

THE THEORY OF ORTHOTROPIC
PLATES UNDER LATERAL
PRESSURE AND ITS APPLICATION
TO CORRUGATED PLATES

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by

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A B S T R A C T

The theory of orthotropic plates under lateral pressure and its application to corrugated plates

A finite difference solution is given for the large deflection behaviour of rectangular orthotropic (and isotropic) plates under any symmetrical system of transverse loading. The equations are programmed for the Atlas computer and the thesis includes a few examples of the solutions which can be obtained by inserting appropriate parameters in the program. The program allows the following quantities to be specified:

1. Any length to breadth ratio.
2. Any symmetrical system of distributed or patch loading.
3. Flexural and extensional rigidities of the plate in the x and y directions.
4. Torsional and shear rigidities of the plate.
5. Rotational, normal extensional and tangential elastic restraints at the boundaries.

Solutions are compared with existing solutions where these are available, and with experimental results, including two tests on swedged plates which form part of the present investigation. There is good agreement in all cases.

The investigation shows the importance of membrane action in the plate, and the influence of the various boundary restraints. The deflection of a plate with complete boundary restraint against in-plane movement, for example, may be only half as much as when the restraint is absent. With the help of the solutions given it is now possible to state the errors involved in the usual assumptions that the restraints are either completely rigid or completely absent.

The method of calculating the local stresses and deformations in the corrugations, presented in this thesis, makes use of the solution for the overall behaviour of the plate obtained by the orthotropic plate theory and is applicable for any degree of membrane action in the plate. The validity of the method has been satisfactorily demonstrated by the two tests on swedged plates.

TO MY PARENTS

A C K N O W L E D G E M E N T S

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CHAPTER I

I N T R O D U C T I O N

The investigation described in this thesis was undertaken with the support of the British Ship Research Association, originally with a view to studying the behaviour of "swedged" and corrugated plates under lateral pressure. (Swedged plates are plates having corrugations with intervening bands of flat plating). Corrugations or swedges in a plate serve the same purpose as stiffeners in a conventionally stiffened plate, but they usually prove to be more economical than the latter in respect of weight-strength ratio because the corrugations cover area as well as providing stiffness.

The lateral bending behaviour of corrugated plates is substantially similar to that of conventionally stiffened plates, though somewhat more complicated because of the arbitrary shape and deformability of the corrugation profile. The two main aspects of the problem are the problem of overall behaviour and the problem of local behaviour.

The study of the overall behaviour of a corrugated plate is necessary not only when deflections are small but also when deflections are large enough to develop membrane stresses in the plate, because, unless the corrugations are very large, the failure of the plate is unlikely to occur without the development of some membrane action, which must, therefore, be known in order that the failure load can be correctly estimated.

The study of the local behaviour is also important. Local deformations in the plate may lead to instability of the corrugation profile and eventually to premature failure of the plate. Local stresses in the plate may be very severe and acting in conjunction with the stresses due to overall bending, they may greatly influence the collapse of the structure.

A rigorous theoretical solution to the problem being practically impossible, an approximate theoretical solution and confirmatory tests are presented in this thesis.

Preparatory to undertaking a research programme the existing literature on corrugated plates was reviewed in some detail. Previous investigations were found to have treated the corrugations in a plate as isolated beams without considering the overall behaviour of the plate. For a fuller understanding of the overall behaviour a plate theory was considered necessary and 'the orthotropic plate theory' was judged to be suitable for the purpose. The theory was subsequently developed in a sufficiently generalised form to be applicable not only to corrugated plates but also to plates with other forms of stiffener and to ideally orthotropic and isotropic plates. The resulting change in the purpose of the investigation is reflected in the final title of the thesis. The following review is, however, presented in its original form with its greater emphasis on corrugated plates.

CORRUGATED PLATES UNDER LATERAL LOADING

Work done in Russia in the 1930's by Stel'makh on folded metal roofs subjected to uniformly distributed lateral load⁽¹⁾ revealed that each panel of the system has two main stress-systems to carry: the in-plane stresses due to longitudinal bending of the corrugations and the stresses in the transverse direction due to bending of the panels away from their plane. For the calculation of the longitudinal stresses each corrugation unit may be considered as an independent beam and for calculating the transverse stresses each panel must be considered as rigidly connected with the adjacent ones, forming a spatial frame. The longitudinal bending moment of the plate elements and torques over the longitudinal and transverse sections of the panels are neglected because of their low magnitudes.

Jaeger⁽²⁾, Elovkov⁽³⁾ and others have derived expressions for stresses in corrugated plates with symmetrical trapezoidal corrugations. For calculating the longitudinal stresses, each corrugation unit was considered to act as a beam with appropriate end conditions and subjected to lateral load. The section modulus is calculated in the usual way assuming complete effectiveness of the flanges. Shear stresses associated with longitudinal bending were calculated for each half of the trough in the same way as for Z-sections. In such calculations, sufficient transverse restraint was assumed to be present to prevent distortion of the corrugations and to ensure their purely vertical deflection. The appropriateness of the beam theory applied to corrugated plates depends on the equality of deflection of all the troughs; such equality may be obtained with the inner corrugations but cannot be achieved close to the edges if the corrugated plate is supported along the edges. Vasilev's experiments⁽¹⁾ have shown that the edge corrugations cannot be covered by the simplified beam theory, but this is not a serious matter as the edge corrugations are not as severely stressed as the interior ones. The deflection and longitudinal stresses for the interior corrugations so calculated were in reasonable agreement with the results of experiments^{(2),(4)}. The experiments on corrugated bulkheads of actual tankers reported by Getz⁽⁵⁾ also prove the validity of the simplified beam theory, provided that a reduced width is considered effective in bending and provided also that the deflection due to shear is taken into consideration. Getz stated that in the case of short spans, the reduction in moment of inertia and section modulus of the corrugation may be as high as 30 to 40 per cent due to the reduced effective width of flange. The effect of shear-deflection must be considered when analysing corrugated plates with cross-girders running transverse to the corrugations and acting as yielding supports to the troughs running as continuous beams over them.

The transverse bending stresses in a symmetrical trapezoidal corrugated plate under lateral pressure are calculated^{(2),(3)} by isolating a transverse strip of unit width and considering this to be an infinitely long beam on unyielding simple supports at each corner and subjected to uniformly distributed load. The only indeterminate transverse bending moment is obtained by considering the condition of no change in slope at the points of symmetry. The transverse axial stresses on the longitudinal sections are statically determinate and are obtained from the consideration of symmetry and of equilibrium of these stresses with the applied pressure and the shear stresses on the transverse section. The transverse axial stresses are only 8 to 10 per cent of the transverse bending stresses. The expressions for these stresses near the supports will be affected if the ends of the corrugations are encastered. At some distance away from the supports, for a plate under uniform pressure, the transverse stresses should be constant throughout the span. This method of calculating the transverse stresses is also independent of the angle between the web and flange of the corrugations. Vasilev⁽¹⁾, however, considered that for an angle of 60° , the transverse bending moments at the corners of the corrugations should be 1.4 to 1.6 times the magnitude given by the method suggested above. Jaeger⁽²⁾ reported that the transverse stresses given by the simplified theory were not corroborated sufficiently by experimental results.

The longitudinal bending behaviour of corrugated plates with symmetrical trapezoidal corrugations including rectangular and triangular sections has been studied in great detail by Caldwell⁽⁴⁾, both theoretically and experimentally. The first series of experiments involved 21 panels of corrugated sheet subjected to pure bending moment. Both mild steel and NS6 5 per cent magnesium alloy 1/4 H state were employed as materials. The tests were carried out in most cases to failure and it was

noted that unless premature flange buckling occurred, the bending moment-deflection relationship was that for a beam. The behaviour in the elasto-plastic range and the final plastic moment were in close agreement with the theoretical results obtained by assuming continued beam behaviour; the theoretical results have been provided by the author in the form of moment-curvature relationships for mild steel and duralumin troughs having any type of symmetrical trapezoidal shape. The experiments showed that the full cross-section of a corrugated trough was effective in carrying bending moments and that unless premature buckling occurred, the reserve of strength after first yield of the corrugations would be governed by the shape factor, which might be as high as 1.5 with triangular profiles. Six aluminium alloy and two steel specimens exhibited either elastic or inelastic instability and had premature collapse. This led the author to investigate the problem of flange buckling in greater detail.

The difficulty in calculating the buckling stress for the compression flange is in assessing the restraint provided along its two edges by the webs. Caldwell assumed that the real restraint could be replaced by an equivalent restraint provided by a narrower plate, which had its longitudinal edges away from the flange simply supported and which was subjected to uniform compression of the same magnitude as the flange stress. The width of this narrow plate was found by experiments on single corrugations made of aluminium alloy and subjected to pure bending, to be 0.45 of the web width. The buckling stresses so calculated agreed well with the results of the first set of experiments on panels. It was also observed that if buckling occurred in the elastic range, there would be some reserve strength left in the corrugations; beyond elastic limit, the buckling load would itself be the collapse load.

Finally, eight model corrugated bulkheads were tested by water pressure under conditions similar to those occurring in practice. The dimensions of the bulkheads between the bolted edge connections were 35 x 22 in. Four of the bulkheads were made of aluminium alloy and four of mild steel. One aluminium alloy and one mild steel bulkhead were tested with the corrugations running in the longer direction. For these two plates, the virtual side ratio was 4.15, whilst for the other specimens the ratio varied from 7.6 to 10.9. The deflection measurements of all the test plates indicated that the inner corrugations carried the load coming on them as independent beams and the pressure-deflection relationship of the central corrugations agreed with the theoretical results obtained by assuming simple beam behaviour and full effectiveness of the sections in carrying bending moments. In all the specimens, failure was precipitated by buckling of the compression flanges. The test plates having corrugations parallel to the longer edges showed a greater rate of deflection with pressure, but before the deflections became large enough to develop over-all membrane stresses, elastic buckling occurred, reducing the stiffness of the plates. However, due to the increase in deflections, membrane stresses developed, which increased the stiffness of the plate, and the plates continued to carry increasing pressure until excessive leakage caused by the large deflection prevented further loading. In the case of the shorter spans buckling and yielding occurred at small deflections and excessive permanent set and severe reduction in stiffness had occurred before membrane stresses could develop. In these cases, therefore, a sufficient increase in deflections occurred at constant load (in most cases due to instability in the plastic range) to indicate failure, unlike the first two cases, where at no time did an increase in deflection occur without some increase in pressure. Stresses were not measured anywhere in the plate and the validity of the expressions for stresses could not be checked.

In the case of corrugated plates having symmetrical corrugations formed by arcs of cylindrical surfaces, longitudinal stresses in the inner corrugations can be calculated on the assumption that each corrugation acts like a beam^{(1),(6)}. With this assumption, the stresses in the longitudinal sections of cylindrical corrugations can be determined without the necessity of using the general shell theory equations. The method is similar to that used for trapezoidal corrugations. A strip of unit width is isolated by two parallel transverse sections. From symmetry, the shear stresses are zero at crests and troughs. The transverse in-plane stresses are obtained directly from statical equilibrium. The transverse moment at crests and troughs on the longitudinal sections are equal and are obtained by using the condition that slopes in the transverse direction at these points remain unchanged when the plate deflects. The shear stress distribution on transverse sections is assumed to be governed by the simple beam theory applied to a corrugation. The transverse in-plane stress is zero at the mid-height axis and maximum at crests and troughs and even there it is small compared to the transverse bending stresses⁽⁶⁾.

For plates with unsymmetrical trapezoidal corrugations having different widths for the top and bottom flange, the longitudinal and transverse stresses can be calculated in the same manner as for symmetrical trapezoidal corrugations⁽¹⁾.

No theoretical or experimental investigation has been made on the distribution and magnitude of transverse stresses in "swedged" plates. Results⁽⁷⁾ of hydrostatic testing on swedged bulkheads conducted by the British Ship Research Association showed that for small deflection, the central swedge with its full share of flat plating deflected like a beam with some degree of restraint at the ends. As only longitudinal strain gauges were used only on the dry side of the bulkhead, the strain readings are difficult to interpret. Since the virtual side ratio of

the test bulkheads was greater than 3 and the deflections were small, the deflection of the central swedges was practically unaffected by the longitudinal boundaries. Lateral loading tests on a number of plate panels with three equally spaced swedges and having longitudinal edges unsupported showed that the flat plating associated with the swedges was fully effective and that the swedges behaved like beams and exhibited no instability of the profile.

From the foregoing summary, it is apparent that for corrugated or swedged plates having high virtual side ratio, the longitudinal and transverse stresses near the centre of the plate due to small deflection bending (without buckling) can be determined without considering two-dimensional plate behaviour. However, for plates with small virtual side ratios and also for investigating the stresses and deflections at points away from the longitudinal central line, some form of plate theory has to be used. One of the available methods is to consider the corrugated plate as equivalent to an orthotropic plate, that is, a plate having different rigidities, equal to those of the corrugated plate, in two orthogonal directions. This method has been applied to investigate the bending behaviour of stiffened plates and the buckling behaviour of stiffened and corrugated plates. Provided that the spacing of the stiffeners or corrugations is small compared with the dimensions of the plate, the actual overall behaviour of such plates is in good agreement with the predictions by the orthotropic plate theory; the local behaviour is however, dependent on the actual form and spacing of the stiffeners or corrugations. A short summary of investigations made on rectangular orthotropic plates under distributed loading is given in the following paragraphs.

Fundamental investigations on orthotropic plates were made by Huber⁽⁸⁾, who derived the differential equation for bending of such plates. Huber gave solutions for a number of cases of simply supported orthotropic plates, by starting with an infinitely long plate subjected to a sinusoidally distributed line load. Schade^{(9),(10),(11)} applied orthotropic plate theory to stiffened plates with particular reference to a ship's bottom structure. Schade also provided a number of design curves for the calculation of maximum deflections, bending moments and shear forces in orthotropic plates having two opposite edges simply supported with the other two edges simply supported or fixed, for three different values of the torsional rigidity parameter. Several other authors have also investigated the problem of lateral bending of orthotropic plates⁽¹²⁾⁻⁽²⁰⁾. In some of these investigations on stiffened plates, the theory has been modified to incorporate the effect of the unsymmetrical arrangement of stiffeners attached only to one side of the plate.

Some work has also been done on large deflection behaviour of orthotropic plates. For the sake of clarity and completeness the investigations on isotropic plates are also being reviewed.

The equations governing the large deflection behaviour of isotropic plates were first derived by von Karman⁽²¹⁾ from considerations of equilibrium and compatibility. The derivation of corresponding equations for orthotropic plates has been attributed to G.G. Rostovcev⁽²²⁾. Soper⁽²³⁾ has also given the derivation as applicable to stiffened plates.

Three principal methods available for the solution of large deflection problems are outlined in the following paragraphs:

Ritz Energy Method. The displacements u , v , w , or the deflection w and the stress function f , are assumed in terms of known functions with unknown coefficients, the functions being

chosen so that they satisfy the boundary conditions of the problem. The unknown coefficients are obtained by minimising with respect to the coefficients the total potential energy of the plate. This results in a set of non-linear algebraic equations which must be solved for the coefficients. For good accuracy the number of unknown coefficients has to be fairly large and this leads to a very great amount of numerical work. Way⁽²⁴⁾ has used this method for analysing uniformly loaded rectangular isotropic plates having all edges clamped against movements and rotation, and has obtained numerical results for the aspect ratios, 1, 1.5 and 2. No attempt has so far been made to use this method for other boundary conditions and non-uniform loading, nor has it been applied to orthotropic plates.

Finite Difference Method. The governing equations and boundary conditions are converted to finite difference form and the resulting non-linear algebraic equations are solved by a method of iteration. Green and Southwell⁽²⁵⁾ have used the relaxation method for solving the finite difference equations in terms of the displacements u , v , and w . Kaiser⁽²⁶⁾ obtained the finite difference equations in terms of the deflection w and a stress function f and worked out in detail, but only for one particular intensity of uniformly distributed load the solution of a square isotropic plate having simply supported edges which are also free from any membrane stresses. Wang^{(27),(28)} extended this method for plates with other boundary conditions and, using a suitable method for the iterative solution of the finite difference equations, found solutions for simply supported rectangular plates having all edges clamped against movement, for the aspect ratios, 1, 1.5 and 2. The finite difference method has not until now been applied to orthotropic plates.

Fourier Series Method. The deflection and the stress function are assumed in terms of two infinite Fourier series, the coefficients of which are subsequently obtained by substituting the series in the governing equations and solving a set of simultaneous cubic algebraic equations. Developed by Levy⁽²⁹⁾,⁽³⁰⁾ and Greenman⁽³¹⁾, this method involves a tremendous amount of numerical work. Moreover, the assumption of a Fourier series for the stress function imposes restrictions on the boundary conditions for membrane stresses, and this method cannot be applied to plates having zero normal stress, or non-zero shearing stresses, at the edges. Solutions have been obtained by this method for a few cases of simply supported and clamped square and rectangular isotropic plates under uniformly distributed load. Levy has also used the method for analysing simply supported isotropic plates under the combined action of lateral pressure and in-plane compression. Soper⁽²³⁾ has extended the method for solving rectangular stiffened plates having edges restrained elastically against rotation, though the limitations inherent in the assumption of a Fourier series for stress function remained. Soper's numerical computation was directed towards corroborating results obtained from the lateral loading test of a particular stiffened plate, the elastic edge restraints of which were systematically varied.

There is one more approximate method available for the analysis of large deflection behaviour of plates. This method, originally developed by Berger⁽³²⁾, is based on the assumption that the "second strain invariant" of the middle surface of the plate can be neglected and this leads to a pair of non-conjugate differential equations in place of the two simultaneous differential equations obtained by the exact method. The solution of the pair of non-conjugate approximate equations is much easier than the solution of the exact equations. Sinha⁽³³⁾ has used this method for solving a number of cases of plates

supported on an elastic foundation, the particular case of zero foundation modulus being the case under discussion in the present paper. Iwinski and Nowinski⁽²²⁾ have derived the corresponding approximate equations for orthotropic plates.

This method is, however, unreliable because the assumption involved in the simplification is of a mathematical nature and cannot be demonstrated to have any physical basis. Only in those cases where the "second strain invariant" happens to be small in most areas of the plate would this method be expected to give reasonable results.

CORRUGATED PLATES UNDER IN-PLANE LOADING

Bergmann, Reissner and Seydel⁽³⁴⁾ applied the orthotropic plate theory to the problem of buckling of corrugated and stiffened plates subjected to pure shear. Seydel solved the problem of an infinitely long plate under pure shear with free and clamped edges⁽³⁵⁾ and also of a finite plate with simply supported boundary conditions⁽³⁴⁾. Some experiments⁽³⁶⁾ made with corrugated and stiffened rectangular plates subjected to shear agreed satisfactorily with the results given by the orthotropic plate theory.

Thielemann⁽³⁷⁾ solved the problem of initial buckling of infinitely long rectangular "generally orthotropic" plates with the long edges simply supported or clamped and subjected to either pure compression in the longitudinal direction or to pure shear. (A "generally orthotropic" rectangular plate is one that has its mutually perpendicular main axes of rigidity making any angle to the edges, the case of the axes of rigidity running parallel to the edges being a special case of such a plate.) Solutions were obtained by the exact method as well as by the Energy method.

In the exact method the numerical evaluation of buckling coefficients for different relative rigidity properties of a "generally orthotropic" plate involved a considerable amount of

labour. For plates having the main rigidity axes parallel to the edges ("specially orthotropic" plates), much less computational work was involved. Thielemann's solution of shear buckling becomes identical to that obtained by Seydel⁽³⁵⁾ and the solution for pure compression with long edges freely supported reduces to a simple form as given by Timoshenko⁽³⁴⁾. Exact values of buckling coefficients for long "specially orthotropic" rectangular plates with fixed edges under pure longitudinal compression were also given in the paper.

In the Energy method solutions were obtained by assuming approximate expressions for the deflected form of the plate, originally suggested by Timoshenko⁽³⁴⁾. The numerical values of buckling coefficients obtained by the approximate method for "specially orthotropic" plates and "generally orthotropic" plates with the main rigidity axes making an angle of 45° with the edges compared very favourably with the exact solution.

Although Thielemann's work was done with special reference to plywood panels, it can be readily applied to infinitely long rectangular corrugated or stiffened plates even when the corrugations or stiffeners are not parallel to the edges. It therefore permits the determination of the optimum inclination of the stiffeners or corrugations to the edges for obtaining the maximum possible buckling loads.

Yoshiki and Fujita⁽³⁸⁾ applied the orthotropic plate theory to the problem of elastic and inelastic buckling of rectangular "swedged" plates subjected to uniform biaxial stresses. An exact solution was given for plates having all edges simply supported, though the method could also be used for plates having the edges perpendicular to the corrugations elastically restrained against rotation, provided that the edges parallel to the corrugations were freely supported. The expressions for critical buckling stress so obtained were

also valid in the inelastic range, provided some or all of the rigidity properties of the plate were suitably modified. The various theories for such modifications, proposed for application to isotropic plates⁽³⁹⁾ were examined and finally one suitable for orthotropic plates was suggested. Experiments on swedged plates under unidirectional compression in the direction of corrugations showed that most of the theories would predict critical buckling loads too high or too low, depending on plate dimensions, except the simplified tangent modulus theory (which simply reduces the critical stress in the ratio E_t/E), which gave consistent though conservative values of critical stress about 10 to 15 per cent lower than the experimental ones.

In the papers on corrugated plates discussed above, one important property of corrugations has been overlooked, namely, the increase in the effective torsional stiffness due to restraint against warping deformations, which would be caused by variable torque within the plate and also by any in-plane restraint present at the edges. Smith⁽⁴⁰⁾ considered this effect while evaluating the critical shear stress of an infinitely long rectangular plate having square corrugations in the transverse direction. The edges of the plate were clamped and were assumed to provide complete restraint to warping deformations. The corrugated plate was treated as an orthotropic plate and the consideration of warping restraint modified the fourth order partial differential equation to a sixth order one, the new sixth order term having the warping rigidity of the plate as its coefficient. The problem was solved approximately by the Energy method, and the results were found in certain limiting cases to be only 6 per cent higher than the corresponding known exact solutions. The increase in the critical buckling stress due to the restraint against warping deformations was shown to depend on the size of the corrugations relative to the width of the plate and the breadth-thickness ratio of the plate elements

forming the corrugations, being greater for bigger and thinner corrugations. For a plate with square corrugations, the sides of which were one-fifteenth of the width of the plate and fifty times the thickness, the critical shear stress with warping prevented was nearly three times the value with warping unrestrained. Smith also investigated the effect on the buckling load of the additional deflection due to internal transverse shear across the corrugations; the effect was found to be negligible.

Peterson and Card⁽⁴¹⁾ investigated the same problem as Smith, taking into account the restraint against warping deformations, but they assumed the long edges to be elastically restrained against rotation instead of being completely fixed. The authors observed that the transverse bending stiffness of such plates was so high (due to the corrugations) compared to the longitudinal bending stiffness that the shear buckles were short in the longitudinal direction and even relatively short shear webs might be considered infinitely long for the purpose of analysis. The torsional rigidity of plates having the types of corrugations mostly used in practice is very small and the authors confirmed the observation⁽⁴⁰⁾ that the coefficient of the term $\partial^4 w / \partial x^2 \partial y^2$ in the orthotropic plate equation might be taken to be zero with negligible error in the calculated buckling loads. The effect of restraint against warping deformations was seen to be appreciable even with freely supported plates, when the restraint was only due to the variation of torque along the corrugations; an increase of about 50 per cent in the theoretical buckling coefficient was obtained in some cases. A number of beams with aluminium alloy web having vertical corrugations were tested to failure. Due to the initial overall deflections of the webs, the load-deflection curves did not show any sign of critical buckling of the webs and the theoretical values could not therefore be checked. However, the failure loads of the shallower webs having high values of

critical stress were in agreement with the theoretical buckling loads though the deeper webs carried considerably greater stress at failure than the computed critical stresses. The authors suggested a method of estimating the approximate failure load of such webs.

Recently a number of stiffened and swedged plate panels were tested to collapse as columns under axial compression⁽⁷⁾ by the British Ship Research Association. Owing to the presence of initial deformation and premature buckling of the plate elements between the stiffeners or swedges, the ultimate load carrying capacity of the specimens was substantially reduced. Because of the reduced width of flat plating associated with swedges together with the inherently stable form of the swedge, swedged plates exhibited a marked superiority over conventionally stiffened plates. The post buckling behaviour of the local plate elements was investigated in the light of available theoretical solutions; the agreement, in general, is satisfactory. Finally, an approximate method was proposed for calculating the collapse strength of stiffened or swedged plating loaded as columns where local plate buckling occurs before overall collapse.

No general solution is available for the post buckling behaviour of stiffened or corrugated plates. Falconer and Chapman⁽⁴²⁾ investigated the initial and post buckling behaviour of long, simply supported rectangular plates having any degree of longitudinal and transverse stiffening when subjected initially to uniform end thrusts. The plate was regarded as truly orthotropic on the assumption that the stiffeners, if any, were sufficiently closely spaced for overall instability to develop. The authors also considered the effect of initial deformation on the post buckling behaviour.

The solution was obtained by the Energy method and was based on certain appropriate simplifying assumptions. Equations suitable for design purposes were given for calculating the stresses and deflections of the plate for any applied load. The effect of the direction and the degree of stiffening and the influence of initial deformation were studied with the help of graphs and numerical examples, which demonstrated that a transversely stiffened plate, unlike one longitudinally stiffened, is able to carry load in excess of the critical owing to the restrained rate of deflection after buckling, and that the load causing yield is greatly influenced by the magnitude of initial deflection.

In the papers so far discussed, the stress distribution in the corrugated plate was assumed to be given before the buckling load was calculated. In very many cases, however, the first problem is to determine the distribution of applied stress on the corrugated plate. The in-plane rigidity of corrugated plates transverse to the corrugations is very low compared to their rigidity in the direction of the corrugations. A girder having vertically corrugated web, therefore, gets no assistance in carrying bending moment from the web, which is subjected practically to pure shear; a horizontally corrugated web, on the other hand, would carry its full share of bending stress together with shearing stresses⁽¹⁾. The low value of in-plane transverse rigidity also leads to a low value of the shearing rigidity of corrugated plates compared to that of flat plates with conventional stiffeners. This low value of shearing rigidity has been utilised in ship structures by constructing the side walls of the superstructure with vertically or horizontally corrugated plates to reduce the longitudinal stress in the superstructure. However, if the deformation of the corrugations is prevented by transverse stiffeners, the shear stiffness is greatly increased. The behaviour of superstructures constructed with corrugated plates and the effect of stiffening have been studied

both theoretically and experimentally in Japan^{(43),(44)}. Another important property of corrugated plates is that when acting as beam flanges with corrugations transverse to the span, they are virtually ineffective and contribute nothing to the bending strength of the beam. Experiments with trapezoidally corrugated plates have confirmed this and have also revealed severe stress concentration at the corners nearer to the neutral axis of the beams^{(5),(45)}.

DETERMINATION OF RIGIDITIES OF STIFFENED AND CORRUGATED PLATES

An isotropic material has two independent elastic constants, namely, the modulus of elasticity E and Poisson's ratio ν ; the modulus of rigidity G can be expressed as a function of E and ν . An orthotropic material, on the other hand, has four independent elastic constants, the moduli of elasticity E_x and E_y , one of the Poisson's ratios ν_x and ν_y (the other being given by Maxwell's reciprocal theorem) and the modulus of rigidity G_{xy} . It is interesting to note that in an orthotropic material G_{xy} cannot be obtained from E_x , E_y and ν_x except when considering a direction at an angle of 45° to the main orthogonal axes. In a homogeneous orthotropic flat plate, the extensional, shear, bending and torsional rigidity properties can all be expressed in terms of the four elastic constants which can be determined from only one set of tests, either involving tension and shear or involving bending and torsion.

The material for corrugated and stiffened plates is, however, isotropic, having only two independent elastic constants. It is the plate rigidities that are in general different in two orthogonal directions, because of the stiffeners or corrugations. In the application of orthotropic plate theory to such plates, the final values of the rigidities are required and therefore direct determination of these properties by experiments is more useful than the determination of the elastic constants. In such

cases, however, two separate sets of experiments are required to determine the extensional and shear rigidities and the bending and torsion rigidities respectively.

Several authors discussed experimental methods for determining the plate rigidities. The bending and torsional rigidities of a stiffened or corrugated plate can be determined by applying each time only one of the bending or torsional moments and measuring the corresponding deformation of the plate. The bending experiment is originally due to Bergstrasser⁽⁴⁶⁾ who used it for determining the two elastic constants of isotropic plates. In this method, a rectangular plate specimen is subjected to pure bending moment in the longer direction. Thielemann⁽³⁷⁾ suggested a modification of this method for determining the bending rigidities of "generally orthotropic" plates. In this case two plate specimens are required, the longer side of the specimens coinciding with the respective directions of applied moment. Thielemann described how with the help of additional measurements of deflections in these two experiments, the additional rigidity properties of a "generally orthotropic" plate could be directly determined, instead of calculating them from the rigidity properties of the corresponding "specially orthotropic" plate and the angle of inclination of the main rigidity axes to the edges.

The experiment for determining the torsional rigidity is similar to the one used by Nadai⁽⁸⁾ for verifying the small deflection theory of flat isotropic plates in bending. In this experiment, a square plate specimen is subjected to pure torsion by a pair of equal loads at a pair of diagonally opposite corners, the other two corners being supported. Seydel⁽³⁴⁾ used this method for determining the torsional rigidity of corrugated and stiffened plates and Thielemann⁽³⁷⁾ suggested its application for generally orthotropic plates. Yoshiki and Fujita⁽³⁸⁾ have

recently used this method with swedged plates. Several other authors^{(47),(48),(49),(50)} have used substantially these bending and torsion experiments to determine the bending and torsional rigidities of timber panels and stiffened plates.

Hoppmann⁽⁵⁰⁾ described experiments for the determination of the extensional and shear rigidities of stiffened plates; two experiments were conducted involving pure tension in the two orthogonal directions separately and measurements were made of the stretch or shortenings in these two directions for each case. One experiment involving pure shear was also made in which the extension and compression in the two diagonal directions were measured.

Although it is always safe to determine the rigidity properties experimentally, so that the overall deflections of stiffened or corrugated plates may be predicted by the orthotropic plate theory, it is possible to estimate some of these properties by inspection of the geometry of the plate⁽⁸⁾. For the rigidities in the direction of the stiffeners or corrugations, a single stiffener or corrugation with the associated plating is treated as an independent beam and its properties are divided by the spacing to give average rigidities of the plate. For the rigidities in the perpendicular direction, a similar procedure is adopted, considering a strip of unit width. The approximate values of Poisson's ratio are either guessed or assumed to be zero (as some authors have done for corrugated plates, supporting their assumption by experiments). Smith⁽⁴⁰⁾ worked out in detail the expressions for the rigidity properties of symmetrically corrugated plates including the expressions for warping and extensional rigidities. Yoshiki and Fujita⁽³⁸⁾ have also given expressions for bending, torsional and extensional rigidities of swedged plates. Both these papers demonstrate that for corrugated plates the ratio of the two bending rigidities is equal to the ratio of the two corresponding extensional rigidities.

Huffington⁽⁵¹⁾ developed expressions for the bending and torsional rigidities of stiffened plates on the basis of equivalence of strain energies of the actual plate and the hypothetical orthotropic plate. The expressions in the general form are rather involved, but they demonstrate the effect of the spacing of the stiffeners on the rigidities, which reach the values given by other authors only in the limiting case of very close spacing of stiffeners.

From the foregoing review it is evident that only the small deflection behaviour of corrugated plates under lateral load has been investigated so far and that in such investigations the corrugations have been treated as isolated beams without considering the overall behaviour of the plate. However, for calculating the initial buckling load of corrugated plates and for studying the bending and buckling behaviour of stiffened plates, the theory of orthotropic plates has been used with satisfactory results.

Considerable work has been done on the small deflection theory of rectangular orthotropic plates and numerical solutions are available for various cases. The large deflection theory for orthotropic plates is also available, but practically no useful numerical solution exists. Even for isotropic plates numerical solutions are available only for a few cases. Only in one instance has the effect of boundary restraint (which usually exists in practice) been considered.

Methods of calculating local transverse stresses in a corrugated plate are available only for certain types of corrugation profile. These are applicable provided that the deflections are small and that the dimension of the plate transverse to the corrugations is sufficiently large. The validity of the methods has not yet been demonstrated by tests.

Various investigators have tested corrugated and swedged plates under lateral load; the dimensions of the test plates, the size of the corrugations and in some cases the boundary conditions have been such that the behaviour of the central corrugation has not been affected by the overall plate behaviour. Strain measurements have not been made in sufficient detail for a proper estimate to be made of various stresses in the plate.

In view of the above findings, the present investigation has been carried out with the following aims:

- (a) to develop the large deflection theory of rectangular orthotropic plates under lateral load, with the edges elastically restrained against rotation, and against movement both normal and parallel to the edges, in a form suitable for application to stiffened and corrugated plates as well as to ideally orthotropic and isotropic plates
- (b) to obtain numerical solutions for a range of plate dimensions, boundary conditions, rigidities etc. and to compare them in particular cases with the available analytical solutions and experimental results
- (c) to show the method of application of the orthotropic plate theory to corrugated plates
- (d) to develop a method of calculating local deformations and local transverse stresses for the general case of a corrugated plate without any restriction on the transverse plate dimension or on the magnitude of the deflection
- (e) to confirm by experiments the applicability of the orthotropic plate theory to corrugated plates and the validity of the method of calculating local stresses and deformations in the corrugations.

CHAPTER II

T H E O R Y

SECTION A: THE LARGE DEFLECTION THEORY OF RECTANGULAR ORTHOTROPIC PLATES UNDER LATERAL LOAD.

Previous derivations of the equations governing the large deflection behaviour of ideally orthotropic⁽²²⁾ and stiffened plates⁽²³⁾ were based on the considerations of equilibrium and compatibility. In the present thesis the governing equations and boundary conditions for the large deflection behaviour of orthotropic plates (including stiffened and corrugated plates and ideally isotropic plates) with given initial deformations are derived by an energy method. The consideration of initial deformation, and the derivation of the boundary conditions simultaneously with the governing equations by a single application of the theorem of minimum potential energy, appear to be novel.

In practice, plates are seldom used in isolation - they are connected at the boundaries to adjoining structures. The effect of this continuity can be represented approximately by equivalent elastic boundary restraints. In the present theory, therefore, the edges are considered to be elastically restrained against rotation and against movement both normal and parallel to the edges (Fig. 1). Elastic extensional and shearing edge restraints have not previously been considered in any analysis, and elastic rotational edge restraints have been considered in the large deflection analysis only by Soper.

In the present analysis, the governing equations and boundary conditions are formulated directly in terms of the rigidity properties of the plate so that their application to stiffened and corrugated plates is straight-forward. It may be

noted that the large deflection equations for ideally orthotropic plates cannot be directly applied to all stiffened plates, because the ratio of the flexural rigidities in two orthogonal directions of a stiffened plate is, in general, different from the corresponding ratio of the extensional rigidities. In the present theory provision has been made to allow for such differences in the rigidity ratios by the introduction of an additional parameter in the equations for ideally orthotropic plates. However, the simplifying transformations applicable to the equations for ideal plates are still applicable, and the resulting reduction in the number of variables required to describe the plate, permits the analysis of a large number of different plates with the help of a few solutions. The large deflection equations for stiffened plates obtained by Soper are, on the other hand, suitable for the analysis of particular plates only.

Of the three principal methods available for the solution of large deflection problems, the finite difference method, with a digital computer solving the difference equations, is considered most suitable, because it facilitates formulation of the problem in a very generalized form and permits solutions to be obtained to any desired degree of accuracy. The method presented here is applicable to isotropic, ideally orthotropic, corrugated and stiffened plates, including those having a flexural rigidity ratio different from the extensional rigidity ratio. As the boundary conditions have been developed in terms of three types of elastic edge restraint, practically all combinations of boundary conditions can easily be incorporated in the solution. The provision for elastic edge restraints not only permits a better assessment of the effects of continuity of plates over supports than has so far been possible, but also permits the determination of the actual magnitude of restraint that may be considered infinite or zero for practical purposes. Moreover, the formulation takes account of the effect of initial deformation,

which does not appear to have been considered previously.

A short outline of the theory will now be given; details are given in Appendix A.

Derivation of equilibrium equations and boundary conditions in terms of displacements. The equilibrium equations have been derived from the principle of stationary total potential energy, according to which the virtual work associated with each of the three virtual displacements δu , δv and δw is zero. The only boundary condition assumed is that the edges do not deflect ($w = 0$). If it were assumed that at the edges u , v and w , as well as the slope normal to the edges ($\partial w / \partial n$) were prescribed constants (including zero), then the three equilibrium equations could be obtained directly by the Calculus of Variations⁽⁵⁴⁾ from the Eulerian Characteristic Equations. However, in the present case the boundary displacements are not prescribed, and the boundary conditions have been derived from first principles, as a direct result of the principle of stationary potential energy, using a method similar to that used by Timoshenko⁽⁸⁾ for deriving the governing equations and Kirchhoff boundary conditions for small deflection bending of isotropic plates. The three equilibrium equations and boundary conditions so obtained are valid for any distribution of loading and any boundary conditions.

Introduction of Airy Stress Function. The above equations could be converted to finite difference form and solved by a method of iteration. To reduce the amount of labour the number of variables has been reduced from three to two by introducing an Airy stress function to replace u and v . As a result, the boundary conditions are somewhat more complicated in form, especially because of the presence of the three edge restraint coefficients. The boundary conditions are derived on the assumption that the edge restraints as well as the applied loading are symmetrical about both centre-lines. Boundary conditions

for loading symmetrical about one centre-line and anti-symmetrical about the other or anti-symmetrical about both centre-lines can be derived in a similar way, considering only one-quarter of the plate, provided the edge restraints remain symmetrical. However, it should be pointed out that the principle of superposition does not apply for large deflection behaviour and therefore in the general case of arbitrary loading and edge conditions the whole plate has to be considered.

Conversion of Equations to Non-Dimensional Form. The non-dimensional forms chosen for deflection, stress function and intensity of loading are explained in the Appendix. In the particular case of isotropy, these non-dimensional ratios reduce to the forms used by previous investigators except for certain numerical multiplying factors.

As a result of the chosen non-dimensional forms and transformation of the axes of co-ordinates, rectangular orthotropic plates (including stiffened and corrugated plates) can now be described completely in terms of the following few parameters.

a) Virtual side-ratio ($\mathcal{E}$):- This is obtained by multiplying the actual side ratio of the plate by the fourth root of the inverse bending rigidity ratio, with the axes chosen so that $\mathcal{E}$ is numerically equal to or greater than 1. For an orthotropic plate having the same bending rigidities in the two orthogonal directions, and also for an isotropic plate, $\mathcal{E}$ becomes equal to the actual side ratio; otherwise a given virtual side ratio corresponds to different actual side ratios.

b) Torsional rigidity parameter (μ_b): This is a measure of the torsional rigidity of the plate relative to the bending rigidities and varies between 0 and 1. For isotropic plates $\mu_b = 1$. For corrugated plates and for grillages with little or not plating μ_b approaches zero.

c) Stiffening parameter (r): This is equal to the fourth root of the ratio of the two rigidity-ratios (extensional and bending) and is equal to 1 for ideally orthotropic, isotropic and corrugated plates and also for stiffened plates having identical stiffening in the two perpendicular directions. For other cases of stiffened plates r can assume any positive value depending on the form and degree of stiffening in one direction relative to the other. It may be noted that the form of the stiffeners has no effect on the extensional rigidity whilst the flexural rigidity is greatly influenced by it. It is, therefore, possible to consider two stiffened plates having identical flexural rigidities but with different extensional rigidities because of difference in the forms of the stiffeners. These two plates will have identical small deflection behaviour (which is independent of extensional rigidities) but when the deflections are large they will behave differently from each other.

d) Shearing rigidity parameter (μ_m): The reciprocal of μ_m is a measure of the shearing rigidity of the plate relative to the extensional rigidities. μ_m is greater than 1 for orthotropic plates and is equal to 1 for isotropic plates. For corrugated plates, however, μ_m is less than 1.

e) Poisson's ratio parameter ($\sqrt{\nu_{xm} \nu_{ym}}$): For isotropic plates, this becomes equal to ν , the Poisson's ratio of the material. For a stiffened plate the value of this parameter is less than the Poisson's ratio of the material of the plate and for corrugated plates it can be neglected altogether.

f) Six edge restraint co-efficients ($\alpha_x, \alpha_y, k_x, k_y, s_x$ and s_y): These coefficients can assume any value between 0 and ∞ corresponding respectively to the condition of complete freedom from restraint and to the condition of complete fixity at the edges, against rotation or against movement either normal or parallel to the edges.

In the case of plates with initial deformation, additional information will be necessary concerning the magnitude of the initial deflections.

All plates having the same values for the above parameters have identical behaviour when the behaviour is expressed in non-dimensional terms. To obtain the actual behaviour of any particular plate in the group, it is necessary only to multiply the given non-dimensional applied pressures and the non-dimensional deflections and stress functions obtained by the analysis, by suitable numerical factors which are dependent on the actual dimensions and rigidity properties of the plate.

In the derivation as well as in the non-dimensional transformation of the equations, the plate thickness, elastic moduli and stresses do not appear explicitly; instead, the rigidity properties, moments and membrane forces are considered to be the most fundamental quantities. This permits the direct use in the analysis of the rigidity properties of stiffened and corrugated plates (which can if necessary be obtained experimentally) without considering the actual modulus of elasticity of the material and the actual geometry of the stiffeners or corrugations. When the moments and membrane forces have been found, stresses can be calculated by considering an element of the plate subjected to the calculated moments and membrane forces; for stiffened or corrugated plates this element must consist of a stiffener or corrugation with the associated flat plating.

Finite Difference Equations. In obtaining the finite difference equations, first order approximations for the derivatives of W and F have been used, except for the slope at the edges, where more accurate expressions are employed to increase the accuracy of solution for clamped or nearly clamped plates. The accuracy of curvatures and membrane forces obtained by using the first order finite difference operators on the calculated W and F is

less than that of W and F themselves. A greater accuracy for these quantities is attainable by using higher order approximations for their calculation⁽⁵⁵⁾.

The computer program is written for Atlas in Mercury Autocode and is developed for initially flat plates. The finite difference equations are, however, fully developed for initially deformed plates and incorporation of the terms involving initial deformation in the program does not present any difficulty. In practical problems, where the shape of initial deformation cannot be exactly determined, it can be assumed for simplicity that the profile of the initially deformed surface normal to the edges is identical to that of the deflected plate under applied loading and certain terms (T^{III} and T^{IV}) can then be neglected.

The small deflection behaviour of orthotropic plates is only a particular case of the large deflection behaviour discussed so far and is obtained by neglecting the membrane forces in the plate. In the computer program provision has been made to print out the small deflection results; these are calculated by setting $F = 0$ everywhere in the plate and are subsequently used as a first approximation to the finite deflection distribution corresponding to the first (lowest) intensity of loading prescribed in the iterative procedure for determining F and W .

SECTION B : METHOD OF PRESENTING THE THEORETICAL RESULTS FOR THE LARGE DEFLECTION ANALYSIS OF ORTHOTROPIC PLATES.

To facilitate the presentation of the numerical solutions and to express them in suitable forms for their direct application to actual plates the non-dimensional forms of applied loading, deflections, curvatures and membrane forces given in Appendix A have been modified to some extent; details are given in Appendix B.

It may be noted that the ratio of deflection to plate thickness is an important index of membrane action in a plate. For stiffened and corrugated plates thickness of the equivalent plate is not a physical quantity and its value must be expressed in terms of the flexural and extensional rigidity properties of the plate. The concept of equivalent thickness (h_{eq}) has, therefore, been introduced and a formula has been suggested for its evaluation. This formula, by incorporating the extensional and flexural rigidities in both orthogonal directions, recognises the possibility of a difference between the ratio of the flexural rigidities and the ratio of the extensional rigidities, which is usual for stiffened plates. For isotropic and ideally orthotropic plates h_{eq} reduces to the actual plate thickness h .

The non-dimensional forms finally adopted for the presentation of the numerical results contain one or more of the four quantities, namely, the flexural rigidity (D_x), the equivalent thickness (h_{eq}) and the lengths of the sides (l_x and l_y).

Replacing the various rigidities and rigidity ratios by D_x and h_{eq} has the advantage of making the relative magnitudes of the normal forces and the shearing force (or the bending moments and the twisting moment) quite apparent from their non-dimensional magnitudes; it also simplifies the numerical evaluation of the actual forces and moments in a plate from the non-dimensional values.

The bending moments in an orthotropic plate depend on the two Poisson's ratios in bending (ν_{xb} and ν_{yb}). The specification of ν_{xb} and ν_{yb} is not necessary for obtaining a solution (see Section A); if for the calculation of bending moments they are now assigned certain values, considerable generality of the solution will be lost. It is, therefore, convenient to present the bending moments in the orthogonal directions on the assumption that ν_{xb} and ν_{yb} are zero. Actual bending moments in a given plate can

easily be calculated from these two bending moments and the appropriate values of the Poisson's ratio. Support moments in a plate with undeflecting edges are independent of the individual values of ν_{xb} or ν_{yb} and can, therefore, be presented without any difficulty.

Knowledge of the value of the torsional rigidity parameter (μ_b) is not sufficient for expressing the twisting moment in term of D_x ; it is also necessary to know the value of ν_{xb} or ν_{yb} . Generality may, however, be retained in the non-dimensional presentation of the twisting moment if ν_{xb} or ν_{yb} is assumed to be zero. Necessary corrections can be made subsequently when calculating the actual twisting moments in a given plate.

For the particular case of isotropic plates, the non-dimensional forms contain one or more of three quantities, namely, the modulus of elasticity (E), the plate thickness (h) and the length of a side of the plate (l_x). The term l_y does not appear because the ratio l_y/l_x is known, being the virtual side ratio of the plate. The non-dimensional forms are necessarily identical to those obtained by previous authors (except for certain numerical multiplying factors) provided that the moments and forces are expressed in terms of stresses. For isotropic plates, the value of Poisson's ratio (ν) is already specified (in the form of $\sqrt{\nu_{xm} \nu_{ym}}$) at the beginning of the computation and no generality is, therefore, lost in presenting the correct magnitudes of non-dimensional bending stresses and shear stress due to torsion. However, the same method of presentation as for orthotropic plates has been adopted for isotropic plates, because an isotropic plate is considered, in the present investigation, to be only a particular case of an orthotropic plate.

SECTION C : APPLICATION OF THE ORTHOTROPIC PLATE THEORY TO CORRUGATED PLATES.

To apply the theory of orthotropic plates to the analysis of a corrugated plate it is necessary to evaluate the various rigidity properties of the corrugated plate; expressions for rigidity properties are given in Appendix C. These expressions, obtained by Smith⁽⁴⁰⁾, incorporate the effect of Poisson's contraction in a proper manner and have, therefore, been chosen in preference to the expressions given by other authors^{(8),(38)}.

In the appendix the value of the stiffening parameter (r) has been demonstrated to be equal to 1. For the particular case of the corrugated plate used in the tests, numerical calculations confirm that the Poisson's ratio parameter ($\sqrt{\nu_{xm} \nu_{ym}}$) is negligible and that the torsional and shearing rigidity parameters (μ_b and μ_m) are very small being respectively equal to 0.07 and 0.08.

For the test plates, the equivalent thickness (h_{eq}) is found to be about 1 in., which is greater than the depth (3/4 in) of the corrugations. This value of h_{eq} is the result of the particular definition chosen for h_{eq} and is considered to give a ratio of deflection to thickness (w/h_{eq}) which is a proper index of the membrane action in the plate.

In Appendix C the method of evaluating the edge restraint coefficients is also shown with the test plates as examples.

With the parameters required for describing a corrugated plate to the computer known the deflections, bending moments and membrane forces in the equivalent orthotropic plate can be determined for any given value of applied load.

The deflections in the equivalent orthotropic plate correspond to the deflections in the corrugated plate except that certain corrections are necessary to incorporate the effect of the local deformation of the corrugation-profile. Such local effects depend on the shape and size of the corrugations and are discussed in the next section (Section D).

For calculating the longitudinal bending (and membrane) stresses in a corrugation, the resultant longitudinal bending moment (and membrane force) acting on the corrugation is evaluated by the numerical integration of the bending moment (and membrane force) per unit width, over the entire width of the corrugation, the distribution and magnitudes of the bending moments (and membrane forces) over the transverse cross-section being taken from the solution of the equivalent orthotropic plate.

To calculate the stresses in the corrugated plate in the transverse direction the distortion of the profiles of the corrugations has to be considered. The method of calculating the transverse stresses is discussed in Section D below.

SECTION D. METHOD OF CALCULATING LOCAL TRANSVERSE STRESSES AND DEFORMATIONS IN CORRUGATIONS.

Transverse stresses and deformations in the corrugations depend on the shape of the corrugations and increase with the size (and spacing) of the corrugations in much the same way as the deformations and the forces and moments in an arch on a frame under uniformly distributed load. These transverse stresses and deformations are of a local character and cannot be predicted by the orthotropic plate theory, which considers plates made up of corrugations of different size and shape but having the same rigidity properties as identical for the purpose of analysis. It may be noted in this connection that for the ideal case of a corrugated plate having a large number of (small) corrugations,

the transverse stresses are independent of the shape and size of the corrugations and can be directly calculated by the orthotropic plate theory.

A method is now presented for calculating the transverse stresses and deformations in a corrugated plate; it is applicable to any form of corrugation, provided that the corrugation is symmetrical. The method has been demonstrated here for the particular case of a central corrugation, which is likely to have greater longitudinal stresses than the other corrugations; after certain modifications the method can also be used to calculate the transverse stresses and deformations in the other corrugations. A brief outline of the method is given below.

1. Isolate a transverse strip of unit width from the central corrugation at a section where the transverse stresses are to be calculated.
2. The forces and moments acting on the strip are shown in Fig. 4. $\mathcal{V}$ represents the difference of the shear stresses acting on the pair of transverse sections unit width apart. It may be considered to be the shear stress produced by a vertical shearing force of magnitude $2(Bq - V_c)$, in a cross section of the corrugation, which is assumed to behave like a beam. It therefore follows that $\mathcal{V}$ is zero at A and C. Twisting moments on the transverse sections are neglected.
3. The shearing force V_c , the horizontal displacement v_c and the rotation θ_c (Fig. 4) are assumed to be known from the analysis of the plate as a whole. The redundant quantities F_c and M_c and the deflection w_c are then determined with the help of Castigliano's First Theorem, according to which

$$\partial U / \partial F_c = v_c, \quad \partial U / \partial M_c = \theta_c \quad \text{and} \quad \partial U / \partial V_c = w_c.$$

where

U = Strain energy due to bending of one half of the element CAC. (Strain energies due to axial force and transverse vertical shear force are neglected.)

4. V_c , F_c and M_c being known, the bending moment and axial force and hence, the stresses can be calculated at any point in the element.

The method of calculation is presented in Appendix D for the particular type of corrugated plate used in the experiments (Fig. 5a and Chapter IV). Although solutions for moments and axial forces can be obtained in terms of the dimensions R , ψ and k (which completely define the shape and size of the profile), the expressions become more and more unwieldy as the calculation progresses. The method is, therefore, illustrated for profiles geometrically similar to the one used in the experiments, that is, for known numerical values of ψ and k , but for any value of R , which defines the size of the corrugation. The thickness t of the element does not appear in the calculation of bending moments and axial forces (except in the expression for the flexural rigidity of the strip), if the quantity $(t/R)^2$ is very small. [See the expression for I/R^2t in Equation (67)]

After obtaining the general expressions for F_c , M_c and w_c the following two particular cases are considered in the appendix to demonstrate the effect of the transverse membrane action on the local behaviour of corrugated plates.

Case I: $V_c = v_c = \theta_c = 0$. This case corresponds to the small deflection behaviour of a corrugated plate, infinitely wide in the direction perpendicular to the corrugations so that the corrugations away from the short edges deflect identically.

Case II: $V_c = F_c = \theta_c = 0$. This case corresponds to the large deflection behaviour of a corrugated plate, infinitely wide in the direction perpendicular to the corrugations at that stage of

loading when the transverse membrane tension exactly counterbalances the compressive force F_c , given by the small deflection analysis (Case I).

In both cases the overall transverse curvature is zero in the interior of the plate and the corrugations suffer no distortion because of curvature. Nevertheless, the behaviour of the corrugations is found to depend on their size (R), the forces and moments respectively being proportional to R and R^2 . The deflection (w_c) of the centre of the flat plating between a pair of corrugations is proportional to R^4 . According to the orthotropic plate theory this deflection is zero irrespective of the size or shape of the corrugations. These local forces, moments and deformations disappear when $R \rightarrow 0$, that is, when the plate consists of a large number of very small corrugations.

The effect of the transverse membrane action on the local behaviour of the corrugations can be seen by comparing the results of the two cases. In both cases mean stresses due to the transverse in-plane forces can be shown, for corrugations having practical dimensions, to be very small compared to the transverse bending stresses; yet the effect of a membrane tension only large enough to balance the compressive force F_c (Case II) is to increase the maximum transverse bending stress and the local vertical deflection at C respectively to about four and ten times their values corresponding to no membrane action (Case I). The transverse membrane action also affects the distribution and the nature of the transverse bending moments in the corrugation. In Case I, the maximum bending moment occurs at the junction of the curved and flat parts (point D) producing tension on the surface on which the pressure is applied, whilst in Case II the bending moment is maximum at the crest A, again producing tension on the surface on which the pressure is applied; the bending moment at A itself has opposite signs in Case I and Case II.

These observations are true for the particular direction of the applied pressure shown in Fig. 4, and the effect of reversing the direction of the pressure will now be considered.

In the absence of membrane action, the effect of a reversal of applied pressure is only to change the signs of forces, moments and deflections in the corrugation, the magnitudes remaining unaltered. In Case I, for instance, replacing q by $-q$ gives a tensile force and a downward deflection at C, although the deflection is still in the direction of the applied pressure. In the presence of membrane action, however, the local stresses and deformations are altered in a different way by a reversal in the direction of applied pressure, because such a reversal whilst reversing the effect of q itself cannot affect the transverse membrane forces or the effects of these forces on the behaviour of the corrugation.

The expressions for F_c and M_c [Equations (79)] and w_c [Equation (80)] obtained in the appendix are still valid; the only modification necessary is to change the signs of the terms containing q , V_c and Θ_c . The sign of the terms involving v_c remains unaltered, being independent of the direction of q .

To demonstrate the effect of a reversal in applied pressure, the same plate and the same degree of membrane action as in Case II may be considered, but with the pressure acting in the opposite direction. It is evident that F_c can no longer be zero at any stage, being tensile even in the absence of membrane action; instead, the horizontal displacement v_c obtained in Case II can now be assumed to be known; this magnitude of v_c together with $V_c = \Theta_c = 0$ describes a case that is identical to Case II except for the direction of q relative to the corrugations. The solution of this case is given in the appendix. Comparison of these results with those of Case II shows a marked difference in the behaviour of the corrugation. The transverse bending moment

continues to be maximum at the crest A but its magnitude is now greater and it develops compression on the surface on which the pressure is applied. The bending moment and vertical deflection at C are changed in magnitude; the bending moment produces tension on the surface on which the pressure is applied and the deflection is in a direction opposite to the direction of q . It is, therefore, clear that the transverse bending stresses at C and the deflection at C relative to the crest of the corrugation undergo reversal in sign under the action of transverse membrane forces when the pressure is applied on the convex surface of the corrugations; such reversal does not occur when the pressure is acting in the opposite direction. It may be added here that when the transverse membrane displacements are large, only a small fraction of the deflections and transverse moments in the corrugation will be caused by the applied pressure itself, and, therefore, the importance of the direction of pressure on the local behaviour of corrugations will be greatly reduced.

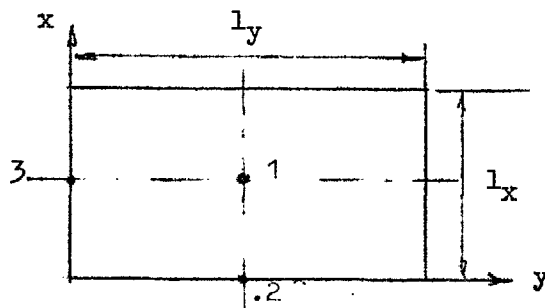
In the general case, the magnitudes of V_c , θ_c and v_c (Fig. 4), required for calculating local stresses and deformations in corrugations, are not known; the methods of calculating them from the overall behaviour of the corrugated plate obtained by the orthotropic plate theory, are outlined in Appendix D.

CHAPTER III

T H E O R E T I C A L R E S U L T S

SMALL DEFLECTION SOLUTIONS

In order to gain some experience in the finite difference method and the technique of writing 'programs' in Mercury Autocode, a program was first written specifically for the small deflection behaviour of rectangular orthotropic plates with elastic edge restraints. This program was subsequently used to obtain solutions for clamped orthotropic plates under uniformly distributed and central patch loading. These solutions are presented here, as they have not been found in the literature.



$$\xi = \frac{l_y}{l_x} \sqrt{4 \frac{D_x}{D_y}}$$

Clamped Plates under Uniform Load (q)

Results are given in Table 1 below; the numbers in parenthesis are the available solutions for isotropic plates⁽⁸⁾ and are given here for comparison. The present solutions were obtained by dividing the plate into sixteen divisions both ways ($i = 8$).

The present results show excellent agreement with the available solutions, except that the error in M_y for $\xi = 3$ is about 6 per cent. This error is probably due to the coarseness of the mesh, because, for $\xi = 3$ the mesh length in the y direction was three times the mesh length in the x direction as a result of dividing the plate in an equal number of divisions in both directions.

Virtual side ratio = ξ	$1000 D_x w_1 / q l_x^4$			$100 M_{x2} / q l_x^2$			$(100 M_{y3} / q l_x^2) \sqrt{D_x / D_y}$		
	$\mu_b = 1$	0.5	0	$\mu_b = 1$	0.5	0	$\mu_b = 1$	0.5	0
1.00	1.26 (1.26)	1.43	1.66	5.13 (5.13)	5.41	5.73	5.13 (5.13)	5.41	5.73
1.25	1.81	2.04	2.33	6.64	7.07	7.59	5.60	5.76	5.89
1.50	2.19 (2.20)	2.40	2.68	7.56 (7.57)	8.01	8.56	5.71 (5.70)	5.76	5.74
1.75	2.41	2.59	2.81	8.05	8.44	8.91	5.70	5.70	5.62
2.00	2.53 (2.54)	-	2.81	8.28 (8.29)	-	8.93	5.66 (5.71)	-	5.59
3.00	2.62	-	2.64	8.40	-	8.44	5.41	-	5.56
∞	(2.60)	-	-	(8.33)	-	-	(5.71)	-	-

Table 1

The central deflection (w_1) and the support moments (M_{x2} and M_{y3}) increase with the decrease in the torsional rigidity parameter (μ_b), and are maximum for $\mu_b = 0$, except that M_{y3} shows a reversal of this trend when the side ratio exceeds a value of about 1.5. A variation of the torsional rigidity parameter has a greater effect on the central deflection than on the support moments; for $\xi = 1$, for example, a reduction of μ_b from 1 to 0 increases w_1 by about 32 per cent, whereas M_{x2} or M_{y3} is increased only by about 12 per cent. The effect of a variation in μ_b can be neglected for plates having ξ greater than or equal to 3.

Figs. 6 and 7 show the deflected shapes of the plate centre-lines parallel to the x and y axis respectively, for $\mu_b = 1$ and $\mu_b = 0$. The deflected shape of the centre-line parallel to the x axis (Fig. 6) is practically independent of μ_b and is affected only a little by the change in the side ratio of the plate, becoming, at $\xi = 2$, identical to the deflected shape of an infinitely long clamped plate. The deflected shape parallel to the y axis (Fig. 7), however, depends on the side ratio of the plate

and become progressively flatter with an increase in the value of ε , showing the diminishing effect of the clamped short edges on the interior of the plate. The effect of a variation in μ_b on the deflected shape (Fig. 7) becomes prominent for longer plates; at $\varepsilon = 3$, the effect of changing μ_b from 1 to 0 is to shift the point of maximum deflection away from the centre and towards the short edges, giving thereby two points where the deflection is a maximum; the maximum deflection is, however, only about 3 per cent greater than the deflection at the centre of the plate.

Clamped Plates under Central Patch load (P)

Results are given in Table 2 below; the numbers in parenthesis are the available solutions for isotropic plates under a concentrated load at the centre⁽⁸⁾ and are given here for comparison. Present solutions were obtained by dividing the plate into sixteen divisions both ways ($i = 8$), giving a patch area equal to $(l_x/16) \times (l_y/16)$.

Virtual side ratio = ε	$1000 D_x w_1 / Pl_x^2$			$100 M_{x2} / P$			$(100 M_{y3} / P) \sqrt{D_x / D_y}$		
	$\mu_b = 1$	0.5	0	$\mu_b = 1$	0.5	0	$\mu_b = 1$	0.5	0
1.00	5.82 (5.60)	6.68	7.88	12.66 (12.57)	13.12	13.51	12.66	13.12	13.51
1.25	6.88	7.84	9.18	15.47	16.24	17.02	8.61	8.43	7.99
1.50	7.35	8.29	9.60	16.62	17.34	18.05	5.26	4.61	3.57
1.75	7.55	8.45	9.71	17.08	17.68	18.24	2.95	2.11	0.84
2.00	7.64 (7.22)	-	9.78	17.30 (16.74)	-	18.28	1.53	-	-0.52
3.00	7.82	-	10.05	18.10	-	19.01	0	-	-0.44
∞	(7.25)			(16.80)					

Table 2

For a virtual side ratio equal to 1, the present solutions for a central patch load agree fairly well with the theoretical solutions for a central concentrated load, the errors in the central deflection and support moments being respectively about 4 per cent and 0.7 per cent. For increasing values of ϵ , the agreement is less satisfactory. For plates having ϵ between 1 and 2, the present solutions may be considered sufficiently accurate for practical purposes.

The effect of a variation in μ_b on deflections and support moments in a centrally loaded plate is similar to that in a plate under uniform load. The bending moment (M_{y3}) at the short edges of a centrally loaded plate decreases progressively with ϵ and can be neglected for plates having $\epsilon \gg 3$.

Variation of μ_b has very little effect on the deflected shape of the plate centre-line (Figs. 8 and 9), except that for plates with high virtual side ratios, the deflected shape of the centre line parallel to the y axis is somewhat affected. It can be seen from Fig. 9 that for a long plate, the areas of the plate near the short edges do not participate in carrying the load applied at the centre of the plate.

LARGE DEFLECTION SOLUTIONS

All the solutions presented in this thesis were obtained with a specified degree of accuracy (ρ) of 1 per cent in the iterative procedure. A greater accuracy would have required more computer time and an accuracy of 1 per cent was considered sufficient for practical purposes.

In most cases the plate was divided into 12 divisions both-ways ($i = 6$). This was considered adequate for obtaining deflections accurate enough for practical purposes; for a simply supported square isotropic plate under uniform load with the edges clamped against in-plane movements the deflections obtained

with a coarser mesh ($i = 4$) were found to be within 0.5 per cent of those obtained with $i = 6$. For the plate under a central patch load a finer mesh ($i = 8$) was employed.

The maximum intensities of pressure for which computations were carried out in the present investigation are, in most cases, much higher than those in the available solutions. The maximum value of the non-dimensional pressure intensity ($ql_x^4/D_x h_{eq}$) in the present solutions is about 4000 for simply supported plates and about 10000 for clamped plates, these values being chosen such that the ratios of the central deflection to the equivalent thickness (w/h_{eq}) of the plates is at least equal to 2 at the maximum pressure.

The solutions were obtained in 10 (or in some cases, 12) equal increments of load; this gave a sufficient number of points for plotting the results properly. Increasing the number of load increments would have required more computer time.

The total number of iterations required to obtain results upto the maximum pressure depends on the problem; for the solutions given here, the number never exceeded 45 and was in most cases in the range 25 - 40, the clamped plates requiring a greater number of iterations than the simply supported plates. Other things remaining the same, an increase in the number of meshes (i) increases the total number of iterations only a little. A maximum number of iterations is required to obtain the solution for the starting pressure; the actual number is only slightly affected by the magnitude of the starting pressure and was 5 in almost all the cases solved. The number of additional iterations required for each subsequent increment of pressure reduces to 3 or less after the first few increments. The total number of iterations required to obtain results upto a certain pressure is appreciably increased if the load increments are (say) halved, because the number of results obtained is doubled. The volume

of output is also doubled, requiring more computer time. A lesser amount of computer time will be required if the number of increments is reduced.

The Manchester University Atlas Computer, employed to obtain the solutions, took about 80 seconds to 'read in' the program and data and for a typical case presented here, the time required for computation and output of results for all the ten pressure levels was about one minute with $i = 6$; this would be increased about three times if a mesh having $i = 8$ were employed.

Of the large number of solutions that can be obtained with the autocode program described in this thesis, about 25 solutions are presented here in detail as examples; those include most of the cases of isotropic plates considered by previous investigators, so that the validity of the present theory and its numerical accuracy can be verified.

Simply Supported Plate under Uniform Load $\epsilon = 1$. $r = 1$.

$$\sqrt{\nu_{xm} \nu_{ym}} = 0.316. \quad \mu_m = \mu_b = 1. \quad (\text{Figs. 10 - 13})$$

This particular case of a rectangular orthotropic plate corresponds to a square isotropic plate with a Poisson's ratio equal to 0.316, the value used by Levy in his numerical calculations. In the figures the non-dimensional terms (pressure, deflection etc) relating to isotropic plates are given in parenthesis.

Solutions for the following in-plane boundary conditions are given in the figures:

- (i) $K_x = K_y = 0$ and $S_x = S_y = 0$
- (ii) $K_x = K_y = 0$ and $S_x = S_y = \infty$
- (iii) $K_x = K_y = \infty$ and $S_x = S_y = 0$
- (iv) $K_x = K_y = \infty$ and $S_x = S_y = \infty$

Where K and S are respectively the stiffnesses of extensional and shear edge restraints.

Fig. 10 shows the central deflection plotted against the intensity of loading. All four curves are tangential to the straight line which describes the linear load-deflection relationship of a simply supported square isotropic plate. As can be expected, the effect of membrane stresses is least for case (i) and greatest of case (iv). It is interesting to note that the central deflection for case (iii) is indistinguishable from that of case (iv). This is because when normal movement of the edges is prevented, the total length of the edges remains unchanged; the local strain along the edges is therefore small even when shear restraint is absent. The deflection for case (ii) is between the two limiting values.

Fig. 10 also shows the results for isotropic plates obtained by previous investigators. The results for case (iv) are in remarkably good agreement with the results obtained by Levy using the Fourier Series Method. There is no solution available for comparison with case (ii). Kaiser's theoretical solution (obtained only at one pressure) and experimental results agree closely with the results for case (i); it may be pointed out here that this is the first complete theoretical corroboration of Kaiser's experimental results, published 28 years ago. Results obtained by Levy for the case where the edges are kept straight and the shear stress and resultant membrane tension are zero at the edges, are also plotted; a comparison with the results of case (i) which assumes the shear stress and the membrane tension at each point on the edges to be zero, shows that the large deflection behaviour is affected by the distribution of the normal edge stresses, even when the resultant normal and shear forces at the edges remain unaltered.

In Figure 11, the curvatures at the centre of the plate have been plotted against pressure and compared with the results obtained by Levy and Kaiser. The maximum discrepancy between

comparable cases is less than 10 per cent. This discrepancy is caused by the maximum error (ρ) of 1 per cent permitted in the deflections, because this small error may lead to a considerably greater error in the curvatures. (A similar error may sometimes be expected in the calculated values of the membrane forces because of a possible error of about 1 per cent in the values of the stress function obtained from the computer). A greater accuracy in the curvatures (and membrane forces) may be consistently obtained by specifying a smaller permissible error (ρ) for the deflections. It may be pointed out here that the 'linear theory curvatures' obtained by the present method are in excellent agreement with the available solutions; these curvatures are calculated from the deflections obtained by an exact solution of the finite difference equations. This confirms that the errors in the large deflection curvatures are due to the error (ρ) in the iterative solution of the deflections and the chosen mesh size ($i = 6$) is adequate for calculating the curvatures with sufficient accuracy.

Figure 12 shows the membrane tensions at the centre of the plate and also the membrane tensions at the midpoints of the edges in a direction normal to the edges (for cases (iii) and (iv)), plotted against pressure. The present solutions are in good agreement with the comparable solutions obtained by Levy and Kaiser. It can be seen that for plates having edges clamped against normal movement, the normal membrane tension is greater at the midpoints of the edges than at the centre of the plate.

In Figure 13, the effect of varying the Poisson's ratio from 0.316 to 0.3 on the central deflection of a square isotropic plate has been considered, for the two extreme in-plane boundary conditions (case (i) and case (iv)). Figure 13(a) shows the effect to be negligible. The results obtained with the Poisson's ratio parameter ($\sqrt{\nu_{xm}\nu_{ym}}$) equal to 0.316 can therefore be

applied to plates having a value of 0.3 for this parameter, with negligible error. Figure 13(b) shows the modifications in the non-dimensional terms whereby the deflections can be made even less dependent on the Poisson's ratio, the curve corresponding to zero in-plane edge restraints ($K = S = 0$) being completely independent of ν .

Clamped Plate under Uniform Load. $\varepsilon = 1. \quad r = 1. \quad \sqrt{\nu_{xm} \nu_{ym}} = 0.316$

$\mu_m = \mu_b = 1.$ (Figs. 14 -16)

This case also corresponds to a square isotropic plate with $\nu = 0.316$. Solutions are given for the four in-plane boundary conditions, discussed in the previous section; the relative effect of the boundary conditions on the deflection, curvatures and membrane forces is similar to that for simply supported plates, the results for case (iii) being indistinguishable from that of case (iv).

There are no solutions available for comparison with case (i) and case (ii). Comparison of case (iii) (and case (iv)) with Levy's Fourier Series solution: shows remarkable agreement for the central deflections and curvatures; the maximum error in the membrane forces is, however, about 8 per cent, probably due to the reasons discussed earlier.

The magnitude of the support moments relative to the moments at the plate-centre is seen to increase progressively with pressure (Fig. 15). Figure 16 shows that for plates having edges clamped against normal movement, the normal membrane tension is less at the midpoints of the edges than at the centre of the plate the reverse of the relation observed in simply supported plates.

Figure 17 shows the effect of the rotational and in-plane edge restraints on the total surface stresses at the centre and at the mid point of the edges, for a square isotropic plate under uniform load. According to small deflection theory the bending stress at the supports of a clamped plate ($A = \infty$) is somewhat greater than the bending stress at the centre of a simply supported plate ($A = 0$); the effect of clamping is thus to increase the maximum surface stresses in the plate, whereas in a beam the maximum bending moment is reduced as a result of end fixity. It can be anticipated therefore that the total tensile surface stresses at the midpoint of the edges of a clamped plate will become progressively greater than the tensile surface stresses at the centre of a simply supported plate, as the membrane stress increases, and this can be seen in Fig. 17.

Of the four cases of rotationally clamped plates ($A = \infty$), the plate having edges free from any in-plane restraint (case (i)) has the highest surface stresses, although the surface stresses are produced by the bending moments alone. For case (iii) and case (iv) the surface bending stresses are augmented by the membrane tension normal to the edges due to the extensional edge restraints; this makes the surface stresses slightly higher than those in case (ii), although the support moments for case (ii) are greater (Fig. 15).

For the simply supported plates ($A = 0$), the relative magnitudes of the surface stresses at the centre are similar to those of the bending stresses (Fig. 11), being a maximum for case (i) ($K = S = 0$) and a minimum for case (iii) and (iv) ($K = \infty, S = 0, \infty$) although the difference is smaller than that in the corresponding bending stresses. It may be noted that the in-plane compressive stress due to membrane action at the mid-points of the edges and in a direction parallel to the edges for a plate with no edge restraints ($A = K = S = 0$) is, beyond a certain magnitude of pressure, greater than the tensile surface

stress at the centre of the plate.

Simply Supported Plate under Uniform Load. $\varepsilon = 1. \quad r = 1,5$

$$\underline{\nu_{xm} \nu_{ym} = 0. \quad \mu_m = \mu_b = 0} \quad (\text{Figs. 18 - 20})$$

The solutions for $r = 1$ correspond approximately to the behaviour of swedged and corrugated plates, whereas the solutions for $r = 5$ are intended to show the effect of a variation in the stiffening parameter (r) on the large deflection behaviour when the other parameters remain unchanged. It may be noted that if the swedges were prevented from spreading by battening, the transverse extensional rigidity of the plate will be greatly increased whilst the other rigidities remain practically unaffected. In such circumstances the stiffening parameter may be equal to 5. A zero value for the shearing rigidity parameter μ_m implies an infinite shearing rigidity relative to the geometric mean of the extensional rigidities of the plate and corresponds approximately to the situation in a swedged plate; when the swedges are prevented from spreading, μ_m can no longer be assumed to be zero. Therefore, the present solutions for $r = 5$ do not correspond to any physical plate.

The following in-plane boundary conditions have been considered with $r = 1$

- (i) $K_x = K_y = 0$ and $S_x = S_y = 0$
- (ii) $K_x = K_y = 0$ and $S_x = S_y = \infty$
- (iii) $K_x = K_y = \infty$ and $S_x = S_y = \infty$

For $r = 5$, solutions have been obtained only for the first boundary condition given above ($K = S = 0$).

Fig. 15 shows the central deflection plotted against the intensity of loading. The small deflection solution (linear theory) is independent of r ; but according to large deflection theory the central deflection is increased considerably when r is changed from 1 to 5.

A comparison with the simply supported isotropic plate (Fig. 10) shows that the relative effect of the in-plane boundary conditions on the central deflection is similar in the two cases. According to linear theory the isotropic plate has a smaller central deflection than the orthotropic plate (Fig. 18), but when membrane action is considered, the relative magnitudes of the central deflections of the two plates become dependent on the in-plane boundary conditions and the intensity of loading. There are no existing solutions available for comparison with the present solutions for orthotropic plates.

Fig. 19 shows that the curvature at the centre of the orthotropic plate with $r = 5$ can be many times larger than the corresponding curvature with $r = 1$, whereas Fig. 20 indicates that the membrane tension at the plate centre is less with $r = 5$. The relative effect of the in-plane boundary conditions on the curvatures and membrane forces in an orthotropic plate is seen to be different from that in an isotropic plate (Figs. 11 and 12).

Clamped Plate under Uniform Load. $\varepsilon = 1. \quad r = 1, 5. \quad \nu_{xm}\nu_{ym} = 0.$
 $\mu_m = \mu_b = 0.$ (Figs. 21 - 23)

Solutions have been obtained for the in-plane boundary conditions discussed in the preceding section; no solutions exist for comparison with any of the present solutions.

Figs. 21, 22 and 23 show that, according to large deflection theory, the effect of increasing r from 1 to 5 is to increase the central deflection and the curvature at the midpoints of the edges, and to decrease the membrane force at the centre of the plate; the effect is, therefore, similar for simply supported and clamped orthotropic plates.

A comparison with the solutions for a clamped isotropic plate (Figs. 14, 15 and 16) shows that the central deflections and the support moments in the isotropic plate may be smaller or larger than the corresponding quantities in the orthotropic plate ($r = 1$) depending on the in-plane boundary conditions and the intensity of loading, although according to small deflection theory, the deflections and moments in the isotropic plate are always smaller. Comparing the membrane forces in the two plates it can be seen that the isotropic plate has smaller membrane forces at the centre and a greater membrane force at the midpoint of the edges (with $K = S = \infty$ and the range of loading considered) than the orthotropic plate having $r = 1$. The relative effect of the in-plane boundary conditions on the central deflection is similar in the two plates; but such similarity does not exist for the curvatures and membrane forces. Figs. 21 and 22 also show that for the orthotropic plate having infinite rotational and shearing edge restraints ($A = S = \infty$) the central deflection and support moments remain practically unaffected by a change in the extensional edge restraint (K) from zero to infinity.

Effect of In-plane Boundary Conditions on the Distribution of the Deflection and Membrane Forces on the Centre-lines of Uniformly Loaded Simply Supported and Clamped Square Isotropic Plates and Rectangular Orthotropic Plates having $\epsilon = 1$, $r = 1$, $\nu_{xm}\nu_{ym} = 0$.

$$\mu_m = \mu_b = 0 \quad (\text{Figs. 24 - 27})$$

Fig. 24 shows the deflected profiles of the plate centre-lines at the maximum intensities of loading for which solutions have been presented in the previous sections. The two limiting conditions of infinite and zero restraints against in-plane movements at the edges have been considered and the results according to the linear theory are also plotted. The intensities of loading and the ratios of the central deflection to the plate thickness corresponding to the deflected profiles are also shown.

In all cases the effect of membrane action is seen to be a flattening of the deflected profiles obtained by neglecting membrane action (linear theory). There is very little difference between an isotropic plate and an orthotropic plate so far as the deflected profiles are concerned.

For simply supported plates at a given load, the profile for the condition of zero in-plane edge restraints is flatter than the corresponding profile for infinite edge restraints, the latter having a smaller central deflection than the former. For a given in-plane boundary condition, the degree of flattening of the profile increases with load (and the consequent central deflection of the plate).

For clamped plates there is very little difference in the effect of the in-plane boundary conditions on the deflected profiles, for a given magnitude of load, although the central deflections are different for the two boundary conditions.

Fig. 25 shows the distribution of the membrane stresses in the plates on the plate centre-lines, the direction of the membrane stress being along the centre-line. Three in-plane boundary conditions have been considered, namely, (i) $K = S = 0$, (ii) $K = 0$, $S = \infty$ and (iii) $K = S = \infty$. The distributions shown in the figure correspond to the maximum intensities of loading, as for the deflected profiles in Fig. 24.

The patterns of the stress distribution are similar for isotropic and orthotropic plates. Irrespective of the rotational edge restraints, membrane stresses are zero at the boundary for case (i) and (ii) because the extensional edge restraints are zero. For case (iii), the membrane stress at the edges of a simply supported plate is greater than the membrane stress at the plate centre; for clamped plates, however, the situation is reversed. The difference between the stress at the edge and the stress at

the centre is greater in orthotropic plates than in isotropic plates, for a given magnitude of non-dimensional loading.

For isotropic and orthotropic plates, simply supported or clamped, the ratio of the membrane stress at any point on the centre line to the stress at the plate centre for case (i) boundary condition is less than the corresponding ratio for case (ii); an increase in the load appears to decrease this ratio slightly, for the simply supported isotropic plate having case (i) boundary condition ($K = S = 0$).

Figs. 26 and 27 show the distribution of the membrane stresses acting across the centre-lines. The in-plane boundary conditions and the magnitudes of loading are the same as for the membrane stresses acting along the centre-lines (Fig. 25).

The patterns of the stress distribution for the simply supported and clamped plates are very similar and it will be sufficient to discuss the simply supported case (Fig. 26).

When the edges of the plate are free from any in-plane restraints ($K = S = 0$), the resultant membrane force normal to any longitudinal or transverse section of the plate must be zero, the tensile stresses in the central part of the section being exactly counterbalanced by the compressive stresses near the edges. The resultant membrane force per unit width of the central section, calculated from the distributions shown in Figs. 26 and 27, is about 1 per cent of the membrane force at the centre; this again confirms the numerical accuracy of the present solutions. Fig. 26 also shows that for the particular intensities of loading chosen, the compressive stress at the edges may be as high as 2 to 3 times the tensile stress at the centre of the plate; the effect of an increase in the load is seen to be an increase in this relative magnitude of the edge compression.

When the shearing restraint at the edges is infinity ($S = \infty$) the strain along the edges of the plate must be zero. For the orthotropic plate the Poisson's ratio parameter ($\nu_{xm}\nu_{ym}$) has been assumed to be zero; the membrane stress parallel to the edges must, therefore, be zero at the edges by Equations (43b) and (44b) whatever the magnitude of the membrane stress normal to the edges may be. For the isotropic plate ($\nu = 0.316$), however, the membrane stress at an edge in a direction parallel to the edge must be equal to 0.316 times the stress normal to the edge at the point, in order that the strain is zero along the edge; when the normal stress is itself zero ($K = 0$), the stress parallel to the edge must also vanish at the edge (Figs. 26 and 27).

Simply Supported Plate under Uniform Load. $\epsilon = 1.5$. $r = 1$.

$$\sqrt{\nu_{xm}\nu_{ym}} = 0.316. \quad \mu_m = \mu_b = 1. \quad (\text{Figs. 28 - 31})$$

This particular case of an orthotropic plate corresponds to a rectangular isotropic plate ($\nu = 0.316$) with a side ratio (l_y/l_x) of 1.5. Solutions have been obtained for three in-plane boundary conditions, namely, (i) $K = S = 0$, (ii) $K = 0$, $S = \infty$ and (iii) $K = S = \infty$. The relative effect of the boundary conditions on the central deflection, central curvatures (in x and y directions) and membrane stresses at the centre (in the x direction) is similar to that for simply supported square isotropic plates. The membrane stresses at the centre in the y direction (Fig. 30) are almost equal for case (i) and (ii) and are greater than the corresponding stress for case (iii) shown in Fig. 31, which also shows that the normal membrane tensions at the midpoints of the edges are greater than the membrane tensions at the plate-centre, acting in the same direction, as in the case of a square simply supported plate. For the rectangular isotropic plate, the central curvatures in the x direction are greater than the corresponding curvatures in the y direction (Fig. 29), for all the three in-plane boundary conditions; so also are the membrane

tensions at the centre and at the midpoints of the edges for case (iii) (Fig. 31). The membrane tensions at the centre for case (i) and (ii) are, however, smaller in the x direction than in the y direction (Fig. 30).

No solutions are available for comparison with case (i) and (ii); comparison of case (iii) with Wang's solution, obtained by the finite difference method using a square mesh of size $l_x/4$, shows good agreement.

Clamped Plate under Uniform Load. $\varepsilon = 1.5.$ $r = 1.$ $\sqrt{\nu_{xm}\nu_{ym}} = 0.316$

$\mu_m = \mu_b = 1$ (Figs. 32 - 35)

This case also corresponds to a rectangular isotropic plate with $l_y/l_x = 1.5$. Solutions have been obtained for the same three in-plane boundary conditions as for the simply supported plate and the relative effects of the boundary conditions on the central deflection, support moments and membrane stresses at the centre are also similar. The normal membrane tension at the midpoint of the edge $y = 0$ is, however, less than the membrane tension at the centre, in the y direction (Fig. 35). The relative magnitudes of the support moments and membrane forces in the orthogonal directions for the clamped isotropic plate are similar to that for the simply supported plate, discussed in the preceding section.

No solutions are available for comparison with case (i) and (ii); comparison of case (iii) ($K = S = \infty$) with Levy's Fourier Series Solution for $K = \infty$ and $S = 0$ shows excellent agreement for the central deflections and support moments (M_x), the maximum discrepancy in the membrane stresses being about 6 per cent. Agreement between the two solutions again demonstrates that when the edges are completely restrained against normal displacements, the shearing restraints at the edges do not materially affect the solution; or in other words, the solutions for the two cases,

namely, $K = S = \infty$ and $K = \infty$, $S = 0$ are almost identical.

Simply Supported Plate under Central Patch Load. $\varepsilon = 1$. $r = 1$.

$$\sqrt{\nu_{xm}\nu_{ym}} = 0.316. \quad \mu_m = \mu_b = 1. \quad (\text{Figs. 36 - 41})$$

This particular case of an orthotropic plate also corresponds to a square isotropic plate with $\nu = 0.316$. Solutions have been obtained only for the condition of zero in-plane restraints at the edges ($K = S = 0$). It was intended to obtain a solution for a patch area as small as possible so that the patch load might be considered as an approximation to a point load; the load was therefore applied only at the central node and the plate was divided into the maximum number of divisions ($i = 8$) allowed by the computer program in its present form.

No large deflection solutions being available for patch loaded plates, the accuracy of the present solution is judged by comparing the small deflection solutions obtained by the present method and by the exact theory ⁽⁸⁾ (Fig. 36). To investigate the effect of the mesh size (and the area of the patch) on the accuracy of the solution, the results for $i = 4$ and $i = 6$ are also compared with the corresponding exact solutions. It is evident from Fig. 36 that the finite difference method consistently overestimates the deflection and curvature at the plate centre. The errors in the central deflections for $i = 4, 6$ and 8 are respectively about 10, 5 and 3 per cent, the corresponding errors in the central curvatures being about 21, 19 and 17 per cent. (These percentages refer to a patch load distributed over one mesh area and, therefore, becoming more concentrated with the increasing values of i . If the area of the patch were kept constant at the value corresponding to $i = 4$, the improvement in the accuracy will be greater for the reduced mesh sizes.) To obtain the solution for a smaller patch area, a greater value for i than 8 has to be used, which will increase the accuracy of the solution only a little but involve the solution of a

much larger number of finite difference equations. This can be avoided, without sacrificing much accuracy, if the solution for the smaller patch area ($i > 8$) is extrapolated from the solutions for $i = 6$ and $i = 8$, noting that at $1/i = 0$ (Fig. 36), the slope to the deflection curve and the reciprocal of the curvature become zero.

The results of the large deflection analysis for the patch loaded plate (Figs. 37 - 39) show similar features to the uniformly loaded plates. The estimated accuracies (based on the accuracy of the small deflection solution) in the deflection and curvature at the centre are respectively about 3 and 17 per cent; the accuracy of the membrane stresses is not known.

In Fig. 40 the deflected profile of the plate centre-line according to linear theory (present solution) is compared with that obtained by the large deflection analysis at the maximum load; the profile under a central point load (exact theory) is also shown. Due to membrane action, the part of the profile near the centre of the plate takes the shape of a membrane under a point load.

Fig. 41 shows the distributions on a centre-line of the membrane stresses acting along and across the centre line for two values of the applied load. At the edges the normal membrane stress is zero and the membrane stress parallel to the edge is compressive; the edge compression for a patch loaded plate appears to be less than the tension at the plate centre, at all magnitudes of the load.

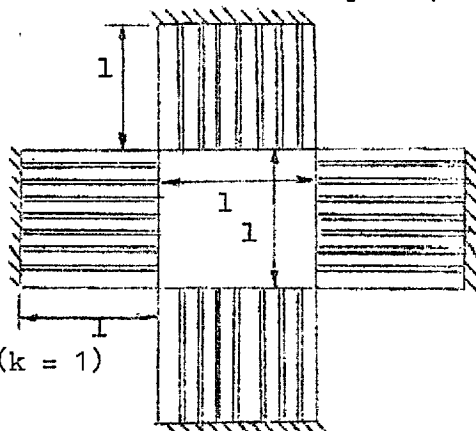
Effect of Varying Edge Restraints on a Square Isotropic Plate under Uniform Load

Extensional Edge Restraint (K)

Fig. 42 shows how the deflection, bending moment and membrane force at the centre of the plate vary with the extensional edge

restraint, when the rotational and shear edge restraints are absent. The results of small deflection analysis are unaffected by a variation in K . According to large deflection theory, the deflection and moment decrease and the membrane force increases with an increase in K . The ratios of the deflection, bending moment and membrane forces at any value of K to those for $K = \infty$, shown in Fig. 42, are to some extent dependent on the intensity of loading, although for higher values of load this dependence is less pronounced. When K is equal to $20Eh/l$, the plate is virtually fully restrained against normal in-plane displacements at the edges and the deflection, bending moment and membrane force at the centre are within 5 per cent of their respective values corresponding to $K = \infty$. When K is small the behaviour of the plate is very sensitive to a change in the value of K . Considering the central deflection at a pressure $q = 200Eh^4/l^4$ the deflection at $K = 0$ is about twice that at $K = \infty$, giving a deflection ratio of 2 at $K = 0$. Increasing the edge restraint coefficient k from 0 to 5 decreases the ratio from 2 to 1.14 (approx.) and a further increase of k from 5 to 10 decreases the deflection ratio from 1.14 to only about 1.07. A similar sensitivity to a change in K for small values of K is noticeable for the bending moment and membrane force as well.

It may be pointed out here that if the extensional edge restraint is provided by closely spaced strips of the same length and thickness and made of the same material as the plate, the edge restraint coefficient k becomes equal to 1, in which case the deflection ratio is about 1.4 at a pressure $q = 200Eh^4/l^4$.

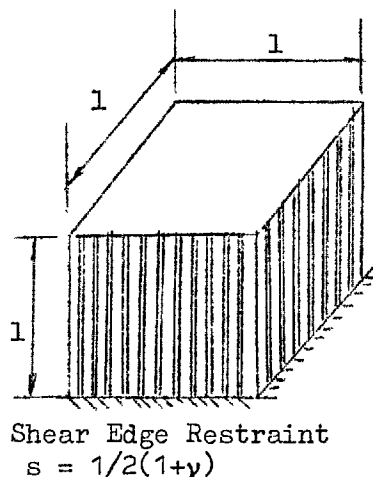


Extensional Edge Restraint ($k = 1$)

Shear Edge Restraint (S)

The effects of varying the shear edge restraint (with the rotational and extensional edge restraints equal to zero) on the deflection, bending moment and membrane force at the plate centre are shown in Fig. 43. The effects of varying S are similar to those of varying K , except that the shear edge restraints have smaller effects on the plate behaviour. When $S = 10Eh/l$, the deflection, bending moment and membrane force are within 5 per cent of their respective values corresponding to $S = \infty$. The sensitivity of the behaviour of the plate to a change in the magnitude of the edge restraint is again seen to be greater when the edge restraint (S) is small.

When the shear edge restraint is provided by closely spaced strips of the same length and thickness and made of the same material as the plate, the strips being at right angles to the plate, the edge restraint coefficient s becomes equal to $1/2(1 + \nu)$ or 0.38 with $\nu = 0.316$.



Rotational Edge Restraint (A)

The variation with the rotational edge restraint (A) of the ratios of the deflection and membrane force at the centre and the bending moment at the centre and midpoints of the edges, to the values corresponding to infinite restraint are shown in Figs. 44 to 46; the in-plane edge restraints are assumed to be zero. The sensitivity of the behaviour of the plate to a change in the magnitude of the edge restraint is again seen to be greater when the edge restraint is small; all the curves shown in the figures have a maximum gradient at $A = 0$.

The deflection and membrane force at the plate centre decrease with increasing values of A (Figs. 44 and 45). The effect of the edge restraint is seen to be more pronounced when the membrane action is less, and for the deflection, the linear theory solutions neglecting membrane action show the maximum effect of the edge restraint; according to linear theory the central deflection of a simply supported plate is reduced to about one third when the edges are clamped.

The edge restraint at which the deflection and membrane force at the plate centre are within 5 per cent of their respective values for $A = \infty$, is greater than $50D/l$; at $50D/l$ ($\alpha = 50$) the maximum departure is, however, less than 15 per cent.

Fig. 46 shows the variation of the central and support bending moments with A . The support moment increases with A and the effect of the edge restraint is a minimum when the membrane action is neglected (linear theory). The situation is, however, reversed for the bending moment at the centre of the plate, which shows a maximum effect of the edge restraint according to linear theory, the effect being a reduction in the moment for an increase in the edge restraint. As the load increases the effect becomes less pronounced, although the bending moments continue to decrease with A , until the load reaches some value between $100Eh^4/l^4$ and $200Eh^4/l^4$ when the moment appears to be unaffected by the edge restraint. When the pressure (and the consequent membrane action) increases still further, the bending moment at the centre increases with the edge restraint; the actual increase is, however, small and at a pressure of about $600Eh^4/l^4$ the effect of the edge restraint appears to reach a maximum, when the bending moment at the centre of a simply supported plate is about 0.85 times that of a clamped plate.

When the edge restraint (A) is equal to $50D/l$ ($\alpha = 50$) the bending moments at the centre and at the midpoints of the edges are respectively within 10 and 20 per cent of their respective values for $A = \infty$.

Fig. 47 shows how the ratio of the bending moment at the midpoint of the edge to that at the centre of the plate varies with the rotational edge restraint and pressure (membrane action) when the in-plane edge restraints are absent; for comparison, the results obtained by the linear theory for the plate and also for a uniformly loaded beam of the same length as the plate are shown in the figure. It is evident that the moment-ratio increases continuously with the edge restraint, for the plate and the beam, reaching the values shown in parentheses in the figure when the edge restraint is infinity. The results by the linear theory for the beam are remarkably close to those of the plate, the moment-ratios for the condition of full fixity being respectively 2 and about 2.2 for the beam and the plate. For a given edge restraint, the moment-ratio in the plate increases with the increase of the membrane action (i.e. pressure), being a minimum when the membrane action is ignored (linear theory). Fig. 47 also shows that the value of edge restraint at which the bending moments at the support and at the centre of the plate are equal, decreases with the pressure, being a maximum (about $6.6D/l$) when the membrane action is negligible; the corresponding end fixity for the beam is $6EI/l$. It should be pointed out here that the edge restraints required to make the total surface stresses equal at the support and the centre of the plate are greater than those obtained by considering the equality of the moments (Fig. 47), because, at the centre the tensile surface stress due to bending is augmented by the membrane tension.

If the rotational edge restraint is provided by closely spaced strips of the same length and thickness and made of the same material as the plate, the strips being clamped at the far ends, the edge restraint coefficient α becomes equal to $4(1-\nu^2)$ or 3.6, with $\nu = 0.316$; at this edge restraint the ratios for the moments at the centre and support and the deflection at the centre are respectively 1.56, 0.46 and 2.1 according to linear theory.

CHAPTER IV

EXPERIMENTS

INTRODUCTION

In order to verify the applicability of the orthotropic plate theory to swedged and corrugated plates, and the validity of the method given for calculating the local transverse stresses and deformations, a series of three tests was planned (Fig. 48). The series was designed to study the small and large deflection behaviour of swedged plates with different virtual side ratios under uniformly distributed lateral load, applied on the concave or convex side of the swedges.

To avoid fabrication difficulties standard "Mansard" sheets were used to represent swedged plates, and a thickness of $1/16$ in. was chosen as being close to the proportions commonly used in ships. Aluminium alloy was accepted as the material for the test plates as it is the only material of which "Mansard" sheets are manufactured; due to its low value of E , aluminium alloy also makes possible the development of large elastic deflections and consequent membrane stresses, and makes local plate buckling more likely. The overall dimensions of the test plates were 70 in. x 30 in., 20 in. x 30 in. and 40 in. x 30 in., there being 5 longitudinal swedges at 6 in. centres (Fig. 48).

The testing rig (Figs. 49 and 77) was designed to provide edge supports practically free of bending moment, and to allow loading by air or fluid pressure. The rig was suitable for testing plates having one side 30 in. long, whilst the other dimension could be varied from 20 in. to 70 in. in steps of 10 in.

The first two tests have so far been completed and are reported here; the third test is in preparation.

TEST 1 (70 in.x 30 in. Plate)

Details of Test Rig

The test plate was connected to the rig with thin sheets (20g) of HS1Q-WP Al-alloy (Fig. 49). Along each long edge of the test plate the side sheet was folded over the plate and fixed to it with screws at about 5 in. centres through a 70 in. long Al-alloy flat bar $\frac{3}{4}$ in. x $\frac{1}{8}$ in. in section; "Bostik 2" sealing compound was used for leak proofing. Along each of the short edges a $1\frac{1}{2}$ in. x $\frac{1}{2}$ in. Al-alloy flat bar 30 in. long and profiled to fit over the corrugations was connected to the top of the end sheet with araldite and rivets and the edge of the test plate was connected to the bar with screws through a layer of "Bostik 2" sealing compound. The vertical corners between side and end sheets were covered with pieces of "neoprene" stuck with "Bostik 1".

The lower edges of the side and end sheets were riveted to 1 in. x 1 in. x $\frac{1}{8}$ in. Al-alloy angle sections, which were bolted to the flats previously welded to the bottom frame (the bottom frame was sufficiently rigid for the deformation under the maximum test pressure to be negligible). For leak proofing the connection was made through a "neoprene" gasket $\frac{1}{16}$ in. thick, which was first stuck to the horizontal legs of the angles with "Bostik 1".

In order to counterbalance the effect of air pressure on the side and end sheets (resulting in the application of undesirable membrane forces and bending moments in the test plate), rubber bags $6\frac{1}{2}$ in. x $\frac{7}{8}$ in. x 10 in. long were placed between $\frac{3}{8}$ in. thick external plates and the side sheets and were fed with the same pressure as acted internally.

The test plate was loaded by air pressure and the arrangement of loading is shown in Fig. 51.

Instrumentation

The arrangement of gauges to measure strain, deflection, slope and swedge spread is shown in Fig. 50.

Four strain gauges were mounted at each location, one pair on each surface of the plate in the longitudinal and transverse directions. The midspan section of the test plate was provided with strain gauges at 11 important stations on one side of the longitudinal central line. This arrangement was designed to determine the longitudinal and transverse stresses on both surfaces of the plate and their distribution across the midspan section and also to estimate the longitudinal bending moments and membrane forces in the swedges. The group of four gauges at station 2 was employed to determine the strain distribution over the depth of the central swedge. Strain gauges were also provided at some locations on the quarter span section.

Deflection were measured with four dial gauges supported from a transverse rail movable over a pair of longitudinal tubes (Plates 1 and 2). The deflections of the crests of the corrugations and of the longitudinal edge of the plate were measured at several transverse sections by moving the cross rail from one position to another and repeating the same loading for each measurement. The deflection of the flat plating between swedges was also measured relative to the swedge crests at the midspan and quarter span sections.

Slopes were measured by two spirit-level inclinometers (Plates 3 and 4) at several locations on the longitudinal and transverse boundaries to estimate the degree of fixity along the edges.

Swedge spread was measured at the midspan and quarter span sections for the central swedge and the two swedges on one side of the longitudinal central line, with a gauge which incorporated a differential transformer transducer which underwent the same

amount of movement as the mouth of the swedge.

The pressure applied to the plate was measured by a pressure transducer which was calibrated beforehand against a U-tube manometer.

Test Procedure

To load the plate, the pump (Fig. 51) was switched on with the needle valve and the inlet valve shut and the outlet valve fully open. The inlet valve was then opened gradually to build up the pressure on the plate and the rate of loading was adjusted by means of the outlet valve. When the required pressure had been attained, the pump was switched off and the inlet valve was shut. Any reduction in pressure due to leakage in the system was compensated by means of a foot pump.

For quick unloading, the needle valve was opened and the motor started with the outlet fully open. This extracted air from the box, and intake from the atmosphere through the filter was prevented with a pinch cock. The motor was switched off when the pressure dropped to zero and any excessive suction was compensated by working the foot pump and then leaving the inlet valve open.

The lateral pressure was applied in increments of 0.1 lb per sq in. up to a pressure of 0.8 lb per sq in.. Pressures were read both by the pressure transducer and the U-tube water manometer up to this pressure; thereafter the manometer was disconnected from the pressure line. No permanent set could be noticed at this pressure. To detect the limiting pressure for elastic behaviour, the pressure increments were reduced, and only the dial gauge at the centre of the plate was read at initial and final zeros after each new increment. When the plate was loaded to 1.0 lb per sq in. and then unloaded, a permanent deflection of about 0.02 in. occurred at the centre of the plate. Readings of all gauges were then taken at this pressure.

The test plate was subjected to further increments of pressure to study its behaviour in the inelastic range. At about 1.5 lb per sq in. the rate of air leakage (aggravated by the large deformations of the plate and the consequent opening of some of the joints) was too great to be compensated by the operation of the foot pump. Oil was therefore pumped into the rig in sufficient quantity to submerge the bottom angles of the side and end sheets. Although oil continued to seep out, the reduction in pressure was retarded to a considerable extent. At about 3.5 lb per sq in., the very large deformations of the plate and consequent rate of leakage made it impossible to keep the pressure constant and the experiment was therefore discontinued.

Test Results and Discussion

Stress-strain Relationship

Stress-strain curves of tensile specimens taken from different parts of the test plate are shown in Fig. 52. The average value of the modulus of elasticity was 10.4×10^6 lb per sq in.. The 0.1 per cent proof stress was 22,000 lb per sq in. and the corresponding strain was 0.003.

Deflections

Deflections of the plate are plotted in Figs. 53 to 61. Figs. 53 to 56 show the very early development of membrane action which reduces the rate of deflection of the plate; non-linearity was noticeable even at the low pressure of 0.1 lb per sq in. when the deflection at the centre of the plate was less than $1/2$ in.. The straight line load-deflection relationships obtained by considering the test plate as an orthotropic plate and neglecting membrane action are tangential at the origin to the experimental load-deflection curves for the midspan and quarter span sections of the central swedge (Figs. 53 and 54). These orthotropic plate deflections are approximately equal to those obtained by considering

the central corrugation as an isolated beam; this is due to the particular side and stiffness ratios of the test plate, and is not a fundamental property of corrugated plates. This is demonstrated by the difference in deflection between the central corrugation and the one adjacent to it. The load-deflection curves of Figs. 53 and 54 also show that the increasing membrane action due to very large deflections more than compensated the reduction in flexural stiffness due to the decreasing tangent modulus of the material.

Figs. 55 and 56 show the comparison of the theoretical and experimental deflections respectively at the centre of the plate, and at quarter span on the central swedge. As the rigidity properties and edge restraints of the plate were difficult to estimate correctly, two limiting cases were solved in addition to the case in which the estimated rigidities and edge restraints (Appendix C) were used. In both of these limiting cases the torsional and shearing rigidity parameters (μ_b and μ_m) and the coefficients of rotational and extensional edge restraints (α and k) were assumed to be zero; the coefficients of shear edge restraint (s_x and s_y) were zero in one case and infinity in the other. It can be seen that the experimental deflections lie between these two limiting cases. Agreement with the solutions for estimated rigidities and edge restraints is fairly good; at the maximum pressure considered (1 lb per sq in.) the experimental deflections are about 6 per cent greater than the theoretical values. This discrepancy may be due to the spread of the swedge at the higher pressures and the consequent reduction in the moment of inertia of the swedge. Comparing the results of the case with estimated edge restraints, with those of the case having infinite shear restraints, it may be observed that an aluminium alloy side sheeting 5.5 in. wide and 1/32 in. thick, giving shear edge restraint coefficients $s_x = 9$ and $s_y = 62$, provided practically complete restraint against movement along the edges, and at a

pressure of 1 lb per sq in. reduced the deflection of an unrestrained plate ($s_x = s_y = 0$) by about 30 per cent. (Estimated accuracy of the theoretical deflections in Figs. 55 and 56 is within 1 per cent).

In Fig. 57 the deflections of the plate on three longitudinal sections are compared with the best fit sine curves (obtained by minimizing the sum of the squares of the deviations) for different values of pressure. It is evident that up to a pressure of 1 lb per sq in. the longitudinal profile of the deflected plate can be adequately expressed by a sine curve. The profiles become flatter however as the pressure increases.

In Fig. 58 deflections on midspan and quarter span transverse sections at different pressure have been compared with the corresponding best fit sine curves. Here a single sine curve is seen to express adequately the deflected transverse profile up to about 0.5 lb per sq in., above which the profiles become flatter.

In Fig. 59 the deflection at high pressure is drawn to the same scale as the plate.

In Fig. 60 the deflections of the plate have been compared with the best fit surfaces given by one or more terms of a Fourier Series. For the purpose of determining the "Fourier Surface", the sum of the squares of the deviations has been minimized. If an error equal to about 5 per cent of the central deflection be considered as the permissible maximum anywhere in the plate, one term of the Fourier Series would seem adequate up to about 1 lb per sq in. Whereas an extra term in the transverse direction would practically cover the whole range of loading.

In Fig. 61 the theoretical and experimental deflected profiles of the plate centre-lines have been compared; agreement is better for the longitudinal than for the transverse profiles, which are seen to be affected to a greater extent by membrane action.

Local Deformation of Swedges and Inter-swedge Flat Plating

The swedge deformations at the midspan and quarter span sections are shown in Fig. 62. For very small pressures, all the swedges closed, but for greater pressures, all except the edge swedge at the quarter span section opened out, due to membrane action. The spread of the central swedge at the midspan section was measured throughout the whole range of test-loading; there was no evidence of swedge instability.

The deflections in Figs. 53 to 61 referred to those measured at the crest of the swedges. To complete the picture of the deflected plate and also to detect any local instability of the compression flange, the deflection of the flat plating was measured relative to the swedges both at midspan and quarter span sections (Fig. 63). These deflections were small compared to the total deflection of the plate (Fig. 59) and there was no evidence of any increased rate of deflection that might be expected if instability occurred.

Fig. 64 shows a comparison of the theoretical and experimental deformations of the central swedge at the midspan section. The theoretical results were obtained by the method given in Appendix D, using the orthotropic plate solution for the corrugated plate having the estimated rigidities and edge restraints (Appendix C). The experimental v in Fig. 64 is equal to half the deformation of the full swedge in Fig. 62. The experimental w in Fig. 64 is the deflection at C relative to the crest of the central swedge and is different from the deflection of the centre of the flat plating (Fig. 63) relative to the points where it meets the pair of adjacent swedges; the experimental w in Fig. 64 has been calculated by deducting from the deflection at C measured relative to the adjacent swedge-crests, an amount equal to half the swedge pitch (3 in.) times the theoretical slope (θ) at C.

The experimental results are in reasonable agreement with the theory. At 0.1 lb per sq in. the experimental v and w are negative, as predicted by the linear theory. The large deflection theory over-estimates the displacement v by about 20 per cent; if the theoretical deflections (w) are calculated using the experimental values of v (Equation (80)), the experimental deflections (w) would show better agreement with theory.

Normal Edge Slopes

The theoretical and experimental normal edge slopes near the centre of the longitudinal and transverse edges are shown plotted against pressure in Fig. 65. The slopes vary non-linearly in the same way as the swedge crest deflections. Agreement between experiment and theory is reasonable, particularly at higher pressures, the theoretical slopes being consistently larger than the experimental slopes; this is probably due to the actual rotational edge restraints being greater than their estimated values due to corner effects.

The variation of slope along the edges is shown in Fig. 66. The normal slopes on the transverse edge were measured on the $1\frac{1}{2}$ in. $\times$ $\frac{1}{2}$ in. flat bar which was one continuous piece 30 in. long. The large slopes at the end of the transverse edge (at the plate corner) represent the twist of the bar at its end. The actual slope of the plate in the longitudinal direction measured at the corner was, however, very small. It is therefore apparent that the fastening between the bar and the plate along the transverse edges allowed relative rotation between them, the twist of the former being mainly governed by the end slope of the central corrugation.

Strains

Distributions of longitudinal strain across the depth of the central swedge at the midspan section are shown in Fig. 67. Up

to a pressure of 0.5 lb per sq in. the departure of the strain distribution from linearity is rather small, but at high pressures the departure is more pronounced. The figure also demonstrates the gradual increase of membrane tension, until eventually the whole section is under tension.

The variation of longitudinal heart of plate strain with pressure also shows the influence of membrane action (Fig. 68). The maximum heart of plate strain at 1.0 lb per sq in. (i.e. the limiting load for elastic behaviour) is about 0.0024 and the surface strains are even higher. The absence of any appreciable permanent deflection of the plate upto this load inspite of such high strains at some points may be due to the permanent deformations being localised and on only one surface of the swedge crests. It is not known whether the von-Mises yield criterion would be expected to apply under these conditions. Strains were recorded after most other quantities had been measured, so that a number of repetitions of the same load had been applied before the strains were measured. It is possible that permanent strains might have been noticed if the strains had been recorded when a new level of pressure was being applied for the first time.

Stresses

Fig. 69 shows the distribution of longitudinal heart of plate stress on the transverse central section. The cyclic nature and the pattern of the stress distribution for the interior swedges indicate that at any transverse section the longitudinal stresses over a swedge and its associated flat plating can be integrated to give a single bending moment and a single membrane force acting on the isolated swedge unit. This is important, as the orthotropic plate theory cannot directly predict the stresses in the swedged plate; the orthotropic plate bending moments and membrane forces must be integrated over a width equal to the pitch of the swedges to give

a single bending moment and a single membrane force acting on the appropriate swedge unit. The edge swedge, being supported by the side sheet along one of its edges, is an exception in so far as the pattern of stress distribution across it is different from that of the interior swedges; this however, is not of much practical importance as the magnitude of stress near the edge is rather small.

Figs. 70(a) and 70(b) show how the longitudinal surface stresses compare with the longitudinal heart of plate stresses. At the swedge crests the bending component of the surface stresses is negligible in comparison to the heart of plate stresses upto a pressure of about 0.25 lb per sq in.; at higher pressures the bending components are, in some cases, as high as 50% of the heart of plate stresses. The situation is different for points on the bottom flange, where the heart of plate stresses themselves are small. Here the bending stresses are of much greater relative importance even at very low pressures. The absolute magnitude of the local bending moment at the swedge crests is, however, of the same order as that in the flat plating everywhere on the mid-span section except very near to the longitudinal edge (Fig. 71).

These local longitudinal bending moments (local, in that they act on the plating forming the swedge rather than on the swedge as a whole) are initially of the same sign as the overall bending moments acting on the swedges, everywhere in the plate; but at about 0.2 lb per sq in. the bending moment at the swedge crests changes sign, though on the midspan section, these secondary moments over a swedge unit continue to be statically equivalent to a single moment in the same direction as the main longitudinal bending moment. Ordinary beam theory ignores completely the existence of the secondary longitudinal bending moments which, in some cases, may increase the stress on the plate surface much above that produced by the main bending moments alone; this will affect the design of such plates based on an allowable working stress not

to be exceeded anywhere in the plate. Nothing can, however, be said as to whether these bending stresses will affect the ultimate carrying capacity of the structure.

Figs. 72(a) and 72(b) show that the transverse membrane stresses are negligibly small; but the transverse surface stresses due to transverse bending alone are as high as and, at some places, even greater than the longitudinal surface stresses. The local transverse curvature everywhere in the plate has initially the same sign as the overall transverse curvature, but at about 0.15 lb per sq in. its sign at the swedge crests changes. At about the same pressure the interior swedges cease to close and start to spread. This demonstrates that the transverse moments are primarily the results of swedge movements, which, in their turn, depend on plate deflections and in-plane movements occurring along the longitudinal edges. Deflections and swedge movements at the quarter span section are smaller than that at the midspan section; so also are the transverse bending stresses. Fig. 73 shows the distribution of transverse bending stresses on the transverse central section.

The change in sign of the secondary longitudinal moments at the swedge crests occurs almost simultaneously with that of the transverse moments, and this confirms that the secondary longitudinal bending moments are associated with the transverse bending moments.

In Fig. 74 the experimental transverse heart of plate and bending stresses at the midspan section of the central swedge have been compared with theory. The theoretical stresses were calculated by the method outlined in Appendix D using the values of V_c , θ_c and v_c obtained from the large deflection solution in which the estimated rigidities and edge restraints of the test plate were used. (The calculated values of v_c and θ_c have been shown in Fig. 64).

Agreement between theory and experiment is on the whole satisfactory. The effect of membrane action on the transverse bending stresses can be seen by comparing the experimental stresses with those given by the linear theory (shown by straight lines tangential to the theoretical curves at the origin). The transverse bending stresses at the crest of the swedge are very appreciably affected by membrane action; at 0.8 lb per sq in. the experimental stress at the crest is about 6 times that predicted by the linear theory. According to the linear theory, the bending stress at the crest is less than the bending stresses in the flat plating; because of membrane action the situation is reversed and the stress at the crest, at 0.8 lb per sq in., becomes about twice the stress in the flat plating. The transverse membrane displacements increase the bending stresses in the flat plating a little at low pressures, but at high pressures these stresses are reduced. The transverse heart of plate stresses are very small throughout the whole width of swedge, according both to theory and experiment.

Longitudinal Membrane Stress in Swedges

Fig. 75 shows the theoretical and experimental membrane stresses in the swedges at the midspan section. The theoretical results have been shown for the case when all the six edge restraint coefficients (α_x , α_y , k_x , k_y , s_x and s_y) and the torsional and shearing rigidity parameters (μ_b and μ_m) are assumed to be zero, in addition to the case where the estimated rigidities and edge restraints are used. The theoretical as well as the experimental stresses were calculated by considering each swedge with 2 in. of flat plating on each side as a unit (of 6 in. overall width) and by dividing the resultant axial force per swedge by the plate area. The theoretical axial force was calculated by integrating the membrane force per unit width over the whole width of the swedge and for calculating the experimental axial

force the variation of the longitudinal stress was assumed to be linear between the centre of the flat plating and the root of the swedge and also between the root and the crest of the swedge.

The membrane stresses are negligible upto about 0.1 lb per sq in., after which they increase rapidly with pressure. The edge swedges are in compression, but as the magnitude is not sufficient to counterbalance the tensions in the inner swedges, it can be surmised that equilibrium is restored by longitudinal shear forces along the longitudinal edges. Comparison of the experimental stresses in the edge swedge with the theoretical stresses shows that the actual shearing edge restraint along the longitudinal edges was probably less than the estimated values. The experimental membrane tension in the central swedge is initially greater than that in the adjacent swedges, but after a pressure of about 0.5 lb per sq in. the situation is reversed. A similar phenomenon can be noticed with both of the alternative theoretical stresses.

Longitudinal Bending Moment in Swedges

Fig. 76 shows the theoretical and experimental average longitudinal bending moments in the swedges at the midspan section. The theoretical values were calculated on the basis of estimated rigidities and edge restraints of the test plate. The average moment per unit width of swedge was obtained by dividing the resultant moment per swedge by the width of swedge (6 in.), the method of calculating the resultant moment being similar to that of calculating the resultant axial force in the swedge, discussed in the previous section.

Agreement between the theoretical and experimental results is satisfactory. The central swedge carries a greater moment than the other interior swedge, throughout the whole range of

pressure. The great effect of membrane action on the moments in all three swedges can be judged by comparing the experimental moments with the results of linear theory shown by the straight lines tangential to the moment-pressure curves at the origin. At higher pressures most of the load is carried by membrane action so that the bending moment increases at a much slower rate. The rate of increase of the experimental bending moment in the central swedge suddenly becomes very slow after a pressure of about 0.6 lb per sq in.; the cause of this abrupt change is not known, although it may be noted that the experimental membrane stresses (Fig. 75) in the inner corrugations exceeded the theoretical stresses calculated on the basis of estimated edge restraints, at about the same pressure.

TEST 2 (20 in. x 30 in. PLATE)

Details of Test Rig

The connection of the test plate 2 (Fig. 77) to the rig was similar to that of the test plate 1, except that the side and end sheets were directly welded to the test plate. The sheets were also welded to each other and, at their lower edges, to 7/8 in. x 3/8 in. Aluminium-alloy flat bars, which were bolted to the mild steel flats previously welded to the bottom frame, through a "neoprene" gasket 1/16 in. thick.

The test plate was loaded by oil pressure and the arrangement of loading is shown in Fig. 79. The rubber bags were used as in the first test to counterbalance the effect of oil pressure on the side and end sheets. In order to eliminate as much air as possible from the chamber under the test plate 5-3/8 in. dia copper tubes (Fig. 77) were used to act as bleeder pipes through which air could be forced out as the chamber was filled with oil; to achieve this, the bottom frame was slightly tilted with the end having the bleeder pipes uppermost.

Instrumentation

The arrangement of gauges to measure strain, deflection, slope, swedge spread and normal in-plane movement at the edges is shown in Fig. 78.

Four strain gauges were mounted at each location, there being 15 such locations at the midspan section and 3 at the quarter span section. The groups of four gauges at stations 2 and 3 were employed to determine the strain distribution over the depth of the central swedge at the midspan section.

Deflections of the central swedge and of the midspan transverse section of the plate were measured with 13 dial gauges supported from an L shaped frame, which was again supported by a transverse rail clamped to a pair of longitudinal tubes (Plate 6); at the midspan transverse section the dial gauges were positioned to measure the deflections at the centre of the inter-swedge flat plating and at the points where the plating meets the swedges.

Slopes were measured by 6 spirit-level clinometers, at several locations on one longitudinal and one transverse boundary to estimate the degree of fixity along the edges.

Swedge spread was measured at the midspan section for the central swedge and the two swedges on one side of the longitudinal central line with the gauge used in the first experiment.

Normal horizontal displacements of one transverse and one longitudinal edge were measured at a number of locations by dial gauges mounted on two angle sections fixed to the bottom frame.

The pressure applied to the plate was measured by a pressure transducer which was calibrated beforehand with a dead-weight tester.

Test Procedure

The plate was loaded by switching on the oil pump (Fig. 79) with the main outlet valve shut and the outlet from the bleeder pipes open. When the level of oil in the chamber reached the top of the bleeder pipes and oil started to flow out, the outlet valve for the bleeder pipes was shut to build up the pressure on the plate, the rate of loading being adjusted by controlling the rate of delivery of oil at the pump. When the required pressure had been attained the pump was switched off.

At the beginning of the experiment the plate was unloaded by opening the main outlet valve, whereby the level of oil in the chamber dropped to the level of the top edge of the outlet pipe, which was about 2 in. from the undersurface of the flat part of the test plate. Further loading would have involved pumping in about 4.5 gallons of oil before any pressure could be built up. To avoid the repeated emptying and refilling of this volume of oil the main outlet valve was not opened at all during the test, and unloading was achieved by letting oil flow out through the bleeder pipes.

The lateral pressure was applied in increments of 1 lb per sq in. upto a pressure of 4 lb per sq in.; there was a permanent set of about 0.004 in. at this pressure. To increase the range of elastic behaviour, the plate was loaded to a higher pressure of about 5 lb per sq in. and then unloaded; the total permanent set at the centre was about 0.015 in..

After taking the initial readings of the gauges, the plate was loaded again in increments of 1 lb per sq in. upto a pressure of 5 lb per sq in.; all the gauges were read at each of the load levels to give the elastic behaviour of the plate. The loading was continued in smaller increments beyond 5 lb per sq in. to study the behaviour of the plate in the inelastic range. At about 6.2 lb per sq in., a small crack developed at the crest of

the central swedge at the midspan section; it was no longer possible to apply any pressure to the plate and the experiment was discontinued.

Test Results and Discussion

Stress-strain Relationship

Stress-strain curves shown in Fig. 52 for the first test plate also apply to the second, as both test plates were cut from one 12 ft long "Mansard" sheet. The average value of the modulus of elasticity used in the calculations was 10.4×10^6 lb per sq in..

Deflections

Deflections of the plate are plotted in Figs. 80 to 84. Fig. 80 shows the theoretical and experimental deflections of the central swedge at the midspan and quarter span sections. The theoretical deflections vary linearly with pressure, as they were obtained by the small deflection orthotropic plate theory, using the estimated torsional rigidity parameter (μ_b) and rotational edge restraint coefficients (α_x and α_y) for the swedged plate (Appendix C). The use of the small deflection theory was justified, as the deflection at the centre of the plate at 5 lb per sq in. (the pressure causing an appreciable amount of permanent set) was 0.22 in. which is less than quarter of the equivalent thickness (h_{eq}) of the plate. Agreement between theory and experiment is satisfactory at low pressures; but at high pressures the experimental deflections are greater than the theoretical deflections, probably due to a reduction in the flexural rigidity of the swedge associated with a loss of the effective width of the inter-swedge flat plating due to buckling (Fig. 90). Beyond 5 lb per sq in. the increased rate of deflection is primarily due to the decreasing tangent modulus of the material.

In Fig. 81 the deflections of the plate have been compared with the best fit surfaces given by one or more terms of a Fourier Series. It can be seen that only one term of the Fourier Series would be very inadequate to describe the deflected surface of the plate, whereas by adding an extra term in the transverse direction the maximum error anywhere in the plate could be limited to about 5 per cent of the central deflection, throughout the whole range of loading.

Figs. 82 and 83 respectively show the deflected shape of the longitudinal and transverse centre-lines of the plate at different values of pressure. Fig. 83 also shows the local deflection of the inter-swedge flat plating drawn to the same scale as the overall plate deflections.

In Fig. 84 the experimental deflected shapes of the plate centre-lines have been compared with the results of the small deflection orthotropic plate theory. Agreement is good for the longitudinal section; but for the transverse section the experimental deflections near the longitudinal edge relative to the centre are smaller than those predicted by the theory. A likely explanation is that the actual effective torsional rigidity of the edge swedge is greater than the value estimated (Appendix C) by neglecting the restraint against warping deformations of the swedge due to the variation of the twisting moments along its length. A reduction in the twist at the midspan section of the edge swedge (relative to its ends) might be expected as a result of an increase in its torsional rigidity and the experimental deflected profile is in agreement with such expectation. The discrepancy between the theory and experiment may also be partly due to the actual rotational restraint at the longitudinal edges being greater than its estimated value, due to corner effects.

Local Deformation of Swedges and Inter-swedge Flat Plating

The swedge deformations measured at the midspan section were very small; at 6 lb per sq in. the interior swedges closed by about 0.002 in. and the edge swedges opened by about 0.004 in. According to theory the deformation of the central swedge should be zero.

In Fig. 85 the deflections at the centre of the inter-swedge flat plating relative to the adjacent swedges are plotted against pressure. The theoretical pressure-deflection relationship also shown in the figure, was obtained by assuming zero values for the vertical shear force, transverse slope and transverse horizontal displacement at 1 (Case 1, Appendix D). Agreement between theory and experiment is satisfactory at low pressures, but at high pressures the experimental deflections increase at a faster rate and become progressively larger than the theoretical deflections. This was probably due to the presence of a high longitudinal compressive strain in the flat plating. At 5 lb per sq in. the average compressive strain in the plating at the midspan section is about 0.0009; considering the plating as a long simply supported plate under end compression and having a breadth-thickness ratio of 64, the critical buckling strain is also about 0.0009. The actual buckling strain is greater than this value because of the edge restraints to the plating and the diminishing magnitude of the compressive strain away from the centre. Nevertheless, the longitudinal compression in the plating would be comparable to the buckling strain and cause an increase in the local deflection of the plating.

Beyond 5 lb per sq in. the local deflections increase at a progressively faster rate due to yielding under the transverse bending moments in the plating.

Horizontal Displacement Normal to Edges

At 6 lb per sq in. the displacement at the centre of the longitudinal edge was about 0.01 in. (inwards); the corresponding displacement for the transverse edge was about 0.02 in. (inwards).

Normal Edge Slopes

In Fig. 86 the experimental and theoretical normal edge slopes at the end of the central corrugation and near the centre of the longitudinal edge are shown plotted against pressure. Agreement between theory and experiment is good for the slope at the end of the central corrugation at low pressures, but at high pressures the experimental end slope increases at a faster rate, like the deflections (Fig. 80), probably due to a reduction in the flexural rigidity of the corrugation. A comparison of the theoretical and experimental slopes at the longitudinal edge indicates that the theory probably underestimated the rotational restraint at the longitudinal edges.

In Fig. 87 the theoretical distributions of the normal edge slopes along the longitudinal and transverse edges are compared with the experimental results for different value of pressure. Agreement is good for the longitudinal edge, but at the transverse edge the experimental slope at the end of the edge corrugation is about 65 per cent of the theoretical slope; this is consistent with the disagreement between the theoretical and experimental transverse deflected profiles of the plate shown in Fig. 84, which shows that the deflection of the edge corrugation at its crest was also about 65 per cent of the value predicted by the theory.

Strains

Fig. 88 shows the linearity of the longitudinal strain distribution across the depth of the central swedge at the mid-span section, for different values of pressure. The experimental neutral axis practically coincides with the theoretical axis calculated by geometry from the nominal dimensions of the corrugation. The swedge section can also be seen to be free from any membrane tension, which justifies the use of the small deflection theory for comparison with experimental results.

Fig. 89 shows the variation with pressure of longitudinal heart of plate strains at the midspan section of the plate and also at the quarter span section of the central corrugation. The magnitude of the strains in the two inner corrugations is practically the same; this is consistent with the transverse deflected profiles (Fig. 83) which show that the inner corrugations deflect almost identically. It can also be seen from Fig. 89 that the longitudinal heart of plate strain at the centre of the inter-swedge flat plating becomes progressively smaller than the strain at the points where the flat plating meets the swedges; this indicates a gradual loss in the effective width of the flat plating.

The maximum longitudinal heart of plate strain occurs (Fig. 89) at the crest of the central swedge at the midspan section and is approximately equal to 0.0021 and 0.0027 respectively, at 4 and 5 lb per sq in.; the corresponding cumulative permanent strains noticed after unloading are respectively about 0.00015 and 0.00030, which are less than half of what would be expected from the stress-strain curve of the material (Fig. 52) for these maximum strain-levels experienced by the plating at its midplane (the strain at the surface being even higher). This may be due to the inapplicability of the uniaxial stress-strain relationship under such conditions of complex biaxial strains as

exist at the crests of the swedges.

Stresses

Fig. 90 shows the distribution of longitudinal heart of plate stress on the transverse central section. The cyclic nature and the pattern of the stress distribution for the interior swedges indicate, as in the first experiment, that at any transverse section the longitudinal stresses over a swedge and its associated flat plating can be integrated to give a single bending moment acting on the isolated swedge unit. The stresses in the interior swedges are almost identical and are greater than those in the edge swedge.

Figs. 91(a) and 91(b) show how the longitudinal surface stresses compare with the longitudinal heart of plate stresses and Fig. 92 shows the distribution of local longitudinal bending stresses on the transverse central section. The local longitudinal bending moments at the swedge crests and at the centres of the inter-swedge flat plating have the same sign as the overall bending moment in the plate, the crests having a smaller bending moment than the flat plating. The bending moments at the edges of the inter-swedge plating are negligibly small.

Figs. 93(a) and 93(b) show that the transverse heart of plate stresses are negligibly small. The transverse surface stresses due to transverse bending are small at the swedge crests, but in the interswedge flat plating they are as high as and, at some places, even greater than the longitudinal surface stresses (Fig. 91(b)).

A comparison of the distribution of transverse bending stresses (Fig. 94) with that of secondary longitudinal bending stresses (Fig. 92) on the transverse central section confirms the observation, made in connection with the first experiment, that the secondary longitudinal bending moments are closely associated with the transverse bending moments.

In Fig. 95 the experimental transverse heart of plate and transverse bending stresses on the top surface of the plate at the midspan and quarter span sections of the central swedge have been compared with theory. The theoretical results are identical for both the sections and are obtained by assuming the vertical shear force (V), the transverse slope (θ) and the transverse horizontal displacement (v) at point 5 to be zero, which is in accordance with the results of small deflection orthotropic plate theory applied to the entire plate; this case is identical to case 1 in Appendix D. The heart of plate stresses are small both by theory and experiment. (The experimental heart of plate stresses at the quarter span section, being almost identical to those at the midspan section, are not shown in the figure.) For the bending stresses agreement between theory and experiment is, in general, better for the quarter span section than for the midspan section; and at both the sections, agreement is better for the swedgecrest than for the flat plating. The possible reason for the disagreement has been given earlier while discussing the local deflection of the inter-swedge flat plating. It may be added here that the magnitude of the longitudinal compressive strain in the flat plating is less at the quarter span section than at the midspan section, which, consequently, shows a greater effect of buckling and therefore a greater disagreement with theory.

Longitudinal Bending Moment in Swedges

Fig. 96 shows the variation with pressure of the theoretical and experimental average longitudinal bending moments in the central swedge at the midspan and quarter span sections. The theoretical results are obtained by the small deflection theory; the method of evaluating the experimental moments from the longitudinal heart of plate stresses in the swedge has already been explained in connection with the first experiment. Agreement between theory and experiment is satisfactory, the

maximum discrepancy occurs at 5 lb per sq in., when the experimental bending moments are about 8 per cent larger than the theoretical values.

In Fig. 97 the experimental distributions of longitudinal swedge moments along the plate centre-lines are compared with theory. Agreement is good for the longitudinal distribution; the transverse distribution shows that the actual longitudinal bending moment in the edge swedge is less than that predicted by theory. (The experimental swedge moments are obtained by considering half the swedge as a unit, so as to get a better picture of the transverse distribution.)

The longitudinal membrane stresses in the interior swedges were negligibly small, throughout the whole range of loading. The average membrane stress in the edge swedge considered as a unit was also negligible; at 5 lb per sq in. the membrane stresses in the inner and outer halves of swedge were respectively about 3,400 (compressive) and 3,900 (tensile) lb per sq in..

CHAPTER V

C O N C L U S I O N S

LARGE DEFLECTION THEORY OF ORTHOTROPIC PLATES

The large deflection orthotropic plate theory presented in the thesis has been formulated in terms of plate stiffnesses, and is readily applicable to rectangular stiffened or corrugated plates as well as to isotropic plates.

By considering initial deformations and by treating the boundaries as elastically restrained against rotation and against movements both normal and parallel to the edges, the theory provides, for the first time, a powerful method for analysing the large deflection behaviour of stiffened and unstiffened plates as incorporated in practical structures.

For numerical solution, the governing equations and boundary conditions have been converted to finite difference equations, which are then solved by a method of iteration using a digital computer. For this purpose a program has been written in Mercury Autocode which requires, as data, only the description of the plate, the edge restraints, the applied loading, the mesh size, and the degree of accuracy required of the iterative procedure. From these basic data, the computer forms matrices and then carries out all iterative calculations, checking automatically that the deflection and the stress function are obtained to the specified degree of accuracy.

The program permits the analysis of large deflection behaviour of a rectangular plate, corrugated, stiffened, or unstiffened, having any side ratio with any degree of rotational, extensional and shear restraints at the boundaries, and subjected to any system of symmetrical lateral loading.

From the cases discussed in the thesis, it is seen that, for uniform loading, the finite difference method gives solutions of sufficient accuracy with a relatively coarse mesh ($i = 6$). For patch loading, the size of the mesh required for a given accuracy depends on the area of the patch; the program in its present form can give solutions of practical usefulness for patch areas not less than $1/256$ times the area of the plate. The convergence of the iterative method is seen to be fairly rapid. Comparisons with some of the available analytical solutions and experimental results show excellent agreement.

The theoretical solutions presented in the thesis show that the large deflection behaviour of plates is considerably influenced by the in-plane boundary conditions, the magnitude of the influence being dependent on the intensity of the applied load. The central deflection of a plate, with given rotational edge restraint and applied pressure, is seen to be respectively a maximum or minimum depending on whether the extensional and shear edge restraints are both zero or both infinity, the maximum value being in some cases; twice the minimum; when the extensional restraints are zero and the shear restraints are infinity, The deflection is between the two limiting values. In isotropic plates, the effects of the in-plane edge restraints on the moments and membrane forces and on the central deflection are similar; for orthotropic plates, no such similarity exists. For isotropic plates having infinite extensional edge restraints, the solutions corresponding to zero and infinite shear edge restraints are almost identical.

The total tensile surface stresses at the midpoint of the edges of a clamped isotropic plate under uniform loading become progressively greater with load than the total tensile surface stresses at the centre of the plate with the edges free to rotate.

According to linear theory, the deflection and bending moment at the centre of an orthotropic plate are greater for smaller values of the torsional rigidity; according to large deflection theory, this is true only for certain boundary conditions.

The importance of membrane action even in stiffened plates has been demonstrated by the solutions obtained for the swedged plate used in the first experiment; the solutions have been corroborated by the experimental observations.

THEORY OF LOCAL TRANSVERSE STRESSES AND DEFORMATIONS IN CORRUGATIONS

The theory for evaluating the local stresses and deformations in the corrugations, presented in this thesis, is applicable to any corrugation profile having a vertical axis of symmetry. The method of calculation has been demonstrated for a corrugation having a profile geometrically similar to that used in the tests, considering the central corrugation; there is no difficulty in extending the method to corrugations away from the centre.

The method makes use of the solution for the overall behaviour of the plate obtained by the orthotropic plate theory and is valid for any degree of membrane action in the plate. The number of corrugations in the plate can be as small as is permissible without rendering the orthotropic plate theory inapplicable to the plate. Existing methods of calculating the local stresses, applicable to symmetrical trapezoidal and cylindrical corrugations, are valid only when the transverse dimension of the plate (and the number of corrugations) is large and the membrane stresses are negligible.

The proposed method assumes that the overall behaviour of the corrugated plate remains unaffected by the local stresses and deformations in the corrugations and that the actual behaviour of the plate can be predicted by combining the overall and local behaviours obtained in two successive stages; the

experimental observations on the two swedged plates show the validity of the assumptions.

EXPERIMENTS

Results of the two tests on swedged plates described in the thesis are in close agreement with the theoretical solutions obtained by considering the plates as equivalent to orthotropic plates.

The first test showed the importance of membrane action even in stiffened plates. Membrane stresses were noticeable at 10 per cent of the pressure which caused permanent set; at the permanent set pressure the central deflection was about 40 per cent of the value given by small deflection orthotropic plate theory. In the inelastic range the rate of deflection continues to decrease, since the membrane effect more than offsets the decreasing value of tangent modulus.

In the second experiment, deflections were small at the elastic limit pressure and no membrane effect was noticed. The elastic deflection at the centre of the plate at high pressures was somewhat greater than that predicted by the linear theory, probably due to buckling of the compression flange (inter-swedge flat plating).

Local transverse bending stresses in the test plates were of the same order of magnitude as the longitudinal stresses and were in reasonable agreement with the theory presented in the thesis. In the first experiment the transverse stresses were primarily due to transverse membrane action. A swedged plate, because of its very small transverse in-plane rigidity, suffers large transverse membrane extensions with practically no membrane tension in the transverse direction. The extension in the transverse direction is made possible by opening of the swedges and large transverse bending stresses develop because of these

swedge deformations. In the second experiment the transverse stresses were the result of the arch-like behaviour of the swedges; at the midspan section of the plate the transverse bending stresses in the inter-swedge flat plating were increased due to buckling.

In both test plates, the transverse stresses resulted in secondary longitudinal swedge stresses which added to the primary stresses.

No instability of the swedge profile was noticeable in either of the tests.

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NOTATION

x, y, z	Rectangular co-ordinates, z measured normal to plane of plate.
u, v, w	Displacements of middle surface of plate in x, y and z directions respectively under applied loading.
u_0, v_0, w_0	Initial deformations of plate.
$\bar{u}$ $\bar{v}$ $\bar{w}$	$\left. \begin{aligned} &= u + u_0 \\ &= v + v_0 \\ &= w + w_0 \end{aligned} \right\} \begin{array}{l} \text{Total displacements of middle} \\ \text{surface of plate.} \end{array}$
l_x	Length of plate in x direction.
l_y	Length of plate in y direction.
h	Thickness of plate.
$q_{x,y}$	Intensity of lateral pressure at the point (x,y) .
M_x	Bending moment in x direction per unit width in y direction.
M_y	Bending moment in y direction per unit width in x direction.
M_{xy}	Twisting moment per unit width of plate.
$\frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial y^2}$	Curvatures in x and y directions respectively $(w \ll l_x, l_y)$

$$\left. \begin{aligned} B_x &= M_x / \frac{\partial^2 w}{\partial x^2} \\ B_y &= M_y / \frac{\partial^2 w}{\partial y^2} \end{aligned} \right\} \begin{array}{l} \text{Bending moments to produce unit} \\ \text{curvature with anticlastic} \\ \text{curvatures unrestrained.} \end{array}$$

ν_{xb}, ν_{yb} Poisson's ratios in bending associated with applied bending in x and y directions.

$D_x = B_x / (1 - \nu_{xb} \nu_{yb})$ Flexural rigidity in x direction.

$D_y = B_y / (1 - \nu_{xb} \nu_{yb})$ Flexural rigidity in y direction.

$D_1 = \nu_{xb} D_y = \nu_{yb} D_x$ (By Maxwell's Reciprocal Theorem).

B_{xy} or D_{xy} Torsional rigidity.

$$H_b = 2D_{xy} + D_1$$

$$\lambda_b = \sqrt{D_y / D_x}$$

$$\mu_b = H_b / \sqrt{D_x D_y}$$

$$\epsilon_0 = \ell_y / \ell_x \quad \text{Actual side ratio of plate.}$$

A_x Stiffness of elastic restraint against rotation at the edges parallel to y axis, being the moment per unit length of edge required for unit rotation of the restraint.

A_y Stiffness of elastic restraint against rotation at the edges parallel to x axis, being the moment per unit length of edge

required for unit rotation of the restraint.

$$\left. \begin{aligned} \alpha_x &= \ell_x A / D_x \\ \alpha_y &= \ell_y A / D_y \end{aligned} \right\} \begin{array}{l} \text{Coefficients of Rotational Edge} \\ \text{Restraints (constant along edges)} \end{array}$$

N_x Tensile force in plate in x direction per unit width in y direction.

N_y Tensile force in plate in y direction per unit width in x direction.

N_{xy} Shearing force per unit width of plate.

e_x Tensile strain in middle surface of plate in x direction.

e_y Tensile strain in middle surface of plate in y direction.

γ_{xy} Shearing strain in middle surface of plate.

$$\left. \begin{aligned} C_x &= N_x / e_x \\ C_y &= N_y / e_y \end{aligned} \right\} \begin{array}{l} \text{Tensile forces to produce unit} \\ \text{strain with transverse deformations} \\ \text{unrestrained.} \end{array}$$

ν_{xm}, ν_{ym} Poisson's ratios in extension associated with applied tension in x and y directions.

$$C_1 = -C_x / \nu_{xm} = -C_y / \nu_{ym} \quad (\text{By Maxwell's Reciprocal Theorem}).$$

$$E_x = C_x / (1 - \nu_{xm} \nu_{ym}) \quad \text{Extensional rigidity in x direction.}$$

$$E_y = C_y / (1 - \nu_{xm} \nu_{ym}) \quad \text{Extensional rigidity in y direction.}$$

$$E_1 = \nu_{xm} E_y = \nu_{ym} E_x \quad (\text{By Maxwell's Reciprocal Theorem}).$$

C_{xy} or E_{xy} Shearing Rigidity.

$$H_m = 1 / \left[\frac{1}{2C_{xy}} + \frac{1}{C_1} \right]$$

$$\lambda_m = \sqrt{4C_y/C_x} = \sqrt{4E_y/E_x}$$

$$\mu_m = \sqrt{C_x C_y} / H_m$$

$$r = \lambda_m / \lambda_b$$

K_x Stiffness of elastic restraint against movement in x direction at the edges parallel to y axis, being the force per unit length of edge required for unit displacement of the restraint.

K_y Stiffness of elastic restraint against movement in y direction at the edges parallel to x axis, being the force per unit length of edge required for unit displacement of the restraint.

$$\left. \begin{aligned} k_x &= \ell_x K_x / C_x \\ k_y &= \ell_y K_y / C_y \end{aligned} \right\} \begin{array}{l} \text{Coefficients of Extensional} \\ \text{Edge Restraints. (Constant} \\ \text{along edges).} \end{array}$$

S_x Stiffness of elastic restraint against movement in x direction at the edges parallel to x axis, being the force per unit length of edge required for unit displacement of the restraint.

S_y Stiffness of elastic restraint against movement in y direction at the edges parallel to y axis, being the force per unit length of edge required for unit displacement of the restraint.

s_x
 s_y

$$= \left. \begin{aligned} &= \ell_x S_x / \lambda_m C_x \\ &= \lambda_m \ell_y S_y / C_y \end{aligned} \right\} \begin{array}{l} \text{Coefficients of Shear Edge} \\ \text{Restrains. (Constant} \\ \text{along edges).} \end{array}$$

V_{bp} Strain energy stored in plate due to bending.

V_{ba} Strain energy stored in Rotational Restraints at the edges.

V_{mp} Strain energy stored in plate due to membrane forces.

V_{mk} Strain energy stored in Extensional Restraints at the edges.

V_{ms} Strain energy stored in Shear Restraints at the edges.

U_q Potential energy of applied load $q_{x,y}$ referred to undeflected plate as datum.

U $= V_{bp} + V_{ba} + V_{mp} + V_{mk} + V_{ms} + U_q$ Total potential energy of deflected plate.

$\delta^u V, \delta^v V, \dots$ Small variations in $V, \dots$ due to small variations $\delta u, \delta v, \dots$ respectively.

$$\int_C$$

Line integral carried out along the closed boundary C.

$$\iint_D$$

Double integral taken over the entire area bounded by C.

(Expression) $\Big|_{x=x_0}$ Value of the expression in brackets on the line $x = x_0$.

f Airy stress function.

L Representative length.

X $= x/L$ } Co-ordinate axes of transformed
Y $= y/\lambda_b L$ } plate.

L_x $= \ell_x/L$ } Lengths of sides of transformed
 L_y $= \ell_y/\lambda_b L$ } plate.

ε $= L_y/L_x = \varepsilon_0/\lambda_b$ Virtual side ratio.

$\bar{W}$ $= \bar{w} \sqrt{C_x/D_x}$ Non-dimensional total deflection.

W_0 $= w_0 \sqrt{C_x/D_x}$ Non-dimensional initial deflection.

Q $= \frac{q \ell_x^4}{D_x} \sqrt{\frac{C_x}{D_x}}$ Non-dimensional load intensity.

F $= f/\lambda_b^2 D_x$ Non-dimensional stress function.

- i Half the number of finite difference meshes in both X and Y directions.
- p $= i^2$ Number of meshes in a quarter of plate.
- p' $= (i+2)^2$
- a $= L_x/2i$ Length of mesh in X direction of transformed plate.
- b $= L_y/2i = \epsilon a$ Length of mesh in Y direction of transformed plate.
- $(Q)^1$ Intensities of non-dimensional loading at p nodes, at the beginning of computation.
- j Multiples of the starting intensity of loading, constant for all the p nodes. ($j = 1, 2, \dots, J$)
- J Number of multiples of starting load for which solutions are to be obtained.
- $(Q)^j$ $= j(Q)^1$ Intensities of loading at p nodes.
- (W_0) Non-dimensional initial deformations of the plate.
- s Number of complete cycles of the iterative procedure.
- $(\bar{W})^{j,s}$ Non-dimensional total deflections at p nodes obtained for a load $(Q)^j$ and after s cycles.
- $(F)^{j,s}$ Values of non-dimensional stress function at p' nodes obtained for a load $(Q)^j$ and after s cycles.

- O_1 Number of variations in virtual side ratio, mesh size, membrane properties and in-plane boundary conditions.
- O_2 Number of variations (for each of O_1) in bending properties and boundary conditions.
- O_3 Number of variations (for each of O_2) in loading and degree of accuracy of solution.
- Θ Key number, to denote the type of loading.
($\Theta = 0$, for uniformly distributed load).
- ρ Acceptable percentage error in the solution of $(\bar{W})$ obtained by the process of iteration.
-

- (1) All stresses, strain deflection, moments, etc., refer to those caused by applied loading, unless otherwise mentioned.
- (2) Sign conventions for forces, moments, etc., are similar to those used by Timoshenko (8).

DERIVATION OF GOVERNING EQUATIONS FOR LARGE DEFLECTION
BEHAVIOUR OF RECTANGULAR ORTHOTROPIC PLATES WITH
INITIAL DEFORMATIONS, UNDER LATERAL LOAD

The differential equations and the boundary conditions governing the large deflection behaviour of rectangular orthotropic plates with initial deformations will first be obtained by using the principle of virtual displacements together with the expression for total potential energy of the bent plate. According to the principle of virtual displacement, if an elastic body is in equilibrium, the total virtual work done during any arbitrary virtual displacement (consistent with prescribed geometric boundary conditions) of the body is zero. Therefore the change in the total potential energy of the body, caused by the particular virtual displacement chosen, is zero. If an elastic plate having large deflection has three components of displacement u , v and w , then it can be given three different types of virtual displacement, each involving one of the three infinitesimally small variations δu , δv and δw . The variation in the total potential energy of the plate (δU) must, therefore, be zero for each of the three variations, δu , δv and δw .

Hence,

$$\left. \begin{aligned} \delta^u U &= \delta^u V_{bp} + \delta^u V_{ba} + \delta^u V_{mp} + \delta^u V_{mk} + \delta^u V_{ms} + \delta^u U_q = 0 \\ \delta^v U &= \delta^v V_{bp} + \delta^v V_{ba} + \delta^v V_{mp} + \delta^v V_{mk} + \delta^v V_{ms} + \delta^v U_q = 0 \\ \delta^w U &= \delta^w V_{bp} + \delta^w V_{ba} + \delta^w V_{mp} + \delta^w V_{mk} + \delta^w V_{ms} + \delta^w U_q = 0 \end{aligned} \right\} \quad (1)$$

The plate is assumed to be supported on all four sides; this gives the geometric condition of zero deflection ($w = 0$) at the boundary. In the present problem, no other geometric restrictions have been imposed on the displacements and the virtual displacements can, therefore, be quite arbitrary except that δw must be zero at the boundary. The other natural boundary conditions for u , v and w will be obtained from Equation (1), which will also give the equilibrium equations for the plate.

Bending Strain Energies (V_{bp} and V_{ba}) and their Variations.

Neglecting the effect of vertical shearing stresses on deflection, the bending strain energy of a plate is given by

$$V_{bp} = \iint_D \left[-\frac{1}{2} M_x \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} M_y \frac{\partial^2 w}{\partial y^2} + M_{xy} \frac{\partial^2 w}{\partial x \partial y} \right] dx dy \quad (2)$$

Substituting in Equation (2) the expressions for M_x , M_y and M_{xy} in terms of curvatures and twist, namely,

$$M_x = - (D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}) \quad (3a)$$

$$M_y = - (D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}) \quad (3b)$$

$$\text{and } M_{xy} = 2D_{xy} \frac{\partial^2 w}{\partial x \partial y} \quad (3c)$$

the bending strain energy of a rectangular orthotropic plate is given by

$$V_{bp} = \frac{1}{2} \int_0^{\ell_y} \int_0^{\ell_x} [D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2H_b \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + 2D_1 \left\{ \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right\}] dx dy \quad (4)$$

By Green's theorem⁽⁵²⁾

$$\iint_D \left(G \frac{\partial \eta}{\partial x} + F \frac{\partial \eta}{\partial y} \right) dx dy = - \iint_D \eta \left(\frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \right) dx dy + \int_C \eta (G dy - F dx) \quad \dots (5)$$

Setting $G = \partial^2 w / \partial y^2$, $\eta = \partial w / \partial x$ and $F = - \partial^2 w / \partial x \partial y$ and noting that the term $(\partial G / \partial x + \partial F / \partial y)$ equals zero, gives

$$\iint_D \left[\frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dx dy = \int_C \left(\frac{\partial w}{\partial x} \left(\frac{\partial^2 w}{\partial y^2} dy + \frac{\partial^2 w}{\partial x \partial y} dx \right) \right) = 0$$

because, for a rectangular plate with $w = 0$ at the boundary

$$\partial w / \partial x = 0 \quad \text{at the edges } y = 0 \quad \text{and } y = \ell_y$$

and $\partial w / \partial y = \partial^2 w / \partial y^2 = 0$ at the edges $x = 0$ and $x = \ell_x$.

Therefore,

$$V_{bp} = \frac{1}{2} \int_0^{\ell_y} \int_0^{\ell_x} [D_x \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + D_y \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2H_b \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2] dx dy \quad (6)$$

and

$$\begin{aligned} \delta^w V_{bp} = & \frac{D_x}{2} \delta \left[\int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx dy \right] + \frac{D_y}{2} \delta \left[\int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 dx dy \right] \\ & + H_b \delta \left[\int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 dx dy \right] \end{aligned} \quad (7)$$

By Green's theorem, ⁽⁵²⁾

$$\iint_D G \frac{\partial^2 \eta}{\partial x^2} dx dy = \iint_D \eta \frac{\partial^2 G}{\partial x^2} dx dy + \int_C \left(G \frac{\partial \eta}{\partial x} - \eta \frac{\partial G}{\partial x} \right) dy \quad (8a)$$

$$\iint_D G \frac{\partial^2 \eta}{\partial y^2} dx dy = \iint_D \eta \frac{\partial^2 G}{\partial y^2} dx dy - \int_C \left(G \frac{\partial \eta}{\partial y} - \eta \frac{\partial G}{\partial y} \right) dx \quad (8b)$$

$$\begin{aligned} \text{and } \iint_D G \frac{\partial^2 \eta}{\partial x \partial y} dx dy = & \iint_D \eta \frac{\partial^2 G}{\partial x \partial y} dx dy + \frac{1}{2} \int_C \left(G \frac{\partial \eta}{\partial y} - \eta \frac{\partial G}{\partial y} \right) dy \\ & - \frac{1}{2} \int_C \left(G \frac{\partial \eta}{\partial x} - \eta \frac{\partial G}{\partial x} \right) dx \end{aligned} \quad (8c)$$

The first term of the R.H.S. of Equation (7) is,

$$\begin{aligned}
\frac{D_x}{2} \delta \left[\int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx dy \right] &= D_x \int_0^{\ell_y} \int_0^{\ell_x} \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 \delta w}{\partial x^2} dx dy \\
&= D_x \int_0^{\ell_y} \int_0^{\ell_x} \delta w \frac{\partial^4 w}{\partial x^4} dx dy + D_x \int_C \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial \delta w}{\partial x} - \delta w \frac{\partial^3 w}{\partial x^3} \right) dy \\
&\quad \text{[By Equation (8a)]} \\
&= D_x \int_0^{\ell_y} \int_0^{\ell_x} \frac{\partial^4 w}{\partial x^4} \delta w dx dy - D_x \int_0^{\ell_y} \frac{\partial^2 w}{\partial x^2} \frac{\partial \delta w}{\partial x} \Big|_{x=0} dy \\
&\quad + D_x \int_0^{\ell_y} \frac{\partial^2 w}{\partial x^2} \frac{\partial \delta w}{\partial x} \Big|_{x=\ell_x} dy \quad (9a)
\end{aligned}$$

Similarly, using Equations (8b) and (8c) and noting that $\delta w = 0$ at the boundary, the other two terms of Equation (7) can be written as

$$\begin{aligned}
\frac{D_y}{2} \delta \left[\int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial^2 w}{\partial y^2} \right)^2 dx dy \right] &= D_y \int_0^{\ell_y} \int_0^{\ell_x} \frac{\partial^4 w}{\partial y^4} \delta w dx dy \\
&\quad - D_y \int_0^{\ell_x} \frac{\partial^2 w}{\partial y^2} \frac{\partial \delta w}{\partial y} \Big|_{y=0} dx + D_y \int_0^{\ell_x} \frac{\partial^2 w}{\partial y^2} \frac{\partial \delta w}{\partial y} \Big|_{y=\ell_y} dx \quad (9b)
\end{aligned}$$

and

$$H_b \delta \left[\int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 dx dy \right] = 2H_b \int_0^{\ell_y} \int_0^{\ell_x} \frac{\partial^4 w}{\partial x^2 \partial y^2} \delta w dx dy \quad (9c)$$

Therefore, from Equations (7), (9a), (9b) and (9c), the w variation is

$$\begin{aligned}
 \delta^w V_{bp} = & \int_0^{\ell_y} \int_0^{\ell_x} \left[D_x \frac{\partial^4 w}{\partial x^4} + 2H_b \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} \right] \delta w \, dx \, dy \\
 & - D_x \int_0^{\ell_y} \left. \frac{\partial^2 w}{\partial x^2} \frac{\partial \delta w}{\partial x} \right|_{x=0} dy + D_x \int_0^{\ell_y} \left. \frac{\partial^2 w}{\partial x^2} \frac{\partial \delta w}{\partial x} \right|_{x=\ell_x} dy \\
 & - D_y \int_0^{\ell_x} \left. \frac{\partial^2 w}{\partial y^2} \frac{\partial \delta w}{\partial y} \right|_{y=0} dx + D_y \int_0^{\ell_x} \left. \frac{\partial^2 w}{\partial y^2} \frac{\partial \delta w}{\partial y} \right|_{y=\ell_y} dx \quad (10c)
 \end{aligned}$$

and evidently

$$\delta^u V_{bp} = 0 \quad (10a)$$

$$\text{and } \delta^v V_{bp} = 0 \quad (10b)$$

because V_{bp} is independent of u and v .

The strain energy stored in the rotational restraints along the boundary is given by

$$\begin{aligned}
 V_{ba} = & \frac{1}{2} \left[\int_0^{\ell_y} A_x \left(\frac{\partial w}{\partial x} \right)^2 \bigg|_{x=0} dy + \int_0^{\ell_y} A_x \left(\frac{\partial w}{\partial x} \right)^2 \bigg|_{x=\ell_x} dy \right. \\
 & \left. + \int_0^{\ell_x} A_y \left(\frac{\partial w}{\partial y} \right)^2 \bigg|_{y=0} dx + \int_0^{\ell_x} A_y \left(\frac{\partial w}{\partial y} \right)^2 \bigg|_{y=\ell_y} dx \right] \quad (11)
 \end{aligned}$$

The variations of this energy are, therefore,

$$\delta^u V_{ba} = 0 \quad (12a)$$

$$\delta^v V_{ba} = 0 \quad (12b)$$

and

$$\begin{aligned}
\delta^w V_{ba} = & \int_0^{l_y} A_x \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \bigg|_{x=0}^{x=l_x} dy + \int_0^{l_y} A_x \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \bigg|_{x=l_x} dy \\
& + \int_0^{l_x} A_y \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \bigg|_{y=0}^{y=l_y} dx + \int_0^{l_x} A_y \frac{\partial w}{\partial y} \frac{\partial \delta w}{\partial y} \bigg|_{y=l_y} dx \quad (12c)
\end{aligned}$$

Membrane Strain Energies (V_{mp} , V_{mk} and V_{ms}) and their Variations.

By definition,

$$\left. \begin{aligned} e_x &= N_x/C_x + N_y/C_l \\ e_y &= N_y/C_y + N_x/C_l \\ \gamma_{xy} &= N_{xy}/C_{xy} \end{aligned} \right\} \quad (13)$$

and

$$\left. \begin{aligned} N_x &= E_x e_x + E_l e_y \\ N_y &= E_y e_y + E_l e_x \\ N_{xy} &= E_{xy} \gamma_{xy} \end{aligned} \right\} \quad (14)$$

Again, strains due to applied loading can be expressed in terms of displacements as follows

$$\left. \begin{aligned} e_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial x} \\ e_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \frac{\partial w_0}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x} \frac{\partial w}{\partial y} \end{aligned} \right\} \quad (15)$$

Eliminating u and v from Equations (15), the following

compatibility equation is obtained

$$\begin{aligned} \frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} &= \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{\partial^2 w_0}{\partial x \partial y} \\ &\quad - \frac{\partial^2 w}{\partial x^2} \cdot \frac{\partial^2 w_0}{\partial y^2} - \frac{\partial^2 w_0}{\partial x^2} \cdot \frac{\partial^2 w}{\partial y^2} \end{aligned} \quad (16)$$

From Equations (14) and (15), the membrane forces can be expressed in terms of displacements

$$N_x = E_x \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \cdot \frac{\partial w_0}{\partial x} \right] + E_1 \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \cdot \frac{\partial w_0}{\partial y} \right] \quad (17a)$$

$$N_y = E_1 \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \cdot \frac{\partial w_0}{\partial x} \right] + E_y \left[\frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \cdot \frac{\partial w_0}{\partial y} \right] \quad (17b)$$

$$N_{xy} = E_{xy} \left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \cdot \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \cdot \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x} \cdot \frac{\partial w}{\partial y} \right] \quad (17c)$$

In deriving Equations (15), (16) and (17), the following relationship has been used

$$\begin{aligned} &\text{Strain due to applied loading (associated with } u, v, w) \\ = &\text{Total strain (associated with } \bar{u}, \bar{v}, \bar{w}) \\ &- \text{Initial strain (associated with } u_0, v_0, w_0). \end{aligned}$$

The strain energy due to membrane forces in the plate is given by

$$V_{mp} = \frac{1}{2} \int_0^y \int_0^x [N_x e_x + N_y e_y + N_{xy} \gamma_{xy}] dx dy \quad (18)$$

where the forces and strains are given by Equations (17)

and (15). To obtain the variations of V_{mp} with virtual displacements δu , δv and δw , it will be convenient to obtain first the expressions for variations of the component forces and strains.

From Equations (15) and (17),

$$\left. \begin{aligned} \delta^u e_x &= \frac{\partial \delta u}{\partial x}, & \delta^u N_x &= E_x \frac{\partial \delta u}{\partial x}, \\ \delta^u e_y &= 0, & \delta^u N_y &= E_1 \frac{\partial \delta u}{\partial x}, \\ \delta^u \gamma_{xy} &= \frac{\partial \delta u}{\partial y}, & \delta^u N_{xy} &= E_{xy} \delta^u \gamma_{xy} \end{aligned} \right\} \quad (19a)$$

$$\left. \begin{aligned} \delta^v e_x &= 0, & \delta^v N_x &= E_1 \frac{\partial \delta v}{\partial y}, \\ \delta^v e_y &= \frac{\partial \delta v}{\partial y}, & \delta^v N_y &= E_y \frac{\partial \delta v}{\partial y}, \\ \delta^v \gamma_{xy} &= \frac{\partial \delta v}{\partial x}, & \delta^v N_{xy} &= E_{xy} \delta^v \gamma_{xy} \end{aligned} \right\} \quad (19b)$$

$$\text{and } \left. \begin{aligned} \delta^w e_x &= \frac{\partial \bar{w}}{\partial x} \cdot \frac{\partial \delta w}{\partial x}, & \delta^w N_x &= E_x \frac{\partial \bar{w}}{\partial x} \cdot \frac{\partial \delta w}{\partial x} + E_1 \frac{\partial \bar{w}}{\partial y} \cdot \frac{\partial \delta w}{\partial y}, \\ \delta^w e_y &= \frac{\partial \bar{w}}{\partial y} \cdot \frac{\partial \delta w}{\partial y}, & \delta^w N_y &= E_1 \frac{\partial \bar{w}}{\partial x} \cdot \frac{\partial \delta w}{\partial x} + E_y \frac{\partial \bar{w}}{\partial y} \cdot \frac{\partial \delta w}{\partial y}, \\ \delta^w \gamma_{xy} &= \frac{\partial \bar{w}}{\partial x} \cdot \frac{\partial \delta w}{\partial y} + \frac{\partial \bar{w}}{\partial y} \cdot \frac{\partial \delta w}{\partial x}, & \delta^w N_{xy} &= E_{xy} \delta^w \gamma_{xy} \end{aligned} \right\} \quad (19c)$$

From Equations (18), (19) and (14) and Green's theorem [Equation (5)], the variation of V_{mp} due to δu is given by

$$\begin{aligned}
\delta^u V_{mp} &= \frac{1}{2} \int_0^{\ell_y} \int_0^{\ell_x} [N_x \delta^u e_x + e_x \delta^u N_x + N_y \delta^u e_y + e_y \delta^u N_y \\
&\quad + N_{xy} \delta^u \gamma_{xy} + \gamma_{xy} \delta^u N_{xy}] dx dy \\
&= \int_0^{\ell_y} \int_0^{\ell_x} [N_x \frac{\partial \delta u}{\partial x} + N_{xy} \frac{\partial \delta u}{\partial y}] dx dy \\
&= - \int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u dx dy + \int_C \delta u (N_x dy - N_{xy} dx) \\
&= - \int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u dx dy - \int_0^{\ell_y} N_x \delta u \Big|_{x=0} dy \\
&\quad + \int_0^{\ell_y} N_x \delta u \Big|_{x=\ell_x} dy - \int_0^{\ell_x} N_{xy} \delta u \Big|_{y=0} dx + \int_0^{\ell_x} N_{xy} \delta u \Big|_{y=\ell_y} dx \\
&\quad \dots (20a)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\delta^v V_{mp} &= - \int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v dx dy - \int_0^{\ell_y} N_{xy} \delta v \Big|_{x=0} dy \\
&\quad + \int_0^{\ell_y} N_{xy} \delta v \Big|_{x=\ell_x} dy - \int_0^{\ell_x} N_y \delta v \Big|_{y=0} dx + \int_0^{\ell_x} N_y \delta v \Big|_{y=\ell_y} dx \quad (20b)
\end{aligned}$$

and

$$\begin{aligned} \delta^w V_{mp} = & - \int_0^{\ell_y} \int_0^{\ell_x} \left[N_x \frac{\partial^2 \bar{w}}{\partial x^2} + 2N_{xy} \frac{\partial^2 \bar{w}}{\partial x \partial y} + N_y \frac{\partial^2 \bar{w}}{\partial y^2} \right. \\ & \left. + \frac{\partial \bar{w}}{\partial x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) + \frac{\partial \bar{w}}{\partial y} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \right] \delta w \, dx \, dy \quad (20c) \end{aligned}$$

The line integral vanishes in Equation (20c) as $\delta w = 0$ at the boundary.

The strain energy stored in the extensional restraints at the boundary is given by

$$\begin{aligned} V_{mk} = \frac{1}{2} \left[\int_0^{\ell_y} K_x u^2 \Big|_{x=0} dy + \int_0^{\ell_y} K_x u^2 \Big|_{x=\ell_x} dy + \int_0^{\ell_x} K_y v^2 \Big|_{y=0} dx \right. \\ \left. + \int_0^{\ell_x} K_y v^2 \Big|_{y=\ell_y} dx \right] \quad (21) \end{aligned}$$

The variations of this energy are, therefore,

$$\delta^u V_{mk} = \int_0^{\ell_y} K_x u \, \delta u \Big|_{x=0} dy + \int_0^{\ell_y} K_x u \, \delta u \Big|_{x=\ell_x} dy \quad (22a)$$

$$\delta^v V_{mk} = \int_0^{\ell_x} K_y v \, \delta v \Big|_{y=0} dx + \int_0^{\ell_x} K_y v \, \delta v \Big|_{y=\ell_y} dx \quad (22b)$$

and

$$\delta^w V_{mk} = 0 \quad (22c)$$

The strain energy stored in the shear restraints at the boundary is given by

$$\begin{aligned}
 V_{ms} = \frac{1}{2} [& \int_0^{\ell_x} S_x u^2 \Big|_{y=0} dx + \int_0^{\ell_x} S_x u^2 \Big|_{y=\ell_y} dx + \int_0^{\ell_y} S_y v^2 \Big|_{x=0} dy \\
 & + \int_0^{\ell_y} S_y v^2 \Big|_{x=\ell_x} dy] \quad (23)
 \end{aligned}$$

The variations of this energy are, therefore,

$$\delta^u V_{ms} = \int_0^{\ell_x} S_x u \delta u \Big|_{y=0} dx + \int_0^{\ell_x} S_x u \delta u \Big|_{y=\ell_y} dx \quad (24a)$$

$$\delta^v V_{ms} = \int_0^{\ell_y} S_y v \delta v \Big|_{x=0} dy + \int_0^{\ell_y} S_y v \delta v \Big|_{x=\ell_x} dy \quad (24b)$$

and

$$\delta^w V_{ms} = 0 \quad (24c)$$

Potential Energy of Applied Load (U_q) and its Variations

$$U_q = - \int_0^y \int_0^x q_{x,y} \bar{w} \, dx \, dy \quad (25)$$

The variations are

$$\delta^u U_q = 0 \quad (26a)$$

$$\delta^v U_q = 0 \quad (26b)$$

$$\text{and } \delta^w U_q = - \int_0^y \int_0^x q_{x,y} \delta w \, dx \, dy \quad (26c)$$

Substituting the expressions for the variations of the component energies [Equations (10), (12), (20), (22), (24) and (26)] in Equations (1), the conditions for zero virtual work are given by

$$\begin{aligned} & - \int_0^y \int_0^x \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) \delta u \, dx \, dy + \int_0^y \left(K_x u - N_x \right) \delta u \Big|_{x=0} dy \\ & + \int_0^y \left(K_x u + N_x \right) \delta u \Big|_{x=\ell_x} dy + \int_0^x \left(S_x u - N_{xy} \right) \delta u \Big|_{y=0} dx \\ & + \int_0^x \left(S_x u + N_{xy} \right) \delta u \Big|_{y=\ell_y} dx = 0 \end{aligned} \quad (27a)$$

$$\begin{aligned}
& - \int_0^{\ell_y} \int_0^{\ell_x} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) \delta v \, dx \, dy + \int_0^{\ell_y} (S_y v - N_{xy}) \delta v \Big|_{x=0} dy \\
& + \int_0^{\ell_y} (S_y v + N_{xy}) \delta v \Big|_{x=\ell_x} dy + \int_0^{\ell_x} (K_y v - N_y) \delta v \Big|_{y=0} dx \\
& + \int_0^{\ell_x} (K_y v + N_y) \delta v \Big|_{y=\ell_y} dx = 0 \quad (27b)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^{\ell_y} \int_0^{\ell_x} \left[D_x \frac{\partial^4 w}{\partial x^4} + 2H_b \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} - (N_x \frac{\partial^2 \bar{w}}{\partial x^2} + 2N_{xy} \frac{\partial^2 \bar{w}}{\partial x \partial y} \right. \\
& \left. + N_y \frac{\partial^2 \bar{w}}{\partial y^2}) - \frac{\partial \bar{w}}{\partial x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) - \frac{\partial \bar{w}}{\partial y} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) - q_{x,y} \right] \delta w \, dx \, dy \\
& + \int_0^{\ell_y} \left(A_x \frac{\partial w}{\partial x} - D_x \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial \delta w}{\partial x} \Big|_{x=0} dy + \int_0^{\ell_y} \left(A_x \frac{\partial w}{\partial x} + D_x \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial \delta w}{\partial x} \Big|_{x=\ell_x} dy \\
& + \int_0^{\ell_x} \left(A_y \frac{\partial w}{\partial y} - D_y \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial \delta w}{\partial y} \Big|_{y=0} dx + \int_0^{\ell_x} \left(A_y \frac{\partial w}{\partial y} + D_y \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial \delta w}{\partial y} \Big|_{y=\ell_y} dx \\
& = 0 \quad (27c)
\end{aligned}$$

As the virtual displacements δu , δv and δw are arbitrary, Equations (27) will be satisfied only if the following equations are satisfied:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (28)$$

$$(K_x u - N_x) \Big|_{x=0} = 0 \quad (28a)$$

$$(K_x u + N_x) \Big|_{x=\ell_x} = 0 \quad (28b)$$

$$(S_x u - N_{xy}) \Big|_{y=0} = 0 \quad (28c)$$

$$(S_x u + N_{xy}) \Big|_{y=\ell_y} = 0 \quad (28d)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (29)$$

$$(K_y v - N_y) \Big|_{y=0} = 0 \quad (29a)$$

$$(K_y v + N_y) \Big|_{y=\ell_y} = 0 \quad (29b)$$

$$(S_y v - N_{xy}) \Big|_{x=0} = 0 \quad (29c)$$

$$(S_y v + N_{xy}) \Big|_{x=\ell_x} = 0 \quad (29d)$$

$$\begin{aligned}
D_x \frac{\partial^4 w}{\partial x^4} + 2H_b \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} \\
= q_{x,y} + (N_x \frac{\partial^2 \bar{w}}{\partial x^2} + 2N_{xy} \frac{\partial^2 \bar{w}}{\partial x \partial y} + N_y \frac{\partial^2 \bar{w}}{\partial y^2}) \quad (30)
\end{aligned}$$

$$\left(A_x \frac{\partial w}{\partial x} - D_x \frac{\partial^2 \bar{w}}{\partial x^2} \right) \bigg|_{x=0} = 0 \quad (30a)$$

$$\left(A_x \frac{\partial w}{\partial x} + D_x \frac{\partial^2 \bar{w}}{\partial x^2} \right) \bigg|_{x=\ell_x} = 0 \quad (30b)$$

$$\left(A_y \frac{\partial w}{\partial y} - D_y \frac{\partial^2 \bar{w}}{\partial y^2} \right) \bigg|_{y=0} = 0 \quad (30c)$$

$$\left(A_y \frac{\partial w}{\partial y} + D_y \frac{\partial^2 \bar{w}}{\partial y^2} \right) \bigg|_{y=\ell_y} = 0 \quad (30d)$$

Equations (28) and (29) have been used to simplify Equation (30). By virtue of Equations (17), Equations (28) to (30d) can be expressed solely in terms of the three displacements u , v and w and initial deflection w_0 . Equations (28), (29) and (30) are the equilibrium equations valid in the interior of the plate and Equations (28a) to (28d), (29a) to (29d) and (30a) to (30d) supply the boundary conditions for the displacements. The three equilibrium equations and the twelve boundary conditions obtained above, together with the prescribed condition of zero vertical deflection at the boundary are adequate for

the unique determination of the displacements of the middle surface of the plate under lateral load.

Stress Function

Instead of solving the three non-linear partial differential equations [Equations (28), (29) and (30)] for the unknown displacements u , v and w , the problem may be reduced to that of finding only two unknown quantities by the introduction of Airy stress function. This stress function f is defined such that

$$N_x = \frac{\partial^2 f}{\partial y^2}, \quad N_y = \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad N_{xy} = - \frac{\partial^2 f}{\partial x \partial y} \quad (31)$$

It can be readily verified that the stress function as defined above satisfies the equilibrium equations [Equations (28) and (29)]. Substituting Equations (31) in Equations (13), the strains can be expressed as

$$\left. \begin{aligned} e_x &= \frac{1}{C_x} \cdot \frac{\partial^2 f}{\partial y^2} + \frac{1}{C_1} \cdot \frac{\partial^2 f}{\partial x^2} \\ e_y &= \frac{1}{C_y} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{C_1} \cdot \frac{\partial^2 f}{\partial y^2} \\ \gamma_{xy} &= - \frac{1}{C_{xy}} \cdot \frac{\partial^2 f}{\partial x \partial y} \end{aligned} \right\} \quad (32)$$

By substituting Equations (32) in Equations (15), the displacements u and v can be expressed as functions of f and $\bar{w}$ as

$$\frac{\partial u}{\partial x} = \frac{1}{C_x} \cdot \frac{\partial^2 f}{\partial y^2} + \frac{1}{C_1} \cdot \frac{\partial^2 f}{\partial x^2} - \frac{1}{2} \left(\frac{\partial \bar{w}}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial w_0}{\partial x} \right)^2 \quad (33a)$$

and

$$\frac{\partial v}{\partial y} = \frac{1}{C_y} \cdot \frac{\partial^2 f}{\partial x^2} + \frac{1}{C_1} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{1}{2} \left(\frac{\partial \bar{w}}{\partial y} \right)^2 + \frac{1}{2} \left(\frac{\partial w_0}{\partial y} \right)^2 \quad (33b)$$

Governing Equations in terms of f and $\bar{w}$

One of the governing equations is obtained by substituting Equations (31) in Equation (30), the equilibrium equation for bending of the plate,

$$\begin{aligned} \frac{\partial^4 \bar{w}}{\partial x^4} + 2\lambda_b^2 \mu_b \frac{\partial^4 \bar{w}}{\partial x^2 \partial y^2} + \lambda_b^4 \frac{\partial^4 \bar{w}}{\partial y^4} \\ = \frac{\bar{q}_{x,y}}{D_x} + \frac{1}{D_x} \left[\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial^2 \bar{w}}{\partial x^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 \bar{w}}{\partial x \partial y} + \frac{\partial^2 f}{\partial x^2} \frac{\partial^2 \bar{w}}{\partial y^2} \right] \end{aligned} \quad (34)$$

where

$$\bar{q}_{x,y} = q_{x,y} + D_x \left(\frac{\partial^4 w_0}{\partial x^4} + 2\lambda_b^2 \mu_b \frac{\partial^4 w_0}{\partial x^2 \partial y^2} + \lambda_b^4 \frac{\partial^4 w_0}{\partial y^4} \right)$$

The other equation is furnished by Equation (16), the compatibility relationship for the strains in the middle surface of the plate. Substituting Equations (32) in

Equation (16), gives

$$\begin{aligned}
 & \frac{\partial^4 f}{\partial x^4} + 2\lambda_m^2 \mu_m \frac{\partial^4 f}{\partial x^2 \partial y^2} + \lambda_m^4 \frac{\partial^4 f}{\partial y^4} \\
 &= C_y \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w_0}{\partial x \partial y} - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} \right. \\
 &\quad \left. - \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \\
 &= C_y \left[\left(\frac{\partial^2 \bar{w}}{\partial x \partial y} \right)^2 - \frac{\partial^2 \bar{w}}{\partial x^2} \frac{\partial^2 \bar{w}}{\partial y^2} \right] - C_y \left[\left(\frac{\partial^2 w_0}{\partial x \partial y} \right)^2 - \frac{\partial^2 w_0}{\partial x^2} \frac{\partial^2 w_0}{\partial y^2} \right] \\
 &\quad \text{---(35)}
 \end{aligned}$$

It may be noted that the governing equations depend on the initial deformation w_0 but are independent of u_0 and v_0 .

Boundary Conditions in terms of f and $\bar{w}$.

In the derivation of the boundary conditions it is assumed that the elastic edge restraints, the loading and the initial deformation w_0 are symmetrical about both the centre-lines of the plate. It will be sufficient, therefore, to consider the edges $x=0$ and $y=0$.

The boundary conditions for bending [Equations (30a) to (30d)] remain unaffected by the introduction of the stress function. They are (after rearrangement of certain terms) at $x=0$, $\bar{w}=0$ and $\frac{\partial^2 \bar{w}}{\partial x^2} = \frac{\alpha_x}{\ell_x} \frac{\partial \bar{w}}{\partial x} + \left(\frac{\partial^2 w_0}{\partial x^2} - \frac{\alpha_x}{\ell_x} \frac{\partial w_0}{\partial x} \right)$ (36a)

and

$$\text{at } y = 0, \quad \bar{w} = 0 \quad \text{and} \quad \frac{\partial^2 \bar{w}}{\partial y^2} = \frac{\alpha_y}{\ell_y} \frac{\partial \bar{w}}{\partial y} + \left(\frac{\partial^2 w_0}{\partial y^2} - \frac{\alpha_y}{\ell_y} \frac{\partial w_0}{\partial y} \right) \quad (36b)$$

From Equations (28a) and (29c) the boundary conditions for membrane forces at the edge $x=0$ are given by

$$u = \frac{N_x}{K_x} \quad (37a)$$

and

$$\frac{\partial v}{\partial y} = \frac{1}{S_y} \frac{\partial N_{xy}}{\partial y} \quad (37b)$$

(Equation (37b) has been obtained by differentiating Equation (29c) with respect to y).

Integrating Equation (33a) with respect to x between the limits $x=0$ and $x = \ell_x/2$ (while y remains constant) and noting that because of symmetry $u = \partial f / \partial x = 0$ at $x = \ell_x/2$,

$$\begin{aligned} -u_{0,y} &= \frac{1}{C_x} \int_0^{\ell_x/2} \left. \frac{\partial^2 f}{\partial y^2} \right|_{x,y} dx - \frac{1}{C_1} \left. \frac{\partial f}{\partial x} \right|_{0,y} - \frac{1}{2} \int_0^{\ell_x/2} \left. \left(\frac{\partial \bar{w}}{\partial x} \right)^2 \right|_{x,y} dx \\ &\quad + \frac{1}{2} \int_0^{\ell_x/2} \left. \left(\frac{\partial w_0}{\partial x} \right)^2 \right|_{x,y} dx \\ &= - \frac{1}{K_x} \left. \frac{\partial^2 f}{\partial y^2} \right|_{0,y} \quad [\text{By Equation (37a)}]. \end{aligned}$$

or

$$\begin{aligned}
 & \frac{1}{C_x} \int_0^{\ell/2} \left. \frac{\partial^2 f}{\partial y^2} \right|_{x,y} dx + \left. \frac{1}{K_x} \frac{\partial^2 f}{\partial y^2} \right|_{0,y} - \left. \frac{1}{C_1} \frac{\partial f}{\partial x} \right|_{0,y} \\
 &= \frac{1}{2} \int_0^{\ell/2} \left. \left(\frac{\partial \bar{w}}{\partial x} \right)^2 \right|_{x,y} dx - \frac{1}{2} \int_0^{\ell/2} \left. \left(\frac{\partial w_0}{\partial x} \right)^2 \right|_{x,y} dx
 \end{aligned} \tag{38a}$$

The other boundary condition at the same point $(0,y)$ is obtained from Equations (37b), (33b) and (31) as follows

$$\left. \frac{1}{C_y} \frac{\partial^2 f}{\partial x^2} \right|_{0,y} + \left. \frac{1}{C_1} \frac{\partial^2 f}{\partial y^2} \right|_{0,y} + \left. \frac{1}{S_y} \frac{\partial^3 f}{\partial x \partial y^2} \right|_{0,y} = 0 \tag{38b}$$

The terms $(\partial \bar{w} / \partial y)^2$ and $(\partial w_0 / \partial y)^2$ vanish because $\bar{w} = w_0 = 0$ along the edge.

Similarly, the boundary conditions for membrane forces at the edge $y=0$ can be shown to be

$$\begin{aligned}
 & \frac{1}{C_y} \int_0^{\ell/2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x,y} dy + \left. \frac{1}{K_y} \frac{\partial^2 f}{\partial x^2} \right|_{x,0} - \left. \frac{1}{C_1} \frac{\partial f}{\partial y} \right|_{x,0} \\
 &= \frac{1}{2} \int_0^{\ell/2} \left. \left(\frac{\partial \bar{w}}{\partial y} \right)^2 \right|_{x,y} dy - \frac{1}{2} \int_0^{\ell/2} \left. \left(\frac{\partial w_0}{\partial y} \right)^2 \right|_{x,y} dy
 \end{aligned} \tag{39a}$$

and

$$\left. \frac{1}{C_x} \frac{\partial^2 f}{\partial y^2} \right|_{x,0} + \left. \frac{1}{C_1} \frac{\partial^2 f}{\partial x^2} \right|_{x,0} + \left. \frac{1}{S_x} \frac{\partial^3 f}{\partial x^2 \partial y} \right|_{x,0} = 0 \quad (39b)$$

Conversion of Equations to Non-dimensional Forms.

The governing equations and the boundary conditions obtained in the preceding sections apply only to a particular plate. By writing the equations in a suitable non-dimensional form, it becomes possible to consider the problem of a large number of different but dimensionally similar plates as a single problem. The different factors, that is, the dimensions and properties of the plate, the elastic edge restraints, etc., can then be incorporated in the problem in terms of a minimum number of non-dimensional plate parameters, which may be varied systematically to cover various cases. The number of such parameters can be further reduced by suitable transformations of the axes of co-ordinates⁽²⁰⁾. The transformations and the non-dimensional forms chosen for the deflection, the stress function, the loading and various plate properties are given in the notation. The resulting governing equations are

$$\begin{aligned}
& \frac{\partial^4 \bar{W}}{\partial X^4} + 2 \mu_b \frac{\partial^4 \bar{W}}{\partial X^2 \partial Y^2} + \frac{\partial^4 \bar{W}}{\partial Y^4} \\
& = \frac{\bar{Q}}{L_x^4} + \left[\frac{\partial^2 F}{\partial Y^2} \frac{\partial^2 \bar{W}}{\partial X^2} - 2 \frac{\partial^2 F}{\partial X \partial Y} \frac{\partial^2 \bar{W}}{\partial X \partial Y} + \frac{\partial^2 F}{\partial X^2} \frac{\partial^2 \bar{W}}{\partial Y^2} \right] \quad (40)
\end{aligned}$$

where

$$\bar{Q} = Q + T^I$$

$$\text{and} \quad T^I = L_x^4 \left[\frac{\partial^4 W_0}{\partial X^4} + 2 \mu_b \frac{\partial^4 W_0}{\partial X^2 \partial Y^2} + \frac{\partial^4 W_0}{\partial Y^4} \right]$$

and

$$\frac{\partial^4 F}{\partial X^4} + 2 \mu_m r^2 \frac{\partial^4 F}{\partial X^2 \partial Y^2} + r^4 \frac{\partial^4 F}{\partial Y^4} = r^4 \left[\left(\frac{\partial^2 \bar{W}}{\partial X \partial Y} \right)^2 - \frac{\partial^2 \bar{W}}{\partial X^2} \frac{\partial^2 \bar{W}}{\partial Y^2} \right] + \frac{T^{II}}{L_x^2 L_y^2}$$

---(41)

where

$$T^{II} = - r^4 L_x^2 L_y^2 \left[\left(\frac{\partial^2 W_0}{\partial X \partial Y} \right)^2 - \frac{\partial^2 W_0}{\partial X^2} \frac{\partial^2 W_0}{\partial Y^2} \right]$$

The bending boundary conditions are

$$\text{at } X=0, \quad \bar{W}=0 \quad \text{and} \quad \frac{\partial^2 \bar{W}}{\partial X^2} = \frac{\alpha_x}{L_x} \frac{\partial \bar{W}}{\partial X} + \frac{T^{III}}{L_x^2} \quad (42a)$$

and

$$\text{at } Y=0, \quad \bar{W}=0 \quad \text{and} \quad \frac{\partial^2 \bar{W}}{\partial Y^2} = \frac{\alpha_y}{L_y} \frac{\partial \bar{W}}{\partial Y} + \frac{T^{IV}}{L_y^2} \quad (42b)$$

$$\text{where} \quad T^{III} = L_x^2 \left(\frac{\partial^2 W_0}{\partial X^2} - \frac{\alpha_x}{L_x} \frac{\partial W_0}{\partial X} \right)$$

$$\text{and } T^{IV} = L_y^2 \left(\frac{\partial^2 W_0}{\partial Y^2} - \frac{\alpha_y}{L_y} \frac{\partial W_0}{\partial Y} \right)$$

The boundary conditions for membrane forces are

at $X=0$

$$\begin{aligned} \int_0^{L_x/2} \left. \frac{\partial^2 F}{\partial Y^2} \right|_{X,Y} dX + \left. \frac{L_x}{k_x} \frac{\partial^2 F}{\partial Y^2} \right|_{0,Y} + \frac{\sqrt{v_{xm} v_{ym}}}{r^2} \left. \frac{\partial F}{\partial X} \right|_{0,Y} \\ = \frac{1}{2} \int_0^{L_x/2} \left. \left(\frac{\partial \bar{W}}{\partial X} \right)^2 \right|_{X,Y} dX + \frac{T \bar{V}}{L_x} \end{aligned} \quad (43a)$$

where

$$T \bar{V} = - \frac{L_x}{2} \int_0^{L_x/2} \left. \left(\frac{\partial W_0}{\partial X} \right)^2 \right|_{X,Y} dX$$

and

$$\left. \frac{\partial^2 F}{\partial X^2} \right|_{0,Y} - r^2 \sqrt{v_{xm} v_{ym}} \left. \frac{\partial^2 F}{\partial Y^2} \right|_{0,Y} + \frac{r L_y}{s_y} \left. \frac{\partial^3 F}{\partial X \partial Y^2} \right|_{0,Y} = 0 \quad (43b)$$

and at $Y=0$

$$\begin{aligned} \int_0^{L_y/2} \left. \frac{\partial^2 F}{\partial X^2} \right|_{X,Y} dY + \left. \frac{L_y}{k_y} \frac{\partial^2 F}{\partial X^2} \right|_{X,0} + r^2 \sqrt{v_{xm} v_{ym}} \left. \frac{\partial F}{\partial Y} \right|_{X,0} \\ = \frac{r^4}{2} \int_0^{L_y/2} \left. \left(\frac{\partial \bar{W}}{\partial Y} \right)^2 \right|_{X,Y} dY + \frac{T \bar{VI}}{L_y} \end{aligned} \quad (44a)$$

where $T^{\overline{VI}} = - \frac{r^4 L_y}{2} \int_0^{L_y/2} \left(\frac{\partial W_0}{\partial Y} \right)^2 \Big|_{X,Y} dY$

and

$$\frac{\partial^2 F}{\partial Y^2} \Big|_{X,0} - \frac{\sqrt{\nu_{xm} \nu_{ym}}}{r^2} \frac{\partial^2 F}{\partial X^2} \Big|_{X,0} + \frac{L_x}{rs_x} \frac{\partial^3 F}{\partial X^2 \partial Y} \Big|_{X,0} = 0 \quad (44b)$$

Equations (40) and (41) are the non-dimensional forms of the two fundamental partial differential equations governing the large deflection behaviour of orthotropic plates (with initial deformations) under lateral load. These two equations together with the boundary conditions [Equations (42a) to (44b)] must be solved for the stress function (F) and deflection ($\overline{W}$) for a given load (Q) and a given initial deformation (T^I to T^{VI}). Curvatures and membrane forces can thereafter be obtained in non-dimensional form from the following

$$\left. \begin{aligned} \ell_x^2 \frac{\partial^2 \overline{W}}{\partial x^2} \sqrt{\frac{C_x}{D_x}} &= L_x^2 \frac{\partial^2 \overline{W}}{\partial X^2} \\ \ell_y^2 \frac{\partial^2 \overline{W}}{\partial y^2} \sqrt{\frac{C_y}{D_y}} &= L_y^2 \frac{\partial^2 \overline{W}}{\partial Y^2} \end{aligned} \right\} \text{(Curvatures)} \quad (45)$$

$$\ell_x \ell_y \frac{\partial^2 \overline{W}}{\partial x \partial y} \sqrt{\frac{C_x}{D_x}} = L_x L_y \frac{\partial^2 \overline{W}}{\partial X \partial Y} \quad \text{(Twist)} \quad (46)$$

$$\left. \begin{aligned}
 \ell^2_x \frac{N_x}{D_x} &= L_x^2 \frac{\partial^2 F}{\partial Y^2} \\
 \ell^2_y \frac{N_y}{D_y} &= L_y^2 \frac{\partial^2 F}{\partial X^2} \\
 \ell_x \ell_y \frac{N_{xy}}{\sqrt{D_x D_y}} &= -L_x L_y \frac{\partial^2 F}{\partial X \partial Y}
 \end{aligned} \right\} \begin{array}{l} \text{(Membrane} \\ \text{Forces)} \end{array} \quad (47)$$

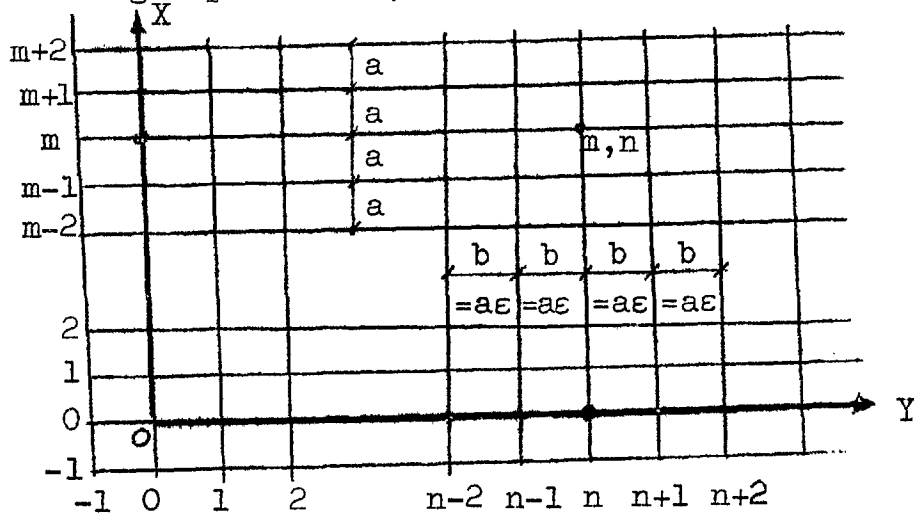
FINITE DIFFERENCE SOLUTION OF LARGE DEFLECTION PROBLEM

There is no general solution available for the differential equations obtained in the previous section. The problem can, however, be solved by converting the differential equations to difference equations and then by solving the resulting non-linear algebraic equations by a method of iteration. In this section the finite difference equations corresponding to the differential equations will be derived.

Finite Difference Approximations

The transformed plate is divided into rectangular meshes of size (a x b). Referring to the diagram below, the finite difference approximations for the derivatives of a function ϕ at the node (m,n) are given by the

following expressions,



$$\frac{\partial \varphi}{\partial X} = \frac{1}{2a} (\varphi_{m+1,n} - \varphi_{m-1,n}) \quad (48a)$$

$$\frac{\partial \varphi}{\partial Y} = \frac{1}{2b} (\varphi_{m,n+1} - \varphi_{m,n-1}) \quad (48b)$$

$$\frac{\partial^2 \varphi}{\partial X^2} = \frac{1}{a^2} (\varphi_{m+1,n} - 2\varphi_{m,n} + \varphi_{m-1,n}) \quad (48c)$$

$$\frac{\partial^2 \varphi}{\partial Y^2} = \frac{1}{b^2} (\varphi_{m,n+1} - 2\varphi_{m,n} + \varphi_{m,n-1}) \quad (48d)$$

$$\frac{\partial^2 \varphi}{\partial X \partial Y} = \frac{1}{4ab} (\varphi_{m+1,n+1} - \varphi_{m-1,n+1} + \varphi_{m-1,n-1} - \varphi_{m+1,n-1}) \quad (48e)$$

$$\frac{\partial^4 \varphi}{\partial X^4} = \frac{1}{a^4} (\varphi_{m+2,n} - 4\varphi_{m+1,n} + 6\varphi_{m,n} - 4\varphi_{m-1,n} + \varphi_{m-2,n}) \quad (48f)$$

$$\frac{\partial^4 \varphi}{\partial Y^4} = \frac{1}{b^4} (\varphi_{m,n+2} - 4\varphi_{m,n+1} + 6\varphi_{m,n} - 4\varphi_{m,n-1} + \varphi_{m,n-2}) \quad (48g)$$

$$\begin{aligned}\frac{\partial^4 \varphi}{\partial X^2 \partial Y^2} &= \frac{1}{a^2 b^2} [4\varphi_{m,n} - 2(\varphi_{m+1,n} + \varphi_{m,n+1} + \varphi_{m-1,n} + \varphi_{m,n-1}) \\ &\quad + (\varphi_{m+1,n+1} + \varphi_{m-1,n+1} + \varphi_{m-1,n-1} + \varphi_{m+1,n-1})]\end{aligned}\quad (48h)$$

$$\begin{aligned}\frac{\partial^3 \varphi}{\partial X \partial Y^2} &= \frac{1}{2ab^2} (\varphi_{m+1,n+1} - 2\varphi_{m+1,n} + \varphi_{m+1,n-1} - \varphi_{m-1,n+1} \\ &\quad + 2\varphi_{m-1,n} - \varphi_{m-1,n-1})\end{aligned}\quad (48i)$$

$$\begin{aligned}\frac{\partial^3 \varphi}{\partial X^2 \partial Y} &= \frac{1}{2a^2 b} (\varphi_{m+1,n+1} - 2\varphi_{m,n+1} + \varphi_{m-1,n+1} - \varphi_{m+1,n-1} \\ &\quad + 2\varphi_{m,n-1} - \varphi_{m-1,n-1})\end{aligned}\quad (48j)$$

The following two more accurate formulae for first derivatives will be used for slopes at the edges $m=0$ and $n=0$ of the deflected plate⁽⁵³⁾

$$\left(\frac{\partial \bar{W}}{\partial X} \right)_{0,n} = \frac{1}{6a} (- \bar{W}_{2,n} + 6\bar{W}_{1,n} - 3\bar{W}_{0,n} - 2\bar{W}_{-1,n}) \quad (49a)$$

and

$$\left(\frac{\partial \bar{W}}{\partial Y} \right)_{m,0} = \frac{1}{6b} (- \bar{W}_{m,2} + 6\bar{W}_{m,1} - 3\bar{W}_{m,0} - 2\bar{W}_{m,-1}) \quad (49b)$$

Finite Difference Equations for Bending Action

From Equations (42a), (42b), (48c), (48d), (49a) and (49b), the boundary conditions are at $X = 0$, (i.e. $m = 0$)

$$\bar{W}_{0,n} = 0 \quad (50a)$$

and
$$\bar{W}_{-1,n} = b_5 \bar{W}_{1,n} + b_6 \bar{W}_{2,n} + t_3 T^{III} \quad (50b)$$

and at $Y = 0$, (i.e. $n = 0$)

$$\bar{W}_{m,0} = 0 \quad (51a)$$

and
$$\bar{W}_{m,-1} = b_7 \bar{W}_{m,1} + b_8 \bar{W}_{m,2} + t_4 T^{IV} \quad (51b)$$

where

$$\left. \begin{aligned} b_5 &= (3\alpha_x - 6i)/(\alpha_x + 6i) \\ b_6 &= -0.5\alpha_x/(\alpha_x + 6i) \\ b_7 &= (3\alpha_y - 6i)/(\alpha_y + 6i) \\ b_8 &= -0.5\alpha_y/(\alpha_y + 6i) \\ t_3 &= 1.5/i(\alpha_x + 6i) \\ \text{and } t_4 &= 1.5/i(\alpha_y + 6i) \end{aligned} \right\}$$

The finite difference form of Equation (40) (Equilibrium Equation) at any internal node point (m,n) (where $m,n = 1,2,3,\dots,i$, because, due to symmetry, it will suffice to consider only a quarter of the plate) is

$$\bar{W} = \frac{\bar{Q} \varepsilon^4}{16i^4} \quad (52)$$

where the left hand side diagram represents an operation on $\bar{W}$ and

$$b_0 = 6\varepsilon^4 + 8\mu_b \varepsilon^2 + 6$$

$$b_1 = \varepsilon^4$$

$$b_2 = 2\mu_b \varepsilon^2$$

$$b_3 = -4\mu_b \varepsilon^2 - 4$$

$$b_4 = -4\varepsilon^4 - 4\mu_b \varepsilon^2$$

$$\beta_x = -\varepsilon^2 (F_{m,n+1} - 2F_{m,n} + F_{m,n-1})$$

$$\beta_y = -\varepsilon^2 (F_{m+1,n} - 2F_{m,n} + F_{m-1,n})$$

$$\text{and } \beta_{xy} = 0.125 \varepsilon^2 (F_{m+1,n+1} - F_{m-1,n+1} + F_{m-1,n-1} - F_{m+1,n-1})$$

The number of internal node points being i^2 , there will be only i^2 such equations [Equation (52)], which will involve as unknown not only deflections at the i^2 internal node points, but also those on lines $m = -1$, $m = 0$, $m = i+1$, $m = i+2$ and $n = -1$, $n = 0$, $n = i+1$ and $n = i+2$. These additional unknown deflections can be expressed in terms of the deflections at the internal points with the help of Equations (50) and (51) and from the consideration of symmetry about both the longitudinal and transverse centre-lines. If the stress function F is known at all points, the set of i^2 equations will be sufficient to determine the i^2 deflections, namely,

$$(\bar{w}_{1,1}), (\bar{w}_{1,2}), \dots (\bar{w}_{1,i})$$

$$(\bar{w}_{2,1}), (\bar{w}_{2,2}), \dots (\bar{w}_{2,i})$$

$$\dots \dots \dots$$

$$(\bar{w}_{i,1}), (\bar{w}_{i,2}), \dots (\bar{w}_{i,i}).$$

For example, Equation (52) for the point (1,1) becomes

$$\begin{aligned}
& b_1 \bar{w}_{3,1} + (b_1 b_6 + b_4 + \beta_x) \bar{w}_{2,1} + (b_2 + \beta_{xy}) \bar{w}_{2,2} \\
& + (b_1 b_5 + b_7 + b_0 - 2\beta_x - 2\beta_y) \bar{w}_{1,1} + (b_8 + b_3 + \beta_y) \bar{w}_{1,2} \\
& + \bar{w}_{1,3} = \frac{\bar{Q}_{1,1} \varepsilon^4}{16i^4} - (b_1 t_3 T^{III} + t_4 T^{IV})
\end{aligned}$$

where

$$\left. \begin{aligned}
\beta_x &= -\varepsilon^2 (F_{1,2} - 2F_{1,1} + F_{1,0}) \\
\beta_y &= -\varepsilon^2 (F_{2,1} - 2F_{1,1} + F_{0,1}) \\
\text{and } \beta_{xy} &= 0.125 \varepsilon^2 (F_{2,2} - F_{0,2} + F_{0,0} - F_{2,0})
\end{aligned} \right\}$$

and the same equation for the centre of the plate, that is, point (i,i) becomes

$$\begin{aligned}
& 2\bar{w}_{i,i-2} + 2(b_3 + \beta_y) \bar{w}_{i,i-1} + (b_0 - 2\beta_x - 2\beta_y) \bar{w}_{i,i} + 4b_2 \bar{w}_{i-1,i-1} \\
& + 2(b_4 + \beta_x) \bar{w}_{i-1,i} + 2b_1 \bar{w}_{i-2,i} = \frac{\bar{Q}_{i,i} \varepsilon^4}{16i^4}
\end{aligned}$$

$$\left. \begin{aligned}
\text{where } \beta_x &= -2\varepsilon^2 (F_{i,i-1} - F_{i,i}) \\
\text{and } \beta_y &= -2\varepsilon^2 (F_{i-1,i} - F_{i,i})
\end{aligned} \right\}$$

In matrix notation, this set of equations can be written as

$$A_f \bar{W} = \bar{Q} \quad (53a)$$

$$\text{and its solution is } \bar{W} = A_f^{-1} \bar{Q} \quad (53b)$$

where

A_f is a square matrix ($p \times p$) dependent on F .

$\bar{W}$ is the unknown column matrix ($p \times 1$)

and $\bar{Q}$ is the given column matrix ($p \times 1$) representing the given loading and the initial deformations.

Finite Difference Equations for Membrane Action

In obtaining the finite difference forms of the boundary conditions, the integrals in Equations (43a) and (44a) are evaluated by Simpson's rule, assuming i to be even.

At $X = 0$, (i.e. at $m = 0$), the boundary conditions obtained from Equations (43a) and (43b) respectively, are

$$\begin{aligned}
& (F_{0,n+1} - 2F_{0,n} + F_{0,n-1}) + 4d_5 \sum_{m=1,3}^{i-1} (F_{m,n+1} - 2F_{m,n} + F_{m,n-1}) \\
& + 2d_5 \sum_{m=2,4}^{i-2} (F_{m,n+1} - 2F_{m,n} + F_{m,n-1}) + d_5 (F_{i,n+1} - 2F_{i,n} + F_{i,n-1}) \\
& + d_6 (F_{1,n} - F_{-1,n}) = d_7 [(b_9 \bar{W}_{1,n} - b_6 \bar{W}_{2,n} - t_3 T^{III})^2 + 4\bar{W}_{2,n}^2 \\
& + 2 \sum_{m=2,4}^{i-2} (\bar{W}_{m+1,n} - \bar{W}_{m-1,n})^2 + 4 \sum_{m=3,5}^{i-1} (\bar{W}_{m+1,n} - \bar{W}_{m-1,n})^2] \\
& + t_5 T^V \tag{54a}
\end{aligned}$$

and

$$\begin{aligned}
& F_{1,n} + d_8 F_{0,n} + d_9 F_{-1,n} + d_{10} (F_{0,n+1} + F_{0,n-1}) \\
& + d_{11} (F_{1,n+1} + F_{1,n-1} - F_{-1,n+1} - F_{-1,n-1}) = 0 \tag{54b}
\end{aligned}$$

where

$$\left. \begin{aligned}
b_9 &= 1 - b_5 \\
d_5 &= k_x / (k_x + 6i) \\
d_6 &= 1.5\epsilon^2 \sqrt{v_{xm} v_{ym}} k_x / r^2 (k_x + 6i) \\
d_7 &= 0.125\epsilon^2 k_x / (k_x + 6i) \\
t_5 &= 1.5\epsilon^2 k_x / i(k_x + 6i) \\
d_8 &= 2s_y (r^2 \sqrt{v_{xm} v_{ym}} - \epsilon^2) / \epsilon(\epsilon s_y - 2ir) \\
d_9 &= (\epsilon s_y + 2ir) / (\epsilon s_y - 2ir) \\
d_{10} &= -s_y r^2 \sqrt{v_{xm} v_{ym}} / \epsilon(\epsilon s_y - 2ir) \\
d_{11} &= ir / (\epsilon s_y - 2ir)
\end{aligned} \right\}$$

The finite difference approximations given by Equations (48a) and (48d) together with the boundary conditions for $\bar{W}$ [Equations (50)] and the condition of symmetry about the centre-line $m=i$ are used to obtain Equation (54a). Equation (54b) is obtained by Equations (48c), (48d) and (48i). In the particular case of $(\epsilon s_y - 2ir) \rightarrow 0$, the term $F_{1,n}$ vanishes from Equation (54b), which then becomes

$$d'_8 F_{0,n} + d'_9 F_{-1,n} + d'_{10} (F_{0,n+1} + F_{0,n-1}) + d'_{11} (F_{1,n+1} + F_{1,n-1} - F_{-1,n+1} - F_{-1,n-1}) = 0 \quad (54b')$$

where

$$\left. \begin{aligned} d'_8 &= 2s_y (r^2 \sqrt{v_{xm} v_{ym}} - \epsilon^2) \\ d'_9 &= \epsilon(\epsilon s_y + 2ir) \\ d'_{10} &= -s_y r^2 \sqrt{v_{xm} v_{ym}} \\ d'_{11} &= \epsilon ir. \end{aligned} \right\}$$

Similarly Equations (44a) and (44b) respectively will give the following two equations at $Y = 0$, (i.e. at $n=0$)

$$\begin{aligned}
& (F_{m+1,0} - 2F_{m,0} + F_{m-1,0}) + 4d_{12} \sum_{n=1,3}^{i-1} (F_{m+1,n} - 2F_{m,n} + F_{m-1,n}) \\
& + 2d_{12} \sum_{n=2,4}^{i-2} (F_{m+1,n} - 2F_{m,n} + F_{m-1,n}) + d_{12}(F_{m+1,i} - 2F_{m,i} + F_{m-1,i}) \\
& + d_{13}(F_{m,1} - F_{m,-1}) = d_{14} [(b_{11}\bar{W}_{m,1} - b_8\bar{W}_{m,2} - t_4 T^{IV})^2 + 4\bar{W}_{m,2}^2 \\
& + 2 \sum_{n=2,4}^{i-2} (\bar{W}_{m,n+1} - \bar{W}_{m,n-1})^2 + 4 \sum_{n=3,5}^{i-1} (\bar{W}_{m,n+1} - \bar{W}_{m,n-1})^2] \\
& + t_6 T^{VI} \tag{55a}
\end{aligned}$$

and

$$\begin{aligned}
& F_{m,1} + d_{15}F_{m,0} + d_{16}F_{m,-1} + d_{17}(F_{m+1,0} + F_{m-1,0}) \\
& + d_{18}(F_{m+1,1} + F_{m-1,1} - F_{m+1,-1} - F_{m-1,-1}) = 0 \tag{55b}
\end{aligned}$$

where

$$\left. \begin{aligned}
b_{11} &= 1 - b_7 \\
d_{12} &= k_y / (k_y + 6i) \\
d_{13} &= 1.5r^2 \sqrt{v_{xm} v_{ym}} k_y / \varepsilon^2(k_y + 6i) \\
d_{14} &= 0.125r^4 k_y / \varepsilon^2(k_y + 6i) \\
t_6 &= 1.5k_y / i\varepsilon^2(k_y + 6i) \\
d_{15} &= 2s_x(\varepsilon^2 \sqrt{v_{xm} v_{ym}} - r^2) / r(rs_x - 2\varepsilon i) \\
d_{16} &= (rs_x + 2\varepsilon i) / (rs_x - 2\varepsilon i) \\
d_{17} &= -\varepsilon^2 s_x \sqrt{v_{xm} v_{ym}} / r(rs_x - 2\varepsilon i) \\
d_{18} &= \varepsilon i / (rs_x - 2\varepsilon i)
\end{aligned} \right\}$$

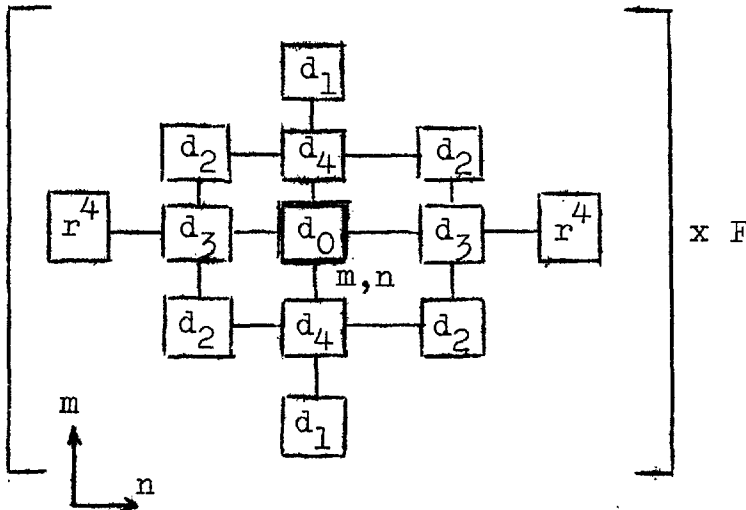
In the particular case of $(rs_x - 2\epsilon i) \rightarrow 0$, Equation (55b) can be written as

$$d'_{15}F_{m,0} + d'_{16}F_{m,-1} + d'_{17}(F_{m+1,0} + F_{m-1,0}) + d'_{18}(F_{m+1,1} + F_{m-1,1} - F_{m+1,-1} - F_{m-1,-1}) = 0 \quad (55b')$$

where

$$\left. \begin{aligned} d'_{15} &= 2s_x(\epsilon^2 \sqrt{v_{xm} v_{ym}} - r^2) \\ d'_{16} &= r(rs_x + 2\epsilon i) \\ d'_{17} &= -\epsilon^2 s_x \sqrt{v_{xm} v_{ym}} \\ d'_{18} &= \epsilon i r \end{aligned} \right\}$$

At any internal node point (m,n) (where $m,n=1,2,\dots,i$), the finite difference form of the compatibility equation [Equation (41)] is given by



$$\begin{aligned}
&= \frac{r^4 \epsilon^2}{16} [(\bar{W}_{m+1,n+1} - \bar{W}_{m-1,n+1} + \bar{W}_{m-1,n-1} - \bar{W}_{m+1,n-1})^2 \\
&- 16(\bar{W}_{m+1,n} - 2\bar{W}_{m,n} + \bar{W}_{m-1,n})(\bar{W}_{m,n+1} - 2\bar{W}_{m,n} + \bar{W}_{m,n-1}) + \frac{T^{II}}{i^4 r^4}]
\end{aligned}
\tag{56}$$

where the left hand side represents an operation on F and

$$\left. \begin{aligned}
d_0 &= 6\epsilon^4 + 8\mu_m \epsilon^2 r^2 + 6r^4 \\
d_1 &= \epsilon^4 \\
d_2 &= 2\mu_m \epsilon^2 r^2 \\
d_3 &= -4\mu_m \epsilon^2 r^2 - 4r^4 \\
d_4 &= -4\epsilon^4 - 4\mu_m \epsilon^2 r^2
\end{aligned} \right\}$$

The set of Equations (56) contains i^2 equations, but the number of unknown F 's in the set is greater than i^2 , as the stress function on and outside the edges, as well as those outside the centre-lines, are involved, although the latter can be eliminated by the conditions of symmetry about both centre-lines. The stress functions on and outside the edges can be expressed in terms of stress functions at the internal node points by solving algebraically the four sets of boundary equations [Equations (54a), (54b), (55a) and (55b)]. In view of the difficulty of this procedure the equations governing the boundary conditions

will be solved simultaneously with the compatibility equations at the internal node points, to determine the values of stress function at internal node points as well as on and outside the edges.

The following table shows the types of equation that will be used for the solution of F .

Type of Eqn.	At points...	No. of Eqn.	Remark
Equation (56)	$m, n=1, 2, \dots, i$	i^2	At internal nodes
Equation (54a)	$m=0; n=0, 1, 2, \dots, i$	$(i+1)$	At boundary $X=0$
Equation (54b)	$m=0; n=1, 2, \dots, i$	i	At boundary $X=0$
Equation (55a)	$m=0, 1, 2, \dots, i; n=0$	$(i+1)$	At boundary $Y=0$
Equation (55b)	$m=1, 2, \dots, i; n=0$	i	At boundary $Y=0$

$$\text{Total number of Equations} = (i+2)^2 - 2$$

The number of unknown F 's involved in the above set of equations is $(i+2)^2 - 1$. They are

$$\left. \begin{array}{l} (F_{-1,0}), (F_{-1,1}), (F_{-1,2}), \dots, (F_{-1,i}) \\ (F_{0,-1}), (F_{0,0}), (F_{0,1}), (F_{0,2}), \dots, (F_{0,i}) \\ (F_{1,-1}), (F_{1,0}), (F_{1,1}), (F_{1,2}), \dots, (F_{1,i}) \\ \dots\dots\dots \\ (F_{i,-1}), (F_{i,0}), (F_{i,1}), (F_{i,2}), \dots, (F_{i,i}) \end{array} \right\}$$

The magnitude of membrane forces remains unaffected if the stress function is given an arbitrary value at a point. Therefore the additional equation required is provided by assuming $F = 0$ at the corner of the plate, that is,

$$F_{0,0} = 0 \quad (57)$$

For the solution of the equations by partitioning, it is convenient to have an even number of equations in the set; the following equation will complete the set without in anyway affecting the solution.

$$F_{-1,-1} = 0 \quad (58)$$

In matrix notation the set of equations can be written as

$$B F = C_{\bar{W}} \quad (59a)$$

$$\text{and its solution is } F = B^{-1} C_{\bar{W}} \quad (59b)$$

where

B is a square matrix ($p' \times p'$)

F is the unknown column matrix ($p' \times 1$)

and $C_{\bar{W}}$ is a column matrix ($p' \times 1$) dependent on $\bar{W}$ and given initial deformations.

SOLUTION BY DIGITAL COMPUTER

In the foregoing sections the problem of large deflection behaviour of orthotropic plates with initial deformations has been reduced to the problem of simultaneous solution of two sets of algebraic equations for any given intensity of loading (Q) and initial deformations (W_0). The solution is obtained by a method of successive approximation which is fairly convergent and which can be carried to any desired degree of accuracy. It may be noted that when numerical values are assigned to (F) , Equations (53a) become linear and similarly Equations (59a) become linear when $(\bar{W})$ is assigned numerical values.

Step 1). Assume $(F) = 0$. Find $(\bar{W})^{1,1}$ from Equation (53b) [Note that $(\bar{W})^{1,1}$ represents the small deflection solution].

Step 2). Assume $(\bar{W}) = (\bar{W})^{1,1}$. Find $(F)^{1,1}$ from Equation (59b).

Step 3). Assume $(F) = (F)^{1,1}$. Find $(\bar{W})^{1,2}$ from Equation (53b).

$$\text{Calculate } (\bar{W})_{\text{mean}}^{1,2} = 0.5(\bar{W})^{1,1} + 0.5(\bar{W})^{1,2}.$$

Step 4). Assume $(\bar{W}) = (\bar{W})_{\text{mean}}^{1,2}$. Find $(F)^{1,2}$ from Equation (59b).

Step 5). Compare each deflection of $(\bar{W})^{1,2}$ with the corresponding deflection $(\bar{W})_{\text{mean}}^{1,2}$. If the difference is in no case greater than the specified percentage of the latter, $(\bar{W})_{\text{mean}}^{1,2}$ and $(F)^{1,2}$ represent the required solution, correct to the degree of accuracy specified.

If, however, the above condition is not satisfied at every point, the calculation must be continued through the following steps.

Step 6). Assume $(F) = (F)^{1,2}$. Find $(\bar{W})^{1,3}$ from Equation (53b). Calculate $(\bar{W})_{\text{mean}}^{1,3} = 0.5(\bar{W})_{\text{mean}}^{1,2} + 0.5(\bar{W})^{1,3}$.

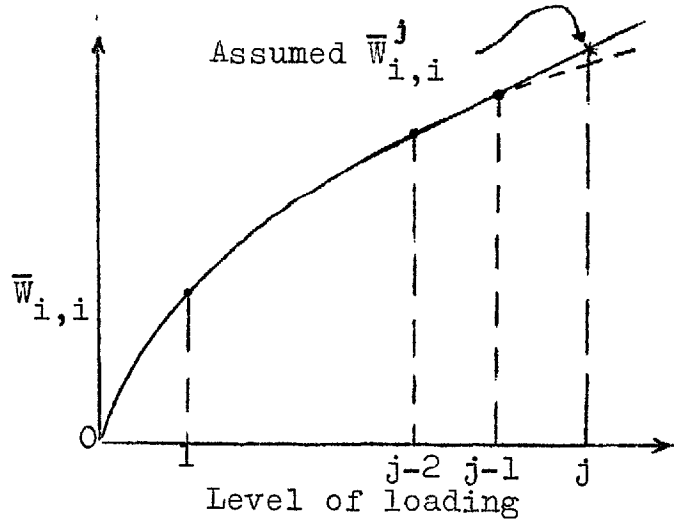
Step 7). Assume $(\bar{W}) = (\bar{W})_{\text{mean}}^{1,3}$. Find $(F)^{1,3}$ from Equation (59b).

Step 8). Compare each deflection of $(\bar{W})^{1,3}$ with the corresponding deflection $(\bar{W})_{\text{mean}}^{1,3}$. If the difference, and so on till the required degree of accuracy is attained.

Now, suppose that the correct deflection and stress function obtained for the level of load intensity $(Q)^{j-1}$ are given by $(\bar{W})^{j-1,s}$ and $(F)^{j-1,s}$. To find the solution for the next increment of load intensity [i.e. for $(Q)^j$], the central deflection $\bar{W}_{i,i}$ is obtained, as a first approximation, by straight line extrapolation of the central deflections for load intensities $(Q)^{j-2}$ and $(Q)^{j-1}$,

i.e.

$$\begin{aligned}\bar{W}_{i,i}^j &= \left[2 - \frac{\bar{W}_{i,i}^{j-2}}{\bar{W}_{i,i}^{j-1}} \right] \bar{W}_{i,i}^{j-1} \\ &= Z \bar{W}_{i,i}^{j-1}\end{aligned}$$



After calculating Z the procedure can be summarised as follows

- 1) Calculate $(\bar{W})^{j,s} = Z(\bar{W})^{j-1,s}$ and $(F)^{j,s} = Z^2(F)^{j-1,s}$.
- 2) Assume $(F) = (F)^{j,s}$. Find $(\bar{W})^{j,s+1}$ from Equation (53b). Calculate $(\bar{W})_{\text{mean}}^{j,s+1} = 0.5(\bar{W})^{j,s} + 0.5(\bar{W})^{j,s+1}$.
- 3) Assume $(\bar{W}) = (\bar{W})_{\text{mean}}^{j,s+1}$. Find $(F)^{j,s+1}$ from Equation (59b).
- 4) Compare each deflection of $(\bar{W})^{j,s+1}$ with the corresponding deflections $(\bar{W})_{\text{mean}}^{j,s+1}$ and so on as in the previous step 5).

All the calculations detailed in the preceding paragraphs are done with the help of a digital computer. The computer 'program' has been developed for the particular case of an initially flat plate. The computer not only solves the two sets of linear equations, it also forms the necessary matrices themselves and carries out all the intermediate calculations and checks automatically. In short, given the fundamental plate properties, the type and magnitude of loading, size of mesh and degree of accuracy desired, the computer prints out the final non-dimensional deflections and stress functions for each level of load intensity. The scheme of the computer program is shown in Fig. 3 with the help of a simplified flow diagram.

Finally, the non-dimensional curvatures, twist and membrane forces at (m,n) can be obtained from the following expressions

$$\begin{aligned}
 \ell_x^2 \partial^2 w / \partial x^2 \cdot \sqrt{C_x / D_x} &= 4i^2 (W_{m+1,n} - 2W_{m,n} + W_{m-1,n}) \\
 \ell_y^2 \partial^2 w / \partial y^2 \cdot \sqrt{C_x / D_x} &= 4i^2 (W_{m,n+1} - 2W_{m,n} + W_{m,n-1}) \\
 \ell_x \ell_y \partial^2 w / \partial x \partial y \cdot \sqrt{C_x / D_x} &= i^2 (W_{m+1,n+1} - W_{m-1,n+1} + W_{m-1,n-1} - W_{m+1,n-1}) \\
 \ell_x^2 N_x / D_x &= 4i^2 / \varepsilon^2 \cdot (F_{m,n+1} - 2F_{m,n} + F_{m,n-1}) \\
 \ell_y^2 N_y / D_y &= 4i^2 \varepsilon^2 (F_{m+1,n} - 2F_{m,n} + F_{m-1,n}) \\
 \ell_x \ell_y N_{xy} / \sqrt{D_x D_y} &= -i^2 (F_{m+1,n+1} - F_{m-1,n+1} + F_{m-1,n-1} - F_{m+1,n-1})
 \end{aligned}$$

APPENDIX - BMETHOD OF PRESENTING THE THEORETICAL
RESULTS OF THE ORTHOTROPIC PLATE ANALYSISNOTATION

h_{eq}	Equivalent thickness of stiffened or corrugated plate.	
m_x	$= - D_x(\partial^2 w / \partial x^2)$	} Bending moments in x and y direction and twisting moment, neglecting Poisson's ratio in bending.
m_y	$= - D_y(\partial^2 w / \partial y^2)$	
m_{xy}	$= H_b(\partial^2 w / \partial x \partial y)$	
$\sigma'_{xb}, \sigma'_{yb}$	Extreme fibre bending stresses due to m_x and m_y .	
τ_b'	Extreme fibre shear stress due to twisting moment m_{xy} .	
σ_{xb}, σ_{yb}	Extreme fibre bending stresses due to M_x and M_y .	
τ_b	Extreme fibre shear stress due to twisting moment M_{xy} .	
σ_{xm}, σ_{ym}	Membrane stresses in x and y directions.	
τ_m	Membrane shear stress.	
σ_{xs}	$= \sigma_{xb} + \sigma_{xm}$	} Total surface stresses due to bending and membrane action.
σ_{ys}	$= \sigma_{yb} + \sigma_{ym}$	
τ_s	$= \tau_b + \tau_m$	
P	$= q \ell_x \ell_y / 4i^2$	Magnitude of patch load at any node of the finite difference mesh, distributed over one mesh area.

(For other notation, see APPENDIX A).

All the rigidity terms appearing in the non-dimensional forms for pressure, deflection, curvatures, twist and membrane forces (APPENDIX A) can be replaced by a single rigidity term (D_x) and the plate thickness (h), for isotropic and ideally orthotropic plates. For stiffened and corrugated plates the thickness of the equivalent plate (h_{eq}) is not a physical quantity and the following formula is suggested for its evaluation

$$h_{eq} = \sqrt{12} \sqrt[4]{D_x D_y / E_x E_y} = [\sqrt{12}(1 - \nu_{xm} \nu_{ym}) D_x / C_x] / r \quad -- (60)$$

It can be readily verified that for isotropic and ideally orthotropic plates, h_{eq} reduces to h .

According to this definition [Equation (60)], the non-dimensional pressure, deflection, etc., are given by the following equations,

$$\frac{q \ell_x^4}{D_x h_{eq}} = \frac{r}{\sqrt{12}(1 - \nu_{xm} \nu_{ym})} Q \quad (\text{Distributed load}) \quad -- (61a)$$

$$\frac{P \ell_x^2}{D_x h_{eq}} \left(\frac{\ell_x}{\ell_y} \right) = \frac{r}{4i^2 \sqrt{12}(1 - \nu_{xm} \nu_{ym})} Q \quad (\text{Patch load}) \quad -- (61b)$$

$$\frac{w}{h_{eq}} = \frac{r}{\sqrt{12}(1 - \nu_{xm} \nu_{ym})} W \quad -- (62)$$

$$\frac{m_x \ell_x^2}{D_x h_{eq}} = \frac{4i^2 r}{\sqrt{12}(1 - \nu_{xm} \nu_{ym})} (-W_{m+1,n} + 2W_{m,n} - W_{m-1,n}) \quad -- (62a)$$

$$\frac{m_y \ell_x^2}{D_x h_{eq}} \left(\frac{\ell_x}{\ell_y} \right)^2 = \frac{4i^2 r}{\varepsilon^4 \sqrt{12(1 - \nu_{xm} \nu_{ym})}} (-W_{m,n+1} + 2W_{m,n} - W_{m,n-1})$$

--- (62b)

$$\frac{m_{xy} \ell_x^2}{D_x h_{eq}} \left(\frac{\ell_x}{\ell_y} \right) = \frac{\mu_b i^2 r}{\varepsilon^2 \sqrt{12(1 - \nu_{xm} \nu_{ym})}} (W_{m+1,n+1} - W_{m-1,n+1} + W_{m-1,n-1} - W_{m+1,n-1})$$

--- (62c)

$$\frac{f}{D_x} \left(\frac{\ell_x}{\ell_y} \right)^2 = \frac{F}{\varepsilon^2}$$

--- (63)

$$\frac{N_{xx} \ell_x^2}{D_x} = \frac{4i^2}{\varepsilon^2} (F_{m,n+1} - 2F_{m,n} + F_{m,n-1})$$

--- (63a)

$$\frac{N_{yy} \ell_x^2}{D_x} \left(\frac{\ell_x}{\ell_y} \right)^2 = \frac{4i^2}{\varepsilon^2} (F_{m+1,n} - 2F_{m,n} + F_{m-1,n})$$

--- (63b)

$$\frac{N_{xy} \ell_x^2}{D_x} \left(\frac{\ell_x}{\ell_y} \right) = \frac{i^2}{\varepsilon^2} (-F_{m+1,n+1} + F_{m-1,n+1} - F_{m-1,n-1} + F_{m+1,n-1})$$

--- (63c)

The following formulae are used to calculate the correct values of bending and twisting moments in a given plate

$$M_x = m_x + \nu_{xb} m_y$$

$$M_y = m_y + \nu_{yb} m_x$$

and $M_{xy} = (2D_{xy}/H_b) m_{xy}.$

At the edges of the plate

$$M_x = m_x$$

and $M_y = m_y.$

For isotropic plates

$$D_x = H_b = Eh^3/12(1-\nu^2)$$

$$\ell_y / \ell_x = \epsilon$$

$$h_{eq} = h$$

and $\nu_{xb} = \nu_{yb} = \nu$.

Expressing the bending and twisting moments and the membrane forces in terms of the appropriate stresses, the left-hand-side terms of Equations (61a) to (63c) respectively become

$$\begin{aligned} & 12(1-\nu^2) \frac{q \ell_x^4}{Eh^4}, \quad \frac{12(1-\nu^2)}{\epsilon} \frac{P \ell_x^2}{Fh^4}, \\ & \frac{W}{h}, \quad 2(1-\nu^2) \frac{\sigma'_{xb} \ell_x^2}{Eh^2}, \quad \frac{2(1-\nu^2)}{\epsilon^2} \frac{\sigma'_{yb} \ell_x^2}{Eh^2}, \quad \frac{2(1-\nu^2)}{\epsilon} \frac{\tau'_b \ell_x^2}{Eh^2}, \\ & \frac{12(1-\nu^2)}{\epsilon^2} \frac{f}{Eh^3}, \quad 12(1-\nu^2) \frac{\sigma_{xm} \ell_x^2}{Eh^2}, \quad \frac{12(1-\nu^2)}{\epsilon^2} \frac{\sigma_{ym} \ell_x^2}{Eh^2} \\ & \text{and } \frac{12(1-\nu^2)}{\epsilon} \frac{\tau_m \ell_x^2}{Eh^2}. \end{aligned}$$

The following formulae are used to calculate the correct values of the bending stresses and the shear stress due to torsion in a given plate

$$\sigma_{xb} = \sigma'_{xb} + \nu \sigma'_{yb}$$

$$\sigma_{yb} = \sigma'_{yb} + \nu \sigma'_{xb}$$

and $\tau_b = (1-\nu)\tau'_b$

At the edges of the plate

$$\sigma_{xb} = \sigma'_{xb} \quad \text{and} \quad \sigma_{yb} = \sigma'_{yb}$$

APPENDIX - CAPPLICATION OF THE ORTHOTROPIC PLATE THEORY
TO CORRUGATED PLATESExpressions for Plate Rigidities

The following expressions for the rigidities and Poisson's ratios for corrugated plates with corrugations running in the x direction, have been taken from ref.(40) after appropriate alterations in notation. These expressions enable the corrugated plate to be represented as an equivalent orthotropic plate.

$$\begin{array}{rcl}
 B_x & = & EI \\
 B_y & = & Et^3/12\beta \\
 D_{xy} \text{ or } B_{xy} & = & (1 + \beta)Et^3/48(1 + \nu) \\
 \nu_{xb} & = & \nu \\
 C_x & = & Et\beta \\
 C_y & = & Et^4/12 I \\
 \nu_{xm} & = & \nu
 \end{array} \quad \left. \vphantom{\begin{array}{rcl} B_x \\ B_y \\ D_{xy} \text{ or } B_{xy} \\ \nu_{xb} \\ C_x \\ C_y \\ \nu_{xm} \end{array}} \right\} \quad (64)$$

where

t = thickness of the plating

β = ratio of the developed width to projected width
of a corrugation.

I = Moment of inertia of a corrugation per unit width,
and E, ν are respectively the modulus of elasticity and
Poisson's ratio of the material of the corrugated
plate.

Now,

$$\lambda_b = \sqrt[4]{B_y/B_x} = \sqrt[4]{t^3/12\beta I} \quad (65a)$$

$$\text{and } \lambda_m = \sqrt[4]{C_y/C_x} = \sqrt[4]{t^3/12\beta I} \quad (65b)$$

$$\text{Hence } r = \lambda_m/\lambda_b = 1$$

and also, by Maxwell's reciprocal theorem,

$$\left. \begin{aligned} v_{yb} &= v_{xb} B_y/B_x = vt^3/12\beta I \\ \text{and } v_{ym} &= v_{xm} C_y/C_x = vt^3/12\beta I \end{aligned} \right\} \quad (65c)$$

The following formula may be used to evaluate the shearing rigidity of the equivalent orthotropic plate

$$E_{xy} \text{ or } C_{xy} = Gt/\beta = Et/2\beta(1+\nu) \quad (66)$$

This formula implies that in a corrugated plate, the shear deformation of the whole developed width of plate occurs over the projected width of a corrugation thereby increasing the shear strain by a factor β .

Numerical Evaluation of Rigidity Parameters for the Corrugated Plates used in the Experiments.

For the type of corrugated plate tested in the present investigation (Fig. 5a), the following geometrical relationships apply

$$\left. \begin{aligned} \beta &= (k + \psi - \sin \psi)/k \\ \bar{Z}/R &= (\sin \psi - 0.5 \sin 2\psi + k \cos \psi)/\beta k \\ Z_c/R &= (\sin \psi - \psi \cos \psi)/\beta k \\ I/R^2 t &= 0.25(2\psi - \sin 2\psi)/k - \beta(\bar{Z}/R)(Z_c/R) \\ &\quad + (t/R)^2(k - \sin \psi)/12k. \end{aligned} \right\} \quad (67)$$

The last term in the expression for (I/R^2t) represents the moment of inertia of the flat plating about its own axis; it contributes very little to the total moment of inertia, for the type of corrugations under consideration and can, therefore, be neglected in subsequent calculations.

For the experimental plate

$$\psi = 1.294 \text{ radians}$$

$$k = 2.907$$

$$t = 0.064 \text{ in.}$$

$$R = 1.032 \text{ in.}$$

$$E = 10.4 \times 10^6 \text{ lb per sq. in.}$$

$$\nu = 0.32$$

Substituting these values in Equations (67), (65), (64) and (66) gives

$$\beta = 1.114$$

$$I = 0.00552 \text{ in.}^3 \text{ (the term neglected would have increased the value of } I \text{ by about } 0.25\%)$$

$$\lambda_b = \lambda_m = 0.244$$

$$\nu_{yb} = \nu_{ym} = 0.00114$$

$$\nu_{xb} \nu_{yb} = \nu_{xm} \nu_{ym} = 0.00036 \simeq 0$$

$$D_x = B_x = 5.74 \times 10^4 \text{ lb in.}$$

$$D_{xy} = 91 \text{ lb in.}$$

$$C_x = 74.15 \times 10^4 \text{ lb /in.}$$

$$C_{xy} = 22.63 \times 10^4 \text{ lb /in.}$$

By definition

$$\mu_b = H_b / \sqrt{D_x D_y} = (2D_{xy} + \nu_{yb} D_x) / D_x \lambda_b^2$$

$$\text{and } \mu_m = \sqrt{C_x C_y} / H_m = \left(\frac{1}{2C_{xy}} - \frac{\nu_{xm}}{C_x} \right) C_x \lambda_m^2$$

Hence $\mu_b = 0.07$

$\mu_m = 0.08$

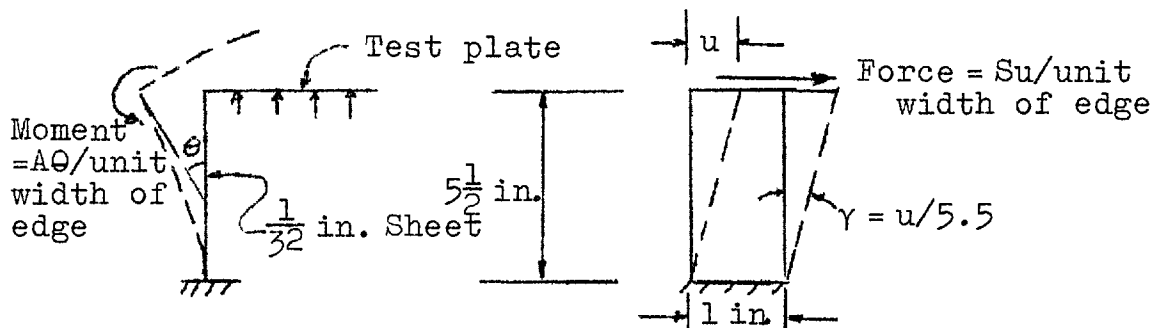
From Equation (60)

$h_{eq} = 0.964 \text{ in.}$

The important rigidity properties of the corrugated plate can now be summarized as follows

D_x	$=$	$5.74 \times 10^4 \text{ lb in.}$
C_x	$=$	$74.15 \times 10^4 \text{ lb /in.}$
$\lambda_b = \lambda_m$	$=$	0.244
$\nu_{xb} = \nu_{xm}$	$=$	0.32
$\nu_{xm} \nu_{ym}$	$=$	0
r	$=$	1
μ_b	$=$	0.07
μ_m	$=$	0.08
h_{eq}	$=$	0.964 in.

To calculate the virtual side ratio and the six edge restraint coefficients, the actual dimensions of the test plates and the attachments at the edges are to be considered.



Bending deformation of
a side sheet

Shearing deformation
of a side sheet

The test plates were attached on all four sides to vertical aluminium alloy side sheets $1/32$ in. thick and approximately $5\frac{1}{2}$ in. wide (see CHAPTER IV). Assuming $E = 10.4 \times 10^6$ lb per sq in. and $\nu = 0.32$ for the material of the side sheets and assuming the sheets to be free to sway at the top edges and clamped at the bottom, the elastic restraints against rotation at the edges of the test plates are given by

$$A_x = A_y = \frac{10.4 \times 10^6}{(1 - .32 \times .32)} \times \frac{1}{12} \times \left(\frac{1}{32}\right)^3 \times \frac{1}{5.5} = 5.36$$

lb in./in. width

The actual magnitude of the edge restraints depends on the horizontal displacement imposed by the deflected plate at the top edges of the side sheets and can even be greater than three times the value given above. At the time of writing this thesis, calculations for these higher values of A_x and A_y are incomplete.

The elastic restraints against movement parallel to the edges can be calculated by considering the shearing deformation of a unit width of side sheet (see sketch above) and are given by

$$S_x = S_y = \frac{1}{5.5} \times \frac{10.4 \times 10^6}{2(1 + .32)} \times \frac{1}{32} = 2.24 \times 10^4 \text{ lb in./in. width}$$

The elastic restraints against movement normal to the edges are neglected giving

$$K_x = K_y = 0.$$

With the dimensions of the test plates and the

magnitudes of edge restraints known, the virtual side ratio and edge restraint coefficients can be calculated from the relationships defined in APPENDIX A and rigidity properties calculated earlier in this appendix, giving for Test Plate 1 ($\ell_x = 70$ in., $\ell_y = 30$ in.)

ε	=	1.76
α_x	=	0.007
α_y	=	0.788
s_x	=	8.659
s_y	=	62.192
$k_x = k_y$	=	0

and for Test Plate 2 ($\ell_x = 20$ in., $\ell_y = 30$ in.)

ε	=	6.15
α_x	=	0.002
α_y	=	0.788
s_x	=	2.474
s_y	=	62.192
$k_x = k_y$	=	0

APPENDIX - DCALCULATION OF LOCAL TRANSVERSE STRESSES
AND DEFORMATIONS IN CORRUGATIONS

Calculation for a Corrugation of the type shown in Fig. 5a
having $\psi = 1.294$ radians and $k = 2.907$.

Because of symmetry only one half of the transverse strip need be considered. The geometrical properties of the half strip ADC are given by Equations (67),

k'	$=$	1.945
$\bar{Z}/R$	$=$	0.461
Z_c/R	$=$	0.188
$\bar{I}/R^3t$	$=$	0.235

$$\text{and } \left. \begin{aligned} J_p &= \text{Sin}\alpha - 0.461\alpha \text{ (when P is in AD)} \\ &= 0.188x/R \text{ (when P is in CD)} \end{aligned} \right\} \quad (68)$$

where

$\bar{I}$ = BI Moment of inertia of the half corrugation
ADC about the neutral axis.

and J_p = (First moment of the length AP or CP about the
neutral axis)/ R^2 .

The differential shear flow at P (Fig. 5b) is given by

$$(\tau)_p = t(Bq - V_c)R^2J_p/\bar{I} = (kq - V_c/R)J_p/(\bar{I}/R^3t).$$

Substituting Equations (68) and the numerical values of the corrugation properties in the above expression gives

$$(\tau\tau)_p = (12.365q - 4.254 V_c/R)(\sin\alpha - 0.461\alpha) \quad (69a)$$

(when P is in AD)

$$\text{and } (\tau\tau)_p = (2.322q - 0.799 V_c/R)(x/R) \quad (69b)$$

(when P is in CD).

It can be shown that the sum of the vertical components of $(Rd\alpha)(\tau\tau)$ acting over the length AD is equal to $(Bq - V_c)$.

Considering the equilibrium of forces in the horizontal direction

$$\begin{aligned} F_c + F_a &= qR(1 - \cos\psi) + R \int_0^\psi (\tau\tau)_p \cos\alpha \, d\alpha + \int_0^{k'R} (\tau\tau)_p \, dx \\ &= 7.886 qR - 2.463 V_c \end{aligned}$$

$$\text{Hence } F_a = 7.886 qR - 2.463 V_c - F_c \quad (70)$$

Again, considering the equilibrium of moments about O

$$\begin{aligned} M_a - M_c + RF_a + RF_c \cos\psi - kRV_c \\ &= -k'(k - 0.5k')qR^2 + R^2 \int_0^\psi (\tau\tau)_p \, d\alpha + R \cos\psi \int_0^{k'R} (\tau\tau)_p \, dx \\ &= 1.650 qR^2 - 1.862 RV_c \end{aligned}$$

Eliminating F_a from the above equation with the help of Equation (70)

$$M_a = M_c + 0.727 RF_c + 3.508 RV_c - 6.236 qR^2 \quad (71)$$

In the curved part AD the bending moment M_ϕ at an angle ϕ from A is given by

$$\begin{aligned}
M_{\varphi} &= M_a + (1 - \cos\varphi)RF_a - R^2 \int_0^{\varphi} [(\tau\tau)_p \{1 - \cos(\varphi - \alpha)\} \\
&\quad + q \sin(\varphi - \alpha)] d\alpha \\
&= M_c + RF_c(-0.273 + \cos\varphi) + RV_c(7.260 - 3.752 \cos\varphi \\
&\quad - 2.127 \varphi \sin\varphi - 0.981 \varphi^2) + qR^2(-17.416 + 11.180 \cos\varphi \\
&\quad + 6.183 \varphi \sin\varphi + 2.851 \varphi^2) \quad \text{--- (72)}
\end{aligned}$$

Equations (69a), (70) and (71) have been used to obtain the above expression.

From Equation (72)

$$\partial M_{\varphi} / \partial V_c = R(7.260 - 3.752 \cos\varphi - 2.127 \varphi \sin\varphi - 0.981 \varphi^2) \quad (73a)$$

$$\partial M_{\varphi} / \partial F_c = R(-0.273 + \cos\varphi) \quad (73b)$$

$$\partial M_{\varphi} / \partial M_c = 1 \quad (73c)$$

In the straight part CD, the bending moment M_x at a distance x from C is given by

$$M_x = M_c + (x/R)RV_c - 0.5(x/R)^2 qR^2 \quad (74)$$

and

$$\partial M_x / \partial V_c = R(x/R) \quad (75a)$$

$$\partial M_x / \partial F_c = 0 \quad (75b)$$

$$\partial M_x / \partial M_c = 1 \quad (75c)$$

The bending moment at D can be calculated by putting $\varphi = \psi = 1.294$ radians in Equation (72) or $(x/R) = k' = 1.945$ in Equation (74) and is given by

$$M_d = M_c + 1.945 RV_c - 1.892 qR^2 \quad (76)$$

The strain energy due to bending in the strip ADC is given by

$$U = \int_a^c \frac{M^2 ds}{2D} = \frac{1}{2D} \left[R \int_0^\psi M_\phi^2 d\phi + \int_0^{k'R} M_x^2 dx \right]$$

where $D = Et^3/12(1 - \nu^2)$ Flexural rigidity per unit width of the strip.

Equating the partial derivatives of U with respect to V_c , F_c and M_c to w_c , v_c and Θ_c respectively, the following equations are obtained

$$R \int_0^\psi M_\phi (\partial M_\phi / \partial V_c) d\phi + \int_0^{k'R} M_x (\partial M_x / \partial V_c) dx = Dw_c \quad (77a)$$

$$R \int_0^\psi M_\phi (\partial M_\phi / \partial F_c) d\phi + \int_0^{k'R} M_x (\partial M_x / \partial F_c) dx = Dv_c \quad (77b)$$

$$R \int_0^\psi M_\phi (\partial M_\phi / \partial M_c) d\phi + \int_0^{k'R} M_x (\partial M_x / \partial M_c) dx = D\Theta_c \quad (77c)$$

Substituting the expressions for M_ϕ , M_x and their derivatives [Equations (72) to (75c)] in Equations (77) and integrating gives

$$13.816RV_c + 1.918RF_c + 5.674M_c = 20.066qR^2 + Dw_c/R^2 \quad (78a)$$

$$1.918RV_c + 0.349RF_c + 0.608M_c = 3.190qR^2 + Dv_c/R^2 \quad (78b)$$

$$5.674RV_c + 0.608RF_c + 3.239M_c = 7.188qR^2 + D\Theta_c/R \quad (78c)$$

Equations (78b) and (78c) can now be solved simultaneously for the redundant quantities F_c and M_c giving

$$F_c = 7.837qR + 4.258Dv_c/R^3 - 0.799D\Theta_c/R^2 - 3.631V_c \quad (79a)$$

and

$$M_c = 0.748qR^2 - 0.799Dv_c/R^2 + 0.459D\Theta_c/R - 1.070RV_c \quad (79b)$$

Substituting Equations (79a) and (79b) in Equation (78a) gives

$$w_c = - 0.790 q R^4 / D + 3.631 v_c + 1.070 R \Theta_c + 0.779 R^3 V_c / D \quad (80)$$

For known values of q , V_c , v_c and Θ_c , the forces and moments in the transverse strip of the corrugation and its deformations can now be calculated.

Two particular cases are discussed below.

Case I: $V_c = v_c = \Theta_c = 0$.

From Equations (79)

$$F_c = 7.837 q R$$

$$\text{and } M_c = 0.748 q R^2.$$

Substituting these values in Equations (70), (71) and (76) gives

$$F_a = 0.049 q R$$

$$M_a = 0.209 q R^2$$

$$\text{and } M_d = - 1.144 q R^2$$

The deflection w_c (+ve downward) is obtained from Equation (80)

$$w_c = - 0.790 q R^4 / D$$

(Upward, in the direction of q).

Case II: $V_c = F_c = \Theta_c = 0$.

In this case, the value of F_c being known, M_c is the only redundant quantity and is obtained from Equation (78c)

$$M_c = 2.219 q R^2$$

From Equations (70), (71) and (76)

$$F_a = 7.886 q R$$

$$M_a = - 4.017 q R^2$$

$$M_d = 0.327 q R^2$$

Substituting $M_c = 2.219 qR^2$ and $V_c = F_c = 0$ in Equations (78a) and (78b) gives

$$w_c = - 7.474 qR^4/D$$

(upward, in the direction of q)

$$\text{and } v_c = - 1.841 qR^4/D$$

(opening of corrugation).

The results for a plate identical to the plate in Case II in dimensions and transverse membrane action, but having the direction of q reversed can be obtained from the results of Case I and II, by separating the effects of applied pressure and membrane forces on the corrugation. The effect of the membrane action can be calculated by subtracting the results of Case I from those of Case II, giving

$$F_c = - 7.837 qR$$

$$F_a = 7.837 qR$$

$$M_c = 1.471 qR^2$$

$$M_a = - 4.226 qR^2$$

$$M_d = 1.471 qR^2$$

$$w_c = - 6.684 qR^4/D$$

$$v_c = - 1.841 qR^4/D$$

The results of Case I with their signs changed are now to be added to the above results to give the required solution as follows

$$F_c = - 15.674 qR$$

$$F_a = 7.788 qR$$

$$M_c = 0.723 qR^2$$

$$M_a = -4.435 q R^2$$

$$M_d = 2.615 q R^2$$

$$w_c = -5.894 q R^4/D$$

(Upward, opposite to the direction of q)

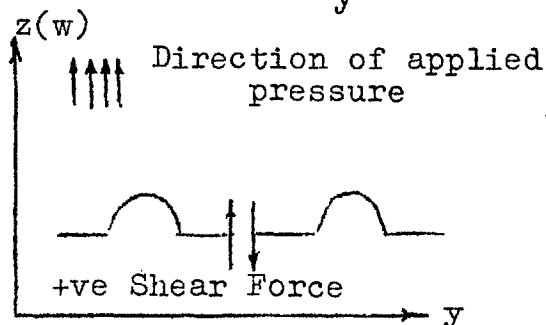
Evaluation of V_c , θ_c and v_c from the Results of Orthotropic Plate Analysis.

Evaluation of V_c :

At any point in the corrugated plate, the transverse shear force per unit length acting on a longitudinal section is given by

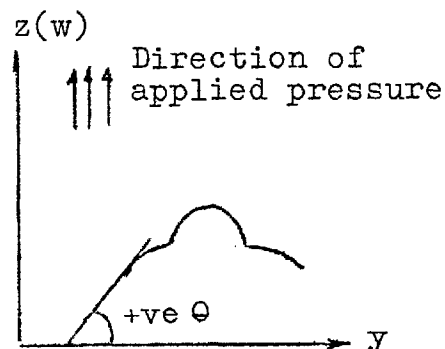
$$V_y = -\frac{\partial}{\partial y} \left(D_y \frac{\partial^2 w}{\partial y^2} + H_b \frac{\partial^2 w}{\partial x^2} \right) = D_y \frac{\partial}{\partial y} \left(-\frac{\partial^2 w}{\partial y^2} - \frac{\mu_b}{\lambda_b^2} \frac{\partial^2 w}{\partial x^2} \right) \quad (81)$$

where w is the deflection given by the orthotropic plate theory. Assuming that the transverse stresses and deformations are required at a section coinciding with a row of nodes of the finite difference mesh, and if the point C (Fig. 4) is itself a node point, the shear force at C (V_c) can be obtained directly from Equation (81) and the appropriate finite difference approximations [Equations (48b), (48c) and (48d)]. If, however, C is not a node point, V_c has to be interpolated from the calculated values of V_y at the surrounding nodes. The adjacent sketch shows the sign convention used for V_y .



Evaluation of Θ_c

The transverse slope Θ_c is equal to $\partial w / \partial y$ and can be calculated directly from Equation (48b) if C is a node point; otherwise, Θ_c is interpolated from the calculated values of $\partial w / \partial y$ at the surrounding nodes. The adjoining sketch shows the sign convention for Θ_c .



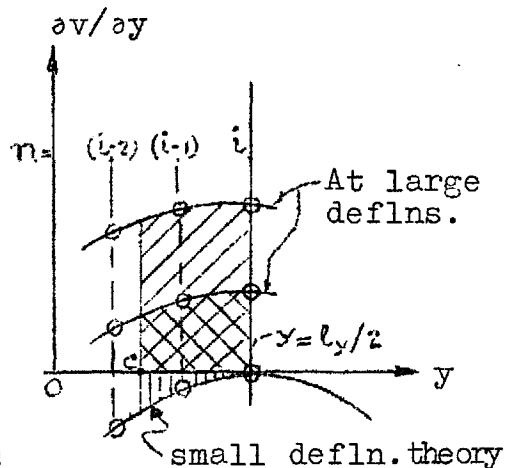
Evaluation of v_c

From Equation (33b) the derivative of the horizontal displacement v is given by

$$\frac{\partial v}{\partial y} = \frac{1}{\lambda_m^4 C_x} \left[\frac{\partial^2 f}{\partial x^2} - \nu_{ym} \frac{\partial^2 f}{\partial y^2} \right] - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \quad (82)$$

Assuming the transverse section under consideration to coincide with a row of finite difference nodes, the magnitude of $(\partial v / \partial y)$ at each node can be calculated from Equation (82) using appropriate finite difference approximations and the values of f and w obtained by the orthotropic plate theory.

v_c is equal to the area of the $(\partial v / \partial y)$ vs. y diagram between the point $y = \ell_y / 2$ and the point C (because $v = 0$ at $y = \ell_y / 2$). A positive area implies a movement of the point C away from the longitudinal centre-line.



It may be noted that for small deflections ($f = 0$) the point C moves towards the centre, indicating that the swedge closes, whereas for large deflections the swedge opens.

When the transverse stresses are to be calculated at a section between two transverse rows of nodes, interpolation will be necessary for calculating v_c .

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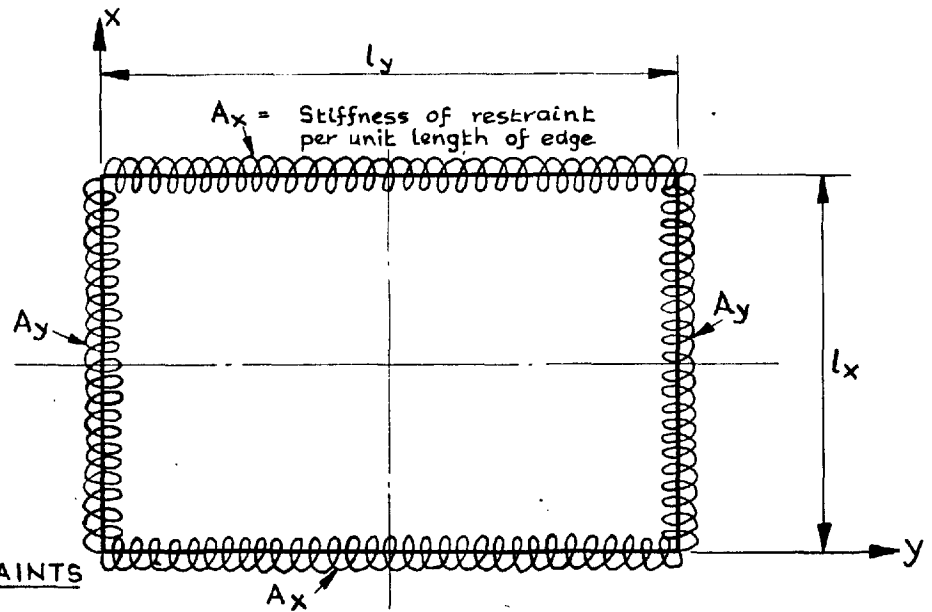
TEST 2

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79	Details of arrangement for loading by oil
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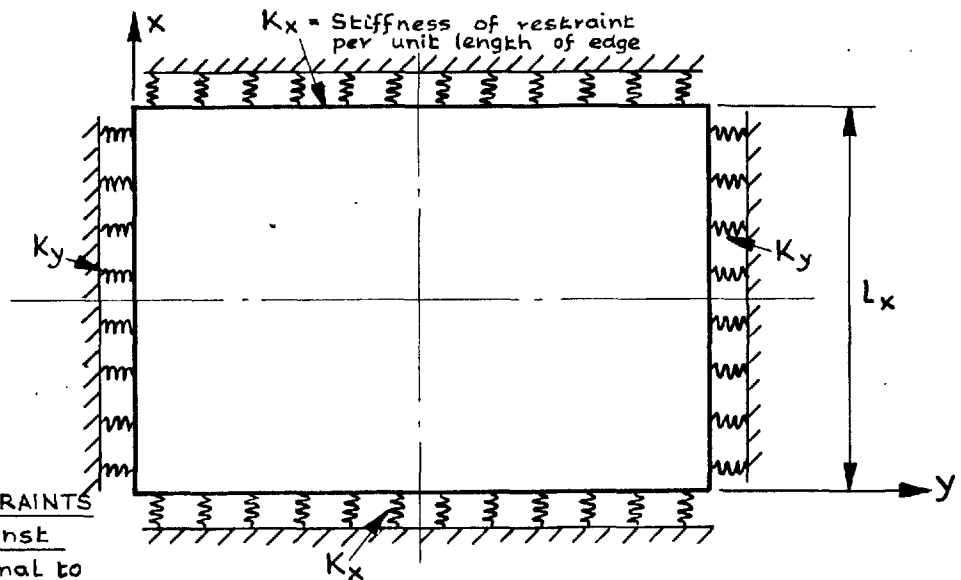
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ROTATIONAL
ELASTIC RESTRAINTS



EXTENSIONAL
ELASTIC RESTRAINTS
(Restraint against
movement normal to
edges.)



SHEAR
ELASTIC RESTRAINTS
(Restraint against
movement parallel
to edges.)

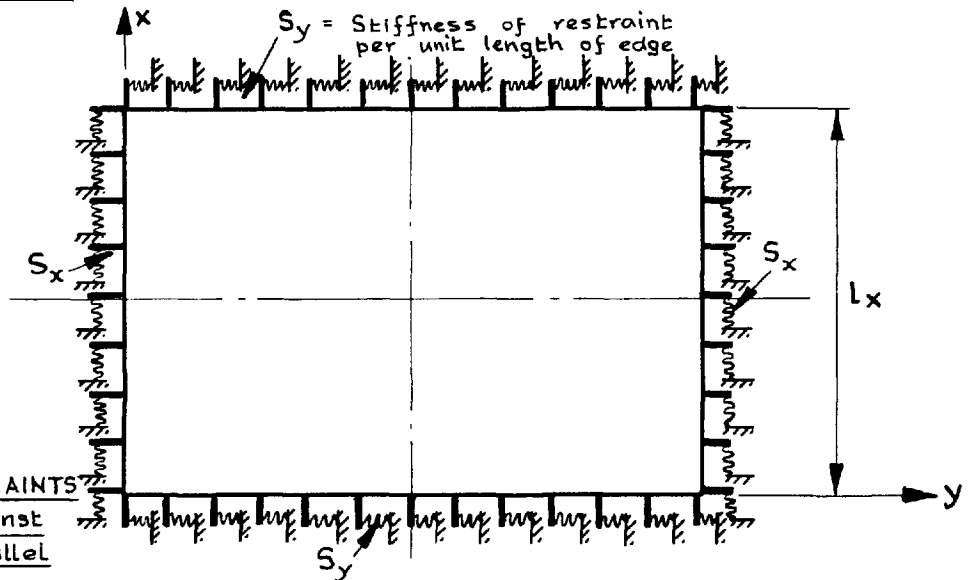


FIG. 1: PLATE EDGE RESTRAINTS

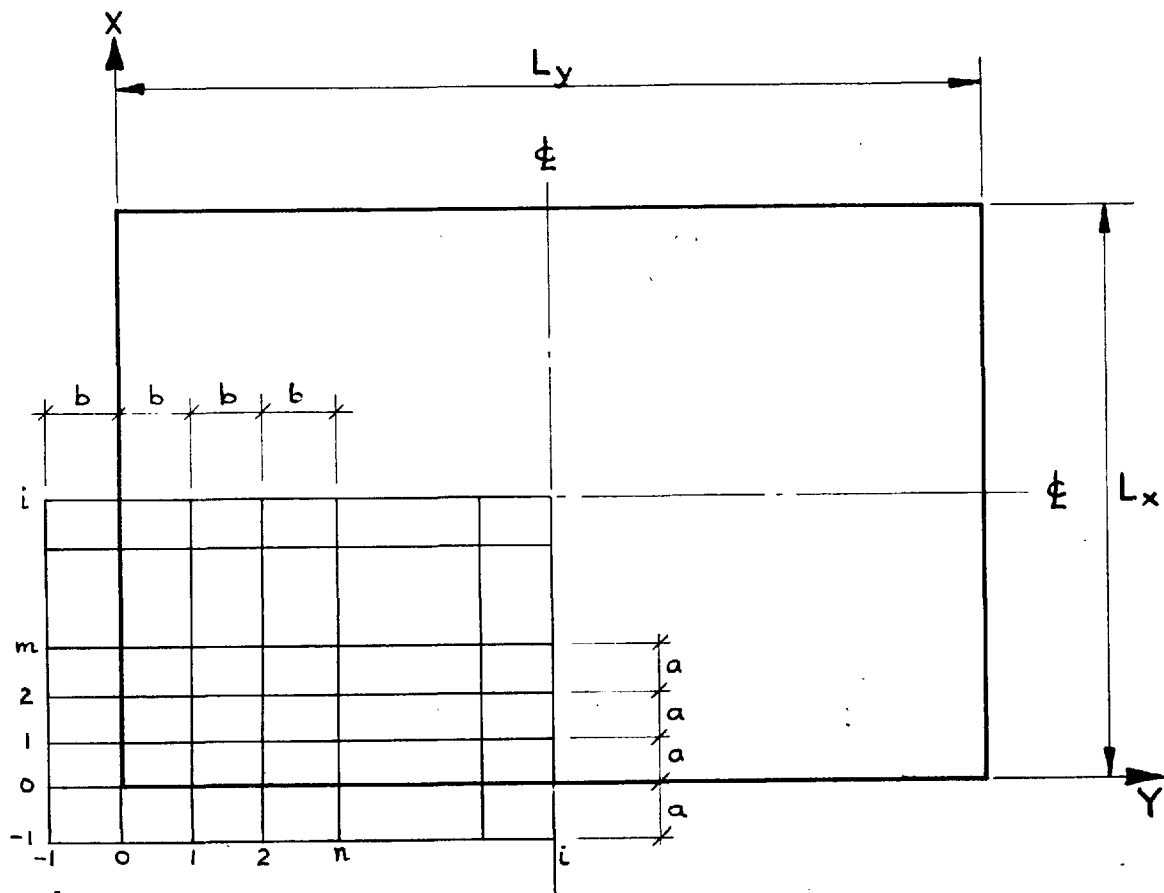
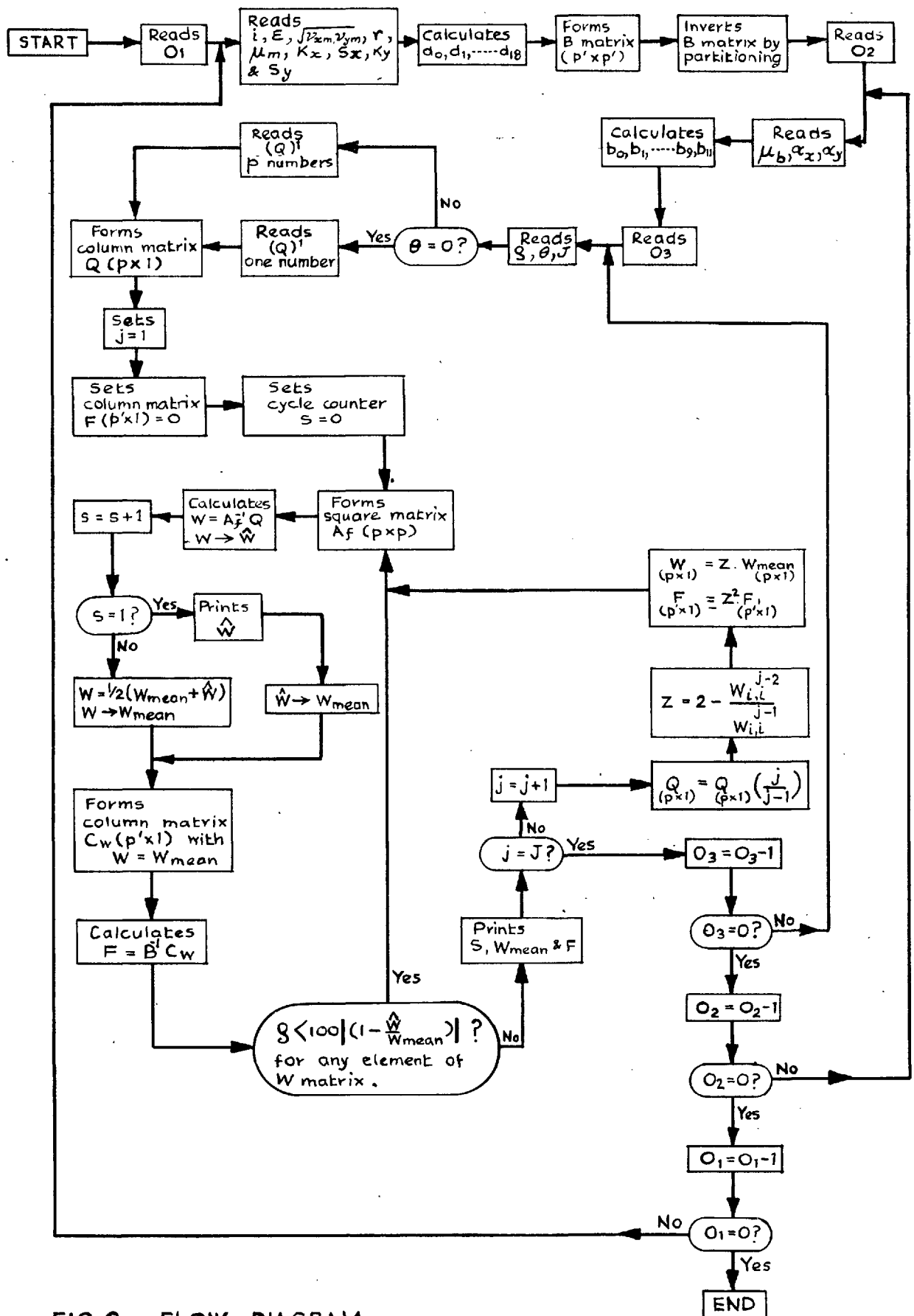


FIG. 2 : TRANSFORMED PLATE



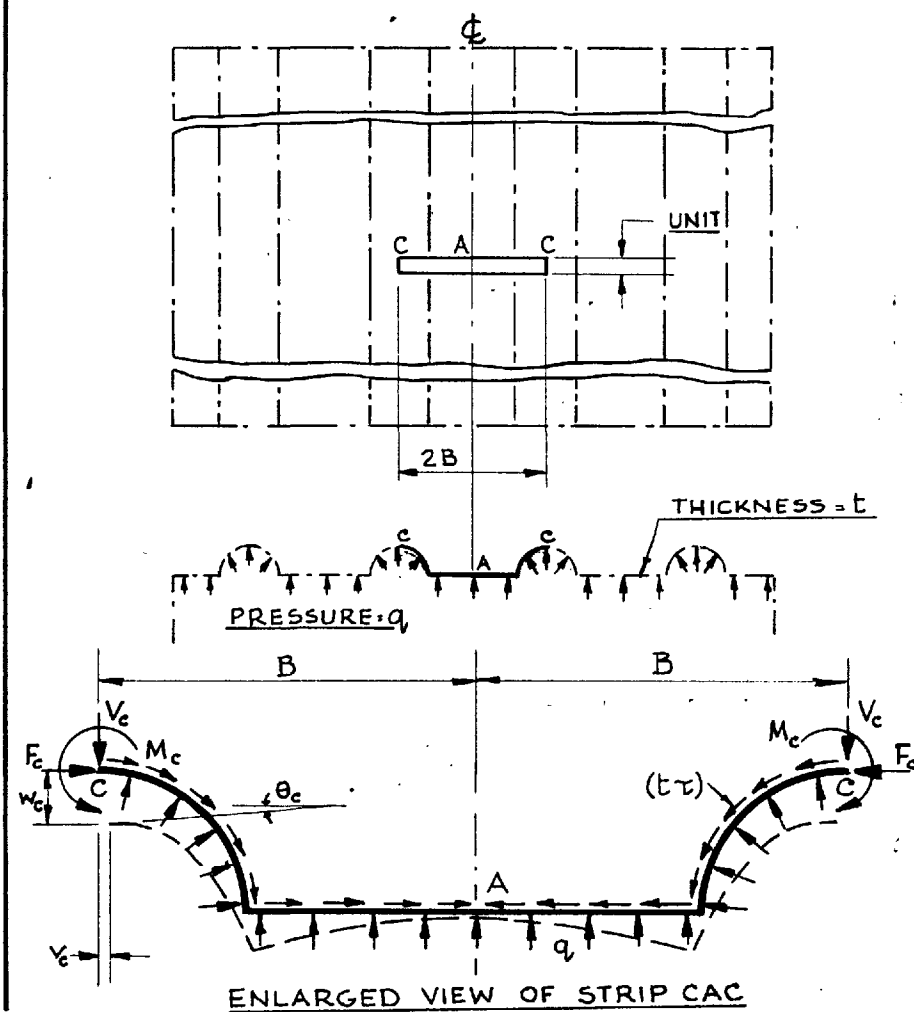
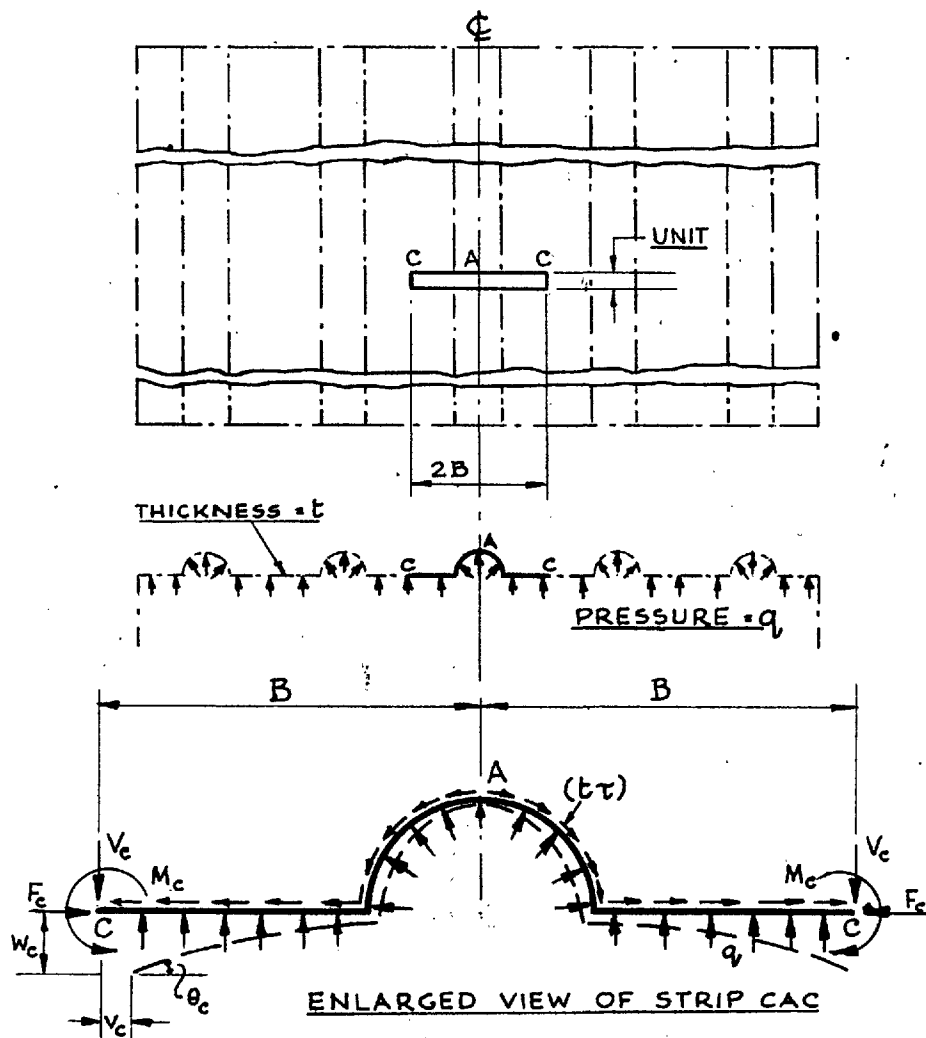
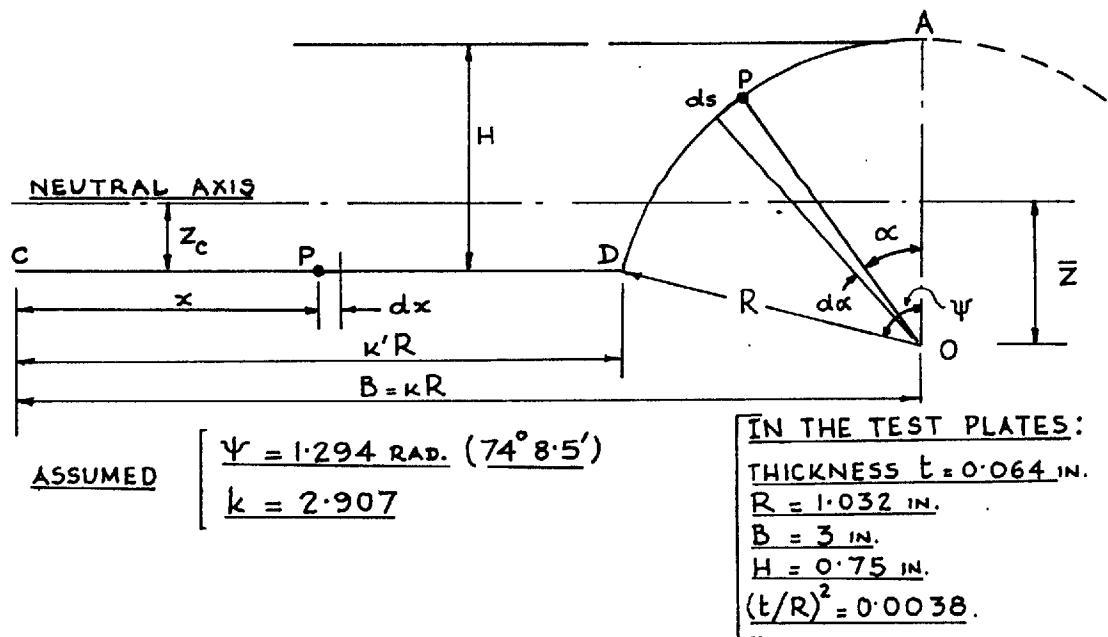
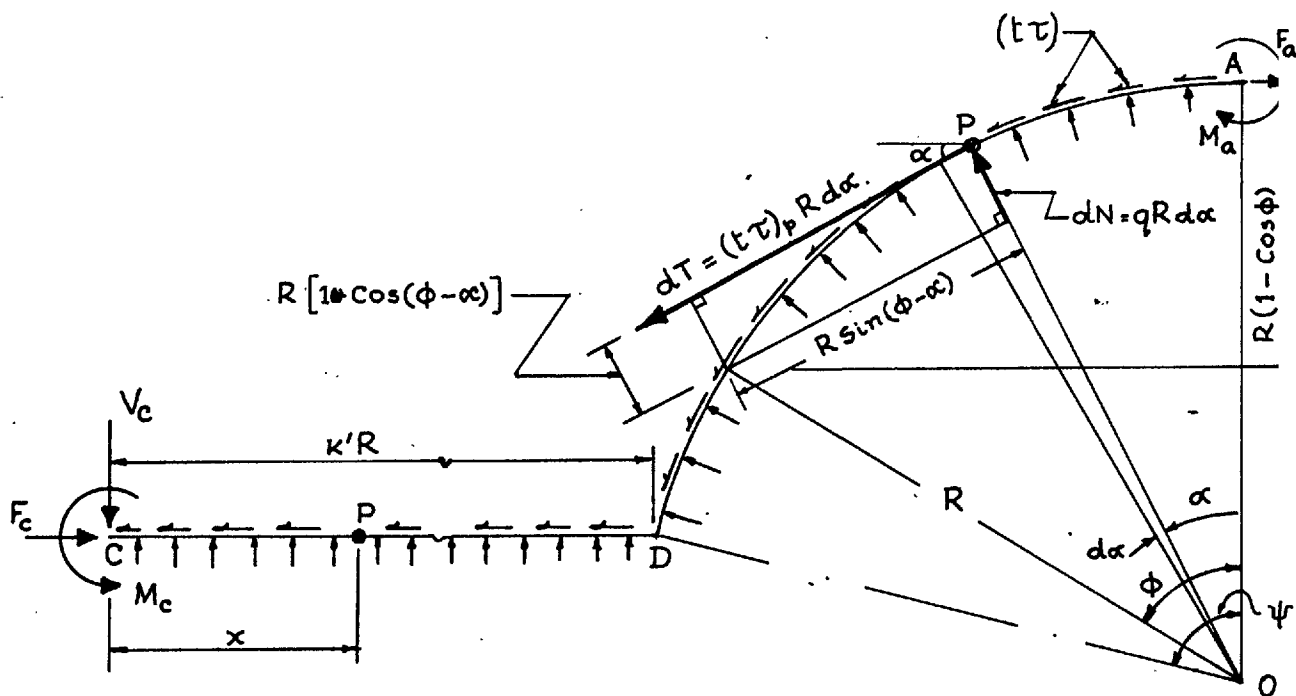


FIG. 4: ALTERNATIVE FORMS OF CENTRAL TRANSVERSE STRIP



(a)



(b)

FIG.5. CORRUGATION UNDER INVESTIGATION
DIMENSIONS AND LOADING

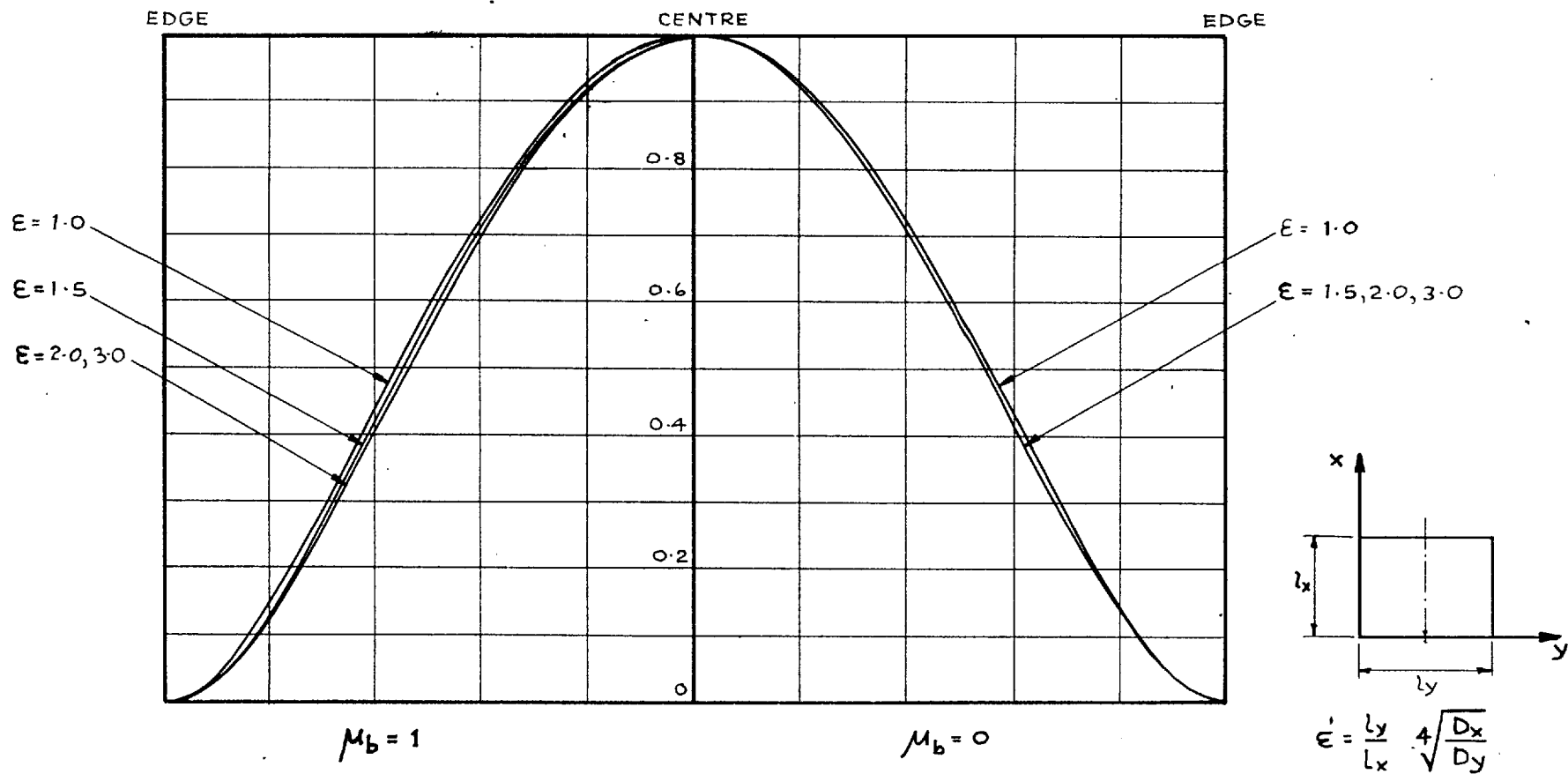


FIG. 6: RECTANGULAR ORTHOTROPIC PLATE, CLAMPED, UNIFORM LOAD.
DEFLECTED SHAPES OF PLATE CENTRE-LINES PARALLEL TO X-AXIS (LINEAR THEORY)

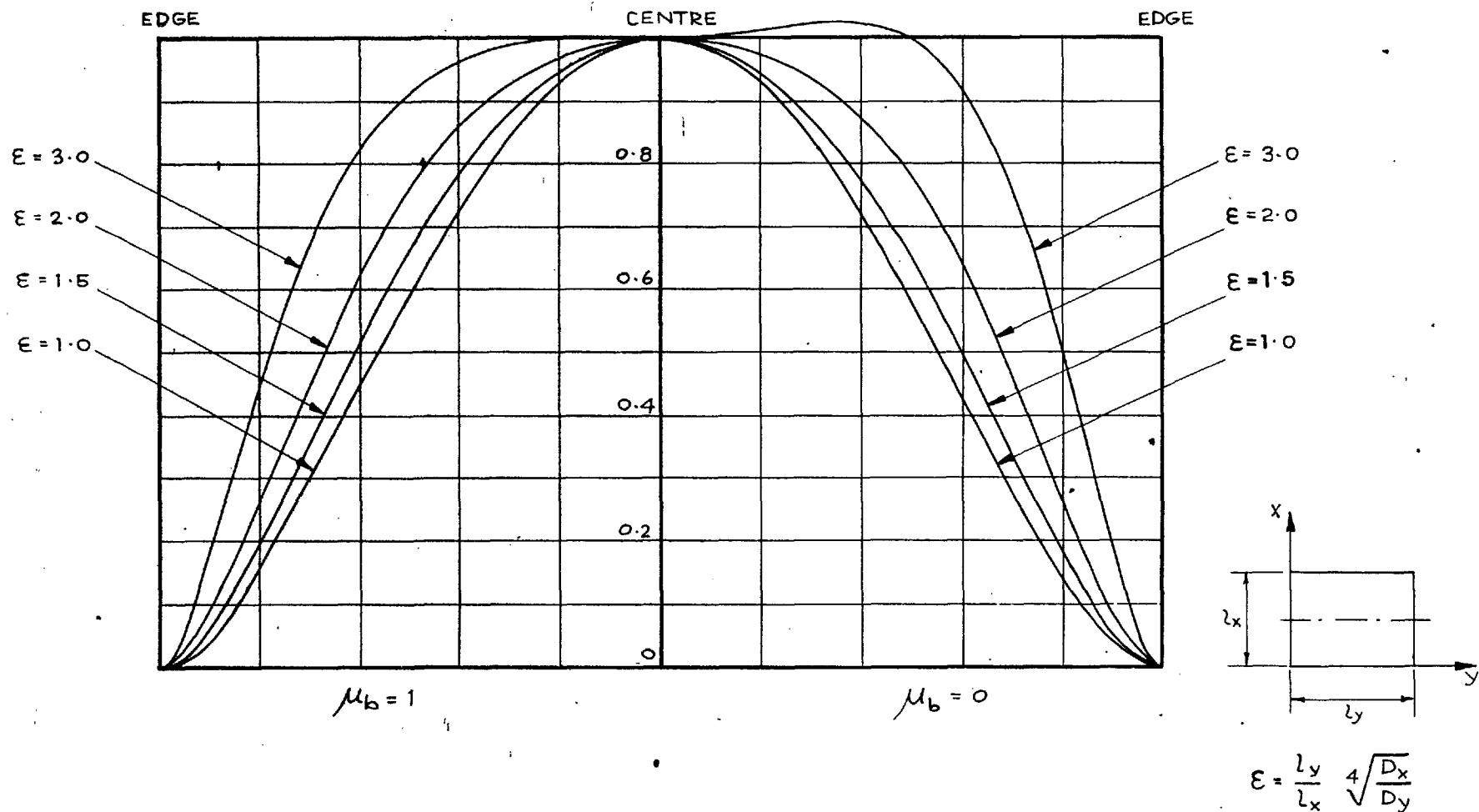


FIG. 7: RECTANGULAR ORTHOTROPIC PLATE, CLAMPED, UNIFORM LOAD.

DEFLECTED SHAPES OF PLATE CENTRE-LINES PARALLEL TO Y-AXIS (LINEAR THEORY)

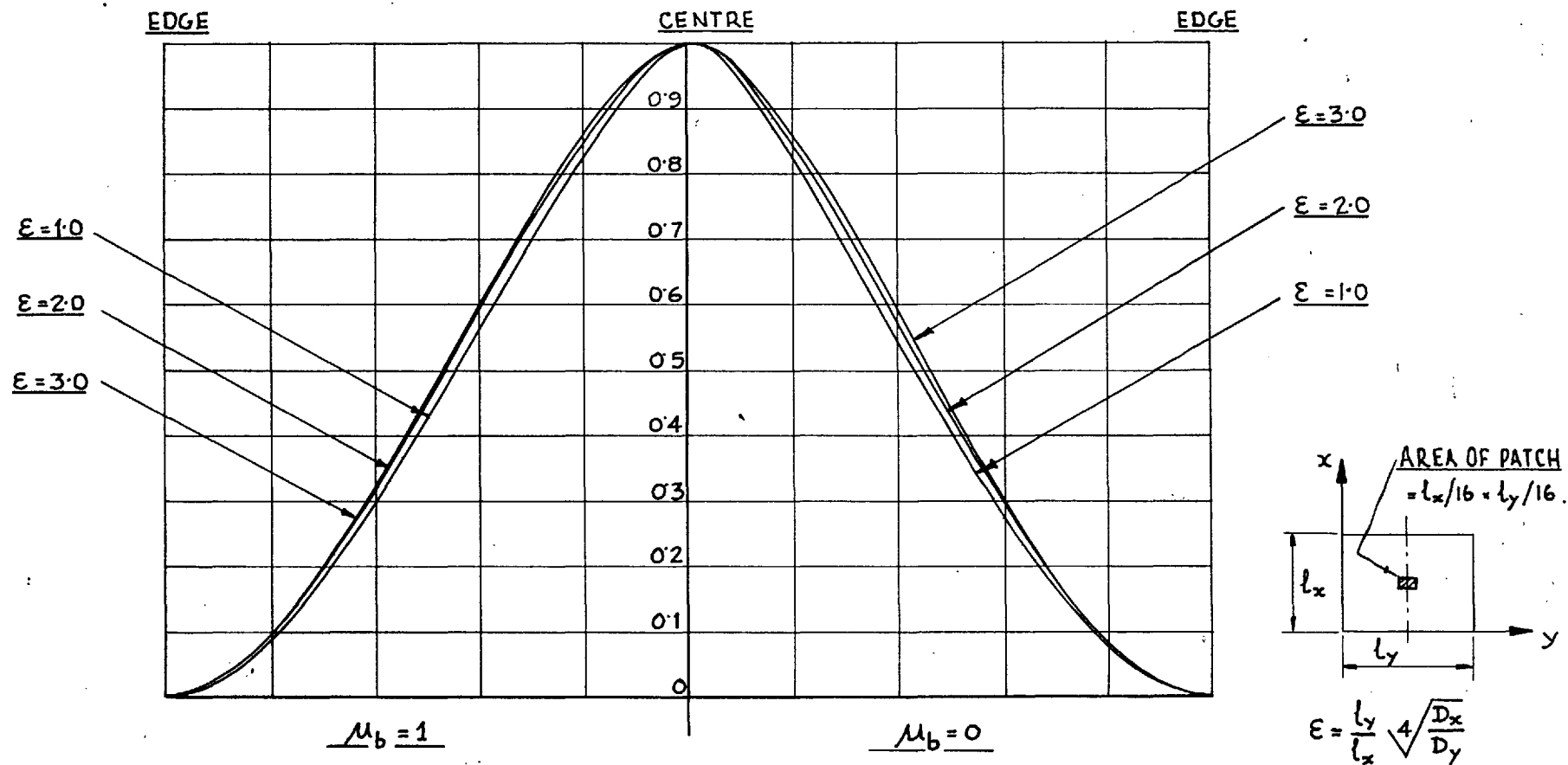


FIG. 8: RECTANGULAR ORTHOTROPIC PLATE. CLAMPED. CENTRAL PATCH LOAD. DEFLECTED SHAPE OF PLATE CENTRE-LINES PARALLEL TO X-AXIS. (LINEAR THEORY)

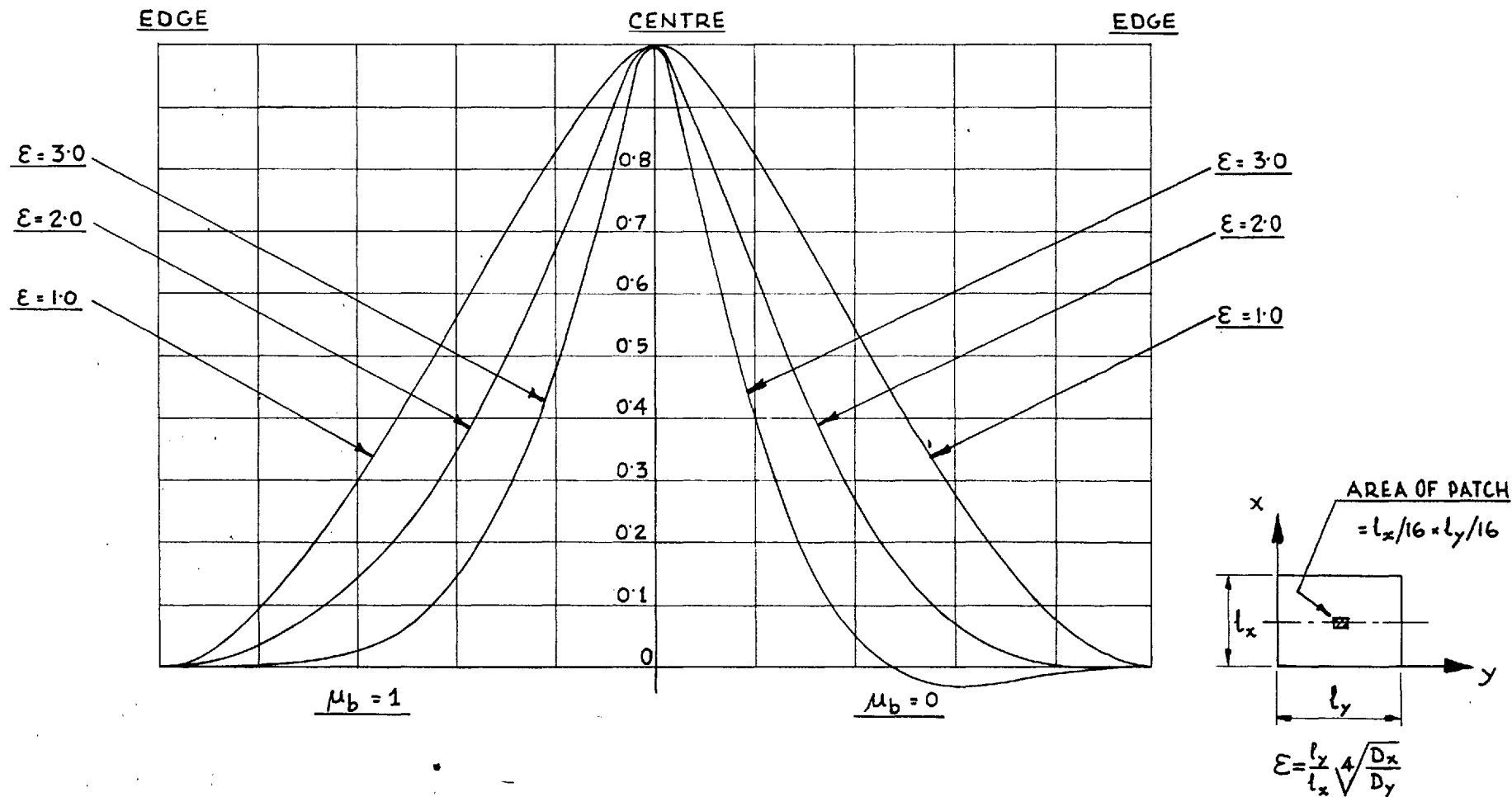


FIG. 9: RECTANGULAR ORTHOTROPIC PLATE . CLAMPED . CENTRAL PATCH LOAD .
 DEFLECTED SHAPE OF PLATE CENTRE-LINES PARALLEL TO Y-AXIS .
 (LINEAR THEORY)

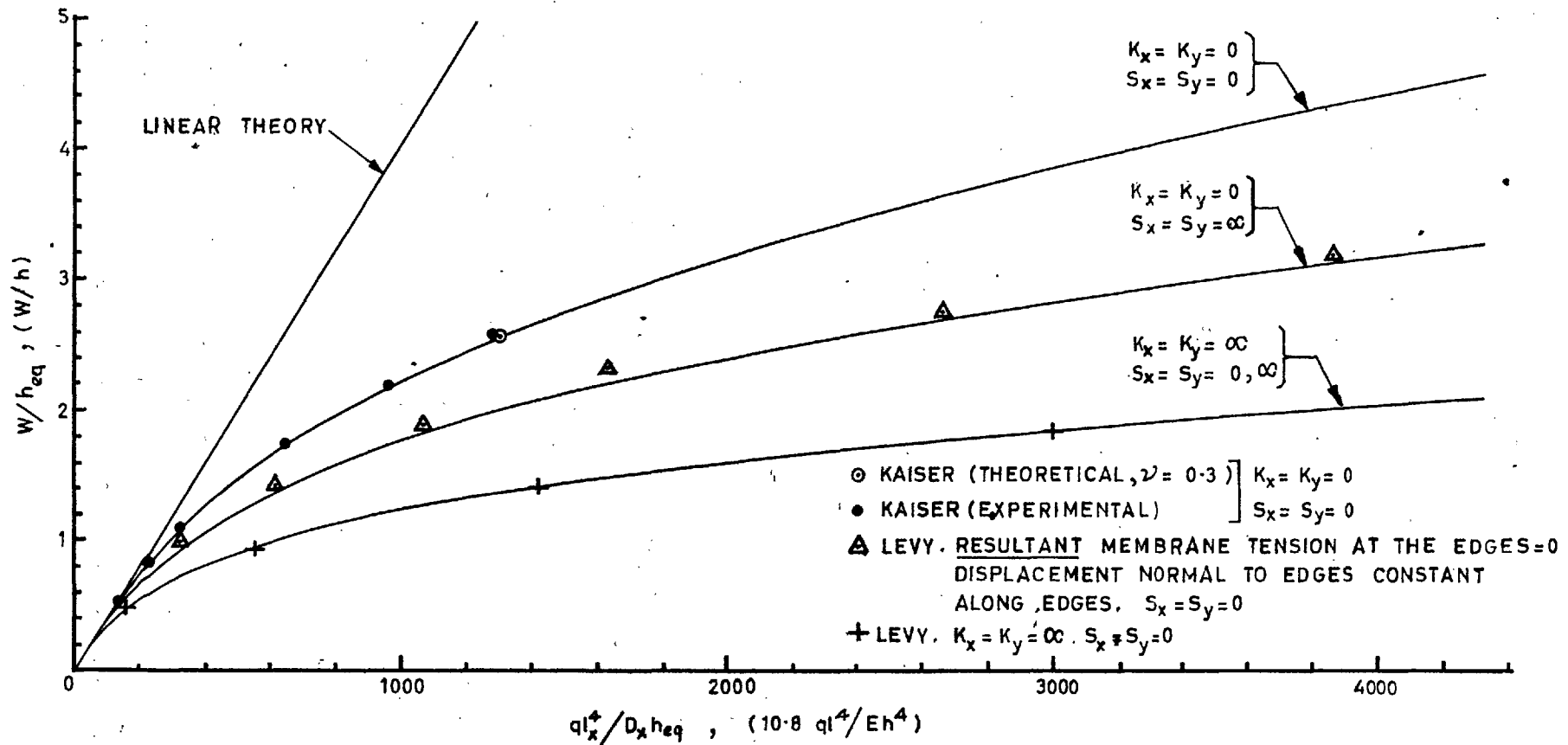


FIG.10: RECTANGULAR ORTHOTROPIC PLATE, $E = 1$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD.

DEFLECTION AT CENTRE

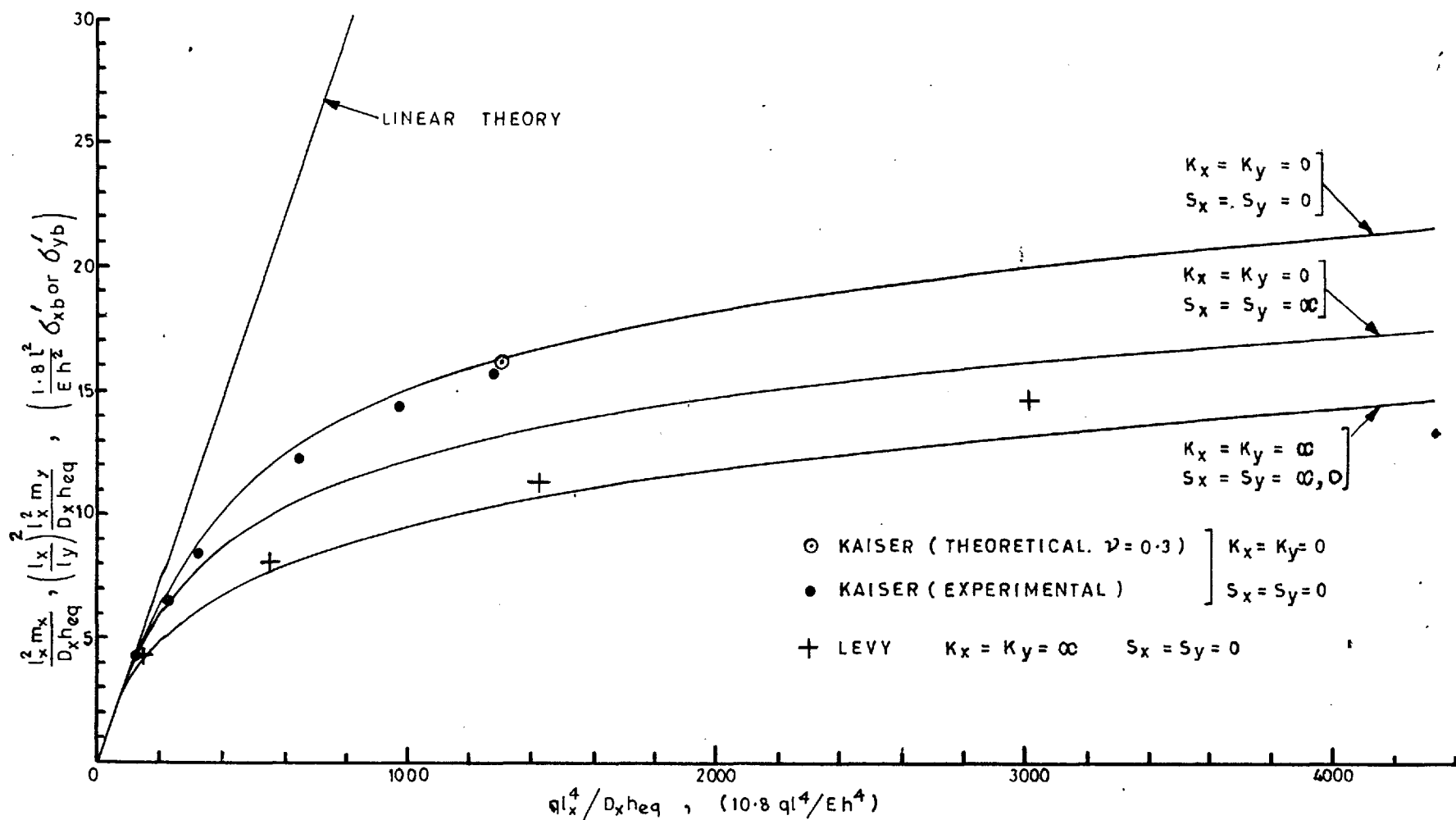


FIG. 11: RECTANGULAR ORTHOTROPIC PLATE, $E = 1, r = 1, \sqrt{\nu_{xm} \nu_{ym}} = 0.316, \mu_m = \mu_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD
CURVATURES AT CENTRE

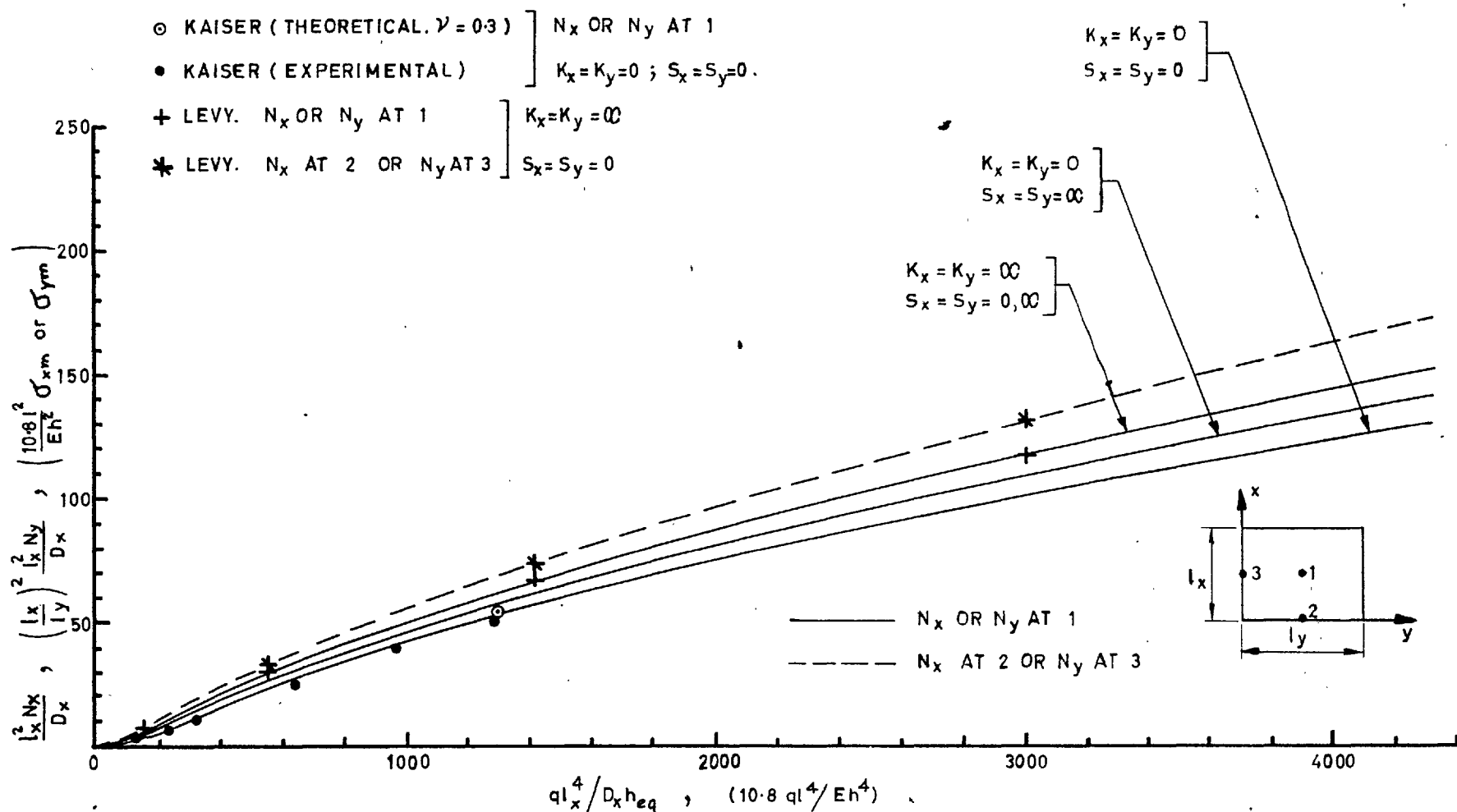


FIG.12: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon=1$, $r=1$, $\sqrt{\nu_{xm}\nu_{ym}}=0.316$, $M_m=M_b=1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu=0.316$) $A_x=A_y=0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD.

MEMBRANE FORCES (TENSION)

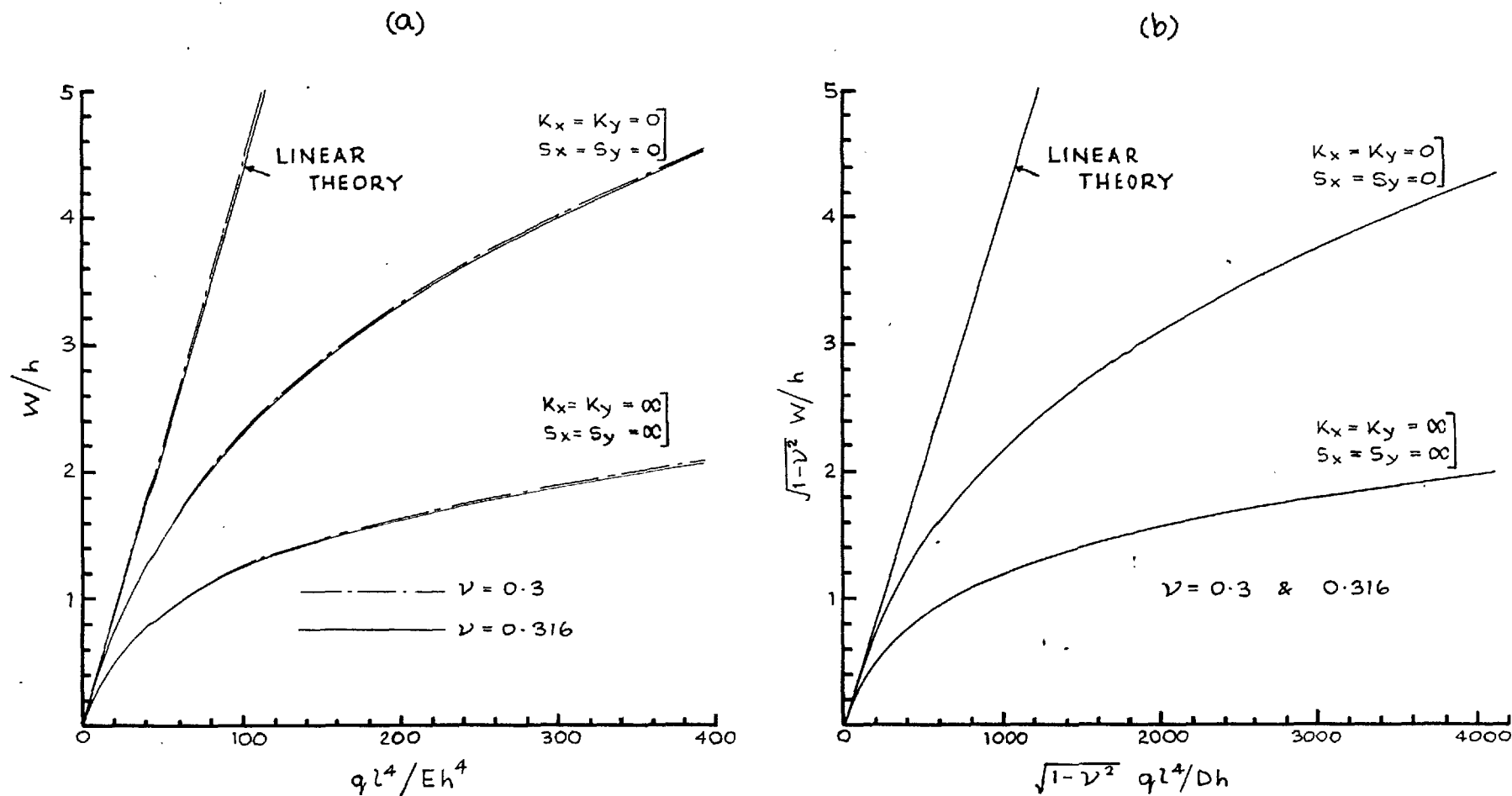


FIG. 13: SQUARE ISOTROPIC PLATE, $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD.
EFFECT OF VARYING POISSON'S RATIO ν ON CENTRAL DEFLECTION

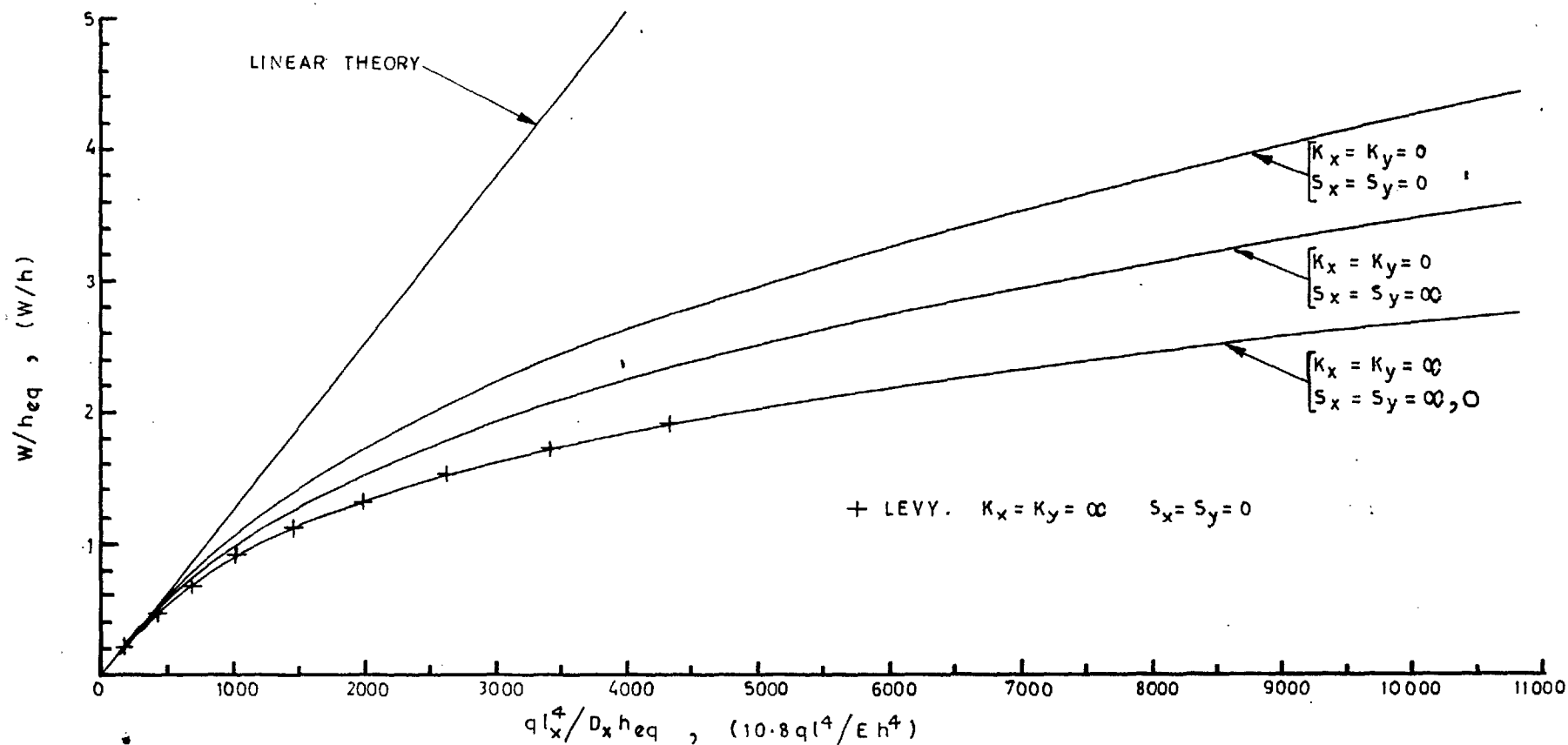


FIG.14: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1, r = 1, \sqrt{\nu_{xm}\nu_{ym}} = 0.316, \mu_m = \mu_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD

DEFLECTION AT CENTRE

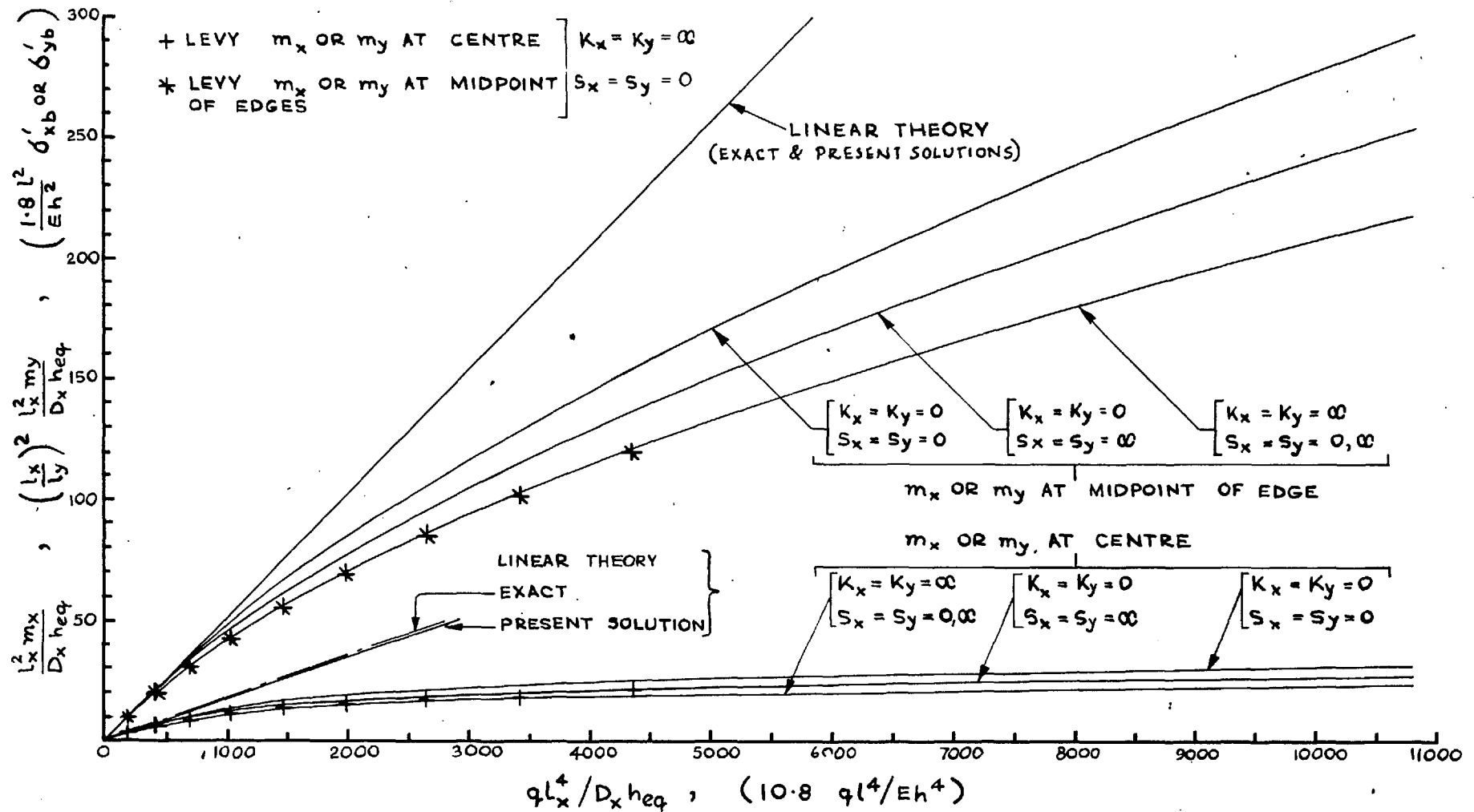


FIG.15: RECTANGULAR ORTHOTROPIC PLATE, $E = 1, r = 1, \sqrt{\nu_{xm} \nu_{ym}} = 0.316, M_m = M_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD CURVATURES.

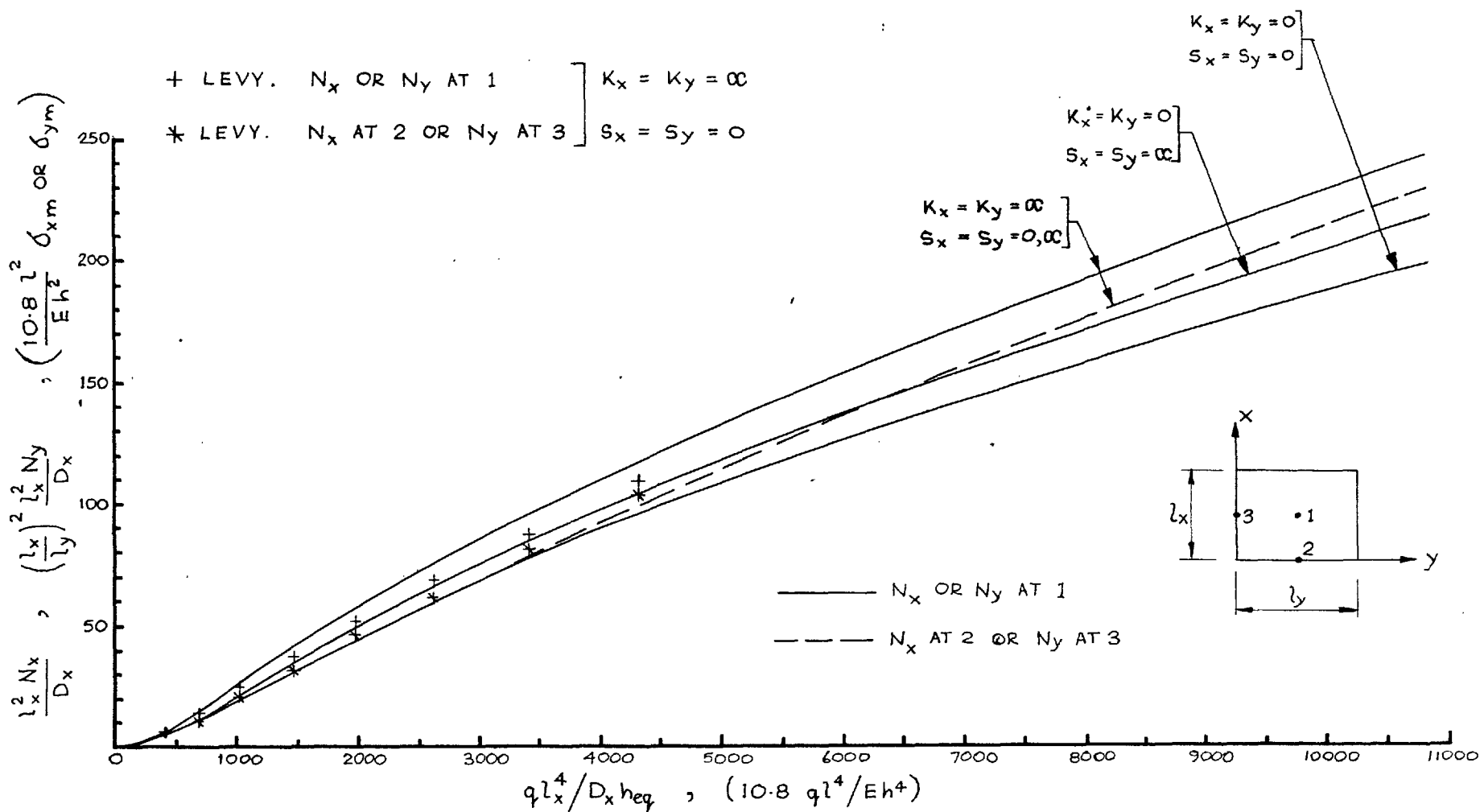


FIG. 16: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon = 1$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD
MEMBRANE FORCES (TENSION)

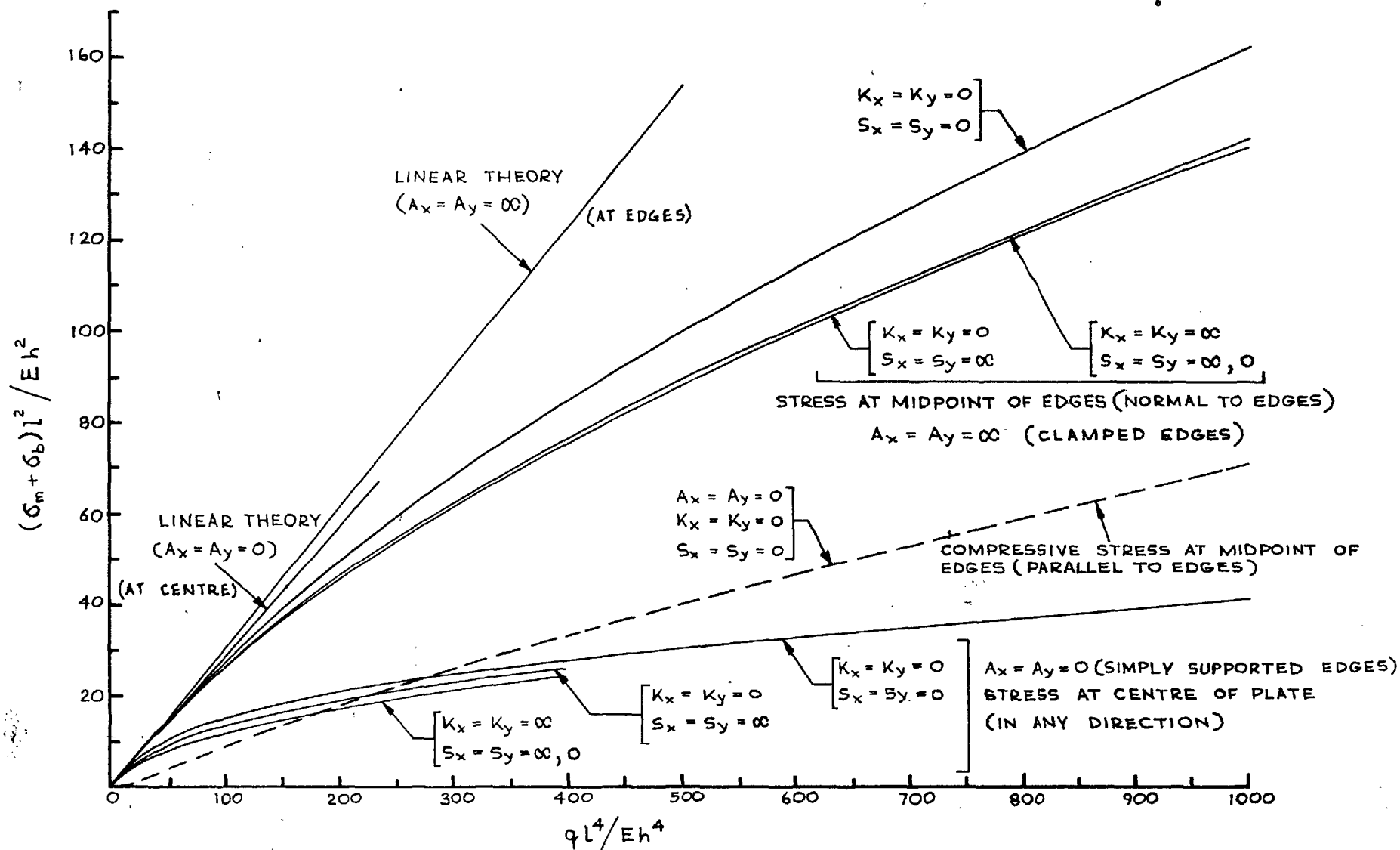


FIG. 17: SQUARE ISOTROPIC PLATE, $\nu = 0.316$, UNIFORM LOAD.
EFFECT OF EDGE RESTRAINTS ON TOTAL SURFACE STRESSES.

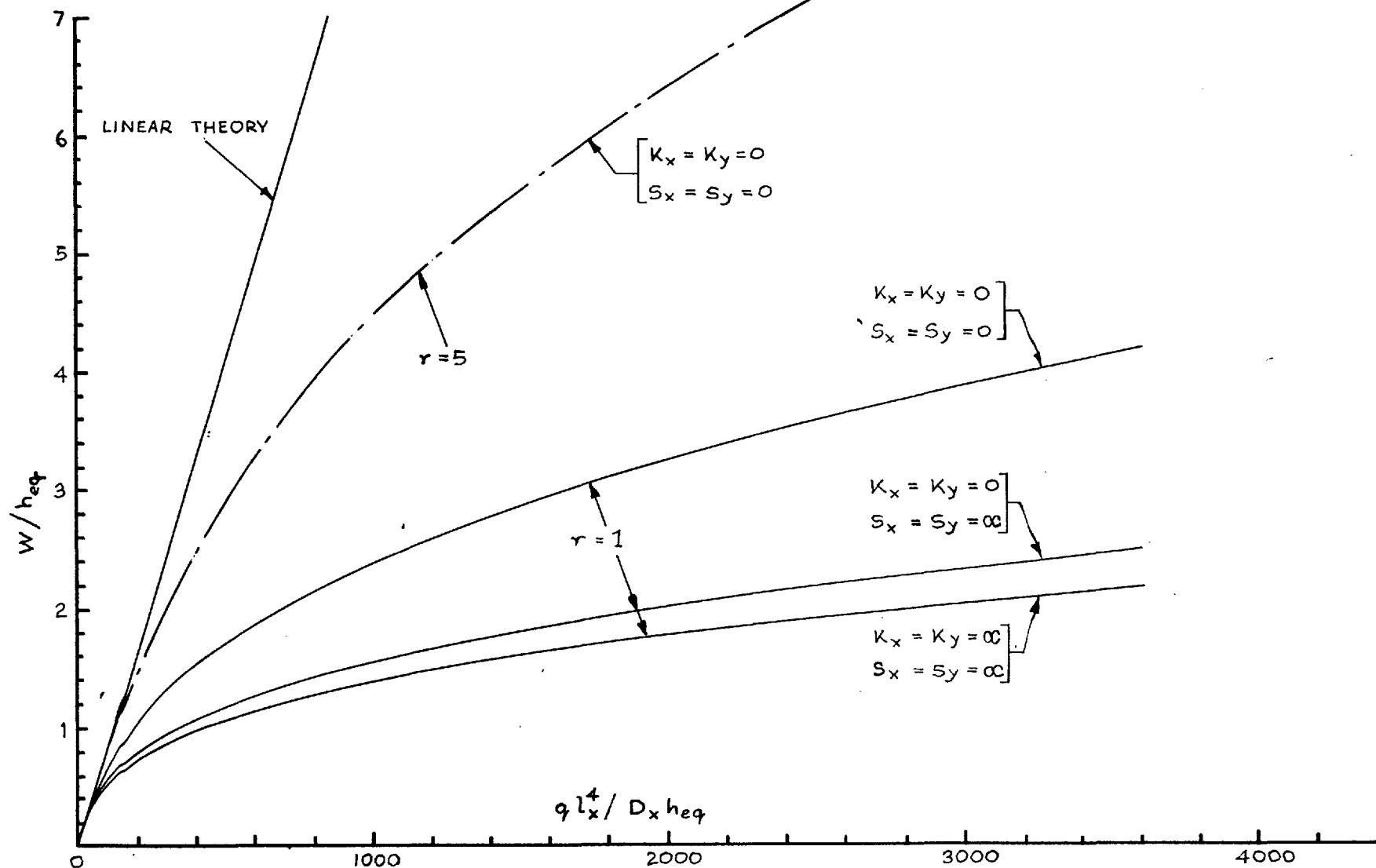


FIG. 18: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1$, $\nu_{xm} \nu_{ym} = 0$, $M_m = M_b = 0$, $A_x = A_y = 0$
 (SIMPLY SUPPORTED EDGES) UNIFORM LOAD
DEFLECTION AT CENTRE

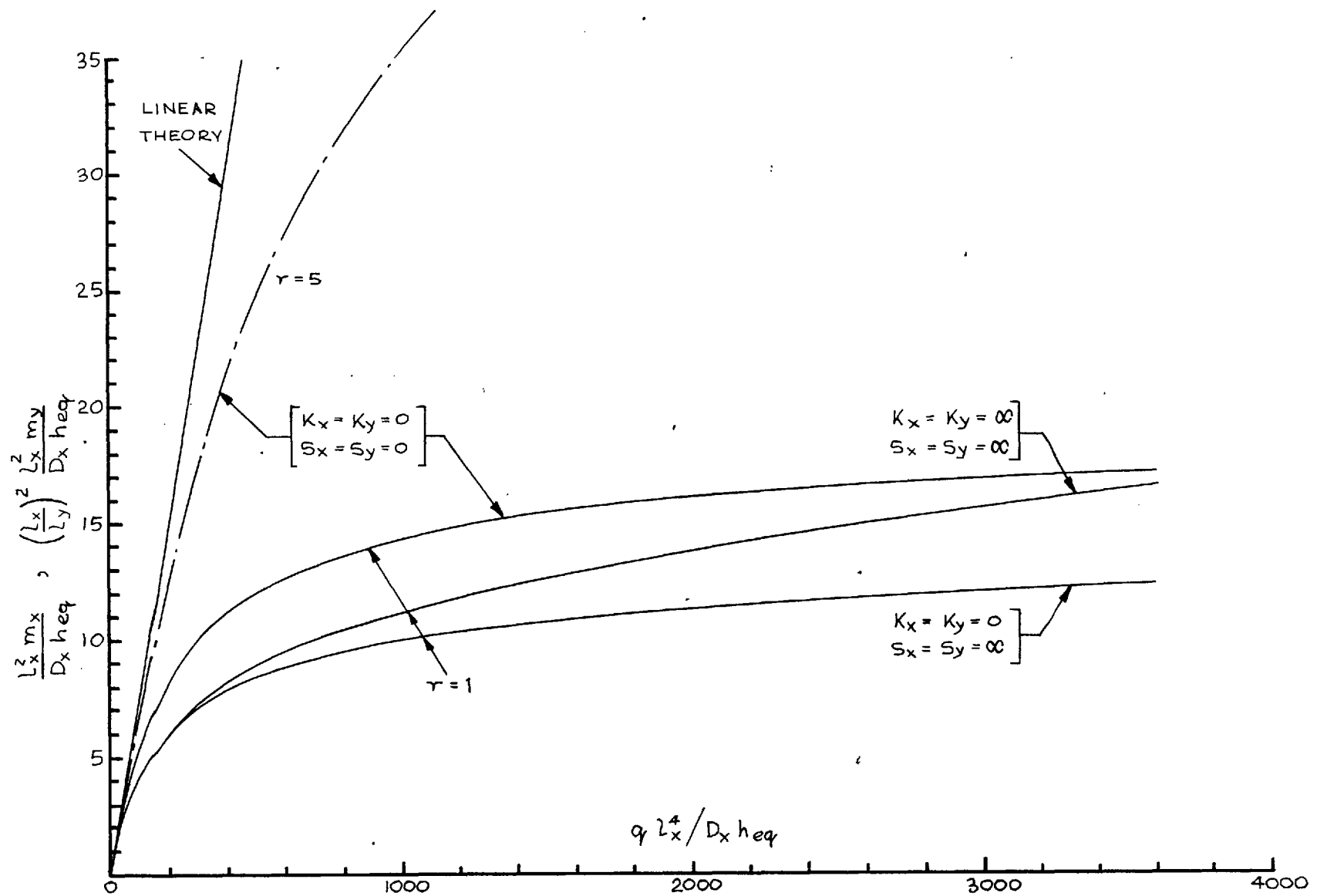


FIG. 19: RECTANGULAR ORTHOTROPIC PLATE, $E = 1, \sqrt{\nu_{xm} \nu_{ym}} = 0, \mu_m = \mu_b = 0, A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD.

CURVATURES AT CENTRE

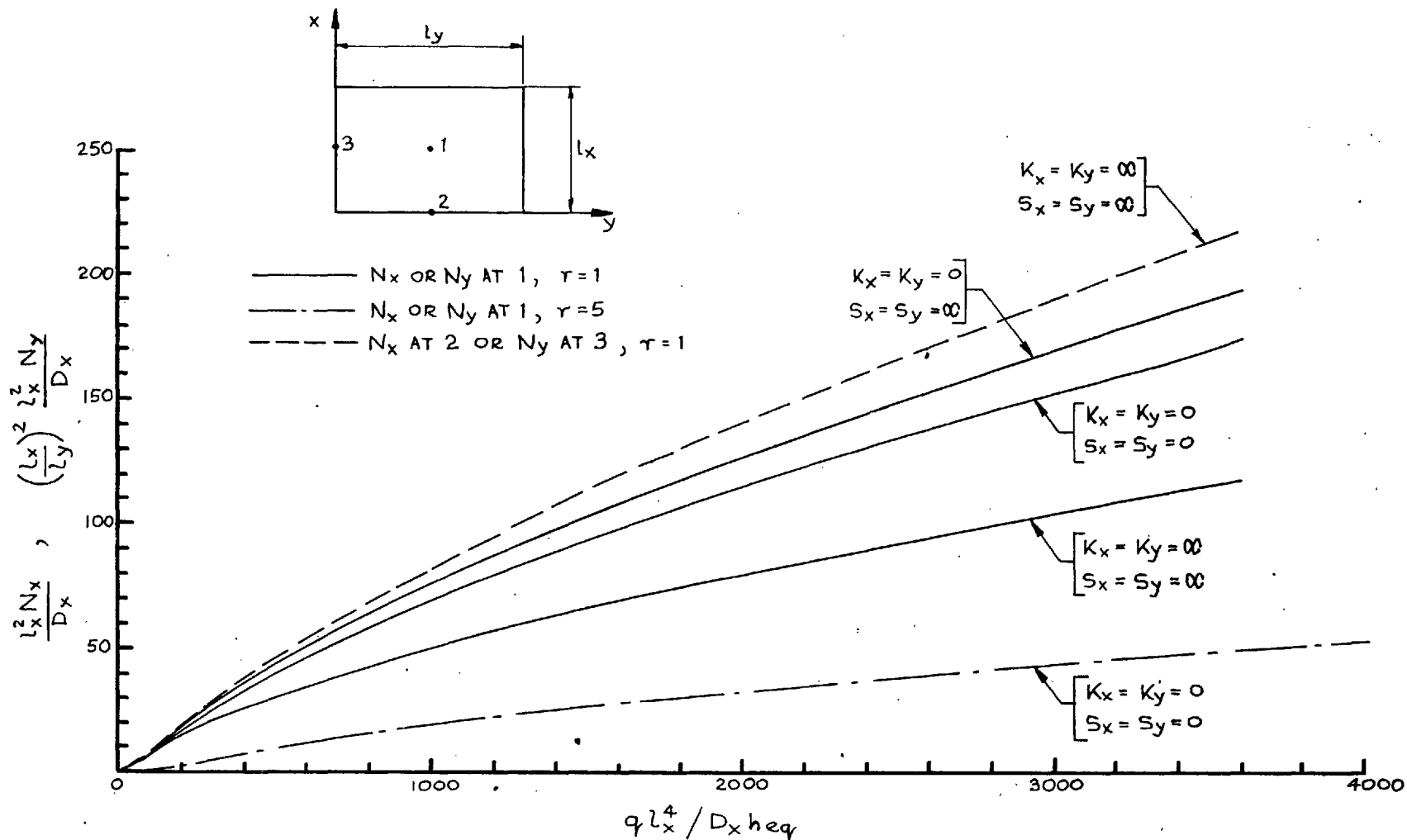


FIG. 20: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon=1$, $\nu_{xm}\nu_{ym}=0$, $\mu_m=\mu_b=0$, $A_x=A_y=0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD.

MEMBRANE FORCES (TENSION)

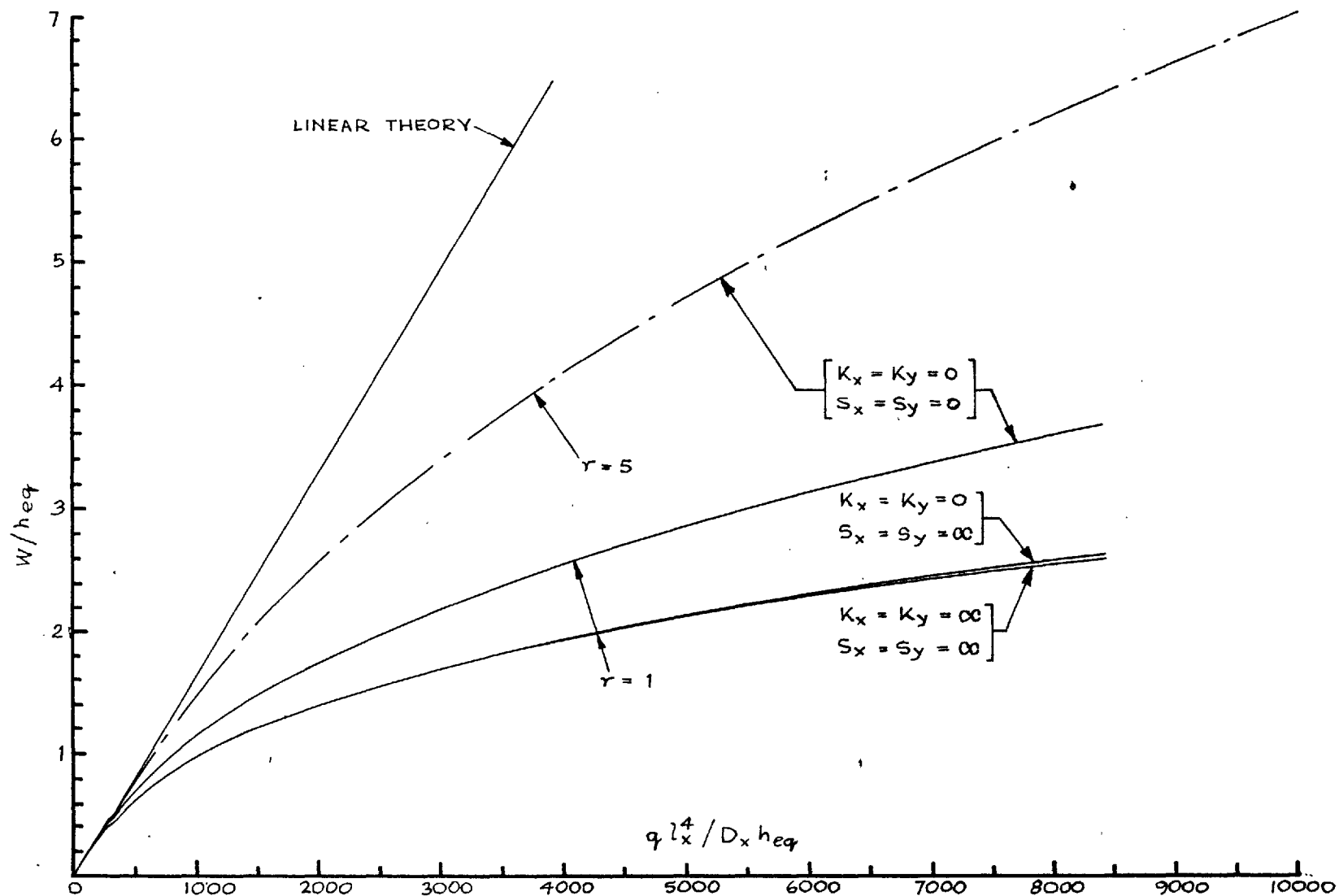


FIG.21: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon = 1$, $\nu_{xm} \nu_{ym} = 0$, $\mu_m = \mu_b = 0$, $A_x = A_y = \infty$
(CLAMPED EDGES) UNIFORM LOAD.

DEFLECTION AT CENTRE

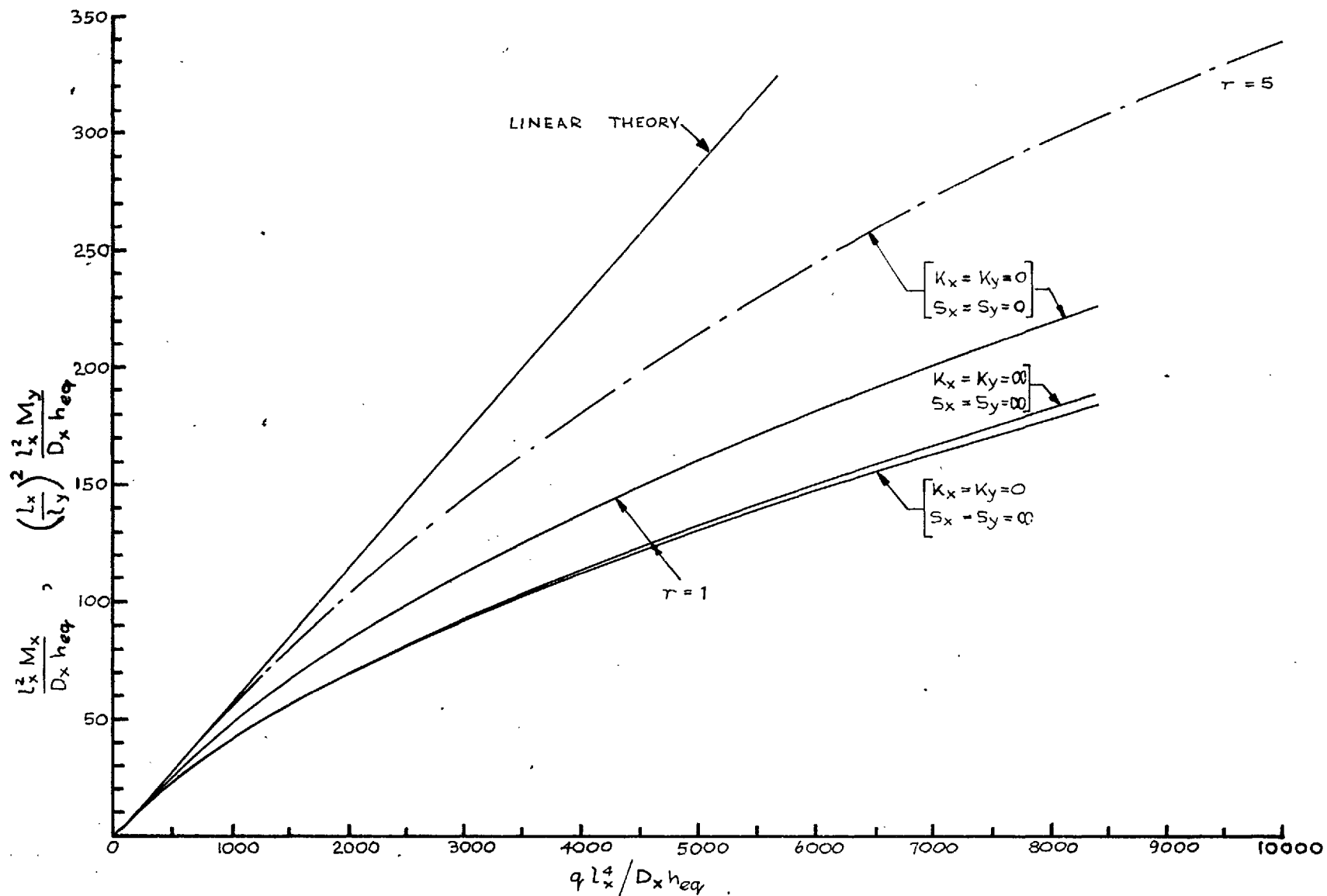


FIG.22: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon=1$, $\nu_{xm}\nu_{ym}=0$, $\mu_m=\mu_b=0$, $A_x=A_y=\infty$ (CLAMPED EDGES) UNIFORM LOAD.

CURVATURES AT MIDPOINTS OF EDGES.

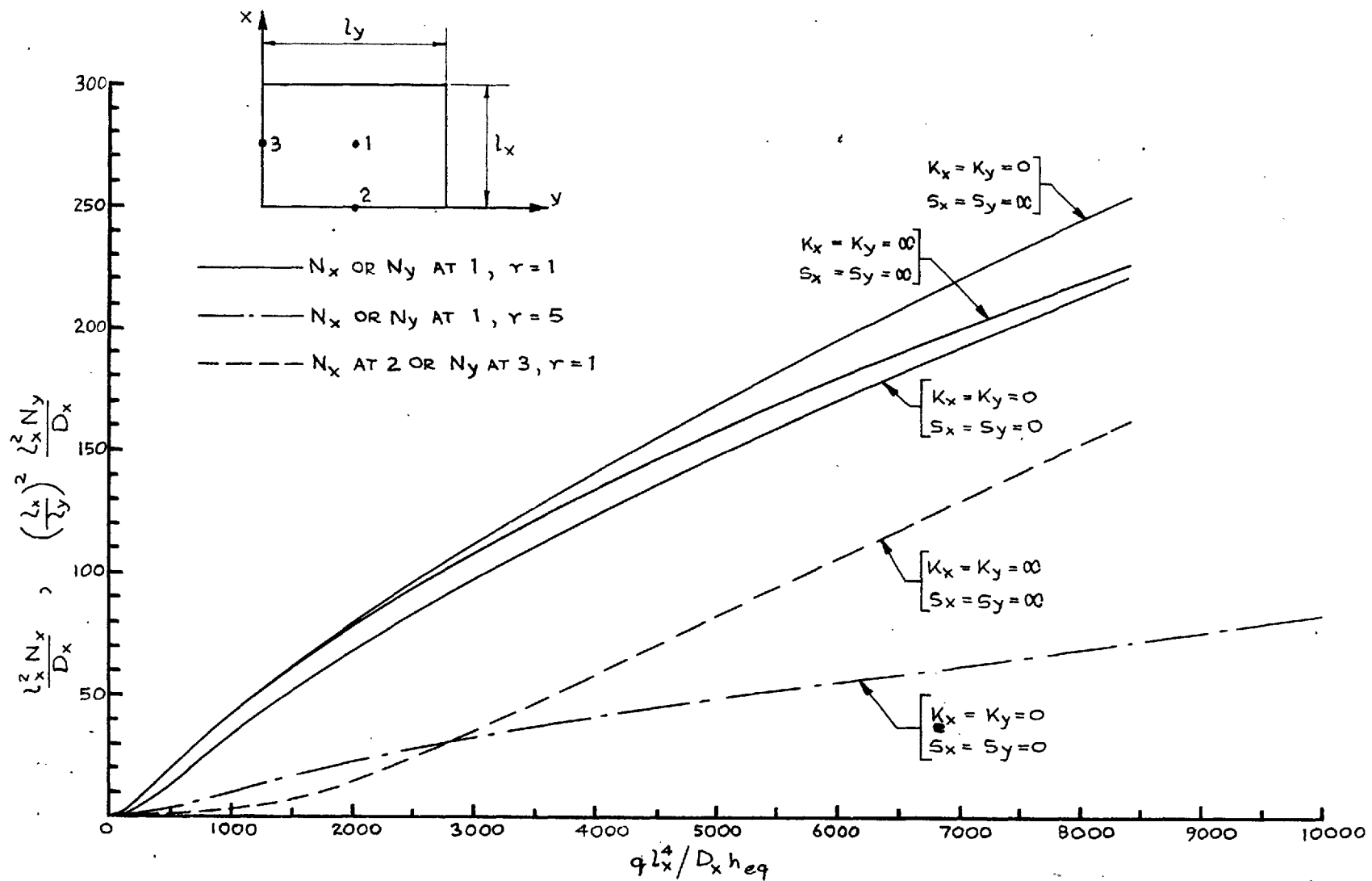


FIG.23: RECTANGULAR ORTHOTROPIC PLATE, $E = 1$, $\nu_{xm} \nu_{ym} = 0$, $M_m = M_b = 0$, $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD.

MEMBRANE FORCES (TENSION)

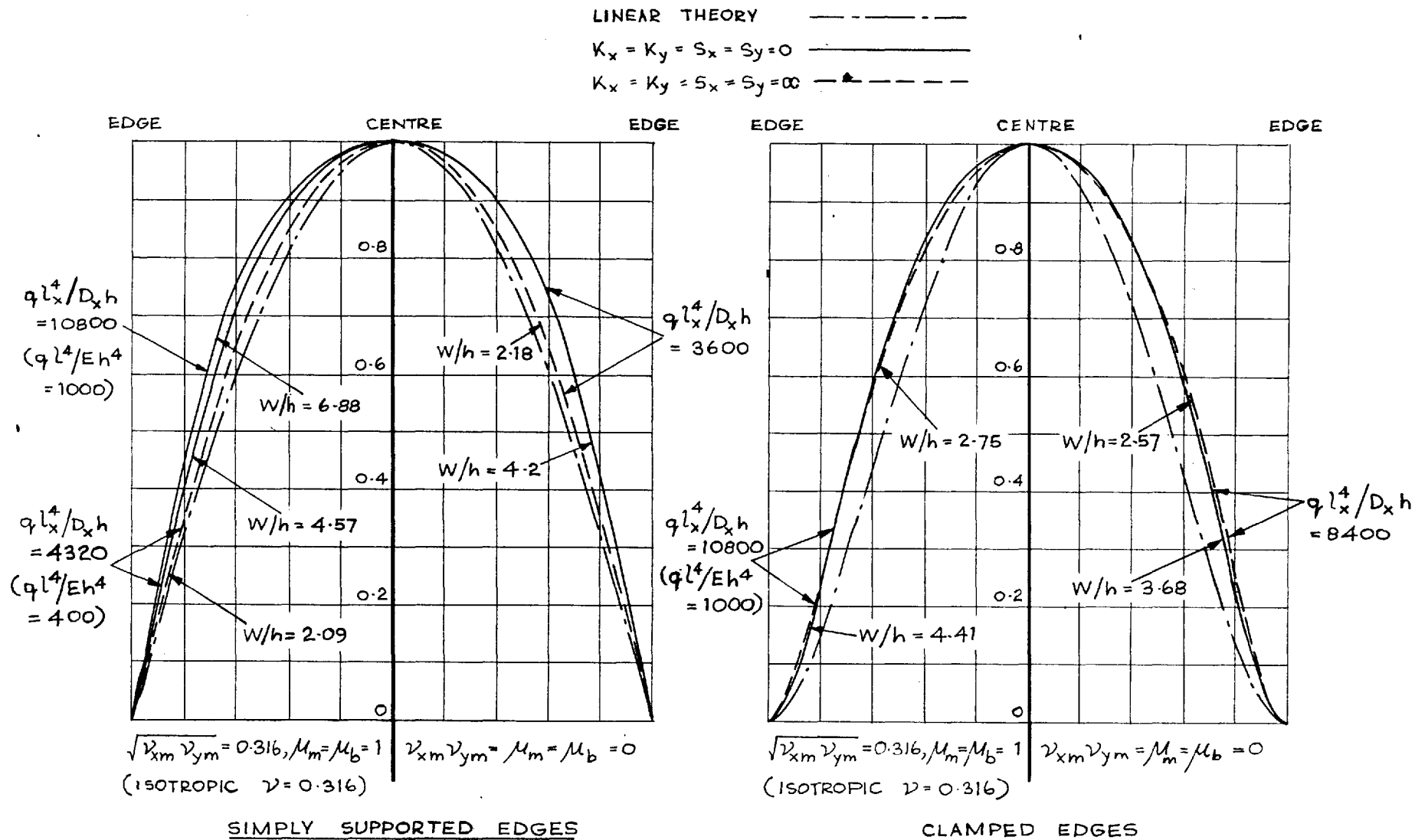
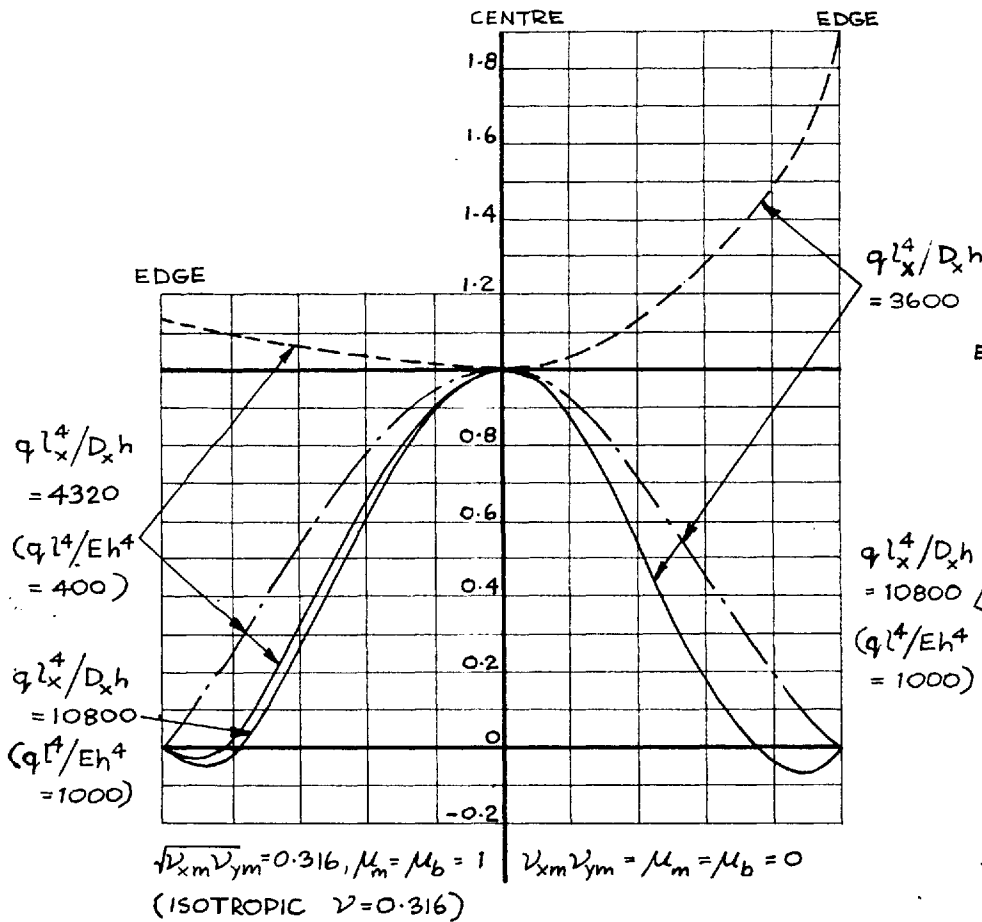
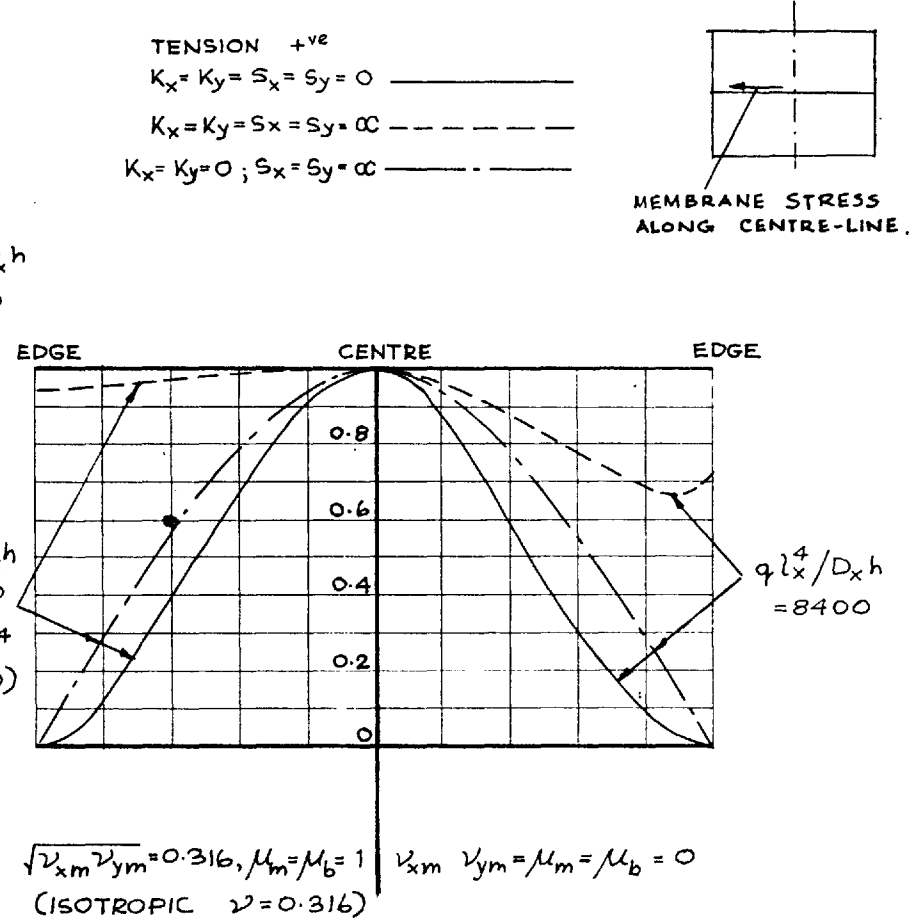


FIG.24: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon=1$ (INCLUDING THE SQUARE ISOTROPIC PLATE)
 UNIFORM LOAD
COMPARISON OF DEFLECTED SHAPES OF PLATE CENTRE-LINES



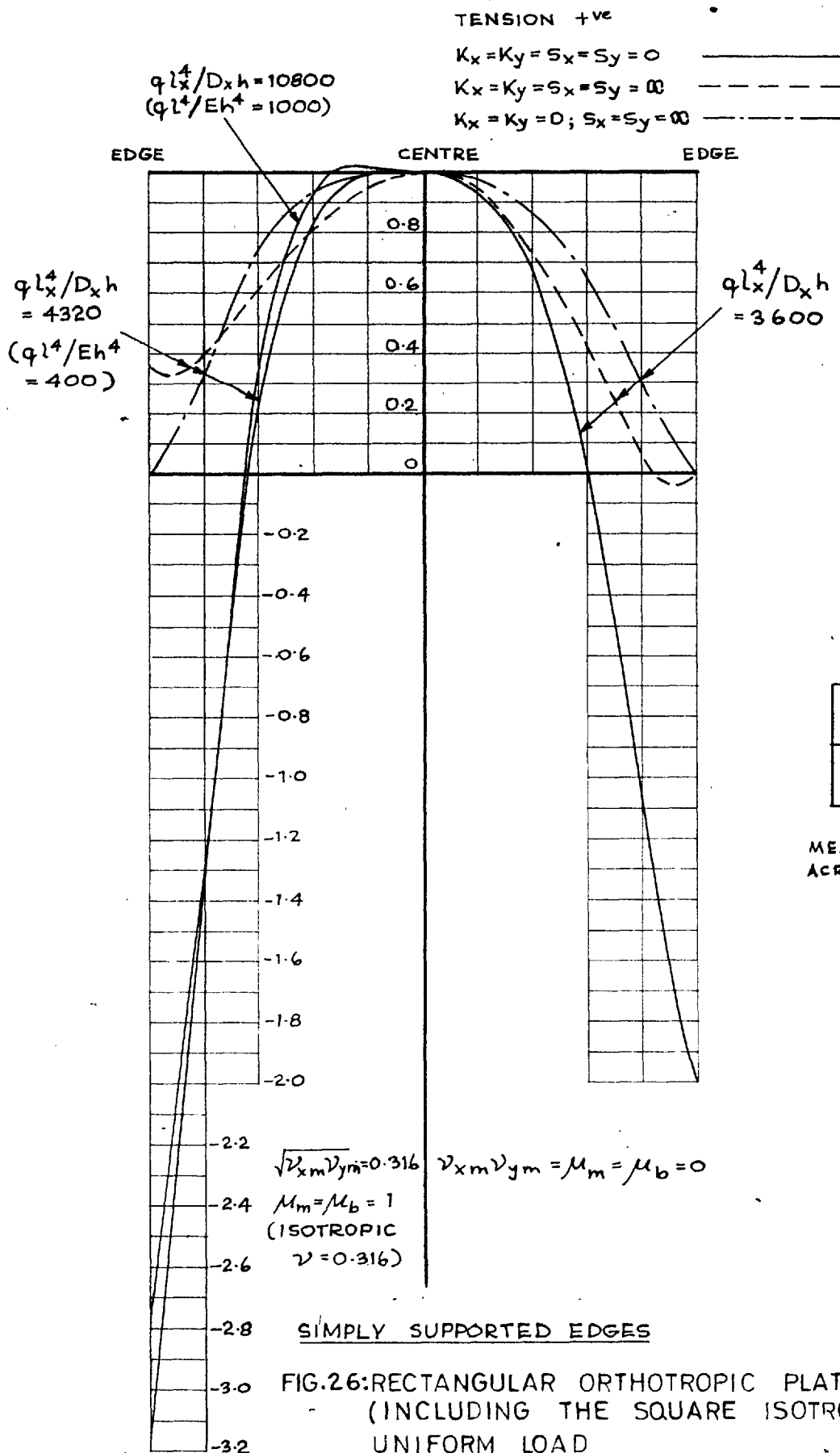
SIMPLY SUPPORTED EDGES



CLAMPED EDGES

FIG.25: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1$, (INCLUDING THE SQUARE ISOTROPIC PLATE) UNIFORM LOAD

COMPARISON OF DISTRIBUTION OF MEMBRANE STRESS ALONG CENTRE LINE ON CENTRAL SECTION.



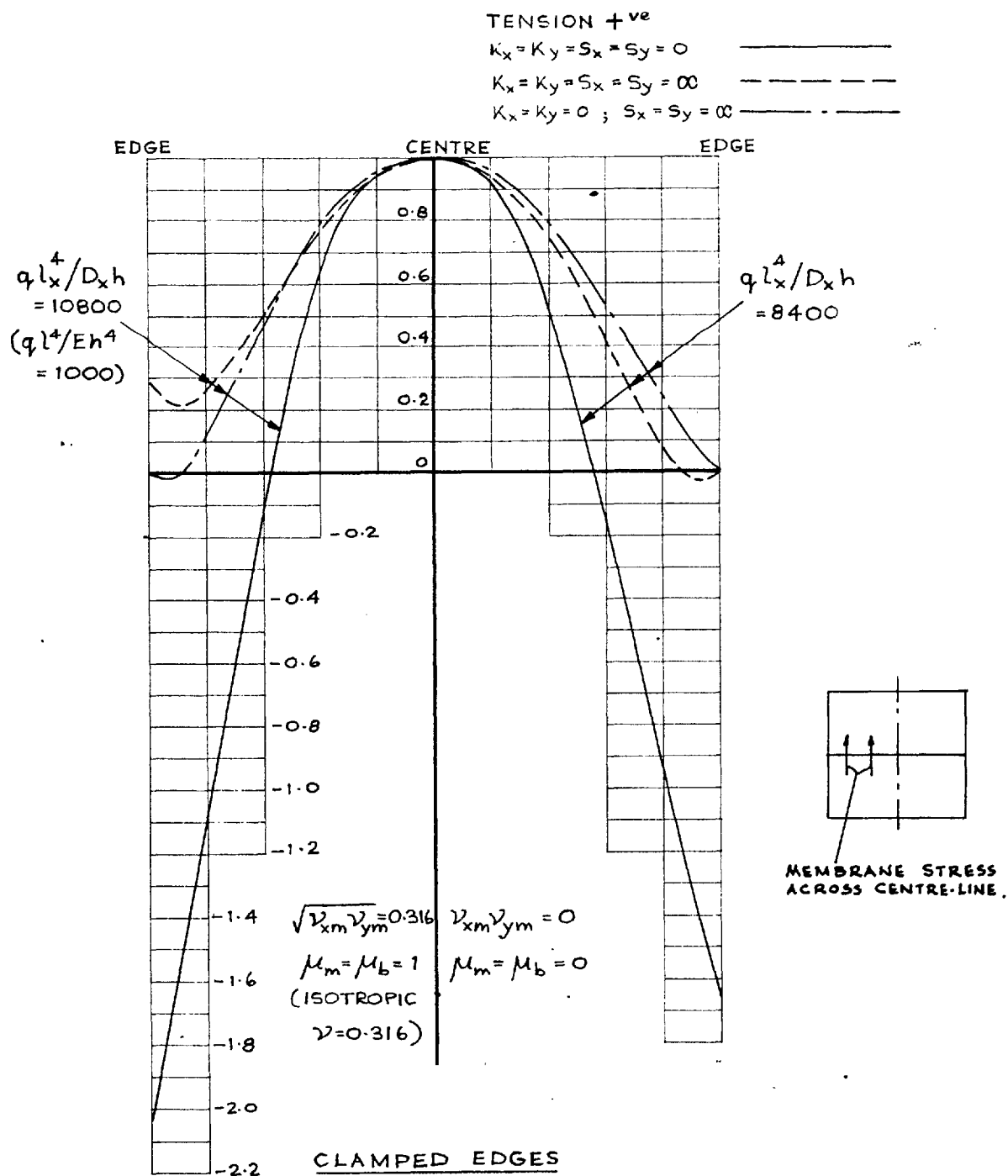


FIG.27: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon=1$ (INCLUDING THE SQUARE ISOTROPIC PLATE) UNIFORM LOAD.

COMPARISON OF DISTRIBUTION OF MEMBRANE STRESS ACROSS CENTRE-LINE ON CENTRAL SECTION.

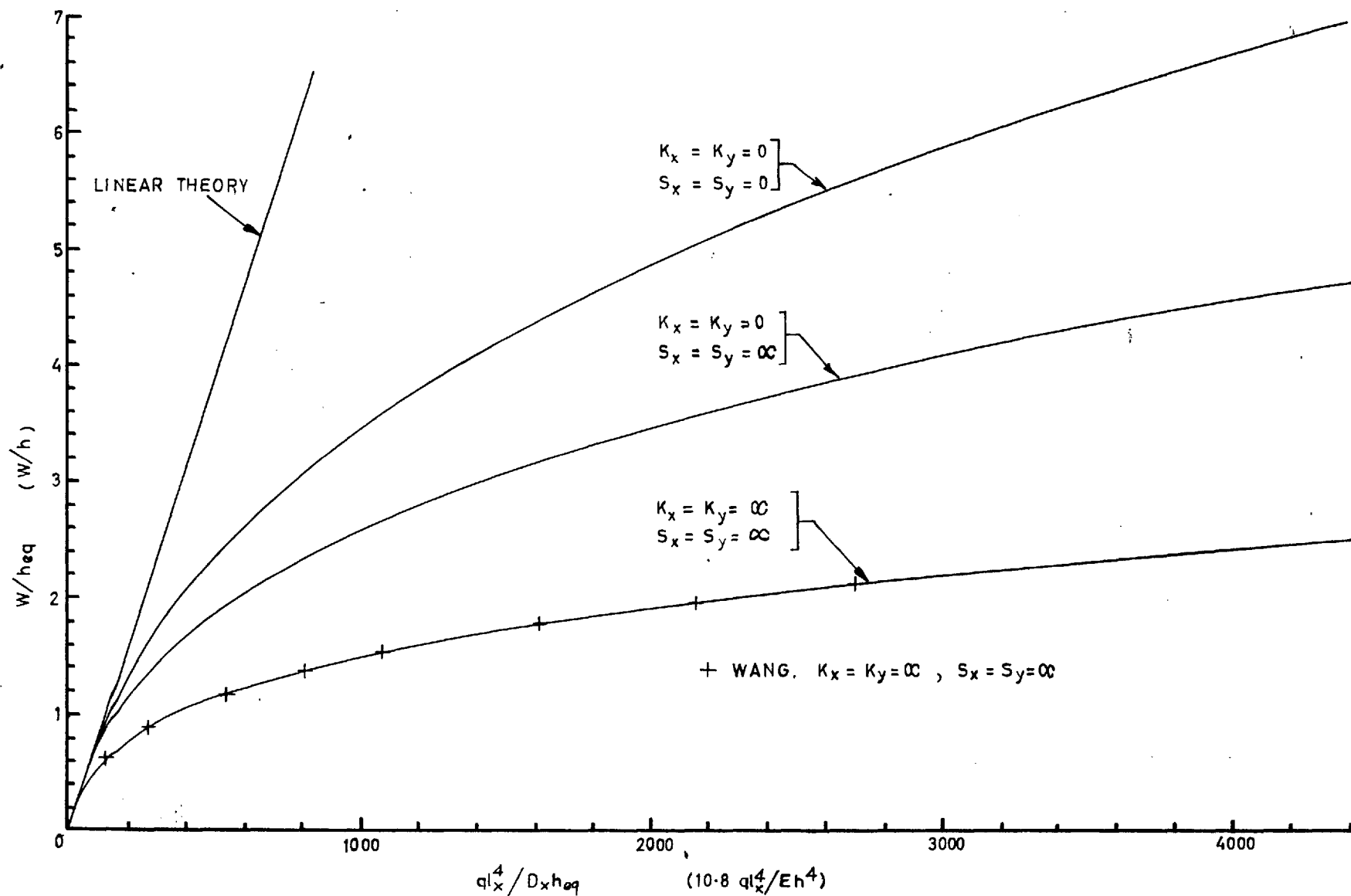


FIG. 28: RECTANGULAR ORTHOTROPIC PLATE $\varepsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm}\nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$
(INCLUDING THE RECTANGULAR ISOTROPIC PLATE $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY
SUPPORTED EDGES) UNIFORM LOAD DEFLECTION AT CENTRE

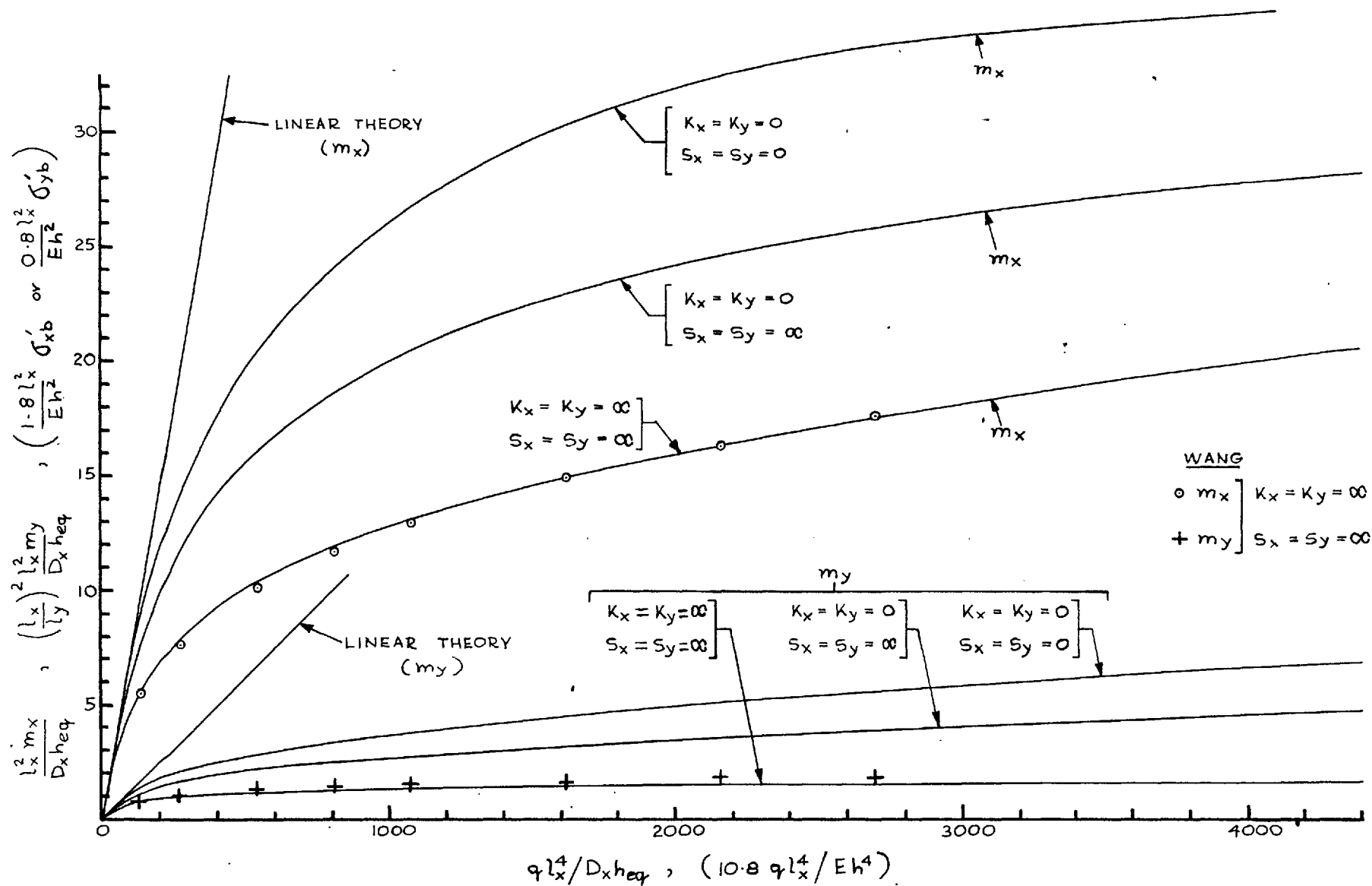


FIG.29: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE ISOTROPIC PLATE, $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD

CURVATURES AT CENTRE

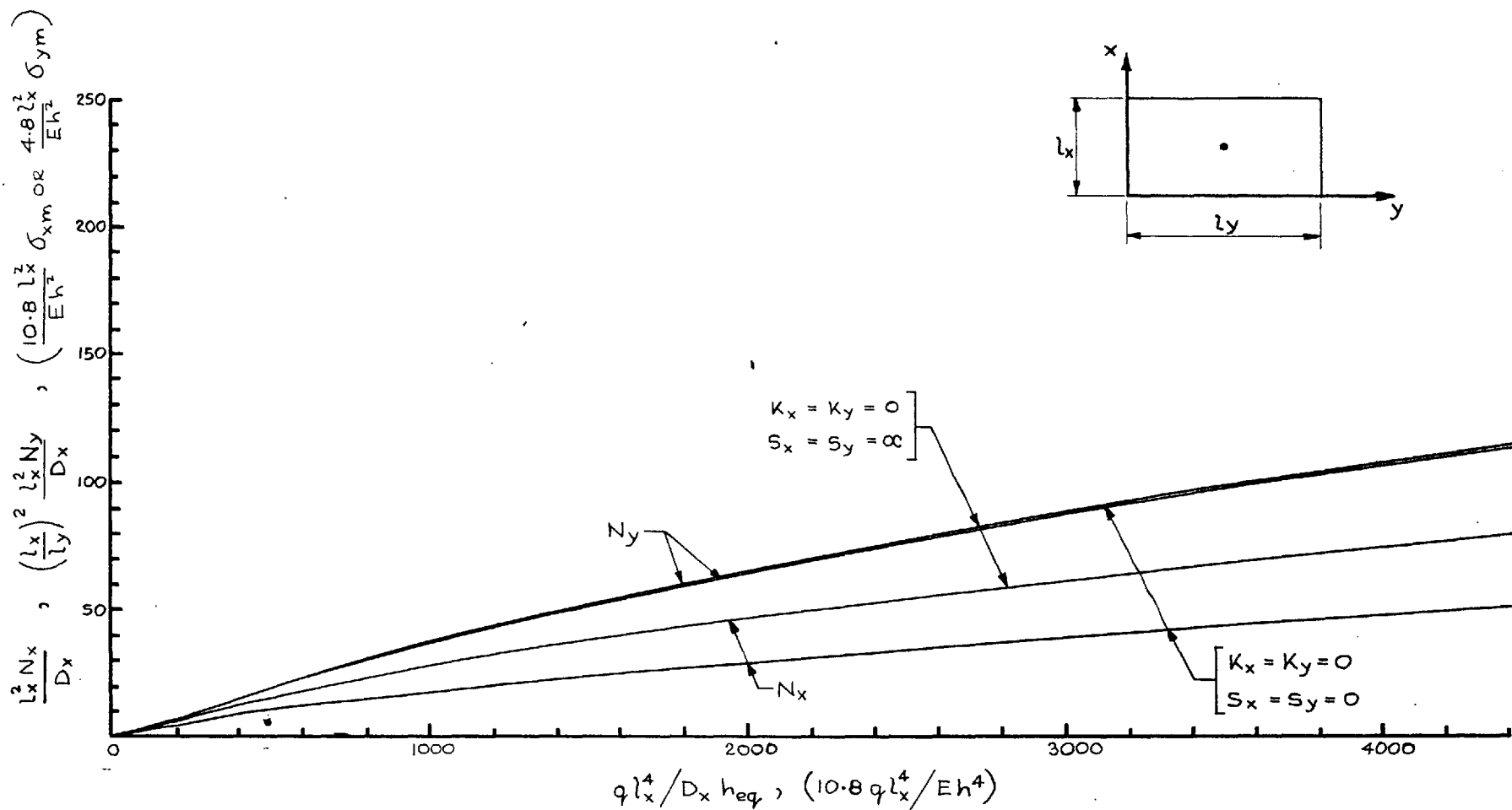


FIG. 30: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE RECTANGULAR ISOTROPIC PLATE, $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD.

MEMBRANE FORCES AT CENTRE (TENSION)

WANG

$\begin{array}{l} + N_x \text{ AT } 2 \\ * N_x \text{ AT } 1 \\ \circ N_y \text{ AT } 3 \\ \square N_y \text{ AT } 1 \end{array} \left[\begin{array}{l} K_x = K_y = \infty \\ S_x = S_y = \infty \end{array} \right.$

--- EDGE FORCES
 — CENTRE FORCES

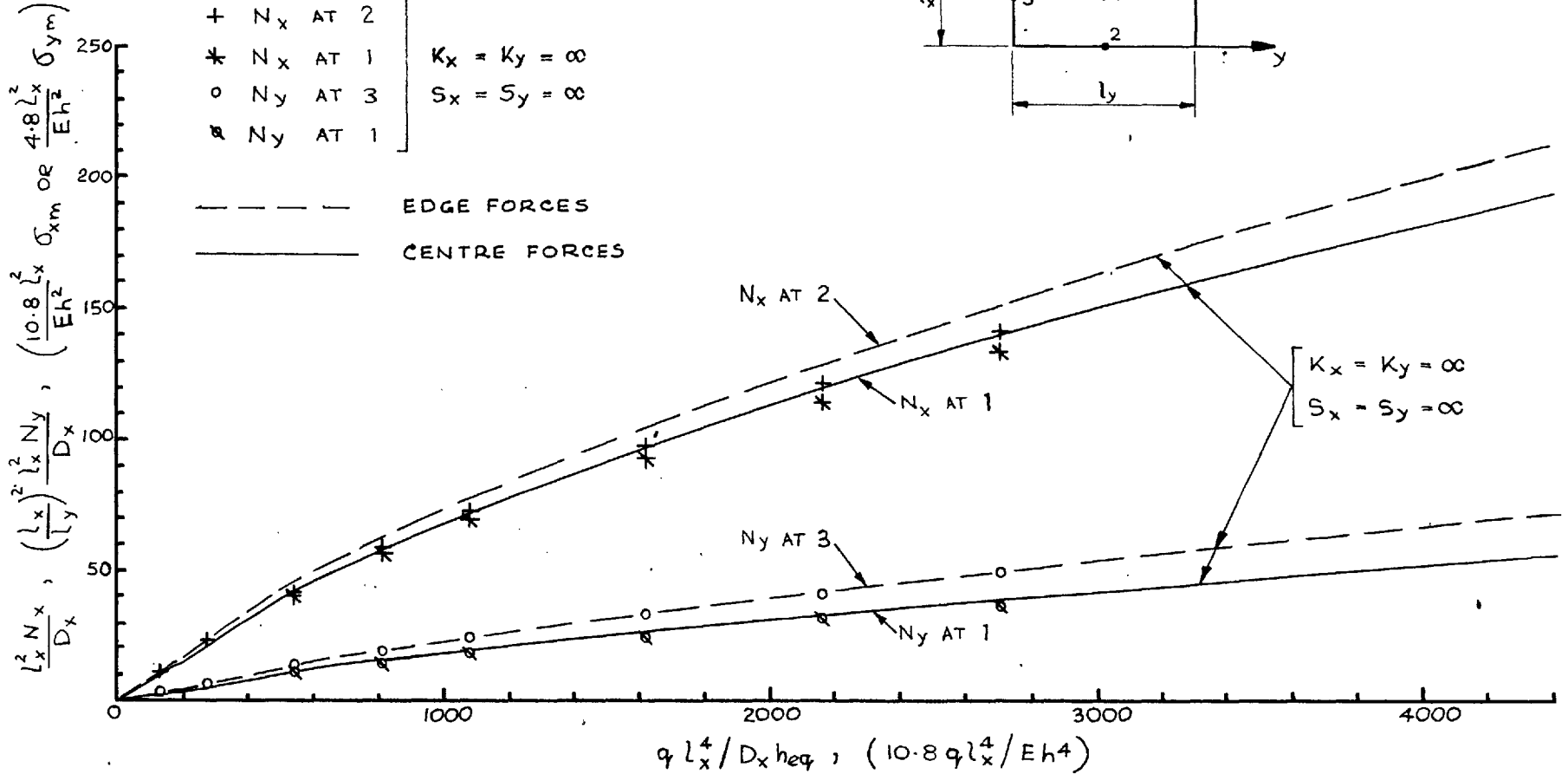
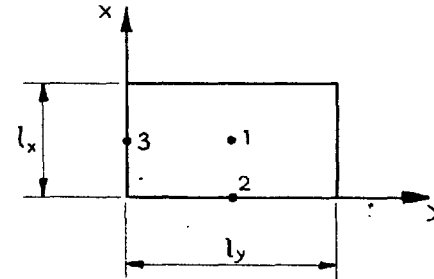


FIG.31: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $M_m = M_b = 1$ (INCLUDING THE RECTANGULAR ISOTROPIC PLATE, $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) UNIFORM LOAD

MEMBRANE FORCES (TENSION)

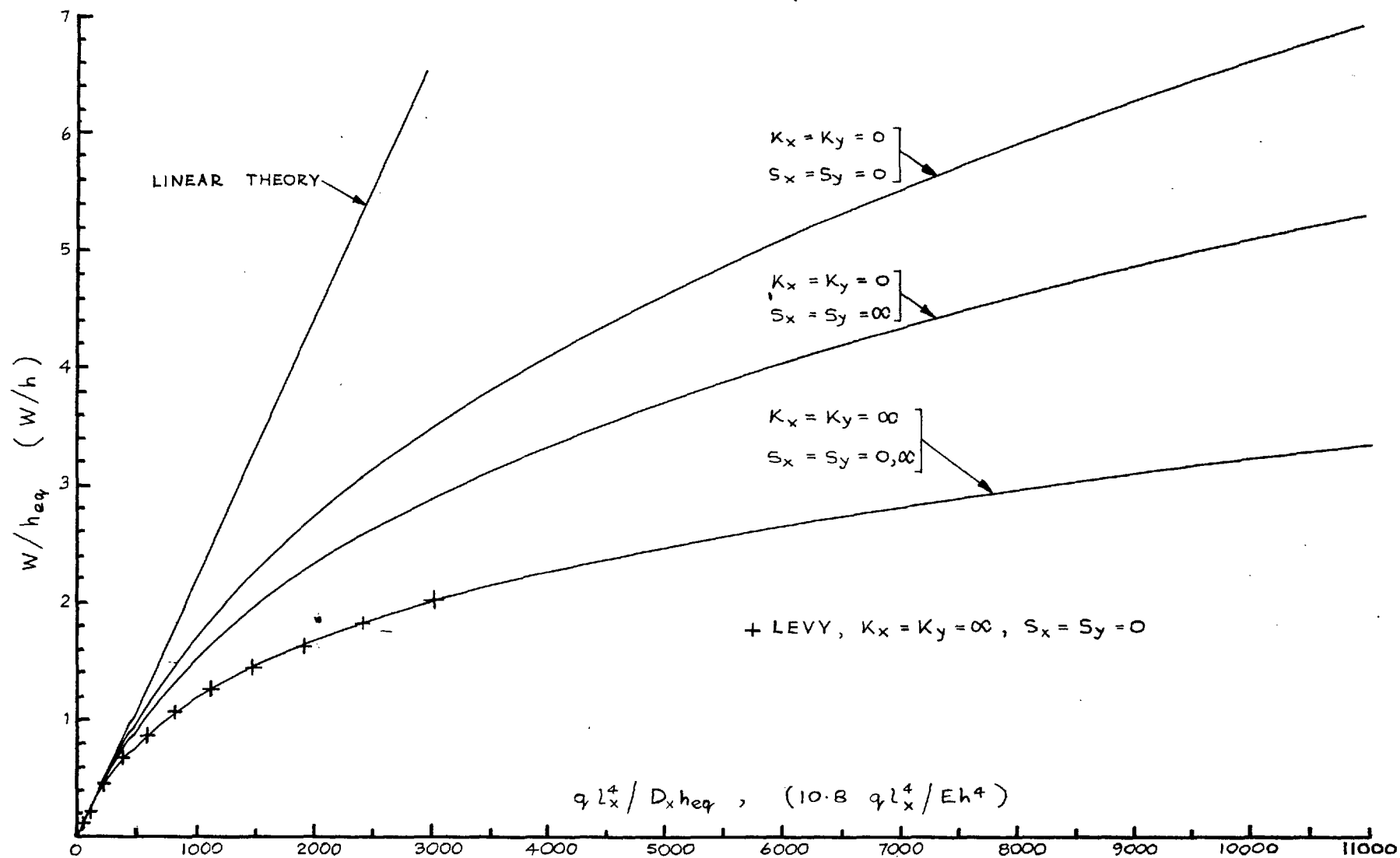


FIG.32: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE RECTANGULAR ISOTROPIC PLATE, $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD

DEFLECTION AT CENTRE

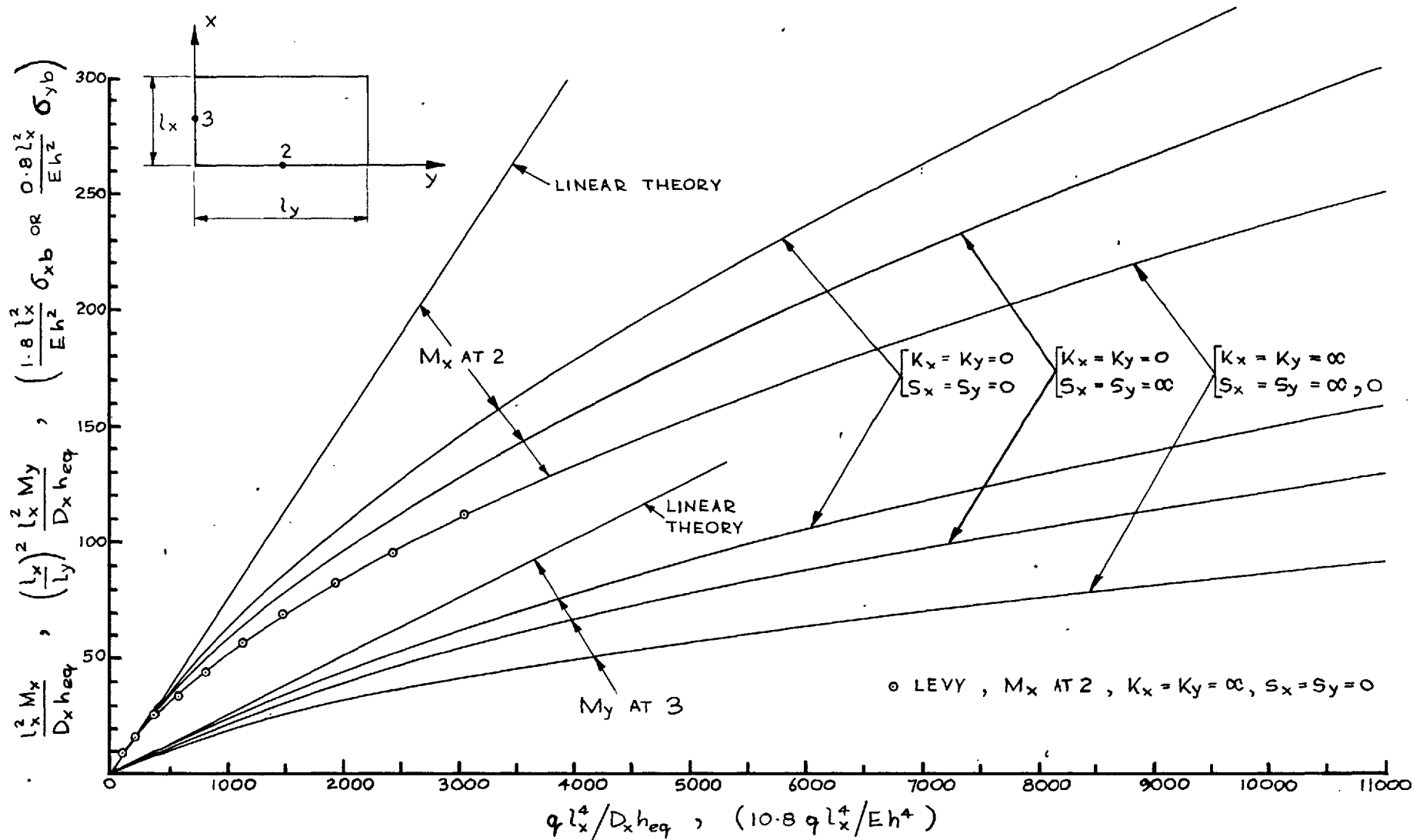


FIG. 33: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE RECTANGULAR ISOTROPIC PLATE, $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD

CURVATURES AT MIDPOINTS OF EDGES

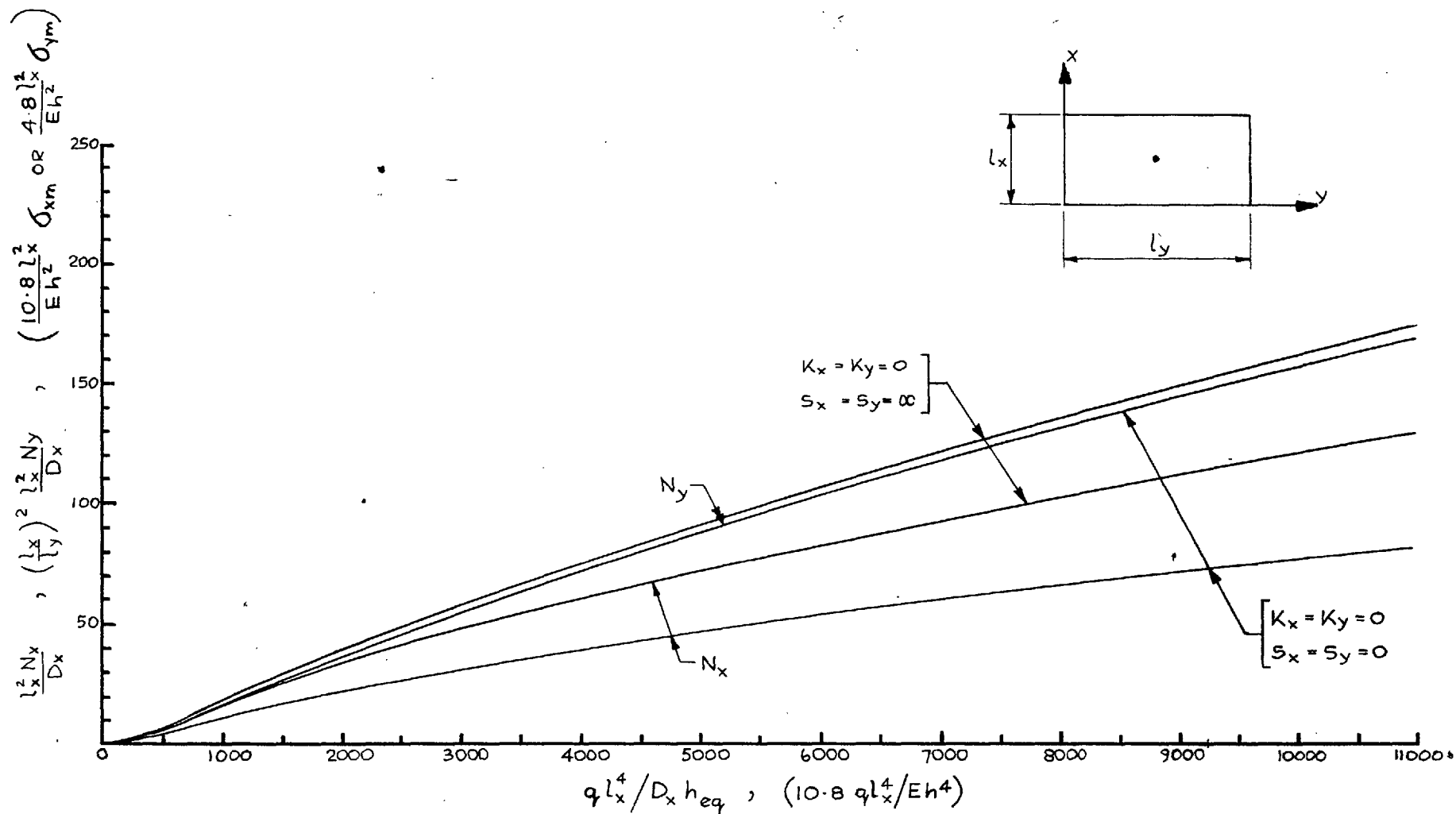


FIG. 34: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1.5$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE RECTANGULAR ISOTROPIC PLATE $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD.

MEMBRANE FORCES AT CENTRE (TENSION)

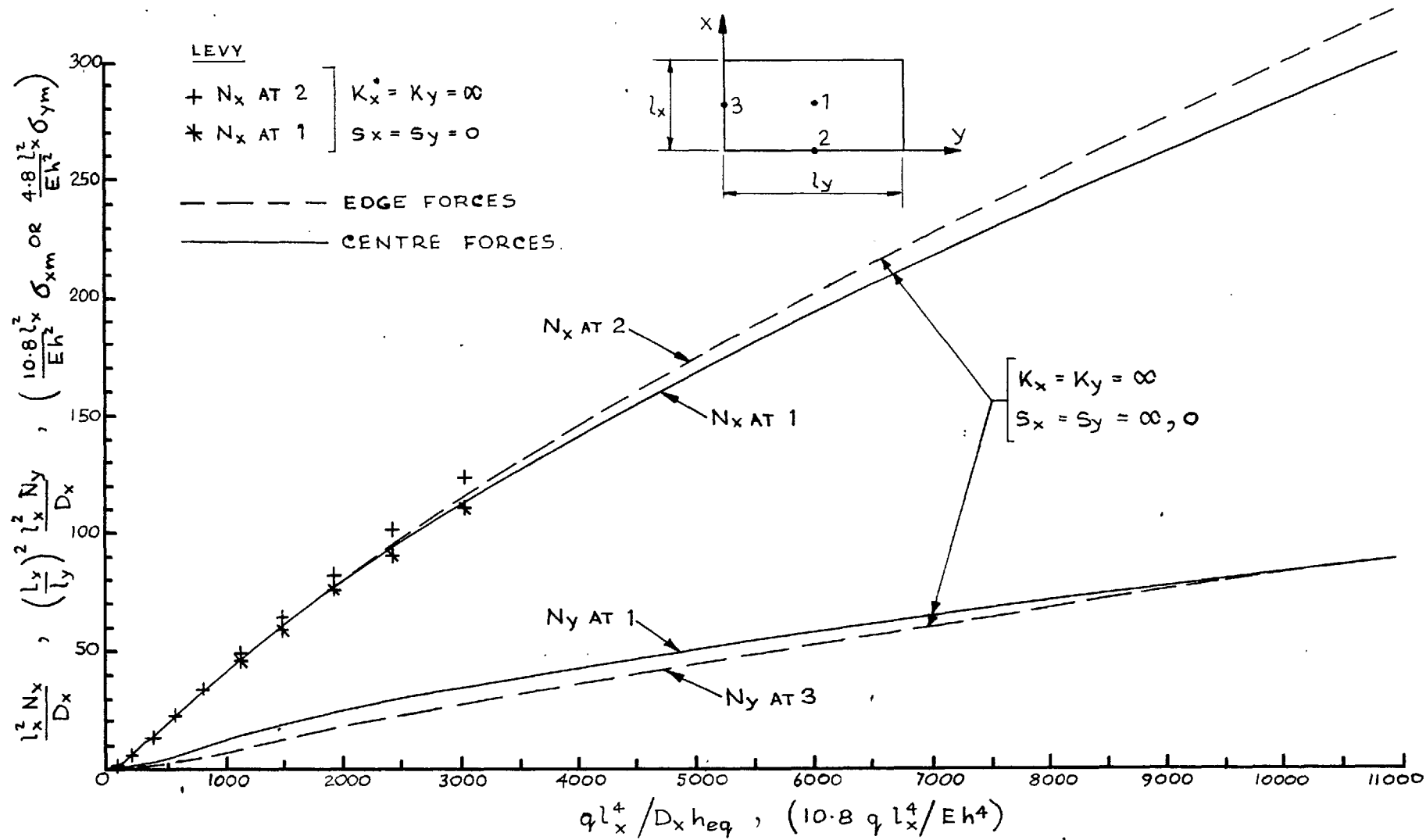


FIG. 35: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon=1.5$, $r=1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE RECTANGULAR ISOTROPIC PLATE. $l_y/l_x = 1.5$, $\nu = 0.316$) $A_x = A_y = \infty$ (CLAMPED EDGES) UNIFORM LOAD.

MEMBRANE FORCES (TENSION)

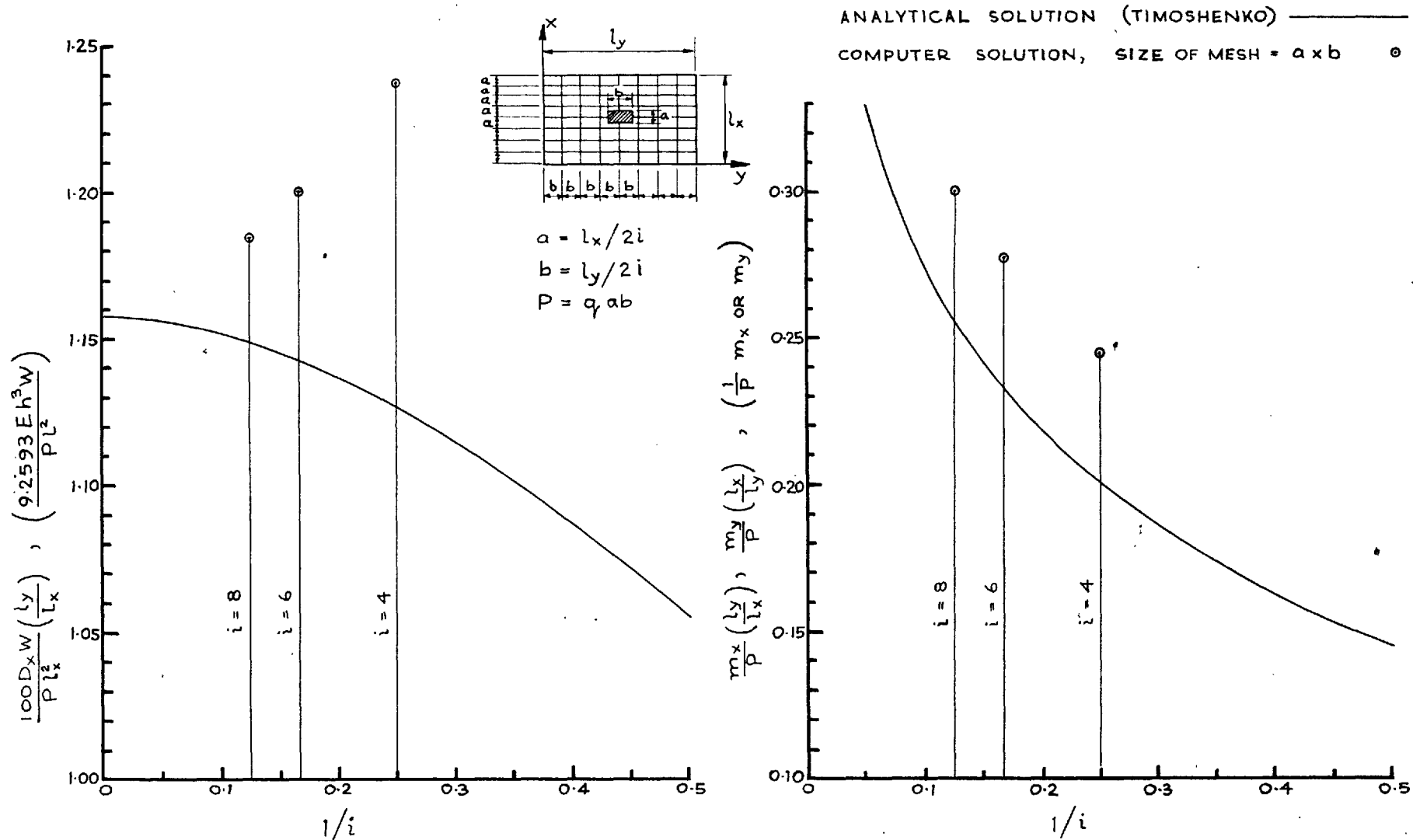


FIG.36: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) CENTRAL PATCH LOAD.

DEFLECTION AND CURVATURE AT CENTRE ACCORDING TO LINEAR THEORY

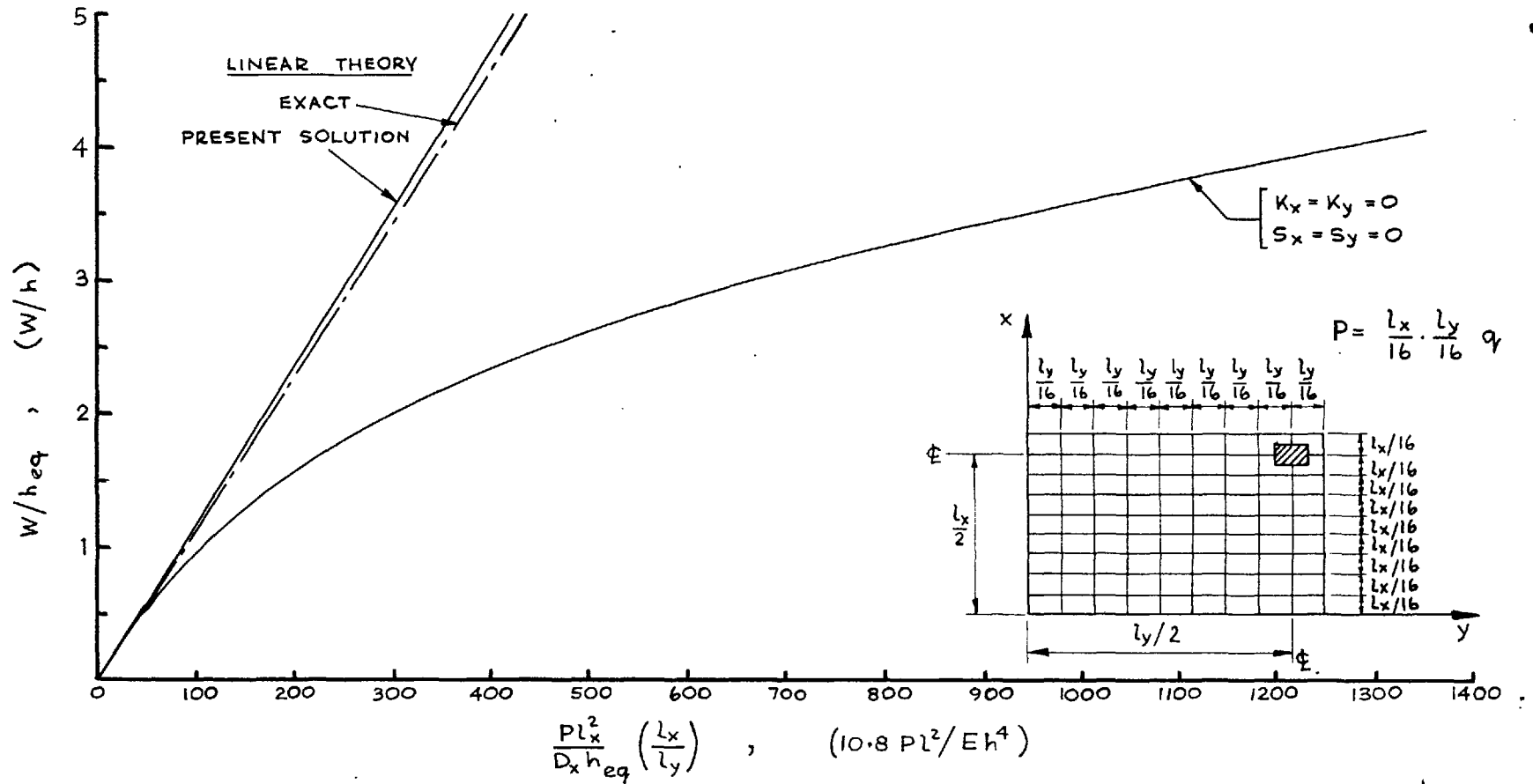


FIG.37: RECTANGULAR ORTHOTROPIC PLATE, $E = 1, \nu = 1, \sqrt{\nu_{xm}\nu_{ym}} = 0.316, M_m = M_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) CENTRAL PATCH LOAD.

DEFLECTION AT CENTRE

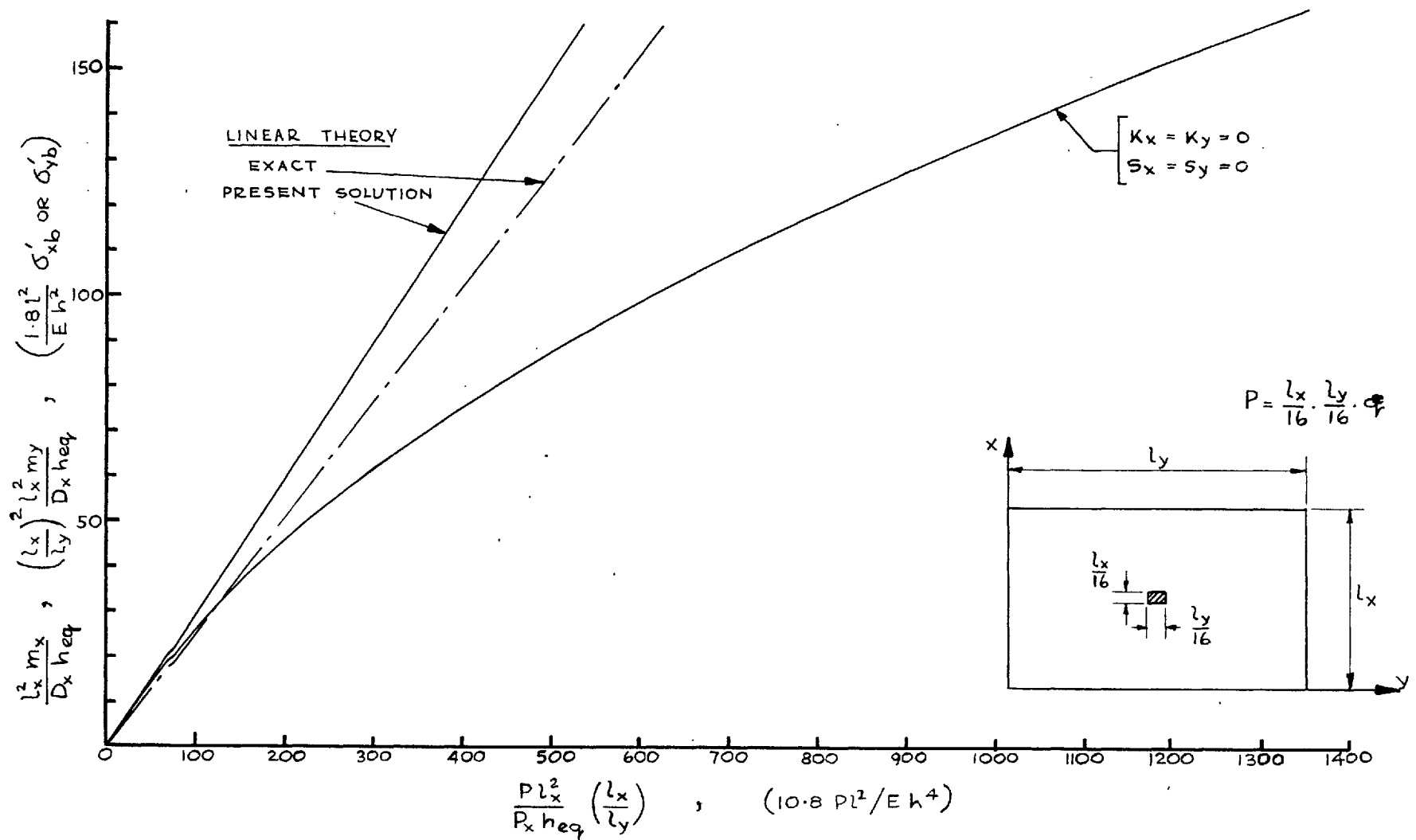


FIG.38: RECTANGULAR ORTHOTROPIC PLATE, $\varepsilon=1$, $r=1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) CENTRAL PATCH LOAD.

CURVATURE AT CENTRE

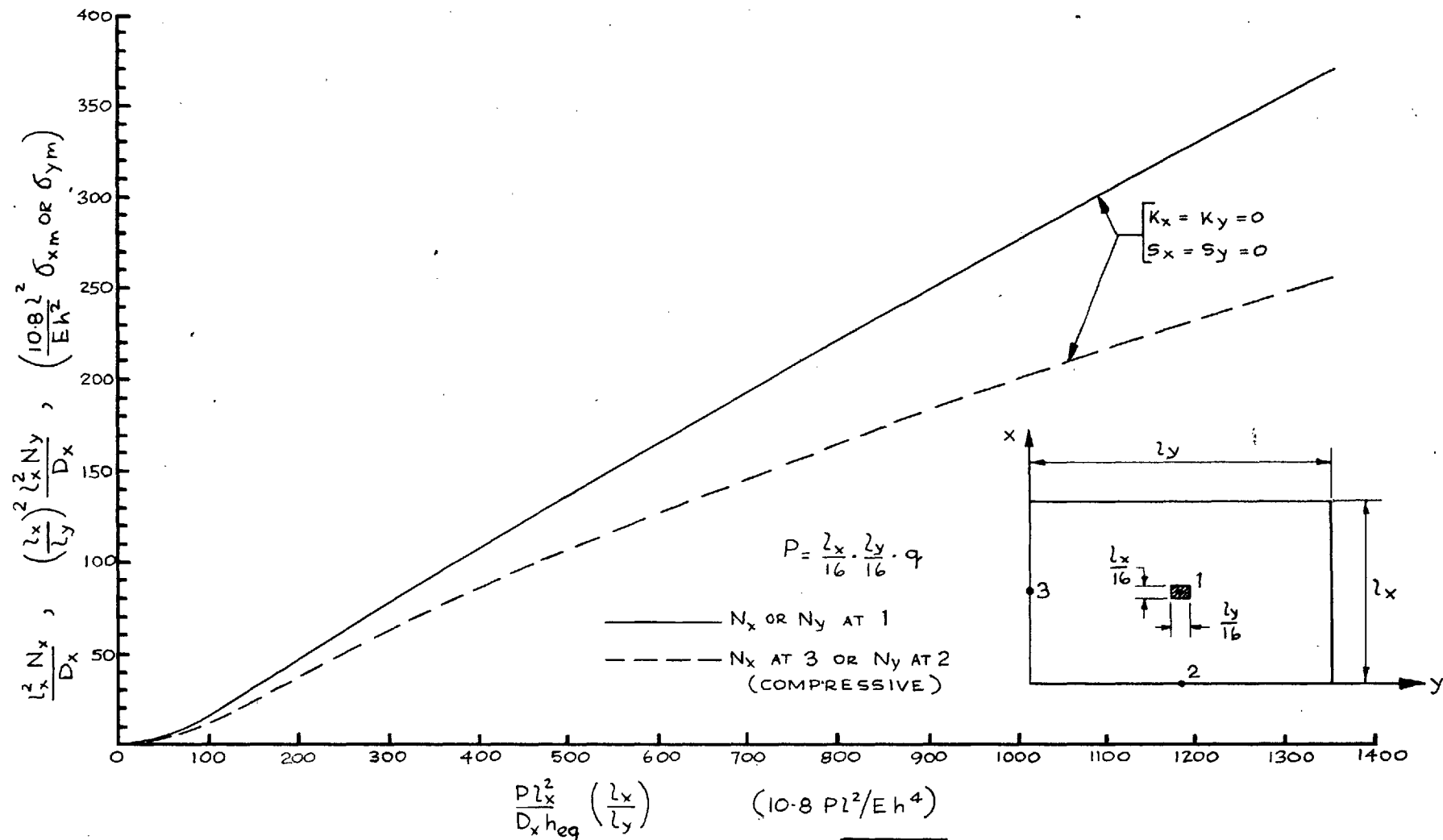


FIG. 39: RECTANGULAR ORTHOTROPIC PLATE, $E = 1$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $\mu_m = \mu_b = 1$, (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) CENTRAL PATCH LOAD.

MEMBRANE FORCES

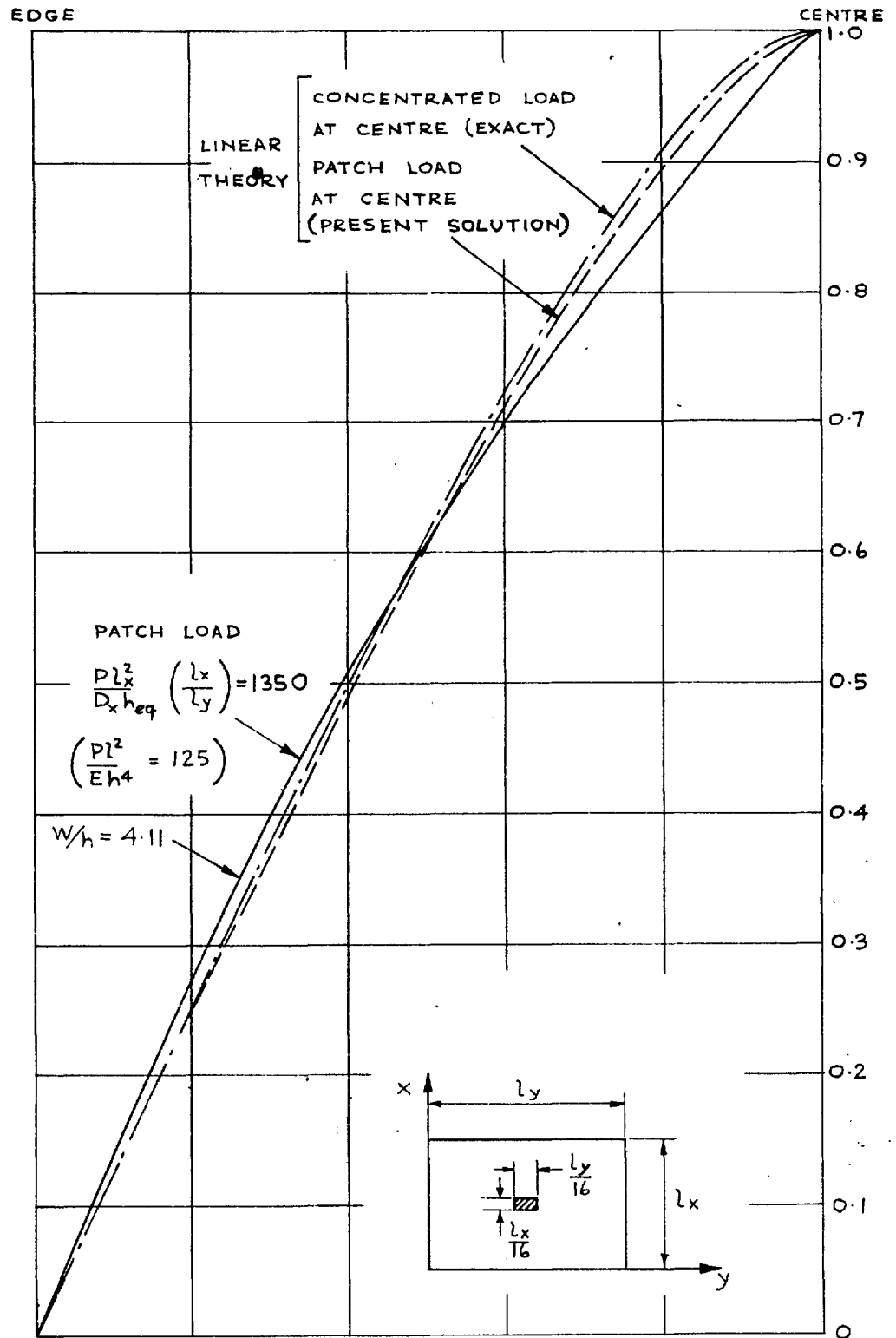


FIG. 40: RECTANGULAR ORTHOTROPIC PLATE, $E=1$, $r=1$, $\sqrt{\nu_{xm}\nu_{ym}}=0.316$, $M_m=M_b=1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu=0.316$) $A_x=A_y=0$ (SIMPLY SUPPORTED EDGES) $K_x=K_y=S_x=S_y=0$. CENTRAL PATCH LOAD. DEFLECTED SHAPE OF PLATE CENTRE LINE

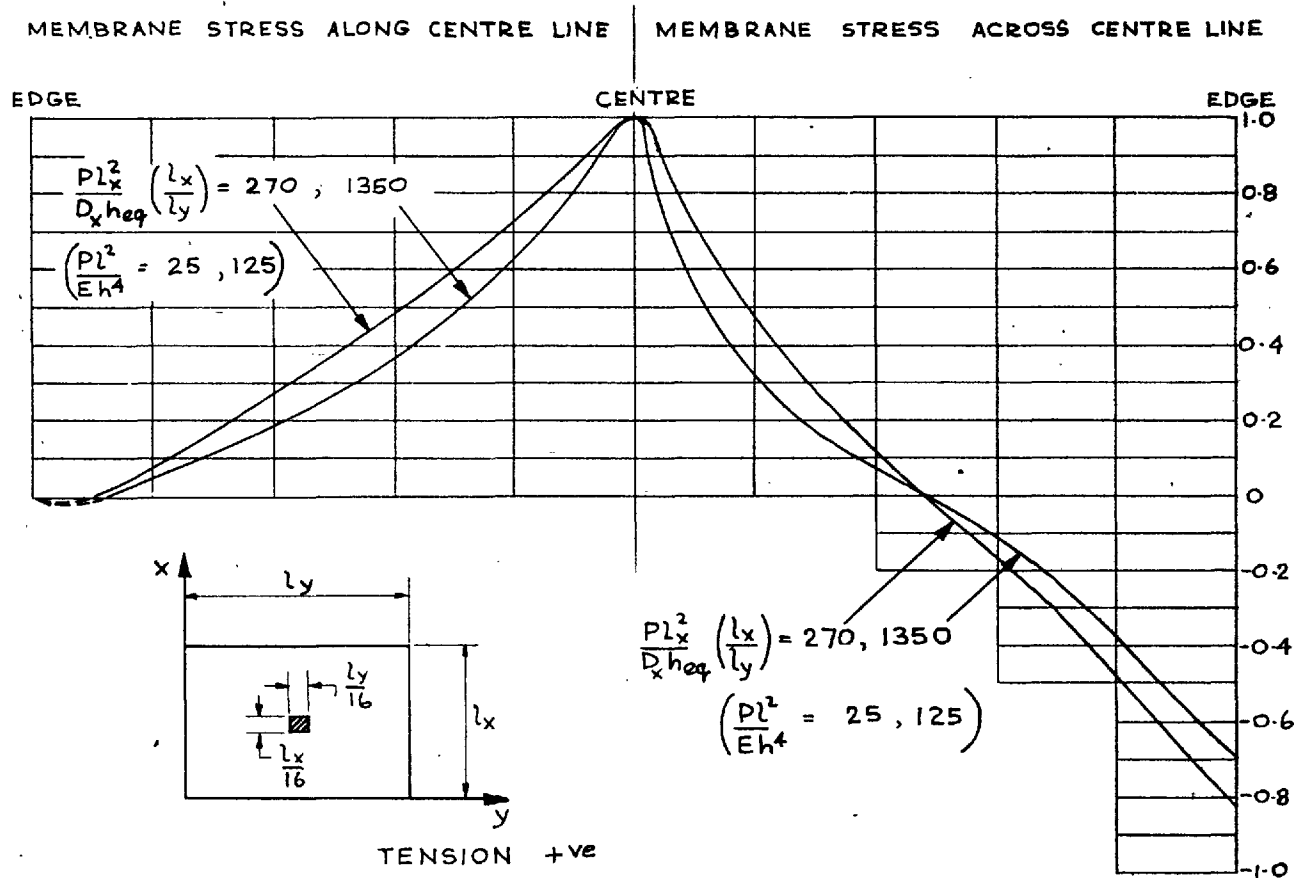


FIG.41: RECTANGULAR ORTHOTROPIC PLATE, $\epsilon = 1$, $r = 1$, $\sqrt{\nu_{xm} \nu_{ym}} = 0.316$, $M_m = M_b = 1$ (INCLUDING THE SQUARE ISOTROPIC PLATE, $\nu = 0.316$) $A_x = A_y = 0$ (SIMPLY SUPPORTED EDGES) $K_x = K_y = S_x = S_y = 0$, CENTRAL PATCH LOAD.

DISTRIBUTION OF MEMBRANE STRESS AT THE CENTRAL SECTION

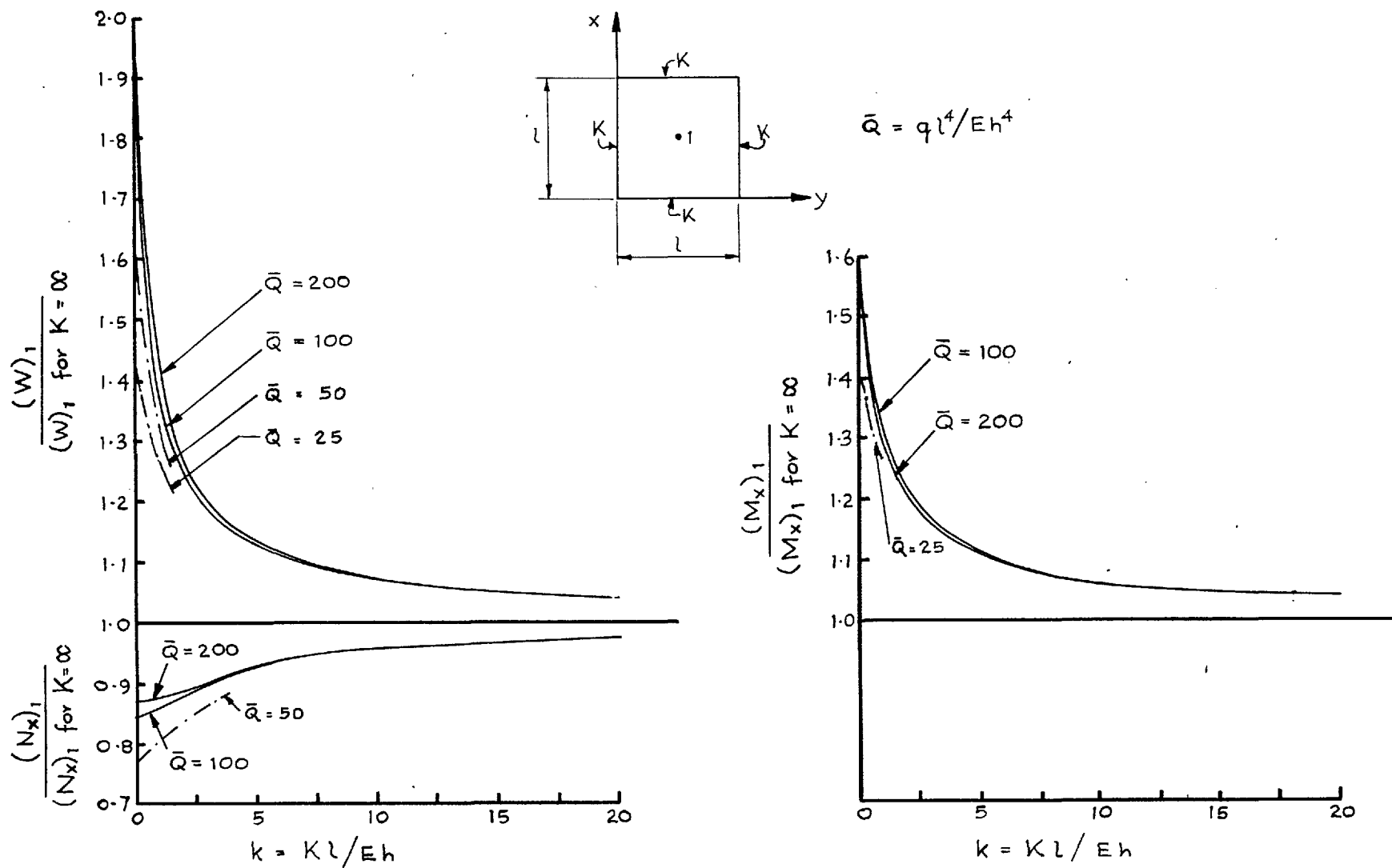
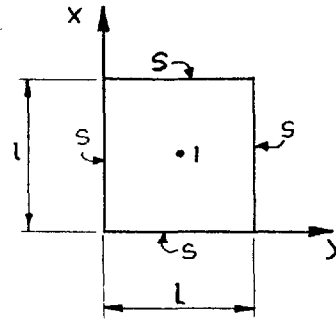


FIG. 42: SQUARE ISOTROPIC PLATE, $\nu = 0.316$, $A_x = A_y = 0$, $S_x = S_y = 0$, UNIFORM LOAD.
EFFECT OF VARYING EXTENSIONAL EDGE RESTRAINT K



$$\bar{Q} = ql^4/Eh^4.$$

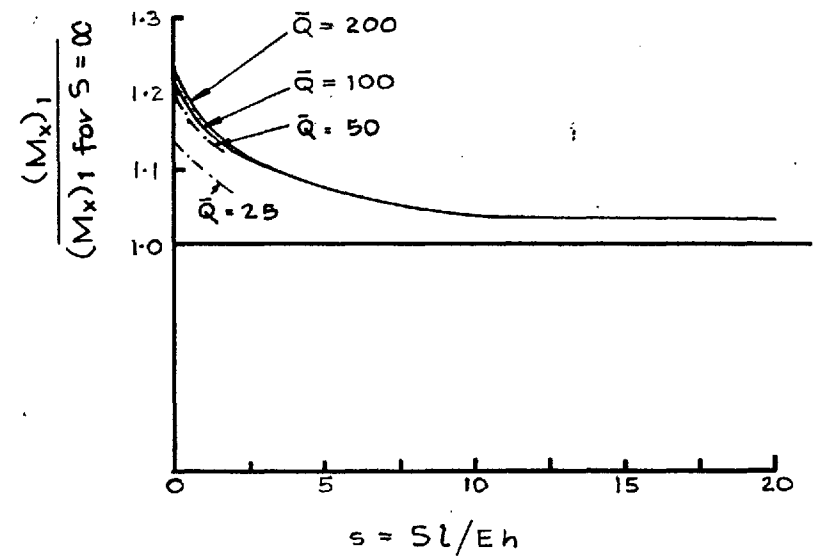
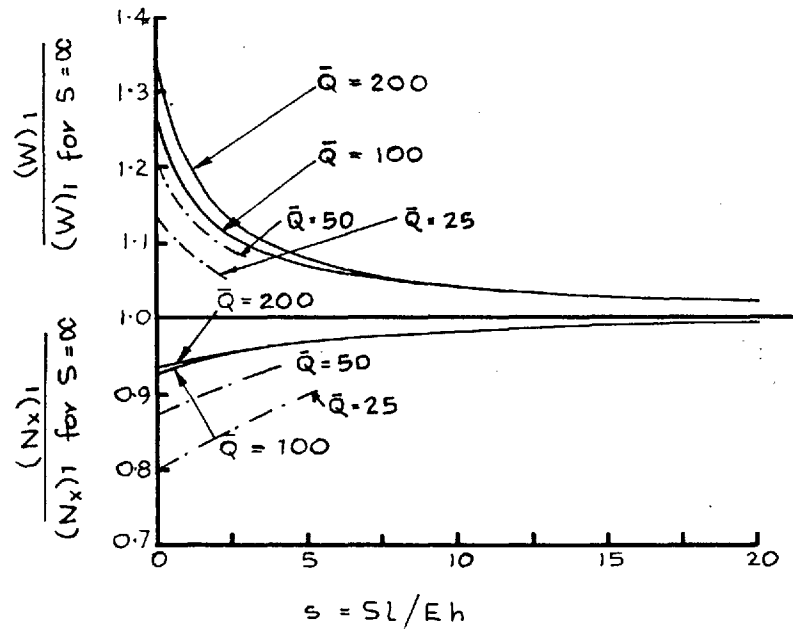


FIG. 43: SQUARE ISOTROPIC PLATE, $\nu = 0.316$, $A_x = A_y = 0$, $K_x = K_y = 0$, UNIFORM LOAD
EFFECT OF VARYING SHEAR EDGE RESTRAINT S

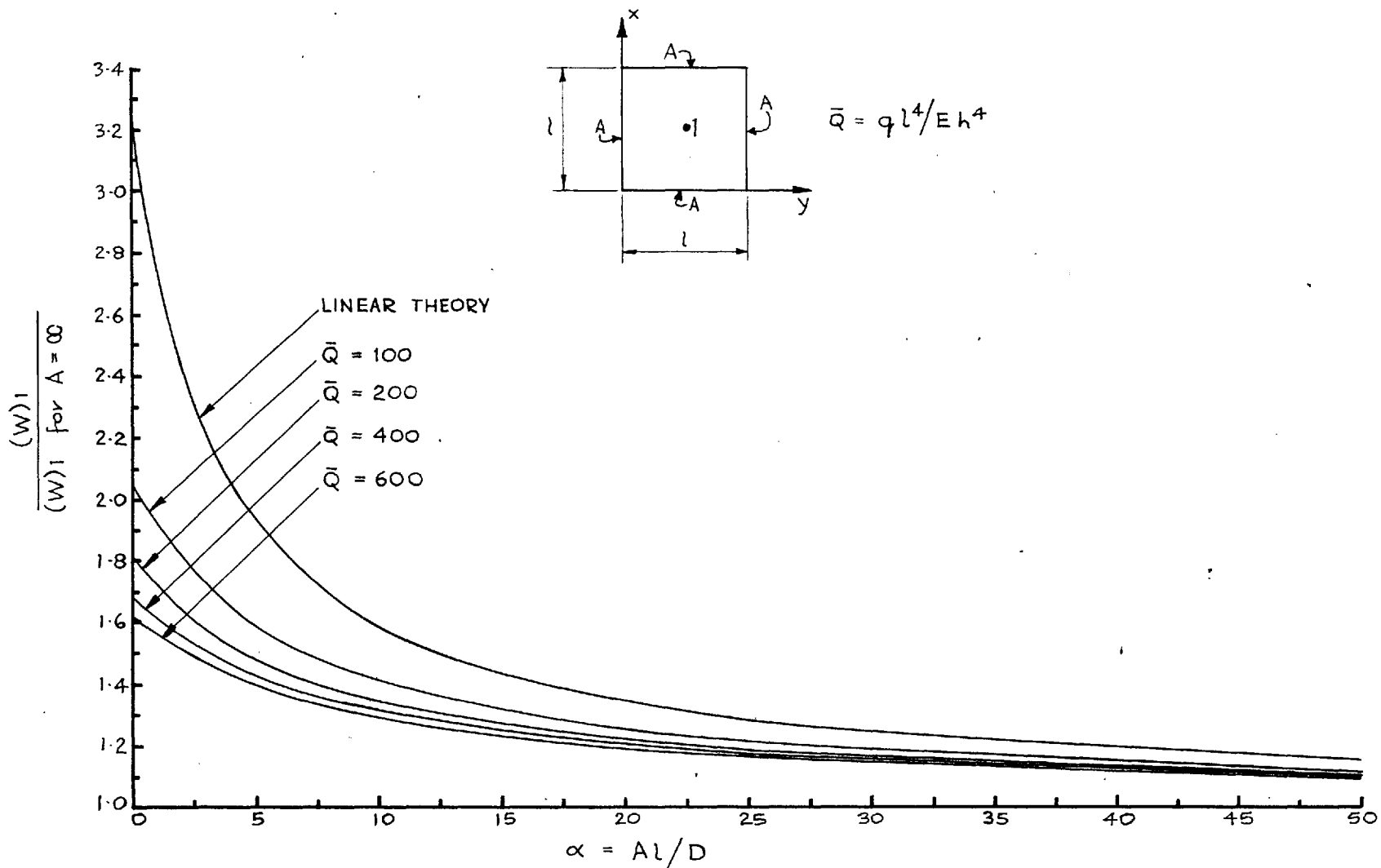


FIG.44: SQUARE ISOTROPIC PLATE, $\nu = 0.316$, $K_x = K_y = 0$, $S_x = S_y = 0$, UNIFORM LOAD
EFFECT OF VARYING ROTATIONAL EDGE RESTRAINT A

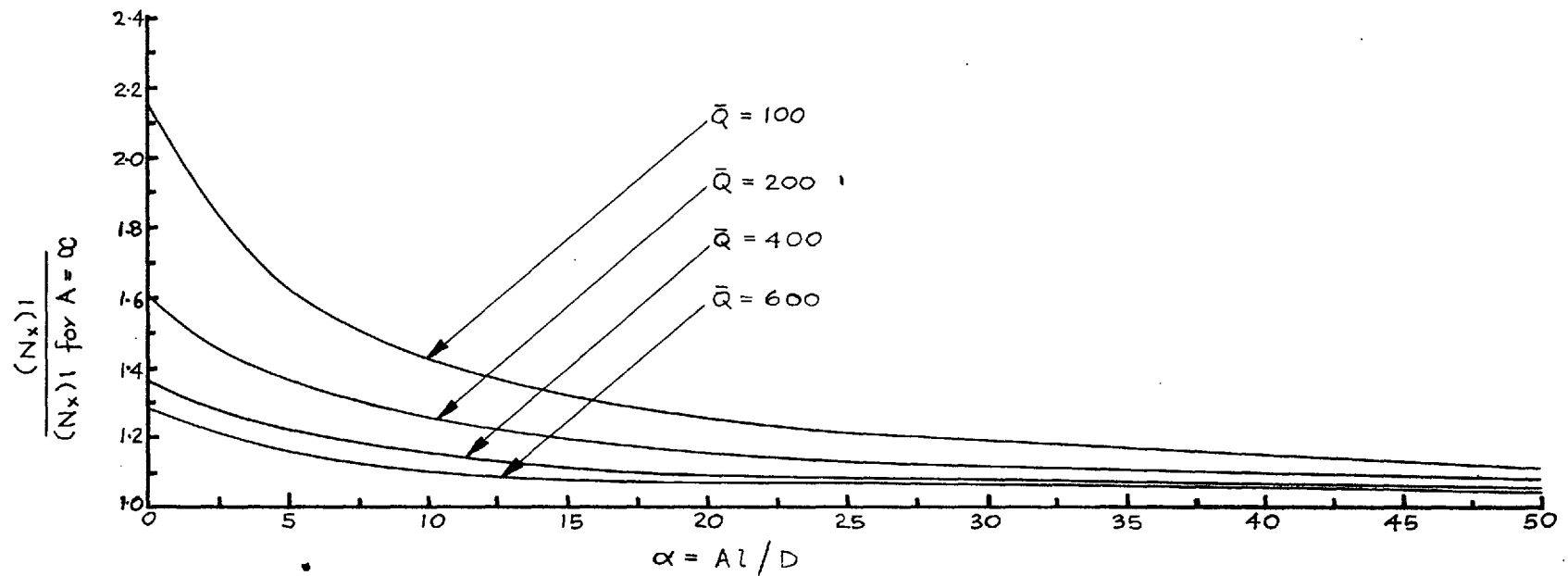
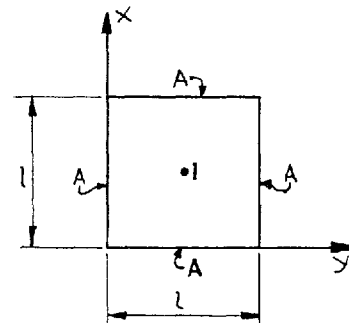


FIG.45: SQUARE ISOTROPIC PLATE, $\nu = 0.316$, $K_x = K_y = 0$, $S_x = S_y = 0$, UNIFORM LOAD
EFFECT OF VARYING ROTATIONAL EDGE RESTRAINT A

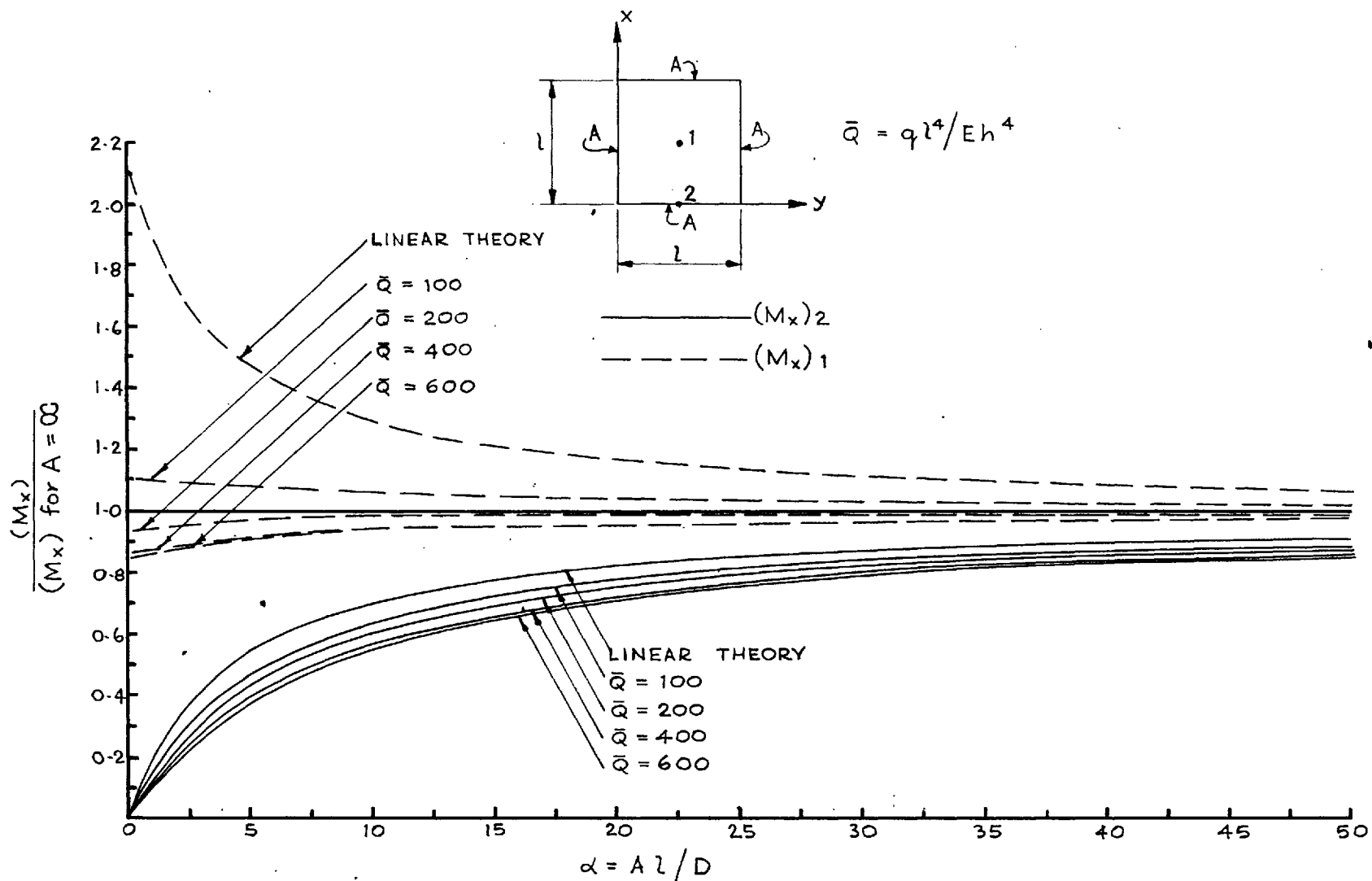


FIG. 46: SQUARE ISOTROPIC PLATE, $\nu = 0.316$, $K_x = K_y = 0$, $S_x = S_y = 0$, UNIFORM LOAD.

EFFECT OF VARYING ROTATIONAL EDGE RESTRAINT A

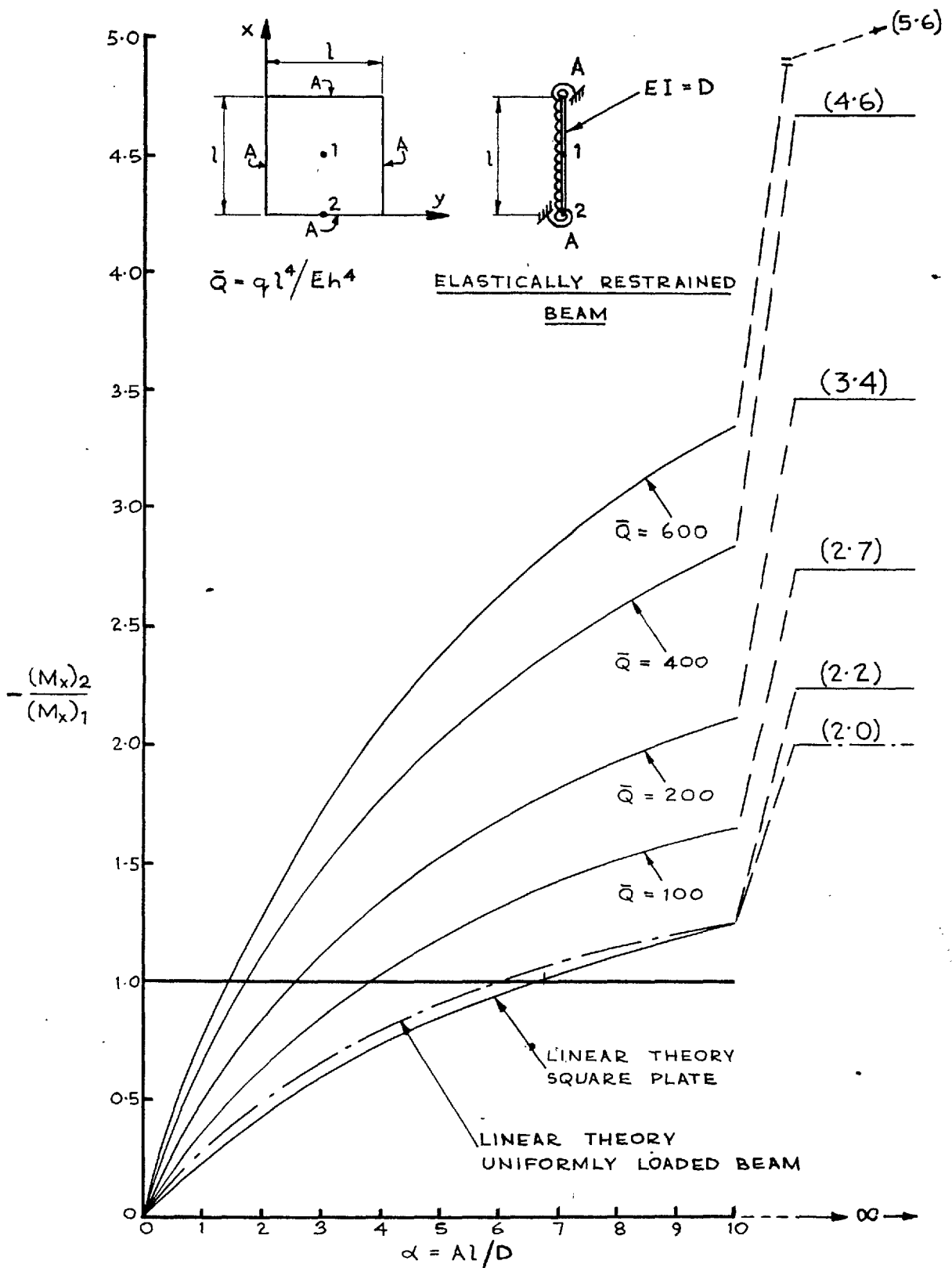
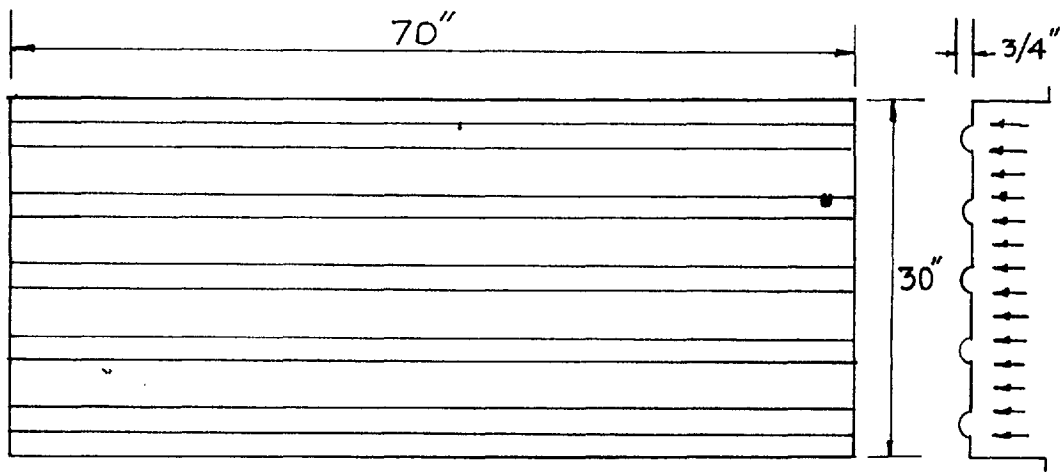
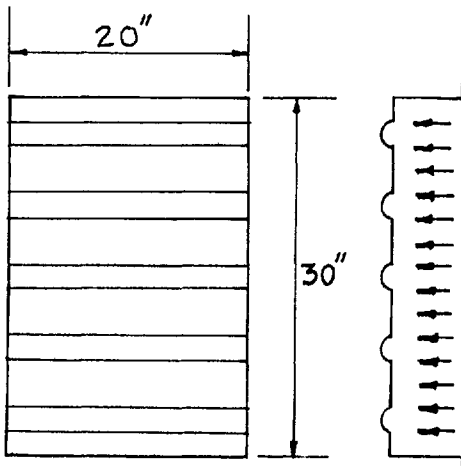


FIG.47: SQUARE ISOTROPIC PLATE, $\nu=0.316$, $K_x=K_y=0$, $S_x=S_y=0$, UNIFORM LOAD
EFFECT OF VARYING ROTATIONAL EDGE RESTRAINT A.



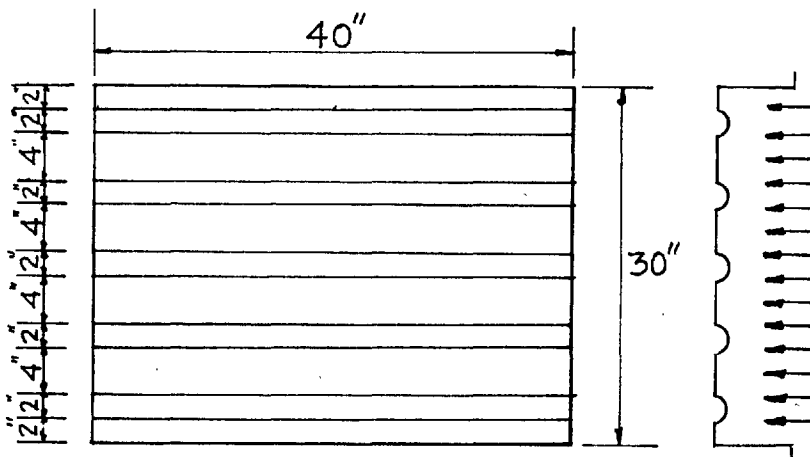
TEST 1

VIRTUAL SIDE RATIO = 1.76



TEST 2

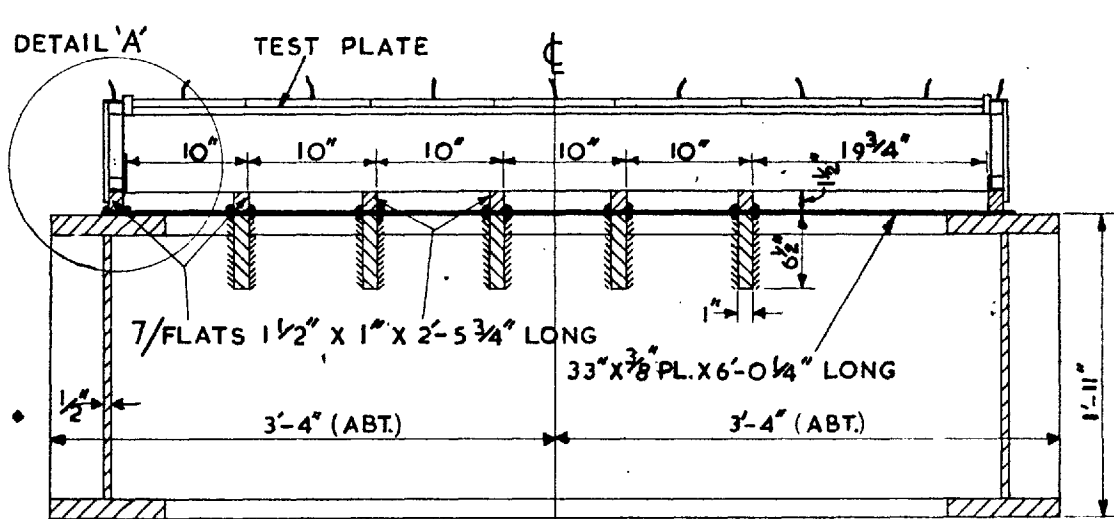
VIRTUAL SIDE RATIO = 6.15



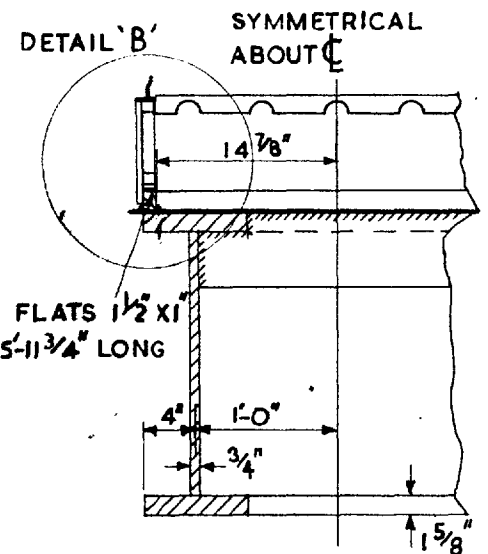
TEST 3

VIRTUAL SIDE RATIO = 3.08

FIG. 48. TEST SERIES



SECTIONAL ELEVATION



SECTIONAL END VIEW

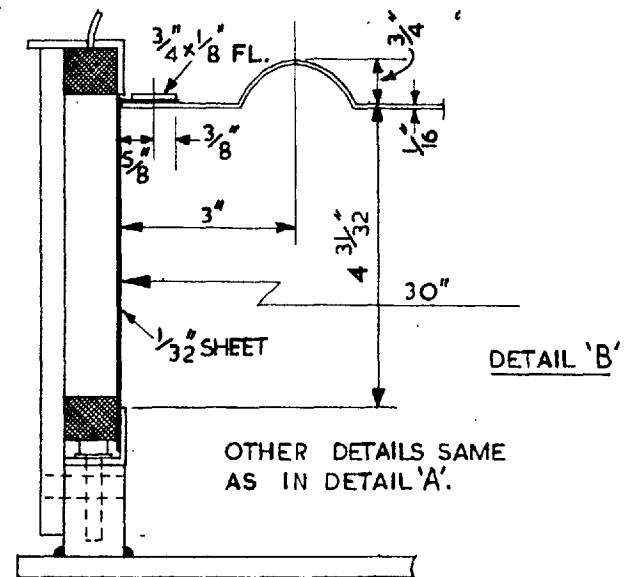
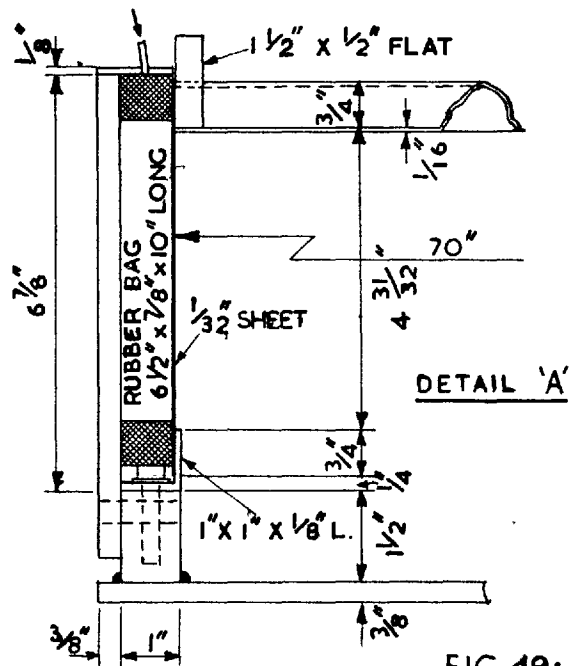
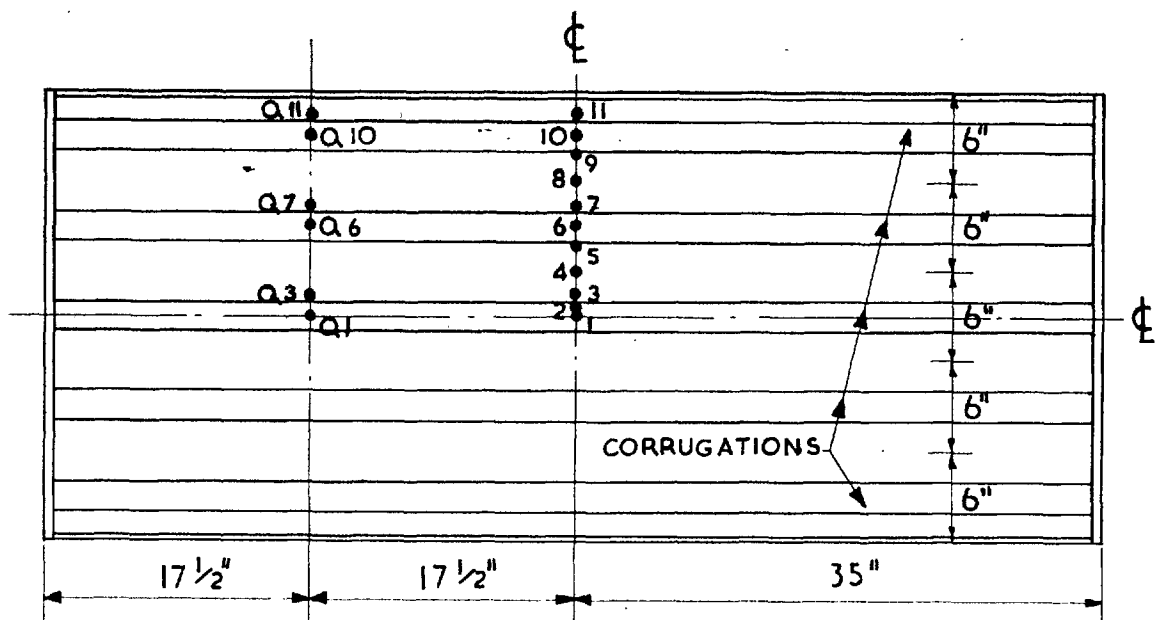


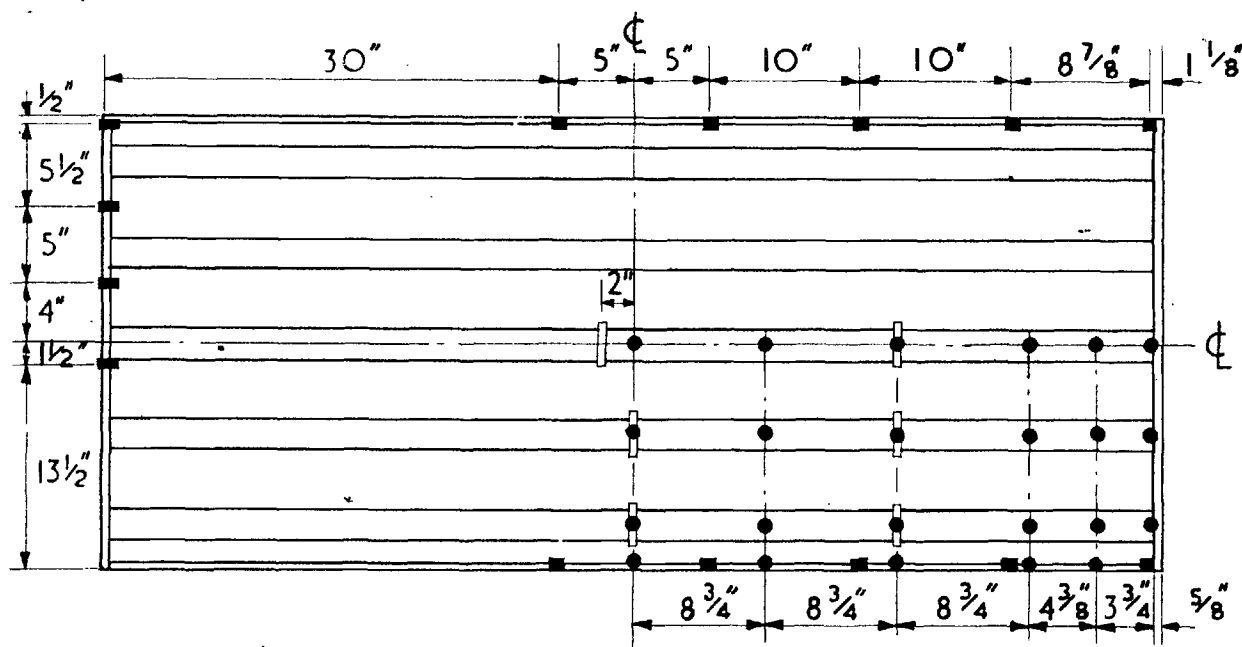
FIG.49:DETAILS OF TEST RIG. (TEST1)



SEE ALSO FIG. 67

ARRANGEMENT
OF STRAIN GAUGES.

EACH POINT REPRESENTS A GROUP
OF 4 STRAIN GAUGES, 2 (LONGITUDINAL
AND TRANSVERSE) ON EACH SURFACE
LONGITUDINAL GAUGES 1 CM.
TRANSVERSE GAUGES 0.5 CM. }



ARRANGEMENT OF DEFLECTION,
SLOPE & SWEDGE SPREAD GAUGES.

DEFLECTION GAUGE ●
SLOPE GAUGE ■
SWEDGE SPREAD GAUGE □

FIG. 50: INSTRUMENTATION DETAILS. (TEST 1)

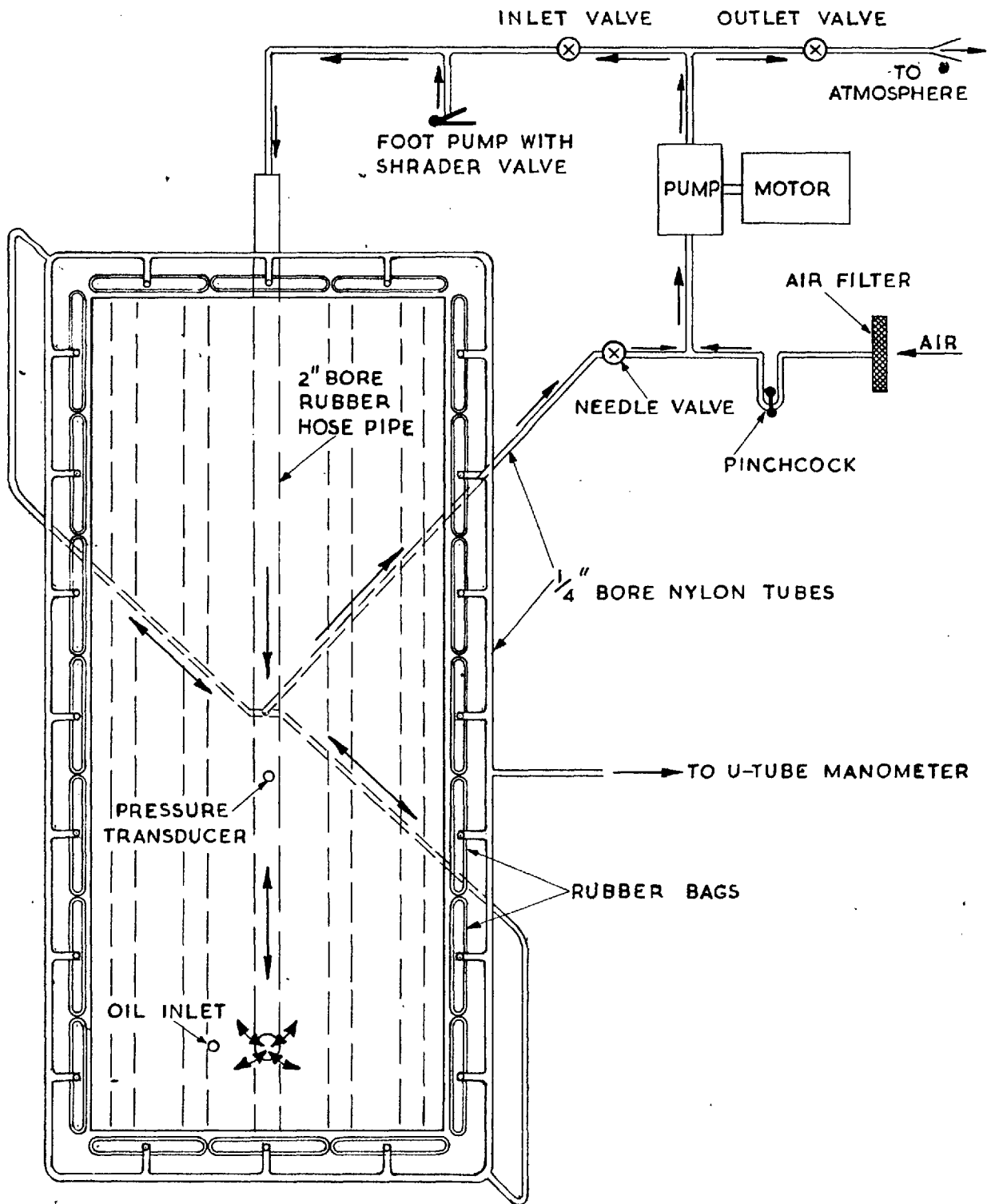


FIG. 51: DETAILS OF ARRANGEMENT FOR LOADING BY AIR.
(TEST 1)

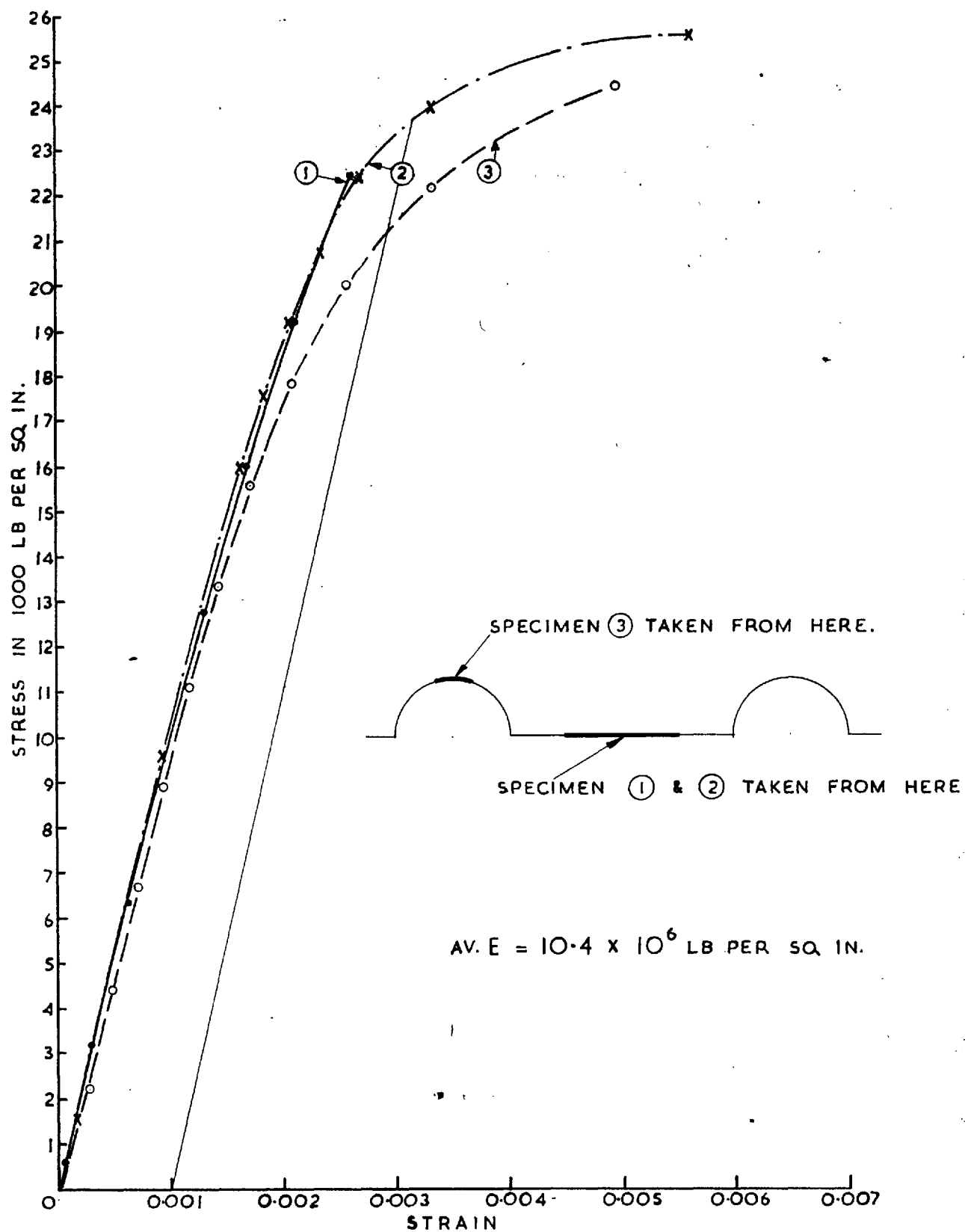


FIG. 52: STRESS-STRAIN CURVES FOR TENSION SPECIMENS.
(TEST 1 & 2)

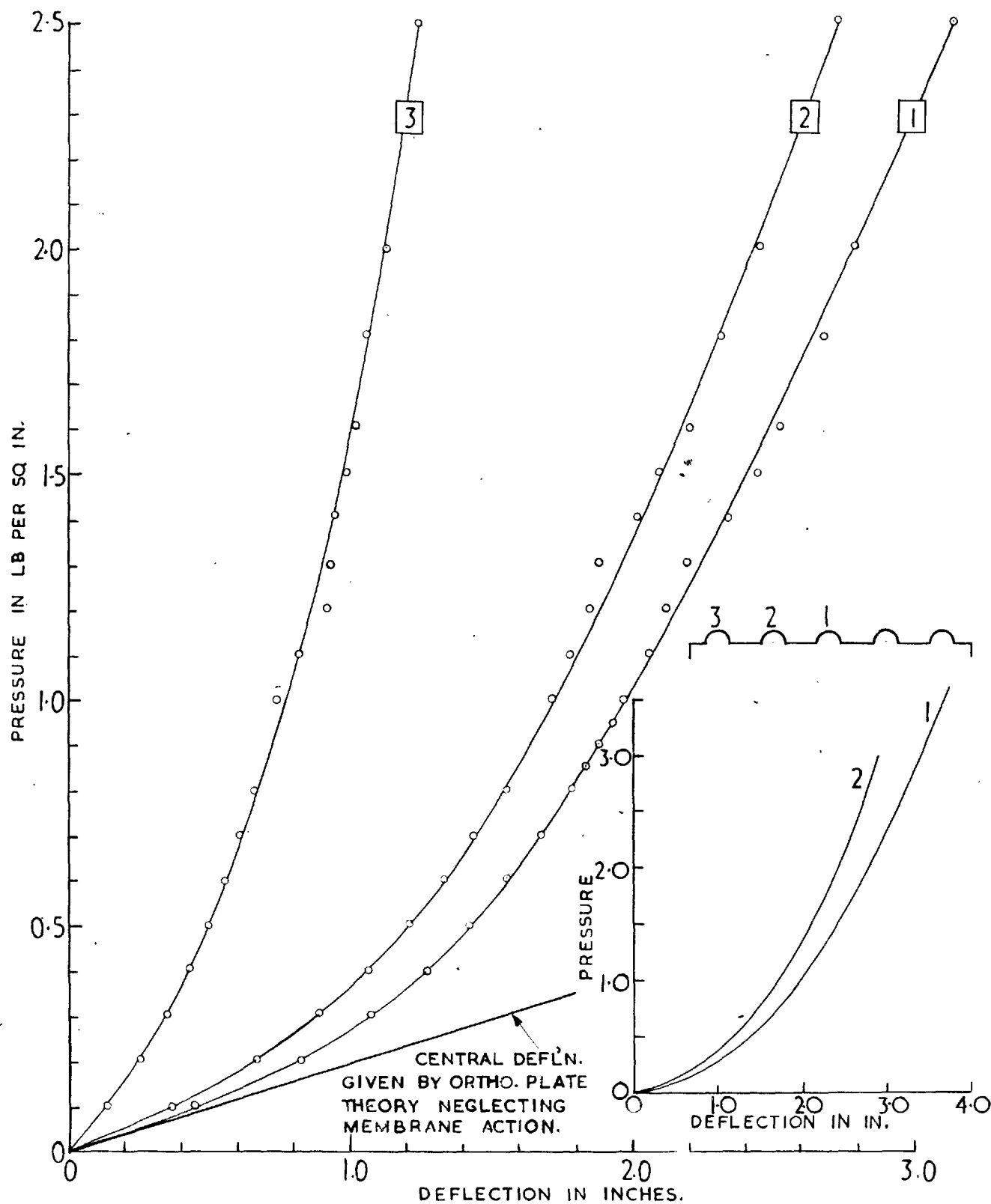


FIG.53 : DEFLECTIONS AT THE CENTRAL SECTION, (TEST 1)

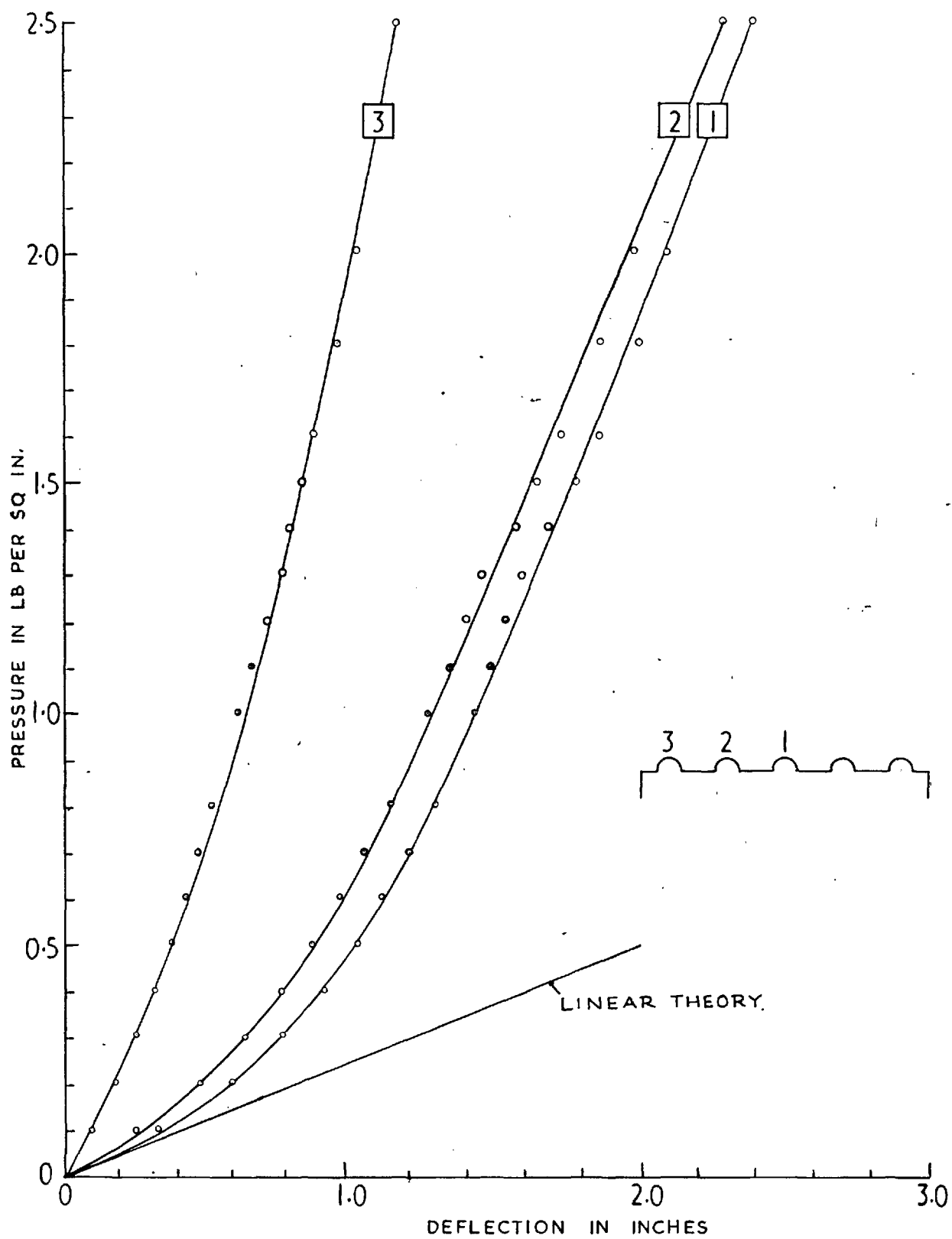


FIG.54:DEFLECTIONS AT QUARTER SPAN. (TEST 1)

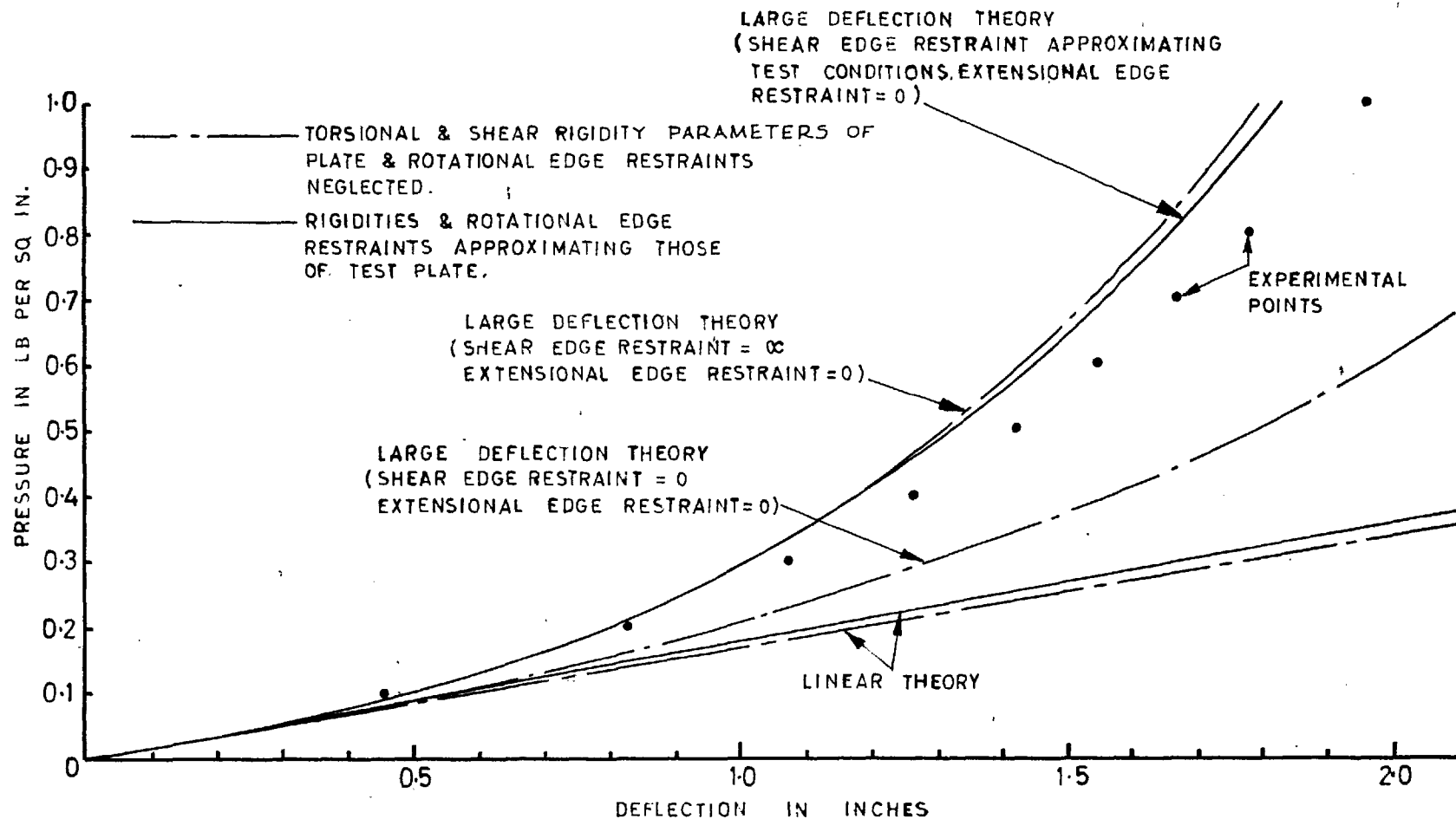


FIG. 55: DEFLECTION OF CENTRAL SWEDGE AT MIDSPAN (TEST 1)

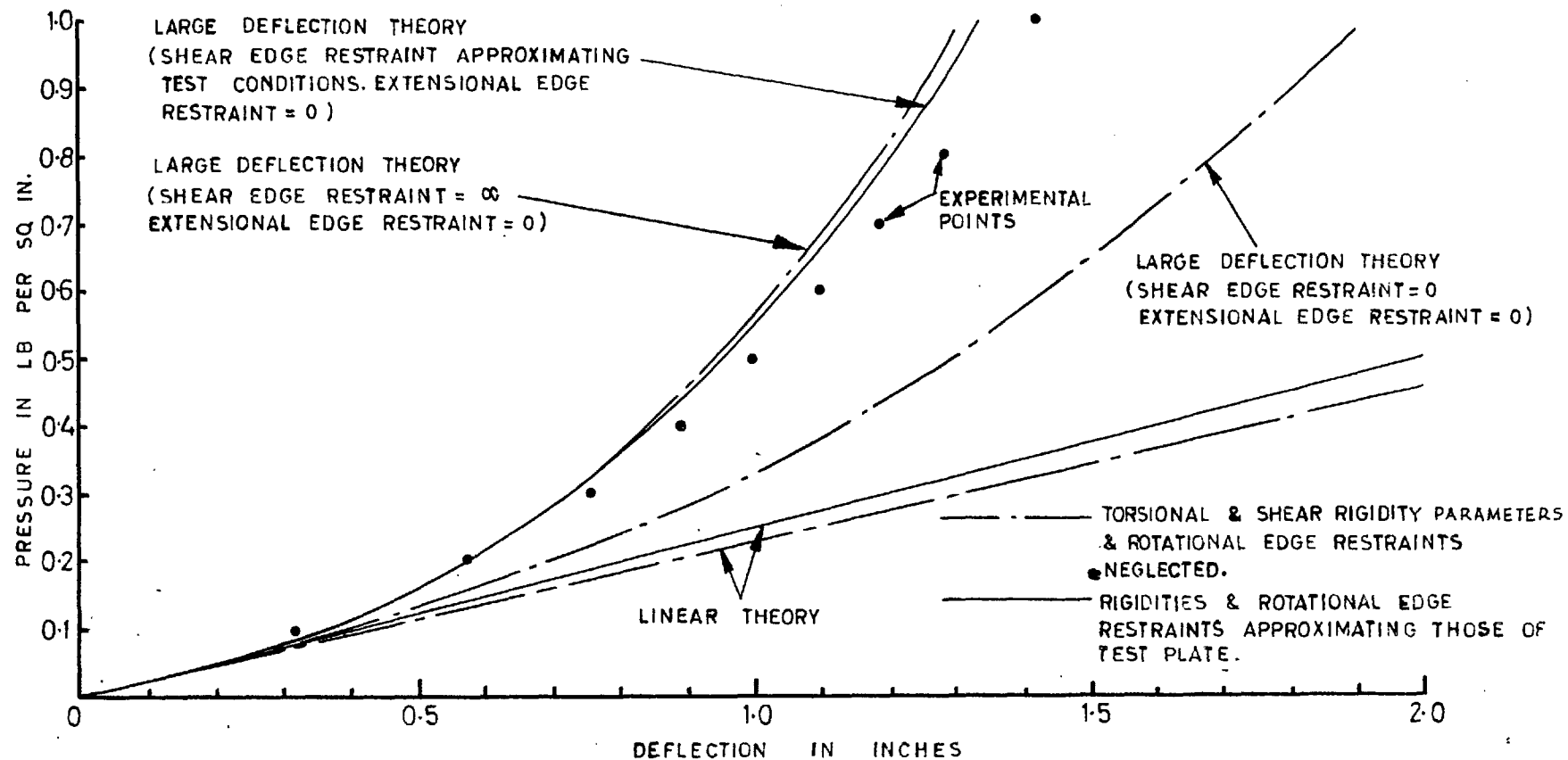


FIG.56:DEFLECTION OF CENTRAL SWEDGE AT QUARTER SPAN (TEST 1)

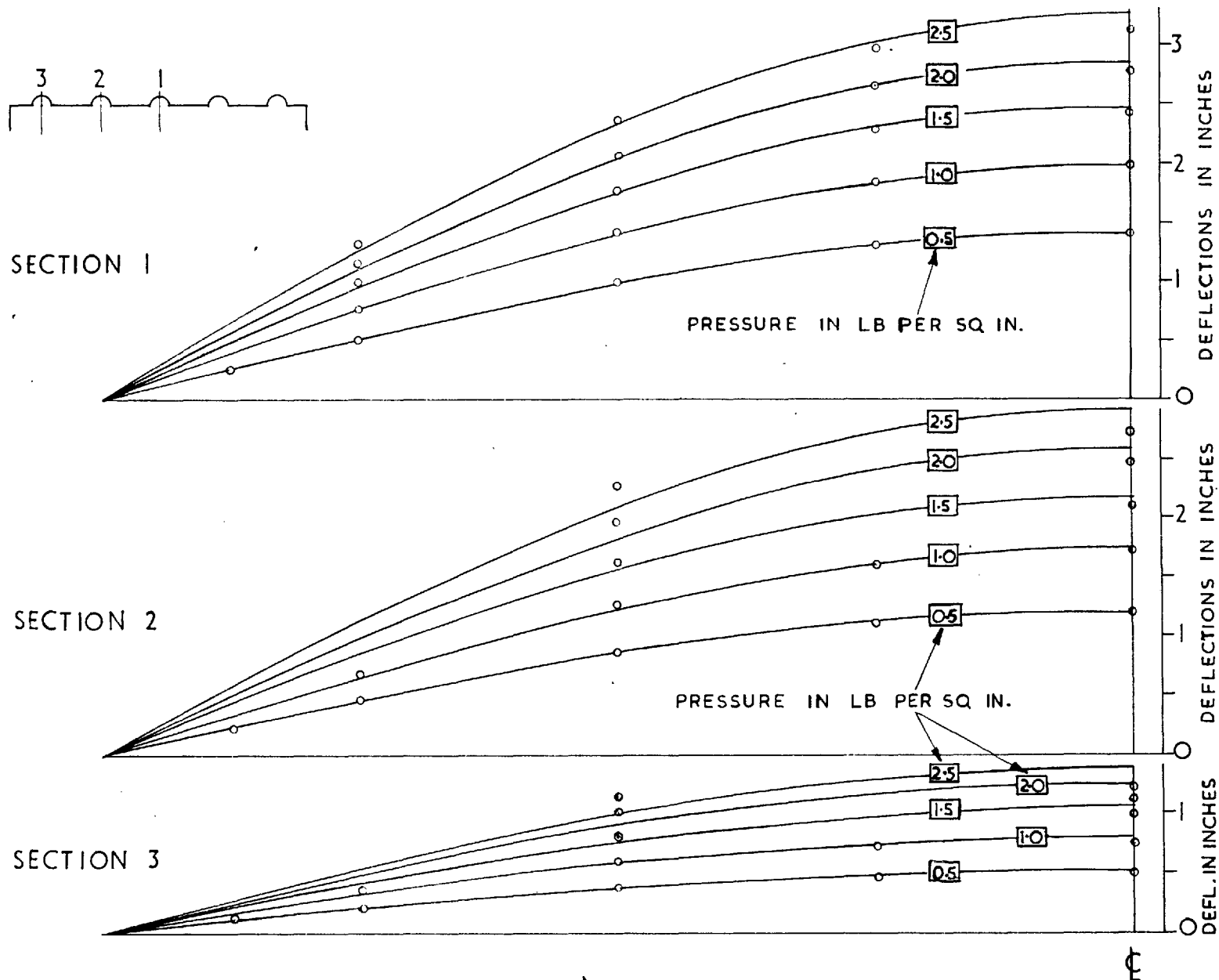


FIG.57: LONGITUDINAL SECTIONS OF THE DEFLECTED PLATE. (TEST1)

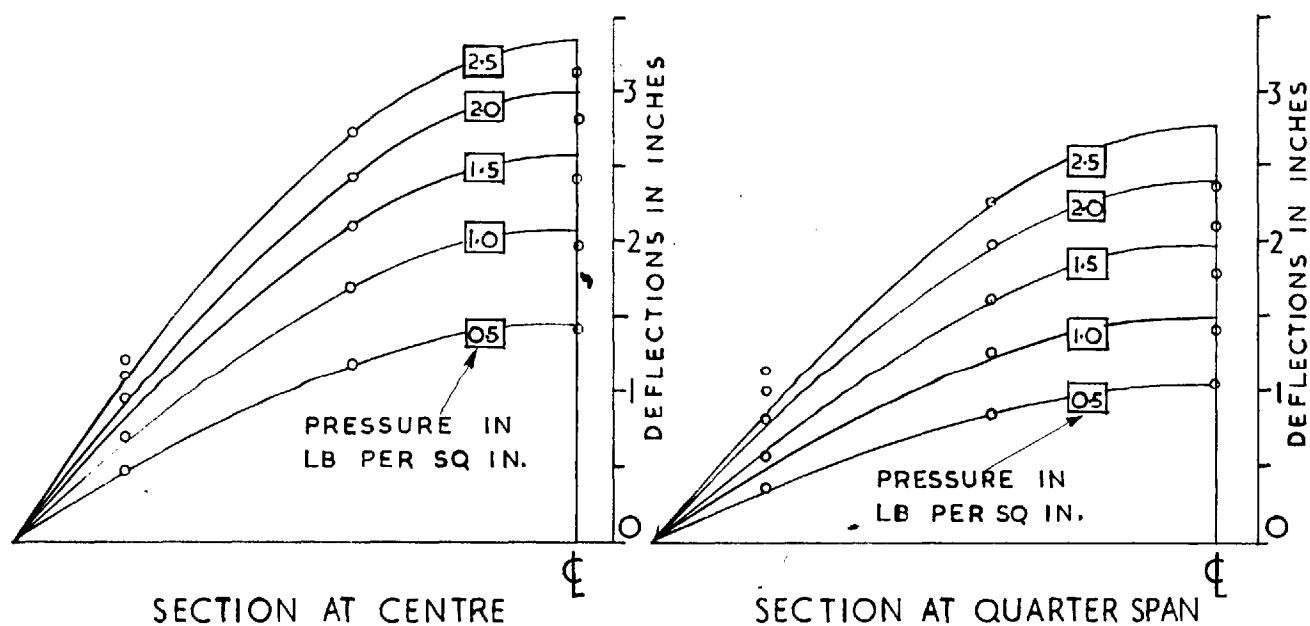


FIG. 58:
TRANSVERSE SECTIONS OF THE DEFLECTED PLATE. (TEST 1)

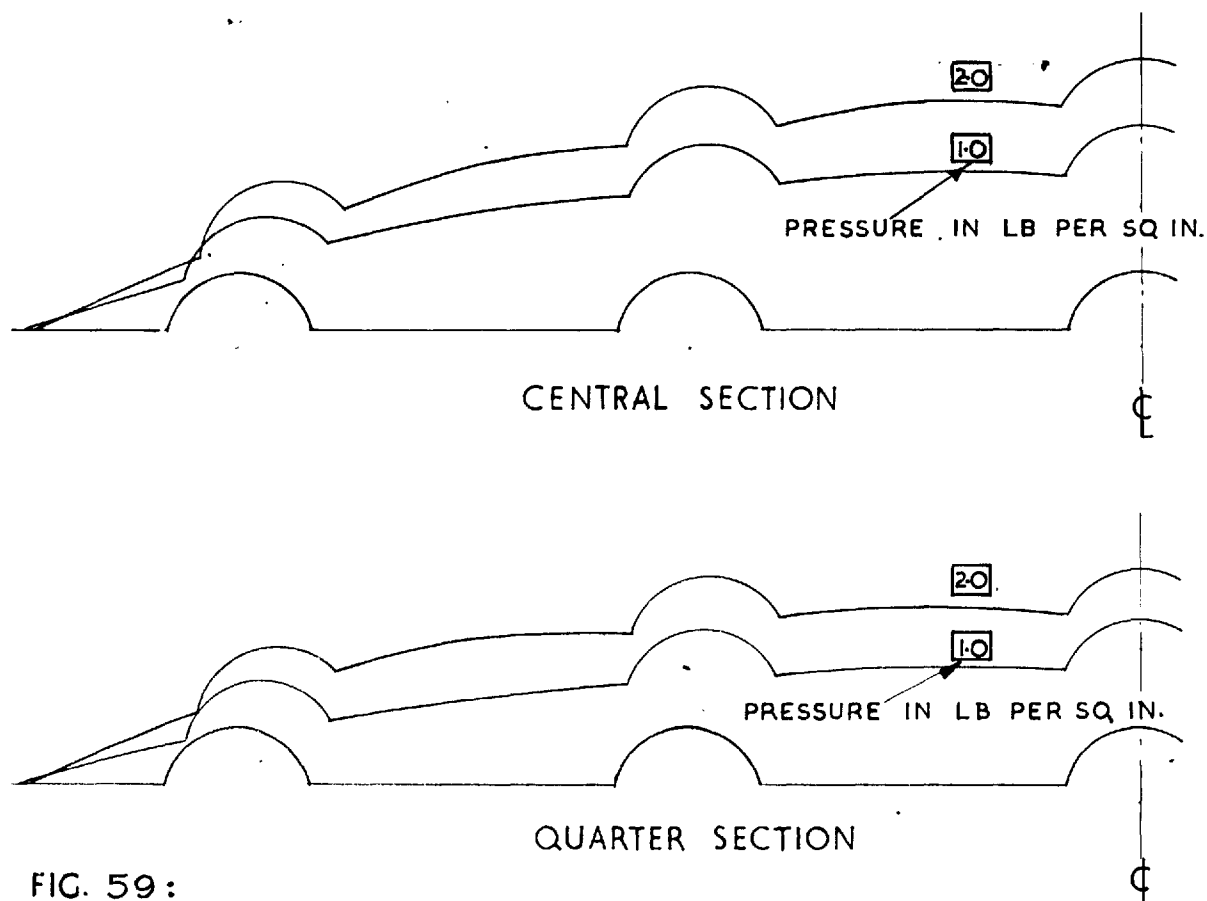


FIG. 59:
PLATE PROFILES ON TRANSVERSE SECTIONS. (TEST 1)

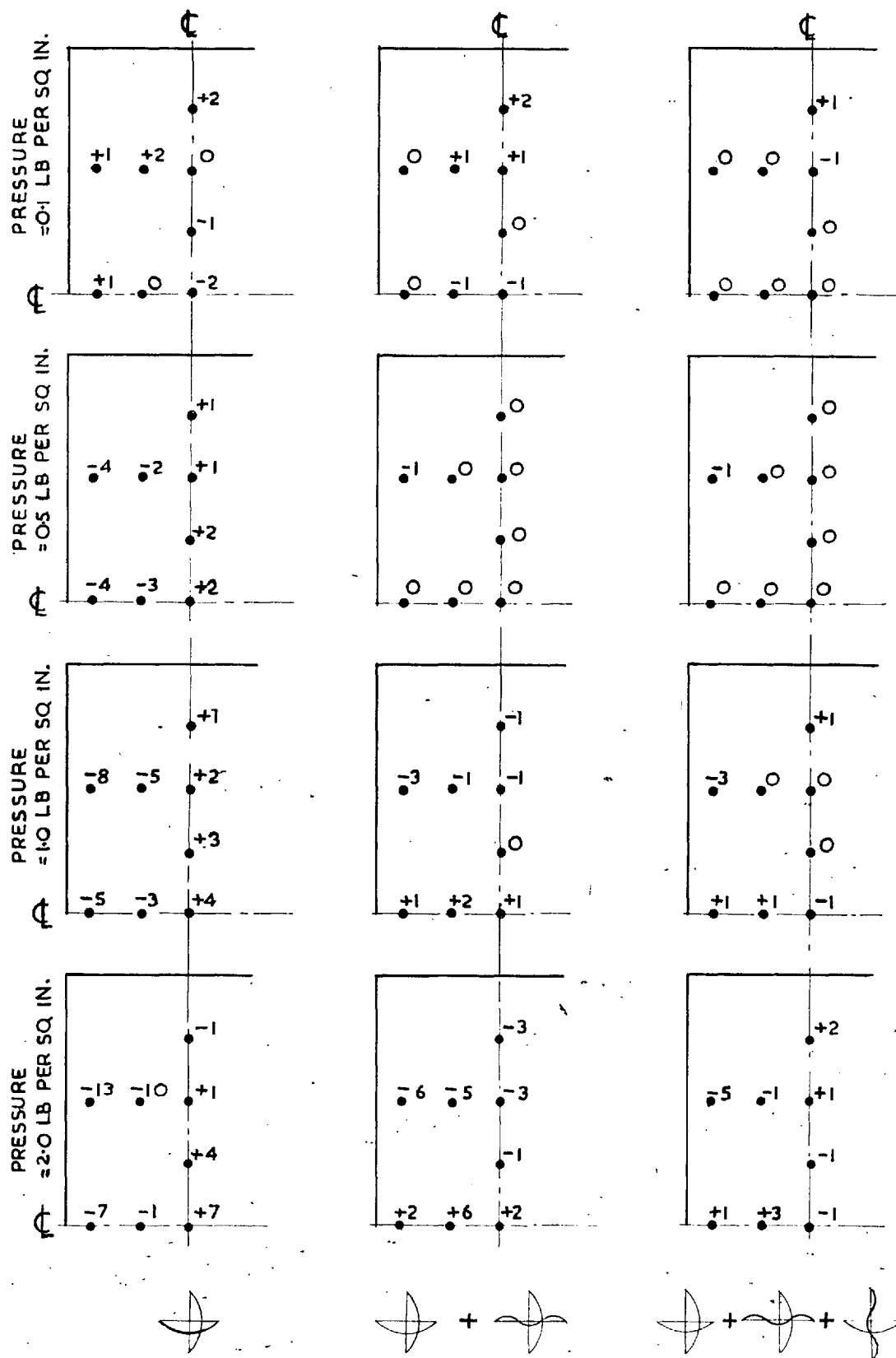


FIG. 60: DEPARTURE OF "BEST" FOURIER DEFLECTED FORM FROM OBSERVED DEFLECTION. ERRORS ARE EXPRESSED AS PERCENTAGES OF OBSERVED CENTRAL DEFLECTION.

(TEST 1)

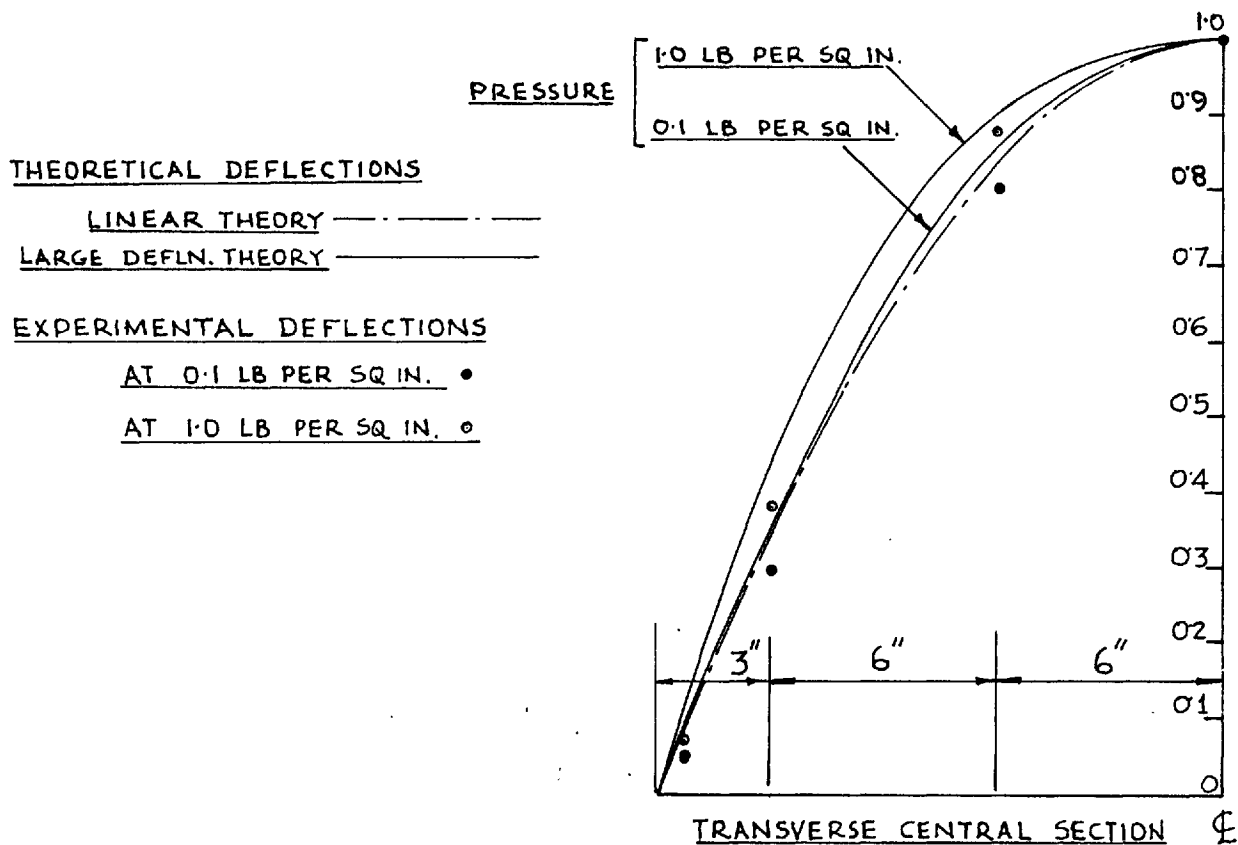
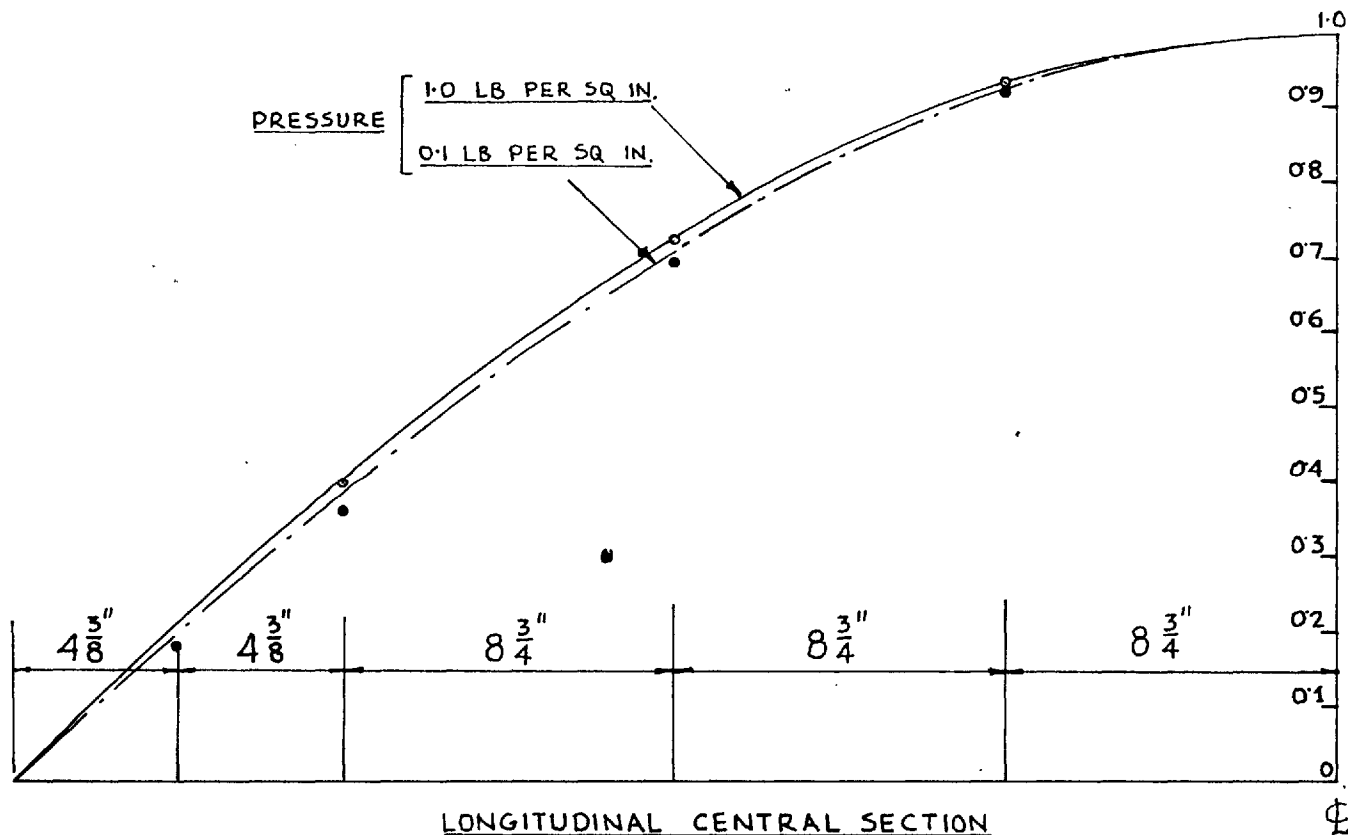


FIG. 61: DEFLECTED PROFILES OF THE PLATE CENTRE-LINES (TEST 1)
COMPARISON WITH THEORY

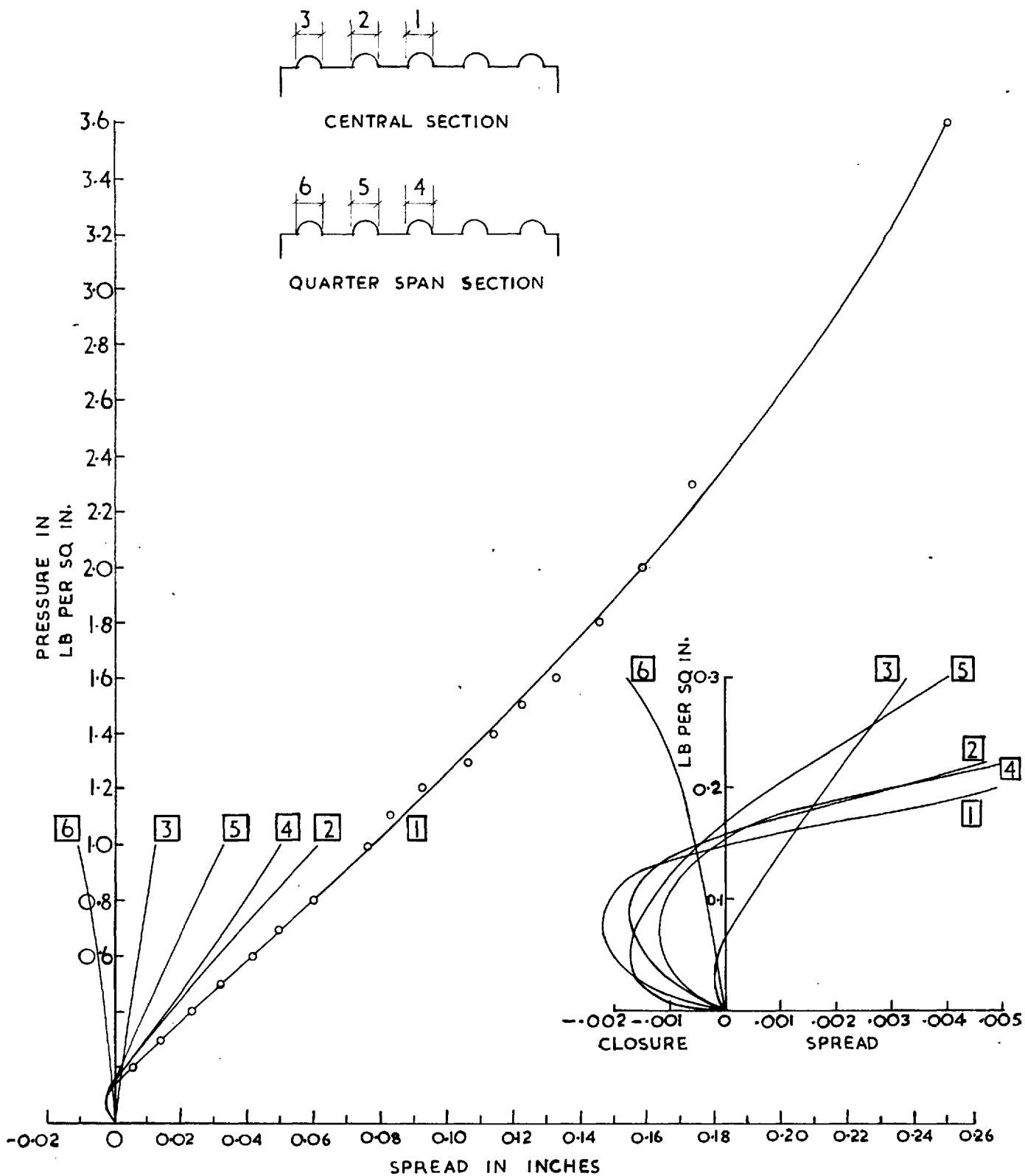


FIG.62: DEFORMATION OF SWEDGES. (TEST 1)

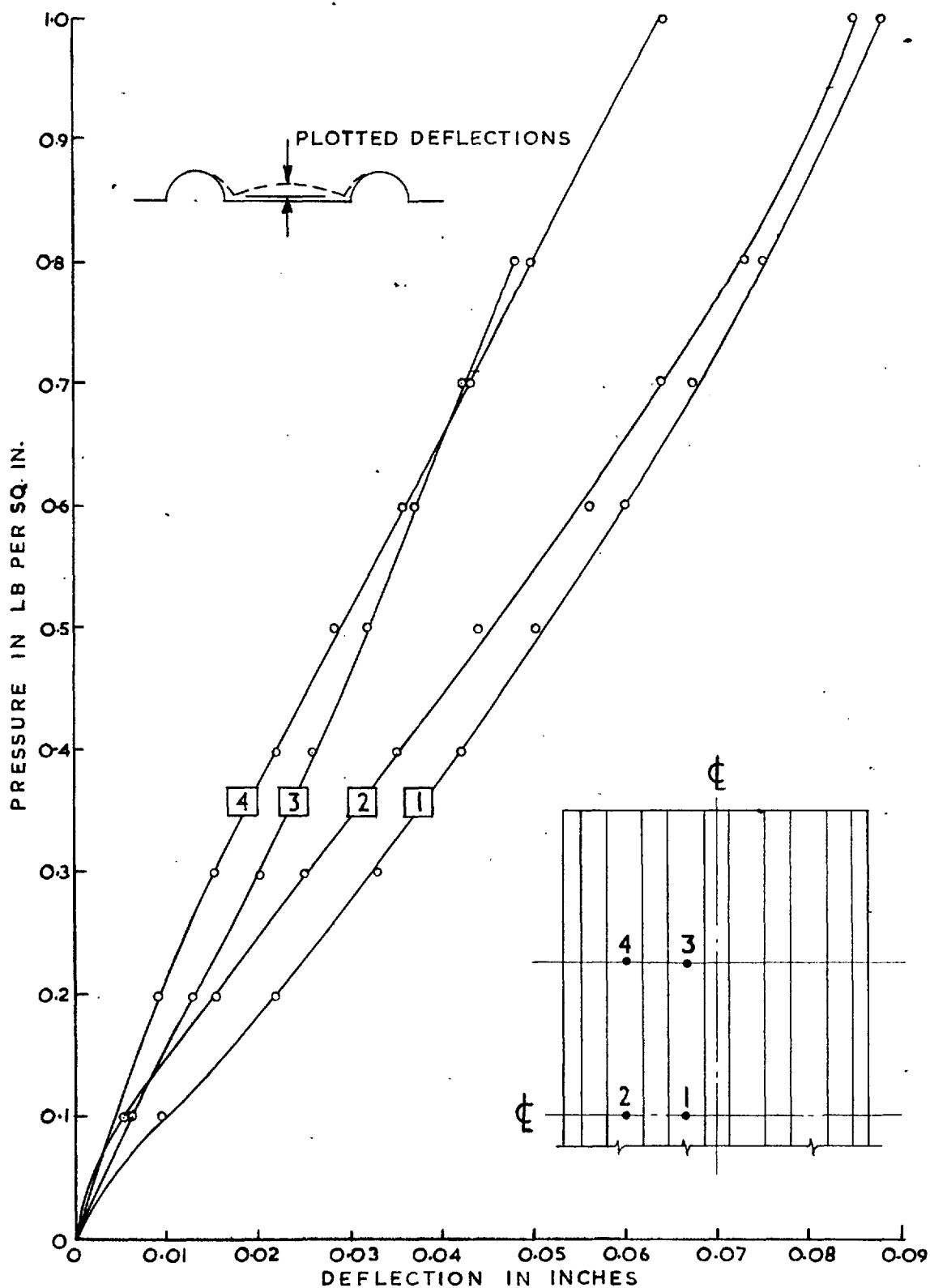


FIG.63: DEFLECTION OF PLATE RELATIVE TO SWEDGE.
(TEST 1)

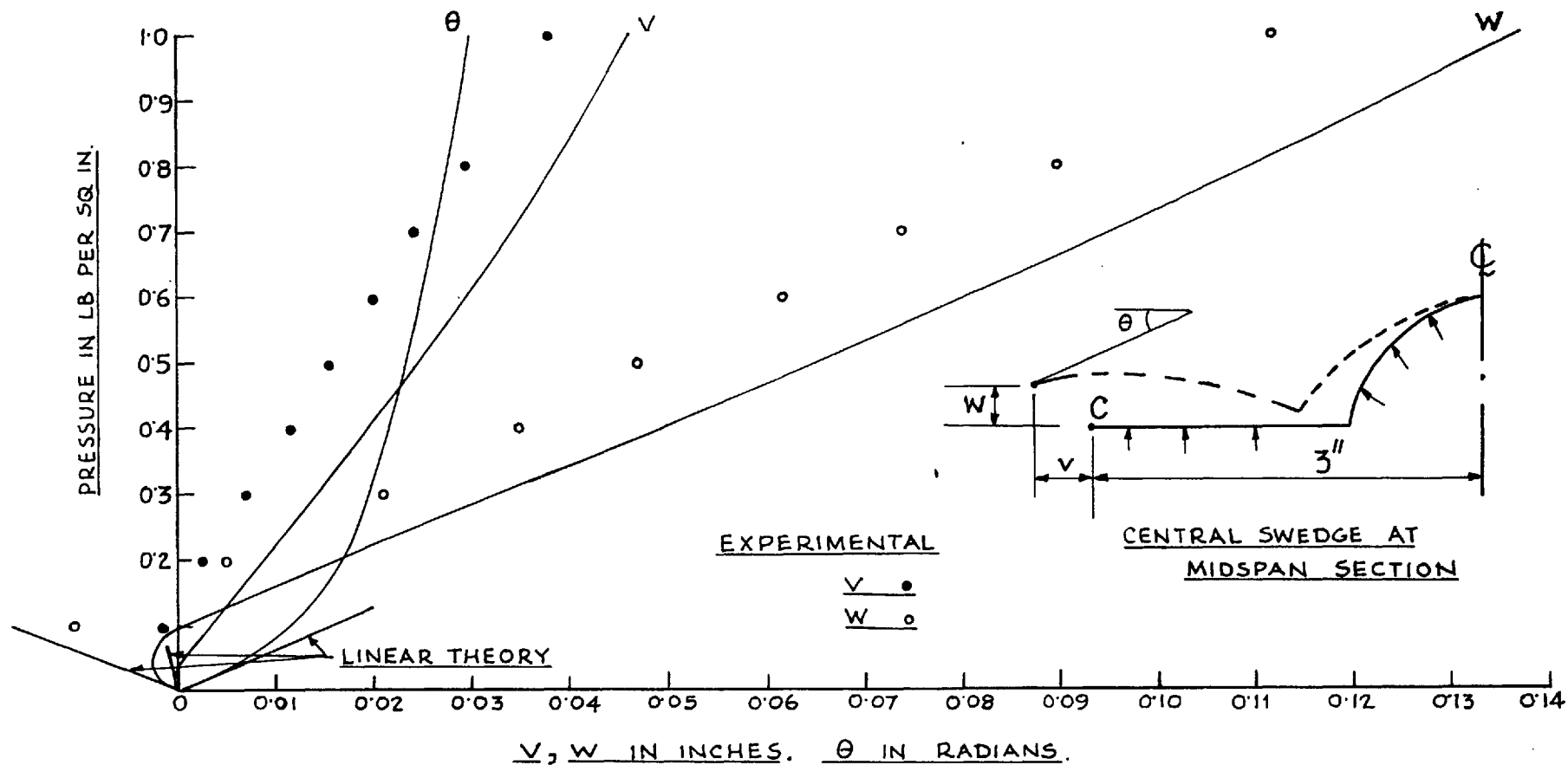


FIG. 64. SWEDGE DEFORMATIONS. COMPARISON WITH THEORY (TEST 1)

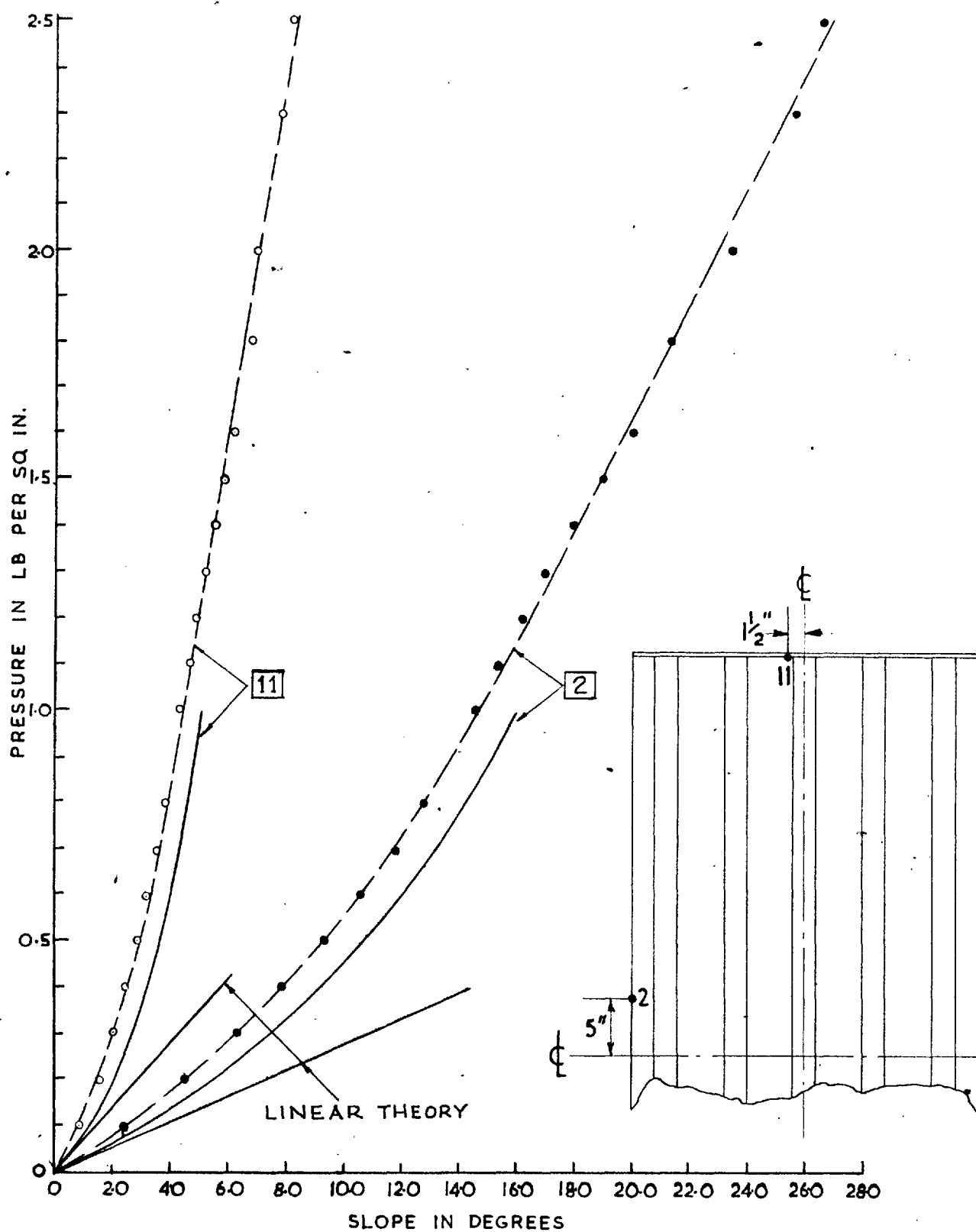


FIG.65:SLOPES ON PLATE EDGES. (TEST 1)

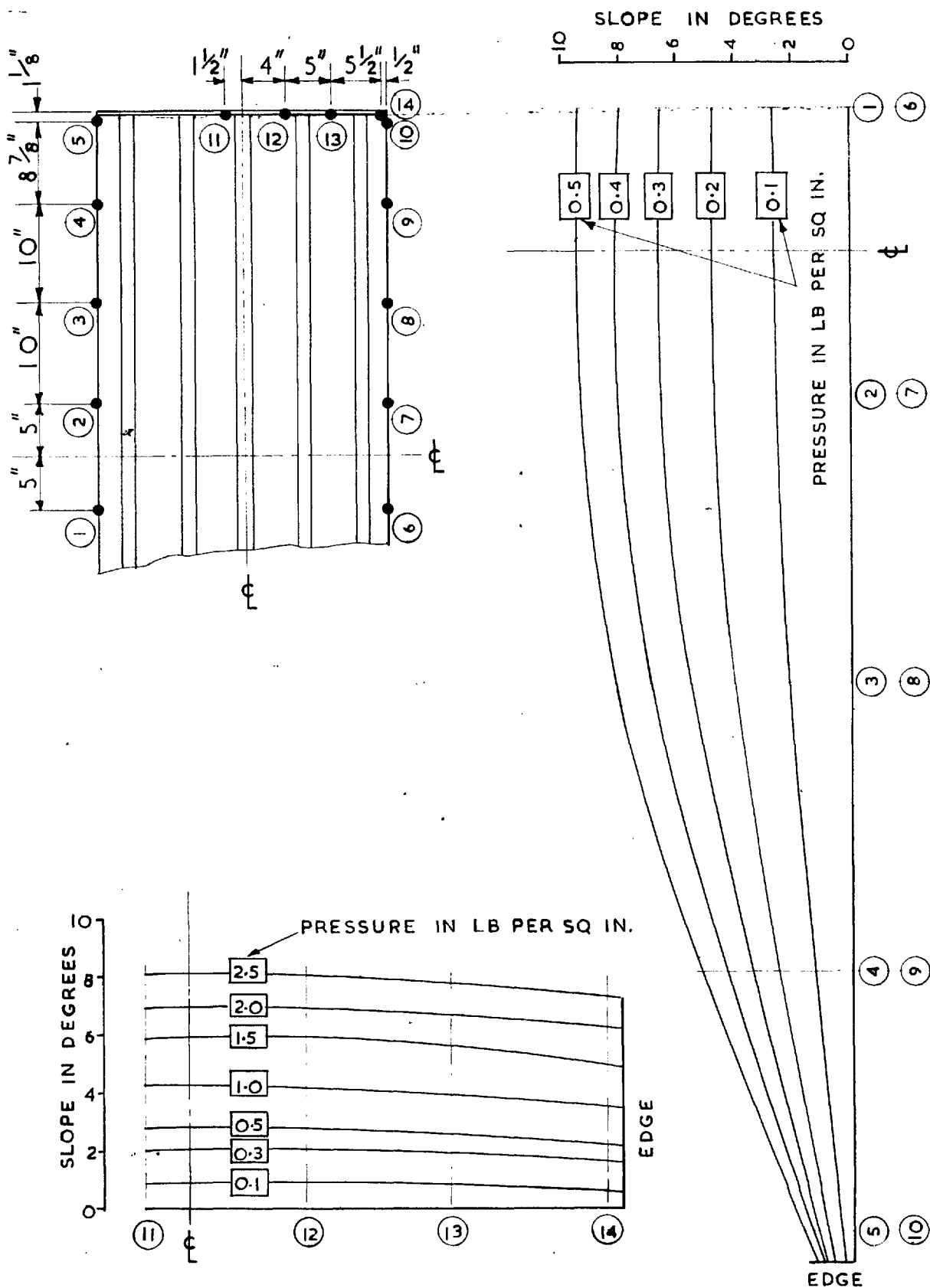


FIG.66: VARIATION OF NORMAL EDGE SLOPE (TEST 1)

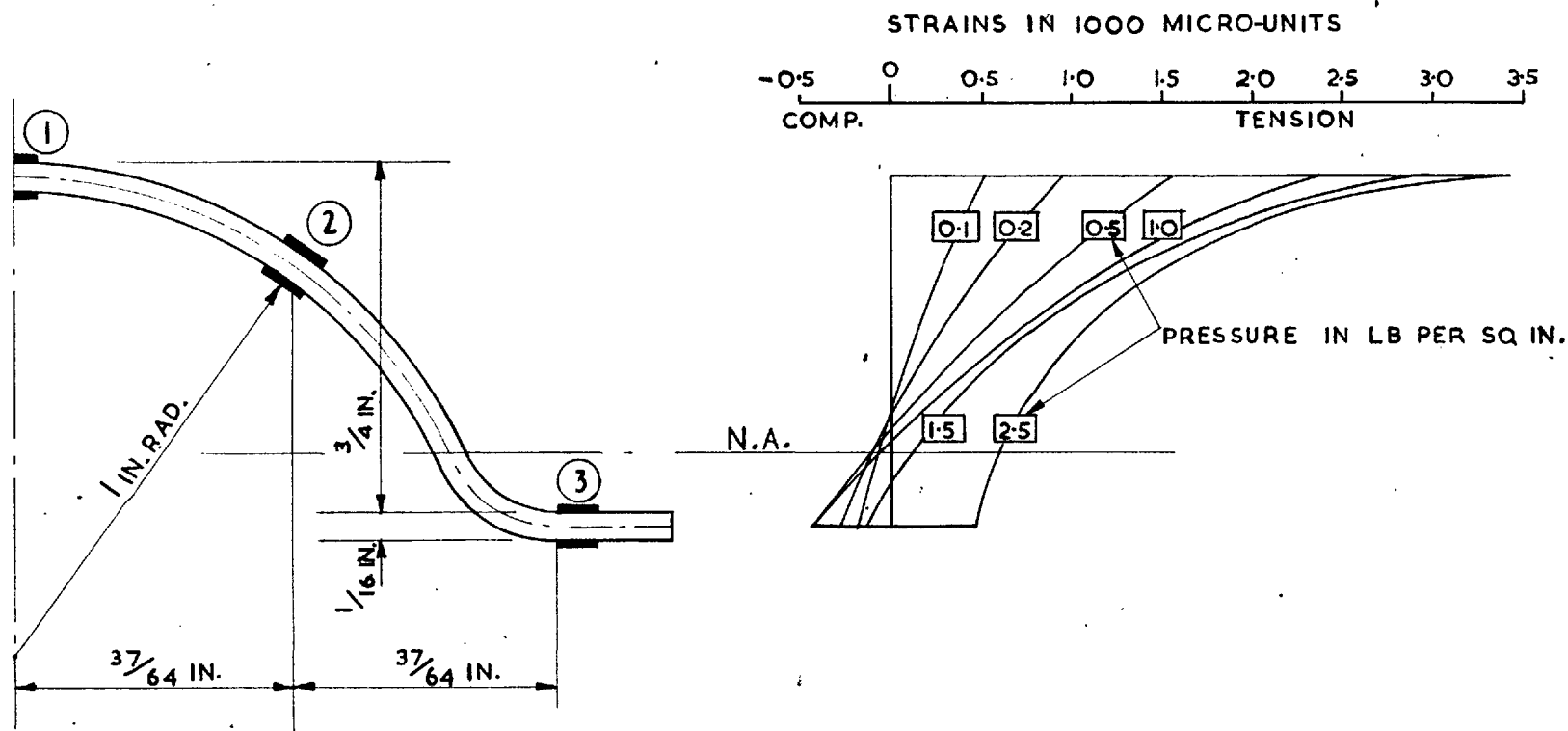


FIG.67: LONGITUDINAL STRAIN DISTRIBUTION OVER DEPTH OF CENTRE SWEDGE AT MIDSPAN. (TEST 1)

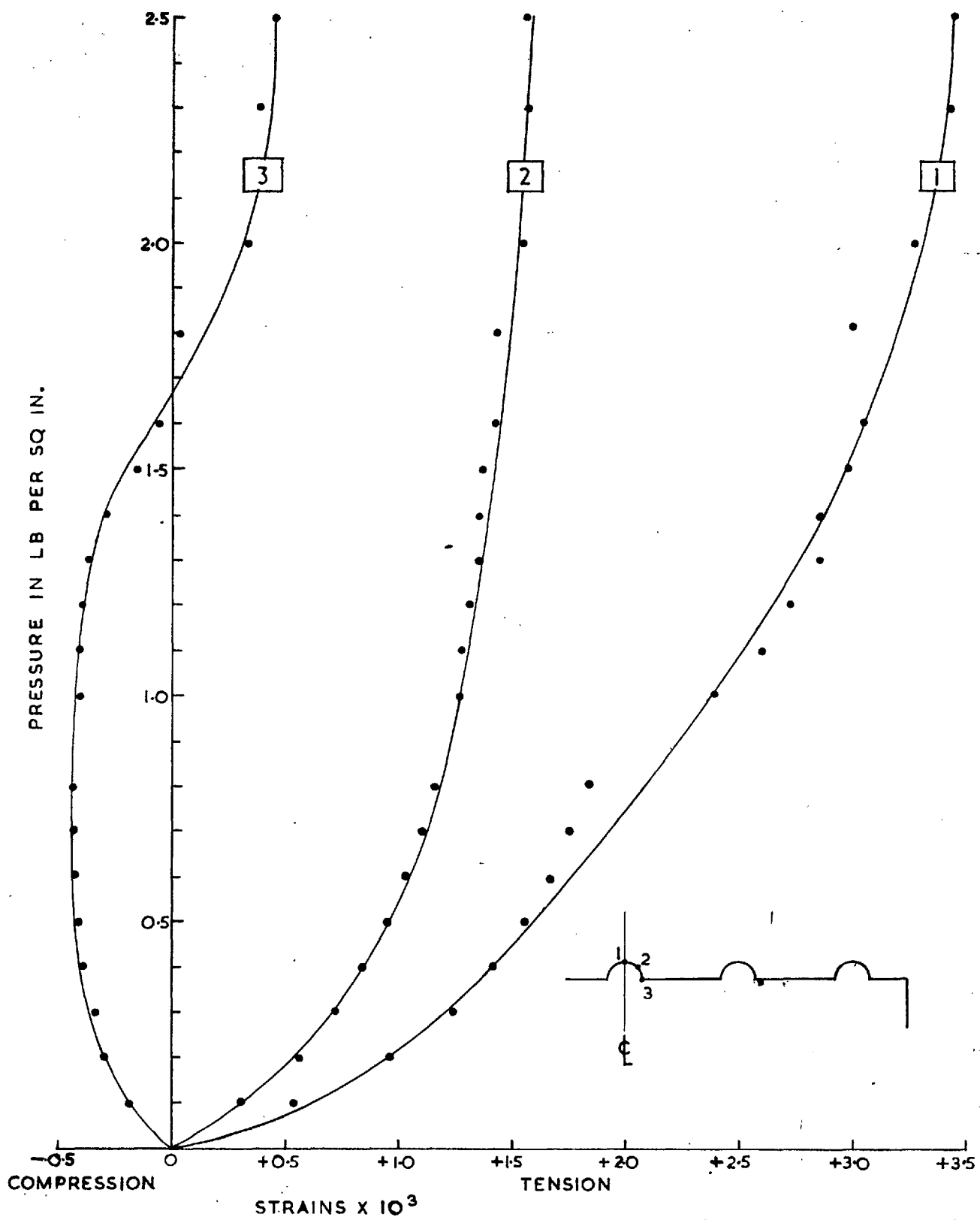


FIG. 68: LONGITUDINAL HEART OF PLATE STRAINS (CENTRAL SECTION)
(TEST 1)

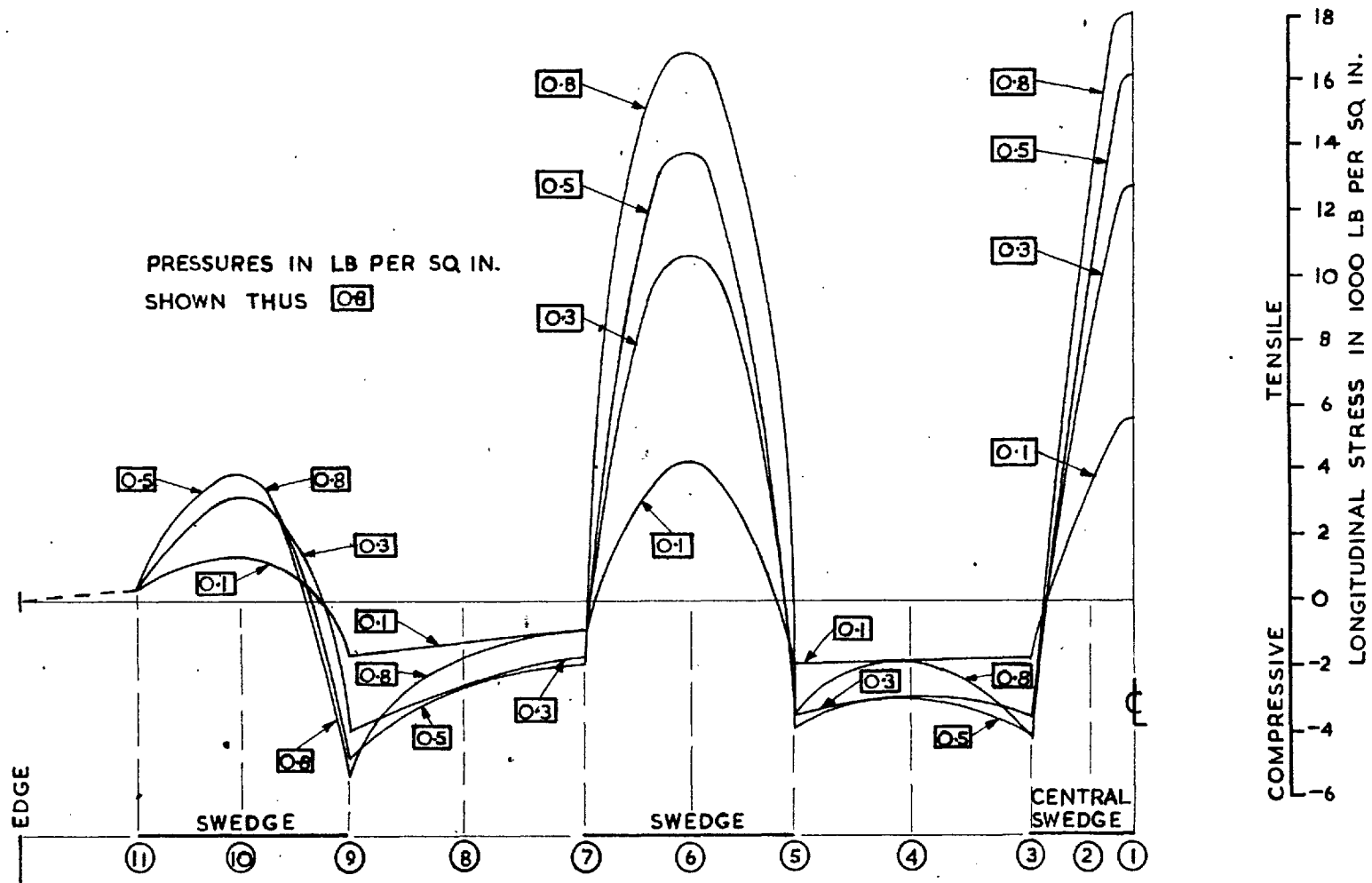


FIG.69: DISTRIBUTION OF LONGITUDINAL HEART OF PLATE STRESS ON TRANSVERSE
 CENTRAL SECTION (DEVELOPED VIEW). (TEST 1)

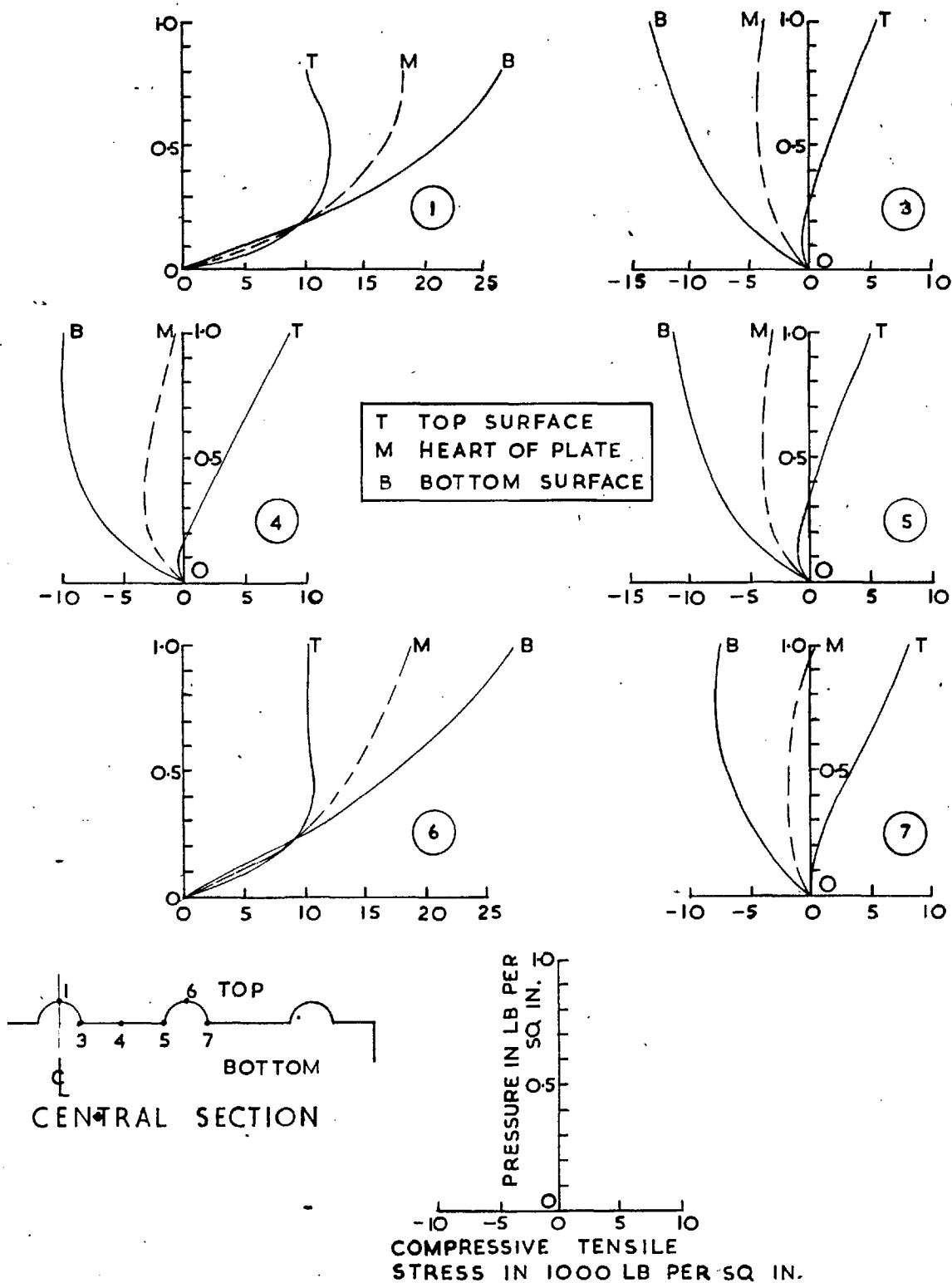


FIG. 70a: LONGITUDINAL SURFACE AND HEART OF PLATE STRESSES. (TEST 1)

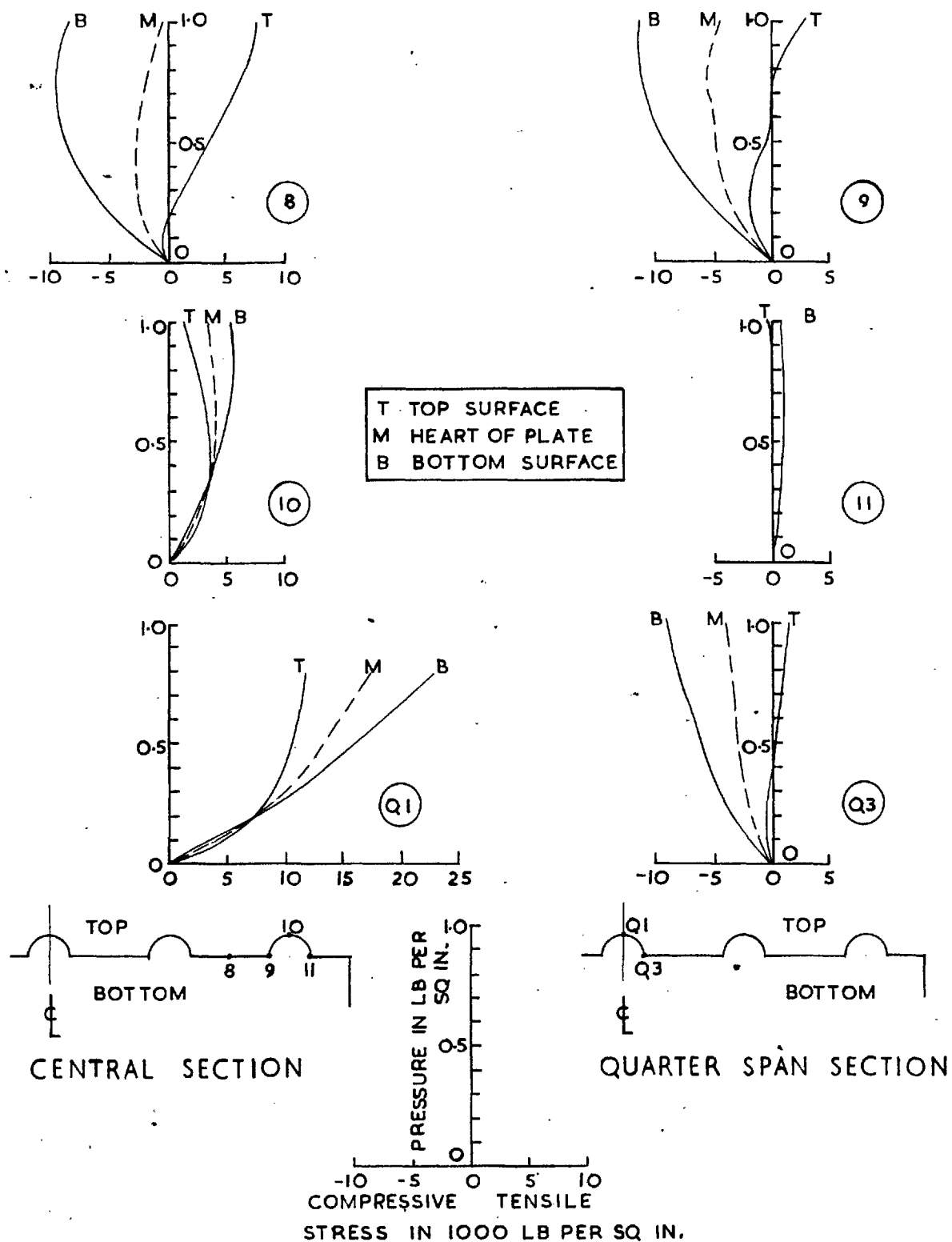


FIG.70b: LONGITUDINAL SURFACE AND HEART OF PLATE STRESSES.
(TEST 1)

PRESSURE IN LB PER SQ IN.

SHOWN THUS 0.8

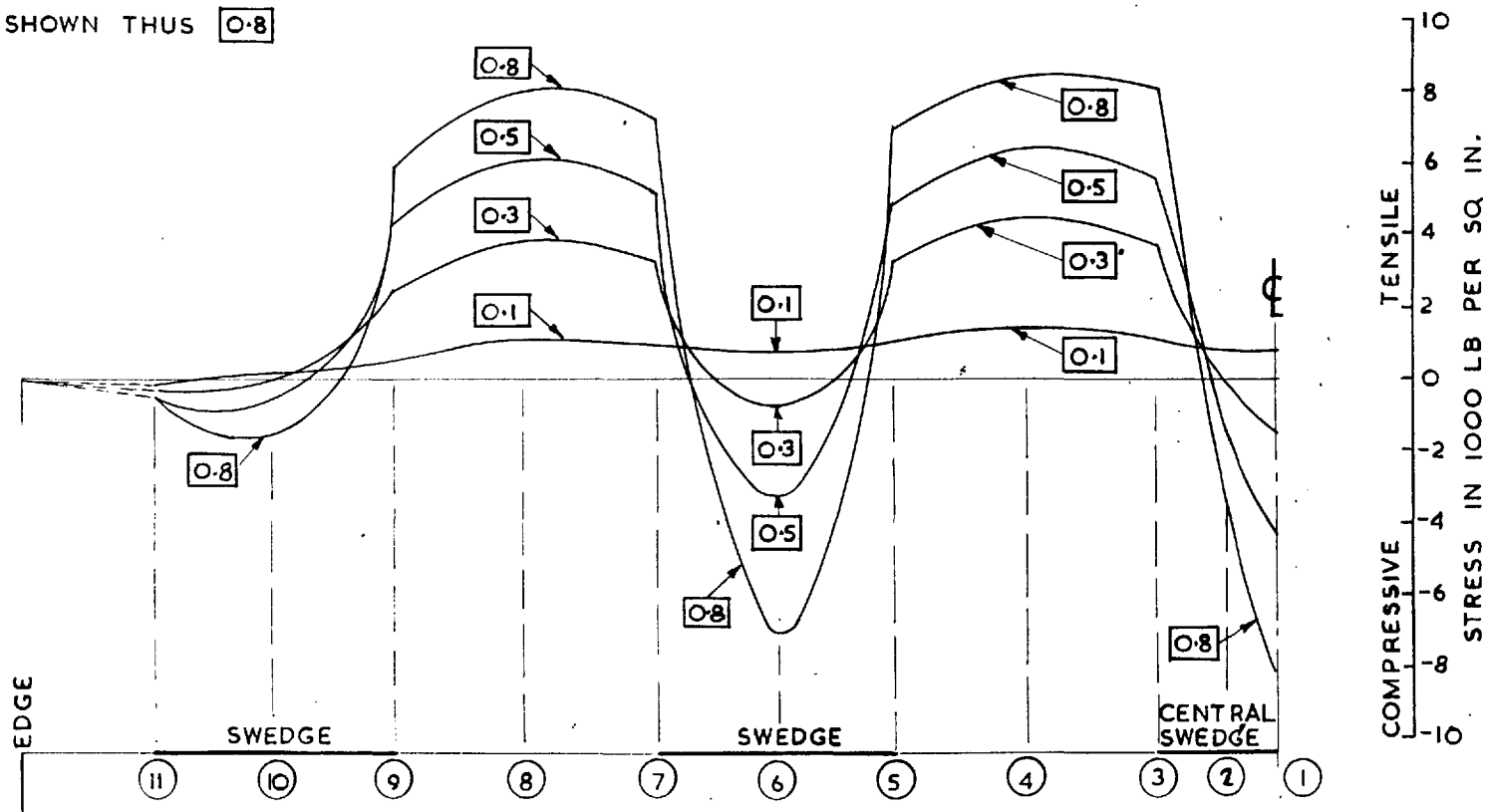
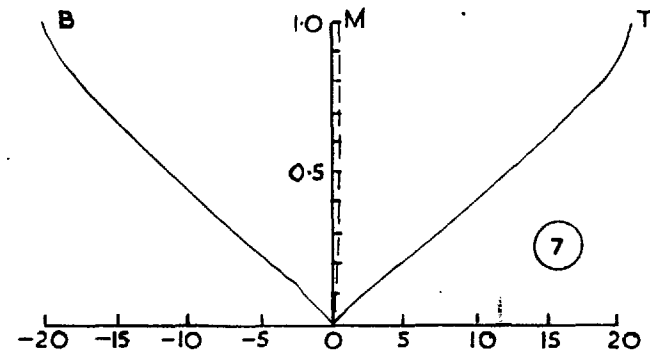
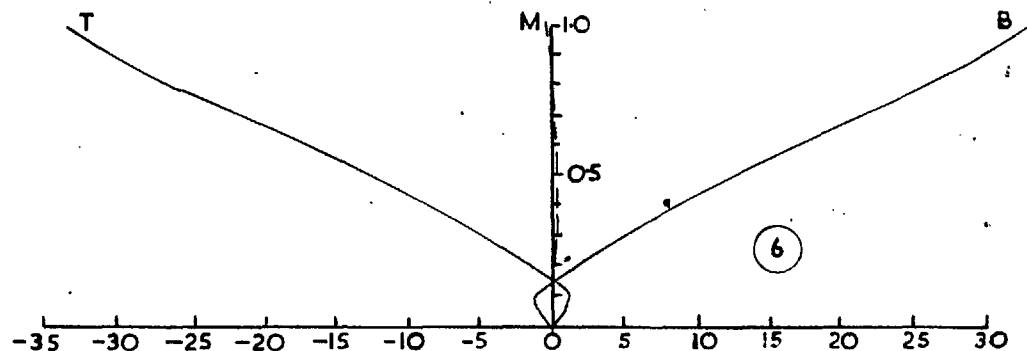
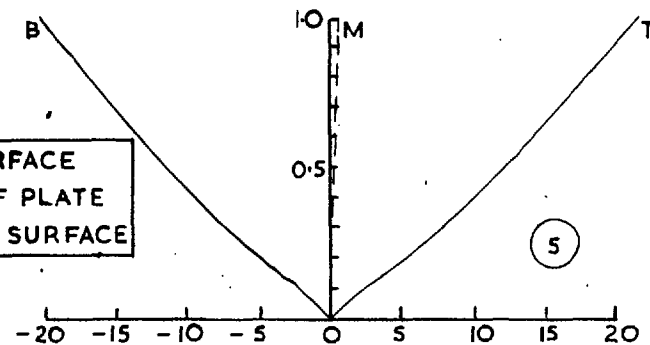
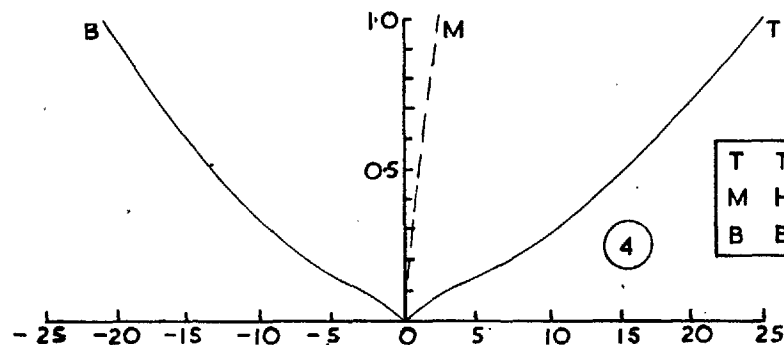
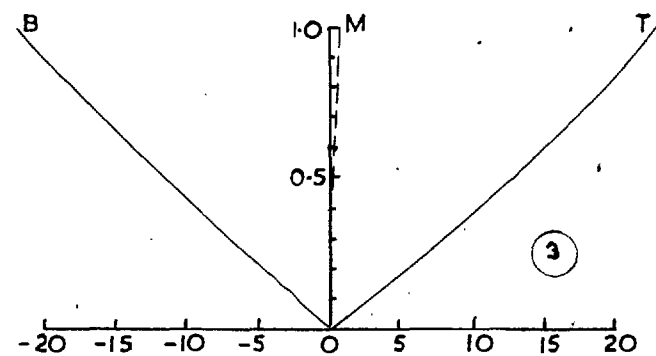
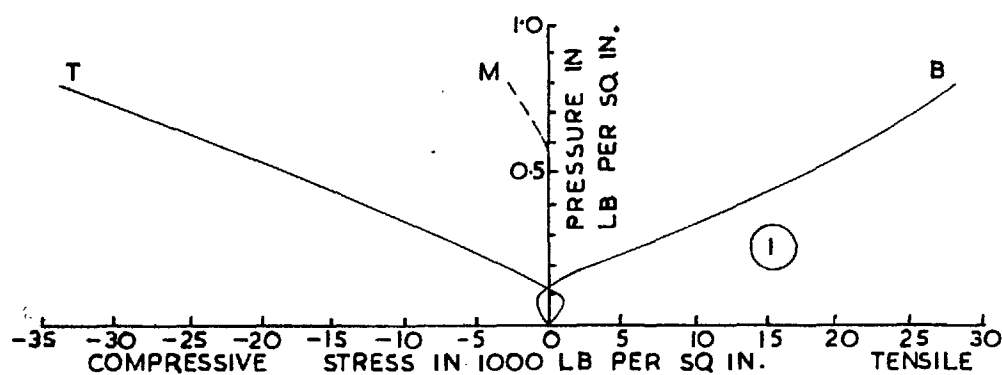


FIG. 71: DISTRIBUTION OF SECONDARY LONGITUDINAL BENDING STRESS ON TOP SURFACE OF PLATE ON TRANSVERSE CENTRAL SECTION (DEVELOPED VIEW) (TEST 1)



T TOP SURFACE
M HEART OF PLATE
B BOTTOM SURFACE

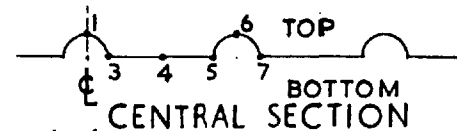


FIG. 72a. TRANSVERSE SURFACE AND HEART OF PLATE STRESSES. (TEST 1)

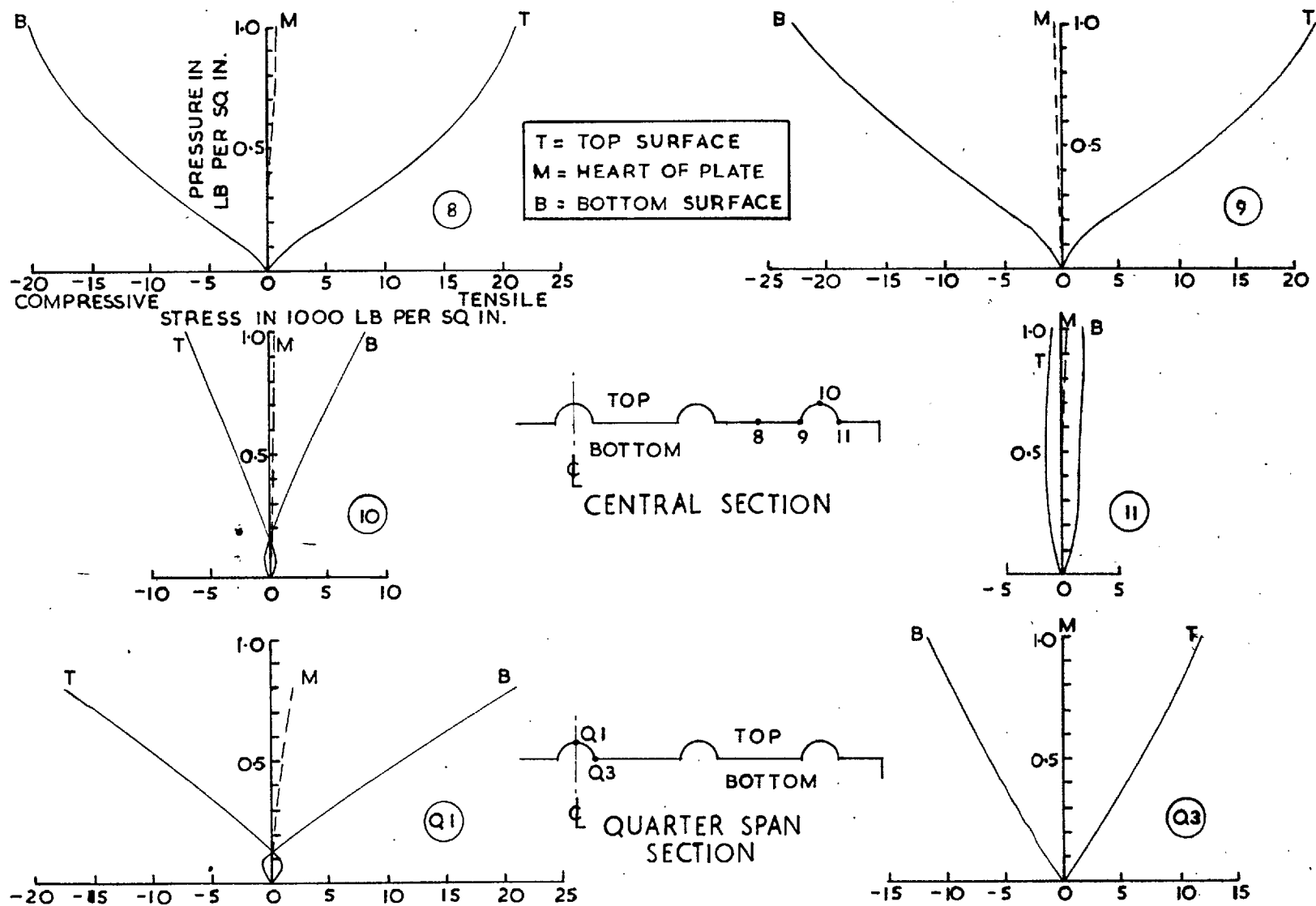


FIG. 72b: TRANSVERSE SURFACE AND HEART OF PLATE STRESSES. (TEST 1)

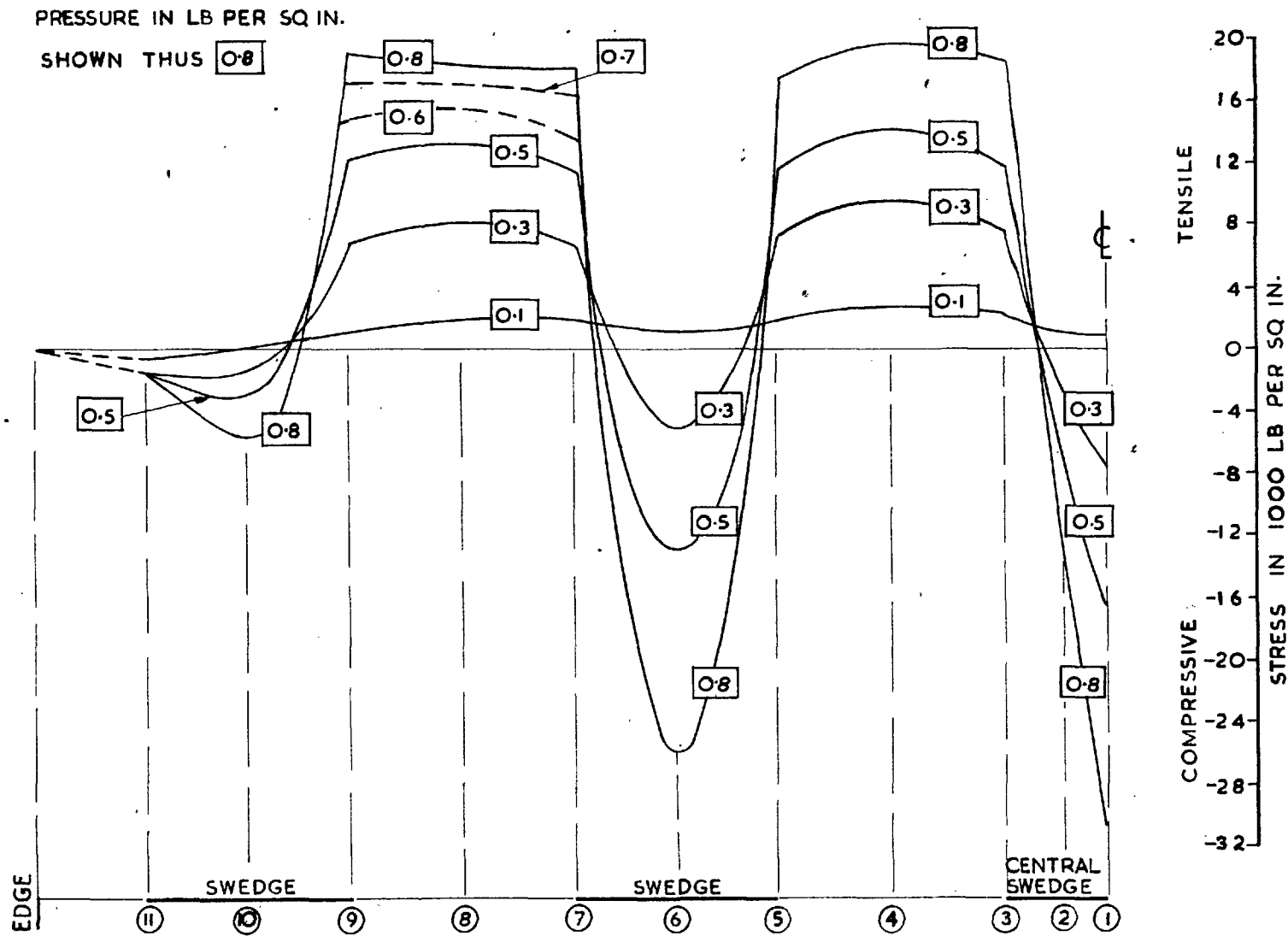


FIG. 73: DISTRIBUTION OF TRANSVERSE BENDING STRESS ON TOP SURFACE OF PLATE ON TRANSVERSE CENTRAL SECTION (DEVELOPED VIEW). (TEST 1)

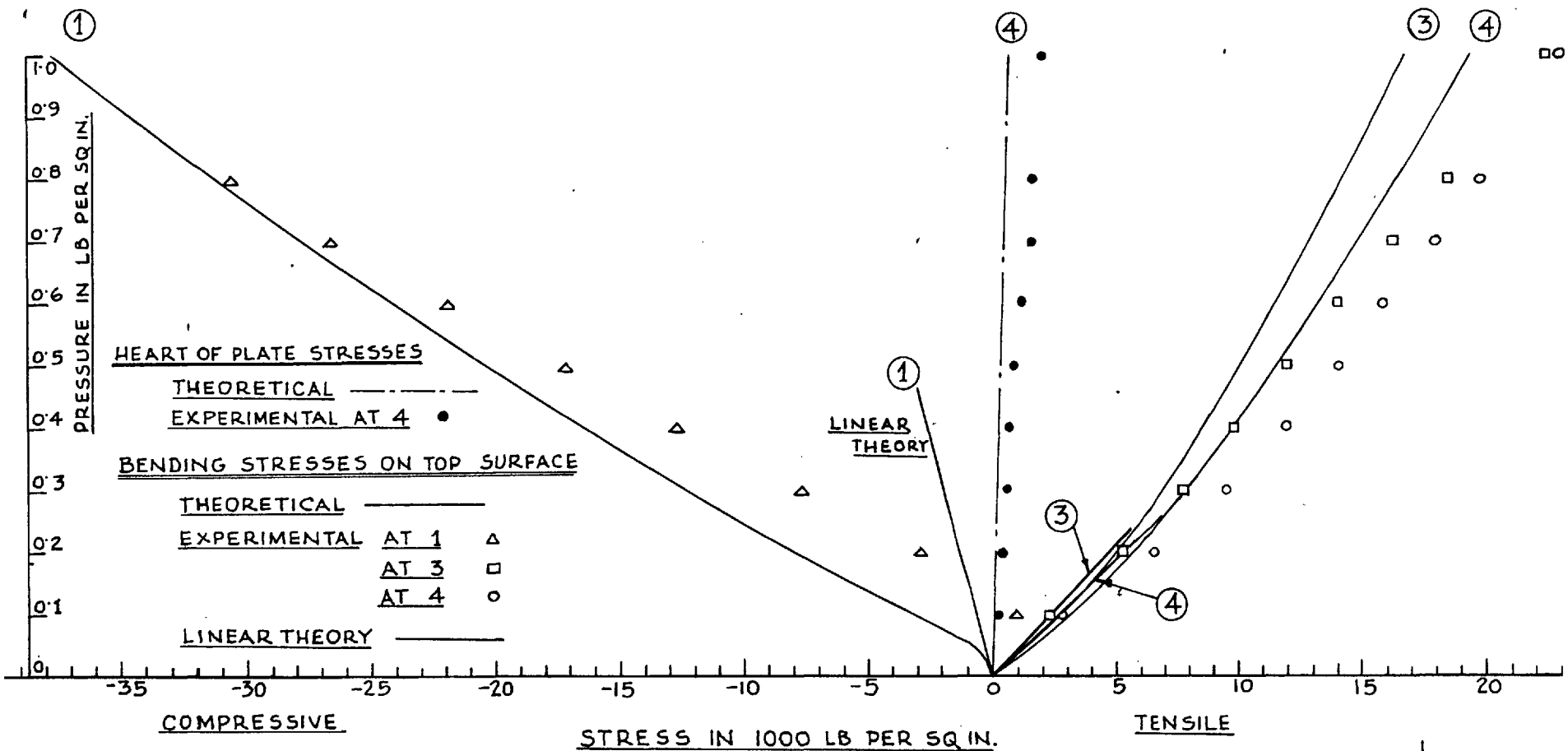
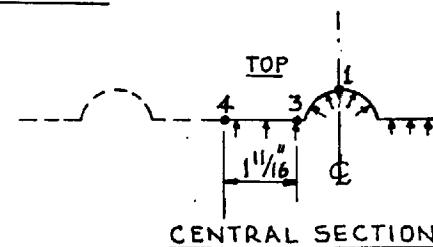


FIG. 74. TRANSVERSE HEART OF PLATE AND BENDING STRESSES
COMPARISON WITH THEORY (TEST 1)



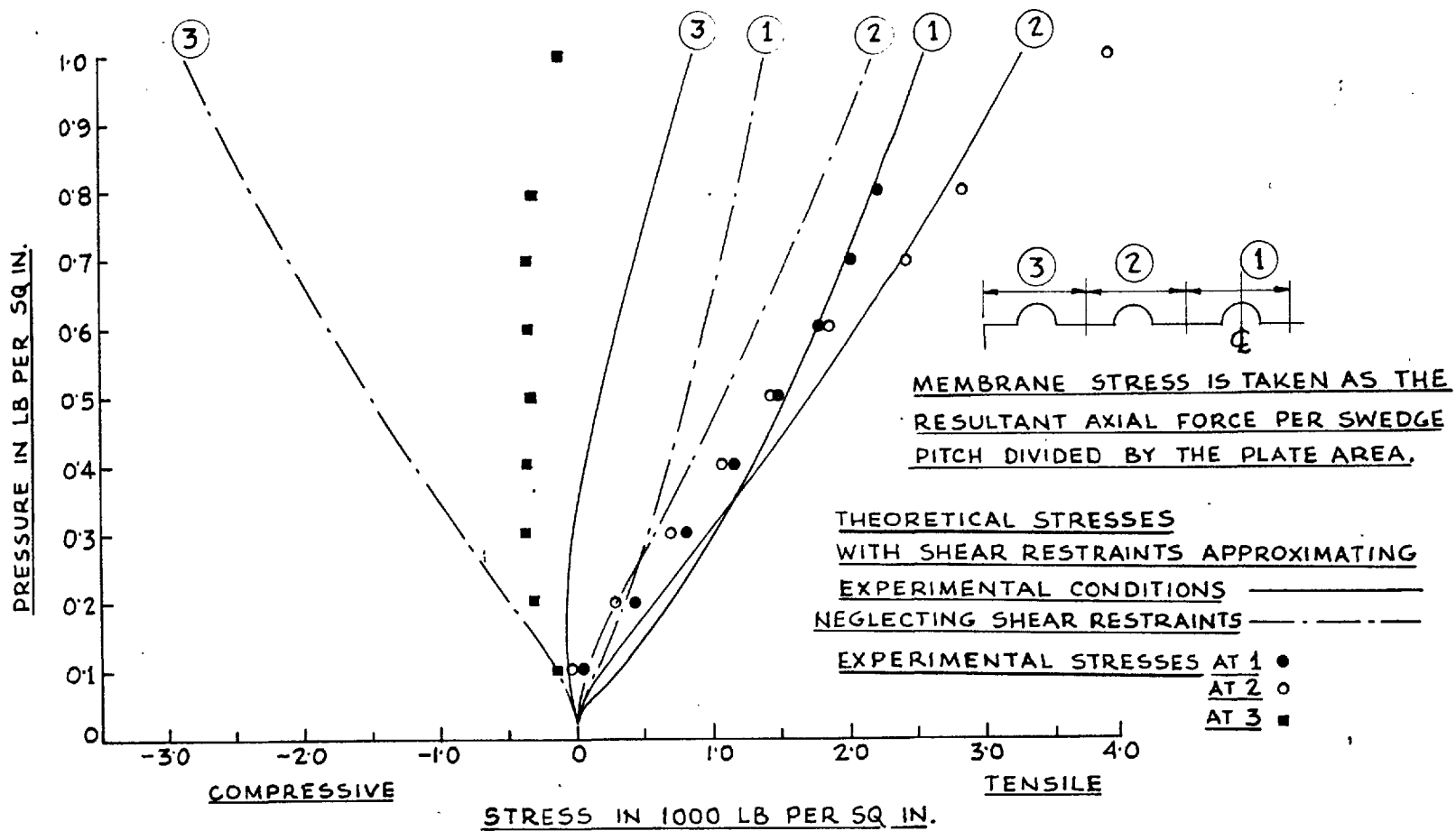


FIG. 75: LONGITUDINAL MEMBRANE STRESSES IN SWEDGES AT MIDSPAN SECTION (TEST 1)

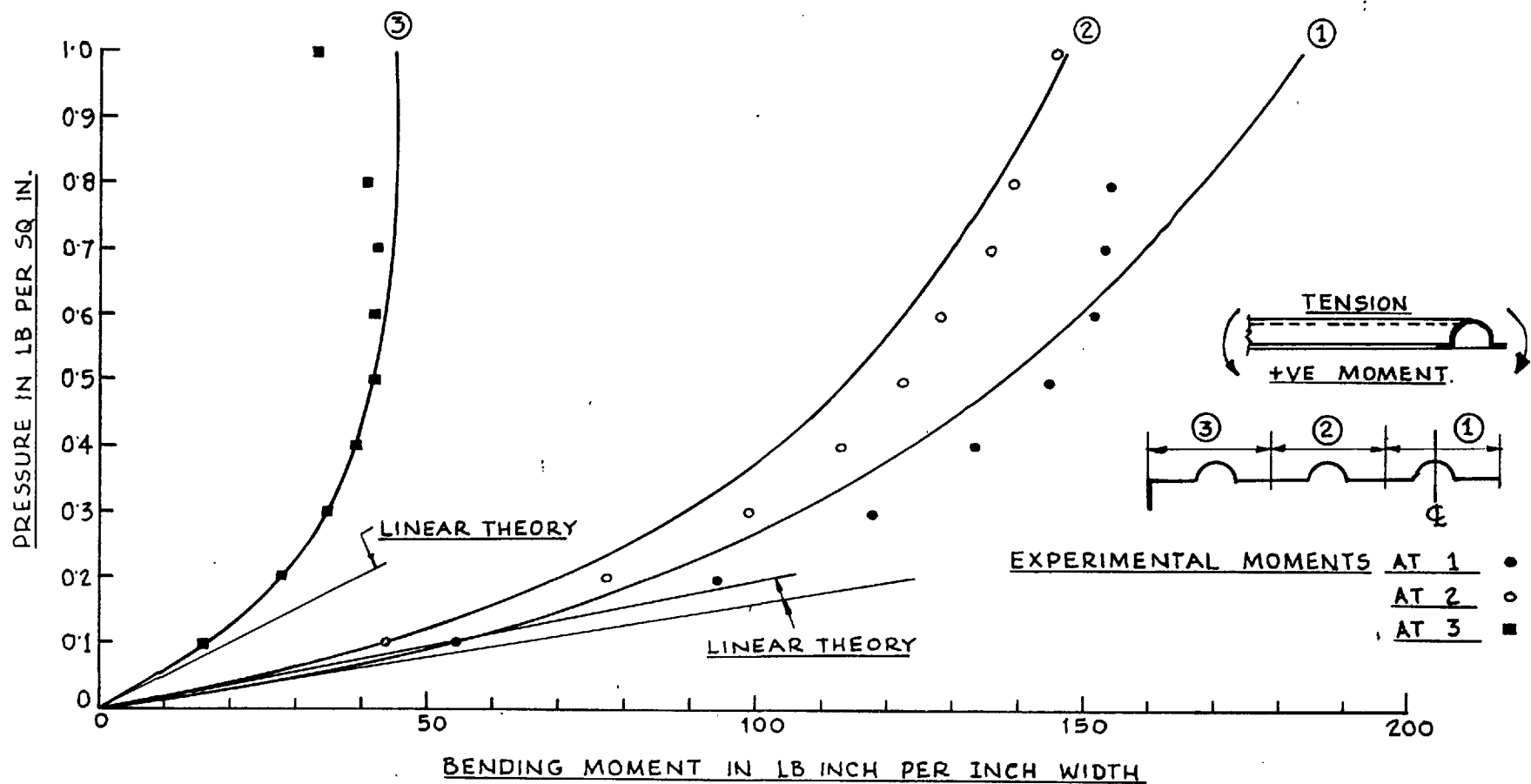
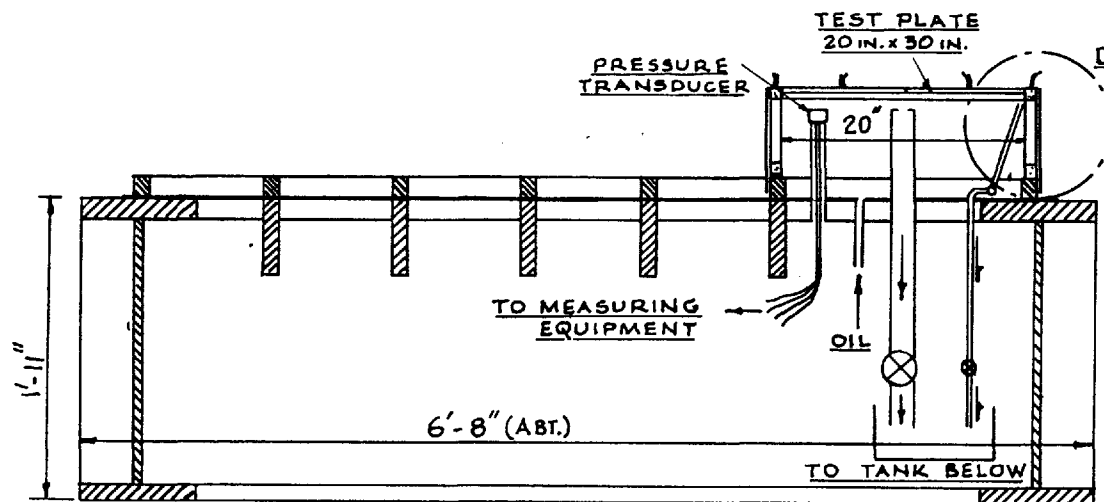
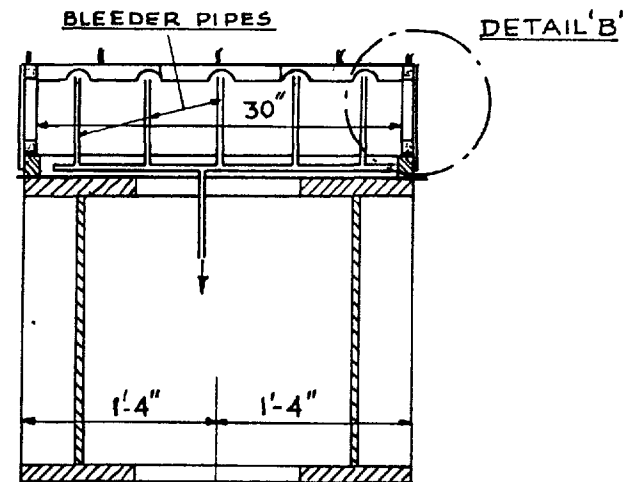


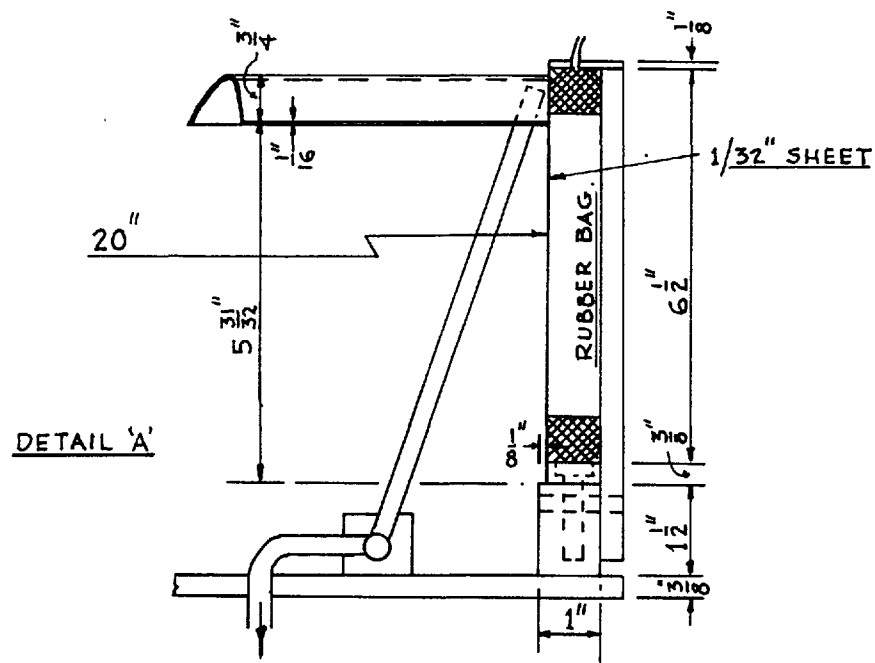
FIG. 76. AVERAGE LONGITUDINAL BENDING MOMENT IN SWEDGES AT MIDSPAN SECTION (TEST 1)



SECTIONAL ELEVATION



SECTIONAL END VIEW



DETAIL 'B'
OTHER DETAILS SAME
AS IN DETAIL 'A'

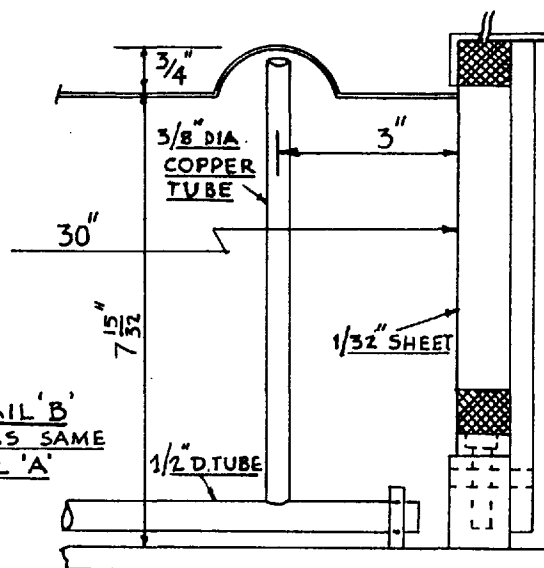
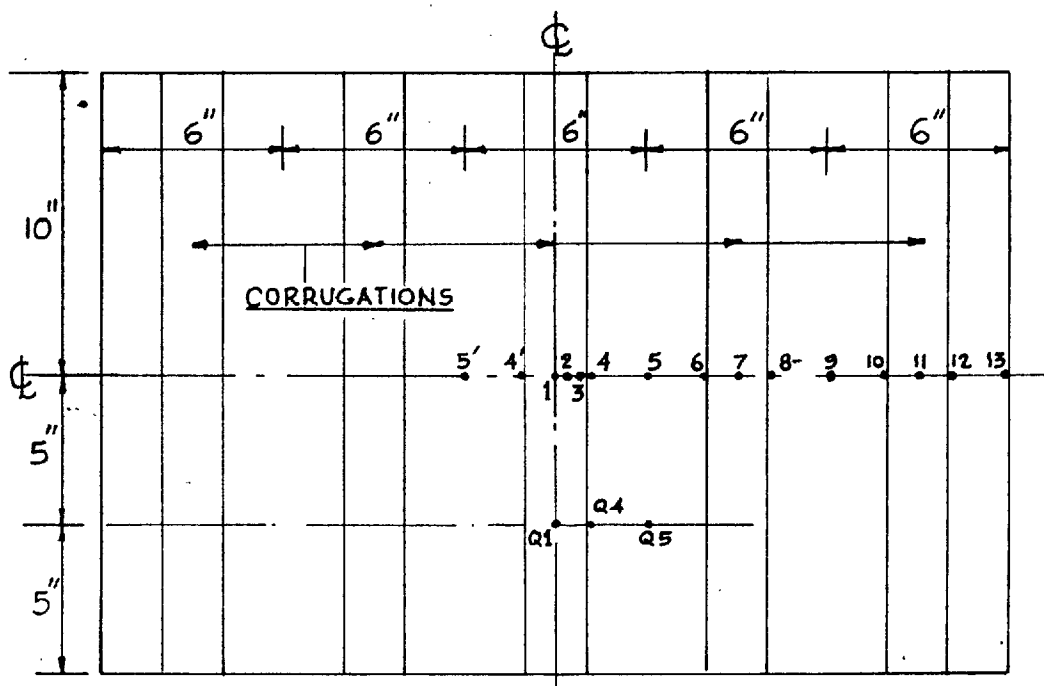


FIG. 77. DETAILS OF TEST RIG (TEST 2)

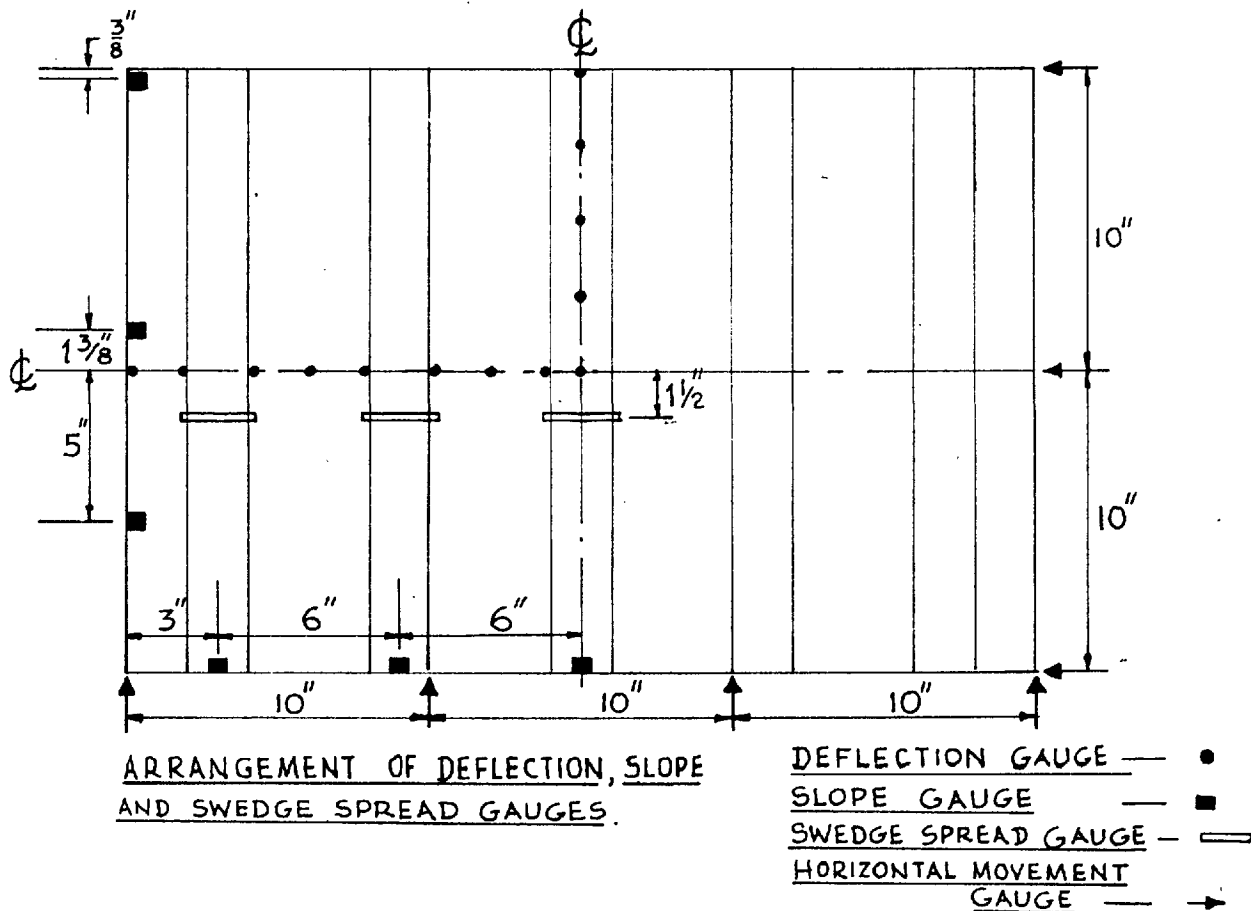


SEE ALSO FIG. 88

EACH POINT REPRESENTS A GROUP OF 4
STRAIN GAUGES, 2 (LONGITUDINAL AND
TRANSVERSE) ON EACH SURFACE

ARRANGEMENT OF
STRAIN GAUGES.

LONGITUDINAL GAUGES 1/4 IN.
TRANSVERSE GAUGES 1/8 IN.



ARRANGEMENT OF DEFLECTION, SLOPE
AND SWEDGE SPREAD GAUGES.

DEFLECTION GAUGE — ●
SLOPE GAUGE — ■
SWEDGE SPREAD GAUGE — ▭
HORIZONTAL MOVEMENT
GAUGE — →

FIG. 78. INSTRUMENTATION DETAILS
(TEST 2)

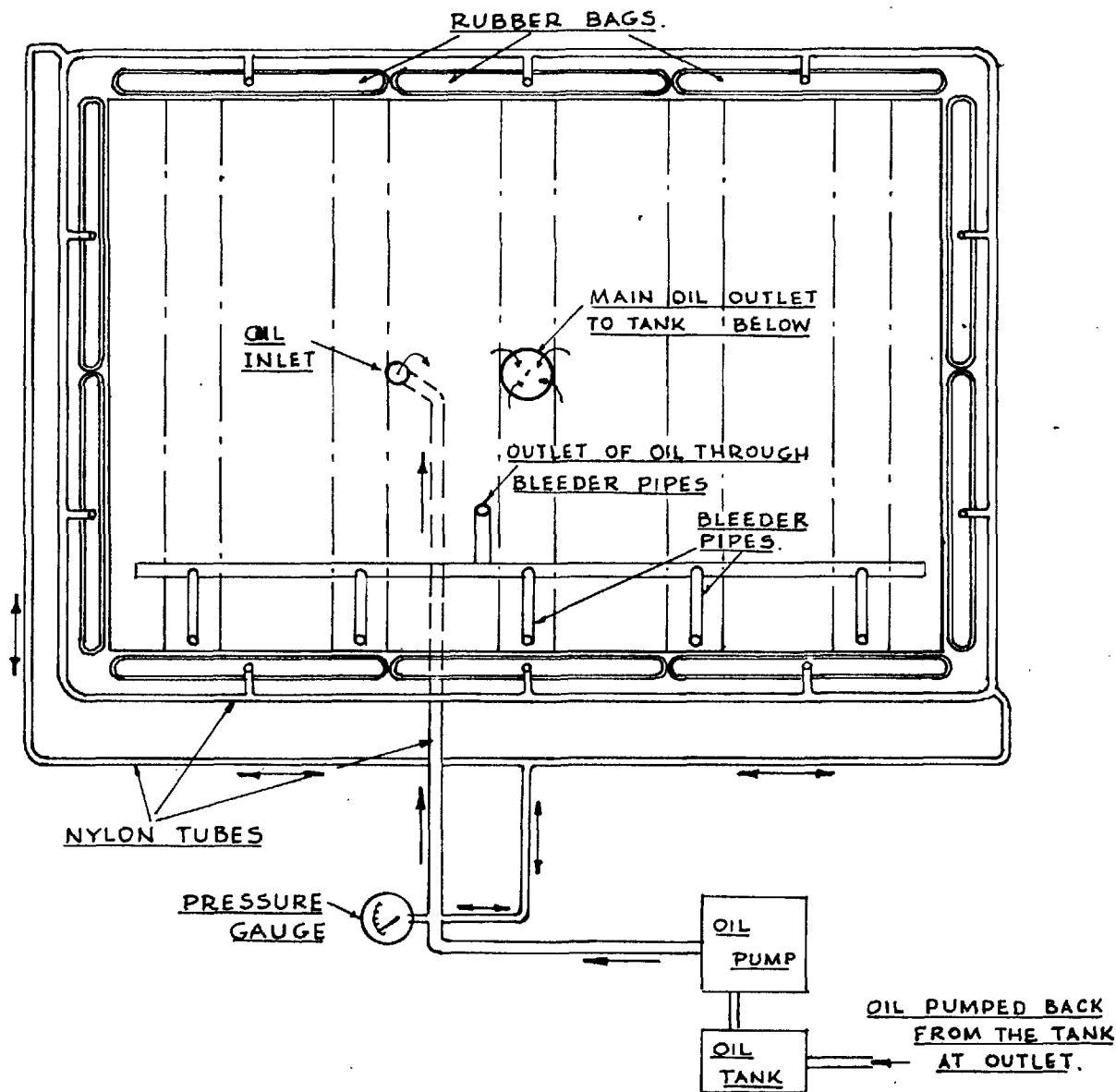


FIG. 79. DETAILS OF ARRANGEMENT FOR LOADING BY OIL
(TEST 2)

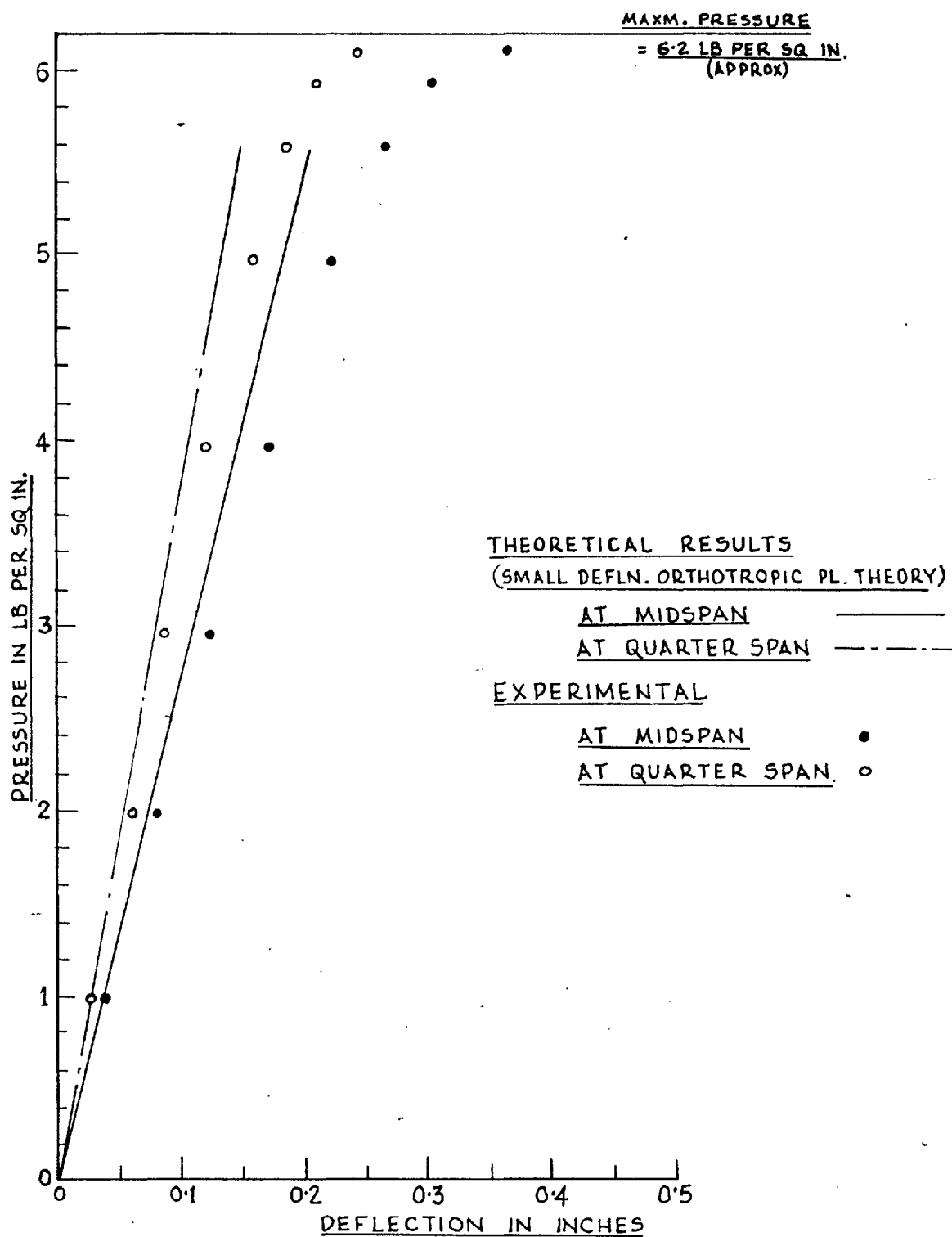


FIG. 80. DEFLECTIONS OF CENTRAL SWEDGE (TEST 2)

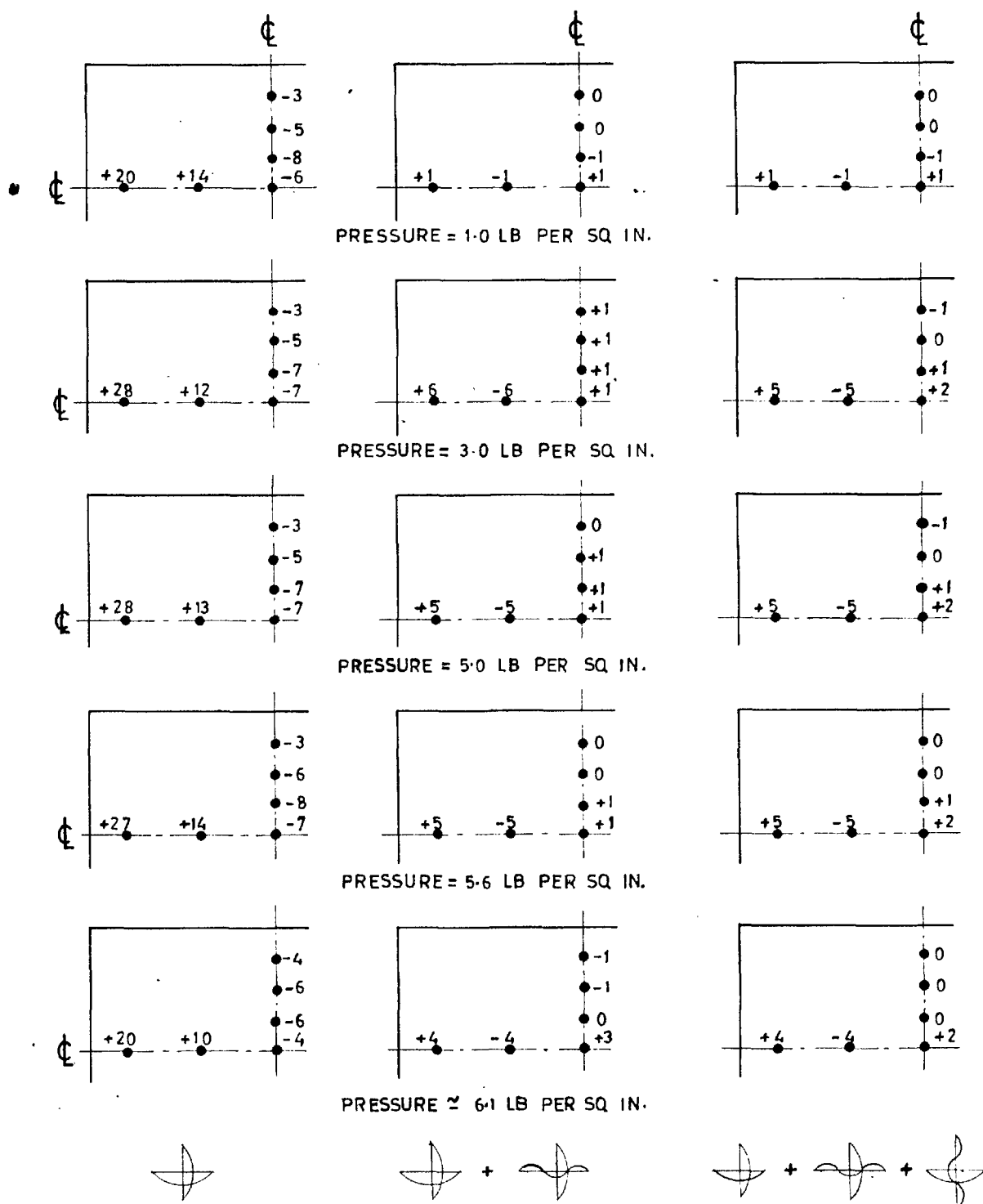


FIG.81:DEPARTURE OF "BEST" FOURIER DEFLECTED FORM FROM OBSERVED DEFLECTION. ERRORS ARE EXPRESSED AS PERCENTAGES OF OBSERVED CENTRAL DEFLECTION. (TEST 2)

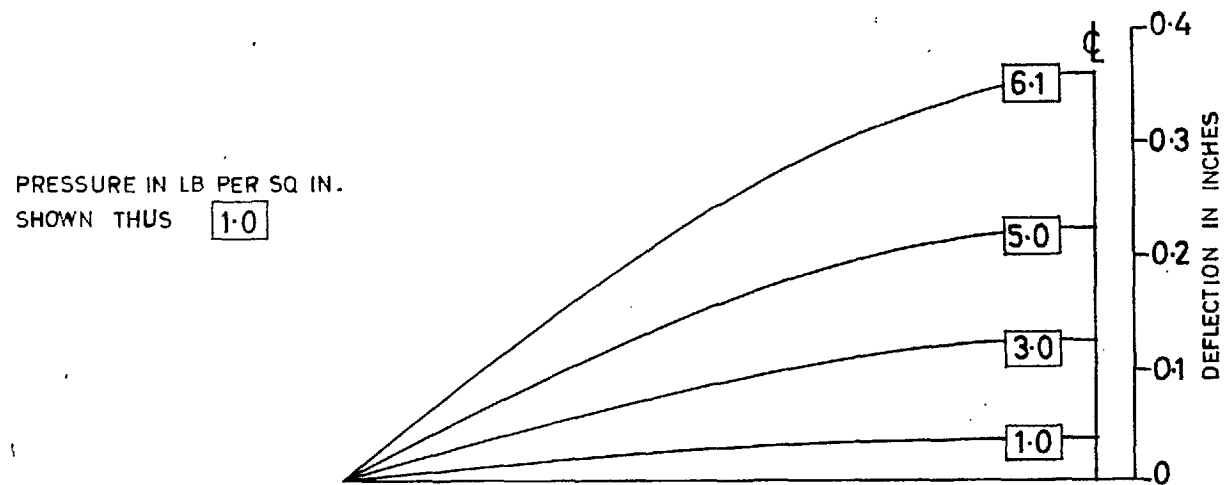


FIG.82:LONGITUDINAL CENTRAL SECTION OF THE DEFLECTED PLATE (TEST 2)

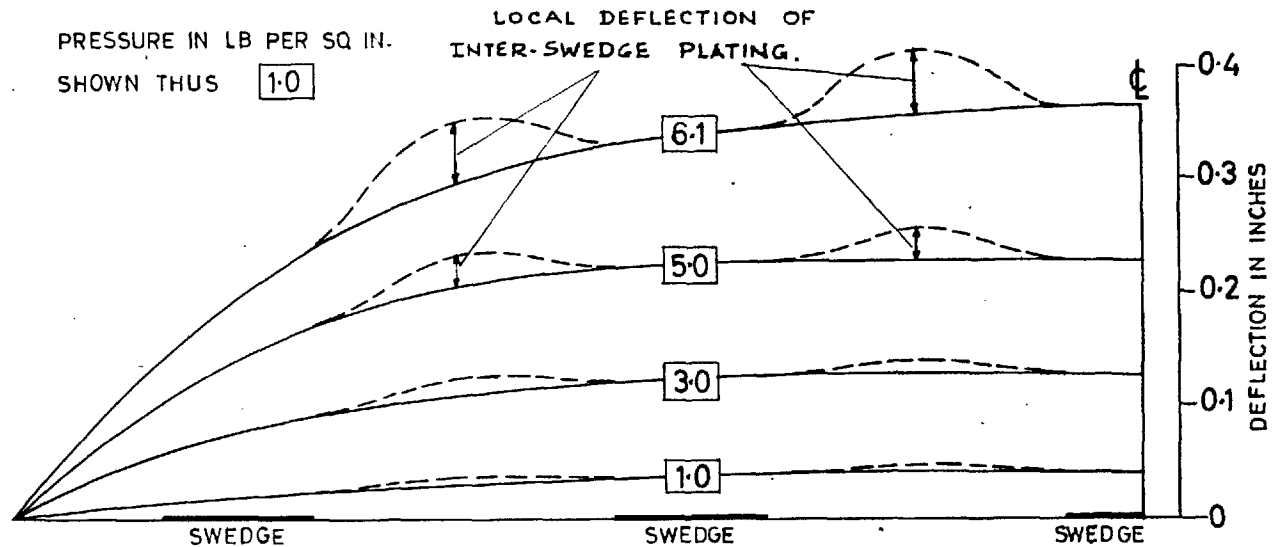


FIG.83:TRANSVERSE CENTRAL SECTION OF THE DEFLECTED PLATE (TEST 2)

EXPERIMENTAL

AT 1 LB PER SQ IN. •

AT 5 LB PER SQ IN. ◦

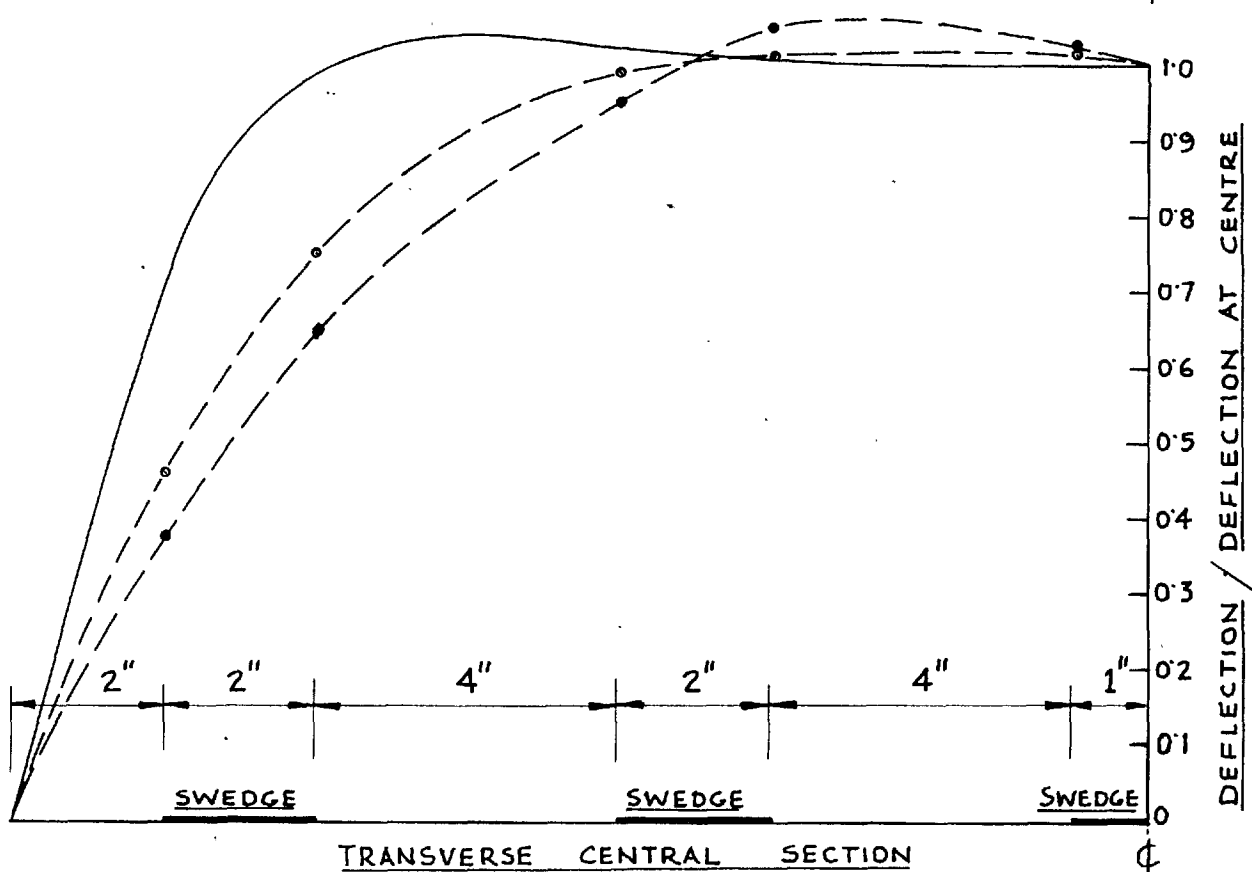
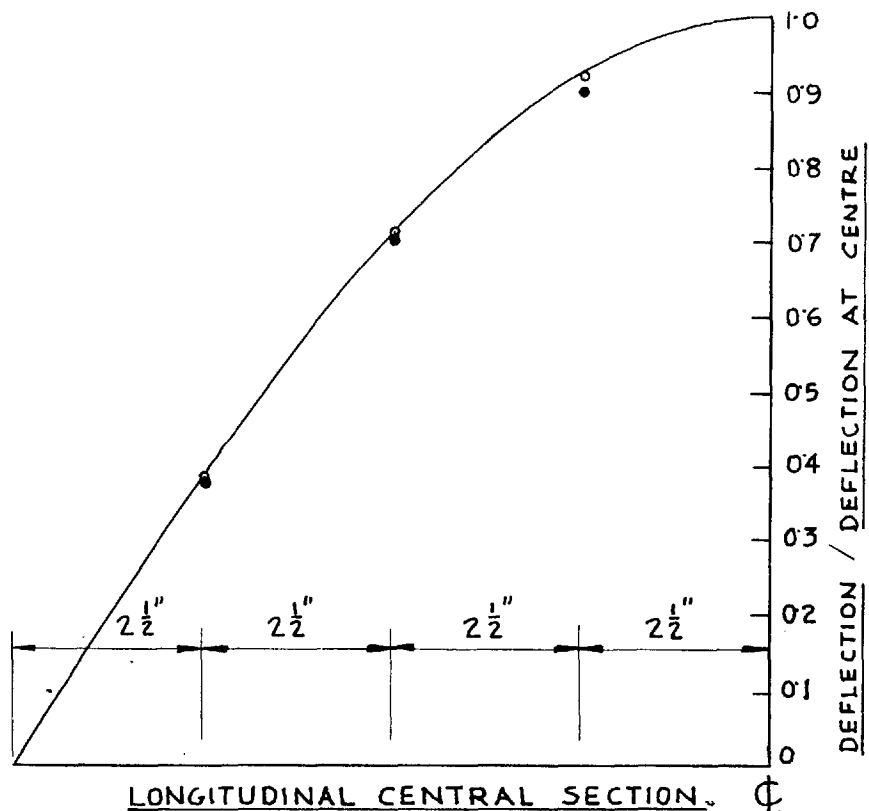


FIG. 84: DEFLECTED SHAPES OF PLATE CENTRE-LINES (TEST 2)
COMPARISON WITH THEORY

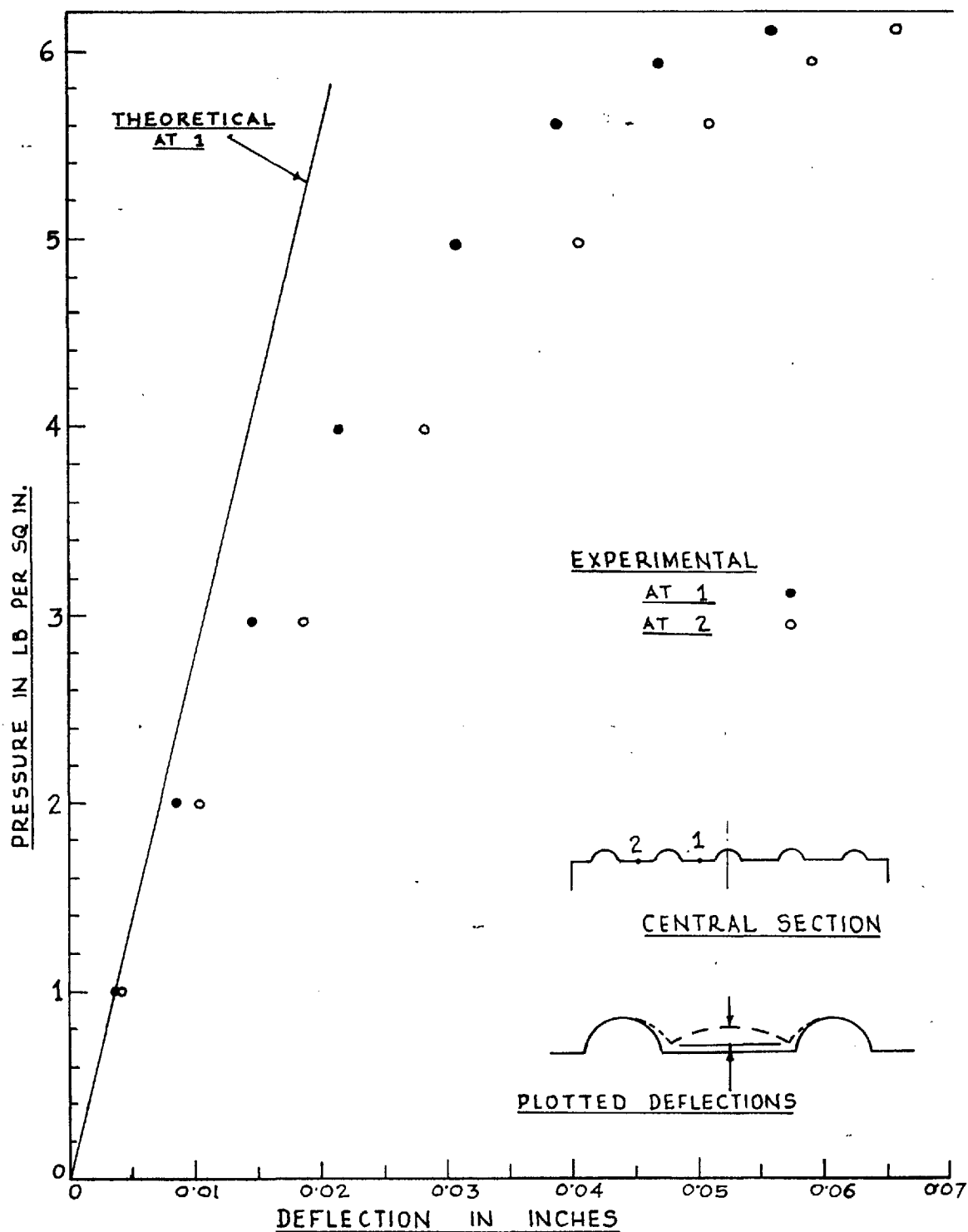


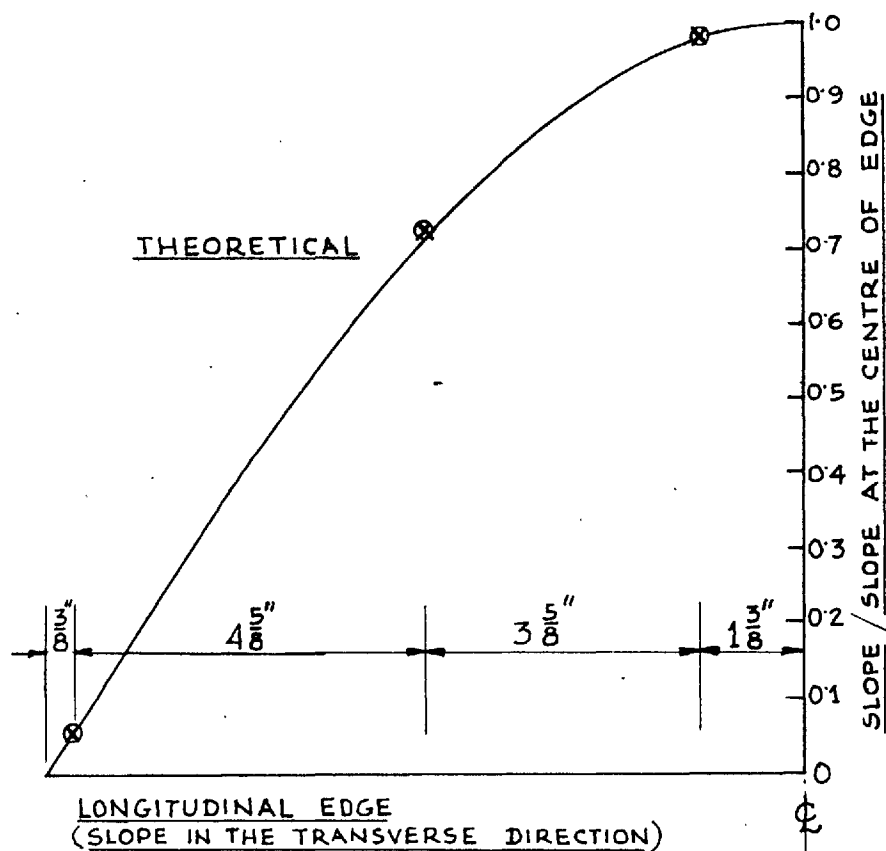
FIG. 85: DEFLECTION OF PLATE RELATIVE TO SWEDGE
(TEST 2)

EXPERIMENTAL

AT 1 LB PER SQ IN. •

AT 3 LB PER SQ IN. X

AT 5 LB PER SQ IN. ⊙



THEORETICAL

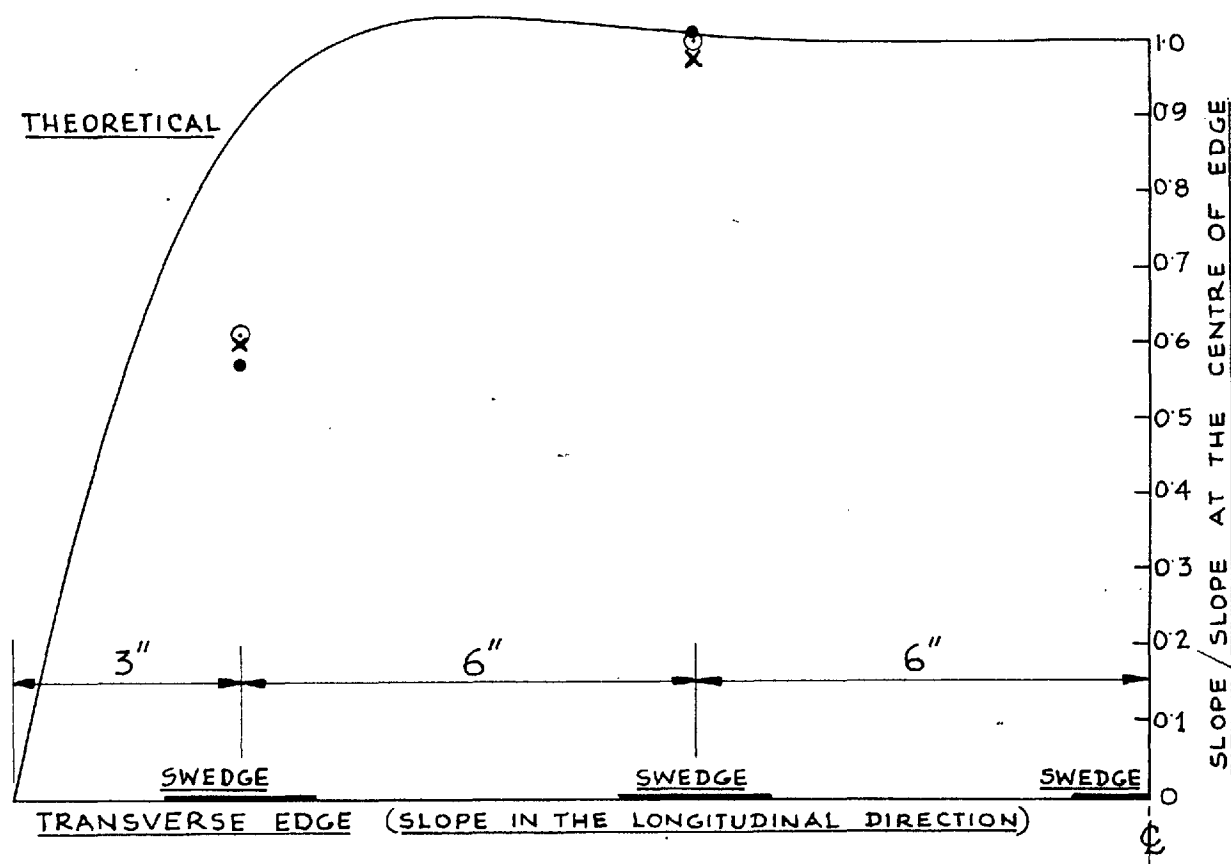


FIG. 87. DISTRIBUTION OF NORMAL EDGE SLOPES ALONG THE EDGES
(TEST 2)

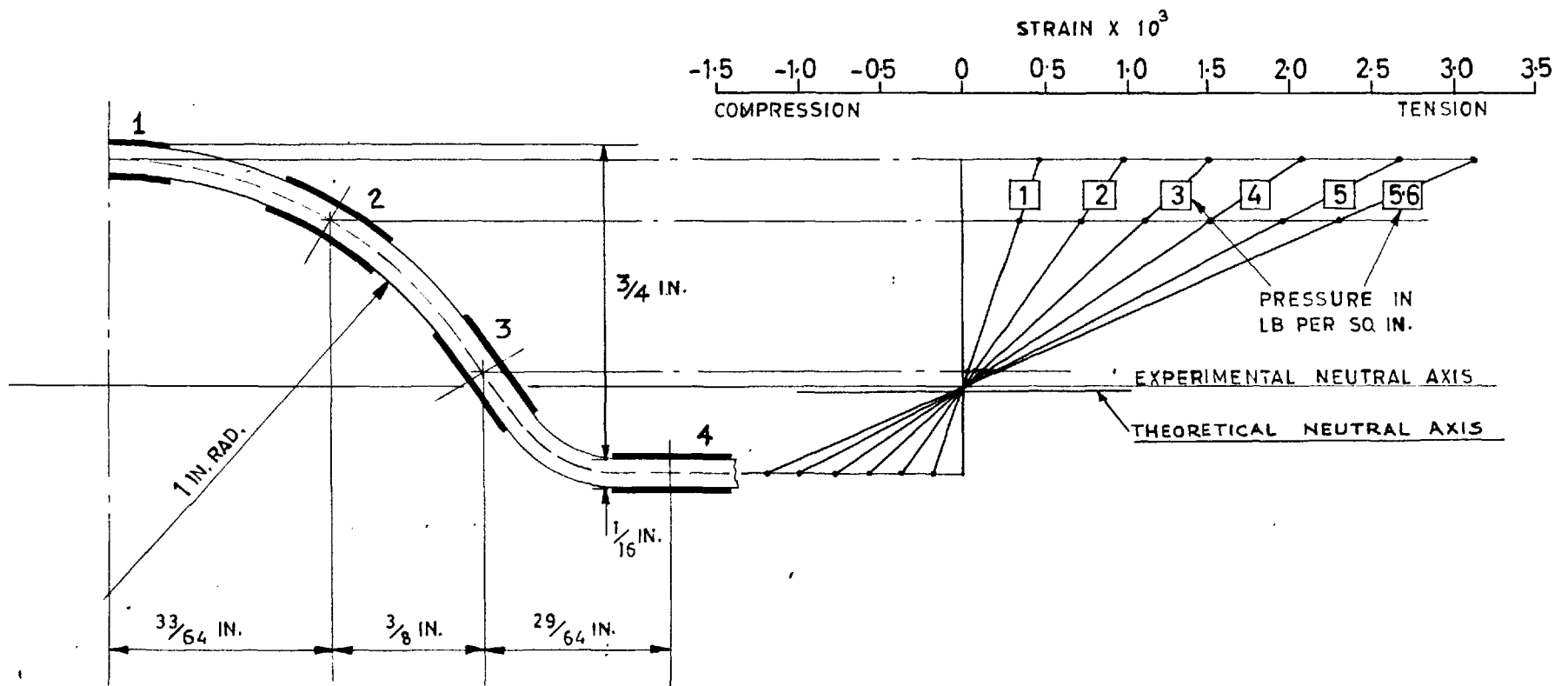


FIG.88:LONGITUDINAL STRAIN DISTRIBUTION OVER DEPTH OF CENTRE SWEDGE AT MIDSPAN
(TEST 2)

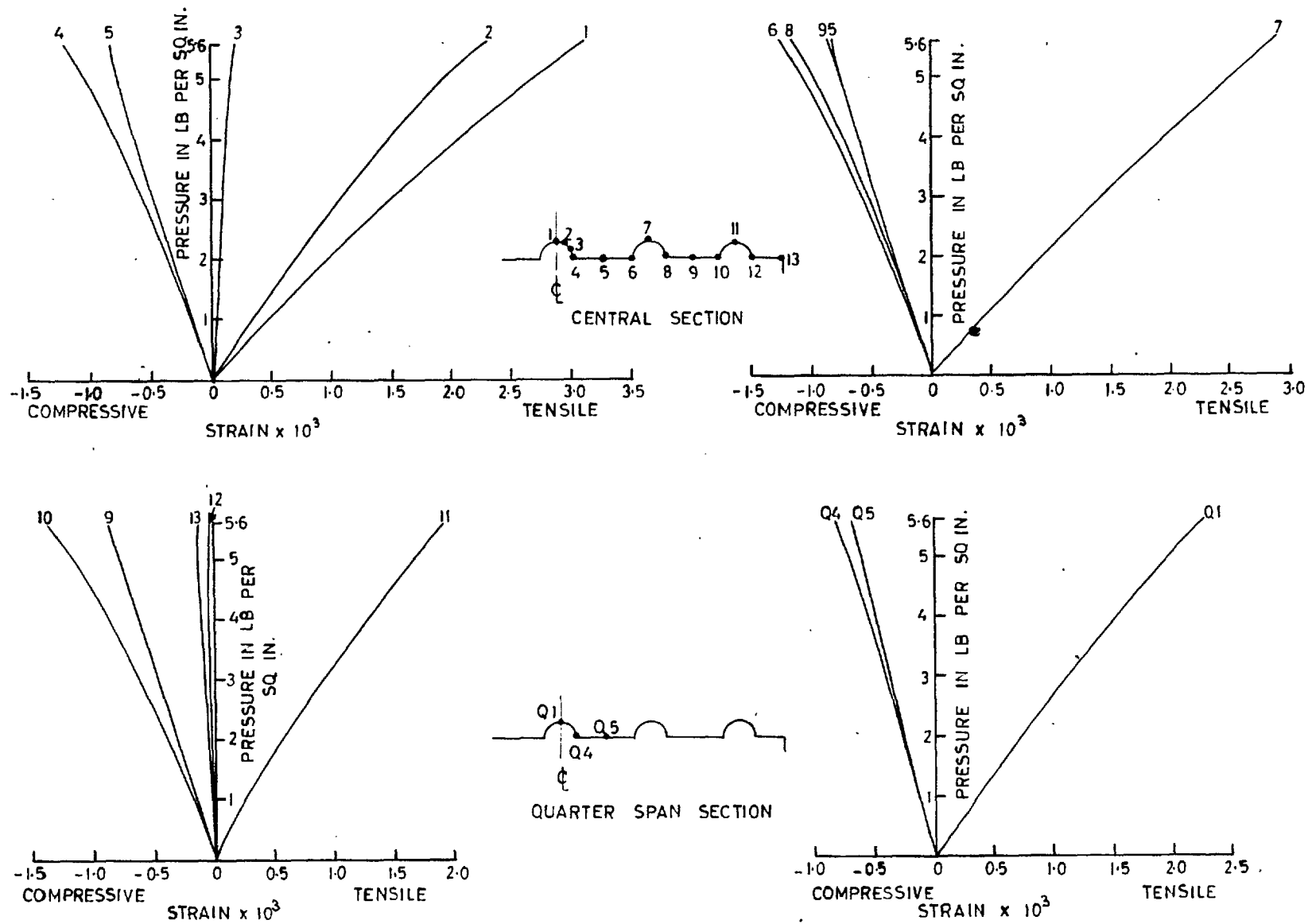


FIG.89:LONGITUDINAL HEART OF PLATE STRAIN (TEST 2)

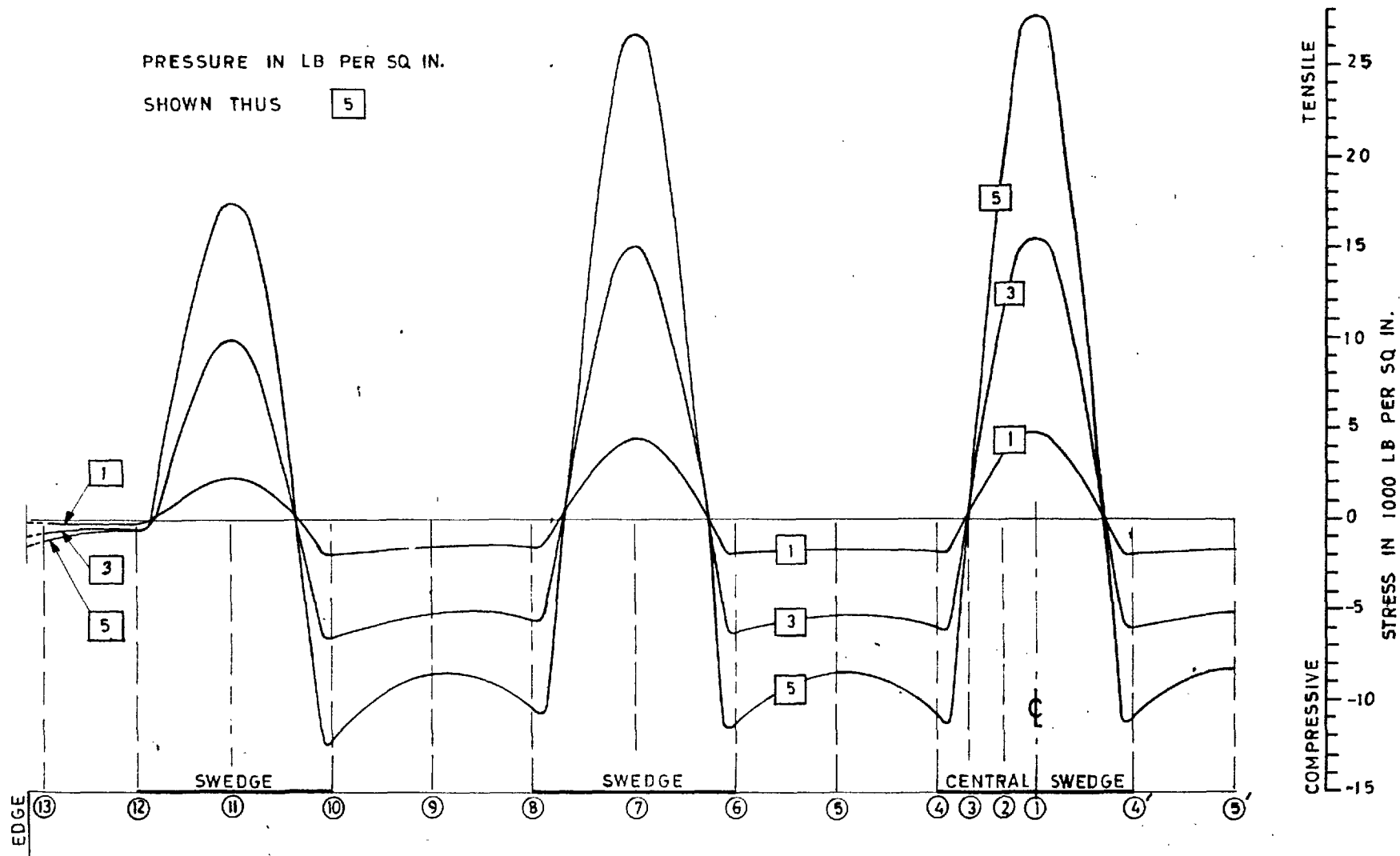
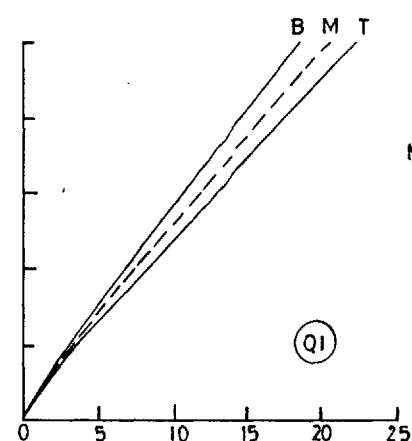
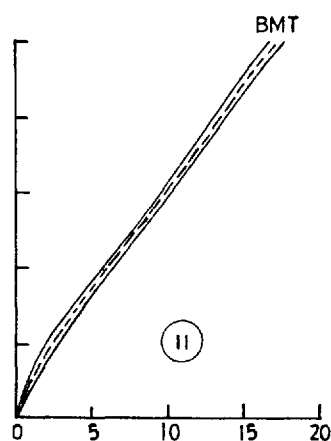
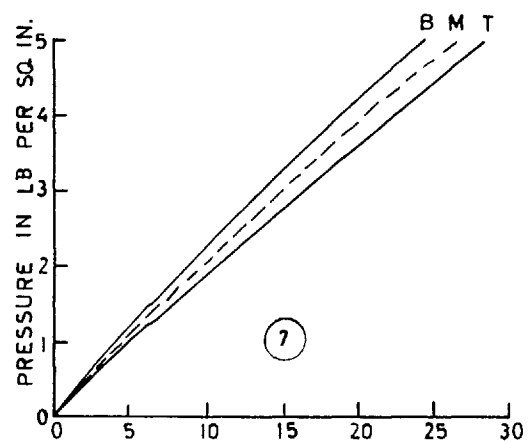
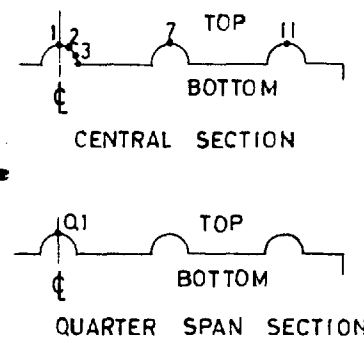
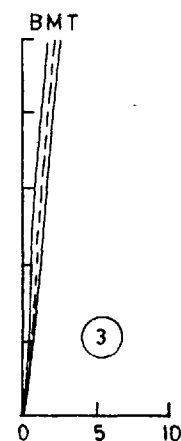
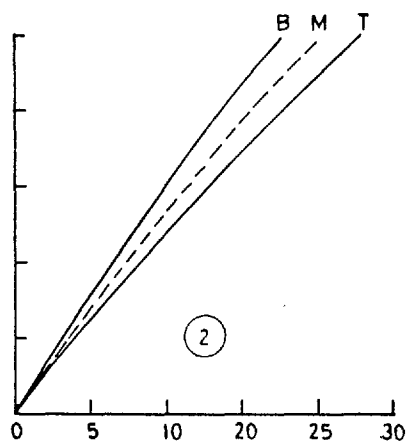
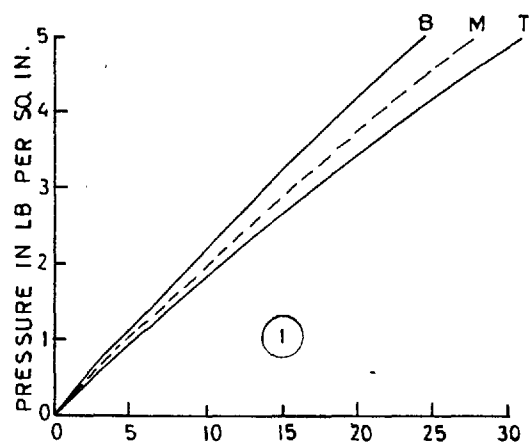
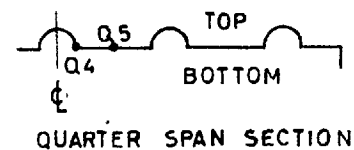
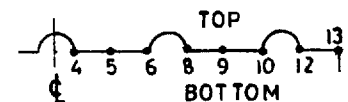
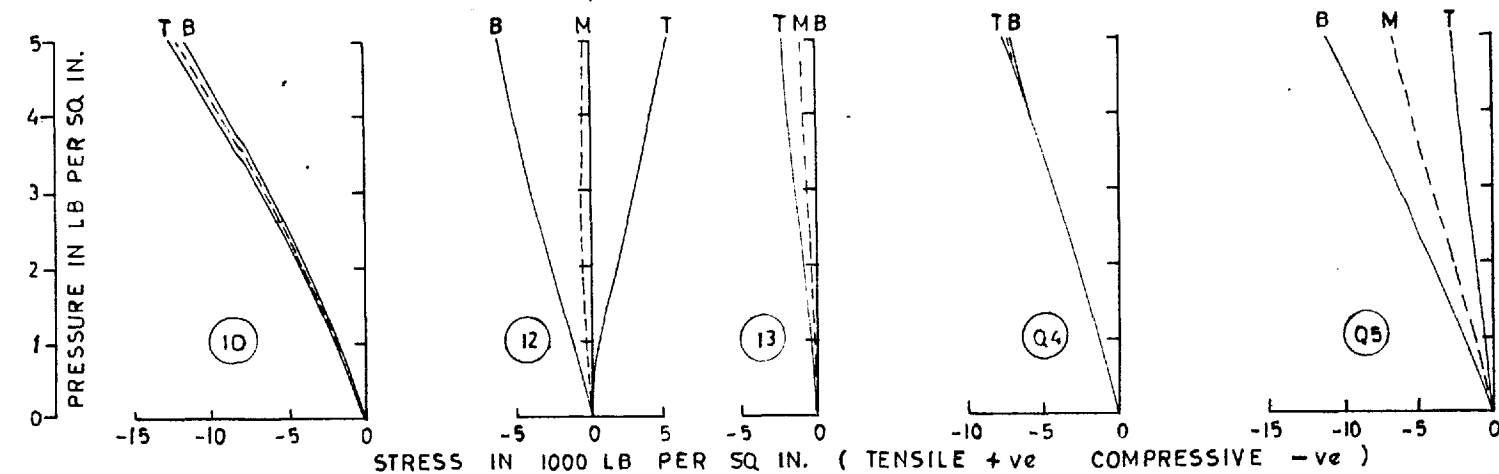
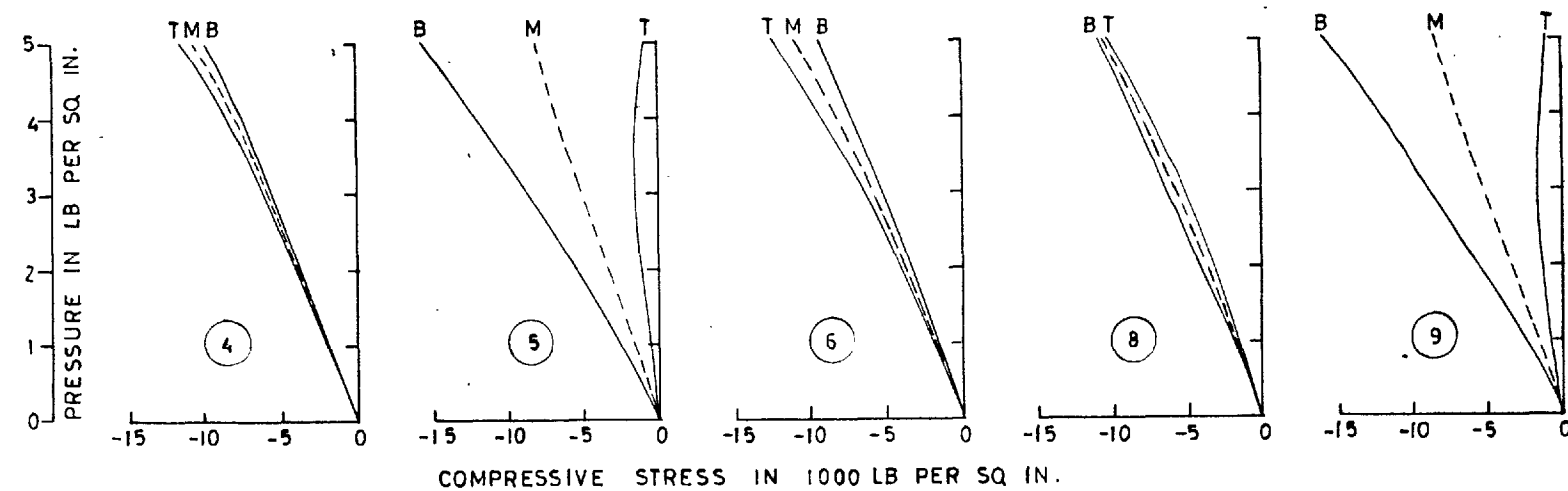


FIG. 90: DISTRIBUTION OF LONGITUDINAL HEART OF PLATE STRESSES ON TRANSVERSE
 CENTRAL SECTION (DEVELOPED VIEW) (TEST 2)



T - TOP SURFACE
M - HEART OF PLATE
B - BOTTOM SURFACE

FIG.91a: LONGITUDINAL SURFACE AND HEART OF PLATE STRESSES (TEST 2)



T — TOP SURFACE
M — HEART OF PLATE
B — BOTTOM SURFACE

FIG.91b: LONGITUDINAL SURFACE AND HEART OF PLATE STRESSES. (TEST 2)

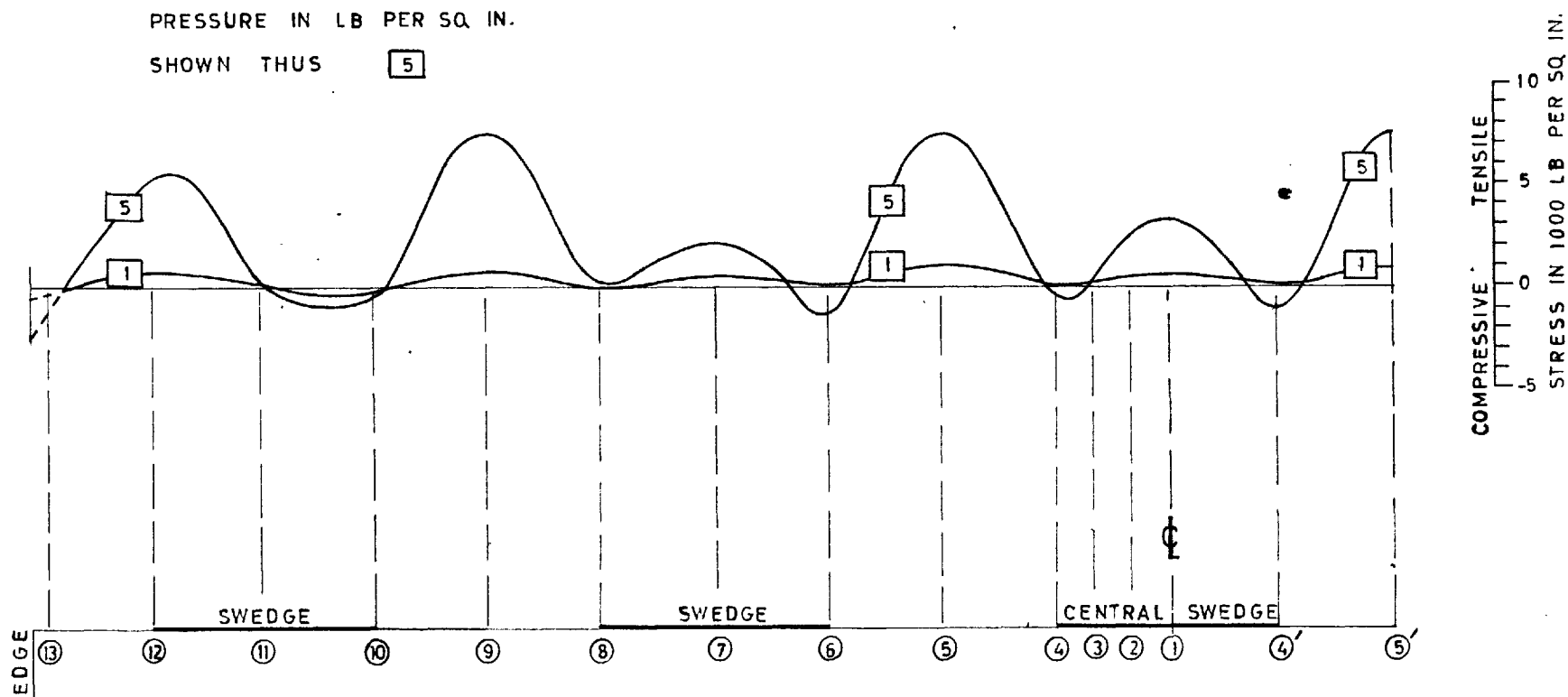


FIG.92: DISTRIBUTION OF SECONDARY LONGITUDINAL BENDING STRESS ON TOP SURFACE OF
 PLATE ON TRANSVERSE CENTRAL SECTION (DEVELOPED VIEW) (TEST 2)

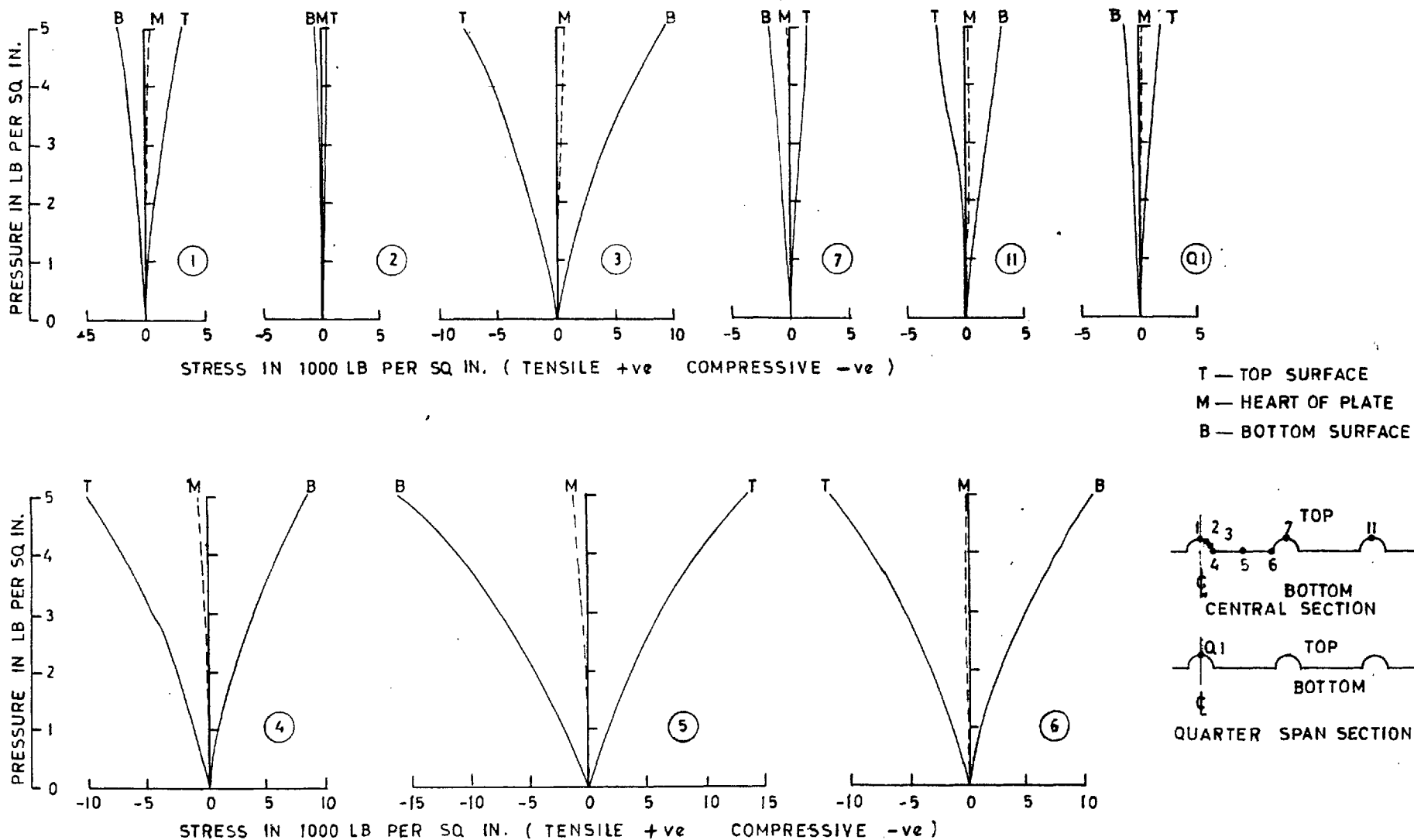
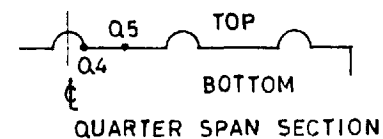
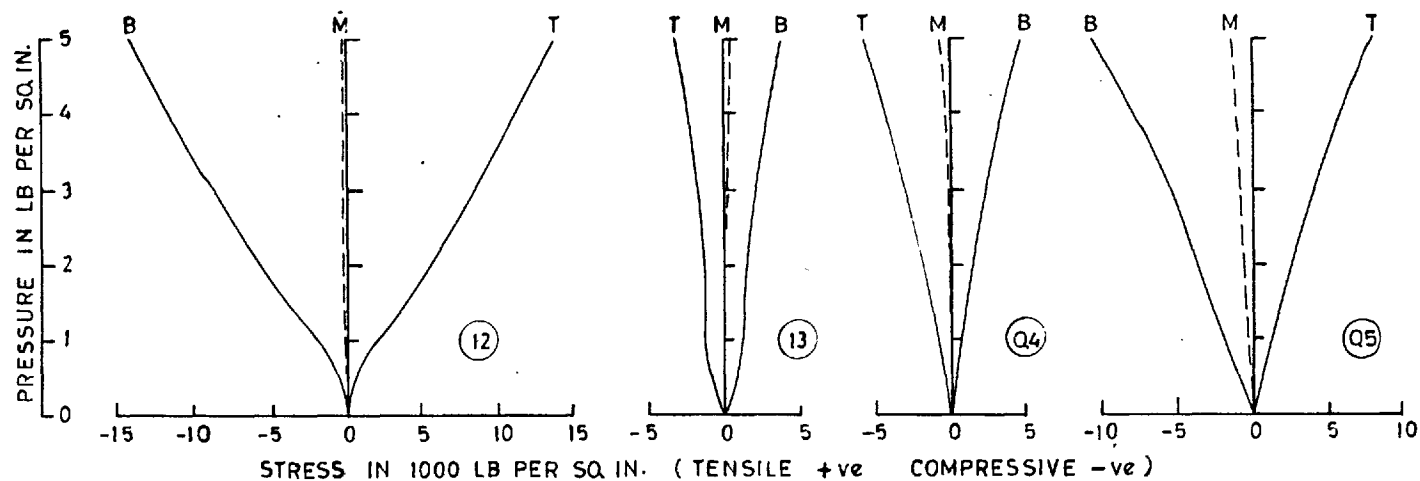
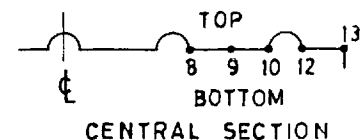
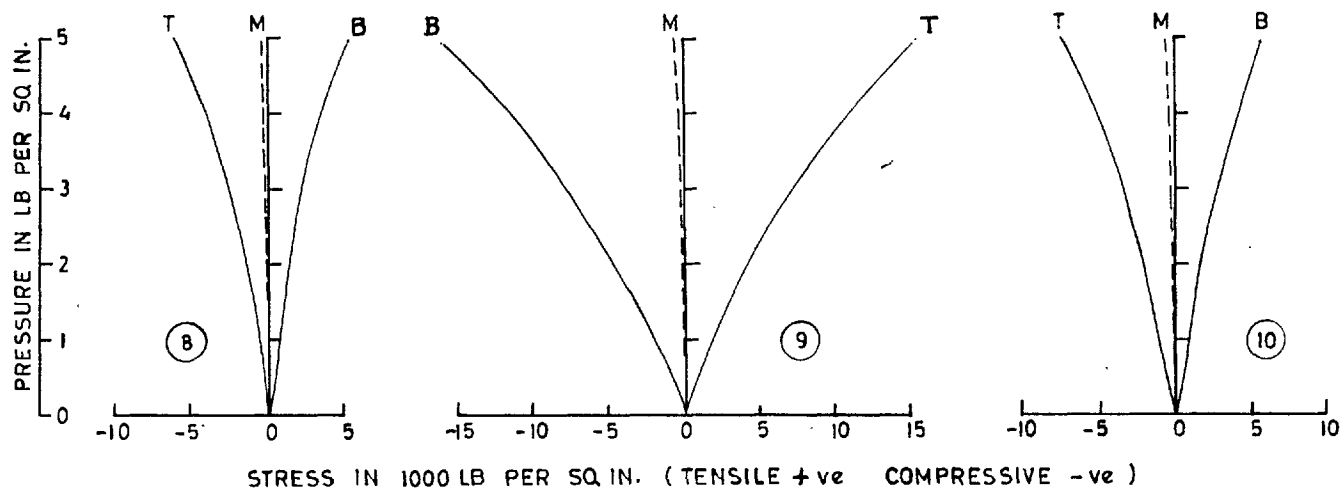


FIG.93a: TRANSVERSE SURFACE AND HEART OF PLATE STRESSES. (TEST 2)



T — TOP SURFACE
M — HEART OF PLATE
B — BOTTOM SURFACE

FIG. 93b: TRANSVERSE SURFACE AND HEART OF PLATE STRESSES. (TEST 2)

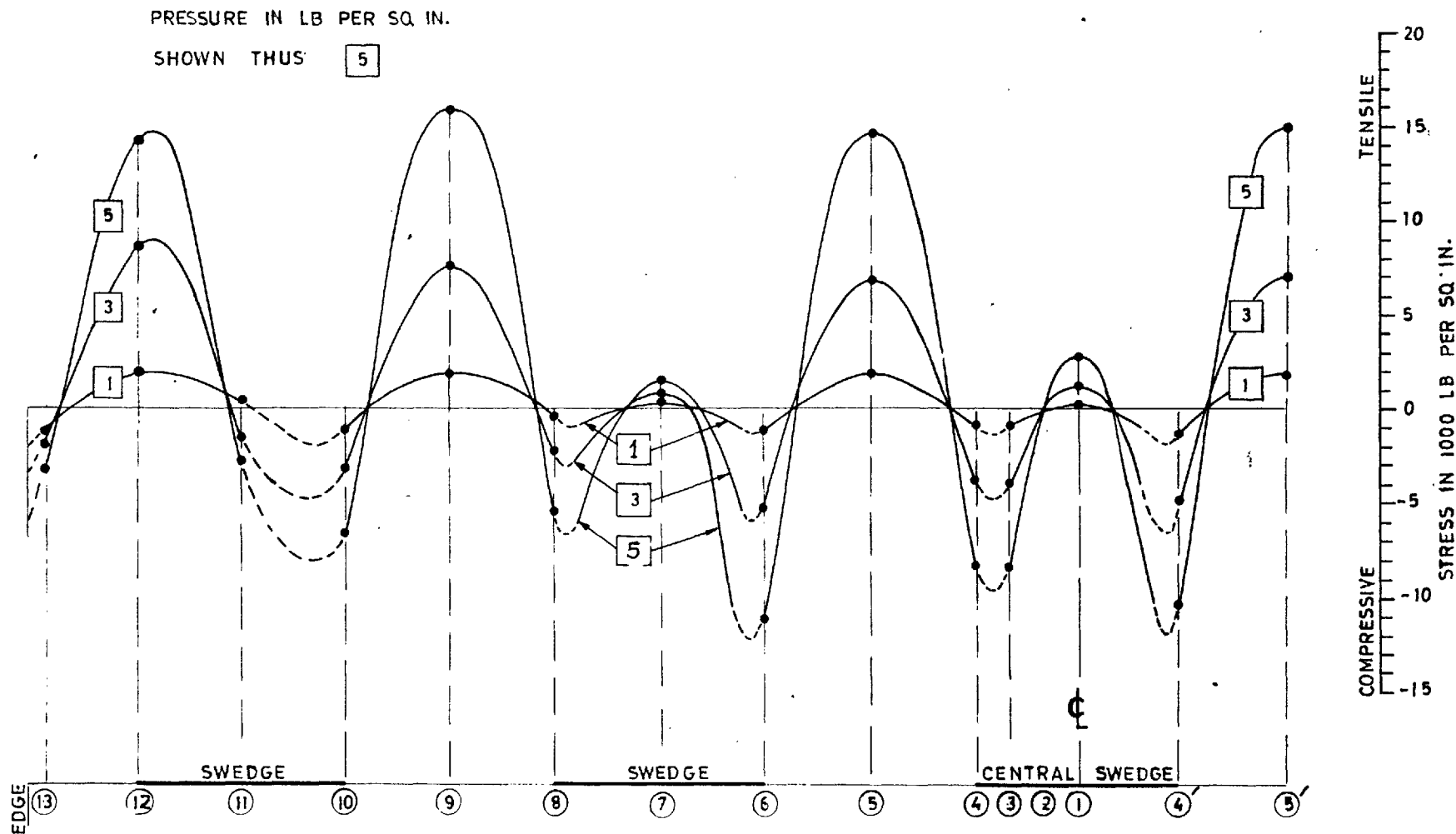


FIG. 94: DISTRIBUTION OF TRANSVERSE BENDING STRESS ON TOP SURFACE OF PLATE ON TRANSVERSE CENTRAL SECTION (DEVELOPED VIEW) (TEST 2)

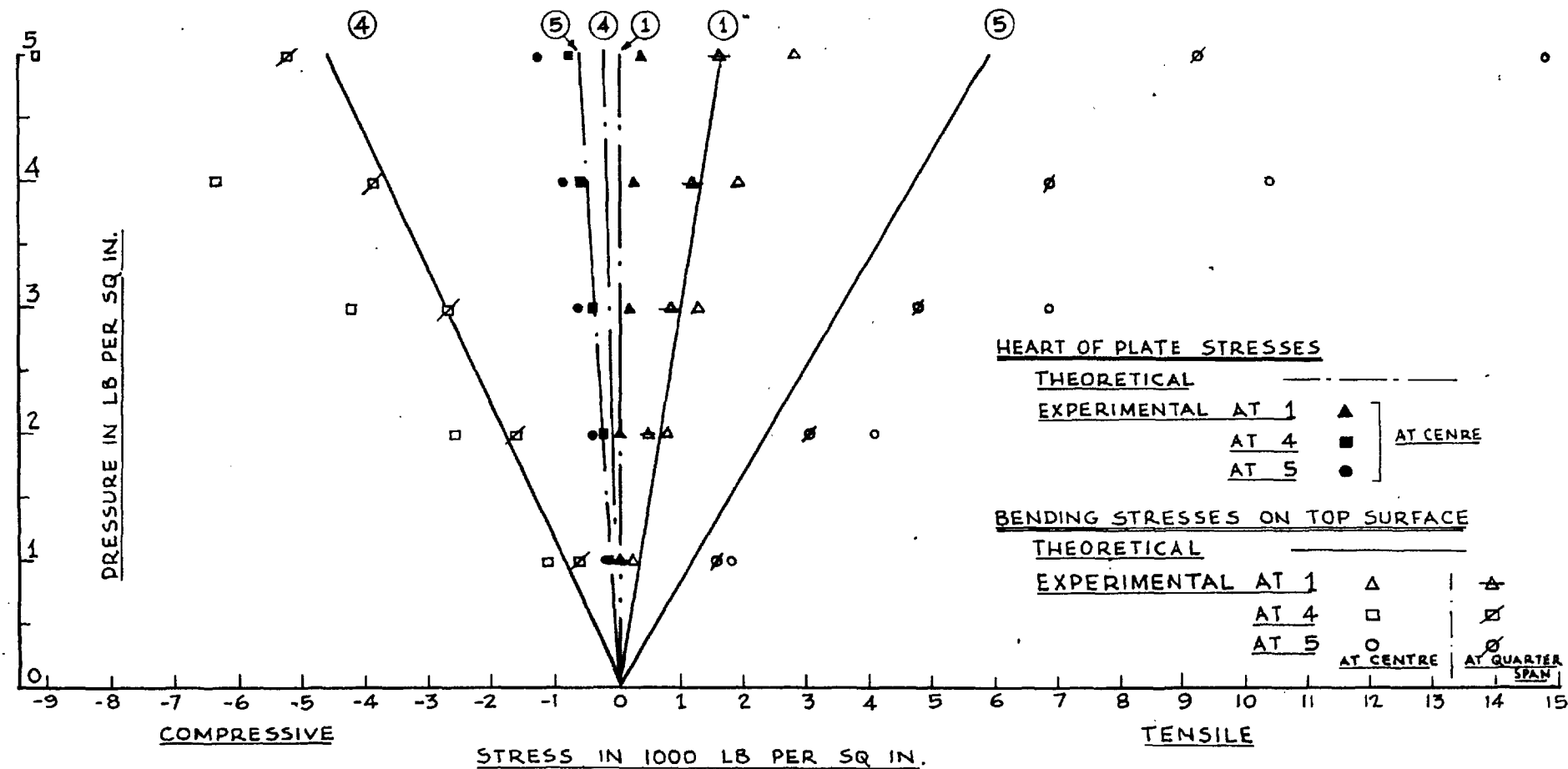


FIG. 95: TRANSVERSE HEART OF PLATE AND BENDING STRESSES (TEST 2)
COMPARISON WITH THEORY

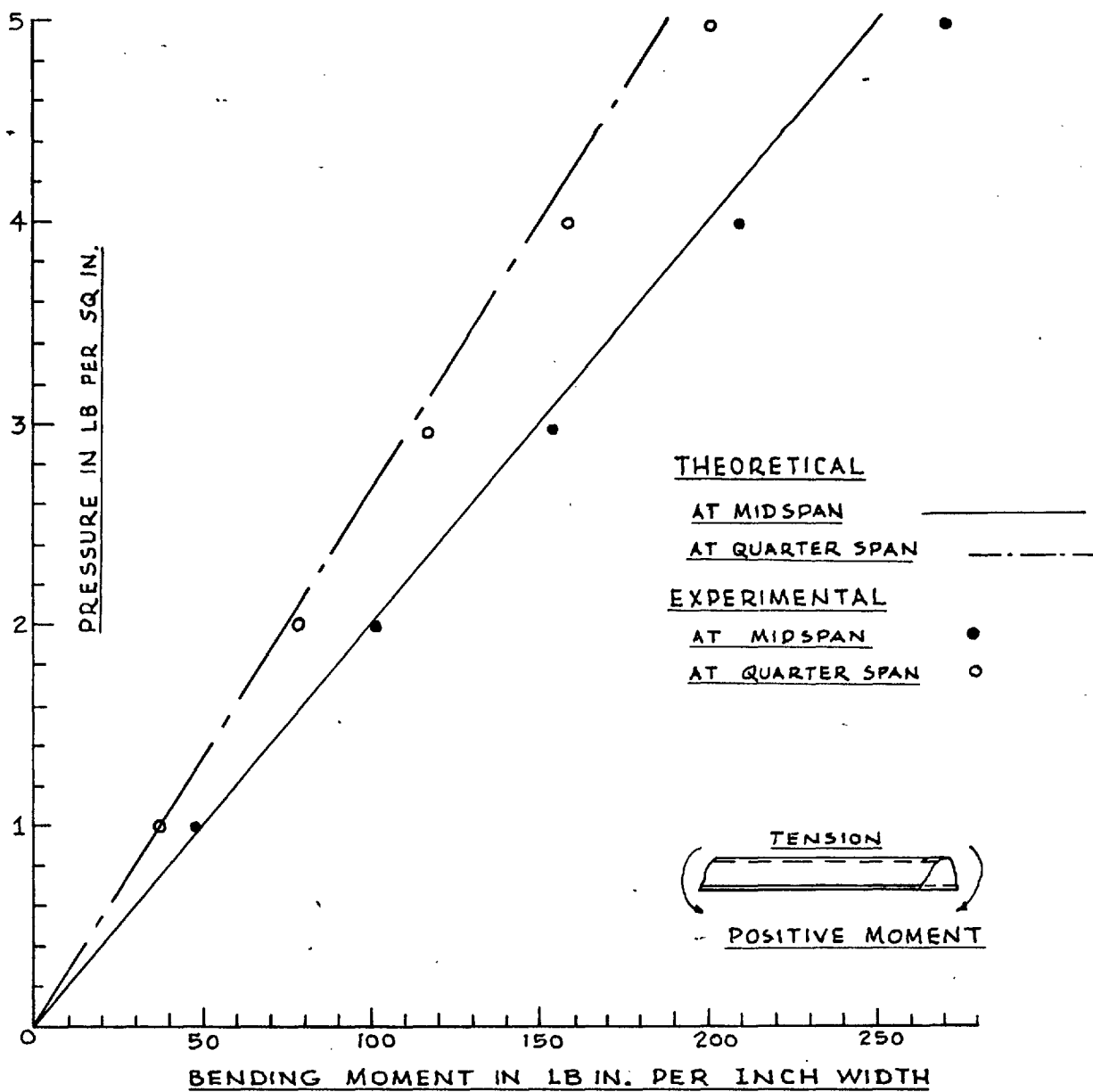


FIG 96 AVERAGE LONGITUDINAL BENDING MOMENTS IN CENTRAL SWEDGE (TEST 2)

EXPERIMENTAL

AT 1 LB PER SQ IN. •

AT 3 LB PER SQ IN. x

AT 5 LB PER SQ IN. o

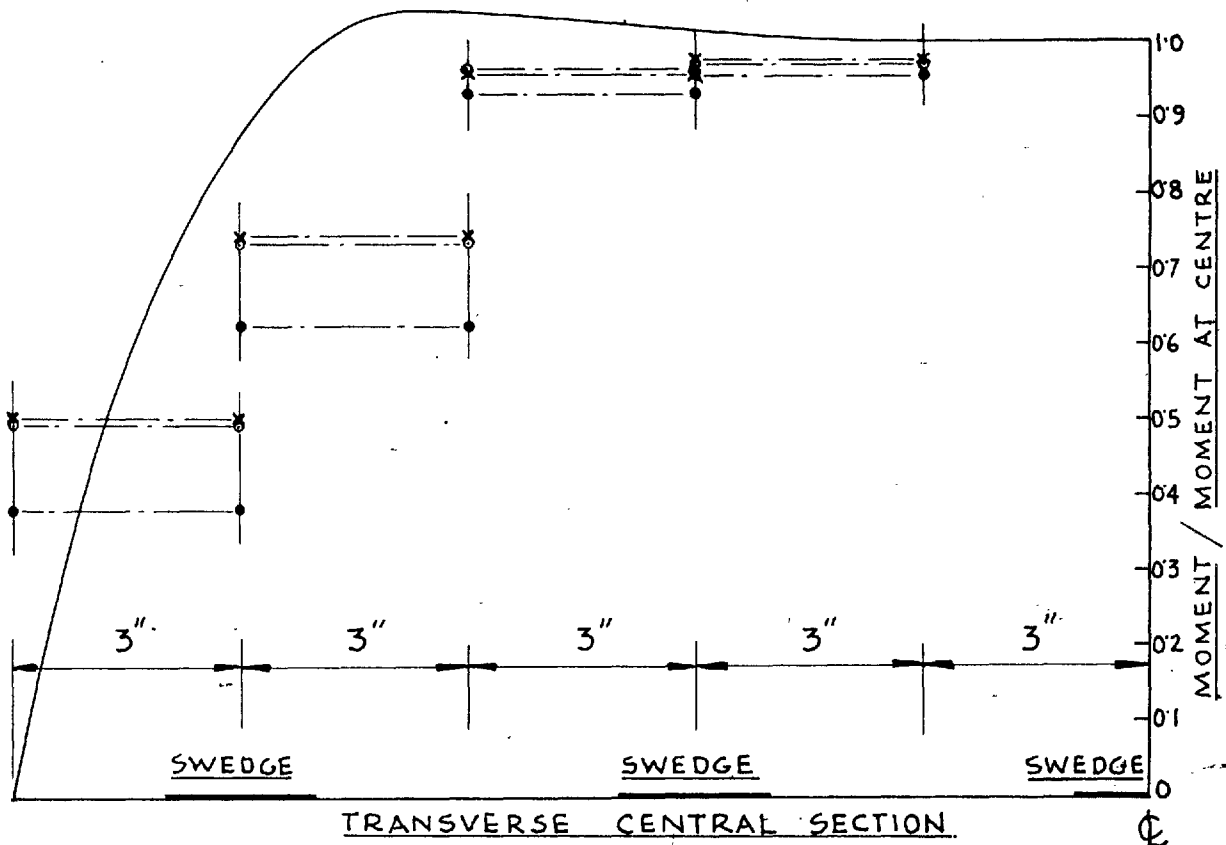
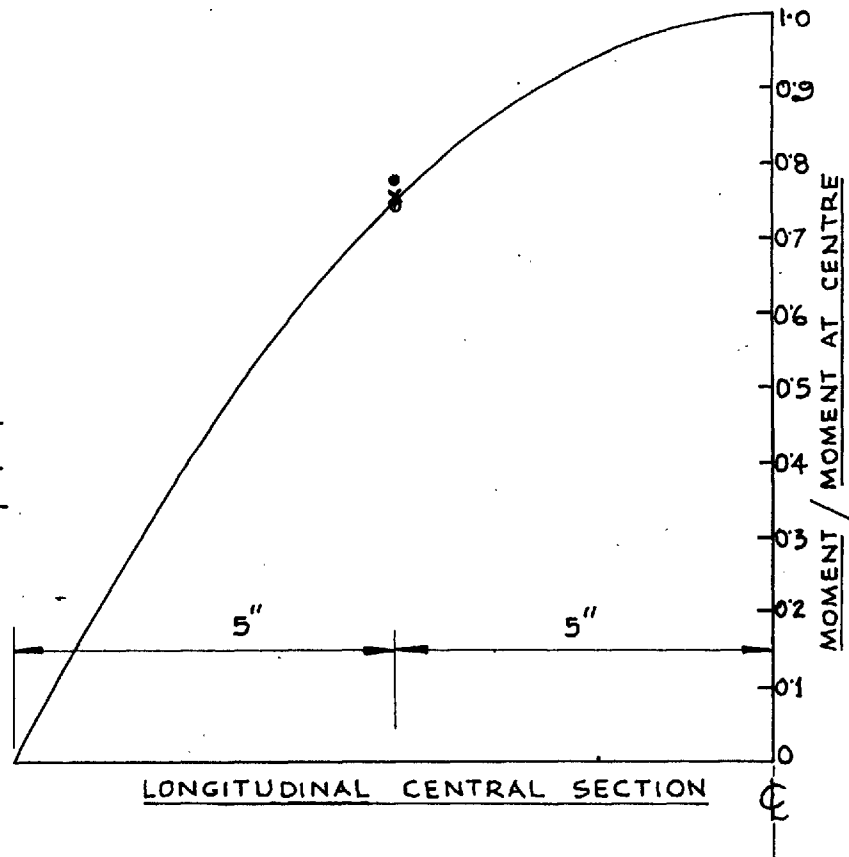


FIG.97: DISTRIBUTION OF LONGITUDINAL BENDING MOMENT ALONG PLATE CENTRE- LINES (TEST 2)

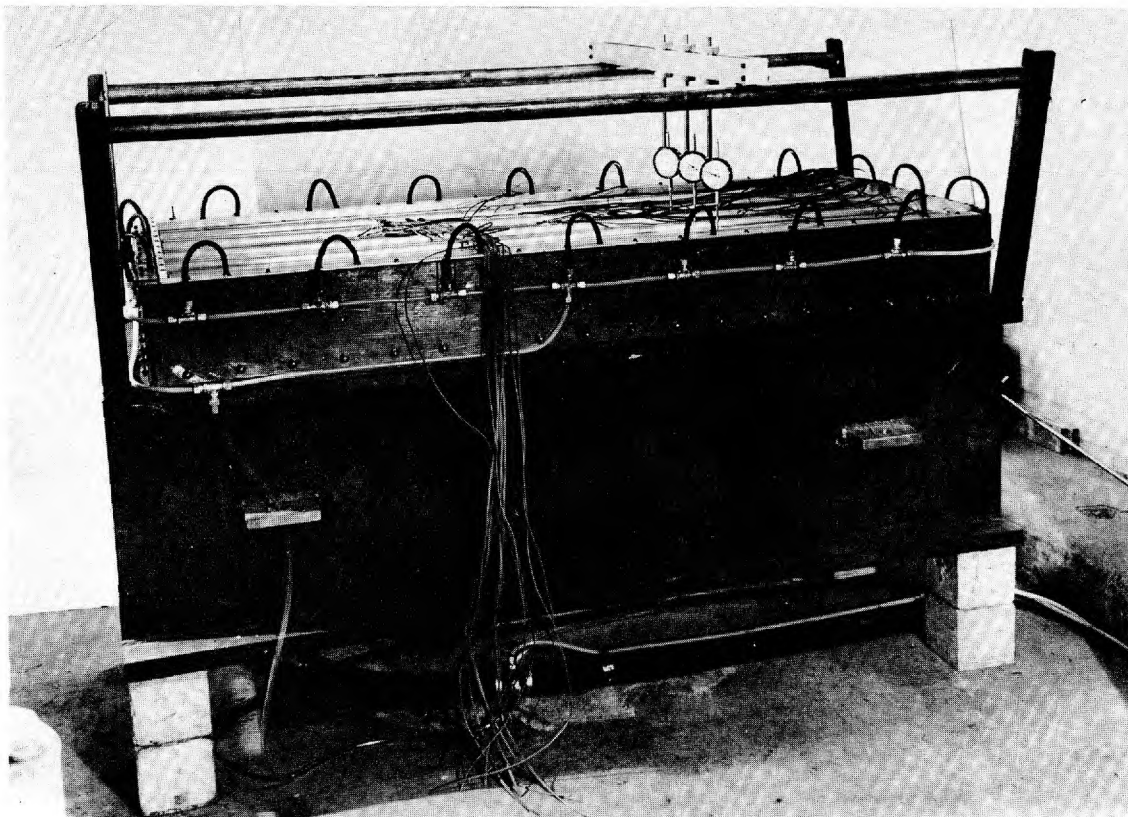


PLATE 1 (TEST 1)

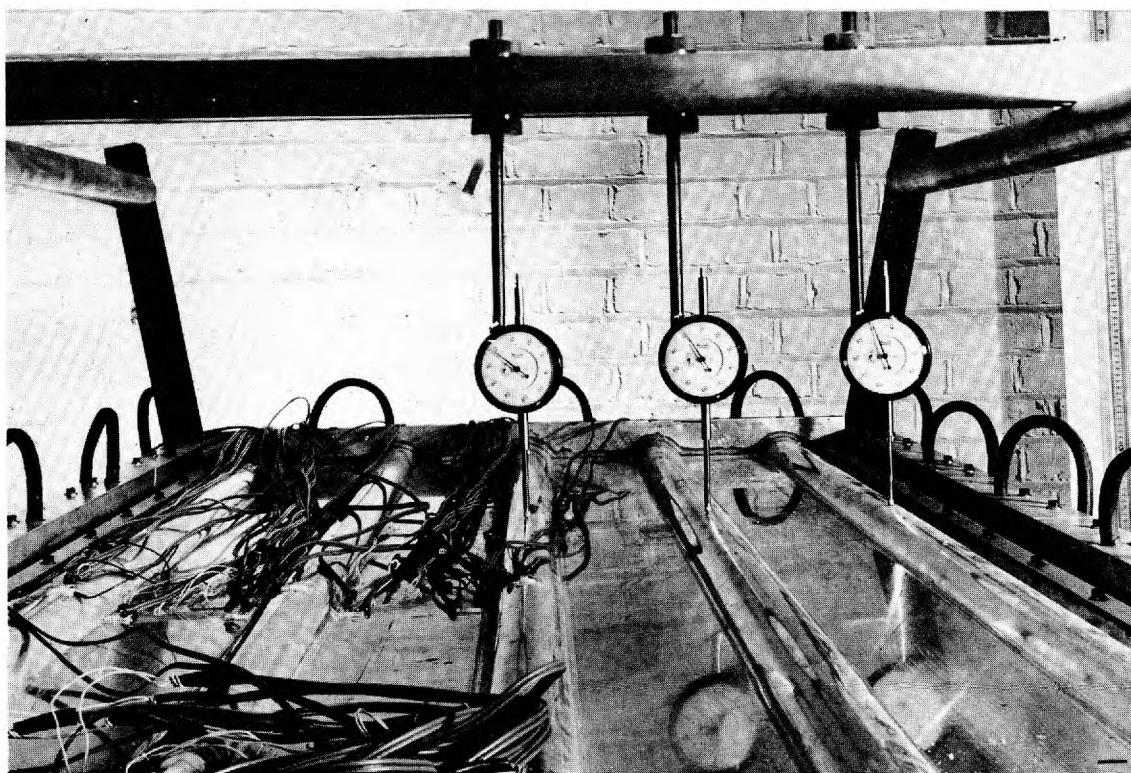


PLATE 2 (TEST 1)

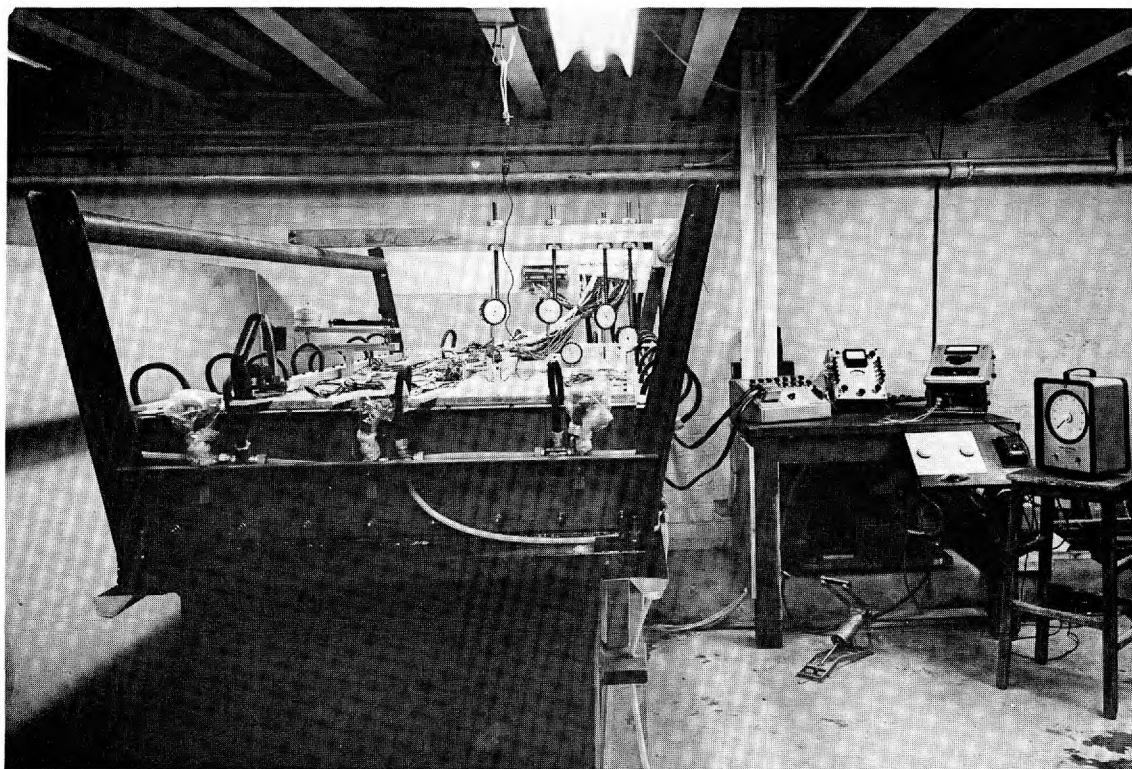


PLATE 3 (TEST 1)

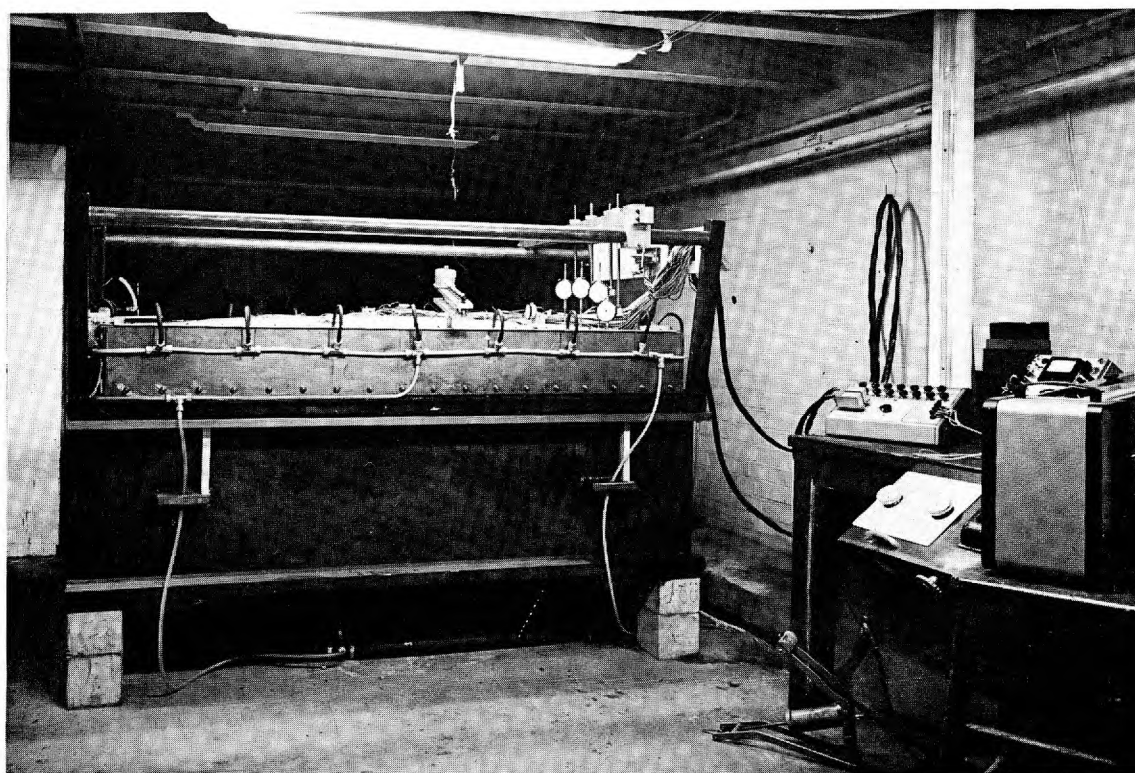


PLATE 4 (TEST 1)

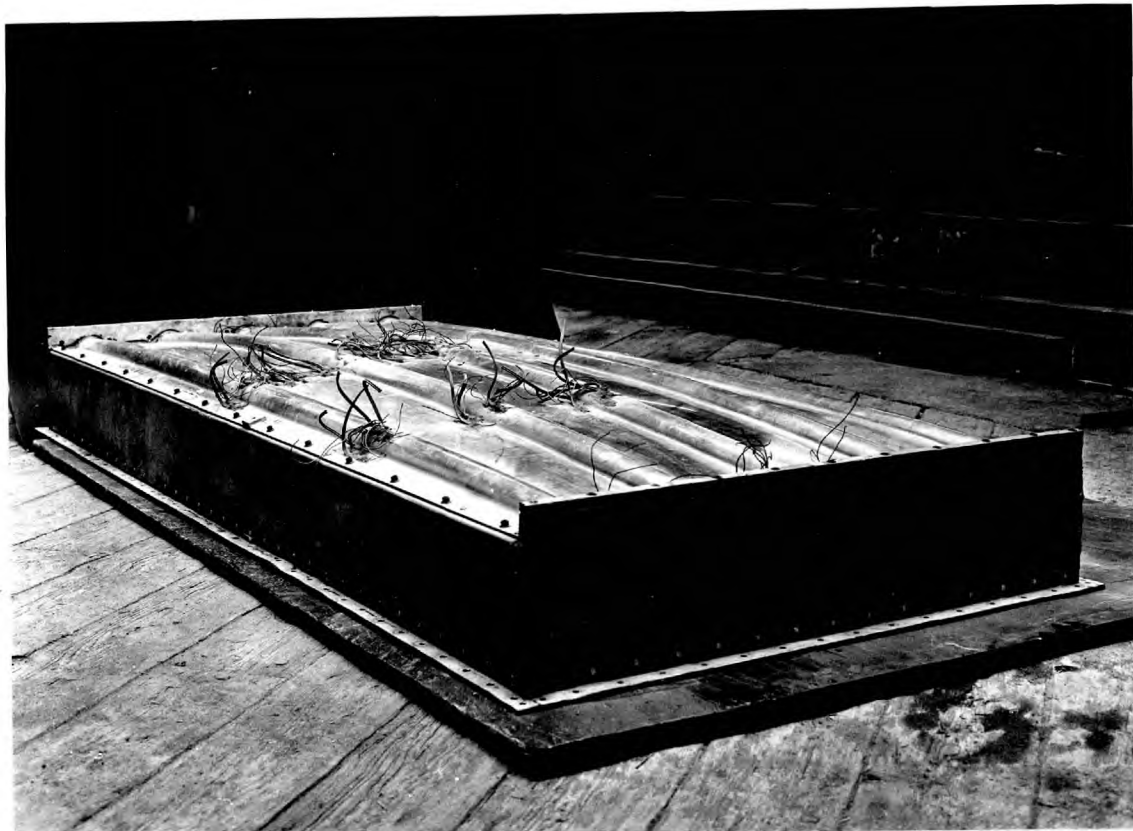


PLATE 5 (TEST 1)

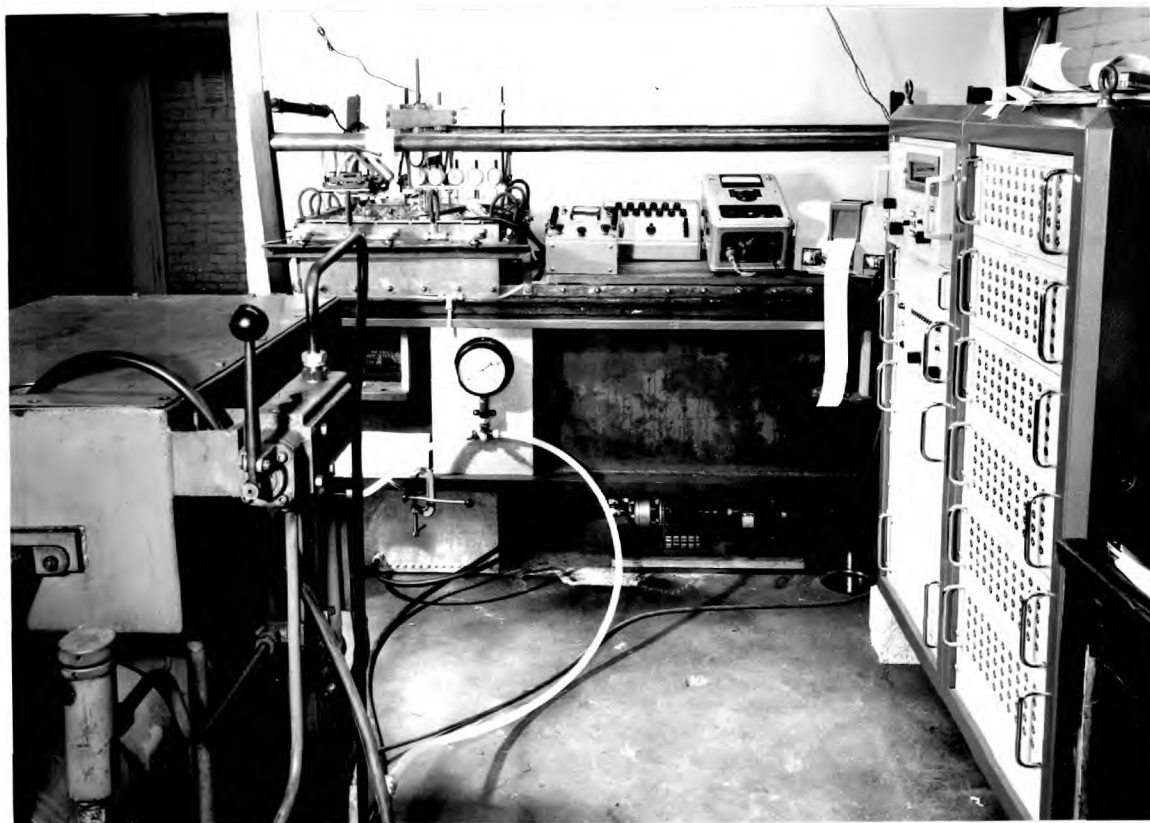


PLATE 6 (TEST 2)

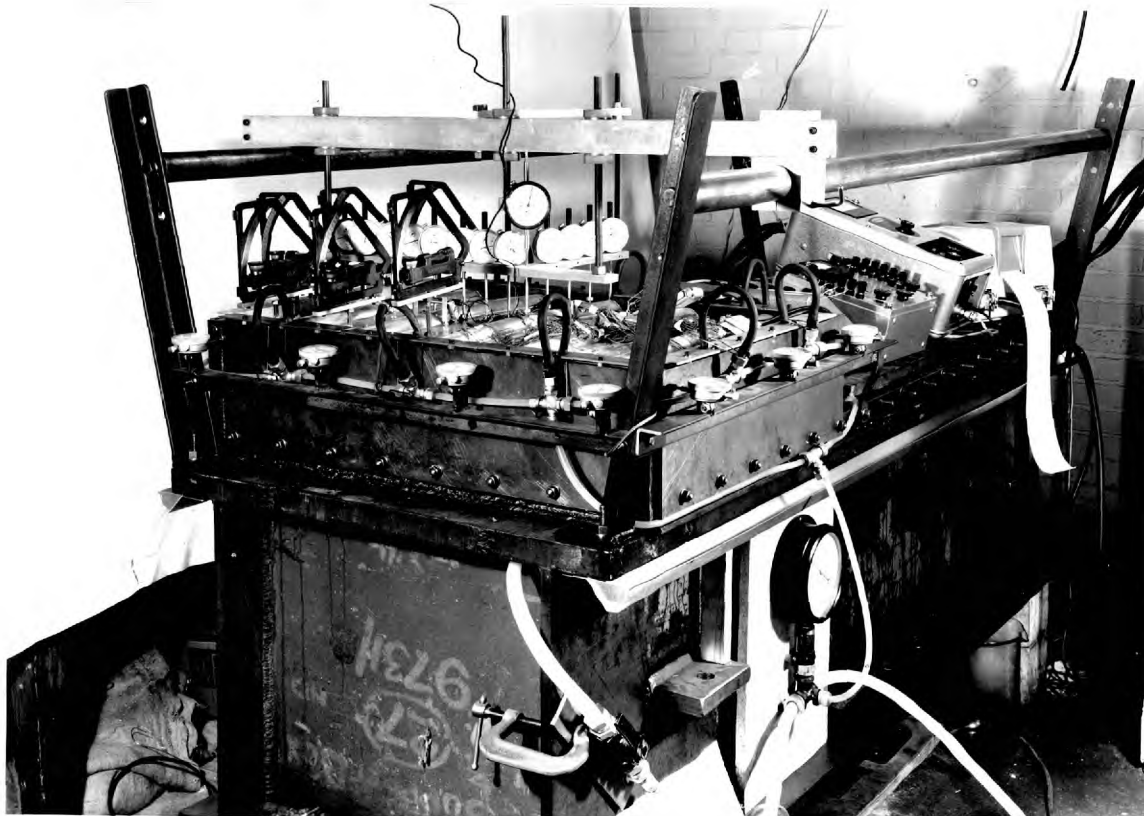


PLATE 7 (TEST 2)



PLATE 8 (TEST 2)