

ON SOME VISCOUS-INVISCID INTERACTION PROBLEMS
IN INCOMPRESSIBLE FLUID FLOW

by

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DEDICATED TO
MY PARENTS, GRANDPARENTS
MY BROTHER AND MY SISTERS

Abstract

The work in this thesis discusses several problems arising in incompressible fluid flow at large Reynolds number and involving viscous-inviscid interactions, in both steady flows and in the context of hydrodynamic stability.

Fully developed and undeveloped free surface flows are investigated within a structural framework, essentially involving ideas similar to those in channel flow and triple-deck theory. For fully developed flows, the model problem of liquid layer flow over a convex corner is considered and linear and nonlinear solutions obtained for various corner angles and inclinations. For undeveloped flows an interactive problem for determining the free surface is obtained: the boundary layer equations are coupled with the inviscid equations. Linear solutions are presented for the uniform flow over an infinite step and various limits discussed. The possible relevance of the above structures, and in particular the free interactions, to the hydraulic jump problem is mentioned.

The stability of the above flows is examined, again within the structural framework, and contains some new results. The upper-branch stability of channel flow, boundary layer flow, and liquid layer flow, is strongly influenced by the properties of nonlinear critical layers and both steady and unsteady (with a moving critical layer) effects are analysed.

Finally, the flow near a wing-body junction comprising a flat plate body and a thin, two or three dimensional, symmetric wing, is investigated. In particular the study takes into account the interaction between the body boundary layer and the wing. Linear solutions are presented for various wing shapes.

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CHAPTER ONE

GENERAL INTRODUCTION

The many significant advances recently in the theory of high Reynolds number flows, with particular reference to viscous-inviscid interactions, have stemmed directly from the important initial contributions due to Neiland (1969), Stewartson & Williams (1969), Messiter (1970). (One is referred to the excellent reviews by Stewartson (1974, 1981), Messiter (1979), Smith (1982), for details of the earlier and the most recent developments of this subject). The key idea to emerge from their work, and which has formed the basis for many of the later applications, was that of the 'triple deck'. This concept helped to explain the adjustment near the trailing edge of a flat plate from the Blasius solution upstream to the Goldstein (1930) near wake downstream, and also to provide a rational description of the interaction between a shock wave and the boundary layer. In both cases the local flow response was described by a three-tiered structure with each region characterizing the different physical balances holding there: inviscid irrotational, inviscid rotational, viscous. The central point of this triple-deck structure, and which distinguished it from the classical heirarchical structure, was that the boundary layer could exert a significant influence on the external flow. In fact the unknown pressure driving the boundary layer was intimately related to the (also unknown) boundary layer displacement, via the solutions in the inviscid regions. This interaction enabled the boundary layer to adjust in anticipation of a disturbance downstream and allowed regular separation to take place, thus avoiding the well known classical Goldstein (1948) singularity.

The importance of this idea can be judged by the subsequent applications in the many diverse areas of hypersonic flows, transonic flows, internal flows, breakaway separation, turbulent flow calculations, hydrodynamic stability etc.. Although some of the details are different, (e.g. the governing structures may not be of the triple-deck form but multideck), the principal features are the same in that the boundary layer equations are coupled with the inviscid equations, thus allowing an interaction (see aforementioned references).

We will not attempt to elaborate any further on the details of triple-deck theory. The basic principles have been outlined here, and the reader is referred to the review papers for further details.

The present work is aimed at extending and applying some of these new ideas to some other situations: high Reynolds number fully developed and undeveloped free surface flows; hydraulic jumps; hydrodynamic stability and nonlinear critical layers; wing-body junctions.

Generally free surface flows have received extensive coverage in the literature. One of the best basic accounts to date is still Lamb's (1932) 'Hydrodynamics'. This covers many aspects of free surface flows: wave motion, wave reflection, refraction, dispersion, stability, flow past and over submerged bodies, etc.. Another good account can be found in Stoker (1957), and Wehausen and Laitone (1960) review some work on surface waves. Mathematically there are several difficulties associated with free surface flows, one major one being

that the fully nonlinear boundary conditions make the problem of solving the Navier-Stokes equations even more intractable both analytically and numerically . As in other areas of fluid mechanics, from good assumptions and approximations relevant to the particular problem a great deal is known about various facets of free surface flows.

Our concern here is with high Reynolds number flows and we note that most theoretical works to date on free surface flows have concentrated on inviscid flows. The effects of viscosity and flow separation on free surface flows may be broadly understood from an engineering viewpoint, but a satisfactory rational description of high Reynolds number motion past, say, a submerged obstacle is still lacking. This may not be too surprising since as mentioned above it is only in the last fifteen years that a consistent account of regular separation has been obtained. For sufficiently deep fluids it does indeed seem plausible to neglect viscous effects; however for shallow depths this is more questionable because of the likelihood of viscous boundary layer separation influencing the flow field. For a discussion of shallow-water theory, see Lamb (1932), Wehausen and Laitone (1960), Peregrine (1972).

The work in the first few chapters, 2-4, concerns high Reynolds free surface flows under the assumption of steadiness. We have taken the most obvious route in trying to include viscous effects, by appropriately adapting the scaling ideas of triple-deck and channel flow theory. There are perhaps many other scalings and assumptions which emphasize different features, but the particular scalings

studied here highlight the viscous-inviscid interaction between the boundary layer and the free surface. The applications to liquid layer flows past corners or humps are considered in chapters 2, ³₁. It is also suggested (chapter 4) that, in certain circumstances, hydraulic jumps may be viewed as predominantly steady free interactions, and this contrasts with Benjamin's (1970) comments which stress the role of unsteadiness rather than viscosity in the phenomenon of upstream influence. The concept of the 'free interaction' is based on ideas originally put forward by Lighthill (1953). In many problems, e.g. the supersonic separation problem of Stewartson and Williams (1969), the basic viscous-inviscid equations contain eigensolutions which allow the oncoming flow to evolve nonlinearly (and uniquely) with distance, without change of local boundary conditions, as required for example in a hydraulic jump. The ultimate form of the free interaction, as this is termed, is interpreted as describing how the flow adjusts and responds in anticipation of a severe obstacle far downstream.

The work here is extended to a discussion (chapters 5-7) on certain types of viscous-inviscid stability of some free surface flows, and the results are different from and additional to those from inviscid theory. The inviscid theory as proposed by Thompson (1949), Burns (1953), (see also the survey by Peregrine (1976)), gives interesting results for gravity waves which allow upstream propagation. In the present approach however, as in Velthuisen and van Wijngaarden's (1969), the long and short waves considered depend on the properties of critical layers and have wavespeeds which are asymptotically small so that all disturbances necessarily propagate

downstream.

The remaining chapters are essentially self contained - a brief description is given below - and chapters 4, 7, 8 and 9 are joint papers.*

Chapter 2 is concerned with fully developed free surface flows, exemplified by the model problem of a liquid layer negotiating a convex corner. The work is based largely on the study by Smith (1977) of upstream interactions in channel flows. The interaction here occurs over the same long lengthscale and is governed by a similar structure. Linear and nonlinear solutions are presented. A special case of the present work reduces to the problem considered by Merkin & Smith (1982)

* My co-authors are willing to testify that the work in these joint papers has been equally shared out. J.G.

for free convection boundary layers.

Chapter 3 discusses undeveloped free surface flows. As with fully developed flows, this can be considered as a simple extension of channel flows, in this case developing channel flows, Smith (1976). The problem of determining the free surface is reduced to solving the boundary-layer equations but with a complicated pressure-displacement-free surface relation. Linearized solutions for the uniform flow past obstacles are obtained and some limiting cases discussed. The free interactions for this problem may be relevant to hydraulic jumps.

A special case of the problems considered in chapters 2 and 3 occurs when the pressure is proportional to the local displacement. The free interaction occurring (for supercritical flows only) in this instance, with the pressure and free surface rising and the skin friction falling, is studied in chapter 4. A novel structure for the flow far downstream, involving a forward moving shear layer above a reversed flow region and with the pressure growing algebraically, is proposed. The primary application here is to hydraulic jumps, but the same free interaction problem is relevant to hypersonic flows (Brown, Stewartson & Williams (1975)).

Chapter 5 is concerned with the 'lower-branch' stability properties of fully developed and undeveloped free surface flows. The work here follows closely the important contributions by Smith (1979^{ab}) who showed that the same interactive structures also have strong connections with the hydrodynamic stability of the basic flow, see also Hall & Smith (1982). The structural approach is useful here and highlights, for fully

developed flows, the new structure which emerges when the Froude number and wavelength become $O(1)$. The results for fully developed and developing flow in a channel are special cases of the neutral modes found here.

Chapters 6, 7, 8 discuss the 'upper-branch' stability properties of liquid-layer or parallel flows, and boundary layer flows, (and which depend heavily on the properties of nonlinear critical layers, see Benny & Bergeron (1969), Davis (1969), Haberman (1972), and for a review by Stewartson (1981)). The work in chapter 6, concerning liquid layer flow, is similar to and complements Smith & Bodonyi's (1982) study for accelerating boundary layer flow. A special case of the results is for plane Poiseuille flow. As the disturbance size, which is initially small but finite, is increased further, a new Rayleigh-type structure emerges with ^{the} wavelength $O(1)$ and the critical layer occurring in the interior of the flow, as in Smith & Bodonyi. This is studied in chapter 7. In chapter 8 we consider growing modes, but for the smaller disturbance size which in the steady case would lead to the Haberman problem of Smith & Bodonyi or chapter 6. The amplitude and also the wavenumber and frequency, depend on slow spatial and temporal variables. This leads to some novel equations for the moving critical layer and the growth or decay of the amplitude, and some of the limiting properties are analysed.

In chapter 9 we study the flow near a wing-body junction, modelled by a flat plate body with a projecting thin symmetric wing. The essential point here is that the study takes into account the interaction between the body boundary layer and the wing. Linearized solutions are presented for various wing shapes.

Finally in chapter 10 we make some concluding remarks.

CHAPTER 1

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CHAPTER 2

FULLY DEVELOPED FREE SURFACE FLOWS -
LIQUID LAYER FLOW OVER A CONVEX CORNER

SUMMARY

The paper discusses the steady flow of a liquid layer over a convex corner at high Reynolds numbers (R). A double decked structure for the flow is proposed with the displaced free surface equal to the local boundary layer displacement ($-A$), and the pressure P being determined by the law $P = -sA - A''$. Here s is proportional to the angle of inclination of the initial plane. Linearized solutions are obtained for small angles and numerical calculations are carried out for much larger angles, with the plane at various inclinations. The unusual breakdown of the linear theory downstream is also discussed.

The computed results and the asymptotic description far downstream show that the flow does not return to its undisturbed state; instead the layer continues to get thinner with the fluid moving much faster. It is suggested that this novel feature may be resolved on a longer $O(R)$ lengthscale.

The analysis presented applies equally to a more general fully developed profile at large Froude numbers, and extra features such as surface tension can be incorporated into the basic formulation without formal difficulty.

1. INTRODUCTION

Our concern here is with the behaviour of liquid layer flows near sudden changes in boundary conditions and in the presence of sizable obstacles. One can ask the question, what happens in the corresponding situations when a free surface is present. How does the boundary layer influence the free surface and vice-versa, and also what happens when flow separation is present? How does separation affect the free surface? The questions are of theoretical as well as practical interest, certainly when one considers fluid flow in familiar contexts, e.g. river flows, waterfalls, flow over weirs, liquid layers, flow out of a teapot etc.. Our aim in this paper is to address some of these questions and try and extend some of the modern ideas of high Reynolds number flow theory to free surface flows. In external and internal flows the behaviour of boundary layers has received much attention and is extensively reviewed in Stewartson (1974, 81) and Smith (1982). We concentrate on the case when the flow sufficiently far upstream is fully developed. The theory for the undeveloped profile is considered by Gajjar (1983), and Gajjar and Smith (1983) discuss the hypersonic free interaction problem which is also shown to apply to the start of the hydraulic jump.

We start with the case in which the oncoming profile is fully developed. The Reynolds number (R) is assumed to be large throughout, and the motion steady, laminar, and two dimensional. Surface tension effects are ignored. To be specific we concentrate on the flow of a liquid layer down an inclined plane which has angles β^* , $\beta^* + \alpha^*$ to the horizontal, upstream and downstream respectively, see fig. (1).

We note that far upstream the simple solution given by the half-Poiseuille form satisfies the Navier-Stokes equations exactly and that it is reasonable to expect that the flow sufficiently far downstream is also given by a similar half Poiseuille profile. Global arguments e.g. mass flux considerations, can then be used deduce useful relations between the upstream and downstream velocities and depths. However as mentioned before we are primarily interested in how the boundary layer and the free surface first anticipate the presence of an obstacle such as a corner when the oncoming profile is fully developed, and our interest therefore will be centered more on this initial adjustment regime.

Before discussion of the scalings it is appropriate to comment on a few of the assumptions made below. If we think of a more general distortion of the bed, then provided that the distortion locally is small, of $O(\delta)$ say ($\delta \ll 1$), then it is reasonable to suppose that the fluid is also displaced by an amount $O(\delta)$, since the oncoming profile is fully developed and viscous effects extend up to the free surface. This it should be noted is then equivalent to the assumption that the displacement of the free surface is of the same order as that of the bed, i.e. that the basic flow has not been modified by an $O(1)$ amount due to the presence of an infinitesimally small obstacle, and this assumption is crucial to the analysis and scalings given below. It should be noted that this simple assumption allows us to make use of the techniques of boundary layer theory to study directly the interaction between the boundary layer and the free surface. Also, unlike

* Dr. D.H. Peregrine is thanked for pointing this out.

shallow-water theory (see Lamb (1932), Wehausen and Laitone (1960), Peregrine (1972)), we do not assume that the vertical accelerations of the fluid can be neglected. This is also equivalent to the previous assumption. In fact in regions where large transverse changes occur, e.g. near a corner, the hydrostatic assumption becomes more questionable since significant normal pressure gradients are generated there. Also the above excludes any possible standing wave solutions. Alternative scalings and analysis of interest such as those for standing waves arise from different physical assumptions.

We consider how the fluid negotiates the corner. It is useful to note the upstream response occurring in channel flow theory (Smith 1977). There a displacement of the core induces a pressure gradient across the channel which in turn drives the viscous wall layers, and provides a self-sustaining mechanism for the upstream response. Here, the fully developed profile can be thought of as half a channel flow profile, and essentially the same ideas as those in channel flow theory are applied in §2, leading to a double deck structure with the same $R^{1/7}$ stream-wise interaction lengthscale. Again, the mechanism for the upstream response is similar although the pressure displacement law stems from the condition of zero stress at the free surface. A displacement of the main body of flow induces a transverse pressure gradient across the liquid layer which drives the viscous wall layer. Also the displaced free surface causes a pressure imbalance which is then cancelled out by the induced pressure, and this balance is expressed in the interesting law ((2.16) below), for determining the unknown pressure, the boundary layer displacement and the free surface. This law is a combination of a special case of that occurring in hypersonic flow theory Brown-Stewartson-Williams (1975), (1974/75), and the 'jet law' of Smith-Duck (1977), Merkin-Smith (1982).

In the above regime the first angle β^* must be $O(R^{-5/7})$ strictly. However if the plane is nearly horizontal, with $\beta^* \ll R^{-5/7}$, then to leading order the hypersonic law applies with the pressure proportional to the local boundary layer displacement. Also in this regime the upstream response is pushed upstream and occurs over a much longer interaction

length scale. If, on the other hand, the inclined plane is much steeper, so that $\beta^* \gg R^{-5/7}$, then the jet law of Smith-Duck holds. Solutions for convex and concave corners, humps and indentations, there, have been obtained by Merkin-Smith (1982), Merkin (1983), although in a different context. In particular regular separation is found to occur with the flow returning to its 'undisturbed' state sufficiently far downstream.

We will be concerned solely with the convex corner problem ^{with $\alpha^* \propto O(R^{-3/7})$} . The concave corner problem presents some difficulties which will be outlined below. In §2 the basic problem is formulated; §3 discusses the upstream eigensolutions; §4, §5 discuss the linear theory and the numerical procedures and results for much larger angles; and §6 presents the asymptotic form for the flow far downstream.

Finally it should be emphasized that the analysis presented applies equally to a more general fully developed profile in fluid flow at large Froude numbers ^{$\propto O(R^{2/7})$} with little modification.

§2 Basic Formulation

We take a set of cartesian axes Ox^*, y^* with $x^*=0$ chosen to be the start of the change in slope of the plane, and y^* representing distances normal to the plane for $x^*<0$. See fig.(1). The basic flow far upstream is given by velocity (U_B^*, V_B^*) , and pressure P_B^* , where

$$U_B^* = \frac{g \sin \beta^*}{2\nu} \left[h^2 - (y-h)^2 \right], \quad V_B^* = 0$$

$$P_B^* = \rho g \cos \beta^* (h - y^*) \quad . \quad (2.1)$$

Here h is the undisturbed depth of the liquid layer, ν the kinematic viscosity of the fluid, g the acceleration due to gravity, and ρ the density of the fluid. Nondimensionalization with respect to the length scale h , velocity $U_c = 2U_B^*(h)$, pressure ρU_c^2 , in the usual notation, leaves the Navier-Stokes equations as

$$\text{div } \underline{u} = 0 \quad (2.2)$$

$$\underline{u} \cdot \nabla \underline{u} = -\nabla p + \left(\frac{1}{R}, -\frac{1}{R \tan \beta^*} \right) + \frac{\nabla^2 \underline{u}}{R}$$

where $R = \frac{U_c h}{\nu}$ is the Reynolds number assumed to be large throughout.

In addition to (2.2) we have the boundary conditions of no slip at any fixed solid surface, and zero stress at the free surface $y=1+\eta(x)$, expressed by

$$\left(-p + \frac{2}{R} \frac{\partial u}{\partial x} \right) \frac{d\eta}{dx} - \frac{1}{R} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0 \quad (2.3)$$

$$-p + \frac{2}{R} \frac{\partial v}{\partial y} - \frac{1}{R} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \frac{d\eta}{dx} = 0$$

$$v = u \frac{d\eta}{dx}$$

Consider a perturbation of $O(\delta)$ to the oncoming flow. It is postulated that the interaction here occurs because of two factors,

(a) the displacement of the free surface which induces a pressure of

$O\left(\frac{\delta}{R \tan \beta^*}\right)$ from (2.1), (2.3), to counteract the change in the hydro-

static pressure, and (b) the displacement of the free surface which sets

up a transverse pressure gradient of $O(\lambda)$, as in channel flow Smith (1977).

The various regimes in which (a) and (b) are significant are dictated by the size of the angle β^* . We consider the central case when (a) and (b) are comparable. From the y momentum equation the transverse pressure is $O(\frac{\delta}{L^2})$, where L is the unknown streamwise length-scale and assumed to be large. Hence

$$\lambda \sim \frac{\delta}{R \tan \beta^*} \sim \frac{\delta}{L^2} \quad . \quad (2.4)$$

The rest of the argument is identical to that given in Smith (1977) and fuller details may be found there. Viscous forces come into play when $y \sim O(\delta)$ to reduce the slip velocity of $O(\delta)$ to zero, and then viscous - inertial and pressure balances imply

$$\frac{\delta^2}{L} \sim \frac{\lambda}{L} \sim \frac{\delta}{R \delta^2} \quad . \quad (2.5)$$

Thus (2.4) (2.5) yield the crucial scalings

$$\delta \sim R^{-2/7}, \quad L \sim R^{1/7}, \quad \lambda \sim R^{-4/7}, \quad \tan \beta^* \sim R^{-5/7}. \quad (2.6)$$

With regard to (a) and (b) above, if $\beta^* \ll R^{-5/7}$ then the hydrostatic induced pressure is important and if $\beta^* \gg R^{-5/7}$ then the transverse pressure gradient plays the dominant role.

It is convenient to set $\epsilon = R^{-1/7}$, $\tan \beta^* = \frac{R^{-5/7}}{\bar{s}}$, and $y = -\alpha \epsilon^2 \bar{F}(\epsilon x)$ where $\bar{F}(x)$ is the reduced wall shape, and $\bar{F}(x) \rightarrow x$ for $x \gg 1$. Hence far downstream the angle α^* is of $O(R^{-3/7})$.

Expansions

Let $\bar{X} = \epsilon x$ be the scaled streamwise coordinate, and $y = 1 + \epsilon^2 \bar{\eta} + \dots$ be the free surface. Then in the majority of the flow I where $y \sim O(1)$

$$u \sim U_B + \epsilon^2 u_1 + O(\epsilon^4)$$

$$v \sim \epsilon^3 v_1 + O(\epsilon^5) \quad (2.7)$$

$$p \sim \epsilon^2 \bar{s} (1-y) + \epsilon^4 p_1 + O(\epsilon^6)$$

Here $U_B = y - y^2/2$ is the basic flow. Substitution into (2.2) gives the simple displacement solution

$$u_1 = \bar{A}(\bar{X}) U_{By}, \quad v_1 = -\bar{A}'(\bar{X}) U_B$$

$$p_1 = \bar{P}(\bar{X}) + \bar{A}'' \int_0^y U_B^2 dt \quad (2.8)$$

$\bar{A}(\bar{X})$, $\bar{P}(\bar{X})$ are unknown and $\bar{A}(\bar{X})$, $\bar{P}(\bar{X}) \rightarrow 0$ as $\bar{X} \rightarrow -\infty$. Applying the conditions at the free surface $y = 1 + \epsilon^2 \bar{\eta}$, gives

$$\bar{\eta}(\bar{X}) = -\bar{A}(\bar{X})$$

$$\bar{P}(\bar{X}) = \bar{s} \bar{\eta}(\bar{X}) - \bar{A}'' \gamma \quad (2.9a,b)$$

$$\text{where } \gamma = \int_0^1 U_B^2 dy$$

A viscous wall layer II of thickness $O(\varepsilon^2)$ is required to reduce the slip velocity in (2.7) to zero, and there with $y = \varepsilon^2 \bar{Y}$, $\bar{Y} \sim O(1)$

$$u \sim \varepsilon^2 \bar{U}_I + \dots$$

$$v \sim \varepsilon^5 \bar{V}_I + \dots$$

$$p \sim \varepsilon^2 \bar{S} + \varepsilon^4 (-s\bar{Y} + \bar{P}_I) + \dots \quad (2.10)$$

$\bar{U}_I, \bar{V}_I, \bar{P}_I$ satisfy the usual boundary layer equations

$$\bar{U}_{I,\bar{X}} + \bar{V}_{I,\bar{Y}} = 0$$

$$\bar{U}_I \bar{U}_{I,\bar{X}} + \bar{V}_I \bar{U}_{I,\bar{Y}} = -\bar{P}_{I,\bar{X}} + \bar{U}_{I,\bar{Y}\bar{Y}}, \quad \bar{P}_{I,\bar{Y}} = 0, \quad (2.11)$$

with the boundary conditions

$$\bar{U}_I = \bar{V}_I = 0 \quad \text{at } \bar{Y} = -\alpha \bar{F}(\bar{X})$$

$$\bar{U}_I \rightarrow \bar{Y} + \bar{A}(\bar{X}) \quad \text{as } \bar{Y} \rightarrow \infty$$

$$\bar{U}_I \rightarrow \bar{Y} \quad \text{as } \bar{X} \rightarrow -\infty \quad (2.12)$$

$$\bar{P}_I = \bar{P}(\bar{X})$$

to match with I, and for no slip at the wall.

The factor γ in (2.9) can be effectively set equal to unity with the normalization

$$(U, V, X, Y, A, \eta, P, s, F(X)) \quad (2.13)$$

$$= (\gamma^{-1/7} \bar{U}, \gamma^{1/7} \bar{V}, \gamma^{-3/7} \bar{X}, \gamma^{-1/7} \bar{Y}, \gamma^{-1/7} \bar{A}, \gamma^{-1/7} \bar{\eta}, \gamma^{-2/7} \bar{P}, \gamma^{-1/7} \bar{s}, \gamma^{-1/7} \bar{F}(\bar{X}))$$

and with the Prandtl Transformation applied to (2.12) we obtain

$$U_x + V_y = 0$$

$$UU_x + VU_y = -P_x + U_{yy}, \quad P = P(X) \quad (2.14 \text{ a-c})$$

$$U = V = 0 \text{ on } Y = 0$$

$$U \rightarrow Y + A - \alpha F(X) \text{ as } Y \rightarrow \infty \quad (2.15 \text{ a-c})$$

$$U \rightarrow Y \text{ as } X \rightarrow -\infty$$

$$P = -sA - A'', \quad \eta = -A \quad (2.16 \text{ a-b})$$

Although surface tension effects have been ignored, it should be mentioned that a term $-\chi\eta''$ would be added to the right hand side of (2.9b), $\chi > 0$ here being a scaled surface tension parameter. Hence (2.16) covers this case also.

§3 Eigensolutions Upstream

The governing equations (2.14), (2.15), (2.16), as with similar pressure displacement laws, allow eigensolution^s to occur upstream when X is large and negative. This can be seen as follows. If we put

$$U = Y + \hat{p} e^{\theta \sigma x} f'(\gamma) + \dots, \quad V = -\theta \sigma e^{\theta \sigma x} f(\gamma) \hat{p} + \dots$$

$$P = \hat{p} e^{\sigma x \theta} + \dots, \quad A = \hat{A} e^{\sigma x \theta} + \dots$$

$$\theta = (-3 \text{Ai}'(0))^{3/7} = .8972 \quad (3.1)$$

we obtain, after substituting into (2.14), (2.15), (2.16) and neglecting terms of $O(e^{2\sigma x \theta})$,

$$f'(\gamma) = \frac{(\sigma \theta)^{2/3}}{\text{Ai}'(0)} \int_0^\gamma \text{Ai}((\sigma \theta)^{1/3} t) dt$$

$\text{Ai}(x)$, here, is the Airy function. The boundary conditions now determine σ as the positive root of the equation

$$\sigma^{1/3} (\sigma^2 + a) - 1 = 0, \quad a = \frac{s}{\theta^2} \quad (3.2)$$

The equation (3.2) has three roots, one real and positive, and two complex conjugate pairs lying in $\text{Re}(\sigma) < 0$.

The constant $\hat{p}$ remains unknown, and is determined by the downstream conditions. The case $\hat{p} < 0$ is relevant in (eg.) flow past

convex corners (see later), the overfall problem. If $\hat{p} < 0$ then the pressure falls, the level of the free surface drops and the fluid accelerates.

The case $\hat{p} > 0$ is also interesting, and is applicable to concave shapes, e.g. humps. Here the pressure and hence the free surface level rises, in anticipation of some severe obstacle downstream, and this is the familiar phenomenon of the hydraulic jump. It is discussed more fully in Gajjar & Smith (1983).

Finally, regarding (3.2) if s is large then $\sigma \sim (\frac{s}{\theta})^{-3}$, and hence the interaction is pushed further upstream. On this length-scale then, with

$$X=s^3X_1, U=su_1, A=sA_1, P=s^2P_1, Y=sy_1, s \gg 1, \quad (3.3)$$

u_1, v_1, p_1 satisfy the same equations as (u, v, p) except that to leading order the pressure displacement law is

$$P_1 = -A_1 \quad (3.4)$$

Formally if s increases to $O(\epsilon^{-2})$, then since $X = \epsilon \gamma^{-3/7} x$ the interaction lengthscale increases to $O(R)$ and the double deck structure disappears, with the boundary layer equations holding across the entire layer together with the zero tangential stress and the kinematic condition, to be satisfied at the free surface. The same problem is also discussed in Gajjar & Smith (1983).

We return now to the convex corner problem and discuss the linearized solution.

§4 Linearized Solution for Small Angles

For $O(1)$ angles α , (2.14), (2.15), (2.16) requires a numerical approach and this is discussed in §5. However we can proceed further analytically, as in Stewartson (1970a), Smith (1982), Merkin & Smith (1982), if α is small. Suppose $\alpha \ll 1$ and let

$$U = Y + \alpha \tilde{U} + O(\alpha^2)$$

$$V = \alpha \tilde{V} + \dots, \quad A = \alpha \tilde{A} + \dots, \quad P = \alpha \tilde{P} + \dots$$

Substituting into (2.14), (2.15) and neglecting $O(\alpha^2)$ terms, $\tilde{U}$, $\tilde{V}$, $\tilde{P}$ satisfy linear equations which can be solved by the usual transform methods.

Denoting the Fourier Transform of $q(X)$ by $q^* = \int_{-\infty}^{\infty} q(X) e^{-iwx} dX$, we obtain

$$\tilde{A}^*(w) = \frac{F^*(w)}{1 - \left(\frac{iw}{\theta}\right)^{1/3} \left(a - \left(\frac{w}{\theta}\right)^2\right)}$$

$$\tilde{P}^*(w) = - (s - w^2) A^*$$

$$\tau_\ell^* = \left(\tilde{U}_y^*\right)_{y=0} = \frac{Ai(0)}{Ai'(0)} (iw)^{2/3} P^*(w) \quad (4.1abc)$$

τ_ℓ represents the change in the shear stress at the wall from its initial value of unity. For arbitrary wall shapes $F(x)$, (4.1) can be inverted as a convolution integral. However we restrict ourselves to the case when $F(X) = XH(X)$.

Using the inversion formula and a simple substitution, we find

$$\tilde{A}(X) = \frac{1}{2\pi i \theta} \int_L \frac{e^{\theta x z} dz}{z^2 (1-z)^{1/3} (a+z^2)} \quad (4.2)$$

with similar expressions for $\tilde{P}(X)$, τ_ℓ .

In (4.2) the path of integration, L lies to the right of the imaginary axis, and we take a cut along the negative real axis.

As mentioned earlier the equation (3.2) has exactly three roots, two lying in $\text{Re}(z) < 0$. Denoting the roots by

$$q_1, q_{2,3} = r e^{\pm i\phi}$$

where $q_1 > 0$ and $\pi/2 < \phi < \pi$ we obtain from (4.2),

$$\begin{aligned} \tilde{A}(X) &= \frac{3e^{Xq_1\theta}}{q_1^{4/3} \theta (a+7q_1^2)} \\ \tilde{P}(X) &= \frac{-3\theta e^{\theta Xq_1}}{q_1^{5/3} (a+7q_1^2)} \\ \tau_\ell &= \frac{9\theta^{-2/3} \text{Ai}(0) e^{\theta Xq_1}}{q_1 (a+7q_1^2)} \end{aligned} \quad (4.3)$$

and for $X > 0$

$$\tilde{A}(X) = X + \frac{3}{\theta q_1^{4/3} (a+7q_1^2)} + \frac{6(a \cos(4\phi) + 7r^2 \cos(10\phi))}{3\theta r^{4/3} (a^2 + 14ar^2 \cos(2\phi) + 49r^4)}$$

$$\begin{aligned}
& - \frac{6 e^{\theta x r \cos \phi}}{\theta r^{4/3}} \left\{ \frac{a \cos(\frac{4\phi}{3} - \theta x r \sin \phi) + 7 r^2 \cos(\frac{10\phi}{3} - \theta x r \sin \phi)}{(a^2 + 14 a r^2 \cos(2\phi) + 49 r^4)} \right\} \\
& + \frac{\sqrt{3}}{2\pi\theta} \int_0^\infty \frac{(1-e^{-\theta x z}) z^{1/3} (a+z^2)}{z^2 (1-z^{1/3} (a+z^2) + z^{2/3} (a+z^2)^2)} dz \\
\tilde{P}(X) = & -s\tilde{A} + \frac{\sqrt{3}\theta}{2\pi} \int_0^\infty \frac{e^{-\theta x z} z^{1/3} (a+z^2)}{(1-z^{1/3} (a+z^2) + z^{2/3} (a+z^2)^2)} dz \\
& + \frac{6\theta r^{2/3} e^{x r \theta \cos \phi} \left\{ a \cos(\frac{2\phi}{3} + x r \theta \sin \phi) + 7 r^2 \cos(x r \theta \sin \phi - \frac{4\phi}{3}) \right\}}{(a^2 + 14 a r^2 \cos(2\phi) + 49 r^4)} \quad (4.4a, b) \\
\tau_e = & \frac{3A_i(0)}{\theta^{2/3}} \left[\frac{3}{q_1 (a+7q_1^2)} + \frac{\sqrt{3}}{2\pi} \int_0^\infty \frac{(1-e^{-\theta x z}) (a+z^2) (1-z^{1/3} (a+z^2)) dz}{z^{4/3} (1-z^{1/3} (a+z^2) + z^{2/3} (a+z^2)^2)} \right. \\
& - \frac{6a}{r} \left\{ \frac{e^{x\theta r \cos \phi} \cos(\theta x r \sin \phi - \phi) - \cos \phi}{(a^2 + 14 a r^2 \cos 2\phi + 49 r^4)} \right. \\
& \left. \left. - \frac{42r \{ e^{\theta x r \cos \phi} \cos(\theta x r \sin \phi - 3\phi) - \cos 3\phi \}}{(a^2 + 14 a r^2 \cos 2\phi + 49 r^4)} \right\} \right] \quad (4.4c)
\end{aligned}$$

For $s=0$ (4.3abc), (4.4abc) simplify to the expressions given by Merkin-Smith (1983). Graphs of the solution for various values of s are given in figs. 2(a-e).

The linear solution breakdown for large s and this can be seen clearly in fig. 2c. This breakdown occurs because all three roots of (3.2) approach the imaginary axis for $s \gg 1$, specifically

$$q_1 \sim a^{-3} \quad q_{2,3} \sim \pm a^{1/2} i + \frac{1}{2} a^{2/3} e^{-2\pi i/3}.$$

This suggests that nonlinear terms will become important ahead of the corner, even for small angles.

The graphs also indicate that $\tilde{A}$, $\tilde{P}$, τ_c become unbounded for large X (for $s \neq 0$). From (4.1abc), we obtain for $X \gg 1$

$$\tilde{A} \sim X + \frac{aX^{2/3}}{\theta^{1/3} \Gamma(5/3)} + \frac{a^2 X^{1/3}}{\theta^{2/3} \Gamma(4/3)} + \dots$$

$$\tilde{P} \sim -sX - \frac{asX^{2/3}}{\theta^{1/3} \Gamma(5/3)} + \dots \quad (4.5abc)$$

$$\tau_c \sim \frac{3A_i(0)a}{\theta^{2/3}} \left[\frac{X^{1/3}}{\Gamma(4/3)} + \frac{a}{\theta^{1/3}} + \dots \right].$$

Since the skin friction is

$$\tau = 1 + \alpha \tau_c + \dots,$$

linear theory is clearly not valid for all X , for any $\alpha \neq 0$, and in fact $\alpha \tau_c \sim O(1)$ when $X \sim \alpha^{-3}$, from (4.5c). If $\alpha < 0$ then $\tau \sim 0$ when $X \sim |\alpha|^{-3}$

for any nonzero α or $s(>0)$! This is in contrast with the corner problem for similar pressure displacement laws where no such difficulty arises. We note also that $\alpha < 0$ would imply that the fluid is moving uphill, ~~and this is physically unrealistic.~~

If $\alpha > 0$ then nonlinear terms cannot be neglected when $X \sim O(\alpha^{-3})$. Hence from (2.14) (4.5b), $U \sim (P)^{1/2} \sim \alpha^{-1}$ and $Y \sim (\frac{X}{U})^{1/2} \sim \alpha^{-1}$.

Thus, setting

$$X = \alpha^{-3} X_1, \quad Y = \alpha^{-1} Y_1, \quad U = \alpha^{-1} U_1, \quad V = \alpha V_1,$$

$$P = -s\alpha X + \alpha^{-1} p_1, \quad A = \alpha X + \alpha^{-1} A_1,$$

we get

$$U_{,X_1} + V_{,Y_1} = 0$$

$$U_1 U_{,X_1} + V_1 U_{,Y_1} = s + U_{,Y_1} Y_1, \quad p_1 = -s A_1$$

$$U_1 = V_1 = 0 \quad \text{on } Y = 0 \tag{4.6}$$

$$U \rightarrow Y_1 + A_1 \quad \text{as } Y_1 \rightarrow \infty$$

$$A_1 \rightarrow \frac{a X_1^{2/3}}{\theta^{1/3} \Gamma(5/3)} \quad \text{as } X_1 \rightarrow 0.$$

The asymptotic form of (2.14), (2.15), (2.16) far downstream, (see later) also implies that $A_1 \sim O(X_1^{1/2})$ for $X_1 \gg 1$. We won't attempt to

obtain the solutions of (4.6).

Before discussing the full numerical solution of (2.14), (2.15), (2.16) for $O(1)$ angles, a few points concerning the breakdown of the linear solution. Firstly if $s=0$, then the aforementioned difficulties disappear, and regular solutions have been obtained for concave corners by Merkin-Smith (1982), with separation occurring for larger angles. However, in this case, the fluid is negotiating a corner with initial angle comparable with the turning angle $\mathcal{O}(\epsilon^3)$, and the problem is physically sensible. Also for arbitrary shapes $F(X)$, with $F(\infty) = 0$, the above inconsistencies do not occur, as can be seen from (4.1).

§5 Numerical Solution for $O(1)$ Angles

Basically three different numerical procedures were adopted to tackle the problem (2.14), (2.15), (2.16), two using a shooting technique, and the third based on global iteration. The need to use three methods arose from the fact that severe difficulties were encountered with the shooting approach in coping with the downstream conditions. The third method did overcome some of these problems but this too had its limitations. However, the three methods proved useful in checking the self consistency of the results.

Of the two involving shooting, one (M1) was based on Chebychev polynomials. The details are very similar to those given by Stewartson & Williams (1969) and can be referenced there. The other methods employed finite difference techniques.

The following difference scheme was used to discretize equation (2.14), where ψ below, denotes the stream

function, $U = \psi_y$, $V = -\psi_x$.

$$\frac{(U_j^{n+1} + U_j^n)(U_j^{n+1} - U_j^n)}{2 \Delta X} - \frac{(\psi_j^{n+1} - \psi_j^n)}{\Delta X} \frac{1}{2} \left[\frac{(U_{j+1}^{n+1} - U_{j-1}^{n+1}) + (U_{j+1}^n - U_{j-1}^n)}{2 \Delta y} \right]$$

$$= -\frac{(p_j^{n+1} - p_j^n)}{\Delta X} + \frac{1}{2} \left[\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1} + U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta y^2} \right]$$

$$\frac{U_j^{n+1} + U_j^n}{2} = \frac{1}{2} \frac{\psi_{j+1}^{n+1} - \psi_{j-1}^{n+1} + \psi_{j+1}^n - \psi_{j-1}^n}{2 \Delta y} \quad (5.1a, b)$$

Here $U_j^n = U(n\Delta X, j\Delta y)$ is the n, j th grid point, $\Delta X, \Delta y$ being the step sizes in the X, Y directions respectively.

The pressure-displacement law was handled by

$$\frac{p_j^{n+1} + p_j^{n-1}}{2} = -s \frac{(A_j^{n+1} + A_j^{n-1})}{2} - \frac{(A_j^{n+1} - 2A_j^n + A_j^{n-1})}{\Delta X^2} \quad (5.1c)$$

The boundary conditions were replaced by the equivalent set

$$U = \psi = 0 \quad \text{at } Y = 0$$

$$\psi \rightarrow \frac{1}{2} (Y + A - \alpha F(X))^2 + P(X) \quad \text{as } Y \rightarrow \infty$$

$$U \rightarrow Y + A - \alpha F(X) \quad \text{as } Y \rightarrow \infty$$

with the additional requirement that

$$A' \rightarrow \alpha \quad \text{as } X \rightarrow \infty$$

These too were differenced appropriately. This latter constraint expresses the fact that the free surface to leading order (see results later), is parallel to the plane downstream.

The above equations were linearized and Newton iteration was performed, with a tolerance usually 10^{-6} or 10^{-7} . The large banded matrix resulting from the above problem was again tackled by the usual methods, see e.g. Smith (1974).

For the two methods using shooting (M1, M2), the calculation was initiated at $X = X_0$ by giving the pressure a small negative kick, and marching forward up to the corner $X = 0$ where the resulting profiles were stored. After the corner, the calculation was continued with a guess for the value of the angle α and $A(X)$ was checked at successive downstream stations to ensure that the downstream condition was being satisfied. The angle α could thus be adjusted appropriately until this condition was met.

The third method (M3) was based on ideas used successfully by Veldman (1980), Smith & Merkin (1982), and is more efficient and can be much faster than the shooting method. Here the pressure displacement law is differenced differently. Instead of (5.1c) we used

$$P_{(k)}^{n+1} = -s A_{(k)}^{n+1} - \frac{(A_{(k-1)}^{n+2} - 2A_{(k)}^{n+1} + A_{(k)}^n)}{\Delta x^2} \quad (5.2)$$

k being the k th iteration sweep. The idea of using (5.2) is to incorporate existing information into the calculation, and this overcomes one of the main difficulties with the shooting approach, i.e. that of the solutions tending to unwanted eigensolutions downstream. The values of U were stored over the whole computational domain, and in the subsequent Newton iteration step these values were used as initial guesses rather than the previous step values, as in the shooting methods. Also only a fixed number of Newton iterations (usually 2) were performed. This enabled the computational time to be reduced. The calculation was initiated either by using results from a previous computation as a guess, or by initially guessing for $A(X)$ and marching forward to obtain the flow field, and then proceeding as before.

Both finite difference schemes used a stretched grid in ^{the} y direction to improve the resolution near the wall and to allow the condition at y_∞ to be imposed satisfactorily. Correction terms were added to (5.1a,b) to ensure that the y differences remained second order accurate with the transformations used.

Various checks were made including grid size checks, checks with linear theory, and checks with the results by Merkin-Smith (1982) for the case $\xi = 0$.

Calculations with the sharp corner $F(X) = XH(X)$ ($H(X)$ the Heaviside step function), proved to be difficult with all three methods, as the angle α was increased. A fine mesh was needed near the corner, to

deal with the oscillations appearing in the solutions. A mesh refinement in the X direction, was tried with M3 but this introduced extra spurious effects upstream of the corner. M1 was found to be the best here. The problems encountered could be because of the singular nature of the solution near $X = 0 + (\frac{d^2 \tilde{p}}{dx^2} \sim X^{-2/3}$ from (4.1b)). Some typical solutions are presented in fig.(3) . It can be seen that the fluid accelerates up to the corner, is retarded immediately afterwards, and then continues to accelerate, with the pressure falling and the layer thinning out. For $s = .1$ the drop in the skin friction becomes more pronounced and there is a hint that separation will occur for sufficiently large α . For both $s = .1$ and $s = 1$ there were additional difficulties with the shooting techniques for larger angles, with the methods becoming very sensitive to small changes in α .

It was felt that the sharp corner was not the central issue and that better progress could be made with a smooth corner. In fact the gross features, such as separation would also be presented for the smooth corner, and that the results would give good qualitative insight into the flow properties for large α . With this in mind calculations were also performed for the case $F(X) = XH(X) \tanh(X^2)$.

The previous difficulties near $X = 0$ disappeared although the problems with shooting remained. Also M1 ran into problems as separation was approached, with many terms required in the Chebychev expansion. M3 was found to be the best and most efficient, if a reasonable

starting guess was incorporated, and then converged in typically 300 sweeps. Typical results are shown in figs.(4) . Again the overall features are similar to those for the sharp corner. For $s = .1$ separation was found to occur and the familiar trend as the angle is further increased can be noticed in fig.(4.b). The separation and reattachment points move upstream and downstream respectively with separation and reattachment becoming distinct local processes. Again the fluid continues to accelerate far beyond reattachment.

For $s = 1$ the results are quite revealing and exemplify the rapid deceleration of the fluid as the corner is negotiated. There is a hint that separation will occur when $\alpha \sim 15$ and it was felt that the present method (M3) was inadequate to proceed that far. Certainly the outer boundary has to be increased substantially and this requires a much bigger grid. Also a good initial guess is needed to start the calculation and consequently many computations need to be performed with α being increased very slowly.

A comparison with the linear theory is made in fig.(5) , for $F(X) = X H(X)$. For $s = .1$ there is good agreement for ^{small} values of α , but for $s = 1$ the numerical results bear out the earlier discussions on the breakdown of the linear theory. However these calculated values do lie on a parabola passing close to the origin.

Finally it is worth emphasizing that all of the results presented show that the disturbances to the original flow, do not decay far

downstream. This is quite novel and in contrast with results for different pressure-displacement laws in similar problems. This feature is probably just characteristic for this particular problem as the asymptotic description in §6 suggests, and may not arise if a different wall shape, e.g. a hump, instead of a corner, is used.

§6 Asymptotic forms for $X \gg 1$

The conditions $A \rightarrow \alpha X$, $P \rightarrow -s\alpha X$ for $X \gg 1$, imply the following self similar forms for downstream. The boundary layer splits up into two distinct regions.

Let $A = \alpha X + a_0 X^n + \dots$. Then $P = -s\alpha X - sa_0 X^n + \dots$. A pressure-inertial balance now gives $U \sim |P|^{\frac{1}{2}} \sim X^{\frac{1}{2}}$.

From (2.15), in the majority of the flow $U \sim Y \sim A - \alpha X \sim X^n \sim X^{\frac{1}{2}}$. Hence $n = \frac{1}{2}$. As the wall is approached, the slip velocity of $O(X^{\frac{1}{2}})$ has to be reduced to zero, and a viscous-inertial balance there suggests

$$Y \sim \left(\frac{X^{\frac{1}{2}}}{U} \right) \sim X^{\frac{1}{4}}, \quad \text{since } U \sim O(X^{\frac{1}{2}}).$$

Having established the scalings, the expansions follow readily.

Firstly, in the upper sublayer (i) where $y = X^{\frac{1}{2}} \eta$,

$$A = \alpha X + a_0 X^{\frac{1}{2}} + a_1 X^{\frac{1}{4}} + a_2 + \dots$$

$$P = -s\alpha X - sa_0 X^{\frac{1}{2}} - sa_1 X^{\frac{1}{4}} + \dots$$

$$\psi = XG_1(\eta) + X^{\frac{3}{4}}G_2(\eta) + X^{\frac{1}{2}}G_3(\eta) + \dots$$

$$U = X^{\frac{1}{2}}G_1' + X^{\frac{1}{4}}G_2' + G_3' + \dots \quad (6.1)$$

Substituting into (2.14), the G_i satisfy

$$\frac{G_1'^2}{2} - G_1 G_1'' = s\alpha$$

$$\frac{3}{4} G_1' G_2' - G_1 G_2'' - \frac{3}{4} G_2 G_1'' = 0$$

$$\frac{G_2'^2}{4} + \frac{G_3' G_1'}{2} - G_1 G_3'' - \frac{3}{4} G_2 G_2'' - \frac{G_3 G_1''}{2} = \frac{sa_0}{2} + G_1''' \quad (6.2)$$

with the boundary conditions

$$G_1' \sim \eta + a_0, \quad G_2' \sim a_1, \quad G_3' \sim a_2 \dots, \quad \eta \gg 1$$

$$G_1(0) = 0$$

These equations can be solved by first differentiating and then

integrating and making use of the boundary conditions.

$$\text{This gives } G_1 = \eta^2_{/2} + a_0 \eta \quad . \quad (6.3)$$

$$\text{From (6.2), } G'_1(0) = (2s\alpha)^{\frac{1}{2}}$$

$$\text{Hence } a_0 = (2s\alpha)^{\frac{1}{2}}.$$

We note that this too requires $s\alpha > 0$, and the analysis is not applicable to the concave corner problem.

Solving for G_2 , G_3 yields the simple solutions

$$G_2 = a_1 \eta + g_{20} \quad , \quad g_{20} = a_0 a_1$$

$$G_3 = a_2 \eta + g_{30} \quad , \quad g_{30} = \frac{a_1^2}{2} + a_2 a_0 - s a_0$$

where the constants g_{20} , g_{30} are determined by matching with the inner sublayer.

Further terms in (6.1) can be calculated with eigensolutions appearing at higher order.

Turning now to the inner sublayer (ii) where $y = \xi X^{\frac{1}{4}}$, the expansions (6.1) and (6.3) imply

$$\begin{aligned} \psi &= X^{\frac{3}{4}} F_0(\xi) + X^{\frac{1}{2}} F_1(\xi) + \dots \\ V &= X^{\frac{1}{2}} F'_0 + X^{\frac{1}{4}} F'_1 + \dots \end{aligned} \quad (6.5)$$

Here F_0, F_1 satisfy

$$\frac{F_0'^2}{2} - \frac{3}{4} F_0 F_0'' = s\alpha + F_0''' \quad (6.6)$$

$$\frac{3}{4} F_0' F_1' - \frac{3}{4} F_0 F_1'' - \frac{F_1 F_0''}{2} = F_1'''$$

with the boundary conditions

$$F_0(0) = F_0'(0) = 0, \quad F_1(0) = F_1'(0) = 0$$

and $F_0' \rightarrow a_0, \xi \rightarrow \infty$

$$F_1' \rightarrow \xi + a_1, \quad \xi \rightarrow \infty \quad (6.7)$$

to match with (i). The transformation

$$F = \frac{2^{5/4}}{3^{1/2}} (s\alpha)^{1/4} \tilde{F}_0(\tilde{\xi}), \quad \xi = \frac{2^{3/4}}{3^{1/2} (s\alpha)^{1/4}} \tilde{\xi} \quad (6.8)$$

converts (6.6) into a standard Falkner-Skan equation for $\tilde{F}_0$, with $\beta = 2/3$ in the notation of Jones & Watson (1963). This was solved numerically using a standard second order-centered difference scheme, giving

$$\tilde{F}_0 - \tilde{\xi} = -.74022 \quad (\tilde{\xi} > 1), \quad \tilde{F}_0''(0) = 1.039 \quad (6.9)$$

The displacement matches with ^{the} outer layer (i) provided

$$g_{20} = -\frac{2}{\sqrt{3}} (2s\alpha)^{\frac{1}{4}} 0.74022 \quad \text{Hence} \quad a_1 = \frac{g_{20}}{a_0} = -0.71874 (s\alpha)^{-\frac{1}{4}} .$$

$a_2, a_3 \dots$ can be similarly determined. Combining the above results we obtain the asymptotes far downstream

$$-\eta = A \sim \alpha X + (2s\alpha X)^{\frac{1}{2}} - 0.71874 \left(\frac{X}{s\alpha}\right)^{\frac{1}{4}} + \dots$$

$$p \sim -sA - A'' \quad (X \gg 1)$$

$$\tau = (U_y)_{y=0} \sim 1.039 \frac{3^{\frac{1}{2}}}{2^{\frac{1}{4}}} (s\alpha)^{\frac{3}{4}} X^{\frac{1}{4}} + \dots \quad (6.10)$$

Comparisons with the forms (6.10) and the numerical results are made in table ^(2.1) for various values of s, α and X , and they show reasonable agreement with the discrepancies decreasing for large X and α .

Inspection of (6.10) shows that the expansion is nonuniform when α (or s) is small, and the second and third terms become comparable when

$$(\alpha s X)^{\frac{1}{2}} \sim \left(\frac{X}{s\alpha}\right)^{\frac{1}{4}}$$

$$\text{i.e. when } X \sim (\alpha s)^{-3} .$$

and then $A \sim \alpha X + O((\alpha s)^{-1})$.

A scaling argument then establishes the same scalings as in §4, and the problem (4.6) is recovered. The nonuniformity above provides the downstream condition on A_1 in (4.6) as $X_1 \rightarrow \infty$, i.e.

$$A_1 \rightarrow (2sX_1)^{\frac{1}{2}} \quad \text{as } X_1 \rightarrow \infty.$$

A more unusual feature is the form of the asymptote for $X \gg 1$. In terms of the original variables, the free surface is given by

$$1 + \varepsilon^2 \bar{\eta} + \sim 1 - \gamma^{-2/7} \alpha^* x - \varepsilon^{5/2} (2s\alpha\gamma^{-4/7} x)^{\frac{1}{2}} + .71874 \varepsilon^{9/4} \left(\frac{x\gamma^{-2/7}}{s\alpha} \right)^{\frac{1}{4}} + \dots \quad (6.11)$$

The first two terms here just represent the liquid layer turning through an angle α^* . However the third term shows that the liquid layer does not attain this fixed level (fixed, in terms of the variable normal to the plane) but continues to get thinner with the flow becoming strongly attached. Again this is in sharp contrast to corresponding situations in external flows where the disturbances to the basic flow die down as the fluid negotiates the corner, (see e.g. Smith 1982).

The above expansions and structure, breakdown when this decrease in the free surface becomes comparable with the original depth. From (6.11) we see that this occurs when

$$\varepsilon^{5/2} x^{1/2} \sim O(1) \quad (6.12)$$

i.e. when $x \sim O(\epsilon^{-5})$. On this length scale, the outer sublayer (i) of original thickness $O(\epsilon^2 X^{\frac{1}{2}})$ merges with majority of the flow, and the inner layer has thickness $\sim O(\epsilon^2 X^{\frac{1}{4}}) \sim O(\epsilon)$. In (2.7) the perturbation of $O(\epsilon^2 U)$ becomes $O(1)$ in view of (6.10) and (6.12).

The above argument suggests the following expansions on this much longer lengthscale. It is convenient to set $\gamma=1$, and to work with the variables

$$\begin{aligned} z &= y + \epsilon^3 \alpha x & , \quad x &= \epsilon^{-5} \hat{x} \\ \hat{v} &= v + \epsilon^3 \alpha u & . \end{aligned} \quad (6.13)$$

Then in the majority of the flow

$$\begin{aligned} u &\sim \hat{u}_0 + \epsilon \hat{u}_1 + \dots \\ \hat{v} &\sim \epsilon^5 \hat{v}_0 + \dots \\ p &\sim \epsilon^2 \hat{p}_0 + \epsilon^5 \hat{p} + \dots \end{aligned} \quad (6.14)$$

and the free surface is $z=1+\hat{\eta}_0 + \epsilon \hat{\eta}_1 + \dots$. Substituting into the equations (2.2) with the transformations (6.13) gives

$$\begin{aligned} \hat{u}_{0x} + \hat{v}_{0z} &= 0 \\ \hat{u}_c \hat{u}_{0x} + \hat{v}_c \hat{u}_{0z} &= -\alpha \hat{p}_{0z} \\ \hat{p}_{0z} &= -s \end{aligned} \quad (6.15)$$

From (6.15) $\hat{p}_0 = s(1-z) + \hat{p}_0(\hat{x})$ and the boundary conditions give

$$\hat{p}_0(\hat{x}) = s\hat{\eta}_0(\hat{x}) .$$

Hence $\hat{u}_0, \hat{v}_0$ satisfy

$$\hat{u}_{0\hat{x}} + \hat{v}_{0z} = 0 ,$$

$$\hat{u}_0 \hat{u}_{0\hat{x}} + \hat{v}_0 \hat{u}_{0z} = \alpha s , \quad (6.16)$$

with the free surface conditions

$$\hat{v}_0 = \hat{u}_0 \hat{\eta}_{0\hat{x}} \quad \text{on } z=1 + \hat{\eta}_{0\hat{x}} ,$$

$$\hat{u}_{0z} = 0 \quad \text{on } z = 1 + \hat{\eta}_{0\hat{x}} , \quad (6.17)$$

and $\hat{v}_0 = 0$ on $z = 0$. Also $\eta_0 \rightarrow -(2s\alpha\hat{x})^{\frac{1}{2}}$ as $\hat{x} \rightarrow 0$.

(6.16) gives a slip velocity of $(2s\alpha\hat{x})^{\frac{1}{2}}$ and a viscous layer of thickness $O(\epsilon)$ is needed to reduce this to zero.

Thus setting $z = \epsilon \hat{z}$, (6.14) imply

$$U \sim \bar{v}_0 + \dots$$

$$\hat{v} \sim \epsilon \bar{v}_0 + \dots$$

$$p \sim \epsilon^2 \bar{p}_0 + \dots$$

giving

$$\bar{U}_0 \bar{U}_0 \hat{x} + \bar{V}_0 \bar{U}_0 \hat{z} = \alpha s + \bar{U}_{0\hat{z}\hat{z}}$$

$$\bar{p}_0 = \hat{p}_0(\hat{x}) \quad (6.18)$$

and the boundary conditions

$$\bar{U}_0 \rightarrow (2s\alpha\hat{x})^{\frac{1}{2}} \text{ as } \hat{z} \rightarrow \infty$$

$$\bar{U}_0 = \bar{V}_0 = 0 \quad \text{on } \hat{z} = 0 \quad (6.19)$$

and $\bar{U}_0 \rightarrow \hat{x}^{\frac{1}{2}} F_0' \left(\frac{\hat{z}}{\hat{x}^{\frac{1}{4}}} \right)$ as $\hat{x} \rightarrow 0$

to match with (6.5).

The solution for $\bar{U}_0, \bar{V}_0$ is as before, and in terms of the similarity variable $\xi = \hat{z}/\hat{x}^{\frac{1}{4}}$

$$\bar{U}_0 = \hat{x}^{\frac{1}{2}} F_0'(\xi), \quad \bar{V}_0 = \hat{x}^{-\frac{1}{4}} \left[\frac{3F_0}{4} - \frac{1}{4} \xi F_0' \right] \quad \forall \hat{x}$$

The ultimate form of the solution of (6.16), (6.17), for large $\hat{x}$, is of interest, although this problem is left unresolved. We note, however, that a return to the half-Poiseuille profile is not possible on this lengthscale but may require a much longer lengthscale.

57 CONCLUSIONS

We have attempted to discuss the flow of a liquid layer over a convex corner. With the various assumptions outlined earlier, we have shown that for a fully developed oncoming profile, a distortion to the bed of $O(R^{-2/3})$ laterally (in y), provokes an adjustment to the basic flow at distances of $O(R^{1/3})$ upstream of the distortion. For smaller distortions only a linear adjustment takes place usually, although for some distortions such as the convex corner considered here, a nonlinear adjustment still emerges sufficiently far downstream. For the convex corner this is because the change in the free surface is eventually much greater than $O(R^{-2/3})$ downstream. This can be seen by considering the mass flux upstream and downstream, assuming that the profile is the half-Poiseuille profile in each case, giving $\sin(\beta^*)h^3 = \sin(\alpha^* + \beta^*)H^3$. Here H is the uniform depth far downstream. By contrast much larger disturbances provoke a free interaction further upstream, as is evident in the numerical calculations for the convex corner presented here.

At the outset one of the questions posed was how does the boundary layer influence the free surface. The interaction involved is expressed concisely by the pressure-displacement laws together with the problem (2.14), (2.15). These show that for fully developed flows, while the leading the order displacement of the free surface is the same as the boundary layer displacement, the surface pressure which drives the boundary layer is made up of (a), the induced hydrostatic pressure due to the displacement of the free surface, and (b), the induced pressure due to the vertical acceleration of the fluid. For the convex corner this latter component

(b) ($\phi A''$) is important closest to the corner, whereas the other component (a) (ϕA) is ultimately responsible for the adjustment to the new downstream state, which is achieved on a lengthscale much longer than that, $O(R^{\frac{1}{2}})$, considered here.

Although the solutions and the asymptotic form far downstream give some insight into the physical processes pertaining, there are still some problems outstanding, many having been cited already. There are also others. One such problem is the mathematical description of the flow properties as the corner angle is increased. Here we would expect the free interaction (with the pressure falling) to be significant, as in Stewartson (1970b), Stewartson (1974). An analysis as outlined in these papers would help greatly in understanding the flow properties for $O(1)$ angles.

The problem (2.14), (2.15), (2.16) with other shapes e.g. a hump is also interesting, especially as the obstacle size is increased. From experience with similar problems we would expect the free interaction with $\hat{p} > 0$ to take place, together with the hydraulic jump occurring far ahead of the obstacle, (see also Gajjar and Smith 1983). Of course numerical calculations are needed to substantiate this.

An equally important question is the role of surface tension in all these problems. The transformation (2.13) would indicate that the above description would be significantly altered as the parameter γ is varied. However the present scalings and structure set out would permit such an investigation.

Finally, the stability of these flows is another vast topic. The structural approach as in Smith (1979)^{ab} is ideally suited for these problems. A start on such an investigation, for fully developed and undeveloped flows, has been made in Gajjar (1983).

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CHAPTER 2

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Table 2.1

Comparisons between the numerical results and the asymptotic form (A_{as}) (6.10), for A.

X	A/A_{as}	s, α
6	1.81	$s = .1$
4.5	2.3	$\alpha = .583$
6.5	1.025	$s = .1$
5	1.035	$\alpha = 3.28$
3	1.021	
6.5	1.006	$s = .1$
5	1.009	$\alpha = 6.15$
3	0.96	
7	1.04	$s = 1$
5	1.05	$\alpha = 3$
3	1.08	
7	1.016	$s = 1.$
5	1.022	$\alpha = 7.7$
3	1.033	
6	1.27	$s = 1$
3	1.38	$\alpha = .494$

Fig. 1

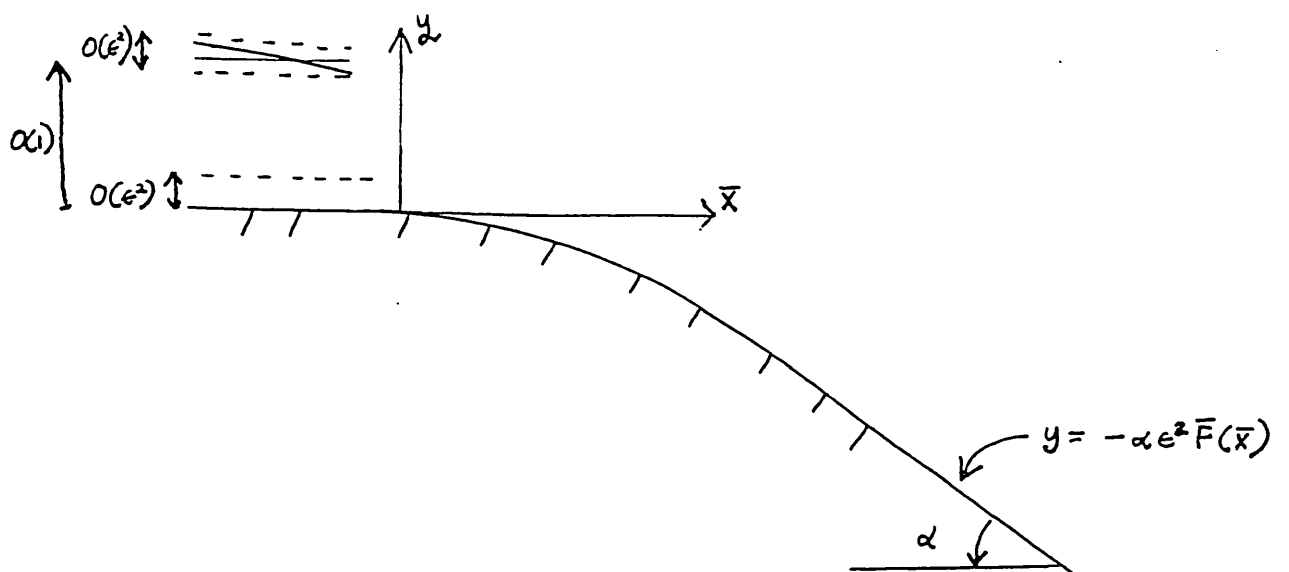
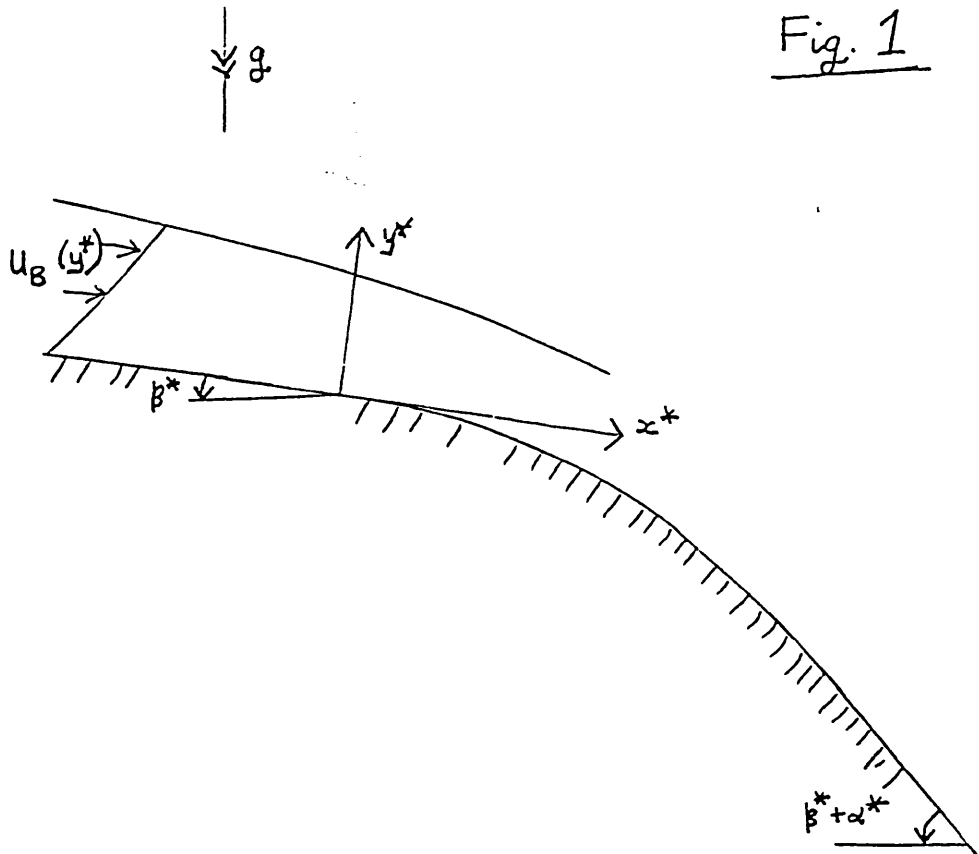
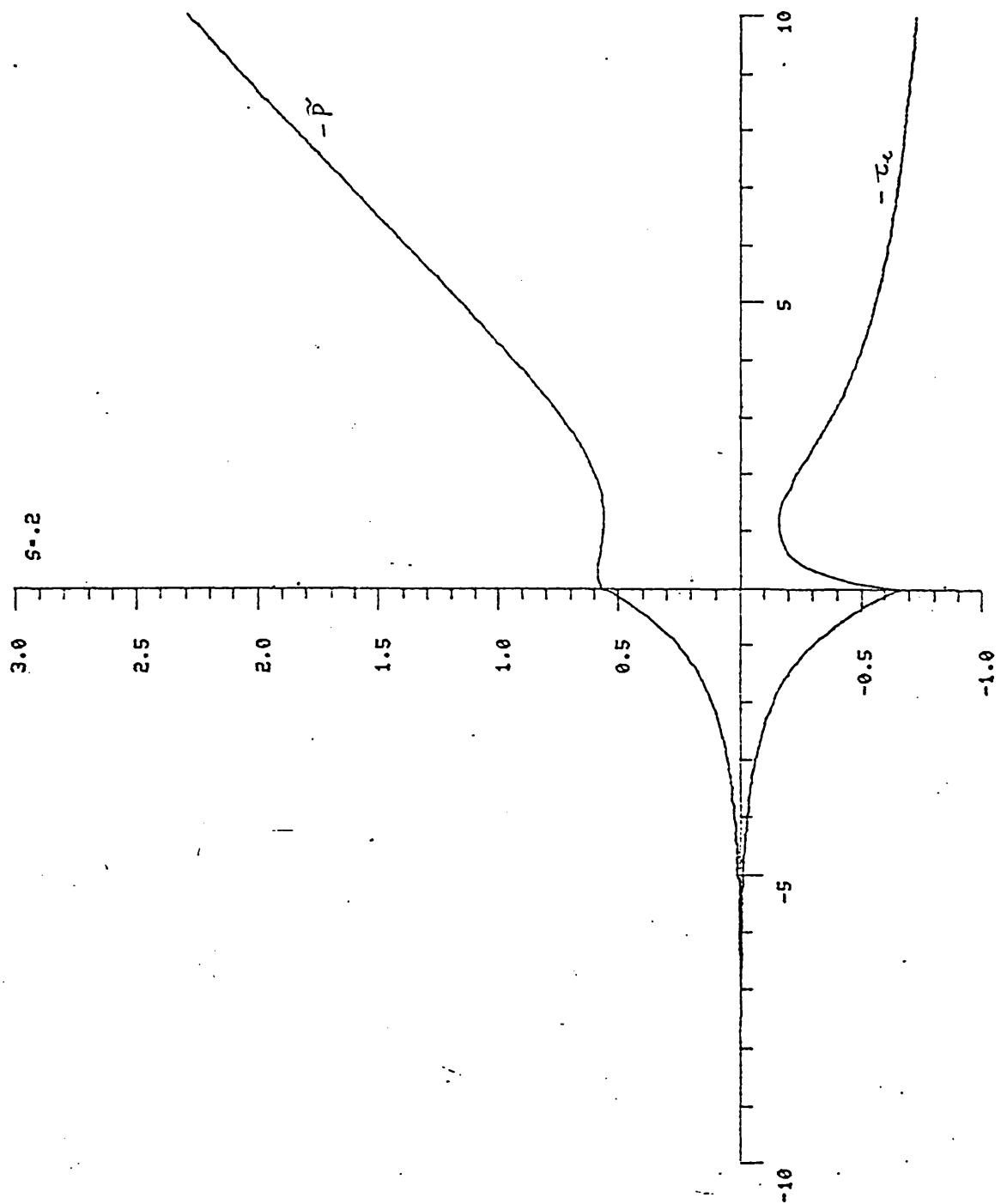


Fig. 2a



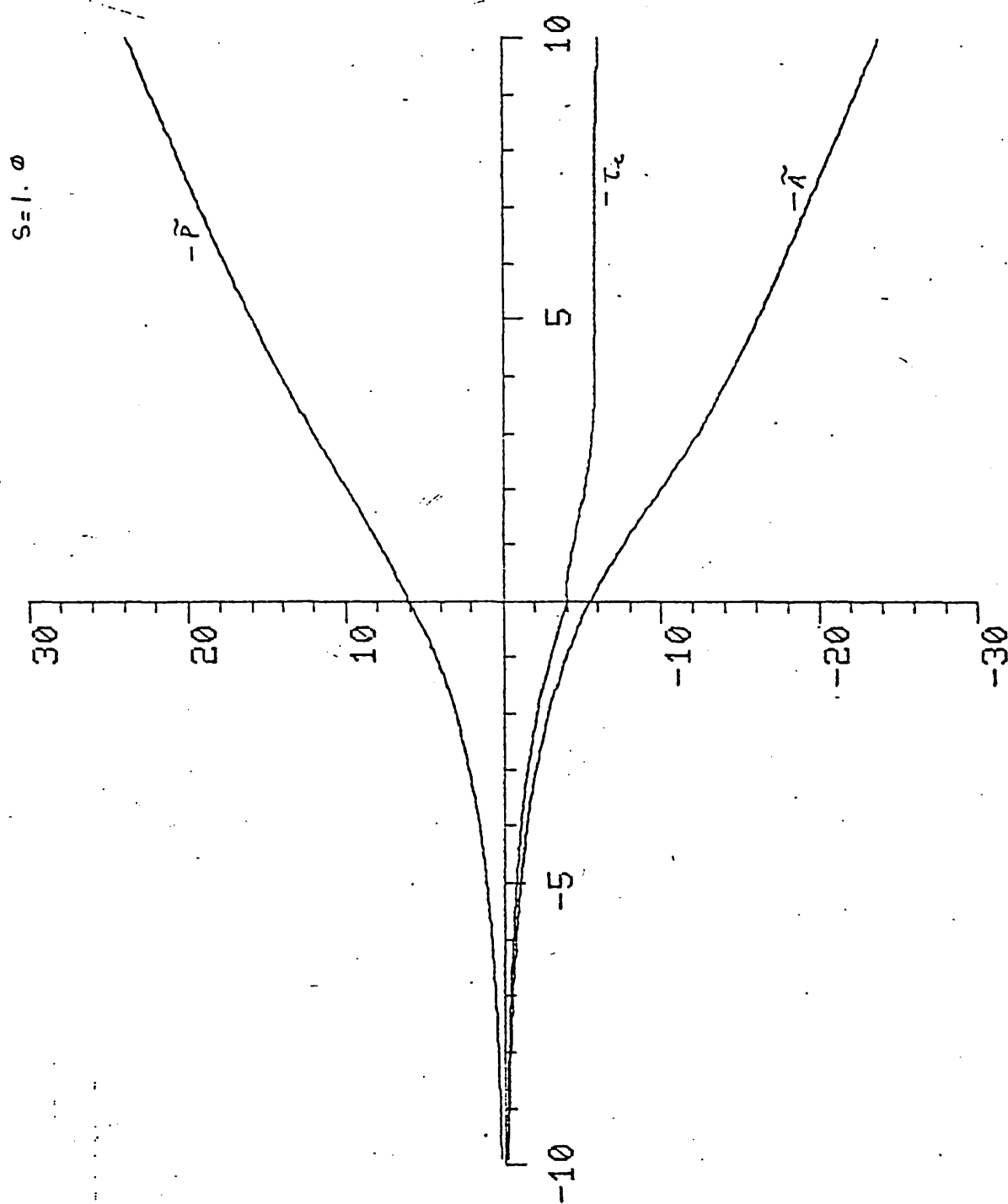


Fig. 2c

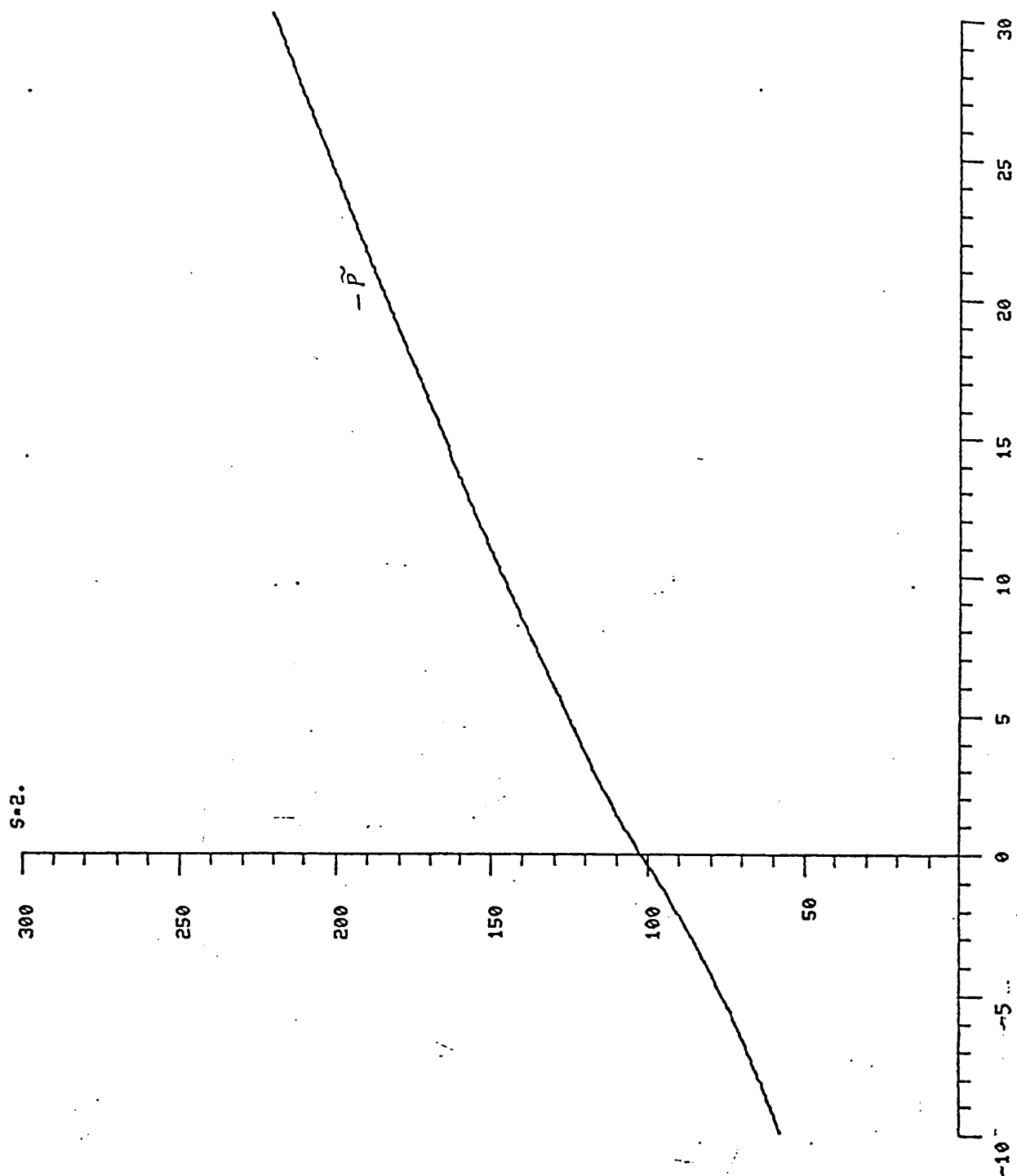


Fig. 2d

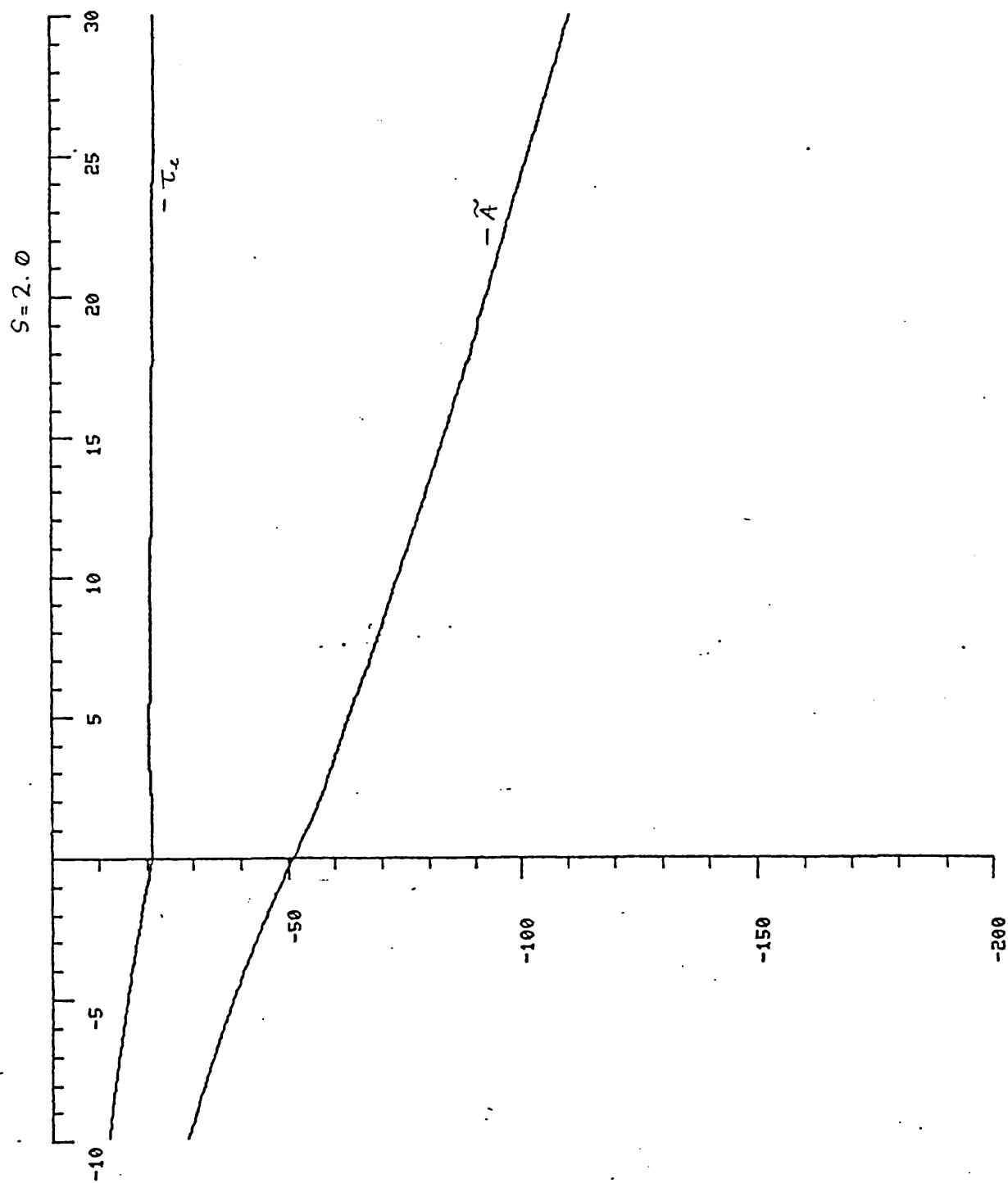


Fig. 29

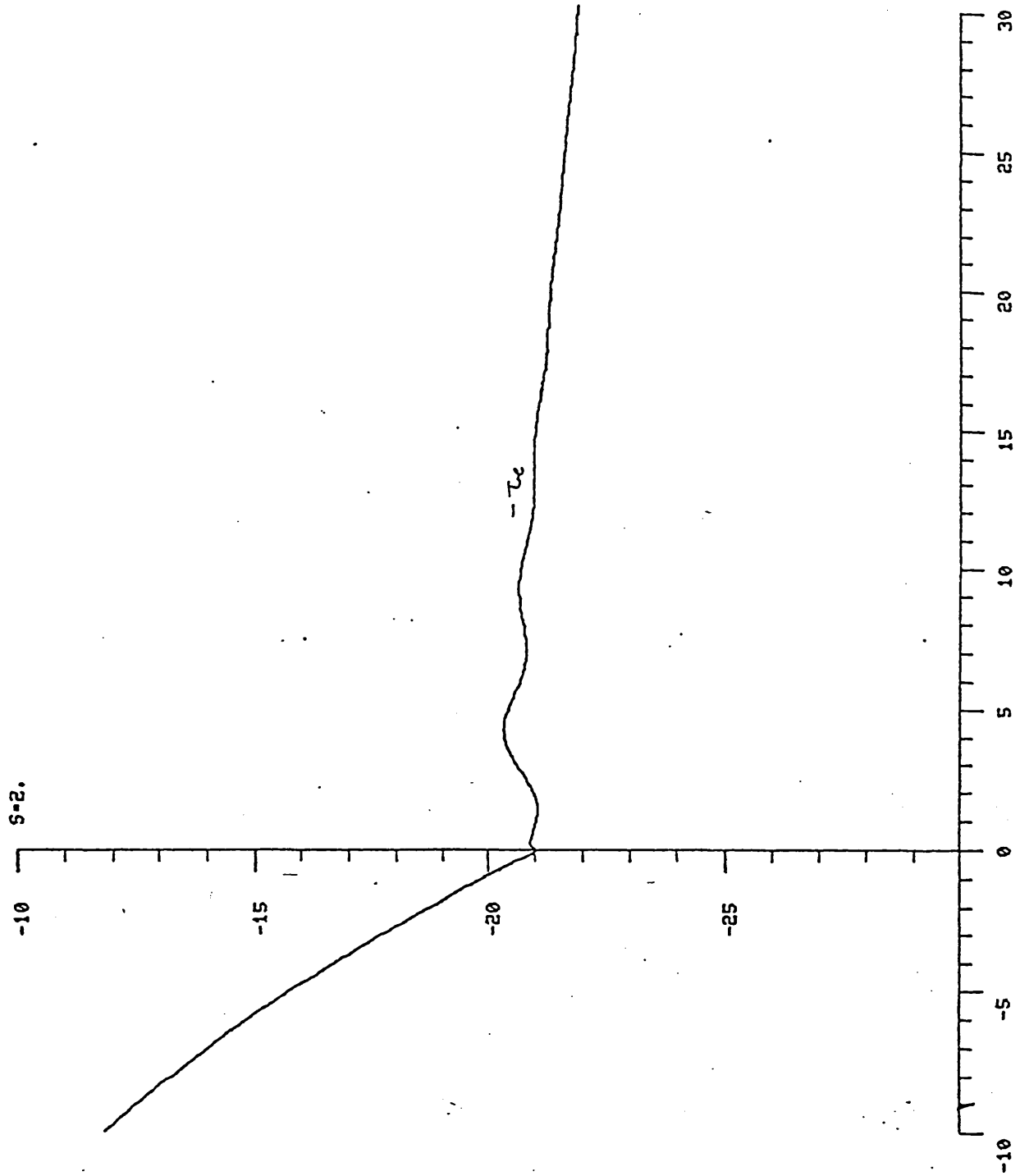


Fig. 3a

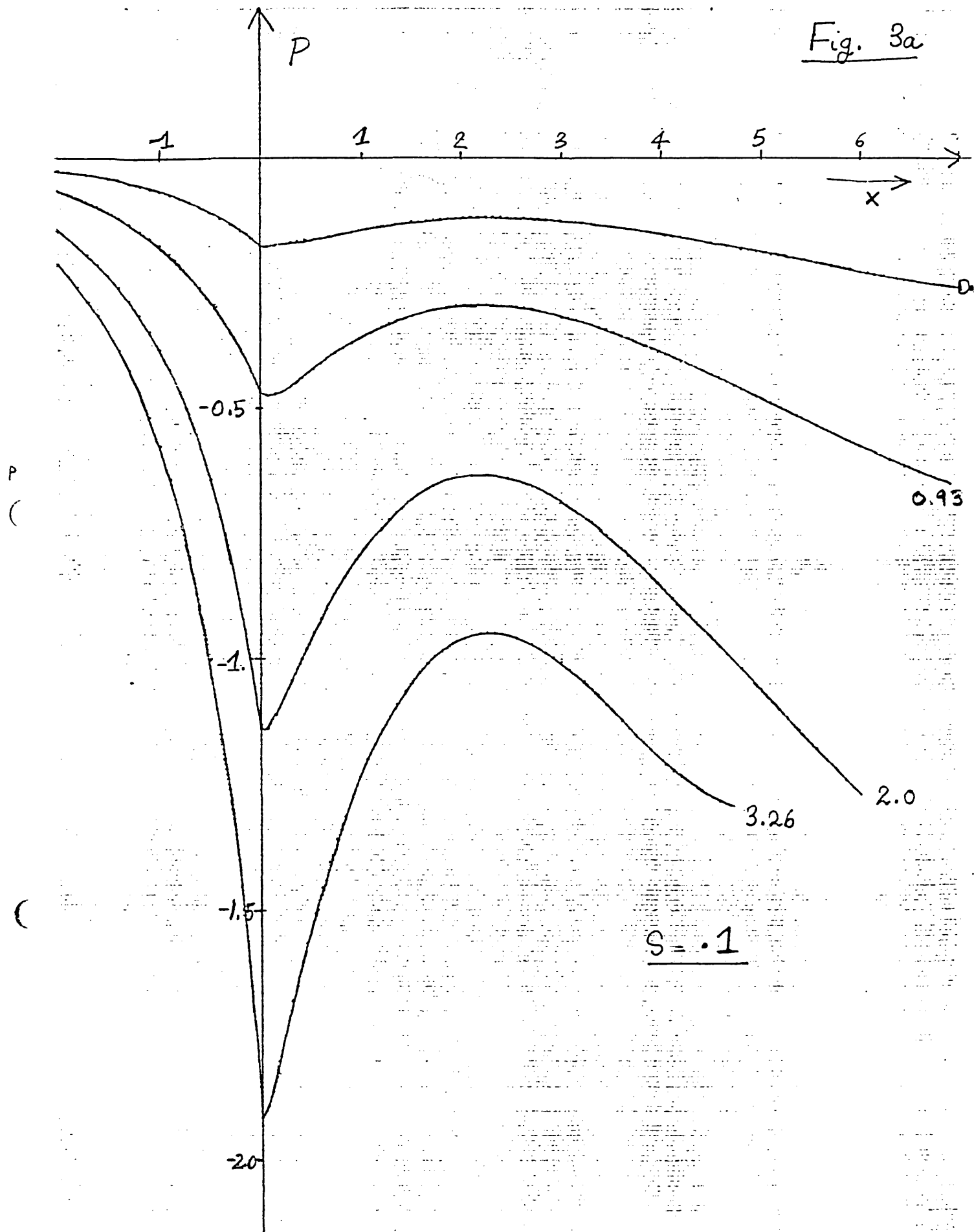


Fig. 3b

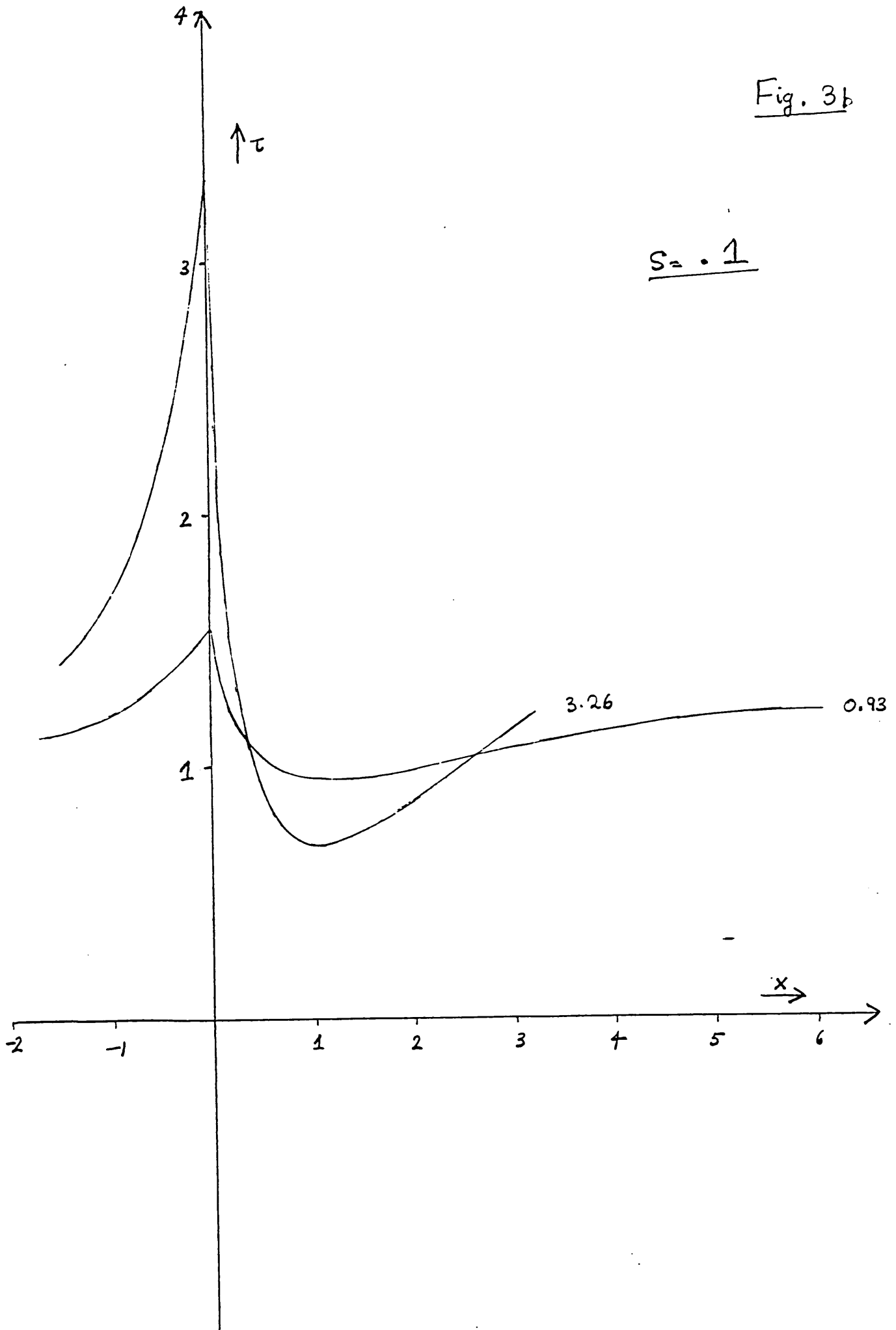
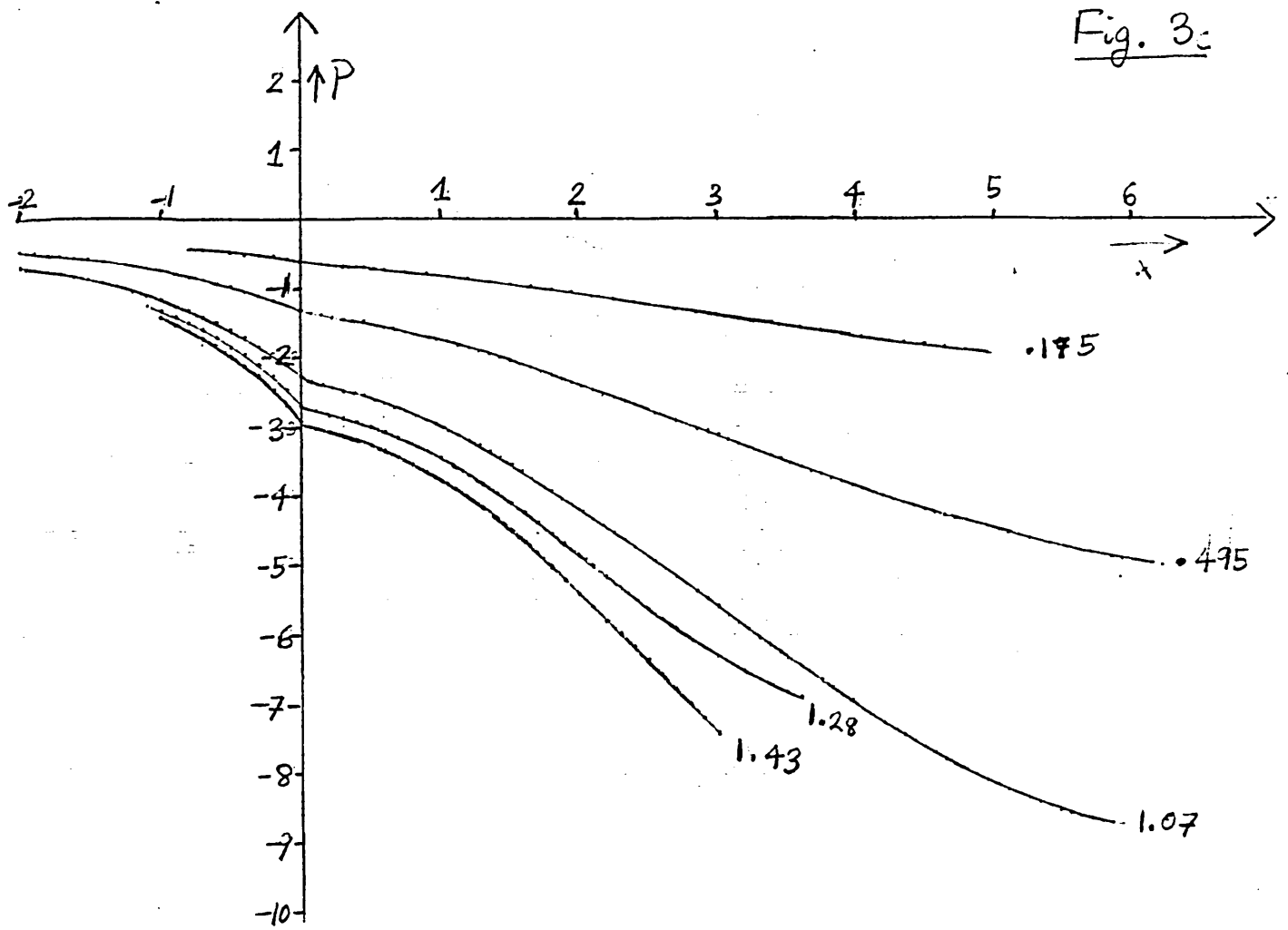


Fig. 3c



$S = 1. \phi$

Fig. 3d

$S = 1. \phi$

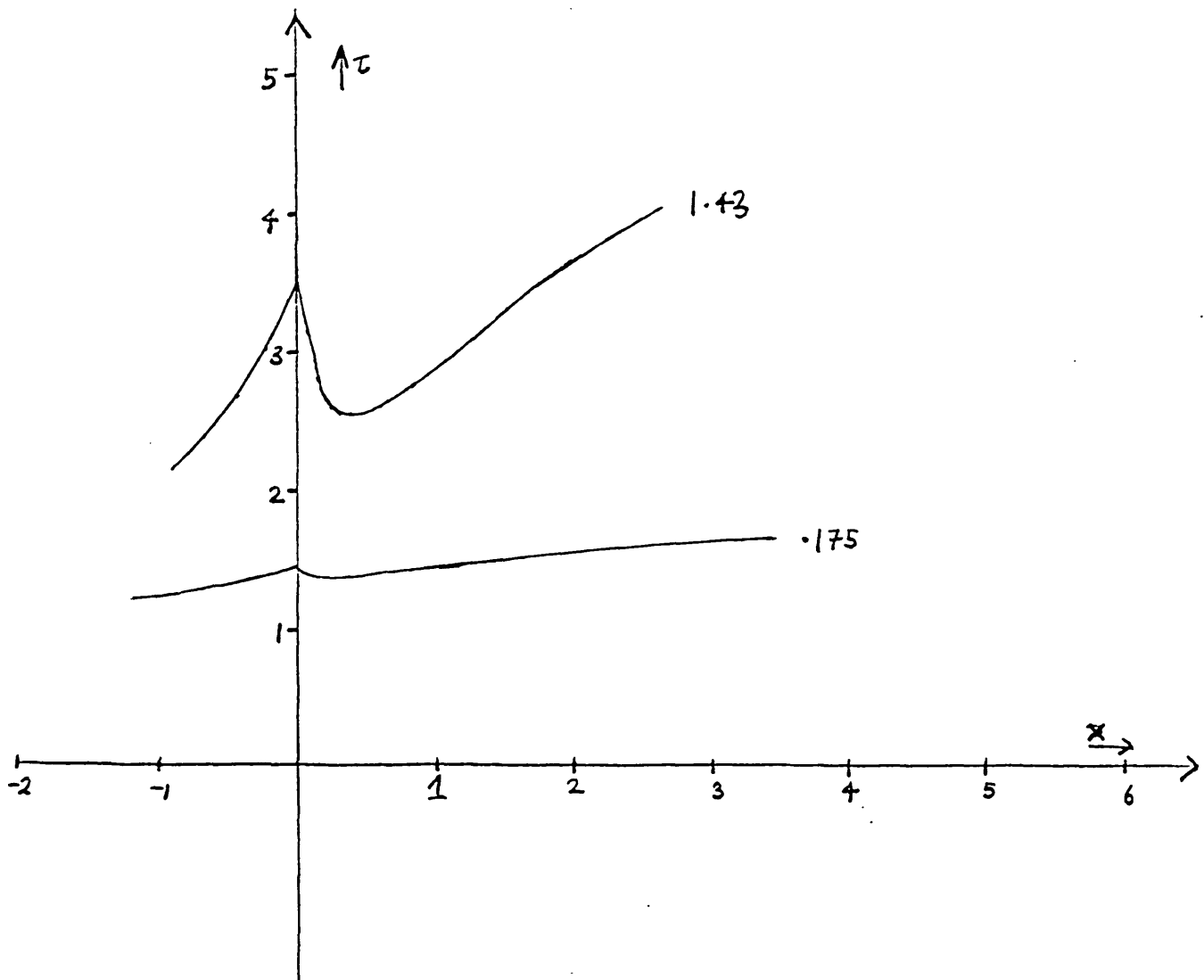
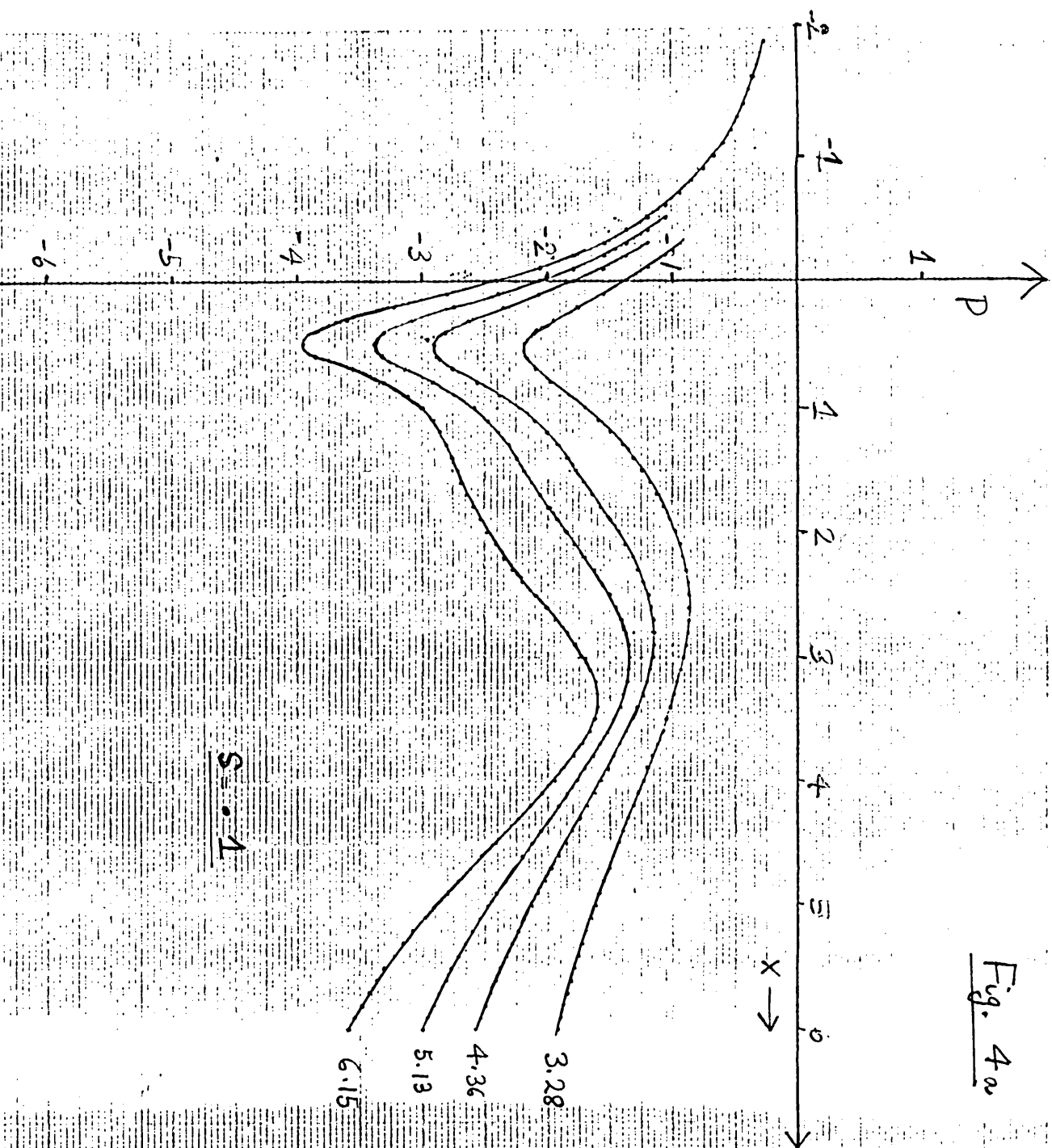


Fig. 4a



$S=0.1$

Fig. 4b

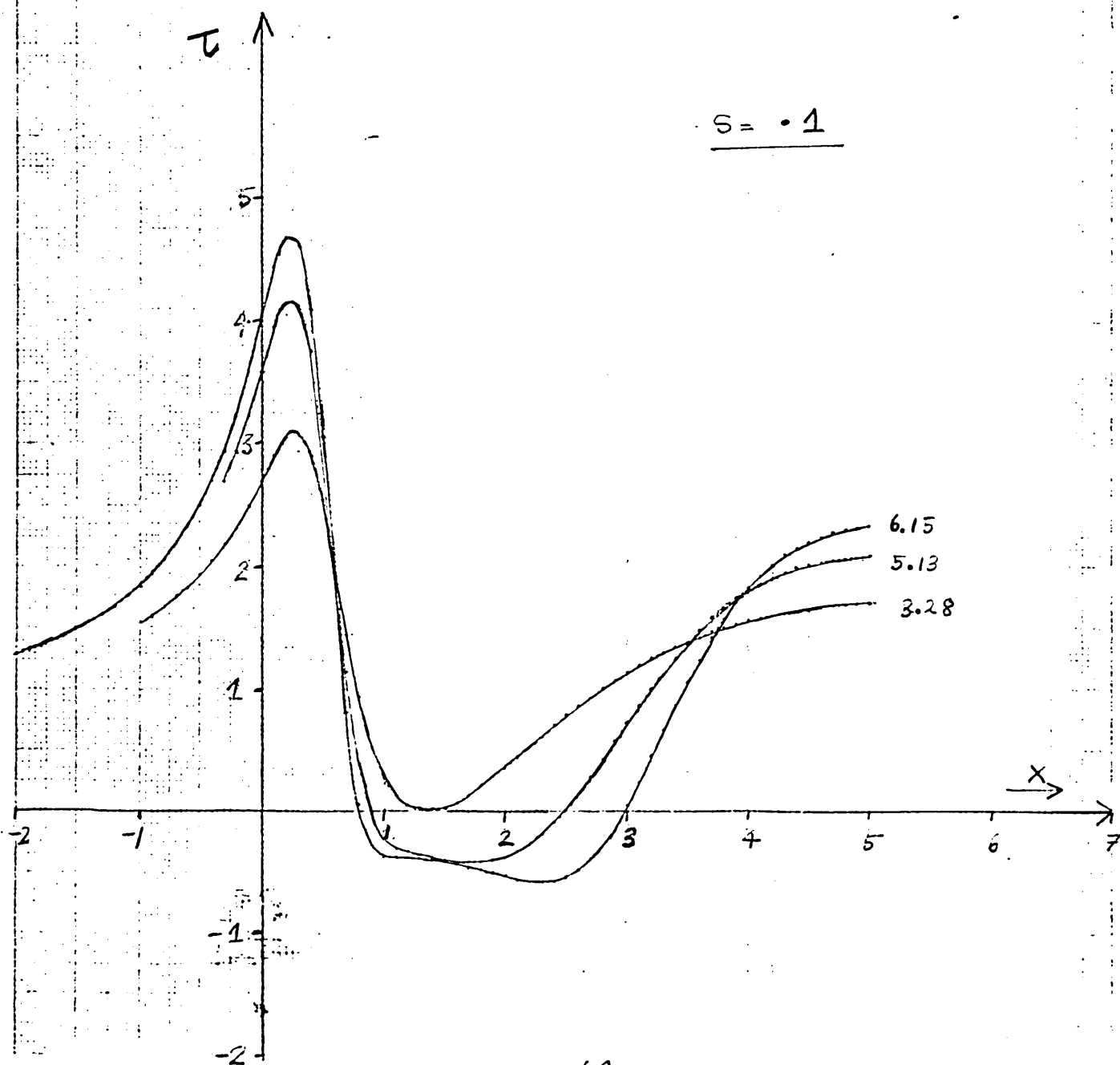


Fig. 4c

$$S = 0.1$$

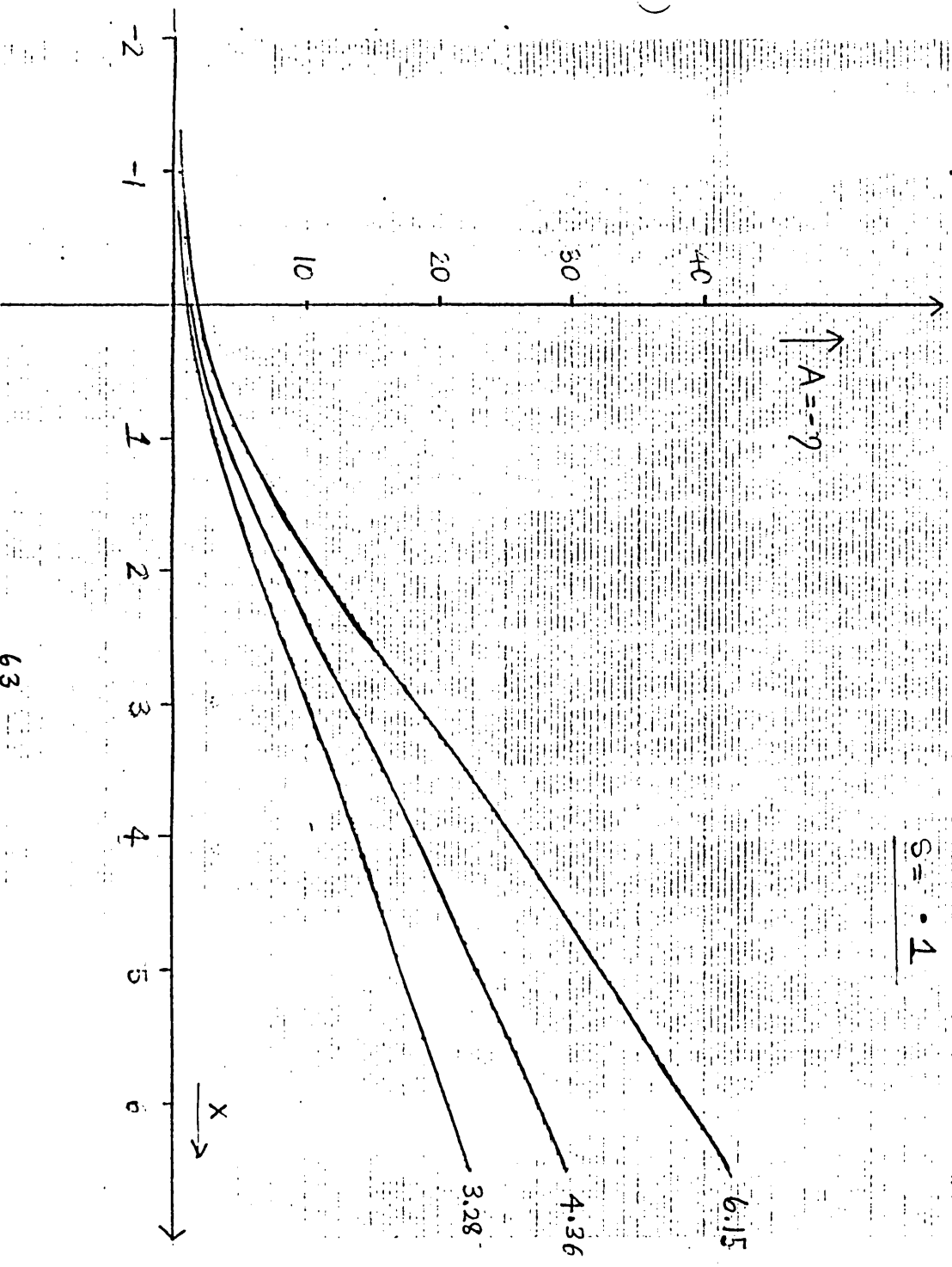
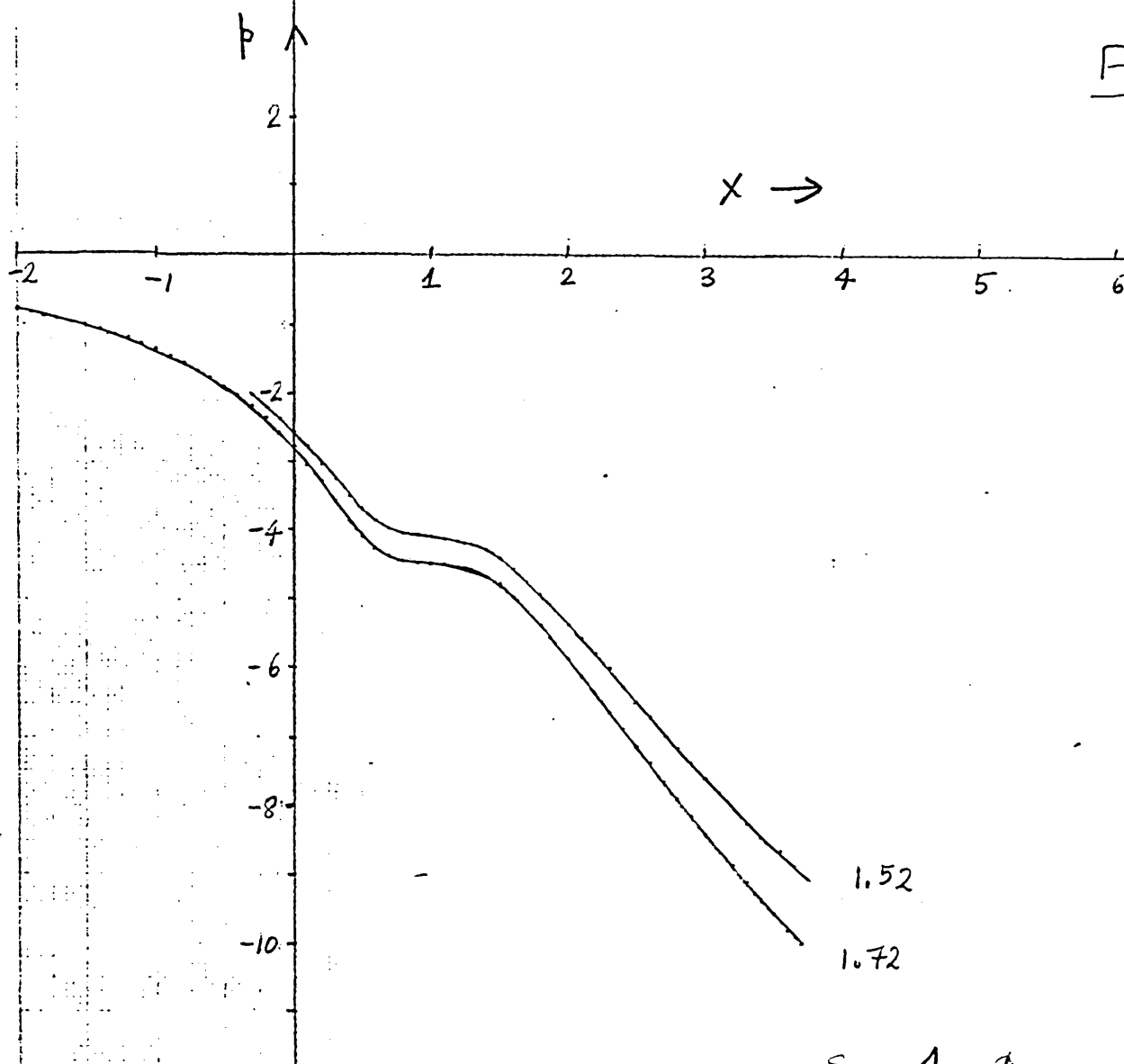


Fig. 4d



$S = 1. \phi$

Fig. 4c

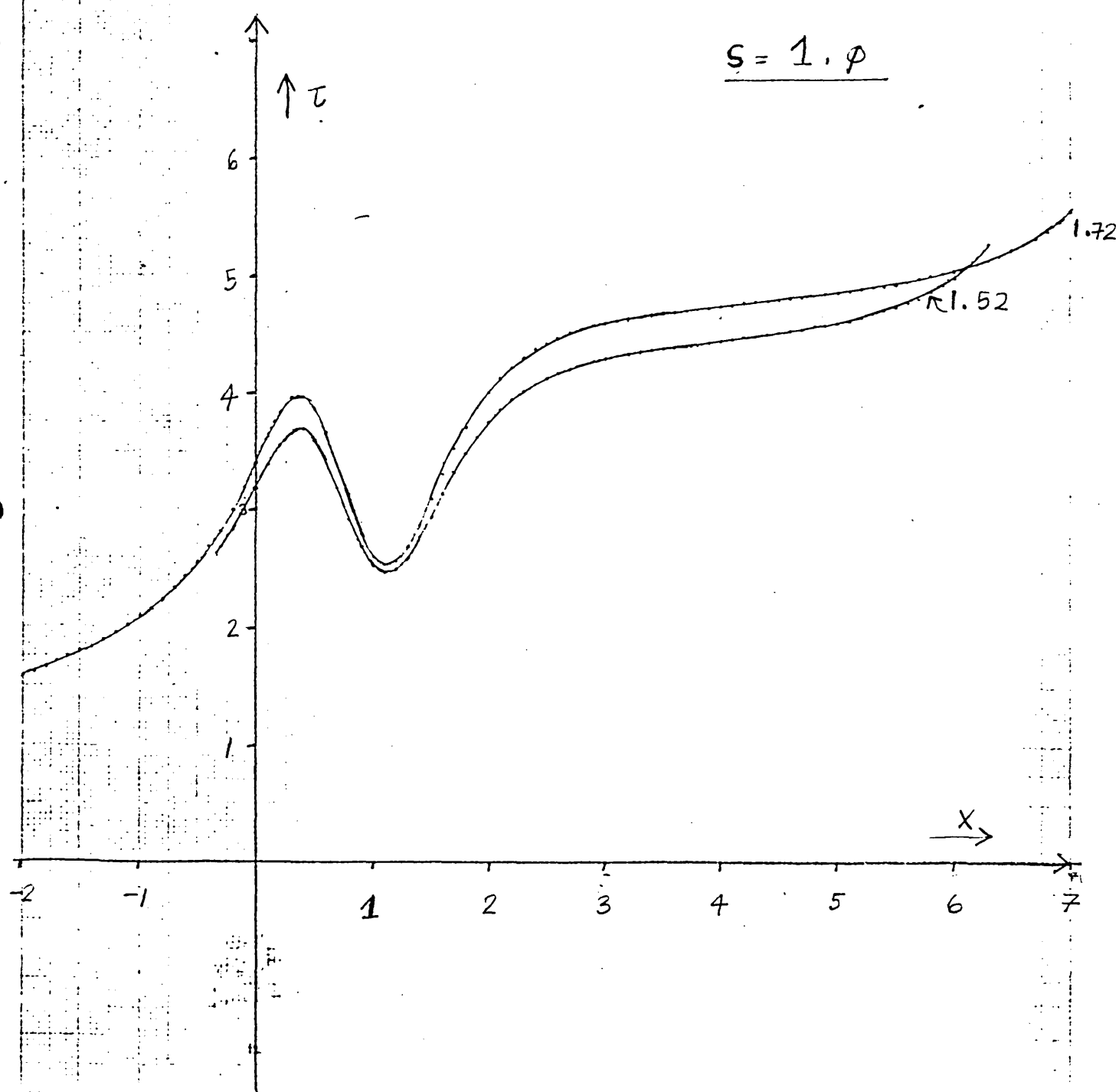


Fig. 4f

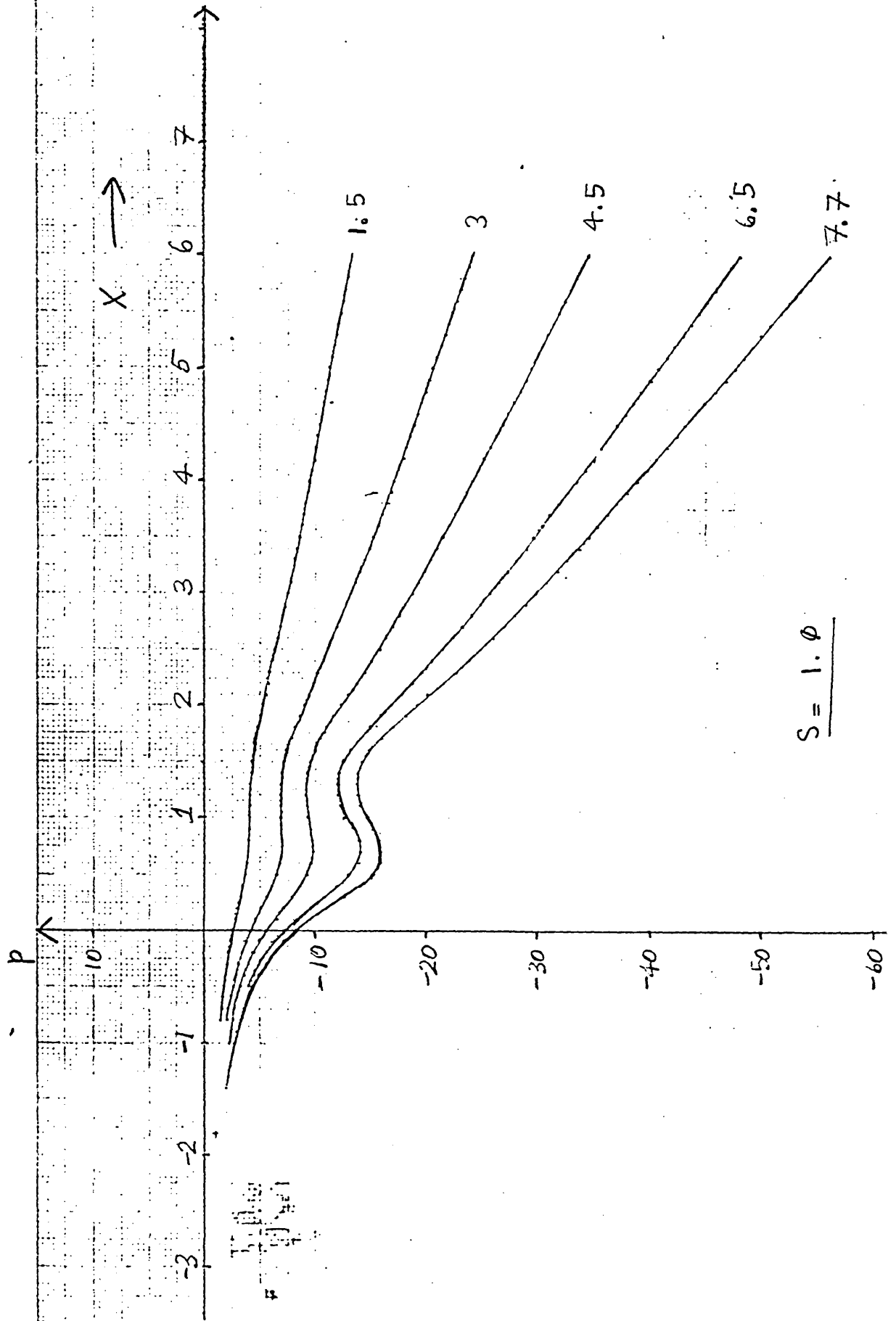


Fig. 4g

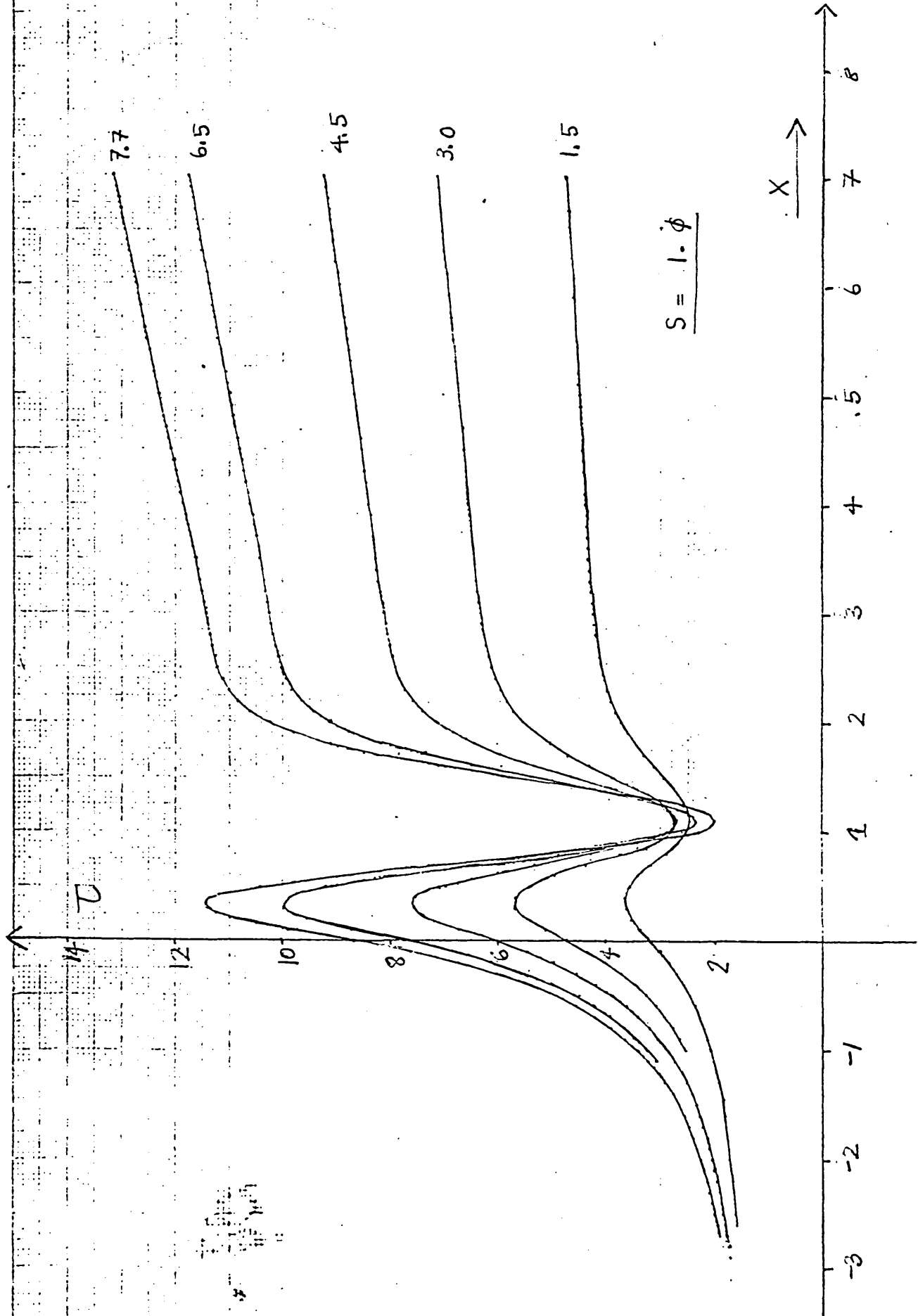


Fig. 4h

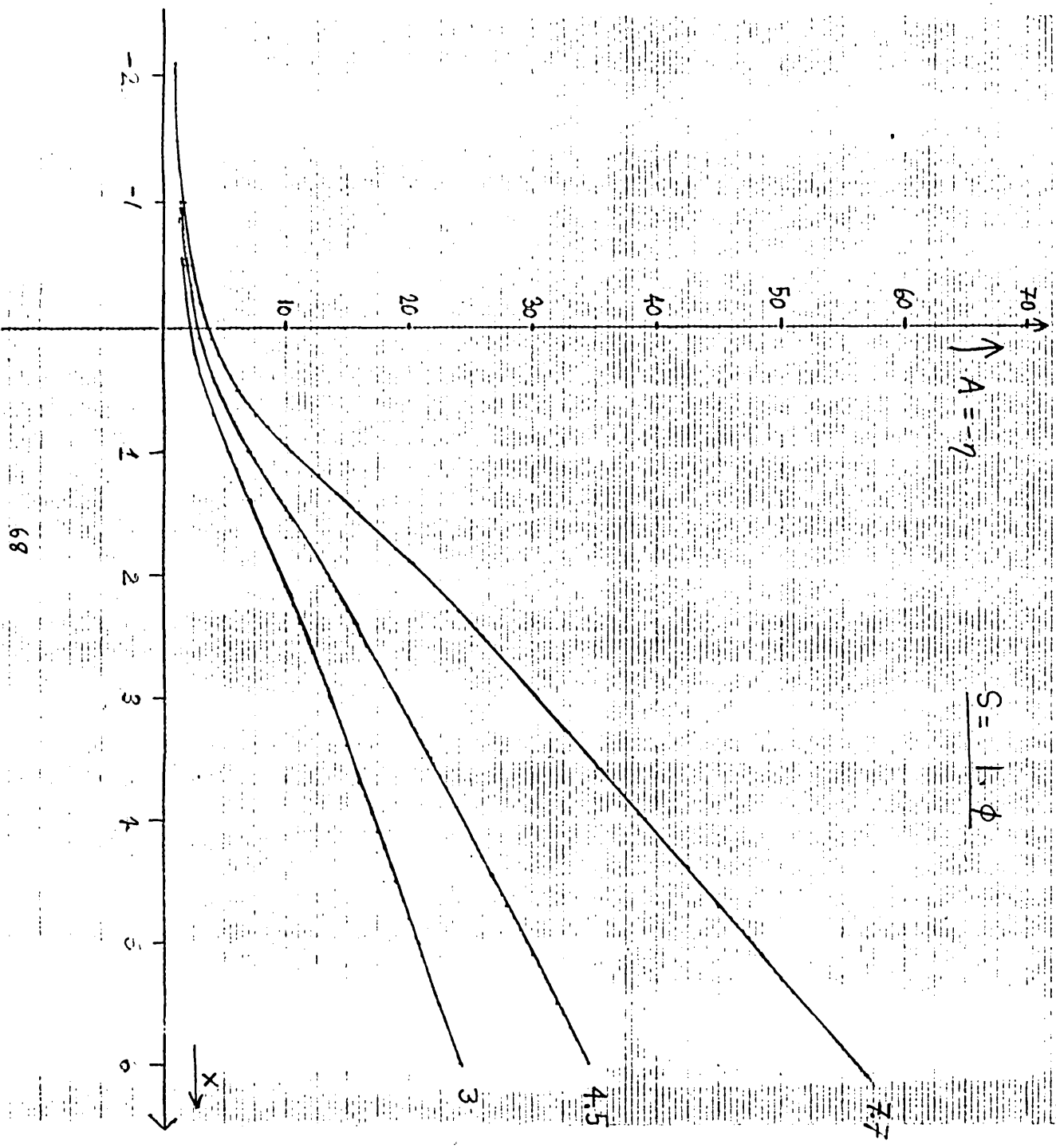
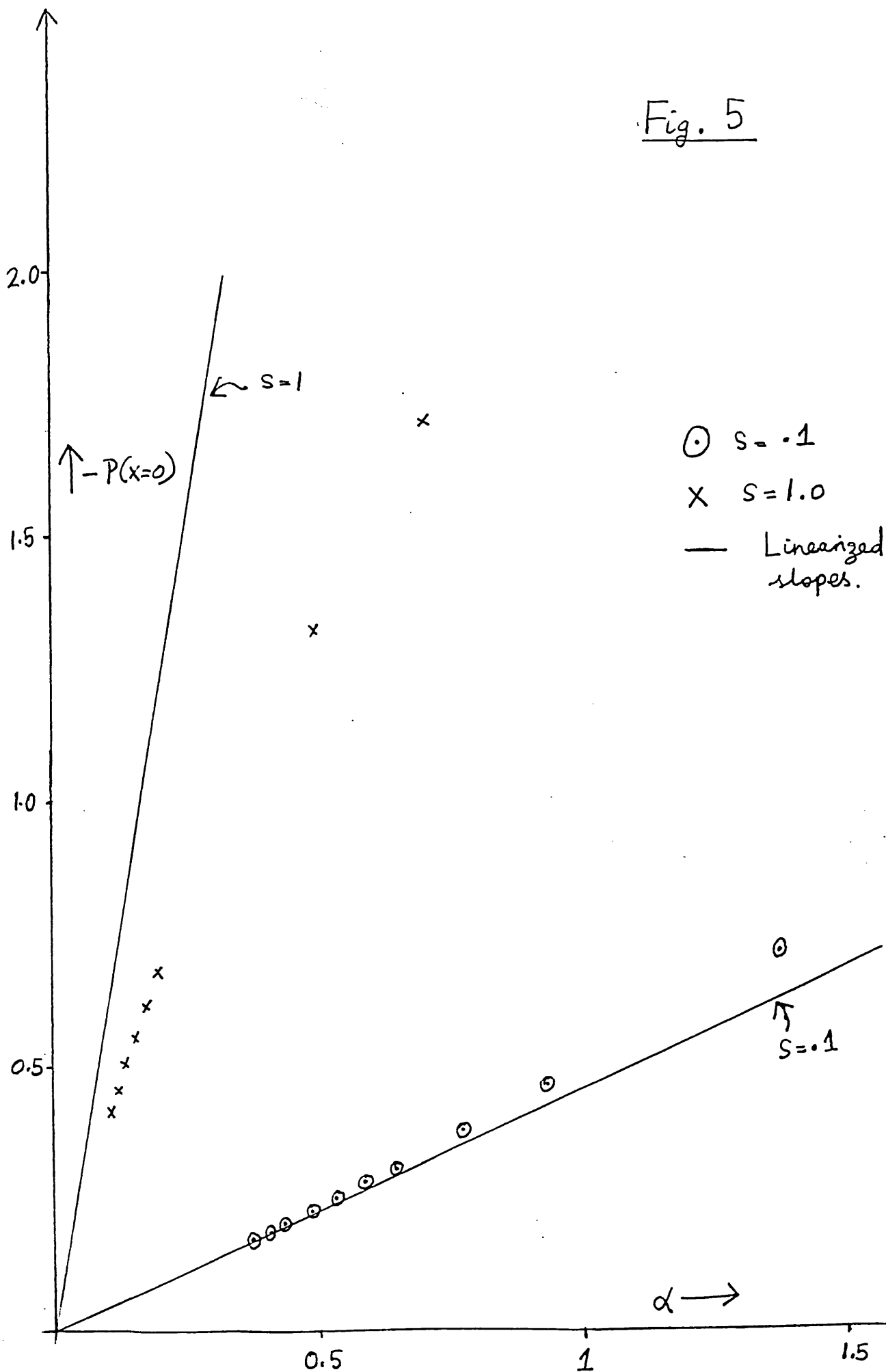


Fig. 5



Captions

Fig. 1

Sketch of the basic coordinate system and geometry.

Fig. 2(a-e).

Linearized solutions for $\tilde{P}, \tilde{A}, \tilde{\tau}$ for various values of s , see § 4.

Fig. 3.

Nonlinear solutions of (2.14-2.16) for the sharp corner $F(X) = XH(X)$, for various values of α , see § 5.

(a), (b) Pressure (P), and skin friction (τ), for $s=.1$, (M1).

(c), (d) P, τ for $s=1.0$, (M1).

Fig. 4

Nonlinear solutions of (2.14-2.16) for the smooth corner $F(X) = XTanh(X^2)$. $-A$ denotes the boundary layer displacement, P, τ as above.

(a)-(c) P, τ, A for $s=.1$, (M2).

(d), (e) P, τ for $s=1.0$, (M2)

(f)-(h) P, τ, A for $s=1.0$, (M3).

Fig. 5

Comparisons between the linearized and nonlinear solutions for small angles α .

CHAPTER 3

UNDEVELOPED FREE SURFACE FLOWS

§1 Introduction

Having considered fully developed flows in the previous chapter, we turn our attention next to undeveloped free surface flows. There is a major difference here in that the majority of the flow consists of a uniform stream with a thin boundary layer near the wall, whereas for fully developed flows the boundary layer extends right up to the free surface and viscous effects, in some sense, are more significant. This distinction between fully developed and undeveloped flows is important and manifests itself in the controlling equations and relations for the displaced free surface, induced pressure and boundary layer displacement.

The various aspects of inviscid free surface flows over submerged obstacles have received appreciable attention in the literature. An excellent concise account can be found in Lamb (1932), and Wehausen and Laitone (1960) and Peregrine (1976) who give comprehensive surveys of much of the field. Concerning the present problem, the linear inviscid theory for streaming flows past uneven beds was first considered by Kelvin (1886); see also Lamb (1932). For more recent typical work one is referred to the papers by Forbes (1981), Forbes and Schwartz (1982), and the references therein. These last two papers consider the inviscid irrotational flow over a submerged semicircular obstruction, and linear and nonlinear solutions are presented for various values of the Froude number, Fr , and the relative size of the obstruction.

All these flow studies show the marked change in character between

supercritical ($Fr > 1$), and subcritical ($Fr < 1$) flows. The rise or fall in the mean surface level depends on the Froude number and in particular for subcritical flows waves are induced immediately after the start of the obstacle. Also if the flow is nearly critical then large changes in the mean surface level are produced. For supercritical motions the upstream influence becomes enhanced.

To the best of our knowledge, very little attempt has been made to set up a rational theory to study the influence of viscosity and flow separation on the flow properties and the free surface. The question arises of whether viscous effects on free surface flows are indeed negligible. For deep fluids neglecting the effects of the bottom boundary layer may be justifiable, but for moderately deep or shallow fluids this assumption is open to contention and needs to be seriously investigated, we believe. For support one is referred to the experiments on axisymmetric hydraulic jumps by Craik et al. (1981), who found that the scale and extent of the flow separation was interrelated to the overall flow properties.

One of the aims of the present investigation is to include viscous effects in free surface flows past obstacles and to try to address some of the above questions in a more consistent fashion. Since also we want to consider the question of regular separation this suggests the use of triple-deck theory and its various extensions. For an approach based on the classical ideas i.e. of a passive boundary layer to deal with problems such as those considered by Forbes (1981), Forbes and Schwartz (1982), would necessarily fail in regions of large adverse pressure

gradients such as those generated in fig.5 of Forbes and Schwartz (1982). As in external flow (Smith (1979), Smith (1982), Peregrine (1982)) this leads one to question the validity of some of the inviscid models which fail to include possible free streamlines, although admittedly the problem would then become very difficult.

We emphasize our concern here will be with a steady flow although an exactly steady flow may not necessarily be attainable in practice. One is referred to the comments by Benjamin (1970) on upstream influence and the possibility of unsteady long-wave disturbances ahead of the obstacle in streaming flows past an obstacle affixed to the bed.

Consider then the steady motion of a uniform stream of undisturbed depth H flowing past an obstacle on an initially flat or slightly inclined bed, (see fig.1). The fluid is incompressible of density ρ and as before we ignore surface tension. We take a set of cartesian coordinates (x,y) , velocity components (u,v) , and pressure p , nondimensionalized with respect to a characteristic lengthscale L of the obstacle, velocity scale U_c , say the surface speed, and pressure ρU_c^2 . The Reynolds number $R=UL/\nu$ is large and the Froude number $Fr=U_c^2/gH$ is $O(1)$. Here ν is the kinematic viscosity of the fluid and g is the acceleration due to gravity. We note that for infinitely long obstacles we replace L by H above. Generally the flow properties will depend on the Reynolds number, the nondimensional parameter $h=H/L$, the basic skin friction, λ , (or equivalently the development lengthscale of the boundary layer), and the Froude number.

The scalings below are derived or chosen so that we can apply again the ideas of high Reynolds number flow theory to study the interaction between the boundary layer and the free surface. We have assumed again (as in chapter 2) that an infinitesimally small obstacle of $O(\delta)$ ($\delta \ll 1$) only distorts the basic flow and the free surface by an amount $O(\delta)$. We do not rule out the possibility of other, different, assumptions, of course. In fact this last assumption may not hold for small but finite disturbances. Nevertheless we take the view that by studying the small infinitesimal obstacle problem the relevant scalings for small finite obstacles should emerge as a limiting case (c.f. the supersonic ramp problem Stewartson (1974, 1981), Smith (1982)). We ignore again any possible standing wave solutions as in McIver (1982).

For fully developed flows a double-deck structure was found earlier to govern the viscous-inviscid interaction, and essentially involved ideas of internal channel flow theory. It seems quite natural and indeed obvious to try and extend and apply the ideas of external flow theory to consider undeveloped flows. In this regard we note that the triple-deck structure of Stewartson-Williams (1969), Messiter (1970), Neiland (1969), plays a crucial role in providing a rational description of regular separation taking place in streaming flows past humps, corners, etc.

Here, as in external flows, the first significant nonlinear-viscous response occurs within a triple-deck structure, although the scalings are modified because of the zero stress conditions at the free surface. In fact, the interaction lengthscales in the streamwise and lateral directions are now $O(1)$ (as in ---

developing flow in a channel Smith (1976)) instead of the more usual $R^{-3/8}$. This leads to an interactive problem for determining the free surface (see §2 below) and the boundary layer equations are coupled with ^{the} inviscid equations, enabling the boundary layer to influence the free-surface and vice-versa. As with fully developed flows, the interaction here occurs principally because the displacement of the free-surface induces streamwise and lateral pressure gradients which in turn drive the boundary-layer. However, the unknown pressure, free surface, and boundary layer displacement, are related in a much more complicated manner than before, but in the various limits, large depth, small depth, etc. they reduce to the more familiar relations for external flow and fully developed flows (see §3).

The linearized problem for small distortions of the bed is discussed in §4 and shows the much greater upstream response of the fluid occurring for supercritical flows (with the Froude number $Fr > 1$). Also for subcritical flows, waves are produced behind the obstacle, again in line with the inviscid theory (of Kelvin). The major differences between the latter and the linearized theory in §4, occur when the flow is either supercritical or nearly critical. It is emphasized that the linear theory is valid only for small obstacles, and the full problem, (2.3), with (2.5) below, governs the flow response for much larger obstacles, and this may alter some of the conclusions. Certainly the linear results for the flow past a forward

facing step show that an adverse pressure gradient is generated ahead of the step for both supercritical and subcritical flows (and ^{is} much greater in the former case). This would suggest that for larger steps separation will occur, ahead of the step. However this requires a numerical treatment of the full nonlinear problem below.

Finally in §5 further comments are made including a brief discussion on the relevance of free interactions to the hydraulic jump.

§2 The Fundamental Interactive Problem

⌈ The scalings involved are established as in triple-deck theory, or developing flow in a channel Smith (1976), see also chapter 4. The main difference here being that the streamwise and lateral dimensions are now $O(1)$. In addition, closer to the wall the main deck now has thickness $O(R^{-1/5})$, (see chapter 4 for more details), and the much thinner lower deck has thickness $O(R^{-2/5})$ to invoke a nonlinear-viscous response. If we think of the undeveloped free surface flow here as being half of a developing flow in a channel, then the scalings follow directly also

from Smith's (1976) analysis of the latter. Formally then, with $\varepsilon = R^{-1/5}$, the expansions are in the main deck, where $y = \varepsilon \bar{y}$,

$$\begin{aligned} u &= \bar{U}_0(\bar{y}) + \varepsilon A(x) \bar{U}_{0\bar{y}} + \dots \\ v &= -\varepsilon^2 A'(x) \bar{U}_0 + \dots \\ p &= sh - s\varepsilon \bar{y} + \varepsilon^2 P(x) + \dots \end{aligned} \quad (2.1)$$

Here $\bar{U}_0(\bar{y})$ is the oncoming profile with $\bar{U}_0(\bar{y}) \rightarrow 1, \bar{y} \rightarrow \infty$, $\bar{U}_0(\bar{y}) \sim \lambda \bar{y} + \dots$, $\bar{y} \rightarrow 0$, where $\lambda(>0)$ is the incident skin friction, and $-A(x), P(x)$ are the unknown induced boundary layer displacement and pressure respectively, with $A(x), P(x) \rightarrow 0$ as $x \rightarrow -\infty$. $h = H/L$ is the non-dimensional undisturbed depth and $(sh)^{-1}$ the Froude number Fr . The displacement in (2.1) feeds into the upper-deck where $y \sim O(1)$ and

$$u = 1 + \varepsilon^2 \hat{U}_0 + \dots, \quad v = \varepsilon^2 \hat{V}_0 + \dots, \quad p = s(h-y) + \varepsilon^2 \hat{P}_0 + \dots \quad (2.2)$$

Here $\hat{U}_0, \hat{V}_0, \hat{P}_0$ satisfy the linearized inviscid equations and free surface conditions i.e.

$$\nabla^2 \hat{P}_0 = 0 \quad (2.3a)$$

$$\hat{P}_0(x, h) = s\eta(x), \quad \hat{P}_{0y}(x, h) = -\eta_{xx} \quad (2.3bcde)$$

$$\hat{P}_0(x, 0) = P(x), \quad \hat{P}_{0y}(x, 0) = A_{xx}$$

where $y = h + \epsilon^2 \eta(x) + \dots$ is the free surface. (2.3de) follow from matching with the displacement from (2.1) and (2.3bc) reflect the conditions of zero normal stress and the kinematic condition respectively, at the free surface. It is noted that a much thinner $O(R^{-1/2})$ boundary layer is also required to satisfy the additional condition of zero tangential stress at the free surface. Closer to the bed (2.1) become invalid when $\bar{Y} = O(\epsilon)$ and there in the lower deck, with $y = \epsilon^2 Y$,

$$u = \epsilon U + \dots, \quad v = \epsilon^3 V + \dots, \quad p = sh - \epsilon^2 Y_s + \epsilon^2 P(x) + \dots \quad (2.4)$$

Substitution into the Navier-Stokes equations shows that (U, V, P) satisfy the usual boundary layer equations,

$$\frac{U \partial U}{\partial x} + V \frac{\partial U}{\partial Y} = - \frac{dP}{dx} + \frac{\partial^2 U}{\partial Y^2}, \quad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0 \quad (2.5a)$$

with the boundary conditions

$$U = V = 0 \text{ at } Y = dF(x) \quad (2.5b)$$

$$U \rightarrow \lambda (Y + A(x)) \quad \text{as} \quad Y \rightarrow \infty \quad (2.5c)$$

$$U \rightarrow \lambda Y \quad \text{as} \quad x \rightarrow -\infty \quad (2.5d)$$

to satisfy no slip at the bed $y = \epsilon^2 dF(x)$ in scaled terms, and to match with the main deck.

Hence the determination of the unknown pressure, displacement, and free surface requires a solution of the coupled problems (2.3), (2.5). Several remarks are in order here. Firstly, the fundamental difference between (2.3), (2.5) and the triple-deck problems of external flows, is that here the potential flow equation in the upper deck, holds in the finite region $0 < y < h$, whereas for incompressible flows, $\eta = 0$, and $h = \infty$, effectively, in (2.3bc), which then leads to the well known Cauchy-Hilbert integral relation for $P(x)$, $A(x)$. Secondly we have not, as yet, commented too much on the basic flow. We make the (reasonable) assumption (although this needs to be justified) that the basic flow is half the basic flow of developing flow in a channel. In view of Smith's (1976) analysis, the η , (2.1), (2.2), (2.4) above would need to be modified (see his (2.4), (2.5)), but this does not alter the central problem (2.3), (2.5). In addition the obstacle is required to be at a large distance $O(R^{3/5})$ from the source. Specifically $x^* = R^{3/5} x_0 L + xL$, where x is the $O(1)$ non dimensional coordinate, $x_0 \sim O(1)$ and $R^{3/5} x_0$ is ^{the} distance of the obstacle (or the start of the distortion of the bed) from the source. The incident skin friction λ is then given by $\lambda = \hat{\lambda} x_0^{-1/2}$, where $\hat{\lambda} = 0.332$.

The parameter λ occurring in (2.5) can be scaled out of the problem by writing

$$(x, Y, U, V, P, A, \eta, y, s, h) \rightarrow (\lambda^{-5/4} x, \lambda^{-3/4} Y, \lambda^{1/4} U, \lambda^{3/4} V, \lambda^{1/2} P, \lambda^{-3/4} A, \lambda^{-3/4} \eta, \lambda^{-5/4} y, \lambda^{5/4} s, \lambda^{-5/4} h) \quad (2.6)$$

eg. $\hat{x} = \lambda^{-5/4} x$ and we drop the λ .

in (2.3) (2.5), and henceforth we set $\lambda = 1$. Thus (2.6) shows that there are two independent parameters in the problem h (or ξ), and $(sh)^{-1}$ the Froude number.

We note too that the inviscid theory as discussed by Kelvin (1886), Lamb (1932) is equivalent to taking $-A$ equal to $dF(x)$ in (2.3). This greatly simplifies the problem and essentially suppresses the interaction between the boundary layer and the free surface. Finally we note that (2.3), (2.5) is a special case of the general problem in Smith's (1976) paper (if we take $s\tilde{A}(x) = \tilde{P}(x)$ in his notation).

§3 The Pressure-Displacement and Free Surface Relations in the Various Limits

The problem (2.3) may be solved formally with Fourier Transforms, giving the pressure displacement and free surface relations as

$$P^*(\alpha) = \alpha A^* \left[\frac{s - \alpha \tanh(\alpha h)}{s \tanh(\alpha h) - \alpha} \right] \quad (3.1a)$$

$$P^*(\alpha) = \left[s - \alpha \tanh(\alpha h) \right] \cosh(\alpha h) \eta^* \quad (3.1b)$$

where $g^*(\alpha) = \int_{-\infty}^{\infty} e^{-i\alpha x} g(x) dx$ defines the Fourier Transform of $g(x)$.

The expressions (3.1a b) can now be inverted and expressed as convolution integrals after suitably rearranging (3.1) to give

$$p(x) - G^A(x) = \frac{h}{\pi} \int_{-\infty}^{\infty} \frac{s\eta(\xi) + G^\eta(\xi)}{(\xi - x)^2 + h^2} d\xi \quad (3.2a)$$

$$\frac{h}{\pi} \int_{-\infty}^{\infty} \frac{p(\xi) + G^A(\xi)}{(\xi - x)^2 + h^2} d\xi = s\eta(x) - G^h(x) \quad (3.2b)$$

$$\text{where } G^A(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A'(\xi) d\xi}{\xi - x}, \quad G^h(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta'(\xi) d\xi}{(\xi - x)} \quad (3.2c,d)$$

These formulae are not particularly convenient to work with when considering the limiting cases and we will instead stick to the relations in Fourier Transform space.

For h small (3.1) gives

$$p^* = s\eta^*, \quad A^* = (sh-1)\eta^*$$

which imply

$$p(x) = s\eta(x) = \frac{s}{(sh-1)} A(x) \quad (3.3)$$

For large Froude numbers, $sh \ll 1$, these expressions agree with those obtained previously for fully developed flows (see also chapter 4). Since $h=H/L \ll 1$, the interaction lengthscale is large compared to the depth, as e.g. in the liquid layer problem. (3.3) admits eigensolutions upstream only for $sh < 1$, and hence on this much longer lengthscale, free interactions can take place for super-critical ^{flows} only.

If $h \gg 1$, $s \approx 0(1)$, $\alpha = 0(1)$, then from (3.1)

$$P^* = |\alpha| A^* \quad , \eta^* = \frac{2e^{-|\alpha|} P^*}{s - |\alpha|}$$

which on inversion yield

$$P(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A'(\xi) d\xi}{(\xi-x)} \quad , \quad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{2hP(\xi) d\xi}{h^2 + (\xi-x)^2}$$

$$- \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\eta'(\xi) d\xi}{(\xi-x)} = s\eta(x) \quad (3.4a,b)$$

The P-A relation (3.4a) is the familiar Cauchy-Hilbert integral occurring in external incompressible flow. The second expression (3.4b) is just the interactive version of the standard inviscid result connecting η and P for large depths. It is important to note that (3.4a,b) hold only for $x \approx O(1)$. For $x \gg O(h)$ (i.e. $|\alpha|h \ll 1$) the relations (3.3) are operative.

If $sh = 1$, $h \gg 1$ then (3.1) (3.2) again reduce to (3.4).

If $sh = 1$, $h \ll 1$ (3.1), (3.2) give

$$p^* \sim \frac{-3A^*}{\alpha^2 h^2} \quad , \eta^* \sim \frac{-3A^*}{\alpha^2 h^2}$$

which yield

$$A = \frac{h^3}{3} P'' \quad , \quad A = \frac{h^2 \eta''}{3} \quad (3.5a,b)$$

and no eigensolutions exist upstream. $sh = 1$ corresponds to $U_c = \sqrt{gH}$, and this is the velocity with which long waves travel on a fluid of

finite depth (from an inviscid analysis), see Lamb (1932).

§4 Linearized Solution for $|d| \ll 1$

The full problem (2.3) (2.5) poses a formidable numerical task which has still to be tackled seriously. In this section we consider the linearized solution valid strictly for $|d| \ll 1$.

If we write

$$(U, V, P, A, \eta) = (Y, 0, 0, 0, 0) + d(\tilde{U}, \tilde{V}, \tilde{P}, \tilde{A}, \tilde{\eta}) + O(d^2) \quad (4.1)$$

then $\tilde{U}, \tilde{V}, \tilde{P}$ satisfy the linearized version of (2.5)

$$Y \tilde{U}_x + \tilde{V} = -\tilde{P}_x + \tilde{U}_{YY}, \quad \tilde{U}_x + \tilde{V}_Y = 0 \quad (4.2)$$

with the boundary conditions,

$$\tilde{U} = -F(x), \quad \tilde{V} = 0 \quad \text{at} \quad \tilde{Y} = 0 \quad (4.3a)$$

$$\tilde{U} \rightarrow \tilde{A} \quad \text{as} \quad Y \rightarrow \infty \quad (4.3b)$$

$$\tilde{U}, \tilde{V}, \tilde{P}, \tilde{A}, \tilde{\eta} \rightarrow 0 \quad \text{as} \quad x \rightarrow -\infty. \quad (4.3c)$$

As usual the solution of (4.2) (4.3) is obtained using Fourier Transforms, and we find

$$\tilde{P}^* = - \frac{\alpha F^*(\alpha) (s - \alpha \tanh(\alpha h))}{\square} \quad (4.4a)$$

$$\tilde{\eta}^* = \frac{-\alpha F^*(\alpha)}{\cosh(\alpha h)} \square \quad (4.4b)$$

$$\tilde{A}^* = \frac{-(s \tanh(\alpha h) - \alpha) F^*(\alpha)}{\square} \quad (4.4c)$$

$$\frac{(\tilde{U}_y^* - 1)_{y=0}}{d} = \tilde{U}^* = \frac{-\alpha F^*(\alpha) (i\alpha)^{2/3} Ai(0) (s - \alpha \tanh(\alpha h))}{Ai'(0) \square} \quad (4.4d)$$

$$\square(\alpha, s, sh) = \alpha (i\alpha)^{1/3} \gamma (s - \alpha \tanh(\alpha h)) + s \tanh(\alpha h) - \alpha \quad (4.4e)$$

where $\gamma = (-3Ai'(0))^{-1} = 1.2878$, and $Ai(x)$ is Airy's function.

We take a branch cut along the positive imaginary axis.

In order to invert the above expressions we first need to know the zeros of the denominator $\square$ occurring in (4.4a-d). The properties of ^{the} equation $\square = 0$ are discussed in the appendix and in particular we note that

$$\square(\alpha, s, sh) = 0 \quad (4.5)$$

has an infinite set of simple zeros along the negative imaginary axis. If $sh \geq 1$, the roots α_n , are at $\alpha_n h = -i\theta_n$ $n = 1, 2, 3, \dots, (n - \frac{1}{2})\pi < \theta_n < (n + \frac{1}{2})\pi$. If $sh < 1$, there is an additional root α_0 at $\alpha_0 h = -i\theta_0$ where $0 < \theta_0 < \frac{\pi}{2}$. For $\text{Im}(\alpha) > 0$ there exists an infinite set of complex conjugate roots of (4.5).

The expressions (4.4a-d) can now be inverted as a convolution for a general $F(x)$, although to be more specific we will take the case when $F(x) = H(x)$ where $H(x)$ is the Heaviside unit step function.

(4.4a-d) and the above comments now show that

$x < 0$:

$$\tilde{P}(x) = \sum_{n=0}^{\infty} \frac{(\zeta + \frac{\theta_n}{h} \tan(\theta_n)) e^{\frac{\theta_n x}{h}}}{B_{n1}}$$

$$\tilde{\eta}(x) = \sum_{n=0}^{\infty} \frac{e^{\frac{\theta_n x}{h}}}{B_{n2}}$$

(4.6a-d)

$$\tilde{A}(x) = \sum_{n=0}^{\infty} \frac{(sh \tan \theta_n - \theta_n) e^{\frac{\theta_n x}{h}}}{\theta_n B_{n1}}$$

$$\frac{-A_i'(0) \tilde{\tau}(x)}{A_i(0)} = - \sum_{n=0}^{\infty} \frac{\theta_n^{2/3}}{h^{5/3}} \frac{(sh + \theta_n \tan \theta_n)}{B_{n1}} e^{\frac{\theta_n x}{h}}$$

where the first term in the summation is omitted if $sh \geq 1$. Here

$$B_{n1} = \frac{4}{3} \left(\frac{\theta_n}{h}\right)^{1/3} \gamma\left(\zeta + \frac{\theta_n}{h} \tan \theta_n\right) + \left(\frac{\theta_n}{h}\right)^{4/3} \gamma \left[\tan \theta_n + \theta_n \sec^2 \theta_n \right]$$

$$+ sh \sec^2 \theta_n - 1 \quad n=0, 1, 2, \dots$$

$$B_{n2} = \frac{4}{3} \left(\frac{\theta_n}{h}\right)^{1/3} \gamma(\sec \theta_n + \frac{\theta_n}{h} \sin \theta_n)$$

$$\begin{aligned}
& + \left(\frac{\theta_n}{h}\right)^{4/3} \gamma \left[(1-sh) \sin \theta_n + \theta_n \cos \theta_n \right] \\
& + (sh-1) \cos \theta_n + \theta_n \sin \theta_n \quad n=0,1,2,\dots \quad (4.7a-b)
\end{aligned}$$

For $X > 0$ assuming that the complex roots are simple and given by

$$i\alpha_n = \frac{r_n}{h} e^{\pm i\phi_n}, \quad \pi/2 < \phi_n < \pi \quad n=0,1,2,\dots$$

If $sh \neq 1$

$$\tilde{P}(x) = 2\text{Re} \left[\sum_{n=0}^{\infty} \frac{(s + \frac{r_n}{h} e^{i\phi_n} \text{Tan}(r_n e^{i\phi_n})) e^{\frac{x r_n}{h} e^{i\phi_n}}}{C_{n1}} \right]$$

$$- \frac{s}{(sh-1)} + \frac{\sqrt{3}}{2\pi} \int_0^{\infty} \frac{e^{-zx} (s+z \text{Tan}(zh))^2 z^{4/3}}{\gamma e^{i\pi/3} \{ |z^{4/3} \gamma e^{i\pi/3} (s+z \text{Tan}(zh)) + s \text{Tan}(zh) - z|^2 \}} dz,$$

$$\tilde{n}(x) = 2\text{Re} \left[\sum_{n=0}^{\infty} \frac{e^{\frac{x r_n}{h} e^{i\phi_n}}}{C_{n2}} \right] - \frac{1}{(sh-1)},$$

$$+ \frac{\sqrt{3}}{2\pi} \int_0^{\infty} \frac{e^{-zx} [s+z \text{Tan}(zh)] z^{4/3}}{\gamma e^{i\pi/3} \{ |z^{4/3} \gamma e^{i\pi/3} (s+z \text{Tan}(zh)) + s \text{Tan}(zh) - z|^2 \} \cos(zh)} dz$$

$$\tilde{A}(x) = -2\text{Re} \left[\sum_{n=0}^{\infty} \frac{(\text{sh Tan}(r_n e^{i\phi_n}) - r_n e^{i\phi_n}) e^{\frac{r_n x e^{i\phi_n}}{h}}}{r_n e^{i\phi_n} C_{n1}} \right]$$

$$-1 + \frac{\sqrt{3}}{2\pi} \gamma \int_0^{\infty} e^{-zx} \frac{z^{1/3} (s \text{ Tan}(zh) - z) (s + z \text{ Tan}(zh)) dz}{\{ |z^{4/3} e^{i\pi/3} \gamma (s + z \text{ Tan}(zh)) + s \text{ Tan}(zh) - z|^2 \}}$$

$$\frac{-A_i'(0)}{A_i(0)} \tilde{u}(x) = 2 \text{Re} \left[\sum_{n=0}^{\infty} \frac{(\frac{r_n}{h})^{2/3} e^{2i\phi_n/3} (s + \frac{r_n}{h} e^{i\phi_n} \text{Tan}(r_n e^{i\phi_n})) e^{\frac{r_n x e^{i\phi_n}}{h}}}{C_{n1}} \right]$$

$$+ \frac{\sqrt{3}}{2\pi} \int_0^{\infty} \frac{e^{-zx} z^{2/3} (s + z \text{ Tan}(zh)) (z^{4/3} \gamma (s + z \text{ Tan}(zh)) + s \text{ Tan}(zh) - z) dz}{\{ |z^{4/3} e^{i\pi/3} \gamma (s + z \text{ Tan}(zh)) + e^{2\pi i/3} (s \text{ Tan}(zh) - z)|^2 \}}$$

(4.8a-d)

If sh=1, then

$$\tilde{P}(x) = 2\text{Re} \left[\sum_{n=0}^{\infty} \frac{-(s + \frac{r_n}{h} e^{i\phi_n} \text{Tan}(r_n e^{i\phi_n})) e^{\frac{x r_n e^{i\phi_n}}{h}}}{C_{n1}} \right]$$

$$+ \tilde{p}(0) + \frac{\sqrt{3}}{2\pi} \gamma \int_0^{\infty} \frac{(1 - e^{-zx}) z^{4/3} (s + z \text{ Tan}(zh))^2 dz}{\{ |z^{4/3} \gamma e^{i\pi/3} (s + z \text{ Tan}(zh)) + s \text{ Tan}(zh) - z|^2 \}}$$

$$\tilde{\eta}(x) = 2 \operatorname{Re} \left[\sum_{n=0}^{\infty} \frac{e^{\frac{x \Gamma_n}{h}} e^{i \phi_n}}{C_{n2}} \right]$$

$$+\tilde{\eta}(0) + \frac{\sqrt{3}\gamma}{2\pi} \int_0^{\infty} \frac{(1-e^{-zx}) [s+Z \tan(zh)] z^{4/3} dz}{\{|z|^{4/3} e^{i\pi/3} \gamma(s+Z \tan(zh)) + s \tan(zh) - z\}^2}$$

$$\tilde{A}(x) = \hat{A}(x)+1$$

$$\tilde{\tau}(x) = \hat{\tau}(x) \quad . \quad (4.9a-d)$$

Here $\hat{A}(x)$, $\hat{\tau}(x)$ are the same expressions as in (4.8ad); C_{n1} , C_{n2} are identical to B_{n1} , B_{n2} except that θ_n in B_{n1} , B_{n2} is replaced by $r_n e^{i\phi_n}$ to obtain C_{n1} , C_{n2} respectively; Re denotes the real parts of the expressions.

$\gamma = 0$ in (4.6ab) (4.7ab) effectively gives the inviscid solution of Kelvin (1886) for the pressure and the free surface (the summations occurring in the expression for $x > 0$ are modified because the branch cut disappears). The form $F(x) = H(x) \tanh(x^2)$ was chosen, and the expressions (4.4a-d) were inverted directly using the Fast Fourier Transform. The results are shown in figs. (2a-d). There are several interesting features present here which are worth mentioning. (i) Firstly for $x < 0$, the linearized solutions for both

supercritical and subcritical flows have the familiar exponential form which occurs in supersonic flows, channel flows, liquid layer flows etc., and is responsible for any free interactions taking place in these flows, (the exponential factor is essentially the eigensolution occurring for upstream). The pertinent question then is do free interactions occur here, and this is discussed further in §5.

(ii) The upstream influence occurring for both subcritical and supercritical flows is much greater in the latter case (corresponding to the extra root in $\text{Im}(\alpha) < 0$, above). (iii) Large changes in the mean surface level are produced when the flow is nearly critical.

(iv) For $Fr = 1$, the pressure and the free surface become unbounded for large x , whereas the boundary layer displacement oscillates and falls to zero, a surprising feature since for $Fr \neq 1$, $\tilde{A} \rightarrow 1$ for large x , and the skin friction also oscillates and tends to its initial

value of unity. (v) For subcritical flows, with $Fr \sim 1$, the solution for $\tilde{\tau}$ and $\tilde{A}$, suggest that the fluid is being continually accelerated and decelerated as it negotiates the step. (vi) For $h \gg 1$,

(subcritical flows) waves are produced behind the step, as in inviscid theory. Also note that in contrast to the flow over a forward^{facing} step in external flow (Smith-Merkin (1982)), the fluid returns to its uniform state $U = Y$, $A = -1$, slowly downstream, but the pressure tends to a nonzero constant. This quite clearly shows the influence of the free surface, since it is the pressure induced by the change in the surface level which feeds into and drives the boundary layer.

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In summary then, for both supercritical and subcritical flow over the step, the pressure rises ahead of the step, and the fluid is retarded there. Downstream, the pressure falls and tends to a constant value (if $Fr \neq 1$), while the fluid is sharply accelerated just ahead of the step, and the skin friction slowly returns to its initial value of unity. Also the free surface falls or rises according as whether the flow is subcritical or supercritical. Waves are produced in the former case and die down ultimately. The boundary layer displacement rises ahead of the step. In supercritical flow it overshoots and then falls downstream, while for subcritical flows it is more oscillatory but is increasing to its constant value of 1.

Finally far downstream, the following asymptotes are implied from (4.4(a-d)). (For $F(x) = H(x)$)

$$sh \neq 1$$

$$\tilde{\eta} \sim \frac{-1}{(sh-1)} + \frac{sy}{(sh-1)^2} x^{-1/3} \Gamma(2/3) + \dots$$

$$\tilde{p} \sim \frac{-s}{(sh-1)} + \frac{s^2 y}{(sh-1)^2} x^{-1/3} \Gamma(2/3) + \dots$$

$$\tilde{A} \sim -1 + \frac{s\gamma}{(sh-1)\Gamma(2/3)} x^{-1/3} + \dots$$

$$\frac{-A_i'(0)\tilde{\tau}}{A_i(0)} \sim \frac{s x^{-2/3}}{(sh-1)\Gamma(1/3)} + \dots$$

$$sh = 1$$

$$\tilde{\eta} \sim \frac{-x^{1/3}}{s\gamma\Gamma(4/3)} + \frac{h^3 x^{-4/3}}{3\gamma^2 s\Gamma(-1/3)} + \dots$$

$$\tilde{p} \sim \frac{-x^{1/3}}{\gamma\Gamma(4/3)} + \frac{h^3 x^{-4/3}}{3\gamma^2\Gamma(-1/3)} + \dots$$

$$\tilde{A} \sim \frac{-h^3 x^{-5/3}}{3\gamma\Gamma(-2/3)} + \dots$$

$$\frac{-A_i'(0)\tilde{\tau}}{A_i(0)} \sim \frac{-x^{-1/3}}{\gamma\Gamma(2/3)} + \dots \quad (4.11a-d)$$

§5 Further Comments

Returning to the point mentioned in §4 concerning the existence of the eigensolutions, if we write

$$U = Y + e^{\sigma x} g'(Y) + \dots, \quad V = -\sigma e^{\sigma x} g(y),$$

$$\tilde{p}_0 = \tilde{p}_0(Y) e^{\theta x}, \quad P = \tilde{P} e^{\theta x}, \quad A = \tilde{A} e^{\theta x}, \quad \eta = \tilde{\eta} e^{\theta x} \quad (5.1)$$

with x large and negative ($\sigma > 0$), and substitute into (2.3a-e) (2.5a) (with $F(x) = 0$), neglecting terms of $O(e^{2\sigma x})$, a solution for $g(y)$ exists with $\tilde{p}_0, \tilde{A}, \tilde{\eta}$ undertermined provided σ satisfies the equation,

$$\square (-i \sigma, s, sh) = 0 \quad . \quad (5.2)$$

As shown in the appendix, (5.2) has simple zeros given by

$$\sigma = \sigma_n = \frac{\theta_n}{h}.$$

In contrast to e.g. supersonic flows, this result however does not directly imply that eigensolutions exist. There, the pressure and displacement are related locally, enabling the boundary layer equations to be marched forward given the initial form (5.1), whereas here the integral relations quite clearly demonstrate the elliptic nature of the problem. In general then the above question can only be addressed numerically akin to the problem of incompressible breakaway separation (Smith (1977)). It is conjectured that

eigensolutions and hence free interactions, do exist, and take the form (5.1) as $x \rightarrow -\infty$.

Physically the hydraulic jump is one obvious example of a free interaction taking place: in supercritical flow the fluid adjusts locally in anticipation of a severe obstacle downstream, by changing to a much slower flow with a raised surface level. It is worth stating some of the above results which are relevant to the hydraulic jump problem. Firstly much bigger upstream influence occurs for supercritical flow. If h is large, the interaction length scale, ($\propto L$), is shortened with inviscid and external flow properties dominating; if h is small then the interaction occurs over a much longer lengthscale, (see also chapter 4 where the free interaction problem with the laws (3.3) is discussed). If the flow were to return to a uniform state downstream (with $P = 5\eta_0$, $\eta = \eta_0$, $A = 0$, η_0 constant) then separation would also imply reattachment, with the extent of the reversed flow, $\propto$ dependent on the interaction lengthscale. These conclusions tie in with the experimental findings of Craik et al. (1982), on their experiments on axisymmetric jumps.

On another matter, the linearized solution breaks down when $sh \rightarrow 1\pm$. For $sh \rightarrow 1-$, the first root θ_0 , approaches the origin ($\theta_0 \sim (1-sh)^3$) with $B_{01}, B_{02} \propto (sh-1)$, and the situation is similar to that in chapter 2, in the liquid layer problem. This suggests that the nonlinear terms in (2.5) will become important ahead of the step. Similar conclusions also apply to the limit $sh \rightarrow 1+$, but

nonlinearly will become important first after the start of the step. For $sh = 1$ on the other hand, the linearized results show that for $x \gg 1$, the influence of the boundary layer diminishes. Instead since $\tilde{u}_{0x}(X,0) = -\tilde{p}_0'(x,0)$ in the upper deck, (and $\tilde{p} \sim -x^{1/3}$), the nonlinear terms cannot be neglected there far downstream.

It is interesting also to compare the asymptotic forms (4.10a-d) downstream with those from linear inviscid theory (for $sh \neq 1$). The first term in (4.10) is the same in both cases, but the next few terms show clearly the contributions from viscous effects (and which are important for $sh \sim 1$).

The full nonlinear problem (2.3) (2.5) merits further attention, for both $O(1)$, and bigger wall shapes. Certainly the latter should indicate the trend in the solution for physically realistic wall profiles, and also give an indication of whether free interactions occur or not.

Finally the analysis presented can be immediately generalised and extended to include three-dimensionality, surface tension, etc. An important problem is to investigate the interaction between submerged obstacles and the bed. If the obstacle is relatively small (of $O(R^{-2/5})$) then the problem (2.3) (2.5) is unchanged apart from the additional tangential flow constraint on the body, in the potential flow region, (see also Smith (1976)). However, even for simple bodies, the finite depth, inviscid problem is difficult both analytically and numerically.

In summary, under the various assumptions stated earlier, we have shown that an obstacle of relative height $O(R^{-2/5})$ placed at a distance of $O(R^{3/5})$ from an upstream source will provoke a nonlinear viscous-inviscid interaction and this is expressed by the coupled problems (2.3), (2.5). The nature of the interaction depends on the Froude number and the ratio $h=(H/L)$, of the depth to a characteristic streamwise lengthscale. If $h \gg 1$ for subcritical flows, the interaction is somewhat diminished and takes a form similar to that in subsonic flow. If $h \ll 1$ and the flow is supercritical then the boundary layer influence is very much enhanced. In fact the linear solution investigated here suggests that boundary layer separation is much more likely to occur in a supercritical flow rather than a subcritical flow. But a sufficiently large object affixed to the bed may well cause flow separation to occur in both supercritical and subcritical flows since an adverse pressure gradient is generated ahead of the obstacle.

CHAPTER 3

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APPENDIX A

In this appendix we derive some results concerning some roots of the equation

$$\square (\alpha, s, sh) \equiv \alpha(i\alpha)^{1/3} \gamma (s - \alpha \tanh(\alpha h)) + s \tanh(\alpha h) - \alpha = 0$$

where $(i\alpha)^{1/3}$ has a branch cut along the positive imaginary axis.

We will make use of the result from complex analysis that if $f(z)$ is analytic inside and on a closed contour C , apart from a finite number of poles P , and zeros N , inside C and is not zero on the contour, then

$$\frac{1}{2\pi} \Delta_c (\arg f(z)) = N - P \quad (A1)$$

where $\Delta_c (\arg f(z))$ is the variation in the argument of the function f around C , see e.g. Titchmarsh pp115.

First we write $ih\alpha = U$ in (4.4e), and consider the variation in the argument of the function

$$f(U) = Uh \cdot \left[\frac{h^{1/3} - \gamma s U^{1/3}}{sh^{2/3} + u^{2/3} \gamma} \right] - \tanh U$$

around a closed contour C_A consisting of a small semicircle C_δ

of radius δ ($\delta \ll 1$) around the origin, and the straight line paths L_1, L_2, L_3, L_4, L_5 , as shown in fig. (3a).

$$\text{On } C_\delta, U = \delta e^{i\theta}, \quad -\pi/2 < \theta < \pi/2$$

$$f(U) \sim \delta e^{i\theta} \left[\frac{1-\text{sh}}{\text{sh}} \right] - \delta^{4/3} e^{4i\theta/3} \frac{\gamma}{h^{4/3}} + O(\delta^3) \quad (A2)$$

$$\text{Hence if } \text{sh} \neq 1 \quad \Delta_{C_\delta} (\arg f(Z)) = -\pi. \quad (A3a)$$

$$\text{If } \text{sh} = 1 \quad \Delta_{C_\delta} (\arg f(Z)) = -\frac{4\pi}{3}. \quad (A3b)$$

$$\text{On } L_1 \quad U = Ze^{i\pi/2} \quad \delta \leq Z \leq n\pi \quad (1 \ll n)$$

$$\text{Re}(f) = \frac{-Z^{4/3} h^{4/3} \gamma [Z^2 - s^2 h^2]}{|\text{sh}^{7/3} + Z^{7/3} e^{i7\pi/6} \gamma|^2} \quad (A4a)$$

$$\text{Hence } \text{Re}(f) \text{ changes from being negative for large } Z \text{ to positive for small } Z. \text{ Also } \text{Im}(f(\text{sh} e^{i\pi/2})) = i(1 - \text{Tanh}(\text{sh})) \quad (A4b)$$

$$\text{and } f(Ze^{i\pi/2}) \sim -i \quad (Z \gg 1) \quad (A4c)$$

A(2, 3, 4) together show that

$$\begin{aligned}
 \Delta_{L_1}(\arg f) &= -\pi & \text{sh} < 1 \\
 &= -2\pi & 1 < \text{sh} \\
 &= -\frac{11\pi}{6} & \text{sh} = 1
 \end{aligned} \quad (A5a-b)$$

Similarly on L_2 , where $u = ze^{-i\pi/2}$ $\delta \leq z \leq n\pi$

$$\begin{aligned}
 \Delta_{L_2}(\arg f) &= -\pi & \text{sh} < 1 \\
 &= -2\pi & 1 < \text{sh} \\
 &= -\frac{11\pi}{6} & \text{sh} = 1
 \end{aligned} \quad (A6a-c)$$

On L_3 $u = z - in\pi = R_n e^{i\phi_n}$ where

$$\delta \leq z \leq n\pi, \quad n\pi \leq R_n = (z^2 + n^2\pi^2)^{1/2} \leq \sqrt{2}n\pi, \quad -\pi/2 \leq \phi_n \leq \pi/4$$

$$f(u) = i + O\left(\frac{1}{R_n}\right)$$

Hence $\Delta_{L_3}(\arg f) = 0$ (A7a)

$$\text{Similarly } \Delta_{L_5}(\arg f) = 0 \quad (A7b)$$

$$\text{On } L_4, U = n\pi + iz = R_n e^{i\phi_n}$$

$$\text{where } -n\pi \leq z \leq n\pi, n\pi \leq R_n \leq \sqrt{2}n\pi, -\pi/4 \leq \phi_n \leq \pi/4$$

$$f(u) \sim \frac{-sh}{R_n} e^{-i\phi_n} - i \tanh z$$

$$\text{Hence } \Delta_{L_4}(\arg f) = \pi \quad (A8)$$

Combining the above results together (A3, A5, A6, A7, A8) we get

$$\begin{aligned} \Delta_{C_n}(\arg f) &= -2\pi & sh < 1 \\ &= -4\pi & sh \geq 1 \end{aligned} \quad (A9)$$

Now in $0 < \operatorname{Re}(U) < n\pi$, $f(U)$ has n poles where $\cos U = 0$, and 2 poles where $sh^{7/3} + U^{7/3} \gamma = 0$. This together with (A1, A9) then gives

$$\begin{aligned} \text{no. of zeros in } C_n \text{ of } f(U) &= n + 1 \text{ if } sh < 1 \\ &= n \text{ if } sh \geq 1 \end{aligned} \quad (A10)$$

Since the zeros of $f(u)$ are identical to those of $\square(-\frac{iu}{h}, s, sh)$,

we obtain the result that $\square(\alpha, s, sh)$ has an infinite set of zeros in $\text{Im}(\alpha) < 0$. Furthermore, if $sh \geq 1$, then the zeros are at $h\alpha_n = -i\theta_n$, $(n-1)\pi < \theta_n < (n+\frac{1}{2})\pi$, $(n=1, 2, \dots)$, and if $sh < 1$ there is an additional zero at $h\alpha_c = -i\theta_c$, $0 < \theta_c < \pi/2$. This follows from (A10) and from graphical considerations of the function $f(u)$. The leading zeros for several values of sh , h are given in Table (3.1).

Results for the zeros of $\square(\alpha, s, sh)$ in $\text{Im}(\alpha) > 0$ are derived similarly, although we will just state the result. If we consider the variation in the argument of the function $f_2(u')$, where

$$f_2(u) = Uh \cos u \left[h^{1/3} - s\gamma u^{1/3} \right] - \sin u \left[sh^{7/3} + u^{7/3}\gamma \right]$$

taken around the contour C'_n consisting of a small circle C'_δ around the origin, and the straight line paths $L_4, L'_1, L'_2, L'_3, L'_5, L'_6, L'_7$, as shown in fig. (3b), then

$$\begin{aligned} \Delta_{c,n}(\arg f) &= (6n + 4)\pi && \text{if } sh \geq 1 \\ &= (6n + 6)\pi && sh < 1 \end{aligned} \quad (A11ab)$$

This together with (A10) now gives the result that, the number of zeros of $f(u)$ in $U < 0 = 2n + 2$. Hence in $\text{Im}(\alpha) > 0$

$\square(\alpha, s, sh)$ has an infinite (countable) set of complex conjugate roots.

The following additional properties of the roots in $\text{Im}(\alpha) < 0$ for various limits, are readily established;

$$\theta_0 = \frac{h}{(s\gamma)^3} + O(h^2) \quad \text{as } h \rightarrow 0 ,$$

$$\theta_0 = \left(\frac{1-sh}{\gamma}\right)^3 h^4 \quad \text{for } sh \sim 1^- ,$$

$$\theta_1 \sim \pi \left(1 + \frac{1}{sh}\right) \quad h \gg 1 ,$$

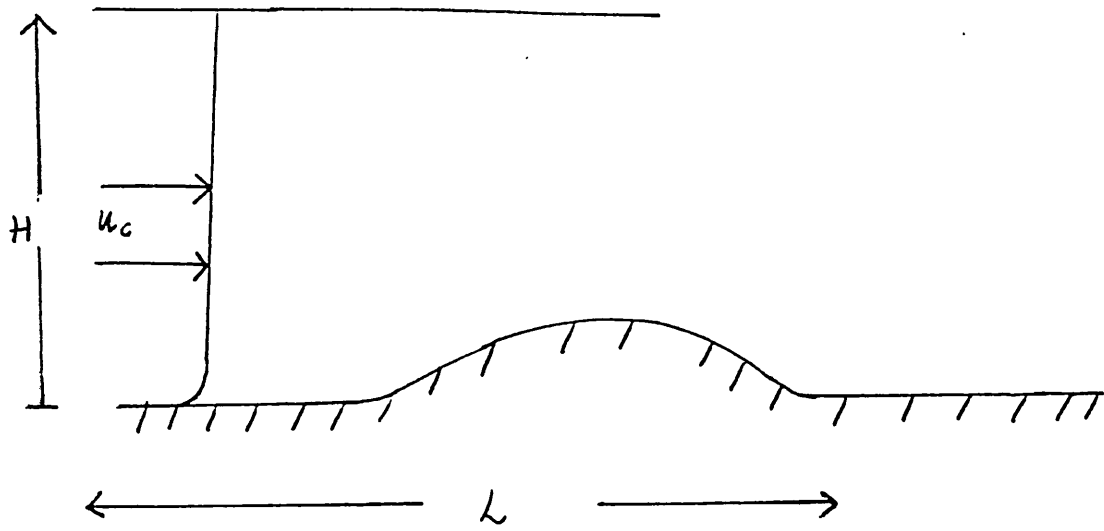
$$\theta_n \sim n\pi - \frac{sh}{n\pi} \quad \text{for } n \gg 1 . \quad (\text{A12a-d})$$

Table 3.1

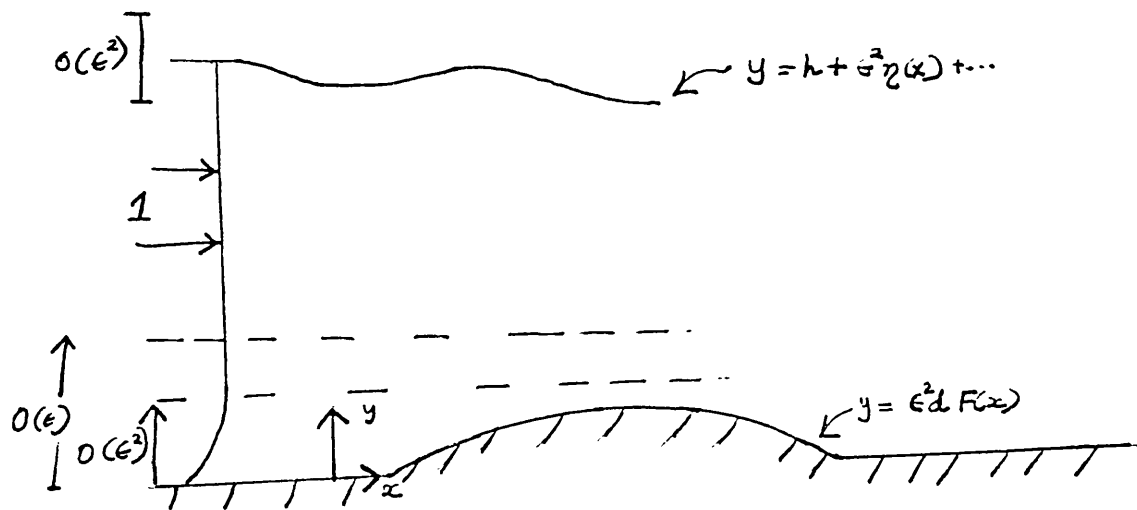
Some leading zeros of $f(U) = 0$, and $\square = 0$, see Appendix.
 θ denotes θ_0 (or θ_1), and α denotes α_0 (or α_1).

s	h	sh	θ	$i\alpha$
.4	1.0	.4	.42533	.42533
.5	1.0	.5	.27781	.27781
.6	1.0	.6	.12531	.12531
.7	1.0	.7	3.653E-2	3.653E-2
.8	1.0	.8	7.323E-3	7.323E-3
1.0	1.0	1.0	2.9974	2.9974
1.1	1.0	1.1	2.9667	2.9667
1.2	1.0	1.2	2.9363	2.9363
1.3	1.0	1.3	2.9061	2.9061
1.4	1.0	1.4	2.8762	2.8762
1.5	1.0	1.5	2.8468	2.8468
1.0	.5	.5	2.909E-2	5.818E-2
1.0	.6	.6	1.795E-2	2.991E-2
1.0	1.1	1.1	2.9910	2.7191
1.0	1.2	1.2	2.9858	2.4882
1.0	1.3	1.3	2.9817	2.2936
1.0	1.4	1.4	2.9787	2.1277
1.0	1.5	1.5	2.9767	1.9845
1.0	2.0	2.0	2.9786	1.4893
1.0	3.0	3.0	3.0157	1.0052

Fig. 1



(a)



(b)

Fig. 2a

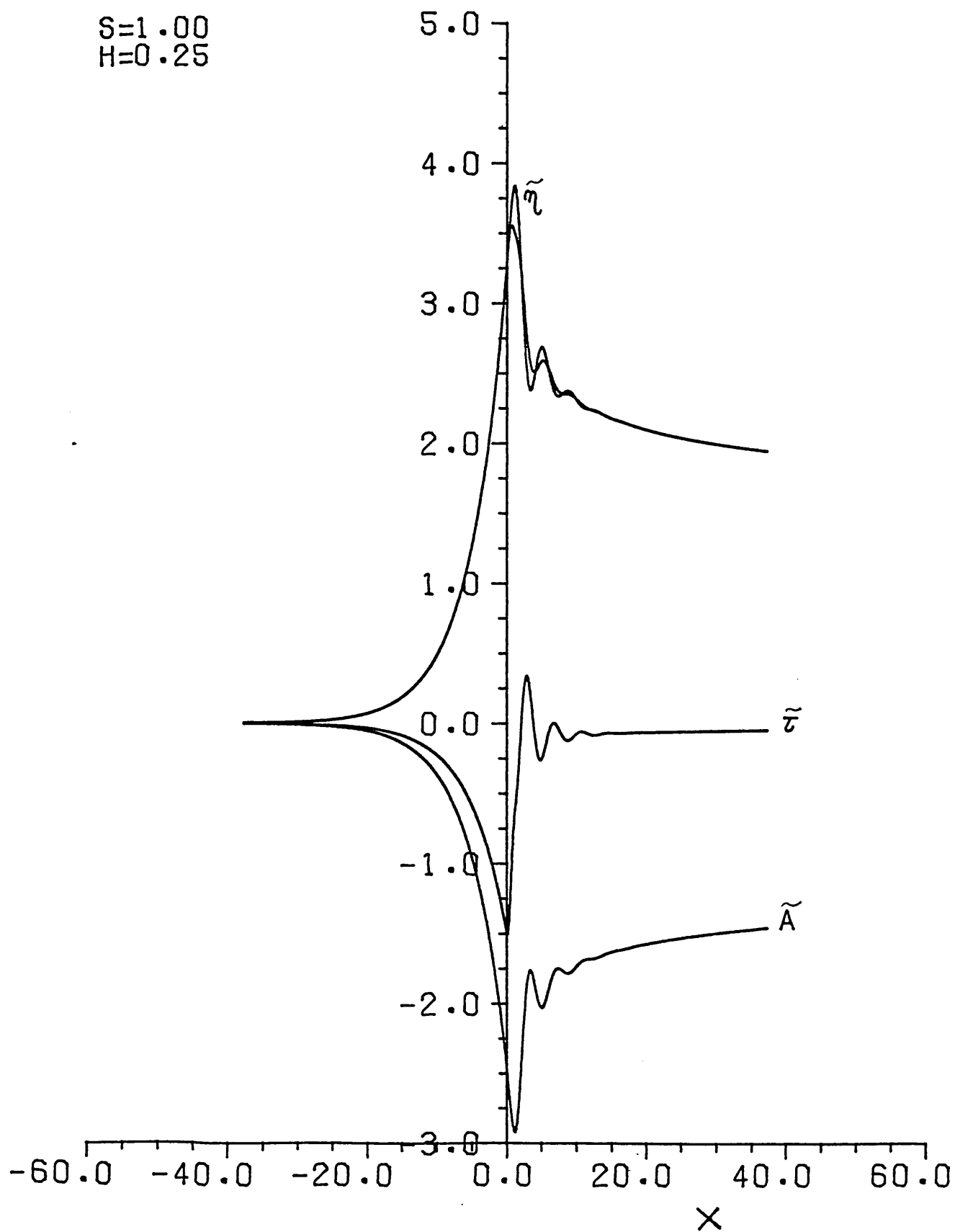


Fig. 2b

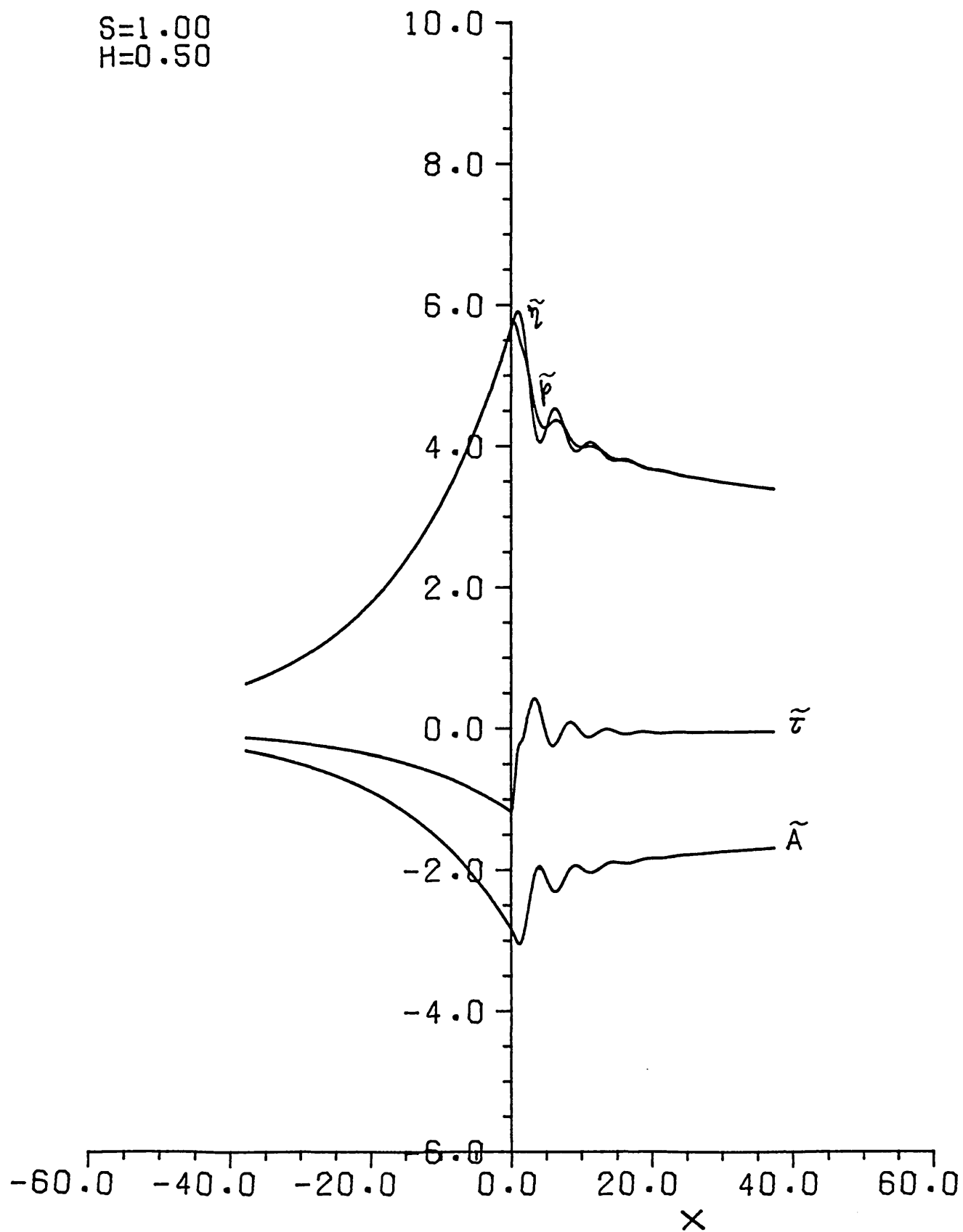


Fig. 2c

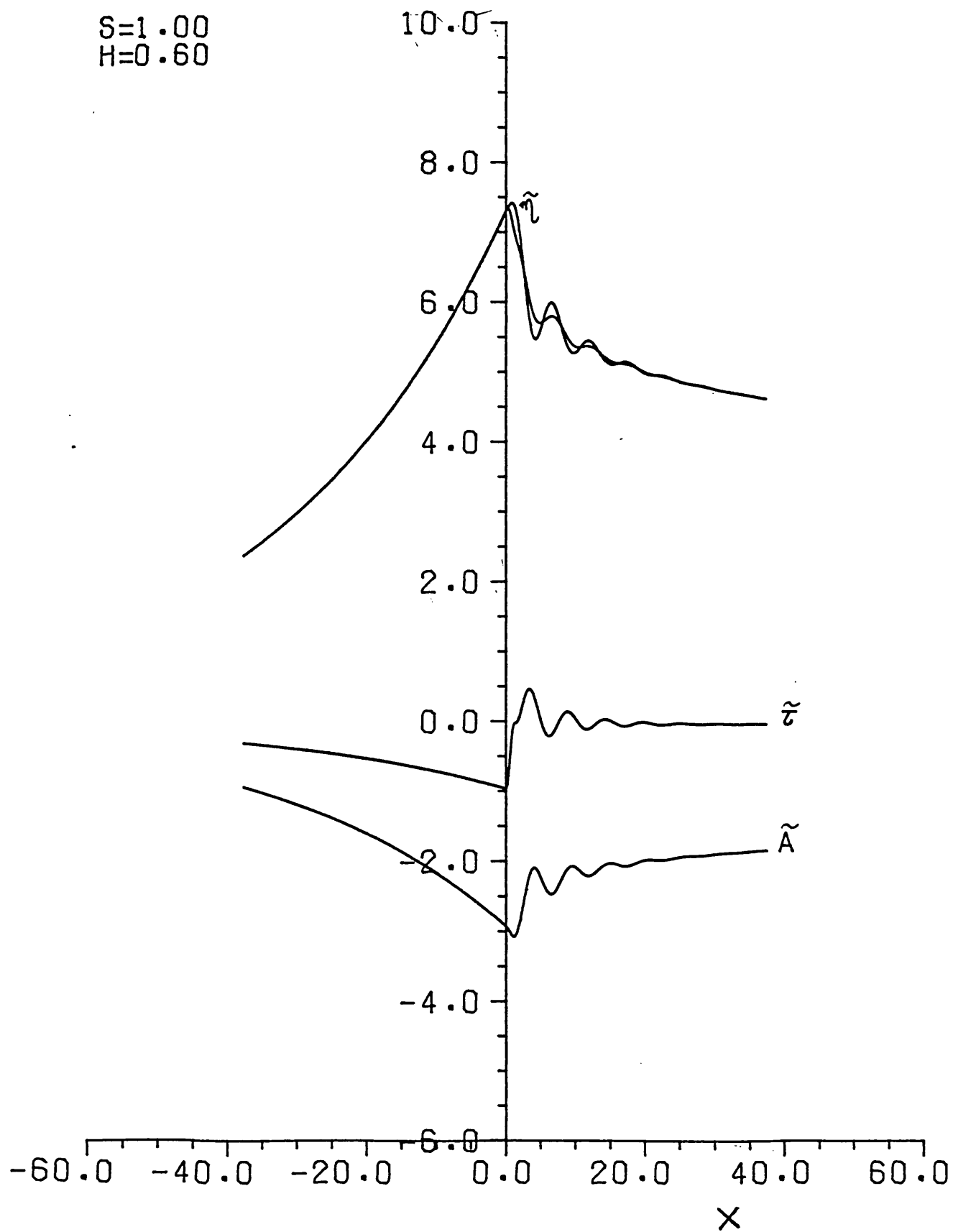


Fig. 2d

$S=0.50$
 $H=1.00$

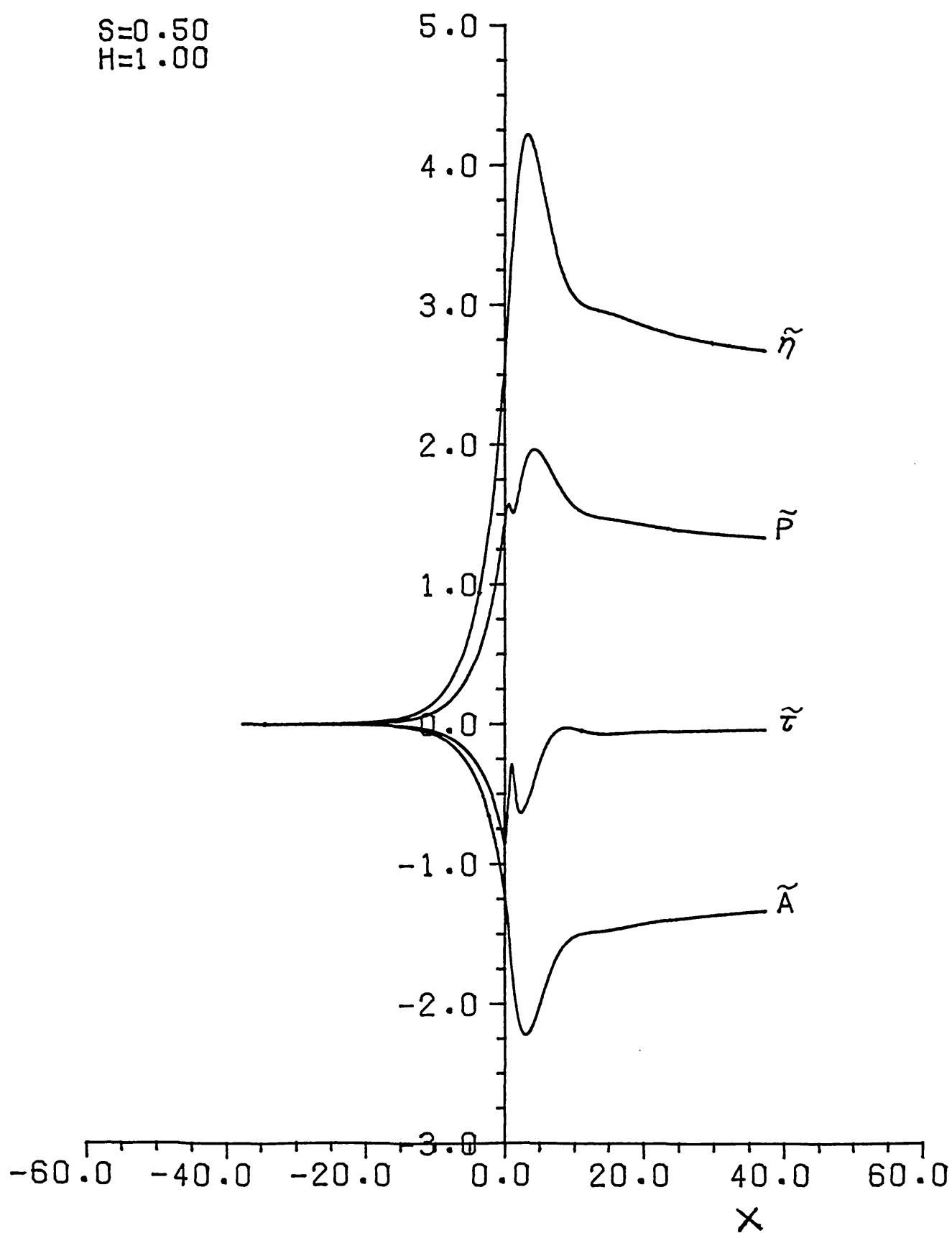


Fig. 2e

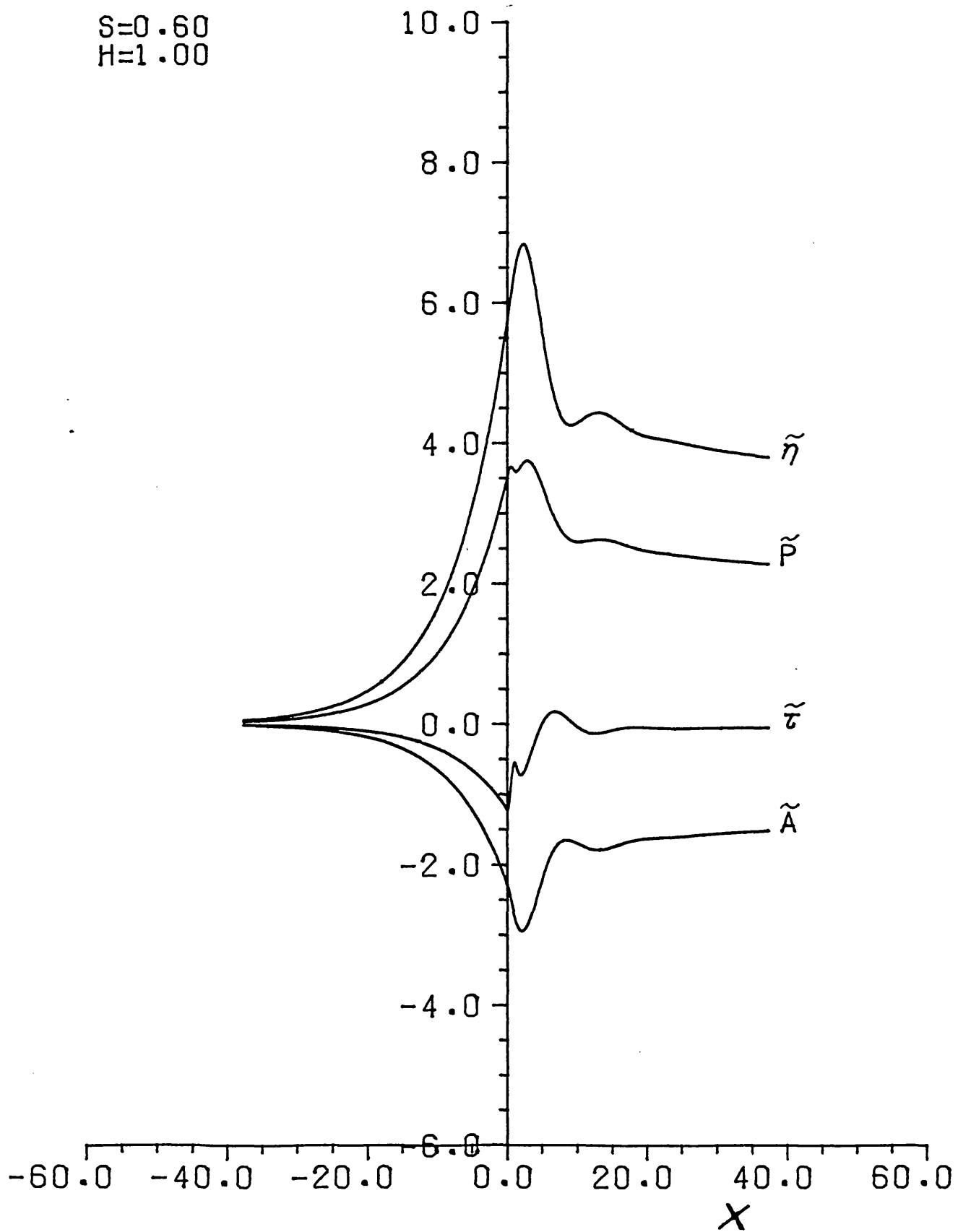


Fig. 2f

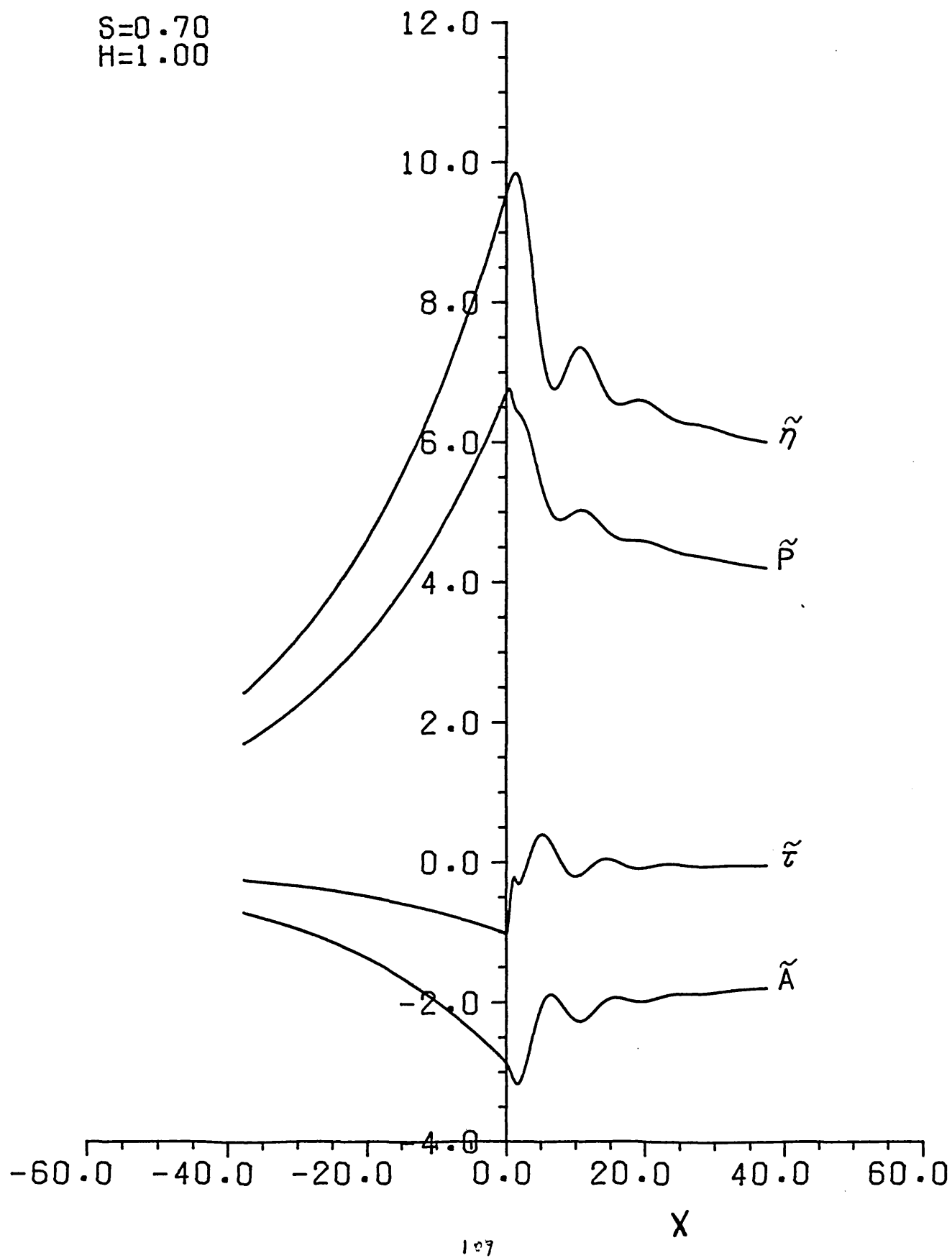


Fig. 2g

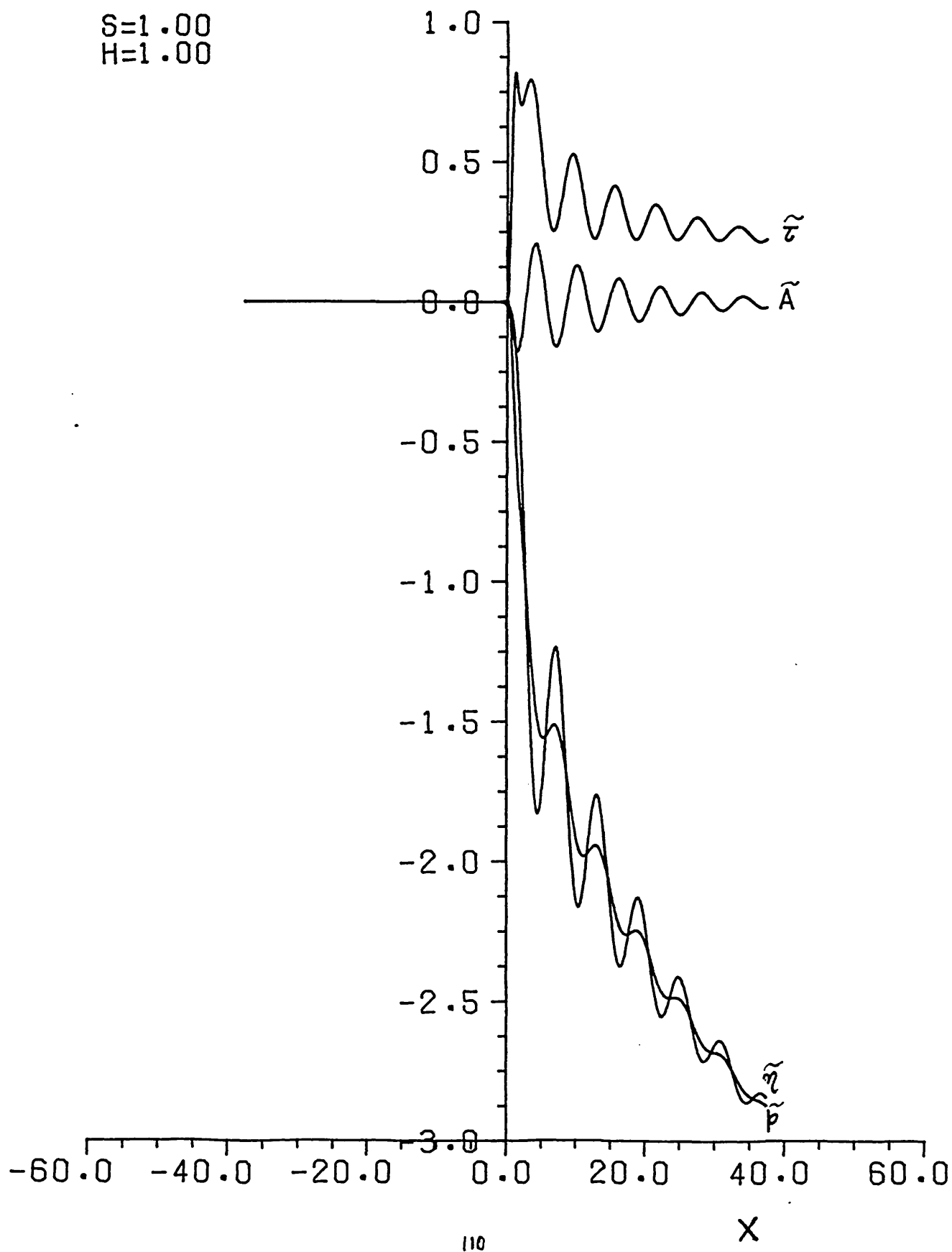


Fig. 2h

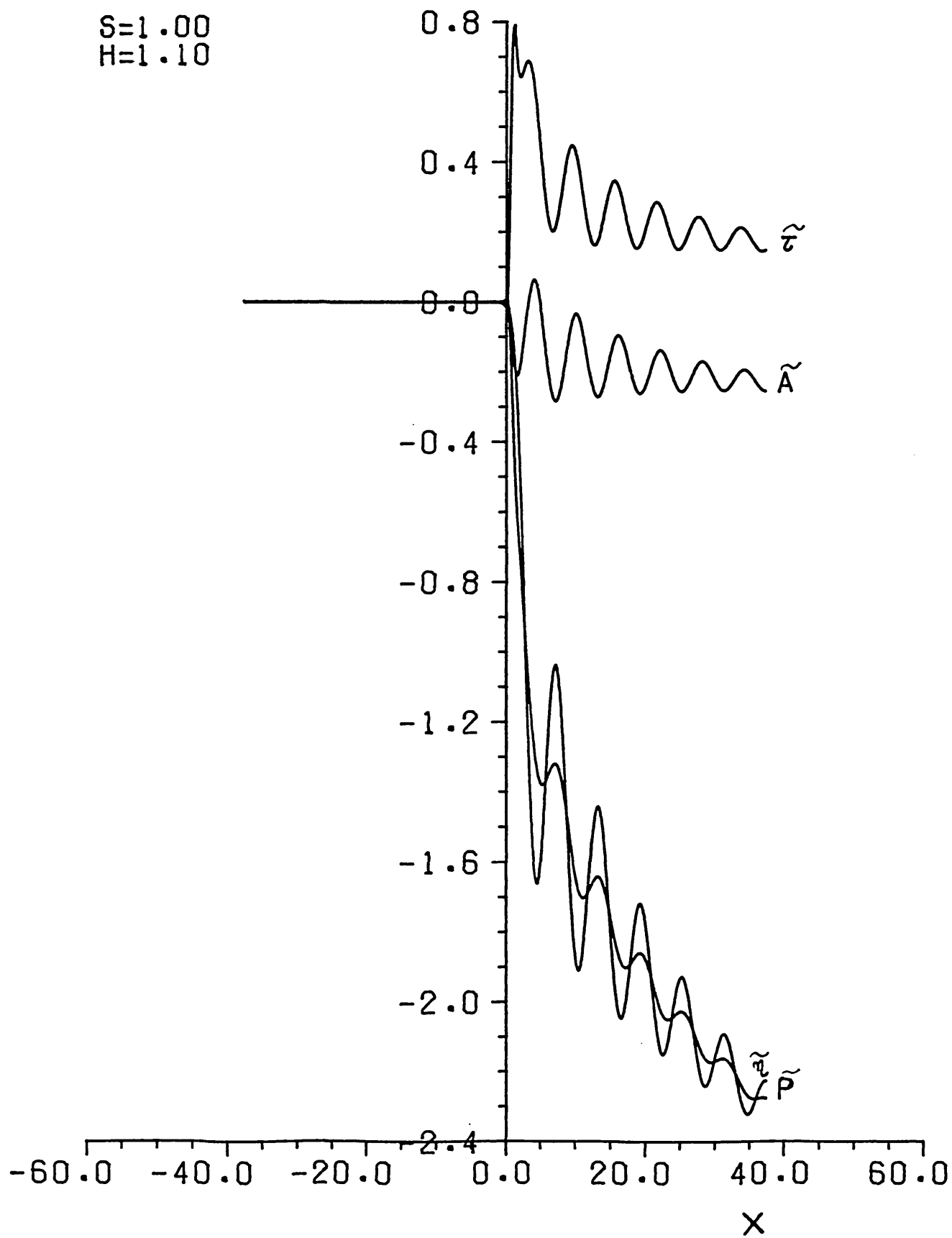


Fig. 2i

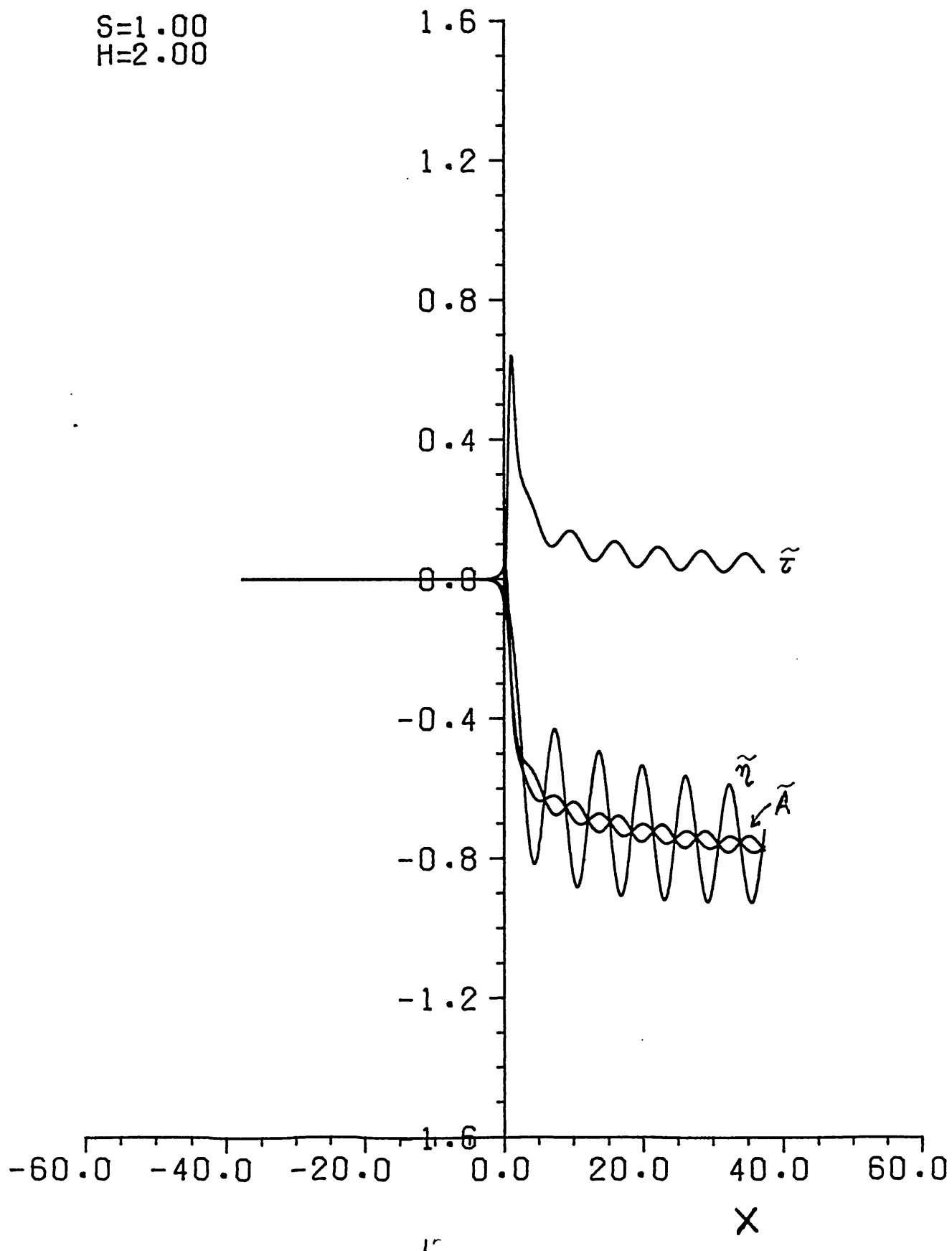


Fig. 2j

S=1.00
H=3.00

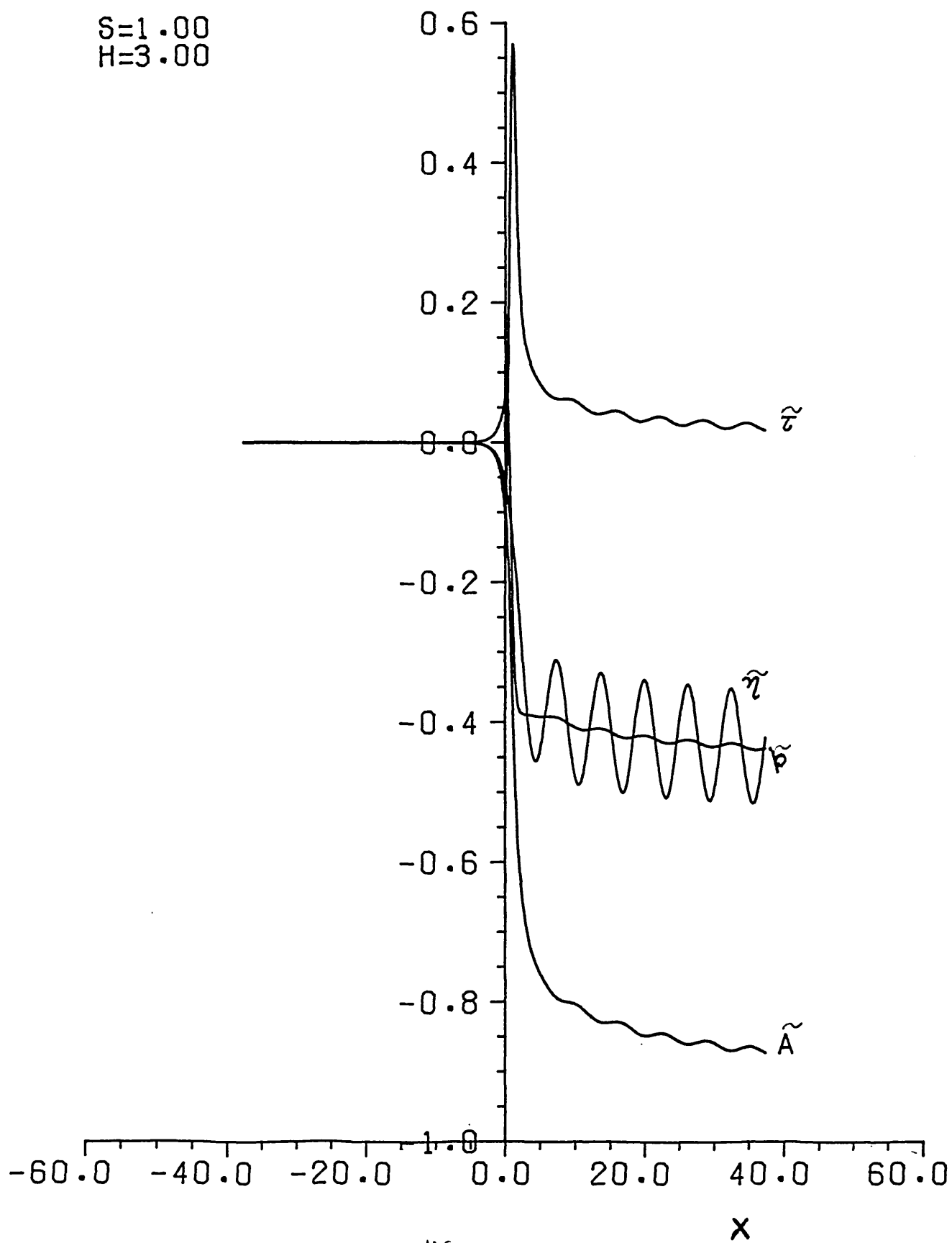


Fig. 2k

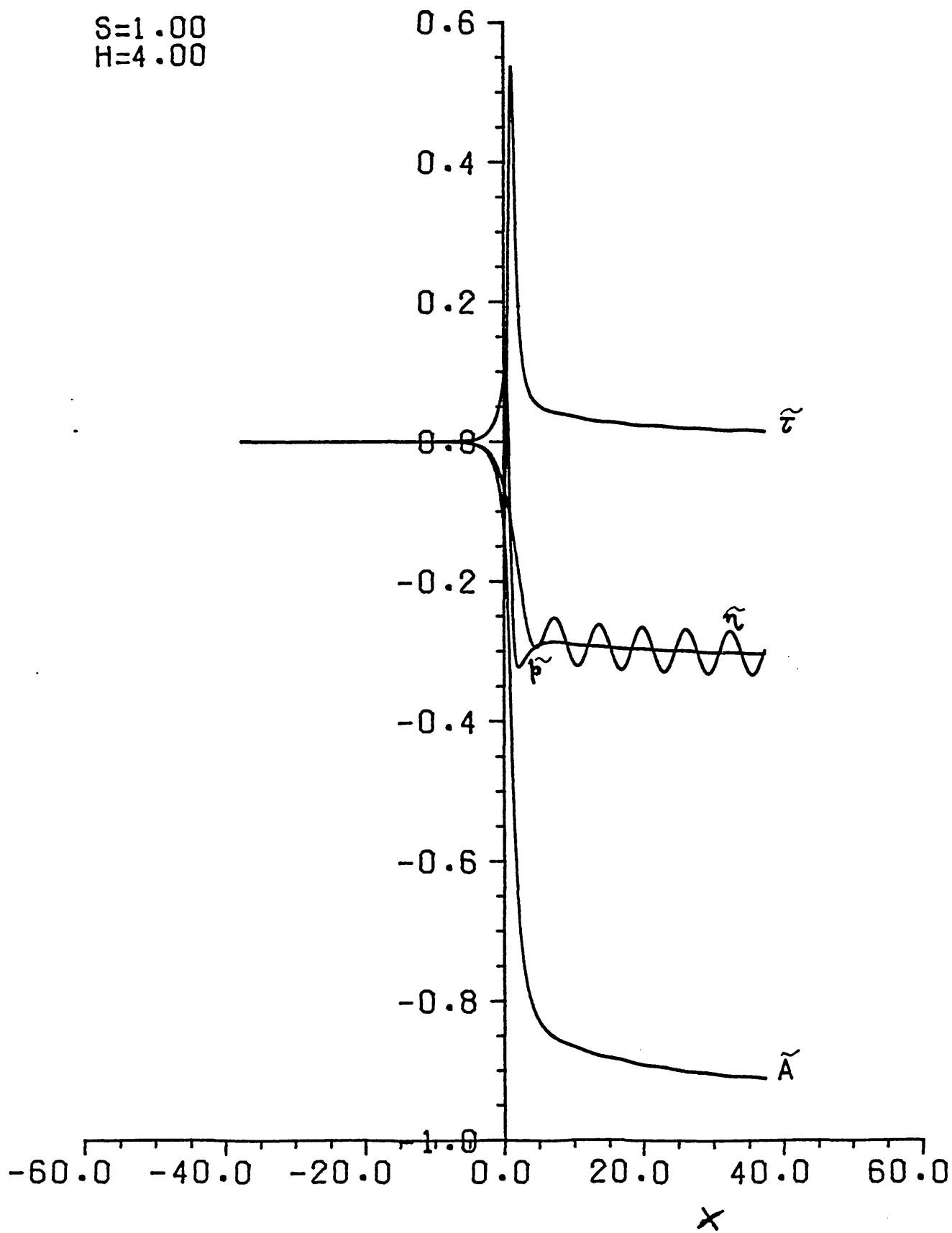


Fig. 21

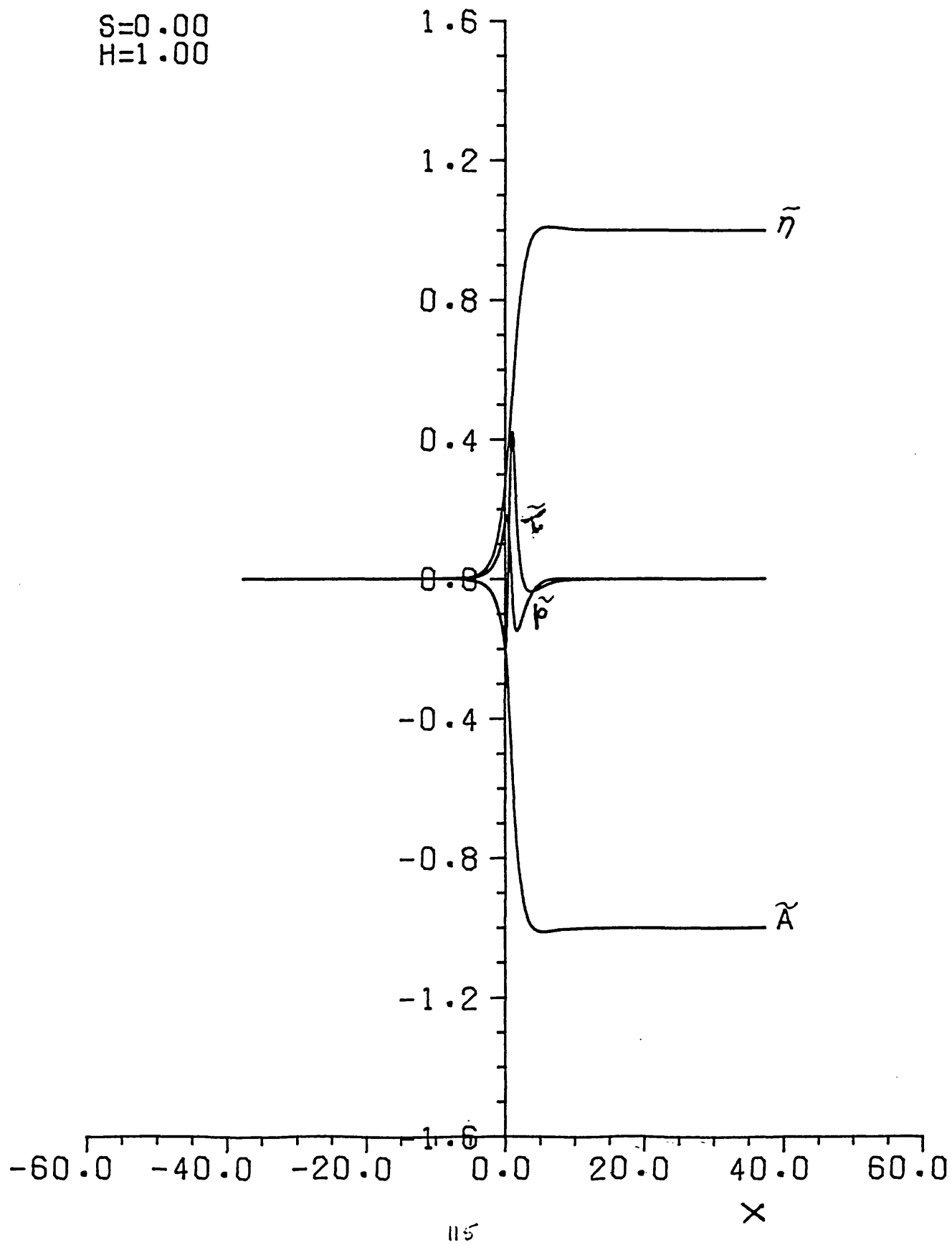
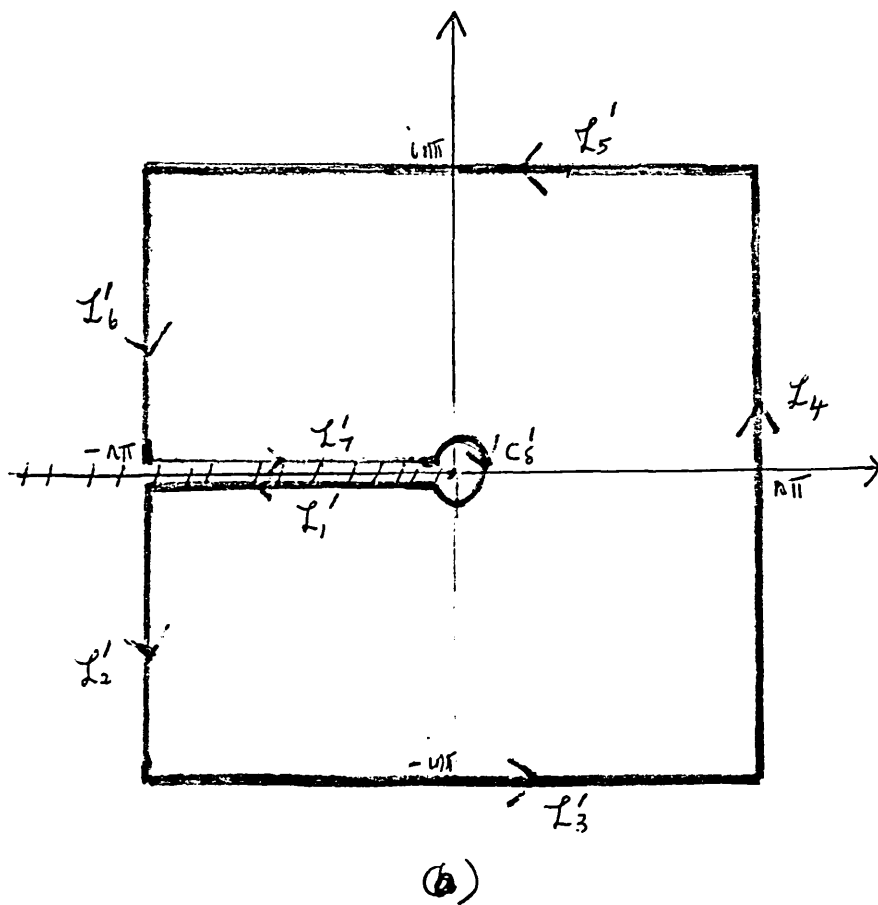
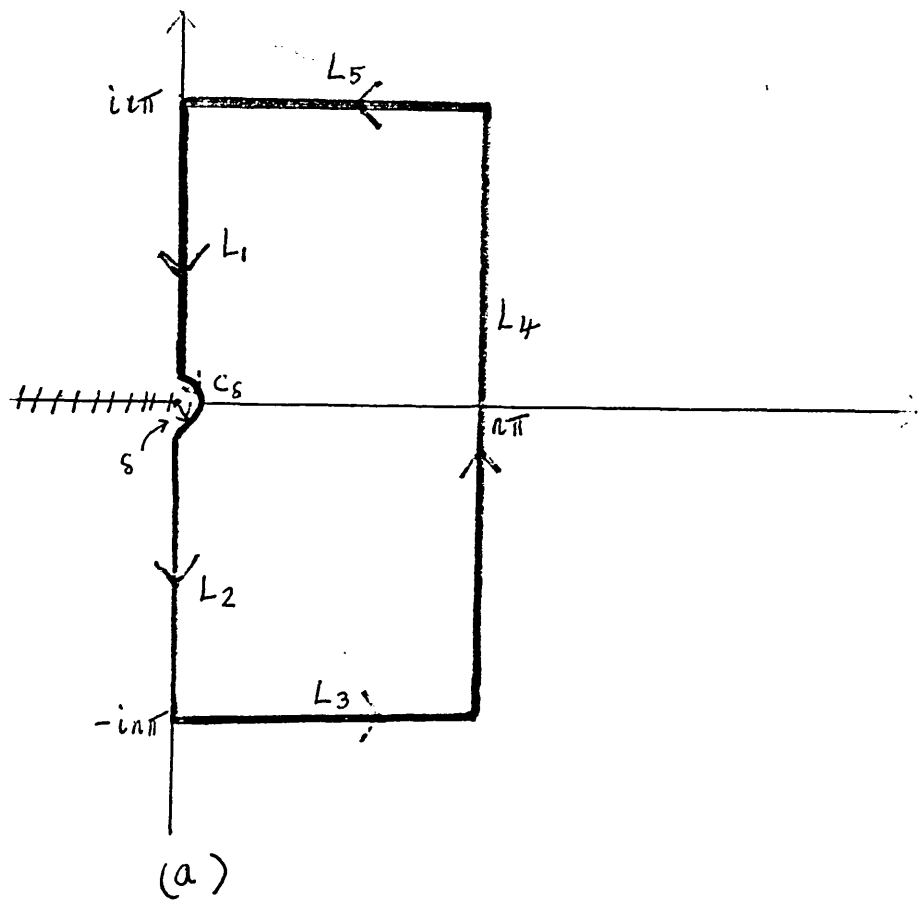


Fig. 3



Captions

Fig. 1(ab)

A sketch of the geometry and the triple-deck structure.

Fig. 2(a-1)

Linearized solutions for the flow over an infinite step $F(X) = \tanh(X^2)H(X)$, for several values of s and h .

Fig. 3

(a) Contour C_n , (b) contour C'_n , see appendix.

CHAPTER 4

ON HYPERSONIC SELF-INDUCED SEPARATION, HYDRAULIC JUMPS
AND BOUNDARY LAYERS WITH ALGEBRAIC GROWTH

To be published in Mathematika (1983).

Preliminary Introduction

The work of the previous chapters suggests studying the free interaction problem when the streamwise lengthscale is large compared to the depth since then the interaction laws take on a much simpler form and the free interaction problem is consequently much more amenable to some further analysis. In fact for the flow over an inclined plane when the angle of inclination of the plane to the horizontal $\beta^* \ll R^{-5/7}$, where R is a characteristic Reynolds number, the induced pressure is proportional to the local displacement, as is the case for the undeveloped free surface flow when the depth is small compared with the streamwise lengthscale.

Another motivation for studying this problem is, as has been mentioned already, the possible application to hydraulic jumps. In hydraulic jumps the mean depth of the oncoming stream changes more or less spontaneously without change of local conditions, from one level to another. The change can be gradual or more abrupt. In some cases an undular jump is formed. There are in fact many different types of jumps each characterized by different properties; see Rajaratnam (1967) for a comprehensive review on experimental and some theoretical aspects of hydraulic jumps. There are also numerous theoretical models of hydraulic jumps. The simplest assumes uniform depths of fluid before and after the jump, and momentum and mass flux considerations then fix relations between the upstream and downstream velocities and depths. Given the upstream conditions; the downstream conditions can be obtained readily, as in Lamb (1932), Rajaratnam (1967), Benjamin and Lighthill (1954). Another approach or viewpoint is to assume a long streamwise lengthscale

and then apply the ideas of shallow-water theory (Peregrine (1972)), accounts of which are given by Peregrine (1974), Hibberd and Peregrine (1979). However these studies do not include viscous effects and do not allow the details or structure of separating flow. In the theories based on the shallow water approximation, the boundary layer which would be needed to reduce the slip velocity to zero is driven by an adverse pressure gradient, since the pressure gradient here is effectively the slope of the free surface, for steady flows. Consequently the assumption that the boundary layer influence is small is open to debate, in a rational if not a practical sense. There have also been studies discussing the flow in a turbulent jump; see Rajaratnam (1967), Narayanam (1975), Madsen and Svendsen (1983).

Here we approach the problem from a different viewpoint, i.e. from that of interactive boundary layer theory. The hydraulic jump is described by a free interaction, and the work highlights the complicated separated flow structure which emerges beneath the jump. Far downstream the present structure also breaks down. In fact the free surface is still increasing algebraically downstream, and a new more complex structure must come into operation on some still longer lengthscale dependent on the Reynolds number.

Our objective throughout is not necessarily to obtain a complete description of the hydraulic jump, merely the beginnings of one. In fact the description strictly applies only as the Reynolds tends to infinity, but the work does demonstrate that a small frictional force can have a significant effect on the flow properties. It may be that the

interacting Navier- Stokes or boundary layer equations will give a better description of the hydraulic jump applicable at large but finite Reynolds numbers.

ON HYPERSONIC SELF-INDUCED SEPARATION, HYDRAULIC
JUMPS AND BOUNDARY LAYERS WITH ALGEBRAIC GROWTH

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SUMMARY

Analytical and numerical properties are described for the free interaction and separation arising when the induced pressure and local displacement are equal, in reduced terms, for large Reynolds number flow. The interaction, known to apply to hypersonic flow, *supercritical (Froude number > 1)* is shown to have possible relevance also to the origins of hydraulic jumps in liquid layers flowing along horizontal walls. The main theoretical task is to obtain the ultimate behaviour far beyond the separation. An unusual structure is found to emerge there, involving a backward-moving wall layer with algebraically growing velocity at its outer edge, a detached shear layer moving forward and, in between, reversed inertial flow uninfluenced directly by the adverse pressure gradient. As a result the pressure increases like $(\text{distance})^m$ then [with $m = 2(\sqrt{7}-2)/3 (= 0.43050\dots)$] and does not approach a plateau. Some more general properties of *(Falkner - Skan)* boundary layers with algebraic growth are also described.

§1. INTRODUCTION

The bulk of the present paper is concerned with the free interaction that occurs with the pressure proportional to the local boundary layer displacement, in fluid flow at large Reynolds number. This free interaction, which in essence describes upstream influence in forward flow, arises in two major contexts, the first being for hypersonic flow (Brown, Stewartson & Williams (1974/75, 1975), Brown & Williams (1975), Stewartson (1974)). The problem is to solve the boundary layer equations, holding in a viscous sub-layer on a fixed plane wall, with the unknown pressure being equal to the unknown displacement in scaled terms. There Brown *et al* (1975) point out that the determination of the ultimate form of the free interaction beyond separation is a main task: our aim is to address the task.

The second main context is in the motion of a liquid layer along a fixed horizontal wall, in particular the phenomenon of the hydraulic jump. There, without change of local boundary conditions, the depth of an oncoming liquid layer adjusts itself in anticipation of a new downstream state. [See, e.g., the experiments or analysis in Rajaratnam 1967, Henderson 1966, Craik *et al* 1981, Watson 1964, Tani 1948, concerning classical, moving, planar, circular, forced, submerged or other types of jumps.] This is characteristic of a free interaction taking place upstream and in fact it is found to be controlled initially, in certain regimes dependent upon the length scale and Froude number Fr , by the same problem as the hypersonic separation above, provided that the flow is supercritical, $Fr > 1$. The pressure-displacement law stems from the traditional predominantly inviscid boundary condition at the free surface. Our intention here is to achieve or start an account of the hydraulic jump consistent with the Navier-Stokes equation^S_X, that is, accommodating the influence of viscous forces, which are seldom inconsequential in high Reynolds number flows, rather than following the classical inviscid lines as,

for example, in the (nonetheless interesting) works based on Lamb's (1932) account: see also Rajaratnam (1967) and Henderson (1966).

The problem of the hypersonic separation/free interaction is stated in §2, while §3 shows its relevance to the supercritical hydraulic jump also. No such interaction occurs in subcritical or critical flow, $Fr \leq 1$. Then §4 presents the numerical procedure adopted, and the numerical results, for the interaction problem. The ultimate form found to emerge for the free interaction, far downstream, is described analytically in §5 and further comments are given in §6.

Concerning §5, the ultimate form has certain novel aspects and turns on the properties of boundary layers with algebraic *growth*. A viscous detached shear layer emerges, supported by an inertial layer of reversed flow where both viscous and pressure forces are secondary, an unusual feature which yields a power law for the velocity profile. Closer to the wall a reversed viscous sublayer is formed whose streamwise velocity must grow algebraically at the outer extremes. This also is unusual and in contrast to the algebraic decay of boundary layers which has received appreciable attention. Again, the reduced pressure P does not tend to a uniform plateau far downstream; c.f. all previously studied interactions; instead we find that

$$P \propto (\text{distance})^m, \text{ where } m = \frac{2}{3} (\sqrt{7} - 2) (= 0.43050 \dots), \quad (1.1)$$

is unbounded there. Another point is that the shear layer position although far from the wall is governed by the solution of the boundary layer with algebraic growth and hence by viscous forces. Since algebraic growth in boundary layers is of some interest in itself — — —

— — — certain further properties are considered in an Appendix.

The work below is expressed in terms of the velocity components u, v in cartesian coordinates x, y and the pressure p , non dimensionalized with respect to $u_\infty, u_\infty, l_\infty, l_\infty, \rho_\infty u_\infty^2$ in turn. Here $u_\infty, l_\infty, \rho_\infty$ are representative velocity, length and density scales of the motions. For the possible application to hydraulic jumps the fluid is taken to be incompressible of density ρ_∞ , and surface tension effects are ignored. The motion is two-dimensional, steady and laminar, and the Reynolds number $Re = u_\infty l_\infty / \nu_\infty$ (ν_∞ being the kinematic viscosity of the fluid) is large.

§2. THE HYPERSONIC INTERACTIVE BOUNDARY LAYER

Interaction in hypersonic flow occurs by means of a triple-deck structure, as shown by Brown *et al* (1975) and Rizzetta *et al* (1978), who present the derivation and scalings involved. The central problem reduces to solving the viscous boundary layer equations

$$U = \frac{\partial \Psi}{\partial Y}, \quad U \frac{\partial U}{\partial X} - \frac{\partial \Psi}{\partial X} \frac{\partial U}{\partial Y} = - \frac{dP}{dX} + \frac{\partial^2 U}{\partial Y^2}, \quad (2.1a)$$

for the reduced velocity $U(X, Y)$, stream function $\Psi(X, Y)$ and pressure $P(X)$, subject to the boundary conditions

$$U = \Psi = 0 \quad \text{at} \quad Y = 0 \quad (2.1b)$$

$$U \sim Y + A(X) \quad \text{as} \quad Y \rightarrow \infty \quad (2.1c)$$

$$(U, \Psi, P) \rightarrow (Y, \frac{1}{2}Y^2, 0) \quad \text{as} \quad X \rightarrow -\infty \quad (2.1d)$$

where the displacement $-A(X)$ is also unknown. The pressure-displacement relation for hypersonic flow requires in addition that

$$P(X) = -A(X) \quad , \quad (2.1e)$$

in the hypersonic limit: see Brown *et al* (1975).

A free interaction can take place, when (2.1a-e) hold, starting in the form

$$P \sim be^{\kappa X} \quad \text{for } X \rightarrow -\infty : \quad \kappa = (-3Ai'(0))^3 = 0.4681. \quad (2.2)$$

Here the constant b is unknown. If $b < 0$ the flow becomes strongly attached as X increases, a singularity being encountered at a finite value of X : see Brown *et al* (1974/75). Our concern is with $b > 0$ however, where instead the flow proceeds to separate, and the ensuing numerical and analytical features then will be described in §4, §5 below. In particular the main difficulty (see Brown *et al*'s (1975) comments), which concerns the unknown ultimate form taken by the free interaction as X increases beyond separation, will be addressed.

§3. THE APPLICATION IN HYDRAULIC JUMPS

It is argued below that the same problem (2.1a-e) may arise in the context of hydraulic jumps.

Suppose that a uniform flow $u = 1$ of liquid of depth 1 is approaching horizontally from upstream, supplemented by a thin boundary layer of thickness $O(\epsilon)$ to satisfy the no slip condition at the wall $y = 0$. ^{also} see fig. 1. If an eventual hydraulic jump in the liquid depth is to be produced downstream then the local flow must admit upstream influence, enabling its depth to deviate slightly at first from the original depth 1 . We postulate that the length scale L governing this upstream influence is large,

The boundary layer effects on the uniform stream are negligible to our order of working below:

$L \gg 1$, and we seek a self-consistent account of the Navier-Stokes equations for Re large. Over the bulk ^(I) of the local motion only a small perturbation of the oncoming stream is induced then, with

$$\psi = y + \delta \left[\tilde{A}(\tilde{X}) \right] + O(\delta^2) \quad (3.1a)$$

$$\Rightarrow p = \delta \tilde{P}(\tilde{X}) + O(\delta^2) \quad (3.1b)$$

where δ is an unknown small parameter and $\tilde{A}(\tilde{X})$, $\tilde{P}(\tilde{X})$ are unknown functions of $\tilde{X} = (x-l)/L$ ($l = \text{constant}$, see below). The boundary conditions at the unknown free surface, $y = 1 + \eta$ say, require

$$\left. \begin{aligned} \psi &= 1, \\ (-\tilde{p} + 2Re^{-1} \frac{\partial u}{\partial x} \frac{d\eta}{dx} - Re^{-1} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})) &= 0 \\ -\tilde{p} - 2Re^{-1} \frac{\partial u}{\partial x} - Re^{-1} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{d\eta}{dx}) &= 0 \end{aligned} \right\} \quad (3.2a,b,c)$$

however, for continuity of normal velocity and both stress components, with $\tilde{p} \equiv p - g\lambda_\infty(y-1)u_\infty^{-2}$ being the modified pressure. The density and viscosity of the fluid above the free surface are neglected.

Hence $\eta = \delta(\tilde{P} - \tilde{A})(\tilde{X}) + O(\delta^2)$ from (3.1a), and (3.2b,c) both lead to the standard inviscid result for a free surface, $p = s\eta + O(\delta^2)$ where $Fr = \frac{u_\infty^2}{g\lambda_\infty}$ is the Froude number. Hence

$$\tilde{P}(\tilde{X}) = - \frac{\tilde{A}(\tilde{X})}{(Fr-1)} \quad (3.3)$$

The minor corrections needed to satisfy (3.2b,c) exactly are produced in a thin surface layer of thickness $O(Re^{-\frac{1}{2}}L^{\frac{1}{2}})$, requiring that $L \ll Re$.

Next, the displacement effect of (3.1a,b) feeds into the majority of the thin boundary layer (II) wherein $y = \epsilon \bar{Y}$ with $\bar{Y}$ of $O(1)$. There, since the motion upstream has a profile $\psi = \epsilon \psi_0(\bar{Y})$ with $\psi_0'(\infty) = 1$, we have

$$\psi = \epsilon \psi_0(\bar{Y}) + \delta \tilde{\lambda}(\tilde{X}) \psi_0'(\bar{Y}) + \dots, \quad (3.4)$$

matching with (3.1a) provided $\delta \ll \epsilon$. The result (3.1b) for p is valid since $(u\partial v/\partial x)$ is only $O(\delta L^{-2})$. Finally, (3.4) becomes invalid near the wall (III) when $\bar{Y} \rightarrow O(\delta/\epsilon)$, since $\psi_0(\bar{Y}) \sim \frac{1}{2} \lambda \bar{Y}^2$ where λ is the $O(1)$ oncoming reduced skin friction, and a nonlinear inertial-pressure-viscous force balance also emerges then if $\delta^2/\epsilon^2 L = \delta/L = \delta/\epsilon \delta^2 Re$, since y, u are $O(\delta), O(\delta/\epsilon)$ respectively. Hence $L = \epsilon^5 Re, \delta = \epsilon^2$. So the restrictions above require only that

$$1 \gg \epsilon \gg Re^{-1/5}. \quad (3.5)$$

Given (3.5), then, the boundary layer equations hold in an $O(\epsilon^2)$ thick sublayer III subject to the no slip condition at the wall and to a relative displacement $-\tilde{A}(\tilde{X})$ and pressure force $\tilde{P}(\tilde{X})$ linked by the law (3.3). Provided $Fr > 1$ a normalization to allow for the factors Fr, λ therefore reduces the problem to the form (2.1a - e), which admits upstream influence through (2.2) and so permits the gradual growth (or drop) in the liquid depth for $\tilde{X}$ large and negative, as required by a hydraulic jump. Hence the need for the solution properties of (2.1a - e) is reinforced by the application to supercritical hydraulic jumps where $Fr > 1$ [whereas subcritical or critical flows, where $Fr \leq 1$, do not admit such upstream influence because of the resultant sign change, or zero, in (3.3) and hence in (2.1e): see also §6].

Before the solution of (2.1a - e) is investigated it is worth considering the restrictions in (3.5). At one extreme, if ϵ is $O(Re^{-1/5})$ the interaction length scale L falls to $O(1)$, comparable — — —

--- with the liquid depth (as in Smith 1976), so that the outermost zone becomes one of classical linearized potential flow, replacing (3.1a,b), *and allowing sub- or supercritical upstream influence.* This regime includes the present case of (3.1a,b), (3.3) as a limit, shows the influence of viscosity on linear and nonlinear water waves for instance and is studied by Gajjar (1983). Again, if the interaction occurs at a distance ℓ ($\gg L$) from an upstream source of the motion then typically the boundary layer thickness is $\epsilon = O(Re^{-\frac{1}{2}} \ell^{\frac{1}{2}})$; so the regime above has $\ell = O(Re^{3/5})$, while (3.5) requires that $Re^{3/5} \ll \ell \ll Re$. At the other extreme, if ϵ is $O(1)$ then both ℓ , L rise to $O(Re)$ and the interaction has no subscales in the y -direction. Instead the boundary layer equations apply across the entire depth, with ψ , u , p , $\hat{X} \equiv x/Re$, y , n all of $O(1)$ and the uniform stream $u = 1$, $\psi = y$ holds at the start $\hat{X} = 0 +$ for $0 < y \leq 1$. For $\hat{X} > 0$ the boundary conditions are $\psi = u = 0$ at $y = 0$ (no slip) and $\psi = 1$, $\partial u / \partial y = 0$ at $y = 1 + \eta(\hat{X})$ (free surface, from (3.2a,b)), with the law $p(\hat{X}) = s\eta(\hat{X})$ following from (3.2c). This system also admits ^{supercritical} upstream influence, with the flow able to deviate in an initially exponentially small form $\propto \exp(-\hat{k} \hat{X}^{3/2})$ ($\hat{k}$ a positive constant) from the uniform stream allied to a Blasius boundary layer holding at $\hat{X} = 0 +$. Thus the solution is not unique and can branch. Essentially similar equations were presented by Kurihara (1946) and Tani (1948) for the axisymmetric hydraulic jump but without acknowledgement of the branching, and branching is also absent in Watson's (1964) classical boundary layer approach; *see also Chester (1966)*. Further comment on the branching is given in §6 below. Two final points are the following (see Gajjar 1983). First, if the flat wall is slightly inclined to the horizontal then a curvature term $\propto d^2 A / dX^2$ is added to the pressure-displacement law (3.3), depending on the angle of inclination. Second, the same

additional term can become significant if the upstream source is so strong that the Froude number is large (s small, $u_{\infty}^2 \gg g l_{\infty}$) as an argument analogous to that of (3.1)-(3.5) shows. We concentrate henceforth on the ^(supercritical) regime where (2.1a-e) holds, however.

§4. NUMERICAL SOLUTIONS

The problem (2.1a-e) was solved numerically by a finite-difference box scheme of second order accuracy. The method is fairly standard now and need not be described in much detail. A small pressure kick was given to start the interaction and its subsequent non linear evolution was followed step by step, forward in X , using Newton iteration per step. Typical values taken for the steplengths ΔX in X and iterative tolerance were $0.1-0.2$, 10^{-7} respectively. For the outer constraint (2.1c), which proved a significant factor further downstream, we used ^{either a uniform grid in Y or} a stretching of Y to cover the range $0 \leq Y \leq Y_{\infty}$ with, typically, $Y_{\infty} \gg 80$ but a minimum step length $\Delta Y \leq 0.2$ near the wall $Y = 0$. In reversed flow $U < 0$, where strictly a forward marching scheme is inappropriate, two techniques were employed ~~which nowadays might be described best as tongue-in-cheek~~, namely FLARE (where $U \partial U / \partial X$ is neglected if $U < 0$, from Reyhner & Flügge-Lotz 1968) and extended backward differencing (where $U \partial U / \partial X$ is evaluated from U values at X , $X - \Delta X$, $X - 2\Delta X$, $X - 3\Delta X$ if $U < 0$, from Dijkstra 1978). Neither of these is really satisfactory, the first because it changes the governing equations, the second because it cannot suppress the reversed direction of influence for very small steplengths, and indeed a forward-backward marching or windward differencing procedure would seem best suited at present - but of course a consistent boundary condition downstream

was not known at the outset of this study. Our major aim was to deduce that downstream boundary condition, i.e. the ultimate form of the free interaction, and so we hoped that, as so often, the forward-marching ~~long-range in stream~~ approaches would give a good guide there.

The results are presented in fig. 2, previous solutions being given by Brown *et al* (1975) over a smaller range of X . Upstream the trend of (2.2) is observed satisfactorily, regular separation then occurs ($\tau \equiv \partial U / \partial Y (X, 0) \rightarrow 0$), at $X = 0$ say, and beyond that the pressure P and hence the displacement $-A$ continue to increase and $\dot{\tau}$ continues to decrease. There is as yet no hint of a minimum in τ being reached and P shows no sign of approaching a plateau value, in contrast with the usual developments in free interactions. The numerical trends, especially for P , led to the account below of the rather unusual nature of the solution far downstream.

§5. THE ULTIMATE FORM FAR DOWNSTREAM

For $X \rightarrow \infty$ suppose that

$$P \sim P_1 X^m \quad (5.1)$$

with m, P_1 both unknown positive constants, so that the pressure gradient is adverse but no pressure plateau is attained. Then

$-A \sim P_1 X^m$ also, from (2.1e). Assume next ^{(fig. 3(a))} that a detached shear layer ⁽ⁱ⁾ is formed far from the wall, of the usual thickness $O(X^{1/3})$

in view of (2.1c), so that we require $m > 1/3$. The shear layer has

$Y = -A(X) + X^{1/3} \xi$ with ξ of $O(1)$, $\psi = X^{2/3} G_0(\xi)$ to leading

order and provided $|P| \ll |U|^2$, i.e. $m < 2/3$, G_0 satisfies

$3G_0''' + 2G_0 G_0'' - G_0'^2 = 0$ and $2G_0 \sim \xi^2$ as $\xi \rightarrow \infty$. Also

$G'_0(-\infty) = 0$ provided that $|U| \ll X^{1/3}$ below the shear layer: see below. Then G_0 has the solution of Stewartson & Williams (1973), giving $G_0(-\infty) = -C_0$, where $C_0 = 1.2521$, ~~λ~~ ^{i.e.} the entrainment condition

$$\psi \sim -C_0 X^{2/3} \quad \text{as} \quad Y \rightarrow P_1 X^m, \quad (5.2a)$$

provided (so far) that

$$\frac{1}{3} < m < \frac{2}{3}. \quad (5.2b)$$

Between the shear layer and the wall the majority of the motion ⁽ⁱⁱ⁾ ~~λ~~ has ψ of $O(X^{2/3})$, U of $O(X^{2/3-m})$ then and viscous forces are negligible from (5.2b). Usually the pressure force matters substantially here, but not in the present case: thus $P_1 X^m \ll |U|^2$, leading to $m < 4/9$ (see below). As inertial forces alone dominate the only possible solution of (2.1a,b) here is then a power law profile,

$$\psi = -\hat{b} Y^K = O(X^{2/3}) \quad (5.3)$$

($\hat{b} > 0$ an unknown constant) where the inviscid constraint $\psi \rightarrow 0$ as $Y \rightarrow 0+$ is imposed. Since Y is $O(X^m)$ therefore

$$K = \frac{2}{3m} \quad \text{and} \quad \hat{b} P_1^K = C_0 \quad (5.4)$$

from (5.2a). That leaves a viscous sublayer ⁽ⁱⁱⁱ⁾ ~~λ~~ near the wall, to ensure that the no slip condition is met. Given (5.3) the sublayer thickness is $Y = O(X^{1/(K+1)})$ with ψ , U of the orders $X^{K/(K+1)}$, $X^{(K-1)/(K+1)}$ in turn. We expect the pressure force to come into play at 1^{st} ast, so that $|P| = O(U^2) = O(X^{(2K-2)/(K+1)})$. Comparing

this with (5.1) we have

$$\frac{2K-2}{K+1} = m \quad (5.5)$$

and hence, referring to (5.4), we require that $(2K-2)/(K+1) = 2/3K$.

Consequently

$$\begin{aligned} K &= \frac{2 + \sqrt{7}}{3} \quad (= 1.54858 \dots) , \\ m &= \frac{2(\sqrt{7}-2)}{3} \quad (= 0.43050 \dots) , \end{aligned} \quad (5.6)$$


the choice $+$ $\sqrt{\quad}$ being forced by (5.2b).

A check confirms that all the assumptions made above (basically that $\frac{1}{3} < m < \frac{4}{9}$) are satisfied with (5.6). The whole structure holds together (fig. 3(a)) *provided* that the similarity problem in the reversed flow sublayer adjoining the wall,

$$f'''' - \frac{K}{K+1} f f'' + \left(\frac{K-1}{K+1}\right) (f'^2 + 2P_1) = 0 \quad (5.7a)$$

with

$$f(0) = f'(0) = 0, \quad f \sim \eta^{\wedge K} \quad \text{as } \eta \rightarrow \infty, \quad (5.7b,c,d)$$

has a solution with $P_1 > 0$. Here $\eta = X^{-\frac{1}{K+1}} Y$,

$$\psi = -X^{\frac{1}{K+1}} [f(\eta) + o(1)].$$

The similarity problem (5.7a-d) is the first example we know of where a (reversed) boundary layer solution with algebraic growth is sought: *see also Smith (1983)*. In the standard notation of Falkner-Skan forms (see Jones & Watson 1963) (5.7a-d) corresponds to $\alpha = -K/(K+1) < 0$, $\beta = -(K-1)/(K+1) < 0$, allowing no bounded solution for f' as $\eta \rightarrow \infty$.

It can be shown (see Appendix) that solutions of (5.7a-d) exist for $1 < K < 2$ in fact, with $P_1 > 0$, while for the value (5.6) in question here a numerical solution gives the graph in fig. 3(b) and the properties

$$\left. \begin{aligned} f &\sim C_0 P_1^{-K} \eta^K \quad \text{as } \eta \rightarrow \infty, \\ P_1 &= 0.94796, \quad f''(0) = 1.5098. \end{aligned} \right\} \quad (5.8)$$

Therefore the main predictions are

$$-A = P \sim P_1 X^{2(\sqrt{7}-2)/3}, \quad P_1 = 0.94796, \quad (5.9a)$$

$$\tau \sim -f''(0)X^{-(3-\sqrt{7})/2} \quad (5.9b)$$

as $X \rightarrow \infty$. A comparison with the calculations described in §4 proves fairly encouraging, as fig. 4 shows by means of a log-log plot for P . Comparisons with the velocity profiles of fig. 2 and other features seem not discouraging, although hindered again by the slow algebraic decay towards (5.9a,b) (see below) and possibly by the nature of the numerical schemes in reversed flow.

Further terms in the essentially three-regioned structure (i)-(iii) above can be developed in principle. The major relative corrections are algebraically small, stemming from the wall layer (5.7a-d), but the exponentially small terms stemming from the shear layer (i) are really of more concern as a test of the proposed structure. From (i) we have

$$G_0 \sim -C_0 + D_0 e^{\gamma \xi} \quad \text{as } \xi \rightarrow -\infty \quad (5.10)$$

where D_0 is a constant and $\gamma = 2C_0/3$. This forces an underlying

form

$$\psi = -\frac{1}{2}Y^K + \dots + X^\sigma a(\theta) \exp\left[\frac{1}{3}C_0 P_1 X^{m-\frac{1}{3}} g(\theta)\right] + \dots \quad (5.11a)$$

into the inviscid zone (ii) where $0 < \theta \equiv Y/(P_1 X^m) < 1$, while from substitution into (2.1a) $g(\theta)$ satisfies the nonlinear differential equation, in normalized form,

$$-(2-K)\theta^{K-1}g + 2\theta^K g' = g'^2 \quad (5.11b)$$

and $a(\theta)$ satisfies a linear equation. Here the normalized condition $g(1-) = 0$ holds, from (5.10), and self-consistency demands that $g \leq 0$. The solution of (5.11b) is then *e.g. from use of θ^{K+1} as independent variable,*

$$\left. \begin{aligned} (2-K)g &= 2\theta s - \theta^{1-K} s^2 \quad \text{where} \\ \left[\frac{(1+K)s - 3K\theta^K}{2-K} \right]^{\frac{4K-2}{K+1}} \cdot s^2 &= 4\theta \end{aligned} \right\} \quad (5.11c)$$

giving $g \leq 0$ as required. As $\theta \rightarrow 0$, $s \propto \theta^{\frac{K-1}{2}}$ and g tends to a negative constant from above. This implies that certain much higher order terms in the viscous layer (iii) can grow exponentially as $\eta \rightarrow \infty$, to match with (5.11a-c). A like analysis for the supersonic self-induced separation, for instance, leads to similar features.

In the Appendix we generalize the structure above, to demonstrate a link with the asymptotic forms found previously for other *interacting* flows, to indicate why this structure does not apply in those cases, and to demonstrate further properties of boundary layers with algebraic growth.

As a final point, the effect of the FLARE approximation is to convert (5.7a) into

$$f'''' - [kf - \eta f'] \frac{f''}{(k+1)} + p_1 m = 0$$

and a solution satisfying (5.7b-d) ^{may} exist, in contrast with the analogous situation in the supersonic free interaction (Stewartson & Williams 1973).

§6. FURTHER COMMENTS

First, given that the terminal structure downstream in §5 is self-consistent and is attained, as we believe, it is interesting that the rising pressure does not approach a plateau value and that, nevertheless, the pressure force has only a secondary role in most of the flow downstream. Neither the detached shear layer (i) nor the inviscid reversed flow region (ii) beneath it is controlled significantly by the adverse pressure gradient. This is an unusual feature, as is the viscous reversed sublayer (iii) where the pressure force becomes significant and the similarity layer has to admit algebraic *growth*. Second, we note that all the layers (i)-(iii) far downstream are inter-dependent and strongly affected by viscosity, despite their quite different characters. For the value of the constant P_1 , which fixes the position of layer (i) and the back flow in layer (ii), can be determined only from the properties of the viscous sublayer (iii) at the wall. This too is a novel feature. It implies that beyond the present structure, as the disturbance responsible for the upstream interaction and separation is approached, the flows near and far from the wall must remain inter-dependent, and in particular the properties there cannot be governed solely by inviscid mechanics.

A third point is that, in consequence, the determination of the separated motion near the disturbance downstream should be of considerable interest theoretically as well as practically. A complete account of an hydraulic jump for instance remains to be found, if it can be in steady flow (see below), although we believe the initial upstream response, separation and breakaway above may form the basis for such an account^{*}. Similar comments apply to the hypersonic flow past a sizeable surface-mounted obstacle downstream, for example.

Fourth, concerning hydraulic jumps, two other main length scales can enter into consideration as noted in §3. The shorter scale is the subject of a study by Gajjar (1983), while the longer $O(\text{Re})$ scale poses an intriguing boundary layer problem (involving branching of solutions and presumably the possibility of separation and reattachment) which has still to be tackled seriously. We note that if the wall downstream becomes slightly inclined downwards the flow can approach a fully developed half-Poiseuille form ultimately with a uniform liquid depth there. On a horizontal wall, however, or with a slight upward inclination, such an ultimate form is not attainable and neither are Watson's (1964). Fifth, therefore, it is very unlikely that a steady hydraulic jump from one uniform level to another can be maintained, with a fluid of any nonzero viscosity, if the wall does not slope downwards *ultimately*.

Finally, the predicted blunt shape ($\eta \propto X^m$, $m = 0.43050$) of the free surface beyond the start of the interaction seems not inconsistent with the experience of hydraulic jumps in reality, e.g. in the kitchen sink or in rivers. Three-dimensionality and unsteadiness always affect the practical situation of course, as does the finiteness

*

We refer here to an account consistent with the Navier-Stokes equations, rather than the traditional inviscid descriptions which clearly do not yield such consistency.

of the Reynolds number, but a local quasi-planar quasi-steady view does not seem unreasonable for the start of the jump. A gradual raising of the liquid level downstream, for instance, as in Craik et al's (1981) interesting experiments on axisymmetric hydraulic jumps, would tend to force our supercritical ^{*} interaction to move further upstream and hence to shorten its length scale in response (from §3), until the super/subcritical ^{*} stage $\lambda = O(Re^{3/5})$ of Gajjar (1983) is entered, in which significant waves and possibly reattachment can appear. All of this could take place in an effectively steady planar fashion locally although the time scale (see also below) need not be remarkably short, $O(\lambda_\infty^2/Re_\infty)$, to alter the structure of (3.1a) - (3.5), whereas the shortness of the streamwise length scale relative to the development length scale would tend to diminish most three-dimensional effects. The theoretical features above bear some resemblance to Craik et al's observations although as yet only qualitative agreement with any experiments could properly be claimed since the continuation downstream of the form described in §5 has still to be studied. Again, concerning unsteadiness, it may be significant that the unsteady version of the present interaction admits the Tollmien-Schlichting instability at a sufficiently large distance from the original source of fluid.

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FOOTNOTE

^{*} Here supercritical and subcritical refer to the Froude number ($Fr \gtrless 1$ respectively) as distinct from referring to the Reynolds number for instability, the latter being considered by Craik et al (their experiments suggest stability for the longer scale interactions).

Final comments

In addition to the comments in ^{§6}₁ we would make the following brief comments.

(1). The above shows that steady theory can account for the start of a hydraulic jump and on any large lengthscale. This start of course is equivalent to a mechanism for upstream influence quite different from unsteadiness as in Benjamin (1970).

(2). It is necessary however to acknowledge the viscous-inviscid interaction involved, in which the induced pressure is proportional to the displaced free surface and the boundary layer displacement.

(3). The above process implies separation (contrary to most inviscid theories but in line with the experiments of Craik et al.)

(4). Beyond separation the eventual displacement and free surface, though small, grow algebraically like X^m . Therefore some new physical balances must come into operation on a longer streamwise lengthscale downstream before a complete account of the hydraulic jump is achieved.

(5). There exists some qualitative experimental evidence for the " X^m " shape, the general picture proposed in this chapter.

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A P P E N D I X

The structure discussed in §5 also holds in a more general sense when the pressure-displacement law is

$$\mathcal{P} = - \frac{A(X)}{X^N} \quad (A1)$$

or $\mathcal{P} = -(d/dX)^N \Lambda$, say. We take the case (A1). Then $A \sim -P_1 X^m$ for $X \gg 1$ provided that M, N satisfy

$$m - N = 2 \frac{(2-3m)}{(2+3m)} = 2 \frac{(K-1)}{(K+1)} \quad (A2)$$

$$\frac{1}{3} < m < 2/3, \quad -\frac{1}{3} < N < 2/3 \quad (A3)$$

and provided also that a solution of (5.7 a-c) exists with P_1 positive.

For $1 < K < 2$ the similarity problem (5.7a-c) requires a numerical treatment, although the limiting cases $K = 1+, 2-$ can be further analyzed.

First, for $K = 2-$, write

$$f = \lambda \bar{f}(\theta), \quad \theta = \lambda \gamma, \quad \gamma = \frac{2P_1}{\lambda^4} \quad (A4)$$

$$\text{where } \lambda = \left(\frac{C_0}{P_1 K} \right)^{\frac{1}{K+1}}.$$

This transforms (5.7a-c) into the form

$$\bar{f}''' - \frac{K}{K+1} \bar{f} \bar{f}'' + \frac{(K-1)}{(K+1)} (\gamma + \bar{f}'^2) = 0 \quad (A5)$$

$$\bar{f}(0) = \bar{f}'(0) = 0, \quad \bar{f} \sim \theta^K \text{ as } \theta \rightarrow \infty. \quad (A6,7)$$

Let $K = 2 - \Omega$ ($0 < \Omega < 1$) and expand $\bar{f} = \bar{f}_0 + \Omega \bar{f}_1 + \dots$, $\gamma = \gamma_0 + \Omega \gamma_1 + \dots$.

Substitution into (A 5-7) then yields $\bar{f}_0 = \theta^2$, $\gamma_0 = 0$, ←

← leaving the problem for $\bar{f}_1$:

$$\bar{f}_1''' - \frac{2\theta^2}{3} \bar{f}_1'' + \frac{4\theta}{3} \bar{f}_1' - \frac{4}{3} \bar{f}_1 = \frac{-\gamma_1}{3} + \frac{2\theta^2}{3} \quad (A8)$$

$$\bar{f}_1(0) = \bar{f}_1'(0) = 0, \quad \bar{f}_1 \sim -\theta^2 \log \theta \quad \text{as } \theta \rightarrow \infty. \quad (\text{A9,10})$$

A series solution $\bar{f}_1 = \sum_{n=0}^{\infty} a_n \theta^n$ gives

$$\begin{aligned} \bar{f}_1 = a_2 \theta^2 + 9a_3 \left\{ \theta^2 \int_0^{\theta} t^{-3} (e^{\frac{2}{9}t^3} - 1) dt - \theta \int_0^{\theta} t^{-2} (e^{\frac{2}{9}t^3} - 1) dt \right\} \\ + 12 a_5 \left\{ \theta^2 \int_0^{\theta} t^2 M(1, \frac{8}{3}, \frac{2}{9}t^3) dt - \theta \int_0^{\theta} t^3 M(1, \frac{8}{3}, \frac{2}{9}t^3) dt \right\}. \end{aligned} \quad (\text{A11})$$

$$\text{Here } 6a_3 = -\frac{\gamma_1}{3}, \quad 60 a_5 = \frac{2}{3}, \quad (\text{A12})$$

and $M(a,b,x)$ is the confluent hypergeometric function (see Abramowitz and Stegun, p.504). To avoid exponential growth we thus require

$$9 a_3 = -12 a_5 \Gamma(8/3) \left(\frac{2}{9}\right)^{-5/3}, \quad \text{giving}$$

$$\gamma_1 = 6 \Gamma(2/3) \left(\frac{2}{9}\right)^{1/3} > 0. \quad (\text{A13})$$

Also noting that

$$\theta^{-3} \Gamma(8/3) \left(\frac{2}{9}\right)^{5/3} e^{\frac{2}{9}\theta^3} - \theta^2 M(1, 8/3, \frac{2}{9}\theta^3)$$

$$\sim \frac{15}{2} \theta^{-1} (1 + 3\theta^{-3} + \dots) \quad \text{for } \theta \gg 1,$$

we obtain from (A8), using (A12), $\bar{f}_1 \sim -\theta^2 \log \theta + \frac{\gamma_1}{4} + O(\frac{1}{\theta})$ as required.

Then a_2 is determined by the requirement that there is no θ^2 term in the expansion of (A11) for large θ . A perhaps simpler argument leading to (A13) is to differentiate (A8) and then use a boundedness condition on $\bar{f}_1'''$.

For $K = 1+$ we first transform (5.7a-c) by writing

$$f = \sigma \hat{f}(\phi), \quad \phi = \sigma \eta, \quad \sigma = (2P_1)^{1/4}, \quad (\text{A14})$$

$$\text{so that} \quad \hat{f}''' - \frac{K}{K+1} \hat{f} \hat{f}'' + \frac{(K-1)}{(K+1)} (\hat{f}^{1/2} + 1) = 0, \quad (\text{A15})$$

$$\hat{f}(0) = \hat{f}'(0) = 0, \quad \hat{f} \sim \frac{C_0}{\sigma^{1+K}} \left(\frac{\phi}{P_1}\right)^K \quad \text{as } \phi \rightarrow \infty. \quad (\text{A16,17})$$

Let $K = 1 + \beta$ ($0 < \beta \ll 1$). Then (A15-17) splits into two sublayers: see also Smith (1980). In the inner sublayer, $\hat{f} = \beta^{1/4} \xi_1 + \dots$ and

$$\phi = \beta^{1/4} \xi_1 + \dots, \quad \text{giving}$$

$$g_1''' - \frac{1}{2} g_1'' g_1 + \frac{1}{2} = 0 \quad (\text{A18})$$

with $g_1(0) = g_1'(0) = 0$, and we have to impose $g_1 \sim \xi_1 (2 \log \xi_1)^{\frac{1}{2}}$ as $\xi_1 \rightarrow \infty$

instead of (A17). The solution of (A18) has been obtained by Smith (1980),

yielding $g_1''(0) = \frac{1.436}{\sqrt{2}}$. In the outer sublayer

$$\phi = \xi^{-1/4} e^{\xi/\beta}, \quad \frac{\Lambda}{f} = \phi g_2 + \dots \quad (\text{A19})$$

The equation for g_2 is then $g_2 g_2' - 1 - g_2^2 = 0$, with the solution being

$g_2 = (e^{2\xi/\beta} - 1)^{\frac{1}{2}}$ to match with the inner sublayer. The outer condition now gives $\frac{\Lambda}{f} \sim \phi e^{\beta \log(\phi \beta^{1/4})}$. ←

Hence, from (A14), (A17),

$$P_1 = \left[\frac{C_0}{\sqrt{2}} \frac{\xi^{-1/4}}{\beta} \right]^{2/3} \quad (\text{A20})$$

[fig. 5]

For $1 < K < 2$ the problem (5.7a-c) was solved numerically with the equations written as a first order scheme and with central differences, λ for $0 \leq \eta \leq \eta_\infty$. The equivalent forms (A8-10), (A15-17) were also tried and a stretched grid was used to handle (5.7c). This latter constraint, allowing algebraic growth, caused a lot of difficulties and was replaced by an equivalent condition involving f, f', f'' , eliminating η_∞ and leading to a more stable scheme. Checks were made by changing the grid sizes and altering the value of η_∞ .

Comparisons with the asymptote (A13) near $K = 2$ showed good agreement. However, calculations for values of K near 1 proved to be difficult. There η_∞ had to be increased substantially, but this still left P_1 difficult to determine directly. This was probably because of the exponentially growing outer sublayer as $K \rightarrow 1 +$; see (A19). Fig. 5^(c) shows a graph of η_∞ vs. P_1 for some values of K near 1.

Finally it is worth re-emphasising that in all other known free interaction studies of separation the pressure tends to a constant value downstream, whereas here solutions of the similarity problem allowing algebraic growth have been obtained with the pressure growing algebraically, $P \sim P_1 x^{m-N}$. In the limiting cases, as $m \rightarrow \frac{1}{3}+$, $N \rightarrow -\frac{1}{3}+$, the constant P_1 tends to zero [(A4,A13)]; at the other extreme, $m \rightarrow \frac{2}{3}-$, $N \rightarrow \frac{2}{3}-$, the pressure P tends to a constant, from (A19), but the above structure breaks down then because $|P|$ and $|u^2|$ become comparable in the inertial layer. Thus our algebraic form holds for $-\frac{1}{3} < N < \frac{2}{3}$, whereas a constant pressure plateau is found to be appropriate for $N > \frac{2}{3}$ (including the supersonic case where $N = 1$). The case $N = 0$ reduces to that discussed in §5.

FIG. 1

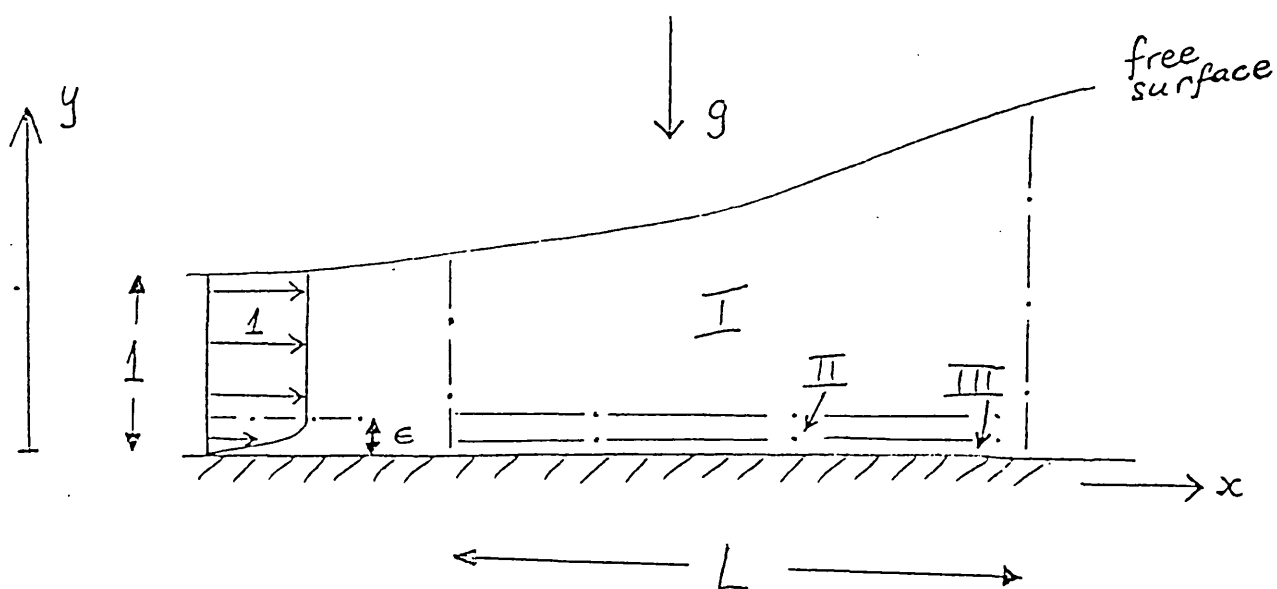


FIG. (2a)

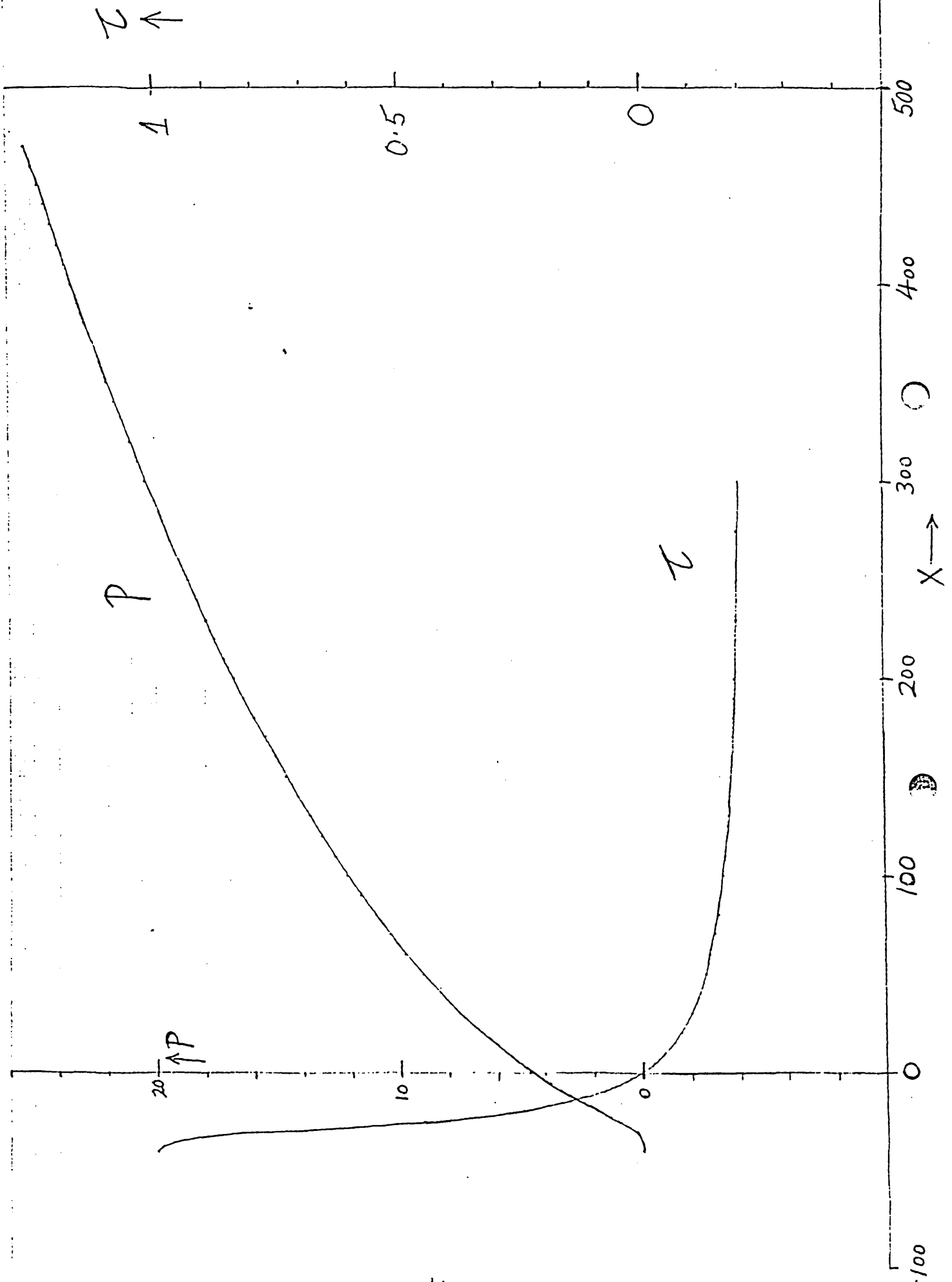


FIG.

2b

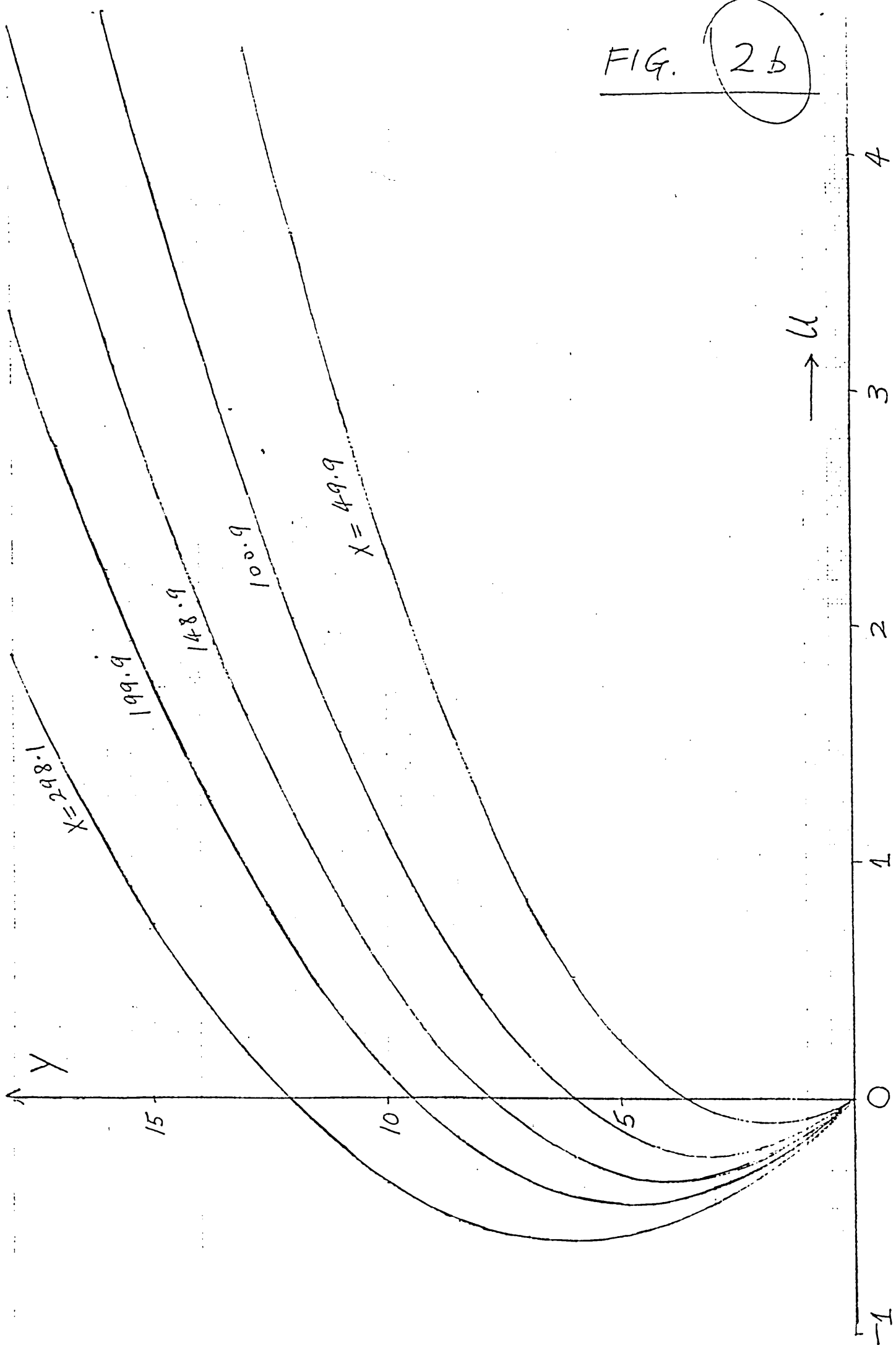


FIG. (3a)

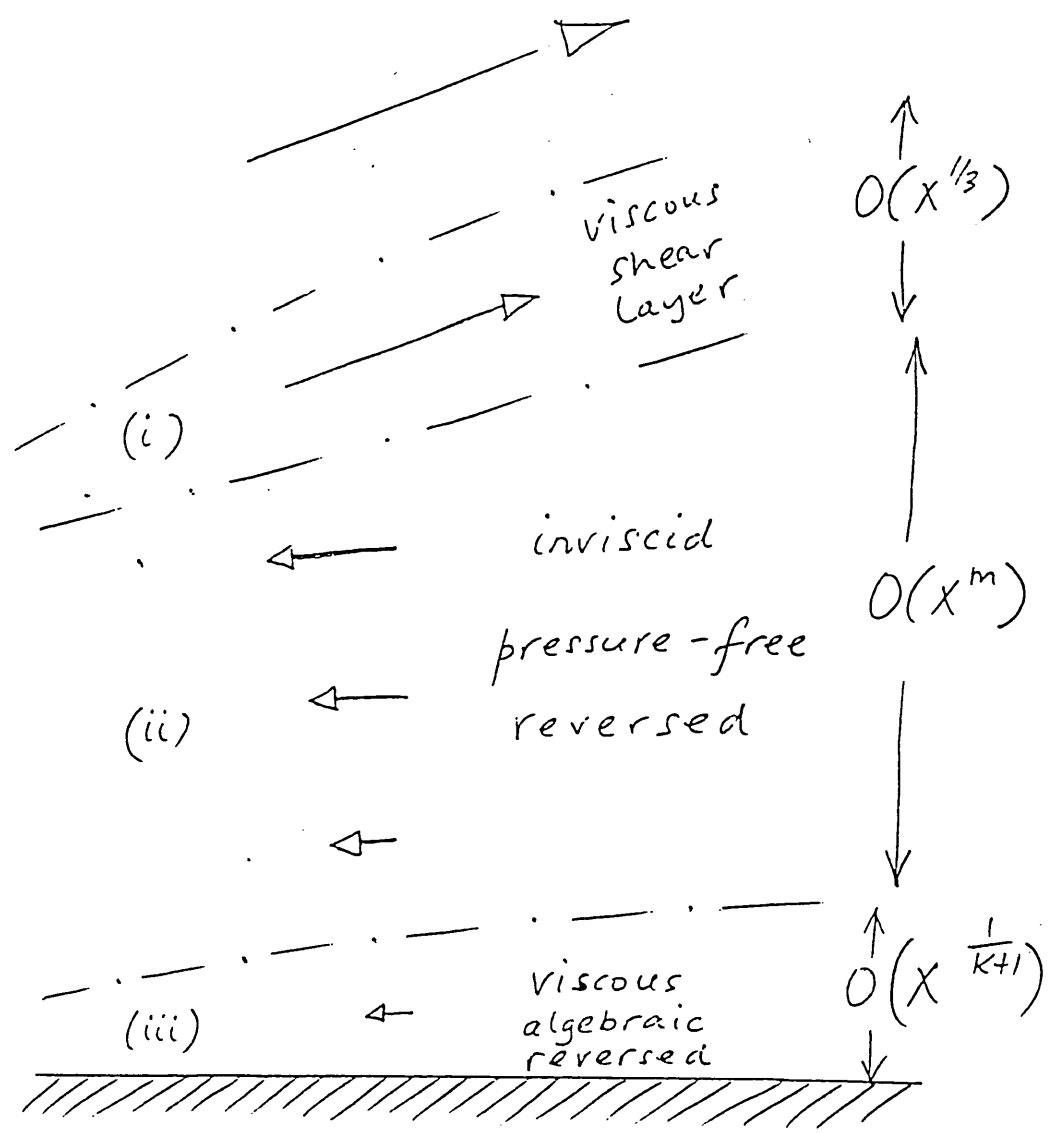
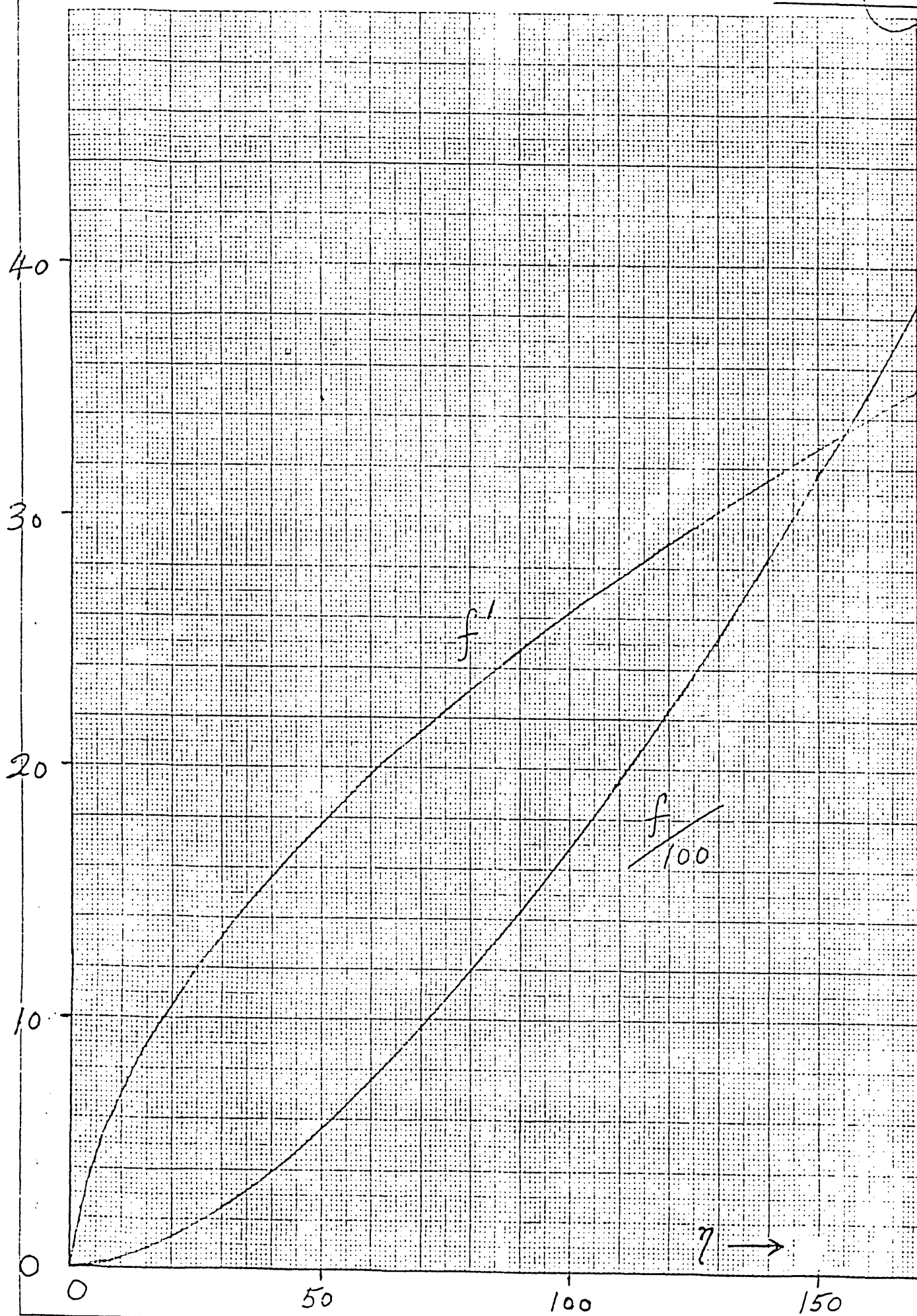


FIG. (35)



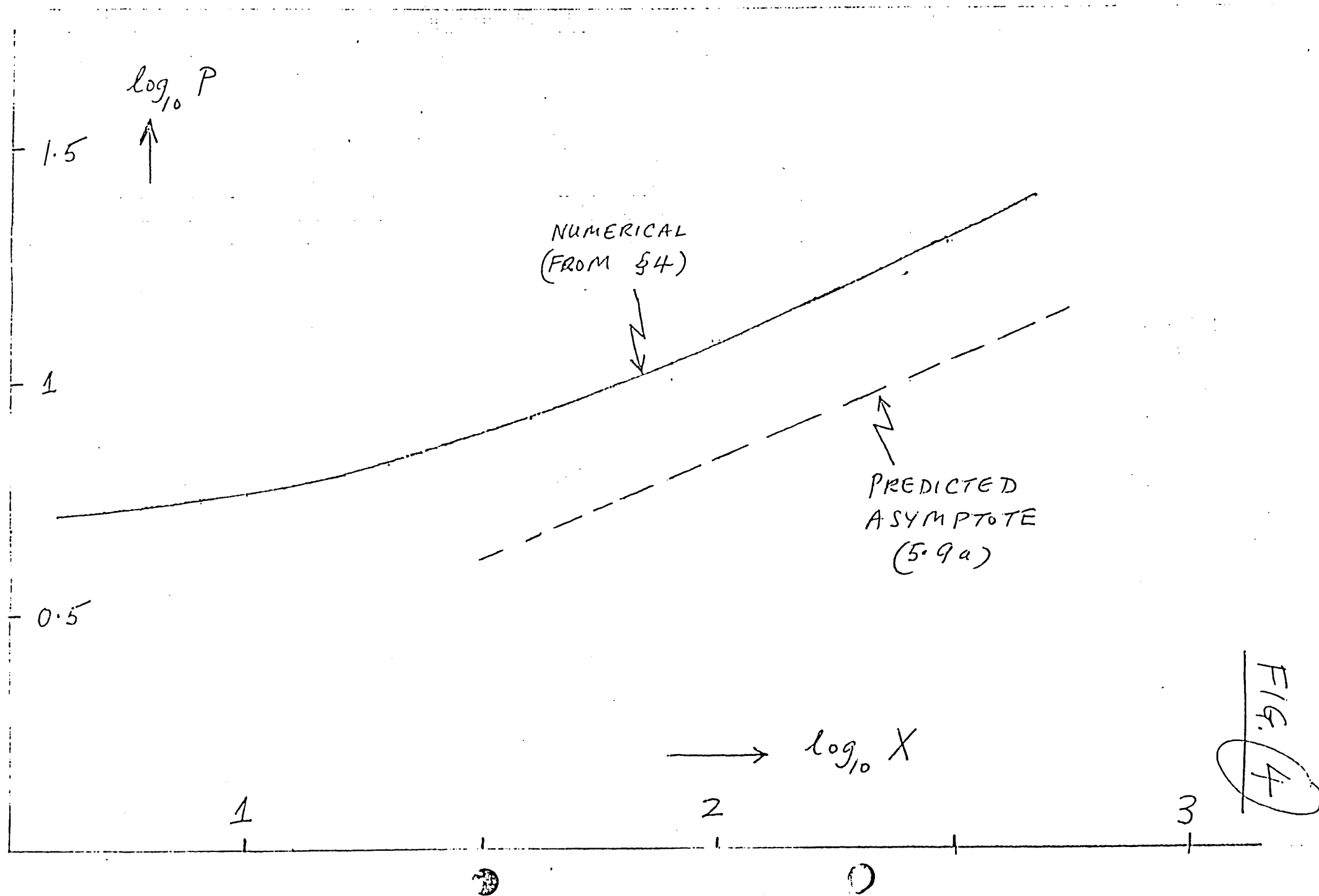


FIG. 4

Fig 5a

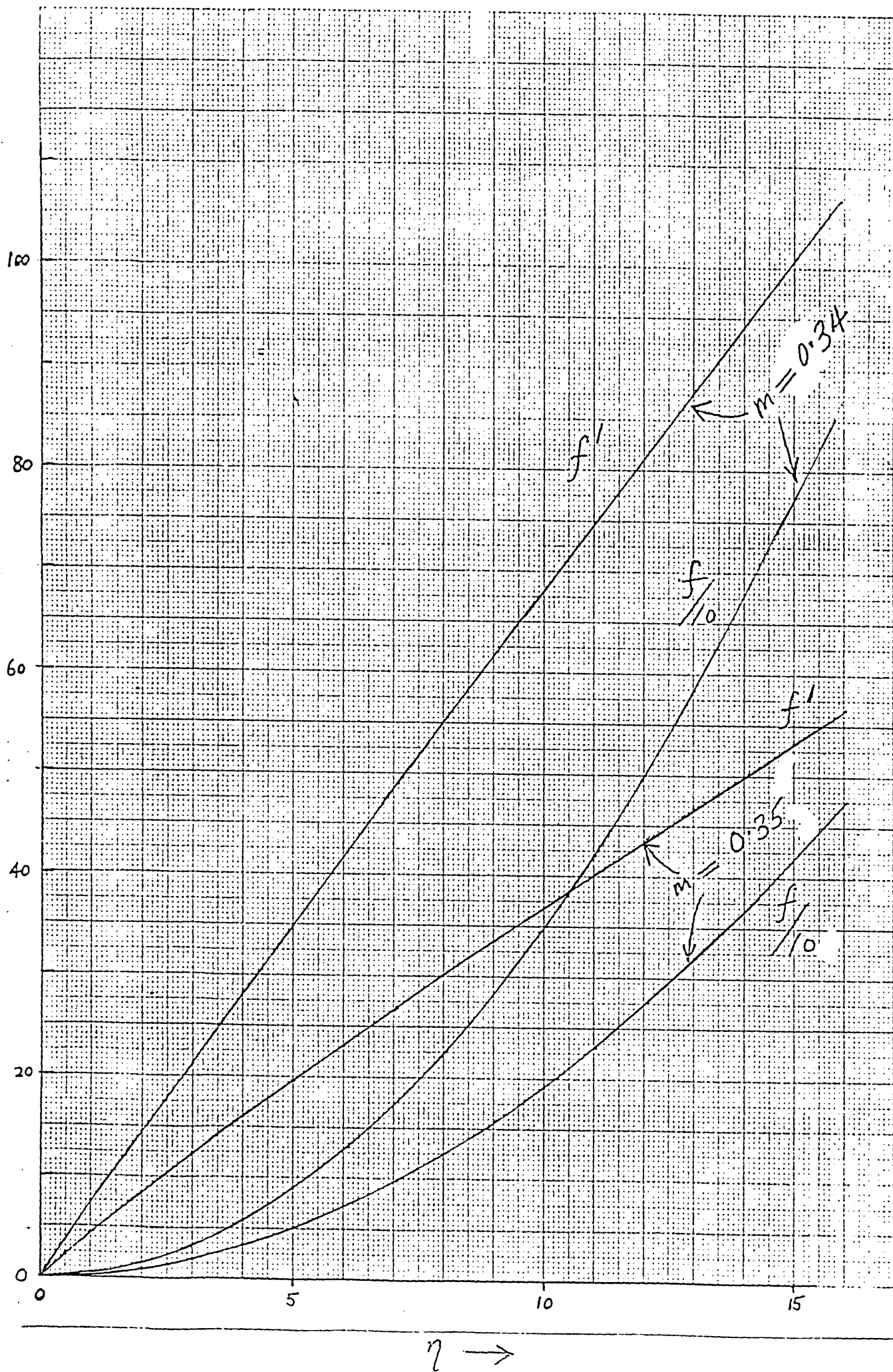
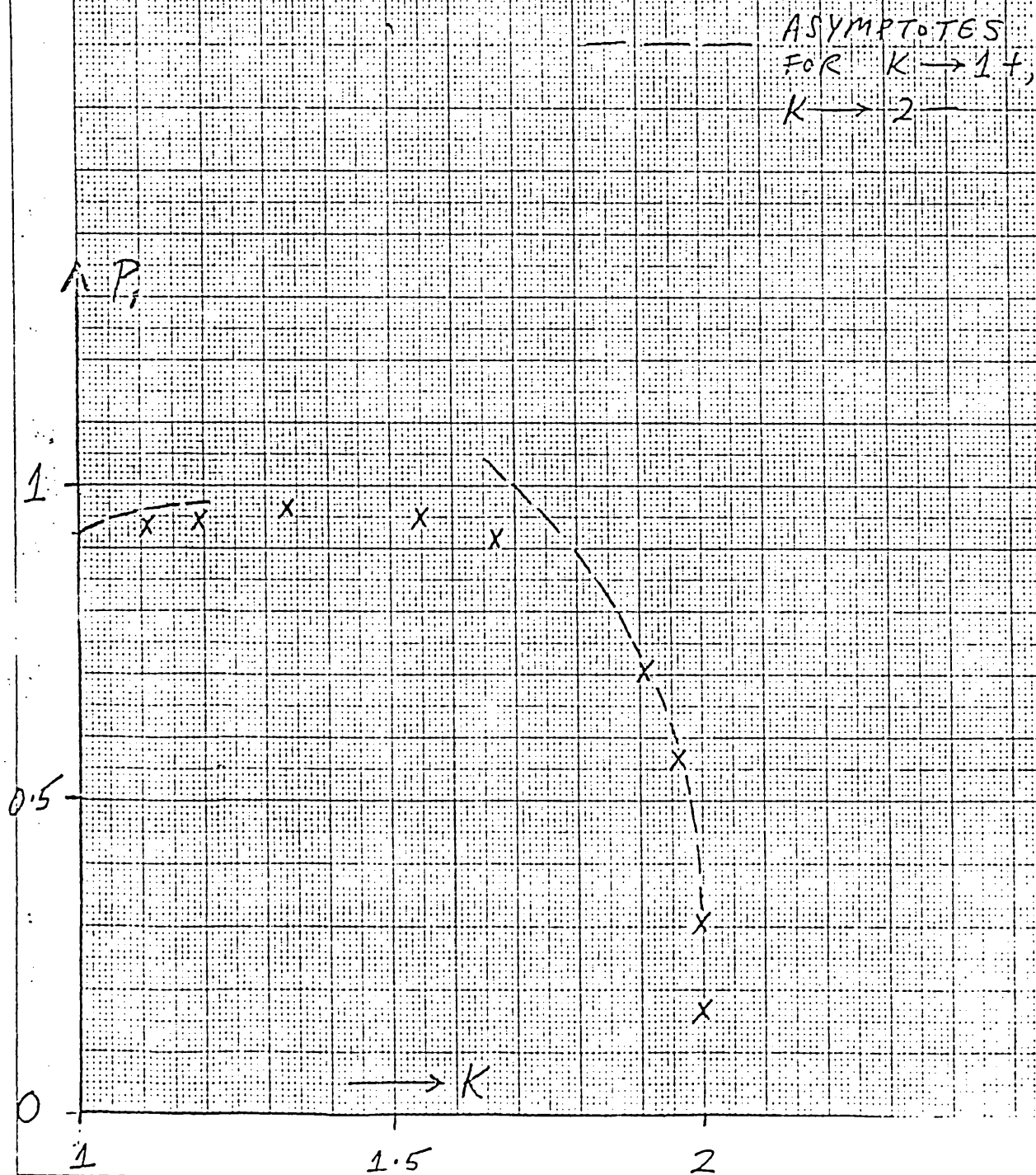


FIG. 56



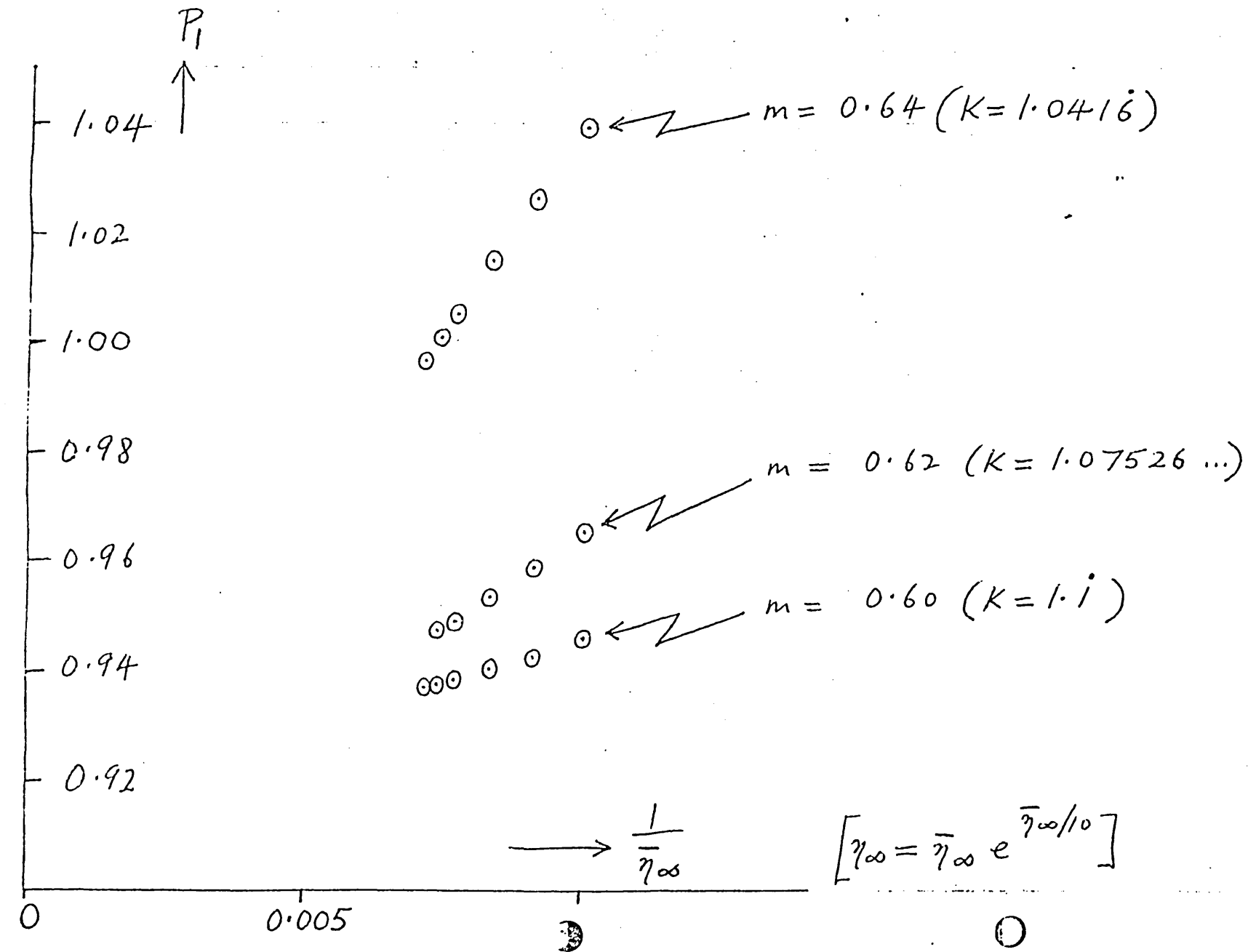
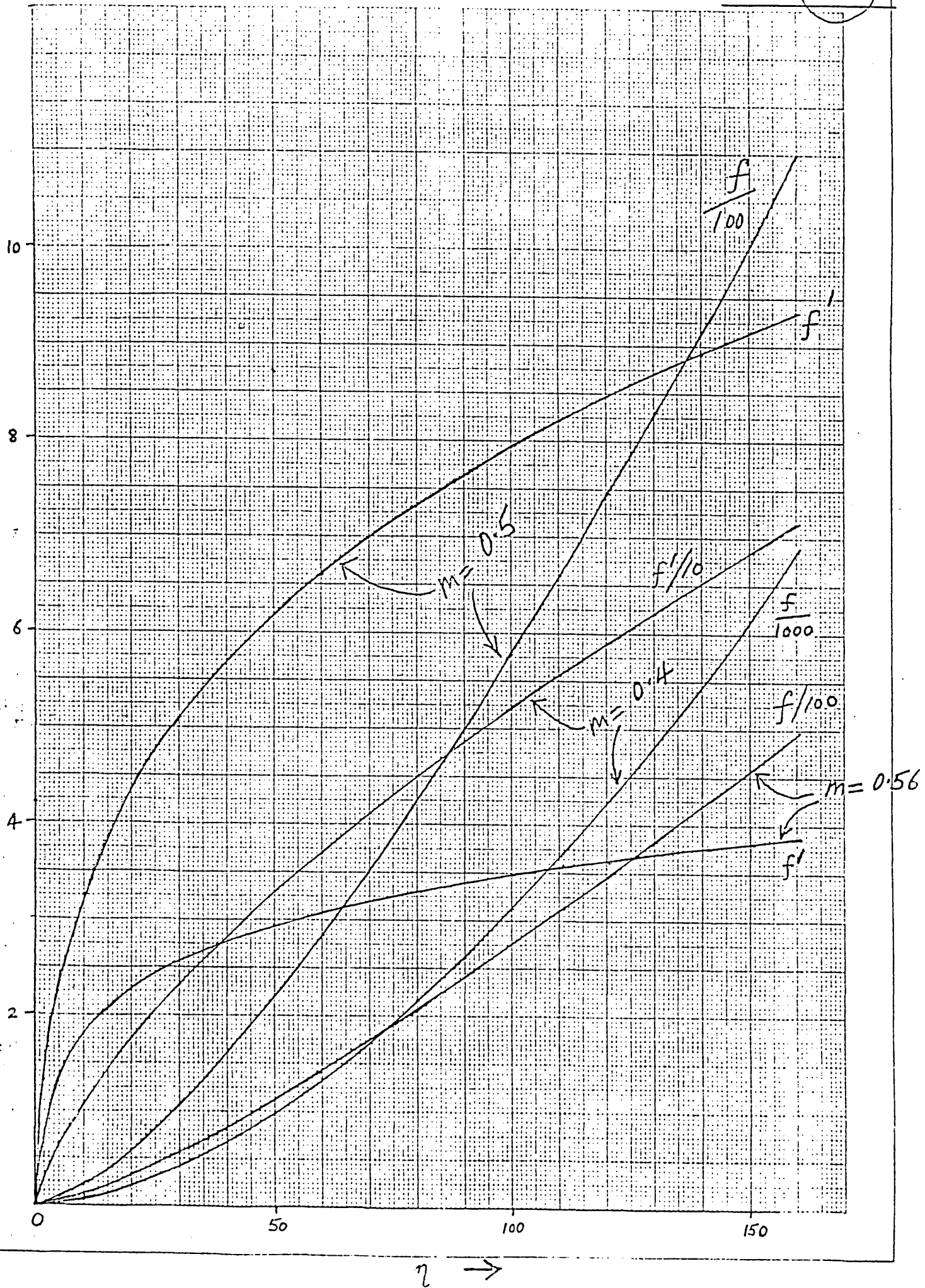


FIG. 5c

FIG. 5a₂



CAPTIONS

- Fig.1 The structure of the free interaction (§3) possible in liquid layer flow along a horizontal wall ($y=0$) when the oncoming boundary layer thickness δ is small.
- Fig.2 Numerical results (§4) for (a) the pressure P and skin friction τ , versus X , and (b) sample velocity profiles.
- Fig.3 (a) The proposed ultimate structure (i)-(iii) as $X \rightarrow \infty$: see §5.
(b) The numerical solution of (5.7a-d) for $f(\eta)$, $f'(\eta)$, versus η , when (5.6) holds (see also fig. 5 below).
- Fig.4 Comparison between the numerical values (§4) for P and the predicted asymptote (5.9a) as $X \rightarrow \infty$, in log-log form.
- Fig.5 (a) Numerical solutions of (5.7a-d) for $f(\eta)$, $f'(\eta)$, versus η , for various values of K between 1 and 2. (b) The corresponding solutions (X) for the constant P_1 , and comparisons with the limiting analytic forms holding as $K \rightarrow 1+$, $K \rightarrow 2-$.
(c) The dependence of P_1 on the outer boundary position η_∞ (for K near 1), where $\eta_\infty = \bar{\eta}_\infty \exp(\bar{\eta}_\infty/10)$ is used and the limit as $\frac{1}{\bar{\eta}_\infty} \rightarrow 0$ is required. See Appendix. Here $m = 2/3K$ from (A2).

CHAPTER 5

LOWER BRANCH STABILITY OF FULLY DEVELOPED AND
UNDEVELOPED FREE SURFACE FLOWS

§1 Introduction

In this chapter we examine briefly the lower branch stability properties of the fully developed and undeveloped flows considered in chapters 2, 3. The lower-branch, as opposed to the upper-branch, is used to emphasize that the critical layer (where the local speed is equal to the wavespeed), is close to and is not asymptotically distinct from the wall layer. The upper-branch stability of liquid layer flow (or fully-developed flow) is considered in the next chapter.

The stability of liquid layer flow over an inclined plane has been studied by various people based largely, on the major contribution due to Yih (1954, 1963), and Benjamin (1957), (see also Yih (1965)). However all these papers are concerned primarily with the stability at low Reynolds numbers, (or more strictly with the case $\alpha R \ll 1$, where α is the wavenumber and R the Reynolds number). The main conclusions from these studies are, (ignoring surface tension), that (i) $\alpha = 0$ is part of the neutral stability curve, and (ii) there is a neutral branch from the $\alpha = 0$ axis at $R = \frac{5}{6} \cot \beta^*$, (β^* is the inclination of the plane to the horizontal). A possible neutral curve is sketched in Yih (1965). It is questionable whether their work is applicable at large Reynolds numbers and also when the inclination of the plane to the horizontal is much smaller.

Our concern here is instead with the stability properties at large Reynolds number, and we follow closely the structural approach as for example in Smith (1979a,b). It is found in §2 that long wave neutral modes exist for fully developed flows, and, not too surprisingly,

the asymptotic lower-branch properties are similar to those of channel flow or plane Poiseuille flow, (see ^{Smith (1979c),} Hall & Smith (1982)). The neutral wavenumber and wavespeed depend on the Froude number (or the angle of inclination of the plane in the liquid layer flow). For large Froude numbers a match with channel flow properties is achieved, but as the Froude number decreases to $O(1)$, (or the plane becomes more horizontal), the wavelength shortens, and the wavespeed decreases with the critical layer moving closer to the wall. When the wavelength becomes $O(1)$ a new structure emerges and this is addressed in §3. The stability properties there are governed by the shorter Rayleigh-type modes but with a major difference in that the waves are non-simple and the spatial and temporal variations of the wave-amplitude depend critically on the interactions between the inviscid and viscous regimes. This structure, which may also be relevant to the stability of the Rayleigh modes in channel-flows, does not seem to have been recognised before.

(Thompson (1949) has discussed the stability of long and $O(1)$ length waves on a non-uniform flow by assuming a velocity profile made up of piecewise linear shears. He considers only inviscid modes, and rather questionably dismisses any solutions which would give critical layers in the interior of the flow. Burns (1953) has derived a dispersion relation for long waves on a non-uniform flow by making the shallow water approximations. In his work the wavespeed is assumed to be $O(1)$ and outside the range of basic flow. In fact the results he obtains would allow upstream propagating waves: see also Fenton (1972), and the survey by Peregrine (1976). Yih (1972) has discussed the linear inviscid stability of free surface flows and has proved various interesting results. One such result is that if the basic flow $U_g(y)$ is monotonic within the flowfield, and $U_g''(y)$ does not change sign then critical layers cannot occur and the flow is stable. Velthuisen and van Wijngaarden (1969) have attempted to include viscous effects with critical layers occurring in the interior of the flow. However their work can be regarded as incomplete and they do not discuss the stability of the flow. For, it is not clear in their work what assumptions are made, e.g., is the wavespeed $O(1)$, or dependent on some power of the Reynolds number? They assume a jump of $-i\frac{1}{2}$ across the critical layer, which needs to be justified since the velocity jump across a critical layer can depend on various factors such as the amplitude of the wave, the wavespeed and the wavelength.) A match with the long-wave modes considered in 12 is shown to be possible.

Neutral modes exist also for undeveloped free surface flows, and this is considered in 14. The results presented may be relevant

to the stability of open channel flow, far downstream of the inlet. An interesting point is that a special case of the neutral modes here corresponds to the 'odd-mode' of developing flow in a channel Smith & Bodonyi (1980). For supercritical flows one mode exists. For subcritical flows there is an additional mode. This latter mode, in the limiting cases, may be identified with the Tollmien-Schlichting mode associated with boundary layer instability. The other mode (again in the limiting cases), seems to be connected with the wavy solution occurring for steady subcritical flow.

Finally additional remarks are made in §5.

§2 Fully Developed Flows-Long Wavelength Modes

We start first with fully developed flows. The essential structure has been set out in chapter 2, and consequently we shall omit some of the details. It suffices to consider the unsteady version of the basic lower deck problem in chapter 2, viz.

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0 \quad , \quad \frac{\partial P}{\partial Y} = 0 \quad (2.1a,b)$$

$$\frac{\partial U}{\partial T} + U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = - \frac{\partial P}{\partial X} + \frac{\partial^2 U}{\partial Y^2} \quad (2.1c)$$

$$U \rightarrow \lambda(Y + A(X,T)) \left(\begin{array}{l} \infty Y \rightarrow \infty \\ U = V = 0 \text{ at } Y = 0 \end{array} \right), \quad (2.2a,b)$$

$$P = -sA(X,T) - \gamma A_{xx}, \quad \eta(X,T) = -A(X,T) \quad (2.2c,d)$$

Here the $O(1)$ time scale $T = Re^{-3/7} t$, has been introduced, and it may be verified that the leading order solution in the inviscid region is unchanged apart from the additional time dependence of the unknown quantities A , P , η . The skin friction parameter λ of the basic flow $U_B(y)$ has been introduced in (2.2a), so that we are considering a more general fully-developed flow. See chapter 2 for the rest of the notation. It is noted that $U_B(y)$ satisfies,

$$U_{By}(1) = 0, \quad U_B \sim \lambda y + \lambda_2 y^2 \text{ as } y \rightarrow 0.$$

As in Smith (1979b) we consider a disturbance to the basic flow $U = \lambda Y$, $A = P = \eta = 0$, and write

$$(U, V, P, A, \eta) = (\lambda Y, 0) + \Delta(\tilde{U}, \tilde{V}, \tilde{P}, \tilde{A}, \tilde{\eta})E + \dots \quad (2.3)$$

where $\Delta \ll 1$ is the infinitesimal disturbance size, and we define $E = \exp(i\alpha_0 X - i\beta_0 T)$, where $\alpha = R^{-1/7} \alpha_0 + \dots$, $\beta = R^{-3/7} \beta_0$ are the wavenumber and frequencies respectively, and also the unknown constants $\tilde{A}$, $\tilde{P}$, $\tilde{\eta}$ are independent of X , T , and $(\tilde{U}, \tilde{V})$ depend on Y only. $C_0 = \beta_0 / \alpha_0$ is the scaled wavespeed. The scalings are suggested by the basic structure in chapter 2, as e.g. in Smith (1979b), Smith & Bodonyi (1980).

Substituting (2.3) into (2.1), (2.2) we obtain the linear problem

$$i\alpha_0 \tilde{U} + \tilde{V}_Y = 0, \quad -i\beta_0 \tilde{U} + i\alpha_0 \lambda Y \tilde{U} + \lambda \tilde{V} = -i\alpha_0 \tilde{P} + \tilde{U}_{YY}$$

$$\tilde{U} = \tilde{V} = 0 \quad \text{at } Y = 0, \quad \tilde{U} \rightarrow \lambda \tilde{A} \quad \text{as } Y \rightarrow \infty. \quad (2.4)$$

(2.4) is solved as in Smith (1979b) and leads to the relation

$$\frac{Ai'(\xi_0)}{K \xi_0} = - \frac{\tilde{P}}{C_0 \lambda} \quad (2.5)$$

where Ai is the Airy function, $K = \int_{\xi_0}^{\infty} Ai(t) dt$ and $\xi_0 = -i\beta_0 (i\alpha_0 \lambda)^{-2/3}$.

Here the branch cut is defined by $\arg(i\alpha_0)^n = e^{in\pi/2}$ if α_0 is real. From the inviscid region and the free surface conditions, or equivalently from (2.2cd) we obtain

$$\tilde{\eta} = -\tilde{A}, \quad \tilde{p} = -(\mathfrak{s} - \gamma \alpha_0^2) \tilde{A}. \quad (2.6)$$

Hence from (2.5), (2.6) we obtain the eigenrelation

$$\frac{Ai'(\xi_0)}{K \xi_0} \equiv -1 + F(\beta_0 (\alpha_0 \lambda)^{-2/3}) = \frac{(\mathfrak{s} - \gamma \alpha_0^2)}{\lambda C_0}. \quad (2.7)$$

Here $F(Z)$ is the Tietjens function, Reid (1965). The neutral requirements i.e. α_o, β_o , real, then gives (Reid (1965))

$$\beta_{on} (\alpha_{on} \lambda)^{-2/3} = 2.297, \quad \left(\frac{s - \gamma \alpha_{on}^2}{\lambda c_{on}} \right) = -0.4355 \quad (2.8)$$

which leads to the dispersion relation

$$\alpha_{on}^{1/3} (\gamma \alpha_{on}^2 - s) - 1.001 \lambda^{5/3} = 0 \quad (2.9)$$

for the neutral wavenumber α_{on} , with the wavespeed c_{on} , and frequency β_{on} , determined from (2.8).

It is shown in the appendix that if $\frac{P}{A} = \frac{g(\alpha)}{\alpha}$, then the departure from the neutral states in (2.7) is fixed by the criteria

$$\text{Im}(c_o - c_{on}) \geq 0 \quad \text{if} \quad (\alpha_o - \alpha_{on}) \left(g'(\alpha_{on}) - \frac{2}{3} \frac{g(\alpha_{on})}{\alpha_{on}} \right) \geq 0 \quad (\text{assuming } \alpha_o \text{ is real}).$$

Applying this condition with $g(\alpha)/\alpha = -(s - \alpha^2)$ then gives $\text{Im}(c_o - c_{on}) \geq 0$ if $(\alpha_o - \alpha_{on}) \geq 0$. Hence the lower branch of the neutral curve is similar to that of channel flow.

The equation (2.9) (with $\gamma > 0$) has only one real root, and fig.(1) gives a graph of the roots for different values of s . If $s=0$ from (2.9) we obtain

$$\alpha \sim R^{-1/7} \quad \alpha_{on} = 1.000 \lambda^{5/7} \gamma^{-3/7} R^{-1/7}, \quad (2.10)$$

and this is the odd mode occurring in channel flow, after allowing for the different notation, see Smith & Bodonyi (1980). (We note

that $s=0$ corresponds to $\beta^* \sim O(R^{-3/7})$ from chapter 2). The results of Yih-Benjamin cannot be recovered here because of the condition $\alpha R \gg 1$ implicit in the above analysis. If $s \gg 1$ (i.e. $\beta^* \ll O(R^{-5/7})$), then

$$\alpha_{on} = \left(\frac{s}{\gamma}\right)^{1/2} + \frac{1.001\lambda}{2\gamma^{1/3}} s^{5/3} s^{-2/3} + \dots, \quad c_o = \frac{2.297\lambda^{2/3}}{\alpha_{on}^{1/3}}. \quad (2.11)$$

Hence the wavelength shortens and the critical layer retreats towards the wall. Thus for decreasing Froude numbers (or as the plane becomes more horizontal) (2.9), (2.11) show that the dominant balance occurs between the induced hydrostatic force and the force due to the non-zero transverse pressure gradient.

Formally, since the effective Froude number is $O(R \frac{2/7}{s})$, as s increases to $O(R^{2/7})$, the Froude number decreases to $O(1)$, and from (2.11) the wavenumber and frequency are given by

$$\alpha \sim \left(\frac{R^{-2/7}s}{\gamma}\right)^{1/2} + O(R^{-1/3}), \quad \beta \sim O(R^{-1/3}). \quad (2.12)$$

Also an examination of the disturbance velocities and pressure in the inviscid region shows that they are of the same order. We turn next to a discussion of the new structure which emerges on this shortened lengthscale.

§3 The Stability Properties for $O(1)$ Wavelengths

The scalings and expansions below are suggested by the above discussion. In view of (2.12), we introduce the slow scales $\tilde{x} = R^{-1/3} x$, $\tau = R^{-1/3} t$, and the disturbance quantities now depend on x , $\tilde{x}$, y , and τ (to our order of working).

Firstly, in the inviscid region, the expansions are

$$U = U_E(y) + R^{-1/3} \tilde{U}_0 + R^{-2/3} \tilde{U}_1 + \dots, \quad V = R^{-1/3} \tilde{V}_0 + R^{-2/3} \tilde{V}_1 + \dots,$$

$$p = \frac{1}{2}(1-y) R^{-1/3} \tilde{p}_0 + R^{-2/3} \tilde{p}_1 + \dots, \quad (3.1a-c)$$

and the free surface is given by

$$y = 1 + R^{-1/3} \tilde{\eta}_0 + R^{-2/3} \tilde{\eta}_1 + \dots \quad (3.1d)$$

It will be seen below that it is necessary to look at the second order inviscid problem to effect a match with the boundary layer.

Substituting (3.1) into the Navier-Stokes equations and free surface conditions, yields at the lowest order the problem

$$\tilde{U}_{0x} + \tilde{V}_{0y} = 0$$

$$U_B \tilde{U}_{0x} + U_{By} \tilde{V}_0 = -\tilde{p}_{0x}, \quad U_B \tilde{V}_{0x} = -\tilde{p}_{0y}, \quad (3.2a-c)$$

$$\text{and } \tilde{p}_0 = s\tilde{\eta}_0, \quad \tilde{V}_0 = U_B \tilde{\eta}_{0x} \quad \text{at } y = 1 \quad (3.3ab)$$

In addition to (3.3ab) we have the wall condition

$$\tilde{V}_0 = 0 \quad \text{at } y = 0. \quad (3.3c)$$

We note that the condition of zero tangential stress at the free surface, has been relaxed. A much thinner $O(R^{-1/2})$ layer (similar to the Blasius boundary layer), is needed to satisfy this.

We now seek solutions dependent on

$$\tilde{E} = \exp(i\alpha_0 x), \text{ (with } \alpha_0 \text{ real), and write}$$

$$\tilde{V}_0 = \tilde{V}_{01} \tilde{E} + \text{c.c.}, \quad \tilde{\eta} = \tilde{\eta}_{01} \tilde{E} + \text{c.c. etc.} \quad (3.4)$$

where $\tilde{V}_{01}$, $\tilde{\eta}_{01}$ are independent of x , and c.c. denotes the complex conjugate. (3.2), (3.3) with (3.4) can be manipulated to yield

$$\tilde{V}_{01yy} - (\alpha_0^2 + \frac{U_B''}{U_B}) \tilde{V}_{01} = 0$$

$$\text{with } \tilde{V}_{01} = 0 \text{ at } y = 0, \quad U_B \tilde{V}_{01y} = s \frac{\tilde{V}_{01y}}{U_B} \text{ at } y = 1. \quad (3.5a-c)$$

(3.5) constitutes an eigenvalue problem for determining α_0 as a function of the Froude number, and (3.5a) is in fact the Rayleigh equation with zero wavespeed. After multiplying (3.5a) by $U_B(y)$ and integrating from $y = 0$ to $y = 1$, we obtain

$$\alpha_0^2 = \frac{s (\tilde{V}_{01})_{y=1}}{U_B(1) \int_0^1 U_B \tilde{V}_{01} dy} \quad (3.6)$$

We normalize and write $\tilde{V}_{o1} = A \bar{V}_{o1}(y)$, $\tilde{U}_{o1} = A \bar{U}_{o1}(y)$, $\tilde{p} = A \bar{p}_{o1}(y)$ where the unknown A is independent of y , but depends on $\tau, \tilde{X}, \dots$, and also $\bar{V}_{o1}(1) = -i\alpha_o U_B(1)$. Hence from (3.3), (3.5), $\tilde{\eta}_{o1} = -A$.

The problem (3.5) fixes $\alpha_o = \alpha(s)$ and gives an unknown slip velocity $\tilde{U}_{oe} = -\frac{A \bar{V}_{o1}(y)}{i\alpha_o} \Big|_{y=0} \tilde{E} + \text{c.c.}$. Also from (3.2bc), since

$\tilde{p}_o \Big|_{y=0} = \tilde{p}_{oy} \Big|_{y=0} = 0$, in the wall layer, where $Y = R^{1/3} y$ is $O(1)$

we have the expansion there,

$$\begin{aligned} U &= R^{-1/3} U_o + \dots, \quad V = R^{-2/3} V_o + \dots, \\ p &= s - R^{-1/3} Y_s + R^{-2/3} P_o + \dots \end{aligned} \quad (3.7)$$

This yields the unsteady problem

$$U_{o\tau} + U_o U_{ox} + V_o U_{oy} = -P_{ox} + U_{oy\tau}$$

$$U_{ox} + V_{oy} = 0, \quad P_{oy} = 0 \quad (3.8a-c)$$

$$U_o \rightarrow \lambda Y + \tilde{U}_{oe} \quad \text{as } Y \rightarrow \infty, \quad U_o = V_o = 0 \quad \text{at } Y = 0 \quad (3.9a,b)$$

From (3.8c) $P_o = \tilde{p}_1 \Big|_{y=0}$. A further relation however, is necessary to close the above problem and to determine the unknown wave amplitude

A. This requires an examination of the second order problem in the inviscid region. There, from the Navier-Stokes equations using (3.1) we obtain the set of equations ,

$$\tilde{U}_{1x} + \tilde{V}_{1y} = -\tilde{U}_{ox}$$

$$U_B \tilde{U}_{1x} + \tilde{V}_1 \tilde{U}_{By} = -\tilde{p}_{1x} - \tilde{U}_{o\tau} - \tilde{U}_o \tilde{U}_{ox} - \tilde{V}_o \tilde{U}_{oy} - U_B \tilde{U}_{ox} \tilde{p}_{ox}$$

$$U_B \tilde{V}_{1x} = -\tilde{p}_{1y} - \tilde{V}_{o\tau} - \tilde{U}_o \tilde{V}_{ox} - \tilde{V}_o \tilde{V}_{oy} - U_B \tilde{V}_{ox} \quad . \quad (3.10a-c)$$

The free-surface conditions give

$$\tilde{V}_1 + \tilde{V}_{oy} \tilde{\eta}_o = U_B (\tilde{\eta}_{1x} + \tilde{\eta}_{ox}) + (\tilde{U}_o + U_{oy} \tilde{\eta}_o) \tilde{\eta}_{ox} + \tilde{\eta}_{o\tau} , \text{ at } y = 1$$

$$\tilde{p}_1 + \tilde{p}_{oy} \tilde{\eta}_o = s \tilde{\eta}_1 \quad \text{at } y = 1 \quad . \quad (3.11ab)$$

To match with the wall layer we also need

$$\tilde{V}_1 = V_{oBe} , \quad \tilde{p}_1 = P_o \quad \text{at } y = 0 \quad . \quad (3.11cd)$$

Here V_{oBe} denotes the finite part of V_o in (3.8), (3.9) as $Y \rightarrow \infty$.

The forcing terms in (3.10), (3.11) give rise to terms proportional to $\tilde{E}^0$, $\tilde{E}^{\pm 1}$, $\tilde{E}^{\pm 2}$. Hence writing

$$\tilde{V}_1 = \sum_{n=0}^{\infty} \tilde{V}_{1n} \tilde{E}^{(n)} + \text{c.c.}, \quad \tilde{\eta}_1 = \sum_{n=0}^{\infty} \tilde{\eta}_{1n} \tilde{E}^{(n)} + \text{c.c.}, \text{ etc.} \quad (3.12)$$

where ^{the} coefficients of $\tilde{E}^{(n)}$ are independent of y , we obtain the following set of equations from substitution into (3.10) (3.11).

$\tilde{E}^{(0)}$:

$$2\tilde{V}_{10y} = 0, \quad 2\tilde{V}_{10} U_{0y} = -(\tilde{V}_{01} \tilde{U}_{01y}^{(c)} + \tilde{V}_{01}^{(c)} \tilde{U}_{01y})$$

$$0 = -2\tilde{p}_{10y} - (\tilde{V}_{01} \tilde{V}_{01y}^{(c)} + \tilde{V}_{01}^{(c)} \tilde{V}_{01y}) \quad (3.13a-c)$$

$$2\tilde{V}_{10} + \tilde{V}_{01y} \tilde{\eta}_{01}^{(c)} + \tilde{V}_{01y}^{(c)} \tilde{\eta}_{01} = -i\alpha_0 (\tilde{U}_{01} \tilde{\eta}_{01}^{(c)} - \tilde{U}_{01}^{(c)} \tilde{\eta}_{01}) \text{ at } y = 1$$

$$\tilde{p}_{0y} \tilde{\eta}_{01}^{(c)} + 2\tilde{p}_{10} + \tilde{p}_{01y}^{(c)} \tilde{\eta}_{01} = 2s\tilde{\eta}_{10} \text{ at } y = 1 \quad (3.13d-g)$$

and

$$2\tilde{V}_{10} = \tilde{V}_{0Be}^{(0)}, \quad \tilde{p}_{10} = \tilde{p}_{00} \text{ at } y=0,$$

($\tilde{V}_{0Be}^{(n)}$ denoting the coefficient of the $\tilde{E}^{(n)}$ term).

$\tilde{E}^{(1)}$:

$$i\alpha_0 \tilde{V}_{011} + \tilde{V}_{11y} = -\tilde{U}_{10x}$$

$$i\alpha_0 U_B \tilde{U}_{11} + \tilde{V}_{11} U_{By} = -i\alpha_0 \tilde{p}_{011} - \tilde{U}_{01\tau} - U_B \tilde{U}_{01x} - \tilde{p}_{01} \tilde{x}$$

$$i\alpha_0 U_B \tilde{V}_{11} = -\tilde{p}_{11y} - \tilde{V}_{01\tau} - U_B \tilde{V}_{01x} \quad (3.14a-c)$$

$$\text{with } \tilde{V}_{11} = U_B (i\alpha_o \tilde{\eta}_{11} + \tilde{\eta}_{01x}^{(c)}) + \tilde{\eta}_{01\tau} \text{ at } y = 1$$

$$\tilde{p}_{21} = s\tilde{\eta}_{11} \quad \text{at } y = 1, \quad (3.15a-d)$$

$$\text{and } \tilde{V}_{11} = \tilde{V}_{oBe}^{(o)}, \quad \tilde{p}_{21} = P_{01} \quad \text{at } y = 0.$$

$$\tilde{E}^{(2)}:$$

$$2i\alpha_o \tilde{U}_{12} + \tilde{V}_{12y} = 0$$

$$2i\alpha_o U_B \tilde{U}_{12} + \tilde{V}_{12} U_{By} = -2\tilde{p}_{12} i\alpha_o - i\alpha_o \tilde{U}_{01}^2 - \tilde{V}_{01} \tilde{U}_{01y} \quad (3.16a-c)$$

$$2i\alpha_o U_B \tilde{V}_{12} = -\tilde{p}_{12y} - \tilde{U}_{01} \tilde{V}_{01} i\alpha_o - \tilde{V}_{01} \tilde{V}_{01y}$$

$$\text{and, } \tilde{V}_{12} + \tilde{V}_{oy} \tilde{\eta}_{01} = 2U_B i\alpha_o \tilde{\eta}_{12} + i\alpha_o \tilde{U}_{01} \tilde{\eta}_{01},$$

$$\tilde{p}_{12} + \tilde{p}_{01y} \tilde{\eta}_{01} = s\tilde{\eta}_{12} \quad \text{at } y = 1. \quad (3.16d-e)$$

$$\text{Also } \tilde{V}_{12} = \tilde{V}_{oBe}^{(2)}, \quad \tilde{p}_{12} = P_{02} \quad \text{at } y = 0.$$

The superscript (c) denotes the complex conjugate. We consider first (3.14), (3.15). After substituting for $\tilde{U}_{11}$ from — — —

(3.14a) into (3.14b) and using (3.14c) we obtain

$$\tilde{V}_{11} - (\alpha_0^2 + \frac{U_B''}{U_B}) \tilde{V}_{11} = \frac{i U_B''}{U_B^2} \tilde{V}_{01\tau} + 2 \frac{\tilde{p}_{01yx}}{U_B} \quad (3.17)$$

The left hand side here is the same as in the eigenvalue problem (3.5a). Hence multiplying (3.17) by $\tilde{V}_{01}$ and integrating from $y = 0$ to $y = 1$, we obtain the solvability condition,

$$\begin{aligned} (\tilde{V}_{11y} \tilde{V}_{01} - \tilde{V}_{01y} \tilde{V}_{11})_1 - (\tilde{V}_{01y} \tilde{V}_{11})_0 &= \frac{i}{\alpha_0} \int_0^1 \frac{U_B''}{U_B^2} \tilde{V}_{01} \tilde{V}_{01\tau} dy \\ &+ 2 \int_0^1 \frac{\tilde{p}_{01yx}}{U_B} \tilde{V}_{01} dy \end{aligned} \quad (3.18)$$

From (3.8a), for large Y , we find

$$\lambda V_{0Be}^{(1)} = -i\alpha_0 P_{01} + \frac{A_\tau \tilde{V}_{01y}}{i\alpha_0} \Big|_{y=0} \quad (3.19)$$

The boundary conditions (3.3), (3.5ab), (3.15), and the normalizations introduced above, finally simplifies (3.18) to yield the equation

$$P_{01} = A_\tau d_1 + A_\tau d_2 \quad (3.20)$$

relating the $\tilde{E}^1$ component of the wall pressure and A. Here the constants d_1, d_2 depend only on the solution of the normalized eigenvalue problem (2.5) and are given by

$$\alpha_o^2 d_1 = \frac{1}{\bar{u}_{o1}(0)} \left[-\frac{2i\alpha_o \lambda}{U_B(1)} - (\bar{u}_{o1} \bar{v}_{o1y})_{y=0} - \frac{i\lambda}{\alpha_o} \int_0^1 \frac{U_B''}{U_B^2} \bar{v}_{o1}^2 dy \right]$$

$$\alpha_o^2 d_2 = -\frac{2\lambda}{\bar{u}_{o1}(0)} \int_0^1 \frac{\bar{p}_{o1y} \bar{v}_{o1}}{U_B} dy \quad (3.21ab)$$

Turning next to the other equations (3.13), (3.16), firstly from (3.13ab), $\tilde{V}_{10} = 0$. (3.13cef) now give

$$s\tilde{\eta}_{10} - \frac{1}{2} (\tilde{p}_{o1y} \tilde{\eta}_{o1}^{(c)} + \tilde{p}_{o1y}^{(c)} \tilde{\eta}_{o1}) = -\frac{1}{2} \int_0^1 (\tilde{v}_{o1} \tilde{v}_{o1y}^{(c)} + \tilde{v}_{o1}^{(c)} \tilde{v}_{o1y}) dy + p_{oo}.$$

This connects p_{oo} with the unknown $\tilde{\eta}_{10}$, although as far as the wall layer problem is concerned an explicit determination of p_{oo} is not necessary.

From (2.16a-c) we obtain

$$\tilde{V}_{12yy} - (4\alpha_o^2 + \frac{U_B''}{U_B}) \tilde{V}_{12} = -\frac{1}{U_B} (\tilde{u}_{o1} \tilde{v}_{o1yy} - \tilde{v}_{o1} \tilde{u}_{o1yy}) \text{ at } y = 1 \quad (3.22)$$

with the boundary conditions ,

$$\frac{\tilde{V}_{12}}{U_B} - \frac{\tilde{V}_{12y} U_B}{U_B^2} = -2s \frac{\tilde{V}_{o1y} \tilde{\eta}_{o1}}{U_B} - i\alpha_o \frac{\tilde{U}_{o1}^2}{U_B} - \frac{\tilde{V}_{o1}}{U_B} \tilde{U}_{o1y} \text{ at } y = 1$$

$$\tilde{V}_{12} = -2i\alpha_o P_{o2} - i\alpha_o A^2 \left[(\bar{V}_{o1y})_{y=0} \right]^2 \quad \text{at } y = 0, \quad (3.23 a-c)$$

$$P_{o2} + \frac{\tilde{p}_{o1y}}{U_B i\alpha_o} \tilde{\eta}_{o1} - s \frac{\tilde{V}_{o1y} \tilde{\eta}_{o1}}{2U_B i\alpha_o} = \frac{s \tilde{V}_{12}}{2U_B i\alpha_o} + \int_0^1 2i\alpha_o U_B \tilde{V}_{12} dy \quad \text{at } y = 1.$$

Here we have used $\lambda \tilde{V}_{o1}^{(2)} = -2i\alpha_o P_{o2} - i\alpha_o A^2 \left[(\bar{V}_{o1y})_{y=0} \right]^2$ from (3.8), (3.9). (3.22), (3.23) show that $\tilde{V}_{12}$ is proportional to A^2 , and thus (3.22), (3.23) and (3.8), (3.9) then determine the unknown (constant) $\frac{P_{o2}}{A^2} = d_3$.

Briefly summarizing then, the growth (or decay) in time and space of the non-simple waves for $O(1)$ Froude numbers is governed by, (i) the solution of the eigenvalue problem (3.5) which determines the wavenumber $\alpha_o = \alpha(s)$, (ii) the solutions of the first and second order inviscid problems together with, (iii) the solution of the unsteady boundary layer problem with the pressure-displacement relations depending on (i) and (ii), as shown above. The full problem poses a complex numerical task. However, a limiting case ($\xi \rightarrow 0$) can be investigated further and this also provides a match with the results of §2.

For $s \ll 1$ the solution of (3.5), (3.6) is given by

$$\alpha = \alpha_{o1} s^{1/2} + s^{3/2} \alpha_{o2} + \dots, \quad \bar{V}_{o1} = \bar{V}_{o1}^{(0)} s^{1/2} + s^{3/2} \bar{V}_{o1}^{(1)} + \dots,$$

$$\bar{U}_{o1} = \bar{U}_{o1}^{(0)} + s \bar{U}_{o1}^{(1)} + \dots \quad (3.24)$$

Substitution in (3.5), (3.6) yields

$$\alpha_{01}^2 = \frac{1}{\int_0^1 U_B^2 dy}, \quad \bar{V}_{01}^{(0)} = -i\alpha_{01} U_B, \quad \bar{U}_{01}^{(0)} = U_{By}$$

$$\bar{V}_{01}^{(1)} = -i\alpha_{02} U_B - i\alpha_{01}^3 U_B \int_0^y \frac{1}{U_B^2} \int_0^{y_1} U_B^2 dy_1 dy_2, \quad ,$$

$$\alpha_{02} = -\alpha_{01}^5 \left(\int_0^1 U_B^2 \int_0^y \frac{1}{U_B^2} \int_0^{y_2} U_B^2 dy_1 dy_2 dy \right) .$$

(3.21ab) now give

$$d_1 = \frac{-2is^{1/2}}{\alpha_{01}} \left[\frac{1}{U_B(1)} + \alpha_{01}^2 \left(\int_0^1 \frac{U_B'}{U_B^2} \int_0^y U_B^2 dy_1 dy \right) \right] + O(s^{3/2})$$

$$d_2 = -2i\alpha_{01} s^{1/2} \left(\int_0^1 U_B^2 dy \right) + O(s^{3/2}) \quad . \quad (3.25a,b)$$

Similarly from (3.22), (3.23)

$$\bar{V}_{12} = \alpha_{01} s^{1/2} A^2 \left[-iU_B' + C_1 U_B \right] + \dots$$

where C_1 is an unknown constant. Hence

$$\frac{\bar{P}_{02}}{A^2} = d_3 = s C_1 \left(-\frac{1}{2}i + 2i \right) + \dots .$$

If in addition we assume A is small and

$A = \bar{A} \exp (i\alpha_1 \tilde{x} - i\beta_0 \tau)$, then from (3.20) (3.25)

$$\frac{p_{01}}{\bar{A}} = \frac{-2\beta_0 s^{1/2}}{\alpha_{01}} \left[\frac{1}{U_B(1)} + \alpha_{01}^2 \left(\int_0^1 \frac{U_B'}{U_B^2} \int_0^y U_B^2 dy dy_1 \right) \right] + 2\alpha_1 \alpha_{01} s^{1/2} \left(\int_0^1 U_B^2 dy \right) \quad (3.26)$$

If α_1 is real and $\alpha_1 = O(s^{-2/3})$ then this is consistent with (3.26) and (2.5), and matches exactly with the results of §2 giving β_0 , α_1 , as before. d_3 is determined from the higher order terms.

In the next section we consider undeveloped flows.

§4 Lower Branch Stability of Undeveloped Flows

Again, the basic structure has been set out in chapter 3 and, (to our order of working), it is sufficient to consider the linearized version of the unsteady lower-deck problem of chapter 3, (see also Smith & Bodonyi (1980)), i.e.

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial Y} = 0, \quad \frac{\partial P}{\partial y} = 0$$

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial Y} = - \frac{\partial P}{\partial x} + \frac{\partial^2 U}{\partial Y^2}$$

$$U \rightarrow \lambda(Y + A(x, t)) \text{ as } Y \rightarrow \infty, \quad U = V = 0 \text{ at } Y = 0 \quad (4.1a-e)$$

$\tilde{p}(X, T)$, $A(X, T)$, $\eta(X, T)$ are related through the outer inviscid problem

$$\begin{aligned} \nabla^2 \tilde{p} &= 0, \quad \tilde{p} = s\eta \quad \text{at } y = h, \quad \tilde{p}_y = -\eta_{xx} \quad \text{at } y = h \\ \tilde{p} &= P, \quad \tilde{p}_y = A_{xx} \quad \text{at } y = 0 \end{aligned} \quad (4.2a-e)$$

In the above, $T = R^{-1/5} t$ is $O(1)$, and the rest of the notation is as in chapter 3. (We note that in the local skin $\lambda = 0.332 x_0^{-1/2}$, where we are considering disturbances locally at $x = x_0$). The parallel flow approximation has been used in the analysis below and this implies a restriction on the wavelength of the modes we are considering. (See Smith & Bodonyi (1980) for a further discussion on this point.)

We linearize about the basic flow by writing

$$(U, V, P, A, \eta, \tilde{p}) = (\lambda Y, \underline{0}) + \Delta(\bar{U}, \bar{V}, \bar{P}, \bar{A}, \bar{\eta}, \bar{p}) \bar{E} \quad (4.3)$$

Here $\bar{E} = \exp(i\alpha_0 X - i\beta_0 T)$ where $\alpha = \alpha_0 + O(R^{-1/5})$, $\beta = \beta_0 R^{-1/5} + O(R^{-2/5})$, are the wavenumber and frequency respectively, and we define the scaled wavespeed $c_0 = \beta_0 / \alpha_0$. Substitution in (4.1a-e) then leads to the same eigenrelation (2.5)

$$\frac{A'(\xi_0)}{K \xi_0} = - \frac{\bar{P}}{c_0 \lambda \bar{A}} \quad (4.4)$$

where ξ_0 , K are as in §2. From the outer deck problem (4.2ae), we obtain

$$\frac{\bar{f}}{\bar{A}} = \frac{\alpha(s - \alpha \tanh(\alpha h))}{(s \tanh(\alpha h) - \alpha)} \quad (4.5)$$

(4.4), (4.5), leads to the eigenrelation

$$\frac{A'(\xi_0)}{K \xi_0} = - \frac{\bar{f}}{c_e \lambda \bar{A}} = - \frac{\alpha_o (s - \alpha_o \tanh(\alpha_o h))}{c_o \lambda (s \tanh(\alpha_o h) - \alpha_o)} \quad (4.6)$$

Hence for neutral modes with $\alpha_o = \alpha_{on}$, $\beta_o = \beta_{on}$ real, we obtain the dispersion relations

$$\omega = \alpha_{on} \cdot h, \beta_{on} = 2.297 (\alpha_{on} \lambda)^{2/3}, \tanh \omega = \omega H \left[\frac{\bar{\gamma} H^{1/3} + S \omega^{1/3}}{\bar{\gamma} S H^{7/3} + \omega^{7/3}} \right] \quad (4.7abc)$$

$$\text{where } h = \lambda^{-5/4} H, s = \lambda^{5/4} S, \bar{\gamma} = 1.001 \quad (4.8a-c)$$

It should be noted that $(sh)^{-1} = (SH)^{-1} = Fr$ (the Froude number) is independent of λ .

It is readily established, as in appendix A of chapter 3, that for $\omega > 0$, the equation (4.7c) has exactly one real root if $Fr > 1$ and two real roots if $Fr < 1$. This can also be seen graphically. Figs. (2q-f)

depict graphs of the roots plotted against $(Fr)^{-1}$ for several values of H .

If $S = 0$ (4.7c) reduces to the 'odd mode' of developing flow in a channel, Smith & Bodonyi (1980) (with h interpreted as being half the width of the channel).

Before discussing the properties of the neutral curves, it is worth deriving some additional results concerning the roots of the eigenrelation (4.7c).

Firstly for $H \ll 1$, $S = O(1)$, from (4.7c)

$$\omega = (SH)^{1/2} + \frac{\bar{\gamma}}{2} \left(\frac{H}{S} \right)^{2/3} - \frac{5}{24} \bar{\gamma}^2 \frac{H^{5/6}}{S^{11/6}} + \frac{\bar{\gamma}^3}{6S^2} \left(\frac{H}{S} \right) + O(H^{7/6})$$

$$\text{giving } \alpha_{on} \sim \left(\frac{S}{h} \right)^{1/2} + \frac{\bar{\gamma} \lambda^{5/3}}{2h^{1/3} S^{2/3}} - \frac{5\bar{\gamma}^2 \lambda^{10/3}}{24S^{11/6} h^{1/6}} + \frac{\bar{\gamma}^3 \lambda^5}{6S^3} + O(h^{1/6}) \text{ as } h \rightarrow 0. \quad (4.9)$$

If $h \gg 1$, writing $\omega = H\bar{\omega}$ with $\bar{\omega}$ and $S = O(1)$, we obtain $\bar{\omega} = S$, or $\bar{\omega} = \bar{\gamma}^{3/4}$. Hence

$$\alpha_{on} \sim S, \quad \alpha_{on} \sim \bar{\gamma}^{3/4} \lambda^{5/4} \quad \text{for } h \gg 1 \quad (4.10)$$

Also for $SH = 1+$, again (4.7c) yields

$$\omega = H^4 \bar{\gamma}^3 (SH - 1)^3 + \dots, \text{ giving}$$

$$\alpha_0 \sim \bar{\gamma}^3 \lambda^5 h^3 \cdot (Sh-1)^3 + \dots \quad \text{as } sh \rightarrow 1+ \quad (4.11)$$

The result (4.10) is interesting since the first mode can be identified with the wavy solution occurring in steady flow (chapter 3), and the second mode is the Tollmien-Schlichting mode associated with boundary layer instability (Smith 1979b). ~~The mode given by (4.11) is again the same as in the steady case.~~ The result (4.9) is very similar to that for fully developed flows in §2, (see 2.11). In fact, if we replace h by γ in (4.9), the first few terms of (4.9) are the same as (2.11). (This makes sense since $\gamma = \int_0^1 U_b^2 dy$ in §2, and for undeveloped flows $U_b \sim 1$ in the majority of the flow $0 < y < 1$).

It is noted that $\alpha_0 = 0$ satisfies the dispersion relations (4.7). However it is implicit in the above analysis that $\alpha_0 \neq c$, and we can demonstrate that for $Fr \neq 1$, $\alpha_0 = 0$ is not part of the neutral curve. Firstly with $\bar{p}/\bar{A}$ given by (4.5) the right hand side of (4.6) $\sim -\frac{s}{c_0 \lambda (sh-1)}$ for small α_0 . But the neutral requirement would give the left hand side ~ -0.4355 . These two assertions imply c_0 is an $O(1)$ constant. This contradicts the other neutral requirement $c_0 \propto 2.297 \alpha_0^{-1/3} \lambda^{2/3}$ as $\alpha_0 \rightarrow 0$. The same argument also demonstrates that in the limit $Fr \rightarrow 1 -$, $\alpha_0 \rightarrow 0$.

The various regimes in which, for real frequencies, $\text{Im}(\alpha) > 0$ (or equivalently for real wavenumbers $\text{Im}(\beta) < 0$), are obtained by using the results of the appendix which restated are,

$\text{Im} (\alpha_o - \alpha_{on}) > 0$, according as

$$\{ (\beta_o - \beta_{on}) (g'(\alpha_{on}) - \frac{2}{3} \frac{g(\alpha_{on})}{\alpha_{on}}) \} < 0 \quad (4.12)$$

where $\frac{g(\alpha)}{\alpha} = \frac{p}{A} = \alpha \frac{(s - \alpha \tanh(\alpha h))}{(s \tanh(\alpha h) - \alpha)}$. (4.13)

Hence $g'(\alpha) = \frac{2}{3} \frac{g(\alpha)}{\alpha}$

$$= \frac{2\alpha s - 3\alpha^2 \tanh(\alpha h) - \alpha^3 h \text{sech}^2(\alpha h)}{(s \tanh(\alpha h) - \alpha)} - \alpha^2 \frac{(s - \alpha \tanh(\alpha h))(s h \text{sech}^2(\alpha h) - 1)}{(s \tanh(\alpha h) - \alpha)^2}$$

$$= \frac{2\alpha}{3} \frac{(s - \alpha \tanh(\alpha h))}{(s \tanh(\alpha h) - \alpha)} \quad (4.14)$$

For convenience we shall refer to mode 1 for the mode which occurs for both subcritical and supercritical flows, and mode 2 for the other mode.

For mode 1 it is enough to consider the case $s = 0$, for then

$$\frac{g(\alpha_{on})}{\alpha_{on}} = \alpha_{on} \tanh(\alpha_{on} h) = \gamma \lambda^{5/3} \alpha_{on}^{-1/3} ,$$

and

$$g'(\alpha_{on}) - \frac{2}{3} \frac{g(\alpha_{on})}{\alpha_{on}} = \frac{1}{3\alpha_o} \left[4\alpha_o^{2/3} \gamma \lambda^{5/3} + 3\alpha_o^2 h \operatorname{sech}^2(\alpha_o h) \right] > 0 .$$

Hence for mode 1, $\operatorname{Im}(\alpha - \alpha_{on}) > 0$ as $(\beta - \beta_{on}) < 0$. (4.15)

For mode 2, for small α_{on} , from (4.14)

$$\frac{g'(\alpha_{on}) - 2g(\alpha_{on})}{3\alpha_{on}} \sim \left[\frac{s}{3(sh-1)} + O(\alpha_{on}^2) \right] .$$

Hence (4.15) holds also for mode 2.

The neutral curves are sketched in figs (2a-d) and are based on the above results. A few features concerning the neutral curves are worth mentioning; (i) firstly, for supercritical flows, the limit as $h \rightarrow 0$ represents a stabilising effect; (ii) subcritical flows are destabilised by viscosity and there is a large region of instability as $Fr \rightarrow 1-$; (iv) for $h \gg 1$, the Tollmien-Schlichting mode appears to be the dominant mode; (iv) also, for $h \gg 1$, the 'steady mode' (mode 2) is insensitive to the x_o station; (v) for certain values of s , h , the neutral curves cross.

§5 Further Comments

In this chapter we have considered the lower-branch stability of fully-developed and undeveloped flows. Only a linearized account

has been discussed, although the results here form a basis for a weakly-nonlinear account along the lines of Smith (1979a), Hall & Smith (1982).

For fully-developed flows the solution of the eigenvalue problem needs to be addressed since this would indicate the stability characteristics for both $O(1)$ and shorter waves.

Finally we note that there is a dramatic change in the stability properties for nearly critical undeveloped flows. The limit $Fr \rightarrow 1$ deserves special attention since from §4 a new structure is indicated in this limit. In fact from (4.11) the wavespeed $c \sim R^{-1/5} c_0 \propto \frac{R^{-1/5}}{(1-Fr)}$, and hence as $(1 - Fr) \rightarrow O(R^{-1/5})$ the wavespeed increases to $O(1)$ with the critical layer occurring in the interior of the flow. Also the x length scale lengthens to $O(R^{3/5})$ since the neutral wavenumber $\alpha(1-Fr)^3$.

CHAPTER 5

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APPENDIX A

In this appendix we derive some relations for nearly neutral wavenumbers α_0 and frequencies β_0 satisfying the eigenrelation (4.5) but for a more general $\bar{p}/\bar{A}$ relation.

We take $\bar{p}/\bar{A} = \frac{g(\alpha)}{\alpha}$ and the eigenrelation then becomes

$$\frac{A'(\xi_0)}{K\xi_0} = \frac{-\bar{p}}{c_0 \lambda \bar{A}} = \frac{g(\alpha_0)}{c_0 \lambda \alpha_0} \quad (A1)$$

where $\xi = -i\beta_0 (i\alpha_0 \lambda)^{-2/3}$, (α^n is real is α is real), $c_0 = \beta_0/\alpha_0$.

λ , usually, can be scaled out of the problem and for convenience we set $\lambda = 1$. (A1) can also be written as

$$-1 + F(\beta_0 \alpha_0^{-2/3}) = \frac{g(\alpha_0)}{\beta_0} \quad (A1')$$

In the near neutral state $\alpha_0 = \alpha_n + \tilde{\alpha}$, $\beta = \beta_n + \tilde{\beta}$

and, $-1 + F(z_n + \tilde{z}) \sim a_0 + \tilde{z}(a_1 + ia_2) + O(\tilde{z}^2)$

where $z_n = 2.297$, $a_0 + 1 = .5645$, $a_1 = -.1197$, $a_2 = .2307$ (A2)

from Reid (1965) . α_n, β_n are real and $\tilde{\alpha}, \tilde{\beta}$ complex.

Thus expanding (A1') for small $\tilde{\alpha}, \tilde{\beta}$, we obtain

$$\beta_n \alpha_n^{-2/3} \left[\frac{-2}{3} \frac{\tilde{\alpha}}{\alpha_n} + \frac{\tilde{\beta}}{\beta_n} \right] (a_1 + i a_2) = \frac{-g(\alpha_n)}{\beta_n} \left[\frac{\tilde{\alpha}}{g(\alpha_n)} - \frac{\tilde{\beta}}{\beta_n} \right] \quad (A3)$$

Taking the real and imaginary parts of A3 yields

$$\beta_n \alpha_n^{-2/3} \left\{ \left[\frac{-2}{3} \frac{\tilde{\alpha}_r}{\alpha_n} + \frac{\tilde{\beta}_r}{\beta_n} \right] a_1 - \left[\frac{-2}{3} \frac{\tilde{\alpha}_i}{\alpha_n} + \frac{\tilde{\beta}_i}{\beta_n} \right] a_2 \right\} = \frac{-g(\alpha_n)}{\beta_n} \left[\frac{\tilde{\alpha}_r g'(\alpha_n)}{g(\alpha_n)} - \frac{\tilde{\beta}_r}{\beta_n} \right]$$

$$\beta_n \alpha_n^{-2/3} \left\{ \left[\frac{-2}{3} \frac{\tilde{\alpha}_r + \tilde{\beta}_r}{\alpha_n \beta_n} \right] a_2 + \left[\frac{-2}{3} \frac{\tilde{\alpha}_i + \tilde{\beta}_i}{\alpha_n \beta_n} \right] a_1 \right\} = \frac{-g(\alpha_n)}{\beta_n} \left[\frac{\tilde{\alpha}_i g'(\alpha_n)}{g(\alpha_n)} - \frac{\tilde{\beta}_i}{\beta_n} \right] \quad (A4ab)$$

Here the suffices r, i denote the real and imaginary parts, and it is assumed that $g(\alpha)$ is real if α is real.

Substitution $\tilde{\beta}$ from (A4a) into (A4b) and rearranging

$$\begin{aligned} & -\tilde{\alpha}_r \left\{ \frac{2}{3} \frac{g(\alpha_n)}{\alpha_n} - g'(\alpha_n) \right\} \frac{a_2}{\beta_n} - \tilde{\alpha}_i \left\{ \frac{g'(\alpha_n)}{\beta_n} - \frac{2}{3} \beta_n \alpha_n^{-2/3} \right\} \left(a_1 - \frac{g(\alpha_n)}{\beta_n^2} \alpha_n^{2/3} \right) - \frac{2}{3} a_2^2 \alpha_n^{-5/3} \beta_n \left\{ \right. \\ & \left. = \tilde{\beta}_i \left[\left(\frac{g(\alpha_n)}{\beta_n^2} \alpha_n^{2/3} - a_1 \right)^2 + a_2^2 \right] \right\} \cdot \alpha_n^{-2/3} \end{aligned} \quad (A5)$$

Thus for real wavenumbers $\tilde{\alpha}_i = 0$ and (A5) shows that $\tilde{\beta}_i \geq 0$ if

$$\frac{\tilde{\alpha}_r \alpha_n^{2/3}}{\beta_n} \left(-\frac{2}{3} \frac{g(\alpha_n)}{\alpha_n} + g'(\alpha_n) \right) \geq 0 \quad (A6)$$

Similarly substitution for $\tilde{\alpha}_r$ from (A4a) into (A4b) yields

$$\begin{aligned} & \tilde{\alpha}_i \left[\left(\frac{g'(\alpha_n)}{\beta_n} - \frac{2}{3} \alpha_n^{-5/3} a_1 \right)^2 + \frac{4}{9} a_2^2 \beta_n^2 \alpha_n^{-10/3} \right] \\ & + \tilde{\beta}_r \cdot \frac{a_2 \alpha_n^{-2/3}}{\beta_n} \left\{ g'(\alpha_n) - \frac{2}{3} \frac{g(\alpha_n)}{\alpha_n} \right\} \\ & = \tilde{\beta}_i \left[\left(\frac{g(\alpha_n) - \alpha_n^{-2/3} a_1}{\beta_n^2} \right) \left(\frac{-2\alpha_n^{-5/3} a_1 + g'(\alpha_n)}{\beta_n} \right) + \frac{2}{3} a_2^2 \alpha_n^{-2/3} \beta_n \right] \quad (A7) \end{aligned}$$

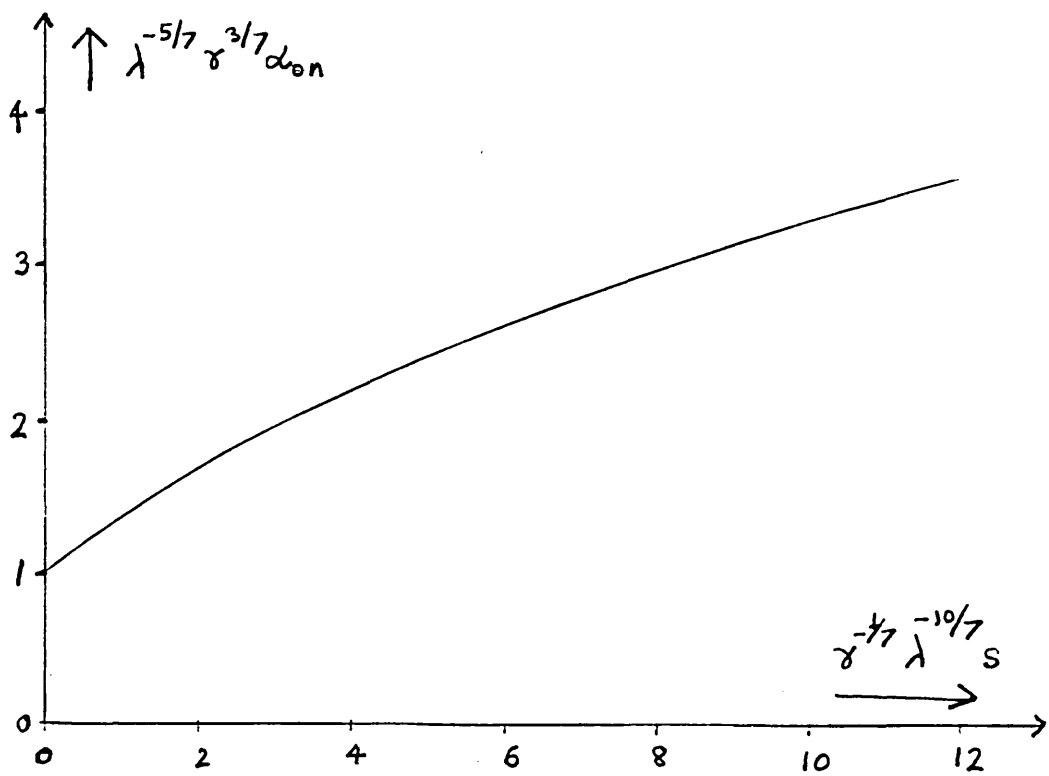
Hence for real frequencies $\tilde{\beta}_i = 0$ and (A7) gives the condition

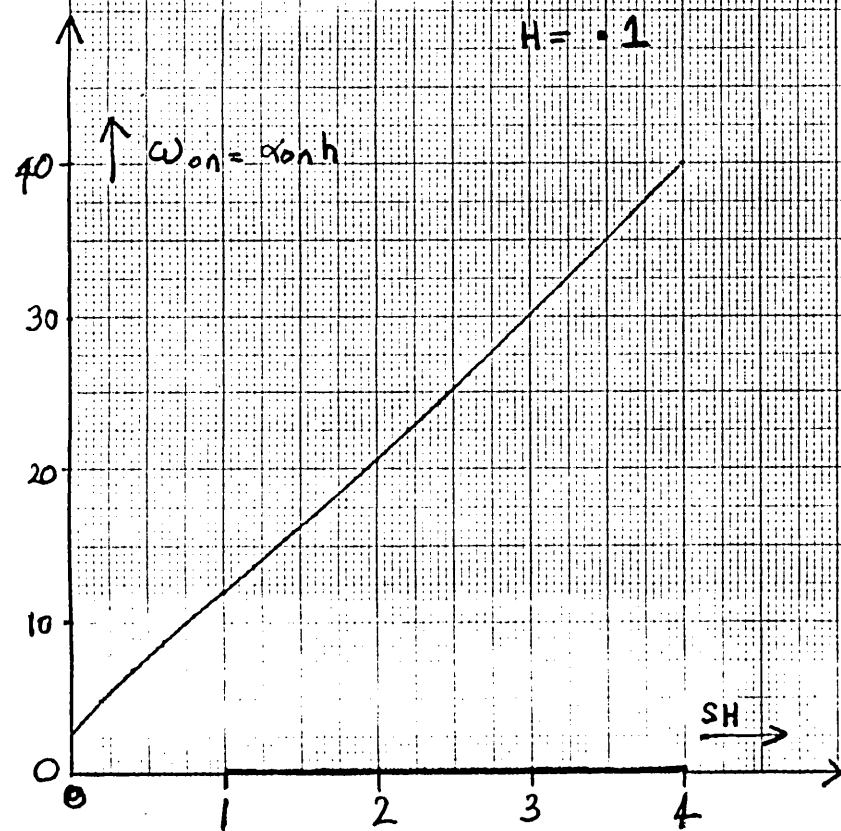
$$\tilde{\alpha}_i \geq 0 \text{ if } \frac{\tilde{\beta}_r \alpha_n^{-2/3}}{\beta_n} \left[\frac{2g}{3} \frac{(\alpha_n)}{\alpha_n} - g'(\alpha_n) \right] \geq 0 \quad (A8)$$

We note that $g'(\alpha_n) = \frac{2}{3} \frac{g(\alpha_n)}{\alpha_n}$ implies coalescent modes since in

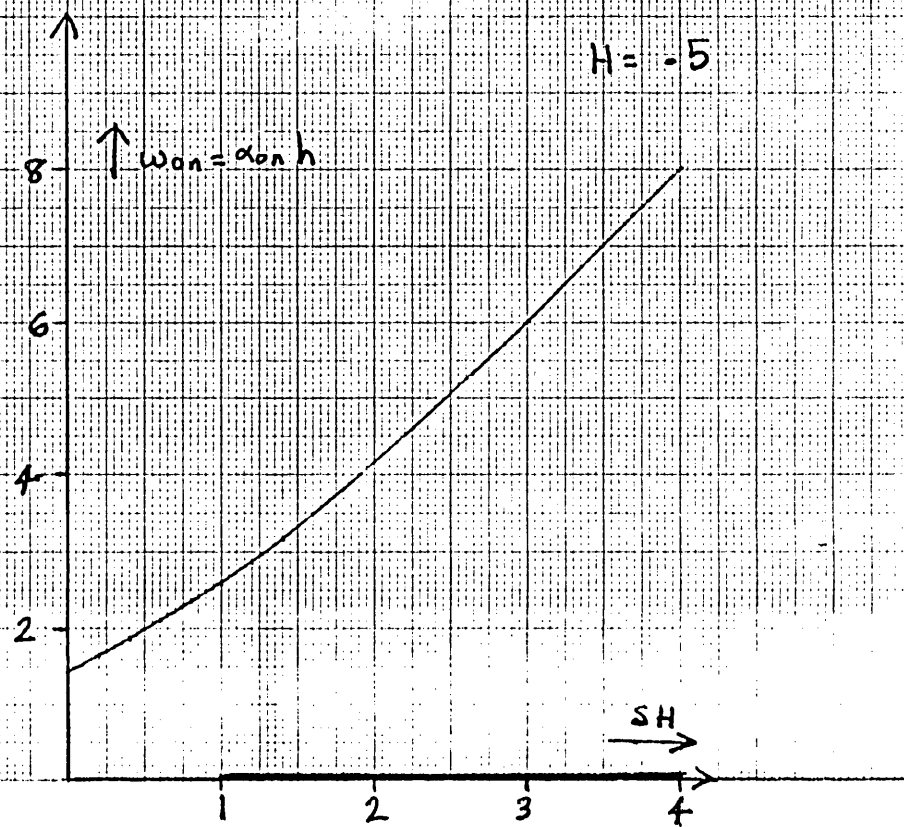
the neutral state $g(\alpha_n) = 1.001 \alpha_n^{2/3}$.

Fig. 1



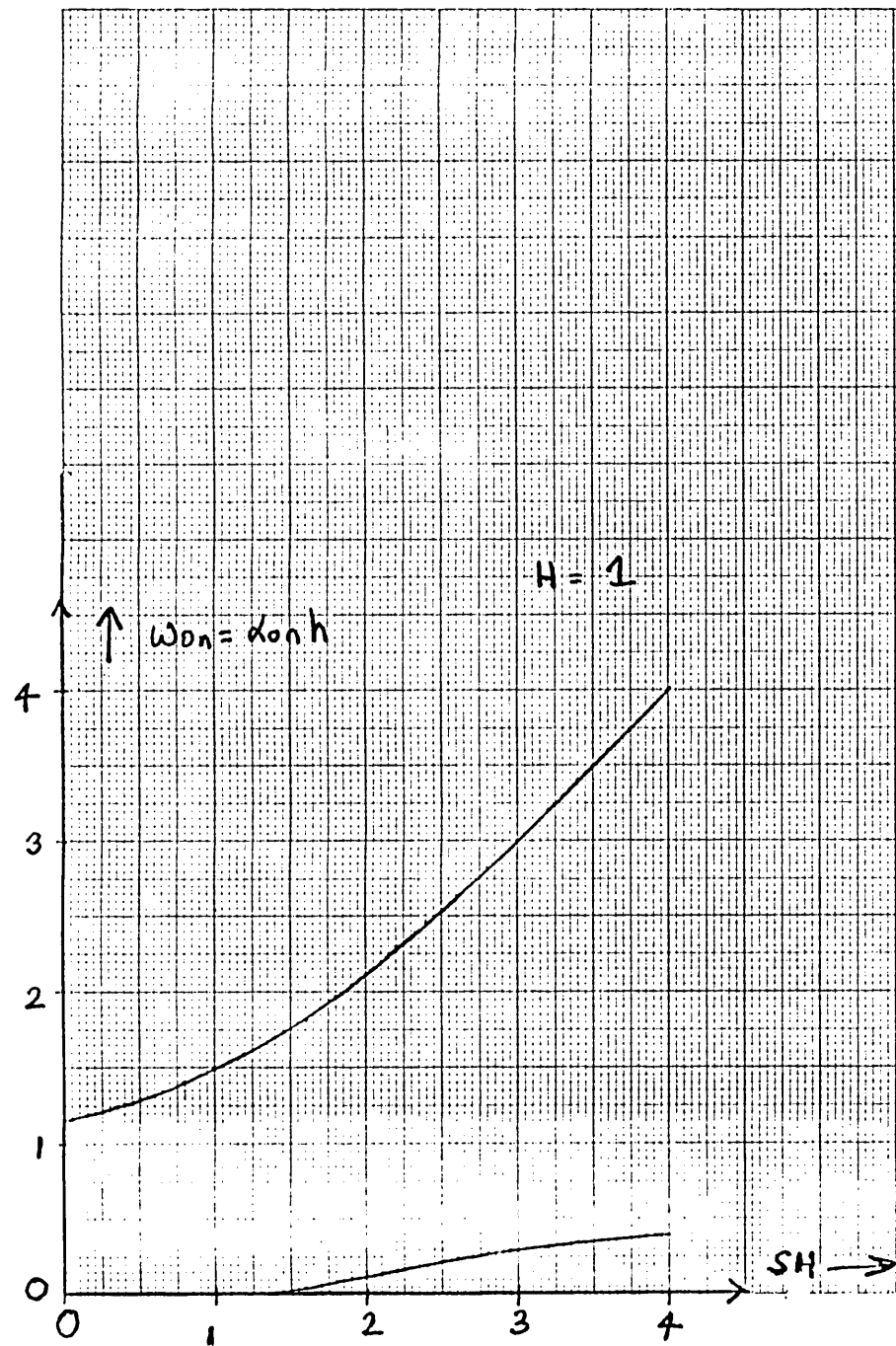


(a)

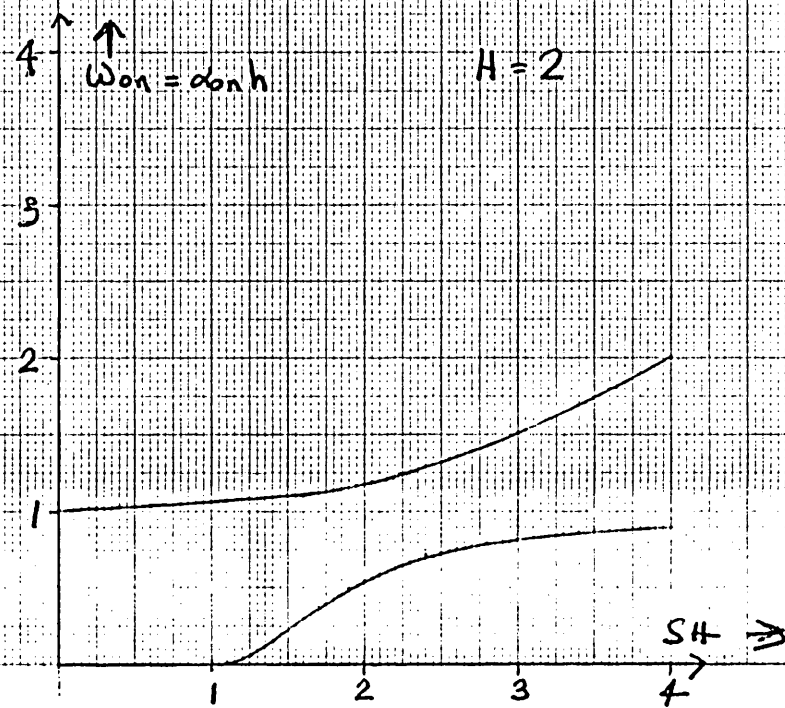


(b)

Fig. 2ab

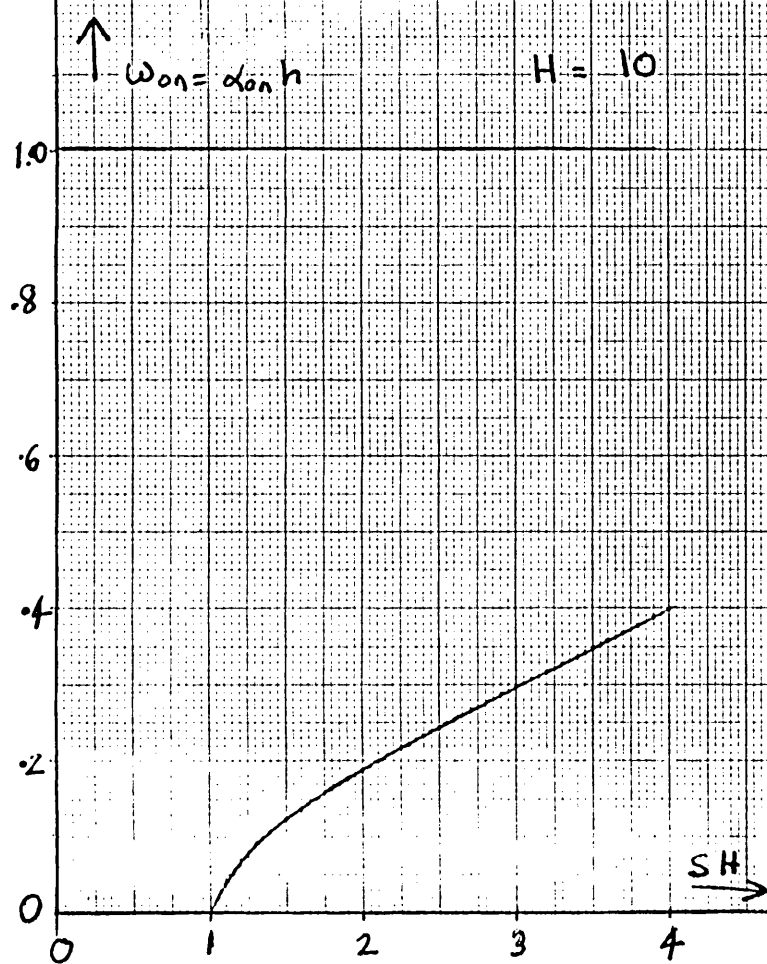


(c)

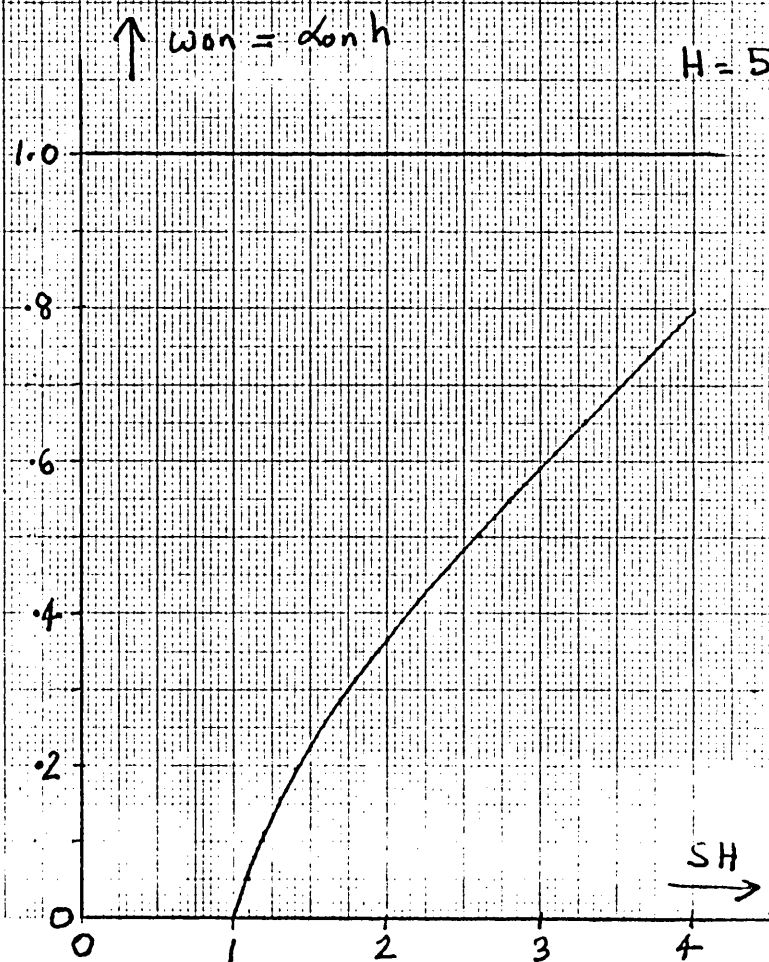


(d)

Fig. 2c1



(e)



(f)

Fig. 2ef

Fig. 3a

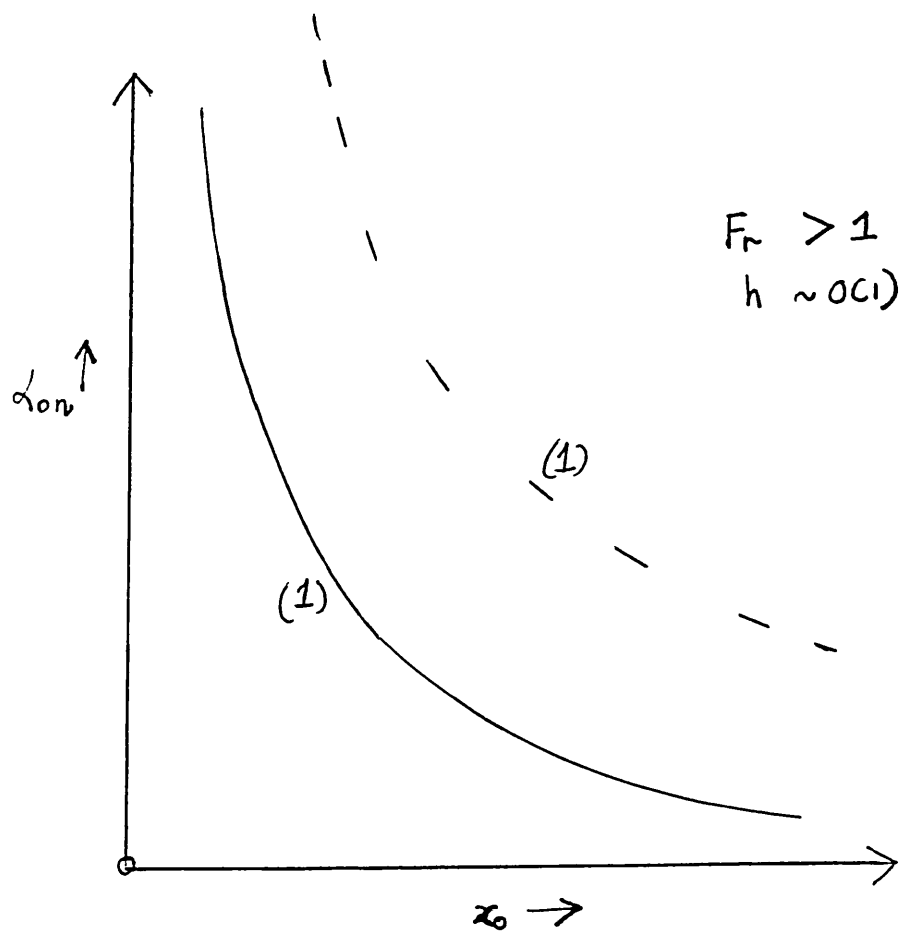


Fig. 3b

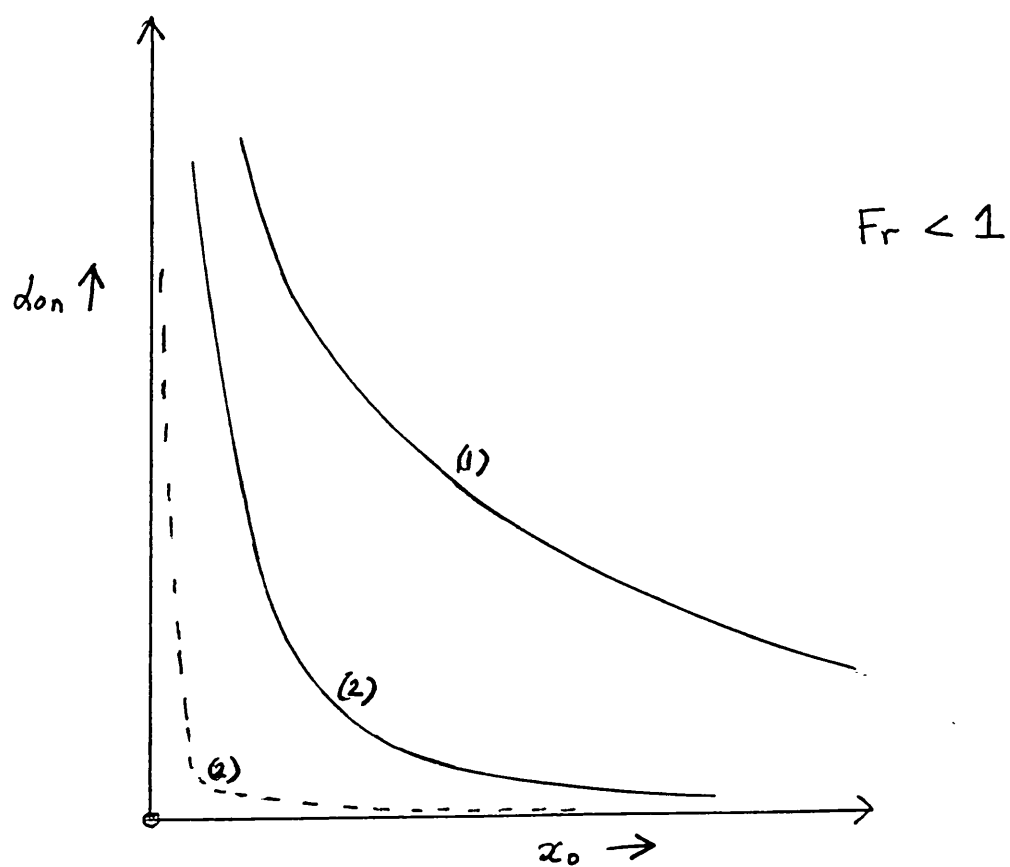
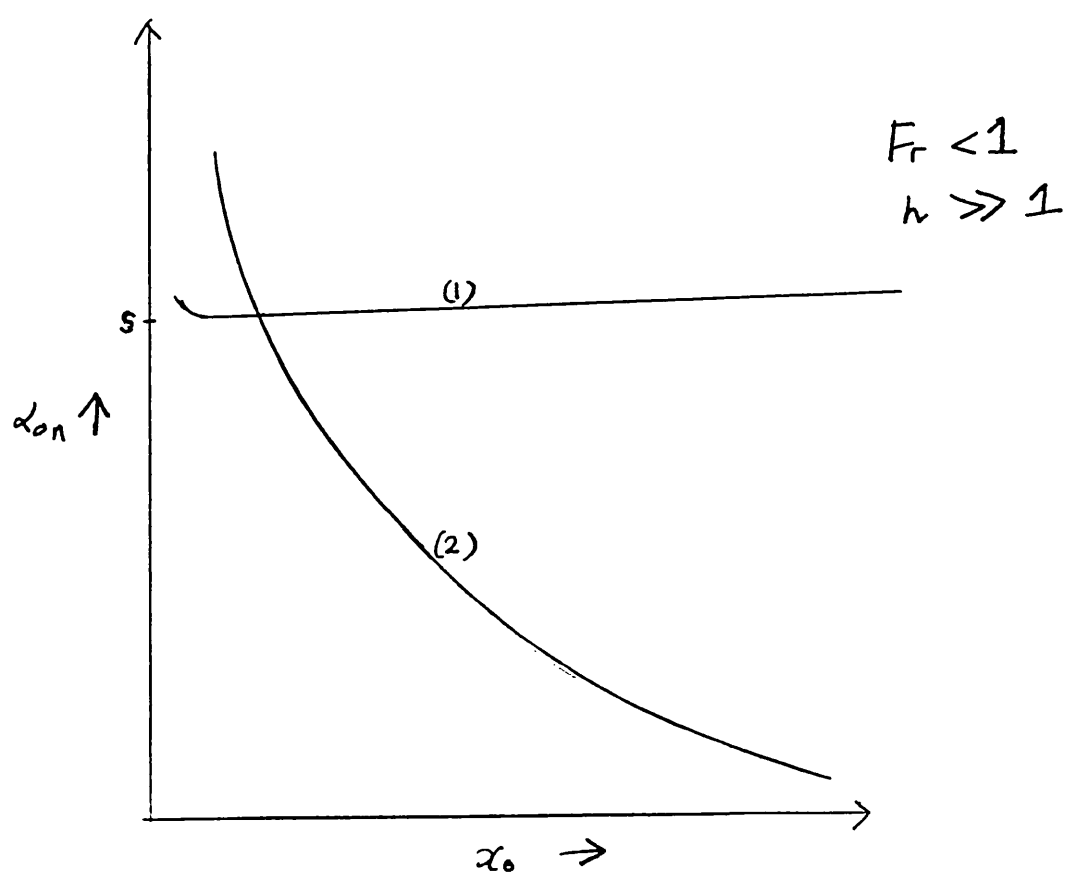


Fig. 3c



Captions

Fig. 1

Neutral wavenumber $\alpha_{on} = \alpha_{on}(s)$ from (2.9), for fully developed free surface flows. Note $\alpha_{on} \propto R^{1/4}$.

Fig. 2(a-f)

Neutral wavenumbers α_{on} for undeveloped free surface flows, plotted against $(Fr)^{-1} = SH$, for several values of H , from (4.7c). See § 4.

Fig. 3(a-c)

Sketch of the possible neutral curves for undeveloped free surface flows.

(1), (2), refer to mode 1, mode 2, see § 4.

(a) Dashed line represents behaviour of mode 2 as $h \rightarrow 0$.

(b) Dashed line represents the limit $Fr \rightarrow 1^-$.

CHAPTER 6

UPPER-BRANCH STABILITY OF LIQUID LAYER OR
FULLY DEVELOPED FREE SURFACE FLOWS.

§1 Introduction

The linear theory for the upper-branch stability of parallel flows at large Reynolds number is well known, and has been discussed by various authors, see e.g. Lin (1955), Stuart (1963), Reid (1965), Drazin & Reid (1981), and recently Smith (1979). The earlier approaches, although leading to the correct results, can be described at best as being good approximate methods. Reid (1965) was the first to discuss the theory with more formal asymptotic methods, and Smith (1979) showed how these ideas could be put into a more consistent structural framework.

Velthuisen and Van Wijngaarden (1969) have attempted to use some of the classical ideas of parallel flow stability to discuss the propagation of small amplitude waves over a non-uniform free surface flow. However they do not discuss neutral modes, and in addition there are certain assumptions implicit in their work which need careful consideration.

It is reasonable to assume that the stability of liquid layer, or fully developed free surface flows at large Reynolds number is similar to that of channel flows. In §2 we consider briefly the linear theory for two-dimensional long-wave disturbances to the basic flow, at large Froude numbers. In fact the disturbance structure and analysis is more or less identical to that of Smith (1979), and the results for channel flows can be regarded as a special case here. As in Smith & Bodonyi (1981) we consider the next stage (§3) when the non-

linear terms first come into play. This happens when the disturbance size attains the crucial ordering $O(R^{-14/33})$ (where $R \gg 1$, is the Reynolds number), and there linear properties continue to hold except in the critical layer where the governing equations are those derived by Haberman (1972), (see Stewartson (1981) for a review on critical layers). The analysis in §3 follows closely the work for the upper-branch stability of boundary layer flows by Smith & Bodonyi (1982).

There are two interesting aspects of the results derived in §2 and §3. Firstly, if the disturbance size in §3, is further increased, then (again as in Smith & Bodonyi) the Rayleigh regime, with the disturbance size now $O(R^{-1/3})$ ^{and} λ with the wavelength and wave-speed both $O(1)$, is obtained, (for $O(1)$ or large Froude numbers). Secondly, if in §2 the Froude number becomes $O(1)$, the wavelength shortens and becomes $O(1)$, and the critical layer retreats towards the wall (but is still asymptotically distinct from the wall layer). A new structure emerges in which the outer inviscid regions give rise to eigenvalue problems for determining the wavenumbers, and the critical and wall layer serve to determine the wavespeed. In fact the leading order eigenvalue problem for $O(1)$ Froude numbers is identical to that of the previous chapter. The nonlinear counterpart of this structure (in which the critical layer properties are again those governed by the Haberman equation), is briefly examined in §4. The crucial disturbance size in this instance $O(R^{-7/15})$, can be obtained from the results of §3 (or also from chapter 8, §2).

Finally we make a few more comments in §5.

§2 Linearized Theory for Infinitesimal Disturbances

The Navier-Stokes equations are suitably non-dimensionalized (as e.g. in chapter 2) and we work with cartesian coordinates (x,y) , velocity (u,v) time t , and pressure p . h is the undisturbed dimensional depth of the liquid, and $Fr = (s^*)^{-1}$ the Froude number. For the liquid layer flow the basic flow is given by the velocity $(U_B, 0)$ and pressure p_B , where $U_B(y) = y(1-y/2)$, $p_B = s^*(1-y)$, $s^* = (\cot \Theta^*)/R$, Θ^* being the inclination of the plane to the horizontal. The main results below are appropriate for a more general fully-developed profile, so instead we take a general basic flow satisfying $U_{By}(1)=0$, $U_B \sim \lambda y + \lambda_2 y^2 + \dots$ as $y \rightarrow 0$. λ is the skin-friction and $\lambda_2 (<0)$ the curvature of the oncoming profile. (Strictly the basic flow needs to be expanded also in the ensuing expansions, but we avoid this extra complication since it does not alter the main results).

The velocities and pressure are expanded as

$$(U, V, p) = (U_B, 0, p_B) + \delta \left[\tilde{U}, \tilde{V}, \tilde{p} \right] (y) E + \text{c.c.} + \dots \quad (2.1)$$

where $E \equiv \exp(i\alpha(x-ct))$, and $\delta (\ll 1)$ is the infinitesimal disturbance size. The free surface is given by

$$y = 1 + \eta = 1 + \delta \tilde{\eta} E + \text{c.c.} + \dots \quad (2.2)$$

We expect the upper-branch solutions here (for long waves) to be similar to those for channel flows, Smith (1979), and hence we set

$$\epsilon = R^{-\frac{1}{11}} \ll 1 \text{ and}$$

$$s^* = \epsilon^2 s, \quad \alpha = \epsilon \alpha_0 + \epsilon^3 \alpha_1 + \dots, \quad c = \epsilon^2 c_0 + \epsilon^4 c_1 + \dots \quad (2.3a-c)$$

α_0, c_0 are real for neutral modes.

The disturbance structure, comprising four zones Z1 - Z4, see fig. (1a) ^{is} identical to that in Smith (1979) and consequently we omit most of the details.

In Z1 where $y \sim O(1)$, the disturbance quantities are expanded as

$$\tilde{u} = u_0 + \epsilon^2 u_1 + \dots, \quad \tilde{v} = \epsilon v_0 + \epsilon^3 v_1 + \dots, \quad (2.4a-d)$$

$$\tilde{p} = \epsilon^2 p_0 + \epsilon^4 p_1 + \dots, \quad \tilde{\eta} = \eta_0 + \epsilon^2 \eta_1 + \dots$$

Substituting (2.1 - 2.4) into the linearized Navier Stokes equations then yields

$$u_0 = A_0 \frac{dU_B}{dy}, \quad v_0 = -i\alpha_0 U_B A_0, \quad (2.5a-c)$$

$$p_0 = p_{00} - \alpha_0^2 A_0 \int_0^y U_B^2(y_1) dy_1$$

$$\text{and } v_1 = i\alpha_o U_B \int_1^y \frac{p_o(t)dt}{U_B^2(t)} + i\alpha_o A_o c_o - i\alpha_o A_1 U_B \quad (2.6a)$$

$$p_1 = p_{10} + \int_0^y \alpha_o^2 U_B^2(t) dt \int_1^{t_1} \frac{p_o(t)dt}{U_B^2(t)}$$

$$-\alpha_o (\alpha_1 A_o + \alpha_o A_1) \int_0^y U_B^2(t) dt + 2\alpha_o^2 A_o c_o \int_0^y U_B^2 dt \quad (2.6b)$$

where A_o , A_1 , p_{o0} , p_{10} are unknown constants, (and assumed to be real, without loss of generality). We note that (2.6ab) are different from ^{the} expressions in Smith (1979) because of misprints in the latter.

Application of the first order free surface conditions which require

$$p_o = s\eta_o \text{ at } y = 1, \quad v_o = iU_B \eta_o \alpha_o \text{ at } y = 1, \quad (2.7)$$

now yield

$$\eta_o = -A_o, \quad p_o = s\eta_o + \alpha_o^2 \cdot A_o \int_0^1 U_B^2 dy \quad (2.8)$$

(The tangential stress constraint does not give anything extra).

The solutions in the remaining zones are as in Smith (1979), and we will just state some of the important results. First in $\mathbb{R}^2$

where $y = \varepsilon^2 Y$ with $Y \sim O(1)$, we have from (2.4 - 2.6) the expansions

$$\begin{aligned}\tilde{U} &= U^{(0)} + \varepsilon^2 U^{(1)} + \dots, \quad \tilde{V} = \varepsilon^3 V^{(0)} + \varepsilon^5 V^{(1)} + \dots \\ \tilde{P} &= \varepsilon^2 P^{(0)} + \varepsilon^4 P^{(1)} + \dots\end{aligned}\quad (2.8)$$

The solution for $V^{(0)}$ together with the tangential flow constraint $(V^{(0)})_{y=0} = 0$, gives

$$P^{(0)} = P_{00} = A_0 c_0 \lambda \quad (2.9)$$

The solution for $V^{(1)}$ contains a logarithmic singularity at $Y = \frac{c_0}{\lambda}$, the critical level, and this requires a viscous layer, (Z3), of thickness $O(\varepsilon^{10/3})$ to smooth out the irregularity. There with $y = \frac{\varepsilon^2 c_0}{\lambda} + \varepsilon^{10/3} \bar{Y}$, the expansions

$$\begin{aligned}\tilde{U} &= \bar{U}^{(0)} + \varepsilon^2 \bar{U}^{(1)} + \dots, \quad \tilde{V} = \varepsilon^3 \bar{V}^{(0)} + \varepsilon^{13/3} \bar{V}^{(1)} + \\ &\varepsilon^5 \bar{V}^{(2)} + \varepsilon^{19/3} \bar{V}^{(3)} + \dots\end{aligned}\quad (2.10)$$

hold. $\bar{U}^{(0)}$ is constant, and $\bar{U}^{(1)}$ satisfies a forced Airy equation which provides the necessary match with the logarithmic irregularity in Z2, and gives the condition that $\ln(Y - \frac{c_0}{\lambda})$ be replaced by

($\ln |Y - \frac{c_0}{\lambda}| - i\pi$) for $Y < \frac{c_0}{\lambda}$ in $U^{(1)}$, $V^{(1)}$ in $Z2$. The displacement from the wall layer ($Z4$), where $y = \epsilon^4 \bar{y}$, and

$$\tilde{U} = \bar{U}_0 + \dots, \quad \tilde{V} = \epsilon^5 \bar{V}_0 + \dots, \quad \tilde{p} = \epsilon^2 \bar{p}_0 \quad (2.11)$$

then matches with the phase-shift induced in $Z2$ and leads to the eigenrelation,

$$\frac{2\pi(-\lambda_2) c_0^2}{\lambda^3} = \frac{1}{(2\alpha_0 c_0)^{1/2}} \quad (2.12)$$

for real wavenumbers and frequencies, see Smith (1979) or §3. (The solutions for $Z2 - Z4$ above can in fact be obtained by neglecting the harmonic terms in §3).

Hence (2.8), (2.9), (2.12) lead to the relations

$$c_0 = \frac{(\alpha_0^2 I_1 - s)}{\lambda}, \quad \alpha_0^{1/2} (\alpha_0^2 I_1 - s)^{5/2} = \frac{\lambda^{11/2}}{\sqrt{2} \pi (-\lambda_2)} \quad (2.13ab)$$

for the neutral wavenumbers and wavespeeds. Here $I_1 = \int_0^1 U_B^2 dy$.

If $\tilde{s} = 0$, (2.13ab) reduce to the expressions for the wavenumbers and frequencies for the upper-branch of the linear neutral curve in

channel flows, see Stuart (1963), Reid (1965), Smith (1979). For liquid layer flow, s small corresponds to a much greater inclination of the plane, specifically $\tan \theta \gg R^{-9/11}$.

If s is large, then (2.13ab) gives

$$\alpha_o \sim \left(\frac{s}{I_1} \right)^{1/2}, \quad c_o \sim \left[\frac{\lambda^3}{2\pi(-\lambda_2)\sqrt{2}} \right]^{2/5} I_1^{+1/10} s^{-1/10} \quad (2.14ab)$$

It is interesting that the wavenumber here is the same as in the previous chapter. Formally if s becomes $O(\epsilon^{-2})$ so that the Froude number falls to $O(1)$, the wavelength $\alpha \sim O(1)$ and the wave-speed $c \sim R^{-1/5}$. A new structure is thus indicated for $O(1)$ Froude numbers, and this is addressed in §4.

Having established the linear theory, we now consider much larger disturbance sizes. Formally, by substituting the expansions into the Navier-Stokes equations we find that nonlinearity first comes into play, inside the critical layer, when δ attains the crucial size $O(R^{-14/33} = \epsilon^{14/3})$, and then the $V \frac{\partial u}{\partial y}$ term cannot be neglected there. This regime in which nonlinearity and viscosity are equally important, was first studied by Haberman (1972). A similar situation occurs in the study by Smith & Bodonyi (1982) where as the disturbance size is increased, significant alterations take place within the critical layer.

Hence summarizing linear theory continues to hold for $\delta \ll R^{-14/33}$, but when $\delta \sim R^{-14/33}$ — — —

the linear theory becomes invalid. We consider the nonlinear theory next.

§3 The Nonlinear Critical Layer*

The expansions below with the disturbance size $\delta = \epsilon^{14/3}$, are suggested by the linear theory, and in fact the main balances are linear except in the critical layer (as in Smith & Bodonyi). Also we now seek periodic solutions dependent on $\tilde{X}$, where

$$\frac{\partial \tilde{X}}{\partial x} = \epsilon \alpha, \quad \frac{\partial \tilde{X}}{\partial t} = -\epsilon \alpha c. \quad (3.1)$$

The free surface is given by

$$y = 1 + \eta = 1 + \epsilon^{14/3} \eta_0 + \epsilon^{20/3} \eta_1 + \dots \quad (3.2)$$

In §1, then we have from (2.4a-d)

$$U = U_B + \epsilon^{14/3} U_0 + \epsilon^{16/3} U_{\text{MF}}(y) + \epsilon^{20/3} U_1 + \dots, \quad V = \epsilon^{17/3} V_0 + \epsilon^{23/3} V_1 + \dots, \quad (3.3a-c)$$

$$p = \epsilon^2 s(1-y) + \epsilon^{14/3} p_0 + \epsilon^{20/3} p_1 + \dots$$

* It should be noted that the main results and solutions here can be obtained by appropriately adapting the unsteady solutions of chapter 8. To avoid too much duplication therefore, some of the details omitted can be found there, and vice-versa.

The extra mean flow term U_{MF} is needed because of the properties of steady nonlinear critical layers (see Stewartson (1981)). Substitution into the Navier-Stokes equations then yields the solutions

$$U_B = A_0 \frac{dU_0}{dy} \cos \tilde{X}, \quad V_0 = U_B A_0 \alpha_0 \sin \tilde{X} \quad (3.4ab)$$

$$p_0 = P_{00}(\tilde{X}) - A_0 \alpha_0^2 \hat{C}_1 \int_0^y U_B^2 dy_1 \quad (3.4c)$$

and

$$V_1 = \alpha_0 U_B \int_1^y \frac{1}{U_B^2} \frac{\partial p_0}{\partial \tilde{X}} d\eta - A_0 \alpha_0 c_0 \hat{S}_1 + \alpha_0 U_B \hat{g}_{\tilde{X}}(\tilde{X})$$

$$p_1 = P_{10}(\tilde{X}) - \alpha_0^2 \int_0^y U_B^2 \frac{\partial^2}{\partial \tilde{X}^2} \int_1^y \frac{p_0}{U_B^2} d\eta + 2\alpha_0^2 c_0 A_0 \hat{C}_1 \int_0^y U_B dy_1$$

$$- \left\{ \alpha_0^2 \frac{\partial^2 \hat{g}}{\partial \tilde{X}^2} + \alpha_0 \alpha_1 A_0 \hat{C}_1 \right\} \int_0^y U_B^2 dy \quad (3.4de)$$

where $\hat{C}_n = \cos(n\tilde{X})$, $\hat{S}_n = \sin(n\tilde{X})$ $n = 0, 1, 2, \dots$, and A_0 is a constant, P_{00} , P_{10} , $\hat{g}$ are unknown periodic functions of $\tilde{X}$. The

latter are expanded in Fourier series, i.e.

$$P_{10} = \sum_0^{\infty} P_{10}^{(n)} \hat{C}_n + \tilde{P}_{10}^{(n)} \hat{S}_n, \quad \hat{g} = \sum_0^{\infty} g^{(n)} \hat{C}_n + \tilde{g}^{(n)} \hat{S}_n \quad (3.5a-c)$$

$$P_{00} = P_{00}^{(1)} \hat{C}_1$$

Here (3.5c) is written in anticipation of Z2, and the harmonic terms in (3.5ab) are needed because of the non-monochromatic phase jump across the critical layer. The free surface conditions requiring

$$p_0 = s\eta_0, \quad p_1 = s\eta_1 \quad \text{at } y = 1 \quad (3.6)$$

$$V_0 = \alpha_0 U_B \eta_{ox}^{\sim}, \quad V_1 = \alpha_0 U_B \eta_{1x}^{\sim} + \alpha_1 U_B \eta_{ox}^{\sim} - \alpha_0 c_0 \eta_{ox}^{\sim} \quad \text{at } y=1$$

now yield the relations from (3.4), (3.5), (3.6)

$$\eta_0 = \eta_0^{(1)} \hat{C}_1 = -A_0 \hat{C}_1, \quad P_{00}^{(1)} = s\eta_0^{(1)} + A_0 \alpha_0^2 I_1 \quad (3.7ab)$$

$$n\alpha_0 g^{(n)} = \alpha_0 \eta_i^{(n)} n + \delta_{1n} \alpha_1 \eta_0^{(1)} \quad n = 1, 2, \dots \quad (3.7c)$$

$$\tilde{g}^{(n)} = \tilde{\eta}_1^{(n)} \quad n = 1, 2, 3, \dots \quad (3.7d)$$

$$s\tilde{\eta}_1^{(n)} = \tilde{p}_{10}^{(n)} + \alpha_0^2 n^2 I_1 \tilde{g}^{(n)} \quad n = 0, 1, 2, \dots \quad (3.7e)$$

$$s\tilde{\eta}_1^{(n)} = p_{10}^{(n)} + \alpha_0^2 n^2 I_1 g^{(n)} + A_0 \delta_{1n} D_0 \quad n=0, 1, 2, \dots \quad (3.7f)$$

Here δ_{ij} is the Kronecker delta and

$$\eta_1 = \sum_0^\infty \eta_1^{(n)} \hat{C}_n + \tilde{\eta}_1^{(n)} \hat{S}_n$$

$$I_1 = \int_0^1 U_B^2 dy, \quad I_2 = \int_0^1 U_B^2 \int_1^y \frac{1}{U_B^2} \int_0^{y_1} U_B^2 dy_2 dy_1 dy$$

$$I_3 = \int_0^1 U_B^2 \int_1^y \frac{1}{U_B^2} dy_1 dy, \quad I_4 = \int_0^1 U_B dy$$

$$D_0 = \alpha_0^2 \lambda c_0 I_3 - \alpha_0^4 I_2 + 2\alpha_0^2 c_0 I_4 - \alpha_0 \alpha_1 I_1.$$

(3.7) (a-f) serve to determine a relation between P_{10} , and $\hat{g}$.

As $y \rightarrow 0$, V_1 contains a logarithmic singularity and this can be seen clearly by considering the much thinner region (Z2) where $y = \varepsilon^2 Y$ is comparable with the wavespeed. There from (3.3), (3.4)

we have the expansions

$$U = \lambda \varepsilon^2 Y + \lambda_2 \varepsilon^4 Y^2 + \varepsilon^{14/3} U^{(0)} + \varepsilon^{16/3} \bar{U}_{MF} + \varepsilon^{20/3} U^{(1)} + \dots$$

$$V = \varepsilon^{2/3} V^{(0)} + \varepsilon^{29/3} V^{(1)} + \dots, \quad p = \varepsilon^{20/3} p^{(0)} + \varepsilon^{20/3} p^{(1)} + \dots \quad (3.8a-c)$$

(3.8) substituted into the Navier-Stokes equations and matching with ZI no_A^w gives $p^{(0)} = p_{00}$, $p^{(1)} = p_{10}$ and

$$V^{(0)} = -\frac{1}{\lambda} \frac{dp^{(0)}}{dx} + (\lambda Y - c_0) A_0 \alpha_0 \hat{S}_1, \quad U^{(0)} = A_0 \lambda \hat{C}_1 \quad (3.9ab)$$

The tangential flow constraint $V^{(0)} = 0$ at $y = 0$ (or equivalently by considering the solutions in the wall layer), yields from (3.9a)

$$p_{00}^{(1)} = A_0 \lambda c_0 \quad (3.10)$$

At the next order we find

$$V^{(1)} = -\frac{1}{\lambda} \left[\alpha_0 p_{\lambda}^{(1)} - \alpha_0 c_1 U_{\lambda}^{(0)} + \alpha_1 p_{\lambda}^{(0)} \right] - \frac{A_0 \alpha_0 \hat{S}_1}{\lambda^2} \left[c_0^2 \lambda_2 - 2\lambda_2 c_0 \xi \ln \xi - \lambda_2 \xi^2 \right]$$

$$- \xi \alpha_0 a_{\lambda} \quad (3.11ab)$$

$$\alpha_0 U^{(1)} = -\alpha_1 U^{(0)} + \frac{2A_0 \alpha_0 \lambda \hat{C}_1}{\lambda} \left[\xi + c_0 (1 + \ln \xi) \right] + \alpha_0 a \lambda$$

where $\xi = \lambda Y - c_0$, and $a(\tilde{x})$ is some unknown function of $\tilde{x}$. Hence decomposing $a(\tilde{x})$ into a Fourier series $a = \sum_0^\infty a^{(n)} \hat{C}_n + \tilde{a}^{(n)} \hat{S}_n$, we obtain after matching the linear terms to $O(\varepsilon^{29/3})$ in V between $Z1$ and $Z2$,

$$p_{00z} \tilde{\alpha}_0 \left[\lambda \int_1^0 \left(\frac{1}{U_B^2} - \frac{1}{\lambda^2 y^2} + \frac{2\lambda_2}{\lambda^3 y} \right) dy + \left(-\frac{1}{\lambda} + \frac{\lambda_2}{\lambda^2} \right) \right] + \alpha_0 \lambda \hat{g}_x^{\tilde{v}}$$

$$- A_0 \alpha_0^3 \lambda \hat{S}_1 \left(\int_0^1 \frac{1}{U_B^2} \int_0^1 U_B^2 dy_1 dy \right) = - \lambda \alpha_0 a_x - \frac{2A_0 \alpha_0 \hat{S}_1 c_0 \lambda_2}{\lambda} \quad (3.12)$$

giving the relations

$$\tilde{g}^{(n)} = -\tilde{a}^{(n)}, \quad g^{(n)} = -a^{(n)} + A_0 D_1 \delta_{1n} \quad n = 1, 2, \dots, \quad (3.13)$$

where

$$D_1 = c_0 \left[\lambda \int_0^1 \left(\frac{1}{U_B^2} - \frac{1}{\lambda^2 y^2} + \frac{2\lambda_2}{\lambda^3 y} \right) dy - \left(-\frac{1}{\lambda} + \frac{\lambda_2}{\lambda^2} \right) \right]$$

$$- \alpha_0^2 \cdot \left[\int_0^1 \frac{1}{U_B^2} \int_0^y U_B^2 dy_1 dy \right] + \frac{2c_0 \lambda_2}{\lambda^2}.$$

The logarithmic terms above require an examination of ^{the} critical layer, Z3, where $y = \frac{\varepsilon^2 c_o}{\lambda} + \varepsilon^{1/3} \bar{Y}$. The solutions in Z2, thus imply the expansions there,

$$U = \varepsilon^2 c_o + \varepsilon^{1/3} \lambda \bar{Y} + \frac{\lambda_2 c_o^2}{\lambda^2} + \varepsilon^{1/3} \bar{U}_0 + \varepsilon^{16/3} \left(\bar{U}_m + \frac{2\lambda_2 c_o \bar{Y}}{\lambda} \right) + (\bar{U}_1 - \lambda_2 \bar{Y}^2) \varepsilon^{2/3} + \dots \quad (3.14)$$

$$V = \varepsilon^{2/3} A_o \alpha_o c_o \hat{S}_1 + \varepsilon^{2/3} \bar{V}_0 + \varepsilon^{29/3} \bar{V}_{1/2} + \varepsilon^{33/3} \bar{V}_1 + \dots$$

$$p = \varepsilon^{2/3} \bar{p}_0 + \varepsilon^{26/3} \bar{p}_1 + \varepsilon^{28/3} \bar{p}_2 + \varepsilon^{30/3} \bar{p}_3 + \dots$$

After substituting into the Navier-Stokes equations we find that $\bar{U}_0, \bar{U}_m, \bar{V}_{1/2}, \bar{p}_0, \bar{p}_1, \bar{p}_2, \bar{p}_3$ are independent of $\bar{Y}$, and can be obtained from matching, e.g. $\bar{U}_0 = A_o \lambda \hat{C}_1$. The first non-trivial balance yields the equation

$$\alpha_o \lambda \bar{Y} \bar{I}_{1x} + A_o \alpha_o c_o \hat{S}_1 \bar{\zeta}_{1\bar{Y}} + 2\lambda_2 A_o \alpha_o c_o \hat{S}_1 - \bar{\zeta}_{1\bar{Y}\bar{Y}} = 0 \quad (3.15a)$$

where $\bar{\zeta}_1 = \bar{U}_{1\bar{Y}}$, together with the matching constraints

$$\bar{U}_1 \sim \frac{2A_o \alpha_o \lambda_2 \hat{C}_1}{\lambda} \ln|\bar{Y}| + \tilde{\lambda}_{\pm} \bar{Y} + \bar{U}_{\pm}(\tilde{X}) \text{ as } \bar{Y} \rightarrow \pm \infty. (3.15b)$$

The transformation

$$\bar{U}_1 = -\lambda_2 \bar{Y}^2 + \frac{2\lambda_2 c_o A_o}{\lambda} \frac{\partial \psi^*}{\partial Y^*}, \quad \bar{Y} = \left[\frac{A_o c_o}{\lambda} \right]^{\frac{1}{2}} Y^* \quad (3.17)$$

now reduces (3.15ab) to the canonical form

$$Y^* \frac{\partial^3 \psi^*}{\partial \tilde{X} \partial Y^{*2}} + \sin(\tilde{X}) \frac{\partial^3 \psi^*}{\partial Y^{*3}} = \gamma_c \frac{\partial^4 \psi^*}{\partial Y^{*4}} \quad (3.18)$$

$$\psi^*_{Y^*} \sim \frac{1}{2} Y^{*2} + U^*_{\pm}(\tilde{X}) + \cos(\tilde{X}) \ln|Y^*| + 2H_{\pm} Y^* \text{ as } Y^* \rightarrow \infty.$$

$$\text{Here } H_{\pm} = \frac{\tilde{\lambda}_{\pm} \lambda^{\frac{1}{2}}}{4\lambda_2 (c_o A_o)^{\frac{1}{2}}}, \quad \gamma_c = (\alpha_o \lambda)^{-1} \left(\frac{c_o A_o}{\lambda} \right)^{-3/2}$$

$$\text{and } U^*_+ - U^*_- = \frac{\lambda}{2\lambda_2 c_o A_o} (\bar{U}_+ - \bar{U}_-) = \frac{\lambda^2}{2\lambda_2 c_o A_o} (a_+ - a_-) \quad (3.19a-c)$$

The equation (3.18) was first derived by Haberman (1972), and has been extensively studied by various authors, e.g. Haberman (1976), Brown & Stewartson (1978), Smith & Bodonyi (1982), Stewartson (1981). Briefly

its main properties are; (i) the velocity jump $U_+^* - U_-^* = \sum_0^\infty F_n \hat{C}_n + \tilde{F}_1 \hat{S}_n$ is non-monochromatic; (ii) the phase jump defined by $\phi = \hat{F}_1 = \frac{1}{\pi} \int_0^{2\pi} (U_+^* - U_-^*) \sin(\hat{X}) d\hat{X}$, is related to the vorticity jump by $\phi = 4\gamma_c (H_+ - H_-)$, and consequently a non-zero mean vorticity is generated (hence the need for the terms of $O(\epsilon^{16/3})$ above); the parameter γ_c describes in effect the linearity, nonlinearity of the disturbance. As $\gamma_c \rightarrow \infty$, $\phi \rightarrow -\pi$ and this corresponds to the phase jump of linear theory. If $\gamma_c \rightarrow 0$ $\phi \sim \tilde{\tau} \gamma_c$, where $\tilde{\tau}$ is an $O(1)$ constant. The phase jump decreases monotonically with γ_c .

For a further discussion of the above equation and its properties one is referred to the references mentioned above.

The continuation below the critical layer ($\xi < 0$) in Z_2 is thus given by $\ln \xi \rightarrow \ln |\xi|$ (for $\xi < 0$), and a_- given by (3.19c).

In the wall layer, which is required to reduce the disturbance velocity to zero, $y = \epsilon^4 \bar{y}$, and we have from (3.3), (3.4)

$$U = \lambda \epsilon^4 \bar{y} + \epsilon^{14/3} \hat{U}_0 + \dots, \quad V = \epsilon^{29/3} \hat{V}_0 + \dots, \quad p = \epsilon^{20/3} \hat{p}_0 + \dots \quad (3.20)$$

We find that

$$\hat{U}_0 = A_0 \lambda \left[\cos \hat{X} - e^{-m_1 \bar{y}} \cos(\hat{X} + m_1 \bar{y}) \right]$$

$$\hat{V}_0 = \frac{A_0 \lambda \alpha_0}{\sqrt{2} m_1} \left\{ -\sin(\hat{X} + \pi/4) + e^{-m_1 \bar{y}} \sin(\hat{X} + m_1 \bar{y} + \pi/4) + \sqrt{2} m_1 \sin \hat{X} \right\}$$

$$\hat{p}_0 = p_{00}, \quad \text{and } m_1 = \left(\frac{\alpha_0 c_0}{2} \right)^{1/2} \quad (3.20a-d)$$

This provides the necessary displacement to match with the phase jump, and in fact we have

$$\lim_{\bar{y} \rightarrow 0} V^{(1)} = - \frac{A_o \lambda \alpha_o}{\sqrt{2} m_1} \sin(\tilde{x} + \pi/4) .$$

Hence from (3.11a) we obtain

$$\begin{aligned} & - \frac{1}{\lambda} \left[\alpha_o p_{\tilde{x}}^{(1)} - \alpha_o c_1 U_{\tilde{x}}^{(o)} + \alpha_1 p_{\tilde{x}}^{(o)} \right] - \frac{2A_o \alpha_o}{\lambda^2} \lambda^2 c_o^2 \ln c_o \hat{S}_1 + c_o \alpha_o a_{\tilde{x}} \\ & = - \frac{A_o \lambda \alpha_o}{\sqrt{2} m_1} \sin(\tilde{x} + \pi/4) \end{aligned}$$

which gives the additional relations for the Fourier components, using (3.19c)

$$- \frac{\alpha_o n}{\lambda} \tilde{p}_{10}^{(n)} + c_o \alpha_o a_{\tilde{x}}^{(n)} n - \frac{2\lambda A_o \alpha_o c_o^2 n \tilde{F}_n}{\lambda^2} = - \frac{A_o \lambda \alpha_o}{2m_1} \delta_{1n}$$

$$n = 1, 2, 3, \dots \quad (3.22a)$$

$$\frac{\alpha_o n \tilde{p}_{10}^{(n)}}{\lambda} - c_o \alpha_o n a_{\tilde{x}}^{(n)} + \frac{2\lambda A_o \alpha_o c_o^2 n F_n}{\lambda^2} = A_o D_2 \delta_{1n} \quad n=1, 2, \dots \quad (3.22b)$$

Here

$$D_2 = \alpha_0 c_1 - \alpha_1 c_0 + \frac{2\lambda \alpha_0 c_0^2 \ln c_0}{\lambda^2} - \frac{\lambda \alpha_0}{2m_1}$$

Thus combining the above results we obtain first from (3.7ab),
(3.10) the eigenrelation

$$c_0 \lambda = \alpha_0^2 I_1 - s, \quad (3.23)$$

and from (3.7c-f), (3.13), (3.22)

$$- \tilde{P}_{10}^{(n)} \left[\frac{1}{\lambda} + \frac{c_0}{(s - \alpha_0^2 n^2 I_1)} \right] - \frac{2\lambda A_0 c_0^2 F_n}{\lambda^2} = - \frac{A_0 \lambda}{2m_1} \delta_{1n} \quad n=1,2,\dots$$

$$P_{10}^{(n)} \left[\frac{1}{\lambda} + \frac{c_0}{(s - \alpha_0^2 n^2 I_1)} \right] + \frac{2\lambda A_0 c_0^2 F_n}{\lambda^2} \quad (3.24ab)$$

$$= A_0 \delta_{1n} \cdot \left[\frac{D_2}{\alpha_0} - c_0 \left(\frac{D_0 - s \alpha_1}{\alpha_0} + \lambda D_1 \right) \right] \quad n=1,2,\dots$$

The constant terms in the Fourier series can be taken to be zero, without loss of generality. The harmonic coefficients are determined

from (3.24ab), and the other relations. With $n=1$ in (3.24ab), the $p_{10}^{(1)}$, $\bar{p}_{10}^{(1)}$, and hence $\eta_{10}^{(1)}$, $\bar{\eta}_{10}^{(1)}$, $a^{(1)}$, $\bar{a}^{(1)}$, $g^{(1)}$, $\bar{g}^{(1)}$ remain indeterminate, (since they multiply the basic eigenfunction), and instead (3.2.b) gives an eigenrelation connecting α_1, c_1 . (3.24a) yields

$$\frac{2\lambda_2 c_o^2 \phi}{\lambda^2} = \frac{\lambda}{(2c_o \alpha_o)^{\frac{1}{2}}} \quad (3.25)$$

Briefly summarizing, the neutral wavenumber and frequency are determined as a function of the amplitude A_o , and the Froude number, from (3.23), (3.25) and the solution of the Haberman critical layer problem (3.18). For general values of A_o , this can be done numerically as in Smith & Bodonyi, who also give a graph of $\phi = \phi(\gamma_c)$.

The limiting cases are of some interest, and their properties are easily deduced. (i) We note that $s=0$ gives the results for channel flows, (see Haberman (1972)). (ii) For $s=O(1)$, the linear theory is recovered as $A_o \rightarrow 0$. (iii) For $A_o \gg 1$, with again $s=O(1)$, $\phi \sim \gamma_c$, (3.19b), (3.23), (3.25), show that $\alpha_o \sim A_o \sim c_o^{\frac{1}{2}}$. Hence, formally if A_o increases by $O(\epsilon^{-1})$, the wavenumber and wavespeed become $O(1)$, and the effective disturbance amplitude is now $O(\epsilon^{\frac{1}{3}} A_o) \sim O(R^{-\frac{1}{3}})$. Thus for this Rayleigh structure with the critical layer occurring in the interior of

of the flow, and dependent more on nonlinearity than viscosity, (see Benney & Bergeron (1969)), the disturbance amplitude must be $O(R^{-1/3})$. The same conclusion was obtained by Smith & Bodonyi (1982) from their study of boundary layer flows. (iv) If in (iii) we take s to be large, specifically $s \sim c$, the above conclusion holds also for $O(1)$ Froude numbers. (v) If we take $s \gg 1$, but keep $\phi = O(1)$, then from (3.19b), (3.23), (3.25), we find that

$$\alpha \left(\frac{s}{I_1} \right)^{1/2} + O(s^{-2/3}), \quad c_0 \sim \left(\frac{\lambda^3}{2 \phi \lambda_2 \sqrt{2}} \right)^{2/5} I_1^{1/10} s^{-1/10}, \quad (3.26)$$

$$A_0 \sim s^{-7/30}.$$

Hence formally if s increases by $O(\epsilon^{-2} = R^{2/11})$ the wavelength shortens and becomes $O(1)$, the wavespeed $c \sim O(R^{-1/5})$, and the disturbance amplitude $\epsilon^{1/3} A_0 \sim O(R^{-7/15})$. We briefly examine this new structure in §4.

The Rayleigh structure mentioned in (iii) above is discussed in the next chapter, for the case $s = 0$, i.e. channel flows.

§4 The Disturbance Structure When $s^* = O(1)$, $\alpha = O(1)$,
 $c = O(R^{-1/5})$.

Following the above discussion, we now set

$$s^* = s, \alpha = \bar{\alpha}_0 R^{-1/5} + \bar{\alpha}_1 + \dots, c = \bar{c}_0 R^{-1/5} + \bar{c}_1 R^{-2/5} + \dots \quad (4.1)$$

$$\beta = \bar{\beta}_0 R^{-1/5} + \bar{\beta}_1 R^{-2/5} + \dots$$

with the disturbance size $\delta = O(R^{-7/15})$ in (2.1). The disturbance structure, fig. (1.b), again consists of four zones $\bar{Z}1 \rightarrow \bar{Z}4$, (five if we include the much thinner region, which is needed in order to satisfy the zero tangential stress constraint, at the free surface). In zone $\bar{Z}1$, with $y = O(1)$, from (2.1), (3.2), (3.26), we have the expansions

$$u = U_B R^{-7/15} + U_O R^{-10/15} + U_1 R^{-13/15} + U_2 + \dots, V = V_O R^{-7/15} + V_1 R^{-10/15} + V_2 + \dots$$

$$p = s(1-y) + R^{-7/15} P_O + R^{-10/15} P_1 + R^{-13/15} P_2 + \dots, \quad (4.2a-d)$$

$$\eta = R^{-7/15} \bar{\eta}_0 + R^{-10/15} \bar{\eta}_1 + \dots$$

We look at disturbances periodic in $\bar{X}$, where

$$\frac{\partial}{\partial x} = \alpha \frac{\partial}{\partial \bar{X}}, \quad \frac{\partial}{\partial t} = -\alpha c \frac{\partial}{\partial \bar{X}} \equiv -\beta \frac{\partial}{\partial \bar{X}} \quad (4.3)$$

and take the fundamental to be given by

$$U_0 = U_{01}(y) \cos \bar{X}, \quad V_0 = V_{01}(y) \sin \bar{X}, \quad P_0 = P_{01}(y) \cos \bar{X},$$

$$\bar{\eta}_0 = \bar{\eta}_{01} \cos \bar{X} \quad (4.4)$$

After substituting (4.2), (4.4) into the Navier-Stokes equation we find that $U_{01}, V_{01}, \dots$, satisfy

$$\bar{\alpha}_0 U_{01} - V_{01} y = 0, \quad \bar{\alpha}_0 U_{01} U_{01} - U_{01} V_{01} = -\bar{\alpha}_0 P_{01}, \quad \bar{\alpha}_0 U_{01} V_{01} = -P_{01} y \quad (4.5a-c)$$

with the free surface conditions

$$P_{01} = s \bar{\eta}_{01}, \quad V_{01} = -\bar{\alpha}_0 U_{01} \bar{\eta}_{01} \quad \text{at } y=1 \quad (4.5de)$$

and the inviscid tangential flow constraint at the wall,

$$V_{01} = 0 \quad \text{at } y = 0 \quad (4.5f)$$

The eigenvalue problem (4.5), giving the Rayleigh equation with zero wavespeed, is identical to that discussed in the previous chapter (see (3.5) of Chapter 5). We normalize so that $U_{01}(0) = A_0 \lambda$, where

A_0 represents a scaled disturbance size.

At the next order, $U_1, V_1, P_1, \bar{\eta}_1$, satisfy

$$\bar{\alpha}_0 U_{1\bar{x}} + \bar{\alpha}_1 U_{0\bar{x}} + V_{1y} = 0, \quad -\bar{\beta}_0 V_{0\bar{x}} + (\bar{\alpha}_0 V_{1\bar{x}} + \bar{\alpha}_1 V_{0\bar{x}}) U_B = -P_{1y}$$

$$-\bar{\beta}_0 U_{0\bar{x}} + U_B (\bar{\alpha}_0 U_{1\bar{x}} + \bar{\alpha}_1 U_{0\bar{x}}) + U_{By} V_1 = -(\bar{\alpha}_0 P_{1\bar{x}} + \bar{\alpha}_1 P_{0\bar{x}}) \quad (4.6a-c)$$

with the boundary conditions

$$P_1 = s\bar{\eta}_1, \quad V_1 = \bar{\alpha}_0 U_B \bar{\eta}_{1\bar{x}} + \bar{\alpha}_1 U_B \eta_{1\bar{x}} - \bar{\beta}_0 \bar{\eta}_{0\bar{x}} \quad \text{at } y=1 \quad (4.6de)$$

together with matching with $\bar{z}2$. We note that V_1 behaves logarithmically as $y \rightarrow 0$. In $\bar{z}2$ $y = R^{-1/5} Y$, with the basic flow comparable with the wavespeed, we have from (4.2) (4.5), the expansions

$$U = R^{-1/5} \lambda \bar{Y} + R^{-2/5} \lambda_2 Y^2 + R^{-7/15} \bar{U}_0 + R^{-8/15} \bar{U}_{MF}(Y) + R^{-3/5} \lambda_3 Y^3 + R^{-10/15} \bar{U}_1 + \dots, \quad (4.7a-c)$$

$$V = R^{-10/15} \bar{V}_0 + R^{-13/15} \bar{V}_1 + \dots, \quad p = s(1-y) + R^{-10/15} \bar{P}_1 + R^{-13/15} \bar{P}_2 + \dots$$

The Navier-Stokes equations now give the solutions,

$$\bar{P}_1 = P_1)_{y=0}, \quad \bar{P}_2 = P_2)_{y=0}, \quad (4.8ab)$$

$$\bar{U}_0 = A_0 \lambda \cos \bar{X}, \quad \bar{V}_0 = -\frac{1}{\lambda} \frac{d\bar{P}}{d\bar{X}} + (\lambda Y - \bar{C}_0) A_0 \bar{\alpha}_0 \sin \bar{X} \quad (4.8cd)$$

$$\bar{V}_1 = -\frac{1}{\lambda} \left[\bar{\alpha}_0 \bar{P}_{2\bar{X}} - \bar{\alpha}_0 \bar{C}_1 \bar{U}_{0\bar{X}} + \bar{\alpha}_1 \bar{P}_{1\bar{X}} \right] - \xi \bar{\alpha}_0 a_{\bar{X}}$$

$$-\frac{A_0 \bar{\alpha}_0 \sin \bar{X}}{\lambda^2} \cdot \left[\bar{C}_0^2 \lambda_2 - 2\lambda_2 \bar{C}_0 \xi \ln \xi - \lambda_2 \xi^2 \right] \quad (4.8e)$$

where $\xi = \lambda Y - \bar{C}_0$. Here the mean flow term U_{MF} is again necessary from the critical layer analysis and $a(\bar{X})$ is unknown. From (4.8d) we require $\bar{V}_0 = 0$ at $Y = 0$, (this again follows from the momentum balance in the wall layer) giving

$$\frac{1}{\lambda} \frac{d\bar{P}_1}{d\bar{X}} = -A_0 \bar{C}_0 \sin \bar{X} \quad (4.9)$$

The analysis for the critical layer ($\bar{Z}3$ where $y = R^{-1/5} \frac{\bar{C}_Q}{\lambda} + R^{-1/3} \bar{y}$) and the wall layer ($\bar{Z}4$, with $y = R^{-2/5} \bar{Y}$) is very similar to that in §3, and we will just use the results. First the continuation (of 4.8e) below the critical layer is as given by (3.19c). Second, the wall layer yields the same displacement condition, i.e. $\bar{V}_1)_{y=0} =$

$-\frac{A_o \lambda \bar{\alpha}_o}{(\bar{\alpha}_o \bar{c}_o)^{\frac{1}{2}}} \sin(\bar{X} + \pi/4)$. This latter result together with (4.8e) now

determines a relation for the $\sin \bar{X}$ components of $\bar{P}_2$, a , i.e.,

$$-\frac{\bar{\alpha}_o}{\lambda} \tilde{P}_{21} + \bar{c}_o \bar{\alpha}_o \tilde{a}_1 - \frac{2\lambda A_o \bar{\alpha}_o \bar{c}_o^2 \tilde{F}_1}{\lambda^2} = -\frac{A_o \bar{\alpha}_o \lambda}{(2\bar{\alpha}_o \bar{c}_o)^{\frac{1}{2}}} \quad (4.10)$$

Here $\tilde{P}_{21}$, $\tilde{a}_1$ are the $\sin \bar{X}$ components of $\bar{P}_2$, a respectively. Hence to complete the problem we need another relation between $\tilde{P}_{21}$, $\tilde{a}_1$. To obtain this, we need to reconsider the second and third order problems in $\bar{Z}1$.

From (4.6a-e), (4.8a), (4.9), we see that the $\sin \bar{X}$ component of $\tilde{V}_1$, $\tilde{V}_{11}$, satisfies the eigenvalue problem

$$\tilde{V}_{11yy} - \left(\frac{\bar{\alpha}_o^2 + U_{Byy}}{U_B^2} \right) \tilde{V}_{11} = \frac{\bar{\beta}_o U_{Byy} V_{01}}{\bar{\alpha}_o U_B^2} - \frac{2\bar{\alpha}_o P_{01y}}{U_B}$$

$$\tilde{V}_{11y} - \frac{\tilde{V}_{11}}{U_B^2} = -\frac{2\bar{\beta}_o}{U_B^2} P_{01} \quad \text{at } y=1, \quad \tilde{V}_{11}=0 \quad \text{at } y=0$$

Also (from inspection of the relevant equations) the $\cos \bar{X}$ component

of V_2 , i.e. V_{21} , satisfies the same equation as $\tilde{V}_{11}$, but with the condition at the wall given by $V_{21}|_{y=0} = -\frac{\bar{\alpha}_0}{\lambda} \tilde{P}_{21} + C_0 \bar{\alpha}_0 \tilde{a}_1$.

Hence a solution exists only provided $V_{21}|_{y=0} = 0$ giving, from (4.10), the eigenrelation

$$\frac{2\lambda \phi \bar{C}_0^2}{\lambda^2} = \frac{\lambda}{(2\bar{\alpha}_0 \bar{C}_0)^{\frac{1}{2}}} \quad (4.11)$$

In addition the solvability condition for $\tilde{V}_{11}$ (as in chapter 5) yields the relation for $\bar{\alpha}_1$,

$$\frac{2\bar{\alpha}_0 \bar{\beta}_0 \bar{\eta}_{01}^2}{U_B(1)} s = -2\bar{\alpha}_1 \int_0^1 \frac{p_{01y}}{U_B} V_{01} dy + \frac{\bar{\beta}_0}{\bar{\alpha}_0} \frac{U_B''}{U_B^2} V_{01}^2 dy$$

In summary then, the eigenvalue problem (4.5) determines the neutral wavenumber $\bar{\alpha}_0$, and the critical layer problem, together with (4.11) give the neutral wavespeed and wavenumber as functions of the amplitude A_0 , and the Froude number s^{-1} . For large Froude numbers, the results of chapter ⁵_A show that a match with (3.26) is obtained. The problem for $O(1)$ and small Froude numbers requires a numerical solution of the relevant equations, and is left unresolved.

§5 Additional Comments

Regarding the stability of liquid layer flow over an inclined plane, it is important to note that the modes presented here are different from those discussed by Benjamin and Yih. Their results are appropriate for small and finite Reynolds numbers, and also when the inclination of the plane to the horizontal is much greater than that discussed here. Numerical solutions of the Orr-Sommerfeld equation would help in clarifying the properties of the neutral curve(s) at large Reynolds numbers .

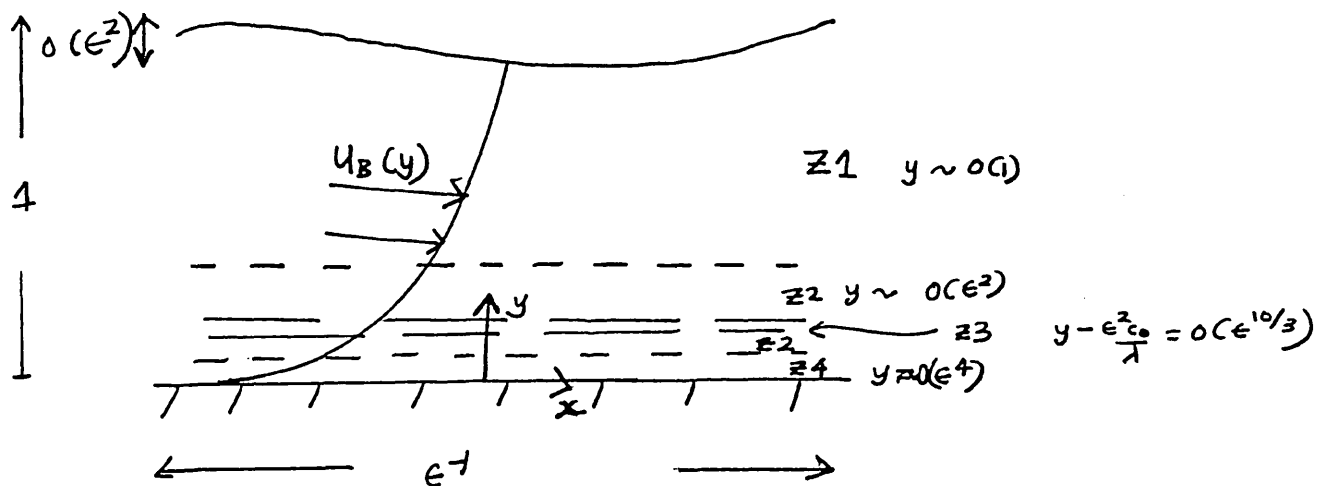
Finally, it is interesting that for $O(1)$ Froude numbers, with the critical layer occurring in the interior of the flow, i.e. the Rayleigh stage, the boundary layer at the free surface is comparable with the wall layer, $O(R^{-1/2})$, and hence will directly influence the stability properties.

CHAPTER 6

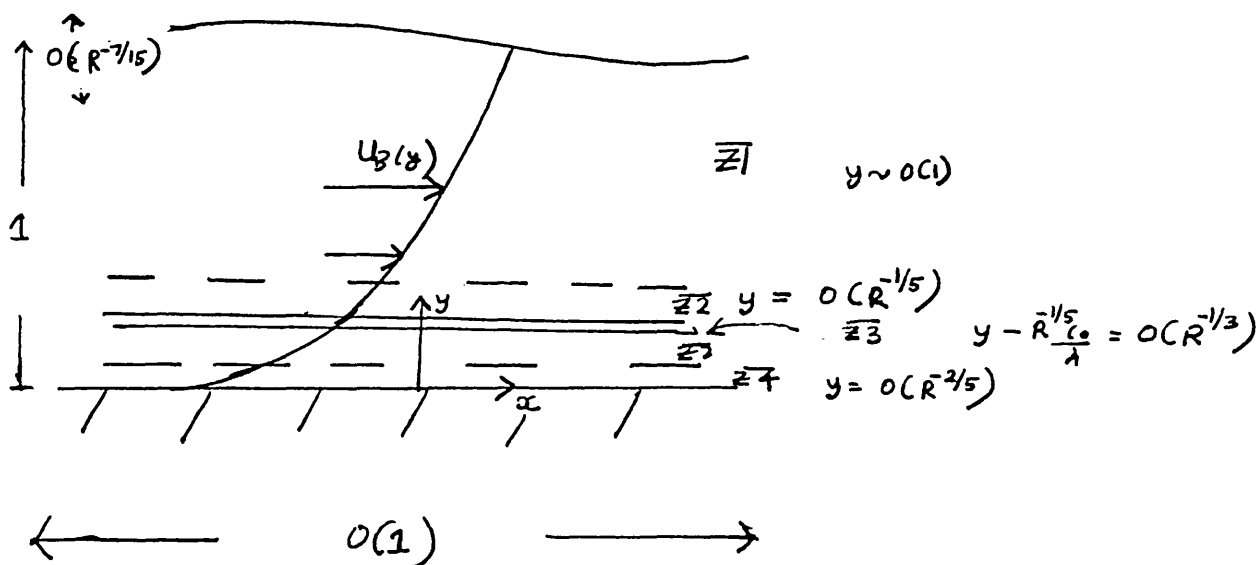
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Fig 1



(a)



(b)

Captions

Fig. 1

(a) Disturbance structure for long wave modes with $s^* = O(\epsilon^2)$.
Here $\epsilon = R^{-1/11}$, see §2, §3.

(b) Disturbance structure for $\alpha = O(1)$, $s^* = O(1)$, $c = O(R^{-1/5})$.
See §4.

CHAPTER 7

AMPLITUDE-DEPENDENT STABILITY OF BOUNDARY LAYER FLOW
WITH A STRONGLY NONLINEAR CRITICAL LAYER

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AMPLITUDE-DEPENDENT STABILITY OF BOUNDARY LAYER FLOW

WITH A STRONGLY NONLINEAR CRITICAL LAYER

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Summary

The amplitude-dependent neutral stability properties, mainly of an accelerating boundary-layer flow, are studied theoretically for large Reynolds numbers when the disturbance size δ is sufficiently large to provoke a strongly nonlinear critical layer within the flow field. The theory has a rational basis aimed at a detailed understanding of the delicate physical balances controlling stability. It shows that when the fundamental disturbances size δ is of $O(R^{-1/3})$, where R is the Reynolds number based on the boundary-layer thickness, the neutral wavelength and wavespeed scale in such a way that they become comparable with the typical thickness and speed, respectively, of the basic flow. In this Rayleigh-like situation a new feature emerges, that of a substantial pressure variation across the critical layer, which strongly affects the jump conditions on the Rayleigh solutions holding outside the critical layer. As a result of the strong nonlinearity the total velocity jump is affected nonlinearly by the critical layer vorticity, while in contrast the phase shift remains linearly dependent on the vorticity. Furthermore, it is shown that the phase shift, not the total velocity jump, dictates the neutral stability criteria.

Also, flow reversal occurs near the wall where the disturbance is greater than the basic flow. The link between the viscous effects in the wall layers and in the critical layer fixes the amplitude-dependence of the neutral modes throughout. As the disturbance amplitude increases

the critical layer with vorticity trapped within it moves toward the edge of the boundary layer and is forced to leave the boundary layer when δ exceeds $O(R^{-1/3})$. This departure is rather abrupt, involving a dependence on (scaled amplitude)⁻¹².

1. Introduction

The stability properties of boundary layers and channel flows depend significantly on the amplitude of the unsteady disturbance introduced, if that disturbance is not sufficiently small. This is especially so when the Reynolds number is large. In this paper we consider mainly the amplitude-dependent planar stability of an accelerated boundary layer (on a plane wall) subjected to a small fundamental disturbance whose relative amplitude δ is $O(R^{-1/3})$, where R is the Reynolds number based on the boundary layer thickness. This particular size, $O(R^{-1/3})$, arises from Smith & Bodonyi's (1982a) study of smaller disturbances. They found that if δ is $O(R^{-7/18})$ for an accelerating boundary layer then the neutral wavelength is greater than the boundary layer thickness by a factor $R^{1/6}$, the wavespeed is small so that the critical layer lies close to the wall and as in the works of Haberman (1972) and Brown & Stewartson (1978, 1980) the critical layer is influenced as much by nonlinear interactions as by viscosity. However, contrary to previous suggestions, the mean flow correction due to the nonlinear critical layer there is smaller than the fundamental disturbance. Then as δ is increased from $O(R^{-7/18})$ the neutral wavelength falls and the wavespeed increases in such a way that just as the wavespeed rises to the order of the average flowspeed the wavelength decreases to the order of the boundary layer thickness and δ reaches the present size, $O(R^{-1/3})$. Similar amplitude-dependent adjustments from linear theory, which in the above case holds for $\delta \ll R^{-7/18}$, take place in other planar motions such as Blasius or plane Poiseuille flow (Gajjar 1982) where again the $O(R^{-1/3})$ amplitude is the crucial one which produces the Rayleigh-like situation of (i) a critical layer lying in the midst of the basic flow, with (ii) the neutral wavelength then being comparable with the typical flow thickness.

The conditions (i), (ii) are assumed in almost all previous related studies of sizeable disturbances. It is often suggested also that then (iii) the mean flow correction is of order $\delta^{1/2}$, greater than the fundamental, and (iv) the normal pressure variation within the critical layer has negligible effect. However, the present study supports Smith & Bodonyi (1982a) in finding that (i)-(iv) are incompatible. Specifically, when (iv) holds (i)-(iii) do not, since (iv) requires δ to be less than $O(R^{-1/3})$ which is the Smith & Bodonyi case where the neutral wavespeed and wavenumber and the mean flow correction are relatively small. Conversely, when (i), (ii) do hold, which is the regime of present concern, then $\delta \sim R^{-1/3}$ and (iii) holds but (iv) is found to be invalid. The new feature emerging, that of a substantial pressure variation across the critical layer, strongly affects the jump conditions on the Rayleigh solutions holding outside the critical layer for $\delta = O(R^{-1/3})$. In §2 below the overall flow structure (Fig. 1a) is described. Then the form of the strongly nonlinear critical layer, as opposed to the nonlinear viscous type considered by Haberman (1972), Brown & Stewartson (1978, 1980) and Smith & Bodonyi (1982a), is studied in §3. As a result of the strong nonlinearity the total velocity jump, emphasized by Brown & Stewartson (1978), is affected nonlinearly by the critical layer vorticity. In contrast the phase shift, emphasized by Haberman (1972, 1976) and Smith & Bodonyi (1982a), remains linearly dependent on the vorticity and it is the phase shift, not the total velocity jump, which dictates the neutral stability criteria.

This phase shift ϕ across the critical layer has to be in tune with the shift produced by the wall layers, which are discussed in §4. The disturbance is greater than the basic flow near the wall and so reversed flow is present there. Balancing of the phase shifts between these layers then requires

ϕ to be $O(R^{-1/2})$, which gives the underlying reason for the particular size $\delta \sim R^{-1/3}$ being singled out, as distinct from the estimate $\delta \sim R^{-1/3}$ of Benney & Bergeron (1969). For when $\delta \ll R^{-1/3}$ then $\phi \gg R^{-1/2}$ and no neutral Rayleigh modes exist if the basic velocity profile is inflexion-free: disturbances of amplitude less than $O(R^{-1/3})$ have small wavenumbers and wave-speeds and are much less unstable. The amplitude-dependence of the neutral nondimensional wavespeed c_0 and wavenumber α are determined in §5. As the disturbance increases in size within the $O(R^{-1/3})$ scaling both α and c_0 increase: see §5 and the Appendix. Ultimately for relatively large disturbances c_0 approaches the free stream speed, implying (see Fig. 4 below) that the critical layer with vorticity trapped inside it then starts to emerge from the boundary layer. This emergence is found to be a rather fast affair involving a dependence on (amplitude)⁻¹².

Although the present analysis concentrates on boundary layer flows, the analysis is broadly applicable to plane Poiseuille flow as well. It concerns the upper branch of the amplitude-dependent neutral curve, wherein the critical layer remains distinct from the wall layers, as opposed to the lower branch stability problem of amplitude dependence studied by Smith (1979). Figure 1b gives a summary of the effects of disturbance size on the neutral curve for flows of the type considered here.

In what follows we consider a Cartesian coordinate system with stream-wise and lateral coordinates x, y and the corresponding velocity components $\bar{u}, \bar{v}$ nondimensionalised with respect to a typical boundary layer thickness a^* and the free stream speed U_∞^* , respectively, and time is written as a^*t/U_∞^* . With ρ^*, ν^* denoting the density and kinematic viscosity of the incompressible fluid, the pressure is given by $\rho^*U_\infty^{*2}p$ and the Reynolds number $R = U_\infty^*a^*/\nu^* \gg 1$. Thus the global Reynolds number Re used by Smith and Bodonyi (1982a) is of order R^2 . For the most important aspects the basic flow can be regarded as essentially parallel i.e.,

$\bar{u} = U_0(y)$, $\bar{v} = \bar{p} = 0$ effectively, since the influence of nonparallelism is negligible to the order we consider here. Also, we have $U_0(\infty) = 1$, $U_0(0) = 0$ and $U_0'(0) = \lambda$, $U_0''(0) = 2\lambda_2$, where λ , λ_2 give the reduced wall shear stress and profile curvature, respectively. Finally, the nondimensional stream function is denoted by $\bar{\psi}$.

§2. The Rayleigh Zones

When the disturbance size increases to $O(R^{-1/3})$ the critical layer moves to a new location in the midst of the main part of the boundary layer as discussed by Smith and Bodonyi (1982a) and we approach exactly the classical Rayleigh scalings wherein the majority of the inviscid characteristics of the disturbance are controlled over length scales comparable with the typical lateral scale of the basic flow as shown schematically in Fig. 1a. The disturbance structure subdivides into four regions denoted by I, II, III, and IV. We begin our study by considering the Rayleigh zones I⁺.

The unsteady fundamental part of the disturbance considered here has size δ of $O(R^{-1/3})$. Since, from §1, this induces a larger, $O(R^{-1/6})$, correction to the mean flow the disturbed motion takes the form

$$\begin{aligned} (\bar{u}, \bar{v}, \bar{p}) = & [u_0(y), 0, 0] + R^{-1/6} [u_{1s}(y), 0, 0] + R^{-1/3} [u_2, v_2, p_2](\bar{x}, y) \\ & + R^{-1/3} [u_{2s}(y), 0, 0] + \dots + R^{-5/6} [u_5, v_5, p_5](\bar{x}, y) + \dots \end{aligned} \quad (2.1a)$$

over the bulk of the flowfield. We seek neutral travelling wave forms dependent on $\bar{x} \equiv x - c_0 t$ and y rather than on x, t, y , and the successive mean flow corrections u_{1s}, u_{2s} are dependent only on y . We note that the lateral and streamwise dimensions of the flow structure are comparable with the reduced boundary layer thickness of $O(1)$, in contrast with the earlier work of Smith & Bodonyi (1982a). The unsteady fundamental disturbance in (2.1a) is expressed in the form

$$(u_2, v_2, p_2) = [f(y)C, g(y)S, p(y)C] \quad (2.1b)$$

where $C \equiv \cos(\alpha \bar{x})$, $S \equiv \sin(\alpha \bar{x})$ and the $O(1)$ constants c_0 ($0 < c_0 < 1$), α , the dominant wavespeed and the streamwise wavenumber, respectively, are to be determined in terms of the effective disturbance size Λ , where Λ is a typical value of p , say the wall value $p(0+)$, i.e., $\Lambda = p(0+)$.

Substitution of (2.1a,b) into the Navier-Stokes equations leaves the mean flow corrections u_{1s} , u_{2s} undetermined to leading order but yields the inviscid equations

$$\alpha f - g' = 0; (U_0 - c_0)\alpha f - gU_0' = -\alpha p; (U_0 - c_0)\alpha g = -p' \quad (2.2a,b,c)$$

controlling the fundamental disturbance, with ' denoting the appropriate ordinary derivative. Hence the Rayleigh equation

$$(U_0 - c_0)(g'' - \alpha^2 g) = U_0'' g \quad (2.3a)$$

holds for $g(y)$. The boundary conditions on (2.3a) are, first,

$$|g(\infty)| < \infty; \quad g(0+) = 0 \quad (2.3b,c)$$

to avoid exponential growth at the edge of the boundary layer and to give the typical inviscid condition of tangential flow as the wall is approached in I^- : see also §4 below. Second, since $0 < c_0 < 1$ there is an internal critical level, at $y = y_c$ where the flowspeed $U_0(y_c)$ equals the wavespeed c_0 , across which jump conditions on f , g , p are necessary. These are given as follows. As $y \rightarrow y_c+$ we have

$$g \sim \sum_{n=0}^{\infty} g^{(n)} (y - y_c)^n + \ln(y - y_c) \sum_{n=1}^{\infty} G^{(n)} (y - y_c)^n \quad (2.4a)$$

from (2.3a), where the constants $g^{(0)} (>0)$, $g^{(1)}$ are unknowns upon which the other coefficients $g^{(n)} (n \geq 2)$, $G^{(n)} (n \geq 1)$ depend, for a given value of y_c . For instance.

$$G^{(1)} = 2b_2 g^{(0)} / b_1, \quad G^{(2)} = 2b_2^2 g^{(0)} / b_1^2, \quad \text{where } n!b_n \equiv (d^n U_0 / dy^n)(y_c) \quad (2.4b)$$

for $n \geq 1$. Here b_1, b_2 give the shear and curvature of the basic flow at the critical layer. The continuation of (2.4a) on the other side of the critical layer, $y = y_c^-$, depends on the flow properties within the thin critical layer which surrounds $y = y_c$ and is required to smooth out the logarithmic irregularity in (2.4a). With our disturbance size $\delta = O(R^{-1/3})$, the critical layer is strongly nonlinear, in contrast both with the nonlinear viscous kind studied by Habermann (1972), Brown & Stewartson (1978), Smith & Bodonyi (1981), and others, and with the classical linear viscous kind of Reid (1965) for example. The strongly nonlinear critical layer here has the new structure tentatively suggested by Smith & Bodonyi's (1982a) §5 and it is considered in detail in §3 below. Its principal feature which we anticipate here is the smallness of the velocity jump and phase shift produced across it. Thus the local behaviour (2.4a) just above the critical layer, and the accompanying form

$$f \sim \frac{1}{G} \sum_{n=1}^{\infty} [G^{(n)} + n g^{(n)} + n G^{(n)} \ln(y - y_c)] (y - y_c)^{n-1}, \quad y \rightarrow y_c^+, \quad (2.4c)$$

for f and that for p implied by (2.2a-c), (2.4a), are reproduced as $y \rightarrow y_c^-$, just below the critical layer, with

$$\ln(y - y_c)[y \rightarrow y_c^+] \text{ replaced by } \ln(y_c - y)[y \rightarrow y_c^-]. \quad (2.4d)$$

This result for sizeable disturbances is essentially that proposed by Benney & Bergeron (1969) and Davis (1969) although they did not determine the amplitude.

To find the amplitude dependence of α , c_0 requires a more precise knowledge of the velocity jump rather than just knowing (from (2.4d)) that it is small. The solution of the eigenvalue problem (2.3a-c), (2.4a,d) fixes only the dependence of α on c_0 , not the dependence of α , c_0 on the scaled disturbance size, $A (=p(0+))$. So next we consider in detail the critical layer (§3) and wall layer (§4) properties.

3. The Strongly Nonlinear Critical Layer

The critical layer II surrounds $y = y_c$ and has thickness $O(R^{-1/6})$, with $y = y_c + R^{-1/6}\tilde{Y}$, say. The underlying development of the flow solution here is given by

$$\begin{aligned}\bar{u} &= c_0 + R^{-1/6}b_1\tilde{Y} + R^{-1/3}\ln(R)\tilde{U}_{2L} + R^{-1/3}\tilde{U}_2 + R^{-1/2}\tilde{U}_3 + R^{-2/3}\tilde{U}_4 + R^{-5/6}\tilde{U}_5 + \dots \\ \bar{v} &= R^{-1/3}g^{(C)}S + R^{-1/2}\ln(R)\tilde{V}_{2L} + R^{-1/2}\tilde{V}_2 + R^{-2/3}\tilde{V}_3 + R^{-5/6}\tilde{V}_4 + R^{-1}\tilde{V}_5 + \dots \quad (3.1a,b,c) \\ \bar{p} &= R^{-1/3}g^{(O)}\frac{b_1}{\alpha}C + R^{-1/2}\tilde{P}_2 + R^{-2/3}\ln(R)\tilde{P}_{3L} + R^{-2/3}\tilde{P}_3 + R^{-5/6}\tilde{P}_4 + R^{-1}\tilde{P}_5 + \dots\end{aligned}$$

from §2. Although additional terms, e.g. of $O(R^{-1/2} \ln R)$ in $\bar{u}$, are introduced into (3.1a-c) and into some of the results below by the nonlinear interactions they do not disturb significantly the main balances of forces controlling the critical layer features nor do they alter the jump properties to be found. Again, without loss of generality it is assumed that the mean flow correction u_{1s} vanishes at $y = y_c$, although a nonzero value of $u_{1s}(y_c)$ could be accommodated by adding a constant to $\tilde{Y}$. To match with the Rayleigh zone I⁺ solution we require

$$\tilde{V}_2 \sim (g^{(2)} + G^{(2)} \ln \tilde{Y})\tilde{Y}S \quad \text{as } \tilde{Y} \rightarrow \infty \quad (3.2a)$$

$$\tilde{U}_2 \sim b_2\tilde{Y}^2 + \tilde{\lambda}^+\tilde{Y} + \alpha^{-1}(g^{(1)} + G^{(1)} + G^{(1)} \ln|\tilde{Y}|)C \quad \text{as } \tilde{Y} \rightarrow \infty \quad (3.2b)$$

from (2.1a,b), (2.4a,c) where $\tilde{\lambda}^+ = u_{1s}'(y_c)$. Similar matching governs the higher order velocities and the pressure terms. Also

$$\tilde{U}_{2L} = -a^{-1} G^{(1)} C/6, \quad \tilde{V}_{2L} = -G^{(1)} \tilde{Y} S/6. \quad (3.3)$$

The aim of the critical layer analysis below, which stems from Haberman's original approach and its amendments, is to remove the logarithmic behaviour in (3.2a,b) and to provide the corresponding asymptotic conditions as $\tilde{Y} \rightarrow -\infty$ which allow the features of the lower Rayleigh zone I^- to be related to those of I^+ . To fulfill that aim requires examination of all the terms up to and including $\tilde{U}_5, \tilde{V}_5, \tilde{P}_5$ in (3.1a-c).

From the Navier-Stokes equations the balances controlling the successive contributions $\tilde{U}_n, \tilde{V}_n, \tilde{P}_n$ in (3.1a-c) are predominately inviscid:

$$b_1 \tilde{Y} \frac{\partial \tilde{U}_n}{\partial \tilde{x}} + b_1 \tilde{V}_n + g^{(0)} S \frac{\partial \tilde{U}_n}{\partial \tilde{Y}} = - \frac{\partial \tilde{P}_n}{\partial \tilde{x}} + F_n(\tilde{x}, \tilde{Y}), \quad (3.4a)$$

$$G_n(\tilde{x}, \tilde{Y}) = - \frac{\partial \tilde{P}_n}{\partial \tilde{Y}}, \quad (3.4b)$$

for $n \geq 2$, from the streamwise and lateral momentum equations respectively.

The first four of the forcing terms F_n, G_n in (3.4a,b) are given by

$$F_2 = 0, \quad F_3 = - \frac{\partial}{\partial \tilde{x}} \left(\frac{1}{2} \tilde{U}_2^2 \right) - \tilde{V}_2 \frac{\partial \tilde{U}_2}{\partial \tilde{Y}}, \quad F_4 = - \frac{\partial}{\partial \tilde{x}} (\tilde{U}_2 \tilde{U}_3) - \left(\tilde{V}_2 \frac{\partial \tilde{U}_3}{\partial \tilde{Y}} + \tilde{V}_3 \frac{\partial \tilde{U}_2}{\partial \tilde{Y}} \right), \quad (3.4c)$$

$$F_5 = \partial^2 \tilde{U}_2 / \partial \tilde{Y}^2 - \frac{\partial}{\partial \tilde{x}} (\tilde{U}_2 \tilde{U}_4 + \frac{1}{2} \tilde{U}_3^2) - \left(\tilde{V}_2 \frac{\partial \tilde{U}_4}{\partial \tilde{Y}} + \tilde{V}_3 \frac{\partial \tilde{U}_3}{\partial \tilde{Y}} + \tilde{V}_4 \frac{\partial \tilde{U}_2}{\partial \tilde{Y}} \right), \quad (3.4d)$$

$$G_2 = 0, \quad G_3 = b_1 \tilde{Y} g^{(0)} \alpha C, \quad G_4 = b_1 \tilde{Y} \frac{\partial \tilde{V}_2}{\partial \tilde{x}} + \tilde{U}_2 g^{(0)} \alpha C + g^{(0)} S \frac{\partial \tilde{V}_2}{\partial \tilde{Y}}, \quad (3.4e)$$

$$G_5 = b_1 \tilde{Y} \frac{\partial \tilde{V}_3}{\partial \tilde{x}} + \tilde{U}_2 \frac{\partial \tilde{V}_2}{\partial \tilde{x}} + \tilde{U}_3 g^{(0)} \alpha C + \tilde{V}_2 \frac{\partial \tilde{V}_2}{\partial \tilde{Y}} + g^{(0)} S \frac{\partial \tilde{V}_3}{\partial \tilde{Y}}, \quad (3.4f)$$

and are due to the nonlinear streamwise and lateral inertial responses except for the contribution $\partial^2 \tilde{U}_2 / \partial \tilde{Y}^2$ in Γ_5 which marks the first appearance of any viscous effect. Also, the continuity equation is satisfied by use of the stream function terms $\tilde{\Psi}_n(\tilde{x}, \tilde{Y})$, where $\tilde{U}_n = \partial \tilde{\Psi}_n / \partial \tilde{Y}$, $\tilde{V}_n = -\partial \tilde{\Psi}_n / \partial \tilde{x}$ for $n \geq 2$, while we define

$$\tilde{\Psi}_1 = b_1 \tilde{Y}^2 / 2 + \frac{g^{(0)}}{\alpha} C \quad (3.5)$$

from (3.1a,b).

It is convenient to renormalize the critical layer variables now by writing

$$\begin{aligned} \tilde{x} &= \alpha^{-1} x, \tilde{Y} = M_1^{1/2} Y, \tilde{\Psi}_1 = b_1 M_1 \psi_1, \tilde{\Psi}_2 = 2b_2 M_1^{3/2} \psi_2, \\ \tilde{\Psi}_3 &= \frac{g^{(0)2}}{b_1} \psi_3, \tilde{\Psi}_4 = 2b_2 M_1 \alpha^2 \psi_4, \tilde{\Psi}_5 = \frac{4b_2^2 g^{(0)3}}{\alpha b_1^4} \psi_5, \end{aligned} \quad (3.6)$$

where $M_1 = g^{(0)} / \alpha b_1$. Then the solution of the first order problem gives

$$\psi_{2YY} = K(\eta), \quad \text{where } \eta = \psi_1 = \frac{1}{2} Y^2 + C \quad (3.7a)$$

from (3.4c,e) and eliminating $\tilde{P}_2$ between (3.4a,b) with $n = 2$. The main streamlines $\eta = \psi_1 = \text{constant}$ give the Kelvin cat's eyes: when $\eta < 1$ the streamlines are closed [in Π^0] so that vorticity $K(\eta)$ it must have a uniform unknown value K_0 (the Prandtl-Batchelor theorem). When $\eta > 1$ [$|Y| > 2^{1/2} (1 - c)^{1/2}$] however, $K(\eta)$ remains an unknown function of η . To determine it we need to proceed to the fourth order contributions, since the second and third-order solutions for the vorticity can be worked out explicitly and are

$$\psi_{3YY} = C + M_2 K'(\eta) \psi_2 + K_3(\eta) \quad (3.7b)$$

$$\psi_{4YY} = -\psi_{2XX} + \frac{1}{2} M_2 K''(\eta) \psi_2^2 + K_3'(\eta) \psi_2 + K'(\eta) \psi_3 + K_4(\eta) \quad (3.7c)$$

from (3.4a,b) with $n = 3, 4$ and from (3.4c,d), and these impose no restrictions on $K(\eta)$. In (3.7b,c) the vorticity corrections $K_3(\eta)$, $K_4(\eta)$ are also unknown, and the parameter $M_2 \equiv (2b_2/\alpha b_1)^2$.

At the fourth order, then, we obtain the expression⁺ for the vorticity

$$\begin{aligned} \psi_{5YY} = & \frac{\partial}{\partial \eta} \int_0^\xi [M_3 \psi_{2YY} + Y^2 (S \psi_{4\eta} \psi_{2\eta\eta} + \psi_{2\xi} \psi_{4\eta\eta} - \psi_{2\eta} \psi_{4\eta\xi} - \psi_{4\eta} \psi_{2\eta\xi}) \\ & + \psi_{4\eta} \psi_{2X} + K(\eta) \psi_{4X} + M_2^{-1} (Y^2 \psi_{3\xi} \psi_{3\eta\eta} - Y^2 \psi_{3\eta} \psi_{3\eta\xi} + \psi_{3\eta} \psi_{3X})] d\xi \\ & + \int_0^\xi [\psi_{2\eta} (C \psi_{2\eta} - \psi_{2\xi X}) + \psi_{2\xi} \psi_{2\eta X} - M_2^{-1} \psi_{3\xi X} \psi_{3\eta} + K_5(\eta)] d\xi \end{aligned} \quad (3.8a)$$

where $\xi = X$, but the subscripts X, ξ denote differentiation with Y, η fixed respectively, $K_5(\eta)$ is an unknown function of η and

$$M_3 \equiv b_1^3 / 2b_2 g^{(0)^3} \quad (3.8b)$$

is an $O(1)$ parameter. It is shown below that the finite part of the $O(R^{-5/6})$ velocity jump, denoted $((U_5))_{-\infty}^\infty$, across the critical layer, governs the neutral criteria for α , c_0 . Since the integral of (3.8a) with respect to Y gives the contributions to this jump from outside the cat's eye, the jump is influenced as much by nonlinear inertial forces [and in particular by the non-zero lateral pressure gradients in (3.4b)] as by the main linear viscous force

+ FOOTNOTE: SEE NEXT PAGE.

Footnote for preceding page.

³ An expression alternative to (3.8a) is

$$\begin{aligned} \psi_{5\gamma\gamma} = & -\frac{\psi_{3\gamma\gamma}}{M_2} + K'(\eta) \psi_4 + K''(\eta) \psi_2 \psi_3 + \frac{K_3'(\eta)}{M_2} \psi_3 + M_2 \frac{K'''(\eta)}{6} \psi_2^3 \\ & + \frac{K_3''(\eta)}{2} \psi_2^2 + K_4'(\eta) \psi_2 + M_3 \int_0^\xi \frac{\partial}{\partial \eta} (\psi_{2\gamma\gamma}) d\xi \end{aligned}$$

$\propto M_2$ in (3.8a). The former forces involve $K^2(\eta)$ from (3.7a-c), while the latter force involves only $K(\eta)$. By contrast, only the latter force plays a significant role when the disturbance considered is smaller as in the earlier work of Haberman (1972, 1976), Brown & Stewartson (1978) and Smith & Bodonyi (1982a). Again, the extra inertial forces are neglected in the Benney & Bergeron (1969) analysis. However, the condition which effectively fixes the dominant vorticity $K(\eta)$ in $\eta > 1$ is that ψ_{5YY} must be periodic of period 2π in X . Whereas ψ_{2YY} , ψ_{3YY} , ψ_{4YY} in (3.7a-c) are automatically periodic, ψ_{5YY} is not because of the first term on the right hand side of (3.8a) at least. This term is due to the main small viscous effect present, namely $\tilde{u}_{2YY}$ in (3.4d). It is a significant property of the strongly nonlinear critical layer that the periodicity requirement

$$\psi_{5YY}(2\pi) - \psi_{5YY}(0) = 0 \quad (3.9a)$$

now gives, from (3.8a), (3.7a), the condition

$$\frac{d}{d\eta} \left\{ K'(\eta) \int_0^{2\pi} (\eta - c)^{1/2} d\xi \right\} = 0 \quad (3.9b)$$

for any M_2, M_3 . Only the linear viscous term $\propto M_3$ in the integrands of (3.8a) contributes nontrivially to (3.9a) because all the other nonlinear terms, being inertial responses, are odd about $X = \pi$. Thus, for example, ψ_{2YY} is even about $X = \pi$ from (3.7a); integration and the asymptotic conditions give ψ_{2Y} , ψ_2 being likewise even; so in (3.8a) the terms $\psi_{2\eta}(C\psi_{2\eta} - \psi_{2\xi}^2) + \psi_{2\xi}^2$ are even, their X -derivatives are odd and hence the integral with respect to

from 0 to 2π is identically zero. Similar reasoning applies to the other terms, given that ψ_{3yy}, ψ_{4yy} in (3.7b,c) are both even about $X = \pi$. The total inertial contribution to (3.9a) turns out to be zero. So the result (3.9b), found in the earlier critical layer studies effectively for M_3 small, continues to hold for neutral modes even when M_3 is $O(1)$, despite the new balances controlling the critical layer. On the other hand, if growing modes were considered then the new balances for M_3 of $O(1)$ would influence the growth. From (3.9b) and (3.2b) it follows that

$$K'(\eta) = \pm 2^{1/2} \pi / \int_0^{2\pi} (\eta - C)^{1/2} dx \quad (3.10)$$

above and below the cat's eye $y \lesseqgtr 2^{1/2} (1 - C)^{1/2}$, respectively. Then continuity of the vorticity at the edges of the cat's eye, $\eta = 1\pm$, gives

$$K(\eta) = N_0 + \int_1^{\eta} K'(\bar{\eta}) d\bar{\eta} \quad (3.11)$$

for $\eta > 1$. The velocity ψ_{2y} and streamfunction ψ_2 are also continuous (the latter being constant) at $\eta = 1$: see (3.13a,b) below. From letting $y \rightarrow \pm\infty$ in (3.11) and examining the finite parts it can be seen that

$$N_0 = H_+ + H_- \quad \text{and}$$

$$2(H^+ - H^-) = 2 \int_1^{\infty} \{ |K'(\eta)| - (2\eta)^{-1/2} \} d\eta - s^{1/2} \left[\frac{1}{2} C^{(1)} \right] \quad (3.12)$$

where $\frac{1}{4} C^{(1)} = -1.379$. Here (3.12) fixes the vorticity jump $2(\tilde{\lambda}^+ - \tilde{\lambda}^-)$, where $\tilde{\lambda}^{\pm} = 4b_2 M_1^{1/2} H^{\pm}$, and gives the jump conditions $u_{1s}'(y_c \pm) = \tilde{\lambda}^{\pm}$ on the mean flow correction of (2.1a). We have still to consider the more important

jump $((\tilde{U}_5))_{-\infty}$. However, to fix the neutral stability conditions we require only the $\sin(\alpha \bar{x})$ component of $((\tilde{U}_5))_{-\infty}$, or the phase shift (see §5 below). From (3.7a-c), no phase shift is induced in the three highest order terms $\tilde{U}_2, \tilde{U}_3, \tilde{U}_4$ since they are solely inertial influences. In fact the behaviour (3.2b) of $\tilde{U}_2$ is reproduced as $\tilde{Y} \rightarrow -\infty$ (with $\tilde{\lambda}_2^-$ replacing $\tilde{\lambda}_2^+$), so that $((U_2))_{-\infty}^{\infty} = 0$, in keeping with the exterior jump condition anticipated in (2.4d). The dominant phase shift is caused by the viscous influence on ψ_{5YY} in (3.8a) and is obtained on multiplying (3.8a) by $\sin(X)$, integrating with respect to Y and then integrating with respect to X from 0 to 2π (Haberman 1972, Smith & Bodonyi (1982a)). In this process the extra inertial effects cannot contribute again because of their evenness about $X = \pi$. So the phase shifts across the inviscid parts $\eta > 1, \eta < 1$ of the critical layer are as in Smith & Bodonyi (1982a).

Finally the phase shifts across the thinner viscous layers $III^\pm$ at the edges of the cat's eye can be established by using the method of the last-named paper. The viscous layers smooth out the discontinuities in the derivatives of the vorticity above at $\eta = 1 \pm$. Within III^+ , $y = y_c + R^{-1/6} \sqrt{2M_1} \sqrt{1 - \zeta} + R^{-5/12} z$ with z of $O(1)$ and the underlying form of the solution is

$$\bar{\psi} = \text{constant} + \sum_{n=2}^{\infty} R^{-n/12} \psi^{(n)}(\bar{x}, z); \quad (3.13a)$$

although again extra logarithmic terms are strictly necessary they do not affect the ultimate results and need not be considered here. In (3.13a)

$$\psi^{(2)} = c_0 (2M_1)^{1/2} (1-C)^{1/2}, \quad \psi^{(3)} = 0, \quad \psi^{(4)} = 1, \quad \psi^{(5)} = c_0, \quad \psi^{(6)} = 0,$$

$$\psi^{(7)} = b_1 (2M_1)^{1/2} (1-C)^{1/2} z, \quad (3.13)$$

$$\psi^{(n)} = d^{(n)} z^2 + e^{(n)}(x)z + f^{(n)}(x) \quad \text{for } 8 \leq n \leq 14$$

where $d^{(n)}$ are constants $d^{(8)} = d^{(9)} = 0$, $d^{(10)} = \frac{b_1}{2}$ and

$e^{(n)}, f^{(n)}$ ($e^{(8)} \equiv C$) are functions of $\bar{x}$ given by the properties in II⁺.

The first nontrivial contribution is $\psi^{(15)}$ which satisfies the vorticity equation

$$\begin{aligned} (2/M_1 b_1^2)^{1/2} \psi_{zzz}^{(15)} &= 2(1-C)^{1/2} \psi_{z\bar{x}}^{(15)} + \frac{\alpha S}{(1-C)^{1/2}} (\psi_z^{(15)} - z \psi_{zz}^{(15)}) \\ &\quad + A^{(15)}(\bar{x}) + B^{(15)}(\bar{x})z \end{aligned} \quad (3.14a)$$

from substitution into the Navier-Stokes equations. Here $A^{(15)}, B^{(15)}$ are prescribed functions of $\bar{x}$, while the boundary conditions are

$$\psi_{zzz}^{(15)}(\infty) = 2^{-1/2} \pi (1-C)^{1/2} b_2, \quad \psi_{zzz}^{(15)}(-\infty) = 0 \quad (3.14b)$$

because of the discontinuity in the vorticity derivative outside III⁺. Hence

if

$$\psi_z^{(15)} \sim \frac{1}{2} \bar{x}^2 + \bar{x}_1 \bar{x} + \bar{x}_0 \quad \text{as } z \rightarrow \infty \quad (3.14c)$$

then $\mathcal{L}_2^+ = 2^{-1/2} \pi(1 - C)^{1/2} b_2$, $\mathcal{L}_2^- = 0$ and (3.14c) substituted into (3.14a)

gives the scaled velocity jump condition

$$\mathcal{L}_0^+ - \mathcal{L}_0^- = -2\pi b_1^{-1} b_2 M_1^{-1/2} \alpha^{-1} \cot\left(\frac{1}{2} \alpha \bar{x}\right) \quad [0 < \alpha \bar{x} < 2\pi] \quad (3.14d)$$

as in Smith & Bodonyi (1982a), again despite the extra inertial forces present here. The irregularity as $\alpha \bar{x} \rightarrow 0+, 2\pi-$ is smoothed out within a length $O(R^{-1/8})$ in $\bar{x}$ by means of Brown & Stewartson's (1978, 1980) solution which provides a higher order correction to (3.14d). The viscous layer III^- near the lower edge of the cat's eye produces the same jump and both jumps are of the same order as the velocity jumps in $\tilde{U}_5$ produced in $II^\pm$ outside. In particular the total phase shift remains as in the Smith & Bodonyi case. An alternative method of derivation of the phase shift is given by Smith and Bodonyi (1982b).

In summary, the total velocity jumps across the strongly nonlinear critical layer are

$$\begin{aligned} ((U_n))_{y_c}^+ &= ((\tilde{U}_n))_{-\infty}^\infty \quad (n \geq 2), \quad \text{where} \quad ((\tilde{U}_2))_{-\infty}^\infty = 0, \\ ((\tilde{U}_n))_{-\infty}^\infty &= \sum_{m=0}^{\infty} \left[F_m^{(n)} \sin(m\alpha \bar{x}) + \tilde{F}_m^{(n)} \cos(m\alpha \bar{x}) \right] \quad (n \geq 3) \end{aligned} \quad (3.15a)$$

for $0 < \alpha \bar{x} < 2\pi$, and $F_m^{(n)}, \tilde{F}_m^{(n)}$ are constants with

$$F_1^{(3)} = F_1^{(4)} = 0, \quad F_1^{(5)} = \frac{2C^{(1)} b_2}{b_1 M_1^{1/2} \alpha} \quad (3.15b)$$

The phase shift is of order $R^{-1/2}$, since it first appears only through $F_1^{(5)}$ in $((U_5))_{-\infty}^\infty$, and it enables the amplitude-independence of the neutral modes to be found once the wall layer characteristics are known.

§4. The Wall Layers

We devote a small separate section to the flow properties near the wall to emphasize the presence of unsteady reversed flow there. Near the wall $y = 0$ the first distinct layer IV_1 encountered is nonlinear and has width $O(R^{-1/3})$ since then the basic flow U_0 and the fundamental u_2 in (2.1a) become comparable. So, in IV_1 , $y = R^{-1/3} z_1$ say and

$$(\bar{u}, \bar{v}, \bar{p}) = (R^{-1/3} \hat{u}, R^{-4/3} \hat{v}, R^{-1/3} \hat{p}) + \dots \quad (4.1)$$

if $u_{1s}(0) = 0$ and provided that $|u_{1s}'(0)| < \infty$. The Navier-Stokes equations reduce to the unsteady inviscid equations

$$\frac{\partial \hat{u}}{\partial \bar{x}} + \frac{\partial \hat{v}}{\partial z_1} = 0, \quad -c_0 \frac{\partial \hat{u}}{\partial \bar{x}} = -\frac{\partial \hat{p}}{\partial \bar{x}}, \quad 0 = -\frac{\partial \hat{p}}{\partial z_1} \quad (4.1a)$$

however, and since $\hat{p} = p_2(\bar{x}, 0+)$ and $\hat{u} \sim \lambda z_1 + u_2(\bar{x}, 0+)$ as $z_1 \rightarrow \infty$ to match with §2 we have the additive solution

$$\hat{u} = \lambda z_1 + u_2(\bar{x}, 0+), \quad \hat{v} = -z_1 \frac{\partial u_2}{\partial \bar{x}}(\bar{x}, 0+). \quad (4.2)$$

So the nonlinearity is simply an additive effect and neither the fundamental nor the basic flow is distorted in IV_1 .

The second wall layer IV_2 lying between IV_1 and the wall then has thickness $O(R^{-1/2})$ and is controlled by an unsteady-pressure-viscous force balance.

Here

$$(\bar{u}, \bar{v}, \bar{p}) = (R^{-1/3} \hat{u}, R^{-5/6} \hat{v}, R^{-1/3} \hat{p}_2(x, 0+)) + \dots \quad (4.3a)$$

with $y = R^{-1/2} z_2$ and the controlling equations and boundary conditions are

$$\frac{\partial \hat{U}}{\partial \bar{x}} + \frac{\partial \hat{V}}{\partial z_2} = 0, \quad -c_0 \frac{\partial \hat{U}}{\partial \bar{x}} = -\frac{dp_2}{d\bar{x}}(\bar{x}, 0+) + \frac{\partial^2 \hat{U}}{\partial z_2^2}, \quad (4.4a)$$

$$\hat{U} \rightarrow u_2(\bar{x}, 0+) \quad \text{as} \quad z_2 \rightarrow \infty, \quad (4.4b)$$

$$\hat{U} = \hat{V} = 0 \quad \text{at} \quad z_2 = 0, \quad (4.4c)$$

i.e., the linear problem of Smith & Bodonyi (1982a) is retrieved. The solution of (4.4a-c) is

$$\begin{aligned} \hat{U} &= \frac{\Lambda}{c_0} \left[C - e^{-m_1 z_2} \cos(\alpha \bar{x} - m_2 z_2) \right], \\ \hat{V} &= \frac{\alpha^{1/2} \Lambda}{c_0^{3/2}} \left[\alpha^{1/2} c_0^{1/2} z_2 S - \sin(\alpha \bar{x} + \frac{\pi}{4}) + e^{-m_1 z_2} \sin(\alpha \bar{x} + \frac{\pi}{4} - m_2 z_2) \right], \end{aligned} \quad (4.5a,b,c)$$

therefore, with $m_1 = -m_2 = (\alpha c_0/2)^{1/2}$, and we have the displacement condition

$$\bar{v}(\bar{x}, 0+) \sim -R^{-5/6} \left(\frac{\alpha^{1/2} \Lambda}{c_0^{3/2}} \right) \sin(\alpha \bar{x} + \frac{\pi}{4}) \quad (4.6)$$

i.e., a nonzero wall value required for v_5 in (2.1a), to match between zones II and IV. Once again, it is noted that the results of the earlier study continue to hold despite the extra effects of nonlinearity introduced in layer IV₁. The major difference from the earlier work is that near the wall the flow in layer IV₂ is mainly oscillatory in time, with the fundamental disturbance being greater than the basic flow. So in particular the skin friction at the wall is relatively large, and purely oscillatory to leading order,

$$(\partial \bar{u} / \partial y)_{y=0} = R^{1/6} \Lambda (\frac{\alpha}{c_0})^{1/2} \sin (\alpha \bar{x} + \frac{\pi}{4}) + O(1) \quad (4.7)$$

and flow reversal is present (as in inviscid water waves in a sufficiently weak mean flow).

§5. The Amplitude-Dependence of the Neutral Conditions

The conditions of a small velocity jump in (3.15a,b) and the small displacement in (4.6) fix the amplitude-dependence as follows. As in Smith & Bodonyi (1982a) what matters most is the small phase shift giving rise to a $\cos(\alpha\bar{x})$ component in the Fourier series for v_5 in the Rayleigh zones $I^\pm$. The $\sin(\alpha\bar{x})$ component in v_5 merely introduces an $O(R^{-1/2})$ correction in the neutral wavespeed, while any $\cos(n\alpha\bar{x})$, $\sin(n\alpha\bar{x})$ components for $n \geq 2$ can be determined explicitly as in the Smith & Bodonyi case. Therefore we define

$$v_5^*(y) = \frac{\alpha}{\pi} \int_0^{2\pi/\alpha} v_5(\bar{x}, y) \cos(\alpha\bar{x}) d\bar{x}, \quad (5.1)$$

so that in $I^\pm$ the Fourier component v_5^* satisfies the same Rayleigh equation as the fundamental $g(y)$ in §2,

$$(U_0 - c_0)(v_5^{*''} - \alpha^2 v_5^*) = U_0'' v_5^* \quad (5.2a)$$

from (2.1a) and the Navier-Stokes equations, while the boundary and jump conditions on v_5^* are

$$v_5^*(\infty) = 0, \quad ((v_5^*))_{y_c^-}^{y_c^+} = -2C^{(1)} b_1^{-1} b_2 M_1^{-1/2}, \quad (5.2b, c, d)$$

$$v_5^*(0+) = - \left(\frac{\alpha}{2c_0^3} \right)^{1/2} \Lambda.$$

Here (5.2b) is necessary for the boundedness condition in the farfield, the jump condition (5.2c) follows from the critical layer results (3.15a,b) and (5.2d) is from the wall displacement condition (4.6). The problem (5.2a-d)

fixes the amplitude-dependence of α , c_0 since (5.2a-d) is soluble if and only if a certain compatibility relation is satisfied. This is derived by multiplying (5.2a) by $g(y)/(U_0 - c_0)$ and integrating with respect to y from 0 to ∞ , using the equation (2.3a) for $g(y)$. The result

$$((gv_5^{*'} - g'v_5^*))_{y_c^+}^\infty + ((gv_5^{*'} - g'v_5^*))_0^{y_c^-} = 0 \quad (5.3)$$

for the wronskian of g , v_5^* is obtained. Then the boundary and jump conditions (2.3b,c) (2.4a-d) on g and (5.2c-d) on v_5^* imply the amplitude-dependence equation

$$\alpha A^2 / (2c_0^5)^{1/2} = 2C^{(1)} b_2 (g^{(0)} / b_1)^{1/2} \quad (5.4)$$

from (5.3), since $g'(0+) = \alpha p(0+)/c_0$ from (2.2a,b).

The determination of the amplitude dependence of α , c_0 requires just two successive steps, then: (i) solve (2.3a-c), (2.4a-d) for the fundamental $g(y)$. This fixes α in terms of c_0 , say, and $g(y_c) = g^{(0)}$ in terms of the wall pressure $p(0+) = A$ but A remains unknown because the problem (2.3a-c), (2.4a-d) is homogeneous. (ii) Use (5.4) to determine A in terms of c_0 and hence by inversion determine c_0 , α in terms of A . The step (i) here was performed by Benney & Bergeron (1969), without an explicit statement of the disturbance size required to produce the Rayleigh structure of §2. The present analysis shows that the required size is $O(R^{-1/3})$ and that the step (ii) gives the dependence on size explicitly. It is noteworthy that this amplitude dependence, (5.4), comes directly from the small viscous effects

operating in the strongly nonlinear critical layer of §3 and in the wall layers of §4. Although the flow properties almost throughout are predominately inviscid, due to the largeness of the disturbance size, the interplay between the two $O(R^{-1/2})$ phase shifts produced by viscous action within the critical layer and the wall layers nevertheless controls the stability properties.

Calculations for steps (i), (ii) were performed by use of a 4th order Runge-Kutta shooting procedure similar to Benney & Bergeron's, supplemented by a series expansion near $y = y_c$. Our results for step (i) agree at least graphically with theirs and combined with step (ii) the results give the amplitude dependence of the neutral stability criteria.

In Fig. 2 we present the results of our calculations for the neutral mode criteria for a number of basic profiles $U_0(y)$. Also, in Fig. 3 the corresponding amplitude dependence of the neutral stability criteria are given. As shown both the neutral wavespeed c_0 and wave number α increase monotonically with the scaled disturbance amplitude $A(=p(0+))$. These computed curves are consistent with the trends of the asymptotic analyses for small and large α as well as for $c_0 \approx 1$ to be discussed below.

When the amplitude A is small for the accelerating boundary layer $\alpha \ll 1$ and the solution of (2.3a-c), (2.4a-d) for g develops a three-tiered structure in which y takes on the orders α^{-1} , 1 , α . To leading order in the outermost tier the solution takes on the form

$$g = p = \alpha^{4/3} B \exp(-\alpha y)$$

while in the second and third inner tiers it is found that

$$g = \alpha^{4/3} B U_0(y) \quad \text{and} \quad p = \alpha^{4/3} B$$

where the unknown constant B is $O(1)$ and $c_0 \sim \alpha c_{01}$ with c_{01} of $O(1)$.

Then (2.2a-c) give $\lambda c_{01} = 1$. Hence it follows that $A \sim \alpha^{4/3} B$, $g^{(0)} \sim \lambda^{-1} \alpha^{7/3} B$, while $b_2 \rightarrow \lambda_2$, $b_1 \rightarrow \lambda$ since the critical layer retreats towards the wall.

So the amplitude equation (5.4) is in balance at order $\alpha^{7/6}$, with

$$B = 2^{1/3} (2c^{(1)} \lambda_2)^{2/3} \lambda^{-7/3}, \quad (5.5a)$$

$$\text{so that } \alpha \sim B^{-3/4} A^{3/4}, \quad c_0 \sim \lambda^{-1} B^{-3/4} A^{3/4} \quad \text{for } A \ll 1. \quad (5.5b)$$

The results (5.5a,b) for a disturbance small on the $O(R^{-1/3})$ scale agree with Smith & Bodonyi's (1982a) (5.1), (5.2) for a disturbance large on their $O(R^{-7/18})$ scale since their relative disturbance size Λ_0 is equal to $R^{1/18} \Lambda/2\alpha$. Likewise the smoothness of g to leading order and the enlarged $O(\alpha^{-1})$ extent of the flow structure are in line with their scalings for slightly smaller disturbances. Similar analyses for small A apply to the Blasius boundary layer and to plane Poiseuille flow ($U_0 = 1 - y^2$: $|y| \leq 1$) for example, whereas for the decelerating boundary layer and other flow profiles containing inflexion points $\alpha \rightarrow O(1)$ generally as $A \rightarrow 0$.

When A is large the solution features depend on the form of $U_0(y)$ near the external stream, as in Benney & Bergeron (1969). For most boundary layer profiles $U_0(y) \sim 1 - q_1 \exp(-q_2 y)$ as $y \rightarrow \infty$, with $q_1, q_2 > 0$ nonzero constants, and then the wavenumber α approaches a limiting $O(1)$ value as the wavespeed c_0 approaches the freestream speed 1 in the manner described by Benney & Bergeron. Below we show that this limiting process $c_0 \rightarrow 1$ is associated with A being large. If $c_0 \sim 1 - \bar{c}q_1$, where $\bar{c} \ll 1$, then

$\alpha \sim \alpha_2 = o(1)$ and a two-tiered structure comes into play. The first, outer, tier has $q_2 y = -\ln \bar{c} + \bar{y}$ where $\bar{y}$ is $O(1)$, so that $U_0 \sim 1 - q_1 \bar{c} \exp(-\bar{y})$. Supposing that $g \sim \Delta \bar{g}$, with $\Delta \gg 1$ to be found and $\bar{g}$ of $O(1)$, we have the equation

$$\bar{g}'' = \left(\frac{\alpha^2}{1 - e^{-\bar{y}}} - \frac{e^{-\bar{y}}}{1 - e^{-\bar{y}}} \right) \bar{g} \quad (5.6a)$$

for $\bar{g}(\bar{y})$, from (2.3a). The appropriate boundary conditions are $\bar{g}(\pm\infty) = 0$ and the jump condition at the critical level $\bar{y} = 0^\pm$ requires zero phase shift, from (2.4a-d). The calculations by Benney and Bergeron gave the eigenvalue $\bar{\alpha} \doteq 0.75$, but the analysis in the Appendix below shows that in fact $\bar{\alpha} = 3/4$. So $g^{(0)} \sim \Delta \gamma$, say, where $\gamma \equiv \bar{g}(0)$ is $O(1)$, while

$$g \sim B_1 \gamma e^{(1+\bar{\alpha}^2)^{1/2} \bar{y}} \quad \text{as} \quad \bar{y} \rightarrow -\infty \quad (5.6b)$$

where the $O(1)$ constant B_1 is known in principle from Benney & Bergeron's calculations of (5.6a) or from the analysis in the Appendix. Next, the second inner tier has y of $O(1)$ and the disturbance amplitude is smaller here,

$$g = \Delta \bar{c} (1+\bar{\alpha}^2)^{1/2} \hat{g} + \dots \quad (5.7a)$$

where $\hat{g}(y)$ is $O(1)$, from (5.6b). Hence (2.3a) reduces to the equation

$$(U_0 - 1)(\hat{g}'' - q_2^2 \bar{\alpha}^2 \hat{g}) = U_0'' \hat{g} \quad (5.7b)$$

for $\hat{g}(y)$. The appropriate boundary conditions are now

$$\hat{g}(0+) = 0, \quad \hat{g} \sim B_1 \gamma e^{(1+\bar{\alpha}^2)^{1/2} q_2 y} \quad \text{as} \quad y \rightarrow \infty, \quad (5.7c)$$

from (2.3c) and from the exponential growth required to effect the match with (5.6b) above. The solution for $\hat{g}$ is proportional to $B_1 \gamma$; hence (5.7a-c) give the expression

$$\Lambda = \Delta \bar{c} (1+\bar{\alpha}^2)^{1/2} B_1 \gamma B_2 + \dots \quad (5.8)$$

for the amplitude $\Lambda \equiv g'(0+)c_0/\alpha$, where B_2 is $O(1)$. Also, $b_1 \sim q_1 q_2 \bar{c}$, $b_2 \sim -q_1 q_2^2 \bar{c}$, from the form of $U_0(y)$ at $y = y_c \sim -q_2^{-1} \ln \bar{c}$. So to leading order the amplitude equation (5.4) yields

$$\Delta = \bar{c}^{-1/3} (4\sqrt{1+\bar{\alpha}^2}-1), \quad \gamma = 2q_2^{1/3} q_1^{1/3} \left(\frac{-c(1)}{\bar{\alpha} B_1^2 B_2^2} \right)^{2/3} \quad (5.9)$$

and then from (5.8) the asymptote

$$c_0 = 1 - q_1 \bar{c} \sim 1 - q_1 \left(\frac{\Lambda}{B_1 \gamma B_2} \right)^{3/2} \left[\frac{-3}{(-1+\sqrt{1+\bar{\alpha}^2})} \right], \quad \text{for } \Lambda \gg 1 \quad (5.10)$$

gives the amplitude dependency of the wavespeed for the larger disturbances.

Furthermore since $\bar{\alpha} = 3/4$ from the Appendix we have immediately

$$c_0 \sim 1 - q_1 (B_1 \gamma B_2)^{1/2} \cdot \Lambda^{-1/2} \quad \text{for } \Lambda \gg 1. \quad (5.11)$$

We note therefore that the approach of the wavespeed to its freestream value of 1 is very rapid, being proportional to $\Lambda^{-1/2}$ for Λ large. A similar analysis applies to (e.g.) Blasius flow and plane Poiseuille flow for both

of which $c_0 \rightarrow 1$ again but $\alpha \rightarrow \infty$ as $A \rightarrow \infty$! see also fig. 3

The result (5.10) or (5.11) is significant because it suggests that if the disturbance size δ becomes greater than $O(R^{-1/3})$, i.e. $A \gg 1$, then the neutral disturbance must penetrate outside the boundary layer into the free stream. Since that penetration includes the strongly nonlinear critical layers situated near $y = y_c$ [$\sim (1/2/q_2) \ln A$] $\gg 1$, we have cat's eyes of uniform vorticity being forced to come out of the boundary layer into the external stream. Moreover, the eyes become thicker (in y) and shorter (in x) as this happens, in view of the transformations (3.6) and the behaviour of α in Fig. 2. It would be interesting if these developments, and in particular the suggestion (Fig. 4) of a penetration... of vorticity into the external stream, as the small disturbance increases in size, were reproduced analytically for a single growing mode or a group as well as for the present neutral states. We believe/conjecture that the balances shown above for neutrally stable modes can form the basis of such an analysis. Also, it would be interesting to know what happens next to the neutral or growing states as the critical layer emerges from the boundary layer and eventually additional dynamical influences come into play. In this connection we note first that the (amplitude)³ effect associated with weakly nonlinear stability theory (e.g., Stuart 1960, Watson 1960, Stewartson & Stuart 1971) has exerted only a minor effect on the flow properties so far but clearly this effect is growing as δ increases from the $O(R^{-1/3})$ stage. Second, the amplitude dependency retains the balance between the small viscous effects, within the critical layer and the wall layer, even when the critical layer starts to penetrate into the external stream. So the behaviour and positioning of the critical layer far from the wall remain crucially influenced by the wall layer properties throughout.

Related experiments on boundary layer stability have been instigated by Dr. M. Gaster at the National Maritime Institute, Teddington.

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APPENDIX

To show that $\bar{\alpha} = 3/4$ is the eigenvalue associated with (5.6a).

A nontrivial solution of (5.6a) is required satisfying

$$\bar{g}(\pm\infty) = 0 \quad (A1)$$

and with zero plane shift across $\bar{y} = 0$. Without loss of generality we normalize $\bar{g}$ so that $\bar{g}(0) = 1$. Then (5.6a) yields the property that $\bar{g} \sim 1 - \bar{y} \ln|\bar{y}| + 0(\bar{y})$ as $\bar{y} \rightarrow 0\pm$. Hence the condition of zero phase shift is that

$$G(0) = 1 \text{ and } G' \text{ is continuous across } \bar{y} = 0, \quad (A2)$$

$$\text{where } G \equiv \bar{g} + \bar{y} \ln|\bar{y}| \quad (A3)$$

for all $\bar{y}$. To solve (5.6a) with (A1) - (A3) for $\bar{\alpha}$, $\bar{g}$ we consider $\bar{y} \gtrless 0$ in turn below.

For $\bar{y} > 0$ a series solution in powers of $s \equiv \exp(-\bar{y})$ is found to hold, in the form

$$s^{-\bar{\alpha}} \bar{g}(s)/\mu_+ = \frac{3s}{2} - \frac{5}{2} - \frac{1}{4} (1-s)(1+3s) \sum_{m=0}^{\infty} \frac{s^m}{(m+3/2)} \quad [0 \leq s < 1] \quad (A4)$$

where we have guessed that $\bar{\alpha} = 3/4$. Here, first, $\mu_+ = -1$

since $\bar{g} = 1$ at $\bar{y} = 0$ ($s \rightarrow 1^-$). Second,

$$\begin{aligned} dG/d\bar{y} = & -\frac{1}{2} s^{7/4} (1-3s) \left[\sum_{m=1}^{\infty} s^m \left(\frac{1}{m+3/2} - \frac{1}{m} \right) + \frac{2}{3} - \ln(1-s) \right] + \frac{3}{2} s^{7/4} \\ & - \frac{3}{4} \bar{g}(s) + \ln \bar{y} + 1 - \frac{1}{4} s^{7/4} (1-s)(1+3s) \left[\frac{1}{(1-s)} + \frac{5}{2} \ln(1-s) \right. \\ & \left. - \frac{3}{5} + \frac{15}{4} \sum_{m=2}^{\infty} \frac{s^{m-1}}{(m+3/2)(m-1)} \right] \end{aligned} \quad (A5)$$

for $\bar{y} > 0$, $0 < s < 1$, from (A3), (A4). Therefore letting $s \rightarrow 1$ - we have

$$(dG/d\bar{y})_{\bar{y}=0+} = \frac{5}{4} - \frac{5}{2} \sum_{m=1}^{\infty} \frac{1}{(m+3/2)m} + \frac{2}{3} \quad (A6)$$

from (A5) and since $\ln(1-s) \sim \ln \bar{y} + 0(\bar{y})$ as $\bar{y} \rightarrow 0+$.

For $\bar{y} < 0$ on the other hand the appropriate series solution is in powers of $\exp(\bar{y}) \equiv \sigma$, namely

$$\sigma^{-\kappa} \bar{g}(\sigma)/\mu_- = \frac{1}{6}\sigma + \frac{5}{6} - \frac{1}{4}(1-\sigma)(\sigma+3) \sum_{m=0}^{\infty} \frac{\sigma^m}{(m+\frac{5}{2})} \quad (A7)$$

where $\kappa = (1+\bar{\alpha}^{-2})^{\frac{1}{2}}$, $0 \leq \sigma < 1$, $\bar{y} < 0$ and $\bar{\alpha} = 3/4$ again. Here $\mu_- = 1$ since $\bar{g} = 1$ at $\bar{y} = 0$ ($\sigma \rightarrow 1-$). Then (A7) with (A5) yields the result

$$\begin{aligned} dG/d\bar{y} = & \frac{1}{2} \sigma^{9/4} (1+\sigma) \left[\sum_{m=1}^{\infty} \sigma^m \left(\frac{1}{m+\frac{5}{2}} - \frac{1}{m} \right) + \frac{2}{5} - \ln(1-\sigma) \right] + \frac{1}{6} \sigma^{9/4} \\ & + \frac{5}{4} \bar{g}(\sigma) + \ln|\bar{y}| + 1 - \frac{1}{4} \sigma^{9/4} (1-\sigma)(\sigma+3) \left[\frac{1}{1-\sigma} + \frac{5}{2} \ln(1-\sigma) \right. \\ & \left. - \frac{5}{7} + \frac{35}{4} \sum_{m=2}^{\infty} \frac{\sigma^{m-1}}{(m+\frac{5}{2})(m-1)} \right] \end{aligned} \quad (A8)$$

for $\bar{y} < 0$, $0 < \sigma < 1$. Hence from letting $\sigma \rightarrow 1-$ in (A8) we have

$$(dG/d\bar{y})_{\bar{y}=0-} = -\frac{5}{2} \sum_{m=1}^{\infty} \frac{1}{(m+\frac{5}{2})m} + \frac{2}{5} + \frac{1}{6} + \frac{5}{4} \quad (A9)$$

since $\ln(|\bar{y}|/(1-\sigma)) \rightarrow 0$ as $\sigma \rightarrow 1-$.

The values (A6), (A9) for $dG/d\bar{y}$ at $\bar{y} = 0\pm$ are exactly equal, from partial fractional rearrangements of the two convergent series involved there. Hence the conditions in (A2) are satisfied

and the eigenvalue $\bar{\alpha} = 3/4$ is verified. The above argument can be generalised to prove the uniqueness or otherwise of this eigenvalue. A referee has taken this up, kindly noting that (A4), (A7) can be replaced by

$$\bar{g} = e^{-\bar{\alpha}\bar{y}} \frac{F(\kappa+\bar{\alpha}, \bar{\alpha}-\kappa; 1+2\bar{\alpha}; e^{-\bar{y}})}{F(\kappa+\bar{\alpha}, \bar{\alpha}-\kappa; 1+2\bar{\alpha}; 1)}, \quad \bar{g} = e^{\kappa\bar{y}} \frac{F(\kappa+\bar{\alpha}, \kappa-\bar{\alpha}; 1+2\kappa; e^{\bar{y}})}{F(\kappa+\bar{\alpha}, \kappa-\bar{\alpha}; 1+2\kappa; 1)}$$

respectively, for general $\bar{\alpha}$, where F is the hypergeometric function (e.g. Abramowitz & Stegun 1972). The jump condition (A2) then requires that

$$\psi(1-\kappa+\bar{\alpha}) - \psi(1+\kappa-\bar{\alpha}) = -\frac{1}{(\kappa-\bar{\alpha})},$$

giving $\cot(\pi(\kappa-\bar{\alpha})) = 0$ from properties of the ψ -function (Abramowitz & Stegun). Therefore the solution $\bar{\alpha} = 3/4$ is unique.

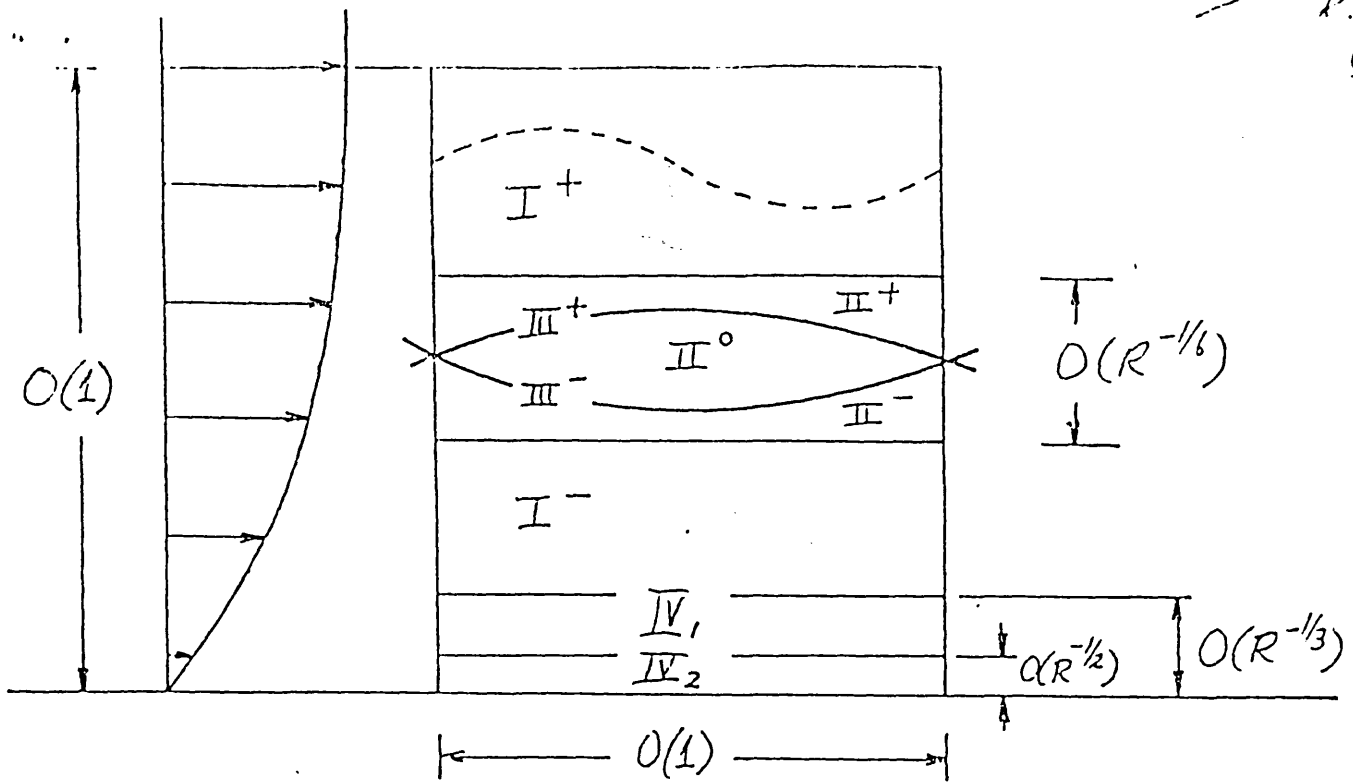


Fig. 1a

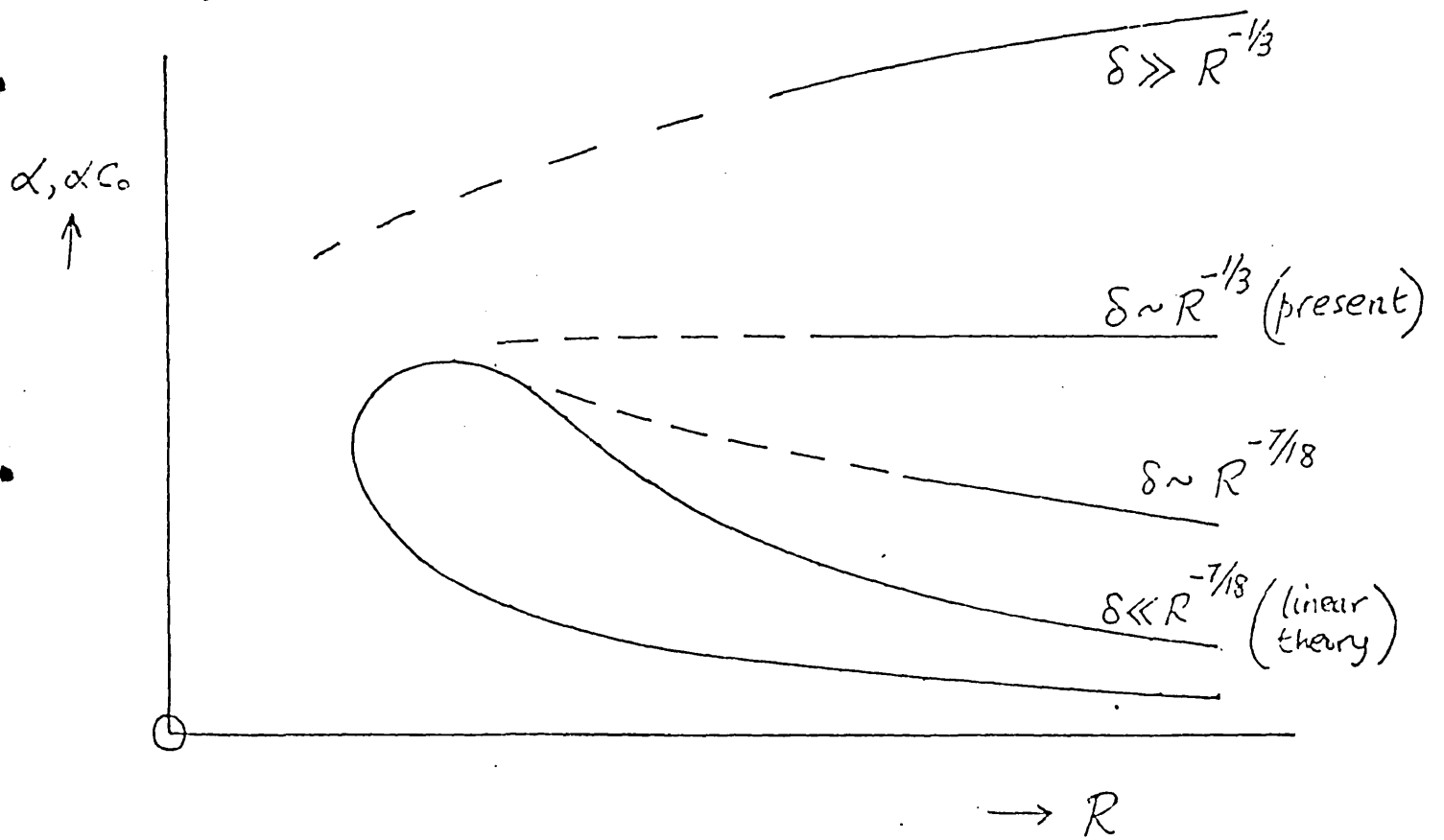


Fig. 1b

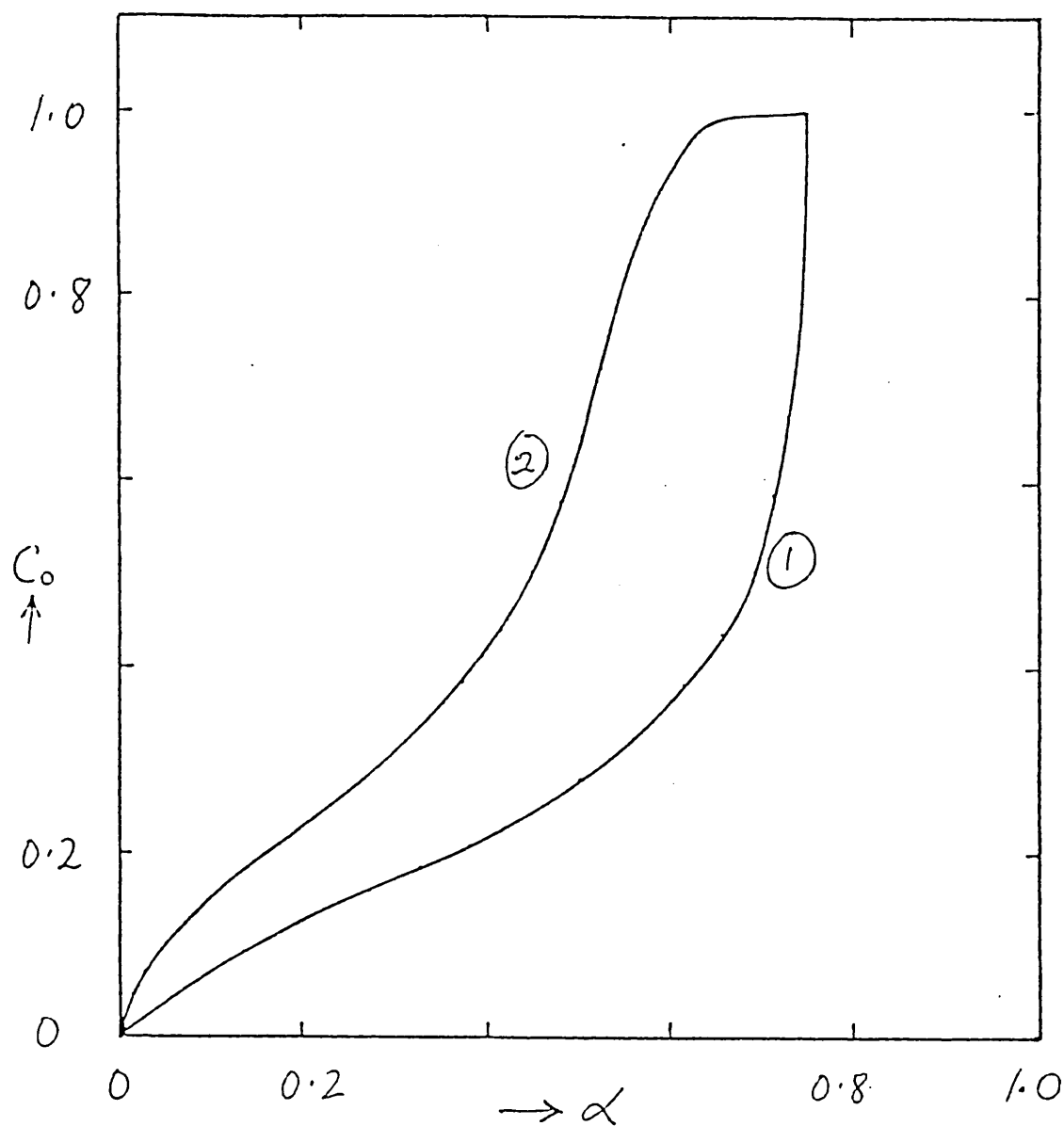


Fig. 2b

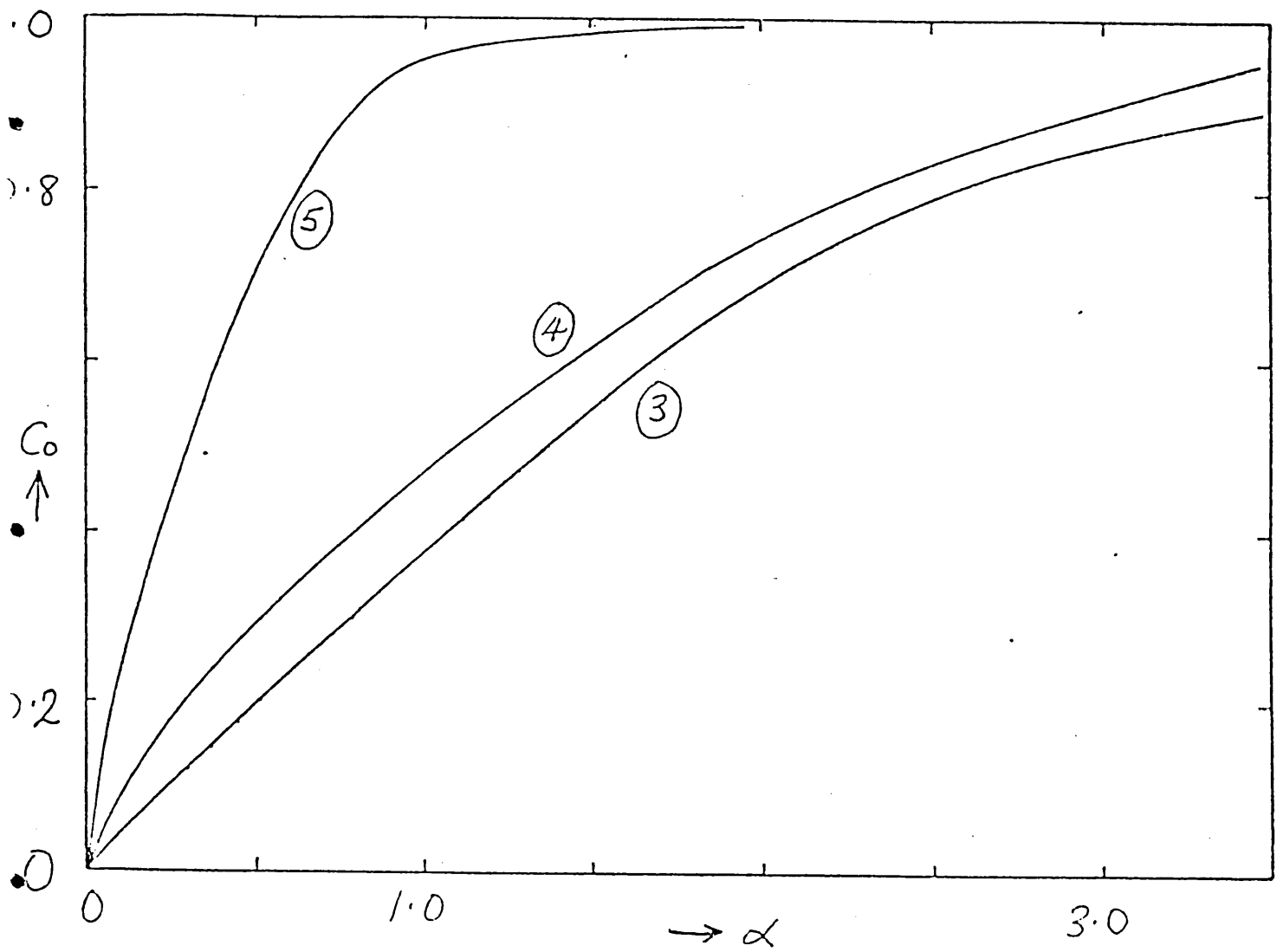
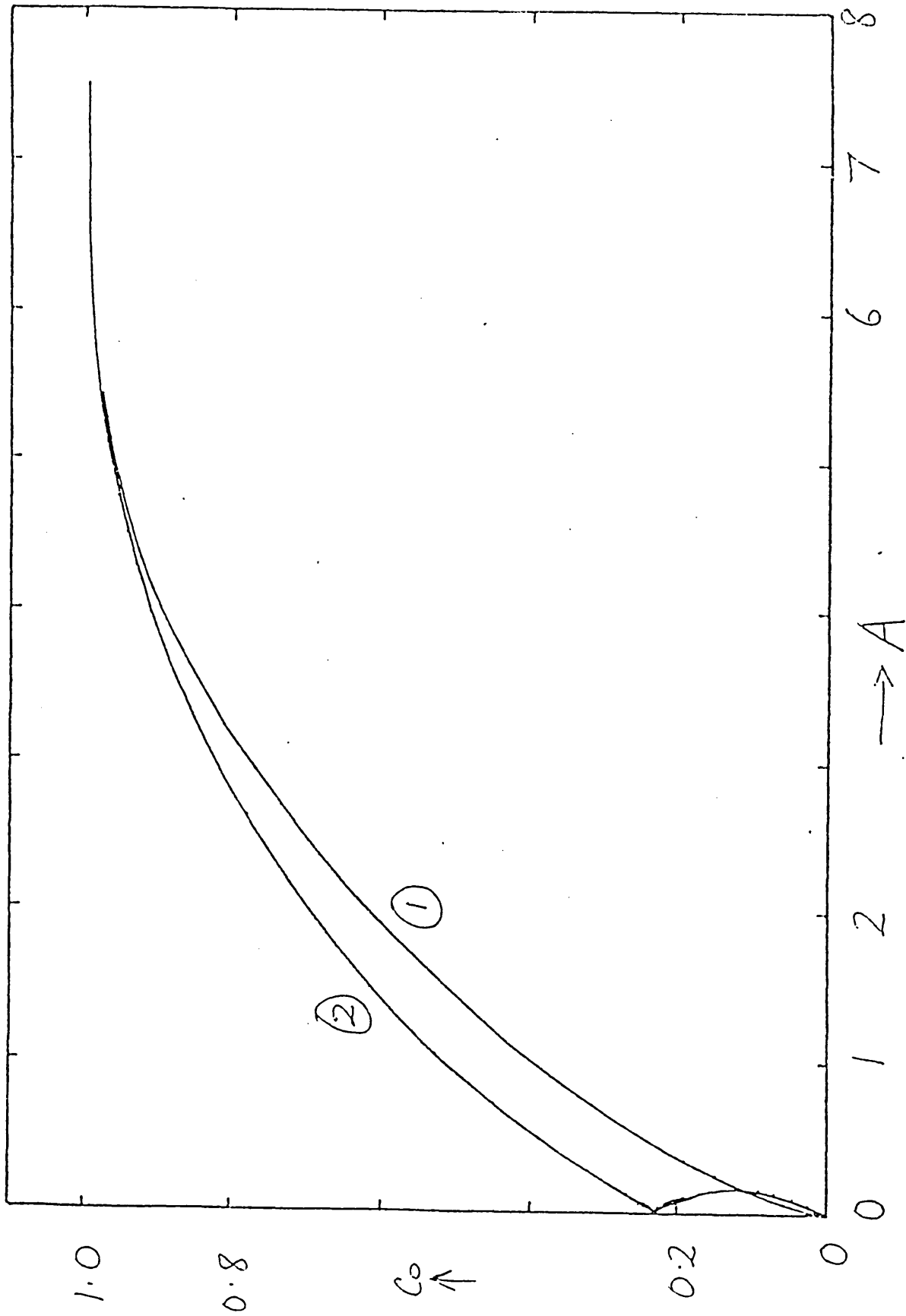


Fig 3a
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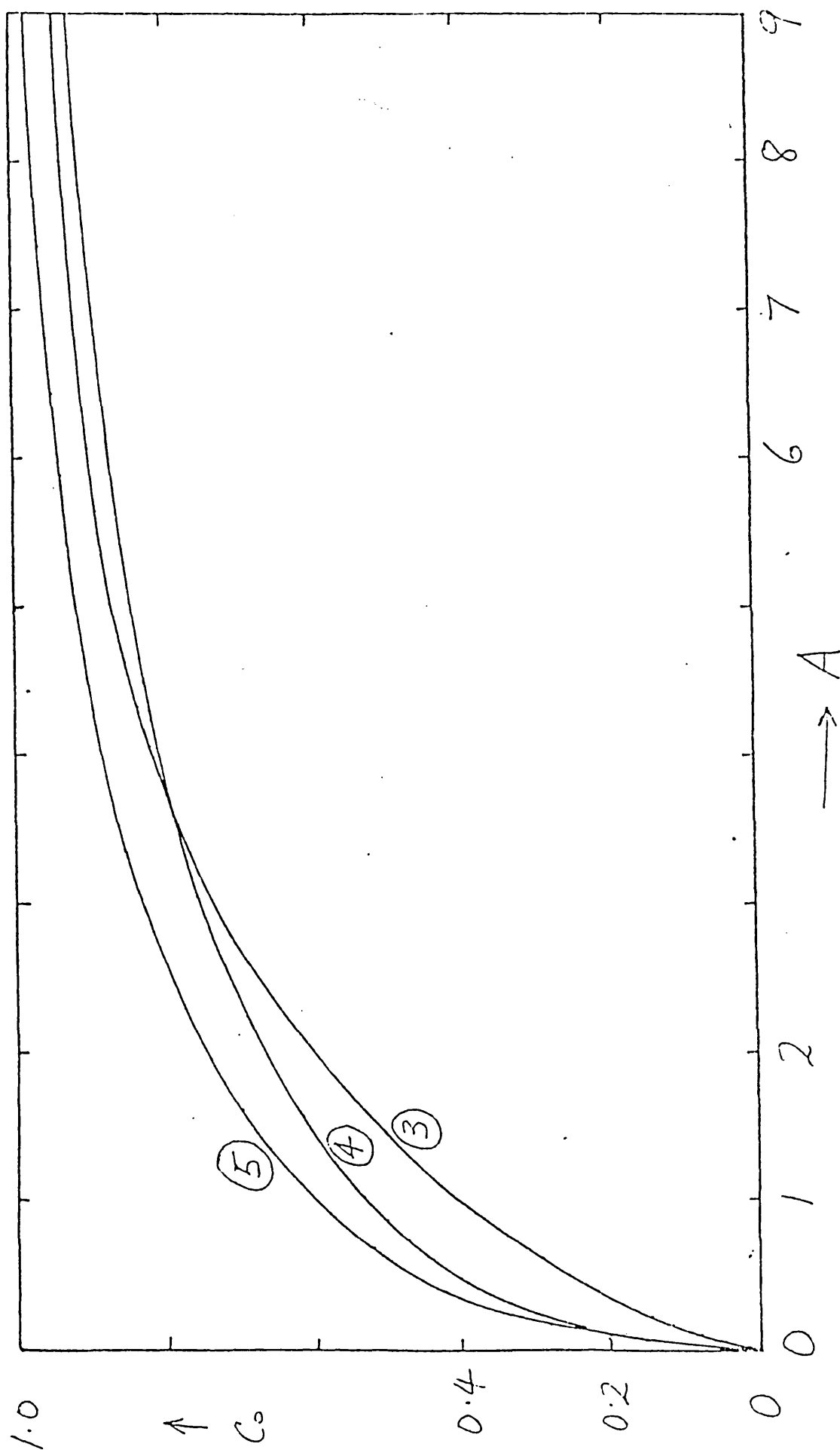
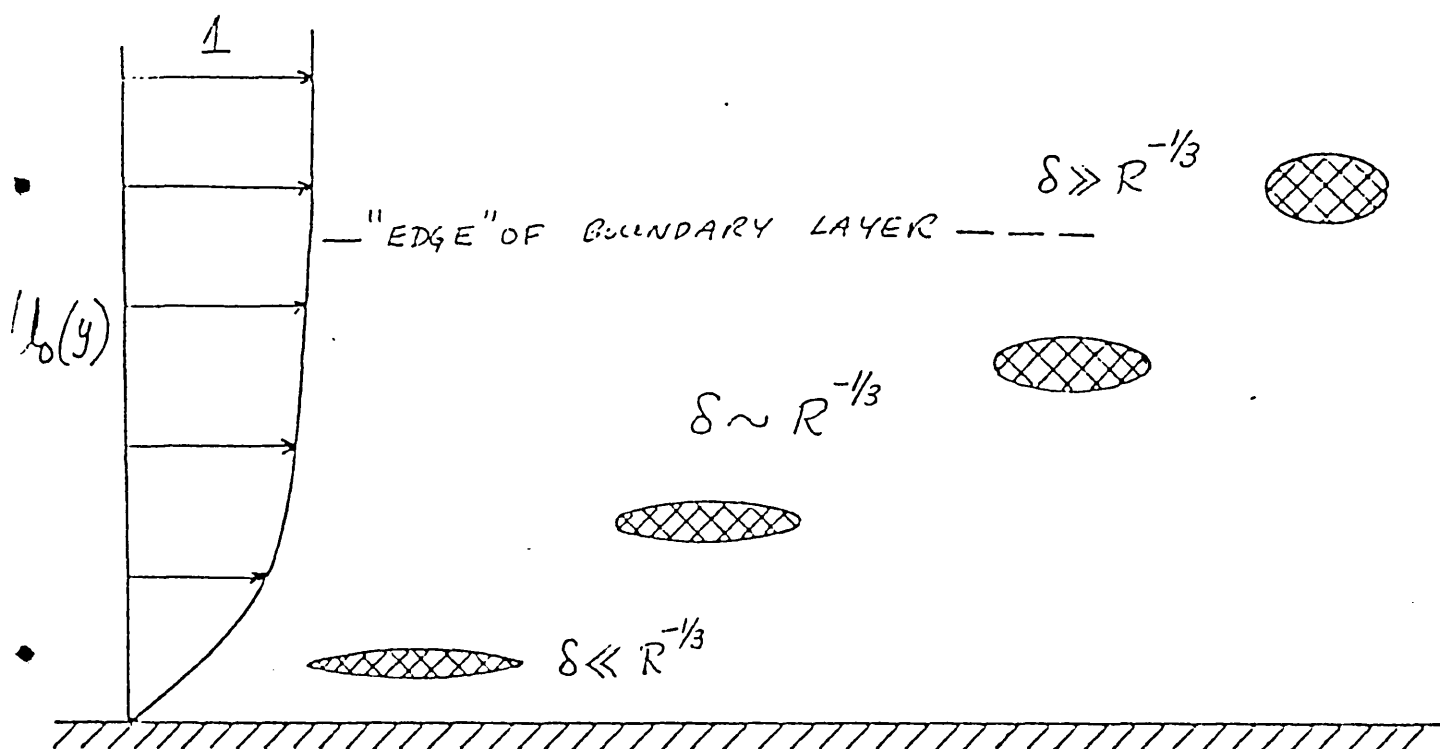


Fig. 4

R. P. Tokoyi

et al



CAPTIONS

Fig. 1 (a) Diagram (not to scale) of the disturbed flow structure for the disturbance size $\delta = O(R^{-1/3})$, indicating the Rayleigh zones I^+ , the inviscid parts II^0, II^+ and viscous parts III^+ of the strongly nonlinear critical layer, and the wall layers IV_1, IV_2 .

(b) Showing the suggested sudden effect of disturbance size on the upper branch of the neutral stability curve, as δ increases from less than $R^{-7/18}$ (linear theory) to greater than $R^{-1/3}$ (see §5 below) through the stages $\delta \sim R^{-7/18}$ (Smith and Bodonyi (1982a)) and $\delta \sim R^{-1/3}$ (present study). The dashes indicate the envisaged displacements of the neutral wavenumber α and frequency αc_0 at finite R .

Fig. 2 (a) Neutral curves $c_0(\alpha)$ for (1): suction profile $U_0 = 1 - \exp(-y)$; (2): inflection point profile $U_0 = 1 - (1 + 3y/4) \cdot \exp(-y)$. See also fig. 3(a).

(b) Neutral curves $c_0(\alpha)$ for (3): parabolic boundary layer model $U_0 = 2y - y^2$, $0 \leq y \leq 1$, $U_0 = 1$, $y > 1$; (4): Blasius model profile $U_0 = 3((-2/3)!)^{-1} \int_0^y \exp(-\xi^3) d\xi$; (5): Blasius flow profile $U_0 = f'$, where $f''' + ff'' = 0$, $f(0) = f'(0) = 0$, $f'(\infty) = 1$. See also fig. 3(b).

Fig. 3 (a) Amplitude-dependence of the neutral modes showing c_0 as a function of the relative amplitude A for (1) the suction profile and (2) the inflection point profile of fig. 2(a). Note that when A is large, for (1), $y_c \sim 12 \ln A$, $c_0 \sim 1 - O(A^{-12})$, $\alpha \rightarrow 3/4$ as in (5.6a) - (5.11) while for (2) $y_c \sim 12 \ln A$, $c_0 \sim 1 - O(A^{-12} \ln A)$, $\alpha \rightarrow 3/4$.

(b) Amplitude-dependence of the neutral modes showing c_0 as a function of the relative amplitude A , for (3) the parabolic boundary layer model, (4) the Blasius model profile and (5) the Blasius flow profile. Note that when A is large, for (3), $y_c \sim 1 - 1/\alpha\tilde{\beta}$, $c_0 \sim 1 - (3\tilde{\beta} \ln A)^{-2}$, $\alpha \sim 3 \ln A$ where $\tilde{\beta}$ is a positive constant, while for (4) $y_c \sim \frac{2}{3}\alpha^{\frac{1}{2}}$, $c_0 \sim 1 - 9/[4\Gamma(1/3)\alpha \exp(8\alpha^{3/2}/27)]$, $\alpha \sim [(81/22) \ln A]^{2/3}$.

Fig. 4 Schematic diagram of the position and shape of the typical cat's eye, regarded as a lump of vorticity, as the disturbance size is increased. Note that the length of the eye decreases, while its thickness increases, as the disturbance size increases, but the thickness remains smaller than the boundary layer thickness throughout.

CHAPTER 8

ON UNSTEADY NONLINEAR VISCOUS CRITICAL LAYERS.

ON UNSTEADY NONLINEAR VISCOUS CRITICAL LAYERS

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SUMMARY

A theoretical study is presented concerning the nonlinear growth or decay, in space and time, of an unsteady non-neutral disturbance in the presence of an unsteady nonlinear viscous critical layer. The basic flow considered is a steady, quasi-parallel, channel flow, boundary layer or liquid layer flow, at high Reynolds number. The unsteadiness with regard to the critical layer manifests itself mainly in the relatively slow movement of the layer across the flow, the temporal and spatial rate of movement discussed being sufficient to affect the nonlinear viscous balance in the layer. The disturbance amplitude, in contrast, responds nonlinearly on faster time - and space - scales, inside and outside the critical layer. The slower and faster scale properties are coupled together in general. The work addresses first the structure and nonlinear governing equations for the growing or decaying disturbance and the critical layer; but preliminary analysis in special cases suggests that, among other things, the results of previous studies based on quasi-neutral assumptions give threshold amplitudes, above which increasingly fast disturbance growth takes place.

s1 INTRODUCTION

Our concern here is with the nonlinear growth or decay, in time and space, of an initially small disturbance to a given parallel or almost parallel flow, at high Reynolds number. At such Reynolds numbers the critical layer, where the basic flow speed coincides with the local effective wavespeed, can play a vital role in determining the behaviour of the disturbance, particularly when nonlinearity exerts a significant influence. The effects of increased nonlinearity, as the disturbance size increases, are felt primarily within the critical layer itself, which gradually undergoes a change in character. There have been many theoretical studies made of the properties of nonlinear critical layers (e.g. Benney & Bergeron 1969, Davis 1969, Haberman 1972, 1976, Brown & Stewartson 1978, 1980, Smith & Bodonyi 1982a,b, Bodonyi, Smith & Gajjar 1983), and Stewartson (1980) has given an interesting review of many aspects of the subject. To the best of our knowledge, however, virtually all theoretical interest so far (apart from purely inviscid theory) has been focussed on disturbances which are neutral travelling waves outside of the critical layer, with non-neutral states of growth or decay having been considered only with respect to the motion within the critical layer. This amounts to an assumption, or restriction, that the critical layer is *fixed*.

In this paper the aim is to remove that restriction, in order to discuss theoretically the temporal - spatial growth or decay of the disturbance amplitude throughout the flowfield, not solely inside the critical layer. It would seem that such a discussion is almost certainly more relevant to the behaviour of unsteady disturbances observed experimentally, in boundary layers, channel flows and liquid

layers for instance. The study is also of interest as a test on the tentative suggestions of Smith & Bodonyi (1982a) and Bodonyi *et al.* (1983) which are based on the quasi-neutral assumption. They suggest formally that an increasing disturbance amplitude can lead to amplitude-dependent instabilities occurring with shortened wavelengths, enhanced wavespeeds and (hence) movement of the critical layer, and that, in line with Benney & Bergeron's (1969) suggestions, this may be related ultimately to the shedding of vorticity from a disturbed boundary layer. A similar approach for three-dimensional disturbances suggests (Smith & Bodonyi 1982b) amplitude-dependent instabilities arising in Hagen-Poiseuille flow. So the evolution of a nonlinear disturbance, or wave-packet, rather than the succession of nonlinear neutral states considered previously, is our concern. Associated with the space-time evolution of the disturbance through the flowfield, the effects of a *moving* critical layer have to be taken into account, and these prove to be of some consequence, more so than unsteady effects confined within the critical layer. For, as the critical layer is relatively thin, its actual movement across the flow produces more change in the internal flow properties than does the inherent unsteadiness of the disturbance velocity. In addition, a relatively slow movement of the critical layer position is found (below) to be accompanied generally by a much faster temporal and spatial response in the typical disturbance amplitude, inside and outside the critical layer. The study below concentrates on the (upper branch) structure, governing equations and certain preliminary properties of both the non-stationary critical layer and the non-neutral amplitude.

Section 2 discusses the basic principles involved when a disturbance is "unsteady", i.e. non-neutral, growing or decaying in time and space, in the presence of a nonlinear critical layer. The

importance of even slight critical layer movement is confirmed and leads to a novel equation describing the behaviour of the unsteady nonlinear critical layer. The multiple scales introduced in §2 are supplemented in §3, where the specific contexts of the accelerating boundary layer, channel flow and liquid layer flow are considered and the faster scale response of the amplitude becomes evident. As a result a set of coupled nonlinear evolutionary equations, controlling the amplitude and effective wavespeed or group velocity of a wave packet, is derived. Section 4 describes the solution in the special case of fixed frequency disturbances. These exhibit a threshold phenomenon whereby initial disturbances of amplitude above/below the threshold value amplify/decay nonlinearly at later times, further downstream. In §5 limiting analytical solutions are considered for the unsteady nonlinear critical layer problem, which otherwise requires a numerical treatment. Finally, a brief summary is given in §6. Only two-dimensional disturbances are examined, for an incompressible fluid of constant density ρ and kinematic viscosity ν . The Reynolds number is written $Re = u_\infty l_\infty \nu^{-1}$ and is assumed to be asymptotically large: see Smith, Papageorgiou & Elliott (1983) for application of the asymptotic theory to finite Re properties. Here u_∞ , l_∞ are typical speed and length scale of the basic motion, l_∞ being the streamwise development scale of the boundary layer, the thickness of channel or the width of the liquid layer. The time, cartesian coordinates, associated velocity components and pressure are $l_\infty u_\infty^{-1} t$, $l_\infty(x, y)$, $u_\infty(u, v)$ and $\rho_\infty u_\infty^2 p$, the stream function being $l_\infty u_\infty \psi$. In the cases considered the basic flow can be taken to be parallel along the x-axis, to the order of working necessary.

§2. GENERAL ARGUMENTS ON UNSTEADY NONLINEAR CRITICAL LAYERS

It seems appropriate to start with the general argument leading to the central unsteady critical layer problem below, without reference to a specific flow yet. This delays the use of expansions (necessary eventually) in somewhat strange powers of the Reynolds number, and some of the initial scalings involved can be identified more readily with the classical scalings, (e.g. Reid, 1965). The formal expansions, for the accelerating boundary layer, channel and liquid layer flows in particular, will be verified in the later sections.

To allow for the slow spatial and temporal variations we introduce the multiple scales replacement of $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial t}$ by

$$\begin{aligned}\frac{\partial}{\partial x} &\rightarrow \alpha \frac{\partial}{\partial X} + \gamma^{(x)} \frac{\partial}{\partial X_2} + \dots, \\ \frac{\partial}{\partial t} &\rightarrow -\beta \frac{\partial}{\partial X} + \gamma^{(t)} \frac{\partial}{\partial T_2} + \dots\end{aligned}\tag{2.1}$$

Here α is the wavenumber, $\beta = \alpha c$ is the frequency and c is the wavespeed of the disturbance non-simple wave solutions, which are periodic in X , but not in the slow variables $x = \gamma^{(x)-1} X_2$, $t = \gamma^{(t)-1} T_2$.

Consider first then the *steady* problem for *infinitesimal* disturbances. The majority of the motion consists of the basic flow $(u, v, p) = (u_B, 0, p_B)$ together with the small disturbances (u_1, v_1) , p_0 proportional to e^{iX} . This holds except in the neighbourhood of the Stokes wall layer and the critical layer, which are assumed to be asymptotically distinct but with the critical layer still lying close to the wall. Just outside the critical layer where the basic flow is comparable with the wavespeed c , the flow expansions take

the form

$$\begin{aligned} u &\sim \lambda c_R \tilde{Y} + c_R^2 \frac{\lambda_2 \tilde{Y}^2}{2} + \delta(u_1 + c_R u_2 + \dots) \\ v &\sim \alpha \delta \sigma_R c_R (v_1 + c_R v_2 + \dots) \\ p &\sim \delta c_R (p_0 + c_R p_1 + \dots) \end{aligned} \quad (2.2)$$

Here $c_R \tilde{Y} = \sigma_R^{-1} y$ defines the boundary layer coordinate, and $c = c_0 c_R$, $\alpha = \alpha_0 \alpha_R$, where the unknown orderings c_R , α_R , σ_R depend on the Reynolds number. Also δ is the infinitesimal disturbance size and λ is the local skin friction. If the local profile curvature λ_2 is non zero, then a logarithmic singularity in u_2 is induced usually, and requires a viscous critical layer of thickness $\sigma_R (\alpha_R \sigma_R^2)^{-1/3}$ to smooth out the irregularity. The familiar phase shift of $-\pi$ in the logarithm below the critical layer results. The Stokes layer of thickness of order $\sigma_R (\alpha_R c_R \sigma_R^2)^{-1/2}$ produces a phase shift in the basic disturbance and provides the necessary displacement condition to match with the critical layer shift. This fixes effectively the Reynolds number dependence of the wavespeed and wavenumber, i.e.

$$c = O[(\alpha \sigma_R^2)^{-1/5}] \quad (2.3)$$

Additional relations dependent on the particular basic flow then determine the neutrally stable modes explicitly.

The above description holds for infinitesimal disturbances. As the disturbance size is increased *nonlinear* interactions first come into play inside the critical layer, and streamwise momentum balances show that the critical layer properties are altered significantly when δ obtains the critical size

$$\delta = 0 \left[\frac{(\alpha_R \sigma_R^2)^{-2/3}}{c} \right] = \delta_c \quad . \quad (2.4)$$

The governing equations in that case are those derived by Haberman (1972). Outside the critical layer the main change produced is that the jump in the streamwise velocity is no longer monochromatic and in particular the phase shift ϕ depends on the disturbance size. Only for $\delta \ll \delta_c$ are classical linear properties recovered, with $\phi \rightarrow -\pi$.

Let us suppose now that non-neutral *unsteadiness* is also present. To accommodate growing or decaying nonlinear modes, we take the disturbance to have a slowly varying amplitude A_0 . Therefore in the development of the critical layer the wavespeed and wavenumber become slowly varying. The unsteadiness first becomes important inside the critical layer. Since the critical layer is defined by

$$y = y_c + \sigma_R (\alpha_R \sigma_R^2)^{-1/3} Y \quad (2.5)$$

where $y_c = 0(c)$ determines the critical layer position and is dependent on the slow variables, we have in addition to (2.1)

$$\gamma^{(x)} \frac{\partial}{\partial X_2} \rightarrow \gamma^{(x)} \left[\frac{\partial}{\partial X_2} - \sigma_R^{-1} (\alpha_R \sigma_R^2)^{1/3} \frac{\partial y_c}{\partial X_2} \frac{\partial}{\partial Y} \right] \quad (2.6)$$

$$\gamma^{(t)} \frac{\partial}{\partial T_2} \rightarrow \gamma^{(t)} \left[\frac{\partial}{\partial T_2} - \sigma_R^{-1} (\alpha_R \sigma_R^2)^{1/3} \frac{\partial y_c}{\partial T_2} \frac{\partial}{\partial Y} \right]$$

The terms in $\frac{\partial}{\partial Y}$ reflect the *movement* of the critical layer itself, and are much larger than the basic slow variations $\frac{\partial}{\partial X_2}$, $\frac{\partial}{\partial T_2}$ here.

Previous studies on unsteady nonlinear critical layers have concentrated on the case where the $\frac{\partial}{\partial Y}$ contributions in (2.6) are negligible. With a moving critical layer, however, these terms dominate and cannot be neglected. Similar terms appear in Cowley's (1982) inviscid analysis of moving critical layers, although in a different context and in which the basic flow is also time dependent.

An unsteady-viscous-inertial balance now gives the appropriate scalings when these new effects become significant: namely when

$$\gamma^{(t)} = O(\alpha_R \delta_c) \quad , \quad \gamma^{(x)} = O\left(\frac{\alpha_R \delta_c}{c_R}\right) \quad . \quad (2.7)$$

The inclusion of nonlinearity and unsteadiness modifies the Haberman equation and instead the vorticity $\zeta^* = \partial U / \partial Y^*$ satisfies the equation

$$\mu \left(1 - \frac{\partial \zeta^*}{\partial Y^*}\right) + Y^* \frac{\partial \zeta^*}{\partial X^*} + \sin X^* \frac{\partial \zeta^*}{\partial Y^*} = \gamma_c \frac{\partial^2 \zeta^*}{\partial Y^{*2}} \quad (2.8)$$

in normalized form. Here $\mu = 0$ gives the steady Haberman equation. The extra term $\mu(1 - \frac{\partial \zeta^*}{\partial Y^*})$ arises because of the slow variations present which force the critical layer to move. It can be seen that $\mu \propto (\partial c_0 / \partial T_2 + c_0 \partial c_0 / \partial X_2) / A_0$ since the basic flow $u_B \sim c$. Again ζ^* satisfies the same boundary conditions as in the steady case and the leading order properties outside the critical layer remain largely unaltered.

It should be emphasized that, so far, the temporal and spatial variations appear only parametrically in (2.8). The properties of this unsteadiness equation strongly influence the amplitude dependence, via the velocity jump, but the explicit slow-time and slow-space variations of the amplitude require an examination of the

higher order terms, both inside and outside the critical layer. Finally the slow variations in time and space chosen above are those which first significantly alter the steady Haberman problem. It can be expected however that the limiting solutions of this unsteady problem, for μ large or small, will indicate what the different structures are for alternative spatial and temporal variations, both weaker and stronger.

§3. THE BOUNDARY LAYER, CHANNEL FLOW, AND LIQUID LAYER FLOW

With the general arguments for unsteady, moving, nonlinear critical layers established in the previous section, we turn to three specific examples of perhaps widest concern, the boundary layer, channel flow and liquid layer flow. The main novel parts of the nonlinear non-neutral disturbance structures for these three basic flows are set out in (a)-(c) below. An extra common feature here is that the disturbance amplitude usually must vary also on temporal and spatial scales which, although still slow, are faster than the time and space scales of the critical layer movement described in §2 and which are significant with regard to the nonlinear development of the disturbance from an initially small state.

(a) The boundary layer

In the five-zoned account (Smith & Bodonyi) of the disturbance structure necessary here, the major changes due to unsteadiness arise near the wall, in the thin predominantly inviscid zone (IZ) surrounding the critical layer and within the thinner nonlinear critical layer (CL) itself. In IZ we have now the expansions, implied mainly by the time and space scales introduced above,

$$\begin{aligned} \bar{u} = & \epsilon \lambda \tilde{Y} + \epsilon^2 \lambda_2 \tilde{Y}^2 + \epsilon^{7/3} \bar{u}^{(0)} + \epsilon^{8/3} u_{MF} \\ & + \epsilon^{10/3} \bar{u}^{(1)} + \epsilon^{11/3} \bar{u}^{(2)} + \dots \end{aligned} \quad (3.1a)$$

$$\bar{v} = \epsilon^{13/3} \bar{v}^{(0)} + \epsilon^{16/3} \bar{v}^{(1)} + \epsilon^{17/3} \bar{v}^{(2)} + \dots \quad (3.1b)$$

$$\bar{p} = \epsilon^{10/3} \bar{p}^{(0)} + \epsilon^{13/3} \bar{p}^{(1)} + \epsilon^{14/3} \bar{p}^{(2)} + \dots \quad (3.1c)$$

Here the mean flow correction u_{MF} is independent of $\tilde{x}$ and the constants λ , λ_2 are the skin friction and curvature of the basic boundary layer near the wall, $\lambda > 0$, $\lambda_2 < 0$, for an accelerating boundary layer, while $\epsilon^{12} = Re^{-1} \ll 1$ and $(x, y, t) = (\epsilon^5 \tilde{x}, \epsilon^7 \tilde{y}, \epsilon^4 \tilde{t})$ with $\tilde{x}$, $\tilde{y}$, $\tilde{t}$ of order unity. For other notation see Smith & Bodonyi. The slower spatial and temporal scales $x = \epsilon^5 (\epsilon^{-1} X_1) = \epsilon^5 (\epsilon^{-4/3} X_2)$ and $t = \epsilon^4 (\epsilon^{-1} T_1) = \epsilon^4 (\epsilon^{-4/3} T_2)$ are also present, so that

$$\frac{\partial}{\partial x} \rightarrow \epsilon^{-5} \left(\frac{\partial}{\partial \tilde{x}} + \epsilon \frac{\partial}{\partial X_1} + \epsilon^{4/3} \frac{\partial}{\partial X_2} \right) \quad (3.2a)$$

$$\frac{\partial}{\partial t} \rightarrow \epsilon^{-4} \left(-c_0 \frac{\partial}{\partial \tilde{x}} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^{4/3} \frac{\partial}{\partial T_2} \right), \quad (3.2b)$$

for an underlying but varying wavespeed c_0 . Hence the fundamental disturbance in (3.1a-c) has the form

$$\begin{aligned} \bar{u}^{(0)} &= 2\lambda A_0 \cos \theta, \quad \bar{v}^{(0)} = 2\alpha_0 \lambda A_0 \tilde{y} \sin \theta, \\ \bar{p}^{(0)} &= 2P_0 \cos \theta \end{aligned} \quad (3.3)$$

satisfying the tangential flow condition as the wall is approached, with A_0 , P_0 dependent on X_1 , T_1 , X_2 , T_2 and, like the effective wavenumber α_0 and wavespeed $c_0 = \beta_0 \alpha_0^{-1}$, unknown. It is necessary for consistency however that α_0 , c_0 and hence the frequency $\beta_0 = \alpha_0 c_0$ remain independent of X_1 , T_1 , depending instead on X_2 , T_2 principally. In other words, physically, a slow variation in α_0 , c_0 , β_0 , corresponding to a slow movement of the critical layer on the X_2 , T_2 scales, tends to force a much faster scale (X_1, T_1) response in the disturbance amplitude A_0 itself. In (3.3) the argument θ satisfies

$$\frac{\partial \theta}{\partial x} = \alpha_0(x_2, T_2) \quad , \quad \frac{\partial \theta}{\partial t} = -\beta_0(x_2, T_2) \quad , \quad (3.4a)$$

to our order of working, and so we have a *distorted* travelling wave. For compatibility (3.4a) then requires that α_0 , β_0 must be related by

$$\frac{\partial \alpha_0}{\partial T_2} = - \frac{\partial \beta_0}{\partial X_2} \quad . \quad (3.4b)$$

In addition the streamwise momentum balance yields the requirement

$$\lambda A_0 \beta_0 = \alpha_0 P_0 \quad , \quad (3.4c)$$

because of (3.3); while the potential flow holding outside the boundary layer forces the classical pressure-displacement relation

$$P_0 = \alpha_0 A_0 \quad (3.4d)$$

to hold. So from (3.4b-d) we have the equations

$$\left[\frac{\partial}{\partial T_2} + 2c_0(x_2, T_2) \frac{\partial}{\partial X_2} \right] \alpha_0 = \left[\frac{\partial}{\partial T_2} + 2c_0(x_2, T_2) \frac{\partial}{\partial X_2} \right] c_0 = 0 \quad ,$$

with $\alpha_0 = \lambda c_0 \quad , \quad (3.4e)$

governing the slow variation of α_0 , c_0 in time and space. As expected, the group velocity $c_g = \partial \beta_0 / \partial \alpha_0 = 2c_0$ controls the group propagation, at *twice* the local wavespeed.

It is significant also that if disturbances of *fixed* frequency β_0 were under consideration this would correspond immediately to a fixed wavenumber α_0 and a fixed wavespeed c_0 , since $\alpha_0 = \beta_0^{\frac{1}{2}} \lambda^{\frac{1}{2}}$

and $c_0 = \beta_0^{\frac{1}{2}} \lambda^{-\frac{1}{2}}$ from (3.4e). Our concern here, however, is with non-uniform conditions of frequency, wavespeed and wavenumber.

At the next order, from the Navier-Stokes equations, (3.1a-c) yield the solution for $\bar{v}^{(1)}$:

$$\begin{aligned} \bar{v}^{(1)} = & -\lambda^{-1} (\partial_{\tilde{p}} \bar{v}^{(1)} / \partial \tilde{x} + \partial_{\tilde{p}} \bar{v}^{(0)} / \partial X_1) - \lambda \xi \frac{\partial \hat{a}}{\partial \tilde{x}} \\ & + 2\alpha_0 \lambda_2 A_0 (\xi^2 + 2\lambda^{-1} c_0 \xi \ln \xi - \lambda^{-2} c_0^2) \sin \theta \\ & - \lambda^{-1} \left(\frac{\partial}{\partial T_1} + c_0 \frac{\partial}{\partial X_1} \right) (\bar{u}^{(0)}) \end{aligned} \quad (3.5a)$$

with

$$\partial_{\tilde{p}} \bar{v}^{(1)} / \partial \tilde{Y} = 0, \quad (3.5b)$$

the function $\hat{a}$ being independent of $\tilde{Y}$ and as in Smith & Bodonyi, and $\xi \equiv \tilde{Y} - \lambda^{-1} c_0$. Here the influences of the profile curvature λ_2 and of the first slow scales X_1 , T_1 first take effect; the scales are selected for that reason. Also, the logarithm in (3.5a) is appropriate to $\tilde{Y} > \lambda^{-1} c_0$, above the critical layer. Beneath it the replacement

$$(\ln \xi) \sin \theta \rightarrow (\ln |\xi|) \sin \theta + \phi \cos \theta (\tilde{Y} < \lambda^{-1} c_0) \quad (3.5c)$$

is necessary, in view of the (unknown) phase shift ϕ across the critical layer (see below) at $\tilde{Y} \approx \lambda^{-1} c_0$, together with a replacement for the function $\hat{a}$ akin to that in Smith & Bodonyi. So the inviscid displacement effect produced near the wall as $\tilde{Y} \rightarrow 0+$ is $O(\epsilon^{16/3})$, given by $\bar{v}^{(1)}(\tilde{Y}=0+)$. This matches the classical viscous wall layer displacement

$$\bar{v}(\tilde{Y}=0+) = \epsilon^{16/3} \left[-\frac{2\alpha_0^{1/2}}{c_0^{3/2}} P_0 \sin(\theta + \frac{\pi}{4}) + O(\epsilon) \right] \quad (3.6)$$

provided that, from the $\cos \theta$ components of (3.5a-c) and (3.6), the phase balance

$$-2 \left[\frac{1}{\lambda} \frac{\partial P_0}{\partial X_1} + \left(\frac{\partial}{\partial T_1} + c_0 \frac{\partial}{\partial X_1} \right) A_0 \right] = 4\alpha_0 \lambda_2 \frac{c_0^2}{\lambda^2} A_0 \phi - \frac{2^{1/2} \alpha_0^{1/2}}{c_0^{3/2}} P_0 \quad (3.7)$$

holds; the other components play a passive role at this stage. Equation (3.7) then leads ultimately to the nonlinear amplitude equation for A_0 . Note that in a neutral state the left hand side of (3.7) is zero and the right hand side fixes the steady neutral amplitude, via the phase shift. Again, for infinitesimal disturbances, where $\phi = -\pi$, (3.7) determines the small growth rate holding across the majority of the neutral curve at large Reynolds numbers.

Similar reasoning applies to the first appearance of the still slower scales X_2 , T_2 , in the forced solution

$$\bar{v}^{(2)} = -\frac{1}{\lambda} \left[\frac{\partial \bar{p}^{(0)}}{\partial X_2} + \left(\frac{\partial}{\partial T_2} + c_0 \frac{\partial}{\partial X_2} \right) \bar{u}^{(0)} \right] - \lambda \xi \frac{\partial B}{\partial \tilde{x}} \quad (3.8a)$$

in IZ, with B independent of $\tilde{Y}$. There are two other sources of forcing invoked at this stage. One is from the pressure-displacement interaction, which requires that

$$B = \frac{P_0}{\alpha_0} \frac{\partial \alpha_0}{\partial X_2} \sin \theta \quad .$$

The other is from a phase shift $\propto \phi_2$ in B across CL: see below. There is no corresponding viscous displacement from the wall layer however, from (3.6), so that the constraint $\bar{v}^{(2)}(\tilde{Y}=0+) = 0$ holds. Consequently (3.8a) leads to the result

$$\frac{1}{\lambda} \frac{\partial}{\partial X_2} (\alpha_0 A_0) + \left(\frac{\partial}{\partial T_2} + c_0 \frac{\partial}{\partial X_2} \right) A_0 + \frac{1}{2} c_0 \frac{A_0}{\alpha_0} \frac{\partial \alpha_0}{\partial X_2} = (\quad) \phi_2 \quad (3.8b)$$

in view of the relation (3.4d).

The slower scales X_2 , T_2 seem to play a fairly minor role in IZ, at first sight, therefore. Their importance is enhanced within the critical layer CL, by contrast. In CL, where $\tilde{Y} = \lambda^{-1} c_0 + \epsilon^{2/3} Y$ with Y of order unity, and the flow speed and effective wavespeed nearly coincide, the flow solution has the form

$$\begin{aligned} \bar{u} = & \epsilon c_0 + \epsilon^{5/3} \lambda Y + \epsilon^2 C^{(1)} + 2\epsilon^{7/3} \lambda A_0 \cos \theta \\ & + \epsilon^{8/3} C^{(2)} Y + \epsilon^{10/3} (\ln \epsilon) C^{(3)} A_0 \cos \theta + \epsilon^{10/3} U_1 \\ & + \epsilon^{11/3} U_2 + \dots \end{aligned} \quad (3.9a)$$

$$\begin{aligned} \bar{v} = & 2\epsilon^{13/3} \alpha_0 A_0 c_0 \sin \theta + 2\epsilon^5 \alpha_0 \lambda A_0 Y \sin \theta \\ & + \epsilon^{16/3} C^{(4)} + \epsilon^{17/3} C^{(5)} + \epsilon^6 (\ln \epsilon) C^{(6)} Y \sin \theta \\ & + \epsilon^6 V_1 + \epsilon^{19/3} V_2 + \dots \end{aligned} \quad (3.9b)$$

$$\begin{aligned} \bar{p} = & 2\epsilon^{10/3} P_0 \cos \theta + \epsilon^{13/3} C^{(7)} + \epsilon^{14/3} C^{(8)} + \epsilon^5 P_1 \\ & + \epsilon^{16/3} P_2 + \dots \end{aligned} \quad (3.9c)$$

where $c^{(n)}$, $n \geq 1$, are functions independent of Y and are determined directly by matching with the solution in IZ . For example, $c^{(1)} = \lambda_2 c_0^2 \lambda^{-2}$, $c^{(2)} = 2\lambda_2 c_0 \lambda^{-1}$. More significantly, since the critical layer is not stationary here, the multiple scaling becomes

$$\frac{\partial}{\partial x} \rightarrow \varepsilon^{-5} \left[\frac{\partial}{\partial \tilde{x}} + \varepsilon \frac{\partial}{\partial X_1} + \varepsilon^{4/3} \left(-\frac{\varepsilon^{-2/3}}{\lambda} \frac{\partial c_0}{\partial X_2} \frac{\partial}{\partial Y} + \frac{\partial}{\partial X_2} \right) + \dots \right] \quad (3.10a)$$

$$\frac{\partial}{\partial t} \rightarrow \varepsilon^{-4} \left[-c_0 \frac{\partial}{\partial \tilde{x}} + \varepsilon \frac{\partial}{\partial T_1} + \varepsilon^{4/3} \left(-\frac{\varepsilon^{-2/3}}{\lambda} \frac{\partial c_0}{\partial T_2} \frac{\partial}{\partial Y} + \frac{\partial}{\partial T_2} \right) + \dots \right] \quad (3.10b)$$

together with $\partial/\partial y \rightarrow \varepsilon^{-23/3} \partial/\partial Y$. So, as noted before, unsteadiness is dominated by the induced Y -derivative, in CL .

In CL , the Navier-Stokes equations subject to (3.9a-c), (3.10a,b) reduce to the controlling equations for U_1 , V_1 , P_1 :

$$\frac{\partial U_1}{\partial \tilde{x}} + 2\lambda \frac{\partial A_0}{\partial X_1} \cos\theta + \frac{\partial V_1}{\partial Y} = 0 \quad (3.11a)$$

$$\begin{aligned} & - \left(\frac{\partial \kappa}{\partial T_2} + c_0 \frac{\partial \kappa}{\partial X_2} \right) \left(\frac{\partial U_1}{\partial Y} - 2\lambda_2 Y \right) + \lambda Y \left(\frac{\partial U_1}{\partial \tilde{x}} + 2\lambda \frac{\partial A_0}{\partial X_1} \cos\theta \right) \\ & + 2\alpha_0 A_0 c_0 \sin\theta \frac{\partial U_1}{\partial Y} + \lambda V_1 = -\frac{\partial P_1}{\partial \tilde{x}} + \frac{\partial^2 U_1}{\partial Y^2} \end{aligned} \quad (3.11b)$$

$$0 = -\frac{\partial P_1}{\partial Y} \quad (3.11c)$$

Here $\kappa \equiv \lambda^{-1} c_0$ fixes the critical layer position and is an unknown function of X_2 , T_2 . Hence the vorticity equation (2.8) is obtained, from the Y -derivative of (3.11b) with (3.11a), with X^* , ζ^* , Y^* in (2.8) defined by

$$X^* \equiv \theta, \quad \zeta^* \equiv (2\kappa A_0)^{-\frac{1}{2}} (2\lambda_2)^{-1} \frac{\partial U_1}{\partial Y}, \quad Y^* \equiv (2\kappa A_0)^{-\frac{1}{2}} Y, \quad (3.12a)$$

and

$$\mu \equiv \frac{(\partial c_0 / \partial T_2 + c_0 \partial c_0 / \partial X_2)}{2\lambda \alpha_0 c_0 A_0}, \quad \gamma_c \equiv \lambda^{\frac{1}{2}} (2c_0 A_0)^{-3/2} \alpha_0^{-1} \quad (3.12b)$$

in the present case. We need also the next order terms within CL, however, to complete the determination of the disturbance. These are governed by the equation

$$\begin{aligned} & - \left(\frac{\partial \kappa}{\partial T_2} + c_0 \frac{\partial \kappa}{\partial X_2} \right) \frac{\partial \zeta_2}{\partial Y} + \lambda Y \frac{\partial \zeta_2}{\partial x} + 2\alpha_0 A_0 c_0 \sin(\theta) \frac{\partial \zeta_2}{\partial Y} - \frac{\partial^2 \zeta_2}{\partial Y^2} \\ & = - \left[\left(\frac{\partial}{\partial T_1} + c_0 \frac{\partial}{\partial X_1} \right) \frac{\partial U_1}{\partial Y} + c^{(1)} \frac{\partial^2 U_1}{\partial x \partial Y} \right] \end{aligned} \quad (3.12c)$$

for the vorticity contribution $\zeta_2 \equiv \partial U_2 / \partial Y$. In normalized form this becomes

$$\begin{aligned} & \mu \frac{\partial \zeta_2^*}{\partial Y^*} + Y^* \frac{\partial \zeta_2^*}{\partial X^*} + \sin(X^*) \frac{\partial \zeta_2^*}{\partial Y^*} - \gamma_c \frac{\partial^2 \zeta_2^*}{\partial Y^{*2}} \\ & = - \frac{1}{\mu_2} \left[\gamma_2 \left(\frac{\partial}{\partial T_1} + c_0 \frac{\partial}{\partial X_1} \right) + \frac{\partial}{\partial X^*} \right] (\mu_2 \zeta^*) \end{aligned} \quad (3.12d)$$

where $\zeta_2 = 2\lambda_2^2 c_0^2 \lambda^{-3} \zeta_2^*$, $\gamma_2 = \lambda^2 \lambda_2^{-1} c_0^{-2} \alpha_0^{-1}$ and $\mu_2 = 2\lambda_2 (2\kappa A_0)^{\frac{1}{2}}$.

The matching condition here is

$$\zeta_2^* \sim O\left(\frac{1}{Y^*}\right), \quad Y^* \rightarrow \pm \infty \quad (3.12e)$$

and the solution of (3.12d,e) is required to fix the secondary phase shift ϕ_2 , where ϕ_2 is the jump in the velocity, i.e. in the integral of ζ_2^* with respect to Y^* , as far as the $\sin(\theta)$ component only is concerned.

The phase shift ϕ is determined by the solution of (2.8) with the asymptotic constants involving an unknown vorticity jump $2(H^+ - H^-)$,

$$\zeta^* \sim Y^* + 2H^{\pm} + O(Y^{*-1}) \quad , \quad (3.13a)$$

followed by integration to give

$$U^* \sim \frac{1}{2}Y^{*2} + 2H^{\pm}Y^* + \cos \theta \ln|Y^*| + U_{\pm}^* \quad (3.13b)$$

where $\zeta^* = \partial U^* / \partial Y^*$. Here ϕ is the $\sin \theta$ component of the velocity jump,

$$\phi = \pi^{-1} \int_0^{2\pi} (U_+^* - U_-^*) \sin \theta \, d\tilde{x} \quad . \quad (3.14)$$

Thus ϕ (and hence ϕ_2) depends on the values of μ , γ_c , which act as quasi-steady parameters in the governing equation (2.8) itself (and in (3.12d)). With $\phi(\mu, \gamma_c)$ and $\phi_2(\mu, \gamma_c)$ determined, the spatial and temporal growth or decay of the amplitude A_0 follows from (3.7) with (3.4c,d), i.e.

$$\begin{aligned} c_0 \frac{\partial}{\partial X_1} (A_0) + \left(\frac{\partial}{\partial T_1} + c_0 \frac{\partial}{\partial X_1} \right) A_0 \\ = -2 \frac{\lambda_2}{\lambda} c_0^3 A_0 [\phi(\mu, \gamma_c) - \phi_s] \end{aligned} \quad (3.15a)$$

where $\phi_s = -\lambda^{5/2} 2^{-3/2} (-\lambda_2)^{-1} c_s^{-3}$ is a negative constant, corresponding to the phase shift in a neutral steady state with quasi-wavespeed $c_0 \rightarrow c_s$, whether such a state is achieved or not. The nonlinearity in (3.15a) is due in particular to the dependence of γ_c on A_0 . Unsteadiness appears explicitly in the LHS of (3.15a) and implicitly through μ in the RHS. In addition to (3.15a), the slower scale

behaviour of $A_0(X_1, T_1, X_2, T_2)$ is governed by

$$\frac{\partial}{\partial X_2} (c_0 A_0) + \left(\frac{\partial}{\partial T_2} + c_0 \frac{\partial}{\partial X_2} \right) A_0 = () \phi_2, \quad (3.15b)$$

from (3.8b), and by the slow scale response of $c_0(X_2, T_2)$, given by

$$\left(\frac{\partial}{\partial T_2} + 2c_0 \frac{\partial}{\partial X_2} \right) c_0 = 0 \quad (3.15c)$$

from (3.4e). In (3.15a), a steady neutral state, if it is attainable, is achieved when the spatial and temporal dependence decay to zero ($\partial/\partial X_1, \partial/\partial T_1, \partial/\partial X_2, \partial/\partial T_2 \rightarrow 0$): then $\mu \rightarrow 0$ and $c_0 \rightarrow c_s$.

(b) Channel flow.

The structure of a flow disturbance with a nonlinear critical layer here involves primarily the long length scale $x = O(\text{Re}^{1/11})$ along the channel and four zones (Z1 to Z4) laterally, the channel being given by $-d \leq y \leq d$ for all x and the basic channel flow by a general symmetric profile $\bar{u} = u_B(y)$ with $\bar{v} = 0$, $\bar{p} + \text{constant} \propto \text{Re}^{-1}x$ in effect. Here $u_B(y) \sim \lambda(y+d) + \lambda_2(y+d)^2$ near the lower wall $y = -d$, with λ positive but λ_2 negative. An example is plane Poiseuille flow.

Zone Z1 occupies the majority of the channel, $-d < y < d$, and the flow solution there has the form

$$\bar{u} = u_B(y) + \epsilon^{14/3} u_0 + \epsilon^{16/3} u_{MF} + \epsilon^{20/3} u_1 + \epsilon^{22/3} u_2 + \dots \quad (3.16a)$$

$$\bar{v} = \epsilon^{17/3} v_0 + \epsilon^{23/3} v_1 + \epsilon^{25/3} v_2 + \dots \quad (3.16b)$$

$$\bar{p} = \epsilon^{20/3} p_0 + \epsilon^{26/3} p_1 + \epsilon^{28/3} p_2 + \dots \quad (3.16c)$$

In addition the multiple scales

$$\frac{\partial}{\partial x} \rightarrow \epsilon \alpha_0 \frac{\partial}{\partial X} + \epsilon^3 \frac{\partial}{\partial X_1} + \epsilon^{11/3} \frac{\partial}{\partial X_2} + \dots \quad (3.17a)$$

$$\frac{\partial}{\partial t} \rightarrow -\epsilon^3 \alpha_0 c_0 \frac{\partial}{\partial X} + \epsilon^5 \frac{\partial}{\partial T_1} + \epsilon^{17/3} \frac{\partial}{\partial T_2} + \dots \quad (3.17b)$$

apply, and now $\epsilon^{11} \equiv \text{Re}^{-1} \ll 1$. In (3.16a), u_{MF} is independent of X . The Navier-Stokes equations then yield the successive solutions

$$u_0 = A_0 u_B(y) C, \quad v_0 = \alpha_0 A_0 u_B(y) S \quad (3.18a-c)$$

$$p_0 = p_{00} - C \alpha_0^2 A_0 \int_{-d}^y u_B^2(y) dy$$

and

$$v_1 = \alpha_0 u_B \int_0^y \frac{\partial p_0}{\partial X} \frac{dy}{u_B^2} - \alpha_0 c_0 A_0 S + \alpha_0 u_B \frac{\partial \hat{g}}{\partial X}$$

$$p_1 = p_{10} - \int_{-d}^y u_B^2 \frac{\partial^2}{\partial X^2} \left\{ \int_0^y \frac{p_0 dy}{u_B^2} \right\} dy + 2\alpha_0^2 c_0 A_0 C \int_{-d}^y u_B dy \quad (3.18d,e)$$

$$- \left\{ \alpha_0^2 \frac{\partial^2 \hat{g}}{\partial X^2} + \alpha_0 \frac{\partial A_0}{\partial X_1} S \right\} \int_{-d}^y u_B^2 dy$$

and

$$v_2 = -\alpha_0 \frac{\partial G}{\partial X} u_B, \quad (3.18f,g)$$

$$p_2 = p_{20} + \left\{ \alpha_0^2 \frac{\partial^2 G}{\partial X^2} - \frac{\partial(\alpha_0 A_0)}{\partial X_2} S \right\} \int_{-d}^y u_B^2 dy$$

where $S \equiv \sin(X)$, $C \equiv \cos(X)$. Here, as in (a), the fundamental wave, in (3.18a-c), is *non-simple* since α_0 , c_0 vary with both X_2 , T_2 . The other unknown functions arising, A_0 , p_{00} , $\hat{g}$, p_{10} , G , p_{20} , depend on the faster scales X_1 , T_1 as well as

X_2 , T_2 and all but A_0 also vary with X . In the solution (3.18a-c) $-A_0$ is the inviscid displacement effect, while (3.18c) reflects the importance of the cross-channel pressure gradient produced by the curvature of that displacement. The time and space scales X_2 , T_2 above, which govern the critical layer movement, are inferred from the general argument of §2, since in the present case (2.3) is $O(\epsilon^2)$. The faster scales X_1 , T_1 , on which the typical disturbance amplitude A_0 then reacts, are induced in response to the traditional logarithmic singularity below due to the curvature λ_1 of the basic flow.

Next, zone Z2 near the lower wall $y = -d$ is a continuation of Z1 but details the position of the critical layer (Z3) . In Z2 , $y = -d + \epsilon^2 \tilde{Y}$ with $\tilde{Y}$ of $O(1)$ and the expansions

$$\begin{aligned} \bar{u} = \epsilon^2 \lambda \tilde{Y} + \lambda_2 \epsilon^4 \tilde{Y}^2 + \epsilon^{14/3} \bar{u}_0 + \epsilon^{16/3} \bar{u}_{MF} \\ + \epsilon^{20/3} \bar{u}_1 + \epsilon^{22/3} \bar{u}_2 + \dots \end{aligned} \quad (3.19a)$$

$$\bar{v} = \epsilon^{23/3} \bar{v}_0 + \epsilon^{29/3} \bar{v}_1 + \epsilon^{31/3} \bar{v}_2 + \dots \quad (3.19b)$$

$$\bar{p} = \epsilon^{20/3} \bar{p}_0 + \epsilon^{26/3} \bar{p}_1 + \epsilon^{28/3} \bar{p}_2 + \dots \quad (3.19c)$$

hold, with $\bar{u}_{MF}$ independent of x . Coupled with (3.17a,b) still, (3.19a-c) yield the solutions

$$\bar{u}_0 = A_0 \lambda C \quad , \quad \bar{v}_0 = \lambda \tilde{Y} A_0 \alpha_0 S \quad , \quad \partial \bar{p}_0 / \partial X = -c_0 A_0 \lambda S \quad (3.20a)$$

at leading order. These match with the forms in Z1 , and give tangential flow at the wall ($\tilde{Y} \rightarrow 0+$) , as required. Next we obtain

$$\begin{aligned} \bar{v}_1 = & -\frac{1}{\lambda} \left(\alpha_0 \frac{\partial \bar{p}_1}{\partial X} + \frac{\partial \bar{p}_0}{\partial X_1} + \frac{\partial \bar{u}_0}{\partial T_1} + c_0 \frac{\partial \bar{u}_0}{\partial X_1} \right) \\ & - 2\lambda_2 \frac{\alpha_0 A_0 S}{\lambda^2} \left(\frac{c_0^2}{2} - c_0 \xi \ln \xi - \frac{\xi^2}{2} \right) \\ & - \xi \alpha_0 \frac{\partial \hat{a}}{\partial X} \end{aligned} \quad (3.20b)$$

followed by

$$\begin{aligned} \bar{v}_2 = & -\frac{1}{\lambda} \left(\alpha_0 \frac{\partial \bar{p}_2}{\partial X} + \frac{\partial \bar{p}_0}{\partial X_2} + \frac{\partial \bar{u}_0}{\partial T_2} + c_0 \frac{\partial \bar{u}_0}{\partial X_2} + \alpha_0 \bar{u}_0 \frac{\partial \bar{u}_0}{\partial X} \right) \\ & - \xi \alpha_0 \frac{\partial B}{\partial X} \end{aligned} \quad (3.20c)$$

again from the Navier-Stokes equations. Here $\xi \equiv \lambda \tilde{Y} - c_0$, $\bar{p}_0$, $\bar{p}_1$, $\bar{p}_2$ are independent of $\tilde{Y}$ and $\hat{a}$, B are further unknown displacement functions of X , X_1 , T_1 , X_2 , T_2 , ... Beneath the critical layer, for $\xi < 0$, the replacement (3.5c) deals with the logarithmic contributions in (3.20b), with ϕ again unknown and S , C replacing $\sin \theta$, $\cos \theta$ now. There are also, as in (a), non-monochromatic jumps in the functions $\hat{a}$, B , due to $Z3$, but not involving any C contributions in $\hat{a}$.

Before addressing the nonlinear critical layer $Z3$, we note that the $O(\epsilon^4)$ thick viscous wall layer, $Z4$, remains of classical linear form and so produces the viscous displacement condition

$$\bar{v}(\tilde{Y} \rightarrow 0+) = \epsilon^{29/3} \left[-\lambda A_0 \left(\frac{\alpha_0}{c_0} \right)^{1/2} \sin(X + \frac{\pi}{4}) + O(\epsilon^2) \right] \quad (3.21a)$$

Therefore the join between $Z2$, $Z4$ requires the two relations

$$\begin{aligned}
 & - \frac{1}{\lambda} (\alpha_0 \tilde{p}_{11} + \frac{\partial \tilde{p}_{00}}{\partial X_1}) - (\frac{\partial A_0}{\partial T_1} + c_0 \frac{\partial A_0}{\partial X_1}) + \alpha_0 c_0 [\tilde{a}_1 - 2\lambda_2 \frac{c_0 A_0}{\lambda^2} \phi] \\
 & = - \lambda A_0 (\frac{\alpha_0}{2c_0})^{\frac{1}{2}}
 \end{aligned}$$

$$- \frac{1}{\lambda} (\alpha_0 \tilde{p}_{21} + \frac{\partial \tilde{p}_{00}}{\partial X_2}) - (\frac{\partial A_0}{\partial T_2} + c_0 \frac{\partial A_0}{\partial X_2}) + \alpha_0 c_0 [\tilde{B}_1 - \phi_2] = 0 \quad (3.21b,c)$$

to be satisfied, as regards the all-important C contributions. In the above $\tilde{p}_{00}$, $\tilde{p}_{11}$, are the C components of $\bar{p}_0$, $\bar{p}_1$, respectively and $\tilde{p}_{21}$, $\tilde{a}_1$, $\tilde{B}_1$ are the S components of the functions $\bar{p}_2$, $\hat{a}$, B above Z3; in fact $\bar{p}_0 = \tilde{p}_{00}C$ and the x-momentum balance in Z2 gives the relation

$$(C^{-1}p_{00} =) \tilde{p}_{00} = \lambda c_0 A_0 \quad (3.21d)$$

Below the critical layer Z3, $\tilde{a}_1$ is unchanged but $\tilde{B}_1$ suffers a jump, given by ϕ_2 , to be determined from the critical layer properties.

Along with (3.21b-d), the relations

$$\tilde{p}_{00} = \alpha_0^2 A_0 I_1 \quad (3.22a)$$

$$\tilde{p}_{11} = (\alpha_0^2 \tilde{a}_1 + \alpha_0 \frac{\partial A_0}{\partial X_1}) I_1 \quad (3.22b)$$

$$\tilde{p}_{21} = (\alpha_0^2 \tilde{B}_1 + \frac{\partial (\alpha_0 A_0)}{\partial X_2}) I_1 \quad (3.22c)$$

also apply, from the merging between Z2 and Z1, where

$$I_1 \equiv \int_{-d}^0 u_B^2(y) dy \quad (3.22d)$$

is a given positive constant. The required condition of anti-symmetry about the channel centre-line $y = 0$ has been used in Z1, for the disturbance pressure. Again, compatibility ($\partial^2/\partial x \partial t = \partial^2/\partial t \partial x$) requires that $\partial \alpha_0 / \partial T_2 + \partial(\alpha_0 c_0) / \partial X_2 = 0$. So, combining this with (3.21d), (3.22a), we obtain the results

$$c_0 = \frac{\alpha_0^2 \Gamma_1}{\lambda} \quad \text{and} \quad \frac{\partial \alpha_0}{\partial T_2} + 3c_0 \frac{\partial \alpha_0}{\partial X_2} = 0 \quad (3.22e)$$

affecting the critical layer movement. The group velocity

$c_g = 3c_0$ is *thrice* the local quasi-wavespeed.

Addressing finally the nonlinear unsteady viscous critical layer Z3, wherein $y = -d + \epsilon^2 \lambda^{-1} c_0 + \epsilon^{10/3} Y$ with Y of order unity, we have there the multiple scalings (3.17a,b) but supplemented by

$$\frac{\partial}{\partial X_2} \rightarrow -\epsilon^{-4/3} \lambda^{-1} \frac{\partial c_0}{\partial X_2} \frac{\partial}{\partial Y} + \frac{\partial}{\partial X_2} \quad (3.23a)$$

$$\frac{\partial}{\partial T_2} \rightarrow -\epsilon^{-4/3} \lambda^{-1} \frac{\partial c_0}{\partial T_2} \frac{\partial}{\partial Y} + \frac{\partial}{\partial T_2}, \quad (3.23b)$$

where once again the dependence on critical layer movement is dominant.

The flow solution in Z3 has the form

$$\begin{aligned} \bar{u} = & \epsilon^2 c_0 + \epsilon^{10/3} \lambda Y + \lambda_2 \lambda^{-2} c_0^2 \epsilon^4 + \epsilon^{14/3} \tilde{u}_0 \\ & + \epsilon^{16/3} (\tilde{u}_1 + 2\lambda_2 \lambda^{-1} c_0 Y) + \epsilon^{20/3} (\tilde{u}_1 + \lambda_2 Y^2) \\ & + \epsilon^{22/3} \tilde{u}_2 + \dots \end{aligned} \quad (3.24a)$$

$$\begin{aligned} \bar{v} = & \epsilon^{23/3} \tilde{v}_{-\frac{1}{2}} + \epsilon^{27/3} \tilde{v}_0 + \epsilon^{29/3} \tilde{v}_{\frac{1}{2}} + \epsilon^{31/3} \tilde{v}_{31} \\ & + \epsilon^{33/3} \tilde{v}_1 + \epsilon^{35/3} \tilde{v}_2 + \dots \end{aligned} \quad (3.24b)$$

$$\bar{p} = \epsilon^{20/3} \tilde{p}_0 + \epsilon^{26/3} \tilde{p}_1 + \epsilon^{28/3} \tilde{p}_2 + \dots \quad (3.24c)$$

where $\tilde{v}_{-\frac{1}{2}} = -\alpha_0 \lambda^{-1} \partial \tilde{p}_0 / \partial X$.

Substitution into the Navier-Stokes equations provides the successive governing equations, of continuity and x-momentum,

$$\begin{aligned} \alpha_0 \frac{\partial \tilde{u}_0}{\partial X} + \frac{\partial \tilde{v}_0}{\partial Y} &= \alpha_0 \frac{\partial \tilde{u}_1}{\partial X} + \frac{\partial \tilde{u}_0}{\partial X_1} + \frac{\partial \tilde{v}_1}{\partial Y} \\ &= \alpha_0 \frac{\partial \tilde{u}_2}{\partial X} + \frac{\partial \tilde{u}_1}{\partial X_1} + \frac{\partial \tilde{u}_0}{\partial X_2} + \frac{\partial \tilde{v}_2}{\partial Y} = 0 \end{aligned} \quad (3.25a-c)$$

and

$$\mathcal{I}(\tilde{u}_0, \tilde{v}_0) = 0 \quad ;$$

$$\begin{aligned} \mathcal{I}(\tilde{u}_{\frac{1}{2}}, \tilde{v}_{\frac{1}{2}}) + \left(\frac{\partial \tilde{u}_0}{\partial T_1} + c_0 \frac{\partial \tilde{u}_0}{\partial X_1} \right) - \frac{\alpha_0 \tilde{g} c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_0}{\partial X} - \frac{\tilde{v}_{-\frac{1}{2}} \tilde{g} c_0}{\lambda} \\ = -\alpha_0 \frac{\partial \tilde{p}_1}{\partial X} - \frac{\partial \tilde{p}_0}{\partial X_1} \quad ; \end{aligned}$$

$$\begin{aligned} \mathcal{I}(0, \tilde{v}_{31}) + \left(\frac{\partial \tilde{u}_0}{\partial T_2} + c_0 \frac{\partial \tilde{u}_0}{\partial X_2} \right) - \frac{\alpha_0 \tilde{g} c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial X} \\ + \left(\alpha_0 u_0 \frac{\partial u_0}{\partial X} + v_0 \frac{\partial u_0}{\partial Y} \right) = -\alpha_0 \frac{\partial \tilde{p}_2}{\partial X} - \frac{\partial \tilde{p}_0}{\partial X_2} \quad ; \end{aligned}$$

$$\begin{aligned} \mathcal{I}(\tilde{u}_1, \tilde{v}_1) + \left(\frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial T_2} + c_0 \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial X_2} \right) + \lambda Y \frac{\partial u_0}{\partial X_1} - Y \frac{\partial c_0}{\partial X_2} \frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial Y} \\ + \alpha_0 \left(\tilde{u}_{\frac{1}{2}} - \frac{\tilde{g} c_0 Y}{\lambda} \right) \frac{\partial \tilde{u}_0}{\partial X} - \tilde{g} Y \tilde{v}_{-\frac{1}{2}} + \tilde{v}_0 \left(\frac{\partial \tilde{u}_{\frac{1}{2}}}{\partial Y} - \frac{\tilde{g} c_0}{\lambda} \right) + \tilde{v}_{\frac{1}{2}} \frac{\partial \tilde{u}_0}{\partial Y} \end{aligned}$$

$$= -\alpha_0 \frac{\partial \tilde{p}_3}{\partial X} ;$$

$$\begin{aligned} \mathcal{L}(\tilde{u}_2, \tilde{v}_2) + \left(\frac{\partial \tilde{u}_1}{\partial T_1} + c_0 \frac{\partial \tilde{u}_1}{\partial X_1} \right) - \frac{\tilde{g}c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_0}{\partial X_1} + \lambda Y \left(\frac{\partial \tilde{u}_0}{\partial X_2} + \frac{\partial \tilde{u}_1}{\partial X_1} \right) \\ - \frac{\alpha_0 \tilde{g}c_0^2}{2\lambda^2} \frac{\partial \tilde{u}_1}{\partial X} - \tilde{v}_{\frac{1}{2}} \frac{\tilde{g}c_0}{\lambda} = -\alpha_0 \frac{\partial \tilde{p}_4}{\partial X} - \frac{\partial \tilde{p}_1}{\partial X_1} ; \end{aligned} \quad (3.25d-h)$$

where the operator $\mathcal{L}$ is defined by

$$\begin{aligned} \mathcal{L}(u, v) \equiv -\frac{1}{\lambda} \left(\frac{\partial c_0}{\partial T_2} + c_0 \frac{\partial c_0}{\partial X_2} \right) \frac{\partial u}{\partial Y} + \lambda Y \alpha_0 \frac{\partial u}{\partial X} \\ + \tilde{v}_{-\frac{1}{2}} \frac{\partial u}{\partial Y} + \lambda v - \frac{\partial^2 u}{\partial Y^2} , \end{aligned} \quad (3.26)$$

and $\tilde{g} = -2\lambda_2$. The y-momentum balances simply give $\partial \tilde{p}_n / \partial Y = 0$ for $0 \leq n \leq 4$. Hence we obtain

$$\begin{aligned} \tilde{v}_{-\frac{1}{2}} = \alpha_0 c_0 A_0 S , \quad \tilde{u}_0 = A_0 \lambda C , \quad \tilde{v}_0 = \lambda Y A_0 \alpha_0 S , \\ \lambda \tilde{v}_{\frac{1}{2}} = -\left(\alpha_0 \frac{\partial \tilde{p}_1}{\partial X} + \frac{\partial \tilde{p}_0}{\partial X_1} + \frac{\partial \tilde{u}_0}{\partial T_1} + c_0 \frac{\partial \tilde{u}_0}{\partial X_1} \right) \\ + \frac{1}{2} \lambda^{-1} \tilde{g} c_0^2 A_0 \alpha_0 S , \quad (3.27) \\ \lambda \tilde{v}_{\frac{3}{2}} = -\left(\alpha_0 \frac{\partial \tilde{p}_2}{\partial X} + \frac{\partial \tilde{p}_0}{\partial X_2} + \frac{\partial \tilde{u}_0}{\partial T_2} + c_0 \frac{\partial \tilde{u}_0}{\partial X_2} \right) \\ - \alpha_0 \tilde{u}_0 \frac{\partial \tilde{u}_0}{\partial X} , \end{aligned}$$

and $\tilde{u}_1$ is independent of X, Y . The first non-simple problem in (3.25) is that for $\tilde{u}_1$, therefore, and takes exactly the form in (2.8), after the transformations

$$\tilde{u}_1 = - \frac{\tilde{g}^c_0 A_0}{\lambda} U^* \quad , \quad Y = \left(\frac{A_0 c_0}{\lambda} \right)^{\frac{1}{2}} Y^* \quad ,$$

$$\frac{\partial \tilde{u}_1}{\partial Y} - \tilde{g} Y = - \tilde{g} \left(\frac{A_0 c_0}{\lambda} \right)^{\frac{1}{2}} \zeta^* \quad , \quad (3.28a-c)$$

$$X = X^*$$

are introduced in the Y-derivative of (3.25g). Here

$$\mu = \left(\frac{\partial c_0}{\partial T_2} + c_0 \frac{\partial c_0}{\partial X_2} \right) / (\lambda A_0 \alpha_0 c_0) \quad , \quad (3.28d)$$

and the farfield conditions (3.13a,b) are retrieved. The second non-simple problem is for $\tilde{u}_2$. In turn, it acquires the form (3.12d),

where now

$$\zeta_2^* \equiv \frac{\tilde{g}^{-2} c_0^{-2} \lambda^3}{2} \left(\frac{\partial \tilde{u}_2}{\partial Y} \right) , \quad \gamma_2 \equiv - \frac{\lambda^2 \tilde{g}^{-1} c_0^{-2} \alpha_0^{-1}}{2} , \quad \left(\begin{array}{l} \mu_2 = - \tilde{g} \left(\frac{2 c_0 A_0}{\lambda} \right)^{\frac{1}{2}} \\ (3.28e) \end{array} \right.$$

and the conditions (3.12e) again apply here. So ζ^* , ζ_2^* are responsible for the phase shifts ϕ , ϕ_2 , respectively, across Z3, which appear in (3.21b,c).

In summary, the unsteady nonlinear disturbance amplitude A_0 and the moving critical layer position $(\lambda^{-1} c_0)$ are controlled by the equations

$$\frac{\partial A_0}{\partial T_1} + 3c_0 \frac{\partial A_0}{\partial X_1} = \sigma A_0 (\phi - \phi_s) \quad (3.29a)$$

$$\frac{\partial A_0}{\partial T_2} + 3c_0 \frac{\partial A_0}{\partial X_2} + \frac{3}{2} A_0 \frac{\partial c_0}{\partial X_2} = -\alpha_0 c_0 \phi_2 \quad (3.29b)$$

$$\frac{\partial c_0}{\partial T_2} + 3c_0 \frac{\partial c_0}{\partial X_2} = 0 \quad (3.29c)$$

where now

$$\sigma \equiv -2\lambda_2 \frac{c_0^2 \alpha_0}{\lambda^2} > 0, \quad \phi_s \equiv \frac{-\lambda \alpha_0}{(2\alpha_0 c_0)^{\frac{1}{2}} \sigma} < 0, \quad (3.30)$$

with α_0 given by (3.22e). Further comments are covered by those at the end of (a).

(c) Liquid layer flow

We consider liquid layer flow over an inclined plane and take the plane to be inclined at an angle ω^* to the horizontal, with $\omega^* = O(R^{-9/11})$, and $s = 1/R^{9/11} \tan \omega^*$ is $O(1)$. The disturbance structure then takes exactly the same form as in (b) above, the only difference being that the free surface conditions lead to the eigenrelation

$$\lambda c_0 = (\alpha_0^2 I_1 - s) \quad (3.22e')$$

instead of (3.22e). (See Gajjar (1983) for further details on the neutral case, i.e. without a moving critical layer). The fundamental problem (2.8) and the amplitude equations (3.29) remain unaltered.

For steeper planes ($\omega^* \gg R^{-9/11}$), $s = 0$, effectively, in (3.22e').

§4. THE AMPLITUDE EQUATION FOR THE SPECIAL CASE $\mu = 0$

If we consider fixed frequency disturbances, with $\beta_0 = \text{const}$, then as mentioned previously, the eigerelations (3.4e) (3.22d) imply that both c_0 , α_0 are constant and hence $\mu = 0$. The phase shift ϕ occurring in the amplitude equations is then determined by solving the steady Haberman problem, the properties of which are well known: $\phi(A)$ is monotonic, with $\phi \rightarrow -\pi$ as $A \rightarrow 0$ and $\phi \propto -A^{-3/2}$ for $A \gg 1$ (see e.g. Haberman (1972), Smith & Bodonyi (1982)). These properties come mainly from a numerical treatment. Qualitatively, however, they may be represented by a model with $\phi(A)$ given by

$$\phi(A) = \frac{-\pi}{1 + k_0 A^{3/2}} \quad , \quad (4.1)$$

where k_0 is a positive constant. If we consider A as a function of T_1 , and $Z = X_1 - T_1 c_g$, where c_g is the group velocity, then (3.15a), (3.27a) with (4.1) take the form,

$$\frac{\partial A}{\partial T_1} = \sigma A \left[\frac{-\pi}{1 + k_0 A^{3/2}} - \phi_s \right] \quad . \quad (4.2)$$

This integrates to give the implicit solution

$$\frac{|A|}{|\phi_s + \pi + \phi_s k_0 A^{3/2}|^{-\frac{2}{3}} \cdot \frac{\pi}{\phi_s}} = \exp\{-\sigma T_1 (\phi_s + \pi)\} M(z) \quad (4.3)$$

where $M(z)$ (> 0) is an arbitrary function of integration and $\phi_s = -\pi$ corresponds to the linear neutral modes, α_{0_n} , c_{0_n} .

If $\phi_s < -\pi$, i.e. $\alpha_0 < \alpha_{0n}$, then $c_0 < c_{0n}$, and the initially small disturbance lies inside the linear neutral curve. The non-linear solution is sketched in fig. (1a). It has the behaviours as $T_1 \rightarrow \pm \infty$,

$$A \sim \exp(-\sigma T_1 (\phi_s + \pi)) |\phi_s + \pi|^{-\frac{2\pi}{3\phi_s}} \quad \text{as} \quad T_1 \rightarrow -\infty \quad (4.4a)$$

governing its initial response, and

$$A \sim \exp(-\sigma T_1 \phi_s) \quad \text{as} \quad T_1 \rightarrow +\infty, \quad (4.4b)$$

which describes its ultimate form. Here (4.4a,b) imply that a mode which initially is small and linearly unstable will amplify, become nonlinear and then become unbounded at larger times (as $T_1 \rightarrow \infty$).

On the other hand if $\phi_s > -\pi$, corresponding to the linearly stable modes, there is an unstable equilibrium amplitude A_s , given by $\phi_s = \phi(A_s)$ as $T_1 \rightarrow -\infty$: see fig. (1b). If the amplitude A is below A_s then the disturbance amplitude decays to zero as time T_1 increases, whereas if A is above A_s , then A becomes unbounded, attaining the behaviour (4.4b).

The above conclusions happen to agree (qualitatively only) with those from the weakly nonlinear stability theory of Stuart-Watson (1960), for smaller disturbances.

Finally it is noted again that the above results are for the special case $\mu = 0$ only. For a moving critical layer $\mu \neq 0$, and the phase shift ϕ is related in a much more complicated manner to A , and not necessarily monotonically as in the model problem above. Even a heuristic discussion of the amplitude equations and

the implications for the neutral stability curve would first require an explicit determination of the functional relation of ϕ on A_0 , and this needs a full numerical solution of the problem (2.8).

We turn next to some limiting cases of (2.8), therefore.

§5. PRELIMINARY PROPERTIES OF THE UNSTEADY NONLINEAR CRITICAL LAYER EQUATION

The properties of the nonlinear critical layer equation when dominated by viscosity, nonlinearity, or unsteadiness are dictated by the various limits of μ , γ_c , large or small, and we devote this section mostly to some preliminaries on these limiting solutions.

First however an interesting general property concerning the mean vorticity jump and the phase shift ϕ is noted. Integrating (2.8) with respect to Y^* we obtain

$$\begin{aligned} \mu(Y^* - \zeta^*) + Y^* U_{X^*}^* - \psi_{X^*}^* + \sin X^* U_{Y^*}^* \\ = - P_{X^*}^* + \gamma_c U_{Y^* Y^*}^* \end{aligned} \quad (5.1)$$

with the boundary conditions, for $Y^* \rightarrow \pm \infty$,

$$\begin{aligned} \psi^* &\sim \frac{1}{6} Y^{*3} + H^{\pm} Y^{*2} + \cos X^* (Y^* \ln |Y^*| - Y^*) \\ &\quad + B^{\pm}(X) Y^* + G^{\pm}(X) + O\left(\frac{1}{Y^*}\right) \\ U^* &\sim \frac{1}{2} Y^{*2} + 2H^{\pm} Y^* + \cos X^* \ln |Y^*| + B^{\pm}(X) + O\left(\frac{1}{Y^*}\right) \\ \zeta^* &\sim Y^* + 2H^{\pm} + \frac{\cos X^*}{Y^*} + O\left(\frac{1}{Y^3}\right) \end{aligned} \quad (5.2a, b, c)$$

Integration of (5.1) with respect to X^* , from $X^* = 0$ to $X^* = 2\pi$, and application of the boundary conditions therefore gives

$$-4\mu\pi(H^+ - H^-) = 0 \quad (5.3)$$

A further integration of (5.1) with respect to Y^* and then with respect to X^* over a period yields

$$\begin{aligned} -\mu \int_0^{2\pi} (B^+ - B^-) dX + \int_0^{2\pi} (B^+ - B^-) \sin X dX \\ = 4\pi\gamma_c (H^+ - H^-) \end{aligned} \quad (5.4a)$$

after use again of (5.2a,b,c). Here (5.3) is a remarkable result, showing that, for any non-zero μ , there is no vorticity jump across the critical layer. This is in contrast with the steady equation where the mean vorticity jump is nonzero and is directly related to the phase shift ϕ , as can be seen from (5.4) if $\mu = 0$. Instead, however, (5.4) determines the mean velocity jump in terms of the phase shift ϕ . The above result also demonstrates that, concerning the mean vorticity jump, a direct match with the steady solution is not possible. Nevertheless a connection with the steady solutions may be effected in all other respects. For μ small, with Y^* of $O(1)$, to leading order the steady Haberman equation holds. For large Y^* , this gives $\zeta^* \sim Y^* + 2H^+ + O(\frac{1}{Y^*})$, therefore, with $H^+ \neq H^-$. Hence writing

$$\zeta^* = Y^* + \tilde{\zeta}_m(Y^*) + O(\frac{1}{Y^*}) \quad (5.5)$$

we have that $\tilde{\zeta}_m$ satisfies

$$-\mu \tilde{\zeta}_{mY^*} = \tilde{\zeta}_{mY^*Y^*} \quad (5.6)$$

after substitution into (2.8) and integrating with respect to X^* from $X^* = 0$ to $X^* = 2\pi$. Here (5.6) yields

$$\tilde{\zeta}_m = h_0 + h_1 e^{-\mu Y^*} \quad (5.7)$$

If $\mu > 0$, the boundary conditions give $h_0 = 2H^-$, $2H^+ = h_0 + h_1$, and if $\mu < 0$ $h_0 = 2H^+$, $2H^- = h_0 + h_1$. Hence an outer region wherein $Y^* \sim O(\frac{1}{\mu})$ is necessary, for μ small, to adjust the vorticity jump back to zero. This shows that the critical layer unsteadiness spreads out like $Y^* \sim \mu^{-1}$ *above* the critical layer if $\mu > 0$, but *below* it if $\mu < 0$. Formally for boundary layer flow, if μ becomes as small as $O(\epsilon^{2/3})$ then the critical layer width $\epsilon^{23/3} Y^* \rightarrow O(\epsilon^7)$ and the unsteadiness then enters the outer inviscid zone $\mathcal{I}Z$. Also (3.12b), (3.10a,b) show that the slow variations now become $O(\epsilon^{-5}(\epsilon^2 X_2))$, $O(\epsilon^{-4}(\epsilon^2 T_2))$ in space and time respectively. Similar comments apply to channel flow and liquid layer flow, where if $\mu \rightarrow O(\epsilon^{4/3})$ the X_2 , T_2 variations increase by a factor $\epsilon^{4/3}$, and the unsteadiness again spreads out into the outer zone.

We turn next to the limiting cases of (2.8). A balance of the inertial and viscous terms suggests that viscous or unsteady effects are significant when either $\gamma_c \gg 1$ and $Y^* \sim O(\gamma_c^{1/3})$, or $\mu \gg 1$ and $Y^* \sim O(\mu^{1/2})$ respectively. By contrast, if μ and γ_c are both small then nonlinearity is important. We consider the different limiting regimes in turn below, but only briefly for now since it is felt that subsequent numerical solutions of (2.8) would probably provide firmer evidence of the solution characteristics emerging in such limits.

i) The linear viscous unsteady critical layer, when $\gamma_c \gg 1$, $\mu \gg 1$ with $\mu = O(\gamma_c^{2/3})$.

Let $Y^* = \gamma_c^{1/3} \eta$ and $\mu = \gamma_c^{2/3} \bar{\mu}$ where η , $\bar{\mu}$ are $O(1)$.

The boundary conditions (5.2a,b,c) and (2.8) suggest an expansion of the form

$$\begin{aligned}
 U^* = & \gamma_c^{2/3} \frac{\eta^2}{2} + 2\gamma_c^{1/3} H^{\pm} \cdot \eta + \frac{\cos X}{3} \ln \gamma_c + u_1 \\
 & + \gamma_c^{-2/3} u_2 + \gamma_c^{-5/3} u_3 + \dots
 \end{aligned}
 \tag{5.8}$$

$$\zeta^* = \gamma_c^{1/3} \eta + \gamma_c^{1/3} 2H^{\pm} + \gamma_c^{-1/3} \zeta_1 + \gamma_c^{-1} \zeta_2 + \dots$$

Substitution in (2.8) gives the sequence of equations:

$$-\bar{\mu} \zeta_{1\eta} - \zeta_{1\eta\eta} + \eta \zeta_{1X^*} = -\sin X^* \tag{5.9a,b}$$

$$-\bar{\mu} \zeta_{k,\eta} - \zeta_{k,\eta\eta} + \eta \zeta_{k,X^*} = -\sin X^* \zeta_{k-1,\eta} \quad (k=2,3,\dots)$$

Here if $\bar{\mu}$ is zero the equations are identical with those discussed by Haberman and can be solved similarly. For nonzero $\bar{\mu}$ analogous results for integral representations of the solutions of (5.9) are stated in the appendix.

Making use of the results in the appendix, we find

$$\zeta_1 = -\text{Im}(e^{iX^*} \int_0^\infty e^{-i\eta t} e^{-(t^3/3 + i \frac{t^2 \bar{\mu}}{2})} dt)$$

(5.10)

and

$$\begin{aligned}
 U_1 = & -\text{Im}(e^{iX^*} \int_0^\infty \frac{e^{-i\eta t} - 1}{-it} e^{-(t^3/3 + i \frac{t^2 \bar{\mu}}{2})} dt) \\
 & + f_1(X^*)
 \end{aligned}$$

where $U_1 = f_1$ at $\eta = 0$. The asymptotic expansion of U_1 for large η now gives

$$\begin{aligned}
 U_1 = & -\text{Im}(e^{iX^*} [-i \log |\eta| + \frac{\pi}{2} \text{sgn}(\eta) \\
 & + i \int_0^\infty \frac{\cos t - e^{-(t^3/3 + i \frac{t^2 \bar{\mu}}{2})}}{t} dt + O(\frac{1}{\eta})] + f_1(X))
 \end{aligned}$$

as $\eta \rightarrow \pm \infty$ (5.11)

where $\text{sgn}(\eta) = \pm 1$ for $\eta \gtrless 0$. In particular (5.11), determines the leading order velocity jump across the critical layer:

$$U_1^+ - U_1^- = -\text{Im}(e^{iX}\pi) \quad (5.12)$$

The solution for ζ_2 is given by

$$\begin{aligned} \zeta_2 = & -\frac{\text{Im}}{2} \left(\int_0^\infty \frac{ie^{-it\eta} e^{-(t^3/3+i\bar{\mu}t^2/2)}}{(\bar{\mu}-it)} dt \right) \\ & - \frac{\text{Im}}{4} \left[e^{2iX^*} \int_0^\infty e^{-int} e^{-(t^3/6+i\bar{\mu}t^2/4)} \right. \\ & \left. \cdot \int_0^t \tau e^{-(\tau^3/6+i\bar{\mu}\tau^2/4)} d\tau dt \right] . \end{aligned} \quad (5.13)$$

For $\bar{\mu} \neq 0$ (5.13) shows that there is no vorticity jump, as required by (5.3), but there is a jump in the mean velocity.

Similarly ζ_3 can be determined, and after some tedious but straightforward algebra we find that

$$\begin{aligned} \zeta_3 = & \text{Im}(e^{iX^*} \{ \frac{1}{4} \int_0^\infty e^{-int} e^{-(t^3/3+i\bar{\mu}t^2/2)} dt \\ & + \frac{i\bar{\mu}}{4} \log(i\bar{\mu}) \int_0^\infty e^{-int} e^{-(t^3/3+i\bar{\mu}t^2/2)} dt \\ & + \frac{1}{8} \int_0^\infty e^{-int} e^{-(t^3/3+i\bar{\mu}t^2/2)} \int_0^t \tau e^{-(\tau^3/6+i\bar{\mu}\tau^2/4)} \\ & \cdot \int_0^\tau T e^{-(T^3/6+i\bar{\mu}T^2/4)} dTd\tau dt \} - \frac{\text{Im}}{24} (e^{3iX^*} \int_0^\infty e^{-int} \times \\ & \times e^{-(t^3/9+i\bar{\mu}t^2/6)} \int_0^t \tau e^{-(\tau^3/18+i\bar{\mu}\tau^2/12)} \\ & \cdot \int_0^\tau T e^{-(T^3/6+i\bar{\mu}T^2/4)} dTd\tau dt) \end{aligned}$$

$$\begin{aligned}
 & - \frac{\text{Im}}{4} (e^{-iX} \{ \int_{-\infty}^0 e^{-int} e^{t^3/6+i\bar{\mu}t^2/2} dt \times \\
 & \times \int_0^{\infty} \frac{t}{(t+i\bar{\mu})} e^{-2(t^3/3+i\bar{\mu}t^2/2)} dt \\
 & - \int_0^{\infty} e^{-int} e^{t^3/3+i\bar{\mu}t^2/2} \int_{-\infty}^t \frac{\tau}{\tau+i\bar{\mu}} \times \\
 & \times e^{-2(\tau^3/3+i\bar{\mu}\tau^2/2)} d\tau dt \} \quad . \quad (5.14)
 \end{aligned}$$

Then U_3 can be obtained by integration and an asymptotic expansion for large η yields a correction for the phase shift ϕ . Hence collecting the above results together we find that

$$\begin{aligned}
 \phi = & -\pi + \gamma_c^{-4/3} \frac{\pi}{2} \left[\int_0^{\infty} \frac{t^2 \cos(\bar{\mu}t^2) - \bar{\mu}t \sin(\bar{\mu}t^2)}{t^2 + \bar{\mu}^2} e^{-2t^3/3} dt - \frac{\pi\bar{\mu}}{4} \right] \\
 & + \dots \quad . \quad (5.15)
 \end{aligned}$$

As $\bar{\mu} \rightarrow 0$, with η fixed, the above expressions reduce to those given by Haberman (1976), although as discussed above for any non-zero $\bar{\mu}$, (2.8), gives no vorticity jump, whereas the Haberman equation entails a discontinuity in the vorticity across the critical layer. This apparent inconsistency however, can be identified with a closer examination of the mean flow term in (5.13) which when rewritten is

$$\zeta_2^m = -\frac{1}{2} \int_0^{\infty} \frac{[\bar{\mu} \cos(t\eta + \bar{\mu}t^2/2) + t \sin(t\eta + \bar{\mu}t^2/2)]}{\bar{\mu}^2 + t^2} e^{-t^3/3} dt \quad . \quad (5.16)$$

For small $\bar{\mu}$

$$\zeta_2^m \sim -\frac{1}{2} \left[\int_0^{\infty} \frac{\cos(\bar{\mu}t\eta)}{1+t^2} dt - \frac{1}{\bar{\mu}} \frac{\partial}{\partial \eta} \int_0^{\infty} \frac{\cos(\bar{\mu}t\eta)}{1+t^2} dt \right] + \dots \quad .$$

Now

$$\int_0^{\infty} \frac{\cos \sigma t}{1+t^2} dt = \frac{\pi}{2} e^{-|\sigma|} ;$$

hence

$$\zeta_2^m \sim -\frac{\pi}{4} [e^{-|\bar{\mu}\eta|} - \text{sgn}(\eta)e^{-|\bar{\mu}\eta|}] \quad . \quad (5.17)$$

For $\eta \gg \frac{1}{\bar{\mu}}$ we recover the zero vorticity jump and for η large, but $\bar{\mu}\eta \ll 1$, $\zeta_2^m \sim -\frac{\pi}{4} [1 - \text{sgn}(\eta)]$, giving a jump of $\frac{\pi}{2} \gamma_c^{-1}$ in the vorticity, as in the steady case. Thus for μ small an outer region where $\eta \sim O(\frac{1}{\bar{\mu}})$ is necessary to adjust the vorticity jump to zero, and this agrees with the earlier comments concerning the emergence of the steady solution as $\mu \rightarrow 0$.

ii) The linear inviscid unsteady critical layer, i.e. $\mu \gg 1$, $\mu \gg \gamma_c^{2/3}$.

Let $Y^* = \mu^{1/2} \hat{\eta}$, and $\gamma_c = \bar{\gamma}_c \mu^{3/2}$ but $\bar{\gamma}_c \ll 1$. Again (2.8) (5.2a,b,c) indicates expansions of the form

$$\begin{aligned} U^* &\sim \mu \hat{U}_0 + \mu^{\frac{1}{2}} \hat{U}_1 + \mu^{-\frac{1}{2}} \hat{U}_2 + \mu^{-\frac{1}{2}} \frac{\ln \mu}{2} \cos X^* + \dots \\ \zeta^* &\sim \mu^{\frac{1}{2}} \hat{\zeta}_0 + \hat{\zeta}_1 + \mu^{-\frac{1}{2}} \hat{\zeta}_2 + \dots \end{aligned} \quad (5.18)$$

Substitution into (2.8) yields the equations

$$\begin{aligned} 1 - \hat{\zeta}_{0\tilde{\eta}} + \tilde{\eta} \hat{\zeta}_{0X^*} - \bar{\gamma}_c \hat{\zeta}_{0\tilde{\eta}\tilde{\eta}} &= 0 \\ -\hat{\zeta}_{1\tilde{\eta}} + \tilde{\eta} \hat{\zeta}_{1X^*} - \bar{\gamma}_c \hat{\zeta}_{1\tilde{\eta}\tilde{\eta}} &= 0 \\ -\hat{\zeta}_{2\tilde{\eta}} + \tilde{\eta} \hat{\zeta}_{2X^*} - \bar{\gamma}_c \hat{\zeta}_{2\tilde{\eta}\tilde{\eta}} &= -\sin X^* \hat{\zeta}_{0\tilde{\eta}} \end{aligned} \quad (5.19)$$

If we write $\hat{\zeta}_0 = \tilde{\eta} + \hat{\zeta}_{00}$, then $\hat{\zeta}_{00}$ satisfies $-\hat{\zeta}_{00\tilde{\eta}} + \tilde{\eta}\hat{\zeta}_{00X^*} - \bar{\gamma}_c \hat{\zeta}_{00\tilde{\eta}\eta} = 0$ and $\hat{\zeta}_{00} \rightarrow \text{const.}$ as $\tilde{\eta} \rightarrow \pm \infty$. For any non-zero $\bar{\gamma}_c$ all non-trivial periodic solutions for $\hat{\zeta}_{00}$ become unbounded as $|\tilde{\eta}| \rightarrow \infty$ and thus $\hat{\zeta}_{00} = h_{00}$ where h_{00} is some constant. Similarly, since $\hat{\zeta}_1 \rightarrow h_{10}$ (h_{10} constant) as $\eta \rightarrow \pm \infty$ $\hat{\zeta}_1 = h_{10}$. Then the solution for $\hat{\zeta}_2$ can be obtained as in (i) and is given by

$$\begin{aligned} \hat{\zeta}_2 = & - \text{Im}(e^{iX^*} \int_0^\infty e^{-i\tilde{\eta}t} e^{-(\bar{\gamma}_c t^3/3 + it^2/2)} dt) \\ & (5.20) \\ \hat{U}_2 = & - \text{Im}(e^{iX^*} \int_0^\infty \frac{(e^{-i\tilde{\eta}t} - 1)}{-it} e^{-(\bar{\gamma}_c t^3/3 + it^2/2)} dt) \\ & + \hat{g}_2(X^*) \end{aligned}$$

where $\hat{g}_2$ is some function of X^* which can be determined from the boundary conditions. For $\bar{\gamma}_c \neq 0$, then, we obtain the same velocity jump as before:

$$\hat{U}_2^+ - \hat{U}_2^- = -\pi \sin X^* \quad (5.21)$$

together with a zero vorticity jump. The limit $\bar{\gamma}_c \rightarrow 0$, $\tilde{\eta} \rightarrow \pm \infty$, is interesting and shows the existence of an outer region *below* the critical layer necessary to reduce the vorticity jump to zero.

For, if we change variables in (5.20) we have

$$\hat{\zeta}_2 = - \text{Im}(\gamma_c^{-1/3} e^{iX^*} \int_0^\infty e^{-t^3/3} e^{-i[\tilde{\eta}t\gamma_c^{-1/3} + \gamma_c^{-2/3}t^2/2]} dt) \quad (5.22)$$

As $\tilde{\eta} \rightarrow +\infty$, there are no stationary points in the integrand of (5.22), and the major contribution occurs at the end point $t = 0$, giving $\hat{\zeta}_2 = O(\frac{1}{\tilde{\eta}})$. In contrast, as $\tilde{\eta} \rightarrow -\infty$, the dominant

contribution to $\hat{\zeta}_2$ comes from the stationary point at $t = -\tilde{\eta}\gamma_c^{1/3}$ and this gives

$$\hat{\zeta}_2 = -\operatorname{Im}\left(e^{i(X^* + \tilde{\eta}^2/2)} e^{\tilde{\eta}^3\gamma_c/3} \sqrt{2\pi} e^{-i\pi/4}\right) + O\left(\frac{1}{\tilde{\eta}}\right). \quad (5.23)$$

If $\eta^* = X^* + \frac{\tilde{\eta}^2}{2}$ is fixed, $|\tilde{\eta}| \gg 1$, and $|\tilde{\eta}| \ll \gamma_c^{-1/3}$, then (5.23) shows that there is a jump in the vorticity with

$$\hat{\zeta}_2^+ - \hat{\zeta}_2^- = \operatorname{Im}(e^{i\eta^*} \sqrt{2\pi} e^{-i\pi/4}).$$

If, however, $|\tilde{\eta}| \gg \gamma_c^{-1/3}$, then there is no difference in the mean vorticity above and below the critical layer. Hence for η^* fixed, there is an outer region below the critical layer, with $Y^* \sim O(\mu\gamma_c^{-1/3})$ in terms of the original variables, in which vorticity diffusion is important.

iii) The unsteady nonlinear critical layer, i.e. $\mu \sim O(1)$, $\gamma_c \ll 1$.

To leading order from (2.8) ζ^* then satisfies

$$\mu(1 - \zeta_{Y^*}^*) + Y^* \zeta_{X^*}^* + \sin X^* \zeta_{Y^*}^* = 0. \quad (5.24)$$

Hence

$$\zeta^* = \int_{\chi \text{ fixed}}^{X^*} \frac{-\mu dX'}{\sqrt{2[X - \mu X' - \cos X']^2}} + K(\chi), \quad (5.25)$$

where $\chi = \frac{Y^{*2}}{2} + \mu X^* + \cos X^*$ is the basic streamfunction, and $K(\chi)$ is an unknown function of χ . The basic streamlines given by $\chi = \text{constant}$ are sketched in fig. (2ab). For $\mu \geq 1$ all the streamlines are open, whereas for $\mu < 1$ there exist at most, a

finite number of *closed* eddies, and the streamlines become open again for large enough X^* . With χ fixed, the integral in (5.25) is assumed to be periodic for all μ . Certainly for μ large the results of (ii) above in the limit $\bar{\gamma}_c \rightarrow 0$ are recovered; and for μ small the integral is again periodic. It is suggested tentatively that a viscous correction to (5.25) with a periodicity requirement then determines the unknown vorticity function $K(\chi)$, and this gives

$$\int_0^{2\pi} \frac{\partial}{\partial \chi} [(\chi - \mu x - \cos x)^{\frac{1}{2}} \zeta^*(x, \chi)] dx = 0 \quad (5.26)$$

where $\zeta^*(x, \chi)$ is defined by (5.25). An explicit determination of the function $K(\chi)$, and more significantly of the velocity jump resulting from (5.25), is left unresolved here, however; it is expected that future numerical results for the full problem (2.8) will indicate more clearly what the trends are in this important regime. This regime is particularly interesting because it corresponds to increased disturbance sizes. Likewise, the regime (ii) above is of physical importance since it is associated with faster time scales. Both regimes therefore merit further analytical and numerical study, to gain more insight into the evolution and movement of unsteady nonlinear critical layers.

§6. A BRIEF SUMMARY

We would emphasize the following points concerning the present study.

(a) In general the space-time evolution of the disturbances considered is controlled by the nonlinear equations presented at the ends of §3(a),(b). These form a set coupling the slower scale dependence X_2 , T_2 (of the amplitude A and wavespeed c_0) nonlinearly with the faster scale dependence X_1 , T_1 (of the amplitude) and with the critical layer movement (via the unknown nonlinear function $\mu(A)$). It is necessary therefore that in general solutions of the unsteady nonlinear critical layer problem (2.8) be available, a task requiring numerical work. The limiting solutions in §5 then provide some preliminary comparisons for the numerical work, which should be of interest.

(b) The special case of fixed frequency disturbances discussed in §4 is encouraging for further research. It shows that the previous quasi-neutral estimates produce in effect threshold amplitudes for given frequencies. An amplitude initially below the threshold value subsequently decays, whereas nonlinear growth occurs for initial amplitudes above the threshold value. Moreover, the growth is unbounded, in the present terms, and can be analyzed. Whether similar fast growth can occur more generally, for varying frequency, wavespeed and wavenumber, or not, remains to be seen from the general equations referred to in (a) above and this in turn supports the value of a numerical study of (2.9). If fast growth is induced then the limits studied in §5 come back into play, and

these should point ultimately to the next stage in the evolution of the growing disturbance.

(c) The equation governing the effective local wavespeed of the wave packet is the inviscid Burger equation. Hence shock-like behaviour can take place.

(d) We notice that the governing equations, derived here from formal substitution into the Navier-Stokes equations, are contained also within the interacting boundary layer framework of Smith, Papageorgiou & Elliott (1983), for finite Re . Thus the nonlinear instability features indicated analytically in this paper should be representable at finite Re within that framework.

(e) Three-dimensional disturbances seem especially worthy of investigation along similar lines, to increase the possible relevance to observed instability processes in boundary layers and channel flows for instance. Larger disturbances are also a major concern. Again, the structures and properties discussed here may be adapted to the study of non parallel basic flows, which force added movement of the critical layer across the flow, and to stratified fluid flows.

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APPENDIX A

Consider the equation

$$\eta \zeta_X - \zeta_{\eta\eta} - \mu \zeta_\eta = \sum_{\omega=-\infty}^{\infty} e^{i\omega x} \int_{-\infty}^{\infty} f(\omega, t) e^{-it\eta} dt \quad (A1)$$

where the right hand side is some known periodic function of x .

For periodic solutions the related homogenous equation

$$i\omega\eta\zeta - \zeta_{\eta\eta} - \mu\zeta_\eta = 0 \quad (A2)$$

has only solutions which become unbounded as $\eta \rightarrow \pm\infty$. Hence the bounded solution to the forced equation (A1) is given uniquely by

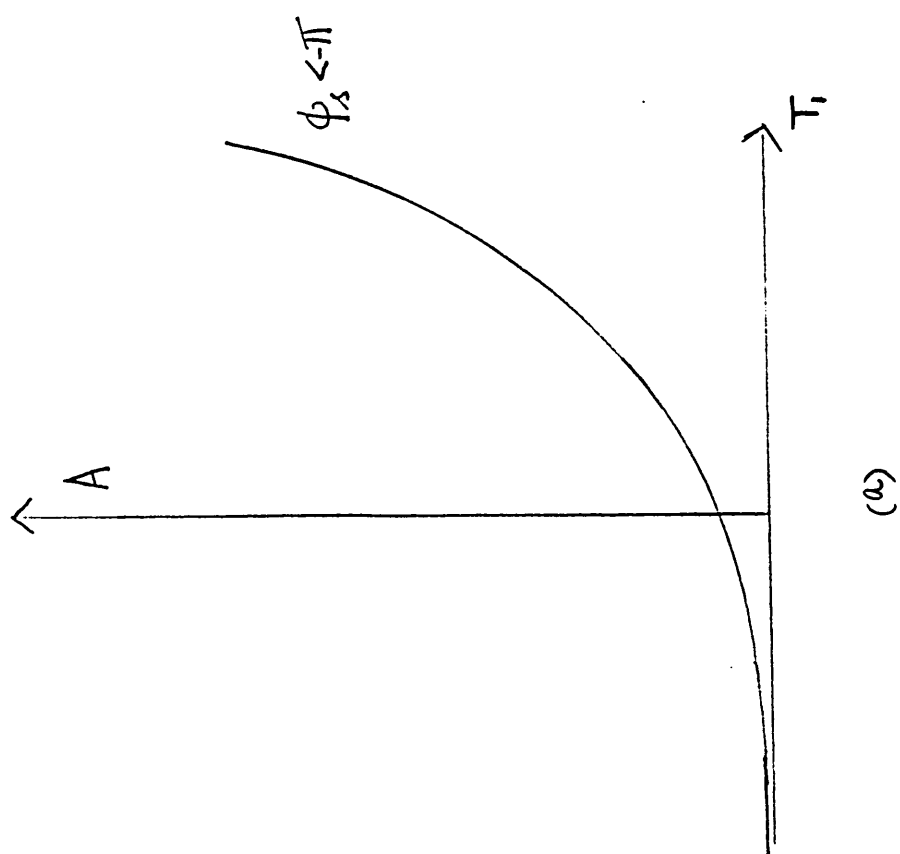
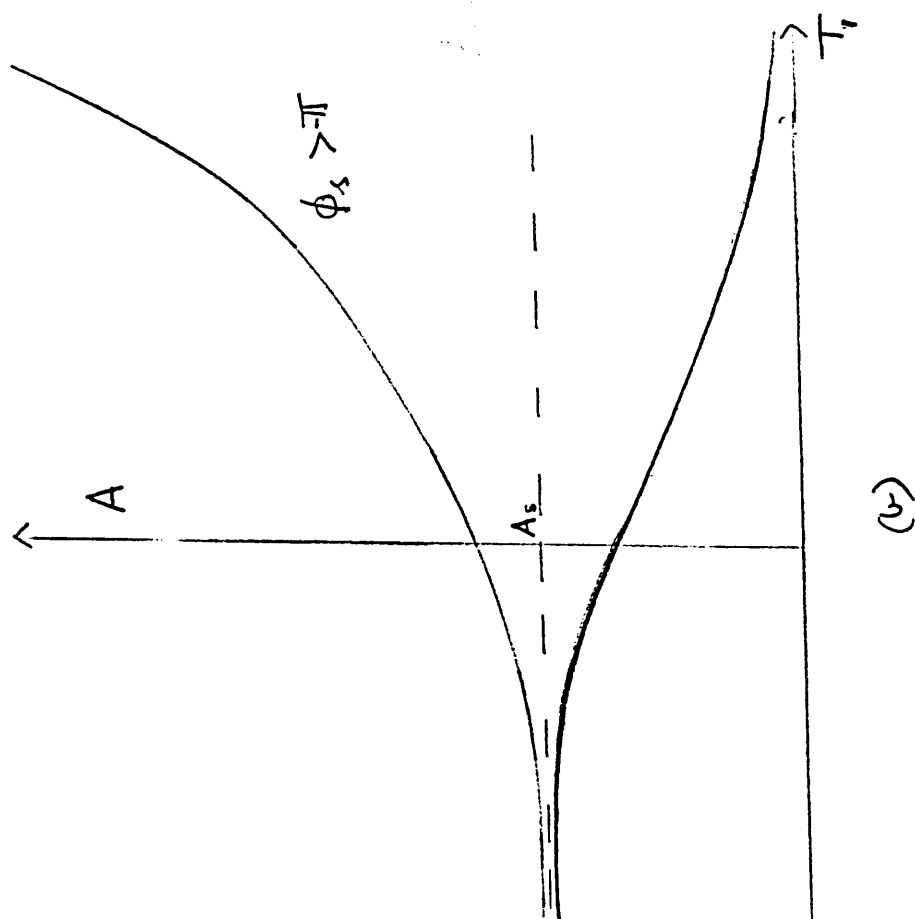
$$\zeta = \sum_{\omega=-\infty}^{\infty} e^{i\omega x} \int_{-\infty}^{\infty} f(\omega, t) S(\eta, t, \omega) dt$$

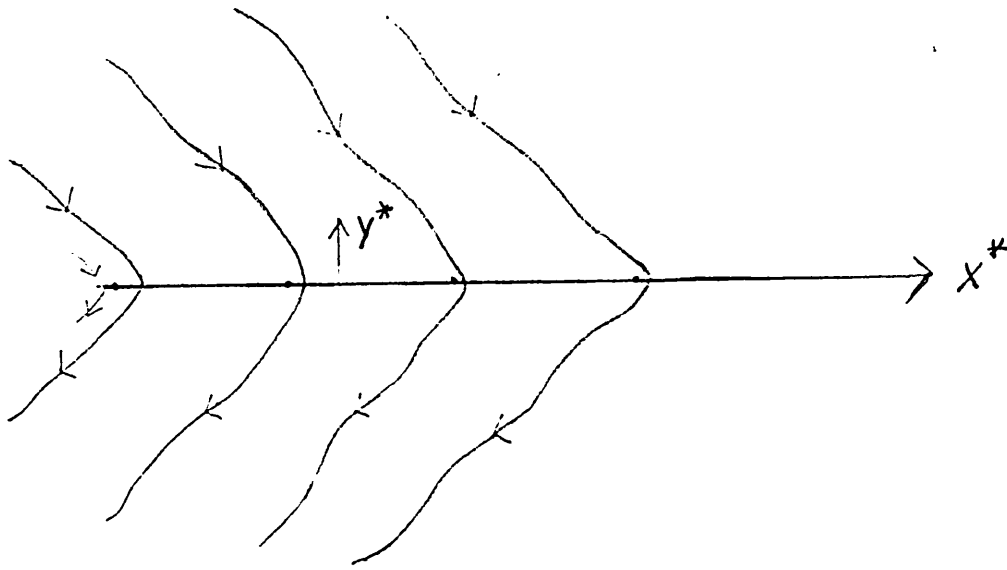
where

$$\begin{aligned} S(\eta, t, \omega) &= \frac{1}{\omega} e^{t^3/3\omega + i\mu t^2/2\omega} \int_t^{\infty} e^{-i\eta t} \times \\ &\quad \times e^{-(t^3/3\omega + i\mu t^2/2\omega)} dt \quad (\omega > 0) \\ &= -\frac{1}{\omega} e^{t^3/3\omega + i\mu t^2/2\omega} \int_{-\infty}^t e^{-i\eta t} \times \\ &\quad \times e^{-(t^3/3\omega + i\mu t^2/2\omega)} dt \quad (\omega < 0) \quad (A3) \end{aligned}$$

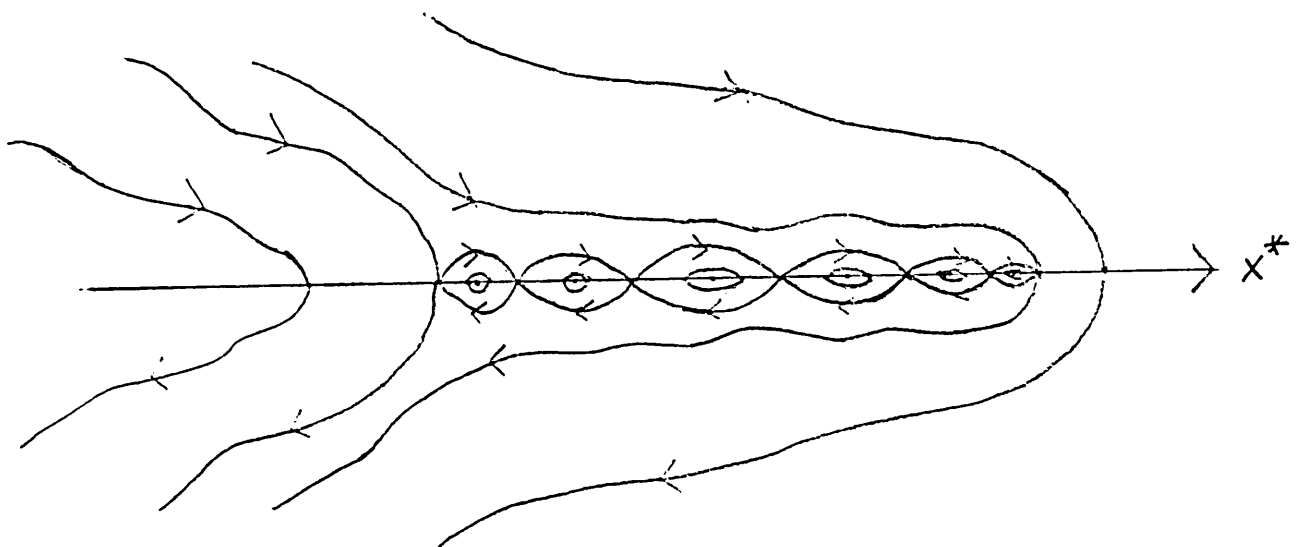
The latter results are derived as in Haberman (1976).

Fig. 1ab





(a)



(b)

Captions

Fig. 1ab.

Sketch of the nonlinear solutions of the amplitude equations for the special case $\mu = 0$, $\phi(A) = -\pi/(1 + k_0 A^{3/2})$ see §4. (a) $\phi_s < -\pi$, $\phi_s > -\pi$.

Fig. 2ab

Sketch of basic streamlines of $\chi = \frac{1}{2}Y^{*2} + \mu X^* + \cos X^* = \text{const.}$, see §5(iii). (a) $\mu \geq 1$, (b) $\mu < 1$.

CHAPTER 9

FLOW PAST WING-BODY JUNCTIONS

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FLOW PAST WING-BODY JUNCTIONS

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SUMMARY

The three-dimensional laminar flow past the junction formed by a thin wing protruding normally from a locally flat body surface is considered, for wings of finite span but short or long chord. The Reynolds number is taken to be large. The leading-edge interaction for a long wing has the triple-deck form, with the pressure due to the wing thickness forcing a three-dimensional flow response on the body surface. The same interaction describes the flow past an entire short wing. Linearized solutions are presented and discussed for long and short two-dimensional wings and for certain three-dimensional wings. The trailing-edge interaction for a long wing is different, however, in that the three-dimensional motions on the wing and on the body are coupled together and in general the coupling is nonlinear. Linearized properties are retrieved only for reduced chord lengths. The overall flow structure for a long wing is also discussed, including the traditional three-dimensional corner layer which is shown to have an unusual singular starting form near the leading edge. Qualitative comparisons with experiments are made also.

§1 INTRODUCTION

The three-dimensional motion of fluid past a wing-body junction is of much concern in aerodynamics, with regard to both the local and the global scale of the viscous effects produced and in particular to any pronounced secondary flows set up. As has proved to be the case in planar wing calculations, understanding of these effects and their scales may form a significant element in numerical schemes, addressing the Navier-Stokes equations or approximations for flow past an entire aircraft, and the question arises of whether there can be an important influence on the aircraft performance overall or not. A number of interesting experimental studies of basically symmetric flow near wing-body junction models have been made (1-3), usually for a very blunt, e.g. circular, wing protruding normally from a flat body surface. A common feature observed is the occurrence of three-dimensional separation upstream, with a curved separation line present on the body surface ahead of the wing.

Such truly nonlinear and complicated flowfields, involving three-dimensional boundary layer collisions for example, would seem to be well beyond the realms of much analysis, of course. Indeed they appear rather remote from the basic, perhaps most obvious, model problem we consider below, namely the junction formed by a flat body surface and a thin symmetric two- or three-dimensional wing of finite chord. Nevertheless, it is found that certain of the theoretical properties which emerge are not far removed from the experimental observations for blunt wings, in qualitative terms at least, including the nature of the motion upstream and the suggested separation line there. Furthermore, the model geometry discussed would seem to be more closely linked with the junctions

of real life than is that associated with the similarity flow along a streamwise corner, which has received much attention (e.g. 5-7). Lastly, we believe that an understanding of the flow past the model geometry here may be of benefit to the understanding and control of the complex flows that occur in practice. The theory below is concerned initially with the flow features arising near the start of the junction, that is, near the leading edge of a long wing or for an entire short wing. The incident boundary layer on the body surface is driven by the locally uniform external stream, is well-formed, attached and planar, and so in a sense its influence dominates the inviscid-viscous interaction between the flow on the wing and the flow on the body surface. For the attached Blasius boundary layer just forming on the wing is much thinner. The interaction provoked then has the triple-deck form, with the motion past the thin wing provoking, on the body surface, an unknown pressure force which interacts with the unknown displacement there. This causes the boundary layer on the body surface to become three-dimensional and to exhibit upstream influence. Nonlinearity results if the typical slope of the wing in the interaction stage is of the order $Re^{-1/4}$, where Re is the large Reynolds number: see §2 below. For thinner wings, including the fundamental case of the flat plate where the Blasius displacement on the wing is responsible for the interaction, linearized analysis is possible and is described in §3. Linearized results are given in §3, §4 for two- and three-dimensional wings respectively, examples of the latter being swept wings and tapered wings.

Further comments are presented in §5, along with a study of the trailing edge of the junction. The model problems studied here have possible relevance also to certain other strongly three-dimensional motions including the flow past fins, tail-body junctions, missiles, fishes, bridges, past high-rise buildings and in branching flow in pipes. An interesting feature is that even the thin wing can provoke a secondary flow component of velocity which is comparable with the streamwise component near the body surface. Although for a very thin wing this nonlinear feature can be suppressed somewhat in the initial leading-edge interaction, it persists in the subsequent trailing-edge interaction for any long wing. The trailing-edge adjustment is always nonlinear, therefore, unless the wing is relatively short, as noted in §5. We believe that experimental and/or computational studies of (e.g.) the fundamental, flat plate, wing-body junction model here could be most illuminating.

In what follows the motion is taken to be laminar and steady and the fluid is assumed to be incompressible.

§2 FOR A THIN PLANAR WING

We start by discussing the fluid motion past a wing-body junction formed by a thin planar symmetric wing projecting normally from the aircraft body. The latter is taken to have a flat surface since our initial concern is with the localized flow features. Moreover, the wing is placed at zero incidence to the oncoming main stream, the resultant motion is assumed to be symmetric about the wing's centre-plane $y = 0$, for convenience, and so we study only the upper half of the motion, $y \geq 0$. Here, in cartesian coordinates x, y, z , the wing starts at $x = 0$ and is given (for all $z > 0$) by $y = \tilde{f}(x)$ for $x > 0$, $y = 0$ for $x < 0$, where the typical wing thickness $|\tilde{f}|$ is small in some sense, whereas the body occupies the flat surface $z = 0$ for all x, y . See Fig.1. The lengths (x, y, z) above, and the corresponding velocity components (u, v, w) , are non-dimensionalized with respect to l_∞, u_∞ respectively, where l_∞ is the characteristic length scale of the body, e.g. the streamwise distance from the aircraft nose to the wing, and u_∞ is the speed of the main stream which flows in the positive x direction. The pressure is written $p_\infty + \rho u_\infty^2 p$, with p_∞, ρ denoting the main stream pressure and the fluid density in turn. The typical global Reynolds number is then $Re = u_\infty l_\infty \nu^{-1}$, if ν is the kinematic viscosity of the fluid, and Re is supposed to be large.

On an inviscid basis, with the above wing-body geometry, the uniform stream $u = 1, v = w = 0$, continues broadly undisturbed apart from a small, $O(|\tilde{f}|)$, perturbation. Viscous effects, however, alter the flow substantially, near the wing and body surfaces at least, where viscous-inviscid interaction occurs. There appear to be two aspects to consider then. First,

sufficiently near the start of the junction, for small x , and for a sufficiently thin wing, the predominant interaction involves the effect, on the flow close to the body surface, of the motion past the wing. The opposite effect, that of the flow near the body surface on the flow near the wing, is much less. The reason for this is that for small x the boundary layer on the wing is only just formed and so is thin compared with the more developed $O(Re^{-1/2})$ oncoming boundary layer, on the body, which by contrast has its origins at an $O(1)$ distance ahead of the wing-body junction. Hence the wing's boundary layer is the more attached one then and is less affected by a given size of induced pressure field than is the thicker boundary layer on the body. The latter layer must be the first to feel the effects of any interaction and is the one where three-dimensional separation for instance is to be expected at first.

That brings us to the second main aspect, which concerns the type and scale of interaction. Nonlinear responses such as separation in an external developed boundary layer, whether three-dimensional (8) or not, are controlled largely by the triple-deck interaction, as is well known. Formally, therefore, the critical size of induced pressure p , i.e. the smallest size which provokes a nonlinear interactive response in the body's boundary layer, is of the order $Re^{-1/4}$ and the streamwise length scale invoked is $O(Re^{-3/8})$. This emphasizes initially the role played by a wing which produces an $O(Re^{-1/4})$ typical pressure field within a distance $O(Re^{-3/8})$ of its leading edge. There are perhaps two main categories of such wings. One concerns wings of $O(1)$ chord length in x ("long wings"), the other concerns wings of chord length $O(Re^{-3/8})$ ("short wings").

Other chord lengths tend to be covered by limiting cases of the above. The pressure p induced near the leading edge or nose of a long wing depends on the local wing shape there. For a typical blunt nose of the parabolic form $y = \tilde{f}(x) \sim H x^{\frac{1}{2}} (x \rightarrow 0+)$, where H is the thickness-to-chord ratio, $p \propto H x^{-\frac{1}{2}}$ as $x \rightarrow 0+$. So when x falls to $O(\text{Re}^{-3/8})$ the critical pressure of $O(\text{Re}^{-1/4})$ is reached if H is $O(\text{Re}^{-7/16})$. The boundary layer on such a wing has the Blasius form, of thickness $O(\text{Re}^{-1/2} x^{\frac{1}{2}})$ ($x > 0$), and so is always thinner than the wing thickness H (see also below). Again, the leading edge zone of non-uniformity is of extent $O(H^2) = O(\text{Re}^{-7/8})$, small compared with the $x \sim \text{Re}^{-3/8}$ interaction scale. Other local shapes can be regarded similarly. Thus if $\tilde{f}(x) \sim H x^n$ as $x \rightarrow 0+$, and $0 < n < 1$ for a blunt nose, then locally $p \propto H x^{n-1}$ tends to $O(H \text{Re}^{3/8(1-n)})$ in the interaction scale, pointing to the critical thickness $H = O(\text{Re}^{-(5-3n)/8})$ for nonlinear interaction. This is self-consistent provided the $O(\text{Re}^{-1/2})$ displacement thickness of the wing layer remains negligible for all x , implying that $n > 1/3$. Conversely if $n \leq 1/3$ the displacement thickness then has a leading order or even dominant influence downstream on the induced pressure although not necessarily within the interaction length. The same can occur if, for a given value of n , H is below the critical thickness, in which case only a linearized wing-body interaction is induced. Examples are given later in §3. The central point is that the induced pressure from any planar blunt long wing is dominated by the local wing shape, and unaffected by the shape further downstream, as regards the interaction near the leading-edge. On the other hand, the critical thickness for a short wing is always $O(\text{Re}^{-5/8})$ since then its typical slope, and hence pressure effect, is $O(\text{Re}^{-1/4})$, as

required for interaction with the body's boundary layer. The above arguments assume suitable smoothness of the thin wing surface, of course, and essentially attached flow thereon. Attached flow is achieved if the trailing-edge is cusped for instance but it becomes open to question for less ^(A)sharp trailing-edges, especially blunt ones. Nevertheless, for the long wing the leading-edge flow properties are unaltered to leading order by any breakaway separation from the wing downstream, while for the short wing trailing-edge separation can remain a relatively confined phenomenon. So the attached flow restriction can be relaxed if necessary.

Two-dimensional flow past the thin, short or long, wing above produces an $O(\text{Re}^{-1/4})$ pressure field dependent only on x, y . This holds for distances greater than $z = O(\text{Re}^{-3/8})$ from the body surface $z = 0$, since when z is $O(\text{Re}^{-3/8})$ the three-dimensional interaction takes place, with the $O(\text{Re}^{-1/4})$ pressure provoking in the body boundary layer an $O(\text{Re}^{-5/8})$ displacement comparable with the local wing thickness. The three-dimensional linearized potential flow just outside the boundary layer then is governed by Laplace's equation $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial \bar{z}^2)\bar{p} = 0$ with $\partial \bar{p}/\partial Y(X, 0+, \bar{z}) = -h d^2 f/dX^2$, $\partial \bar{p}/\partial \bar{z}(X, Y, 0+) = -\partial^2 \delta(X, Y)/\partial X^2$ and $\bar{p}(X, Y, 0+) = P(X, Y)$, for tangential flow on the scaled wing surface $Y = h \text{Re}^{-1/4} f(X)$ and to match the body's unknown scaled displacement thickness $\delta(X, Y)$ and unknown scaled pressure $P(X, Y)$. Here $(x, y, z) = \text{Re}^{-3/8}(X, Y, \bar{z})$, $p = \text{Re}^{-1/4}\bar{p}$ and $\bar{p}, f, \delta, P$ and the parameter h are generally of order unity, while $-\infty < X < \infty$, $0 < Y < \infty$, $0 < \bar{z} < \infty$ by symmetry. The solution for $\bar{p}$ consists of the two-dimensional wing solution

$$\bar{p}(X,Y,\bar{Z}) = -\frac{h}{\pi} \int_{-\infty}^{\infty} \frac{df(\xi)}{d\xi} \frac{(X-\xi)}{(X-\xi)^2 + Y^2} d\xi, \quad (2.1)$$

which gives no displacement on the body surface $\bar{z} = 0+$, and the three-dimensional solution corresponding to the body displacement $\delta(X,Y)$ with symmetry adding no displacement on the wing $Y \rightarrow 0+$. Hence we have the forced pressure-displacement relation

$$P(X,Y) = -\frac{h}{\pi} \int_{-\infty}^{\infty} \frac{df(\xi)}{d\xi} \frac{(X-\xi)d\xi}{(X-\xi)^2 + Y^2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\partial^2 \delta / \partial \xi^2) d\xi d\eta}{[(X-\xi)^2 + (Y-\eta)^2]^{\frac{1}{2}}} \quad (2.2a)$$

controlling the flow induced on the body surface. This flow is governed by the lower deck properties, where $z = \text{Re}^{-5/8} Z$ and the three-dimensional boundary layer equations

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0 \quad (2.2b)$$

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} + W \frac{\partial U}{\partial Z} = -\frac{\partial P}{\partial X}(X,Y) + \frac{\partial^2 U}{\partial Z^2} \quad (2.2c)$$

$$U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} + W \frac{\partial V}{\partial Z} = -\frac{\partial P}{\partial Y}(X,Y) + \frac{\partial^2 V}{\partial Z^2} \quad (2.2d)$$

apply, with the usual triple-deck scalings (8). The boundary conditions here are

$$U = V = W = 0 \quad \text{at } Z = 0 \quad (2.2e)$$

$$U \sim \lambda(Z - \delta(X,Y)), \quad V \rightarrow 0 \quad \text{as } Z \rightarrow \infty \quad (2.2f)$$

$$(U,V,W,P) \rightarrow (\lambda Z, 0, 0, 0) \quad \text{as } X \rightarrow -\infty \quad (2.2g)$$

for no slip on the flat body surface, for matching with the displaced main part of the $O(\text{Re}^{-1/2})$ boundary layer, and for joining with the planar boundary layer upstream, in turn. The constant λ is the incident $O(1)$ skin friction, taken to be unity henceforth, for convenience. Also, by symmetry, $V = 0$ at $y = 0$.

The viscous-inviscid three-dimensional flow interaction for the wing-body junction is given by the solution of (2.2a-g) for a prescribed wing-shape $h_f(X)$. The problem is nonlinear in general and so requires a numerical treatment. Subsequently we consider the linearized versions, valid strictly if the wing thickness parameter h is small or for the basic problem of a wing which is a finite flat plate: see §3. Meanwhile it is worth remarking that the triple-deck interaction emphasized above does not cover all the features of the three-dimensional flow. Other subregions are present, including the thin $O(\text{Re}^{-11/16})$ (in y) boundary layer developing on the locally $O(\text{Re}^{-5/8})$ thick wing and a small corner zone $O(\text{Re}^{-5/8})$ by $O(\text{Re}^{-5/8})$ (in y, z) where y - and z -diffusion ~~and~~ are equally significant. All these sub-regions are believed to be passive relative to the triple-deck interaction, however. In particular the wing's boundary layer is driven forward in a quasi-Blasius fashion, apart from in the corner zone and in the traditional leading-edge zone for smaller x , and so does not affect (2.2a-g). Other more general sizes for the wing dimensions or for the incident boundary layer thickness can also be considered but these are merely limiting cases of the fundamental triple-deck problem, unless large-scale separation occurs, which is a case beyond our current scope. Accordingly, we focus attention on (2.2a-g), for a number of wing shapes $f(X)$, both short and long.

§3 LINEARIZED SOLUTIONS FOR THIN, OR FLAT, PLANAR WINGS

When h in (2.2a) is small a linearized version of (2.2a-g) applies and the analysis in (8) can be taken over. To shorten matters we may work in terms of an "effective hump shape" then, since the double Fourier transform in X, Y (see notation in (9)) of (2.2a) gives

$$\frac{P^{**}(k, \ell)}{h} = -2 \frac{k^2 f^*(k)}{(k^2 + \ell^2)} - \frac{k^2 \delta^{**}(k, \ell)}{h(k^2 + \ell^2)^{\frac{1}{2}}}. \quad (3.1)$$

Here $f^*(k)$ is the transform of $f(X)$ with respect to X , and δ stands for $-A$ in (8). So a comparison with §3 of (8) shows that our linearized problem, for a wing shape $h f(X)$ and flat surface $Z = 0$, corresponds to the linearized problem of (8) for flow over a hump given by $Z = h F(X, Y)$, where the effective hump shape F satisfies

$$F^{**}(k, \ell) = 2 (k^2 + \ell^2)^{-\frac{1}{2}} f^*(k). \quad (3.2)$$

The same analogue holds in the nonlinear regime. The linearized solutions for the transforms then follow from (8). In particular,

$$\frac{P^{**}(k, \ell)}{h} = 2 (ik)^2 f^*(k) (k^2 + \ell^2)^{-1} / \Delta$$

$$\frac{\delta^{**}(k, \ell)}{h} = 2 \gamma^{-4/3} (ik)^{1/3} f^*(k) / \Delta$$

$$\frac{\tau_X^{**}(k, \ell) - 1}{h} = 2 \cdot \frac{3Ai(0)(ik)^{2/3} f^*(k)}{(k^2 + \ell^2)\Delta} \left[\frac{k^2 + \ell^2}{\gamma^{4/3}} - \frac{\ell^2}{9Ai^2(0)} \right]$$

$$\begin{aligned}\frac{\tau_Y^{**}(k, \ell)}{h} &= \frac{-2(i\ell)(ik)^{5/3} f^*(k)}{3Ai(0)(k^2 + \ell^2)\Delta} \\ \frac{D^{**}(k, \ell)}{h} &= \frac{-2(i\ell)(ik) f^*(k)}{(k^2 + \ell^2)\Delta}\end{aligned}\quad (3.3a-e)$$

where

$$\Delta = 1 + \gamma^{-4/3} (ik)^{1/3} (k^2 + \ell^2)^{\frac{1}{2}}.$$

In (3.3) $\tau_X \equiv \partial U / \partial Z(X, Y, 0)$, $\tau_Y \equiv \partial V / \partial Z(X, Y, 0)$ denote the surface shear stresses in the X, Y directions, and $D(X, Y)$ satisfies $\partial D / \partial X = -\partial P / \partial Y$ and helps to describe the motion more removed from the body surface, since $V \sim D(X, Y)Z^{-1}$ for large Z.

Fast Fourier-transform (FFT) calculations were used to determine the corresponding real quantities, and the results were checked on different grid sizes and integration ranges. The following short and long wing shapes were considered:

$$f(X) = X^{\frac{1}{2}} (X > 0) \quad (\text{long}) \quad (3.4b)$$

$$f(X) = X^{\frac{1}{2}} \exp(-X^2) (X > 0) \quad (\text{long}) \quad (3.4a)$$

$$f(X) = X(1-X) (0 < X < 1) \quad (\text{short}) \quad (3.4c)$$

$$f(X) = X^{\frac{1}{2}}(1-X)^{\frac{1}{2}} (0 < X < 1) \quad (\text{short}) \quad (3.4d)$$

with $f(X) \equiv 0$ otherwise. Their solution properties are presented

in Figs. 2-5. It should be noted that rather extensive grids were needed to accommodate the farfield behaviour of (3.3a-e), especially for the long wing shapes. Results from different grids indicate that graphical accuracy is achieved usually, and our figures present the most accurate results obtained in each case, although some distortion persists in the farfields (see also (3.5) ff. below) and in the symmetry or anti-symmetry of the solutions, as well as in a few local features, e.g. wiggles. Such distortions all shrink under grid refinement but computer storage limitations have prevented us from removing them totally from the results.

The main features apparent in the flow solutions seem sensible physically and are as follows. Upstream of the wing, along the symmetry line $Y = 0$, a common occurrence is the rise in pressure and the associated drop in the X-shear stress τ_X . This is much like a two-dimensional flow response ahead of the effective

hump shape. Associated with it there is a generally positive Y-shear stress τ_Y and Y "edge-velocity" D , as might be expected, but the displacement δ falls, which is a surprising feature at first sight. Thus fluid is forced away (in the Y direction) from the symmetry line but tends to be drawn (in the negative Z direction) towards the body surface there, ready to negotiate the wing beyond. This and other initially unexpected properties of the displacement are caused by the pressure from the wing, contributing the f^* term in (3.1). The fluid can only go around the wing, not over it, in broad terms, and so there is here a significant difference from the flow past (and over) humps. Near the front of the wing there is a particularly marked fall in the displacement, in fact. As soon as the wing position is encountered, however, the displacement adjusts rather fast, rising to become positive eventually downstream; the pressure rises more rapidly, then falls rapidly to a minimum before slowly recovering; and the shear stress τ_X increases fairly abruptly to a maximum, also followed by a slower recovery downstream. So there the flow is accelerated, whilst the fluid ultimately is drawn back towards the symmetry line ($\tau_Y, D < 0$), in the Y-direction, and moves away from the body surface ($\partial\delta/\partial x > 0$), in the positive Z-direction. Trends roughly opposite to those near the wing's leading edge occur in the neighbourhood of the trailing edge, for a short wing. The trailing edge acts like a sink for the three-dimensional flow compared with the source-like role of the leading edge. See also §5 below. Finally, sufficiently far downstream the pressure overshoots to become positive again before decreasing back to zero, while the shear stress τ_X undershoots to a minimum, then increases to unity, and the displacement has an overshoot to a

positive maximum before its fall back to zero. Thus fluid is eventually drawn towards the body surface ($\partial\delta/\partial X < 0$), in the Z direction, whereas it moves away from the symmetry line near the body surface ($\tau_Y > 0$) although still approaching the symmetry (X-Z) plane at the edge of the viscous zone ($D < 0$). There is no evidence of the corridor phenomenon (8) far downstream. Instead the flow returns to its original upstream state quite smoothly in general, a property which can be verified from an inspection of the transforms in (3.3).

In the farfield, for $(X^2 + Y^2)$ large, inviscid characteristics tend to dominate usually, since then in effect $k, \ell \rightarrow 0$. Thus (3.3a-e) give, for example,

$$P/h \propto f^*(0)(X^2 - Y^2) / (X^2 + Y^2)^2 \quad (3.5a)$$

$$D/h \propto f^*(0)(-XY) / (X^2 + Y^2)^2 \quad (3.5b)$$

[c.f. 4]

with the wing acting as a point source ($f^*(0)$), in those cases where $\int_{-\infty}^{\infty} f(X)dX$ is finite. The implied asymptotes for the curves of zero

$P, \delta, \tau_X, \tau_Y, D$ are noted in the captions for figures 2-5

and agree reasonably well with the FFT calculations. The only nonuniformity in the farfield account occurs in the direct wake of the wing, when $k \rightarrow 0$ but $\ell \rightarrow \infty$ with $\ell = O(|k|^{-1/3})$, in view of (3.3), i.e. as $X \rightarrow \infty$ but with $Y = O(X^{-1/3}) \rightarrow 0$. There viscous effects reassert their influence, but in a region which thins like $X^{-1/3}$ in the Y direction while expanding (as usual) like $X^{1/3}$ in Z. This contrast with the flow over a three-dimensional hump is due to the two-dimensionality of the wing shape and the corresponding nature of the effective hump shape in (3.2): c.f. §4 below.

The secondary flows induced are also of concern and are sketched in Fig. 2. The basis for this figure is the behaviour of δ , D , τ_X , τ_Y as X, Y vary, since these fix the flow patterns near and relatively far from the body surface. The overall trend initially is one of motion around the wing, as noted earlier. Upstream the secondary flow is away from the symmetry line but towards the body surface. Beyond the ~~front~~^{front} of the wing the flow returns toward the symmetry line but tends to be away from the body surface. Further downstream secondary vortices are induced producing outward motion near the body surface supplied by inward motion further out and a wash of fluid down the wing surface/symmetry line.

Finally here, we turn to the central problem of the thin flat-plate symmetric wing, taking the wing to be "long". Comments on shorter cases are presented later in §5. The flat plate induces an interaction by means of its boundary layer's (Blasius) parabolic displacement, which acts as an effective wing thickness. The thickness is $O(\text{Re}^{-1/2} x^{1/2})$ however and so provokes a pressure of only $O(\text{Re}^{-1/2} x^{-1/2})$, or $O(\text{Re}^{-5/16})$, within the interactive scale at the leading edge. This is too small, by a factor $O(\text{Re}^{-1/16})$, to induce a nonlinear wing-body interaction, therefore. Instead, a linearized response occurs with the local wing shape being, in effect, exactly (3.4b) (for all $X > 0$) and with the gauge factor h replaced by $\beta_1 \text{Re}^{-1/16}$. Here $\beta_1 = 1.7208$, from the Blasius layer, and the flow structure remains intact with such a value of h . So the properties in Fig. 3 cover the flat-plate case also.

§4 FOR THREE-DIMENSIONAL WINGS (e.g. swept, tapered)

Swept-back or -forward wings are of considerable interest in practice and so an investigation of the influence of sweep at a wing-body junction is desirable. Again, a smoother more three-dimensional version of the junction would seem to resemble more closely many of the geometries that are made or occur in practice. Therefore below we take the thin wing to have a general three-dimensional form

$$y = \tilde{f}(x, z) \quad (4.1)$$

whilst preserving the symmetry about the x-z plane (Fig.6).

The critical scalings of §2 remain intact for this case, whether the wing is short or long, but the results (2.1), (2.2a) apply no longer, since the wing condition on $\bar{p}$ is now $\partial \bar{p} / \partial Y(X, 0+, \bar{z}) = -h \partial^2 \tilde{f} / \partial X^2(X, \bar{z})$. Here $\tilde{f}(x, z) \equiv h \operatorname{Re}^{-5/8} f(X, \bar{z})$. So the solution of Laplace's equation for $\bar{p}$ gives the pressure-displacement interaction

$$P(X, Y) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\partial^2 F / \partial \xi^2 - \partial^2 \delta / \partial \xi^2) d\xi d\eta}{[(X-\xi)^2 + (Y-\eta)^2]^{\frac{1}{2}}} \quad (4.2)$$

on the body surface, where

$$F^{**}(k, \ell) = 2h \int_{X=-\infty}^{\infty} \int_{\bar{z}=0}^{\infty} e^{-(k^2 + \ell^2)^{\frac{1}{2}} \bar{z} - ik X} f(X, \bar{z}) dX d\bar{z} \quad (4.3)$$

fixes the Fourier transform of the function $F(X, Y)$. Then

(4.2), (4.3), with the boundary layer equations (2.2b-g), determine the local nonlinear flow interaction. Here in fact (4.3) defines the effective hump shape F , as in §3.

For linearized motion, where h is small, we may use (4.3) to determine F , for a given three-dimensional wing shape f , and then appeal directly to the results in (8). Hence (3.3a-e) are retrieved but with F^{**} now given by (4.3). Two examples of interest are the swept wing and the tapered wing or junction. For the swept wing of sweep angle β , where $f(X, \bar{z}) = g(X - \bar{z} \tan \beta)$ for $X > \bar{z} \tan \beta$, $f(X, \bar{z}) = 0$ for $X < \bar{z} \tan \beta$, and $g(X)$ is the wing shape adjoining the body surface $\bar{z} = 0+$, (4.3) yields

$$F^{**}(k, l) = 2 \frac{g^*(k)}{ik \tan \beta + (k^2 + l^2)^{\frac{1}{2}}} \quad (4.4)$$

The solutions for the corresponding interaction are presented in Fig. 7, for a particular shape $g(X)$. Here again no corridor emerges downstream. It is interesting that $O(1)$ sweep angles β can be analyzed. For the tapered wing, typically $f(X, \bar{z}) = g_1(X) g_2(\bar{z})$ with $g_2(\bar{z})$ controlling the degree of tapering. As an example we took the exponential form $g_2(\bar{z}) = \exp(-\sigma \bar{z})$ ($\bar{z} > 0$), with σ a positive constant, for which (4.3) then gives

$$F^{**}(k, l) = \frac{2 g_1^*(k)}{\sigma + (k^2 + l^2)^{\frac{1}{2}}} \quad (4.5)$$

Fig. 7 also shows the interactive solutions in this case, for a particular wing shape $g_1(X)$. A corridor effect for Y of $O(1)$ is present downstream now as X increases, but its properties are different from those in (8) since here it involves no change in the orders of magnitude of $P, \delta, D, \tau_X, \tau_Y$. There is only a slight, logarithmic, singularity as $Y \rightarrow 0$ which is smoothed out in the zone of $O(X^{-1/3})$ width mentioned earlier. The secondary flows induced are summarized in Fig. 7.

Two unusual aspects of these flow solutions, compared with those of §3, are, first, the occurrence of a positive displacement effect in most of the flow ahead of each of the three-dimensional wings, before an abrupt decrease takes place immediately before the wing, and, second, the closed loops of constant  displacement , found for the junction with a tapered wing. The farfields can be deduced as for the earlier, two-dimensional, wings.

§5 FURTHER COMMENTS

The model studied above for the wing-body junction takes a rather simplistic geometry of course, comprising a thin two- or three-dimensional wing protruding normally from a flat body surface, with or without sweep. Yet such a model would seem to provide a reasonable first step towards understanding analytically the three-dimensional flow interactions possible. Indeed the flat plate example noted in §3 may be regarded as an obvious central problem to study. One would perhaps not expect these thin-wing models to bear much relation to the bluff wing-body configurations examined experimentally (1-3) but nevertheless the measure of agreement on certain flow features is surprisingly good. In particular the linearized solutions always show ~~two~~ ^{local} maximum/deficits in the stream-wise (X) surface shear stress on the body taking place, just ahead ^{and beyond the front of} of the wing, on the axis of symmetry, which suggests that (regular) separation or flow reversal would tend to occur there first, in the nonlinear régime. That, allied with the corresponding Y surface shear stress patterns, forms a fairly clear connection with the experimental observation (1-3) of a separation line starting upstream ^{and of separation downstream} of the wing and bending round it. Also, the strong secondary flow patterns found here appear to be sensible physically and to tie in generally with those reported in (1-3), as fluid is forced to move around the wing. Specifically, for turbulent flow experiments, Fig. 3 of ref. 1 compares most favourably with our theoretical finding (Figs. 2-5) of a pronounced downwash on the wing sufficiently far beyond the leading edge. For laminar flow the experiments of ref. 2 (e.g. their Figs. 11-13, 17) also give cross-flows similar to those found theoretically here. Qualitative

agreement continues to hold in comparisons with the classical boundary layer calculations of ref. 4, concerning both the separation line upstream (see their Figs. 1, 5, 23) and the cross flow (their Figs. 8, 10, 28, 29), although there is a puzzling feature in the displacement thickness (their fig. 11) which shows an increase ahead of the wing rather than the decreases predicted theoretically (see our Figs. above). The further experiments reported in ref. 4 also tend to agree qualitatively with the present theoretical study.

In view of the particularly promising measure of possible agreement overall above, it would be very interesting to see the results of a numerical study of the nonlinear régime (2.2a-g) when h is $O(1)$ or larger. Some other improvements in the models taken can be made also, for instance (a) by adding in a hump shape on the body surface to produce a smoother junction, (b) by considering wings at nonzero angles of incidence, (c) by assuming three-dimensional oncoming flow both along the body surface and away from it, and (d) by considering wings with nonzero angles of dip. Here, in (b), the critical angle of incidence α_1 (the angle between the plane of the wing and the $x-z$ plane) is small, of order $Re^{-7/16}$, for a long thin wing since the induced pressure is of order $\alpha_1 x^{-1/2}$ near the leading edge; whereas the effect of an angle of dip α_2 (the angle between the body surface and the plane of the wing), in (d) above, is less drastic in the sense that the flow structures found earlier in §2-4 remain appropriate even for $O(1)$ values of α_2 , if the wing is thin. Both incidence and dip introduce nonsymmetry into the wing-body interaction, however.

The development of the motion downstream, beyond the triple-deck at the leading edge, is also of concern. Many studies have been

made (e.g. 5-7) of the well-developed similarity flow believed to be achieved only as $x \rightarrow \infty$ in a configuration such as that in Fig.1, e.g. the flat-plate wing, but little or no research into the developing flow properties holding for finite $x > 0$ is evident yet. There the main flow structure becomes at first sight a relatively simple one, with Blasius boundary layers of typical thickness $O(\text{Re}^{-1/2})$ (but different origins) astride the wing and the body except in a classical $O(\text{Re}^{-1/2})$ by $O(\text{Re}^{-1/2})$ zone (Z1, say) near the x-axis where y- and z-diffusion become comparable. This picture emerges from the leading edge properties of §2-4 as $X \rightarrow +\infty$, since there viscous interactive effects concentrate in a zone (Z2) whose Z scale increases like $X^{+1/3}$ but whose Y scale decreases like $X^{-1/3}$ (see §3). So the y,z scales of Z2 are of order $\text{Re}^{-3/8}(\text{Re}^{3/8}_x)^{-1/3}$, $\text{Re}^{-5/8}(\text{Re}^{3/8}_x)^{1/3}$ then, respectively, and both become $O(\text{Re}^{-1/2})$ as x becomes $O(1)$. Thus zone Z2 merges into zone Z1 downstream. It is striking that the thickness of Z1 in the y-direction must therefore appear unbounded ($\infty x^{-1/3}$) initially, as $x \rightarrow 0+$. The remainder of the triple-deck interaction, outside zone Z2, has less influence downstream but its effects do persist nonetheless and they force a secondary flow in the two Blasius layers in $x > 0$. This interactive secondary flow is not mentioned in previous studies.

In addition, in cases where the leading-edge interaction produces a corridor (wherein $Y = O(1)$) downstream, an extra $O(\text{Re}^{-3/8})$ by $O(\text{Re}^{-3/8})$ zone is produced along the x-axis for $x > 0$, forming a buffer between the classical zone Z1 and the Blasius layers. This corridor, which has been observed experimentally in flow over humps, could have a significant impact on the flow downstream.

Finally, again for the flat-plate wing, it is of practical

interest to examine the trailing-edge motion. Suppose the wing has chord length L , of order unity (see Fig. 8). For $0 < x < L$, and outside Z_1 , the two Blasius layers BL_1 , BL_2 grow along the wing and the body surface, after the mainly linear adjustment (§3) near the leading edge $x = 0$. Near the trailing edge, $x = L$, more triple-deck interaction therefore comes into operation. It has a predominantly two-dimensional form (as in 9) on most of the wing, the prime exception being at distances z of order $Re^{-3/8}$ from the body surface, where the flow interaction is three-dimensional and nonlinear for both the wing and body motions. There the flow adjoining the body surface satisfies the three-dimensional boundary layer equations (2.2b-g) again but with

$$\lambda, X \text{ replaced by } \lambda_2, \tilde{X} \equiv Re^{3/8}(x-L), \quad (5.1)$$

in turn, where λ_2 is the incident reduced skin friction of BL_2 as $x \rightarrow L^-$, and $0 < \lambda_2 < \lambda$. The viscous layer on the wing ($\tilde{X} < 0$) and in its wake ($\tilde{X} > 0$) is also controlled by (2.2b-d,f,g), subject to the replacements

$$(U, V, W, P, \delta, X, Y, Z, \lambda) \rightarrow (\hat{U}, \hat{W}, \hat{V}, \hat{P}, \hat{\delta}, \tilde{X}, \tilde{Y}, \lambda_1) \quad (5.2)$$

for the rotation of axes, and to the usual lower deck scalings again. Here $\lambda_1 > 0$ is the reduced skin friction of BL_1 at $x = L^-$, and instead of (2.2e) we have the trailing-edge conditions

$$\hat{U} = \hat{W} = \hat{V} = 0 \quad \text{at} \quad \hat{Y} = 0, \quad \tilde{X} < 0 \quad (5.3a)$$

$$\frac{\partial \hat{U}}{\partial \hat{Y}} = \frac{\partial \hat{W}}{\partial \hat{Y}} = \hat{V} = 0 \quad \text{at} \quad \hat{Y} = 0, \quad \tilde{X} > 0. \quad (5.3b)$$

The interaction between the two nonlinear layers of (5.1), (5.2) arises through their respective unknown pressures and displacements, $P(\tilde{X}, Y)$, $\delta(\tilde{X}, Y)$ on the body and $\hat{P}(\tilde{X}, \bar{Z})$, $\hat{\delta}(\tilde{X}, \bar{Z})$ on the wing, in scaled terms. These are linked via the three-dimensional potential flow in the upper deck, which requires that

$$P(\tilde{X}, Y) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\partial^2 F / \partial \xi^2 - \partial^2 \delta / \partial \xi^2) d\xi d\eta}{[(\tilde{X}-\xi)^2 + (Y-\eta)^2]^{\frac{1}{2}}} \quad (5.4a)$$

$$\hat{P}(\tilde{X}, \bar{Z}) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\partial^2 \hat{F} / \partial \xi^2 - \partial^2 \hat{\delta} / \partial \xi^2) d\xi dv}{[(\tilde{X}-\xi)^2 + (\bar{Z}-v)^2]^{\frac{1}{2}}} \quad (5.4b)$$

where $F(\tilde{X}, Y)$, $\hat{F}(\tilde{X}, \bar{Z})$ are given by the transforms

$$F^{**}(k, l) = 2 \int_{\tilde{X}=-\infty}^{\infty} \int_{\bar{Z}=0}^{\infty} e^{-(k^2 + l^2)^{\frac{1}{2}} \bar{Z} - ik\tilde{X}} \hat{\delta}(\tilde{X}, \bar{Z}) d\tilde{X} d\bar{Z}, \quad (5.4c)$$

$$\hat{F}^{**}(k, m) = 2 \int_{\tilde{X}=-\infty}^{\infty} \int_{Y=0}^{\infty} e^{-(k^2 + m^2)^{\frac{1}{2}} Y - ik\tilde{X}} \delta(\tilde{X}, Y) d\tilde{X} dY, \quad (5.4d)$$

similar to §4. In consequence the wing and body surface flows near the trailing edge affect each other about equally, in general (c.f. the leading edge, in §2-4, where the major interaction is one-way only) and a challenging, nonlinear, coupled three-dimensional problem is posed for the local flow adjustment. Some useful analytical properties can be derived however if the wing is relatively short, so that L is small. For then the Blasius layer BLI has its skin friction enhanced, since

$$\lambda_1 = \tau_1 L^{-\frac{1}{2}} \quad (5.5)$$

where $\tau_1 = 0.33206\dots$. Hence the three-dimensional triple-deck associated with the wing surface, in (5.2), shrinks, by factors $L^{5/8}$ in all the $\tilde{X}, Y, \bar{z}$ directions, and for $\bar{z}$ of order unity the trailing-edge motion becomes virtually planar (and as in 9). Indeed, for $\tilde{X}, Y, \bar{z}$ all of $O(1)$, i.e. in the three-dimensional upper deck which controls the body surface motion, we have potential flow past the classical planar displacement

$$\hat{\delta}(\tilde{X}, \bar{z}) = \begin{cases} 0 & \text{for } \tilde{X} < 0 \\ -\beta_1 L^{1/6} \tilde{X}^{1/3} & \text{for } \tilde{X} > 0, \end{cases} \quad (5.6a)$$

$$(5.6b)$$

when L is small. Here β_1 is a finite positive constant, and (5.6b) describes the Goldstein (10) near-wake. The irregularity at $\tilde{X} = 0 \pm$ in (5.6a,b) is smoothed out when $\tilde{X}$ is smaller, of order $L^{5/8}$. The wing-body interaction of (5.1) - (5.4d) becomes mostly one-way again, and linear, since (5.4a) now yields (2.2a) with

$$hf(X) \rightarrow \hat{\delta}(\tilde{X}, \bar{z}) \quad (5.6c)$$

The reverse effect, of the body on the wing flow is now diminished. Hence the three-dimensional trailing-edge adjustment considered ^e~~reduces~~ to the linearized analysis of §3. The corresponding solutions for the body surface flow near the trailing edge are presented in fig. 9. As expected, they are similar to those of §3 except that the - sign in (5.6b) alters all the signs involved. Thus, on the body surface, we obtain

increased displacement	}	ahead of ~~the~~ trailing	(5.7a)
decreased pressure		edge, close to the	
increased streamwise shear		junction ,	
inward cross-flow shear			

decreased displacement	}	in the wake,	(5.7b)
increased pressure		almost directly	
decreased streamwise shear		behind the wing .	
outward cross flow shear			

This tends to confirm the roughly equal but opposite effects of the leading and trailing edges described in §3. Also, there is no corridor, and the $\tilde{X}^{-1/3}$ narrowing downstream is covered by earlier comments. Note that the above applies for L small but greater than $O(Re^{-3/8})$, so that the trailing-edge interaction remains distinct from the earlier one at the leading edge. It also applies near the trailing edges of short flat-plate wings like those in §3, if the scaled chord length ℓ there is large.

The displacement decrease in (5.7b), which is pronounced, and the associated pressure increase there, would appear to agree with physical expectations for the flow behind the wing.

A computational treatment for the nonlinear version of the trailing-edge junction flow, in (5.1)-(5.4d), would be of considerable interest. Experimental studies of the central, flat plate, wing-body junction model (Fig. 8) could also be an aid to furthering our theoretical and physical understanding.

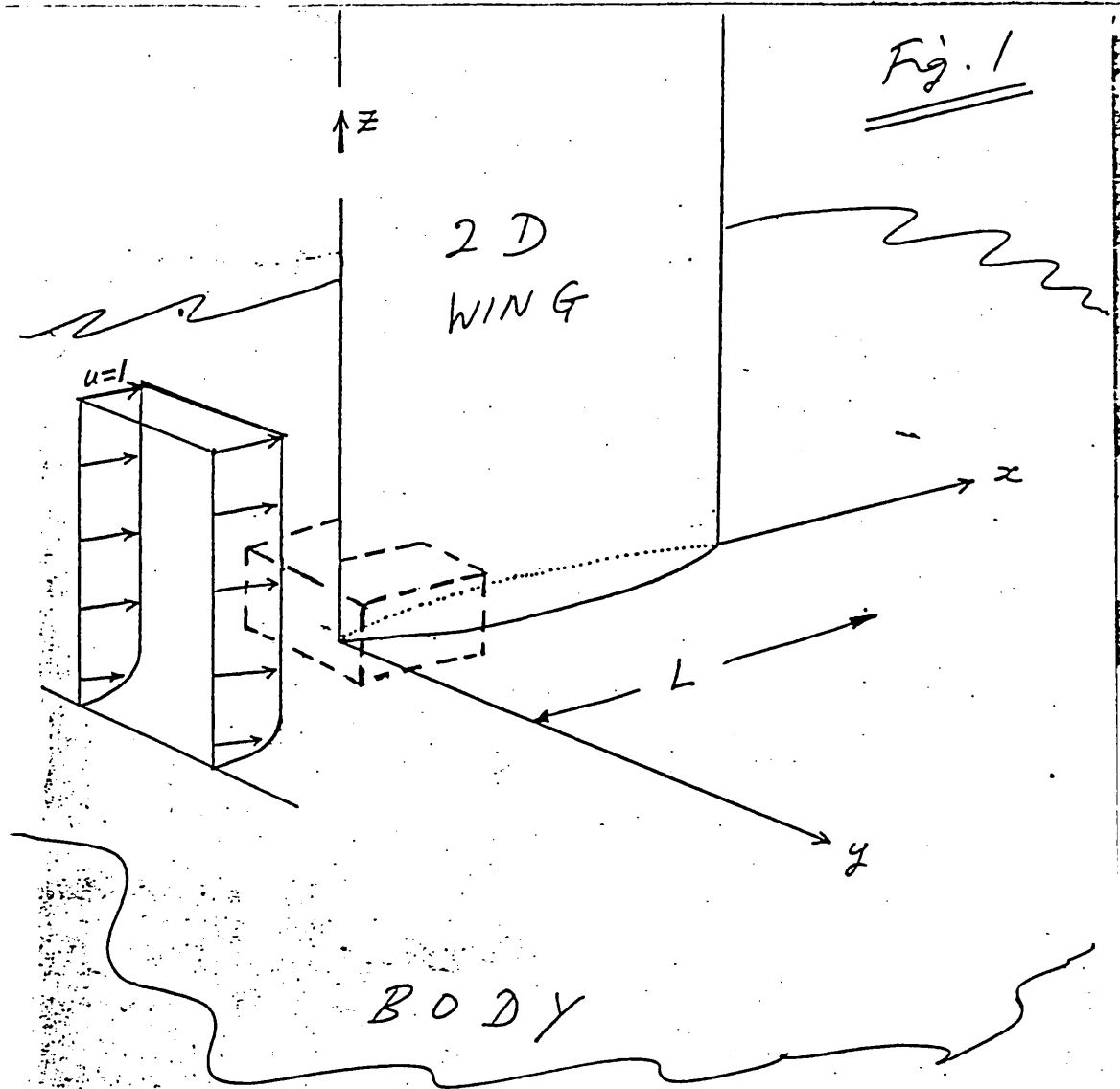
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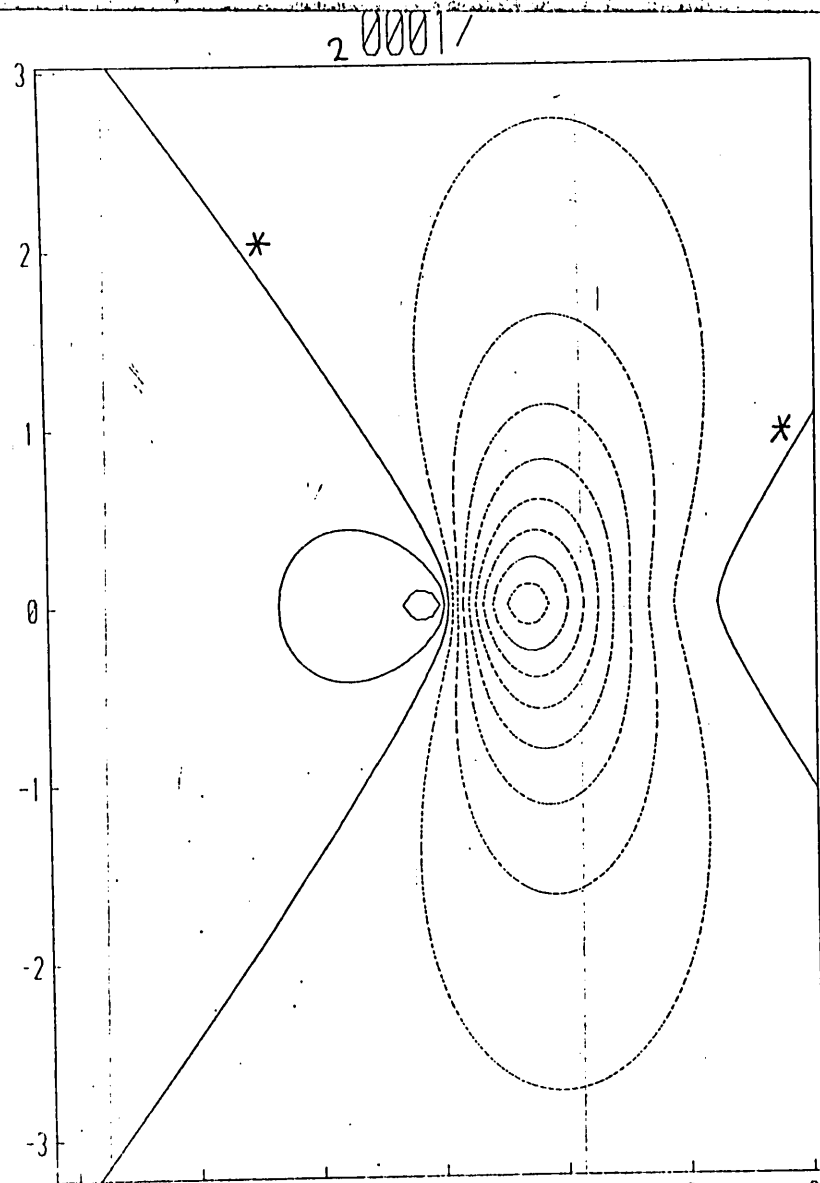
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Fig. 1



349



CONTOUR KEY	
1	-0.14E+00
2	-0.12E+00
3	-0.10E+00
4	-0.87E-01
5	-0.70E-01
6	-0.52E-01
7	-0.35E-01
8	-0.17E-01
9	0.00E+00
10	0.17E-01
11	0.35E-01

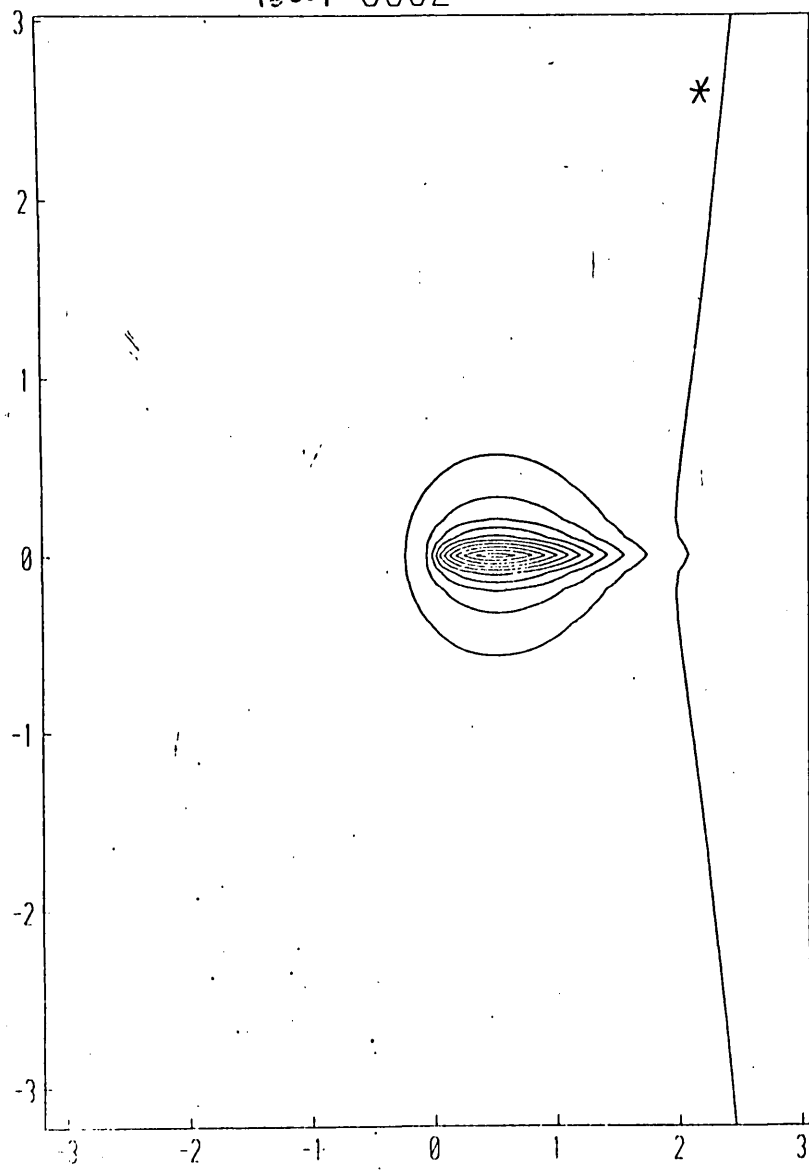
(*)

P

Fig. 2(a)

350

19601 000-71

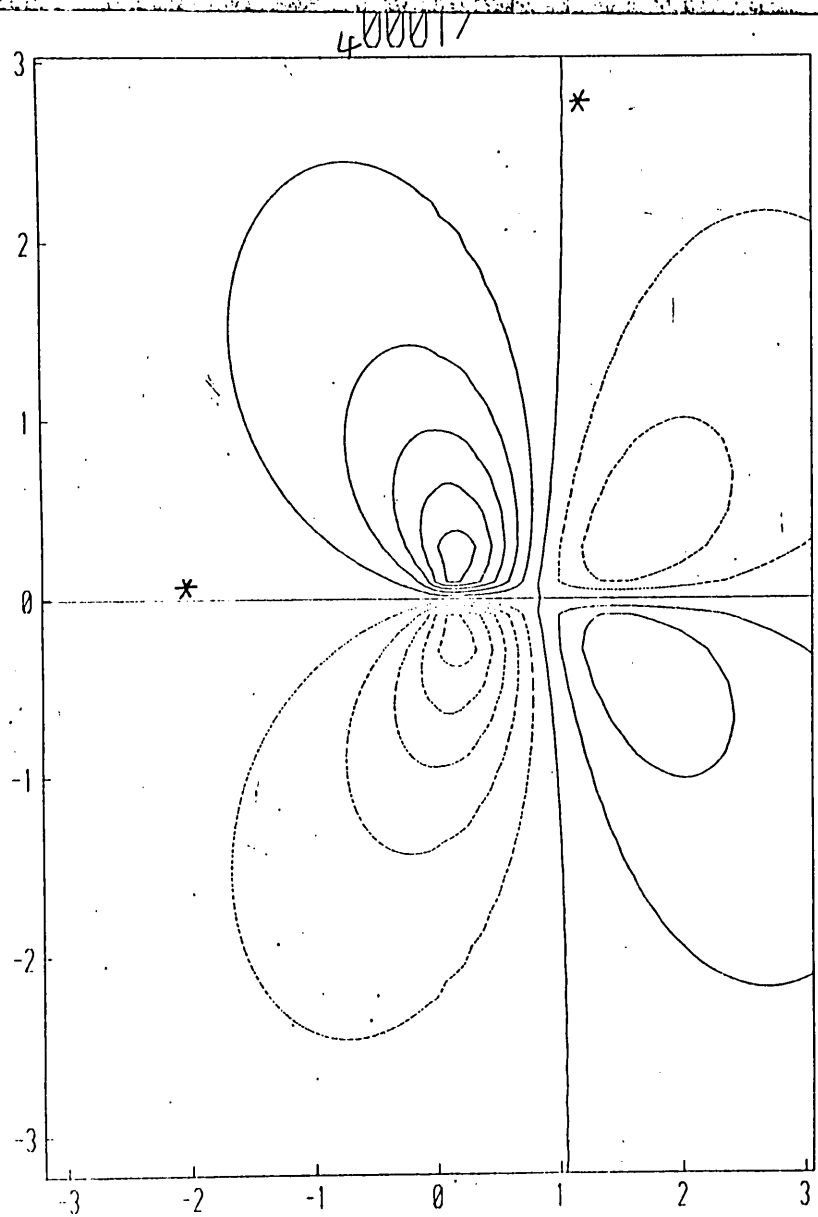


CONTOUR KEY	
1	0.00E+00
2	0.10E+00
3	0.21E+00
4	0.31E+00
5	0.42E+00
6	0.52E+00
7	0.63E+00
8	0.73E+00
9	0.83E+00
10	0.94E+00
11	0.10E+01

(*)

2(6)
-8

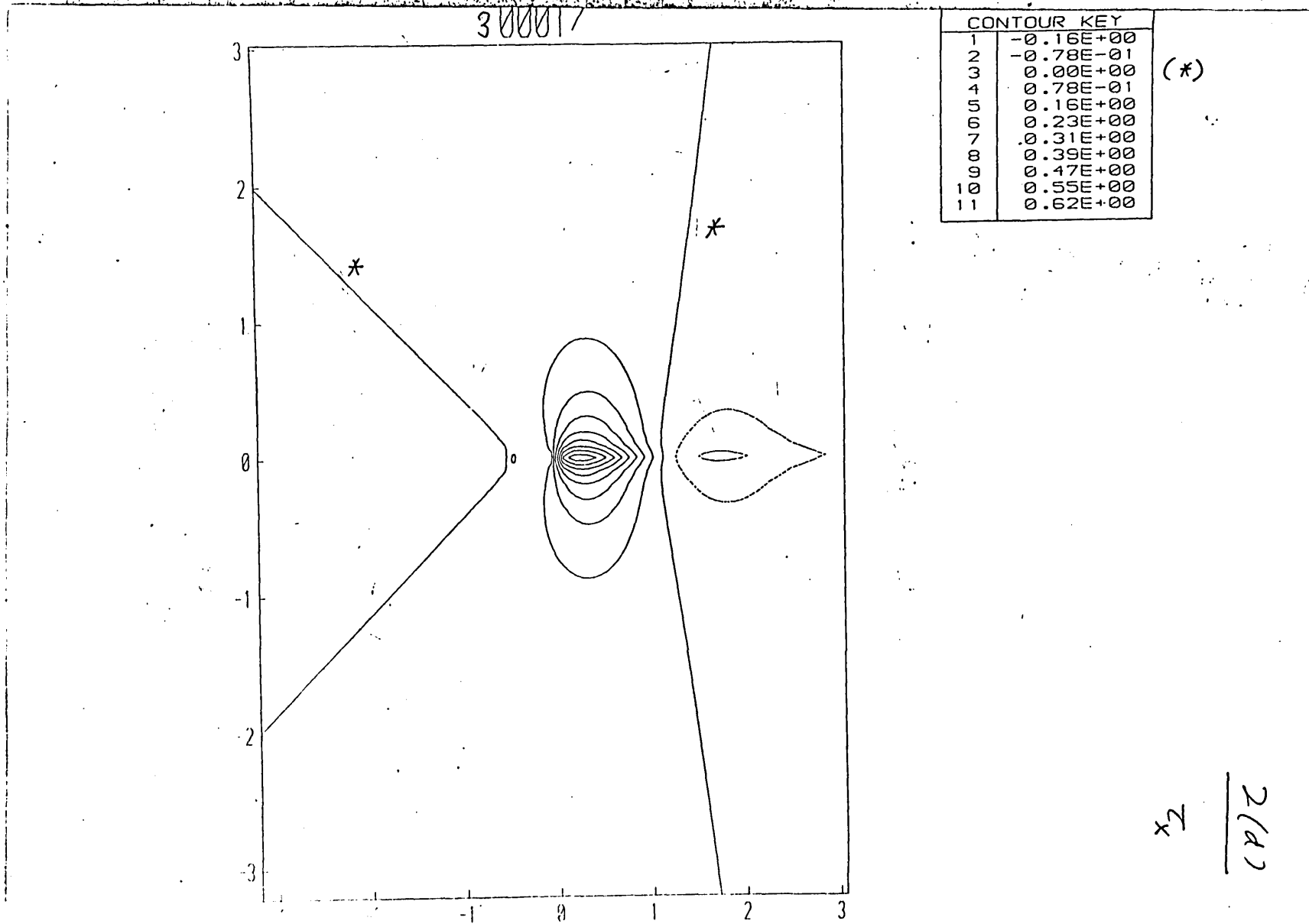
351



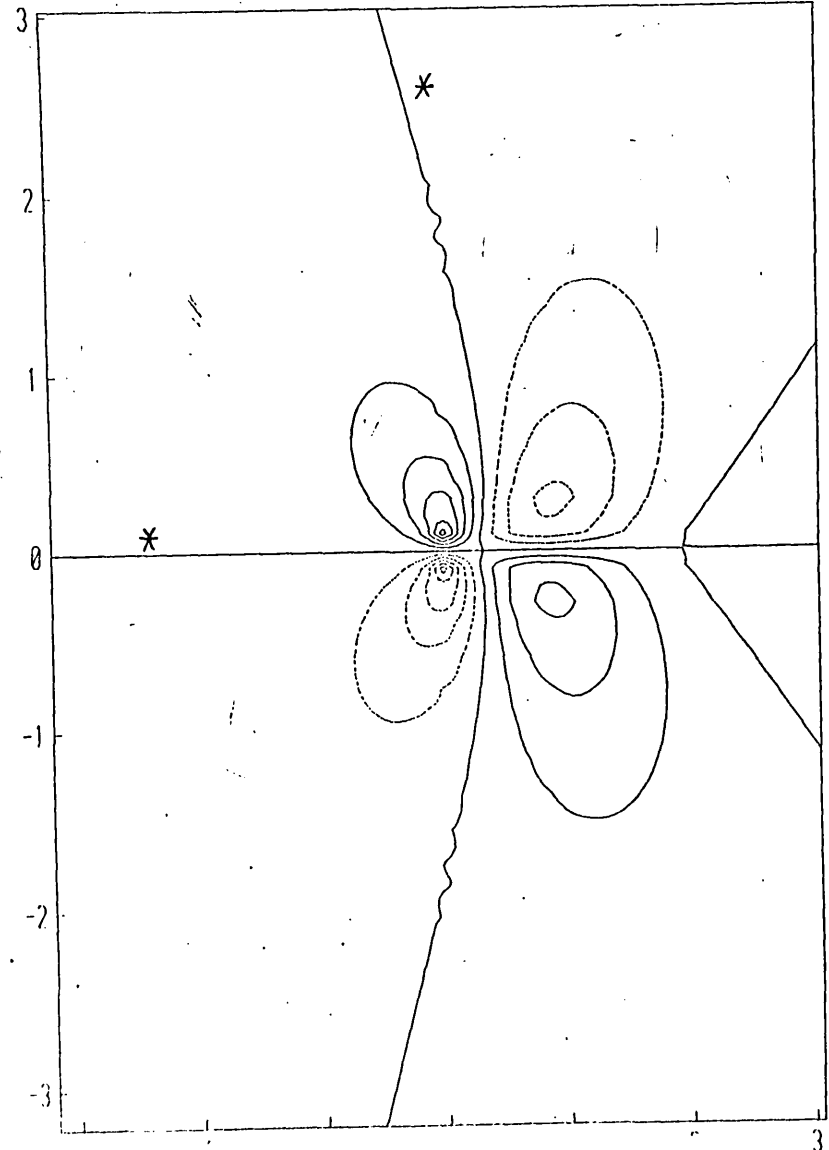
CONTOUR KEY	
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3	-0.38E-01
4	-0.25E-01
5	-0.13E-01
6	0.00E+00
7	0.13E-01
8	0.25E-01
9	0.38E-01
10	0.51E-01
11	0.64E-01

(*)

2(c)
D



500017

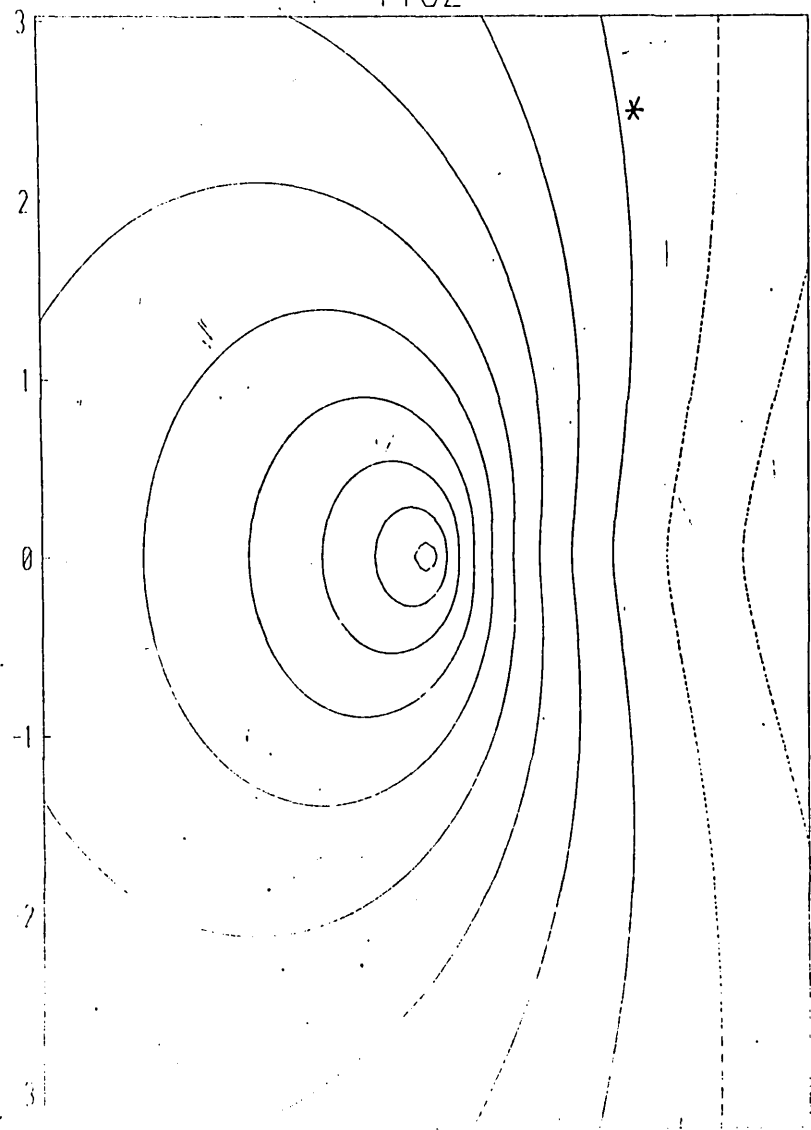


CONTOUR KEY	
1	-0.12E+00
2	-0.99E-01
3	-0.75E-01
4	-0.50E-01
5	-0.25E-01
6	0.00E+00
7	0.25E-01
8	0.50E-01
9	0.75E-01
10	0.99E-01
11	0.12E+00

(*)

$\frac{2(e^2)}{2\gamma}$

1152



CONTOUR KEY	
1	-0.68E-01
2	-0.34E-01
3	0.00E+00
4	0.34E-01
5	0.68E-01
6	0.10E+00
7	0.14E+00
8	0.17E+00
9	0.20E+00
10	0.24E+00
11	0.27E+00

(*)

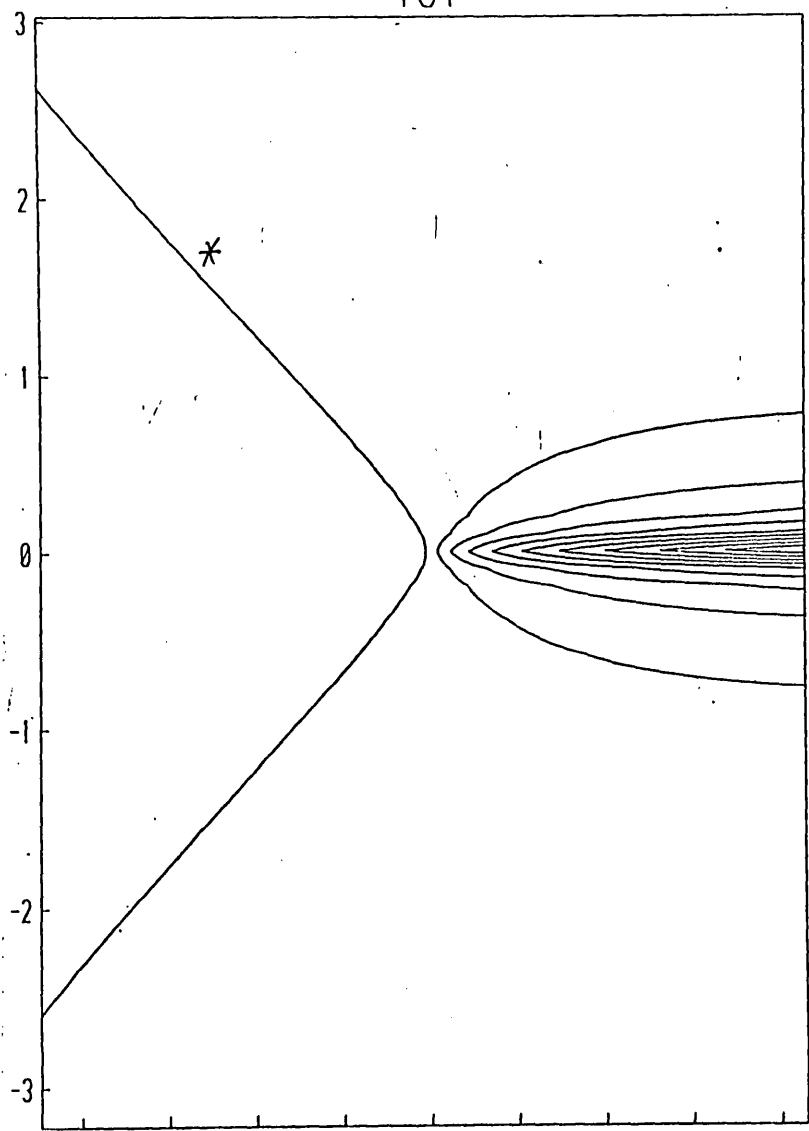
Fig. 3(a)

P

354

355

151

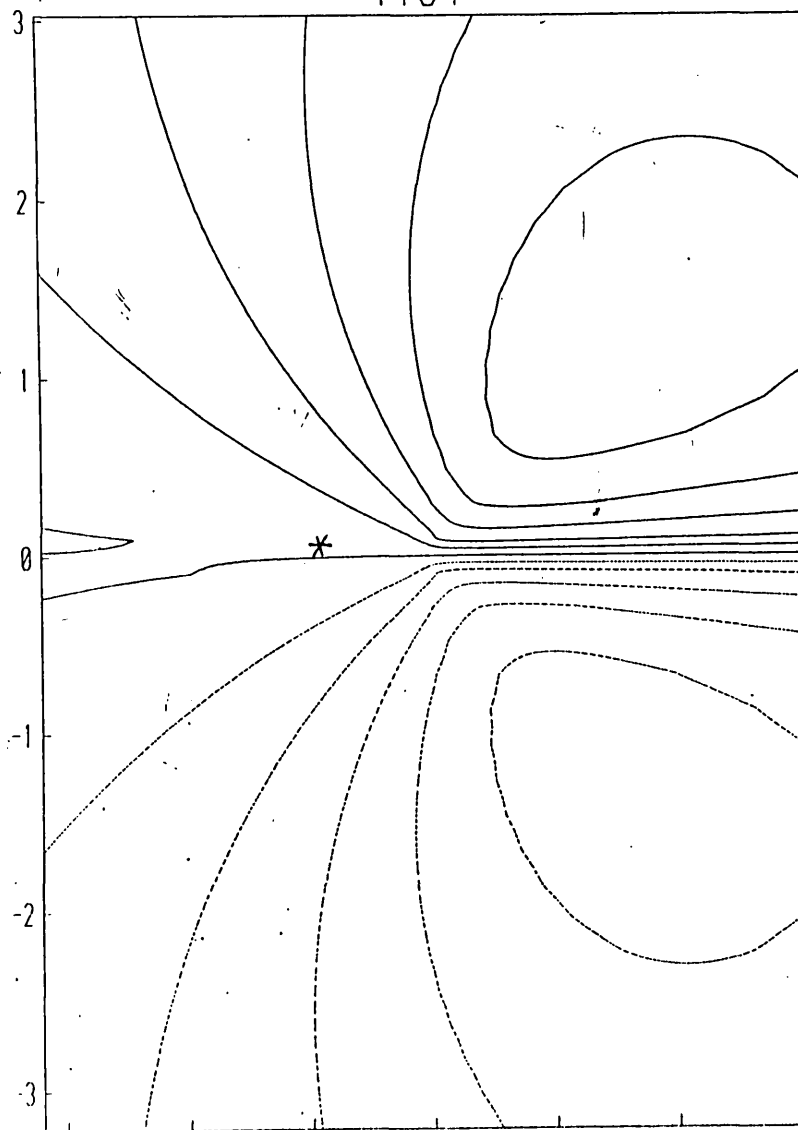


(*)

CONTOUR KEY	
1	0.00E+00
2	0.40E+00
3	0.80E+00
4	0.12E+01
5	0.16E+01
6	0.20E+01
7	0.24E+01
8	0.28E+01
9	0.32E+01
10	0.36E+01
11	0.40E+01

$\frac{3(b)}{-8}$

1154

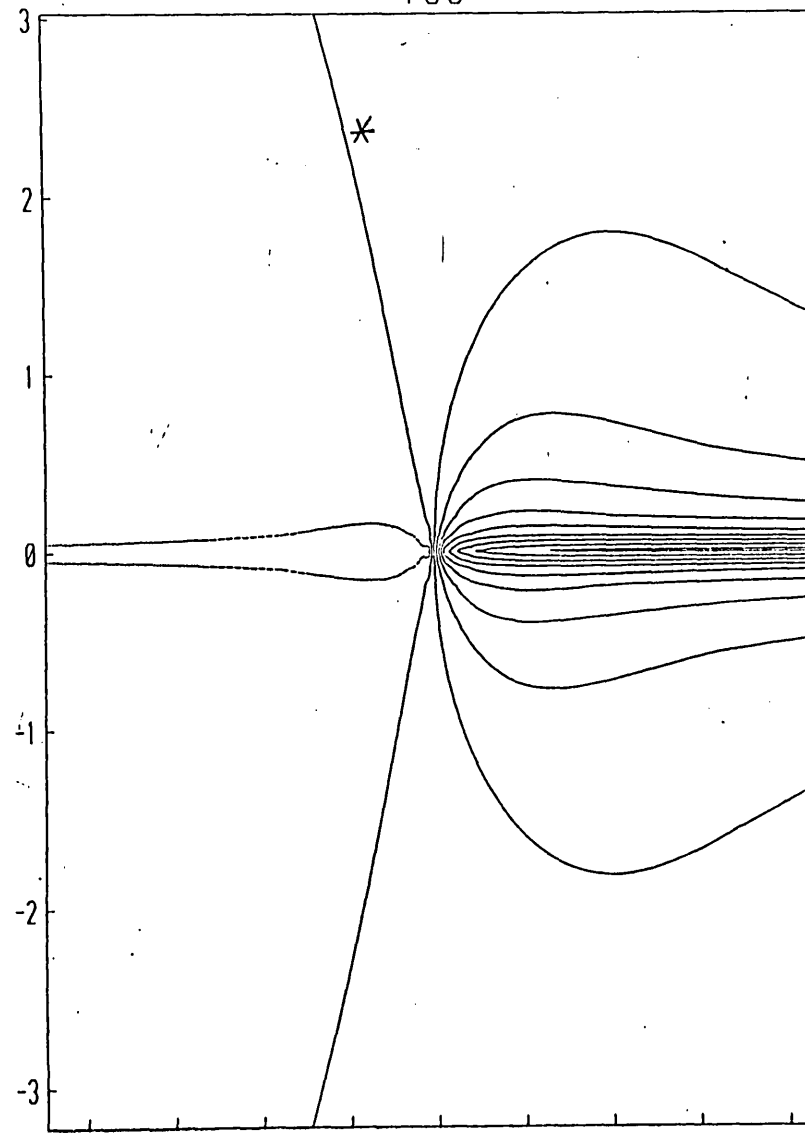


CONTOUR KEY	
1	-0.19E+00
2	-0.15E+00
3	-0.12E+00
4	-0.77E-01
5	-0.39E-01
6	0.00E+00
7	0.39E-01
8	0.77E-01
9	0.12E+00
10	0.15E+00
11	0.19E+00

(*)

3(c)
D

153



(*)

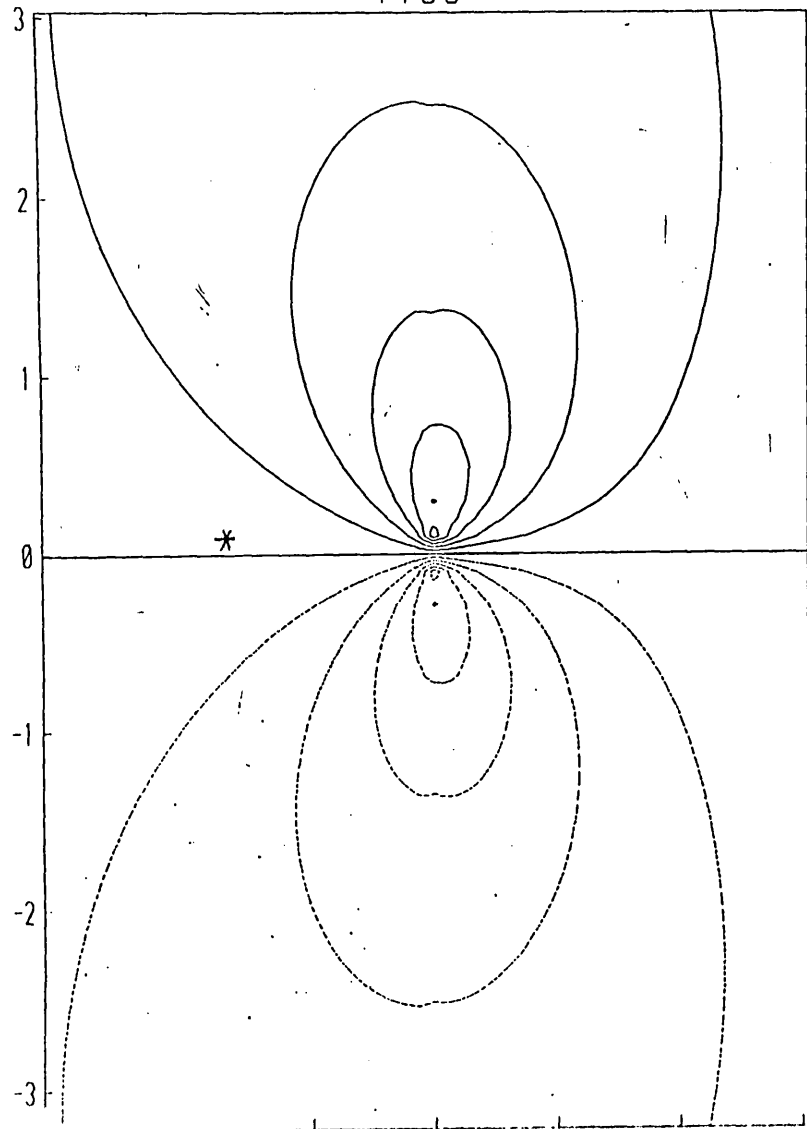
CONTOUR KEY	
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2	0.00E+00
3	0.87E-01
4	0.17E+00
5	0.26E+00
6	0.35E+00
7	0.44E+00
8	0.52E+00
9	0.61E+00
10	0.70E+00
11	0.79E+00

$\frac{3(4)}{x^2}$

453

358

1155

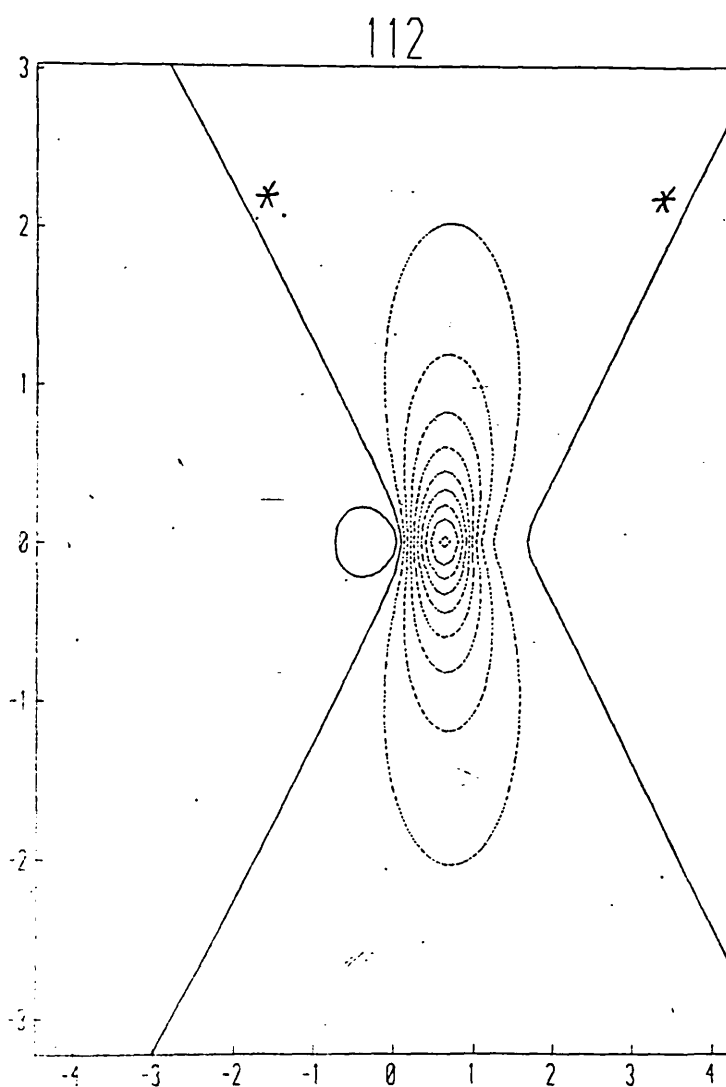


CONTOUR KEY	
1	-0.15E+00
2	-0.12E+00
3	-0.90E-01
4	-0.60E-01
5	-0.30E-01
6	0.00E+00
7	0.30E-01
8	0.60E-01
9	0.90E-01
10	0.12E+00
11	0.15E+00

(*)

$\frac{3(4)}{2}$

Fig. 4(a)

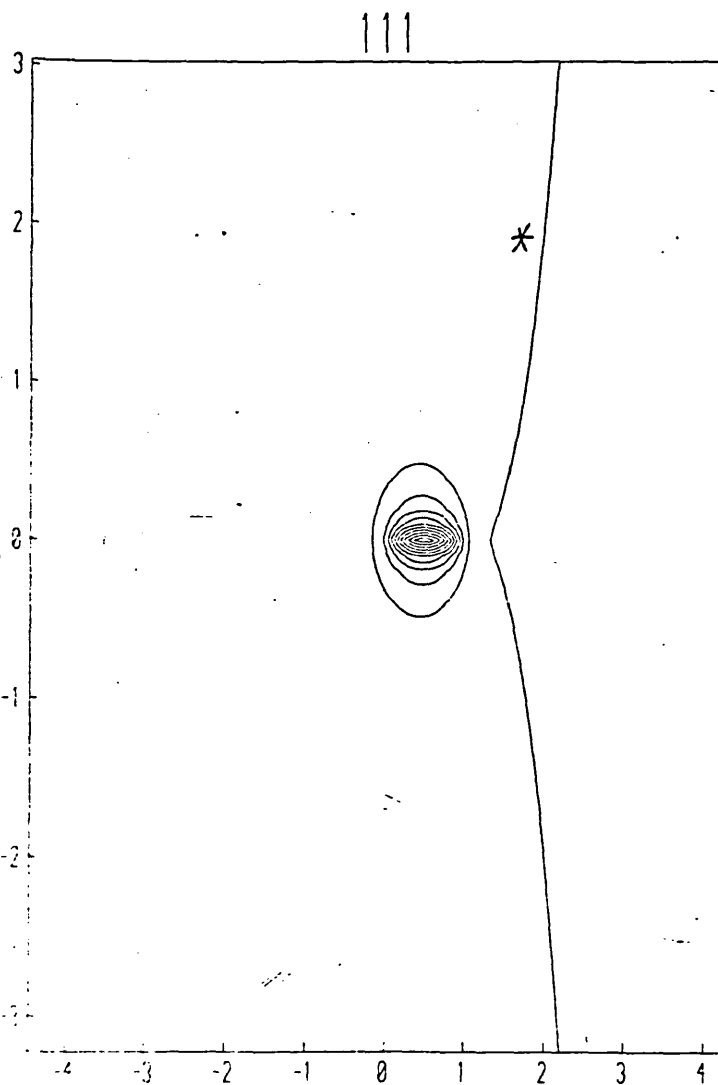


(*)

CONTOUR KEY	
1	-0.69E-01
2	-0.62E-01
3	-0.54E-01
4	-0.46E-01
5	-0.38E-01
6	-0.31E-01
7	-0.23E-01
8	-0.15E-01
9	-0.77E-02
10	0.20E+00
11	0.77E-02

P

4(b)



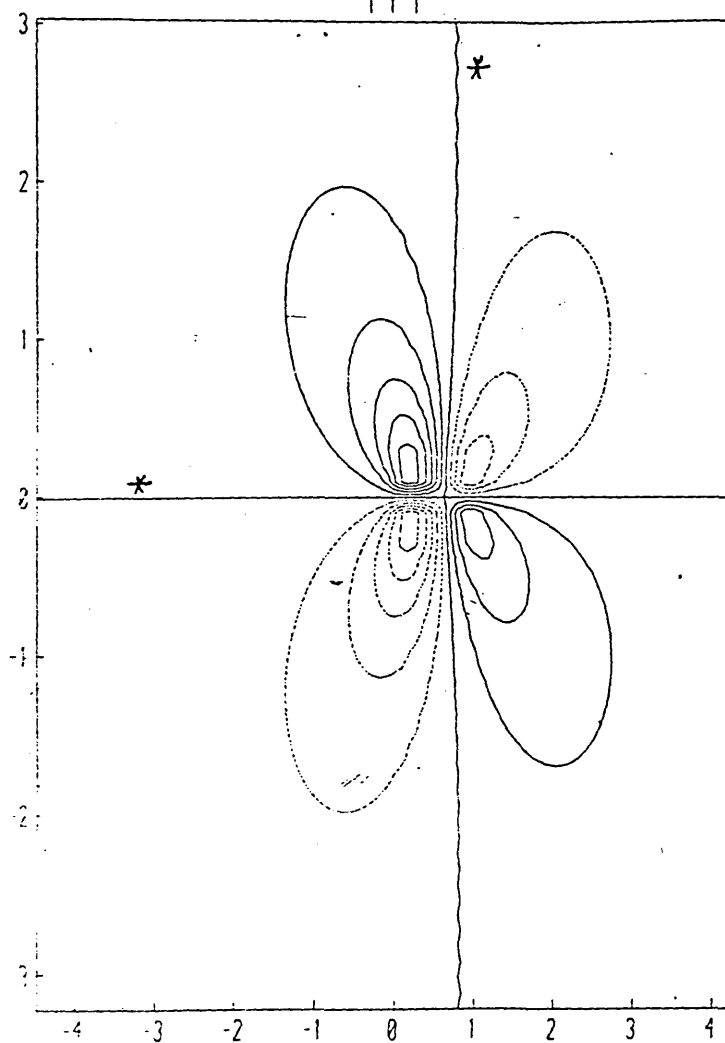
(*)

CONTOUR KEY	
1	0.02E+20
2	0.44E+01
3	0.87E+01
4	0.13E+20
5	0.17E+20
6	0.22E+20
7	0.26E+20
8	0.31E+20
9	0.35E+20
10	0.39E+20
11	0.44E+20

-8

4(c)

114

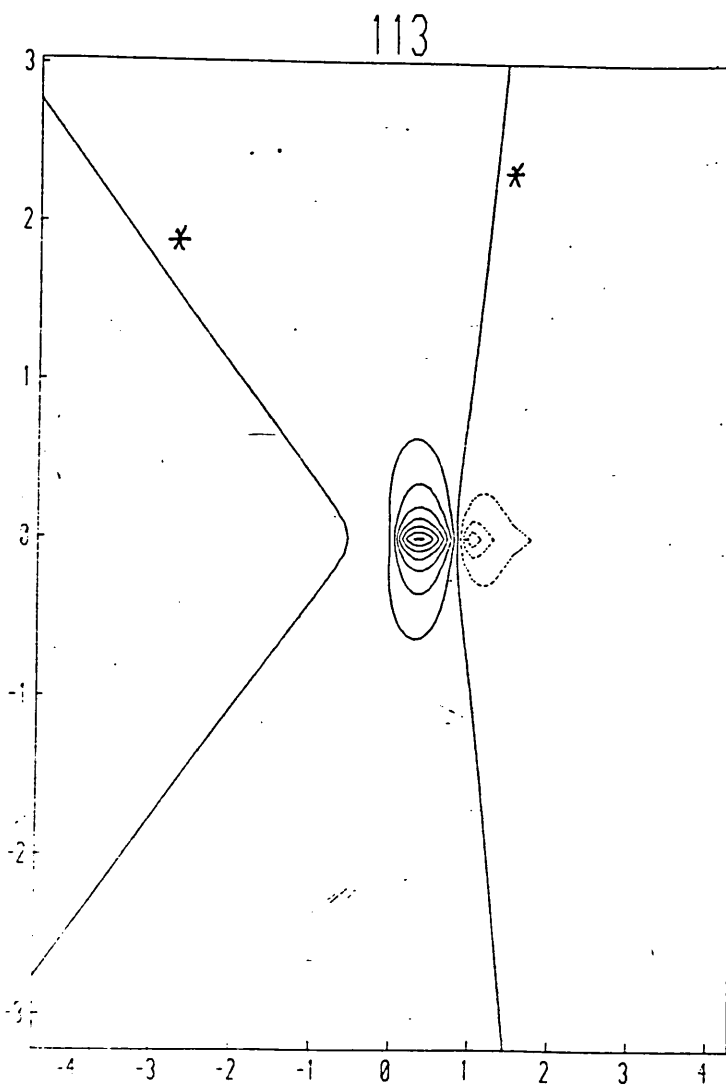


(*)

CONTOUR KEY	
1	-0.24E-01
2	-0.20E-01
3	-0.15E-01
4	-0.98E-02
5	-0.49E-02
6	0.00E+00
7	0.49E-02
8	0.98E-02
9	0.15E-01
10	0.20E-01
11	0.24E-01

D

4(d)

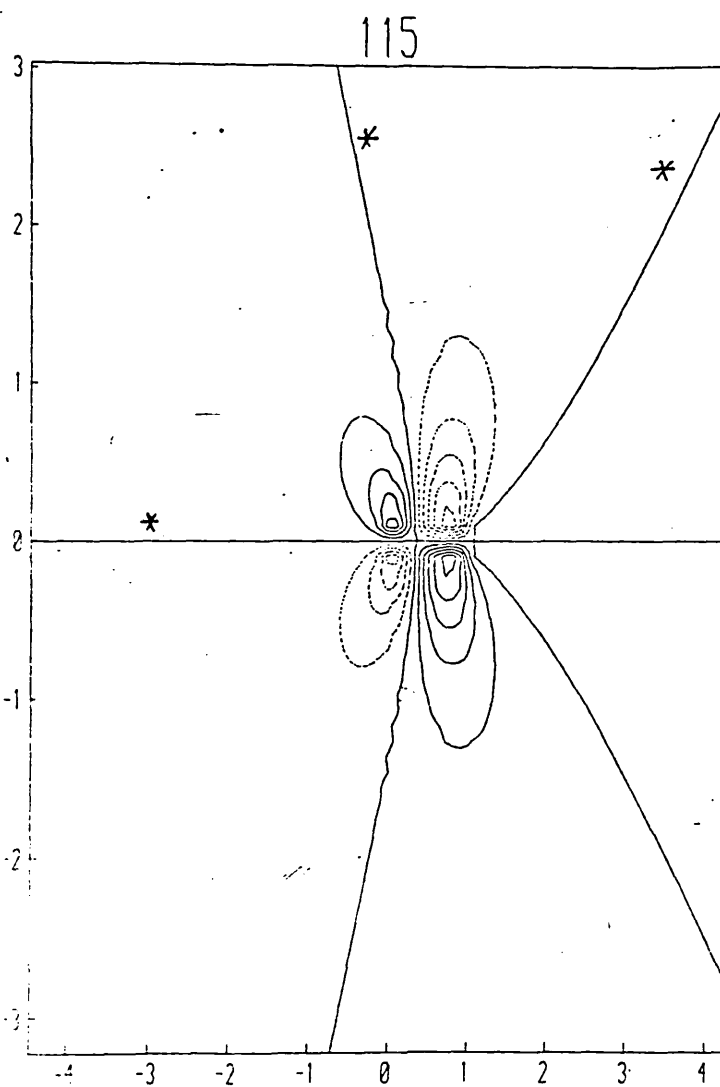


(*)

CONTOUR KEY	
1	-0.13E+20
2	-0.87E-01
3	-0.43E-01
4	0.00E+00
5	0.43E-01
6	0.87E-01
7	0.13E+00
8	0.17E+00
9	0.22E+00
10	0.26E+00
11	0.30E+00

z_x

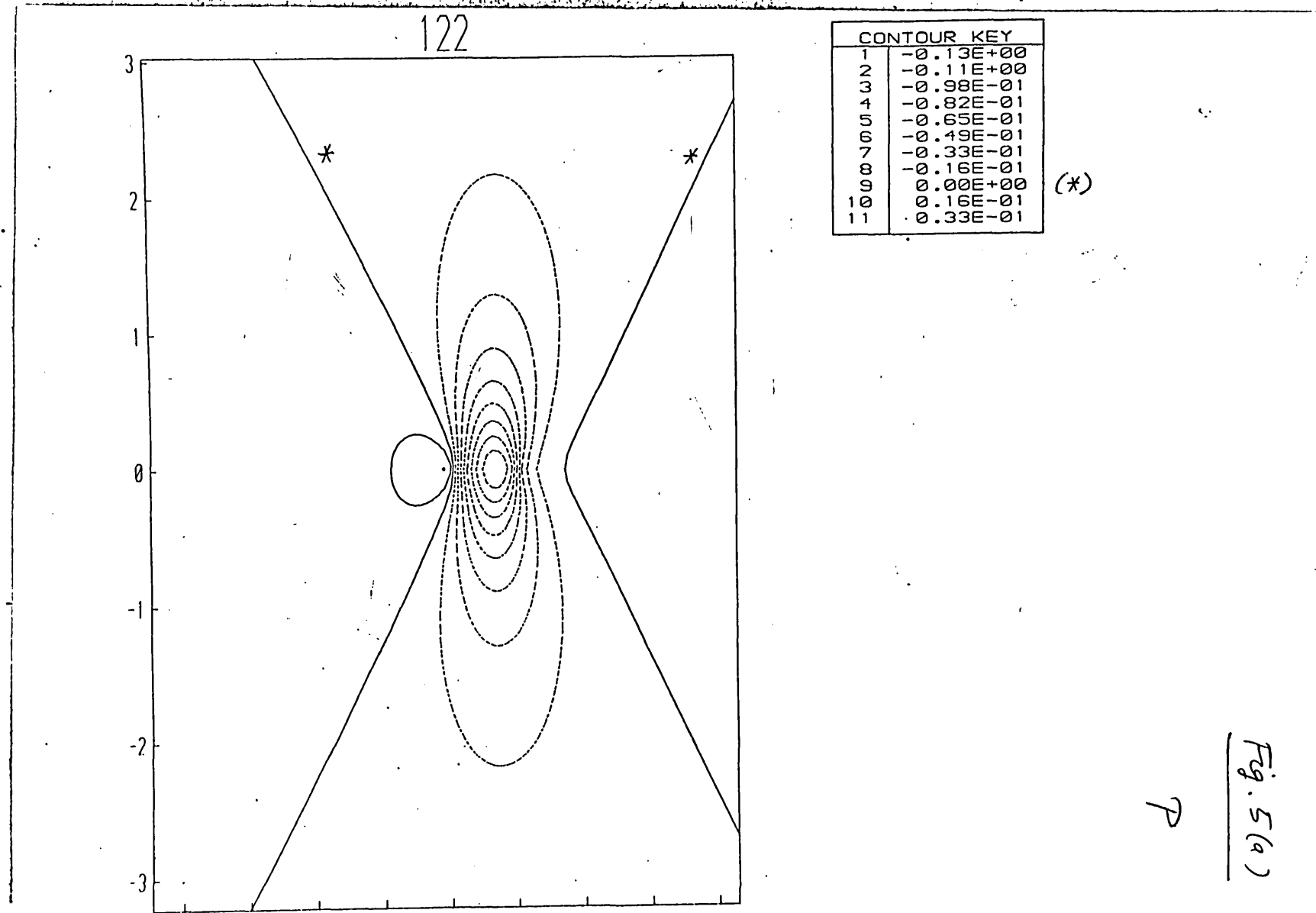
4(e)



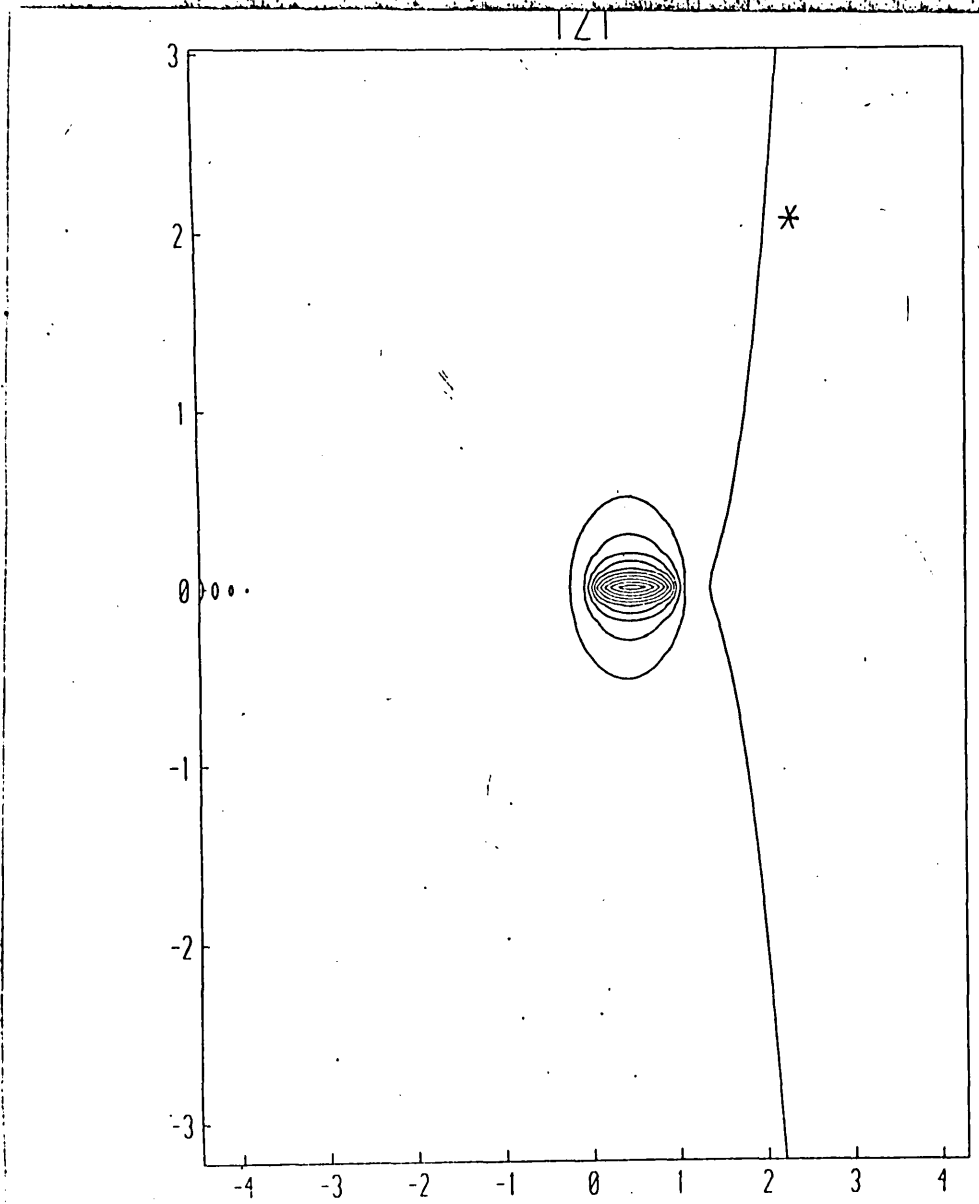
CONTOUR KEY	
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2	-0.41E-01
3	-0.31E-01
4	-0.21E-01
5	-0.12E-01
6	0.00E+00
7	0.10E-01
8	0.21E-01
9	0.31E-01
10	0.41E-01
11	0.51E-01

(*)

z_y



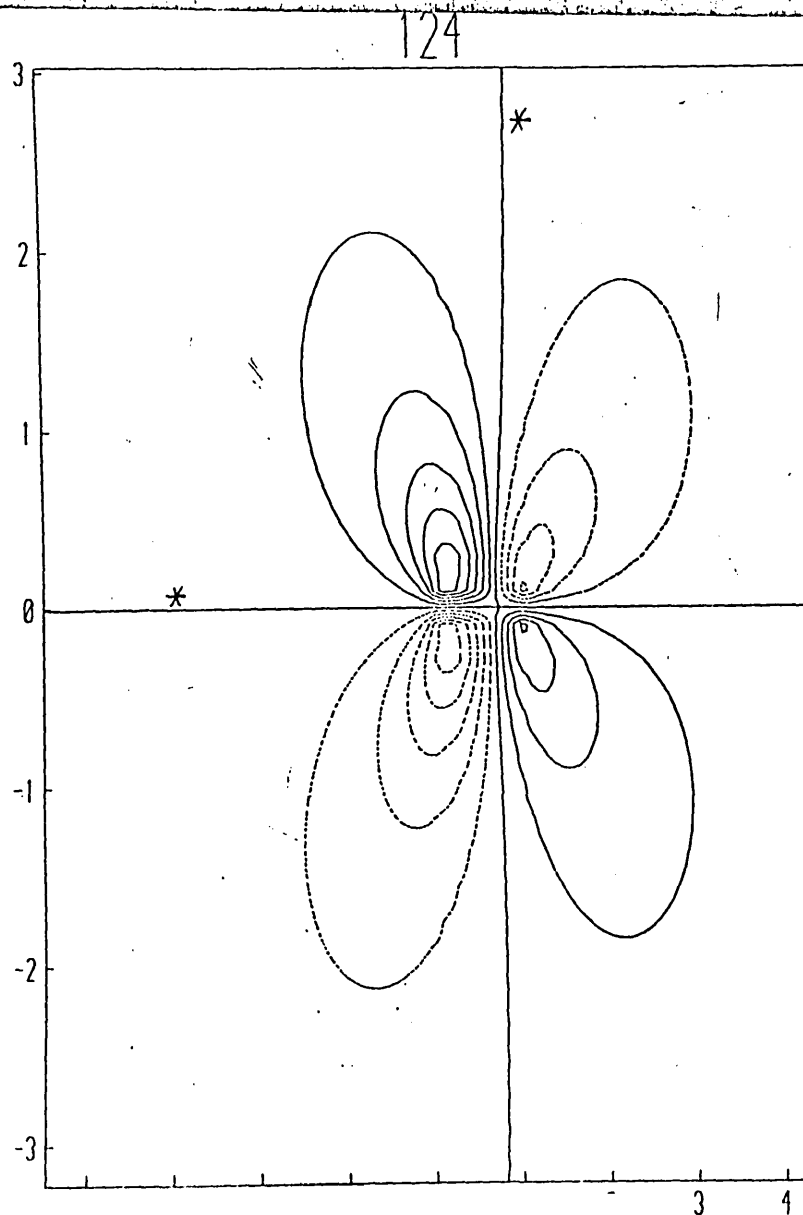
365



CONTOUR KEY	
1	0.00E+00
2	0.91E-01
3	0.18E+00
4	0.27E+00
5	0.37E+00
6	0.46E+00
7	0.55E+00
8	0.64E+00
9	0.73E+00
10	0.82E+00
11	0.91E+00

(*)

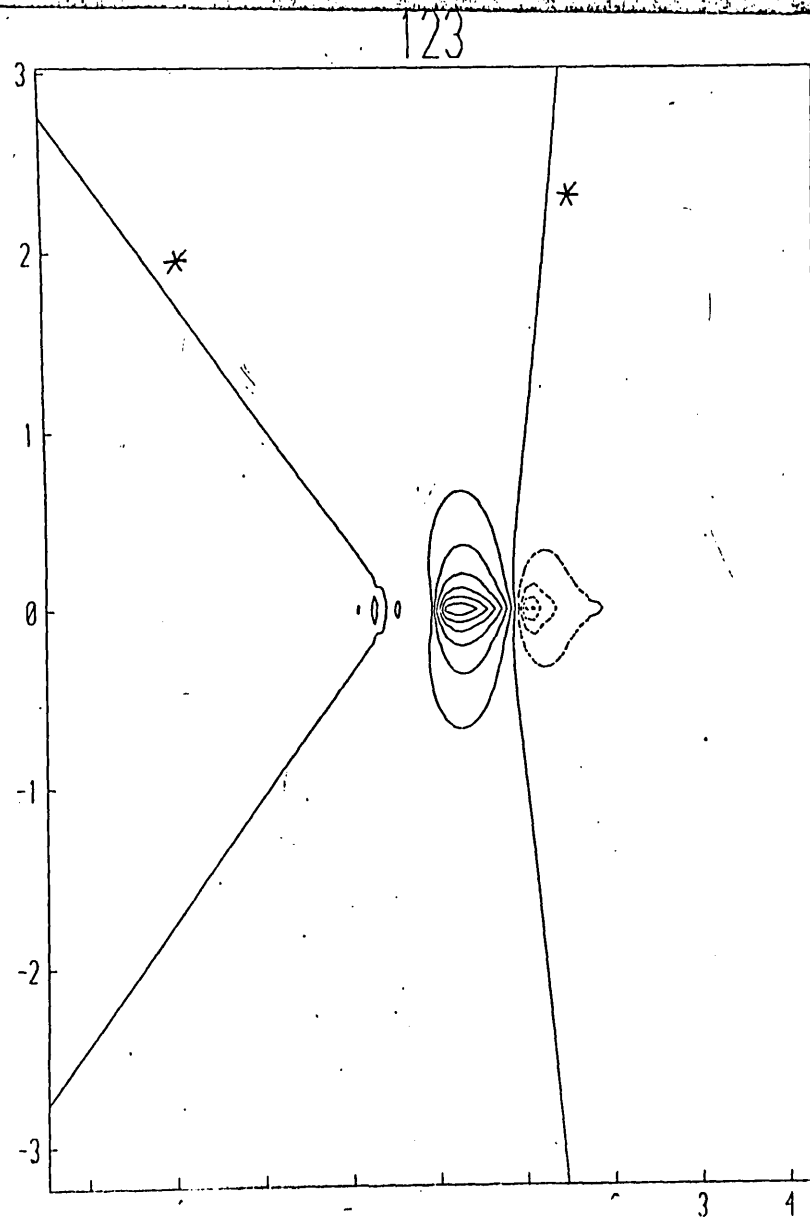
5(4)
8



CONTOUR KEY	
1	-0.52E-01
2	-0.42E-01
3	-0.31E-01
4	-0.21E-01
5	-0.10E-01
6	0.00E+00
7	0.10E-01
8	0.21E-01
9	0.31E-01
10	0.42E-01
11	0.52E-01

(*)

567
D

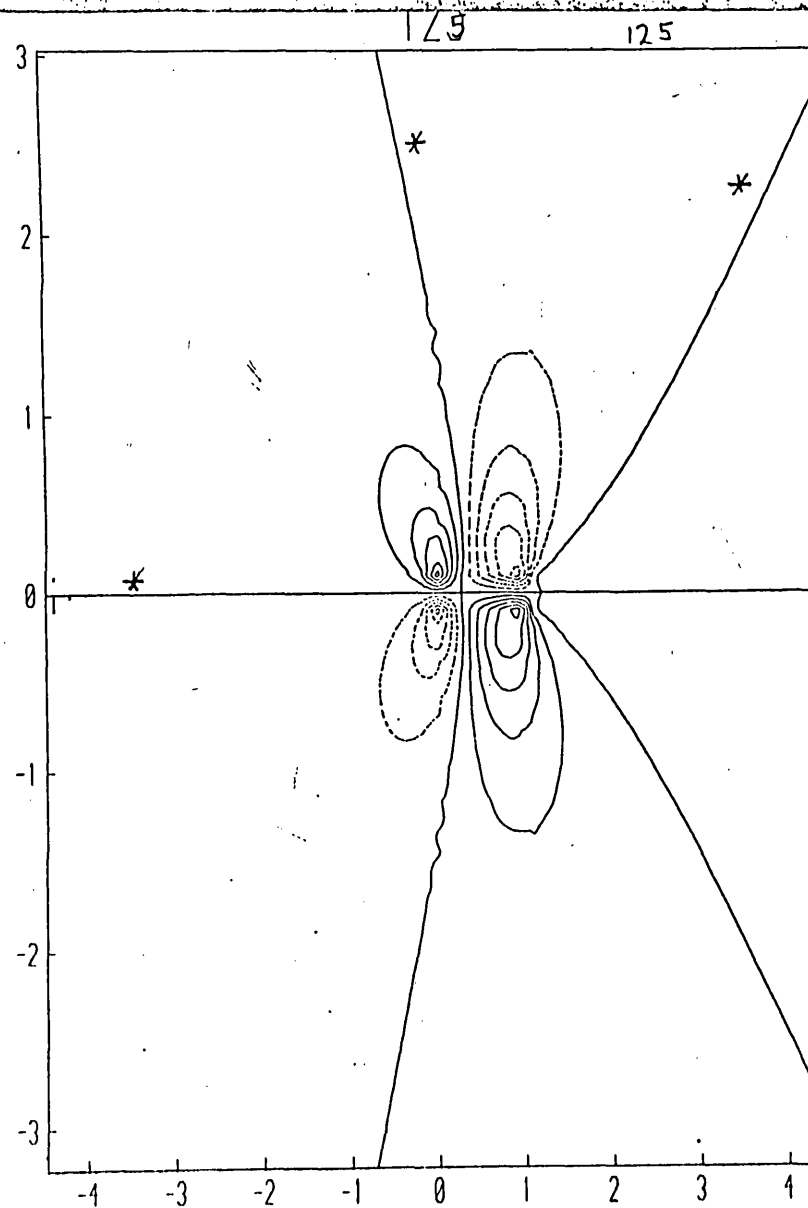


CONTOUR KEY	
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2	-0.27E+00
3	-0.18E+00
4	-0.90E-01
5	0.00E+00
6	0.90E-01
7	0.18E+00
8	0.27E+00
9	0.36E+00
10	0.45E+00
11	0.54E+00

(*)

x₂

5(a)



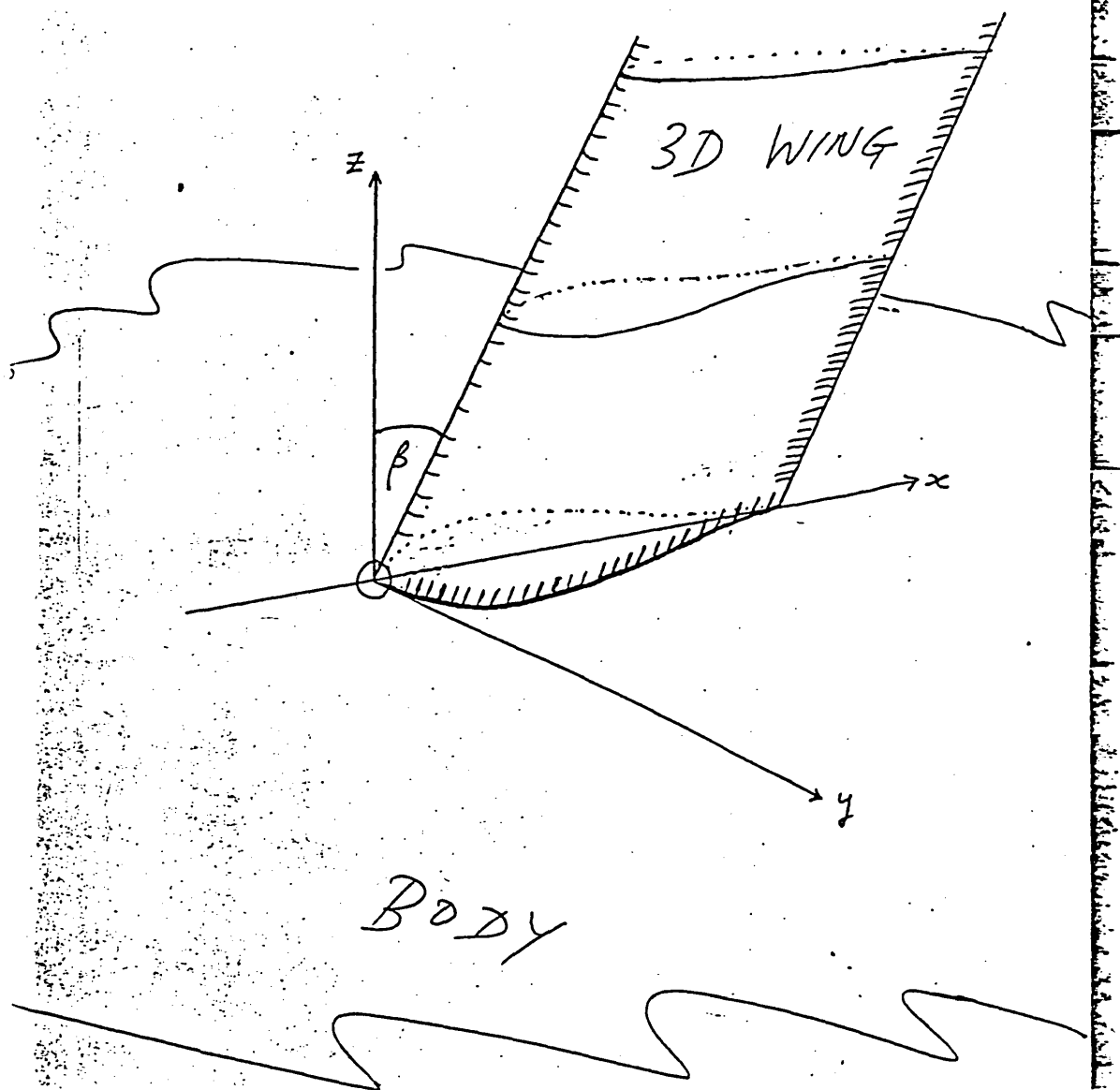
CONTOUR KEY	
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2	-0.89E-01
3	-0.67E-01
4	-0.45E-01
5	-0.22E-01
6	0.00E+00
7	0.22E-01
8	0.45E-01
9	0.67E-01
10	0.89E-01
11	0.11E+00

(*)

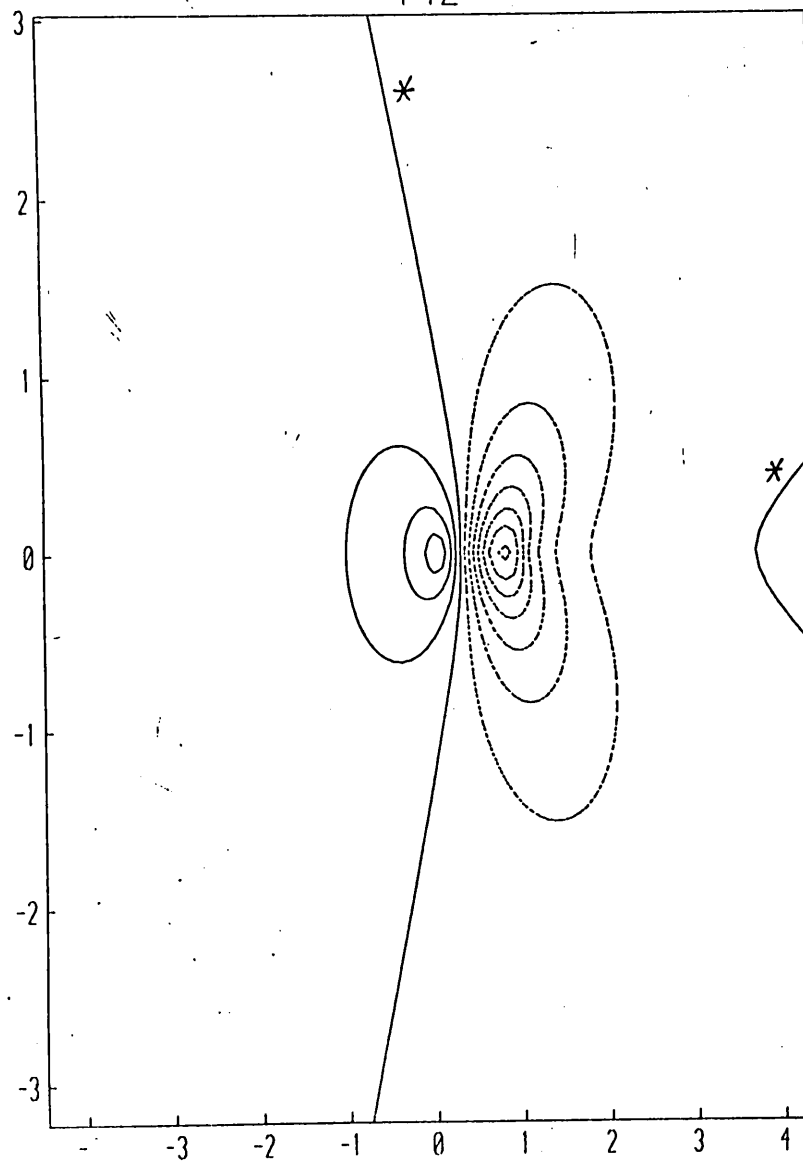
γ_2

5(e)

Fig. 6.



142



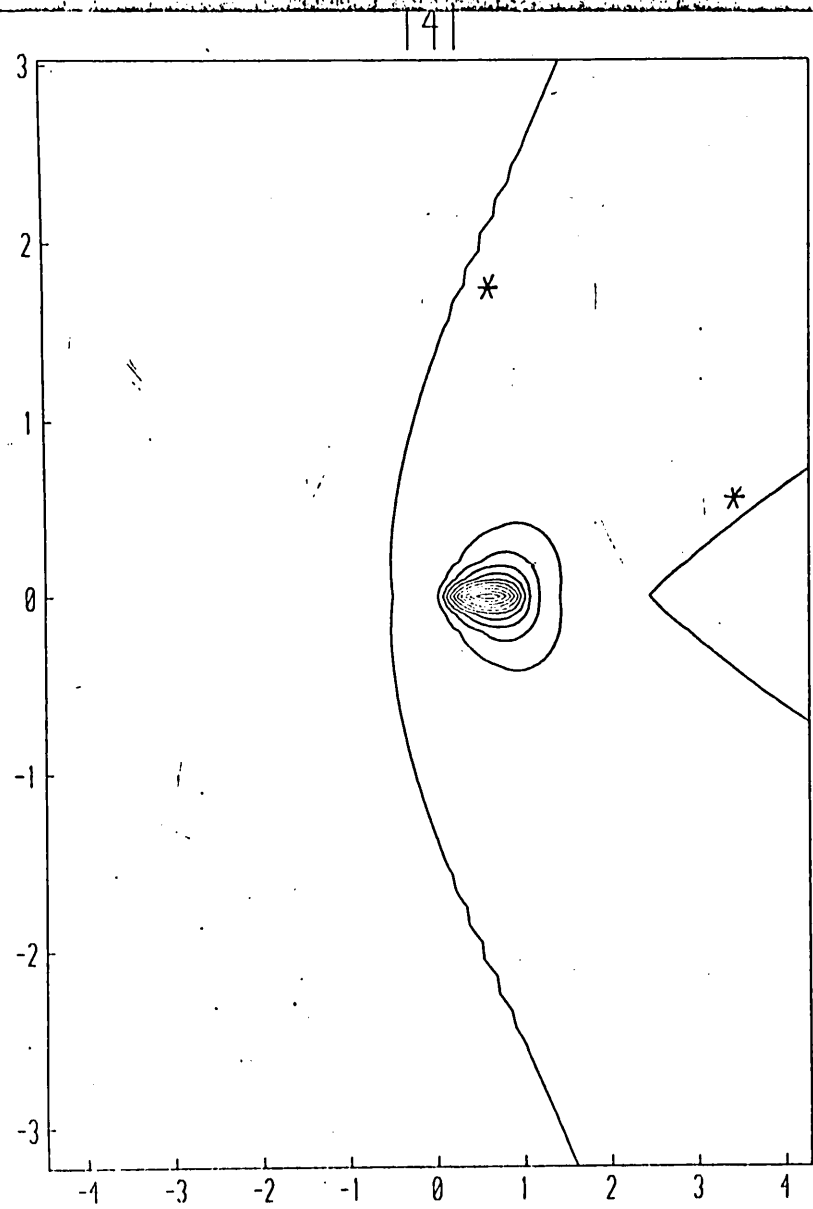
CONTOUR KEY	
1	-0.50E-01
2	-0.43E-01
3	-0.36E-01
4	-0.28E-01
5	-0.21E-01
6	-0.14E-01
7	-0.71E-02
8	0.00E+00
9	0.71E-02
10	0.14E-01
11	0.21E-01

(*)

Fig. 7(a)

P

123

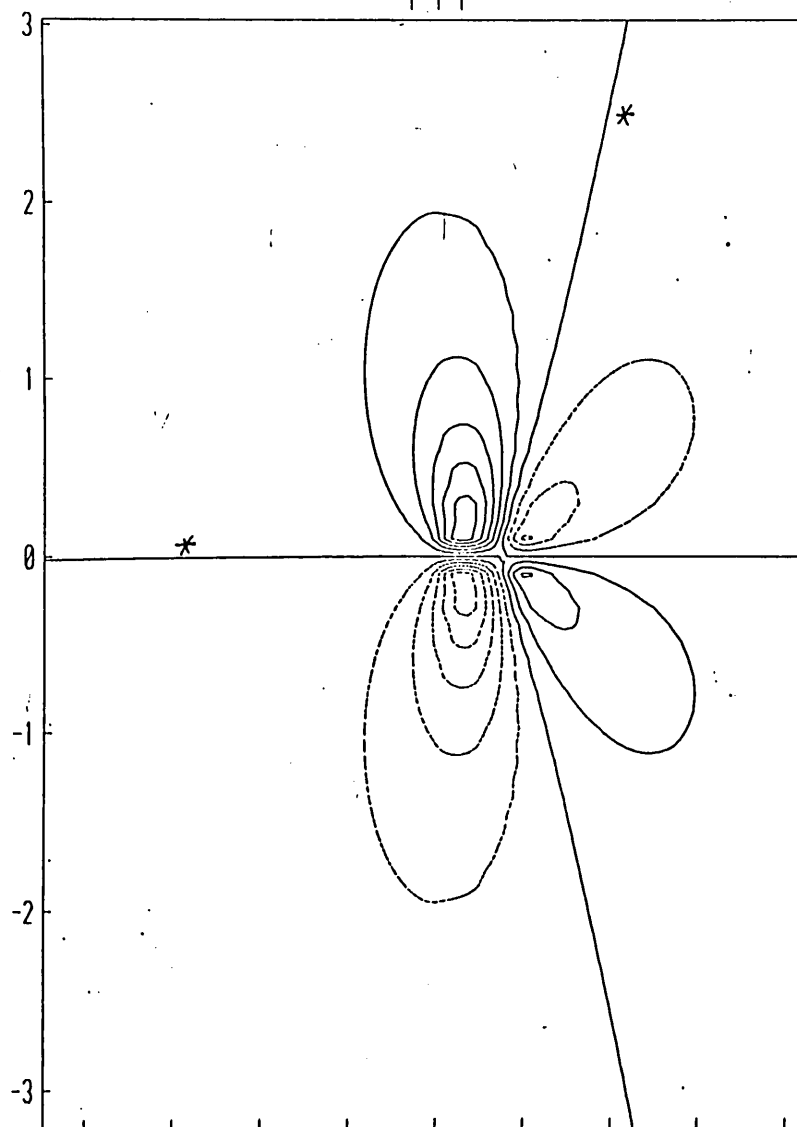


CONTOUR KEY	
1	0.00E+00
2	0.39E-01
3	0.77E-01
4	0.12E+00
5	0.15E+00
6	0.19E+00
7	0.23E+00
8	0.27E+00
9	0.31E+00
10	0.35E+00
11	0.39E+00

(*)

7(6)
—
8

144



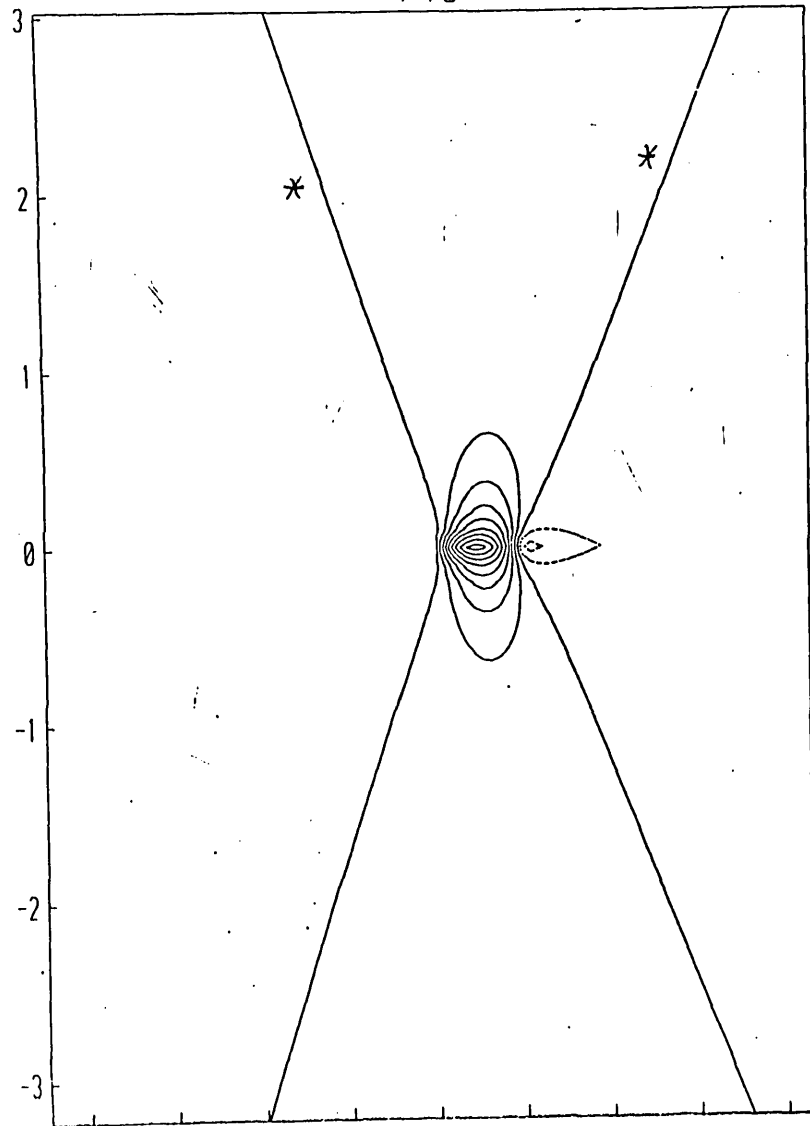
(*)

CONTOUR KEY	
1	-0.23E-01
2	-0.18E-01
3	-0.14E-01
4	-0.91E-02
5	-0.46E-02
6	0.00E+00
7	0.46E-02
8	0.91E-02
9	0.14E-01
10	0.18E-01
11	0.23E-01

7/c)
D

373

143



CONTOUR KEY	
1	-0.62E-01
2	-0.31E-01
3	0.00E+00
4	0.31E-01
5	0.62E-01
6	0.94E-01
7	0.12E+00
8	0.16E+00
9	0.19E+00
10	0.22E+00
11	0.25E+00

(*)

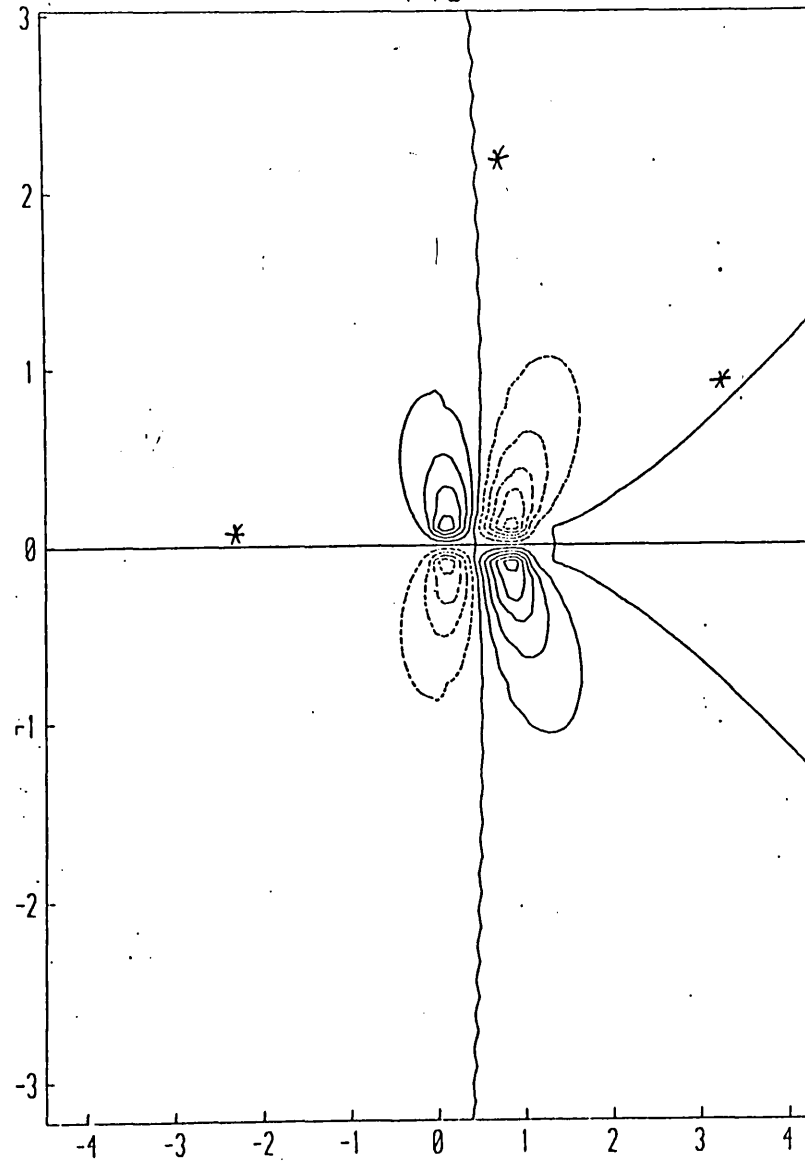
x2

7(4)

145

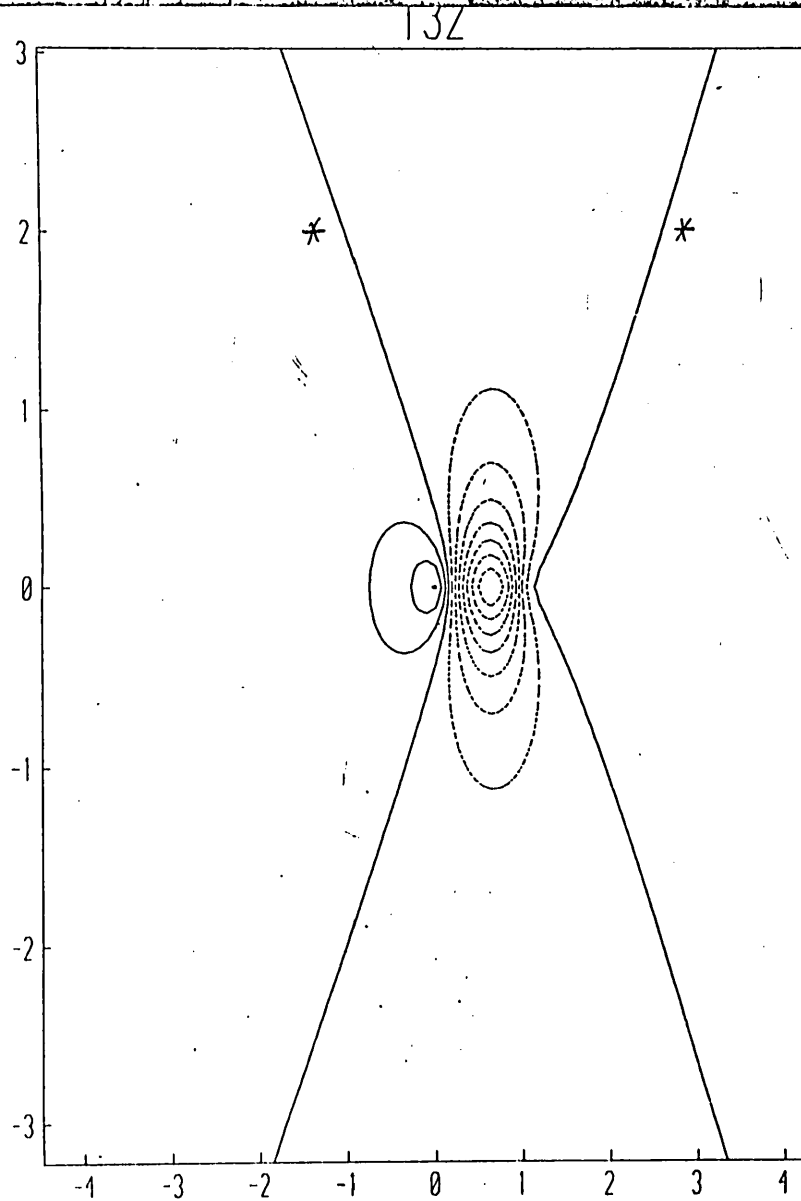
CONTOUR KEY	
1	-0.46E-01
2	-0.37E-01
3	-0.28E-01
4	-0.18E-01
5	-0.92E-02
6	0.06E+00
7	0.92E-02
8	0.18E-01
9	0.28E-01
10	0.37E-01
11	0.46E-01

(*)



$\frac{7/e}{\lambda_2}$

724

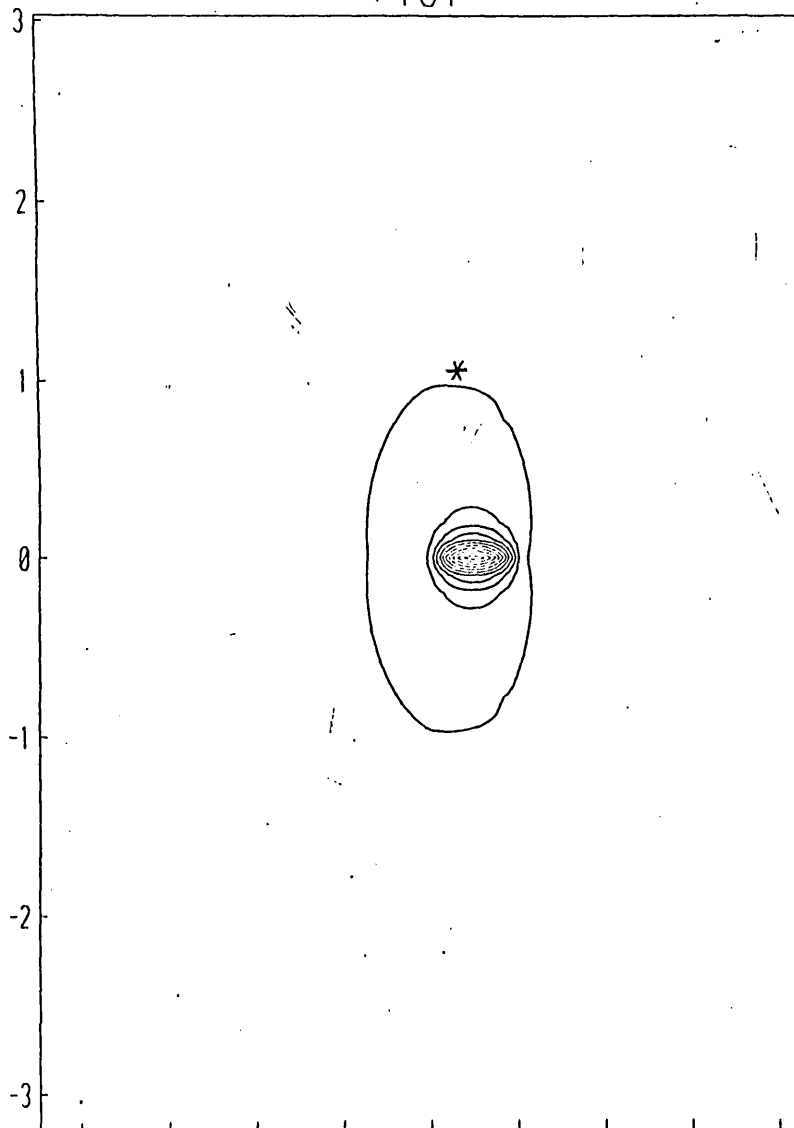


CONTOUR KEY	
1	-0.37E-01
2	-0.32E-01
3	-0.26E-01
4	-0.21E-01
5	-0.16E-01
6	-0.11E-01
7	-0.53E-02
8	0.00E+00
9	0.53E-02
10	0.11E-01
11	0.16E-01

(*)

7(f)
P

131

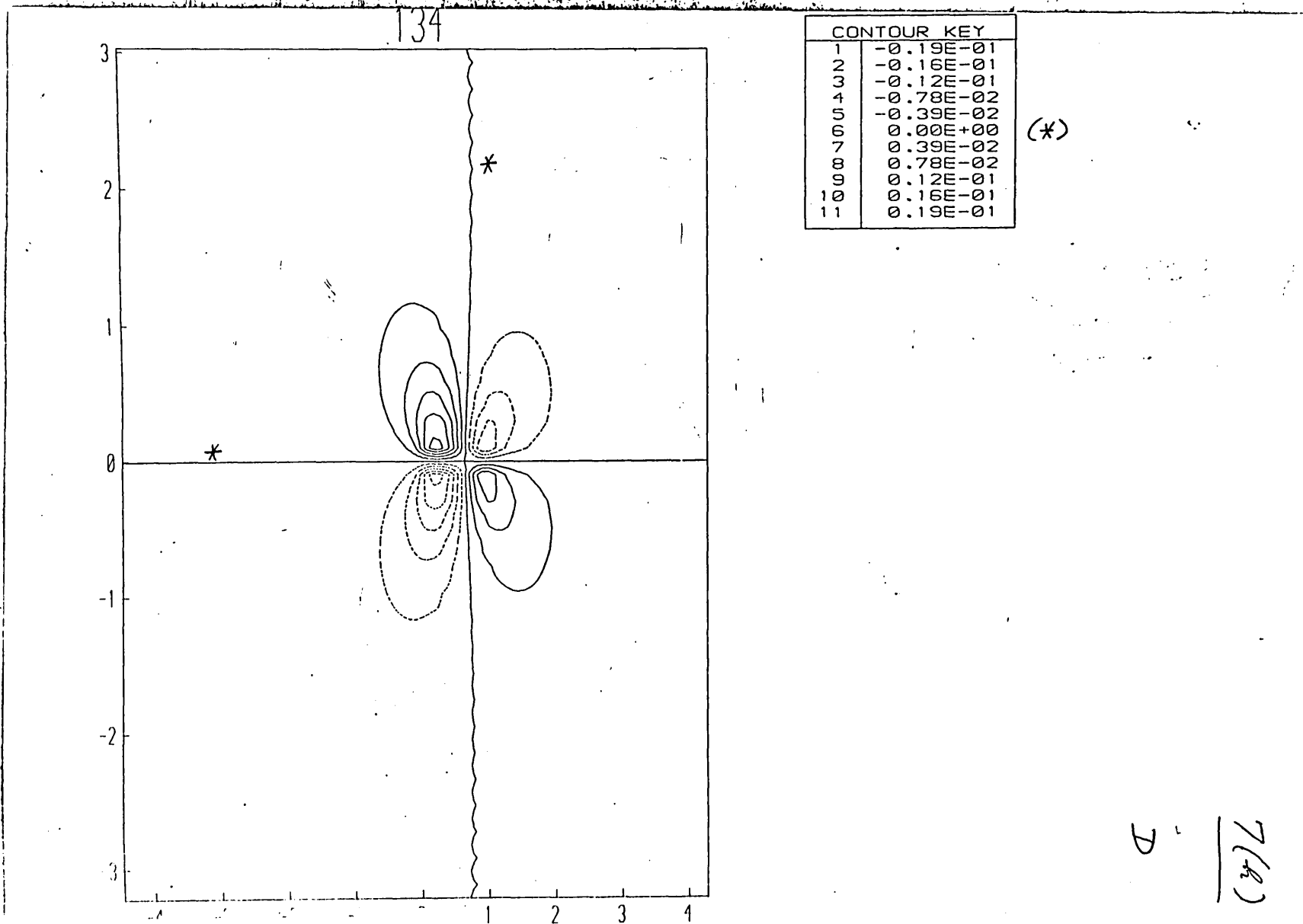


CONTOUR KEY	
1	0.00E+00
2	0.33E-01
3	0.65E-01
4	0.98E-01
5	0.13E+00
6	0.16E+00
7	0.20E+00
8	0.23E+00
9	0.26E+00
10	0.29E+00
11	0.33E+00

(*)

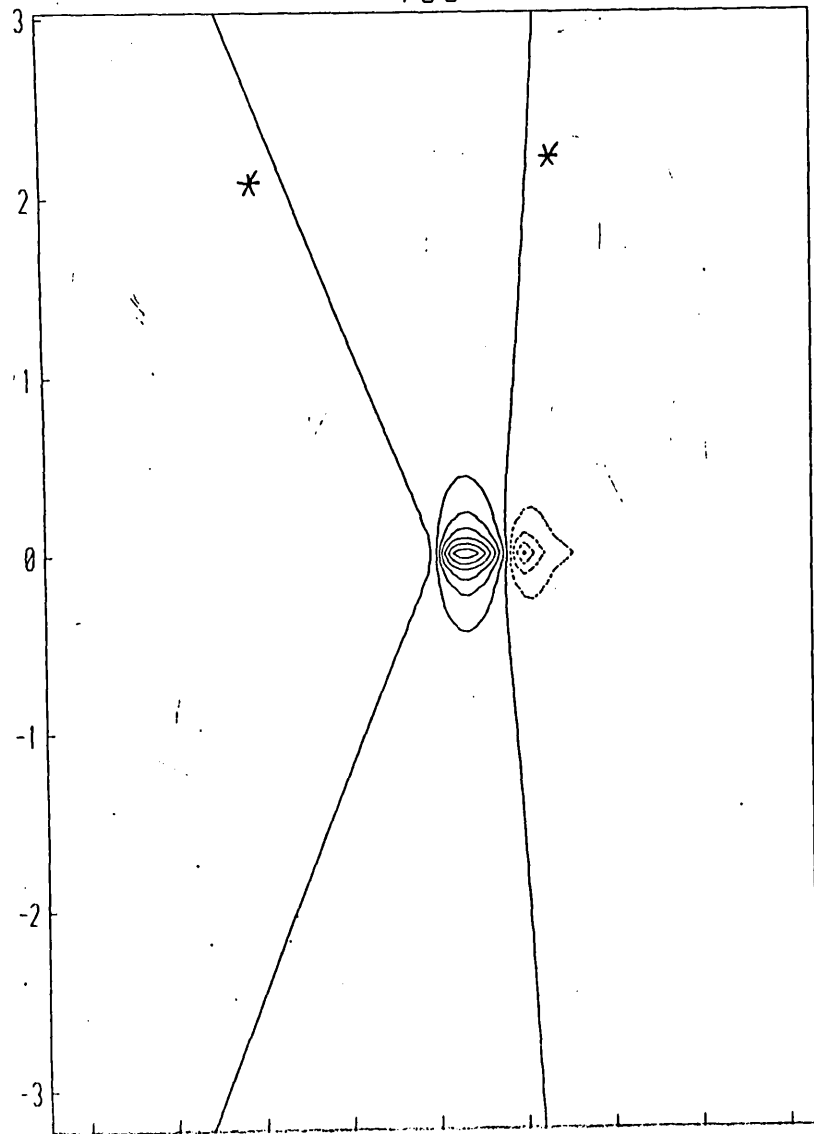
7/8)
-8

377



378

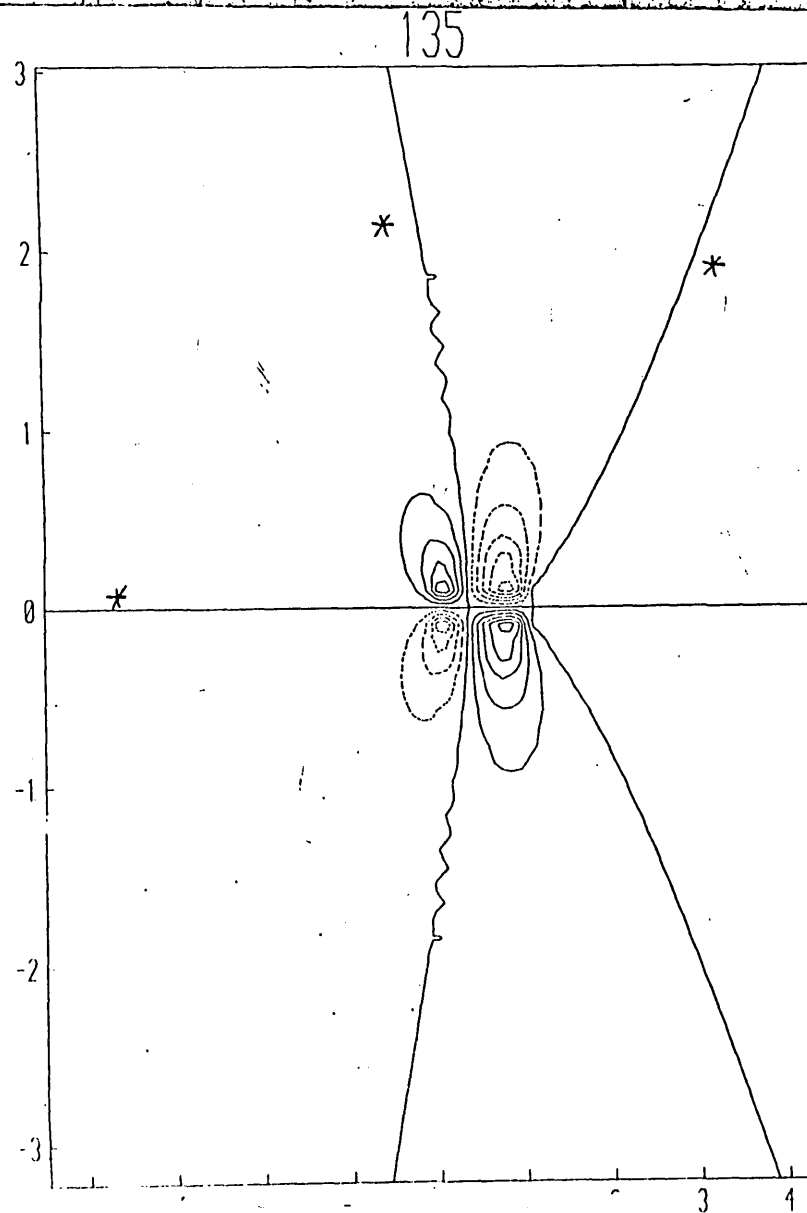
133



CONTOUR KEY	
1	-0.13E+00
2	-0.10E+00
3	-0.66E-01
4	-0.33E-01
5	0.00E+00
6	0.33E-01
7	0.66E-01
8	0.10E+00
9	0.13E+00
10	0.17E+00
11	0.20E+00

(*)

$\frac{7(i)}{x_2}$



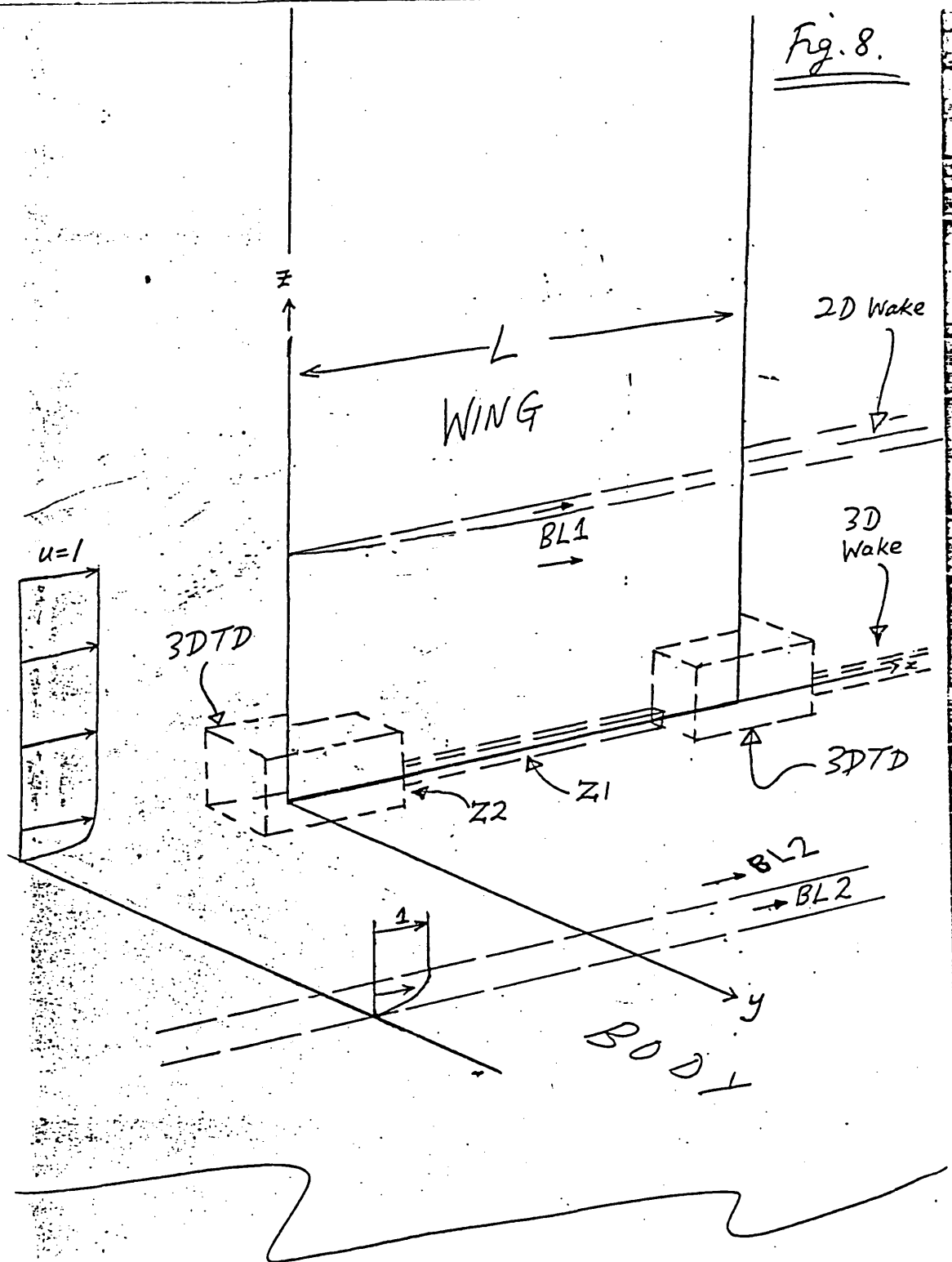
CONTOUR KEY	
1	-0.42E-01
2	-0.34E-01
3	-0.25E-01
4	-0.17E-01
5	-0.84E-02
6	0.00E+00
7	0.84E-02
8	0.17E-01
9	0.25E-01
10	0.34E-01
11	0.42E-01

(*)

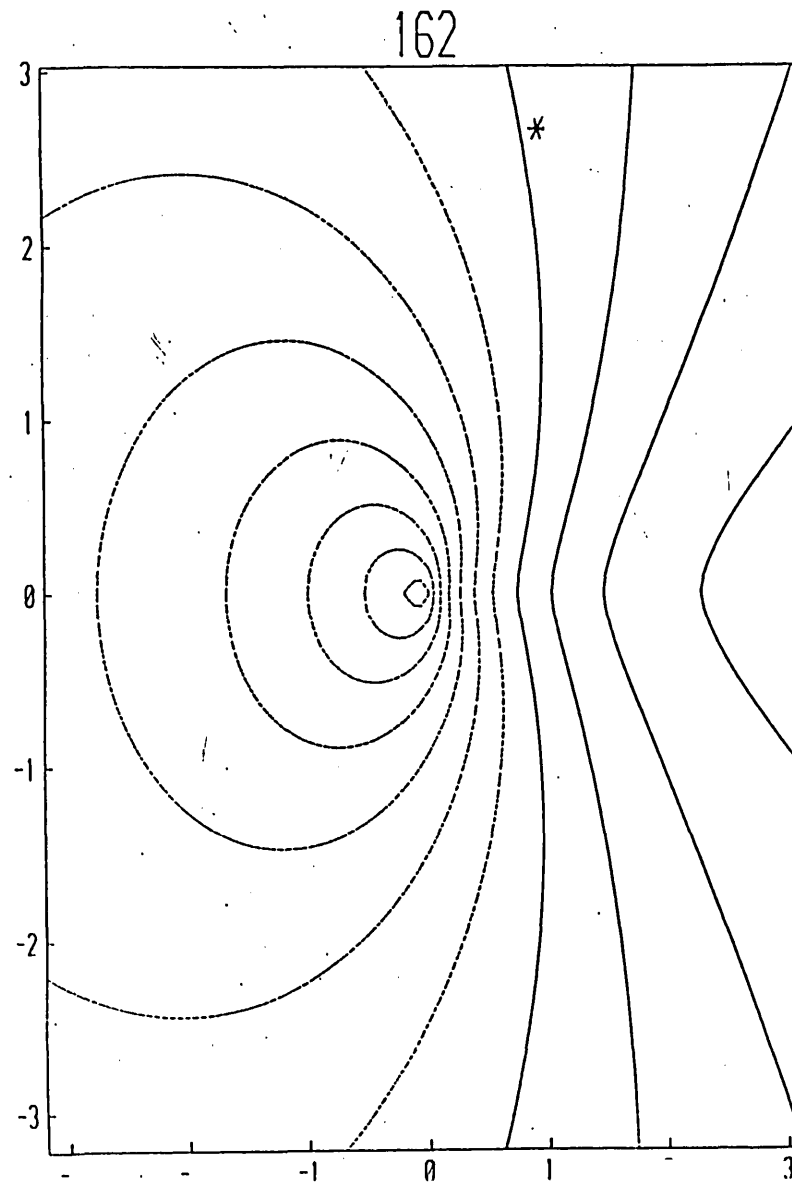
$\frac{7(j)}{7}$

379

Fig. 8.



361



CONTOUR KEY	
1	-0.22E+00
2	-0.19E+00
3	-0.16E+00
4	-0.13E+00
5	-0.96E-01
6	-0.64E-01
7	-0.32E-01
8	0.00E+00
9	0.32E-01
10	0.64E-01
11	0.96E-01

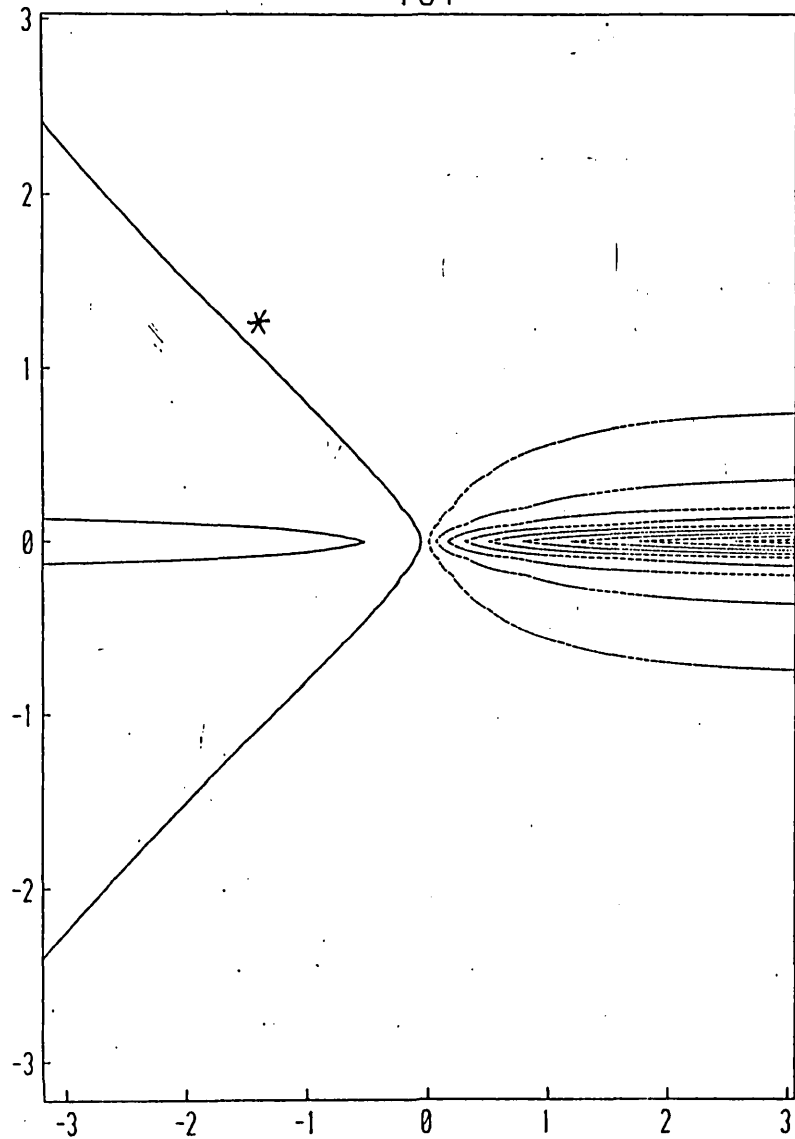
(*)

Fig. 9(a).

P

082

161



(*)

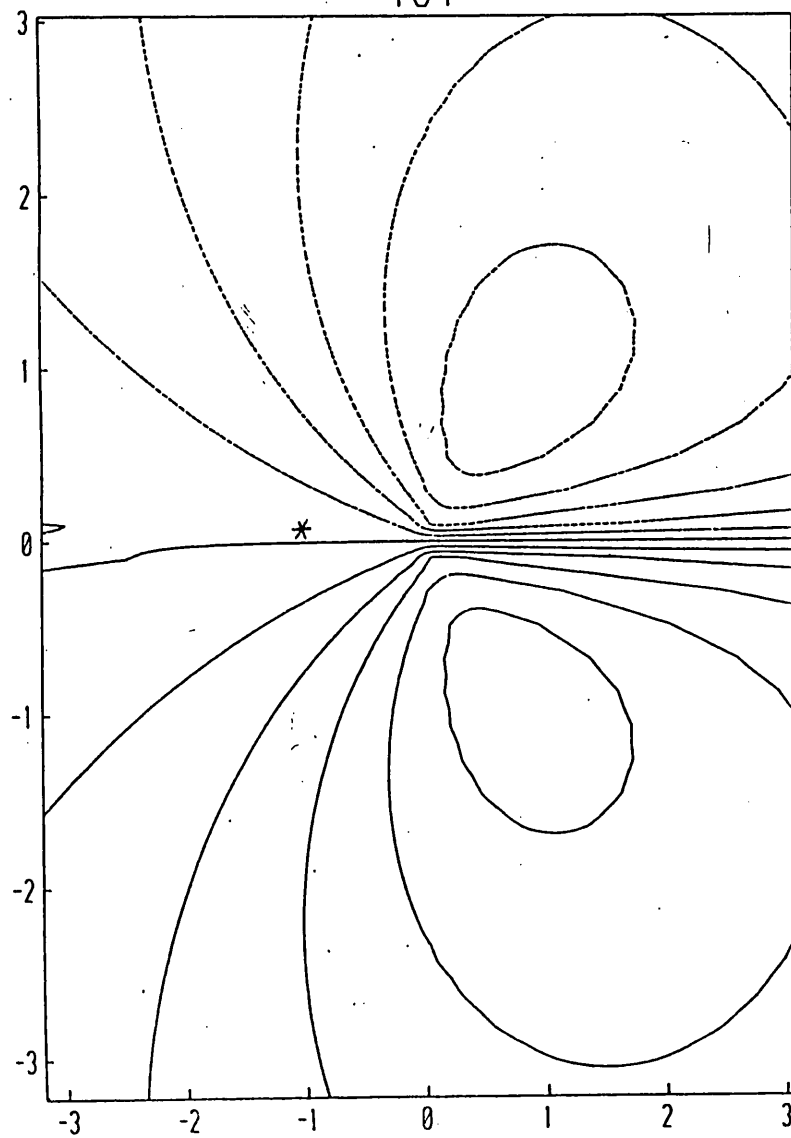
CONTOUR KEY	
1	-0.26E+01
2	-0.23E+01
3	-0.20E+01
4	-0.17E+01
5	-0.14E+01
6	-0.11E+01
7	-0.85E+00
8	-0.57E+00
9	-0.28E+00
10	0.00E+00
11	0.28E+00

-8

9(6)

323

164

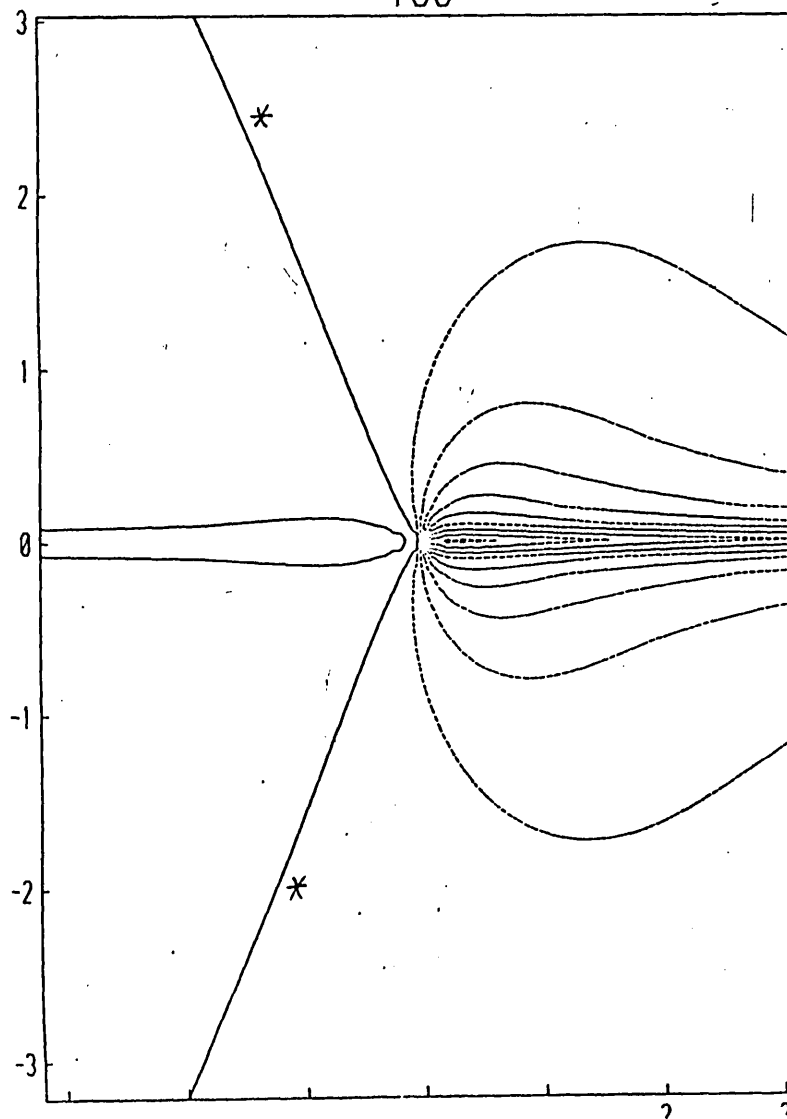


(*)

CONTOUR KEY	
1	-0.16E+00
2	-0.13E+00
3	-0.96E-01
4	-0.64E-01
5	-0.32E-01
6	0.00E+00
7	0.32E-01
8	0.64E-01
9	0.96E-01
10	0.13E+00
11	0.16E+00

9(c)
D

163

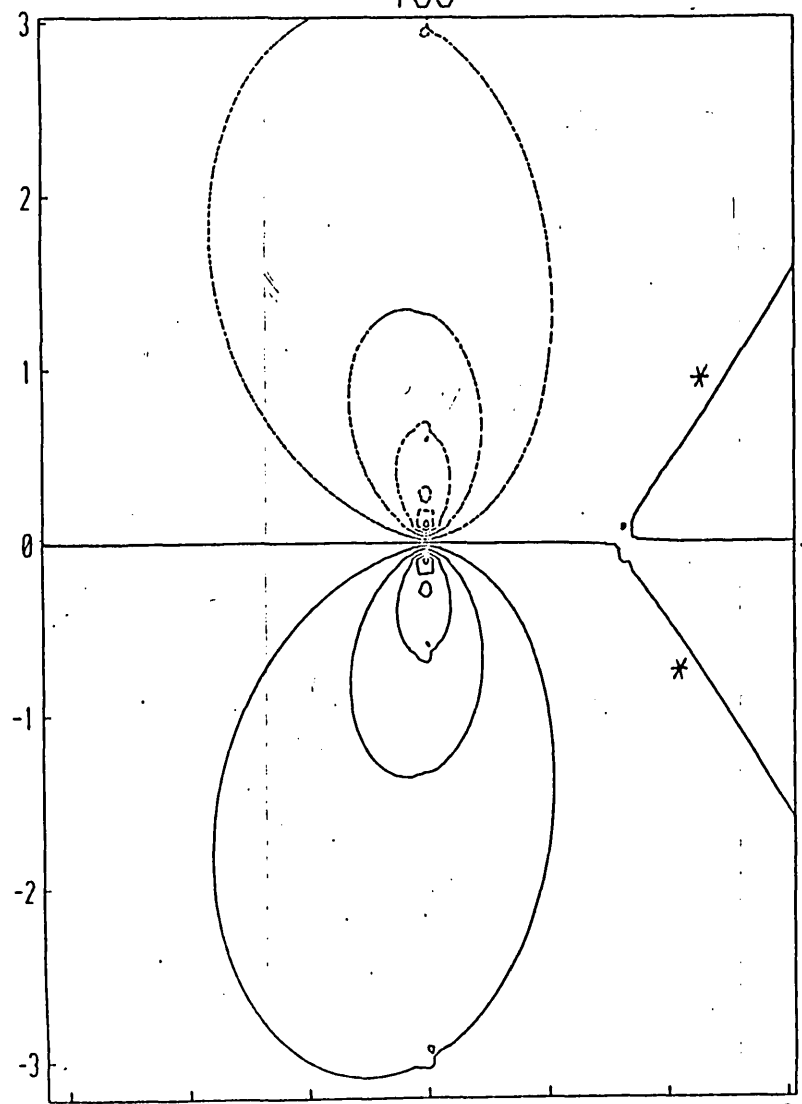


(*)

CONTOUR KEY	
1	-0.75E+00
2	-0.67E+00
3	-0.58E+00
4	-0.50E+00
5	-0.42E+00
6	-0.33E+00
7	-0.25E+00
8	-0.17E+00
9	-0.83E-01
10	0.00E+00
11	0.83E-01

$\frac{q(x)}{x^2}$

165



CONTOUR KEY	
1	-0.19E+00
2	-0.15E+00
3	-0.12E+00
4	-0.77E-01
5	-0.39E-01
6	0.00E+00
7	0.39E-01
8	0.77E-01
9	0.12E+00
10	0.15E+00
11	0.19E+00

(*)

q(e)
12

285

FIGURE CAPTIONS

Fig. 1 The junction formed by a thin symmetric two-dimensional (2D) wing of length L normal to the body surface ($z = 0$) and aligned with the oncoming 2D uniform stream and body surface boundary layer ($O(\text{Re}^{-\frac{1}{2}})$). The triple-deck "box", describing the 3D flow around the junction's leading edge for a long wing ($L = O(1)$), or about the entire junction for a short wing ($L = O(\text{Re}^{-3/8})$), is shown by dashed lines. See also Fig. 8.

Fig. 2 Linearized 3D flow solutions, for the 2D wing-body junction of (3.4a), showing curves of constant $P, \delta, D, \tau_X, \tau_Y$ and the typical secondary flows and vortices produced. For the purposes of comparison, the farfield asymptotes for the contours of level zero are at slopes (Y/X) of: ± 1 for P ; ∞ for δ ; 0 and ∞ for D ; approximately 4.8 and -0.67 for τ_X ; and approximately -2.4 and 1 for τ_Y . Here the asymptotic slopes for zero P, D, δ ^{stem} directly from (3.3b), (3.5a,b), ^{while} those for τ_X, τ_Y come from numerical evaluation of the real convolution integrals implied by (3.3c,d) when $k, \ell \rightarrow 0$.

Fig. 3 Linearized 3D flow solutions for (3.4b). Notation and secondary flow are as in Fig. 2.

Fig. 4 Linearized 3D flow solutions for (3.4c). Notation, asymptotes and secondary flow are as in Fig. 2.

Fig. 5 Linearized 3D flow solutions for (3.4d). Notation, asymptotes and secondary flow are as in Fig. 2.

Fig.6 A characteristic body - 3D wing symmetric junction, with sweep (angle β) and tapering: see §4. Delta wings and other 3D effects can also be dealt with.

Fig.7 Linearized flow solutions for the examples of 3D wings considered in §4, the swept wing ((a) - (e)) and the tapered wing ((f) - (j)). The sweep angle is 45° , with $g(X)=X(1-X)$ for $0 < X < 1$, $g(X) = 0$ otherwise, in (a)-(e). The taper in (f)-(j) is for $\sigma = 2$, with $g_1(X) = X(1-X)$ for $0 < X < 1$, $g_1(X) = 0$ otherwise.

Fig.8 Flow past the junction of a symmetric 2D flat-plate wing, of $O(1)$ chord L , and a flat body surface, showing the flow structure [dashed lines]. This is dominated by the leading edge and trailing edge 3D triple-decks (3D TD), the thinner classical 2D boundary layers BL_1 , BL_2 and classical 3D boundary layer Z_1 , and the thin 2D and 3D wakes. See §5. Similar structures apply for thicker and/or 3D wing-body junctions.

Fig.9 Linearized 3D solutions for $P, \delta, D, \tau_X, \tau_Y$ near the trailing edge of the wing-body junction when the chord L is small (§5).

CHAPTER 10

CONCLUSIONS

Briefly, we have considered several problems, and, with the aid of modern high Reynolds number flow theory, attempted to elucidate the governing structures and some of the properties for these flows. The work on free surface flows makes a start at addressing some of the fundamental questions in this area, such as the effects of viscosity and flow separation on the flow; the stability work presents some new results; lastly the work offers a rational means for discussing and gaining an insight into the complex flow near a realistic wing-body junction.

It was mentioned in the initial introduction that the triple-deck, or multi-deck, approaches highlight the different physical balances and delineate the flow field into various regions. It is worth stressing that the overall flow properties depend on the interaction between these different regimes, and not solely on say, inviscid mechanics.

Finally, we hope that the present work will be improved and extended. In particular the nonlinear versions of the undeveloped free surface flow, and the wing-body junction, problems, merit further investigation.