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Enumerative Geometry of GIT Quotients

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Declaration of Originality

I declare that the work in this thesis is my own and that any collaboration is properly acknowledged and referenced. In particular, Chapter 2 is joint work with Tom Coates and Wendelin Lutz from [CLS21], which has been accepted for publication in Forum of Mathematics, Sigma.

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Abstract

In this thesis we study the enumerative geometry of certain GIT quotients. Chapter 2 details work (joint with T. Coates and W. Lutz) on the comparison between genus-zero Gromov–Witten invariants of X and of a blow up of X. We do this by rewriting a blow up as a subvariety of a certain GIT quotient and extending the Abelian/non-Abelian correspondence, which compares genus-zero Gromov–Witten invariants of a GIT quotient by a non-abelian group with the corresponding quotient by the maximal torus. We also give a reformulation of the Abelian/non-Abelian correspondence in terms of Givental's Lagrangian cones, which suggests a relationship in higher genus. In Chapter 3 we build a theory of logarithmic quasimaps for smooth projective toric varieties relative a simple normal crossings divisor in any genus. To do this we construct a proper Deligne–Mumford moduli stack compactifying the space of quasimaps from smooth marked curves of fixed degree and genus satisfying prescribed tangency conditions to the divisor at the markings. We give a sketch of the construction of the virtual fundamental class using the formalism of Behrend and Fantechi. We then look at how this theory interacts with the local-logarithmic correspondence.

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Contents

Chapter 1

Introduction

1.1 Enumerative Geometry

Enumerative geometry is one of the oldest subjects in mathematics. Throughout the years it has undergone a transformation, with the emergence of moduli and intersection theory and later from developments inspired by theoretical physics. During the 18th century mathematicians came to realise, through results like Bézout's Theorem, that there were alternative methods for answering enumerative questions than direct computation. This culminated in the advent of Schubert calculus, which hinted that questions should obey the 'principle of conservation of number'. The development of moduli theory involved switching perspective to thinking about parameter spaces whose points represent the geometric objects being considered. This allowed enumerative questions to be reframed in terms of counting intersection points of various subsets of a large parameter space. A necessary ingredient for this change in perspective was a rigorous framework for intersection theory, provided by Fulton and Macpherson. We give a simple example of this philosophy below. Suppose we want to count the number of (smooth) conics in $\mathbb{P}^2$ passing through 5 general points. A conic C in $\mathbb{P}^2$ is the vanishing locus of a quadratic equation

$$a_0x^2 + a_1xy + a_2y^2 + a_3xz + a_4yz + a_5z^2$$

so the space of conics in $\mathbb{P}^2$ is $\mathbb{P}^5$ with coordinates $[a_0 : a_1 : a_2 : a_3 : a_4 : a_5]$. The condition to pass through a point is a linear condition on $\mathbb{P}^5$. For example, for C to pass through $[1 : 0 : 0]$ the condition is that $a_0 = 0$. Therefore, the condition of passing through each of the 5 points defines a hyperplane in $\mathbb{P}^5$, the intersection of all 5 hyperplanes defines a single point, so the number of conics in $\mathbb{P}^2$ passing through 5 points is 1.

1.2 Gromov–Witten Theory

Gromov–Witten theory is one of the modern curve counting theories which follows a similar approach. The origins of this mathematics came, surprisingly, out of string theory, which led physicists to make astonishing predictions in enumerative geometry. Gromov–Witten theory was the result of mathematicians attemping to provide rigour to these predictions.

Fix a smooth projective variety X, non-negative integers g, n and a curve class $\beta \in A_1(X)$. In Gromov–Witten theory, rather than studying embedded curves, one considers parametrised curves $f \colon C \to X$, with C a nodal curve of (arithmetic) genus g such that $f_*[C] = \beta$. In order to impose conditions on these curves, one fixes n points $p_1, \ldots, p_n \in C$ called *marked* points. Subject to a stability condition, these objects together form

$$\overline{\mathcal{M}}_{g,n}(X, \beta)$$

the *moduli space of stable maps*. This moduli space is not a scheme in general, but is a proper Deligne–Mumford stack. Even though X is required to be smooth, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is rarely smooth; it may often have excess components of high dimension. For this reason there is often no fundamental class of this moduli space, at least not of the expected dimension. However, it does admit a substitute for this class, *the virtual fundamental class* in the Chow group (or homology group) of the expected or *virtual dimension*

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X, \beta)).$$

For each $i \in \{1, \ldots, n\}$ there is an evaluation map

$$\mathrm{ev}_i \colon \overline{\mathcal{M}}_{g,n}(X, \beta) \to X$$

where $(C, p_1 \ldots, p_n, f \colon C \to X) \mapsto f(p_i)$ Let $Y_1, \ldots, Y_n$ be subvarieties of X. If $\overline{\mathcal{M}}_{g,n}(X, \beta)$ was smooth of the expected dimension and $\alpha_i = \mathrm{PD}(Y_i)$ with $\sum_i \deg \alpha_i = \dim(\overline{\mathcal{M}}_{g,n}(X, \beta))$ then the number of curves in X passing through each Y_i at p_i would be

$$\deg\left(\mathrm{ev}_1^{-1}(Y_1) \cap \cdots \cap \mathrm{ev}_n^{-1}(Y_n)\right) = \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]} \mathrm{ev}_1^* \alpha_1 \smile \cdots \smile \mathrm{ev}_n^* \alpha_n.$$

For this reason one defines *Gromov–Witten invariants* as integrals

$$\int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \alpha_1 \smile \cdots \smile \mathrm{ev}_n^* \alpha_n.$$

1.3 Quasimap Theory

There is another modern curve counting theory which was developed later [MOP11, CFK10, CFKM14]. Unlike Gromov–Witten theory it places some restrictions on the target X; most generally X can be a certain GIT quotient. The idea is to use the extra structure which comes with this GIT presentation. Once again, fix g, n non-negative integers and a curve class $\beta \in A_1(X)$. Quasimaps are roughly parametrised curves $C \dashrightarrow X$, where C is again, a nodal, n-marked curve of arithmetic genus g, but the dashed arrow indicates that the morphism to X is allowed to acquire some basepoints. Subject to a stronger stability condition, these objects form

$$\mathcal{Q}_{g,n}(X, \beta)$$

the *moduli space of stable quasimaps*. As with Gromov–Witten theory, the space admits a virtual class $[\mathcal{Q}_{g,n}(X, \beta)]^{\mathrm{vir}}$ and evaluation maps so that one can define quasimap invariants as integrals

$$\int_{[\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \alpha_1 \smile \cdots \smile \mathrm{ev}_n^* \alpha_n.$$

In general quasimap moduli spaces are simpler than their Gromov–Witten counterparts, for example, they often have fewer components. This makes for easier computations on the quasimap side. Moreover, one of the most powerful applications of quasimaps is that the invariants can be related to Gromov–Witten invariants as follows. Although the moduli spaces for quasimap theory and Gromov–Witten theory differ, one can interpolate between the two, depending on a parameter $\epsilon \in \mathbb{Q}_{>0}$, where large ϵ recovers Gromov–Witten theory and small ϵ recovers quasimap theory. After fixing numerical data, there are finitely many chambers which give genuinely different compactifications. There are *wall-crossing formulas* which relate the invariants for different values of ϵ [CFK20, Zho22], which has provided a two step procedure for computing Gromov–Witten invariants. These wall-crossing formulas show up fundamentally in the computations coming from string theory which motivated the beginnings of Gromov–Witten theory.

1.4 Mirror Symmetry and Wall Crossing

Mirror symmetry is a branch of mathematics with its origins in string theory, which originally predicted, for every Calabi–Yau manifold X, the existence of a *mirror* Calabi–Yau $\check{X}$ such that certain precise geometries get exchanged when passing to the mirror. Gromov–Witten theory grew out of an attempt make rigorous the predictions of string theory. Specifically, in 1991, Candelas, de la Ossa, Green and Parkes [CdlOGP92] conjectured that certain path integral calculations on the mirror to the quintic threefold gave rise to predictions for numbers of degree d rational curves on the quintic which agreed with the known calculations by mathematicians up to degree 3. This inspired the modern breakthrough in enumerative geometry giving the moduli space of stable maps and Gromov–Witten theory. These predictions were rigorously verified by Alexander Givental [Giv96, Giv98] using two cohomology valued functions, the I-function and J-function. Givental's J-function is defined to be a certain generating function of Gromov–Witten invariants for, in this case, the quintic threefold. The I-function, on the other hand, is a function determined by periods of the mirror to the quintic. The formula for the I-function (valued in the cohomology of the ambient $\mathbb{P}^4$) is

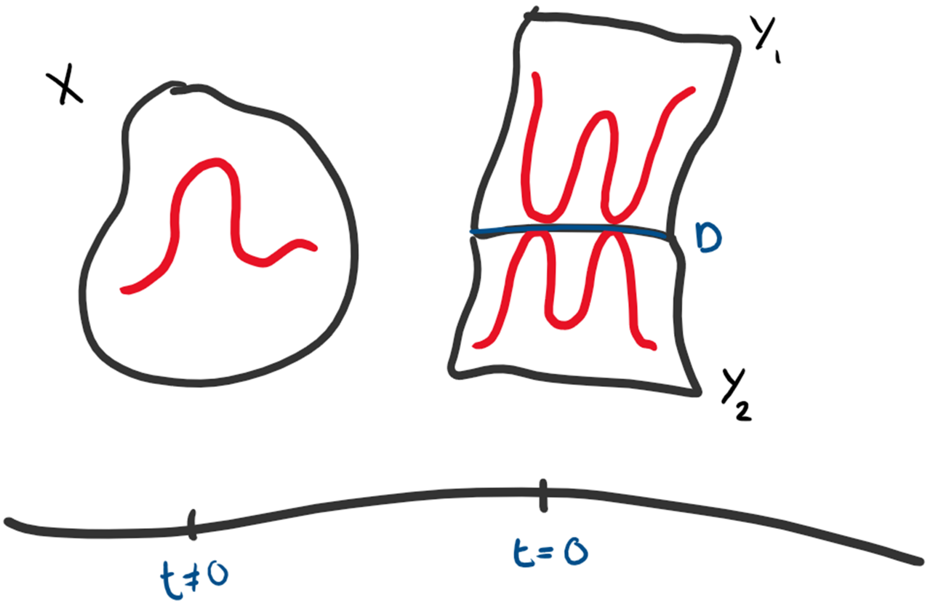
$$I(t,z) = e^{\frac{tH}{z}} \sum_{d=0}^{\infty} e^{td} \frac{\prod_{m=1}^{5d} 5H + mz}{\prod_{m=1}^{d} (H+mz)^5} \tag{1.1}$$

where H is the hyperplane class on $\mathbb{P}^4$. *Givental's Mirror Theorem* states that I and J coincide after a certain change of variables. It turns out that that this mysterious change of variables is exactly the wall-crossing formula between quasimap invariants and Gromov–Witten invariants, which tells us that the I-function, built out of information from the mirror is really a generating function for quasimap invariants.

Another framework which fits in nicely with this perspective is *Givental's quantisation formalism*. Givental showed that the the genus-zero Gromov–Witten theory of a target X can be encoded in a Lagrangian cone in a certain infinite dimensional symplectic vector space. The J-function of X is the unique function lying on the Lagrangian cone satisfying certain properties (see §2.3). On the other hand, the I-function can also be shown to be a function lying on the Lagrangian cone. Therefore, from the perspective of Gromov–Witten theory, one can get at certain Gromov–Witten invariants by showing that certain cohomology valued functions lie on the corresponding Lagrangian cone. We take this perspective in chapter 2.

1.5 Degeneration Formula and Relative Invariants

One of the key properties of enumerative invariants is that they are *deformation invariant*. Suppose we are given a flat family over a curve, where the general fibre is smooth and the special fibre is a simple normal crossings divisor in the total space consisting of smooth components glued along divisors. It is tempting to say the invariants of the general fibre are equal to the invariants of the special fibre, yet since the special fibre is singular, the corresponding moduli space is not virtually smooth. One can ask what the correct notion of 'curves in the special fibre' is, which makes this comparison true. In the case where the special fibre is a union of two smooth components glued along a shared smooth divisor this was worked out by Jun Li [Li02]. A necessary condition on the curves in the special fibre is that they are *predeformable* which means that the preimage of the singular locus consists only of nodes, and moreover, the tangencies of each branch of the curve to D at the node must be equal. This naturally leads one to a notion of *relative invariants* i.e. a framework for counting curves in X with prescribed tangencies to some fixed divisor D. In order to do this one needs to compactify the locus of curves with fixed numerical data and prescribed tangencies to D. There have been many iterations of solutions to this problem in the case of Gromov–Witten theory. In Chapter 3, our aim is to give a general solution to this problem in the quasimap setting.



Chapter 2

Gromov–Witten invariants of Blow-Ups and the Abelian/Non-Abelian Correspondence

The content of this chapter is joint work with Tom Coates and Wendelin Lutz, first appearing in [CLS21]. There are few effective tools for computing the Gromov–Witten invariants of blow-ups. In this chapter we determine how genus-zero Gromov–Witten invariants change when a smooth projective variety X is blown up in certain degeneracy loci which include the case of a complete intersection of convex line bundles. In the case where the blow-up $\widetilde{X}$ is Fano, a special case of our result gives closed form expressions for genus-zero one-point descendant invariants of $\widetilde{X}$ in terms of invariants of X.

2.1 Introduction

2.1.1 $\mathrm{Gr}(2,5)$ and $\mathbb{P}^4 \times \mathbb{P}^4$

One of the main ideas underpinning this chapter is the *Abelian/non-Abelian Correspondence*. This is a collection of results which allows one to express genus-zero Gromov–Witten invariants of a quotient by a non-abelian group in terms of a related quotient, this time by an abelian group. If the abelian quotient is, for example, a toric variety, it admits an action by a torus. This induces an action on the corresponding moduli space of stable maps. It is exactly this kind of torus localisation computation which allows one to explicitly compute (genus-zero) Gromov–Witten invariants of this abelian quotient. On the other hand, the non-abelian quotient does not necessarily admit such a torus action and so these techniques are not available. The Abelian/non-Abelian Correspondence circumvents this problem. For concreteness we will go through an example.

Let $V = \mathbb{C}^{10} = \mathbb{C}^5 \times \mathbb{C}^5$. Any element of V can be thought of as two vectors in $\mathbb{C}^5$, which, most of the time, will together span a two dimensional subspace of $\mathbb{C}^5$. On the other hand, many different pairs of vectors will also span the same two dimensional subspace. To reduce this ambiguity we consider the action of $G = \mathrm{GL}_2(\mathbb{C})$

on $\mathbb{C}^{10}$ by

$$A \cdot g = Ag$$

where A is being thought of as a 5×2 matrix, whose columns give the spanning vectors. There is a linearisation for this action, corresponding to the determinant of the matrix, which identifies as the stable (and semistable) points, the matrices where the two vectors are linearly independent. Moreover, the resulting quotient $V /\!/ G = \mathbb{C}^{10} /\!/ \mathrm{GL}_2(\mathbb{C}) = \mathrm{Gr}(2,5)$, the Grassmannian of two dimensional subspaces of $\mathbb{C}^5$. Inside $\mathrm{GL}(2,\mathbb{C})$, there is a copy of $(\mathbb{C}^*)^2$, a maximal torus, as the set of diagonal matrices. Restricting the action on $\mathbb{C}^{10}$ to $(\mathbb{C}^*)^2$ gives an action where the first generator scales the first column, and the second generator scales the second column. The linearisation from before induces a linearisation of this action, and the resulting quotient can readily be seen to be $\mathbb{P}^4 \times \mathbb{P}^4$. The Abelian/non-Abelian Correspondence will relate genus-zero Gromov–Witten invariants of $\mathrm{Gr}(2,5)$ with those of $\mathbb{P}^4 \times \mathbb{P}^4$ and in this case can be formulated in terms of the J-functions of these quotients, mentioned in the introduction.

First, there is an obvious S_2 action on $\mathbb{P}^4 \times \mathbb{P}^4$, exchanging factors, which induces an action on

$$H^\star(\mathbb{P}^4 \times \mathbb{P}^4) = \mathbb{C}[H_1, H_2]/(H_1^5, H_2^5) \tag{2.1}$$

In this context S_2 is the *Weyl group*, and one can actually relate the cohomology of $\mathrm{Gr}(2,5)$ to the cohomology of $\mathbb{P}^4 \times \mathbb{P}^4$ as

$$H^\star(\mathrm{Gr}(2,5)) \cong \frac{\mathbb{C}[H_1, H_2]^{\mathrm{Sym}}}{\{p(H_1, H_2) : (H_1 - H_2)p \in (H_1^5, H_2^5)\}}$$

as the quotient of the *Weyl-invariant* part of the cohomology of $\mathbb{P}^4 \times \mathbb{P}^4$ by the annihalator of the ideal $(H_1 - H_2)$. Therefore, we can project any symmetric polynomial in $H^\star(\mathbb{P}^4 \times \mathbb{P}^4)$ to the cohomology of $\mathrm{Gr}(2,5)$, using this isomorphism.

In the introduction we mentioned that for any X, there is a certain cohomology valued generating function of genus zero Gromov–Witten invariants called the J-function. This is given by the formula

$$J_X(\tau, z) = e^{\tau/z} \left(1 + \sum_{0 \neq d \in \mathrm{NE}(X)} q^d e^{\langle d, \tau \rangle} \left\langle \frac{\phi^\alpha}{z - \psi} \right\rangle_{0,1,d} \phi_\alpha \right) \tag{2.2}$$

for $\tau \in H^2(X)$ and $\{\phi_\alpha\}_\alpha$ is a basis for cohomology. This is strictly speaking the *small* J-function, and can be obtained from the (big) J-function by letting the parameter τ lie in $H^2(X)$ and using the divisor axiom.

For $\mathbb{P}^4 \times \mathbb{P}^4$ there is a nice formula for the J-function (related to §1.1, as the I-function and J-function for $\mathbb{P}^4 \times \mathbb{P}^4$ coincide).

$$J_{\mathbb{P}^4 \times \mathbb{P}^4}(t_1, t_2, z) = e^{\frac{t_1 H_1 + t_2 H_2}{z}} \sum_{d_1, d_2 \geq 0} \frac{q^{d_1} q^{d_2} e^{t_1 d_1 + t_2 d_2}}{\prod_{i=1}^2 \prod_{m=1}^{d_i} (H_i + mz)^5}. \tag{2.3}$$

The abelian/non-abelian correspondence for $\mathrm{Gr}(2,5)$ can be stated as the fact that the J-function for $\mathrm{Gr}(2,5)$ is

$$J_{\mathrm{Gr}(2,5)}(t,z) = e^{\frac{t(H_1+H_2)}{z}} \sum_{d \geq 0} q^d e^{td} \sum_{d_1 + d_2 = d} \frac{1}{\prod_{i=1}^2 \prod_{m=1}^{d} (H_i + mz)^5} \frac{\prod_{m=-\infty}^{d_1 - d_2} H_1 - H_2 + mz}{\prod_{m=-\infty}^{0} H_1 - H_2 + mz} \frac{\prod_{m=-\infty}^{d_2 - d_1} H_2 - H_1 + mz}{\prod_{m=-\infty}^{0} H_2 - H_1 + mz}. \tag{2.4}$$

Since this expression is symmetric in H_1 and H_2 we can 'project' to the cohomology of $\mathrm{Gr}(2,5)$ and view it as a function valued there.

2.1.2 Bundles and Blow-Ups

Our strategy in this chapter will be to use and extend the Abelian/non-Abelian Correspondence to a situation that allows us to calculate J-functions of certain blow-ups. This will require three ingredients

1. A way to rewrite the blow-up as a subvariety of a non-abelian quotient (a Grassmann bundle)

2. A way to talk about the abelian/non-abelian correspondence for certain subvarieties of quotients

3. An extension of the Abelian/non-Abelian Correspondence to subvarieties of Grassmann bundles

For the first part, suppose that $Y \subset X$ is the degeneracy locus of a map of vector bundles $\varphi\colon E^m \to F^n$ (with $m \geq n$ and write $r = m - n$). We show that in good situations the blow-up $Bl_Y(X)$ of X along Y is a subvariety of $\mathrm{Gr}(r, E)$, cut out as the zero locus of the regular section $s \in \Gamma(\mathrm{Hom}(S, \pi^* F))$ where $\pi\colon \mathrm{Gr}(r, E) \to X$ be the Grassmann bundle of E on X, and S is the tautological bundle. Furthermore, as in the case for the Grassmannian, $\mathrm{Gr}(r, E)$ can be expressed as a GIT quotient by $\mathrm{GL}_r(\mathbb{C})$.

For the second part, which is well known, this will rely on the notion of *twisted* Gromov–Witten invariants. Suppose we have some smooth projective variety X and a subvariety Z defined by a vector bundle E. If X say admits a torus action then we have mentioned that one can compute genus-zero Gromov–Witten invariants using torus localisation. On the other hand Z need not admit a torus action. We would like to somehow turn the computation of Gromov–Witten invariants of Z into a computation on X, so that one can utilise the techniques available there. It turns out that this is possible if E is *convex* [CK99, 7.17].

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ Z \arrow[r, hook, "i"] & X \arrow[u, dashed, bend left, "s"] \end{tikzcd} \qquad \begin{tikzcd} & E_{0,n,\beta} \arrow[d] \\ \overline{\mathcal{M}}_{0,n}(Z, \beta) \arrow[r, hook, "\tilde{i}"] & \overline{\mathcal{M}}_{0,n}(X, i_*\beta) \arrow[u, dashed, bend left, "\tilde{s}"] \end{tikzcd} \tag{2.5}$$

It is always true that $i_*[Z] = e(E) \cap [X]$. In [KKP03], the authors show that if E is convex (and stated here under the assumption that i_* is an isomorphism on H_2) then

$$\tilde{i}_*[\overline{\mathcal{M}}_{0,n}(Z, \beta)]^{\mathrm{vir}} = e(E_{0,n,\beta}) \cap [\overline{\mathcal{M}}_{0,n}(X, i_*\beta)]^{\mathrm{vir}}.$$

This allows one to turn integrals against $[\overline{\mathcal{M}}_{0,n}(Z, \beta)]^{\mathrm{vir}}$ (or its pushforward) into integrals against $e(E_{0,n,\beta}) \cap [\overline{\mathcal{M}}_{0,n}(X, i_*\beta)]^{\mathrm{vir}}$. The latter are called twisted Gromov–Witten invariants, and since they are integrals on the space of maps to X one can often use the techniques that were available before.

The third part is to prove a result which allows us to relate twisted Gromov–Witten invariants of a Grassmannian bundle $\mathrm{Gr}(r, E)$ for $E = L_1 \oplus \cdots \oplus L_n$ (or more generally a partial flag bundle) to the twisted Gromov–Witten invariants of the corresponding abelian quotient (under a convexity assumption). This abelian quotient is a also a fibre bundle over the base, where the fibers are products of projective space.

2.1.3 The Abelian/Non-Abelian Correspondence and Lagrangian Cones

Along the way we also reformulate the Abelian/non-Abelian Correspondence in the framework of Givental's formalism of Lagrangian Cones.

Givental showed how the genus-zero Gromov–Witten theory of a target X can be encoded in a certain Lagrangian submanifold, which lives inside an infinite dimensional symplectic vector space associated to the cohomology of X. We will explain this more carefully in 2.3. Moreover, the J-function of X is the unique function lying on the Lagrangian cone satisfying certain properties. The abelian/non-abelian correspondence has typically been phrased in terms of J-functions (and later Frobenius manifolds [CFKS08]). We produce a reformulation which involves only Lagrangian cones.

One of the most striking features of Givental's formalism is that relationships between higher-genus Gromov–Witten invariants of different spaces can often be expressed as the quantisation, in a precise sense, of the corresponding relationship between the Lagrangian cones that encode genus-zero invariants [Giv04]. Our version of the Abelian/non-Abelian Correspondence hints, therefore, at a higher-genus generalisation. It would be very interesting to develop and prove a higher-genus analog of the conjecture.

2.1.4 Connection to Earlier Work

Our formulation of the Abelian/non-Abelian Correspondence very roughly says that, for genus-zero Gromov–Witten theory, passing from an Abelian quotient $A /\!/ T$ to the corresponding non-Abelian quotient $A /\!/ G$ is almost the same as twisting by the non-convex bundle $\Phi \to A /\!/ T$ defined by the roots of G. This idea goes back to the earliest work on the subject, by Bertram–Ciocan-Fontanine–Kim, and indeed our Conjecture is very much in the spirit of the discussion in [BCFK08, §4]. These ideas were given a precise form in [CFKS08], in terms of Frobenius manifolds and Saito's period mapping; the main difference with the approach that we take here is that in [CFKS08] the authors realise the cohomology $H^\star(A /\!/ G)$ as the Weyl-anti-invariant subalgebra of the cohomology of the Abelian quotient $A /\!/ T$, whereas we realise it as a quotient of the Weyl-invariant part of $H^\star(A /\!/ T)$. The latter approach seems to fit better with Givental's formalism.

Ruan was the first to realise that there is a close connection between quantum cohomology (or more generally Gromov–Witten theory) and birational geometry [Rua99], and the change in Gromov–Witten invariants under blow-up forms an important testing ground for these ideas. Despite the importance of the topic, however, Gromov–Witten invariants of blow-ups have been understood in rather few situations. Early work here focussed on blow-ups in points, and on exploiting structural properties of quantum cohomology such as the WDVV equations and Reconstruction Theorems [Gat96, GP98, Gat01]. Subsequent approaches used symplectic methods pioneered by Li–Ruan [LR01, HLR08, Hu00, Hu01], or the Degeneration Formula following Maulik–Pandharipande [MP06, HHKQ18, CDW20], or a direct analysis of the moduli spaces involved and virtual birationality arguments [Man12a, Lai09, AW18a]. In each case the aim was to prove 'birational invariance': that certain specific Gromov–Witten invariants remain invariant under blow-up. We take a different approach. Rather than deform the target space, or study the geometry of moduli spaces of stable maps explicitly, we give an elementary construction of the blow-up $\tilde{X} \to X$ in terms that are compatible with modern tools for computing Gromov–Witten invariants, and extend these tools so that they cover the cases we need. This idea – of reworking classical constructions in birational geometry to make them amenable to computations using Givental formalism – was pioneered in [CCGK16], and indeed Lemma E.1 there gives the codimension-two case of our Proposition 2.6.2.

Compared to explicit invariance statements

$$\langle \pi^*\phi_{i_1}, \ldots, \pi^*\phi_{i_n} \rangle^{\tilde{X}}_{0,n,\pi^!\beta} = \langle \phi_{i_1}, \ldots, \phi_{i_n} \rangle^{X}_{0,n,\beta}$$

as in [Lai09, Theorem 1.4], we pay a price for our increased abstraction: the range of invariants for which we can extract closed-form expressions is different (see Corollary 2.5.19) and in general does not overlap with Lai's. But we also gain a lot by taking a more structural approach: our results determine, via a Birkhoff factorization procedure as in [CG07, CFK14], genus-zero Gromov–Witten invariants of the blow-up $\tilde{X}$ for curves of arbitrary degree (not just proper transforms of curves in the base) and with a wide range of insertions that can include gravitional descendant classes. See Remark 2.5.26. Furthermore in general one should not expect Gromov–Witten invariants to remain invariant under blow-ups. The correct statement – cf. Ruan's Crepant Resolution Conjecture [CIT09, CR13, Iri10, Iri09] and its generalisation by Iritani [Iri20] – is believed to involve analytic continuation of Givental cones, and we hope that our formulation here will be a step towards this.

In [LLW17], Lee, Lin, and Wang sketch a construction of blow-ups that is very similar to Proposition 2.6.2, and use this to compute Gromov–Witten invariants of blow-ups in complete intersections. The methods they use are different: they rely on a very interesting extension of the Quantum Lefschetz theorem to certain non-split bundles, which they will prove in forthcoming work [LLW]. At first sight, their result [LLW17, Theorem 5.1] is both more general and less explicit than our results. In fact, we believe neither is true. Their theorem as stated applies to blow-ups in complete intersections defined by arbitrary line bundles whereas we require these line bundles to be convex; however, discussions with the authors suggest that both results apply under the same conditions, and the convexity hypothesis was omitted from [LLW17, Theorem 5.1] in error. Furthermore, Lee, Lin, and Wang extract genus-zero Gromov–Witten invariants by combining their generalised Quantum Lefschetz theorem with an inexplicit Birkhoff factorisation procedure whereas we use the formalism of Givental cones. We believe, though, that one can rephrase their argument entirely in terms of Givental's formalism, and after doing so their results become explicit in exactly the same range as ours. The explicit formulas are different, however, and it would be interesting to see if one can derive non-trivial identities from this. Note that Proposition 2.6.2 below is more general than the construction in [LLW17, Section 5]: the fact that we consider Grassmann bundles rather than projective bundles allows us to treat blow-ups in certain degeneracy loci. Combining this with the methods in Section 2.7 allows one to compute genus-zero Gromov–Witten invariants of blow-ups in such degeneracy loci.

2.2 GIT Quotients and Flag Bundles

2.2.1 The topology of quotients by a non-Abelian group and its maximal torus

Let G be a complex reductive group acting on a smooth quasi-projective variety A with polarisation given by a linearised ample line bundle L. Let $T \subset G$ be a maximal torus. One can then form the GIT-quotients $A /\!/ G$ and $A /\!/ T$. We will assume that the stable and semistable points with respect to these linearisations coincide, and that all the isotropy groups of the stable points are trivial; this ensures that the quotients $A /\!/ G$ and $A /\!/ T$ are smooth projective varieties. As we have indicated, the Abelian/non-Abelian Correspondence [CFKS08] relates the genus zero Gromov–Witten invariants of these two quotients. Let $A^s(G)$, and respectively $A^s(T)$, denote the subsets of A consisting of points that are stable for the action of G, and respectively T. The two geometric quotients $A/\!/G$ and $A/\!/T$ fit into a diagram

$$\begin{tikzcd} A/\!/T & A^s(G)/T \arrow[l, hook', "j"'] \arrow[d, "q"] \\ & A/\!/G \end{tikzcd} \tag{2.6}$$

where j is the natural inclusion and π the natural projection.

A representation $\rho\colon G \to \mathrm{GL}(V)$ induces a vector bundle $V(\rho)$ on $A/\!/G$ with fiber V. Explicitly, $V(\rho) = (A \times V)/\!/G$ where G acts as

$$g\colon (a, v) \mapsto (ag, \rho(g^{-1})v).$$

Similarly, the restriction $\rho|_T$ of the representation ρ induces a vector bundle $V(\rho|_T)$ over $A/\!/T$. Note that since T is Abelian, $V(\rho|_T)$ splits as a direct sum of line bundles, $V(\rho|_T) = L_1 \oplus \cdots \oplus L_k$ These bundles satisfy

$$j^*V(\rho|_T) \cong q^*V(\rho). \tag{2.7}$$

When the representation $\rho\colon G \to \mathrm{GL}(V)$ is clear from context, we will suppress it from the notation, writing V^G for $V(\rho)$ and V^T for $V(\rho|_T)$.

We will now describe the relationship between the cohomology rings of $A/\!/G$ and $A/\!/T$, following [Mar00]. This will generalise 2.1. Let W be the Weyl group of G. W acts on $A/\!/T$ and hence on the cohomology ring $H^\star(A/\!/T)$. Restricting the adjoint representation $\rho\colon G \to \mathrm{GL}(\mathfrak{g})$ to T, we obtain a splitting $\rho|_T = \oplus_\alpha \rho_\alpha$ into 1-dimensional representations, i.e. characters, of T. The set Δ of characters appearing in this decomposition is the set of roots of G, and forms a root system. Write L_α for the line bundle on $A/\!/T$ corresponding to a root α. Fix a set of positive roots Φ^+ and define

$$\omega = \prod_{\alpha \in \Phi^+} c_1(L_\alpha).$$

Theorem 2.2.1 (Martin) *There is a natural ring homomorphism*

$$H^\star(A/\!/G) \cong \frac{H^\star(A/\!/T)^W}{\mathrm{Ann}(\omega)}$$

under which $x \in H^\star(A/\!/G)$ maps to $\tilde{x} \in H^\star(A/\!/T)$ if and only if $q^*x = j^*\tilde{x}$.

Theorem 2.2.1 shows that any cohomology class $\tilde{x} \in H^\star(A/\!/T)^W$ is a lift of a class $x \in H^\star(A/\!/G)$, with $\tilde{x}$ unique up to an element of $\mathrm{Ann}(\omega)$.

Corollary 2.2.2 *There is a quotient map*

$$H^\star(A/\!/T)^W \twoheadrightarrow H^\star(A/\!/G) \tag{2.8}$$

Assumption 2.2.3 *Throughout we will assume that the G-unstable locus $A \setminus A^s(G)$ has codimension at least* 2.

This implies that elements of $H^2(A/\!/G)$ can be lifted uniquely:

Proposition 2.2.4 *Pullback via q gives an isomorphism $H^2(A/\!/G) \cong H^2(A/\!/T)^W$, and induces a map $\varrho\colon \mathrm{NE}(A/\!/T) \to \mathrm{NE}(A/\!/G)$ where* NE *denotes the Mori cone.*

Proof 2.2.5 *The assumption that $A \setminus A^s(G)$ has codimension at least 2 implies that $(A^s(T)/T) \setminus (A^s(G)/T)$ has codimension at least 2, so j induces an isomorphism $\mathrm{Pic}(A^s(G)/T) \cong \mathrm{Pic}(A^s(T)/T)$. This gives an isomorphism $H^2(A^s(G)/T) \cong H^2(A^s(T)/T)$ since the cycle class map is an isomorphism for both spaces. Since q^* always induces an isomorphism between $H^2(A/\!/G)$ and $H^2(A^s(G)/T)^W$ [Bor53], the first claim follows. Consequently, the lifting of divisor classes is unique and can be identified with the pullback map $q^*\colon \mathrm{Pic}(A/\!/G) \to \mathrm{Pic}(A^s(G)/T)$. Since the pullback of a nef divisor class along a proper map is nef, we obtain by duality a map $\varrho : \mathrm{NE}(A/\!/T) \to \mathrm{NE}(A/\!/G)$.*

Definition 2.2.6 *We say that $\tilde{\beta} \in \mathrm{NE}(A/\!/T)$ lifts $\beta \in \mathrm{NE}(A/\!/G)$ if $\varrho(\tilde{\beta}) = \beta$. Note that any effective β has finitely many lifts.*

2.2.2 Partial flag varieties and partial flag bundles

Notation

We will now specialise to the case of flag bundles and introduce notation used in the rest of the chapter. Fix once and for all:

- a positive integer n and a sequence of positive integers $r_1 < \cdots < r_\ell < r_{\ell+1} = n$;
- a vector bundle $E \to X$ of rank n on a smooth projective variety X which splits as a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n$.

We write Fl for the partial flag manifold $\mathrm{Fl}(r_1, \ldots, r_\ell; n)$, and $\mathrm{Fl}(E)$ for the partial flag bundle $\mathrm{Fl}(r_1, \ldots, r_\ell; E)$. These are (respectively) the spaces whose points are

$$F_1 \subset F_2 \subset \cdots \subset F_\ell$$

Where F_i is a subspace of $\mathbb{C}^n$ (resp. a fibre of E) of dimension r_i.

Set $N = \sum_{i=1}^{\ell} r_i r_{i+1}$ and $R = r_1 + \cdots + r_\ell$ It will be convenient to use the indexing $\{(1,1), \ldots (1, r_1), (2,1), \ldots, (\ell, r_\ell)\}$ for the set of positive integers smaller or equal than R.

Partial flag varieties and partial flag bundles as GIT quotients

The partial flag manifold Fl arises as a GIT quotient, as follows. Consider $\mathbb{C}^N$ as the space of homomorphisms

$$\bigoplus_{i=1}^{\ell} \mathrm{Hom}\left(\mathbb{C}^{r_i}, \mathbb{C}^{r_{i+1}}\right). \tag{2.9}$$

The group $G = \prod_{i=1}^{\ell} \mathrm{GL}_{r_i}(\mathbb{C})$ acts on $\mathbb{C}^N$ by

$$(g_1, \ldots, g_\ell) \cdot (A_1, \ldots, A_\ell) = (g_2^{-1} A_1 g_1, \ldots, g_\ell^{-1} A_{\ell-1} g_{\ell-1}, A_\ell g_\ell).$$

Let $\rho_i\colon G \to \mathrm{GL}_{r_i}(\mathbb{C})$ be the representation which is the identity on the ith factor and trivial on all other factors. Choosing the linearisation $\chi = \bigotimes_{i=1}^{\ell} \det(\rho_i)$, we have that $\mathbb{C}^N /\!/_\chi G$ is the partial flag manifold Fl. More generally, the partial flag bundle also arises as a GIT quotient, of the total space of the bundle of homomorphisms

$$\bigoplus_{i=1}^{\ell-1} \mathrm{Hom}\left(\mathcal{O}^{\oplus r_i}, \mathcal{O}^{\oplus r_{i+1}}\right) \oplus \mathrm{Hom}\left(\mathcal{O}^{\oplus r_\ell}, E\right) \tag{2.10}$$

with respect to the same group G and the same linearisation. $\mathrm{Fl}(E)$ carries ℓ tautological bundles of ranks $r_1, \dots, r_\ell$, which we will denote $S_1, \dots, S_\ell$. These bundles restrict to the usual tautological bundles on Fl on each fibre. The bundle S_i is induced by the representation ρ_i.

Definition 2.2.7 *Let*

$$p_i(t) = t^{r_i} - c_1(S_i)t^{r_i-1} + \cdots + (-1)^{r_i} c_{r_i}(S_i)$$

be the Chern polynomial of $S_i^\vee$. We denote the roots of p_i by $H_{i,j}$, $1 \le j \le r_i$. The $H_{i,j}$ are in general only defined over an appropriate ring extension of $H^\star(\mathrm{Fl}(E), \mathbb{C})$, but symmetric polynomials in the $H_{i,j}$ give well-defined elements of $H^\star(\mathrm{Fl}(E), \mathbb{C})$.

The maximal torus $T \subset G$ is isomorphic to $(\mathbb{C}^*)^R$. The corresponding Abelian quotient

$$\mathrm{Fl}(E)_T := \mathrm{Hom}\left(\cdots \right) /\!/_\chi (\mathbb{C}^*)^R,$$

where $\mathrm{Hom}\left(\cdots \right)$ is the bundle of homomorphisms (2.10), is a fibre bundle over X with general fibre isomorphic to the toric variety $\mathrm{Fl}_T := \mathbb{C}^N /\!/_\chi (\mathbb{C}^*)^R$. The space $\mathrm{Fl}(E)_T$ also carries natural cohomology classes:

Definition 2.2.8 *Let $\rho_{i,j}\colon (\mathbb{C}^*)^R \to \mathrm{GL}_1(\mathbb{C})$ be the dual of the one-dimensional representation of $(\mathbb{C}^*)^R$ given by projection to the (i,j)th factor $\mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$; here we use the indexing of the set $\{1, 2, \dots, R\}$ specified in §2.2.2. We define $L_{i,j} \in H^2(\mathrm{Fl}_T, \mathbb{C})$ to be the line bundle on $\mathrm{Fl}(E)_T$ induced by $\rho_{i,j}$ and denote its first Chern class by $\tilde{H}_{i,j}$. Similarly, we define $h_{i,j}$ to be the first Chern class of the line bundle on Fl_T induced by the represenation $\rho_{i,j}$. Equivalently, $h_{i,j}$ is the restriction of $\tilde{H}_{i,j}$ to a general fibre Fl_T of $\mathrm{Fl}(E)_T$.*

Recall that, for a representation ρ of G, the corresponding vector bundle V^T splits as a direct sum of line bundles $F_1 \oplus \cdots \oplus F_k$. It is a general fact that if f is a symmetric polynomial in the $c_1(F_i)$, then f can be written as a polynomial in the elementary symmetric polynomials $e_r(c_1(F_1), \dots, c_1(F_k))$, that is, in the Chern classes $c_r(V^T)$. By (2.7) we have that $j^* c_r(V^T) = q^* c_r(V^G)$, and so replacing any occurrence of $c_r(V^T)$ by $c_r(V^G)$ gives an expression $g \in H^\star(A /\!/ G)$ which satisfies $q^* g = j^* f$. That is, f is a lift of g. Applying this to the dual of the standard representation ρ_i of the ith factor of G shows that any polynomial p which is symmetric in each of the sets $\tilde{H}_{i,j}$ for fixed i projects to the same expression in $H^\star(\mathrm{Fl}(E))$ with any occurrence of $\tilde{H}_{i,j}$ replaced by the corresponding Chern root $H_{i,j}$.

Lemma 2.2.9 *Let $(\mathbb{C}^*)^R$ act on $\mathbb{C}^N$, arrange the weights for this action in an $R \times N$-matrix $(m_{i,k})$ and consider $E = L_1 \oplus \cdots \oplus L_N \xrightarrow{\pi} X$ a direct sum of line bundles. Form the associated toric fibration $E /\!/ (\mathbb{C}^*)^R$ with general fibre $\mathbb{C}^N /\!/ (\mathbb{C}^*)^R$ and let h_i (respectively H_i) be the first Chern class of the line bundle on $\mathbb{C}^N /\!/ (\mathbb{C}^*)^R$ (respectively on $E /\!/ (\mathbb{C}^*)^R$ induced by the dual of the representation which is standard on the ith factor of $(\mathbb{C}^*)^R$ and trivial on the other factors. Then*

- *the Poincaré duals u_k of the torus invariant divisors of the toric variety $\mathbb{C}^N /\!/ (\mathbb{C}^*)^R$ are:*

$$u_k = \sum_{k=1}^{R} m_{i,k} h_i$$

- *the Poincaré duals U_k of the torus invariant divisors of the total space of the toric fibration $E /\!/ (\mathbb{C}^*)^R \xrightarrow{\pi} X$ are:*

$$U_k = \sum_{k=1}^{R} m_{i,k} H_i + \pi^* c_1(L_k)$$

When applying Lemma 2.2.9 to our situation (2.10) it will be convenient to define $H_{\ell+1,j} := \pi^* c_1(L_j^\vee)$. Then the set of torus invariant divisors is

$$H_{i,j} - H_{i+1,j'} \qquad\qquad 1 \le i \le \ell,\ 1 \le j \le r_i,\ 1 \le j' \le r_{i+1}$$

We will also need to know about the ample cone of a toric variety $\mathbb{C}^N /\!/ (\mathbb{C}^*)^R$. This is most easily described in terms of the secondary fan, that is, by the wall-and-chamber decomposition of $\mathrm{Pic}(\mathbb{C}^N /\!/ (\mathbb{C}^*)^R) \otimes \mathbb{R} \cong \mathbb{R}^R$ given by the cones spanned by size $R-1$ subsets of columns of the weight matrix. The ample cone of $\mathbb{C}^N /\!/ (\mathbb{C}^*)^R$ is then the chamber that contains the stability condition χ. Moreover, for a subset $\alpha \subset \{1, \ldots, N\}$ of size R the cone in the secondary fan spanned by the classes u_k, $k \in \alpha$, contains the stability condition (and therefore also the ample cone) iff the intersection $u_\alpha = \bigcap_{k \notin \alpha} u_k$ is nonempty. In this case, $U_\alpha = \bigcap_{k \notin \alpha} U_k$ restricts to a torus fixed point on every fibre and, since E splits as a direct sum of line bundles, U_α is the image of a section of the toric fibration π. We denote this section by s_α. By construction, the torus invariant divisors U_k, $k \in \alpha$, do not meet U_α, so that $s_\alpha^*(U_k) = 0$ for all $k \in \alpha$. For the toric variety Fl_T one can easily write down the set of R-dimensional cones containing $\chi = (1, \ldots, 1)$. For each index (i,j), choose some $j' \in \{1, \ldots, r_{\ell+1}\}$. Then the cone spanned by

$$h_{i,j} - h_{i+1,j'} \qquad 1 \le i < \ell - 1,\ 1 \le j \le r_i \qquad h_{\ell,j},\ 1 \le j \le r_\ell \tag{2.11}$$

contains χ and every cone containing χ is of that form.

2.3 Givental's Formalism

In this section we review Givental's geometric formalism for Gromov–Witten theory, concentrating on the genus-zero case. The main reference for this is [Giv04]. Let Y be a smooth projective variety and consider

$$\mathcal{H}_Y = H^\star(Y, \Lambda)[z, z^{-1}]\!] = \Big\{ \sum_{i=-\infty}^{m} a_i z^i \colon a_i \in H^\star(Y, \Lambda),\ m \in \mathbb{Z} \Big\}$$

where z is an indeterminate and Λ is the Novikov ring for Y. After picking a basis $\{\phi_1, \ldots, \phi_N\}$ for $H^\star(Y; \mathbb{C})$ with $\phi_1 = 1$ and writing $\{\phi^1, \ldots, \phi^N\}$ for the Poincare dual basis, we can write elements of $\mathcal{H}_Y$ as

$$\sum_{i=0}^{m} \sum_{\alpha=1}^{N} q_i^\alpha \phi_\alpha z^i + \sum_{i=0}^{\infty} \sum_{\alpha=1}^{N} p_{i,\alpha} \phi^\alpha (-z)^{-1-i} \tag{2.12}$$

where $q_i^\alpha, p_{i,\alpha} \in \Lambda$. The q_i^α, $p_{i,\alpha}$ then provide coordinates on $\mathcal{H}_Y$. The space $\mathcal{H}_Y$ carries a symplectic form

$$\begin{aligned} \Omega \colon \mathcal{H}_Y \otimes \mathcal{H}_Y &\to \Lambda \\ f \otimes g &\to \mathrm{Res}_{z=0}(f(-z), g(z))\, dz \end{aligned}$$

where $(\cdot, \cdot)$ denotes the Poincaré pairing, extended $\mathbb{C}[z, z^{-1}]\!]$-linearly to $\mathcal{H}_Y$. By construction, Ω is in Darboux form with respect to our coordinates:

$$\Omega = \sum_i \sum_\alpha dp_{i,\alpha} \wedge dq_i^\alpha$$

We fix a Lagrangian polarisation of $\mathcal{H}$ as $\mathcal{H}_Y = \mathcal{H}_+ \oplus \mathcal{H}_-$, where

$$\mathcal{H}_+ = H^\star(Y; \Lambda)[z], \quad \mathcal{H}_- = z^{-1} H^\star(Y; \Lambda)[\![z^{-1}]\!]$$

This polarisation $\mathcal{H}_Y = \mathcal{H}_+ \oplus \mathcal{H}_-$ identifies $\mathcal{H}_Y$ with $T^*\mathcal{H}_+$. We now relate this to Gromov–Witten theory.

Definition 2.3.1 *The* genus-zero descendant potential *is a generating function for genus-zero Gromov–Witten invariants:*

$$\mathcal{F}^0_Y = \sum_{n=0}^{\infty} \sum_{d\in \mathrm{NE}(Y)} \frac{Q^d}{n!} t^{\alpha_1}_{i_1} \dots t^{\alpha_n}_{i_n} \langle \phi_{\alpha_1}\psi^{i_1}, \dots, \phi_{\alpha_n}\psi^{i_n}\rangle_{0,n,d}$$

Here t^α_i is a formal variable, $\mathrm{NE}(Y)$ *denotes the Mori cone of Y, and Einstein summation is used for repeated lower and upper indices.*

After setting

$$t^\alpha_i = q^\alpha_i + \delta^i_1 \delta^1_\alpha, \tag{2.13}$$

where δ^j_i denotes the Kronecker delta, we obtain a (formal germ of a) function $\mathcal{F}^0_Y \colon \mathcal{H}_+ \to \Lambda$.

Definition 2.3.2 *The Givental cone $\mathcal{L}_Y$ of Y is the graph of the differential of $\mathcal{F}^0_Y \colon \mathcal{H}_+ \to \Lambda$:*

$$\mathcal{L}_Y = \left\{ (\mathbf{q}, \mathbf{p}) \in T^*\mathcal{H}_Y = \mathcal{H}_+ \oplus \mathcal{H}_- : p_{i,\alpha} = \frac{\partial \mathcal{F}^0_Y}{\partial q^\alpha_i} \right\}$$

Note that $\mathcal{L}_Y$ is Lagrangian by virtue of being the graph of the differential of a function. Moreover, it has the following special geometric properties [Giv04, CCIT09, CG07]

- *$\mathcal{L}$ is preserved by scalar multiplication, i.e. it is (the formal germ of) a cone*
- *the tangent space T_f of $\mathcal{L}_Y$ at $f \in \mathcal{L}_Y$ is tangent to $\mathcal{L}$ exactly along zT_f. This means:*
  1. *$zT_f \subset \mathcal{L}_Y$*
  2. *for $g \in zT_f$, we have $T_g = T_f$*
  3. *$T_f \cap \mathcal{L}_Y = zT_f$*

A general point of $\mathcal{L}_Y$ can be written, in view of the dilaton shift (2.13), as

$$\begin{aligned} &-z + \sum_{i=0}^{\infty} t^\alpha_i \phi_\alpha z^i + \sum_{n=0}^{\infty} \sum_{d\in\mathrm{NE}(Y)} \frac{Q^d}{n!} t^{\alpha_1}_{i_1} \dots t^{\alpha_n}_{i_n} \langle \phi_{\alpha_1}\psi^{i_1}, \dots, \phi_{\alpha_n}\psi^{i_n}, \phi_\alpha \psi^i \rangle_{0,n+1,d} \phi^\alpha (-z)^{-i-1} \\ =& -z + \sum_{i=0}^{\infty} t^\alpha_i \phi_\alpha z^i + \sum_{n=0}^{\infty} \sum_{d\in\mathrm{NE}(Y)} \frac{Q^d}{n!} t^{\alpha_1}_{i_1} \dots t^{\alpha_n}_{i_n} \langle \phi_{\alpha_1}\psi^{i_1}, \dots, \phi_{\alpha_n}\psi^{i_n}, \frac{\phi_\alpha}{-z-\psi} \rangle_{0,n+1,d} \phi^\alpha \end{aligned}$$

Thus knowing $\mathcal{L}_Y$ is equivalent to knowing all genus-zero Gromov–Witten invariants of Y. Setting $t^\alpha_k = 0$ for all $k > 0$, we obtain the *J-function* of Y:

$$J(\tau, -z) = -z + \tau + \sum_{n=0}^{\infty} \sum_{d\in\mathrm{NE(X)}} \frac{Q^d}{n!} \left\langle \tau, \dots \tau, \frac{\phi_\alpha}{-z-\psi} \right\rangle_{0,n+1,d} \phi^\alpha$$

where $\tau = t^1_0 \phi_1 + \dots + t^N_0 \phi_N \in H^\star(Y)$. The J-function is the unique family of elements $\tau \mapsto J(\tau, -z)$ on the Lagrangian cone such that

$$J(\tau, -z) = -z + \tau + O(z^{-1}).$$

We will need a generalisation of all of this to twisted Gromov–Witten invariants [CG07], which were mentioned in 2.5. Let F be a vector bundle on Y and consider the universal family over the moduli space of stable maps

$$\begin{tikzcd} C_{0,n,d} \arrow[r, "f"] \arrow[d, "\pi"'] & Y \\ Y_{0,n,d} \end{tikzcd}$$

Let $\pi_!$ be the pushforward in K-theory. We define

$$F_{0,n,d} = \pi_! f^* F = R^0\pi_* f^* F - R^1\pi_* f^* F$$

(the higher derived functors vanish). In general $F_{0,n,d}$ is a class in K-theory and not an honest vector bundle. This means that in order to evaluate a characteristic class $\mathbf{c}(\cdot)$ on $F_{0,n,d}$ we need $\mathbf{c}(\cdot)$ to be *multiplicative* and *invertible*. We can then set

$$\mathbf{c}(F_{0,n,d}) = \mathbf{c}(R^0\pi_* f^* F) \smile \mathbf{c}(R^1\pi_* f^* F)^{-1}$$

where $\mathbf{c}(R^i\pi_* f^* F)$ is defined using an appropriate locally free resolution.

Definition 2.3.3 *Let F be a vector bundle on Y and let $\mathbf{c}(\cdot)$ be an invertible multiplicative characteristic class. We will refer to the pair $(F, \mathbf{c})$ as twisting data. Define $(F, \mathbf{c})$-twisted Gromov–Witten invariants as*

$$\langle \alpha_1\psi_1^{i_1}, \ldots \alpha_n\psi_n^{i_n}\rangle_{0,n,d}^{F,\mathbf{c}} = \int_{[Y_{0,n,d}]^{\mathrm{vir}} \cap \mathbf{c}(F_{0,n,d})} \mathrm{ev}_1^* \alpha_1 \smile \cdots \smile \mathrm{ev}_n^* \alpha_n \smile \psi_1^{i_1} \smile \cdots \smile \psi_n^{i_n}$$

Any multiplicative invertible characteristic class can be written as $\mathbf{c}(\cdot) = \exp(\sum_{k\geq 0} s_k \mathrm{ch}_k(\cdot))$, where ch_k is the kth component of the Chern character and $s_0, s_1, \ldots$ are appropriate coefficients. So we work with cohomology groups $H^\star(X, \Lambda_s)$, where Λ_s is the completion of $\Lambda[s_0, s_1, \ldots]$ with respect to the valuation

$$v(Q^d) = \langle c_1(\mathcal{O}(1)), d\rangle, \quad v(s_k) = k+1.$$

Most of the definitions from before now carry over. We have the twisted Poincaré pairing $(\alpha, \beta)^{F,\mathbf{c}} = \int_Y \mathbf{c}(F) \cup \alpha \cup \beta$ which defines the basis $\phi^1, \ldots \phi^N$ dual to our chosen basis $1 = \phi_1, \ldots, \phi_N$ for $H^\star(Y)$. The Givental space becomes $\mathcal{H}_Y = H^\star(Y, \Lambda_s) \otimes \mathbb{C}[z, z^{-1}]]$ with the twisted symplectic form

$$\Omega^{F,\mathbf{c}}(f(z), g(z)) = \mathrm{Res}_{z=0}\big(f(-z), g(z)\big)^{F,\mathbf{c}} dz.$$

This form admits Darboux coordinates as before which give a Lagrangian polarisation of $\mathcal{H}_Y$. Then the twisted Lagrangian cone $\mathcal{L}_{F,\mathbf{c}}$ is defined, via the dilaton shift (2.13), as the graph of the differential of the generating function $\mathcal{F}_Y^{0,F,\mathbf{c}}$ for genus zero *twisted* Gromov–Witten invariants. Finally, just as before, we can define a twisted J-function:

Definition 2.3.4 *Given twisting data $(F, \mathbf{c})$ for Y, the twisted J-function is:*

$$J_{F,\mathbf{c}}(\tau, -z) = -z + \tau + \sum_{n=0}^{\infty} \sum_{d\in \mathrm{NE}(Y)} \frac{Q^d}{n!} \left\langle \tau, \ldots \tau, \frac{\phi_\alpha}{-z-\psi} \right\rangle_{0,n+1,d}^{F,\mathbf{c}} \phi^\alpha$$

This is once again characterised as the unique family $\tau \mapsto J_{F,\mathbf{c}}(\tau, -z)$ of elements of the twisted Lagrangian cone of the form

$$J_{F,\mathbf{c}}(\tau, -z) = -z + \tau + O(z^{-1})$$

Note that we can recover the untwisted theory by setting $\mathbf{c} = 1$.

In what follows we take $\mathbf{c}$ to be the $\mathbb{C}^*$-equivariant Euler class

$$\mathbf{c}(V) = \sum_{k=0}^{d} \lambda^{d-k} c_k(V) \qquad \text{where } d \text{ is the rank of the vector bundle } V. \tag{2.14}$$

which is multiplicative and invertible. The $\mathbb{C}^*$-action here is the canonical $\mathbb{C}^*$-action on any vector bundle given by rescaling the fibres. We write F_λ for the twisting data $(F, \mathbf{c})$, where F is equipped with the $\mathbb{C}^*$-action given by rescaling the fibres with equivariant parameter λ. In this setting, Gromov–Witten invariants (and the coefficients s_k) take values in the fraction field $\mathbb{C}(\lambda)$ of the $\mathbb{C}^*$-equivariant cohomology of a point. Here λ is the hyperplane class on $\mathbb{CP}^\infty$, so that $H^\star_{\mathbb{C}^*}(\{\mathrm{pt}\}) = \mathbb{C}[\lambda]$, and we work over the field $\mathbb{C}(\lambda)$.

Remark 2.3.5 *As we have set things up, the twisted cone $\mathcal{L}_{F_\lambda}$ is a Lagrangian submanifold of the symplectic vector space $\left(\mathcal{H}_Y, \Omega^{F_\lambda}\right)$, so as λ varies both the Lagrangian submanifold and the ambient symplectic space change. To obtain the picture where all the Lagrangian submanifolds $\mathcal{L}_{F_\lambda}$ lie in a single symplectic vector space $\left(\mathcal{H}_Y, \Omega\right)$, one can identify $\left(\mathcal{H}_Y, \Omega\right)$ with $\left(\mathcal{H}_Y, \Omega^{F_\lambda}\right)$ by multiplication by the square root of the equivariant Euler class of F. See [CG07, §8] for details.*

2.3.1 Twisting the I-function

We will now prove a general result following an argument from [CCIT09]. We say that a family $\tau \mapsto I(\tau)$ of elements of $\mathcal{H}_Y$ *satisfies the Divisor Equation* if the parameter domain for τ is a product $U \times H^2(Y)$ and $I(\tau)$ takes the form

$$I(\tau) = \sum_{\beta \in \mathrm{NE}(Y)} Q^\beta I_\beta(\tau, z)$$

where

$$z \nabla_\rho I_\beta = \left(\rho + \langle \rho, \beta \rangle z\right) I_\beta \qquad \text{for all } \rho \in H^2(Y). \tag{2.15}$$

Here ∇_ρ is the directional derivative along ρ. Let F' be a vector bundle on Y, and consider any family $\tau \mapsto I(\tau) \in \mathcal{L}_{F'_\mu}$ that satisfies the Divisor Equation. Given another vector bundle F which splits as a direct sum of line bundles $F = F_1 \oplus \cdots \oplus F_k$, we explain how to modify the family $\tau \mapsto I(\tau)$ by introducing explicit hypergeometric factors that depend on F. We prove that (1) this modified family can be written in terms of the *Quantum Riemann-Roch operator* and the original family; and (2) the modified family lies on the twisted Lagrangian cone $\mathcal{L}_{F_\lambda \oplus F'_\mu}$.

Definition 2.3.6 *Define the element $G(x, z) \in \mathcal{H}_Y$ by*

$$G(x, z) := \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} s_{l+m-1} \frac{B_m}{m!} \frac{x^l}{l!} z^{m-1}$$

where B_m are the Bernoulli numbers and the s_k are the coefficients obtained by writing the $\mathbb{C}^*$-equivariant Euler class (2.14) *in the form* $\exp\left(\sum_{k \geq 0} s_k \mathrm{ch}_k(\cdot)\right)$.

Remark 2.3.7 *The discussion in this section is valid for any invertible multiplicative characteristic class, not just the equivariant Euler class, but we will neither need nor emphasize this.*

Definition 2.3.8 *Let F be a vector bundle – not necessarily split – and let f_i be the Chern roots of F. Define the Quantum Riemann-Roch operator, $\Delta_{F_\lambda} : \mathcal{H}_Y \to \mathcal{H}_Y$ as multiplication by*

$$\Delta_{F_\lambda} = \prod_{i=1}^{k} \exp(G(f_i, z))$$

Theorem 2.3.9 ([CG07]) *Δ_{F_λ} gives a linear symplectomorphism of $(\mathcal{H}_Y, \Omega_Y)$ with $(\mathcal{H}_Y, \Omega_Y^{F_\lambda})$ such that*

$$\Delta_{F_\lambda}(\mathcal{L}_Y) = \mathcal{L}_{F_\lambda}$$

Since $\Delta_{F_\lambda} \circ \Delta_{F'_\mu} = \Delta_{F_\lambda \oplus F'_\mu}$, it follows immediately that

$$\Delta_{F_\lambda}(\mathcal{L}_{F'_\mu}) = \mathcal{L}_{F_\lambda \oplus F'_\mu}.$$

Lemma 2.3.10 *Let F be a vector bundle and let $f_1, \ldots, f_k$ be the Chern roots of F. Let*

$$D_{F_\lambda} = \prod_{i=1}^{k} \exp\left(-G(z\nabla_{f_i}, z)\right)$$

and suppose that $\tau \mapsto I(\tau)$ is a family of elements of $\mathcal{L}_{F'_\mu}$. Then $\tau \mapsto D_{F_\lambda}(I(\tau))$ is also a family of elements of $\mathcal{L}_{F'_\mu}$.

Proof 2.3.11 *This follows [CCIT09, Theorem 4.6]. Let $h = -z + \sum_{i=0}^{m} t_i z^i + \sum_{j=0}^{\infty} p_j(-z)^{-j-1}$ be a point on $\mathcal{H}_Y$. The Lagrangian cone $\mathcal{L}_{F'_\mu}$ is defined by the equations $E_j = 0$, $j = 0, 1, 2, \ldots$ where*

$$E_j(h) = p_j - \sum_{n \geq 0} \sum_{d \in \mathrm{NE}(Y)} \frac{Q^d}{n!} t_{i_1}^{\alpha_1} \ldots t_{i_n}^{\alpha_n} \langle \phi_{\alpha_1}\psi^{i_1}, \ldots, \phi_{\alpha_n}\psi^{i_n}, \phi_\alpha \psi^j \rangle_{0,n+1,d} \phi^\alpha$$

We need to show that $E_j(D_{F_\lambda}(I)) = 0$. Note that $D_{F_\lambda}(I) = \prod_{i=1}^{k} \exp(-G(z\nabla_{f_i}, z))I$ depends on the parameters s_i. For notational simplicity assume that $k = 1$, so that

$$D_{F_\lambda}(I) = \exp\left(-G(z\nabla_f, z)\right) I$$

Set $\deg s_i = i + 1$. We will prove the result by inducting on degree. Note that if $s_0 = s_1 = \cdots = 0$ then $D_{F_\lambda}(I) = I$ so that $E_j(D_{F_\lambda}(I)) = 0$. Assume by induction that $E_j(D_{F_\lambda}(I))$ vanishes up to degree n in the variables $s_0, s_1, s_2, \ldots$ Then

$$\frac{\partial}{\partial s_i} E_j(D_{F_\lambda}(I)) = d_{D_{F_\lambda}(I)} E_j(z^{-1} P_i(z\nabla_f, z) D_{F_\lambda}(I))$$

where

$$P_i(z\nabla_f, z) = \sum_{m=0}^{i+1} \frac{1}{m!(i+1-m)!} z^m B_m (z\nabla_f)^{i+1-m}$$

By induction there exists $D_{F_\lambda}(I)' \in \mathcal{L}_{F'_\mu}$ such that

$$\frac{\partial}{\partial s_i} E_j(D_{F_\lambda}(I)) = d_{D_{F_\lambda}(I)'} E_j(z^{-1} P_i(z\nabla_f, z) D_{F_\lambda}(I)')$$

up to degree n. But the right hand side of this expression is zero, since the term in brackets lies in the tangent space to the Lagrangian cone. Indeed, applying ∇_f to $D_{F_\lambda}(I_Y)'$ – or to any family lying on the cone – takes it to the tangent space of the cone at the point. And then applying $z\nabla_f$ preserves that tangent space.

Corollary 2.3.12 *Let $\tau \mapsto I(\tau)$ be a family of elements of $\mathcal{L}_{F'_\mu}$. Then $\tau \mapsto \Delta_{F_\lambda}(D_{F_\lambda}(I(\tau)))$ is a family of elements of $\mathcal{L}_{F_\lambda \oplus F'_\mu}$.*

Proof 2.3.13 *This follows immediately by combining Theorem 2.3.9 and Lemma 2.3.10*

Corollary 2.3.12 produces a family of elements on the twisted Lagrangian cone $\mathcal{L}_{F_\lambda \oplus F'_\mu}$, but in general it is not obvious whether the nonequivariant limit $\lambda \to 0$ of this family exists. However, in the case when F is split and $\tau \mapsto I(\tau)$ satisfies the Divisor Equation we will show that the family $\Delta_{F_\lambda}(D_{F_\lambda}(I(\tau, -z)))$ is equal to the *twisted I-function $I_{F'_\mu \oplus F_\lambda}$* given in Definition 2.3.14. This has an explicit expression, which makes it easy to check whether the nonequivariant limit exists. We make the following definitions.

Definition 2.3.14 *Let $\tau \mapsto I(\tau)$ be a family of elements of $\mathcal{L}_{F'_\mu}$. Let $F = F_1 \oplus \cdots \oplus F_k$ be a direct sum of line bundles, and let $f_i = c_1(F_i)$. For $\beta \in \mathrm{NE}(Y)$, we define the modification factor*

$$M_\beta(z) = \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{\langle f_i, \beta \rangle} \lambda + f_i + mz}{\prod_{m=-\infty}^{0} \lambda + f_i + mz}$$

The associated twisted I-function is

$$I^{\mathrm{tw}}(\tau) = \sum_{\beta \in \mathrm{NE}(Y)} Q^\beta I_\beta(\tau, z) \cdot M_\beta(z)$$

To relate $M_\beta(z)$ to the Quantum Riemann–Roch operator we will need the following Lemma:

Lemma 2.3.15

$$M_\beta(-z) = \Delta_{F_\lambda} \left(\prod_{i=1}^{k} \exp(-G(f_i - \langle f_i, \beta \rangle z, z)) \right)$$

Proof 2.3.16 *Define*

$$\mathbf{s}(x) = \sum_{k \geq 0} s_k \frac{x^k}{k!}$$

By [CCIT09, equation 13] we have that

$$G(x + z, z) = G(x, z) + \mathbf{s}(x) \tag{2.16}$$

We can rewrite

$$M_\beta(z) = \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{\langle f_i, \beta \rangle} \lambda + f_i + mz}{\prod_{m=-\infty}^{0} \lambda + f_i + mz} = \prod_{i=1}^{k} \frac{\prod_{m=-\infty}^{\langle f_i, \beta \rangle} \exp[\mathbf{s}(f_i + mz)]}{\prod_{m=-\infty}^{0} \exp[\mathbf{s}(f_i + mz)]}$$

and so

$$\begin{aligned} M_\beta(-z) &= \prod_{i=1}^{k} \exp\left(\sum_{m=-\infty}^{\langle f_i, \beta \rangle} \mathbf{s}(f_i - mz) - \sum_{m=-\infty}^{0} \mathbf{s}(f_i - mz) \right) \\ &= \prod_{i=1}^{k} \exp(G(f_i, z) - G(f_i - \langle f_i, \beta \rangle z, z) \end{aligned}$$

where for the second equality we used (2.16).

Proposition 2.3.17 *Let $\tau \mapsto I(\tau)$ be a family of elements of $\mathcal{L}_{F'_\mu}$ that satisfies the Divisor Equation, and let $F = F_1 \oplus \cdots \oplus F_k$ be a direct sum of line bundles. Then*

$$I^{\mathrm{tw}} = \Delta_{F_\lambda}(D_{F_\lambda}(I)). \tag{2.17}$$

As a consequence, $\tau \mapsto I^{\mathrm{tw}}(\tau)$ is a family of elements on the cone $\mathcal{L}_{F_\lambda \oplus F'_\mu}$.

Proof 2.3.18 *Lemma 2.3.15 shows that*

$$I^{\mathrm{tw}}(\tau) = \Delta_{F_\lambda} \left(\sum_{\beta \in \mathrm{NE}(Y)} \prod_{i=1}^{k} \exp(-G(f_i - \langle f_i, \beta \rangle z, z)) I_\beta(\tau, z) \right) \tag{2.18}$$

Applying the Divisor Equation, we can rewrite this as

$$I^{\mathrm{tw}} = \Delta_{F_\lambda}(D_{F_\lambda}(I)) \tag{2.19}$$

as required. The rest is immediate from 2.3.12.

Proposition 2.3.19 *If the line bundles F_i are nef, then the nonequivariant limit $\lambda \to 0$ of $I^{\mathrm{tw}}(\tau)$ exists.*

Proof 2.3.20 *This is immediate from Definition 2.3.14.*

2.4 The Givental–Martin cone

We now restrict to the situation described in the Introduction, where the action of a reductive Lie group G on a smooth quasiprojective variety A leads to smooth GIT quotients $A/\!/G$ and $A/\!/T$. As discussed, the roots of G define a vector bundle $\Phi = \oplus_\rho L_\rho \to Y$, where $Y = A/\!/T$, and we consider twisting data $(\Phi, \mathbf{c})$ for Y where $\mathbf{c}$ is the $\mathbb{C}^*$-equivariant Euler class. We call the modification factor in this setting the *Weyl modification factor*, and denote it as

$$W_\beta(z) = \prod_\alpha \frac{\prod_{m=-\infty}^{\langle c_1(L_\alpha),\beta\rangle} c_1(L_\alpha) + \lambda + mz}{\prod_{m=-\infty}^{0} c_1(L_\alpha) + \lambda + mz} \tag{2.20}$$

where the product runs over all roots α. For any family $\tau \mapsto I(\tau) = \sum_{\beta\in\mathrm{NE}(Y)} Q^\beta I_\beta(\tau,z)$ of elements of $\mathcal{H}_Y$, the corresponding twisted I-function is

$$I^{\mathrm{tw}}(\tau) = \sum_{\beta\in\mathrm{NE}(Y)} Q^\beta I_\beta(\tau,z)\cdot W_\beta(z) \tag{2.21}$$

Since the roots bundle Φ is not convex, in general the non-equivariant limit $\lambda \to 0$ of I^{tw} will not exist. However the limit will exist after projecting to $\mathcal{H}_{A/\!/G}$.

To account for the fact that there are fewer curve classes on $A/\!/G$ than there are on $A/\!/T$, we combine the quotient map (2.8) with the map on Novikov rings 2.2.4 to give a map

$$p\colon \mathcal{H}^W_{A/\!/T} \to \mathcal{H}_{A/\!/G} \tag{2.22}$$

between the Weyl-invariant part of the Givental space for the Abelian quotient and the Givental space for the non-Abelian quotient. Here, and also below when we discuss Weyl-invariant functions, we consider the Weyl group W to act on $\mathcal{H}_{A/\!/T}$ through the combination of its action on cohomology classes and its action on the Novikov ring.

Lemma 2.4.1 *Suppose that I is Weyl-invariant. Then $p\circ I^{\mathrm{tw}}$ has a well-defined limit as $\lambda \to 0$.*

Proof 2.4.2 *Since $I(\tau)$ is Weyl-invariant, $I^{\mathrm{tw}}(\tau)$ is also Weyl invariant and so, after applying ϱ, the coefficient of each Novikov term Q^β in $\tau \mapsto I^{\mathrm{tw}}(\tau)$ lies in $H^\star(A/\!/T;\mathbb{C})^W$. The composition $p\circ I^{\mathrm{tw}}$ is therefore well-defined.*

The Weyl modification (2.20) *contains many factors*

$$\frac{c_1(L_\alpha) + \lambda + mz}{-c_1(L_\alpha) + \lambda - mz}$$

which arise by combining the terms involving roots α and $-\alpha$. Such factors have a well-defined limit, -1, as $\lambda \to 0$. Therefore the limit of $p\circ I^{\mathrm{tw}}$ as $\lambda \to 0$ is well-defined if and only if the limit of

$$p\left(\sum_{\beta\in\mathrm{NE}(Y)} Q^\beta I_\beta(\tau,z)\cdot(-1)^{\epsilon(\beta)} \prod_{\alpha\in\Phi^+} \frac{c_1(L_\alpha) \pm \lambda + \langle c_1(L_\alpha),\beta\rangle z}{c_1(L_\alpha) \mp \lambda}\right) \tag{2.23}$$

as $\lambda \to 0$ is well-defined, and the two limits coincide. Here Φ^+ is the set of positive roots of G, and $\epsilon(\beta) = \sum_{\alpha\in\Phi^+}\langle c_1(L_\alpha),\beta\rangle$; cf. [CFKS08, equation 3.2.1]. The limit $\lambda \to 0$ of the denominator terms

$$\prod_{\alpha\in\Phi^+} \left(c_1(L_\alpha) - \lambda\right)$$

in (2.23) *is the fundamental Weyl-anti-invariant class ω from the discussion before Theorem 2.2.1. Furthermore*

$$\sum_{\beta \in \mathrm{NE}(Y)} Q^{\beta} I_{\beta}(\tau, z) \cdot (-1)^{\epsilon(\beta)} \prod_{\alpha \in \Phi^{+}} \big(c_1(L_\alpha) + \lambda + \langle c_1(L_\alpha), \beta \rangle z\big)$$

has a well-defined limit as $\lambda \to 0$ which, as it is Weyl-anti-invariant, is divisible by ω. The quotient here is unique up to an element of $\mathrm{Ann}(\omega)$*, and therefore the projection of the quotient along Martin's map $H^\star(A /\!/ T; \mathbb{C})^W \to H^\star(A /\!/ G; \mathbb{C})$ is unique. It follows that the limit as $\lambda \to 0$ of $p \circ I^{\mathrm{tw}}$ is well-defined.*

Definition 2.4.3 *Let $\tau \mapsto I(\tau)$ be a Weyl-invariant family of elements of $\mathcal{H}_Y$ and let I^{tw} denote the twisted I-function as above. We call the nonequivariant limit of $\tau \mapsto p\big(I^{\mathrm{tw}}(\tau)\big)$ the* Givental–Martin modification *of the family $\tau \mapsto I(\tau)$, and denote it by $\tau \mapsto I_{\mathrm{GM}}(\tau)$*

Recall that we have fixed a representation ρ of G on a vector space V, and that this induces vector bundles $V^T \to A /\!/ T$ and $V^G \to A /\!/ G$. Since the bundle $\Phi \to A /\!/ T$ is not convex, one cannot expect the non-equivariant limit of $\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$ to exist. Nonetheless, the projection along (2.22) of the Weyl-invariant part of $\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$ does admit a non-equivariant limit.

Theorem 2.4.4 *The non-equivariant limit $\lambda \to 0$ of $p\left(\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu} \cap \mathcal{H}^W_{A /\!/ T}\right)$ exists.*

We call this non-equivariant limit the *twisted Givental–Martin cone* $\mathcal{L}_{\mathrm{GM}, V^T_\mu} \subset \mathcal{H}^W_{A /\!/ T}$.

Proof 2.4.5 (Proof of Theorem 2.4.4) *Recall the twisted J-function $J_{V^T_\mu}(\tau, -z)$ from Definition 2.3.4. By [CG07] a general point*

$$-z + t_0 + t_1 z + \cdots + O(z^{-1})$$

on $\mathcal{L}_{V^T_\mu}$ can be written as

$$J_{V^T_\mu}(\tau(\mathbf{t}), -z) + \sum_{\alpha=1}^{N} C_\alpha(\mathbf{t}, z) z \frac{\partial J_{V^T_\mu}}{\partial \tau^\alpha}(\tau(\mathbf{t}), -z)$$

for some coefficients $C_\alpha(\mathbf{t}, z)$ that depend polynomially on z and some $H^\star(A /\!/ T)$-valued function $\tau(\mathbf{t})$ of $\mathbf{t} = (t_0, t_1, \ldots)$. The Weyl modification $\tau \mapsto I^{\mathrm{tw}}(\tau)$ of $\tau \mapsto J_{V^T_\mu}(\tau, -z)$ satisfies $I^{\mathrm{tw}}(\tau) \equiv J_{V^T_\mu}(\tau, -z)$ modulo Novikov variables, and $I^{\mathrm{tw}}(\tau) \in \mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$ by Proposition 2.3.17, so a general point

$$-z + t_0 + t_1 z + \cdots + O(z^{-1}) \tag{2.24}$$

on $\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$ can be written as

$$I^{\mathrm{tw}}(\tau(\mathbf{t})^\dagger, -z) + \sum_{\alpha=1}^{N} C_\alpha(\mathbf{t}, z)^\dagger z \frac{\partial I^{\mathrm{tw}}}{\partial \tau^\alpha}(\tau(\mathbf{t})^\dagger, -z)$$

for some coefficients $C_\alpha(\mathbf{t}, z)^\dagger$ that depend polynomially on z and some $H^\star(A /\!/ T)$-valued function $\tau(\mathbf{t})^\dagger$. Since the twisted J-function is Weyl-invariant, so is $I^{\mathrm{tw}}(\tau)$, and thus if (2.24) *is Weyl-invariant then we may take $C_\alpha(\mathbf{t}, z)^\dagger$ to be such that $\sum_\alpha C_\alpha(\mathbf{t}, z)^\dagger \phi_\alpha$ is Weyl-invariant. Projecting along* (2.22) *we see that a general point*

$$-z + t_0 + t_1 z + \cdots + O(z^{-1}) \tag{2.25}$$

on $p\left(\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu} \cap \mathcal{H}^W_{A /\!/ T}\right)$ can be written as

$$p \circ I^{\mathrm{tw}}(\tau(\mathbf{t})^\ddagger, -z) + \sum_{\alpha=1}^{N} C_\alpha(\mathbf{t}, z)^\ddagger z \frac{\partial (p \circ I^{\mathrm{tw}})}{\partial \tau^\alpha}(\tau(\mathbf{t})^\ddagger, -z)$$

for some coefficients $C_\alpha(\mathbf{t},z)^\ddagger$ that depend polynomially on z and some $H^\star(A/\!/T)$-valued function $\tau(\mathbf{t})^\ddagger$. Furthermore, since $p \circ I^{\mathrm{tw}}(\tau)$ has a well-defined non-equivariant limit $I_{\mathrm{GM}}(\tau)$, we see that $C_\alpha(\mathbf{t},z)^\ddagger$ also admits a non-equivariant limit. Hence a general point (2.25) *on $p\left(\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu} \cap \mathcal{H}^W_{A/\!/T}\right)$ has a well-defined limit as $\lambda \to 0$.*

Corollary 2.4.6 *The non-equivariant limit $\lambda \to 0$ of $p\left(\mathcal{L}_{\Phi_\lambda} \cap \mathcal{H}^W_{A/\!/T}\right)$ exists.*

We call this non-equivariant limit the *Givental–Martin cone* $\mathcal{L}_{\mathrm{GM}} \subset \mathcal{H}^W_{A/\!/T}$.

Proof 2.4.7 *Take the vector bundle V^T in Theorem 2.4.4 to have rank zero.*

Corollary 2.4.8 *If $\tau \mapsto I(\tau)$ is a Weyl-invariant family of elements of $\mathcal{L}_{V^T_\mu}$ that satisfies the Divisor Equation* (2.15) *then the Givental–Martin modification $\tau \mapsto I_{\mathrm{GM}}(\tau)$ is a family of elements of $\mathcal{L}_{\mathrm{GM},V^T_\mu}$*

Proof 2.4.9 *Proposition 2.3.17 implies that $\tau \mapsto I^{\mathrm{tw}}(\tau,-z)$ is a family of elements on $\mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$. Projecting along* (2.22) *and taking the limit $\lambda \to 0$, which exists by Lemma 2.4.1, proves the result.*

We can now state our conjectures.

Conjecture 2.4.10 (The Abelian/non-Abelian Correspondence) $\mathcal{L}_{\mathrm{GM}} = \mathcal{L}_{A/\!/G}$.

This is a reformulation of [CFKS08, Conjecture 3.7.1]. The analogous statement for twisted Gromov–Witten invariants is the Abelian/non-Abelian Correspondence with bundles; this is a reformulation of [CFKS08, Conjecture 6.1.1].

Conjecture 2.4.11 (The Abelian/non-Abelian Correspondence with bundles) $\mathcal{L}_{\mathrm{GM},V^T_\mu} = \mathcal{L}_{V^G_\mu}$.

As in [CFKS08], the Abelian/non-Abelian Correspondence implies the Abelian/non-Abelian Correspondence with bundles.

Proposition 2.4.12 *Conjectures 2.4.10 and 2.4.11 are equivalent.*

Proof 2.4.13 *Conjecture 2.4.10 is the special case of Conjecture 2.4.11 where the vector bundles involved have rank zero. To see that Conjecture 2.4.10 implies Conjecture 2.4.11, observe that the projection of the Quantum Riemann–Roch operator $\Delta_{V^T_\mu}$ under the map* (2.22) *is $\Delta_{V^G_\mu}$: see Definition 2.3.8. Now apply the Quantum Riemann–Roch theorem [CG07].*

The following reformulations will also be useful.

Conjecture 2.4.14 (a reformulation of Conjecture 2.4.10) *Let $t \mapsto I(t)$ be a Weyl-invariant family of elements of $\mathcal{L}_{A/\!/T}$ that satisfies the Divisor Equation. Then the Givental–Martin modification $t \mapsto I_{\mathrm{GM}}(t)$ is a family of elements of $\mathcal{L}_{A/\!/G}$.*

Conjecture 2.4.15 (a reformulation of Conjecture 2.4.11) *Let $t \mapsto I(t)$ be a Weyl-invariant family of elements of $\mathcal{L}_{V^T_\mu}$ that satisfies the Divisor Equation. Then the Givental–Martin modification $t \mapsto I_{\mathrm{GM}}(t)$ is a family of elements of $\mathcal{L}_{V^G_\mu}$.*

2.5 The Abelian/non-Abelian Correspondence for Flag Bundles

2.5.1 The Work of Brown and Oh

In this section we will review results by Brown [Bro14] and Oh [Oh16], and situate their work in terms of the Abelian/non-Abelian Correspondence (Conjecture 2.4.14). In particular, we show that the Givental–Martin modification of the Brown I-function is the Oh I-function. We freely use the notation introduced in Section 2.2.2.

Let X be a smooth projective variety. We will decompose the J-function of X, defined in §2.3, into contributions from different degrees:

$$J_X(\tau, z) = \sum_{D \in \mathrm{NE}(X)} J_X^D(\tau, z) Q^D. \tag{2.26}$$

Recall that we have a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_n \xrightarrow{\pi} X$, and that $\mathrm{Fl}(E) = \mathrm{Fl}(r_1, \ldots, r_\ell, E) = A /\!/ G$ is the partial flag bundle associated to E. As in §2.2.2, we form the toric fibration $\mathrm{Fl}(E)_T = A /\!/ T$ with general fibre $\mathbb{C}^N /\!/ (\mathbb{C}^*)^R$. We denote both projection maps $\mathrm{Fl}(E) \to X$ and $\mathrm{Fl}(E)_T \to X$ by π. For the sake of clarity, we will denote homology and cohomology classes on $\mathrm{Fl}(E)_T$ with a tilde and classes on $\mathrm{Fl}(E)$ without. Recall the cohomology classes $\tilde{H}_{\ell+1,j} = -\pi^* c_1(L_j)$ on $\mathrm{Fl}(E)_T$, and $H_{\ell+1,j} = -\pi^* c_1(L_j)$ on $\mathrm{Fl}(E)$. For a fixed homology class $\tilde{\beta}$ on $\mathrm{Fl}(E)_T$ define $d_{\ell+1,j} = \langle -\pi^* c_1(L_j), \tilde{\beta} \rangle$, and for a fixed homology class β on $\mathrm{Fl}(E)$ define $d_{\ell+1,j} = \langle -\pi^* c_1(L_j), \beta \rangle$. We use the indexing of the set $\{1, \ldots, R\}$ defined in Section 2.2.2, and denote the components of a vector $\underline{d} \in \mathbb{Z}^R$ by $d_{i,j}$. Similarly, we denote components of a vector $\underline{d} \in \mathbb{Z}^\ell$ by d_i.

In [Oh16], the author proves that a certain generating function, the I-function of $\mathrm{Fl}(E)$, lies on the Lagrangian cone for $\mathrm{Fl}(E)$.

Theorem 2.5.1 *Let $\tau \in H^\star(X)$, $t = \sum_i t_i c_1(S_i^\vee)$, and define the I-function of* $\mathrm{Fl}(E)$ *to be*

$$I_{\mathrm{Fl}(E)}(t, \tau, z) =$$
$$e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t \rangle} \pi^* J_X^{\pi_* \beta}(\tau, z) \sum_{\substack{\underline{d} \in \mathbb{Z}^R: \\ \forall i \sum_j d_{i,j} = \langle \beta, c_1(S_i^\vee) \rangle}} \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j} - H_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j} - d_{i+1,j'}} H_{i,j} - H_{i+1,j'} + mz}$$
$$\times \prod_{i=1}^{\ell} \prod_{j \neq j'} \frac{\prod_{m=-\infty}^{d_{i,j} - d_{i,j'}} H_{i,j} - H_{i,j'} + mz}{\prod_{m=-\infty}^{0} H_{i,j} - H_{i,j'} + mz}$$

Then $I_{\mathrm{Fl}(E)}(t, \tau, -z) \in \mathcal{L}_{\mathrm{Fl}(E)}$ for all t and τ.

In [Bro14], the author proves an analogous result for the corresponding Abelian quotient $\mathrm{Fl}(E)_T$.

Theorem 2.5.2 *Let $\tau \in H^\star(X)$, $t = \sum_{i,j} t_{i,j} \tilde{H}_{i,j}$, and define the Brown I-function of* $\mathrm{Fl}(E)_T$ *to be*

$$I_{\mathrm{Fl}(E)_T}(t, \tau, z) =$$
$$e^{\frac{t}{z}} \sum_{\tilde{\beta} \in H_2 \mathrm{Fl}(E)_T} Q^{\tilde{\beta}} e^{\langle \tilde{\beta}, t \rangle} \pi^* J_X^{\pi_* \tilde{\beta}}(\tau, z) \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} + mz}{\prod_{m=-\infty}^{\langle \tilde{\beta}, \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} \rangle} \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} + mz}$$

Then $I_{\mathrm{Fl}(E)_T}(t, \tau, -z) \in \mathcal{L}_{\mathrm{Fl}(E)_T}$ for all t and τ.

Remark 2.5.3 *We have chosen to state Theorem 2.5.2 in a different form than in Brown's original paper. The equivalence of the two versions follows from Lemma 2.5.4 below. The classes $H_{i,j}$ here were denoted in [Bro14] by P_i, and the classes $H_{i,j} - H_{i+1,j'}$ here were denoted there by U_k.*

Lemma 2.5.4 *Writing $I_{\mathrm{Fl}(E)_T} = \sum_{\tilde{\beta}} I^{\tilde{\beta}}_{\mathrm{Fl}(E)_T} Q^{\tilde{\beta}}$, any nonzero $I^{\tilde{\beta}}$ must have $\tilde{\beta} \in \mathrm{NE}(\mathrm{Fl}(E)_T)$.*

Proof 2.5.5 *To see this we temporarily adopt the notation of Brown and denote the torus invariant divisors by U_k, as in Lemma 2.2.9. Then $I_{\mathrm{Fl}(E)_T}$ takes the form*

$$I_{\mathrm{Fl}(E)_T} = \sum_{\substack{\tilde{\beta} \in H_2\mathrm{Fl}(E)_T: \\ \pi_*\tilde{\beta} \in \mathrm{NE}(X)}} (\dots) \prod_{k=1}^{N} \frac{\prod_{m=-\infty}^{0} U_k + mz}{\prod_{m=-\infty}^{\langle \tilde{\beta}, U_k \rangle} U_k + mz}$$

Let $\alpha \subset \{1, \dots N\}$ be a subset of size R which defines a section of the toric fibration as in Section 2.2.2. We have that

$$s_\alpha^* I_{\mathrm{Fl}(E)_T} = (\dots) \prod_{k \in \alpha} \frac{\prod_{m=-\infty}^{0} (0) + mz}{\prod_{m=-\infty}^{\langle \tilde{\beta}, U_k \rangle} (0) + mz} \prod_{k \notin \alpha} \frac{\prod_{m=-\infty}^{0} s_\alpha^* U_k + mz}{\prod_{m=-\infty}^{\langle \tilde{\beta}, U_k \rangle} s_\alpha^* U_k + mz}$$

since $s_\alpha^*(U_k) = 0$ if $k \in \alpha$. Therefore, if $\langle \tilde{\beta}, U_k \rangle < 0$ for some $k \in \alpha$, the numerator contains a term (0) and vanishes. We conclude that any $\tilde{\beta} \in H_2\mathrm{Fl}(E)_T$ which gives a nonzero contribution to $s_\alpha^* I_{\mathrm{Fl}(E)_T}$ must satisfy the conditions

$$\pi_*\tilde{\beta} \in \mathrm{NE}(X), \langle \tilde{\beta}, U_k \rangle \geq 0 \, \forall k \in \alpha.$$

The section s_α gives a splitting $H_2(\mathrm{Fl}(E)_T) = H_2(X) \oplus H_2(\mathrm{Fl}_T)$, via which we may write $\tilde{\beta} = s_{\alpha_*} D + \iota_* d$ where ι is the inclusion of a fibre. We have

$$\langle \tilde{\beta}, U_k \rangle = \langle D, s_\alpha^* U_k \rangle + \langle d, \iota^* U_k \rangle = \langle d, \iota^* U_k \rangle \geq 0$$

for all $k \in \alpha$. However, the cone in the secondary fan spanned by the line bundles $\iota^* U_k$ contains the ample cone of Fl_T (see Section 2.2.2), so this implies $d \in \mathrm{NE}(\mathrm{Fl}_T)$. It follows that any $\tilde{\beta}$ which gives a nonzero contribution to $s_\alpha^* I_{\mathrm{Fl}(E)_T}$ is effective. We now use the Atiyah-Bott localization formula

$$I_{\mathrm{Fl}(E)_T} = \sum_\alpha s_{\alpha_*} \left(\frac{s_\alpha^* I_{\mathrm{Fl}(E)_T}}{e^\alpha} \right), \quad \text{where } e^\alpha = \prod_{k \notin \alpha} s_\alpha^* U_k$$

where α ranges over the torus fixed point sections of the fibration, to conclude that the same is true for $I_{\mathrm{Fl}(E)_T}$.

Lemma 2.5.6 *Brown's I-function satisfies the Divisor Equation. That is,*

$$z \nabla_\rho I^{\tilde{\beta}}_{\mathrm{Fl}(E)_T} = (\rho + \langle \rho, \tilde{\beta} \rangle z) I^{\tilde{\beta}}_{\mathrm{Fl}(E)_T}$$

for any $\rho \in H^2(\mathrm{Fl}(E)_T)$.

Proof 2.5.7 *Decompose $\rho = \rho_F + \pi^* \rho_B$ into fibre and base part. Basic differentiation and the divisor equation for J_X show that*

$$z \nabla_\rho I^{\tilde{\beta}}_{\mathrm{Fl}(E)_T} = \left(\rho_F + \langle \rho_F, \tilde{\beta} \rangle z + (\pi^* \rho_B + \langle \pi^* \rho_B, \tilde{\beta} \rangle z) \right) e^{t/z} e^{\langle \tilde{\beta}, t \rangle} \pi^* J_X^{\pi_* \tilde{\beta}}(\tau, z) \cdot \mathbf{H}$$

where $\mathbf{H}$ is a hypergeometric factor with no dependence on t or τ. The right-hand simplifies to

$$(\rho + \langle \rho, \tilde{\beta} \rangle z) I^{\tilde{\beta}}_{\mathrm{Fl}(E)_T}$$

as required.

Lemma 2.5.8 *If we restrict t to lie in the Weyl-invariant locus $H^2(\mathrm{Fl}(E)_T)^W \subset H^2(\mathrm{Fl}(E)_T)$ then $(t,\tau) \mapsto I_{\mathrm{Fl}(E)_T}(t,\tau,z)$ takes values in $H^\star(\mathrm{Fl}(E)_T)^W$.*

Proof 2.5.9 *This is immediate from the definition of $I_{\mathrm{Fl}(E)_T}(t,\tau,z)$, in Theorem 2.5.2.*

Proposition 2.5.10 *Restrict t to lie in the Weyl-invariant locus $H^2(\mathrm{Fl}(E)_T)^W \subset H^2(\mathrm{Fl}(E)_T)$ and consider the Brown I-function $(t,\tau) \mapsto I_{\mathrm{Fl}(E)_T}(t,\tau,z)$. The Givental–Martin modification $I_{\mathrm{GM}}(t,\tau)$ of this family is equal to Oh's I-function $I_{\mathrm{Fl}(E)}(t,\tau)$.*

Proof 2.5.11 *Lemma 2.5.8 and Lemma 2.4.1 imply that the Givental–Martin modification $I_{\mathrm{GM}}(t,\tau)$ exists. We need to compute it. Note that the restrictions to the fibre of the classes $\tilde{H}_{i,j}$ form a basis for $H^2(\mathrm{Fl}_T)$. Since the general fibre Fl_T of $\mathrm{Fl}(E)_T$ has vanishing first homology, the Leray–Hirsch theorem gives an identification $\mathbb{Q}[H_2(\mathrm{Fl}(E)_T,\mathbb{Z})] = \mathbb{Q}[H_2(X,\mathbb{Z})][q_{1,1},\dots,q_{\ell,r_\ell}]$ via the map*

$$Q^{\tilde{\beta}} \mapsto Q^{\pi_*\tilde{\beta}} \prod_{i,j} q_{i,j}^{\langle \tilde{H}_{i,j},\tilde{\beta}\rangle} \tag{2.27}$$

By Lemma 2.5.4, the summation range in the sum defining $I_{\mathrm{Fl}(E)_T}$ is contained in $\mathrm{NE}(\mathrm{Fl}(E)_T)$. We can therefore write the corresponding twisted I-function (2.21) *as*

$$I^{\mathrm{tw}}(t,\tau,z) = e^{\frac{t}{z}} \sum_{\substack{D\in\mathrm{NE}(X)\\ \underline{d}\in\mathbb{Z}^R}} Q^D \prod_{i,j} q_{i,j}^{d_{i,j}} e^{t\cdot\underline{d}} \pi^* J_X^D(\tau,z) \prod_{i=1}^{\ell}\prod_{j=1}^{r_i}\prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} \tilde{H}_{i,j} - \tilde{H}_{i+1,j'} + mz}$$
$$\times \prod_{i=1}^{\ell}\prod_{j\neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} \tilde{H}_{i,j} - \tilde{H}_{i,j'} + \lambda + mz}{\prod_{m=-\infty}^{0} \tilde{H}_{i,j} - \tilde{H}_{i,j'} + \lambda + mz}$$

where the $t_{i,j} \in \mathbb{C}$, $t = \sum_{i=1}^{\ell}\sum_{j=1}^{r_i} t_{i,j}\tilde{H}_{i,j}$, and $t\cdot\underline{d} = \sum_{i,j} t_{i,j}d_{i,j}$. For the Weyl modification factor we used the fact that the roots of G are given by $\rho_{i,j}\rho_{i,j'}^{-1}$, where the character $\rho_{i,j}$ was defined in section 2.2.2. By Lemma 2.5.4 the effective summation range for the vector $\underline{d}$ here is contained in the set $S \subset \mathbb{Z}^R$ consisting of $\underline{d}$ such that $\langle\tilde{\beta},\tilde{H}_{i,j}\rangle = d_{i,j}$ for some $\tilde{\beta} \in \mathrm{NE}(\mathrm{Fl}(E)_T)$.

We can identify the group ring $\mathbb{Q}[H_2(\mathrm{Fl}(E))]$ with $\mathbb{Q}[H_2(X,\mathbb{Z})][q_1,\dots,q_\ell]$ via the map

$$Q^{\beta} \mapsto Q^{\pi_*\beta} \prod_{i} q_{i}^{\langle c_1(S_i^\vee),\beta\rangle} \tag{2.28}$$

Via (2.27) *and* (2.28) *the map on Mori cones $\varrho : \mathrm{NE}(\mathrm{Fl}(E)_T) \to \mathrm{NE}(\mathrm{Fl}(E))$ becomes*

$$Q^D \prod_{i,j} q_{i,j}^{d_{i,j}} \mapsto Q^D \prod_i q_i^{\sum_j d_{i,j}}$$

Restricting t to the Weyl-invariant locus $H^2(\mathrm{Fl}(E)_T)^W$ corresponds to setting $t_{i,j} = t_i$ for all i and j, which gives $e^{t\cdot\underline{d}} = e^{\sum_i t_i d_i}$ where $d_i = \sum_j d_{i,j}$. The identification $H^2(\mathrm{Fl}(E)_T)^W \cong H^2(\mathrm{Fl}(E))$ sends $\sum_{i,j} t_i\tilde{H}_{i,j}$ to $\sum_i t_i c_1(S_i^\vee)$, so projecting along (2.22) *and taking the limit as $\lambda = 0$ we obtain*

$$e^{\frac{t}{z}} \sum_{\substack{D\in\mathrm{NE}(X)\\ \underline{\delta}\in\mathbb{Z}^{\ell}}} Q^D \prod_i q_i^{\delta_i} e^{t\cdot\underline{\delta}} \pi^* J_X^D(\tau,z) \sum_{\substack{\underline{d}\in\mathbb{Z}^R:\\ \forall i \sum_j d_{i,j}=\delta_i}} \prod_{i=1}^{\ell}\prod_{j=1}^{r_i}\prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j} - H_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} H_{i,j} - H_{i+1,j'} + mz}$$
$$\times \prod_{i=1}^{\ell}\prod_{j\neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} H_{i,j} - H_{i,j'} + mz}{\prod_{m=-\infty}^{0} H_{i,j} - H_{i,j'} + mz}$$

where now $t = \sum_i t_i c_1(S_i^\vee)$. *The effective summation range here is contained in* $\mathrm{NE}(\mathrm{Fl}(E))$ *by construction. Using* (2.28) *again we may rewrite this as*

$$e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_* \beta}(\tau, z) \sum_{\substack{d \in \mathbb{Z}^R: \\ \forall i \sum_j d_{i,j} = \langle \beta, c_1(S_i^\vee)\rangle}} \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j} - H_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j}-d_{i+1,j'}} H_{i,j} - H_{i+1,j'} + mz}$$
$$\times \prod_{i=1}^{\ell} \prod_{j \neq j'} \frac{\prod_{m=-\infty}^{d_{i,j}-d_{i,j'}} H_{i,j} - H_{i,j'} + mz}{\prod_{m=-\infty}^{0} H_{i,j} - H_{i,j'} + mz}$$

This is $I_{\mathrm{Fl}(E)}(t, \tau, z)$*, as required.*

Remark 2.5.12 *In view of* (2.11)*, we see that the effective summation range in* $I_{\mathrm{Fl}(E)}$ *is contained in the subset of vectors satisfying*

$$d_{i,j} \geq \min_{j'} d_{\ell+1,j'} \;\forall\, i, j$$

This will prove useful in calculations in Section 2.7.

Notice that 2.5.10 establishes the statement of Conjecture 2.4.15, not for an arbitrary Weyl-invariant family, but for Brown's I-function. As we will now explain, this is quite close to a proof of Conjecture 2.4.14 in the flag bundle case, and similarly Theorem 2.5.16 is close to a proof of Conjecture 2.4.15. We will discuss only the former, as the latter is very similar. If Oh's I-function were a *big I-function*, in the sense of [CFK16], then Conjecture 2.4.14 would follow. The special geometric properties of the Lagrangian submanifold $\mathcal{L}_Y$ described in [Giv04] and [CCIT09, Appendix B], taking $Y = \mathrm{Fl}(E)$, would then imply that any family $t \mapsto I(t)$ such that $I(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$ can be written as

$$I(t) = I_{\mathrm{Fl}(E)}(\tau(t)) + \sum_\alpha C_\alpha(t, z) z \frac{\partial I_{\mathrm{Fl}(E)}}{\partial \tau_\alpha}(\tau(t)) \tag{2.29}$$

for some coefficients $C_\alpha(t, z)$ that depend polynomially on z and some change of variables $t \mapsto \tau(t)$. Furthermore the same geometric properties imply that any family of the form (2.29) satisfies $I(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$. But $\mathcal{L}_{\mathrm{GM}}$ has the same special geometric properties as $\mathcal{L}_Y$ – it inherits them from the Weyl-invariant part of $\mathcal{L}_{\Phi_\lambda}$ by projection along (2.22) followed by taking the non-equivariant limit – and so if $t \mapsto I_{\mathrm{Fl}(E)}$ is a big I-function then any family of elements $t \mapsto I^\dagger(t)$ on $\mathcal{L}_{\mathrm{GM}}$ can be written as

$$I^\dagger(t) = I_{\mathrm{Fl}(E)}(\tau^\dagger(t)) + \sum_\alpha C_\alpha^\dagger(t, z) z \frac{\partial I_{\mathrm{Fl}(E)}}{\partial \tau_\alpha}(\tau^\dagger(t))$$

That is, $I^\dagger(t)$ can be written in the form (2.29). It follows that $I^\dagger(t) \in \mathcal{L}_{\mathrm{Fl}(E)}$. Applying this with $I^\dagger = I_{\mathrm{GM}}$ from Conjecture 2.4.14 proves that Conjecture; note that we know that the family $t \mapsto I_{\mathrm{GM}}(t)$ here lies in $\mathcal{L}_{\mathrm{GM}}$ by Corollary 2.4.8.

If the Brown and Oh I-functions were big I-functions then Theorem 2.5.10 would continue to hold (with the same proof) and Conjecture 2.4.14 would therefore follow. In reality the Brown and Oh I-functions are only small I-functions, not big I-functions, but Ciocan-Fontanine–Kim have explained in [CFK16, §5] how to pass from small I-functions to big I-functions, whenever the target space is the GIT quotient of a vector space. To apply their argument, and hence prove Conjecture 2.4.14 for partial flag bundles, one would need to check that the Brown I-function arises from torus localization on an appropriate quasimap graph space [CFKM14, §7.2]. The analogous result for the Oh I-function is [Oh16, Proposition 5.1].

Webb has proved a 'big I-function' version of the Abelian/non-Abelian Correspondence for target spaces that are GIT quotients of vector spaces [Web18], and this immediately implies Conjectures 2.4.14 and 2.4.15.

Proposition 2.5.13 *Conjecture 2.4.14 holds when A is a vector space and G acts on A via a representation $G \mapsto \mathrm{GL}(A)$.*

Proof 2.5.14 *Combining [Web18, Corollary 6.3.1] with [CFK16, Theorem 3.3] shows that there are big I-functions $t \mapsto I_{A/\!/T}(t)$ and $t \mapsto I_{A/\!/G}(t)$ such that $I_{A/\!/T}(t) \in \mathcal{L}_{A/\!/T}$ and $I_{A/\!/G}(t) \in \mathcal{L}_{A/\!/G}$. Furthermore it is clear from [Web18, equation 62] that the Givental–Martin modification of the Weyl-invariant part of $t \mapsto I_{A/\!/T}(t)$ is $t \mapsto I_{A/\!/G}(t)$. Now argue as above.*

2.5.2 The Abelian/non-Abelian Correspondence with bundles

We are now ready to prove the theorem which will allow us to calculate twisted Gromov–Witten invariants of Flag bundles, see 2.5.16. Assume we have fixed a representation $\rho\colon G \to \mathrm{GL}(V)$ where $G = \prod_i \mathrm{GL}_{r_i}(\mathbb{C})$, and that this determines vector bundles $V^G \to \mathrm{Fl}(E)$ and $V^T \to \mathrm{Fl}(E)_T$. Since T is Abelian, V^T splits as a direct sum of line bundles

$$V^T = F_1 \oplus \cdots \oplus F_k$$

The Brown I-function gives a family

$$(t, \tau) \mapsto I_{\mathrm{Fl}(E)_T}(t, \tau, -z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W, \tau \in H^\star(X)$$

of elements of $\mathcal{H}_{\mathrm{Fl}(E)_T}$, and Theorem 2.5.2 shows that $I_{\mathrm{Fl}(E)_T}(t, \tau, -z) \in \mathcal{L}_{\mathrm{Fl}(E)_T}$. Twisting by $(F, \mathbf{c})$ where $\mathbf{c}$ is the $\mathbb{C}^*$-equivariant Euler class with parameter μ gives a twisted I-function, as in Definition 2.3.14, which we denote by

$$(t, \tau) \mapsto I_{V^T_\mu}(t, \tau, -z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W, \tau \in H^\star(X)$$

Applying Proposition 2.3.17 shows that $I_{V^T_\mu}(t, \tau, -z) \in \mathcal{L}_{V^T_\mu}$. Twisting again, by $(\Phi, \mathbf{c}')$ where $\Phi \to \mathrm{Fl}(E)_T$ is the roots bundle from the Introduction and $\mathbf{c}'$ is the $\mathbb{C}^*$-equivariant Euler class with parameter λ gives a twisted I-function, as in Definition 2.3.14, which we denote by

$$(t, \tau) \mapsto I_{\Phi_\lambda \oplus V^T_\mu}(t, \tau, -z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W, \tau \in H^\star(X)$$

Applying Proposition 2.3.17 again shows that $I_{\Phi_\lambda \oplus V^T_\mu}(t, \tau, -z) \in \mathcal{L}_{\Phi_\lambda \oplus V^T_\mu}$. We now project along (2.22) and take the non-equivariant limit $\lambda \to 0$, obtaining the Givental–Martin modification of $I_{V^T_\mu}$. This is a family

$$(t, \tau) \mapsto I_{\mathrm{GM}}(t, \tau, -z) \qquad\qquad t \in H^2(\mathrm{Fl}(E)_T)^W, \tau \in H^\star(X)$$

of elements of $\mathcal{H}_{\mathrm{Fl}(E)}$. Explicitly:

Definition 2.5.15 (which is a specialisation of Definition 2.4.3 to the situation at hand)

$$\begin{aligned} &I_{\mathrm{GM}}(t, \tau, z) = \\ &e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_* \beta}(\tau, z) \sum_{\substack{\underline{d} \in \mathbb{Z}^R: \\ \forall i \sum_j d_{i,j} = \langle \beta, c_1(S_i^\vee)\rangle}} \prod_{i=1}^{\ell} \prod_{j=1}^{r_i} \prod_{j'=1}^{r_{i+1}} \frac{\prod_{m=-\infty}^{0} H_{i,j} - H_{i+1,j'} + mz}{\prod_{m=-\infty}^{d_{i,j} - d_{i+1,j'}} H_{i,j} - H_{i+1,j'} + mz} \\ &\qquad \times \prod_{i=1}^{\ell} \prod_{j \neq j'} \frac{\prod_{m=-\infty}^{d_{i,j} - d_{i,j'}} H_{i,j} - H_{i,j'} + mz}{\prod_{m=-\infty}^{0} H_{i,j} - H_{i,j'} + mz} \prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d}} f_s + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \mu + mz} \end{aligned}$$

Here $J^D_X(\tau, z)$ *is as in* (2.26), $f_s \cdot \underline{d} = \sum_{i,j} f_{s,i,j} d_{i,j}$, *and* $f_s = \sum_{i,j} f_{s,i,j} H_{i,j}$, *where*

$$c_1(F_s) = \sum_{i=1}^{\ell} \sum_{j=1}^{r_i} f_{s,i,j} \tilde{H}_{i,j}$$

Lemma 2.4.1 shows that this expression is well-defined despite the presence of

$$\omega = \prod_i \prod_{j<j'} (H_{i,j} - H_{i,j'})$$

in the denominator. Corollary 2.4.8 shows that $I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{\mathrm{GM}, V^T_\mu}$. Note that $I_{\mathrm{GM}}(t, \tau)$ is *not* the V^G-twist of Oh's I-function $I_{\mathrm{Fl}(E)}$. Indeed V^G need not be a split bundle, so the twist may not even be defined.

Theorem 2.5.16 *Let* I_{GM} *be as in Definition 2.5.15. Then:*

$$I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{V^G_\mu} \qquad \text{for all } t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\star(X).$$

Proof 2.5.17 *Before projecting and taking the non-equivariant limit, we have*

$$I_{\Phi_\lambda \oplus V^T_\mu} = \Delta_{V^T_\mu}(D_{V^T_\mu}(I_{\Phi_\lambda}))$$

by Proposition 2.17. Projecting along (2.22) *gives*

$$p \circ I_{\Phi_\lambda \oplus V^T_\mu} = \Delta_{V^G_\mu}(D_{V^G_\mu}(p \circ I_{\Phi_\lambda}))$$

and taking the limit $\lambda \to 0$, *which is well-defined by Lemma 2.4.1, gives*

$$I_{\mathrm{GM}} = \Delta_{V^G_\mu}(D_{V^G_\mu}(I_{\mathrm{Fl}(E)}))$$

by Proposition 2.5.10. The result now follows from Proposition 2.3.17.

Exactly the same argument proves:

Corollary 2.5.18 *Let* $L \to X$ *be a line bundle with first Chern class* ρ, *and define the vector bundle* $F \to \mathrm{Fl}(E)$ *to be* $F = V^G \otimes \pi^* L$. *Let* I_{GM} *be as in Definition 2.5.15, except that the factor*

$$\prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d}} f_s + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \mu + mz} \qquad \text{is replaced by} \qquad \prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot \underline{d} + \langle \rho, \pi_* \beta \rangle} f_s + \pi^*\rho + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \pi^*\rho + \mu + mz}$$

Then:

$$I_{\mathrm{GM}}(t, \tau, -z) \in \mathcal{L}_{F_\mu} \qquad \text{for all } t \in H^2(\mathrm{Fl}(E)_T)^W,\ \tau \in H^\star(X).$$

The following Corollary gives a closed-form expression for genus-zero Gromov–Witten invariants of the zero locus of a generic section Z of F in terms of invariants of X.

Corollary 2.5.19 *With notation as in Corollary 2.5.18, let* Z *be the zero locus of a generic section of* $F \to \mathrm{Fl}(E)$. *Suppose that* $-K_Z$ *is the restriction of an ample class on* $\mathrm{Fl}(E)$ *and that* $\tau \in H^2(X)$. *Then*

$$J_{F_\mu}(t + \tau, z) = e^{-C(t)/z} I_{\mathrm{GM}}(t, \tau, z)$$

where

$$C(t) = \sum_{\beta} n_\beta Q^\beta e^{\langle \beta, t\rangle}$$

for some constants $n_\beta \in \mathbb{Q}$ and the sum runs over the finite set

$$S = \{\beta \in \mathrm{NE}(\mathrm{Fl}(E)) : \langle -K_{\mathrm{Fl}(E)} - c_1(F), \beta\rangle = 1\}$$

If Z is of Fano index two or more then this set is empty and $C(t) \equiv 0$. Regardless, if the vector bundle F is convex then the non-equivariant limit $\mu \to 0$ of J_{F_μ} exists and

$$J_Z\big(i^* t + i^*\tau, z\big) = i^* J_{F_0}(t + \tau, z)$$

where $i\colon Z \to \mathrm{Fl}(E)$ is the inclusion map.

Proof 2.5.20 (Proof of Corollary 2.5.19) *The statement about Fano index two or more follows immediately from the Adjunction Formula*

$$K_Z = \big(K_{\mathrm{Fl}(E)} + c_1(F)\big)\big|_Z$$

We need to show that

$$I_{\mathrm{GM}}(t, \tau, z) = z + t + \tau + C(t) + O(z^{-1}) \tag{2.30}$$

Everything else then follows from the characterisation of the twisted J-function just below Definition 2.3.4, the String Equation

$$J_{F_\mu}(\tau + a, z) = e^{a/z} J_{F_\mu}(\tau, z) \qquad\qquad a \in H^0(\mathrm{Fl}(E))$$

and [Coa14]. To establish (2.30)*, it will be convenient to set* $\deg(z) = \deg(\mu) = 1$, $\deg(\phi) = k$ *for* $\phi \in H^{2k}(\mathrm{Fl}(E))$, *and* $\deg(Q^\beta) = \langle -K_X, \beta\rangle$ *if* $\beta \in H_2(X)$. *The degree axiom for Gromov–Witten invariants then shows that $J_X^{\pi_*\beta}$ is homogeneous of degree $\langle K_X, \pi_*\beta\rangle + 1$. Write*

$$I_{\mathrm{GM}}(t, \tau, z) = e^{\frac{t}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Fl}(E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_*\beta}(\tau, z) \times I_\beta(z) \times M_\beta(z)$$

where

$$M_\beta(z) = \prod_{s=1}^{k} \frac{\prod_{m=-\infty}^{f_s \cdot d + \langle \rho, \pi_*\beta\rangle} f_s + \pi^*\rho + \mu + mz}{\prod_{m=-\infty}^{0} f_s + \pi^*\rho + \mu + mz}$$

A straightforward calculation shows that

$$I_\beta(z) = z^{\langle K_{\mathrm{Fl}(E)} - \pi^* K_X, \beta\rangle} i_\beta(z)$$
$$M_\beta(z) = z^{\langle c_1(F), \beta\rangle} m_\beta(z)$$

where $i_\beta(z), m_\beta(z) \in \mathcal{H}_{\mathrm{Fl}(E)}$ are homogeneous of degree 0*. It follows that $\pi^* J_X^{\pi_*\beta}(\tau, z) \times I_\beta(z) \times M_\beta(z)$ is homogeneous of degree $\langle K_{\mathrm{Fl}(E)} + c_1(F), \beta\rangle + 1$ which is nonpositive for $\beta \neq 0$ by the assumptions on $-K_Z$. Since $\tau \in H^2(X)$, any negative contribution to the homogenous degree must come from a negative power of z, so that $\pi^* J_X^{\pi_*\beta}(\tau, z) \times I_\beta(z) \times M_\beta(z)$ is $O(z^{-1})$, unless $\beta = 0$ or $\beta \in S$. In the latter case, the expression has homogeneous degree* 0 *and is therefore of* *the form $c_0 + \frac{c_1}{z} + O(z^{-2})$ with c_i independent of z and of degree i. Relabeling $n_\beta = c_0$ and expanding I_{GM} in powers of z, we obtain*

$$I_{\mathrm{GM}}(t,\tau,z) = \big(1 + tz^{-1} + O(z^{-2})\big)\big(\pi^* J_X^0 \times I_0 \times M_0 + \Big(\sum_{\beta \in S} n_\beta Q^\beta e^{\langle \beta, t\rangle} + O(z^{-1})\Big) + \sum_{0 \neq \beta \notin S} O(z^{-1})\big)$$
$$= (z + \tau + t + C(t) + O(z^{-1}))$$

where $C(t)$ is as claimed. This proves (2.30)*, and the result follows.*

We restate Corollary 2.5.19 in the case where the flag bundle is a Grassmann bundle, i.e $\ell = 1$, relabelling $H_{1,j} = H_j$, $d_{1,j} = d_j$ and $r_1 = r$. The rest of the notation here is as in §2.2.2.

Corollary 2.5.21 *Let $V^G \to \mathrm{Gr}(r,E)$ be a vector bundle induced by a representation of G, let $L \to X$ be a line bundle with first Chern class ρ, and let $F = V^G \otimes \pi^* L$. Let Z be the zero locus of a generic section of F. Suppose that F is convex, that $-K_{\mathrm{Gr}(E,r)} - c_1(F)$ is ample, and that $\tau \in H^2(\mathrm{Gr}(r,E))$. Then the non-equivariant limit $\mu \to 0$ of the twisted J-function J_{F_μ} exists and satisfies*

$$J_Z\big(i^* t + i^* \tau, z\big) = i^* J_{F_0}(t + \tau, z)$$

where $i\colon Z \to \mathrm{Gr}(r,E)$ is the inclusion map. Furthermore

$$J_{F_0}(t+\tau,z) = e^{\frac{t - C(t)}{z}} \sum_{\beta \in \mathrm{NE}(\mathrm{Gr}(r,E))} Q^\beta e^{\langle \beta, t\rangle} \pi^* J_X^{\pi_*\beta}(\tau,z)$$
$$\sum_{\substack{\underline{d} \in \mathbb{Z}^r:\\ d_1 + \cdots + d_r = \langle \beta, c_1(S^\vee)\rangle}} (-1)^{\epsilon(\underline{d})} \prod_{i=1}^{r} \prod_{j=1}^{n} \frac{\prod_{m=-\infty}^{0} H_i + \pi^* c_1(L_j) + mz}{\prod_{m=-\infty}^{d_i + \langle \pi_*\beta, c_1(L_j)\rangle} H_i + \pi^* c_1(L_j) + mz}$$

$$\times \prod_{i<j} \frac{H_i - H_j + (d_i - d_j)z}{H_i - H_j} \times \prod_{s=1}^{k} \prod_{m=1}^{f_s \cdot \underline{d} + \langle \rho, \pi_*\beta\rangle} \big(f_s + \pi^*\rho + mz\big) \quad (2.31)$$

Here the Abelianised bundle V^T splits as a direct sum of line bundles $F_1 \oplus \cdots \oplus F_k$ with first Chern classes that we write as $c_1(F_s) = \sum_{i=1}^r f_{s,i} \tilde{H}_i$, $J_X^D(\tau,z)$ is as in (2.26)*, $\epsilon(\underline{d}) = \sum_{i<j} d_i - d_j$, $f_s \cdot \underline{d} = \sum_i f_{s,i} d_i$, $f_s = \sum_i f_{s,i} H_i$, and $C(t) \in H^0(\mathrm{Gr}(r,E),\Lambda)$ is the unique expression such that the right-hand side of* (2.31) *has the form $z + t + \tau + O(z^{-1})$.*

Remark 2.5.22 *For a more explicit formula for $C(t)$, see Corollary 2.5.19; in particular if Z has Fano index two or greater then $C(t) \equiv 0$. By Remark 2.5.12 the summand in* (2.31) *is zero unless for each i there exists a j such that $d_i + \langle \pi_*\beta, c_1(L_j)\rangle \geq 0$*

Proof 2.5.23 (Proof of Corollary 2.5.21) *We cancelled terms in the Weyl modification factor, as in the proof of Lemma 2.4.1, and took the non-equivariant limit $\mu \to 0$.*

Remark 2.5.24 *The relationship between I-functions (or generating functions for genus-zero quasimap invariants) and J-functions (which are generating functions for genus-zero Gromov–Witten invariants) is particularly simple in the Fano case [Giv98] [CFK14, §1.4], and for the same reason Corollary 2.5.19 holds without the restriction $\tau \in H^2(X)$ if $Z \to X$ is relatively Fano*[1]*. This never happens for blow-ups $\tilde{X} \to X$, however, and it is hard to construct examples where $Z \to X$ is relatively Fano and the rest of the conditions of Corollary 2.5.19 hold. We do not know of any such examples.*

[1]That is, if the relative anticanonical bundle $-K_{Z/X}$ is ample.

Remark 2.5.25 *Corollary 2.5.19 gives a closed-form expression for the small J-function of Z – or, equivalently, for one-point gravitional descendant invariants of Z – in the case where Z is Fano. But in general (that is, without the Fano condition on Z) one can use Birkhoff factorization, as in [CG07, CFK14] and [CCIT19, §3.8], to compute any twisted genus-zero gravitional descendant invariant of* $\mathrm{Fl}(E)$ *in terms of genus-zero descendant invariants of X. The twisting here is with respect to the* $\mathbb{C}^*$*-equivariant Euler class and the vector bundle F. Thus Corollary 2.5.19 determines the Lagrangian submanifold* $\mathcal{L}_{F_\mu}$ *that encodes twisted Gromov–Witten invariants. Applying [Coa14, Theorem 1.1], we see that Corollary 2.5.19 together with Birkhoff factorization allows us to compute any genus-zero Gromov–Witten invariant of the zero locus Z of the form*

$$\langle \theta_1 \psi^{i_1}, \ldots, \theta_n \psi^{i_n} \rangle_{0,n,d} \tag{2.32}$$

where all but one of the cohomology classes θ_i *lie in* $\mathrm{im}(i^*) \subset H^\star(Z)$ *and the remaining* θ_i *is an arbitrary element of* $H^\star(Z)$*. Here* $i\colon Z \to \mathrm{Fl}(E)$ *is the inclusion map.*

Remark 2.5.26 *Applying Remark 2.5.25 to the blow-up* $\tilde{X} \to X$ *considered in the introduction, we see that Corollary 2.5.19 together with Birkhoff factorization allows us to compute arbitrary invariants of* $\tilde{X}$ *of the form* (2.32) *in terms of genus-zero gravitional descendants of X. In this case* $\mathrm{im}(i^*) \subset H^\star(\tilde{X})$ *contains all classes from* $H^\star(X)$ *and also the class of the exceptional divisor.*

2.6 The Main Geometric Construction

2.6.1 Main Geometric Construction

Let F be a locally free sheaf on a variety X. We denote by $F(x)$ its fibre over x, a vector space over the residue field $\kappa(x)$. A morphism φ of locally free sheaves induces a linear map on fibres, denoted by $\varphi(x)$. We make the following definition:

Definition 2.6.1 *Let* $\varphi\colon E^m \to F^n$ *a morphism of locally free sheaves of rank m and n respectively. The k-th degeneracy locus is the subvariety of X defined by*

$$D_k(\varphi) = \{x \in X \colon \mathrm{rk}\, \varphi(x) \leq k\}$$

Note that $D_k(\varphi) = X$ *if* $k \geq \min\{m, n\}$*; if* $k = \min\{m, n\} - 1$ *we simply call* $D_k(\varphi)$ *the degeneracy locus of* φ*.*

We have the following results:

- Scheme-theoretically, $D_k(\varphi)$ may be defined as the zero locus of the section $\wedge^k \varphi$; this shows that locally the ideal of $D_k(\varphi)$ is defined by the $(k+1) \times (k+1)$-minors of φ.

- If $E^\vee \otimes F$ is globally generated, then $D_k(\varphi)$ of a generic φ is either empty or has expected codimension $(m-k)(n-k)$, and the singular locus of $D_k(\varphi)$ is contained in $D_{k-1}(\varphi)$. In particular, if φ is generic and $\dim X < (m-k+1)(n-k+1)$, then $D_k(\varphi)$ is smooth [Ott95, Theorem 2.8].

- We may freely assume that $m \geq n$ in what follows, since we can always replace φ with its dual map whose degeneracy locus is the same.

Proposition 2.6.2 *Let X be a smooth variety, and $\varphi\colon E^m \to F^n$ a generic morphism of locally free sheaves on X. Suppose that $m \geq n$ and write $r = m - n$. Let $Y = D_{n-1}(\varphi)$ be the degeneracy locus of φ, and assume that φ has generically full rank, that Y has the expected codimension $m - n + 1$ and that Y is smooth. Let $\pi\colon \mathrm{Gr}(r, E) \to X$ be the Grassmann bundle of E on X. Then the blow-up $Bl_Y(X)$ of X along Y is a subvariety of $\mathrm{Gr}(r, E)$, cut out as the zero locus of the regular section $s \in \Gamma(\mathrm{Hom}(S, \pi^*F))$ defined by the composition*

$$S \hookrightarrow \pi^*E \xrightarrow{\pi^*\varphi} \pi^*F$$

where the first map is the canonical inclusion.

Proof 2.6.3 *We write points in $\mathrm{Gr}(r, E)$ as (p, V), where $p \in X$ and V is a r-dimensional subspace of the fibre $E(x)$. At (p, V), the section s is given by the composition*

$$V \hookrightarrow E(x) \xrightarrow{\varphi(x)} F(x)$$

so s vanishes at (p, V) if and only if $V \subset \ker \varphi(x)$.

The statement is local on X, so fix a point $P \in X$ and a Zariski open neighbourhood $U = \mathrm{Spec}(A)$ with trivialisations $E|_U = A^m, F|_U = A^n$. We will show that the equations of $Z(s) \cap U$ and $Bl_{U\cap Y}U$ agree. Under these identifications φ is given by a $n \times m$ matrix with entries in A. Since φ has generically maximal rank and Y is nonsingular, after performing row and column operations and shrinking U if necessary, we may assume that φ is given by the matrix

$$\begin{pmatrix} x_0 & \dots & x_r & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Note that the ideal of the minors of this matrix is just $I = (x_0, \dots x_r)$ and that $x_0, \dots, x_r$ form part of a regular system of parameters around P, so we may assume that $n = 1, m = r + 1$. Writing y_i for the basis of sections of $S^\vee$ on $\mathrm{Gr}(r, A^{r+1})$, we see that $Z(s)$ is given by the equation

$$x_0y_0 + \dots + x_ry_r = 0$$

Under the Plücker isomorphism

$$\mathrm{Gr}(r, A^{r+1}) \to \mathbb{P}(\wedge^r A^{r+1}) \cong U \times \mathbb{P}^r_{y_0,\dots,y_r}$$

$Z(s)$ maps to the variety cut out by the minors of the matrix

$$\begin{pmatrix} x_0 & \dots & x_r \\ y_0 & \dots & y_r \end{pmatrix}$$

i.e the blowup of $Y \cap U$ in U.

2.7 Examples

We close by presenting three example computations that use Proposition 2.6.2 and Corollary 2.5.21, calculating genus-zero Gromov–Witten invariants of blow-ups of projective spaces in various high-codimension complete intersections. Recall, as we will need it below, that if $E \to X$ is a vector bundle of rank n then the anticanonical divisor of $\mathrm{Gr}(r, E)$ is

$$-K_{\mathrm{Gr}(r,E)} = \pi^*\left(-K_X + r(\det E)\right) + n(\det S^\vee) \tag{2.33}$$

where $S \to \mathrm{Gr}(r, E)$ is the tautological subbundle. Recall too that the *regularised quantum period* of a Fano manifold Z is the generating function

$$\widehat{G}_Z(x) = 1 + \sum_{d=2}^{\infty} d! c_d x^d$$

for genus-zero Gromov–Witten invariants of Z, where

$$c_d = \sum_{\beta} \langle \theta \psi_1^{d-2} \rangle_{0,1,\beta} \qquad \text{for } \theta \in H^{\mathrm{top}}(Z) \text{ the class of a volume form}$$

and the sum runs over effective classes β such that $\langle \beta, -K_Z \rangle = d$.

Example 2.7.1 *We will compute the regularised quantum period of $\tilde{X} = \mathrm{Bl}_Y \mathbb{P}^4$ where Y is a plane conic. Consider the situation as in §2.2.2 with:*

- $X = \mathbb{P}^4$
- $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(-1)$
- $G = \mathrm{GL}_2(\mathbb{C})$, $T = (\mathbb{C}^*)^2 \subset G$

Then $A /\!/ G$ is $\mathrm{Gr}(2, E)$, and $A /\!/ T$ is the $\mathbb{P}^2 \times \mathbb{P}^2$-bundle $\mathbb{P}(E) \times_{\mathbb{P}^4} \mathbb{P}(E) \to \mathbb{P}^4$. By Proposition 2.6.2 the zero locus $\tilde{X}$ of a section of $S^\vee \otimes \pi^*(\mathcal{O}(1))$ on $\mathrm{Gr}(2, E)$ is the blowup of $\mathbb{P}^4$ along the complete intersection of two hyperplanes and a quadric. We identify the group ring $\mathbb{Q}[H_2(A /\!/ T, \mathbb{Z})]$ with $\mathbb{Q}[Q, Q_1, Q_2]$, where Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^4$ and Q_i corresponds to $\tilde{H}_i$. Similarly, we identify $\mathbb{Q}[H_2(A /\!/ G, \mathbb{Z})]$ with $\mathbb{Q}[Q, q]$, where again Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^4$ and q corresponds to the first Chern class of $S^\vee$.

We will need Givental's formula [Giv96] for the J-function of $\mathbb{P}^4$:

$$J_{\mathbb{P}^4}(\tau, z) = z e^{\tau/z} \sum_{D=0}^{\infty} \frac{Q^D e^{D\tau}}{\prod_{m=1}^{D} (H + mz)^5} \qquad \tau \in H^2(\mathbb{P}^4)$$

In the notation of §2.2.2, we have $\ell = 1$, $r_\ell = r_1 = 2$, $r_{\ell+1} = 3$. We relabel $\tilde{H}_{\ell,j} = \tilde{H}_j$ and $d_{\ell,j} = d_j$. We have that $\tilde{H}_{\ell+1,1} = \tilde{H}_{\ell+1,2} = 0$, $\tilde{H}_{\ell+1,3} = \pi^* H$ and $d_{\ell+1,1} = d_{\ell+1,2} = 0$, $d_{\ell+1,3} = D$. Write $F = S^\vee \otimes \pi^* \mathcal{O}(1)$. Corollary 2.5.21 and Remark 2.5.22 give

$$J_{F_0}(t, \tau, z) = z e^{\frac{t+\tau}{z}} \sum_{D=0}^{\infty} \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} \frac{(-1)^{d_1 - d_2} Q^D q^{d_1+d_2} e^{D\tau} e^{(d_1+d_2)t} \prod_{i=1}^{2} \prod_{m=1}^{d_i+D} (H_i + H + mz)}{\prod_{m=1}^{D} (H + mz)^5 \prod_{m=1}^{d_1} (H_1 + mz)^2 \prod_{m=1}^{d_2} (H_2 + mz)^2}$$
$$\times \prod_{i=1}^{2} \frac{\prod_{m=-\infty}^{0} (H_i - H + mz)}{\prod_{m=-\infty}^{d_i - D} (H_i - H + mz)} \frac{(H_1 - H_2 + z(d_1 - d_2))}{H_1 - H_2}$$

To obtain the quantum period we need to calculate the anticanonical bundle of $\tilde{X}$. Equation (2.33) *and the adjunction formula give*

$$-K_{\tilde{X}} = 3H + 3 \det S^\vee - (2H + \det S^\vee) = H + 2 \det S^\vee.$$

To extract the quantum period from the non-equivariant limit J_{F_0} of the twisted J-function, we take the component along the unit class $1 \in H^\star(A/\!/G;\mathbb{Q})$, set $z=1$, and set $Q^\beta = x^{\langle \beta, -K_{\tilde{X}}\rangle}$. That is, we set $\lambda = 0$, $t=0$, $\tau = 0$, $z=1$, $q = x^2$, $Q = x$, and take the component along the unit class, obtaining

$$G_{\tilde{X}}(x) = \sum_{n=0}^{\infty}\sum_{l=n+1}^{\infty}\sum_{m=l}^{\infty}(-1)^{l+m-1}x^{l+2m+2n}\tfrac{(l+n)!(l+m)!(l-n-1)!}{(l!)^5(m!)^2(n!)^2(n-l)!}(n-m)$$
$$+\sum_{l=0}^{\infty}\sum_{m=l}^{\infty}\sum_{n=l}^{\infty}(-1)^{m+n}x^{l+2m+2n}\tfrac{(l+n)!(l+m)!}{(l!)^5(m!)^2(n!)^2(n-l)!(m-l)!}\Big(1+(n-m)(-2H_n+H_{l+n}-H_{n-l})\Big)$$

Thus the first few terms of the regularized quantum period are:

$$\widehat{G}_{\tilde{X}}(x) = 1 + 12x^3 + 120x^5 + 540x^6 + 20160x^8 + 33600x^9 + 113400x^{10} + 2772000x^{11} + 2425500x^{12} + \cdots$$

This strongly suggests that $\tilde{X}$ coincides with the quiver flag zero locus with ID 15 in [Kal19], although this is not obvious from the constructions.

Example 2.7.2 *We will compute the regularised quantum period of $\tilde{X} = \mathrm{Bl}_Y \mathbb{P}^6$, where Y is a 3-fold given by the intersection of a hyperplane and two quadric hypersurfaces. Consider the situation as in §2.2.2 with:*

- $X = \mathbb{P}^6$
- $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1)$
- $G = \mathrm{GL}_2(\mathbb{C})$, $T = (\mathbb{C}^*)^2 \subset G$

Then $A/\!/G$ is $\mathrm{Gr}(2,E)$, and $A/\!/T$ is the $\mathbb{P}^2\times\mathbb{P}^2$-bundle $\mathbb{P}(E)\times_{\mathbb{P}^6}\mathbb{P}(E) \to \mathbb{P}^6$. By Proposition 2.6.2 the zero locus $\tilde{X}$ of a section of $S^\vee \otimes \pi^*(\mathcal{O}(2))$ on $\mathrm{Gr}(2,E)$ is the blowup of $\mathbb{P}^6$ along the complete intersection of a hyperplane and two quadrics. We identify the group ring $\mathbb{Q}[H_2(A/\!/T,\mathbb{Z})]$ here with $\mathbb{Q}[Q,Q_1,Q_2]$, where Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and Q_i corresponds to $\tilde{H}_i$. Similarly, we identify $\mathbb{Q}[H_2(A/\!/G,\mathbb{Z})]$ with $\mathbb{Q}[Q,q]$, where again Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and q corresponds to the first Chern class of $S^\vee$.

The J-function of $\mathbb{P}^6$ is [Giv96]:

$$J_{\mathbb{P}^6}(\tau,z) = ze^{\tau/z}\sum_{D=0}^{\infty}\frac{Q^D e^{D\tau}}{\prod_{m=1}^{D}(H+mz)^7} \qquad \tau \in H^2(\mathbb{P}^6)$$

In the notation of §2.2.2, we have $\ell = 1$, $r_\ell = r_1 = 2$, $r_{\ell+1} = 3$. We relabel $\tilde{H}_{\ell,j} = \tilde{H}_j$ and $d_{\ell,j} = d_j$. We have that $\tilde{H}_{\ell+1,1} = \tilde{H}_{\ell+1,2} = 0$, $\tilde{H}_{\ell+1,3} = -\pi^* H$ and $d_{\ell+1,1} = d_{\ell+1,2} = 0$, $d_{\ell+1,3} = -D$. Write $F = S^\vee \otimes \pi^*\mathcal{O}(2)$. Corollary 2.5.21 and Remark 2.5.22 give

$$J_{F_0}(t,\tau,z) = ze^{\frac{t+\tau}{z}}\sum_{D=0}^{\infty}\sum_{d_1=-D}^{\infty}\sum_{d_2=-D}^{\infty}\frac{Q^D q^{d_1+d_2}e^{D\tau}e^{(d_1+d_2)t}}{\prod_{m=1}^{D}(H+mz)^7}\prod_{i=1}^{2}\frac{\prod_{m=-\infty}^{0}(H_i+mz)^2}{\prod_{m=-\infty}^{d_i}(H_i+mz)^2}$$
$$\times\prod_{i=1}^{2}\frac{\prod_{m=1}^{d_i+2D}(H_i+2H+mz)}{\prod_{m=1}^{d_i+D}(H_i+H+mz)}(-1)^{d_1-d_2}\frac{(H_1-H_2+z(d_1-d_2))}{H_1-H_2}$$

Again we will need the anticanonical bundle of $\tilde{X}$, which by (2.33) and the adjunction formula is

$$-K_{\tilde{X}} = 9H + 3\det(S^*) - (4H+\det(S^*)) = 5H + 2\det(S^*).$$

To extract the quantum period from J_{F_0}, we take the component along the unit class $1 \in H^\star(A/\!/G;\mathbb{Q})$, set $z=1$, and set $Q^\beta = x^{\langle \beta, -K_{\tilde{X}}\rangle}$. That is, we set $\lambda=0$, $t=0$, $\tau=0$, $z=1$, $q=x^2$, $Q=x^5$, and take the component along the unit class, obtaining

$$G_{\tilde{X}}(x) = \sum_{D=0}^{\infty}\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty} (-1)^{d_1+d_2} x^{5D+2d_1+2d_2} \frac{(d_1+2D)!(d_2+2D)!}{(D!)^7(d_1!)^2(d_2!)^2(d_1+D)!(d_2+D)!} \times \Big(1+(d_1-d_2)(-2H_{d_1}+H_{d_1+2D}-H_{d_1+D})\Big)$$

The first few terms of the regularized quantum period are:

$$\widehat{G}_{\tilde{X}}(x) = 1 + 480x^5 + 5040x^7 + 4082400x^{10} + 119750400x^{12} + 681080400x^{14} + \cdots$$

Example 2.7.3 *We will compute the regularised quantum period of $\tilde{X} = \mathrm{Bl}_Y \mathbb{P}^6$, where Y is a quadric surface given by the intersection of 3 generic hyperplanes and a quadric hypersurface. Consider the situation as in §2.2.2 with:*

- $X = \mathbb{P}^6$
- $E = \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$
- $G = \mathrm{GL}_3(\mathbb{C})$, $T = (\mathbb{C}^*)^3 \subset G$

Then $A/\!/G$ is $\mathrm{Gr}(3,E)$, and $A/\!/T$ is $\mathbb{P}(E) \times_{\mathbb{P}^6} \mathbb{P}(E) \times_{\mathbb{P}^6} \mathbb{P}(E) \to \mathbb{P}^6$. By Proposition 2.6.2 the zero locus $\tilde{X}$ of a section of $S^\vee \otimes \pi^*(\mathcal{O}(1))$ on $\mathrm{Gr}(3,E)$ is the blowup of $\mathbb{P}^6$ along the complete intersection of three hyperplanes and a quadric. We identify the group ring $\mathbb{Q}[H_2(A/\!/T,\mathbb{Z})]$ with $\mathbb{Q}[Q,Q_1,Q_2,Q_3]$, where Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and Q_i corresponds to $\tilde{H}_i$. Similarly, we identify $\mathbb{Q}[H_2(A/\!/G,\mathbb{Z})]$ with $\mathbb{Q}[Q,q]$, where again Q corresponds to the pullback of the hyperplane class of $\mathbb{P}^6$ and q corresponds the first Chern class of $S^\vee$.

In the notation of §2.2.2, we have $\ell = 1, r_\ell = r_1 = 3, r_{\ell+1} = 4$. We relabel $\tilde{H}_{\ell,j} = \tilde{H}_j$ and $d_{\ell,j} = d_j$. We have that $\tilde{H}_{\ell+1,1} = \tilde{H}_{\ell+1,2} = \tilde{H}_{\ell+1,3} = 0$, $\tilde{H}_{\ell+1,4} = -\pi^* 2H$ and $d_{\ell+1,1} = d_{\ell+1,2} = d_{\ell+1,3} = 0$, $d_{\ell+1,4} = -2D$. Write $F = S^\vee \otimes \pi^*\mathcal{O}(1)$. Corollary 2.5.21 and Remark 2.5.22 give

$$J^{F_0}(t,\tau,z) = ze^{\frac{t+\tau}{z}} \sum_{D=0}^{\infty}\sum_{d_1=-2D}^{\infty}\sum_{d_2=-2D}^{\infty}\sum_{d_3=-2D}^{\infty} \frac{Q^D q^{d_1+d_2+d_3} e^{D\tau} e^{(d_1+d_2+d_3)t}}{\prod_{m=1}^{D}(H+mz)^7} \times \prod_{i=1}^{3} \frac{\prod_{m=-\infty}^{0}(H_i+mz)^3}{\prod_{m=-\infty}^{d_i}(H_i+mz)^3} \prod_{i=1}^{3} \frac{1}{\prod_{m=1}^{d_i+2D}(H_i+2H+mz)} \prod_{i=1}^{3} \frac{\prod_{m=-\infty}^{d_i+D}(H_i+H+mz)}{\prod_{m=-\infty}^{0}(H_i+H+mz)} \times \frac{(H_1-H_2+z(d_1-d_2))}{H_1-H_2} \frac{(H_1-H_3+z(d_1-d_3))}{H_1-H_3} \frac{(H_2-H_3+z(d_2-d_3))}{H_2-H_3}$$

Arguing as before,

$$-K_{\tilde{X}} = 11H + 4\det(S^*) - (3H + \det(S^*)) = 8H + 3\det(S^*).$$

To extract the quantum period from J_{F_0}, we set $\lambda=0$, $t=0$, $\tau=0$, $z=1$, $q=x^3$, $Q=x^8$, and take the component along the unit class. The first few terms of the regularised quantum period are:

$$\widehat{G}_{\tilde{X}}(x) = 1 + 108x^3 + 17820x^6 + 5040x^8 + 5473440x^9 + 56364000x^{11} + 1766526300x^{12} + 117076459500x^{14} + 672012949608x^{15} + \cdots$$

Remark 2.7.4 *Strictly speaking the use of Theorem 2.5.16 in the examples just presented was not necessary. Whenever the base space X is a projective space, or more generally a Fano complete intersection in a toric variety or flag bundle, then one can replace our use of Theorem 2.5.16 (but not 2.6.2) by [CFKS08, Corollary 6.3.1]. However there are many examples that genuinely require both Proposition 2.6.2 and Theorem 2.5.16: for instance when X is a toric complete intersection but the line bundles that define the center of the blow-up do not arise by restriction from line bundles on the ambient space. (For a specific such example one could take X to be the three-dimensional Fano manifold* $\mathrm{MM}_{3\text{–}9}$*: see [CCGK16, §62].) For notational simplicity we chose to present examples with* $X = \mathbb{P}^N$*, but the approach that we used applies without change to more general situations.*

Chapter 3

Logarithmic Toric Quasimaps

In this chapter we build a theory of quasimaps relative to a simple normal crossings divisor. We do this by exploiting the technology coming from logarithmic geometry developed in the construction of the corresponding theory for stable maps [Che14, AC14, GS13, AW18b].

Fix X a smooth projective toric variety associated to a fan Σ and let $D = D_1 + \cdots + D_r$ be any simple normal crossings divisor. Let g, n be non-negative integers and let β be a curve class in X. Fix $\alpha = (\alpha_{i,j})_{i,j}$, an $r \times n$ matrix of non-negative integers such that $\sum_j \alpha_{i,j} = D_i \cdot \beta$. We produce a moduli space $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ compactifying the space of quasimaps from smooth n-marked curves of genus g and degree β with tangency $\alpha_{i,j}$ to D_i at the j^{th} marking, which we show to be a proper Deligne–Mumford stack. We show, subject to the condition that a lifting criterion for logarithmic morphisms in terms of logarithmic cotangent complexes generalises to certain algebraic stacks, that this moduli space admits a perfect obstruction theory relative to $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r])$, the stack of logarithmic maps from n-marked prestable curves of genus g to $[\mathbb{A}^r/\mathbb{G}_m^r]$ with the same tangency conditions. We sketch a proof that where the comparison makes sense, this theory gives the same invariants as the theory of [BN21], which is defined in genus-zero, when D is smooth and very ample.

3.1 Absolute Quasimaps

As explained in the introduction, quasimap theory is an alternative curve counting framework to Gromov–Witten theory. With it, comes a different compactification of the space of maps from smooth, n-marked curves of fixed degree and genus to a certain target variety X. The notion of quasimaps can be motivated by the following example from [Web21].

Example 3.1.1 *Consider the family of rational, 2-marked, degree 2 stable maps over $\mathbb{G}_m$ given by*

$$\begin{aligned} \mathbb{G}_m \times \mathbb{P}^1 &\to \mathbb{P}^2 \\ [z_0 : z_1] &\mapsto [tz_0^2 : z_0 z_1 : z_1^2] \end{aligned}$$

where the markings on $\mathbb{P}^1$ are the points $[1:1]$ and $[2:1]$. This is a family of stable maps in $\mathcal{M}_{0,2}(\mathbb{P}^2, 2)$. What happens as $t \to 0$? The family defines a rational map $\mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$. Resolving the rational map means blowing up $\mathbb{A}^1 \times \mathbb{P}^1$ at the point $(0, [1:0])$. The introduction of the exceptional divisor means the fibre over $0 \in \mathbb{A}^1$ is a reducible *nodal curve with two branches and the induced map to $\mathbb{P}^2$ is degree* 1 *on each of the components. This defines the limit in* $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2)$.

On the other hand there is an arguably simpler limit, if we allow our morphism to acquire a basepoint at $[1 : 0]$ *we can just set the limit to be*

$$\mathbb{P}^1 \dashrightarrow \mathbb{P}^2$$
$$[z_0 : z_1] \mapsto [0 : z_0 z_1 : z_1^2]$$

This is in fact the limit of the family in the moduli space $\mathcal{Q}_{0,2}(\mathbb{P}^2, 2)$. The theory of quasimaps generalises this idea. Notice that this quasimap limit contained more than the data of a rational map but rather the data of a line bundle and three sections (which happen to simultaneously vanish at a certain point). This used the special fact about $\mathbb{P}^2$ that its functor of points can be alternatively described in terms of line bundles and sections. More generally, this has to do with projective space being a GIT quotient $\mathbb{A}^{N+1} /\!/ \mathbb{G}_m$.

3.1.1 Quasimaps to GIT Quotients

Quasimap theory has a long history starting with work due to Drinfeld, Morrison-Plesser, Givental (and many others), as well as the development of the theory of stable quotients [MOP11]. This was generalised to the unifying theory of quasimaps, due to Ciocan-Fontanine, Kim and Maulik [CFKM14]. This theory applies with the following setup. Let G be a complex reductive group acting on an affine algebraic variety $W = \operatorname{Spec} A$, and let $\theta \in \chi(G)$, giving a linearisation of the action. Let $W /\!/ G$ be the induced GIT quotient, projective over $\operatorname{Spec} A^G$. We insist that

- $\emptyset \neq W^s = W^{ss}$
- W^s is nonsingular
- G acts freely on W^s

Then there is well-defined quasimap moduli space for $X = W /\!/ G$. There are many varieties which admit a presentation of this form, for example, toric varieties, Grassmanniann*s*, Nakajima quiver varieties. In this setup, X naturally sits as an open set inside the full stack quotient $[W/G]$ as $[W^s/G]$. Stable maps were maps from marked curves to X itself, and quasimaps are supposed to be maps to X where the map is allowed to have some basepoints. The way to make sense of this in the most general case is to consider maps to $[W/G]$, where the basepoints correspond to points of the curve mapping to the unstable locus of W.

Definition 3.1.2 *A k-pointed, genus g, quasimap of class* $\beta \in \operatorname{Hom}(\operatorname{Pic}^G W, \mathbb{Z})$ *to* $W /\!/ G$ *consists of the data*

$$(C, p_1, \ldots, p_k, u)$$

where

- $(C, p_1, \ldots, p_k)$ *is a connected, at most nodal, k-pointed projective curve of genus g,*
- $u : C \to [W/G]$ *of class* β *(i.e.* $\beta(L) = \deg_C u^* L$*)*

Such that there is a finite (possibly empty) set $B \subset C$ such that $\forall p \in C \setminus B$ we have that $u(p) \in W^s$.

Example 3.1.3 $\mathbb{P}^N = \mathbb{A}^{N+1} /\!/ \mathbb{G}_m$ *where the action is weight one on each coordinate with the linearisation* $\mathrm{id} \in \chi(\mathbb{G}_m)$. *A morphism*

$$C \to [\mathbb{A}^{N+1}/\mathbb{G}_m]$$

is by definition a principal $\mathbb{G}_m$-bundle P, over C, together with a $\mathbb{G}_m$-equivariant morphism $P \to \mathbb{A}^{N+1}$. This is equivalent data to the associated fibre bundle $P \times_{\mathbb{G}_m} \mathbb{A}^{N+1}$, which is $N+1$ copies of the line bundle $L = P \times_{\mathbb{G}_m} \mathbb{A}^1$ and a section $u = (u_0, \dots, u_N) \in H^0(C, L)^{\oplus N}$. This recovers the data from the motivating example 3.1.1.

Example 3.1.4 *Generalising 2.1.1 (or using 2.9 where $\ell = 1$), the Grassmannian* $\mathrm{Gr}(r, n) = \mathbb{A}^{rn} /\!/ \mathrm{GL}_r(\mathbb{C})$. *As before a morphism from a curve*

$$C \to [\mathbb{A}^{rn}/\mathrm{GL}_r(\mathbb{C})]$$

is a principal $\mathrm{GL}_r(\mathbb{C})$-bundle with an equivariant morphism to $\mathbb{A}^{rn}$. This time the associated fibre bundle $P \times_{\mathrm{GL}_r(\mathbb{C})} \mathbb{A}^{rn}$ is n-copies of a rank r vector bundle $V = P \times_{\mathrm{GL}_r} \mathbb{A}^r$, which comes with a section $u = (u_1, \dots, u_n) \in H^0(C, V)^{\oplus n}$.

Definition 3.1.5 *The quasimap is called* prestable *if the basepoints are disjoint from the nodes and the markings on C. It is called* stable *if it is prestable and $\omega_C(\sum_{i=1}^k p_i) \otimes \mathcal{L}^\epsilon$ is ample for all $\epsilon \in \mathbb{Q}_{>0}$, where $\mathcal{L}$ is the line bundle induced by the linearisation.*

Remark 3.1.6 *Just as in Gromov–Witten theory, stability can be reinterpreted as a numerical condition on each component. Namely, a prestable quasimap is stable if and only if*

- *Every rational component contains at least 2 special points*
- *The line bundle $\mathcal{L}$ has positive degree on every rational component containing exactly 2 special points and every elliptic component containing no special points.*

One of the key features of this characterisation we will use in later sections is that stable quasimaps cannot have rational tails i.e. curve components of genus-zero with a single special point.

Generalising this definition leads to the notion of a *family of stable quasimaps* over a scheme S, and one can define a morphism of such families, leading to

$$\mathcal{Q}_{g,n}(W /\!/ G, \beta)$$

the moduli stack of stable quasimaps.

Theorem 3.1.7 *[CFK10, 4.03] The moduli stack $\mathcal{Q}_{g,n}(W /\!/ G, \beta)$ is a Deligne–Mumford stack of finite type. If $W /\!/ G$ is projective, then $\mathcal{Q}_{g,n}(W /\!/ G, \beta)$ is proper over* $\mathrm{Spec}(\mathbb{C})$. *In general, it has a natural proper morphism*

$$\mathcal{Q}_{g,n}(W /\!/ G, \beta) \to \mathrm{Spec}(A^G).$$

The moduli space $\mathcal{Q}_{g,n}(W /\!/ G, \beta)$ also admits a perfect obstruction theory relative to $\mathfrak{M}_{g,n}$ as in the stable maps case. Recall in the case of the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X, \beta)$ the perfect obstruction theory is given by

$$(\mathbf{R}\pi_* f^* T_X)^\vee \to \mathbb{L}_{\overline{\mathcal{M}}_{g,n}(X,\beta)/\mathfrak{M}_{g,n}}$$

where $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,n}(X, \beta)$ is the universal curve and $f : \mathcal{C} \to X$ is the universal map. In the case of quasimaps there is still a universal curve $\pi : \mathcal{C} \to \mathcal{Q}_{g,n}(W /\!/ G, \beta)$, but the universal morphism goes to $[W/G]$ instead.

Theorem 3.1.8 *If W has only l.c.i. singularities, then*

$$(\mathbf{R}\pi_* u^* \mathbb{T}_{[W/G]})^\vee \to \mathbb{L}_{\mathcal{Q}_{g,n}(W/\!/G,\beta)/\mathfrak{M}_{g,n}}$$

is a perfect obstruction theory, where $\mathbb{T}_{[W/G]}$ denotes the tangent complex *of $[W/G]$. This leads to the virtual fundamental class $[\mathcal{Q}_{g,n}(W /\!/ G, \beta)]^{\mathrm{vir}} \in A_{\mathrm{vdim}}(\mathcal{Q}_{g,n}(W /\!/ G, \beta))$.*

Since the basepoints are not allowed to coincide with markings, there are evaluation maps

$$\mathrm{ev}_i : \mathcal{Q}_{g,n}(W /\!/ G, \beta) \to W /\!/ G.$$

As in the Gromov–Witten case, one can define ψ-classes as the first chern classes of the pullbacks via the sections of the relative dualising sheaf ω_π. Then, quasimap invariants (with descendants) are defined as

$$\int_{[\mathcal{Q}_{g,n}(X,\beta)]^{\mathrm{vir}}} \mathrm{ev}_1^* \alpha_1 \smile \cdots \smile \mathrm{ev}_n^* \alpha_n \smile \psi_1^{i_1} \smile \dots \psi_n^{i_n}.$$

There is a variant on this theory, first studied in [Tod11], known as ϵ-quasimaps, which takes inspiration from and is related to Hassett spaces [Has03]. The idea is that although the moduli spaces for quasimap theory and Gromov–Witten theory differ, one can interpolate between the two, depending on a parameter $\epsilon \in \mathbb{Q}_{>0}$, where large ϵ recovers Gromov–Witten theory and small ϵ recovers quasimap theory. After fixing numerical data, there are finitely many chambers which give genuinely different compactifications. Roughly speaking, as ϵ grows the moduli space allows for more rational tails and fewer basepoints, as long as there is enough degree on the component. The main reason for introducing this variant is that there are wall-crossing formulas which relate the invariants of ϵ-quasimaps for different ϵ [CIT09, Zho22], which in particular relate ordinary quasimap invariants to Gromov–Witten invariants. In the genus-zero case, this gives an enumerative interpretation for the mirror map.

3.1.2 Toric Quasimaps

For the rest of the chapter we will restrict to the case of X a smooth projective toric variety. Recall that a fan for X induces a presentation as a GIT quotient $X = \mathbb{A}^{|\Sigma(1)|} /\!/ \mathbb{G}_m^s$ [Cox95, Theorem 2.1]. A quasimap to X is given by a morphism

$$C \to [\mathbb{A}^{|\Sigma(1)|} / \mathbb{G}_m^s].$$

By taking the associated fibre bundle to the defining principal bundle we can see that this morphism is defined by $|\Sigma(1)|$ line bundle-section pairs (L_i, u_i) on C

$$\begin{tikzcd} L_i = P \times_{\mathbb{G}_m^s} \mathbb{A}^1 \arrow[d] \\ C \arrow[u, bend left, "u_i"] \end{tikzcd}$$

where the action of $\mathbb{G}_m^s$ is given by $\underline{a}_i$, the i^{th} column vector of the weight matrix defining the action of $\mathbb{G}_m^s$ on $\mathbb{A}^{|\Sigma(1)|}$. These line bundles are not independent, there are isomorphisms $c_m : \bigotimes_{i=1}^{|\Sigma(1)|} L_i^{\otimes m_i} \cong \mathcal{O}_C$ for $m = (m_1, \ldots, m_{|\Sigma(1)|})$ such that $\sum_{i=1}^{|\Sigma(1)|} m_i \underline{a}_i = 0$. On the other hand there is an exact sequence on X

$$0 \to M \to \mathbb{Z}^{|\Sigma(1)|} \to \operatorname{Pic} X \to 0$$

where the first map is defined by the rays of the fan and the second map is defined by the weight matrix. So we have isomorphisms $c_m : \bigotimes_{i=1}^{|\Sigma(1)|} L_i^{\otimes m_i} \cong \mathcal{O}_C$ for $m = (m_1, \ldots, m_{|\Sigma(1)|})$ in the kernel of $\mathbb{Z}^{|\Sigma(1)|} \to \operatorname{Pic} X$, in other words for each $m \in M$. In fact, in the earlier set up of *toric quasimaps* [CFK10], the authors define toric quasimaps as follows.

Definition 3.1.9 *Let N be a lattice, let Σ be a (complete) fan in N. Let $X = X_\Sigma$ be the toric variety associated to this fan and assume it is smooth and projective. Let $\mathcal{O}_X(1) = \bigotimes_\rho \mathcal{O}_X(D_\rho)^{\otimes \alpha_\rho}$ be a fixed polarisation of X. Fix non-negative integers g, n and an effective curve class β. An n-marked stable toric quasimap of genus g and degree β is*

$$\Big((C, p_1, \ldots, p_n), \{L_\rho, u_\rho\}_{\rho \in \Sigma(1)}, \{c_m\}_{m \in M}\Big)$$

1. *$(C, p_1, \ldots, p_n)$ is an n-marked prestable curve of genus g*

2. *the L_ρ are line bundles on C of degree $D_\rho \cdot \beta$*

3. *the u_ρ are global sections of L_ρ*

4. *c_m are isomorphisms $c_m : \bigotimes_\rho L_\rho^{\langle m, v_\rho \rangle} \cong \mathcal{O}_C$*

which satisfy

1. *(Non-degeneracy) $\exists$ a finite set $B \subset C$ distinct from the nodes and markings such that $\forall c \in C \setminus B, \exists$ maximal cone $\sigma \in \Sigma_{\max}$ such that $\forall \rho \not\subset \sigma$ we have that $u_\rho(c) \neq 0$*

2. *(Stability) $\omega_C(p_1 + \cdots + p_n) \otimes L^\epsilon$ is ample $\forall \epsilon \in \mathbb{Q}_{>0}$, where $L = \bigotimes_\rho L_\rho^{\otimes \alpha_\rho}$*

3. *(Compatibility) $c_m \otimes c_{m'} = c_{m+m'}, \forall m, m' \in M$*

One can show [CFK10, 3.1.3.] that this definition doesn't depend on the polarisation of X, and moreover the non-degeneracy condition coincides with the condition in Definition 3.1.2. In what follows will use both interpretations of toric quasimaps.

3.1.3 Toric Quasimaps with Divisors

In this section we will spell out how we treat tangency in the case of quasimaps. Let $f : C \to X$ be a morphism, with C a prestable curve. Suppose $D \subset X$ is a (smooth) divisor in X defined by the line bundle-section pair $(\mathcal{O}_X(D), s_D)$. When we say that f has *tangency* α at $p \in C$, with $f(p) \in D$, we mean that $f^* s_D = \lambda \cdot t^\alpha$ for t a local parameter for C around p and λ a scalar. In the case of quasimaps we have no morphism to X so we cannot pullback the line-bundle section pair $(\mathcal{O}_X(D), s_D)$ to the curve. We do have a morphism to the full quotient stack $[\mathbb{A}^{|\Sigma(1)|}/\mathbb{G}_m^s]$. In fact $(\mathcal{O}_X(D), s_D)$ defines a line bundle-section pair on $[\mathbb{A}^{|\Sigma(1)|}/\mathbb{G}_m^s]$ which we can pullback. We take this perspective later but now we use a more concrete construction, explained in [BN21, 2.3].

First, note that

$$H^0(X, \mathcal{O}_X(D)) = \left\langle \prod_\rho x_\rho^{\alpha_\rho} : \sum_\rho a_\rho [D_\rho] = [D] \right\rangle$$

where the x_ρ for $\rho \in \Sigma(1)$ are the generators of the homogeneous coordinate ring of X and the a_ρ are non-negative integers. We can therefore write s_D as

$$s_D = \sum_{\underline{a}=(a_\rho)} c_{\underline{a}} \prod_\rho x_\rho^{a_\rho}$$

where the $a = (a_\rho) \in \mathbb{N}^{\Sigma(1)}$ are exponents and the $c_{\underline{a}}$ are scalars. This allows one to define

$$u_D = \sum_{\underline{a}=(a_\rho)} c_{\underline{a}} \prod_\rho u_\rho^{\alpha_\rho} \in H^0(C, \otimes L_\rho^{\otimes a_\rho})$$

where the summands *a priori* are sections of different line bundles, but are in fact all isomorphic by the exact sequence

$$0 \to M \to \mathbb{Z}^{|\Sigma(1)|} \to \operatorname{Pic}(X) \to 0.$$

This gives us a line bundle-section pair (L_D, u_D) which is well-defined up to isomorphism.

Definition 3.1.10 *Let (X, D) be as above and suppose we have a family of stable quasimaps over a scheme S from a family of curves C to X. Then we define the morphism induced by D to be the morphism*

$$C \to [\mathbb{A}^1/\mathbb{G}_m]$$

defined by the above line bundle-section pair on C.

Remark 3.1.11 *If instead of a smooth divisor we have a simple normal crossings divisor $D = D_1 + \cdots + D_r$, then each component is given by a line bundle-section pair $(\mathcal{O}_X(D_i), s_{D_i})$. Thus, we get r induced maps to $[\mathbb{A}^1/\mathbb{G}_m]$. Together these give a morphism*

$$C \to [\mathbb{A}^r/\mathbb{G}_m^r]$$

which we once again call the morphism induced by D.

Suppose we have fixed a fan giving X, a smooth projective toric variety and $D = D_1 + \cdots + D_r$ a simple normal crossings divisor. Fix the discrete data g, n and β which determine the quasimap moduli space $\mathcal{Q}_{g,n}(X, \beta)$. Now we want to consider quasimaps with tangency to D, so choose $(\alpha_{i,j})_{i,j}$, an $r \times n$ matrix of non-negative integers such that $\sum_{j=1}^n \alpha_{i,j} = D_i \cdot \beta$. We could consider the substack of quasimaps which hit D_i with tangency $\alpha_{i,j}$ at the marked point p_j. However, this substack is not proper.

Example 3.1.12 *Consider the family of genus* 0*, degree* 2 *quasimaps (in fact stable maps) given by*

$$\begin{aligned} \mathbb{G}_m \times \mathbb{P}^1 &\to \mathbb{P}^2 \\ [z_0 : z_1] &\mapsto [tz_0^2 : z_0^2 : z_1^2] \end{aligned}$$

with $p_1 = [0 : 1]$ and $p_2 = [1 : 1]$. Let D be the hyperplane $Z(x_0)$ and $(\alpha_1, \alpha_2) = (2, 0)$. It is clear that for $t \neq 0$ the (quasi)map has tangency 2 at p_1 since tz_0^2 vanishes to order 2. On the other hand the limit is given by

$$[z_0 : z_1] \mapsto [0 : z_0^2 : z_1^2]$$

and so the entire curve is now mapped into D. The limit does not hit D with tangency 2 at p_1 as the pullback of x_0 vanishes identically on the curve, so this substack will not satisfy the valuative criterion of properness.

Since we require a proper moduli space in order to define invariants, we will need to compactify this locus in some way.

In [BN21], the authors restrict to genus zero, and the case where the divisor D is smooth and very ample, by mimicking the solution of Gathmann for stable maps [Gat02]. Namely, they take the closure of the locus described above in the space $\mathcal{Q}_{0,n}(X, \beta)$. This closure is not smooth, however, since the divisor is very ample, there is an embedding $X \hookrightarrow \mathbb{P}^N$ inducing a map $\mathcal{Q}_{0,n}(X, \beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$, the target of which is smooth (which is true only in genus-zero). This fact can be used to put a virtual fundamental class on $\mathcal{Q}_{0,n}(X, \beta)$.

In order to build a more general theory we will instead appeal to the modern solution of [GS13, Che14, AC14] using *logarithmic structures*. Roughly this endows both our curve and our target with extra structure which can encode 'tangency' even if the curve component maps into the divisor. More comprehensive introductions to logarithmic geometry include [Gro11, Nab19, ACG$^+$13].

3.2 Logarithmic Geometry

The precise definition of a logarithmic structure on a scheme X is a certain sheaf of monoids with a map to the structure sheaf. The idea is to mimic toric geometry. The sheaf of monoids picks out certain functions or generalised functions which behave like monomials on a toric variety.

Example 3.2.1 *Consider the toric variety $\mathbb{A}^1$. Picking a coordinate function on $\mathbb{A}^1$ means giving a monomial which cuts out the toric boundary (just the origin). All* toric *morphisms $\mathbb{A}^1_x \to \mathbb{A}^1_y$ are of the form*

$$\begin{aligned} \mathbb{A}^1_x &\to \mathbb{A}^1_y \\ x &\mapsto x^a. \end{aligned}$$

Since monomials must pull-back to monomials under toric morphisms, if we think of $(\mathbb{A}^1_x, 0)$ as our curve (C, p) with marking p, and we think of $(\mathbb{A}^1_y, 0)$ as our target (X, D) then this provides the connection to tangency.

Take any smooth variety X of dimension n, with a smooth divisor D. Suppose we have a marked curve (C, p) and a morphism $f : (C, p) \to (X, D)$ with $f^{-1}(D) = p$. Étale locally around $f(p)$ on X we can choose a chart $X \supset U \cong \mathbb{A}^n_{y_1,\ldots,y_n}$ such that in these coordinates D is given by $\{y_1 = 0\}$. The toric structure on $\mathbb{A}^n$ singles out all monomials i.e. of the form $y_1^{i_1} \ldots y_n^{i_n}$ but there is a *logarithmic structure* on $\mathbb{A}^n$ which distinguishes functions of the form λy_1^i (for λ a scalar). The curve (C, p) is locally isomorphic to $(\mathbb{A}^1, 0)$ and in these charts f is a *logarithmic morphism* if $f : \mathbb{A}^1 \to \mathbb{A}^n$ is such that $f^* y_1 = \lambda x^a$ for some scalar λ.

On both the curve and the target the collection of these distinguished functions glue together to form a sheaf which naturally includes into the sheaf of functions. It is a sheaf of monoids because you can multiply two monomials together and still get a monomial but you can't add them.

Definition 3.2.2 *Let X be a scheme. A logarithmic structure on X is a sheaf of monoids $\mathcal{M}_X$ together with a morphism $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ (with the operation of multiplication) such that $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$. We call $(X, \mathcal{M}_X)$ (or just X) a* logarithmic scheme*.*

Remark 3.2.3 *If we have $\alpha' : \mathcal{M} \to \mathcal{O}_X$ but $\alpha'^{-1}(\mathcal{O}_X^*) \ncong \mathcal{O}_X^*$, one can form the* associated logarithmic structure *by taking*

$$\mathcal{M}_X = \frac{\mathcal{M} \oplus \mathcal{O}_X^*}{\{(m, \alpha'(m)^{-1}) : \alpha'(m) \in \mathcal{O}_X^*\}}$$

mapping via $\alpha([m, f]) = \alpha'(m) \cdot f$.

Example 3.2.4 *Let X be a variety and D a closed subset of pure dimension 1. Then the* divisorial logarithmic structure *is given by the sheaf of monoids*

$$\mathcal{M}_X(U) = \{f \in \mathcal{O}_X(U) : f|_{X \setminus D} \in \mathcal{O}_X^*\}$$

and the morphism $\alpha : \mathcal{M}_X \hookrightarrow \mathcal{O}_X$ is just the inclusion of functions.

If the function is invertible away from D then it vanishes only along D, which means this sheaf picks out the local functions cutting out D. Although this example produced a subsheaf of the structure sheaf, the general definition does not require $\mathcal{M}_X$ to be a subsheaf of the sheaf of functions. It is this flexibility which allows logarithmic geometry to account for tangency in the limit.

Definition 3.2.5 *Given $f : X \to Y$ and logarithmic structure $(Y, \mathcal{M}_Y)$ we can define the* pullback logarithmic structure *$(X, f^*\mathcal{M}_Y)$ as the logarithmic structure associated to*

$$f^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X.$$

Example 3.2.6 *Consider $\mathbb{A}^1$ with the divisorial logarithmic structure $\mathcal{M}_{\mathbb{A}^1}$ with respect to the origin. Consider the inclusion of the origin $i : \operatorname{Spec} \mathbb{C} \hookrightarrow \mathbb{A}^1$. The pullback logarithmic structure remembers the coordinate function on $\mathbb{A}^1$ even though we have restricted to a subscheme where that function is identically 0. Concretely, in this example*

$$i^*\mathcal{M}_{\mathbb{A}^1} = \mathbb{G}_m \oplus \mathbb{N} \to \mathbb{C}$$
$$(\lambda, n) \mapsto \begin{cases} \lambda \text{ if } n = 0 \\ 0 \text{ if } n \neq 0 \end{cases}$$

Thinking of (λ, n) as '$\lambda \cdot x^n$', the logarithmic structure remembers the coordinate monomial x, encoded as the element $(1, 1) \in \mathbb{G}_m \oplus \mathbb{N}$.

Now we can understand the idea behind how logarithmic geometry can 'remember tangency'. First we include the definition of a morphism of logarithmic schemes, which is just a morphism of schemes, together with a compatible way to pullback monomials.

Definition 3.2.7 *A morphism $(f, f^\flat) : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of logarithmic schemes is a morphism of schemes $f : X \to Y$ together with a morphism of sheaves of monoids $f^\flat : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the following diagram commutes*

$$\begin{array}{ccc} f^{-1}\mathcal{M}_Y & \xrightarrow{f^\flat} & \mathcal{M}_X \\ \Big\downarrow{\scriptstyle f^{-1}\alpha_Y} & & \Big\downarrow{\scriptstyle \alpha_X} \\ f^{-1}\mathcal{O}_Y & \xrightarrow{f^\#} & \mathcal{O}_X. \end{array}$$

If the logarithmic structure on X is isomorphic to the pullback logarithmic structure from Y, we call the map strict.

Example 3.2.8 *Consider $\pi : \mathbb{A}^2_{t,x} \to \mathbb{A}^1_t$, thought of as a family of (affine) curves for varying t. If we equip $\mathbb{A}^2$ and $\mathbb{A}^1$ with the divisorial logarithmic structures coming from their toric boundaries then the projection π is a logarithmic (in fact toric) morphism since $\pi^* t = t$. Now upgrade this to a family of maps to the target $\mathbb{A}^1_y$ (also with the divisorial logarithmic structure with respect to the origin) given by*

$$\begin{array}{ccc} \mathbb{A}^2_{t,x} & \xrightarrow{(t,x)\mapsto tx^a} & \mathbb{A}^1_y \\ \big\downarrow & & \\ \mathbb{A}^1_t & & \end{array}.$$

Scheme-theoretically this is comparable to Example 3.1.12. For any $t \neq 0$ the fibre maps with tangency a, but when $t = 0$ the entire fibre maps into the origin so there is no notion of tangency. On the other hand if we think of this as a diagram of logarithmic morphisms, *then we can restrict to the fibre over $t = 0$ and pull back the logarithmic structure. This logarithmic structure remembers not only the distinguished function x but also the function t coming from the ambient $\mathbb{A}^2$. Moreover, the data of the logarithmic morphism includes with it the map of monoids $\mathbb{N} \to \mathbb{N}^2, 1 \mapsto (1, a)$ which remembered that the pullback of y is equal to $t^1 x^a$. This means the extra data of the logarithmic morphism can be defined to be a generalised tangency and it knows when it is the limit of a family which genuinely had that tangency.*

Remark 3.2.9 *The map of monoids $\mathbb{N} \to \mathbb{N}^2$ came from the map $f^\flat$ from Definition 3.2.7, by modding out by invertible functions. Since tangency just depends on the exponent it is often useful to do this. In general given $(X, \mathcal{M}_X)$ we let $\overline{\mathcal{M}}_X := \mathcal{M}_X / \mathcal{O}_X^*$ and call this the* ghost sheaf. *The pullback $f^\flat$ induces a map $f^{-1}\overline{\mathcal{M}}_Y \to \overline{\mathcal{M}}_X$.*

General logarithmic structures can be quite wild so one almost always restricts to a certain class of logarithmic structures which have a close connection to toric geometry.

Definition 3.2.10 *Consider a monoid P, and form* $\operatorname{Spec} \mathbb{C}[P]$. *There is a natural map $P \to \mathbb{C}[P]$. Let* $\operatorname{Spec}(P \to \mathbb{C}[P])$ *be the logarithmic scheme given by taking the associated logarithmic structure. If $P = \sigma^\vee \cap M$, for σ a rational polyhedral cone in a lattice dual to M, then* $\operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$ *is the associated affine toric variety and the associated logarithmic structure is the divisorial logarithmic structure with respect to the toric boundary.*

Toric monoids, corresponding to rational polyhedral cones, satisfy certain special properties.

Definition 3.2.11 *We say that a monoid P is* integral *if $p + r = q + r \Rightarrow p = q, \forall p, q, r \in P$. We say it is* finitely generated *if there is a surjective monoid homomorphism $\mathbb{N}^r \to P$. We say it is* saturated *if for any element $p \in P^{gp}$, the groupification of P, and $n \in \mathbb{N}$ we have that $np \in P \Rightarrow p \in P$.*

Remark 3.2.12 *If $P = \sigma^\vee \cap M$, then P is finitely generated, integral and saturated. In fact if P satisfies these three properties and in addition is torsion free, then P comes from a rational polyhedral cone.*

Definition 3.2.13 *A logarithmic structure on X is* fine *if locally on X there are morphisms $P \to \mathcal{O}_X$ where P is finitely generated and integral, whose associated logarithmic structure is the one on X. Equivalently, the induced morphism $X \to$* $\operatorname{Spec}(P \to \mathbb{C}[P])$ *is strict. We call these morphisms* charts. *The logarithmic structure is* fine and saturated *(or f.s.) if P is also saturated.*

Example 3.2.14 *Let X be a smooth variety, D a smooth divisor and endow X with the divisorial logarithmic structure w.r.s.t. D. X can be covered by opens U, where D is cut out by a function f. This function defines a morphism*

$$\begin{aligned} U &\to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}]) \\ p &\mapsto f(p) \end{aligned}$$

Which gives charts for the logarithmic structure.

We could have done away with charts in this instance. It is true that there is not necessarily a *global* morphism $X \to \mathbb{A}^1$ which cuts out D, because the local functions from 3.2.14 may differ on overlaps by an invertible function. However, there is a global morphism

$$X \to [\mathbb{A}^1/\mathbb{G}_m]$$

Furthermore, the logarithmic structure on X is pulled back from the divisorial logarithmic structure on $[\mathbb{A}^1/\mathbb{G}_m]$ w.r.s.t the stacky origin. This is the first instance of what is known as an *Artin fan*. For more on Artin fans see [ACM+16].

Remark 3.2.15 *In fact since a morphism $X \to [\mathbb{A}^1/\mathbb{G}_m]$ is literally the data of a line bundle-section pair (L, s), one can always pull back the logarithmic structure on $[\mathbb{A}^1/\mathbb{G}_m]$ to get the logarithmic structure* defined by (L, s). *If X is smooth and (L, s) is just $(\mathcal{O}_X(D), s_D)$ for D some divisor then we recover Example 3.2.4. However it may be that s vanishes entirely on X, or on components of X. In general the logarithmic structure is the one associated to the sheaf which on a trivialising open set for the line bundle L is (n, a) where $n \in \mathbb{N}$ and a a local generator for $(L^\vee)^{\otimes n}$. The morphism to $\mathcal{O}_X$ is given by $(s^\vee)^{\otimes n} : (L^\vee)^{\otimes n} \to \mathcal{O}_X$.*

3.2.1 Logarithmic Curves

When attempting to translate the theory of stable maps (and quasimaps) into the language of logarithmic geometry one needs to first understand how the theory of (pre)stable curves fits into this framework. Prestable curves are not arbitrary projective curves but can have at worst nodal singularities, which, in families can give rise to toric singularities. This fits in well with logarithmic geometry. Namely, a family of prestable curves $C \to S$ is not necessarily smooth but can always be made to be *logarithmically smooth* with appropriate logarithmic structures on C and S. The logarithmic structure on the base is crucial, for example a nodal curve over $\mathrm{Spec}\,\mathbb{C}$ can never be made to be logarithmically smooth over $\mathrm{Spec}\,\mathbb{C}$ if the latter is equipped with the trivial logarithmic structure. Smoothness can be defined using the infinitesimal lifting property for maps of schemes, logarithmic smoothness is characterised similarly, but for maps of logarithmic schemes.

Definition 3.2.16 *A morphism $X \to Y$ of fine logarithmic schemes is* logarithmically smooth *if it is locally of finite presentation and satifies the infinitesimal lifting property in the logarithmic category for strict square zero extensions.*

Theorem 3.2.17 *(Kato's Criterion) Let $f : X \to Y$ be a morphism of fine logarithmic schemes. Assume we have a chart $Q \to \mathcal{O}_Y$, where Q is a finitely generated integral monoid. Then f is logarithmically smooth if and only if, étale locally* *on X and Y, there is a diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{C}[P] \arrow[d] \\ Y \arrow[r] & \operatorname{Spec}\mathbb{C}[Q] \end{tikzcd}$$

Where the top horizontal row comes from $P \to \mathcal{O}_X$ a chart, and the righthand vertical arrow comes from $Q \to P$ such that the induced morphism $X \to \operatorname{Spec}\mathbb{C}[P] \times_{\operatorname{Spec}\mathbb{C}[Q]} Y$ is smooth.

Logarithmic curves will be defined to be logarithmic schemes over a base with the structure morphism being logarithmically smooth. There is also a description of what this can look like locally.

Theorem 3.2.18 *[Kat00, 1.8] Let $(C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ be a logarithmically smooth and integral morphism of fine and saturated logarithmic schemes. Suppose $(W, \mathcal{M}_W) = (\operatorname{Spec} A, Q)$ given by the chart $\sigma : Q \to A$ for $(A, \mathfrak{m})$ a strictly Henselian ring. Then étale locally C is isomorphic to one of the following.*

- *$\mathbb{A}^1_x$ with the logarithmic structure induced from the homomorphism*

$$Q \to A[x],\ q \mapsto \sigma(q)$$

- *$\mathbb{A}^1_x$ with the logarithmic structure induced from the homomorphism*

$$Q \oplus \mathbb{N} \to A[x],\ (q, n) \mapsto \sigma(q)x^n$$

- *$\operatorname{Spec}(A[x,y]/(xy - t))$ with $t \in \mathfrak{m}$ and with the logarithmic structure induced from the homomorphism*

$$Q \oplus_{\mathbb{N}} \mathbb{N}^2 \to (A[x,y]/(xy - t)),\ (q, (a, b)) \mapsto \sigma(q)x^a y^b$$

  Where $Q \oplus_{\mathbb{N}} \mathbb{N}^2$ defined by $\mathbb{N} \to \mathbb{N}^2$, $1 \mapsto (1,1)$ and $\mathbb{N} \to Q$ given by some element $\rho_q \in Q$.

In the first case there is no interesting logarithmic information, everything is pulled back from the base and scheme-theoretically this is just a neighbourhood of a smooth point. The second is also a neighbourhood of a smooth point but now there is a section given by z which marks the point $z = 0$. The third case is of a node and the includes the information of an infinitesimal smoothing.

Example 3.2.19 *Consider $\operatorname{Spec}(\mathbb{C}[x,y]/(xy))$ and $Q = \mathbb{N}$ with the chart $\mathbb{N} \to \mathbb{C}$ sending $1 \mapsto 0$. If $\rho_q = k \in \mathbb{N}$ then $\mathbb{C}[\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2] \cong \mathbb{C}[x, y, t]/(xy - t^k)$ and so we obtain the diagram*

$$\begin{tikzcd} \operatorname{Spec}(\mathbb{C}[x,y]/(xy)) \arrow[r, hook] \arrow[d] & \operatorname{Spec}(\mathbb{C}[x,y,t]/(xy - t^k)) \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, hook] & \operatorname{Spec}\mathbb{C}[t] \end{tikzcd}$$

Giving a smoothing of the node.

Definition 3.2.20 *An n-marked, genus g* logarithmic curve *is a logarithmically smooth, integral morphism of fine and saturated logarithmic schemes $(C, \mathcal{M}_C) \to (W, \mathcal{M}_W)$ with sections $s_i : W \to C$ such that every geometric fibre is a reduced and connected curve and such that the points corresponding to case (2) coincide with the markings.*

Remark 3.2.21 *If we drop the saturated condition then* $\operatorname{Spec}(\mathbb{N} \setminus \{1\} \to \mathbb{C}[\mathbb{N} \setminus \{1\}])$ *is logarithmically smooth but* $\operatorname{Spec}(\mathbb{C}[\mathbb{N} \setminus \{1\}]) = \operatorname{Spec}(\mathbb{C}[\langle 2, 3\rangle]) = \operatorname{Spec}(\mathbb{C}[t^2, t^3]) = \operatorname{Spec}(\mathbb{C}[x, y]/(y^2 - x^3))$ *is the cuspidal cubic.*

Fixing g and n, the moduli space $\mathfrak{M}_{g,n}$ is an Artin stack parametrising genus g, n-marked, projective curves with at worst nodal singularities. The locus of singular curves defines a normal crossings divisor on $\mathfrak{M}_{g,n}$ which determines a logarithmic structure on $\mathfrak{M}_{g,n}$, defined using the smooth topology. Similarly, the universal curve $\mathfrak{C} \to \mathfrak{M}_{g,n}$ admits a canonical logarithmic structure determined by the preimage of the locus of singular curves together with the sections corresponding to the markings. This makes the morphism

$$(\mathfrak{C}, \mathcal{M}_{\mathfrak{C}}) \to (\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}})$$

into a logarithmically smooth, integral morphism.

This produces a way to produce a canonical logarithmic structure on a prestable curve $C \to S$ by pulling back the logarithmic structures via the diagram

$$\begin{array}{ccc} C & \longrightarrow & \mathfrak{C} \\ \downarrow & & \downarrow \\ S & \longrightarrow & \mathfrak{M}_{g,n} \end{array}$$

making $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ into a logarithmic curve.

Concretely, if $S = \operatorname{Spec}\mathbb{C}$ so that C is some n-marked, genus g, prestable curve, let k be the number of nodes on C. Then the corresponding logarithmic structure on $\operatorname{Spec}\mathbb{C}$ is the one associated to the chart

$$\mathbb{N}^k \to \mathbb{C} \tag{3.1}$$

$$e_i \mapsto 0. \tag{3.2}$$

Each generator corresponds to a different independent parameter t_i which smooths the i^{th} node locally as $xy = t_i$.

On the other hand there are many *different* logarithmic structures on $C \to S$ making it into a logarithmic curve. The one pulled back from $\mathfrak{C} \to \mathfrak{M}_{g,n}$ is called the *basic* or *minimal* one and has the property that any other logarithmic structures making $C \to S$ into a logarithmic curve is pulled back from this one.

Theorem 3.2.22 *[ACG⁺13, Theorem 4.11] Let $C \to S$ be a (pre)stable genus g curve with n marked points and endow it with the canonical log structure above. If $(C', \mathcal{M}'_C) \to (S', \mathcal{M}'_S)$ is a logarithmic curve and $a\colon S' \to S$ and $b : C' \to C$ are morphisms such that $C' \cong C \times_S S'$ and such that the divisors of marked points in C' are sent scheme-theoretically to the divisors of marked points in C, there are unique lifts of a and b to morphisms of logarithmic schemes such that*

$$\begin{array}{ccc} C' & \xrightarrow{b} & C \\ \downarrow & & \downarrow \\ S' & \xrightarrow{a} & S \end{array}$$

becomes cartesian in the category of f.s. logarithmic schemes.

3.3 Moduli Space of Logarithmic Quasimaps

Let X be a smooth projective toric variety associated to a fan Σ and $D = D_1 + \cdots + D_r$ a simple normal crossings divisor. In this section we will define the moduli space of logarithmic quasimaps to $X|D$. We will mimic the strategy of the construction of the corresponding moduli space of stable logarithmic maps in [Che14, AC14]. Namely, we will construct the moduli space in the case when D is smooth and use this to produce the general moduli space. The key point will be that although there is no genuine map from a curve to X, we have the *induced morphism* 3.1.11 which contains all the data concerning tangency. It is this morphism which we will make logarithmic in order to compactify.

An alternative way to approach the problem is to equip the quotient stack containing X with the divisorial logarithmic structure with respect to the closure of D and form a moduli space of logarithmic maps to this quotient stack. This is essentially equivalent, but phrasing it as above allows us to take advantage of existing moduli spaces. Specifically, in this section we can take advantage of the moduli space of logarithmic maps to $[\mathbb{A}^1/\mathbb{G}_m]$.

Definition 3.3.1 *Let g, n be non-negative integers and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ a tuple of non-negative integers. A family of logarithmic maps to $[\mathbb{A}^1/\mathbb{G}_m]$ over $(S, \mathcal{M}_S)$, a fine and saturated logarithmic scheme is a diagram*

$$\begin{tikzcd} C_S \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d] \\ S \arrow[r] & \mathrm{pt} \end{tikzcd}$$

where $C_S \to S$ is an n-marked family of genus g, logarithmic curves and $C_S \to [\mathbb{A}^1/\mathbb{G}_m]$ is a logarithmic morphism to $[\mathbb{A}^1/\mathbb{G}_m]$ equipped with the divisorial logarithmic structure with respect to the origin, with contact orders α at the markings.

As in 3.2.1, with the case of logarithmic curves, there is a notion of minimal logarithmic map, satisfying a universal property, and such logarithmic maps form a logarithmic algebraic stack. We will use the word minimal to refer to the maps case and the word basic when referencing just curves.

Theorem 3.3.2 *Minimal logarithmic maps form a logarithmic algebraic stack $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ [AW18b, Proposition 3.3].*

Remark 3.3.3 *The logarithmic structure on $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ is the divisorial logarithmic structure with respect to the complement of the locus of maps to $[\mathbb{A}^1/\mathbb{G}_m]$ from smooth curves which hit the origin in finitely many points. With respect to this logarithmic structure $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ is logarithmically smooth [AW18b, Propoisition 1.6.1].*

Any logarithmic map to $[\mathbb{A}^1/\mathbb{G}_m]$ (not neccessarily minimal) has certain discrete data coming from the induced map on ghost sheaves. Given

$$\begin{tikzcd} C \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \\ (\operatorname{Spec}\mathbb{C}, Q) & \end{tikzcd}$$

and suppose the scheme-theoretic map corresponds to the line bundle section pair (L, s). We get an induced map for any $p \in C$

$$\overline{\mathcal{M}}_{L,p} \to \overline{\mathcal{M}}_{C,p}$$

where $\mathcal{M}_L$ is the logarithmic structure on C associated to (L, s). The source is either 0 or $\mathbb{N}$ depending on whether or not $s(p) = 0$. This means for each $p \in C$ we automatically get an element $\overline{\mathcal{M}}_{C,p}$ which is either 0 or the image of 1. Below are the possible cases

1. If p is a smooth point or the generic point of a curve component $C' \subset C$ on which the section vanishes identically then we call the image of 1, $e_{C'} \in \overline{\mathcal{M}}_{C,p} = Q$.

2. If $p = p_i \in C$ is a marked point then $\overline{\mathcal{M}}_{C,p} = Q \oplus \mathbb{N}$ by 3.2.18. The image of 1 is $(e_{C'}, \alpha_i)$, where C' is the component containing p.

3. If $p \in C$ is a node, if $s(p) = 0$ and the two adjacent branches are C_1 and C_2 then

$$\overline{\mathcal{M}}_{L,p} = \mathbb{N} \to Q \oplus_{\mathbb{N}} \mathbb{N}^2 = \overline{\mathcal{M}}_{C,p}$$

is determined by the maps from (1) on the two branches. Moreover, we can choose an ordering of the branches such that we get a non-negative integer c_ℓ such that

$$e_{C_1} - e_{C_2} = c_\ell \rho_\ell$$

where ρ_ℓ is the element giving the smoothing of the node.

Remark 3.3.4 *Note that in case (2), if s does not vanish entirely along C' then α_i is the order of vanishing of s at p_i, in other words, the genuine tangency of C' along the origin at p_i.*

Lemma 3.3.5 *In case (3), suppose the section s vanishes entirely along C_1 but not C_2. Then c_q is the tangency of C_2 along the origin at p.*

Proof 3.3.6 *For simplicity assume $Q = \mathbb{N}$. Then locally around the node the logarithmic morphism $(C, \mathcal{M}_C) \to (\operatorname{Spec} \mathbb{C}, \mathbb{N})$ is pulled back via the diagram 3.2.19. By the geometric description we know that s is given by the regular function x^a for some $a \in \mathbb{N}$ where $Z(x,t) = C_1$ and $Z(y,t) = C_2$. In fact a is exactly the contact order of C_2. On the another hand there is a diagram given by generization maps*

$$\begin{array}{ccc} \mathbb{N} = \overline{\mathcal{M}}_{L,\eta_1} & \longrightarrow & \overline{\mathcal{M}}_{C,\eta_1} = \mathbb{N} \\ \Uparrow & & \uparrow \\ \mathbb{N} = \overline{\mathcal{M}}_{L,0} & \longrightarrow & \overline{\mathcal{M}}_{C,0} = \mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2 \end{array}.$$

Since the bottom horizontal map is given by $1 \mapsto [(0,(a,0)]$ and the right vertical map sends $[0,(1,0)] \mapsto k$ we have that the $e_{C_1} = a \cdot k = c_\ell \cdot \rho_\ell$, so $c_\ell = a$ is the tangency.

Remark 3.3.7 *The minimality condition can be phrased as follows. Consider the dual graph G of the underlying curve C. Consider the monoid $\langle e_v, e_\ell \rangle$ for v a vertex of G and ℓ an edge of G, subject to the following equations*

1. $e_v = 0$ *if* $e_{C_{v'}} = 0$

2. $e_{v_2} - e_{v_1} = c_\ell e_\ell$ *(oriented appropriately).*

Let $\overline{\mathcal{M}(G)}$ be the monoid obtained my modding out by torsion and then saturating. Then the logarithmic map is minimal *if $Q \cong \overline{\mathcal{M}(G)}$.*

3.3.1 Construction for D smooth

Definition 3.3.8 *Let g, n be non-negative integers, let β be an effective curve class on X and let D be a* smooth *divisor. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a tuple of non-negative integers such that $\sum_{i=1}^n \alpha_i = D \cdot \beta$. Define $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ as the fibre product of stacks*

$$\begin{tikzcd}
\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \arrow[r] \arrow[d] & \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m]) \arrow[d, "\pi_2"] \\
\mathcal{Q}_{g,n}(X,\beta) \arrow[r, "\pi_1"] & \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])
\end{tikzcd}$$

Here $\mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])$ is the stack of maps from n-marked curves, genus g prestable curves to $[\mathbb{A}^1/\mathbb{G}_m]$. The morphism π_1 is given by associating to a stable quasimap the induced map to $[\mathbb{A}^1/\mathbb{G}_m]$, as in 3.1.10 and π_2 is given by forgetting the logarithmic structure.

Remark 3.3.9 *We equip $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ with the logarithmic structure defined by pullback via the morphism $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \to \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$.*

Proposition 3.3.10 *The moduli stack $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ is a Deligne–Mumford stack.*

Proof 3.3.11 *Since both $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ is a fibre products of algebraic stacks it is automatically algebraic. The fact that it is a Deligne–Mumford stack will follow from the fact that the morphism π_2 is representable, which is true by [GS13, Proposition 1.25]. Given any cartesian diagram of algebraic stacks*

$$\begin{tikzcd}
\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r] \arrow[d] & \mathcal{Y} \arrow[d, "\pi_2"] \\
\mathcal{X} \arrow[r, "\pi_1"] & \mathcal{Z}
\end{tikzcd}$$

with $\mathcal{X}$ Deligne–Mumford and π_2 representable, then we can use the criterion [Cad07, Lemma 3.3.2] to show that any object (x, y, λ), where $x \in \mathrm{Ob}(\mathcal{X}), y \in \mathrm{Ob}(\mathcal{Y}), \lambda : \pi_1(x) \simeq \pi_2(y)$ must have finitely many automorphisms $(\varphi_x, \varphi_y) \in \mathrm{Aut}_{\mathcal{X}}(x) \times \mathrm{Aut}_{\mathcal{Y}}(y)$. There are finitely many automorphisms, φ_x, as $\mathcal{X}$ is Deligne–Mumford, but that fixes $\pi_2(\varphi_y) = \lambda \circ \pi_1(\varphi_x) \circ \lambda^{-1}$. On the other hand, π_2 is representable so it is an injection on automorphism groups, which determines φ_y.

Lemma 3.3.12 *The moduli stack $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ is of finite type.*

Proof 3.3.13 *By [AW18b, 3.2] $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ is locally of finite type, therefore it suffices to show that $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \to \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ factors through a finite type substack of $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$. But this follows from the fact that once we have fixed numerical data g, n, β, the number of components of the curve occurring in $\mathcal{Q}_{g,n}(X,\beta)$ is bounded.*

Proposition 3.3.14 *The moduli space $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ is proper.*

Proof 3.3.15 *Since $\mathcal{Q}_{g,n}(X,\beta)$ is proper by 3.1.7, to prove properness of $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ it suffices to prove properness of the morphism $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m]) \to \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])$. In fact, properness of π_2 essentially follows from [Che14, Theorem 6.1.1.] (as well as [GS13, Theorem 4.1]) using the weak valuative criterion. The weak valuative criterion is as follows. Let R be a discrete valuation ring with quotient field K. Suppose we have a diagram*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m]) \arrow[d] \\
\operatorname{Spec} R \arrow[r] & \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])
\end{tikzcd}$$

Then after potentially a base change via an injection $R \hookrightarrow R'$ of DVRs inducing a finite extension of fraction fields there is a unique morphism $\operatorname{Spec} R' \to \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ *making the diagram commute. We include a summary of the steps of the argument as in [Che14, 6.1.1]. We can separate the criterion into showing existence and uniqueness.*

Existence

First we spell out what the diagram entails. The morphism $\operatorname{Spec} K \to \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$ *is a family of minimal logarithmic prestable maps to* $[\mathbb{A}^1/\mathbb{G}_m]$*, in other words an n-marked genus g logarithmic curve* $(C, \mathcal{M}_C) \to (\operatorname{Spec} K, Q)$ *for some f.s. monoid Q, a line bundle-section pair* (L, s) *on C and a morphism* $\mathcal{M}_L \to \mathcal{M}_C$ *where* $\mathcal{M}_L$ *is the logarithmic structure induced by* (L, s) *3.2.15. The composition* $\operatorname{Spec} K \to \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m]) \to \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])$ *is just the data of* (C, L, s) *(with the markings). The morphism* $\operatorname{Spec} R \to \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])$ *is* $(\mathcal{C}, \mathcal{L}, s')$ *with* $\mathcal{C} \to \operatorname{Spec} R$ *and* $(\mathcal{L}, s')$ *a line bundle-section pair which together restrict to* (C, L, s) *over* $\operatorname{Spec} K$*.*

Step 1

The first step is to put a logarithmic stucture on the base $\operatorname{Spec} R$*. To simplify things we take the minimal logarithmic prestable map over* $(\operatorname{Spec} K, Q)$ *and base change to* $(\operatorname{Spec} K, \mathbb{N})$ *via a morphism* $Q \to \mathbb{N}$*. Consequently, the element* $1 \in \mathbb{N}$ *under the morphism* $\mathbb{N} \to \overline{\mathcal{M}}_C$ *gets sent to the parameter corresponding to the smoothing of each of the nodes in the local models 3.2.19. We get a new family of (not necessarily minimal) logarithmic maps to* $[\mathbb{A}^1/\mathbb{G}_m]$ *which we still label* $(C, \mathcal{M}_C) \to (\operatorname{Spec} K, \mathbb{N})$ *and* $\mathcal{M}_L \to \mathcal{M}_C$ *where* $\mathcal{M}_L$ *is the logarithmic structure induced by* (L, s)*. It suffices to construct a logarithmic map over* $\operatorname{Spec} R$ *(not necessarily minimal) which restricts to this one over* $\operatorname{Spec} K$*. This is due to the universal property of minimality, which states that any logarithmic map can be pulled back from a minimal logarithmic map, and the pullback is unique up to unique isomorphism. Now define the logarithmic structure* $\mathcal{M}_R$ *on* $\operatorname{Spec} R$ *via the chart*

$$\mathbb{N}^2 \to R,\ e_K \mapsto 0,\ e_t \mapsto t$$

where $e_K = (1, 0), e_t = (0, 1)$ *and t is the uniformizer of R.*

Step 2

The next step is to make $\mathcal{C}$ *into a logarithmic curve over* $(\operatorname{Spec} R, \mathcal{M}_R)$ *extending* $(C, \mathcal{M}_C) \to (\operatorname{Spec} K, \mathbb{N})$*. Since every basic logarithmic curve has its logarithmic structure pulled back from the canonical logarithmic structure from* $\mathfrak{C} \to \mathfrak{M}_{g,n}$*, giving* $(\operatorname{Spec} R, \mathcal{M}_{\mathcal{C}/R})$*, it suffices to provide a morphism* $\mathcal{M}_{\mathcal{C}/R} \to \mathcal{M}_R$ *extending the morphism from* $\mathcal{M}_{C/K}$ *defining the logarithmic structure on* $C \to \operatorname{Spec} K$*. But we know by 3.1 that* $\overline{\mathcal{M}}_{\mathcal{C}/R,(t)}$ *is generated by* e_ℓ *for* ℓ *corresponding to a node. Either the node is smoothed out over* $\operatorname{Spec} R$ *or not. If it is smoothed out then locally it is of the form* $xy = t^k$ *and so define*

$$e_\ell \mapsto k e_t$$

if not then we use the morphism $\overline{\mathcal{M}}_{C/K} \to \overline{\mathcal{M}}_K = \mathbb{N}$ *which must send* $e_\ell \mapsto k'$ *for some* k'*. This allows us to define*

$$e_\ell \mapsto k' e_K$$

After choosing charts for the logarithmic structures this induces a map $\mathcal{M}_{\mathcal{C}/R} \to \mathcal{M}_R$*.*

Step 3

Finally, we have to build a morphism of logarithmic structures $\mathcal{M}_{\mathcal{L}} \to \mathcal{M}_{\mathcal{C}}$ where $\mathcal{M}_{\mathcal{L}}$ is the logarithmic structure associated to $(\mathcal{L}, s')$ and $\mathcal{M}_{\mathcal{C}}$ is logarithmic structure on $\mathcal{C}$ from Step 2. This is done by locally around a point p in the central fibre, noticing that over the generic point, the image of the (local) generator corresponding to s has contribution from the base, say $k \cdot e_K$ and contribution from a local function u_p. If p is not a node which persists over $\operatorname{Spec} K$, then Chen shows that $u_p = t^n \cdot h'$ where $n \in \mathbb{N}$ and h' is some function. In fact, Chen shows [Che14, Lemma 6.2.1] what the possibilities for h' are.

1. *If p is a non-distinguished point, then the local function contribution is $u_p = t^n \cdot h$ where h is invertible*
2. *If p is a marked point, with contact order $c \in \mathbb{N}$ then the local function contribution is $u_p = t^n \cdot x^c \cdot h$ where x is the section cutting out p and h is invertible*
3. *If p is a node which is smoothed out over $\operatorname{Spec} K$ then the local function contribution is $u_p = t^n \cdot x^c \cdot h$ where $c \in \mathbb{N}$, x is the local coordinate cutting out one of the branches and h is invertible*

Then the only way to extend this over $\operatorname{Spec} R$ near p is to send the generator of $\mathcal{M}_{\mathcal{L}}$ to

$$k \cdot e_K + n \cdot e_t + \log h'$$

using the additive notation of [Che14]. The case where p is a node which persists over $\operatorname{Spec} K$ is similar, see [Che14, 6.3.4]. It is then shown that after lifting to charts, this construction can be glued together to define a global morphism of logarithmic structures.

Uniqueness

Recall from 3.3.7 that a logarithmic map over $(\operatorname{Spec} \mathbb{C}, Q)$ is minimal if $Q \cong \overline{\mathcal{M}}(G)$. Suppose we have a limit of our family of logarithmic maps. Since the underlying prestable curve is fixed, the dual graph G of the central fibre is also fixed. If v is a vertex of G corresponding to a component C_v on which $s \neq 0$ identically then $e_{C_v} = 0$. Also for any edge ℓ of G corresponding to some node, we have that c_ℓ is determined (it is either 0 or the tangency of the external component). But by the equations in 3.3.7, this determines $\overline{\mathcal{M}(G)} \cong Q$ (the orientation is also determined). Chen shows [Che14, 6.5] that this implies that any extension is uniquely determined.

3.3.2 From smooth to s.n.c. divisors

Now we will use the moduli spaces constructed above to build the moduli space for $D = D_1 + \cdots + D_r$ a simple normal crossings divisor. More precisely, let X be a smooth projective toric variety. Let g, n be non-negative integers, let β be an effective curve class on X and let $D = D_1 + \cdots + D_r$. Let α be an $r \times n$ matrix of non-negative integers $\alpha = (\alpha_{i,j})_{i,j}$ such that $\sum_j \alpha_{i,j} = D_i \cdot \beta$, and let α_i denote the i^{th} row of the matrix. For simplicity, we will make the additional assumption that the intersection of any subset of the components of D is connected.

Definition 3.3.16 *Define $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ to be the fibre product (in the fine and saturated category)*

$$\begin{tikzcd} \mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \arrow[r] \arrow[d] & \mathcal{Q}^{\log}_{g,\alpha_1}(X|D_1,\beta)\times\cdots\times\mathcal{Q}^{\log}_{g,\alpha_r}(X|D_r,\beta) \arrow[d] \\ \mathcal{Q}_{g,n}(X,\beta) \arrow[r, "\Delta"] & \mathcal{Q}_{g,n}(X,\beta)\times\cdots\times\mathcal{Q}_{g,n}(X,\beta) \end{tikzcd} \tag{3.3}$$

where the logarithmic structure on $\mathcal{Q}_{g,n}(X,\beta)$ is pulled back from $\mathfrak{M}_{g,n}$.

Proposition 3.3.17 *$\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ is a proper, Deligne–Mumford logarithmic algebraic stack*

Proof 3.3.18 *This follows from Proposition 2.6.2 of [AW18b].*

Lemma 3.3.19 *We could have defined $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ as the natural generalisation of 3.3.8. Namely as the fibre product*

$$\begin{tikzcd} \mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \arrow[r] \arrow[d] & \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r]) \arrow[d, "\pi_2"] \\ \mathcal{Q}_{g,n}(X,\beta) \arrow[r, "\pi_1"] & \mathfrak{M}_{g,n}([\mathbb{A}^r/\mathbb{G}_m^r]) \end{tikzcd}$$

and indeed this is equivalent and gives a logarithmic algebraic stack with logarithmic structure pulled back from $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r])$.

Proof 3.3.20 *This is essentially the same argument as in [BNR22, 2.2]. By 3.3.8 we can rewrite 3.3 as*

$$\begin{tikzcd} \mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \arrow[r] \arrow[d] & \mathfrak{M}^{\log}_{g,\alpha_1}([\mathbb{A}^1/\mathbb{G}_m])\times\cdots\times\mathfrak{M}^{\log}_{g,\alpha_r}([\mathbb{A}^1/\mathbb{G}_m]) \arrow[d] \\ \mathcal{Q}_{g,n}(X,\beta) \arrow[r] & \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])\times\cdots\times\mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m]) \end{tikzcd}$$

Which fits into a diagram

$$\begin{tikzcd} \mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \arrow[r] \arrow[d] & \prod_{i=1}^r \mathfrak{M}^{\log}_{g,\alpha_i}([\mathbb{A}^1/\mathbb{G}_m]) \times_{\mathfrak{M}^r_{g,n}} \mathfrak{M}_{g,n} \arrow[r] \arrow[d] & \prod_{i=1}^r \mathfrak{M}^{\log}_{g,\alpha_i}([\mathbb{A}^1/\mathbb{G}_m]) \arrow[d] \\ \mathcal{Q}_{g,n}(X,\beta) \arrow[r] & \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])^r \times_{\mathfrak{M}^r_{g,n}} \mathfrak{M}_{g,n} \arrow[r] \arrow[d] & \mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])^r \arrow[d] \\ & \mathfrak{M}_{g,n} \arrow[r] & \mathfrak{M}^r_{g,n} \end{tikzcd}$$

From which we can conclude that the top left commutative square is cartesian. On the other hand we have that $\mathfrak{M}_{g,n}([\mathbb{A}^1/\mathbb{G}_m])^r \times_{\mathfrak{M}^r_{g,n}} \mathfrak{M}_{g,n} = \mathfrak{M}_{g,n}([\mathbb{A}^r/\mathbb{G}^r_m])$ and $\prod_{i=1}^r \mathfrak{M}^{\log}_{g,\alpha_i}([\mathbb{A}^1/\mathbb{G}_m]) \times_{\mathfrak{M}^r_{g,n}} \mathfrak{M}_{g,n} = \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}^r_m])$.

Remark 3.3.21 *The entire construction could have equally been used to build a moduli space parametrising logarithmic ϵ-quasimaps for any $\epsilon \in \mathbb{Q}_{>0}$. This would just involve replacing $\mathcal{Q}_{g,n}(X,\beta)$ with $\mathcal{Q}^{\epsilon}_{g,n}(X,\beta)$ in the definitions. Moreover, the construction of the virtual fundamental class 3.5 also works for any ϵ. We will not make use of this here but it will be important in any future application for wall-crossing 3.8.5.*

3.4 Examples

Having built these moduli spaces, we now examine their geometry in a few simple examples and compare them with the corresponding moduli spaces in Gromov–Witten theory. We will examine a smooth divisor example, a simple normal crossings divisor example and a higher genus example. The first example is strictly speaking covered by [BN21], but we include it here anyway to show that even in the cases where the moduli space is as nice as possible, the quasimap moduli spaces are less complex.

Example 3.4.1 *We begin with the simplest case. Let $X = \mathbb{P}^N$, let $D = H$ a hyperplane and suppose we are in genus 0, degree d. We let $n = 2$ be the number of markings and let $\alpha = (d, 0)$ i.e. we are considering degree d (quasi)maps to $\mathbb{P}^N$ from rational curves with maximal tangency to the hyperplane H at the first marking. We will explain the difference between $\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H, d)$ and $\overline{\mathcal{M}}^{\log}_{0,(d,0)}(\mathbb{P}^N|H, d)$.*

$\overline{\mathcal{M}}^{\log}_{0,(d,0)}(\mathbb{P}^N|H, d)$ is irreducible of the expected dimension and we will see in 3.6.7 that $\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H, d)$ is irreducible of the same dimension for the same reason. More concretely, both spaces (or rather their images in $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^N, d)$ resp. $\mathcal{Q}_{0,2}(\mathbb{P}^N, d)$) are the closures of the locus of <u>maps</u>

$$(\mathbb{P}^1, p_1, p_2) \to \mathbb{P}^2$$

which are not mapped entirely into the hyperplane and hit H with maximal tangency at p_1. Comparing these spaces with their images in the corresponding absolute spaces is a reasonable thing to do as the maps here are finite and generically injective. On the other hand, the quasimap spaces do not allow rational tails i.e. curves with rational components with a single special point. The consequence of this is that the closure in the quasimap space is 'smaller' than in the stable map space. We can see this directly on $\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H, d)$ and $\overline{\mathcal{M}}^{\log}_{0,(d,0)}(\mathbb{P}^N|H, d)$ by enumerating their boundary divisors in each case. In both cases the boundary divisors are comb loci*. These are loci where the source curve is of the form $C_0 \cup \cdots \cup C_r$ with C_i smooth, $C_i \cap C_0$ is a single nodal point $i \neq 0$ and $C_i \cap C_j = \emptyset$ for $i, j \neq 0$. Moreover C_0 contains the non-trivial tangency marking and gets mapped entirely into H, whereas C_i only hits H at the connecting node for $i \neq 0$.*

A comb locus in the quasimap space can have at most two components for stability reasons. There is a divisor corresponding to when $C = C_0$ which falls entirely into the hyperplane and d distinct comb loci with two components $C = C_0 \cup C_1$ corresponding to the different possible degrees on each branch. However in the Gromov-Witten space there can be up to $d + 1$ components on the source curve of a comb locus. Each external component C_i, $i \neq 0$ must have positive degree and so there are comb loci where each of the d external components have degree 1 and the interior component carrying the maximal tangency marking is contracted. Below we include the number of boundary divisors in the quasimap and Gromov-Witten case to make the point that the quasimap spaces are much simpler.

d	$\mathcal{Q}^{\log}_{0,(2,0)}(\mathbb{P}^N\|H, d)$	$\overline{\mathcal{M}}^{\log}_{0,(2,0)}(\mathbb{P}^N\|H, d)$
1	2	3
2	3	7
3	4	14
4	5	26
5	6	45
6	7	75

Example 3.4.2 *We now move on to a simple normal crossings example. In the stable maps case this example comes from [NR22, 1.2] and the ideas come from [Nab19, 3]. Let $X = \mathbb{P}^2$ and $D = H_1 + H_2$ be the union of two hyperplanes. Let*

$g = 0, d = 2$ and let α be given by the matrix

$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

In other words we are considering degree 2 (quasi)maps to $\mathbb{P}^2$ with maximal tangency to H_1 at the first marking and maximal tangency to H_2 at the second marking. In this case we could compare $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ and $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ like in the previous example, and we would get a similar comparison. Namely, both spaces are irreducible by essentially the same reasoning as the previous example but they have differing numbers of boundary strata. Instead we will make a different comparison that requires some explanation. Recall that $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ is defined via the fibre product

$$\begin{tikzcd}
\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2) \arrow[r] \arrow[d] & \mathcal{Q}^{\log}_{0,\alpha_2}(\mathbb{P}^2|H_2, 2) \arrow[d] \\
\mathcal{Q}^{\log}_{0,\alpha_1}(\mathbb{P}^2|H_1, 2) \arrow[r] & \mathcal{Q}_{0,2}(\mathbb{P}^2, 2)
\end{tikzcd}$$

where $\alpha_1 = (2, 0)$ and $\alpha_2 = (0, 2)$. This is the same as 3.3.16, we don't need to use the diagonal because we only have two divisors. Crucially, this fibre product is taken in the category of fine and saturated logarithmic stacks. This is not in general the same as the fibre product of the underlying stacks. The same fact is true in the case of stable maps, that $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ can be *defined as the fibre product (in the fine and saturated category).*

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2) \arrow[r] \arrow[d] & \overline{\mathcal{M}}^{\log}_{0,\alpha_2}(\mathbb{P}^2|H_2, 2) \arrow[d] \\
\overline{\mathcal{M}}^{\log}_{0,\alpha_1}(\mathbb{P}^2|H_1, 2) \arrow[r] & \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2)
\end{tikzcd}$$

Recall from the previous example that $\overline{\mathcal{M}}^{\log}_{0,\alpha_i}(\mathbb{P}^2|H_i, 2) \to \overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2)$ is finite and generically injective. So we will momentarily confuse $\overline{\mathcal{M}}^{\log}_{0,\alpha_i}(\mathbb{P}^2|H_i, 2)$ with its image in $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2)$. The ordinary fibre product would be the intersection of these two spaces. Recall also that these spaces were just the closure of the loci corresponding to maps from irreducible curves which don't map into H_i. So the ordinary fibre product is something like the intersection of the closure of these loci. On the other hand $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ is also irreducible and maps finitely and generically injectively into $\overline{\mathcal{M}}_{0,2}(\mathbb{P}^2, 2)$. This image is also the closure of the locus of maps from irreducible curves which don't map entirely to either hyperplane. So the difference between the ordinary fibre product and $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ is like the difference between the intersection of the closures and the closure of the intersection. In this case the ordinary fibre product is strictly larger than $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$. First, by a dimension count we know that $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ is 3-dimensional. On the other hand there is a locus in the ordinary fibre product corresponding to source curves of the form $C_0 \cup C_1 \cup C_2$, where C_0 is attached to C_1 and C_2 at nodes, both markings lie on C_0 and this component gets contracted to the point of intersection in $H_1 \cap H_2$. This locus has dimension 3 and forms the only other irreducible component of the ordinary fibre product. So in this case $\overline{\mathcal{M}}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ is not the same as the ordinary fibre product.

We could try to make the same comparison in the quasimap case between $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ and the ordinary fibre product of $\mathcal{Q}^{\log}_{0,\alpha_1}(\mathbb{P}^2|H_1, 2)$ and $\mathcal{Q}^{\log}_{0,\alpha_1}(\mathbb{P}^2|H_2, 2)$ over $\mathcal{Q}_{0,2}(\mathbb{P}^2, 2)$. The dimension counts are all identical but due to the quasimap stability condition, even in the ordinary fibre product there can be no component with source curve $C_0 \cup C_1 \cup C_2$ with both markings on C_0, because C_1 and C_2 contain only a single special point. The consequence is that unlike in the stable maps case, $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^2|H_1 + H_2, 2)$ is the same underlying space as the ordinary fibre product.

One interesting reason for pointing out this difference is that the ordinary fibre product can be equipped with a 'virtual' class leading to invariants. These invariants are equal to the orbifold invariants of the multi root stack given by rooting $\mathbb{P}^2$ along H_1 and H_2. The orbifold Gromov–Witten invariants of a root stack are known to coincide in genus-zero with the logarithmic Gromov–Witten invariants when the divisor is smooth but differ when the divisor is simple normal crossings. In the case above the difference can be attributed to the excess component in the ordinary fibre product. On the other hand there is no difference in the quasimap setting, which suggests that the relationship between logarithmic and orbifold quasimap invariants may be less pronounced. The same phenomenon is present in the local-logarithmic correspondence (the orbifold invariants and local invariants coincide where this makes sense) and we will explore this in 3.7.

Example 3.4.3 *Let $X = \mathbb{P}^N$ and $D = H_0 + \cdots + H_N$ be the full toric boundary, where H_i denotes the i^{th} coordinate hyperplane. Then for any valid discrete data g, n, d, α we have that $\overline{\mathcal{M}}^{\log}_{g,\alpha}(\mathbb{P}^N|D, d) = \mathcal{Q}^{\log}_{g,\alpha}(\mathbb{P}^N|D, d)$. Stable maps without rational tails are stable quasimaps, and stable quasimaps without basepoints are stable maps, so it suffices to show that neither of these are possible here. Since any non-constant morphism from a rational curve must hit the boundary in at least two points we must have that this rational component contains at least two special points. On the other hand, any curve component of a logarithmic stable quasimap which is external to one of the H_i cannot have basepoints, because if it did, that basepoint would necessarily have to be at a point of intersection with H_i, but these points are either marked or nodes. Since $\cap_{i=0}^N H_i = \emptyset$, any curve component has to be external to at least one H_i, so no such quasimap can contain basepoints.*

3.5 Logarithmic Quasimap Invariants

In some of our examples from the previous section we found that the moduli space $\mathcal{Q}^{\log}_{g,\alpha}(X|D, \beta)$ was irreducible, and so has an actual fundamental class, in general this is not true. In order to define invariants we will need a substitute for the fundamental class. In this section we will sketch the construction of a *virtual fundamental class* on this moduli space of the expected dimension using the formalism of Behrend and Fantechi [BF97]. Precisely, we provide a morphism to the relative cotangent complex of the morphism $\mathcal{Q}^{\log}_{g,\alpha}(X|D, \beta) \to \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r])$ which we prove defines an obstruction theory subject to a lifting criterion of logarithmic morphisms in terms of logarithmic cotangent complexes generalising to certain algebraic stacks 3.5.9. We then show that this complex is of perfect amplitude contained in $[-1, 0]$. For more on virtual classes see 3.9.

Recall from 3.1.2 that if X is a smooth projective toric variety associated to a complete fan Σ, inducing a GIT presentation $X = \mathbb{A}^M // \mathbb{G}_m^s$, a stable toric quasimap to X is a morphism

$$C \to [\mathbb{A}^M/\mathbb{G}_m^s]$$

satisfying the non-degeneracy and stability conditions.

Lemma 3.5.1 *The divisor D induces a logarithmic structure on the quotient stack $[\mathbb{A}^M/\mathbb{G}_m^s]$ pulled back from the toric logarithmic structure on $[\mathbb{A}^r/\mathbb{G}_m^r]$. Moreover, a logarithmic quasimap to $X|D$ from a curve C, induces a logarithmic morphism $C \to [\mathbb{A}^M/\mathbb{G}_m^s]$.*

Proof 3.5.2 *Taking the closure of D inside $[\mathbb{A}^M/\mathbb{G}_m^s]$ induces a logarithmic structure defined by a morphism*

$$[\mathbb{A}^M/\mathbb{G}_m^s] \to [\mathbb{A}^r/\mathbb{G}_m^r].$$

Moreover, this morphism is strict, so the map from the underlying curve of C to $[\mathbb{A}^M/\mathbb{G}_m^s]$ together with a logarithmic morphism

$$C \to [\mathbb{A}^r/\mathbb{G}_m^r]$$

defines a logarithmic morphism to $[\mathbb{A}^M/\mathbb{G}_m^s]$.

Therefore we have a diagram

$$\begin{tikzcd}
\mathcal{C}_{\mathcal{Q}} \arrow[r] \arrow[d, "\pi"'] \arrow[rr, bend left, "u"] & \mathcal{C}_{\mathfrak{M}} \arrow[d] & {[\mathbb{A}^M/\mathbb{G}_m^s]} \\
\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta) \arrow[r] & \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r]) &
\end{tikzcd}$$

Here $\mathcal{C}_{\mathcal{Q}}$ and $\mathcal{C}_{\mathfrak{M}}$ are the universal curves over the moduli spaces $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ and $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r])$ and u is the universal map defined by 3.5.1. Furthermore, the square is cartesian in both the fine, saturated and the ordinary category.

In [Ols03] the author introduces the stack of logarithmic structures. Namely, if $(S,\mathcal{M}_S)$ is a fine logarithmic scheme, then there is a fibered category $\mathrm{Log}_{(S,\mathcal{M}_S)}$, where the objects over a morphism of schemes $T \to S$ is an enhancement of this morphism to fine logarithmic schemes $(T,\mathcal{M}_T) \to (S,\mathcal{M}_S)$. The main result of [Ols03] is that $\mathrm{Log}_{(S,\mathcal{M}_S)}$ is an algebraic stack, locally of finite presentation (although without the condition that the diagonal is separated and quasicompact). Using this, Olsson defines the logarithmic cotangent complex as follows.

Definition 3.5.3 *[Ols05] Let $X \to Y$ be a morphism of fine logarithmic schemes. Let $X \to \mathrm{Log}_Y$ be the induced morphism. Define $\mathbb{L}^{\log}_{X/Y}$, the* logarithmic cotangent complex*, to be the ordinary cotangent complex of this morphism.*

We will need an extension of this theory to certain algebraic stacks. In [GS13], the authors did this for Deligne–Mumford stacks by reducing to an étale cover, but instead we repeat the definition to this slightly more general context.

Let $\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s]}$ denote the *logarithmic cotangent complex* of $[\mathbb{A}^M/\mathbb{G}_m^s]$. This is defined as the ordinary cotangent complex of the morphism to the stack of logarithmic structures

$$[\mathbb{A}^M/\mathbb{G}_m^s] \to \mathrm{Log}_{\mathbb{C}}.$$

Remark 3.5.4 *The morphism $[\mathbb{A}^M/\mathbb{G}_m^s] \to \mathrm{Log}_{\mathbb{C}}$ factors through the morphism $[\mathbb{A}^M/\mathbb{G}_m^s] \to [\mathbb{A}^r/\mathbb{G}_m^r]$. Moreover, since $[\mathbb{A}^r/\mathbb{G}_m^r]$ is logarithmically étale, the logarithmic cotangent complex is equal to the cotangent complex of*

$$[\mathbb{A}^M/\mathbb{G}_m^s] \to [\mathbb{A}^r/\mathbb{G}_m^r].$$

Lemma 3.5.5 *There is a morphism in the derived category of $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$*

$$\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \otimes \omega_\pi) \to \mathbb{L}_{\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)/\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r]}.$$

Proof 3.5.6 *By the functoriality properties of [Ols05], we get a morphism in the derived category*

$$\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]} \to \mathbb{L}^{\log}_{\mathcal{C}_{\mathcal{Q}}/\mathcal{C}_{\mathfrak{M}}}$$

The fact that the morphism $\mathcal{C}_{\mathcal{Q}} \to \mathcal{C}_{\mathfrak{M}}$ is strict tells us that the logarithmic cotangent complex is actually the ordinary cotangent complex. Combining this with base change properties of the cotangent complex [LMB00, 17.3 (4)] we have that

$$\mathbb{L}^{\log}_{\mathcal{C}_{\mathcal{Q}}/\mathcal{C}_{\mathfrak{M}}} \cong \mathbb{L}_{\mathcal{C}_{\mathcal{Q}}/\mathcal{C}_{\mathfrak{M}}} \cong \mathbf{L}\pi^*\mathbb{L}_{\mathcal{Q}/\mathfrak{M}}$$

where $\mathcal{Q} = \mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ and $\mathfrak{M} = \mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$. Tensoring this morphism with the relative dualising sheaf gives

$$\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]} \otimes \omega_\pi \to \mathbf{L}\pi^*\mathbb{L}_{\mathcal{Q}/\mathfrak{M}} \otimes \omega_\pi$$

so, by applying $\mathbf{R}\pi_*$ and using adjunction, we get a morphism in the derived category

$$\phi : \mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]} \otimes \omega_\pi) \to \mathbb{L}_{\mathcal{Q}/\mathfrak{M}}$$

Now we need to show two things:

1. ϕ defines an obstruction theory

2. $\mathcal{E}^\bullet := \mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]} \otimes \omega_\pi)$ is of perfect amplitude contained in $[-1,0]$

Proposition 3.5.7 *The morphism ϕ defines an obstruction theory (under the condition of Remark 3.5.9).*

Proof 3.5.8 *This argument follows exactly as in [GS13, Proposition 5.1]. Let $T \hookrightarrow \bar{T}$ be a square zero extension with ideal J and let $g : T \to \mathcal{Q}$ be a morphism. Endow T and $\bar{T}$ with logarithmic structures pulled back from $\mathfrak{M}$ and pullback the universal curve to T and $\bar{T}$, which we will denote $\mathcal{C}_T$ and $\mathcal{C}_{\bar{T}}$. We have a commutative diagram*

$$\begin{tikzcd}
& \mathcal{C}_T \arrow[rr, "\tilde{g}"] \arrow[dl, "p"'] \arrow[dd] \arrow[rrrr, bend left, "u_T"] & & \mathcal{C}_{\mathcal{Q}} \arrow[rr, "u"] \arrow[dl, "\pi"'] \arrow[dd] & & {[\mathbb{A}^M/\mathbb{G}^s_m]} \\
T \arrow[rr, "g"] \arrow[dd] & & \mathcal{Q} \arrow[dd] & & & \\
& \mathcal{C}_{\bar{T}} \arrow[rr] \arrow[dl] & & \mathcal{C}_{\mathfrak{M}} \arrow[dl] & & \\
\bar{T} \arrow[rr] & & \mathfrak{M} & & &
\end{tikzcd} \quad . \tag{3.4}$$

Recall from [Ill71, III 2.2.4] and [Ols06, 2.21] that g extends to a morphism $\bar{T} \to \mathcal{Q}$ if and only if $\omega(g) \in$ $\mathrm{Ext}^1(\mathbf{L}g^*\mathbb{L}_{\mathcal{Q}/\mathfrak{M}}, J)$ *is 0 where $\omega(g)$ is defined by*

$$Lg^*\mathbb{L}_{\mathcal{Q}/\mathfrak{M}} \to \mathbb{L}_{T/\bar{T}} \to \tau_{\geq -1}\mathbb{L}_{T/\bar{T}} = J[1]$$

To show that ϕ defines an obstruction theory we use [BF97, Proposition 4.53]. We show that an extension exists if and only if $\phi^*\omega(g) = 0$ and moreover if an extension exists, the set of isomorphism classes of extensions form a torsor under $\mathrm{Hom}(\phi^*\mathcal{E}^\bullet, J)$*. As in [GS13, Proposition 5.1], a lift exists if and only if u_T extends logarithmically to $\mathcal{C}_{\bar{T}}$. But using*

[Ols05, Theorem 8.53] there is a class $o \in \mathrm{Ext}^1(\mathbf{L}u_T^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}, p^*J)$ which vanishes if and only if there is a lift (see Remark 3.5.9). Moreover the lifts form a torsor under $\mathrm{Hom}(\mathbf{L}u_T^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}, p^*J)$. However, just as in [GS13] we have

$$\begin{aligned}
&\mathrm{Ext}^k(\mathbf{L}g^*\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \otimes \omega_\pi), J) \\
&= \mathrm{Ext}^k(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \otimes \omega_\pi, \mathbf{L}\pi^!\mathbf{R}g_*J) \\
&= \mathrm{Ext}^k(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}, \mathbf{R}\tilde{g}_*p^*J) \\
&= \mathrm{Ext}^k(\mathbf{L}u_T^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}, p^*J)
\end{aligned}$$

Which when $k = 1$ sends $\phi^*\omega(g)$ to the obstruction class o.

Remark 3.5.9 *[Ols05, Theorem 8.53] applies to logarithmic schemes, whereas we require this to generalise to the algebraic stack $[\mathbb{A}^M/\mathbb{G}_m^s]$, but we expect this to be standard.*

Proposition 3.5.10 $\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \otimes \omega_\pi)$ *is of perfect amplitude contained in $[-1, 0]$*

The divisor $D = D_1 + \cdots + D_r$ on X is defined by line bundle-section pairs $(L_1, s_1), \ldots, (L_r, s_r)$. In the discussion preceding 3.1.11 it was explained that each s_i can be written as $s_i = \sum_{\underline{b}^i = (b^i_\rho)_\rho} c_{\underline{b}^i} \prod_\rho x_\rho^{b^i_\rho}$. We will denote this $s_i = s_i(x_1, \ldots, x_M)$. These expressions also define the morphism

$$\begin{aligned}
S : [\mathbb{A}^m/\mathbb{G}_m^s] &\to [\mathbb{A}^r/\mathbb{G}_m^r] \\
(x_1, \ldots, x_M) &\mapsto (s_1(x_1, \ldots, x_M), \ldots, s_r(x_1, \ldots, x_M))
\end{aligned}$$

There is a distinguished triangle in the derived category of $[\mathbb{A}^M/\mathbb{G}_m^s]$

$$\mathbf{L}S^*\mathbb{L}_{[\mathbb{A}^r/\mathbb{G}_m^r]} \to \mathbb{L}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbf{L}S^*\mathbb{L}_{[\mathbb{A}^r/\mathbb{G}_m^r]}[1].$$

The action of $\mathbb{G}_m^s$ on $\mathbb{A}^M$ is given by the weight matrix of the toric variety X which we will present as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1M} \\ \vdots & \ddots & \ddots & \vdots \\ a_{s1} & a_{s2} & \cdots & a_{sM} \end{pmatrix}$$

By [Beh05] the *tangent complex* of $[\mathbb{A}^M/\mathbb{G}_m^s]$ is given in degrees $[-1, 0]$ by

$$\begin{aligned}
\mathcal{O}_{\mathbb{A}^M}^{\oplus s} &\to \mathcal{O}_{\mathbb{A}^M}^{\oplus M} \\
e_i &\mapsto (a_{i1}x_1, \ldots, a_{iM}x_M).
\end{aligned}$$

These are regarded as morphisms of equivariant sheaves on $\mathbb{A}^M$, where the action is trivial on the source and given by the weight matrix on the target.

The same reasoning tells us that the tangent complex of $[\mathbb{A}^r/\mathbb{G}_m^r]$ is given in degrees $[-1, 0]$ by

$$\begin{aligned}
\mathcal{O}_{\mathbb{A}^r}^{\oplus r} &\to \mathcal{O}_{\mathbb{A}^r}^{\oplus r} \\
e_i &\mapsto (0, \ldots, 0, x_i, 0, \ldots, 0).
\end{aligned}$$

Pulling this back to $[\mathbb{A}^M/\mathbb{G}_m^s]$ gives the complex

$$\begin{aligned}\mathcal{O}_{\mathbb{A}^M}^{\oplus r} &\to \mathcal{O}_{\mathbb{A}^M}^{\oplus r}\\ e_i &\mapsto (0,\ldots,s_i(x_1,\ldots,x_M),\ldots,0).\end{aligned}$$

Since both are complexes of locally free sheaves, dual to the morphism between cotangent complexes there is a morphism $\mathbb{T}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbf{L}S^*\mathbb{T}_{[\mathbb{A}^r/\mathbb{G}_m^r]}$.

Lemma 3.5.11 *This morphism $\mathbb{T}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbf{L}S^*\mathbb{T}_{[\mathbb{A}^r/\mathbb{G}_m^r]}$ is given by the following commutative diagram*

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{A}^M}^{\oplus s} \arrow[r, bend left, "e_i\mapsto(a_{i,1}x_1,\ldots,a_{i,M}x_M)"] \arrow[d, "e_i\mapsto(\deg_i s_j)_j"'] & \mathcal{O}_{\mathbb{A}^M}^{\oplus M} \arrow[d, "e_i\mapsto\left(\frac{\partial s_j}{\partial x_i}\right)_j"] \\
\mathcal{O}_{\mathbb{A}^M}^{\oplus r} \arrow[r, bend right, "e_i\mapsto s_i"'] & \mathcal{O}_{\mathbb{A}^M}^{\oplus r}
\end{tikzcd} \tag{3.5}$$

Proof 3.5.12 *The diagram is certainly commutative. This is equivalent to the statement that for any $i = 1,\ldots,s$ we have that for any $k = 1,\ldots,r$*

$$\deg_i(s_k)s_k = a_{i1}x_1\frac{\partial s_k}{\partial x_1} + \cdots + a_{iM}x_M\frac{\partial s_k}{\partial x_k}$$

where $\deg_i(s_k)$ denotes the homogeneous degree of s_k with respect to the i^{th} torus factor. But this follows from differentiating both sides of the equation

$$s(0,\ldots,\lambda,\ldots,0)\cdot(x_1,\ldots,x_M)) = \lambda^{\deg_i(s_k)}s(x_1,\ldots,x_M)$$

with respect to λ and setting $\lambda = 1$.

The morphism $[\mathbb{A}^M/\mathbb{G}_m^s] \to [\mathbb{A}^r/\mathbb{G}_m^r]$ is given by the data of a group homomorphism $\varphi : \mathbb{G}_m^s \to \mathbb{G}_m^r$, together with an equivariant morphism $\mathbb{A}^M \to \mathbb{A}^r$, given by $s_1,\ldots,s_r$. The group homomorphism is given by the degrees of the s_i, namely $(0,\ldots,\lambda_i,\ldots,0) \mapsto (\lambda_i^{\deg_i(s_1)}, \lambda_i^{\deg_i(s_2)},\ldots,\lambda_i^{\deg_i(s_r)})$. The differential of these maps give the vertical maps in 3.5.

Definition 3.5.13 *Define the* logarithmic tangent complex $\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$ *as the derived dual of $\mathbb{L}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$.*

Lemma 3.5.14 *The logarithmic tangent complex $\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$ has cohomology supported in $[0,1]$.*

Proof 3.5.15 *Taking the dual distinguished triangle*

$$\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbb{T}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbf{L}S^*\mathbb{T}_{[\mathbb{A}^r/\mathbb{G}_m^r]} \to \mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}[1] \tag{3.6}$$

tells us that we can build $\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$ as the shift of the mapping cone of $\mathbb{T}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbf{L}S^*\mathbb{T}_{[\mathbb{A}^r/\mathbb{G}_m^r]}$. Since the top horizontal morphism in 3.5 is injective it follows that the mapping cone is actually concentrated in $[-1,0]$ and so after shifting, $\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$ is concentrated in $[0,1]$.

By the description as in [CFK10, (5.2.2)] we can also conclude that $\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$ is also supported in $[0,1]$. Furthermore we also know that $\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{L}^{\log}_{[\mathbb{C}^M/(\mathbb{C}^*)^r]} \otimes \omega_\pi)$ is quasi-isomorphic to $\left(\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]})\right)^\vee$, see for example [FHM03, 4.1].

Proof 3.5.16 (of 3.5.10) *We need to show that* $\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]})$ *is of perfect amplitude contained in* $[0,1]$*. First we show that* $\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]})$ *is concentrated in degrees* $[0,1]$*. This will follow from the argument of [CFKM14, Theorem 4.5.2]. Let C be a geometric fibre of* $\mathcal{C}_\mathcal{Q}$*, corresponding to a geometric point* $i_p : \operatorname{Spec}\mathbb{C} \hookrightarrow \mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$*. Let* $F^\bullet$ *denote the restriction of* $\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]}$ *to C so that*

$$\mathbf{L}i_p^*\mathbf{R}\pi_*\left(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]}\right) \cong \mathbf{R}\Gamma(F^\bullet)$$

in the derived category. We show that $\mathbf{R}^2\Gamma(F^\bullet) = 0$*. We have that* $H^1(F^\bullet)$ *is a torsion sheaf. This follows from the fact that away from the set B of finitely many basepoints on C the morphism u factors through X. Moreover, the restriction* $F^\bullet|_{C\setminus B}$ *is quasi-isomorphic to* $\mathcal{T}^{\log}_X$*. We have the spectral sequence*

$$E_2^{p,q} = \mathbf{R}^p\Gamma(H^q(F^\bullet)) \Rightarrow \mathbf{R}^{p+q}\Gamma(F^\bullet) = E^{p+q}$$

The second page looks like

$$\begin{matrix} H^1(C, H^0(F^\bullet)) & \quad & H^1(C, H^1(F^\bullet)) \\ & & \\ H^0(C, H^0(F^\bullet)) & \quad & H^0(C, H^1(F^\bullet)) \end{matrix}$$

But by the above we know that $H^1(C, H^1(F^\bullet)) = 0$*. Consequently* $R^2\Gamma(F^\bullet) = 0$*.*

Using the criterion of [Ill05, §8.3.6.3], the complex $\mathbf{R}\pi_*(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]})$*, which is cohomologically supported in* $[0,1]$*, is of perfect amplitude contained in* $[0,1]$ *if and only if for every point p we have*

$$H^{-1}\left(\mathbf{L}i_p^*\mathbf{R}\pi_*\left(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]}\right)\right) = H^{-1}(\mathbf{R}\Gamma(F^\bullet)) = 0$$

But this is also clear by the above.

Theorem 3.5.17 *There is a perfect obstruction theory on* $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ *leading to a virtual fundamental class* $[\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)]^{\mathrm{vir}}$*.*

Proof 3.5.18 *Under the assumption of 3.5.9, then combining 3.5.7 with 3.5.10 we get a virtual fundamental class on* $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ *by [BF97].*

3.5.1 Toric Divisors

On the other hand 3.5.10 becomes significantly simpler in the case where $D = D_1 + \cdots + D_r$ is a subset of the toric boundary.

Lemma 3.5.19 *If* $D = D_1 + \cdots + D_r$ *is a subset of the toric boundary corresponding to rays* $\rho_{j_1}, \ldots, \rho_{j_r} \in \Sigma(1)$ *(set* $J = \{j_1, \ldots, j_r\}$*), then there is a* logarithmic Euler sequence

$$0 \to \mathcal{O}_X^{\oplus s} \to \bigoplus_{\Sigma(1)\setminus\rho_J} \mathcal{O}_X(D_\rho) \bigoplus_J \mathcal{O}_X \to \mathcal{T}^{\log}_X \to 0 \tag{3.7}$$

Moreover, in this situation $\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]}$ *is quasi-isomorphic to a sheaf.*

Proof 3.5.20 *Consider again the diagram 3.5. In this case* $s_1, \ldots s_r$ *are just* $x_{j_1}, \ldots, x_{j_r}$ *corresponding to the homogeneous coordinates on X. But since the right-hand vertical map is given by differentiating the sections this map becomes* $e_i \mapsto$

$(0,\ldots,1,\ldots,0)$ *where* 1 *is in the* j_i^{th} *place. Consequently, by forming the mapping cone (and shifting) we have that* $\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]}$ *is given by the three term complex*

$$\mathcal{O}^{\oplus s}_{\mathbb{A}^M} \to \mathcal{O}^{\oplus M}_{\mathbb{A}^M} \oplus \mathcal{O}^{\oplus r}_{\mathbb{A}^M} \to \mathcal{O}^{\oplus r}_{\mathbb{A}^M}$$

But the right-hand map is now surjective so this complex is quasisomorphic to a two term complex

$$\mathcal{O}^{\oplus s}_{\mathbb{A}^M} \to \mathcal{O}^{\oplus M-r}_{\mathbb{A}^M} \oplus \mathcal{O}^{\oplus r}_{\mathbb{A}^M} \tag{3.8}$$

Where in the second term the first $M-r$ *copies have action according to the weight matrix and the latter* r *copies have the trivial action. Since the first map is injective the second part follows. For the first part we just pullback this complex under the inclusion* $X \hookrightarrow [\mathbb{A}^M/\mathbb{G}_m^s]$.

Corollary 3.5.21 *If* D *is a subset of the toric boundary then the complex* $(\mathbf{R}\pi_*\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}_m^s]})^\vee$ *is of perfect amplitude contained in* $[-1,0]$.

Corollary 3.5.22 *Under the assumption of 3.5.9, then combining 3.5.7 with 3.5.21 we get a virtual fundamental class on* $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$ *by [BF97].*

3.6 Comparison with Battistella–Nabijou Theory

In [BN21] the theory of relative quasimaps is developed in the case where X is a smooth projective toric variety, $D \subset X$ is a smooth, very ample divisor (not necessarily toric) and in genus-zero. This is achieved by mimicking the Gromov–Witten construction in [Gat01] and so the relative quasimap moduli space is defined as a closed substack of the space of absolute quasimaps.

Let X be a smooth projective toric variety associated to a complete fan Σ and D a smooth (very ample) divisor, cut out by a section $s_D \in H^0(X, \mathcal{O}_X(D))$. Recall that given an ordinary quasimap from a (marked) curve C to X recall there is an *induced* line bundle section pair (L_D, u_D) and in the case where there are no basepoints these are just the pullbacks of $\mathcal{O}_X(D)$ and s_D respectively along the map $C \to X$.

Definition 3.6.1 *Let* $n \geq 2$ *be the number of marked points, let* β *be an effective curve class and* $\alpha = (\alpha_1,\ldots,\alpha_n)$ *such that* $\sum_i \alpha_i \leq D\cdot\beta$. *Define the moduli space of relative quasimaps* $\mathcal{Q}^{\text{rel}}_{0,\alpha}(X|D,\beta)$ *to be the locus of absolute quasimaps* $\Big((C,p_1,\ldots,p_n), \{L_\rho,u_\rho\}_{\rho\in\Sigma(1)}, \{c_m\}_{m\in M}\Big)$ *in* $\mathcal{Q}_{0,n}(X,\beta)$ *such that for every* Z*, a connected component of* $u_D^{-1}(0)$ *we have*

1. *If* Z *is a point, then either it is unmarked or a marked point* p_i *such that the multiplicity of* f *at* p_i *is at least* α_i
2. *If* Z *is one dimensional let* $C^{(i)}$ *for* $1 \leq i \leq r$ *be the irreducible components of* C *not in* Z*, but intersecting* Z*, and let* $m^{(i)}$ *be the multiplicity of* u_D *at the node* $C^{(i)} \cap Z$ *along* D*. Then we must have* $\deg L_D|_Z + \sum\limits_i m^{(i)} \geq \sum\limits_{i\in Z} \alpha_i$

Remark 3.6.2 *let* $X = \mathbb{P}^N$ *and let* $D = H \cong \mathbb{P}^{N-1}$ *be the hyperplane given in coordinates by* $\{x_0 = 0\}$*. Then* $\mathcal{Q}^{\text{rel}}_{0,\alpha}(\mathbb{P}^N|H,d)$ *is irreducible of dimension* $\dim \mathcal{Q}_{0,n}(\mathbb{P}^N,d) - \sum\limits_i \alpha_i$ *and so has an actually fundamental class with which one can define relative quasimap invariants. For the general case of* (X,D) *note that* $\mathcal{O}_X(D)$ *defines a map* $i : X \hookrightarrow \mathbb{P}^N$*. Battistella and Nabijou show that the following diagram is cartesian (with* $d = i_*\beta$*)*

$$\begin{tikzcd}
\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D,\beta) \arrow[r] \arrow[d, hook] & \mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(\mathbb{P}^N|H,d) \arrow[d, hook] \\
\mathcal{Q}_{0,n}(X,\beta) \arrow[r] & \mathcal{Q}_{0,n}(\mathbb{P}^N,d)
\end{tikzcd}$$

then they use diagonal pull-back to define a virtual fundamental class on $\mathcal{Q}(X|Y,\beta)$ and define relative quasimap invariants.

Remark 3.6.3 *From now on we restrict to the case where $\sum_i \alpha_i = D\cdot\beta$ in which case the inequalities above become equalities.*

Theorem 3.6.4 *Let $g=0$ and $D\subset X$ be a smooth, very ample divisor inside a smooth projective toric variety. There is a morphism $g:\mathcal{Q}^{\log}_{0,\alpha}(X|D,\beta)\to\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D,\beta)$ and*

$$g_*[\mathcal{Q}^{\log}_{0,\alpha}(X|D,\beta)]^{\mathrm{vir}} = [\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D,\beta)]^{\mathrm{vir}}.$$

We will only sketch the proof, but the strategy for proving theorem 3.6.4 is as follows. We will first prove the existence of the morphism g. Then we prove theorem 3.6.4 for the case of $X=\mathbb{P}^N$ and $D=H$ a hyperplane. We will then use the ampleness condition to pullback this result to the general case.

Proposition 3.6.5 *The natural morphism $\mathcal{Q}^{\log}_{0,\alpha}(X|D,\beta)\to\mathcal{Q}_{0,n}(X,\beta)$ given by forgetting the logarithmic map to $[\mathbb{A}^1/\mathbb{G}_m]$ factors through the inclusion $\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D,\beta)\hookrightarrow\mathcal{Q}_{0,n}(X,\beta)$.*

Proof 3.6.6 *Certainly on the locus where $C\cong\mathbb{P}^1$ is irreducible and the section u_0 vanishes only at the marked points, this is true. Suppose now we have logarithmic quasimap which contains a connected component $Z\subset C$ on which $u_0\equiv 0$. Then we need to show that*

$$\deg L|_Z + \sum_i m^{(i)} = \sum_{i\in Z}\alpha_i$$

We have a logarithmic morphism $C\to[\mathbb{A}^1/\mathbb{G}_m]$. This induces a morphism of the tropicalisations. As in toric geometry a piecewise linear function on the tropicalisation induces a Cartier divisor. Alternatively, a logarithmic structure can be characterised by an association of a Cartier divisor to each element of the ghost sheaf. The identity function on $\mathbb{R}_{\geq 0}$ induces the divisor $B\mathbb{G}_m$. The fact that we have a logarithmic morphism necessarily implies that the pull back of this line bundle via the morphism is the same as the line bundle associated to the pull-back piecewise linear function. On the one hand the line bundle pulls back to L, when restricting to Z we get $\deg L|_Z$. On the other hand the pull back piecewise linear function is defined by the slopes of the tropicalisation map on each ray. The associated line bundle on the component Z is known to be $\mathcal{O}(\sum_{i\in Z}\alpha_i p_i - \sum_i m^{(i)} q_i)$ where q_i are the nodes. Since these line bundles are isomorphic, taking degrees gives us the desired equality.

Lemma 3.6.7 *Suppose $(X|D)=(\mathbb{P}^N|H)$ where H is the hyperplane defined (in coordinates) by $\{x_0=0\}$. Then $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H,d)$ is irreducible of the expected dimension $N\cdot(d+1)+n-3$.*

Proof 3.6.8 *Recall the obstruction theory on $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H,d)$ is defined by the complex $\left(\mathbf{R}\pi_*\left(\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^{N+1}/\mathbb{G}_m]}\right)\right)^\vee$ Because $\mathbb{P}^N|H$ admits a logarithmic Euler sequence 3.7*

$$0\to\mathcal{O}_{\mathbb{P}^N}\to\mathcal{O}_{\mathbb{P}^N}\oplus\bigoplus_{i=1}^N\mathcal{O}_{\mathbb{P}^N}(1)\to\mathcal{T}^{\log}_{\mathbb{P}^N}\to 0$$

It follows from 3.5.19 that $\mathbf{L}u^*\mathbb{T}^{\log}_{[\mathbb{A}^{N+1}/\mathbb{G}_m]}$ is quasi-isomorphic to a sheaf $\mathcal{F}$ which fits into an exact sequence on $\mathcal{C}$

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathcal{O}_{\mathcal{C}} \oplus \bigoplus_{i=1}^{N} \mathcal{L} \to \mathcal{F} \to 0$$

taking the long exact sequence in cohomology it follows that

$$\mathrm{H}^1(C, \mathcal{F}) = 0$$

and so this moduli space is unobstructed over $\mathfrak{M}^{\log}_{0,\alpha}([\mathbb{A}^1/\mathbb{G}_m])$, which is logarithmically smooth.

To distinguish from a general (X, D), denote the morphism $\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d) \to \mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(\mathbb{P}^N|H, d)$ by f.

Proposition 3.6.9

$$f_*[\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d)]^{\mathrm{vir}} = [\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d)]$$

Proof 3.6.10 *By 3.6.7 it follows that $[\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d)]^{\mathrm{vir}} = [\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d)]$. So the proposition reduces to a statement about* fundamental classes. *By 3.6.7 and [BN21] we know that the locus where the source curve is irreducible and the u_0 only vanishes at the marked points is dense in both spaces. Furthermore, on this locus the map is an isomorphism so the result follows.*

We now use proposition 3.6.9 to give sketch of the proof of theorem 3.6.4.

Lemma 3.6.11 *There is a cartesian diagram*

$$\begin{tikzcd}
\mathcal{Q}^{\log}_{0,\alpha}(X|D, \beta) \arrow[r, "i'"] \arrow[d, "g"] & \mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d) \arrow[d, "f"] \\
\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D, \beta) \arrow[r, "i"] & \mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(\mathbb{P}^N|H, d)
\end{tikzcd}$$

Proof 3.6.12 *This follows from the fact that there are cartesian diagrams*

$$\begin{tikzcd}
\mathcal{Q}^{\log}_{0,\alpha}(X|D, \beta) \arrow[r, "i'"] \arrow[d] & \mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H, d) \arrow[d] \\
\mathcal{Q}_{0,n}(X, \beta) \arrow[r] & \mathcal{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D, \beta) \arrow[r, "i"] \arrow[d] & \mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(\mathbb{P}^N|H, d) \arrow[d, "h"] \\
\mathcal{Q}_{0,n}(X, \beta) \arrow[r] & \mathcal{Q}_{0,n}(\mathbb{P}^N, d)
\end{tikzcd}$$

Proof 3.6.13 (Sketch proof of 3.6.4) *One can show that the morphism i' has a perfect obstruction theory which induces the virtual class on $\mathcal{Q}^{\log}_{0,\alpha}(X|D, \beta)$. This obstruction theory comes from the distinguished triangle*

$$\mathbb{T}_j \to \mathbb{T}^{\log}_{[\mathbb{A}^M/\mathbb{G}^s_m]} \to \mathbf{L}j^*\mathbb{T}^{\log}_{[\mathbb{A}^{N+1}/\mathbb{G}_m]} \to \mathbb{T}_j[1]$$

where $j : [\mathbb{A}^M/\mathbb{G}^s_m] \to [\mathbb{A}^{N+1}/\mathbb{G}_m]$ is the morphism extending the embedding $X \to \mathbb{P}^N$ coming from $\mathcal{O}_X(D)$. Pulling this back to the universal curve and pushing down to the moduli space gives a compatible triple in the sense of [Man12b, Definition 4.5], which tells that $(\mathbf{R}\pi_*\mathbf{L}u^*\mathbb{T}_j)^\vee$ defines a perfect obstruction theory for i'. On the other hand, the virtual class on $\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(X|D, \beta)$ is essentially defined via the natural perfect obstruction theory of the morphism $\mathcal{Q}_{0,n}(X, \beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N, d)$ (the authors show [BN21, Lemma A.0.1] that virtual and diagonal pullback coincide). The *perfect obstruction theory for $\mathcal{Q}_{0,n}(X,\beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N,d)$ is* also *given by pulling back $\mathbb{T}_j$ to the universal curve and pushing down to the moduli space. This can be seen from the distinguished triangle*

$$\mathbb{T}_j \to \mathbb{T}_{[\mathbb{A}^M/\mathbb{G}_m^s]} \to \mathbf{L}j^*\mathbb{T}_{[\mathbb{A}^{N+1}/\mathbb{G}_m]} \to \mathbb{T}_j[1]$$

This hints that the perfect obstruction theory for i' is pulled back from the perfect obstruction theory for $\mathcal{Q}_{0,n}(X,\beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N,d)$. If this were true then 3.6.4 would follow from

$$\begin{aligned}
&g_*[\mathcal{Q}^{\log}_{0,\alpha}(X|D,\beta)]^{\mathrm{vir}}\\
&= g_* i'^![\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H,d)]\\
&= i^! f_*[\mathcal{Q}^{\log}_{0,\alpha}(\mathbb{P}^N|H,d)]\\
&= i^![\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(\mathbb{P}^N|H,d)]\\
&= [\mathcal{Q}^{\mathrm{rel}}_{0,\alpha}(\mathbb{P}^N|H,d)]^{\mathrm{vir}}
\end{aligned}$$

by [Man12b, Proposition 5.29]. The complication is that it is not obvious that perfect obstruction theory of $\mathcal{Q}_{0,n}(X,\beta) \to \mathcal{Q}_{0,n}(\mathbb{P}^N,d)$ pulls back to the one for i'. It is not even clear how to show $\mathbf{L}g^*\mathbb{L}_i \cong \mathbb{L}_{i'}$ as g is not flat. However, we expect this to be true.

3.7 Local-Logarithmic Correspondence

In this section we will explore the local-logarithmic correspondence [vGGR19] in the quasimap setting. The key point will come from a generalisation of the observation from 3.4.2. In some sense the comparisons made in this chapter have little do with local quasimap theory, but rather of the difference between the logarithmic quasimap space for a reducible divisor D, and the logarithmic quasimap spaces for each of the components of D.

Let X be a smooth projective variety, D a smooth nef divisor and β a curve class on X with $d := D\cdot\beta > 0$. The local-logarithmic correspondence relates two sets of natural Gromov–Witten one can build from this data. We could form on the one hand $\overline{\mathcal{M}}^{\log}_{0,(d)}(X|D,\beta)$, the moduli space of stable logarithmic maps to $X|D$, with maximal tangency at the single marking. On the other hand we could form $\overline{\mathcal{M}}_{0,0}(\mathcal{O}_X(-D),\beta)$ the moduli space of stable maps to the total space of the line bundle (dual) associated to D. This moduli space is actually isomorphic to $\overline{\mathcal{M}}_{0,0}(X,\beta)$, but the virtual class differs by an obstruction bundle. The original local-logarithmic correspondence is the striking equality that

Theorem 3.7.1 *[vGGR19, Theorem 1.1]*

$$[\overline{\mathcal{M}}^{\log}_{0,(d)}(X|D,\beta)]^{\mathrm{vir}} = (-1)^{d+1}\cdot d\cdot[\overline{\mathcal{M}}_{0,0}(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}}$$

where the left-hand side is pushed forward to the absolute space. There is also a stronger form of the local-logarithmic correspondence which is the equality that

$$[\overline{\mathcal{M}}^{\log}_{0,(d)}(X|D,\beta)]^{\mathrm{vir}} = (-1)^{d+1}\cdot \mathrm{ev}^* D\cdot[\overline{\mathcal{M}}_{0,1}(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}}$$

which is equivalent, in this case, to the previous version. The authors also conjectured that if instead $D = D_1+\cdots+D_r$ is a simple normal crossings divisor, then there is a generalisation of this result.

Conjecture 3.7.2

$$[\overline{\mathcal{M}}^{\log}_{0,((d_1,\dots,d_r))}(X|D,\beta)]^{\mathrm{vir}} = \prod_{i=1}^{r}(-1)^{d_i+1}\cdot d_i \cdot [\overline{\mathcal{M}}_{0,0}(\oplus_{i=1}^{r}\mathcal{O}_X(-D_i),\beta)]^{\mathrm{vir}}$$

The corresponding strong form generalisation is the equality

$$[\overline{\mathcal{M}}^{\log}_{0,((d_1,\dots,d_r))}(X|D,\beta)]^{\mathrm{vir}} = \prod_{i=1}^{r}(-1)^{d_i+1}\cdot \mathrm{ev}_i^* D_i \cdot [\overline{\mathcal{M}}_{0,r}(\oplus_{i=1}^{r}\mathcal{O}_X(-D_i),\beta)]^{\mathrm{vir}}$$

Although this generalisation often holds numerically [BBvG22, BBvG21], at the level of virtual classes it is not true in complete generality, due to counterexamples in [NR22]. However in [NR22, Remark 5.4], the authors note that the difference appears when there are components in the moduli space containing rational tails and so may not be present in the quasimap setting.

To explore this relationship in the quasimap theory there is an unimportant technical point that in genus zero, quasimap spaces with less than two markings don't exist due to stability. On the other hand the local-logarithmic correspondence in Gromov–Witten theory holds if both sides carry redundant extra markings. This means the cleanest way to formulate the local-logarithmic correspondence in quasimap theory is using the strong form and adding an extra marking to both sides.

Conjecture 3.7.3

$$[\mathcal{Q}^{\log}_{0,(d,0)}(X|D,\beta)]^{\mathrm{vir}} = (-1)^{d+1}\cdot \mathrm{ev}^* D \cdot [\mathcal{Q}_{0,2}(\mathcal{O}_X(-D),\beta)]^{\mathrm{vir}}$$

The way one could try to prove this correspondence is by proving a degeneration formula for logarithmic quasimap invariants applying it to the degeneration given by taking the total space $\mathcal{O}(\mathcal{D})$, where $\mathcal{D}$ is the strict transform of $D \times \mathbb{A}^1$ in the deformation to the normal cone of X in D (assuming the quasimap theory of the pieces can be made sense of). This is the strategy used in [vGGR19].

For now we will just try to investigate this result in the simplest case where $X = \mathbb{P}^N$ and $D = H$ a hyperplane. In this case we can try to take advantage [MOP11, Theorem 3] which states that there is a comparison morphism $\chi : \overline{\mathcal{M}}_{0,2}(\mathbb{P}^N, d) \to \mathcal{Q}_{0,2}(\mathbb{P}^N, d)$ which is birational, i.e.

$$\chi_*[\overline{\mathcal{M}}_{0,2}(\mathbb{P}^N, d)] = [\mathcal{Q}_{0,2}(\mathbb{P}^N, d)]$$

A cheap way to verify the local-logarithmic correspondence for quasimaps for $\mathbb{P}^N|H$ would be to show that the Gromov–Witten invariants and quasimap invariants coincide in both logarithmic and local case. Then we could just appeal to the local-logarithmic correspondence for Gromov–Witten invariants. We already know that $\overline{\mathcal{M}}^{\log}_{0,(d,0)}(\mathbb{P}^N|H,d)$ and $\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H,d)$ are irreducible of the same dimension as they are unobstructed. Moreover, they are isomorphic on their interior so it follows that

$$\chi_*[\overline{\mathcal{M}}^{\log}_{0,(d,0)}(\mathbb{P}^N|H,d)] = [\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H,d)]$$

suppressing the pushforwards to the absolute spaces. So the logarithmic Gromov–Witten and quasimap invariants coincide in this case. On the other hand it is not as obvious that the local Gromov–Witten and quasimap invariants coincide even in this case. The local virtual classes are given by

$$[\overline{\mathcal{M}}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H),d)]^{\mathrm{vir}} = e(R^1\pi'_* f^*\mathcal{O}_{\mathbb{P}^N}(-H)) \cap [\overline{\mathcal{M}}_{0,2}(\mathbb{P}^N,d)]$$
$$[\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H),d)]^{\mathrm{vir}} = e(R^1\pi_*\mathcal{L}^\vee) \cap [\mathcal{Q}_{0,2}(\mathbb{P}^N,d)]$$

One would like to use a projection formula argument to deduce that

$$\chi_*[\overline{\mathcal{M}}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H),d]^{\mathrm{vir}} = [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H),d]^{\mathrm{vir}}$$

but the universal curve over stable maps space is *not* pulled back from the universal curve over the quasimap space. In other words the diagram

$$\begin{tikzcd} \mathcal{C}_{\overline{\mathcal{M}}} \arrow[r, "\chi_{\mathcal{C}}"] \arrow[d, "\pi'"] & \mathcal{C}_{\mathcal{Q}} \arrow[d, "\pi"] \\ \overline{\mathcal{M}}_{0,2}(\mathbb{P}^N,d) \arrow[r, "\chi"] & \mathcal{Q}_{0,2}(\mathbb{P}^N,d) \end{tikzcd}$$

is not cartesian. The comparison map will contract rational components with only a single special point.

For now, we will just assume the result to be true to open up techniques of rank reduction.

Proposition 3.7.4 *Let H_1, H_2 be hyperplanes in $\mathbb{P}^N$. Assume that 3.7.3 is true for $\mathbb{P}^N|H$. Then*

$$[\mathcal{Q}^{\log}_{0,((d,d))}(\mathbb{P}^N|H_1+H_2,d)]^{\mathrm{vir}} = (-1)^{2(d+1)}\prod_{i=1}^{2}\mathrm{ev}_i^*\,H_i\cdot[\mathcal{Q}_{0,2}(\oplus_{i=1}^2\mathcal{O}_{\mathbb{P}^N}(-H_i),d)]^{\mathrm{vir}}$$

Where the notation $((d,d))$ is the 2×2 matrix of contact orders

$$\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$$

Proof 3.7.5 *We are interpreting the local virtual class as*

$$[\mathcal{Q}_{0,2}(\oplus_{i=1}^2\mathcal{O}_{\mathbb{P}^N}(-H_i),d)]^{\mathrm{vir}} = e(R^1\pi_*(\mathcal{L}^\vee)^{\oplus 2})\cap[\mathcal{Q}_{0,2}(\mathbb{P}^N,d)]$$

But because of the Whitney sum formula, and the fact that pushforward commutes with direct sum, we have that

$$e(R^1\pi_*(\mathcal{L}^\vee)^{\oplus 2})\cap[\mathcal{Q}_{0,2}(\mathbb{P}^N,d)] = \prod_{i=1}^{2}e(R^1\pi_*\mathcal{L}^\vee)\cap[\mathcal{Q}_{0,2}(\mathbb{P}^N,d)] = \prod_{i=1}^{2}[\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H_i),d)]^{\mathrm{vir}}$$

So by assumption

$$[\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H_1,d)]\cdot[\mathcal{Q}^{\log}_{0,(0,d)}(\mathbb{P}^N|H_2,d)] = (-1)^{2(d+1)}\prod_{i=1}^{2}\mathrm{ev}_i^*\,H_i\cdot[\mathcal{Q}_{0,2}(\oplus_{i=1}^2\mathcal{O}_{\mathbb{P}^N}(-H_i),d)]^{\mathrm{vir}}$$

Therefore it suffices to show that

$$[\mathcal{Q}^{\log}_{0,((d,d))}(\mathbb{P}^N|H_1+H_2,d)]^{\mathrm{vir}} = [\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H_1,d)]\cdot[\mathcal{Q}^{\log}_{0,(0,d)}(\mathbb{P}^N|H_2,d)]$$

The moduli space $\mathcal{Q}^{\log}_{0,((d,d))}(\mathbb{P}^N|H_1+H_2,d)$ is also unobstructed so it is irreducible of the expected dimension, and so the proposition is just about fundamental classes. The point is that there is a cartesian diagram (in the category of ordinary stacks)

$$\begin{tikzcd} \mathcal{Q}^{\log}_{((d,d))}(\mathbb{P}^N|H_1+H_2,d) \arrow[r] \arrow[d] & \mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H_1,d) \arrow[d] \\ \mathcal{Q}^{\log}_{0,(0,d)}(\mathbb{P}^N|H_2,d) \arrow[r] & \mathcal{Q}_{0,2}(\mathbb{P}^N,d) \end{tikzcd}$$

Since we know that $\mathcal{Q}^{\log}_{((d,d))}(\mathbb{P}^N|H_1+H_2,d)$ is irreducible of the expected dimension we just need to check that the fibre product contains no components of excess dimension coming from the intersection of the boundaries of the smooth divisor *spaces. However by 3.4.1, we know that boundary divisors of either space must correspond to reducible curves with one marking on each component with the defining section vanishing on the component which contains the non-trivial tangency marking. The intersection of any pair of these boundary divisors consists of quasimaps which correspond to curves where the section corresponding to H_1 vanishes on the component with the first marking and the section for H_2 vanishes on the component with the second marking. Necessarily, both sections must vanish at the node. We can directly compute the dimension of this locus since it can be expressed as the fibre product*

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{M}^{\log}_{0,(0,d_2)}(H_2, d_2) \\ \downarrow & & \downarrow \\ \mathcal{M}^{\log}_{0,(0,d_1)}(H_1, d_1) & \longrightarrow & H_1 \cap H_2 = \mathbb{P}^{N-2} \end{array}$$

Where we omit the bars because we are talking about the interior of either the stable map or quasimap moduli spaces. But the dimension here is

$$\dim E = N - 2 + (N-1) \cdot d_1 + N - 2 + (N-1) \cdot d_2 - (N-2) = (N-1) \cdot (d+1) - 1$$

On the other hand the dimension of $\mathcal{Q}^{\log}_{0,((d,d))}(\mathbb{P}^N | H_1 + H_2, d)$ is $(N-1) \cdot (d+1)$. This locus defines a divisor in $\mathcal{Q}^{\log}_{((d,d))}(\mathbb{P}^N | H_1 + H_2, d)$.

Remark 3.7.6 (What goes wrong in higher rank) *The above shows that the counterexamples to the local-logarithmic correspondence for Gromov–Witten theory given in [NR22, Section 1] are not counterexamples in the quasimap setting, assuming the result in the smooth divisor case. However, this is expected to go wrong if you increase the rank of the divisor. For example, consider $\mathcal{Q}^{\log}_{((d,d,d))}(\mathbb{P}^3 | H_1 + H_2 + H_3, d)$. Once again, this space is irreducible, this time of dimension $3 + d$ and is given by a fibre product in the fine and saturated category. On the other hand the ordinary fibre product*

$$\begin{array}{ccc} \mathcal{Q}^{\mathrm{snc}}_{0,((d,d,d))}(\mathbb{P}^3, d) & \longrightarrow & \prod_{i=1}^{3} \mathcal{Q}^{\log}_{0,(d,0,0)}(\mathbb{P}^3 | H_i, d) \\ \downarrow & & \downarrow \\ \mathcal{Q}_{0,3}(\mathbb{P}^3, d) & \xrightarrow{\quad\Delta\quad} & \mathcal{Q}_{0,3}(\mathbb{P}^3, d)^3 \end{array}$$

may not agree with this. Roughly, this has to do with the fact that the morphism $\mathcal{Q}^{\log}_{0,(d,0,0)}(\mathbb{P}^3 | H_i, d) \to \mathcal{Q}_{0,3}(\mathbb{P}^3, d)$ is not logarithmically flat. This can be seen by the fact that there are divisors in $\mathcal{Q}^{\log}_{0,(d,0,0)}(\mathbb{P}^3 | H_i, d)$ where the source curve has more than two components.

We have just learned that the local-logarithmic correspondence as stated in 3.7.2 will also fail in generality for quasimap theory. 3.7.2 was a generalisation which involved maximal tangency to each divisor component at different markings. Another generalisation would be to consider maximal tangency to *all* divisor components at a single marking. In the Gromov–Witten case this conjecture would look like

$$[\overline{\mathcal{M}}^{\log}_{0,(\underline{d})}(X|D, \beta)]^{\mathrm{vir}} = \prod_{i=1}^{r} (-1)^{d_i + 1} \cdot \mathrm{ev}^* D_i \cdot [\overline{\mathcal{M}}_{0,1}(\oplus_{i=1}^{r} \mathcal{O}_X(-D_i), \beta)]^{\mathrm{vir}} \tag{3.9}$$

Where $(\underline{d})$ indicates the contact order $d_i = D_i \cdot \beta$ with each divisor. This generalisation is also false in the Gromov–Witten case. However, we expect that this generalisation is true in the quasimap setting. Once again we need to add a redundant marking in order for the statement to make sense.

Conjecture 3.7.7 *Let X be a smooth projective toric variety and $D = D_1 + \cdots + D_r$ a simple normal crossings divisor with $D_1 \cap \cdots \cap D_r \neq \emptyset$. Then*

$$[\mathcal{Q}^{\log}_{0,(\underline{d},0)}(X|D,\beta)]^{\mathrm{vir}} = \prod_{i=1}^{r}(-1)^{d_i+1} \cdot \mathrm{ev}_1^* D_i \cdot [\mathcal{Q}_{0,2}(\oplus_{i=1}^{r}\mathcal{O}_X(-D_i),\beta)]^{\mathrm{vir}} \tag{3.10}$$

Let $H_1,\ldots,H_r$ be hyperplanes in $\mathbb{P}^N$ for $r < N+1$ and set $D = H_1 + \cdots + H_r$.

Proposition 3.7.8 *Assuming 3.7.3 for $\mathbb{P}^N|H$ we have that*

$$[\mathcal{Q}^{\log}_{0,(\underline{d},0)}(\mathbb{P}^N|D,d)] = (-1)^{r(d-1)}\prod_{i=1}^{r}\mathrm{ev}_1^*(H_i) \cdot [\mathcal{Q}_{0,2}(\mathcal{O}_{\mathbb{P}^N}(-H_i),d)]^{\mathrm{vir}} \tag{3.11}$$

Proof 3.7.9 *Once again, assuming the smooth divisor case, it suffices to show that*

$$[\mathcal{Q}^{\log}_{0,(\underline{d},0)}(\mathbb{P}^N|D,d)] = \prod_{i=1}^{n}[\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H_i,d)]$$

Similarly to before we want to show that there is a cartesian diagram, in the category of ordinary stacks

$$\begin{array}{ccc} \mathcal{Q}^{\log}_{0,(\underline{d},0)}(\mathbb{P}^N|D) & \longrightarrow & \prod_{i=1}^{r}\mathcal{Q}^{\log}_{0,(d,0)}(\mathbb{P}^N|H_i,d) \\ \downarrow & & \downarrow \\ \mathcal{Q}_{0,2}(\mathbb{P}^N,d) & \xrightarrow{\quad\Delta\quad} & \mathcal{Q}_{0,2}(\mathbb{P}^N,d)^r \end{array}$$

This time the only way the boundary divisors can intersect non-trivially is if we have a quasimap whose curve has two components, each of which contains a marking. The section for each hyperplane vanishes identically on the component containing the non-trivial tangency condition and then none of the sections vanish on the other component except at the node where it is maximally tangent to the intersection $H_1 \cap \cdots \cap H_r$. This locus can be expressed as the fibre product

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{M}^{\log}_{0,(0,d_2)}(\mathbb{P}^N|D,d_2) \\ \downarrow & & \downarrow \\ \mathcal{M}_{0,2}(\mathbb{P}^{N-r},d_1) & \longrightarrow & H_1 \cap \cdots \cap H_r = \mathbb{P}^{N-r} \end{array}$$

The dimension is therefore

$$N-1+(N+1-r)\cdot d_2 + N-1-r+(N+1-r)\cdot d_1 - (N-r) = N-2+(N+1-r)\cdot d$$

On the other hand the dimension of $\mathcal{Q}^{\log}_{0,(\underline{d},0)}(\mathbb{P}^N|D,d)$ is $N-1+(N+1-r)\cdot d$, so the result follows.

Remark 3.7.10 *In fact both simple normal crossings generalisations of the local-logarithmic correspondence we have given are actually instances of a formulation that one could make using [TY20, Conjecture 1.8].*

3.8 Future Directions

3.8.1 Virtual Class

The most pressing thing left to do is to show that [Ols05, Theorem 8.5.3] generalises to the case that we need so that we can conclude we have a genuine obstruction theory on $\mathcal{Q}^{\log}_{g,\alpha}(X|D,\beta)$. Related to this is to fill in the missing details of the proof of 3.6.4 to deduce that the corresponding logarithmic quasimap invariants agree with the relative quasimap invariants of [BN21].

3.8.2 More General Targets

In this chapter we decided to restrict our theory to toric varieties, but the most general theory of absolute quasimaps [CFK20] works for much more general targets. One could ask whether the theory developed in this chapter naturally extends to these more general ambient spaces. We make a few comments on this. For ambient spaces of the form $V /\!/ G$ where V is a vector space and G is a reductive group (under the assumptions 3.1.1) the construction of the moduli space should go through without change. The only minor difference is that the description in 3.1.3 may have no analogue. In the construction of the virtual class, to prove perfect amplitude, we computed the logarithmic tangent complex of the quotient stack. We expect that one can control this for $[V/G]$ using the 'generalised Euler sequence' in [CFK20, 5.1].

On the other hand, this class includes other targets of the form, complete intersections in toric varieties, representation theoretic subvarieties of flag varieties etc. In these types of situations we expect there to be more difficulty. Restrict to the case of a hypersurface in $\mathbb{P}^N$ for simplicity. The moduli space could certainly be built in the same way, at least for divisors pulled back from $\mathbb{P}^N$. On the other hand the $[W/G]$ is the stack quotient of the affine cone by $\mathbb{G}_m$, which has l.c.i. singularities. So now the divisor has singularities at the singularity of the ambient space. This could cause issues for showing perfect amplitude of the obstruction theory.

3.8.3 Degeneration Formula

One of the most useful results for the computation of Gromov–Witten invariants is the degeneration formula [Li02] and more generally [Ran19]. This gives access to possibly calculating absolute quasimap invariants for more complicated targets, for example, the quintic threefold. Additionally it should allow access to the local-logarithmic correspondence. The strategy for proving a degeneration formula should be to mimic the ideas of [Ran19], which necessarily takes advantage of the expanded formalism. Therefore we would need to provide the corresponding expanded space in quasimap theory. However, in [Ran19], the author first produces a logarithmic modification of the space of maps to the artin fan, and since we have built our quasimap space in a similar way, (using $\mathfrak{M}^{\log}_{g,\alpha}([\mathbb{A}^r/\mathbb{G}_m^r])$) we can also do this. A complication, related to the previous section, is that sometimes natural degenerations involve pieces which don't necessarily support a quasimap theory. For example, if we take the deformation to the normal cone for a smooth but not toric divisor D inside a toric variety, then a piece of the central fibre is the projective completion of the normal bundle of D, which is of course not toric, but only relatively toric over D.

3.8.4 Local-Logarithmic Correspondence

As we have mentioned, the proof for the local-logarithmic correspondence in Gromov–Witten theory [vGGR19] uses the degeneration formula. Gaining access to this in the quasimap setting would allow us to verify 3.7.3. Then we could also investigate whether the generalisation 3.7.7 holds in all generality in the quasimap setting. Due to the local-orbifold correspondence [BNTY22], there is a parallel story which compares logarithmic and orbifold quasimap invariants.

3.8.5 Wall Crossing

The most powerful applications of quasimaps are the wall-crossing formulas relating quasimap invariants to Gromov–Witten invariants. The original motivation for developing a full theory of logarithmic quasimaps [BN21], was to prove wall-crossing results in the relative setting, as a means of computing relative/logarithmic Gromov–Witten invariants. The authors also provide some evidence for this. Namely, they show that a generating function for genus-zero (one-pointed descendant) relative quasimap invariants of a smooth (very ample) pair is equal to what is referred to as a 'relative I-function'. In an earlier paper [FTY19], the authors show that the corresponding generating function of relative Gromov–Witten invariants can be obtained from this relative I-function by a change of variables, which is likely to be the wall-crossing map. Since (see 3.3.21) this chapter could also be used to construct logarithmic ϵ-quasimaps, this is an interesting direction for the future.

3.9 Appendix: Virtual Classes

In this appendix will give a very brief introduction to virtual classes, stressing the fundamental example of a space given by the vanishing of a section of a vector bundle. The main references are [Ful98, CK99, BCM20, Nab15].

Fundamental Example

Suppose that we are interested in putting a virtual class on a space $\mathcal{M}$ and that $\mathcal{M}$ is presented as the vanishing locus of a section of a vector bundle E of rank r over a smooth scheme Y of dimension n

$$\begin{tikzcd} & E \arrow[d, "\pi"] \\ \mathcal{M} \arrow[r, hook, "i"] & Y \arrow[u, bend left, dotted, "s"] \end{tikzcd}$$

The vanishing locus of a *generic* section of E would give a space of dimension $n-r$. For this reason, we will call this the *expected dimension* of $\mathcal{M}$. However it may be that $\mathcal{M}$ is not of pure dimension $n-r$ and may have components of strictly greater dimension. The goal is then to provide a good substitute for the fundamental class of $\mathcal{M}$, which behaves nicely in families. If we were just looking for a class on Y, the candidate would be evident, since the class of the vanishing of a *generic* section of E in Y is

$$c_{\text{top}}(E) \cap [Y] \in A_{n-r}(Y)$$

We are looking for a class in $A_*(\mathcal{M})$ which pushes forward to $c_{\text{top}}(E) \cap [Y]$. Fortunately this is provided for us in [Ful98].

Let $C_{\mathcal{M}}Y$ be the *normal cone* of $\mathcal{M}$ in Y i.e.

$$\underline{\text{Spec}}\left(\bigoplus_{n=0}^{\infty} \mathcal{I}^n/\mathcal{I}^{n+1}\right)$$

This coincides with the normal bundle when $\mathcal{M}$ is regularly embedded. On the other hand, the normal cone is always of pure dimension n if Y is. Moreover, the surjection

$$\mathcal{O}(E^*) \to \mathcal{I}$$

given by multiplication by s induces an embedding

$$C_{\mathcal{M}}Y \hookrightarrow E|_{\mathcal{M}} \tag{3.12}$$

But since $A_*(\mathcal{M}) \cong A_{*+r}(E|_{\mathcal{M}})$ we get by intersecting with the 0 section

$$0^*[C_{\mathcal{M}}Y] \in A_{n-r}(\mathcal{M})$$

Moreover we have that

$$i_*0^*[C_{\mathcal{M}}Y] = c_{\text{top}}(E) \cap [Y]$$

So a good definition for virtual class in this case is $[\mathcal{M}]^{\text{vir}} = 0^*[C_{\mathcal{M}}Y]$.

Example 3.9.1 *Consider $\mathcal{M} = Z(xz, yz) \subset \mathbb{P}^3_{[x:y:z:w]}$. $\mathcal{M}$ is defined by two non-transverse sections of $\mathcal{O}_{\mathbb{P}^3}(2)$ which cut out the line $\ell = \{x = y = 0\}$ and the plane $H = \{z = 0\}$, intersecting in a single point. In the chart where $w \neq 0$, $\mathcal{M}$ has coordinate ring $R = \mathbb{C}[x, y, z]/(xz, yz)$ and the normal cone is*

$$C_{\mathcal{M}}\mathbb{P}^3|_{\{w\neq 0\}} = \operatorname{Spec}(R[A, B]/(yA - xB))$$

This can be seen from considering the surjective morphism

$$\begin{aligned} R[A, B] &\to \bigoplus_{n=0}^{\infty}(xz, yz)^n/(xz, yz)^{n+1} \\ A &\mapsto xz \\ B &\mapsto yz \end{aligned}$$

Which has kernel given by the ideal $(yA - xB)$. This provides an embedding $C_{\mathcal{M}}\mathbb{P}^3|_{\{w\neq 0\}} \hookrightarrow \mathcal{M}|_{\{w\neq 0\}} \times \mathbb{A}^2_{(A,B)}$. Local descriptions glue to give an embedding

$$C_{\mathcal{M}}\mathbb{P}^3 \hookrightarrow \mathcal{O}(2) \oplus \mathcal{O}(2)|_{\mathcal{M}}$$

Where the rank over the plane $\{z = 0\}$ minus the origin is 1 and the rank over the line $\{x = y = 0\}$ is 2, which provides some evidence for the fact that $C_{\mathcal{M}}\mathbb{P}^3$ is of pure dimension 3. We will see below that in this case the virtual class is given by the fundamental class of ℓ and the class of a cubic curve in the plane H.

Remark 3.9.2 *This is an instance of a more powerful intersection product. Consider the fibre square*

$$\begin{tikzcd} \mathcal{M} = Z(s) \arrow[r, hook] \arrow[d, hook, "i"] & Y \arrow[d, hook, "s"] \\ Y \arrow[r, hook, "0"] & E \end{tikzcd}$$

Then the Fulton-Macpherson intersection product [Ful98, 6.1] provides a class in $A_*(\mathcal{M})$ given by intersecting Y and Y in E via the embeddings given by the section s and the zero section. In this case the construction is actually called *the localized top chern class. In general the bottom row is not an embedding into a vector bundle, but an arbitrary regular embedding. Using deformation to the normal cone [Ful98, 5] one can replace this embedding with an embedding into the normal bundle. For similar reasons to 3.12, the normal cone of the top row embeds into this normal bundle and intersecting this cone with the 0 section of the normal bundle gives the intersection product.*

Remark 3.9.3 *In the case of a localized top chern class where $\mathcal{M} = Z(s)$ contains an effective Cartier divisor D on Y, Fulton provides a Residual Formula for $[\mathcal{M}]^{\mathrm{vir}}$ in terms of E, D and s' which is obtained from s by dividing out the equation for D. More precisely*

$$[\mathcal{M}]^{\mathrm{vir}} = [Z(s)]^{\mathrm{vir}} = [Z(s')] + \sum_{i=1}^{r} (-1)^{i-1} c_{r-i}(E) \cap D^{i-1} \cdot [D]$$

Example 3.9.4 *(Continued) Consider again $\mathcal{M} = Z(xz, yz) \subset \mathbb{P}^3_{[x:y:z:w]}$. Then $\mathcal{M}$ contains the cartier divisor $H = \{z = 0\}$ and s' is the section of $\mathcal{O}(1) \oplus \mathcal{O}(1)$ given by (x, y), i.e. cutting out the line $\ell = \{x = y = 0\}$. Then the residual formula for top Chern classes gives the following class in $A_1(\mathcal{M}) = A_1(\ell) \oplus A_1(H)$*

$$[\mathcal{M}]^{\mathrm{vir}} = [\ell] + c_1(\mathcal{O}(2) \oplus \mathcal{O}(2)) \cap [H] - H \cdot [H] = [\ell] + 3 \cdot [\mathrm{line}] \in A_1(\ell) \oplus A_1(H) = A_1(\mathcal{M})$$

In other words the virtual class of $\mathcal{M}$ is given by the fundamental class of the line $\ell = \{x = y = 0\}$ and the class of a cubic curve on the plane $H = \{z = 0\}$. Moreover, this clearly pushes forward to $c_2(\mathcal{O}(2) \oplus \mathcal{O}(2)) \cap [\mathbb{P}^3] = 4 \cdot [\mathrm{line}]$.

General Case

In general one doesn't have a description of $\mathcal{M}$ as the zero locus of a section of a vector bundle on a smooth space. In some sense a perfect obstruction theory provides locally, this description. We will first motivate the definition by returning to the fundamental example. If $\mathcal{M}$ is cut out of Y by a section of E then there are dual diagrams

$$\begin{tikzcd} T_Y|_{\mathcal{M}} \arrow[r] & E|_{\mathcal{M}} \\ T_Y|_{\mathcal{M}} \arrow[r] \arrow[u, "\mathrm{id}"] & N_{\mathcal{M}/Y} \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} E^*|_{\mathcal{M}} \arrow[r] \arrow[d] & \Omega_Y|_{\mathcal{M}} \arrow[d, "\mathrm{id}"] \\ \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega_Y|_{\mathcal{M}} \end{tikzcd}$$

Note that one can read off the virtual or expected dimension of $\mathcal{M}$ as $\dim Y - \operatorname{rk} E = T_Y|_{\mathcal{M}} - \operatorname{rk} E|_{\mathcal{M}}$. Moreover, as pointed out in [Nab15], this number is equal to the difference between the ranks of the cokernel and kernel of the top row, which give the *cohomology* of this complex. One of the insights of a perfect obstruction theory is that just as the virtual dimension can be read off from the data of the above diagram(s), in actual fact one can define a virtual class using a generalisation of this diagram, and crucially this will exist even when an embedding as the zero section of a vector bundle does not.

Definition 3.9.5 *(Warm Up) A perfect obstruction theory is roughly a diagram, presented as a morphism of complexes in degrees -1 and 0*

$$\begin{tikzcd} \mathcal{E}^{-1} \arrow[r] \arrow[d] & \mathcal{E}^0 \arrow[d] \\ \mathcal{I}/\mathcal{I}^2 \arrow[r] & \Omega_Y|_{\mathcal{M}} \end{tikzcd}$$

Which is an isomorphism on cohomology in degree 0 and surjective in degree -1, where $\mathcal{M} \hookrightarrow Y$ is an embedding, inducing the bottom row.

In actual fact, the bottom row is the truncated cotangent complex of $\mathcal{M}$, which, as an object in the derived category of $\mathcal{M}$, is independent of the choice of embedding.

Definition 3.9.6 *A perfect obstruction theory on $\mathcal{M}$ is a complex $\mathcal{E}^\bullet \in D^b(\mathcal{M})$, which is locally quasi-isomorphic to a complex of locally free sheaves supported in $[-1,0]$, together with a morphism $\phi : \mathcal{E}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}}$ such that $h^0(\phi)$ is an isomorphism and $h^{-1}(\phi)$ is surjective.*

In our fundamental example, the virtual class came from a certain cone embedding in a vector bundle, and intersecting it with the zero section. At this point it is not really clear how this procedure generalises to this more abstract setup ...

As above we mentioned that the (truncated) cotangent complex didn't depend on the embedding $\mathcal{M} \hookrightarrow Y$. This hints that we also need some sort of normal cone which doesn't depend on an embedding. Behrend and Fantechi define exactly that, the *intrinsic normal cone* $\mathcal{C}_{\mathcal{M}}$ which is a stack of pure dimension 0 over $\mathcal{M}$. Locally, one can describe this cone by choosing an embedding $U \hookrightarrow W$ for W smooth and ideal sheaf $\mathcal{I}$. The differential map

$$\mathcal{I} \to \Omega_W$$

which sends f to df induces a morphism

$$T_W|_U \to C_W U.$$

Taking the *stack quotient* gives the intrinsic normal cone locally as $[C_U W/T_W|_U]$. Now, give a perfect obstruction theory $\mathcal{E}^\bullet \to \mathbb{L}^\bullet_{\mathcal{M}}$, a similar notion allows one to define a vector bundle stack $h^1/h^0((\mathcal{E}^\bullet)^\vee)$. Moreover the conditions on the morphism ϕ induces an embedding of $C_{\mathcal{M}} \hookrightarrow h^1/h^0((\mathcal{E}^\bullet)^\vee)$ and so one defines the virtual class via intersection with the zero section as before

$$[\mathcal{M}]^{\mathrm{vir}} = 0^*[C_{\mathcal{M}}].$$

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