Aspects of the topological dynamics of sparse graph automorphism groups

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Statement of originality:

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Abstract:

We examine sparse graph automorphism groups from the perspective of the Kechris-Pestov-Todorčević (KPT) correspondence. The sparse graphs that we discuss are Hrushovski constructions: we consider the 'ab initio' Hrushovski construction M_0, the Fraïssé limit of the class of 2-sparse graphs with self-sufficient closure; M_1, a simplified version of M_0; and the ω-categorical Hrushovski construction M_F. We prove a series of results that show that the automorphism groups of these Hrushovski constructions demonstrate very different behaviour to previous classes studied in the KPT context. Extending results of Evans, Hubička and Nešetřil, we show that $\mathrm{Aut}(M_0)$ has no coprecompact amenable subgroup. We investigate the fixed points on type spaces property, a weakening of extreme amenability, and show that for a particular choice of control function F, $\mathrm{Aut}(M_F)$ does not have any closed oligomorphic subgroup with this property. Next we consider the $\mathrm{Aut}(M_1)$-flow of linear orders on M_1, and show that minimal subflows of this have all $\mathrm{Aut}(M_1)$-orbits meagre. We give partial analogous results for the $\mathrm{Aut}(M_0)$-flow of linear orders on M_0, and find the universal minimal flow of the automorphism group of the "dimension 0" part of M_0.

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Notation and terminology

Classes of graphs:

Class	Fraïssé limit	Automorphism group	Definition
$\mathcal{C}_1$	M_1	G_1	finite graphs B s.t. $\forall\, A \subseteq B$, $\exists\, a \in A$, $\mathrm{d}(a) \leq 2$
$\mathcal{C}_0$	M_0	G_0	2-sparse graphs
$\mathcal{C}_{00}$	M_{00}	G_{00}	$A \in \mathcal{C}_0$, $\delta(A) = 0$
$\mathcal{C}_F$	M_F	G_F	$B \in \mathcal{C}_0$, $\delta(A) \geq F(\lvert A\rvert)$ $\forall\, A \subseteq B$, with Fraïssé limit M_F ω-categorical

Classes of oriented graphs:

Class	Fraïssé limit	Automorphism group	Definition
$\mathcal{D}_1$	N_1	K_1	2-oriented graphs with no directed cycles
$\mathcal{D}_0$	$\times$	$\times$	2-oriented graphs
$\mathcal{D}_{\mathrm{fin}}$	$\times$	$\times$	finely 2-oriented graphs
$\mathcal{D}_{\mathcal{E}}$	$N_{\mathcal{E}}$	$K_{\mathcal{E}}$	digraph reducts of $\mathcal{E}_{\mathrm{fin}}$
$\mathcal{D}_{00}$	$\times$	$\times$	2-oriented graphs with $\delta = 0$

Classes of ordered oriented graphs:

Class	Fraïssé limit	Automorphism group	Definition
$\mathcal{E}_1$	(N_1, α)	H_1	admissibly ordered 2-oriented graphs with no directed cycles
$\mathcal{E}_{\mathrm{fin}}$	$(N_{\mathcal{E}}, \alpha)$	$H_{\mathcal{E}}$	admissibly ordered finely 2-oriented graphs
$\mathcal{E}_{00}$	(N_{00}, α)	H_{00}	admissibly ordered 2-oriented graphs with $\delta = 0$

In Chapter 4, where we study linear orders on M_1, we write:

$$G = G_1, K = K_1, H = H_1.$$

In Chapter 5, where we study linear orders on M_0, we write

$$G = G_0, K = K_{\mathcal{E}}, H = H_{\mathcal{E}}.$$

Other notation:

Notation	Description
$\mathcal{K}^{\prec}$	(A, γ) with $A \in \mathcal{K}$ and γ a linear order on A
$\mathcal{LO}(M)$	the flow of linear orders on M
$\mathrm{Or}(M)$	the flow of 2-orientations on M
$\delta(A)$	the predimension $\delta(A) = 2\|A\| - \|\,\mathrm{E}(A)\|$
$\mathrm{V}(A)$	vertex set of the graph A
$\mathrm{E}(A)$	edge set $\mathrm{E}(A) \subseteq A^{(2)}$ of the graph A
E_A	symmetric, irreflexive edge relation $E_A \subseteq A^2$
$A \leq_s B$	$\delta(C) \geq \delta(A)\ \forall\, A \subseteq C \subseteq B$
$A \sqsubseteq_s B$	A is successor-closed in B
$(A, \gamma_A) \sqsubseteq_s (B, \gamma_B)$	for A, B oriented graphs with orders γ_A, γ_B: $A \sqsubseteq_s B$, $\gamma_A = \gamma_B\|_A$
$A \leq_d B$	$\delta(C) > \delta(A)$ for all $A \subsetneq C \subseteq B$
$A \leq_1 B$	there exists a 2-orientation of B w/o directed cycles in which A is successor closed
scc	strongly connected component
scl	successor-closure
$Q_\rho(x)$	the cone of x in the orientation ρ
G_x	the stabiliser of x in the group G
Gx	the G-orbit of x

Attribution of results. Results due to other authors will usually be explicitly referenced. New results due to the author (in collaboration with the author's PhD supervisor David Evans) will be indicated as follows:

(*) indicates a new result which is a straightforward translation of earlier results.

(**) indicates a new result whose proof contains an original idea.

If a result is not explicitly referenced and is not indicated by asterisks, then it is either folklore, or comes from [**6**], [**7**], [**8**], the key paper which provides the starting point for this PhD thesis (of which we use results from several versions).

Introduction

This thesis examines the automorphism groups of sparse graphs from the perspective of the Kechris-Pestov-Todorčević correspondence (KPT), a series of results linking the topological dynamics of automorphism groups of countable homogeneous structures with structural Ramsey theory. Classes of sparse graphs demonstrate different behaviour to previous classes studied in the KPT context.

Sparse graphs.

A graph A is *k-sparse* if for all finite $B \subseteq A$, the number of edges of B is at most k times the number of vertices of B. We will take $k = 2$ throughout this thesis for presentational simplicity.

Predimension. The classes of sparse graphs that we study are examples of Hrushovski constructions, an important source of examples in model theory. One way to phrase 2-sparsity is in terms of *graph predimension*. For A a finite graph, the predimension $\delta(A)$ of A is given by $2|\,\mathrm{V}(A)| - |\,\mathrm{E}(A)|$, i.e. twice the number of vertices minus the number of edges. A graph is then 2-sparse iff all its finite subgraphs have non-negative predimension. The predimension we use here is a particular case of a more general notion of predimension used in Hrushovski constructions, where $\delta(A) = c|\,\mathrm{V}(A)| - |\,\mathrm{E}(A)|$ with $c \in \mathbb{R}_+$ (see [**5**] for more details, and [**15**], [**16**] for the original papers of Hrushovski where these constructions were introduced).

Strong classes. The classes of sparse graphs that we study will also be *strong* classes: they will have a distinguished notion of embedding, which we call *strong* embeddings.

Let $\mathcal{C}_0$ denote the class of finite 2-sparse graphs. For $A \subseteq B \in \mathcal{C}_0$, we will say A is *self-sufficient* in B, written $A \leq_s B$, if for $A \subseteq C \subseteq B$, $\delta(C) \geq \delta(A)$, and $\mathcal{C}_0$ together with this distinguished notion of $\leq_s$-substructure gives the strong class $(\mathcal{C}_0, \leq_s)$. The classical Fraïssé theory (see [**14**], Section 7.1), which gives a correspondence between countable homogeneous structures and amalgamation classes, carries over to strong classes, where in our definition of homogeneity, the amalgamation property and so on we restrict to strong embeddings.

We then have that $(\mathcal{C}_0, \leq_s)$ is a $\leq_s$-free amalgamation class, with Fraïssé limit M_0.

Orientations. A central fact in the analysis of sparse graphs is that a graph is k-sparse iff it is k-orientable: its edges may be oriented so that each vertex has at most k out-edges. This fact is well known to

graph theorists, and the proof is by Hall's Marriage Theorem. Writing $\mathcal{D}_0$ for the class of finite 2-oriented graphs, we thus have that $\mathcal{C}_0$ is the class of graph reducts of $\mathcal{D}_0$. For $A \subseteq B \in \mathcal{D}_0$, we write $A \sqsubseteq_s B$ if A is successor-closed in B, and $(\mathcal{D}_0, \sqsubseteq_s)$ gives another example of a $\sqsubseteq_s$-free amalgamation class. In fact, as shown in Lemma 1.5 of [**11**], we may again use Hall's Marriage Theorem to show that for $A \subseteq B \in \mathcal{C}_0$, $A \leq_s B$ iff there exists a 2-orientation B^+ of B in which $A \sqsubseteq_s B^+$, so the strong classes $(\mathcal{C}_0, \leq_s), (\mathcal{D}_0, \sqsubseteq_s)$ are very closely linked. Throughout this thesis, orientations will be a key tool in studying sparse graphs.

More classes of sparse graphs. We will also study two more examples of Hrushovski constructions. The first is M_1, a "simplified" version of M_0. Let $\mathcal{D}_1$ denote the class of finite 2-oriented graphs where the orientation has no directed cycles. $(\mathcal{D}_1, \sqsubseteq_s)$ is a $\sqsubseteq_s$-free amalgamation class. Let $\mathcal{C}_1$ be the class of graph reducts of $\mathcal{D}_1$. For $A \subseteq B \in \mathcal{C}_1$, write $A \leq_1 B$ if there exists an expansion $B^+ \in \mathcal{D}_1$ of B with $A^+ \sqsubseteq_s B^+$, where we write A^+ for the orientation induced on A by B^+. We then have that $(\mathcal{C}_1, \leq_1)$ is a $\leq_1$-free amalgamation class, whose Fraïssé limit we denote by M_1. We can regard $(\mathcal{C}_1, \leq_1)$ as, in some sense, a simplified version of $(\mathcal{C}_0, \leq_s)$: often the same results are true for M_1 and M_0 but easier to prove for M_1, and the simpler proof for M_1 can then provide a proof idea for the M_0 case.

The second example of a Hrushovski construction that we study is the ω-categorical M_F (first introduced in [**15**]). The details of the construction are somewhat more technical than for M_1 and M_0, and we provide an outline. Let $\mathcal{C}_{>0}$ be the class of graphs in $\mathcal{C}_0$ whose non-empty subgraphs all have positive predimension. For $A \subseteq B \in \mathcal{C}_{>0}$, write $A \leq_d B$ if $\delta(C) > \delta(A)$ for all $A \subsetneq C \subseteq B$. Then $(\mathcal{C}_{>0}, \leq_d)$ is a $\leq_d$-free amalgamation class, but its Fraïssé limit is not ω-categorical. To obtain an ω-categorical Fraïssé limit, we specify a uniform bound on the d-closure, which we do by means of a control function F. Let F be a function starting at zero whose growth rate is that of $\log(x)$ (with a few additional assumptions), and let $\mathcal{C}_F$ be the class of $A \in \mathcal{C}_{>0}$ such that for $B \subseteq A$, $\delta(B) \geq F(|B|)$. $(\mathcal{C}_F, \leq_d)$ will then have a uniformly bounded d-closure, so writing M_F for the Fraïssé limit, $\mathrm{Aut}(M_F)$ is oligomorphic (i.e. has finitely many orbits on M_F^n for $n \geq 1$), and so by the Ryll-Nardzewski theorem, M_F is ω-categorical.

The KPT correspondence

A *G-flow* is a continuous action of a Hausdorff topological group G on a (non-empty) compact Hausdorff space X. The study of G-flows is a central topic in topological dynamics. A *G-flow morphism* is a continuous, G-equivariant map. Every G-flow contains a *minimal* G-flow:

one with no proper subflows. A well-known result in topological dynamics states that every Hausdorff topological group G has a *universal minimal flow*, unique up to isomorphism: a G-flow $M(G)$ which has a surjective G-flow morphism onto any other minimal G-flow (see, for example, [**1**], Ch. 8).

We call G *extremely amenable* if every G-flow has a G-fixed point, or equivalently, if $M(G)$ is trivial (i.e. a singleton). G is *amenable* if every G-flow has a G-invariant Borel probability measure. (We have that extreme amenability implies amenability by taking the Dirac measure on the fixed point).

For Polish groups G, $M(G)$ may be very complicated - i.e. $M(G)$ may be non-metrisable. However, in the case of $M(G)$ being trivial, there is a connection to structural Ramsey theory, as seen in the KPT correspondence, first developed by Kechris-Pestov-Todorčević in [**19**] and further extended in [**24**], [**25**], [**2**] by Nguyen Van Thé, Zucker, Ben Yaacov, Melleray, Tsankov and others. We rephrase it here in terms of strong classes.

THEOREM 1.62 ([**19**], Th. 4.8; [**24**], Th. 1). *Let M be a Fraïssé limit of a strong amalgamation class $(\mathcal{K}, \leq)$. Then* $\mathrm{Aut}(M)$ *is extremely amenable iff $(\mathcal{K}, \leq)$ is Ramsey and rigid.*

Here, as usual in the context of homogeneous structures, $\mathrm{Aut}(M)$ has the pointwise convergence topology, where an open basis is given by left cosets of pointwise stabilisers of finite $A \leq M$. Rigidity means that elements of $\mathcal{K}$ have trivial automorphism groups.

We now describe some further results in the context of the KPT correspondence.

Let $(\mathcal{K}, \leq)$ be an amalgamation class of L-structures, with Fraïssé limit M and $G = \mathrm{Aut}(M)$. Let $L \subseteq L^+$ be an expanded language, and let $\mathcal{D}$ be a class of finite L-structures where $\mathcal{K}$ is the class of L-reducts of $\mathcal{D}$ and where each structure in $\mathcal{K}$ has finitely many expansions in $\mathcal{D}$ (together with a few other basic axioms - see Definition 1.63). We call $\mathcal{D}$ a *reasonable* class of expansions of $(\mathcal{K}, \leq)$ (following [**25**]).

Let

$$X(\mathcal{D}) = \{M^+ \text{ an } L^+\text{-expansion of } M : \text{for all finite } A \leq M, A^{M^+} \in \mathcal{D}\}.$$

Here, A^{M^+} denotes the L^+-structure induced on A by M^+. Basic open sets are given by fixing some A^{M^+}. We then have that $X(\mathcal{D})$ is an $\mathrm{Aut}(M)$-flow (Lemma 1.69). For an example: $\mathcal{D}_0$ is a reasonable class of expansions of $(\mathcal{C}_0, \leq_s)$, and $X(\mathcal{D}_0)$ is the flow $\mathrm{Or}(M_0)$ of 2-orientations of M_0.

For $\mathcal{D}$ reasonable over $(\mathcal{K}, \leq)$, $\mathcal{D}$ has the *expansion property* over $(\mathcal{K}, \leq)$ if for all $A \in \mathcal{K}$, there exists $A \leq B \in \mathcal{K}$ such that for all expansions $A^+, B^+ \in \mathcal{D}$, B^+ contains a $\leq$-copy of A^+.

THEOREM 1.73 ([**24**], Th. 4). *The G-flow $X(\mathcal{D})$ is minimal iff $\mathcal{D}$ has the expansion property over $(\mathcal{K}, \leq)$.*

(An analogous result is Th. 7.4 of [**19**], but this only covers the case where L is expanded by a single binary relation symbol. In this simpler case the expansion property is referred to as the *ordering property*. The paper [**24**] introduces the expansion property in full generality.)

For $J \leq \mathrm{Aut}(M)$, we say J is a *coprecompact* subgroup of $\mathrm{Aut}(M)$ if each $\mathrm{Aut}(M)$-orbit on M^n splits into finitely many J-orbits. The importance of coprecompactness was first explored in [**24**].

We then have the following extended KPT-type result, adapted for strong classes (see [**19**], Th. 10.8, [**24**], Th. 5 and [**25**], Th. 5.7):

THEOREM 1.76 (KPT, Nguyen Van Thé, Zucker). *Let $(\mathcal{K}, \leq)$ be an amalgamation class with Fraïssé limit M, and let $(\mathcal{K}^+, \leq^+)$ be a reasonable strong expansion of $(\mathcal{K}, \leq)$ which is an amalgamation class with Fraïssé limit N. Suppose $(\mathcal{K}^+, \leq^+)$ is rigid, Ramsey and has the expansion property, and suppose $\mathrm{Aut}(N)$ is a coprecompact subgroup of $\mathrm{Aut}(M)$.*

Then $X(\mathcal{K}^+)$ is the universal minimal flow for $\mathrm{Aut}(M)$, and has a comeagre orbit consisting of expansions of M isomorphic to N.

So, if G is the automorphism group of a countable homogeneous structure M and there exists a coprecompact extremely amenable subgroup of G, then we control to some extent the universal minimal flow of G.

New behaviour in the KPT context: papers of Evans, Hubička and Nešetřil

Evans, Hubička and Nešetřil (EHN) showed in [**8**] that such a coprecompact extremely amenable subgroup need not exist in the case M ω-categorical (specifically, proving this for the Hrushovski construction M_F) and also in the cases $M = M_1, M_0$:

THEOREM 2.2 ([**8**], Th. 1.2, 3.7). *The automorphism group $\mathrm{Aut}(M_F)$ has no coprecompact extremely amenable subgroup.*

THEOREM 2.3 ([**8**], Th. 3.16). *Let $M = M_1$ or M_0. Then $\mathrm{Aut}(M)$ has no coprecompact extremely amenable subgroup.*

The proof of Theorem 2.2 uses the $\mathrm{Aut}(M_F)$-flow of orientations $\mathrm{Or}(M_F)$. This flow of orientations will be a key tool throughout this thesis. Specifically, the result in [**8**] states:

PROPOSITION 2.1 ([**8**], Th. 3.7). *Let M be an infinite 2-sparse graph in which all vertices have infinite valency. Let $G = \mathrm{Aut}(M)$.*

Consider the G-flow $G \curvearrowright \mathrm{Or}(M)$. If $J \leq G$ fixes a 2-orientation of M, then J has infinitely many orbits on M^2.

(This result applies to $M = M_F$ with a few basic assumptions on the control function F to guarantee that all vertices have infinite valency - see Section 1.5.)

The proof technique for Theorem 2.3 is quite similar.

The paper [**8**] contains two further results relevant to this thesis, which demonstrate that the automorphism groups of the Hrushovski constructions M_1, M_0, M_F display dramatically different behaviour to the case where the automorphism group G has a coprecompact extremely amenable subgroup.

THEOREM ([**8**], Cor. 3.11). $\mathrm{Aut}(M_F)$ *has no coprecompact amenable subgroup.*

THEOREM ([**8**], Th. 5.2). *Let* $M = M_1, M_0, M_F$, $G = \mathrm{Aut}(M)$. *Let* Y *be a minimal subflow of the* G*-flow of* 2*-orientations* $\mathrm{Or}(M)$. *Then all* G*-orbits of* Y *are meagre in* Y.

New results

The new results in this thesis are contained in Chapters 2 to 6, and we summarise the most important results below:

Chapter 2: this contains the new result that $\mathrm{Aut}(M_0)$ does not have a coprecompact amenable subgroup, via a straightforward combination of arguments in Section 3 of [**8**].

Chapter 3: we investigate the fixed points on type spaces property (FPT), and show that M_F does not have any ω-categorical expansion whose automorphism group has FPT.

Chapter 4: we investigate the $\mathrm{Aut}(M_1)$-flow $\mathcal{LO}(M_1)$ of linear orders on M_1, and show that all orbits on minimal subflows of $\mathcal{LO}(M_1)$ are meagre.

Chapter 5: we attempt to carry out the same analysis for the $\mathrm{Aut}(M_0)$-flow $\mathcal{LO}(M_0)$ as we did for $\mathcal{LO}(M_1)$ in the previous chapter, but without complete success. We obtain some results on stabilisers of linear orders, and a result that clarifies the difficulty of the M_0 case.

Chapter 6: we find the universal minimal flow of $\mathrm{Aut}(M_{00})$, the automorphism group of the predimension zero part of M_0.

We briefly describe these new results in the rest of this introduction.

Fixed points on type spaces

In Chapter 3, we investigate a weakening of extreme amenability, the *fixed points on type spaces* (FPT) property.

Let M be a first-order structure, and let $G = \mathrm{Aut}(M)$ with the pointwise convergence topology. Then G acts continuously on the Stone

spaces $S_n(M)$ of n-types with parameters in M, via the action

$$g \cdot p(\overline{x}) = \{\phi(g\overline{m}, \overline{x}) : \phi(\overline{m}, \overline{x}) \in p(\overline{x})\}.$$

This gives a natural source of G-flows.

G has the *fixed points on type spaces* property (FPT) if every subflow of $S_n(M)$ ($n \geq 1$) has a fixed point.

Investigating the fixed points on type spaces property was suggested by David Evans, the PhD supervisor of the author. The motivation arose both from the simple fact that type spaces give a natural flow, and also from some unpublished examples (personal communication, Evans) where Ramsey expansions of certain classes had been found by noting that the automorphism groups of the Fraïssé limits of these Ramsey expansions would have to fix points on type spaces.

An additional motivation was the paper [**8**] of Evans, Hubička and Nešetřil. David Evans raised the following question:

QUESTION. M_F does not have an ω-categorical expansion whose automorphism group is extremely amenable (Th. 2.2). Does M_F have an ω-categorical expansion whose automorphism group has the fixed points on type spaces property?

In Chapter 3 we show directly, as an introduction, that the automorphism group of the random graph has FPT, without using strong Ramsey results. This introductory example demonstrates interesting behaviour (FPT for the random graph is equivalent to the pigeonhole property of the random graph: any 2-colouring of the random graph must have a monochromatic copy of the random graph itself).

We then answer the question of Evans in the negative:

THEOREM 3.6. *M_F does not have any ω-categorical expansion N with* $\mathrm{Aut}(N)$ *having FPT.*

PROOF IDEA. Encode each orientation of M_F as a 1-type. We then define an $\mathrm{Aut}(M_F)$-map $\gamma : S_1(M_F) \to 2^{M_F{}^2}$ which sends our encoded 1-types back to the original orientations. Any $H \leq \mathrm{Aut}(M_F)$ with FPT fixes a point in the subflow of encoded types, so fixes an orientation. We then use Proposition 2.1 to see that H has infinitely many orbits on M_F^2. □

Linear orders on sparse graphs

This topic forms the greater part of this thesis. Th. 5.2 from [**8**] shows that $\mathrm{Aut}(M_0), \mathrm{Aut}(M_F)$ have metrisable minimal flows all of whose orbits are meagre. Tsankov asked the following question ([**8**], concluding remarks):

QUESTION (Tsankov). Do $\mathrm{Aut}(M_0)$ or $\mathrm{Aut}(M_F)$ have a (non-trivial) metrisable minimal flow with a comeagre orbit?

David Evans suggested that the author investigate the $\mathrm{Aut}(M)$-flow $\mathcal{LO}(M)$ of linear orders on $M = M_1, M_0, M_F$. We obtain the following result for M_1:

THEOREM 4.18. *Let $Y \subseteq \mathcal{LO}(M_1)$ be a minimal subflow. Then all $\mathrm{Aut}(M_1)$-orbits on Y are meagre.*

The proof involves using the *admissible orders* from Section 3.1 of [**9**], a paper which accompanies [**8**] and proves new Ramsey theorems for structures with relations and *set-valued* functions. This framework of languages with set-valued functions is used to encode the strong closures associated with Hrushovski constructions.

For $A \in \mathcal{D}_1$, we may define the level $\mathrm{l}(x)$ of a vertex $x \in A$ as the maximal length of an out-path from x. A linear order $\prec$ on A is *admissible* if:

- for $x, y \in A$ with $\mathrm{l}(x) < \mathrm{l}(y)$, $x \prec y$;
- for $x, y \in A$ of the same level, $x \prec y$ if the descending chain of successors of x is lexicographically before the descending chain of successors of y.

The class $(\mathcal{E}_1, \sqsubseteq_s)$ of $(A, \prec)$, with $A \in \mathcal{D}_1$ and $\prec$ admissible, is a Ramsey class with free amalgamation (and indeed also has the expansion property over $(\mathcal{D}_1, \sqsubseteq_s)$), as we show in Proposition 4.10. This is a particular case of the more general Theorem 1.4 of [**9**], though in this thesis we will keep our presentation self-contained.

Let the Fraïssé limit of $(\mathcal{E}_1, \sqsubseteq_s)$ be (N_1, α), where α is the linear order of the Fraïssé limit and N_1 is the oriented graph, and let $H = \mathrm{Aut}(N_1, \alpha)$. H is extremely amenable, so fixes some linear order β on Y. We may then show the failure of the weak amalgamation property (see Section 1.11) for $\mathrm{Age}(M_1, \beta)$, which shows that all Y-orbits are meagre by an adapted result of Kechris & Rosendal (Theorem 3.4 of [**20**], Lemma 1.78 here). We do this by using H-automorphisms to find out information about β (β agrees with α or its reverse on sets of α-indiscernible vertices), and then, assuming the weak amalgamation property and seeking a contradiction, use β to force certain incompatible orientations.

We may define admissible orders in a similar (but more complicated) way for M_0, M_F (indeed, [**9**] defines admissible orders in a general fashion so that they may also be defined on e.g. Steiner systems and bowtie-free graphs). The author has not yet extended Theorem 4.18 to M_0 and M_F due to unanticipated technical difficulties, but has some partial results.

For M_0, we will show in Chapter 5, with an analogous setup to the previous chapter, that $\mathrm{Aut}(M_0, \alpha) = \mathrm{Aut}(M_0, \beta)$. We will also show in Chapter 6 that:

THEOREM 6.13. $\mathrm{Aut}(M_0)$ *has a non-trivial metrisable minimal flow with comeagre orbit.*

However, this particular flow is not faithful, so does not answer the question of Tsankov in a satisfying way. The flow results from the "dimension 0" part of M_0. Let $\mathcal{C}_{00}$ be the class of graphs A with $A \in \mathcal{C}_0$ and $\delta(A) = 0$, and let $M_{00} \subseteq M_0$ be the Fraïssé limit of $(\mathcal{C}_{00}, \leq_s)$. Then $G_{00} = \mathrm{Aut}(M_{00})$ has a coprecompact extremely amenable subgroup, given by the automorphism group of the class of admissibly ordered orientations of $\mathcal{C}_{00}$, so we may use Theorem 1.76 to produce the universal minimal flow of G_{00} with a comeagre orbit, and from this we may produce a G_0-flow. (We have coprecompactness here because for $A \in \mathcal{C}_{00}$, any orientation of A must have the same strongly connected components and orientation of edges between strongly connected components.)

These partial results represent some progress towards extending Theorem 4.18 to M_0 and M_F, but the question is still open for these cases.

Chapter 1

Background

In this chapter, we introduce the background material for this thesis. A reader familiar with Hrushovski constructions and the KPT correspondence should hopefully have seen most of the results in this chapter before - though we reformulate the KPT correspondence for strong classes, and the Ramsey result of Section 1.7 will possibly be unfamiliar.

We assume that the reader is familiar with the classical Fraïssé theory, the pointwise convergence topology on automorphism groups of first-order structures, and the Ryll-Nardzewski theorem. (The background for these three topics can be found in Ch. 7 of [**14**] and Sections 1-2 of [**5**].)

None of the material in this chapter is new. The presentation is strongly influenced by [**8**], [**7**] and [**5**], which are key references for this background material.

All first-order languages considered in this thesis will be countable and relational.

1.1 Graphs and oriented graphs: notation and setup

A *graph* $(A, \mathrm{E}(A))$ consists of a set A, the *vertex set*, and a set $\mathrm{E}(A) \subseteq A^{(2)}$, the *edge set*. (Here, $A^{(2)}$ denotes the set of unordered pairs of distinct elements of A.) We will usually just write A to denote the graph $(A, \mathrm{E}(A))$ when this is clear from context. We will sometimes write $\mathrm{V}(A)$ for the vertex set. By the above definition, here we only work with *simple* graphs: graphs having no loops on a single vertex or multiple edges between two vertices.

Definition 1.1. Let $(A, \mathrm{E}(A))$ be a graph. A set $\rho_A \subseteq A^2$ is an *orientation* of $(A, \mathrm{E}(A))$ if:

- for $xy \in \mathrm{E}(A)$, exactly one of $(x, y), (y, x)$ is in ρ_A;
- for $(x, y) \in \rho_A$, $xy \in \mathrm{E}(A)$.

Note that this implies that ρ_A contains no directed loops or directed 2-cycles. We will refer to $(A, \mathrm{E}(A), \rho_A)$ as an *oriented graph*.

(We use ρ as a mnemonic for o**r**ientation.)

DEFINITION 1.2. Let $(A, \mathrm{E}(A), \rho_A)$ be an oriented graph. (See Figure 1.1 for an example.)

If $(x, y) \in \rho_A$, we refer to (x, y) as an *out-edge* of x and an *in-edge* of y. We call y an *out-vertex* or *successor* of x, and x an *in-vertex* or *predecessor* of y.

The *out-neighbourhood* $\mathrm{N}_+(x)$ of x consists of the out-vertices of x. The *in-neighbourhood* $\mathrm{N}_-(x)$ of x consists of the in-vertices of x. The *out-degree* $\mathrm{d}_+(x)$ of x is defined to be $\mathrm{d}_+(x) = |\mathrm{N}_+(x)|$, and the *in-degree* $\mathrm{d}_-(x)$ of x is defined similarly.

If $x_1, \cdots, x_n$ is a path of the graph A and $(x_i, x_{i+1}) \in \rho_A$ for $i < n$, we will say that $x_1, \cdots, x_n$ is an *out-path* of ρ_A. We define in-paths in the corresponding fashion.

For $B \subseteq A$, we write $B \sqsubseteq_s A$ to mean that B is *successor-closed* in A: for $b \in B$, if $(b, a) \in \rho_A$ then $a \in B$. Note that intersections of successor-closed subsets are also successor-closed.

For $B \subseteq A$, $\mathrm{scl}(B)$, the *successor-closure* of B, is defined to be the smallest successor-closed subset of A containing B. Equivalently, $\mathrm{scl}(B)$ consists of the vertices of A reachable by an out-path starting in B.

For $x, y \in A$, we write $x \sim y$ if $\mathrm{scl}(x) = \mathrm{scl}(y)$. Then $\sim$ is an equivalence relation, and we call the equivalence classes the *strongly connected components* (sccs) of A. We write $\mathrm{scc}(x)$ for the strongly connected component of x. (As a reminder to the reader, we note the difference between $\mathrm{scl}(x)$ and $\mathrm{scc}(x)$). We have that $x \sim y$ iff there exist out-paths from x to y and from y to x.

Let S be a scc of A. We define the *out-neighbourhood* $\mathrm{N}_+(S)$ of S to be the set

$\mathrm{N}_+(S) = \{v \in A : v \notin S \text{ and there exists } x \in S \text{ with } (x, v) \in \rho_A\}$.

Any out-edge from a vertex of S to the out-neighbourhood of S will be called an *exiting out-edge from* S (note that this excludes out-edges whose vertices both lie in S).

DEFINITION 1.3. Let $(A, \mathrm{E}(A), \rho_A)$ be a *finite* oriented graph.

For $x \in A$, we inductively define the *level* $\mathrm{l}_A(x)$ of x as follows. If $\mathrm{scl}(x) = \mathrm{scc}(x)$, then $\mathrm{l}_A(x) = 0$. Otherwise, let y be a vertex of $\mathrm{scl}(x) - \mathrm{scc}(x)$ of maximum level, and then $\mathrm{l}_A(x) = \mathrm{l}_A(y) + 1$. For $B \subseteq A$, we define the level $\mathrm{l}_A(B)$ of B to be the maximum of the levels of its vertices.

We write $\mathrm{L}_n(A)$ $(n \geq 0)$ for the set of vertices of A of level n. We define $A^{\uparrow n}$ (read 'A up to n') to be the oriented subgraph of A whose vertex set consists of the vertices of A of level $\leq n$, and $A^{\downarrow n}$ to be the oriented subgraph of A whose vertex set is all the vertices of A of level $\geq n$.

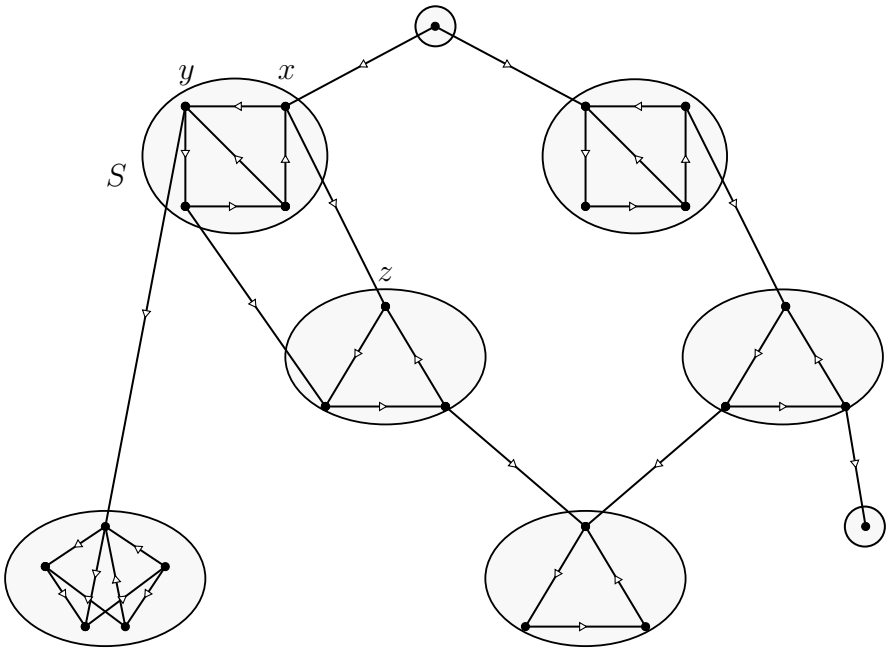


FIGURE 1.1. An example of a 2-oriented graph. Sccs are in grey. The out-neighbourhood of x is $\{y, z\}$. (x, z) is an exiting out-edge from the scc S, and z lies in $\mathrm{N}_+(S)$, but y does not. x has level 2, and z has level 1.

REMARK 1.4. The definitions of successor-closure, scc and levels of vertices have been taken from [**6**], an early, unpublished version of [**8**], but are well-known to graph theorists. These definitions are a specific case of the more general notions of closure, closure-components and levels of vertices found in Section 3.1 of [**9**].

We will work with graphs and oriented graphs in first-order languages as follows. The *language of graphs* $L_{\mathrm{graph}} = \{E\}$ consists of a single binary relation E. A L_{graph}-structure (A, E_A) is then a graph if $E_A \subseteq A^2$, the interpretation of E, is symmetric and irreflexive. (It is clear how to switch between $\mathrm{E}(A)$ and E_A. We will be flexible and switch between the standard graph theory notation and the first order formalism without comment.)

We expand L_{graph} to the language of oriented graphs $L_{\mathrm{or}} = \{E, \rho\}$, where ρ is a binary relation. A L_{or}-structure (A, E_A, ρ_A) is an oriented graph if ρ_A is an orientation of the graph (A, E_A).

When we refer to a subgraph of a graph, or an oriented subgraph of an oriented graph, we mean an L_{graph}- or L_{or}-substructure respectively. This is standard in model theory, but for graph theorists these substructures might be referred to as *induced* subgraphs.

(We use the full notation for structures in this section for clarity, but henceforth we will usually denote graphs (A, E_A) by A, and oriented graphs (A, E_A, ρ_A) by A or (A, ρ_A).)

1.2 Sparse graphs: $\mathcal{C}_0$ and $\mathcal{C}_1$

We will now define the classes of sparse graphs that we will be concerned with in this thesis.

Definition 1.5. Take $k \in \mathbb{N}_+$. A graph A is *k-sparse* if $\forall\, B \subseteq_{\text{fin.}} A$, $|\,\mathrm{E}(B)| \leq k|B|$.

Definition 1.6. Let (A, ρ_A) be an oriented graph. Take $k \in \mathbb{N}_+$. We call ρ_A a *k-orientation* if for $x \in A$, $|\,\mathrm{N}_+(x)| \leq k$. We refer to (A, ρ_A) as a *k-oriented graph*, and if a graph A has a k-orientation, we say it is *k-orientable*.

The following proposition is well known in graph theory ([**22**]), and will be a key tool here.

Proposition 1.7 ([**8**], Th. 3.4). *A graph A is k-orientable iff it is k-sparse.*

Proof. $\Rightarrow$: easy. $\Leftarrow$: We will first prove the statement for finite A. We wish to produce a k-orientation of A, and so to do this, we must direct each edge. We will use Hall's Marriage Theorem ([**3**], III.3), which for the convenience of the reader we briefly restate: for a finite bipartite graph with left set X and right set Y, then there is an X-saturated matching iff $|W| \leq |\,\mathrm{N}(W)|$ for $W \subseteq X$.

Form a bipartite graph B with left set $\mathrm{E}(A)$ and right set $A \times [k]$, and place an edge between $e \in \mathrm{E}(A)$ and $(x, i) \in A \times [k]$ if $x \in e$. If there is a left-saturated matching, then if e is matched to (x, i), we orient e so that x is the start-vertex of e, and this gives a k-orientation of A.

To see that a left-saturated matching exists, take $W \subseteq \mathrm{E}(A)$. Let V be the set of vertices of the edges which lie in W. Then $|\,\mathrm{N}_B(W)| = k|V|$, and as A is k-sparse, we have that $k|V| \geq |\,\mathrm{E}_A(V)|$, where $\mathrm{E}_A(V)$ is the set of edges in A whose vertices lie in V. As $|\,\mathrm{E}_A(V)| \geq |W|$, by Hall's Marriage Theorem there exists a left-saturated matching of the bipartite graph B. □

For presentational simplicity, we will usually work with $k = 2$. Our results generalise straightforwardly for $k > 2$.

Note: in this thesis, we may occasionally say "oriented graph" to in fact mean "2-oriented graph", as most of the oriented graphs we are concerned with are 2-oriented. We will try to avoid this in general, but when this does occur the meaning will be clear from context.

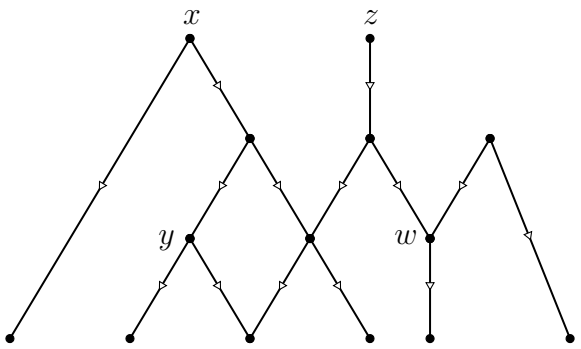


FIGURE 1.2. An example of an element A of $\mathcal{D}_1$: a 2-oriented graph with no directed cycles. x has level 3, and y has level 1. The roots of A of multiplicity 1 are z, w. The roots of A of multiplicity 2 are the 6 vertices on level 0. We have $\delta(A) = 14$.

DEFINITION 1.8. We let $\mathcal{C}_0$ be the class of finite 2-sparse graphs. We let $\mathcal{D}_0$ be the class of finite 2-oriented graphs.

By Proposition 1.7, $\mathcal{C}_0$ is the class of graph reducts of $\mathcal{D}_0$.
We now introduce a "simplified" version of $\mathcal{D}_0$ ([**7**], Section 3.4 and [**10**]), for which it will be usually easier to prove results.

DEFINITION 1.9. We define $\mathcal{D}_1$ to be the class of finite 2-oriented graphs with no directed cycles. (See Figure 1.2.) By a slight abuse of terminology, we will refer to a 2-oriented graph with no directed cycles as an *acyclic 2-oriented graph*. We define $\mathcal{C}_1$ to be the class of graph reducts of $\mathcal{D}_1$.

We may also define $\mathcal{C}_1$ purely in terms of graphs, as per the lemma below.

LEMMA 1.10 ([**10**], Lem. 1.3). *A finite graph A has a k-orientation with no directed cycles iff every non-empty subgraph B has a vertex of degree $\leq k$ in B.*

PROOF. $\Rightarrow$: Take $\varnothing \neq B \subseteq A$ a subgraph of A where A has an acyclic k-orientation. Consider the induced orientation on B. As this orientation is acyclic, B contains a vertex v which has no in-edges in B, and therefore the degree of v in B is equal to its out-degree in B, which is $\leq k$.
$\Leftarrow$: We prove the claim by induction on $|A|$. For $|A| = 1$, the claim is trivial. For the inductive step, take a vertex $a \in A$ of degree $\leq k$. By the induction assumption, we may give $A - \{a\}$ an acyclic k-orientation. Then orient the edges of a outwards from a to produce an acyclic k-orientation of A. $\square$

PROPOSITION 1.11. *$\mathcal{C}_1$ consists of the finite graphs A where every non-empty subgraph $B \subseteq A$ has a vertex of degree ≤ 2.*

PROOF. Clear by Lemma 1.10. □

1.2.1 Graph predimension

One way to characterise k-sparsity is in terms of *graph predimension*.

DEFINITION 1.12. Let A be a finite graph. We define the *predimension* $\delta(A)$ of A to be $\delta(A) = 2|A| - |\,\mathrm{E}(A)|$.
For $B \subseteq A$, we define the *relative predimension of A over B* to be $\delta(A/B) = \delta(A) - \delta(B)$.

We can therefore characterise $\mathcal{C}_0$ as the class of finite graphs A such that all subgraphs of A have non-negative predimension. (This forms part of the 'ab initio' Hrushovski construction M_0 described in [**16**]. An accessible treatment is in Section 3.2 of [**5**].)

DEFINITION 1.13. Let $A \in \mathcal{D}_0$. Take $a \in A$. a is a *root* of A if $\mathrm{d}_+(a) < 2$.
For $a \in A$ a root of A, we define the *multiplicity* m_a of a to be $m_a = 2 - \mathrm{d}_+(a)$.

LEMMA 1.14. *Let $A \in \mathcal{D}_0$. Then we have that $\delta(A)$ is the sum of the multiplicities of the roots of A.*

The proof is straightforward. (This is from Section 4 of [**8**].)

1.3 Strong classes and Fraïssé limits

1.3.1 Strong classes

The classes of graphs and oriented graphs detailed in the previous two sections will each come with certain particular distinguished embeddings between structures in the class.
(Throughout this section, L will be a countable relational first-order language.)

DEFINITION 1.15. Let $\mathcal{K}$ be a class of finite L-structures closed under isomorphisms. Let $\mathcal{S} \subseteq \mathrm{Emb}(\mathcal{K})$ be a class of embeddings between structures in $\mathcal{K}$ satisfying the following:

(S1) $\mathcal{S}$ contains all isomorphisms;
(S2) $\mathcal{S}$ is closed under composition;
(S3) if $f : A \to C$ is in $\mathcal{S}$ and $f(A) \subseteq B \subseteq C$ with $B \in \mathcal{K}$, then $f : A \to B$ is in $\mathcal{S}$.

Then we call $(\mathcal{K}, \mathcal{S})$ a *strong class*, and call the elements of $\mathcal{S}$ *strong embeddings*.

(This is originally due to Hrushovski: see [**15**] and [**16**]. An accessible exposition of strong classes is in Section 3 of [**5**].)

If $A, B \in \mathcal{K}$, $A \subseteq B$ and the inclusion map $\iota : A \hookrightarrow B$ is in $\mathcal{S}$, then we write $A \leq B$ and say A is a *strong substructure* of B. We have that:

(L1) $\leq$ is reflexive;
(L2) $\leq$ is transitive;
(L3) if $A \leq C$ and $A \subseteq B \subseteq C$ with $B \in \mathcal{K}$, then $A \leq B$.

We will often write $(\mathcal{K}, \leq)$ instead of $(\mathcal{K}, \mathcal{S})$, and we will use different symbols resembling $\leq$ (e.g. $\leq_1, \leq_s, \sqsubseteq_s$) to indicate different classes of embeddings.

Remark 1.16. If we start with a distinguished class of strong substructures of elements of $\mathcal{K}$, where we write $A \leq B$ to mean that A is a strong substructure of B, we could define a distinguished class of embeddings $\mathcal{S}$ by stating that $f : A \to B$ is in $\mathcal{S}$ if $f(A) \leq B$. However, if $\leq$ satisfies (L1), (L2), (L3), this does not necessarily formally imply that $\mathcal{S}$ satisfies (S1), (S2), (S3), and embeddings in $\mathcal{S}$ do not necessarily preserve $\leq$.

However, in any examples of strong classes in this thesis, this somewhat pedantic technical distinction will not appear, and defining $\mathcal{S}$ will be equivalent to defining $\leq$ because of the particular details of the example.

If $(\mathcal{K}, \leq)$ is a strong class (i.e. $\mathcal{S}$ satisfies (S1), (S2), (S3)), then we have that for $f : A \to B$ in $\mathcal{S}$, if $X \leq A$, then $f(X) \leq B$.

1.3.2 Strong classes: infinite structures

Suppose $(\mathcal{K}, \leq)$ is a strong class. Let $A_1 \leq A_2 \leq \cdots$ be an increasing $\leq$-chain of structures in $\mathcal{K}$, and let $M = \bigcup_{i \geq 1} A_i$. Let $A \subseteq_{\text{fin.}} M$. Then we write $A \leq M$ to mean that there is some A_i $(i \geq 1)$ with $A \leq A_i$. Say M is also the union of the elements of the increasing $\leq$-chain $B_1 \leq B_2 \leq \cdots$ of $\mathcal{K}$-structures. Take some A_i $(i \geq 1)$. Then $A_i \subseteq B_j$ for some $j \geq 1$, and $B_j \subseteq A_k$ for some $k \geq i$. As $A_i \leq A_k$, by (S3) $A_i \leq B_j$. Thus we see that when we write $A \leq M$, this does not depend on any particular $\leq$-chain.

Take $g \in \mathrm{Aut}(M)$. Take some $A_i \leq A_j$ $(i < j)$. Then $g|_{A_j} : A_j \to gA_j$ is an isomorphism, so $g|_{A_j} \in \mathcal{S}$, and so $gA_i \leq gA_j$. Thus M is also the union of the increasing $\leq$-chain $gA_1 \leq gA_2 \leq \cdots$. So if $A \leq M$, then $gA \leq M$: that is, all $g \in \mathrm{Aut}(M)$ preserve $\leq$.

1.3.3 Fraïssé theory for strong classes

We now develop an analogue of the classical Fraïssé theory for strong classes. As the details are entirely analogous to the classical case, we omit the proofs and state the relevant material as a series of definitions and lemmas. (For the classical Fraïssé theory, originally developed in [**13**], see [**14**], and for a more complete treatment of Fraïssé theory for strong classes, see Section 3 of [**5**].)

DEFINITION 1.17. Let $(\mathcal{K}, \leq)$ be a strong class of L-structures.

- $(\mathcal{K}, \leq)$ has the *joint embedding property* (JEP) if for A_1, A_2 in $\mathcal{K}$, there is $B \in \mathcal{K}$ with $\leq$-embeddings $f_1 : A_1 \to B, f_2 : A_2 \to B$.
- $(\mathcal{K}, \leq)$ has the *amalgamation property* (AP) if, for any $\leq$-embeddings $f_1 : A \to C_1, f_2 : A \to C_2$ of $A \in \mathcal{K}$, there exists $D \in \mathcal{K}$ with $\leq$-embeddings $g_1 : C_1 \to D, g_2 : C_2 \to D$ such that $g_1 f_1 = g_2 f_2$.
- For $A, C_1, C_2 \in \mathcal{K}$ with $A \leq C_1, C_2$, the *free amalgam* F of C_1, C_2 over A is the L-structure F whose domain is the disjoint union of C_1, C_2 over A and whose relations R_F are exactly the unions $R_{C_1} \cup R_{C_2}$ of the relations R_{C_1}, R_{C_2} on C_1, C_2 (for R a relation symbol in L). If for all L-structures $A \leq C_1, C_2$ in $\mathcal{K}$ we have that the free amalgam F of C_1, C_2 over A is in $\mathcal{K}$ with $C_1, C_2 \leq F$, then we say that $(\mathcal{K}, \leq)$ is a *free amalgamation class*.

We will usually not mention the distinguished class of embeddings in our terminology, as it will be clear from context and the fact that we are working with strong classes. For instance, we say that $(\mathcal{K}, \leq)$ has the amalgamation property - even though perhaps more strictly we should say that $(\mathcal{K}, \leq)$ has the $\leq$-amalgamation property.

In the following definitions and lemmas, let $(\mathcal{K}, \leq)$ be a strong class, and let M be the union of an increasing $\leq$-chain $A_1 \leq A_2 \leq \cdots$ in $(\mathcal{K}, \leq)$.

DEFINITION 1.18. The $\leq$-*age* of M, written $\mathrm{Age}_{\leq}(M)$, is the class of $A \in \mathcal{K}$ such that there is a $\leq$-embedding $A \to M$.

$(\mathrm{Age}_{\leq}(M), \leq)$ is a hereditary strong subclass of $(\mathcal{K}, \leq)$, and has the joint embedding property.

DEFINITION 1.19. M has the $\leq$-*extension property* if for all $A, B \in \mathrm{Age}_{\leq}(M)$ and $\leq$-embeddings $f : A \to M, g : A \to B$, there exists a $\leq$-embedding $h : B \to M$ with $hg = f$.

M is $\leq$-*homogeneous* if each isomorphism $f : A \to A'$ between strong substructures A, A' of M extends to an automorphism of M.

(Again, to avoid presentational clutter, we will often omit the $\leq$- prefix and just say that M has the extension property or is homogeneous, when it is clear from context that M is the union of an increasing $\leq$-chain $A_1 \leq A_2 \leq \cdots$ in a strong class $(\mathcal{K}, \leq)$.)

LEMMA 1.20. *Let M' also be a union of an increasing $\leq$-chain in $\mathcal{K}$. Suppose M, M' have the same $\leq$-age and both have the $\leq$-extension property. Then M, M' are isomorphic.*

LEMMA 1.21. *M is $\leq$-homogeneous iff M has the $\leq$-extension property.*

LEMMA 1.22. *Suppose M is $\leq$-homogeneous. Then $(\mathrm{Age}_{\leq}(M), \leq)$ has the amalgamation property.*

DEFINITION 1.23. Let $(\mathcal{K}, \leq)$ be a strong class. We say that $(\mathcal{K}, \leq)$ is an *amalgamation class* or *Fraïssé class* if $(\mathcal{K} \leq)$ contains countably many isomorphism types, and has the joint embedding and amalgamation properties.

THEOREM 1.24 (Fraïssé-Hrushovski). *Let $(\mathcal{K}, \leq)$ be an amalgamation class. Then there is a structure M which is a union of an increasing $\leq$-chain, unique up to isomorphism, such that M is $\leq$-homogeneous and $\mathrm{Age}_{\leq}(M) = \mathcal{K}$.*
We call this structure the Fraïssé limit *or* generic structure *of $\mathcal{K}$.*

1.3.4 Strong expansions

We will often be concerned with the situation where we have a strong class of L-structures together with a strong class of L^+-structures (with a potentially different notion of distinguished embedding), where $L^+ \supseteq L$ is a relational language expanded from L.

DEFINITION 1.25 ([**8**], Def. 2.9). Let $(\mathcal{K}, \leq)$ be a strong class of L-structures, and let $(\mathcal{K}^+, \leq^+)$ be a strong class of L^+-structures.
$(\mathcal{K}^+, \leq^+)$ is a *strong expansion* of $(\mathcal{K}, \leq)$ if:

(1) $\mathcal{K}$ is the class of L-reducts of $\mathcal{K}^+$;
(2) for $\leq^+$-strong $f : A^+ \to B^+$, $f : A^+|_L \to B^+|_L$ is $\leq$-strong;
(3) for $\leq$-strong $f : A \to B$ and $A^+ \in \mathcal{K}^+$ an expansion of A, there exists an expansion $B^+ \in \mathcal{K}^+$ of B such that $f : A^+ \to B^+$ is $\leq^+$-strong.

LEMMA 1.26. *Let $(\mathcal{K}^+, \leq^+)$ be a strong expansion of the strong class $(\mathcal{K}, \leq)$. If $(\mathcal{K}^+, \leq^+)$ is a (free) amalgamation class, then so is $(\mathcal{K}, \leq)$. If M^+ is the Fraïssé limit of $(\mathcal{K}^+, \leq^+)$, then $M = M^+|_L$ is the Fraïssé limit of $(\mathcal{K}, \leq)$, and $\mathrm{Aut}(M^+)$ is a closed subgroup of $\mathrm{Aut}(M)$.*

The proof is straightforward. (To show that $\mathrm{Aut}(M^+)$ is closed in $\mathrm{Aut}(M)$, show that the complement is open by witnessing failure to be in $\mathrm{Aut}(M^+)$ on a finite set.)

DEFINITION 1.27. A particular case of Definition 1.25 is the *order expansion* of a strong class.

Let $(\mathcal{K}, \leq)$ be a strong class of L-structures, and let L^+ be an expansion of L by a binary relation symbol.

Let $\mathcal{K}^\prec$ be the class of L^+-structures (A, γ_A), where $A \in \mathcal{K}$ and γ_A is a linear order on A (here interpreting the additional binary relation symbol of L^+).

For $(A, \gamma_A), (B, \gamma_B) \in \mathcal{K}^\prec$, write $(A, \gamma_A) \leq (B, \gamma_B)$ if $A \leq B$ and $\gamma_A = \gamma_B|_A$. We then have that $(\mathcal{K}^\prec, \leq)$ is a strong class and a strong expansion of $(\mathcal{K}, \leq)$. We call $(\mathcal{K}^\prec, \leq)$ the *order expansion* of the strong class $(\mathcal{K}, \leq)$.

Note: Throughout this thesis, we will tend to use letters at the start of the Greek alphabet (α, β, γ) for linear orders. This will avoid confusion with the notation $A \leq B$ for strong substructure, and also enables us to conveniently handle the situation where there are several different linear orders on the same structure.

1.4 Sparse graphs: $\leq_s$, $\sqsubseteq_s$ and $\leq_1$

We will now describe the distinguished notions of embedding involved in defining particular strong classes for $\mathcal{C}_0$ and $\mathcal{D}_0$, and for $\mathcal{C}_1$ and $\mathcal{D}_1$.

1.4.1 Self-sufficiency: $\leq_s$

DEFINITION 1.28. Let $A, B \in \mathcal{C}_0, A \subseteq B$. We say that A is *self-sufficient* in B, written $A \leq_s B$, if for $A \subseteq C \subseteq B$, $\delta(C) \geq \delta(A)$.

(This is a part of the 'ab initio' Hrushovski construction from [**16**], as are the other lemmas in this subsection. The presentation here is from Section 3.2 of [**5**].)

LEMMA 1.29 (Submodularity). *Take $A \in \mathcal{C}_0$, $B, C \subseteq A$. Then we have that $\delta(B \cup C) \leq \delta(B) + \delta(C) - \delta(B \cap C)$. We have equality iff $\mathrm{E}(B \cup C) = \mathrm{E}(B) \cup \mathrm{E}(C)$, i.e. B, C are freely amalgamated over $B \cap C$ in A.*

PROOF. Straightforward. □

LEMMA 1.30. *Let $B \in \mathcal{C}_0$.*

(1) $A \leq_s B$, $X \subseteq B \Rightarrow A \cap X \leq_s X$.
(2) $A \leq_s C \leq_s B \Rightarrow A \leq_s B$.
(3) $A_1, A_2 \leq_s B \Rightarrow A_1 \cap A_2 \leq_s B$.

(Recall that for $B \in \mathcal{C}_0$, if $A \subseteq B$ then $A \in \mathcal{C}_0$. Therefore we only need to prove self-sufficiency in the above.)

PROOF.

(1) Take $A \cap X \subsetneq Y \subseteq X$. Note that $A \cap Y = A \cap X$. By submodularity,

$$\begin{aligned}\delta(A \cup Y) &\leq \delta(A) + \delta(Y) - \delta(A \cap Y) \\ &= \delta(A) + \delta(Y) - \delta(A \cap X),\end{aligned}$$

so $\delta(Y) - \delta(A \cap X) \geq \delta(A \cup Y) - \delta(A) \geq 0$.

(2) Take $A \subsetneq X \subseteq B$. By (1), $C \cap X \leq_s X$. Also $A \subseteq C \cap X \subseteq C$. So $\delta(A) \leq \delta(C \cap X) \leq \delta(X)$.

(3) By (1), $A_1 \cap A_2 \leq A_1$. Then use (2).

□

The previous lemma shows us that $(\mathcal{C}_0, \leq_s)$ satisfies (S1), (S2), (S3) - that is:

LEMMA 1.31. *$(\mathcal{C}_0, \leq_s)$ is a strong class.*

For $B \in \mathcal{C}_0$, by (3) we see that for $A \subseteq B$ we have that

$$\bigcap \{A' : A \subseteq A' \leq_s B\} \leq_s B,$$

so we can define the *closure* of A in B as this intersection, written $\mathrm{cl}^s_B(A)$. This is a closure operation.[1].

LEMMA 1.32. *Let $B \in \mathcal{C}_0$ and let $A \subseteq B$. Then $\delta(A) \geq \delta(\mathrm{cl}^s_B(A))$.*

PROOF. Amongst all $A \subseteq X \subseteq B$, consider those for which $\delta(X)$ is smallest, and then out of these choose a C of greatest size. By the first stage of selection, we have $\delta(C) \leq \delta(A)$, and by the second stage, if $C \subseteq D \subseteq B$ then $\delta(C) < \delta(D)$, so $C \leq_s B$. So $\mathrm{cl}^s_B(A) \subseteq C \subseteq B$, and as $\mathrm{cl}^s_B(A) \leq_s B$, $\delta(\mathrm{cl}^s_B(A)) \leq \delta(C)$. □

1.4.2 Self-sufficiency and successor-closure: $\leq_s$ and $\sqsubseteq_s$

It is straightforward to see that Lemma 1.30 holds for $\mathcal{D}_0$ with $\leq_s$ replaced by $\sqsubseteq_s$, and thus we have:

LEMMA 1.33. *$(\mathcal{D}_0, \sqsubseteq_s)$ is a strong class.*

We have the following link between the strong classes $(\mathcal{C}_0, \leq_s)$ and $(\mathcal{D}_0, \sqsubseteq_s)$ ([**11**], Lemma 1.5):

PROPOSITION 1.34. *Take $A, B \in \mathcal{C}_0$, $A \subseteq B$. Then $A \leq_s B$ iff there exists a 2-orientation B^+ of B for which the induced 2-orientation A^+ on A is successor-closed.*

[1]Let X be a set, and let $\mathrm{cl} : \mathcal{P}(X) \to \mathcal{P}(X)$ be a function on the power set of X. cl is a *closure operator* if it is (1) *extensive*: $A \subseteq \mathrm{cl}(A)$, (2) *increasing*: $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ and (3) *idempotent*: $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$.

PROOF. $\Rightarrow$: This is an extension of the proof technique of Proposition 1.7. As $A \in \mathcal{C}_0$, by Proposition 1.7 there is a 2-orientation A^+ of A. Define a bipartite graph (X, Y) as follows. Let $X = \mathrm{E}(B) - \mathrm{E}(A)$ (so X consists of edges with both vertices not in A together with edges with exactly one vertex in A). Let $Y = (B - A) \times [2]$. Place an edge between $e \in X$ and $(b, i) \in Y$ if $b \in e$. If there is an X-saturated matching of (X, Y), we can produce a 2-orientation B^+ of B in which $A^+ \sqsubseteq_s B^+$: take A^+, and to orient the remaining edges of B, if $e \in X$ matches with $(b, i) \in Y$, direct e so that b is the start-vertex of e.

We show that there is an X-saturated matching using Hall's Marriage Theorem. Take $W \subseteq X$, and let

$$V = \{v \in B - A : v \in w \text{ for some } w \in W\}.$$

Let $C = A \cup V$, considering C as a subgraph of B. Then $\delta(C/A) \geq 0$ as $A \leq_s B$, so

$$2|V| \geq |\,\mathrm{E}(V)| + |\text{edges between } V \text{ and } A| \geq |W|,$$

and as $|\,\mathrm{N}(W)| = 2|V|$, we are done. □

LEMMA 1.35. *$(\mathcal{D}_0, \sqsubseteq_s)$ is a strong expansion of $(\mathcal{C}_0, \leq_s)$.*

PROOF. $\mathcal{C}_0$ is the class of graph reducts of $\mathcal{D}_0$ by Proposition 1.7, and if $A \sqsubseteq_s B$, $A, B \in \mathcal{D}_0$, then $A \leq_s B$ by Proposition 1.34.

Take $A \leq_s B \in \mathcal{C}_0$, and let $A^+ \in \mathcal{D}_0$ be an expansion of A. As $A \leq_s B$, there exists an orientation B' of B in which the induced orientation A' on A has $A' \sqsubseteq_s B'$. Let B^+ be the orientation of the edges of B given by taking B' and replacing A' with A^+ on A. Then B^+ is still a 2-orientation, and we have $A^+ \sqsubseteq_s B^+$. □

In the above proof, we produce a new orientation B^+ by replacing an orientation on a successor-closed subset, and this argument will be used frequently in this thesis - often without explicit comment.

REMARK 1.36. We could have easily proved Lemma 1.30 and shown that $(\mathcal{C}_0, \leq_s)$ was a strong class using Proposition 1.34, rather than using the original definition of $\leq_s$ in terms of predimension. (To show that $A \leq_s B \leq_s C \Rightarrow A \leq_s C$, we use the method of replacing an orientation on a successor-closed subset, as in the proof of Lemma 1.35.)

However, the disadvantage of this is that the proof in terms of $\sqsubseteq_s$ is specific to our particular choice of predimension. For a generalised notion of predimension (see [**5**]), our first proof using submodularity of δ is still valid.

LEMMA 1.37. *$(\mathcal{C}_0, \leq_s)$ and $(\mathcal{D}_0, \sqsubseteq_s)$ are free amalgamation classes.*

PROOF. It is clear that $(\mathcal{D}_0, \sqsubseteq_s)$ is a free amalgamation class. Using Lemma 1.26, we have that $(\mathcal{C}_0, \leq_s)$ is a free amalgamation class. □

DEFINITION 1.38. Let M_0 denote the Fraïssé limit of $(\mathcal{C}_0, \leq_s)$.

As M_0 can be written as the union of an increasing $\leq_s$-chain of structures in $\mathcal{C}_0$, we already have a definition of $A \leq_s M_0$ for finite A from Section 1.3.2. Namely, writing M_0 as the union of an increasing $\leq_s$-chain $A_0 \leq_s A_1 \leq_s \cdots$, for finite $A \subseteq M_0$ we say that $A \leq_s M_0$ if $A \leq_s A_i$ for some $i \geq 0$. This definition is not dependent on the particular $\leq_s$-chain we take - this is shown in Section 1.3.2.

For finite $A \subseteq M_0$, we define $\mathrm{cl}^s_{M_0}(A)$ to be the smallest finite $\leq_s$-closed subset of M_0 containing A. (We know that there exists finite $B \leq_s M$ with $A \subseteq B$, as we can write M_0 as the union of an increasing $\leq_s$-chain.)

DEFINITION 1.39. For $A \subseteq M_0$ with A possibly infinite, we will say $A \leq_s M_0$ if $A \cap X \leq_s X$ for all finite $X \subseteq M_0$. (Note that this is consistent with the definition for finite A by part (1) of Lemma 1.30.)

As $A \leq_s M_0$, $B \leq_s M_0$ implies that $A \cap B \leq_s M_0$ by the above definition (using Lemma 1.30), we can define $\mathrm{cl}^s_{M_0}(A)$ for $A \subseteq M_0$ as the smallest $\leq_s$-closed subset of M_0 containing A. Note that this is consistent with our definition of $\mathrm{cl}^s_{M_0}(A)$ for finite A: if $A \subseteq M_0$ is finite, then $\mathrm{cl}^s_{M_0}(A)$ is finite.

1.4.3 The acyclic case: $\leq_1$

DEFINITION 1.40. Let $A, B \in \mathcal{C}_1$ with $A \subseteq B$. We write $A \leq_1 B$ if there exists an acyclic 2-orientation $B^+ \in \mathcal{D}_1$ of B in which the induced orientation $A^+ \in \mathcal{D}_1$ on A has $A^+ \sqsubseteq_s B^+$.

We have an analogue of Lemma 1.30:

LEMMA 1.41. *Let $B \in \mathcal{C}_1$.*

(1) $A \leq_1 B$, $X \subseteq B \Rightarrow A \cap X \leq_1 X$.
(2) $A \leq_1 C \leq_1 B \Rightarrow A \leq_1 B$.
(3) $A_1, A_2 \leq_1 B \Rightarrow A_1 \cap A_2 \leq_1 B$.

PROOF. (1) is immediate. (2) follows by replacing an orientation on a successor-closed subset, as in the proof of Lemma 1.35. (3) follows from (1) and (2). □

Therefore we have that:

LEMMA 1.42. $(\mathcal{C}_1, \leq_1)$ *is a strong class.*

LEMMA 1.43. $(\mathcal{D}_1, \sqsubseteq_s)$ *is a strong class, and is a strong expansion of* $(\mathcal{C}_1, \leq_1)$.
$(\mathcal{C}_1, \leq_1)$ *and* $(\mathcal{D}_1, \sqsubseteq_s)$ *are free amalgamation classes.*

The proofs are as in the previous section.

(The material in this section is based on Section 3.4 of [**7**], and was originally developed in [**11**].)

1.5 Sparse graphs: $\leq_d$ and the ω-categorical M_F

We now construct an amalgamation class of sparse graphs whose Fraïssé limit is ω-categorical (specifically, the ω-categorical Hrushovski construction M_F, first presented by Hrushovski in [**15**]). We will do this by defining a new closure (i.e. a new notion of strong substructure), d-closure, which will be uniformly bounded. The relevance of this can be seen in the lemma below.

LEMMA 1.44. *Let $(\mathcal{K}, \leq)$ be an amalgamation class such that for $n \in \mathbb{N}$, $(\mathcal{K}, \leq)$ contains only finitely many isomorphism types of structures of size n. Suppose there is a function $F : \mathbb{N} \to \mathbb{N}$ such that for $B \in \mathcal{K}$, $A \subseteq B$ with $|A| \leq n$, then there exists $A \subseteq C \leq B$ with $|C| \leq F(n)$. (F will be a uniform bound on the size of $\leq$-closures.)*
Then the Fraïssé limit M of $(\mathcal{K}, \leq)$ is ω-categorical.

PROOF. By the Ryll-Nardzewski theorem, it is enough to show that $\mathrm{Aut}(M)$ is oligomorphic, i.e. that, for $n \geq 1$, $\mathrm{Aut}(M)$ has finitely many orbits on M^n. Take $n \geq 1$. As there are only finitely many isomorphism types of structures of size $\leq F(n)$ in $\mathcal{K}$ and M is $\leq$-homogeneous, $\mathrm{Aut}(M)$ has finitely many orbits on $\{\overline{c} \in M^{F(n)} : \overline{c} \leq M\}$. We can extend any $\overline{a} \in M^n$ to an element of this set (note that when we work with ordered tuples, we can have repeats). If $\overline{a}, \overline{a'}$ are not in the same orbit, then nor will their extensions be, so we are done. □

DEFINITION 1.45. Let $\mathcal{C}_{>0}$ be the class of finite graphs A such that for $B \subseteq A$, $\delta(B) > 0$.

DEFINITION 1.46. Take $A, B \in \mathcal{C}_{>0}$ with $A \subseteq B$. We say that A is d-closed in B, written $A \leq_d B$, if for $A \subsetneq C \subseteq B$, $\delta(A) < \delta(C)$.

As in the cases of $\leq_s$ and $\leq_1$, we have the following lemma:

LEMMA 1.47. *Let $B \in \mathcal{C}_{>0}$.*

(1) $A \leq_d B$, $X \subseteq B \Rightarrow A \cap X \leq_d X$.
(2) $A \leq_d C \leq_d B \Rightarrow A \leq_d B$.
(3) $A_1, A_2 \leq_d B \Rightarrow A_1 \cap A_2 \leq_d B$.

The proof is similar to that of Lemma 1.30. Similarly to the case of $\leq_s$, for $A \subseteq B \in \mathcal{C}_{>0}$ we may define $\mathrm{cl}^d_B(A)$ as the intersection of all d-closed substructures of B which contain A, and as before (by an analogous proof), we have that $\delta(A) \geq \delta(\mathrm{cl}^d_B(A))$.

LEMMA 1.48. *$(\mathcal{C}_{>0}, \leq_d)$ is a free amalgamation class.*

PROOF. It only remains to check the free amalgamation property (which implies JEP). We prove a stronger claim: given $A \leq_d B_1$, $A \subseteq B_2$ with $B_1, B_2 \subseteq E$, where E is the free amalgam of B_1, B_2 over A, we claim that $B_2 \leq_d E$. Once we have the claim, note that $\varnothing \leq_d B_2 \leq_d E$ implies that $E \in \mathcal{C}_{>0}$. Take $B_2 \subsetneq X \subseteq E$. Then letting $Y = X \cap B_1 \supsetneq A$, $X = B_2 \cup Y$, and X is the free amalgam of B_2, Y over A. So

$$\delta(X) = \delta(B_2 \cup Y) = \delta(B_2) + \delta(Y) - \delta(A),$$

and so

$$\delta(X) - \delta(B_2) = \delta(Y) - \delta(A) > 0,$$

as $A \leq_d B_1$. □

The generic structure $M_{>0}$ of $(\mathcal{C}_{>0}, \leq_d)$ is not ω-categorical, as for $A \subseteq_{\text{fin.}} M_{>0}$, there is no uniform bound on $|\operatorname{cl}^d(A)|$ in terms of $|A|$, and we have the same problem for M_0.

To construct ω-categorical examples, as mentioned at the start of this section, we consider subclasses of $\mathcal{C}_{>0}$ in which d-closure is uniformly bounded.

DEFINITION 1.49. Let $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous, increasing function with $F(0) = 0$ and $F(x) \to \infty$ as $x \to \infty$. We define

$$\mathcal{C}_F := \{B \in \mathcal{C}_{>0} : \delta(A) \geq F(|A|) \; \forall \, A \subseteq B\}.$$

LEMMA 1.50.

(1) If $B \in \mathcal{C}_F$, $A \subseteq B$, then $|\operatorname{cl}^d_B(A)| \leq F^{-1}(2|A|)$.
(2) If $(\mathcal{C}_F, \leq_d)$ is an amalgamation class, then the generic structure M_F is ω-categorical.

PROOF.

(1) From Lemma 1.32 and the fact that $\operatorname{cl}^d_B(A) \in \mathcal{C}_F$, we have $F(|\operatorname{cl}^d_B(A)|) \leq \delta(\operatorname{cl}^d_B(A)) \leq \delta(A) \leq 2|A|$.
(2) This follows from Lemma 1.44.

□

DEFINITION 1.51. Analogously to Definition 1.38 and Definition 1.39, for $A \subseteq M_F$ with A possibly infinite, we say that $A \leq_d M_F$ if $A \cap X \leq_d X$ for all finite $X \subseteq M_F$. Similarly we define $\operatorname{cl}^d_{M_F}(A)$ as the smallest $\leq_d$-closed subset of M_F containing A.

LEMMA 1.52. *Let F be as in Definition 1.49, and assume additionally that F is piecewise smooth, its right derivative F' is decreasing and $F'(x) \leq 1/x$ for $x > 0$. Then $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class.*

PROOF. Let $A, B_1, B_2 \in \mathcal{C}_F$, $A \leq_d B_1, B_2$. Let E be the free amalgam of B_1, B_2 over A. By Lemma 1.48, $E \in \mathcal{C}_{>0}$ and $B_1, B_2 \leq_d E$. We need to show that $E \in \mathcal{C}_F$. Assuming $E \neq B_1, B_2$, we have $A \neq B_1, B_2$. Suppose $X \subseteq E$: we need to show that $\delta(X) \geq F(|X|)$. As X is the

free amalgam of $B_1 \cap X$, $B_2 \cap X$ over $A \cap X$ and as $A \cap X \leq_d B_i \cap X$, it suffices to check just for $X = E$.

We have that

$$\delta(E) = \delta(B_1) + \delta(B_2) - \delta(A) = \delta(B_1) + (|B_2| - |A|)\frac{\delta(B_2) - \delta(A)}{|B_2| - |A|}.$$

Without loss of generality,

$$\frac{\delta(B_2) - \delta(A)}{|B_2| - |A|} \geq \frac{\delta(B_1) - \delta(A)}{|B_1| - |A|},$$

and as δ is integer-valued and $A \leq_d B_1$, $A \neq B_1$, the latter is $\geq 1/|B_1|$. So

$$\delta(E) \geq \delta(B_1) + \frac{|B_2| - |A|}{|B_1|} \geq F(|B_1|) + \frac{|B_2| - |A|}{|B_1|},$$

and the conditions on F ensure that

$$(*) \qquad F(x + y) \leq F(x) + y/x,$$

so

$$\delta(E) \geq F(|B_1| + |B_2| - |A|) = F(|E|).$$

□

LEMMA 1.53. *Let F be as in Definition 1.49, and suppose that $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class. Assume in addition that F is strictly increasing, $F(1) = 2$ and $F(2) = 3$. Then:*

(1) $\mathcal{C}_F$ contains a point and an edge;
(2) if $a \in A \in \mathcal{C}_F$, then $a \leq_d A$;
(3) if $ab \subseteq A \in \mathcal{C}_F$ is an edge, then $ab \leq_d A$;
(4) each vertex of M_F has infinite valency.

PROOF. (1): As $F(1) = 2$, if x is a point then $\delta(\{x\}) = 2 = F(|\{x\}|)$, so $\{x\} \in \mathcal{C}_F$. If ab is an edge, then $\delta(ab) = 3 = F(2)$, so $ab \in \mathcal{C}_F$.

(2), (3): These result from the fact that F is strictly increasing.

(4): Let $k \geq 1$. Let $ax \in \mathcal{C}_F$ be an edge. a is d-closed in ax, and so by taking the free amalgamation of k copies $ax_1, \cdots, ax_k$ of ax over a, we have that the star graph S_k is in $\mathcal{C}_F$ (where S_k is the complete bipartite graph $K_{1,k}$). Using the $\leq_d$-extension property of M_F, this implies that each vertex of M_F has infinite valency. □

Note: For all examples of M_F in this thesis, we will take F satisfying the conditions of Lemma 1.53. We thus eliminate trivial examples such as the case where M_F has no edges.

EXAMPLE 1.54. We will illustrate the flexibility of the above construction. We construct a connected ω-categorical graph with a vertex-transitive and edge-transitive automorphism group whose smallest cycle is a 5-cycle (this example originally appears in [**15**]).

Take F piecewise linear for $0 \leq x \leq 5$ where $F(0) = 0$, $F(1) = 2$, $F(2) = 3$, $F(5) = 5$, and let $F(k) = log(k) + 5 - log(5)$ for $k \geq 5$. So F is as in Definition 1.49, but only satisfies the conditions of Lemma 1.52 for $k \geq 5$. Then we have:

- $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class (we can use lemma 1.52 if $|B_1|$ or $|B_2| \geq 5$ - this is what is needed for $(*)$ to be true - and check the other cases individually);
- the smallest cycle in $\mathcal{C}_F$ is a 5-cycle (just check), so the same is true for M_F;
- if $a \in A \in \mathcal{C}_F$ then $a \leq_d A$ (as $F(k) \geq 2$ for $k \geq 1$), so $\mathrm{Aut}(M_F)$ is vertex-transitive;
- if $ab \subseteq B \in \mathcal{C}_F$ is an edge then $ab \leq_d B$ (as F(2) = 3), so $\mathrm{Aut}(M_F)$ is edge-transitive;
- M_F is connected: for non-adjacent $a, b \in M_F$, let $A = cl^d_{M_F}(ab)$. $\delta(A) \leq \delta(ab) = 4$, so $|A| \leq 3$. So either A is a path of length two with endpoints a, b or $A = ab$, so $ab \leq_d M_F$. In the latter case, consider B a path of length 3 between a, b. We have $ab \leq_d B$, so by the extension property there is a $\leq_d$-copy of B in M_F over ab, and so a, b have distance 3 in M_F.

(The material in this section is based on Section 3.2 of [**5**].)

1.6 The Ramsey property

We will now define the Ramsey property for strong classes. (This section is based on Section 2.2 of [**8**], which is based on [**9**] and [**17**], where general Ramsey theorems are developed for classes of structures with closures.)

Definition 1.55. Let $(\mathcal{K}, \leq)$ be a strong class. For $A, B \in \mathcal{K}$, we write $\binom{B}{A} = \{A' \leq B : A' \cong A\}$ for the set of $\leq$-copies of A inside B. (Note that by parts (S2), (S3) in the definition of a strong class, if $B \leq C \in \mathcal{K}$, then $\binom{B}{A} = \binom{C}{A} \cap \mathcal{P}(B)$.)

For $A, B \in \mathcal{K}$, $r \in \mathbb{N}_+$, an *r-colouring* of the set $\binom{B}{A}$ is a function $\chi : \binom{B}{A} \to \{1, \cdots, r\}$. We say that $\binom{B}{A}$ is *monochromatic* in the r-colouring χ if χ is constant on $\binom{B}{A}$.

For $A, B, C \in \mathcal{K}$, $r \in \mathbb{N}_+$, we write $C \to (B)^A_r$ if for every r-colouring χ of $\binom{C}{A}$, there exists $B' \in \binom{C}{B}$ such that $\binom{B'}{A}$ is monochromatic in χ.
We say that $(\mathcal{K}, \leq)$ has the *Ramsey property* if for $r \in \mathbb{N}_+$, $A, B \in \mathcal{K}$, there exists $C \in \mathcal{K}$ with $C \to (B)^A_r$.

The following is a well-known observation of Nešetřil ([**23**]) which shows a strong connection between Fraïssé theory and structural Ramsey theory:

PROPOSITION 1.56. *Let $(\mathcal{K}, \leq)$ be a strong, rigid class with JEP and the Ramsey property. Then $(\mathcal{K}, \leq)$ has the amalgamation property.*

PROOF. Let $A, B, C \in \mathcal{K}$ with $f : A \to B$, $g : A \to C$ $\leq$-embeddings. By JEP, find $E \in \mathcal{K}$ which B, C $\leq$-embed into. Then, by the Ramsey property, find $D \in \mathcal{K}$ such that $D \to (E)^A_4$. Define a 4-colouring $c : \binom{D}{A} \to \{\varnothing, \{B\}, \{C\}, \{B, C\}\}$ by:

$$B \in c(A_0) \Leftrightarrow \text{there is a } \leq\text{-embedding } r : B \to D \text{ with } rf(A) = A_0,$$

and similarly for C. Take $E_0 \in \binom{D}{E}$ monochromatic. By considering the $\leq$-embeddings $A \xrightarrow{f} B \to E_0$ and $A \xrightarrow{g} C \to E_0$, we see that for all $A_0 \in \binom{E_0}{A}$, $c(A_0) = \{B, C\}$. Take $A_0 \in \binom{E_0}{A}$. Then there are $\leq$-embeddings $r : B \to D$, $s : C \to D$ with $rf(A) = sg(A) = A_0$, so rf, sg are isomorphisms $A \xrightarrow{\sim} A_0$, so $rf = sg$ as A is rigid. So we have the amalgamation property with r, s the $\leq$-embeddings into D. □

1.7 A Ramsey result

We now prove a Ramsey result that will be a fundamental tool in Chapters 4, 5 and 6: we will use it to construct Ramsey expansions of $(\mathcal{D}_1, \sqsubseteq_s)$ and $(\mathcal{D}_0, \sqsubseteq_s)$, namely the *admissibly ordered orientations* of these classes. The novelty here is that this result is for *strong* classes - the classical Fraïssé theory generalises straightforwardly to strong classes, but the construction of Ramsey objects does not.

We will keep our presentation of this Ramsey result self-contained, but in [**9**], the authors prove Ramsey theorems for strong classes as an example of a more general framework, which we now briefly sketch. Theorems 1.3 and 1.4 of [**9**] give Ramsey-type theorems for structures in a language consisting of relation and *set-valued* function symbols, i.e. the codomain of each function is the power set of the structure. In Section 5 of [**9**], these results are then applied to $(\mathcal{D}_0, \sqsubseteq_s)$ by encoding the notion of $\sqsubseteq_s$-substructure using an expanded language with set-valued functions. We expand the language of oriented graphs L by a single unary function symbol F, calling the expanded language L^+, and take the class $\mathcal{D}_0^+$ of L^+-structures (A, F_A) where $A \in \mathcal{D}_0$ and $F_A : A \to \mathcal{P}(A)$ is a unary set-valued function taking each vertex of A to its out-neighbourhood. We then have that L^+-embeddings between elements of $\mathcal{D}_0^+$ are $\sqsubseteq_s$-embeddings when considered in the language L, and so the Ramsey result of Theorem 1.3 of [**9**] applied to $\mathcal{D}_0^+$ will give an equivalent Ramsey result for the strong class $(\mathcal{D}_0, \sqsubseteq_s)$.

We will not take this approach, as the Ramsey theorem that we require is simple enough to be proved independently. The below is an adaptation of Theorem 6.6 of [**7**] (a version of Theorem 4.29 of [**17**]):

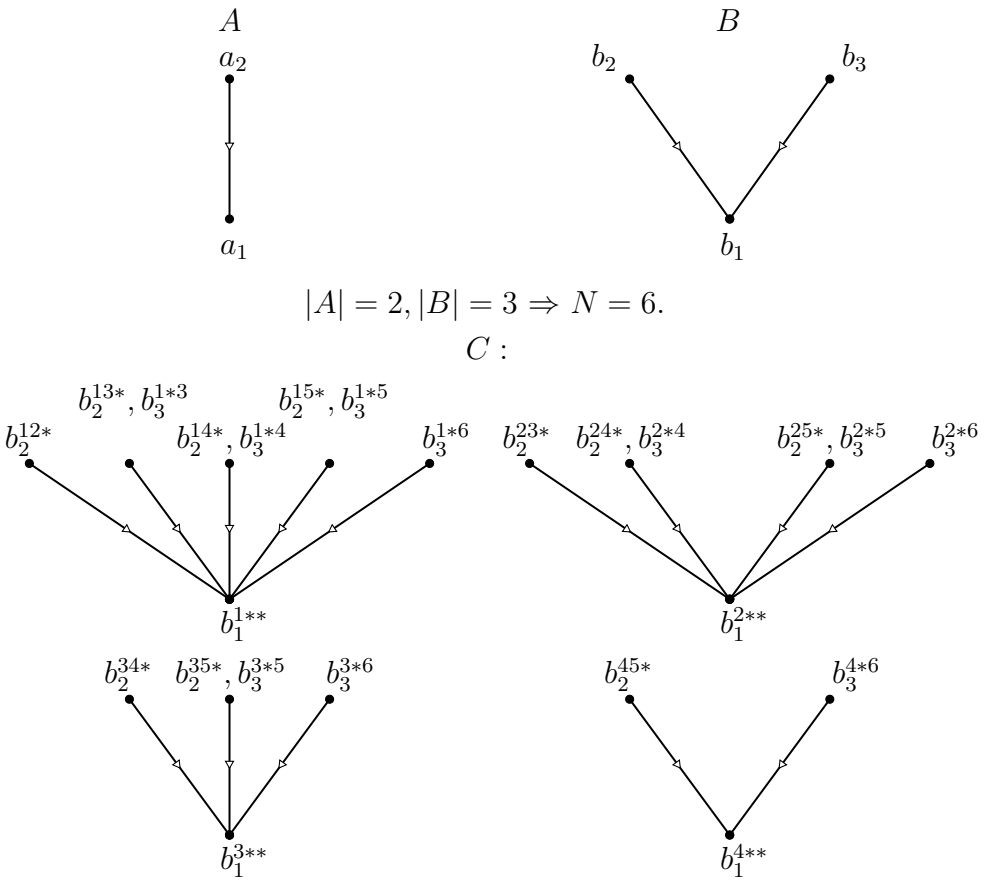


FIGURE 1.3. An example of the construction of the Ramsey witness C for A, B in the proof of Theorem 1.57. The order on A, B is by increasing index. In the above figure we label the vertices of C by the vertices contained in each equivalence class of $\sim$ on P, where, for example, b_2^{34*} means the equivalence class contains $\{b_2^{345}, b_2^{346}\}$.

THEOREM 1.57. *Let $(\mathcal{K}, \sqsubseteq_s)$ be a hereditary subclass of $(\mathcal{D}_0^{\prec}, \sqsubseteq_s)$, the class of linearly ordered elements of $\mathcal{D}_0$. Suppose $(\mathcal{K}, \sqsubseteq_s)$ has free amalgamation for any completion of the linear order, i.e.*

(∗) *for $(A, \gamma_A) \sqsubseteq_s (B_1, \gamma_1), (B_2, \gamma_2) \in \mathcal{K}$, if C is the free amalgamation of B_1, B_2 over A and γ is a linear order on C extending γ_1, γ_2, then $(C, \gamma) \in \mathcal{K}$.*

Then $(\mathcal{K}, \sqsubseteq_s)$ is a Ramsey class.

PROOF. Take $A \sqsubseteq_s B \in \mathcal{K}$ (where A, B are ordered oriented graphs - we will not indicate the order in the notation, for presentational simplicity). We will construct $C \in \mathcal{K}$ with $C \to (B)_2^A$.

We label B as $B = \{b_1, \cdots, b_k\}$ with $b_1 < \cdots < b_k$ in the order of B.

Using the Ramsey theorem for sets, take $N \in \mathbb{N}$ with $N \to (k)_2^{|A|}$.

Let L denote the language of ordered oriented graphs. Expand L to a new language L^+ by adding unary relation symbols R_i for $1 \leq i \leq N$. Construct an L^+-structure P as follows (where we also write R_i for the interpretation of the symbol R_i):

(1) for each k-tuple $\overline{v} = (v_1, \cdots, v_k)$ with $v_i \in \{1, \cdots, N\}$ $(1 \leq i \leq k)$ and $v_1 < \cdots < v_k$, put a disjoint copy $B^{\overline{v}} = \{b_1^{\overline{v}}, \cdots, b_k^{\overline{v}}\}$ of the ordered oriented graph B in P, where we place $b_i^{\overline{v}}$ in R_{v_i};
(2) complete the order on P in an arbitrary manner so that $\mathrm{V}(R_1) < \cdots < \mathrm{V}(R_N)$, where $\mathrm{V}(R_i)$ denotes the vertices inside R_i.

Define an equivalence relation $\sim$ on P where for $x, y \in P$, $x \sim y$ if $\mathrm{scl}_P(x), \mathrm{scl}_P(y)$ with the induced order and unary relations are L^+-isomorphic.

Let Q be the set of $\sim$-equivalence classes, and use the quotient map $\pi : P \to Q$ to induce an L^+-structure on Q (which we also denote by Q): it is straightforward to check that this L^+-structure is well-defined. π is then a L^+-homomorphism which is an L^+-embedding on every $B^{\overline{v}}$.

Let C be the reduct of Q to the language L of ordered oriented graphs, forgetting the unary relations. (See Figure 1.3.)

Recall that $(\mathcal{K}, \sqsubseteq_s)$ has free amalgamation for any completion of the linear order. As the L-reduct of P is a disjoint union of copies of B, thus $P|_L \in \mathcal{K}$. We then obtain C from $P|_L$ by taking a quotient over a set of successor-closed substructures. We can therefore reconstruct C via a sequence of free amalgamations of successor-closed substructures of B, and thus $C \in \mathcal{K}$.

Define a map $f : Q \to \{1, \cdots, N\}$ which, for $x \in Q$, sends x to the index i of the unary relation R_i for which $x \in R_i$. (It is easy to check that f is well-defined: recall that equivalence classes were defined via L^+-isomorphisms.) For $X \sqsubseteq_s C$ (here we work with L-structures), we can then regard f as a map from $\binom{C}{X}$, the set of $\sqsubseteq_s$-copies (via L-embeddings) of X in C, to $\binom{N}{|X|}$, the set of subsets of $\{1, \cdots, N\}$ of size $|X|$. It is an easy check that $f : \binom{C}{X} \to \binom{N}{|X|}$ is an injection.

Also, by the construction of Q, we have that $f : \binom{C}{B} \to \binom{N}{|B|}$ is a bijection.

Therefore, given a 2-colouring $\chi : \binom{C}{A} \to 2$, we may induce a 2-colouring on a subset of $\binom{N}{|A|}$, which we extend to a colouring χ' of $\binom{N}{|A|}$. By our choice of N, we have a χ'-monochromatic subset $\{v_1, \cdots, v_k\}$ (where $v_1 < \cdots < v_k$), and this corresponds to a $\sqsubseteq_s$-copy B' of B in C with $\binom{B'}{A}$ monochromatic. $\square$

1.8 Topological dynamics: G-flows

A central object of study in topological dynamics is the following (see [**1**] for more of a background):

DEFINITION 1.58. A *G-flow* is a continuous action $G \curvearrowright X$ of a Hausdorff topological group G on a nonempty compact Hausdorff topological space X.

We will often simply write X to refer to the G-flow $G \curvearrowright X$ when this is clear from context. Given a G-flow on X, $\overline{G \cdot x}$, the orbit closure of a point $x \in X$, is a G-invariant compact subset of X. In general, a nonempty compact G-invariant subset $Y \subseteq X$ defines a *subflow* by restricting the G-action to Y. A G-flow on X is *minimal* if it contains no proper subflows. A G-flow is minimal iff every orbit is dense.

Zorn's lemma shows that every G-flow contains a minimal subflow. (The proof is straightforward, remembering that everything is Hausdorff.)

Let X, Y be G-flows. A *G-flow morphism* $X \to Y$ is a continuous map $\pi : X \to Y$ such that $\pi(g \cdot x) = g \cdot \pi(x)$ (this property is called *G-equivariance*). Bijective G-flow morphisms are isomorphisms, as they are between compact Hausdorff spaces.

If Y is minimal, then any G-flow morphism $X \to Y$ is surjective, as the image is a subflow.

We now state a well-known theorem from topological dynamics that we shall use without proof (for more details see, for example, [**1**], Ch. 8):

THEOREM 1.59. *Let G be a Hausdorff topological group. Then there is a minimal G-flow $M(G)$ such that for any minimal G-flow X, there is a surjective G-flow morphism $M(G) \to X$, and $M(G)$ is uniquely determined up to isomorphism by this property.*

$M(G)$ is called the *universal minimal flow* of G.

DEFINITION 1.60. Let G be a Hausdorff topological group. If every G-flow has a G-fixed point, we call G *extremely amenable*.

We immediately have that G is extremely amenable iff $M(G)$ is a singleton.

DEFINITION 1.61. Let G be a Hausdorff topological group. If every G-flow has a G-invariant Borel probability measure, we call G *amenable*.

Note that extreme amenability implies amenability by taking the Dirac probability measure on the fixed point given by extreme amenability.

We have the fact that if G is separable metrisable, then it is sufficient for metrisable G-flows to have a fixed point for G to be extremely amenable (see [**19**]).

1.9 The KPT correspondence

In many cases, for a general Hausdorff topological group G, $M(G)$ can be quite complicated - i.e. non-metrisable (see the introduction of [**19**] for an exploration of the broader context).

We will be concerned in this thesis with the case where G is the automorphism group of a countable first-order structure, with the pointwise convergence topology whose open basis is given by left cosets of pointwise stabilisers of finite sets. G is thus a Polish group. (For the details - which are not technically difficult - see the notes [**5**].) In this particular context, there is a connection between triviality of the universal minimal flow $M(G)$ and structural Ramsey theory.

This was first shown by Kechris-Pestov-Todorčević in [**19**], and then further extended by work of Nguyen Van Thé, Zucker, Ben Yaacov, Melleray, Tsankov and others, e.g. in [**24**], [**25**], [**2**]. These extensions of the results in [**19**] will be discussed in the next section. Our presentation will be based on Section 2 of [**8**].

THEOREM 1.62 ([**19**], Th. 4.8; [**24**], Th. 1). *Let $(\mathcal{K}, \leq)$ be an amalgamation class of rigid L-structures, with Fraïssé limit M and $G = \mathrm{Aut}(M)$. Then G is extremely amenable iff $(\mathcal{K}, \leq)$ has the Ramsey property.*

Here, a finite L-structure is *rigid* if it has trivial automorphism group.

1.10 Further KPT: the expansion property

Throughout this section, L, L^+ will be first order relational languages, with $L^+ \supseteq L$ an expansion of L.

1.10.1 Reasonable expansions

Here we follow Section 2.3 of [**8**], which takes the notion of a reasonable expansion from [**25**].

DEFINITION 1.63. Let $(\mathcal{K}, \leq)$ be an amalgamation class of L-structures. A class $\mathcal{D}$ of finite L^+-structures is a *reasonable expansion* ([**25**]) of $(\mathcal{K}, \leq)$ if $\mathcal{D}$ is closed under isomorphisms and satisfies the following:

(1) $\mathcal{K}$ is the class of L-reducts of $\mathcal{D}$;
(2) for $A \in \mathcal{K}$, A has finitely many expansions in $\mathcal{D}$ (*weak coprecompactness*);

(3) for $B^+ \in \mathcal{D}$, if $A^+ \subseteq B^+$ and $A^+|_L \leq B^+|_L$, then $A^+ \in \mathcal{D}$;
(4) for $f : A \to B$ a strong embedding in $(\mathcal{K}, \leq)$, if $A^+ \in \mathcal{D}$ is an expansion of A, then there exists an expansion $B^+ \in \mathcal{D}$ of B such that $f : A^+ \to B^+$ is an embedding.

LEMMA 1.64. *For any amalgamation class $(\mathcal{K}, \leq)$ of L-structures, the order expansion $\mathcal{K}^\prec$ of $(\mathcal{K}, \leq)$ (see Definition 1.27) is a reasonable expansion of the class $(\mathcal{K}, \leq)$.*

PROOF. Straightforward. □

DEFINITION 1.65. Let $\mathcal{D}_F$ be the class of all 2-orientations of graphs in $\mathcal{C}_F$.

LEMMA 1.66. *The classes $\mathcal{D}_1$, $\mathcal{D}_0$ and $\mathcal{D}_F$ are reasonable expansions of $(\mathcal{C}_1, \leq_1)$, $(\mathcal{C}_0, \leq_s)$ and $(\mathcal{C}_F, \leq_d)$ respectively.*

PROOF. We check parts (1)-(4) of Definition 1.63 for $\mathcal{D}_0$. Part (1) results from Proposition 1.7. Parts (2) and (3) are immediate. Part (4) follows by a similar argument to that used in the proof of Lemma 1.35.

The proof for $\mathcal{D}_F$ is similar.

For $\mathcal{D}_1$, part (1) of Definition 1.63 follows by the definition of $\mathcal{C}_1$ as the class of graph reducts of $\mathcal{D}_1$. Parts (2)-(4) are similar to the $\mathcal{D}_0$ case. □

DEFINITION 1.67. Let $\mathcal{D}$ be a reasonable L^+-expansion of the amalgamation class $(\mathcal{K}, \leq)$. Then we define a topological space $X(\mathcal{D})$ as follows.

Let $X(\mathcal{D}) = \{(M, s)$ an L^+-expansion of $M : \forall\, A \leq M,\ (A, s|_A) \in \mathcal{D}\}$.

Each basic open set of $X(\mathcal{D})$ is given by first fixing $B \leq M, (B, r_B) \in \mathcal{D}$, and then taking $U(r_B) = \{(M, s) \in X(\mathcal{D}) : s|_B = r_B\}$ as the basic open set. It is straightforward to check that this does in fact form a basis. We will see below that $X(\mathcal{D})$ gives a G-flow.

LEMMA 1.68. *Let $\mathcal{D}$ be a reasonable L^+-expansion of the amalgamation class $(\mathcal{K}, \leq)$. Let M be the Fraïssé limit of $(\mathcal{K}, \leq)$.*

Then for $A \leq M, (A, r_A) \in \mathcal{D}$, there is an expansion $(M, r) \in X(\mathcal{D})$ of M with $r|_A = r_A$.

PROOF. Write M as the union of an increasing $\leq$-chain $A = A_1 \leq A_2 \leq \cdots$ starting at A. Let $r_1 = r_A$. By part (4) of reasonableness, we may inductively define r_i on A_i $(i \geq 2)$ such that $(A_{i-1}, r_{i-1}) \leq (A_i, r_i) \in \mathcal{D}$. We then take $r = \bigcup_{i \geq 1} r_i$. For $B \leq M$, $B \leq A_i$ for some i, so by part (3) of reasonableness, $(B, r) \in \mathcal{D}$. So $(M, r) \in X(\mathcal{D})$. □

LEMMA 1.69 (after Prop. 5.3, [**25**]). *Let $\mathcal{D}$ be a reasonable expansion of the amalgamation class $(\mathcal{K}, \leq)$. Let M be the Fraïssé limit of $(\mathcal{K}, \leq)$ and let $G = \mathrm{Aut}(M)$. Then $X(\mathcal{D})$ is a G-flow with the natural action.*

PROOF. By Lemma 1.68, $X(\mathcal{D})$ is non-empty.

Checking the continuity of the action $G \curvearrowright X(\mathcal{D})$, $g \cdot (M, s) = (M, gs)$ is straightforward, as is checking that $X(\mathcal{D})$ is Hausdorff.

To see that $X(\mathcal{D})$ is compact, consider the topological space $P = \prod_{A \leq M} \{\text{expansions } A^+ \text{ of } A\}$, where each set of the product has the discrete topology and P has the product topology. Basic open sets of P are given by fixing $A_1, \cdots, A_n \leq M$ and then fixing expansions of each A_i.

We define a map $\gamma : X(\mathcal{D}) \to P$, $\gamma((M, s)) = (s|_A)_{A \leq M}$, and it is straightforward to see that γ is a homeomorphism onto its image.

Let $Q = \prod_{A \leq M} \{\text{expansions } A^+ \in \mathcal{D} \text{ of } A\}$. Using part (2) of reasonableness (weak coprecompactness) and the fact that finite discrete topological spaces are compact, $Q \subseteq P$ is compact. As $\mathrm{Im}(\gamma) \subseteq Q$, it suffices to show that $\mathrm{Im}(\gamma)$ is a closed subspace of Q. We have that

$$\mathrm{Im}(\gamma) = \{(p_A)_{A \leq M} \in Q : \forall\, B \leq C \leq M,\, p_C|_B = p_B\},$$

and so if an element of Q lies in the complement of $\mathrm{Im}(\gamma)$, this is witnessed on a finite set. □

DEFINITION 1.70. Let $(\mathcal{K}, \leq)$ be an amalgamation class with Fraïssé limit M and $G = \mathrm{Aut}(M)$. Then, as the order expansion $\mathcal{K}^\prec$ is a reasonable expansion of $(\mathcal{K}, \leq)$, we have that $X(\mathcal{K}^\prec)$ is a G-flow: we denote this by $\mathcal{LO}(M)$ and call it the *flow of linear orders* on M.

As $\mathcal{D}_1, \mathcal{D}_0$ and $\mathcal{D}_F$ are reasonable expansions of $(\mathcal{C}_1, \leq_1), (\mathcal{C}_0, \leq_s)$ and $(\mathcal{C}_F, \leq_d)$ respectively, we have that $X(\mathcal{D}_1)$ is a G_1-flow, $X(\mathcal{D}_0)$ is a G_0-flow and $X(\mathcal{D}_F)$ is a G_F flow. We will denote these flows by $\mathrm{Or}(M_1)$, $\mathrm{Or}(M_0)$ and $\mathrm{Or}(M_F)$ respectively, and call them *flows of orientations*.

LEMMA 1.71 (Lem. 2.16, [**8**]). *Let $\mathcal{D}$ be a reasonable expansion of the amalgamation class $(\mathcal{K}, \leq)$. Let M be the Fraïssé limit of $(\mathcal{K}, \leq)$ and let $G = \mathrm{Aut}(M)$.*

Let $Y \subseteq X(\mathcal{D})$ be a subflow. Then there exists $\mathcal{D}' \subseteq \mathcal{D}$ which is a reasonable expansion of $(\mathcal{K}, \leq)$ with $X(\mathcal{D}') = Y$.

PROOF. Let

$$\mathcal{D}' = \bigcup_{(M,s) \in Y} \mathrm{Age}_{\leq}(M, s),$$

i.e. $\mathcal{D}'$ is the class of finite L^+-structures which $\leq$-embed into an element of Y. So $\mathcal{D}' \subseteq \mathcal{D}$. Parts (1), (2), (3) of reasonableness of $\mathcal{D}'$ are straightforward, and (4) follows from the $\leq$-homogeneity of M and the G-invariance of Y. Clearly $Y \subseteq X(\mathcal{D}')$. To see that $X(\mathcal{D}') \subseteq Y$, take $(M, s) \in X(\mathcal{D}')$ and show that $(M, s) \in \mathrm{cl}_{X(\mathcal{D}')}(Y) = Y$: this follows from the $\leq$-homogeneity of M and the G-invariance of Y. □

DEFINITION 1.72. Let $\mathcal{D}$ be a reasonable expansion of the amalgamation class $(\mathcal{K}, \leq)$. We say that $\mathcal{D}$ has the *expansion property* over

$(\mathcal{K}, \leq)$ if for $A \in \mathcal{K}$, there exists B in $\mathcal{K}$ with $A \leq B$ such that for all expansions A^+, B^+ of A, B in $\mathcal{D}$, there exists a $\leq$-embedding $A^+ \to B^+$.
It is straightforward to see that to prove the expansion property for $\mathcal{D}$, it suffices to show that for $A^+ \in \mathcal{D}$, there exists $B \in \mathcal{K}$ such that for all expansions $B^+ \in \mathcal{D}$ of B, there exists a $\leq$-embedding $A^+ \to B^+$ (use weak coprecompactness of $\mathcal{D}$ and JEP for $\mathcal{K}$).

(The above notion of expansion property first appears in [**24**], generalising the *ordering property* found in [**19**] where L is expanded by a single binary relation symbol.)

THEOREM 1.73 ([**24**], Th. 4). *Let $\mathcal{D}$ be a reasonable expansion of the amalgamation class $(\mathcal{K}, \leq)$ with Fraïssé limit M and $G = \mathrm{Aut}(M)$.*
Then the G-flow $X(\mathcal{D})$ is minimal iff $\mathcal{D}$ has the expansion property over $(\mathcal{K}, \leq)$.

1.10.2 Coprecompact expansions

We now define coprecompactness, the final definition we need to state our extended KPT result.

DEFINITION 1.74 (above Th. 2.20, [**8**]). Let $(\mathcal{K}^+, \leq^+)$ be an amalgamation class of L^+-structures which is a strong expansion of $(\mathcal{K}, \leq)$. Let M^+ denote the Fraïssé limit of $(\mathcal{K}^+, \leq^+)$. By Lemma 1.26, $(\mathcal{K}, \leq)$ is an amalgamation class with Fraïssé limit $M = M^+|_L$.
We will say that $(\mathcal{K}^+, \leq^+)$ is a *coprecompact* expansion of $(\mathcal{K}, \leq)$ if every $\mathrm{Aut}(M)$-orbit on M^n $(n \geq 1)$ splits into finitely many $\mathrm{Aut}(M^+)$ orbits (equivalently, $\mathrm{Aut}(M^+)$ is a coprecompact subgroup of $\mathrm{Aut}(M)$).

In the context of strong classes, this definition of coprecompactness (first formulated for strong classes in [**8**]) is stronger than the weak coprecompactness seen in the definition of a reasonable expansion. In the simpler non-strong case of classes with no distinguished notion of embedding (i.e. the standard classical Fraïssé theory), these two notions coincide.

LEMMA 1.75. *Let $(\mathcal{K}^+, \leq^+)$ be an L^+-amalgamation class which is a strong expansion of $(\mathcal{K}, \leq)$. Then $(\mathcal{K}^+, \leq^+)$ is a coprecompact strong expansion of $(\mathcal{K}, \leq)$ iff:*
for $A \in \mathcal{K}$, there exist finitely many $\leq$-embeddings $f_i : A \to B_i^+$ $(1 \leq i \leq n)$, $B_i^+ \in \mathcal{K}^+$, such that for any $\leq$-embedding $g : A \to C^+$, $C^+ \in \mathcal{K}^+$, there exists $1 \leq i \leq n$ and a $\leq^+$-embedding $h : B_i^+ \to C^+$ with $g = hf_i$.

The proof is straightforward.

1.10.3 The extended KPT correspondence

The following collates the results of Theorem 10.8 of [**19**], Theorem 5 of [**24**] and Theorem 5.7 of [**25**], reformulating these for strong classes as in Section 2 of [**8**]:

THEOREM 1.76. *Let $(\mathcal{K}, \leq)$ be an amalgamation class with Fraïssé limit M and $G = \mathrm{Aut}(M)$. Let $(\mathcal{K}^+, \leq^+)$ be a strong, reasonable, coprecompact expansion of $(\mathcal{K}, \leq)$ consisting of rigid L^+-structures, with Fraïssé limit M^+.*

If $(\mathcal{K}^+, \leq^+)$ is a Ramsey class and $\mathcal{K}^+$ has the expansion property over $(\mathcal{K}, \leq)$, then $X(\mathcal{K}^+)$ is the universal minimal flow of G. $X(\mathcal{K}^+)$ has a comeagre orbit consisting of the expansions of M which are isomorphic to M^+.

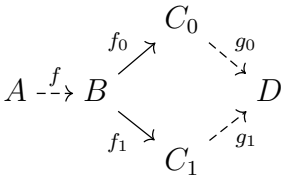
1.11 Meagre orbits

We now introduce the weak amalgamation property, which will be essential in the proof of Theorem 4.18, one of the main results of this thesis.

(A similar property (the almost amalgamation property) was introduced by Ivanov ([**18**]). This was then adapted to the weak amalgamation property by Kechris & Rosendal in [**20**].)

DEFINITION 1.77 ([**20**]). Let $\mathcal{D}$ be a reasonable class of L^+-expansions of an L-amalgamation class $(\mathcal{K}, \leq)$. We say that $(\mathcal{D}, \leq)$ has the *weak amalgamation property* (WAP) if:

for all $A \in \mathcal{D}$, there exists $B \in \mathcal{D}$ and a $\leq$-strong L^+-embedding $f : A \to B$ such that, for any $\leq$-strong L^+-embeddings $f_i : B \to C_i \in \mathcal{D}$ $(i = 0, 1)$, there exists $D \in \mathcal{D}$ and $\leq$-strong L^+-embeddings $g_i : C_i \to D$ $(i = 0, 1)$ with $g_0 \circ f_0 \circ f = g_1 \circ f_1 \circ f$. (Note that here we specify only that the diagram commutes for A.)



LEMMA 1.78 (Lem. 2.23, [**8**] - adapting Th. 3.4 of [**20**]). *Let $\mathcal{D}$ be a reasonable class of L^+-expansions of an L-amalgamation class $(\mathcal{K}, \leq)$. Let M be the Fraïssé limit of $(\mathcal{K}, \leq)$ and let $G = \mathrm{Aut}(M)$. Suppose that $X(\mathcal{D})$ is a minimal flow.*

If $(\mathcal{D}, \leq)$ does not have the weak amalgamation property, then all G-orbits on $X(\mathcal{D})$ are meagre.

PROOF. We assume $(\mathcal{D}, \leq)$ does not have the weak amalgamation property. Suppose $(A, r_A) \in \mathcal{D}$ witnesses the failure of WAP for $(\mathcal{D}, \leq)$. We may assume $A \leq M$.

Recall that $X(\mathcal{D})$ has a basis where we specify $P \leq M, (P, r_P) \in \mathcal{D}$, and then take $U(r_P) = \{(M, s) \in X(\mathcal{D}) : s|_P = r_P\}$ as the basic open set.

Take $(M, t) \in X(\mathcal{D})$. As $X(\mathcal{D})$ is minimal, $X(\mathcal{D})$ is the G-orbit closure of (M, t). By Lemma 1.68, we may extend r_A to an element of $X(\mathcal{D})$, and so $U(r_A)$ is non-empty. Then there is $g \in G$ such that $g^{-1}(M, t) \in U(r_A)$, and thus g is a $\leq$-embedding $(A, r_A) \to (M, t)$. Let $A' = gA$.

Let $J = G_{(A')}$, the pointwise stabiliser of A' in G. As M is countable, there are countably many $\leq$-copies of A' in M, and so J is of countable index in G. We will show that Jt is nowhere dense in $X(\mathcal{D})$, and as $|G : J| = \aleph_0$, thus we will have that Gt is meagre in $X(\mathcal{D})$, and given that (M, t) was an arbitrary element of $X(\mathcal{D})$, this will complete the proof.

For a contradiction, say Jt is *not* nowhere-dense in $X(\mathcal{D})$, i.e. Jt is dense in some non-empty open set of $X(\mathcal{D})$, which we may take to be a basic open set $U(r_P)$ for some $(P, r_P) \in \mathcal{D}, P \leq M$. As $A', P \leq M$, there is finite $B \leq M$ with $A', P \leq B$, and by part (4) of reasonableness there is an expansion $(B, r_B) \in \mathcal{D}$ with $r_B|_P = r_P$. So Jt is dense in $U(r_B)$, and $A' \leq B$. By Lemma 1.68, we may extend (B, r_B) to an element of $U(r_B)$, so there exists $j \in J$ with $jt \in U(r_B)$, and as j fixes A' pointwise, $r_B|_{A'} = t|_{A'}$.

For $(B, r_B) \leq (C, r_C) \in \mathcal{D}$, by the extension property of M and Lemma 1.68, there is $r \in U(r_B)$ and a $\leq$-embedding $f : (C, r_C) \to (M, r)$ which is the identity on (B, r_B).

As Jt is dense in $U(r_B)$, there is $j \in J$ such that $(jt)|_{f(C)} = r|_{f(C)}$. So then $j^{-1}f : (C, r_C) \to (M, t)$ is a $\leq$-embedding which is the identity on (A', t). As we have found a $\leq$-embedding $(C, r_C) \to (M, t)$ which is the identity on (A', t) for (C, r_C) any arbitrary element of $\mathcal{D}$ with $(B, r_B) \leq (C, r_C)$, we therefore have that (A', t) has WAP, and thus so does (A, r_A) - contradiction. $\square$

Chapter 2

Amenable and extremely amenable subgroups of the automorphism groups of sparse graphs

In this chapter, we first prove that $\mathrm{Aut}(M_F), \mathrm{Aut}(M_0)$ have no coprecompact extremely amenable subgroup, restating results from [**8**]. Theorem 3.8 of [**8**] also provides a criterion for non-amenability, and uses this to show that $\mathrm{Aut}(M_F)$ has no coprecompact amenable subgroup. We use this criterion to prove the new result that $\mathrm{Aut}(M_0)$ has no coprecompact amenable subgroup, extending proof techniques from [**8**].

2.1 Extremely amenable subgroups

PROPOSITION 2.1 (adaptation of [**8**], Th. 3.7). *Let M be an infinite 2-sparse graph in which all vertices have infinite valency. Let $G = \mathrm{Aut}(M)$.*

Consider the G-flow $G \curvearrowright \mathrm{Or}(M)$. (Recall that $\mathrm{Or}(M)$ denotes the space of 2-orientations of M, introduced in Definition 1.70.)

If $J \leq G$ fixes a 2-orientation of M, then J has infinitely many orbits on M^2.

PROOF. Let $\tau \in \mathrm{Or}(M)$ be an orientation of M fixed by J. So $J \leq \mathrm{Aut}(M, \tau)$. Let $K = \mathrm{Aut}(M, \tau)$. It suffices to show that K has infinitely many orbits on M^2.

Seeking a contradiction, suppose K has a finite number m of orbits on M^2. Note that for $a, x, y \in M$, if there is no element of K_a taking x to y, then there is no element of K taking (a, x) to (a, y)). So for all $a \in M$, K_a has finitely many orbits on M, and the number of orbits on M of K_a is bounded uniformly in a by the number of orbits m of K on M^2.

Take $a \in M$. Say $x, y \in \mathrm{scl}_\tau(a)$ (the successor-closure of a in the orientation τ) and let p, q be the lengths of the shortest out-paths from a to x, y respectively. If $p \neq q$, then x, y lie in different K_a-orbits.

Also, if there is an out-path $ax_1 \cdots x_r$ from a to x_r of minimal length r amongst out-paths from a to x_r, then the out-path $ax_1 \cdots x_i$ is of minimal length i amongst out-paths from a to x_i.

So all vertices in $\mathrm{scl}_\tau(a)$ must be reachable via an out-path from a of length at most $m-1$. As τ is a 2-orientation, this means that $|\mathrm{scl}_\tau(a)| \leq 1 + 2 + \cdots + 2^{m-1} = 2^m - 1$. This bound is uniform in a. Let $l \leq 2^m - 1$ be the minimal uniform upper bound, and let $a \in M$ be a vertex that attains this. (So for $b \in M$, $|\mathrm{scl}_\tau(b)| \leq l$, and in particular for a, $|\mathrm{scl}_\tau(a)| = l$.)

As a has infinite valency and τ is a 2-orientation, a has an in-vertex c. But then $|\mathrm{scl}_\tau(c)| \geq l + 1$, contradiction. □

THEOREM 2.2 (adaptation of [**8**], Th. 1.2, Th. 3.7). $\mathrm{Aut}(M_F)$ *has no coprecompact extremely amenable subgroup.*

(Recall that, as stated after Lemma 1.53, throughout this thesis we assume that we have taken the control function F such that $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class, vertices and edges lie in $\mathcal{C}_F$ and are d-closed in M_F, and every vertex of M_F has infinite valency.)

PROOF. Write $M = M_F$, $G = \mathrm{Aut}(M)$. Let J be an extremely amenable subgroup of G. As J must fix an orientation of M, by Proposition 2.1 we have that J has infinitely many orbits on M^2.

As G is oligomorphic, G has finitely many orbits on M^2, and so any coprecompact subgroup of G must have finitely many orbits on M^2. So J is not coprecompact. □

THEOREM 2.3 ([**8**], Th. 3.16). $\mathrm{Aut}(M_0)$ *has no coprecompact extremely amenable subgroup.*

PROOF. We write $M = M_0$, $G = \mathrm{Aut}(M_0)$. Suppose $J \leq G$ is extremely amenable. Consider the flow $G \curvearrowright \mathrm{Or}(M)$. As J is extremely amenable, J fixes some orientation $\sigma \in \mathrm{Or}(M)$. So $J \leq \mathrm{Aut}(M, \sigma)$. We will show that there is a G-orbit P on M^2 which splits into infinitely many J-orbits, so J is not a coprecompact subgroup of G.

Take non-adjacent $x, y \in M$ with $\{x, y\} \leq_s M$. Let $P \subseteq M^2$ be the G-orbit of (x, y). Suppose, seeking a contradiction, that P splits into finitely many J-orbits.

Take $m \in \mathbb{N}$, and take a rooted tree T of height $m+1$ with each non-leaf vertex having 3 children, where the shortest path from each leaf vertex to the head has exactly $m+1$ edges. (To avoid terminological confusion with the definition of roots in Definition 1.13, what is ordinarily referred to in graph theory terminology as the root vertex of a rooted tree will be called the *head* vertex. Specifically, the head vertex is the vertex which has no parent, i.e. the vertex furthest away from the leaves.)

By directing the edges of T from leaves towards head, we obtain a 2-orientation τ of T, so $T \in \mathcal{C}_0$, and we may take $T \leq_s M$.

As each non-leaf vertex $a \in T$ has 3 children and σ is a 2-orientation, some child of a must be an in-vertex of a in σ, and so there is an out-path $v = v_{-1}, v_0, \cdots, v_m$ of length $m+1$ in (T, σ) going from some leaf vertex to the head. We denote this by (Q, σ).

As σ, τ agree on Q and (Q, τ) is successor-closed in (T, τ), we have that $Q \leq_s M$ by Proposition 1.34. For $1 \leq i \leq m$, we may reverse τ on $(v_{-1}, v_0), (v_i, v_{i+1})$ to produce an orientation of Q in which $\{v, v_i\}$ is successor-closed, and so $\{v, v_i\} \leq_s M$. Hence $(v, v_i) \in P$.

To sum up, for $m \in \mathbb{N}$ we have shown that there exists a $(m+1)$-length path $v = v_{-1}, v_0, \cdots, v_m$ in M with $(v_i, v_{i+1}) \in \sigma$ $(i \geq -1)$ and $(v, v_i) \in P$ $(i \geq 1)$.

Let $k = 2$. Define a function $s : \mathbb{Z}_+ \to \mathbb{Z}_+$ by $s(1) = 1$, $s(t+1) = k^{s(t)+2}$. We claim that for $t \geq 1$, $m \geq s(t)$, we have that $v_1, \cdots, v_m$ lie in at least t different J_v-orbits. This will contradict P splitting into finitely many J-orbits.

First, let $B(v, j)$ be the set consisting of vertices w which are reachable via an out-directed path in σ from v of length at most j $(j \in \mathbb{N})$. So $|B(v, j)| \leq 1 + k + \cdots + k^j < k^{j+1} - 1$. We then prove the claim by induction. Note that the case $t = 1$ is trivial. Seeking a contradiction, assume the claim is true for t, but not for $t+1$. Then $v_{s(t)+1}, \cdots, v_{s(t+1)}$ lie in the same J_v-orbits as $v_1, \cdots, v_{s(t)}$, and so as $J \leq \mathrm{Aut}(M, \sigma)$, $\{v_1, \cdots, v_{s(t+1)}\} \subseteq B(v, s(t)+1)$. But $s(t+1) = k^{s(t)+2}$ and $|B(v, s(t)+1)| < k^{s(t)+2} - 1$, contradiction. This completes the proof of the claim. □

2.2 A criterion for non-amenability

LEMMA 2.4 ([**8**], Th. 3.8). *Let M be M_1, M_0 or M_F, and let $J \leq \mathrm{Aut}(M)$. Suppose there is an edge ab of M such that $J_a b$, $J_b a$ are both infinite. Then J is not amenable.*

PROOF. Seeking a contradiction, suppose J is amenable. So the J-flow $\mathrm{Or}(M)$ has a J-invariant Borel probability measure μ. For $x, y \in M$, let $U_{xy} = \{\sigma \in \mathrm{Or}(M) : (x, y) \in \sigma\}$. U_{xy} is open in $\mathrm{Or}(M)$. We have that $U_{ab} \cup U_{ba} = \mathrm{Or}(M)$, so without loss of generality, $\mu(U_{ab}) = p > 0$.

For $r \geq 1$, let $b_1, \cdots, b_r$ be distinct elements of $J_a b$. Let χ_i be the characteristic function of U_{ab_i} in $\mathrm{Or}(M)$. As μ is J-invariant, for each b_i, $\mu(U_{ab_i}) = p$. So $\int_{\sigma \in \mathrm{Or}(M)} \chi_i(\sigma)\, \mathrm{d}\mu(\sigma) = p$.

Also, for $\sigma \in \mathrm{Or}(M)$, $\sum_{i \leq r} \chi_i(\sigma) \leq 2$, as a has at most 2 out-vertices in σ. So $\int_{\sigma \in \mathrm{Or}(M)} \sum_{i \leq r} \chi_i(\sigma)\, \mathrm{d}\mu(\sigma) \leq 2$, which implies that $rp \leq 2$. As r is an arbitrary positive integer, this is a contradiction. □

2.3 Amenable subgroups

The following is stated in [**8**] but not proved explicitly (the construction of σ in the below proof was rediscovered by the author of this thesis, though it is implied in [**8**] that the authors already know how to do this.)

THEOREM 2.5 ([**8**], Cor. 3.11). *$G = \mathrm{Aut}(M_F)$ has no coprecompact amenable subgroup.*

PROOF. Write $M = M_F$. For a contradiction, suppose J is a coprecompact amenable subgroup of G.

Define a binary relation σ on M by $\sigma = \{(a, b) \in E_M : J_a b \text{ is finite}\}$. By Lemma 2.4, σ is a direction of the edges of M, where we may possibly direct some edges in both directions (i.e. we may have $(a, b) \in \sigma, (b, a) \in \sigma$).

Take $(a, b) \in E_M, j \in J$. Then $J_{ja} = jJ_aj^{-1}$, so $J_{ja}jb = jJ_ab$, and thus $|J_{ja}jb| = |J_ab|$. So if $(a, b) \in E_M, (a', b') \in E_M$ lie in the same J-orbit, $|J_ab| = |J_{a'}b'|$. Specifically, if $(a, b) \in \sigma$, then for (a', b') in the J-orbit of (a, b), $|J_{a'}b'| = |J_ab| < \infty$. So $J \leq \mathrm{Aut}(M, \sigma)$.

Also as G is oligomorphic and J is a coprecompact subgroup of G, J has finitely many orbits on M^2.

Thus for $(a, b) \in \sigma$, there is a uniform bound k, independent of a, on the number of out-edges of a in σ. (We think of σ as a k-orientation where some edges are oriented in both directions.)

The rest of the proof follows that of Proposition 2.1, with σ replacing the 2-orientation. □

We will now combine Lemma 2.4, used in a similar way to the proof of Theorem 2.5, with the proof argument of Theorem 2.3 to obtain the new result in this chapter below.

THEOREM 2.6 (**). *$G = \mathrm{Aut}(M_0)$ has no coprecompact amenable subgroup.*

PROOF. Seeking a contradiction, say $J \leq G$ is coprecompact and amenable.

Let x, y be adjacent vertices of M_0 with $\{x, y\} \leq_s M_0$ (a single edge is 2-sparse and thus in $\mathcal{C}_0$), and let X be the G-orbit of $(x, y) \in {M_0}^2$.

For $(a, b) \in X$, a and b are adjacent, and so as J is amenable, by Lemma 2.4 at least one of J_ab, J_ba is finite. Define a binary relation σ on M_0 by $\sigma = \{(a, b) \in X : J_ab \text{ is finite}\}$. σ is a direction of the edges of X, where we may possibly direct some edges in both directions (i.e. we may have $(a, b) \in \sigma, (b, a) \in \sigma$).

Take $(a, b) \in X, j \in J$. Then $J_{ja} = jJ_aj^{-1}$, so $J_{ja}jb = jJ_ab$, and thus $|J_{ja}jb| = |J_ab|$. So if $(a, b) \in X, (a', b') \in X$ lie in the same J-orbit,

$|J_a b| = |J_{a'} b'|$. Specifically, if $(a, b) \in \sigma$, then for (a', b') in the J-orbit of (a, b), $|J_{a'} b'| = |J_a b| < \infty$. So $J \leq \mathrm{Aut}(M_0, \sigma)$.

Also, as J is coprecompact, X splits into finitely many J-orbits. Thus for $(a, b) \in \sigma$, there is a uniform bound k, independent of a, on the number of out-edges of a in σ.

Let x', y' be non-adjacent vertices of M_0 with $\{x', y'\} \leq_s M_0$, and let Y be the G-orbit of $(x', y') \in {M_0}^2$. As J is coprecompact, Y splits into finitely many J-orbits.

Take $m \in \mathbb{N}$. Take a rooted tree T of height $m + 1$, with each non-leaf vertex having $k + 1$ children (here, the terminology is as in the proof of Theorem 2.3). Let τ be the direction of T given by directing the tree "upside down" from the leaves to the head, i.e. if $a \in T$ has child vertex b, then $(b, a) \in \tau$. Each non-head vertex of T has out-degree 1 in τ and the head vertex has out-degree 0, so τ is a 2-orientation, and so $T \in \mathcal{C}_0$. We may take $T \leq_s M_0$. For $(b, a) \in \tau$, if a has a successor c in τ (i.e. $(a, c) \in \tau$), then we may reverse the orientation of (a, c) to produce a 2-orientation of T in which $\{a, b\}$ is successor-closed, and so $\{a, b\} \leq_s T \leq_s M$. Thus for ab an edge of T, $(a, b) \in X$.

As each non-leaf vertex $a \in T$ has $k + 1$ in-edges in τ and a has at most k out-edges in σ, a must have some τ-in-vertex b for which $(b, a) \in \sigma$. Therefore, (T, τ) contains an out-directed path of length $m + 1$, successor-closed in τ, where each out-edge also lies in σ. Label this path as $v = v_{-1}, v_0, \cdots, v_m$, where $(v_i, v_{i+1}) \in \tau$ for $i \geq -1$. We have that $\{v, v_i\} \leq_s M_0$ for $i \geq 1$ and so $(v, v_i) \in Y$, as we may simply reverse the orientation τ on $(v_{-1}, v_0), (v_i, v_i + 1)$ to produce an orientation of the path in which $\{v, v_i\}$ is successor-closed.

To sum up, for $m \in \mathbb{N}$ we have shown that there exists a $(m + 1)$-length path $v = v_{-1}, v_0, \cdots, v_m$ in M_0 with $(v_i, v_{i+1}) \in \sigma$ $(i \geq -1)$ and $(v, v_i) \in Y$ $(i \geq 1)$.

Define a function $s : \mathbb{Z}_+ \to \mathbb{Z}_+$ by $s(1) = 1$, $s(t + 1) = k^{s(t)+2}$. We claim that for $t \geq 1$, $m \geq s(t)$, we have that $v_1, \cdots, v_m$ lie in at least t different J_v-orbits. This will contradict Y splitting into finitely many J-orbits, meaning that G has no coprecompact amenable subgroup.

First, let $B(v, j)$ be the set consisting of vertices w which are reachable via an out-directed path in σ from v of length at most j $(j \in \mathbb{N})$. So $|B(v, j)| \leq 1 + k + \cdots + k^j < k^{j+1} - 1$. We then prove the claim by induction. Note that the case $t = 1$ is trivial. Seeking a contradiction, assume the claim is true for t, but not for $t+1$. Then $v_{s(t)+1}, \cdots, v_{s(t+1)}$ lie in the same J_v-orbits as $v_1, \cdots, v_{s(t)}$, and so as $J \leq \mathrm{Aut}(M_0, \sigma)$, $\{v_1, \cdots, v_{s(t+1)}\} \subseteq B(v, s(t)+1)$. But $s(t+1) = k^{s(t)+2}$ and $|B(v, s(t)+1)| < k^{s(t)+2} - 1$, contradiction. This completes the proof of the claim.

□

Chapter 3

Fixed points on type spaces

3.1 Introduction and definition

We investigate a weakening of extreme amenability, which we call the *fixed points on type spaces property* (FPT).

The following is folklore:

LEMMA 3.1. *Let M be an L-structure, and let $G = \mathrm{Aut}(M)$ with the pointwise convergence topology. Then G acts continuously on the Stone spaces $S_n(M)$, with the action given by*

$$g \cdot p(\bar{x}) = \{\phi(g\bar{m}, \bar{x}) : \phi(\bar{m}, \bar{x}) \in p(\bar{x})\}.$$

That is, $G \curvearrowright S_n(M)$ with the action defined above is a G-flow.

Note that we define the action of G on $L(M)$-formulae as

$$g \cdot \phi(\bar{m}, \bar{x}) = \phi(g\bar{m}, \bar{x}).$$

PROOF. We need to show that $g \cdot p(\bar{x})$ is a complete theory in $L(M) \cup \{\bar{x}\}$ containing the diagram $\mathrm{Th}_{L(M)}(M)$ of M. For consistency, by compactness we need only show the consistency of a finite collection of formulae $\{\phi_1(g\bar{m}, \bar{x}), \cdots, \phi_r(g\bar{m}, \bar{x})\}$ in $g \cdot p(\bar{x})$. We have that

$$g \cdot (\phi_1(\bar{m}, \bar{x}) \wedge \cdots \wedge \phi_r(\bar{m}, \bar{x})) = \phi_1(g\bar{m}, \bar{x}) \wedge \cdots \wedge \phi_r(g\bar{m}, \bar{x}),$$

so it suffices to show the consistency of $\phi(g\bar{m}, \bar{x})$ given $\phi(\bar{m}, \bar{x}) \in p(\bar{x})$. The type $p(\bar{x})$ is realised in some elementary extension $M' \succeq M$, so $M' \models (\exists\, \bar{x})\phi(\bar{m}, \bar{x})$ implies that $M \models (\exists\, \bar{x})\phi(\bar{m}, \bar{x})$, and thus $M \models \phi(\bar{m}, \bar{n})$ for some $\bar{n} \in M$. As automorphisms of structures preserve functions, relations and constants, $M \models \phi(g\bar{m}, g\bar{n})$, and so we have consistency (formally, this is by induction on the complexity of the formula).

Take a formula $\phi(\bar{m}, \bar{x})$. Either $\phi(g^{-1}\bar{m}, \bar{x})$ or $\neg\phi(g^{-1}\bar{m}, \bar{x})$ is in $p(\bar{x})$, and as the action of G commutes with Boolean connectives, we have that $g \cdot p(\bar{x})$ is a complete theory - it is clear that it contains the diagram of M.

To show that the action is continuous, take a basic open set

$$\langle \psi(\bar{m}, \bar{x}) \rangle = \{p(\bar{x}) : p(\bar{x}) \ni \psi(\bar{m}, \bar{x})\}$$

of $S_n(M)$. Then the preimage of this set under the action is

$$\{(g, p(\bar{x})) : g \cdot p(\bar{x}) \ni \psi(\bar{m}, \bar{x})\},$$

and taking g_0, $p_0(\bar{x})$ such that $g_0 \cdot p_0(\bar{x}) \ni \psi(\bar{m}, \bar{x})$, we have $p_0 \ni \psi(g_0^{-1}\bar{m}, \bar{x})$, and then $\{g : g^{-1}\bar{m} = g_0^{-1}\bar{m}\} \times \langle \psi(g_0^{-1}\bar{m}, \bar{x})\rangle$ is the open neighbourhood we seek. □

Definition 3.2 (**). Let M be an L-structure with automorphism group $G = \mathrm{Aut}(M)$. We say that G has the *fixed points on type spaces property* (FPT) if every subflow of $G \curvearrowright S_n(M)$, $n \geq 1$, has a fixed point.

Note that FPT is equivalent to every orbit closure $\overline{G \cdot p(\bar{x})}$ in $S_n(M)$ having a fixed point.

Remark 3.3. $p(\bar{x}) \in S_n(M)$ is a G-invariant type (i.e. a fixed point of the action) iff for all formulae $\phi(\bar{y}, \bar{x})$, $\{\bar{m} : \phi(\bar{m}, \bar{x}) \in p(\bar{x})\}$ is G-invariant, and for ω-categorical structures, this is equivalent to $p(\bar{x})$ being $\varnothing$-definable.

3.2 An example: the random graph.

In this section, let M denote the random graph (i.e. the Fraïssé limit of the class of finite graphs). We will prove that:

Theorem 3.4. *The random graph M has FPT for subflows of $S_1(M)$.*

In fact, it is a straightforward generalisation to show that the random graph has FPT for subflows of $S_n(M)$, $n \geq 1$, and therefore has FPT in the full sense. The full argument will appear in the published version of this material.

Before proving Theorem 3.4, we recall the following:

Proposition 3.5. *The random graph M is* indivisible*, i.e. for any 2-colouring of the vertices of M, there is a monochromatic copy of M.*

A very short proof of a stronger fact (the *pigeonhole property* of the random graph) may be found in [**4**].

Proof of Theorem 3.4. The random graph M is homogeneous in a finite relational language, and so is ω-categorical. Therefore M admits quantifier elimination. Let $G = \mathrm{Aut}(M)$.

Consider the space $S_1(M)$ of (1-)types of M. Given a G-invariant type $p(x)$, we can realise it in some elementary extension of M and use quantifier elimination to see that we just have a point a outside M which is adjacent to some of the vertices of M. If a were adjacent to a vertex of M and not adjacent to another, then by extending the local isomorphism sending the first vertex to the second to an automorphism of M (by homogeneity), we can see that $p(x)$ would not be G-invariant. Therefore the only G-invariant 1-types of M are $p_0(x)$, the type of the

point not adjacent to any vertices of M, and $p_1(x)$, the type of the point adjacent to all vertices of M.

From the above, we see that G has FPT for 1-types iff $p_0(x)$ or $p_1(x)$ belongs to every $\overline{Gp(x)}$. Let $p(x)$ be a type, realised as a point a outside M. Let X_0 be the set of vertices of M not adjacent to a, and let X_1 be the set of vertices of M adjacent to a. Then $p_1(x) \in \overline{Gp(x)} \Leftrightarrow$ for every basic open set $\langle f(\bar{m}, x)\rangle \ni p_1(x)$, there exists $g \in G$ with $gp(x) \in \langle f(\bar{m}, x)\rangle$, i.e. $f(g^{-1}\bar{m}, x) \in p(x)$. By quantifier elimination, we see that $f(\bar{m}, x)$ specifies a finite subgraph $A \subseteq M$ that the realisation point of $p_1(x)$ is adjacent to, and so $p_1(x) \in \overline{Gp(x)} \Leftrightarrow$ for all finite subgraphs $A \subseteq M$, there exists $g \in G$ with $gA \subseteq X_1$. Clearly a corresponding statement is true for $p_0(x)$ and X_0.

We can regard X_0 and X_1 as a colouring of M. Thus, FPT for 1-types is equivalent to the statement that

($*$) for every colouring $c : M \to \{0, 1\}$, there is a colour i such that for all finite $A \subseteq M$, there is a copy of A of colour i.

The statement ($*$) follows immediately from the indivisibility of the random graph.

□

3.3 An ω-categorical structure having no ω-categorical expansion with FPT

THEOREM 3.6 ($**$). *There is a countable ω-categorical structure M which does not have any ω-categorical expansion M' with* $\mathrm{Aut}(M')$ *having FPT, the fixed points on type spaces property.*

The ω-categorical structure M in the above theorem will be a particular case of the ω-categorical Hrushovski construction M_F (see Section 1.5).

Recall that, as stated after Lemma 1.53, throughout this thesis we assume that we have taken the control function F such that $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class, vertices and edges lie in $\mathcal{C}_F$ and are d-closed in M_F, and every vertex of M_F has infinite valency. The particular control function F used in the proof of Theorem 3.6 will satisfy additional conditions on top of this (these additional conditions only apply in this chapter).

The proof will depend on Proposition 2.1, which we recall below:

PROPOSITION 2.1. *Let M be an infinite 2-sparse graph in which all vertices have infinite valency. Let $G = \mathrm{Aut}(M)$.*

Consider the G-flow $G \curvearrowright \mathrm{Or}(M)$. If $J \leq G$ fixes a 2-orientation of M, then J has infinitely many orbits on M^2.

Before giving the details of the proof of Theorem 3.6, we first give an informal general outline.

For each orientation $\tau \in \mathrm{Or}(M_F)$, we will construct a 1-type $p_\tau(x) \in S_1(M_F)$ which "encodes" the orientation, by use of the label structure N_τ described below.

We will construct a G-morphism $u : S_1(M_F) \to 2^{{M_F}^2}$, with $u(p_\tau(x)) = \tau$ for $\tau \in \mathrm{Or}(M_F)$. We have that $\mathrm{Or}(M_F)$ is a subflow of $2^{{M_F}^2}$, and so $D = u^{-1}(\mathrm{Or}(M_F))$ is a subflow of $S_1(M_F)$.

Say we have an expansion M' of M_F with automorphism group $H = \mathrm{Aut}(M')$ having FPT. As there is a natural H-morphism $w : S_1(M') \to S_1(M_F)$ (where we forget the formulae that use relation symbols from the expanded language), $w^{-1}(D)$ is a subflow of the H-flow $S_1(M')$, and as H has FPT, H will fix a point of $w^{-1}(D)$. So via the H-morphism $u \circ w$, H will fix an orientation of M_F. By Theorem 2.1, H has infinitely many orbits on ${M_F}^2$, so is not oligomorphic, and thus by the Ryll-Nardzewski theorem (see chapter 4 of [**21**]) M' is not ω-categorical, concluding the proof of Theorem 3.6.

We now start the formal details of the proof of Theorem 3.6.

We begin with a description of the control function F and properties of the class $\mathcal{C}_F$.

Lemma 3.7 (*). *Let F be a control function for the class $\mathcal{C}_F$ satisfying the conditions of Definition 1.49, and additionally assume:*

- *F is strictly increasing;*
- *F is piecewise smooth, and its right derivative $F'(x)$ is decreasing;*
- *$F(1) = 2$, $F(2) = 3$;*
- *$F'(x) \leq \frac{2}{8x+1}$ for $x \geq 2$, where F' denotes the right derivative.*

Then:

(1) for $a \in M_F$, we have $a \in \mathcal{C}_F$ and $a \leq_d M_F$;
(2) for $ab \in \mathrm{E}(M_F)$, we have $ab \in \mathcal{C}_F$ and $ab \leq_d M_F$;
(3) $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class;
(4) if $a_0 a_1 \cdots a_{n-1} \subseteq M_F$ is a path, then $a_0 a_1 \cdots a_{n-1} \in \mathcal{C}_F$;
(5) $F(4) < 4$, $F(5) < 4, F(6) < 4$;
(6) if $abcd \subseteq M_F$ is a 4-cycle, then $abcd \in \mathcal{C}_F$.

Proof.

(1) This follows from Lemma 1.53.
(2) This also follows from Lemma 1.53.
(3) Take $A, B_1, B_2 \in \mathcal{C}_F$ with $A \leq_d B_1, B_2$. Then as $F'(x) \leq \frac{2}{8x+1} < 1/x$ for $x \geq 2$, Lemma 1.52 states that the free amalgam of B_1, B_2 over A lies in $\mathcal{C}_F$ if $|B_1| \geq 2$ or $|B_2| \geq 2$. The

only remaining possibility is $|B_1|, |B_2| \leq 1$, and the only non-trivial case is where $A = \varnothing$: if b_1, b_2 are non-adjacent points then $\delta(\{b_1, b_2\}) = 4 > F(2)$. So $(\mathcal{C}_F, \leq_d)$ is a free amalgamation class.

(4) Proceed by induction, and obtain $a_0 \cdots a_{n-1} \in \mathcal{C}_F$ by the free amalgamation of $a_0 \cdots a_{n-2}$, $a_{n-2}a_{n-1}$ over a_{n-2}.

(5) F is strictly increasing, and so it suffices to show $F(6) < 4$. $F(6) \leq F(2) + \int_2^6 \frac{2}{8x+1}\,\mathrm{d}x = 3 + \frac{1}{4}\log(49) - \frac{1}{4}\log(17) < 4$.

(6) Let $abcd \subseteq M_F$ be a 4-cycle. Then $\delta(abcd) = 4 > F(4)$. For $C \subsetneq abcd$, C either consists of a path of length 2, an edge, two non-adjacent points or a single point. All of these lie in $\mathcal{C}_F$.

□

Throughout the rest of this chapter, we will assume F is a control function satisfying the conditions of Lemma 3.7.

Note that control functions satisfying the conditions of the above lemma exist: take F piecewise linear with $F(0) = 0$, $F(1) = 2$, $F(2) = 3$, and then for $x \geq 2$ define $F(x) = \frac{1}{4}\log(8x + 1) + 3 - \frac{1}{4}\log(17)$.

We now describe how to encode orientations of M_F as 1-types.

Take an orientation $\tau \in \mathrm{Or}(M_F)$. Define a graph N_τ with distinguished point c as follows:

- N_τ includes M_F as a substructure (in the language of graphs);
- add a new vertex c to N_τ, with $c \notin M_F$;
- for $(a, b) \in \tau$ (i.e. the edge ab is oriented from a to b in the orientation τ), add to N_τ four new vertices

$$l_1^{(a,b)}, l_2^{(a,b)}, l_3^{(a,b)}, l_4^{(a,b)}$$

and new edges

$$cl_1^{(a,b)}, l_1^{(a,b)}l_2^{(a,b)}, l_2^{(a,b)}l_3^{(a,b)}, l_3^{(a,b)}l_4^{(a,b)}, l_4^{(a,b)}l_1^{(a,b)},$$

and attach two edges from $l_2^{(a,b)}, l_4^{(a,b)}$ to a (the "start vertex") and one edge from $l_3^{(a,b)}$ to b (the "end vertex").

The resulting structure N_τ is depicted in Figure 3.1. For $(a, b) \in \tau$, let $L_{(a,b)} = \{c, l_1^{(a,b)}, l_2^{(a,b)}, l_3^{(a,b)}, l_4^{(a,b)}\}$. Informally, each $(a, b) \in \tau$ has its orientation labelled by $L_{(a,b)}$. So we have that

$$N_\tau = \bigcup\{L_{(a,b)} : (a, b) \in \tau\} \cup M_F,$$

and the $L_{(a,b)}$ intersect only in c.

Let $A_0 \leq_d A_1 \leq_d \cdots$ be an increasing $\leq_d$-chain with $M_F = \cup_{i\in\mathbb{N}} A_i$. Let $L_i = \bigcup\{L_{(a,b)} : (a, b) \in \tau|_{A_i}\}$, and let $B_i = A_i \cup L_i$, regarding B_i as a substructure of N_τ.

Lemma 3.8 (**). *For $i \in \mathbb{N}$, we have $A_i \leq_d B_i$.*

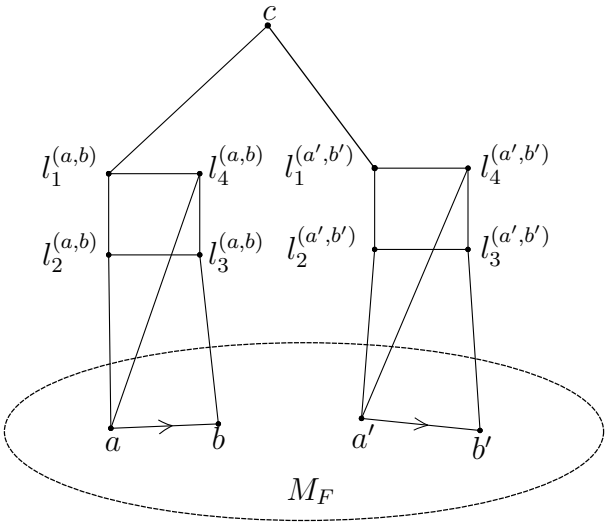


FIGURE 3.1

PROOF. Take $i \in \mathbb{N}$ and write $A = A_i$, $B = B_i$. For $A \subsetneq C \subseteq B$, we need to show $\delta(C) > \delta(A)$.

First consider the case where A consists of a single edge ab, with $(a, b) \in \tau$. Then, suppressing subscripts for notational convenience, we have $B = \{c, l_1, l_2, l_3, l_4, a, b\}$. We calculate the relative predimension of some $A \subsetneq C \subseteq B$ in the table below.

$C - A$	$\delta(C/A)$
l_2	1
l_3	1
l_4	1
l_1, l_2	2
l_1, l_4	2
l_2, l_3	1
l_3, l_4	1
c, l_1	3
l_1, l_2, l_3	2
l_1, l_2, l_4	2
l_1, l_3, l_4	2
l_2, l_3, l_4	1
l_1, l_2, l_3, l_4	1
c, l_1, l_2, l_3, l_4	2

The remaining cases result from free amalgamations over A, and so also have positive predimension (if X, Y are freely amalgamated over Z, then $\delta((X \cup Y)/Z) = \delta(X/Z) + \delta(Y/Z)$). The remaining cases are where $C - A$ is equal to $\{l_1\}, \{c\}, \{l_1, l_3\}, \{l_2, l_4\}$ or $\{c\} \cup X$, where $X \subseteq \{l_2, l_3, l_4\}$.

Now consider general $A \leq_d M_F$. Given $A \subsetneq C \subseteq B$, the vertices of C consist of A together with subsets $J_{(a,b)}$ of $L_{(a,b)}$ for each $(a,b) \in \tau|_A$. For $(a,b) \in \tau|_A$, let $J'_{(a,b)} = J_{(a,b)} \cup A$.

If $c \notin C$, then the $J'_{(a,b)}$ are freely amalgamated over A, and so from the single-edge case we see that $\delta(C/A) > 0$.

We now consider the case where $c \in C$. If $l_1^{(a,b)} \notin J_{(a,b)}$ for all $(a,b) \in \tau|_A$, then C consists of a vertex c with no neighbours together with a free amalgamation over A of each of the $J'_{(a,b)} - \{c\}$, $(a,b) \in \tau|_A$. So, from the single-edge case and the fact that $\delta(\{c\}) = 2$, we have that $\delta(C/A) > 0$.

If $c \in C$ and there exists $(a',b') \in \tau|_A$ with $l_1^{(a',b')} \in J_{(a',b')}$, then C is a free amalgamation over A of each of the $J'_{(a,b)} - \{c\}$, $(a,b) \in \tau|_A$, for which $l_1^{(a,b)} \notin J_{(a,b)}$, together with $\bigcup\{J'_{(a,b)} : (a,b) \in \tau|_A,\ l_1^{(a,b)} \in J_{(a,b)}\}$. Therefore we need only consider the case where $l_1^{(a,b)} \in J_{(a,b)}$ for all $(a,b) \in \tau|_A$. The single-edge calculation shows that $\delta(J_{(a,b)} - \{c\}/A) \geq 1$ for each $J_{(a,b)}$, and these $J'_{(a,b)} - \{c\}$ are freely amalgamated over A. Each addition of an edge $l_1^{(a,b)}c$ reduces the predimension by one, but the single addition of the vertex c adds two to the predimension, so in total $\delta(C/A) > 0$. □

Lemma 3.9 (**). *For $(a,b) \in \tau$, we have $\{a, b, l_1^{(a,b)}, l_2^{(a,b)}, l_3^{(a,b)}, l_4^{(a,b)}\} \in \mathcal{C}_F$ and $L_{(a,b)} \in \mathcal{C}_F$, where these structures are considered as substructures of N_τ.*

Proof. We write l_1, l_2, l_3, l_4, suppressing superscripts.

To show that $\{a, b, l_1, l_2, l_3, l_4\} \in \mathcal{C}_F$, we consider each subset $C \subseteq \{a, b, l_1, l_2, l_3, l_4\}$ and show that $\delta(C) \geq F(|C|)$. To speed up the process of checking each subset C, in the below table we show that certain subsets $C \subseteq \{a, b, l_1, l_2, l_3, l_4\}$ lie in $\mathcal{C}_F$, and therefore every $C' \subseteq C$ must satisfy $\delta(C') \geq F(|C'|)$.

C	Proof that $C \in \mathcal{C}_F$
$l_1l_2l_3l_4, l_2l_3ab, l_3l_4ab, l_1l_2l_4a, l_2l_3l_4a$	C is a 4-cycle
$l_1l_2l_3ab$	free amalgam of l_2l_3ab, l_1l_2 over l_2
$l_1l_3l_4ab$	free amalgam of l_3l_4ab, l_1l_4 over l_4
$l_1l_2l_4ab$	free amalgam of $l_1l_2l_4a$, ab over a
$l_1l_2l_3l_4b$	free amalgam of $l_1l_2l_3l_4$, l_3b over l_3

We now check the remaining subsets $C \subseteq \{a, b, l_1, l_2, l_3, l_4\}$ by directly calculating the predimension:

C	$\delta(C)$	$F(\lvert C\rvert)$
$l_2l_3l_4ab$	4	$F(5) < 4$
$l_1l_2l_3l_4a$	4	$F(5) < 4$
$l_1l_2l_3l_4ab$	4	$F(6) < 4$

In the above we showed that $\{a, b, l_1, l_2, l_3, l_4\} \in \mathcal{C}_F$, and so for the second part of the lemma, we obtain $L_{(a,b)} \in \mathcal{C}_F$ via the free amalgam of $L_{(a,b)}$ and cl_1 over l_1 (recalling that we have defined our control function F so that points are always d-closed). □

LEMMA 3.10 (**). *For $i \in \mathbb{N}$, we have that $B_i \in \mathcal{C}_F$.*

PROOF. Take $i \in \mathbb{N}$ and write $A = A_i$, $B = B_i$. For a finite graph X, recall that we write $|X|$ and $|E(X)|$ for the number of vertices and edges of X respectively.

We have to show that $\delta(C) \geq F(|C|)$ for $C \subseteq B$. The vertices of C consist of $C \cap A$ together with subsets $J_{(a,b)}$ of $L_{(a,b)}$ for each $(a, b) \in \tau|_A$ (some of these $J_{(a,b)}$ may be empty). For $(a, b) \in \tau|_A$, let $J'_{(a,b)} = J_{(a,b)} \cup (C \cap A)$.

First we consider the case where $c \notin C$. C is then the free amalgam of the $J'_{(a,b)}$, $(a, b) \in \tau|_A$, over $C \cap A$. Given that $\mathcal{C}_F$ is a free amalgamation class and $C \cap A \leq_d C$, it therefore suffices to show that $J'_{(a,b)} \in \mathcal{C}_F$ for $(a, b) \in \tau|_A$. Fix $(a, b) \in \tau|_A$. To show that $J'_{(a,b)} \in \mathcal{C}_F$, as $J'_{(a,b)}$ is a free amalgam of $J_{(a,b)} \cup (\{a, b\} \cap C)$ and $C \cap A \in \mathcal{C}_F$ over $\{a, b\} \cap C \in \mathcal{C}_F$, it suffices to show that $J_{(a,b)} \cup (\{a, b\} \cap C)$ lies in $\mathcal{C}_F$, and we have already checked this in Lemma 3.9.

Now we consider the case where $c \in C$. If $l_1^{(a,b)} \notin J_{(a,b)}$ for each $(a, b) \in \tau|_A$, then C consists of a vertex c with no neighbours together with the free amalgam over $C \cap A$ of each $J'_{(a,b)} - \{c\}$, $(a, b) \in \tau|_A$, and so we are done by the first case in the previous paragraph. Otherwise, C is the free amalgam over $C \cap A$ of $\bigcup\{J'_{(a,b)} : l_1^{(a,b)} \in J_{(a,b)}, (a, b) \in \tau|_A\}$ with each $J'_{(a,b)} - \{c\}$ for which $l_1^{(a,b)} \notin J_{(a,b)}$, and so using the first case considered above we may reduce to the case where each non-empty $J_{(a,b)}$ contains $l_1^{(a,b)}$.

Similarly, we may exclude the case where C contains sets $J_{(a,b)}$ for which $J_{(a,b)} = \{c, l_1^{(a,b)}, l_3^{(a,b)}\}$, as C is the free amalgam over $C \cap A$ of

$$\bigcup\{J'_{(a,b)} : (a, b) \in \tau|_A, J_{(a,b)} \neq \{c, l_1^{(a,b)}, l_3^{(a,b)}\}\} \cup$$
$$\bigcup\{\{c, l_1^{(a,b)}\} \cup (C \cap A) : J_{(a,b)} = \{c, l_1^{(a,b)}, l_3^{(a,b)}\}\}$$

with each $\{l_3^{(a,b)}\} \cup (C \cap A)$ (in $\mathcal{C}_F$ by Lemma 3.9) for which $J_{(a,b)} = \{c, l_1^{(a,b)}, l_3^{(a,b)}\}$. We may likewise free amalgamate over c to exclude the

cases where C contains sets $J_{(a,b)}$ for which $J_{(a,b)} = \{c, l_1^{(a,b)}\}$, or for which $J_{(a,b)}$ is any subset of $L_{(a,b)}$ but $a, b \notin C \cap A$.

So, the case remaining is where C consists of $C \cap A$ together with sets $J_{(a,b)}$ containing $c, l_1^{(a,b)}$ and at least one of $l_2^{(a,b)}, l_4^{(a,b)}$, where each $J_{(a,b)}$ has some edge to $C \cap A$. We need to show that $\delta(C) \geq F(|C|)$.

We now calculate the relative predimension over $A \cup \{c\}$ of each remaining possible $J_{(a,b)} \cup X$, $X \subseteq \{a, b\}$, in the following table, where we label each structure as Y_i, $1 \leq i \leq 11$:

$J_{(a,b)} \cup X$	Label	$\delta(J_{(a,b)} \cup X / A \cup \{c\})$
cl_1l_2a	Y_1	1
cl_1l_4a	Y_2	1
$cl_1l_2l_3a$	Y_3	2
$cl_1l_2l_3b$	Y_4	2
$cl_1l_2l_3ab$	Y_5	1
$cl_1l_3l_4a$	Y_6	2
$cl_1l_3l_4b$	Y_7	2
$cl_1l_3l_4ab$	Y_8	1
$cl_1l_2l_3l_4a$	Y_9	1
$cl_1l_2l_3l_4b$	Y_{10}	2
$cl_1l_2l_3l_4ab$	Y_{11}	0

We write k_i for how many times Y_i occurs in C. We also write $\delta_i = \delta(Y_i / A \cup \{c\})$. Let $\lambda_i = |\{l_1, l_2, l_3, l_4\} \cap Y_i|$.

Then, recalling that the vertex c also adds 2 to the predimension, we have that

$$\delta(C) = \sum_{1 \leq i \leq 11} \delta_i k_i + 2 + \delta(C \cap A).$$

Now,

$$\begin{aligned} F(|C|) &= F(1 + |C \cap A| + \sum_{1 \leq i \leq 11} \lambda_i k_i) \\ &\leq F(1 + |C \cap A| + 4 \sum_{1 \leq i \leq 11} k_i) \\ &= F(1 + |C \cap A| + 4(k_4 + k_7 + k_{10}) + 4 \sum_{i \leq 11,\, i \notin \{4,7,10\}} k_i). \end{aligned}$$

As $\tau|_{C \cap A}$ is a 2-orientation, we have that each $a \in C \cap A$ can have at most two label structures with a as the starting vertex (i.e. with edges to a from l_2, l_4), and so

$$\sum_{i \leq 11,\, i \notin \{4,7,10\}} k_i \leq 2|C \cap A|.$$

So

$$F(|C|) \leq F(8|C \cap A| + 4(k_4 + k_7 + k_{10}) + 1 + |C \cap A|).$$

As $F(u+v) \leq F(u) + vF'(u)$ and $F'(x) \leq \frac{2}{8x+1}$ for $x \geq 2$, we have that if $|C \cap A| \geq 2$, then

$$\begin{aligned} F(|C|) &\leq F(|C \cap A|) + \frac{2}{8|C \cap A| + 1}(8|C \cap A| + 4(k_4 + k_7 + k_{10}) + 1) \\ &< F(|C \cap A|) + 2 + k_4 + k_7 + k_{10} \\ &\leq \delta(C). \end{aligned}$$

If $|C \cap A| = 1$, then

$$\begin{aligned} F(|C|) &\leq F(1 + |C \cap A|) + (8|C \cap A| + 4(k_4 + k_7 + k_{10}))F'(1 + |C \cap A|) \\ &= 3 + \frac{2}{8 \cdot 2 + 1}(8|C \cap A| + 4(k_4 + k_7 + k_{10})) \\ &< 4 + \frac{8}{17}(k_4 + k_7 + k_{10}) \\ &\leq \delta(C) \end{aligned}$$

(as $\delta(C \cap A) = 2$). □

Lemma 3.11 (**). *For $i \in \mathbb{N}$, we have $B_i \leq_d B_{i+1}$.*

Proof. Take $B_i \subsetneq C \subseteq B_{i+1}$. Then, by counting vertices and edges and using the definition of the predimension δ,

$$\delta(C/B_i) = \delta(C \cap A_{i+1}/B_i) + \delta(C - A_{i+1}/B_i \cup (C \cap A_{i+1})).$$

We have that $\delta(C \cap A_{i+1}/B_i) = \delta(C \cap A_{i+1}/A_i)$, and as $A_i \leq_d A_{i+1}$, if $C \cap A_{i+1} \neq \varnothing$ then $\delta(C \cap A_{i+1}/A_i) > 0$.
Also

$$\delta(C - A_{i+1}/B_i \cup (C \cap A_{i+1})) \geq \delta(C - A_{i+1}/A_{i+1} \cup \{c\}),$$

as we are just adding extra edges to A_{i+1}. From the calculations in the proof of Lemma 3.8, we see that $\delta(C - A_{i+1}/A_{i+1} \cup \{c\}) \geq 0$, and so if $C \cap A_{i+1} \neq \varnothing$, then $\delta(C/B_i) > 0$.
If $C \cap A_{i+1} = \varnothing$, then $(C - B_i) \cup \{c\}$ is a substructure of $\bigcup\{L_{(a,b)} : (a,b) \in \tau|_{A_{i+1}}\}$, which consists of disjoint 4-cycles $\{l_1^{(a,b)}, l_2^{(a,b)}, l_3^{(a,b)}, l_4^{(a,b)}\}$ for $(a,b) \in \tau|_{A_{i+1}}$, with each 4-cycle having an extra edge $l_1^{(a,b)}c$. It is then a straightforward calculation to see that any substructure of $\bigcup\{L_{(a,b)} : (a,b) \in \tau|_{A_{i+1}}\}$ has positive predimension over c, so $\delta(C/B_i) > 0$. □

For $a, b \in M_F$, we define the *label formula* $f(a,b,x)$ in the language $L(M_F)$ with variable x to be:

$$f(a,b,x) \equiv (x \neq a \wedge x \neq b \wedge a \neq b \wedge a \sim b) \wedge$$
$$(\exists\, l_1, l_2, l_3, l_4)((\bigwedge_{i<j} l_i \neq l_j) \wedge (\bigwedge_{i} l_i \neq x \wedge l_i \neq a \wedge l_i \neq b) \wedge$$
$$(x \sim l_1 \wedge l_1 \sim l_2 \wedge l_2 \sim l_3 \wedge l_3 \sim l_4 \wedge l_4 \sim l_1 \wedge l_2 \sim a \wedge l_4 \sim a \wedge l_3 \sim b)).$$

Informally, $f(a,b,x)$ will test if (a,b) has a label structure $L_{(a,b)}$ attached with $x = c$.
Define a map $u : S_1(M_F) \to 2^{{M_F}^2}$ by

$$u(p(x)) = \{(a,b) \in {M_F}^2 : f(a,b,x) \in p(x)\}.$$

Note that we will often use subset notation when formally we in fact mean the characteristic function of that subset within ${M_F}^2$.

Lemma 3.12 (**). *The map u is a G-flow morphism.*

Proof. First we show G-equivariance.

$$\begin{aligned} u(g \cdot p(x)) &= u(\{\phi(g\overline{m}, x) : \phi(\overline{m}, x) \in p(x)\}) \\ &= \{(a,b) \in {M_F}^2 : f(g^{-1}a, g^{-1}b, x) \in p(x)\} \\ &= g \cdot \{(c,d) \in {M_F}^2 : f(c,d,x) \in p(x)\}. \end{aligned}$$

Now we show continuity. Let $S \subseteq 2^{{M_F}^2}$ be a subbasic open set, i.e. for some $(a,b) \in {M_F}^2$,

$$S = \{\psi \in 2^{{M_F}^2} : \psi((a,b)) = 1\}$$

or

$$S = \{\psi \in 2^{{M_F}^2} : \psi((a,b)) = 0\}.$$

If $S = \{\psi \in 2^{{M_F}^2} : \psi((a,b)) = 1\}$, then

$$u^{-1}(S) = \{p(x) : f(a,b,x) \in p(x)\},$$

an open set in the Stone space $S_1(M_F)$.
If $S = \{\psi \in 2^{{M_F}^2} : \psi((a,b)) = 0\}$, then

$$u^{-1}(S) = \{p(x) : f(a,b,x) \notin p(x)\} = \{p(x) : \neg f(a,b,x) \in p(x)\},$$

also an open set. □

Proposition 3.13 (**). *For $\tau \in \mathrm{Or}(M_F)$, there exists $p_\tau(x) \in S_1(M_F)$ with $u(p_\tau(x)) = \tau$.*

Proof. From Lemma 3.11, we have that N_τ is a union of the chain of d-closed structures $B_0 \leq_d B_1 \leq_d \cdots$. Let $N = M_F$. We will construct a $\leq_d$-embedding $r : N_\tau \to N$ using a standard "forth" proof (only the forward direction of the back and forth construction).

N is the Fraïssé limit of $\mathcal{C}_F$, and as such has $\mathrm{Age}_{\leq_d}(N) = \mathcal{C}_F$ and the extension property.

We construct r by induction. As $B_0 \in \mathcal{C}_F$, there exists a $\leq_d$-embedding $r_0 : B_0 \to N$. For the inductive step, assume we have already constructed compatible $\leq_d$-embeddings $r_i : B_i \to N$ for $i \leq k$. Then by the extension property, there exists a $\leq_d$-embedding $r_{k+1} : B_{k+1} \to N$ which extends r_k. This completes the inductive definition of the r_k, $k \in \mathbb{N}$. Let $r = \bigcup_{k\in\mathbb{N}} r_k$. Then r is a $\leq_d$-embedding $N_\tau \to N$.

We now show that $r(M_F) \preceq N$ using Tarski's test (see Prop 2.3.5 of [**21**]). It suffices to show that for $\overline{m} \in r(M_F)^k$, $n \in N$, there exists $g \in \mathrm{Aut}(N)$ with $g\overline{m} = \overline{m}$, $gn \in r(M_F)$.

Let $X = \mathrm{cl}^d_{r(M_F)}(\overline{m})$, and let $Y = \mathrm{cl}^d_N(X \cup \{n\})$. Then $Y \in \mathcal{C}_F$, and so using the extension property of $r(M_F)$, there is a $\leq_d$-embedding $s : Y \to r(M_F)$ extending the inclusion $X \subseteq r(M_F)$. We have that $s(Y) \leq_d r(M_F)$, so $s(Y) \leq_d N$ as r is a $\leq_d$-embedding, and we can then extend the $\leq_d$-local isomorphism $s : Y \to s(Y)$ to $g \in \mathrm{Aut}(N)$. We have that g is the identity on X and $gn \in r(M_F)$, so g satisfies the conditions we require.

We now have a $\leq_d$-embedding $r : N_\tau \to N$, such that $r(M_F) \preceq N$. We may therefore construct a structure N' with $N_\tau \subseteq N'$, with the set inclusion $N_\tau \hookrightarrow N'$ being a $\leq_d$-embedding, and with $M_F \subseteq N_\tau$ as an elementary substructure $M_F \preceq N'$. We do this purely for notational convenience.

We now have $c \in N_\tau \leq_d N'$. Let $p_\tau(x) = \mathrm{tp}_{N'}(c/M_F)$. As $M_F \preceq N'$, we have $p_\tau(x) \in S_1(M_F)$. From the construction of N_τ, we have that $\tau \subseteq u(p_\tau(x))$. To see that $u(p_\tau(x)) \subseteq \tau$, note that it suffices to show that for $(a, b) \in M_F^2$, if $N' \models f(a, b, c)$ then the l_i, $1 \leq i \leq 4$, that $f(a, b, c)$ specifies must lie in $\mathrm{cl}^d_{N'}(\{a, b, c\})$, and therefore in N_τ, as $N_\tau \leq_d N'$. We show that $\{l_1, l_2, l_3, l_4\} \subseteq \mathrm{cl}^d_{N'}(\{a, b, c\})$ in the table below.

X/Y	$\delta(X/Y)$
$l_1, l_2, l_3, l_4/a, b, c$	0
$l_1, l_2, l_3/l_4, a, b, c$	-1
$l_1, l_2, l_4/l_3, a, b, c$	-1
$l_1, l_3, l_4/l_2, a, b, c$	-1
$l_2, l_3, l_4/l_1, a, b, c$	-1
$l_1, l_2/l_3, l_4, a, b, c$	-1
$l_1, l_3/l_2, l_4, a, b, c$	-2
$l_1, l_4/l_2, l_3, a, b, c$	-1
$l_2, l_3/l_1, l_4, a, b, c$	-1
$l_2, l_4/l_1, l_3, a, b, c$	-2
$l_3, l_4/l_1, l_2, a, b, c$	-1
$l_1/l_2, l_3, l_4, a, b, c$	-1
$l_2/l_1, l_3, l_4, a, b, c$	-1
$l_3/l_1, l_2, l_4, a, b, c$	-1
$l_4/l_1, l_2, l_3, a, b, c$	-1

This completes the proof of Proposition 3.13. □

Proof of Theorem 3.6. Let M' be an expansion of M_F whose automorphism group $H = \mathrm{Aut}(M')$ has FPT.

We have a natural H-morphism $w : S_1(M') \to S_1(M_F)$ given by $w(p(x)) = \{\phi(x) \in p(x) : \phi(x) \text{ is a formula in the language } L(M_F)\}$.

$u : S_1(M_F) \to 2^{{M_F}^2}$ is a G-morphism, and $\mathrm{Or}(M_F)$ is a subflow of $2^{{M_F}^2}$. So $w^{-1}u^{-1}(\mathrm{Or}(M_F))$ is a subflow of the H-flow $S_1(M')$, and thus has an H-fixed point $p(x)$. Therefore, as $u \circ w$ is an H-morphism, $(u \circ w)(p(x))$ is an H-fixed point, i.e. H fixes an orientation of M_F. By Theorem 2.1, H has infinitely many orbits on ${M_F}^2$, and so is not oligomorphic. Therefore M' is not ω-categorical, by the Ryll-Nardzewski theorem. □

Chapter 4

Linear orders and orientations on M_1

In this chapter, we investigate the G_1-flow $\mathcal{LO}(M_1)$ of linear orders on M_1 (where $G_1 = \mathrm{Aut}(M_1)$). The main theorem of this chapter is Theorem 4.18, which states that minimal subflows of $\mathcal{LO}(M_1)$ have all G_1-orbits meagre. This complements results in Section 5 of [**8**], which show that minimal subflows of $\mathrm{Or}(M)$ ($M = M_1, M_0, M_F$) have all G-orbits meagre - in fact the proof for $\mathcal{LO}(M_1)$ is strongly inspired by the proof for $\mathrm{Or}(M_0)$.

The main tool here is a Ramsey expansion $(\mathcal{E}_1, \sqsubseteq_s)$ of $(\mathcal{C}_1, \leq_1)$, the class of *admissibly ordered orientations* of graphs in $\mathcal{C}_1$ ([**9**]). Writing (N_1, α) for the Fraïssé limit of $(\mathcal{E}_1, \sqsubseteq_s)$, where N_1 is the oriented graph and α is the order of the Fraïssé limit, and also writing $H_1 = \mathrm{Aut}(N_1, \alpha)$, we have that H_1 is extremely amenable. Given a minimal subflow $Y \subseteq \mathcal{LO}(M_1)$, H_1 therefore fixes a point β of Y, and by minimality we have that Y is the G_1-orbit-closure of β. Via H_1-automorphisms, we then use information about α to give us some knowledge of β, and then we use the linear order β to force certain orientations and prove Theorem 4.18. This interplay between linear orders and orientations will be typical of the results in this chapter.

After proving Theorem 4.18, in Section 4.5 we then find out more information about β: we show, independently of Theorem 4.18, that H_1, equal to the G_1-stabiliser of α in the G_1-flow $\mathcal{LO}(M_1)$, is also the G_1-stabiliser of β. This was in fact the author's original approach in an unsuccessful attempt to prove Theorem 4.18 - though the author has not been able to use this result on stabilisers to prove the main theorem, it is still of interest. We then conclude the chapter with an example of a minimal subflow of $\mathcal{LO}(M_1)$.

4.1 Orientations on M_1

We reprove the below result from [**7**], as later proofs in this chapter of the Expansion Property will have a similar proof method.

PROPOSITION 4.1 ([**7**], Th. 6.1). $\mathrm{Or}(M_1)$ *is a minimal* G_1*-flow.*

PROOF. Note that, by Lemma 1.66, $\mathcal{D}_1$ is a reasonable expansion of $(\mathcal{C}_1, \leq_1)$, and also recall that $\mathrm{Or}(M_1) = X(\mathcal{D}_1)$ (see Definition 1.70). Thus, to show that $\mathrm{Or}(M_1)$ is a minimal G_1-flow, by Lemma 1.73 it

suffices to show that $\mathcal{D}_1$ has the expansion property over $(\mathcal{C}_1, \leq_1)$, i.e. for $(A, \tau_A) \in \mathcal{D}_1$, there exists $B \in \mathcal{C}_1$ such that for any expansion $(B, \tau_B) \in \mathcal{D}_1$, there is a $\leq_1$-embedding $f : (A, \tau_A) \to (B, \tau_B)$. (Here we use the second paragraph of Definition 1.72.)

We proceed by induction on $|A|$. The base case $|A| = 1$ is trivial.

As τ_A is acyclic, there exists $a \in A$ with no in-edge. Let $X = A - \{a\}$, and let $\tau_X = \tau_A|_X$. Then $(X, \tau_X) \sqsubseteq_s (A, \tau_A)$, so $(X, \tau_X) \in \mathcal{D}_1$ and $X \leq_1 A$, so by the induction assumption there exists $Y \in \mathcal{C}_1$ such that any expansion of Y in $\mathcal{D}_1$ contains a $\leq_1$-copy of (X, τ_X).

Let $X_1, \cdots, X_n$ be the $\leq_1$-copies of X in Y. Let $Y_0 = Y$, and inductively define Y_i $(1 \leq i \leq n)$ to be the free amalgam of Y_{i-1} with 5 copies of A over X_i (where we take each copy A' of A with $X_i \leq A'$). Let $Z = Y_n$. As $\mathcal{C}_1$ is a free amalgamation class, $Z \in \mathcal{C}_1$, $Y_i \leq_1 Z$ $(0 \leq i \leq n)$, and each copy of A in the inductive sequence of free amalgamations is $\leq_1$-closed in Z. We will show that Z witnesses the expansion property for (A, τ_A).

Let (Z, τ_Z) be an expansion of Z in $\mathcal{D}_1$. As τ_Z induces an acyclic 2-orientation on any subgraph of Z, $(Y, \tau_Z) \in \mathcal{D}_1$, and so there is a $\leq_1$-embedding $f : (X, \tau_X) \to (Y, \tau_Z)$. $f(X)$ is equal to some X_i, and so as τ_Z is a 2-orientation and Z contains 5 copies of A freely amalgamated over X_i, there is a $\leq_1$-copy A' of A in Z for which $X_i \sqsubseteq_s A'$. We may then extend f to an isomorphism $(A, \tau_A) \to (A', \tau_Z)$, and as $A' \leq_1 Z$, f is a $\leq_1$-embedding into (Z, τ_Z). □

4.2 $\mathcal{LO}(M_1)$ is not minimal

Before investigating minimal subflows of $\mathcal{LO}(M_1)$, we first check that $\mathcal{LO}(M_1)$ is not in fact minimal itself.

PROPOSITION 4.2 (**). *$\mathcal{LO}(M_1)$ is not a minimal flow.*

PROOF. Let $\mathcal{Q}_1$ denote the class of ordered graphs (A, γ) where $A \in \mathcal{C}_1$ and the linear order γ induces an acyclic 2-orientation τ_γ on A, i.e. $\tau_\gamma = \{(x, y) \in E_A : x >_\gamma y\}$ is an acyclic 2-orientation.

We will show that $\mathcal{Q}_1$ is a reasonable class of expansions of $(\mathcal{C}_1, \leq_1)$ (see Definition 1.63).

For part (1) of reasonableness, take $A \in \mathcal{C}_1$, and let τ be an acyclic 2-orientation of A. Let $\gamma_0 = \{(x, y) \in A^2 : x \neq y$ and there exists an out-path from y to x in $\tau\}$. Then γ_0 is irreflexive and transitive, and as τ is acyclic, γ_0 is antisymmetric. So γ_0 is a strict partial order. Extend γ_0 arbitrarily to a linear order γ on A. Let $\tau_\gamma = \{(a, b) \in E_A : a >_\gamma b\}$. Then $\tau_\gamma = \tau$, so $(A, \gamma) \in \mathcal{Q}_1$, and so $\mathcal{C}_1$ is the class of graph reducts of $\mathcal{Q}_1$.

Part (2) of reasonableness follows from the fact that there are only finitely many linear orders on a finite set, and part (3) results from the fact that a sub-digraph of an acyclic 2-orientation is still an acyclic 2-orientation.

For part (4), take $A, B \in \mathcal{C}_1$, $A \leq_1 B$, with $(A, \gamma_A) \in \mathcal{Q}_1$. Let τ_A be the acyclic 2-orientation induced by γ_A on A. As $A \leq_1 B$, there exists an acyclic 2-orientation τ_B of B extending τ_A. Let $\gamma_0 = \{(x, y) \in B^2 : x \neq y$ and there exists an out-path from y to x in $\tau_B\}$. Then as before, γ_0 is a strict partial order. γ_A and γ_0 are compatible, and so we may extend the partial order $\gamma_A \cup \gamma_0$ arbitrarily to a linear order γ_B on B. Then γ_B induces τ_B, so $(B, \gamma_B) \in \mathcal{Q}_1$. This concludes the proof that $\mathcal{Q}_1$ is a reasonable class of expansions of $(\mathcal{C}_1, \leq_1)$.

By Lemma 1.69, we therefore have that $X(\mathcal{Q}_1)$ is a subflow of $\mathcal{LO}(M_1)$. To see that it is a proper subflow, we produce a linear order on M_1 which does not induce an acyclic 2-orientation. $(\mathcal{C}_1^{\prec}, \leq_1)$ is an amalgamation class and a strong expansion of $(\mathcal{C}_1, \leq_1)$, so let γ be the generic linear order of the Fraïssé limit (M_1, γ). By genericity, there exists a graph $A \leq_1 M_1$ consisting of vertices $a, b_1, \cdots, b_3$ and edges ab_i with $a >_\gamma b_i$ $(1 \leq i \leq 3)$, so γ does not induce a 2-orientation. □

4.3 Admissibly ordered orientations on M_1

We now describe how to construct a Ramsey expansion of $(\mathcal{C}_1, \leq_1)$: the class of *admissibly ordered orientations* $(\mathcal{E}_1, \sqsubseteq_s)$. This Ramsey expansion is taken from [**7**], Section 6. It is a simple case of the admissible orders defined in [**9**] (see Theorem 1.4 and Section 3 of [**9**]). However, we will keep our presentation self-contained.

Definition 4.3. Take $A \in \mathcal{D}_1$. For $a \in A$, the *level of a in A*, denoted $\mathrm{l}_A(a)$, was defined in Definition 1.2 for general oriented graphs. In the case of $\mathcal{D}_1$, the 2-oriented graphs with no directed cycles, the definition of $\mathrm{l}_A(a)$ simplifies to the following:

- if a has no successor, $\mathrm{l}_A(a) := 0$;
- otherwise $\mathrm{l}_A(a) := \max\{\mathrm{l}_A(b) : b$ a successor of $a\} + 1$.

Note that if $A \sqsubseteq_s B \in \mathcal{D}_1$ and $a \in A$, then $\mathrm{l}_A(a) = \mathrm{l}_B(a)$.

Definition 4.4 ([**7**], Def. 6.7). Take $A \in \mathcal{D}_1$. A linear order γ on A is *admissible* if for all $x, y \in A, x \neq y$, we have:

(1) if $\mathrm{l}_A(x) < \mathrm{l}_A(y)$, then $x <_\gamma y$;
(2) if $\mathrm{l}_A(x) = \mathrm{l}_A(y)$, then $x <_\gamma y$ if the decreasing chain of successors of x in γ is lexicographically less than the decreasing chain of successors of y in γ.

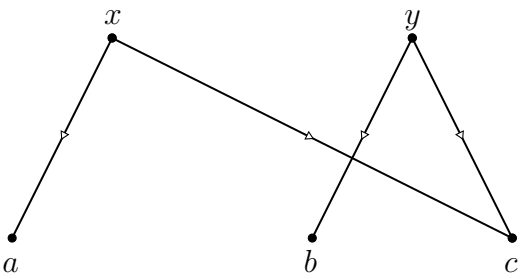


FIGURE 4.1. $a < b < c < x < y$ is an admissible order on the above acyclic 2-oriented graph.

EXAMPLE 4.5. (See Figure 4.1.) Let $A \in \mathcal{D}_1$ be the 2-oriented graph with vertex set $\{a, b, c, x, y\}$ and out-edges xa, xc, yb, yc. Let γ be the linear order $a < b < c < x < y$.

Then a, b, c are on level 0 of A, and x, y are on level 1. γ is admissible: (2) is vacuously true for a, b, c, (1) is satisfied by $a, b, c < x, y$, and finally the decreasing chain (c, a) of successors of x is lexicographically less than the decreasing chain (c, b) of successors of y and we have $x < y$ in γ, so (2) is satisfied for x, y.

DEFINITION 4.6. Take $A \in \mathcal{D}_1$, $x \in A$. Let the *cone* $Q_A(x)$ of x in A be the set $Q_A(x) = \{y \in A : \mathrm{N}^A_+(y) = \mathrm{N}^A_+(x)\}$, i.e. the set of vertices of A with the same successors as x. Let the *base* x° of x be the set $x^\circ = \mathrm{scl}_A(x) - \{x\}$.

LEMMA 4.7 (*). *Take $(A, \tau_A) \in \mathcal{D}_1$ and let γ be an admissible order on (A, τ_A). Then γ induces an acyclic 2-orientation τ_γ on A, and $\tau_\gamma = \tau_A$.*

PROOF. If $xy \in \tau_A$, then $\mathrm{l}_A(x) > \mathrm{l}_A(y)$, so $x >_\gamma y$ and so $xy \in \tau_\gamma$. If $xy \in \tau_\gamma$, then xy is an edge of A and $x >_\gamma y$, so $\mathrm{l}_A(x) > \mathrm{l}_A(y)$, and so $xy \in \tau_A$. □

We next show that, given $A \in \mathcal{D}_1$, it suffices to specify an order on cones to define an admissible order on A.

LEMMA 4.8 (*). *Take $A \in \mathcal{D}_1$. Let $\{\gamma_Q : Q$ a cone of $A\}$ be a set of linear orders on the cones of A. Then there exists a unique admissible order γ on A such that for each cone Q, γ agrees with γ_Q.*

PROOF. We will show by induction on n that there exists a unique admissible order γ_n on $A^{\uparrow n}$ such that for each cone Q of $A^{\uparrow n}$, γ_n agrees with γ_Q. For $n = 0$, note that all the vertices of $A^{\uparrow 0} = \mathrm{L}_0(A)$ lie in a single cone Q_0, and have no successors. So γ_{Q_0} is trivially the unique admissible order on $A^{\uparrow 0}$ agreeing with itself.

Now assume the induction claim for $n-1$, so we have a unique admissible order γ_{n-1} on $A^{\uparrow n-1}$ agreeing with γ_Q for each cone Q of $A^{\uparrow n-1}$.

To show existence for $A^{\uparrow n}$, we define a linear order γ_n as follows:

(1) γ_n extends γ_{n-1};
(2) if $x \in A^{\uparrow n-1}$ and $y \in \mathrm{L}_n(A)$, then $x < y$ in γ_n;
(3) if $x, y \in \mathrm{L}_n(A)$ and the decreasing chain of successors of x is lexicographically less than the decreasing chain of successors of y, then $x < y$ in γ_n;
(4) if $x, y \in \mathrm{L}_n(A)$ and x, y have the same successors, then x, y lie in some cone Q, so let $x < y$ in γ_n if $x < y$ in γ_Q.

γ_n is clearly admissible and agrees with all γ_Q on cones Q of $A^{\uparrow n}$ by definition, and uniqueness is clear. □

Let $\mathcal{E}_1$ denote the class of ordered digraphs (A, γ) where $A \in \mathcal{D}_1$ and γ is an admissible order on A. For $(A_1, \gamma_1), (A_2, \gamma_2) \in \mathcal{E}_1$, we write $(A_1, \gamma_1) \sqsubseteq_s (A_2, \gamma_2)$ to mean $A_1 \sqsubseteq_s A_2$, $\gamma_1 = \gamma_2|_{A_1}$.

LEMMA 4.9. *$(\mathcal{E}_1, \sqsubseteq_s)$ has the hereditary property, i.e. for $(B, \gamma_B) \in \mathcal{E}_1$, if $(A, \gamma_A) \in \mathcal{D}_1^{\prec}$ and $(A, \gamma_A) \sqsubseteq_s (B, \gamma_B)$, then $(A, \gamma_A) \in \mathcal{E}_1$.*

PROOF. As $A \sqsubseteq_s B$, for $x \in A$, $\mathrm{l}_A(x) = \mathrm{l}_B(x)$. So for $x, y \in A$ with $\mathrm{l}_A(x) < \mathrm{l}_A(y)$, we have that $\mathrm{l}_B(x) < \mathrm{l}_B(y)$, so $x < y$ in γ_B, and thus $x < y$ in γ_A, showing that γ_A satisfies part (1) of Definition 4.4.

For $x, y \in A$ with $\mathrm{l}_A(x) = \mathrm{l}_A(y)$, as x, y have the same successors in A and in B, if the decreasing chain of successors of x in A is lexicographically less than the decreasing chain of successors of y in A, then the same is true in B, so $x < y$ in γ_B and therefore in γ_A, showing part (2) of Definition 4.4. □

PROPOSITION 4.10 ([**7**], Th. 6.9). *$(\mathcal{E}_1, \sqsubseteq_s)$ is:*

(1) a free amalgamation class, i.e. for $(A, \gamma_A), (B_1, \gamma_1), (B_2, \gamma_2) \in \mathcal{E}_1$ with $(A, \gamma_A) \sqsubseteq_s (B_1, \gamma_1), (B_2, \gamma_2)$, there exists $(C, \gamma_C) \in \mathcal{E}_1$ with C the free amalgam of B_1, B_2 over A and γ_C extending γ_1, γ_2;
(2) a Ramsey class;
(3) a reasonable class of expansions of $(\mathcal{D}_1, \sqsubseteq_s)$, and has the expansion property over $(\mathcal{D}_1, \sqsubseteq_s)$.

PROOF. (1): We first show that in the case where B_1 has a single vertex x of maximum level k, $A = x^\circ$ and $\mathrm{l}(B_1) \geq \mathrm{l}(B_2)$, then (C, γ) exists and is unique up to isomorphism (*).

Let C be the free amalgam of B_1, B_2 over A. We will define a linear order γ on C as follows. We require that γ extend γ_1, γ_2, and then we need only define γ for x. Take x greater than all vertices of B_2 of level $< k$, and amongst the vertices of B_2 of level k, order x according to (2) of Definition 4.4, ordering x within its cone arbitrarily. Then γ is admissible, and uniqueness up to isomorphism is clear. (We may reorder x within its cone, but these reorderings will be ordered digraph-isomorphic.)

For the general case, we use induction on the total number n of vertices of the free amalgam of B_1, B_2 over A. The base case $n = 1$ is trivial, as are the cases $B_1 = A, B_2 = A$. So we assume B_1, B_2 have vertices outside A. Without loss of generality, $\mathrm{l}(B_1) \geq \mathrm{l}(B_2)$.

In the case $\mathrm{l}(A) < \mathrm{l}(B_1)$, let $x \in B_1 - A$ be of maximum level in B_1. As $(\mathcal{E}_1, \sqsubseteq_s)$ has the hereditary property, $(B_1 - \{x\}, \gamma_1) \in \mathcal{E}_1$, and so by the induction assumption, there exists $(D, \gamma_D) \in \mathcal{E}_1$ with D the free amalgam of $B_1 - \{x\}, B_2$ over A and γ_D extending $\gamma_1|_{B_1-\{x\}}, \gamma_2$. By (*), there is $(C, \gamma) \in \mathcal{E}_1$ with C the free amalgam of $\mathrm{scl}_{B_1}(x), D$ over x° and γ extending $\gamma_1|_{\mathrm{scl}_{B_1}(x)}, \gamma_D$. (C, γ) contains the free amalgam of $(\mathrm{scl}_{B_1}(x), \gamma_1), (B_1 - \{x\}, \gamma_1)$ over (x°, γ_1), so by the uniqueness part of (*), (C, γ) contains an isomorphic copy of (B_1, γ_1).

In the case $\mathrm{l}(A) = \mathrm{l}(B_1)$, let x be a vertex of A of maximum level, so x has no predecessors. Then using the hereditary property of $(\mathcal{E}_1, \sqsubseteq_s)$ and the induction assumption, there is $(D, \gamma_D) \in \mathcal{E}_1$ with D the free amalgam of $B_1 - \{x\}, B_2 - \{x\}$ over $A - \{x\}$ and γ_D extending $\gamma_1|_{B_1-\{x\}}, \gamma_2|_{B_2-\{x\}}$. Then use (*) and let (C, γ) be the free amalgam of $\mathrm{scl}_A(x), D$ over x° with γ extending γ_A, γ_D. By the uniqueness part of (*), (C, γ) is the free amalgam of isomorphic copies of $(B_1, \gamma_1), (B_2, \gamma_2)$ over an isomorphic copy of (A, γ_A), with γ extending γ_1, γ_2.

(2): Take $(A, \gamma_A) \sqsubseteq_s (B, \gamma_B) \in \mathcal{E}_1$. By Theorem 1.57 applied to $\mathcal{D}_1^\prec$, there is $(C, \gamma_C) \in \mathcal{D}_1^\prec$ with $(C, \gamma_C) \to ((B, \gamma_B))_2^{(A,\gamma_A)}$. Let $C' = \cup\binom{C}{B}$. Then $C' \sqsubseteq_s C$ and $(C', \gamma_C) \in \mathcal{D}_1^\prec$ still has $(C', \gamma_C) \to ((B, \gamma_B))_2^{(A,\gamma_A)}$.

So it will suffice to show that there is a linear order γ on C' with $(C', \gamma) \in \mathcal{E}_1$ such that γ agrees with γ_C on each $\sqsubseteq_s$-copy of (B, γ_B) inside (C', γ_C), as then (C', γ) will still witness the Ramsey property for (A, γ_A) and (B, γ_B).

That is, it suffices to prove the following statement:

> $(*)$ if $(X, \gamma_X) \in \mathcal{E}_1$ and $(Y, \gamma_Y) \in \mathcal{D}_1^\prec$ is such that every vertex of (Y, γ_Y) lies in a $\sqsubseteq_s$-copy of (X, γ_X), then there exists a linear order γ on Y with $(Y, \gamma) \in \mathcal{E}_1$ such that γ agrees with γ_Y on every $\sqsubseteq_s$-copy of (X, γ_X).

To prove $(*)$, we will define γ inductively. Index the vertices of Y as $y_1, \cdots, y_n$, where if $i < j$ then $\mathrm{l}(y_i) \leq \mathrm{l}(y_j)$. Let γ be the trivial linear order on $\{y_1\}$. Now suppose that γ has already been defined on $\{y_1, \cdots, y_{i-1}\}$ such that $(\{y_1, \cdots, y_{i-1}\}, \gamma) \in \mathcal{E}_1$ and γ agrees with γ_Y on every $\sqsubseteq_s$-copy of (X, γ_X) where γ is defined. We define γ on $\{y_1, \cdots, y_i\}$. Let $k = \mathrm{l}(y_i)$. We specify that:

> (A) y_i is greater in γ than all vertices of level $< k$;
> (B) for y_j $(j < i)$ of level k, if the descending chain of successors of y_j is lexicographically less than the descending chain of successors of y_i in γ, then $y_j < y_i$ in γ;

(C) for y_j ($j < i$) of level k in the same cone as y_i, y_j, y_i are ordered in γ according to γ_Y.

Then clearly $(\{y_1, \cdots, y_i\}, \gamma) \in \mathcal{E}_1$. We show that γ, γ_Y agree on $\sqsubseteq_s$-copies of (X, γ_X). By the induction assumption, we know that this is the case on $\{y_1, \cdots, y_{i-1}\}$. Say y_j, y_i ($j < i$) lie in a $\sqsubseteq_s$-copy $(X', \gamma_{X'})$ of (X, γ_X). If $\mathrm{l}(y_j) < k$, then by part (1) of Definition 4.4, $y_j < y_i$ in $\gamma_{X'}$, and by (A) $y_j < y_i$ in γ. If $\mathrm{l}(y_j) = k$, then as $\gamma_{X'}, \gamma$ agree on levels $< k$, the lexicographic preorders agree by (B), and if y_j, y_i are in the same cone, then $\gamma_{X'}, \gamma$ agree by (C).

(3): We now show that $\mathcal{E}_1$ is a reasonable class of expansions of $(\mathcal{D}_1, \sqsubseteq_s)$. To show (1) and (4) in the definition of reasonableness, it suffices to show that if $A \sqsubseteq_s B \in \mathcal{D}_1$ and $A^+ \in \mathcal{E}_1$ is an expansion of A, then there exists an expansion $B^+ \in \mathcal{E}_1$ of B with $A^+ \sqsubseteq_s B^+$ (we allow $A = \varnothing$). Fix $A^+ \in \mathcal{E}_1$, and proceed by induction on $|B|$. The base case is trivial. For the induction step, take $b \in B - A$ of maximum level. b has no predecessors in B. Let $C = B - \{b\}$, and let $C^+ \in \mathcal{E}_1$ be an expansion of C with $A^+ \sqsubseteq_s C^+$. To define an order γ on B extending C^+, we need only define the order between b and each $c \in C^+$, which we do according to (1), (2) of Definition 4.4, where we order b arbitrarily within its cone $Q_B(b)$. γ is clearly admissible.

Part (2) of reasonableness results from the fact that there are finitely many linear orders on a finite set. Part (3) is clear: successor-closed subsets preserve levels, and part (2) of Definition 4.4 is preserved for successor-closed subsets (formally, we prove (3) by induction on $|A|$).

To show that $\mathcal{E}_1$ has the expansion property over $(\mathcal{D}_1, \sqsubseteq_s)$, the proof is similar to that of Proposition 4.1, except that we take $X = A - Q_A(a)$. □

Definition 4.11. Let (M_1, ρ, α) be the Fraïssé limit of $(\mathcal{E}_1, \sqsubseteq_s)$, where ρ is the orientation and α is the linear order of the Fraïssé limit.

Let $N_1 = (M_1, \rho)$. As $(\mathcal{E}_1, \sqsubseteq_s)$ is a strong expansion of $(\mathcal{D}_1, \sqsubseteq_s)$, we have that N_1 is the Fraïssé limit of $(\mathcal{D}_1, \sqsubseteq_s)$.

Recall that $G_1 = \mathrm{Aut}(M_1)$. Let $K_1 = \mathrm{Aut}(N_1)$, and let $H_1 = \mathrm{Aut}(N_1, \alpha)$. Then by Proposition 4.10, H_1 is extremely amenable.

We will write $G = G_1, K = K_1, H = H_1$ in this chapter for notational brevity.

Lemma 4.12. *For $x \in N_1$, $\mathrm{scl}_{N_1}(x)$ is finite.*

(We will usually just write $\mathrm{scl}_\rho(x)$ or $\mathrm{scl}(x)$, when this is clear from context. Recall that ρ is the orientation of the oriented graph $N_1 = (M_1, \rho)$.)

Proof. Take $x \in N_1$. As N_1 is the Fraïssé limit of $(\mathcal{D}_1, \sqsubseteq_s)$, we can write N_1 as the union of an increasing $\sqsubseteq_s$-chain $A_0 \sqsubseteq_s A_1 \sqsubseteq_s \cdots$ of

elements of $\mathcal{D}_1$. Note that each element of $\mathcal{D}_1$ is a finite oriented graph. As there is $i \in \mathbb{N}$ such that x is in A_i, and as $\mathrm{scl}_{N_1}(x)$ is contained in every successor-closed subset of N_1 containing x, we have that $\mathrm{scl}_{N_1}(x)$ is contained in the finite set A_i, and so $\mathrm{scl}_{N_1}(x)$ is finite. □

DEFINITION 4.13. Let $x \in N_1$. We define the *level* $\mathrm{l}_{N_1}(x)$ of x in N_1, usually just denoted $\mathrm{l}_\rho(x)$ or $\mathrm{l}(x)$ when clear from context, to be the level of x in $\mathrm{scl}_{N_1}(x)$. (Note that the level of a vertex in a finite oriented graph was defined in Definition 1.3, and we know that $\mathrm{scl}_{N_1}(x)$ is finite by the previous lemma.)

Note that for $A \sqsubseteq_s N_1$ with $x \in A$, as $\mathrm{scl}_{N_1}(x) \sqsubseteq_s A$, we have that $\mathrm{l}_{N_1}(x) = \mathrm{l}_A(x)$.

DEFINITION 4.14. Let $x \in N_1$. Define the *cone* $Q_{N_1}(x)$ of x in N_1, usually just denoted $Q_\rho(x)$ or $Q(x)$, to be the set $Q_{N_1}(x) = \{y \in N_1 : \mathrm{N}_+^{N_1}(y) = \mathrm{N}_+^{N_1}(x)\}$, i.e. the set of vertices of N_1 with the same successors as x. Let the *base* x° of x be the set $x^\circ = \mathrm{scl}_{N_1}(x) - \{x\}$.

Note that $Q_{N_1}(x)$ is the orbit of x in N_1 under the pointwise stabiliser of x° in $\mathrm{Aut}(N_1)$, by $\sqsubseteq_s$-homogeneity of N_1.

LEMMA 4.15 (*). *Consider the G-flow $G \curvearrowright \mathcal{LO}(M_1)$. Let G_α denote the G-stabiliser of α for this flow. Then $H = G_\alpha$.*

PROOF. Clearly $H \leq G_\alpha$. Take $g \in G_\alpha$. We must show that g preserves the generic orientation ρ. Take an out-edge $xy \in \rho$. Then as $\mathrm{l}_\rho(x) > \mathrm{l}_\rho(y)$, $x > y$ in α, and so $gx > gy$. g is a graph automorphism, so $\{gx, gy\}$ is an edge of M_1, and so we have that $(gx, gy) \in \rho$. So $g \in H$. □

LEMMA 4.16 (**). *Let $\beta \in \mathcal{LO}(M_1)$ be an H-fixed point in the flow $G \curvearrowright \mathcal{LO}(M_1)$, and let Q be a cone of N_1. Then β agrees with either α or α' on Q, where α' denotes the reverse of the linear order α.*

PROOF. Take $x_0, y_0 \in Q$ with $x_0 <_\alpha y_0$. Then for $x, y \in Q$ with $x <_\alpha y$, there exists an ordered digraph isomorphism $f : \mathrm{scl}(x_0, y_0) \to \mathrm{scl}(x, y)$ with $f(x_0) = x, f(y_0) = y$, and by homogeneity we may extend to an element $f \in H$.

As $H \subseteq G_\beta$, f is β-preserving. If $x_0 <_\beta y_0$, then $f(x_0) <_\beta f(y_0)$, i.e. $x <_\beta y$, and so as $x <_\alpha y$ implies $x <_\beta y$, β agrees with α on Q. If $x_0 >_\beta y_0$, then β agrees with α' on Q. □

We now show that $(\mathcal{E}_1, \sqsubseteq_s)$ is in some sense an "optimal" Ramsey expansion: the automorphism group H of its Fraïssé limit is maximal amongst extremely amenable subgroups of G.

PROPOSITION 4.17 ([**7**], Th. 6.10). *If $H \leq J \leq G$ and J is extremely amenable, then $J = H$.*

PROOF. Recall that $H = \mathrm{Aut}(M_1, \rho, \alpha)$. Say $\sigma \in \mathrm{Or}(M_1)$ is an H-fixed point. If $\sigma \neq \rho$, then there exist adjacent $a, b \in M_1$ such that $(a, b) \in \rho, (b, a) \in \sigma$. As $(a, b) \in \rho$, the H_b-orbit of a is infinite. But H fixes σ, so this contradicts σ being a 2-orientation. So ρ is the only orientation of M_1 fixed by H. As J is extremely amenable, J must fix ρ. So $J \leq \mathrm{Aut}(N_1) = K$.

As $\mathcal{E}_1$ is a reasonable class of expansions of $(\mathcal{D}_1, \sqsubseteq_s)$, $X(\mathcal{E}_1)$ is a K-flow. $X(\mathcal{E}_1)$ is the space of admissible orders on N_1. $J \leq K$, so J fixes some admissible order γ on N_1. Thus H preserves γ on N_1, and therefore by Lemma 4.16, γ agrees with α or α' on cones of N_1. But if $k \in K$ preserves a linear order on a cone, it also preserves its reverse. So J preserves α on cones, and so by Lemma 4.8, J preserves α. Therefore $J \leq H$. □

4.4 Minimal subflows of $\mathcal{LO}(M_1)$ and their orbits

We now state the main new theorem of this chapter.

THEOREM 4.18 (**). *Let $Y \subseteq \mathcal{LO}(M_1)$ be a minimal subflow of $\mathcal{LO}(M_1)$. Then all G-orbits on Y are meagre.*

4.4.1 Setup and proof notation

Before beginning the proof, we first need to set up our approach.

Let $Y \subseteq \mathcal{LO}(M_1)$ be a minimal subflow of $\mathcal{LO}(M_1)$, and using Lemma 1.71, let $\mathcal{J}$ be the class of finite ordered graphs such that $Y = X(\mathcal{J})$, i.e. $\mathcal{J}$ is the class of isomorphism types of (A, λ), where $A \leq M_1$ and λ is a linear order induced on A by an element of Y.

Recall that $H = \mathrm{Aut}(N_1, \alpha)$ is extremely amenable. So H fixes an element (M_1, β) of Y. As Y is a minimal G-flow, $Y = \overline{G(M_1, \beta)}$.

Take $(M_1, \gamma) \in \overline{G(M_1, \beta)}$. Then for $A \leq M_1$ finite, there exists $g \in G$ such that $g\beta$ and γ agree on A, i.e. there exists a $\leq_1$-ordered graph embedding $(A, \gamma) \to (M_1, \beta)$. Conversely, if $(M_1, \gamma) \in \mathcal{LO}(M_1)$ is such that for all finite $A \leq M_1$, there exists a $\leq_1$-ordered graph embedding $(A, \gamma) \to (M_1, \beta)$, then by homogeneity we can extend these embeddings to elements of G.

Thus we see that $\mathcal{J}$ is exactly the class of finite ordered graphs (A, λ) where $A \in \mathcal{C}_1$ and there exists a $\leq_1$-ordered graph embedding $(A, \lambda) \to (M_1, \beta)$ - concisely, we have that $\mathcal{J} = \mathrm{Age}_{\leq}(M_1, \beta)$. (Note that here we do not assume homogeneity of (M_1, β)).

We will show that $(\mathcal{J}, \leq_1)$ does not have the weak amalgamation property (WAP), which implies that all G-orbits on Y are meagre by Lemma 1.78.

To sum up: given a minimal subflow $Y \subseteq \mathcal{LO}(M_1)$, H fixes a linear order β inside Y. To show that Y has all G-orbits meagre, it suffices to show that $\mathcal{J} = \mathrm{Age}_{\leq}(M_1, \beta)$ does not have the weak amalgamation property with $\leq_1$-embeddings.

We will now use the above notation throughout the rest of section 4.4.

4.4.2 Proof idea - informal overview

We will assume $(\mathcal{J}, \leq_1)$ has WAP, seeking a contradiction. Take some vertex a_0. Let $A_0 = \{a_0\}$, and let λ_0 be the trivial order on A_0. By assumption (A_0, λ_0) has a WAP-witness (A, λ). We will then construct $\leq_1$-embeddings of (A, λ) into two ordered graphs $(C_0, \gamma_0), (C_1, \gamma_1) \in \mathcal{J}$ which are WAP-incompatible: it will not be possible to have (D, γ) completing the WAP commutative diagram for (A_0, λ_0) with the two embeddings, and thereby we will obtain our contradiction.

The incompatibility of the two ordered graphs (C_i, γ_i) in $\mathcal{J}$ will result from them forcing incompatible orientations: it turns out that we can use the order β to force certain orientations of edges in ρ (see Figure 4.3). The incompatible orientations will essentially consist of a binary out-directed tree T_0 and a binary out-directed tree with the successor-closures of two vertices identified, which we denote by T_1: these cannot start from the same point of a 2-orientation, as one contains a 4-cycle and the other does not.

The idea to use two incompatible orientations in the WAP commutative diagram and thereby obtain a contradiction comes from the proof of Theorem 5.2 of [**8**]:

THEOREM (Th. 5.2, [**8**]). *Let $M = M_1$, M_0 or M_F, and let $G = \mathrm{Aut}(M)$. Let Y be a minimal subflow of the G-flow $\mathrm{Or}(M)$. Then all G-orbits on Y are meagre.*

Theorem 4.18 will not depend on the above result, though, as stated, several aspects of the proof are inspired by it. The key difficulties in the proof of Theorem 4.18 are showing that we can use β (specifically, particular finite ordered graphs in $\mathcal{J} = \mathrm{Age}_{\leq_1}(M_1, \beta)$) to force orientations of edges in ρ (Lemma 4.20), and also showing that the ordered graphs that we construct to force orientations of edges in ρ do in fact lie in $\mathcal{J}$ (Lemma 4.21).

4.4.3 Attaching trees and near-trees

For $q \in \mathbb{N}$, let $T_0(q)$ be a binary tree of height $2q+1$, oriented outwards towards the leaves and with head vertex c. Let $T_1(q)$ be the digraph

given by taking $T_0(q)$ and identifying the successor-closures of two vertices at height $q+2$ whose paths to the head vertex c meet at height q. We have $T_0(q), T_1(q) \in \mathcal{D}_1$.

Let T be one of the digraphs $T_0(q)$ or $T_1(q)$, for some $q \in \mathbb{N}$. Take $C \in \mathcal{D}_1$ with each vertex having out-degree 2 or 0. Let D_T be the digraph consisting of C together with, for each vertex $v \in C$ with $\mathrm{d}_+(v) = 0$, a copy of T attached at v, where we identify c and v. Let Z_T denote the sub-digraph of D_T whose vertices are the vertices of the copies of T attached to C in D_T. Let D_T^- denote the graph reduct of D_T. (We will use this notation throughout this section.)

We have that D_T is still 2-oriented and has no directed cycles, and so $D_T \in \mathcal{D}_1$. Let ${D_T}'$ be the acyclic reorientation of D_T where the copies of T have been oriented so that the non-head vertices of each copy of T are directed towards the head vertex c, leaving the orientation on vertices of C unchanged. Then we have $C \sqsubseteq_s {D_T}'$ in this reorientation, and so $C^- \leq_1 D_T^-$.

Definition 4.19. Let $C \in \mathcal{D}_1$ with each vertex having out-degree 2 or 0, and let D_T be defined as above.

An ordered graph $(C_T, \gamma) \in \mathcal{J}$ is a *T-witness ordered graph for C* if:

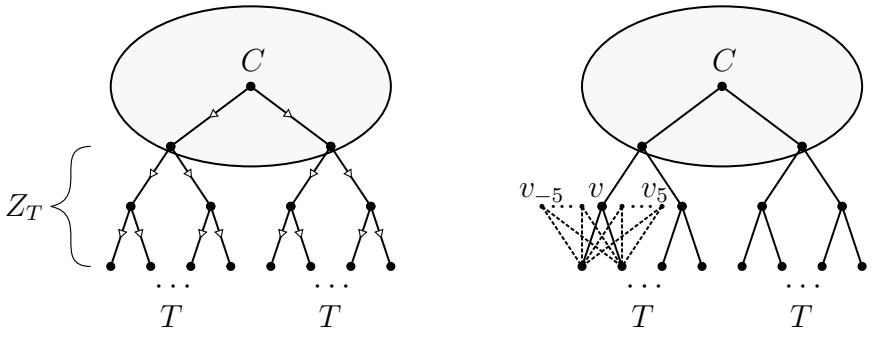
- C_T consists of the graph reduct ${D_T}^-$ of D_T together with, for each non-leaf tree vertex v of D_T, an additional 10 copies of $\mathrm{scl}_{D_T}(v)$ freely amalgamated (as graphs) over $\mathrm{scl}_{D_T}(v)^\circ$, and $C \leq_1 C_T$;
- for each non-leaf tree vertex $v \in D_T$, the additional 10 copies of v may be labelled as $v_{-5}, \cdots, v_{-1}, v_1, \cdots, v_5$ so that $v_{-5} < \cdots < v_{-1} < v < v_1 < \cdots < v_5$ in γ. (We call these v_i the *witness vertices* of v.)

(See Figure 4.2.)

The following is the key lemma here.

Lemma 4.20 (**). *Let $C \in \mathcal{D}_1$ with each vertex having out-degree 2 or 0, and let $(C_T, \gamma) \in \mathcal{J}$ be a T-witness ordered graph for C (as defined above). As (C_T, γ) is an element of $\mathcal{J}$, there exists a $\leq_1$-ordered graph embedding $\theta : (C_T, \gamma) \to (M_1, \beta)$. Then, considering the digraph structure on Z_T induced by D_T, $\theta|_{Z_T} : Z_T \to (M_1, \rho)$ is also a digraph embedding. (See Figure 4.3.)*

Proof. Take $v \in Z_T$ with out-edges $(v, x), (v, y)$, where $x, y \in Z_T$. We need to show that $\theta(v)$ has out-edges $(\theta(v), \theta(x)), (\theta(v), \theta(y))$ in the orientation ρ of M_1. Let $v_{-5}, \cdots, v_{-1}, v_1, \cdots, v_5$ be the witness vertices of v in (C_T, γ), and let $v_0 = v$. As θ is a $\leq_1$-ordered graph embedding, we have that $\theta(v_i) <_\beta \theta(v_j)$ for $i < j$, and we have undirected edges $\theta(v_i)\theta(x), \theta(v_i)\theta(y)$ for $-5 \leq i \leq 5$.



the oriented graph D_T

the ordered graph (C_T, γ), with witness vertices v_i indicated on one vertex v of Z_T

FIGURE 4.2

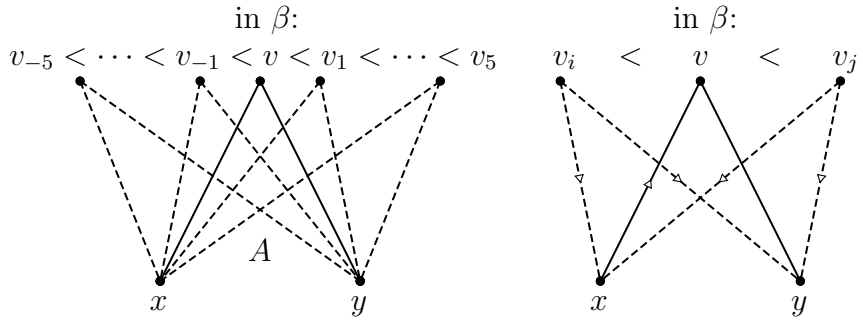


FIGURE 4.3. The key idea in the proof of Theorem 4.18: we can use the linear order β to force orientations of edges. If the ordered graph $(A, \beta) \leq (M_1, \beta)$, then considering A in the generic orientation ρ of M_1, the right-hand oriented graph in the figure cannot occur, and we must have that vx, vy are out-edges in ρ.

As ρ is a 2-orientation, for some i with $-5 \leq i \leq -1$ we must have that $(\theta(v_i), \theta(x)), (\theta(v_i), \theta(y))$ are out-edges of ρ, and likewise for some j with $1 \leq j \leq 5$ we must have that $(\theta(v_j), \theta(x)), (\theta(v_j), \theta(y))$ are out-edges of ρ. If either $(\theta(x), \theta(v_0)) \in \rho$ or $(\theta(y), \theta(v_0)) \in \rho$, then $\theta(v_0) \in \mathrm{scl}_\rho(\theta(x), \theta(y))$, and as $\theta(v_i), \theta(v_j)$ lie in the same α-cone, there exists $h \in G_\alpha$ with $h\theta(v_i) = \theta(v_j)$ and h fixing $\theta(v_0)$. As $G_\alpha \subseteq G_\beta$, we have that $h \in G_\beta$. But $\theta(v_i) <_\beta \theta(v_0)$, so $h\theta(v_i) <_\beta h\theta(v_0)$, thus $\theta(v_j) <_\beta \theta(v_0)$ - contradiction. So therefore both $(\theta(v_0), \theta(x)) \in \rho$ and $(\theta(v_0), \theta(y)) \in \rho$. □

LEMMA 4.21 (**). *Let $C \in \mathcal{D}_1$ with each vertex having out-degree 2 or 0. Then there exists a T-witness ordered graph (C_T, γ) for C, with $(C_T, \gamma) \in \mathcal{J}$.*

PROOF. Let $d_1, \cdots, d_k$ be an enumeration of the non-leaf tree vertices of D_T which preserves the order of levels, i.e. for $i < j$, $\mathrm{l}_{D_T}(d_i) \leq \mathrm{l}_{D_T}(d_j)$. We will show, by induction on i, that for $0 \leq i \leq k$ there exists an ordered graph $(C_i, \gamma_i) \in \mathcal{J}$ such that:

(1) C_i consists of D_T together with, for $1 \leq j \leq i$, an additional 10 copies of $\mathrm{scl}_{D_T}(d_j)$ freely amalgamated (as graphs) over $\mathrm{scl}_{D_T}(d_j)^\circ$;
(2) for $1 \leq j \leq i$, the additional 10 copies of d_j may be labelled as $d_{j,-5}, \cdots, d_{j,-1}, d_{j,1}, \cdots, d_{j,5}$ such that $d_{j,-5} < \cdots < d_{j,-1} < d_j < d_{j,1} < \cdots < d_{j,5}$ in γ_i. We will call these the *witness* vertices of d_j, and let W_j denote the set of witness vertices of d_j.

For the base case $i = 0$, take $C_0 = {D_T}^-$. As $C_0 \in \mathcal{C}_1$ and $\mathcal{J}$ is a reasonable class of expansions of $(\mathcal{C}_1, \leq_1)$, there exists a linear order γ_0 on C_0 such that $(C_0, \gamma_0) \in \mathcal{J}$. (C_0, γ_0) satisfies (1) and (2) vacuously.

For the induction step, assume we have $(C_i, \gamma_i) \in \mathcal{J}$ satisfying (1) and (2). Let

$$X = \mathrm{L}_0(D_T) \cup \bigcup_{1 \leq j \leq i} \mathrm{scl}_{D_T}(d_j) \cup \bigcup_{1 \leq j \leq i} W_j.$$

There is an acyclic orientation τ_i of C_i in which X is successor-closed: take the orientation of D_T, and orient the two edges of each witness vertex $d_{j,m}$ outwards from $d_{j,m}$. Thus $X \leq_1 C_i$. Note that for $j' > i \geq j$, $\mathrm{l}_{D_T}(d_{j'}) \geq \mathrm{l}_{D_T}(d_j)$, so $d_{j'} \notin X$ for $j' > i$.

Let (E, τ) be the free amalgam of (C_i, τ_i) 11 times over (X, τ_i). As $\mathcal{D}_1$ is a free amalgamation class, $(E, \tau) \in \mathcal{D}_1$. Hence $E \in \mathcal{C}_1$, and we have $X \leq E$.

Let $\gamma_X = \gamma_i|_X$. We have that $(X, \gamma_X) \in \mathcal{J}$, so let $\theta_X : (X, \gamma_X) \to (M_1, \beta)$ be a $\leq_1$-ordered graph embedding. By the extension property of M_1, we have a $\leq_1$-graph embedding $\theta : E \to M_1$ extending θ_X.

Let $\zeta = \theta^{-1}(\beta)$, i.e. for $x, y \in E$, $x <_\zeta y$ iff $\theta(x) <_\beta \theta(y)$. We have that ζ is a linear order on E extending γ_X on X, and that $\theta : (E, \zeta) \to (M_1, \beta)$ is a $\leq_1$-ordered graph embedding.

We may label the 11 copies of C_i in E as $C_{i,m}$ $(-5 \leq m \leq 5)$, with $\leq_1$-embeddings $\eta_m : C_i \to C_{i,m} \leq E$, and the corresponding copies of d_{i+1} as $d_{i+1,m} \in C_{i,m}$ $(-5 \leq m \leq 5)$, such that $d_{i+1,-5} < \cdots < d_{i+1,5}$ in ζ. Let ${C_{i+1}}' = C_{i,0} \cup \{d_{i+1,m} : -5 \leq m \leq 5\}$. We have that $({C_{i+1}}', \tau) \sqsubseteq_s (E, \tau)$, so ${C_{i+1}}' \leq E$. So $\theta : ({C_{i+1}}', \zeta) \to (M_1, \beta)$ is a $\leq_1$-ordered graph embedding.

We have that ${C_{i+1}}'$ consists of a copy $C_{i,0} = \eta_0(C_i)$ of C_i, where $\eta_0|_X = \mathrm{id}_X$ and $\eta_0|_X : (X, \gamma_X) \to (X, \zeta)$ is order-preserving, together with witness vertices $d_{i+1,-5}, \cdots, d_{i+1,-1}, d_{i+1,1}, \cdots, d_{i+1,5}$ for $d_{i+1,0} = \eta(d_{i+1})$.

Recall that C_i consists of D_T together with, for $1 \leq j \leq i$, the witness vertices for d_j, and also that X consists of $\mathrm{L}_0(D_T)$ together with, for $1 \leq j \leq i$, d_j and its witness vertices.

Therefore $({C_{i+1}}', \zeta)$ consists of a graph-isomorphic copy $\eta_0(D_T)$ of D_T, together with witness vertices in ζ for $\eta_0(d_1) = d_1, \cdots, \eta_0(d_i) = d_i$ and witness vertices in ζ for an additional vertex $\eta_0(d_{i+1})$. We can therefore construct an ordered graph (C_{i+1}, γ_{i+1}) isomorphic to $({C_{i+1}}', \zeta) \in \mathcal{J}$ such that C_{i+1} consists of D_T together with witness vertices for d_j, $1 \leq j \leq i+1$. This completes the induction step. We then let $(C_T, \gamma) = (C_k, \gamma_k)$. □

4.4.4 $(\mathcal{J}, \leq_1)$ does not have WAP

PROPOSITION 4.22 (**). *The class $(\mathcal{J}, \leq_1)$ does not have the weak amalgamation property.*

PROOF. Suppose $(\mathcal{J}, \leq_1)$ has WAP, seeking a contradiction. Let a_0 be a point. Take $A_0 = \{a_0\}$ and λ_0 the trivial order on A_0, so $(A_0, \lambda_0) \in \mathcal{J}$. Then there exists $(A_0, \lambda_0) \leq_1 (A, \lambda) \in \mathcal{J}$ with (A, λ) witnessing the weak amalgamation property for (A_0, λ_0). Take $A \leq_1 B \in \mathcal{C}_1$ witnessing for A the expansion property of $\mathcal{J}$ over $(\mathcal{C}_1, \leq_1)$. (Here we use Theorem 1.73, recalling that $Y = X(\mathcal{J})$ is a minimal G-flow.)

Take $B^+ \in \mathcal{D}_1$ such that the undirected reduct of B^+ is B. For each $v \in B^+$ with $\mathrm{d}_+(v) = 1$, add to B^+ a new vertex v' and out-edge vv', and call the resulting digraph $C \in \mathcal{D}_1$. Note that each vertex of C has out-degree 0 or 2. We have that $B \leq_1 C^-$.

Let q be the maximum number of levels in any acyclic reorientation of C (i.e. if C when reoriented has levels $0, \cdots, n$, then $q = n + 1$).

For $i = 0, 1$, let $(C_i, \gamma_i) \in \mathcal{J}$ be $T_i(q)$-witness ordered graphs for C, using Lemma 4.21, and let D_i, Z_i denote $D_{T_i(q)}, Z_{T_i(q)}$ (the notation here is introduced just above Definition 4.19).

As $B \leq C_i$ witnesses the expansion property for A, there exist $\leq_1$-ordered graph embeddings $\zeta_i : (A, \lambda) \to (B, \gamma_i) \leq (C_i, \gamma_i)$ $(i = 0, 1)$. As (A, λ) witnesses WAP for (A_0, λ_0), there exists $D \leq M_1$ and $\leq_1$-ordered graph embeddings $\theta_i : (C_i, \gamma_i) \to (D, \beta)$ with $\theta_0\zeta_0(a) = \theta_1\zeta_1(a)$. By Lemma 4.20, $\theta_i|_{Z_i} : Z_i \to (D, \rho)$ are also digraph embeddings.

If r is a vertex of C of out-degree 0 in C, then as $\theta_i|_{Z_i}$ is a digraph embedding, $\theta_i(r)$ has out-degree 0 in $\theta_i(C)$. Also θ_i preserves predimension, being a graph embedding, so $\delta(\theta_i(C)) = \delta(C)$. Thus the roots of $\theta_i(C)$ are exactly the $\theta_i(r)$ for r a root of C, and they all have out-degree 0. (Recall that a root vertex is one that has out-degree less than 2 - see Definition 1.13.)

Let $d = \theta_i\zeta_i(a)$, and let U_n be the set of vertices of (M_1, ρ) that can be reached from d by an outward-directed path of length $\leq n$. As the

only vertices of $\theta_i(D_i)$ of out-degree less than 2 are the leaves of the copies of T_i, $U_{2q+1} \subseteq \theta_i(D_i)$ $(i = 0, 1)$.

We now obtain a contradiction by comparing the two cases $i = 0$ and $i = 1$. As $U_{2q+1} \subseteq \theta_0(D_0)$, we have that $U_{2q+1} - U_{q-1}$ does not contain any (undirected) cycles. But as $U_{2q+1} \subseteq \theta_1(D_1)$, we have that $U_{2q+1} - U_{q-1}$ contains a 4-cycle - contradiction. □

This completes the proof of Theorem 4.18.

4.5 Stabilisers of H-fixed points

Recall that we write $G = G_1 = \mathrm{Aut}(M_1)$, $K = K_1 = \mathrm{Aut}(N_1)$, $H = H_1 = \mathrm{Aut}(N_1, \alpha)$.

In this section, β is an arbitrary linear order on M_1. We will show, as the main result of this section, that if $G_\alpha \subseteq G_\beta$, then $G_\alpha = G_\beta$. (Here G_α, G_β denote the G-stabilisers of α, β in the flow $G \curvearrowright \mathcal{LO}(M_1)$.) Note that we may easily obtain examples of such $\beta \in \mathcal{LO}(M_1)$ with $G_\alpha \subseteq G_\beta$ by permuting the order of the levels of α.

This result originally formed part of the author's first attempt at proving Theorem 4.18.

First attempt at a proof approach: let $Y \subseteq \mathcal{LO}(M_1)$ be a minimal subflow, and let β be an H-fixed point inside Y as before. So $G_\alpha \subseteq G_\beta$. If we have that $G_\alpha = G_\beta$ (as proved below), then we may define a surjective map between orbits $p : G\beta \to G\alpha$, $g \cdot \beta \mapsto g \cdot \alpha$. By Lemma 4.15, there is a well-defined continuous map $q : G\alpha \to \mathrm{Or}(M_1)$, $g \cdot \alpha \mapsto g \cdot \rho$, and if we could show that $q \circ p : G\beta \to \mathrm{Or}(M_1)$ were uniformly continuous, we could continuously extend $q \circ p$ to $Y = \overline{G\beta} \to \mathrm{Or}(M_1)$, and perhaps use the fact that minimal subflows of $\mathrm{Or}(M_1)$ have all G-orbits meagre to show that minimal subflows of $\mathcal{LO}(M_1)$ have all G-orbits meagre. Unfortunately, the author was not able to show that $q \circ p$ was uniformly continuous.

Nonetheless, the results in this section are still of interest, as they give us some more information about what β can be.

LEMMA 4.23 (**). *For $\beta \in \mathcal{LO}(M_1)$, if $K_\alpha \subseteq K_\beta$ then $K_\alpha = K_\beta$.*

PROOF. Note that given an edge xy of M_1, $(x, y) \in \rho$ iff $x >_\alpha y$, so we have that $H = K_\alpha = G_\alpha$.

Take $k \in K - K_\alpha$. k preserves the orientation ρ but not the order α. So there exist $x, y \in M_1$ with $x >_\alpha y$ and $kx <_\alpha ky$. Take x of minimal level n satisfying this. So k preserves α on levels below n.

As k is a digraph isomorphism, it preserves the level of vertices, i.e. $\mathrm{l}(kv) = \mathrm{l}(v)$ for $v \in N_1$. If x, y are not in the same cone, then as $y <_\alpha x$, either $\mathrm{l}(y) < \mathrm{l}(x)$, or $\mathrm{l}(y) = \mathrm{l}(x)$ and the descending chain

of successors of y is lexicographically before that of x. If $\mathrm{l}(y) < \mathrm{l}(x)$, then $\mathrm{l}(ky) < \mathrm{l}(kx)$, contradicting $kx <_\alpha ky$. If $\mathrm{l}(y) = \mathrm{l}(x)$ and the decreasing chain of successors of y is lexicographically before that of x, given that k preserves α on levels below n, the decreasing chain of successors of ky must be lexicographically before that of kx, again contradicting $kx <_\alpha ky$.

So x, y lie in the same cone, and as k is a digraph isomorphism, kx, ky also lie in the same cone. As k preserves α on levels below n, we have that $\mathrm{scl}(x), \mathrm{scl}(kx)$ are isomorphic as ordered digraphs, and similarly for $\mathrm{scl}(y), \mathrm{scl}(ky)$. So, as $y <_\alpha x$ and $kx <_\alpha ky$, there exists an ordered digraph automorphism $h \in H$ taking $\mathrm{scl}(y)$ to $\mathrm{scl}(kx)$ and $\mathrm{scl}(x)$ to $\mathrm{scl}(ky)$. We have that $h(y) = kx, h(x) = ky$, so $k^{-1}h(y) = x, k^{-1}h(x) = y$, and thus $k^{-1}h$ does not preserve any linear order. Therefore, as h preserves β, k cannot preserve β, i.e. $k \notin K_\beta$. □

LEMMA 4.24 (**). *For $\beta \in \mathcal{LO}(M_1)$, if $G_\alpha \subseteq G_\beta$ then $G_\alpha = G_\beta$.*

PROOF. Seeking a contradiction, say there exists $g \in G_\beta - G_\alpha$. We have $K_\alpha = G_\alpha$ and $K_\alpha \subseteq K \cap G_\beta = K_\beta$, so $K_\alpha = K_\beta$ by Lemma 4.23. So $g \notin K$, i.e. there exist $x, y \in M_1$ such that $(x, y) \in \rho$ and $(gy, gx) \in \rho$.

We will eliminate this possibility case by case.

Case 1: In ρ, x has out-degree 2, with out-edges to y, z.

Consider the cone $Q_\rho(x)$ of x in the orientation ρ. As $(Q_\rho(x), \alpha)$ is order-isomorphic to $\mathbb{Q}$, there exist $v_{-5}, \cdots, v_{-1}, v_1, \cdots, v_5$ in $Q_\alpha(x)$ with $v_{-5} < \cdots < v_{-1} < x < v_1 < \cdots < v_5$ in α. We have that ρ is a 2-orientation, and so there exists some $-5 \leq i \leq -1$ such that $(gv_i, gy) \in \rho$ and $(gv_i, gz) \in \rho$, i.e. g does not reverse the orientation of the out-edges $(v_i, y), (v_i, z)$. Likewise there exists some $1 \leq j \leq 5$ with $(gv_j, gy) \in \rho$, $(gv_j, gz) \in \rho$.

So gv_i, gv_j lie in the same cone, and so by homogeneity of (N_1, α) there exists $h \in H$ sending $(\mathrm{scl}(gv_i), \alpha)$ to $(\mathrm{scl}(gv_j), \alpha)$ with $hgv_i = gv_j$ and $\mathrm{scl}(gv_i)^\circ = \mathrm{scl}(gv_j)^\circ$ fixed. As $gy \in \mathrm{scl}(gv_i)^\circ$ and $(gy, gx) \in \rho$ by assumption, $hgx = gx$.

As β agrees with α or α' on $Q_\alpha(x)$ by Lemma 4.16, we have $v_i < x < v_j$ or $v_j < x < v_i$ in β, and as $g \in G_\beta$, therefore we have $gv_i < gx < gv_j$ or $gv_j < gx < gv_i$ in β.

But $h \in G_\beta$, $hgx = gx$ and $hgv_i = gv_j$, so we obtain a contradiction.

Case 2: In ρ, x has out-degree 1, with out-edge to y.

As $Q_\alpha(x)$ is order-isomorphic to $\mathbb{Q}$, there exist $v_{-3}, \cdots, v_{-1}, v_1, \cdots, v_3$ in $Q_\alpha(x)$ with $v_{-3} < \cdots < v_{-1} < x < v_1 < \cdots < v_3$ in α. As ρ is a 2-orientation, we have some $-3 \leq i \leq -1$ such that $(gv_i, gy) \in \rho$, and likewise we have some $1 \leq j \leq 3$ such that $(gv_j, gy) \in \rho$.

Take $u \in M_1$ with out-edge $(u, v_i) \in \rho$. If g reverses the orientation of (u, v_i), i.e. $(gv_i, gu) \in \rho$, then gv_i has out-degree 2. But Case 1 applied to $g^{-1} \in G_\beta - G_\alpha$ shows that g^{-1} preserves the orientation of out-edges of vertices of out-degree 2, and so $(v_i, u) \in \rho$, contradiction. Thus g preserves the orientation of the in-edges of v_i, and likewise also of v_j. So gv_i, gv_j have out-degree 1. Thus gv_i, gv_j lie in the same cone, and the rest of the argument for Case 2 is the same as for Case 1. □

4.6 An example of a minimal subflow of $\mathcal{LO}(M_1)$

We now give an example of a minimal subflow of $\mathcal{LO}(M_1)$.

Let $\mathcal{A}_1 \subseteq \mathcal{C}_1^\prec$ be the class of ordered graphs (A, γ) where $A \in \mathcal{C}_1$ and:

(1) γ induces an acyclic 2-orientation τ_γ on A;
(2) γ is admissible for (A, τ_γ).

$\mathcal{A}_1$ is the reduct of $\mathcal{E}_1$ to the language of ordered graphs. We will refer to an order γ satisfying (1), (2) as an admissible order on a graph.

(This section is essentially a straightforward translation of section 6.4 of [**7**] from $\mathcal{E}_1$ to $\mathcal{A}_1$, with some minor modifications.)

Note that if $(B, \gamma_B) \in \mathcal{A}_1$ and $A \leq_1 B$, the restriction of γ_B to A may not be an admissible order. Consider the ordered graph (B, γ_B) consisting of two disjoint edges x_0x_1, y_0y_1 where $x_0 < y_0 < x_1 < y_1$. Then γ_B is admissible, so $(B, \gamma_B) \in \mathcal{A}_1$. Let (A, γ_A) be the ordered subgraph of (B, γ_B) with vertex set $\{x_0, x_1, y_1\}$. Then by reversing the orientation of $y_1y_0 \in \tau_{\gamma_B}$ we see that $A \leq_1 B$, but γ_A is not admissible: y_1 is on level 0 of τ_{γ_A}, x_1 is on level 1 of τ_{γ_A}, but $x_1 < y_1$ in γ_A.

As $\mathcal{E}_1$ satisfies parts (1), (2) and (4) of reasonableness over $(\mathcal{C}_1, \leq_1)$, so does $\mathcal{A}_1$, but as we have just seen, $\mathcal{A}_1$ does not satisfy (3). The same problem occurs with $\mathcal{E}_1$. So let $\mathcal{E}_1'$ be the class of ordered oriented graphs A^+ such that $A^+ \leq_1 B^+$ for some $B^+ \in \mathcal{E}_1$, and let $\mathcal{A}_1'$ be the class of ordered graphs (A, γ_A) such that $(A, \gamma_A) \leq_1 (B, \gamma_B)$ for some $(B, \gamma_B) \in \mathcal{A}_1$. That is, $\mathcal{E}_1'$ and $\mathcal{A}_1'$ are the $\leq_1$-closures of $\mathcal{E}_1, \mathcal{A}_1$.

LEMMA 4.25 (*). *$\mathcal{E}_1'$ is a reasonable class of expansions of $(\mathcal{C}_1, \leq_1)$.*

PROOF. Parts (1), (2) of reasonableness for $\mathcal{E}_1'$ follow from the fact that $\mathcal{E}_1$ satisfies (1) and (2), and (3) follows by the definition of $\mathcal{E}_1'$ as the $\leq_1$-closure of $\mathcal{E}_1$. To check (4), take $f : A_1 \to B$ a strong map of $(\mathcal{C}_1, \leq_1)$, and let $A_1^+ = (A_1, \rho_1, \gamma_1) \in \mathcal{E}_1'$ be an expansion of A_1. (A_1, ρ_1, γ_1) is a $\leq_1$-substructure of some $(A_2, \rho_2, \gamma_2) \in \mathcal{E}_1$. Let D be the free amalgam of B and a copy C of A_2 over $f(A_1)$ (where we take C with $f(A_1) \leq_1 C$. Give C the orientation ρ_C and order γ_C induced from (A_2, ρ_2, γ_2) (so $f : A_1^+ \to (C, \rho_C, \gamma_C)$ is an ordered digraph embedding). As $C \leq_1 D$, there is an expansion $(D, \rho_D) \in \mathcal{D}_1$ with $(C, \rho_C) \sqsubseteq_s (D, \rho_D)$. As $\mathcal{E}_1$ is a

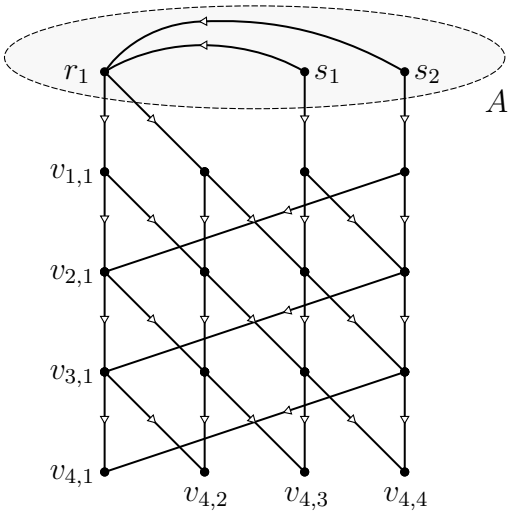


FIGURE 4.4

reasonable class of expansions of $(\mathcal{D}_1, \sqsubseteq_s)$, there is an admissible order γ_D on (D, ρ_D) extending γ_C. $B \leq_1 D$, so $(B, \gamma_D) \in \mathcal{E}_1'$. □

LEMMA 4.26 (*). *$\mathcal{A}_1'$ is a reasonable class of expansions of $(\mathcal{C}_1, \leq_1)$.*

The proof is analogous to that of the previous lemma.

LEMMA 4.27. *Take $C \in \mathcal{E}_1$ and $k \geq 0$. Then $C^{\downarrow k} \in \mathcal{E}_1$.*

PROOF. Straightforward. □

LEMMA 4.28. *Let $A \in \mathcal{D}_1$. Then there exists $B \in \mathcal{D}_1$, $A \leq_1 B$, such that for $C \in \mathcal{D}_1$, if $B \subseteq C$, then there exists $k \geq 0$ such that $A \sqsubseteq_s C^{\downarrow k}$.*

PROOF. Let $r_1, \cdots, r_n$ be the vertices of A of out-degree 0, and let $s_1, \cdots, s_m$ be the vertices of A of out-degree 1. Let $N = 2n + m$.

Let B consist of A together with new vertices $v_{i,j}$ $(1 \leq i, j \leq N)$ and new out-edges:

- $r_i v_{1,2i-1}, r_i v_{1,2i}$ $(1 \leq i \leq n)$;
- $s_i v_{1,2n+i}$ $(1 \leq i \leq m)$;
- $v_{i,j} v_{i+1,j}$ $(1 \leq i < N, 1 \leq j \leq N)$;
- $v_{i,j} v_{i+1,j+1 \pmod N}$ $(1 \leq i < N, 1 \leq j \leq N)$.

See Figure 4.4 for an example.

To see that $A \leq_1 B$, note that we may reorient B by reversing the orientation of all new out-edges not in A, forming an acyclic orientation in which A is successor-closed.

Now take $B \subseteq C \in \mathcal{D}_1$. Note that it is possible for the $v_{N,j}$ to be on different levels of C. However, we will show that the r_i all lie on the same level of C.

Take some particular $v_{N,b}, v_{1,c}$. Then any out-path from $v_{1,c}$ to $v_{N,b}$ must be of length $N-1$, as an out-edge from any new vertex $v_{i,j}$ goes to an out-vertex with i component increased by one.

There is an out-path from $v_{1,c}$ to $v_{N,b}$ of length $N-1$, namely:

- for $b < c$: $v_{1,c}, v_{2,c+1 \pmod N}, \cdots, v_{1+(N-c)+b,b}, \cdots, v_{N,b}$;
- for $b \geq c$: $v_{1,c}, \cdots, v_{1+(b-c),b}, \cdots, v_{N,b}$.

(Informally, in these specified paths, we move diagonally first.)

Thus all the $v_{1,j}$ lie on the same level l of C, where $l = \max\{\mathrm{l}_C(v_{N,j}) : 1 \leq j \leq N\} + N - 1$. Let $k = l + 1$. Then all the r_i lie on level k, and each s_a is of level $> k$, with one successor in A and one successor being one of the $v_{1,j}$ on level $k-1$. Each vertex of A of out-degree 2 is of level $> k$ and its successors lie in A and are of level $\geq k$. So $A \sqsubseteq_s C^{\downarrow k}$. $\square$

PROPOSITION 4.29. *$\mathcal{E}_1'$ has the expansion property over $(\mathcal{C}_1, \leq_1)$.*

PROOF. Take $(A', \gamma_{A'}) \in \mathcal{E}_1'$. There is $(A, \gamma_A) \in \mathcal{E}_1$ with $(A', \gamma_{A'}) \leq_1 (A, \gamma_A)$. We will find $B \in \mathcal{C}_1$ such that for any expansion $(B^+, \gamma) \in \mathcal{E}_1'$ (where $B^+ \in \mathcal{D}_1$), (B^+, γ) contains a $\leq_1$-copy of (A, γ_A), and therefore of $(A', \gamma_{A'})$. We do this in three steps.

(1) By Proposition 4.10, $\mathcal{E}_1$ has the expansion property over $(\mathcal{D}_1, \sqsubseteq_s)$, so take $X \in \mathcal{D}_1$ such that every expansion to $\mathcal{E}_1$ contains a $\sqsubseteq_s$-copy of (A, γ_A).
(2) Then, using Lemma 4.28, take $Y \in \mathcal{D}_1$ with $X \leq_1 Y$ such that for any $C \in \mathcal{D}_1$ with $f : Y \to C$ an embedding (not necessarily strong), there exists $k \geq 0$ with $f(X) \sqsubseteq_s C^{\downarrow k}$.
(3) Finally, as $\mathcal{D}_1$ has the expansion property over $(\mathcal{C}_1, \leq_1)$ by Proposition 4.1, take $B \in \mathcal{C}_1$ such that any expansion to $\mathcal{D}_1$ contains a $\leq_1$-copy of Y.

We now show that B witnesses the expansion property for (A, γ_A) of $\mathcal{E}_1'$ over $(\mathcal{C}_1, \leq_1)$. Let $(B, \rho_B, \gamma_B) \in \mathcal{E}_1'$ be an expansion of B. There is $(C, \rho_C, \gamma_C) \in \mathcal{E}_1$ with $(B, \rho_B, \gamma_B) \leq_1 (C, \rho_C, \gamma_C)$. By (3), there is a $\leq_1$-embedding $f : Y \to (B, \rho_B)$, and we may regard f as a $\leq_1$-embedding into (C, ρ_C). By (2), there is $k \geq 0$ such that $f(X) \sqsubseteq_s (C, \rho_C)^{\downarrow k}$. We have that $(f(X), \gamma_B) = (f(X), \gamma_C) \sqsubseteq_s (C, \rho_C, \gamma_C)^{\downarrow k}$, so as $(C, \rho_C, \gamma_C) \in \mathcal{E}_1$, by Lemma 4.27 we have that $(C, \rho_C, \gamma_C)^{\downarrow k} \in \mathcal{E}_1$, so $(f(X), \gamma_B) \in \mathcal{E}_1$. So by (1), $(f(X), \gamma_B)$ contains a $\sqsubseteq_s$-copy of (A, γ_A), and as $(f(X), \gamma_B) \leq_1 (B, \gamma_B)$, B witnesses the expansion property for (A, γ_A). $\square$

PROPOSITION 4.30 (*). *$\mathcal{A}_1'$ has the expansion property over $(\mathcal{C}_1, \leq_1)$.*

PROOF. Take $(A', \gamma_{A'}) \in \mathcal{A}_1'$. Then there is $(A, \gamma_A) \in \mathcal{A}_1$ such that $(A', \gamma_{A'}) \leq_1 (A, \gamma_A)$. Let ρ_A be the acyclic 2-orientation induced on A by γ_A, so $(A, \rho_A, \gamma_A) \in \mathcal{E}_1$. By Proposition 4.29, take $B \in \mathcal{C}_1$ such that any expansion $(B, \rho_B, \gamma_B) \in \mathcal{E}_1'$ contains a $\leq_1$-copy of (A, ρ_A, γ_A).

Let $(B, \gamma_B) \in \mathcal{A}'_1$ be an expansion of B. Then $(B, \gamma_B) \leq_1 (C, \gamma_C)$ for some $(C, \gamma_C) \in \mathcal{A}_1$. Let ρ_C be the acyclic 2-orientation induced on C by γ_C, so $(C, \rho_C, \gamma_C) \in \mathcal{E}_1$. Let $\rho_B = \rho_C|_B$. Then $(B, \rho_B, \gamma_B) \leq_1 (C, \rho_C, \gamma_C)$, so $(B, \rho_B, \gamma_B) \in \mathcal{E}'_1$, and so (B, ρ_B, γ_B) contains a $\leq_1$-copy of (A, ρ_A, γ_A). $\square$

Chapter 5

Linear orders and orientations on M_0

In this chapter, we attempt to carry out the same analysis for M_0 as we did for M_1 in the previous chapter. As before, we describe a Ramsey expansion $(\mathcal{E}_{\text{fin}}, \sqsubseteq_s)$ of $(\mathcal{C}_0, \leq_s)$, the class of *admissibly ordered fine orientations* of $\mathcal{C}_0$. We write $(N_\mathcal{E}, \alpha)$ for the Fraïssé limit and $H_\mathcal{E} = \text{Aut}(N_\mathcal{E}, \alpha)$, where α is the linear order and $N_\mathcal{E}$ is the oriented graph of the Fraïssé limit. We then have that $H_\mathcal{E}$ is extremely amenable, so, as in Chapter 4, any minimal subflow $Y \subseteq \mathcal{LO}(M_0)$ of the G_0-flow $\mathcal{LO}(M_0)$ is the G_0-orbit closure of a $H_\mathcal{E}$-fixed point β.

The author has not been able to prove that any minimal subflow of $\mathcal{LO}(M_0)$ has all orbits meagre, due to a difficulty elaborated in Section 5.4: when we try to prove an analogue of Theorem 4.18, and want to find information about β using $H_\mathcal{E}$-automorphisms (i.e. information about α), we do not know anything about β inside strongly connected components (the details here are somewhat technical). However, we do manage to prove the equivalent result on stabilisers: we show that the G_0-stabiliser of β in the G_0-flow $\mathcal{LO}(M_0)$ is $H_\mathcal{E}$, the G_0-stabiliser of α, and this at least gives us some information about β.

5.1 $\mathcal{LO}(M_0)$ is not minimal

As in the previous chapter, we first check that $\mathcal{LO}(M_0)$ is not itself a minimal flow.

LEMMA 5.1 ([**10**], Lem. 1.14). *Let $A, B \in \mathcal{C}_0$ with $A \leq_s B$. Then there is an acyclic 4-orientation of B in which A is successor-closed.*

PROOF. Take $\varnothing \neq C \subseteq A$. The average vertex degree of C is $\frac{2|E(C)|}{|C|}$, which is ≤ 4 as $\delta(C) = 2|C| - |E(C)| \geq 0$. So C has a vertex of degree ≤ 4, and so by Lemma 1.10, A has an acyclic 4-orientation τ_A.

Take $A \subseteq C \subseteq B$. We'll show by induction on $|C|$ that C has an acyclic 4-orientation extending τ_A in which A is successor-closed. $|C| = |A|$ is trivial. We now do the inductive step. The average degree in C of $C - A$ is

$$\frac{2|E(C)| - 2|E(A)| - |E(C - A, A)|}{|C - A|} \leq \frac{2|E(C)| - 2|E(A)|}{|C - A|} \leq 4$$

(where to show the rightmost inequality we recall that $\delta(C) \geq \delta(A)$). So there exists $c \in C - A$ with degree ≤ 4 in C. By the induction assumption we may give $C - \{c\}$ an acyclic 4-orientation extending τ_A in which A is successor-closed. Then we orient the edges of c outwards from c. □

PROPOSITION 5.2 (**). *$\mathcal{LO}(M_0)$ is not a minimal flow.*

PROOF. Let $\mathcal{Q}_0$ denote the class of ordered graphs (A, γ) where $A \in \mathcal{C}_0$ and the linear order γ induces an acyclic 4-orientation on A, i.e. $\tau_\gamma = \{(x, y) \in E_A : x >_\gamma y\}$ is an acyclic 4-orientation.

The proof that $\mathcal{Q}_0$ is a reasonable class of expansions of $(\mathcal{C}_0, \leq_s)$ is entirely analogous to the proof of Proposition 4.2, replacing acyclic 2-orientations with acyclic 4-orientations and using Lemma 5.1.

We therefore have that $X(\mathcal{Q}_0)$ is a subflow of $\mathcal{LO}(M_0)$. To see that it is a proper subflow, we produce a linear order on M_0 which does not induce an acyclic 4-orientation. $(\mathcal{C}_0^\prec, \leq_s)$ is an amalgamation class and a strong expansion of $(\mathcal{C}_0, \leq_s)$, so let γ be the linear order of the Fraïssé limit (M_0, γ). By genericity, there exists a graph $A \leq_s M_0$ consisting of vertices $a, b_1, \cdots, b_5$ and edges ab_i with $a >_\gamma b_i$ $(1 \leq i \leq 5)$, so γ does not induce a 4-orientation. □

5.2 Admissibly ordered fine orientations: constructing a Ramsey expansion of $\mathcal{C}_0$

We will now construct a Ramsey expansion $(\mathcal{E}_{\text{fin}}, \sqsubseteq_s)$ of $(\mathcal{C}_0, \leq_s)$. The material in this section is a modified and elaborated version of Section 7 of [**6**], an unpublished early version of the paper [**8**]. We can also view this Ramsey expansion construction as a particular case of the more general treatment of admissible orders seen in Section 3 of [**9**]. (In fact, the material from Section 7 of the unpublished paper [**6**] was removed before the final published version in [**8**], in order to be generalised and published in [**9**] instead).

However, here we keep our presentation self-contained. We use the standalone Ramsey result of Theorem 1.57.

(The new material in this section is a characterisation of fine extensions in terms of predimension, and a new definition of similarity. The new definition of similarity requires a straightforward modification of the construction of the Ramsey expansion.)

Throughout this section, we will use the notation and terminology of Definition 1.2: we refer the reader to this for the definition of scl (successor-closure), scc (strongly connected component) and $\mathrm{N}_+(A)$ (the out-neighbourhood of A).

5.2.1 Out-degree-preserving graph isomorphisms

In order to construct a Ramsey expansion of $(\mathcal{C}_0, \leq_s)$, we will need to know which graph isomorphisms preserve the "scc structure" of elements of $\mathcal{D}_0$. The following is folklore:

LEMMA 5.3 (Lem. 7.11, [**6**]). *Take $A \in \mathcal{C}_0$, and let $A_1, A_2 \in \mathcal{D}_0$ be two orientations of A such that the out-degree of each vertex is the same in each orientation. Then A_1, A_2 have the same successor-closed subsets, sccs and orientation of edges between sccs.*

PROOF. $C \subseteq A$ is successor-closed in an orientation iff the number of edges of C is equal to the sum of the out-degrees in A of the vertices of C in that orientation. Thus A_1, A_2 have the same successor-closed subsets.

For $C, D \subseteq A$, if $C \sqsubseteq_s D \sqsubseteq_s A'$ in some orientation A' of A and there does not exist $E \subseteq A$, $E \neq C, D$ with $C \sqsubseteq_s E \sqsubseteq_s D$, we will say that $C \sqsubseteq_s D$ is an *unrefinable $\sqsubseteq_s$-chain* of A'. So A_1, A_2 have the same unrefinable $\sqsubseteq_s$-chains.

We will show by induction on n that:

- $C \subseteq A$ is a scc of A_1 of level n iff it is a scc of A_2 of level n;
- if $C \subseteq A$ is a scc of A_1 or A_2 of level n, the out-edges from C are the same in either orientation.

For the base case $n = 0$, observe that $C \subseteq A$ is a level 0 scc of A_i iff $C \sqsubseteq_s A_i$ and $\varnothing \sqsubseteq_s C$ is an unrefinable $\sqsubseteq_s$-chain of A_i. Assume the induction claim for levels $< n$, and let C be a scc of A_1 of level n. Let $X = \mathrm{scl}_{A_1}(C) - C$. Then $X \sqsubseteq_s A_1$, so $X \sqsubseteq_s A_2$. So if (c, x) is an out-edge from C to X in A_1, then (c, x) is an out-edge in A_2. By the induction assumption, as X has level $n-1$ in A_1, X has the same sccs and orientation of edges between them in A_1 and A_2. So C has level $\geq n$ in A_2. $X \sqsubseteq_s \mathrm{scl}_{A_1}(C)$ is unrefinable in A_1, and thus in A_2, and so C is a scc of level n in A_2. The reverse direction is identical, swapping A_1 and A_2. □

LEMMA 5.4 (*). *Take $A \in \mathcal{C}_0$ with $\delta(A) = 0$. Then any orientation of A has the same successor-closed subsets, sccs and orientation of edges between sccs.*

PROOF. As $\delta(A)$ is the sum of the multiplicities of the roots in any orientation (Lemma 1.14), A has no roots in any orientation, i.e. every vertex is of out-degree two. Then use Lemma 5.3. □

5.2.2 Fine orientations

DEFINITION 5.5 (Def. 7.6, [**6**]). Take $A \in \mathcal{D}_0$. A is an *extension* if it has a unique scc C of maximum level and $A = \mathrm{scl}_A(C)$.

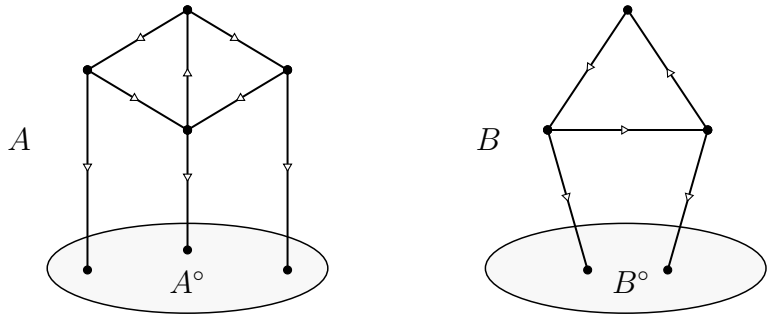


FIGURE 5.1. The extension A is fine. The extension B is not fine.

If A is an extension, we denote its unique scc of maximum level by $\mathrm{hd}(A)$ (the *head scc*). We define the *base* A° of A to be $A^\circ = A - \mathrm{hd}(A)$. Note that $A^\circ \sqsubseteq_s A$.

It is straightforward to see that any element of $\mathcal{D}_0$ can be built via a sequence of free amalgamations of extensions over successor-closed substructures (formally, we proceed by induction on the number of sccs). If required, we can also carry out the sequence of free amalgamations in order of increasing level.

Our Ramsey expansion will orient elements of $\mathcal{C}_0$ in a particular way: we will work with *fine* orientations, which we now define.

DEFINITION 5.6. Let $A \in \mathcal{D}_0$ be an extension. A is a *fine* extension if, in any 2-reorientation A' of A with $A^\circ \sqsubseteq_s A'$, we have that $A' - A^\circ$ is strongly connected.

If A is a fine extension, we will also say that the head scc $\mathrm{hd}(A)$ of the extension is fine.

LEMMA 5.7 (*). *Let $A \in \mathcal{D}_0$ be an extension. A is fine iff $A^\circ \leq_s A$ is an unrefinable $\leq_s$-chain.*

PROOF. $\Rightarrow$: If $A^\circ \leq_s B \leq_s A$, then there exists an orientation A' of A such that $B \sqsubseteq_s A'$ (where B has the induced orientation from A'), and there exists an orientation B^+ of B such that $A^\circ \sqsubseteq_s B^+$. Reorient $B \sqsubseteq_s A'$ with the orientation from B^+, and call the resulting orientation A''. Then $A^\circ \sqsubseteq_s B \sqsubseteq_s A''$, so by fineness $B = A^\circ$ or $B = A$.

$\Leftarrow$: Let A' be a reorientation of A with $A^\circ \sqsubseteq_s A'$. If $A' - A^\circ$ is not strongly connected, there exists $y \in A' - A^\circ$ with $\mathrm{scl}_{A'}(y) \neq A'$. Then $\mathrm{scl}_{A'}(y) \cup A^\circ$ refines $A^\circ \leq_s A$, contradiction. □

DEFINITION 5.8. Take $B \in \mathcal{D}_0$. We say that B is a *fine orientation* if, for every extension A with $A \sqsubseteq_s B$, A is fine.

We let $\mathcal{D}_{\mathrm{fin}}$ denote the class of fine orientations.

LEMMA 5.9 (*). *Take $A, B \in \mathcal{C}_0$ with $A \neq B$ and $A \leq_s B$ unrefinable. Suppose there is an expansion $A^+ \in \mathcal{D}_{fin}$ of A. Then there exists an*

expansion $B^+ \in \mathcal{D}_{fin}$ of B with $A^+ \sqsubseteq_s B^+$, and B^+ consists of the free amalgamation of A^+ and a fine extension D over D°, where $D^\circ \sqsubseteq_s A^+$.

PROOF. Let $B^+ \in \mathcal{D}_0$ be an expansion of B with $A^+ \sqsubseteq_s B^+$. As $A \neq B$, B^+ has some scc C disjoint from A^+. As $A \leq_s B$ is unrefinable, we have that C is the only scc of B^+ not contained in A^+. Let $D = \mathrm{scl}_{B^+}(C)$. Then $D^\circ \sqsubseteq_s A^+$, and as $A \leq_s B$ is unrefinable, $D^\circ \leq_s D$ must be unrefinable, so D is fine. B^+ is the free amalgamation over D° of D and A^+, and so as $(\mathcal{D}_{\mathrm{fin}}, \sqsubseteq_s)$ is a free amalgamation class, $B^+ \in \mathcal{D}_{\mathrm{fin}}$. □

LEMMA 5.10 (*). *$(\mathcal{D}_{fin}, \sqsubseteq_s)$ is a strong expansion of $(\mathcal{C}_0, \leq_s)$.*

PROOF. We have to show that for $A \leq_s B \in \mathcal{C}_0$, if $A^+ \in \mathcal{D}_{\mathrm{fin}}$ is an expansion of A, then there exists an expansion $B^+ \in \mathcal{D}_{\mathrm{fin}}$ of B with $A^+ \sqsubseteq_s B^+$.

It suffices to show that for $A^+ \in \mathcal{D}_{\mathrm{fin}}$ with a sccs, then for $A^+ \sqsubseteq_s B' \in \mathcal{D}_0$ with $n \geq a$ sccs, there exists a reorientation $B^+ \in \mathcal{D}_{\mathrm{fin}}$ of B' such that $A^+ \sqsubseteq_s B^+$.

We use induction on $n \geq a$. The base case is trivial. For the induction step, take C to be a scc of $B' - A^+$ of maximum level. Let $D = B - C$, and let D' denote the orientation induced on D by B'. We have that $D' \sqsubseteq_s D' \cup C = B'$. $A^+ \sqsubseteq_s D'$, so by the induction assumption we may reorient D' to $D^+ \in \mathcal{D}_{\mathrm{fin}}$ such that $A^+ \sqsubseteq_s D^+$. Let B'' be the reorientation of B' obtained by replacing D' with D^+. So $D^+ \sqsubseteq_s D^+ \cup C = B''$. We may refine $D \leq_s B$ to an unrefinable $\leq_s$-chain $D = D_0 \leq_s \cdots \leq_s D_k = B$. By Lemma 5.9 applied to $D_0 \leq_s D_1$ with the expansion D^+, we obtain an expansion $D_1^+ \in \mathcal{D}_{\mathrm{fin}}$ of D_1 with $A^+ \sqsubseteq_s D^+ \sqsubseteq_s D_1^+$. Assuming the expansion D_i^+ has already been defined, we likewise apply Lemma 5.9 up the chain on $D_i \leq_s D_{i+1}$ to obtain an expansion D_{i+1}^+, and thus by induction we obtain the reorientation B^+ required. □

We now partially characterise fine extensions in terms of predimension.

LEMMA 5.11 (**). *Let $A \in \mathcal{D}_0$ be an extension. Then:*

(1) if $\mathrm{l}(A) > 0$, then A is fine iff $|A - A^\circ| = 1$ or $\delta(A) = \delta(A^\circ)$;
(2) if $\mathrm{l}(A) = 0$ and A is fine, then $|A| = 1$ or $\delta(A) \leq 1$;
(3) if $\mathrm{l}(A) = 0$, then if $|A| = 1$ or $\delta(A) = 0$ then A is fine.

(See Example 5.12.)

PROOF. (1): $\Rightarrow$: Let $C = \mathrm{hd}(A)$. If C has no roots of A, then by $\delta(A) = \delta(A^\circ)$ (here we use Lemma 1.14). Otherwise, let $r \in C$ be a root of A. r must have out-degree 1. If the out-vertex of r lies in A°, then as C is a scc, $|C| = 1$. Seeking a contradiction, if the out-vertex of r does not lie in A°, then there must exist $x \in C$, $x \neq r$, with an

out-edge into A°, and as C is strongly connected, there is an out-path from x to r (so x is of out-degree 2). Reverse the orientation of the out-edges of this out-path, giving another 2-orientation as r is a root. In this new orientation, there is no out-path from x to r, and so C is no longer strongly connected, contradicting fineness.

$\Leftarrow$: If $|A - A^\circ| = 1$ then it is trivial that A is fine. Suppose $\delta(A) = \delta(A^\circ)$ and $|A - A^\circ| > 1$. Take $A^\circ \subsetneq B \subsetneq A$. Let C be the head scc of A. Then B must have a vertex v with an out-edge to a vertex of $C - B$, as C is strongly connected, and so v is a root of B (considering out-degrees only within B). Therefore $\delta(B) > \delta(A^\circ)$, and so we cannot have $A^\circ \leq_s B \leq_s A$.

(2): If A contains two roots r_1, r_2, then the orientation obtained by reversing the out-path from r_1 to r_2 is not strongly connected, contradiction. So A has at most one root. If $|A| > 1$, then if there is a root r of A, as A is strongly connected r must have an out-edge, so $\delta(A) = 1$.

(3): The argument is the same as for the right-to-left direction of (1). □

Example 5.12 (**).

(1) For an example of a level 0 fine extension A with $\delta(A) = 0$, let A be any orientation of a K_5. It is straightforward to check that any orientation of K_5 is strongly connected.
(2) For an example of a level 0 fine extension A with $\delta(A) = 1$, take any orientation of a K_5 with one edge deleted. (It is straightforward case-checking to see that any orientation is strongly connected. For case-checking, it is convenient to begin by categorising the orientations into those that have the root at a vertex of the deleted edge, and those that do not.)
(3) We give an example of a level 0 extension A with $\delta(A) = 1$ which is not fine. Let B be an orientation of a K_5 with an edge removed. Let r be the root vertex of B, and let b be a non-root vertex of B. Let A be the oriented graph B together with a new vertex a with out-edges ra, ab. Then A is strongly connected and $\delta(A) = 1$. By reversing the orientation of the out-edge ra, making $\{a\}$ a scc, we may refine A, so A is not fine.

5.2.3 Admissibly ordered fine orientations

Definition 5.13. Take $A \in \mathcal{D}^{\prec}_{\text{fin}}$. Let S be an ordered scc of A. The *cone* of S, denoted $Q_A(S)$, is the set consisting of the ordered sccs S' of A for which:

(1) $\text{scl}_A(S)^\circ = \text{scl}_A(S')^\circ$;

(2) there exists an ordered digraph isomorphism $f_{S'} : \mathrm{scl}(S) \to \mathrm{scl}(S')$ fixing $\mathrm{scl}(S)^\circ = \mathrm{scl}(S')^\circ$ and sending S to S'.

Note that by the rigidity of finite linearly ordered sets, $f_{S'}$ is unique.

Take a vertex $x \in A$, and let $S = \mathrm{scc}_A(x)$. The *cone* of x is $Q_A(x) := \{f_{S'}(x) : S' \in Q_A(S)\}$.

DEFINITION 5.14 (**). Let $A, B \in \mathcal{D}^{\prec}_{\mathrm{fin}}$ be ordered extensions. A, B are *similar* if one of the following holds:

(1) $\mathrm{l}(A), \mathrm{l}(B) > 0$ and there exists an out-degree-preserving graph isomorphism $A \to B$ which is also an ordered digraph isomorphism $A^\circ \to B^\circ$;
(2) $\mathrm{l}(A) = \mathrm{l}(B) = 0$ and there exists a graph isomorphism $A \to B$.

Our class of admissibly ordered fine orientations will only contain one element (up to ordered digraph isomorphism) of each similarity class of fine ordered extensions.

DEFINITION 5.15 (adapted from Lem. 7.9, [**6**]). We fix an arbitrary linear order $\trianglelefteq$ between isotypes of ordered extensions in $\mathcal{D}^{\prec}_0$.

Let $(C, \gamma_C) \in \mathcal{D}^{\prec}_{\mathrm{fin}}$ be of level k and such that each scc of C is an interval in γ_C. We will define a preorder λ_C, which we call the *level-lex preorder*, on the set of level k sccs of C. This preorder λ_C will depend on $\trianglelefteq$ (which is arbitrary, fixed and will not change throughout this section) and γ_C.

Let C_1, C_2 be sccs of C of level k. Let $C_i^l = (\mathrm{scl}_C(C_i))^{\downarrow l}$ ($i = 1, 2$, $0 \le l \le k$). For $0 \le l \le k$, we define preorders $\trianglelefteq_l, \preceq_l$:

- $C_1 \trianglelefteq_l C_2$ if $\mathrm{isotype}(C_1^l, \gamma_C) \trianglelefteq \mathrm{isotype}(C_2^l, \gamma_C)$;
- $C_1 \preceq_l C_2$ if the decreasing chain of successor sccs of C_1^l ordered by γ_C is lexicographically before the decreasing chain of successor sccs of C_2^l ordered by γ_C.

We define the level-lex preorder λ_C to be the lexicographic preorder $(\trianglelefteq_k, \preceq_{k-1}, \trianglelefteq_{k-1}, \cdots, \preceq_0, \trianglelefteq_0)$.

If C_1, C_2 are equivalent in λ_C, then we write $C_1 \sim_{\lambda_C} C_2$.

LEMMA 5.16. *Let $(C, \gamma_C) \in \mathcal{D}^{\prec}_{fin}$ be of level k and such that each scc of C is an interval. Let C_1, C_2 be sccs of C of level k. Then $C_1 \sim_{\lambda_C} C_2$ iff $(C_1, \gamma_C), (C_2, \gamma_C)$ lie in the same cone of (C, γ_C).*

PROOF. $\Rightarrow$: $C_1 \sim_{\lambda_C} C_2$ implies that $C_1 \preceq_0 C_2, C_2 \preceq_0 C_1$, so C_1, C_2 have the same successors, i.e. $\mathrm{scl}(C_1)^\circ = \mathrm{scl}(C_2)^\circ$. As $C_1 \trianglelefteq_0 C_2, C_2 \trianglelefteq_0 C_1$, we have that $(\mathrm{scl}(C_1), \gamma_C), (\mathrm{scl}(C_2), \gamma_C)$ have the same ordered digraph isomorphism type, i.e. there exists an ordered digraph isomorphism $f : (\mathrm{scl}(C_1), \gamma_C) \to (\mathrm{scl}(C_2), \gamma_C)$. As f preserves levels, $f(\mathrm{scl}(C_1)^\circ) = \mathrm{scl}(C_2)^\circ$, and so as f preserves a linear order on a finite set, $f|_{\mathrm{scl}(C_1)^\circ}$ is the identity. So f is an ordered digraph isomorphism taking $\mathrm{scl}(C_1)$ to

$\mathrm{scl}(C_2)$ which is the identity on $\mathrm{scl}(C_1)^\circ = \mathrm{scl}(C_2)^\circ$, so $(C_1, \gamma_C), (C_2, \gamma_C)$ are in the same cone. The $\Leftarrow$ direction is clear. $\square$

DEFINITION 5.17 (* - adapted and modified from Lem. 7.9 of [**6**]). Let $\mathcal{E}_{\mathrm{fin}} \subseteq \mathcal{D}_{\mathrm{fin}}^{\prec}$. We will say that $(\mathcal{E}_{\mathrm{fin}}, \sqsubseteq_s)$ is a *class of admissibly ordered fine orientations of* $(\mathcal{C}_0, \leq_s)$ if the following hold:

(1) $\mathcal{C}_0$ is the class of graph reducts of $\mathcal{E}_{\mathrm{fin}}$;
(2) if $A \leq_s B$ in $\mathcal{C}_0$ and $A^+ \in \mathcal{E}_{\mathrm{fin}}$ is an expansion of A, then there exists an expansion $B^+ \in \mathcal{E}_{\mathrm{fin}}$ of B with $A^+ \sqsubseteq_s B^+$;
(3) if $B \in \mathcal{E}_{\mathrm{fin}}$ and $A \in \mathcal{D}_{\mathrm{fin}}^{\prec}$ with $A \sqsubseteq_s B$, then $A \in \mathcal{E}_{\mathrm{fin}}$;
(4) if $(B_1, \gamma_1), (B_2, \gamma_2) \in \mathcal{E}_{\mathrm{fin}}$, and $(A, \gamma_A) \in \mathcal{E}_{\mathrm{fin}}$ with $(A, \gamma_A) \sqsubseteq_s (B_1, \gamma_1), (B_2, \gamma_2)$, then:
   (a) there exists $(C, \gamma) \in \mathcal{E}_{\mathrm{fin}}$ with C the free amalgam of B_1, B_2 over A and γ extending γ_1, γ_2;
   (b) if B_1 is an extension with $A = B_1^\circ$ and $\mathrm{l}(B_1) \geq \mathrm{l}(B_2)$, then (C, γ) is unique up to isomorphism;
(5) for ordered extensions $A, B \in \mathcal{E}_{\mathrm{fin}}$, if A, B are similar then A, B are isomorphic as ordered digraphs (the isomorphism need not be the same as the similarity isomorphism);
(6) for $(A, \gamma_A) \in \mathcal{E}_{\mathrm{fin}}$, sccs form intervals in γ_A, and
   (a) if C_1, C_2 are sccs of A with $\mathrm{l}(C_1) < \mathrm{l}(C_2)$, then $C_1 < C_2$ in γ_A;
   (b) the linear order γ_A between sccs of the same level extends the level-lex preorder λ_A;
(7) if $(A, \gamma_A) \in \mathcal{E}_{\mathrm{fin}}$ and $(B, \gamma_B) \in \mathcal{D}_{\mathrm{fin}}^{\prec}$ is such that every vertex of (B, γ_B) lies in a $\sqsubseteq_s$-copy of (A, γ_A), then there exists a linear order γ on B with $(B, \gamma) \in \mathcal{E}_{\mathrm{fin}}$ such that γ agrees with γ_B on every $\sqsubseteq_s$-copy of (A, γ_A).

PROPOSITION 5.18 (* - modified from Lem. 7.9 of [**6**]). *There exists a class* $(\mathcal{E}_{fin}, \sqsubseteq_s)$ *of admissibly ordered fine orientations of* $(\mathcal{C}_0, \leq_s)$.

PROOF. We define inductively a family $\mathcal{X}_k$ of ordered fine extensions of level k, which we use to build a family $\mathcal{Y}_k$ of ordered oriented graphs of level k, again defined inductively. We will then take $\mathcal{E}_{\mathrm{fin}} = \bigcup_{k \geq 0} \mathcal{Y}_k$.

To define $\mathcal{X}_0$, take a single element of each similarity class of ordered extensions of level 0, and close under ordered digraph isotypes. Let $\mathcal{Y}_0$ consist of (C, γ_C) where (C, γ_C) is a disjoint union of ordered extensions in $\mathcal{X}_0$, with γ_C completed to a linear order where:

(Str0) every scc of C is an interval in γ_C, and γ_C extends the level-lex preorder λ_C.

(In this particular case, λ_C is just $\trianglelefteq$.)

Suppose $\mathcal{X}_i, \mathcal{Y}_i$ have already been defined for $i < k$.

Let $\mathcal{X}_k'$ consist of ordered extensions $(A, \gamma_A) \in \mathcal{D}_{\mathrm{fin}}^{\prec}$ of level k such that $(A^\circ, \gamma_A) \in \mathcal{Y}_{k-1}$ and $\mathrm{hd}(A) > A^\circ$ in γ_A. Define $\mathcal{X}_k$ by taking one

element of each similarity class of $\mathcal{X}'_k$, closing under ordered digraph isotypes. So $\mathcal{X}_k$ satisfies:

(Ext1) $\mathcal{X}_k$ contains ≤ 1 element from each similarity class of level k extensions of $\mathcal{D}^{\prec}_{\text{fin}}$;
(Ext2) for $(A, \gamma_A) \in \mathcal{X}_k$, $(A^\circ, \gamma_A) \in \mathcal{Y}_{k-1}$;
(Ext3) for $(A, \gamma_A) \in \mathcal{D}^{\prec}_{\text{fin}}$ an ordered extension of level k such that $(A^\circ, \gamma_A) \in \mathcal{Y}_{k-1}$, there exists $(B, \gamma_B) \in \mathcal{X}_k$ similar to (A, γ_A);
(Ext4) for $(A, \gamma_A) \in \mathcal{X}_k$, $\text{hd}(A) > A^\circ$ in γ_A.

We now define $\mathcal{Y}_k$. Let $\mathcal{Y}_k$ consist of $(C, \gamma_C) \in \mathcal{D}^{\prec}_{\text{fin}}$ of level k such that (C, γ_C) results from a sequence of free amalgamations of elements of $\mathcal{X}_k \cup \bigcup_{i<k} \mathcal{Y}_i$, with γ_C completed to a linear order where:

(Str1) $(C^{\upharpoonright k-1}, \gamma_C) \in \mathcal{Y}_{k-1}$;
(Str2) the vertices of C of level k are greatest in γ_C;
(Str3) each scc of C of level k is an interval in γ_C, and γ_C extends the level-lex preorder λ_C on sccs of level k.

We let $\mathcal{E}_{\text{fin}} = \bigcup_{k \geq 0} \mathcal{Y}_k$, and now check properties (1)-(7) in the definition of admissibility.

We prove (3), the $\sqsubseteq_s$-hereditary property. Say $A \neq B$. We use induction on n the number of sccs of B. The base case $n = 1$ is trivial. Let $k = \text{l}(B)$. B is a free amalgam of elements of $\mathcal{X}_k \cup \bigcup_{i<k} \mathcal{Y}_i$. Let S be the head scc of a level k ordered extension which is part of the sequence of free amalgamations forming B, where $S \cap A = \varnothing$. So $\text{scl}_B(S) \in \mathcal{X}_k$, and so $\text{scl}_B(S)^\circ \in \mathcal{Y}_{k-1}$. So $B - S$ is a free amalgam of elements of $\mathcal{X}_k \cup \bigcup_{i<k} \mathcal{Y}_i$, and also still satisfies (Str1), (Str2), (Str3), so $B - S \in \mathcal{E}_{\text{fin}}$. As $A \sqsubseteq_s B - S$, by the induction assumption $A \in \mathcal{E}_{\text{fin}}$.

For (4), the free amalgamation property, we start by proving (4)(b). Let $(B_1, \gamma_1) \in \mathcal{E}_{\text{fin}}$ be an ordered extension and let $(B_2, \gamma_2) \in \mathcal{E}_{\text{fin}}$ with $({B_1}^\circ, \gamma_1) \sqsubseteq_s (B_2, \gamma_2)$ and $\text{l}(B_1) \geq \text{l}(B_2)$.

If $\text{l}(B_1) = 0$, then $(B_1, \gamma_1) \in \mathcal{X}_0$, ${B_1}^\circ = \varnothing$ and $(B_2, \gamma_2) \in \mathcal{Y}_0$. (B_2, γ_2) is a disjoint union of ordered extensions in $\mathcal{X}_0$. Let (C, γ) be the disjoint union of $(B_1, \gamma_1), (B_2, \gamma_2)$ where we complete γ according to (Str0). Then $(C, \gamma) \in \mathcal{Y}_0$, and uniqueness up to ordered digraph isomorphism is clear.

For the case $\text{l}(B_1) = k > 0$, we have that $(B_1, \gamma_1) \in \mathcal{X}_k$. By (Ext4), the vertices of $\text{hd}(B_1)$ are greatest in γ_1. By (Str2), the vertices of maximum level $\text{l}(B_2)$ are greatest in γ_2. Let (C, γ) be the free amalgamation of $(B_1, \gamma_1), (B_2, \gamma_2)$ over $({B_1}^\circ, \gamma_1)$, where we complete the linear order γ as follows. Note that as $({B_1}^\circ, \gamma_1) \sqsubseteq_s (B_2, \gamma_2)$ and $\text{l}(B_1) \geq \text{l}(B_2)$, all elements of C of level $< k$ are already comparable in γ, and we need only complete γ for level k. We do this by taking all vertices of $\text{hd}(B_1)$ to be greater than all vertices of level $< k$ in B_2, taking $\text{hd}(B_1)$ to be an interval, and, if $\text{l}(B_2) = k$, ordering $\text{hd}(B_1)$ amongst the sccs of level k in B_2 via λ_B, where we order $\text{hd}(B_1)$ amongst the sccs of its cone in C

arbitrarily. Then (C, γ) satisfies (Str1), (Str2), (Str3), and uniqueness up to isomorphism is clear. (We may reorder sccs within a cone, but these reorderings will be ordered digraph-isomorphic.)

We now prove (4)(a). We use induction on the total number n of sccs of the free amalgamation of B_1, B_2 over A. The base case $n = 1$ is trivial. The case $B_1 = A$ or $B_2 = A$ is also trivial, so we assume B_1, B_2 have sccs outside A. Without loss of generality $\mathrm{l}(B_1) \geq \mathrm{l}(B_2)$.

In the case $\mathrm{l}(A) < \mathrm{l}(B_1)$, take a scc S of B_1 of maximum level, with S outside A. By (3), $(B_1 - S, \gamma_1) \in \mathcal{E}_{\mathrm{fin}}$. By the induction assumption, there exists $(D, \gamma_D) \in \mathcal{E}_{\mathrm{fin}}$ with D the free amalgam of $B_1 - S, B_2$ over A and γ_D extending $\gamma_1|_{B_1 - S}, \gamma_2$. Then use (4)(b) and let (C, γ) be the free amalgam of $(\mathrm{scl}_{B_1}(S), \gamma_1), (D, \gamma_D)$ over $(\mathrm{scl}_{B_1}(S)^\circ, \gamma_1)$ with γ extending $\gamma_1|_{\mathrm{scl}_{B_1}(S)}, \gamma_D$. (C, γ) contains the free amalgam of $(\mathrm{scl}_{B_1}(S), \gamma_1), (B_1 - S, \gamma_1)$ over $(\mathrm{scl}_{B_1}(S)^\circ, \gamma_1)$, and so by the uniqueness up to isomorphism part of (4)(b), (C, γ) contains an isomorphic copy of (B_1, γ_1).

In the case $\mathrm{l}(A) \geq \mathrm{l}(B_1)$, let S be a scc of A with no predecessors. Then, using (3) and the induction assumption, there is $(D, \gamma_D) \in \mathcal{E}_{\mathrm{fin}}$ with D the free amalgam of $B_1 - S, B_2 - S$ over $A - S$ and γ_D extending $\gamma_1|_{B_1 - S}, \gamma_2|_{B_2 - S}$. Then use (4)(b) and let (C, γ) be the free amalgam of $(\mathrm{scl}_A(S), \gamma_A), (D, \gamma_D)$ over $(\mathrm{scl}_A(S)^\circ, \gamma_A)$ with γ extending γ_A, γ_D. By the uniqueness up to isomorphism part of (4)(b), (C, γ) is the free amalgam of isomorphic copies of $(B_1, \gamma_1), (B_2, \gamma_2)$ over an isomorphic copy of (A, γ_A), with γ extending γ_1, γ_2.

We now prove (2). We allow $A = \varnothing$ and thereby also prove (1). It suffices to show that for $A^+ \in \mathcal{E}_{\mathrm{fin}}$, with a sccs, that if $A^+ \sqsubseteq_s B' \in \mathcal{D}^{\prec}_{\mathrm{fin}}$, where B' has $n \geq a$ sccs, then there exists a reordering and reorientation of $B^+ \in \mathcal{E}_{\mathrm{fin}}$ of B', preserving sccs and the orientation of edges between them, with $A^+ \sqsubseteq_s B^+$. We show this by induction on $n \geq a$. The base case $n = a$ is trivial. For the induction step, let S be a scc of $B' - A^+$ of maximum level k, so $B' - S \sqsubseteq_s B'$. Let $D \in \mathcal{E}_{\mathrm{fin}}$ be a reordering and reorientation of $B' - S$ with $A^+ \sqsubseteq_s D$, using the induction assumption. Let B'' be the reordering and reorientation of B' given by replacing B' on $B' - S$ with D. We then use (Ext3) to obtain $E \in \mathcal{X}_k$ similar to $\mathrm{scl}_{B''}(S)$, and by (4) we use free amalgamation of $E, B'' - S$ over E° to obtain a reordering and reorientation $B^+ \in \mathcal{E}_{\mathrm{fin}}$ with $A^+ \sqsubseteq_s B^+$.

(5) and (6) are clear by the construction of $\mathcal{E}_{\mathrm{fin}}$ above.

We now prove (7). We will define γ inductively, and also check that $(B, \gamma) \in \mathcal{E}_{\mathrm{fin}}$ and that γ agrees with γ_B on $\sqsubseteq_s$-copies of (A, γ_A) inductively.

Index the sccs of B as $C_1, \cdots, C_n$, where if $i < j$ then $\mathrm{l}(C_i) \leq \mathrm{l}(C_j)$. For C_1, observe that for any orders γ_1, γ_2 on C_1 such that $(C_1, \gamma_1), (C_1, \gamma_2) \in \mathcal{E}_{\mathrm{fin}}$, we have that $\gamma_1 = \gamma_2$ by (5). So take γ on C_1 such that $(C_1, \gamma) \in$

$\mathcal{E}_{\text{fin}}$, and then γ must agree with any $\sqsubseteq_s$-copy of γ_A on C_1, as $(C, \gamma_A) \in \mathcal{E}_{\text{fin}}$.

Now suppose that γ has already been defined on $C_1, \cdots, C_{i-1}$ such that $(C_1 \cup \cdots \cup C_{i-1}, \gamma) \in \mathcal{E}_{\text{fin}}$ and γ agrees with γ_B on every $\sqsubseteq_s$-copy of (A, γ_A) where γ is defined. We define γ on $D := C_1 \cup \cdots \cup C_i$. Let k be the level of C_i. Let $X = \text{scl}_B(C_i)$. Then $(X^\circ, \gamma) \in \mathcal{E}_{\text{fin}}$, so take γ on C_i such that $(X, \gamma) \in \mathcal{X}_k$.

Then, to define γ on D, specify that:

(i) C_i is an interval in γ;
(ii) C_i is greater than all vertices of level $< k$;
(iii) for C_j $(j < i)$ of level k, $C_j < C_i$ in γ if $C_j < C_i$ in the level-lex preorder λ_D (which depends only on γ on levels $< k$);
(iv) for C_j $(j < i)$ of level k with C_j, C_i lying in the same cone, then order C_j, C_i by their first vertices in γ_B.

Then we have that $(D, \gamma) \in \mathcal{E}_{\text{fin}}$, as we satisfy (Str1), (Str2), (Str3). We now show that γ agrees with γ_B on $\sqsubseteq_s$-copies of (A, γ_A). By the induction assumption, we know that this is the case on $C_1 \cup \cdots \cup C_{i-1}$. C_i lies in some $\sqsubseteq_s$-copy $(A', \gamma_{A'})$ of (A, γ_A). As $\gamma, \gamma_{A'}$ agree on X° and as $(X, \gamma), (X, \gamma_{A'}) \in \mathcal{E}_{\text{fin}}$, by (Ext1) we know that $\gamma, \gamma_{A'}$ agree on C_i.

Each C_j $(j \leq i)$ is an interval in γ, and also in any $\sqsubseteq_s$-copy of (A, γ_A) by (Str3).

Say C_j, C_i $(j < i)$ lie in a $\sqsubseteq_s$-copy $(A', \gamma_{A'})$ of (A, γ_A). If C_j is of level $< k$, then by (Str2) $C_j < C_i$ in $\gamma_{A'}$, and by (ii) $C_j < C_i$ in γ. If C_j is of level k, then as $\gamma_{A'}, \gamma$ agree on levels $< k$, the level-lex preorders $\lambda_{\gamma_{A'}}, \lambda_\gamma$ agree, and if C_j, C_i lie in the same cone, then by (iv) $C_j < C_i$ in $\gamma_{A'}$ iff $C_j < C_i$ in γ. □

PROPOSITION 5.19 (Cor. 7.16, [**6**]). *$(\mathcal{E}_{fin}, \sqsubseteq_s)$ is a Ramsey class.*

PROOF. This is analogous to the proof for $(\mathcal{E}_1, \sqsubseteq_s)$ - i.e. the proof of part (2) of Proposition 4.10. We apply Theorem 1.57 to $\mathcal{D}_{\text{fin}}^{\prec}$, and use part (7) in the definition of $\mathcal{E}_{\text{fin}}$. □

PROPOSITION 5.20 (adapted from Lem. 7.9, [**6**]). *We have that:*

(1) $(\mathcal{E}_{fin}, \sqsubseteq_s)$ is a strong expansion of $(\mathcal{D}_{fin}, \sqsubseteq_s)$;
(2) $\mathcal{E}_{fin}$ is a reasonable expansion of $(\mathcal{D}_{fin}, \sqsubseteq_s)$;
(3) $\mathcal{E}_{fin}$ has the expansion property over $(\mathcal{D}_{fin}, \sqsubseteq_s)$.

PROOF. (1), (2): Considering Definition 5.17 (the definition of $\mathcal{E}_{\text{fin}}$) and its seven parts, we have that:

- part (1) implies part (1) of reasonableness and part (1) of the definition of strong expansion;
- part (2) implies part (4) of reasonableness and part (3) of the definition of strong expansion;
- part (3) implies part (3) of reasonableness.

Part (2) of reasonableness and part (2) of the definition of strong expansions are clear.

(3): The proof of the expansion property is analogous to that of part (3) of Proposition 4.10, with a scc with no predecessors replacing a point. For a very similar proof, see the proof of Lemma 6.10. □

DEFINITION 5.21. We denote the Fraïssé limit of $(\mathcal{E}_{\text{fin}}, \sqsubseteq_s)$ by (M_0, ρ, α), where ρ is the generic orientation and α is the generic linear order. (The notation is the same as in the previous chapter.)

We let $N_\mathcal{E} = (M_0, \rho)$, $K_\mathcal{E} = \text{Aut}(N_\mathcal{E})$ and $H_\mathcal{E} = \text{Aut}(N_\mathcal{E}, \alpha)$.

5.3 Stabilisers of H-fixed points

In this section, we write $H = H_\mathcal{E}, K = K_\mathcal{E}, G = G_0 = \text{Aut}(M_0)$ for brevity. We carry out the same analysis for M_0 as we did for M_1 in Section 4.5. We consider the G-flow $G \curvearrowright \mathcal{LO}(M_0)$, and show that for $\beta \in \mathcal{LO}(M_0)$, if $G_\alpha \subseteq G_\beta$ then $G_\alpha = G_\beta$. (Here, as before, G_α, G_β denote the G-stabilisers of α and β in this flow.) This gives us some information about what β can be in the case where β is an H-fixed point inside a minimal subflow of $\mathcal{LO}(M_0)$.

As just mentioned, we are able to prove the equivalent results on stabilisers of H-fixed points for M_0 as for M_1. The proofs of these results on stabilisers in the M_1 case were similar to the proof of Theorem 4.18 - unfortunately the author has not been able to prove an equivalent of Theorem 4.18 in the case of M_0. There is an added difficulty in the M_0 case which will be explained in the next section.

In the lemma below, recall that H is the automorphism group of $(N_\mathcal{E}, \alpha)$, the Fraïssé limit of $(\mathcal{E}_{\text{fin}}, \sqsubseteq_s)$. (See Definition 5.21.)

LEMMA 5.22 (**). *Consider the G-flow $G \curvearrowright \mathcal{LO}(M_0)$, and let G_α denote the stabiliser of α in this flow. Then $G_\alpha = H$.*

PROOF. Seeking a contradiction, suppose there exists $g \in G_\alpha - H$. Then g must reverse the direction of some out-edge $(x, y) \in \rho$, i.e. $(gy, gx) \in \rho$.

Suppose $x \notin \text{scc}(y)$. So $\text{l}(x) > \text{l}(y)$, and thus $x >_\alpha y$. As g preserves α, $gx >_\alpha gy$. As $(gy, gx) \in \rho$, we must have that $gy \in \text{scc}(gx)$. But the closed interval $[y, x]$ in α is infinite, and sccs are convex in α, so this would imply that $\text{scc}(gx)$ were infinite - as sccs are finite, this gives a contradiction, and so we have that $x \in \text{scc}(y)$.

So we know that g preserves the orientation of out-edges between sccs, and by the same argument, so does g^{-1}.

Take $(x, y) \in \rho$ with y of minimal level k such that $(gy, gx) \in \rho$. So $x \in \text{scc}(y)$ and $gy \in \text{scc}(gx)$. Let $A = \text{scl}(x)$. By the minimality of k,

g does not flip any out-edge within or into A°. By fineness, $g \cdot \mathrm{scc}(x)$ is strongly connected, so is the subset of a scc. Any out-edge into $\mathrm{scc}(x)$ must have its orientation preserved by g, so $g \cdot \mathrm{scc}(x)$ is a scc. So gA is an extension of level k with head scc equal to $g \cdot \mathrm{scc}(x)$.

If $k = 0$, then as g is a graph isomorphism, case (2) of similarity (Definition 5.14) is satisfied. If $k > 0$, then by fineness (part (1) of Lemma 5.11), $\delta(A) = \delta(A^\circ)$, and g as a graph isomorphism preserves predimension, so $\delta(gA) = \delta(g(A^\circ)) = \delta((gA)^\circ)$. Thus g is out-degree-preserving, so case (1) of similarity is satisfied.

So $(A, \alpha), (gA, \alpha)$ are similar extensions, and so are isomorphic as ordered digraphs. By the rigidity of finite linear orders and the fact that g preserves α, g must in fact be the ordered digraph isomorphism given by similarity, and cannot reverse (x, y) - contradiction. So $G_\alpha = H$. $\square$

In the lemma below, recall that K is the automorphism group of $N_\mathcal{E} = (M_0, \rho)$, the oriented graph in the Fraïssé limit $(N_\mathcal{E}, \alpha) = (M_0, \rho, \alpha)$ of $(\mathcal{E}_{\mathrm{fin}}, \sqsubseteq_s)$. (See Definition 5.21.)

LEMMA 5.23 (**). *Consider the G-flow $G \curvearrowright \mathcal{LO}(M_0)$.*
For $\beta \in \mathcal{LO}(M_0)$, if $K_\alpha \subseteq K_\beta$, then $K_\alpha = K_\beta$.

PROOF. From Lemma 5.22, we know that $H = K_\alpha = G_\alpha$. Take $g \in K - K_\alpha$. We will show that $g \notin K_\beta$.

g preserves the orientation ρ but not the order α. So there exist $x, y \in M_0$ with $x >_\alpha y, gx <_\alpha gy$, and take x of minimal level k satisfying this. So g preserves α on levels below k.

Thus $(\mathrm{scl}(x), \alpha), (\mathrm{scl}(gx), \alpha)$ are similar, and so are isomorphic as ordered digraphs (note that the ordered digraph isomorphism is not necessarily g), and likewise for $(\mathrm{scl}(y), \alpha), (\mathrm{scl}(gy), \alpha)$.

If $\mathrm{scc}(x), \mathrm{scc}(y)$ are not in the same cone, then as $x >_\alpha y$, we must either have that (i) $\mathrm{l}(y) < \mathrm{l}(x)$, in which case $\mathrm{l}(gy) < \mathrm{l}(gx)$, so $gy <_\alpha gx$, contradiction; or (ii) $\mathrm{scc}(y) < \mathrm{scc}(x)$ in the level-lex preorder $\lambda = (\trianglelefteq_k, \trianglerighteq_{k-1}, \trianglelefteq_{k-1}, \cdots, \trianglerighteq_0, \trianglelefteq_0)$. If the first coordinate of λ for which $\mathrm{scc}(y)$ is less than $\mathrm{scc}(x)$ is $\trianglelefteq_l$ or $\trianglerighteq_l$, then the same is true for $\mathrm{scc}(gy)$ and $\mathrm{scc}(gx)$, so $gy <_\alpha gx$, contradiction. So $\mathrm{scc}(x), \mathrm{scc}(y)$ lie in the same cone.

Therefore there exists $h \in H$ taking $\mathrm{scl}(y)$ to $\mathrm{scl}(gx)$ and $\mathrm{scl}(x)$ to $\mathrm{scl}(gy)$. $h^{-1}g$ cannot be the identity on $\mathrm{scl}(y) \cup \mathrm{scl}(x)$, as $y <_\alpha x$ but $h^{-1}gy >_\alpha h^{-1}gx$, and therefore $h^{-1}g$ does not fix a linear order. So, as $H = K_\alpha \subseteq K_\beta$, g cannot fix β. $\square$

THEOREM 5.24 (**). *Consider the G-flow $G \curvearrowright \mathcal{LO}(M_0)$.*
For $\beta \in \mathcal{LO}(M_0)$, if $G_\alpha \subseteq G_\beta$ then $G_\alpha = G_\beta$.

PROOF. Seeking a contradiction, say there exists $g \in G_\beta - G_\alpha$. We have that $K_\alpha = G_\alpha$ and $K_\alpha \subseteq K \cap G_\beta = K_\beta$, so $K_\alpha = K_\beta$. So $g \notin K$, i.e. there exist $x, y \in M_0$ such that $(x, y) \in \rho, (gy, gx) \in \rho$.

We will eliminate this possibility case-by-case.

Case 1: x has out-degree 2, with out-edges to y, z, and y is not in the same scc as x.

Let $C = \mathrm{scc}(x)$, and let $A = \mathrm{scl}(C)$. As C is fine and contains a vertex of out-degree 2 in $N_\mathcal{E}$, we have that $\delta(A) = \delta(A^\circ)$ by Lemma 5.11.

Let $n = 2|\,\mathrm{N}_+(C)| + 1$. The cone $Q_\alpha(C)$ is order-isomorphic to $\mathbb{Q}$, and so as $\mathrm{scl}(gA)$ is finite, there exist sccs $C_{-n}, \cdots, C_{-1}, C_1, \cdots, C_n \in Q_\alpha(C)$ with $C_{-n} < \cdots < C_{-1} < C < C_1 < \cdots < C_n$ in α and each gC_i disjoint from $\mathrm{scl}(gA)$. Let $f_i \in H$ denote the (unique) ordered digraph isomorphism sending C to C_i, and let $x_i = f_i(x)$.

As ρ is a 2-orientation, each vertex of $g \cdot \mathrm{N}_+(C)$ has out-degree ≤ 2, and so as $n = 2|\,\mathrm{N}_+(C)| + 1$, there is $-n \leq i \leq -1$ such that the orientation of out-edges from C_i to $\mathrm{N}_+(C)$ is preserved by g, i.e. for $(v, w) \in \rho$ with $v \in C_i, w \in \mathrm{N}_+(C)$, then $(gv, gw) \in \rho$. Likewise there is $1 \leq j \leq n$ such that the orientation of out-edges from C_j to $\mathrm{N}_+(C)$ is preserved by g.

As g is a graph isomorphism, it preserves predimension. We have that $\delta(C_i/\,\mathrm{N}_+(C)) = \delta(C/\,\mathrm{N}_+(C)) = 0$, and so $\delta(gC_i/g\,\mathrm{N}_+(C)) = 0$. Hence if $v \in gC_i, w \in M_0$ and $(v, w) \in \rho$, then $w \in gC_i \cup g\,\mathrm{N}_+(C)$. So for $v \in C_i$, v has out-degree k within C_i iff gv_i has out-degree k within gC_i. Therefore $(C_i, g^{-1}\rho)$ is a reorientation of C_i in which out-degrees within C_i have been preserved, and so $(C_i, g^{-1}\rho)$ is strongly connected by fineness. Thus gC_i is strongly connected in ρ, and therefore is a subset of a scc. As all out-edges from gC_i lie in $g\,\mathrm{N}_+(C) \subseteq gA$ and gC_i is disjoint from $\mathrm{scl}(gA)$, gC_i is a scc. Likewise, gC_j is a scc.

We have that gC_i is a scc and for $v \in gC_i, w \in M_0$, (v, w) is an exiting out-edge from gC_i iff $(g^{-1}v, g^{-1}w)$ is an exiting out-edge from C_i - and likewise for gC_j. Therefore we have that the ordered extensions $(\mathrm{scl}(gC_i), \alpha), (\mathrm{scl}(gC_j), \alpha)$ with head sccs gC_i, gC_j are similar, so are isomorphic as ordered digraphs by part (5) of Definition 5.15, and so by homogeneity of $(N_\mathcal{E}, \alpha)$ there exists $h \in H$ sending $(\mathrm{scl}(gC_i), \alpha)$ to $(\mathrm{scl}(gC_j), \alpha)$ with $h(gC_i) = gC_j$. As $G_\alpha \subseteq G_\beta$, we have $gf_if_j^{-1}g^{-1}h \in G_\beta$, and as $gf_if_j^{-1}g^{-1}h$ fixes gC_i setwise, $gf_if_j^{-1}g^{-1}h$ is the identity on gC_i. Therefore $hgx_i = gx_j$.

The orientation of out-edges from C_i, C_j is preserved by g, so $(gx_i, gy) \in \rho$, $(gx_j, gy) \in \rho$, and since by assumption $(gy, gx) \in \rho$, we have that $gx \in \mathrm{scl}(gC_i)^\circ = \mathrm{scl}(gC_j)^\circ$. Therefore h fixes gx.

We have that either $x_i < x < x_j$ or $x_j < x < x_i$ in β, and as $g \in G_\beta$, therefore $gx_i < gx < gx_j$ or $gx_j < gx < gx_i$ in β. As h fixes gx,

$hgx_i = gx_j$ and $h \in G_\beta$, we obtain our contradiction, completing the elimination of Case 1.

We have shown that the orientation of out-edges from sccs of relative predimension zero is preserved by g. Before eliminating Case 2, we will prove the following claim:

Claim: g preserves sccs.

Proof of claim: first we show that if C is a scc, then gC is strongly connected. There are three cases for C.

Consider the case where C is a scc of relative predimension zero (i.e. $\delta(C/C^\circ) = 0$). We know from Case 1 that g preserves the orientation of out-edges from C. As $\delta(C/\operatorname{N}_+(C)) = 0$ and g preserves predimension, we have that $\delta(gC/g\operatorname{N}_+(C)) = 0$. Thus if $v \in gC, w \in M_0$ and $(v, w) \in \rho$ then $w \in gC \cup g\operatorname{N}_+(C)$. So for $v \in C$, v has out-degree k within C iff gv has out-degree k within gC, and so by fineness (as in Case 1), gC is strongly connected.

In the case $|C| = 1$, then gC is trivially strongly connected.

In the final case where $\mathrm{l}(C) = 0$ (including the case $\delta(C) = 1$), then $(C, g^{-1}\rho)$ is a reorientation of C, so is strongly connected by fineness - i.e. gC is strongly connected in ρ. This completes the proof that if C is a scc, then gC is strongly connected.

Hence, for any scc C, $|C| \leq |\operatorname{scc}(gC)|$. The same argument for $g^{-1} \in G_\beta - G_\alpha$ applied to $\operatorname{scc}(gC)$ shows that $g^{-1}\operatorname{scc}(gC)$ is strongly connected. But $C \subseteq g^{-1}\operatorname{scc}(gC)$ is a scc, and so $g^{-1}\operatorname{scc}(gC) = C$. Hence $|\operatorname{scc}(gC)| \leq |C|$, so $|C| = |\operatorname{scc}(gC)|$ and thus gC is a scc, proving the claim.

Case 2: x has out-degree 1, with out-edge to y, where $y \notin \operatorname{scc}(x)$.

The cone $Q_\alpha(x)$ is order-isomorphic to $\mathbb{Q}$, and so as $\operatorname{scl}(gy)$ is finite, there exist $x_{-3}, x_{-2}, x_{-1}, x_1, x_2, x_3 \in Q_\alpha(x)$ with $x_{-3} < x_{-2} < x_{-1} < x < x_1 < x_2 < x_3$ in α and each $gx_i \notin \operatorname{scl}(gy)$. As ρ is a 2-orientation, there are i, j with $-3 \leq i \leq -1$ and $1 \leq j \leq 3$ such that g preserves the orientation of $(x_i, y), (x_j, y)$, i.e. $(gx_i, gy) \in \rho, (gx_j, gy) \in \rho$.

Let $(w, x_i) \in \rho$ be an in-edge of x_i. Seeking a contradiction, suppose $(gx_i, gw) \in \rho$. If $gw \notin \operatorname{scc}(gx_i)$, then (gx_i, gw) is an out-edge of $\operatorname{scc}(gx_i)$ which is of relative predimension zero, so by Case 1 applied to g^{-1}, $(x_i, w) \in \rho$, contradiction. If $gw \in \operatorname{scc}(gx_i)$, as g^{-1} preserves sccs, $w \in \operatorname{scc}(x_i)$, contradiction. Thus g preserves the orientation of in-edges of x_i, and so $\{gx_i\}$ is a scc, as (gx_i, gy) is the only out-edge of gx_i and $gx_i \notin \operatorname{scl}(gy)$. Likewise $\{gx_j\}$ is a scc.

Hence, the ordered extensions $(\operatorname{scl}(gx_i), \alpha), (\operatorname{scl}(gx_j), \alpha)$ with head sccs $\{gx_i\}, \{gx_j\}$ are similar, so are isomorphic as ordered digraphs by part (5) of Definition 5.15, and so by homogeneity of $(N_\mathcal{E}, \alpha)$ there exists $h \in H$ sending $(\operatorname{scl}(gx_i), \alpha)$ to $(\operatorname{scl}(gx_j), \alpha)$ with $hgx_i = gx_j$ and $\operatorname{scl}(gx_i)^\circ =$

$\mathrm{scl}(gx_j)^\circ$ fixed. Specifically, as $(gx_i, gy) \in \rho, (gy, gx) \in \rho$, we have that h fixes gx.

We have that either $x_i < x < x_j$ or $x_j < x < x_i$ in β, and as $g \in G_\beta$, thus $gx_i < gx < gx_j$ or $gx_j < gx < gx_i$ in β. As $h \in G_\beta$, h fixes gx and $hgx_i = hgx_j$, we obtain a contradiction.

Case 3: x has out-degree 2, with out-edges to y, z, and y is in the same scc as x.

Let $C = \mathrm{scc}(x)$, and assume C is of minimum level - that is, C is a scc of minimum level such that there are $x, y \in C$ with $(x, y) \in \rho, (gy, gx) \in \rho$. Let $A = \mathrm{scl}(C)$. We have already shown that g preserves sccs and out-edges from sccs, and by the minimality of the level of C we know that g preserves the orientation of edges within sccs of A°. Therefore (gA, α) is an ordered extension with head scc gC which is similar to (A, α), and so as before, by part (5) of Definition 5.15 and homogeneity of $(N_\mathcal{E}, \alpha)$, there exists $h \in H$ sending (A, α) to (gA, α). In particular, h sends C to gC, and as $h \in G_\beta$, by the rigidity of finite linear orders h agrees with g on C. But h is a digraph isomorphism, so $(x, y) \in \rho \Rightarrow (hx, hy) \in \rho$, i.e. $(gx, gy) \in \rho$, contradiction.

Case 4: x has out-degree 1, with out-edge to $y \in \mathrm{scc}(x)$.

Let $C = \mathrm{scc}(x)$. By Lemma 5.11, we have that $\mathrm{l}(C) = 0, \delta(C) = 1$. As g preserves sccs and out-edges from sccs, gC is also a scc with $\mathrm{l}(gC) = 0$, and as g preserves predimension $\delta(gC) = 1$. So $(C, \alpha), (gC, \alpha)$ are similar (by case (2) of similarity, where we need only have a graph isomorphism). The rest of the argument is as for Case 3. □

5.4 A difficulty in extending results from M_1 to M_0

We now explain why the proof of Theorem 4.18 does not immediately carry over to the case of M_0.

The problem is as follows. In the proof of Theorem 4.18, we use a finite ordered graph in $\mathrm{Age}(M_1, \beta)$ to force a certain orientation in the generic orientation ρ, and we do this using α-automorphisms and the fact that $G_\alpha \subseteq G_\beta$ (see Figure 4.3). However, in the M_0 case, it could be that an element A of $\mathrm{Age}(M_0, \beta)$, when $\leq_s$-embedded in ρ, is inside a scc - as seen in the below proposition. This seems to prevent us from using α-automorphisms and $\sqsubseteq_s$-homogeneity to find out information about the order β on A.

More specifically, the below proposition shows that we cannot use the graph structure alone to guarantee that A does not lie inside a scc. It may be that the order β on A can be used to show that A cannot lie inside a scc, but it is not clear to the author how to do this.

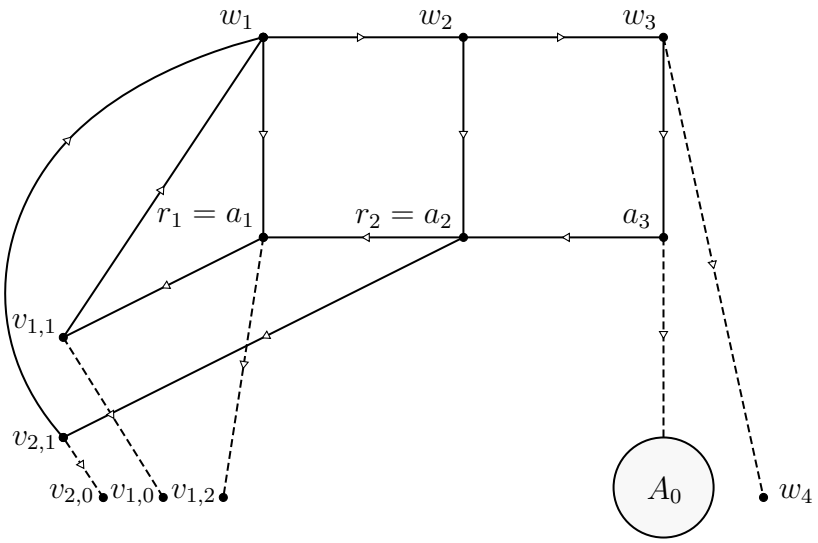


FIGURE 5.2

PROPOSITION 5.25 (**). *Take $A \in \mathcal{D}_{fin}$ with $\delta(A) > 0$. Let A_0 be the maximal successor-closed substructure of A with $\delta(A_0) = 0$. Then there exists $B \in \mathcal{D}_{fin}$ and a $\leq_s$-digraph embedding $f : A \to B$ such that $f(A - A_0)$ is contained within a scc of B.*

PROOF. Let $a_1, \cdots, a_n$ be the vertices of $A - A_0$. As $\delta(A) > 0$, A must have at least one root. Let $r_1, \cdots, r_k$ be the roots of A, with multiplicities $m_1, \cdots, m_k$. As $\delta(A_0) = 0$, $r_1, \cdots, r_k$ lie in $A - A_0$.

Let B be the 2-oriented graph given by the following (see Figure 5.2 for an example):

- B contains A;
- for each root r_i of A with multiplicity m_i, add new vertices $v_{i,0}, \cdots, v_{i,m_i}$ and out-edges $(r_i, v_{i,j})$ $(1 \leq j \leq m_i)$, $(v_{i,1}, v_{i,0})$;
- add new vertices $w_1, \cdots, w_{n+1}$ and out-edges $(v_{i,1}, w_1)$ $(1 \leq i \leq k)$, (w_j, a_j) (for $1 \leq j \leq n$), (w_j, w_{j+1}) (for $1 \leq j \leq n$).

We have that $B \in \mathcal{D}_0$, and we can reverse the orientations of the out-edges $(v_{i,1}, v_{i,0})$, $(r_i, v_{i,j})$ $(1 \leq i \leq k, 1 \leq j \leq m_i)$ to produce a new orientation B' of B in which $A \sqsubseteq_s B'$, so $A \leq_s B$.

Let $C = (A - A_0) \cup \{v_{i,1} : 1 \leq i \leq k\} \cup \{w_j : 1 \leq j \leq n\}$. First we show that C is a scc of B. For $1 \leq i \leq k$, the sequence of out-edges $(v_{i,1}, w_1), (w_1, w_2), \cdots, (w_{j-1}, w_j)$ gives an out-path from $v_{i,1}$ to w_j $(1 \leq j \leq n)$, which with the additional out-edge (w_j, a_j) gives an out-path from $v_{i,1}$ to a_j $(1 \leq j \leq n)$. As each r_l $(1 \leq l \leq k)$ is a vertex of $A - A_0$ (i.e. $r_i = a_j$ for some j), the additional out-edge $(r_l, v_{l,1})$ gives an out-path from $v_{i,1}$ to $v_{l,1}$. We therefore have an out-path from $v_{i,1}$ to each vertex of C. For $a \in A - A_0$, $\mathrm{scl}_A(a)$ must contain some root r_i, and so with the out-edge $(r_i, v_{i,1})$ we must have an out-path

from a to $v_{i,1}$, and thus from a to each vertex of C. Finally, for each w_j $(1 \leq j \leq n)$, the out-edge (w_j, a_j) gives an out-path from w_j to a_j, and therefore we have an out-path from w_j to each vertex of C. Thus C is a scc.

As each vertex of C has out-degree 2 in B, C is a fine scc, and as each out-edge from C either has its out-vertex in $A_0 \in \mathcal{D}_{\text{fin}}$ or has its out-vertex forming a scc of size one (as in the case of the out-edges $(v_{i,1}, v_{i,0})$ $(1 \leq i \leq k)$, $(r_i, v_{i,j})$ $(1 \leq i \leq k, 2 \leq j \leq m_i)$, (w_n, w_{n+1})), we have that $B \in \mathcal{D}_{\text{fin}}$. □

Note that the proof of the above proposition can easily be adapted to show an analogous result for $\mathcal{D}_0$ in place of $\mathcal{D}_{\text{fin}}$.

Chapter 6

The universal minimal flow of $\mathrm{Aut}(M_{00})$

In this chapter, we find a coprecompact (reasonable, strong) expansion $\mathcal{E}_{00}$ of $\mathcal{C}_{00}$ (the class of 2-sparse graphs of predimension zero), which is rigid and has the Ramsey property and the expansion property, and therefore specifies the universal minimal flow of G_{00}, the automorphism group of the Fraïssé limit of $(\mathcal{C}_{00}, \leq_s)$. We do this by using the admissibly ordered orientations from the previous chapter, which behave in a much simpler fashion for the predimension zero case. We then use a simplified version of a result of Evans, Ghadernezhad and Tent ([**12**]), which states that any element of G_{00} may be extended to an element of G_0, to obtain a G_0-flow with a comeagre orbit.

6.1 A key observation

Definition 6.1. Let $\mathcal{C}_{00}$ be the class of finite 2-sparse graphs of predimension zero - that is, $\mathcal{C}_{00}$ consists of $A \in \mathcal{C}_0$ with $\delta(A) = 0$.

Let $\mathcal{D}_{00}$ be the class of finite 2-oriented graphs of predimension zero. As predimension is dependent only on the underlying graph structure, by Proposition 1.7 we have that $\mathcal{C}_{00}$ is the class of graph reducts of $\mathcal{D}_{00}$.

Note that for $B \in \mathcal{C}_{00}$ and $A \subseteq B$, we do not necessarily have $A \in \mathcal{C}_{00}$, unlike in the case of $\mathcal{C}_0$.

Lemma 6.2.

(1) $(\mathcal{C}_{00}, \leq_s)$ *and* $(\mathcal{D}_{00}, \sqsubseteq_s)$ *are strong classes.*
(2) $(\mathcal{D}_{00}, \sqsubseteq_s)$ *is a strong expansion of* $(\mathcal{C}_{00}, \leq_s)$.
(3) $(\mathcal{D}_{00}, \sqsubseteq_s)$ *and* $(\mathcal{C}_{00}, \leq_s)$ *are free amalgamation classes.*

Proof. (1): this follows by a straightforward check of (S1)-(S3) in Definition 1.15. (Note that we cannot use Lemma 1.30.)

(2): this is analogous to the proof of Lemma 1.35.

(3): Recall the fact that for $A, B, C \in \mathcal{C}_0$, if B, C are freely amalgamated over A then $\delta(B \cup C) = \delta(B) + \delta(C) - \delta(A)$ (this is the equality case of Lemma 1.29). Therefore $(\mathcal{D}_{00}, \sqsubseteq_s)$ is immediately a free amalgamation class. Using part (2), by Lemma 1.26 we have that $(\mathcal{C}_{00}, \leq_s)$ is a free amalgamation class. □

DEFINITION 6.3. Let M_{00} denote the Fraïssé limit of $(\mathcal{C}_{00}, \leq_s)$. As $\mathcal{C}_{00} \subseteq \mathcal{C}_0$, by the extension property of M_0 we may take $M_{00} \subseteq M_0$. Let $G_{00} = \mathrm{Aut}(M_{00})$.

Many aspects of the analysis in the preceding chapter are rather simplified in the predimension zero case.

Every oriented graph in $\mathcal{D}_{00}$ consists entirely of vertices of out-degree 2, and so all graph isomorphisms between elements of $\mathcal{D}_{00}$ are out-degree-preserving. By Lemma 5.4, any two orientations of $A \in \mathcal{C}_{00}$ must have the same successor-closed subsets, sccs and orientation of edges between sccs. So, with the same definition of extension as before, there is no distinction between orientations and fine orientations within $\mathcal{D}_{00}$: all orientations within $\mathcal{D}_{00}$ are fine.

The below key observation shows that $\leq_s$, $\sqsubseteq_s$ are essentially synonymous in the predimension zero case:

LEMMA 6.4 (*). *Let $A \leq_s B \in \mathcal{C}_{00}$, and let $B^+ \in \mathcal{D}_{00}$ be an orientation of B. Then $A \sqsubseteq_s B^+$, i.e. the orientation induced on A by B^+ is successor-closed in B^+.*

PROOF 1. As $A \leq_s B$, there exists an orientation B' of B in which $A \sqsubseteq_s B'$. As any two orientations of B have the same successor-closed subsets, $A \sqsubseteq_s B^+$. □

PROOF 2. As $\delta(A) = 0$, A has no roots (considered in A). So there are no out-edges from A to $B^+ - A$. □

6.2 Admissibly ordered orientations on $\mathcal{C}_{00}$

The definition of similarity in $\mathcal{D}^{\prec}_{\mathrm{fin}}$ reduces to the following:

DEFINITION 6.5. Let $A, B \in \mathcal{D}^{\prec}_{00}$ be ordered extensions. A, B are *similar* if there exists a graph isomorphism $A \to B$ which is also an ordered digraph isomorphism $A^\circ \to B^\circ$.

DEFINITION 6.6. $\mathcal{E}_{00} \subseteq \mathcal{D}^{\prec}_{00}$ is a *class of admissibly ordered orientations of* $\mathcal{C}_{00}$ if $\mathcal{E}_{00}$ satisfies parts (1)-(7) of Definition 5.17 with $\mathcal{C}_{00}, \mathcal{D}^{\prec}_{00}$ replacing $\mathcal{C}_0, \mathcal{D}^{\prec}_{\mathrm{fin}}$.

The proofs in the previous chapter straightforwardly restricted to the case of graphs of predimension zero give:

PROPOSITION 6.7. *There exists a class $(\mathcal{E}_{00}, \sqsubseteq_s)$ of admissibly ordered orientations of $\mathcal{C}_{00}$.*

$(\mathcal{E}_{00}, \sqsubseteq_s)$ is a Ramsey class, and a strong, reasonable expansion of $(\mathcal{D}_{00}, \sqsubseteq_s)$.

Differently to the case of $\mathcal{E}_{\mathrm{fin}}$, we have that:

LEMMA 6.8. *$\mathcal{E}_{00}$ is a reasonable expansion of $(\mathcal{C}_{00}, \leq_s)$.*

PROOF. The only part of reasonableness left to check is part (3): we need to show that $\mathcal{E}_{00}$ is $\leq_s$-closed. This follows immediately from Lemma 6.4 and the fact that $(\mathcal{E}_{00}, \sqsubseteq_s)$ has the hereditary property. □

We will now show that $(\mathcal{E}_{00}, \sqsubseteq_s)$ has the properties required for $X(\mathcal{E}_{00})$ to be the universal minimal flow of G_{00}.

LEMMA 6.9 (**). *$\mathcal{E}_{00}$ is a coprecompact expansion of $\mathcal{C}_{00}$.*

PROOF. For $A \in \mathcal{C}_{00}$, $C^+ \in \mathcal{E}_{00}$ with $A \leq_s C^+$, let $A^+ \in \mathcal{E}_{00}$ be the ordered orientation induced on A by C^+. Then $A^+ \sqsubseteq_s C^+$. As $A \in \mathcal{C}_{00}$ has finitely many expansions in $\mathcal{E}_{00}$, this shows coprecompactness by Lemma 1.75. □

LEMMA 6.10 (**). *$\mathcal{E}_{00}$ has the expansion property over $(\mathcal{C}_{00}, \leq_s)$.*

PROOF. We must show that for $A^+ = (A, \rho_A, \gamma_A) \in \mathcal{E}_{00}$, there exists $B \in \mathcal{C}_{00}$ such that for any expansion $(B, \rho_B, \gamma_B) \in \mathcal{E}_{00}$, there exists a $\leq_s$-embedding $A^+ \to (B, \rho_B, \gamma_B)$. We use induction on $|A|$, and the base case $|A| = 1$ is trivial.

There exists a scc S of A^+ with no predecessors. Let $X^+ = A^+ - Q_{A^+}(S)$, with the induced ordered orientation from A^+. Then $X^+ \sqsubseteq_s A^+$, so $X^+ \in \mathcal{E}_{00}$, and by the induction assumption there exists $Y \in \mathcal{C}_{00}$ such that any expansion $Y^+ \in \mathcal{E}_{00}$ of Y contains a $\leq_s$-copy of X^+. Let X be the underlying graph of X^+.

Let $X_1, \cdots, X_n$ be the $\leq_s$-copies of X in Y. Let $Y_0 = Y$, and inductively define Y_i $(1 \leq i \leq n)$ to be the free amalgam of Y_{i-1} with a copy of A over X_i (where we take the copy A' of A with $X_i \leq_s A'$). Let $Z = Y_n$. As $\mathcal{C}_{00}$ is a free amalgamation class, $Z \in \mathcal{C}_{00}$. We have that $Y_i \leq_s Z$ $(0 \leq i \leq n)$, and each copy of A in the sequence of free amalgamations is $\leq_s$-closed in Z. We will show that Z witnesses the expansion property for A^+.

Let $Z^+ = (Z, \rho_Z, \gamma_Z) \in \mathcal{E}_{00}$ be an expansion of Z. As $Y \leq_s Z$, by Lemma 6.4 $(Y, \rho_Z, \gamma_Z) \sqsubseteq_s Z^+$, so $(Y, \rho_Z, \gamma_Z) \in \mathcal{E}_{00}$, and so there is a $\leq_s$-embedding $f : X^+ \to (Y, \rho_Z, \gamma_Z)$. $f(X)$ is equal to some X_i, and $X_i \leq_s A'$ for some copy $A' \leq_s Z$ of A in Z.

By Lemma 6.4, f is a $\sqsubseteq_s$-embedding and $f(X^+) \sqsubseteq_s (A', \rho_Z, \gamma_Z) \sqsubseteq_s Z^+$.

By Lemma 5.4, there is a graph isomorphism $g : A \to A'$ which extends f and preserves sccs and the orientation of edges between them for A^+ and (A', ρ_Z).

Let T be a scc of (A', ρ_Z) lying outside $f(X^+)$. Then we have that $(\mathrm{scl}_{A'}(T), \gamma_Z)$, $(\mathrm{scl}_{A^+}(S), \gamma_A)$ are similar, and so they are ordered digraph isomorphic. We may therefore take g to be an ordered digraph isomorphism. We have that $g : A^+ \to (A', \rho_Z, \gamma_Z) \leq_s Z$, so g witnesses the expansion property for A^+. □

THEOREM 6.11. *$X(\mathcal{E}_{00})$ is the universal minimal flow of G_{00}, and has a comeagre orbit consisting of the expansions of M_{00} which are isomorphic to the Fraïssé limit of $(\mathcal{E}_{00}, \sqsubseteq_s)$.*

PROOF. This follows immediately by Theorem 1.76. □

6.3 A G_0-flow with a comeagre orbit

The following lemma is a simple particular case of [**12**], Lemma 4.2.1:

LEMMA 6.12. *Let $j \in G_{00}$. Then j can be extended to an element of G_0.*

PROOF. Take $A_0 \leq_s M_0$ with $\delta(A_0) = 0$, and take A_i, $i \geq 1$, such that $A_0 \leq_s A_1 \leq_s \cdots$ is an increasing $\leq_s$-chain with $M_0 = \bigcup_{i\geq 0} A_i$. Let $B_0 = jA_0$, and take B_i, $i \geq 1$, such that $B_0 \leq_s B_1 \leq_s \cdots$ is an increasing $\leq_s$-chain with $M_0 = \bigcup_{i\geq 0} B_i$.

We will find $g \in \mathrm{Aut}(M_0)$ extending j via a back-and-forth argument. Specifically, we will define a sequence of partial isomorphisms $g_i : C_i \to D_i$, $C_i, D_i \leq_s M_0$, $i \in \mathbb{N}$, such that, for $i \in \mathbb{N}$,

(1) g_{i+1} extends g_i;
(2) $A_i \subseteq C_i$ and $B_i \subseteq D_i$;
(3) g_i, j agree on $C_i \cap M_{00}$.
(Note that because graph isomorphisms preserve predimension, $g_i(C_i \cap M_{00}) = D_i \cap M_{00}$.)

Once this sequence $(g_i)_{i\in\mathbb{N}}$ has been defined, we will let $g = \bigcup_{i\in\mathbb{N}} g_i$, and then g will be the automorphism of M_0 extending j that we seek.

We define $(g_i)_{i\in\mathbb{N}}$ by induction. Let $g_0 = j|_{A_0} : A_0 \to B_0$ (so here we have $C_0 = A_0$, $D_0 = B_0 = jA_0$). Suppose we have already defined $g_0, \cdots, g_{n-1}$ satisfying (1), (2), (3). We now define g_n.

Let $P = \mathrm{cl}^s(C_{n-1} \cup A_n)$. Then $C_{n-1} \cup (P \cap M_{00}) \leq_s P$: we may see this by noting that as $C_{n-1} \leq_s M_0$, therefore $C_{n-1} \leq_s P$, and so there exists a 2-orientation of P in which C_{n-1} is successor-closed; as $\delta(P \cap M_{00}) = 0$, $P \cap M_{00}$ is successor-closed in this orientation.

By (3), $g_{n-1} : C_{n-1} \to D_{n-1}$ and $j|_{P\cap M_{00}}$ are compatible functions. We have that $D_{n-1} \cup j(P \cap M_{00}) \leq_s M_0$ by a similar argument to the previous paragraph, and therefore we have a partial isomorphism $g_{n-1} \cup j|_{P\cap M_{00}}$ between $\leq_s$-substructures of M_0. By $\leq_s$-homogeneity of M_0, there is $h_n \in \mathrm{Aut}(M_0)$ extending $g_{n-1} \cup j|_{P\cap M_{00}}$. Let $Q = h_n(P) \leq_s M_0$.

Then $h_n : P \to Q$ is a partial isomorphism between $\leq_s$-substructures of M_0 extending g_{n-1} with $A_n \subseteq P$ such that h_n, j agree on $P \cap M_{00}$. This completes the "forth" step of the back-and-forth construction.

The "back" step is similar. We let $D_n = \mathrm{cl}^s(Q \cup B_n)$, and as before, we have $Q \cup (D_n \cap M_{00}) \leq_s D_n$. As h_n, j agree on $P \cap M_{00}$ and $h_n(P \cap M_{00}) = Q \cap M_{00}$ (as graph isomorphisms preserve predimension), we have that h_n^{-1}, j^{-1} agree on $Q \cap M_{00}$, and so $h_n^{-1} : Q \to P$, $j^{-1}|_{D_n \cap M_{00}}$ are compatible functions. As before, $P \cup j^{-1}(D_n \cap M_{00}) \leq_s M_0$, and so we have a partial isomorphism $h_n^{-1} \cup j^{-1}|_{D_n \cap M_{00}}$ between $\leq_s$-substructures of M_0. By $\leq_s$-homogeneity of M_0, there is $g_n \in \mathrm{Aut}(M_0)$ such that g_n^{-1} extends $h_n^{-1} \cup j^{-1}|_{D_n \cap M_{00}}$. Let $C_n = g_n^{-1}(D_n) \leq_s M_0$. Then $g_n : C_n \to D_n$ is a partial isomorphism between $\leq_s$-substructures of M_0 with $B_n \subseteq D_n$ and $A_n \subseteq P \subseteq C_n$. As g_n^{-1}, j^{-1} agree on $D_n \cap M_{00}$ and $C_n \cap M_{00} = g_n^{-1}(D_n \cap M_{00})$, we have that g_n, j agree on $C_n \cap M_{00}$, completing the "back" step of the back-and-forth construction. □

Thus we have that:

THEOREM 6.13. *There is a G_0-flow with a comeagre orbit.*

PROOF. Let $r : G_0 \to G_{00}$, $r(g) = g|_{M_{00}}$ be the restriction map. By the above lemma r is surjective, and it is straightforward to see that r is continuous. Therefore, taking the universal minimal flow $X(\mathcal{E}_{00})$ of G_{00} and precomposing with r, we obtain a G_0-flow $G_0 \curvearrowright X(\mathcal{E}_{00})$ with a comeagre G_0-orbit. □

REMARK 6.14. Note that the G_0-flow $G_0 \curvearrowright X(\mathcal{E}_{00})$ is not faithful.

Chapter 7

Further questions

QUESTION 7.1. Do minimal subflows of $\mathcal{LO}(M_0)$ or $\mathcal{LO}(M_F)$ have all G-orbits meagre?

QUESTION 7.2. Can we adapt the construction of $\mathcal{E}_{\text{fin}}$ to force certain orientations within sccs, and thereby extend Theorem 4.18 from M_1 to M_0?

QUESTION 7.3. Is there an "acyclic" version of M_F? Can we then apply the proof method of Theorem 4.18?

Specifically, for $A, B \in \mathcal{C}_1$, say $A \leq_e B$ if there exists an acyclic 2-orientation B^+ of B in which $A \sqsubseteq_s B^+$ and there does not exist $b \in B$ with two out-edges into A. Can we define a subclass $\mathcal{C}_G$ of $\mathcal{C}_1$ such that $(\mathcal{C}_G, \leq_e)$ is a free amalgamation class with ω-categorical Fraïssé limit?

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