both calculate the induced drag by considering downwash at the precise location of spanwise vorticity, hence the good agreement. The Katz and Plotkin solution methodology using only one chordwise panel is equivalent to the method of Weissinger [151], developed shortly after the second world war for the analysis of swept planforms.

Unsteady Induced-Drag

The simulations here are of flat-plates exhibiting small-amplitude oscillations with fully-developed linear wakes. Garrick [37] wrote a closed-form solution for the drag, or thrust, generated by oscillations in plunge, h , positive down, and pitch, α , positive nose-up. While the assumptions he made relating to the aerofoil kinematics are the same as Theodorsen's [141] (linearity) the drag force is quadratic in the aerofoil degrees of freedom. For harmonic plunging motions, $h = -\bar{h} \cos(ks)$, Garrick's solution can be written in non-dimensional form as [49]

$$C_d(s) = -2\pi k^2 \frac{\bar{h}^2}{b^2} [G(k) \cos(ks) + F(k) \sin(ks)]^2, \quad (5.1)$$

where the sectional drag coefficient, C_d , is presented as a function of non-dimensional time $s = \frac{V_\infty t}{2c_{ref}}$, and reduced frequency $k = \frac{\omega c_{ref}}{2V_\infty}$, which is the same definition used by Theodorsen and in the discrete-time aerodynamics of this thesis (see Section 2.4 and equation (2.20)). The transfer functions $F(k)$ and $G(k)$ are, respectively, the real and imaginary parts of Theodorsen's function [141], and the plunging amplitude is nondimensionalized with the semi-chord, $b = \frac{c_{ref}}{2}$, for consistency. Inspection of (5.1) reveals that there is a propulsive force for plunging motions at all reduced frequencies that is proportional to both the square of the reduced frequency and the square of the plunging amplitude.

For pitching, an additional parameter defines the chordwise location of the center of rotation, a , which is non-dimensionalized by the semi-chord and measured from the mid-chord as positive in the aft direction. The induced drag is then given by

$$C_d(s) = \alpha C_l(s) - C_s(s), \quad (5.2)$$

where the suction force and lift coefficients, for pitching motions of the type $\alpha = \bar{\alpha} \sin(ks)$,

are, respectively,

$$C_s(s) = \frac{\pi \bar{\alpha}^2}{2} [\Upsilon_1 \sin(ks) + \Upsilon_2 \cos(ks)]^2, \quad (5.3)$$

and

$$\begin{aligned} C_l(s) = \pi \bar{\alpha} \bigg[& k \cos(ks) + ak^2 \sin(ks) \\ & + 2F(k) \left(\sin(ks) + \left(\frac{1}{2} - a \right) k \cos(ks) \right) \\ & + 2G(k) \left(\cos(ks) - \left(\frac{1}{2} - a \right) k \sin(ks) \right) \bigg], \end{aligned} \quad (5.4)$$

where $\Upsilon_1 = 2 [F(k) - kG(k) (\frac{1}{2} - a)]$, and $\Upsilon_2 = 2 [G(k) + kF(k) (\frac{1}{2} - a)] - k$. The results of (5.1) and (5.2) are compared to those computed with the unsteady vortex-lattice-based aerodynamics method in **SHARPy**, which will be referred to as the UVLM, in Figure 5.3. Here phase plots of induced drag against kinematics are shown for a given discretisation at two reduced frequencies. The agreement between the UVLM results and those of the analytical theory is good with the exception of the Katz & Plotkin method for $k = 1$ plunging (5.3(b)) which requires a finer mesh for convergence.

This is investigated further through a convergence exercise on both force calculation methods with respect to the chordwise discretization. The results for both plunging and pitching motions can be found in Figure 5.4, where the error is defined as

$$error = \frac{\text{RMS}(C_{d_{UVLM}} - C_{d_{Garrick}})}{\max_t |C_{d_{Garrick}}|}. \quad (5.5)$$

The convergence of both methods for plunging oscillations at $k = 0.1$ is shown in Figure 5.4(a). In this case the convergence is very fast, with errors less than 1% for all discretizations shown. For higher reduced frequencies the convergence pattern is similar, with the Joukowski method converging marginally faster than the Katz and Plotkin method (Figure 5.4(b)). Convergence is also shown for pitching oscillations at $k = 0.1$ in Figure 5.4(a) using the same error metric, (5.5). Convergence of the Katz & Plotkin method and the Joukowski method are similar in this case. Increased spatial resolution is required at higher reduced frequencies for both pitch and plunge motions, as shown in Figure 5.4(b). For a given discretisation, both methods show increasing discrepancies with the analytical theory as the frequency increases.

A low aspect-ratio finite wing ($AR = 4$) is simulated with the UVLM to further investigate

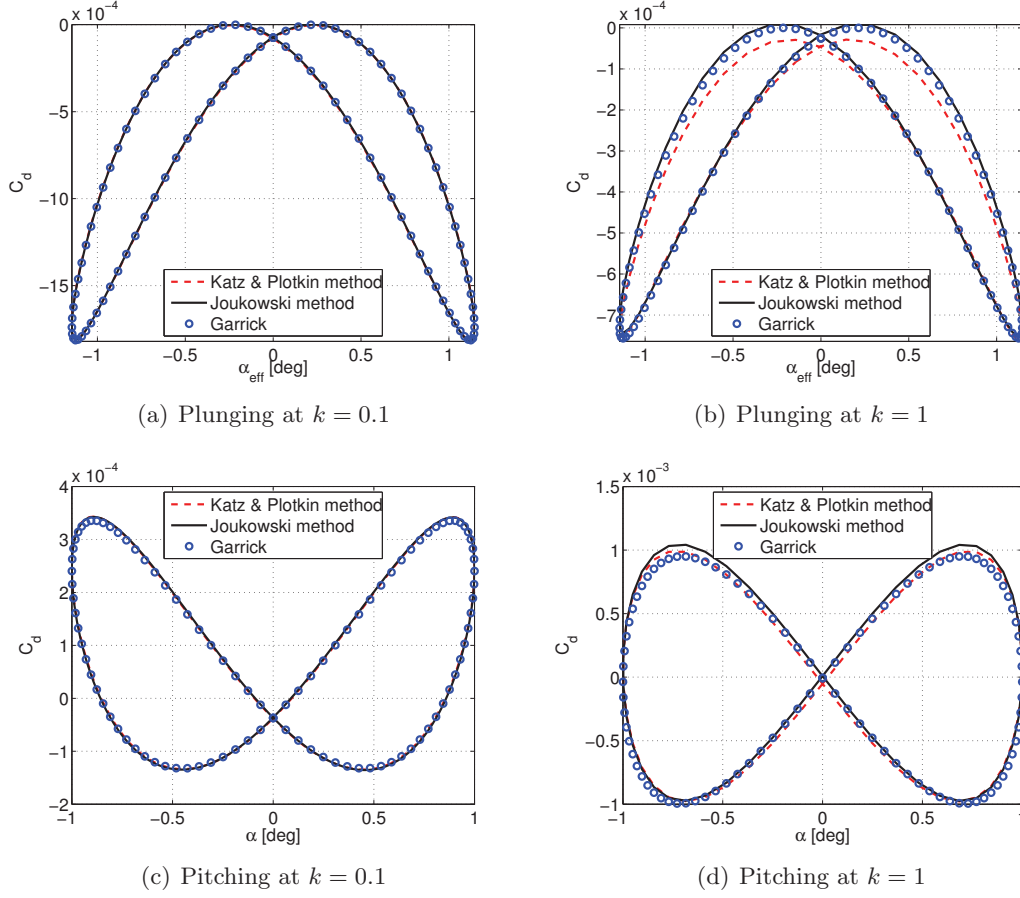


Figure 5.3: Unsteady induced-drag for sinusoidal oscillations in plunge and pitch. In plunge $\alpha_{\text{eff}} = \tan^{-1} \left(k \frac{\bar{h}}{b} \sin(ks) \right)$ with $\frac{\bar{h}}{b} = 0.2$ at $k = 0.1$, and $\frac{\bar{h}}{b} = 0.02$ at $k = 1$. In pitch $\bar{\alpha} = 1$ deg, and $a = -0.5$. $M = 16$ and $\frac{\Delta t V_\infty}{c} = \frac{1}{16}$ in all cases.

the convergence of the force calculation methods in a three-dimensional problem. The steady results, at a constant angle of attack, are shown alongside results from pitching oscillations about the quarter-chord at different reduced frequencies (Figure 5.5) in order to ascertain any additional effect that reduced frequency may have on convergence. In this case the error has been normalized with respect to the Joukowski method result from the most densely panelled simulation at each frequency.

The aerodynamics of a finite wing undergoing harmonic oscillations is subject to three length-scales [1]: chord, c ; span, B ; and the wake wavelength, $\lambda = \frac{\pi c}{k}$. Typically, low-aspect-ratio wings constitute a problem that is very three-dimensional as $B \approx c$, and care must be taken to discretise the wing appropriately in both the spanwise and chordwise directions. The steady result (at $k = 0$) where $\lambda \rightarrow \infty$, and therefore $\lambda \gg c$, shows the convergence of drag

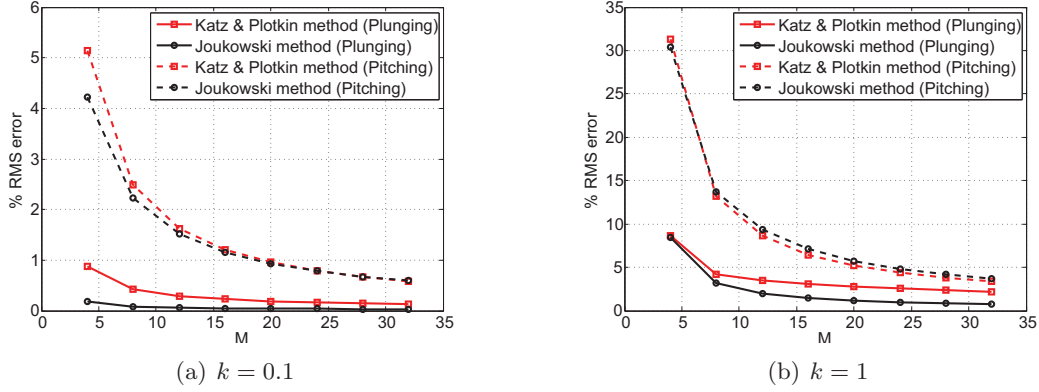


Figure 5.4: RMS error in aerofoil drag as a function of M , the number of chordwise panels. Plunging motion with $\bar{h}/b = 0.2$ at $k = 0.1$, and $\bar{h}/b = 0.02$ at $k = 1$. Pitching motion with $\bar{\alpha} = 1$ deg, and $a = -0.5$. $\frac{\Delta t U_\infty}{c} = \frac{1}{M}$ in all cases.

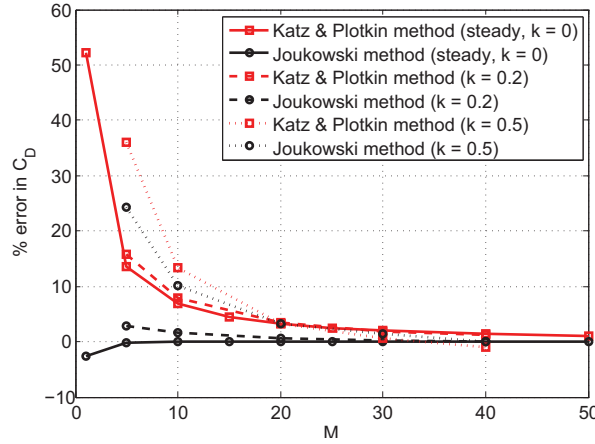


Figure 5.5: Convergence of steady drag, and unsteady mean drag, with respect to chordwise panelling for a flat-plate of aspect-ratio 4. Results for $\alpha = 5^\circ \cos(ks)$, $\frac{\Delta t U_\infty}{c} = \frac{1}{M}$, and $N = 50$.

calculations with the Katz & Plotkin method is slower than the Joukowski method in such a problem. This is likely due to the details of the discretisation highlighted in the results of Figure 5.2(b), above. In the case of a high reduced frequency (i.e when $\lambda \approx c$) the problem becomes more two-dimensional [1]. This effect is observed in the results of Figure 5.5 as the coalescence of error curves with increasing reduced frequency. The methods converge very similarly with respect to spanwise panelling for the case studied, although in general this cannot be expected.

The comparisons made above, for the induced-drag of plunging and pitching flat-plate

aerofoils, show that both UVLM force calculation methods converge to the analytical solutions as finer discretisations are used. The three-dimensional results of this section go further and show that the Joukowski method offers the same level of accuracy for a much coarser discretisation compared to the Katz & Plokin method. Therefore, for three-dimensional problems the Joukowski method presents itself as a desirable option, especially for applications featuring low-frequency unsteadiness, such as the flight dynamics of flexible aircraft.

The results of Figure 5.5 were calculated with prescribed wakes, and the effects of a free-wake, or *wake roll-up*, which may be included in the wake convection equation of (2.4), are left out. For this particular single-surface problem, where there is no interference from the wake of an upstream surface, the inclusion of free-wake non-linearity was found to have a negligible effect up to reduced frequencies of $k = 1$. This is probably because the influence of wake vorticity on the surface it is shed from decays rapidly as it is convected downstream. A comparison of prescribed and free wakes is shown in Figure 5.6 for the same surface geometry used in the convergence analysis above. A large difference in wake geometry is observed when wake roll-up is included, although this doesn't necessarily imply a significant effect on the resulting aerodynamic forces. Quantification of the effect of wake roll-up on flexible aircraft dynamics in which multiple surfaces are present, and wake-interference may occur, remains largely untreated due to the bottleneck presented by the roll-up procedure. The efficient solutions to this problem, which are discussed in Section 1.1.2, have been found not to be critical for tackling the problems presented in this thesis and are left to future research efforts.

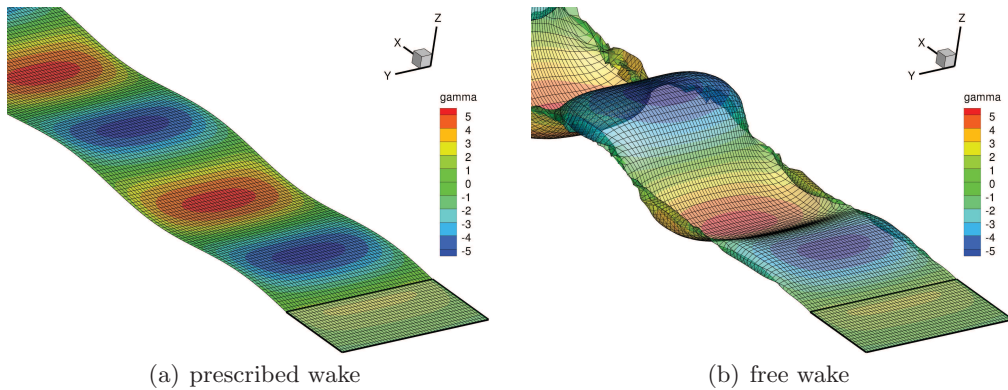


Figure 5.6: An illustration of wake geometries in the UVLM. Pitching of an aspect-ratio 4 wing about its quarter-chord with $\alpha = 5^\circ \sin(ks)$, $k = 1$, $M = 16$, $N = 50$, and $\frac{\Delta t U_\infty}{c} = \frac{1}{M}$ is shown.

5.1.2 Linearized vortex-lattice-based unsteady aerodynamics of a 2-D aerofoil

The frequency-domain aerodynamic response of aerofoils is now investigated using the linearized model of Section 2.4, defined by equations (2.32) and (2.40). The general inputs of this linear model, which are perturbations of the lattice geometry and velocities, are written in terms of the pitch angle, α (with time-derivative α'), about the quarter chord, and the angle of a trailing-edge flap, β (with time-derivative β'), hinged at the three-quarter-chord, using the relationships defined in (4.7). Magnitude and phase plots of the lift and moments (about the quarter-chord) that result from oscillations in these degrees-of-freedom are presented in Figure 5.7 over a range of reduced frequencies. As the number of vortex rings, M , discretizing the chord is increased the response obtained from the vortex-based method converges to Theodorsen's analytical solution. In some cases a very fine discretization is required to model the phase of the response accurately, particularly in the response of pitching moment to pitch angle, shown in Figure 5.7(b). In all other cases, and under a reduced frequency of 0.5, using 20 vortex rings to discretize the chord gives a close match to Theodorsen in the magnitude and phase of the response.

The aerofoil is also subject to a travelling vertical gust that moves with the airspeed V_∞ and has an instantaneous known upwash $\hat{w}_g(t)$ at the leading edge. If the upwash due to gusts on the airfoil at time t_n is $\boldsymbol{\nu}_g^n \in \mathbb{R}^{M+1}$, the upwash at time t_{n+1} will be obtained by the lag operator

$$\begin{aligned} \nu_{g,1}^{n+1} &= w_g(t_{n+1}) \\ \nu_{g,j}^{n+1} &= \nu_{g,j-1}^n \quad \text{for } j = 2, \dots, M+1, \end{aligned} \tag{5.6}$$

where the non-dimensional gust velocity $w_g := \frac{\hat{w}_g}{V_\infty}$ has been defined. For a given time-history of the gust (5.6) can be solved alongside the aerodynamic system equations. In this work the state-space aerodynamics model is augmented with the gust input w_g , gust states $\boldsymbol{\nu}_g$, and the dynamics of (5.6). The lift response to a sinusoidal gust at the leading-edge is calculated and compared to the analytical solution of Sears [132] in Figure 5.7(e). Similar convergence properties are observed in the response to those of the aerofoil degrees-of-freedom in the rest of Figure 5.7.

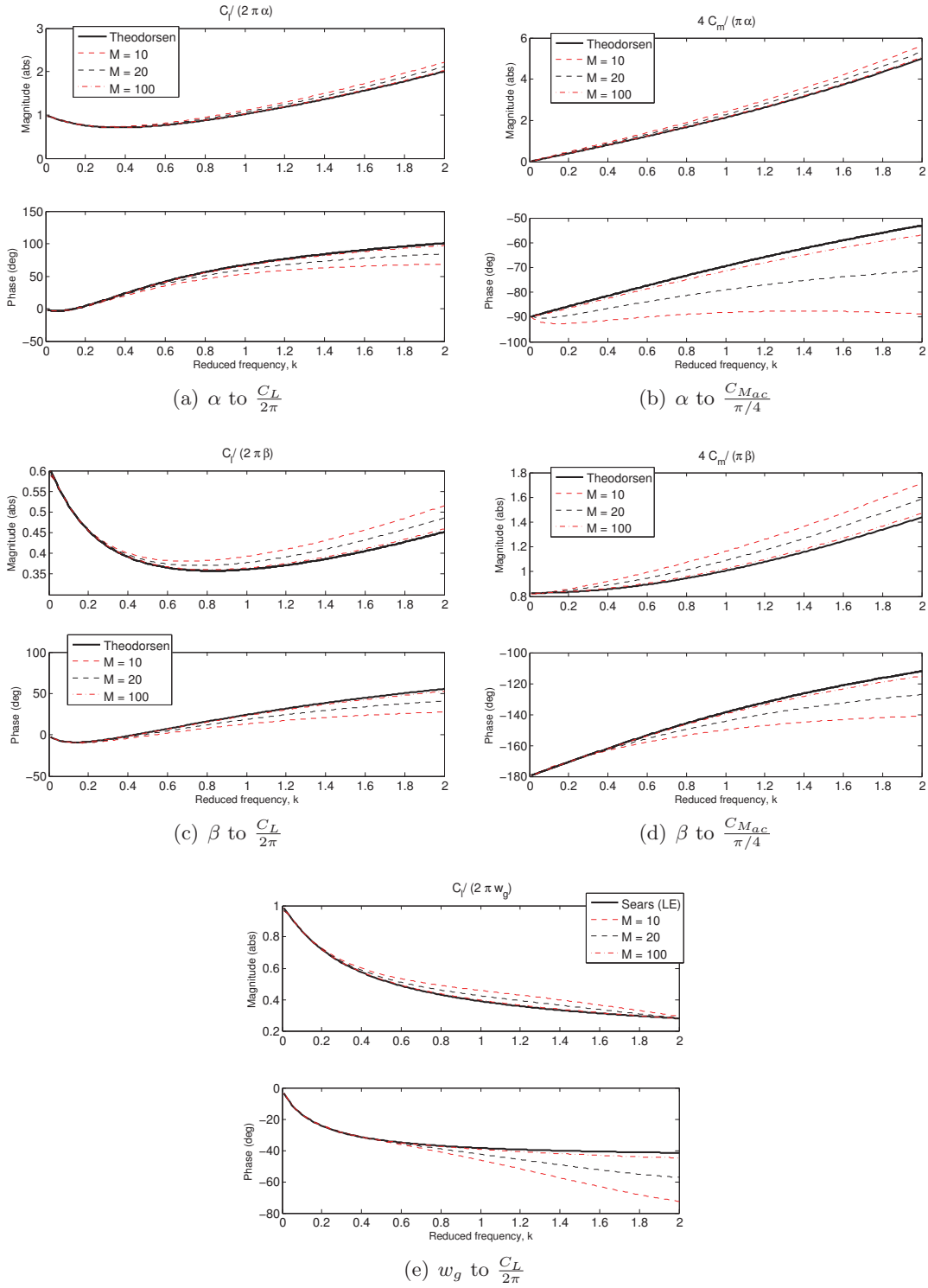


Figure 5.7: Frequency-response functions between incident gust velocity, angle of incidence and flap deflection (inputs), and lift and moment coefficients (outputs). Results are shown from analytical models, and the linearized UVLM with $M_w/M=30$ and $\Delta t = \frac{2}{M}$. $x_{ea} = 1/4$ and $x_{fh} = 3/4$.

5.2 Static aeroelasticity of high-aspect-ratio wings

High-aspect-ratio wings are analysed using the nonlinear beam finite element method described in Chapter 3, implemented in the `PyBeam` module of the `SHARPy` framework. After an analysis of the convergence of the beam finite-elements in situations featuring large rotations in 3-D, the nonlinear aerodynamic solver is coupled with the beam model to provide nonlinear static aeroelastic analysis of a representative high-altitude long-endurance (HALE) aircraft wing, which is used as a verification case.

5.2.1 Objectivity considerations in the structural model

As mentioned above, the structural model has been extensively tested by Hesse [58]. However, the flexible aircraft flight dynamics analysis of that work was carried out using 2-noded (linear) elements, which are known to have very slow convergence. Higher-order elements are therefore investigated here.

The test case used for the work presented here is defined in the book by Geradin and Cardona [40]. It is a cantilever beam of length $L = 5$ m, stiffness constants: $EA = 4.8 \times 10^8$ N, $GA = 3.231 \times 10^8$ N (in both axes), $GJ = 1.0 \times 10^6$ Nm, $EI = 9.346 \times 10^6$ Nm (in both axes). Table 5.1 shows the geometrically-nonlinear tip displacements for different discretizations under a vertical *dead-load* at the tip of -600 kN. All results were obtained with 10 load sub-steps and the convergence parameter, ϵ_1 in Section 4.2.1, was set to 10^{-5} . Convergence of computed tip displacements and rotations with published results is observed for both 2- and 3-noded elements as the number of elements increases.

Model	ΔR_1 (m)	ΔR_3 (m)	Ψ_2 (rad)
5 elements, 2 nodes	0.586	-2.147	-0.6745
10 elements, 2 nodes	0.594	-2.156	-0.6726
20 elements, 2 nodes	0.596	-2.159	-0.6722
2 elements, 3 nodes	0.550	-2.070	-0.6576
5 elements, 3 nodes	0.589	-2.144	-0.6700
10 elements, 3 nodes	0.596	-2.159	-0.6719
(Geradin and Cardona,2001)		-2.159	-0.6720

Table 5.1: Tip displacements and rotations under tip dead force (600 kN)

In the case above the beam deformation occurs in-plane, but specifically in a plane defined by the spatial (as opposed to material) coordinate system (in this case the $x - z$ plane).

Consequently, the Cartesian rotation vector (CRV) has only one component and interpolation of rotations is not problematic. On inspection of Table 5.1 it appears that using either element-type leads to rapid convergence for such problems. However, it will be seen that for practical problems, in which deformations are outside of any one plane defined by the spatial axes, the choice of element has a significant effect on the accuracy of interpolated rotations.

As mentioned above, the interpolation of rotations should guarantee *objectivity*, that is, invariance of the strain field under rigid-body rotations [27, 13]. This effect is investigated in the results of Table Table 5.2, which are obtained solving the cantilever beam under a tip follower-force of 3000 kN for different azimuth locations on the $x - y$ plane of the undeformed beam. The CRV in the solution includes the azimuth angle, and results in the table are rotated back to the plane of the undeformed beam. Both 2- and 3-noded elements are used in the analysis, and it should be noted that a model with 50×3 -noded elements has the same number of degrees of freedom as one with 100×2 -noded elements. As it can be seen, the quadratic interpolation on the three-noded element provides a much better approximation to the CRV than the linear interpolation on the two-noded element. The 3-noded quadratic elements are therefore favorable for analysis of structures with large deflections using the nonlinear beam method in this work.

Azimuth	Model	Ψ_1 [rad]	Ψ_2 [rad]	Ψ_3 [rad]
0°	50 elements, 3 nodes	0	-2.7614	0
90°	50 elements, 3 nodes	5.577×10^{-8}	-2.7614	-4.085×10^{-7}
180°	50 elements, 3 nodes	1.408×10^{-8}	-2.7614	-6.938×10^{-7}
0°	100 elements, 2 nodes	0	-2.7613	0
90°	100 elements, 2 nodes	9.625×10^{-5}	-2.7613	-1.024×10^{-3}
180°	100 elements, 2 nodes	-3.812×10^{-5}	-2.7612	-1.336×10^{-3}
0°	10 elements, 3 nodes	0	-2.7553	0
90°	10 elements, 3 nodes	2.339×10^{-5}	-2.7553	-1.621×10^{-4}
180°	10 elements, 3 nodes	4.858×10^{-6}	-2.7553	-3.010×10^{-4}
0°	20 elements, 2 nodes	0	-2.7597	0
15°	20 elements, 2 nodes	5.306×10^{-4}	-2.7596	-4.583×10^{-3}
90°	20 elements, 2 nodes	2.333×10^{-3}	-2.7589	-2.453×10^{-2}
180°	20 elements, 2 nodes	-7.965×10^{-4}	-2.7562	-3.235×10^{-2}

Table 5.2: Tip rotations (in the plane of the undeformed beam) under a tip follower force of 3000 kN for varying azimuth orientations of the undeformed beam in the $x - y$ plane.

5.2.2 Nonlinear static aeroelastic response of a very flexible wing

To verify the aeroelastic coupling, a representative HALE wing, introduced in [106] and later analyzed in [137] using a nonlinear-beam / Euler-equation-CFD combination, is analyzed using the present method. The pertinent wing properties are given in table 5.3.

Chord	1 m
Semi-span	16 m
Elastic axis	50% chord
Center of gravity	50% chord
Mass per unit length	0.75 kg/m
Moment of inertia	0.1 kg·m
Torsional stiffness	1×10^4 N·m ²
Bending stiffness	2×10^4 N·m ²
Chordwise bending stiffness	5×10^6 N·m ²

Table 5.3: HALE wing properties.

The static deflection of the wing is analyzed at two angles of attack in a free-stream flow of velocity $V_\infty = 25 \text{ ms}^{-1}$. The free-stream density is set as $\rho_\infty = 0.08891 \text{ kg}\cdot\text{m}^{-3}$ corresponding to the U.S Standard Atmosphere at an altitude of 20 km. 25 load steps are used to approach full aerodynamic load conditions, and the minimum required static residual is set as $\epsilon_1 = 10^{-4}$. 20, 3-noded beam elements are required for a converged static tip-deflection, corresponding to 40 spanwise panels in the aerodynamic model. 10 chordwise panels are also required, and the steady wake is approximated using vortex rings that stretch 100 chord-lengths downstream. The Joukowski force calculation method is used in the aerodynamic analysis.

The results of Figure 5.8(a) show excellent agreement with the work of Smith [137]. This is perhaps to be expected as the free-stream velocity is so low that compressibility effects, present in the Euler equation CFD used in [137] but not in the vortex-lattice method, are negligible. Perhaps the most significant modeling difference is in the treatment of the wing geometry (NACA0012 section in CFD versus flat-plate in UVLM) especially near the tip – this may explain the discrepancy observed at $\alpha = 4$ deg in the twist results of 5.8(c), which are more sensitive to the location and orientation of forces at the local centre-of-pressure. Since the effect of gravitational forces can be very significant in flexible wings, they have been included in Figures 5.8(b) and 5.8(d) for comparison. The acceleration due to gravity used in those results also correspond to that of the U.S Standard Atmosphere at 20 km.

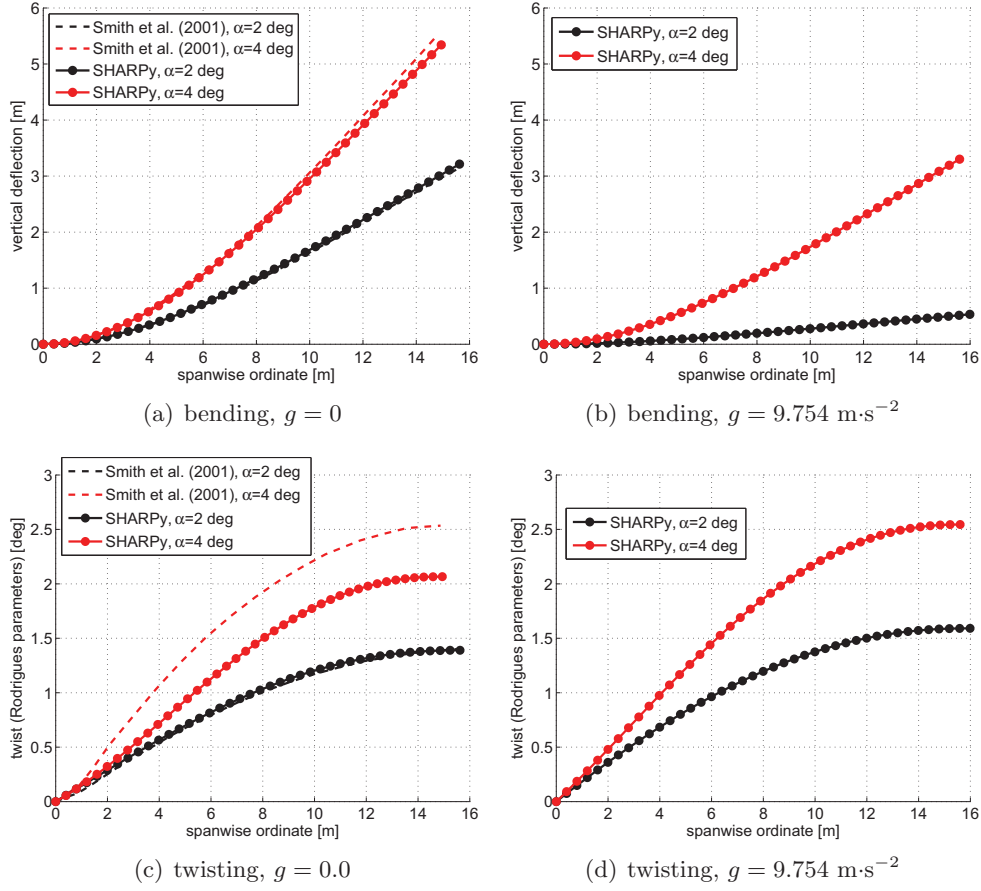


Figure 5.8: Bending and twisting deflections of the HALE wing at two angles of attack, with and without the force of gravity. $M = 10$, $N = 41$ (20, 3-noded elements).

5.3 Goland Wing Flutter

So far in this chapter the nonlinear form of the aerodynamic model has been verified for two- and three-dimensional, steady and unsteady flows and in steady aeroelasticity with large deflections. The linearized model has so far been verified in problems in two-dimensional unsteady aerofoil dynamics, and will now be verified in three-dimensional problems by predicting the flutter characteristics of the Goland wing [44, 45].

Aeroelastic models are constructed here by coupling the linearized aerodynamics model described by (2.32) & (2.40) with the modal beam dynamics equations of (3.40) in a feedback loop, illustrated in the block diagram of Figure 4.1. In doing so it is convenient to modify the aerodynamics model so that the inputs and outputs are the generalized coordinates and forces of the cantilever beam model, respectively, as subsequent balancing of the aerodynamics can take advantage of these relatively few modal degrees of freedom in the input/output of the

system, as will be demonstrated in Section 6.2.

To show the modular nature of the analysis framework, the beam modes in this section will be computed using an alternative approach, specifically by the direct solution of the beam differential equations in [101]. For that purpose, let $\mathbf{R}(s, t)$ and $\mathbf{\Psi}(s, t)$ be the vectors of beam displacement and linear rotations along the beam arc length s , respectively, and $\mathbf{M}(s)$ and $\mathbf{C}(s)$ the 6×6 symmetric mass and compliance matrix, respectively. The mode shapes are obtained in terms of the linear velocities, $(\mathbf{v}; \boldsymbol{\omega}) = \frac{d}{dt}(\mathbf{R}; \mathbf{\Psi})$, and stress resultants $(\mathbf{f}; \mathbf{m}) = \mathbf{C}^{-1} \frac{d}{ds}(\mathbf{R}; \mathbf{\Psi})$. If we define the vector $\boldsymbol{\Phi} = (\mathbf{v}; \boldsymbol{\omega}; \mathbf{f}; \mathbf{m})$, they are the non-trivial solutions of [101]

$$\frac{d}{ds} \boldsymbol{\Phi}_j = \begin{bmatrix} \mathbf{E}^\top & -\omega_j \mathbf{C} \\ \omega_j \mathbf{M} & -\mathbf{E} \end{bmatrix} \boldsymbol{\Phi}_j,$$

that satisfy the problem boundary conditions. The j -th mode shape is $\boldsymbol{\Phi}_j(s)$ and the corresponding natural frequency ω_j . $\mathbf{E}$ is a constant matrix to include the moments from shear forces and whose exact definition depends on the orientation of the reference axis and the local initial curvature [154].

As mentioned above, the wing model defined by Goland is used [44]. This is a prismatic cantilever wing with a structural model given by a beam of constant sections described by the parameters in Table 5.4. The first four natural velocity modes, $\boldsymbol{\Phi}_1 = (\mathbf{v}; \boldsymbol{\omega})$, are shown in Figure 5.9, where the wing surface has been constructed under the assumption of rigid cross-sections. Due to the local pitch-plunge inertial coupling introduced by the offset of the inertial axis from the elastic axis, all of the modes are coupled bending/torsion modes.

chord	1.8288 m
semi-span	6.096 m
elastic axis	33% chord
center of gravity	43% chord
mass per unit length	35.71 kg/m
polar moment of inertia	8.64 kg·m
torsional stiffness	$0.99 \times 10^6 \text{ N}\cdot\text{m}^2$
bending stiffness	$9.77 \times 10^6 \text{ N}\cdot\text{m}^2$

Table 5.4: Goland wing properties.

Using these velocity modes as bases for the beam description the aerodynamic model of

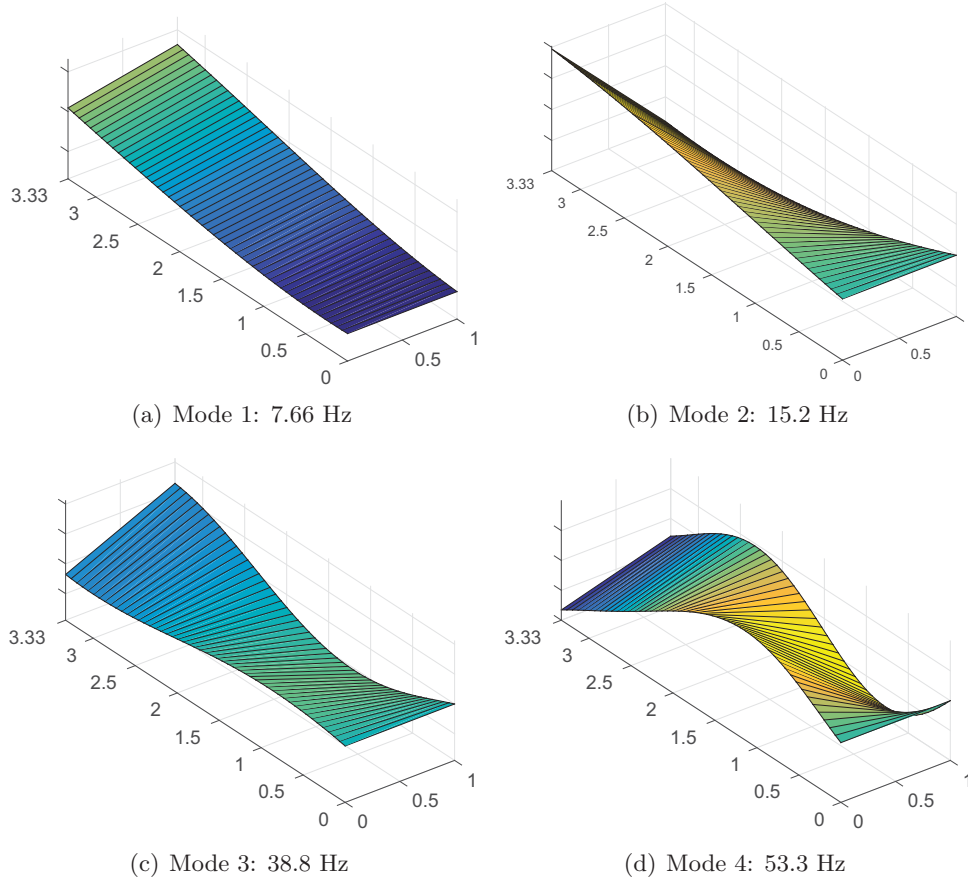


Figure 5.9: The four lowest frequency natural modes of the Goland wing.

(2.32) can be modified as

$$\begin{aligned}
 E_a \mathbf{x}_a^{n+1} &= F_a \mathbf{x}_a^n + G_a T_{as} \Phi_{sm} \mathbf{x}_m^n \\
 \mathbf{Q}^n &= \Phi^\top T_{sa} C_a \mathbf{x}_a^n + \Phi_1^\top T_{sa} D_a T_{as} \Phi_{sm} \mathbf{x}_m^n
 \end{aligned} \tag{5.7}$$

where T_{as} is defined by (4.10) and

$$\Phi_{sm} = \begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_1 \end{bmatrix},$$

so that the modal amplitudes $\mathbf{x}_m = [\mathbf{q}'; \mathbf{q}]$ and nondimensional generalized forces $\mathbf{Q}$ are, respectively, inputs and outputs to the system. This modified system is then coupled directly with the state-space structural equations, (3.40), to form an aeroelastic model.

Convergence of the aerodynamic discretization can now be done in the context of the modal coordinates of the beam. For this purpose the input/output response of (5.7) is studied in

the frequency-domain for different spanwise discretizations in the aerodynamic model, i.e. the number of spanwise vortex rings, N , as shown in Figures 5.10 & 5.11. In all combinations of generalized forces and modal amplitudes 40 spanwise vortex rings are required to achieved a converged magnitude and phase response in the modal coordinates.

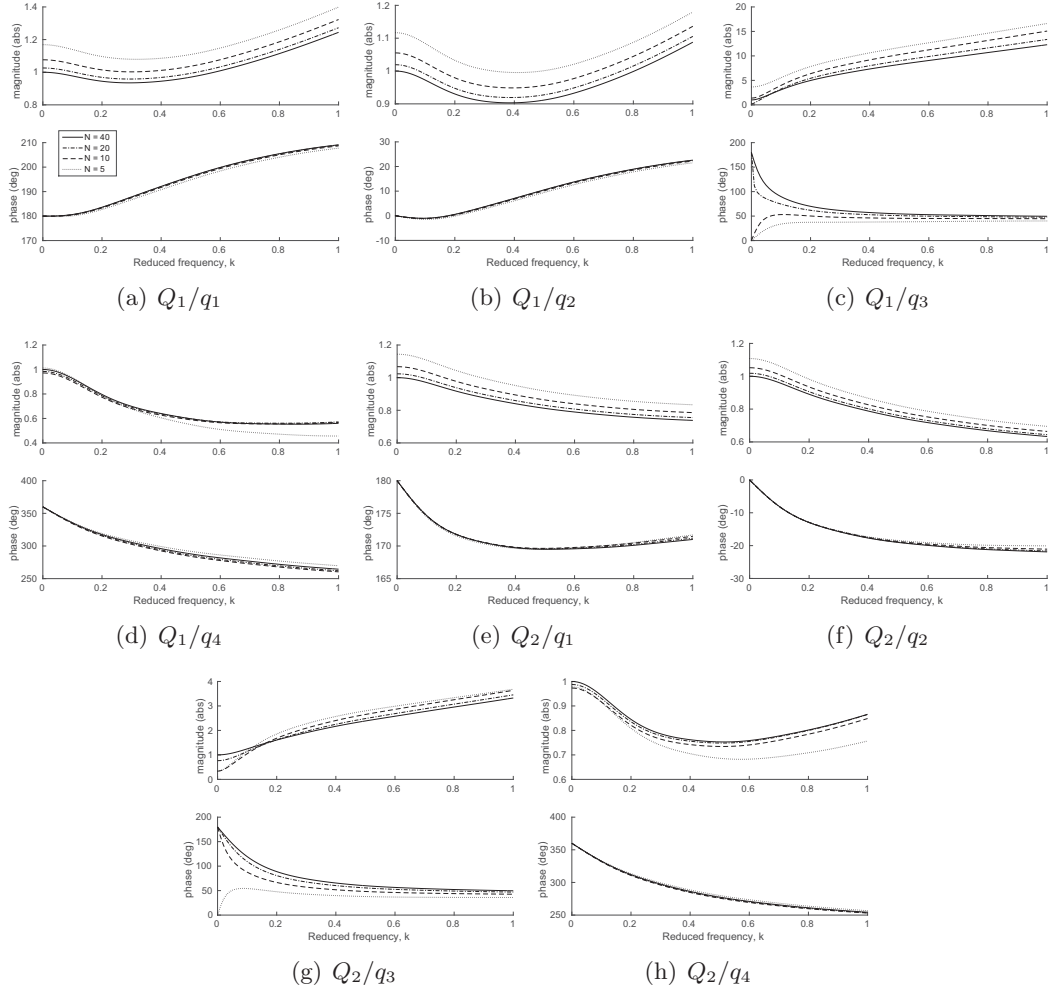


Figure 5.10: Full-order model frequency responses of the Goland wing aerodynamics modified for four structural modes. Q_1 and Q_2 are shown here, with amplitudes normalized by the $N = 40$ model steady-state values. Aerodynamic discretization: $M = 20$, $M_w = 200$.

For verification purposes the flutter characteristics of the Goland wing were then calculated, and are compared with the analytical and computational results of other authors in Table 5.5. The results show that at the finest aerodynamic discretization the flutter speed and frequency are converged with less than 1% error, and compare well to those calculated by other authors who used three-dimensional aerodynamics (Wang *et al.* and Hesse). In this test

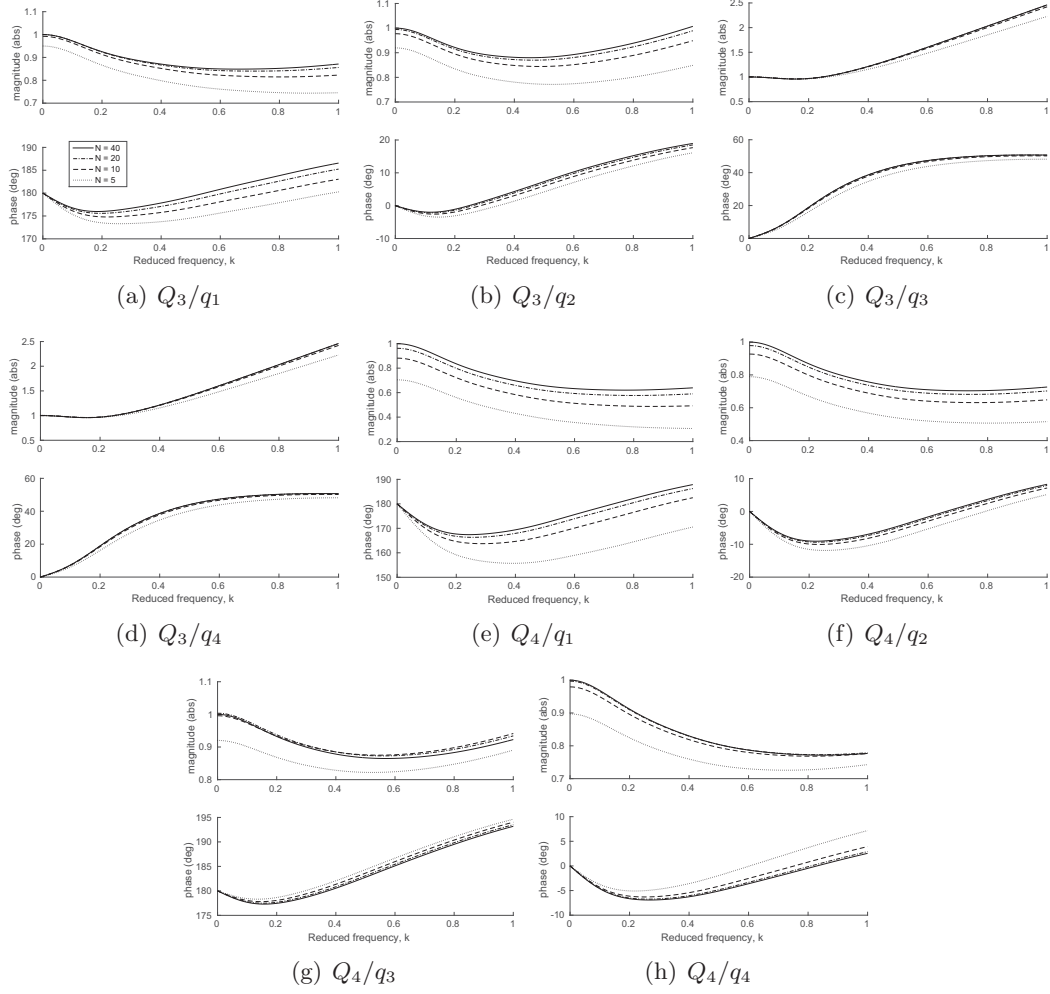


Figure 5.11: Full-order model frequency responses of the Goland wing aerodynamics modified for four structural modes. Q_3 and Q_4 are shown here, with amplitudes normalized by the $N = 40$ steady-state values. Aerodynamic discretization: $M = 20, M_w = 200$.

case the air density is $\rho_\infty = 1.02 \text{ kg}\cdot\text{m}^{-3}$, and the wing is at zero incidence. Consequently, as there is zero loading at all flight velocities, only one modified aerodynamic model was required to construct aeroelastic models for the entire flight envelope owing to the model's nondimensional form. Thus, in computing successive non-trivial solutions to the eigenproblem

$$\mathbf{A}(V_\infty) \mathbf{x}_i = e^{\Delta t \lambda_i} \mathbf{x}_i \quad \text{for } i = 1, \dots, n, \quad (5.8)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is the discrete-time state-transfer matrix of the aeroelastic model with eigenvectors $\mathbf{x}_i = (\mathbf{x}_{a_i}; \mathbf{x}_{m_i})$, only the relatively-small structural dynamics sub-matrices need to be updated. The terms λ_i are continuous-time eigenvalues of the aeroelastic system, which when

plotted for a range of free-stream velocities yield root-loci of the coupled system dynamics, as shown in Figure 5.12.

The coupled system has dimension $n = 8808$, with very many highly-damped, predominantly-aerodynamic eigenvalues visible across the whole frequency spectrum, while the critical mode, found in the low-frequency region near the origin in Figure 5.12(c), originates from the second *in vacuo* structural mode. While capturing all of the eigenvalues shown in Figure 5.12(a) is perhaps unnecessary, it is a consequence of the aerodynamic method itself, which to its advantage directly gives discrete-time models that may be obtained around arbitrary reference states. Moreover, these models are also shown to be very amenable to model reduction techniques, as will be shown in Section 6.2.

Source	Discretization ($M \times N$)	States	V_f , $\text{m}\cdot\text{s}^{-1}$	ω_f , Hz
Present	10×20	2200	166.2	10.4
Present	15×30	4950	169.3	10.5
Present	20×40	8800	170.0	10.6
Goland[44]	N/A	N/A	137.2	11.25
Wang <i>et al.</i> [149]	-	-	174.3	-
Hesse[58]	16×26	4550	169	11.1

Table 5.5: Flutter characteristics of the Goland wing calculated with different aerodynamic discretizations, and a comparison with other authors. In the present analysis the first four *in vacuo* velocity modes were used to approximate the beam dynamics, and the aerodynamic wake was ten chord-lengths long.

5.4 Aeroelasticity of swept flying-wings

Having verified the beam and aerodynamics models in static problems featuring large deflections, and used them to calculate the dynamic stability of the Goland wing, this section aims to take the verification further — to static and dynamic aeroelastic cases involving wing sweep. Then, after comparing the results of SHARPy to the numerical and experimental work of other authors on a set of cantilever wing models, a representative swept flying-wing aircraft configuration is defined, inspired by the X-56 aeroelastic test aircraft [19, 129]. The unconstrained natural modes and the dynamic stability characteristics of the configuration are investigated while varying the aircraft’s mass distribution. In doing so the phenomenon of body-freedom-flutter (BFF) is observed, which is shown to be sensitive to the position of lumped masses/inertias representing the aircraft’s payload/powerplant.

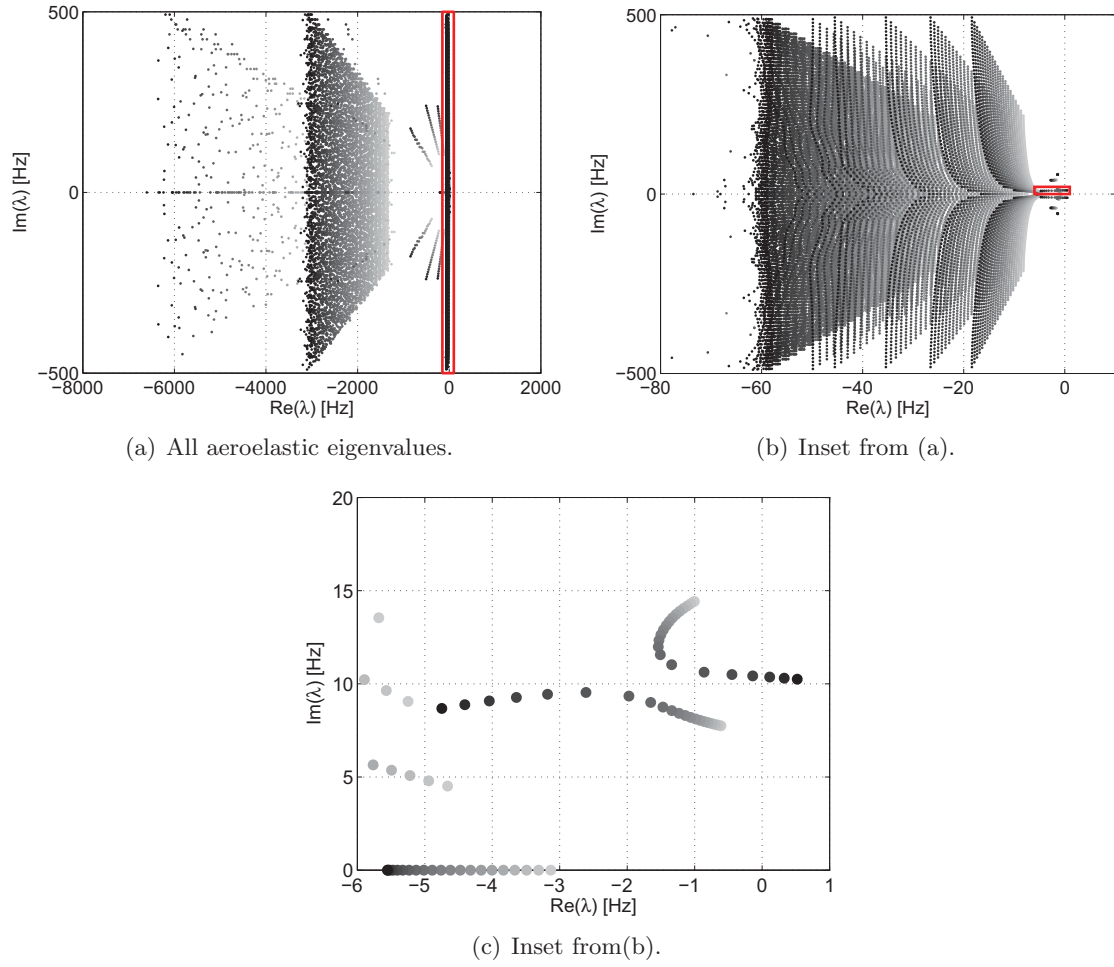


Figure 5.12: Continuous-time eigenvalues of the aeroelastic Goland wing model calculated at free-stream velocities $V_\infty = 80, 85, \dots, 175, 180 \text{ m}\cdot\text{s}^{-1}$ (light to dark). Aerodynamic discretization: $M = 10$, $N = 20$, $M_w = 100$. 4 beam modes.

5.4.1 Verification of aeroelastic analyses

The aeroelastic phenomena of divergence and flutter were investigated numerically and experimentally on swept configurations by Ricketts and Doggett [122]. In their experiments wind-tunnel models of forward-swept, isotropic, cantilever wings (with the structure being a flat plate) were subjected to dynamic pressures equal to, or in some cases above, their divergence dynamic pressures. In some cases flutter occurred prior to the onset of divergence, which was reported and predicted numerically in their work. Their wings were swept in the *oblique* sense, as illustrated in the planform sketches of Figure 5.13. The material properties are given in Table 5.6.

The natural modes of vibration of the wing models were calculated using the linearized beam equations of (3.33) and the resulting modal frequencies are compared to published values in Table 5.7. The small difference between this work and the reference data comes from the beam approximation in this relatively short-aspect-ratio ($AR = 4$) wing.

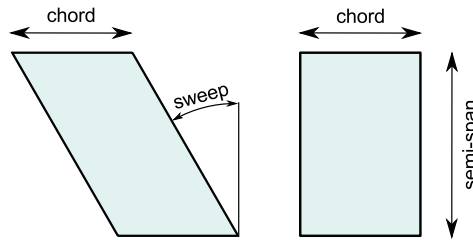


Figure 5.13: Planforms of the family of swept wings from Ricketts and Doggett.

Chord	0.127 m
Semi-span	0.508 m
Thickness	0.00229 m
Density	2712 kg·m ³
Shear Modulus	26 GPa
Poisson's ratio	0.32

Table 5.6: Ricketts and Doggett's wing properties.

Divergence

Divergence analysis was carried out by conducting static analysis of the aeroelastic bending response at different angles of attack and sub-critical dynamic pressures using the static aeroelastic solver described in Section 4.2.1. The tip deflections arising from simulation of

Sweep (deg)	Ref. experimental			Ref. computational			Current		
	f_1	f_2	f_3	f_1	f_2	f_3	f_1	f_2	f_3
0	7.0	43.8	59.8	7.4	46.3	59.8	7.21	45.4	55.0
-7.5	6.9	42.8	57.2	7.3	45.5	60.0	7.09	44.7	55.0
-15	6.6	41.2	56.4	7.0	43.2	60.4	6.73	42.4	55.0
-30	5.5	33.4	57.4	5.8	35.3	60.7	5.41	34.1	55.0

Table 5.7: Natural vibration frequencies of the Ricketts and Doggett wings (in Hz). Negative sweep indicates fore sweep, i.e. towards the oncoming flow.

this parameter space were then used to extrapolate to the critical condition using a Divergence Index Projection [57]. Results of this divergence analyses are compared with the experimental and computational work of Ricketts and Doggett in Figure 5.14. There is good agreement with the experimental and numerical results in the reference at all sweep angles.

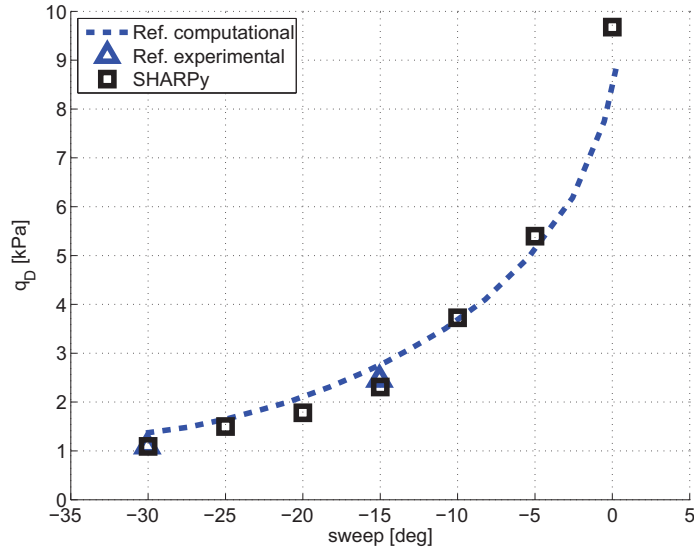


Figure 5.14: Comparison of divergence dynamic pressures, q_D , with those reported in [122] for wings of varying sweep angle. Negative sweep indicates fore sweep, i.e. towards the oncoming flow.

Flutter

Flutter results were also obtained, using $\rho_\infty = 1 \text{ kg}\cdot\text{m}^{-3}$, for sweep angles of 0, -7.5 and -20 degrees. Note that for -20 deg sweep, divergence occurs before flutter, but the flutter onset can still be identified from visualizations of the linear stability analyses (by plotting

the continuous- or discrete-time system eigenvalues). The calculated flutter velocities and frequencies are given in Table 5.8. With zero sweep, a relatively low-frequency (28.6Hz) first-bending/torsion flutter mechanism is observed, whereas in the forward swept cases a high frequency ($> 42\text{Hz}$) but lower speed second-bending/torsion flutter is the critical dynamic instability. A comparison is made with the experimental and computational analysis of Ricketts and Doggett [122] in Figure 5.15, which shows the current method to be in good, albeit conservative, agreement with both analyses.

Model	0 deg sweep		-7.5 deg sweep		-20 deg sweep	
	V_f [$\text{m}\cdot\text{s}^{-1}$]	ω_f [Hz]	V_f [$\text{m}\cdot\text{s}^{-1}$]	ω_f [Hz]	V_f [$\text{m}\cdot\text{s}^{-1}$]	ω_f [Hz]
Current	109.5	28.6	71.0	46.5	78.0	42.3

Table 5.8: Flutter results on forward swept wing (AR=4)

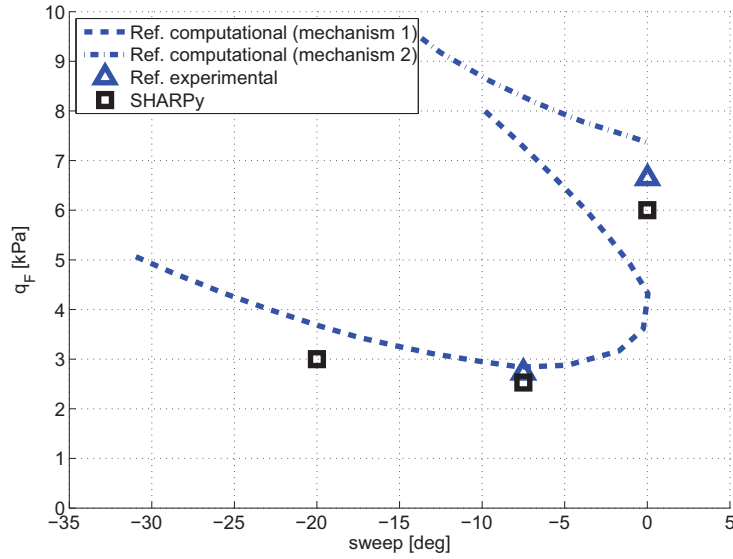


Figure 5.15: Comparison of flutter dynamic pressures, q_F ($\rho_\infty = 1 \text{ kg}\cdot\text{m}^{-3}$), calculated by SHARPy and those reported in [122].

5.4.2 Swept flying-wing analysis

In this section an unconstrained linear beam, (3.38), and linearized aerodynamics model, (2.32) & (2.40), are defined to model the unconstrained dynamic aeroelastic behaviour of a swept flying-wing (SFW) configuration, inspired by the X-56 test aircraft [19, 129].

Model definition

The geometry, mass distribution, and stiffness distribution of the model are symmetrical across the aircraft center line and are described by a swept beam segment and a lattice geometry, shown in Figure 5.16. The mass and stiffness properties required in the beam description are given in Table 5.9. In addition to the distributed mass of Table 5.9, which is defined at an inertial axis 10% of the local chord aft of the elastic axis, a lumped mass of 40kg was added 0.1m aft of the elastic axis at 10% span on each wing. These masses represent the aircraft powerplant/payload, and will be varied with effects on the unconstrained natural modes, and the aeroelastic stability of the aircraft.

In order to create a similar aeroelastic problem to that of the X-56, the mass and stiffness properties of the beam were chosen so that the unconstrained modes were qualitatively-similar to those published in recent work on the aircraft [19]. The first eight non-zero-frequency modes of the model, calculated using (3.33), are presented in Figure 5.17. As mentioned in Section 3.2.1, the first nine modes are zero-frequency modes corresponding to the rigid-body degrees-of-freedom.

	Root	Tip
chord [m]	1.15	0.65
$\bar{m}$ [kg·m ⁻¹]	15	2
$\mathcal{I}_1$ [kg·m ²]	0.5	0.1
EA [N]	1.0×10^7	1.0×10^7
GJ [N·m ²]	7.5×10^4	3.0×10^4
EI_2 [N·m ²]	4.5×10^4	2.0×10^4
EI_3 [N·m ²]	2.4×10^6	1.0×10^6

Table 5.9: SFW geometry, mass and stiffness properties. Parameters vary linearly from root to tip.

Mass variations and stability analyses

Firstly, the dynamic stability of the SFW model described above was investigated. From the similarity of this model to that of the X-56 it was expected that a coupled longitudinal rigid-body / 1st symmetric bending mode, known as body-freedom flutter (BFF), would be the critical flutter mode [19]. This behaviour is consistent with other recent work on the X-56 [129] and other swept flying-wing configurations [80]. In addition, subsequent instability of, at first the coupled symmetric bending / torsion mode, and finally the coupled antisymmetric

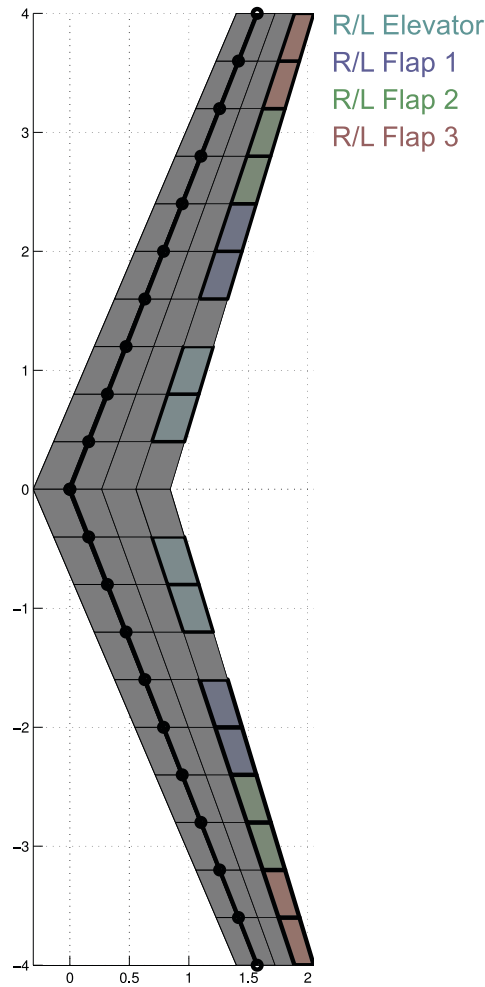
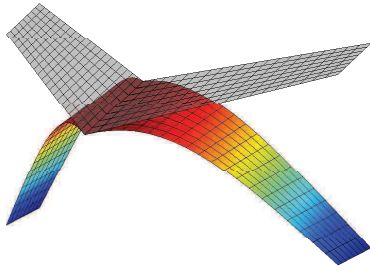


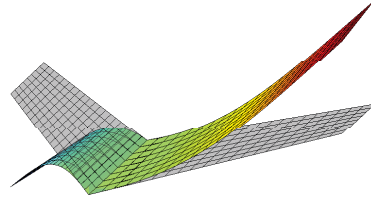
Figure 5.16: SFW geometry in plan view. Coarse discretization corresponding to beam finite elements (dotted black lines) and UVLM (gray/color quadrilateral lattice). The coloured panels represent potential trailing-edge flap locations for control studies on this configuration.

bending / torsion mode, was expected [19].

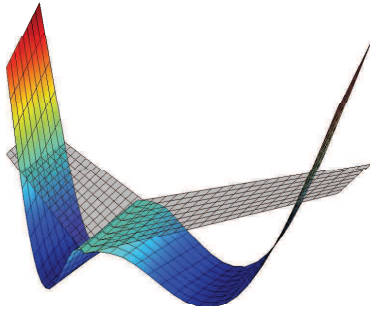
To visualize the stability of the aircraft model throughout the flight envelope, linearized models were constructed for a range of velocities from $1 - 50 \text{ m}\cdot\text{s}^{-1}$ at sea-level air density ($\rho_\infty = 1.225 \text{ kg}\cdot\text{m}^{-3}$). Continuous-time eigenvalues of the resulting coupled state-transfer matrices were then used to construct root loci illustrating the evolution of the system dynamics with velocity. Figure 5.18 illustrates the dynamics of the aircraft short-period and 1st symmetric bending modes in this way. The short-period is identified as rising steadily in frequency and damping from close to the origin to a frequency of above 2.5 Hz at the left-hand edge of the plot. This is accompanied by the first bending mode (which starts at around 5 Hz) dropping in frequency and showing a region of decreasing damping (becoming less stable)



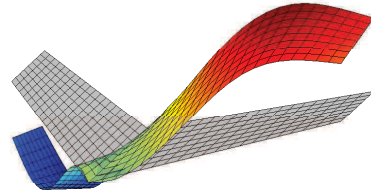
(a) Mode 10: 5.15 Hz



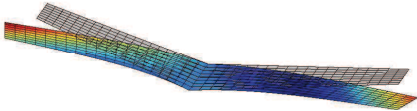
(b) Mode 11: 8.31 Hz



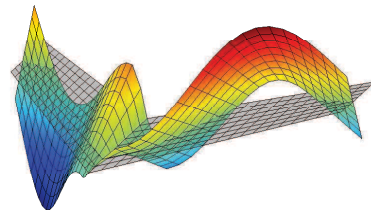
(c) Mode 12: 15.8 Hz



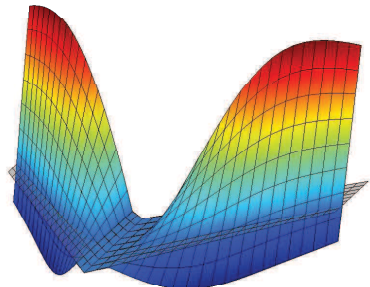
(d) Mode 13: 20.3 Hz



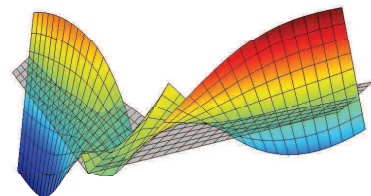
(e) Mode 14: 22.2 Hz



(f) Mode 15: 23.7 Hz



(g) Mode 16: 34.2 Hz



(h) Mode 17: 35.4 Hz

Figure 5.17: Unconstrained structural modes of the SFW aircraft.

after $30 \text{ m}\cdot\text{s}^{-1}$. This root migrates into the right-hand plane at around $40 \text{ m}\cdot\text{s}^{-1}$, indicating the eigenvalue representation of BFF instability. The coalescence of modal frequencies and divergence of damping — one mode becoming unstable while the other, in this case the aircraft short period mode, rapidly becoming very highly damped — is analogous to the classical bend-twist flutter of unswept cantilever wings. The only other unstable mode, which sits just to the right of the origin in Figure 5.18, is the spiral mode, which is unstable at all reference conditions, and very slow.

To investigate the effect of mass distribution on the aircraft dynamic stability, two modified SFW models were constructed. This was done by moving the lumped masses described in the definition of the baseline to 20% span, in the first case, and 30% in the second. Moving the concentrated mass outboard increases the rolling and yawing inertias, and decreases the pitch inertia, as shown in Table 5.10, which changes the coupling between bending and torsional degrees-of-freedom and hence the natural mode frequencies.

Finally, the results of Figure 5.19 show the aeroelastic stability of the baseline model (black), 20% span model (red) and 30% span model (green). In general the BFF mode critical velocity is increased as the lumped masses are moved outboard: $40 \text{ m}\cdot\text{s}^{-1}$ for the baseline case, $42 \text{ m}\cdot\text{s}^{-1}$ for the 20% span case, and $54 \text{ m}\cdot\text{s}^{-1}$ for the 30% span case. Moving the masses also has the side-effect of introducing a second, higher-frequency, instability in the 2nd antisymmetric bending / torsion mode, which occurs at $150 \text{ m}\cdot\text{s}^{-1}$ and 14.3 Hz in the 20% span case. This effect is more pronounced for the 30% span model where the same mode becomes unstable, but at $121 \text{ m}\cdot\text{s}^{-1}$ and 12.45 Hz.

Mode description	Baseline	20% span	30% span
1sfb	5.15	5.00	5.02
1asfb	8.31	7.05	6.77
2sfb	15.8	16.8	13.9
2asfb/1asT	20.3	17.3	16.9
1sllb	22.2	21.7	23.4
3asfb/2asT	23.7	22.9	24.9
1sT	34.2	35.7	30.3
3asT/3asfb	35.4	37.7	41.1
m_{total} [kg]	153.1	153.1	153.1
CoG [% chord]	71.90	79.02	86.17
$\mathcal{I}_{roll}$ [kg·m ²]	255.4	293.8	357.8
$\mathcal{I}_{pitch}$ [kg·m ²]	28.18	24.35	22.42
$\mathcal{I}_{yaw}$ [kg·m ²]	271.0	305.6	367.6

Table 5.10: Natural vibration frequencies (in Hz) and rigid-body properties of the unconstrained SFW aircraft models. Mode descriptions are made using acronyms, for example, the first symmetric flapwise bending mode is denoted 1sfb and the first antisymmetric torsional mode is denoted 1asT. The CoG is expressed as the chordwise center-line ordinate aft of the aircraft nose apex. The rigid-body inertias are calculated around the CoG.

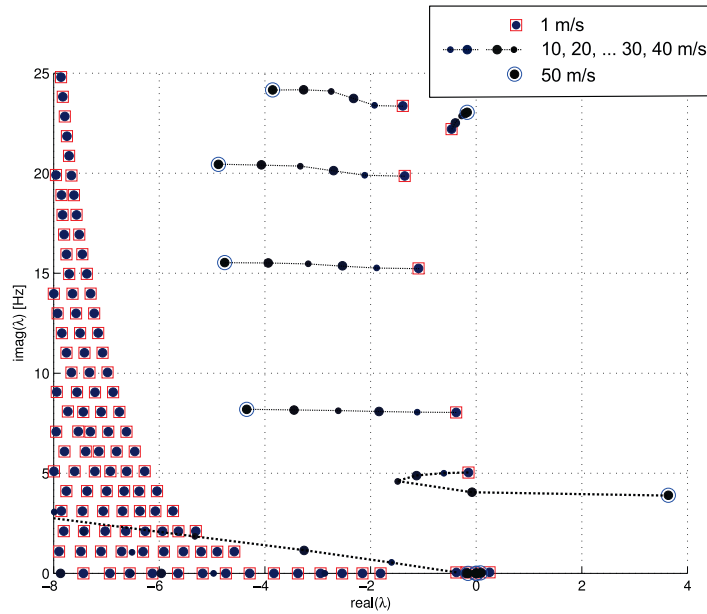


Figure 5.18: Aeroelastic stability plot of the unconstrained SFW aircraft at sea-level ($\rho_\infty = 1.225 \text{ kg}\cdot\text{m}^{-3}$).

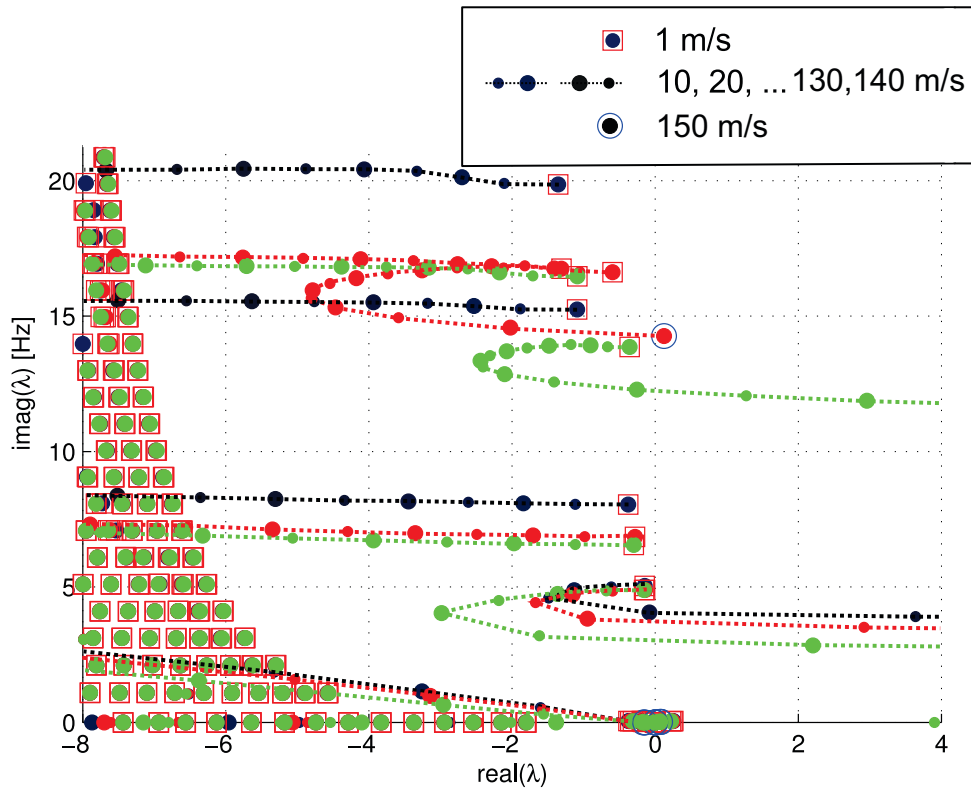


Figure 5.19: Aeroelastic stability plot of the unconstrained SFW aircraft with baseline mass configuration (black), lumped mass at 20% span (red), and lumped mass at 30% span (green). Sea-level altitude ($\rho_\infty = 1.225 \text{ kg}\cdot\text{m}^{-3}$).

Chapter 6

Model Reduction and Predictive Control

In this chapter the hierarchy of aeroelastic models introduced and verified so far in this thesis is supplemented with reduced-order models obtained from balanced realizations [43, 90] of the relevant linearized dynamics. Predictive capability is therefore possible using a family of nonlinear, linearized (about arbitrary reference conditions) and reduced-order (linear) models, which are exercised accordingly. The effectiveness of the reduction approach is investigated, and the utility of the resulting low-order systems is explored in the context of aeroelastic stability analysis and active control of aeroelastic systems subjected to gust loads.

Two approaches are used for the reduction of aeroelastic models in this work, although both rely on the consistent linearizations of aerodynamic and structural models, described in Sections 2.4 and 3.2 respectively, which can, in general, include the effect of arbitrary reference states obtained from nonlinear equilibria. The first approach is to use the common method of modal truncation for the structural dynamics, and a subsequent balanced realization and residualization of aerodynamic models that have been modified so that the input/output is written in the modal degrees-of-freedom of the structure. Since this form of the aerodynamics has a reduced set of inputs and outputs, it is amenable to balancing and yields an efficient method for flutter and gust-load analysis. In this approach a single aerodynamic model can be used for aeroelastic stability analysis throughout the whole (incompressible) flight envelope, provided the aircraft static equilibrium does not vary greatly between reference conditions. The second method is a simple balanced realization and truncation of the monolithic aeroe-

lastic model, which, although lacking the utility of the first approach, is potentially more effective as a model-order reduction technique in the context of feedback control design. The reduced monolithic models are then exercised as the plant model in active gust-load alleviation using model-predictive and linear-quadratic control schemes applied in linear and nonlinear simulations.

6.1 Balanced residualization

Reduced-order aerodynamic models are obtained in this work by truncating a balanced realization using residualization to preserve the steady-state gains. Assuming a stable, linear, time-invariant discrete-time state-space dynamics model, the controllability and observability Gramians are the symmetric matrices obtained from solution to the following Lyapunov equations [76]

$$\begin{aligned} \mathbf{A}\mathbf{W}_c\mathbf{A}^\top - \mathbf{W}_c + \mathbf{B}\mathbf{B}^\top &= 0, \\ \mathbf{A}^\top\mathbf{W}_o\mathbf{A} - \mathbf{W}_o + \mathbf{C}^\top\mathbf{C} &= 0. \end{aligned} \tag{6.1}$$

The controllability Gramian $\mathbf{W}_c$ is inversely correlated to the energy required by an input signal to reach a certain state of the system, while $\mathbf{W}_o$ relates to the ability of non-zero initial conditions of the state vector to reach a certain output of system. The square root of the eigenvalues of the product $\mathbf{W}_c\mathbf{W}_o$ are the Hankel singular values (HSV) of the system, σ_j for $j = 1, \dots, n$, which will be ordered such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$. The balanced realization is then defined as the similarity transformation, $\mathbf{T}$, in which the controllability and observability Gramians are equal and diagonal, that is, $\mathbf{T}^{-1}\mathbf{W}_c\mathbf{T} = \mathbf{\Sigma}$ and $\mathbf{T}^{-1}\mathbf{W}_o\mathbf{T} = \mathbf{\Sigma}$, with $\mathbf{\Sigma} = \text{diag}(\sigma_1, \dots, \sigma_n)$. These realizations were obtained using the `balreal` function in [82], which uses the algorithm described in [76].

The HSVs provide a metric to rank the states in order of their contribution to the input/output behaviour of the system, so that states with small HSVs can be removed. For that purpose the balanced equations are partitioned as

$$\begin{aligned} \tilde{\mathbf{x}}_1^{n+1} &= \tilde{\mathbf{A}}_{11}\tilde{\mathbf{x}}_1^n + \tilde{\mathbf{A}}_{12}\tilde{\mathbf{x}}_2^n + \tilde{\mathbf{B}}_1\mathbf{u}^n, \\ \tilde{\mathbf{x}}_2^{n+1} &= \tilde{\mathbf{A}}_{21}\tilde{\mathbf{x}}_1^n + \tilde{\mathbf{A}}_{22}\tilde{\mathbf{x}}_2^n + \tilde{\mathbf{B}}_2\mathbf{u}^n, \\ \mathbf{y}^n &= \tilde{\mathbf{C}}_1\tilde{\mathbf{x}}_1^n + \tilde{\mathbf{C}}_2\tilde{\mathbf{x}}_2^n + \mathbf{D}\mathbf{u}^n, \end{aligned} \tag{6.2}$$

where $\tilde{\mathbf{x}}_1 \in \mathbb{R}^r$ are the states that we wish to preserve corresponding to the first r HSVs. Finally, in the reduction of aerodynamics models, the partitioned system is residualized on the preserved states, by assuming $\tilde{\mathbf{x}}_2^{n+1} = \tilde{\mathbf{x}}_2^n$, writing $\tilde{\mathbf{x}}_2^n = \left(\mathbf{I} - \tilde{\mathbf{A}}_{22}\right)^{-1} \left(\tilde{\mathbf{A}}_{21}\tilde{\mathbf{x}}_1^n + \tilde{\mathbf{B}}_2\mathbf{u}^n\right)$ and substituting this expression in the first and last equations in (6.2). This residualization step is preferred to a simple truncation of $\tilde{\mathbf{x}}_2$ for unsteady aerodynamics problems as it preserves the quasi-steady aerodynamic solution, which the DLM cannot provide. If $(\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \mathbf{D})$ is the residualized system, the constant output $\mathbf{y}_0$ due to a constant input $\mathbf{u}_0$ will satisfy

$$\mathbf{y}_0 = \left(\tilde{\mathbf{C}} \left(\mathbf{I} - \tilde{\mathbf{A}}\right)^{-1} \tilde{\mathbf{B}} + \mathbf{D}\right) \mathbf{u}_0. \quad (6.3)$$

Since the approach above is limited to stable systems, reducing monolithic aeroelastic models obtained at super-critical reference conditions (beyond divergence or flutter) is not possible, although this can be achieved using coprime factorization if necessary [58, 87]. Alternatively, since the linearized aerodynamics of Section 2.4 are stable [55, 56, 58] the aerodynamic subsystem can be reduced with the above method, which is the approach taken when building models for aeroelastic stability analysis, discussed next.

6.2 Model reduction for efficient aeroelastic stability analysis

In this section aeroelastic models are constructed for the purposes of stability analysis by coupling the modal form of linearized beam dynamics, (3.40), with the linearized aerodynamics model of Section 2.4 modified for the beam modal degrees-of-freedom, as in (5.7), in a similar approach to that of Section 5.3. The resulting coupled state-space equations governing the wing dynamics have the states

$$\mathbf{x}_{sys} = [\mathbf{x}_a; \mathbf{x}_m] = [\mathbf{x}_a; \mathbf{q}'; \mathbf{q}], \quad (6.4)$$

where $\mathbf{x}_a$ are the full-order states of the vortex-lattice based aerodynamics, and $\mathbf{q}$ are the beam modal amplitudes with $(\cdot)'$ denoting their rate-of-change with respect to nondimensional time, defined in (2.20). In the stability analysis of Section 5.3 an aerodynamic system with 8800 states was required to predict the converged flutter characteristics of the Goland wing, which is used again as a test case here.¹ Since the dynamic stability of these systems is

¹Pertinent properties of this unswept, planar, cantilever wing are given in Table 5.4.

obtained through eigenanalysis, which is a computation of complexity $\mathcal{O}(n^3)$ where n is the size of the coupled system, it is extremely desirable to find a reduced-order representation of the aerodynamics. As the majority of the aerodynamic states are related to the circulation distribution in the wake, and the dynamics of wake convection were captured by a single transfer function by Theodorsen [141] for two-dimensional flows, and a few coefficients in the rational function approximation of frequency-domain-based 3-D models [11, 35], it appears likely that substantial reductions in the model-order of the discrete-time aerodynamics is possible.

Hankel singular values (HSV) of the modified aerodynamics model (5.7) are shown in Figure 6.1. The steep initial drop in HSVs indicates that it may be possible to obtain accurate aerodynamics models with very few states. To test this, the frequency-domain responses of the modified aerodynamic model, i.e. the transfer functions relating beam modal coordinates $\mathbf{q}$ to the generalized aerodynamic forces $\mathbf{Q}$, are obtained for reduced-order models of different order, r . Convergence of the reduced-order frequency responses to that of the full-order model is shown in phase plots of Figure 6.2. Across the frequency range investigated as little as 12 states ($r = 12$) were required to closely-match the full-order model response, which suggests that for the purposes of aeroelastic analysis the full-order aerodynamics could be replaced by an equivalent model that is at almost three orders-of-magnitude smaller than the original.

6.2.1 Flutter analysis using reduced-order aerodynamics models

Flutter analysis of the Goland wing, originally performed in Section 5.3, is repeated here using reduced-order aerodynamics models obtained through balanced realization and residualization. As described in Section 4.2.3, the stability of aeroelastic systems is determined in this work by tracking eigenvalues of the coupled system through a range of different reference conditions. In this study the density is held constant at $\rho_\infty = 1.02 \text{ kg}\cdot\text{m}^{-3}$ as before, while the free-stream velocity V_∞ is varied and the evolution of the system eigenvalues (*root loci*) monitored.

Since the aerodynamic models are written directly in a nondimensional (independent of dynamic pressure, $\frac{1}{2}\rho_\infty V_\infty^2$) discrete-time form, a change in reference condition implies that only the relatively-small and cheap-to-compute structural dynamics matrices need to be updated. Thus, a single reduced-order aerodynamic model of this kind (5.7) can enable extremely-efficient stability analysis at all incompressible dynamic pressures of interest (altitudes and

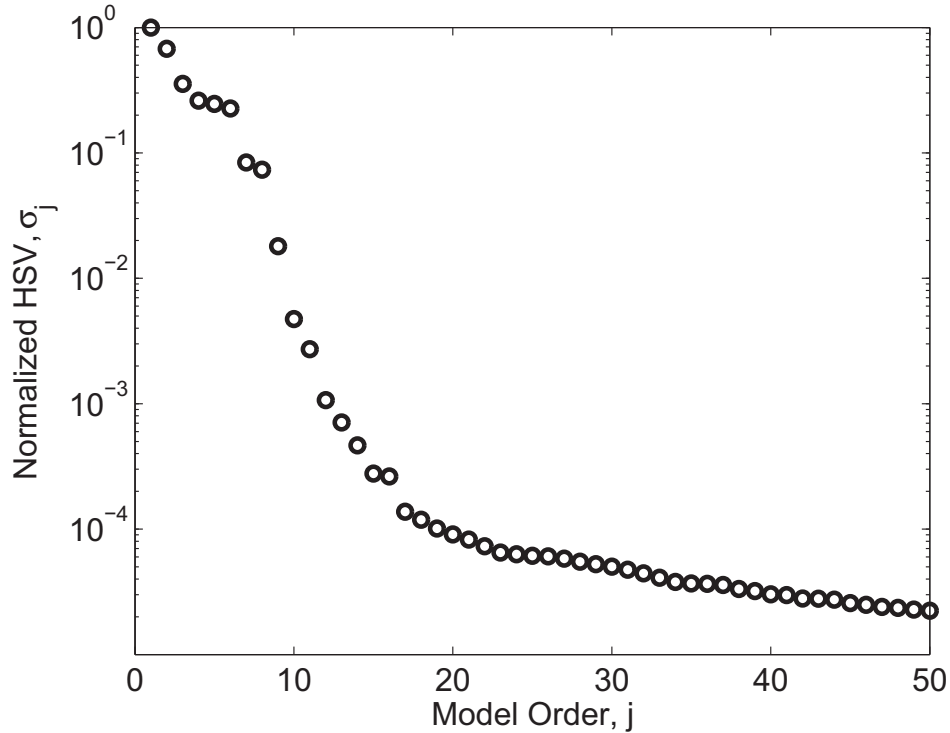


Figure 6.1: First 50 Hankel singular values of the aerodynamic model modified for a 4 mode approximation in the beam dynamics. Aerodynamic discretization: $M = 20$, $N = 40$, $M_w = 200$.

airspeeds). It is worth noting that the cost of balancing the system is $\mathcal{O}(n^3)$, which is a one-time up-front cost.²

The root loci of Figure 6.3 show the evolution of aeroelastic eigenvalues of different coupled systems that are constructed using reduced aerodynamics models of varying order r . With increasing model order, the loci converge to that of the full-order model, starting from the static gain model ($r = 0$) and ending with a 9 state model that closely preserves the dynamics of the full-order system. A table of the flutter characteristics obtained with models of varying order is given in Table 6.1. Large jumps in the calculated flutter speed occur between the $r = 2$ and $r = 6$ models, and, crucially, between the $r = 8$ and $r = 9$ models, the latter of which is almost converged to the full-order dynamics. As expected, the large changes in system dynamics correspond to large drops in the HSVs in Figure 6.1. Including the dynamics of the 9-th balanced state seems to be the last piece of aerodynamic information required to

²This, however, assumes no change in reference loading or geometry between flight conditions, which in general is not the case, and may affect the system dynamics and stability boundaries [95]. The models in this work are capable of capturing the effects of arbitrary non-zero reference states on the coupled system dynamics, although results here are restricted to a zero reference. If large changes in the reference states do occur then interpolation of reduced-order models could be considered [3, 4] for efficient stability analysis.

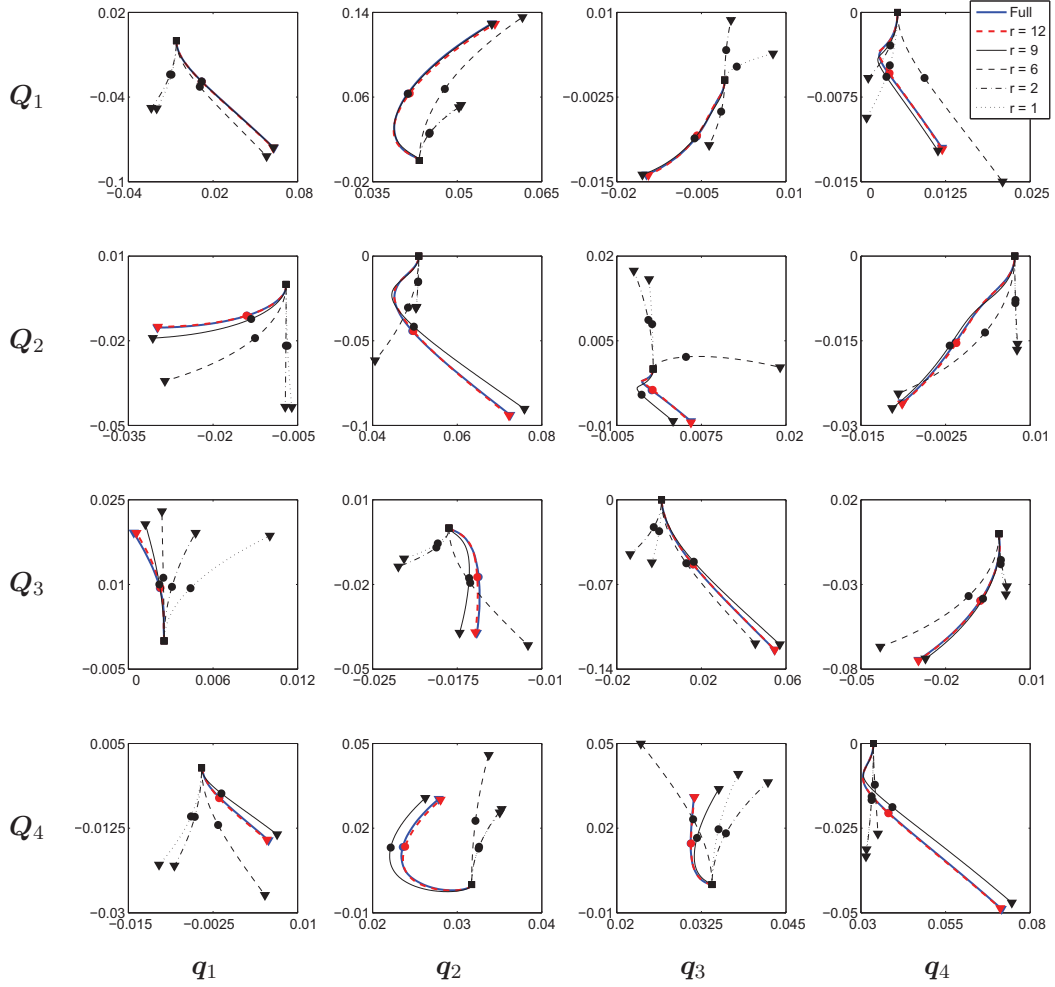


Figure 6.2: Transfer functions of the modified aerodynamic model inputs and outputs (Q_i/q_j) as a function of reduced frequency for models of varying order. $k = 0$ (■), $k = 1$ (●) and $k = 2$ (▼) are indicated by symbols. Aerodynamic discretization: $M = 20$, $N = 40$, $M_w = 200$.

accurately predict flutter in this case.

The eigenvalues and eigenvectors of the $r = 9$ aeroelastic system at the critical condition, $V_\infty = 170.6 \text{ m}\cdot\text{s}^{-1}$, are investigated further. The normalized eigenvectors are visualized in Figure 6.4 by shading each element according to its real part, and can be split roughly into 2 groups: those with terms predominantly in the structural states, eigenvectors 7, 8, 9 & 10; and those with predominantly aerodynamic terms, eigenvectors 1 – 6. All of the eigenvectors contain coupling between the reduced-order aerodynamic states $\tilde{\mathbf{x}}_a$, modal coordinates $\mathbf{q}$, and their rates of change $\mathbf{q}'$ (although the scales makes this difficult to see in the case of eigenvectors 1 – 5). The continuous-time eigenvalues corresponding to each eigenvector are

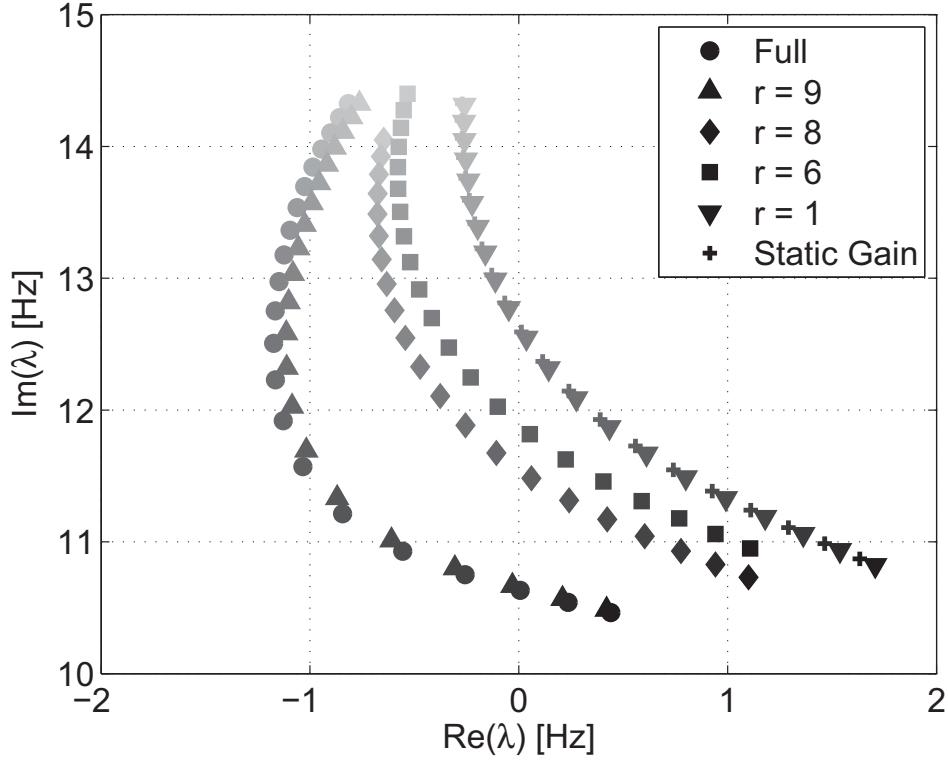


Figure 6.3: Critical eigenvalue evolution of full-order, reduced and static gain models for $V_\infty = 80, 85, \dots, 175, 180 \text{ m}\cdot\text{s}^{-1}$ (light to dark). Aerodynamic discretization: $M = 20$, $N = 40$, $M_w = 200$.

Order	$V_f, \text{ m}\cdot\text{s}^{-1}$	$\omega_f, \text{ Hz}$
8800 (full)	170.0	10.6
100	170.0	10.6
12	170.4	10.6
9	170.6	10.6
8	148.3	11.5
6	148.4	11.9
2	128.4	12.6
1	128.0	12.6
0 (static gain)	129.3	12.6

Table 6.1: Flutter characteristics of the Goland wing calculated using aerodynamics models of varying order. The first four *in vacuo* velocity modes were used to approximate the beam dynamics. Aerodynamic discretization: $M = 20$, $N = 40$, $M_w = 200$.

also reported, in Table 6.2.

The critical mode, corresponding to eigenvector 8, can be seen to have a similar shape to mode 7, and is of comparable frequency — the modal frequencies have coalesced (this is also visible in the full-order results of Figure 5.12(c)). Further insight can be gained by using the modal and balancing truncations to transform, respectively, the reduced-order structural

dynamic and aerodynamic states in the critical eigenvector back into physical coordinates, which allows the visualization of Figure 6.5. The shape of this critical mode is similar to that of the second *in vacuo* mode of the wing (shown in Figure 5.9) from which it originates, and contains the bend-twist coupling expected of this instability.

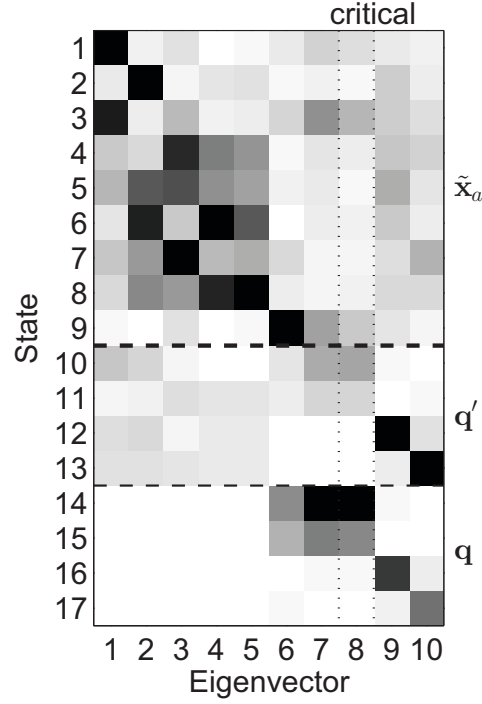


Figure 6.4: Eigenvectors of the $r = 9$ aeroelastic model (4 structural modes) at the flutter point, $V_\infty = 170.6 \text{ m}\cdot\text{s}^{-1}$. The absolute value of each element is normalized by the largest component in its corresponding vector, and is shaded from white (0) to black (1).

Eigenvector	Eigenvalue, Hz		Description
	Real	Imag	
1	-406.5	± 456.8	conjugate pair
2	-471.9	± 456.3	conjugate pair
3	-617.6	± 405.1	conjugate pair
4	-774.3	+ 932.9	complex singleton
5	-555.9	-	real
6	-7.013	-	real
7	-3.337	± 9.164	conjugate pair
8	0.000	± 10.65	conjugate pair
9	-2.318	± 37.97	conjugate pair
10	-0.908	± 53.92	conjugate pair

Table 6.2: Eigenvalues corresponding to the eigenvectors of Figure 6.4. Results correspond to the $r = 9$ aeroelastic model (4 structural modes) at the flutter point, $V_\infty = 170.6 \text{ m}\cdot\text{s}^{-1}$.

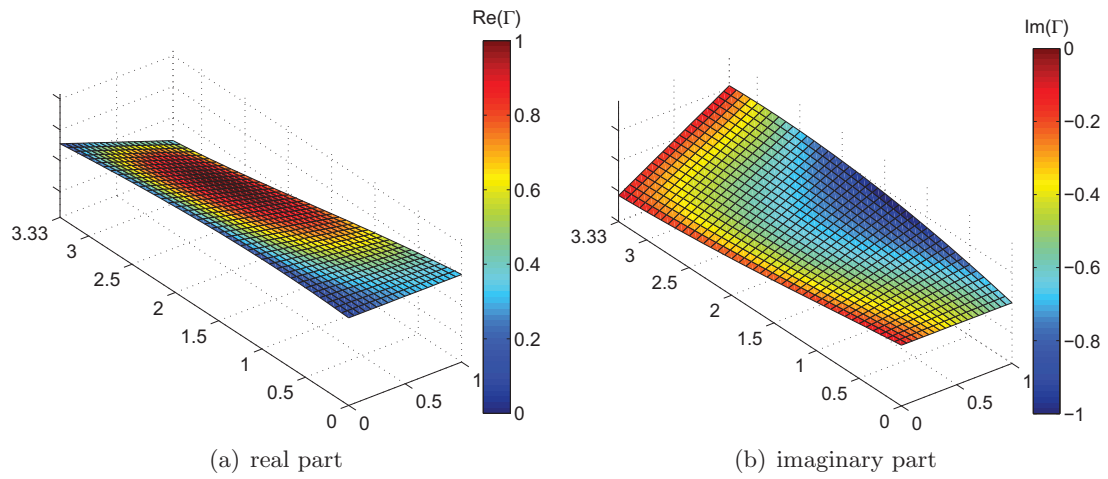


Figure 6.5: The critical eigenvector at flutter transformed back into physical coordinates. The wing displacements and normalized circulation distribution are shown.

6.3 Predictive control for gust load alleviation

In order to test the nonlinear, linearized and reduced-order modelling approaches described above, and the synthesis and application of predictive controllers for gust load alleviation using these models, two cantilever wing test cases are used which are based on the aeroelastic wing model introduced by Goland. The wings will be subjected to external disturbances in the form of gust loads, which will be rejected using an active control scheme that uses a trailing-edge flap for actuation. For this purpose, predictive controllers will be synthesized using the linearized aeroelastic models described in Section 4.2.3. Since low-order systems are necessary in this context, the aeroelastic models will be transformed using the balancing criteria introduced in Section 6.2. Since preserving the steady-state gain is not necessary for models used in synthesis of feedback control systems, simple truncation of the balanced states is used to reduce the model order.

The model predictive control (MPC) scheme presented here can be described as a time-invariant implicit control law of the form, $\mathcal{K}(\hat{x}, \hat{\mathbf{x}}^{ref}, \mathbf{u}^{ref})$, where $\hat{x} \in \mathbb{R}^r$ is the reduced-order state which will be sampled at discrete time steps $t(k)$. The state and control trajectories are, respectively, $\hat{\mathbf{x}}^{ref} = \{\hat{x}_0^{ref}, \dots, \hat{x}_{H-1}^{ref}\}$ and $\mathbf{u}^{ref} = \{u_0^{ref}, \dots, u_{H-1}^{ref}\}$, where the integer $H \geq 2$ is the number of steps in the prediction horizon. This control law is defined as the first control action, $u(k+1) = u_1$, in the series of actions found by online solution of the optimization problem

$$\min_{\substack{\hat{x}_i, u_i \\ i=0, \dots, H-1}} \sum_{i=0}^{H-1} \left(\left\| \hat{x}_i - \hat{x}_i^{ref} \right\|_Q^2 + \left\| u_i - u_i^{ref} \right\|_R^2 \right) + \left\| \hat{x}_H - \hat{x}_H^{ref} \right\|_P^2 \quad (6.5)$$

s.t.

$$\hat{x}_{i+1} = \hat{A}\hat{x}_i + \hat{B}u_i \quad (6.6)$$

$$\hat{x}_0 = \hat{x}(k) \quad (6.7)$$

$$E_x \hat{x}_i + E_u u_i \leq \mathbf{1} \quad \forall i \in \{0, \dots, H-1\} \quad (6.8)$$

$$F\hat{x}_H \leq \mathbf{1} \quad (6.9)$$

where the matrices Q , R , and P are the state, input, and terminal cost weighting matrices, respectively. Equation 6.6 contains the dynamics of the reduced-order coupled aeroelastic system in which the inputs correspond to deployment of a trailing-edge flap. The initial reduced-state $\hat{x}_0$ is measured at the current discrete time step k under the assumptions that

full-state feedback is available from the plant and the balancing transformation $\mathbf{T}$, defined in Section 6.1, is still valid. In the case where the plant is nonlinear, which is to be expected, or in the case of unknown disturbances to the plant input and output, offset-free tracking can be facilitated by comparing the measured plant state with the current state as predicted at the previous time-step. The resulting error is assumed to be constant along the prediction horizon and the reference trajectory is augmented to reflect this [81]; when MPC schemes in this work include this form of offset-free tracking they will be referred to as *MPC with prediction enhancement (PE)* in the style of [50]. Mixed constraints are specified with matrices E_x and E_u , which are typically sparse, and the terminal state, $\hat{x}_H$, is constrained to lie within the polyhedron (possibly open) defined by the matrix F and the vector of ones, $\mathbf{1}$, as in (6.9).

The MPC optimization problem of (6.5)–(6.9) is cast in the form of a general quadratic programming (QP) problem using the open-source package μ AO-MPC [158]. In their Python-based implementation the Hessian matrix, gradient vector, and constraint equations associated with the optimization of (6.5)–(6.9) are calculated by considering the problem in condensed form. In addition to this, μ AO-MPC automatically generates library-free C-code to solve the condensed problem. The result is an extremely fast, low-memory implementation (the original intended application was for microcontrollers). An Augmented-Lagrangian approach using a fixed number of iterations is used to solve the optimization in (nearly) deterministic time, and although the result may be sub-optimal it is still likely to give good controller performance [158].

In addition to predictive control, equivalent linear quadratic regulators (LQRs) are synthesized using the same Q and R matrices as in (6.5). The optimization problem in LQR control is simply an infinite-horizon ($H \rightarrow \infty$) unconstrained version of this problem, i.e, at time step k :

$$\begin{aligned} \min_{\substack{\hat{x}_i, u_i \\ i=0,1,\dots}} \sum_{i=0}^{\infty} & \left(\left\| \hat{x}_i - \hat{x}_i^{ref} \right\|_Q^2 + \left\| u_i - u_i^{ref} \right\|_R^2 \right) \\ \text{s.t.} & \\ \hat{x}_{i+1} &= \hat{A}\hat{x}_i + \hat{B}u_i \\ \hat{x}_0 &= \hat{x}(k) \end{aligned} \tag{6.10}$$

where the terminal cost and constraints, and mixed constraints, are no longer present. Solu-

tion of this problem yields a constant optimal gain matrix, K_{LQR} and corresponding control action $u(k) = -K_{LQR} \hat{x}(k)$. For comparison of relative closed-loop performance, when input constraints are present in an MPC control law the corresponding LQR controller is saturated.

In the following numerical studies cantilever wings are subjected to a family of vertical ‘ $1 - \cos$ ’ gusts; details of these gusts, the open-loop response, and identification of a worst-case gust are presented in Section 6.3.1. Identification of reduced-order models (ROMs) suitable for synthesis of predictive control schemes are then presented in Section 6.3.2. Subsequently, the capabilities of predictive and LQR control schemes synthesized using these reduced-models are tested in nonlinear, closed-loop simulations of the plant in Section 6.3.3. Finally in Section 6.3.4, a second, relatively-flexible test case based on a stretched version of the Goland wing is used to demonstrate the same reduced-order modeling and control synthesis approach for a case involving the large (nonlinear) dynamic deflections expected of VFA wings.

6.3.1 Worst-case gust loads

The first test case is the relatively stiff, low-aspect-ratio wing described by Goland introduced in Section 5.3 (details in Table 5.4). Reference conditions for the open- and closed-loop calculations of this section are given in Table 6.3 alongside details of the control system which will be used for gust-load alleviation. The wing is now subjected to a ‘ $1 - \cos$ ’ family of gusts similar to those described in Federal Aviation Regulation (FAR) §25.341. The disturbance takes the form

$$U_g = \frac{U_{design}}{2} \left(1 - \cos \left(\frac{2\pi d}{L_g} \right) \right), \quad \text{where } 0 \leq d \leq L_g, \quad (6.11)$$

and U_g is the vertical gust velocity, U_{design} is the design gust velocity, and d is the distance into the gust which has length L_g . The largest design gust velocity in the requirements, $U_{design} = 17.07 \text{ m}\cdot\text{s}^{-1}$, was chosen for open-loop tests at a range of gust lengths, the results of which are shown in Figure 6.6.

Results of the linearized model, shown in Figure 6.6(a), show the root strains in torsion and bending due to gusts with lengths in the range 10–80m. Results from a 10m gust are shown (despite this being outside the requirements) because the high-frequency content of such disturbances excites the relatively stiff torsional degrees-of-freedom of the Goland wing and produces the maximum torsional strain of all the gust lengths investigated. The critical gust for maximum bending strain was found to be approximately 20m, and will be referred

Free stream velocity	140 m·s ⁻¹
Angle-of-attack	0 deg
Air density	1.02 kg·m ⁻³
Control surfaces	1 TE flap at tip
Flap size	25% chord, 20% span
Measurements	root torsion and bending strains
Sampling interval, Δt	1.633×10^{-3} s
Model discretization	$M = 8, N = 20$

Table 6.3: Reference condition and control system details.

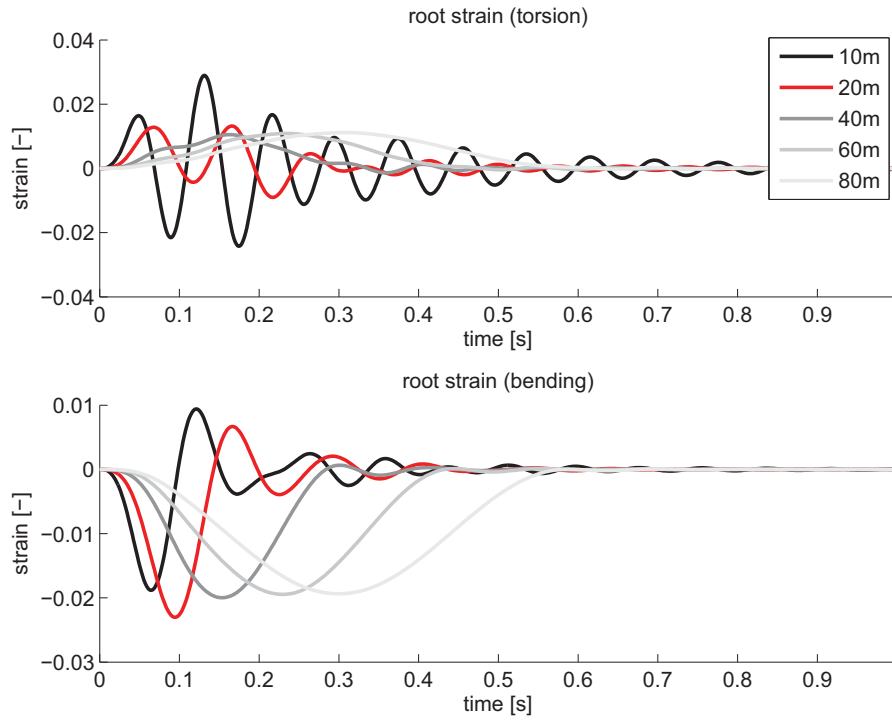
to as the critical gust length. Gust lengths of 60m and above produced a response that was effectively quasi-static, with minimal oscillatory behavior and lower peak strains than the critical case.

Results of the nonlinear model in Figure 6.6(b) show very similar open-loop strain responses when compared to the linear model in Figure 6.6(a). Oscillations in the nonlinear model appear slightly more damped than in the linear case at this temporal discretization despite both models having the same Newmark damping (0.5%); this is probably due to staggering of aerodynamic and structural solutions in the nonlinear simulation. A final test of the critical gust response at a fine spatial and temporal discretization ($M = 16, N = 40, \Delta t = 8.164 \times 10^{-4}$ s) shows that the nonlinear and linear models are effectively converged (Figure 6.7).

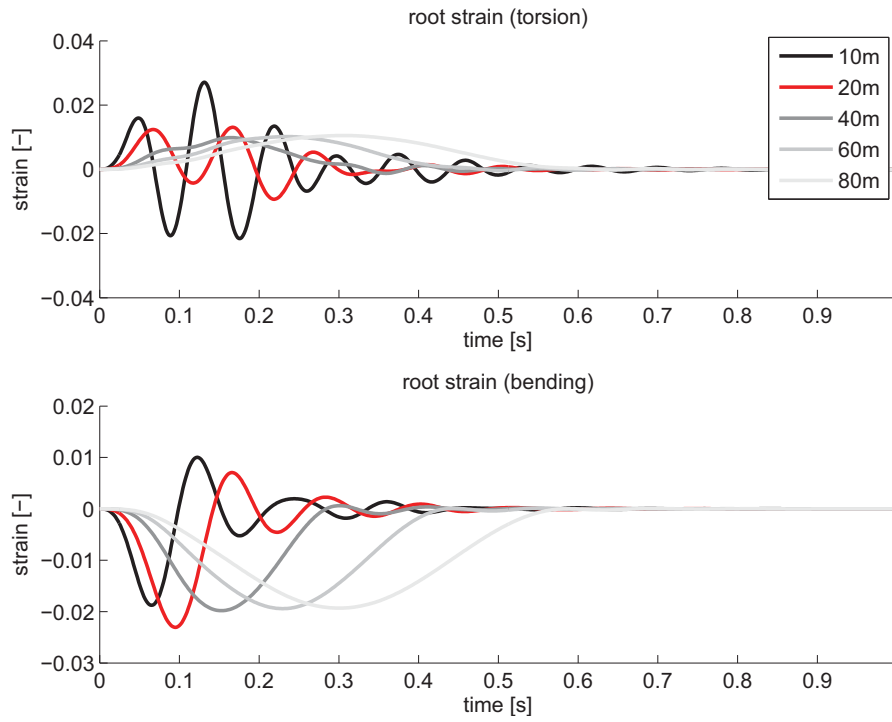
6.3.2 Aeroelastic model reduction

Model-order reduction of the linear system was performed using truncation of the balanced realizations described in Section 6.1. The flap deflection, β , was chosen as the sole control input to the system, and the root bending and torsional strains were chosen as the outputs. The quality of reduced models of various model orders was assessed by comparing their relative time- and frequency-domain forced responses. Time-domain wing responses to flap oscillations at 15 and 30 rad·s⁻¹ are shown in Figure 6.8. Here the nonlinear and full-order linear response is very similar in both the 15 rad·s⁻¹, Figure 6.8(a), and 30 rad·s⁻¹, Figure 6.8(b), cases. The reduced-order model with 10 states is shown to approximate the forced response of the system well in both cases too, however when the model order is reduced to 8 states small discrepancies are observed in the magnitude and phase of the predicted oscillations, as shown in Figures 6.8(a) and 6.8(b).

To further investigate the characteristics of the reduced models they were subjected to



(a) full-order linear model



(b) nonlinear model

Figure 6.6: Open-loop gust response at a range of gust lengths.

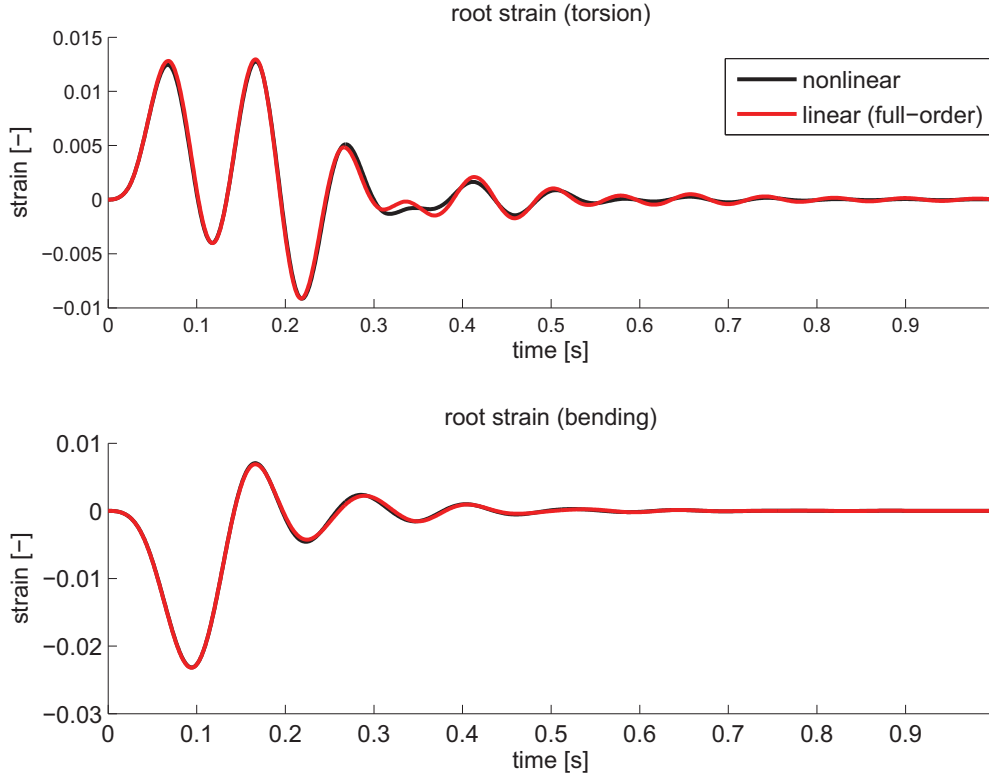


Figure 6.7: Comparison of linear and nonlinear models for the critical gust length, $L = 20\text{m}$, using a fine discretization.

frequency-domain forced response analysis. Bode plots of the response of a range of reduced models are shown in Figure 6.9 with the model order varied from 8 to 100 states. The reduced frequency shown on the horizontal axes is defined as $k = \omega c / 2V_\infty$, where ω is the flap angular velocity and V_∞ is the free-stream flow velocity. For reduced systems of size 10 to 100 the gain and phase plots are very similar up to a reduced frequency of approximately 1.5. The 8 state model shows departures in gain and phase from the higher-order models even at low reduced frequencies which seems to corroborate the relatively poor time-domain comparison in Figure 6.8. Despite these small discrepancies, the 8 state model will allow faster MPC and is appropriate for control synthesis, as the results of the following section demonstrate.

6.3.3 Closed-loop simulations

In this section predictive and LQR control schemes formulated with an 8 state reduced model, which provides a compromise between fidelity and performance, are applied to the nonlinear plant simulation. Unless otherwise stated the closed-loop response of the plant under MPC

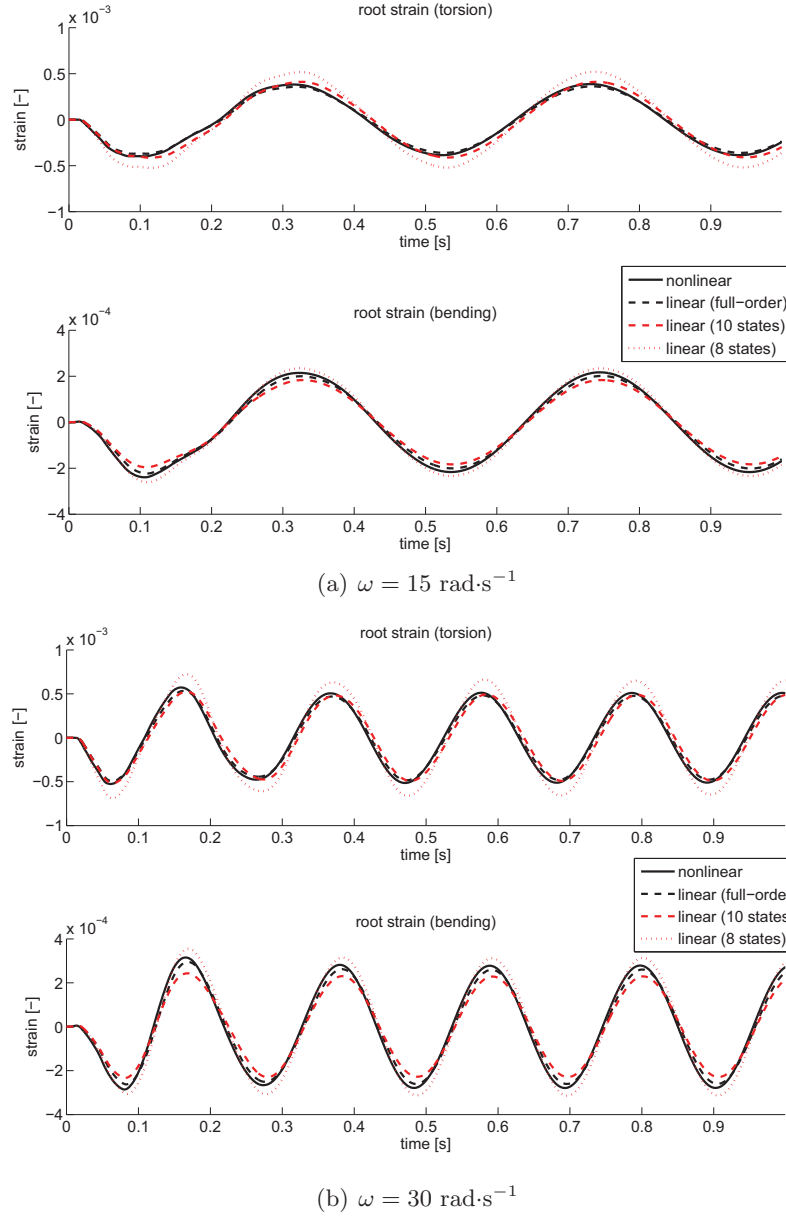


Figure 6.8: Time-domain forced response of nonlinear, full-order linear and reduced-order linear models of the Goland wing with trailing-edge flap oscillations.

was obtained using $H = 100$ steps in the prediction horizon and a sample interval of 1.633 ms as stated in Table 6.3. The reference conditions on both the states and control inputs is the origin, hence the control systems are designed to provide state regulation. In addition, input constraints are specified that correspond to maximum flap deflections of ± 10 deg.

The first control schemes applied here are based on unit weighting of the reduced-order (balanced) plant states and inputs, i.e. the matrices P , Q and R in (6.5) and (6.10) are

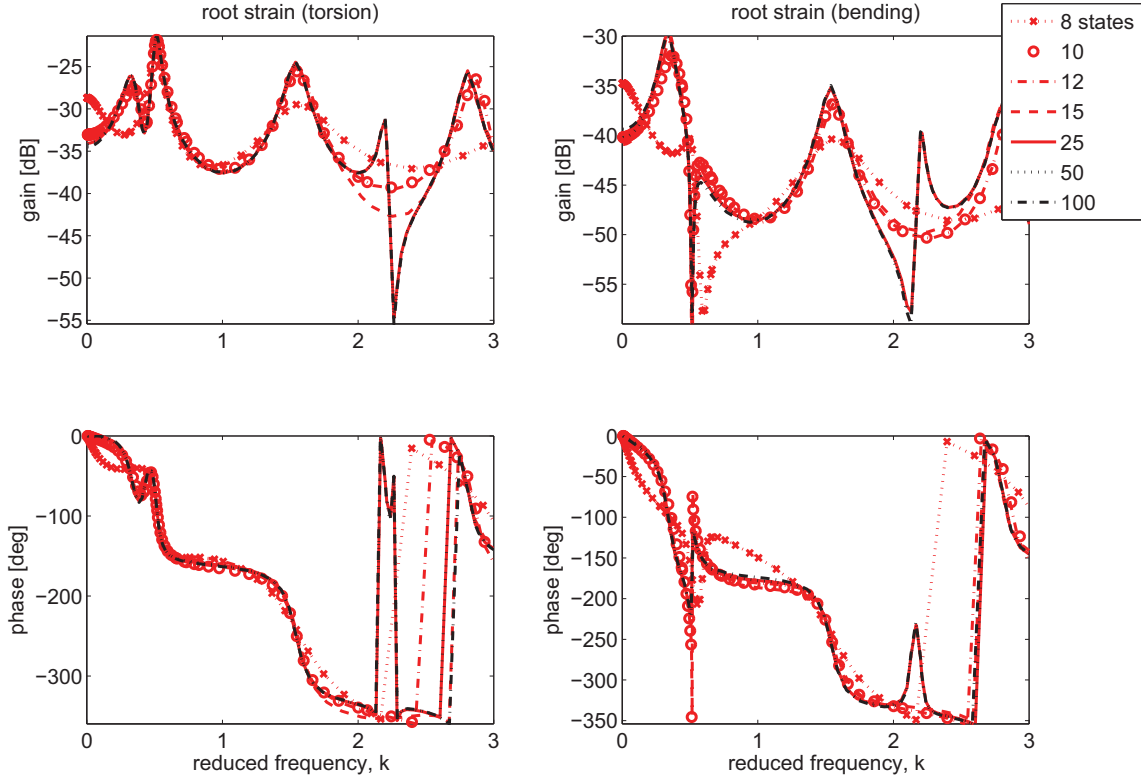


Figure 6.9: Frequency response of the Goland wing to flap inputs calculated using balanced models of varying order.

identity matrices, since these states have no immediate physical interpretation. The open- and closed-loop responses, using both MPC and LQR, to the critical gust are shown in Figure 6.10. Regulating the model states serves to reduce peak torsional strain with either MPC or LQR by approximately 25% with a corresponding increase in maximum bending strain of approximately 13%. The control action required to produce these results is shown in Figure 6.11. While the LQR controller is saturated for two periods at the start of the gust response, the MPC controller does not owing to the finite horizon in the control formulation. In general an MPC controller can anticipate future constraint violations and provide an optimal control response that acknowledges the presence of these constraints as they are included in the online optimization of (6.5)–(6.9).

The second and third control schemes demonstrated in this work are MPC schemes based on unit weighting of the system outputs, i.e. bending and torsional strains, and unit weighting of bending strain alone. This is justified on the basis that results of the gust-loads analysis show the outputs have the same order of magnitude. In both cases the control inputs are also

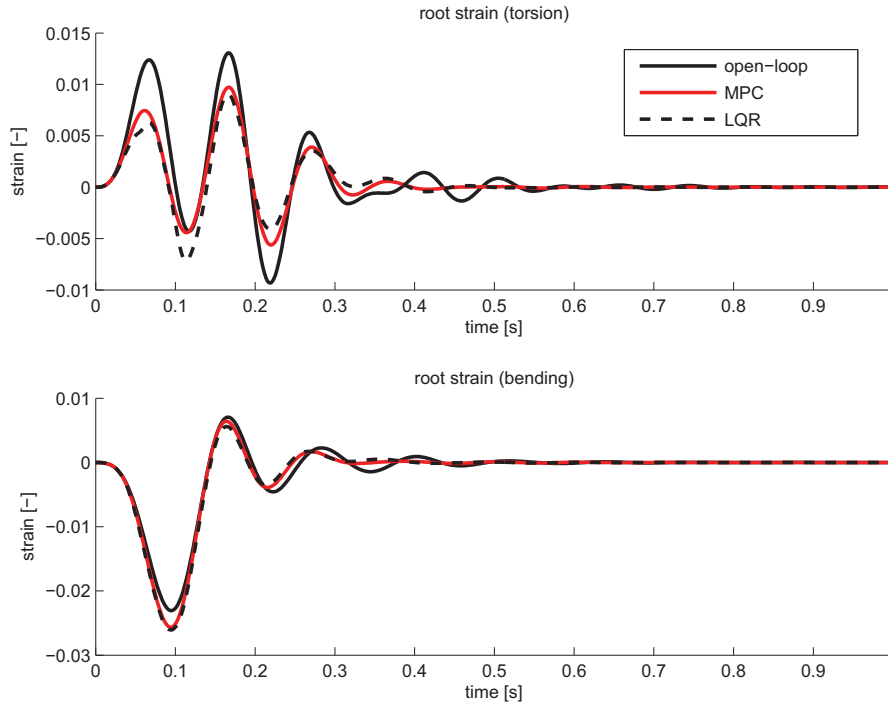


Figure 6.10: Gust load alleviation on the Golland wing using MPC and LQR with unit weighting on the reduced states and inputs. Regulating the model states serves to reduce peak torsional strain with a corresponding increase in bending.

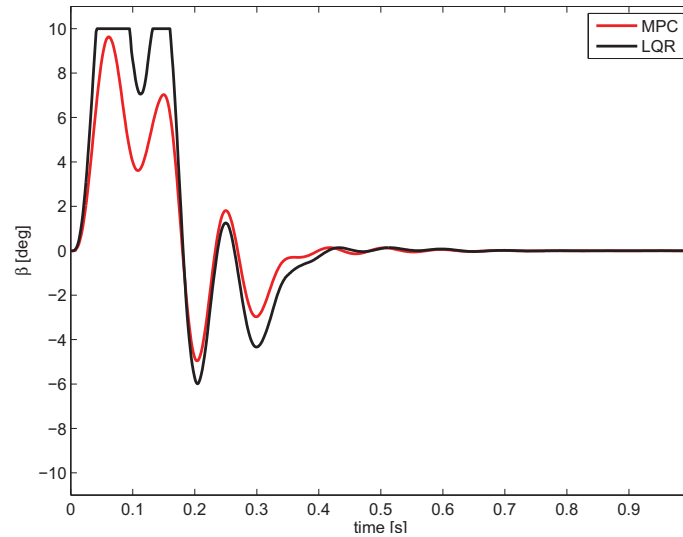


Figure 6.11: Control response commanded in response to critical gust with unit weighting on the reduced states and inputs.

given unit weighting. The results of Figure 6.12 demonstrate that the closed-loop performance of system could be effectively tuned using this output weighting. If both outputs are weighted then a 2% reduction in peak bending strain is observed alongside 8% reduction in peak torsional strain. If reduction of peak bending is prioritized then a 10% reduction is realized using unit weighting on bending strain alone, however there is a relatively large corresponding increase in torsional strain.

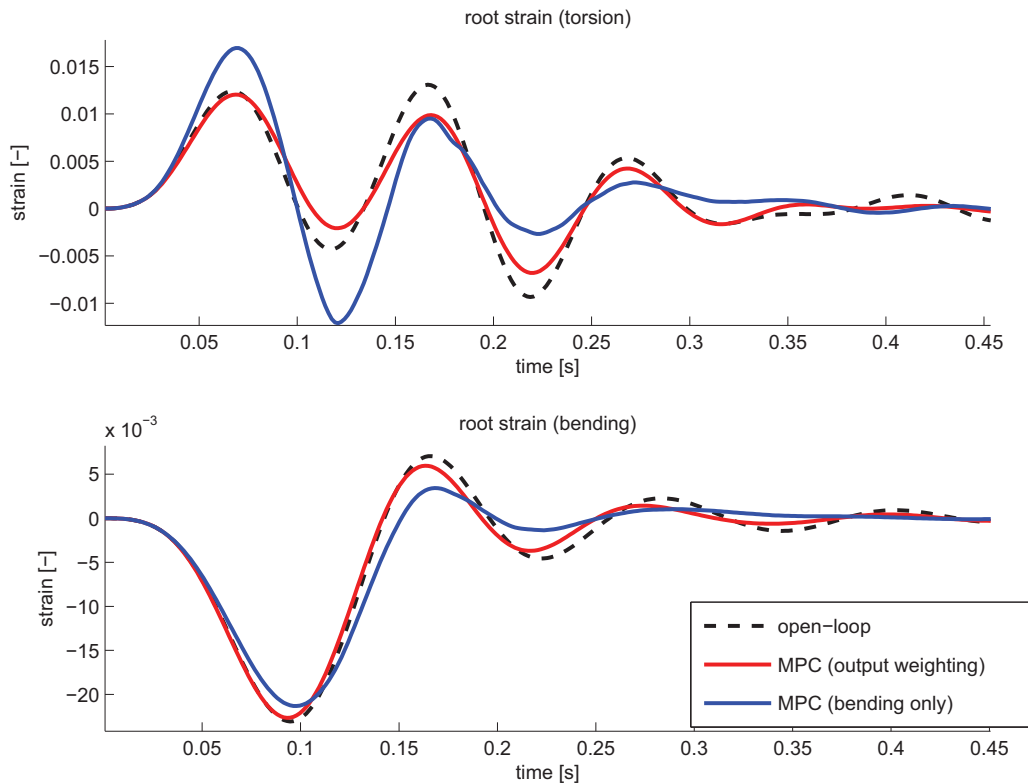


Figure 6.12: Closed-loop performance for critical gust on the Goland wing when unit weighting on the reduced system outputs is used (red line), and when only the bending strain is prescribed unit weighting (blue line).

The effects of horizon length on the closed-loop performance of the bending-only MPC schemes presented above are now investigated in detail. Performance metrics, which are chosen as the 2- and ∞ -norm of the bending strain time-histories, are shown in Figures 6.13(a) and 6.13(b) respectively, and are normalized with respect to the performance of the infinite-horizon LQR controller. For the critical gust used in this exercise this LQR scheme reduced the 2-norm of the bending response by 2.3% and the ∞ -norm by 1.4%. Moreover, since there is no saturation in the control output the closed-loop behaviour of the MPC

controller converges to that of the LQR as H is increased. Short horizons cause the controller to act more aggressively in response to the gust disturbance, as shown in the flap angle time-histories of Figure 6.14(a), which has the effect of increased controller performance for all but the shortest horizon lengths. Prediction enhancement (PE), as described in Section 6.3, is beneficial to the performance of the MPC schemes in all cases; this is because discrepancies between the nonlinear plant simulation and the internal model, due in part to the plant's response to the gust disturbance, are accounted for in the MPC reference trajectory. As a result the corresponding control action is more aggressive as shown in Figure 6.14(a). The highest flap deflection rate of any MPC controller was $234.6 \text{ deg} \cdot \text{s}^{-1}$ ($H = 150$) while the LQR commanded $234.3 \text{ deg} \cdot \text{s}^{-1}$ – these rates are very fast due to the high-frequency dynamics of this test case and are perhaps beyond the capabilities of any real-life actuator.

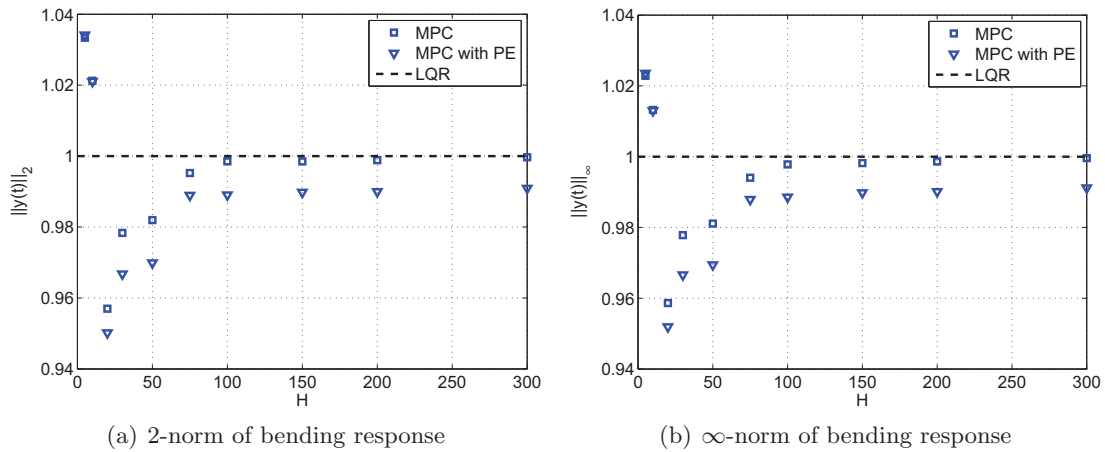


Figure 6.13: Norms of the bending response to critical gust with unit weighting on root bending strain and flap deflection. A range of prediction horizons are used in the MPC schemes which are implemented with and without prediction enhancement (PE). Results are normalized against the closed-loop response using an LQR control scheme.

To investigate the relative behaviour of MPC and LQR in situations featuring saturation of both schemes the closed-loop response to the critical gust is simulated using controllers synthesized with $R = 0.1$ weighting on the control inputs. The resulting control response is shown in Figure 6.14(b) in which the MPC schemes are seen to saturate relatively-quickly in anticipation of meeting the constraints. Performance of MPC schemes using a range of horizon lengths are compared to that of the LQR controller using 2-norm and ∞ -norm metrics in Figure 6.15. The performance of MPC schemes in this case is generally worse than the

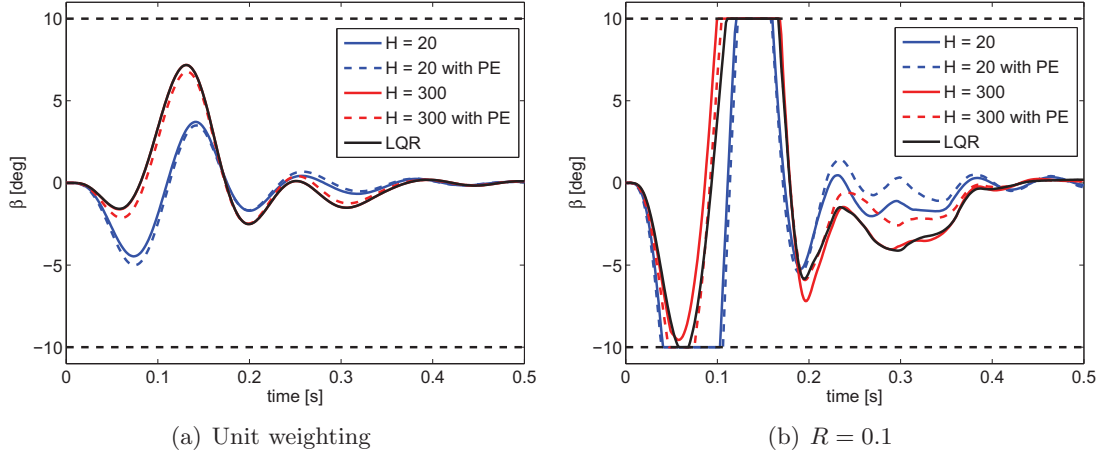


Figure 6.14: Commanded flap response to the critical gust using bending-only control schemes.

LQR schemes for large horizon lengths but marginally better using short horizons with or without prediction enhancement, and once again the prediction enhancement improves the controller performance according to both metrics. Relative to the open-loop gust response the reference LQR response reduces the 2-norm of the bending strain response by 11.5% and the ∞ -norm by 1.0% in this case.

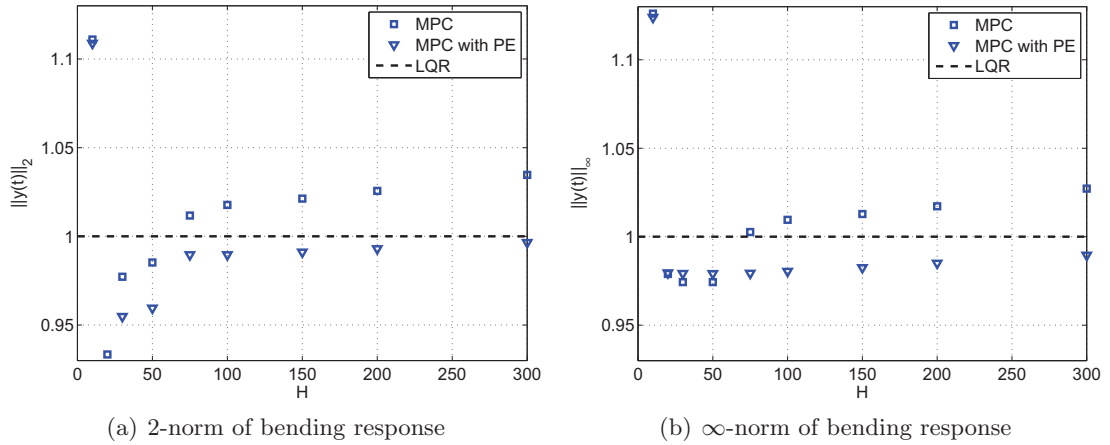


Figure 6.15: Norms of the bending response to critical gust with $R = 0.1$. A range of prediction horizons are used in MPC controllers with and without prediction enhancement (PE). Results are normalized against the closed-loop response using an LQR control scheme.

Finally the timing characteristics of the aforementioned MPC schemes are explored. Since the optimization of (6.5) - (6.9) must be solved online, it must have a time-to-solve that

is less than the sample time of the system. Figure 6.16 shows the minimum, maximum and mean times taken to solve the MPC control problem using muAO-MPC[158] generated, single-thread C-code accessed through a Python interface on a 2.93GHz Intel® Core™ i7 desktop CPU. A range of prediction horizons were investigated resulting in varying closed-loop performance and an increase in time-to-solve as the prediction horizon is increased. Using a horizon of 60 steps gives acceptable closed-loop performance with a mean time-to-solve of 1.31 ms, although there were occasional maxima reaching up to 2.9 ms. These results are very much hardware dependent and superior performance would likely be achieved using hardware designed specifically for the task. It is also worth noting that the system used to compute the control response was running an operating system and the nonlinear plant simulation at the same time, which probably contributed to some of the maxima observed in controller time-to-solve. While the Goland wing is relatively stiff it can be expected that the structural response of very flexible aircraft will be an order of magnitude lower in frequency, which is advantageous for control system design and operation. However, this comes at the cost of increased geometric nonlinearity in the plant due to larger dynamic deformations.

6.3.4 Gust-load alleviation on a flexible wing

Chord	1.8288 m
Semi-span	30.48 m
Elastic axis	33% chord
Center of gravity	23% chord
Mass per unit length	35.71 kg·m ⁻¹
Moment of inertia	8.64 kg·m
Torsional stiffness	0.99 × 10 ⁶ N·m ²
Bending stiffness	9.77 × 10 ⁶ N·m ²
Free stream velocity	28 m·s ⁻¹
Angle-of-attack	0 deg
Air Density	1.02 kg·m ⁻³
Control surfaces	1 TE flap at tip
Flap size	25% chord, 20% span
Measurements	Root torsion and bending strains
Sampling interval, Δt	8.164 × 10 ⁻³ s
Model discretization	$M = 8, N = 20$

Table 6.4: Extended Goland wing properties.

In the following example the same modeling and control synthesis procedure is applied to an extended version of the aforementioned Goland wing. The modified wing is a factor

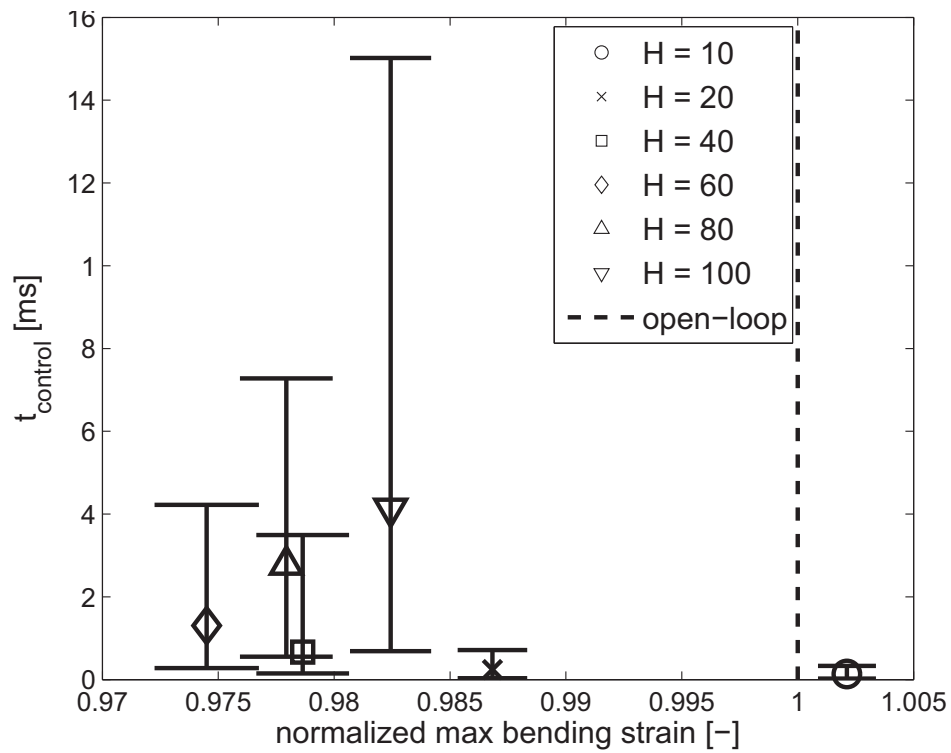


Figure 6.16: Minimum, maximum and mean time taken to solve the online MPC problem for a range of horizon lengths, plotted against closed-loop performance. In this case unit weighting was prescribed to the system outputs and input.

of 5 longer than the standard definition, which results in a planform with an aspect-ratio of approximately 17. In addition, the center-of-gravity of the wing section is moved fore of the elastic axis and the free-stream velocity is reduced to $28 \text{ m}\cdot\text{s}^{-1}$ to avoid flutter. The wing geometry, mass properties, reference condition and control system details are given in Table 6.4 for completeness. Because the wing is now very slender, large deflections occur when it is subjected to the same 20 m long ‘ $1 - \cos$ ’ gust used in previous simulations.³ Vertical tip deflections of the modified wing, plotted in Figure 6.17, show that deflections of up to 24% of the span occur in this case, indicating that the dynamics may be geometrically nonlinear. Plotting the open-loop response predicted by the linear model against that of the nonlinear plant simulation (Figure 6.18) illustrates significant differences in predicted root strains which are probably caused by nonlinear coupling between bending and torsion degrees-of-freedom due to large displacements and finite rotations. Despite this nonlinear behavior the control schemes, synthesized using 8 state reduced-order linear models, are still effective at reducing the peak tip-deflection, shown in Figure 6.17, and bending and torsional strains, shown in Figure 6.19. It is worth noting that these closed loop results are obtained using naïvely tuned controllers for which unit weighting on the bending strain and control outputs was chosen. Better performance could probably be obtained through tuning of the weighting matrices in the MPC and LQR optimization problems.

³This results in a peak effective angle-of-attack far beyond the stall of the wing (which is not captured using this aerodynamics method) however the intention here is simply to test the control approach in a situation with nonlinear deflections (accepting the fluid mechanics are not accurate).

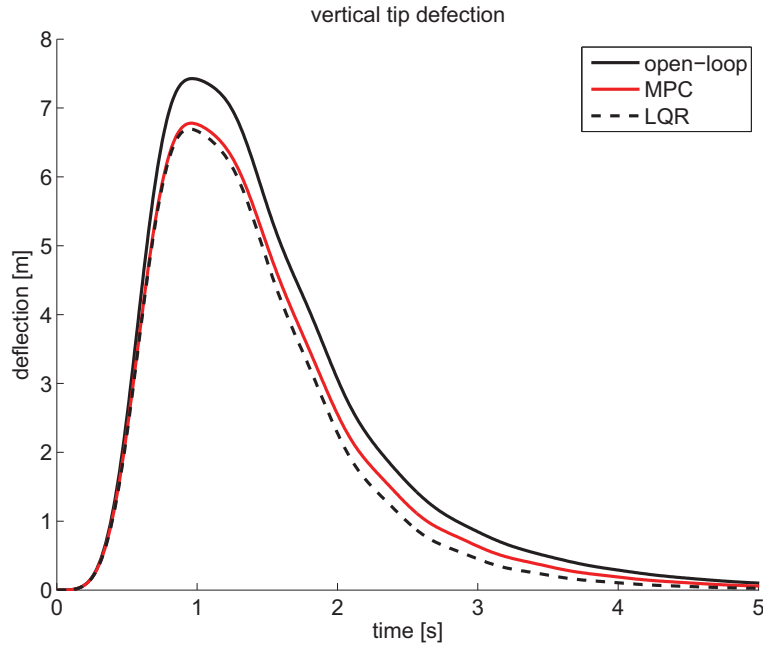


Figure 6.17: Time history of the extended Goland wing tip deflection in response to a 20m ‘1 – cos’ gust in open- and closed-loop simulations.

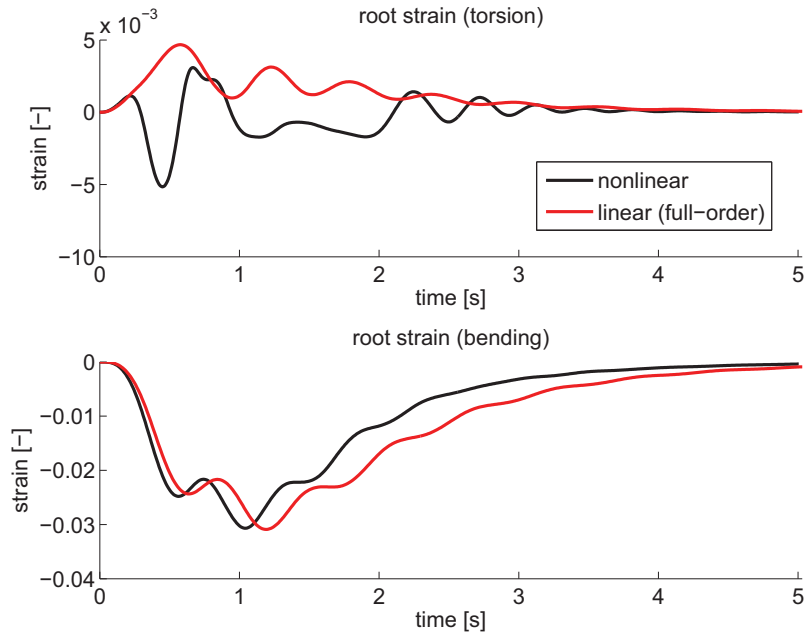


Figure 6.18: Time history of the extended Goland wing root strains in response to a 20m ‘1 – cos’ gust calculated using full-order linear and nonlinear models.

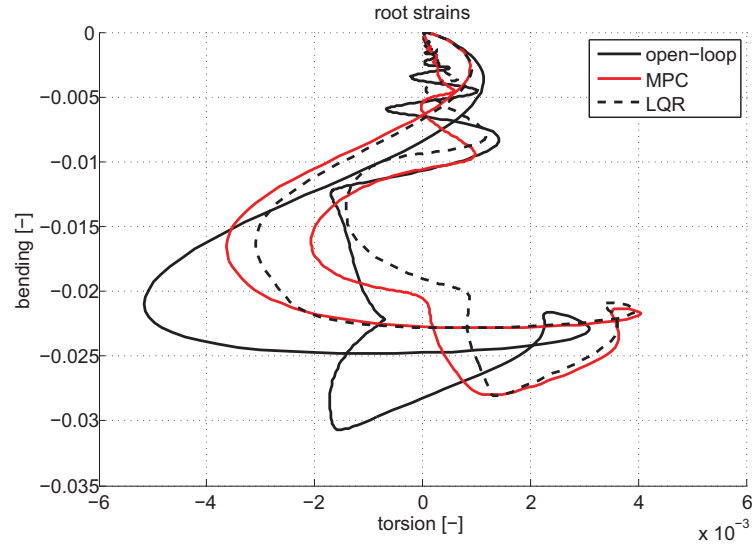


Figure 6.19: Loci of bending versus torsional strain at the root of the modified Goland wing in open- and closed-loop simulations of a gust encounter.

Chapter 7

Conclusions

This chapter concludes the thesis by highlighting the main outcomes of the research effort. An overall summary of the developments made is given in Section 7.1 which comments on the main theoretical developments, development of the modelling environment (Simulation of High-Aspect-Ratio Planes in Python, **SHARPy**), and the results of numerical simulations carried out with these tools. These are then concisely summarized in the key contributions of Section 7.2. Finally, some future applications of the modelling framework are suggested in Section 7.3 alongside suggested improvements to the existing methods.

7.1 Overall conclusions

The theories described in this work are implemented in a flexible aircraft dynamics analysis framework called Simulation of High-Aspect-Ratio Planes in Python (**SHARPy**). In doing so, a hierarchy of nonlinear, consistently-linearized, and reduced-order (linear) aeroelastic models have been created. Aerodynamic analysis is provided by a discrete-time vortex-lattice-based model, described in Chapter 2, that can capture nonlinear effects expected of flexible aircraft, such as large dynamic deformations, the effect of a force-free wake, and the interference between lifting-surfaces and their wakes. The nonlinear geometrically-exact beam dynamics model described in Chapter 3 captures large, dynamic displacements and rotations of aircraft primary structures in a moving frame-of-reference, and includes the effect of nonlinear rigid-body translations and rotations in a flexible-body dynamics formulation. Consistent linearizations of these methods, which can capture the effects of arbitrary non-zero reference conditions and all the linear couplings in their respective degrees-of-freedom, have also

been described. Coupling of these models at their various levels of fidelity has been achieved through the fluid-structure interfaces described in Chapter 4, in which static/dynamic nonlinear and linearized aeroelastic modelling approaches are described. The methods comprising the SHARPy framework were then extensively verified in Chapter 5, followed by the construction of reduced-order models and their application to aeroelastic stability analysis and active control of aeroelastic systems in Chapter 6.

7.1.1 Theoretical developments

The main theoretical focus of this thesis has been the development of an aerodynamic theory that can capture pertinent effects in the modelling of very flexible aircraft dynamics. A three-dimensional, discrete-time, vortex-lattice-based approach was chosen. In contrast to 2-D theories [77, 115] this approach captures finite-wing effects and the interference of aerodynamic surfaces and their wakes, and enforces the fluid boundary conditions on the instantaneous deformed geometry of the aircraft, unlike the doublet-lattice method (DLM) [2], which is the most common three-dimensional method.

Leading-edge suction effects have also been included in this model which allows accurate prediction of unsteady induced-drag forces. Although the concept itself is certainly not novel, expression of the two existing methods that do this in a form suitable for the complex kinematics expected in flexible aircraft applications is. The *Katz & Plotkin method*, which has its origins in the book by those authors [71], has been generalized for complex motions. An alternative approach is also described, namely the *Joukowski method*, which is based on the three-dimensional (vector) form of the Joukowski theorem and so resolves all components of the aerodynamic forcing, and naturally includes the induced-drag components, including those from in-plane motions.

Critically, for dynamic stability analysis of flexible aircraft and the synthesis of model-based control schemes, a linearization of the aerodynamics must be obtained. This work proposes a consistent linearization (one which includes all of the dynamics related to arbitrary non-zero reference states) of the general nonlinear vortex-lattice aerodynamics formulated using the Joukowski force calculation method. No assumptions are made relating to the kinematics of the fluid-structure interface (inputs) and use of the three-dimensional Joukowski relation naturally resolves all components of the unsteady aerodynamic forcing (outputs). This model is nondimensionalized (independent of dynamic pressure) and is formulated such

that it is completely independent from any structural model, with as general input/output as possible, i.e. the inputs are linear displacements and velocities of the underlying lattice geometry, and the outputs are the aerodynamic forces distributed on the lattice. The method has been implemented in an efficient framework, and has been verified against unsteady aerodynamic theories and through flutter predictions made with the model.

As opposed to the industry-standard DLM, which samples unsteady aerodynamic responses at discrete frequencies, the discrete-time method presented here captures all frequencies up to a Nyquist limit defined by the spatio-temporal discretization of the wake. Instead of using rational function approximations in frequency space, which involve a curve fit, and a subsequent inverse Laplace transform to obtain time-domain models, the method presented in this thesis directly provides a discrete-time model of the aerodynamics; the state-space equations developed here also have a rational function representation, which can be obtained without any approximation.

As with the DLM, and other frequency-domain based approaches [72], the linearized aerodynamics of this work can be written in terms of the modal coordinates and generalized forces of an underlying structural dynamics model. So, given the increased fidelity of the vortex-lattice-based method presented here, and the fact that it is written directly in the time-domain as a discrete-time state-space model, with no further assumptions in its frequency-domain representation either, the new method could replace or improve upon the DLM in any aeroelastic problem (with incompressible flow). Moreover, for the purposes of very efficient aeroelastic stability analysis, gust load modelling, or as the model in an active control scheme, reduced-order balanced realizations of the discrete-time aerodynamics are available with an optimal model-order from an input/output energy point-of-view.

7.1.2 Numerical simulations

Simulations of the steady and unsteady aerodynamics of aerofoils and finite-wings carried out using the SHARPy framework have shown good agreement with analytical theories, experiment and numerical methods of other authors, which has served to verify the implementation of the aerodynamic model. In particular, the convergence properties of the force calculation methods show that the Joukowski method provides higher-accuracy induced-drag results using coarse discretizations on problems featuring low-frequency unsteadiness, like those expected in flexible aircraft dynamics. Linearized aerodynamics models of aerofoils undergoing pitch,

plunge and trailing-edge flap oscillations and a traveling vertical gust were also shown to converge to linear theories with increasing number of vortex rings used to discretize the domain. Under the assumptions of incompressible, attached flow the vortex-lattice-based methods presented in this work were found to be accurate, and are of increased fidelity and utility compared to two-dimensional theories and linear three-dimensional aerodynamics provided by the DLM.

These aerodynamics models were used in aeroelastic calculations of the static response of a representative high-altitude long-endurance (HALE) aircraft wing. In this nonlinear setting the convergence of the geometrically-exact beam finite elements, which contain an approximation in the spatial interpolation of finite rotations, were found to converge rapidly using quadratic (3-noded) elements. The nonlinear response of the HALE wing was found to closely match those of authors who had used volume-based Euler equation fluid mechanics model and a similar nonlinear beam model. This vindicates the medium-fidelity approach taken in this thesis, and verifies the implementation in cases featuring large wing deflections.

Dynamic aeroelasticity, specifically flutter instability, was investigated using the aforementioned nondimensional aerodynamics model, and a state-space modal model of beam dynamics written in the same reduced-time basis as the aerodynamics. The new method was found to predict very similar flutter characteristics of the Goland wing to the analysis carried out by other authors using a numerical approach based on three-dimensional aerodynamics. Also, the flutter of forward-swept isotropic wings was predicted in agreement with previous computational work and experiments. The divergence of these models was also predicted using a divergence index method, and served to verify the implementation further.

As a demonstration of the methods, the body-freedom flutter (BFF) characteristics of swept-flying wings were then investigated using linearized aeroelastic models of the unconstrained vehicle dynamics constructed using the aforementioned methods. Changing the mass distribution of the configuration was found to affect the body-freedom flutter mode quite strongly. Specifically, moving a pair of concentrated masses outboard on each wing was found to delay the BFF instability but lower the velocity at which a higher-frequency antisymmetric bending-torsion flutter would occur. It is likely that the preliminary design of aircraft of this type would benefit enormously from the medium-fidelity unconstrained aeroelastic analysis methods presented in this work.

It was then demonstrated that balanced realizations [43, 90] can be used to obtain very-

low-order representations of the aforementioned discrete-time aerodynamics models. Initially, this was investigated in the context of efficient stability analysis. When the models were written in the modal coordinates of a cantilever beam model the aerodynamic frequency response could be reproduced at all relevant reduced frequencies using 12 balanced states, as compared to the 8800 physical states required for a converged result with the full-order model. Flutter characteristics calculated using a 9-state reduced-order aerodynamics were converged to within less than 0.5% of those predicted by the full-order system. Coupled with the model reduction approach explored in this work the discrete-time form of the aerodynamics is an attractive alternative to existing methods in stability analysis, especially in the context of the dynamics of flexible aircraft where complex non-planar kinematics are expected and there may be significant changes to the system dynamics due to the reference condition.

Reduced-order aeroelastic models were also calculated, and used as the basis of predictive controllers and linear-quadratic regulators applied in linear and nonlinear simulations for gust load alleviation. It was found that an 8-state aeroelastic model was adequate for synthesizing controllers applied to the nonlinear plant model for the purposes of alleviating gust loads using a trailing-edge flap. Reductions in the root strain measures of a cantilever wing configuration were realized, with the predictive controllers avoiding saturation through inclusion of flap angle constraints in the control formulation. The time it took to calculate control actions by solving the online optimization required for model predictive control was in the order of milliseconds, which, given the relatively-slow dynamics of flexible aircraft, gives some indication that the approach is feasible from a timing perspective. Finally, similar control schemes were applied to a flexible wing featuring deflections of 24% of the span under gust conditions, and were once again found to be effective, this time in situations featuring the large dynamic deflections expected of flexible aircraft.

7.2 Key contributions

Three-dimensional time-domain models of flexible aircraft aerodynamics have been shown to be directly obtainable through linearization of the UVLM equations that use the Joukowski force calculation method. In doing so it is not necessary to make any assumptions relating to the kinematics of the fluid-structure interface (other than that they are small perturbations on a potentially nonlinear reference configuration) and all components of the unsteady aerody-

dynamic forcing are naturally resolved. The large number of states in the wake of these models, when converged, makes them unwieldy for routine aeroelastic stability analysis, however this can be overcome by balanced realizations of the aerodynamics. This is most effective when a reduced basis for the aerodynamic input/output is found, which, in-keeping with traditional frequency-domain methods, is often the modal coordinates obtained from *in vacuo* vibration analysis of the corresponding structural model, which may be obtained at non-zero reference conditions. If the aerodynamic equations are written in a form that removes their dependency on dynamic pressure, using a nondimensionalization scheme, then the same reduced model can be used for all (incompressible) dynamic pressures of interest, yielding a very efficient method for aeroelastic stability analysis. The reduced-order aerodynamics are also very useful for synthesizing active control schemes based on coupled aeroelastic models.

The following key steps were required to realize the conclusions summarized above:

1. Full articulation and evaluation of two force calculation methods in the UVLM that resolve all force components (including induced-drag) under arbitrary kinematics.
2. A novel linearization of the general geometrically-nonlinear UVLM equations that yields directly a set of nondimensional (independent of dynamic pressure) linear time-invariant (LTI) discrete-time state-space equations of motion. The linearization is formulated around a non-zero reference state and naturally resolves all components of the linear aerodynamic forcing, with no assumptions on the kinematics of the fluid-structure interface.
3. Integration of the above aerodynamic models in a hierarchy of aeroelastic analysis tools, brought together in a new efficient modular aeroelastic modelling framework, **SHARPy**, which was developed in this work and will form the basis of future research.
4. Balanced reduction of modally-augmented *time-domain* unsteady aerodynamics; thereby avoiding the approximations in converting traditional frequency-domain methods to the time domain, and providing extremely efficient and relatively-high-fidelity models for stability analysis and time-domain simulations.
5. Application of predictive and static control schemes for gust-load alleviation, synthesized using reduced-order realizations of the aforementioned aeroelastic models, on nonlinear simulations.

7.3 Recommendations for future work

The efficient, modular framework developed in this thesis has manifold future applications as a research tool, and is open-source, so will ideally become a point of collaboration between researchers, institutions and even commercial interests. There is still much work to be done, not only on the methods and the framework, but also in the design of aerospace (and wind energy) systems with these tools. The main themes envisaged are as follows:

Flight dynamics of high-altitude long-endurance aircraft, including, but not limited to, swept flying-wing configurations, including the effects of trim/manoeuvre loading on the system dynamics. The work of [93] and [58] has already made significant contributions to this field. However, recent developments have created opportunities for verification of the methods with flight test data from aeroelastic test beds, namely the X-56 configurations, now at NASA [129], and the X-HALE test aircraft under development at the University of Michigan [21].

Active control of flexible aircraft dynamics for stabilization [62], load control [61] and trajectory control. The gust environments relevant to very large, slow, lightweight designs are yet to be fully defined and very interesting discrete gust encounters may be important, such as wind-shear [100] and thermals within the length-scale of the vehicle span. The coupled lateral dynamics in such conditions are relatively unexplored. For this purpose the model predictive control schemes synthesized in this work, on flexible cantilever wings, could be developed further and applied to the full-vehicle dynamics, with a view to obtaining stabilizing, optimal schemes.

Multi-disciplinary optimization. This could include optimization of wing structures for aeroelastic criteria using aeroelastic tailoring of composite materials (since the beam model can handle material anisotropy) or co-design (simultaneous optimization) of flexible aircraft configurations and their control systems, which at the time of writing is already underway using the SHARPy framework. Of particular interest is the design of solar-powered aircraft, the design of which is based on power considerations – a coupled airframe, control, and trajectory optimization problem is envisaged.

Improved flight dynamic fidelity using, primarily, viscous information such as the steady para-

sitic drag, which is important for calculating the dynamics of the phugoid mode, for example. This could be achieved by offsetting the relevant aerodynamic loading in the reference condition prior to linearization of the coupled system dynamics. Such effects would no doubt affect the trim state of flexible aircraft and therefore the dynamic characteristics of the aircraft powerplants. Dynamic engine models, including the angular momenta of rotating machinery, could also be explored in this context.

Inclusion of nonlinearities possibly leading to limit-cycle oscillations and chaotic super-critical behaviour. Possible sources of nonlinearity include static and dynamic stall [111], and local material effects such as hardening or free-play in articulated systems.

Further leverage of model reduction methods for design and testing. Model reduction of unstable aeroelastic systems, using coprime factorization, for example, and interpolation of reduced-order models [3, 4] for efficient stability analysis and design. Proper-orthogonal decomposition [152] or other snapshot-based methods could be used, although these lack the predictive capability of model-based approaches like the one used in this thesis. This could facilitate real-time simulation of flexible aircraft dynamics to enable pilot-in-the-loop assessment of control systems.

Appendices

Appendix A

Connectivity and Interpolation in the UVLM

In order to facilitate expressions that relate the various degrees of freedom in the method a relationship must be found between indices that describe the following: the relative chordwise and spanwise locations of panels used to discretize each body/wake geometry, the position of each panel in a list of all the panels in each body/wake lattice, and the position of the corner points of each panel in a list of all of the nodes that make up the vertices of each lattice. In addition to this, it is useful to be able to interpolate and distribute quantities between or to the corners of each panel, particularly for the purposes of interfacing the UVLM with methods for structural analysis. This appendix describes the schemes used to meet both these requirements.

Each surface in the UVLM is discretized using a lattice topology whereby the chordwise extent is split using M panels, and the spanwise extent using N panels. The wake of each surface may have considerably more chordwise “panels,” denoted M_w , but must have the same number of spanwise panels. Therefore the total panels on the body and wake lattices are $K = MN$ and $K_w = M_w N$, respectively. The relative position of panels within each lattice geometry is described using lattice indices, i and j , using the convention illustrated in Figure A.1.

Using this convention the chordwise lattice index takes the values $i = 1, \dots, M$, while the spanwise index takes $j = 1, \dots, N$. The panel index, which describes the position of each panel in a list of all the panels in each body or wake, takes the values $k = 1, \dots, K$,

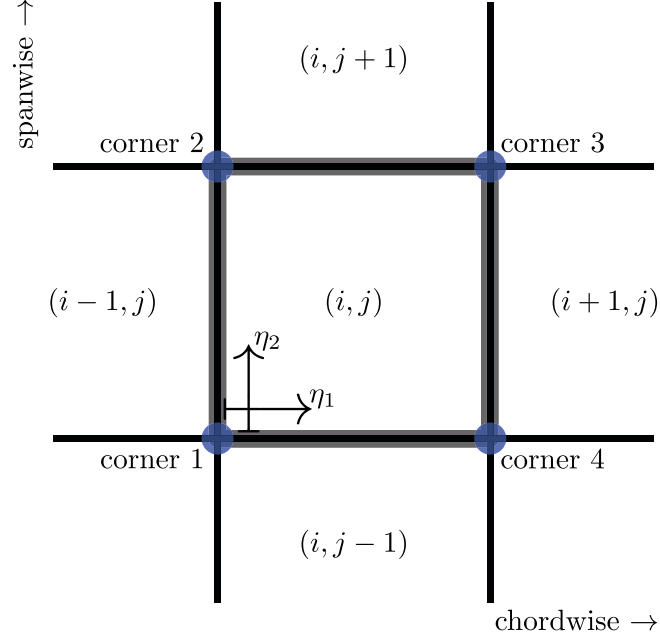


Figure A.1: Lattice indices (i, j) , computational coordinates (η_1, η_2) , and panel corner points.

and is related to the lattice indices through the relations

$$\begin{aligned}
 k &= N(i-1) + j \\
 i &= (k-1) \setminus N + 1 \\
 j &= (k-1) \bmod N + 1
 \end{aligned}
 \tag{A.1}$$

where $\setminus$ is the integer division operator and $\bmod$ is the modulo operator. This is to say that the panel index counts predominantly in the spanwise direction.

The same counting convention is employed for the grid index which is given by $q = 1, \dots, K_\zeta$, where $K_\zeta = (M+1)(N+1)$ is the total number of vertices on each lattice. Expressing the grid indices of the four panel corner points (see Figure A.1) in terms of the

lattice indices yields

$$\begin{aligned}
q_{ij_1} &= (N+1)(i-1) + j \\
q_{ij_2} &= (N+1)(i-1) + j + 1 \\
q_{ij_3} &= (N+1)i + j + 1 \\
q_{ij_4} &= (N+1)i + j
\end{aligned} \tag{A.2}$$

where, for example, q_{ij_1} , is the grid index of corner 1 on the i - j th panel. Consequently, the relationships between grid indices of the panel corner points and the panel index are

$$\begin{aligned}
q_{k_1} &= (N+1)((k-1) \setminus N) + (k-1) \bmod N+1 \\
q_{k_2} &= (N+1)((k-1) \setminus N) + (k-1) \bmod N+2 \\
q_{k_3} &= (N+1)((k-1) \setminus N+1) + (k-1) \bmod N+2 \\
q_{k_4} &= (N+1)((k-1) \setminus N+1) + (k-1) \bmod N+1
\end{aligned} \tag{A.3}$$

which are necessary for describing the relationships between panel and grid quantities in the method's degrees-of-freedom, for example, between the downwash at a collocation point and the motion of its vertices.

Interpolation of quantities within each panel is performed through bilinear interpolation based on the local computational coordinates η_1 and $\eta_2 \in [0 \ 1]$, illustrated in Figure A.1. The interpolated value of a vector quantity, for example $f \in \mathbb{R}^3$, at a point P within the panel is given by

$$f_P = [(1-\eta_2)(1-\eta_1)\mathcal{I} \quad \eta_2(1-\eta_1)\mathcal{I} \quad \eta_2\eta_1\mathcal{I} \quad (1-\eta_2)\eta_1\mathcal{I}] f_{1-4}, \tag{A.4}$$

where $f_{1-4} := [f_1; f_2; f_3; f_4]$, and $\mathcal{I} \in \mathbb{K}^3$ is an identity matrix. The quantity f_P can also be distributed between the four corners by multiplying it by the transpose of the (squarely) parenthesised term in (A.4), obtaining f_{1-4} .

Consequently, an interpolation matrix $\Xi(\eta_1, \eta_2) \in \mathbb{R}^{3K \times 3K_\zeta}$, which interpolates between the vectors at every panel's four corner points to an interior point (η_1, η_2) on that panel, can

now be expressed by it's submatrices

$$\begin{aligned}
\Xi_{3k,3q} = & \delta_{q_{k_1}q} (1 - \eta_2)(1 - \eta_1) \mathcal{I} + \dots \\
& \delta_{q_{k_2}q} \eta_2(1 - \eta_1) \mathcal{I} + \dots \\
& \delta_{q_{k_3}q} \eta_2\eta_1 \mathcal{I} + \dots \\
& \delta_{q_{k_4}q} (1 - \eta_2)\eta_1 \mathcal{I},
\end{aligned} \tag{A.5}$$

where the index notation ${}^3(\cdot) = 3((\cdot) - 1) + 1 : 3((\cdot) - 1) + 3$ is used as shorthand, and the δ terms are Kronecker deltas.

Similarly, the matrix $H \in \mathbb{R}^{3K_\zeta \times 12K}$, which distributes the forces calculated at the midpoint of the vortex segments onto the lattice vertices can be expressed as

$$H_{3q,3s} = \Xi_{3k,3q}^l \tag{A.6}$$

where the segment index $s = 4(k - 1) + l = 1, \dots, 4K$, the sub-panel segment index $l = 1, \dots, 4$, and the submatrix $\Xi_{3k,3q}^l \in \mathbb{R}^{3 \times 3}$ is calculated using computational coordinates (η_1, η_2) that correspond to the midpoint of the l -th segment.

Appendix B

Tensors in the Linearized UVLM Formulation

This section contains a detailed description of tensors from the linearized description of the UVLM in Section 2.4. The derivatives of matrix-vector products required for the state equation (2.32) are described first, followed by the terms in the output equation appearing in (2.40). Firstly, the submatrices of (2.25), $\frac{\partial(\mathcal{A}\Gamma_0)}{\partial\zeta} \in \mathbb{R}^{K \times 3K_\zeta}$, have the form

$$\frac{\partial(\mathcal{A}_{1k,2k} \Gamma_{0_{2k}})}{\partial\zeta_{3q}} \in \mathbb{R}^{(1 \times)3}, \quad (\text{B.1})$$

and can be found by setting $f = r_{0l}^\top \left(\frac{r_{1l}}{|r_{1l}|} - \frac{r_{2l}}{|r_{2l}|} \right) \left(\frac{\widetilde{r_{1l}r_{2l}}}{|\widetilde{r_{1l}r_{2l}}|^2} \right)^\top n_{1k}^3 \in \mathbb{R}$, which isolates the objects in the AIC matrix kernel that are dependent on the geometry of the problem. Taking partial derivatives of f and using the chain rule to expose the ζ degrees of freedom results in the expression

$$\frac{\partial f}{\partial\zeta_{3q}} = \frac{\partial f}{\partial r_{0l}} \frac{\partial r_{0l}}{\partial\zeta_{3q}} + \frac{\partial f}{\partial r_{1l}} \frac{\partial r_{1l}}{\partial\zeta_{3q}} + \frac{\partial f}{\partial r_{2l}} \frac{\partial r_{2l}}{\partial\zeta_{3q}} + \frac{\partial f}{\partial n_{1k}^3} \frac{\partial n_{1k}^3}{\partial\zeta_{3q}}. \quad (\text{B.2})$$

Using the indexing convention in Appendix A the following relationships can be found for the vectors in f :

$$r_{0l} = \zeta_{3q_{2k_l+1}} - \zeta_{3q_{2k_l}}, \quad (\text{B.3})$$

$$r_{1l} = \Xi_{1k,3q} \zeta_{3q} - \zeta_{3q_{2k_l}}, \quad (\text{B.4})$$

$$r_{2l} = \Xi_{1k,3q} \zeta_{3q} - \zeta_{3q_{2k_l+1}}, \quad (\text{B.5})$$

where, if $l = 4$, then $q_{2k_{l+1}} = q_{2k_1}$. The derivatives of these terms with respect to the lattice geometry are therefore

$$\frac{\partial r_{0_l}}{\partial \zeta_{3q}} = \left(\delta_{q_{2k_{l+1}}, q} - \delta_{q_{2k_l}, q} \right) \mathcal{I}, \quad (\text{B.6})$$

$$\frac{\partial r_{1_l}}{\partial \zeta_{3q}} = \Xi_{1k, 3q} - \delta_{q_{2k_l}, q} \mathcal{I}, \quad (\text{B.7})$$

$$\frac{\partial r_{2_l}}{\partial \zeta_{3q}} = \Xi_{1k, 3q} - \delta_{q_{2k_{l+1}}, q} \mathcal{I}. \quad (\text{B.8})$$

where $\mathcal{I}$ are identity matrices $\in \mathbb{Z}^3$, in this case.

Subsequently, the sub-matrices defined in (B.1) can be expressed as

$$\begin{aligned} \frac{\partial(\mathcal{A}_{1k, 2k} \Gamma_{0_{2k}})}{\partial \zeta_{3q}} &= \frac{\Gamma_{0_{2k}}}{4\pi} \sum_{l=1}^4 \left(\left(\frac{\partial f}{\partial r_{1_l}} + \frac{\partial f}{\partial r_{2_l}} \right) \Xi_{1k, 3q} \dots \right. \\ &\quad - \left(\frac{\partial f}{\partial r_{0_l}} + \frac{\partial f}{\partial r_{1_l}} \right) \delta_{q_{2k_l}, q} \dots \\ &\quad + \left(\frac{\partial f}{\partial r_{0_l}} - \frac{\partial f}{\partial r_{2_l}} \right) \delta_{q_{2k_{l+1}}, q} \dots \\ &\quad + \frac{\partial f}{\partial n_{3k}} \frac{\partial n_{1k}}{\partial d_{1k}} \left(\delta_{q_{1k_3}, q} - \delta_{q_{1k_1}, q} \right) \dots \\ &\quad \left. + \frac{\partial f}{\partial n_{3k}} \frac{\partial n_{1k}}{\partial e_{1k}} \left(\delta_{q_{1k_2}, q} - \delta_{q_{1k_4}, q} \right) \right), \end{aligned} \quad (\text{B.9})$$

where the implicit sum through $2k$ necessary on the right-hand-side is obscured by the subscript notation on the vectors used in the Biot-Savart law, i.e r_{0_l} , r_{1_l} and r_{2_l} . The wake analogue of this matrix, $\frac{\partial(\mathcal{A}_w \Gamma_{w_0})}{\partial \zeta} \in \mathbb{R}^{K \times 3K_\zeta}$, is calculated by letting the source lattice be ζ_w , the index $2k$ run to K_w , and ignoring any terms from variations of the source lattice, i.e. setting all $\delta(2k) = 0$.

The last required term in the definition of the state equation is (2.27), $\frac{\partial(W \zeta'_0)}{\partial \zeta} \in \mathbb{R}^{K \times 3K_\zeta}$, whose constituent sub-matrices are defined as

$$\frac{\partial(W_{k, 1q} \zeta'_{0_{3q}})}{\partial \zeta_{3q}} = \zeta'_{0_{3q}}{}^\top \Xi_{3k, 3q}^\top \frac{\partial n_{3k}}{\partial \zeta_{3q}}. \quad (\text{B.10})$$

Turning to the output equation (2.40), the derivatives of matrix-vector products appearing

in (2.38) are similar to the previously derived (B.9) and have the form

$$\begin{aligned} \frac{\partial(\mathcal{A}_{1k,2k}^c \Gamma_{0_{2k}})}{\partial \zeta_{3q}} &= \frac{\Gamma_{0_{2k}}}{4\pi} \sum_{l=1}^4 \left(\left(\frac{\partial f^m}{\partial r_{1l}} + \frac{\partial f^m}{\partial r_{2l}} \right) \Xi_{1k,3q}^3 \dots \right. \\ &\quad \left. - \left(\frac{\partial f^m}{\partial r_{0l}} + \frac{\partial f^m}{\partial r_{1l}} \right) \delta_{q_{2k_l}, q} \dots \right. \\ &\quad \left. + \left(\frac{\partial f^m}{\partial r_{0l}} - \frac{\partial f^m}{\partial r_{2l}} \right) \delta_{q_{2k_{l+1}}, q} \right), \end{aligned} \quad (\text{B.11})$$

where $f^m = r_{0l}^\top \left(\frac{r_{1l}}{|r_{1l}|} - \frac{r_{2l}}{|r_{2l}|} \right) \left(\frac{\widetilde{r_{1l}r_{2l}}}{|\widetilde{r_{1l}r_{2l}}|^2} \right) \in \mathbb{R}^3$. Similarly the wake analogue of this matrix, $\frac{\partial(\mathcal{A}_w^c \Gamma_{w_0})}{\partial \zeta} \in \mathbb{R}^{3K \times 3K_\zeta}$, is calculated by letting the source lattice be ζ_w , the index $2k$ run to K_w , and ignoring any terms from variations of the source lattice, i.e. setting all $\delta(2k) = 0$.

Finally, the terms $Y_{(1-5)}$, introduced in (2.41) and (2.42), are described here to complete the mathematical description of the method. The first of these terms, $Y_1 \in \mathbb{R}^{12K \times K}$, can be expressed as

$$Y_1 = \underset{(12 \times 1)}{\text{diag}} \left\{ \left[\widetilde{v_{0_1}^m} l_{0_1}; \widetilde{v_{0_2}^m} l_{0_2}; \widetilde{v_{0_3}^m} l_{0_3}; \widetilde{v_{0_4}^m} l_{0_4} \right] \right\}_k, \quad (\text{B.12})$$

where the *diag* notation indicates the matrix is block diagonal with blocks of dimension 12×1 , each corresponding to the index k . Note that k is a panel index and therefore takes the values $1, \dots, K$, and $v_{0_{1-4}}^m, l_{0_{1-4}} \in \mathbb{R}^3$ (the superscript $^3(\cdot)$ notation on the indices has been dropped for clarity) represent the four midpoint velocities and segment length vectors on each panel, respectively. The matrix $Y_2 \in \mathbb{R}^{12K \times 3K_\zeta}$ can be expressed by submatrices

$$Y_{2(3s,3q)} = \Gamma_{0_k} \widetilde{v_{0_{3s}}^m} \frac{\partial l_{3s}}{\partial \zeta_{3q}}, \quad (\text{B.13})$$

where $s = 1, \dots, 4K$ is an index through all of the segments of all of the vortex rings. Note that $k = (s - 1) \setminus 4 + 1$. The matrix $Y_3 \in \mathbb{R}^{12K \times 12K}$ can be written as

$$Y_3 = \underset{(3 \times 3)}{\text{diag}} \left\{ \left[\Gamma_{0_k} \widetilde{l_{0_l}} \right] \right\}_s, \quad (\text{B.14})$$

where k and l – which are, respectively, the panel index and intra-panel segment index – are inferred from s . The matrix $Y_4 \in \mathbb{R}^{3K \times K}$ is simply

$$Y_4 = \underset{(3 \times 1)}{\text{diag}} \{ A_k n_k \}_k, \quad (\text{B.15})$$

where A_k is the panel area and n_k the panel normal vector. Lastly, the matrix $Y_5 \in \mathbb{R}^{3K \times 3K_\zeta}$ can be expressed by submatrices

$$Y_{5(3_k, 3_q)} = \frac{\Gamma'_{0_k}}{2} \left(\widetilde{d_{0_{3_k}}} \frac{\partial e_{3_k}}{\partial \zeta_{3_q}} - \widetilde{e_{0_{3_k}}} \frac{\partial d_{3_k}}{\partial \zeta_{3_q}} \right), \quad (\text{B.16})$$

where d and e are the (diagonal) vectors joining opposite corners of each vortex ring.

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