



Maximum non-bounded difference method for overbounding Global Navigation Satellite System errors

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Abstract

Global Navigation Satellite Systems (GNSS) support numerous applications, including mission-critical ones that require high integrity for safe operations such as air, maritime, and land-based navigation. These applications require ensuring the upper bound of the positioning computation, including overbounding any assumptions made in positioning algorithms. One of the main assumptions in the GNSS positioning algorithm is that the error follows a Probability Density Function (PDF), widely assumed to be Gaussian. This assumption should be overbounded to ensure safety. Two mathematical definitions of distribution overbounding have been proposed: Cumulative Distribution Function (CDF) overbounding and paired overbounding. Achieving these definitions has been an open problem for the past 20 years, and to date, no methods have effectively achieved either definition in multimode datasets. This paper proposes the first overbounding method, the Maximum non-Bounded Difference (MnBD) method, capable of achieving both definitions. Its effectiveness is assessed by comparison with the combined CDF bounding and paired overbounding approach. The results show that the MnBD method successfully achieves overbounding in both real datasets and 1000 simulated multimode data, tested on Gaussian and Generalized Extreme Value (GEV) distributions. The latter demonstrates MnBD's ability to handle asymmetric distributions. The developed approach can be utilized in overbounding the GNSS error at the system level and user level to ensure the system integrity.

Keywords GNSS · Mission-critical applications · Error distribution overbounding · SBAS

Introduction

Global Navigation Satellite Systems (GNSS) supports a wide range of mission-critical applications such as autonomous driving, drones, visually impaired people navigation, air navigation, maritime operation, and farm machinery. Integrity has special importance for mission-critical applications, which are monitored at the user level through the Receiver Autonomous Integrity Monitoring (RAIM) algorithm. One of the main outputs of the RAIM algorithm is the protection level, which is an upper bound that a position

error must not exceed without being detected with a given probability based on integrity risk requirements.

The protection level is computed by considering the overbounded error distribution, nominal bias, probabilities of missed detection and false alert, and other factors that may vary depending on the positioning algorithms (e.g., ambiguity resolution overbounding). Figure 1 illustrates the main elements that should be considered when computing the protection level. Given that the error distribution is assumed to follow a Probability Density Function (PDF) (e.g., Gaussian), this assumption must be overbounded to ensure the safety of the navigation system. For example, Fig. 2 presents a Gaussian distribution that best fits the measurement errors, along with an overbounded distribution that has a higher sigma to ensure safety. If the protection level is computed from the original Gaussian distribution, the probability that the true error exceeds the protection level might be higher than the integrity risk due to the Gaussian assumption. Therefore, the overbounded distribution should be used to ensure safety/integrity.

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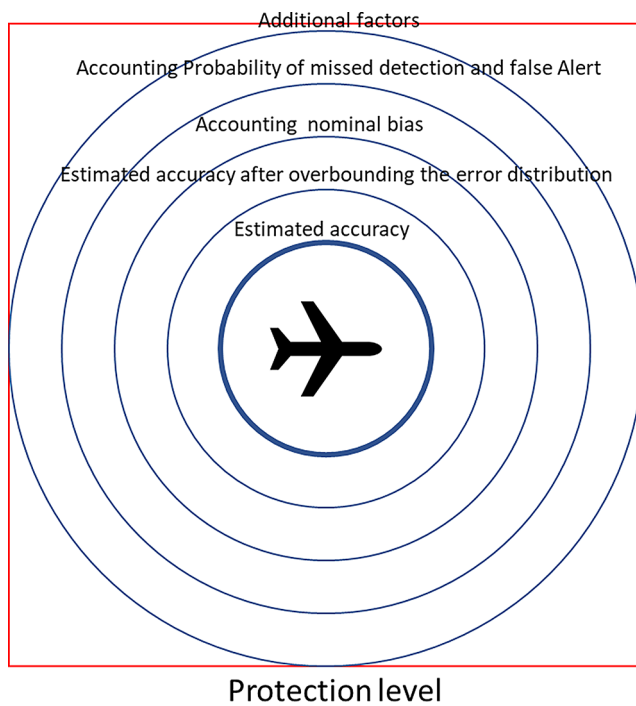


Fig. 1 Protection level elements

Overbounding can be applied at the system level, user level, or both. In system-level integrity monitoring, the overbounded error distribution can be computed using the integrity reference station and then sent to the user via an Integrity Support Message (ISM), such as those sent by the Satellite-Based Augmentation System (SBAS) and Ground-Based Augmentation System (GBAS). According to the

EU-U.S. Cooperation on Satellite Navigation 2016, the Advanced RAIM (ARAIM) ISM includes the Sigma user range accuracy (σ_{URA}) in Type 1 A and Type 1B messages. The σ_{URA} value multiplies the broadcast user range accuracy value from the satellite to determine the overbounding integrity sigma, which is then input into the user algorithm.

However, system-level integrity monitoring does not account for user-level errors (e.g., multipath), so using ISM information alone cannot ensure bounding for errors caused by user-level sources. Consequently, user-level error overbounding must be utilized, particularly in complex environments (e.g. urban environment).

In theory, there are four potential operational options for implementing overbounding. The first option is utilizing ISM information alone; while this cannot ensure complete overbounding, it can be used to some extent in open-sky environments. The second option is using ISM information with additional factors to overbound user-level errors, which can improve system integrity and safety compared to using ISM information alone but may impact system availability. The third option involves developing a mechanism to overbound user-level errors while also using ISM information, which can improve system availability compared to the second option while ensuring system integrity. Finally, the fourth option is developing a mechanism that can overbound all error sources at the user level without using ISM messages, maintaining system availability while reducing investment in SBAS and GBAS systems. However, to date, the development of overbounding has focused on system-level overbounding.

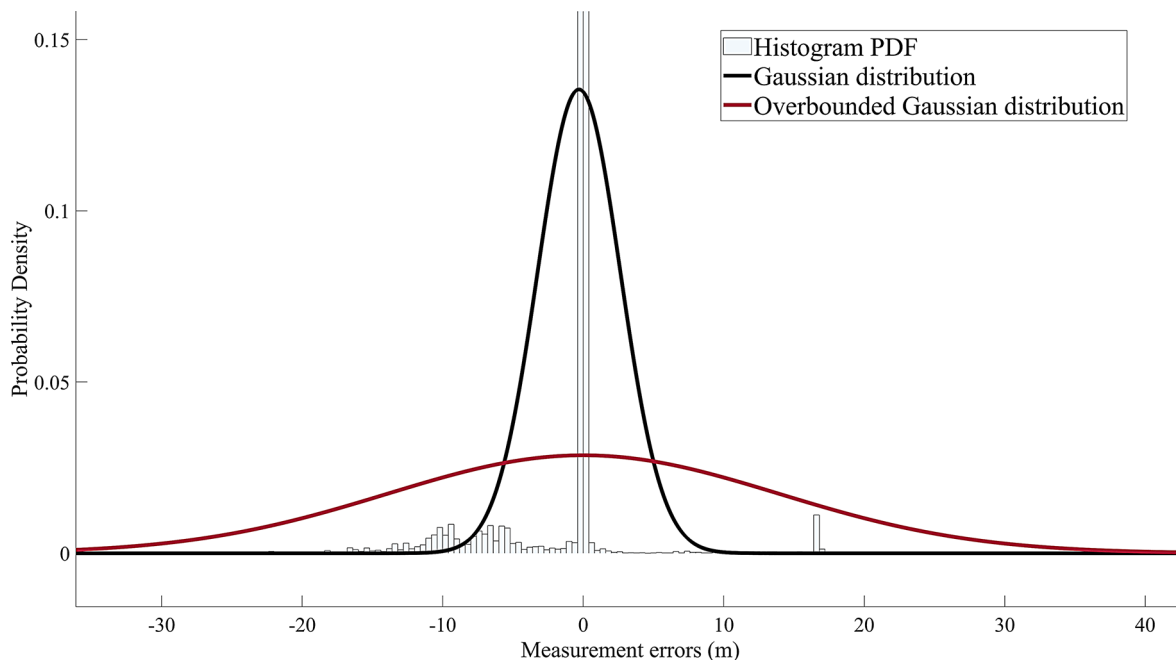


Fig. 2 Gaussian distribution vs. overbounded Gaussian distribution

Overbounding methods are classified as either measurement domain or position domain. The former is based on bounding measurement data, where errors are modelled separately for each satellite and mapped or projected to the position domain. In contrast, position domain overbounding is based on bounding the position errors without the need for transformation or mapping of the errors (Osechas and Rife 2013). Figure 3 presents the overbounding in the measurement and position domains within the positioning architecture.

Position domain bounding was introduced by Markin and Shively (1995) as an alternative to range or measurement domain overbounding, as it could reduce the protection level and simplification of the related computations. Since then, significant efforts have been made to improve position domain overbounding methods (Lee 2004; Ahmad et al. 2014; Lee et al. 2009; Osechas and Rife 2013). However, overbounding in the position domain has limitations, including not considering temporal correlation and multipath. In addition, position domain overbounding does not ensure the bounding for the position itself (Osechas and Rife 2013).

In the measurements domain overbounding, the first true mathematical definition of overbounding was given by DeCleene (2000), called the Cumulative Distribution

Function (CDF) overbounding theorem. Although this definition effectively links measurement and position domain overbounding, it has limitations in overbounding multimodal distributions. Therefore, significant research has been undertaken since the introduction of the CDF overbounding theorem to deal with multimodal distribution and minimize the overbounding region to increase system availability. Of these efforts, moment bounding (Raytheon 2002), operates in the moment domain and accounts for asymmetric, non-zero mean, and multimodal distributions. According to Rife et al. (2004b), moment bounding provides poor performance for small biases and can only efficiently bound large biases.

In addition to the CDF overbounding definition, Rife et al. (2004a) introduced the Paired CDF overbounding mathematical definition, which extends the CDF overbounding definition. Paired CDF overbounding is based on two CDF bounds: left and right CDF overbounds. Paired CDF overbounding effectively links measurement domain with position domain overbounding. In addition, it allows bounding of asymmetric, shifted-median, and multimodal distributions.

Several methods have been developed to achieve the overbounding definitions (CDF and paired). More recent efforts by Blanch et al. (2017) and Blanch et al. (2018)

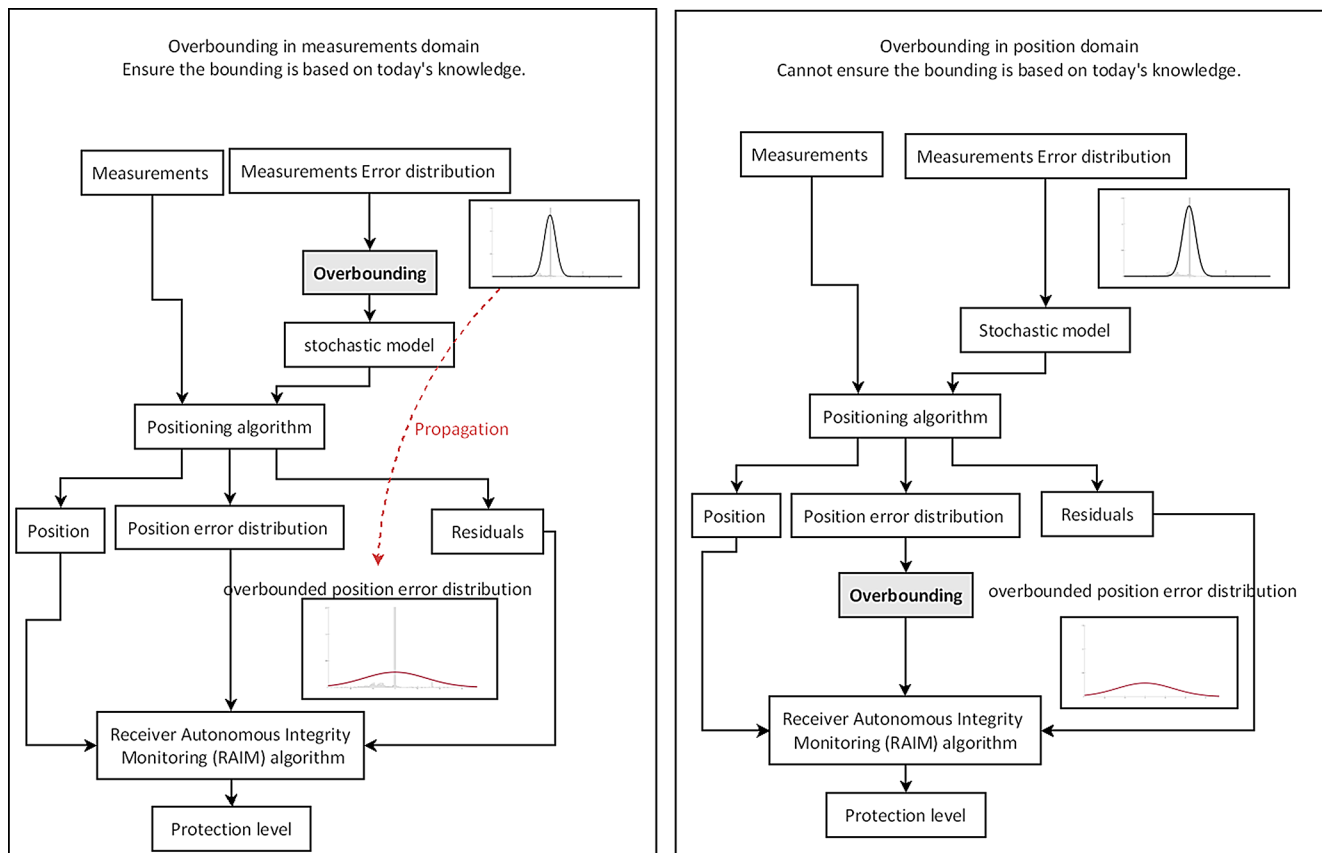


Fig. 3 The overbounding in the measurement and position domains within the positioning architecture

introduced combining CDF bounding and paired overbounding technique. This technique aims to achieve the paired overbounding definition, by applying the CDF overbounding independently to each side. Therefore, this technique can be considered a mechanism to achieve the paired overbounding definition. However, this technique has been tested in this paper, and the results show that it cannot ensure bounding in all datasets and can cause a significant shift in the distribution mean.

This paper proposes, for the first time, an overbounding method that can achieve overbounding for any dataset, including multimodal datasets. The proposed method, called the Maximum non-Bounded Difference (MnBD) overbounding method, can achieve both CDF and paired overbounding definitions. It can effectively overbound both Gaussian and non-Gaussian distributions, as demonstrated using the Gaussian and Generalized Extreme Value (GEV) distributions. The method has been tested using both real and simulated datasets.

Overbounding methods

This section reviews the two overbounding definitions: CDF overbounding and paired CDF overbounding. In addition, this section reviews combining CDF bounding and paired overbounding, which aims to achieve paired CDF overbounding.

Cumulative distribution function overbounding

The CDF overbounding is considered the first true overbounding definition, and it effectively links measurement and position domain overbounding. If G_0 and $G_a(x)$ are CDFs, then G_0 overbound $G_a(x)$ if:

$$G_0(x) \geq G_a(x), \quad \forall G_a < \frac{1}{2} \quad (1)$$

$$G_0(x) \leq G_a(x), \quad \forall G_a > \frac{1}{2} \quad (2)$$

The CDF overbounding results in a single distribution that overbounds the empirical distribution.

Paired CDF overbounding

Paired overbounding is based on two CDF bounds: left and right CDF overbounds. The first bounds the left side of the distribution, while the second bounds the right side of the distribution. The left and right CDF overbounds for the CDF (G_a) can be achieved by:

$$G_L(x) \geq G_a(x) \quad \forall G_L \leq \frac{1}{2} \quad (3)$$

$$G_R(x) \leq G_a(x) \quad \forall G_L \geq \frac{1}{2} \quad (4)$$

where: $G_L(x)$ and $G_R(x)$ are two CDF overbounds (G_a) from the left and right sides respectively. Then the overbounded distribution G_0 can be given by:

$$G_0 = \begin{cases} G_L(x) & \forall G_L < \frac{1}{2} \\ \frac{1}{2} & otherwise \\ G_R(x) & \forall G_R > \frac{1}{2} \end{cases} \quad (5)$$

Unlike CDF overbounding, paired overbounding ends up with two distributions—one to overbound the right tail and another to overbound the left tail. This introduces complexity in the positioning algorithm, especially when the solution separation RAIM algorithm is implemented, making it less reliable to execute based on today's computer power.

Combining CDF bounding and paired overbounding

Combining CDF bounding and paired overbounding involves two stages: determining overbounded Gaussian distributions for the right and left tails, and then bounding both overbounded distributions with a single Gaussian distribution. The first stage includes two steps: forming a unimodal and symmetric distribution that bounds the empirical distribution and determining a Gaussian overbounding distribution. Since this method involves overbounding raw data with an empirical distribution and then overbounding the empirical distribution, it can result in extra bounds.

Testing the Combining CDF bounding and paired overbounding technique (Blanch et al. 2017) using MATLAB Gaussian overbounding scripts (The Board of Trustees of the Leland Stanford Junior University 2017), which is presented in the [Results](#) section, reveals significant shifts in the distribution's mean values across many datasets. Furthermore, the tests show that bounding is not always ensured in the datasets.

Maximum non-bounded difference method

The MnBD method is based on utilizing a search algorithm to compute the maximum non-bounded difference between the CDF and the empirical CDF (eCDF). Specifically, the method calculates this difference, and if the maximum non-bounded difference indicates that the overbounding does not achieved, the distribution parameters are updated. The

search process continues until the maximum difference in the non-bounded regions ideally equals zero.

For example, in the case of overbounding a Gaussian distribution, the method computes the maximum non-bounded difference between the CDF of the fitted Gaussian and the eCDF. Initially, this difference is typically not zero for the fitted distribution. The search algorithm then incrementally increases the standard deviation values until the maximum non-bounded difference becomes zero, thereby achieving overbounding.

This method is designed to achieve overbounding as defined by CDF overbounding (DeCleene 2000) or paired overbounding (Rife et al. 2004a). Consequently, the MnBD method is categorized in this paper as CDF-based MnBD and paired-based MnBD.

CDF-based MnBD overbounding

To derive the mathematical model of the CDF overbounding-based MnBD method, we start with the CDF overbounding definition. Overbounding is achieved using Eqs. (1–2) if:

$$F(x) \geq eCDF(x), \quad \forall eCDF < \frac{1}{2} \quad (6)$$

$$eCDF(x) \geq F(x), \quad \forall eCDF > \frac{1}{2} \quad (7)$$

where: $F(x)$ is the CDF of the overbounded distribution and $eCDF$ is the empirical CDF.

Then, Eqs. (6–7) can be rewritten as:

$$F(x) - eCDF(x) \geq 0, \quad \forall eCDF < \frac{1}{2} \quad (8)$$

$$F(x) - eCDF(x) \leq 0, \quad \forall eCDF > \frac{1}{2} \quad (9)$$

To achieve the above condition on the left side, the maximum value of $eCDF(x_i) - F(x_i)$ must be equal or lower than zero, which refers to the Maximum non-Bounded Difference on the left side (MnBD_L). This can be represented by:

$$\text{MnBD}_L = \max(eCDF(x_i) - F(x_i)), \quad \forall eCDF < \frac{1}{2} \quad (10)$$

where: x_i is the row of data points used to create the eCDF. Distribution overbounding is achieved in the left side bounding if the MnBD_L is equal to or lower than zero. In case the MnBD_L is higher than zero, this value then represents the

maximum non-bounded difference between the CDF and eCDF

To achieve the above condition on the right side, the maximum value of $eCDF(x) - F(x)$ must be equal or lower than zero, which refers to MnBD on the right side (MnBD_R). This can be given by:

$$\text{MnBD}_R = \max(F(x) - eCDF(x)), \quad \forall eCDF > \frac{1}{2} \quad (11)$$

From the above equations, the MnBD between the eCDF and CDF, computed in each step of the empirical distribution (n steps) is represented by:

$$\text{MnBD} = \max(\text{MnBD}_L, \text{MnBD}_R) \quad (12)$$

Distribution overbounding is achieved if the assessment MnBD is equal to or greater than zero. The MnBD method is an iterative method, using Eq. 12 as a search window. Two main conditions must exist to achieve the full overbounding: the MnBD equal to or lower than zero, and the median of the overbounded distribution equal to the row data median. The overbounding must be achieved for distribution overall in the measurement's domain, then the position domain overbounding can be achieved.

The MnBD methods can be used to overbound the tail at various significance levels (α). Then the MnBD is given by:

$$\text{MnBD}_L = \max(eCDF(x_i) - F(x_i)), \quad \forall eCDF < \frac{\alpha}{2} \quad (13)$$

$$\text{MnBD}_R = \max(F(x) - eCDF(x)), \quad \forall eCDF > \frac{\alpha}{2} \quad (14)$$

$$\text{MnBD}_\alpha = \max(\text{MnBD}_{L_{\frac{\alpha}{2}}}, \text{MnBD}_{R_{\frac{\alpha}{2}}}) \quad (15)$$

Although the distribution median must equal the raw data median to ensure full overbounding, it can be replaced by the mean in case bounding is required at a specific significance level, such as bounding in the position domain. In summary, the CDF-based MnBD overbounding search algorithm is based on Eq. 12, with the condition that the median of the overbounded distribution must always equal the median of the raw data. For overbounding at a specific significance level, Eq. 15 can be used instead of Eq. 12.

If overbounding is not achieved, which may occur in multimodal datasets, then paired overbounding can be used.

Paired based-MnBD overbounding

The paired overbounding is based on generating two distributions that overbound the eCDF. From Eqs. (10–11),

the distribution (H) that overbounds the right tail can be achieved when the $MnBD_L$ value on the right side is lower than or equal to zero, and the distribution (Q) that overbounds the right tail can be achieved when the $MnBD_R$ value on the right side is lower than or equal to zero, as follows:

$$MnBD_L = \max(eCDF(x_i) - H(x_i)), \forall eCDF < \frac{1}{2} \quad (16)$$

$$MnBD_R = \max(F(Q) - eCDF(x)), \forall eCDF > \frac{1}{2} \quad (17)$$

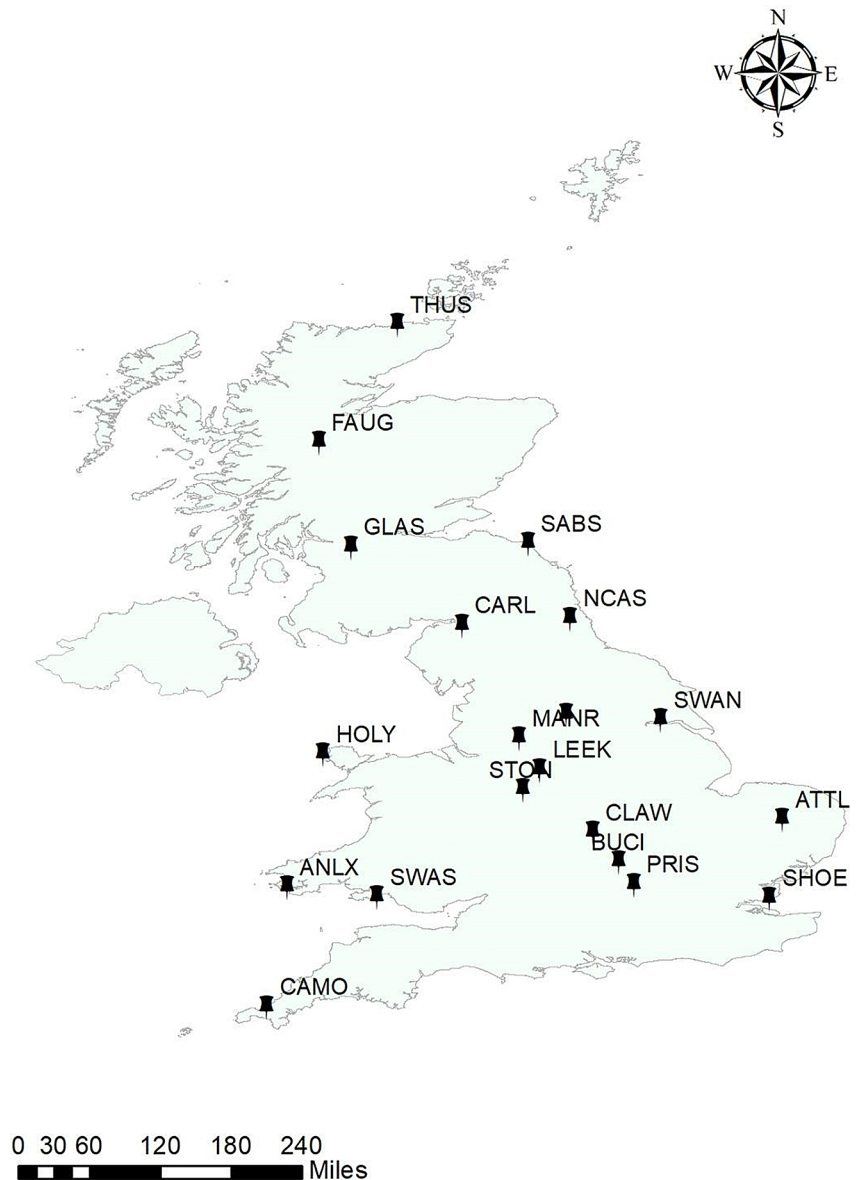
If overbounding is not achieved using the median, which may occur in multimodal datasets, the median of the distribution shall be shifted. For the right overbounding distribution, the shift shall be applied to the right side, while the left

side distribution will be shifted to the left. Put differently, in multimodal datasets, the search should be applied to the distribution median to achieve the conditions in Eqs. (16–17). In summary, the paired based-MnBD overbounding search algorithm is based on Eqs. 16 and 17.

Results

In this section, pseudorange errors were computed using data from 20 Ordnance Survey (OS) National GNSS reference stations in the UK, as presented in Fig. 4. Observations were collected over a 3-hour period with a 30-second sampling interval. Positioning for these stations was computed with millimetre-level accuracy using the static positioning mode (carrier-based static positioning) in the RTKlib

Fig. 4 Map of OS stations utilised in this paper



software, using known reference station. The pseudorange residuals from the least square's method were assumed to be the true error values, given that the pseudorange errors are at the meter level while the position accuracy was computed at the millimetre level. The GPS L1-C/A pseudorange data were extracted and used for the analysis in this section.

Overbounded Gaussian distribution using the MnBD method

The MnBD method utilised a search algorithm to find the standard deviation for the overbounded Gaussian distribution $F_0 \sim (\text{median}, \sigma)$. The error data were combined from 20 stations, resulting in a dataset with 142,651 rows. Overbounding was applied to the combined dataset, and the results show that the proposed method achieves overbounding, as shown in Fig. 5. While the standard deviation of the fitted Gaussian distribution is 0.53 m, the overbounded distribution has a standard deviation of 9.53 m, which is more than 17 times higher than the fitted one.

The MnBD overbounding was also applied to the first 1000 rows of the dataset, as overbounding is highly impacted by measurement distribution and failure measurements. The results show that the standard deviation of the fitted Gaussian distribution is 0.46 m, while the overbounded Gaussian distribution has a standard deviation of 1.46 m, which is 3.17 times the fitted standard deviation. Figure 6 presents the overbounding results for the sample size of 1000.

Comparing the standard deviations of the fitted Gaussian distributions between the 1000 and 142,651 samples, the standard deviation increased by 1.15 times, whereas the overbounded Gaussian distribution increased by more than 6.5 times. This demonstrates that overbounding is a complex problem with behaviour significantly different from simply fitting a distribution and is highly impacted by failure measurements. These findings highlight the importance of Outlier Detection and Exclusion (ODE) at the system level within mission-critical applications.

Overbounded GEV distribution using the MnBD method

Since the GEV scale and shape parameters impact the bounding domain, one or two of these parameters can be used in the MnBD searching algorithm. In this section, the scale parameter has been used in the MnBD searching algorithm. As noted earlier, the median of the distribution must equal the median of the row data to achieve the full overbounding. Therefore, the value of the mean parameter has been updated by changing the scale parameter within MnBD searching, to ensure that the distribution median equal to the row data medium.

The results demonstrate that the MnBD method achieves overbounding using the GEV distribution, as shown in Fig. 7. While the scale and shape parameters of the fitted GEV distribution are -0.2283 m and 1.0779 m, respectively, the overbounded GEV distribution has scale and

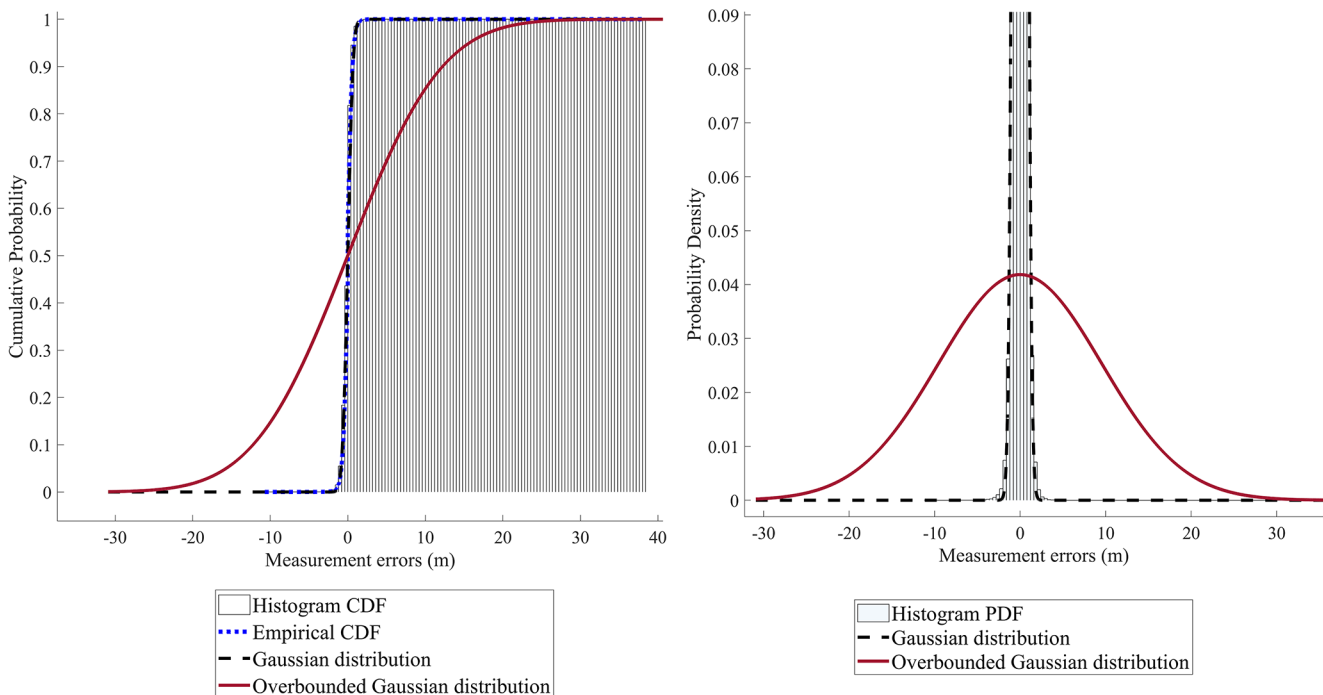


Fig. 5 MnBD overbounding results from the combined dataset

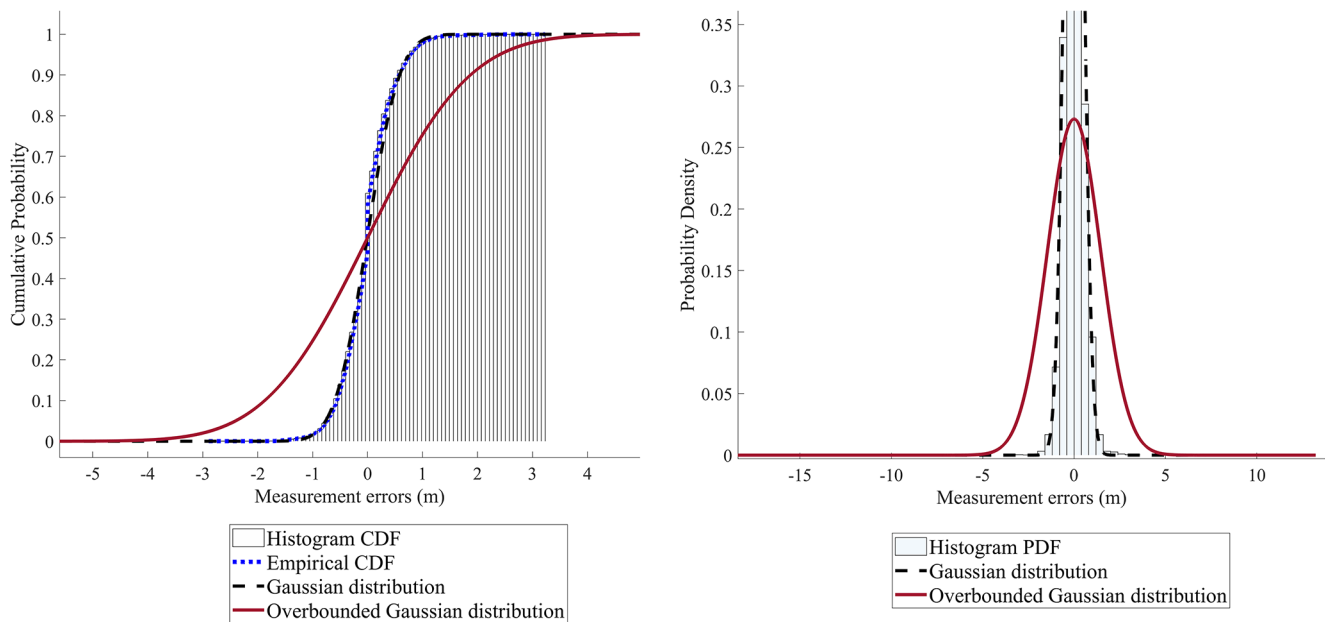


Fig. 6 MnBD overbounding results from the first 1000 row

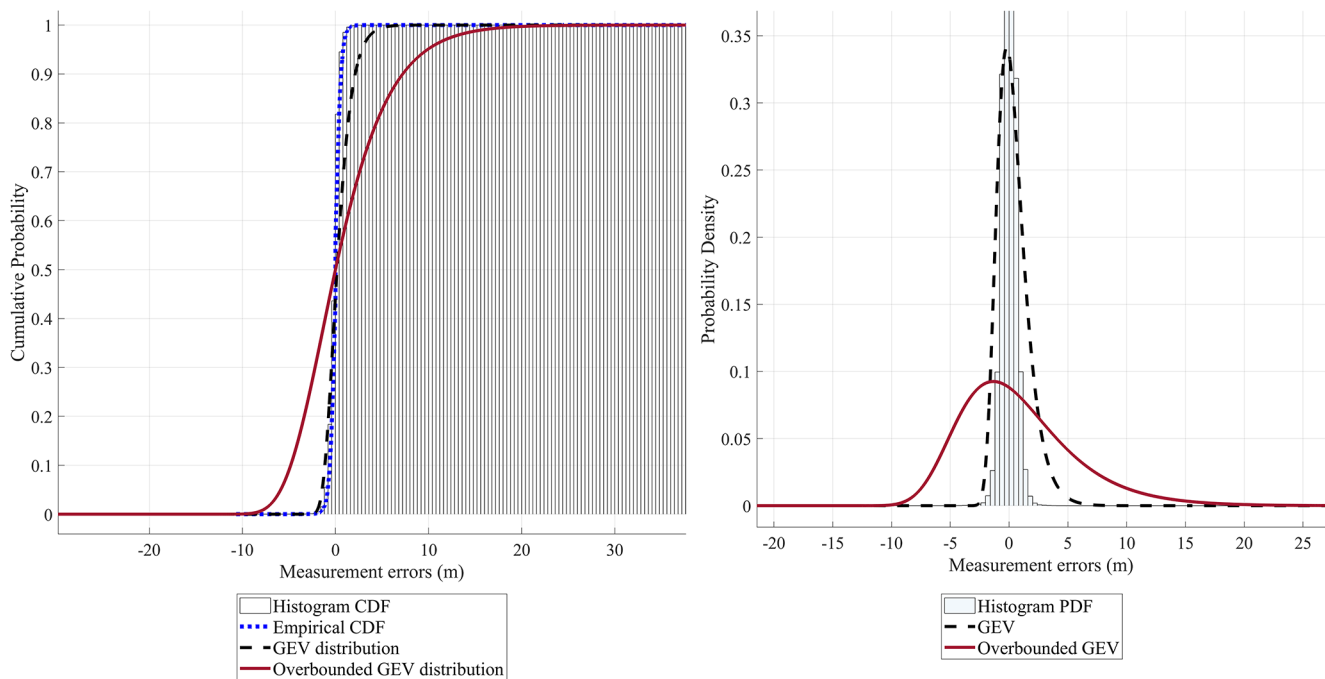


Fig. 7 Overbounding results using the MnBD method with the GEV distribution

shape parameters of -0.0278 m and 3.9779 m. By comparing the overbounded GEV distribution with the overbounded Gaussian distribution, the results show that the GEV distribution provides tighter bounds than the overbounded Gaussian, as illustrated in Fig. 8. This suggests that using the GEV distribution can improve system availability in some cases.

Paired-based MNDB overbounding results

The MnBD method was applied to the combined dataset to assess the impact of paired overbounding on system availability. The results indicate that paired overbounding can significantly enhance system availability compared to CDF overbounding, as shown in Fig. 9. Specifically, the 95% confidence interval length computed from the two paired

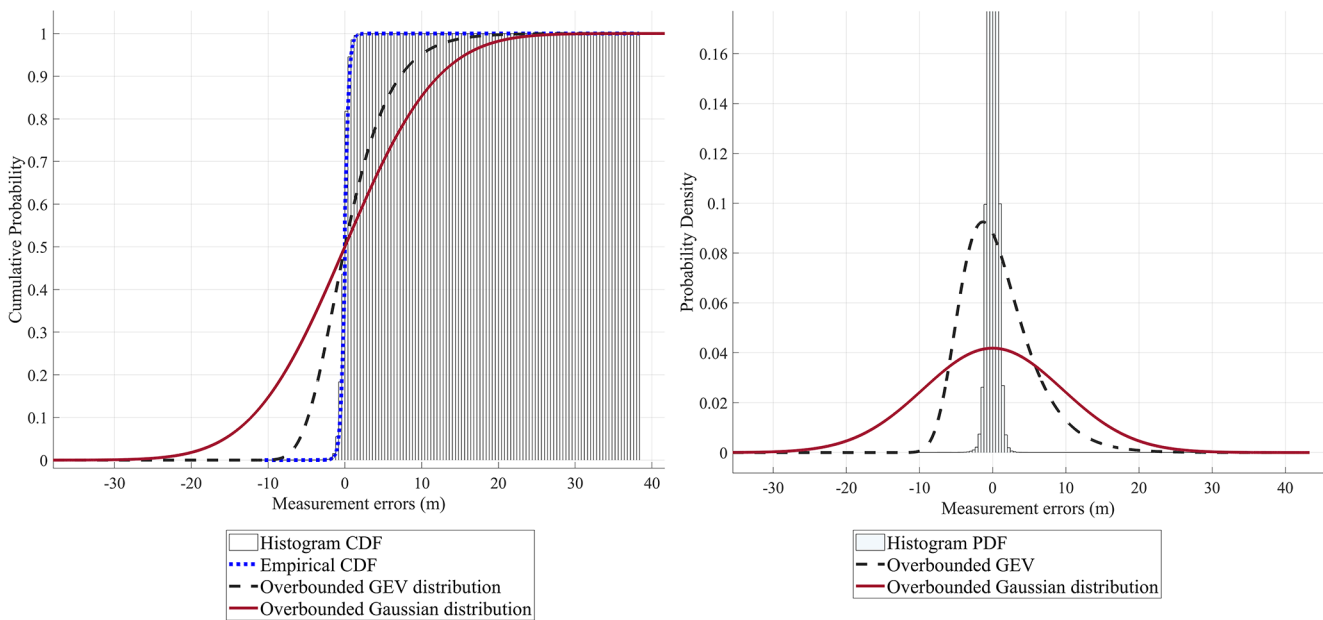


Fig. 8 Comparison of overbounded GEV distribution and overbounded Gaussian distribution

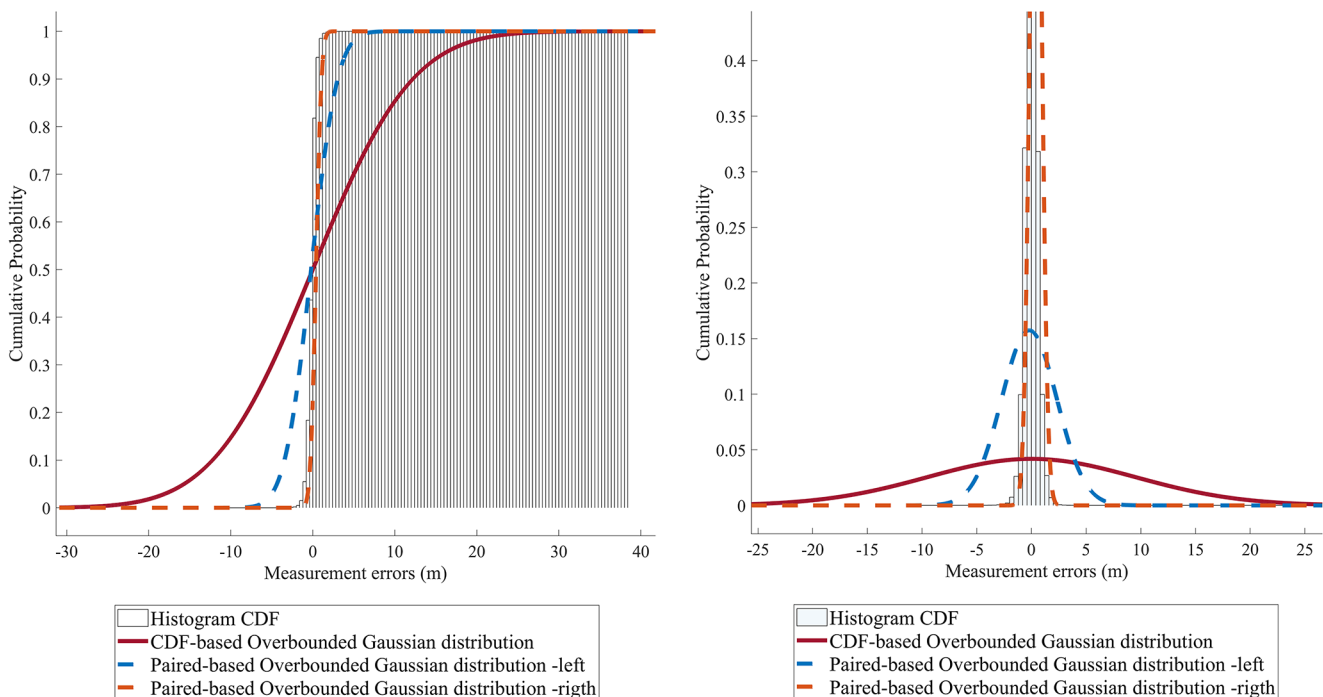


Fig. 9 Comparison of system availability using paired overbounding and CDF overbounding with the MnBD method

overbounded distributions is 6.32 m, while the interval computed from the CDF overbounding is 38.12 m. Table 1 summarizes the overbounded distribution parameters and the length of the 95% confidence interval.

Since CDF overbounding has limitations in overbounding multimodal asymmetric data, a random sample of 1000 multimodal asymmetric datasets was simulated to test the proposed MnBD method. The results demonstrate that the

MnBD method successfully achieved overbounding for all 1000 simulated datasets. These results were compared with those obtained using the CDF bounding and paired overbounding techniques proposed by Blanch et al. 2017; using MATLAB Gaussian overbounding scripts (The Board of Trustees of the Leland Stanford Junior University 2017), and utilising configuration parameters of 100 bins, 1000 steps on the CDF, and an epsilon value of 0.0025. Further

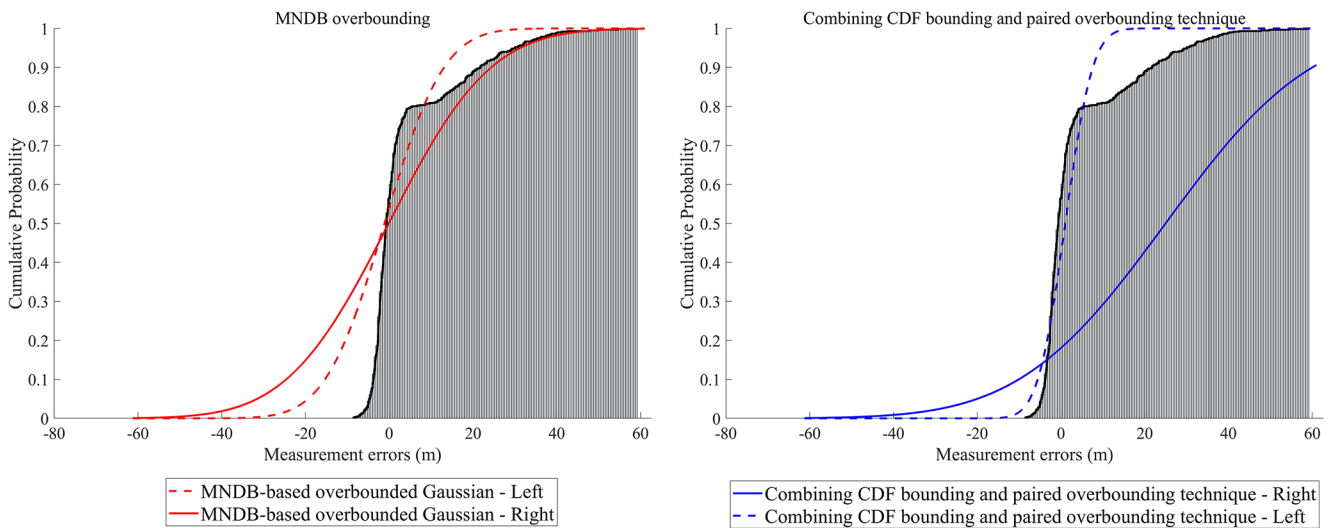
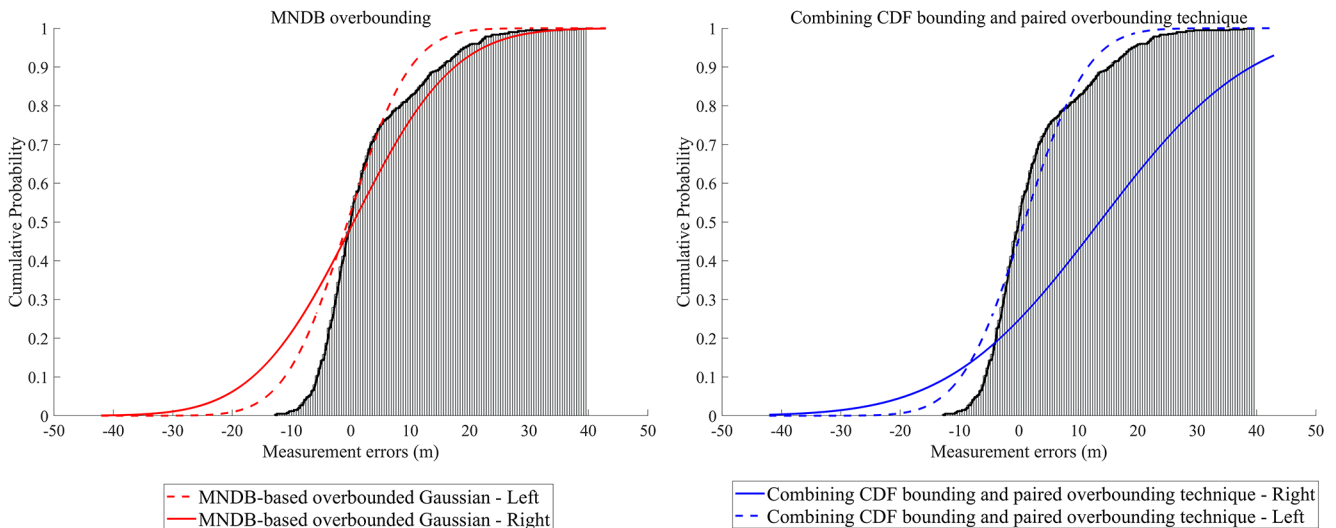
Table 1 Parameters of the overbounded distributions and the length of the 95% confidence interval for paired overbounding and CDF overbounding using the MnBD method

	Mean [m]	Standard deviation [m]	Length of 95% confidant interval [m]
Fitted Gaussian distribution	≈ 0	0.53	2.12
CDF-based overbounding	≈ 0	9.53	38.12
Paired-based overbounding			
Overbounded Gaussian – left	-0.2	2.53	6.32
Overbounded Gaussian – right	0.4	0.53	

sensitivity analysis of these parameters is conducted in this section.

The findings indicate that the CDF bounding and paired overbounding techniques failed to achieve overbounding

on the left side in 389 datasets. Figure 10, 11 and 12 present key examples of datasets where overbounding is not achieved. In addition, the MnBD method produced two distributions that ensured overbounding with minimal shifting, while maintaining the standard deviation of the distribution. In contrast, the CDF bounding and paired overbounding techniques resulted in significant shifting and/or large standard deviations in many datasets. Specifically, the shift in the mean of the right distribution using the CDF bounding and paired overbounding techniques was 10 times greater than that of the MnBD method in 235 datasets, and 20 times greater in 108 datasets. Furthermore, the standard deviation of the right distribution computed using the CDF bounding and paired overbounding techniques was twice as high as that of the MnBD method in 378 datasets, and three times as high in 123 datasets. Figures 10, 11 and 12 illustrate

**Fig. 10** MnBD overbounding against combined CDF bounding and paired overbounding, dataset 1**Fig. 11** MnBD overbounding against combined CDF bounding and paired overbounding, dataset 2

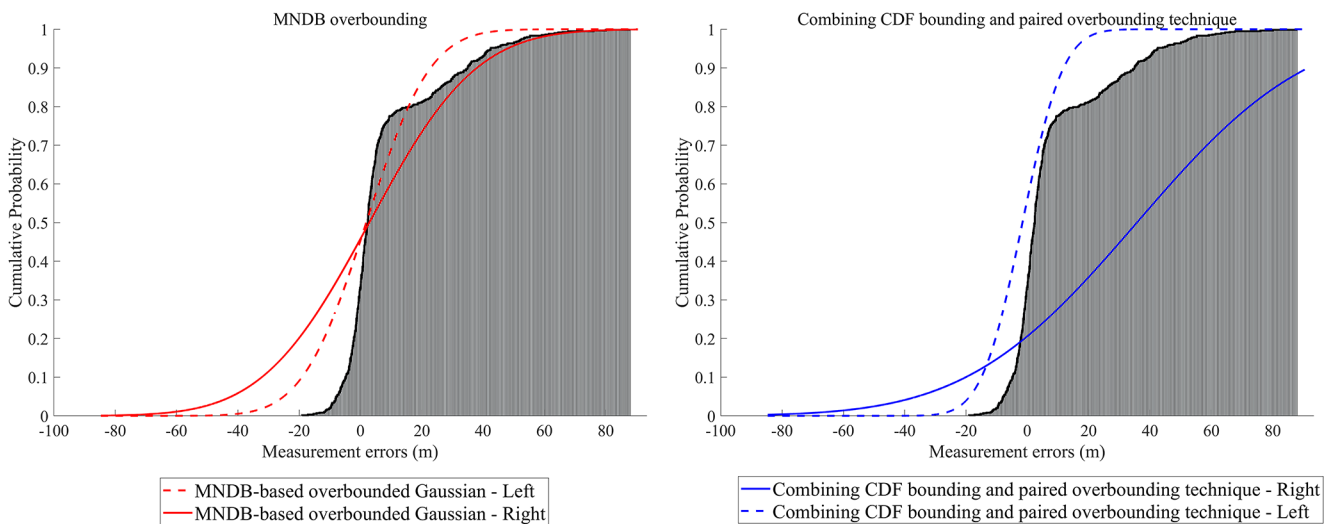


Fig. 12 MnBD overbounding against combined CDF bounding and paired overbounding, dataset 3

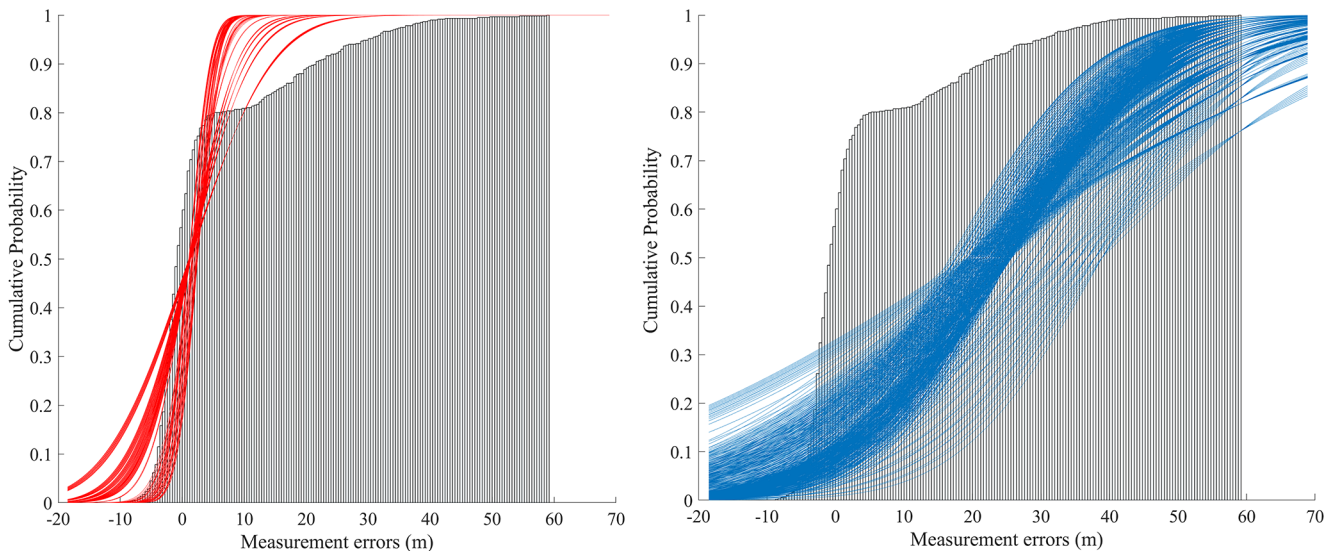


Fig. 13 Overbounded distributions on the left and right sides generated from various combinations of configuration parameters for the CDF bounding and paired overbounding techniques. In total, 7497 overbounding solutions were tested

the significant variance in the standard deviation observed when comparing the CDF bounding and paired overbounding techniques with the MnBD method.

Since CDF bounding and paired overbounding techniques are functions of three configuration parameters—number of bins, steps in the CDF, and epsilon—a sensitivity analysis has been conducted on these parameters for the dataset presented in Fig. 13. The sensitivity analysis involved generating overbounded distributions using various combinations of these parameters: the number of bins ranging from 10 to 1010 (in increments of 50, resulting in 21 values within the range), the number of steps in the CDF ranging from 10 to 1010 (in increments of 50, resulting in 21 values within the range), and epsilon values ranging from 0.001 to 0.05

(in increments of 0.003, resulting in 21 values within the range).

In total, 7497 overbounding solutions were generated in this test. Figure 13 presents the overbounded distributions on the left and right sides. The results show that bounding was not achieved on the left side in all solutions. The standard deviation of the left overbounded distributions ranged between 2.4 m and 10.6 m, while the mean ranged between 1 m and 2.5 m. For the right overbounded distributions, the results show that all overbounded distributions exhibited a high level of mean shifting, ranging between 16 m and 39 m, with standard deviations ranging from 13 m to 44 m. These findings indicate that the CDF bounding and paired overbounding technique cannot achieve bounding in some datasets, even when searching for the ideal configuration

parameters. In addition, this technique is sensitive to configuration parameters, as evidenced by the wide range of standard deviations and means.

Overall, the results demonstrate that the paired-based MnBD method can achieve overbounding in multimodal asymmetric data, maintaining a low level of mean shifting while preserving the standard deviation. This has been validated with 1000 simulated datasets. In addition, the paired-based MnBD method can improve system availability compared to the CDF-based MnBD method. The effectiveness of the MnBD methods has been demonstrated against CDF bounding and paired overbounding technique.

Implementation of the MnBD Method

The MnBD method shall be used in mission-critical applications that require a high level of integrity across various sectors, including but not limited to aviation, maritime, transport, and farming. The method can be implemented in both system-level and user-level integrity monitoring.

The following key points summarize the considerations for the MnBD overbounding implementation in the system level includes key futures works:

- The overbounded distribution can be computed at the system level and sent to the user via an integrity support message in the SBAS and GBAS. SBAS shall consider sending overbounding information to the users, as suggested by the EU-U.S. Cooperation on Satellite Navigation 2016.
- In the case of using CDF-based MnBD, one overbounded distribution shall be sent to the users. However, when using paired-based MnBD, the information of the two distributions must be sent to the users. The transfer of the two distributions can be simplified by sending one mean parameter (b_i) and one standard deviation ($\sigma_{bounded_i}$) to the users, which provides the upper bound for the two distributions. This approach is suggested in the context of ARAIM development. For example, if the mean and standard deviation of the two distributions is ($\mu_1 = 0.5$ m, $\sigma_1 = 2$ m) and ($\mu_2 = -0.6$ m, $\sigma_2 = 1.5$ m), the upper levels of the mean and standard deviation can be sent to users is ($b_i = 0.6$ m, $\sigma_{bounded} = 2$ m). At the user level, the user would use ($\mu_{left} = -b_i = -0.6$ m, $\sigma_{left} = \sigma_{bounded} = 2$ m) and ($\mu_{right} = b_i = 0.6$ m, $\sigma_{right} = \sigma_{bounded} = 2$ m) as the left and right overbounded distributions, which ensures safety. This simplifies the transfer of data but may reduce system availability.
- While the paired-based MnBD has advantages in terms of improving system availability, it adds complexity to the system, especially when solution separation RAIM is implemented. Future works shall reduce the ARAIM complicity and conduct a cost-benefit analysis of utilising paired-based MnBD.
- The overbounding shall be evaluated when the SBAS integrity monitoring stations are within a specific area. For example, in the UK, a trial for UK SBAS has been developed using the OS net reference stations (Ordnance Survey 2024). Since these stations are within the UK, the system level may have complexity in mapping and computing the overbounded distribution, due to the poor geometry.

Since the system level does not account for user-level error sources, the MnBD overbounding implementation should be applied at the user level. In complex environments (e.g., urban areas), user-level MnBD is particularly important, as the level of user-level error is higher due to the impact of multipath. The following points summarize considerations for overbounding implementation at the user level, along with key areas for future work:

- Apply overbounding to the measurement population (by signal ID) that includes all environments. This results in factors that ensure the bounding. This requires defining the characteristics of the measurement population; however, this approach may significantly reduce system availability.
- Develop an adaptive overbounding model that can dynamically update the overbounding distribution. This can be achieved by using measurement populations, each corresponding to a specific environment. The model can then update the overbounding distribution based on the environment. The environment here might refer to a physical one (e.g., open sky, urban, semi-urban) or to Quality Indicators (QIs)-based one. The latter can be defined using advanced statistical models (e.g., machine learning and/or deep learning) to determine the environment based on the QIs. In this case, the environment would not refer to a physical context but to QIs.
- Apply paired-based MnBD can improve system availability; however, it can add complexity to the system. Therefore, a cost-benefit analysis at the user level should be conducted to evaluate this approach.
- Develop a model that can integrate the user-level and system-level overbounding distribution. This required developing a user-level overbounding model that does not account for the system-level error, which then can be integrated with the overbounded distribution coming from the system-level.
- Develop an outlier detection and exclusion model at both the user and system levels, since overbounding is

highly impacted by outliers and faulty measurements, as demonstrated in the [Results](#) section. Advanced statistical models, such as machine learning utilizing QIs, can be applied to improve outlier detection performance.

Conclusion

Overbounding of the error distribution is essential for mission-critical applications to ensure safe navigation. To date, there are two mathematical definitions for overbounding: CDF overbounding and paired overbounding. Achieving these definitions has been an open problem for the past 20 years. This paper proposes, for the first time, an overbounding method called MnBD, which can accomplish both definitions. The proposed method can handle non-Gaussian, asymmetric, shifted-median, and multimodal distributions.

The results of this paper demonstrate that the MnBD method can achieve overbounding based on both CDF overbounding and paired overbounding, with a low level of mean shifting and maintenance of the standard deviation. The method has been tested on both real datasets and 1000 simulated multimodal datasets. The results show that the MnBD method was always able to achieve overbounding in all tested datasets.

The MnBD method can be applied to overbound the error distribution at both the system level and the user level. At the system level, our results suggest utilizing MnBD in future SBAS systems, allowing the overbounded distribution to be sent to the user to compute the protection level.

In addition to the future works outlined in the implementation of the MnBD section, future research endeavours can extend the application of the MnBD method to different distribution types and varying sets of parameters. This will enable a comprehensive evaluation of the influence of overbounded distributions on system availability.

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Data availability The raw data (RINEX files) utilized in this study was downloaded from OS Net stations and is available upon request. It is open-source, free data available at <https://www.ordnancesurvey.co.uk/gps/rinex/2024/>. The satellite navigation data used in the processing was downloaded from the IGS station (https://cddis.gsfc.nasa.gov/Data_and_Derived_Products/GNSS/broadcast_ephemeris_data.html). The simulated data used in the study is available upon request and was generated using MATLAB. All the results for 1000 simulated scenarios are available upon request.

Declarations

Ethics approval and consent to participate Given the nature of this research, it doesn't involve human participants, human data, or human tissue. Consequently, no ethics. Approval was required as per applicable institutional and national guidelines and regulations. Similarly, the concept of 'consent to participate' is not relevant to the context of this study since it involves the analysis of technical data that does not contain any personal or sensitive information.

Consent for publication All authors have reviewed the manuscript and consent to its publication.

Competing interests The authors declare no competing interests.

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