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From the Monster to Majorana: A study of the 3A-axes

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Declaration of Originality

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We are such stuff
As dreams are made on,
and our little life
Is rounded with a sleep.

The Tempest
William Shakespeare

Abstract

The $3A$-axes are one of four famous families of vectors which in union span the acclaimed Monster algebra. Existing in 4-dimensional subalgebras generated by a pair of $2A$-axes, they are idempotents of length $\frac{8}{5}$. Inside the Monster algebra, these idempotents have a special association with, and are indexed by the $3A$-elements of the Monster. It is therefore paramount to understand these axes in order to further understand the Monster. This thesis sets out to uncover the many properties and profound consequences of the $3A$-axes. We present three main accomplishments. The first is an axiomatic approach. Properties of the $3A$-axes in the Monster algebra are first proven. These properties are then axiomatized as the definition of what we call a standard $3A$-axis. The second is on the $(2A, 3A)$-configurations. This is the study of subalgebras of the Monster algebra generated by a $2A$- and $3A$-axis. The algebra products between a $2A$- and $3A$-axis for three new cases are discovered. We also present the structures of several subalgebras generated by a $2A$- and $3A$-axis for the very first time. The third central result is the successful formulation of a methodology for determining all values of inner products between two $3A$-axes contained in a very prominent Majorana algebra. There has been much interest especially in Majorana theory in this algebra associated with A_{12}. The inner product classification achieved in this thesis contributes towards this open topic notably in the study of linear spans of axes.

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I would also like to thank my family whom I love for their patience and affection. Friends and colleagues, thank you for the good times and for making life a little more bearable.

Preface

Ever since I was a child, I have often wondered why certain things could be explained and certain things could not. I was convinced that everything had an explanation and it was only a matter of time before someone would come up with a solution. What's more, I believed that these solutions would ultimately uncover the great mysteries of life and that order may be realized from the chaos and uncertainty we live in. I have always hoped to play some part in this grand scheme. Through mathematics have I seen the existence of perplexing and abstract entities born out of an idea. Through mathematics have I also seen these problems addressed in a logical and irrefutable manner. As counter intuitive as it may seem, abstraction does sometimes give meaning to reality. The study of mathematics is a connected and continuing process. Discoveries often lead to more discoveries and topics which seem completely unrelated bizarrely are. It is this view which I take that inclines me towards this path.

The concept of a group is one of the most basic ideas in mathematics derived from observing symmetries. Though it may appear simple, delve deeper and one will only see the enormous complexities and deep consequences it promises. Most perplexing to me is the existence of the finite simple group known as the Monster. Why are there such groups not following a uniform pattern and why is there a largest one. These are among the many philosophical unanswered questions about the Monster. This thesis is the culmination of my work for the past three and a half years on the Monster group and the algebra which was constructed to prove its existence. The work follows a novel axiomatic approach introduced by Ivanov which goes by the name 'Majorana Theory'. Although relatively new, this theory has already been proven to be very effective in studying the Monster.

It is my strong belief that presentation is key to delivery. This thesis should be accessible to anyone with an introduction and familiarity to the concepts of group theory. Although the main work presented here may be separated into three parts, the topics studied are interrelated. The reader is therefore recommended to read it in the order that it is presented. I hope the reader will find delight in the contents of this thesis as much as I have had researching it.

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Preliminaries and Notations

The results stated in this section will not be proven. For further details, kindly refer to the references given.

Let G always denote a finite group unless otherwise stated. The identity element of G shall be denoted 1. For $g \in G$, the order of g shall be denoted $o(g)$. Let $h \in G$. The product g composed with h is denoted gh, while the conjugate of g by h is

$$g^h := h^{-1}gh.$$

The conjugacy class of g in G is denoted g^G. Let X be a non-empty subset of G. The smallest subgroup of G containing X (or the subgroup of G generated by X) is denoted $\langle X \rangle$. The set of conjugates of g by elements of X is denoted g^X, while X^g denotes the set conjugate to X by g:

$$g^X := \{g^x \mid x \in X\} \quad \text{and} \quad X^g := \{x^g \mid x \in X\}.$$

Let Y be another non-empty subset of G. Then X^Y denotes the set of elements of X conjugated by elements of Y:

$$X^Y := \{x^y \mid x \in X \text{ and } y \in Y\}.$$

The centralizer of X in G is denoted $C_G(X)$, while the normalizer of X in G is denoted $N_G(X)$. If X contains a single element say x, then the centralizer of X in G is simply denoted $C_G(x)$.

Let G act on a finite set Ω. The cardinality of Ω is denoted $|\Omega|$. The notation for the action of G on itself by conjugation is adopted to any action of G on Ω. For example, ω^g denotes the image of $\omega \in \Omega$ under the action of $g \in G$ while ω^G denotes the orbit of ω. The image of a subset $\Delta \subseteq \Omega$ under g is denoted Δ^g. The stabilizer of ω is denoted G_ω. The number of orbits of G on Ω shall be denoted $orb(G, \Omega)$.

The following well known result relating orbits and stabilizers may be seen in *Proposition 29.3* of [JL01].

Theorem 0.1 (**Orbit-Stabilizer Theorem**). *Let G act on Ω. Then for any $\omega \in \Omega$,*

$$|\omega^G| = \frac{|G|}{|G_\omega|}.$$

The trivial character of G shall be denoted 1_G. Let χ and ψ be characters of G. The inner product of characters between χ and ψ is denoted $\langle \chi, \psi \rangle$, and is defined as

$$\langle \chi, \psi \rangle := \frac{1}{|G|} \sum_{g \in G} \chi(g)\psi(g^{-1}).$$

The following result, often referred as **Burnside's Lemma**, establishes a connection between orbits and characters. As it is not due to Burnside but rather Cauchy and Frobenius, it is also referred as the **Cauchy–Frobenius Lemma**. Here it shall be called the **Orbit-Counting Theorem** (see *Proposition 29.4* of [JL01] for proof).

Theorem 0.2 (**Orbit-Counting Theorem**). *Let G act on Ω and let χ be the permutation character of G corresponding to this action. Then*

$$orb(G, \Omega) = \langle \chi, 1_G \rangle.$$

The following result which may be seen in *Proposition 29.6* of [JL01], is a variation of the orbit-counting theorem concerning the actions of G on two sets.

Theorem 0.3. *Let G act on finite sets Ω_1 and Ω_2 with permutation characters χ_1 and χ_2 respectively. Then*

$$orb(G, \Omega_1 \times \Omega_2) = \langle \chi_1, \chi_2 \rangle$$

where $orb(G, \Omega_1 \times \Omega_2)$ is the number of orbits of G on the set of ordered pairs $\Omega_1 \times \Omega_2$.

The next result is straightforward.

Theorem 0.4. *Let G act on finite sets Ω_1 and Ω_2. If G acts transitively on Ω_1, then the orbits of G on $\Omega_1 \times \Omega_2$ are in bijective correspondence with the orbits of G_ω on Ω_2 for any fixed $\omega \in \Omega_1$.*

Since representations may be viewed as modules and vice versa, they will be used interchangeably. The next four results are standard results in character theory. Their proofs may be found in *Chapters 8, 9, 17 and 30* respectively of [JL01].

Theorem 0.5 (**Maschke's Theorem**). *Let F be a field of characteristic zero (e.g. $\mathbb{R}$ or $\mathbb{C}$). Let V be an FG-module and suppose U is an FG-submodule of V. Then there exists an FG-submodule W of V such that $V = U \oplus W$.*

As a consequence of Maschke's Theorem, every FG-module may be decomposed as a direct sum of irreducible FG-submodules.

Theorem 0.6 (**Schur's Lemma**). *Let V and W be irreducible $\mathbb{C}G$-modules. If $\phi : V \to W$ is a $\mathbb{C}G$-homomorphism, then either ϕ is a $\mathbb{C}G$-isomorphism or ϕ is the zero map. Furthermore, if $\phi : V \to V$, then ϕ is multiplication by a scalar.*

The following result is also known as the **First Orthogonality Theorem**.

Theorem 0.7 (**Character Orthogonality Theorem**). *The characters of the irreducible complex representations (or irreducible characters) of G form an orthonormal set with respect to the inner product of characters $\langle\ ,\ \rangle$.*

As a consequence to the character orthogonality theorem, if V is a $\mathbb{C}G$-module and W is an irreducible $\mathbb{C}G$-submodule of V, then the multiplicity of W in the decomposition of V, is the inner product between the characters of these modules.

Theorem 0.8 (**Class Algebra Constant Formulae**). *For $i = 1, \dots r$, let C_i and χ_i be the conjugacy classes and irreducible characters, respectively of G. Then for any $i, j, k \in \{1, \dots, r\}$, and for some $z \in C_k$,*

$$a_{ijk} = |\{(x, y) \mid x \in C_i, y \in C_j, xy = z\}| = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{l=1}^{r} \frac{\chi_l(g_i)\chi_l(g_j)\overline{\chi_l(g_k)}}{\chi_l(1)}$$

*where g_i, g_j and g_k are elements in C_i, C_j and C_k respectively. The integers a_{ijk} are known as the **class algebra constants** of G.*

The direct product of $n \in \mathbb{N}$ copies of G shall be denoted G^n. A cyclic group of order $m \in \mathbb{N}$ is simply denoted m. If a group G is denoted $A.B$, then A is a normal subgroup of G, and B is the factor group of G over A. The notation $A : B$ refers to a semidirect product between normal subgroup A and B.

Remark. The notation $A.B$ does not uniquely determine the group. Two non isomorphic groups may both have this same notation. However, when such a notation is used repeatedly in this thesis, it shall be referring to the same group up to isomorphism.

The largest sporadic group known as the Monster shall be denoted $\mathbb{M}$. It has 194 conjugacy classes each of which is denoted NX, a notation used in [Atl85]. The letter N represents the order of elements in the conjugacy class while the letter X is substituted with an alphabet, i.e. $X \in \{A, B, C, \dots\}$. Elements in an NX conjugacy class are called NX-elements. If $\mathbb{M}$ contains more than one conjugacy class of elements of a particular order, then the alphabetical ordering of X which depends on the orders of centralizers, differentiates the conjugacy classes. For example, $\mathbb{M}$ has two conjugacy classes of involutions denoted $2A$ and $2B$. Let t and z be involutions in $\mathbb{M}$. Then

$$t \in 2A \text{ and } z \in 2B \text{ if } |C_{\mathbb{M}}(t)| > |C_{\mathbb{M}}(z)|.$$

Let V be a vector space over a field F where F is either $\mathbb{R}$ or $\mathbb{C}$. For a subset $W \subseteq V$, denote by $\langle W \rangle$ the subspace of V spanned by all vectors in W.

Let ϕ be a linear transformation on V. Denote by v^ϕ and W^ϕ the images under ϕ, of $v \in V$ and $W \subseteq V$ respectively. For an eigenvalue λ of ϕ, the λ-eigenspace of ϕ shall be denoted V_λ^ϕ. The set of vectors in V fixed by ϕ is denoted $C_V(\phi)$:

$$V_\lambda^\phi := \{v \in V \mid v^\phi = \lambda v\} \quad \text{and} \quad C_V(\phi) := \{v \in V \mid v^\phi = v\}.$$

Let $(\ ,\ ) : V \times V \to F$ be a symmetric positive definite bilinear form (or inner product) on V. The (quadratic) length of a vector $v \in V$ (with respect to this inner product), defined as (v, v) shall be denoted as $l(v)$. The next result is from *Chapter 8* of [Shi61].

Theorem 0.9 (**Gram Determinant**). *Let $S := \{v_i \mid 1 \leq i \leq n\}$ be a finite set of vectors in V, and let Gr_S be the Gram matrix of S (i.e. the matrix where its (i,j)-entry is the value of the inner product (v_i, v_j)). Then the determinant of Gr_S is non-zero if and only if S is linearly independent.*

Let $\cdot : V \times V \to V$ be a bilinear map. Then V is said to have an algebra product $\cdot$ and $(V, \cdot)$ is called an algebra. The identity of V (if it exists) shall be denoted id. For a subset $W \subseteq V$, denote by $\langle\langle W \rangle\rangle$ the smallest subalgebra of V containing W (or subalgebra of V generated by W).

For any vector $v \in V$, multiplying V by v on the left or on the right induces endomorphisms of V. If V is a commutative algebra, this common endomorphism known as the adjoint action of v is denoted ad_v.

$$u^{ad_v} := v \cdot u \quad \text{for all } u \in V.$$

If μ is an eigenvalue of ad_v, the notation for the μ-eigenspace of ad_u shall be abbreviated to V_μ^v instead of $V_\mu^{ad_v}$. The eigenvalues and eigenvectors of the adjoint action of a vector $v \in V$ shall be referred for short as the eigenvalues and eigenvectors of v respectively.

The Conway-Griess-Norton algebra or more commonly known as the Griess algebra, is a commutative, non-associative algebra whose underlying vector space over $\mathbb{R}$ has dimension 196884. It was Tits who first proved that the Monster is the full automorphism group of the Griess algebra. In this thesis, this algebra shall simply be called the Monster algebra and shall be denoted $V_\mathbb{M}$.

$$\mathbb{M} = Aut(V_\mathbb{M}).$$

As an $\mathbb{M}$-module, $V_\mathbb{M}$ isomorphic to the direct sum of $V_{1_\mathbb{M}}$, the trivial 1-dimensional module, and $V_{2_\mathbb{M}}$, the smallest non-trivial complex module of the Monster:

$$V_\mathbb{M} \cong V_{1_\mathbb{M}} \oplus V_{2_\mathbb{M}}.$$

The algebra product on $V_\mathbb{M}$ shall be denoted $\cdot_\mathbb{M}$. The Monster algebra is also equipped with an $\mathbb{M}$-invariant inner product $(\ ,\ )_\mathbb{M}$.

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Chapter 1

Introduction to Majorana theory

Majorana theory, introduced by Ivanov in his book [Iva09], encapsulates an axiomatic approach to studying the Monster and its 196884-dimensional algebra. An immediate outcome to this approach was the discovery of a number of new explicitly constructed subalgebras of the Monster algebra. A crucial advantage of Majorana theory is that one could bypass the many complexities of the Monster and avoid performing calculations within the entirety of the Monster algebra.

This chapter should serve as a concise introduction to Majorana theory. In Section 1.1, the main definitions required for a Majorana representation are stated followed by some basic deductions. The next section contains a description of an important family of Majorana algebras known as the Norton-Sakuma algebras. Then in Section 1.3, we outline a strategy developed for explicitly constructing Majorana representations of arbitrary groups generated by involutions. Section 1.4 contains a brief overview of some of the achievements by various authors towards advancing Majorana theory. The last section may not be relevant to the heading of this chapter and so can be regarded as a separate section entirely. It is dedicated to extending a result by Castillo-Ramirez regarding the automorphism groups of Majorana representations of S_4.

1.1 Majorana algebras and representations

This first definition was first stated in *Chapter 8* of [Iva09].

Definition 1.1. *Let V be a real vector space equipped with a commutative, non-associative algebra product $\cdot$, and an inner product $(\ ,\ )$ which associates in the sense that*

$$(u \cdot v, w) = (u, v \cdot w) \text{ for all } u, v, w \in V.$$

*A vector $a \in V$ is called a **Majorana axis** if it is an idempotent of length 1, such that*

(i) V is a direct sum of eigenspaces of a, of which the eigenvalues are in the set $\{1, 0, \frac{1}{4}, \frac{1}{32}\}$, and 1 is a simple eigenvalue;

(ii) The linear transformation $\tau(a)$ of V which negates every $\frac{1}{32}$-eigenvector of a and fixes the remaining eigenvectors, preserves the algebra product:

$$(u \cdot v)^{\tau(a)} = u^{\tau(a)} \cdot v^{\tau(a)} \text{ for all } u, v \in V;$$

(iii) The linear transformation $\sigma(a)$ of $C_V(\tau(a))$ which negates every $\frac{1}{4}$-eigenvector of a and fixes the remaining eigenvectors, preserves the algebra product restricted to $C_V(\tau(a))$.

*If V contains a set A of Majorana axes which generate V under the algebra product, then $(V, A, \cdot, (\ ,\ ))$ is known as a **Majorana algebra**. The linear transformation $\tau(a)$ is known as a **Majorana involution**.*

Remark. It was proven by Ivanov in *Section 8.6* of [Iva09] that the Monster algebra is a Majorana algebra. Its generating set of Majorana axes is the set of $2A$-axes defined by Conway in *Section 14* of [Con84]. The set of $2A$-axes in $V_{\mathbb{M}}$ shall be denoted $A_{\mathbb{M}}$.

For short, we say V or (V, A) is a Majorana algebra when A or its products are already fixed or when referring to some general Majorana algebra. Throughout the remaining of this chapter, $(V, A, \cdot, (\ ,\ ))$ shall denote a Majorana algebra.

Definition 1.2. *The **dimension** of a subalgebra of V is its dimension as a vector space. For $X \subseteq A$, the subalgebra $\langle\langle X \rangle\rangle$ is **k-closed** (with respect to X) if it is the linear span of k-long products, i.e.*

$$\langle\langle X \rangle\rangle = \langle x_1 \cdot x_2 \cdot \cdots \cdot x_k \mid x_i \in X \rangle$$

where $x_1 \cdot x_2 \cdot \cdots \cdot x_k$ denotes all possible bracketing of $k \in \mathbb{N}$ vectors.

For example, $V = \langle\langle A \rangle\rangle$ is 2-closed if

$$V = \langle a_i \cdot a_j \mid a_i, a_j \in A \rangle.$$

The following seven results were stated in [I+10]. They are restated here to draw comparisons with later results in this thesis.

Lemma 1.3. *The decomposition of V as a direct sum of eigenspaces of $a \in A$ is orthogonal, i.e.*

$$(u, v) = 0 \text{ for all } u \in V_\mu^a, \text{ and } v \in V_\lambda^a \text{ if } \mu \neq \lambda.$$

Proof. This follows from associativity of the inner product and commutativity of the algebra product from Definition 1.1. □

Lemma 1.4. *For every $a \in A$, the Majorana involution $\tau(a)$ is an isometry of V (preserves the inner product on V), i.e.*

$$(u^{\tau(a)}, v^{\tau(a)}) = (u, v) \text{ for all } u, v \in V.$$

Proof. The vectors u and v may be written as linear combinations of eigenvectors of a. Then $\tau(a)$ negates the $\frac{1}{32}$-eigenvectors and fixes everything else. The inner product between these expressions may then be expanded linearly and the result follows. □

Corollary 1.5. *For every $a \in A$, the Majorana involution $\tau(a)$ is an automorphism of the Majorana algebra $(V, A, \cdot, (\ ,\ ))$.*

Proof. The Majorana involution $\tau(a)$ preserves the algebra product by Definition 1.1 and the inner product by Lemma 1.4. □

Lemma 1.6. *For any $v \in V$ and $a \in A$, the vector $(a, v)a$ is the projection of v onto $V_1^a = \langle a \rangle$, the 1-eigenspace of a in V.*

Proof. Let λa be the projection of v to V_1^a where $\lambda \in \mathbb{R}$. Then v is the sum of λa and some non 1-eigenvectors of a. By orthogonality of eigenvectors and since $l(a) = 1$, the inner product (a, v) equals to λ. □

The following result was known both in the context of Vertex Operator Algebras (VOAs) (*Lemma 9.1* of [Miy04]) and in the Monster algebra (*Section 19* of [Con84]). A proof for the Monster algebra version may be found in the reference given below.

Lemma 1.7. *If a_1 and a_2 are distinct Majorana axes, then*

$$0 \leq (a_1, a_2) \leq \frac{1}{3}.$$

Proof. See *Section 8.6* of [Iva09]. □

$\lambda \diagdown \mu$	1	0	$\frac{1}{4}$	$\frac{1}{32}$
1	1	0	$\frac{1}{4}$	$\frac{1}{32}$
0	0	0	$\frac{1}{4}$	$\frac{1}{32}$
$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1, 0	$\frac{1}{32}$
$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{32}$	1, 0, $\frac{1}{4}$

Table 1.1: The fusion rules

Lemma 1.8. *The algebra products between eigenvectors of a Majorana axis $a \in A$ obey the following rule:*

$$u \cdot v \in \bigoplus_{\gamma \in S(\lambda,\mu)} V_\gamma^a$$

where $u \in V_\lambda^a$, $v \in V_\mu^a$ and $S(\lambda, \mu)$ is the (λ, μ)-entry in Table 1.1.

Proof. This follows from the definitions of $\tau(a)$ and $\sigma(a)$ in Definition 1.1. For example, let $u \in V^a_{\frac{1}{4}}$ and $v \in V^a_{\frac{1}{32}}$. Then $u^{\tau(a)} = -u$ and $v^{\tau(a)} = v$. As $\tau(a)$ preserves the algebra product,

$$(u \cdot v)^{\tau(a)} = u^{\tau(a)} \cdot v^{\tau(a)} = -u \cdot v.$$

Hence $u \cdot v \in V^a_{\frac{1}{32}}$. The other cases are similar. □

Lemma 1.8 is known as the **Fusion rules of Majorana axes**.

The following result may be traced back to [Sak07] but was formulated as it is seen here in [I+10].

Lemma 1.9. *Let $a \in A$ and let X be an a-stable subspace of V, i.e. $a \cdot x \in X$ for all $x \in X$. Let $v \in V$ and suppose that*

$$\alpha_v = v + x_\alpha \text{ and } \beta_v = v + x_\beta$$

are 0- and $\frac{1}{4}$-eigenvectors of a respectively for some $x_\alpha, x_\beta \in X$. Then

$$v = -[4a \cdot (x_\alpha - x_\beta) + x_\beta].$$

In particular, $v \in X$.

Proof. Firstly, $x_\alpha - x_\beta = \alpha_v - \beta_v$. Multiplying both sides of this equation by a gives

$$a \cdot (x_\alpha - x_\beta) = a \cdot (\alpha_v - \beta_v) = 0 - \frac{1}{4}\beta_v = -\frac{1}{4}(v - x_\beta).$$

The expression for v follows from rearranging the equation above. □

Remark. Lemma 1.9 is known as the **Resurrection principle**. Although v disappears from the term $\alpha_v - \beta_v$, it reappears after multiplying by a. This result may appear simple but it has been very useful in determining the algebra product between two vectors in V.

The following definition was first introduced in *Section 3* of [I+10].

Definition 1.10. *Let G be a finite group containing a set T of involutions such that T generates G, and T is a union of conjugacy classes of G. If there is a linear representation*

$$\varphi : G \to GL(V),$$

and a bijective map

$$\psi : T \to A,$$

such that

(i) $\varphi(G)$ permutes $\psi(T)$ the same way G permutes T by conjugation:

$$\psi(t)^{\varphi(g)} = \psi(t^g) \text{ for all } g \in G \text{ and } t \in T;$$

(ii) the Majorana involution associated to $\psi(t)$ equals to the image of t under φ:

$$\tau(\psi(t)) = \varphi(t) \text{ for all } t \in T,$$

*then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is a **Majorana representation** of (G, T).*

The Majorana axis $\psi(t)$ will often be denoted a_t. When there are several involutions subscripted by integers, e.g. $t_i \in T$, the Majorana axes $\psi(t_i)$ will often be denoted simply as a_i. When we speak of the corresponding involution of a Majorana axis $a \in A$, we mean $t \in T$ such that $\psi(t) = a$. Conversely, when we speak of the corresponding Majorana axis of $t \in T$, we mean $\psi(t)$.

Let $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ be a Majorana representation of (G, T) throughout the remaining of this chapter. For short, we may say V is a Majorana representation of G. All other parameters are assumed to be present.

Definition 1.11. *The **dimension** of a Majorana representation is the dimension of the underlying Majorana algebra.*

Remark. It was proven by Ivanov (*Chapter 8* of [Iva09]) that the Monster algebra is a Majorana representation of the Monster with respect to its $2A$-involutions. The Majorana involutions for this representation are exactly the $2A$-involutions in the Monster.

Denote by $\varphi_{\mathbb{M}} : \mathbb{M} \to GL(V_{\mathbb{M}})$ the linear representation and $\psi_{\mathbb{M}} : 2A \to A_{\mathbb{M}}$ the bijective map for the Majorana representation of the Monster. Then $(V_{\mathbb{M}}, A_{\mathbb{M}}, \cdot_{\mathbb{M}}, (\ ,\ )_{\mathbb{M}}, \varphi_{\mathbb{M}}, \psi_{\mathbb{M}})$ is a Majorana representation of $(\mathbb{M}, 2A)$. To simplify notation, the $\mathbb{M}$ subscript on the algebra and inner products for $V_{\mathbb{M}}$ is dropped later on in this thesis.

For a $2A$-axis $a \in A_{\mathbb{M}}$, the involution associated to a refers to the $2A$-involution $\tau(a) \in 2A$. Conversely, the $2A$-axis associated to $t \in 2A$ refers to $a \in A_{\mathbb{M}}$ such that $\tau(a) = t$.

Lemma 1.12. *If the linear representation $\varphi : G \to GL(V)$ is faithful, then G is isomorphic to a subgroup of $Aut(V)$, the automorphism group of the Majorana algebra V.*

Proof. By Corollary 1.5, $\varphi(t) \in Aut(V)$ for all $t \in T$. Since T generates G, the image of G under φ is therefore a subgroup of $Aut(V)$. However, since φ is faithful, G is isomorphic to $\varphi(G)$. □

An arbitrary group may not necessarily possess a Majorana representation (see Step 1 of the generic strategy in Section 1.3). The following argument demonstrates the default existence of certain Majorana representations.

Let $H \leq G$ such that there exist a set T_H in H which generates H, is a union of conjugacy classes of H, and $T_H \subseteq T$. Then (H, T_H) has a Majorana representation. Let

$$A_H := \psi(T_H) \text{ and } V_H := \langle\langle A_H \rangle\rangle.$$

Denote by $\cdot_{|_{V_H}}$ and $(\ ,\ )_{|_{V_H}}$ the restriction of $\cdot$ and $(\ ,\ )$ to V_H, respectively. Then

$$(V_H, A_H, \cdot_{|_{V_H}}, (\ ,\ )_{|_{V_H}})$$

is a Majorana algebra and a subalgebra of V. Moreover,

$$(V_H, A_H, \cdot_{|_{V_H}}, (\ ,\ )_{|_{V_H}}, \varphi_{|_H}, \psi_{|_{T_H}})$$

is a Majorana representation of (H, T_H).

This observation brought about the need to differentiate representations which are embedded this way in the Monster's Majorana representation. Hence the following definition

was introduced in [I+10].

Definition 1.13. *Let (V, A) be a Majorana representation of (G, T). Suppose there is a group embedding $\varrho : G \to \mathbb{M}$ such that $\varrho(T) \subseteq 2A$ and the map $a_t \in A \mapsto \varphi_{\mathbb{M}}(\varrho(t)) \in A_{\mathbb{M}}$, for all $t \in T$ defines an isomorphism of algebras between and $\langle\langle \varphi_{\mathbb{M}}(\varrho(T))\rangle\rangle \leq V_{\mathbb{M}}$. If moreover there is an isomorphism of modules between the $\mathbb{R}G$-module V and the $\mathbb{R}\varrho(G)$-module $\langle\langle \varphi_{\mathbb{M}}(\varrho(T))\rangle\rangle$, then the Majorana represention of (G, T) is said to be* ***based on an embedding in the Monster****.*

1.2 The Norton-Sakuma algebras

The following well known result was obtained from calculations using character values of the Monster. See *Section 14* of [Con84] for further details.

Theorem 1.14. *The Monster with respect to its $2A$-involutions is a 6-transposition group, i.e. the $2A$ conjugacy class generates the Monster and the product of any two $2A$-involutions has order at most 6. Moreover, the product of any two $2A$-involutions belong to one of the following 9 conjugacy classes of the Monster:*

$$1A,\ 2A,\ 2B,\ 3A,\ 3C,\ 4A,\ 4B,\ 5A \text{ and } 6A.$$

Norton then proved the following result, implicitly stated in [Nor96] using more theoretical calculations.

Theorem 1.15. *Let $t_0, t_1 \in 2A$ with associated $2A$-axes $a_0, a_1 \in A_{\mathbb{M}}$, respectively. The subalgebra $\langle\langle a_0, a_1\rangle\rangle \leq V_{\mathbb{M}}$ is uniquely determined up to isomorphism by the conjugacy class in the Monster of the product $\rho := t_0 t_1$. If ρ belongs to the NX conjugacy class ($N \in \{1, \ldots, 6\}$ and $X \in \{A, B, C\}$), then the subalgebra $\langle\langle a_0, a_1\rangle\rangle$ is isomorphic to the algebra of type NX described in Table 1.2.*

Type	Basis	Algebra and inner products
$2A$	a_0, a_1, a_ρ	$a_0 \cdot a_1 = \frac{1}{8}(a_0 + a_1 - a_\rho)$, $a_0 \cdot a_\rho = \frac{1}{8}(a_0 + a_\rho - a_1)$, $(a_0, a_1) = (a_0, a_\rho) = (a_1, a_\rho) = \frac{1}{8}$
$2B$	a_0, a_1	$a_0 \cdot a_1 = 0$, $(a_0, a_1) = 0$
$3A$	a_0, a_1, a_{-1}, u_ρ	$a_0 \cdot a_1 = \frac{1}{32}(2a_0 + 2a_1 + a_{-1}) - \frac{135}{2048}u_\rho$, $a_0 \cdot u_\rho = \frac{1}{9}(2a_0 - a_1 - a_{-1}) + \frac{5}{32}u_\rho$, $u_\rho \cdot u_\rho = u_\rho$, $(a_0, a_1) = \frac{13}{256}$, $(a_0, u_\rho) = \frac{1}{4}$, $(u_\rho, u_\rho) = \frac{8}{5}$
$3C$	a_0, a_1, a_{-1}	$a_0 \cdot a_1 = \frac{1}{64}(a_0 + a_1 - a_{-1})$, $(a_0, a_1) = \frac{1}{64}$
$4A$	a_0, a_1, a_{-1}, a_2, v_ρ	$a_0 \cdot a_1 = \frac{1}{64}(3a_0 + 3a_1 + a_{-1} + a_2 - 3v_\rho)$, $a_0 \cdot v_\rho = \frac{1}{16}(5a_0 - 2a_1 - 2a_{-1} - a_2 + 3v_\rho)$, $a_0 \cdot a_2 = 0$, $v_\rho \cdot v_\rho = v_\rho$, $(a_0, a_1) = \frac{1}{32}$, $(a_0, v_\rho) = \frac{3}{8}$, $(a_0, a_2) = 0$, $(v_\rho, v_\rho) = 2$
$4B$	a_0, a_1, a_{-1}, a_2, a_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{64}(a_0 + a_1 - a_{-1} - a_2 + a_{\rho^2})$, $a_0 \cdot a_2 = \frac{1}{8}(a_0 + a_2 - a_{\rho^2})$, $(a_0, a_1) = \frac{1}{64}$, $(a_0, a_2) = (a_0, a_{\rho^2}) = \frac{1}{8}$
$5A$	a_0, a_1, a_{-1}, a_2, a_{-2}, w_ρ	$a_0 \cdot a_1 = \frac{1}{128}(3a_0 + 3a_1 - a_{-1} - a_2 - a_{-2}) + w_\rho$, $a_0 \cdot a_2 = \frac{1}{128}(3a_0 + 3a_2 - a_1 - a_{-1} - a_{-2}) - w_\rho$, $a_0 \cdot w_\rho = \frac{7}{4096}(a_1 + a_{-1} - a_2 - a_{-2}) + \frac{7}{32}w_\rho$, $w_\rho \cdot w_\rho = \frac{175}{524288}(a_0 + a_1 + a_{-1} + a_2 + a_{-2})$, $(a_0, a_1) = \frac{3}{128}$, $(a_1, w_\rho) = 0$, $(w_\rho, w_\rho) = \frac{875}{524288}$
$6A$	a_0, a_1, a_{-1}, a_2, a_{-2}, a_3 a_{ρ^3}, u_{ρ^2}	$a_0 \cdot a_1 = \frac{1}{64}(a_0 + a_1 - a_{-1} - a_2 - a_{-2} - a_3 + a_{\rho^3}) + \frac{45}{2048}u_{\rho^2}$, $a_0 \cdot a_2 = \frac{1}{32}(2a_0 + 2a_2 + a_{-2}) - \frac{135}{2048}u_{\rho^2}$, $a_0 \cdot u_{\rho^2} = \frac{1}{9}(2a_0 - a_2 - a_{-2}) + \frac{5}{32}u_{\rho^2}$, $a_0 \cdot a_3 = \frac{1}{8}(a_0 + a_3 - a_{\rho^3})$, $a_{\rho^3} \cdot u_{\rho^2} = 0$, $(a_0, a_1) = \frac{5}{256}$, $(a_0, a_2) = \frac{13}{256}$, $(a_0, a_3) = \frac{1}{8}$, $(a_{\rho^3}, u_{\rho^2}) = 0$

Table 1.2: Norton-Sakuma algebras

The case where $t_0 = t_1$ is omitted from Table 1.2 as the resulting algebra of type $1A$, generated by $a_0 = a_1$ is just 1-dimensional. The column labelled 'Basis' consists of a set of basis vectors for $\langle\langle a_0, a_1\rangle\rangle$ as a vector space. The vector $a_{\epsilon+2i}$ is a $2A$-axis and is defined as

$$a_{\epsilon+2i} := \psi_{\mathbb{M}}(\rho^{-i} t_\epsilon \rho^i) \text{ for } \epsilon \in \{0, 1\}.$$

The vector a_{ρ^i} is also a $2A$-axis and is defined as

$$a_{\rho^i} := \psi_{\mathbb{M}}(\rho^i) \text{ for } i \in \{1, 2, 3\}.$$

The vectors u_ρ, v_ρ and w_ρ on the other hand are not $2A$-axes. They may be defined in terms of $2A$-axes.

Definition 1.16. *Let $a_0, a_1 \in A_{\mathbb{M}}$ with associated $2A$-involutions $t_0, t_1 \in 2A$ respectively, and let $\rho := t_0 t_1$. If $\rho \in 3A$, then $\langle\langle a_0, a_1\rangle\rangle$ contains a vector u_ρ known as a* ***3A-axis****, where*

$$u_\rho := \frac{64}{135}(2a_0 + 2a_1 + a_{\rho t_0} - 32 a_0 \cdot a_1).$$

If $\rho \in 4A$, then $\langle\langle a_0, a_1\rangle\rangle$ contains a vector v_ρ known as a ***4A-axis****, where*

$$v_\rho := a_0 + a_1 + \frac{1}{3}(a_{\rho t_0} + a_{t_1 \rho} - 64 a_0 \cdot a_1).$$

If $\rho \in 5A$, then $\langle\langle a_0, a_1\rangle\rangle$ contains a vector w_ρ known as a ***5A-axis****. The $5A$-axis however may be only defined this way up to sign:*

$$w_\rho := -\frac{1}{128}(3a_0 + 3a_1 - a_{\rho t_0} - a_{t_1 \rho} - a_{\rho^2 t_0}) + a_0 \cdot a_1;$$

$$-w_\rho := \frac{1}{128}(3a_0 + 3a_{t_1\rho} - a_1 - a_{\rho t_0} - a_{\rho^2 t_0}) + a_0 \cdot a_{t_1 \rho}.$$

The sign depends on the conjugacy class of ρ in $\langle t_0, t_1\rangle$, i.e. if $t_2, t_3 \in \langle t_0, t_1\rangle$, then $w_{t_2 t_3} = w_\rho$ if $t_2 t_3$ is conjugate to ρ, and $w_{t_2 t_3} = -w_\rho$ otherwise.

Given two $2A$-axes $a_0, a_1 \in A_{\mathbb{M}}$ such that $\langle\langle a_0, a_1\rangle\rangle$ has type NA ($3 \leq N \leq 5$), the NA-axis contained in $\langle\langle a_0, a_1\rangle\rangle$ refers to the NA-axis defined in terms of a_0 and a_1 as in Definition 1.16.

Let $t_0, t_1 \in 2A$. The NA-axis in $\langle\langle a_{t_0}, a_{t_1}\rangle\rangle$ is often indexed by the product $\rho := t_0 t_1$ as a subscript. Like the $2A$-axes, the Monster acts on these axes by permuting the subscripts. For example, let $g \in \mathbb{M}$ and u_ρ a $3A$-axis in $\langle\langle a_{t_0}, a_{t_1}\rangle\rangle$. Then

$$u_\rho^g = u_{\rho^g}.$$

In other words, g sends the $3A$-axis in $\langle\langle a_{t_0}, a_{t_1}\rangle\rangle$ to the $3A$-axis in $\langle\langle a_{t_0^g}, a_{t_1^g}\rangle\rangle$. This is clear from the definition of the NA-axes and the action of the $2A$-involutions (which generate $\mathbb{M}$) on $V_\mathbb{M}$.

Remark. Table 1.2 is not exactly as in [Nor96]. The $2A$-, $3A$- and $4A$-axes have been rescaled to be idempotents. The $5A$-axis on the other hand was rescaled so that its coefficient in the product of two $2A$-axes (as a linear combination of axes) generating an algebra of type $5A$ is 1. The inner product on $V_\mathbb{M}$ has also been rescaled so that the $2A$-axes have length 1.

It is observed that certain types of subalgebras in Table 1.2 contain other types of two $2A$-axes generated subalgebras. There are only four such inclusions: A $4A$-type contains a $2B$-type subalgebra, a $4B$-type contains a $2A$-type subalgebra and a $6A$-type contains both $2A$- and $3A$-type subalgebras. These inclusions shall be written as:

$$2B \hookrightarrow 4A,\ 2A \hookrightarrow 4B,\ 2A \hookrightarrow 6A \text{ and } 3A \hookrightarrow 6A.$$

Remark. Although Table 1.2 does not contain all pairwise algebra and inner products between the basis vectors, they can be deduced by symmetry under automorphisms induced by $\langle t_0, t_1\rangle \cong D_{2n}$ where $n = o(t_0 t_1)$, or by the subalgebra inclusions described above.

By swapping the positions of a_0 and a_1, or by exchanging them with another pair of $2A$-axes in $\langle\langle a_0, a_1\rangle\rangle$ whose associated $2A$-involutions also generate the dihedral group $\langle t_0, t_1\rangle$, the following relations are obtained:

$$u_\rho = u_{\rho^{-1}},\ v_\rho = v_{\rho^{-1}} \text{ and } w_\rho = -w_{\rho^2} = -w_{\rho^3} = w_{\rho^4}.$$

From these relations, it is more accurately the cyclic subgroup $\langle\rho\rangle \leq \mathbb{M}$ rather than $\rho \in NA$, that a $3A$- and $4A$-axis is associated to. The $5A$-axis has an additional change of sign depending on the choice of generators. Note that the same NA-axis ($3 \leq N \leq 5$) may be defined differently according to Definition 1.16 and the relations regarding the subscripts.

In the context of VOAs, a Majorana axis is a conformal vector of central charge $\frac{1}{2}$. Sakuma in [Sak07] classified the algebras generated by pairs of these conformal vectors. Sakuma's result is therefore equivalent to the classification of algebras generated by pairs of Majorana axes. The following result is a version of Sakuma's theorem written in the context of Majorana theory and which was proven in [I+10].

Theorem 1.17 (**Sakuma's Theorem**). *The dihedral groups have exactly nine Majorana representations, all of which are based on an embedding in the Monster. These nine representations match exactly Norton's subalgebras described in Table 1.2.*

The algebras in Table 1.2 may now be viewed not only as subalgebras of the Monster algebra but as abstract algebras generated by a pair of Majorana axes. They are known in this sense as the **Norton-Sakuma algebras**. Let V_{NX} denote the Norton-Sakuma algebra of type NX.

The description for the NA-axes ($3 \leq N \leq 5$) in Definition 1.16 are adapted to the context of Majorana theory. A $3A$-, $4A$- and $5A$-axis in a Majorana algebra is defined as in Definition 1.16 but with the $2A$-axes replaced with Majorana axes.

For example, let a_1 and a_2 be Majorana axes generating a Norton-Sakuma algebra of type $3A$. Then $\langle\langle a_1, a_2\rangle\rangle = \langle a_1, a_2, a_3, u\rangle$ where u is a $3A$-axis defined in terms of a fixed pair of generators $\{a_1, a_2\}$ as

$$u := \frac{64}{135}(2a_1 + 2a_2 + a_3 - 32a_1 \cdot a_2).$$

The vector a_3 is a Majorana axis in $\langle\langle a_1, a_2\rangle\rangle$, and is defined as

$$a_3 := -\frac{78}{7}a_1 + a_2 - \frac{512}{7}a_1 \cdot a_2 + \frac{2048}{7}a_1 \cdot (a_1 \cdot a_2),$$

a consequence of the algebra products in Table 1.2.

Recall that $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is a Majorana representation of (G, T).

Let $t_0, t_1 \in T$ with corresponding Majorana axes $a_0, a_1 \in A$ respectively. If $\langle\langle a_0, a_1\rangle\rangle$ is of type NA $(3 \leq N \leq 5)$, then the NA-axis in $\langle\langle a_0, a_1\rangle\rangle$ is indexed by t_0t_1 as a subscript. The NA-axis corresponding to t_0t_1 refers to the NA-axis in $\langle\langle a_0, a_1\rangle\rangle$. Conversely, the element corresponding to the NA-axis in $\langle\langle a_0, a_1\rangle\rangle$ refers to t_0t_1.

Definition 1.18. *Let $(V, \cdot)$ be a real algebra equipped with a bilinear form $(\ ,\ ) : V \times V \to \mathbb{R}$. Then V is said to obey **Norton's inequality** if*

$$(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v) \text{ for all } u, v \in V,$$

with equality exactly when $(u \cdot w) \cdot v = u \cdot (w \cdot v)$ for all $w \in V$.

Remark. Sakuma utilized Norton's inequality in the proof of his version of Theorem 1.17. It was not required when reproving Sakuma's theorem under the context of Majorana theory in [I+10]. Norton's inequality was initially included as one of the defining properties of a Majorana algebra but has since been excluded due to its lack of application. It is unclear if Norton's inequality is a consequence of the already stated Majorana definitions.

Remark. The Monster algebra obeys Norton's inequality (see *Proposition 8.9.5* of [Iva09]) and hence so does any of its subalgebras. Therefore if a Majorana representation does not obey Norton's inequality, it cannot be based on an embedding in the Monster.

Lemma 1.19. *Let u and v be idempotents in $V_{\mathbb{M}}$. Then*

$$(u, v) \geq 0.$$

Proof. This follows from Norton's inequality and by the positive definiteness of $(\ ,\ )$. □

1.3 The generic strategy

Since the classification of Majorana representations of the dihedral groups, it was possible to consider Majorana representations of arbitrary groups generated by involutions. The first group to be considered was S_4, the symmetric group of degree 4. There were four Majorana representations of S_4 constructed in *Section 4* of [I+10], all of which are based on embeddings in the Monster.

The methods used in [I+10] were then successfully applied to other groups as well(see [ISe12] and [ISh12]). It was then implemented into a GAP computer program by Seress [Ser12], resulting in many more Majorana representations of relatively small groups obtained quickly and efficiently. This procedure is often referred to as the generic strategy for constructing 2-closed Majorana representations. In this section, we outline the generic strategy in a series of steps as was described by Seress.

Remark. One of the S_4-representations was not 2-closed with respect to a set of Majorana axes but was obtained nevertheless in an ad hoc manner. Generally it is difficult to construct Majorana representations which are not 2-closed. This area has been worked on by Shpectorov.

Before describing the generic strategy, the following additional conditions are introduced.

Definition 1.20. *Let $t_1, t_2, t_3, t_4 \in T$ with corresponding Majorana axes $a_1, a_2, a_3, a_4 \in A$, respectively. If the following conditions hold:*

(i) $t_1t_2 \in T$ if and only if $\langle\langle a_1, a_2\rangle\rangle \cong V_{2A}$;

(ii) if $t_1t_2 = t_3$, then a_3 coincides with $a_{t_1t_2}$, i.e. $a_3 = a_1 + a_2 - 8a_1 \cdot a_2$,

then the Majorana representation is said to satisfy the **2A-*condition***.

If $\langle t_1t_2\rangle = \langle t_3t_4\rangle$, both $\langle\langle a_1, a_2\rangle\rangle$ and $\langle\langle a_3, a_4\rangle\rangle$ are of type $3A$, $4A$ or $5A$, and $u_{t_1t_2} = u_{t_3t_4}$, $v_{t_1t_2} = v_{t_3t_4}$ or $w_{t_1t_2} = \pm w_{t_3t_4}$ respectively, then the Majorana representation is said to satisfy the **3A-**, **4A-** *or* **5A-*condition*** *respectively.*

Remark. The Monster algebra satisfies all the NA-conditions (see comment on *Page 367* of [Iva11a]). A weaker version of the $2A$-condition was used by Seress in [Ser12] whereby conditions (i) and (ii) in Definition 1.20 were replaced with

(i) If* $t_1t_2 = t_3 \in T$ *and* $\langle\langle a_1, a_2\rangle\rangle \cong V_{2A}$*, then* a_3 *coincides with* $a_{t_1t_2}$*;.*

(ii) If* $t_1t_2 = t_3t_4$ *and both* $\langle\langle a_1, a_2\rangle\rangle$ *and* $\langle\langle a_3, a_4\rangle\rangle$ *are of type* $2A$*, then* $a_{t_1t_2} = a_{t_3t_4}$*.*

The following result is a consequence of the $2A$-condition and the inclusion of a Norton-Sakuma algebra of type $2A$ as a subalgebra of Norton-Sakuma algebras of types $4A$ and $6A$.

Lemma 1.21. *Assume a Majorana representation satisfies the* $2A$*-condition. Let* $t_0, t_1 \in T$ *with corresponding Majorana axis* $a_0, a_1 \in A$ *respectively, and let* $\rho := t_0t_1$*. If* $\langle\langle a_0, a_1\rangle\rangle \cong V_{4A}$ *or* V_{6A}*, then* ρ^2 *or* ρ^3 *belongs to* T*, and* a_{ρ^2} *or* a_{ρ^3} *coincides with* $\psi(\rho^2)$ *or* $\psi(\rho^3)$ *respectively.*

When classifying Majorana representations, it is important and useful to differentiate the types of Norton-Sakuma algebras contained in the representations. The following definition was introduced in [I+10] for this purpose.

Definition 1.22. *The* ***shape*** *of a Majorana representation is a rule which assigns to each pair in the set of Majorana axes generating the Majorana algebra, the type of Norton-Sakuma algebra the pair generates.*

Remark. The shape of a Majorana representation is subject to subalgebra inclusions and has to obey the symmetries of the algebra induced by the generators.

The shape is often denoted in the form $(N_1X_1, \ldots, N_nX_n)$ where N_iX_i are the types of Norton-Sakuma algebras present. Types which are present by default will be omitted. For example, if the types present are $2A$, $3A$ and $6A$, we say V is a Majorana representation of G with shape $(2A, 3A)$.

The generic strategy

Let G be a finite group containing a set T of involutions such that $\langle T\rangle = G$, and T is a union of conjugacy classes of G. The tuple $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is assumed to be a Majorana representation of (G, T), of which we aim to construct. It is assumed that the Majorana representation obeys all NA-conditions from Definition 1.20.

Step 1.

Determine the shape of the representation.

For each pair $t_1, t_2 \in T$, the type of Norton-Sakuma algebra generated by the corresponding Majorana axes $a_1, a_2 \in A$ is decided. The choices for the types are not completely independent. For example, if $|t_1t_2| = N$ then $\langle\langle a_1, a_2\rangle\rangle$ has type NX for some X. Moreover, $\langle\langle a_1, a_2\rangle\rangle \cong \langle\langle a_1^{\varphi(g)}, a_2^{\varphi(g)}\rangle\rangle$ as $\varphi(g) \in Aut(V)$ for all $g \in G$. The $2A$-condition and subalgebra inclusions also play an important role in determining all the types.

Remark. If there exists a pair $t_1, t_2 \in T$ such that $|t_1t_2| > 6$, then (G, T) does not possess a Majorana representation.

Step 2.

Collect a spanning set for V.

Let X denote the set consisting of all Majorana axes in A together with the NA-axes $(3 \leq N \leq 5)$ contained in the Norton-Sakuma algebras generated by pairs of Majorana axes in A as dictated by the shape in Step 1. Since the NA-axes are defined as linear combinations of at most 2-long products of Majorana axes in A, if X spans $V = \langle\langle A\rangle\rangle$, then the Majorana representation is 2-closed. To prove this, one needs to show that the algebra product between any two vectors in X remain in the linear span of X.

Step 3.

Identify pairs of vectors in X whose products are unknown.

Some of the algebra and inner products between vectors in X are already known by the Norton-Sakuma algebras. Suppose G contains a subgroup H which is generated by $H \cap T$. If the Majorana representation of $(H, H \cap T)$ which has the same shape in Step 1, has already been constructed and is unique subject to the shape, then the products between vectors in $\langle\langle \psi(H \cap T)\rangle\rangle \leq \langle\langle A\rangle\rangle$ are known. The pairs with unknown products may be

grouped into orbits on pairs under the action of $\varphi(G)$. Since $\varphi(G)$ preserve the products, determining the product for one orbit representative is sufficient to deduce the products of the remaining pairs in the orbit.

Step 4.

Record the eigenvectors of the Majorana axes.

Eigenvectors of Majorana axes contained in Norton-Sakuma algebras generated by pairs in A, and from subalgebras corresponding to subgroups of G, are already known. By the fusion rules in Table 1.1, additional eigenvectors may be obtained provided there is sufficient information to calculate the algebra products between eigenvectors. Additionally, if $\rho \in G$ corresponds to $x_\rho \in X$, then $x_\rho - x_\rho^{\varphi(t)} = x_\rho - x_{t\rho t}$ is a $\frac{1}{32}$-eigenvector of a_t for $t \in T$ (see *Lemma 3.3* of [ISe12] for proof).

Step 5.

Compute the unknown inner products between vectors in X.

Any vector in the kernel of the Gram matrix of X corresponds to the zero vector in V. Subspaces of the kernel may be known from previously constructed Majorana representations of subgroups of G. The unknown inner products may then be computed by solving a system of linear equations arising from the orthogonality condition between eigenvectors as in Lemma 1.3. Additional linear equations may be acquired if there are a non-trivial vectors in the kernel since the inner product between the zero vector and any other vector is 0. Once all inner products between vectors in X are obtained, the Gram matrix of X is computed and its null space is determined.

Step 6.

Compute the unknown algebra products between vectors in X.

Similar to computing unknown inner products in Step 5, the unknown algebra products may be computed by solving a system of linear equations arising from the algebra product between the zero vector and any other vector, and from the algebra products between Majorana axes and its eigenvectors in Step 4. Additionally, unknown algebra products may be computed by applying the Resurrection principle in Lemma 1.9 on appropriate eigenvectors.

Step 7.

Verify the results.

Check that the fully constructed algebra $(V, A, \cdot, (\ ,\ ))$ does obey all the definitions of a Majorana algebra. Then check that $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is a Majorana representation of (G, T) as in Definition 1.10.

Remark. In some cases, not all algebra products may not be computed. This is either because there is not enough information to solve the system of linear equations or because the product does not appear in any linear equation. In such cases, failure of the generic strategy is reported.

1.4 Advances in Majorana theory

It was mentioned earlier that reproving Sakuma's theorem in the context of Majorana theory and constructing Majorana representations of S_4 were achieved in [I+10]. The shapes of the S_4-representations are $(2B, 3A)$, $(2A, 3A)$, $(2B, 3C)$ and $(2A, 3C)$, and are of dimensions 13, 13, 6 and 9 respectively. All of these representations are based on embeddings in the Monster. A similar approach was used to construct Majorana representations of A_5, $L_3(2)$, A_6 and A_7, in [ISe12], [ISh12] and [Iva11b], respectively. These representations are also based on embeddings in the Monster. The dimensions of some of these Majorana algebras were conjectured early on in [Iva09] but their explicit construction was only achieved through Majorana theory. In [Iva11a], it was established that there exists a 70-dimensional Majorana representation of A_6 with shape involving Norton-Sakuma algebras of type $3C$. However this representation is not embeddable into the Monster algebra, nor does it appear in any known vertex operator algebra.

Decelle in her PhD thesis [Dec13] aimed to classify Majorana algebras generated by three Majorana axes a_1, a_2, a_3, where the Majorana algebra also contained the Majorana axis whose Majorana involution coincided with $\tau(a_1)\tau(a_2)$. This was achieved through the classification of quotients of certain Coxeter groups. She proved that there are only eleven universal 6-transposition Coxeter groups of type $G^{(n,m,p)}$. The group $L_2(11)$ which

is one of the eleven groups known to be generated by $2A$-involutions of the Monster and hence it possesses a Monster embeddable Majorana representation. The full Gram matrix of this representation with respect to a spanning set of Majorana, $3A$- and $5A$-axes was computed. Calculating the rank of this Gram matrix together with a few other results, it was deduced that this Majorana representation was 101-dimensional.

Castillo-Ramirez in his PhD thesis [CR14] worked on a Majorana representation of A_{12} which is based on an embedding in the Monster. This representation contains a significantly larger number of Majorana axes generators compared to previously considered representations. It was established in his thesis that the dimension of this represention is between 3960 and 4689. By using the inner product structure together with information from known subalgebras of this representation, he also proved that the $3A$- and $4A$-axes are not contained in the span of Majorana axes. In a separate section of his thesis, Castillo-Ramirez worked on classifying idempotents of low dimensional Majorana algebras, in particular the Norton-Sakuma algebras and the S_4-representations with shapes $(2B, 3C)$ and $(2A, 3C)$. This was successfully done and as a result, the automorphism groups of those algebras could be determined. Another consequence of the idempotent classification together with an application of a theorem by Mayer and Neutsch, was the identification of maximal associative subalgebras of these low dimensional Majorana algebras in [CR13a].

In the previous section, we discussed the generic strategy for constructing Majorana representations. The procedure was implemented into a GAP computer program by Seress in [Ser12]. As a result, Majorana representations for a number of relatively small groups were shown to exist (see *Table 3* of [Ser12]). However, the full details (e.g. basis, algebra and inner products) for most of these computer verified representations were not published. It was also mentioned in the previous section that a weaker version of the $2A$-condition was proposed by Seress. This resulted with three more Majorana representations of S_4 but these are not based on embeddings in the Monster. It is obvious that weaker conditions result in possibly larger families of algebras. Therefore relaxing the definitions of Majorana algebras led to the study of what is known as axial algebras. These are non-associative algebras controlled by fusion rules of idempotents. Among other results in this thesis [Reh15], Rehren worked jointly with Hall and Shpectorov, classifying axial algebras with fusion rules of Jordan type related to 3-transposition groups.

The Harada-Norton group which is one of the 26 sporadic simple groups was also worked on in the context of Majorana theory by Franchi, Ivanov and Mainardis. They proved that all $2A$-Majorana representations of the Harada-Norton group have the same shape [IFM16a]. Furthermore, they determined its dimension and irreducible constituents of the linear span of Majorana axes by utilizing the theory of association schemes. This approach was then extended, establishing a new method for computing the dimensions of irreducible constituents of Majorana representations [IFM16b]. Applying this method to the symmetric groups, it was shown that S_n has a Majorana representation in which every permutation of cycle shape (2^2) in S_n corresponds to a Majorana axis if and only if $n \leq 12$.

1.5 Automorphism groups of the S_4-representations

This section may not be relevant to this chapter as it is about deducing the automorphism groups of certain Majorana algebras arising from representations of S_4. The formulas stated here are obtained from [Map16] and may seem long and messy. They should serve primarily as a reference.

In [CR13b], Castillo-Ramirez classified the idempotents in the Majorana algebras underlying the representations of S_4 with shapes $(2A, 3C)$ and $(2B, 3C)$, by solving a system of non-linear equations using [Map16]. As the dimensions of these algebras were relatively small, it was possible to solve the system of equations and hence identify all the idempotents. It was then manifest that the Majorana axes in the bases of these algebras as stated in [I+10] were the only idempotents of length 1. As a corollary, the automorphism groups for both of these algebras are isomorphic to S_4.

For the other larger dimensional representations of S_4, specifically with shapes $(2A, 3A)$ and $(2B, 3A)$, it was then and still is computationally not possible to solve the system of equations with [Map16]. The systems of non-linear equations arising from these 13-dimensional representations are considered too large to solve. In this section, we deduce the automorphism groups for the Majorana algebras of these representations without completely classifying all of its idempotents, but by identifying all idempotents of a particular

length.

Let $G \cong S_4$ be realized as the group of permutations of $\{1,2,3,4\}$. There are only two conjugacy classes of involutions in G, namely the transpositions and double transpositions. These sets shall be denoted as T_1 and T_2 respectively. Some notations will slightly be abused in what follows. Although the set of generators T of G has not been fixed, let the following nine a_is be Majorana axes corresponding to the following involutions in T_1 and T_2, sent by the bijective map ψ from T to a set A of Majorana axes:

$$a_1 := \psi((1,2)), \qquad a_2 := \psi((1,3)), \qquad a_3 := \psi((1,4)),$$
$$a_4 := \psi((2,3)), \qquad a_5 := \psi((2,4)), \qquad a_6 := \psi((3,4)),$$
$$a_7 := \psi((1,2)(3,4)), \qquad a_8 := \psi((1,3)(2,4)), \qquad a_9 := \psi((1,4)(2,3)).$$

Denote by u_i, for $1 \leq i \leq 4$, the $3A$-axes in the following subalgebras:

$$u_1 := u_{(1,2,3)} \in \langle\langle a_1, a_2 \rangle\rangle, \qquad u_2 := u_{(1,2,4)} \in \langle\langle a_1, a_3 \rangle\rangle,$$
$$u_3 := u_{(1,3,4)} \in \langle\langle a_2, a_3 \rangle\rangle, \qquad u_4 := u_{(2,3,4)} \in \langle\langle a_4, a_5 \rangle\rangle.$$

Denote by i_j, for $1 \leq j \leq 4$, the following vectors:

$$i_1 := \frac{16}{21}(a_1 + a_2 + a_4) + \frac{9}{14}u_1, \qquad i_2 := \frac{16}{21}(a_1 + a_3 + a_5) + \frac{9}{14}u_2,$$
$$i_3 := \frac{16}{21}(a_2 + a_3 + a_6) + \frac{9}{14}u_3, \qquad i_4 := \frac{16}{21}(a_4 + a_5 + a_6) + \frac{9}{14}u_4.$$

1.5.1 $(2A, 3A)$-shape S_4-representation

Let

$$T := T_1 \cup T_2 \quad \text{and} \quad A := \{a_i \mid \ 1 \leq i \leq 9\}.$$

Then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is the Majorana representation of (G, T) with shape $(2A, 3A)$. There are four distinct Norton-Sakuma subalgebras in V of type $3A$ generated by pairs of Majorana axes in A. Let $U := \{u_i \mid 1 \leq i \leq 4\}$ be the set of $3A$-axes contained in these subalgebras. Then $A \cup U$ is the basis of V provided in [I+10]. The images of involutions

in T under the linear representation $\varphi : G \to GL(V)$ may be written as permutations of axes in $A \cup U$. For example:

$$\varphi((1,2)) = (a_2, a_4)(a_3, a_5)(a_8, a_9)(u_3, u_4) \text{ and}$$
$$\varphi((1,2)(3,4)) = (a_2, a_5)(a_3, a_4)(u_1, u_2)(u_3, u_4).$$

Lemma 1.23. *The linear representation $\varphi : G \to GL(V)$ is faithful.*

Proof. Both non-trivial normal subgroups of $G \cong S_4$, i.e. A_4 and V_4, contain double transpositions. Since $\varphi((1,2)(3,4)) = (a_2, a_5)(a_3, a_4)(u_1, u_2)(u_3, u_4)$, $ker(\varphi)$ is trivial. □

The first step is to find a basis of V which is acted on by $Aut(V)$. As automorphisms of V preserve the algebra and inner products, an idempotent of a particular length would either be fixed or sent to another idempotent of that same length by an automorphism.

The $3A$-axes are idempotents of length $\frac{8}{5}$. However, a $3A$-axis is not the only idempotent of its length in a Norton-Sakuma algebra of type $3A$ (see *Proposition 3.4* of [CR13b]). It will be shown later (in Proposition 1.26) that U does not contain all idempotents of length $\frac{8}{5}$ in V. It is therefore not a given that $Aut(V)$ acts on $A \cup U$.

The identity of the Norton-Sakuma algebra V_{3A} is the only idempotent in V_{3A} of length $\frac{116}{35}$ (see *Proposition 3.4* of [CR13b]). Let $I_{3A} := \{i_j \mid 1 \leq j \leq 4\}$. Then I_{3A} is the set of identities for the four distinct Norton-Sakuma subalgebras of type $3A$ in V.

Lemma 1.24. *The set $A \cup I_{3A}$ is a basis of V.*

Proof. This follows by checking that $A \cup I_{3A}$ is linearly independent. □

The approach used in [CR13b] to find all the idempotents is as follows. A general vector v in an algebra of dimension n is written as a linear sum in terms of a given basis. Assuming that v is an idempotent, it would have to satisfy the following equation:

$$v \cdot v - v = 0.$$

Given that all pairwise algebra products between basis vectors are known, the equation above produces a system of $n \times n$ non-linear equations (equate the coefficients of each basis vector to zero). The real solutions to this system corresponds to the coefficients of the basis vectors in the general expression of v, making v an idempotent.

The command in [Map16] used by Castillo-Ramirez to solve the system of non-linear equations is **RootFinding[Isolate]**. This command isolates the real roots of a polynomial system by computing a *Groebner basis* followed by a *Rational Univariate Representation*. This command computes isolating intervals for each of the roots and the output evaluates the midpoints of those intervals numerically at a desired precision. This command is only applicable to polynomial systems having a finite number of complex solutions and will return an error otherwise. Unlike purely numerical methods, no roots are lost are ever lost through this algorithm. (Details regarding this command was obtained from [Map16a].)

Suppose one is only interested in obtaining idempotents of a particular length say k, then an additional equation may be added to form an $(n+1) \times n$ system of non-linear equations. This additional equation arises from:

$$l(v) - k = 0,$$

which may be expanded given that all pairwise inner products between basis vectors are known. Solving the $(n + 1) \times n$ system using the same command in [Map16] mentioned above, all idempotents of length k should be obtained.

The non-linear equations shall be described in terms of polynomials $p(\lambda_1, \ldots, \lambda_n)$ in $\mathbb{R}[\lambda_1, \ldots, \lambda_n]$, the polynomial ring in n variables over $\mathbb{R}$. For this notation, we define an action of $\sigma \in S_n$ on $p(\lambda_1, \ldots, \lambda_n)$ as

$$p(\lambda_1, \ldots, \lambda_n)^\sigma = p(\lambda_{\sigma(1)}, \ldots, \lambda_{\sigma(n)}).$$

The following three propositions are proved using the method described above.

Proposition 1.25. *The Majorana axes in A are the only idempotents of length 1 in V.*

Proof. Let $v \in V$ be an arbitrary vector. Then v may be expressed in terms of the basis

$A \cup U$:

$$v = \sum_{i=1}^{9} \lambda_i a_i + \sum_{i=1}^{4} \lambda_{i+9} u_i, \text{ where } \lambda_i \in \mathbb{R}.$$

Assuming v is an idempotent, it would satisfy

$$v \cdot v - v = 0.$$

Expand this equation by multiplying out all basis vectors pairwise. Then equating the coefficients of a_i to 0 for $1 \leq i \leq 6$, the following equations are obtained:

$$p_1(\lambda_1, \ldots, \lambda_{13})^{\sigma_1} = 0,$$

where

$$\sigma_1 \in \{1, (1,2)(5,6)(7,8)(11,12), (1,3)(4,6)(7,9)(10,12), (1,4)(3,6)(7,9)(11,13),$$
$$(1,5)(2,6)(7,8)(10,13), (1,6)(2,5)(10,13)(11,12)\}$$

and

$$\begin{aligned} p_1 =& \lambda_1^2 - \lambda_1 + \frac{2}{45}(\lambda_1(\lambda_{12} + \lambda_{13}) + \lambda_6(\lambda_{10} + \lambda_{11})) + \frac{4}{9}\lambda_1(\lambda_{10} + \lambda_{11}) \\ &- \frac{1}{45}(\lambda_2(\lambda_{11} + \lambda_{13}) + \lambda_3(\lambda_{10} + \lambda_{13}) + \lambda_4(\lambda_{11} + \lambda_{12}) + \lambda_5(\lambda_{10} + \lambda_{12})) \\ &- \frac{2}{9}(\lambda_{10}(\lambda_2 + \lambda_4) + \lambda_{11}(\lambda_3 + \lambda_5)) + \frac{1}{8}\lambda_1(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) \\ &\frac{1}{16}(\lambda_2\lambda_4 + \lambda_3\lambda_5) + \frac{1}{32}(\lambda_1 - \lambda_6)(\lambda_8 + \lambda_9) + \frac{1}{4}\lambda_1(\lambda_6 + \lambda_7) - \frac{1}{4}\lambda_6\lambda_7. \end{aligned}$$

Equating the coefficients of a_i to 0 for $7 \leq i \leq 9$, the following equations are obtained:

$$p_2(\lambda_1, \ldots, \lambda_{13})^{\sigma_2} = 0,$$

where

$$\sigma_2 \in \{1, (1,2)(5,6)(7,8)(11,12), (1,3)(4,6)(7,9)(10,12)\}$$

and

$$\begin{aligned} p_2 =& \lambda_7^2 - \lambda_7 + \frac{2}{9}\lambda_7(\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13}) - \frac{1}{32}(\lambda_8(\lambda_3 + \lambda_4) + \lambda_9(\lambda_2 + \lambda_5)) \\ &+ \frac{1}{32}(\lambda_7(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + (\lambda_1 + \lambda_6)(\lambda_8 + \lambda_9)) - \frac{1}{4}(\lambda_8\lambda_9 + \lambda_1\lambda_6) \\ &- \frac{2}{45}(\lambda_1(\lambda_{10} + \lambda_{11}) + \lambda_6(\lambda_{12} + \lambda_{13})) + \frac{1}{4}\lambda_7(\lambda_1 + \lambda_6 + \lambda_8 + \lambda_9) \\ &- \frac{128}{675}(\lambda_{10} + \lambda_{11})(\lambda_{12} + \lambda_{13}) + \frac{256}{2025}(\lambda_{10}\lambda_{11} + \lambda_{12}\lambda_{13}). \end{aligned}$$

Equating the coefficients of u_i to 0 for $1 \leq i \leq 4$, the following equations are obtained:

$$p_3(\lambda_1, \ldots, \lambda_{13})^{\sigma_3} = 0,$$

where

$$\sigma_3 \in \{1, (2,3)(4,5)(10,11)(12,13), (1,6)(3,4)(10,12)(11,13), (1,6)(2,5)(10,13)(11,12)\}$$

and

$$\begin{aligned} p_3 =& \lambda_{10}^2 - \lambda_{10} - \frac{1}{32}(\lambda_3\lambda_{13} + \lambda_5\lambda_{12} + \lambda_6\lambda_{11}) - \frac{135}{1024}(\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4) \\ &+ \frac{1}{32}(\lambda_{10}(\lambda_3 + \lambda_5 + \lambda_6) + \lambda_{11}(\lambda_2 + \lambda_4) + \lambda_{12}(\lambda_1 + \lambda_4) + \lambda_{13}(\lambda_1 + \lambda_2)) \\ &+ \frac{5}{32}\lambda_{10}(\lambda_7 + \lambda_8 + \lambda_9) + \frac{5}{16}\lambda_{10}(\lambda_1 + \lambda_2 + \lambda_4) + \frac{2}{5}\lambda_{10}(\lambda_{11} + \lambda_{12} + \lambda_{13}) \\ &- \frac{1}{8}(\lambda_7(\lambda_{12} + \lambda_{13}) + \lambda_8(\lambda_{11} + \lambda_{13}) + \lambda_9(\lambda_{11} + \lambda_{12})) \\ &- \frac{1}{9}(\lambda_{11}\lambda_{12} + \lambda_{11}\lambda_{13} + \lambda_{12}\lambda_{13}) + \frac{3}{32}(\lambda_7\lambda_{11} + \lambda_8\lambda_{12} + \lambda_9\lambda_{13}). \end{aligned}$$

Assuming v has length 1, it would satisfy $l(v) - 1 = 0$, which produces the following equation:

$$\begin{aligned} p_4 =& \sum_{i=1}^{9} \lambda_i^2 + \frac{8}{5}\sum_{i=10}^{13} \lambda_i^2 + \frac{13}{128}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \lambda_1\lambda_4 + \lambda_1\lambda_5 + \lambda_4\lambda_5 \\ &+ \lambda_2\lambda_4 + \lambda_2\lambda_6 + +\lambda_4\lambda_6 + \lambda_3\lambda_5 + \lambda_3\lambda_6 + \lambda_5\lambda_6) + \frac{1}{4}(\lambda_1\lambda_6 + \lambda_1\lambda_7 + \lambda_6\lambda_7 \\ &+ \lambda_2\lambda_5 + \lambda_2\lambda_8 + \lambda_5\lambda_8 + \lambda_3\lambda_4 + \lambda_3\lambda_9 + \lambda_4\lambda_9 + \lambda_7\lambda_8 + \lambda_7\lambda_9 + \lambda_8\lambda_9) \\ &+ \frac{1}{2}(\lambda_{10}(\lambda_1 + \lambda_2 + \lambda_4) + \lambda_{11}(\lambda_2 + \lambda_4 + \lambda_6) + \lambda_{12}(\lambda_2 + \lambda_3 + \lambda_6) \\ &+ \lambda_{13}(\lambda_4 + \lambda_5 + \lambda_6)) + \frac{1}{18}(\lambda_{10}(\lambda_3 + \lambda_5 + \lambda_6) + \lambda_{11}(\lambda_2 + \lambda_4 + \lambda_6) \\ &+ \lambda_{12}(\lambda_1 + \lambda_4 + \lambda_5) + \lambda_{13}(\lambda_1 + \lambda_2 + \lambda_3)) + \frac{1}{32}(\lambda_7(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) \\ &+ \lambda_8(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6) + \lambda_9(\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6) + \frac{272}{405}\sum_{10 \leq i < j \leq 13} \lambda_i\lambda_j \\ &+ \frac{2}{9}(\lambda_7 + \lambda_8 + \lambda_9)(\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13}) = 1. \end{aligned}$$

The command **RootFinding[Isolate]** in [Map16] identifies exactly 9 real roots to this system. It is then checked that these correspond to the 9 Majorana axes in A. □

Proposition 1.26. *There are exactly* 16 *idempotents of length* $\frac{8}{5}$ *in* V*.*

Proof. This follows from solving the system of equations from the proof of the previous proposition except $p_4 = 1$ is replaced with $p_4 = \frac{8}{5}$. □

Remark. The 16 idempotents of length $\frac{8}{5}$ in V arise from the four Norton-Sakuma subalgebras of type $3A$. Each $3A$-subalgebra contributes one $3A$-axis and three other idempotents of that length (see *Proposition 3.4* of [CR13b]).

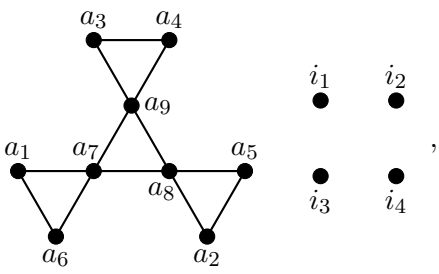
Proposition 1.27. *The vectors in* I_{3A} *are the only idempotents of length* $\frac{116}{35}$ *in* V*.*

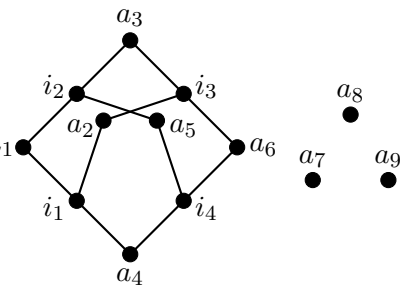
Proof. This follows from solving the system of equations from the proof of the Proposition 1.25 except $p_4 = 1$ is replaced with $p_4 = \frac{116}{35}$. □

Corollary 1.28. *The automorphism group* $Aut(V)$ *acts faithfully on* $A \cup I_{3A}$*.*

Proof. The action of $Aut(V)$ is well defined by Propositions 1.25 and 1.27. It is faithful as $A \cup I_{3A}$ is a basis of V. □

Let Γ_1 and Γ_2 be graphs, both with 13 vertices labelled by the vectors in $A \cup I_{3A}$. For Γ_1, an edge is drawn between two vertices if the inner product between the corresponding vectors is 1. For Γ_2, an edge is drawn between two vertices if the inner product between the corresponding vectors is $\frac{1}{8}$.

$\Gamma_1 =$ 

, $\Gamma_2 =$ 

Proposition 1.29. *The automorphism group* $Aut(\Gamma_1)$ *of the graph* Γ_1 *is isomorphic to* $2 \times S_4^2$*, and* $Aut(\Gamma_2)$ *is isomorphic to* $S_4 \times S_3$*. The intersection of these two automorphism groups,* $Aut(\Gamma_1) \cap Aut(\Gamma_2)$*, is isomorphic to* S_4*.*

Proof. It may be verified using the **GRAPE** package on [GAP16] that

$$\begin{aligned} Aut(\Gamma_1) =& \langle (a_1, a_6)(a_2, a_5)(a_3, a_4) \rangle \times \\ & \langle (a_1, a_2, a_4)(a_3, a_6, a_5)(a_7, a_8, a_9), (a_1, a_2)(a_3, a_4)(a_5, a_6)(a_7, a_8) \rangle \times \\ & Sym(\{i_1, i_2, i_3, i_4\}) \cong 2 \times S_4^2 \end{aligned}$$

and

$$\begin{aligned} Aut(\Gamma_2) =& \langle (a_1, a_2)(a_5, a_6)(i_2, i_3), (a_1, a_4)(a_3, a_6)(i_2, i_4), (a_1, a_3)(a_4, a_6)(i_1, i_3) \rangle \times \\ & Sym(\{a_7, a_8, a_9\}) \cong S_4 \times S_3. \end{aligned}$$

It it then straightforward to check that

$$\begin{aligned} Aut(\Gamma_1) \cap Aut(\Gamma_2) =& \langle (a_2, a_3)(a_4, a_5)(a_8, a_9)(i_1, i_2), \\ & (a_1, a_2, a_4)(a_3, a_6, a_5)(a_7, a_8, a_9)(i_2, i_3, i_4) \rangle \cong S_4. \end{aligned}$$

□

Corollary 1.30. $Aut(V) \cong S_4$.

Proof. $Aut(V)$ acts faithfully on $A \cup I_{3A}$ by Corollary 1.28, preserving the inner products. As $Aut(V)$ preserves both Γ_1 and Γ_2, $Aut(V) \leq Aut(\Gamma_1) \cap Aut(\Gamma_2) \cong S_4$ by Proposition 1.29. Since the representation is faithful by Lemma 1.23, $Aut(V) = \varphi(S_4) \cong S_4$. □

1.5.2 $(2B, 3A)$-shape S_4-representation

Let

$$T := T_1 \quad \text{and} \quad A := \{a_i \mid 1 \leq i \leq 6\}.$$

Then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is the Majorana representation of (G, T) with shape $(2B, 3A)$. There are four distinct Norton-Sakuma subalgebras in V of type $3A$ generated by pairs of Majorana axes in A. Let $U := \{u_i \mid 1 \leq i \leq 4\}$ be the set of $3A$-axes contained in these subalgebras.

This representation of S_4 is based on an embedding in the Monster where the transpositions are embedded into the $2A$-conjugacy class while the 3-cycles are embedded into the $3A$-conjugacy class. Although the 4-cycles in S_4 are embedded into the $4A$-conjugacy class of the Monster, no pair of Majorana axes in A generates a Norton-Sakuma algebra of type $4A$. In [I+10], a basis of V was given as $A \cup \{v_1, v_2, v_3\} \cup U$. The vectors v_1, v_2 and v_3 are $4A$-axes in the embedding corresponding to elements $(1,2,3,4)$, $(1,2,4,3)$ and $(1,3,2,4)$ respectively. They are contained in Norton-Sakuma algebras of type $4A$ generated by pairs of $2A$-axes in the Monster algebra but not in A (*Section 5* of [I+10]).

Lemma 1.31. *The linear representation $\varphi : G \to GL(V)$ is faithful.*

Proof. This is similar to the proof of Lemma 1.23 □

Let $I_{3A} := \{i_j \mid 1 \leq j \leq 4\}$. Then I_{3A} is the set of identities for the four distinct Norton-Sakuma subalgebras in V of type $3A$. Let $I_{4A} := \{i_j \mid 5 \leq j \leq 7\}$ where

$$i_5 := \frac{68}{69}(a_2 + a_5) + \frac{8}{69}(a_1 + a_3 + a_4 + a_6) - \frac{15}{92}\sum_{i=1}^{4} u_i + \frac{22}{23}v_1 + \frac{4}{23}(v_2 + v_3),$$

$$i_6 := \frac{68}{69}(a_3 + a_4) + \frac{8}{69}(a_1 + a_2 + a_5 + a_6) - \frac{15}{92}\sum_{i=1}^{4} u_i + \frac{22}{23}v_2 + \frac{4}{23}(v_1 + v_3),$$

$$i_7 := \frac{68}{69}(a_1 + a_6) + \frac{8}{69}(a_2 + a_3 + a_4 + a_5) - \frac{15}{92}\sum_{i=1}^{4} u_i + \frac{22}{23}v_3 + \frac{4}{23}(v_1 + v_2).$$

Proposition 1.32. *The vectors in I_{4A} are idempotents of length 4.*

Proof. This follows from calculations with the algebra and inner products on V. □

Remark. A Norton-Sakuma algebra of type $4A$ contains only one idempotent of length 4, which is its identity (see *Proposition 3.8* of [CR13b]).

Lemma 1.33. *The set $A \cup I_{3A} \cup I_{4A}$ is a basis of V.*

Proof. This follows from checking that $A \cup I_{3A} \cup I_{4A}$ is linearly independent. □

The same technique from the previous subsection for solving a system of non-linear equations and obtaining all idempotents of a specific length is applied here.

Proposition 1.34. *The Majorana axes in A are the only idempotents of length 1 in V.*

Proof. Let $v \in V$ be an arbitrary vector Then v may be expressed in terms of the basis $A \cup \{v_1, v_2, v_3\} \cup U$:

$$v = \sum_{i=1}^{6} \lambda_i a_i + \sum_{i=1}^{3} \lambda_{i+6} v_i + \sum_{i=1}^{4} \lambda_{i+9} u_i, \quad \lambda_i \in \mathbb{R}.$$

Assuming v is an idempotent, it would satisfy

$$v \cdot v - v = 0.$$

Expand this equation by multiplying out all basis vectors pairwise. Equating the coefficients of a_i to 0 for $1 \leq i \leq 6$, the following equations are obtained:

$$p_1(\lambda_1, \ldots, \lambda_{13})^{\sigma_1} = 0,$$

where

$$\begin{aligned}\sigma_1 \in \{&1, (1,2)(5,6)(8,9)(11,12), (1,3)(4,6)(8,9)(10,12), (1,4)(3,6)(8,9)(11,13),\\ &(1,5)(2,6)(7,9)(10,13), (1,6)(3,4)(10,12)(11,13)\}\end{aligned}$$

and

$$\begin{aligned}p_1 =&\lambda_1^2 - \lambda_1 + \frac{1}{135}(\lambda_{10}(\lambda_3+\lambda_5) + \lambda_{11}(\lambda_2+\lambda_4) + \lambda_{12}(\lambda_4+\lambda_5) + \lambda_{13}(\lambda_2+\lambda_3)\\
&- \frac{5}{54}\lambda_9(\lambda_7+\lambda_8) - \frac{1}{144}(\lambda_7(\lambda_3+\lambda_4) + \lambda_8(\lambda_2+\lambda_5) + \lambda_8(\lambda_2+\lambda_3+\lambda_4+\lambda_5))\\
&+ \frac{22}{135}\lambda_1(\lambda_{12}+\lambda_{13}) - \frac{1}{72}\lambda_6\lambda_9 - \frac{256}{6075}(\lambda_{10}+\lambda_{11})(\lambda_{12}+\lambda_{13}) + \frac{5}{72}\lambda_1\lambda_9\\
&- \frac{1}{18}(\lambda_7(\lambda_2+\lambda_5) + \lambda_8(\lambda_3+\lambda_4)) - \frac{2}{135}\lambda_6(\lambda_{10}+\lambda_{11}) + \frac{85}{288}\lambda_1(\lambda_7+\lambda_8)\\
&+ \frac{2}{135}(\lambda_7+\lambda_8)(\lambda_{10}+\lambda_{11}) - \frac{2}{9}(\lambda_{10}(\lambda_2+\lambda_4) + \lambda_{11}(\lambda_3+\lambda_5)) + \frac{5}{27}\lambda_7\lambda_8\\
&+ \frac{7}{288}\lambda_6(\lambda_7+\lambda_8) + \frac{512}{6075}(\lambda_{10}\lambda_{11} + \lambda_{12}\lambda_{13}) - \frac{4}{135}\lambda_9(\lambda_{10}+\lambda_{11})\\
&- \frac{26}{135}(\lambda_{12}+\lambda_{13})(\lambda_7+\lambda_8) + \frac{4}{45}\lambda_9(\lambda_{12}+\lambda_{13}) + \frac{1}{16}(\lambda_2\lambda_4 + \lambda_3\lambda_5)\\
&+ \frac{1}{8}\lambda_1(\lambda_2+\lambda_3+\lambda_4+\lambda_5) + \frac{4}{9}\lambda_1(\lambda_{10}+\lambda_{11}).\end{aligned}$$

Equating the coefficients of v_i to 0 for $1 \leq i \leq 3$, the following equations are obtained:

$$p_2(\lambda_1, \ldots, \lambda_{13})^{\sigma_2} = 0,$$

where

$$\sigma_2 \in \{1, (1,5)(2,3)(7,8)(10,11), (1,5)(2,6)(7,9)(10,13)\}$$

and

$$\begin{aligned} p_2 =& \lambda_7^2 - \lambda_7 + \frac{4}{45}(\lambda_{10}(\lambda_3 + \lambda_6) + \lambda_{11}(\lambda_4 + \lambda_6) + \lambda_{12}(\lambda_1 + \lambda_4)) + (\lambda_8 + \lambda_9) \\ & (\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13}) + \frac{1}{36}\lambda_8\lambda_9 + \frac{11}{96}(\lambda_8(\lambda_1 + \lambda_6) + \lambda_9(\lambda_3 + \lambda_4)) \\ & - \frac{1}{12}(\lambda_8(\lambda_3 + \lambda_4) + \lambda_9(\lambda_1 + \lambda_6)) + \frac{256}{2025}(\lambda_{10}\lambda_{12} + \lambda_{11}\lambda_{13}) + \frac{2}{45}(\lambda_2(\lambda_{11} + \lambda_{13}) \\ & + \lambda_5(\lambda_{10} + \lambda_{12})) + \frac{1}{24}\lambda_7(\lambda_2 + \lambda_5) - \frac{7}{96}(\lambda_2 + \lambda_5)(\lambda_8 + \lambda_9) - \frac{128}{2025}(\lambda_{10} + \lambda_{12}) \\ & (\lambda_{11} + \lambda_{13}) + \frac{17}{96}\lambda_7(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6) + \frac{2}{5}\lambda_7(\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13}) \\ & + \frac{19}{36}\lambda_7(\lambda_8 + \lambda_9). \end{aligned}$$

Equating the coefficients of u_i to 0 for $1 \leq i \leq 4$, the following equations are obtained:

$$p_3(\lambda_1, \ldots, \lambda_{13})^{\sigma_3} = 0,$$

where

$$\sigma_3 \in \{1, (2,3)(4,5)(7,8)(10,11), (1,3)(4,6)(8,9)(10,12), (1,5)(2,6)(7,9)(10,13)\}$$

and

$$\begin{aligned} p_3 =& \lambda_{10}^2 - \lambda_{10} - \frac{1}{96}(\lambda_{11}(\lambda_2 + \lambda_4) + \lambda_{12}(\lambda_1 + \lambda_4) + \lambda_{13}(\lambda_1 + \lambda_2)) - \frac{125}{768}(\lambda_7\lambda_8 \\ & + \lambda_7\lambda_9 + \lambda_8\lambda_9) + \frac{5}{512}(\lambda_7(\lambda_1 + \lambda_4) + \lambda_8(\lambda_1 + \lambda_2) + \lambda_9(\lambda_2 + \lambda_4)) \\ & + \frac{5}{64}(\lambda_7(\lambda_2 + \lambda_5) + \lambda_8(\lambda_3 + \lambda_4) + \lambda_9(\lambda_1 + \lambda_6)) - \frac{7}{135}(\lambda_{11}\lambda_{12} + \lambda_{11}\lambda_{13} \\ & + \lambda_{12}\lambda_{13}) + \frac{14}{135}\lambda_{10}(\lambda_{11} + \lambda_{12} + \lambda_{13}) + \frac{5}{96}(\lambda_3\lambda_{13} + \lambda_5\lambda_{12} + \lambda_6\lambda_{11}) \\ & + \frac{11}{96}(\lambda_{10}(\lambda_3 + \lambda_5 + \lambda_6) + \lambda_7\lambda_{12} + \lambda_8\lambda_{13} + \lambda_9\lambda_{11}) - \frac{35}{256}(\lambda_7(\lambda_3 + \lambda_6) \\ & + \lambda_8(\lambda_5 + \lambda_6) + \lambda_9(\lambda_3 + \lambda_5)) + \frac{15}{32}\lambda_{10}(\lambda_7 + \lambda_8 + \lambda_9) - \frac{1}{12}(\lambda_{11}(\lambda_7 + \lambda_8) \\ & + \lambda_{12}(\lambda_8 + \lambda_9) + \lambda_{13}(\lambda_7 + \lambda_9)) + \frac{5}{16}\lambda_{10}(\lambda_1 + \lambda_2 + \lambda_4) \\ & - \frac{135}{1024}(\lambda_1\lambda_2 + \lambda_1\lambda_4 + \lambda_2\lambda_4). \end{aligned}$$

Assuming v has length 1, it would satisfy $l(v) - 1 = 0$, which produces the following equation:

$$\begin{aligned}
p_4 = &\sum_{i=1}^{6} \lambda_i^2 + 2\sum_{i=7}^{9} \lambda_i^2 + \frac{8}{5}\sum_{i=10}^{13} \lambda_i^2 + \frac{9}{8}(\lambda_7\lambda_8 + \lambda_7\lambda_9 + \lambda_8\lambda_9) + \frac{1}{12}(\lambda_7(\lambda_2 + \lambda_5) + \\
&\lambda_8(\lambda_3 + \lambda_4) + \lambda_9(\lambda_1 + \lambda_6)) + \frac{112}{675}\sum_{10\leq i<j\leq 13} \lambda_i\lambda_j + \frac{31}{96}(\lambda_7(\lambda_1 + \lambda_3 + \lambda_4 + \lambda_6) \\
&+ \lambda_8(\lambda_1 + \lambda_2 + \lambda_5 + \lambda_6) + \lambda_9(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)) + \frac{13}{90}(\lambda_{10}(\lambda_3 + \lambda_5 + \lambda_6) \\
&+ \lambda_{11}(\lambda_2 + \lambda_4 + \lambda_6) + \lambda_{12}(\lambda_1 + \lambda_4 + \lambda_5) + \lambda_{13}(\lambda_1 + \lambda_2 + \lambda_3)) \\
&+ \frac{22}{27}(\lambda_7 + \lambda_8 + \lambda_9)(\lambda_{10} + \lambda_{11} + \lambda_{12} + \lambda_{13}) + \frac{1}{2}(\lambda_{10}(\lambda_1 + \lambda_2 + \lambda_4) \\
&+ \lambda_{11}(\lambda_1 + \lambda_3 + \lambda_5) + \lambda_{12}(\lambda_2 + \lambda_3 + \lambda_6) + \lambda_{13}(\lambda_4 + \lambda_5 + \lambda_6)) + \frac{13}{128}(\lambda_1\lambda_2 \\
&+ \lambda_1\lambda_4 + \lambda_2\lambda_4 + \lambda_1\lambda_3 + \lambda_1\lambda_5 + \lambda_3\lambda_5 + \lambda_2\lambda_3 + \lambda_2\lambda_6 + \lambda_3\lambda_6 + \lambda_4\lambda_5 \\
&+ \lambda_4\lambda_6 + \lambda_5\lambda_6) = 1.
\end{aligned}$$

The command **RootFinding[Isolate]** in [Map16] identifies exactly 6 real roots to this system. It is then checked that these correspond to the 6 Majorana axes in A. □

Proposition 1.35. *The vectors in I_{3A} are the only idempotents of length $\frac{116}{35}$ in V.*

Proof. This follows from solving the system of equations from the proof of the previous proposition except $p_4 = 1$ is replaced with $p_4 = \frac{116}{35}$. □

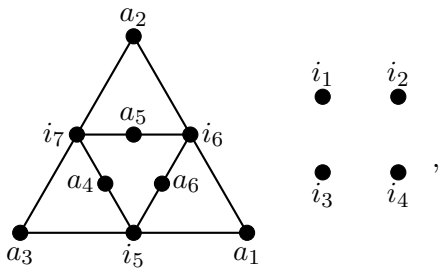
Proposition 1.36. *The vectors in I_{4A} are the only idempotents of length 4 in V.*

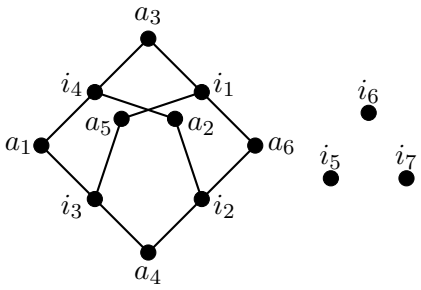
Proof. This follows from solving the system of equations from the proof of the Proposition 1.34 except $p_4 = 1$ is replaced with $p_4 = 4$. □

Corollary 1.37. *The automorphism group $Aut(V)$ acts faithfully on $A \cup I_{3A} \cup I_{4A}$.*

Proof. The action of $Aut(V)$ is well defined by Propositions 1.34, 1.35 and 1.36. It is also faithful as $A \cup I_{3A} \cup I_{4A}$ is a basis of V. □

Let Γ_1 and Γ_2 be graphs, both with 13 vertices labelled by the vectors in $A \cup I_{3A} \cup I_{4A}$. For Γ_1, an edge is drawn between two vertices if the inner product between the corresponding vectors is $\frac{5}{16}$. For Γ_2, an edge is drawn between two vertices if the inner product between the corresponding vectors is $\frac{13}{105}$.

$\Gamma_1 =$ 

, $\Gamma_2 =$ 

Proposition 1.38. *The automorphism group $Aut(\Gamma_1)$ of Γ_1 is isomorphic to $2 \times S_4^2$. The automorphism group $Aut(\Gamma_2)$ of Γ_2 is isomorphic to $S_4 \times S_3$. The intersection of these two automorphism groups, $Aut(\Gamma_1) \cap Aut(\Gamma_2)$, is isomorphic to S_4.*

Proof. It may be verified using the **GRAPE** package on [GAP16] that

$$\begin{aligned} Aut(\Gamma_1) =& \langle (a_1, a_6)(a_2, a_5)(a_3, a_4) \rangle \times \\ & \langle (a_1, a_2, a_3)(a_4, a_6, a_5)(i_5, i_6, i_7), (a_1, a_3)(a_2, a_5)(a_4, a_6)(i_6, i_7) \rangle \times \\ & Sym(\{i_1, i_2, i_3, i_4\}) \cong 2 \times S_4^2 \end{aligned}$$

and

$$\begin{aligned} Aut(\Gamma_2) =& \langle (a_1, a_5)(a_2, a_6)(i_1, i_4), (a_1, a_3)(a_4, a_6)(i_1, i_3), (a_1, a_4)(a_3, a_6)(i_2, i_4) \rangle \times \\ & Sym(\{i_5, i_6, i_7\}) \cong S_4 \times S_3. \end{aligned}$$

It it then easy to check that

$$\begin{aligned} Aut(\Gamma_1) \cap Aut(\Gamma_2) =& \langle (a_2, a_3)(a_4, a_5)(i_1, i_2)(i_6, i_7), \\ & (a_1, a_2, a_4)(a_3, a_6, a_5)(i_2, i_3, i_4)(i_5, i_6, i_7) \rangle \cong S_4. \end{aligned}$$

□

Corollary 1.39. *$Aut(V) \cong S_4$.*

Proof. This is similar to the proof of Corollary 1.30. □

By Corollaries 1.30, 1.39 and by results of Castillo-Ramirez in [CR13b], the next result is clear.

Theorem 1.40. *The automorphism groups of all four Majorana algebras arising from the four Majorana representations of S_4 constructed in [I+10], are isomorphic to S_4.*

Chapter 2

The $3A$-axes

The role of the $2A$-axes in the Monster algebra and more generally of the Majorana axes in a Majorana algebra has been central to Majorana theory. Though the NA-axes ($3 \leq N \leq 5$) seen initially in the Norton-Sakuma algebras are important vectors which together with Majorana axes span Majorana algebras, much is still unknown about them nor are they applied extensively in Majorana theory. As a next stage in developing Majorana theory, we propose a more independent definition of a $3A$-axis by means of axiomatization. For this we would first need to understand some of its key features.

This chapter begins with the deduction of properties of a $3A$-axis contained in the Monster algebra, the same way it was done for the $2A$-axes in Chapter 1. There is a special association between the $3A$-elements and the $3A$-axes. We shall see how a $3A$-element in the Monster may be described simply in terms of scalar multiplication on the eigenspaces of the associated $3A$-axis in the Monster algebra. The following two sections look into the fusion rules obeyed by the eigenvectors of a $3A$-axis. Then in Section 2.4, we define what we call a standard $3A$-axis in an arbitrary algebra. This is an axiomatic approach to studying the $3A$-axes as opposed to it being dependent on the existence of pairs of $2A$-axes. The last section is an application of this axiomatic approach by which we construct an algebra using the properties of the Majorana axes together with our definitions of a standard $3A$-axis. This algebra shall also be assumed to be 2-closed with respect to a certain set of axes.

2.1 Properties of a $3A$-axis in $V_{\mathbb{M}}$

In the previous chapter, a $3A$-axis was first defined as a vector in the Monster algebra in terms of a pair of $2A$-axes generating a Norton-Sakuma algebra of type $3A$ (see Definition 1.16). This definition was then generalized to the Majorana setting by replacing the $2A$-axes with Majorana axes:

Definition 2.1. *Let a_1 and a_2 be Majorana axes in some algebra V. If $\langle\langle a_1, a_2\rangle\rangle \cong V_{3A}$, then the vector u defined as*

$$u := \frac{64}{135}(2a_1 + 2a_2 + a_3 - 32a_1 \cdot a_2),$$

is known as a **3A-*axis***. *The vector a_3 is a Majorana axis in $\langle\langle a_1, a_2\rangle\rangle$ defined as*

$$a_3 := -\frac{78}{7}a_1 + a_2 - \frac{512}{7}a_1 \cdot a_2 + \frac{2048}{7}a_1 \cdot (a_1 \cdot a_2).$$

For convenience, a Norton-Sakuma algebra of type $3A$ shall be described in terms of the notations used in the definition above, i.e. it is spanned by $\{a_1, a_2, a_3, u\}$. The full algebra and inner products of V_{3A} are in Table 1.2.

A $3A$-axis in a Majorana representation $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ of (G, T) is defined in terms of a pair $\{a_{t_1}, a_{t_2}\}$ of Majorana axes in A such that $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{3A}$. The element $\rho := t_1t_2$ has order 3 and it corresponds to the $3A$-axis u. Therefore u is often indexed by ρ as a subscript, i.e. u_ρ, and is defined as:

$$u_\rho := \frac{64}{135}(2a_{t_1} + 2a_{t_2} + a_{t_3} - 32a_{t_1} \cdot a_{t_2}),$$

where $t_3 := t_1t_2t_1$. By commutativity of the algebra product, the following relation is obtained:

$$u_\rho = u_{\rho^{-1}}.$$

Hence the same $3A$-axis may be defined differently depending on the choice of Majorana axes generators. The image of G under φ acts on u_ρ by conjugating ρ.

By Table 1.2 of the Norton-Sakuma algebras, the $3A$-axes are idempotents of length $\frac{8}{5}$. This length of idempotent is not unique to the $3A$-axis in a Norton-Sakuma algebra of

type $3A$, as seen in the following result from *Proposition 3.4* of [CR13b].

Proposition 2.2. *A Norton-Sakuma algebra of type* $3A$ *contains exactly* 4 *idempotents of length* $\frac{8}{5}$*. Besides the* $3A$*-axis* u*, the vectors* x_i *defined below in terms of the spanning set* $\{a_1, a_2, a_3, u\}$ *of* V_{3A} *are also idempotents of length* $\frac{8}{5}$*.*

$$x_i := \frac{2}{9}a_i + \frac{8}{9}(a_j + a_k) - \frac{1}{4}u,$$

where $\{i, j, k\} = \{1, 2, 3\}$*.*

Lemma 2.3. *There is a unique* $3A$*-axis in* V_{3A} *.*

Proof. By *Proposition 3.4* of [CR13b], the only Majorana axes in V_{3A} are a_1, a_2 and a_3. All $3A$-axes defined in terms of pairs of these Majorana axes coincide with u. □

In the case of the $2A$-axes in $V_{\mathbb{M}}$, each $2A$-involution is associated to a $2A$-axis. However for the case of the $3A$-axes, the association is between the cyclic subgroup generated by a $3A$-element in $\mathbb{M}$ and a $3A$-axis. This association is bijective as the action of the Monster on the Monster algebra obeys the $3A$-condition. Let h be a $3A$-element in the Monster. Then there exist $2A$-involutions t_1, t_2 such that $h = t_1t_2$. The $3A$-axis associated to $\langle h \rangle$ is the $3A$-axis u_h in $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle$. For simplicity, we say h is associated to u_h. As $V_{\mathbb{M}}$ is a Majorana representation of $\mathbb{M}$, the Monster acts on the u_hs by conjugating the hs.

Remark. To simplify notations in this chapter, the $\mathbb{M}$ subscript shall be dropped from the algebra and inner products on the Monster algebra.

The following result was known for some time. See [Nor96] for a reference.

Theorem 2.4. *The centralizer in the Monster of a* $3A$*-element* h *is isomorphic to a triple cover of the Fisher group* Fi_{24}'*:*

$$C_{\mathbb{M}}(h) \cong 3.Fi_{24}'.$$

In terms of modules, the Monster algebra is an $\mathbb{R}\mathbb{M}$-module where $\mathbb{R}\mathbb{M}$ is the group algebra of the Monster over the field of real numbers. However, it may be convenient to extend $\mathbb{R}$ to the field of complex numbers $\mathbb{C}$, as we shall see in due course.

Lemma 2.5. *Let $h \in 3A$ with corresponding $3A$-axis $u_h \in V_{\mathbb{M}}$. The λ-eigenspace $V_{\mathbb{M},\lambda}^{u_h}$ of u_h in $V_{\mathbb{M}}$ for some $\lambda \in \mathbb{R}$, is an $\mathbb{R}C_{\mathbb{M}}(h)$-module.*

Proof. It is sufficient to show that $v^g \in V_{\mathbb{M},\lambda}^{u_h}$ for any $g \in C_{\mathbb{M}}(h)$ and $v \in V_{\mathbb{M},\lambda}^{u_h}$.

$$\begin{aligned} u_h \cdot v^g &= u_{g^{-1}hg} \cdot v^g && \text{(as } g \text{ centralizes } h\text{)} \\ &= u_h^g \cdot v^g && \text{(as } g \text{ acts on } u_h \text{ by conjugating } h\text{)} \\ &= (u_h \cdot v)^g && \text{(as } g \text{ preserves the algebra product)} \\ &= (\lambda v)^g && \text{(as } v \text{ is a } \lambda\text{-eigenvector of } u_h\text{)} \\ &= \lambda v^g. \end{aligned}$$

Hence v^g is a λ-eigenvector of u_h. □

The notation in the previous lemma is also used for the next lemma.

Lemma 2.6. *Let $N_{\mathbb{M}}(\langle h \rangle)$ be the normalizer in the Monster of the cyclic subgroup generated by h. Then $V_{\mathbb{M},\lambda}^{u_h}$ is an $\mathbb{R}N_{\mathbb{M}}(\langle h \rangle)$-module.*

Proof. Let $g \in N_{\mathbb{M}}(\langle h \rangle)$. Then $h^g = h$ or h^{-1}. But since $u_h = u_{h^{-1}}$, we have that $u_h = u_h^g$. The rest is similar to the proof of the previous lemma. □

In the section of 'Preliminaries and Notations', it was noted that the Monster algebra is isomorphic to the direct sum of the trivial 1-dimensional module and the smallest non-trivial complex module of the Monster:

$$V_{\mathbb{M}} \cong V_{1_{\mathbb{M}}} \oplus V_{2_{\mathbb{M}}}.$$

Let h be a $3A$-element in $\mathbb{M}$. Extending $\mathbb{R}$ to $\mathbb{C}$, the Monster algebra may be viewed as a $\mathbb{C}C_{\mathbb{M}}(h)$-module. As a consequence of Maschke's theorem (Theorem 0.5), $V_{\mathbb{M}}$ may be

decomposed as a direct sum of irreducible $\mathbb{C}3.Fi_{24}'$-modules.

Lemma 2.7. *There are* 5 *irreducible complex representations of* $3.Fi_{24}'$ *of dimension less than* 196883*. These representations have dimensions* 1*,* 783*,* 8671*,* 57477 *and* 64584*.*

Proof. This may be verified from the character table of $3.Fi_{24}'$ in [Atl85]. □

Lemma 2.8. *The multiplicity of the trivial* 1*-dimensional module of* $3.Fi_{24}'$ *in the decomposition of* $V_{2_{\mathbb{M}}}$ *into irreducible* $\mathbb{C}3.Fi_{24}'$*-modules is* 1*.*

Proof. This may be verified by taking inner products between characters of the Monster and $3.Fi_{24}'$ from [Atl85], an application of the character orthogonality theorem (Theorem 0.7). □

Lemma 2.9. *The following equation describes the decomposition of* $V_{2_{\mathbb{M}}}$ *as irreducible* $\mathbb{C}3.Fi_{24}'$*-modules:*

$$196883 = 1 \cdot 1 + 2 \cdot 783 + 1 \cdot 8671 + 1 \cdot 57477 + 2 \cdot 64584.$$

The numbers multiplying from the left hand side, the dimensions of the irreducible $\mathbb{C}3.Fi_{24}'$*-modules are their multiplicities.*

Proof. By Lemma 2.8, the multiplicity of the trivial module is 1. The remaining multiplicities may be deduced by using some simple programming to verify that the equation above is the only positive integral combination for 196883 in terms of the dimensions of the 5 irreducible $3.Fi_{24}'$-representations from Lemma 2.7. □

Corollary 2.10. *A* $3A$*-axis has at most* 7 *eigenvalues in* $V_{2_{\mathbb{M}}}$*.*

Proof. By Lemma 2.5 and Maschke's theorem (Theorem 0.5), an eigenspace of a $3A$-axis may be decomposed as a direct sum of irreducible $\mathbb{C}3.Fi_{24}'$-modules. By Lemma 2.9, $V_{2_{\mathbb{M}}}$ decomposes into exactly 7 irreducible modules. □

The following result was proven by Norton in [Nor96] by computing the traces of various powers of the adjoint action of a $3A$-axis. It states explicitly the eigenvalues of a $3A$-axis together with the dimensions of the corresponding eigenspaces. The trivial module $V_{1_{\mathbb{M}}}$ is spanned by a 0-eigenvector of a $3A$-axis.

Lemma 2.11. *A $3A$-axis in $V_{\mathbb{M}}$ has exactly 5 eigenvalues which are 1, 0, $\frac{1}{5}$, $\frac{1}{3}$ and $\frac{1}{30}$. The dimensions of the corresponding eigenspaces are 1, 57478, 8671, 1566 and 129168 respectively.*

Remark. The rescaling of the $3A$-axes to the Majorana setting affects the eigenvalues. The eigenvalues stated in [Nor96] are by a factor of 90 the eigenvalues in the above lemma.

From here on, denote by ε, one of the two primitive cube roots of unity. Then ε^{-1}, which is its complex conjugate is the other primitive cube root of unity.

Lemma 2.12. *Let $h \in 3A$ and let V be an irreducible $\mathbb{C}C_{\mathbb{M}}(h)$-submodule of $V_{\mathbb{M}}$. The map*

$$\vartheta(h) : V \to V \text{ where } v \mapsto v^h \text{ for all } v \in V$$

is a $\mathbb{C}C_{\mathbb{M}}(h)$-isomorphism. Moreover $\vartheta(h)$ is a scalar multiplication by λ for some $\lambda \in \{1, \varepsilon, \varepsilon^{-1}\}$.

Proof. The map $\vartheta(h)$ is a $\mathbb{C}C_{\mathbb{M}}(h)$-homomorphism as it is linear and for any $g \in C_{\mathbb{M}}(h)$,

$$(v^{\vartheta(h)})^g = v^{hg} = v^{gh} = (v^g)^{\vartheta(h)}.$$

Moreover $\vartheta(h^{-1})$ is the inverse of $\vartheta(h)$. By Schur's lemma (Theorem 0.6), $\vartheta(h)$ is a scalar multiple of the identity endomorphism. Let $v^{\vartheta(h)} = \lambda v$ for some $\lambda \in \mathbb{C}$. Then

$$v = v^{h^3} = ((v^{\vartheta(h)})^{\vartheta(h)})^{\vartheta(h)} = \lambda^3 v \Rightarrow \lambda^3 = 1.$$

Hence $\lambda \in \{1, \varepsilon, \varepsilon^{-1}\}$. □

Corollary 2.13. *Each $h \in 3A$ acts on the 1-dimensional trivial module as the identity map.*

Proof. This is obvious. □

Theorem 2.14. *Let $h \in 3A$ with associated $3A$-axis $u \in V_{\mathbb{M}}$. Then h acts as the identity map on the 1-, 0- and $\frac{1}{5}$-eigenspaces of u in $V_{\mathbb{M}}$. Each of the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces split into pairs of irreducible $\mathbb{C}C_{\mathbb{M}}(h)$-modules. On one irreducible module h acts as multiplication by ε, and on the other, h acts as multiplication by ε^{-1}.*

Proof. By Lemmas 2.11 and 2.12, h acts on the 1-, 0- and $\frac{1}{5}$-eigenspaces of u_h in $V_{2_{\mathbb{M}}}$ as multiplication by a scalar. The $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces split into two copies of isomorphic irreducible $\mathbb{C}C_{\mathbb{M}}(h)$-modules. Then h acts on each copy as multiplication by a scalar. Let X_λ denote the subspace of $V_{2_{\mathbb{M}}}$ for which h acts as multiplication by $\lambda \in \{1, \varepsilon, \varepsilon^{-1}\}$. Then

$$V_{2_{\mathbb{M}}} = X_1 \oplus X_\varepsilon \oplus X_{\varepsilon^{-1}}.$$

Let x_λ denote the dimension of X_λ. Then

$$196883 = x_1 + x_\varepsilon + x_{\varepsilon^{-1}}.$$

The character of h in $V_{2_{\mathbb{M}}}$ is 782 (see [Atl85]). Hence

$$782 = x_1 + \varepsilon x_\varepsilon + \varepsilon^{-1} x_{\varepsilon^{-1}}.$$

But for $\varepsilon x_\varepsilon + \varepsilon^{-1} x_{\varepsilon^{-1}}$ to be real, it must be that $x_\varepsilon = x_{\varepsilon^{-1}}$. Let $y := x_\varepsilon = x_{\varepsilon^{-1}}$. Then

$$196883 = x_1 + 2y, \text{ and} \tag{1}$$

$$782 = x_1 - y. \tag{2}$$

Solving simultaneously equations (1) and (2), we get that $x_1 = 66149$ and $y = 65367$. The only way for x_1 to be written as a positive integral sum of the dimensions of irreducible $\mathbb{C}C_{\mathbb{M}}(h)$-modules is

$$66149 = 1 + 8671 + 57447.$$

For y, we have

$$65367 = 783 + 64584.$$

Together with Corollary 2.13, the result follows. □

Remark. A $3A$-axis in the Monster algebra is a conformal vector of central charge $\frac{4}{5}$ in the VOA context. For this conformal vector, an automorphism of the vertex operator algebra was defined in [Miy01]. The automorphism of $V_{\mathbb{M}}$ in Theorem 2.14 coincides with the restriction of the automorphism in [Miy01] to $V_{\mathbb{M}}$.

Denote by $V^u_{\mathbb{M},\eta\gamma}$, the subspace of the η-eigenspace of u in $V_{\mathbb{M}}$ for which h acts as scalar multiplication by γ, for $\eta \in \{\frac{1}{3}, \frac{1}{30}\}$ and $\gamma \in \{\varepsilon, \varepsilon^{-1}\}$.

$$V^u_{\mathbb{M},\eta\gamma} := \{v \in V^u_{\mathbb{M},\gamma} \mid v^h = \gamma v\}$$

Like in the proof of Lemma 1.3, the next result follows from associativity of the inner product and commutativity of the algebra product.

Lemma 2.15. *The decomposition of $V_{\mathbb{M}}$ as a direct sum of eigenspaces of a $3A$-axis u is orthogonal, i.e.*

$$(v, w) = 0 \text{ for all } v \in V^u_{\mathbb{M},\mu},\ w \in V^u_{\mathbb{M},\lambda} \text{ and } \mu \neq \lambda.$$

Proposition 2.16. *The $3A$-axis in V_{3A} displays only eigenvalues 1, 0 and $\frac{1}{3}$. The corresponding eigenspaces have dimensions 1, 1 and 2 respectively.*

Proof. This is verified by determining the characteristic polynomial of ad_u. □

Proposition 2.17. *The eigenvalues of x_i from Proposition 2.2 in V_{3A} are 1, 0, $\frac{1}{3}$ and $\frac{13}{16}$.*

Proof. This is similar to the proof of the preceding proposition. □

Although like the $3A$-axes being idempotents of length $\frac{8}{5}$, the vectors x_i do not have the same set of eigenvalues as a $3A$-axis in the Monster algebra. By the above proposition, x_i has eigenvalue $\frac{13}{16}$.

Proposition 2.18. *All eigenvalues (*1, 0, $\frac{1}{5}$, $\frac{1}{3}$ *and* $\frac{1}{30}$*) of a* $3A$*-axis in* $V_{\mathbb{M}}$ *appear in the Monster embeddable Majorana representations of* S_4 *with shape involving type* $3A$*. In the* $(2A, 3A)$*-shape* S_4*-representation, the corresponding eigenspaces have dimensions* 1, 3, 1, 4 *and* 4 *respectively. The same is true for the* $(2B, 3A)$*-shape* S_4*-representation.*

Proof. This is similar to the proof of Proposition 2.16 □

Lemma 2.19. *For any* $v \in V_{\mathbb{M}}$ *and* $3A$*-axis* $u \in V_{\mathbb{M}}$*, the vector* $\frac{5}{8}(u, v)u$ *is the projection of* v *to* $V^u_{\mathbb{M},1} = \langle u \rangle$*, the* 1*-eigenspace of* u *in* $V_{\mathbb{M}}$*.*

Proof. The vector v may be written as $v = \lambda u + w$ for some $w \in V_{\mathbb{M}} \setminus V^u_{\mathbb{M},1}$ and $\lambda \in \mathbb{R}$. Then

$$(u, v) = (u, \lambda u + w) = \lambda(u, u) + (u, w) = \frac{8}{5}\lambda + 0$$

as w is orthogonal to u by Lemma 2.15. Hence $\lambda = \frac{5}{8}(u, v)$. □

Lemma 2.20. *Let* $a, u \in V_{\mathbb{M}}$ *be a* $2A$*- and* $3A$*-axis respectively. Then*

$$0 \leq (a, u) \leq \frac{2}{5}.$$

Proof. The $2A$-axis a may be written as $a = \lambda u + v$ for some $v \in V_{\mathbb{M}} \setminus V^u_{\mathbb{M},1}$ and $\lambda \in \mathbb{R}$. As a and u are idempotents,

$$a = a \cdot a = (\lambda u + v) \cdot (\lambda u + v) = \lambda^2 u \cdot u + 2\lambda u \cdot v + v \cdot v = \lambda^2 u + 2\lambda u \cdot v + v \cdot v.$$

By Lemma 2.19, $\lambda = \frac{5}{8}(a, u)$. Hence

$$\lambda = \frac{5}{8}(a, u) = \frac{5}{8}(\lambda^2 u + 2\lambda u \cdot v + v \cdot v, u).$$

Expanding linearly the inner product in the above equation. By associativity of the inner product, we have

$$\lambda - \lambda^2 = \frac{5}{8}(u \cdot v, v). \tag{1}$$

Since a and u have length 1 and $\frac{8}{5}$ respectively, and u is orthogonal to v,

$$1 = (a, a) = (\lambda u + v, \lambda u + v) = \lambda^2(u, u) + 2\lambda(u, v) + (v, v) = \frac{8}{5}\lambda^2 + (v, v).$$

Hence

$$(v, v) = 1 - \frac{8}{5}\lambda^2. \tag{2}$$

In the decomposition of v as a linear sum of eigenvectors of u, the largest eigenvalue is $\frac{1}{3}$ as $v \in V_{\mathbb{M}} \setminus V^u_{\mathbb{M},1}$, Hence $(u \cdot v, v) \leq (\frac{1}{3}v, v)$. Replacing this inequality in equation (1) and substituting with equation (2) gives:

$$\lambda - \lambda^2 = \frac{5}{8}(u \cdot v, v) \leq \frac{5}{8}(\frac{1}{3}v, v) = \frac{5}{24}(1 - \frac{8}{5}\lambda^2) = \frac{5}{24} - \frac{1}{3}\lambda^2.$$

Rearrange to get

$$\frac{2}{3}\lambda^2 - \lambda + \frac{5}{24} \geq 0. \tag{3}$$

Solving equation (3), either $\lambda \leq \frac{1}{4}$ or $\lambda \geq \frac{5}{4}$. Assuming the latter, $(v, v) \leq -\frac{3}{2}$ contradicting the positive definiteness of $(\ ,\ )$. Hence $(a, u) = (\lambda u + v, u) = \frac{8}{5}\lambda \leq \frac{2}{5}$. Also $(a, u) \geq 0$ by Lemma 1.19.

□

Lemma 2.21. *Let $u_1, u_2 \in V_{\mathbb{M}}$ be two distinct $3A$-axes. Then*

$$0 \leq (u_1, u_2) \leq \frac{4}{5}.$$

Proof. This is similar to the proof of the preceding lemma. □

Remark. The previous two propositions are only true in the Monster algebra as the proof utilized the additional information regarding the eigenvalues of a $3A$-axis in $V_{\mathbb{M}}$. These results may not be extended to the $3A$-axes defined in an arbitrary Majorana algebra just yet. This is because nothing is yet deduced regarding the set of eigenvalues of a $3A$-axis based on its definition.

The next result gives a recursive formula for the algebra product of an arbitrary vector multiplied repeatedly by a $3A$-axis. This would ease calculations within the algebra provided that algebra products contained in the recursive formula are already known.

Proposition 2.22. *Let $u \in V_{\mathbb{M}}$ be a $3A$-axis and $v \in V_{\mathbb{M}}$ an arbitrary vector. Let ${}^{n}u$ be the function on $V_{\mathbb{M}}$ defined as*

$$v^{{}^{n}u} := (\cdots((v\underbrace{\cdot u)\cdot u)\cdots u)}_{n \text{ times}}$$

for any $v \in V_{\mathbb{M}}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} v^{{}^{n}u} =& \left(\frac{145}{5^n} - \frac{58}{3^n} - \frac{320}{30^n} + 1\right)\frac{5}{8}(u,v)u + \left(-\frac{5}{2\cdot 5^n} + \frac{1}{2\cdot 3^n} + \frac{40}{30^n}\right)v\cdot u + \\ & \left(\frac{165}{2\cdot 5^n} - \frac{35}{2\cdot 3^n} - \frac{320}{30^n}\right)(v\cdot u)\cdot u + \left(-\frac{225}{5^n} + \frac{75}{3^n} + \frac{600}{30^n}\right)((v\cdot u)\cdot u)\cdot u. \end{aligned}$$

Proof. It is first verified from the above formula, that $v^{{}^{n}u} = v\cdot u$, $(v\cdot u)\cdot u$ and $((v\cdot u)\cdot u)\cdot u$ for $n = 1, 2$ and 3, respectively. By Lemmas 2.11 and 2.19, v may be expressed as

$$v = \frac{5}{8}(u,v)u + \alpha + \beta + \gamma + \delta \tag{1}$$

where $\alpha \in V^u_{\mathbb{M},0}$, $\beta \in V^u_{\mathbb{M},\frac{1}{3}}$, $\gamma \in V^u_{\mathbb{M},\frac{1}{5}}$ and $\delta \in V^u_{\mathbb{M},\frac{1}{30}}$. Multiplying both sides of equation (1) on the right by u consecutively gives the following equations:

$$v^{{}^{1}u} = \frac{5}{8}(u,v)u + \frac{1}{3}\beta + \frac{1}{5}\gamma + \frac{1}{30}\delta; \tag{2}$$

$$v^{{}^{2}u} = \frac{5}{8}(u,v)u + \frac{1}{9}\beta + \frac{1}{25}\gamma + \frac{1}{900}\delta; \tag{3}$$

$$v^{{}^{3}u} = \frac{5}{8}(u,v)u + \frac{1}{27}\beta + \frac{1}{125}\gamma + \frac{1}{27000}\delta. \tag{4}$$

Solving the system of linear equations arising from equations (1), (2), (3) and (4) gives:

$$\alpha = v + 145(u,v)u - 38v^{{}^{1}u} + 255v^{{}^{2}u} - 450v^{{}^{3}u}; \tag{5}$$

$$\beta = -\frac{145}{4}(u,v)v + \frac{1}{2}v^{{}^{1}u} - \frac{35}{2}v^{{}^{2}u} + 75v^{{}^{3}u}; \tag{6}$$

$$\gamma = \frac{725}{8}(u,v)v - \frac{5}{2}v^{{}^{1}u} + \frac{165}{2}v^{{}^{2}u} - 225v^{{}^{3}u}; \tag{7}$$

$$\delta = -200(u,v)u + 40v^{{}^{1}u} - 320v^{{}^{2}u} + 600v^{{}^{3}u}. \tag{8}$$

Now multiply equation (4) both sides on the right by u and replace β, γ and δ with their expressions in equations (6), (7) and (8) respectively. This gives the following equation

$$v^{{}^{4}u} = \frac{29}{90}(u,v)u + \frac{1}{450}v^{{}^{1}u} - \frac{19}{225}v^{{}^{2}u} + \frac{17}{30}v^{{}^{3}u}.$$

The result follows by using the command **rsolve** in [Map16] to solve the recursive relation:

$$v^{{}^{n+3}u} = \frac{29}{90}(u,v)u + \frac{1}{450}v^{{}^{n}u} - \frac{19}{225}v^{{}^{n+1}u} + \frac{17}{30}v^{{}^{n+2}u}$$

with base case $v^{{}^{1}u} = v\cdot u$. □

Corollary 2.23. *Let ${}^nu$ be as in Proposition 2.22. Then $v^{{}^nu} \to \frac{5}{8}(u,v)v$ as $n \to \infty$ for all $v \in V_{\mathbb{M}}$.*

Proof. All terms except $\frac{5}{8}(u,v)v$ in Proposition 2.22 tend to 0 as n tends to infinity. □

The same process in the proof of Proposition 2.22 may be applied to any vector $x \in V_{\mathbb{M}}$ to obtain a formula for ${}^nx$ given that its full set of eigenvalues are known.

The following result is by Conway in [Con84] (*alternation implies association*). It is a direct consequence of Nortons inequality.

Lemma 2.24. *Let $x \in V_{\mathbb{M}}$ be an idempotent and y a 0-eigenvector of x. Then x and y associate in the sense that*

$$(x \cdot z) \cdot y = x \cdot (z \cdot y) \text{ for all } z \in V_{\mathbb{M}}.$$

Corollary 2.25. *Let $u \in V_{\mathbb{M}}$ be a $3A$-axis. Let v, w be a 0- and λ-eigenvector of u respectively where $\lambda \neq 1$. Then $w \cdot v$ is also a λ-eigenvector of u.*

Proof. By Lemma 2.24, u and v associate. Hence

$$u \cdot (w \cdot v) = (u \cdot w) \cdot v = (\lambda w) \cdot v = \lambda(w \cdot v).$$

□

2.2 Fusion rules

In the previous chapter, we saw how the eigenvectors of a Majorana axis obeyed certain fusion rules. These fusion rules dictate the linear span of eigenspaces that the product of two eigenvectors lie in. In the Monster algebra, these fusion rules are obeyed by the $2A$-axes (as the Monster algebra is a Majorana algebra with Majorana axes being precisely the $2A$-axes). In this section, we determine the fusion rules obeyed by the eigenvectors in the Monster algebra of a $3A$-axis, and deduce some of its implications.

The initial approach that we shall use to determine the fusion rules is by calculating them directly from low dimensional Majorana algebras which are isomorphic to subalgebras of the Monster algebra. Although the Norton-Sakuma algebra of type $3A$ is an elementary example of a Majorana algebra embedded in the Monster algebra and containing a $3A$-axis, it does not display the full spectrum of the eigenvalues (see Proposition 2.16) and hence the fusion rules in V_{3A} would not reflect the fusion rules in $V_{\mathbb{M}}$. Both of the $3A$-shape S_4-representations embed into the Monster algebra and contain $3A$-axes. These algebras display all the eigenvalues of a $3A$-axis present in the Monster algebra (Proposition 2.18). Working within these algebras, the following result is obtained:

Proposition 2.26. *Let V denote both of the 13-dimensional Majorana representations of S_4 with shape involving type $3A$ from Section 1.5, and let $u \in V$ be a $3A$-axis. The algebra products between the eigenvectors of u obey the following rule:*

$$v \cdot w \in \bigoplus_{\gamma \in S(\lambda,\mu)} V^u_\gamma$$

where $v \in V^u_\lambda$, $w \in V^u_\mu$ and $S(\lambda, \mu)$ is the (λ, μ)-entry of Table 2.1.

$\lambda \diagdown \mu$	1	0	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{30}$
1	1	0	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{30}$
0	0	0	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{30}$
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$1, 0$	$\frac{1}{30}$	$\frac{1}{3}, \frac{1}{30}$
$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{30}$	$1, 0, \frac{1}{3}$	$\frac{1}{5}, \frac{1}{30}$
$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{30}$	$\frac{1}{3}, \frac{1}{30}$	$\frac{1}{5}, \frac{1}{30}$	$1, 0, \frac{1}{5}, \frac{1}{3}, \frac{1}{30}$

Table 2.1: Fusion rules of a $3A$-axis (in S_4-representations).

Proof. It is sufficient to check for one $3A$-axis as all $3A$-axes in V lie in a single orbit under the action of $Aut(V) \cong S_4$. A basis B of V consisting of eigenvectors of u is first obtained. Let $\{x_1, x_2\}$ be a pair of vectors in B. Then the algebra product $x_1 \cdot x_2$ as a linear combination of vectors in B is worked out. It is then checked which linear span of eigenspaces does the linear combination lie in. This process is then repeated for all pairs in B. These calculations may be done efficiently on [Map16]. □

Remark. As the $3A$-shape representations of S_4 are based on embeddings in the Monster, the fusion rules in Table 2.1 may be viewed as a 'lower bound' for the fusion rules in the Monster algebra in the sense that there may be additional entries added to Table 2.1 for the complete fusion rules in $V_{\mathbb{M}}$.

It is neither convenient nor wise to use the method in the proof of Proposition 2.26 to calculate the fusion rules within the Monster algebra. The dimension is simply too large and it is unknown how to explicitly calculate within the Monster algebra. To obtain an 'upper bound' for the fusion rules, we look at results proven in the VOA context.

Remark. A self contained explanation to the theory of VOAs will not be included in this thesis. We shall only attempt to give a flavour to what it is.

A graded vertex operator algebra $V = \bigoplus_{i=0}^{\infty} V_i$ is an $\mathbb{N}$-graded vector space with a vacuum element $\mathbf{1} \in V$, equipped with infinitely many bilinear products $(n) : V \times V \to V$, and satisfying certain axioms (see [Roi08]). The famous Moonshine VOA $V_\natural$ constructed by Frenkel, Lepowsky and Meurman in [FLM84] is a specific VOA where $dim(V_0) = 1$ and $V_1 = 0$ (such VOA's are called 'One-Zero') and its weight 2 part $V_2^\natural$ is the Monster algebra. (In fact, the weight 2 part of any One-Zero VOA V is called the Griess algebra of V.) Therefore, results proven in the VOA context, when restricted appropriately may be applied to the Monster algebra.

A $2A$-axis in the Monster algebra is a conformal vector with central charge $\frac{1}{2}$ lying in $V_2^\natural$. Its fusion rules in the VOA context may be observed in *Section 3.1* of [Miy94], where $L(c, h)$ is an irreducible Vir-module with central charge c and highest weight h. The irreducible modules $L(\frac{1}{2}, 0)$, $L(\frac{1}{2}, \frac{1}{2})$ and $L(\frac{1}{2}, \frac{1}{16})$ of the simple Virasoro VOA $L(\frac{1}{2}, 0)$ may be interpreted as the 1- and 0-eigenspace, the $\frac{1}{4}$-eigenspace, and the $\frac{1}{32}$-eigenspace respectively of the $2A$-axis.

On the other hand, a $3A$-axis in the Monster algebra is a conformal vector with central charge $\frac{4}{5}$ in $V_2^\natural$. The simple Virasoro module $L(\frac{4}{5}, 0)$ known also as the 3-state Potts model, has 10 irreducible modules (see [Miy01]). The fusion rules involving the 1-and 0-, $\frac{1}{5}-$, $\frac{1}{3}$-, and $\frac{1}{30}$-eigenspaces may be extracted from the fusion rules in the VOA involving only the

irreducible modules $L(\frac{4}{5},0)$, $L(\frac{4}{5},\frac{2}{5})$, $L(\frac{4}{5},\frac{2}{3})$ and $L(\frac{4}{5},\frac{1}{15})$ respectively. Miyamoto listed the fusion rules involving these modules and two more in *Table A* of [Miy01]. Therefore, one only needs to exclude the irrelevant modules in Miyamoto's table to obtain the fusion rules in the Monster algebra.

$\lambda \diagdown \mu$	1	0	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{1}{30}$
1	$\star$	$\star$	$\star$	$\star$	$\star$
0	$\star$	$\star$	$\star$	$\star$	$\star$
$\frac{1}{5}$	$\star$	$\star$	1, 0	$\frac{1}{30}$	$\frac{1}{3}$, $\frac{1}{30}$
$\frac{1}{3}$	$\star$	$\star$	$\frac{1}{30}$	1, 0, $\frac{1}{3}$	$\frac{1}{5}$, $\frac{1}{30}$
$\frac{1}{30}$	$\star$	$\star$	$\frac{1}{3}$, $\frac{1}{30}$	$\frac{1}{5}$, $\frac{1}{30}$	1, 0, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{1}{30}$

Table 2.2: Restriction of Miyamoto's fusion rules.

Proposition 2.27. *Table 2.2 is the result of restricting Table A in [Miy01] to the* 5 *eigenvalues of a* $3A$*-axis in the Monster algebra.*

Although the first two rows and columns of Table 2.2 are not displayed, their entries for the Monster algebra may easily be deduced. The entries in the first row (and column) are obvious. The entries in the second row (and column) follow from Corollary 2.25 and they match the second row (and column) of Table 2.1. Hence the following result is deduced.

Theorem 2.28. *The fusion rules of the eigenvectors of a* $3A$*-axis in the Monster algebra is as in Table 2.1.*

Corollary 2.29. *Let* $u \in V_{\mathbb{M}}$ *be a* $3A$*-axis. Then*

$$V^u_{\mathbb{M},1},\quad V^u_{\mathbb{M},0},\quad V^u_{\mathbb{M},1}\oplus V^u_{\mathbb{M},0},\quad V^u_{\mathbb{M},1}\oplus V^u_{\mathbb{M},0}\oplus V^u_{\mathbb{M},\frac{1}{5}}\quad \textit{and}\quad V^u_{\mathbb{M},1}\oplus V^u_{\mathbb{M},0}\oplus V^u_{\mathbb{M},\frac{1}{3}}$$

are subalgebras of the Monster algebra of dimensions 1, 57478, 57479, 66149 *and* 59044 *respectively.*

Proof. This follows from Table 2.1 and Lemma 2.11. □

Corollary 2.30. *Let $u_h \in V_{\mathbb{M}}$ be a $3A$-axis and let $V = C_{V_{\mathbb{M}}}(h) = V^u_{\mathbb{M},1} \oplus V^u_{\mathbb{M},0} \oplus V^u_{\mathbb{M},\frac{1}{5}}$. Define $\sigma(u_h) : V \to V$ as the linear transformation negating every $\frac{1}{5}$-eigenvector of u_h and fixing the remaining eigenvectors. Then $\sigma(u_h)$ preserves the algebra product of the Monster algebra restricted to V.*

Proof. This follows from the fusion rules of u_h in Table 2.1. □

2.3 Splitting of the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces

In Theorem 2.14, we deduced how a $3A$-element in the Monster acts on the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces of the associated $3A$-axis in the Monster algebra. These eigenspaces split into two subspaces of equal dimension. The $3A$-element acts on one subspace as multiplication by ε, and on the other by ε^{-1}. Note that any future reference to the splitting of these eigenspaces refers to this description.

The fusion rules obeyed by vectors in these eigenspaces have now been established (see Theorem 2.28). Now that the action of a $3A$-element is known, the fusion rules with respect to the splitting may also be deduced.

Remark. The fusion rules of the splittings are known in the VOA context. See *Table B* of [Miy01].

Proposition 2.31. *Let $h \in 3A$ with associated $3A$-axis $u \in V_{\mathbb{M}}$. Table 2.3 describes the fusion rules of u in $V_{\mathbb{M}}$ with respect to the splitting of the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces.*

$\lambda \diagdown \mu$	1	0	$\frac{1}{5}$	$\frac{1}{3}\varepsilon$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon^{-1}$
1	1	0	$\frac{1}{5}$	$\frac{1}{3}\varepsilon$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon^{-1}$
0	0	0	$\frac{1}{5}$	$\frac{1}{3}\varepsilon$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon^{-1}$
$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	1, 0	$\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon$, $\frac{1}{30}\varepsilon$	$\frac{1}{3}\varepsilon^{-1}$, $\frac{1}{30}\varepsilon^{-1}$
$\frac{1}{3}\varepsilon$	$\frac{1}{3}\varepsilon$	$\frac{1}{3}\varepsilon$	$\frac{1}{30}\varepsilon$	$\frac{1}{3}\varepsilon$	1, 0	$\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{5}$
$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{30}\varepsilon^{-1}$	1, 0	$\frac{1}{3}\varepsilon^{-1}$	$\frac{1}{5}$	$\frac{1}{30}\varepsilon$
$\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon$	$\frac{1}{3}\varepsilon$, $\frac{1}{30}\varepsilon$	$\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{5}$	$\frac{1}{3}\varepsilon^{-1}$, $\frac{1}{30}\varepsilon^{-1}$	1, 0, $\frac{1}{5}$
$\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{3}\varepsilon^{-1}$, $\frac{1}{30}\varepsilon^{-1}$	$\frac{1}{5}$	$\frac{1}{30}\varepsilon$	1, 0, $\frac{1}{5}$	$\frac{1}{3}\varepsilon$, $\frac{1}{30}\varepsilon$

Table 2.3: Extended fusion rules of a $3A$-axis in $V_{\mathbb{M}}$.

Proof. It is sufficient to prove for one case as all other cases are similar. Let $v, w \in V^u_{\mathbb{M},\frac{1}{3}\varepsilon}$. Then by Table 2.1 and Theorem 2.14,

$$v \cdot w = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\alpha_3 + \lambda_4\alpha_4$$

where $\alpha_1 \in V^u_{\mathbb{M},1}$, $\alpha_2 \in V^u_{\mathbb{M},0}$, $\alpha_3 \in V^u_{\mathbb{M},\frac{1}{3}\varepsilon}$, $\alpha_4 \in V^u_{\mathbb{M},\frac{1}{3}\varepsilon^{-1}}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. Then

$$(v \cdot w)^h = \lambda_1\alpha_1^h + \lambda_2\alpha_2^h + \lambda_3\alpha_3^h + \lambda_4\alpha_4^h = \lambda_1\alpha_1 + \lambda_2\alpha_2 + \lambda_3\varepsilon\alpha_3 + \lambda_4\varepsilon^{-1}\alpha_4.$$

Also,

$$v^h \cdot w^h = (\varepsilon v) \cdot (\varepsilon w) = \varepsilon^{-1}(v \cdot w) = \lambda_1\varepsilon^{-1}\alpha_1 + \lambda_2\varepsilon^{-1}\alpha_2 + \lambda_3\varepsilon^{-1}\alpha_3 + \lambda_4\varepsilon^{-1}\alpha_4.$$

Since h preserves the algebra product,

$$\lambda_1(1-\varepsilon^{-1})\alpha_1 + \lambda_2(1-\varepsilon^{-1})\alpha_2 + (\varepsilon - \varepsilon^{-1})\lambda_3\alpha_3 = 0.$$

As the eigenvectors α_1, α_2 and α_3 are linearly independent, $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Hence $v \cdot w = \lambda_4\alpha_4 \in V^u_{\mathbb{M},\frac{1}{3}\varepsilon^{-1}}$. □

Lemma 2.32. *Let $u \in V_{\mathbb{M}}$ be a $3A$-axis. The bilinear form on $V_{\mathbb{M}}$ extended to $\mathbb{C}$ and restricted to the $\frac{1}{3}$- or $\frac{1}{30}$-eigenspace of u, is orthogonal with respect to the splitting, i.e.*

$$(v, w) = 0$$

for all $v, w \in V^u_{\mathbb{M},\eta\gamma}$, $\eta \in \{\frac{1}{3}, \frac{1}{30}\}$ and $\gamma \in \{\varepsilon, \varepsilon^{-1}\}$.

Proof. Let h be the $3A$-element associated to u. As h preserves the bilinear form,

$$\begin{aligned} &(v,w) = (v^h, w^h) = (\gamma v, \gamma w) = \gamma^{-1}(v,w) \\ \Rightarrow\ &(1-\gamma^{-1})(v,w) = 0 \\ \Rightarrow\ &(v,w) = 0. \end{aligned}$$

□

Remark. By the previous lemma, it is observed that the bilinear form on the Monster algebra extended to the complex numbers and restricted to a splitting of the $\frac{1}{3}$- or $\frac{1}{30}$-eigenspace of a $3A$-axis is no longer positive definite.

Lemma 2.33. *Let V be a subalgebra of $V_{\mathbb{M}}$ over $\mathbb{R}$ and which contains a $3A$-axis u. Let $h \in \mathbb{M}$ be the $3A$-element associated to u. Suppose also that V is an $\mathbb{R}\langle h\rangle$-module. Then $V \cap V^u_{\mathbb{M},\eta}$ is even dimensional for $\eta \in \{\frac{1}{3}, \frac{1}{30}\}$. Moreover,*

$$dim(V \cap V^u_{\mathbb{M}\eta\varepsilon}) = dim(V \cap V^u_{\mathbb{M}\eta\varepsilon^{-1}}).$$

Proof. By Lemma 2.11, V may be decomposed into eigenspaces of u:

$$V = (V \cap V^u_{\mathbb{M},1}) \oplus (V \cap V^u_{\mathbb{M},0}) \oplus (V \cap V^u_{\mathbb{M},\frac{1}{5}}) \oplus (V \cap V^u_{\mathbb{M},\frac{1}{3}}) \oplus (V \cap V^u_{\mathbb{M},\frac{1}{30}}).$$

Since the $3A$-element h acts on each of these eigenspaces separately, each eigenspace is also an $\mathbb{R}\langle h\rangle$-module. Therefore there exists a basis B of $V \cap V^u_{\mathbb{M},\eta}$ such that the entries of $[h]_B$, the matrix of the action of h on $V \cap V^u_{\mathbb{M},\eta}$ with respect to B, are real. By Theorem 2.14, $V \cap V^u_{\mathbb{M},\eta}$ may be decomposed as

$$V \cap V^u_{\mathbb{M},\eta} = (V \cap V^u_{\mathbb{M},\eta\varepsilon}) \oplus (V \cap V^u_{\mathbb{M},\eta\varepsilon^{-1}}).$$

Let C be a basis of $V \cap V^u_{\mathbb{M},\eta}$ with respect to this splitting. Then $[h]_C$ is diagonal with entries in $\{\varepsilon, \varepsilon^{-1}\}$. Since $[h]_B$ is similar to $[h]_C$ over $\mathbb{C}$, they must have the same determinant. The determinant of $[h]_C$ is real if and only if $V \cap V^u_{\mathbb{M},\eta\varepsilon}$ has the same dimension as $V \cap V^u_{\mathbb{M},\eta\varepsilon^{-1}}$. The dimension of $V \cap V^u_{\mathbb{M},\eta}$ is therefore even. □

Proposition 2.34. *The orthogonality condition from Lemma 2.32 does not uniquely determines how the $\frac{1}{3}$- or $\frac{1}{30}$-eigenspaces of a $3A$-axis split.*

Proof. Let $u \in V_{\mathbb{M}}$ be a $3A$-axis. Assume $V^u_{\mathbb{M},\eta}$ for $\eta \in \{\frac{1}{3}, \frac{1}{30}\}$ splits into two subspaces of equal dimension:

$$V^u_{\mathbb{M},\eta} = V \oplus W,$$

such that

$$(v_1, v_2) = (w_1, w_2) = 0$$

for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$. We will show that V and W are not necessary $V^u_{\mathbb{M},\eta\varepsilon}$ and $V^u_{\mathbb{M},\eta\varepsilon^{-1}}$. Without loss of generality, let $\{e_1, \dots, e_m\}$ and $\{f_1, \dots, f_m\}$ be bases for $V^u_{\mathbb{M},\eta\varepsilon}$ and $V^u_{\mathbb{M},\eta\varepsilon^{-1}}$ respectively such that $(e_i, f_j) = \delta_{i,j}$ for all i, j. Define $\tilde{e_i} := e_i$ for $1 \leq i \leq m$. Define also the vectors $\tilde{f_i}$ as:

$$\begin{aligned}
\tilde{f_1} &:= \sum_{k=2}^{m} e_k + f_1; \\
\tilde{f_i} &:= -\sum_{k=1}^{i-1} e_k + \sum_{k=i+1}^{m} e_k + f_i, \quad 2 \leq i \leq m-1; \\
\tilde{f_m} &:= -\sum_{k=1}^{m-1} e_k + f_m.
\end{aligned}$$

Let V be the subspace spanned by $\{\tilde{e_1}, \dots, \tilde{e_m}\}$, and W the subspace spanned by $\{\tilde{f_1}, \dots, \tilde{f_m}\}$. Then V and W satisfy the conditions of the proposition. Moreover $(\tilde{e_i}, \tilde{f_j}) = \delta_{i,j}$ for all i, j. However, the $3A$-element associated to u clearly does not act on W as scalar multiplication by ε^{-1}. □

By the previous proposition, the trivial condition for the bilinear form does not uniquely determine the splitting of $V^u_{\mathbb{M},\eta}$. Given a basis of the $\frac{1}{3}$- or $\frac{1}{30}$-eigenspace of a $3A$-axis in the Monster algebra, we conjecture that the fusion rules in Table 2.3 are sufficient to determine the splitting. This is motivated by the following two results which concerns subalgebras of the Monster algebra isomorphic to Majorana algebras of S_4-representations.

Proposition 2.35. *Let V denote the 13-dimensional Majorana representation of S_4 with shape $(2A, 3A)$, and let $u \in V$ be a $3A$-axis. The $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces of u in V split into two subspaces, both 2-dimensional:*

$$V^u_{\frac{1}{3}} = V^u_{\frac{1}{3}\varepsilon} \oplus V^u_{\frac{1}{3}\varepsilon^{-1}} \quad \textit{and} \quad V^u_{\frac{1}{30}} = V^u_{\frac{1}{30}\varepsilon} \oplus V^u_{\frac{1}{30}\varepsilon^{-1}}.$$

The 3-cycle in S_4 corresponding to u acts on $V^u_{\eta\gamma}$ as scalar multiplication by γ for $\eta \in \{\frac{1}{3}, \frac{1}{30}\}$ and $\gamma \in \{\varepsilon, \varepsilon^{-1}\}$.

Proof. This may be verified by hand. □

The notation in the previous proposition is used for the next proposition.

Proposition 2.36. *Let V be from Propositon 2.35. The fusion rules in Table 2.3 are sufficient to determine the splitting of $V^u_{\frac{1}{3}}$. If the splitting of $V^u_{\frac{1}{3}}$ is known explicitly, then this together with the fusion rules are sufficient to determine the splitting of $V^u_{\frac{1}{30}}$.*

Proof. Let $\{b_1, b_2, b_3, b_4\}$ be a basis of $V^u_{\frac{1}{3}}$, and let v an arbitrary $\frac{1}{3}$-eigenvector of u. Then

$$v = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 + \lambda_4 b_4$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. Assume v is in only one subspace of the splitting of $V^u_{\frac{1}{3}}$. Then by Table 2.3, $v \cdot v$ is also a $\frac{1}{3}$-eigenvector of u. Hence

$$u \cdot (v \cdot v) - \frac{1}{3}(v \cdot v) = 0.$$

Equating the coefficients of a basis of V containing $\{b_1, b_2, b_3, b_4\}$ to 0 in the equation above then solving the system of equations, either $v \in V^u_{\frac{1}{3}\varepsilon}$ or $v \in V^u_{\frac{1}{3}\varepsilon^{-1}}$.

Now let $\{d_1, d_2, d_3, d_4\}$ be a basis of $V^u_{\frac{1}{30}}$, and let w be an arbitrary $\frac{1}{30}$-eigenvector of u. Then

$$w = \mu_1 d_1 + \mu_2 d_2 + \mu_3 d_3 + \mu_4 d_4$$

where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{C}$. Let v_1 and v_2 be arbitrary vectors in $V^u_{\frac{1}{3}\varepsilon}$ and $V^u_{\frac{1}{3}\varepsilon^{-1}}$, respectively. Firstly assume $w \in V^u_{\frac{1}{30}\varepsilon}$. Then by Table 2.3, $v_1 \cdot w \in V^u_{\frac{1}{30}\varepsilon^{-1}}$. Hence

$$u \cdot (v_1 \cdot w) - \frac{1}{30}(v_1 \cdot w) = 0.$$

Equating the coefficients of a basis of V containing $\{d_1, d_2, d_3, d_4\}$ to 0 in the equation above then solving the system of equations, v lies in a 3-dimensional subspace of $V^u_{\frac{1}{30}}$. Also by Table 2.3, $v_2 \cdot w \in V^u_{\frac{1}{5}}$. Hence

$$u \cdot (v_2 \cdot w) - \frac{1}{5}(v_2 \cdot w) = 0.$$

Solve this equation to get $w \in V^u_{\frac{1}{30}\varepsilon}$. Similarly if w is assumed to satisfy the fusion rules affecting $V^u_{\frac{1}{30}\varepsilon^{-1}}$, then $w \in V^u_{\frac{1}{30}\varepsilon^{-1}}$. □

2.4 The standard $3A$-axes

In the spirit of the axiomatic approach of Majorana theory, we introduce the notion of a standard $3A$-axis. The defining properties of a standard $3A$-axis are taken from some of the key properties of the $3A$-axes in the Monster algebra deduced in the previous sections of this chapter.

Definition 2.37. *Let V be a real vector space equipped with a commutative, non-associative algebra product $\cdot$ and an inner product $(\ ,\ )$ which associates with $\cdot$. A vector $u \in V$ is said to be a* ***standard 3A-axis*** *if it is an idempotent of length $\frac{8}{5}$, such that*

(i) V is a direct sum of eigenspaces of u, of which the eigenvalues are from the set $\{1, 0, \frac{1}{5}, \frac{1}{3}, \frac{1}{30}\}$, and 1 is a simple eigenvalue;

(ii) Extending to the field $\mathbb{C}$, the eigenspaces $V^u_{\frac{1}{3}}$ and $V^u_{\frac{1}{30}}$ split into two subspaces of equal dimension:

$$V^u_{\frac{1}{3}} = V^u_{\frac{1}{3}\varepsilon} \oplus V^u_{\frac{1}{3}\varepsilon^{-1}} \quad \text{and} \quad V^u_{\frac{1}{30}} = V^u_{\frac{1}{30}\varepsilon} \oplus V^u_{\frac{1}{30}\varepsilon^{-1}};$$

(iii) The linear transformation $\phi(u)$ of V which acts as a scalar multiplication by γ on $V^u_{\eta\gamma}$ for $\eta \in \{\frac{1}{3}, \frac{1}{30}\}$ and $\gamma \in \{\varepsilon, \varepsilon^{-1}\}$, and fixes the remaining eigenvectors, preserves the algebra product:

$$(v \cdot w)^{\phi(u)} = v^{\phi(u)} \cdot w^{\phi(u)} \text{ for all } v, w \in V;$$

(iv) The linear transformation $\theta(u)$ of $C_V(\phi(u))$ which negates every $\frac{1}{5}$-eigenvector of u and fixes the remaining eigenvectors, preserves the algebra product restricted to $C_V(\phi(u))$;

(v) The algebra product between eigenvectors of u obey the fusion rules in Table 2.3.

Remark. This axiomatization of a $3A$-axis in the Monster algebra as a standard $3A$-axis in an arbitrary algebra may be justified as it is far from clear if these properties may be deduced from a $3A$-axis described in terms of Majorana axes in Definition 2.1.

Corollary 2.38. *Let (V, A) be a Majorana representation of (G, T) containing a $3A$-axis u. If V is based on an embedding in the Monster, then u is also a standard $3A$-axis in V.*

Proof. There is a group embedding $\varrho : G \to \mathbb{M}$ such that $\varrho(T) \subseteq 2A$ and the map $a_t \in A \mapsto \varphi_{\mathbb{M}}(\varrho(t)) \in A_{\mathbb{M}}$, $t \in T$, defines an isomorphism of algebras between V and $\bar{V} := \langle\langle \varphi_{\mathbb{M}}(\varrho(T))\rangle\rangle \leq V_{\mathbb{M}}$. Since u is defined in terms of Majorana axes in A, it is embedded to a $3A$-axis $\bar{u} \in V_{\mathbb{M}}$. By Lemma 2.33, the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces of $\bar{u}$ in $\bar{V}$ split into two subspaces of equal dimension. The $3A$-element $h \in \mathbb{M}$ associated to $\bar{u}$ acts on $\bar{V}$, preserving the algebra product on $\bar{V}$. It also acts on the two subspaces of the split eigenspaces as scalar multiplication by ε and ε^{-1}, respectively. Moreover, h acts on the direct sum of the 1- 0- and $\frac{1}{5}$-eigenspaces of $\bar{u}$ in $\bar{V}$ as in Corollary 2.30. Since there is an isomorphism of modules between V and $\bar{V}$, these properties are also true for u in V. □

Since most of the results regarding the $3A$-axes in the Monster algebra proven in previous sections of this chapter used only the properties in Definition 2.37, those results are also true for the standard $3A$-axes in some algebra V. The linear transformation $\phi(u)$ replaces the action of the associated $3A$-element on the Monster algebra. Since $\mathbb{M}$ is the automorphism group of $V_{\mathbb{M}}$, it is obvious that a $3A$-element acts as an isometry with respect to the inner product on $V_{\mathbb{M}}$. This is also true for the standard $3A$-axes.

Lemma 2.39. *Let $u \in V$ be a standard $3A$-axis. Then $\phi(u)$ preserves the inner product on V:*

$$(v^{\phi(u)}, w^{\phi(u)}) = (v, w) \text{ for all } v, w \in V.$$

Proof. This follows from Definition 2.37. □

2.5 The A_4-algebra

In this section, we use a different approach to construct an algebra containing Majorana axes but which is not necessarily a Majorana algebra. We aim to construct an algebra associated to A_4, the alternating group of degree 4. This algebra denoted V_{A_4} shall be

constructed from the properties of Majorana and standard $3A$-axes. In addition, we shall assume that V_{A_4} is spanned by the Majorana and standard $3A$-axes introduced. Since A_4 is not generated by involutions, by definition, it does not have a Majorana representation. There is an A_4-subgroup of the Monster acting on some subalgebra of the Monster algebra as seen from Majorana representations of S_4 with shape $(2A, 3A)$. In this representation, involutions in A_4 are embedded into the $2A$ conjugacy class while elements of order 3 are embedded into the $3A$ conjugacy class.

Let $A_4 := \langle (1,2)(3,4), (2,3,4) \rangle$. For each involution $t \in A_4$, introduce a Majorana axis a_t. Let

$$a_1 := a_{(1,2)(3,4)},\ a_2 := a_{(1,3)(2,4)} \text{ and } a_3 := a_{(1,4)(2,3)}.$$

For each cyclic subgroup $\langle h \rangle$ of order 3 in A_4, introduce a standard $3A$-axis u_h. Let

$$u_1 := u_{(2,3,4)},\ u_2 := u_{(1,3,4)},\ u_3 := u_{(1,2,3)} \text{ and } u_4 := u_{(1,2,4)}.$$

Let V_{A_4} be a real vector space spanned by $X := \{a_1, a_2, a_3, u_1, u_2, u_3, u_4\}$. It is also equipped with a commutative, non-associative algebra product $\cdot$ and and inner product $(\ ,\ )$. It is assumed that V_{A_4} is closed under the algebra product, i.e. the algebra product between any two vectors in X lies in the linear span of X. This is motivated by the construction of the S_4-representation with shape $(2B, 3A)$, whereby $4A$-axes were added into the representation for each cyclic subgroup of order 4 in S_4. Together with the Majorana and $3A$-axes already present as described in the generic strategy, these vectors were sufficient to span the whole algebra.

For each Majorana axis a_i, $i \in \{1, 2, 3\}$, the linear transformation $\tau(a_i)$ negates the $\frac{1}{32}$-eigenvectors of a_i and fixes the remaining eigenvectors in V_{A_4}. For each standard $3A$-axis u_j, $j \in \{1, 2, 3, 4\}$, the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces of u_j split into two subspaces of equal dimension. Extending to the complex numbers, the linear transformation $\phi(u_j)$ acts on one subspace as scalar multiplication by ε, and on the other by ε^{-1}. All other eigenvectors of u_j are fixed by $\phi(u_j)$.

Additionally, $\tau(a_i)$ and $\phi(u_j)$ are set to act on X by permuting the subscripts. In terms

of permutations, they are:

$$\tau(a_1) = (u_1, u_2)(u_3, u_4),\ \tau(a_2) = (u_1, u_4)(u_2, u_3),\ \tau(a_3) = (u_1, u_3)(u_2, u_4),$$

$$\phi(u_1) = (a_1, a_2, a_3)(u_2, u_4, u_3),\ \phi(u_2) = (a_1, a_3, a_2)(u_1, u_4, u_3),$$

$$\phi(u_3) = (a_1, a_3, a_2)(u_1, u_2, u_4),\ \phi(u_4) = (a_1, a_2, a_3)(u_1, u_2, u_3).$$

Based on this embedding of A_4 in the Monster, set $\{a_1, a_2, a_3\}$ to span a Norton-Sakuma algebra of type $2A$. Then

$$a_i \cdot a_j = \frac{1}{8}(a_i + a_j - a_k) \text{ and } (a_i, a_j) = \frac{1}{8}$$

for $i, j, k \in \{1, 2, 3\},\ i \neq j,\ k \neq i, j$.

What needs to be determined are the unknown algebra products, i.e. $a_i \cdot u_j$ and $u_i \cdot u_j$ for all i, j. Since τ and ϕ preserve the algebra product, the pairs of axes in X may be grouped into orbits under the action of the group generated by all $\tau(a_i)$ and $\phi(u_j)$. There is a single orbit for all pairs $\{a_i, u_j\}$. There is also a single orbit for all pairs $\{u_i, u_j\}$ where $i \neq j$. It is therefore sufficient to determine the products for one representative in these orbits, for example $\{a_1, u_1\}$ and $\{u_1, u_2\}$.

The first step is to determine the multiplicities of the eigenvalues of a Majorana axis. In the Norton-Sakuma subalgebra of type $2A$ spanned by $\{a_1, a_2, a_3\}$, the 1-, 0- and $\frac{1}{4}$-eigenspaces of a_1 are all 1-dimensional. In V_{A_4}, the $\frac{1}{32}$-eigenspace of a_1 is spanned by $\{v - v^{\tau(a_1)} \mid v \in V_{A_4}\} = \langle u_1 - u_2, u_3 - u_4 \rangle$. As 1 is a simple eigenvalue, there can be only three possibilities: Let $\{a, b\}$ be the multiplicities of the 0- and $\frac{1}{4}$-eigenvalues. Then $\{a, b\} = \{3, 1\}$, $\{1, 3\}$ or $\{2, 2\}$.

The unknown products $a_1 \cdot u_1$ and $u_1 \cdot u_2$ may be expressed as a linear combination in X with unknown coefficients. Through a series of arguments, e.g. fusion rules, decomposition of V_{A_4} into eigenspaces, comparing coefficients, determinants, solving polynomial equations etc (refer to Appendix A for full details), it may be deduced that only the last possibility is valid, i.e. the multiplicity of the 0- and $\frac{1}{4}$-eigenvalues are both 2. Some of the unknown coefficients in the expression for $a_1 \cdot u_1$ would also be deduced.

The next step is to determine the multiplicities of the eigenvalues of u_1. Through a similar process, it may be deduced that the 1-, 0- and $\frac{1}{5}$-eigenvalues all have multiplicity 1. The $\frac{1}{3}$- and $\frac{1}{30}$-eigenvalues both have multiplicity 2. The algebra products $a_1 \cdot u_1$ and $u_1 \cdot u_2$ are also known by this point. They are:

$$a_1 \cdot u_1 = \frac{1}{9}a_1 + \frac{5}{64}u_1 + \frac{3}{64}u_4 - \frac{1}{16}(u_2 + u_3);$$
$$u_1 \cdot u_2 = \frac{128}{2025}a_1 - \frac{64}{675}(a_2 + a_3) + \frac{1}{5}(u_1 + u_2) - \frac{1}{18}(u_3 + u_4).$$

All other unknown algebra products may be deduced from the products above by permuting the axes under τ and ϕ.

The inner products may be deduced from orthogonality between eigenvectors of either a Majorana or a standard $3A$-axis. They are:

$$(a_1, u_1) = \frac{1}{9} \quad \text{and} \quad (u_1, u_2) = \frac{136}{405}.$$

As τ and ϕ are isometries, the inner product between any Majorana and standard $3A$-axis or between any two distinct standard $3A$-axis, are the values above. The set X is also linearly independent and hence is a basis of V_{A_4}.

It is observed that V_{A_4} is isomorphic to the subalgebra of the Majorana representation of S_4 with shape $(2A, 3A)$ spanned by Majorana axes associated to double transpositions and $3A$-axes associated to three cycles.

The method described in the previous chapter for classifying idempotents of a particular length may be applied to V_{A_4}. For this algebra, it was computationally possible to identify all the idempotents of all lengths.

Proposition 2.40. *V_{A_4} has exactly* 104 *idempotents. The Majorana axes a_1, a_2, a_3 and standard $3A$-axes u_1, u_2, u_3, u_4 are the only idempotents of length* 1 *and* $\frac{8}{5}$ *respectively.*

Proof. This may be shown on [Map16] using the method described in Section 1.5. □

Remark. The construction of V_{A_4} from properties of the Majorana and standard $3A$-axes is tedious and is not ideal especially with larger dimensional algebras.

By definition, V_{A_4} is not a Majorana algebra since it is not generated by Majorana axes. By Proposition 2.40, a_1, a_2 and a_3 are the only Majorana axes which together generate a Norton-Sakuma algebra to type $2A$. It was also mentioned earlier that A_4 does not have a Majorana representation as A_4 is not generated by involutions. However, there is a natural representation of the group $\langle \tau(a_1), \tau(a_2), \tau(a_3), \phi(u_1), \phi(u_2), \phi(u_3), \phi(u_4) \rangle \cong A_4$ on V_{A_4} and this may be embedded into the action of the Monster on the Monster algebra. The generic strategy does not address such representations directly. They may however be obtained as subrepresentations of subgroups of a group whose Majorana representation has already been constructed, just as in this case: V_{A_4} as a subrepresentation of the $(2A, 3A)$-shape representation of S_4.

Suppose it is known explicitly how a group G embeds into the Monster and we want to construct representations of G based on this embedding. Including a standard $3A$-axes into the representation to correspond to $3A$-cyclic subgroups in G could be a worthwhile improvement to the generic strategy. For this to be effective, we would need a classification of subalgebras of the Monster algebra generated by a pair consisting of a Majorana and $3A$-axis, or at least know how to multiply between these two axes. This shall be investigated in the next chapter.

Chapter 3

The $(2A, 3A)$-pairs

The classification of algebras generated by pairs of Majorana axes has been the main ingredient for constructing new Majorana representations. In many cases, determining the types of Norton-Sakuma algebras generated by all pairs of Majorana axes in the generating set determines the structure of the whole algebra. Once we fix the shape, there will be instances where $3A$-axes become present. It seems a natural step forward would be to construct algebras generated by a $2A$- and $3A$-axis and obtain some sort of classification. Unlike the Norton-Sakuma algebras, we shall see that in algebras generated by a $2A$- and $3A$-axis, the eigenspaces of the $3A$-axis does not necessarily split and there are no automorphisms of order 3.

The first section of this chapter restates an important result by Norton regarding the orbits of the Monster on pairs in $\mathbb{M}$ consisting of a $2A$-involution and $3A$-element. From this, we shall deduce information regarding $(2A, 3A)$-pairs in a Majorana representation of A_{12} based on an embedding in the Monster. Constructing this Majorana representation is an open and ongoing problem. Then in Section 3.3, the generic strategy will be used to explicitly construct Majorana representations of certain groups generated by, or containing groups generated by a $2A$-involution or $3A$-element. From this, we obtain a description for the algebra products between a $2A$- and $3A$-axis for three new cases. Finally in Section 3.5, we present subalgebras of the Monster algebra generated by a $2A$- and a $3A$-axis.

3.1 Norton's list

In [Nor96], Norton classified the orbits of the centralizer in the Monster of a $3A$-element, on the conjugacy class of $2A$-involutions. This is equivalent to classifying the orbits of the Monster on pairs consisting of a $2A$-involution and a $3A$-element (or $(2A, 3A)$-pairs). There are only 22 such orbits.

t	h	$\langle t, h\rangle$	$(th)^{\mathbb{M}}$	(a_t, u_h)
$(1,2)(4,5)$	$(1,2,3)$	S_3	$2A$	$\frac{1}{4}$
$(4,5)(6,7)$	$(1,2,3)$	6	$6A$	0
$(1,2)(3,4)$	$(1,2,3)$	A_4	$3A$	$\frac{1}{9}$
$(1,4)(5,6)$	$(1,2,3)$	S_4	$4B$	$\frac{1}{36}$
$(1,4)(2,5)$	$(1,2,3)$	A_5	$5A$	$\frac{1}{18}$
$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$	$(1,2,3)$	$2 \times A_4$	$6C$	$\frac{1}{45}$
$(1,4)(2,5)(3,6)(7,8)(9,10)(11,12)$	$(1,2,3)$	$3 \times S_3$	$6A$	$\frac{1}{20}$
$(1,7)(2,4)$	$(1,2,3)(4,5,6)$	$L_2(7)$	$7A$	$\frac{1}{24}$
$(1,7)(2,8)$	$(1,2,3)(4,5,6)$	$3 \times A_5$	$15A$	$\frac{11}{360}$
$(1,7)(4,8)$	$(1,2,3)(4,5,6)$	S_4	$4A$	$\frac{13}{180}$
$(1,7)(8,9)$	$(1,2,3)(4,5,6)$	$3 \times S_4$	$12C$	$\frac{1}{36}$
$(1,2)(3,4)(5,7)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$2 \times L_2(7)$	$14A$	$\frac{11}{360}$
$(1,2)(3,7)(4,5)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$2 \times A_4$	$6A$	$\frac{2}{45}$
$(1,2)(3,7)(4,8)(5,9)(6,10)(11,12)$	$(1,2,3)(4,5,6)$	$S_3 \times A_4$	$6B$	$\frac{17}{360}$
$(1,4)(2,7)(3,8)(5,9)(6,10)(11,12)$	$(1,2,3)(4,5,6)$	$2 \times A_5$	$10A$	$\frac{1}{30}$
$(1,4)(2,8)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$3^3.A_4$	$9A$	$\frac{13}{360}$
$(1,5)(2,4)(3,7)(6,10)(8,9)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$4^2.S_3$	$8B$	$\frac{1}{36}$
$(1,2)(3,4)(5,7)(6,10)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$L_2(11)$	$11A$	$\frac{1}{30}$
†	†	$3 \times S_3$	$6D$	$\frac{1}{40}$
†	†	$GL_2(3)$	$8C$	$\frac{1}{36}$
†	†	$SL_2(3) : 2$	$12A$	$\frac{1}{30}$
†	†	$2.4^2.S_3$	$8A$	$\frac{7}{180}$

Table 3.1: Orbits of $\mathbb{M}$ on its $(2A, 3A)$-pairs.

Norton listed representatives for these orbits in *Table 3* of [Nor96]. His results are restated in Table 3.1. Norton also listed the centralizers of $\langle t, h\rangle$ in $\mathbb{M}$. One may check using the orbit-stabilizer theorem (Theorem 0.7) that the sum of indices of these subgroups in $3.Fi'_{24}$ is the number of $2A$-involutions in $\mathbb{M}$. Of the 22 orbits, the first 18 have representatives contained in a subgroup isomorphic to A_{12}, which is the centralizer of another subgroup of the Monster isomorphic to A_5. Representatives for the last four orbits may be found in subgroups S_{12}, $2S_4$ which centralizes a Tits group, $2^{1+24}.Co_1$ and $2^2.M_{21}.2$ which centralizes a $14A$-element respectively (see *Section 6* of [Nor96]).

The first two columns of Table 3.1 are the representatives for $(2A, 3A)$-pairs (t is a $2A$-involution and h is a $3A$-element). The column $\langle t, h\rangle$ is the isomorphism class of the group generated by t and h, while $(th)^{\mathbb{M}}$ is the conjugacy class of the product th in the Monster. The last column is the value of the inner product between the $2A$-axis a_t associated to t, and the $3A$-axis u_h associated to h.

Representatives in the first 18 rows of Table 3.1 are presented as elements of the standard form of A_{12}, i.e. as even permutations of the set $\{1, \ldots, 12\}$.

† Representative for the 19th to 22nd orbits are omitted from Table 3.1 as it is the A_{12}-subgroup which we are mainly concerned in this thesis. They may however be referred in [Nor96].

3.2 The A_{12}-subgroup

In Section 3.1, we mentioned that an A_{12}-subgroup of the Monster centralizing an A_5-subgroup contains representations of exactly 18 orbits of the Monster on its $(2A, 3A)$-pairs. The A_5-subgroup in question is in a unique class of A_5-subgroups of the Monster containing $2A$-, $3A$- and $5A$-elements (see *Table 3* of [Nor98]). In fact, there are only two conjugacy classes of A_5-subgroups in the Monster containing $2A$-involutions, as seen in the following result. A proof may be found in *Proposition 2.2* of [ISe12].

Theorem 3.1. *There are precisely two conjugacy classes of monomorphisms of A_5 into the Monster under the condition that the involutions in A_5 are mapped to the $2A$-conjugacy class. One monomorphism maps the elements of order 3 to $3A$ while the other to $3C$.*

Remark. Since the two types of A_5-subgroups from Theorem 3.1 are generated by $2A$-involutions, they possess Majorana representations. These representations were constructed in [ISe12] using the generic strategy. One is 26-dimensional while the other is 20-dimensional.

Let $H \leq \mathbb{M}$ such that $H \cong A_5$ and the non-trivial elements of H are in conjugacy classes $2A$, $3A$ and $5A$. Denote by A, the centralizer of H in $\mathbb{M}$ throughout the remaining of this thesis. This subgroup will be a main focus of study. The following result may be referred in *Table 1* of [Nor98] or *Section 5.8.4* of [Wil09].

Theorem 3.2. *A is isomorphic to A_{12}, and H is the centralizer of A in $\mathbb{M}$. Hence H and A form a pair of mutually centralizing subgroups in the Monster:*

$$A := C_{\mathbb{M}}(H) \cong A_{12} \quad \textit{and} \quad C_{\mathbb{M}}(A) = H \cong A_5.$$

Let $X := N_{\mathbb{M}}(A)$ $(= N_{\mathbb{M}}(H))$. Then X is maximal in $\mathbb{M}$ and

$$X = (A \times H) : 2 \cong (A_{12} \times A_5) : 2.$$

There has been much interest of late in constructing the Majorana representation of A_{12} based on the embedding of A in the Monster (A is generated by $2A$-involutions, a corollary of Theorem 3.3). The symmetric and alternating groups are a very accessible class of groups to work with, and 12 is the largest degree of alternating groups contained in the Monster. There are also certain relations established in [CRI14], which are only true in the Majorana algebra associated to A_{12}. These relations are not true in Majorana algebras associated to lower degree alternating groups. This phenomena shows that much of the linear independence between axes begin to collapse in this algebra. Therefore constructing this representation is of great significance as it is regarded as the key to unlocking the Monster algebra.

The following result is partially from *Lemma 6* of [Nor96], but is also clear from Table 3.1. For convenience, A shall always be presented as the group of even permutations of $\{1, \ldots, 12\}$.

Theorem 3.3. *The $2A$-involutions in A are involutions with cycle shape (2^2) or (2^6). The $3A$-elements in A are elements with cycle shape (3), (3^2) or (3^4).*

Since the Majorana representation of A_{12} is based on the embedding of A the Monster, it is important to know how the orbits of A on its $(2A, 3A)$-pairs fuse into orbits of the Monster. We may deduce these fusions from certain observations in Table 3.1.

It appears that the isomorphism classes of the groups generated by pairs in the 18 orbits intersecting A are mostly different. There are only two cases where pairs in two different orbits generate the same group up to isomorphism. This happens when the group generated is isomorphic to S_4 or $2 \times A_4$. For all other cases, the isomorphism class of the group generated distinguishes the orbit. Let t_1 and t_2 be $2A$-involutions in A, and h_1 and h_2, $3A$-elements in A. If

$$\langle t_1, h_1 \rangle \cong \langle t_2, h_2 \rangle \ncong S_4 \text{ or } 2 \times A_4,$$

then $\{t_1, h_1\}$ is conjugate to $\{t_2, h_2\}$ in the Monster.

For cases where $\langle t_1, h_1 \rangle$ is isomorphic to S_4 or $2 \times A_4$, it is clear from Table 3.1 that what distinguishes the orbits is the conjugacy class of the product $t_1 h_1$. For the S_4 case, the product lies in either the $4A$ or $4B$ conjugacy class, and for the $2 \times A_4$ case, it is either $6A$ or $6C$. The following result identifies the conjugacy classes in the Monster of these elements by their cycle shapes.

Lemma 3.4. *Let $t, h \in A$ be a $2A$-involution and a $3A$-element respectively.*

(i) Suppose $\langle t, h \rangle \cong S_4$. If th has cycle shape (4^2) or $(4^2, 2^2)$, then $th \in 4A$. If th has cycle shape $(4, 2)$ or $(4, 2^3)$, then $th \in 4B$.

(ii) Suppose $\langle t, h \rangle \cong 2 \times A_4$. If th has cycle shape $(3^2, 2^2)$, $(6, 2^3)$ or (6^2), then $th \in 6A$. If th has cycle shape $(3, 2^4)$ or $(6, 2)$, then $th \in 6C$.

Proof. Let $x \in A$ have cycle shape (4^2) or $(4^2, 2^2)$. Then x^2 has cycle shape (2^4) which is not $2A$ by Theorem 3.3. It is therefore $2B$ since the Monster has only two conjugacy classes of involutions. It may be referred from the conjugacy class table of the Monster in [AgrV3], that the square of a $4A$-element is a $2B$-involution. Hence elements of cycle shape (4^2) or $(4^2, 2^2)$ are $4A$-elements. Similarly for the other cycle shapes. Their conjugacy classes may be deduced by looking at the cycle shapes of squares or cubes of the element. The conjugacy classes for all powers of elements in the Monster are listed in [AgrV3]. □

Lemma 3.5. *Let $t, h \in A$ be a $2A$-involution and $3A$-element respectively. Then $\{t, h\}$ is conjugate to $\{t, h^{-1}\}$ in the Monster.*

Proof. By Norton's list, the isomorphism class of the group $\langle t, h\rangle$ and the conjugacy class of th are sufficient to determine the orbit. Since $\langle t, h\rangle = \langle t, h^{-1}\rangle$, it remains to show that th is conjugate to th^{-1} in the Monster. The subgroup A is contained in a subgroup of $\mathbb{M}$ isomorphic to S_{12} (from the normalizer X in Theorem 3.2). Since conjugacy classes in S_{12} are characterized by cycle shapes, th is conjugate to its inverse $h^{-1}t$, which in turn is conjugate to th^{-1} by t. □

By Lemma 3.5, to classify the orbits of A on its $(2A, 3A)$-pairs, it is sufficient to classify instead the orbits of A on pairs consisting of a $2A$-involution in A and a cyclic subgroup generated by a $3A$-element in A (or $(2A, \langle 3A\rangle)$-pairs).

Proposition 3.6. *There are exactly* 60 *orbits of A on its $(2A, \langle 3A\rangle)$-pairs.*

Proof. Let T be the set of involutions in A of cycle shape (2^2) and (2^6), and let $h_1 = (1, 2, 3)$, $h_2 = (1, 2, 3)(4, 5, 6)$ and $h_3 = (1, 2, 3)(4, 5, 6)(7, 8, 9)(10, 11, 12)$. Let H_i be the normalizer of the cyclic subgroup $\langle h_i\rangle$ in A for $i \in \{1, 2, 3\}$. The orbits of A on its $(2A, \langle 3A\rangle)$-pairs are in bijection with the union of orbits of H_i on T, $i \in \{1, 2, 3\}$. The number of orbits may be determined with [GAP16] or by taking the inner products of characters corresponding to the actions of A (Theorem 0.3). □

Remark. A list of representatives for the orbits of A on its $(2A, \langle 3A\rangle)$-pairs may be obtained using [GAP16]. This list is in Appendix B.

3.3 Application of the generic strategy

The Norton-Sakuma algebras are the Majorana representations of the dihedral groups. Unlike the dihedral groups, not all groups in Norton's list are generated by involutions let alone $2A$-involutions. For these cases, the generic strategy is therefore not applicable. In this section, we use the generic strategy to construct 2-closed Majorana representations for certain groups which are or contain as subgroups, groups in Norton's list. Although it is routine to obtain these representations, none of them have been explicitly described before.

3.3.1 $(2A, 2B, 3A)$-shape $2 \times S_4$-representation

Let G be a group with the following presentation:

$$G := \langle t_\alpha, t_\beta, h_\alpha \mid t_\alpha^2, t_\beta^2, h_\alpha^3, (t_\alpha h_\alpha)^4,\ t_\alpha^{t_\beta} t_\alpha,\ h_\alpha^{t_\beta} h_\alpha^{-1} \rangle.$$

Then G is isomorphic to $2 \times S_4$. There are exactly 5 conjugacy classes of involutions in G with representatives t_α, $t_\gamma := (t_\alpha h_\alpha)^2$, $t_\gamma t_\beta$, t_β and $t_\alpha t_\beta$. Let T be the union of the first three conjugacy classes:

$$T := t_\alpha^G \cup t_\gamma^G \cup (t_\gamma t_\beta)^G.$$

Then $|T| = 12$ and T generates G.

For each $t \in T$, introduce a Majorana axis a_t. Let $A := \{a_t \mid t \in T\}$. Then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is assumed to be a Majorana representation of (G, T) where V is the algebra generated by A, $\varphi : G \to GL(V)$ and $\psi : T \to A$.

Shape of the representation

There are exactly 7 orbits of G on $T \times T$ (unordered pairs) such that the product of the pair is an involution. Below are representatives $\{t_1, t_2\}$ for these orbits:

$$\{t_\alpha, t_\alpha^{t_\gamma}\},\ \{t_\alpha, t_\gamma^{h_\alpha}\},\ \{t_\gamma, t_\gamma^{t_\alpha}\},\ \{t_\gamma, (t_\gamma t_\beta)^{h_\alpha}\},\ \{t_\gamma t_\beta, (t_\gamma t_\beta)^{t_\alpha}\},\ \{t_\alpha^{t_\gamma}, t_\gamma^{h_\alpha} t_\beta\},\ \{t_\gamma, t_\gamma t_\beta\}.$$

For $\{t_1, t_2\}$ in the first 5 orbits, $t_1 t_2 \in T$. By the $2A$-condition, $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{2A}$ for all $\{t_1, t_2\}$ in these orbits. Moreover the following equation holds:

$$a_{t_1} \cdot a_{t_2} = \frac{1}{8}(a_{t_1} + a_{t_2} - a_{t_1 t_2}).$$

For $\{t_1, t_2\}$ in the last two orbits, set $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{2B}$.

There are exactly two orbits of G on $T \times T$ such that the product of the pair has order 4. Representatives for these orbits are

$$\{t_\alpha, t_\gamma\}, \text{ and } \{t_\alpha, t_\gamma t_\beta\}.$$

For $\{t_1, t_2\}$ in both these orbits, the group $\langle t_1, t_2\rangle \cong D_8$ contains a subgroup isomorphic to D_4 with involutions associated to a $2A$-type Norton-Sakuma algebra. By the subalgebra inclusion $2A \hookrightarrow 4B$, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{4B}$ for all $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 4$.

There is only one orbit of G on $T \times T$ such that the product of the pair has order 3, namely $\{t_\alpha, t_\alpha^h\}^G$. Set $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{3A}$ for all $\{t_1, t_2\}$ in this orbit.

Spanning set

There are exactly four distinct Norton-Sakuma algebras of type $3A$ generated by pairs of Majorana axes in A. Let U denote the set of $3A$-axes contained in these subalgebras, and let H be a set of elements of order 3 in G corresponding to the $3A$-axes in U. The spanning set is $X := A \cup U$ as the Majorana representation is assumed to be 2-closed.

Known subrepresentations

The subgroup of G generated by $t_\alpha^G \cup t_\gamma^G$ is isomorphic to S_4. The shape of this subrepresentation restricted to $\psi(t_\alpha^G \cup t_\gamma^G)$ is $(2A, 3A)$. This unique representation is known and was constructed in [I+10]. The $3A$-axes in U are also contained in this subrepresentation. Therefore all products between axes in $X \setminus \psi((t_\gamma t_\beta)^G)$ are known.

Determining the remaining products

The only unknown products between axes in X are cases between a_t and u_h where $\langle t, h\rangle \cong 2 \times A_4$ for which there is only one orbit of G on $T \times H$. The pair $\{t_\gamma t_\beta, h_\alpha\}$ is in this orbit.

The inner product for this case may be determined by orthogonality between eigenvectors. The result is

$$(a_t, u_h) = \frac{1}{45}$$

for all $\{t, h\} \in \{t_\gamma t_\beta, h_\alpha\}^G$.

The algebra product may be determined using the Resurrection principle. For $a_t \in A$ and $u_h \in U$ such that $\langle t, h\rangle \cong 2 \times A_4$, the algebra product is

$$a_t \cdot u_h = \frac{1}{90}(2a_t + a_{t_1} + a_{t_2} - a_{t_3} - a_{t_4}) + \frac{1}{64}(u_h - u_{h_1}),$$

where $h_1 = h^t$ and $\{t_3, t_4\} = t^{\langle t,h\rangle} \setminus t$. The order of $h_1 h h_1$ is assumed to be 2. If not, replace h_1 with its inverse. Then $\{t_1, t_2\} = (h_1 h h_1)^{\langle t,h\rangle} \setminus h_1 h h_1$.

This completes the construction of the unique 2-closed Majorana representation of (G, T) with shape $(2A, 2B, 3A)$. By calculating the rank of the Gram matrix, the dimension of V is 16 and X is a basis of V. The identity vector of V is

$$\frac{8}{15}\sum_{t_i \in t_\alpha^G} a_{t_i} + \frac{2}{5}\sum_{t_j \in t_\gamma^G} a_{t_j} + \frac{2}{3}\sum_{t_k \in (t_\gamma t_\beta)^G} a_{t_k} + \frac{3}{8}\sum_{h_l \in H} u_{h_l}.$$

It has length $\frac{44}{5}$.

3.3.2 $(2A, 3A)$-shape $(3 \times A_4) : 2$-representation

Let G be a group with the following presentation:

$$G := \langle h_\alpha, h_\beta, h_\gamma, t_\alpha \mid (h_i)^3, t_\alpha^2, (h_\alpha h_\beta)^2,\ {h_i}^{h_\gamma} h_i^{-1},\ {h_i}^{t_\alpha} h_i,\ i \in \{\alpha, \beta, \gamma\}\rangle.$$

Then G is isomorphic to $(3 \times A_4) : 2$ or $(Sym\{1, 2, 3\} \times Sym\{4, 5, 6, 7\}) \cap A_7$. There are only two conjugacy classes of involutions in G with representatives t and $h_\alpha h_\beta$. Let T be the union of both of these classes:

$$T := t^G \cup (h_\alpha h_\beta)^G.$$

Then $|T| = 21$ and T generates G.

For each $t \in T$, introduce a Majorana axis a_t. Let $A := \{a_t \mid t \in T\}$. Then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is assumed to be a Majorana representation of (G, T) where V is the algebra generated by A, $\varphi : G \to GL(V)$ and $\psi : T \to A$.

Shape of the representation

Since T is the set of all involutions in G, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{2A}$ for all $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 2$. As a consequence, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{4B}$ for all $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 4$.

There are exactly 4 orbits of G on $T \times T$ such that the product of the pair has order 3. Below are representative for these orbits:

$$\{t, t^{h_\alpha}\},\ \{t, t^{h_\alpha h_\gamma}\},\ \{t, t^{h_\beta h_\gamma}\},\ \{t, t^{h_\gamma}\}.$$

Since t and t^{h_γ} are contained in a subgroup of G isomorphic to D_{12} which is generated by a pair of involutions in T, by the subalgebra inclusion $3A \hookrightarrow 6A$, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{3A}$ for all $\{t_1, t_2\}$ in $\{t, t^{h_\gamma}\}^G$. Set $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle \cong V_{3A}$ for $\{t_1, t_2\}$ in the first three orbits.

Spanning set

There are exactly 13 distinct Norton-Sakuma algebras of type $3A$ generated by pairs of Majorana axes in A. Let U denote the set of $3A$-axes contained in these subalgebras, and let H be a set of elements of order 3 in G corresponding to the $3A$-axes in U. The spanning set is $X := A \cup U$ as the Majorana representation is assumed to be 2-closed.

Known subrepresentations

All subgroups of G isomorphic to S_4 are generated by its intersection with T. The subrepresentations of these S_4-subgroups have shape $(2A, 3A)$.

Subgroups of G isomorphic to $3^2 : 2$ are generated by involutions in T. The restricted shape of the subrepresentations are all $(2A, 3A)$. This representation may be seen in [Iva11b]. In this representation and hence in the representation of G, for all commuting $h_1, h_2 \in H$, the inner product and algebra product between the corresponding $3A$-axes are 0.

There is also a relation involving the axes in $3^2 : 2$-subrepresentations. This is known as **Pasechnik's relation**. Let p_x be the following vectors:

$$p_x := \frac{32}{45} \sum_{t_i \in t_\alpha^{\langle t_\alpha, h_x, h_\gamma\rangle}} a_{t_i} - (u_{h_x} + u_{h_\gamma} + u_{h_x h_\gamma} + u_{h_x h_\gamma^{-1}})$$

where $h_x \in \{h_\alpha, h_\beta, h_\alpha^{h_\alpha h_\beta}, h_\beta^{h_\alpha h_\beta}\}$. Then p_x are zero vectors in the linear span of X.

Determining the remaining products

The only unknown products are the cases between u_{h_1} and u_{h_2} where $\langle h_1, h_2\rangle \cong 3\times A_4$. The inner products for these cases may be determined by orthogonality between eigenvectors. The result is

$$(u_{h_1}, u_{h_2}) = \frac{64}{405}$$

for all $h_1, h_2 \in H$ such that $\langle h_1, h_2\rangle \cong 3 \times A_4$.

The algebra product for these cases may be determined using the Resurrection principle and the result is

$$\begin{aligned} u_{h_1} \cdot u_{h_2} =& \frac{64}{2025}(2(a_{t_1} + a_{t_2} + a_{t_3}) + a_{t_4} + a_{t_5} + a_{t_6}) + \frac{16}{2025}(3(a_{t_7} + a_{t_8}) - 4a_{t_9}) \\ &+ \frac{1}{90}(7(u_{h_1} + u_{h_2}) - 3(u_{h_3} + u_{h_4}) - (u_{h_5} + u_{h_6}) - 5(u_{h_7} + u_{h_8}) - 4u_{h_9}). \end{aligned}$$

The indices in the above expression may be described as follows. Without loss of generality, assume $o(h_1h_2) = 6$ (if not, replace either h_1 or h_2 with its inverse). Then $t_7 = (h_1h_2)^3$. Let $H_1 = \{h \in H \mid h_1^h = h_1\}$ and $H_2 = \{h \in H \mid h_2^h = h_2\}$. Then $h_9 = H_1 \cap H_2$. Let $H_3 = (H_1 \cup H_2) \setminus \{h_1, h_2, h_7\}$. Then $\{h_5, h_6\} \subseteq H_3$ such that $\langle h_5, h_6\rangle \cong A_4$, and $\{h_3, h_4\} = H_3 \setminus \{h_5, h_6\}$. Let $T_{1,2} = \{t \in T \mid \{h_1, h_2\}^t = \{h_3, h_4\}\}$. Then $t_9 \in T_{1,2}$ such that $t^{t_9} = t$ for all $t \in T_{1,2}$, and $\{t_4, t_5, t_6\} = T_{1,2} \setminus t_9$. Also, $t_8 = t_7^{t_9}$, $\{h_7, h_8\} = \{h_5, h_6\}^{t_7}$ and $\{t_1, t_2, t_3\} = \{t_4, t_5, t_6\}^{t_7}$.

This completes the construction of the unique 2-closed Majorana representation of (G, T) with shape $(2A, 3A)$. By calculating the rank of the Gram matrix, the dimension of V is 30. The zero space in the linear span of X is spanned by the four p_x vectors. The identity vector of V is

$$\frac{8}{27}\sum_{t_i \in t_\alpha^G} a_{t_i} + \frac{16}{45}\sum_{t_j \in T \setminus t_\alpha^G} a_{t_j} + \frac{5}{24}\sum_{h_k \in H \setminus h_\gamma} u_{h_k} + \frac{1}{6}u_{h_\gamma}.$$

It has length $\frac{32}{5}$.

3.3.3 $(2A, 3A)$-shape S_3^2-representation

Let G be a group with the following presentation:

$$G := \langle t_\alpha, t_\beta, h_\alpha, h_\beta \mid t_\alpha^2, t_\beta^2, h_\alpha^3, h_\beta^3, t_\alpha^{t_\beta} t_\alpha, t_\alpha^{h_\beta} t_\alpha, h_\alpha^{t_\beta} h_\alpha^{-1}, h_\alpha^{h_\beta} h_\alpha^{-1} \rangle.$$

Then G is isomorphic to S_3^2 where $\langle t_\alpha, h_\alpha \rangle \cong \langle t_\beta, h_\beta \rangle \cong S_3$. There are exactly 3 conjugacy classes of involutions in G with representatives t_α, t_β and $t_\alpha t_\beta$. Let T be the union of all these classes:

$$T := t_\alpha^G \cup t_\beta^G \cup (t_\alpha t_\beta)^G.$$

Then $|T| = 21$ and T generates G.

For each $t \in T$, introduce a Majorana axis a_t. Let $A := \{a_t \mid t \in T\}$. Then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is assumed to be a Majorana representation of (G, T) where V is the algebra generated by A, $\varphi : G \to GL(V)$ and $\psi : T \to A$.

Shape of the representation

Since T is the set of all involutions in G, $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{2A}$ for all $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 2$.

There are exactly 5 orbits of G on $T \times T$ such that the product of the pair has order 3. Below are representatives for these orbits:

$$\{t_\alpha, t_\alpha^{h_\alpha}\},\ \{t_\beta, t_\beta^{h_\beta}\},\ \{t_\alpha t_\beta, (t_\alpha t_\beta)^{h_\alpha}\},\ \{t_\alpha t_\beta, (t_\alpha t_\beta)^{h_\beta}\},\ \{t_\alpha t_\beta, (t_\alpha t_\beta)^{h_\alpha h_\beta}\}.$$

Pairs in the first four orbits are contained in a subgroup isomorphic to D_{12} generated by two involutions in T. By the subalgebra inclusion $3A \hookrightarrow 6A$, $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{3A}$ for all $\{t_1, t_2\}$ in the first four orbits. Set $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{3A}$ for $\{t_1, t_2\}$ in the last orbit.

Spanning set

There are exactly four distinct Norton-Sakuma algebras of type $3A$ generated by pairs of Majorana axes in A. Let U denote the set of $3A$-axes contained in these subalgebras, and let H be a set of elements of order 3 in G corresponding to the $3A$-axes in U. The spanning set is $X := A \cup U$ as the Majorana representation is assumed to be 2-closed.

Known subrepresentations

There is a unique subgroup of G isomorphic to $3^2:2$ and it is generated by its intersection with T. The shape of this subrepresentation is $(2A, 3A)$. By the previous section, the Pasechnik's relation appearing in this subrepresentation is

$$p := \frac{32}{45} \sum_{t_i \in (t_\alpha t_\beta)^G} a_{t_i} - (u_{h_\alpha} + u_{h_\beta} + u_{h_\alpha h_\beta} + u_{h_\alpha h_\beta^{-1}}) = 0.$$

Determining the remaining products

The only unknown products between axes in X are between a_t and u_h where $\langle t, h\rangle \cong 3 \times S_3$.

The inner products for these cases may be determined by taking for example, the inner product between t_α and p. Then for all $t \in T$ and $h \in H$ such that $\langle t, h\rangle \cong 3 \times S_3$,

$$(a_t, u_h) = \frac{1}{20}.$$

The algebra product may be attained by multiplying for example, a_{t_α} with p and solving the equation for $a_{t_\alpha} \cdot u_{h_\alpha h_\beta}$, since $u_{h_\alpha h_\beta} - u_{h_\alpha h_\beta^{-1}}$ is a $\frac{1}{32}$-eigenvector of a_{t_α}. For all $t \in T$ and $h \in H$ such that $\langle t, h\rangle \cong 3 \times S_3$,

$$a_t \cdot u_h = \frac{1}{90}(5a_t + 2(a_{t_1} + a_{t_2}) - 3(a_{t_3} + a_{t_4} + a_{t_5}) + 3(a_{t_6} + a_{t_7} + a_{t_8})) + \frac{1}{64}(u_h - u_{h_1} - 2u_{h_2})$$

where $t_2 = t_1^{h_1}$, $t_3 = t_1^{h_1^{-1}}$, $\{t_6, t_7, t_8\} = \{t_x \in T \mid o(tt_x) = 2\} \cap \{t_y \in T \mid o(ht_y) = 2\}$, $\{t_3, t_4, t_5\} = \{tt_6, tt_7, tt_8\}$, $h_2 = h_1^{t_1}$ and $h_3 \in \{h_1h_2h_1, h_1h_2^{-1}h_1\}$ such that $o(th_3) = 2$.

This completes the construction of the unique Majorana representation of (G, T) with shape $(2A, 3A)$ (It was brought to the attention by Shpectorov that this representation may in fact be constructed without the assumption of 2-closeness). By calculating the rank of the Gram matrix, the dimension of V is 18. The zero space in the linear span of X is spanned by p. The identity vector of V is

$$\frac{16}{33} \sum_{t_i \in t_\alpha^G \cup t_\beta^G} a_{t_i} + \frac{40}{99} \sum_{t_j \in (t_\alpha t_\beta)^G} a_{t_j} + \frac{9}{44}(u_{h_\alpha} + u_{h_\beta}) + \frac{15}{44}(u_{h_\alpha h_\beta} + u_{h_\alpha h_\beta^{-1}}).$$

It has length $\frac{456}{55}$.

3.3.4 $(2A, 2B, 3A)$-shape $S_3 \times S_4$-representation

Let G be a group with the following presentation:

$$G := \langle t_\alpha, t_\beta, t_\gamma, h_\alpha \mid t_\alpha^2, t_\beta^2, t_\gamma^2, h_\alpha^3, (t_\alpha t_\beta)^2, (t_\gamma h_\alpha)^4,\ t_\gamma^{t_\alpha} t_\gamma,\ h_\alpha^{t_\alpha} h_\alpha^{-1}, t_\gamma^{t_\beta} t_\gamma,\ h_\alpha^{t_\beta} h_\alpha^{-1} \rangle.$$

Then G is isomorphic to $S_3 \times S_4$ where $\langle t_\alpha, t_\beta \rangle \cong S_4$ and $\langle t_\gamma, h_\alpha \rangle \cong S_4$. There are exactly 5 conjugacy classes of involutions in G with representatives t_α, t_γ, $t_\alpha t_\gamma$, $t_\delta := (t_\gamma h)^2$ and $t_\alpha t_\delta$. Let T be the union of the first four conjugacy classes:

$$T := t_\alpha^G \cup t_\gamma^G \cup (t_\alpha t_\gamma)^G \cup t_\delta^G.$$

Then $|T| = 30$ and T generates G.

For each $t \in T$, introduce a Majorana axis a_t. Let $A := \{a_t \mid t \in T\}$. Then $(V, A, \cdot, (\ ,\ ), \varphi, \psi)$ is assumed to be a Majorana representation of (G, T) where V is the algebra generated by A, $\varphi : G \to GL(V)$ and $\psi : T \to A$.

Shape of the representation

For any $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 2$ and $t_1 t_2 \in T$, $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{2A}$. If $o(t_1 t_2) = 2$ but $t_1 t_2 \notin T$, then set $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{2B}$.

For all $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 4$, the subgroup $\langle t_1, t_2 \rangle$ intersects T at three involutions whose corresponding Majorana axes generate a Norton-Sakuma algebra of type $2A$. By the subalgebra inclusion $2A \hookrightarrow 4B$, $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{4B}$ for all $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 4$.

For $\{t_1, t_2\} \in T \times T$ such that $o(t_1 t_2) = 3$, set $\langle\langle a_{t_1}, a_{t_2} \rangle\rangle \cong V_{3A}$.

Spanning set

There are exactly 13 distinct Norton-Sakuma algebras of type $3A$ generated by pairs of Majorana axes in A. Let U denote the set of $3A$-axes contained in these subalgebras, and let H be a set of elements of order 3 in G corresponding to the $3A$-axes in U. The spanning set is $X := A \cup U$ as the Majorana representation is assumed to be 2-closed.

Known subrepresentations

The subgroup of G generated by $(t_\alpha t_\gamma)^G \cup t_\delta^G$ is isomorphic to $(3 \times A_4) : 2$. This subrepresentation was constructed in Section 3.3.2. Moreover, every $3A$-axis in U appears in this subrepresentation. Hence the products between any two axes in $X \setminus \psi(t_\alpha^G \cup t_\gamma^G)$ are known.

G has a subgroup isomorphic to S_3^2, generated by its intersection with T. We've seen its representation in Section 3.3.3. Subrepresentations of S_4-subgroups have shape $(2A, 3A)$.

Determining the remaining products

The unknown products are between a_t and u_h where $\langle t, h\rangle \cong 3 \times S_4$. For example $t = t_\gamma^h$ and $h = t_\alpha t_\beta t_\gamma t_\gamma^{h_\alpha^{-1}}$. The inner products can be determined by orthogonality between eigenvectors. We get

$$(a_t, u_h) = \frac{1}{36}$$

for all $\{t, h\} \in T \times H$ such that $\langle t, h\rangle \cong 3 \times S_4$.

The algebra products on the other hand may be obtained using the Resurrection principle. For all $t \in T$ and $h \in H$ such that $\langle t, h\rangle \cong 3 \times S_4$,

$$\begin{aligned} a_t \cdot u_h =& \frac{1}{90}((2a_t - a_{t_1} - a_{t_2}) + a_{t_3} - (a_{t_4} + a_{t_5} + a_{t_6}) + (a_{t_7} + a_{t_8} + a_{t_9}) \\ & - (a_{t_{10}} + a_{t_{11}} + a_{t_{12}})) + \frac{1}{192}(3(u_h - u_{h_1}) - u_{h_2} + u_{h_3} + 3(u_{h_4} + u_{h_5})) \end{aligned}$$

where $t_1 = t^h$, $t_2 = t^{h^{-1}}$, $t_3 = (th)^6(th^{-1})^6$, $t_4 = tt_3$, $t_5 = t_4^h$, $t_6 = t_4^{h^{-1}}$, $\{t_7, t_8, t_9\} = \{t_x \in T \mid \langle t_x, h\rangle \cong 3 \times S_3\} \setminus \{t_4, t_5, t_6\}$, $\{t_{10}, t_{11}, t_{12}\} = \{tt_7, tt_8, tt_9\}$, $h_1 = h^t$, $h_2 = (th)^4$, $h_3 = t_1 t_2$, $h_4 = h_3^h$ and $h_5 = h_3^{h^{-1}}$.

This completes the construction of the unique 2-closed Majorana representation of (G, T) with shape $(2A, 2B, 3A)$. By calculating the rank of the Gram matrix, the dimension of V is 39. The zero space in the linear span of X is spanned by the four vectors from Pasechnik's relation in the subrepresentation of $(3 \times A_4) : 2$. The identity vector of V is

$$\frac{16}{57}\sum_{t_i \in t_\alpha^G} a_{t_i} + \frac{8}{19}\sum_{t_j \in t_\gamma^G} a_{t_j} + \frac{88}{285}\sum_{t_k \in t_\delta^G} a_{t_k} + \frac{32}{57}\sum_{t_l \in (t_\alpha t_\gamma)^G} a_{t_l} - \frac{15}{152}\sum_{h_m \in h_\alpha^G} u_{h_m} - \frac{27}{38}u_{t_\alpha t_\beta}$$

It has length $\frac{240}{19}$.

3.4 Algebra products between a and u

Knowing how to multiply between a $2A$- and $3A$-axis is fundamental to construct Majorana representations. If the representation is based on an embedding in the Monster, then these axes multiply in the same way it does in the Monster algebra. By Norton's list, there are only 22 orbits for the Monster on its $(2A, 3A)$-pairs. Therefore there are also only 22 orbits for the action of the Monster on pairs of $2A$- and $3A$-axes.

Let $\{t_1, h_1\}$ and $\{t_2, h_2\}$ be $(2A, 3A)$-pairs conjugate in the Monster. Then there exists some $g \in \mathbb{M}$ such that $t_1^g = t_2$ and $h_1^g = h_2$. The algebra product $a_{t_2} \cdot u_{h_2}$ is equal to $(a_{t_1} \cdot u_{h_1})^g$ since $\mathbb{M}$ preserve the algebra product and acts by conjugating the indices. If there is an explicit expression for $a_{t_1} \cdot u_{h_1}$ as a linear combination of axes, then $a_{t_2} \cdot u_{h_2}$ has a similar expression but with the axes conjugated by g. It will become evident that these expressions are crucial for the work in Chapter 4.

Throughout this section, let $\{t, h\}$ be a $(2A, 3A)$-pair in the Monster.

Proposition 3.7. *If $\langle t, h\rangle \cong S_3$, then*

$$a_t \cdot u_h = \frac{1}{9}(2a_t - a_{t_1} - a_{t_2}) + \frac{5}{32}u_h$$

where $t_1 = t^h$ and $t_2 = t^{h^{-1}}$.

If $\langle t, h\rangle \cong 6$ then

$$a_t \cdot u_h = 0.$$

Proof. By Norton's list there is only one orbit for which $\langle t, h\rangle \cong S_3$ or 6. These products are seen in a Norton-Sakuma algebra of type $6A$, which is contained in the Monster algebra. □

Proposition 3.8. *Let $\langle t, h\rangle \cong S_4$. If $th \in 4A$, then*

$$\begin{aligned} a_t \cdot u_h =& \frac{1}{135}(11a_t - a_{t_1}) + \frac{1}{270}(a_{t_2} + a_{t_3} + a_{t_4} + a_{t_5}) \\ &+ \frac{1}{192}(11u_h + 5u_{h_1} - u_{h_2} - u_{h_3}) + \frac{1}{45}(v_{x_1} - 2v_{x_2} - 2v_{x_3}) \end{aligned}$$

where $\{t_1\} = \{t_x \in t^{\langle t,h\rangle} \mid o(tt_1) = 2\}$, $\{t_2, t_3, t_4, t_5\} = t^{\langle t,h\rangle} \setminus \{t, t_1\}$, $h_1 = h^t$ *and* $\{h_2, h_3\} = \{h^{h_1}, h^{h_1^{-1}}\}$. *The vectors labelled* v_{x_i} *are* $4A$*-axes in the Monster algebra. Then* $\{x_1\} = \{x \in \langle t, h\rangle \mid o(x) = 4, x_1^2 = tt_1\}$ *and* $\{x_2, x_3\} = \{x_1^h, x_1^{h^{-1}}\}$.

If $th \in 4B$, *then*

$$a_t \cdot u_h = \frac{1}{45}(a_t + a_{t_1} - a_{t_2}) - \frac{1}{90}(a_{t_3} + a_{t_4} + a_{t_5} + a_{t_6}) + \frac{1}{64}(u_h - u_{h_1} + u_{h_2} + u_{h_3})$$

where $\{t_1\} = \{t_x \in t^{\langle t,h\rangle} \mid o(t_1 h) = 2\}$. *Then* $t_2 = tt_1$ *and* $\{t_3, t_4, t_5, t_6\} = t^{\langle t,h\rangle} \setminus \{t, t_1\}$. *Also* $h_1 = h^t$ *and* $\{h_2, h_3\} = \{h_1^h, h_1^{h^{-1}}\}$.

Proof. By Norton's list, there is only one orbit for which $\langle t, h\rangle \cong S_4$ and $th \in 4A$ or $4B$. The algebra products are seen in Majorana representations of S_4 with shapes $(2B, 3A)$ and $(2A, 3A)$ respectively. □

Proposition 3.9. *If* $\langle t, h\rangle \cong A_4$, *then*

$$a_t \cdot u_h = \frac{1}{9}a_t + \frac{1}{64}(5u_h + 3u_{h_1} - 4u_{h_2} - 4u_{h_3})$$

where $h_1 = h^t$, $h_2 = h_1^h$ *and* $h_3 = h_1^{h^{-1}}$.

Proof. Again there is only one orbit for this case. This algebra product is seen in the Majorana representation of S_4 with shape $(2A, 3A)$. □

Proposition 3.10. *If* $\langle t, h\rangle \cong A_5$, *then*

$$\begin{aligned} a_t \cdot u_h =& \frac{1}{45}(2a_t + a_{t_1} - a_{t_2} - a_{t_3} - a_{t_4} - a_{t_5} - a_{t_6}) + \frac{1}{64}(2u_h + 2u_{h_1} + 2u_{h_2} - u_{h_3} - u_{h_4}) \\ &+ \frac{1}{90}(a_{t_7} + a_{t_8} + a_{t_9} + a_{t_{10}} - a_{t_{11}} - a_{t_{12}} - a_{t_{13}} - a_{t_{14}}) + \sigma_{th}\frac{32}{135}w. \end{aligned}$$

The indices in the above expression may be described as follows: For $r \in \{2, 3, 5\}$ *let* $T^{(r)}$ *be a set of elements of order* r *in* $\langle t, h\rangle$ *containing one representative from every subgroup of order* r. *Moreover* $T^{(5)}$ *is in a single conjugacy class of* $\langle t, h\rangle$. *For* $g \in \langle t, h\rangle$, *let* $T_s^{(r)}(g) = \{x \in T^{(r)} \mid o(gx) = s\}$.

Then $\{t_1\} = T_2^{(2)}(t) \setminus T_2^{(2)}(h)$, $\{t_2\} = T_2^2(t) \cap T_2^2(h)$ *and* $\{t_3, t_4, t_5, t_6\} = T_3^{(2)}(t)$. *Also* $\{h_1, h_2\} = T_2^{(3)}(t)$ *and* $\{h_3, h_4\} = T_2^{(3)}(t_1)$. *Let* $\{g_1, g_2\} = T_5^{(3)}(t) \cap H_2^{(3)}(t_1)$ *and* $\{g_3, g_4\} = T_5^{(3)}(t) \cap H_2^{(3)}(t_2)$. *Then* $\{t_7, t_8, t_9, t_{10}\} = \{t_1 g_1, t_1 g_1^{-1}, t_1 g_2, t_1 g_2^{-1}\}$ *and* $\{t_{11}, t_{12}, t_{13}, t_{14}\} = \{t_2 g_3, t_2 g_3^{-1}, t_2 g_4, t_2 g_4^{-1}\}$. *The symbol* σ_{th} *is defined to be* 1 *if* th *is conjugate to elements in* $T^{(5)}$ *and* -1 *otherwise. The vector* w *is defined as the sum of all* $5A$ *axes* w_x *where* $x \in T^{(5)}$.

Proof. There is also one orbit for this case. This algebra product is seen in the 26-dimensional Majorana representation of A_5 with shape $(2A, 3A)$ which is based on an embedding in the Monster (see [ISe12]). □

Remark. There is one orbit for $\langle t, h\rangle \cong L_2(7)$. This case is seen in the Majorana representation of $L_3(2)$ with shape $(2A, 3A)$ constructed in [ISh12]. The expression for $a_t \cdot u_h$ shall be omitted here. One may however be refer to [ISh12]. It is a linear combination of Majorana and $3A$-axes.

Remark. There is also only one orbit for $\langle t, h\rangle \cong L_2(11)$. This case would be seen in the Majorana representation of $L_2(11)$, worked on by Decelle in [Dec13]. Seress also constructed this representation around the same time but he did not publish his results. The explicit algebra products were not described by Decelle. This representation may be spanned by a set of Majorana, $3A$- and $4A$-axes. Therefore the algebra product $a_t \cdot u_h$ would be some linear combination of these axes.

The Majorana representations of $2 \times S_4$, $(3 \times A_4) : 2$, S_3^2 and $S_3 \times S_4$ with shapes constructed in Section 3.3 are all based on an embeddings in the Monster. This may be proven by showing that these groups appear as subgroups of A, and that the conjugacy class of products of $2A$-involutions coincide with the shape. Since fixing the shape determines the representation, it is the Monster embeddable representation.

Lemma 3.11. *Let* H *be the following subgroup of* A*:*

$$H := \langle (1,2)(3,4), (1,3)(2,4)(6,9)(7,10), (1,4,6)(2,9,3)\rangle.$$

Then the subalgebra of the Monster algebra generated by the $2A$-axes associated to $2A$-involutions in H, is isomorphic to the Majorana algebra arising from the representation of $G \cong 2 \times S_4$ in Section 3.3.1.

Proof. $H \cong 2 \times S_4$. Let T_H be the set of $2A$-involutions in H. Then T_H coincides with the set T of generators of G in Section 3.3.1. The shape of the representation of G in Section 3.3.1 respects the conjugacy classes in the Monster of the product of pairs of involutions in T_H. Since determining the shape fixes the structure of the algebra, and since there is only one orbit by Norton's list, these algebras are isomorphic. □

Corollary 3.12. *Let t_1 be a $2A$-involution and h_1 a $3A$-element in H as in Lemma 3.11. If $\langle t_1, h_1 \rangle \cong 2 \times A_4$, then $t_1 h_1 \in 6C$.*

Proof. The cycle shape of $t_1 h_1$ is $(6, 2)$. By the Lemma 3.4, it is a $6C$-element. □

Proposition 3.13. *If $\langle t, h \rangle \cong 2 \times A_4$ and $th \in 6C$, then*

$$a_t \cdot u_h = \frac{1}{90}(2a_t + a_{t_1} + a_{t_2} - a_{t_3} - a_{t_4}) + \frac{1}{64}(u_h - u_{h_1})$$

where $h_1 = h^t$, $\{t_3, t_4\} = t^{\langle t,h \rangle} \setminus \{t\}$. The order of $h_1 h h_1$ is assumed to be 2, if not, replace h_1 with its inverse. Then $\{t_1, t_2\} = (h_1 h h_1)^{\langle t,h \rangle} \setminus \{h_1 h h_1\}$.

Proof. There is only one orbit for this case on Norton's list. The algebra product may be seen in the Majorana representation of $2 \times S_4$ with shape $(2A, 2B, 3A)$ constructed in Section 3.3.1 and which is based on an embedding in the Monster by Lemma 3.11. □

Lemma 3.14. *Let H be the following subgroup of A:*

$$H := \langle (1, 2, 10), (1, 2, 7), (3, 4, 9), (12)(3, 4) \rangle.$$

Then the subalgebra of the Monster algebra generated by the $2A$-axes associated to $2A$-involutions in H, is isomorphic to the Majorana algebra arising from the representation of $G \cong (3 \times A_4) : 2$ in Section 3.3.2.

Proof. This is similar to the proof of Lemma 3.11. □

Lemma 3.15. *Let H be the following subgroup of A:*

$$H := \langle (1,2)(3,4), (1,2)(10,11), (3,4,5), (10,11,12)) \rangle.$$

Then the subalgebra of the Monster algebra generated by the $2A$-axes associated to $2A$-involutions in H, is isomorphic to the Majorana algebra arising from the representation of $G \cong S_3^2$ in Section 3.3.3.

Proof. This is similar to the proof of Lemma 3.11. □

Corollary 3.16. *Let t_1 be a $2A$-involution and h_1 a $3A$-element in H of Lemma 3.15. If $\langle t_1, h_1 \rangle \cong 3 \times S_3$, then $t_1 h_1 \in 6A$.*

Proof. By Norton's list, there is only one orbit of the Monster on its $(2A, 3A)$-pairs such that the pair is in A and it generates a group isomorphic to $3 \times S_3$. Also by Norton's list, $t_1 h_1$ is in $6A$. □

Proposition 3.17. *If $\langle t, h \rangle \cong 3 \times S_3$ and $th \in 6A$, then*

$$a_t \cdot u_h = \frac{1}{90}(5a_t + 2(a_{t_1} + a_{t_2}) - 3(a_{t_3} + a_{t_4} + a_{t_5}) + 3(a_{t_6} + a_{t_7} + a_{t_8})) + \frac{1}{64}(u_h - u_{h_1} - 2u_{h_2})$$

where $t_2 = t_1^{h_1}$, $t_3 = t_1^{h_1^{-1}}$, $\{t_6, t_7, t_8\} = \{t_x \in T \mid o(tt_x) = 2\} \cap \{t_y \in T \mid o(ht_y) = 2\}$, $\{t_3, t_4, t_5\} = \{tt_6, tt_7, tt_8\}$, $h_2 = h_1^{t_1}$ and $h_3 \in \{h_1 h_2 h_1, h_1 h_2^{-1} h_1\}$ such that $o(th_3) = 2$.

Proof. There is only one orbit for this case on Norton's list. The algebra product may be seen in the Majorana representation of S_3^2 with shape $(2A, 3A)$ constructed in Section 3.3.3. This representation is based on an embedding in the Monster by Lemma 3.15. □

Corollary 3.18. *Let G be a subgroup of the Monster isomorphic to $3 \times S_3$ and which can be generated by a $2A$-involution t and $3A$-element h. If $th \in 6A$, then G is also contained in a subgroup of the Monster isomorphic to S_3^2.*

Proof. Let $G = \langle t_1, h_1 \rangle$. Then $a_{t_1} \cdot u_{h_1}$ is a linear combination of Majorana and $3A$-axes by Proposition 3.17. The elements associated to these axes generate a subgroup of the Monster isomorphic to S_3^2 and which contain G as a subgroup. □

Lemma 3.19. *Let H be the following subgroup of A:*

$$H := \langle (1,2)(3,4), (1,2)(3,5), (1,2)(11,12), (8,10,11) \rangle.$$

Then the subalgebra of the Monster algebra generated by the $2A$-axes associated to $2A$-involutions in H, is isomorphic to the Majorana algebra arising from the representation of $G \cong S_3 \times S_4$ in Section 3.3.4.

Proof. This is similar to the proof of Lemma 3.11. □

Proposition 3.20. *If $\langle t, h \rangle \cong 3 \times S_4$, then*

$$\begin{aligned} a_{t_1} \cdot u_{h_1} =& \frac{1}{90}((2a_{t_1} - a_{t_2} - a_{t_3}) + a_{t_4} - (a_{t_5} + a_{t_6} + a_{t_7}) + (a_{t_8} + a_{t_9} + a_{t_{10}}) \\ & - (a_{t_{11}} + a_{t_{12}} + a_{t_{13}})) + \frac{1}{192}(3(u_{h_1} - u_{h_2}) - u_{h_3} + (u_{h_4} + 3(u_{h_5} + u_{h_6}))) \end{aligned}$$

where $t_2 = t_1^{h_1}$, $t_3 = t_1^{h_1^{-1}}$, $t_4 = (t_1 h_1)^6 (t_1 h_1^{-1})^6$, $t_5 = t_1 t_4$, $t_6 = t_5^{h_1}$, $t_7 = t_5^{h_1^{-1}}$, $\{t_8, t_9, t_{10}\} = \{t \in T \mid \langle t, h_1 \rangle \cong 3 \times S_3\} \setminus \{t_5, t_6, t_7\}$, $\{t_{11}, t_{12}, t_{13}\} = \{t_1 t_8, t_1 t_9, t_1 t_{10}\}$, $h_2 = h_1^{t_1}$, $h_3 = (t_1 h_1)^4$, $h_4 = t_2 t_3$, $h_5 = h_4^{h_1}$ and $h_6 = h_4^{h_1^{-1}}$.

Proof. There is only one orbit for this case on Norton's list. The algebra product may be seen in the Majorana representation of $S_3 \times S_4$ with shape $(2A, 2B, 3A)$ constructed in Section 3.3.4. This representation is based on an embedding in the Monster by Lemma 3.19. □

Corollary 3.21. *Let G be a subgroup of the Monster isomorphic to $3 \times S_4$ and which can be generated by a $2A$-involution and $3A$-element. Then G is also contained in a subgroup of the Monster isomorphic to $S_3 \times S_4$.*

Proof. This is similar to the proof of Corollary 3.18. □

In the case of algebra products between two $2A$-axes, all elements associated to the axes appearing in the algebra product are contained in the dihedral group generated by the associated involutions. Compared to the algebra products between a_t and u_h in some cases, e.g. two of the three new cases: Propositions 3.17 and 3.20, the elements associated to the axes in the expression for $a_t \cdot u_h$ are not solely generated by the $2A$-involution and $3A$-element. It requires a larger group containing $\langle t, h \rangle$.

3.5 Subalgebras generated by a and u

In this section, we look at Majorana algebras containing a Majorana and $3A$-axis, and then generate the subalgebra using these two axes. Such subalgebras have not been looked at before. It would be interesting to see what structure they hold and if they may be used in the inductive process of constructing new Majorana representations.

Let a and u be a Majorana and $3A$-axis respectively in some Majorana algebra V contained in the Monster algebra. Let $G_{a,u}$ be the subgroup of the Monster generated by the $2A$-involution and $3A$-element associated to a and u respectively. Denote by $V_{a,u}$ the subalgebra of V generated by a and u.

Proposition 3.22. *Let $V = V_{3A}$. Then $G_{a,u} \cong S_3$. The subalgebra $V_{a,u}$ of V is 3-dimensional and has the set $B := \{a, u, \tilde{u}\}$ of idempotents as a basis. The algebra and inner products with respect to this basis are:*

$$a \cdot u = \frac{1}{8}(u - \tilde{u}) + \frac{1}{4}a,$$
$$a \cdot \tilde{u} = \frac{1}{8}(\tilde{u} - u) + \frac{1}{4}a,$$
$$u \cdot \tilde{u} = \frac{1}{12}(u + \tilde{u}) - \frac{1}{6}a,$$
$$(\tilde{u}, \tilde{u}) = \frac{8}{5},\ (a, \tilde{u}) = \frac{1}{4},\ (u, \tilde{u}) = \frac{1}{10}.$$

The automorphism group of $V_{a,u}$ has order 2. It is generated by the involution τ swapping u with $\tilde{u}$, and fixing a.

Proof. There are three possibilities for a and one for u by *Section 3.2* of [CR13b]. For each of these cases, $G_{a,u} \cong S_3$. The algebra products are worked out inside V using [Map16] by multiplying a and u in different ways until a linearly independent set closed under the algebra product is obtained. By classifying the idempotents, it turns out that there are no other idempotents in $V_{a,u}$ of lengths 1 or $\frac{8}{5}$ besides those in B. It is now clear from B and the products that τ is the only non-trivial automorphism of $V_{a,u}$. □

Proposition 3.23. *Let V be the Majorana representation of S_4 with shape $(2A, 3A)$, and let a, u be axes in V such that $G_{a,u} \cong A_4$. The subalgebra $V_{a,u}$ of V is 5-dimensional and has the set $B := \{a, u_1 := u, u_2, x_1, x_2\}$ of idempotents as a basis. The algebra and inner products with respect to this basis are:*

$$\begin{aligned}
a \cdot u_i &= \frac{1}{9}a + \frac{1}{64}(u_i - u_j) - \frac{13}{180}(x_1 + x_2),\\
a \cdot x_i &= \frac{5}{13}a + \frac{1}{8}(x_i - x_j),\\
u_i \cdot x_1 &= -\frac{20}{117}a + \frac{85}{832}(u_i - u_j) + \frac{13}{90}x_1 + \frac{1}{18}x_2,\\
u_i \cdot x_2 &= -\frac{4}{39}a + \frac{261}{832}(u_i - u_j) + \frac{1}{30}x_1 + \frac{7}{10}x_2,\\
u_1 \cdot u_2 &= -\frac{16}{81}a - \frac{7}{18}(u_1 + u_2) + \frac{182}{2025}x_1 + \frac{338}{405}x_2,\\
x_1 \cdot x_2 &= -\frac{20}{169}a + \frac{1}{26}(x_1 + x_2),
\end{aligned}$$

$$(a, u_i) = \frac{1}{9},\ (a, x_1) = (a, x_2) = \frac{5}{13},\ (u_2, u_2) = \frac{8}{5},\ (u_i, x_1) = \frac{25}{117},$$

$$(u_i, x_2) = \frac{17}{13},\ (u_1, u_2) = \frac{136}{405},\ (x_i, x_i) = \frac{25}{13},\ (x_1, x_2) = \frac{5}{169},$$

where $\{i, j\} = \{1, 2\}$. The automorphism group of $V_{a,u}$ has order 2. It is generated by the involution τ swapping u with $\tilde{u}$, and fixing everything else.

Proof. This is similar to the proof of Proposition 3.22. It may be checked that there are no other idempotents in $V_{a,u}$ which have lengths 1, $\frac{8}{5}$ or $\frac{25}{13}$ besides those in B. □

Remark. For the cases where V is the Majorana representation of S_4 with shape $(2A, 3A)$ or $(2B, 3A)$, and $G_{a,u}$ is isomorphic to S_4, both $V_{a,u}$ are 9-dimensional. This was verified in [Map16] and the details are omitted here. The automorphism groups of these subalgebras were both identified to be isomorphic to 2^2. In the 26-dimensional Majorana representation of A_5, the subalgebra $V_{a,u}$ where $G_{a,u} \cong A_5$ could not be identified due to computational constraints. From initial calculations, it is at least 14-dimensional. These details are also omitted.

It is clear from this section that considering subalgebras generated by a $2A$- and $3A$-axis alone loses vital information. A $3A$-axis in the Majorana algebra is no longer a $3A$-axis in

$V_{a,u}$ as there are no pairs of $2A$-axes generating a Norton-Sakuma algebra of type $3A$. It is also not a standard $3A$-axis as the $\frac{1}{3}$-eigenspace of u is 1-dimensional.

Unlike the dihedral groups and the Norton-Sakuma algebras, the algebras generated by a $2A$- and $3A$-axis do not correspond to the subgroups of the Monster generated by a $2A$-involution and $3A$-element. Another way of viewing the Norton-Sakuma algebras are as algebras generated by all NA-axes associated to all NA-elements in the dihedral group. This is also the case in the $(2A, 3A)$ generated A_4 and V_{A_4} in Section 2.5. This interpretation may be taken for the groups in Norton's list. However, to obtain these algebras, we may have to consider Majorana representations of larger groups which contain as subgroups, groups on Norton's list. This question remains open.

Chapter 4

Inner products between $3A$-axes

Let A be the A_{12}-subgroup of the Monster from Chapter 3, and let V_A be the subalgebra of the Monster algebra generated by all $2A$-axes associated to the $2A$-involutions in A. This subalgebra is essentially what Majorana theory aims to construct axiomatically. It was proven in [CRI14] that the $3A$-axes in V_A do not lie in the span of the $2A$-axes. By the Gram Determinant theorem (Theorem 0.9), the linear span of $3A$-axes in V_A depends on the values of inner products between pairs of $3A$-axes in V_A. The determination of these inner products shall be what we achieve in this chapter.

In Section 4.1, we first obtain a list of orbits of A on its unordered pairs of $3A$-cyclic subgroups. These orbits are in bijection with the orbits of A on pairs of $3A$-axes in V_A. Then in Section 4.2, we formulate the methods and steps which we will use to determine all inner products between pairs of $3A$-axes in V_A. The results are then recorded in Table 4.2. The values of the inner products are invariant under the action of the Monster. Pairs having different inner product values therefore cannot be conjugate in the Monster. Nevertheless, we will identify some orbits of A which fuse under the action of the Monster in Section 4.4. Finally in Section 4.5, we present a new approach to obtaining certain relations between axes.

4.1 Orbits for A on pairs of $3A$-axes in V_A

Let A be the centralizer of a $(2A, 3A, 5A)$ A_5-subgroup H of $\mathbb{M}$ from Chapter 3. Presented as even permutations of the set $\{1, \ldots, 12\}$, the $2A$-involutions in A have cycle shape (2^2) or (2^6), while the $3A$-elements have cycle shape (3), (3^2) or (3^4) (see Theorem 3.3). It was further shown in Lemma 3.4 that elements in A with certain cycle shapes are $4A$-, $4B$-, $6A$- or $6C$-elements in the Monster. This may in fact be extended to the whole of A.

Lemma 4.1. *Table 4.1 shows the conjugacy classes of elements in A based on their cycle shapes.*

A	$\mathbb{M}$	A	$\mathbb{M}$	A	$\mathbb{M}$	A	$\mathbb{M}$
(1^{12})	$1A$	$(4, 2)$	$4B$	$(6, 2)$	$6C$	$(4, 3, 2)$	$12C$
(2^2)	$2A$	$(4, 2^3)$	$4B$	(7)	$7A$	$(4, 3^2, 2)$	$12C$
(2^6)	$2A$	(5)	$5A$	$(8, 2)$	$8B$	$(6, 4)$	$12C$
(2^4)	$2B$	(5^2)	$5A$	$(8, 4)$	$8B$	$(7, 2^2)$	$14A$
(3)	$3A$	$(3, 2^2)$	$6A$	(9)	$9A$	$(5, 3)$	$15A$
(3^2)	$3A$	$(3^2, 2^2)$	$6A$	$(9, 3)$	$9A$	$(5, 3^2)$	$15A$
(3^4)	$3A$	$(6, 2^3)$	$6A$	$(5, 2^2)$	$10A$	$(5, 4, 2)$	$20B$
(3^3)	$3B$	(6^2)	$6A$	$(10, 2)$	$10A$	$(7, 3)$	$21A$
(4^2)	$4A$	$(6, 3, 2)$	$6B$	(11)	$11A$	$(5, 3, 2^2)$	$30B$
$(4^2, 2^2)$	$4A$	$(3, 2^4)$	$6C$	$(4^2, 3)$	$12A$	$(7, 5)$	$35A$

Table 4.1: Conjugacy classes of elements of A in $\mathbb{M}$

Proof. Firstly by Table 3.1, the conjugacy class of the product of a $2A$-involution and $3A$-element is known if the isomorphism class of the group generated by the $(2A, 3A)$-pair is known and is not S_4 or $2 \times A_4$. Using the list of representatives for the orbits of A on its $(2A, \langle 3A \rangle)$-pairs (see Appendix B), most of the cycle shapes of elements in A are covered. The remaining 5 cycle shapes may be deduced by identifying the cycle shapes of various powers of the element. The conjugacy classes of powers are listed in [AgrV3]. □

Remark. The fusions of conjugacy classes of A in $\mathbb{M}$ seen in Table 4.1 was already proven by Decelle in [Dec13] but by a different approach. It may be deduced from Table 4.1 that A intersects 24 of the Monster's 194 conjugacy classes.

Let V_A be the subalgebra of the Monster algebra generated by the $2A$-axes associated to the $2A$-involutions in A. Then V_A has 11880 $2A$-axes generators (1485 associated to (2^2) + 10395 associated to (2^6)). The dimension of V_A is still unknown. However, the dimension of the linear span of all NA-axes ($2 \leq N \leq 5$) in V_A was determined to be 3960 with its codimension bounded by 1191 [CRI14].

The main aim of this chapter is to determine the values of all inner products between two $3A$-axes in V_A. Since the action of the Monster on its algebra and hence on V_A preserves the inner product, so does the action of A on V_A. Hence it is sufficient to determine the inner product for a representative of each orbit of A on unordered pairs of $3A$-axes in V_A. A $3A$-axis in V_A is associated to a $3A$-cyclic subgroup in A. This association is bijective by the $3A$-condition. Therefore, to classify the orbits of A on pairs of $3A$-axes in V_A, it is equivalent to classify the orbits of A acting by conjugation on its pairs of $3A$-cyclic subgroups.

Proposition 4.2. *There are a total of* 191 *orbits for the conjugation action of A on ordered pairs of its $3A$-cyclic subgroups.*

Proof. There are 6 cases to consider due to the three cycle shapes of $3A$-elements in A. Let $x_1 := (1,2,3)$, $x_2 := (1,2,3)(4,5,6)$ and $x_3 := (1,2,3)(4,5,6)(7,8,9)(10,11,12)$. For $i \in \{1,2,3\}$, let H_i be the following normalizers in A, and let X_i be the following sets of conjugates of $3A$-cyclic subgroups:

$$H_i := N_A(\langle x_i \rangle) \quad \text{and} \quad X_i := \langle x_i \rangle^A.$$

Then the orbits of A on ordered pairs of its $3A$-cyclic subgroups are in bijection with the union of orbits of H_j on X_k, where $j,k \in \{1,2,3\}$ and $j \leq k$. The number of orbits may be determined with [GAP16] or by taking the inner products of characters corresponding to these actions of A (Theorem 0.3).

The number of orbits are 4, 6, 3, 32, 34 and 112, for H_1 on X_1, H_1 on X_2, H_1 on X_3, H_2 on X_2, H_2 on X_3, and H_3 on X_3 respectively. This gives a total of 191 orbits. □

Remark. A list of representatives for these 191 orbits is omitted from this thesis as it is the unordered pairs which are relevant (inner product is symmetric). However it would be useful to have a record of these orbit representative for the next proposition. This may be obtained using [GAP16].

Proposition 4.3. *There are a total of* 153 *orbits for the conjugation action of* A *on unordered pairs of its* $3A$*-cyclic subgroups.*

Proof. The only cases which need to be considered are of H_i on X_i for $i \in \{1, 2, 3\}$ from the proof of Proposition 4.2. For each of these cases, an ordered pair in each orbit is reversed and checked if it is contained in any other orbit. This may be done efficiently in [GAP16]. For $i = 2$, there are 5 ordered pairs conjugate as unordered pairs. For $i = 3$, there are 33 such pairs. There are none for $i = 1$. □

A list of representatives for the 153 orbits of A on unordered pairs of it $3A$-cyclic subgroups is recorded in Table 4.2 of Section 4.3. This list shall be called L throughout this chapter.

4.2 Methods for determining the inner products

Let h_1 and h_2 be two $3A$-elements in A.

The principal idea behind calculating most inner products (u_{h_1}, u_{h_2}) in V_A is by expressing it as a linear combination of other inner products. If the values of all the inner products in the expression are known, then so will the value of (u_{h_1}, u_{h_2}).

Norton determined all the inner products between a $2A$- and $3A$-axis in the Monster algebra. These values are in Table 3.1. The inner products between any two $2A$-axes are

also known by the Norton-Sakuma algebras (see Table 1.2).

Therefore, if (u_{h_1}, u_{h_2}) can be expressed as a linear combination of inner products between $2A$-axes and/or between $2A$- and $3A$-axes, one needs only to identify these values (by looking at the isomorphism class of group generated, or by looking at the conjugacy classes of products) to determine the value of (u_{h_1}, u_{h_2}). However, not all of the 153 orbit representatives in L shall be expressed this way and other techniques will be applied.

For some cases, (u_{h_1}, u_{h_2}) will be expressed as a linear combination of known inner products, some of which between pairs of $3A$-axes. The values of these inner products may be known by some previous result. It is then possible to determine (u_{h_1}, u_{h_2}). One would also expect cases where partial results are obtained (e.g. some inner products are known while some are not). This produces linear equations involving unknown inner products. The values of the unknown inner products may be worked out if there are a sufficient number of linear equations to solve the system.

Computing the values for all inner products between two $3A$-axes in V_A is therefore an iterative process. Inner products which were initially obtained are then used to compute other inner products until all 153 orbit representatives have been covered. Before beginning, we remark on the following notation.

Remark. If a $(2A, 3A)$-pair $\{t, h\}$ in A generates a subgroup isomorphic to S_4 or $2 \times A_4$, to distinguish which orbit of the Monster this pair belongs to, denote next the group generated, the conjugacy class of the product th in the Monster. For example, $\langle t, h\rangle \cong S_4(4B)$.

Lemma 4.4. *If there exists a $2A$-involution $t_1 \in A$ such that $\langle t_1, h_1\rangle \cong S_3$, then*

$$(u_{h_1}, u_{h_2}) = \frac{64}{135}(2(a_{t_1}, u_{h_2}) + 2(a_{t_2}, u_{h_2}) + (a_{t_3}, u_{h_2})) - \frac{2048}{135}(a_{t_1}, a_{t_2} \cdot u_{h_2})$$

where $t_2 := t_1^{h_1}$ and $t_3 := t_1^{h_1^{-1}}$.

Proof. Since $\langle t_1, t_2\rangle = \langle t_1, h_1\rangle \cong S_3$, and t_2 is also a $2A$-involution, $\langle\langle a_{t_1}, a_{t_2}\rangle\rangle$ is a Norton-Sakuma algebra of type $3A$. Then u_{h_1} may be expressed in terms of $2A$-axes:

$$u_{h_1} = \frac{64}{135}(2a_{t_1} + 2a_{t_2} + a_{t_3}) - \frac{2048}{135}a_{t_1} \cdot a_{t_2}.$$

Taking the inner product between this expression for u_{h_1} with u_{h_2}, then expanding linearly gives

$$(u_{h_1}, u_{h_2}) = \frac{64}{135}(2(a_{t_1}, u_{h_2}) + 2(a_{t_2}, u_{h_2}) + (a_{t_3}, u_{h_2})) - \frac{2048}{135}(a_{t_1} \cdot a_{t_2}, u_{h_2}).$$

The result follows by associativity of the inner product. □

Remark. By commutativity of the algebra product, the last term in the expression for (u_{h_1}, u_{h_2}) in Lemma 4.4 may be replaced with $-\frac{2048}{135}(a_{t_2}, a_{t_1} \cdot u_{h_2})$.

By Lemma 4.4, if the algebra product $a_{t_2} \cdot u_{h_2}$ can be expressed as a linear combination of $2A$- and/or $3A$-axes, expanding linearly, the inner product (u_{h_1}, u_{h_2}) is a linear combination of inner products between pairs of $2A$-axes and/or between $2A$- and $3A$-axes. By identifying the values of these inner products, the value for (u_{h_1}, u_{h_2}) may be determined. Such a scenario happens if $\langle t_2, h_2 \rangle$ (or $\langle t_1, h_2 \rangle$ by the remark above) is isomorphic to 6, S_3, A_4 or $S_4(4B)$ (see Propositions 3.7, 3.8 and 3.9).

The next four lemmas assumes the hypothesis of Lemma 4.4, i.e there exists a $2A$-involution $t_1 \in A$ such that $\langle t_1, h_1 \rangle \cong S_3$ and $t_2 := t_1^{h_1}$.

Lemma 4.5. *If $\langle t_2, h_2 \rangle \cong 6$, then*

$$(u_{h_1}, u_{h_2}) = \frac{64}{135}(2(a_{t_1}, u_{h_2}) + 2(a_{t_2}, u_{h_2}) + (a_{t_3}, u_{h_2})).$$

Proof. By Proposition 3.7, $a_{t_2} \cdot u_{h_2} = 0$. This cancels the last term in the expression for (u_{h_1}, u_{h_2}) in Lemma 4.4. □

Lemma 4.6. *If $\langle t_2, h_2 \rangle \cong S_3$, then*

$$\begin{aligned}(u_{h_1}, u_{h_2}) =& \frac{64}{135}(2(a_{t_1}, u_{h_2}) + 2(a_{t_2}, u_{h_2}) + (a_{t_3}, u_{h_2})) \\ &- \frac{2048}{1215}(2(a_{t_1}, a_{t_2}) - (a_{t_1}, a_{t_4}) - (a_{t_1}, a_{t_5})) - \frac{64}{27}(a_{t_1}, u_{h_2})\end{aligned}$$

where $\{t_4, t_5\} := {t_2}^{\langle t_2, h_2 \rangle} \setminus t_2$.

Proof. The result follows by substituting the expression for $a_{t_2} \cdot u_{h_2}$ from Proposition 3.7 into the expression in Lemma 4.4, then expanding linearly. □

Lemma 4.7. *If* $\langle t_2, h_2 \rangle \cong A_4$, *then*

$$\begin{aligned}(u_{h_1}, u_{h_2}) =& \frac{64}{135}(2(a_{t_1}, u_{h_2}) + 2(a_{t_2}, u_{h_2}) + (a_{t_3}, u_{h_2})) - \frac{2048}{1215}(a_{t_1}, a_{t_2}) \\ & - \frac{32}{135}(5(a_{t_1}, u_{h_2}) + 3(a_{t_1}, u_{h_3}) - 4(a_{t_1}, u_{h_4}) - 4(a_{t_1}, u_{h_5}))\end{aligned}$$

where $h_3 := h_2^{t_2}$ *and* $\{h_4, h_5\} := {h_2}^{\langle t_2, h_2 \rangle} \setminus \{h_2, h_3\}$.

Proof. This is similar to the proof of Lemma 4.6 except that the expression for $a_{t_2} \cdot u_{h_2}$ is from Proposition 3.9. □

Lemma 4.8. *If* $\langle t_2, h_2 \rangle \cong S_4(4B)$, *then*

$$\begin{aligned}(u_{h_1}, u_{h_2}) =& \frac{64}{135}(2(a_{t_1}, u_{h_2}) + 2(a_{t_2}, u_{h_2}) + (a_{t_3}, u_{h_2})) \\ & - \frac{2048}{6075}((a_{t_1}, a_{t_2}) + (a_{t_1}, a_{t_4}) - (a_{t_1}, a_{t_5})) \\ & + \frac{1024}{6075}((a_{t_1}, a_{t_6}) + (a_{t_1}, a_{t_7}) + (a_{t_1}, a_{t_8}) + (a_{t_1}, a_{t_9})) \\ & - \frac{32}{135}((a_{t_1}, u_{h_2}) - (a_{t_1}, u_{h_3}) + (a_{t_1}, u_{h_4}) + (a_{t_1}, u_{h_5}))\end{aligned}$$

where $t_4 \in {t_2}^{\langle t_2, h_2 \rangle}$ *such that* $o(t_2 t_4) = 2$, $t_5 := t_2 t_4$, $\{t_6, t_7, t_8, t_9\} := {t_2}^{\langle t_2, h_2 \rangle} \setminus \{t_2, t_4\}$, $h_3 := h_2^{t_2}$ *and* $\{\langle h_4 \rangle, \langle h_5 \rangle\} := \langle h_2 \rangle^{\langle t_2, h_2 \rangle} \setminus \{\langle h_2 \rangle, \langle h_3 \rangle\}$.

Proof. Again this is similar to the proof of Lemma 4.6 except the expression for $a_{t_2} \cdot u_{h_2}$ is from Proposition 3.8. □

Now we are ready to determine some of the inner products between $3A$-axes in V_A. For a pair $\{u_{h_1}, u_{h_2}\}$ of $3A$-axes, there will be instances where there are two $2A$-involutions $t_1, \tilde{t_1} \in A$ satisfying Lemma 4.4. In the first case, t_2 may satisfy Lemma 4.5 while in the second, $\tilde{t_2} := \tilde{t_1}^{h_1}$ may satisfy Lemma 4.6. Since there are fewer terms in the expression for (u_{h_1}, u_{h_2}) in Lemma 4.5 compared to that of Lemma 4.6, it is obviously more convenient to pick t_1. Therefore, running through the list L of the 153 orbit representatives, we check

if Lemmas 4.5 to 4.8 are satisfied in that particular order.

Implementation of Lemmas 4.5 to 4.8 and methods introduced later on may be done efficiently in [GAP16]. The common strategy is to identify $2A$-involutions which fit the criteria of the lemma and [GAP16] is very efficient at finding said $2A$-involution from the set of 11880 $2A$-involutions in A. This process may be described as follows. Firstly, the set of $2A$-involutions in A is defined as T_A. Then a code is written such that one may input a pair of $3A$-elements from L and [GAP16] finds a $2A$-involution from T_A which satisfies the conditions of the lemma. If there are no involutions which meet the conditions, the pair is skipped and will be tested again with a different code corresponding to a different lemma. After we have found a suitable involution and have an expression for the inner product, identifying the known inner product values in the expression is a technical matter.

Remark. Of the 153 orbits representatives in L, there are exactly three pairs $\{h_1, h_2\}$ such $h_1 = h_2$ (one for each $3A$ cycle shape in A). By the $3A$-condition, the $3A$-axis u_{h_1} coincides with u_{h_2}. Hence

$$(u_{h_1}, u_{h_2}) = \frac{8}{5}.$$

Proposition 4.9. *Of the remaining* 150 *orbit representatives in L, there are exactly* 31 *pairs whose inner products may be determined by Lemma 4.5. The values for the inner products may be then computed by the given expression in Lemma 4.5 and is listed in Table 4.2.*

Proof. This may be shown in [GAP16]. □

Proposition 4.10. *Of the remaining* 119 *orbit representatives in L, there are exactly* 24 *pairs whose inner products may be determined by Lemma 4.6. The values for the inner products may be then computed by the given expression in Lemma 4.6 and is listed in Table 4.2.*

Proof. This may be shown in [GAP16]. □

Proposition 4.11. *Of the remaining* 95 *orbit representatives in* L*, there are exactly* 45 *pairs whose inner products may be determined by Lemma 4.7. The values for the inner products may be then computed by the given expression in Lemma 4.7 and is listed in Table 4.2.*

Proof. This may be shown in [GAP16]. □

Proposition 4.12. *Of the remaining* 50 *orbit representatives in* L*, there are exactly* 12 *pairs whose inner products may be determined by Lemma 4.8. The values for the inner products may be then computed by the given expression in Lemma 4.8 and is listed in Table 4.2.*

Proof. This may be shown in [GAP16]. □

This leaves 38 orbits to consider.

In Lemma 4.4, a $3A$-axis u_{h_1} was expressed only in terms of $2A$-axes. The following lemma gives a condition where a $3A$-axis u_{h_1} may be expressed in terms of both $2A$- and $3A$-axes. The expression for the inner product (u_{h_1}, u_{h_2}) would then include inner products between pairs of $3A$-axes. Here is when the iterative process begins.

Lemma 4.13. *If there exists a* $2A$*-involution* $t_1 \in A$ *such that* $\langle t_1, h_1 \rangle \cong A_4$*, then*

$$\begin{aligned}(u_{h_1}, u_{h_2}) = &-\frac{64}{45}(a_{t_1}, u_{h_2}) - \frac{1}{5}(3(u_{h_3}, u_{h_2}) - 4(u_{h_4}, u_{h_2}) - 4(u_{h_5}, u_{h_2})) \\ &+ \frac{64}{5}(u_{h_1}, a_{t_1} \cdot u_{h_2})\end{aligned}$$

where $h_3 := h_1^{t_1}$ *and* $\{h_4, h_5\} := h_1^{\langle t_1, h_1 \rangle} \setminus \{h_1, h_3\}$.

Proof. By Proposition 3.9, u_{h_1} may be expressed in terms of $2A$- and $3A$-axes:

$$u_{h_1} = -\frac{64}{45}a_{t_1} - \frac{1}{5}(3u_{h_3} - 4u_{h_4} - 4u_{h_5}) + \frac{64}{5}a_{t_1} \cdot u_{h_1}.$$

Taking the inner product between u_{h_1} in the above expression with u_{h_2}, then expanding linearly gives:

$$(u_{h_1}, u_{h_2}) = -\frac{64}{45}(a_{t_1}, u_{h_2}) - \frac{1}{5}(3(u_{h_3}, u_{h_2}) - 4(u_{h_4}, u_{h_2}) - 4(u_{h_5}, u_{h_2}))$$

$$+ \frac{64}{5}(a_{t_1} \cdot u_{h_1}, u_{h_2}).$$

The result follows by commutativity of the algebra product and associativity of the inner product. □

If the algebra product $a_{t_1} \cdot u_{h_2}$ from Lemma 4.13 can be written as a linear combination of $2A$- and $3A$-axes, then (u_{h_1}, u_{h_2}) can be written as a linear combination of inner products between pairs of $2A$-axes, between $2A$- and $3A$-axes, and between pairs of $3A$-axes. If the inner products between the pairs of $3A$-axes are already known by previous results, then the value for (u_{h_1}, u_{h_2}) may also be determined.

The next two lemmas assumes the hypothesis of Lemma 4.13, i.e. there exist a $2A$-involution $t_1 \in A$ such that $\langle t_1, h_1 \rangle \cong A_4$.

Lemma 4.14. *If* $\langle t_1, h_2 \rangle \cong 6$, *then*

$$(u_{h_1}, u_{h_2}) = \frac{1}{5}(-3(u_{h_3}, u_{h_2}) + 4(u_{h_4}, u_{h_2}) + 4(u_{h_5}, u_{h_2})).$$

Proof. The first and last terms of the expression for (u_{h_1}, u_{h_2}) in Lemma 4.13 vanishes since $(a_{t_1}, u_{h_2}) = a_{t_1} \cdot u_{h_2} = 0$ when $\langle t_1, h_2 \rangle \cong 6$. □

Lemma 4.15. *If* $\langle t_1, h_2 \rangle \cong S_4(4B)$, *then*

$$\begin{aligned}(u_{h_1}, u_{h_2}) = &- \frac{16}{9}(a_{t_1}, u_{h_2}) - \frac{1}{4}(3(u_{h_3}, u_{h_2}) - 4(u_{h_4}, u_{h_2}) - 4(u_{h_5}, u_{h_2})) \\ &+ \frac{16}{45}((a_{t_1}, u_{h_1}) + (a_{t_2}, u_{h_1}) - (a_{t_3}, u_{h_1})) \\ &- \frac{8}{45}((a_{t_4}, u_{h_1}) + (a_{t_5}, u_{h_1}) + (a_{t_6}, u_{h_1}) + (a_{t_7}, u_{h_1})) \\ &+ \frac{1}{4}(-(u_{h_6}, u_{h_1}) + (u_{h_7}, u_{h_1}) + (u_{h_8}, u_{h_1}))\end{aligned}$$

where $t_2 \in t_1^{\langle t_1, h_2 \rangle}$ *such that* $o(t_1 t_2) = 2$, $t_3 := t_1 t_2$, $\{t_4, t_5, t_6, t_7\} := t_1^{\langle t_1, h_2 \rangle} \setminus \{t_1, t_2\}$, $h_6 := h_2^{t_1}$ *and* $\{\langle h_7 \rangle, \langle h_8 \rangle\} := \langle h_2 \rangle^{\langle t_2, h_2 \rangle} \setminus \{\langle h_2 \rangle, \langle h_6 \rangle\}$.

Proof. Substituting $a_{t_1} \cdot u_{h_2}$ from Proposition 3.8 into the expression for (u_{h_1}, u_{h_2}) in Lemma 4.13 gives an equation where (u_{h_1}, u_{h_2}) appears on both sides. The result follows after rearrangement. □

Remark. If $\langle t_1, h_2 \rangle \cong S_3$, then using the method in Lemma 4.15, there is a similar expression for (u_{h_1}, u_{h_2}). This is omitted for the reason that the condition was not met for any of the remaining 38 orbit representatives. If $\langle t_1, h_2 \rangle \cong A_4$, the term (u_{h_1}, u_{h_2}) vanishes.

Proposition 4.16. *Of the remaining* 38 *orbit representatives in L, there is only* 1 *pair whose inner product may be determined by Lemma 4.14. All inner products in the expression for (u_{h_1}, u_{h_2}) are already known by previous methods.*

Proof. This may be shown in [GAP16]. The only remaining pair in L for which there exists an involution t_1 satisfying the conditions of Lemma 4.14 is:

$$\{h_1, h_2\} = \{(1,2,3)(4,5,6)(7,8,9)(10,11,12), (1,2,3)(4,5,7)(6,8,10)(9,11,12)\}.$$

An example for t_1 is $(1,10)(2,6)(3,8)(4,12)(5,9)(7,11)$. Then $\{\langle h_2 \rangle, \langle h_3 \rangle\}$ is conjugate to $\{\langle h_1 \rangle, \langle h_2 \rangle\}$ (by $(1,6,11,7,2,8,3,10,5,9)(4,12)$), and $(u_{h_4}, u_{h_2}) = (u_{h_5}, u_{h_2}) = \frac{28}{405}$ by Lemma 4.8. □

Proposition 4.17. *Of the remaining* 37 *orbit representatives in L, there are exactly* 10 *pairs whose inner products may be determined by Lemma 4.15. All inner products in the expression for (u_{h_1}, u_{h_2}) are already known by previous methods.*

Proof. This may be shown in [GAP16]. □

There are some cases where the hypotheses of Lemmas 4.14 and 4.15 are satisfied but not all inner products in the expression are known. For these cases, linear equations in unknown inner products are obtained. Although there is not enough information to determine (u_{h_1}, u_{h_2}), these linear equations are recorded.

Proposition 4.18. *Of the remaining* 27 *orbit representatives in L, there are* 2 *cases where the inner products may be determined by solving linear equations arising from Lemmas 4.14 and 4.15.*

Proof. Solving the system of linear equations is straightforward. □

This leaves 25 orbits to consider.

The next result shows thats any two pairs of $3A$-cyclic subgroups are conjugate in the Monster if they generate a certain subgroup. If pairs are conjugate in the Monster, the inner products between the associated pairs of $3A$-axes would have the same value.

Lemma 4.19. *Let h_1, h_2, h_3 and h_4 be $3A$-elements in $\mathbb{M}$. Suppose that*

$$\langle h_1, h_2\rangle \cong \langle h_3, h_4\rangle \cong A_4.$$

If moreover the involutions in $\langle h_1, h_2\rangle$ and $\langle h_1, h_2\rangle$ are $2A$-involutions, then $\{\langle h_1\rangle, \langle h_2\rangle\}$ is conjugate to $\{\langle h_3\rangle, \langle h_4\rangle\}$ in the Monster.

Proof. Without loss of generality, h_1 and h_2 may be assumed to be in the same conjugacy class in $\langle h_1, h_2\rangle$. The same goes for h_3 and h_4 in $\langle h_3, h_4\rangle$. There exist $2A$-involutions t_1 and t_2 in $\langle h_1, h_2\rangle$ and $\langle h_3, h_4\rangle$ respectively such that

$$h_1^{t_1} = h_2 \quad \text{and} \quad h_3^{t_2} = h_4.$$

Then

$$\langle t_1, h_1\rangle = \langle h_1, h_2\rangle \cong A_4 \cong \langle h_3, h_4\rangle = \langle t_2, h_3\rangle.$$

By Norton's list, there is only orbit for which a $(2A, 3A)$-pair generates A_4. Hence there exists $g \in \mathbb{M}$ such that

$$t_1^g = t_2 \quad \text{and} \quad h_1^g = h_3.$$

Then h_2 is also conjugate to h_4 in $\mathbb{M}$ by g:

$$h_2^g = h_1^{t_1 g} = h_3^{t_2} = h_4.$$

□

Lemma 4.20. *Let h_3 be another $3A$-element in A. If $\langle h_1, h_3\rangle \cong A_4$ and it contains $2A$-involutions, then*

$$u_{h_1} \cdot u_{h_3} = \frac{1}{5}(u_{h_1} + u_{h_3}) - \frac{1}{18}(u_{h_4} + u_{h_5}) + \frac{64}{2025}(2a_{t_1} - 3a_{t_2} - 3a_{t_3})$$

where $\{h_4, h_5\} := h_1^{\langle h_1, h_3\rangle} \setminus \{h_1, h_3\}$, $t_1 := h_1 h_3 h_1$ and $\{t_2, t_3\} := t_1^{\langle h_1, h_3\rangle} \setminus t_1$. It is assumed without loss of generality that h_1 and h_3 belong to the same conjugacy class in $\langle h_1, h_3\rangle$.

Proof. By Lemma 4.19, there is a unique orbit for the Monster on pairs of its $3A$-cyclic subgroups such that the pair satisfies the conditions of the lemma. The equation for the algebra product $u_{h_1} \cdot u_{h_3}$ may be obtained from the Majorana representation of S_4 with shape $(2A, 3A)$. □

Lemma 4.21. *Assuming the hypothesis and notations of Lemma 4.20, if* $(u_{h_2}, u_{h_3}) = 0$*, Then*

$$(u_{h_1}, u_{h_2}) = \frac{5}{18}((u_{h_2}, u_{h_4}) + (u_{h_2}, u_{h_5})) - \frac{64}{2025}(2(u_{h_2}, a_{t_1}) - 3(u_{h_2}, a_{t_2}) - 3(u_{h_2}, a_{t_3})).$$

Proof. By Lemma 4.20, u_{h_1} may be written as a linear combination of $2A$- and $3A$-axes. Taking the inner product between this expression for u_{h_1} with u_{h_2}, then expanding linearly and cancelling all zero terms gives the desired expression. □

Proposition 4.22. *Of the remaining* 25 *orbit representatives in* L*, there are exactly* 4 *pairs whose inner products can be determined by Lemma 4.21.*

Proof. This may be shown in [GAP16]. □

Remark. From the methods which have been used so far to determine the inner products between $3A$-axes in V_A, it is observed that when $(u_{h_1}, u_{h_2}) = 0$, then h_1 and h_2 commute. The converse is not true as seen in the following result.

Proposition 4.23. *If two* $3A$*-elements commute, the inner product between the associated* $3A$*-axes is not necessarily zero.*

Proof. This is best seen with an example. Let $h_1 = (4, 7, 5)(6, 9, 8)$, $h_2 = (1, 2, 3)(4, 5, 6)$, and $h_3 = (4, 5, 6)(7, 8, 9)$. Then h_2 and h_3 commute. Assume for contradiction that $(u_{h_2}, u_{h_3}) = 0$. By Lemma 4.21,

$$(u_{h_1}, u_{h_2}) = \frac{88}{2025}.$$

Now let $t_1 = (5, 7)(8, 9)$. Then $\langle t_1, h_1 \rangle \cong S_3$ and $\langle t_1^{h_1}, h_2 \rangle \cong A_4$. By Lemma 4.7,

$$(u_{h_1}, u_{h_2}) = \frac{124}{2025}.$$

□

Proposition 4.24. *Of the remaining* 21 *orbit representatives in* L*, there are* 5 *pairs where the inner products may be determined by solving linear equations arising from Lemma 4.15 and Lemma 4.21.*

Proof. This may be shown in [GAP16]. □

Lemma 4.25. *Suppose there exists a* $2A$*-involution* $t_1 \in A$ *such that* $\langle t_1, h_1\rangle \cong S_4(4B)$*. Let* $h_3 := h_1^{t_1}$*. Then*

$$u_{h_1} = \frac{64}{45}(a_{t_1} + a_{t_2} - a_{t_3}) - \frac{32}{45}(a_{t_4} + a_{t_5} + a_{t_6} + a_{t_7}) + u_{h_3} + u_{h_4} + u_{h_5} - 64a_{t_1} \cdot u_{h_3}$$

where $t_2 \in t_1^{\langle t_1, h_3\rangle}$ *such that* $o(t_1t_2) = 2$*,* $t_3 := t_1t_2$*,* $\{t_4, t_5, t_6, t_7\} := t_1^{\langle t_1, h_3\rangle} \setminus \{t_1, t_2\}$ *and* $\{\langle h_4\rangle, \langle h_5\rangle\} := \langle h_3\rangle^{\langle t_1, h_3\rangle} \setminus \{\langle h_1\rangle, \langle h_3\rangle\}$*.*

Proof. Since $\langle t_1, h_3\rangle = \langle t_1, h_1\rangle \cong S_4(4B)$, the algebra product $a_{t_1} \cdot u_{h_3}$ by Proposition 3.9 is given as

$$a_{t_1} \cdot u_{h_3} = \frac{1}{45}(a_{t_1} + a_{t_2} - a_{t_3}) - \frac{1}{90}(a_{t_4} + a_{t_5} + a_{t_6} + a_{t_7}) + \frac{1}{64}(u_{h_3} - u_{h_1} + u_{h_4} + u_{h_5}).$$

The result follows by rearranging the equation above. □

Lemma 4.26. *Assuming the hypothesis and notations of Lemma 4.25, if* $\langle h_2, h_3\rangle \cong A_4$ *and the involutions in* $\langle h_2, h_3\rangle$ *are in* $2A$*, then*

$$\begin{aligned}(u_{h_1}, u_{h_2}) =& \frac{64}{45}((a_{t_1}, u_{h_2}) + (a_{t_2}, u_{h_2}) - (a_{t_3}, u_{h_2}))\\ &- \frac{32}{64}((a_{t_4}, u_{h_2}) + (a_{t_5}, u_{h_2}) + (a_{t_6}, u_{h_2}) + (a_{t_7}, u_{h_2}))\\ &+ (u_{h_3}, u_{h_2}) + (u_{h_4}, u_{h_2}) + (u_{h_5}, u_{h_2})\\ &- \frac{64}{5}((a_{t_1}, u_{h_3}) + (a_{t_1}, u_{h_2})) + \frac{32}{9}((a_{t_1}, u_{h_6}) + (a_{t_1}, u_{h_7}))\\ &- \frac{4096}{2025}(2(a_{t_1}, a_{t_8}) - 3(a_{t_1}, a_{t_9}) - 3(a_{t_1}, a_{t_{10}}))\end{aligned}$$

where $\{h_6, h_7\} := h_3^{\langle h_2, h_3\rangle} \setminus \{h_2, h_3\}$*,* $t_8 := h_3h_2h_3$ *and* $\{t_9, t_{10}\} := t_8^{\langle h_2, h_3\rangle} \setminus t_8$*. It is assumed without loss of generality that* h_2 *and* h_3 *belong to the same conjugacy class in* $\langle h_2, h_3\rangle$*.*

Proof. Firstly, u_{h_1} may be expressed in terms of $2A$- and $3A$-axes by Lemma 4.25. Then the inner product between this expression for u_{h_1} with u_{h_2} may be expanded linearly. By associativity of the inner product, the term $u_{h_3} \cdot u_{h_2}$ is present. This term is replaced with the expression in Lemma 4.20. The result follows after linear expansion. □

Proposition 4.27. *Of the remaining* 16 *orbit representatives in L, there is only* 1 *pair for which the inner product may be determined by Lemma 4.26.*

Proof. This may be shown in [GAP16]. The only remaining pair in L for which there exists an involution t_1 satisfying the conditions of Lemmas 4.25 and 4.26 is

$$\{h_1, h_2\} = \{(1,2,3)(4,5,6)(7,8,9)(10,11,12), (1,2,3)(4,5,7)(6,10,11)(8,9,12)\}.$$

An example for t_1 is $(1,6)(2,5)(3,4)(7,12)(8,9)(10,11)$. □

The next method for determining the values of the remaining inner products is by showing that two pairs of $3A$-cyclic subgroups in A are conjugate in the Monster. The inner products between the associated pairs of $3A$-axes would then have the same value.

Lemma 4.28. *Let h_1, h_2, h_3 and h_4 be $3A$-elements in A. Suppose there exist $2A$-involutions t_1, t_2 (not necessarily in A) such that $\{t_1, h_1\}$ is conjugate to $\{t_2, h_3\}$ in the Monster. Suppose also that*

$$h_2 = h_1^{t_1} \quad \textit{and} \quad h_4 = h_3^{t_2}.$$

Then $\{h_1, h_2\}$ is conjugate to $\{h_3, h_4\}$ in the Monster.

Proof. There exists $g \in \mathbb{M}$ such that

$$t_1^g = t_2 \quad \text{and} \quad h_1^g = h_3.$$

Now g also conjugates h_2 to h_4:

$$h_2^g = h_1^{t_1 g} = t_1^g h_1^g t_1^g = t_2 h_3 t_2 = h_3^{t_2} = h_4.$$

□

Remark. If t_1 and t_2 from Lemma 4.28 are both in A, then h_1 has the same cycle shape as h_2, and h_3 has the same cycle shape as h_4.

Proposition 4.29. *Of the remaining* 15 *orbit representatives in L, there are* 3 *pairs such that there exists $2A$-involutions $t_1, t_2 \in A$ satisfying the conditions of Lemma 4.28. By this lemma, these pairs are conjugate to pairs whose inner products were already calculated by previous methods.*

Proof. This may be shown in [GAP16]. Since the involutions are from A, the remaining pairs which can be considered are only pairs with the same cycle shape. □

There are now 12 orbits left to consider. To determine these values, we apply Pasechnik's relation.

Lemma 4.30. *Let h_3 and h_4 be $3A$-elements in A which commute. If there exists a $2A$-involution $t_1 \in A$ inverting both h_3 and h_4, the vector*

$$\frac{32}{45}\sum_{i=1}^{9} a_{t_i} - (u_{h_3} + u_{h_4} + u_{h_3h_4} + u_{h_3h_4^{-1}})$$

where $\{t_1, \ldots, t_9\} := t_1^{\langle h_3, h_4, t_1\rangle}$, is the zero vector.

Proof. The algebra generated by $\{a_{t_1}, \ldots, a_{t_9}\}$ is isomorphic to the Majorana representation of $3^2 : 2$ with shape $(2A, 3A)$ in [Iva11b]. By Pasechnik's relation, the vector above is the zero vector. □

Lemma 4.31. *If h_1 has cycle shape (3^2) or (3^4), then*

$$(u_{h_1}, u_{h_2}) = \frac{32}{45}\sum_{i=1}^{9}(a_{t_i}, u_{h_2}) - ((u_{h_3}, u_{h_2}) + (u_{h_4}, u_{h_2}) + (u_{h_3h_4^{-1}}, u_{h_2}))$$

where $h_3, h_4 \in A$ are $3A$-elements which commute such that $h_1 = h_3h_4$. The $2A$-involution $t_1 \in A$ inverts both h_3 and h_4, and $\{t_1, \ldots, t_9\} := t_1^{\langle h_3, h_4, t_1\rangle}$.

Proof. There are two cases to consider. For the first case, let

$$h_1 = (1,2,3)(4,5,6),\ h_3 = (1,2,3),\ h_4 = (4,5,6) \text{ and } t_1 = (2,3)(5,6).$$

For the second, let

$$h_1 = (1,2,3)(4,5,6)(7,8,9)(10,11,12),\ h_3 = (1,2,3)(4,5,6),\ h_4 = (7,8,9)(10,11,12)$$

$$\text{and } t_1 = (1,4)(2,6)(3,5)(7,10)(8,12)(9,11).$$

For both cases, h_3 and h_4 commute, they are both inverted by t_1, and $h_1 = h_3h_4$.

Any other $3A$-element in A with cycle shape (3^2) is conjugate to $(1,2,3)(4,5,6)$. Then the choices for h_3, h_4 and t_1 are the corresponding conjugate elements from the first case. Similarly for cycle shapes (3^4) and the second case.

By Lemma 4.30,

$$\frac{32}{45}\sum_{i=1}^{9} a_{t_i} - (u_{h_3} + u_{h_4} + u_{h_1} + u_{h_3h_4^{-1}})$$

is the zero vector. Hence the inner product between the vector above and u_{h_2} is zero. Expanding and rearranging this inner product gives the result. □

Proposition 4.32. *The inner products for the remaining* 12 *orbit representatives in L may be determined using Lemma 4.31.*

Proof. Since for the remaining cases, h_1 is one of the two cases in Lemma 4.31, we only need to determine the inner products in the expression for (u_{h_1}, u_{h_2}). They are all known by previous results. □

This completes the aim of determining all inner products between $3A$-axes in V_A.

Theorem 4.33. *Let h_3 and h_4 be any pair of $3A$-elements in A. Then the unordered pair $\{\langle h_3\rangle, \langle h_4\rangle\}$ is conjugate in A to one of the* 153 *pairs in Table 4.2. Say it is conjugate to $\{\langle h_1\rangle, \langle h_2\rangle\}$. Then the inner product (u_{h_3}, u_{h_4}) which is equal to (u_{h_1}, u_{h_2}), is listed in Table 4.2.*

4.3 The results

no.	h_1	h_2	l	(u_{h_1}, u_{h_2})	$C_{1,2}$	$\langle h_1, h_2 \rangle$
1	3	$(1,2,3)$	-	$\frac{2^3}{5}$	$\{1A, 3A\}$	3
2	3^2	$(1,2,3)(4,5,6)$	-			
3	3^4	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	-			
4	3	$(10,12,11)$	4.5	0	$\{3A\}$	3^2
5	3^2	$(7,8,9)(10,11,12)$	4.5			
6	3^2	$(1,3,2)(4,5,6)$	4.5			
7	3^4	$(1,3,2)(4,6,5)(7,8,9)(10,11,12)$	4.5			
8	3	$(1,2,3)(4,5,6)$	4.6			
9	3^2	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	4.6			
10	3	$(7,8,9)(10,11,12)$	4.5	0	$\{3B\}$	
11	3^2	$(1,2,3)(4,6,5)(7,8,9)(10,11,12)$	4.5			
12	3	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	4.31	$\frac{2^2}{5^2}$	$\{3A, 3B\}$	
13	3^2	$(4,5,6)(10,11,12)$	4.31			
14	3^4	$(1,2,3)(4,5,6)(7,8,9)(10,12,11)$	4.31			
15	3^4	$(1,2,3)(4,7,10)(5,8,11)(6,9,12)$	4.31			
16	3	$(2,3,4)$	4.6	$\frac{2^3 \cdot 17}{3^4 \cdot 5}$	$\{2A, 3A\}$	A_4
17	3^2	$(2,7,3)(4,5,6)$	4.6			
18	3^2	$(1,4,2)(3,6,5)$	4.6			
19	3^4	$(1,2,3)(4,5,6)(7,8,10)(9,11,12)$	4.6			
20	3^4	$(1,8,4)(2,11,7)(3,5,10)(6,9,12)$	4.7			
21	3^2	$(2,3,7)(5,6,8)$	4.6	$\frac{2^3 \cdot 7}{3^3 \cdot 5^2}$	$\{2B, 3A\}$	
22	3^4	$(1,2,4)(3,5,6)(7,8,10)(9,11,12)$	4.6			
23	3^2	$(2,4,5)(3,6,7)$	4.7	$\frac{2^2}{3^3}$	$\{3A, 7A\}$	F_{21}
24	3^2	$(2,4,8)(3,7,6)$	4.8	$\frac{2^3 \cdot 7}{3^3 \cdot 5^2}$	$\{4A, 6C\}$	$SL_2(3)$
25	3^2	$(1,7,4)(2,9,6)(3,8,5)(10,12,11)$	4.7	$\frac{2^4}{3 \cdot 5^2}$	$\{3A\}$	$3^2 : 3$

26	3^4	$(1,2,3)(4,5,7)(6,10,11)(8,9,12)$	4.26			
27	3^4	$(1,2,3)(4,6,10)(5,7,9)(8,12,11)$	4.31	$\frac{2^2}{3\cdot5^2}$	$\{3B\}$	
28	3^4	$(1,2,3)(4,6,7)(5,10,12)(8,11,9)$	4.31			
29	3	$(2,4,3)(10,12,11)$	4.6	$\frac{2^6}{3^4\cdot5}$	$\{3A,6A\}$	$3\times A_4$
30	3^2	$(1,2,4)(3,5,6)(7,8,9)(10,11,12)$	4.6			
31	3^2	$(2,3,7)(4,5,6)$	4.6			
32	3^4	$(1,2,3)(4,5,6)(7,9,11)(8,10,12)$	4.6			
33	3^2	$(5,6,7)(10,11,12)$	4.15	$\frac{2^2\cdot17}{3^4\cdot5^2}$	$\{3B,6A\}$	
34	3^4	$(1,2,3)(4,5,7)(6,8,9)(10,12,11)$	4.28			
35	3^4	$(1,2,4)(3,7,12)(5,8,9)(6,10,11)$	4.15			
36	3^4	$(1,2,4)(3,7,8)(5,6,10)(9,11,12)$	4.6	$\frac{2^3}{3^2\cdot5}$	$\{3A,4A\}$	$4^2:3$
37	3^4	$(1,4,9)(2,5,10)(3,7,11)(6,8,12)$	4.6			
38	3	$(3,4,5)$	4.5	$\frac{2^4}{3^4\cdot5}$	$\{5A\}$	A_5
39	3^2	$(1,2,4)(3,6,5)$	4.5			
40	3^4	$(1,4,2)(3,9,7)(5,6,10)(8,12,11)$	4.6			
41	3^2	$(3,9,7)(6,10,8)$	4.5	$\frac{2^4\cdot13}{3^4\cdot5^2}$	$\{5A\}$	
42	3^4	$(1,2,4)(3,6,5)(7,8,10)(9,12,11)$	4.5			
43	3^4	$(1,2,4)(3,6,5)(7,9,12)(8,10,11)$	4.5			
44	3	$(1,4,7)(2,5,8)(3,6,9)(10,11,12)$	4.7	$\frac{2^4}{3^2\cdot5^2}$	$\{9A\}$	$3^3:3$
45	3^4	$(1,3,2)(4,10,7)(5,12,8)(6,11,9)$	4.7			
46	3^2	$(1,2,3)(4,7,10)(5,8,11)(6,9,12)$	4.7	$\frac{2^3}{3^2\cdot5^2}$	$\{9A\}$	
47	3^2	$(1,4,7)(2,5,8)(3,6,9)(10,11,12)$	4.7			
48	3^4	$(1,2,4)(3,5,7)(6,10,9)(8,11,12)$	4.7	$\frac{2^3\cdot17}{3^4\cdot5^2}$	$\{4B,6A\}$	$2^3.A_4$
49	3^2	$(2,7,3)(5,6,8)$	4.5	$\frac{2^7}{3^4\cdot5^2}$	$\{6A\}$	A_4^2
50	3^4	$(1,2,4)(3,5,6)(7,9,11)(8,10,12)$	4.5			
51	3^2	$(3,4,5)(6,7,8)$	4.7	$\frac{2^2\cdot19}{3^3\cdot5^2}$	$\{4A,7A\}$	$L_2(7)$
52	3^2	$(2,4,3)(5,6,7)$	4.6	$\frac{2^5}{3^4\cdot5}$	$\{4B\}$	
53	3^2	$(2,4,6)(3,5,7)$	4.7	$\frac{2^2}{3^4}$	$\{4B,7A\}$	

54	3^2	$(2,4,7)(3,5,8)$	4.6	$\frac{2^5}{3^2\cdot 5^2}$	$\{4A\}$	
55	3^2	$(3,4,7)(5,6,8)$	4.7	$\frac{2^2}{3\cdot 5^2}$	$\{6C,7A\}$	$(2^3:7):3$
56	3	$(3,4,5)(10,11,12)$	4.5	$\frac{2^4}{3^4\cdot 5}$	$\{15A\}$	$3\times A_5$
57	3^2	$(1,2,4)(3,6,5)(7,8,9)(10,11,12)$	4.5			
58	3^2	$(3,7,8)(4,5,6)$	4.8	$\frac{2^2\cdot 47}{3^4\cdot 5^2}$	$\{5A,15A\}$	
59	3^4	$(1,2,3)(4,5,6)(7,8,10)(9,12,11)$	4.8			
60	3^4	$(1,2,3)(4,5,7)(6,9,11)(8,10,12)$	4.7	$\frac{2^4}{3^3\cdot 5}$	$\{4A,6B\}$	$3^2.SL_2(3)$
61	3^4	$(1,3,2)(4,10,5)(6,11,7)(8,9,12)$	4.7			
62	3^4	$(1,2,4)(3,7,12)(5,8,11)(6,10,9)$	4.7			
63	3^4	$(1,4,2)(3,12,7)(5,9,10)(6,11,8)$	4.7			
64	3^4	$(1,2,3)(4,5,7)(6,8,10)(9,11,12)$	4.14	$\frac{2^2\cdot 7}{3^4\cdot 5}$	$\{6A,9A\}$	$3^3.A_4$
65	3^4	$(1,2,4)(3,7,12)(5,10,11)(6,8,9)$	4.28			
66	3^4	$(1,8,4)(2,12,6)(3,11,7)(5,9,10)$	4.28			
67	3	$(1,4,2)(3,6,5)$	4.6	$\frac{2^5}{3^4\cdot 5}$	$\{4B,5A\}$	A_6
68	3^2	$(1,2,7)(3,8,11)(4,5,9)(6,10,12)$	4.6	$\frac{2^8}{3^4\cdot 5^2}$	$\{4A,5A\}$	
69	3^2	$(1,2,7)(3,8,12)(4,5,9)(6,10,11)$	4.6			
70	3^2	$(6,7,8)(10,11,12)$	4.5	$\frac{2^4}{3^4\cdot 5}$	$\{15A\}$	$3^2\times A_5$
71	3^4	$(1,3,2)(4,7,5)(6,8,9)(10,11,12)$	4.5			
72	3^2	$(1,4,2)(3,9,7)(5,6,8)(10,12,11)$	4.15,4.21	$\frac{2^3\cdot 7}{3^3\cdot 5^2}$	$\{6B,12A\}$	$3^3.SL_2(3)$
73	3^2	$(1,4,2)(3,9,7)(5,6,8)(10,11,12)$	4.15,4.21			
74	3^4	$(1,3,2)(4,7,5)(6,10,8)(9,11,12)$	4.15,4.21	$\frac{2^5}{3^3\cdot 5^2}$	$\{6C,12A\}$	
75	3^4	$(1,2,4)(3,6,8)(5,7,10)(9,11,12)$	4.6	$\frac{2^3}{3\cdot 5^2}$	$\{5A,6A\}$	$L_2(11)$
76	3^4	$(1,2,4)(3,7,11)(5,9,8)(6,10,12)$	4.7	$\frac{2^3\cdot 7}{3^3\cdot 5^2}$	$\{5A,11A\}$	
77	3^4	$(1,4,8)(2,5,10)(3,7,12)(6,9,11)$	4.5	$\frac{2^5}{3^4\cdot 5}$	$\{6A\}$	
78	3^2	$(3,7,9)(5,6,8)$	4.8	$\frac{2^2\cdot 31}{3^4\cdot 5^2}$	$\{10A,15A\}$	$A_4\times A_5$
79	3^4	$(1,4,2)(3,6,5)(7,10,8)(9,11,12)$	4.8			
80	3^4	$(1,2,3)(4,6,8)(5,7,10)(9,12,11)$	4.31	$\frac{2^4\cdot 11}{3^4\cdot 5^2}$	$\{6A,9A\}$	$3^4.A_4$
81	3^4	$(1,4,9)(2,6,12)(3,7,11)(5,10,8)$	4.15			

82	3	$(1,2,4)(3,5,6)(7,8,9)(10,11,12)$	4.7	$\frac{2^2\cdot 31}{3^4\cdot 5^2}$	$\{12C, 15A\}$	$3 \times A_6$
83	3^2	$(4,7,5)(6,9,8)$	4.7			
84	3^2	$(3,4,7)(6,8,9)$	4.15	$\frac{2^4}{3^2\cdot 5^2}$	$\{9A\}$	$L_2(8) : 3$
85	3^4	$(1,2,3)(4,5,7)(6,9,12)(8,10,11)$	4.31	$\frac{2^2\cdot 13}{3^3\cdot 5^2}$	$\{7A, 9A\}$	$3 \times L_2(8)$
86	3^4	$(1,3,2)(4,9,6)(5,10,7)(8,11,12)$	4.15	$\frac{2^3}{3^3\cdot 5}$	$\{9A, 21A\}$	
87	3^2	$(1,8,4)(2,9,5)(3,11,7)(6,12,10)$	4.6	$\frac{2^4\cdot 11}{3^4\cdot 5^2}$	$\{6A\}$	$2^6.(3^2 : 3)$
88	3^4	$(1,2,4)(3,5,7)(6,8,10)(9,11,12)$	4.6			
89	3^4	$(1,2,4)(3,6,10)(5,7,11)(8,9,12)$	4.7	$\frac{2^4\cdot 11}{3^4\cdot 5^2}$	$\{5A, 10A\}$	$2^5 : A_5$
90	3^4	$(1,4,9)(2,6,10)(3,7,12)(5,8,11)$	4.5	$\frac{2^4}{3^4\cdot 5}$	$\{10A\}$	
91	3	$(2,4,6)(3,5,7)$	4.5	$\frac{2^5}{3^4\cdot 5}$	$\{5A, 6A\}$	A_7
92	3^2	$(2,3,4)(5,6,7)$	4.6	$\frac{2^3}{3^4}$	$\{7A\}$	
93	3^2	$(1,2,3)(4,5,7)(6,8,9)(10,11,12)$	4.21	$\frac{2^3\cdot 11}{3^4\cdot 5^2}$	$\{12C, 15A\}$	$3^2 \times A_6$
94	3^2	$(1,2,3)(4,5,7)(6,8,9)(10,12,11)$	4.21			
95	3^2	$(1,3,2)(4,8,6)(5,9,7)(10,12,11)$	4.21			
96	3^2	$(1,2,3)(4,6,8)(5,7,9)(10,12,11)$	4.21			
97	3^2	$(1,2,4)(3,5,7)(6,8,9)(10,11,12)$	4.7	$\frac{2^3\cdot 7}{3^3\cdot 5^2}$	$\{6B, 9A\}$	$(L_2(8) : 3)$
98	3^2	$(1,2,4)(3,5,7)(6,8,9)(10,12,11)$	4.7			$\times 3$
99	3^2	$(1,7,2)(3,8,4)(5,6,9)(10,12,11)$	4.7	$\frac{2^6}{3^3\cdot 5^2}$	$\{6B\}$	
100	3^2	$(5,9,7)(6,10,8)$	4.5	$\frac{2^3\cdot 11}{3^4\cdot 5^2}$	$\{21A\}$	$3 \times A_7$
101	3^2	$(1,4,8)(2,6,10)(3,7,11)(5,9,12)$	4.5	$\frac{2^3\cdot 17}{3^4\cdot 5^2}$	$\{9A\}$	$A_4^3 : 3$
102	3^2	$(1,4,9)(2,7,11)(3,8,5)(6,10,12)$	4.7	$\frac{2^5}{3^3\cdot 5^2}$	$\{8B, 10A\}$	$2^5 : A_6$
103	3^2	$(2,4,7)(3,6,8)$	4.5	$\frac{2^3\cdot 11}{3^4\cdot 5^2}$	$\{15A\}$	A_8
104	3^4	$(1,4,2)(3,7,5)(6,11,10)(8,12,9)$	4.8	$\frac{2^2}{3\cdot 5^2}$	$\{9A, 12C\}$	$A_4^3.A_4$
105	3^4	$(1,2,4)(3,5,10)(6,7,8)(9,11,12)$	4.8			
106	3^4	$(1,4,2)(3,10,7)(5,11,8)(6,12,9)$	4.8	$\frac{2^2\cdot 7}{3^4\cdot 5}$	$\{9A, 12A\}$	
107	3^4	$(1,2,4)(3,5,7)(6,8,10)(9,12,11)$	4.7	$\frac{2^3\cdot 7}{3^3\cdot 5^2}$	$\{6B, 11A\}$	M_{12}
108	3^4	$(1,4,2)(3,10,5)(6,7,11)(8,9,12)$	4.7			
109	3^4	$(1,2,4)(3,5,7)(6,9,12)(8,10,11)$	4.7	$\frac{2^2}{3\cdot 5^2}$	$\{8B\}$	

110	3^4	$(1,2,4)(3,5,10)(6,7,11)(8,9,12)$	4.7			
111	3^4	$(1,2,4)(3,5,7)(6,10,12)(8,9,11)$	4.7	$\frac{2^2}{3^4}$	$\{8B,11A\}$	
112	3^4	$(1,4,2)(3,7,5)(6,11,10)(8,9,12)$	4.7			
113	3^4	$(1,2,4)(3,7,10)(5,8,12)(6,9,11)$	4.7			
114	3^4	$(1,2,4)(3,7,11)(5,8,12)(6,10,9)$	4.7			
115	3^4	$(1,2,4)(3,5,7)(6,10,12)(8,11,9)$	4.7	$\frac{2^4\cdot 13}{3^4\cdot 5^2}$	$\{5A,6B\}$	
116	3^4	$(1,2,4)(3,5,10)(6,7,9)(8,12,11)$	4.7			
117	3^4	$(1,2,4)(3,6,9)(5,7,10)(8,12,11)$	4.5	$\frac{2^3\cdot 17}{3^4\cdot 5^2}$	$\{11A\}$	
118	3^4	$(1,2,4)(3,6,7)(5,10,9)(8,12,11)$	4.5			
119	3^4	$(1,4,2)(3,12,7)(5,11,9)(6,8,10)$	4.7	$\frac{2^5}{3^4\cdot 5}$	$\{6B,10A\}$	
120	3^4	$(1,2,4)(3,7,11)(5,10,9)(6,8,12)$	4.7			
121	3^2	$(1,7,2)(3,11,8)(4,10,6)(5,12,9)$	4.5	$\frac{2^5}{3^3\cdot 5^2}$	$\{20B\}$	A_6^2
122	3^2	$(1,2,7)(3,8,12)(4,6,10)(5,9,11)$	4.5			
123	3^2	$(1,4,2)(3,8,6)(5,9,7)(10,12,11)$	4.7	$\frac{2^7}{3^4\cdot 5^2}$	$\{9A,15A\}$	$3\times A_9$
124	3^2	$(1,4,2)(3,8,6)(5,9,7)(10,11,12)$	4.7			
125	3^2	$(1,2,7)(3,4,8)(5,6,9)(10,11,12)$	4.8	$\frac{2^4\cdot 7}{3^4\cdot 5^2}$	$\{21A,30B\}$	
126	3^2	$(1,8,4)(2,9,6)(3,5,7)(10,12,11)$	4.7	$\frac{2^4\cdot 7}{3^4\cdot 5^2}$	$\{12C\}$	
127	3^2	$(1,4,8)(2,5,9)(3,7,6)(10,11,12)$	4.7	$\frac{2^3\cdot 19}{3^4\cdot 5^2}$	$\{9A\}$	
128	3^2	$(1,4,2)(3,9,7)(5,8,6)(10,12,11)$	4.15,4.21	$\frac{2^3\cdot 11}{3^4\cdot 5^2}$	$\{12C,21A\}$	
129	3^2	$(1,2,4)(3,7,9)(5,6,8)(10,12,11)$	4.15,4.21			
130	3^4	$(1,2,3)(4,5,7)(6,10,12)(8,9,11)$	4.31	$\frac{2^2\cdot 37}{3^4\cdot 5^2}$	$\{7A,15A\}$	
131	3^4	$(1,3,2)(4,10,5)(6,8,7)(9,11,12)$	4.31			
132	3^4	$(1,2,3)(4,5,7)(6,10,12)(8,11,9)$	4.31	$\frac{2^4\cdot 7}{3^4\cdot 5^2}$	$\{15A,21A\}$	
133	3^4	$(1,2,3)(4,7,12)(5,8,11)(6,10,9)$	4.15	$\frac{2^3\cdot 31}{3^4\cdot 5^2}$	$\{10A,30B\}$	
134	3^2	$(1,4,2)(3,10,7)(5,11,8)(6,12,9)$	4.7	$\frac{2^4\cdot 7}{3^4\cdot 5^2}$	$\{8B,11A\}$	A_{12}
135	3^2	$(1,2,7)(3,4,8)(5,9,11)(6,10,12)$	4.7	$\frac{2^3\cdot 17}{3^4\cdot 5^2}$	$\{10A,11A\}$	
136	3^2	$(1,2,7)(3,4,8)(5,9,12)(6,10,11)$	4.7			
137	3^2	$(1,4,9)(2,7,5)(3,8,11)(6,10,12)$	4.5	$\frac{2^5}{3^3\cdot 5^2}$	$\{35A\}$	
138	3^4	$(1,2,4)(3,5,7)(6,9,11)(8,10,12)$	4.5	$\frac{2^5}{3^3\cdot 5^2}$	$\{20B\}$	

139	3^4	$(1,4,2)(3,7,5)(6,8,10)(9,11,12)$	4.5		
140	3^4	$(1,2,4)(3,5,7)(6,10,9)(8,12,11)$	4.15	$\frac{2^7}{3^4\cdot 5^2}$	$\{11A,14A\}$
141	3^4	$(1,2,4)(3,5,10)(6,7,12)(8,11,9)$	4.15		
142	3^4	$(1,2,4)(3,6,10)(5,7,8)(9,11,12)$	4.7	$\frac{2^2\cdot 7}{3^4\cdot 5}$	$\{9A\}$
143	3^4	$(1,4,2)(3,10,6)(5,8,7)(9,11,12)$	4.7	$\frac{2^2}{3\cdot 5^2}$	$\{11A,12C\}$
144	3^4	$(1,2,4)(3,6,7)(5,10,12)(8,9,11)$	4.7		
145	3^4	$(1,2,4)(3,6,10)(5,7,9)(8,12,11)$	4.14,4.15	$\frac{2^3}{3^3\cdot 5}$	$\{12A,21A\}$
146	3^4	$(1,2,4)(3,7,8)(5,6,10)(9,12,11)$	4.31	$\frac{2^2\cdot 11}{3^3\cdot 5^2}$	$\{9A,15A\}$
147	3^4	$(1,2,4)(3,7,9)(5,6,10)(8,12,11)$	4.5	$\frac{2^4}{3^4\cdot 5}$	$\{21A\}$
148	3^4	$(1,2,4)(3,7,11)(5,8,9)(6,10,12)$	4.15	$\frac{2^3}{3^3\cdot 5}$	$\{11A,21A\}$
149	3^4	$(1,2,4)(3,7,10)(5,9,12)(6,8,11)$	4.8	$\frac{2^3}{3^3\cdot 5}$	$\{11A,35A\}$
150	3^4	$(1,2,4)(3,7,11)(5,9,12)(6,10,8)$	4.14,4.15	$\frac{2^3}{3^3\cdot 5}$	$\{10A,14A\}$
151	3^4	$(1,4,2)(3,11,7)(5,9,10)(6,8,12)$	4.15	$\frac{2^2}{3^4}$	$\{15A,35A\}$
152	3^4	$(1,4,8)(2,5,10)(3,7,11)(6,9,12)$	4.8	$\frac{2^3\cdot 17}{3^4\cdot 5^2}$	$\{9A,10A\}$
153	3^4	$(1,8,4)(2,10,5)(3,11,7)(6,9,12)$	4.8	$\frac{2^2}{3^4}$	$\{30B,35A\}$

Table 4.2: Inner products between $3A$-axes in V_A.

There are 153 orbits for A on unordered pairs of its $3A$-cyclic subgroups represented by $\{\langle h_1\rangle, \langle h_2\rangle\}$. The second and third columns of Table 4.2 lists these representatives h_1 and h_2 respectively. For the column labelled h_1, 3 represents the permutation $(1,2,3)$, 3^2 represents $(1,2,3)(4,5,6)$ and 3^4 represents $(1,2,3)(4,5,6)(7,8,9)(10,11,12)$. The column labelled l is the numbering of the Lemma(s) used in the previous section to determine the value of the inner product (u_{h_1}, u_{h_2}), of which is recorded on the next column to the right. The last column lists the isomorphism class of $\langle h_1, h_2\rangle$.

Let h_i and h_j be two $3A$-elements. Then define

$$C_{i,j} := \{(h_ih_j)^{\mathbb{M}}, (h_ih_j^{-1})^{\mathbb{M}}\}.$$

If $(h_ih_j)^{\mathbb{M}} = (h_i^{-1}h_j)^{\mathbb{M}}$ then $C_{i,j} = \{(h_ih_j)^{\mathbb{M}}\}$. For h_1 and h_2 in Table 4.2, $C_{1,2}$ is determined by identifying the cycle shapes of the products and referring to Table 4.1.

Proposition 4.34. *Let h_1, h_2, h_3 and h_4 be $3A$-elements in A. If $\{\langle h_1\rangle, \langle h_2\rangle\}$ is conjugate to $\{\langle h_3\rangle, \langle h_4\rangle\}$ as unordered pairs in the Monster, then $C_{1,2} = C_{3,4}$.*

Proof. Since $\{\langle h_1\rangle, \langle h_2\rangle\}$ is conjugate to $\{\langle h_3\rangle, \langle h_4\rangle\}$, there exists $g \in \mathbb{M}$ conjugating one set to the other. As the pairs are conjugate as unordered pairs, there are two cases to consider. Either

$$\langle h_1\rangle^g = \langle h_3\rangle \quad \text{and} \quad \langle h_2\rangle^g = \langle h_4\rangle,$$

or

$$\langle h_1\rangle^g = \langle h_4\rangle \quad \text{and} \quad \langle h_2\rangle^g = \langle h_3\rangle.$$

The proofs for both cases are similar so only the first case is shown. Now

$$h_1^g = h_3 \text{ or } h_3^{-1} \quad \text{and} \quad h_2^g = h_4 \text{ or } h_4^{-1}.$$

Hence

$$\{(h_1h_2)^{\mathbb{M}}, (h_1h_2^{-1})^{\mathbb{M}}\} = \{(h_1^gh_2^g)^{\mathbb{M}}, (h_1^g(h_2^{-1})^g)^{\mathbb{M}}\} \subseteq \{(h_3h_4)^{\mathbb{M}}, (h_3h_4^{-1})^{\mathbb{M}}, (h_3^{-1}h_4)^{\mathbb{M}}, (h_3^{-1}h_4^{-1})^{\mathbb{M}}\}.$$

As $o(h_3) = 3$, h_3h_4 is conjugate to h_4h_3. The cycle shape of h_4h_3 is the same as the cycle shape of its inverse $h_3^{-1}h_4^{-1}$. Therefore $(h_3h_4)^{\mathbb{M}} = (h_3^{-1}h_4^{-1})^{\mathbb{M}}$. Similarly $(h_3h_4^{-1})^{\mathbb{M}} = (h_3^{-1}h_4)^{\mathbb{M}}$. Hence

$$\{(h_3h_4)^{\mathbb{M}}, (h_3h_4^{-1})^{\mathbb{M}}, (h_3^{-1}h_4)^{\mathbb{M}}, (h_3^{-1}h_4^{-1})^{\mathbb{M}}\} = \{(h_3h_4)^{\mathbb{M}}, (h_3h_4^{-1})^{\mathbb{M}}\},$$

and therefore $C_{1,2} = C_{3,4}$. □

Consider the action of the Monster on the set of pairs of $3A$-cyclic subgroups of A. Then certain orbits of A will fuse as orbits under the action of the Monster. The least number of orbits the Monster has on this set may be enumerated from Table 4.2 since $C_{1,2}$, the value of the inner product and the isomorphism class of the group generated are $\mathbb{M}$-invariant properties. One may check that this number is 87.

Proposition 4.35. *For all conjugacy classes NX of the Monster except*

$$5B,\ 13B \text{ and } 15D,$$

there is a pair of $3A$-elements in the Monster such that the product of the pair lies in NX.

Proof. This is verified using the class algebra constant formula (Theorem 0.8). □

Corollary 4.36. *There are at least* 84 *orbits for the action of the Monster on its unordered pairs of* $3A$*-cyclic subgroups which do not intersect* A*.*

Proof. A lower bound for the number of orbits may be obtained by calculating a lower bound for the number of possibilities for $C_{i,j}$ where both conjugacy classes are not in A. The Monster has 194 conjugacy classes for which 3 is excluded by Proposition 4.35 and 24 intersecting A. Divide by two assuming the two conjugacy classes in $C_{i,j}$ are distinct. Then

$$\left\lceil \frac{194 - 3 - 24}{2} \right\rceil = 84.$$

□

Given a random pair of $3A$-elements in A, it may be tedious to check which pair in the list L it is conjugate to, in order to check the value of the inner product between the corresponding $3A$-axes from Table 4.2. However by observing Table 4.2, there is a more convenient way to cross reference the inner product.

Proposition 4.37. *Let* h_1 *and* h_2 *be* $3A$*-elements in* A*. The inner product* (u_{h_1}, u_{h_2}) *may be readily identified from Table 4.2 if* $\langle h_1, h_2 \rangle$ *is isomorphic to one of the following* 21 *groups:*

$$3, \quad F_{21}, \quad SL_2(3), \quad 4^2 : 3, \quad 2^3.A_4, \quad A_4^2, \quad (2^3 : 7) : 3,$$

$$3^3.A_4, \quad 3^2.SL_2(3), \quad 3^2 \times A_5, \quad A_4 \times A_5, \quad 3^4.A_4, \quad 3 \times A_6, \quad L_2(8) : 3,$$

$$2^6.(3^2 : 3), \quad 3^2 \times A_6, \quad 3 \times A_7, \quad A_4^3 : 3, \quad 2^5 : A_6, \quad A_8, \quad A_6^2.$$

Proof. By Table 4.2, if $\langle h_1, h_2 \rangle$ is isomorphic to one of these groups, there is only one value for the inner product. □

Proposition 4.38. *Let* h_1 *and* h_2 *be* $3A$*-elements in* A*. The inner product* (u_{h_1}, u_{h_2}) *may be readily identified from Table 4.2 if* $C_{1,2}$ *is known and* $\langle h_1, h_2 \rangle$ *is isomorphic to one of*

the following 17 *groups:*

$$3^2, \quad A_4, \quad 3^2:3, \quad 3\times A_4, \quad L_2(7), \quad 3\times A_5,$$

$$A_6, \quad 3^2.SL_2(3), \quad L_2(11), \quad 3\times L_2(8), \quad 2^5:A_5, \quad A_7$$

$$(L_2(8):3)\times 3, \quad A_4^3.A_4, \quad M_{12}, \quad 3\times A_9, \quad A_{12}.$$

Proof. See Table 4.2. □

Proposition 4.39. *Let h_1 and h_2 be 3A-elements in A. If $\langle h_1, h_2\rangle \cong A_5$ and the Monster conjugacy class of the involutions in $\langle h_1, h_2\rangle$ is known, then (u_{h_1}, u_{h_2}) may be identified from Table 4.2. If $\langle h_1, h_2\rangle \cong 3^2:3$ and the cycle shapes of h_1 and h_2 are known, then (u_{h_1}, u_{h_2}) may be identified from Table 4.2.*

Proof. By Table 4.2, there are only two values for (u_{h_1}, u_{h_2}) if $\langle h_1, h_2\rangle \cong A_5$ or $3^2:3$. It may be verified from the representatives of these cases that the conjugacy class of involutions and cycle shapes respectively, differentiate the values of the inner products. □

4.4 Fusions of orbits

Table 4.2 restricts the possible fusions of orbits under the action of the Monster. It was mentioned in the previous section that the 153 orbits of A fuse into at most 87 orbits of the Monster by enumerating Table 4.2. In this section, we shall identify some of these fusions.

Proposition 4.40. *The* 5 *pairs in rows* 16 *to* 20 *of Table 4.2 belong to a single orbit of the Monster on unordered pairs of its* 3A*-cyclic subgroups.*

Proof. This follows from Lemma 4.19. □

To show that certain inner products had the same value, we used Lemma 4.28 in Proposition 4.29 on cases where the inner product was not yet determined. Lemma 4.28 may

of course be applied to any two pairs $\{h_1, h_2\}$ and $\{h_3, h_4\}$ such that $\langle h_1, h_2\rangle \cong \langle h_3, h_4\rangle$, $C_{1,2} = C_{3,4}$ and $(u_{h_1}, u_{h_2}) = (u_{h_3}, u_{h_4})$. If the conditions of Lemma 4.28 are met, then the pairs are conjugate in the Monster.

Proposition 4.41. *The following sets $\{*, \ldots, *\}$, indicate the row numbers in Table 4.2 whose pairs are conjugate in the Monster.*

$$\{4,5,6,7\},\quad \{21,22\},\quad \{31,32\},\quad \{33,34,35\},$$
$$\{38,39,40\},\quad \{41,42,43\},\quad \{58,59\},\quad \{64,65,66\}.$$

Proof. This follows from Lemma 4.28 and using [GAP16] to identify the pair of involutions t_1 and t_2 from the pool of $2A$-involutions in A. □

The next method for showing two pairs are conjugate is rather straightforward. Take a larger subgroup of the Monster containing A which is still managable to compute using [GAP16] and see which pairs are conjugate in this larger subgroup. The first subgroup considered is S_{12}. By Theorem 3.2, the normalizer $N_{\mathbb{M}}(A)(= N_{\mathbb{M}}(H)) = (A \times H) : 2$. This subgroup of the Monster contains an S_{12}-subgroup $A : 2$ containing A.

Proposition 4.42. *The following sets $\{*, *\}$, indicate the row numbers in Table 4.2 whose pairs are conjugate by the S_{12}-subgroup of the Monster containing A described above.*

$$\{27,28\},\quad \{42,43\},\quad \{60,61\},\quad \{62,63\},\quad \{68,69\},$$
$$\{72,73\},\quad \{93,94\},\quad \{95,96\},\quad \{97,98\},\quad \{104,105\},$$
$$\{107,108\},\quad \{109,110\},\quad \{111,112\},\quad \{113,114\},\quad \{115,116\},$$
$$\{117,118\},\quad \{119,120\},\quad \{121,122\},\quad \{123,124\},\quad \{128,129\},$$
$$\{130,131\},\quad \{135,136\},\quad \{138,139\},\quad \{140,141\},\quad \{143,144\}.$$

Proof. This may be verified in [GAP16] by first defining the normalizer of h_1 in S_{12}. Then the command **RepresentativeAction** is used to obtain an element in the normalizer conjugating one h_2 to the other. For example, $(7,10)(8,11)(9,12)$ sends $\{h_1, h_2\}$ in row 27 of Table 4.2 to $\{h_1, h_2\}$ in row 28. □

Besides S_{12}, we may also work with another larger subgroup of $\mathbb{M}$. Denote by H_1 the A_4-subgroup of $H \cong A_5$ from Theorem 3.2, and denote by O, the centralizer of H_1 in $\mathbb{M}$. The following result may be referred from *Table 1* of [Nor75] and [Atl85].

Theorem 4.43. *O is isomorphic to $O_{10}^-(2)$ and contains A as a subgroup. Additionally, H_1 is the centralizer of O in $\mathbb{M}$. Hence H_1 and O form a pair of mutually centralizing subgroups in the Monster. The normalizer of O (and of H_1) in $\mathbb{M}$ is $(H_1 \times O).2$. Also, A is in a unique class of subgroup in O, which is maximal.*

The subgroup $O.2$ contains O which in fact contains A. This $O.2$ may be represented as a group of permutations of 495 elements generated by an involution c and an element d of order 11. Below are the generators listed in [AgrV3].

```
c:=(  2,  4)(  5,  7)(  8, 12)( 10, 15)( 14, 21)( 19, 27)( 20, 29)( 22, 31)( 28, 39)( 34, 47)
   ( 35, 49)( 36, 51)( 41, 57)( 46, 63)( 48, 66)( 50, 69)( 55, 75)( 58, 74)( 62, 84)( 67, 91)
   ( 71, 97)( 72, 99)( 73,101)( 77,107)( 81,111)( 82,113)( 83,112)( 86,118)( 87,120)( 93,127)
   (100,117)(102,139)(103,141)(105,144)(114,155)(115,156)(116,157)(122,164)(124,167)(126,170)
   (130,175)(135,150)(137,182)(140,186)(147,195)(148,151)(149,197)(162,210)(163,212)(165,215)
   (166,217)(168,221)(171,200)(172,226)(174,228)(176,231)(179,235)(187,243)(192,248)(193,250)
   (198,253)(205,257)(207,259)(209,262)(211,266)(213,270)(218,261)(225,286)(233,296)(234,263)
   (246,281)(249,275)(258,316)(260,320)(268,327)(271,304)(273,333)(276,314)(287,340)(288,343)
   (290,346)(291,348)(298,354)(303,310)(306,362)(308,363)(309,365)(313,367)(319,373)(325,378)
   (326,380)(328,374)(329,383)(334,388)(338,393)(351,404)(352,405)(353,389)(356,408)(358,401)
   (359,377)(361,395)(364,413)(369,417)(371,419)(372,407)(379,426)(398,423)(399,418)(409,446)
   (410,447)(415,450)(416,452)(427,434)(430,461)(444,471)(445,473)(458,478)(462,470)(464,480);

d:=(  1,  2,  4,  6,  9, 14, 20, 13,  8,  5,  3)(  7, 10, 16, 24, 34, 48, 50, 35, 25, 17, 11)
   ( 12, 18, 26, 36, 47, 65, 77, 56, 40, 28, 19)( 15, 22, 32, 45, 62, 85, 88, 64, 46, 33, 23)
   ( 21, 30, 43, 60, 82,114,115, 83, 61, 44, 31)( 27, 37, 52, 72,100,136,140,102, 73, 53, 38)
   ( 29, 41, 58, 80,110,151,153,112, 81, 59, 42)( 39, 54, 74,103,142,189,194,146,106, 76, 55)
   ( 49, 67, 92,126,113,154,200,173,128, 93, 68)( 51, 70, 96,131,177,233,236,179,133, 98, 71)
   ( 57, 78,108,149,198,186,160,118,150,109, 79)( 63, 86,119,161,209,263,271,213,163,121, 87)
   ( 66, 89,122,165,216,275,278,218,166,123, 90)( 69, 94,129,174,229,292,295,232,176,130, 95)
   ( 75,104,143,190,246,306,309,251,193,145,105)( 84,116,158,207,260,250,235,261,208,159,117)
   ( 91,124,168,222,283,315,257,284,223,169,125)( 97,132,178,234,297,353,312,254,199,152,111)
   ( 99,134,180,237,298,355,356,299,238,181,135)(101,137,183,240,301,326,267,211,162,120,138)
   (107,147,196,252,310,307,248,293,230,175,148)(127,171,164,214,272,332,386,345,289,227,172)
   (139,184,241,243,303,359,393,358,302,242,185)(141,187,244,304,360,409,410,361,305,245,188)
   (144,191,247,253,311,366,414,364,308,249,192)(155,201,195,231,294,350,403,368,313,255,202)
   (156,203,256,314,262,321,377,344,288,226,204)(157,205,258,317,371,357,300,239,182,197,206)
   (167,219,279,286,341,396,363,394,338,280,220)(170,224,285,323,265,210,264,322,342,287,225)
   (212,268,328,370,316,369,418,405,384,329,269)(215,273,334,389,434,463,464,435,390,335,274)
   (217,276,336,391,436,465,466,437,392,337,277)(221,281,339,346,380,419,452,438,395,340,282)
   (228,290,347,399,440,468,469,442,402,349,291)(259,318,372,420,427,379,325,266,324,374,319)
   (270,330,385,376,320,375,423,431,387,333,331)(296,351,398,343,397,439,467,472,444,406,352)
   (327,381,428,422,373,421,456,479,460,429,382)(348,400,365,354,407,445,470,443,404,441,401)
   (362,411,448,476,486,491,485,475,447,449,412)(367,415,451,477,487,488,481,459,426,453,416)
   (378,424,457,455,417,454,478,471,480,458,425)(383,408,446,474,450,413,433,388,432,462,430)
   (461,482,473,484,490,493,495,494,492,489,483);
```

Proposition 4.44. *The following elements (x_1 and x_2) have order 3 and generate a subgroup of $O.2$ conjugate to A in the Monster. These elements expressed as words in terms of c and d are:*

```
x1:=d*c^-1*d^-2*(c^-1*d)^3*c*d^-3*c*d^-1*c^-1*d^5*(c^-1*d^-1)^2*d^-3*c^-1
    *d*(d*c^-1)^2*d^-4*(c^-1*d)^3*d*c^-1*d^-1*c^-1*d^4*c*d^-1*c*d^5*c*d^2*
    c^-1*d^-4*c^-2*d^-2*(c^-1*d^-1*c^-1)^2*d^-5*(d^-1*c^-1*d^-3*c^-1)^2*d^4*
    c^-1*d^-9*c^-1*d^2*(c^-1*d^-2)^2*c^-1*d^-1*c^-2*d^2*c*d^2*(c^-1*d^-2)^2*
    d^-1*c^-1*d^3*c^-1*(d^-1*c^-1*d^-3)^2*d^-1*c^-1;

x2:=d^-2*c^-1*d^3*c^-1*d^-2*c*(d^5*c*d^-1*c*d^2)^2*(d*c*d)^2*d*c*d^-4*c^-1*
    d^-9*c*d^7*c^2*(d*c*d)^2*d*c*d^-2*c*d^9*c*d^-3*c*(c*d*(d*c)^2*d^-3)^2*
    d^-1*c*d^2*c*d^-5*c*d^3*c*d^-2*c*d^5*c*d^-3*c*d^2*(c*d^-1)^2*d^-2*c*d^2;
```

Proof. It may be verified that $\langle x_1, x_2 \rangle$ is isomorphic to A_{12} and is contained in the subgroup O of $O.2$ with [GAP16]. □

Proposition 4.45. *The following sets $\{*, \ldots, *\}$, indicate the row numbers in Table 4.2 whose pairs are conjugate in $O.2$.*

$$\{36, 37\}, \quad \{60, 61, 62, 63\}, \quad \{111, 112, 113, 114\}.$$

Proof. This is verified in [GAP16] with an isomorphism between $\langle x_1, x_2 \rangle$ and A being first established. The rest is similar to the proof of Proposition 4.42. □

Corollary 4.46. *Let x be the number of orbits of the Monster on unordered pairs of $3A$-cyclic subgroups in A. Then*

$$87 \leq x \leq 106.$$

Proof. The lower bound of 87 was enumerated from Table 4.2. The upper bound follows from taking into consideration all the orbits which we have shown fuses in the Monster. □

The range in Corollary 4.46 tells us that while there may be many pairs which are not conjugate in A but conjugate in the Monster, there is still a significant number of orbits of pairs of $3A$-axes in V_A. Each orbit is a case which has to be considered separately and each will have a different algebra product (up to the action by $\mathbb{M}$).

4.5 Relations

Pasechnik's relation in the Monster algebra involves a collection of $2A$- and $3A$-axes, whereby a vector expressed in terms of these axes is the zero vector. This relation involves 13 axes in total (four $3A$-axes and nine $2A$-axes). Let u_{h_i} for $1 \leq i \leq 4$ be four $3A$-axes and a_{t_j} for $1 \leq j \leq 9$ be nine $2A$-axes involved in a Pasechnik's relation. Then the inner products between these axes are:

$$\begin{aligned} (a_{t_i}, a_{t_j}) &= \frac{13}{256} && \text{for } 1 \leq i \neq j \leq 9; \\ (a_{t_i}, u_{h_j}) &= \frac{1}{4} && \text{for all } i \text{ and } j; \\ (u_{h_i}, u_{h_j}) &= 0 && \text{for } 1 \leq i \neq j \leq 4. \end{aligned}$$

Let Gr be the Gram matrix with respect to these axes. Then

$$Gr = \begin{pmatrix} \frac{13}{256}J_{9,9} + (1 - \frac{13}{256})I_9 & \frac{1}{4}J_{9,4} \\ \frac{1}{4}J_{4,9} & \frac{8}{5}I_4 \end{pmatrix}$$

where I_n is the $n \times n$ identity matrix and $J_{m,n}$ is the $m \times n$ matrix with all entries being 1. It may be calculated that the rank of the Gr is 12. The kernel of Gr is therefore one dimensional and it is spanned by

$$v = (32, 32, 32, 32, 32, 32, 32, 32, 32, -45, -45, -45, -45)^T.$$

Pasechnik's relation is simply

$$(a_{t_1}, \ldots, a_{t_9}, u_{h_1}, \ldots, u_{h_4}) \cdot v = 0.$$

In this section, we search for new relations with constant coefficients similar to that of Pasechniks'. We shall also assume that the relations contains only $2A$- and $3A$-axes. Let K be a set of n axes in the Monster algebra, containing n_1 $2A$-axes and n_2 $3A$-axes where

$$1 \leq n_1 \leq n - 1 \quad \text{and} \quad n_1 + n_2 = n.$$

Let $a_{t_1}, \ldots, a_{t_{n_1}}$ be the $2A$-axes in K, and $u_{h_1}, \ldots, u_{h_{n_2}}$ the $3A$-axes. For simplicity, we assume that the inner products are constants:

$$(a_{t_i}, a_{t_j}) = x \qquad \text{for } 1 \leq i \neq j \leq n_1;$$

$$(a_{t_i}, u_{h_j}) = y \qquad \text{for all } i \text{ and } j;$$

$$(u_{h_i}, u_{h_j}) = z \qquad \text{for } 1 \leq i \neq j \leq n_2.$$

Let Gr_K be the Gram matrix with respect to K. Then

$$Gr_K = \begin{pmatrix} xJ_{n_1,n_1} + (1-x)I_{n_1} & \frac{1}{4}J_{n_1,n_2} \\ \frac{1}{4}J_{n_2,n_1} & zI_{n_2} \end{pmatrix}.$$

If there exist a linear combination of axes in K which equal the zero vector, then the rank of Gr_K is strictly less than n. Such linear combinations are obtained from the kernel of Gr_K as explained in the example with Pasechnik's relation.

Since the axes in K are assumed to be in the Monster algebra, the possible values for x are from the Norton-Sakuma algebras. The possible values for y are from Table 3.1 by Norton. We do not know all possible values for z so we may only pick those from Table 4.2. Using [GAP16], we run through these possible values while fixing a relatively small n. The following results were obtained.

Proposition 4.47. *For $n \leq 15$, there are only 7 relations excluding Pasechnik's relation. They are:*

(i)

$$\sum_{i=1}^{4} a_{t_i} - \frac{5}{8}\sum_{j=1}^{7} u_{h_j} = 0$$

where $x = \frac{1}{32}$, $y = \frac{1}{4}$ and $z = 0$;

(ii)

$$\sum_{i=1}^{7} a_{t_i} - \frac{35}{32}\sum_{j=1}^{4} u_{h_j} = 0$$

where $x = \frac{1}{64}$, $y = \frac{1}{4}$ and $z = 0$;

(iii)

$$\sum_{i=1}^{7} a_{t_i} - \frac{35}{48}\sum_{j=1}^{6} u_{h_j} = 0$$

where $x = \frac{1}{64}$, $y = \frac{1}{4}$ and $z = \frac{4}{25}$;

(iv)

$$\sum_{i=1}^{8} a_{t_i} - \frac{5}{4}\sum_{j=1}^{6} u_{h_j} = 0$$

where $x = \frac{1}{8}$, $y = \frac{1}{4}$ *and* $z = 0$*;*

(v)

$$\sum_{i=1}^{2} a_{t_i} - \frac{5}{16}\sum_{j=1}^{13} u_{h_j} = 0$$

where $x = \frac{1}{64}$, $y = \frac{1}{4}$ *and* $z = 0$*;*

(vi)

$$\sum_{i=1}^{9} a_{t_i} - \frac{15}{16}\sum_{j=1}^{6} u_{h_j} = 0$$

where $x = \frac{13}{256}$, $y = \frac{1}{4}$ *and* $z = \frac{4}{25}$*;*

(vii)

$$\sum_{i=1}^{12} a_{t_i} - \frac{25}{16}\sum_{j=1}^{3} u_{h_j} = 0$$

where $x = \frac{1}{64}$, $y = \frac{1}{4}$ *and* $z = \frac{4}{9}$.

At this stage, it has not yet been confirmed if any of the configurations in Proposition 4.47 exist in the Monster algebra. However, if there is such a set K where the inner products are as stated in the above proposition, then the linear combination of axes equals the zero vector.

Lemma 4.48. *Let* $u_{h_1}, \ldots, u_{h_x}$ *be* $3A$*-axes in* V_A *such that* $(u_{h_i}, u_{h_j}) = 0$ *or* $\frac{4}{25}$ *for all* i *and* j*. Then* $h_1, \ldots, h_x$ *generate an elementary abelian* 3*-group:*

$$\langle h_1, \ldots, h_x \rangle \cong 3^x.$$

Proof. By Table 4.2, if $(u_{h_i}, u_{h_j}) = 0$ or $\frac{4}{25}$, then h_i and h_j commute. □

Proposition 4.49. *The largest elementary abelian* 3*-subgroup of* A_{12} *has order* 3^4.

Proof. Let Q be the following subgroup of A_{12}:

$$Q := \langle (1,12,7,2,11,8,3,10,9), (4,5,6), (10,11,12) \rangle.$$

Then Q has order 3^5 and is not abelian. Since the 5 is highest power of 3 dividing the order of A_{12}, all Sylow 3-subgroups of A_{12} are conjugate to Q. However, the subgroup

$$\langle (1,2,3), (4,5,6), (7,8,9), (10,11,12) \rangle$$

is an elementary abelian 3-subgroup of A_{12} of order 3^4. □

Proposition 4.50. *None of the configurations in Proposition 4.47 exists in V_A.*

Proof. By Lemma 4.48 and Proposition 4.49, the cases which have more than four $3A$-axes may be excluded. This leaves cases (ii) and (vii).

For case (ii), let $\{h_1, h_2, h_3, h_4\} := \{(1,2,3), (4,5,6), (7,8,9), (10,11,12)\}$. Since $(a_{t_i}, u_{h_j}) = \frac{1}{4}$ for all i, j, each t_i $(1 \leq i \leq 7)$ must invert h_1, h_2, h_3 and h_4. It may be checked that there does not exist any $2A$-involutions in A inverting all these elements.

For case (vii), let $\{h_1, h_2\} := \{(1,2,3)(4,5,6), (2,4,5)(3,6,7)\}$ since by Table 4.2, there is only one orbit such that $(u_{h_1}, u_{h_2}) = \frac{4}{9}$. Similar to case (ii), there must be 12 $2A$-involutions inverting h_1, h_2 and h_3. It may be checked that there does not exist any $2A$-involutions in A inverting all these elements. □

Although we have proven that none of the configurations of Proposition 4.47 exist in V_A, it remains unproven that these configurations exist in the Monster algebra.

Chapter 5

Conclusion

In this chapter, we shall discuss and summarize some of the results of this thesis. It is emphasized that the three main achievements presented in this thesis are the axiomatic approach in Chapter 2, the $(2A, 3A)$-configurations in Chapter 3, and the inner product methodology in Chapter 4. We shall also discuss some of the implications of these results, insights they provide and possible areas for future work.

In Section 1.5, the automorphism groups of the 13-dimensional Majorana algebras arising from the Majorana representations of S_4 with shapes $(2A, 3A)$ and $(2B, 3A)$ were deduced. This was achieved by identifying and classifying idempotents of certain lengths, an approach similar to that of Castillo-Ramirez in [CR13b]. This result together with the results in [CR13b] demonstrate that the automorphism groups of all Majorana algebras underlying the Majorana representations of S_4 constructed in [I+10], are just S_4. As these representations were faithful, the images of the linear transformations are the full automorphism groups. It is uncertain under what conditions this happens or that if it is a consequence of the methods used to construct the algebra. In the case of the Norton-Sakuma algebras, this is not true. For the $2A$, $2B$ and $5A$ type Norton-Sakuma algebras, their full automorphism groups are not the dihedral groups (see [CR13b]).

In Chapter 2, an axiomatic approach to studying the $3A$-axis was pursued. Instead of defining a $3A$-axis in the Monster algebra or an arbitrary Majorana algebra based on the existence of certain pairs of $2A$-axes, we want to define it based on its properties in the algebra it is in, similar to that of the definition of a Majorana axis. To do this we first proved properties of a $3A$-axis in the Monster algebra. Like the $2A$-involutions and $2A$-axes, we may define a $3A$-element in the Monster simply in terms of scalar multiplication on the eigenspaces of the the associated $3A$-axis. However things get a little more complicated. Scalar multiplication on the $\frac{1}{3}$- and $\frac{1}{30}$-eigenspaces may only be defined once it splits into two subspaces of equal dimensions. Moreover the scalar is complex. A disadvantage of extending to the complex field is that the $\mathbb{M}$-invariant bilinear form is no longer positive definite (see Lemma 2.32).

The inner product between any two vectors contained in one splitting of an eigenspace was shown to be zero. This orthogonality condition we proved does not uniquely determine how the eigenspaces split. What did determine the splitting in the cases of the S_4-representations were the fusion rules and the action of the $3A$-element. It may therefore be conjectured at this stage that this is also true in the Monster algebra. The properties of a 3A-axis in the Monster algebra were then axiomatized as the definition of what we call a standard $3A$-axis in an arbitrary algebra. In Section 2.5, we demonstrated that the subrepresentation of A_4 based on an embedding in the $(2A, 3A)$-shape representation of S_4 may be constructed using the properties of the axes. Under the assumption that

multiplication was 2-closed with respect to the set of Majorana and standard $3A$-axes, it was then established that V_{A_4} is the unique algebra containing the said axes as they are defined. This demonstrates that the defining properties of the axes is a strong criteria which restricts the structure of the whole algebra. The construction of V_{A_4} implemented a different strategy to constructing algebras containing Majorana axes. The main difference is that we included $3A$-axes to correspond to $3A$-cyclic subgroups in A_4. This is something which may be considered in future especially when dealing with non 2-closed Majorana representations.

In Chapter 3, we demonstrated the application of the generic strategy to construct 2-closed Majorana representations of certain groups with certain shapes. These cases though implicitly existing in representations known to exist, have not been explicitly expressed before. A consequence of these Majorana representations is that we now know how to multiply between a $2A$-axis and $3A$-axis in the Monster algebra for 3 new cases. By Norton's list, there are only 22 orbits for these types of algebra products in the Monster algebra. In some cases, the elements associated to the axes in the expression for the algebra product may be expressed only in terms of the associated $2A$-involution and $3A$-element. In other cases, elements of a larger subgroup containing the $(2A, 3A)$ generated group is required. This demonstrates the phenomenon whereby properties regarding the Monster algebra indirectly proves properties regarding the Monster (see Corollaries 3.18 and 3.21).

Next we considered subalgebras $V_{a,u}$ of Majorana algebras generated by a Majorana axis a and $3A$-axis u. We discovered that these subalgebras do not have much structure and loses vital information. The $\frac{1}{3}$- or $\frac{1}{30}$-eigenspaces do not split in all the examples we looked at. The subalgebra may not be defined in terms of a basis of axes. The disadvantage of the definition of a $3A$-axes in terms of Majorana axes also becomes clear in this subalgebra as the $3A$-axis generator may no longer be defined as a $3A$-axis. There are no pairs of Majorana axes generators in $V_{a,u}$. What more, the $3A$-axis generator u is also not a standard $3A$-axis in $V_{a,u}$ as it does not satisfy the properties in Definition 2.37. This brings to question whether it is the right approach to study subalgebras generated by a $2A$- and $3A$-axis. The $2A$-involution generated dihedral groups are generated by a pair of $2A$-involutions while the Norton-Sakuma algebras are generated by a pair of $2A$-axes. Such a correspondence cannot be replicated with the $(2A, 3A)$ groups in Norton's list and

subalgebras in the Monster algebra generated by a $2A$- and $3A$-axis

In Chapter 4, we successfully determined the values for the inner products between any pair of $3A$-axes in V_A. This was achieved through formulating the inner products as linear sums of other known inner products. These formulas are subject to the existence of elements in A satisfying certain conditions. Finding these elements was done mainly with the GAP computer program. Since we already knew the inner products between two $2A$-axes and between $2A$- and $3A$-axes, most cases were deducible. For the remaining cases, we either formulated an expression for the inner product in terms of other inner products including between two $3A$-axes, or we showed that certain pairs were conjugate in the Monster. We then used the values which were determined previously to deduce the remaining inner products.

Some of the techniques used in this chapter may be adapted to determine the values of inner products between other types of axes in V_A, certainly between $2A$-axes and $4A$- or $5A$-axes. An immediate implication of this inner product classification is that when classifying Majorana representations of subgroups of A, all inner products between $3A$-axes are already known. This would certainly be required if the construction of V_A were to be achieved through the induction process of classifying Majorana representations of subgroups of A. A current topic of research in Majorana theory is the classification of Majorana representations of the Mathieu groups. In the case of M_{11} in A, we now know that there are no pairs of $3A$-elements generating M_{11}. The values of the inner products also play an important role in determining the linear span of vectors. In theory, it is now possible to determine the linear span of the full set of $3A$-axes in V_A. It is equal to rank of the corresponding Gram matrix. The kernel of this matrix (which is non-empty) would then automatically reveal some interesting relations between $3A$-axes.

I believe that studying the Monster algebra and the Monster through Majorana theory is an effective approach. The NA-axes are excellent choices as bases vectors of Majorana algebras and we know how the NA-elements acts on them. Exploiting the properties of the Majorana axes has proven to be very successful. I also believe that the properties of the $3A$-axes, some of which were proven in this thesis are the key to uncovering even more insights into the elusive Monster.

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Appendix A

A_4-algebra

The products to be determined are between a_1 and u_1, and between u_1 and u_2. Let

$$a_1 \cdot u_1 = p_1a_1 + p_2a_2 + p_3a_3 + q_1u_1 + q_2u_2 + q_3u_3 + q_4u_4 \quad \text{and}$$
$$u_1 \cdot u_2 = r_1a_1 + r_2a_2 + r_3a_3 + s_1u_1 + s_2u_2 + s_3u_3 + s_4u_4$$

where p_i, q_j, r_i and s_j for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$ are real. The aim is to determine these 14 unknown scalars.

Deducing the multiplicities of the eigenvalues of a_1 in V_{A_4}

Firstly, 1 is a simple eigenvalue. As a_1 is contained in a Norton-Sakuma algebra of type $2A$, it has eigenvalues 0 and $\frac{1}{4}$ each with multiplicity at least 1. The $\frac{1}{32}$-eigenspace is spanned by vectors of the form $v - v^{\varphi(a)}$. By the permutations for φ given in Section 2.5, it is 2-dimensional and is spanned by $u_1 - u_2$ and $u_3 - u_4$. Hence the multiplicities for the $\frac{1}{32}$-eigenvalue is 2. Now there are 3 possibilities to consider:

$$\{1, 0^3, \frac{1}{4}, \frac{1}{32}^2\}, \quad \{1, 0, \frac{1}{4}^3, \frac{1}{32}^2\} \quad \text{and} \quad \{1, 0^2, \frac{1}{4}^2, \frac{1}{32}^2\}.$$

Case 1.
Let $v \in V_{A_4}$ be an arbitrary vector. The determinant of the matrix of $a_1 \cdot v - \lambda v$ with respect to X is calculated. It is a polynomial in λ of degree 7 which should equal

$$(\lambda - 1)(\lambda - 0)^3(\lambda - \frac{1}{4})(\lambda - \frac{1}{32})^2.$$

Comparing coefficients for the appropriate powers of λ, a system of equations is obtained. This system may be solved to give possible values for q_j, $j \in \{1,2,3,4\}$. There are 6 possibilities:

no.	q_1	q_2	q_3	q_4	*no.*	q_1	q_2	q_3	q_4
1	$\frac{1}{64}$	$\frac{1}{64}$	0	0	4	$\frac{1}{64}$	$-\frac{1}{64}$	0	0
2	$\frac{1}{64}$	0	$\frac{1}{64}$	0	5	$\frac{1}{64}$	0	$-\frac{1}{64}$	0
3	$\frac{1}{64}$	0	0	$\frac{1}{64}$	6	$\frac{1}{64}$	0	0	$-\frac{1}{64}$

Only *no.* 4 has $u_1 - u_2$ and $u_3 - u_4$ as $\frac{1}{32}$-eigenvectors for a_1. Therefore the other 5 possibilities may be excluded.

For the dimension of the 0-eigenspace to equal the multiplicity of the 0-eigenvalue, $p_3 = -p_2$. For the fusion rule $V^0_{A_4,a_1} \cdot V^0_{A_4,a_1} \subseteq V^0_{A_4,a_1}$ to hold, $p_2 = 0$, $s_4 = s_3$, $s_2 = s_1$, $r_3 = r_2$, $r_2 = 8p_1^2 - 8p_1s_1 - *p_1s_3 - 4p_1 - 4r_1$, $r_1 = \frac{64}{35}p_1^2 - \frac{8}{5}p_1s_1 - \frac{8}{5}p_1s_3 - \frac{36}{35}p_1$. This leaves 3 unknown scalars: p_1, s_1 and s_3.

Let $a, b, c, d \in \mathbb{N}$ where

$$(\lambda - 1)(\lambda - 0)^a(\lambda - \frac{1}{5})^b(\lambda - \frac{1}{3})^{2c}(\lambda - \frac{1}{30})^{2d} \tag{A.1}$$

$$a + b + 2c + 2d = 6. \tag{A.2}$$

The characteristic polynomial of ad_{u_1} in λ is calculated, It is a polynomial in λ of degree 7 which should equal (A.1). By these conditions, there are 30 possibilities for the set $\{a, b, c, d\}$. Solving simultaneous equations, only 8 possibilities give values for p_1, s_1 and s_3:

no.	a	b	c	d	p_1	s_1	s_3
1	3	1	1	0	0	$\frac{13}{45}$	$-\frac{2}{45}$
2	4	0	1	0	0	$\frac{2}{9}$	$-\frac{1}{9}$
3	3	1	0	1	0	$\frac{4}{45}$	$\frac{1}{18}$

4	4	0	0	1	0	$\frac{1}{45}$	$-\frac{1}{90}$
5	3	3	0	0	0	$\frac{1}{5}$	0
6	4	2	0	0	0	$\frac{2}{15}$	$-\frac{1}{15}$
7	5	1	0	0	0	$\frac{1}{15}$	$\frac{1}{15}$
8	6	0	0	0	0	0	0

For all these cases, it may be verified that either the fusion rules do not hold of the eigenspaces do not span V_{A_4}. Therefore Case 1 does not hold.

Case 2.

Performing calculations similar to that of the first part of Case 1, there are again 6 possibilities for the set $\{q_i \mid 1 \leq i \leq 4\}$:

no.	q_1	q_2	q_3	q_4	*no.*	q_1	q_2	q_3	q_4
1	$\frac{9}{64}$	$\frac{7}{64}$	0	0	4	$\frac{9}{64}$	$-\frac{7}{64}$	0	0
2	$\frac{9}{64}$	0	$\frac{7}{64}$	0	5	$\frac{9}{64}$	0	$-\frac{7}{64}$	0
3	$\frac{9}{64}$	0	0	$\frac{7}{64}$	6	$\frac{9}{64}$	0	0	$-\frac{7}{64}$

Only *no.* 1 has $u_1 - u_2$ and $u_3 - u_4$ as $\frac{1}{32}$-eigenvectors for a_1. Therefore the other 5 possibilities may be excluded.

For the dimension of the $\frac{1}{4}$-eigenspace to equal the multiplicity of the $\frac{1}{4}$-eigenvalue, $p_3 = p_2$. For the fusion rule $V^{\frac{1}{4}}_{A_4,a_1} \cdot V^{\frac{1}{4}}_{A_4,a_1} \subseteq V^0_{A_4,a_1} \oplus V^1_{A_4,a_1}$ to hold, $p_1 = -4p_2$, $s_3 = s_4 = -\frac{7}{4}p_2$, $s_1 = s_2 = -\frac{1}{2} + \frac{17}{4}p_2$, $r_3 = r_2$ and $p_2 = -\frac{1}{6}$. This leaves 2 unknown scalars: r_1 and r_2.

The characteristic polynomial of ad_{u_1} in λ is calculated, It is a polynomial in λ of degree 7 which should equal (A.1). The coefficient of λ^6 is $\frac{29}{8}$. This is equal to the sum of roots

of the equation above. Hence

$$1 + \frac{b}{5} + \frac{2c}{3} + \frac{2d}{30} = \frac{29}{8}.$$

For (A.2) to also hold, there are no solutions to this equation . Therefore Case 2 is excluded.

Case 3.

As before, the characteristic polynomial of ad_{u_1} is calculated, coefficients are compared and the system of equations is solved for q_j, $j \in \{1, 2, 3, 4\}$. There are 12 possibilities:

no.	q_1	q_2	q_3	q_4	*no.*	q_1	q_2	q_3	q_4
1	$\frac{5}{64}$	$-\frac{3}{64}$	$\frac{1}{16}$	$-\frac{1}{16}$	7	$\frac{5}{64}$	$\frac{3}{64}$	$\frac{1}{16}$	$\frac{1}{16}$
2	$\frac{5}{64}$	$-\frac{3}{64}$	$-\frac{1}{16}$	$\frac{1}{16}$	8	$\frac{5}{64}$	$\frac{1}{16}$	$\frac{3}{64}$	$\frac{1}{16}$
3	$\frac{5}{64}$	$\frac{1}{16}$	$-\frac{3}{64}$	$-\frac{1}{16}$	9	$\frac{5}{64}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{64}$
4	$\frac{5}{64}$	$-\frac{1}{16}$	$-\frac{3}{64}$	$\frac{1}{16}$	10	$\frac{5}{64}$	$\frac{3}{64}$	$-\frac{1}{16}$	$-\frac{1}{16}$
5	$\frac{5}{64}$	$\frac{1}{16}$	$-\frac{1}{16}$	$-\frac{3}{64}$	11	$\frac{5}{64}$	$-\frac{1}{16}$	$\frac{3}{64}$	$-\frac{1}{16}$
6	$\frac{5}{64}$	$-\frac{1}{16}$	$\frac{1}{16}$	$-\frac{3}{64}$	12	$\frac{5}{64}$	$-\frac{1}{16}$	$-\frac{1}{16}$	$\frac{3}{64}$

All cases except *no.* 7 and *no.* 10 does not have $u_1 - u_2$ and $u_3 - u_4$ as $\frac{1}{32}$-eigenvectors of a_1 and hence may be excluded. These cases are labelled Case 3*a* and 3*b* respectively.

Case 3a.

For the dimension of the $\frac{1}{4}$-eigenspace to be 2, $p_3 = p_2$. For the fusion rule $V^0_{A_4,a_1} \cdot V^0_{A_4,a_1} \subseteq V^0_{A_4,a_1}$ to hold, $s_1 = \frac{1}{2} - s_3$, $s_2 = \frac{1}{2} - s_4$, $r_2 = r_3$ and $r_1 = \frac{8}{3}r_3$. For the fusion rule $V^{\frac{1}{4}}_{A_4,a_1} \cdot V^{\frac{1}{4}}_{A_4,a_1} \subseteq V^0_{A_4,a_1} \oplus V^1_{A_4,a_1}$ to hold, $p_1 = -4p_2$ and $p_2 = \frac{1}{6}$. For the fusion rule $V^{\frac{1}{32}}_{A_4,a_1} \cdot V^{\frac{1}{32}}_{A_4,a_1} \subseteq V^0_{A_4,a_1} \oplus V^1_{A_4,a_1} \oplus V^{\frac{1}{4}}_{A_4,a_1}$ to hold, $s_4 = s_3$. For the fusion rule $V^0_{A_4,a_1} \cdot V^{\frac{1}{4}}_{A_4,a_1} \subseteq V^{\frac{1}{4}}_{A_4,a_1}$ to hold, $s_3 = \frac{11}{24}$. This leaves 1 unknown scalar: r_3.

The characteristic polynomial of ad_{u_1} in λ is calculated. The coefficient of λ^6 is $\frac{25}{8}$. This

is equal to the sum of roots of the characteristic polynomial. Hence

$$1 + \frac{b}{5} + \frac{2c}{3} + \frac{2d}{30} = \frac{25}{6}.$$

There are no solutions to this equation given that (A.2) holds. Therefore this case is excluded and Case 3b is assumed to hold.

For the fusion rule $V^0_{A_4,a_1} \cdot V^0_{A_4,a_1} \subseteq V^0_{A_4,a_1}$ to hold, $p_2 = 0$, $s_2 = s_1$ and $s_3 = s_4$. This leaves 6 unknown scalars: p_1, r_1, r_2, r_3, s_1 and s_3.

Deducing the multiplicities of the eigenvalues of u_1 in V_{A_4}

Comparing the coefficients of the characteristic polynomial of ad_{u_1} with equation (A.1) and solving the system of equations, there are 30 possibilities for the set $\{a, b, c, d\}$. All but 6 possibilities are not valid as the eigenspaces of u_1 do not span V_{A_4}.

no.	a	b	c	d	*no.*	a	b	c	d
1	1	1	1	1	4	1	3	0	1
2	1	3	1	0	5	3	1	0	1
3	3	1	1	0	6	3	3	0	0

For these 6 cases, $r_3 = r_2$ and the remaining unknown scalars may be written in terms of p_1. For the fusion rule $V^0_{A_4,a_1} \cdot V^0_{A_4,a_1} \subseteq V^0_{A_4,a_1}$ to hold, there are 2 possible values for p_1 for all 6 cases. Hence there are 12 cases in total to consider.

Eigenvalues for u_1	*no.*	p_1	r_1	r_2	s_1	s_3
$\{1, 0, \frac{1}{5}, \frac{1}{3}^2, \frac{1}{30}^2\}$	1	0	$-\frac{64}{945}$	$\frac{32}{945}$	$\frac{14}{45}$	$-\frac{1}{18}$
	2	$\frac{1}{9}$	$\frac{128}{2025}$	$-\frac{64}{675}$	$\frac{1}{5}$	$-\frac{1}{18}$
$\{1, 0, \frac{1}{5}^3, \frac{1}{3}^2\}$	3	0	$-\frac{128}{315}$	$\frac{64}{315}$	$\frac{!9}{45}$	$-\frac{1}{9}$
	4	$\frac{1}{18}$	$-\frac{3952}{14175}$	$\frac{416}{4725}$	$\frac{11}{30}$	$-\frac{1}{9}$

$\{1, 0^3, \frac{1}{5}, \frac{1}{3}^2\}$	5	0	0	0	$\frac{13}{45}$	$-\frac{2}{45}$
	6	$\frac{1}{9}$	$\frac{11}{90}$	$\frac{8272}{70875}$	$-\frac{2816}{23625}$	$\frac{1}{6}$
$\{1, 0, \frac{1}{5}^3, \frac{1}{30}^2\}$	7	0	$-\frac{64}{1575}$	$\frac{32}{1575}$	$\frac{2}{9}$	$-\frac{1}{90}$
	8	$\frac{7}{45}$	$\frac{256}{70875}$	$-\frac{1088}{23625}$	$\frac{1}{!5}$	$-\frac{1}{90}$
$\{1, 0^3, \frac{1}{5}, \frac{1}{30}^2\}$	9	0	0	0	$\frac{4}{45}$	$\frac{1}{18}$
	10	$\frac{2}{9}$	$-\frac{3328}{14175}$	$\frac{704}{4725}$	$-\frac{2}{15}$	$\frac{1}{18}$
$\{1, 0^3, \frac{1}{5}^3\}$	11	0	0	0	$\frac{1}{5}$	0
	12	$\frac{1}{6}$	$\frac{16}{1575}$	$-\frac{64}{1575}$	$\frac{1}{30}$	0

Only for Case 2 does the fusion rules for the eigenspaces of u_1 hold true. This completes determining the algebra products $a_1 \cdot u_1$ and $u_1 \cdot u_2$. The inner products may be deduced easily by orthogonality between eigenvectors. It may be verified that all properties are satisfied in V_{A_4}.

Appendix B

Orbits of A on its $(2A, \langle 3A \rangle)$-pairs

The table below lists the orbits of A on its $(2A, \langle 3A \rangle)$-pairs $\{t, \langle h \rangle\}$. The column labelled $c(th)$ is the cycle shape of th. There are 60 orbits in total.

no.	t	h	$c(th)$	$\langle t, h \rangle$
1	$(2,3)(11,12)$	$(1,2,3)$	(2^2)	S_3
2	$(2,3)(5,6)$	$(1,2,3)(4,5,6)$	(2^2)	
3	$(1,4)(2,6)(3,5)(7,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	(2^6)	
4	$(1,4)(2,6)(3,5)(7,10)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(2^6)	
5	$(9,10)(11,12)$	$(1,2,3)$	$(3,2^2)$	6
6	$(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(3^2,2^2)$	
7	$(1,4)(2,5)(3,6)(7,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(6,2^3)$	
8	$(1,4)(2,5)(3,6)(7,10)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(6^2)	
9	$(1,2)(3,4)$	$(1,2,3)$	(3)	A_4
10	$(4,5)(6,7)$	$(1,2,3)(4,5,6)$	(3^2)	
11	$(2,4)(3,5)$	$(1,2,3)(4,5,6)$	(3^2)	
12	$(8,10)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(3^4)	
13	$(1,4)(2,7)(3,10)(5,12)(6,8)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(3^4)	
14	$(3,4)(11,12)$	$(1,2,3)$	$(4,2)$	S_4
15	$(3,7)(5,6)$	$(1,2,3)(4,5,6)$	$(4,2)$	
16	$(2,4)(3,6)$	$(1,2,3)(4,5,6)$	$(4,2)$	

17	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(4,2^3)$	
18	$(1,2)(3,4)(5,6)(7,10)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(4,2^3)$	
19	$(2,4)(2,5)$	$(1,2,3)$	(5)	A_5
20	$(3,4)(5,6)$	$(1,2,3)(4,5,6)$	(5)	
21	$(1,2)(3,4)(5,7)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(5^2)	
22	$(1,2)(3,4)(5,10)(6,11)(7,8)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(5^2)	
23	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$	$(1,2,3)$	$(3,2^4)$	$2 \times S_4$
24	$(3,4)(11,12)$	$(1,2,3)(4,5,6)$	$(6,2)$	
25	$(1,4)(2,5)(3,6)(7,8)(9,10)(11,12)$	$(1,2,3)$	$(6,2^3)$	$3 \times S_3$
26	$(5,6)(11,12)$	$(1,2,3)(4,5,6)$	$(3,2^2)$	
27	$(1,7)(2,8)(3,9)(4,10)(5,11)(6,12)$	$(1,2,3)(4,5,6)$	(6^2)	
28	$(1,7)(2,8)(3,9)(4,10)(5,12)(6,11)$	$(1,2,3)(4,5,6)$	(6^2)	
29	$(8,9)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(3^2,2^2)$	
30	$(1,4)(2,5)(3,6)(7,10)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(6,2^3)$	
31	$(3,4)(6,7)$	$(1,2,3)(4,5,6)$	(7)	$L_2(7)$
32	$(5,7)(6,8)$	$(1,2,3)(4,5,6)$	$(5,3)$	$3 \times A_5$
33	$(9,10)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(5,3^2)$	
34	$(3,7)(6,8)$	$(1,2,3)(4,5,6)$	(4^2)	S_4
35	$(1,4)(2,5)(3,7)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(4^2,2^2)$	
36	$(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(4^2)	
37	$(1,4)(2,6)(3,7)(5,10)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(4^2,2^2)$	
38	$(1,4)(2,6)(3,10)(5,7)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(4^2,2^2)$	
39	$(6,7)(11,12)$	$(1,2,3)(4,5,6)$	$(4,3,2)$	$3 \times S_4$
40	$(8,10)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(4,3^2,2)$	
41	$(1,2)(3,4)(5,6)(7,10)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(6,4)$	
42	$(1,2)(3,4)(5,7)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(7,2^2)$	$2 \times L_2(7)$
43	$(1,2)(3,7)(4,5)(6,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(3^2,2^2)$	$2 \times A_4$
44	$(1,4)(2,6)(3,7)(5,8)(9,10)(11,12)$	$(1,2,3)(4,5,6)$	$(6,2^3)$	

45	$(3,4)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(6^2)	
46	$(1,4)(2,5)(3,7)(6,10)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(6^2)	
47	$(1,4)(2,5)(3,10)(6,7)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	(6^2)	
48	$(1,2)(3,7)(4,8)(5,9)(6,10)(11,12)$	$(1,2,3)(4,5,6)$	$(6,3,2)$	$S_3 \times A_4$
49	$(6,7)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(6,3,2)$	
50	$(1,2)(3,4)(5,7)(6,10)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(6,3,2)$	
51	$(1,4)(2,7)(3,8)(4,9)(6,10)(11,12)$	$(1,2,3)(4,5,6)$	$(10,2)$	$2 \times A_5$
52	$(1,4)(2,5)(3,7)(6,10)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(10,2)$	
53	$(1,4)(2,5)(3,10)(6,7)(8,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(10,2)$	
54	$(6,7)(9,10)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(9,3)$	$3^3.A_4$
55	$(1,2)(3,4)(5,7)(6,10)(8,9)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12$	(9)	
56	$(1,4)(2,7)(3,10)(5,8)(6,12)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(9,3)$	
57	$(1,2)(3,4)(5,7)(6,9)(8,10)(11,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(8,2)$	$4^2.S_3$
58	$(1,2)(3,4)(5,510)(6,12)(7,8)(9,11)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12$	$(8,2)$	
59	$(1,4)(2,7)(3,10)(5,8)(6,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12)$	$(8,4)$	
60	$(1,2)(3,4)(5,7)(6,10)(8,11)(9,12)$	$(1,2,3)(4,5,6)(7,8,9)(10,11,12$	(11)	$L_2(11)$

Table B.1: Orbits of A on its $(2A, \langle 3A \rangle)$-pairs.