

NONLINEAR EXACT COHERENT STRUCTURES IN HIGH REYNOLDS NUMBER SHEAR FLOWS

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Declaration of Originality

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Liam Joseph Dempsey

March 2016

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Abstract

Recently a dynamical systems approach to the process of transition to turbulence has arisen in which so called exact coherent structures play a key role. Such structures are equilibrium states which are exact solutions of the Navier-Stokes equations. Given the nature of the governing equations such states can be difficult to find due to the size of the state space and nonlinear effects. Techniques such as homotopy and artificial forcing have been used to overcome these issues. However, here we will focus on finding such states at large Reynolds numbers where an asymptotically large parameter allows the identification of the leading order effects, from which the structure of the states can be described. The states described in this thesis will be split into two general types.

The first are of vortex-Tollmien-Schlichting-wave type and these states arise via the viscous instability mechanism. The flows studied here (plane Poiseuille flow and the asymptotic suction boundary layer) both exhibit a linear instability and hence we can find nonlinear travelling wave equilibrium states which bifurcate directly from the basic state. Importantly the class of states studied here have a large spanwise wavelength. We will find and investigate these states, finding that they appear to terminate in a singularity caused by the lack of spanwise diffusion at leading order.

The second type are characterised by the presence of structures both in the freestream and close to the wall, and were first found by Deguchi & Hall (*J. Fluid Mech*, vol. 752, 2014, pp. 602–625). We will consider two variations of these states. The first is the effect of wall curvature which leads to the imperfect bifurcation of Görtler vortices from the basic state. The second is the spontaneous generation of near wall structures when freestream structures are assumed to appear impulsively at some downstream location.

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1. Introduction

It is typical in many fluid dynamics problems of physical interest that a simple basic state known as the laminar state exists. In practice, whether in experiments or in numerical simulations, this state is rarely observed for realistic values of the physical parameters in the problem - such as the flux for example. The most famous example of this phenomenon comes from Reynolds pipe flow experiments (1883). The laminar basic state in this flow is radially symmetric uni-directional flow. Reynolds found that altering the physical parameters in the problem determined whether the laminar state was observed or not. This led him to identify the important non-dimensional parameter which has become known as the Reynolds number, is denoted by Re , and is defined as the ratio of inertial and viscous forces.

In many problems of interest, such as the flow of a river or the flight of a plane the Reynolds number is large. A consequence of this is that the flow appears to be dominated by inertial forces. This is problematic because the large Reynolds number form of the governing equations, which are the incompressible Navier-Stokes equations, is a singular perturbation problem. That is, the highest order terms which are due to viscous forces, are multiplied by the small parameter Re^{-1} . Taking the limit $Re \rightarrow \infty$, so $Re^{-1} \rightarrow 0$, these equations become the Euler equations. Due to the singular nature of the problem, we are unable to satisfy all of the boundary conditions. This leads to the apparent slip of a fluid over a surface contrary to common wisdom, a peculiarity which was not resolved until the development of boundary layer theory by Prandtl (1904). While the flow may be effectively inviscid throughout the majority of the domain, a thin layer known as the boundary layer exists near a solid surface. Throughout this thin layer, viscous forces are important and act to reduce the velocity of the fluid to the velocity of the boundary. The characteristic size of this layer depends on the Reynolds number. For example, for the flow over a flat plate the boundary layer has size $O(Re^{-1/2})$.

The laminar state is characterised by its organised behaviour. An example is the flow between two infinite parallel flat plates driven by an imposed pressure gradient. Alternatively this flow can be defined via an imposed flux. The laminar state is uni-directional

flow through the channel. This flow depends only on the wall normal coordinate, and is in the direction of the favourable pressure gradient, with no slip on the walls. It is known as plane Poiseuille flow and will appear in chapter 2. In many experiments and numerical simulations of practical interest a disordered state dominated by irregular fluctuations is much more likely to be observed. This disordered state is known as turbulence. These fluctuations are attributed to instabilities in the flow where small disturbances to the laminar state are amplified resulting in a flow which is now a (disordered) finite perturbation to the laminar state. Numerical simulation of realistic flows which involve turbulence is very difficult, in particular due to the wide range of scales involved. Typically methods for numerically simulating turbulent flows involve separating the large and small flow scales. Equations can be derived which govern the large scale motions. However the small scale motions, which capture effects such as the classical energy cascade, have to be modelled. Our interest here is on the process of transition to turbulence - the questions of how and why a turbulent flow develops rather than actual turbulent flow. To put the above discussion more precisely, consider the flow of an incompressible fluid characterised by a Reynolds number denoted by Re . If the Reynolds number is small then the laminar state is observed and the flow is likely to be linearly stable. In contrast, if the Reynolds number is sufficiently large then turbulent flow is observed. Phenomena such as intermittency and turbulent spots are observed for Reynolds numbers in between the two extremes. In certain scenarios the transition between the two states can be explained by the existence of linear instabilities, whereby transition occurs through a series of bifurcations originating from the basic state.

Often so called subcritical transition is observed whereby transition, via the instability in question, is observed even when the parameter values lie outside of the region of linear instability. For example the flow could be linearly unstable for large Reynolds numbers, say $Re > Re_c$, but in fact instability or even transition to turbulence can be observed for smaller Reynolds numbers, $Re < Re_c$, where the flow is supposedly linearly stable. Such occurrences can be ascribed to nonlinear effects directly via finite amplitude disturbances. Alternatively, non-normal growth in the linear regime can push the system into the nonlinear regime. It should be pointed out that the existence of a linear instability does not necessarily lead to turbulence. Examples of this can be found in centrifugal instabilities. Increasing the curvature parameter into the linearly unstable regime leads to a different, more complex flow such as in the cases of Taylor or Görtler vortices which bifurcate from the basic state. A linear stability analysis identifies the parameter values at which these fundamental changes in the behaviour of the system occur.

Some flows, such as plane Couette flow and Hagen-Poiseuille flow, are found to be linearly stable. Yet transition to turbulence is observed for Reynolds numbers on the order of a couple of hundred (depending on the details of the non-dimensionalisation). The particular case when the basic state is linearly stable provided quite a conundrum. In this case (and the cases of subcritical transition) it becomes important to consider nonlinear effects if the disturbance to the basic state is of finite amplitude. The flow may be linearly stable to arbitrarily small disturbances but by including nonlinear effects, a critical amplitude can be identified which separates disturbances which will eventually decay to zero from disturbances which will grow and ultimately lead to some other state from the basic (laminar) state. These critical amplitudes correspond to equilibrium solutions of the Navier-Stokes equations. Throughout this work we will encounter different types of equilibrium solutions including Taylor and Görtler vortices. The main type of interest in this work are travelling wave states. We will only consider the large Reynolds number form of each of these states. In the analysis which follows in this thesis we will identify critical scales for the disturbances in terms of the Reynolds number. For example the disturbances to the basic state will have size (in some suitably defined norm, the maximum denoted by L_∞ for example) say $O(Re^{-\gamma})$ for some $\gamma > 0$. This corresponds to the basin of attraction of the laminar state decaying algebraically as the Reynolds number is increased, in line with our expectation that we are more likely to observe instability and ultimately turbulence at larger Reynolds numbers.

Throughout this thesis we will consider multiple flow geometries and their corresponding basic states. In chapter 2 we will focus on the flow between two parallel infinite flat plates driven by an imposed pressure gradient. In parts of this work we will allow the walls to slide - this flow is known as plane Poiseuille-Couette flow. Almost all of the chapter will be restricted to the case of small wall sliding velocities, and the majority to where the wall sliding velocity is zero (plane Poiseuille flow). Chapters 3 and 4 concern external flows at high Reynolds number. Consider for a moment the Blasius boundary layer which is the flow which develops over a semi-infinite flat plate with a non-zero freestream velocity and zero imposed pressure gradient. The basic state here is written in terms of a similarity variable, which is the the wall normal coordinate divided by the square root of the streamwise distance from the edge of the plate. The key difficulty here is that the basic state develops with downstream distance. The streamwise development of this flow can be suppressed by applying constant wall suction. Now consider the flow over an infinite flat plate with constant freestream velocity and wall suction. A solution exists where the wall normal velocity is a constant (equal to the wall suction velocity) and the streamwise velocity depends only on the wall normal coordinate. The amount of

suction determines the wall normal scale over which the streamwise velocity approaches the freestream velocity. This flow is known as the asymptotic suction boundary layer. The independence of the basic state from the streamwise coordinate makes it a prototype parallel boundary layer upon which it is much easier to develop and test ideas and theories. Wall suction can have a strongly stabilising effect. Applying suction to the Blasius profile will lead to the asymptotic suction boundary layer. This means that in some cases this flow can be realised and used as a model of the flat plate boundary layer stabilised by suction.

1.1. Linear stability at finite Reynolds numbers

The type of nonlinear states of interest in this thesis rely on the existence of a linear instability. We will track states which bifurcate from the basic state via these linear instabilities - this will be discussed in more detail in section 1.3 and chapters 2 and 3. A precursor to an understanding of the nonlinear interaction is a study of the linear stability properties of the basic states of interest in this thesis. Firstly let us consider the essence of a linear stability analysis. We denote the basic state velocity and pressure by $(\mathbf{u}_b, p_b)$, where we assume the basic flow to be steady and parallel - we assume $\mathbf{u} = \mathbf{u}(y)$. The question we would like to address is whether small perturbations to this basic state grow or decay indicating linear instability or stability respectively. The first step is to linearise the Navier-Stokes equations around this basic state, assuming that perturbations to the basic state are sufficiently small that any nonlinear terms (in the perturbations) can be ignored. The coefficients of the resulting linearised equations depend only on y . This system can be examined using a modal analysis, i.e. the disturbances can be written in the form

$$f(y) \exp(i\alpha(x - ct) + i\beta z) + c.c. \quad (1.1)$$

where *c.c.* denotes the complex conjugate, α is the streamwise wavenumber, c is the corresponding wavespeed and β is the spanwise wavenumber. The linear system of equations, with homogeneous boundary conditions, yields a relationship between α, c and β which must hold for non-zero solutions to exist. This relation is called the dispersion relation and can be denoted by $\mathcal{D}(\alpha, c, \beta) = 0$. In practice this dispersion relation cannot be written down explicitly because the resulting linear system of equations forms a nonlinear eigenvalue problem in the form of a fourth order differential equation known as the Orr-Sommerfeld equation. Usually this equation can only be solved numerically. This eigenvalue problem determines the dispersion relation. For instance, suppose we are

interested in the temporal stability problem. Setting $\alpha, \beta \in \mathbb{R}$, this analysis determines a family of $c \in \mathbb{C}$ (and corresponding eigenmodes $f(y)$). Within this family of solutions, we are normally interested in the fastest growing or in the case of stable flow, the slowest decaying mode. From the form of the disturbances we see that the sign of $\text{Im}(c)$ determines the stability of such a mode, and the size of $\text{Im}(c)$ gives the growth (or decay) rate.

Consider the linear stability of channel flow which is well known. Interestingly the instability here is caused by the presence of viscosity which would normally be associated with a stabilising effect. The instability waves which this gives rise to are known as Tollmien-Schlichting waves. In the case of plane Poiseuille flow, where the flow is driven purely by a prescribed streamwise pressure gradient with stationary walls, the flow is known to be linearly unstable for sufficiently large Reynolds numbers. This was shown by Orszag (1971) who solved the Orr-Sommerfeld equation using a Chebyshev spectral method. For a fixed spanwise wavenumber, the region of linear instability in (α, Re) space begins at a critical Reynolds number Re_c and stretches to infinity. The boundary of this region separates modes which grow and modes which decay, and thus corresponds to the linear neutral modes i.e. those with zero growth rate. This bounding curve can be split into two segments, $\alpha_L(Re)$ and $\alpha_U(Re)$ such that $\alpha_L(Re) \leq \alpha_U(Re)$ - see figure 2.1 for example. The subscripts denote lower and upper parts of the linear neutral curve which are referred to as the lower and upper branches, respectively. In the other extreme if we take plane Couette flow which is the flow driven by sliding walls with zero prescribed pressure gradient, this flow is linearly stable and a proof of this fact was indicated by Romanov (1973).

To determine the critical Reynolds number for a particular flow we can appeal to Squires theorem. This says that the most unstable (with regards to the lowest critical Reynolds number) mode is two-dimensional. Thus we only need to consider the two-dimensional stability problem, i.e. we set the spanwise wavenumber β to be zero. In plane Poiseuille flow, Re_c was determined by Orszag (1971) to be approximately 5772 (when non-dimensionalised on the channel half-height and centerline velocity). In the asymptotic suction boundary layer the linear stability problem was investigated by Hughes & Reid (1965) and Hocking (1975), and Re_c was found to be approximately 54700 (non-dimensionalised on the length scale ν/V where V is the wall suction, and the freestream velocity). The existence of this linear instability has been verified experimentally by Fransson & Alfredsson (2003).

Now we will consider plane Poiseuille-Couette flow with a fixed pressure gradient (and

hence fixed Poiseuille component) and a variable wall sliding velocity denoted by u_w say. Then $u_w = 0$ corresponds to plane Poiseuille flow and the limit $u_w \rightarrow \infty$ corresponds to plane Couette flow. We expect the linear neutral curve to disappear as the wall speed is increased from zero so the flow is linearly stable as $u_w \rightarrow \infty$. In fact the linear neutral curves disappear as u_w tends to some finite critical value u_w^c . As u_w increases from zero the neutral curve recedes to larger Reynolds number, disappearing completely by the time $u_w = u_w^c$ - see Potter (1966) and Hains (1967). Further to this there is some other complicated behaviour involving neutral curve splitting for $u_w \in (0, u_w^c)$ which was predicted by the asymptotic work of Cowley & Smith (1985) and will be detailed in chapter 2.

1.2. Linear stability at large Reynolds numbers

In the next section (and throughout this thesis) we will consider nonlinear travelling wave equilibrium states at large Reynolds number. As a precursor to this we must describe the large Reynolds number form of travelling wave neutral modes in the flows of interest. We will now describe the types of asymptotic structures which arise. Classically asymptotic descriptions of Tollmien-Schlichting waves are composed of multiple regions in physical space where different effects dominate in different regions. Near boundaries viscosity is important in reducing the velocity of the flow to the velocity of the boundary. Away from boundaries the flow is usually dominated by inviscid dynamics. In the far field (in the freestream for example) the leading order dynamics may be a balance between freestream convection and the pressure gradient leading to the linearised Euler equations. These asymptotic structures are often complicated by the presence of so called critical layers which are active internal layers. The existence of such critical layers can be explained by considering the inviscid stability problem which is governed by Rayleigh's equation. This equation has a singularity along curves where the streamwise velocity is equal to the wavespeed, and this singularity must be regularised by viscosity which smooths out the singularity in a thin layer. In the flows of interest here we will only consider the asymptotic structure of the lower branch neutral modes and such modes have the property that the wavespeed is small. Thus the critical layers sit within the viscous wall layers and do not have to be considered separately.

With this discussion in mind, the structure associated with lower branch modes in plane Poiseuille flow (and plane Poiseuille-Couette flow when the wall sliding velocity is sufficiently small) is three tiered. The dynamics of the wave are inviscid throughout the

majority of the channel, away from the walls. There are thin layers attached to each of the walls where viscosity is a leading order effect.

The asymptotic suction boundary layer follows more closely the classical triple deck structure of the flat plate boundary layer, albeit with different scalings. Throughout the majority of the boundary layer the wave is governed by inviscid dynamics. In the far field, where the basic state is close to the freestream velocity, the leading order governing equations become the linearised Euler equations. Close to the wall there is a thin layer where viscous effects are important.

We will save a full derivation of the critical sizes of each of the layers and the slow wave variables (streamwise wavenumber, etc.) until the relevant sections. To illustrate the results, the streamwise wavenumber is found to be of order $Re^{-1/7}$ in plane Poiseuille flow and $Re^{-1/4}$ in the asymptotic suction boundary layer, as found by Lin (1945). Note that in the asymptotic suction boundary layer, Re is the Reynolds number based on the boundary layer thickness. Since the wall suction is assumed to be small, the linear stability problem is essentially that of semi-infinite bounded flow such as the Blasius boundary layer. The triple deck scaling in classical asymptotic linear stability calculations of this kind are $LR^{-3/8}$ in the streamwise direction and $L(R^{-5/8}, R^{-1/2}, R^{-3/8})$ in the wall normal direction, where L is the length scale of interest and R is the Reynolds number based on the freestream velocity and this length scale L . Contrast this to the scalings applied here where the length scale is δ (kinematic viscosity divided by the size of the wall suction velocity). Then the scalings are $\delta Re^{1/4}$ in the streamwise direction and $\delta(Re^{-1/4}, 1, Re^{1/4})$. These two formulations are equivalent and are related by $\delta = R^{-1/2}L$ and $Re = R^{1/2}$. Finally, a detailed analysis of the linear stability problem in the asymptotic suction boundary layer can be found in Hughes & Reid (1965).

1.3. Equilibrium states: self-sustained processes and vortex-wave interaction theory

Following the linear stability analysis one might be inclined to believe that transition to turbulence should not be observed unless the Reynolds number is greater than the critical Reynolds number, which is dependent on the flow in question. For example, plane Poiseuille flow is linearly stable if the Reynolds number is less than approximately 5772. In practice transition to turbulence can be observed at much lower Reynolds numbers, for example as low as around 1000 in plane Poiseuille flow (Carlson *et al.* 1982) and a few

hundred in plane Couette flow. This discrepancy could be caused by nonlinear effects, so we focus on the effect of nonlinearity assuming finite amplitude disturbances to the flow. Various investigations have demonstrated the instability of plane Poiseuille flow to finite amplitude disturbances at subcritical Reynolds numbers, such as Zahn *et al.* (1974), Herbert (1976,1977) and Orszag & Kells (1980) who have identified nonlinear equilibrium states in plane Poiseuille flow which bifurcate from the basic state confirming the importance of nonlinear effects on the instability properties of plane Poiseuille flow.

1.3.1. Nonlinear equilibrium states

A dynamical systems picture of transition has emerged in which nonlinear equilibrium states play a key role. These are solutions to the Navier-Stokes equations which are strongly nonlinear in that they are an $O(1)$ perturbation from the basic state. In some flows transition can be observed to occur through a sequence of bifurcations. For example in Taylor-Couette flow, vortices bifurcate from the laminar flow, followed by secondary bifurcations into wavy vortices before breakdown to turbulence (see Coles 1965). It is well known that even in the turbulent regime Taylor vortices are observed. This demonstrates the importance of equilibrium states in laminar, transitional and turbulent flows. Such states rely on the existence of curvature so for flows such as plane Poiseuille flow, plane Couette flow and the asymptotic suction boundary layer we have to look elsewhere for such equilibrium states.

In flows which lack an external mechanism (such as curvature) to induce equilibrium states, one can seek states from inviscid and viscous instabilities in the flow. These instabilities naturally give rise to travelling wave states, and will be described in more detail in the next subsection. It turns out that states of this type exist in plane Poiseuille flow, plane Couette flow and the asymptotic suction boundary layer.

Prior to the dynamical systems interpretation of transition to turbulence, an explanation of why a linearly stable flow such as plane Couette flow could experience turbulence at Reynolds numbers as low as 500 was lacking. This is explained by the existence of so called edge-states which are nonlinear equilibrium states which sit on an unstable manifold with only one stable direction. As depicted in figure 1.1a, the stable manifold acts as the boundary in state space between states which will become turbulent, and states which will eventually decay back to the laminar basic state - see Toh & Itano (2003) for example. Due to the nature of these states, they can be computed using edge tracking algorithms. As the Reynolds number increases the edge state decreases in size

resulting in a shrinking of the basin of attraction of the laminar state, and an increase in the basin of attraction for turbulence in line with our intuition that turbulence is more likely to be observed at larger Reynolds numbers. A nice illustration of the state space for plane Couette flow and the edge of chaos as discussed in Skufca *et al.* (2006) can be found therein and also in Schneider *et al.* (2008) and Gibson *et al.* (2008). The lower branch equilibrium states in plane Couette flow have been identified with the edge states.

1.3.2. Travelling wave states and self-sustained processes

Both plane Poiseuille flow and the asymptotic suction boundary layer are linearly unstable to viscous Tollmien-Schlichting waves. We can conclude the existence of linear neutral waves, at least when the amplitude of the wave is small (or alternatively the streamwise wavenumber is close to its linear neutral value). We can view such a solution as a pair consisting of the basic state $(\mathbf{u}_b, p_b)$, which we refer to as the vortex flow, superimposed with an associated infinitesimally small travelling wave of size Δ (so we assume Δ is infinitesimally small) where we denote the wave flow by $\Delta(\mathbf{u}^w, p^w)$. The vortex flow is independent of time and streamwise distance. The wave has properties corresponding to a linear neutral mode of the flow i.e. the linear neutral streamwise wavenumber and wavespeed. One can then imagine increasing the value of Δ from zero, or changing the streamwise wavenumber from its linear neutral value. Then the nonlinear self-interaction of the wave drives the vortex flow away from the basic state, and this modified vortex flow now has different linear stability properties i.e. a different neutral streamwise wavenumber and wavespeed. For small Δ the vortex flow will be a small perturbation to the basic state, and the wave variables will be close to their linear neutral values - this is essentially the weakly nonlinear analysis of Stuart (1960). Increasing Δ further, eventually we find the vortex flow becomes an $O(1)$ perturbation to the basic state at which point we conclude that the interaction is strongly nonlinear. In essence these are the nonlinear equilibrium states we would like to find in this thesis. In practice the details of the interaction are more complicated. This procedure can be performed numerically at finite Reynolds numbers - see for example Ehrenstein & Koch (1991) and Dempsey *et al.* (2016). We will be interested in the large Reynolds number limit of these states.

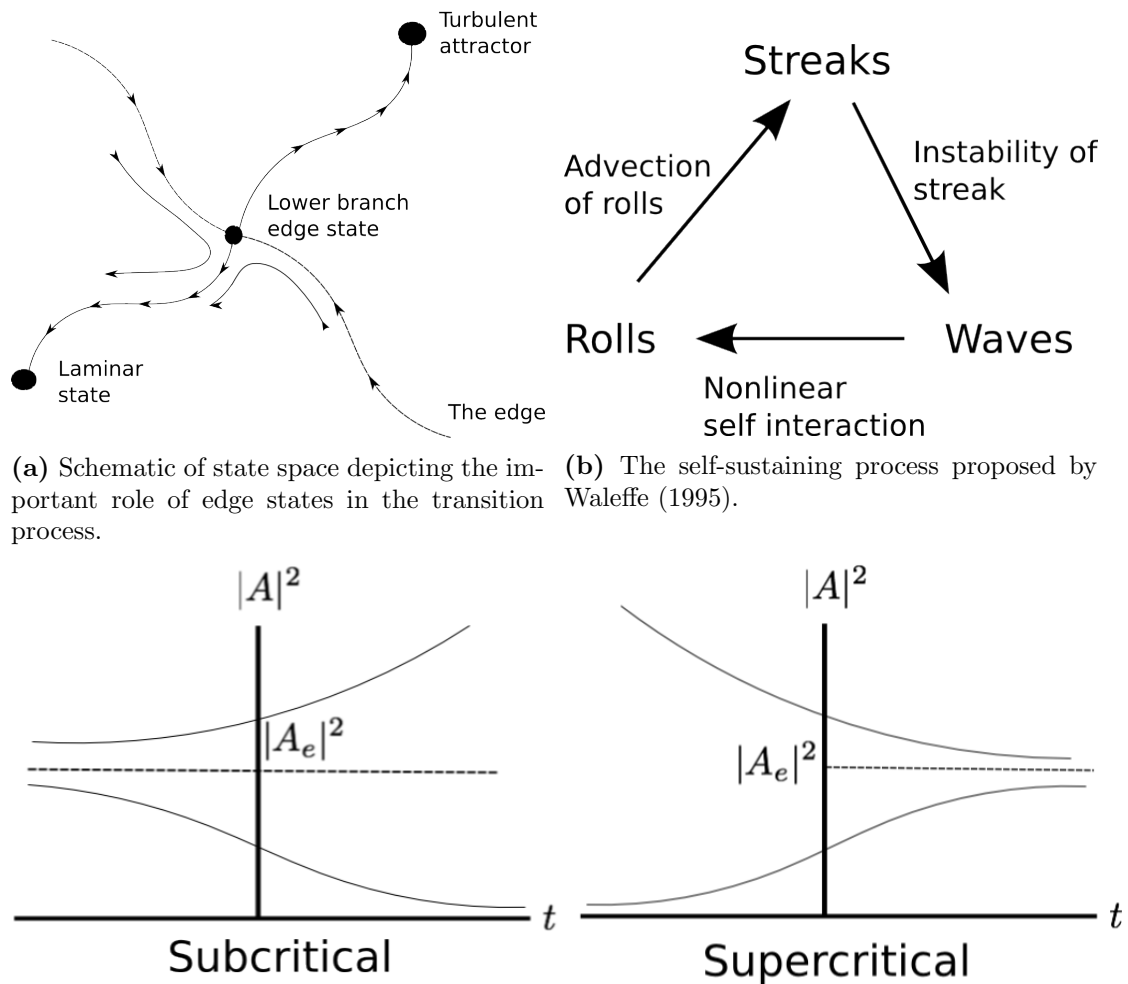
Let us turn our attention towards plane Couette flow and Hagen-Poiseuille flow for a moment. Although not directly relevant to this research, these flows have been the subject of some pioneering research and successes within the field. These flows present a

unique challenge since they are linearly stable and thus not susceptible to the reasoning of the previous paragraph. Early attempts to generalise the above ideas to flows without a linear stability were at first unsuccessful with uncertainty about the existence of such states in plane Couette flow as outlined by Saffman (1983). The important point here is that although these flows are linearly stable, a perturbation to these flows of sufficient size can render them unstable. Of course, finding these states is still difficult but such states were eventually found by Nagata (1990). Many more states have been found since then by for example Waleffe (1997, 1998, 2001) and Deguchi *et al.* (2013). States were also found in pipe flow by for example Wedin & Kerswell (2004).

The driving mechanism behind these states has been described by Waleffe (1995) using the language of self-sustaining processes, these states being labelled exact coherent structures. This process is composed of three interacting parts denoted in figure 1.1b. The first is known as the streak, and this is the steady, streamwise independent part of the streamwise velocity. The instability of the streak supports a three-dimensional finite amplitude wave. The nonlinear self-interaction of this three dimensional wave, through the Reynolds stresses for example, creates a forcing on cross-stream vortex flow, which are known as the rolls. Finally, the rolls provide a forcing on the streak through the advection of mean shear. If the different flow parameters (wave amplitude, wavenumbers, etc.) are just right, then this process is self-sustaining.

Travelling wave states have also been found in the asymptotic suction boundary layer. Nonlinear two dimensional finite amplitude states were first found by Milinazzo & Saffman (1985) which bifurcate from the linear neutral point $Re_c \approx 54700$. Interestingly these states were found to persist down to Reynolds numbers of a few thousand where even here they have a small perturbative energy suggesting that the basic state is only weakly stable even at Reynolds numbers much below critical. More recently three dimensional edge states have been identified by for example Kreilos *et al.* (2013) and Khapko *et al.* (2012).

Comparisons have been made between states found numerically in the full Navier-Stokes equations and experiments. In pipe flow, promising agreement between states found numerically by Faisst & Eckhardt (2003) and observations in experiments by Hof *et al.* (2004) has been observed.



(c) Subcritical and supercritical bifurcations. The small amplitude dynamics of the equilibrium states depends on how they bifurcate from the basic state.

Figure 1.1. Self-sustained processes, equilibrium states and sub/super-critical bifurcations.

1.3.3. Vortex-wave interaction

The self-sustaining processes approach by Waleffe (1995) is inspired by the work of Benney (1984), whose work also led to the development of an asymptotic approach known as vortex-wave interaction theory. This theory was developed through a sequence of papers by Hall & Smith (1988, 1989, 1990, 1991), Bennett, Hall & Smith (1991) and Walton & Smith (1989, 1992), amongst others. Within this approach the steady, streamwise independent part of the flow, which is composed of the streak and the rolls using the terminology of the self-sustaining process community, is called the vortex flow. The wave is described asymptotically and is either of Rayleigh-type or Tollmien-Schlichting-type. The type of wave affects not only the wave system but also how the nonlinear self-interaction of the wave forces the vortex flow to generate streamwise streaks. Throughout this work we will only consider interactions of the lower branch Tollmien-Schlichting-type wave.

In plane Poiseuille flow, which is the main focus of chapter 2, the nonlinear equations governing the interaction were first set out by Bennett, Hall & Smith (1991) in the context of curved channel flows. Here the basic state is linearly unstable to Tollmien-Schlichting-type waves and hence states bifurcate directly from the basic state. The wave forcing is directly into the channel core via the Reynolds stresses. The asymptotic suction boundary layer, of chapters 3 and 4, is also linearly unstable to Tollmien-Schlichting-type waves. In this case the asymptotic structure of the wave is different. In fact the wave forcing manifests itself through a strong near wall vortex flow.

Although we will only consider the Tollmien-Schlichting-type interaction, consider for a moment the Rayleigh-type interaction in plane Couette flow. The relevant equations were solved and states found by Hall & Sherwin (2010). In this case the basic state is linearly stable, so a sufficiently large amplitude wave is required to produce a driving mechanism for the streamwise vortex flow. One of the most important properties of Rayleigh waves is the presence of a critical layer. It turns out that the wave forces the vortex flow by driving a discontinuity in the roll shear and pressure across this critical layer. The solutions found by those authors at large Reynolds number were found to be in excellent agreement with previous work by other authors, in particular the numerical simulations of Wang *et al.* (2007), even down to relatively small Reynolds numbers. From this we conclude that states found using vortex-wave interaction are the large Reynolds number limit of full numerical solutions of the Navier-Stokes equations. While vortex-Rayleigh-wave interaction has received considerable attention, vortex-Tollmien-Schlichting-wave interaction has largely been neglected. For example see Bennett, Hall

& Smith (1991) for computations performed for curved channel flows, and Hall & Smith (1991) and Walton & Patel (1999) for computations performed for the flat plate boundary layer.

Chapters 2 and 3 will concentrate on vortex-Tollmien-Schlichting-wave interaction in flows which exhibit linear instability. We will track nonlinear equilibrium states which bifurcate from the basic laminar states. Details of the bifurcation in plane Poiseuille flow and a comparison with states computed at finite Reynolds numbers can be found in Dempsey *et al.* (2016), and are summarised in chapter 2.

1.3.4. Small amplitude states

For equilibrium states which bifurcate from the basic state via a linear instability mechanism, the bifurcation can be of two types. These types are known as subcritical and supercritical, and describe the behaviour of small amplitude states, close to the bifurcation point. Supposing that these conditions hold, then the amplitude of the disturbance evolves over a slow time denoted here by τ . The perturbation from neutral (from the bifurcation) is represented by $s \in \mathbb{R}$ which is assumed to be small. For example, s could represent the perturbation of the streamwise wavenumber from its linear neutral point in the context of vortex-wave interactions. Or it could represent the perturbation to the Taylor number from its critical value in the context of curved channel flows. The evolution can then be described by the Stuart-Landau equation (see Landau 1944 and Stuart 1960) which is of the form

$$\frac{dA}{d\tau} = sa_1A + a_3|A|^2A, \quad (1.2a)$$

where a_1 and a_3 are complex constants. The squared amplitude of the disturbance $B = |A|^2$ satisfies

$$\frac{dB}{d\tau} = sb_1B + b_3B^2, \quad (1.2b)$$

where $(b_1, b_3) = 2 \operatorname{real}(a_1, a_3)$. A solution for B is found to be

$$|A|^2 = B = \frac{sb_1B_0e^{sb_1\tau}}{sb_1 + b_3B_0(1 - e^{sb_1\tau})}, \quad (1.2c)$$

where the initial value of B is denoted by $B_0 = B(\tau = 0)$.

From this we see that there are several different types of behaviour depending on the values of b_1, b_3 and B_0 . This means that the eventual fate of $B(\tau)$ depends not only

on the parameters b_1 and b_3 but also on the initial amplitude of the disturbance, B_0 . The two different types of behaviour are shown in figure 1.1c. Both types contain two equilibrium amplitudes, the first of which is $|A|^2 = 0$ and the second is denoted by $|A|^2 = |A_e|^2$. In the first type the bifurcation is subcritical. The zero state is stable and A_e acts as an amplitude barrier between states which decay to zero and states which blow up in finite time (presumably developing into some fully nonlinear state), as can be seen from equation (1.2c). In the second type the bifurcation is supercritical. The zero state is unstable but now the state A_e is stable and all disturbances (within this framework) now converge to this state in the long term.

1.3.5. Transition and turbulent spots

There are different routes to transition. One such route is bypass transition when the disturbances to the laminar state are large - see Morkovin (1969). Another route to transition is via the growth of turbulent spots, which were observed as early as the pioneering experiments of Reynolds (1883) which famously displayed intermittency in pipe flow. This route concerns the growth and subsequent breakdown of Tollmien-Schlichting waves. It is known to be particularly important in low levels of background turbulence (Nishioka *et al.* 1975). Initially Tollmien-Schlichting waves are generated in the receptivity process - see section 1.6. These waves then grow exponentially according to linear theory - see section 1.1. Through experiments in boundary layers Emmons (1951) introduced the idea that islands of turbulence known as turbulent spots form in a transitional flow, however the underlying mechanism was unknown. Later, Klebanoff *et al.* (1962) associated spot formation with the breakdown of three-dimensional Tollmien-Schlichting waves. This route, through the emergence of turbulent spots which grow as they propagate downstream, is also observed in plane Poiseuille flow and it was found by Alavyoon *et al.* (1986) in an experimental study that Tollmien-Schlichting-type waves are present at the leading and trailing edges of the spots. These are believed to play a role in their spanwise spreading. Spots of this type have also been generated numerically as in Henningson *et al.* (1987) where initiation occurs via a localised disturbance.

The importance of turbulent spot formation and spreading in the transition process generates interest in localisation mechanisms. As we will see in chapters 2 and 3, the states found within this thesis consisting of the strongly nonlinear interaction of a vortex flow with a pair of oblique Tollmien-Schlichting waves, experience strong spanwise focusing. These factors suggest such states could have some relevance to the formation of turbulent spots.

1.4. Centrifugal instabilities

In the previous section we required the wave to be of sufficient size to generate streamwise streaks. In the absence of a wave, streamwise streaks can be generated by other effects such as curvature. These can be studied analytically and numerically in a variety of flows. A common numerical method is continuation in some parameter, which is used to transform states in one flow to states in the flow of interest. For example, Nagata (1990) used curvature to generate nonlinear equilibrium states. In this case the continuation parameter is the curvature, and reducing the curvature to zero these states can be continued to non-trivial states in plane Couette flow. Alternatively one could use states in one configuration to generate states in another by for example continuation in the boundary conditions as in Waleffe (1998). Similar methods include using an artificial body forcing used to support states, and then applying a homotopy method (continuation in the forcing amplitude for example) to find non-trivial states in the absence of such a forcing.

In chapter 2 we will briefly consider the effect that the presence of curvature has on the vortex-Tollmien-Schlichting-wave interaction, the case studied by Bennett, Hall & Smith (1991). When the curvature of the channel walls is small a unique basic state exists, plane Poiseuille flow. At some critical value of the curvature there is a bifurcation into Taylor vortices. The effect of curvature on vortex-wave interaction states is two-fold. Firstly any finite amplitude state computed in plane Poiseuille flow is altered by the presence of curvature. Also there is a non-uniqueness in the zero-wave state and hence there are more (zero-wave) states to bifurcate from i.e. vortex-wave interaction states can bifurcate from plane Poiseuille flow, or from the Taylor vortex state, when it exists. In chapter 4 we will investigate the effect curvature has on so called freestream coherent structures (see section 1.5 for more details) in the asymptotic suction boundary layer.

The physical mechanism behind curvature driven states can be reasoned in terms of the centrifugal effect. The centrifugal forces on the flow over a concave surface have a destabilizing effect which results in the formation of counter-rotating vortices as observed by Clauser & Clauser (1937). The theoretical explanation for these vortices was given by Görtler (1940) after which they are named. The streamwise development of Görtler vortices in non-parallel boundary layers has been studied by Hall (1988, 1990) where the receptivity problem is of particular importance given the non-existence of a unique linear neutral curve for $O(1)$ wavelengths as shown by Hall (1983). In chapter 4 we will be interested in the generation of Görtler vortices in response to freestream coherent

structures in a parallel boundary layer, where the concept of a linear neutral curve is well defined. These freestream coherent structures effectively act as a freestream forcing on the near wall Görtler vortex problem. How these freestream perturbations penetrate into the wall layer is the focus of receptivity theory as discussed in section 1.6. Introducing forcing to a natural bifurcation problem leads to imperfect bifurcations, which will be discussed in section 1.7.

Consider the flow in a curved geometry such as over a curved plate or in a curved channel with a parallel basic state denoted by $\mathbf{u}_b(y)$. We will consider the linear stability of this basic state to spanwise modulations, at a fixed spanwise wavenumber β , due to curvature. For this discussion we require some measure of the curvature. The situation is generic but for this discussion we will use T , which usually denotes the Taylor number in channel flow, as the required measure. In the resulting set of linear governing differential equations, the coefficients depend only on the wall normal coordinate, and hence disturbances to this basic state can be written in the form

$$f(y)e^{\sigma x + i\beta z}, \quad (1.3)$$

for some function f , where σ is the streamwise growth rate. To understand what typically occurs we will discuss an example, Taylor vortices in channel flow in the small β limit - see section 2.10.1. We find that $\sigma < 0$ for Taylor numbers say less than some critical value which depends on the spanwise wavenumber, $T_c(\beta)$ and $\sigma > 0$ for $T > T_c(\beta)$. Thus at $T = T_c(\beta)$ the basic state is neutrally stable to some disturbances with spanwise wavenumber β . At this critical value of T there is a bifurcation into finite amplitude equilibrium states associated with the curvature instability. These can be described for small amplitudes by extending the linear analysis to include weakly nonlinear effects - see the previous section, in particular equation (1.2a), and section 1.7. This bifurcation can take many forms but in this case it is of Stuart-Landau type. A description of the bifurcation can be reasoned as follows by assuming that T is close to $T_c(\beta)$, i.e. $|T - T_c(\beta)| \ll 1$. The leading order term is the first harmonic like $e^{i\beta z}$ with the amplitude A of this mode undetermined at this point. The leading order analysis requires the linearised operator to be degenerate and this defines $T_c(\beta)$, since $T = T_c(\beta)$ at leading order in A . The quadratic nonlinearity generates the zeroth and second harmonics, 1 and $e^{2i\beta z}$, at second order A^2 . At third order the nonlinear interaction of the first two orders of terms yields harmonics $e^{i\beta z}$ and $e^{3i\beta z}$. It turns out that the governing equation for the first harmonic at this order is the degenerate operator at leading order, but now with forcing terms. This yields a solvability condition at this order, and thus determines

the critical perturbation to the curvature to be $T - T_c(\beta) \sim A^2$, thereby confirming the quadratic nature of the bifurcation. This describes how curvature induced equilibrium states bifurcate from the basic state via a linear instability. In general to track vortices to larger amplitudes outside of the range of validity of the weakly nonlinear analysis one must resort to fully nonlinear computations on the governing equations. In chapter 2 we will compute nonlinear long wavelength Taylor vortices in plane Poiseuille flow and chapter 4 we will compute nonlinear Görtler vortices in the asymptotic suction boundary layer.

1.5. Freestream coherent structures

Chapter 3, on vortex-wave interaction in the asymptotic suction boundary layer, yields equilibrium states with structures concentrated near the wall. Typically in boundary layer studies structures are observed both near the walls (see Marusic *et al.* 2010 for example for a general discussion, and figures 3.8 and 3.9 for a particular example in this work) and in the freestream leading to a debate about the relationship between the two, in particular their interdependence. Deguchi & Hall (2014) documented a new type of exact coherent structure which those authors found in the asymptotic suction boundary layer by homotopy methods. This work was later generalised to growing boundary layers in Deguchi & Hall (2015), demonstrating that these ideas can be carried across to non-parallel boundary layers. Here we will focus exclusively on parallel boundary layers.

This structure consists of a strongly nonlinear interaction in a layer of $O(1)$ dimensions sitting in the freestream a distance $\ln Re$ from the wall. Waves within this layer propagate downstream at close to the freestream velocity. Those authors named this layer “the production layer”. The appearance of a layer a distance $\ln Re$ away from the wall may be intriguing but it should be noted that this is not the first time that a structure with such a layer has been found. For example Smith, Doorly & Rothmayer (1990) found such a structure in an attempt to explain some of the phenomena observed in turbulent boundary layers over a flat plate - although the details are different. The governing equations within this layer are in fact the full Navier-Stokes equations at unit Reynolds number on an unbounded domain. Above this layer the flow decays to the freestream velocity. For a range of spanwise wavenumbers, the strongly nonlinear interaction within the production layer excites an exponentially growing streak as the wall is approached and this generates near wall structures. The fact that near wall structures can be induced by freestream structures further complicates the question of the interdependence of near

wall and freestream structures in turbulent flows.

More details regarding freestream coherent structures will be given in chapter 4. We will find that although the dynamics within the production layer can only be determined by solving the full Navier-Stokes equations it is possible to identify the form of the flow at the bottom of the production layer towards the wall. This can be determined up to an unknown parameter (representing the amplitude of the disturbance) which must be determined from the full numerical solution of the production layer equations. In effect this means that we can ignore the production layer dynamics (we know that solutions exist from Deguchi & Hall 2014) and we can concentrate on determining the near wall structures. In chapter 4 we will extend this work to study two effects, the first of which is the effect of replacing the flat wall with a curved wall. The effect of curvature turns out to be a higher order effect (in terms of Re) in the production layer and hence this part of the problem is unaltered at leading order. Considering the near wall problem, with the introduction of wall curvature we now have the possibility of bifurcations of Görtler vortices from the basic state. The production layer structures act as a forcing on the near wall problem and this forcing acts to break the symmetry of the bifurcation creating imperfect bifurcations - see section 1.7. The second effect we will consider is the streamwise development of near wall structures when the production layer is impulsively started at a specific downstream location. We will find these structures develop over a non-dimensionalised streamwise distance of order $O(Re \ln Re)$.

1.6. Receptivity

Receptivity is the process by which external disturbances excite the eigenmodes of a dynamical system. In boundary layers receptivity is concerned with how external disturbances, such as those arising from the freestream (freestream turbulence impinging on the boundary layer for example) or wall roughness, penetrate into the boundary layer. The boundary layer has an infinite number of discrete eigenmodes, some of which are stable and (in general) others which are unstable. To excite an instability with a particular frequency within the boundary layer, a forcing is required with not only the same frequency but also over the same spatial scales. This is a potential problem given that the disturbances in the freestream are governed by inviscid dynamics, and hence in general act over much different spatial scales. To excite boundary layer instabilities an interaction between the mean flow and some mechanism is required to generate the required rapid variations in the streamwise direction. There are several possibilities in-

cluding for example leading edge effects in Blasius flow, or some surface inhomogeneity on a flat plate such as an isolated roughness elements or blowing/suction.

There are two approaches to receptivity theory. One is a numerical treatment of the linearised Navier-Stokes equations at finite Reynolds numbers - see Choudhari & Streett (1992) for example. The other is an analytical approach at large Reynolds numbers. For Blasius flow over a flat plate this is performed via the triple deck formulation of the Navier-Stokes equations. Triple deck theory was developed by several authors including Stewartson & Williams (1969), Neiland (1969) and Messiter (1970). It is an asymptotic theory which describes short scale instabilities in a variety of flows such as the Blasius boundary layer and the asymptotic suction boundary layer. Coincidentally we will encounter the triple deck structure in the asymptotic suction boundary layer in chapter 3 where we consider the strongly nonlinear interaction of a vortex with a Tollmien-Schlichting-wave. The structure of the wave is described by triple deck theory, albeit with different scalings to the triple deck structure in the flat plate boundary layer. Ruban (1985) and Goldstein (1985) considered receptivity due to surface perturbations in the presence of a freestream acoustic wave, and later this work was extended to cover a much wider class of problems.

The relevance of the receptivity mechanism depends on the nature of the instability, i.e. whether the instability is absolute or convected with the flow. In fact the concept of receptivity relies on the instability in question being convective. Receptivity will be the focus of chapter 4 where we will consider the receptivity of Görtler vortices, which are a near wall phenomenon, to freestream coherent structures in the asymptotic suction boundary layer. The work of Park & Huerre (1988) demonstrates that the instability of the asymptotic suction boundary layer is convective, and this was backed up by experiments performed by Chomax & Perrier (1991).

Within chapter 4 we will consider the asymptotic suction boundary layer but in the presence of wall curvature which lends itself to the development of streamwise vortices, referred to as Görtler vortices - see section 1.4. Ruban (1990) considered the propagation of a wave packet over a curved surface demonstrating the interesting effects which can occur when more than a single wave is present, and then other authors such as Denier, Hall & Seddougui (1991) considered the receptivity problem for wall roughness, with both of these effects being studied in the Blasius boundary layer. In chapter 4 however we wish to consider the receptivity of Görtler vortices to the freestream coherent structures found by Deguchi & Hall (2014) which were discussed in section 1.5. In this case the receptivity problem is concerned with how external disturbances generated due to the

existence of freestream coherent structures, penetrate into the boundary layer and affect the development of near wall structures. It turns out that curvature (on the scale concerned) is small enough that its effect on the freestream structures is negligible (higher order correction in Re). Near the wall the presence of curvature generates natural structures which are forced in the presence of freestream disturbances.

1.7. Perfect and imperfect bifurcations

In section 1.3 we discussed perfect bifurcations, where the amplitude of some state A can be found (in some situations) to satisfy a temporal evolution equation known as a Stuart-Landau equation which is of the form

$$\frac{dA}{d\tau} = sa_1A + a_3|A|^2A. \quad (1.4)$$

In section 1.4 we discussed how this analysis applies to centrifugal instabilities. Recall that in this setting the variable denoted by s would correspond to the perturbation from the linear neutral value of the Taylor or Görtler number, $s = T - T_c$ or $s = G - G_c$. This gives an amplitude equation of the form (1.4) when seeking periodic solutions in the spanwise direction and in the absence of freestream coherent structures. When the Görtler number reaches a critical value denoted by G_c , the one dimensional basic flow undergoes a pitchfork bifurcation from a state which is uniform in the spanwise direction, to states with a non-uniform modulation in the spanwise direction. These states are known as Görtler vortices.

Consider how the situation changes when freestream coherent structures are present in the flow. In the presence of such structures the far field decay condition changes from

$$u \rightarrow 0, \quad y \rightarrow \infty, \quad (1.5)$$

to the matching condition

$$u \sim f(y) \cos \beta z, \quad y \rightarrow \infty, \quad (1.6)$$

which is determined by the freestream coherent structures. This provides an external forcing on the near wall bifurcation problem which effectively breaks the symmetry of the bifurcation. The amplitude evolution equation is now found to be of the form

$$\frac{dA}{d\tau} = a_0 + sa_1A + a_3|A|^2A, \quad (1.7)$$

where a_0 represents the effect of the far field forcing arising from the freestream coherent structures. The result is a so called imperfect bifurcation, which breaks and smooths the bifurcation as we will see in chapter 4.

Imperfect bifurcations have been used to explain a variety of phenomena. For example they have been used in fluid dynamics to explain end effects in convection in a two dimensional box by Hall & Walton (1977, 1979) and end effects in the Taylor vortex problem by Hall (1980). It turns out that the latter of these has some strong similarities with the current work in the near resonant spanwise wavenumber case as will be discussed in section 4.3.3 and at the end of section 4.4.2.

1.8. Numerical methods

Throughout this thesis a variety of numerical methods will be applied to solve problems involving ordinary differential equations and partial differential equations in one and two variables, both of linear and nonlinear type. We will also be required to solve nonlinear eigenvalue problems, in some cases coupled to nonlinear partial differential equations. This is the form of the fully nonlinear interaction equations found in chapters 2 and 3. We will discuss the numerical methods used in more detail when they are encountered later in this thesis. However, we will use this section to provide a brief overview of the numerical methods applied throughout.

1.8.1. Finite difference methods

A method which will be used several times (for some linear problems in this thesis) is the finite difference method. This is a method which can be used to solve differential equations by discretising the solution over the domain. Derivatives can be approximated by taking differences between the points on the discretised grid. There are multiple ways to do this, each with its advantages and disadvantages.

For example, consider approximating the first derivative of a function of one variable $f(x)$. Consider a function f , which is analytic in some neighbourhood around a point x_0 . We can Taylor expand f around the point x_0 ,

$$f(x_0 + h) = f(x_0) + hf'(x_0) + O(h^2), \quad (1.8)$$

valid for sufficiently small h . Rearranging this formula we can find multiple expressions

for $f'(x_0)$. Three possible examples are given by

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0)}{h} + O(h), \quad (1.9a)$$

$$f'(x_0) = \frac{f(x_0) - f(x_0 - h)}{h} + O(h), \quad (1.9b)$$

$$f'(x_0) = \frac{f(x_0 + h) - f(x_0 - h)}{2h} + O(h^2). \quad (1.9c)$$

We have indicated the corresponding errors for each representation using the standard O notation where $f = O(g)$ means “ $|f/g| < \text{a non-zero constant}$ ” in some limit, in this case as h tends to zero. In equations (1.9) the different forms are known as forward, backward and central differences, respectively. Consider a general autonomous one dimensional dynamical system given by $\dot{x} = f(x)$ with some initial condition $x(0) = a$, where the dot is used to denote differentiation with respect to time. We will consider the problem for some interval of time $t \in [0, T]$ where T is some real positive value. We discretise the system sampling at $(N + 1) \in \mathbb{Z}$ points so the time step is $\Delta t = T/N$. Then the n th time step is $t_n = n\Delta t$ for $n = 0, \dots, N$, and we denote the value of x at each time step by $x_n = x(t_n)$. There are advantages and disadvantages to each of the approximations given above in equations (1.9) and we will briefly discuss them here.

Firstly the forward difference scheme (1.9a) corresponds to explicit time marching where at each time step we have

$$x_{n+1} = x_n + \Delta t f(x_n), \quad (1.10)$$

so x_{n+1} is explicitly given at each time step. This is the easiest of the schemes to implement but can suffer from stringent stability requirements in more complicated systems - see for example the Courant-Friedrichs-Lewy condition in the one dimensional heat equation.

The second difference scheme (1.9b) corresponds to implicit time marching where at each time step

$$x_{n+1} = x_n + \Delta t f(x_{n+1}). \quad (1.11)$$

This in general has much more favourable stability properties. It suffers from the complication that for a general function f , solving for x_{n+1} at each time step is more complicated. In general for nonlinear functions f , some type of fixed point iterative method or Newton method would have to be applied at each time step. This will be discussed in more detail shortly.

In contrast to the first two options which have an error at each time step which is linear $O(\Delta t)$, the third option (1.9c) is quadratic in the error. At each time step we have

$$x_{n+1} = x_{n-1} + 2\Delta t f(x_n). \quad (1.12)$$

When using this scheme for time marching, the price paid for the extra accuracy of this scheme at each time step is the difficulty in starting the scheme. It is not possible to apply this scheme at $n = 0$ because two points are required to march this scheme forward, i.e. x_{-1} lies outside the domain. And one cannot simply apply one of the first two schemes in the first time step because the associated error would be $O(\Delta t)$. Thus we would lose the benefit of having $O(\Delta t^2)$ error in the scheme if the first time step has $O(\Delta t)$ error - this larger error would propagate forward in the solution through time leading to an $O(\Delta t)$ error overall.

When applying these types of finite difference methods, in section 2.2 for example, we will opt to use centred differences to reduce the overall error. The problems which will be solved using this method within this thesis are elliptic, and thus we solve for the solution at all points in the domain at once. The complications discussed in the previous paragraph can be overcome by applying higher order one sided differences near domain boundaries. For example, the first order derivative can be approximated by the one sided second order scheme

$$f'(x_0) = \frac{-f(x_0 + 2h) + 4f(x_0 + h) - 3f(x_0)}{h} + O(h^2). \quad (1.13)$$

This allows us to maintain the second order error achieved elsewhere in the domain using the centred difference formula (1.9c).

One can apply multi-step methods where at each time step multiple (pseudo-)steps are taken to decrease the error. One class of examples are the predictor-corrector methods, where the first step predicts the value at x_{n+1} and then the next step updates our prediction reducing the overall error of the scheme. In section 2.6.2 we will apply a method from a class of multi-step methods called Runge-Kutta methods to solve a system of nonlinear ordinary differential equations with a combination of marching and shooting. We have discussed marching in its simplest form but if we have a two point boundary value problem, say $\ddot{x} = f(x)$ with $x(0) = a$ and $x(1) = b$, then in this case we cannot apply the simple marching procedure. However, if we have the extra initial condition $\dot{x}(0) = c$ (in place of the boundary condition $x(1) = b$) then we can apply the simple marching procedure. Shooting consists of solving the second problem with some

fixed c and altering c (usually using the Newton method) until we satisfy the original boundary condition $x(1) = b$. This will be discussed further in appendix A. Alternatively, we could discretise the whole original problem at once and seek a global solution to the problem directly. For nonlinear systems system however, this would involve solving a large set of nonlinear equations simultaneously.

1.8.2. Spectral methods

In the finite difference method, derivatives of a function are approximated using the function values on a grid. This can be justified in multiple ways, one of which is by using an interpolating polynomial. An alternative to this is to use a spectral representation of the function and then calculate derivatives using this representation. This benefits from so called exponential convergence for smooth functions, as the coefficients in the spectral decomposition decay exponentially for smooth functions.

Two of the methods that have been used in this thesis are Chebyshev and Fourier decompositions of functions. Chebyshev polynomials are useful if we wish to solve a problem on the domain $x \in [-1, 1]$. Let T_n denote the n th Chebyshev polynomial,

$$T_n(x) = \cos(n \arccos(x)). \quad (1.14)$$

Then suppose we are interested in the representation of a function f on the domain $[-1, 1]$. Then we can find coefficients a_n such that

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x). \quad (1.15)$$

Derivatives can be calculated easily using this representation so the k th derivative is approximated by some finite truncation of

$$f^{(k)}(x) = \sum_{n=0}^{\infty} a_n T_n^{(k)}(x). \quad (1.16)$$

If the function f is periodic on the domain $x \in [0, L]$, then one could use a Fourier representation of the function. This representation decomposes the function into its

harmonics, so we find

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp \frac{2in\pi x}{L} = \frac{d_0}{2} + \sum_{n=1}^{\infty} d_n \cos \frac{2n\pi x}{L} + e_n \sin \frac{2n\pi x}{L}. \quad (1.17)$$

Derivatives can then be approximated by a finite truncation of

$$f'(x) = \sum_{n=-\infty}^{\infty} \frac{2in\pi}{L} c_n \exp \frac{2in\pi x}{L} = \sum_{n=1}^{\infty} -\frac{2n\pi}{L} d_n \sin \frac{2n\pi x}{L} + \frac{2n\pi}{L} e_n \cos \frac{2n\pi x}{L}. \quad (1.18)$$

Spectral representations can be applied in more than one dimension, as we will see when we apply this method to solve the interaction equations in chapters 2 and 3. In these sections we will exploit symmetries in one of the dimensions to write solutions in the form of, for example:

$$f(x, y) = \sum_{n,k=0}^{\infty} a_{n,k} T_n(x) \cos ky. \quad (1.19)$$

In practice these expansions are truncated at large values of n and k such that the error from ignoring the higher modes, known as the truncation error, is negligible, or in other words less than some set tolerance.

Once a spectral representation has been chosen to approximate derivatives, one must also consider the nature of the residual (of the discretised equations) which leads to different types of methods. Say the equation of interest for example is one dimensional and is given by some generally nonlinear operator $\mathcal{N}[u](x) = 0$, where u is the function to be found. One possibility is to apply the collocation method which gives a pointwise error where we define the residual vector to be $R(j) = \mathcal{N}[u](x_j)$. An alternative is the Galerkin method which exploits the orthogonality of the basis functions to give a modal error, i.e. using a Fourier representation the error would be composed of terms of the form $R(j) = \langle \mathcal{N}[u](x), \cos jx \rangle$.¹ With boundary conditions applied pointwise this second method is known as the tau method.

¹ $\langle \cdot, \cdot \rangle$ denotes the inner product defined by $\langle f, g \rangle = \int f(s)g(s)ds$ where the integral is taken over the domain of the problem.

1.8.3. Solving linear and nonlinear equations

The discussed discretisation methods reduce the system of differential equations to a set of algebraic equations. If the underlying equations are linear, then the discretised algebraic equations will also be linear. In this case they can be represented in matrix form. Once in this form the system can be inverted using a variety of methods, such as LU-decomposition for example. These methods can be implemented using a standard linear algebra package.

If the equations are nonlinear, then solving the system is more difficult. One method which can be applied is a fixed point iterative scheme. For example if one wishes to solve the simple quadratic equation $x^2 - 2x - 1 = 0$ then one could iterate on $x_{n+1}^2 - 2x_n - 1 = 0$ or $x_n^2 - 2x_{n+1} - 1 = 0$, starting with different initial conditions. If such an iterative scheme converges to a finite limit, then such a limit must be a root of the original equation. In general convergence is not guaranteed and even when it occurs, may be slow. This is the type of method that was originally used to solve the full nonlinear interaction equations of chapter 2 - see equations (2.38) and (2.56). This method was found to be unreliable, often failing to converge.

Throughout this thesis we will instead apply the Newton or quasi-Newton method to solve nonlinear equations. Quasi-Newton simply means that we have approximated the Jacobian. Consider solving the equation $f(x) = 0$, where f is nonlinear in general and assumed to be continuously differentiable. The Newton method in one dimension can be reasoned by considering the Taylor expansion of a function around the solution, $x_{n+1} = x_n + \delta x_n$, supposing δx_n is small and x_{n+1} satisfies the equation $f(x) = 0$. Then we find

$$0 = f(x_{n+1}) = f(x_n + \delta x_n) = f(x_n) + \delta x_n f'(x_n) + O(\delta x_n^2). \quad (1.20)$$

Ignoring higher order terms in δx_n , since we assumed that δx_n is small, we find

$$x_{n+1} = x_n + \delta x_n = x_n - \frac{f(x_n)}{f'(x_n)}. \quad (1.21)$$

It turns out that the Newton method converges quadratically. That is, for sufficiently small δx_n , we find that $\delta x_{n+1} = O(\delta x_n^2)$ provided $f'(x) \neq 0$, where x is the desired solution ($x_n \rightarrow x$ to $f(x) = 0$).

The scheme follows analogously in higher dimensions, $\mathbf{0} = \mathbf{f}(\mathbf{x}_{n+1}) = \mathbf{f}(\mathbf{x}_n) + \mathbf{J}\delta\mathbf{x}_n + O(|\delta\mathbf{x}_n|^2)$ where $\mathbf{J}$ is the Jacobian of $\mathbf{f}$ with respect to all of the variables. We find by

ignoring higher order terms in $\delta \mathbf{x}_n$ that

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{J}^{-1} \mathbf{f}(\mathbf{x}_n). \quad (1.22)$$

The convergence is quadratic provided the Jacobian evaluated at the solution is non-singular, i.e. $\det \mathbf{J}(\mathbf{x}) \neq 0$ where $\mathbf{x}$ is the desired solution to $\mathbf{f}(\mathbf{x}) = \mathbf{0}$. The quadratic convergence (and guaranteed convergence if sufficiently close to the solution) make this approach attractive. Also the Jacobian matrix evaluated at the solutions holds information about the system which can be valuable, for example, in identifying bifurcation points. One drawback is the difficulty in computing the Jacobian matrix. It can be difficult, and time consuming, to compute analytically. And computing an approximate Jacobian, such as using finite differences which is the method used in this thesis, can be slow. However, parallelising this procedure the time required to compute the Jacobian can be reduced.

As stated at the beginning of this section, more details will be given in later sections as and when each numerical method is encountered. This section has given a rough outline and description of the numerical methods applied throughout this thesis.

1.9. Summary of this thesis

Chapters 2 and 3 will concentrate primarily on vortex-wave interaction. A number of numerical studies have been performed on the nonlinear vortex-wave interaction equations involving inviscid Rayleigh-type waves, whereas with the exception of a few numerical investigations for boundary layers such as by Hall & Smith (1991) and Walton & Patel (1999), and for channel flows by Bennett, Hall & Smith (1991), the Tollmien-Schlichting-type interaction has received far less attention. Many of the solutions found (at moderate amplitudes) exhibit strong spanwise focusing. This means they would have been unlikely to have been found in studies which lacked a high degree of resolution in the spanwise direction, or by using simple iterative schemes.

In chapter 2 we will consider the strongly nonlinear interaction of Tollmien-Schlichting-type waves with a vortex flow. With the use of a multi-dimensional Newton method we will track the solutions into the strongly nonlinear regime in plane Poiseuille flow. We will also consider the interaction when the walls are allowed to (slowly) slide, so the basic state possesses a weak Couette flow component. Later in the chapter we will revisit the case of curved channel flows originally studied by Bennett, Hall & Smith (1991), where

there can be an interaction between the Tollmien-Schlichting mechanism and the Taylor vortex mechanism. Using a mixture of analytic, semi-analytic and numerical approaches we will shed some light on the states described by this interaction in the weakly and strongly nonlinear regimes.

In chapter 3 we will consider the interaction of Tollmien-Schlichting-type waves with a vortex in the asymptotic suction boundary layer. Again we will apply similar methods and use a mixture of analytic, semi-analytic and numerical approaches to describe the interaction in the strongly nonlinear regime, finding strong spanwise focusing of the solution for finite amplitude interactions.

In chapter 4 we extend our work in the asymptotic suction boundary layer to consider the effect of curvature on equilibrium states in the presence of freestream coherent structures. We start with a study on the effects of curvature in the asymptotic suction boundary layer, identifying Görtler vortices bifurcating from the basic state. Further to this we find many secondary bifurcations, some of which are found to connect states with different spanwise wavenumbers. An understanding of these secondary bifurcations allows us to explain the change in criticality of the bifurcation of Görtler vortices in the asymptotic suction boundary layer. We will review the freestream coherent structures found by Deguchi & Hall (2014) and then add curvature to the system. Since the effect of curvature is small in the freestream, this becomes a receptivity problem for the near wall Görtler vortices which leads to imperfect bifurcations. Later, we remove the assumption of homogeneity in the streamwise direction, and consider the streamwise development of Görtler vortices when the production layer is impulsively started at a specific streamwise location.

2. Vortex-wave interaction in plane Poiseuille-Couette flow

2.1. Introduction

In this chapter we will discuss the strongly nonlinear interaction between a vortex flow (recall from section 1.3 that the vortex is the steady part of the flow) and a finite amplitude wave. The wave is a lower branch viscous neutral TS-wave (Tollmien-Schlichting-wave) and the interaction takes place in a channel flow resulting in fully nonlinear equilibrium travelling wave states. The governing equations for this strongly nonlinear interaction were first derived by Bennett, Hall & Smith (1991) in the context of curved channel flows, and for completeness we will re-derive them in this chapter. It has been shown by Dempsey, Deguchi, Hall & Walton (2016) that the vortex-wave interaction equations describe states which are the large Reynolds number limit of nonlinear equilibrium states¹ in plane Poiseuille flow. Throughout the majority of this chapter we will consider the case of plane Poiseuille flow with straight parallel walls and then later we will extend this work into the analogous strongly nonlinear interaction in plane Poiseuille-Couette flow and curved channel flows. The asymptotic structure of the lower branch neutral TS wave is identical in plane Poiseuille-Couette flow provided the wall sliding velocities are sufficiently small. Whereas in plane Poiseuille flow there is only one lower branch neutral TS wave, with the addition of a small Couette flow component it turns out that there are multiple such lower branch neutral TS waves - see Cowley & Smith (1985) for example. When the channel walls are curved the situation is complicated by a non-unique basic flow. The level of curvature is quantified by the non-dimensional number denoted by T which is known as the Taylor number. It is well known that, in the absence of any other effects such as TS waves, at a critical value of the Taylor number denoted by T_c which is dependent upon the spanwise wavenumber of interest, a symmetric pair of states bifurcates from the basic state, and these states which are a consequence of the

¹These are exact solutions of the Navier-Stokes equations which are found computationally.

wall curvature are known as Taylor vortices.

The existence of a linear instability in the flows of interest here provides a strategy for computing fully nonlinear equilibrium travelling wave states. Linear stability calculations can be performed on the basic state to identify the streamwise wavenumber and wavespeed of the neutral modes as the amplitude of the TS wave tends to zero. This provides us with a first state, essentially the trivial state, which consists of the unperturbed basic state with a wave of infinitesimally small amplitude superimposed whose streamwise wavenumber and wavespeed have been determined from a linear stability calculation. The problem for a finite amplitude TS wave is complicated by the nonlinear self-interaction of the wave which alters the vortex flow from the basic state. However, we can consider bifurcations from the linear neutral point which can be described analytically in the form of a weakly nonlinear analysis. From these analyses we can track these small amplitude states computationally into strongly nonlinear finite amplitude interactions either by varying the amplitude of the TS wave (the strategy adopted here) or the streamwise wavenumber to map out neutral surfaces in the state space which correspond to nonlinear equilibrium travelling wave states. In the past such tracking has been performed by fixed-point iterative schemes; however here we will apply a multi-dimensional Newton method which, while computationally much more expensive, is much more reliable and allows us to track these states to much larger amplitudes and observe phenomena that we would not otherwise have found.

We consider the flow of an incompressible fluid with density ρ , dynamic viscosity μ and kinematic viscosity ν , through a channel with half-height L , driven by two effects. The first is a uniform streamwise pressure gradient $-G$ where $G = 2U_0L^{-2}\mu$ and U_0 is the centreline speed of the streamwise flow generated by the pressure gradient $-G$. The second is sliding walls with velocities $\pm U_0u_w$. We can assume without loss of generality that the upper and lower walls move with equal and opposite velocities - if not we can move into a streamwise translating frame with the average of the two wall velocities where in this frame the two walls move with equal speed but in opposite directions. The full governing equations for the interaction are the incompressible Navier-Stokes equations, non-dimensionalised with the centreline velocity from the Poiseuille component of the velocity U_0 , the channel half height H , and the usual high Reynolds number pressure scale ρU_0^2 . The Reynolds number is defined by $Re = U_0L/\nu$. The non-dimensionalised basic state consists of plane Poiseuille flow with a plane Couette component of size u_w .

The governing equations are thus found to be

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1a)$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + Re^{-1} \nabla^2 \mathbf{u}, \quad (2.1b)$$

with no slip boundary conditions on the walls,

$$\mathbf{u} = \pm u_w \mathbf{e}_x, \quad y = \pm 1. \quad (2.1c)$$

With the chosen non-dimensionalisation the basic state for plane Poiseuille-Couette flow is

$$\mathbf{u}_b = (1 - y^2 + u_w y, 0, 0), \quad p_b = -2Re^{-1}x, \quad (2.2)$$

where u_w is the wall sliding velocity. We can recover plane Poiseuille flow by setting the Couette component to zero, i.e. $u_w = 0$. The states which will be found in this chapter are strongly nonlinear in the sense that the streamwise component of the vortex flow, referred to as the streak in the self-sustaining processes community, is altered at leading order, i.e. $\|u - u_b\| \sim O(1)$. As we will show in the next section, plane Poiseuille flow becomes linearly unstable at a critical Reynolds number $R_c \approx 5772$ as found by for example Orszag (1971) who solved the famous Orr-Sommerfeld equation using a Chebyshev spectral method. In later sections we will discuss the effect of increasing u_w which interestingly, for small wall sliding velocities, increases the number of linear neutral modes (Cowley & Smith 1985). Increasing the wall velocity further, eventually a cut off velocity is reached above which the flow appears to be linearly stable as one might expect given the linear stability of plane Couette flow.

As a brief overview of this chapter, we will firstly consider the linear stability of plane Poiseuille and plane Poiseuille-Couette flow both at finite Reynolds numbers and in the large Reynolds number limit. This serves as a starting point for the interaction allowing us to identify the important scales for the interaction at large Reynolds numbers, from which we can derive the governing equations for the interaction. A weakly nonlinear analysis of the interaction equations reveals the structure of small amplitude travelling wave states which bifurcate from the linear neutral points, and importantly the nature of the bifurcation. To compute states at larger amplitudes we must turn to numerical computations of the fully nonlinear interaction equations, guided by the analyses already performed. We will consider in detail the interaction in plane Poiseuille flow for a variety of spanwise wavenumbers, tracking states computationally from the weakly nonlinear regime into the strongly nonlinear regime. We will then discuss variations of vortex-wave

interaction in plane Poiseuille flow, namely the interaction in plane Poiseuille-Couette flow and finally in a curved channel.

2.2. Linear stability of plane Poiseuille flow at finite Reynolds numbers

In this section we will review the linear stability of plane Poiseuille flow at finite Reynolds numbers. This will include a derivation of the famous Orr-Sommerfeld equation and a description of two methods which have been applied here to solve it which are a second order finite difference scheme and a Chebyshev-collocation method. We will focus on the temporal stability to infinitesimal disturbances in this section, although the spatial stability can be determined in a similar manner. The first step is to add an arbitrarily small perturbation to the basic state, so we set

$$(u, v, w, p) = (u_b, 0, 0, p_b) + [(\bar{u}, \bar{v}, \bar{w}, \bar{p})\hat{E} + c.c.], \quad (2.3)$$

where E denotes the streamwise travelling wave component

$$\hat{E} = \exp(i\hat{\alpha}(x - \hat{c}t) + i\hat{\beta}z), \quad (2.4)$$

with streamwise wavenumber $\hat{\alpha}$, wavespeed $\hat{c}$ and spanwise wavenumber $\hat{\beta}$. We use *c.c.* to denote the complex conjugate, and the barred variables are assumed to only depend on the wall normal coordinate y . The perturbations are assumed to be sufficiently small that all terms which are quadratic in the perturbations to the basic flow are negligible. As we are considering the temporal stability problem, we prescribe the streamwise and spanwise wavenumbers, $\hat{\alpha}$ and $\hat{\beta}$, and we find an eigenvalue problem for the complex wave speed $\hat{c}$. The real part of the complex wavespeed, $\text{Re}(\hat{c})$, denotes the phase speed which is the speed at which the wave propagates. The imaginary part of the complex wavespeed multiplied by the streamwise wavenumber, $\hat{\alpha}\text{Im}(\hat{c})$, denotes the temporal growth rate. Substituting the expansions (2.3) into the Navier-Stokes equations (2.1), and ignoring terms which are quadratic in the perturbations, we find the governing equations are the linearised Navier-Stokes equations decomposed into streamwise and spanwise harmonics

$$i\hat{\alpha}\bar{u} + D\bar{v} + i\hat{\beta}\bar{w} = 0, \quad (2.5a)$$

$$i\hat{\alpha}(u_b - \hat{c})\bar{u} + u_b'\bar{v} = -i\hat{\alpha}\bar{p} + Re^{-1}(D^2 - \hat{\alpha}^2 - \hat{\beta}^2)\bar{u}, \quad (2.5b)$$

$$i\hat{\alpha}(u_b - \hat{c})\bar{v} = -D\bar{p} + Re^{-1}(D^2 - \hat{\alpha}^2 - \hat{\beta}^2)\bar{v}, \quad (2.5c)$$

$$i\hat{\alpha}(u_b - \hat{c})\bar{w} = -i\hat{\beta}\bar{p} + Re^{-1}(D^2 - \hat{\alpha}^2 - \hat{\beta}^2)\bar{w}, \quad (2.5d)$$

where $D \equiv d/dy$. It is possible to eliminate $\bar{u}, \bar{w}, \bar{p}$ to leave a single equation for $\bar{v}$ which together with homogenous boundary conditions (determined by no slip and using (2.5a)),

$$\bar{v} = D\bar{v} = 0, \quad y = \pm 1, \quad (2.6)$$

is an eigenvalue problem for $\hat{c} \in \mathbb{C}$ given $\hat{\alpha}, \hat{\beta}$ and $Re \in \mathbb{R}$. This leaves the famous Orr-Sommerfeld equation

$$[(u_b - \hat{c})(D^2 - \hat{\alpha}^2 - \hat{\beta}^2) - u_b'' - (i\hat{\alpha}Re)^{-1}(D^2 - \hat{\alpha}^2 - \hat{\beta}^2)^2]\bar{v} = 0. \quad (2.7)$$

Decomposing the disturbances to the basic state in this way is referred to as a modal analysis. This is the governing equation for a single harmonic of the perturbation in a normal mode decomposition, and thus if we decompose the disturbance into streamwise and spanwise harmonics then the Orr-Sommerfeld equations tells us which modes will grow and which will decay.

On the other hand we could linearise the Navier-Stokes equations without a decomposition into harmonics, we set

$$(u, v, w, p) = (u_b, 0, 0, p_b) + (\hat{u}, \hat{v}, \hat{w}, \hat{p}), \quad (2.8)$$

where the hatted variables depend on time and all spatial variables.² Once again assuming the disturbances are sufficiently small that we can ignore quadratic terms in the disturbances, we find the governing equations can be reduced to a single equation in the wall normal perturbation

$$(\partial_t + u_b\partial_x)\nabla^2\hat{v} - u_b''\partial_x\hat{v} = Re^{-1}\nabla^4\hat{v}, \quad (2.9)$$

which becomes (2.7) when a normal mode decomposition is taken. While the modal analysis predicts the large time growth rates of disturbances, there are drawbacks of

²To recover the modal analysis in equation (2.3) we would set $(\hat{u}, \hat{v}, \hat{w}, \hat{p}) = [(\bar{u}, \bar{v}, \bar{w}, \bar{p})\hat{E} + c.c.]$.

this method such as its failure to describe short time phenomena such as short time algebraic growth when the operator in the Orr-Sommerfeld equation is non-normal, see for example Schmidt (2007).

For the discussion which follows we assume $\hat{\alpha} > 0$. Focusing back on the Orr-Sommerfeld equation (2.7) we recall that the temporal growth rate is $\hat{\alpha}\text{Im}(\hat{c})$. In some flows such as plane Couette flow and Hagen-Poiseuille flow it is observed that $\text{Im}(\hat{c}) < 0$ for all $(\hat{\alpha}, \hat{\beta}, Re)$, and hence the flow is temporally linearly stable. In plane Poiseuille flow when the Reynolds number is large enough there are both disturbances with $\text{Im}(\hat{c}) < 0$ (stable modes) and disturbances with $\text{Im}(\hat{c}) > 0$ (unstable modes). From this it follows that there are regions in $\hat{\alpha}, \hat{\beta}, Re$ space where $\text{Im}(\hat{c}) = 0$ which correspond to disturbances which neither grow, nor decay, but have a constant non-zero amplitude, and in general travel downstream with a non-zero wave speed $\text{Re}(\hat{c})$. These modes are referred to as neutral modes and the curves which these modes lie on in $\hat{\alpha}, \hat{\beta}, Re$ space are known as neutral curves.

Two methods have been applied here to solve the Orr-Sommerfeld equation (2.7). The first is a second order finite difference scheme. This method considers the solution on a uniform discretised grid of say $(N + 1)$ points with a spacing $h = 1/N$. We denote the j th grid point by y_j where $y_j = 2j/N$, and $\bar{v}(y_j)$ is denoted by $\bar{v}_j$ for $j = 0, \dots, N$. The derivatives are discretised using second order centred differences, so for example the second derivative is discretised by

$$\bar{v}_j'' = (\bar{v}_{j+1} - 2\bar{v}_j + \bar{v}_{j-1})/h^2 + O(h^2). \quad (2.10)$$

Applying the equation at the interior points for $j = 2, \dots, N - 2$ and the boundary conditions at the exterior points, this yields a matrix eigenvalue problem which determines the modes on the grid points y_j through the eigenvectors and their growth rates and wave speeds through the complex eigenvalues. Iterating on the streamwise wavenumber $\hat{\alpha}$ and/or the Reynolds number Re , this method can be used to find the neutral modes or simply to identify the most dangerous (those with the largest growth rates) modes for a specific streamwise wavenumber, spanwise wavenumber and Reynolds number. As mentioned earlier, the other method which has been used here is a Chebyshev spectral method. This involves decomposing $\bar{v}$ into Chebyshev modes, so we write

$$\bar{v} = \sum_k \bar{V}_k T_k(y), \quad (2.11)$$

and then applying the equation at the Chebyshev (Gauss-Lobatto) collocation points,

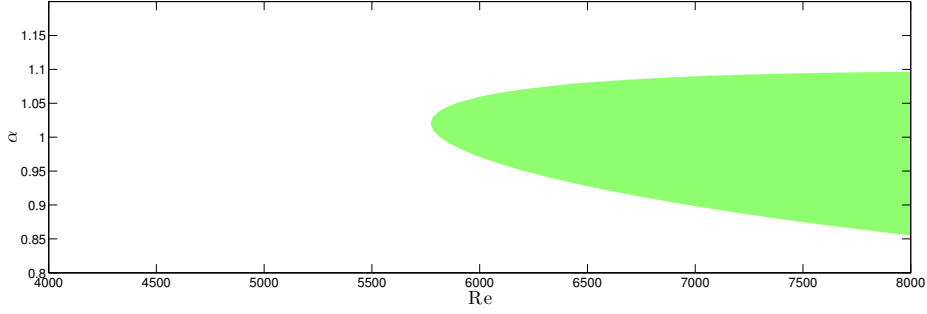


Figure 2.1. Linear neutral curve for plane Poiseuille flow when $\hat{\beta} = 0$. The shaded region represents the unstable region in $(\hat{\alpha}, Re)$ space where $\text{Im}(\hat{c}) > 0$.

$y_j = \cos(j\pi/N)$. Irregular grid points are used to counter Gibbs (or Runge) phenomenon. They have the advantage that they pack points near the boundaries where we often expect increased activity due to boundary conditions. Using the spectral expansion (2.11), we find an eigenvalue problem for the spectral coefficients $\bar{V}_k$, which can be solved as in the finite difference problem to reveal the most dangerous modes and neutral modes.

The linear stability of plane Poiseuille flow at finite Reynolds numbers and for a fixed spanwise wavenumber $\hat{\beta}$ can be summarised in $(\hat{\alpha}, Re)$ space. There is a single region of linear instability extending from the critical Reynolds number Re_c (found here to be approximately 5772.22 for $\hat{\beta} = 0$ in agreement with Orszag (1971)) to infinite Reynolds numbers. The neutral curve for $\hat{\beta} = 0$ can be seen in figure 2.1. The effect of changing the spanwise wavenumber only acts to scale the problem, effectively changing the critical Reynolds number Re_c - this is a consequence of Squire's theorem. The bounding curve between the stable and unstable modes is called the linear neutral curve, and represents the neutral modes. It consists of a lower branch and an upper branch, which are denoted by $\alpha_L(Re)$ and $\alpha_U(Re)$ respectively, and are defined such that $\alpha_L(Re) \leq \alpha_U(Re)$. Thus the flow is linearly unstable to streamwise wavenumbers in the interval $(\alpha_L(Re), \alpha_U(Re))$ for a given Reynolds number Re . It turns out that both $\alpha_L(Re)$ and $\alpha_U(Re)$ tend to zero as $Re \rightarrow \infty$ and in fact both branches can be described asymptotically for large Reynolds numbers (see Lin 1945). The asymptotic scaling of the lower branch neutral TS wave will be the focus of the next section.

2.3. Linear stability of plane Poiseuille flow at large Reynolds number: lower branch

As mentioned in the previous section, the streamwise wavenumber tends to zero for both branches of the linear neutral curve as the Reynolds number increases, and Lin (1945) identified the correct scalings for each of the branches. We will concentrate on the lower branch neutral mode. The proceeding analysis follows from Smith (1979) but for completeness we will derive the scales and the corresponding dispersion relation for the lower branch mode. Before we can deduce the correct scaling, we must consider a scaling analysis of the equations to identify the most important terms in each region of the domain for large Reynolds numbers. To do this we will consider the relative size of the perturbations to the basic flow in relation to the size of the streamwise perturbation to the velocity through the middle of the channel, thus $u_c \sim 1$, where we use subscript c's to denote core variables (throughout the middle of the channels) and subscript w's to denote variables near the walls. If we denote the size of the streamwise wavenumber by $[\alpha]$, the size of the wave speed by $[c]$ and the size of the spanwise wavenumber by $[\beta]$, then the continuity equation $u_x \sim v_y$ applied in the core of the channel yields that $v_c \sim [\alpha]$ and then the wall normal momentum equation $p_y \sim uv_x$ gives $p_c \sim [\hat{\alpha}]^2$. The spanwise momentum equation $p_z \sim uw_x$ yields $w_c \sim p_c[\beta]/[\alpha] = [\alpha][\beta]$. Now consider the wall layer equations, where the wall has size $[y_w]$ and here the basic flow $u_b \sim [y_w]$ by Taylor expanding the basic flow near the wall. Matching the wall streamwise velocity with the core streamwise velocity we have³ $u_w \sim 1$, and then the continuity equation gives $v_w \sim [\alpha][y_w]$ and $w_w \sim [\alpha]/[\beta]$. The streamwise momentum equation $uu_x \sim p_x \sim Re^{-1}u_{yy}$, noting that viscous forces are required to satisfy the boundary conditions on the wall, gives $[\alpha][y_w] \sim [\alpha]p_w \sim Re^{-1}/[y_w]^2$. Matching the pressure between the core and the wall, we have $p_w \sim [\alpha]^2$, and thus $[y_w] = [\alpha]^2$, and hence finally that $[\alpha] = Re^{-1/7}$. From the spanwise normal equation $p_z \sim uw_x$ we find $[\alpha]^2[\beta] \sim [y_w][\alpha]^2/[\beta]$ and thus $[\beta] \sim [\alpha]$. For convenience we denote $\epsilon = Re^{-1/7}$, and then from the argument just given we can introduce the relevant length scales for the problem

$$X = \epsilon x, \quad Z = \epsilon z, \quad T = \epsilon^3 t, \quad Y = \epsilon^{-2}(y + 1), \quad (2.12)$$

where Y is the wall normal variable in the lower wall layer - an analogous upper wall variable could be defined but we will only discuss the lower wall layer because the analysis in the upper wall layer follows similarly. Identifying the important scalings for the lower

³This can be verified from the solution in the core - see equation (2.17).

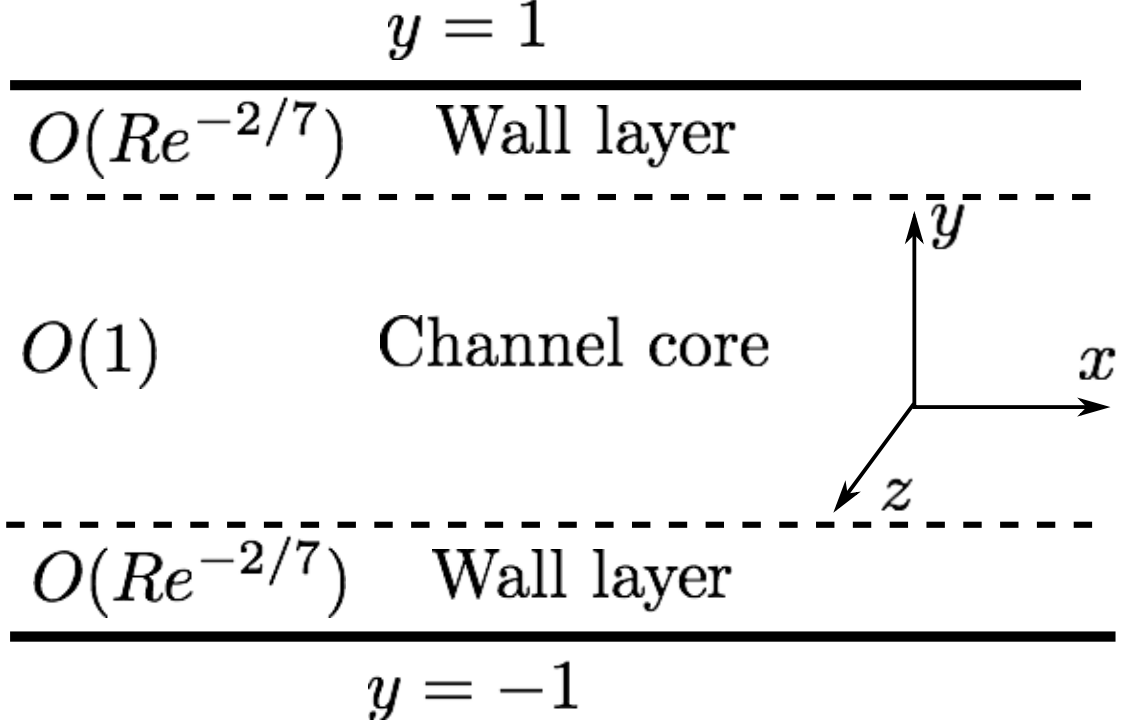


Figure 2.2. Asymptotic structure of lower branch Tollmien-Schlichting-type-waves.

branch wave is the first step in writing down the interaction equations and the derived asymptotic structure of the lower branch wave is illustrated in figure 2.2. We will use the quantity δ to denote the size of the wave which is undetermined at this stage due to the linear nature of the current analysis, the only constraint being that it is sufficiently small that all nonlinear terms are negligible at leading order. We have identified the critical scales for the wave, so for convenience we denote the travelling wave component by

$$E_0 = \exp\{i\alpha(X - cT) + i\beta Z\}. \quad (2.13)$$

Splitting the flow into the basic state and the wave which is of undetermined size δ we can write the flow in the core of the channel in the form

$$u = u_b + \delta[\hat{u}(y)E_0 + c.c.] + \dots, \quad (2.14a)$$

$$v = \epsilon\delta[\hat{v}(y)E_0 + c.c.] + \dots, \quad (2.14b)$$

$$w = \epsilon^2\delta[\hat{w}(y)E_0 + c.c.] + \dots, \quad (2.14c)$$

$$p = -\epsilon^7 2x + \epsilon^2\delta[\hat{p}(y)E_0 + c.c.] + \dots, \quad (2.14d)$$

where we have restricted our attention to plane Poiseuille flow in this section so with zero wall sliding velocity, $u_w = 0$, we have

$$u_b = (1 - y^2). \quad (2.15)$$

Substituting these expansions into the non-dimensionalised Navier-Stokes equations we find a set linear equations at leading order,

$$i\alpha\hat{u} + \hat{v}_y = 0, \quad (2.16a)$$

$$i\alpha u_b \hat{u} + u'_b \hat{v} = 0, \quad (2.16b)$$

$$i\alpha u_b \hat{v} + \hat{p}_y = 0, \quad (2.16c)$$

$$i\alpha u_b \hat{w} + i\beta \hat{p} = 0. \quad (2.16d)$$

Eliminating $\hat{u}$ from (2.16a) and (2.16b), we find $u_b \hat{v}_y = u'_b \hat{v}$ which can be solved up to an arbitrary constant $i\alpha A_1$ where the unknown constant A_1 will be determined later in the analysis. Then we can use (2.16c) to determine $\hat{p}$ up to an arbitrary constant $\hat{p}_0$, and (2.16d) to determine $\hat{w}$. Thus the solution to the core wave system (2.16) can be written in the form

$$\hat{u} = -A_1 u'_b, \quad \hat{v} = i\alpha A_1 u_b, \quad \hat{w} = -\frac{\beta}{\alpha u_b} \hat{p}, \quad \hat{p} = \hat{p}_0 + \alpha^2 A_1 \int u_b^2 dy. \quad (2.17)$$

We now need to consider the flow in the lower wall layer which is of size $O(\epsilon^2)$, and has wall normal variable Y defined in (2.12). The expansions and subsequent equations in the upper wall layer follow similarly to the following analysis. For $|y + 1| \ll 1$, a Taylor expansion of the basic flow gives

$$u_b \sim u'_b(-1)(y + 1) + \frac{1}{2}u''_b(-1)(y + 1)^2 + o[(y + 1)^2]. \quad (2.18)$$

From the streamwise momentum equation applied at $y = -1$ we see that $0 = \epsilon^7 2 + \epsilon^7 u''_b(-1)$ and thus $u''_b(-1) = -2$, and in fact this argument holds even in the nonlinear case which will be discussed later when u_b is replaced by a general function of y and Z . In the interest of keeping the argument as general as possible, we will denote $u'_b(-1)$ by λ_0 where for plane Poiseuille flow $\lambda_0 = 2$. Thus, near the wall

$$u_b \sim \epsilon^2 \lambda_0 Y - \epsilon^4 Y^2. \quad (2.19)$$

Now with the scales of the wave as determined by the scaling analysis described above we find

$$u = \epsilon^2 \lambda_0 \bar{Y} - \epsilon^4 \bar{Y}^2 + \delta[\bar{u}(\bar{Y})E_0 + c.c.] + \dots, \quad (2.20a)$$

$$v = \epsilon^3 \delta[\bar{v}(\bar{Y})E_0 + c.c.] + \dots, \quad (2.20b)$$

$$w = \delta[\bar{w}(\bar{Y})E_0 + c.c.] + \dots, \quad (2.20c)$$

$$p = -\epsilon^7 2x + \epsilon^2 \delta[\bar{p}(\bar{Y})E_0 + c.c.] + \dots \quad (2.20d)$$

Substituting these expansions into the Navier-Stokes equations we find the governing equations for the wall layer are

$$i\alpha \bar{u} + \bar{v}_{\bar{Y}} + i\beta \bar{w} = 0, \quad (2.21a)$$

$$i\alpha(\lambda_0 \bar{Y} - c)\bar{u} + \lambda_0 \bar{v} = -i\alpha \bar{p} + \bar{u}_{\bar{Y}\bar{Y}}, \quad (2.21b)$$

$$0 = -\bar{p}_{\bar{Y}}, \quad (2.21c)$$

$$i\alpha(\lambda_0 \bar{Y} - c)\bar{w} = -i\beta \bar{p} + \bar{w}_{\bar{Y}\bar{Y}}, \quad (2.21d)$$

with the boundary conditions,

$$\bar{u} = \bar{v} = \bar{w} = 0, \quad \bar{Y} = 0, \quad (2.21e)$$

$$\bar{u} \rightarrow -\lambda_0 A_1, \quad \bar{w} \rightarrow 0, \quad \bar{Y} \rightarrow \infty, \quad (2.21f)$$

with the displacement effect in (2.21f) determined by the streamwise velocity in the core (2.17). Equation (2.21c) reveals that the pressure within the lower wall layer is a constant, which we denote by $\bar{p} = P_-$. Following the work of Smith (1979), these equations can be manipulated into a dispersion relation. This is composed of a few steps, and first we note that similar equations to (2.21) hold in the upper wall layer, and if we set the pressure in the upper wall layer to be P_+ then symmetry considerations yield that $P_+ = -P_-$. Considering the difference between the pressure in the upper and lower wall layers, which must be accounted for by a difference in the pressure across the core from (2.17) and hence we find

$$P_+ - P_- = -2P_- = \alpha^2 A_1 \int_{-1}^1 u_b^2 dy. \quad (2.22)$$

This yields one pressure-displacement relation, i.e. a relationship between P_- and A_1 . Examining equation (2.21d), we see that this is a forced Airy equation and thus it is possible to solve for $\bar{w}$ in terms of Airy functions and the Scorer function. Differentiating

(2.21b) and using (2.21a,c) we find $\bar{u}_Y$ also satisfies a forced Airy's equations, and this can be solved in terms of Airy functions, the Scorer function and integrals of Airy functions. Then differentiating $\bar{u}_Y$ and applying (2.21b) on $Y = 0$, and integrating $\bar{u}_Y$ between $Y = 0$ and infinity yields another equation relating P_- and A_1 . This is a system of two homogeneous equations for two unknowns and hence we find a solvability condition, and this is the required dispersion relation which is found to be

$$\mathcal{D}(\alpha, c, \beta) = \lambda_0^{5/3} \mathcal{G}_0 + \alpha^{1/3} (\beta^2 + \alpha^2) J_0 = 0, \quad (2.23)$$

where

$$\lambda_0 = u'_b(-1) = 2, \quad \xi_0 = -\frac{(i\alpha\lambda_0)^{1/3}}{\lambda_0} c, \quad J_0 = \int_{-1}^1 u_b^2 dy = \frac{16}{15}, \quad (2.24)$$

$$Ai'_0 = Ai'(\xi_0), \quad \kappa_0 = \int_{\xi_0}^{\infty} Ai(s) ds, \quad \mathcal{G}_0 = i^{5/3} \frac{Ai'_0}{\kappa_0}. \quad (2.25)$$

Given a spanwise wavenumber β , this complex dispersion relation yields the real valued quantities α and c which are the streamwise wavenumber and wavespeed. For example, when $\beta = 1$ we can solve $\mathcal{D}(\alpha, c, 1) = 0$ to find $(\alpha, c) \simeq (1.9426, 2.9225)$. This says that for $\beta = 1$, the neutral modes on the lower branch have the large Reynolds number form

$$\hat{\alpha} \sim \alpha \epsilon \quad \text{i.e.} \quad \hat{\alpha} \sim 1.9426 Re^{-1/7}, \quad (2.26a)$$

$$\hat{c} \sim c \epsilon^2 \quad \text{i.e.} \quad \hat{c} \sim 2.9225 Re^{-2/7}. \quad (2.26b)$$

where we have used hats to denote the quantities over $O(1)$ non-dimensionalised scales. Figure 2.3 shows a comparison of α for finite Reynolds number found by solving (2.7), with the asymptotic value 1.9426. We see that the value is approaching the asymptotic value, but perhaps unsurprisingly due to the small parameter being $Re^{-1/7}$ (which we note is only 0.1 when $Re = 10^7$), the Reynolds number has to be quite large before agreement to a couple of decimal places can be reached.

2.4. Linear stability of plane Poiseuille-Couette flow

In this section we will discuss both the linear stability at finite Reynolds numbers and the asymptotic form of the lower branch at large Reynolds numbers when a small sliding velocity is added to the basic state. As we have seen in the previous sections, plane Poiseuille flow has a linear instability which develops at a critical Reynolds number Re_c . The critical Reynolds number depends on the spanwise wavenumber of the disturbances.

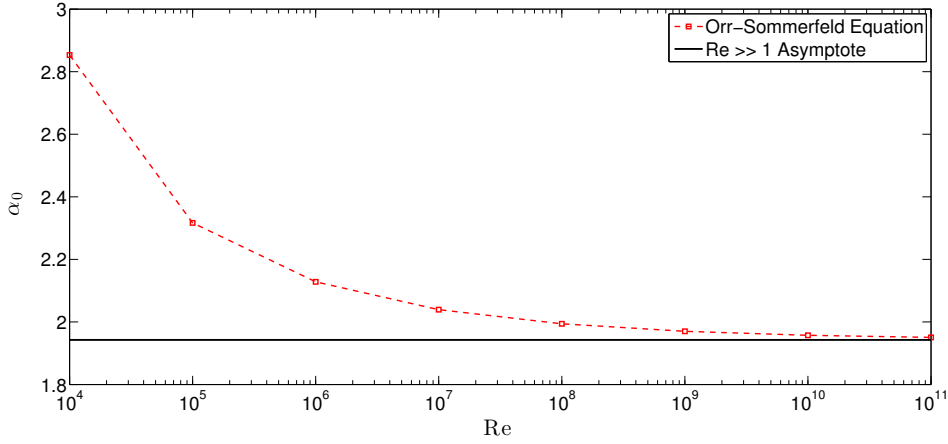


Figure 2.3. Comparison of Orr-Sommerfeld stability results with the asymptotic prediction for large Re , for $\beta = 1$.

It is well known that plane Couette flow, which is the flow through a straight channel where the walls slide at a constant velocity and there is zero pressure gradient, is linearly stable - no linear instability in plane Couette flow has ever been found and in fact a proof that the flow is linearly stable was indicated by Romanov (1973). Now consider plane Poiseuille flow but allowing now for a non-zero (non-dimensional) wall speed $\pm u_w$ on $y = \pm 1$. Given the linear stability of plane Couette flow, we expect as $u_w \rightarrow \infty$ that the flow should become linearly stable. In fact it turns out that the flow becomes linearly stable at a critical wall speed $u_w^c \approx 0.34$ (Potter 1966), with the region of linear instability receding to infinity, i.e. $Re_c \rightarrow \infty$ as $u_w \rightarrow u_w^c$.

It was predicted by Cowley & Smith (1985) that other linear neutral curves exist at large Reynolds numbers for non-zero wall speeds which are below the critical wall speed for cut off u_w^c . The difficulty in finding these new neutral curves is that there is no prediction of how large the Reynolds number must be before these new neutral curves appear, so to the authors knowledge these new neutral curves have not been observed numerically from the Orr-Sommerfeld equation before. A numerical search of the parameter space (α, Re, u_w) undertaken here has found one of these further neutral curves and here we will briefly document the successive growth, development and close up of this neutral curve as can be seen in figure 2.4.

We set $\beta = 0$, and increase u_w from zero. When $u_w = 0$ we have the usual linear stability properties for plane Poiseuille flow as seen in figure 2.1, namely a single region of instability extending for $Re \approx 5772$ to infinity. Increasing u_w from zero, a region of

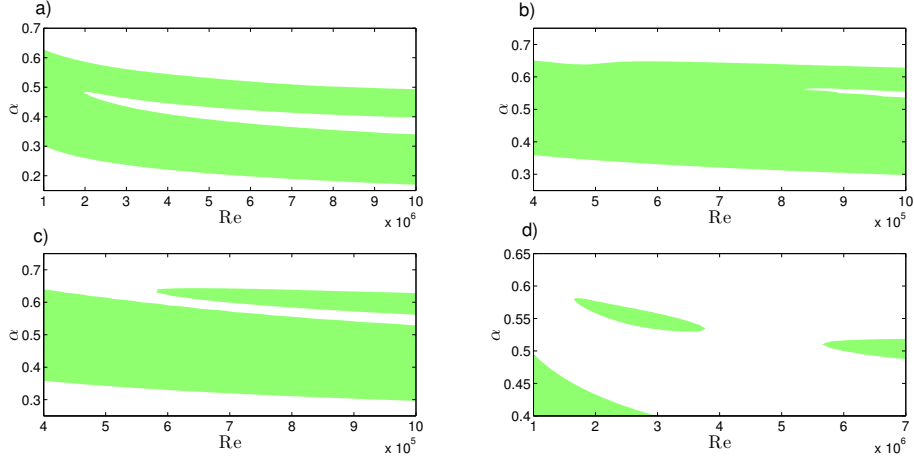


Figure 2.4. a) Birth ($u_w \approx 0.017$), b) - c) Splitting main neutral curve ($u_w \approx 0.0185 - 0.019$), d) Break off, closing and receding of surviving branch ($u_w \approx 0.025$).

linear stability enclosed within the main neutral curve was first observed at $u_w = 0.017$, the region extending from a Reynolds number of around 2,000,000 to infinity - see figure 2.4a. Figures 2.4b, c show as the wall sliding velocity increases to around 0.0185-0.019, a pinching off from the main neutral curve is observed at a Reynolds number of around 600,000. In figure 2.4c, the wall sliding velocity is such that there are now two disjoint regions of linear instability extending from some finite Reynolds numbers to infinity. Increasing the wall speed further to around 0.025, the upper instability region splits into two with one part closing up at a Reynolds number of around 2,000,000 and the other part receding to infinity as u_w is increased. This can be seen in figure 2.4d which focuses on the splitting of the upper region of instability. Increasing u_w further, the remaining (lower) region of instability recedes to infinite Reynolds number as expected from the discussion earlier in this subsection.

Now that we have documented the linear stability of plane Poiseuille-Couette flow we will consider the asymptotic form of the lower branch neutral curve. Given the asymptotic structure in plane Poiseuille flow, since $u_b \sim O(\epsilon^2)$ near the wall, it is clear that the same asymptotic structure can only hold if the non-dimensionalised wall speed $u_w \sim O(\epsilon^2)$. This can be seen by considering for example the combination $u_t + uu_x$ which near the lower wall is approximately given by $i\alpha(\epsilon^2\lambda Y - u_w - \epsilon^2 c)u$, from which we see that the critical size of u_w is $O(\epsilon^2)$ to retain the same structure. We define γ to measure the relative size of the wall sliding velocity on this critical scale by setting $u_w = \epsilon^2\gamma$, and recall that we assumed without loss of generality that the velocities of the

upper/lower walls are $\pm u_w = \pm \epsilon^2 \gamma$. The analysis follows analogously to that in plane Poiseuille flow, except in the lower wall $c \rightarrow c + \gamma$ and in the upper wall $c \rightarrow c - \gamma$, and taking this into account we find the dispersion relation

$$\mathcal{D}_\gamma(\alpha, c, \beta) = \lambda_0^{5/3} [\mathcal{G}(\xi_+) + \mathcal{G}(\xi_-)] + \alpha^{1/3} (\beta^2 + \alpha^2) J_0 = 0, \quad (2.27)$$

where we now have defined,

$$\xi_\pm = \frac{(i\alpha\lambda_0)^{1/3}}{\lambda_0} (\pm\gamma - c), \quad \mathcal{G}(\xi) = i^{5/3} \frac{Ai'(\xi)}{\kappa(\xi)}, \quad \kappa(\xi) = \int_\xi^\infty Ai(s) ds. \quad (2.28)$$

The quantities λ_0 and J_0 are unchanged from the values in plane Poiseuille flow, namely $\lambda_0 = 2$ and $J_0 = 16/15$, since the wall sliding is small so changes in these quantities occur at higher orders. If we set $\gamma = 0$ so the walls are stationary and the flow reduces to plane Poiseuille flow, then this agrees with the dispersion relation for plane Poiseuille flow (2.23) as it should.

We now investigate the solutions to the dispersion relation $\mathcal{D}_\gamma$ for different wall speeds. We know from our previous work that for $\gamma = 0$ which represents plane Poiseuille flow, there is one solution to the dispersion relation $\mathcal{D}_0 = 0$ which corresponds to the lower branch neutral curve. However for non-zero γ there is the possibility of more solutions. To visualise the number of solutions we plot the null curves of $\mathcal{D}_\gamma$, i.e. in (α, c) space we plot the curves $\text{real}(\mathcal{D}_\gamma) = 0$ and $\text{imag}(\mathcal{D}_\gamma) = 0$. Figure 2.5 shows a visualisation of the roots of the dispersion relation as the wall speed varies. There appears to be only one solution for $\gamma = 1$, and then around $\gamma = 2$ two new solutions arise. Note that this only predicts neutral modes of the lower branch type which exist as $Re \rightarrow \infty$, so for example we could not pick up the existence of an isola located at finite Reynolds numbers. As the wall speed increases through 3, 5 and 10 the number of solutions increases. For example for $\gamma = 2$ we see three solutions in this interval, whereas at $\gamma = 10$ there are five solutions, four within this window and another solution around $(\alpha, c) = (6.37, 11.70)$. Figure 2.6 shows the linear neutral values of α and c against the wall speed γ . We see two bifurcations at $\gamma \approx 2$ and $\gamma \approx 6.5$. Solutions 1, 2 and 4 appear to continue to arbitrarily large values of γ , and figure 2.7 shows a comparison of the wavespeed for these solutions with the large wall speed asymptotes computed by Cowley & Smith (1985). Those authors also predicted two other linear neutral points would exist for large wall speeds and it is possible that these are solutions 3 and 5 although for these roots at least one of $\xi_\pm$ becomes unbounded and this caused difficulty tracking these roots to larger wall speeds.

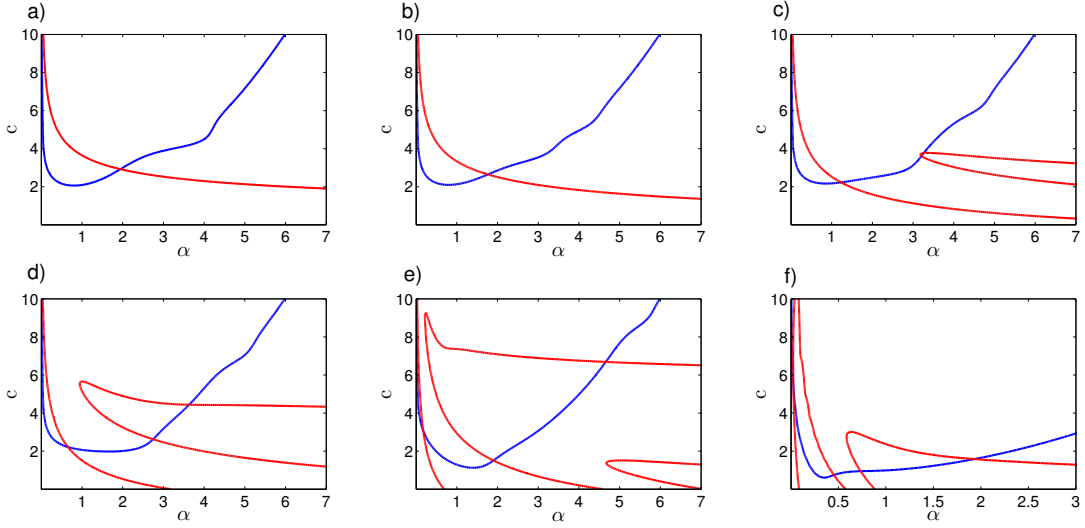


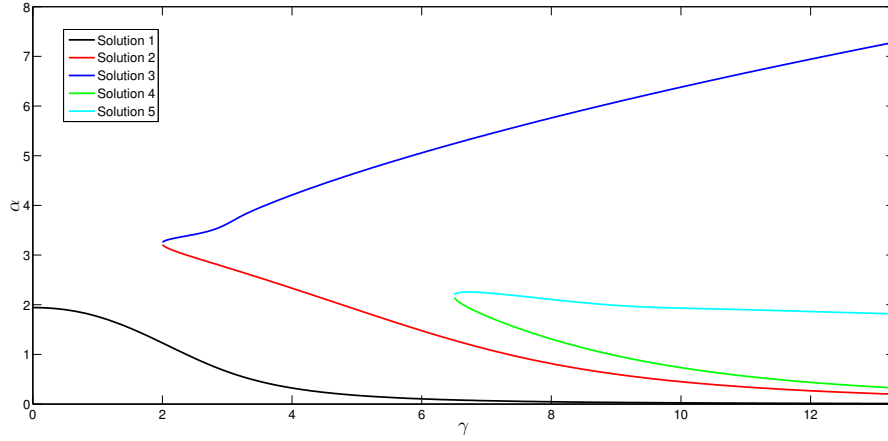
Figure 2.5. Visualisation of solutions to the dispersion relation as the wall speed varies. Zero contours of $\text{real}(\mathcal{D}_\gamma) = 0$ (blue) and $\text{imag}(\mathcal{D}_\gamma) = 0$ (red) against (α, c) . a) $\gamma = 0$, b) $\gamma = 1$, c) $\gamma = 2$, d) $\gamma = 3$, e) $\gamma = 5$, f) $\gamma = 10$.

2.5. Derivation of the nonlinear interaction equations in plane Poiseuille flow

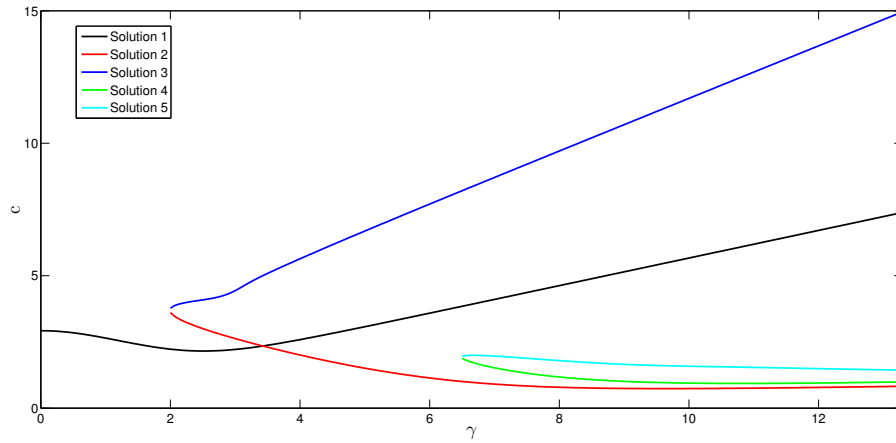
2.5.1. The vortex-wave interaction equations in plane Poiseuille flow

In this section we will formulate the equations governing the nonlinear interaction between a vortex flow and a finite amplitude three dimensional lower branch TS wave at large Reynolds number. The governing equations discussed in this section were first set out by Bennett, Hall & Smith (1991) in the context of curved channel flows. Whereas those authors only provided a study of these equations for large wall curvatures, we will consider these equations in the limit of zero curvature (plane Poiseuille flow), and later we will consider them for various curvatures. When we add a small wall sliding velocity these equations must be modified and this will be discussed later. For completeness we will motivate and rederive the governing equations here.

In the large Reynolds number framework for the three dimensional lower branch linear instability of plane Poiseuille flow (Smith 1979), the disturbance is characterized by the streamwise, spanwise and temporal scales derived in section 2.3 and given in equation (2.12). In section 2.3 we assumed that the wave was of size δ which was sufficiently



(a) (γ, α) : Plot of streamwise wavenumber against the wall sliding velocity.



(b) (γ, c) : Plot of the wave speed against the wall sliding velocity.

Figure 2.6. Evolution of the streamwise wavenumber and wave speed as the wall sliding velocity increases.

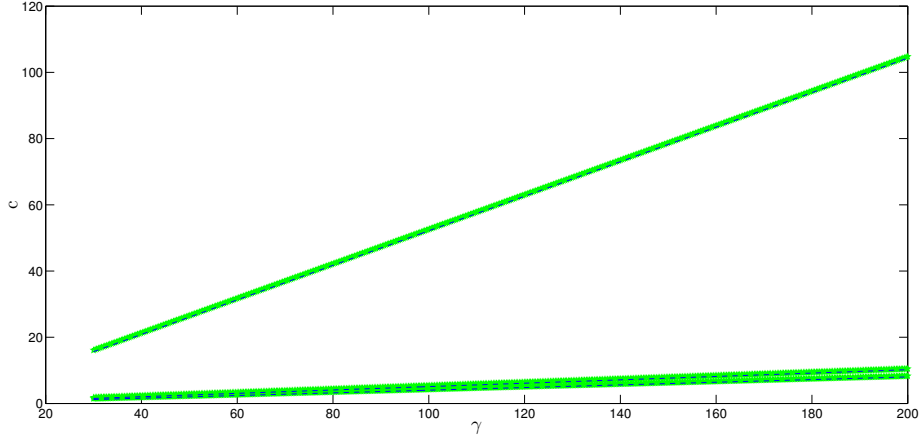


Figure 2.7. (γ, c) : Comparison of linear neutral wavespeed for solutions 1,2 and 4 (green stars) with asymptotic prediction (blue dashed line).

small that all nonlinear effects could be ignored. In this section we will increase δ to the critical size where nonlinear effects, through the self-interaction of the wave, drive the vortex flow at leading order. The underlying vortex flow is now spanwise dependent and hence a normal mode approach in Z (as in section 2.3) is no longer appropriate. We therefore suppose that, for example, the streamwise component of the wave perturbation has the form

$$\delta \hat{u}(y, Z)E + c.c., \quad E \equiv \exp[i\alpha(X - cT)], \quad (2.29)$$

where δ is the wave amplitude which is to be determined, X, Z and T are the fast wave scales defined in (2.12) and (x, y, z, t) are the streamwise, normal, spanwise and temporal coordinates. We will concentrate on nonlinear travelling wave disturbances for which the scaled streamwise wavenumber α and phasespeed c are real $O(1)$ amplitude-dependent quantities which will be determined as part of the solution.

We will now identify the critical size of the wave δ in terms of the Reynolds number Re such that the nonlinear self-interaction of the wave induces an $O(1)$ perturbation to the vortex flow, in particular to the streamwise component of the vortex flow which is known as the streak. Firstly, from an inertial-viscous balance of terms in the streamwise momentum equation $vu_y \sim Re^{-1}u_{yy}$ it follows that $v \sim Re^{-1} = \epsilon^7$ in the core of the channel. Assuming a three-dimensional mean flow, the continuity balance $v_y \sim w_z$ implies a spanwise core velocity $w \sim Re^{1/7}Re^{-1} = \epsilon^6$ in view of the fast spanwise scaling assumed above. A pressure gradient-viscous balance in the spanwise momentum equation $p_z \sim Re^{-1}w_{yy}$ then yields a core pressure scaling $p \sim Re^{1/7}Re^{-1}Re^{-6/7} = \epsilon^{12}$.

We now suppose that this roll-streak flow is forced by the nonlinear self-interaction of the wave. Consideration of the wave forcing terms in the normal momentum equation $p_y \sim [v_w v_{wy}]$ leads to the balance $p \sim \epsilon^2 \delta^2$, in view of (2.29). Implicit here is the assumption that $u \sim O(1)$ since, as we will see, the size of the wave in the core of the channel is proportional to the size of u - see equation (2.31). Thus if u were not $O(1)$ then the nonlinear self-interaction of the wave, which is of size $\epsilon^2 \delta^2 [v_w]^2 = \epsilon^2 \delta^2 [u]^2$, would not force the roll flow. The fact that u is $O(1)$ is also important in the consideration of the wave component in the wall layers where we assume $\lambda(Z)$ is $O(1)$. Comparing this estimate of the core pressure with the earlier estimate for the core pressure, $p \sim \epsilon^{12}$ leads to

$$\delta = Re^{-5/7} = \epsilon^5$$

as the critical amplitude for this strongly nonlinear interaction.

The full core flow expansion can now be expressed as

$$u = U(y, Z) + \epsilon^5 [\hat{u}(y, Z)E + c.c.] + \dots, \quad (2.30a)$$

$$v = \epsilon^6 [\hat{v}(y, Z)E + c.c.] + \epsilon^7 V(y, Z) + \dots, \quad (2.30b)$$

$$w = \epsilon^6 W(y, Z) + \epsilon^7 [\hat{w}(y, Z)E + c.c.] + \dots, \quad (2.30c)$$

$$p = -\epsilon^7 2x + \epsilon^7 [\hat{p}(y, Z)E + c.c.] + \epsilon^{12} P(y, Z) + \dots. \quad (2.30d)$$

where we recall that we have defined $\epsilon = Re^{-1/7}$. Substituting these expansions into the Navier-Stokes equations we find, following a similar process to the linear stability calculation of section 2.3, that the wave component is driven by the vortex flow. In particular the wave is driven by the instability of the streamwise component of the vortex flow, which is known as the streak. The wave components are found to satisfy a set of linear equations which can be solved up to an amplitude function $A(Z)$, which is to be determined, as

$$\hat{u} = -AU_y, \quad \hat{v} = i\alpha AU, \quad \hat{w} = -(i\alpha U)^{-1} \hat{p}_Z, \quad \hat{p}_y = \alpha^2 AU^2. \quad (2.31)$$

Integrating $\hat{p}_y$ across the channel, this leads to a pressure-amplitude (displacement) relation

$$\hat{p}(1, Z) - \hat{p}(-1, Z) = \alpha^2 A \int_{-1}^1 [U(s, Z)]^2 ds, \quad (2.32)$$

which is the nonlinear analogue of (2.22). Next we turn to the equations governing the vortex flow (U, V, W) . In SSP terminology the (V, W) part is referred to as the roll, while the U component represents a streamwise streak. The wall normal momentum equation

reduces to a simple balance between the normal pressure gradient and the nonlinear wave forcing,

$$[i\alpha\hat{v}\hat{u}^* + \hat{v}\hat{v}_y^* + c.c.] = -P_y, \quad (2.33)$$

which upon using (2.31) we find

$$P_y = -4\alpha^2|A|^2UU_y, \quad \implies \quad P = -2\alpha^2|A|^2U^2 + P_0(Z), \quad (2.34)$$

where $P_0(Z)$ is an arbitrary function of the spanwise coordinate. The spanwise momentum equation is a balance between inertial, viscous, pressure and wave forcing terms,

$$VW_y + WW_Z + [i\alpha\hat{w}\hat{u}^* + \hat{v}\hat{w}_y^* + c.c.] = -P_Z + W_{yy}. \quad (2.35)$$

Using (2.31) the wave forcing term in the spanwise momentum equation can be written in the form

$$[i\alpha\hat{w}\hat{u}^* + \hat{v}\hat{w}_y^* + c.c.] = \alpha^2(|A|^2)_Z U^2 + 4\alpha^2|A|^2UU_Z. \quad (2.36)$$

We can eliminate the roll pressure gradient P in the spanwise momentum equation by cross differentiation, so applying $\partial_y(2.35) - \partial_Z(2.33)$, and using the continuity equation $V_y + W_Z = 0$, we find

$$VW_{yy} + WW_{yZ} = 2\alpha^2(|A|^2)_Z UU_y + W_{yyy}. \quad (2.37)$$

With these manipulations we are left with the governing equations for the interaction in the core of the channel,

$$V_y + W_Z = 0, \quad (2.38a)$$

$$VU_y + WU_Z = 2 + U_{yy}, \quad (2.38b)$$

$$VW_{yy} + WW_{yZ} = 2\alpha^2(|A|^2)_Z UU_y + W_{yyy}, \quad (2.38c)$$

where the wave forcing term in (2.38c) is written in terms of A using (2.31). These equations are to be solved subject to no slip boundary conditions on the walls

$$U = V = W = 0, \quad y = \pm 1. \quad (2.38d)$$

These equations are equations (4.3) in Bennett, Hall & Smtih (1991) after taking into account the change in notation for the roll flow (factor of 1/2) and setting the curvature to zero.

To close the system given by equations (2.38) we need to determine the wave amplitude function $A(Z)$ which is found by considering the behaviour of the wave in the viscous wall layers. The dynamics of the wall layers are very similar, and hence it suffices to consider only the lower wall layer at $y = -1$ from which we can easily derive the analogous results for the upper wall layer. Recall the viscous wall layers have thickness $O(\epsilon^2)$, and we defined the lower wall layer variable Y accordingly in equation (2.12). In the wall layer, the relevant size of the wave was determined in section 2.3, and the size of the wall layer variables follows from a combination of matching with the core flow and the dynamics in the wall layer. Through this analysis it is found that the flow variables in the lower wall layer expand as

$$u = \epsilon^2 \lambda_-(Z) Y - \epsilon^4 Y^2 + \epsilon^5 [\bar{u}_-(Y, Z) E + c.c.] + \epsilon^8 \bar{U}_-(Y, Z) + \dots, \quad (2.39a)$$

$$v = \epsilon^8 [\bar{v}_-(Y, Z) E + c.c.] + \epsilon^{11} \bar{V}_-(Y, Z) + \dots, \quad (2.39b)$$

$$w = \epsilon^5 [\bar{w}_-(Y, Z) E + c.c.] + \epsilon^8 \bar{W}_-(Y, Z) + \dots, \quad (2.39c)$$

$$p = -\epsilon^7 2x + \epsilon^7 [\bar{p}_-(Y, Z) E + c.c.] + \epsilon^{12} \bar{P}_-(Y, Z) + \dots, \quad (2.39d)$$

where $\lambda_-(Z)$ is the streak shear stress on the lower wall, defined by

$$\lambda_-(Z) = \left. \frac{\partial U}{\partial y} \right|_{y=-1}. \quad (2.40)$$

The wave components satisfy linearised unsteady boundary-layer-type equations similar to those found in section 2.3

$$i\alpha \bar{u}_- + \bar{v}_- Y + \bar{w}_- Z = 0, \quad (2.41a)$$

$$i\alpha(\lambda_- Y - c) \bar{u}_- + \lambda_- \bar{v}_- + \lambda_- Z Y \bar{w}_- = -i\alpha \bar{p}_- + \bar{u}_- Y Y, \quad (2.41b)$$

$$i\alpha(\lambda_- Y - c) \bar{w}_- = -\bar{p}_- Z + \bar{w}_- Y Y, \quad (2.41c)$$

where $\bar{p}_- = \bar{p}_-(Z) = \hat{p}(1, Z)$, which are driven by the streak shear and must satisfy no slip boundary conditions on the wall

$$\bar{u}_- = \bar{v}_- = \bar{w}_- = 0, \quad Y = 0, \quad (2.41d)$$

and far field matching to the core flow solution (2.31)

$$\bar{u}_- \rightarrow -\lambda_- A, \quad \bar{w}_- \sim -\frac{\bar{p}_- Z}{i\alpha \lambda_- Y}, \quad Y \rightarrow \infty. \quad (2.41e)$$

Equations (2.41) can be manipulated into a single equation for $\bar{p}_-$ following the analysis of section 2.3 or Smith (1979), namely

$$\bar{p}_-'' - \frac{\lambda_-'}{\lambda_-} \mathcal{F}(\xi_-) \bar{p}_-' - \alpha^2 \bar{p}_- = -(\alpha \lambda_-)^{5/3} \mathcal{G}(\xi_-) A, \quad (2.42)$$

where ξ_- is defined by

$$\xi_-(Z) = -\frac{(i\alpha\lambda_-)^{1/3}}{\lambda_-} c, \quad (2.43)$$

and we have also defined

$$\begin{aligned} \mathcal{F}(\xi) &= \frac{3}{2} + \frac{\xi(\xi\kappa(\xi) + Ai'(\xi))}{2Ai(\xi)}, & \mathcal{G}(\xi) &= i^{5/3} \frac{Ai'(\xi)}{\kappa(\xi)}, \\ \kappa(\xi) &= \int_{\xi}^{\infty} Ai(s) ds, & J(Z) &= \int_{-1}^1 [U(y, Z)]^2 dy, \end{aligned} \quad (2.44)$$

with $\lambda_-(Z)$ given in (2.40) and Ai being the usual Airy function. For the analogous problem in the upper wall layer if we introduce the wall variable

$$Y_+ = \epsilon^{-2}(1 - y), \quad (2.45)$$

define (for convenience) λ_+ to be the negative of the wall normal derivative on the upper wall

$$\lambda_+ = -\frac{\partial U}{\partial y} \Big|_{y=1}, \quad (2.46)$$

and negate the wall normal velocity then the analysis follows in a very similar manner. We find the near wall expansions,

$$u = \epsilon^2 \lambda_+(Z) Y_+ - \epsilon^4 Y_+^2 + \epsilon^5 [\bar{u}_+(Y_+, Z) E + c.c.] + \epsilon^8 \bar{U}_+(Y_+, Z) + \dots, \quad (2.47a)$$

$$v = -\epsilon^8 [\bar{v}_+(Y_+, Z) E + c.c.] - \epsilon^{11} \bar{V}_+(Y_+, Z) + \dots, \quad (2.47b)$$

$$w = \epsilon^5 [\bar{w}_+(Y_+, Z) E + c.c.] + \epsilon^8 \bar{W}_+(Y_+, Z) + \dots, \quad (2.47c)$$

$$p = -\epsilon^7 2x + \epsilon^7 [\bar{p}_+(Y_+, Z) E + c.c.] + \epsilon^{12} \bar{P}_+(Y_+, Z) + \dots. \quad (2.47d)$$

Then the wave components $(\bar{u}_+, \bar{v}_+, \bar{w}_+, \bar{p}_+)$ are found to satisfy equations (2.41a-c) with no slip boundary conditions (2.41d) on $Y_+ = 0$. However the far field matching condition is

$$\bar{u}_+ \rightarrow \lambda_+ A, \quad \bar{w}_+ \sim -\frac{\bar{p}_+ Z}{i\alpha \lambda_+ Y_+}, \quad Y_+ \rightarrow \infty. \quad (2.48)$$

Similarly to equation (2.42), the equations governing the wave components in the upper

wall layer can be reduced to a single equation for $\bar{p}_+$ and A ,

$$\bar{p}_+'' - \frac{\lambda_+'}{\lambda_+} \mathcal{F}(\xi_+) \bar{p}_+' - \alpha^2 \bar{p}_+ = (\alpha \lambda_+)^{5/3} \mathcal{G}(\xi_+) A, \quad (2.49)$$

with ξ_+ defined by

$$\xi_+ = -\frac{(\mathrm{i}\alpha\lambda_+)^{1/3}}{\lambda_+} c, \quad (2.50)$$

and other quantities are defined in (2.44).

Equations (2.32, 2.42, 2.49) are a complex nonlinear eigenvalue problem that determine the streamwise wavenumber and wavespeed. These equations, see especially the governing equations (2.42, 2.49) for $\bar{p}_\pm$, are equivalent to equations (4.10, 4.12) from Bennett, Hall & Smith (1991) once A has been eliminated.

The interaction problem has now been reduced to solving the vortex equations (2.38), coupled with the wave equations (2.32, 2.42, 2.49), with the two systems of equations being linked by the amplitude function $A(Z)$, the streak shear stresses $\lambda_\pm(Z)$ and a measure of the kinetic energy density $J(Z)$. As documented throughout their derivation, these equations can be found in Bennett, Hall & Smith (1991), equations (4.3, 4.10, 4.12).

This coupling allows us to describe the self-sustaining mechanism behind the vortex-wave interaction states. Firstly, the streak U drives the wave A in equations (2.32, 2.42, 2.49) through the wall shear stresses $\lambda_\pm(Z)$. Secondly, the wave drives the roll (V, W) in equations (2.38a) and (2.38c) through the nonlinear self-interaction of the wave. Finally, closing the loop, the roll drives the streak through the advection of mean shear in equation (2.38b). This provides a framework in which equilibrium states can be described at large Reynolds numbers.

We will restrict our attention to solutions which have period $2\pi/\beta$ in the spanwise direction, with the spanwise wavenumber β to be specified. To close the system we must also prescribe some measure of the amplitude of the wave. This could be done in multiple ways such as prescribing the flux or the streamwise wavenumber, but here we choose to prescribe the amplitude $\mathcal{A}$ of the wave, which we define as

$$\mathcal{A}^2 = \int_0^{2\pi/\beta} |A(Z)|^2 dZ. \quad (2.51)$$

The behaviour of the solution for small $\mathcal{A}$ is investigated in section 2.6. To give a physical sense to the size of the states we also introduce the average flux deficit of the

perturbation, defined to be

$$\mathcal{Q} = -\frac{\beta}{4\pi} \int_0^{2\pi/\beta} \int_{-1}^1 \tilde{u}(y, Z) dy dZ, \quad (2.52)$$

where $\tilde{u}$ is the perturbation to the basic state streak, and is defined by

$$\tilde{u}(y, Z) = U(y, Z) - (1 - y^2). \quad (2.53)$$

In the current formulation, setting $\mathcal{A} = 0$ (and hence $A(Z) \equiv 0$), we see that there is more than one possible basic state due to the lack of an imposed spanwise pressure gradient in (2.38c). We find a solution of the form $U = u_b = (1 - y^2)$, $V = 0$ and $W = w_b(y)$ where $w_b'''(y) = 0$ with $w_b(\pm 1) = 0$. This has solution $w_b(y) = k(1 - y^2)$ where $k \in \mathbb{R}$ is an arbitrary constant. Thus to uniquely define the basic state, where we require $k = 0$, we also fix the spanwise flux by imposing

$$\int_{-1}^1 W dy = 0. \quad (2.54)$$

When we solve the interaction equations numerically in section 2.7.1, it turns out that this condition can be written neatly in the streamfunction formulation of (V, W) .

If we restrict our attention to modes that are symmetric about $y = 0$ this simplifies the governing equations. Under this symmetry, we denote $\lambda_+ = \lambda_-$ by λ , and $\xi_+ = \xi_-$ by ξ , i.e.

$$\lambda = \frac{\partial U}{\partial y} \Big|_{y=-1}, \quad \xi = -\frac{(i\alpha\lambda)^{1/3}}{\lambda} c. \quad (2.55)$$

In particular we find $\hat{p}(1, Z) = -\hat{p}(-1, Z)$ and thus $\bar{p}_+ = -\bar{p}_-$. Thus using (2.32) we can combine (2.42, 2.49) to find a single equation for A ,

$$(JA)'' - \frac{\lambda'}{\lambda} \mathcal{F}(\xi)(JA)' - \alpha^2(JA) = 2 \frac{(\alpha\lambda)^{5/3}}{\alpha^2} \mathcal{G}(\xi)A. \quad (2.56)$$

In fact all the states which were found in plane Poiseuille flow have this symmetry. However, later when we discuss plane Poiseuille-Couette flow and flow in curved channels this symmetry is obviously broken for all solutions.

2.5.2. Near wall equations

The previous section fully describes the interaction to leading order, determining the wavenumber and wavespeed, and the flow variables throughout the majority of the channel. Since the wall layer is of size $O(\epsilon^2) = O(Re^{-2/7})$ it is likely that any wall layer structure, even at quite large Reynolds numbers, will be observable. Consider the governing equations for the vortex flow near the wall which are found to be

$$\bar{V}_Y + \bar{W}_Z = 0, \quad (2.57a)$$

$$\lambda \bar{V} + \lambda' Y \bar{W} + [\bar{v} \bar{u}_Y^* + \bar{w} \bar{u}_Z^* + c.c.] = \bar{U}_{YY} + \lambda'' Y, \quad (2.57b)$$

$$[i\alpha \bar{u}^* \bar{w} + \bar{v} \bar{w}_Y^* + \bar{w} \bar{w}_Z^* + c.c.] = \bar{W}_{YY}, \quad (2.57c)$$

subject to no slip boundary conditions on the wall,

$$\bar{U} = \bar{V} = \bar{W} = 0, \quad Y = 0, \quad (2.57d)$$

and matching conditions with the core flow as $Y \rightarrow \infty$ which can be found as follows. Taylor expanding the spanwise component of the roll in the core flow near the lower wall, $W = W_y(-1, Z)(y + 1) + O[(y + 1)^2]$. This gives the dominant contribution to the large Y behaviour of $\bar{W}$; however it turns out that although the next order term is subdominant, it is not small. The method which we will use to solve these equations involves truncating the domain at a large but finite value of Y , and thus it is important to capture the large Y behaviour of the solution. As we will see shortly, the wave forcing terms drive a logarithmic growth in $\bar{W}$ as $Y \rightarrow \infty$, i.e. a growth of the form $B(Z) \ln Y$, and we will determine $B(Z)$ in the following analysis. Writing the matching condition for $\bar{W}$ in terms of near wall coordinates we find

$$\bar{W} \sim W_y(-1, Z)Y + B(Z) \ln Y, \quad Y \rightarrow \infty. \quad (2.57e)$$

For the streamwise velocity, it is sufficient to provide the matching condition for the streamwise velocity,

$$\bar{U} \sim \frac{1}{24} U_{yyyy}(-1, Z)Y^4, \quad Y \rightarrow \infty. \quad (2.57f)$$

Once the core flow has been computed from the interaction equations (2.38) and (2.32, 2.42, 2.49), we can solve (2.57) to determine the flow within the wall layers. We will return to these equations in section 2.8.2.

Before we can solve for the vortex flow in the wall layer, we need to compute the wave

components which provide a forcing on the near wall vortex problem. It is possible to make some progress analytically but even then this involves the use of special functions. Due to the similarities between the lower wall and upper wall equations, we consider only the lower wall problem and drop the $\pm$ subscripts. To recap, the governing equations for the lower wall wave components (2.41) are

$$i\alpha\bar{u} + \bar{v}_Y + \bar{w}_Z = 0, \quad (2.58a)$$

$$i\alpha(\lambda Y - c)\bar{u} + \lambda\bar{v} + \lambda'Y\bar{w} = -i\alpha\bar{p} + \bar{u}_{YY}, \quad (2.58b)$$

$$0 = -\bar{p}_{\bar{Y}}, \quad (2.58c)$$

$$i\alpha(\lambda Y - c)\bar{w} = -\bar{p}_Z + \bar{w}_{YY}, \quad (2.58d)$$

with no slip boundary conditions,

$$\bar{u} = \bar{v} = \bar{w} = 0, \quad Y = 0, \quad (2.58e)$$

and far field matching conditions determined from the core wave solutions in (2.31)

$$\bar{u} \rightarrow -\lambda A, \quad \bar{w} \sim -\frac{\bar{p}_Z}{i\alpha\lambda Y}, \quad Y \rightarrow \infty. \quad (2.58f)$$

Where solutions for $(\bar{u}, \bar{v}, \bar{w})$ are required for (2.57) we will solve (2.58) computationally by truncating the domain in the wall normal direction, and applying a Chebyshev/Fourier decomposition of the solution.

To solve for the vortex components $(\bar{U}, \bar{V}, \bar{W})$ in the wall layer, we must first justify the claim of logarithmic growth made earlier in this section so we can determine the correct far field boundary conditions for (2.57). With this in mind, let us consider the large Y behaviour of the wave components in the wall layer. Equation (2.58d) and the far field conditions yields $\bar{w} \sim -(\bar{p}_Z/i\alpha\lambda)Y^{-1} + O(Y^{-2})$. Then from the far field matching condition from the streamwise wave component $\bar{u} \sim -\lambda A + O(Y^{-1})$, and from the continuity equation we find $\bar{v} \sim i\alpha\lambda AY + O(1)$. It turns out that the next order terms are needed to determine the logarithmic growth term, and a little bit more work yields,

$$\bar{u} \sim -\lambda A - \left(\frac{\bar{p}_Z}{\alpha^2\lambda}\right)_Z \frac{1}{Y}, \quad (2.59a)$$

$$\bar{v} \sim i\alpha\lambda AY + \frac{1}{i\alpha\lambda}(\bar{p}_{ZZ} + \alpha^2\bar{p}) - i\alpha cA, \quad (2.59b)$$

$$\bar{w} \sim -\frac{\bar{p}_Z}{i\alpha\lambda} \frac{1}{Y} - \frac{c\bar{p}_Z}{i\alpha\lambda^2} \frac{1}{Y^2}. \quad (2.59c)$$

Plugging this into equation (2.57c) we find that as $Y \rightarrow \infty$,

$$\bar{W}_{YY} \sim \frac{1}{\alpha^2 \lambda^2 Y^2} \frac{d}{dZ} \left(\left| \frac{d\bar{p}}{dZ} \right|^2 + \alpha^2 |\bar{p}|^2 \right), \quad (2.60)$$

and thus combining with the matching condition with the core,

$$\bar{W} \sim W_y(-1, Z)Y - \frac{1}{\alpha^2 \lambda^2} \frac{d}{dZ} \left(\left| \frac{d\bar{p}}{dZ} \right|^2 + \alpha^2 |\bar{p}|^2 \right) \ln Y, \quad (2.61)$$

as $Y \rightarrow \infty$. The wall layer problem is now well defined and it can be solved numerically using the method described in section 2.7.2.

2.6. Preliminary calculations in plane Poiseuille flow

2.6.1. Weakly nonlinear analysis

In general the nonlinear nature of the interaction equations makes any analytical progress impossible. However, when the amplitude of the wave is small the interaction is in the so called weakly nonlinear regime, and we can take advantage of this to construct solutions where the perturbation to the basic flow is small, and the streamwise wavenumber and wavespeed are close to the linear neutral values for the undisturbed flow. This allows us to find analytic expressions which describe how the states bifurcate from the linear neutral point into finite amplitude states and also provides a means of validating numerical results as we will see in section 2.7.3.

We consider small amplitude solutions, $|A| \sim \Delta \ll 1$, where α, c are close to their linear neutral values and Δ is an arbitrarily small parameter. The forcing in the core is $O(\Delta^2)$ from equation (2.38c), so the perturbations from plane Poiseuille flow in (2.38) are also of order $O(\Delta^2)$. Therefore we expand

$$A(Z) = \Delta \cos \beta Z + \Delta^3 (A_3^1 \cos \beta Z + A_3^3 \cos 3\beta Z) + \dots, \quad (2.62)$$

and substituting this into equations (2.38a-2.38c) we find the form of the resulting perturbed vortex flow, so for example the streak expands as

$$U(y, Z) = 1 - y^2 + \Delta^2 U_2(y) \cos 2\beta Z + \dots, \quad (2.63)$$

where U_2 is a polynomial in y that can be determined analytically. Thus the perturbations to the wall shear stress $\lambda(Z)$ and $J(Z)$ are also $O(\Delta^2)$ so from the equation governing the wave (2.56), we see that α, c are $O(\Delta^2)$ from their linear neutral values, i.e. they expand as

$$\alpha = \alpha_0 + \Delta^2 \alpha_2 + \dots, \quad c = c_0 + \Delta^2 c_2 + \dots, \quad (2.64)$$

where (α_0, c_0) corresponds to the linear neutral point which satisfies the linear dispersion relation

$$\mathcal{D}(\alpha_0, c_0, \beta) = 15(2\alpha_0)^{5/3} \mathcal{G}(\xi_0) + 8\alpha_0^2(\alpha_0^2 + \beta^2) = 0, \quad (2.65)$$

from substitution of (2.62, 2.63) into (2.56), where $\xi_0 = -(2i\alpha_0)^{1/3}c_0/2$ is the leading order value of the quantity ξ defined in (2.44). Taking the imaginary part of (2.65), we find that the imaginary part of $\mathcal{G}(\xi_0)$ must be zero and thus ξ_0 and $\mathcal{G}(\xi_0)$ are determined independently of the spanwise wavenumber. Letting $\xi_0 = -i^{1/3}s_0$, i.e. $s_0 = (2\alpha_0)^{1/3}c_0/2$, we find

$$s_0 = 2.297, \quad \text{and} \quad \mathcal{G}(\xi_0) = -1.001, \quad (2.66)$$

with each given to three decimal places - see Lin (1955), Reid (1965) and Stuart (1963). Note that $\mathcal{G}(\xi_0)$ is purely real as required. For example, when $\beta = 1$, we find

$$(\alpha_0, c_0) = (1.943, 2.923), \quad (2.67)$$

to three decimal places.

One of the most important consequences of this analysis is that it describes the nature of the bifurcation from the linear neutral point. The leading order analysis at $O(\Delta)$ gave the dispersion relation $\mathcal{D}$, then at the next order $O(\Delta^3)$ we find, by equating terms proportional to $\cos \beta Z$, the solvability condition which relates (α_2, c_2) , namely

$$\nu_1 \alpha_2 + \nu_2 c_2 = 1, \quad (2.68)$$

where $\nu_{1,2}$ are complex constants which depend on β . For example, we find that for $\beta = 1$, the bifurcation for small amplitudes is described to $O(\Delta^2)$ by

$$\alpha \simeq 1.9426 - 0.0090\Delta^2, \quad c \simeq 2.9225 + 0.0057\Delta^2. \quad (2.69)$$

Numerical computations indicate that the bifurcation appears to be subcritical for all spanwise wavenumbers as in figure 2.8 (i.e. in (2.64), $\alpha_2 < 0$ for all β).

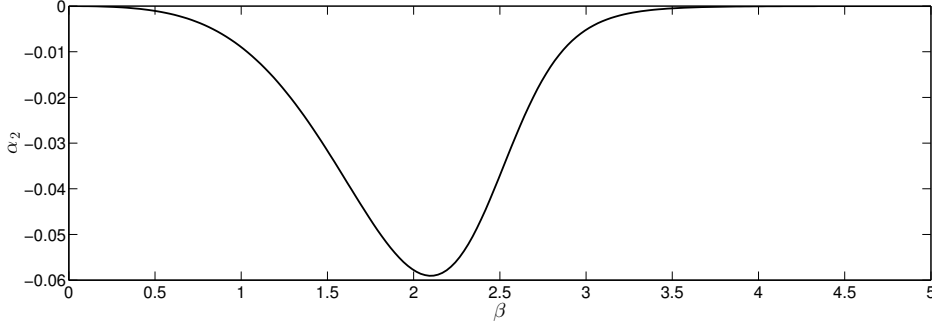


Figure 2.8. (β, α_2) plot indicating the criticality of the bifurcation as the spanwise wavenumber β varies.

2.6.2. Local analysis

A local spanwise analysis of the interaction problem (2.38, 2.32, 2.42, 2.49) can give insight into the nature of the solutions in section 2.8 and can also be used as a validity check on the fully nonlinear computations. To explain what this entails, consider the wave-forcing term in (2.38) which is of the form $\mathbb{F}(Z)UU_y$ where $\mathbb{F}$ denotes the wave contribution to this forcing term,

$$\mathbb{F}(Z) = 2\alpha^2(|A|^2)_Z. \quad (2.70)$$

Given the form of $\mathbb{F}$, we see that this function is periodic in Z (as A is assumed to be periodic) and has zero mean ($\int_0^{2\pi/\beta} \mathbb{F}(Z)dZ = 0$). Therefore it follows that there are spanwise locations which we denote by Z_c where $\mathbb{F}(Z_c) = 0$. Assuming α is non-zero, we see that given $(|A|^2)_Z = A^*A_Z + c.c.$, these zeros are located at spanwise locations where either $A = 0$ or $A_Z = 0$. We will analyse the behaviour of the flow around these points. We will find that it is not possible to completely describe the local solution due to the global nature of the eigenvalue problem given by equations (2.32, 2.42, 2.49). We can however describe the flow around these points subject to the unknown gradient of $\mathbb{F}$ around Z_c . We start by Taylor expanding $\mathbb{F}$ around $Z = Z_c$

$$\mathbb{F}(Z) = (Z - Z_c)F_1 + (Z - Z_c)^3F_3 + \dots \quad (2.71)$$

Assuming all of the flow variables are analytic around $Z = Z_c$, from the streamwise and spanwise momentum equations we find that $U, V \sim O(1)$ and $W \sim O(Z - Z_c)$. It follows

that the vortex flow must assume the local form

$$U(y, Z) = u_0(y) + (Z - Z_c)^2 u_2(y) + \dots, \quad (2.72)$$

$$V(y, Z) = v_0(y) + (Z - Z_c)^2 v_2(y) + \dots, \quad (2.73)$$

$$W(y, Z) = (Z - Z_c) w_1(y) + (Z - Z_c)^3 w_3(y) + \dots \quad (2.74)$$

Substituting these expansions into (2.38), we find a coupled set of ordinary differential equations in y ,

$$v'_0 + w_1 = 0, \quad (2.75a)$$

$$v_0 u'_0 = 2 + u''_0, \quad (2.75b)$$

$$v_0 w''_0 + w_1 w'_1 = F_1 u_0 u'_0 + w'''_1, \quad (2.75c)$$

where primes are used to denote derivatives with respect to y , d/dy . These equations are to be solved subject to no slip boundary conditions

$$u_0 = v_0 = w_1 = 0, \quad y = \pm 1. \quad (2.75d)$$

The solution was obtained using two methods: the first of which is a shooting method with a two stage explicit Runge-Kutta method, and the second is a spectral method. These methods are discussed in more detail in appendix A. The results of these computations are compared with the full numerical calculations in the next section.

2.7. Numerical method

2.7.1. Numerical method for the interaction equations

The equations to be solved are (2.38a-2.38c) subject to (2.38d), and (2.32, 2.42, 2.49)⁴ with some prescribed amplitude, (2.51). We will seek solutions which are periodic in Z with period $2\pi/\beta$ where β is the spanwise wavenumber. The equations are coupled through the wall shear stresses, $\lambda_{\pm}(Z)$ defined in (2.40, 2.46), $J(Z)$ defined in (2.44), and the wave forcing terms. Due to the nonlinear nature of the equations we will use a Newton method to search directly for equilibrium solutions. Equation (2.38a) suggests

⁴Or alternatively (2.56) if we seek only symmetric solutions.

a stream function formulation, so we introduce the stream function Ψ , where

$$(V, W) = (\partial_Z, -\partial_y)\Psi. \quad (2.76)$$

It is convenient to split the flow into the basic flow, plane Poiseuille flow, and a perturbation to this basic flow denoted by $\tilde{u}$, defined in (2.53). Thus the equations to be solved numerically are

$$\Psi_Z(u'_b + \tilde{u}_y) - \Psi_y \tilde{u}_Z = \tilde{u}_{yy}, \quad (2.77a)$$

$$\Psi_Z \Psi_{yyy} - \Psi_y \Psi_{yyZ} = -2\alpha^2(|A|^2)_Z(u_b + \tilde{u})(u'_b + \tilde{u}_y) + \Psi_{yyy}. \quad (2.77b)$$

The boundary conditions for the perturbed variables $(\tilde{u}, \Psi)$ are no slip on the walls,

$$\tilde{u} = \Psi = \Psi_y = 0, \quad y = \pm 1, \quad (2.78)$$

where no spanwise flux,

$$\int_{-1}^1 W dy = - \int_{-1}^1 \Psi_y dy = -[\Psi]_{-1}^1, \quad (2.79)$$

allows us to impose $\Psi = 0$ on both walls.

The most natural choice is to use a spectral method with a sensible choice of basis functions which take into account the geometry and the boundary conditions. With this in mind, we use a Chebyshev/Fourier basis in the wall normal/spanwise direction with T_n denoting the n th Chebyshev polynomial. The flow variables are thus represented by the series,

$$\begin{aligned} \tilde{u} &= \sum_{n,k=1} \tilde{u}_{n,k} T_{n-1}(y) \cos[2(k-1)\beta Z], \\ \Psi &= \sum_{n,k=1} \Psi_{n,k} T_{n-1}(y) \sin(2k\beta Z), \quad A = \sum_{k=1} A_k \cos[(2k-1)\beta Z]. \end{aligned} \quad (2.80)$$

These series are truncated at $(n, k) = (N, K)$, with (N, K) determined by requiring sufficient decay in the spectral coefficients $|\tilde{u}_{n,k}|$, typically to around 10^{-8} . We apply the collocation method in the wall normal direction using the collocation points $y_j = \cos j\pi/N$, and the Galerkin method in the Z direction. A pseudo spectral approach is used to deal with the nonlinear terms, and de-aliasing is performed by zero-padding.

We must solve the core equations (2.38) and the wave equation (2.56) with specified

amplitude (2.51). In the current formulation, if A is a solution, then so is $e^{i\theta}A$ for all real θ which corresponds to an arbitrary complex phase. Thus one more condition is required to close the system - we fix the complex phase of A by setting the imaginary part of A_1 to be zero. The resulting system of equations is solved using a quasi-Newton method, where we approximate the Jacobian with finite differences and iterate until the equations are solved to within a set tolerance, typically taken to be 10^{-8} .

2.7.2. Numerical method for the near wall problem

Let us now consider the lower wall layer problem where the vortex part is governed by (2.57) and the wave part is governed by (2.41). The reason for considering the full wave equations (2.41) rather than the reduced system (2.32, 2.42, 2.49) is that we would like to compute the vortex flow in the wall layer and hence we require all the wave components (rather than just the pressure). One could solve for the pressure first from which $\bar{w}$ could easily be deduced in terms of Airy functions, however more work would be required to find $\bar{u}$ and $\bar{v}$. Thus it was decided to consider the whole wave problem at once given by (2.41). The vortex problem is driven by the wave and thus we must solve the wave part first.

Both of these problems are posed on a semi-infinite domain in the wall normal direction, which must be dealt with. There are multiple ways to deal with a semi-infinite domain including mapping to a finite domain using a Möbius map or an exponential map, or cutting the domain off a large distance from the wall where one might expect the problem on this large domain to well approximate the semi-infinite domain problem.

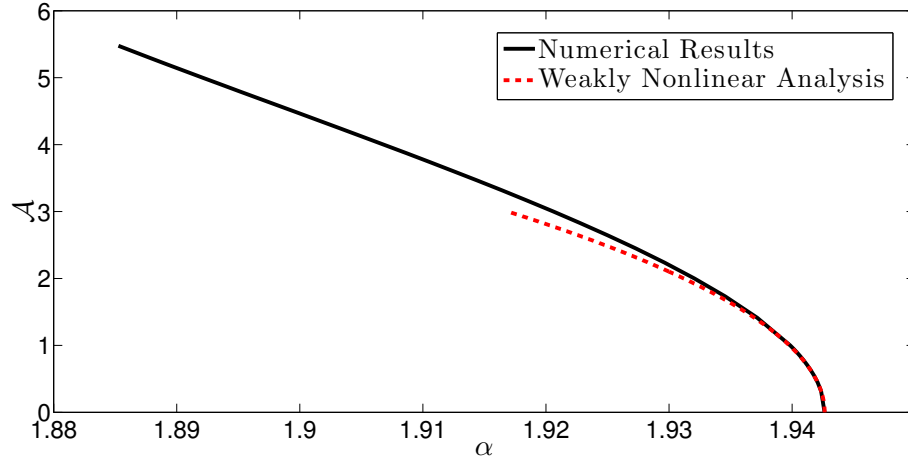
The latter is the method applied here. We truncate the domain at some large $Y = H$ and consider the problem on the domain $(Y, Z) \in [0, H] \times [0, 2\pi/\beta]$. For convenience we would like to apply an adapted version of the method which we have used to solve the interaction equations in section 2.7.1, although this is clearly not the optimal technique to use to solve these linear equations. We map the domain $[0, H]$ linearly onto $[-1, 1]$ where we can use a Chebyshev basis in the wall normal direction. As alluded to before, the idea is to increase the truncation height until the solution is independent of the choice of H . The domain size required for this computation is generally reasonably large, typically $H = 30$ or 50 , to capture the large Y behaviour and thus many more Chebyshev modes (than in the core interaction equations) are required to resolve the solution in the wall normal direction. As noted by Boyd (1982) increasing the accuracy of such a truncation method involves simultaneously increasing both the truncation height

H and the number of Chebyshev modes N . This increase in the required computational load means that using this non-optimal method and with the current computational resources, we can only compute solutions of the wall layer problem at small to moderate amplitudes when the number of Fourier modes required in the spanwise direction to accurately describe the flow is not too large ($\sim O(100)$).

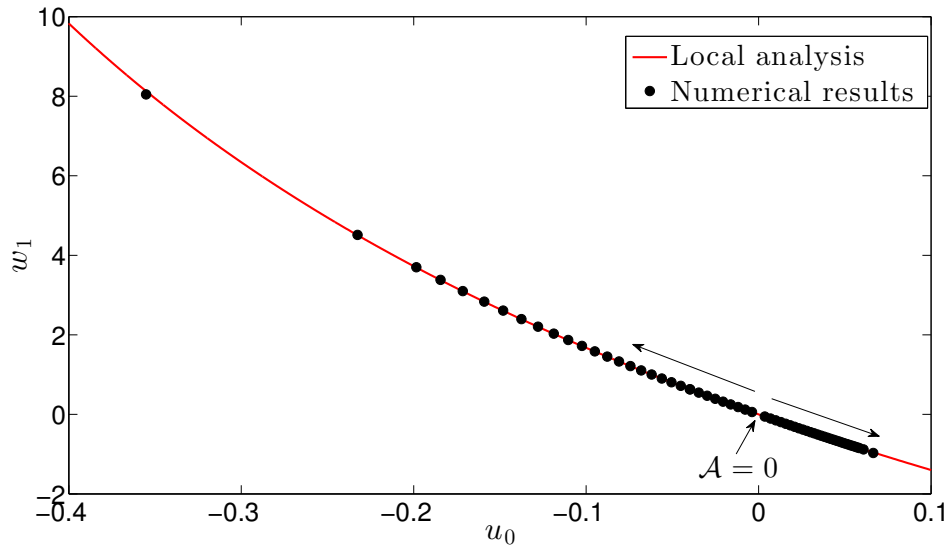
2.7.3. Validation of the numerical method used to solve the interaction equations

Figure 2.9(a) shows a plot of the streamwise wavenumber α versus the wave amplitude $\mathcal{A}$ for unit spanwise wavenumber, $\beta = 1$. We plot the weakly nonlinear solution given in (2.69) and the results of our numerical calculation using the method outlined in the section 2.7.1. The two approaches agree well when $\mathcal{A}$ is small, with similarly good agreement found when comparing other aspects of the solution at this and other values of β . In figure 2.9(b) we present a comparison between the local analysis of section 2.6.2 and the full numerical computations, again obtaining excellent agreement over a large range of amplitudes. Note that for each computation there are two associated local points where $\mathcal{F} = 0$, one where $A = 0$ and one where $A_Z = 0$. These correspond to different points in figure 2.9(b), where one corresponds to $u_0 > 0$ and the other to $u_0 < 0$. This gives us confidence that our numerical method is accurately capturing the full nonlinearity of the interaction equations.

The last validation technique is a resolution test which involves fixing the amplitude $\mathcal{A}$ and the spanwise wavenumber β and varying the numbers of both Fourier and Chebyshev modes to check that the solutions found are independent of the number of spectral modes used. We conduct this test at an amplitude where the state is in the strongly nonlinear regime, in that there is a large perturbation to the basic state. From our computational results we observe that at $\mathcal{A}^2 = 20$, the perturbation to the streak flow is approximately 11% of the basic state, measured along the centreline and thus we conclude that this amplitude is suitable for the resolution test. We denote the number of Fourier modes used in the test by K and the number of Chebyshev modes used by N : we then compare the value of α as N and K vary. For convenience we denote the value of α for $N = 40$ and $K = 110$, the highest resolution used for this test, by $\bar{\alpha}$. The results of the resolution test can be seen in table 2.1 where we record the base 10 logarithm of the weighted change to an accuracy of one decimal place. With a change in the sixth decimal place for the lowest resolution and clear convergence as the number of modes is increased we conclude that the results of the resolution test are satisfactory and the numerical method



(a) Streamwise wavenumber α against amplitude $\mathcal{A}$ for $\beta = 1$. The solid black curve represents numerical results from the VWI equations, and the dashed red curve shows the results of the weakly nonlinear analysis, which is valid for small $\mathcal{A}$.



(b) Comparison of local analysis values (u_0, w_1) at $y = 0$ (solid red curve) and the numerical results from the VWI equations (black dots). The arrows depict increasing amplitude $\mathcal{A}$.

Figure 2.9. Validation of numerical method for solving the interaction equations.

of section 2.7.1 is capturing fully resolved solutions of the interaction equations, even in the strongly nonlinear regime.

N\K	30	50	70	90	110
20	6.3	7.1	7.6	8.3	8.6
30	6.3	7.1	7.7	8.3	12.9
40	6.3	7.1	7.7	8.3	n.a.

Table 2.1. Resolution test results. For varying numbers of Chebyshev modes (N) and varying numbers of Fourier modes (K) we have $-\log_{10} |\alpha - \bar{\alpha}|/\bar{\alpha}$.

2.8. Results in plane Poiseuille flow

2.8.1. VWI results for $\beta = 1$ in the channel core

All of the results presented in this section are for unit scaled spanwise wavenumber $\beta = 1$, but the behaviour for other spanwise wavenumbers, which will be discussed shortly, is qualitatively similar. For small amplitudes, solutions are found which have the structure predicted from the weakly nonlinear analysis. Figure 2.10(a) shows the solution for a representative small amplitude, where the interaction is still in the weakly nonlinear regime with colour denoting the size of the perturbation to the streak. The structure is mild, with a small spanwise modulation in the streak, and the rolls give rise to a fairly uniform vortex structure. Increasing the amplitude further results in a stronger wave forcing, inducing larger perturbations to the flow and fully nonlinear interactions arise. These solutions, seen in figures 2.10(c,e), become highly localised in the spanwise direction, and an ever larger number of Fourier modes is required to achieve satisfactory resolution. On the right of figure 2.10 we see the associated shear stress λ and the squared wave amplitude function $|A|^2$ for each amplitude. From $|A|^2$, we see the two local points for all amplitudes at spanwise locations $Z = 0$ and $Z = \pi/2$ ($+\pi\mathbb{Z}$). The shear stress plot shows extreme localisation only occurs at one of these local points, the one at $Z = \pi/2$ where $A = 0$. The structure is mild at the other local point at $Z = 0$ corresponding to $A_Z = 0$.

Figure 2.11 shows the wavespeed c against the streamwise wavenumber α . This is plotted as the amplitude increases from zero, at the linear neutral point which is marked by the black star, to large amplitudes where localisation in the spanwise direction is observed. Towards the end of the computed solution branch, the value of the wavespeed c

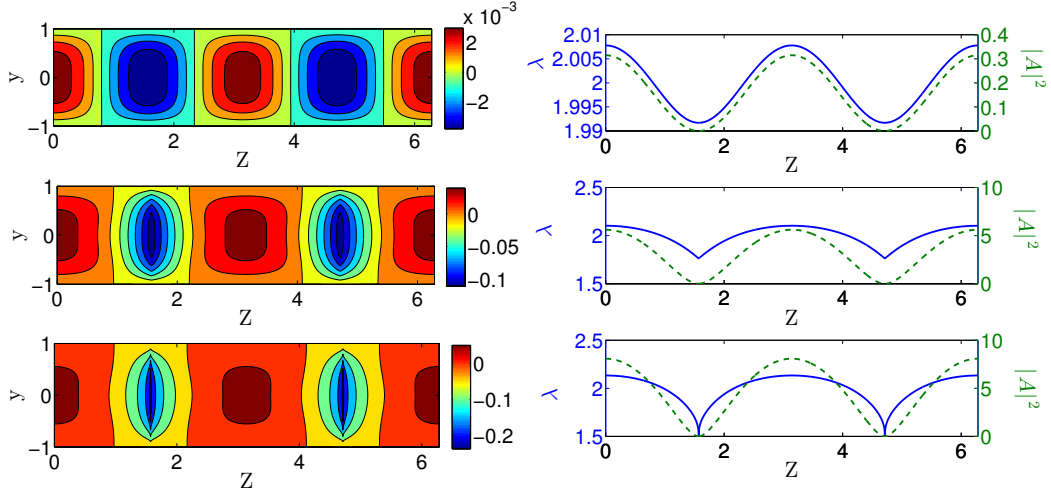


Figure 2.10. Solutions of the interaction equations for various amplitudes. Left: contours of the perturbation to the streak; right: the associated wall shear stress $\lambda(Z)$. Amplitudes plotted: a)-b) $\mathcal{A}^2 = 1$ ($\mathcal{Q} = 5.7 \times 10^{-6}$); c)-d) $\mathcal{A}^2 = 20$ ($\mathcal{Q} = 1.8 \times 10^{-3}$); e)-f) $\mathcal{A}^2 = 35$ ($\mathcal{Q} = 5.0 \times 10^{-3}$).

appears to be tending to a limit, however over the range where this quantity is remaining relatively unchanged the flow is undergoing extreme localisation until eventually the resolution required for these short-scale structures becomes beyond our computational resources.

We see from figure 2.10 that large gradients form in the spanwise direction. To try and gauge whether a singularity exists within the VWI equations we plot the amplitude, $\mathcal{A}$, against the reciprocal of the maximum of the absolute value of the first derivative of the shear stress as seen on the left of figure 2.12. This appears to show that rather than λ' blowing up at finite $\mathcal{A}$, as we would expect if a singularity was present, λ' appears to be blowing up exponentially. To investigate this further we have a log-log plot of the same results on the right of figure 2.12 (the blue solid line). This appears to confirm our suspicions, although interestingly the behaviour appears to change quite suddenly around $\ln \mathcal{A} \approx 1.47$ which corresponds to $\mathcal{A} \approx 4.35$. The red dashed line in this figure shows a line of best fit for the tail of this curve, where we find

$$\ln(\max |\lambda'|)^{-1} \approx 18.125 - 11.75 \ln \mathcal{A}, \quad (2.81)$$

which implies

$$\max |\lambda'| \approx \frac{74.4 \times 10^6}{\mathcal{A}^{11.75}}. \quad (2.82)$$

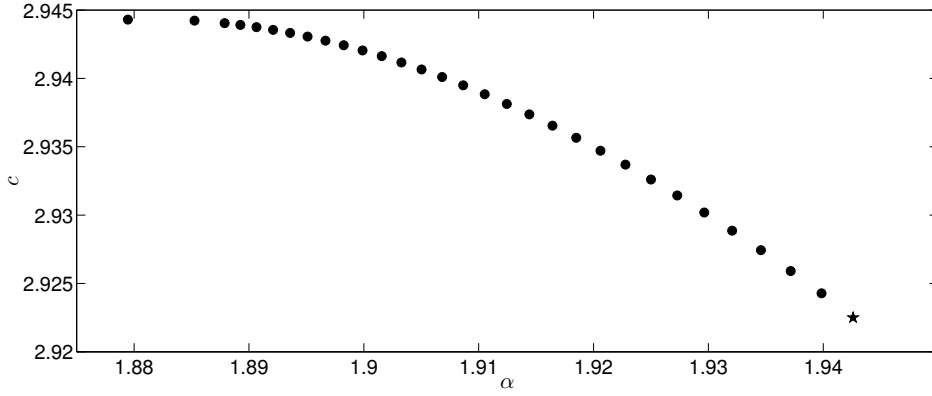


Figure 2.11. (α, c) as $\mathcal{A}$ increases from zero and travelling wave states bifurcate from the linear neutral point $(1.9426, 2.9225)$ denoted by the black star.

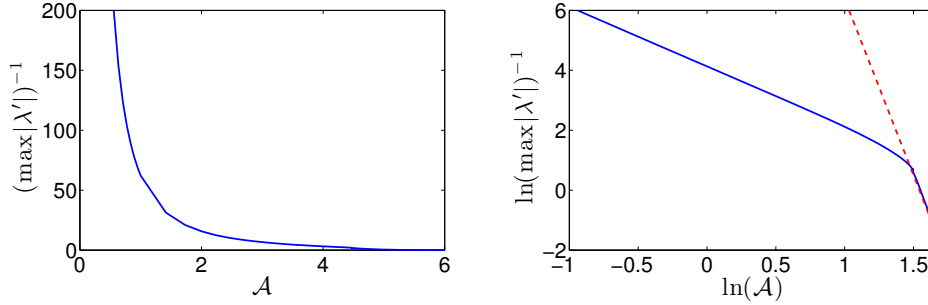


Figure 2.12. Plot of the amplitude $\mathcal{A}$ against some measure of spanwise focusing, via the shear stress derivative λ' .

These results appear to be pointing towards spanwise focusing as $\mathcal{A} \rightarrow \infty$, and no singularity in the VWI equations at finite amplitudes.

Although it is computationally expensive to reach larger amplitudes due to spanwise localisation and the associated large number of Fourier modes required to accurately capture fully resolved solutions, we can get some idea of what the solution looks like at larger amplitudes. In under-resolved computations, the localisation causes some spurious behaviour of the highest frequency modes. Thus applying a high frequency filter to these under-resolved solutions, while not rigorous, gives some indication as to the solution at larger amplitudes. The largest fully resolved solution is found at $\mathcal{A}^2 = 35$ as documented in figures 2.10 and 2.11 and in figure 2.13 we show a filtered solution at $\mathcal{A}^2 = 40$. Such a filtered solution was found following the usual procedure but to eliminate high frequency modes which have been contaminated by the localised nature of the state, we cut off

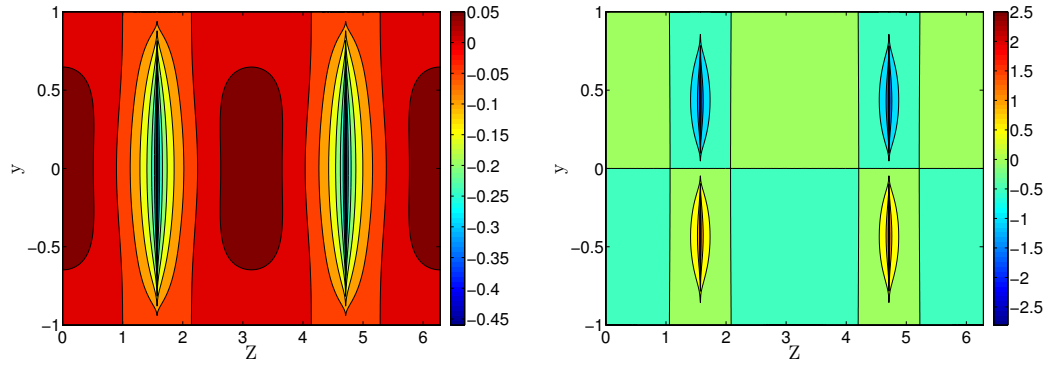
the tail modes in the figure. The continued, and very extreme, spanwise localisation is apparent in this computation, particularly in the perturbation to the streak and in the wall normal velocity. Interestingly, while the wave amplitude function is steepening around the localisation point, it still has a fairly regular structure.

2.8.2. VWI results for $\beta = 1$ near the walls

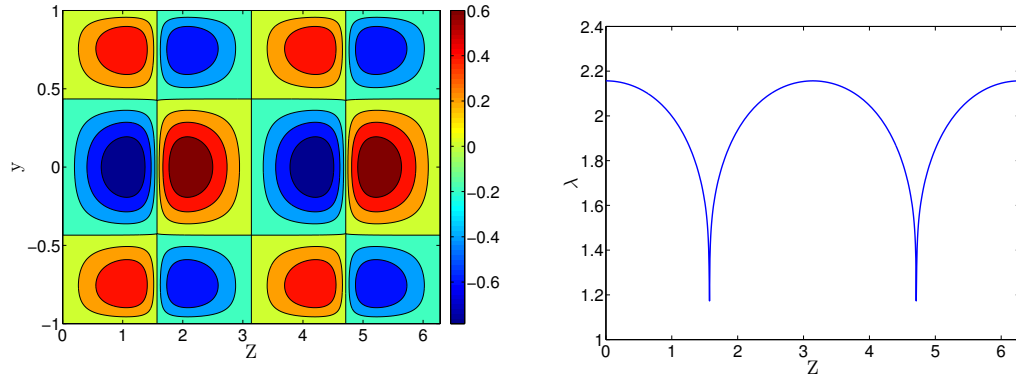
We have discussed how to solve the vortex (2.57) and wave equations (2.41) in the wall layers in section 2.7.2, and we will now provide some solutions at various amplitudes. Our aim here is to identify what type of near wall structure exists for small to moderate amplitudes. Figures 2.14, 2.15 and 2.16 show solutions for amplitudes $\mathcal{A}^2 = 0.01$, $\mathcal{A}^2 = 1$ and $\mathcal{A}^2 = 8$ respectively. The cross sectional flow in the wall layer is dominated by the spanwise wave component with the next order component $O(\epsilon^3)$ smaller, and in fact the streamwise vorticity is concentrated near the walls and comes from this travelling wave component. We are also interested in vortex structures inside the wall layer and for instance we find near wall structures in $\bar{V}$ and $\bar{W}$. When $\mathcal{A}^2 = 8$, we see that although the solution in the core appears to be moderate in the spanwise direction, the spanwise structure of the solution in the wall layer is already fairly extreme. From $\bar{V}$ we see that it also stretches to a much further distance from the wall, compared to the smaller amplitude solutions.

2.8.3. Composite solutions and finite Reynolds numbers

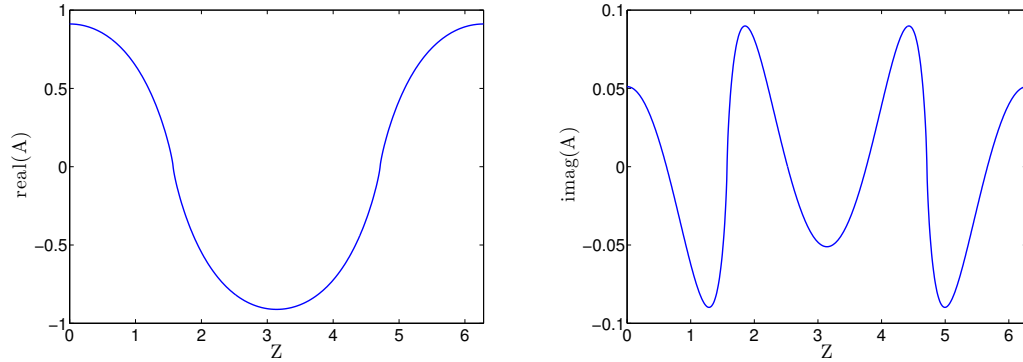
In this section we will discuss composite solutions to the equations and how the results compare with finite Reynolds number computations which can be found in Dempsey *et al.* (2016) - the computations at finite Reynolds numbers were performed by Dr K. Deguchi and details of the numerical scheme can be found in Deguchi, Hall & Walton (2013). The motivation for doing this is that near wall structures are observed in nonlinear travelling wave states computed at large but finite Reynolds numbers, and we would like to ensure that these are captured within this asymptotic framework. We saw in section 2.8.2 that the numerical solution of the near wall equations shows the existence of near wall structures. Now we will consider the solution throughout the whole of the channel. At present we have computed the solution in the core of the channel in section 2.8.1, and the solution in the wall layer in section 2.8.2 which is expressed in terms of the wall layer coordinate Y . To express the solution throughout the entire channel we must form the composite solution and to do this we must choose a Reynolds number so



(a) Contour plot of the perturbation to the streak: $\tilde{u}$. (b) Contour plot of the wall normal component of the roll: V .



(c) Contour plot of the spanwise component of the roll: W . (d) Plot of the wall shear stress: $\lambda(Z)$.



(e) Plot of the real part of the wave amplitude function: $\text{Re}(A)$. (f) Plot of the imaginary part of the wave amplitude function: $\text{Im}(A)$.

Figure 2.13. Large amplitude, $\mathcal{A}^2 = 40$, solution with high frequency filter.

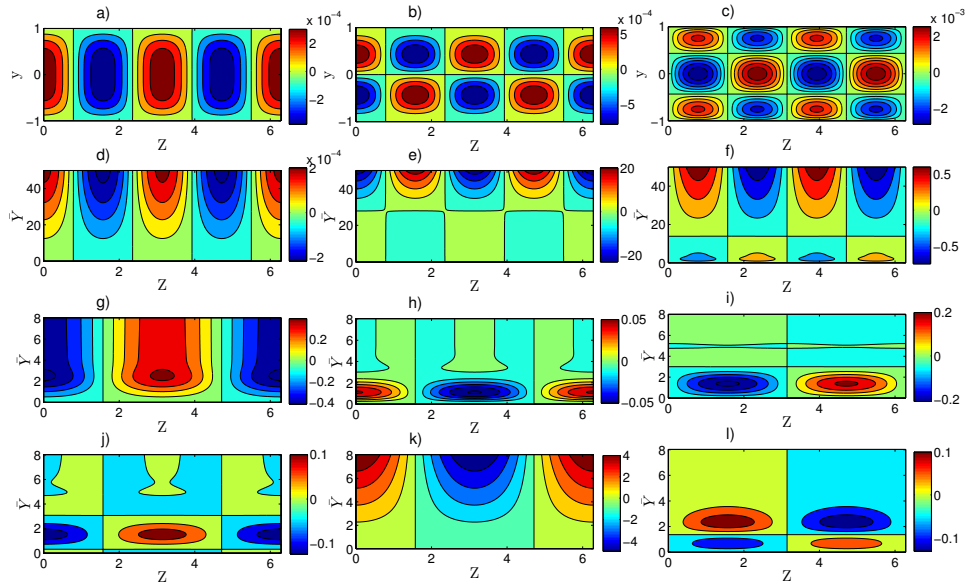


Figure 2.14. Contour plots of the solution for $\mathcal{A}^2 = 0.01$. a-c) Vortex flow in the core of the channel: $\tilde{u}, V, W$. d) Perturbation to the streak in the wall layer: $\bar{U} - \epsilon^2 \lambda_0 \bar{Y} + \epsilon^4 \bar{Y}^2$, for $\epsilon^2 = 0.1$. e-f) Roll flow in the wall layer: $\bar{V}, \bar{W}$. g-i) Real part of the complex wave solution in the wall layer: $\text{real}(\bar{u}, \bar{v}, \bar{w})$. j-l) Imaginary part of the complex wave solution in the wall layer: $\text{imag}(\bar{u}, \bar{v}, \bar{w})$.

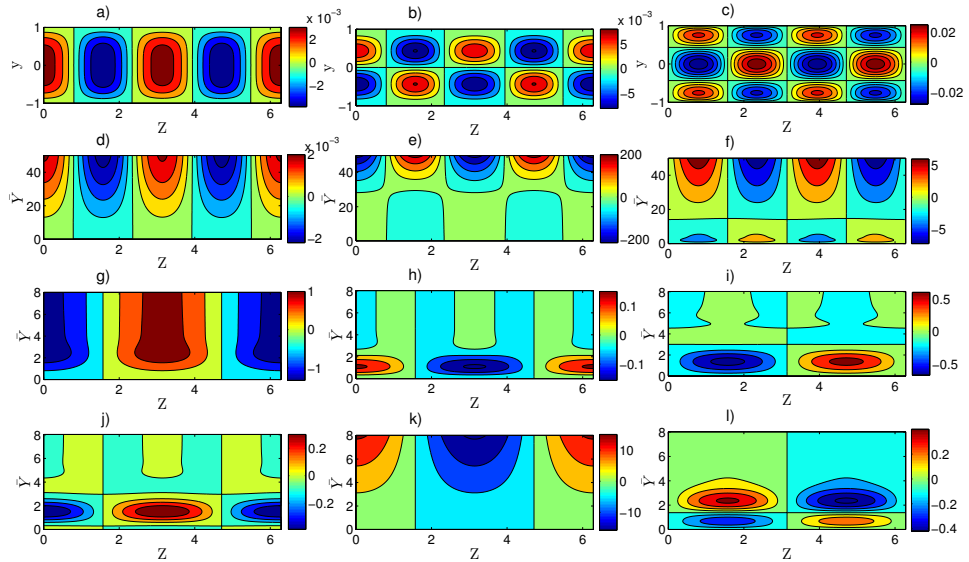


Figure 2.15. Solution for $\mathcal{A}^2 = 1$. See figure (2.14) for labels.

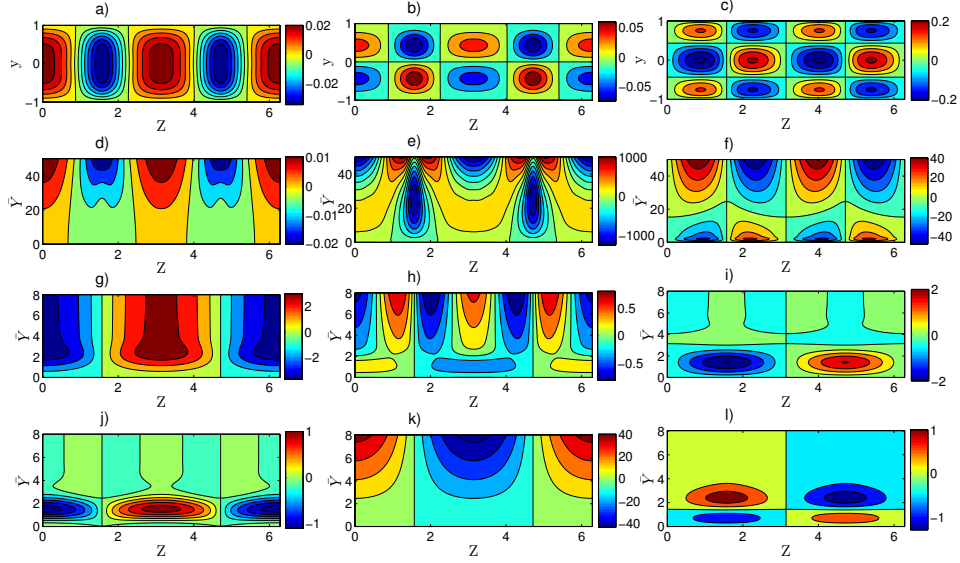
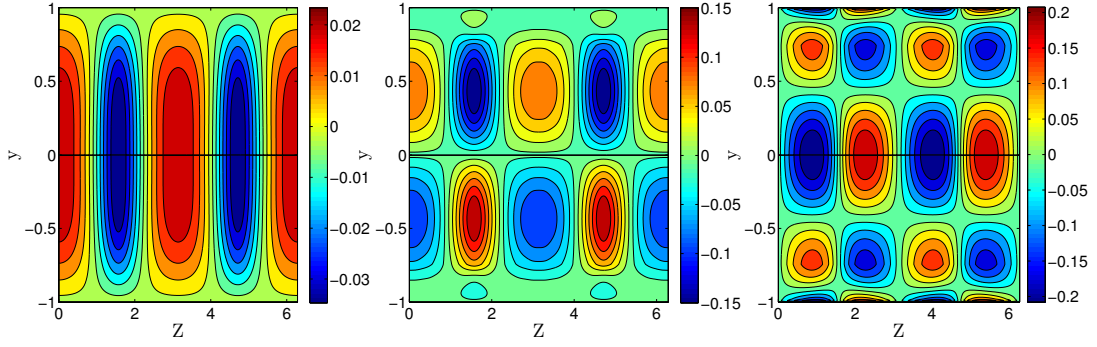


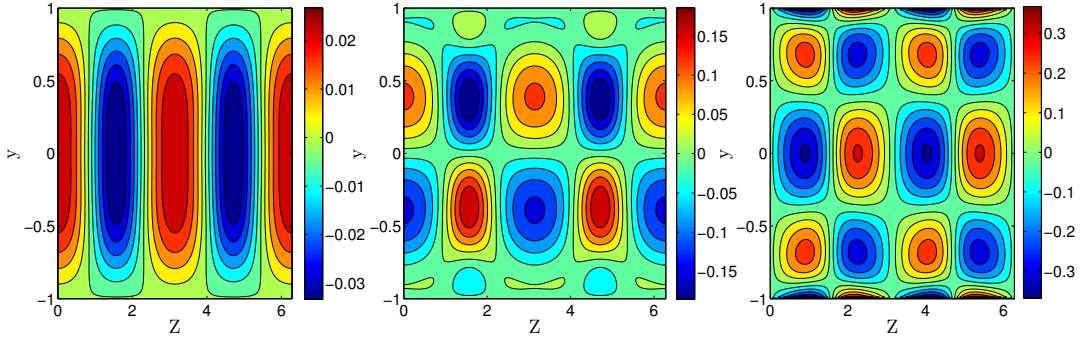
Figure 2.16. Solution for $\mathcal{A}^2 = 8$. See figure (2.14) for labels.

we can evaluate the size of the wall layer.

The procedure for patching the solutions for the core vortex flow and the wall layer vortex flow is essentially to compute the usual composite solution $u_{core} + u_{wall} - u_{matching}$. However, since we only have numerical solutions for each of the components, this matching must be done numerically. Firstly we define a (in this case uniform) grid throughout the entire channel on which we would like to form the composite solution. The core vortex components can be evaluated over this grid using the spectral expansions giving u_{core} on the grid. For the wall components we must pick a Reynolds number which determines the size of the wall units ($Re^{-2/7}$) and then we evaluate the wall solution over a sensible number of wall units (determined by the choice of wall truncation height H). We then interpolate this onto the relevant part of the channel grid giving u_{wall} . Since the matching comes, for example for the lower wall, from $U \sim \lambda(Z)(1 + y)$ we can form $u_{matching}$ from u_{wall} at the edge of the wall layer multiplied by $(1 + y)$ (accounting for the wall layer scaling) for the lower wall but we evaluate this only on the near wall part of the channel grid (the same points as u_{wall} for the purposes of matching the solutions). Then applying any relevant powers of the Re , we form the composite expansion for the streamwise component of the vortex flow for example by taking $u_{core} + u_{wall} - u_{matching}$. We can use this procedure to patch solutions together for each of the flow components.



(a) Composite VWI solution for streak perturbation and roll field at $\mathcal{A}^2 = 8$.



(b) Full Navier-Stokes solution at $Re = 10^8$ from Dempsey *et al.* (2016).

Figure 2.17. Comparison of contour plots for the wave independent part of the Navier-Stokes computational solution at $Re = 10^8$ and the VWI state for the streamwise perturbation, wall normal and spanwise flows, with flux deficit $\mathcal{Q} = 3.23 \times 10^{-4}$ which corresponds to $\mathcal{A}^2 = 8$ in the VWI calculation.

Figure 2.17(a) shows such a composite solution for a Reynolds number of $Re = 10^8$ and a wave amplitude of $\mathcal{A}^2 = 8$, and figure 2.17(b) shows the result of a finite Reynolds number computation of the full Navier-Stokes equations at $Re = 10^8$. In the full Navier-Stokes computations we see near wall structures for the roll components, which we can see from the composite solutions are captured within the VWI framework. With this comparison of the flow field, we are satisfied that vortex-wave interaction is successfully describing the structure of nonlinear travelling wave states bifurcating from the linear neutral curve in plane Poiseuille flow at large Reynolds numbers.

Figure 2.11 showed the bifurcation curve from the linear neutral point in (α, c) space. We have already noted the convergence of the lower branch linear neutral point as the Reynolds number is increased in figure 2.3 for $\beta = 1$, and now we will briefly show that the bifurcation curve (α, c) does indeed converge to the curve predicted by vortex-wave

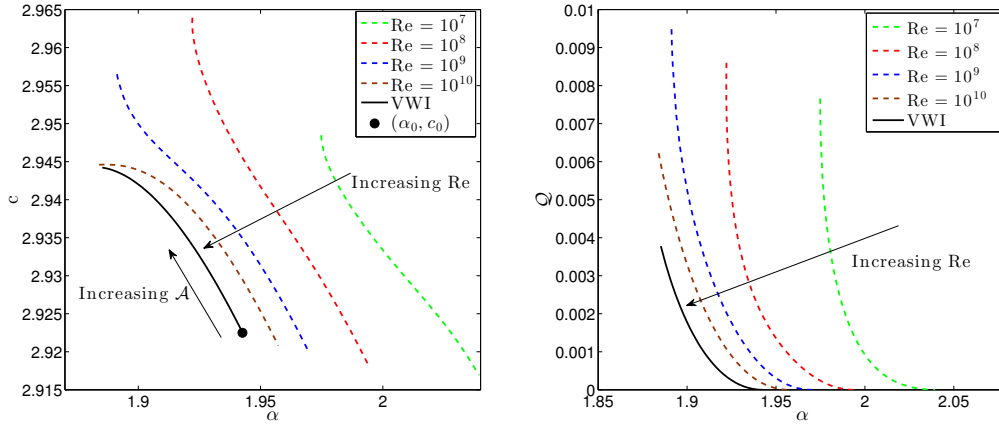


Figure 2.18. VWI (black) and finite Reynolds numbers results from the full Navier-Stokes equations. a) Streamwise wavenumber α against phasespeed c , b) Streamwise wavenumber α against the average flux deficit $\mathcal{Q}$.

interaction as the Reynolds number is increased. Again we will use finite Reynolds number computations of the full Navier-Stokes equations performed by K. Deguchi as documented in Dempsey *et al.* (2016). Figure 2.18(a) shows a comparison of the (scaled) (α, c) curves for VWI states, against finite Reynolds number computations on the full Navier-Stokes equations. Figure 2.18(b) shows a similar comparison in $(\alpha, \mathcal{Q})$ space. In both these plots, convergence to the predictions of vortex-wave interaction can be seen as the Reynolds number is increased.

2.8.4. VWI results for general spanwise wavenumbers and the large spanwise wavenumber limit

We have asserted that the behaviour for general spanwise wavenumbers is similar to the unit spanwise wavenumber case, and in this section we will demonstrate this and show that upon increasing the spanwise wavenumber, the interaction rapidly converges to a large spanwise wavenumber limit whose governing equations will be derived. We will only derive the relevant equations for the symmetric interaction equations, however the generalisation to the non-symmetric case follows without any difficulty. Consider the interaction equations (2.38, 2.56) for a general spanwise wavenumber β , which are to be solved subject to periodic boundary conditions in a box of size $2\pi/\beta$ in the spanwise direction. Computations were performed for various spanwise wavenumbers and in this section we will concentrate on $\beta = 2, 2.5, 3$ and 5 to highlight the results.

Before we consider the results, it is useful to rescale the variables on the spanwise wavenumber. The spanwise wavenumber does not scale out of the equations, but we can motivate the important scales with the following argument. Clearly for large β , $Z \sim \beta^{-1}$, and then the continuity equation $V_y \sim W_Z$ yields $W \sim \beta^{-1}$ and hence the streamfunction $\Psi \sim \beta^{-1}$. Consider the wave equation (2.56), then balancing terms $(JA)_{ZZ} \sim \alpha^{-1/3} \lambda^{5/3} \mathcal{G}(\xi)A$ we find $\alpha \sim \beta^{-6}$. Also from the wave equation we expect ξ to remain $O(1)$ and thus $\alpha^{1/3}c \sim O(1)$, i.e. $c \sim \beta^2$. Finally, considering the spanwise momentum equation $W_{yyy} \sim \alpha^2(|A|^2)_Z U U_y$ we expect $A \sim \beta^5$. Thus we apply the rescaling,

$$\hat{Z} = \beta Z, \quad \hat{\Psi} = \beta \Psi, \quad \hat{\alpha} = \beta^6 \alpha, \quad \hat{c} = \beta^{-2} c, \quad \hat{\mathcal{A}} = \beta^{-5} \mathcal{A}, \quad (2.83)$$

to the interaction equations (2.38, 2.56). Under this rescaling the core equations (2.38) are unchanged (in terms of the newly-scaled variables), except they are now to be solved subject to periodicity on $[0, 2\pi]$. The wave equation however becomes,

$$(J\hat{A})_{\hat{Z}\hat{Z}} - \frac{\lambda \hat{Z}}{\lambda} \mathcal{F}(J\hat{A})_{\hat{Z}} - \beta^{-14} \hat{\alpha}^2 (J\hat{A}) = \hat{\alpha}^{-1/3} \lambda^{5/3} \mathcal{G}\hat{A}. \quad (2.84)$$

This rescaling suggests that in the limit $\beta \rightarrow \infty$ the VWI remains governed essentially by (2.38, 2.56) with $\beta = 1$ but with the final term on the left-hand-side of (2.56) missing,

$$(J\hat{A})_{\hat{Z}\hat{Z}} - \frac{\lambda \hat{Z}}{\lambda} \mathcal{F}(J\hat{A})_{\hat{Z}} = \hat{\alpha}^{-1/3} \lambda^{5/3} \mathcal{G}\hat{A}, \quad (2.85)$$

and we call the interaction equations (2.38) with (2.56) replaced by (2.85), the large β equations - the fact that these equations are indeed the large β limit of (2.84) will be demonstrated shortly. We will solve the interaction equations in these new scaled coordinates with (2.84) for various β and (2.85) for the large β limit. The amplitude is now prescribed via $\hat{\mathcal{A}}$ and we track states which bifurcate from the linear neutral point into finite amplitudes. These states are qualitatively similar to $\beta = 1$ and other values of the spanwise wavenumber, with moderate structure near the linear neutral point and localisation in the spanwise direction as the amplitude increases. A representative solution can be seen in figure 2.19 which shows the perturbation to the streak and the scaled streamfunction $\hat{\Psi}$ throughout the majority of the channel for $\hat{\mathcal{A}} = 0.1$. In this figure convergence to the large β limit can be observed.

To confirm convergence we consider the streamwise wavenumber for varying β and the large β limit. In figure 2.20 we see a plot of scaled $(\alpha, \mathcal{A})$ for $\beta = 2, 2.5, 3$ and 5 , along with solutions from the large β equations. Convergence to the large β limit in α can clearly be seen. It is also worth mentioning here that in the large spanwise wavenumber

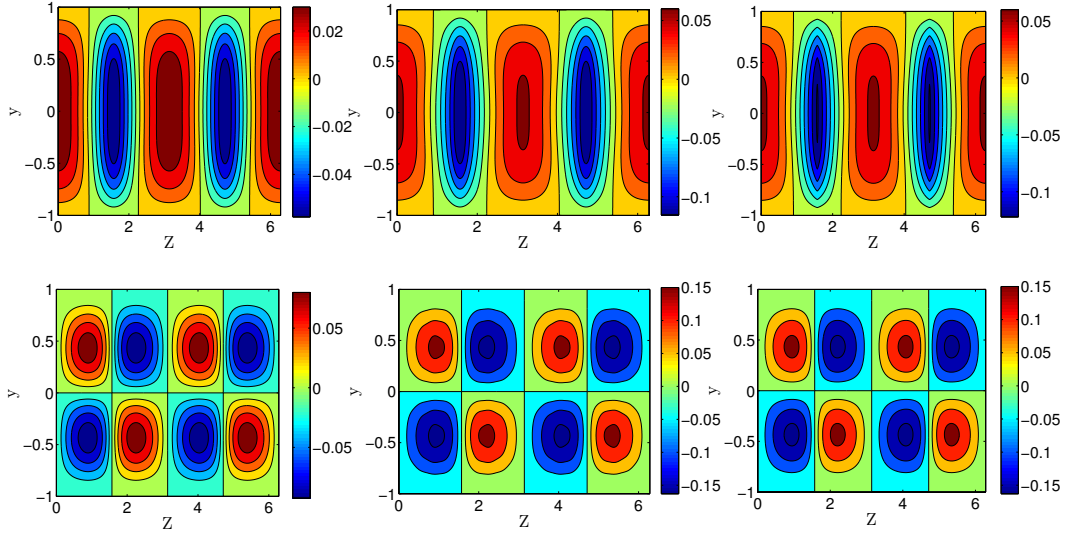


Figure 2.19. Solution for general spanwise numbers and the large spanwise wavenumber limit for a scaled amplitude $\hat{\mathcal{A}} = 0.1$. The top row shows the perturbation to the streak, and the bottom row shows the scaled streamfunction $\hat{\Psi}$ for $\beta = 2$ (left), 3 (middle), large β limit (right), where $\hat{\mathcal{A}}$ and $\hat{\Psi}$ are defined in (2.83).

limit ξ , defined in (2.55), remains order 1. Since we have demonstrated that the vortex flow and the streamwise wavenumber converge to the large β limit, we conclude that the large β equations do indeed represent the large β limit of the interaction equations.

We can derive the large β scalings for the other near wall components from equation (2.57) and (2.41). First we must consider the pressure displacement relation $p_+ - p_- = \alpha^2 J A$ with the large β scalings reveals $\bar{p} \sim \beta^{-7}$. Considering the near wall wave components, the terms $\lambda Y - c$ in (2.41b,c) with (2.83) yields that the wave acts over the wall normal scale $Y \sim c \sim O(\beta^2)$. Then a pressure/viscous/inviscid balance in (2.41c) $\bar{p}_Z \sim \bar{w}_{YY}$ reveals that $\bar{w} \sim \beta^{-2}$. From the continuity equation (2.41a) it follows that $\bar{u} \sim \beta^5$ and $\bar{v} \sim \beta^{-1}$. Consider the vortex equations in the wall layer (2.57). Equation (2.57c) balancing viscosity with the wave forcing $W_{YY} \sim \bar{w} \bar{w}_Z^*$ yields $\bar{W} \sim \beta$, and from the continuity equation (2.57a) we find $\bar{V} \sim \beta^4$. Then the streamwise momentum equation (2.57b) balance $U_{YY} \sim \lambda \bar{V} \sim \bar{w} \bar{u}_Z^*$ yields $\bar{U} \sim \beta^8$. In a moment we will consider the implications of these large β scalings.

Increasing the scaled spanwise wavenumber β corresponds to moving from a large spanwise wavelength interaction back towards an $O(1)$ wavelength. Clearly a distinguished limit is reached when $\beta \sim O(Re^{1/7})$, which corresponds to $O(1)$ spanwise variations. Ap-

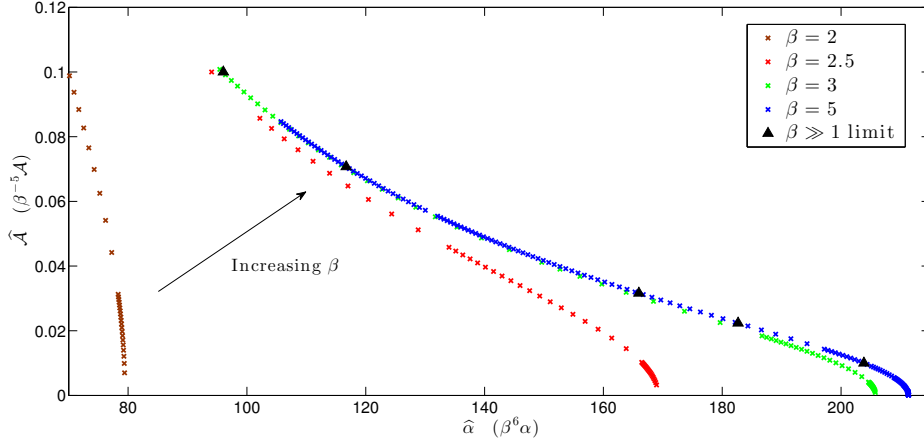


Figure 2.20. Convergence of α to the large β limit.

plying $\beta \sim O(Re^{1/7})$ in (2.83) with the above discussion to the expansions (2.30, 2.39) we find both components of the roll in the channel core and the wall layer are $O(Re^{-1})$, the streamwise wavenumber is $O(Re^{-1})$, the wavespeed is $O(1)$ and the streamwise component of the velocity is $O(1)$. Further to this the wall layer is of size $\beta^2 Re^{-2/7}$ and thus when $\beta \sim Re^{1/7}$ the wall layer merges with the channel core. In this case the asymptotic expansions break down, and the next asymptotic expansions valid throughout the whole channel are

$$u = U_1(X_1, y, z) + \dots, \quad (2.86a)$$

$$(v, w) = Re^{-1}(V_1, W_1) + \dots, \quad (2.86b)$$

$$p = Re^{-2}P_1 + \dots, \quad (2.86c)$$

where $X_1 = Re(x - ct)$ is the travelling wave coordinate. Note that the governing equations with these scalings are the three-dimensional boundary-region equations in which the roll, streak and wave now evolve over a single $O(Re)$ streamwise lengthscale.

2.9. Vortex-wave interaction in plane Poiseuille-Couette flow

As we saw in the asymptotic linear stability calculation of section 2.4, if the wall sliding velocities are $O(\epsilon^2)$ then the lower branch neutral wave has the same structure as the lower branch neutral wave in plane Poiseuille flow. Thus as before we set the wall sliding

velocities to $u(y = \pm 1) = \pm \epsilon^2 \gamma$ where γ is $O(1)$. Similarly to the case of plane Poiseuille flow, we can derive interaction equations but there are a couple of differences. Firstly, near the lower wall (for example) the velocity now looks like $u \sim \epsilon^2(\lambda Y - \gamma)$, and this is where the wall sliding velocity affects the interaction equations - it is a higher order effect on the core equations where the streamwise velocity is $O(1)$. Secondly, the symmetry in y is now broken in the wave pressure equations and we must now consider the shear stress and pressure on each wall separately - it is not possible to eliminate the wall wave pressures to yields a single equation for A as it is in plane Poiseuille flow. This also has the effect of substantially increasing the size of the computational problem for the wave pressure system.

Taking these changes into account, the governing equations can be found and are composed of the core equations which are identical to the plane Poiseuille case (2.38),

$$V_y + W_Z = 0, \quad (2.87a)$$

$$VU_y + WU_Z = 2 + U_{yy}, \quad (2.87b)$$

$$VW_{yy} + WW_{yZ} = 2\alpha^2(|A|^2)_Z U U_y + W_{yyy}, \quad (2.87c)$$

with boundary conditions

$$U = V = W = 0 \quad \text{on} \quad y = \pm 1. \quad (2.87d)$$

These are coupled with the wave equations which now with the broken symmetry, must be considered in their full form

$$p_+'' - \frac{\lambda_+'}{\lambda_+} \mathcal{F}(\xi_+) p_+' - \alpha^2 p_+ = (\alpha \lambda_+)^{5/3} \mathcal{G}(\xi_+) A, \quad (2.88a)$$

$$p_-'' - \frac{\lambda_-'}{\lambda_-} \mathcal{F}(\xi_-) p_-' - \alpha^2 p_- = -(\alpha \lambda_-)^{5/3} \mathcal{G}(\xi_-) A, \quad (2.88b)$$

$$p_+ - p_- = \alpha^2 J A, \quad (2.88c)$$

where for completeness we redefine the various components to include the wall sliding velocity and the asymmetry in the wall normal direction,

$$\xi_{\pm} = \frac{(i\alpha\lambda_{\pm})^{\frac{1}{3}}}{\lambda_{\pm}} (\pm\gamma - c), \quad \lambda_{\pm} = \mp \frac{\partial U}{\partial y}(\pm 1, Z), \quad J = \int_{-1}^1 U^2 dy. \quad (2.88d)$$

As before we are interested in solutions which are periodic in the spanwise direction, with period $2\pi/\beta$ where β is the spanwise wavenumber. We will restrict our attention here

to $\beta = 1$. No computations have been performed at other spanwise wavenumbers but we expect the situation to be similar for other spanwise numbers based on our previous results in plane Poiseuille flow.

These equations have been solved computationally using the same method as for plane Poiseuille flow documented in section 2.7.1. As found in section 2.4 the number of linear neutral points is dependent on the wall sliding velocity. For small γ there is only a single linear neutral point to bifurcate from. At some critical wall speed denoted by $\gamma_c^{(1)}$ there is a bifurcation into two further linear neutral points. Then at a second critical value of the wall speed denoted by $\gamma_c^{(2)}$ there is a further bifurcation into two more neutral points, resulting in a total of five linear neutral points for $\gamma > \gamma_c^{(2)}$.

As in section 2.6 for plane Poiseuille flow we can construct weakly nonlinear solutions when $\mathcal{A}$ is small, extending the linear analysis of section 2.4 where we found the linear dispersion relation (2.27) for D_γ where

$$\mathcal{D}_\gamma(\alpha, c, \beta) = \lambda_0^{5/3}[\mathcal{G}(\xi_+) + \mathcal{G}(\xi_-)] + \alpha^{1/3}(\beta^2 + \alpha^2)J_0 = 0, \quad (2.89)$$

for wall sliding velocities $\pm\gamma$ (see section 2.4 for definitions of the terms and a discussion of the solutions to this dispersion relation). The details follow similarly to section 2.6 except we must now consider the contribution from each wall separately - see appendix B for more details. A comparison of the results from the weakly nonlinear analysis and fully nonlinear computations on the interaction equations can be found in figure 2.21 for $\gamma = 3$. Very good agreement is found for small $\mathcal{A}$, especially for roots 1 and 3. For root 2 we see agreement although the existence of a turning point at a reasonably small amplitude limits the range of applicability of the weakly nonlinear analysis. Overall, good agreement with the weakly nonlinear analysis is observed.

Consider the linear neutral point corresponding to plane Poiseuille flow by setting $\gamma = 0$. As the wall speed increases from zero, the corresponding neutral streamwise wavenumber, α , becomes small. Then as the wave forcing on the core scales with α^2 , a much larger amplitude wave is required to cause a substantial change to the basic state. Figure 2.22 shows the nonlinear states bifurcating from this neutral point at $\gamma = 5$. Comparing the distortion to the basic state at this amplitude with the case for zero wall speeds as in figure 2.10 for plane Poiseuille flow confirms the discrepancy with the size of wave required to cause a particular size perturbation to the basic state. Interestingly in the case given in figure 2.22, moderate localisation is observed even though the changes to the basic state are small suggesting that the localisation process is not a consequence

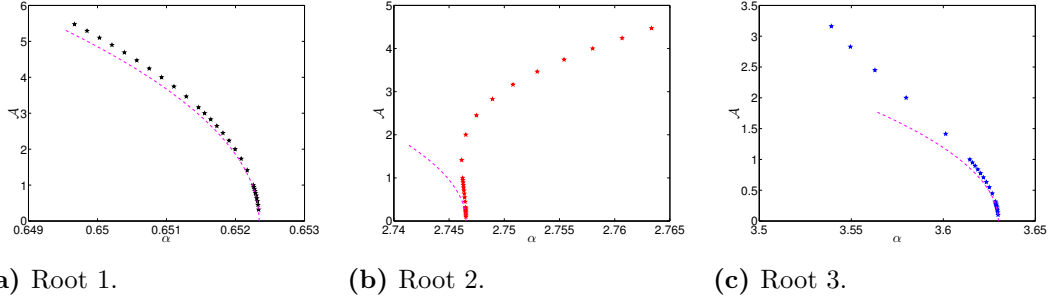


Figure 2.21. Comparison with the weakly nonlinear analysis for the three bifurcations present when $\gamma = 3$. Nonlinear computations are given by stars, and the weakly nonlinear results by the dashed magenta curves.

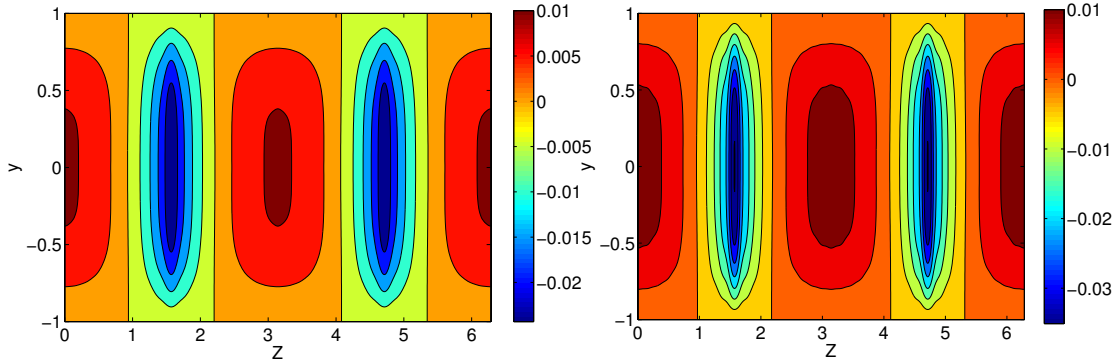


Figure 2.22. VWI state for plane Poiseuille-Couette flow with $\gamma = 5$. In the figures we see the streak perturbation for large amplitude states with $\mathcal{A}^2 =$ (i) 500, (ii) 600.

of highly nonlinear interactions of the vortex flow in the core of the channel, but perhaps instead a phenomenon associated with the wave.

Now consider $\gamma = 2 \gtrapprox \gamma_c^{(1)}$ so there are now three linear neutral points. Figure 2.23 shows the bifurcations from the linear neutral points, into finite amplitude equilibrium states from all three neutral points. The first root, denoted by solution 1, is the original root from plane Poiseuille flow modified by the sliding walls. Solutions 2 and 3 are the new linear neutral points. We find that these two lower branches coalesce at a finite amplitude forming a closed curve in $(\alpha, \mathcal{A})$ space, and this is perhaps an unexpected result. The solutions along this branch are similar to the solutions we have seen before and surprisingly even when they terminate at a finite amplitude there is a limited change to the basic state and no other associated phenomenon.

If we now consider a wall sliding velocity γ such that $(\gamma_c^{(1)} < \gamma < \gamma_c^{(2)})$, e.g. $\gamma = 3$ as in

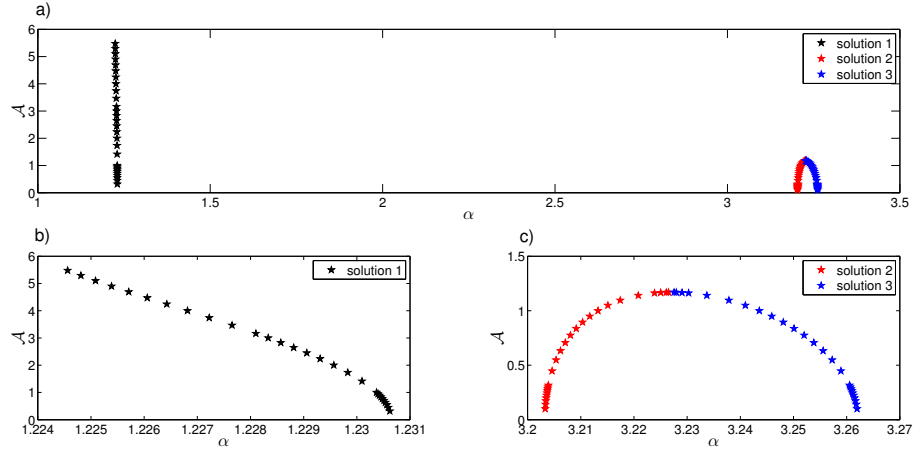


Figure 2.23. $\gamma = 2$ bifurcations for the linear neutral point in $(\alpha, \mathcal{A})$ space. a) All roots. b) Zoomed into solution 1 (original solution). c) Zoomed into solutions 2 and 3, the new solutions.

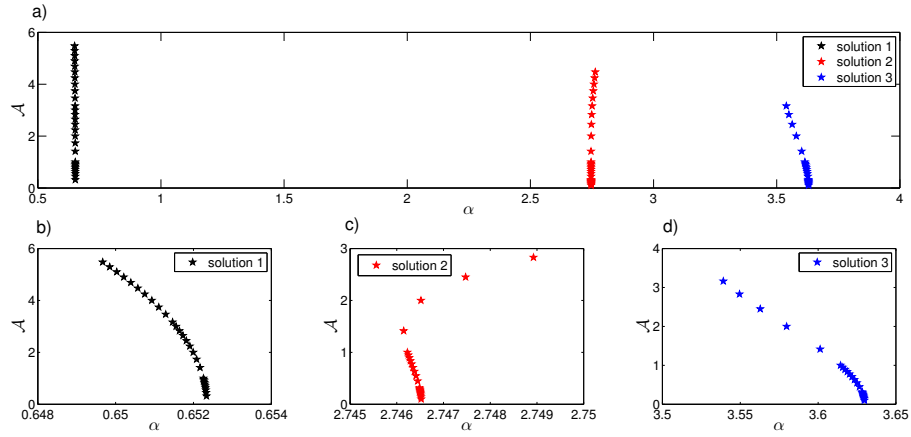


Figure 2.24. $\gamma = 3$ bifurcations for the linear neutral point in $(\alpha, \mathcal{A})$ space. a) All roots. b) Zoomed into solution 1 (original solution). c) Zoomed into solutions 2 and 3, the new solutions.

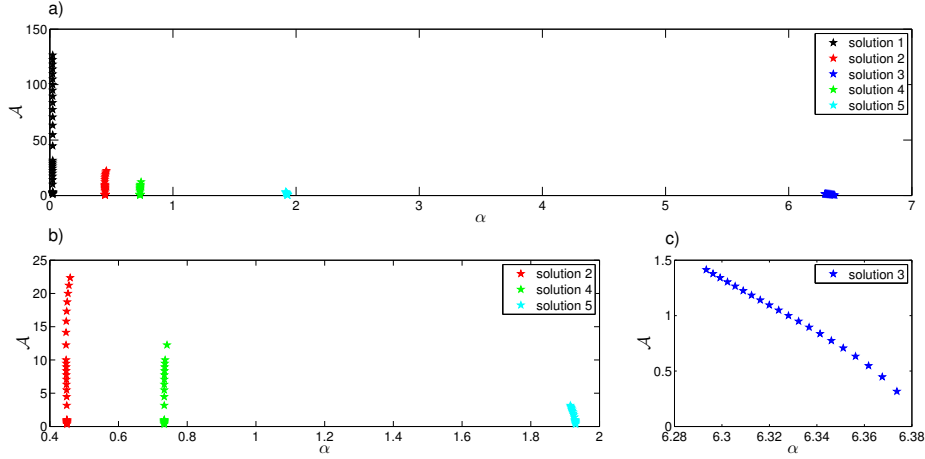


Figure 2.25. $\gamma = 10$ bifurcations for the linear neutral point in $(\alpha, \mathcal{A})$ space. a) All roots. b) Zoomed into solution 2, 4 and 5. c) Zoomed into solution 3.

figure 2.24 then we see a similar bifurcation diagram. However, tracking these states into finite amplitudes the localisation eventually becomes too extreme to track any further although given our results for $\gamma = 2$, we expect that if we could follow them to larger amplitudes that they may coalesce. Now consider $\gamma = 10$, which is greater than $\gamma_c^{(2)}$ and thus there are now five lower branch linear neutral points. There are no surprises here from our previous analysis. We track the bifurcations until the localisation becomes too extreme, and the distance which states can be tracked is inversely proportional to the streamwise wavenumber. It is perhaps worth noting, because it is not obvious from the bifurcation diagrams shown, that all the bifurcations appear to be in the decreasing α direction, i.e. $\alpha_2 < 0$ for all the bifurcations seen here. From our results we expect this may be true for all the linear neutral points, and this could potentially be proven from the weakly nonlinear analysis although this was not attempted here.

2.10. Curved channel flows

We will now consider the effect of adding curvature to a channel with stationary walls. If the curvature is large enough then there is now more than one basic state, the usual spanwise independent basic state and the Taylor vortex state⁵. With the curvature fixed we can introduce an infinitesimally small wave into the flow, and track bifurcations into nonlinear equilibrium travelling wave states in each of the flows. The scales and

⁵Actually a symmetric pair of states.

governing equations for these interactions were set out by Bennett, Hall & Smith (1991) but substantially better computational resources today allow us to expand on their results. Before we can make any progress we need to understand the effect of curvature on the basic state in the absence of a wave. When we add the wave component to the interaction, variations are over the long spanwise wavelength thus we will only consider long wavelength Taylor vortices. To compute Taylor vortices, first we consider the linear stability of the basic state to curvature instabilities as the Taylor number varies. This identifies the critical Taylor number at which Taylor vortices bifurcate from the usual basic state (plane Poiseuille flow), from which we can compute nonlinear Taylor vortices. Following this analysis, we can compute states where both effects are present, i.e. wave and wall curvature are present.

2.10.1. Long wavelength Taylor vortices

We begin with an investigation of long wavelength Taylor vortices. The spanwise lengthscale of the vortex-wave interaction states derived in plane Poiseuille flow is $O(\epsilon^{-1}) = O(Re^{1/7})$ but for now we will consider a general large wavelength, say Λ^{-1} where $\Lambda \ll 1$. We start from the Navier-Stokes equations in body fitted coordinates see appendix C. First we will motivate the relevant scales. Clearly the spanwise scale is $O(\Lambda^{-1})$, and given the basic state is $O(1)$ and varies over the $O(1)$ scale in the wall normal direction we conclude that the streamwise velocity and wall normal lengthscales are $O(1)$. As in many such nonlinear interactions, a viscous-inviscid balance of the streamwise momentum equation $vu_y \sim Re^{-1}u_{yy}$ reveals that the wall normal velocity must be $O(Re^{-1})$. For a dominant balance in the continuity equation $v_y \sim w_z$, the large spanwise scale results in a spanwise velocity scale of $O(\Lambda^{-1}Re^{-1})$. A pressure balance in the spanwise momentum equation $vw_y \sim p_z$ reveals that the pressure is $O(\Lambda^{-2}Re^{-2})$. Thus we set

$$u = U_s, \quad v = Re^{-1}V_s, \quad w = \Lambda^{-1}Re^{-1}W_s, \quad p = \Lambda^{-2}Re^{-2}P_s, \quad z = \Lambda^{-1}Z_s. \quad (2.90)$$

Considering the wall normal momentum equations which provide the dominant contribution from the curvature, we must balance $\kappa u^2 \sim p_y$, where κ is the curvature of the wall. Thus curvature effects first come into play when the curvature of the wall is $O(\Lambda^{-2}Re^{-2})$. We define the Taylor number T to correspond to variations of the curvature on this critical scale

$$\kappa = \Lambda^{-2}Re^{-2}T. \quad (2.91)$$

We will look for equilibrium solutions, so we will assume independence of x and t . Substituting the expansions (2.90) into the Navier-Stokes equations in a curved geometry, we find the governing equations

$$V_s y + W_s Z_s = 0, \quad (2.92a)$$

$$V_s U_s y + W_s U_s Z_s = 2 + U_s yy, \quad (2.92b)$$

$$TU_s^2 = -P_s y, \quad (2.92c)$$

$$V_s W_s y + W_s W_s Z_s = -P_s Z_s + W_s yy. \quad (2.92d)$$

If we cross differentiate to eliminate the pressure from (2.92c,d) we find

$$2TU_s U_s Z_s - V_s W_s yy - W_s W_s y Z_s = -W_s yy y. \quad (2.93)$$

Now we are interested in bifurcations, so we linearise the equations (2.92a,b) and (2.93) around the basic state by setting

$$(U_s, V_s) = (u_b, 0) + (\bar{u}_s, \bar{v}_s) \cos \beta_s Z_s, \quad W_s = \bar{w}_s \sin \beta_s Z_s, \quad (2.94)$$

where we assume the perturbations $\bar{u}_s, \bar{v}_s$ and $\bar{w}_s$ are sufficiently small that we can ignore quadratically small terms, β_s is the scaled spanwise wavenumber and u_b is the basic state for plane Poiseuille flow, $u_b = 1 - y^2$. The linearised equations are

$$\bar{v}_s' + \beta_s \bar{w}_s = 0, \quad (2.95a)$$

$$u_b' \bar{v}_s = \bar{u}_s'', \quad (2.95b)$$

$$-2\beta_s T u_b \bar{u}_s = -\bar{w}_s'''. \quad (2.95c)$$

If we eliminate $\bar{w}_s$ we are left with the problem

$$\bar{u}_s'' - u_b' \bar{v}_s = 0, \quad (2.96a)$$

$$\bar{v}_s'''' + 2\beta_s^2 T u_b \bar{u}_s = 0, \quad (2.96b)$$

with no slip boundary conditions

$$\bar{u}_s, \bar{v}_s, \bar{v}_s' = 0, \quad y = \pm 1. \quad (2.96c)$$

Equations (2.96) can only be solved computationally. This is an eigenvalue problem for T . Solving this problem we find that for $T < T_c(\beta_s)$, the basic state is linearly stable.

However, at $T = T_c(\beta_s)$ there is a bifurcation and the basic state becomes unstable for $T > T_c(\beta_s)$. This represents a bifurcation into finite amplitude non-zero solutions, and these are known as Taylor vortices. In the vortex-wave interaction work we have focused on $\beta = 1$ for the wave. Because this corresponds to a change to the basic state in the second harmonic, we will concentrate on the corresponding spanwise wavenumber which is $\beta_s = 2\beta = 2$. A finite difference computation, using second order centered finite differences, with 500 points in the wall normal direction found $T_c(2) \simeq 104.2237$.

Now that we have identified a single point on the curve $(\beta_s, T_c(\beta_s))$ it turns out we can describe the whole curve. Consider the linear problem given by equations (2.96) for a general spanwise wavenumber β . In particular from equation (2.96b) if we write the term $2\beta_s^2 T u_b \bar{u}_s$ in the form $2(4)(\beta_s^2 T/4) u_b \bar{u}_s$ then it follows, using $T_c(2)$, that

$$T_c(\beta_s) = \frac{4T_c(2)}{\beta_s^2}. \quad (2.97)$$

Thus we have determined $T_c(\beta_s)$ for all β_s from the knowledge of a single point⁶. Further, changing β_s we see that $\bar{u}_s$ and $\bar{v}_s$ do not change as β_s varies, and $\bar{w}_s$ scales with $2\beta_s^{-1}$.

Now we have determined the location of the bifurcation from the basic state into Taylor vortices we are in a position to consider the nonlinear problem which is governed by the equations from (2.92a,b) and (2.93)

$$V_s y + W_s Z_s = 0, \quad (2.98a)$$

$$V_s U_s y + W_s U_s Z_s = 2 + U_s y y, \quad (2.98b)$$

$$V_s W_s y y + W_s W_s y Z_s - 2T U_s U_s Z_s = W_s y y y. \quad (2.98c)$$

with boundary conditions,

$$U_s = V_s = W_s = 0, \quad y = \pm 1. \quad (2.98d)$$

Nonlinear Taylor vortices can be found by solving (2.98) using a method similar to that used to solve the nonlinear vortex-wave interaction equations. We introduce the stream function Ψ_s so $(V_s, W_s) = (\partial_{Z_s}, -\partial_y)\Psi_s$ and then decompose U_s and Ψ_s using a

⁶This is done using $\beta_s = 2$ as the reference point, but we could of course use $\beta_s = 1$ and we would find $T_c(\beta_s) = T_c(1)/\beta_s^2$, and then $\bar{w}_s$ scales with β_s^{-1} .

Chebyshev-Fourier basis taking into account the spanwise symmetries,

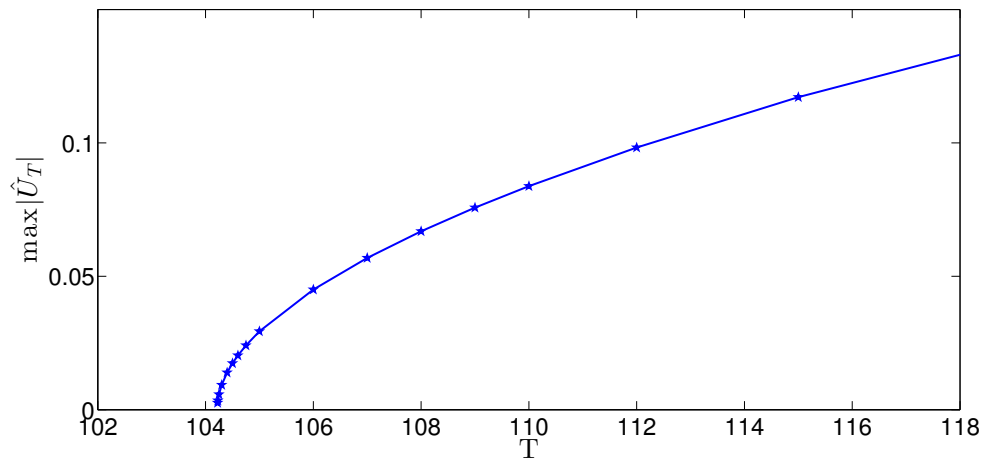
$$U_s = \sum_{n,k=1}^{N,K} U_{s,n,k} T_{n-1}(y) \cos[(k-1)\beta_s Z_s], \quad \Psi_s = \sum_{n,k=1}^{N,K} \Psi_{s,n,k} T_{n-1}(y) \sin(k\beta_s Z_s). \quad (2.99)$$

Applying the collocation method in the wall normal direction at collocation points $y_j = \cos(j\pi/N)$ and the Galerkin method in the spanwise direction we are left with a set of nonlinear equations for the spectral coefficients which are solved using a quasi-Newton method. The basic state is always a solution to this system so we must take some care with how we initialise the Newton method to avoid convergence to the usual basic state. This is where the linear analysis comes in since we have identified the critical Taylor number T_c where the bifurcation into Taylor vortices occurs. We have determined the eigenfunctions which we can use to initialise the Newton method increasing the likelihood of convergence to a non-zero perturbation to the basic state. We could perform a weakly nonlinear analysis to give us more information for starting the Newton iteration but in fact the information from the linear analysis is sufficient to find the bifurcation curve from $T_c(\beta_s)$ and once a single non-trivial solution is found on the bifurcation curve we can track the bifurcation curve by continuing our solutions in the Taylor numbers T . We can compare the values of T_c found in the linear analysis and the nonlinear computations. In the nonlinear computations, non-zero solutions were found down to a Taylor number of $T = 104.225$ in good agreement with the linear analysis, especially given the difficulties in finding small amplitude solutions of the nonlinear equations as the Taylor vortex and basic state solutions are very close to one another in the state space. Figure 2.26(a) shows the bifurcation diagram for Taylor vortices from the nonlinear computations of equations (2.98). The bifurcation parameter used here to measure the amplitude of the Taylor vortices is the maximum of the absolute value of the perturbation to the basic state,

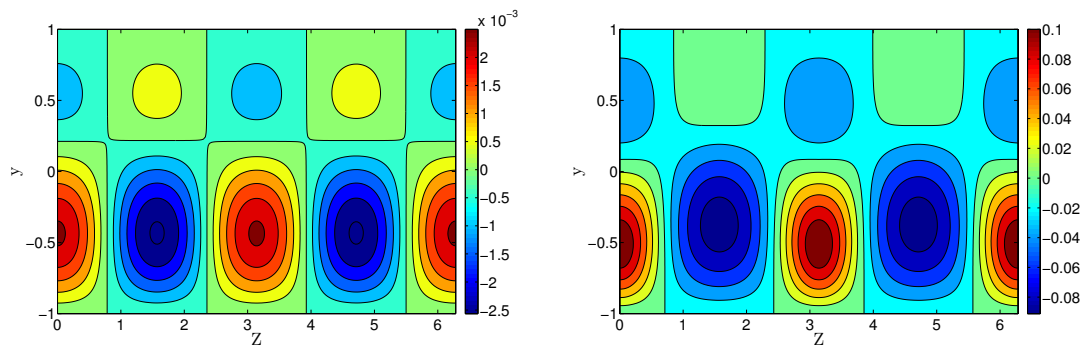
$$\hat{U}_T = U_s - u_b. \quad (2.100)$$

The bifurcation is seen to be supercritical for $\beta_s = 2$. Figures 2.26(b,c) show the structure of long wavelength Taylor vortices for a small amplitude in (b) near the bifurcation point, and for a large amplitude in (c) where the largest change to the basic state is 0.117, which is 11.7% of the centreline velocity of the basic state.

As the Taylor number is increased, structure develops on a small scale and it is computational more expensive to compute solutions when these smaller scales are involved. It is not obvious whether these solutions are localising or whether a singularity develops in the equations but figure 2.27 shows the streamwise perturbation to the basic state



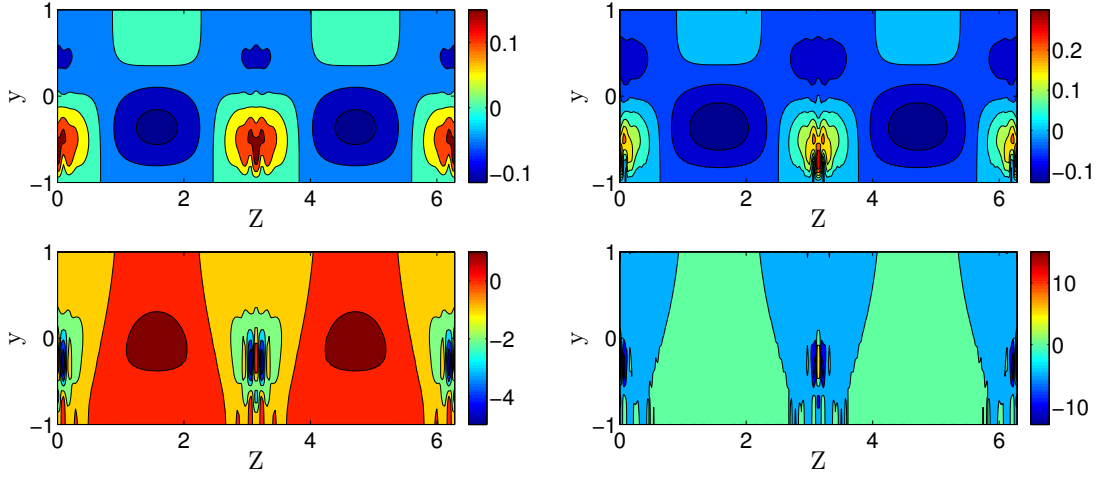
(a) Bifurcation diagram for Taylor vortices in $(T, \max|\hat{U}_T|)$ space.



(b) Contours of $\hat{U}_T$ for $T = 104.225$.

(c) Contours of $\hat{U}_T$ for $T = 115$.

Figure 2.26. Taylor vortex bifurcation.

(a) Contours of $\hat{U}_T$ and V_s for $T = 125$.(b) Contours of $\hat{U}_T$ and V_s for $T = 150$.**Figure 2.27.** Taylor vortices at very large amplitudes.

$\hat{U}_T$, and wall normal perturbation V_s for large amplitudes where quite extreme localisation is apparent in the solutions. To tidy up the solutions we have applied a high frequency filter to eliminate the high frequency noise resulting from the under-resolved high frequency modes.

In the next section when we add a neutral Tollmien-Schlichting-wave to the system. We will restrict our attention to Taylor numbers where the Taylor vortices have a moderate structure where computations can be performed with little difficulty.

2.10.2. Vortex-wave interaction in plane Poiseuille flow at non-zero Taylor numbers

In this section we will consider vortex-wave interaction in plane Poiseuille flow in the presence of wall curvature where the relevant scales of the curvature are defined as in section 2.10.1 but the spanwise wavelength is set to $\Lambda^{-1} = \epsilon^{-1} = Re^{1/7}$, a convenient asymptotic scale which leads to the vortex-wave interaction equations. The governing equations for this interaction were set out by Bennett, Hall & Smith (1991) and are similar to the interaction equations in a straight channel (2.38,2.56), except the presence

of curvature in the core equations alters (2.38) to

$$V_y + W_Z = 0, \quad (2.101a)$$

$$VU_y + WU_Z = 2 + U_{yy}, \quad (2.101b)$$

$$VW_{yy} + WW_{yZ} - 2TUU_Z = 2\alpha^2(|A|)_Z^2 UU_y + W_{yyy}, \quad (2.101c)$$

with no slip boundary conditions

$$U = V = W = 0, \quad y = \pm 1, \quad (2.101d)$$

and the symmetries in the wall normal direction are now broken. Breaking this symmetry changes the shear stress on the walls, and thus we must consider the pressure in each wall layer separately as in equations (2.88).

These equations, in a different form, were solved by Bennett, Hall & Smith (1991). For larger than critical Taylor numbers, those authors documented the variation in the neutral spanwise wavenumber and wavespeed of the Taylor vortex mode as the Taylor number varies. They also produced bifurcation diagrams for $\alpha(\mathcal{A})$ for fixed Taylor numbers, and again for the Taylor vortex mode. For subcritical Taylor numbers, they performed a weakly nonlinear analysis. However, they were unable to find any solutions to the interaction problem for $T \approx T_c$ or $T < T_c$ using the numerical method they adopted, which involved a fixed-point iteration of the core equations (2.101) and wave problems (2.88). In this section we will expand on this work to include solutions at subcritical Taylor numbers and we will also document in some detail the effect on both the basic and vortex mode bifurcations as the Taylor number increases but still below the point where localised long wavelength Taylor vortices are found.

These equations will be solved here using the same method employed for the vortex-wave interaction equations in a channel with straight walls, which is described in section 2.7.1. We will only consider the interaction for unit scaled spanwise wavenumber. From section 2.10.1 we know that, of states which bifurcate from the basic state, the only small amplitude solution for $T < T_c \approx 104.223$ is the basic state, and at $T = T_c$ there is a supercritical bifurcation into Taylor vortices so for $T > T_c$ there is the basic state plus the Taylor vortex which provides more than one state which we can bifurcate from as we increase $\mathcal{A}$ from zero. That is not to say that the presence of curvature does not affect the bifurcation for subcritical Taylor numbers, and we will focus on this first before we consider the changes resulting from the presence of Taylor vortices, around $T = T_c$.

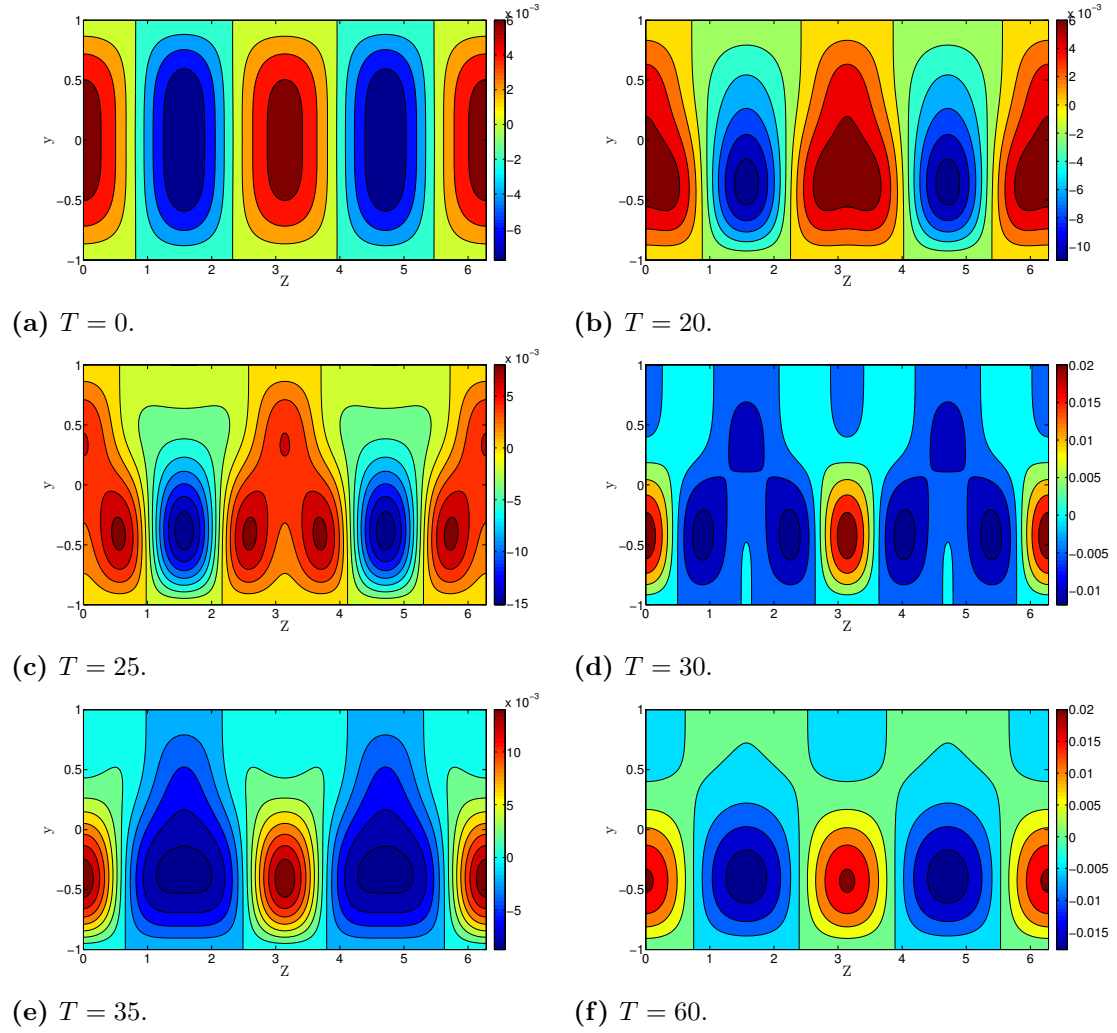


Figure 2.28. Contours of the perturbation to the streak for various Taylor numbers (given in subcaptions), when the Tollmien-Schlichting-wave has size $\mathcal{A}^2 = 2$.

First consider figure 2.28 which shows the perturbation to the streak when the wave is of size $\mathcal{A}^2 = 2$ and we vary the Taylor number to see how the perturbation to the streak changes as the Taylor number varies. When $T = 0$ this corresponds to the flat wall problem for plane Poiseuille flow - see figure 2.10a. For large T where the flow is becoming increasingly dominated by the curvature, the activity is focused near the lower wall. Between these two we see some interesting structures as the solution transitions between the two extremes, even though the Taylor number here is below the critical Taylor number.

We will now consider the effect of wall curvature on the bifurcation curve for the vortex-wave interaction states which bifurcate from the basic state. Figure 2.29 shows the bifurcation in $(\alpha, \mathcal{A})$ space from the linear neutral point for Taylor numbers $T = 0$, corresponding to vortex-wave interaction states in a channel with straight walls, and also $T = 30$ and $T = 60$, representing channels with curved walls. As one might expect, for small T the bifurcation curve is not so different from the $T = 0$ curve. However, at for example $T = 30$ where there is now a significant amount of wall curvature we see that there is now a turning point in the bifurcation curve. Actually the solution here seems to be approaching a period doubling bifurcation as can be seen in figure 2.30, but it was not possible to track this bifurcation curve any further than this point due to the numerical method failing to converge.⁷ For $T = 60$, we must now reach larger amplitudes of the wave before a turning point is reached. This branch was tracked to a fairly large amplitude before it was stopped due to the need to increase the resolution based on the emergence of small scale structures in the spanwise direction as can be seen in figure 2.31. These two examples show different behaviour depending on the value of the Taylor number, i.e. depending on the level of wall curvature.

Next we will consider the effect that the existence of Taylor vortices has on the bifurcation from the TS linear neutral point into mixed TS/Taylor vortex states. From section 2.10.1 we know that the bifurcation into Taylor vortices occurs at a critical Taylor number, denoted by T_c , where $T_c \approx 104.2$. So for $T < T_c$ there is only one branch bifurcating from the linear neutral point. However, for $T > T_c$ on top of the basic state there is also the possibility of the Taylor vortex at $\mathcal{A} = 0$. Thus for $T > T_c$ we must consider the bifurcation from the basic state, and also from the Taylor vortex state. Figure 2.32 shows how the bifurcation changes as T passes T_c . For $T < T_c$ there is a unique basic state and as we see in figures 2.29 and 2.32a, this unique pure TS state

⁷Attempts to track this curve further, by for example reducing the step size or reversing the search direction (to decreasing $\mathcal{A}$), have failed.

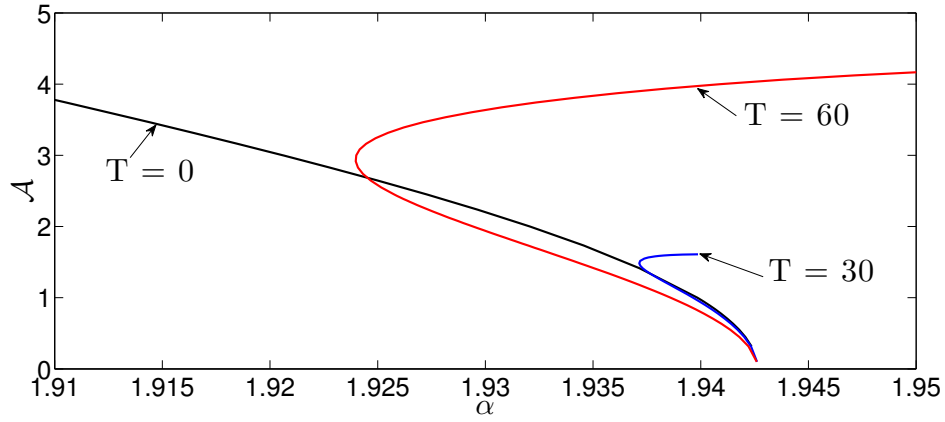
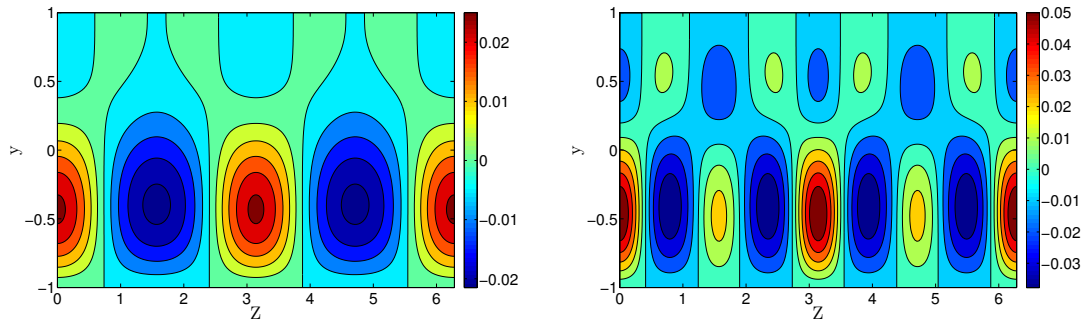


Figure 2.29. Bifurcations of the pure TS modes for various Taylor numbers less than the critical Taylor number, $T < T_c$.



(a) $A^2 = 2.5$.

(b) $A^2 = 2.59$.

Figure 2.30. Contours of the perturbation to the streak at a Taylor number of $T = 30$ near the end of the computed branch suggest a possible period doubling bifurcation, although this could not be found.

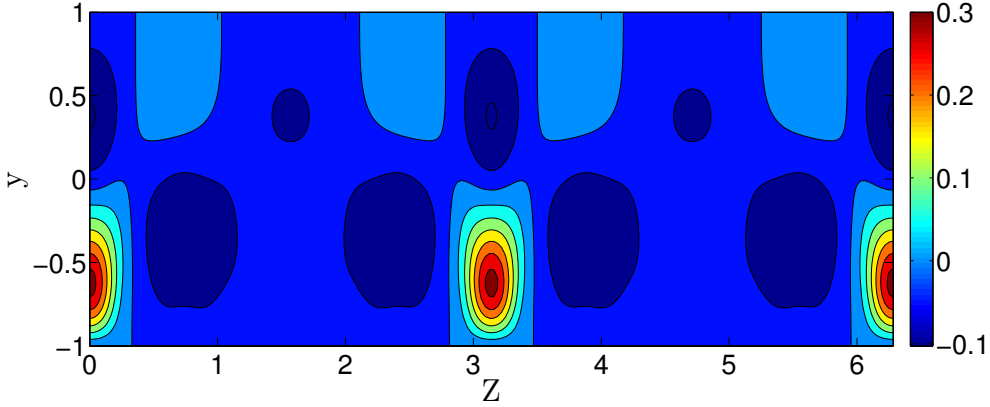
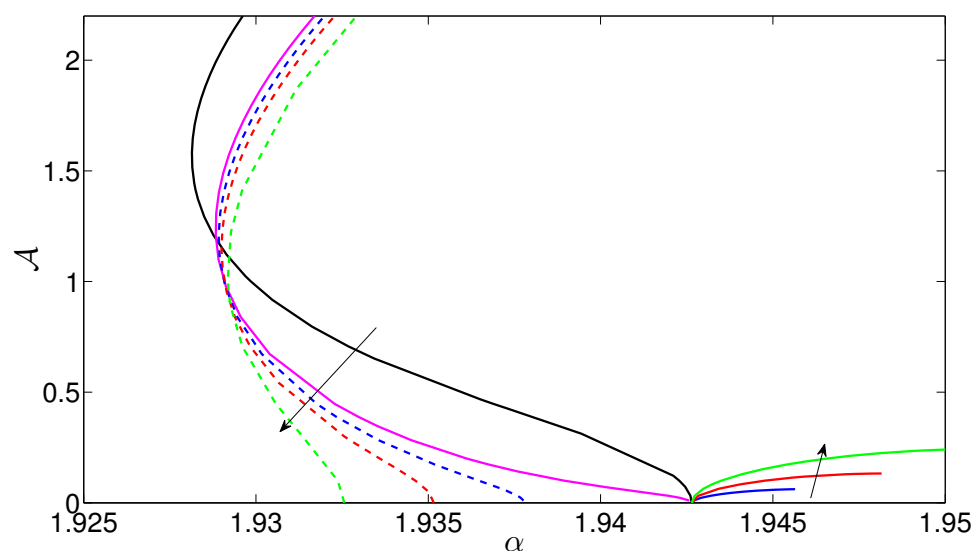


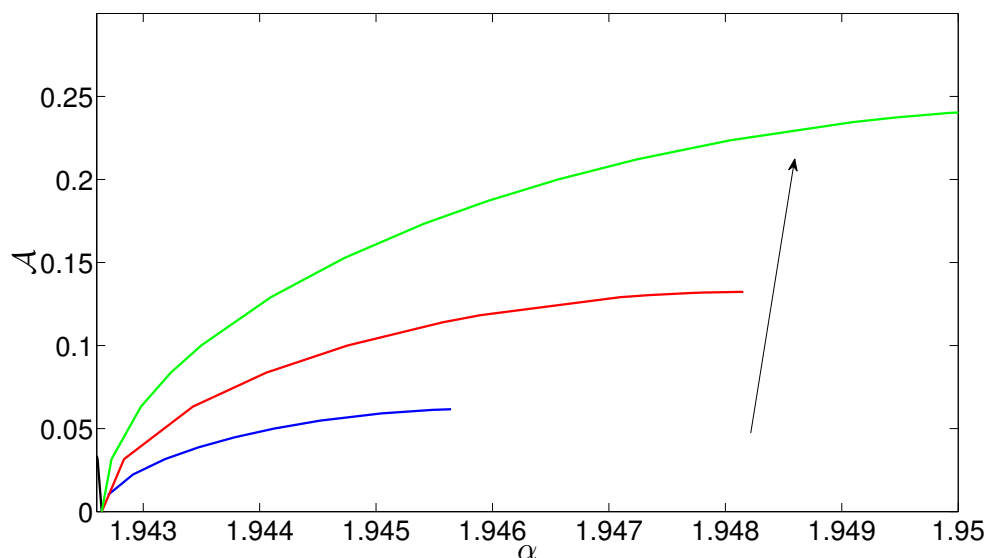
Figure 2.31. Contours of the perturbation to the streak at a Taylor number of $T = 60$ when the wave is of size $\mathcal{A}^2 = 18$.

bifurcates subcritically from its linear neutral point. As $T \rightarrow T_c^-$ this bifurcation curve appears to flatten with $\mathcal{A}'(\alpha) \rightarrow 0$. For $T > T_c$ the situation is very different with two distinct states at $\mathcal{A} = 0$, the basic state and the Taylor vortex. As can be seen from figure 2.32a the bifurcation of the pure TS state changes from subcritical to supercritical as T crosses T_c , and this can be seen from a modification of the argument within Bennett, Hall & Smith (1991) which considers the weakly nonlinear analysis as the Taylor number varies⁸. The pure TS states for $T > T_c$ as seen in figure 2.32b have been tracked as far as our numerical method would allow. This turns out to be to an amplitude where $\mathcal{A}'(\alpha) = 0$ although there does not appear to be a theoretical explanation for why the states would terminate or change dramatically around such a point. It could be an artefact of the numerical method applied, tracking in $\mathcal{A}$, and perhaps tracking the solutions in α would be more fruitful. However, efforts were made using the current method to track these states further, looking for solutions with amplitudes above, below and equal to the maximum amplitude seen in the figure but still these branches could not be continued. Note that the distance these states can be tracked from the linear neutral point decreases as the Taylor number decreases towards T_c which could be important in determining the fate of these states. Given the effort that has been put into finding further solutions past this point, it is likely that whatever bifurcation occurs here cannot be captured by the numerical method applied here. This could be a bifurcation to a wavy vortex mode, or a breaking of the spanwise symmetries imposed

⁸Those authors used this analysis to suggest that there is a change in the criticality of the bifurcation for $T = T_c/4$. However, altering their argument in fact shows that there is a change in the criticality of the bifurcation at $T = T_c$.



(a) Bifurcations for $T = 100$ (black) and $T = 104$ (magenta), where $T < T_c$ and hence there is only the pure TS mode (solid curve). Bifurcations for $T = 104.5$ (blue), $T = 105$ (red) and $T = 106$ (green) where $T > T_c$ and hence for each Taylor number there exists the pure TS mode (solid curve) and the Taylor vortex mode (dashed curve). The arrows indicate the effect of increasing the Taylor number.



(b) Zoomed into supercritical bifurcations for the pure TS mode when $T > T_c$. The arrow indicates the effect of increasing the Taylor number.

Figure 2.32. The effect of varying the Taylor number on the bifurcation of pure TS states (solid curves) and mixed TS/Taylor vortex states (dashed curves).

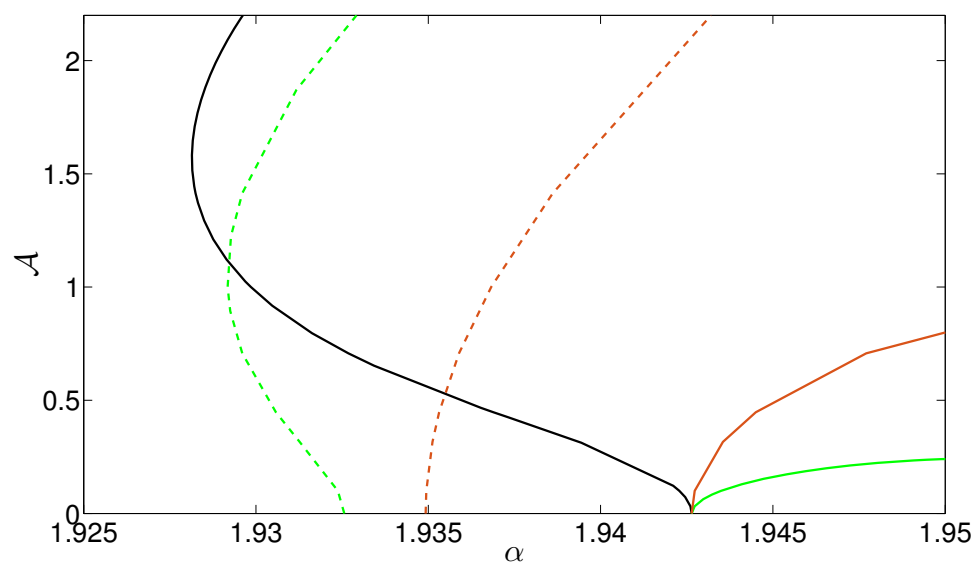


Figure 2.33. Changing the Taylor number over a larger range shows that eventually both the pure TS states and the mixed TS/Taylor vortex states bifurcate supercritically from their linear neutral points. The plot shows pure TS modes (solid curves) and mixed TS/Taylor vortex modes (dashed curves) for $T = 100 < T_c$ (black), $T = 106 > T_c$ (green) and $T = 120 > T_c$ (red).

by the spectral expansion of the flow variables. Alternatively, it could be a singularity in the vortex-wave interaction equations.

Now consider the mixed TS/Taylor vortex mode which exists for $T > T_c$. Here the Taylor vortex is an $O(1)$ perturbation to the basic state, and thus the linear neutral point is different to that for the basic state. Figure 2.32a shows that for $T > T_c$ but close to T_c , the bifurcation is subcritical and the linear neutral point decreases as the Taylor number increases. For fixed T , increasing the amplitude $\mathcal{A}$ we encounter a turning point, and increasing the amplitude further the flow field appears to localise as in figure 2.31. However, we see that as the Taylor number is increased, $|\mathcal{A}\mathcal{A}'|(\mathcal{A} = 0)$ increases⁹ until at some critical Taylor number the mixed TS/Taylor vortex state changes criticality so it also bifurcates supercritically as can be seen in figure 2.33. Thus for sufficiently large Taylor numbers we expect both the pure TS state and the mixed TS/Taylor vortex state to bifurcate supercritically from their respective linear neutral points.

2.11. Conclusions

In this chapter we have formulated the governing equations for the strongly nonlinear interaction between a vortex and a viscous three-dimensional Tollmien-Schlichting wave, as can be found in Bennett, Hall & Smith (1991). We have provided a much more comprehensive study of these equations even when we have considered them in the same context as those authors. Where they relied on fixed-point iterative methods, we were able to make much more progress through the use of a more reliable multi-dimensional Newton method. The states which have been computed are strongly nonlinear in that they are an $O(1)$ perturbation from the basic state. First, we considered these states in plane Poiseuille flow for unit scaled spanwise wavenumber where they are found to bifurcate subcritically from the linear neutral point into finite amplitude states. As the amplitude of the wave is increased, and highly nonlinear interactions arise, the states appear to localise at specific spanwise locations. A local analysis of the interaction equations around these spanwise locations reveals no explanation for the spanwise focusing of the states, although an extension of this analysis in Dempsey *et al.* (2016) suggests that the solutions around these local points lose regularity as the amplitude increases. Computations on the Navier-Stokes equations confirm that the states described by the vortex-wave interaction equations are indeed the large Reynolds number limit

⁹ $\mathcal{A}^2 \sim a(T)(\alpha - \alpha_c)$ for small $\mathcal{A}$ where a depends on the Taylor number T and α_c is the linear neutral value of the streamwise wavenumber. Then $\mathcal{A}' \sim a^{1/2}(\alpha - \alpha_c)^{-1/2}/2 = a/(2\mathcal{A})$ and thus a sensible measure for the size of $a(T)$ is $2\mathcal{A}\mathcal{A}'$.

of finite Reynolds number nonlinear travelling wave states described by the full Navier-Stokes equations - see Dempsey *et al.* (2016). The behaviour was found to be the same for general spanwise wavenumbers, the solution rapidly approaching a large spanwise wavenumber limit - the finite spanwise wavenumber correction to the large spanwise wavenumber limit equations is $O(\beta^{-14})$ which decays to zero very quickly. Solutions to the large spanwise wavenumber equations were also found to have the same behaviour with spanwise focusing, and thus we can conclude that the situation observed for unit spanwise wavenumbers is generic.

We moved on to consider the case of vortex-wave interaction in plane Poiseuille-Couette flow. As outlined by Cowley & Smith (1979), neutral TS waves only have the same structure as in the case of plane Poiseuille flow if the wall sliding velocity is sufficiently small, by which we mean $O(Re^{-2/7})$. Interestingly then there are more linear neutral curves, one of which was found numerically and documented within this thesis, and hence more linear neutral points to bifurcate from. The interaction equations in this case were only considered for unit scaled spanwise wavenumbers. In this case much the same behaviour as in plane Poiseuille was observed with spanwise focusing as the amplitude of the wave is increased. However, interestingly for wall speeds close to the critical values where more neutral modes appear, we observe closed curves in $(\alpha, \mathcal{A})$ space, i.e. the new neutral points are connected through finite amplitude states.

Finally, we considered the situation which was originally the focus of Bennett, Hall & Smith (1991), this being the strongly nonlinear interaction in a curved channel. We only considered this interaction for unit scaled spanwise wavenumbers. For less than critical Taylor numbers, the bifurcation proceeds as in plane Poiseuille flow with a subcritical bifurcation from the linear neutral point and spanwise focusing. However the presence of curvature causes structures to focus near the lower wall, and also a turning point is encountered in the bifurcation curve $\mathcal{A}(\alpha)$ at a finite amplitude. For greater than critical Taylor numbers, the bifurcation curve from the basic state of pure TS type, now appears to terminate at a finite amplitude (dependent upon the Taylor number). Interestingly, this termination does not appear to correspond to anything in particular - the states appear to be completely regular near the alleged termination amplitudes. The mixed TS/Taylor vortex state bifurcate from their linear neutral points, and can be tracked to finite amplitudes where they are observed to localise in the spanwise direction. For an interval of Taylor numbers, starting from T_c , the bifurcation is subcritical and then at a critical Taylor number the bifurcation changes criticality, and it appears to be supercritical for all Taylor numbers greater than this second critical Taylor number.

A recurring theme throughout this work is the spanwise focusing of the finite amplitude states. This appears to be a consequence of the lack of spanwise diffusion, which acts over an $O(1)$ spanwise scale. At sufficiently large amplitudes then it appears we must consider two problems, one on the large spanwise scale where $Z \sim O(1)$ and the flow is governed by the nonlinear vortex-wave interaction equations given here. This must be coupled to a set of equations which govern the interaction over an $O(1)$ spanwise scale z near the localisation points where spanwise diffusion is needed to smooth out the irregularity found in the large scale dynamics.

3. Vortex-wave interaction in the asymptotic suction boundary layer

3.1. Introduction

In this chapter we will consider the strongly nonlinear interaction between a vortex flow and a finite amplitude lower branch viscous Tollmien-Schlichting-wave in the asymptotic suction boundary layer. We will consider the interaction at large Reynolds numbers giving rise to an asymptotic description of nonlinear travelling wave equilibrium states of the Navier-Stokes equations. Firstly we will introduce the asymptotic suction boundary layer. Consider the flow of a viscous incompressible fluid, with density ρ and kinematic viscosity ν , over an infinite flat plate positioned on $\hat{y} = 0$ for $\hat{x} \in \mathbb{R}$ with constant wall suction V_0 and freestream streamwise velocity U_0 , where U_0 and V_0 are positive. In this setup $\hat{x}$ is the streamwise direction, $\hat{y}$ is the wall normal direction and $\hat{z}$ is the spanwise direction, with corresponding velocities $\hat{\mathbf{u}} = (\hat{u}, \hat{v}, \hat{w})$. This flow is governed by the incompressible Navier-Stokes equations

$$\hat{\nabla} \cdot \hat{\mathbf{u}} = 0, \tag{3.1a}$$

$$\hat{\mathbf{u}}_t + (\hat{\mathbf{u}} \cdot \hat{\nabla})\hat{\mathbf{u}} = -\frac{1}{\rho}\hat{\nabla}\hat{p} + \nu\hat{\nabla}^2\hat{\mathbf{u}}, \tag{3.1b}$$

with no slip boundary conditions on the plate

$$\hat{\mathbf{u}} = (0, -V_0, 0), \quad \hat{y} = 0, \tag{3.1c}$$

and far field matching conditions with the freestream velocity,

$$\hat{\mathbf{u}} \rightarrow (U_0, -V_0, 0), \quad \hat{y} \rightarrow \infty. \tag{3.1d}$$

In the above equations hats denote dimensional variables. The simplest solution to these equations which is steady and fully developed ($\partial_t = \partial_x = 0$) is when the wall suction

gives rise to the uniform wall normal velocity $\hat{v} = -V_0$, with no spanwise perturbation to the velocity and zero pressure gradient $\hat{w} = \hat{p} = 0$, and then we find the stream-wise velocity profile $\hat{u} = U_0\{1 - \exp(-V_0\hat{y}/\nu)\}$. This solution identifies characteristic scales motivating the non-dimensionalisation of the Navier-Stokes equations using the characteristic velocity scale U_0 , length scale ν/V_0 , time scale $\nu/(U_0V_0)$ and pressure scale ρU_0^2 . With these scales the Reynolds number is found to be $Re \equiv U_0/V_0$, and the Navier-Stokes equations become

$$\nabla \cdot \mathbf{u} = 0, \quad (3.2a)$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + Re^{-1}\nabla^2\mathbf{u}, \quad (3.2b)$$

with no slip boundary conditions on the plate

$$\mathbf{u} = (0, -Re^{-1}, 0), \quad y = 0, \quad (3.2c)$$

and far field matching conditions,

$$\mathbf{u} \rightarrow (1, -Re^{-1}, 0), \quad y \rightarrow \infty. \quad (3.2d)$$

The basic state written in terms of non-dimensional coordinates is

$$\mathbf{u} = (u_b, Re^{-1}v_b, 0), \quad p = 0, \quad (3.3a)$$

where

$$u_b = 1 - e^{-y}, \quad v_b = -1. \quad (3.3b)$$

The primary objective of this chapter is to find other solutions to equations (3.2) aside from the basic state (3.3). We will do this by tracking travelling wave equilibrium states which bifurcate from the basic state via the viscous linear instability that exists for sufficiently large Reynolds numbers. This can be done at finite Reynolds numbers by looking for travelling wave equilibrium states in the full Navier-Stokes equations, but here we will consider the problem of computing these states at asymptotically large Reynolds numbers where we can identify the leading order dynamics allowing us to gain an understanding of how the state is self sustained. Considering small amplitude states we see that these states must arise through a linear instability of the basic state. More precisely they sit on the linear stability/instability boundary which is known as the linear neutral curve. The basic state has two such linear neutral curves referred to as the upper and lower branch neutral curves and of these we will concentrate on the lower

branch where the so called critical layer sits within a thin viscous wall layer.

The first step is to deduce the asymptotic properties of the lower branch linear neutral modes. Once this is known we can increase the amplitude of the wave until nonlinear effects come into play and the interaction is within the strongly nonlinear regime. The structure for the interaction can be seen to follow from the interaction described by Smith & Walton (1989), but for completeness we will provide a derivation of the key scales and equations here.

To end this section we will provide a brief summary of the chapter. First we consider the asymptotic description of the lower branch neutral curve in section 3.2 and then in section 3.3 we increase the size of the wave so the interaction is strongly nonlinear. Following the derivation of the interaction equations we will construct weakly nonlinear solutions describing how these states bifurcate from the linear neutral point when the amplitude is small but finite. Finally we consider computational solutions of the fully nonlinear problem in the strongly nonlinear regime and discuss the fate of these states as the amplitude is increased.

3.2. Linear stability of the asymptotic suction boundary layer at large Reynolds numbers: lower branch

In this section we will focus on the asymptotic description of the lower branch linear neutral curve. It turns out that the lower branch linear neutral modes at large Reynolds numbers have a triple deck structure with a main deck of $O(1)$ extent in the wall normal direction. We will now motivate the scales for these modes. Due to the linear nature of the system to be considered we will let the size of the perturbation to the streamwise velocity be of some prescribed arbitrarily small size $O(\sigma)$, and we will consider the sizes of the other perturbations in relation to the streamwise velocity perturbation. In the argument that follows we will use subscripts L , M and U to denote the perturbation to the basic state in the lower, main and upper decks respectively. We assume the disturbances possess a streamwise wavenumber α , a spanwise wavenumber β and a wavespeed c , all of which are real. For convenience, in the discussion which follows we let ϵ denote the size of the streamwise wavenumber α .

Now consider the main deck where $y \sim O(1)$. The continuity equation $u_x \sim v_y$ implies that $v_M \sim \epsilon$. Then the spanwise momentum equation $uw_x \sim p_z$ yields that $w_M \sim \beta p_M / \epsilon$. It turns out that the dominant balance in the wall normal momentum

equations is just $p_y = 0$, and thus the pressure is a constant throughout the main deck. By matching the pressure with the upper and lower decks we conclude that the pressure is of the same order throughout the whole structure, i.e. $p_L \sim p_M \sim p_U$. As we will find when we set out the governing equations in the main deck, the streamwise velocity is $u_U = Au'_b$ and the wall normal velocity is $v_U = -i\alpha Au_b$ where A is a constant to be determined following an analysis of the upper and lower decks. Thus as $y \rightarrow \infty$ we find the wall normal velocity tends to $-i\alpha A$, and as $y \rightarrow 0$ the streamwise velocity tends to A .

Now consider the upper deck whose size is to be determined. In the upper deck the basic state velocity is $(1, 0, 0)$ at leading order, and thus $u\partial_x \sim \epsilon$. From the matching condition with the main deck we find $v_U \sim \epsilon$. The momentum equations $u\partial_x(u, v, w) \sim \nabla p_M$ reveal that $p_M \sim u_U, \epsilon v_U y_U, \epsilon w_U / \beta$ respectively. From the continuity equation $u_x \sim v_y \sim w_z$ we find $\epsilon u_U \sim v_U / y_U \sim \beta w_U$, and using the momentum equations balances $\epsilon \sim 1/(\epsilon y_U^2) \sim \beta^2 / \epsilon$. These balances yield $\beta \sim \epsilon$ and $y_U \sim \epsilon^{-1}$. Substituting this back into the momentum equations and using the matching condition with the main deck we find $u_U, v_U, w_U \sim \epsilon$ in the upper deck, and the pressure throughout the whole structure is $p_M \sim \epsilon$.

Finally to close the system consider the lower deck. Taylor expanding the basic state for small y we find $u_b \sim y_L$. From the main deck the streamwise velocity is $u_L \sim 1$. The continuity equation reveals the balances $\epsilon \sim v_L / y_L \sim \beta w_L$ from which we conclude that $w_L \sim 1$ and $v_L \sim \epsilon y_L$. From matching with the main deck $v_L \sim \epsilon^2$ implies that $y_L \sim \epsilon$. From the streamwise momentum equation $u_t \sim uu_x \sim p_x \sim Re^{-1} u_{yy}$ we find $c \sim \epsilon^2 \sim Re^{-1} / y_L^2$, and thus we have the scaling for the wavespeed $c \sim \epsilon^2$ and the size of the lower deck scaling is $y_L \sim \epsilon$. This argument uniquely determines $\epsilon(Re)$ to be

$$\epsilon = Re^{-1/4}. \quad (3.4)$$

Recall the discussion at the end of section 1.2. The above scaling argument differs from the usual triple deck argument only in that here we pick the length scale such that the boundary layer has size $O(1)$ on the chosen length scale. This formulation is equivalent to the triple deck formulation, albeit with a different Reynolds number (discussed in section 1.2). The scalings are essentially due to Lin (1945, 1955) and also see Hughes & Reid (1965) for further discussion.

For convenience we define the relevant streamwise, spanwise and temporal scales for

the lower branch Tollmien-Schlichting wave which consists of the slow variables

$$(X, Z) = \epsilon(x, z), \quad T = \epsilon^2 t. \quad (3.5)$$

Then the travelling wave component can be conveniently written in terms of these slow variables as

$$E_0 = \exp\{i\alpha(X - cT) + i\beta Z\}. \quad (3.6)$$

Note that although we discussed the scalings in terms of the wavenumbers, we have kept the wavenumbers as $O(1)$ quantities by defining the slow variables X, Z and T . Now we will consider the expansions discussed above in each deck.

The main deck expansions

Within the main deck where $y \sim O(1)$ the flow variables expand as

$$u = u_b + \sigma[u_{11}E_0 + c.c.] + \dots, \quad (3.7a)$$

$$v = \epsilon^4 v_b + \sigma\epsilon[v_{11}E_0 + c.c.] + \dots, \quad (3.7b)$$

$$w = \sigma\epsilon[w_{11}E_0 + c.c.] + \dots, \quad (3.7c)$$

$$p = \sigma\epsilon[p_{11}E_0 + c.c.] + \dots, \quad (3.7d)$$

where σ is a sufficiently small parameter that all nonlinear terms can be ignored. Substituting (3.7) into the Navier-Stokes equations (3.2), we find that the governing equations for the perturbations in the main deck are,

$$i\alpha u_{11} + v_{11}y = 0, \quad (3.8a)$$

$$i\alpha u_b u_{11} + u_b' v_{11} = 0, \quad (3.8b)$$

$$i\alpha u_b w_{11} + i\beta p_{11} = 0, \quad (3.8c)$$

where p_{11} is a constant. It is worth mentioning here that equations (3.8) are quasi-steady because the wavespeed is small. As stated before these equations have the solution

$$u_{11} = Au_b', \quad v_{11} = -i\alpha Au_b, \quad w_{11} = -\frac{\beta}{\alpha u_b} p_{11}. \quad (3.9)$$

where A is a constant to be determined. Recall that $u_{11} = A$ on $y = 0$, and $v_{11} \rightarrow -i\alpha A$ as $y \rightarrow \infty$ which motivated the need for both upper and lower decks, layers over which the velocity components alter to match the boundary conditions over the whole domain.

The upper deck expansions

From the scaling analysis we introduce the upper deck wall normal variable $\bar{Y} = \epsilon^{-1}y$, and then the flow variables expand as

$$u = 1 + \sigma\epsilon[\tilde{u}_1 E_0 + c.c.] + \dots, \quad (3.10a)$$

$$v = \epsilon^4 v_b + \sigma\epsilon[\tilde{v}_1 E_0 + c.c.] + \dots, \quad (3.10b)$$

$$w = 1 + \sigma\epsilon[\tilde{w}_1 E_0 + c.c.] + \dots, \quad (3.10c)$$

$$p = 1 + \sigma\epsilon[\tilde{p}_1 E_0 + c.c.] + \dots \quad (3.10d)$$

Substituting the expansions (3.10) into equations (3.2) we find the governing equations in this layer are,

$$i\alpha\tilde{u}_1 + \tilde{v}_{1\bar{Y}} + i\beta\tilde{p}_1 = 0, \quad (3.11a)$$

$$i\alpha\tilde{u}_1 + i\alpha\tilde{p}_1 = 0, \quad (3.11b)$$

$$i\alpha\tilde{v}_1 + \tilde{p}_{1\bar{Y}} = 0, \quad (3.11c)$$

$$i\alpha\tilde{w}_1 + i\beta\tilde{p}_1 = 0. \quad (3.11d)$$

We note here that, as we found for the main deck equations (3.8), the upper deck equations (3.11) are also quasi-steady due to the small wavespeed. Combining these equations to eliminate $\tilde{u}_1, \tilde{v}_1$ and $\tilde{w}_1$ leaves a single equation for the pressure $\tilde{p}_1$,

$$\tilde{p}_{1\bar{Y}\bar{Y}} - \chi^2 \tilde{p}_1 = 0, \quad (3.12)$$

where we have defined $\chi \in \mathbb{R}_+$ to be

$$\chi = \sqrt{\alpha^2 + \beta^2}. \quad (3.13)$$

The general solution to this equation is a linear combination of $e^{\chi\bar{Y}}$ and $e^{-\chi\bar{Y}}$. As $\bar{Y} \rightarrow \infty$, the perturbations should decay, i.e. we require $\tilde{p}_1 \rightarrow 0$ as $\bar{Y} \rightarrow \infty$. We find the solution is $\tilde{p}_1 = \tilde{P}_1 e^{-\chi\bar{Y}}$ where $\tilde{P}_1$ is a constant to be determined. Matching with the main deck pressure and the wall normal velocity we have a further two matching conditions

$$\tilde{p}_1 = p_{11}, \quad \tilde{p}_{1\bar{Y}} = -\alpha^2 A, \quad Y = 0. \quad (3.14)$$

This not only determines the pressure in the upper deck $\tilde{p}_1$, but also gives a relationship between the pressure p_{11} and the displacement A in the main deck. The first of these conditions gives $\tilde{P}_1 = p_{11}$. Applying the second condition we find the pressure

displacement relation,

$$\chi p_{11} = \alpha^2 A. \quad (3.15)$$

This provides an extra equation between the pressure and displacement which is required to close the system when we consider the lower deck equations as we will do next.

The lower deck expansions

We introduce the lower deck wall normal variable $Y = \epsilon^{-1}y$ and motivated by the scaling analysis we expand the flow variables as

$$u = \epsilon Y + \sigma[u_1 E_0 + c.c.] + \dots, \quad (3.16a)$$

$$v = \epsilon^4 v_b + \sigma \epsilon^2[v_1 E_0 + c.c.] + \dots, \quad (3.16b)$$

$$w = \sigma[w_1 E_0 + c.c.] + \dots, \quad (3.16c)$$

$$p = \sigma \epsilon[p_1 E_0 + c.c.] + \dots \quad (3.16d)$$

Substituting the expansions (3.16) into the Navier-Stokes equations (3.2) reveals the governing equations for the perturbations in the lower deck

$$i\alpha u_1 + v_{1Y} + i\beta w_1 = 0, \quad (3.17a)$$

$$i\alpha(Y - c)u_1 + v_1 = -i\alpha p_1 + u_{1YY}, \quad (3.17b)$$

$$i\alpha(Y - c)w_1 = -i\beta p_1 + w_{1YY}, \quad (3.17c)$$

where p_1 is a constant, and matching with the main deck we find $p_1 = p_{11}$. Note that in comparison to the main deck equations (3.8) and the upper deck equations (3.11), the lower deck equations (3.17) are unsteady. This is to be expected because the wavespeed is small and hence temporal effects are only important when the velocities are small which occurs near the wall where they must satisfy the no slip boundary conditions. These equations are to be solved subject to no slip boundary conditions on the plate,

$$u_1 = v_1 = w_1 = 0, \quad Y = 0, \quad (3.17d)$$

and far field matching conditions with the main deck,

$$u_1 \rightarrow A, \quad w_1 \rightarrow 0, \quad Y \rightarrow \infty. \quad (3.17e)$$

The system of equations given by (3.17) can be manipulated following Smith (1979) to

yield a single relationship between the pressure and the displacement,

$$-\chi^2 p_{11} = \alpha^{5/3} \mathcal{G}_0 A. \quad (3.18)$$

In the above we have defined

$$\xi_0 = -i^{1/3} \alpha^{1/3} c, \quad \kappa_0 = \int_{\xi_0}^{\infty} \text{Ai}(s) ds, \quad \text{Ai}'_0 = \text{Ai}'(\xi_0), \quad \mathcal{G}_0 = i^{5/3} \frac{\text{Ai}'_0}{\kappa_0}. \quad (3.19)$$

The linear dispersion relation

Combining the two equations relating the pressure and the displacement, (3.15) and (3.18), we find the linear dispersion relation $\mathcal{D}(\alpha, c, \beta) = 0$ where

$$\mathcal{D}(\alpha, c, \beta) \equiv \mathcal{G}_0 + \alpha^{1/3} \chi, \quad (3.20)$$

and recall χ is defined in (3.13).

This dispersion relation is the same as the dispersion relation for the Blasius boundary layer, albeit with a different shear stress. So we find that in fact the wall suction only affects the stability indirectly through the induced change on the wall shear stress. In fact we expect in general to find a similar dispersion for any flow over an infinite flat plate with non-zero wall shear stress and a non-zero flow in the freestream.

The dispersion relation (3.20) gives a single complex equation which given the spanwise number of interest β , can be solved for the two real unknowns α and c . This completes the linear part of the description of the lower branch neutral curve for large Reynolds numbers. To demonstrate the result, if for example we set $\beta = 1$ we can solve $\mathcal{D}(\alpha, c, 1) = 0$ to find $(\alpha, c) = (0.617, 2.698)$ to three decimal places. So as $Re \rightarrow \infty$ the lower branch neutral curve is described by $(\hat{\alpha}, \hat{c}) \sim (\alpha, c)\epsilon$ i.e.

$$(\hat{\alpha}, \hat{c}) \sim (0.617, 2.698) Re^{-1/4}, \quad (3.21)$$

where we have used hats to denote unscaled non-dimensional variables. For the majority of this work we will focus on $\beta = 1$, however for the purposes of comparing our results with previous work we will briefly consider two dimensional waves. For $\beta = 0$, we can solve the dispersion relation $\mathcal{D}(\alpha, c, 0) = 0$ to find $(\alpha, c) = (1.0005, 2.297)$, and hence

$$\hat{\alpha} \sim 1.0005 Re^{-1/4}. \quad (3.22)$$

This can be compared with the asymptotic behaviour found by Hughes & Reid (1965),

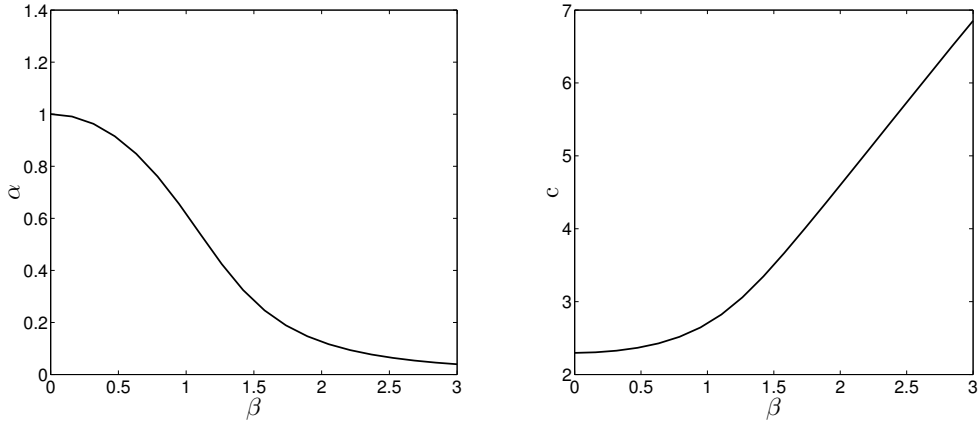


Figure 3.1. Solutions for the scaled streamwise wavenumber and wavespeed (α, c) to the dispersion relation $\mathcal{D}(\alpha, c, \beta) = 0$ for various scaled spanwise wavenumbers β .

$\hat{\alpha} \sim 1.0002Re^{-1/4}$, which is an error of approximately 0.03%. Figure 3.1 shows plots of the neutral streamwise wavenumber $\alpha(\beta)$ and the wavespeed $c(\beta)$ as the spanwise wavenumber varies.

3.3. Vortex-wave interaction in the asymptotic suction boundary layer

3.3.1. Setting up the interaction

In this section we will increase the size of the wave, denoted by σ , to the point where nonlinear effects are no longer negligible. In fact we will find that the self-interaction of the wave creates a strongly nonlinear interaction in that the vortex becomes an $O(1)$ perturbation from the basic state. Whereas in plane Poiseuille flow where the wave forcing on the vortex problem is directly into the core of the channel, it turns out that here the wave forcing actually manifests itself through a spanwise wall slip in the main deck problem.

This formulation is essentially the same as the nonlinear vortex/Tollmien-Schlichting wave interaction discussed in Hall & Smith (1991), with the different length scale taken into account leading to a different definition of the Reynolds number. In fact the governing equations derived in this section can be found in Hall & Smith (1991), equations (2.2-2.4). This nonlinear type of interaction can be seen as originating from the lower

amplitude work of for example Hall & Smith (1984) and Smith & Walton (1989). Although the scalings and equations could be inferred from these previous works, we will motivate them here. Although the equations are the same there are some subtle differences resulting from the particular structure here (since the wall normal velocity is non-zero in the freestream) which will be discussed in this section when we consider the main and upper decks. For example the size of the mean flow pressure correction here differs from the work of Smith & Walton (1989).

A complete derivation of the forcing term will be given shortly. However first let us try to understand this and derive the scalings for the nonlinear interaction. The wave scalings are determined from the linear stability analysis of section 3.2 which differs only from the fact that the wall shear stress is now a function of the spanwise coordinate Z , and then the only degree of freedom in the wave is the choice of σ whose size will now be determined. Consider the vortex in the main deck - recall this is the steady part of the flow. To have a full nonlinear interaction we require $vu_y \sim \epsilon^4 u_{yy}$ and thus $v \sim \epsilon^4$. Over the slow spanwise scale $z \sim O(\epsilon^{-1})$ this means that to balance $v_y \sim w_z$ we require $w \sim \epsilon^3$. Now let us consider the vortex part of the lower deck problem. Here the nonlinear self interaction of the wave drives the vortex flow directly, so if we consider the spanwise momentum equation then we find the wave forcing terms must balance with the wall normal diffusive term $[v_1^* w_1 z + c.c.] \sim \epsilon^4 w_{yy}$ i.e. $\epsilon \sigma^2 \sim \epsilon^2 w$ and thus we find the spanwise component of the vortex in the lower deck has size $w \sim \epsilon^{-1} \sigma^2$. The wave components decay algebraically away from the wall, and it turns out (as we will see shortly) that this causes the spanwise component of the vortex to grow logarithmically away from the wall. Thus coming away from the wall within the lower deck we find the spanwise component of the vortex is of the form $\epsilon^{-1} \sigma^2 \log \epsilon^{-1} = \epsilon^{-1} \sigma^2 (\log Re)/4$. Balancing this with the spanwise component of the vortex in the main deck which is $O(\epsilon^3)$ we find the critical size of the wave to be

$$\sigma = \epsilon^2 \delta, \quad (3.23)$$

where for convenience we have defined δ to be

$$\delta = 2(\log Re)^{-1/2}. \quad (3.24)$$

It is possible for these states to modulate over a slow streamwise lengthscale or slow timescale of $O(Re)^1$ but we will restrict our attention to structures which do not display

¹Slower than the wave scales.

this behaviour. Due to nonlinear effects we must now allow for much more complicated behaviour in the spanwise direction, and with this in mind we set the travelling wave component to be

$$E = e^{i\alpha(X-cT)}, \quad (3.25)$$

but we will still restrict our attention to solutions which are periodic in the spanwise direction with period $2\pi/\beta$ where β is the spanwise wavenumber which will be prescribed.

The main deck expansions

In the above discussion we have motivated the nonlinear extension to the expansions, so here we will just state the full expansions within the main deck which are

$$u = U + \epsilon^2 \delta[u_{11}E + c.c.] + \dots, \quad (3.26a)$$

$$v = \epsilon^3 \delta[v_{11}E + c.c.] + \epsilon^4 V + \dots, \quad (3.26b)$$

$$w = \epsilon^3 W + \epsilon^3 \delta[w_{11}E + c.c.] + \dots, \quad (3.26c)$$

$$p = \epsilon^3 \delta[p_{11}E + c.c.] + \dots \quad (3.26d)$$

Now $U(y, Z)$ is in general a perturbation to the basic state, but when the wave amplitude tends to zero, we recover the basic state u_b given in (3.3). Substituting the expansions (3.26) into the Navier-Stokes equations (3.2) we find the governing equations for the vortex are

$$V_y + W_Z = 0, \quad (3.27a)$$

$$VU_y + WU_Z = U_{yy}, \quad (3.27b)$$

$$VW_y + WW_Z = W_{yy}. \quad (3.27c)$$

In these equations (in comparison to the vortex-wave interaction equations in plane Poiseuille flow) there is no explicit wave forcing - this is a higher order effect with the current scalings. These equations are to be solved subject to far field decay conditions

$$U \rightarrow 1, \quad W \rightarrow 0, \quad y \rightarrow \infty, \quad (3.27d)$$

no slip boundary conditions, and as discussed the spanwise slip velocity

$$U = 0, \quad V = -1, \quad W = q(Z), \quad y = 0, \quad (3.27e)$$

with the spanwise slip velocity $q(Z)$ still to be determined from a detailed analysis of

the lower deck equations.

Substituting the expansions (3.26) into the Navier-Stokes equations (3.2), and considering the travelling wave component we find

$$i\alpha u_{11} + v_{11y} = 0, \quad (3.28a)$$

$$i\alpha U u_{11} + U_y v_{11} = 0, \quad (3.28b)$$

$$p_{11y} = 0, \quad (3.28c)$$

$$i\alpha U w_{11} + p_{11z} = 0, \quad (3.28d)$$

which is the strongly nonlinear analogue of equations (3.8), where U is in general perturbed from the basic state u_b . These equations can be solved up to an arbitrary function of the spanwise coordinate $A(Z)$, the displacement or amplitude function, which will be determined essentially by matching. The wall normal momentum equation yields $p_{11} = p_{11}(Z)$ and then it follows, similarly to (3.9), that

$$u_{11} = AU_y, \quad v_{11} = -i\alpha AU, \quad w_{11} = -\frac{1}{i\alpha U} p'_{11}. \quad (3.29)$$

As before this does not satisfy the boundary conditions throughout the whole domain, the only difference with the linear analysis being that the displacement function A now depends on the spanwise coordinate. In the nonlinear case the matching conditions now take account of the changes to the basic state, so $u_{11} = \lambda A$ on $y = 0$, where $\lambda(Z) = U_y(y = 0, Z)$. Also as $y \rightarrow \infty$, we find $v_{11} \rightarrow -i\alpha A$ as before.

Two things may seem odd about the mean flow equations (3.27). The first is that the mean pressure correction is not present, and wasn't included in the expansions (3.26). This is because the effects of the pressure are a higher order effect thus the pressure is not required to determine the dynamics of the system. For completeness we will discuss this a little here. If we consider wave forcing terms in the Z -momentum equation (3.27c), then we find they are $O(\delta^2 \epsilon^6)$ and are of the form

$$[i\alpha w_{11} u_{11}^* + w_{11y} v_{11}^* + c.c.] \quad (3.30)$$

which turns out to be zero using (3.29). This is important since the given equation (3.27c) is found at $O(\epsilon^7)$. Considering the y -momentum equation, we find the nonlinear wave forcing here is $O(\delta^2 \epsilon^6)$. This is non-zero and determines the size of the mean pressure correction to be

$$\delta^2 \epsilon^6 P. \quad (3.31)$$

Using (3.29) we find

$$-P_y = [i\alpha v_{11}u_{11}^* + v_{11}v_{11y}^* + c.c.] = 4\alpha^2|A|^2UU_y, \quad (3.32)$$

and thus $P = -2\alpha^2|A|^2U^2 + G(Z)$ where G is some as yet arbitrary function. This pressure has no effect on the leading order mean flow correction equations (3.27).

The second perhaps odd thing is that V does not decay to zero as y tends to infinity, and in general we find that V tends to some function of Z as y tends to infinity. This is because V alters to satisfy the far field condition throughout the upper deck. We will return to discuss these points again briefly in the next section when we consider the mean flow correction in the upper deck.

The upper deck expansions

In the upper deck we only need to consider the wave terms to close the system. The analysis is similar to the linear case, except now nonlinear effects come into play leading to a more complicated spanwise dependence. The flow variables expand as

$$u = 1 + \epsilon^3\delta[\tilde{u}_1E + c.c.] + \dots, \quad (3.33a)$$

$$v = \epsilon^3\delta[\tilde{v}_1E + c.c.] + \dots, \quad (3.33b)$$

$$w = \epsilon^3\delta[\tilde{w}_1E + c.c.] + \dots, \quad (3.33c)$$

$$p = \epsilon^3\delta[\tilde{p}_1E + c.c.] + \dots \quad (3.33d)$$

Substituting the expansions (3.33) into the Navier-Stokes equations (3.2), we find the wave components satisfy the linearised Euler equations,

$$i\alpha\tilde{u}_1 + \tilde{v}_{1\bar{Y}} + \tilde{w}_{1Z} = 0, \quad (3.34a)$$

$$i\alpha\tilde{u}_1 + i\alpha\tilde{p}_1 = 0, \quad (3.34b)$$

$$i\alpha\tilde{v}_1 + \tilde{p}_{1\bar{Y}} = 0, \quad (3.34c)$$

$$i\alpha\tilde{w}_1 + \tilde{p}_{1Z} = 0. \quad (3.34d)$$

Comparing this to the linear case given by equations (3.11), we see that the only difference is the more complicated spanwise dependence here due to nonlinear effects elsewhere in the flow. Eliminating the velocity variables we find the equation for the wave component of the pressure in the upper deck can be written in the form

$$\tilde{p}_{1\bar{Y}\bar{Y}} + \tilde{p}_{1ZZ} - \alpha^2\tilde{p}_1 = 0, \quad (3.35a)$$

with the far field condition,

$$\tilde{p}_1 \rightarrow 0, \quad \bar{Y} \rightarrow \infty. \quad (3.35b)$$

Matching with the main deck we find a further two conditions which are the matching conditions for the pressure and the wall normal velocity,

$$\tilde{p}_1 = p_{11}, \quad \tilde{p}_{1\bar{Y}} = -\alpha^2 A \quad \bar{Y} = 0. \quad (3.35c)$$

Comparing these equations to the linear stability problem given by (3.12) and (3.14), we see that this is also an overprescribed boundary value problem which not only determines the pressure (and hence velocities) but also determines the pressure-displacement ($p_{11} - A$) relation.

A study of the mean flow correction in the upper deck is not required to close the system but for completeness we will discuss it briefly here. In the last section we pointed out that in general V in the main deck does not satisfy the far field condition (which would require $V \rightarrow -1$ as $y \rightarrow \infty$) but instead we find V tends to some function of Z , say $V_\infty(Z)$. Then in the upper deck the wall normal velocity must alter from $v = \epsilon^4 V_\infty(Z)$ to satisfy the far field condition $v = -\epsilon^4$. Thus the mean correction to the wall normal velocity is $O(\epsilon^4)$. Since the length scales are equal in the upper deck, we find the mean correction to the spanwise velocity is also $O(\epsilon^4)$. This allows the wall normal velocity to alter throughout the upper deck to satisfy the far field condition.

Consider the size of the terms in the wall normal and spanwise momentum equations. Since we determined the size of the mean flow corrections to the roll flow, we find inertia terms are $O(\epsilon^9)$. However, the wave forcing terms are $O(\delta^2 \epsilon^7)$ and since this dominates the inertia terms, the nonlinear wave forcing terms must balance with the pressure terms which are $O(\epsilon[\text{pressure}])$. This requires the mean flow correction to the pressure to be $\delta^2 \epsilon^6 \tilde{P}$ which is consistent with the scale determined in the main deck, see equation 3.31. The governing equations for the mean flow correction terms (V, W) are determined at higher order, the leading order equations demonstrating only the balance between the nonlinear wave forcing and the pressure gradient. We find the pressure correction satisfies the equations

$$[i\alpha \tilde{v}_1 \tilde{u}_1^* + \tilde{v}_1 \tilde{v}_{1\bar{Y}}^* + \tilde{w}_1 \tilde{v}_{1Z}^* + c.c.] = -\tilde{P}_{\bar{Y}}, \quad (3.36)$$

$$[i\alpha \tilde{w}_1 \tilde{u}_1^* + \tilde{v}_1 \tilde{w}_{1\bar{Y}}^* + \tilde{w}_1 \tilde{w}_{1Z}^* + c.c.] = -\tilde{P}_{\bar{Z}}. \quad (3.37)$$

Using (3.34), the solution for $\tilde{P}$ is found to be

$$\tilde{P} = \tilde{P}_0 - \frac{\alpha^2 |\tilde{p}_1|^2 + |\tilde{p}_{1\bar{Y}}|^2 + |\tilde{p}_{1Z}|^2}{\alpha^2}, \quad (3.38)$$

where $\tilde{P}_0$ is a constant. As discussed before, this is not needed to close the system but it does demonstrate that the system is consistent.

The lower deck expansions

We have already discussed the lower deck expansions and here we will expand on this discussion which will allow us to determine the spanwise slip velocity denoted by $q(Z)$. In our previous discussion we motivated the scale of the spanwise velocity component, $w \sim \epsilon^{-1}\sigma^2 = \epsilon^3\delta^2$. Considering the continuity equation we require $v_y \sim w_z$ and thus $v \sim \epsilon^5\delta^2$. In the streamwise momentum equation the wave forcing terms are of size $[v_1 u_{1y} + c.c.] \sim \epsilon^5\delta^2$ and the viscous term is of size $\epsilon^4 u_{yy} \sim \epsilon^2 u$ from which we find that $u \sim \epsilon^3\delta^2$. Thus the flow variables in the lower deck are found to expand as

$$u = \epsilon\lambda(Z)Y - \epsilon^2\lambda(Z)Y/2 + \epsilon^2\delta[u_1E + c.c.] + \epsilon^3\delta^2u_v + \dots, \quad (3.39a)$$

$$v = -\epsilon^4 + \epsilon^4\delta[v_1E + c.c.] + \epsilon^5\delta^2v_v + \dots, \quad (3.39b)$$

$$w = \epsilon^2\delta[w_1E + c.c.] + \epsilon^3\delta^2w_v + \dots, \quad (3.39c)$$

$$p = \epsilon^3\delta[p_1E + c.c.] + \dots, \quad (3.39d)$$

where $\lambda(Z) = U_y(y=0, Z)$ denotes the shear stress on the wall from the main deck. Substituting the expansions (3.39) into the Navier-Stokes equations (3.2) we find the equations governing both the vortex and wave parts of the flow. Firstly the equations governing the vortex part are found to be

$$v_v Y + w_v Z = 0, \quad (3.40a)$$

$$\lambda(Z)v_v + \lambda'(Z)Yw_v + [v_1 u_{1Y}^* + w_1 u_{1Z}^* + c.c.] = u_v Y Y, \quad (3.40b)$$

$$[i\alpha w_1 u_1^* + v_1 w_{1Y}^* + w_1 w_{1Z}^* + c.c.] = w_v Y Y, \quad (3.40c)$$

and these must be solved subject to no slip boundary conditions on the wall,

$$u_v = w_v = 0, \quad v_v = -1, \quad Y = 0, \quad (3.40d)$$

and far field matching conditions with the main deck which require careful consideration. Take for example the spanwise momentum equation (3.40c) and denote the wave forcing

term by $F_w(Y, Z)$, so

$$F_w(Y, Z) = [\mathrm{i}\alpha w_1 u_1^* + v_1 w_{1Y}^* + w_1 w_{1Z}^* + c.c.]. \quad (3.41)$$

Integrating (3.40c) we find w_{vY} is given by

$$w_{vY}(Y, Z) = w_{vY}(0, Z) + \int_0^Y F_w(s, Z) ds. \quad (3.42)$$

As we will see shortly, $F_w \sim Y^{-2}$ as $Y \rightarrow \infty$, and hence we conclude that

$$w_{vY} \sim \bar{a}(Z) + o(1), \quad (3.43)$$

where

$$\bar{a}(Z) = w_{vY}(0, Z) + \int_0^\infty F_w(s, Z) ds, \quad (3.44)$$

and this integral exists due to the far field behaviour of the integrand. In fact because F_w decays like Y^{-2} , this causes w_{vY} to decay like Y^{-1} and this drives logarithmic growth in w_v for large Y . Now consider $\bar{a}(Z)$. If $\bar{a}(Z)$ is non-zero then matching this term onto the core would be $\epsilon^{-1}\bar{a}(Z)y$ which would dominate the logarithmic growth term at $\log \epsilon^{-1}q(Z)$ and hence for a consistent balance (where the nonlinear self-interaction of the wave drives the roll flow) we require $\bar{a}(Z) = 0$. We find u_v and v_v grow like $Y^3 \log Y$ and $Y \log Y$ respectively, which do not impose a forcing on the leading order problem in the main deck and thus can be ignored in this analysis. The relevant far field condition for the spanwise vortex component in the lower deck is then

$$w_{vY} \rightarrow 0, \quad Y \rightarrow \infty. \quad (3.45)$$

The equations governing the wave component of the flow in the lower deck are

$$\mathrm{i}\alpha u_1 + v_{1Y} + w_{1Z} = 0, \quad (3.46a)$$

$$\mathrm{i}\alpha(\lambda(Z)Y - c)u_1 + \lambda(Z)v_1 + \lambda'(Z)Yw_1 = -\mathrm{i}\alpha p_1 + u_{1YY}, \quad (3.46b)$$

$$0 = -p_{1Y}, \quad (3.46c)$$

$$\mathrm{i}\alpha(\lambda(Z)Y - c)w_1 = -p_{1Z} + w_{1YY}, \quad (3.46d)$$

which should be compared with the analogous linear stability equations (3.17) where $\lambda \equiv 1$. These equations must be solved subject to no slip boundary conditions on the

wall,

$$u_1 = v_1 = w_1 = 0, \quad Y = 0, \quad (3.46e)$$

and far field matching conditions with the main deck wave components,

$$u_1 \rightarrow \lambda A, \quad w_1 \rightarrow 0, \quad Y \rightarrow \infty. \quad (3.46f)$$

Once again matching the pressure with the main deck we find $p_1 = p_{11}$.

Wave forcing: the logarithmic growth of the vortex in the lower deck

The next part of this section will be devoted to identifying the spanwise slip velocity for the main deck vortex problem. This essentially comes down to computing the large Y behaviour of the spanwise component of the vortex in the lower deck. As we see from the governing equation (3.40c), w_v is forced by the wave terms which we denote by F_w . Thus we must first work out the large Y behaviour of the wave terms. Consider the equations governing the wave components in the lower deck (3.46): then using the matching condition (3.46f) we find $u_1 \sim \lambda A + O(Y^{-1})$. Using the continuity equation (3.46a) and noting that $w_1 \rightarrow 0$ we find that $v_1 \sim -i\alpha\lambda AY + O(1)$. Then from the spanwise momentum equation (3.46c) we find $w_1 \sim -(i\alpha\lambda)^{-1}p_{1Z}Y^{-1} + O(Y^{-2})$. Posing such expansions we find that the first two terms in each expansion are

$$u_1 \sim \lambda A - \frac{1}{\alpha^2} \frac{\partial}{\partial Z} \left(\frac{1}{\lambda} \frac{\partial p_1}{\partial Z} \right) \frac{1}{Y} + \dots, \quad (3.47a)$$

$$v_1 \sim -i\alpha\lambda AY - \frac{1}{i\alpha\lambda} \left[\frac{\partial^2 p_1}{\partial Z^2} - \alpha^2 p_1 + \lambda\alpha^2 cA - \frac{2}{\lambda} \frac{\partial \lambda}{\partial Z} \frac{\partial p_1}{\partial Z} \right] + \dots, \quad (3.47b)$$

$$w_1 \sim -\frac{1}{i\alpha\lambda} \frac{\partial p_1}{\partial Z} \frac{1}{Y} - \frac{c}{i\alpha\lambda^2} \frac{\partial p_1}{\partial Z} \frac{1}{Y^2} + \dots \quad (3.47c)$$

Now we can consider the large Y behaviour of w_v given the governing equation $w_{vYY} = F_w$ where F_w is defined in (3.41). We find for large Y that

$$F_w \sim \frac{1}{\alpha^2 \lambda^2} \frac{d}{dZ} \left(\alpha^2 |p_1|^2 + \left| \frac{dp_1}{dZ} \right|^2 \right) Y^{-2} + O(Y^{-3}). \quad (3.48)$$

Recalling that $p_1 = p_{11}$ we conclude that $w_v \sim q(Z) \log Y$ where

$$q(Z) = -\frac{1}{\alpha^2 \lambda^2} \frac{d}{dZ} \left(\alpha^2 |p_{11}|^2 + \left| \frac{dp_{11}}{dZ} \right|^2 \right). \quad (3.49)$$

Hence the boundary conditions for the main deck vortex problem are

$$U = 0, \quad V = -1, \quad W = q(Z), \quad y = 0. \quad (3.50)$$

The lower deck pressure equation

Here we will derive the governing equation for the pressure in the main and lower decks $p_1 = p_{11}$ by considering the lower deck problem. The derivation of the governing equation for p_{11} is similar to the derivation for plane Poiseuille flow in chapter 2, and full details can be found in Smith (1979). To briefly summarise, consider the equations governing the wave components in the lower deck (3.46) where we see that we can solve (3.46c) in terms of Airy functions. To do this we change variables from Y to $\bar{\xi} = (i\alpha\lambda)^{1/3}(\lambda Y - c)/\lambda$. If we denote the value of $\bar{\xi}(Y = 0)$ by ξ , then the boundary conditions for w_1 require $w_1(\bar{\xi} = \xi) = 0$ and $w_1 \rightarrow 0$ as $\bar{\xi} \rightarrow \infty$, and thus we find

$$w_1 = (i\alpha\lambda)^{1/3} \frac{dp_{11}}{dZ} \left(S(\bar{\xi}) - Ai(\bar{\xi}) \frac{S(\xi)}{Ai(\xi)} \right), \quad (3.51)$$

where S is a scaled Scorer function which satisfies $S''(\bar{\xi}) - \xi S(\bar{\xi}) = 1$ and the boundary conditions $S(0) = 0$ and $S(\bar{\xi}) \rightarrow 0$ as $\bar{\xi} \rightarrow \infty$. Now applying ∂_Y to the streamwise momentum equation (3.46b) and using the continuity equation (3.46a) we find the governing equation for u_{1Y} to be

$$(\partial_{\bar{\xi}}^2 - \bar{\xi})u_{1Y} = (i\alpha\lambda)^{-2/3}(\lambda'Y w_{1Y} + \lambda' - \lambda w_{1Z}). \quad (3.52)$$

This equation can be solved for u_{1Y} up to a single degree of freedom in the solution to the homogeneous problem, which is a function of Z multiplied by $Ai(\bar{\xi})$. We now have two equations left, which are $\int_0^\infty u_{1Y} dY = \lambda A$, and from equation (3.46b) applied on the wall, $u_{1Y}(Y = 0) = i\alpha p_{11}$. Following this process we find an equation relating the pressure p_{11} and the displacement A ,

$$p_{11}'' - \frac{\lambda'}{\lambda} \mathcal{F}(\xi) p_{11}' - \alpha^2 p_{11} = (\alpha\lambda)^{5/3} \mathcal{G}(\xi) A, \quad (3.53)$$

where we have defined

$$\xi = -i^{1/3} \frac{(\alpha\lambda)^{1/3} c}{\lambda}, \quad (3.54)$$

and we also have

$$\kappa(\xi) = \int_\xi^\infty Ai(s) ds, \quad \mathcal{F}(\xi) = \frac{3}{2} + \frac{\xi(\xi\kappa(\xi) + Ai'(\xi))}{2Ai(\xi)}, \quad \mathcal{G}(\xi) = i^{5/3} \frac{Ai'(\xi)}{\kappa(\xi)}. \quad (3.55)$$

This closes the system of interaction equations.

3.3.2. The vortex-wave interaction equations

For clarity here we will list the equations governing the interaction which were derived in the last section and come in three interacting parts as will be outlined in what follows. Recall that these equations are identical to equations (2.2-2.4) of Hall & Smith (1991). The first part of the interaction equations is the vortex equations in the main deck which are driven by the logarithmic growth of the vortex in the lower deck. The governing equations are

$$V_y + W_Z = 0, \quad (3.56a)$$

$$VU_y + WU_Z = U_{yy}, \quad (3.56b)$$

$$VW_y + WW_Z = W_{yy}, \quad (3.56c)$$

subject to no slip and spanwise slip on the wall

$$U = 0, \quad V = -1, \quad W = q(Z), \quad y = 0, \quad (3.56d)$$

where $q(Z)$ is defined in (3.49) and far field matching conditions with the upper deck

$$U \rightarrow 1, \quad W \rightarrow 0, \quad y \rightarrow \infty. \quad (3.56e)$$

The second part of the interaction equations is the governing equation for the wave which is in the form of an eigenvalue problem for the streamwise wavenumber and wavespeed pair,

$$p''_{11} - \frac{\lambda'}{\lambda} \mathcal{F}(\xi) p'_{11} - \alpha^2 p_{11} = (\alpha\lambda)^{5/3} \mathcal{G}(\xi) A, \quad (3.57)$$

with definitions of the terms given in (3.55).

The third and final part of the interaction equations is the pressure-displacement relation which is found from a consideration of the upper deck wave problem. The $p_{11} - A$ relation is defined via the problem

$$\tilde{p}_{1\bar{Y}\bar{Y}} + \tilde{p}_{1ZZ} - \alpha^2 \tilde{p}_1 = 0, \quad (3.58a)$$

with matching conditions with the main deck

$$\tilde{p}_1 = p_{11}, \quad \tilde{p}_{1\bar{Y}} = -\alpha^2 A, \quad \bar{Y} = 0, \quad (3.58b)$$

and far field decay conditions

$$\tilde{p}_1 \rightarrow 0, \quad \bar{Y} \rightarrow \infty. \quad (3.58c)$$

The $p_{11} - A$ relation can be simplified by taking a Fourier transform in Z , solving the system in Fourier space to find the relation $\hat{p}_{11}(k) = \alpha^2(\alpha^2 + k^2)^{-1/2} \hat{A}(k)$ where we use $\hat{\cdot}$ to denote the Fourier transform in Z , and k is the Fourier transform variable. Taking the inverse Fourier transform using the convolution property of the Fourier transform we find the $p_{11} - A$ relation can be written down directly. We will not use this form of the $p_{11} - A$ relation here, opting instead to solve the full problem given by equations (3.58).

To close the system we set a spanwise wavenumber β so we look for solutions that are periodic in the spanwise direction with period $2\pi/\beta$, and we must also give some measure of the amplitude of the wave. As in the case of plane Poiseuille flow we do this by defining an amplitude of the wave which we denote by $\mathcal{A}$ but here we define the amplitude in terms of the pressure p_{11} by

$$\mathcal{A}^2 = \frac{\beta}{2\pi} \int_0^{2\pi/\beta} |p_{11}|^2 dZ. \quad (3.59)$$

This now fully defines the vortex-wave interaction states up to the streamwise and spanwise shift invariances.

In the absence of a vortex flow, when $\mathcal{A} \rightarrow 0$, we find $q \rightarrow 0$ and hence $(\lambda, U, V, W) \rightarrow (1, u_b, -1, 0)$. Then the second and third parts of the interaction equations, (3.57) and (3.58), reduce to the linear stability problem of section 3.2. In section 3.4.1 we will compute weakly nonlinear states where we assume $\mathcal{A} \ll 1$, and these states can be described analytically. Then in section 3.6 we move onto finding strongly nonlinear states which must be computed numerically.

3.3.3. Large spanwise wavenumber limit equations

Although we will only solve the full nonlinear interaction equations computationally with unit spanwise wavenumber, here we will briefly consider the large spanwise wavenumber limit of the interaction equations. The spanwise length scale in the interaction equations

is $O(\epsilon^{-1}) = O(Re^{1/4})$ so taking the spanwise wavenumber β to be large corresponds to moving back towards an $O(1)$ scale where spanwise diffusion is comparable to wall normal diffusion throughout the majority of the boundary layer. To derive the governing equations in the large spanwise wavenumber limit consider a scaling analysis of the equations when $\beta \gg 1$. Since solutions are periodic on $Z \in [0, 2\pi/\beta]$ it immediately follows that $Z \sim \beta^{-1}$. Considering the main deck vortex equations (3.56), it is clear from the boundary conditions that we must still have $U, V \sim 1$. From the continuity equation since $V_y \sim 1$ we require $W \sim \beta^{-1}$. Then considering the forcing term $W \sim q(Z)$ where $q(Z)$ is defined in (3.49) we have either $\beta^{-1} \sim \beta p_{11}^2$ or $\beta^{-1} \sim \beta^3 p_{11}^2 / \alpha^2$. From the linear dispersion relation we expect $\xi \sim O(1)$ and $\alpha^{1/3} \beta \sim O(1)$, i.e. $\alpha \sim \beta^{-3}$ and $c \sim \beta$. Thus we see that since $\alpha \rightarrow 0$, the second term of the forcing q is the dominant term and hence $p_{11} \sim \beta^{-5}$. Alternatively we could consider the lower deck wave equation (3.57) to deduce the α and c scalings.

Now consider the upper deck equations (3.58). Comparing $\partial_Y^2, \beta^2, \alpha^2$ we see that the activity in the upper deck is confined to the layer where $Y \sim \beta^{-1}$. The boundary condition matching the pressure $\bar{p}_1 = p_{11}$ requires $\bar{p}_1 \sim \beta^{-5}$ and matching the wall normal velocity $\bar{p}_{1Y} = -\alpha^2 A$ requires $A \sim \beta^2$. We have now identified all of the large spanwise wavenumber scales so we will now define the suitably scaled large β variables which are denoted by subscript l 's as

$$Z_l = \beta Z, \quad Y_l = \beta Y, \quad W_l = \beta W, \quad (3.60)$$

for the length and vortex components and

$$\alpha_l = \beta^3 \alpha, \quad c_l = \beta^{-1} c, \quad p_l = \beta^5 p_1, \quad \bar{p}_l = \beta^5 \bar{p}_1, \quad A_l = \beta^{-2} A, \quad (3.61)$$

for the wave components.

Rather than immediately considering the large spanwise wavenumber limit let us first consider the interaction equations for general spanwise wavenumbers under this rescaling. With the exception of the wave forcing term in the boundary conditions, the core interaction equations (3.56) are invariant under this change of variables. Equations (3.56) become

$$V_y + W_l Z_l = 0, \quad (3.62a)$$

$$V U_y + W_l U_{Z_l} = U_{yy}, \quad (3.62b)$$

$$V W_{ly} + W_l W_{lZ_l} = W_{lyy}, \quad (3.62c)$$

with the far field matching conditions

$$U \rightarrow 1, \quad W_l \rightarrow 0, \quad y \rightarrow \infty, \quad (3.62d)$$

and no slip conditions on the plate where we note the change in the spanwise slip condition

$$U = 0, \quad V = -1, \quad W_l = -\frac{1}{\alpha_l^2 \lambda^2} \frac{d}{dZ_l} \left(\beta^{-8} \alpha_l^2 |p_l|^2 + \left| \frac{dp_l}{dZ_l} \right|^2 \right), \quad y = 0. \quad (3.62e)$$

Considering the lower deck pressure equation (3.57) under this rescaling we find

$$p_l'' - \frac{\lambda'}{\lambda} \mathcal{F}(\xi_l) p_l' - \beta^{-8} \alpha_l^2 p_l = (\alpha_l \lambda)^{5/3} \mathcal{G}(\xi_l) A_l, \quad (3.63)$$

where the α and c scales are such that ξ remains $O(1)$, with the rescaled ξ denoted by ξ_l and defined in terms of the scaled variables as

$$\xi_l = -\frac{(i\alpha_l \lambda)^{1/3}}{\lambda} c_l. \quad (3.64)$$

Finally considering the upper deck problem given by equations (3.58) we found a boundary layer of width $O(\beta^{-1})$ so in Y_l coordinates the problem becomes

$$(\partial_{Y_l}^2 + \partial_{Z_l}^2 - \beta^{-8} \alpha_l^2) \bar{p}_l = 0, \quad (3.65a)$$

with far field decay conditions

$$\bar{p}_l \rightarrow 0, \quad Y_l \rightarrow \infty, \quad (3.65b)$$

and matching conditions with the main deck

$$\bar{p}_l = p_l, \quad \bar{p}_{l Y_l} = -\alpha_l^2 A_l, \quad Y_l = 0. \quad (3.65c)$$

These equations hold for all general spanwise wavenumbers β although we can see that as $\beta \rightarrow \infty$ these equations appear to approach a limit which we will call the large spanwise wavenumber limit with the error in the equations of order $O(\beta^{-8})$. In this limit the governing equations are (3.62) with (3.62e) replaced by

$$U = 0, \quad V = -1, \quad W_l = -\frac{1}{\alpha_l^2 \lambda^2} \frac{d}{dZ} \left| \frac{dp_l}{dZ_l} \right|^2, \quad y = 0, \quad (3.66)$$

the lower deck wave equation (3.63) is replaced by

$$p_l'' - \frac{\lambda'}{\lambda} \mathcal{F}(\xi_l) p_l' = (\alpha_l \lambda)^{5/3} \mathcal{G}(\xi_l) A_l, \quad (3.67)$$

and the upper deck equations (3.65) with (3.65a) replaced by

$$(\partial_{Y_l}^2 + \partial_{Z_l}^2) \bar{p}_l = 0. \quad (3.68)$$

One could find a $p_l - A_l$ relation by using Fourier transforms. However no further analytic progress is possible and one would have to revert to solving these equations computationally even if the $p_l - A_l$ relation is written in this form.

Ideally to confirm that these scalings and equations do in fact represent the large spanwise wavenumber limit of the interaction equations one should perform computations at various spanwise wavenumbers demonstrating convergence to the proposed limit. It was not possible to investigate convergence to this limit here because computations were only performed for $\beta = 1$. However it was confirmed from the linear dispersion relation (3.20) that $\alpha \sim \beta^{-3}$ and $c \sim \beta$ demonstrating that we have identified the correct streamwise wavenumber and wavespeed scales.

As the spanwise wavenumber increases eventually a limit is reached where the interaction equations break down. This can be seen because spanwise diffusion becomes a leading order effect in the main deck vortex equations when $\beta \sim \epsilon^{-1}$ i.e. $z \sim O(1)$ or by considering for example the problem in the upper deck which is on the wall normal scale $O(\epsilon^{-1})$. In the large spanwise wavenumber limit this scale becomes $O(\beta^{-1} \epsilon^{-1})$ so when $\beta \sim \epsilon^{-1}$ the upper deck scale becomes $O(1)$ and it merges with the main deck. Inside the main deck the roll components of the vortex become $O(Re^{-1})$, and the wave components become comparable to the vortex components. Further the streamwise wavenumber becomes of size $O(Re^{-1})$ suggesting streamwise development over a lengthscale $O(Re)$. Thus we introduce new variables with these scalings,

$$t = ReT_r, \quad x = ReX_r, \quad u = U_r, \quad (v, w) = Re^{-1}(V_r, W_r), \quad p = Re^{-2}P_r. \quad (3.69)$$

The wavespeed is $O(1)$ and this suggests that the dynamics satisfy the boundary region equations

$$\nabla_r \cdot \mathbf{U}_r = 0, \quad (3.70a)$$

$$\mathbf{U}_{rT} + (\mathbf{U}_r \cdot \nabla_r) \mathbf{U}_r = -\nabla_{2r} P_r + \nabla_{2r}^2 \mathbf{U}_r, \quad (3.70b)$$

where $\nabla_r = (\partial_{X_r}, \partial_y, \partial_z)$ and $\nabla_{2r} = (0, \partial_y, \partial_z)$. We could look for travelling wave solutions of these equations by setting $\zeta_r = X_r - cT_r$ and looking for solutions in terms of (ζ_r, y, z) where the wavespeed c is now to be determined as part of the solution. These equations can only be solved computationally and this has not been done here. However computations on this problem in plane Couette flow have been performed by Deguchi, Hall & Walton (2013) where streamwise and spanwise localisation were observed as the amplitude of the interaction was increased.

3.4. Preliminary calculations in the asymptotic suction boundary layer

3.4.1. Weakly nonlinear analysis of the vortex-wave interaction equations

Here we consider the situation where a small but finite size wave is present and it turns out that this can be done analytically. We assume $\mathcal{A} \sim \Delta$ where $\Delta \ll 1$ so

$$(A, p_{11}) = \Delta(A_1, P_1) \cos \beta Z + \dots, \quad (3.71)$$

and then the wave forcing in the main deck vortex equations (3.56) is $O(\Delta^2)$. Since the forcing from the wave on the main deck vortex problem is of size $|p_1|^2 \sim O(\Delta^2)$ we find the perturbation to the basic state is also $O(\Delta^2)$ so for example,

$$U = u_b + \Delta^2 u_2 \cos 2\beta Z + \dots, \quad (3.72)$$

and thus the perturbation to the shear stress on the wall is also $O(\Delta^2)$, i.e. we have

$$\lambda = \lambda_0 + \Delta^2 \lambda_2 \cos 2\beta Z + \dots, \quad (3.73)$$

where $\lambda_j = u_{jy}(y = 0)$. Considering the wave pressure equation (3.57), we find that the streamwise wavenumber α and the wavespeed c are perturbed by $O(\Delta^2)$ from their linear neutral values. If we let (α_0, c_0) denote their linear neutral values then we find that (α, c) expand as

$$(\alpha, c) = (\alpha_0, c_0) + \Delta^2(\alpha_2, c_2) + \dots, \quad (3.74)$$

and the correction to A is $O(\Delta^2)$ so we can conclude that A expands as

$$A = \Delta A_1 \cos \beta Z + \Delta^3 [A_3^1 \cos \beta Z + A_3^3 \cos 3\beta Z] + \dots \quad (3.75)$$

From the leading order terms in the the wave pressure equation (3.57) and the $p_{11} - A$ relation from equations (3.58) we find the linear dispersion relation,

$$\mathcal{G}_0 + \alpha_0^{1/3} \chi_0 = 0, \quad (3.76)$$

where $\mathcal{G}$ and κ are defined in (3.55) and

$$\xi_0 = -\frac{(i\alpha_0\lambda_0)^{1/3}}{\lambda_0} c_0, \quad \chi_0 = (\alpha_0^2 + \beta^2)^{1/2}. \quad (3.77)$$

This is the linear dispersion relation (3.20) found in the linear stability section 3.2 as one would expect. Continuing the analysis to higher order we calculate the $O(\Delta^2)$ correction to the vortex in the main deck equations (3.56) and from this the $O(\Delta^2)$ correction to the shear λ . Now consider the wave equations again, (3.57) and (3.58). At the next order we find the solvability condition which describes the bifurcation of weakly nonlinear states from the linear neutral point,

$$\nu_1 \alpha_2 + \nu_2 c_2 + \nu_3 |P_1|^2 = 0, \quad (3.78)$$

where the ν_j are complex constants. There is a degree of freedom in the choice of Δ , so we can set $|P_1| = 1$ without loss of generality. This leaves a single complex equation with two real unknown quantities, α_2 and c_2 , which describes how the wavenumber and wavespeed of the states vary for small amplitude weakly nonlinear states bifurcating from the linear neutral point. This will be used to verify the computational results at small amplitudes as we will see later in the results section 3.6. More details on the weakly nonlinear analysis including the definition of the constants ν_j can be found in appendix D.

Figure 3.2 shows solutions for α_2 and c_2 in (3.78). We see that the bifurcation changes criticality twice within the range of spanwise wavenumbers considered here. Both times that the bifurcation changes criticality α_2 passes through zero (as opposed to blowing up for example like $(\beta - \beta_c)^{-1}$) and thus the weakly nonlinear analysis is valid even around these critical spanwise wavenumbers. For small β the bifurcation is supercritical, then it changes to subcritical, and back to supercritical where it appears to stay. All of the computations performed here will be for $\beta = 1$, so we are expecting a supercritical

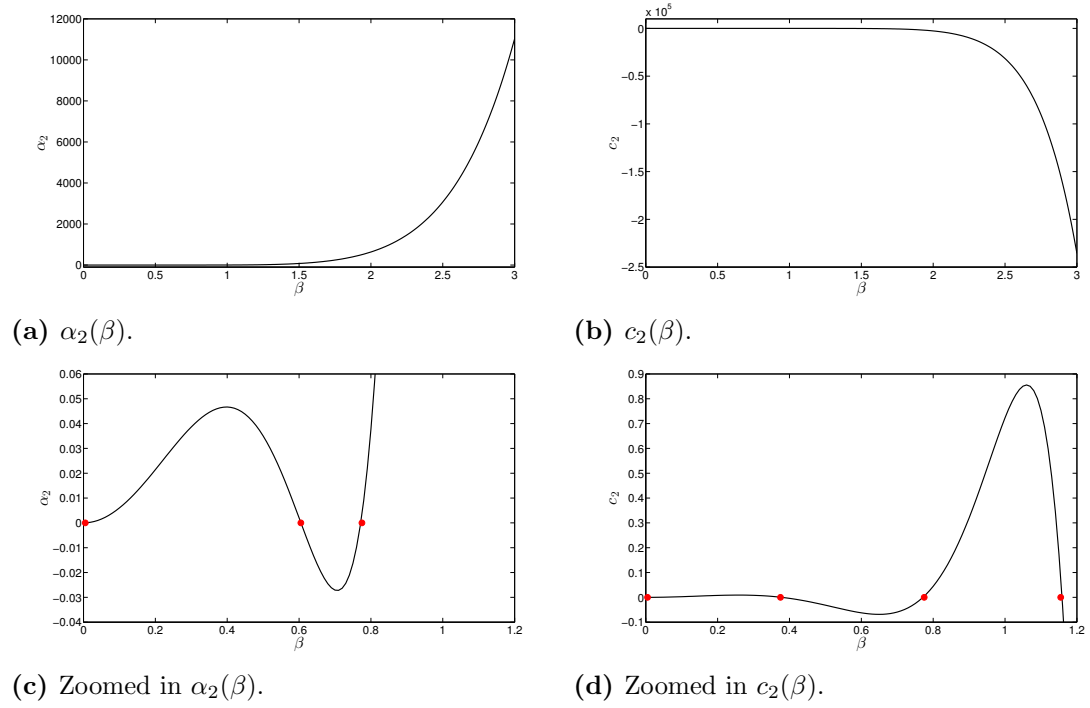


Figure 3.2. The bifurcation parameters α_2 and c_2 from the weakly nonlinear analysis. Figure (c) shows $\alpha_2(\beta)$ zoomed in on $\beta \in (0, 1.2)$ where we see that the bifurcation changes criticality twice for $\beta \in (0, 1)$ - the points where $\alpha_2(\beta) = 0$ are marked with red dots for clarity.

bifurcation with both α_2 and c_2 greater than zero. For example for $\beta = 1$, we find that to three decimal places,

$$(\alpha_0, c_0) = (0.617, 2.698), \quad (\alpha_2, c_2) = (1.629, 0.723), \quad (3.79)$$

so noting that $\mathcal{A}^2 = \Delta^2/2$ at leading order, we find that the bifurcation is described up to $(\mathcal{A}^2)$ by

$$\begin{aligned} \alpha &\approx \alpha_0 + \Delta^2 \alpha_2 = 0.617 + 3.358 \mathcal{A}^2, \\ c &\approx c_0 + \Delta^2 c_2 = 2.698 + 1.446 \mathcal{A}^2, \end{aligned} \quad (3.80)$$

for small $\mathcal{A}$, and we will compare this result with numerical solutions of the full interaction equations in the results section 3.6.

3.4.2. Local analysis of the vortex-wave interaction equations

In this section we will consider the local behaviour of solutions to the vortex equations (3.56) around spanwise locations where the wave forcing is zero i.e. $q = 0$. The leading order local behaviour will depend on the spanwise derivative of q around such zeros which we denote by q_0 . Although it is not possible to predict $q_0(\mathcal{A})$ we expect that $|q_0|$ increases as the amplitude $\mathcal{A}$ of the state is increased. Regardless of whether our expectation of the behaviour of q_0 is realised we can make some predictions about the local behaviour of solutions to the main deck vortex equations which hold for all amplitudes. To see that such points where q vanishes actually exist note that $-(\alpha\lambda)^{-2}$ is negative and non-zero, and the remaining part has zero spanwise mean (and for $\mathcal{A} \neq 0$ is not identically zero). Further such points occur in pairs where in each pair there is one point where $q_0 < 0$ and one point where $q_0 > 0$.

We assume without loss of generality that such a point occurs at $Z = 0$ and has gradient q_0 . From the wall slip velocity in (3.56d) we find $W \sim Z$, then the continuity equations (3.56a) yields $V \sim 1$ and we find that also $U \sim 1$. Thus locally the vortex components in the main deck expand as

$$q = q_0 Z + \dots, \quad (U, V) = (U_0, V_0) + \dots, \quad W = W_0 Z + \dots, \quad (3.81)$$

where U_0, V_0 and W_0 depend on y only, and q_0 is a prescribed constant. Substituting these expansions into the main deck vortex problem given by equations (3.56) we find

the local solutions satisfy

$$V'_0 + W_0 = 0, \quad (3.82a)$$

$$V_0 U'_0 = U''_0, \quad (3.82b)$$

$$V_0 W'_0 + W_0^2 = W''_0, \quad (3.82c)$$

with far field boundary conditions

$$U_0 \rightarrow 1, \quad W_0 \rightarrow 0, \quad y \rightarrow \infty, \quad (3.82d)$$

and no slip and forcing conditions on the wall

$$U_0 = 0, \quad V_0 = -1, \quad W_0 = q_0, \quad y = 0. \quad (3.82e)$$

Perhaps surprisingly these equations have an exact solution given by

$$U_0 = \frac{1 - \exp[(q_0/B^2)(1 - \exp[-By])]}{1 - \exp(q_0/B^2)}, \quad (3.83a)$$

$$V_0 = -1 - \frac{q_0}{B}(1 - e^{-By}), \quad (3.83b)$$

$$W_0 = q_0 e^{-By}, \quad (3.83c)$$

where B is a positive constant (to ensure far field decay) which satisfies

$$B^2 - B - q_0 = 0. \quad (3.83d)$$

The solutions to this equation for $B(q_0)$ are

$$B_{\pm} = \frac{1 \pm \sqrt{1 + 4q_0}}{2}, \quad (3.84)$$

and since we require B to be positive there are two possible solutions for $-1/4 < q_0 < 0$, one solution for $q_0 \geq 0$ and $q_0 = -1/4$ and no solutions for $q_0 < -1/4$. From this we see that at the local point where $q_0 > 0$ we have $B = B_+ > 1$. However at the local point where $-1/4 < q_0 < 0$ we have two admissible solutions $B_- \in (0, 1/2)$ and $B_+ \in (1/2, 1)$. Given the nonlinear nature of the local equations it would not be surprising to discover that there are more solutions. As we will see soon the solutions to the full interaction equations that were computed in this work (as documented in section 3.6) are described by the solutions to the local equations found in this section. At the local point with

$-1/4 < q_0 < 0$ we observed the decay given by B_+ i.e. the more rapid decay of the two possibilities.

As discussed we generally expect $|q_0|$ to increase as $\mathcal{A}$ increases. Considering the spanwise location where q_0 is negative, the analysis in this section raises the question of what happens as $\mathcal{A}$ increases. We expect q_0 to continue decreasing, however the analysis in this section suggests that no solutions of the local equations exists for $q_0 < -1/4$. This could mean one of three things: there is a singularity in the equations when $q_0 = -1/4$, there is no singularity and $q_0 > -1/4$ for all solutions of the interaction equations, or thirdly the local behaviour of solutions of the interaction equations is described by other solutions of the local equations which have not been found here but do exist for $q_0 < -1/4$. Which of these three eventualities holds was not determined in this work, but if it could be then it may shed some light on the ultimate fate of these vortex-wave interaction states as the amplitude of the wave is increased.

3.5. The numerical method for the vortex-wave interaction equations

3.5.1. The numerical method

In this section we will consider the numerical solution of the vortex-wave interaction equations given by (3.56), (3.57) and (3.58) subject to (3.59). The method used here is similar to that used to solve the vortex-wave interaction equations in plane Poiseuille flow in chapter 2, however now the pressure-displacement law is a little more complicated so rather than having one simple relation between p_{11} and A we must also solve the upper deck problem (3.58). Complexity wise this is a minor alteration although it does substantially add to the size of the computational problem.

In plane Poiseuille flow the domain is bounded by straight walls at $y = \pm 1$ which is convenient for the use of a spectral expansion in Chebyshev polynomials. However, both the main deck problem given by (3.56) and the upper deck problem given by (3.58) are to be solved in a semi-infinite domain in the wall normal direction. Thus we must adopt a strategy to deal with the semi-infinite domain. Here we opt to map the semi-infinite domain onto the bounded domain $[-1, 1]$ where we can expand the solution in Chebyshev polynomials. There are multiple ways of performing this mapping, for example we could truncate the domain at some large distance H from the wall, compute solutions on the bounded domain and then increase H until the results are independent of the choice of H

when we can be sure that the domain accurately approximates the semi-infinite domain problem. Instead the method used here is to map the whole semi-infinite domain onto $[-1, 1]$ using an exponential map,

$$\hat{y} = 1 - 2e^{-y}, \quad \hat{Y} = 1 - 2e^{-Y}. \quad (3.85)$$

Now the solution domain is $(\hat{y}, Z) \in [-1, 1] \times [0, 2\pi/\beta]$ where we can apply an expansion in Chebyshev polynomials in the $\hat{y}$ direction, and we can exploit symmetries in the spanwise direction to decompose the solution in Fourier sine and cosine series, so the unknown variables can be written in the form

$$\begin{aligned} (U, V) &= \sum_{n,k=1} (U_{n,k}, V_{n,k}) T_{n-1}(\hat{y}) \cos 2(k-1)\beta Z, \quad W = \sum_{n,k=1} W_{n,k} T_{n-1}(\hat{y}) \sin 2k\beta Z, \\ (A, p_{11}) &= \sum_{k=1} (A_k, p_{11k}) \cos(2k-1)\beta Z, \quad \tilde{p}_1 = \sum_{n,k=1} \tilde{P}_{n,k} T_{n-1}(\hat{Y}) \cos(2k-1)\beta Z. \end{aligned} \quad (3.86)$$

The series expansions are terminated at $n = N$ and $k = K$ which is taken large enough that sufficient decay in the spectral coefficients is observed. From here we substitute the expansions into the governing equations (3.56), (3.57) and (3.58) then apply the collocation method in the wall normal direction and the Galerkin method in the spanwise direction. Nonlinear terms are evaluated using the pseudospectral method - the product is calculated in physical space and then transformed back into Fourier space. Due to the nonlinear nature of the vortex-wave interaction equations we must apply a de-aliasing strategy, the simplest of which is zero padding. This involves computing nonlinear products over say $\bar{K} > K$ points in physical space before computing the K spectral coefficients of the product in Fourier space. Orszag's "2/3 truncation rule" translates into picking $\bar{K} = 3K$ (which is $(3/2) \times 2K$ modes). However here we instead take $\bar{K} = 4K$ because the interaction equations are highly nonlinear - the vortex part of the main deck equations (3.56) only has a quadratic nonlinearity, but the wave pressure equation (3.57) contains Airy functions of the negative two thirds power of the shear stress.

Once the collocation and Galerkin methods have been applied this leaves a residual vector which is a highly nonlinear function of the spectral coefficients. To solve this system of equations we apply a quasi-Newton method, calculating the Jacobian matrix by taking finite differences of the residual vector with respect to each spectral coefficient and the unknown streamwise wavenumber and wavespeed. This is computationally

expensive but since columns of the Jacobian are calculated independently this process can be parallelised. Also the Jacobian can often be reused for a number of iterations as in the chord method - as a rule we reuse the Jacobian while the error decreases from one iterate to the next. The convergence properties of this method allow us to compute highly nonlinear states.

3.5.2. Validation using the local analysis results

In this section we will compare the local solutions from section 3.4.2 with solutions which have been computed from the full interaction equations using the numerical method discussed in the last section - these solutions will be discussed in more detail in the next section. As discussed in section 3.4.2 local points appear in pairs with $q_0 < 0$ at one point and $q_0 > 0$ at the other and here we will consider the agreement with the local analysis at both points for different amplitude states. From the representation of the solution in equation (3.86) we see that the local points are located at the spanwise locations $Z = 0$ and $Z = \pi/2$ ($+\pi\mathbb{Z}$ since the vortex flow has period π/β and $\beta = 1$ for all the solutions here), and this has been confirmed from the full numerical solutions.

First we consider a small amplitude solution where $\mathcal{A}^2 = 0.005$. This solution sits within the range of validity of the weakly nonlinear analysis and the full solution at this amplitude can be seen later in the results section in figure 3.8. First consider the comparison at $Z = 0$ which is shown in figure 3.3 where it was found from the full numerical solution that $q_0 = -0.0301$ from which we find two admissible values for B , $B_1 = 0.0311$ or $B_2 = 0.9689$ and the solution to the full interaction equation corresponds to B_2 . The comparison at $Z = \pi/2$ is shown in figure 3.4 where we found $q_0 = 0.0314$ and hence the only admissible value of B is $B = 1.0305$. Excellent agreement between the two is observed - the results are almost completely indistinguishable with a typical pointwise error of $O(10^{-6})$.

Now we will consider a comparison with the local analysis at a larger amplitude out of the range of the weakly nonlinear analysis where the interaction is now within the strongly nonlinear regime. For this comparison we pick $\mathcal{A}^2 = 0.0242$ and the full solution at this amplitude can be seen later in figure 3.10. First consider the comparison at $Z = 0$ which can be seen in figure 3.5 where we find $q_0 = -0.1282$ from which we find two admissible values of B which are $B_1 = 0.1510$ and $B_2 = 0.8490$ and the solution here corresponds to B_2 . For the comparison at $Z = \pi/2$ which can be seen in figure 3.6 we find $q_0 = 0.1462$ from which it follows that $B = 1.1295$. The agreement again

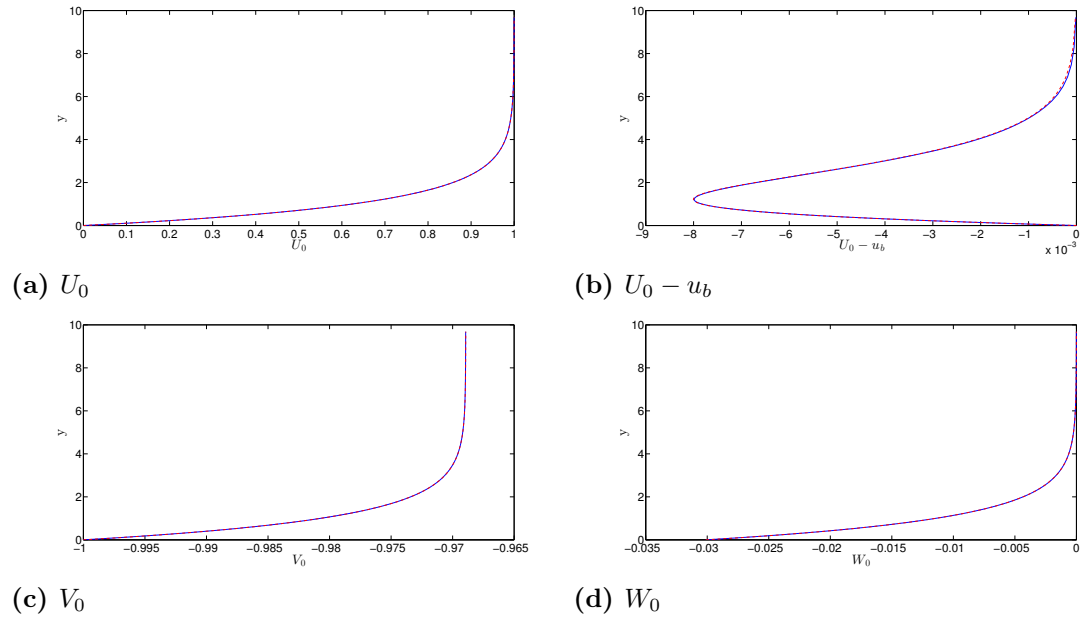


Figure 3.3. Comparison of numerical results (blue solid curve) with local analysis results (red dashed curve) when $\mathcal{A}^2 = 0.005$ at the local point at $Z = 0$.

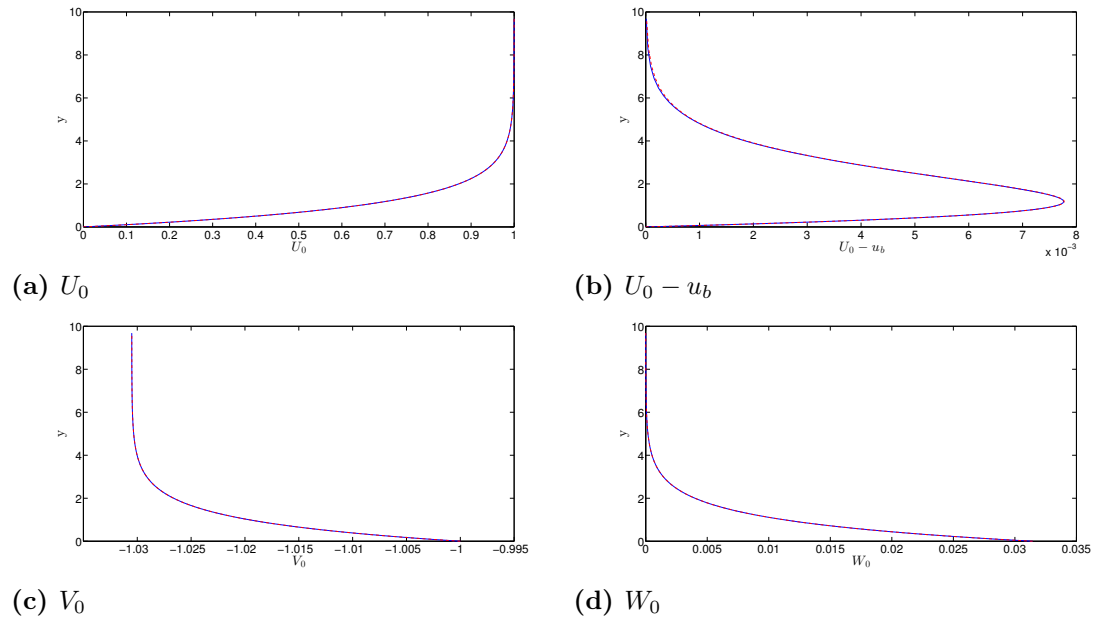


Figure 3.4. Comparison of numerical results (blue solid curve) with local analysis results (red dashed curve) when $\mathcal{A}^2 = 0.005$ at the local point at $Z = \pi/2$.

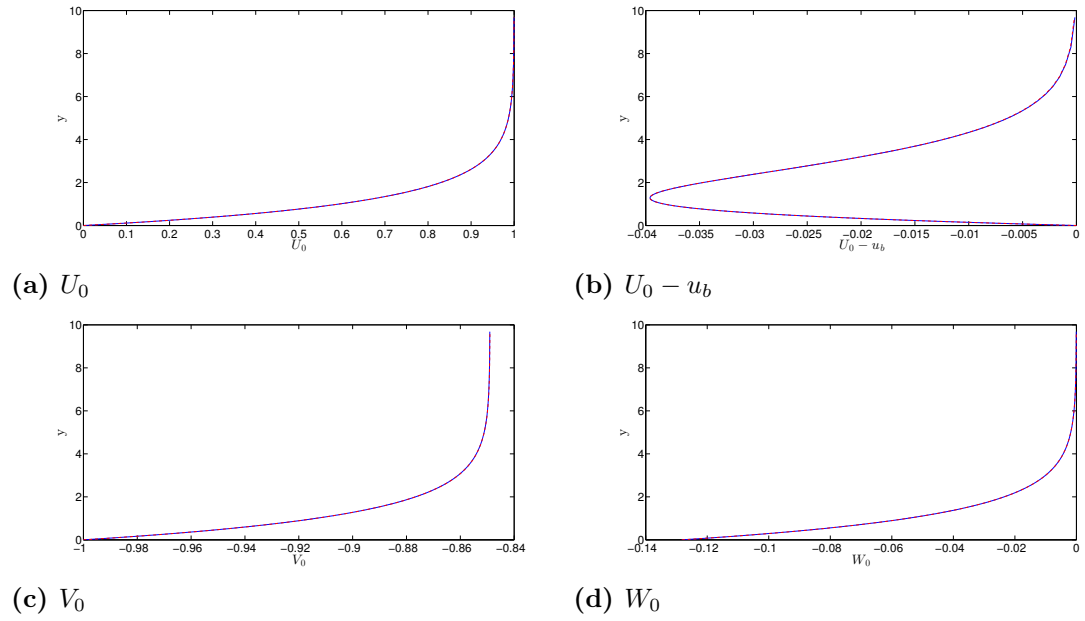


Figure 3.5. Comparison of numerical results (blue solid curve) with local analysis results (red dashed curve) when $\mathcal{A}^2 = 0.0242$ at the local point at $Z = 0$.

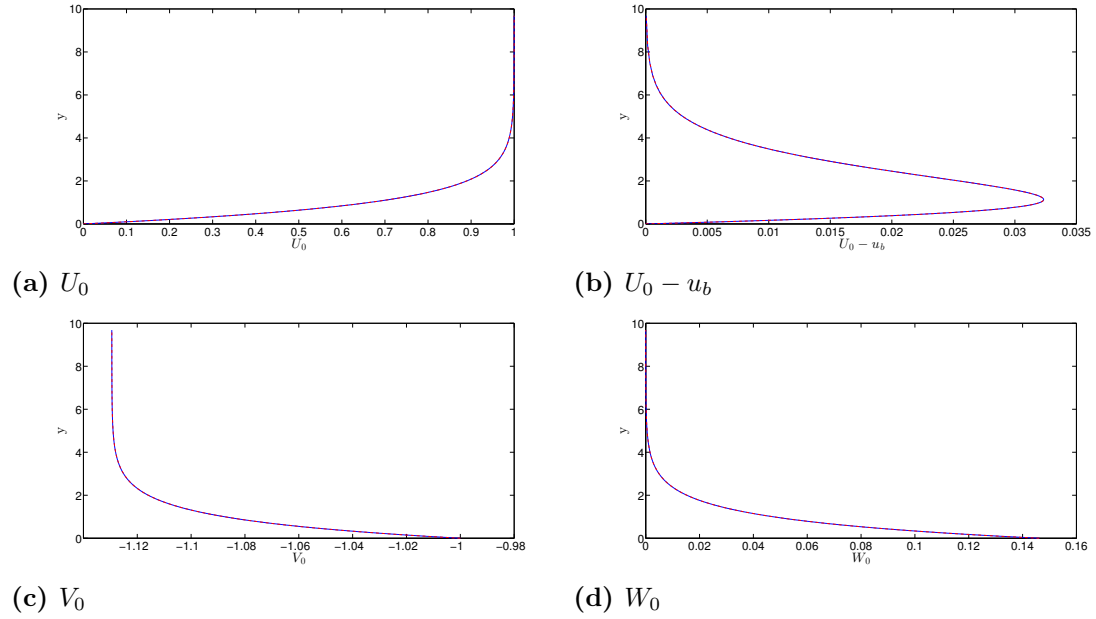


Figure 3.6. Comparison of numerical results (blue solid curve) with local analysis results (red dashed curve) when $\mathcal{A}^2 = 0.0242$ at the local point at $Z = \pi/2$.

between the local solutions and the full numerical solutions is excellent.

3.6. Computational results for unit scaled spanwise wavenumber

In this section we will give some solutions to the vortex-wave interaction equations which were computed using the method described in the last section. We will only consider solutions at unit scaled spanwise wavenumber $\beta = 1$. The expectation is that the states are similar for other spanwise wavenumbers as we found in plane Poiseuille flow but this has not been checked in the fully nonlinear computations. Also the results of the weakly nonlinear analysis cast doubt on this idea given the change in the criticality of the bifurcation as the spanwise wavenumber varies. We will start by computing small amplitude states which can be compared with the weakly nonlinear analysis of section 3.4.1. Then we will move on to compute larger amplitude states where nonlinear effects come into play.

First consider figure 3.7 which shows a comparison of the streamwise wavenumber and wavespeed computed for various amplitudes $\mathcal{A}$ from the full nonlinear equations with the predictions for small amplitudes from the weakly nonlinear analysis. Good agreement is observed for small amplitudes confirming that the numerical method is accurately capturing the small amplitude states. However, we see that the agreement is only good for quite small amplitudes - the range of validity of the weakly nonlinear analysis when used to predict the streamwise wavenumber for example is approximately $\mathcal{A} \lesssim 0.1$ before significant deviations are observed suggesting that nonlinear effects are important even at fairly small amplitudes.

Now we will observe the type of structures found in these states at various amplitudes. In the following figures we present the perturbation to the streak $U - u_b$, the perturbations to the vortex roll component $V - v_b$ and W , and the perturbation to the basic state shear $\bar{\lambda} = \lambda - \lambda_0$ where $\lambda_0 = 1$ is the shear stress of the basic state. To start with, consider a small amplitude solution which is within the range of applicability of the weakly nonlinear analysis. Figure 3.8 shows the state computed at an amplitude of $\mathcal{A}^2 = 0.005$ ($\mathcal{A} \approx 0.0707$). The state at this amplitude has a regular spanwise structure as we would expect from the weakly nonlinear analysis. If we increase the amplitude further to $\mathcal{A}^2 = 0.02$ ($\mathcal{A} = 0.1\sqrt{2} \approx 0.141$) as in figure 3.9, then we are now out of the range of validity of the weakly nonlinear analysis and we see through the spanwise deformation

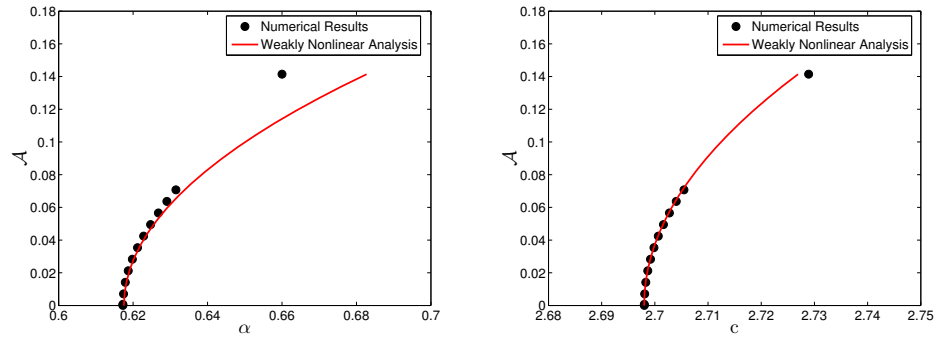


Figure 3.7. Comparison of computational states with the weakly nonlinear analysis which is only valid for $\mathcal{A} \ll 1$.

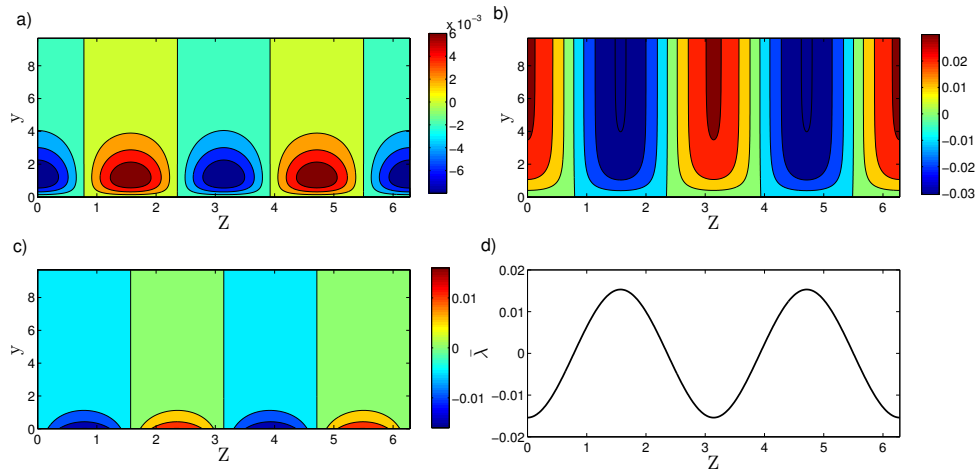


Figure 3.8. $\mathcal{A}^2 = 0.005$. a) The perturbation to the streak basic state, $U - u_b$. b),c) The perturbation to the roll components, $(V - v_b, W)$. d) The perturbation to the shear stress of the basic state, $\bar{\lambda}$.

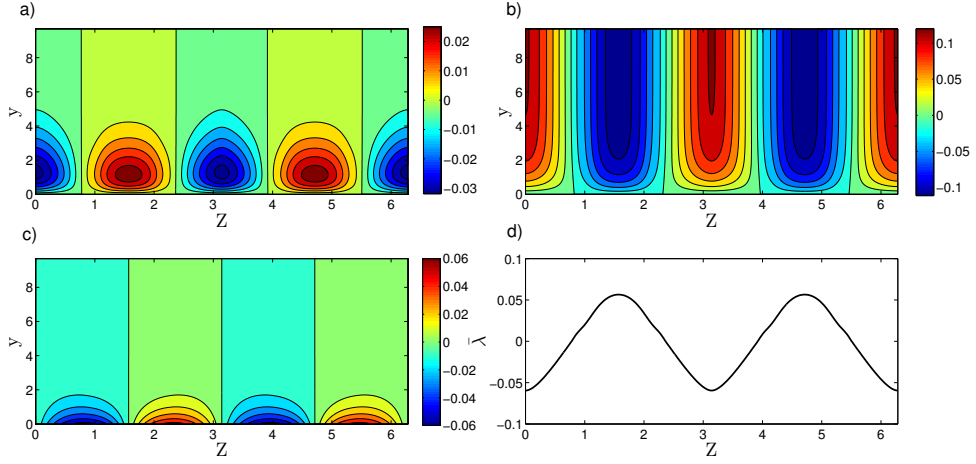


Figure 3.9. $\mathcal{A}^2 = 0.02$. a) The perturbation to the streak basic state, $U - u_b$. b),c) The perturbation to the roll components, $(V - v_b, W)$. d) The perturbation to the shear stress of the basic state, $\bar{\lambda}$.

that nonlinear effects are now present in the solution. Increasing the amplitude of the state just slightly to $\mathcal{A}^2 = 0.0242$ ($\mathcal{A} \approx 0.1556$) we see the change to the basic state has increased significantly, in fact the magnitude of the perturbation to the streak has increased by around 50%. The shear stress appears to be exhibiting some odd behaviour with rapid changes in the derivatives of the shear stress over short ranges at specific spanwise locations, as will be verified in figure 3.13 shortly. Continuing to increase the amplitude it becomes very hard to compute solutions with the Newton method failing to converge in a reasonable number of iterations. Even though the Newton method is much more reliable than other methods, due to the highly nonlinear nature of the interaction equations (involving Airy functions of $\alpha^{1/3}$ for example), if a single iteration moves (ξ for example) a long distance from the solution then difficulties can arise. A solution was found with an amplitude of $\mathcal{A}^2 = 0.0451$ ($\mathcal{A} \approx 0.2124$) as shown in figure 3.11. The solutions appears to be localising around the spanwise locations $\mathbb{Z}\pi$. The shear stress displays odd behaviour at other spanwise locations which have no apparent significance - physical or mathematical. Figure 3.12 shows a measure,

$$E_F(k) = \sum_n |U_{n,k}|, \quad E_C(n) = \sum_k |U_{n,k}|, \quad (3.87)$$

of the decay of the Fourier and Chebyshev spectral coefficients, when the calculation is repeated with different numbers of Fourier modes at fixed amplitude $\mathcal{A}^2 = 0.0451$.

This shows that this state is under-resolved, but also that the decay of the spectral

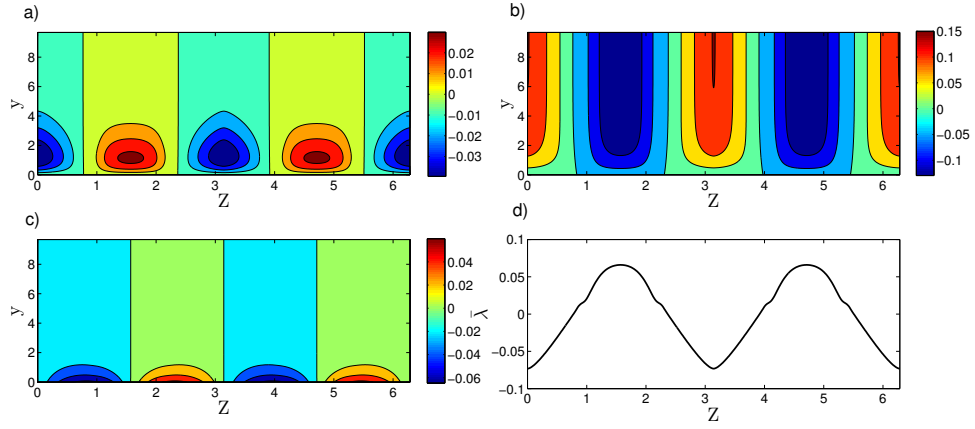


Figure 3.10. $\mathcal{A}^2 = 0.0242$. a) The perturbation to the streak basic state, $U - u_b$. b),c) The perturbation to the roll components, $(V - v_b, W)$. d) The perturbation to the shear stress of the basic state, $\bar{\lambda}$.

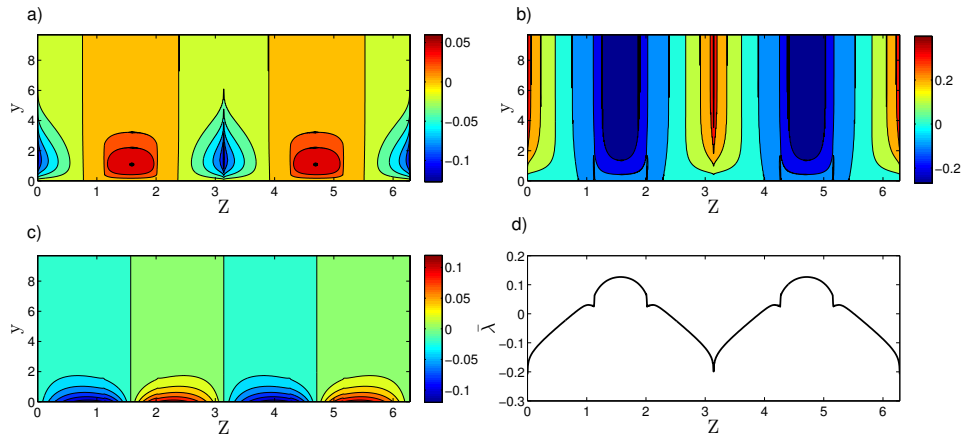


Figure 3.11. $\mathcal{A}^2 = 0.0451$. a) The perturbation to the streak basic state, $U - u_b$. b),c) The perturbation to the roll components, $(V - v_b, W)$. d) The perturbation to the shear stress of the basic state, $\bar{\lambda}$.

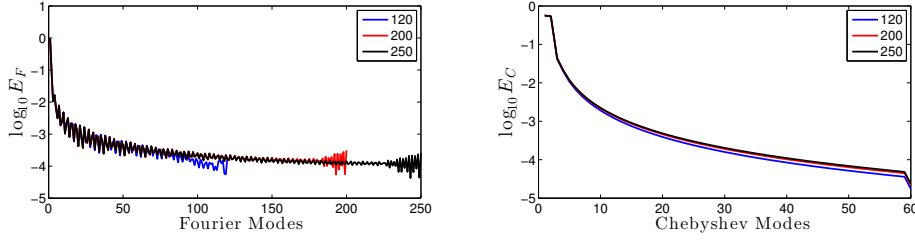


Figure 3.12. $\mathcal{A}^2 = 0.0451$. Decay of the Fourier and Chebyshev spectral coefficients when the computation is repeated with various numbers of Fourier modes, i.e. 120, 200 and 250 Fourier modes.

coefficients is so slow that it is unlikely we would be able to repeat the computation with a sufficient number of Fourier modes to accurately capture the behaviour of the solution at this amplitude. It is possible that there is a singularity in the equations, and perhaps no solution to the vortex-wave interaction equations actually exists at this amplitude. Or it may just be that the solutions are highly localised in the spanwise direction. The analysis performed here does not allow us to conclude the existence or non-existence of solutions to the interaction equations.

We will now revisit the previous solutions to investigate what is actually happening to the shear stress as the amplitude is increased. For this reason we consider the derivatives of the shear stress for various amplitudes. Once again we consider $\bar{\lambda}$ which was defined to be the perturbation to the shear stress from the basic state shear stress $\lambda_0 = 1$. Figure 3.13 shows the first and second derivatives of the shear stress as the amplitude is increased. This confirms the extreme behaviour expected in the derivatives in the shear stress at specific spanwise locations. The sharp regions in this figure suggest that there could be small regions in the spanwise direction where spanwise diffusion becomes important although the ultimate fate of these states is beyond the scope of this thesis.

3.7. Conclusions

In this work we have set out the large Reynolds number scales for lower branch viscous Tollmien-Schlichting waves in the asymptotic suction boundary layer. With the important scales identified we increased the size of the wave until the correction to the vortex flow from the nonlinear self interaction of the wave is large enough to alter the basic state at the leading order. The resulting structure was found to consist of the strongly nonlinear interaction of a viscous wave within the lower deck which generates

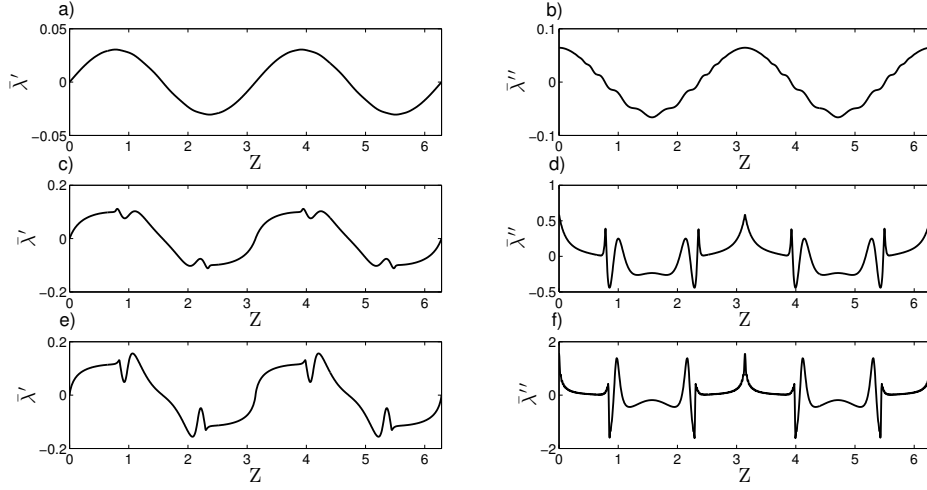


Figure 3.13. Shear stress derivatives for various amplitudes: $\mathcal{A}^2 = 0.005$ (a-b), 0.02 (c-d), 0.0242 (e-f).

a vortex within the lower deck which grows logarithmically with distance from the wall. This generates a vortex within the main deck which is large enough to alter the basic streak throughout the majority of the boundary layer. The perturbed streak suffers an instability which is propagated through the lower deck by the shear stress, sustaining the wave which drove the system away from the basic state in the first place.

Weakly nonlinear states which bifurcate from the linear neutral point were investigated analytically and good agreement with small amplitude states computed from the fully nonlinear vortex-wave interaction equations was found. Further to this a local spanwise analysis of the interaction equations was performed. An exact analytical solution of the local equations was found which is parameterised by the local gradient of the wall forcing. It was found that solutions to the local equations terminate at a particular critical value of this gradient which could indicate a breakdown of the interaction at a particular amplitude, although there are other possible explanations for termination. One such explanation could be that due to the nonlinear nature of the local equations, there could be more than one solution. Another is that solutions to the interaction equations may exist for all amplitudes with the local gradient of the wave forcing less than the critical value indicated by the local analysis.

For large amplitudes, strongly nonlinear states were found and spatial localisation was observed. Interestingly in contrast to the nonlinear states found in plane Poiseuille flow, the nonlinear states found in the asymptotic suction boundary layer also exhibit short

scale variations (as can be seen in the derivatives of the shear stress) at other spanwise locations. The exact nature of these variations is unknown but given the difficulty experienced trying to find solutions past a particular amplitude this could be indicative of a singularity in the governing interaction equations. Although the difficulties observed in the convergence of the Newton method could also be explained by a non-uniqueness of the solution. This is however speculation given no other solutions have been found in this work. At present there is no evidence to suggest anything other than spanwise localisation as the amplitude of the states is increased.

4. The excitation of Görtler vortices by freestream coherent structures

4.1. Introduction

In this chapter we will consider equilibrium states in the asymptotic suction boundary layer. In chapter 3 we identified an example of such states in this flow at large Reynolds number. These states consist of the nonlinear interaction of a Tollmien-Schlichting-wave with a vortex flow, and bifurcate from the basic flow via the viscous instability mechanism. Since the main effect of the wave forcing, generated by the nonlinear self-interaction of the wave, is on the near wall vortex flow this results in states which consist of near wall structures only.

Deguchi & Hall (2014), hereafter denoted by DH, identified a new class of equilibrium states in the asymptotic suction boundary layer. These are travelling wave states, travelling at close to the freestream velocity, which differ from the states which have been found before in that they contain a layer of $O(1)$ dimensions in the freestream where nonlinear structures are found. This layer, referred to as “the production layer” by DH, sits a large distance $\ln Re$ from the wall. The dynamics of this layer, which were discussed in section 1.5 and will be discussed further in section 4.4.1, also cause near wall structures in the flow. We refer to the structures in the production layer as freestream coherent structures.

The main focus of this chapter is to study the structures found by DH when firstly the plate is curved rather than flat, and secondly when the production layer is started impulsively at a specific downstream location. We will start with a study of Görtler vortices in the asymptotic suction boundary layer in the absence of freestream coherent structures. From this we move on to introduce freestream structures and incorporate them into the Görtler vortex bifurcation problem. Lastly, we consider the case of impulsively started freestream structures. We will see that in the presence of both effects, the response is potentially very complicated. By this we mean that depending on the values

of certain parameters in the problem, sometimes the solution could only be described analytically up to a particular streamwise location.

4.2. Formulation of the problem

We consider the asymptotic suction boundary layer when perturbed due to curvature, which is the flow over an infinite plate with curvature Λ , freestream velocity U_0 at infinity, and constant wall suction V_0 . Non-dimensionalising with the velocity scale U_0 , and the length scale ν/V_0 , the flow is governed by the non-dimensionalised Navier-Stokes equations in body-fitted coordinates (see appendix C),

$$\nabla \cdot \mathbf{u} = 0, \quad (4.1a)$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + Re^{-2} G(-uv, u^2, 0) = -\nabla p + Re^{-1} \nabla^2 \mathbf{u}, \quad (4.1b)$$

where we have ignored higher order terms (in Re) that play no part in the analysis which follows. Alternatively, we could consider the flow over a wall of height $h(x)$, and then assuming h is small and applying a Prandtl transformation we would recover the above equations. In the above equations, the streamwise direction is denoted by x , the wall normal direction by y , and the spanwise direction by z . These equations are to be solved subject to no slip conditions on the wall and far field matching conditions,

$$\mathbf{u} = (0, -Re^{-1}, 0), \quad y = 0, \quad \mathbf{u} \rightarrow (1, -Re^{-1}, 0), \quad y \rightarrow \infty. \quad (4.1c)$$

The Reynolds numbers Re is defined to be $Re = U_0/V_0$, and is assumed to be large. The Görtler number, denoted by G , is assumed to be $O(1)$ and is defined to be $G = -Re^2 \Lambda$, where Λ is the curvature of the plate. This flow configuration admits the leading order basic state solution,

$$\mathbf{u}_b = (1 - e^{-y}, -Re^{-1}, 0), \quad p_b = -Re^{-2} G \int_0^y u_b(s)^2 ds, \quad (4.2)$$

where $u_b(y) = 1 - e^{-y}$. In the absence of curvature, $G = 0$, this is an exact solution and is known as the asymptotic suction boundary layer. The presence of curvature does not affect the leading order basic state velocities - only the pressure. Plugging the basic state (4.2) into the Navier-Stokes equations (4.1) we see that the correction to the streamwise velocity for example is $O(Re^{-1})$.

We will restrict our attention to the well known characteristic scales for the develop-

ment of Görtler vortices. A brief justification of the scales can be reasoned as follows: we seek a strongly nonlinear perturbation throughout the wall layer and over an $O(1)$ spanwise length scale so we assume $u, y, z \sim O(1)$. Then a viscous inertial balance, $vu_y \sim Re^{-1}u_{yy}$, requires the wall normal velocity scale to be $v \sim O(Re^{-1})$. The continuity equation $u_x, w_z \sim v_y \sim O(Re^{-1})$ then yields the spanwise velocity and the streamwise length scales to be $w \sim O(Re^{-1})$, and $x \sim O(Re)$. Finally the momentum equation gives the balances $v_t, p_y \sim vv_y \sim O(Re^{-2})$, which in turn yields the pressure scale to be $p \sim O(Re^{-2})$, and the temporal scale to be $t \sim O(Re)$. Thus we set,

$$u = U, \quad (v, w) = Re^{-1}(V, W), \quad p = Re^{-2}P, \quad (t, x) = Re(\bar{t}, \bar{x}). \quad (4.3)$$

It is convenient to consider the perturbation to the basic state, and thus we separate the basic state from the perturbations which are denoted by over bars,

$$(U, V, W) = (u_b, -1, 0) + (\bar{u}, \bar{v}, \bar{w}), \quad (4.4)$$

The nonlinear disturbance equations governing the perturbations $\bar{u}, \bar{v}$ and $\bar{w}$ can be found in Hall (1988) but for the analysis that follows we will write them here in a slightly different form. For convenience we define the linear and nonlinear operators

$$\mathcal{L} = (\partial_y^2 + \partial_y + \partial_z^2 - \partial_{\bar{t}} - u_b \partial_{\bar{x}}), \quad \mathcal{N} = (\bar{u} \partial_{\bar{x}} + \bar{v} \partial_y + \bar{w} \partial_z). \quad (4.5)$$

The nonlinear disturbance equations governing the perturbations to the basic state can now be written in the form

$$\bar{u}_{\bar{x}} + \bar{v}_y + \bar{w}_z = 0, \quad (4.6a)$$

$$\mathcal{L}\bar{u} - u'_b \bar{v} = \mathcal{N}\bar{u}, \quad (4.6b)$$

$$\mathcal{L}(\bar{v}_z - \bar{w}_y) + u'_b \bar{w}_{\bar{x}} - 2Gu_b \bar{u}_z = (\mathcal{N} - \bar{u}_{\bar{x}})(\bar{v}_z - \bar{w}_y) + 2G\bar{u}\bar{u}_z - \bar{u}_y \bar{w}_{\bar{x}} + \bar{u}_z \bar{v}_{\bar{x}}. \quad (4.6c)$$

The equations are to be solved subject to no slip boundary conditions on the wall and far field decay conditions,

$$\bar{u} = \bar{v} = \bar{w} = 0, \quad y = 0, \quad \bar{u}, \bar{v}, \bar{w} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.6d)$$

In the following section we will consider linear, weakly nonlinear and fully nonlinear solutions of (4.6). In later sections we consider the corresponding solutions in the presence of freestream structures, which provide forcing on the near wall problem from

equation (4.6d) through the far field matching condition, as $y \rightarrow \infty$.

4.3. Analysis with no freestream coherent structures

4.3.1. Linear stability of the basic state

In this section we will consider the linear stability of the basic state as the Görtler number varies. We assume that the disturbances to the basic state are sufficiently small that nonlinear effects can be ignored. Solving the resulting set of linearised equations, we find either that the perturbations decay exponentially to zero in which case the flow is linearly stable, or the perturbations grow exponentially until such a point where nonlinear effects are no longer negligible in which case the flow is linearly unstable. Alternatively, the disturbances can remain at a constant small amplitude and in this case the flow is referred to as neutrally stable.

It is important to note here that in contrast to non-parallel boundary layers over the usual scales (see Hall (1983) for a discussion), a parallel flow analysis is valid here. In other words, since the basic state is independent of the streamwise and spanwise coordinates, disturbances to the basic state grow at a uniform rate independent of streamwise and spanwise location, and independent of time.

Linearising equations (4.6) we find

$$\bar{u}_{\bar{x}} + \bar{v}_y + \bar{w}_z = 0, \quad (4.7a)$$

$$\mathcal{L}\bar{u} - u'_b \bar{v} = 0, \quad (4.7b)$$

$$\mathcal{L}(\bar{v}_z - \bar{w}_y) + u'_b \bar{w}_{\bar{x}} - 2Gu_b \bar{u}_z = 0, \quad (4.7c)$$

where $\mathcal{L}$ is defined in (4.5). Because these equations are linear, and u_b only depends the wall normal coordinate y , we can decompose the solution into spanwise harmonics, each with a constant streamwise growth rate. For a spanwise wavenumber β this can be written as

$$(\bar{u}, \bar{v}) = e^{\sigma \bar{t} + \gamma \bar{x}} (\tilde{u}(y), \tilde{v}(y)) \cos \beta z, \quad \bar{w} = e^{\sigma \bar{t} + \gamma \bar{x}} \tilde{w}(y) \sin \beta z, \quad (4.8)$$

where the temporal growth rate σ , streamwise growth rate γ and the behaviour in the wall normal direction are to be determined in the following analysis. For the majority of the work which follows, we will restrict our attention to equilibrium solutions with

$\sigma = \gamma = 0$, i.e. zero temporal and streamwise growth rates. However, we will briefly consider the temporal stability problem (i.e. $\gamma = 0$, and $\sigma = \sigma(\beta, G)$) at the end of this section in comparison with previous work performed by Park & Huerre (1988).

Given the form of the solution (4.8), it is convenient for the linear and weakly nonlinear analysis to denote by $\mathcal{L}_n$ the form that linear operator $\mathcal{L}$ takes when applied to the n th spanwise harmonic,

$$\mathcal{L}_n = \partial_y^2 + \partial_y - n^2\beta^2. \quad (4.9)$$

Substituting (4.8) into equations (4.7) and eliminating $\bar{w}$ we find the eigenvalue problem for the growth rates σ or γ , in terms of the spanwise wavenumber β and the Görtler number G , to be

$$(\mathcal{L}_1 - \sigma - \gamma u_b)\tilde{u} - u_b'\tilde{v} = 0, \quad (4.10a)$$

$$(\mathcal{L}_1 - \sigma - \gamma u_b)(D^2 - \beta^2)\tilde{v} + \gamma u_b''\tilde{v} + 2\beta^2 G u_b\tilde{u} = 0. \quad (4.10b)$$

In the above D denotes derivatives taken in the wall normal direction $D := d/dy$. These equations are to be solved subject to no slip boundary conditions on the wall, and far field decay conditions. The conditions for no slip on $\tilde{u}$ and $\tilde{v}$ are straightforward, but for $\tilde{w}$ we must use the continuity equation, $\gamma\tilde{u} + \tilde{v}' + \beta\tilde{w} = 0$, to see that the corresponding conditions must be applied to $\tilde{v}'$. The boundary conditions are then found to be

$$\tilde{u} = \tilde{v} = \tilde{v}' = 0, \quad y = 0, \quad \tilde{u}, \tilde{v}, \tilde{v}' \rightarrow 0, \quad y \rightarrow \infty. \quad (4.10c)$$

Picking a spanwise wavenumber β and a Görtler number G , (4.10) is either an eigenvalue problem for the streamwise growth rate γ or for the temporal growth rate σ depending on whether we set $\sigma = 0$ or $\gamma = 0$, and the eigenfunctions are $\tilde{u}$ and $\tilde{v}$. Fixing the spanwise wavenumber, the sign of the streamwise/temporal growth rate indicates whether disturbances grow or decay with distance downstream/time passed and both growth and decay are possible. For example if we set $\gamma = 0$, then in solutions of (4.10) we find a critical value of the Görtler number $G_c(\beta)$ where (for the most unstable/slowest decaying mode) $\sigma(\beta, G) < 0$ if $G < G_c$ and $\sigma(\beta, G) > 0$ if $G > G_c$. This special point has the property that $\sigma(\beta, G_c(\beta)) = 0$ i.e. the flow is neutrally stable to some disturbances. We call this special curve where $\sigma = \gamma = 0$, denoted by $G_c(\beta)$, the neutral curve. There is actually a family of such curves $G_c^{(n)}(\beta)$ where we use n to denote the n th such curve, referred to as the n th neutral mode. It turns out that we can order these curves $0 < G_c^{(1)}(\beta) < G_c^{(2)}(\beta) < \dots$, as we see in figures 4.1(a) and 4.1(b). For fixed spanwise wavenumber β , increasing the Görtler number G through some $G_c^{(n)}(\beta)$

signifies the basic state becoming linearly unstable to another family of eigenmodes of (4.10).

Equations (4.10) can only be solved numerically. The solution procedure for finding neutral modes will briefly be described below, although the solution for non-zero growth rates is very similar. Firstly, to deal with the semi-infinite domain in the wall normal direction the domain is truncated at some large distance H from the wall. To be sure that H is large enough to well approximate the semi-infinite domain, the following procedure must be repeated until the solution is independent of the choice of H . For a particular choice of H , we scale the problem linearly onto the domain $[-1, 1]$ where an expansion of the solution in Chebyshev polynomials is sought. The method of domain truncation combined with the use of Chebyshev polynomials was investigated by Boyd (1982) where it was found to work well provided the function to be approximated is analytic, as we expect the solution to be. Although this method is straightforward to implement, one drawback is that improving the accuracy of the solution involves increasing H and whenever this is done the number of Chebyshev polynomials used must also be increased. This does not cause any problems here where the structures are confined to a layer near the wall and decay sufficiently fast. However, in section 4.3.4 when we consider fully nonlinear states of increasing amplitude, these states can now cover a much larger distance in the wall normal direction resulting in the need to increase H until the computation becomes too computationally expensive to run.

Applying the collocation method, the system of equations can now be written as a matrix $\mathbf{M}$ times the vector of Chebyshev coefficients is equal to the zero vector. The matrix $\mathbf{M}$ depends on the spanwise wavenumber and the critical Görtler number, and non-zero solutions are possible only if the determinant of $\mathbf{M}(\beta, G_c)$ is zero. Given a spanwise wavenumber β , this determines G_c and the Chebyshev coefficients of $\bar{u}$ and $\bar{v}$ can be determined by finding the zero eigenvector of $\mathbf{M}(\beta, G_c(\beta))$. Figure 4.1(a) shows a plot of $G_c(\beta)$ for the first mode, $G_c^{(1)}(\beta)$, where the differing line styles indicate whether the bifurcation is supercritical or subcritical which will be discussed in section 4.3.2. Figure 4.1(b) is a plot of the first three modes, $G_c^{(n)}(\beta)$ for $n = 1, 2$ and 3 with the character of the bifurcations identified with differing linestyles as in figure 4.1(a).

Park & Huerre (1995) considered, amongst other things, the linear temporal growth rate of Görtler vortices away from equilibrium - see table 1 of Park & Huerre (1995). Due to differences in the definition of the Görtler number and non-dimensionalisation, if we denote their definition of the Görtler number by $\bar{G}$, spanwise wavenumber by $\bar{\beta}$ and temporal growth rate by $\bar{\sigma}$ then we find they are related to the analogous quantities

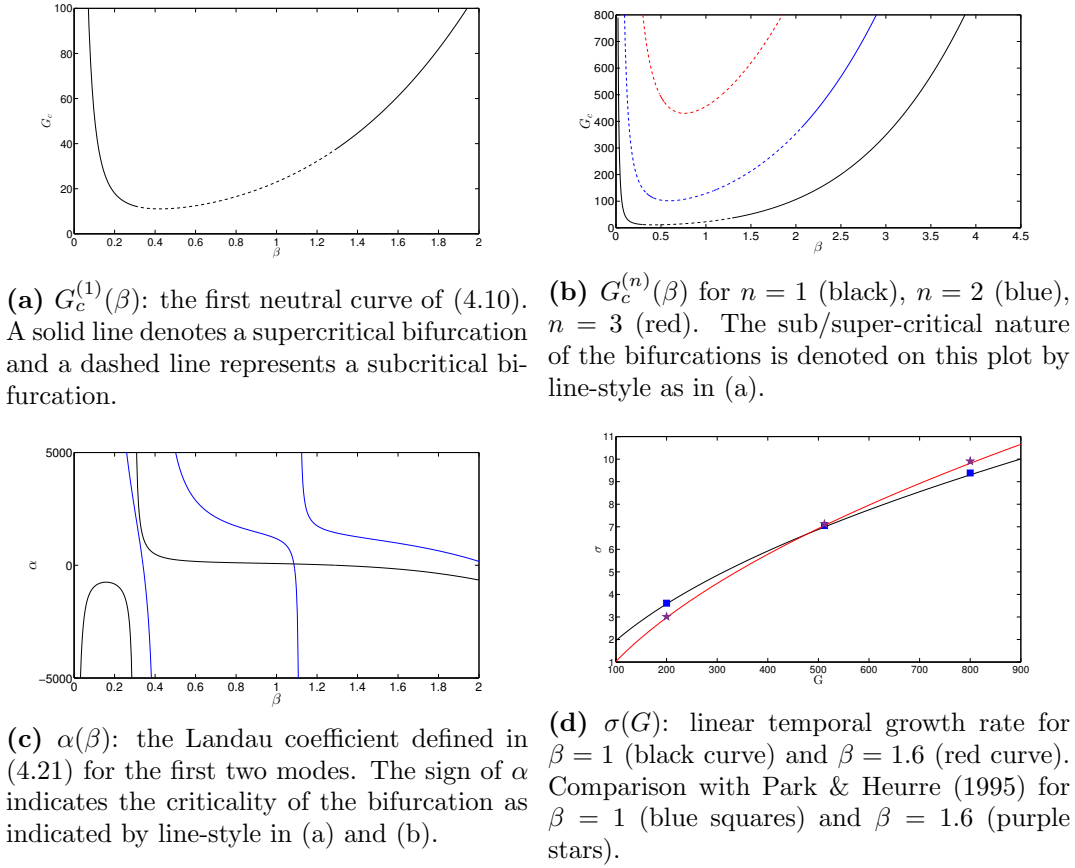


Figure 4.1. Linear and weakly nonlinear analysis results.

used here by $\bar{G} = \sqrt{G/8}$, $\bar{\beta} = \beta/2$ and $\bar{\sigma} = \sigma/4$. Their paper documented the linear temporal growth rate for $\bar{\beta} = 0.5$ ($\beta = 1$) and $\bar{\beta} = 0.8$ ($\beta = 1.6$). A comparison of these results can be found in figure 4.1(d) where the results of Park & Huerre (1988) are given by the blue squares and purple stars, and the results from this work are given by the solid black curve and the solid red curve. Good agreement between the two is observed.

Consider now the problem of finding small but finite amplitude equilibrium solutions. Then the critical Görtler number will lie close to the linear critical Görtler number, which we have denoted by G_c , and in fact small amplitude solutions can be tracked from this linear neutral point. Essentially the point $G_c(\beta)$ marks the Görtler number at which there is a bifurcation into finite amplitude equilibrium states known as Görtler vortices. This will be the focus of the next subsection.

4.3.2. Weakly nonlinear analysis for general spanwise wavenumbers

Consider once again equations (4.6), but for finite disturbances to the basic state. In particular the size of the disturbances will be such that the solutions are weakly nonlinear, and we will restrict our attention to streamwise neutral solutions, requiring $\partial_{\bar{x}} = 0$. Then (4.6a) allows a streamfunction formulation for $\bar{v}$ and $\bar{w}$ where

$$(\bar{v}, \bar{w}) = (\partial_z, -\partial_y)\Psi. \quad (4.11)$$

With this streamfunction formulation the governing equations (4.6) become

$$(\partial_y^2 + \partial_y + \partial_z^2 - \partial_t)\bar{u} - u'_b\Psi_z = (\Psi_z\partial_y - \Psi_y\partial_z)\bar{u}, \quad (4.12a)$$

$$(\partial_y^2 + \partial_y + \partial_z^2 - \partial_t)(\partial_y^2 + \partial_z^2)\Psi - 2Gu_b\bar{u}_z = (\Psi_z\partial_y - \Psi_y\partial_z)(\partial_y^2 + \partial_z^2)\Psi + 2G\bar{u}\bar{u}_z, \quad (4.12b)$$

with no slip boundary conditions on the wall and far field decay given by

$$\bar{u} = \Psi = \Psi_y = 0, \quad y = 0, \quad \bar{u}, \Psi, \Psi_y \rightarrow 0, \quad y \rightarrow \infty. \quad (4.12c)$$

We will seek solutions to these equations which are periodic in the spanwise direction, with spanwise wavenumber β .

Consider the situation where the disturbance to the basic flow is small, say $O(\epsilon)$, and the Görtler number is close to the critical value G_c . Under the assumption that the disturbances are small, the Görtler number must be close to G_c for non-trivial solutions to exist at leading order - see the linearised equations in section 4.3.1. We will now motivate the weakly nonlinear expansion which follows similarly to the analysis of Kelly & Pal (1976) and Hall & Walton (1979). This will lead to a Stuart-Landau equation - see Landau (1944) and Stuart (1960).

This will involve a discussion of the weakly nonlinear expansions for $\bar{u}$ and ψ in terms of the spanwise harmonics, where the n th spanwise harmonic denotes a term of the form $\cos n\beta z, \sin n\beta z$ for $\bar{u}, \Psi$ respectively. Consider the leading order term in the expansions at $O(\epsilon)$, which consists solely of the 1st harmonic. As in section 4.3.1 we find the governing equations are the linearised form of (4.12), and hence non-trivial solutions exist only if the Görtler number G is equal to the critical Görtler number $G_c(\beta)$ at leading order in ϵ . The $O(\epsilon)$ components of $\bar{u}, \Psi$ are also determined up to a one dimensional degree of freedom in the amplitude, denoted by A , which evolves over slow temporal and streamwise scales and will be determined at higher order. In fact,

$\bar{u}$ and Ψ are the zero-eigenfunctions of the now degenerate (through the choice of G_c) linear operator. The analysis so far is identical to section 4.3.1 with $\sigma = \gamma = 0$. This actually reaffirms our initial assumption that there is only one mode at leading order, because any other mode present say with spanwise wavenumber $\bar{\beta}$ would need to satisfy $G_c(\beta) = G_c(\bar{\beta})$ (see section 4.3.3 where we will discuss a special case of this). The components at $O(\epsilon)$ interact nonlinearly to generate terms at $O(\epsilon^2)$ in the 0th and 2nd harmonic, which are thus the terms in the expansions of $\bar{u}, \Psi$ at $O(\epsilon^2)$. At $O(\epsilon^2)$ we find two independent linear problems for the 0th and 2nd harmonics which are forced by the 1st harmonic terms at $O(\epsilon)$. In general this set of equations can be solved (again, see section 4.3.3 for the exception). The nonlinear interaction of the 1st harmonic at $O(\epsilon)$ and the 0th and 2nd harmonics at $O(\epsilon^2)$ generates 1st and 3rd harmonics at $O(\epsilon^3)$. As we will see, it is at $O(\epsilon^3)$ where the interesting dynamics are determined and hence the critical curvature perturbation scale and slow time scale (given leading order terms are $O(\epsilon)$) are $G - G_c \sim O(\epsilon^2)$ and $\partial_{\bar{t}} \sim O(\epsilon^2)$. If we consider only the problem for the 1st harmonic at $O(\epsilon^3)$ then we find that the homogenous part of the operator is identical to the linear operator found at $O(\epsilon)$, which through the determination of G_c is degenerate. However, the problem at $O(\epsilon^3)$ is forced by the nonlinear interaction of the lower order terms, and thus a solvability condition is found which determines the evolution equation for the amplitude A . Motivated by the above argument we introduce new parameters which measure variations on the scales of interest as derived above. We let G_1 denote variations of the Görtler number on the $O(\epsilon^2)$ scale, and τ denote variations in the slow time scale $O(\epsilon^{-2})$. Thus we set

$$G_1 = \epsilon^{-2}(G - G_c), \quad \tau = \epsilon^2 \bar{t}. \quad (4.13)$$

Exploiting symmetries in the spanwise direction the weakly nonlinear expansion, up to $O(\epsilon^3)$, is of the form

$$\begin{aligned}
 (\bar{u}, \Psi) = & \epsilon A(\tau) (U_1^{(1)} \cos(\beta z), \Psi_1^{(1)} \sin(\beta z)) + \epsilon^2 A(\tau)^2 (U_0^{(2)} + U_2^{(2)} \cos(2\beta z), \Psi_2^{(2)} \sin(\beta z)) \\
 & + \epsilon^3 (U_1^{(3)} \cos(\beta z) + U_3^{(3)} \cos(3\beta z), \Psi_1^{(3)} \sin(\beta z) + \Psi_3^{(3)} \sin(3\beta z)). \quad (4.14)
 \end{aligned}$$

We will now discuss the equations found at each order, noting once more that terms due to the variation of curvature or variations in time do not appear until $O(\epsilon^3)$. Substituting

(4.14) into (4.12) we find the governing equations at $O(\epsilon)$ are

$$\mathcal{L}_1 U_1^{(1)} - \beta u_b' \Psi_1^{(1)} = 0, \quad (4.15a)$$

$$\mathcal{L}_1 (\partial_y^2 - \beta^2) \Psi_1^{(1)} + 2G_c \beta u_b U_1^{(1)} = 0, \quad (4.15b)$$

where $\mathcal{L}_n$ is defined in (4.9). These equations are to be solved subject to homogeneous boundary conditions

$$U_1^{(1)}, \Psi_1^{(1)}, \partial_y \Psi_1^{(1)} = 0, \quad y = 0, \quad U_1^{(1)}, \Psi_1^{(1)}, \partial_y \Psi_1^{(1)} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.15c)$$

The leading order problem is identical to the linear stability problem in section 4.3.1 with $\sigma = 0$ in (4.10). As discussed earlier in this section and in section 4.3.1 this problem can be solved numerically and determines $G_c(\beta)$ and $U_1^{(1)}$ and $\Psi_1^{(1)}$ up to a degree of freedom in the scale, which is eliminated by requiring that the maximum of $U_1^{(1)}$ is 1.

Now considering terms at $O(\epsilon^2)$ when (4.14) is substituted into (4.12). We find the equations governing the 0th and 2nd harmonics are

$$\mathcal{L}_0 U_0^{(2)} = \frac{1}{2} \beta (\Psi_1^{(1)} \partial_y U_1^{(1)} + U_1^{(1)} \partial_y \Psi_1^{(1)}), \quad (4.16a)$$

$$\mathcal{L}_2 U_2^{(2)} - 2\beta u_b' \Psi_2^{(2)} = \frac{1}{2} \beta (\Psi_1^{(1)} \partial_y U_1^{(1)} - U_1^{(1)} \partial_y \Psi_1^{(1)}), \quad (4.16b)$$

$$\begin{aligned} \mathcal{L}_2 (\partial_y^2 - 4\beta^2) \Psi_2^{(2)} + 4G_c \beta u_b U_2^{(2)} = & -\frac{1}{2} \beta [U_1^{(1)}]^2 + \frac{1}{2} \beta \Psi_1^{(1)} \partial_y (\partial_y^2 - \beta^2) \Psi_1^{(1)} \\ & - \frac{1}{2} \beta (\partial_y \Psi_1^{(1)}) (\partial_y^2 - \beta^2) \Psi_1^{(1)}, \end{aligned} \quad (4.16c)$$

which are also to be solved subject to homogeneous boundary conditions,

$$U_0^{(2)}, U_2^{(2)}, \Psi_2^{(2)}, \partial_y \Psi_2^{(2)} = 0, \quad y = 0, \quad U_0^{(2)}, U_2^{(2)}, \Psi_2^{(2)}, \partial_y \Psi_2^{(2)} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.16d)$$

The only nonlinear terms in (4.16) are from the $O(\epsilon)$ terms which have already been determined, so this is a linear problem for the 0th and 2nd harmonic terms at $O(\epsilon^2)$ which can in general be solved provided the operator is not degenerate.

Now consider the problem for the first harmonic at $O(\epsilon^3)$, by once again substituting (4.14) into (4.12) to find

$$\mathcal{L}_1 U_1^{(3)} - \beta u_b' \Psi_1^{(3)} = A^3 F_3 + \frac{dA}{d\tau} U_1^{(1)}, \quad (4.17a)$$

$$\mathcal{L}_1 (\partial_y^2 - \beta^2) \Psi_1^{(3)} + 2G_c \beta u_b U_1^{(3)} = A^3 H_3 + AG_1 H_1 + \frac{dA}{d\tau} (\partial_y^2 - \beta^2) \Psi_1^{(1)}, \quad (4.17b)$$

where $F_3(y)$, $H_1(y)$ and $H_3(y)$ represent the contribution to the 1st harmonic by the nonlinear interaction of the different harmonics at $O(\epsilon)$ and $O(\epsilon^2)$ such as $U_1^{(1)}$, etc. These equations are to be solved subject to homogeneous boundary conditions on the wall and at infinity,

$$U_1^{(3)}, \Psi_1^{(3)}, \partial_y \Psi_1^{(3)} = 0, \quad y = 0, \quad U_1^{(3)}, \Psi_1^{(3)} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.17c)$$

The forcing terms from the lower orders take the forms,

$$2\beta^{-1}F_3 = \Psi_1^{(1)}(2\partial_y U_0^{(2)} + \partial_y U_2^{(2)}) + 2\Psi_2^{(2)}\partial_y U_1^{(1)} + 2(\partial_y \Psi_1^{(1)})U_2^{(2)} + (\partial_y \Psi_2^{(2)})U_1^{(1)}, \quad (4.18a)$$

$$\beta^{-1}H_1 = -2u_b U_1^{(1)}, \quad (4.18b)$$

$$2\beta^{-1}H_3 = -2G_c U_1^{(1)}(2U_0^{(2)} + U_2^{(2)}) + \Psi_1^{(1)}\partial_y(\partial_y^2 - 4\beta^2)\Psi_2^{(2)} - \Psi_2^{(2)}\partial_y(\partial_y^2 - \beta^2)\Psi_1^{(1)} \\ + 2(\partial_y \Psi_1^{(1)})(\partial_y^2 - 4\beta^2)\Psi_2^{(2)} - (\partial_y \Psi_2^{(2)})(\partial_y^2 - \beta^2)\Psi_1^{(1)}. \quad (4.18c)$$

As we mentioned in our discussion earlier in this section, the homogeneous part of the problem at $O(\epsilon^3)$ in equations (4.17) is identical to the leading order problem at $O(\epsilon)$ given by equations (4.15). The presence of forcing in (4.17) determines the a priori unknown amplitude evolution equation for A through the solvability condition, which is found to be

$$\bar{\mu}_0 \frac{dA}{d\tau} = \bar{\mu}_1 G_1 A + \bar{\mu}_3 A^3, \quad (4.19)$$

where the coefficients $\bar{\mu}_0$, $\bar{\mu}_1$ and $\bar{\mu}_3$ are constants which depend on the spanwise wavenumber β only. If we restrict our attention to equilibrium solutions, so $\partial_\tau = 0$, then we find the equilibrium amplitudes much satisfy

$$\mu_1 G_1 A + \mu_3 A^3 = 0, \quad (4.20)$$

where μ_1 and μ_3 depend on the spanwise wavenumber β . The equilibrium states are denoted by A_{eq} . Equation (4.19) describes the evolution of the amplitude of non-equilibrium solutions and hence we can deduce the stability of each of the equilibrium solutions by finding the sign of $dA/d\tau$ for $A > A_{eq}$ and $A < A_{eq}$. Both equilibrium solutions A_{eq} and their stability are given later in figure 4.6.

Varying the spanwise wavenumber is found to change the character of the bifurcation in (G_1, A) space. The non-zero solutions for A_{eq} satisfies $A_{eq}^2 = -G_1/\alpha$ where α is the

Landau coefficient defined to be

$$\alpha = \mu_3/\mu_1. \quad (4.21)$$

Thus the criticality of the bifurcation, whether there are solutions for $G_1 < 0$ or $G_1 > 0$, depends on the sign of α . In figure 4.1(a,b) we have indicated the character of the bifurcation (i.e. the sign of α), which is plotted in figure 4.1(c) for the first two modes. A solid curve is used when $\alpha < 0$ and the bifurcation is supercritical, and a dashed line is used when $\alpha > 0$ and the bifurcation is subcritical.

For the primary branch, two sign changes of the Landau coefficient are observed for $\beta \in (0, 2)$, and hence as the spanwise wavenumber increases the bifurcation goes from supercritical, to subcritical and back to supercritical over this range of spanwise wavenumbers. At the second of these critical points, $\beta \approx 1.3$ the Landau coefficient passes through zero smoothly. However, at the first of the critical points, $\beta = \beta_c \approx 0.30$, the change in sign is due to the Landau coefficient blowing up. As we will see this is due to a resonance between the first and second harmonics, and hence the second harmonic blows up. From equations (4.18) we see that this causes μ_3 to blow up. Clearly the weakly nonlinear analysis given in this section breaks down here, and it turns out that we must revisit our assumption that the leading order expansion at $O(\epsilon)$ consists of only a single harmonic.

4.3.3. Weakly nonlinear analysis for near resonant spanwise wavenumbers

Consider the problem for the 2nd harmonic at $O(\epsilon^2)$ given by equations (4.16b,c). Comparing this with the leading order problem at $O(\epsilon)$, we see that the homogeneous part of the equations governing the 2nd harmonic at $O(\epsilon^2)$ (4.16b, c) are identical to the equations governing the 1st harmonic at $O(\epsilon)$ if β is replaced by 2β . This means that the problem for the 2nd harmonic at $O(\epsilon^2)$ is degenerate if $G = G_c(2\beta)$, and this tells us that the expansion will break down if a spanwise wavenumber β_c is found such that $G_c(\beta_c) = G_c(2\beta_c)$. At this critical value of the spanwise wavenumber, the leading order problem supports both the 1st and 2nd harmonics, and in fact both of these terms must appear at leading order $O(\epsilon)$. This critical spanwise wavenumber coincides with the point where the bifurcation first changes from supercritical to subcritical as discussed in section 4.3.2, and the critical value of the spanwise wavenumber β_c is found to be approximately 0.30. Then in the expansion (4.14) there is a resonance of the 2nd harmonic terms at $O(\epsilon^2)$, $U_2^{(2)}$ and $\Psi_2^{(2)}$, which behave like $(\beta - \beta_c)^{-1}$ for β close to β_c ,

which in turn causes μ_3 to behave like $(\beta - \beta_c)^{-1}$ and thus μ_3 changes sign as β crosses β_c resulting in the observed change in character around β_c in section 4.3.2.

For β close to β_c then the expansion (4.14) breaks down. Let us discuss the relevant scaling argument in this case as we did in section 4.3.2 which was used to derive the expansions given in (4.14). Once again we assume the leading order disturbance to the basic state is $O(\epsilon)$ where ϵ is small. In this case, as we have discussed, both the 1st and 2nd harmonic appear at leading order. This gives us two eigenvalue problems at $O(\epsilon)$ which determines β_c and $G_c(\beta_c)$ (because we now require $G_c(\beta_c) = G_c(2\beta_c)$ although we actually already know the values of β_c and G_c from the previous analysis in section 4.3.2). Then the $O(\epsilon)$ terms in $\bar{u}, \Psi$ are determined up to two degrees of freedom represented by the amplitudes of the 1st and 2nd harmonics which are denoted by A and B respectively. The nonlinear interaction of the 1st and 2nd harmonics at $O(\epsilon)$ creates contributions at $O(\epsilon^2)$ to the 0th, 1st, 2nd, 3rd and 4th harmonic. As we will see it is at $O(\epsilon^2)$ where the amplitude evolution equations for A and B are determined, and thus we see that the critical curvature perturbation scale and slow time scale are $G - G_c \sim O(\epsilon)$ and $\partial_{\bar{t}} \sim O(\epsilon)$. Now the perturbation to the critical spanwise wavenumber needs to be considered and this, similarly to the perturbation of curvature from critical and the associated slow time variation, comes into the analysis at $O(\epsilon)$, and hence when $\beta - \beta_c \sim O(\epsilon)$. Considering only the 1st and 2nd harmonics at $O(\epsilon^2)$ we find that the homogeneous parts of the operators are identical to the leading order problems at $O(\epsilon)$, but these linear operators are degenerate and thus we find two solvability conditions which determine the hitherto unknown amplitude evolution equations for A and B .

As before, we introduce the new scales based on the relevant spanwise and curvature perturbation scales, and time evolution scale,

$$\beta_1 = \epsilon^{-1}(\beta - \beta_c), \quad G_2 = \epsilon^{-1}(G - G_c), \quad \hat{\tau} = \epsilon \bar{t}. \quad (4.22)$$

Similarly to (4.14) we can exploit symmetries in the spanwise direction to write the solution up to $O(\epsilon^2)$ in the form

$$\begin{aligned} (\bar{u}, \Psi) = & \epsilon \hat{A}(\tau) (\hat{U}_1^{(1)} \cos(\beta z), \hat{\Psi}_1^{(1)} \sin(\beta z)) + \epsilon \hat{B}(\tau) (\hat{U}_2^{(1)} \cos(2\beta z), \hat{\Psi}_2^{(1)} \sin(2\beta z)) \\ & + \epsilon^2 (\hat{U}_1^{(2)} \cos(\beta z) + \hat{U}_2^{(2)} \cos(2\beta z), \Psi_1^{(2)} \sin(\beta z) + \Psi_2^{(2)} \sin(2\beta z)), \end{aligned} \quad (4.23)$$

where the other harmonics (0th, 3rd and 4th) at $O(\epsilon^2)$ can be ignored because they do not affect the analysis which follows. Before we look at the equations note that because β has now changed from being a general $O(1)$ spanwise wavenumber, to being

a perturbative quantity it is convenient to denote the leading order part of $\mathcal{L}_n$ by $\mathcal{L}_n^c$ so that

$$\mathcal{L}_n^c := (\partial_y^2 + \partial_y - \beta_c^2), \quad \mathcal{L}_n = \mathcal{L}_n^c - 2\beta_c\beta_1\epsilon - \beta_1^2\epsilon^2. \quad (4.24)$$

Substituting the new expansion (4.23) into equations (4.12) we find at $O(\epsilon)$, the equations and boundary conditions, for the 1st harmonic are

$$\mathcal{L}_1^c \hat{U}_1^{(1)} - \beta u'_b \hat{\Psi}_1^{(1)} = 0, \quad (4.25a)$$

$$\mathcal{L}_1^c (\partial_y^2 - \beta_c^2) \hat{\Psi}_1^{(1)} + 2\beta_c G_c u_b \hat{U}_1^{(1)} = 0, \quad (4.25b)$$

$$\hat{U}_1^{(1)} = \hat{\Psi}_1^{(1)} = \partial_y \hat{\Psi}_1^{(1)} = 0, \quad y = 0, \quad \hat{U}_1^{(1)}, \hat{\Psi}_1^{(1)}, \partial_y \hat{\Psi}_1^{(1)} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.25c)$$

Similarly for the second harmonic we find

$$\mathcal{L}_2^c \hat{U}_2^{(1)} - 2\beta u'_b \hat{\Psi}_2^{(1)} = 0, \quad (4.26a)$$

$$\mathcal{L}_2^c (\partial_y^2 - 4\beta_c^2) \hat{\Psi}_2^{(1)} + 4\beta_c G_c u_b \hat{U}_2^{(1)} = 0, \quad (4.26b)$$

$$\hat{U}_2^{(1)} = \hat{\Psi}_2^{(1)} = \partial_y \hat{\Psi}_2^{(1)} = 0, \quad y = 0, \quad \hat{U}_2^{(1)}, \hat{\Psi}_2^{(1)}, \partial_y \hat{\Psi}_2^{(1)} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.26c)$$

Equations (4.25) and (4.26) form a nonlinear eigenvalue problem for (β_c, G_c) , and the eigenfunctions are $\hat{U}_1^{(1)}, \hat{\Psi}_1^{(1)}, \hat{U}_2^{(1)}$ and $\hat{\Psi}_2^{(1)}$.

We can now consider the nonlinear effects at $O(\epsilon^2)$ where time modulations, variations from the critical Görtler number G_c and now also from the critical spanwise wavenumber β_c are important. Substituting (4.23) into (4.12) the equations governing the 1st harmonic at $O(\epsilon^2)$ are

$$\mathcal{L}_1^c \hat{U}_1^{(2)} - \beta u'_b \hat{\Psi}_1^{(2)} = \hat{F}_1^{(1)} \frac{d\hat{A}}{d\hat{\tau}} + \hat{F}_2^{(1)} \beta_1 \hat{A} + \hat{F}_3^{(1)} \hat{A} \hat{B}, \quad (4.27a)$$

$$\mathcal{L}_1^c (\partial_y^2 - \beta_c^2) \hat{\Psi}_1^{(2)} + 2\beta_c G_c u_b \hat{U}_1^{(2)} = \hat{H}_1^{(1)} \frac{d\hat{A}}{d\hat{\tau}} + \hat{H}_2^{(1)} \beta_1 \hat{A} + \hat{H}_3^{(1)} G_2 \hat{A} + \hat{H}_4^{(1)} \hat{A} \hat{B}, \quad (4.27b)$$

$$\hat{U}_1^{(2)} = \hat{\Psi}_1^{(2)} = \partial_y \hat{\Psi}_1^{(2)} = 0, \quad y = 0, \quad \hat{U}_1^{(2)}, \hat{\Psi}_1^{(2)}, \partial_y \hat{\Psi}_1^{(2)} \rightarrow 0, \quad y \rightarrow \infty. \quad (4.27c)$$

Similarly for the second harmonic we find

$$\mathcal{L}_2^c \hat{U}_2^{(2)} - 2\beta u'_b \hat{\Psi}_2^{(2)} = \hat{F}_1^{(2)} \frac{d\hat{B}}{d\hat{\tau}} + \hat{F}_2^{(2)} \beta_1 \hat{B} + \hat{F}_3^{(2)} \hat{A}^2, \quad (4.28a)$$

$$\mathcal{L}_2^c (\partial_y^2 - 4\beta_c^2) \hat{\Psi}_2^{(2)} + 4\beta_c G_c u_b \hat{U}_2^{(2)} = \hat{H}_1^{(2)} \frac{d\hat{B}}{d\hat{\tau}} + \hat{H}_2^{(2)} \beta_1 \hat{B} + \hat{H}_3^{(2)} G_2 \hat{B} + \hat{H}_4^{(2)} \hat{A}^2, \quad (4.28b)$$

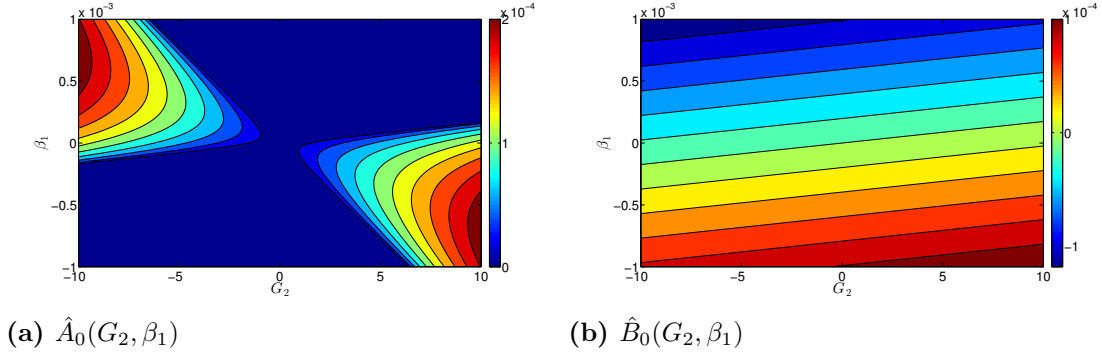


Figure 4.2. Plots of $\hat{A}_0$ and $\hat{B}_0$ for varying G_2 and β_1 .

$$\hat{U}_2^{(2)} = \hat{\Psi}_2^{(2)} = \partial_y \hat{\Psi}_2^{(2)} = 0, \quad y = 0, \quad \hat{U}_2^{(2)}, \hat{\Psi}_2^{(2)}, \partial_y \hat{\Psi}_2^{(2)} \rightarrow 0, \quad y \rightarrow \infty, \quad (4.28c)$$

where the $\hat{F}_i^{(j)}$ and $\hat{H}_i^{(j)}$ are functions of the wall normal variable y , which are defined implicitly through the leading order terms $\hat{U}_1^{(1)}, \hat{\Psi}_1^{(1)}, \hat{U}_2^{(1)}$ and $\hat{\Psi}_2^{(1)}$. Comparing the equations at $O(\epsilon)$ and $O(\epsilon^2)$ we see that the linear operators on the right hand sides of (4.27) and (4.28) are identical to the linear operators at leading order in (4.25) and (4.26) which determined β_c and G_c through the condition that the two linear operators are degenerate at leading order. Thus taking inner products of (4.27) and (4.28) with zero eigenfunctions of the corresponding adjoint operators, we find via the solvability conditions that the evolution equations for A and B are

$$\frac{d\hat{A}}{d\hat{\tau}} = a_1\beta_1\hat{A} + a_2G_2\hat{A} + a_3\hat{A}\hat{B}, \quad (4.29a)$$

$$\frac{d\hat{B}}{d\hat{\tau}} = b_1\beta_1\hat{B} + b_2G_2\hat{B} + b_3\hat{A}^2. \quad (4.29b)$$

In the above, the a_i and b_i are constants determined by the inner product of $\hat{F}_i^{(j)}$ and $\hat{H}_i^{(j)}$ with the zero eigenfunctions of the adjoint of the operators in (4.25) and (4.26). These equations have a quadratic nonlinearity in comparison to the non-resonant case which has a cubic nonlinearity. These equations (4.29) (and the imperfect pair (4.47) which will be discussed in section 4.4.2) were found and investigated by Hall (1980) in the context of the two- and four-cell bifurcation problem in finite containers.

We will briefly consider the equilibrium solutions of (4.29) where $d\hat{A}/d\hat{\tau} = d\hat{B}/d\hat{\tau} = 0$, and their stability will be discussed later in section 4.4.2. There are either one or three solutions depending on the values of β_1 and G_2 . The first is the trivial solution $\hat{A} = \hat{B} = 0$, which is always a solution. The other two possible solutions are the pair of

solutions given by $(\hat{A}, \hat{B}) = (\pm \hat{A}_0, \hat{B}_0)$ where

$$(\hat{A}_0, \hat{B}_0) = \left(\sqrt{\frac{(a_1\beta_1 + a_2G_2)(b_1\beta_1 + b_2G_2)}{a_3 b_3}}, -\frac{(a_1\beta_1 + a_2G_2)}{a_3} \right). \quad (4.30)$$

These solutions exist provided the argument of the square root is positive. This happens if $f_1(G_2) < \beta_1 < f_2(G_2)$, for functions $f_1(G_2)$ and $f_2(G_2)$ which are zeros of the square root in (4.30), given by the straight lines with gradients $-b_2/b_1$ and $-a_2/a_1$. This can be seen from the plots of $\hat{A}_0$ and $\hat{B}_0$ in figure 4.2.

4.3.4. Computation of fully nonlinear Görtler vortices

In this section we consider equations (4.12) once again, but now we will look for fully nonlinear solutions which must be done computationally. These equations were solved using a spectral method. Firstly to deal with the semi-infinite domain, the domain was truncated in the wall-normal direction at some large distance H from the wall, and then mapped linearly onto $[-1, 1]$, where we will use v to denote the new bounded and scaled wall-normal coordinate. This method for dealing with semi-infinite domains was discussed previously in section 4.3.1, but we will recap here. The idea is to repeat the computation increasing the truncation height H until the results are independent of H . At such a point we can be confident that the domain is now large enough to be a good approximation of the semi-infinite domain of interest.

A Chebyshev/Fourier basis was applied with the collocation/Galerkin method in the wall-normal/spanwise direction. We denote the n th Chebyshev polynomial by T_n . We can exploit symmetries in the the governing equations to write the solutions in the form of the series expansions,

$$\bar{u} = \sum_{n,k=1} \bar{u}_{n,k} T_{n-1}(v) \cos(k-1)\beta z, \quad \Psi = \sum_{n,k=1} \Psi_{n,k} T_{n-1}(v) \sin k\beta z. \quad (4.31)$$

These expansions are truncated at suitably large $(n, k) = (N, K)$, determined by requiring sufficient decay in the spectral coefficients, typically to around 10^{-8} . The resulting equations for $\bar{u}_{n,k}$ and $\psi_{n,k}$ are nonlinear, and are thus solved using the Newton method. Starting off the Newton iteration requires some thought because $\bar{u}_{n,k} = \psi_{n,k} = 0$ for all n, k is always a solution of the system. There are multiple ways to get around this issue such as using the predictions of the weakly nonlinear analysis as starting guesses, but here we used a random initial guess (with some sensible restrictions) to find a single

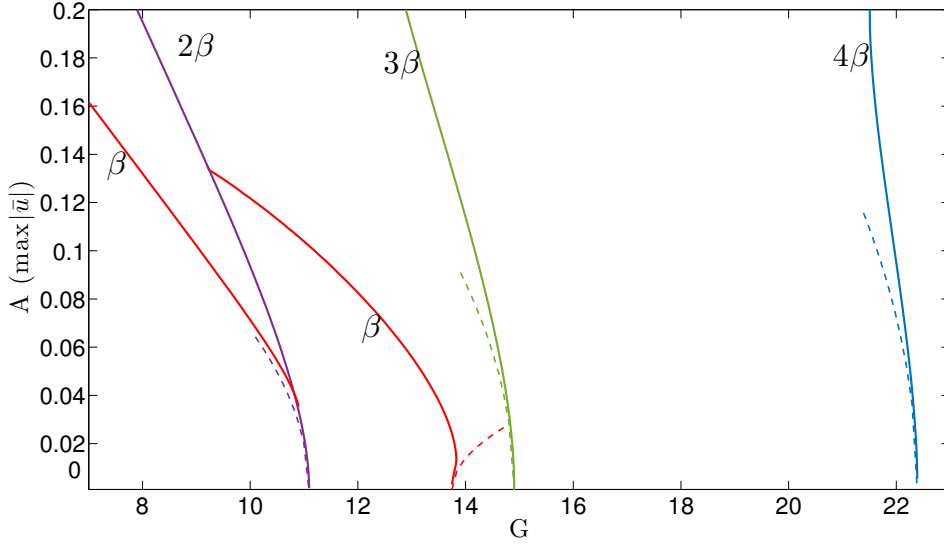


Figure 4.3. Bifurcation curves in (G, A) space for spanwise wavenumbers $\beta, 2\beta, 3\beta$ and 4β as labelled, where $\beta = 0.25$. The solid lines represent bifurcation curves calculated from the nonlinear equations (4.12) and the dashed lines are the corresponding weakly nonlinear analysis from (4.20).

solution. Once we have one solution, it is possible to generate all other solutions which bifurcate from the first mode linear neutral point by tracking the solution in the Görtler number G as we did here. Solutions could also be tracked in the spanwise wavenumber β , or some measure of the amplitude of the state such as the infinity norm $\|\bar{u}\|_\infty$ which will be used to visualise some of the bifurcations.

The solid lines in figure 4.3 show the results of this computation for the spanwise wavenumber $\beta = 0.25$ and the three harmonics, $2\beta, 3\beta$ and 4β . The dashed lines show the results of the weakly nonlinear analysis of section 4.3.2, which captures excellently the small amplitude behaviour of the nonlinear bifurcation curves. For the spanwise wavenumber β the bifurcation from the linear critical point $G_c(\beta) \approx 13.74$ is supercritical but there is a turning point at a small amplitude ($\max |\bar{u}| \approx 0.02$). Increasing the amplitude further, a period halving bifurcation is found signalling the end of the β branch. The bifurcations for the higher harmonics are subcritical and some of these branches have been tracked to turning points when $\max |\bar{u}| \approx 0.3$, although as the amplitude increases the wall normal size of the structures increases requiring an increase in the wall normal truncation height H and resolution adding computational cost. Thus we will not pursue solutions to a larger amplitude than that shown in figure 4.3 of

$\max |\bar{u}| \approx 0.2$. The 2β branch bifurcates from its linear neutral point $G_c(2\beta) \approx 11$ and soon after, at a fairly small amplitude, there is a secondary bifurcation of a branch with spanwise wavenumber β from the 2β branch, after which both branches continue to larger amplitudes. The 3β and 4β branches bifurcate from their respective linear neutral points, and continue into larger amplitudes. As we will discuss shortly, it is possible to test for secondary bifurcations from the branches found so far provided they are of harmonic β or higher. This can be done using the Jacobian, which is calculated in the Newton method. Using this to check for more bifurcations, we conclude that figure 4.3 contains all bifurcations (in the interval shown) with spanwise wavenumber β or a higher harmonic thereof, which are connected to one of the bifurcation curves found. This does not rule out the possibility of subharmonic bifurcations, of spanwise wavenumber $\beta/2$ for example, which couldn't be picked up in this computation because the numerical method looks for solutions which are periodic in a spanwise box of size $2\pi/\beta$. General subharmonics would be difficult to find using the numerical method applied so far, so we will concentrate on the first subharmonic by re-running the computations with β replaced by $\beta/2$. There is no reason to think that we have picked a special wavenumber here ($\beta = 0.25$ in figure 4.3). We expect the situation to be generic provided the spanwise wavenumber is not changed by too much.

To demonstrate this we will consider the spanwise wavenumber 0.295, but to allow for the possibility of picking up the first subharmonic we alter the spanwise wavenumber so that $2\beta = 0.295$, i.e. $\beta = 0.1475$. For the purpose of comparing the results with figure 4.3, we let $\beta^* = 2\beta$. The results of this computation can be seen in figure 4.4. In this figure we have denoted the linear neutral points by brown squares, and the secondary and saddle node bifurcations by black circles for clarity. Taking only the 2β and 4β branches (β^* and $2\beta^*$ branches) into consideration for a moment, we see a very similar bifurcation structure as in figure 4.3. However, our suspicions of subharmonic bifurcations are well founded, as we find branches with spanwise wavenumber β ($\beta^*/2$) connecting the 2β and 4β (β^* and $2\beta^*$) branches, and also the 2β and 5β (β^* and $5\beta^*/2$) branches. This begs the question of whether halving the spanwise wavenumber again would reveal more subharmonic bifurcations but this issue, while interesting, will not be addressed here.

In section 4.3.3 we investigated how the bifurcation behaves around the resonant spanwise wavenumber $\beta = \beta_c$, where the bifurcation changes from supercritical to subcritical. To illustrate this in figure 4.5 we plot the bifurcation curves for β and 2β for $\beta = 0.28, \beta = 0.295, \beta \approx \beta_c$ and $\beta = 0.31$ in (a), (b), (c) and (d), respectively. In figures 4.5(a) and (b) where $\beta < \beta_c$ we see a period halving bifurcation where the β branch,

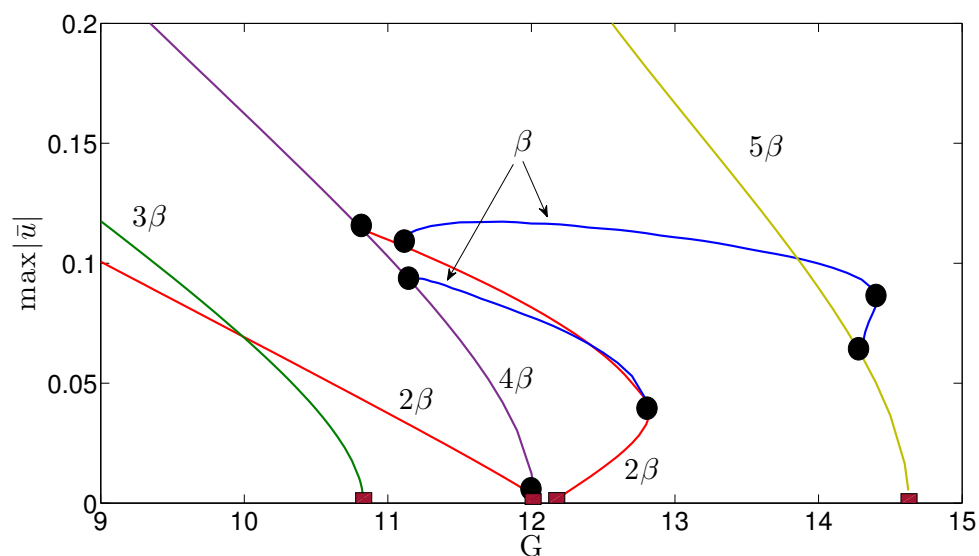


Figure 4.4. Bifurcation curves in (G, A) space for spanwise wavenumbers $\beta, 2\beta, 3\beta, 4\beta$ and 5β as labelled, where $\beta = 0.1475$. The solid lines represent the bifurcation curves with different colours denoting different spanwise wavenumbers as labelled, for clarity. The brown squares denote the linear neutral points for spanwise wavenumbers $2\beta, 3\beta, 4\beta$ and 5β (the linear neutral point for spanwise wavenumber β occurs at a much larger Görtler number). The black circles show the locations of the secondary bifurcations and saddle node bifurcations (turning points).

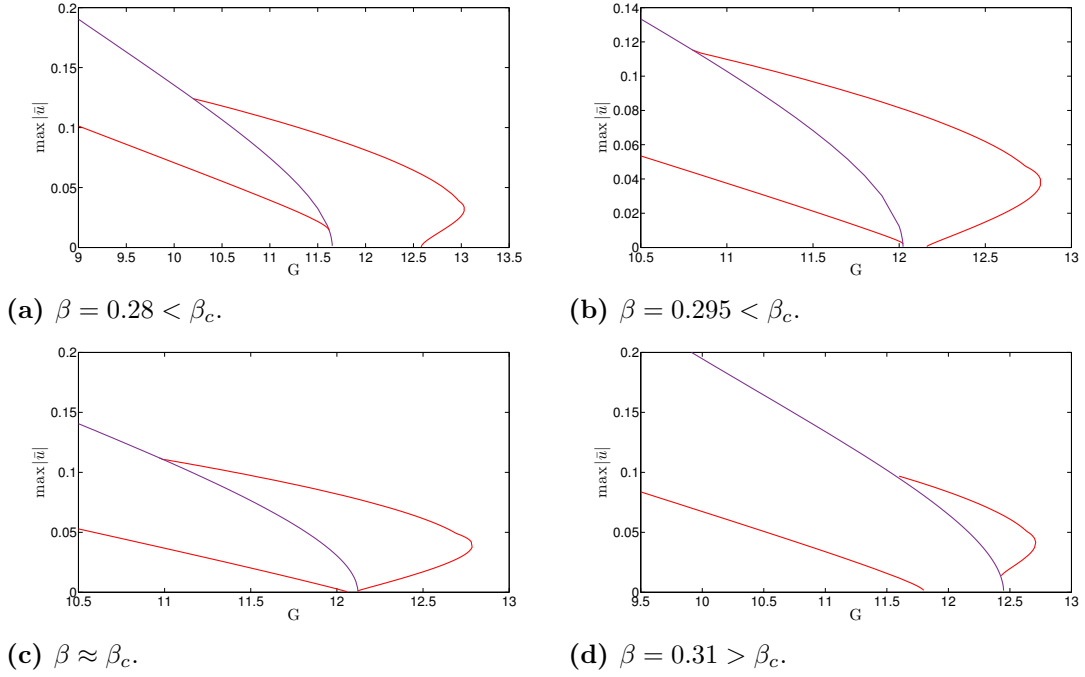


Figure 4.5. Bifurcation curves for β (red) and 2β (purple) modes only, highlighting the change in behaviour as β crosses β_c for nonlinear equilibrium states computed from (4.12).

which bifurcated from its linear neutral point, bifurcates into the 2β branch. Also we see a secondary bifurcation of a β branch from the 2β branch. In figure 4.5(d) where $\beta > \beta_c$, the behaviour is in fact the opposite to (a) and (b), and now a β branch bifurcates from the 2β branch and then joins back onto the 2β branch in a further bifurcation. From these figures we can see exactly how the bifurcation curves evolve as β crosses β_c and the bifurcation changes from supercritical to subcritical through an interaction with the second harmonic.

Given the number of bifurcations in figures 4.3, 4.4 and 4.5, it is natural to ask if we have captured all the secondary bifurcations - we can be sure we have captured all of the primary bifurcations because of the analysis we performed in section 4.3.1. We could perform a linear stability analysis of each solution but in fact the numerical method applied here gives a much easier way to check for secondary bifurcations. Our method applied with a spanwise wavenumber β looks for solutions to the governing equations (4.12) which are periodic in a spanwise box of length $2\pi/\beta$. Provided any other solution is periodic in this box, it could also potentially be picked up using this numerical method and thus at any secondary bifurcation point the Jacobian in the Newton method must

become singular. Thus, to check for secondary bifurcations with spanwise wavenumber β or any higher harmonic thereof, we only need to monitor for example the condition number of the Jacobian along the solution branch. This method is equivalent to looking at the secondary stability of the state since the Jacobian describes the linearised behaviour around the state. When the condition number blows up this corresponds to a change in the sign of one of the eigenvalues and hence a change in the stability of one of the eigenmodes of the state. We in fact look at the reciprocal of the condition number, so we are interested in points where the reciprocal of the condition number is zero. This was the method used to find the location of the secondary bifurcation curves, and then initialising the Newton method with the (suitably scaled) eigenfunctions of the Jacobian yields the new branches. Evidently, this method will not indicate when there are subharmonic bifurcations, but as alluded to before we expect that further subharmonic branches (i.e. $\beta/4$, etc.) are present in the state space.

4.4. Receptivity to exact freestream coherent structures

4.4.1. Production layer analysis

In this subsection we will review the freestream coherent structures found by DH. The only difference between the derivation here and the work of DH is that in the absence of curvature an exact solution can be found for the basic state. Here with the presence of curvature, we must work with the leading order basic state where for example the pressure is $-Re^{-2}Gy$ + higher order terms, but in fact this difference does not affect the leading order analysis. The scales for the interaction can either be deduced by an inspection of the equations, as we will do here to motivate the scales, or they could equally well be deduced from the numerical results of DH. These freestream coherent structures consist of a travelling wave state propagating in the streamwise direction at close to the freestream velocity. Activity is concentrated in two regions. The first of these regions is the subject of this subsection, and the second will be the subject of the next subsection. Firstly, there is a large amount of activity confined to a layer of $O(1)$ extent which sits in the freestream a large distance, denoted by κ , from the wall. The majority of this subsection will be devoted to an examination of this layer, which will reveal the scales over which the interaction occurs, and the distance κ at which the layer sits from the wall. Towards the wall it turns out that the streamwise velocity is growing exponentially. This growth continues until the second region of increased activity is reached, which is another layer of $O(1)$ extent but this time located near the

wall where the velocity must adjust to satisfy the boundary conditions on the wall. Due to the nonlinear interaction in the layer in the freestream, the flow which the wall layer sees in the freestream is some perturbation to the basic state, and this drives near wall structures.

Now we will concentrate on the first region where there is strong activity in a layer contained within the freestream, which is known as the production layer. We are interested in a nonlinear travelling wave structure where all length scales in the interaction are comparable, in this case we assume they are all $O(1)$. For a viscous-inviscid balance $vu_y \sim wu_z \sim Re^{-1}u_{yy}$ so we require $v, w \sim O(Re^{-1})$. Then from a wall normal pressure balance $p_y \sim Re^{-1}v_{yy}$ we find $p \sim O(Re^{-2})$. As this structure sits in the freestream we know that $u \sim 1$, and in fact the continuity equation $u_x \sim v_y$ yields $u - 1 \sim O(Re^{-1})$. The streamwise advection term $\partial_t + u\partial_x$ written in a travelling wave coordinate $X_{TW} = x - ct$ is $(u - c)u_{X_{TW}}$. Since the streamwise momentum equation is at $O(Re^{-1})$, to have a nontrivial interaction we require $u - c \sim O(Re^{-1})$ and thus the wavespeed c to be close to the freestream velocity, $c - 1 \sim O(Re^{-1})$. Further, since u must include the basic state $1 - e^{-y}$ as a solution, it follows that the production layer must be located where $e^{-y} \sim Re^{-1}$, which happens when $y \sim \ln Re$. Given that we have assumed $O(1)$ length scales we find that the production layer is located a distance $\ln Re$ from the wall and covers an $O(1)$ distance in the wall normal direction. This motivates the introduction of the wall normal production layer coordinate, Y , defined by

$$Y = y - \kappa(Re), \quad (4.32)$$

where

$$\kappa(Re) = \ln Re. \quad (4.33)$$

Motivated by the above scaling arguments, we let $c = 1 - Re^{-1}c_1$ and

$$(u, v, w) = (1, 0, 0) + Re^{-1}(\hat{U}, \hat{V}, \hat{W}), \quad p = Re^{-2}(\hat{P} - GY). \quad (4.34)$$

The variables (U, V, W, P) depend on (X_{TW}, Y, z) where X_{TW} is the travelling wave coordinate $X_{TW} = x - ct$. The governing equations in the production layer are found to be the full Navier-Stokes equations at unit Reynolds number in a streamwise travelling frame,

$$\nabla_1 \cdot \hat{\mathbf{U}} = 0, \quad (4.35a)$$

$$(\hat{\mathbf{U}} \cdot \nabla_1)\hat{\mathbf{U}} + c_1\hat{\mathbf{U}}_{X_{TW}} = -\nabla_1\hat{P} + \nabla_1^2\hat{\mathbf{U}}, \quad (4.35b)$$

where $\nabla_1 := (\partial_{X_{TW}}, \partial_Y, \partial_z)$, and we have the far field matching conditions,

$$\hat{\mathbf{U}} \sim \begin{cases} (0, -1, 0) & Y \rightarrow \infty, \\ (-e^{-Y}, -1, 0) & Y \rightarrow -\infty. \end{cases} \quad (4.35c)$$

Equations (4.35) govern the strongly nonlinear interaction of a wave travelling in the streamwise direction and a vortex flow inside the production layer. We note that curvature has no effect in the production layer at leading order, and thus these are exactly the equations solved by DH. In other words, the form of the production layer forcing on the flow below is the same at leading order whether curvature is present or not. We consider the effect of the production layer on the structures below, noting that the boundary conditions (4.35c) allow for exponential growth of the streak provided it is not as fast as e^{-Y} . To do this we linearise in the far field below the production layer as $Y \rightarrow -\infty$ by setting

$$\hat{\mathbf{U}} = (-e^{-Y}, -1, 0) + \hat{\mathbf{u}}, \quad \hat{P} = \hat{p}. \quad (4.36)$$

The wave is observed (by DH) to decay below the production layer so the variables only depend on Y and z in the far field i.e. $\hat{\mathbf{u}}(Y, z)$ and $\hat{p}(Y, z)$ are independent of X_{TW} . Substituting (4.36) into (4.35) and linearising, the governing equations for the perturbations $(\hat{u}, \hat{v})$ are found to be

$$(\partial_Y^2 + \partial_Y + \partial_z^2)\hat{u} - e^{-Y}\hat{v} = 0, \quad (4.37a)$$

$$(\partial_Y^2 + \partial_Y + \partial_z^2)(\partial_Y^2 + \partial_z^2)\hat{v} = 0. \quad (4.37b)$$

There are multiple possible solutions of the form $\hat{u} \propto e^{(\omega-1)Y}$ and $\hat{v} \propto e^{\omega Y}$, where ω satisfies $(\omega^2 + \omega - \beta^2)(\omega^2 - \beta^2) = 0$. The boundary conditions (4.35c) confine ω to lie in the interval $(0, 1)$ because the streak must grow approaching the wall to generate near wall structures - but no faster than e^{-Y} . The largest value of ω which satisfies this criteria and hence the ω of interest is found to be $\omega = (-1 + \sqrt{1 + 4\beta^2})/2$, which dominates in the far field of the production layer and causes exponentially growth towards the wall to drive near wall structures. Recalling that we require $0 < \omega < 1$, we see from the expression for ω that this argument is only valid provided $0 < |\beta| < \sqrt{2}$. The relevant far field behaviour is then found to be

$$(\hat{u}, \hat{v}) = (Je^{-Y}, K)e^{\omega Y} \cos \beta z, \quad (4.38)$$

where J and K are constants. These constants are related by the condition $K = -2\omega J$. The constant J depends on both the streamwise and spanwise wavenumber of the non-linear interaction inside the production layer, and can only be determined by solving the relevant equations (4.35) which must be done computationally. For the purposes of this work it is enough to know that solutions to this set of equations do exist as demonstrated in the work of DH. Noting that $e^{\omega Y} = Re^{-\omega} e^{\omega y}$ we conclude that the matching from the lower region to the production layer requires

$$(U, V) \sim (u_b, -1) + \delta e^{\omega y} (J e^{-y}, K) \cos(\beta z), \quad (4.39)$$

as $y \rightarrow \kappa$, where $\delta = Re^{-\omega}$. No further analysis of the production layer is required for this work. The rest of this chapter will focus on the excitation of Görtler vortices by the freestream coherent structures which are felt through the matching condition (4.39) and act as a forcing term on the near wall problem in section 4.4.2.

4.4.2. Wall layer analysis

As discussed in the previous subsection, the presence of the production layer causes structures to develop near the wall where the flow must adjust to satisfy the no slip boundary conditions on the wall. Recalling the results of section 4.3.2, we saw the formation of near wall structures in the absence of the production layer. These structures were caused purely by wall curvature. In this subsection we will see how the structures observed within the wall layer in section 4.3.2 are affected by the presence of freestream coherent structures. If we now consider the flow near the wall where $y \sim O(1)$, then the governing equations (4.12) are altered by the far field matching condition inherited from the production layer via equation (4.39). The far field matching in the governing equation (4.12c) is replaced by

$$(\bar{u}, \Psi) \sim \delta (J e^{(\omega-1)y} \cos \beta z, \beta^{-1} K e^{\omega y} \sin \beta z), \quad y \rightarrow \infty. \quad (4.40)$$

There are four different regimes to consider here depending on the value of the spanwise wavenumber and the Görtler number G . First for a general β , there are three different regimes to consider depending on whether the Görtler number is less than, close to, or greater than the critical Görtler number.

When the Görtler number is less than the critical Görtler number the solution is of

the form

$$(\bar{u}, \Psi) = \delta(\hat{U}(y) \cos \beta z, \hat{\Psi}(y) \sin \beta z), \quad (4.41)$$

from which we find the governing equations are (4.15) but with the far field matching conditions in equations (4.15c) replaced by $\hat{U} \sim J e^{(\omega-1)y}$ and $\hat{\Psi} \sim \beta^{-1} K e^{\omega y}$ as $y \rightarrow \infty$. The governing equations are linear, and we see overall that the $O(\delta)$ forcing from the production layer results in near wall structures with size $O(\delta)$. As the Görtler number approaches the critical Görtler number, the problem becomes degenerate and the solution grows like $(G - G_c)^{-1}$. Thus we see that the solution breaks down for $G \sim G_c$ which is the second regime and will be discussed in more detail shortly. The third regime is when the Görtler number is larger than the critical Görtler number. In this case the basic state is linearly unstable and we would expect to see a fully nonlinear state of the type seen in section 4.3.4. For the final regime, recall from section 4.3.3 that there is a special value of the spanwise wavenumber denoted by β_c where we find both the 1st and 2nd harmonics are present at leading order. In this scenario the bifurcation structure is altered. This will be discussed later in this section.

Next we turn to the second regime. For what follows, recall the analysis without a production layer in section 4.3.2. We assume that the leading order solution is $O(\epsilon)$ and following the arguments of section 4.3.2 we pose the expansion (4.14) with $\epsilon(\delta)$ to be determined. Now because $G \sim G_c$, the leading order problem is degenerate and thus the forcing, which is on the first harmonic, must be applied at $O(\epsilon^3)$. So we conclude that the relevant expansion in this regime is (4.14) with $\epsilon = \delta^{1/3}$. The analysis at $O(\epsilon)$ and $O(\epsilon^2)$ is identical to that of section 4.3.2. However, at $O(\epsilon^3)$ we now have a new far field matching condition of the form

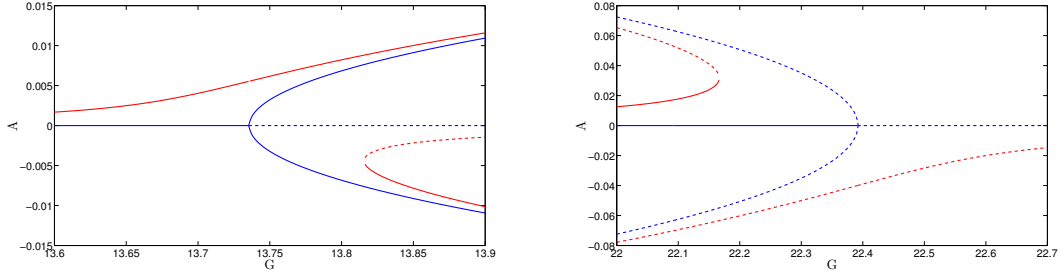
$$U_1^{(3)} \sim J e^{(\omega-1)y}, \Psi_1^{(3)} \sim \beta^{-1} K e^{\omega y}, \quad y \rightarrow \infty, \quad (4.42)$$

which represents the forcing on the near wall flow by the production layer.

The presence of this far field forcing breaks the $A \rightarrow -A$ symmetry, providing a source term in the amplitude equation (4.19), which becomes

$$\bar{\mu}_0 \frac{dA}{d\tau} = \bar{\mu}_1 G_1 A + \bar{\mu}_3 A^3 + \bar{\mu}_4 J, \quad (4.43)$$

where $\bar{\mu}_0, \bar{\mu}_1$ and $\bar{\mu}_3$ are identical to those in equation (4.19). The parameter $\bar{\mu}_4$ is a result of the far field forcing. The analysis is only a minor modification to the zero forcing case discussed in section 4.3.2. Thus equilibrium states are now found to satisfy



(a) $\beta = 0.25$, $G_c = 13.74$.
Blue: $J = 0$, red: $J = 10^{-3}$.

(b) $\beta = 1$, $G_c = 22.39$.
Blue: $J = 0$, red: $J = 10^{-4}$.

Figure 4.6. Bifurcation diagram showing $A(G)$. The blue curve shows the perfect bifurcation when $J = 0$, and the red curve shows the imperfect bifurcation when $J \neq 0$. Stable branches are solid curves, and unstable branches are dashed curves.

the equation

$$J + \mu_1 G_1 A + \mu_3 A^3 = 0, \quad (4.44)$$

where μ_1 and μ_3 depend on the spanwise wavenumber β . The parameter $J(\beta)$ which measures the strength of the freestream structures below the production low, is essentially a measure of the effect of the production layer. Provided $J \neq 0$ (i.e. a production layer exists), it is possible to scale J out of the problem. This is performed by setting

$$\tilde{A} = J^{-1/3} A, \quad \tilde{G}_1 = J^{-2/3} G_1. \quad (4.45)$$

The amplitude equation then becomes

$$1 + \mu_1 \tilde{G}_1 \tilde{A} + \mu_3 \tilde{A}^3 = 0. \quad (4.46)$$

This demonstrates that the effect of changing the level of production layer forcing is simply a rescaling of the variables, although for our illustrations we will use equation (4.44).

Figure 4.6 shows plots in (G, A) space for two different values of the spanwise wavenumber, $\beta = 0.25$ and $\beta = 1$, as used before in figure 4.3. For each spanwise wavenumber we see the bifurcation curve with no freestream coherent structures, $J = 0$, and in the presence of freestream coherent structures, where $J \neq 0$ (values given in the captions). The blue curves for $J = 0$ are identical to the curves in figure 4.3, but by setting $J \neq 0$, given by the red curves, we see how breaking the symmetry creates an imperfect bifurcation. In figure 4.6 we have used linestyle to denote the stability of each branch, applying

the normal convention of solid curves for stable states and dashed curves for unstable states. First we will discuss the $\beta = 0.25$ case. For $J = 0$, the perfect bifurcation is supercritical - we have one stable branch bifurcating into one unstable branch and a pair of stable branches. When $J \neq 0$ and the bifurcation is imperfect, this becomes two disconnected curves. One is a stable branch, then there is also a bifurcation into a stable-unstable pair at a particular value of G which is approximately 13.82. Note that the location of this bifurcation has moved between $J = 0$ and $J \neq 0$, and in the future we will denote the location of the bifurcation in the $J \neq 0$ case by G_b , so for example in this case we have $G_b \approx 13.82$. Now consider the $\beta = 1$ case. For $J = 0$, the perfect bifurcation is subcritical consisting of a bifurcation of one stable and an unstable pair of branches into a single unstable branch. When $J \neq 0$ the bifurcation is imperfect and once again this becomes two disconnected curves one of which is an unstable branch and the other is a stable-unstable pair which terminate at some value of G denoted as before by $G_b \approx 22.17$.

Now we will discuss the fourth regime. This is when the spanwise wavenumber is close to the resonant spanwise wavenumber, similar to the analysis of section 4.3.3. Recalling the analysis of section 4.3.3 we assume a disturbance of size $O(\epsilon)$ and are thus led to the expansion (4.23) with $\epsilon(\delta)$ undetermined at this stage. The leading order equations are degenerate and thus the forcing must appear at $O(\epsilon^2)$. Thus the relevant expansion is (4.23) with $\epsilon = \delta^{1/2}$. The forcing only applies to the 1st harmonic, and hence only appears in the $\hat{A}$ evolution equation. Thus, following the analysis of section 4.3.3 which is unaltered at $O(\epsilon)$, we find the solvability condition (4.29) now becomes a pair of solvability conditions,

$$\frac{d\hat{A}}{d\hat{\tau}} = a_1\beta_1\hat{A} + a_2G_2\hat{A} + a_3\hat{A}\hat{B} + a_4J, \quad (4.47a)$$

$$\frac{d\hat{B}}{d\hat{\tau}} = b_1\beta_1\hat{B} + b_2G_2\hat{B} + b_3\hat{A}^2, \quad (4.47b)$$

where the a_i, b_i for 1,2,3 are as in section 4.3.3. The constant a_4 is determined by the far field behaviour of the adjoint zero eigenfunction which is used to find the solvability condition.

Figure 4.7 shows the equilibrium solutions of (4.47) for both perfect ($J = 0$ in blue) and imperfect ($J \neq 0$ in red) bifurcations. The effect of the forcing from the production layer is to break the symmetry of the system and hence to create imperfect bifurcations as can clearly be seen in figure 4.7. Similarly to figure 4.6 we have denoted stable branches by solid curves and unstable branches by dashed curves. The system is complicated

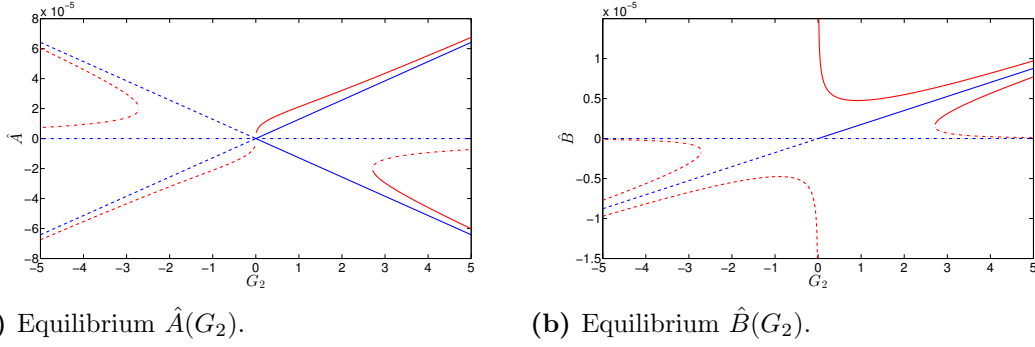


Figure 4.7. Perfect and imperfect bifurcations for near resonant β . Here we have chosen $\beta_1 = 0$ so $\beta = \beta_c$ and have plotted the resulting bifurcations for $J = 0$ (blue curve) and $J = 10^{-4}$ (red curve). We have used line-style to denote the stability of each branch with solid, dashed and dot-dashed for stable, unstable and unstable saddle respectively.

now by the presence of two modes, and so there are also saddles which are marked as dot-dash curves.

4.5. Spontaneous generation of Görtler vortices by freestream structures: receptivity

4.5.1. Adjustment layer analysis

In this section we will consider the situation where the production layer, located at $\kappa = \ln Re$, is started from a specific downstream location, here chosen to be $\bar{x} = 0$. Let us consider this assumption for a moment. What we are assuming is that for all time there is no production layer in $\bar{x} < 0$, and by some hitherto unknown mechanism, a fully developed production layer exists for $\bar{x} \geq 0$. It is worth labouring this point given the production layer consists of structures which travel in the streamwise direction at close to the freestream velocity. This is a potential source of confusion since one might expect this source to move in the streamwise direction at the same velocity as the freestream structures. The assumption that the source stays at $\bar{x} = 0$ is exactly that, an assumption. So throughout this section we will assume that some unknown mechanism sits in the flow at $\bar{x} = 0$ generating the production layer in $\bar{x} \geq 0$.

Due to the presence of the production layer only for $\bar{x} > 0$, the large region of size $O(\kappa)$ between the production layer and the wall layer is no longer passive and we must

consider the non-parallel development of the flow in this region. We refer to this region as the adjustment layer. The boundary conditions for the top of the adjustment layer come from the production layer analysis,

$$u \sim u_b + \delta J e^{(\omega-1)y} \cos(\beta z) \mathcal{H}(\bar{x}), \quad (4.48a)$$

$$(Re)v \sim -1 + \delta K e^{\omega y} \cos(\beta z) \mathcal{H}(\bar{x}), \quad (4.48b)$$

as $y \rightarrow \kappa_-$ where $\mathcal{H}(\bar{x})$ denotes the Heaviside function, and $\delta = Re^{-\omega}$. As we will verify shortly, the curvature has no effect on the leading order dynamics (apart from the pressure but this has no effect on the velocity variables) in the adjustment layer and thus from the boundary conditions (4.48) it follows that the perturbation to the basic state in the adjustment layer is $O(\delta)$ - recall that the curvature is negligible away from the wall and hence there is no resonance of the linearised problem here. Exploiting symmetries in the spanwise direction, it follows that the solution is of the form

$$(U, V, P) = (u_b, -1, -Gy) + \delta(u_s, v_r, p_r) \cos(\beta z), \quad W = \delta w_r \sin(\beta z), \quad (4.49)$$

where u_s, v_r, w_r, p_r are functions of $\bar{x}$ and y . In the previous section this led to the linearised problem, (4.10a-d), and here it leads to similar equations but with σ replaced by $\partial_{\bar{x}}$ because of the now more complicated streamwise evolution of the flow. The governing equations for the linearised perturbations are

$$(\partial_y^2 + \partial_y - \beta^2 - u_b \partial_{\bar{x}})u_s - u_b' v_r = 0, \quad (4.50a)$$

$$(\partial_y^2 + \partial_y - \beta^2 - u_b \partial_{\bar{x}})(\partial_y^2 - \beta^2)v_r + u_b'' \partial_{\bar{x}} v_r + 2\beta^2 G u_b u_s = 0, \quad (4.50b)$$

subject to the production layer forcing

$$u_s \sim J e^{(\omega-1)y}, v_r \sim K e^{\omega y}, \quad y \rightarrow \kappa, \bar{x} > 0. \quad (4.50c)$$

This problem is posed for $y \in [0, \kappa]$. It follows that the characteristic lengthscale in the wall normal direction is $O(\kappa)$. As $u_s, v_r \sim O(1)$ from the boundary conditions and hence within this region, we see from the continuity equation that the characteristic streamwise lengthscale is also $O(\kappa)$. We call coordinates with the characteristic lengthscales of the adjustment layer, adjustment layer variables and denote them by X and ξ where

$$(X, \xi) = \kappa^{-1}(\bar{x}, y). \quad (4.51)$$

The governing equations in the adjustment layer (4.50) written in terms of adjustment layer variables are,

$$(\kappa^{-2}\partial_\xi^2 + \kappa^{-1}\partial_\xi - \beta^2 - \kappa^{-1}u_b\partial_X)u_s - u'_bv_r = 0, \quad (4.52a)$$

$$(\kappa^{-2}\partial_\xi^2 + \kappa^{-1}\partial_\xi - \beta^2 - \kappa^{-1}u_b\partial_X)(\kappa^{-2}\partial_\xi^2 - \beta^2)v_r + \kappa^{-1}u_b''\partial_X v_r + 2\beta^2 Gu_b u_s = 0, \quad (4.52b)$$

subject to matching with the production layer forcing

$$u_s \sim J e^{\kappa(\omega-1)\xi}, v_r \sim K e^{\kappa\omega\xi}, \quad \xi \rightarrow 1_-, X > 0, \quad (4.52c)$$

where in the above u_b denotes the basic flow written in ξ coordinates but primes still denote differentiation with respect to y , so for example $u_b = (1 - e^{-\kappa\xi})$ and $u'_b = e^{-\kappa\xi}$.

Recall that $\kappa = \ln Re$ is assumed to be large throughout this analysis. In the absence of an analytical solution of (4.52), we can exploit the large parameter κ to construct an asymptotic solution. If $\xi \sim O(1)$, then terms of the form $e^{-\kappa(\cdot)}$ are exponentially small (when $(\cdot)$ is positive). We find for example that $u_b \sim 1$, and inspecting equation (4.52a) it follows that $u_s \sim e^{-\kappa\xi}v_r$. Thus for $\xi \sim O(1)$ equations (4.52) become

$$(\kappa^{-2}\partial_\xi^2 + \kappa^{-1}\partial_\xi - \beta^2 - \kappa^{-1}\partial_X)u_s - u'_bv_r = 0, \quad (4.53a)$$

$$(\kappa^{-2}\partial_\xi^2 + \kappa^{-1}\partial_\xi - \beta^2 - \kappa^{-1}\partial_X)(\kappa^{-2}\partial_\xi^2 - \beta^2)v_r = 0, \quad (4.53b)$$

subject to the boundary condition (4.52c). Notice that the Görtler number G is non-zero in this analysis but curvature is a higher order effect in the region where $\xi \sim O(1)$, and in fact we find that it is only important at leading order within the wall layer, when $y \sim O(1)$, which we will consider in section 4.5.2.

Because κ is large, one might be tempted to ignore the derivative terms in $O(\kappa^{-1}, \kappa^{-2})$, however this can not be done due to the boundary conditions in the problem. In fact, from the form of the equations (4.53) and in particular the boundary conditions (4.52c), we seek a WKB solution of the form

$$(u_s, v_r) = (e^{-\kappa\xi}\hat{u}, \hat{v})e^{\kappa\theta}, \quad (4.54)$$

where $\hat{u}$ and $\hat{v}$ are amplitude functions and θ is the WKB phase, all functions of the slow variables X and ξ . The roll is decoupled from the streak in equation (4.53b), so our strategy for solving the system is to solve for the roll first in (4.53b) and then for the streak using (4.53a). We simplify the analysis by first solving (4.53b) for the vorticity

V_r defined by

$$V_r = (\kappa^{-2}\partial_\xi^2 - \beta^2)v_r. \quad (4.55)$$

From (4.53b) we see that the governing equation for the vorticity V_r is

$$(\kappa^{-2}\partial_\xi^2 + \kappa^{-1}\partial_\xi - \beta^2 - \kappa^{-1}\partial_X)V_r = 0, \quad (4.56)$$

and the far field boundary condition (4.52c) is

$$V_r \sim K(\omega^2 - \beta^2)e^{K\omega\xi}, \quad \xi \rightarrow 1-, X > 0. \quad (4.57)$$

The calculation is a little cumbersome but we will provide brief details here, and more details can be found in appendix E.

Seeking a WKB solution of (4.56) with phase θ and amplitude $\hat{V}$, so $V_r = \hat{V}e^{\kappa\theta}$ we find the phase equation

$$\theta_\xi^2 + \theta_\xi - \beta^2 - \theta_X = 0. \quad (4.58)$$

Solving this equation using the method of characteristics we find the solution $\theta = \omega\xi$ for $\bar{\xi}(X) < \xi < 1$ where

$$\bar{\xi}(X) = 1 - (2\omega + 1)X. \quad (4.59)$$

This region is referred to as the parallel region, as this region is identical to the parallel production layer problem considered by DH - it is influenced by the production layer at $\xi = 1$ but not the starting condition as $X = 0$ i.e. the structure is fully developed (independent of X) in this region. However, here the vorticity must decay to zero as $X \rightarrow 0$ and thus we find that around $\xi \approx \bar{\xi}$, there is a diffusion layer.

The width of the diffusion layer is determined by balancing viscous and inviscid forces in the amplitude equation. From this balance it can be seen that the diffusion layer has thickness $\kappa^{-1/2}X^{1/2}$ - see section E of appendix E for more details. Below the diffusion layer, the vorticity is found to be exponentially small. We have now determined the vorticity throughout the whole of the adjustment layer - see appendix E for more details. The vorticity is split into three distinct regions: region 1, the parallel region; region 2, the diffusive region; and region 3, the lower adjustment layer.

Now that we have found the vorticity V_r by solving (4.53b), we can determine the roll v_r by solving (4.55). Given the form of the vorticity, one might conclude that the roll is split into three regions. However, the presence of a WKB turning point in the lower adjustment layer divides this region into three subregions, meaning there are five regions

to consider in all: the parallel and diffusive regions from the vorticity discussion, plus three more regions from the lower adjustment layer which is now subdivided by a WKB turning point layer which is located at $\xi(X)$ where

$$\xi = 1 - (2\beta + 1)X. \quad (4.60)$$

The WKB turning point layer also has thickness of size $\kappa^{-1/2}X^{1/2}$ which is determined by a viscous-inviscid balance of the WKB amplitude equation. Below the WKB turning point layer, in the fifth and final region, the flow is irrotational. It turns out that the regions of interest are 1 and 2, because below the diffusion layer the flow decays exponentially and this can be seen from the WKB phase function ϕ which is given by

$$\theta(X, \xi) = \begin{cases} \omega\xi & \xi \in (\bar{\xi}, 1), \\ \omega\xi - \frac{(\xi - \bar{\xi})^2}{4X} & \xi \in (\xi, \bar{\xi}), \\ \beta X + \omega - \beta + \beta\xi & \xi \in (0, \xi). \end{cases} \quad (4.61)$$

It is not immediately obvious from the form of θ , but using the definition of ω we can show that θ is actually continuous across all the regions. The corresponding WKB phase for the streak is found by subtracting ξ from the WKB phase for the roll.

Figure 4.8 shows plots of the WKB phases of the streak and roll for unit spanwise wavenumber, $\beta = 1$, which is plotted from (4.61). The black lines indicate the positions of the diffusion front (upper line) and the WKB turning point layer (lower line). From this figure, the WKB turning point layer appears to indicate the start of growth of the near wall structures, and the diffusion layer marks the point where the structures become fully developed.

It is worthwhile here, with the aid of figure 4.8, illustrating the different regions in the flow. Progressing from top right to bottom left, the first region is the parallel flow region and is referred to as region 1. As mentioned in the previous paragraph, the upper line is the diffusion front which is referred to as region 2. Between the two lines is region 3.1, and then the lower line is the WKB turning point layer which is referred to as region 3.2. Finally, below this layer is the irrotational flow region, referred to as region 3.3.

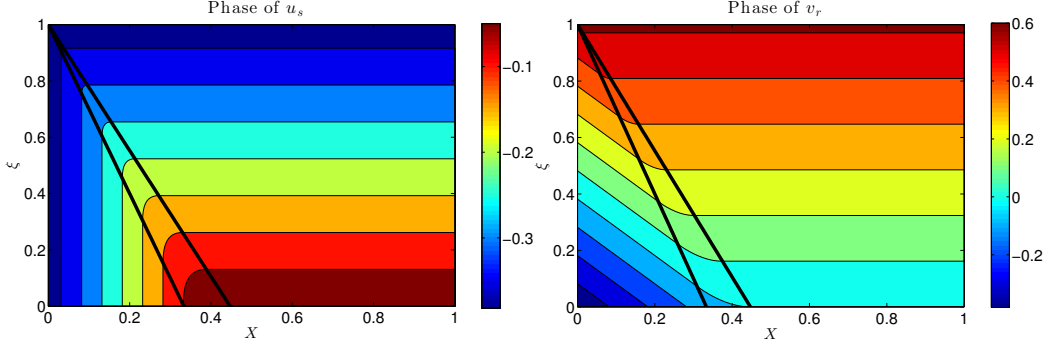


Figure 4.8. WKB phase function θ in the adjustment layer for $\beta = 1$ from equations (4.61). The black lines indicate the location of the diffusion layer and the WKB turning point layer.

4.5.2. Wall layer analysis

This analysis differs depending on the value of the Görtler number. There are two cases where we can make progress analytically. In one case when $G < G_c$ the response is linear and we will discuss this first. In the second case when $G \approx G_c$ the response is in the weakly nonlinear regime throughout the parallel layer and the upper diffusive layer and we must revisit the ideas of section 4.4.2. When $G > G_c$, the flow is linearly unstable. This results in the generation of Görtler vortices before the production layer is reached. Since the purpose of this paper is the receptivity problem we will not focus on this case here.

Solution for the Görtler number less than the critical Görtler number

We will now consider the flow inside the wall layer where $y \sim O(1)$. If the curvature is less than critical, i.e. if $G < G_c(\beta)$ (and $G \not\approx G_c$), then the governing equations are the linearised equations given in (4.50) but with modified far field matching conditions to match with the adjustment layer solution. An analytic solution of these equations can be found in regions 1 and 2 where the phase is independent of the streamwise coordinate and hence the X dependence is purely parametric.

For example, if $G = 0$, then we find in region 1 that the governing equation for v_r from (4.50b) is

$$(\partial_y^2 + \partial_y - \beta^2)(\partial_y^2 - \beta^2)v_r = 0. \quad (4.62)$$

Looking for a solution of the form $v_r \propto e^{my}$ we find $(m^2 + m - \beta^2)(m^2 - \beta^2) = 0$. This equation has the roots $m = \pm\beta$, $m = \omega$, and we denote the other root by $m = \bar{\omega}$. Thus

it follows that the solutions for u_s and v_r in region 1 are

$$u_s = p_1 e^{(\omega-1)y} + p_2 e^{\bar{\omega}y} + p_3 e^{(\bar{\omega}-1)y} + p_4 e^{-(\beta+1)y}, \quad (4.63a)$$

$$v_r = q_1 e^{\omega y} + q_2 e^{\bar{\omega}y} + q_3 e^{-\beta y}. \quad (4.63b)$$

The constants p_i and q_i are given by

$$p_1 = J, \quad p_2 = -(p_1 + p_3 + p_4), \quad p_3 = -J \frac{\omega(\omega + \beta)}{\bar{\omega}(\bar{\omega} + \beta)}, \quad p_4 = -2J \frac{\omega(\omega - \bar{\omega})}{\beta(\bar{\omega} + \beta)}, \quad (4.63c)$$

$$q_1 = K, \quad q_2 = -K \frac{(\omega + \beta)}{(\bar{\omega} + \beta)}, \quad q_3 = K \frac{(\omega - \bar{\omega})}{(\bar{\omega} + \beta)}. \quad (4.63d)$$

Considering the solution within the diffusion front, in the adjustment layer, we find that the relevant streamwise scale in the wall layer below region 2 is $O(\kappa^{1/2})$. We introduce the scaled streamwise coordinate $\hat{x} = \kappa^{1/2}(X - X_0)$, where $X_0 = (1 + 2\omega)^{-1}$ denotes the streamwise location (in X) where the diffusion front meets the wall. On this length scale, derivatives in the streamwise direction are subdominant and the $\hat{x}$ dependence is parametric as alluded to before. The problem in this region is then essentially the same as in region 1 apart from the factor

$$F(\hat{x}) = \frac{1}{2} \operatorname{erfc} \left(-\frac{\hat{x}}{2X_0^{3/2}} \right), \quad (4.64)$$

multiplies the solutions given in (4.63). This factor compensates for the diffusion layer - see section E of appendix E. All that is left to do here is to solve the equations in regions 3.1-3.3 numerically. However, the solution in these regions is exponentially small and the regions of interest are 1 and 2 where we have an analytical expression for the leading order solution.

Solution for near the critical Görtler number

We move on to discuss the case when $G \approx G_c$. In this case there is a resonance within the linearised equations, and the response becomes weakly nonlinear. The resonance does not occur however until region 2 is reached, so in regions 3.1, 3.2 and 3.3 the solution is $O(\delta)$ but it is also exponentially small as can be seen from the phase function (4.61). For these reasons, let us concentrate on regions 1 and 2.

In region 1 the forcing from the production layer is independent of the streamwise coordinate so the flow is parallel, and the problem is identical to that considered in

section 4.4.2. Thus posing expansion (4.14) and following the analysis of section 4.4.2 we find the same solvability condition,

$$J + \mu_1 G_1 A + \mu_3 A^3 = 0. \quad (4.65)$$

The question to be addressed here is, how does the response from the production layer forcing go from exponentially small (effectively zero) in region 3.1, to $O(\delta^{1/3})$ in region 1? In this section we will show how a minor generalisation of section 4.4.2 allowing for the streamwise development of structures reveals the connection between the two regions. For the rest of this section we will consider the flow near the wall in region 2. Importantly, in this region streamwise derivatives are $O(\kappa^{-1/2})$ which is assumed to be small and hence the governing equations reduce to the fully nonlinear governing equations (4.12) of the previous section, but with the production layer present so the far field conditions (4.12c) are replaced by

$$(\bar{u}, \Psi) \sim \delta F(\hat{x})(J e^{(\omega-1)y} \cos \beta z, \beta^{-1} K e^{\omega y} \sin \beta z), \quad y \rightarrow \infty. \quad (4.66)$$

This differs from the condition in region 1 (4.40) only by the presence of $F(\hat{x})$ defined in (4.64). Perhaps unsurprisingly, the solution can be found in the form similar to that given in (4.14),

$$\begin{aligned} (\bar{u}, \Psi) = & \delta^{1/3} A(\hat{x})(U_1^{(1)} \cos(\beta z), \Psi_1^{(1)} \sin(\beta z)) + \delta^{2/3} A(\hat{x})^2 (U_0^{(2)} + U_2^{(2)} \cos(2\beta z), \Psi_2^{(2)} \sin(\beta z)) \\ & + \delta F(\hat{x})(U_1^{(3)} \cos(\beta z) + U_3^{(3)} \cos(3\beta z), \Psi_1^{(3)} \sin(\beta z) + \Psi_3^{(3)} \sin(3\beta z)). \end{aligned} \quad (4.67)$$

Substituting (4.67) into (4.12), we find that the leading order and second order problems are identical to those in section 4.3.2. The problem at $O(\delta)$ is only slightly changed from section 4.4.2 by the presence of $F(\hat{x})$ which now divides the functions F_3, H_1 and H_3 . The ultimate effect of this is a slight modification to the amplitude equation (4.44) which now becomes

$$JF(\hat{x}) + \mu_1 G_1 A(\hat{x}) + \mu_3 A(\hat{x})^3 = 0. \quad (4.68)$$

An inspection of this equation reveals that it matches with (4.20) as $\hat{x} \rightarrow -\infty$ ($F(\hat{x}) \rightarrow 0$ and we recover the perfect bifurcation) and (4.44) as $\hat{x} \rightarrow \infty$ ($F(\hat{x}) \rightarrow 1$ and we recover the imperfect bifurcation (4.65)) as expected. Given a spanwise wavenumber β , we can determine μ_1 and μ_3 , then if we are further given J and G_1 we are left with a cubic equation (4.68) for A at each streamwise location $\hat{x}$.

Before we discuss the solutions to this equation, recall the discussion of the stability of

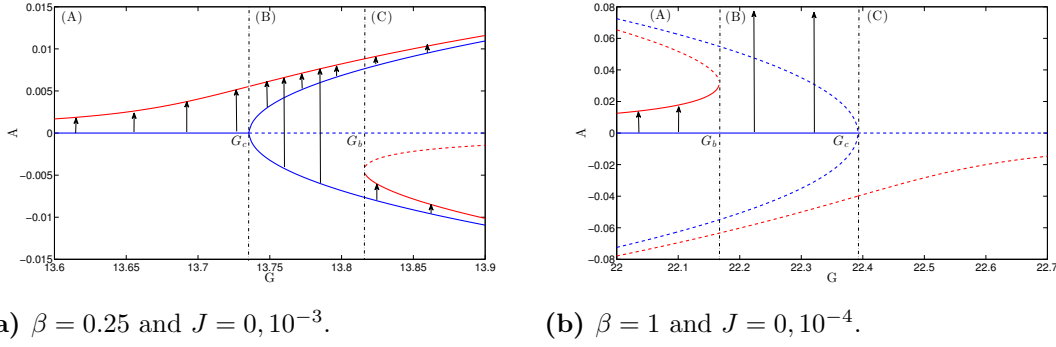


Figure 4.9. Bifurcation diagram showing $A(G)$. The blue curve shows the perfect bifurcation when $J = 0$ ($\hat{x} \rightarrow -\infty$), and the red line shows the imperfect bifurcation when $J \neq 0$ ($\hat{x} \rightarrow \infty$). Stable branches are solid and unstable branches are dashed. The arrows in this figure are trajectories showing $A(\hat{x})$, the solutions of equation (4.68).

the branches for perfect and imperfect bifurcations in section 4.4.2 and in figure 4.6. The perfect bifurcation with $J = 0$ given by the solid curve in figure 4.6 denotes the solutions A as $\hat{x} \rightarrow -\infty$, and the imperfect bifurcation with $J \neq 0$ given by the dashed curve in figure 4.6 denotes the solutions A as $\hat{x} \rightarrow \infty$, or some scaled version of it based on the rescaling given by equation (4.45). As $F(\hat{x})$ varies smoothly, we see that the solutions for A vary smoothly from the perfect bifurcation curve to the imperfect bifurcation curve. Note however that the location of the bifurcation is moved from $G = G_c$ ($G_1 = 0$) when $J \neq 0$, and for what follows, as in section 4.4.2, we will denote the value of G at which the bifurcation occurs (as $\hat{x} \rightarrow \infty$) by G_b . For $\beta = 0.25$, G_b is approximately equal to 13.82, and for $\beta = 1$, G_b is approximately equal to 22.17 as can be seen from figures 4.6 and 4.9.

We will now consider solutions of (4.68) for $A(\hat{x})$ for the two cases of subcritical and supercritical bifurcations given in figure 4.6. We are primarily interested in the case when the Görtler vortices are caused by the production layer. If the flow is linearly unstable as $\hat{x} \rightarrow -\infty$, then Görtler vortices would develop before the production layer is reached, and thus we concentrate on the situation when the basic state is linearly stable as $\hat{x} \rightarrow -\infty$, i.e. the case of interest where the development of Görtler vortices is due to the presence of the production layer. For the discussions which follow we will refer to figure 4.9 which is a replication of figure 4.6 showing the bifurcation curves for $J = 0$ and $J \neq 0$ and different spanwise wavenumbers covering both subcritical and supercritical bifurcations. However, the arrows in figure 4.9 now denote the evolution of $A(\hat{x})$ for $\hat{x} \in (-\infty, \infty)$, with each arrow corresponding to a fixed value of G .

Consider first the supercritical bifurcation in figure 4.9(a) when $\beta = 0.25$ and $J = 10^{-3}$. We must consider three different regions referred to as (A), (B) and (C) in the figure, which denote $G < G_c$, $G \in (G_c, G_b)$ and $G > G_b$ respectively. In regions (B) and (C) the basic flow is linearly unstable as $\hat{x} \rightarrow -\infty$, and thus we would expect to see one of the stable branches i.e. Görtler vortices. Our primary interest is in the case where the production layer prompts the generation of Görtler vortices and hence we are most interested in region (A). In region (A) $A(\hat{x})$ can be seen to smoothly vary between 0 and some non-zero value, with the trajectories given by the arrows. Flow visualisations created by solving (4.68) and using (4.67) for this case with $G_1 = 0$ are given in figure 4.10 where the smooth development of Görtler vortices as $\hat{x}$ increases is observed. For completeness we will briefly discuss regions (B) and (C). In region (C), we see regardless of which branch the Görtler vortices sit on, there is a small, smooth change as $\hat{x}$ increases. From region (B), if we start on the upper branch then the situation is similar to (C). Starting on the lower branch there are two possible types of behaviour depending on the value of J . If J is not too large, then there is a small, smooth transition as $\hat{x}$ increases. However if J is larger then we see that as $\hat{x}$ increases eventually the lower stable branch disappears. When this happens the solution is no longer in equilibrium (see figure 4.9(a)) and the only stable branch is the upper branch. Hence we would expect the streamwise evolution of the solution onto the stable branch. Say this occurs at $\hat{x} = \hat{x}_0$, then the value of A taken at this point, denoted by A_0 , satisfies both $\mu_3 A_0^3 + \mu_1 A_0 + JF(\hat{x}_0) = 0$ and $3\mu_3 A_0^2 + \mu_1 = 0$ (since it is a repeated root). At this point there is a catastrophe and we expect a rapid streamwise response over the streamwise scale $\delta^{-2/3}\kappa^{-1/2}$. Over this scale we expect solutions to develop into fully nonlinear equilibrium states. Since these nonlinear states cannot be described analytically, there is little hope of making progress in this scenario except in a small region around $\hat{x}_0$ where A is close to A_0 . However this analysis would only tell us that for $\hat{x} < \hat{x}_0$, A is in equilibrium and for $\hat{x} > \hat{x}_0$ the solution changes rapidly which we already know.

Now consider the subcritical bifurcation in figure 4.9(b) when $\beta = 1$ and $J = 10^{-4}$. Again we must consider the three different regions (A), (B) and (C). Region (C) is linearly unstable and for $G_1 > 0$ the only equilibrium solutions for both the perfect and imperfect bifurcations are fully nonlinear and hence no progress can be made analytically. In region (A) where $G < G_b$, the basic flow is linearly stable as $\hat{x} \rightarrow -\infty$ and as $\hat{x} \rightarrow \infty$. Thus Görtler vortices develop smoothly as $\hat{x}$ increases in a similar way to the supercritical visualisation given in figure 4.10. In region (B) where $G_b < G < G_c$ the situation is complicated because now the basic state is linearly stable as $\hat{x} \rightarrow -\infty$. However, the equilibrium solution described by the weakly nonlinear analysis as $\hat{x} \rightarrow \infty$ is unstable

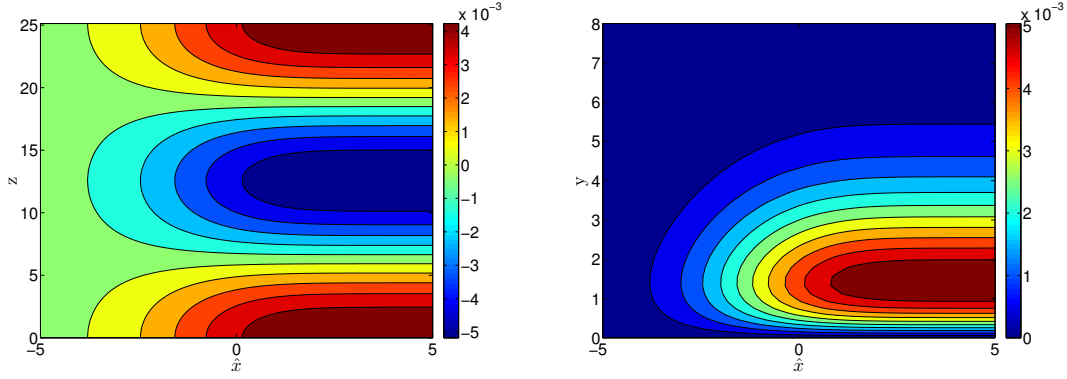
(a) $(\hat{x}, z)$ at $y = 1$.(b) $(\hat{x}, y)$ at $z = 0$.

Figure 4.10. Flow visualisation of leading order streak perturbation $A(\hat{x})u_{11}(y)\cos(\beta z)$ for $(\beta, G_1, J) = (0.25, 0, 10^{-3})$.

and in fact any stable equilibrium solution for the imperfect bifurcation is fully nonlinear in this case. So as $\hat{x}$ increases from $-\infty$, Görtler vortices develop smoothly until a critical location in $\hat{x}$ is reached, where the equilibrium solution disappears. At this point the flow is out of equilibrium and the only stable solutions are fully nonlinear. In this case we would expect the streamwise development of fully nonlinear solutions which are the nonlinear continuation of the upper imperfect branch.

4.6. Conclusions

The first part of this chapter concerned Görtler vortices caused by the centrifugal instability of the asymptotic suction boundary layer in the absence of freestream coherent structures. Park & Huerre (1995) considered this stability problem more generally, calculating growth rates and the streamwise evolution of disturbances in the flow over a curved plate. However, here we focus on equilibrium solutions which are Görtler vortices and can be either stable or unstable. From a linear stability analysis we determined the critical value of the Görtler number, denoted by G , in terms of the spanwise wavenumber β , or in a box of fixed spanwise size $2\pi/\beta$. Crossing this curve, denoted by $G_c(\beta)$, the linear stability of the basic state changes and we see a bifurcation into finite amplitude Görtler vortices. Extending this analysis to account for weakly nonlinear effects we determined the bifurcation structure for small amplitude disturbances where the nature of the bifurcation was found to depend on the spanwise wavenumber. Over the interval of spanwise wavenumbers considered here, the primary mode was found to be

supercritical except for in the region given approximately by $0.30 < \beta < 1.30$. If in fact the bifurcation remains supercritical for all $\beta > 1.30$ then one could conclude that the bifurcation is supercritical in any box of spanwise extent less than $2\pi L/1.30$ where L is the lengthscale, i.e. approximately $< 4.83 \times (\nu/V_0)$.

Fully nonlinear computation of equilibrium structures was performed to extend the states into the strongly nonlinear regime. These computations confirmed the nature of the bifurcation as predicted by the weakly nonlinear analysis. Careful branch tracking (by continuation in the Görtler number, G) revealed a rich bifurcation structure at certain spanwise wavenumbers where intermodal bifurcations are found. In particular subharmonic linking branches were observed. This does hint at the complexity of the state space, but we did not consider further subharmonic bifurcations in this work which could possibly lead to further subharmonic linking branches.

Next we reviewed the freestream coherent structures found by Deguchi & Hall (2014), leading to the main focus of this chapter which is the generation of Görtler vortices in the presence of freestream coherent structures. Consider the situation when there is a freestream coherent structure of size $O(\delta)$, within the production layer, where δ is small. We find the existence of solutions which are homogeneous in the streamwise direction. For a fixed spanwise wavenumber β , if the Görtler number G is less than the critical value, denoted by G_c , then the flow is linearly stable and the $O(\delta)$ forcing from the freestream produces structures of $O(\delta)$ near the wall. Consider now G close to critical its critical value G_c . In the homogeneous problem (no freestream structures), there is a bifurcation into finite amplitude Görtler vortices in a supercritical/subcritical pitchfork bifurcation. When freestream structures are present, this provides a forcing on the near wall bifurcation problem breaking the symmetry of the bifurcation and creating imperfect bifurcations as seen in figure 4.6 - this effect has been used to explain phenomena in convection by Daniels (1977) and in centrifugal instabilities by Hall (1990). In this case much larger near wall structures are observed which are of size $O(\delta^{1/3})$. In the resonant spanwise wavenumber problem a similar phenomenon of imperfect bifurcations is observed although the nature of the bifurcations is different - see figure 4.7.

We moved on to investigate the streamwise development of near wall structures when the freestream structures are impulsively started at some streamwise location. In the intermediate region between the freestream and the wall, the curvature effect is a higher order contribution (in Re) to the solution and a WKB solution is found involving multiple asymptotic regions. For less than critical Görtler numbers the response of the wall layer is linear, whereas for near critical Görtler numbers the response is weakly nonlinear. Struc-

tures develop over the streamwise scale $O(Re(\ln Re)^{1/2})$ a distance $(2\omega + 1)^{-1} Re \ln Re$ downstream from the start of the freestream structures. Recall that $2\omega = -1 + \sqrt{1 + 4\beta^2}$. This is a diffusive layer and beyond this layer the structures are fully developed and hence identical to those structures found in the streamwise homogeneous problem. The exact streamwise development depends on the value of the Görtler number. The development could be a simple smooth quasi-streamwise-independent transition, or in some cases could be more complicated involving a stronger streamwise dependence.

The discovery of freestream coherent structures by Deguchi & Hall (2014) complicates the issue of interdependence between near wall and freestream structures in transitional and turbulent boundary layers. In this work we have further shown how the presence of even very small freestream coherent structures can excite large near wall structures which can develop smoothly downstream or may exhibit some more complicated behaviour. Clearly given the non-trivial state space for strongly nonlinear Görtler vortices, there is scope for much more complicated behaviour which is beyond the scope of this work.

5. Conclusions

5.1. Summary of the research in this thesis

The main focus of this thesis is the identification and description of a variety of travelling wave equilibrium states of the incompressible Navier-Stokes equations. In particular we considered their asymptotic form at large Reynolds numbers where the driving mechanisms can be identified and the states can be described by a nonlinear reduced form of the full Navier-Stokes equations. In chapters 2 and 3 we focused on equilibrium states which bifurcate from the basic state via a viscous instability. The successful asymptotic description of such states relies on the description of the linear instability of the basic state at large Reynolds number, and in particular, the structure and properties of the neutrally stable modes. For the channel flows considered in chapter 2, the asymptotic convergence was found to depend on the small parameter $Re^{-1/7}$ suggesting that the Reynolds number has to be reasonably large before convergence to the large Reynolds number limit is achieved. This was observed in comparisons with finite Reynolds number linear stability calculations and also in finite amplitude states (see Dempsey *et al.* 2016).

In chapter 2 states were found which bifurcate from the lower branch linear neutral point for plane Poiseuille flow. In this interaction the wave drives the roll flow via the nonlinear self-interaction of the wave directly through the Reynolds stresses throughout the channel. These states were continued to large amplitudes where a strongly nonlinear interaction rendered a new travelling wave state which is an $O(1)$ perturbation to plane Poiseuille flow. Such states were observed to localise around spanwise locations where the wave displacement function, denoted by $A(Z)$, vanishes. The same localisation phenomenon is observed in computations of travelling wave equilibrium states found computationally in the full Navier-Stokes equations. Until the localisation becomes too extreme, vortex-wave interaction appears to be accurately describing the large Reynolds number limit of such states (Dempsey *et al.* 2016). This localisation process limits the extent to which these states can be computed given the difficulties associated with

resolving such structures. Eventually the number of Fourier modes required to accurately capture the states becomes too large for the computational resources which were available. Later in the chapter this work was extended to include channels with slowly sliding walls (plane Poiseuille-Couette flow) and channels with curved walls (Taylor vortex flow), which was the scenario originally considered by Bennett *et al.* (1991) and where the equations governing the strongly nonlinear interaction were first derived. Allowing the walls to slide is known (Cowley & Smith 1985) to lead to more lower branch linear neutral points and this leads to the existence of more states - we can track states which bifurcate from each linear neutral point. Such new linear neutral points are found to occur in pairs, resulting in loops in $(\alpha, \mathcal{A})$ space connecting these pairs of equilibrium states. In the presence of sufficiently curved walls (i.e. large enough Taylor number, T), and with stationary walls now, there is more than one zero-wave amplitude state: the basic state of plane Poiseuille flow and the Taylor vortex state. We can track travelling wave states bifurcating from either state. Generally (apart from the closed loops found for near bifurcation wall speeds) the states progress along each branch, in increasing amplitude, in a similar way to in plane Poiseuille flow. They localise at specific spanwise locations as the amplitude of the wave increases until it is not feasible to track the states any further. The ultimate fate of these modes is expected to be a similar structure but which contains an $O(1)$ layer in the spanwise direction ($z \sim O(1)$, i.e. $Z \sim O(Re^{-1/7})$) where spanwise diffusion contributes at leading order. However, this was not further investigated in this work.

Similar states in the asymptotic suction boundary layer were computed in chapter 3. The structure of the interaction is slightly different in this case. The nonlinear self-interaction of the wave drives a near wall vortex which grows logarithmically with distance from the wall, and in turn this drives a non-zero vortex flow throughout the $O(1)$ boundary layer. Although the interaction equations derived in this chapter are valid for all spanwise wavenumbers of size $\sim Re^{-1/4}\beta$, where β is assumed to be $O(1)$, we only investigated the case $\beta = 1$. The weakly nonlinear analysis shows the nature of the bifurcation is dependent on the spanwise wavenumber chosen, and so we might expect different behaviour for different choices of β . Apart from this observed change in the criticality of the bifurcation, we do not expect the case $\beta = 1$ to be atypical. Similarly to the case of plane Poiseuille flow, we find that large amplitude states localise around spanwise locations where the displacement function vanishes. However, in this case we also see spanwise kinks forming at other, seemingly arbitrary, spanwise locations. This phenomenon has not been explained, and the eventual fate of these states for large amplitudes is unknown at this point although it is likely to hinge on the inclusion of

spanwise diffusion at specific spanwise locations.

In chapter 4 we built on the work of Deguchi & Hall (2014) on freestream coherent structures in the asymptotic suction boundary layer. Those authors found numerically, and subsequently asymptotically, equilibrium states which involve both structures in the freestream and near the wall. This is important due to the long standing debate about the interdependence of such structures. Here we investigated the effect that the presence of a curved wall has on the solutions. We started with a study of Görtler vortices in the asymptotic suction boundary layer finding rich bifurcation structures involving non-trivial intermodal linking branches in the strongly nonlinear regime. It turns out that the presence of curvature does not affect the freestream structures at leading order (in Re), and thus the major effects of the curvature are confined to the layer of $O(1)$ extent near the wall. The presence of freestream structures provides a forcing on the near wall problem effectively breaking the symmetry of the bifurcation, and this results in imperfect bifurcations. Further to this, we considered the streamwise development of near wall structures over a flat plate when the freestream structures are assumed to start impulsively at a specific downstream location. Structures were found to develop over a streamwise lengthscale of $O(Re \ln Re)$ with structures fully developed, and hence identical to the streamwise-independent case, by a distance of around $k(\beta)Re \ln Re$ where $k(\beta) = (2\omega(\beta) + 1)^{-1}$ depends on the spanwise wavenumber β . Recall that $\omega(\beta) = (-1 + \sqrt{1 + 4\beta^2})/2$. At close to critical Görtler numbers, the development of near wall structures was found to be sometimes smooth and in quasi-streamwise-equilibrium.¹ However, for some Görtler numbers the solution is forced from equilibrium at some downstream location and the dynamics are complicated by the non-existence of a stable small amplitude solution. In this situation we expect the rapid development of the solution into the fully nonlinear regime as alluded to in chapter 4.

Perhaps the most important part of this thesis is the solution of the vortex-wave (Tollmien-Schlichting-type) interaction equations in plane Poiseuille flow. This allows the comparison of the states described by vortex-Tollmien-Schlichting-wave interaction theory with numerical solutions of the full Navier-Stokes equations. From this we can confirm (see Dempsey *et al.* 2016) that the vortex-wave interaction equations seen in chapter 2, which were first derived by Bennett *et al.* (1991), do indeed describe the large Reynolds number limit of nonlinear travelling wave equilibrium states of the full Navier-Stokes equations.

¹By this we mean dependence on the streamwise direction only in terms of an amplitude, i.e. the solution is of the form $A(x)f(y) \cos \beta z$ at leading order.

5.2. Future work

This thesis raises several questions which have not been addressed here. Perhaps the most pressing of these is the eventual fate of the travelling wave equilibrium states found in chapters 2 and 3. The long spanwise scale causes spanwise diffusion to be neglected at leading order and this appears to cause singularities at particular spanwise locations. It is expected that introducing some spanwise diffusive layers where $z \sim O(1)$ would allow the description of these states to larger amplitudes. The idea being that, at a sufficiently large amplitude active diffusive layers would emerge affecting the solution throughout the whole channel.

It would also be interesting to see how the large spanwise wavenumber limit (in chapters 2 and 3) connect onto states of $O(1)$ spanwise extent which are governed by the so called boundary region equations. Although these equations have not been solved for this flow set up we would expect to see localised states based on the work of Deguchi *et al.* (2013) in plane Couette flow.

Regarding the work contained within chapter 4, there are some questions behind the impulsively started freestream coherent structures over a curved wall when the Görtler number lies in certain ranges where the quasi-streamwise-independent solution breaks down. Presumably this involves some rapid streamwise variations until the solution lies in the fully nonlinear quasi-streamwise-independent state but this has not been fully resolved in this work.

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A. Numerical solution of the local equations

In this section we will analyse the solution around particular special points in Z . First consider the wave forcing term in the core equations, which we denote by $\mathbb{F}$,

$$\mathbb{F}(Z) = \alpha^2(|A|^2)_Z. \quad (\text{A.1})$$

We have restricted our attention to periodic solutions, and thus since A is periodic in Z , so is $\mathbb{F}$. Since it is periodic, in general not identically zero, and has zero mean, it must have zeroes. We will label such a point by $Z = Z_c$, where $\mathbb{F}(Z_c) = 0$, and we will analyse the behaviour of the solution around these points. Given sufficient analyticity of the solution around $Z = Z_c$, we can Taylor expand around this point given $|Z - Z_c| \ll 1$.

Thus we expand the solution in a Taylor series around $Z = Z_c$,

$$U = u_0(y) + \dots, \quad V = v_0(y) + \dots, \quad W = (Z - Z_c)w_1(y) + \dots, \quad \mathbb{F} = (Z - Z_c)F_1 + \dots \quad (\text{A.2})$$

Then the equations governing the leading order solution locally are

$$v'_0 + w_1 = 0, \quad (\text{A.3a})$$

$$v_0 u'_0 = 2 + u''_0, \quad (\text{A.3b})$$

$$v_0 w''_1 + w_1 w'_1 = F_1 u_0 u'_0 + w'''_1, \quad (\text{A.3c})$$

which are to be solved subject to the boundary conditions

$$u_0, v_0, w_1 = 0 \quad \text{on} \quad y = \pm 1. \quad (\text{A.3d})$$

For a given amplitude we do not know what F_1 would be at any such points Z_c , however we expect in general that F_1 increases with amplitude (and this turns out to be true from our numerical results). For this reason we would like to analyse how the possible

local solutions vary as F_1 varies. Next we will briefly document two methods which have been used to solve the local equations.

The first method is a shooting method. We use the boundary conditions we have on the lower wall, and we guess the boundary conditions that we don't know. Then we march using an explicit two-stage Runge-Kutta method to the upper wall, where we compare our result with the prescribed upper boundary condition. We use this to update our guess of the lower wall boundary conditions, using a secant method, and we iterate until the upper wall boundary condition is satisfied. To be more precise, we guess u'_0, w'_1, w''_1 at $y = -1$, then we will iterate until we have satisfied no slip and no flux boundary conditions on u_0, v_0, w_1 on the wall at $y = 1$. The first step is to rewrite the system as a set of first order nonlinear ordinary differential equations,

$$(u_0, p, v_0, w_1, q, r)' = (p, v_0 p - 2, -w_1, q, r, v_0 r + w_1 q - F_1 u_0 p), \quad (\text{A.4a})$$

where we have the boundary conditions

$$u_0 = v_0 = w_0 = 0, \quad \text{on} \quad y = \pm 1. \quad (\text{A.4b})$$

Using the marching procedure we can readily apply the boundary conditions on $y = -1$, however we must guess the other boundary conditions on the bottom wall,

$$P = p(-1), \quad Q = q(-1), \quad R = r(-1), \quad (\text{A.5})$$

and we iterate until $u_0(1) = v_0(1) = w_1(1) = 0$.

The Runge-Kutta method used for the marching is an extension of the *modified Euler method* into more variables. In the modified Euler method, used to solve $x'(t) = f(t, x)$, the update procedure with step size h is given by

$$k_1 = f(t_n, x_n), \quad (\text{A.6a})$$

$$k_2 = f(t_n + \frac{1}{2}h, x_n + \frac{1}{2}hk_1), \quad (\text{A.6b})$$

$$x_{n+1} = x_n + hk_2, \quad (\text{A.6c})$$

$$t_{n+1} = t_n + h. \quad (\text{A.6d})$$

We use an extension of this to multiple variables, denoting the variables at each step with a subscript j (and dropping the previously subscripted 0 just for this description),

we have step 1:

$$(k_u, k_p, k_v, k_w, k_q, k_r) = (p_j, v_j p_j - 2, -w_j, q_j, r_j, v_j r_j + w_j q_j - F_1 u_j p_j), \quad (\text{A.7a})$$

and step 2:

$$(K_u, K_p, K_v, K_w, K_q, K_r) = (p_j + \sigma k_p, (v_j + \sigma k_v)(p_j + \sigma k_p) - 2, -(w_j + \sigma k_w), q_j + \sigma k_q, r_j + \sigma k_r, (v_j + \sigma k_v)(r_j + \sigma k_r) + (w_j + \sigma k_w)(q_j + \sigma k_q) - F_1(u_j + \sigma k_u)(p_j + \sigma k_p)), \quad (\text{A.7b})$$

where $\sigma = h/2$, half the step size. Then the final variable update at each step, step 3:

$$(u, p, v, w, q, r)_{j+1} = (u, p, v, w, q, r)_j + 2\sigma(K_u, K_p, K_v, K_w, K_q, K_r). \quad (\text{A.7c})$$

The second method is spectral. We expand the variables in terms of Chebyshev polynomials $T_n(y)$, and use collocation to generate a pointwise error. This nonlinear set of equations in the Chebyshev coefficients can then be solved using a Newton method. We expand each variable in Chebyshev polynomials,

$$(u_0, v_0, w_1) = \sum_{n=0}^{N-1} (U_n, V_n, W_n) T_n(y). \quad (\text{A.8})$$

Applying the equations throughout the solution domain, and the boundary conditions on the boundaries, gives the required equations for the Chebyshev coefficients, U_n, V_n and W_n .

B. Weakly nonlinear analysis of the vortex-wave interaction equations in plane Poiseuille-Couette flow

Here we will perform a weakly nonlinear analysis of the interaction equations (2.38, 2.88) for plane Poiseuille-Couette flow. We assume the wave is small, so $(A, p_{\pm}) \sim O(\Delta) \cos \beta Z$ say. From the core equations (2.38c) the forcing is $O(\Delta^2) \sin 2\beta Z$ and hence the spanwise velocity W is $O(\Delta^2) \sin 2\beta Z$. It follows from (2.38a, b) that (U, V) are an $O(\Delta^2) \cos 2\beta Z$ perturbation from the basic state. Thus the wall shear stresses and J are also $O(\Delta^2) \cos 2\beta Z$ from their basic state values. Considering the wave part of the interaction equations (2.88), given the size of the perturbations to $\lambda_{\pm}$ and J , it follows that α and c are $O(\Delta^2)$ from their linear neutral values. Further, the next order corrections to $(A, p_{\pm})$ are $O(\Delta^3) \cos \beta Z$ and $O(\Delta^3) \cos 3\beta Z$. We only need to consider the first harmonic to determine the amplitude equation via a solvability condition.

Hence the core vortex components expand as

$$(U, V) = (u_b, 0) + \Delta^2(U_2, V_2) \cos 2\beta Z + \dots, \quad W = \Delta^2 W_2 \sin 2\beta Z + \dots, \quad (\text{B.1a})$$

where $u_b(y) = 1 - y^2$, and (U_2, V_2, W_2) depend only on the wall normal coordinate y . The wall shear stresses and J expand as

$$(\lambda_{\pm}, J) = (2, 16/15) + \Delta^2(\lambda_{\pm 2}, J_2) \cos 2\beta Z + \dots \quad (\text{B.1b})$$

The wave components expand as

$$(A, p_+, p_-) = \Delta(A_1, P_1, Q_1) \cos \beta Z + \Delta^3(A_3, P_3, Q_3) \cos \beta Z + \dots, \quad (\text{B.1c})$$

and

$$(\alpha, c) = (\alpha_0, c_0) + \Delta^2(\alpha_2, c_2) + \dots \quad (\text{B.1d})$$

Substituting these expansions into the core equations (2.38) we find

$$V_2' + 2\beta W_2 = 0, \quad (\text{B.2a})$$

$$-2yV_2 = U_2'', \quad (\text{B.2b})$$

$$0 = -2\beta\alpha_0^2|A_1|^2 u_b u_b' + W_2''', \quad (\text{B.2c})$$

which are to be solved subject to the boundary conditions

$$U_2 = V_2 = W_2 = 0, \quad y = \pm 1. \quad (\text{B.2d})$$

Equation (B.2c) can be integrated to give

$$W_2(y) = \beta\alpha_0^2|A_1|^2 \left(\frac{1}{30}y^6 - \frac{1}{6}y^4 + \frac{1}{2}y^2 \right) + \frac{1}{2}\bar{a}y^2 + \bar{b}y + \bar{c}, \quad (\text{B.3})$$

where $(\bar{a}, \bar{b}, \bar{c})$ are constants determined by the boundary conditions. Applying the boundary conditions on $y = \pm 1$ yields $\bar{b} = 0$, and we also find $\bar{c}$ in terms of $\bar{a}$ and are thus left with

$$W_2(y) = \beta\alpha_0^2|A_1|^2 \left(\frac{1}{30}y^6 - \frac{1}{6}y^4 + \frac{1}{2}y^2 - \frac{11}{30} \right) + \frac{1}{2}\bar{a}(y^2 - 1). \quad (\text{B.4})$$

The, as yet arbitrary constant, $\bar{a}$ is to be determined according to the boundary conditions for V_2 . Considering the continuity equation (B.2a) we find

$$V_2'(y) = -2\beta^2\alpha_0^2|A_1|^2 \left(\frac{1}{30}y^6 - \frac{1}{6}y^4 + \frac{1}{2}y^2 - \frac{11}{30} \right) - \beta\bar{a}(y^2 - 1). \quad (\text{B.5})$$

Integrating once leaves

$$V_2(y) = -2\beta^2\alpha_0^2|A_1|^2 \left(\frac{1}{210}y^7 - \frac{1}{30}y^5 + \frac{1}{6}y^3 - \frac{11}{30}y \right) - \beta\bar{a} \left(\frac{1}{3}y^3 - y \right) + \bar{d}. \quad (\text{B.6})$$

Applying the boundary conditions we find $\bar{d} = 0$ and

$$\bar{a} = -\frac{144}{210}\beta\alpha_0^2|A_1|^2. \quad (\text{B.7})$$

We have now determined $V_2(y)$,

$$V_2(y) = -\frac{1}{105}\beta^2\alpha_0^2|A_1|^2(y^7 - 7y^5 + 11y^3 - 5y). \quad (\text{B.8})$$

and $W_2(y)$,

$$W_2(y) = \frac{1}{210} \beta \alpha_0^2 |A_1|^2 (7y^6 - 35y^4 + 33y^2 - 5). \quad (\text{B.9})$$

Substituting $V_2(y)$ into equation (B.2b) we find

$$U_2''(y) = \frac{2}{105} \beta^2 \alpha_0^2 |A_1|^2 (y^8 - 7y^6 + 11y^4 - 5y^2). \quad (\text{B.10})$$

Integrating this equation twice and applying the boundary conditions we find

$$U_2(y) = \frac{1}{105 \times 180} \beta^2 \alpha_0^2 |A_1|^2 (4y^{10} - 45y^8 + 132y^6 - 150y^4 + 59). \quad (\text{B.11})$$

Now that we have determined the $O(\Delta^2)$ corrections to the streamwise component of the vortex flow in the core U_2 , we can compute the $O(\Delta^2)$ corrections to the shear stresses and J , which are easily found to be

$$\lambda_{\pm 2} = \lambda_2 = K \beta^2 \alpha_0^2 |A_1|^2, \quad J_2 = M \beta^2 \alpha_0^2 |A_1|^2, \quad (\text{B.12})$$

where K and M are constants given by

$$K = \frac{32}{4725}, \quad M = \frac{307200}{42567525}. \quad (\text{B.13})$$

Note that since $U_2(y)$ is symmetric, we find that $\lambda_+ = \lambda_-$.¹

Now consider the pressure equations (2.88) at leading order, $O(\Delta)$, where we find

$$(2\alpha_0)^{\frac{5}{3}} \mathcal{G}(\xi_{0+}) A_1 + (\alpha_0^2 + \beta^2) P_1 = 0, \quad (\text{B.14a})$$

$$(2\alpha_0)^{\frac{5}{3}} \mathcal{G}(\xi_{0-}) A_1 - (\alpha_0^2 + \beta^2) Q_1 = 0, \quad (\text{B.14b})$$

$$\frac{16}{15} \alpha_0^2 A_1 - P_1 + Q_1 = 0, \quad (\text{B.14c})$$

where $\mathcal{G}$ is defined in (2.44) and

$$\xi_{0\pm} = \frac{(2i\alpha_0)^{\frac{1}{3}}}{2} (\gamma_{\pm} - c_0). \quad (\text{B.15})$$

If we let $\mathbf{v}_1 = [A_1, P_1, Q_1]^T$ then we can write equations (B.14) in the form

$$\mathbf{M}_1 \mathbf{v}_1 = \mathbf{0}, \quad (\text{B.16})$$

¹Recall that the shear stress on the lower wall is $\lambda_-(Z) = U_y(y = -1, Z)$, however we have defined $\lambda_+(Z) = -U_y(y = 1, Z)$.

where $\mathbf{M}_1 \in \mathbb{M}_3(\mathbb{C})$. This form turns out to be convenient for the discussion of the $O(\Delta^3)$ terms which follows soon. By eliminating A_1, P_1 and Q_1 from equations (B.14) or setting $\det(\mathbf{M}_1) = 0$, we find the leading order dispersion relation

$$\mathcal{D}_\gamma(\alpha_0, c_0, \beta) = \frac{15}{16} \frac{(2\alpha_0)^{\frac{5}{3}}}{\alpha_0^2} (\mathcal{G}(\xi_{0+}) + \mathcal{G}(\xi_{0-})) + (\alpha_0^2 + \beta^2) = 0. \quad (\text{B.17})$$

The solution of this dispersion relation determines the scaled linear neutral values of the streamwise wavenumber and wavespeed, (α_0, c_0) , given the scaled spanwise wavenumber β . We can also find expressions for P_1 and Q_1 in terms of A_1 ,

$$(P_1, Q_1) = -\frac{(2\alpha_0)^{\frac{5}{3}}}{(\alpha_0^2 + \beta^2)} (\mathcal{G}(\xi_{0+}), -\mathcal{G}(\xi_{0-})) A_1. \quad (\text{B.18})$$

Now consider the next order, $O(\Delta^3)$, analysis of the wave equations (2.88). This will ultimately determine the perturbations of the wave quantities to the linear neutral values (α_2, c_2) for a given amplitude wave, ΔA_1 . Since $|\Delta| \ll 1$ we can Taylor expand $\xi_\pm$ around $\xi_{0\pm}$ as

$$\xi_\pm = \xi_{0\pm} + \Delta^2 \xi_{2\pm} + \dots, \quad (\text{B.19})$$

where $\xi_{0\pm}$ is defined in (B.15) and

$$\xi_{2\pm} = \left(\frac{\alpha_2}{3\alpha_0} - \frac{\lambda_2}{3} - \frac{c_2}{(\gamma_\pm - c_0)} \right) \xi_{0\pm}. \quad (\text{B.20})$$

Now we consider the wave equations (2.88) at $O(\Delta^3)$. Taking the first spanwise harmonic (terms in $\cos \beta Z$) we find from equation (2.88a)

$$\begin{aligned} & -(\beta^2 + \alpha_0^2)P_3 - (2\alpha_0)^{\frac{5}{3}}\mathcal{G}(\xi_{0+})A_3 \\ &= \frac{1}{2}\beta^2\lambda_2\mathcal{F}(\xi_{0+})P_1 + 2\alpha_0\alpha_2P_1 + \frac{5}{3}(2\alpha_0)^{\frac{5}{3}}\mathcal{G}(\xi_{0+})\left(\frac{\alpha_2}{\alpha_0} + \frac{\lambda_2}{4}\right)A_1 \\ & \quad - 2\alpha_0^2(\gamma_+ - c_0)i^{-5/3}\mathcal{G}'(\xi_{0+})\left(\frac{\alpha_2}{3\alpha_0} - \frac{\lambda_2}{6} - \frac{c_2}{(\gamma_+ - c_0)}\right)A_1, \end{aligned} \quad (\text{B.21a})$$

where $\mathcal{F}$ is defined in (2.44), and from equation (2.88b)

$$\begin{aligned} & -(\beta^2 + \alpha_0^2)Q_3 + (2\alpha_0)^{\frac{5}{3}}\mathcal{G}(\xi_{0-})A_3 \\ & = \frac{1}{2}\beta^2\lambda_2\mathcal{F}(\xi_{0-})Q_1 + 2\alpha_0\alpha_2Q_1 - \frac{5}{3}(2\alpha_0)^{\frac{5}{3}}\mathcal{G}(\xi_{0-})\left(\frac{\alpha_2}{\alpha_0} - \frac{\lambda_2}{4}\right)A_1 \\ & \quad + 2\alpha_0^2(\gamma_- - c_0)i^{-5/3}\mathcal{G}'(\xi_{0-})\left(\frac{\alpha_2}{3\alpha_0} - \frac{\lambda_2}{6} - \frac{c_2}{(\gamma_- - c_0)}\right)A_1, \end{aligned} \quad (\text{B.21b})$$

and from equation (2.88c)

$$P_3 - Q_3 - \frac{16}{15}\alpha_0^2A_3 = \alpha_0^2\left(\frac{32\alpha_2}{15\alpha_0} + \frac{J_2}{2}\right)A_1. \quad (\text{B.21c})$$

Recalling the definition of $\mathbf{M}_1$ and defining $\mathbf{v}_3 = [A_3, P_3, Q_3]^T$, with a little rearrangement we can write the above equations (B.21) in the form

$$\mathbf{M}_1\mathbf{v}_3 = \mathbf{f}. \quad (\text{B.22})$$

But we know that $\mathbf{M}_1$ is singular from equation (B.16), and in fact

$$[1, 1, (\alpha_0^2 + \beta^2)]^T \cdot \mathbf{M}_1 = \mathbf{0}, \quad (\text{B.23})$$

and hence for a solution to the system we require the solvability condition

$$[1, 1, (\alpha_0^2 + \beta^2)]^T \cdot \mathbf{f} = \mathbf{0}, \quad (\text{B.24})$$

to be satisfied. We values of $\mathbf{f} = (f_1, f_2, f_3)^T$ can be read off from equations (B.21) as

$$\begin{aligned} f_1 = & -\frac{1}{2}\beta^2\lambda_2\mathcal{F}(\xi_{0+})P_1 - 2\alpha_0\alpha_2P_1 - \frac{5}{3}(2\alpha_0)^{\frac{5}{3}}\mathcal{G}(\xi_{0+})\left(\frac{\alpha_2}{\alpha_0} + \frac{\lambda_2}{4}\right)A_1 \\ & + 2\alpha_0^2(\gamma_+ - c_0)\left(\frac{\alpha_2}{3\alpha_0} - \frac{\lambda_2}{6} - \frac{c_2}{(\gamma_+ - c_0)}\right)i^{-5/3}\mathcal{G}'(\xi_{0+})A_1, \end{aligned} \quad (\text{B.25a})$$

$$\begin{aligned} f_2 = & \frac{1}{2}\beta^2\lambda_2\mathcal{F}(\xi_{0-})Q_1 + 2\alpha_0\alpha_2Q_1 - \frac{5}{3}(2\alpha_0)^{\frac{5}{3}}\mathcal{G}(\xi_{0-})\left(\frac{\alpha_2}{\alpha_0} + \frac{\lambda_2}{4}\right)A_1 \\ & + 2\alpha_0^2(\gamma_- - c_0)\left(\frac{\alpha_2}{3\alpha_0} - \frac{\lambda_2}{6} - \frac{c_2}{(\gamma_- - c_0)}\right)i^{-5/3}\mathcal{G}'(\xi_{0-})A_1, \end{aligned} \quad (\text{B.25b})$$

$$f_3 = -\alpha_0^2\left(\frac{32\alpha_2}{15\alpha_0} + \frac{J_2}{2}\right)A_1. \quad (\text{B.25c})$$

Then the solvability condition written in terms of $\mathbf{f}$ is

$$f_1 + f_2 + (\alpha_0^2 + \beta^2)f_3 = 0. \quad (\text{B.26})$$

Writing this out in full, using equations (B.18) for (P_1, Q_1) and the forms of (λ_2, J) from equation (B.12), we find the amplitude equation (from the solvability condition) relating α_2, c_2 and $|A_1|^2$ can be written as

$$\begin{aligned} \alpha_0 \beta^2 |A_1|^2 & \left\{ \frac{K \beta^2 (2\alpha_0)^{\frac{5}{3}}}{2(\alpha_0^2 + \beta^2)} [\mathcal{G}(\xi_{0+}) \mathcal{F}(\xi_{0+}) + \mathcal{G}(\xi_{0-}) \mathcal{F}(\xi_{0-})] + \frac{4K}{9} \alpha_0^2 (\alpha_0^2 + \beta^2) \right. \\ & \left. - \frac{M}{2} \alpha_0^2 (\alpha_0^2 + \beta^2) - \frac{i^{-5/3} K}{3} \alpha_0^2 [(\gamma_+ - c_0) \mathcal{G}'(\xi_{0+}) + (\gamma_- - c_0) \mathcal{G}'(\xi_{0-})] \right\} \\ & + \alpha_2 \left\{ -\frac{16}{45} (7\alpha_0^2 + \beta^2) + \frac{2i^{-5/3}}{3} [(\gamma_+ - c_0) \mathcal{G}'(\xi_{0+}) + (\gamma_- - c_0) \mathcal{G}'(\xi_{0-})] \right\} \\ & + c_2 \left\{ -2\alpha_0 i^{-5/3} [\mathcal{G}'(\xi_{0+}) + \mathcal{G}'(\xi_{0-})] \right\} = 0. \quad (\text{B.27}) \end{aligned}$$

At present there is a degree of freedom in the weakly nonlinear analysis through the choice of Δ . We can eliminate the degree of freedom by setting A_1 for example, so we set $A_1 = 1$.

Choosing a scaled spanwise wavenumber β and scaled wall sliding velocities $\gamma_{\pm}$, equation (B.17) determines the scaled linear neutral streamwise wavenumber and wavespeed (α_0, c_0) , and (B.27) reveals the nature of the bifurcation from the linear neutral point into finite amplitude states. Note that this analysis is only valid when the amplitude is small, but in this case we find the velocities are determined by (U_2, V_2, W_2) through equations (B.11, B.8, B.9).

C. Body-fitted coordinates

Sometimes it is convenient to write the Navier-Stokes equations in a general set of orthogonal coordinates. In the new orthogonal coordinate system denoted by q_i we have

$$d\mathbf{r} = \sum_{i=1}^3 \frac{\partial \mathbf{r}}{\partial q_i} dq_i = \sum_{i=1}^3 h_i dq_i \mathbf{b}_i, \quad (\text{C.1})$$

where the scale factors are the Lamé coefficients,

$$h_i = \left| \frac{\partial \mathbf{r}}{\partial q_i} \right|, \quad (\text{C.2})$$

and the $\mathbf{b}_i$ are the basis vectors

$$\mathbf{b}_i = \frac{1}{h_i} \frac{\partial \mathbf{r}}{\partial q_i}. \quad (\text{C.3})$$

Using the Lamé coefficients, the Navier-Stokes equations can be written in this new coordinate system - see for example Rosenhead (1963).

We will be interested here with the special case of body fitted coordinates. In the discussion which follows we will focus on channels but analogous statements follow for curved walls. With reference to the channel centreline we use s to denote arclength downstream from some reference point, n to denote distance normal to this centreline, and z is the usual spanwise coordinate. We will assume the channel is curved so that the shape of the centreline depends only on the distance downstream i.e. it is independent of z so the curvature is $\Lambda = \Lambda(s)$. Thus to write the equations in terms of the new coordinate system (s, n, z) we require the Lamé coefficients, (h_s, h_n, h_z) . Clearly $h_z = 1$ since this coordinate is unaffected by the change of coordinates. We will now determine h_n and h_s . If we hold s and z constant, then change n to $n + \delta n$, it follows that

$$\delta \mathbf{r} = -n \mathbf{b}_n(s, n) + (n + \delta n) \mathbf{b}_n(s, n) = \delta n \mathbf{b}_n, \quad (\text{C.4})$$

and thus $h_n = 1$. Now we hold n and z constant as in figure C.1. The difference here is

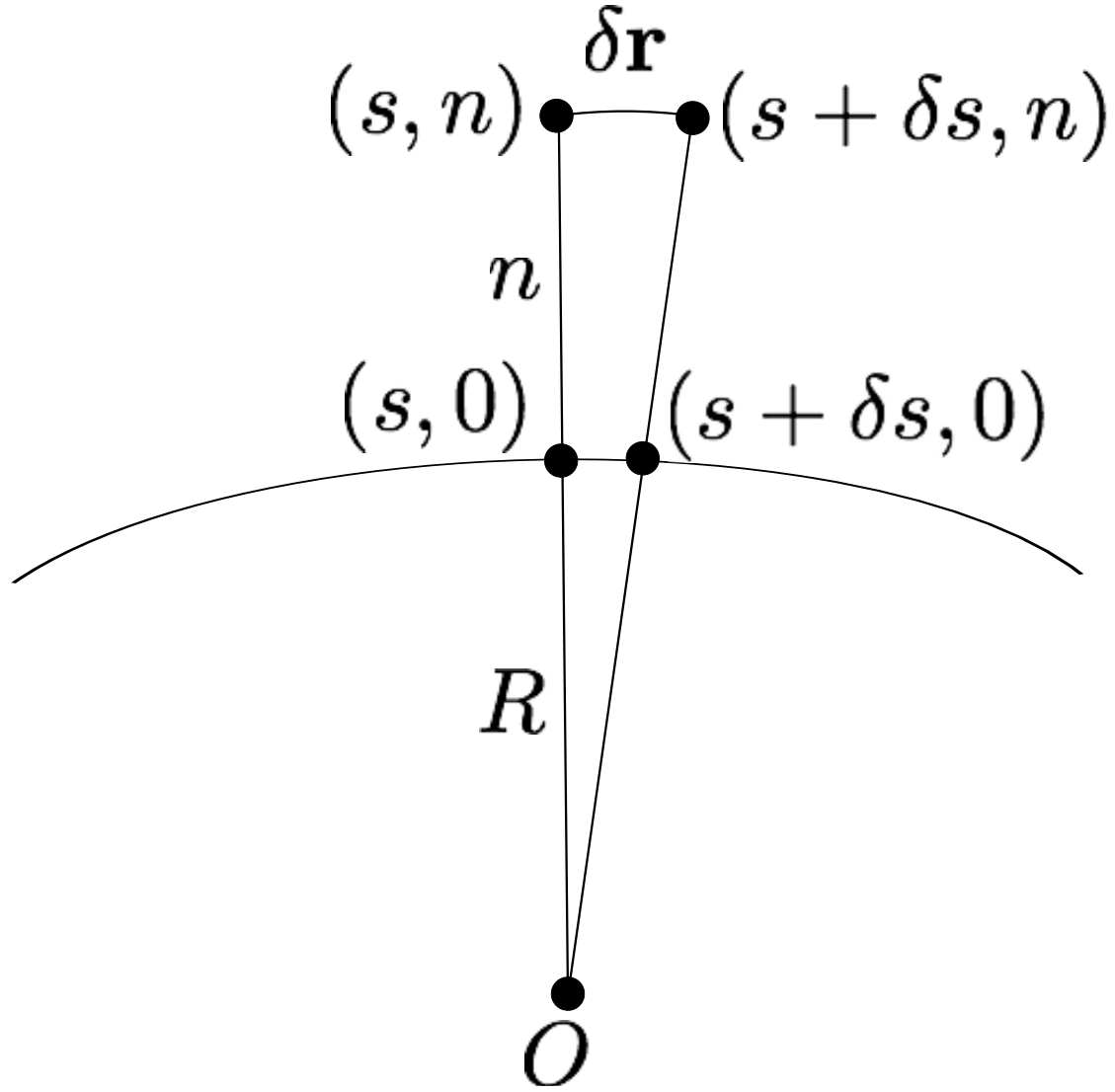


Figure C.1. Body-fitted coordinates. In this diagram points are identified by their location in arclength and wall normal distance coordinates.

that the basis vector for n depends on s and thus

$$\delta \mathbf{r} = -n \mathbf{b}_n(s, n) + n \mathbf{b}_n(s + \delta s, n). \quad (\text{C.5})$$

One could then work out $\mathbf{b}_n(s + \delta s, n)$ and we could calculate h_s . However it is much easier to use a geometrical argument here. Using the arclength of a circle is equal to the angle multiplied by the radius, we find that the angle covered is

$$\delta \theta = \frac{\delta s}{R}. \quad (\text{C.6})$$

Now applying the same rule we find that

$$\delta \mathbf{r} = (R + n) \delta \theta \mathbf{b}_s = \delta s (1 + \Lambda n) \mathbf{b}_s, \quad (\text{C.7})$$

where $\Lambda = 1/R$ is the curvature. Hence it follows that $h_s = 1 + \Lambda n$. Substituting these into the Navier-Stokes equations in general orthogonal coordinates we find the velocities in the (s, n, z) directions given by $\mathbf{v} = (v_s, v_n, v_z)$ satisfy

$$\frac{1}{h_s} \frac{\partial v_s}{\partial s} + \frac{\partial v_n}{\partial n} + \frac{\partial v_z}{\partial z} + \frac{\Lambda}{h_s} v_n = 0, \quad (\text{C.8a})$$

$$\begin{aligned} & \frac{\partial v_s}{\partial t} + \frac{v_s}{h_s} \frac{\partial v_s}{\partial s} + v_n \frac{\partial v_s}{\partial n} + v_z \frac{\partial v_s}{\partial z} + \frac{\Lambda}{h_s} v_s v_n = -\frac{1}{h_s} \frac{\partial p}{\partial s} \\ & + Re^{-1} \left[\frac{1}{h_s} \frac{\partial}{\partial s} \left(\frac{1}{h_s} \frac{\partial v_s}{\partial s} \right) + \frac{\partial^2 v_s}{\partial n^2} + \frac{\partial^2 v_s}{\partial z^2} + \Lambda \frac{\partial}{\partial n} \left(\frac{v_s}{h_s} \right) + \frac{\Lambda}{h_s^2} \frac{\partial v_n}{\partial s} + \frac{1}{h_s} \frac{\partial}{\partial s} \left(\frac{\Lambda}{h_s} v_n \right) \right], \end{aligned} \quad (\text{C.8b})$$

$$\begin{aligned} & \frac{\partial v_n}{\partial t} + \frac{v_s}{h_s} \frac{\partial v_n}{\partial s} + v_n \frac{\partial v_n}{\partial n} + v_z \frac{\partial v_n}{\partial z} - \frac{\Lambda}{h_s} v_s^2 = -\frac{\partial p}{\partial n} \\ & + Re^{-1} \left[\frac{1}{h_s} \frac{\partial}{\partial s} \left(\frac{1}{h_s} \frac{\partial v_n}{\partial s} \right) + \frac{\partial^2 v_n}{\partial n^2} + \frac{\partial^2 v_n}{\partial z^2} + \Lambda \frac{\partial}{\partial n} \left(\frac{v_n}{h_s} \right) - \frac{\Lambda}{h_s^2} \frac{\partial v_s}{\partial s} - \frac{1}{h_s} \frac{\partial}{\partial s} \left(\frac{\Lambda}{h_s} v_s \right) \right], \end{aligned} \quad (\text{C.8c})$$

$$\begin{aligned} & \frac{\partial v_z}{\partial t} + \frac{v_s}{h_s} \frac{\partial v_z}{\partial s} + v_n \frac{\partial v_z}{\partial n} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial z} \\ & + Re^{-1} \left[\frac{1}{h_s} \frac{\partial}{\partial s} \left(\frac{1}{h_s} \frac{\partial v_z}{\partial s} \right) + \frac{\partial^2 v_z}{\partial n^2} + \frac{\partial^2 v_z}{\partial z^2} + \frac{\Lambda}{h_s} \frac{\partial v_z}{\partial n} \right], \end{aligned} \quad (\text{C.8d})$$

Since we will assume that the curvature is a small constant we ignore terms of $O(Re^{-1}\Lambda)$ which do not contribute to the leading order analysis, and relabelling $(s, n, z) \rightarrow (x, y, z)$ and $\mathbf{v} = (v_s, v_n, v_z) \rightarrow \mathbf{u} = (u, v, w)$ we find that the relevant equations to $O(\Lambda)$ are

$$\nabla \cdot \mathbf{u} = 0, \tag{C.9a}$$

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + \Lambda(uv, -u^2, 0) = -\nabla p + Re^{-1} \nabla^2 \mathbf{u}. \tag{C.9b}$$

D. Weakly nonlinear analysis of the vortex-wave interaction equations in the asymptotic suction boundary layer

Here we will discuss the weakly nonlinear analysis of the vortex-wave interaction equations (3.56, 3.57, 3.58) in the asymptotic suction boundary layer. We start by assuming the wave is small with spanwise wavenumber β , so $(A, p_{11}, \tilde{p}_1) = O(\Delta) \cos \beta Z$ where $|\Delta| \ll 1$. From the wall forcing condition (3.56d) we find that the spanwise component of the vortex is of the form $W = O(\Delta^2) \sin 2\beta Z$. From this it follows that $U - u_b$ and V are of the form $O(\Delta^2) \cos 2\beta Z$ and hence the shear stress λ is an $O(\Delta^2)$ perturbation from its basic state value, $\lambda - 1 = O(\Delta^2) \cos 2\beta Z$. Considering equation (3.57) we see that next order perturbations for the wave components must be $O(\Delta^2)$. Thus we can now write the weakly nonlinear expansions,

$$(U, V) = (u_b, v_b) + \Delta^2(u_2, v_2) \cos 2\beta Z, \quad W = \Delta^2 w_2 \sin 2\beta Z, \quad (\text{D.1a})$$

$$\lambda = \lambda_0 + \Delta^2 \lambda_2 \cos 2\beta Z, \quad (\text{D.1b})$$

where $\lambda_0 = 1$ is the basic state value, and for the wave quantities

$$(A, p_{11}, \hat{p}) = \Delta(A_1, P_1, Q_1) \cos \beta Z + \Delta^3[(A_3^1, P_3^1, Q_3^1) \cos \beta Z + (A_3^3, P_3^3, Q_3^3) \cos 3\beta Z], \quad (\text{D.1c})$$

$$(\alpha, c) = (\alpha_0, c_0) + \Delta^2(\alpha_2, c_2). \quad (\text{D.1d})$$

The subsequent analysis is split into the three parts which collectively compose the interaction equations.

We start with the main deck vortex equations (3.56). Substituting in the expansions

(D.1) we find the governing equations are

$$v_2' + 2\beta w_2 = 0, \quad (\text{D.2a})$$

$$-u_2' + e^{-y}v_2 = u_2'', \quad (\text{D.2b})$$

$$-w_2' = w_2'', \quad (\text{D.2c})$$

subject to the boundary conditions,

$$u_2 = v_2 = 0, \quad w_2 = \beta|P_1|^2(1 - \beta^2/\alpha_0^2), \quad y = 0, \quad (\text{D.2d})$$

$$u_2, w_2 \rightarrow 0, \quad y \rightarrow \infty. \quad (\text{D.2e})$$

These equations can be solved to yield the weakly nonlinear perturbation to the basic state flow variables in the main deck,

$$u_2 = 2\beta^2|P_1|^2(1 - \beta^2/\alpha_0^2)(y + 1 + e^{-y}/2 - 3/2)e^{-y}, \quad (\text{D.3a})$$

$$v_2 = -2\beta^2|P_1|^2(1 - \beta^2/\alpha_0^2)(1 - e^{-y}), \quad (\text{D.3b})$$

$$w_2 = \beta|P_1|^2(1 - \beta^2/\alpha_0^2)e^{-y}, \quad (\text{D.3c})$$

from which it follows that the correction to the shear stress is

$$\lambda_2 = \beta^2|P_1|^2(1 - \beta^2/\alpha_0^2). \quad (\text{D.4})$$

Now consider the upper deck equations (3.58) at $O(\Delta)$ and $O(\Delta^3)$. We find

$$Q_1'' - (\alpha_0^2 + \beta^2)Q_1 = 0, \quad (\text{D.5a})$$

$$Q_3^{1''} - (\alpha_0^2 + \beta^2)Q_3^1 = 2\alpha_0\alpha_2Q_1, \quad (\text{D.5b})$$

with boundary conditions on the wall for the $O(\Delta)$ problem,

$$Q_1 = P_1, \quad Q_1' = -\alpha_0^2A_1, \quad \bar{Y} = 0, \quad (\text{D.5c})$$

and on the $O(\Delta^3)$ problem,

$$Q_3^1 = P_3^1, \quad Q_3^{1'} = -\alpha_0^2A_3^1 - 2\alpha_0\alpha_2A_1, \quad \bar{Y} = 0. \quad (\text{D.5d})$$

The far field conditions decay condition is the same for both problems,

$$Q_1, Q_3^1 \rightarrow 0 \quad \text{as} \quad \bar{Y} \rightarrow \infty. \quad (\text{D.5e})$$

If we define $\chi_0 = \sqrt{\alpha_0^2 + \beta^2} \geq 0$ then we find a solution for Q_1 is given by

$$Q_1 = P_1 e^{-\chi_0 \bar{Y}}, \quad (\text{D.6})$$

from which it follows that

$$\chi_0 P_1 = \alpha_0^2 A_1. \quad (\text{D.7})$$

This is the leading order condition as found in the linear stability analysis at asymptotically large Reynolds numbers, equation (3.15). Now concentrating on the second order problem, and looking for a solution of the form $Q_3 = f(\bar{Y})e^{-\chi_0 \bar{Y}}$ we find

$$Q_3^1 = P_3^1 e^{-\chi_0 \bar{Y}} - \frac{\alpha_0 \alpha_2}{\chi_0} P_1 \bar{Y} e^{-\chi_0 \bar{Y}}. \quad (\text{D.8})$$

Applying the condition (D.5d) on $\bar{Y} = 0$ we find,

$$\chi_0 P_3^1 - \alpha_0^2 A_3^1 = \frac{\alpha_2}{\alpha_0 \chi_0} P_1 (2\beta^2 + \alpha_0^2). \quad (\text{D.9})$$

This is the second order pressure-displacement law.

Finally consider the lower deck equations (3.57). Since Δ is small we find that $\xi \simeq \xi_0 + \Delta^2 \xi_2$ where

$$\xi_0 = -i^{1/3} \alpha_0^{1/3} c_0, \quad \xi_2 = -i^{1/3} \alpha_0^{1/3} c_0 \left(\frac{\alpha_2}{3\alpha_0} + \frac{c_2}{c_0} - \frac{2\lambda_2}{3} \cos 2\beta Z \right). \quad (\text{D.10})$$

So we find at leading order $O(\Delta)$

$$-(\beta^2 + \alpha_0^2) P_1 = \alpha_0^{5/3} G(\xi_0) A_1, \quad (\text{D.11})$$

which with the leading order pressure-displacement law leads to the dispersion relation as found in the linear stability calculation at large Reynolds numbers given by equation (3.20),

$$\mathcal{G}(\xi_0) + \alpha_0^{1/3} \chi_0 = 0. \quad (\text{D.12})$$

Considering the next order terms at $O(\Delta^3)$ in equations (3.57,3.58) we find the equations for (A_3^1, P_3^1, Q_3^1) are the inhomogeneous version of the equations governing

(A_1, P_1, Q_1) and hence we find a solvability condition. This condition is an amplitude equation describing the bifurcation of small amplitude states from the linear neutral point, and is of the form

$$\nu_1 \alpha_2 + \nu_2 c_2 + \nu_3 |P_1|^2 = 0, \quad (\text{D.13})$$

where ν_j are complex constants given by

$$\nu_1 = \frac{\beta^2}{3\alpha_0} + \frac{4\alpha_0}{3} + \frac{\chi_0 c_0}{3\alpha_0} \mathcal{H}(\xi_0), \quad (\text{D.14})$$

$$\nu_2 = \chi_0 \mathcal{H}(\xi_0), \quad (\text{D.15})$$

$$\nu_3 = \beta^2 \left(1 - \frac{\beta^2}{\alpha_0^2}\right) \left\{ \beta^2 \mathcal{F}(\xi_0) - \frac{\chi_0 c_0}{3} \mathcal{H}(\xi_0) - \frac{5\chi_0^2}{6} \right\}. \quad (\text{D.16})$$

In the above we have defined $\mathcal{H}(\xi) = \text{i}^{-5/3} \mathcal{G}'(\xi)$. There is a degree of freedom in the definition of Δ and thus we can set for example $|A_1| = 1$ without loss of generality. Equation (D.13) is a complex equation which determines the two real unknowns α_2 and c_2 , thus describing how the wavenumber and wavespeed of these travelling wave states vary for small amplitudes. This will be used to verify the computational results at small amplitudes, and also to determine how the bifurcation varies with spanwise wavenumber β .

E. Computing the roll and streak in the adjustment layer

As mentioned in the text, the streak decouples from the roll in (4.53b), and we consider the equation for the vorticity defined in (4.55). Thus we find the vorticity is governed by the equation

$$(\kappa^{-2}\partial_\xi^2 + \kappa^{-1}\partial_\xi - \beta^2 - \kappa^{-1}\partial_X)V_r = 0, \quad (\text{E.1a})$$

subject to the matching condition

$$V_r \sim K(\omega^2 - \beta^2)e^{\kappa\omega\xi}, \quad \xi \rightarrow 1. \quad (\text{E.1b})$$

Seeking a WKB solution of (E.1) of the form $V_r = \hat{V}e^{\kappa\theta}$ we find that the WKB phase and amplitude functions satisfy the equations

$$\theta_\xi^2 + \theta_\xi - \beta^2 - \theta_X = 0, \quad (\text{E.2a})$$

$$\kappa^{-1}\hat{V}_{\xi\xi} + (2\theta_\xi + 1)\hat{V}_\xi - \hat{V}_X = 0, \quad (\text{E.2b})$$

with the relevant boundary conditions,

$$\hat{V} \sim K(\omega^2 - \beta^2), \quad \theta \sim \omega\xi, \quad \xi \rightarrow 1. \quad (\text{E.2c})$$

Computing the vorticity in the adjustment layer

Firstly, we will consider in more detail the solution for the vorticity in the adjustment layer. The discussion of section 4.5.1 highlighted the fact that there are three distinct regions to consider for the vorticity, which are a parallel region, a diffusive region and a lower adjustment region labelled as regions 1, 2 and 3, respectively.

Region 1

Applying the method of characteristics to (E.2a) subject to $\theta \sim \omega\xi$ as $\xi \rightarrow 1$, we find

that the phase in region 1 is the parallel phase function $\theta = \omega\xi$. Substituting the phase into the amplitude equation (E.2b) we find the amplitude is governed by

$$(2\omega + 1)\hat{V}_\xi - \hat{V}_X = 0, \quad (\text{E.3})$$

with the matching condition $\hat{V} \sim K(\omega^2 - \beta^2)$ as $\xi \rightarrow 1$. A simple application of the method of characteristics yields the solution,

$$\hat{V} = K(\omega^2 - \beta^2), \quad (\text{E.4})$$

for $\bar{\xi}(X) = 1 - (2\omega + 1)X < \xi < 1$.

Regions 2 and 3

We require the solution to decay as $X \rightarrow 0$ and this suggests the existence of a diffusion layer around $\bar{\xi}(X)$. Consider once again the amplitude equation (E.2b). In region 1, $\partial_X \sim \partial_\xi \sim O(1)$, and hence the diffusive term was negligible. A viscous-inviscid balance suggests in the diffusion layer we have $\kappa^{-1}\partial_\xi^2 \sim \partial_X$. Thus we introduce the diffusion layer variables

$$(\eta, X) = (\kappa^{1/2}(\xi - \bar{\xi}), X). \quad (\text{E.5})$$

Writing the amplitude equation (E.2b) in diffusion layer variables, we find

$$\hat{V}_{\eta\eta} - \hat{V}_X = 0, \quad (\text{E.6a})$$

subject to the far field matching conditions

$$\hat{V} \rightarrow \begin{cases} K(\omega^2 - \beta^2), & \eta \rightarrow \infty, \\ 0, & \eta \rightarrow -\infty. \end{cases} \quad (\text{E.6b})$$

This is the one-dimensional diffusion equation on an infinite domain and can be solved in terms of the error function of the similarity variable $\eta/X^{1/2}$. For convenience we write the solution in the form,

$$\hat{V} = \frac{K(\omega^2 - \beta^2)}{2} \text{erfc}\left(-\frac{\eta}{2X^{1/2}}\right), \quad (\text{E.7})$$

where $\text{erfc}(s)$ is the complementary error function, related to the error function $\text{erf}(s)$ via the identity $\text{erfc}(s) = 1 - \text{erf}(s)$. For the analysis of region 3, we note the asymptotic

behaviour of the amplitude as $\eta \rightarrow -\infty$ is

$$\hat{V} \sim -\frac{K(\omega^2 - \beta^2)}{\sqrt{\pi}} \frac{X^{1/2}}{\eta} e^{-\eta^2/4X}. \quad (\text{E.8})$$

Writing (E.8) in terms of the lower adjustment layer variables, we find that this is exactly the solution required in region 3, and thus the solution in region 3 is given by

$$V_r = -\kappa^{-1/2} \frac{K(\omega^2 - \beta^2)}{\sqrt{\pi}} \frac{X^{1/2}}{(\xi - \bar{\xi})} e^{\kappa(\omega\xi - (\xi - \bar{\xi})^2/4X)}. \quad (\text{E.9})$$

Computing the roll in the adjustment layer

Now that we computed the vorticity in the adjustment layer, we can use equation (4.55) to compute the roll. As expressed in section 4.5.1, the solution for the roll is complicated by the presence of a WKB turning point located in region 3, which subdivides this region into three subregions. Once again we seek a WKB solution of the form given in (4.54) with WKB phase function ϕ and amplitude function $\hat{v}$. Until the WKB turning point is encountered, the roll is driven by the vorticity and hence the phase will be identical to the phase of the vorticity, $\phi = \theta$. However, after the WKB turning point has been encountered the vorticity is exponentially small in comparison to the irrotational solution which is of the form $\sim e^{\kappa\beta\xi}$. The WKB region is thus the region where $\phi \approx \beta\xi$, and we will explore this region as part of the analysis to follow.

Regions 1 and 2

Firstly we consider the roll in regions 1 and 2, where the leading order solution comes from the vorticity and hence the phase of the roll is inherited from the vorticity. At leading order then $(\kappa^{-2}\partial_\xi^2 - \beta^2)$ becomes $(\omega^2 - \beta^2)$, and thus we must solve

$$(\omega^2 - \beta^2)\hat{v} = \hat{V} = \begin{cases} K(\omega^2 - \beta^2), & \text{region 1,} \\ K(\omega^2 - \beta^2)\text{erfc}(-\eta/2X^{1/2})/2, & \text{region 2.} \end{cases} \quad (\text{E.10})$$

From this we can conclude that the WKB amplitudes in the parallel and diffusive regions are given by

$$\hat{v} = \begin{cases} K, & \text{region 1,} \\ K\text{erfc}(-\eta/2X^{1/2})/2, & \text{region 2.} \end{cases} \quad (\text{E.11})$$

Region 3.1

Before the WKB turning point layer is encountered, the dominant contribution to the roll is from the vorticity. Thus setting the WKB phase equal to the vorticity phase function, we find the governing equation in this region to be

$$(\phi_\xi^2 - \beta^2)\hat{v} = -\kappa^{-1/2} \frac{K(\omega^2 - \beta^2)}{\sqrt{\pi}} \frac{X^{1/2}}{(\xi - \bar{\xi})}, \quad (\text{E.12})$$

where the WKB phase and ξ derivative of the phase of the vorticity (and hence for the roll in this region) can be seen from equation (E.9) to be

$$\phi = \omega\xi - \frac{(\xi - \bar{\xi})^2}{4X}, \quad \phi_\xi = \omega - \frac{(\xi - \bar{\xi})}{2X}. \quad (\text{E.13})$$

Thus the WKB amplitude of the roll in region 3.1 is given by

$$\hat{v} = -\kappa^{-1/2} \frac{K(\omega^2 - \beta^2)}{\sqrt{\pi}(\phi_\xi^2 - \beta^2)} \frac{X^{1/2}}{(\xi - \bar{\xi})}, \quad (\text{E.14})$$

where ϕ_ξ is given in (E.13). It is apparent that this solution breaks down when $\phi_\xi^2 - \beta^2 = 0$, as the denominator becomes zero. This can be seen to correspond to the location where $\phi = \beta\xi$, i.e. the solution is the same order as the irrotational solution, and formally the irrotational solution becomes the leading order term. Rearranging (E.13), this happens when

$$\xi = \xi(X) = 1 - (2\beta + 1)X, \quad (\text{E.15})$$

and this is the location of the WKB turning point layer.

Region 3.2

In the WKB turning point layer, as in the diffusive layer discussed in section E, viscous terms now become important and we require $\kappa\partial_\xi^2 \sim \partial_X$ to balance viscous-inviscid terms in the amplitude equation. Thus, similarly to the diffusive layer, we define turning point layer coordinates by

$$(\zeta, X) = (\kappa^{1/2}(\xi - \bar{\xi}), X). \quad (\text{E.16})$$

Inside this layer, as we would expect, the WKB phase function is $\phi = \beta\xi$ at leading order and we must compute the amplitude. The analysis is similar to section E, where once again we find the WKB amplitude satisfies a diffusion equation which has a similarity variable $\zeta/X^{1/2}$, and applying the far field matching conditions with region 3.1, the

leading order solution for the WKB amplitude function is found to be

$$\hat{v} = \frac{K(\omega + \beta)}{4\beta} e^{\zeta^2/4X} \operatorname{erfc}\left(\frac{\zeta}{2X^{1/2}}\right). \quad (\text{E.17})$$

For the analysis of the last region, region 3.3, we note that exiting the WKB turning point layer below, as $\zeta \rightarrow -\infty$,

$$\hat{v} \sim \frac{K(\omega + \beta)}{2\beta} e^{\zeta^2/4X}. \quad (\text{E.18})$$

Region 3.3

In this region, below the WKB turning point layer, the contribution from the vorticity is now subdominant to the irrotational flow solution. So to determine the leading order dynamics we only need to solve

$$(\kappa^{-2}\partial_\xi^2 - \beta^2)v_r = 0, \quad (\text{E.19})$$

subject to matching conditions with region 3.2. The solution is of the form, $v_r = F(X)e^{\kappa\beta\xi}$ and applying matching with region 3.2 we find the solution to be,

$$v_r = \frac{K(\omega + \beta)}{2\beta} e^{\kappa(\beta X + \omega - \beta)} e^{\kappa\beta\xi}. \quad (\text{E.20})$$

Computing the streak in the adjustment layer

Now that we have computed the roll throughout the whole of the adjustment layer, we can use equation (4.53a) to compute the streak. Recalling the WKB form of the solution given in (4.54) to account for the exponential factor difference between the streak and the roll, the roll is of the form $v_r = \hat{v}e^{\kappa\phi}$, for different functions $\hat{v}$ and ϕ in each of the five regions, and thus u_s is of the form

$$u_s = \hat{u}e^{\kappa(\phi - \xi)}, \quad (\text{E.21})$$

where ϕ for each region is identical to the phase of the roll in that region, and we must determine the WKB amplitude, $\hat{u}$, in each region. Without labouring the task as it is routine given the previous analysis, we list the solutions in each region. In regions 1, 2

and 3.1

$$\hat{u} = J, \quad \hat{u} = \frac{J}{2} \operatorname{erfc} \left(-\frac{\eta}{2X^{1/2}} \right), \quad \hat{u} = -\kappa^{-1/2} \frac{J\omega(\omega^2 - \beta^2)}{\sqrt{\pi}\phi_\xi(\phi_\xi^2 - \beta^2)} \frac{X^{1/2}}{(\xi - \bar{\xi})}, \quad (\text{E.22})$$

where in region 3.1, ϕ_ξ is given in (E.13). And in regions 3.2 and 3.3 we find

$$\hat{u} = \frac{J\omega(\omega + \beta)}{4\beta^2} e^{\zeta^2/4X} \operatorname{erfc} \left(\frac{\zeta}{2X^{1/2}} \right), \quad \hat{u} = \frac{J\omega(\omega + \beta)}{2\beta^2}. \quad (\text{E.23})$$