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# Local connectivity and quasi-conformal rigidity of non-renormalizable polynomials

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**Abstract**

We prove that topologically conjugate non-renormalizable polynomials are quasi-conformally conjugate. From this we derive that each such polynomial can be approximated by a hyperbolic polynomial. As a by-product we prove that the Julia set of a non-renormalizable polynomial with only hyperbolic periodic points is locally connected, and the Branner-Hubbard conjecture. The main tools are the enhanced nest construction (developed in a previous joint paper with Weixiao Shen [7]) and a lemma of Kahn and Lyubich (for which we give an elementary proof in the real case).

## 1 Statement of Main Results

Let $f\colon \mathbb{C} \to \mathbb{C}$ be a polynomial. Its filled Julia set $K(f)$ is the set of points which do not escape to infinity, i.e. all $z$ so that $|f^n(z)| \not\to \infty$, and its Julia set $J(f)$ is equal to $\partial K(f)$. We call such a polynomial $f$ *hyperbolic*, if each of its critical points is contained in the basin of a hyperbolic periodic attractor. A classical question is whether one can approximate each polynomial by a hyperbolic polynomial of the same degree (i.e. only small changes of the coefficients are needed to 'make' the polynomial hyperbolic). In this paper we give an answer to this question under the additional assumption that the original polynomial is non-renormalizable (or more, generally, only finitely renormalizable) and has only hyperbolic periodic points. We derive this result by proving that topologically conjugate non-renormalizable polynomials (without only hyperbolic periodic orbits) are quasi-conformally

conjugate, while dealing with the additional complication that there may be several critical points. In addition we will show that under the same non-renormalizability assumption each point in the Julia set is contained in arbitrarily small puzzle pieces (and therefore that the Julia set is locally connected) and that there is no measurable invariant linefield supported on the Julia set.

Let us define these notions now. Since $f$ is a polynomial, it is conformally conjugate near $\infty$ to $z \mapsto z^d$ where $d$ is the degree of $f$. curve which under this conformal equivalence near $\infty$ is mapped to a circle (respectively to a line-segment through 0 with rational angle) is called an *equipotential* (respectively a piece of a *periodic ray*). Iterates of an equipotential are all disjoint. If $f$ has a connected Julia set, then the backward orbit of a piece of a periodic ray forms a curve $\gamma$ such that $f^k(\gamma) = \gamma$ and such that $\gamma$ lands on a periodic point; moreover each periodic point has a periodic ray landing on it, see for example [14]. If $f$ has also periodic attractors in $\mathbb{C}$ or $f$ has a Julia set which is not connected, then we also associate to these orbits equipotentials and periodic rays as in Subsection 2.1. A *puzzle piece* $V$ is a bounded set whose boundary consists of pieces of equipotentials and periodic rays such that when $z \in \partial V$ then $f^n(z) \notin \text{int}(V)$ for all $n \geq 0$. We say that a periodic point is *hyperbolic* if its multiplier is not on the unit circle, and *repelling* if its multiplier is outside the unit circle. For quadratic maps with only repelling periodic points it is well-known how to construct puzzle pieces; in Subsection 2.1 we extend this to general polynomials (with only hyperbolic periodic points).

Given a puzzle piece $V$ containing a critical point $c$, define $R_V$ be the first return map to $V$ and let $K_c(R_V)$ be the component of $K(R_V) = \{z; R^k(z) \in V \text{ for all } k \geq 0\}$ containing $c$. We say that $f$ is *finitely renormalizable* if there exists an integer $s$ such that for each critical point $c$ of $f$ and each puzzle piece $V \ni c$ for which the return time of $c$ to $V$ is greater or equal to $s$, the set $K_c(R_V)$ is non-periodic. We say that $f$ is *non-renormalizable*, if $s = 2$.

As usual, we often abbreviate the term 'quasi-conformal conjugacy' to 'qc-conjugacy'. Finally, we say that $f$ has an *invariant linefield* on $J(f)$, if there exists a measurable subset $E \subset J(f)$ and a measurable map which associates to Lebesgue almost every $x \in E$ a line $l(x)$ through $x$ which is $f$-invariant in the sense that $l(f(x)) = Df(x)\, l(x)$. (So the absence of linefields is obvious if the Julia set has zero Lebesgue measure.)

Our first result generalizes the celebrated result of [18] for finitely renormalizable quadratic maps, and of [1] for finitely renormalizable unicritical

maps.

**Theorem 1.1** (Local connectivity and absence of linefields)**.** *Assume that $f$ is a finitely renormalizable polynomial whose periodic points are all hyperbolic. Then*

1. *each point of its Julia set is contained in arbitrarily small puzzle pieces;*
2. *if the Julia set is connected, then it is locally connected;*
3. *$f$ has no invariant linefields on its Julia set;*

Note that the puzzle construction we consider in the above theorem, also consists of rays and equipotentials contained in basins of periodic attractors, see Subsection 2.1.

This theorem implies the following conjecture of Branner and Hubbard (stated in the cubic case in [3]). (While this paper was in the final stages of preparation we received a preprint W. Qiu and Y. Yin [16] also proving this conjecture.)

**Corollary 1.2** (Branner - Hubbard conjecture)**.** *The Julia set of a polynomial is totally disconnected if and only if each critical component of the filled Julia set is aperiodic.*

*Proof.* The only if part is obvious. If each critical component of the filled Julia set is aperiodic, then $f$ is finitely renormalizable but where in the above definition we only need to consider puzzle pieces bounded by preimages of equipotentials. But this implies that in the proof of the above theorem all puzzle pieces are also of this type. □

**Theorem 1.3** (Rigidity of non-renormalizable polynomials)**.** *Assume that $f$ is a finitely renormalizable polynomial whose periodic points are all hyperbolic. Moreover, assume that one of the following conditions hold*

**a)** *$f$ and $\tilde{f}$ are topologically conjugate;*

**b)** *$f, \tilde{f}$ are combinatorially equivalent and restricted to their Fatou sets there exists a compatible conjugacy between $f$ and $\tilde{f}$ (these notions are explained in Definition 1.1) .*

*Then $f$ and $\tilde{f}$ are quasi-conformally conjugate.*

For real polynomials one can drop the assumption that the map is finitely renormalizable: this was shown for real quadratic maps in [4] and [10] and for general real polynomials in [7]. For finitely renormalizable complex quadratic maps, the above result was shown by [18] and for finitely renormalizable unicritical polynomials in [1]. Here combinatorial equivalence is defined as below ([11]); we note that [8] gives the construction of rays associated to a superattracting periodic point (in $\mathbb{C}$ or $\infty$), even if the Julia set is disconnected.

**Definition 1.1** (Combinatorially Equivalence and Compatible Conjugacy)**.** Let us say that $\alpha \sim_f \beta$ for rational numbers $\alpha, \beta$ if the external rays of $f$ with angles $\alpha, \beta$ land on the same (pre-)periodic point. We then say that two polynomials $f, \tilde{f}$ are *combinatorially equivalent* if $\sim_f$ and $\sim_{\tilde{f}}$ define the same equivalence relation on the set of rational numbers.

A topological conjugacy $h$ between $f, \tilde{f}$ on their Fatou sets is called *compatible* if for each periodic component $B$ of the Fatou set, $h$ extends continuously to the closure of $B$ and the correspondence between periodic points in $\partial B$ and periodic points in $\partial h(B)$ coming from this extension of $h$, agrees with the one induced by the combinatorial equivalence.

By Lemma 3.1 below, a conjugacy on a periodic component $B$ of the Fatou set can always be extended to $\partial B$ (for polynomials which are non-renormalizable and with only hyperbolic periodic points); it is not necessarily compatible (since there are several homeomorphisms conjugating the restriction of $z \mapsto z^d$ to $\{z \in \mathbb{C};\, |z| = 1\}$ to itself.) However, if $f$ and $\tilde{f}$ have no escaping critical points and no periodic attractors, then the Böttcher coordinates at infinity induce a (conformal) compatible conjugacy on the basin of infinity, and so the compatibility in assumption (2) in Theorem 1.3 is automatic.

Also note that if $f, \tilde{f}$ are topologically conjugate and their Julia sets are connected, they are combinatorially equivalent: the image of a periodic ray of $f$ by the conjugacy is a periodic curve which lands on a periodic point of $\tilde{f}$, and thereby by Theorem 2 of [8] homotopy equivalent relative to $\mathbb{C} \setminus K(\tilde{f})$ to a periodic ray of $\tilde{f}$. If the Julia set is non-connected, rays need not be smooth, as they can split at critical points. Because of this the connectedness assumption is needed for the implication asserted in this paragraph: all quadratic maps with escaping critical points are topologically conjugate but they need not be combinatorially equivalent.

The following theorem was proved for finitely renormalizable quadratic

maps by Yoccoz [18] and for arbitrary real polynomials in [6].

**Theorem 1.4** (Approximating finitely renormalizable by hyperbolic polynomials)**.** *Any finitely renormalizable polynomial $f$ which has only hyperbolic periodic points can be approximated by a hyperbolic polynomial $g$ of the same degree. If the Julia set of $f$ is connected, then $g$ can be chosen so that its Julia set is also connected.*

As mentioned before, we say that a polynomial is *hyperbolic* if all its critical points are in basins of hyperbolic periodic attractors.

The strategy of the proof is to use the enhanced nest construction in [7] combined with a lemma due to Kahn and Lyubich to obtain complex bounds. As a by-product of the present paper, we obtain a significant simplification of the proof of the Key Lemma from [7]: the main technical part of [7] (Section 8-11) can be replaced by this paper.

In order to prove the above theorems we shall need to work with so-called complex box mappings. For unicritical polynomials the construction of a complex box mapping is standard: It is a map $F : U \to V$, where $V$ is a topological disk, $U$ is a union of (possibly infinitely many) simply connected domains, all components of $U$ except one are mapped univalently onto $V$ by $F$ and $F$ restricted on that other component of $U$ is a cover map onto $V$. When we consider a multicritical polynomial we cannot construct a complex box mapping which has the same nice properties as in the unicritical case. If the domain $V$ is connected and small, then it is possible that the first return map to $V$ has infinitely many critical points, which of course is undesirable. So, the domain $V$ in our construction will have several connected components. In the unicritical case all components of $U$ are compactly contained in $V$, in the multicritical case we have to drop this property as well.

**Definition 1.2** (Complex box mappings)**.** *We say that a holomorphic map $F\colon U \to V$ between open sets $U \subset V$ in $\mathbb{C}$ is a* complex box mapping *if the following hold:*

1. *$F$ has finitely many critical points;*
2. *$V$ is a union of finitely many pairwise disjoint Jordan disks;*
3. *every connected component $V'$ of $V$ is either a connected component of $U$ or the intersection of $V'$ and $U$ is a union of Jordan disks with pairwise disjoint closures which are compactly contained in $V'$,*

*4. for each component $U'$ of $U$, $F(U')$ is a component of $V$.*

This generalizes the well-known notion of a *(generalized) polynomial-like map*: in that case $V$ has only one component and each component of $U$ is compactly contained in $V$. Many authors also require that in addition $U$ has only a finite number of components.

Given a complex polynomial we will construct an induced complex box mapping $F : U \to V$ for this polynomial which has a few particular properties (however, we will not need these properties in what follows). Suppose that all critical points which are in the Julia set of the polynomial are recurrent to each other. Then the domain $V$ can be decomposed as $V = V_0 \cup V_1 \cup \cdots \cup V_k$, where domains $V_1, \ldots, V_k$ are also components of $U$ and $U \setminus (V_1 \cup \cdots \cup V_k) \subset V_0$. Each component $V_i$, $i = 0, \ldots, k$, contains at least one critical point.

The filled Julia set of $F$ is then defined by $K(F) = \{z \in U : f^n(z) \in U \text{ for any } n \in \mathbb{N}\}$ and its *Julia set* as $J(F) = \partial K(F)$. We denote by $Crit(F)$ the set of critical points of $F$ and let $PC(F) = \bigcup_{n \geq 0} f^n(Crit(F))$. A *puzzle piece* is a component of $F^{-n}(V)$ for some $n \geq 0$. If it contains a critical point then it is called a *critical puzzle piece*.

Note that we allow there to be several critical points in a component of the domain, and also that $F$ has no critical points.

**Definition 1.3** (Renormalizable complex box mapping)**.** *The complex box mapping $F$ is called* renormalizable *if there exists $s > 1$ (called the* period*) and a puzzle piece $W$ containing a critical point $c$ of $f$ such that $f^{ks}(c) \in W$, $\forall k \geq 0$ and such that $s$ is the first return time of $c$ to $W$.*

**Definition 1.4** (Itinerary of a puzzle piece relative to some curve family $\Gamma$)**.** *Let $X$ be a collection of points in $\partial V$ intersecting each component of $\partial V$, so that for each point $y \in F^{-1}(X)$ there exists a simple curve in $V \setminus (U \cup PC(F))$ connecting $y$ to a point of the set $X$. Let $\Gamma$ denote this collection of curves. For each component $U'$ of $F^{-n}(U)$ there exists a curve connecting $\partial U'$ to $x$ of the form $\gamma_0 \cdots \gamma_n$ where $F^k(\gamma_k) \in \Gamma$. The word $(\gamma_0, F(\gamma_1), \ldots, F^n(\gamma_n))$ is called a $\Gamma$-itinerary of $U'$. A component of $F^{-n}(U)$ can have several $\Gamma$-itineraries, but different components of $F^{-n}(U)$ have different $\Gamma$-itineraries (as can be easily checked).*

We say that two complex box mappings are combinatorially equivalent, if their critical points have the same itineraries:

**Definition 1.5** (Combinatorial equivalence of complex box mappings)**.** *Non-renormalizable complex box mappings $F\colon U \to V$ and $\tilde{F}\colon \tilde{U} \to \tilde{V}$ are called* combinatorially equivalent *w.r.t. some homeomorphism $H\colon V \to \tilde{V}$ with $H(U) = \tilde{U}$ and with $H(PC(F) \setminus U) = PC(\tilde{F}) \setminus \tilde{U}$ if there exists a curve family $\Gamma$ as in the previous definition, so that each critical point $c \in Crit(F)$ corresponds to a unique critical point $\tilde{c} \in Crit(\tilde{F})$ with the property that for every integer $k, n \geq 0$, the $\Gamma$-itineraries of the component of $F^{-n}(U)$ containing $F^k(c)$ agree with the $\tilde{\Gamma}$-itineraries of the component of $\tilde{F}^{-n}(\tilde{U})$ containing $\tilde{F}^k(\tilde{c})$. Here $\tilde{\Gamma} = H(\Gamma)$.*

For our purposes we will only need to work with non-renormalizable complex box mappings. By defining the notion of renormalization of a complex box mapping, it is not hard to extend the following theorem also to finitely renormalizable complex box mappings.

**Theorem 1.5** (Rigidity for complex box mappings)**.** *Assume that $F\colon U \to V$ is a non-renormalizable complex box mapping whose periodic points are all repelling. Then*

1. *each point of its Julia set is contained in arbitrarily small puzzle pieces;*
2. *$F$ has no measurable invariant linefields on its Julia set;*
3. *Assume that $\tilde{F}\colon \tilde{U} \to \tilde{V}$ is another complex box mapping for which there exists a quasi-conformal homeomorphism $H\colon \mathbb{C} \to \mathbb{C}$ so that $H(V) = \tilde{V}$, $H(U) = \tilde{U}$, $\tilde{F} \circ H = H \circ F$ on $\partial U$ and so that $\tilde{F}$ is combinatorially equivalent to $F$ w.r.t. $H$. Moreover, assume that the boundary of each component of $U, V, \tilde{U}, \tilde{V}$ consists of piecewise smooth arcs. Then $F$ and $\tilde{F}$ are quasiconformally conjugate.*

The maps $F$ and $\tilde{F}$ can have several critical points and are not necessarily real. Also note that it is not necessary to assume that $\tilde{F}$ has only hyperbolic periodic points.

The organisation of this paper is as follows. In the first part of the paper we shall show that rigidity of box mappings (Theorem 1.5) implies the other theorems. From Section 7 we will prove Theorem 1.5 using the enhanced nest construction from [7] and a Lemma by Kahn and Lyubich.

The authors would like to warmly thank Weixiao Shen. At the end of his stay in Warwick, in the spring of 2005, Weixiao indicated that the Kahn-Lyubich lemma combined with the enhanced nest construction from [7] gives complex bounds, see Section 10. The authors also gratefully acknowledge several very useful discussions with Genadi Levin, in particular on the puzzle construction in Section 2.1 and with Tan Lei on the definition of combinatorial equivalence at the beginning of the paper.

# 2 Local connectivity and absence of linefields (Theorem 1.5 implies Theorem 1.1)

First we need to associate a puzzle partition to any polynomial $f$ which only has hyperbolic periodic points, and then use this to construct a complex box mapping $F\colon U \to V$. If $f$ has only repelling periodic points, then the construction is the multi-critical analogue of the usual Yoccoz puzzle partition, but otherwise the construction will make use of internal rays (inside basins of periodic attractors). This means in particular that any critical point of $f$ which is attracted to a periodic attractor, is outside the filled Julia set of the complex box mapping $F$. One of the reasons for setting-up the construction in this way, is that this will allow us to show that puzzle pieces shrink to zero in diameter.

## 2.1 Puzzle construction for polynomials with periodic attractors or escaping critical points

The next construction coincides with the usual Yoccoz puzzle construction in the case where the Julia set is connected and there are no attracting fixed points. First choose a level curve $\Gamma$ of the equipotential function of the Julia set such that the following three properties hold for any connected component

$K'$ of the filled Julia set $K$:

1) if $K'$ contains a critical point and is periodic, i.e. there exists some (minimal) $n > 0$ so that $f^n(K') = K'$, then we want that each component of $\mathbb{C} \setminus \Gamma$ contains at most one of the sets $K', \ldots, f^{n-1}(K')$;

2) if $K'$ contains a critical point and is not periodic, i.e. if $K', f(K'), \ldots$ are all mutually disjoint, then we want that not all these iterates of $K'$ are contained in one component of $\mathbb{C} \setminus \Gamma$;

3) if two critical points lie in one component of $\mathbb{C} \setminus \Gamma$ then they lie in the same component of $K$;

4) if a critical point lies in a bounded component of $\mathbb{C} \setminus \Gamma$ then it lies in $K$.

Since equipotential level curves can be chosen to lie inside an arbitrary small neighbourhood of $K$ it is possible to choose $\Gamma$ as above. Now take a component $K'$ of $K$ which is periodic and contains a critical point; assume its period is $n$ and let $d$ be the degree of $f^n|\mathcal{V}'$ where $\mathcal{V}'$ is the component of $\mathbb{C} \setminus \Gamma$ containing $K'$. By [8], each repelling periodic point in $K'$ has at least one and at most a finite number of external rays landing on it, and the number of rays of period $n$ landing on $K'$ (i.e., on fixed points of $f^n|K'$) is equal to the number of periodic points of $\theta \mapsto d\,\theta \bmod 2\pi$. We are primarily interested in periodic points which are *separating*, i.e. such that there are several rays landing on it; rays landing on such a periodic point divide the complex plane in several components.

**Case 1:** Let us first assume that all fixed points of $f^n|K'$ are repelling. Then, since $f^n|K'$ has $d$ (repelling) fixed points and $\alpha \mapsto d\alpha \bmod 2\pi$ has only $d-1$ fixed points, at least one of the fixed points is separating, i.e., has more than one ray (with period $> n$) landing on it.

**Case 2:** If at least one of the periodic points of $f^n|K'$ is attracting, then the counting argument used in case 1 might not work. By replacing $f^n$ by an iterate we may assume that all periodic attracting points of $f^n\colon K' \to K'$ are fixed points. So consider the immediate basin $B$ of a hyperbolic attracting fixed point $p$ of $f^n|K'$. The following lemma gives that there is a periodic curve in $B$ which has exactly period 2 under $f^n$:

**Lemma 2.1.** *Let $g$ be a polynomial with an attracting hyperbolic fixed point $p$ with immediate basin $B$. Then there exist*

1. *a curve $\gamma$ of period 2 in $B$ which connects a point of period precisely two in $\partial B$ to $p$ such that $\gamma \cap g(\gamma) = \{p\}$ and such that $\gamma \cup g(\gamma)$ divides $B$ into two components;*

2. *a curve* $\Gamma_B \subset B$ *surrounding* $p$ *such that* $(g|B)^i(\Gamma_B)$ *and* $(g|B)^j(\Gamma_B)$ *are disjoint for all* $i, j \in \mathbf{Z}$, $i \neq j$.

3. $\gamma$ *and* $(g|B)^i(\Gamma_B)$ *intersect in a single point, for any* $i \in \mathbf{Z}$.

*Proof.* Taking the Riemann mapping $h\colon \mathbb{D} \to B$ we get that $h^{-1} \circ g \circ h$ is a proper mapping of $\mathbb{D}$ onto itself and so a Blaschke product $\varphi$ with an attracting fixed point $p_\varphi$ in $\mathbb{D}$. Note that all periodic orbits of $\varphi|\partial\mathbb{D}$ are repelling. Hence there exists $N$ so that $|D\varphi^N_{|\partial\mathbb{D}}| > 2$. Let $V$ be a neighbourhood of $\partial\mathbb{D}$ such that $|D\varphi^N_{|V}| > 2$ and such that $\varphi^{-N}(V)$ is compactly contained in $V$.

Next choose a curve $\hat{\gamma} \subset \mathbb{D}$ ending at $p_\varphi$ such that $\varphi^2(\hat{\gamma}) \subset \hat{\gamma}$ and so that $\varphi(\hat{\gamma}) \cap \hat{\gamma}$ is equal to $p_\varphi$. This is easy to do: First take a fundamental annulus $A$ around $p_\varphi$. Then choose a curve $\tau$ in $A$ connecting some point $x$ in the outer boundary of $A$ with a point $y \neq \varphi(x)$ in the inner boundary of $A$ and next choose a curve $\tau'$ in $\varphi(A)$ connecting $y$ to $\varphi^2(x)$ with the property that $\varphi(\tau)$ does not intersect $\tau'$. The curve $\hat{\gamma} = \cup_{i\geq 0}\varphi^{2i}(\tau \cup \tau')$ has the required properties (usually $\hat{\gamma}$ will spiral towards $p_\varphi$). Now consider backward iterates of $\tau \cup \tau'$. Taking appropriate preimages (i.e. components) $\cup_i \varphi^{-2i}(\tau \cup \tau')$ becomes a curve $\tilde{\gamma}$; we can assume that $\tau, \tau'$ are chosen so that $\tilde{\gamma}$ is disjoint from the postcritical set. For $i$ sufficiently large $(\varphi|\mathbb{D})^{-i}(\tau \cup \tau')$ lies in $V$. Hence, by the above expansion properties, the length of the curve $\tilde{\gamma}$ is finite. Since $\varphi^2(\tilde{\gamma}) = \tilde{\gamma}$, it follows that $\tilde{\gamma}$ approaches a point $q \in \partial\mathbb{D}$ which has precisely period two under $\varphi$. Moreover, $\tilde{\gamma}$ approaches $q$ non-tangentially: this holds because $\varphi$ is expanding near $q$ and leaves invariant $\partial\mathbb{D}$. Hence, by [15, Theorem 1], the corresponding curve $\gamma := h(\tilde{\gamma})$ lands on a fixed point of $g^2$. In fact, more holds. *Claim:* $\gamma$ lands on a periodic point of $g$ of period exactly two. Indeed, assume by contradiction that $\gamma$ lands on a fixed point. Then let $B_+$ one of the components of $D \setminus (\gamma \cup g(\gamma))$. Since the boundary of $B_+$ is equal to the Jordan curve $\gamma \cup g(\gamma)$ and since $g^2(\gamma) = \gamma$, the boundary of the set $g(B_+)$ again consists of the Jordan curve $\gamma \cup g(\gamma)$. Since $g(B_+) \neq B_+$ it follows from the Jordan curve theorem that $g(B_+)$ is the unbounded complement of $\gamma \cup g(\gamma)$. This contradicts the assumption that $g(B_+) \subset D$, and finished the proof of the claim.

To complete the proof of the lemma, we notice that the curve $\Gamma_B$ can be constructed using a preimage of the outer boundary of $A$ (which can be constructed by considering the boundary of a neighbourhood of $p$ on which $g$ is linearizable). We can and will choose this curve so that it does not contain iterates of critical points. □

The curve $\gamma$ ('internal ray') together with the external rays landing on the corresponding period two points of $f^n|\partial B$, are called the *rays associated to the attracting fixed point* of $f^n|K'$. These curves are smooth except at the attracting fixed point and its (other) endpoints. Of course the union of such internal and external rays again separates the plane. The curve $\Gamma_B$ is called an *equipotential* of the attracting fixed point. We can choose $\Gamma_B$ so that it bounds a disc containing $p$ and all iterates of critical points of $f$ which lie in $B$.

For each component $K'$ of the filled Julia set containing a critical point, denote its period by $n = n(K')$, and consider the union of rays associated to the attracting fixed points of $f^n|K'$ together with all external rays landing on separating fixed points of $f^n|K'$. Moreover, consider the equipotentials $\Gamma$ and the union of all equipotentials $\Gamma_B$ associated to attracting orbits; the set $\mathcal{V}$ is defined to be the union of the bounded components of $\mathbb{C} \setminus \Gamma$ and minus the bounded components of $\mathbb{C} \setminus \cup\Gamma'$ where the last union runs over all $\Gamma'$ as in Case 2. Let the partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of $\mathcal{V}$ be defined by all such rays, i.e. as the components of $\mathcal{V}$ minus the above collection of rays. Combining Cases 1 and 2 above, we have shown that if a component of $\mathcal{V}$ contains a critical point then it is partitioned in two or more pieces from $\mathcal{P}$. (If there are no components of $K$ containing critical points then the partition does not involve rays.)

Components of $f^{-j}(\mathcal{P})$ map onto a union of components of $f^{-(j-1)}(\mathcal{P})$. Components of $f^{-j}(\mathcal{P})$ are called *puzzle pieces* and the pieces which contain critical points are called *critical puzzle pieces*.

Note that each puzzle piece is the closure of a finite union of open arcs, consisting of equipotentials and rays.

## 2.2 How to associate a complex box mapping to a polynomial

For critical points $c, c'$ we define $c \preceq c'$ if the forward orbit of $f(c')$ intersects each puzzle piece containing $c$. If $c \preceq c$ then $c$ is called *puzzle recurrent*. The partial ordering $\leq$ is defined by $c \leq c'$ if $c = c'$ or $c \preceq c'$.

**Lemma 2.2.** *Let $f$ be a finitely renormalizable polynomial $f$ with all periodic points hyperbolic, and consider the puzzle construction from above. Then for any puzzle recurrent critical point $c$, there is a critical puzzle piece $W$ such that*

1. *the component of the first return map to $W$ containing $c$ is compactly contained in $W$;*

2. *the first return map to $W$ is non-renormalizable;*

3. *take $V$ to be the union of the components of the first entry map to $W$ which contain a critical point, and $U \subset V$ the domain of the first return map to $V$. Then $R\colon U \to V$ is a complex box mapping with the property that $V$ contains no escaping critical point of $f$.*

*Proof.* The proof we give here simplifies an earlier proof, and is due to Genadi Levin. First assume that $f$ is non-renormalizable. We claim that for each critical point $c$ there is a critical piece $P \ni c$, so that its closure is disjoint from all separating periodic points used for the construction of the puzzle partition $\mathcal{P}$. Indeed, otherwise all critical pieces containing $c$ have a (separating) periodic point $a$ in their boundary, and hence the intersection of (all) the critical pieces of $c$ is a continuum which also contains $a$. Because $f$ is non-renormalizable this continuum must be wandering, see [2]. But since the continuum contains the fixed point $a$, this is impossible.

Let us fix $P_0 = P \ni c$. Since the closure of $P$ contains no fixed point, and the boundary of $P$ consists of pieces of equipotentials and rays, the boundary of $P$ contains no periodic points. Now define inductively $P_{i+1}$ to be the first return domain to $P_i$ containing $c$. Assume that $P_{n+1}$ is not compactly contained in $P_n$. Then there are two points $p_1, p_2$ in the boundary of $P_n$, such that $f^k(p_1) = p_2$, where $k$ is the return time of $P_{n+1}$ to $P_n$. Let $r$ be so that $f^r(P_n) = P_0$, and $q_1 = f^r(p_1), q_2 = f^r(p_2)$. Then $q_1, q_2$ are in the boundary of $P_0$, and $f^k(q_1) = f^r(f^k(p_1)) = f^r(p_2) = q_2$. If $n$ is large, then $k$ is large too ($f$ is non-renormalizable), and $f^k(q_1) = q_2$ is impossible, because the boundary of $P_0$ consists of rays and equipotentials (not depending on $k$) and its boundary is disjoint from periodic points. Taking $W = P_n$ with $n$ minimal as above the lemma follows.

Now assume $f$ is renormalizable. In this case we have to work with rays associated to generalized polynomial-like maps (here we allow that the domain $U$ of polynomial-like maps $F\colon U \to V$ we consider has infinitely many components) . Let $P_0$ be as in the non-renormalizable case and define inductively $P_{i+1}$ to be the component containing $c$ of the domain of definition of the first return domain map $R_i\colon \mathrm{Dom}(P_i) \to P_i$. If for all $i$, there exists some $k > 0$ so that $R_i^k(c) \notin P_{i+1}$ then $R_i\colon \mathrm{Dom}(P_i) \to P_i$ is non-renormalizable at $c$ and we are in Case 1. So we may assume that there exists a level $i$ so that

$R_i^k(c) \in P_i$ for all $k \geq 0$. In fact, we can take $i$ so large that each other critical point $c' \neq c$ in $P_i$ also does not leave $P_i$ under iterates of $R_i\colon \mathrm{Dom}(P_i) \to P_i$. Then the filled Julia set $K_i$ of $R_i\colon \mathrm{Dom}(P_i) \to P_i$ is connected and not a singleton. From Theorem $1^\tau$ of [8] each fixed point of $R_i\colon \mathrm{Dom}(P_i) \to P_i$ has external rays landing on it, and there is at least one such fixed point with a ray of some period $q > 1$ landing on it. Let $\gamma$ be the union of rays of fixed points of $R_i\colon \mathrm{Dom}(P_i) \to P_i$ with rays of period $> 1$ landing on it, and consider $P_i \setminus \gamma$. Since the rays have period $> 1$, $c$ and $R_i(c)$ cannot lie in the same component of $P_i \setminus \gamma$. Let $P^1$ be the component of $P_i \setminus \gamma$ containing $c$, and consider the first return map $R\colon \mathrm{Dom}(P^1) \to P^1$. Now repeat the above construction, replacing $R\colon \mathrm{Dom}(P) \to P$ by $R\colon \mathrm{Dom}(P^1) \to P^1$. Since $f$ is finitely renormalizable (say of period $s$) this cannot be repeated more than $k$ times where $2^k \leq s$. □

**Question:** take a polynomial with all periodic points hyperbolic; is it always possible to find puzzle pieces which contain at most one critical point?

We are only able to answer this question once we have complex bounds (so only in the case when the polynomial is only finitely renormalizable). That is the reason why in the definition of complex box mappings we allow several critical points to be in one component of the domain.

Now we will show how to use the previous lemma to construct a complex box mapping.

**Corollary 2.1.** *For a polynomial as in the previous lemma, there exists a collection of puzzle pieces $V$ which contains all critical points which do not escape to infinity or are in the basin of periodic attractors (and only such critical points), such that the first return map into $V$ defines a complex box mapping $F\colon U \to V$.*

*Proof.* Take a critical point $c_1$ which is puzzle recurrent and consider the critical puzzle piece $W_1 \ni c_1$ from the previous lemma (if there is no such critical point then skip this step), and write $\mathcal{W}_1 = W_1$. Now continue by induction and assume $\mathcal{W}_{k-1}$ is chosen. Then choose a critical point $c_k$ which is puzzle recurrent and which is never mapped into $\mathcal{W}_{k-1}$. (If this is impossible, the induction stops, we set $m = k - 1$ and this part of the construction is completed.) Then let $W_k \ni c_k$ be the critical puzzle piece from the previous lemma and whose level is at least the level of each of the puzzle pieces in $\mathcal{W}_{k-1}$. Then write $\mathcal{W}_k = \mathcal{W}_{k-1} \cup W_k$. After a finite number of steps, the only

critical points which are not contained in or eventually mapped into some union of puzzle pieces $\mathcal{W}_m$, are those which are not puzzle recurrent. (If there are no puzzle recurrent critical points, then define $m = 0$ and $\mathcal{W}_0 = \emptyset$).

Next choose a critical point $c_{m+1}$ which is not puzzle recurrent and which is never mapped into $\mathcal{W}_m$, choose a puzzle piece $W_{m+1} \ni c_{m+1}$ of level at least the level of each of the puzzle pieces of $\mathcal{W}_m$ and such that $c_{m+1}$ never re-enters $W_{m+1}$. Then write $\mathcal{W}_{m+1} = \mathcal{W}_m \cup W_{m+1}$. Continue in this way as long as possible (for $m'$ steps). Next let $V$ be the union of $\mathcal{W}_{m'}$ and the domains of the first return map to $\mathcal{W}_{m'}$ containing critical points outside $\mathcal{W}_{m'}$. The set $V$ consists of puzzle pieces and contains all non-escaping critical points. $V$ is *nice*: the boundary of $V$ is never mapped into the closure of $V$. The return map to $V$ therefore defines a box mapping, whose domain $U$ is equal to the set of points in $V$ which eventually return to $V$. □

A critical point which is Misiurewicz (say eventually mapped to a repelling periodic orbit) will be contained in a component of $V$, but might not be in $U$.

Now we will finally show that Theorem 1.5 implies Theorem 1.1. Indeed, Theorem 1.5 implies that all puzzle pieces of this complex box mapping shrink to zero, and that it carries no measurable invariant line field on its Julia set. The set of points which are never mapped into $V$ (and is not contained in the basin of a periodic attractor) is hyperbolic, and therefore puzzle pieces containing such points also shrink to zero. Theorem 1.1 therefore follows.

# 3 QC-conjugacies on basins of periodic attractors

**Lemma 3.1.** *Assume that $f$ and $\tilde{f}$ are finitely renormalizable and only have hyperbolic periodic points.*

1. *If $f$ and $\tilde{f}$ are topologically conjugate on a periodic component of the Fatou set, then this conjugacy extends continuously to the boundary of this component;*

2. *If $h$ is a topological conjugacy between $f$ and $\tilde{f}$ restricted to their Fatou sets, then $h$ can be replaced by a qc conjugacy $h_0$ such that $h$ and $h_0$*

*have the same extension to the boundary of periodic components of the Fatou set. (In particular, the extensions agree on the boundary of the basin of $\infty$, i.e. on the Julia set.)*

Remark that inside the level set associated to the Böttcher coordinates of a super-attracting fixed point there exists a dense set $X$ so that for each $a, b \in X$ there exists $n > 0$ so that $f^n(a) = f^n(b)$. (So the grand orbit of a point in the basin of a super-attracting fixed point is dense in such a topological circle.) It follows that a topological conjugacy maps level sets of Böttcher coordinates to level sets. The same argument also shows that the conjugacy maps rays to rays.

*Proof.* First consider a periodic component $B \subset \bar{\mathbb{C}}$ of the Fatou set which contains a super-attracting periodic point (of, say, period $n$). Consider an equipotential $\Gamma'$ on which $f^n$ is conjugate to $z \mapsto z^d$, $z \in \mathbb{D}$. (Choose $\Gamma'$ so that no iterates of critical points of $f$ are on $\Gamma'$, and so that the disc which $\Gamma'$ bounds contains no other critical point apart from the fixed point.) The curves $\Gamma'$ and $f^n(\Gamma')$ bound a fundamental annulus (which each orbit hits at most twice). By the above remark, the conjugacy $h$ maps this annulus to a similar annulus again bounded by equipotentials, and so $h$ can be written in polar Böttcher coordinates as

$$(r, \phi) \mapsto (Z(r), \phi).$$

Now approximate $Z$ by a smooth map $Z_0$ which agrees with $Z$ on the inner and outer boundary of the annulus and also on circles containing iterates of critical points. By its form $(r, \phi) \mapsto (Z_0(r), \phi)$ induces a qc conjugacy $h_0$ between $f$ and $\tilde{f}$ on the the disc in $B$ bounded by $\Gamma'$. Since $h_0$ agrees with $h$ on iterates of critical points of $f$, one can extend $h_0$ to $B$ by pulling back. Similarly, near an attracting periodic point of period $n$ and with multiplier $\neq 0$, one can do something similar. Indeed, in this case take a curve $\Gamma'$ surrounding the periodic point, which in linearizing coordinates is a circle, and such that there are no iterates of critical points on $\Gamma'$. Also choose $\Gamma'$ so that $f^n$ is univalent on the disc bounded by $\Gamma'$. Then approximate the conjugacy $h$ on the (fundamental) annulus bounded by $\Gamma'$ and $f^n(\Gamma')$ by a smooth homeomorphism $h_0$ which agrees with $h$ at iterates of critical points of $f$. Do this, so that $\tilde{\Gamma}' = h_0(\Gamma')$ is smooth, and so that $h_0$ maps $f^n(\Gamma')$ to $\tilde{f}^n(\tilde{\Gamma}')$. Again this induces a conjugacy $h_0$ between $f$ and $\tilde{f}$ on the basin of this periodic attractor.

Let us show that $h_0$ (or $h$) can be extended continuously to the boundary of $B$. So consider the immediate basin $B$ of an attracting periodic point of period $n$. Let $\phi\colon \mathbb{D} \to B$ be the Riemann mapping. Then $A = \phi^{-1} \circ f^n \circ \phi$ is a Blaschke product (say of degree $d$) and since all periodic points of $f$ are repelling, $A$ is expanding on a neighbourhood $V$ of $\partial\mathbb{D}$ (i.e. there exists $N$ such that $A^N(V) \supset V$ and $|DA^N| \geq 2$ on $V$). By taking components $\Gamma'_i$ of $f^{-ni}(\Gamma')$ such that $\Gamma'_0, \Gamma'_1, \ldots$ lie nested, we obtain a nested sequence of fundamental annuli $Fund_i$ bounded by $\Gamma'_i$ and $\Gamma'_{i+1}$. For $i$ large enough, $\phi^{-1}(Fund_i)$ is contained in $V \cap \mathbb{D}$. For simplicity write $Fund := Fund_i$. It follows that if we consider a smooth curve $\tau$ connecting a point $z$ in the outer boundary of $Fund$ to $f^n(z)$, then the length of each component of $A^{-i}(\phi^{-1}(\tau))$ decreases exponentially with $i$. Hence $\cup_{i\geq 0} A^{-i}(\phi^{-1}(\tau))$ contains a curve $\tau'$ which converges to a fixed point of $A|\partial\mathbb{D}$. Because of this, $A^{-i}(\phi^{-1}(Fund \setminus \tau'))$ consists of $d^i$ topological rectangles with two of its boundaries consisting of curves through preimages of one of the fixed points of $A|\partial\mathbb{D}$ (these two curves are components of $A^{-i}(\tau')$). In other words, each of the $d^i$ preimages of $A^{-i}(z)$ with $z \in \phi^{-1}(Fund)$ is determined by a sequence of $i$ symbols. Now do the same for $\tilde{f}$. Because of the symbolic description given above, and since the size of each of the $d^i$ rectangles shrinks to zero as $i \to \infty$, it follows that $\phi^{-1} \circ h_0 \circ \phi$ extends continuously to a conjugacy on the boundary. But since $J(f)$ and $J(\tilde{f})$ are locally connected, it follows that $h_0$ extends continuously to the closure of $B$.

Of course such an extension is not unique if the degree of $A$ is $> 2$. But note that $h_0|\partial B$ agrees with $h|\partial B$ because $h$ and $h_0$ agree on each of the points $\phi(\tau') \cap \partial f^{-i}(Fund)$, $i = 0, 1, \ldots$, i.e. on a sequence of points in $\phi(\tau')$ converging to $\partial B$.

By doing this for each periodic attractor, and pulling back the resulting map $h_0$ we obtain the required homeomorphism. □

# 4 Rigidity of non-renormalizalble polynomials (Theorem 1.5 implies Theorem 1.3)

Let us now show that combinatorially equivalent polynomials induce combinatorially equivalent complex box mappings.

**Proposition 4.1.** *Assume that $f$ and $\tilde{f}$ are finitely renormalizable polynomials with only hyperbolic periodic points. Moreover, assume that one of the*

*following conditions hold:*

**a)** *$f$ and $\tilde{f}$ are topologically conjugate;*

**b)** *$f, \tilde{f}$ are combinatorially equivalent and restricted to their Fatou sets there exists a compatible conjugacy between $f$ and $\tilde{f}$ (these notions are defined in Definition 1.1).*

*Then associated to $f, \tilde{f}$ are complex box mappings $F\colon U \to V$, $\tilde{F}\colon \tilde{U} \to \tilde{V}$ which are combinatorially equivalent and there exists a qc map $h$ which maps $V$ onto $\tilde{V}$ and for which $\tilde{F} \circ h = h \circ F$ on $\partial U$.*

In other words, the conclusion is that the assumption of Theorem 1.5 is satisfied. As a first step in the proof of this proposition we have:

**Lemma 4.1.** *Assume $f, \tilde{f}$ are as in the proposition above. Then, associated to $f$ and $\tilde{f}$ are complex box mappings $F\colon U \to V$ and $\tilde{F}\colon \tilde{U} \to \tilde{V}$ so that to each puzzle piece $P$ of $F$ is uniquely associated to a puzzle piece $\tilde{P}$ of $\tilde{F}$ with the following properties:*

1. *$P$ contains the same number of critical points as $\tilde{P}$;*
2. *the conjugacy from Assumption 1 or 2 of the above proposition, maps $P \setminus J(f)$ onto $\tilde{P} \setminus J(\tilde{f})$;*
3. *$\widetilde{f^k(P)} = \tilde{f}^k(\tilde{P})$.*

*Proof.* Choose equipotentials $\Gamma$ and $\Gamma_{BS}$ (associated to $\infty$ and super-attracting periodic orbits in the previous subsection) which do not contain iterates of critical points of $f$. Then $h(\Gamma)$ and $h(\Gamma_{BS})$ are also equipotentials, see the remark above, and so define $\tilde{\Gamma} = h(\Gamma)$ and $\tilde{\Gamma}_{BS} = h(\tilde{\Gamma}_{BS})$. For periodic attractors, choose an equipotential $\Gamma_B$ as in Subsection 2.1 and set $\tilde{\Gamma}_B = h(\Gamma_B)$. Similarly, choose rays for $f$ which are mapped by $h$ onto rays. Using these choices, associate to $f$ and $\tilde{f}$ complex box mappings $F$ and $\tilde{F}$ as in Subsection 2.2 (using the same construction for $f$ as for $\tilde{f}$).

For $\alpha \in \mathbb{Q}$, let $p(\alpha, f)$ be the periodic point on which the $f$-ray with angle $\alpha$ lands and define $n(\alpha, f)$ to be the number of rays landing on $p(\alpha, f)$. By definition, $n(\alpha, f) = n(\alpha, \tilde{f})$. So if we determine a puzzle piece by the rays in its boundary, then a collection of rays determines a puzzle piece for $F$ if and only if it determines a puzzle piece for $\tilde{F}$. A puzzle piece contains a critical point if and only if two of its boundary rays are mapped onto a single ray. Hence the lemma follows. □

*Proof of Proposition 4.1.* Let us now show that $F$ and $\tilde{F}$ are combinatorially equivalent (in the sense of Definition 1.5) and that the assumption of Theorem 1.5 holds. Let $U_0, \tilde{U}_0$ be collections of corresponding puzzle pieces of $F, \tilde{F}$ containing the set of critical points of $F, \tilde{F}$ (i.e., the component of $U, \tilde{U}$ containing critical points of $F, \tilde{F}$).

By Lemma 3.1, we can assume that the (compatible) conjugacy on the Fatou set, induces a $K_0$-qc homeomorphism $h\colon V \to \tilde{V}$ mapping $U$ onto $\tilde{U}$ such that $\tilde{F} \circ h = h \circ F$ and which is a combinatorial equivalence (in the sense of Definition 1.5). Of course, $h$ induces a $K_0$-qc map $h\colon V \backslash K(F) \to \tilde{V} \backslash K(\tilde{F})$ which conjugates $F$ and $\tilde{F}$ on these sets.

Let us now cite the Spreading principle (see Section 5.3 of [7]):

**Spreading Principle.** *In the above setting, there exists a $K_0$-qc map $\Phi : \mathbb{C} \to \mathbb{C}$ such that the following hold:*

1. *$\Phi = \phi$ on $U_0$;*
2. *for each $z \notin U_0$ we have $\tilde{f} \circ \Phi(z) = \Phi \circ f(z)$;*
3. *The dilatation of $\Psi$ on $\mathbb{C} - D(U_0)$ is equal to $K_0$, where $D(U_0)$ denotes the domain of the first landing map under $f$ to $U_0$;*
4. *for each puzzle piece $P$ of $F\colon U \to V$ which is not contained in $D(U_0)$, $\Phi(P)$ is equal to the puzzle piece $\tilde{P}$ of $\tilde{F}\colon \tilde{U} \to \tilde{V}$ and $\Phi : P \to \tilde{P}$ agrees with $h$ their boundary.*

*Proof.* Denote by $\mathcal{P}_n$ the collection of level $n$ puzzle pieces. For a puzzle piece $P \in \mathcal{P}_n$, let $k = k(P) \leq n$ be the minimal non-negative integer such that $F^k(P)$ is a critical puzzle piece or has depth 0, and let $\tau(P) = F^k(P)$. Then $F^k : P \to \tau(P)$ is a conformal map, and so is $\tilde{F}^k : \tilde{P} \to \tau(\tilde{P})$. Now define a qc map $\phi_P : P \to \tilde{P}$ by the formula $\tilde{F}^k \circ \phi_P = h \circ F^k$, where $h$ is the $K_0$-qc map from above. Note that the map $\phi_P$ has the same maximal dilatation as $h$.

Let $Y_0$ be the union of all puzzle pieces in $\mathcal{P}_0$. For $n \geq 0$, inductively define $Y_{n+1}$ to be the subset of $Y_n$ consisting of puzzle pieces $P$ of depth $n+1$ so that $P$ is not contained in $D(U_0)$, the domain of the first entry map into $U_0$. Note that each puzzle piece in $Y_n - Y_{n+1}$ of depth $n+1$ is a component of $D(U_0)$.

Define $\Phi_0 = h$. For each $n \geq 0$, assume that $\Phi_n$ is defined and define $\Phi_{n+1}$ so that $\Phi_{n+1} = \Phi_n$ on $V - Y_n$ and for each component $P$ of $Y_n$ define

1. $\Phi_{n+1} = h$ on $P - \bigcup_{Q \in \mathcal{P}_{n+1}} Q$;

2. for each $Q \in \mathcal{P}_{n+1}$ which is contained in $P$, define $\Phi_{n+1} = \phi_Q$.

For each $n \geq 0$, $\Phi_n$ is a $K_0$-qc map. Note that $\Phi_n$ is eventually constant (not dependent on $n$ for large $n$) on $\mathbb{C} - \bigcap_n Y_n$. Since $\bigcap Y_n = E(U_0)$ is a nowhere dense set, $\Phi_n$ converges to a qc map $\Phi$. The properties (1), (2) and (4) follow directly from the construction, and (3) follows from the fact that $E(U_0)$ (which consists of points which are never mapped into $U_0$) has measure zero. □

Let us now complete the proof of Proposition 4.1. Taking for $U_0, \tilde{U}_0$ the collection of critical puzzle piece of level $n$, set $H_n = \Psi$. Then $H_n \circ F = \tilde{F} \circ H_n$ on $V \setminus F^{-n}(V)$. Moreover, $H_n$ agrees with the compatible conjugacy $H$ on $V \setminus F^{-1}(V)$. Since $H$ is $\max(K, K_0)$-qc, the proposition follows. □

It follows that Theorem 1.5 implies Theorem 1.3.

# 5 Robustness of complex box mappings

Let $V$ be the set consisting of a union of puzzle pieces which contains each critical point which is not in the basin of a periodic attractor and let

$$E(V, f) = \{z \in J(f); f^n(z) \notin V \text{ for all } n \geq 0\}.$$

For each $z \in E(V, f)$ the level $k$ puzzle pieces $P_k(z)$ containing $z$ forms a decreasing sequence shrinking to $z$, see Lemma [13, Lemma 1.8]. It follows that $E(V, f)$ is hyperbolic.

So if $f$ has only hyperbolic periodic points, there exists a neighbourhood $O$ of $f$ so that if $V_g$ is the puzzle piece corresponding $V$ for $g \in O$ then $E(V_g, g)$ is hyperbolic and for each $z \in E(V, f)$ these exists an analytic map

$$O \ni g \mapsto z(g) \in E(V_g, g).$$

Take a polynomial $f$ with all periodic points hyperbolic, with critical points $c_1, \ldots, c_d$ and with complex box mapping $F \colon U \to V$ as in Subsection 2.2. Then there exists (locally) a manifold $\hat{\Sigma}(f)$ containing $f$ (also depending on $V$) with the following properties: each $g \in \hat{\Sigma}(f)$ has critical points $c_1(g), \ldots, c_d(g)$ such that

1. $g \mapsto c_i(g)$ is continuous and the degree of $g$ at $c_i(g)$ is the same for each $g \in \hat{\Sigma}(f)$;

2. let $\{A_i\}$ resp. $\{SA_i\}$ be the collection of hyperbolic attracting (resp. super-attracting) periodic points of $f$; then choose $O$ small enough so that $O \ni g \mapsto A_i(g)$ and $O \ni g \mapsto SA_i(g)$ are continuous (but $g$ might have new periodic attracting orbits);

3. if $f^N(c_i(f)) = f^M(c_j(f))$ for some integers $i, j$ and $N, M$ and this point is in the basin of a hyperbolic periodic attractor, then for all $g \in \hat{\Sigma}(f)$ we have $g^N(c_i(g)) = g^M(c_j(g))$; by decreasing $O$ if necessary we can ensure that if for some integers $N, M, i, j, k$ and for some $g \in \hat{\Sigma}(f)$, $g^N(c_i(f)) = g^M(c_j(f))$ and this point is in the basin of a hyperbolic attractor $A_k(g)$ (which is a continuous deformation of an attractor of $f$) , then $f^N(c_i(f)) = f^M(c_j(f))$;

4. if $c_i(f)$ is in the basin of a super-attracting fixed point with the angle of the ray landing on $c_i(f)$ equal to $\alpha_i$ then the same holds for all $g \in \hat{\Sigma}(f)$ (so the angle is the same);

5. if for some integers $i, j$ and $N, M$, $\quad f^N(c_i(f)), f^M(c_j(f))$ are both in the basin of a super-attracting fixed point while hitting the same level set associated to its Böttcher coordinates then the corresponding statement holds for all $g \in \hat{\Sigma}(f)$;

6. if the $k$-th iterate of $c_i(f)$ is equal to $z \in E(V)$ then $g^k(c_i(g)) = z(g)$ where $z(g)$ is as above.

Using quasi-conformal surgery and transversality of unfoldings (see for example [17]) it is not hard to show that $\hat{\Sigma}(f)$ is a manifold (locally near $f$).

**Proposition 5.1.** *Let $f$ be a non-renormalizable polynomial with all periodic points hyperbolic, with critical points $c_1, \ldots, c_d$ and let $F\colon U \to V$ be a complex box mapping associated to $f$ as in Subsection 2.1. Then there exists a neighbourhood $O$ of $f$ with the following properties. If $\tilde{f} \in O \cap \hat{\Sigma}(f)$ then there exists a complex box mapping $\tilde{F}\colon \tilde{U} \to \tilde{V}$ associated to $\tilde{f}$ and a qc map $h\colon \mathbb{C} \to \mathbb{C}$ which maps $U$ to $\tilde{U}$ and such that $\tilde{F} \circ h = h \circ F$ on $\partial U$.*

*Proof of Proposition 5.1.* Let $P_0$ be the puzzle piece partition for $f$ of level 0 constructed using the external rays landing on the hyperbolic periodic point

$a$ and an equipotential level. For all polynomials $g$ close to $f$, the number of rays landing on $a(g)$ is the same. So one can define a puzzle piece $P_0(g)$ corresponding to $P_0$ in an obvious way (taking the rays landing on $a(f)$ and the equipotential of the same level). Clearly, each puzzle piece in $P_0(g)$ converges to a puzzle piece of $P_0$ when $g \to f$ (say in the Hausdorff topology). Now take $\tilde{f}$ so close to $f$ that each periodic point used in the construction of the complex box mapping $F\colon U \to V$ remains hyperbolic. It then follows that there exists a finite collection of puzzle pieces $\tilde{V}$ corresponding to $V$, and that the first return map to $\tilde{V}$ induces a complex box mapping $\tilde{F}\colon \tilde{U} \to \tilde{V}$ corresponding to $F\colon U \to V$. The set of points which never enter $U$ is hyperbolic. Hence since $\tilde{f}$ is close to $f$, any non-hyperbolic or 'new' attracting periodic points of $\tilde{f}$ is inside $\tilde{U}$.

Because of Assumptions 2 and 3 in the space $\hat{\Sigma}(f)$, we can construct a conjugacy between $f$ and $\tilde{f}$ on a neighbourhood of super-attracting periodic points of the form $(r, \phi) \mapsto (Z(r), \phi)$ which maps iterates of critical points to corresponding iterates of critical points. Similarly, one can construct a conjugacy near other periodic attractors which are outside $V$. Using this, and the hyperbolicity of the set $E(V)$, we obtain a qc conjugacy on the complement of $V$. Using the spreading principle from the previous section the result follows. □

# 6 Perturbations to hyperbolic polynomials (Theorem 1.5 implies Theorem 1.4)

Let us say that a polynomial is *semi-hyperbolic* if all periodic points are hyperbolic, and all critical points in the Julia set are eventually mapped into some hyperbolic set. By transversality of the unfolding of such a semi-hyperbolic map (see for example [17]) it follows that it can be approximated by hyperbolic maps.

Let us consider the space $\mathcal{P}_d = \{z^d + a_2 z^{d-2} + \cdots + a_d\}$ of normalized polynomials polynomials of degree $d > 1$; if $f, \tilde{f} \in \mathcal{P}_d$ are conjugate by a Möbius transformation they are equal, and every polynomial is Möbius conjugate to one in $\mathcal{P}_d$.

Let $f \in \mathcal{P}_d$ be a non-renormalizable polynomial of degree $d$ without neutral periodic points, and assume for the moment that all critical points

of $f$ are non-degenerate. The map $f$ can have some critical points whose iterates converge to periodic attractors. Since by Theorem 1.1 $f$ has no measurable invariant linefields on its Julia set, by Theorem 6.3 in [12] the space $QC(f) = \{g \in \mathcal{P}_d; g \text{ is qc conjugate to } f\}$ is a countable union of embedded manifolds $T_i$ each of at most complex dimension $s$, where $s$ is the number of (finite) critical points of $f$ which are in basins of periodic attractors. Of course, $QC(f)$ is contained in the space $\hat{\Sigma}(f)$ introduced in the previous section. To express the dimension of $QC(f)$, we need to look at critical relations: it is possible that there are critical points $c_i, c_j$ of periodic attractors which are critically related, because either $c_i = c_j$ or because

1. some iterates of $c_i$ and $c_j$ in the basin of a periodic attractor could be the same. In this case we say that $c_i$ and $c_j$ are critically related; any map conjugate would need to have the same properties;

2. some iterates of $c_i, c_j$ in the basin of a super-attracting periodic attractor could hit the same level set associated to the Böttcher coordinate. In this case we again say $c_i$ and $c_j$ are critically related. Any conjugate map would also have this property and the 'external angles' of these iterates of $c_i$ and $c_j$ would have to be the same.

By the description in Theorem 6.3 in [12] the dimension of $QC(f)$ is equal to $s'$, where $s'$ is the number of equivalence classes of critical points in basins of hyperbolic periodic attractors. Moreover, $\dim(\hat{\Sigma})$ is the number of equivalence classes of all critical points minus the number of critical points in the Julia set which escape $V$ (see point 4 in the definition of $\hat{\Sigma}$). So the codimension of $QC(f)$ in $\hat{\Sigma}(f)$ is equal to the number of critical points in the Julia set which do not leave $V$.

Near $f$, the set $\hat{\Sigma}(f)$ is a smooth manifold and so take local coordinates in which $\hat{\Sigma} := \hat{\Sigma}(f)$ is a linear space. If $s' = \dim(\hat{\Sigma})$ then all the critical points which are not contained in the basin of a periodic attractor are mapped into a hyperbolic set, and so the map is semi-hyperbolic and can be perturbed to a hyperbolic map as observed in the beginning of this section. So assume $s' < \dim(\hat{\Sigma})$ and consider the Grassman space $G$ of all complex linear subspaces $\Sigma \subset \hat{\Sigma}$ of complex codimension $s'$ containing $f$ (here $\Sigma$ is assumed to be a linear subspace in terms of the local coordinates which make $\hat{\Sigma}$ a linear space). Denote by $B(f; r)$ the *open* ball of radius $r$ through $f$. Since $\Sigma, T_i$ are both subsets of $\hat{\Sigma}$ such that the sum of their dimensions is $\dim(\hat{\Sigma})$, for each $i$, the set of $\Sigma \in G$ and $r > 0$ such that $\Sigma \cap \partial B(f; r) \cap T_i = \emptyset$ is open

and dense in $G \times \mathbb{R}^+$. By the Baire category theorem it follows that there exist (many) $\Sigma_0 \in G$ and $r_0 > 0$ such that $\Sigma_0 \cap \partial B(f; r_0) \cap QC(f) = \emptyset$. So throughout the remainder restrict $\mathcal{P}_d$ to $\Sigma_0$ and define

$$W := \Sigma_0 \cap B(f; 2r_0) \text{ and } W_0 := \Sigma_0 \cap \overline{B(f; r_0)}$$

where we take $r_0$ so small that $W \subset \Sigma_0$ (we can still shrink $r_0$ later in the proof). Let us denote the boundary of $W_0$ as a subset of $\Sigma_0$ by $\partial W$. One reason for choosing $W$ in this way is that $\partial W_0 \cap QC(f) = \emptyset$.

We may assume that $W$ is chosen so small that all critical points of polynomials $g \in W$ are still non-degenerate, and that all critical points of $g$ corresponding to those critical points of $f$ converging to periodic attractors, still converge to periodic attractors. (But it is possible that some additional critical points lie in basins of attracting periodic points of $g$.) The other reason for the above construction (and for introducing $\Sigma_0$) is that, provided we take $r_0 > 0$ small, for each map $g \in \partial W_0$ there exists a qc conjugacy defined on the basins $B(f)$ of periodic attractors of $f$ to the basin $\tilde{B}(g)$ of the corresponding periodic attractor of $\tilde{f}$ with the additional property that the orbit of the critical points in $B(f)$ is mapped to the orbit of the corresponding critical points in $\tilde{B}(g)$. (At this moment we don't say anything about other periodic attractors that $g$ might have.) Decreasing $W$ if necessary we can assume that all periodic points used in the construction of the complex box mapping $F\colon U \to V$ remain repelling for all polynomials from $W$ and that the conclusion Proposition 5.1 holds.

Let $c_1, \ldots, c_N$ be the critical points of $f$ whose iterates do not converge to periodic attractors (and so these points are in the Julia set of $f$) and which also are not mapped into the hyperbolic set mentioned above. As we have shown above, $N$ is the codimension of $QC(f)$ as a subspace of $\hat{\Sigma}(f)$ and is equal to the dimension of $W$. For $g \in W$ the critical point corresponding to $c_i$ will be denoted as $c_i(g)$. Suppose that for all $g \in W$ we have $g^k(c_i(g)) \neq c_1(g)$, $i = 1, \ldots, N$, $k = 1, \ldots$. Then all preimages of $c_1(g)$ move holomorphically with $g \in W$. Therefore we have a holomorphic motion on the set of all preimages of $c_1$ and, since this set is dense in the Julia set, we can extend the holomorphic motion to the whole Julia set. In fact, from Theorem 7.4 of [12] (which is based on the harmonic $\lambda$-lemma), any $g \in W$ is qc conjugate to $f$. But this contradicts the assumption that $QC(f) \cap \partial W_0 = \emptyset$.

So there exists $g \in W$ such that $g^k(c_i(g)) = c_1(g)$ for some $k$ and $i$. If $N = 1$ then $g$ is a semi-hyperbolic map and the theorem follows. Let $N$ be larger

than one. The space $\Sigma_0$ is finite-dimensional and the equation $g^k(c_i(g)) = c_1(g)$ determines an algebraic variety in this space. The singularities of this variety have complex codimension one in it and when we remove them we obtain a manifold of complex codimension one. Take a connected component of the intersection of this manifold with $W$, and denote this manifold by $M_1$.

Now let us restrict ourselves to $M_1$ and consider all the preimages of $c_2(g)$, $g \in M_1$. Arguing as above we can deduce that either all maps in $M_1$ are qc conjugate or there is $g \in M_1$ and $i', k'$ such that $g^{k'}(c_{i'}(g)) = c_2(g)$. In the second case, if $N = 2$, we obtain a semi-hyperbolic map, otherwise we can continue the construction. More precisely, we can consider a manifold $M_2$ which is obtained from the algebraic variety given by the equations $g^k(c_i(g)) = c_1(g)$ and $g^{k'}(c_{i'}(g)) = c_2(g)$ removing singularities from it. The manifold $M_2$ has codimension two in $W$.

Since the dimension of $W$ is equal to the number of critical points of $f$ which are in the Julia set ($= N$), we can continue until we obtain either a semi-hyperbolic map $g \in W$ or a manifold which consists of qc conjugate maps. In the first alternative, all remaining critical points of $g$ either were already in the basin of a periodic attractor, or are mapped into a hyperbolic set. So as before, we again can perturb $g$ to a hyperbolic map. So assume the second alternative holds. Note that the manifold can consist of infinitely renormalizable maps, so we cannot use the Rigidity theorem directly. Also note that this manifold is obtained from an algebraic variety by removing singularities and intersecting this variety with $W$, so the intersection of the boundary of this manifold and the boundary of $W$ is not empty. This is because a complex algebraic variety cannot be contained in a bounded topological ball and if it has non empty intersection with such a ball, it also has non empty intersection with the ball's boundary.

Obviously, we can construct infinitely many manifolds like this and, moreover, we can do it in such a way that these manifolds accumulate on $f$ (by taking a shrinking sequence of neighbourhoods $W_i \ni f$ and finding maps $g$ as above in $W_i$). Since these manifolds extend to the boundary of $W$ there exists $\tilde{f} \in \partial W_0$, $\tilde{f} \neq f$, such that some subsequence of these manifolds accumulates on $\tilde{f}$ as well. So, we have obtained two sequence of maps $\{f_i\}$ and $\{\tilde{f}_i\}$ such that $f_i \to f$, $\tilde{f}_i \to \tilde{f}$ and such that for each $i$ the maps $f_i$ and $\tilde{f}_i$ are qc conjugate.

For any $g \in W$ there exist complex box mapping $G\colon \hat{U} \to \hat{V}$ such that the domains of the box depend continuously on $g$ as long as $g \in W$ (here we use Proposition 5.1). Applying this to $f_i, f, \tilde{f}_i, \tilde{f}$, and using the fact that $\tilde{F}_i$ and

$F_i$ are combinatorially equivalent, it follows that $F$ and $\tilde{F}$ are combinatorially equivalent. Hence, since $F$ is non-renormalizable, Theorem 1.5 implies that $F$ and $\tilde{F}$ are qc conjugate. It follows that $f$ and $\tilde{f}$ are qc conjugate. Since $\tilde{f} \in \partial W_0$ we obtain a contradiction of the assumption that $QC(f) \cap \partial W_0 = \emptyset$.

# 7 Complex bounds implies rigidity for non-renormalizable complex box mappings

Assume that $f\colon U \to V$ and $\tilde{f}\colon \tilde{U} \to \tilde{V}$ are complex box mappings with only repelling periodic orbits, which are non-renormalizable and combinatorially equivalent (in the sense defined in Definition 1.5).

For simplicity, define $\mathcal{L}_x(P)$ to be the domain of the first entry map to $P$ containing $x$ and let $\hat{\mathcal{L}}_x(P)$ is defined to be equal to $P$ if $x \in P$ and otherwise equal to $\mathcal{L}_x(P)$.

Given two critical puzzle pieces $P, Q$, we say that $Q$ is a *child* of $P$ if it is a unimodal pullback of $P$, i.e., if there exists a positive integer $n$ such that $f^{n-1} : f(Q) \to P$ is a diffeomorphism. Given a puzzle piece $P \ni c$, by a *successor* of $P$ we mean a puzzle piece of the form $\hat{\mathcal{L}}_c(Q)$, where $Q$ is a child of $\hat{\mathcal{L}}_{c'}(P)$ for some $c' \in Crit(f)$. The map $f$ is called *persistently recurrent*, if each critical puzzle piece has at most finitely many successors.

**Definition 7.1.** A puzzle piece $P$ is called *$\rho$–nice* if for any $x \in P \cap PC(f)$ one has $\mathrm{mod}(P - \mathcal{L}_x(P)) \geq \rho$ and *$\delta$–fat* if there are puzzle pieces $P^+ \supset P \supset P^-$ such that the set $P^+ - P^-$ does not contain points of the postcritical set of $f$, $\mathrm{mod}(P^+ - P) \geq \delta$ and $\mathrm{mod}(P - P^-) \geq \delta$.

We say that a simply connected domain $U$ has *$\rho$–bounded geometry with respect to $x \in U$* if there are two disk $B(x, r) \subset U \subset B(x, R)$ and $R/r < \rho$. A domain $U$ is said to have *$\rho$–bounded geometry* if there is $x \in U$ such that $U$ has $\rho$–bounded geometry with respect to $x$.

**Theorem 7.1** (Strong complex bounds imply rigidity)**.** *Assume that $f\colon U \to V$ and $\tilde{f}\colon \tilde{U} \to \tilde{V}$ are non-renormalizable complex box mappings with all their periodic orbits repelling. Also assume that the map $H_0 = H_1\colon V \setminus U \to \tilde{V} \setminus \tilde{U}$ from definition 1.5 is qc. Moreover, assume that the first return map $R_P$ to each critical puzzle piece $P$ of $f$ is either*

**a)** *non-persistently recurrent, or,*

**b)** *persistently recurrent and there exists a $\delta > 0$ such that the complex box mapping $g := R_P$ has arbitrarily small critical puzzle pieces $W$ which are $\delta$-nice, $\delta$-fat and have $\delta$-bounded geometry with respect to the critical point in $W$; also assume that the same statement holds for the corresponding puzzle pieces of $\tilde{f}$.*

*Then $f$ and $\tilde{f}$ are quasiconformally conjugate.*

*Proof.* This was proved in Sections 6.3 and 6.4 of [7], using the QC-criterion from the Appendix of that paper and the spreading principle from Section 5.3. Note that since we assume here the conclusion rather than the assumption of the Key Lemma from [7], the proof simplifies in a few places (for example in Lemma 6.7). □

In the persistently recurrent unicritical case, a different proof of the above theorem was recently given in [1] which does not require the bounded geometry condition, but which does require that the sequence of puzzle pieces $W_i$ from the second assumption has the property that $W_{i+1}$ is a pullback of $W_i$ of uniformly bounded degree.

The next theorem asserts that the second assumption in the previous theorem is always satisfied:

**Theorem 7.2** (Complex bounds for non-renormalizale complex box mappings)**.** *Assume that $f\colon U \to V$ is a non-renormalizable complex box mapping with each of its periodic orbits repelling. Moreover, assume that the first return map $R_P$ to each critical puzzle piece $P$ of $f$ is persistently recurrent. Then there exists a $\delta > 0$ such that the complex box mapping $g := R_P$ has arbitrarily small critical puzzle pieces $W$ which are $\delta$-nice, $\delta$-fat and have $\delta$-bounded geometry with respect to the critical point in $W$. The puzzle pieces for which this holds are combinatorially defined (the same statement holds for the corresponding puzzle pieces when $\tilde{f}$ is related to $f$ as in the previous theorem).*

Theorem 1.5 now follows from the previous two theorems. Indeed, if the first return map to a critical puzzle piece is non-persistently recurrent, there exists a sequence of critical puzzle pieces which under some iterates are mapped with bounded degree to a puzzle piece of fixed level. This implies in the non-persistently recurrent case that puzzle pieces shrink to points and absence of invariant linefields (and, in fact, that the Julia set of the complex

box mapping has zero Lebesgue measure). In the persistently recurrent case, the previous theorem gives that puzzle pieces shrink to points and absence of invariant linefields follows as in [11, Theorem 10.3] and [9, Proposition 4.3]. In both cases, Theorem 7.1 implies the qc rigidity of complex box mappings.

It follows that it suffices to prove Theorem 7.2. This will be done in the remainder of this paper.

# 8 Enhanced nest

In the remainder of the paper we will prove Theorem 7.2. So let us assume that $f\colon U \to V$ is persistently recurrent. In [7] we constructed a sequence of puzzle pieces

$$\mathbf{I}_0 \supset \mathbf{I}_1 \supset \mathbf{I}_2 \dots \tag{1}$$

around $c_0$, called the *enhanced nest* for the map $f$. The construction was based on the following lemma (Lemma 8.1 from [7]):

**Lemma 8.1.** *Let $\mathbf{I} \ni c$ be a puzzle piece. Then there is a positive integer $\nu$ with $f^\nu(c) \in \mathbf{I}$ such that the following holds. Let $\mathbf{U}_0 = \mathrm{Comp}_c(f^{-\nu}(\mathbf{I}))$ and $\mathbf{U}_j = f^j(U)$ for $0 \le j \le \nu$. Then*

1. $\#\{0 \le j \le \nu - 1 : \mathbf{U}_j \cap \mathit{Crit}(f) \neq \emptyset\} \le b^2$;
2. $\mathbf{U}_0 \cap PC(f) \subset \mathrm{Comp}_c\left(f^{-\nu}(\mathcal{L}_{f^\nu(c)}(\mathbf{I}))\right)$.

For each puzzle piece $\mathbf{I} \ni c$, let $\nu = \nu(\mathbf{I})$ be the smallest positive integer with the properties specified by Lemma 8.1. We define

$$\mathcal{A}(\mathbf{I}) = \mathrm{Comp}_c(f^{-\nu}(\mathcal{L}_{f^\nu(c)}(\mathbf{I}))),$$
$$\mathcal{B}(\mathbf{I}) = \mathrm{Comp}_c(f^{-\nu}(\mathbf{I})).$$

As $f$ is persistently recurrent, each critical puzzle piece $\mathbf{P}$ has a smallest successor, which we denote by $\Gamma(\mathbf{P})$. Remark that if $\mathbb{Q}$ is an entry domain to $\mathbf{P}$ intersecting $PC(f)$, then $\hat{\mathcal{L}}_c(\mathbb{Q})$ is an successor of $\mathbf{P}$ by definition, and thus $\hat{\mathcal{L}}_c(\mathbb{Q}) \supset \Gamma(\mathbf{P})$. Now define the enhanced nest (1) by $\mathbf{I}_0 = V_0$ and for each $n \ge 0$,

$$\mathbf{I}_{n+1} = \Gamma^T \mathcal{B}\mathcal{A}(\mathbf{I}_n),$$

where $T = 5b$. By construction $\mathbf{I}_{n+1}$ is a pullback of $\mathbf{I}_n$ and the map

$$f^{p_n}\colon \mathbf{I}_{n+1} \to \mathbf{I}_n$$

has degree bounded by $M(d)$, where $M(d)$ only depends on the degree of $f$. This construction is chosen because of the following lemma (see Lemma 8.1 and 8.2 in [7]):

**Lemma 8.2.** *For each $n > 0$ there exists $\epsilon_n > 0$ so that $\mathbf{I}_n$ is $\epsilon_n$-nice and $\epsilon_n$-fat. Moreover, denoting by $r(\mathbf{I}_{n+1})$ the minimal return time from $\mathbf{I}_{n+1}$ to itself, we get*

1. $3\, r(I_{n+1}) \geq p_n$;
2. $p_{n+1} \geq 2p_n$.

Since the degree of $f^{p_n} \colon \mathbf{I}_{n+1} \to \mathbf{I}_n$ is bounded by the $M(d)$, by Koebe's distortion lemma

**Lemma 8.3.** *There exists a universal constant $K$ (depending only on the degree $d$ of $f$) such that if $\mathbf{I}_n$ is $\epsilon$–nice then $\mathbf{I}_{n+1}$ is $K\epsilon$–nice and $K\epsilon$–fat.*

# 9 Pullback lemmas

Let $A$ be an annulus. We will use the two equivalent definitions of its modulus.

Let $P_1(A)$ be the class of non-negative Borel measurable function $\rho : A \to \mathbb{R}$ such that if $\gamma$ is any rectifiable Jordan closed curve separating boundaries of $A$, then $\int_\gamma \rho\, d|z| > 1$. Then

$$\mathrm{mod}(A) = \inf_{\rho \in P_1(A)} \int_A \rho^2\, dz^2.$$

Similarly, let $P_2(A)$ be the class of non-negative Borel measure function $\rho : A \to \mathbb{R}$ such that if $\gamma$ is any rectifiable Jordan curve connecting boundaries of $A$, then $\int_\gamma \rho\, d|z| > 1$. Then

$$\mathrm{mod}(A) = (\inf_{\rho \in P_2(A)} \int_A \rho^2\, dz^2)^{-1}.$$

Write $\mathbb{C}^+ = \{z \in \mathbb{C} : Im(z) \geq 0\}$.

**Lemma 9.1** (Small Distortion of Thin Annuli)**.** *For every $K \in (0,1)$ there exists $\kappa > 0$ such that if $A \subset U$, $B \subset V$ are simply connected domains symmetric with respect to the real line, $F : U \to V$ is a real holomorphic branched covering map of degree $D$ with all critical points real which can be decomposed as a composition of maps $F = f_1 \circ \cdots \circ f_n$ with all maps $f_i$ real and either real univalent or real branched covering maps with just one critical point, the domain $A$ is a connected component of $f^{-1}(B)$ symmetric with respect to the real line and the degree of $F|_A$ is $d$, then*

$$\mathrm{mod}(U - A) \geq \frac{K^D}{2d} \min\{\kappa, \mathrm{mod}(V - B)\}.$$

*Proof.* Let $B \subset V$ be two real-symmetric domains and let $B^+ = B \cap \mathbb{C}^+$. We claim that

$$\frac{1}{2}\mathrm{mod}(V - B^+) \leq \mathrm{mod}(V - B) \leq \mathrm{mod}(V - B^+). \tag{2}$$

The second inequality is obvious since $B^+ \subset B$. Let $\rho$ be in $P_1(V - B)$ such that $\int \rho^2 dz^2$ is almost $\mathrm{mod}(V - B)$. Clearly, $\rho_0(z) = (\rho(z) + \rho(\bar{z}))/2$ is also in $P_1(V - B)$ and $\int \rho_0^2 \, dz^2 \leq \int \rho^2 \, dz^2$ (where we use the inequality $((a+b)/2)^2 \leq (a^2 + b^2)/2$). For $z \in V - B^+$ define $\rho_1$ by $\rho_1(z) = 2\rho_0(z)$ if $Im(z) \geq 0$ and $\rho_1 = 0$ otherwise. Obviously, $\rho_1 \in P_1(V - B^+)$ and

$$2 \int \rho_0^2 \, dz^2 = \int \rho_1^2 \, dz^2 \geq \mathrm{mod}(V - B^+)$$

proving (2).

Before we continue with the proof of Lemma 9.1 we state and prove:

**Sublemma 9.1.** *There exists a universal constant $C > 0$ such that the following holds. Let $\mathbb{D}$ be a unit disk and $B \subset \mathbb{D} \cap \mathbb{C}^+$ be a simply connected domain. Let $A$ be a connected component of $\phi^{-1}(B)$ where $\phi(z) = z^d$. Then*

$$\mathrm{mod}(\mathbb{D} - A) > \frac{\mathrm{mod}(\mathbb{D} - B)}{1 + Cd \, \mathrm{mod}(\mathbb{D} - B)}.$$

*Proof.* Let $\mathbb{E} = \{z \in \mathbb{C} : \arg(z) \in [0, \pi/d]\}$ and $\mathbb{F} = \{z \in \mathbb{C} : \arg(z) \in [-\pi/(2d), 3\pi/(2d)]\}$ (i.e. pullbacks by $z \mapsto z^d$ of the upperhalf plane respectively the complex plane minus the negative imaginary axis). Without loss of generality we can assume that $A$ is in the sector $\mathbb{E}$. Let $\rho \in P_2(\mathbb{D} - B)$

and assume that $\int \rho^2\, dz^2$ is close to $1/\mathrm{mod}(\mathbb{D}-B)$. Pull $\rho$ back by $\phi$ to the domain $(\mathbb{D}-A)\cap\mathbb{F}$ and denote this pullback by $\rho_2$, and on $\mathbb{D}-\mathbb{F}$ we set $\rho_2$ to be zero. Define $\rho_3$ on $\mathbb{D}$ satisfying the following two properties: (i) for any Jordan rectifiable path $\gamma$ connecting the set $[0,1/2\exp(\pi i/d)]\cup[0,\frac{1}{2}]$ and the boundary of $\mathbb{D}$ one has $\int_\gamma \rho_3\, d|z| > 1$; (ii) for any Jordan rectifiable curve $\gamma$ connecting the set $[\frac{1}{2}\exp(\pi i/d),\exp(\pi i/d)]\cup[1/2,1]$ and $\mathbb{C}-\mathbf{F}$ one has $\int_\gamma \rho_3\, d|z| > 1$. It is easy to see that $\rho_3$ exists with $C'=\int_{\mathbb{D}}\rho_3^2\, dz^2$ finite. If we take $\rho_3$ equal to 0 outside the region $F$ and constant inside, we see that $C'/d \le C$ with $C$ independent of $d$.

Set $\rho_4(z)=\max\{\rho_2(z),\rho_3(z)\}$ for $z\in\mathbb{D}-A$. It is easy to check that $\rho_4\in P_2(\mathbb{D}-A)$. Then

$$\frac{1}{\mathrm{mod}(\mathbb{D}-A)} \le \int \max\{\rho_2(z)^2,\rho_3(z)^2\}\, dz^2 \le$$

$$\le \int \rho_2(z)^2\, dz^2 + \int \rho_3(z)^2\, dz^2 \le \int \rho_2(z)^2\, dz^2 + C'.$$

This implies

$$\mathrm{mod}(\mathbb{D}-A) \ge \frac{\left[\int \rho_2(z)^2\, dz^2\right]^{-1}}{1+C'\left[\int \rho_2(z)^2\, dz^2\right]^{-1}} \approx \frac{\mathrm{mod}(\mathbb{D}-B)}{1+C'\mathrm{mod}(\mathbb{D}-B)}.$$

□

Lemma 9.1 is a direct consequence of equation (2) and the sublemma. First, we split $B$ into two parts and then only look at $B^+$. Because of the second inequality in (2), we do not lose any modulus at this point. Then we pullback $B^+$ many times all the way. At each step we do not lose modulus if the pullback is univalent. If the $i$-th pullback is quadratic but the corresponding pullback of $B^+$ does not contain a critical value of the map $f_i$, we apply the sublemma and so lose just a bit of modulus of $\mathbb{D}-B^+$ which is small; in this case set $e_i=1$. If the corresponding pullback of $B^+$ does contain a critical value of $f_i$ of order $d_i$, then we lose a factor $e_i=d_i$. At the end we reconstruct the preimage of $B$ by mirroring the preimage of $U^+$ and use the 1st inequality in (2). This means that we lose a factor 2. So in total we lose a factor $K^D/(2d)$, since $d=e_1\ldots e_n$ where $e_i=1$ or $e_i=d_i$. □

In the real case the previous lemma implies and sharpens the following result of J.Kahn and M.Lyubich, see [5]:

**Lemma 9.2.** *For any $\eta > 0$ and $D > 0$ there is $\epsilon = \epsilon(\eta, D) > 0$ such that the following holds: Let $A \subset A' \subset U$ and $B \subset B' \subset V$ be topological disks in $\mathbb{C}$ and let $F : (A, A', U) \to (B, B', V)$ be a holomorphic branched covering map. Let the degree of $f$ be bounded by $D$ and the degree of $f|_{A'}$ be bounded by $d$. Then*

$$\mathrm{mod}(U \setminus A) > \min(\epsilon,\, \eta^{-1}\mathrm{mod}(B' \setminus B),\, C\eta d^{-2}\mathrm{mod}(V \setminus B)),$$

*where $C > 0$ is some universal constant.*

# 10 Complex bounds for the enhanced nest

In this section we will prove the following proposition

**Proposition 10.1** (Complex bounds)**.** *Let $f$ be a complex box mapping. For any $\epsilon > 0$ there exists $\delta > 0$ such that if $\mathbf{I}_0$ is $\epsilon$–nice, then all $\mathbf{I}_n$, $n = 1, 2, \ldots$, are $\delta$–nice.*

Because of Lemma 9.2, $\mathbf{I}_n$, $n = 2, 3, \ldots$ are also $\delta' > 0$-fat.

*Proof.* Denote $\mu_n$ to be such that $\mathbf{I}_n$ is $\mu_n$-nice. Fix some integer $M > 4$ and suppose that $n > M + 1$. Let $A \subset \mathbf{I}_n$ be some domain of the first return map to $\mathbf{I}_n$ containing a point of the postcritical set and let $r$ be its return time.

**Step 1:** $A$ is contained in $\mathcal{L}_x(\mathbf{I}_{n-4})$ for some $x$. Indeed, write

$$P_{n,M} = p_{n-1} + p_{n-2} + \ldots + p_{n-M} \text{ and } B := f^{P_{n,M}}(A).$$

Note that $f^{P_{n,M}}$ maps $\mathbf{I}_n$ onto $\mathbf{I}_{n-M}$. By the last two inequalities in Lemma 8.2,

$$r \geq r(\mathbf{I}_n) \geq \frac{1}{3}p_{n-1} \geq \frac{16}{3}p_{n-5} \geq p_{n-5} + p_{n-6} + p_{n-7} + \cdots \geq P_{n-4,M-4}$$

and so $s := r - p_{n-4,M-4} \geq 0$. Since $\mathbf{I}_n = f^r(A) = f^s \circ f^{n-4,M-4}(A)$ we get that $f^s(f^{p_{n,M}}(A))$ is equal to

$$f^{p_{n-1}} \circ \cdots \circ f^{p_{n-4}} \circ f^s \circ f^{n-4,M-4}(A) = \mathbf{I}_{n-4}$$

and therefore $f^{P_{n,M}}(A)$ is contained in some $\mathcal{L}_x(\mathbf{I}_{n-4})$.

**Step 2:** There exists $\mathbf{I}_k^+$ with $\mathbf{I}_k \subset \mathbf{I}_k^+ \subset \mathbf{I}_{k-1}$ such that $\mathrm{mod}(\mathbf{I}_k^+ - \mathbf{I}_k) > K_1\mu_{k-1}$ and $\mathbf{I}_k^+ - \mathbf{I}_k$ does not contain postcritical points of $f$. Indeed, the

domain $\mathbf{I}_k$ is $K_1\mu_{k-1}$–fat, where $K_1$ is a constant given by Lemma 8.3. Since $\mathbf{I}_{k-1}$ is $\mu_{k-1}$ nice as well we can assume that $\mathbf{I}_k^+ \subset \mathbf{I}_{k-1}$.

**Step 3:** Fix $x \in f^{P_{n,M}}(A) \cap PC(f)$ and some integer $k \in [n-4, n-M]$. Take $\nu$ so that $R_{\mathbf{I}_k}|_x = f^\nu$. Denote the pullback of $\mathbf{I}_k^+ - \mathbf{I}_k$ by $f^\nu$ by $\mathbf{A}_k$, *i.e.* the point $x$ is surrounded by the annulus $\mathbf{A}_k$ and $f^\nu(\mathbf{A}_k) = \mathbf{I}_k^+ - \mathbf{I}_k$. The degree $d$ of $f^\nu|_{\mathcal{L}_x(\mathbf{I}_k)}$ is bounded by some universal constant $K_2$ depending only on $b$ and the degree of the map $f^\nu|_{\mathrm{Comp}_x(f^{-\nu}(\mathbf{I}_k^+))}$ is also $d$. Hence,

$$\mathrm{mod}(\mathbf{A}_k) > (K_1/K_2)\mu_{k-1}. \tag{3}$$

**Step 4:** Obviously, all annuli $\mathbf{A}_k$, $k = n-4, \ldots, n-M$, are nested and surround $f^{P_{n,M}}(A)$. This implies that

$$\mathrm{mod}(\mathbf{I}_{n-M} - f^{P_{n,M}}(A)) > (K_1/K_2)(\mu_{n-M-1} + \ldots + \mu_{n-5}). \tag{4}$$

**Step 5:** The degree of the map $f^{P_{n,M}}|_A$ is bounded by some constant $d$ which does not depend on $M$, while the degree of $f^{P_{n,M}}|_{\mathbf{I}_n}$ is bounded by some constant $K_4(M)$ depending on $M$. The second assertion is obvious, so let us prove the first one. Indeed, decompose the map $f^{P_{n,M}}|_A$ as $f^{P_{n-8,M-8}} \circ f^{P_{n,8}}$. The degree of $f^{P_{n,8}}$ is bounded by some constant depending only on $b$. The domain $f^{P_{n,M}}(A)$ is contained in $\mathcal{L}_x(\mathbf{I}_{n-4})$ and $M > 8$, hence $f^{P_{n,8}}(A)$ belongs to some other component $\mathcal{L}_y(\mathbf{I}_{n-4})$. Due to Lemma 8.2 we know that $r_{n-4} > P_{n-8,M-8}$, therefore the degree of the map $f^{P_{n-8,M-8}}$ is less or equal the degree of $R_{\mathbf{I}_{n-4}}|_{\mathcal{L}_y(\mathbf{I}_{n-4})}$ which is bounded by some constant depending only on $b$.

**Step 6:** Now apply Lemma 9.2. Let $B$ and $B'$ be bounded by the inner and outer boundary of the annulus $A_{n-4}$ and let $V := \mathbf{I}_{n-M}$. Let $A$, $A'$, $U$ be the corresponding pullbacks of $B$, $B'$, and $V$ by $F^{-1}$ where $F := f^{P_{n,M}}$. Notice that the degree of maps $f^{P_{n,M}}|_A$ and $f^{P_{n,M}}|_{A'}$ are the same because $A_{n-4}$ does not contain postcritical points.

We may assume that $K_1/K_2 < 1$. Fix $\eta = \frac{K_1}{2K_2}$ and let $C$ and $\epsilon(\eta, D)$ be the the constants from Lemma 9.2 associated to $\eta$ and $D$. Now fix $M = \frac{4}{C}d^2K_2/K_1 + 6$ and take $d$ and $D = K_4(M)$ as in Step 5. From equation (4),

$$\mathrm{mod}(V \setminus B) > \frac{K_1}{K_2}(\mu_{n-M-1} + \cdots + \mu_{n-5}) \geq \frac{4}{C}d^2\mu_{n-5,M-5} \tag{5}$$

where $\mu_{k,m} = \min(\mu_k, \mu_{k-1}, \ldots, \mu_{k-m})$. Lemma 9.2 together with equations (3) and (5) gives

$$\mathrm{mod}(U \setminus A) > \min\left(\epsilon(\eta, M),\, \eta^{-1}\mathrm{mod}(B' \setminus B),\, C\eta d^{-2}\mathrm{mod}(V \setminus B)\right)$$

$$> \min\left(\epsilon(\eta, M),\, 2\mu_{n-5},\, 2\mu_{n-5,M-5}\right),$$

and so $\mu_n > \min(\epsilon(\eta, M), 2\mu_{n-5,M-5})$. By Lemma 8.3 we also have $\mu_n > K\mu_{n-1}$. These two inequalities prove the proposition. □

If $f$ is real we can apply Lemma 9.1 instead of Lemma 9.2. This makes the above proof slightly easier.

# 11 Bounded Geometry (Proof of Theorem 7.2)

In this section we will show that the above complex bounds immediately imply that the puzzle pieces from the enhanced nest have bounded geometry.

**Proposition 11.1** (Bounded geometry)**.** *Let $f$ be a complex box mapping (not necessarily real). Let $\mathbf{I}_0$ be $\epsilon$–nice and let it has $\rho$–bounded geometry with respect to $c_0$. Then all $\mathbf{I}_n$, $n = 1, 2, \ldots$, have $\delta$–bounded geometry, where $\delta$ depends only on $\rho$ and $\epsilon$.*

Let us first state and prove the following easy consequences of Koebe's distortion lemma:

**Lemma 11.1.** *Let a domain $U$ have $\rho$–bounded geometry with respect to some point $x$ and let $A \subset U$ be a domain containing $x$. Then $U$ has $K\rho$–bounded geometry with respect to all $y \in A$, where the constant $K$ depends only on* $\mathrm{mod}(U - A)$.

*Proof.* By Koebe's distortion lemma it follows that $d(x, y) \le K d(y, \partial U)$ for any $y \in A$ where $K$ depends on $\mathrm{mod}(U - A)$. So if $d(x, y) \ge (1/2)d(x, \partial U)$ then

$$\frac{\sup_{z\in\partial U} d(y, z)}{d(y, \partial U)} \le 2K \frac{\sup_{z\in\partial U} d(x, z)}{d(x, y)} \le 4K \frac{\sup_{z\in\partial U} d(x, z)}{d(x, \partial U)}$$

and we are done. If $\mathrm{diam}(A) \le (1/2)d(x, \partial U)$ then $d(y, \partial U) \ge (1/2)d(x, \partial U)$ and the assertion follows immediately (without using Koebe's distortion lemma). □

Similarly we have

**Lemma 11.2.** *Let $f : U \to V$ be a holomorphic covering map, $B \subset V$, $A$ is a connected component of $f^{-1}(B)$. Then if $B$ has $\rho$–bounded geometry with*

*respect to some point $y \in B$, then $A$ has $K\rho$–bounded geometry with respect to $x$, where the point $x \in A$ is any preimage of $y$ by $f$ and the constant $K$ depends only on* $\mathrm{mod}(V - B)$ *and on the degree of the map $f$.*

*Proof of Proposition 11.1.* Let $\mathbf{I}_n$ have $\rho_n$–bounded geometry with respect to $c_0$. Since we know that $\mathbf{I}_n$ is $\delta$–fat for some constant $\delta$, $\mathbf{I}_n$ has $K_1\rho_n$–bounded geometry with respect to $f^{p_n}(c_0)$ where $K_1$ is given by Lemma 11.1. Then $f(\mathbf{I}_{n+1})$ has $K_1K_2\rho_n$–bounded geometry with respect to $f(c_0)$ where $K_2$ is given by Lemma 11.2 (notice that the degree of the map $f^{p_n-1}$ depends only on $b$). Therefore, $\mathbf{I}_{n+1}$ has $\sqrt{K_1K_2\rho_n}$–bounded geometry with respect to $c_0$, *i.e.* $\rho_{n+1} \geq \sqrt{K_1K_2\rho_n}$. □

Because of Propositions 10.1 and 11.1, we have completed the proof of Theorem 7.2.

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