

BECCHI-ROUET-STORA-TYUTIN SYMMETRY

gauge
IN HIGHER ORDER THEORIES

BY

ANDREW DAVID BROWNING

A thesis presented for the Degree of Doctor of Philosophy of
the University of London and the Diploma of Membership of
Imperial College.

Department of Physics
Blackett Laboratory
Imperial College
London SW7 2BZ

JUNE 1985

ABSTRACT

Constrained dynamical systems frequently arise in the description of situations of physical interest. One approach to discussing the dynamics of such systems involves the application of B.R.S.T. invariance together with the introduction of ghosts and antighosts. This approach is generalised to include higher order theories where the structure functions are not constant. Examples of such theories are gravity and supergravity. The essential uniqueness of this extension is proven. The relationship between this approach and the generalised B.R.S.T. charge of Fradkin and coworkers is investigated. A homological characterisation of the physical observables is presented. The application of these techniques to the quantisation of higher order theories is explored, together with some indications of possible directions of future work.

PREFACE

The work described in this thesis was carried out in the Theoretical Physics Group, Imperial College, London, between October 1982 and June 1985 under the supervision of Professor C.J. Isham. Except where otherwise stated, the work is original and has not been submitted before for a degree of this or any other university.

I would like to thank my supervisor, Professor C.J. Isham, for helpful advice and encouragement. I would also like to thank the following people for many conversations about this work: Dr. D. McMullan, Professor K. Kuchar, J. Bakas, Dr A. Kakas, and H. La Roche.

I gratefully acknowledge the financial support of the Carnegie Trust for the Universities of Scotland.

TO MY PARENTS

C O N T E N T S

	<u>Page No.</u>
ABSTRACT	2
PREFACE	3
CONTENTS	5
INTRODUCTION	7
CHAPTER ONE: SYMPLECTIC GEOMETRY AND DYNAMICS	
1.1 Introduction	13
1.2 Symplectic Geometry	15
1.3 Constrained Dynamics And The	
Dirac-Bergmann Algorithm	25
CHAPTER TWO: GROUP ACTIONS AND B.R.S.T. SYMMETRY	
2.1 Introduction	45
2.2 The Ghost Sequence	47
2.3 The Antighost Sequence	51
2.4 The B.R.S.T. Operator	58
2.5 Difficulties	63
CHAPTER THREE: B.R.S.T. SYMMETRY IN	
HIGHER ORDER THEORIES	
3.1 Introduction	65
3.2 Structure Functions Associated	
With Constraints	68
3.3 Some Examples	72

	<u>Page No.</u>
3.4 B.R.S.T. Symmetry in Higher Order	
Theories	76
3.5 Uniqueness of Ghost Number Complex	94
3.6 Properties of Ghost Number Complex	101
3.7 Algebra of Observables	115
CHAPTER FOUR: QUANTISATION	
4.1 Introduction	120
4.2 B.R.S.T. Symmetry in Quantisation	122
4.3 Conclusions	131
APPENDIX	133
REFERENCES	136

INTRODUCTION

It has long been appreciated in physics that the presence of symmetries in a theory can help obtain results with comparative ease and without solving the equations of motion of the system. In the classical realm this can be seen in the construction of conserved quantities (8). In the quantum region it can be seen in the prediction of particles from theories like QCD. These predictions allow the theory to be tested experimentally by looking for particles with the appropriate properties e.g. as in the discovery of the W and Z bosons.

However the very presence of such symmetries creates difficulties in the description of the dynamics of a theory. If we have a theory invariant under a local transformation, then by Noether's second theorem (10), there are relationships between the various equations of motion. In the Lagrangian framework this is signalled by

$$\det \left| \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} \right| = 0 \quad (1)$$

i.e. the action is not quadratic in the velocities, so that some modes of the fields have no kinetic part and so do not propagate.

In the canonical framework, the relationship (1) tells us that the Legendre transformation to momentum phase space is no longer a diffeomorphism and so we must take great care in how we try to do this transformation. As is described in Chapter One, we must check for dynamical consistency, and in the end we are left with the physical states restricted to a subspace $\mathcal{K}$ of the naive momentum phase space $\mathcal{S}$.

The presence of constraints greatly complicates any quantisation scheme which is used. In the path integral formalism, the lack of kinetic terms for some modes of the fields obstructs the construction of the quantum propagators (32). In order to overcome this, while still retaining Poincare covariance, a covariant gauge fixing term is added to the action. However this term will not in general correspond to a reduction to the true degrees of freedom, and the resulting S matrix of the theory is not unitary. To get around this, Feynmann (40) has shown that we must add closed fictitious loops to the Feynmann diagrams of the theory. De Witt generalised this to all orders in perturbation theory (45), and Faddeev and Popov showed that

for Yang-Mills, this could be incorporated into the action by the addition of ghost terms. These are fermionic Lorentz scalars (vectors for gravity) which are unphysical, and so do not appear as external lines in the Feynmann diagrams. This procedure guarantees a unitary S matrix.

This approach, involving extension of the phase space, is in extreme contrast to canonical quantisation. Most of the approaches to canonical quantisation rely on finding some phase space associated with $\mathcal{K}$ and quantising on this, then the physical quantum states and observables are determined by some criterion. This is all very formal, and it is not clear how to proceed in general. For example, should we quantise on $\mathcal{J}$ and apply the constraints at a quantum level, or reduce to the true degrees of freedom (if we can) and then quantise? (The equivalence of these two approaches in some circumstances is shown in (44). See (46) for reviews of canonical methods in quantisation).

It would be very desirable to relate these two very different philosophies of quantisation to one another. One way of relating canonical and covariant path integral approaches may be through B.R.S.T. symmetry. This is a global, nilpotent supersymmetry of the path integral action with ghost terms. It is known that this symmetry can be used to give proofs of unitarity and of the ward-Takahashi

identities, and is also of great help in discussing renormalisation of Yang-Mills (47). It has been recently suggested that it is of relevance outside the weak field perturbative methods in which it was discovered (34). However this work is formal, and its justification is its relationship with the path integral methods.

In order to develop the ideas of B.R.S.T. symmetry outside the path integral formalism, it is necessary to have some interpretation of the ghosts and antighosts. There have been a number of attempts at this, but while they all give a nice interpretation of the ghosts, the antighosts cause problems. This can be circumvented by introducing a new symmetry, anti-B.R.S.T. symmetry. However this symmetry is not necessary for unitarity, and so is not as fundamental as the B.R.S.T. symmetry from the physical point of view. (See (41) for a differing opinion.)

Recently McMullan has given a classical phase space discussion of B.R.S.T. symmetry using the language of homological algebra. This allows a nice geometric interpretation of ghosts and antighosts to be developed. This discussion is applicable when the constraints arise from a group action on $\mathcal{S}$. In this case the powerful techniques of momentum maps can be used. This work can be related to the phase space path integral approach of Fradkin

et al. (7) which reproduces the Faddeev-Popov discussion for Yang-Mills, but is slightly different for gravity and supergravity. It is sensitive to the precise structure of the constraints and in particular to the rank of a theory. It is therefore desirable to extend the work of McMullan to theories of arbitrary rank. This thesis will show that such an extension is possible, and that in many ways it is unique. Some of the geometric flavour is lost, compared with the Yang-Mills case, but the analysis of the constraints and observables seems to parallel the sensitivity of the quantum theory on the rank of the constraints used.

For most of this thesis, we will consider finite dimensional constrained systems of arbitrary rank (for example, as would result from the Dirac-Bergmann algorithm). These cases model the structures of field theory fairly accurately, without the technical problems of function space choice etc. (Work is continuing to extend the formalism to the field theory cases.)

In the first chapter, we introduce the phase space methods used to describe dynamical systems with constraints. This will indicate how constrained theories of arbitrary rank may arise.

In Chapter Two, we describe the homological approach to B.R.S.T. symmetry as developed by McMullan. This is the

case of a first order theory, relevant when the constraints arise from a group action on $\mathcal{S}$.

In the third chapter, we extend these results to theories of arbitrary rank, and show that this extension is essentially unique. The characterisation of observables described in Chapter Two is shown to be valid in this generalisation.

In Chapter Four, we discuss the relationship of this approach to quantum theory, and possible directions for extending this work.

There is one appendix, introducing the definitions of homological algebra.

C H A P T E R O N E

SYMPLECTIC GEOMETRY AND DYNAMICS

1.1 Introduction

The description of classical dynamical systems is greatly helped by the introduction of concepts from symplectic geometry in the framework of the phase space approach. This approach has also contributed to the understanding of the covariant path integral in quantization, despite the loss of explicit covariance in the canonical quantization scheme. (An example of this is the occurrence of four ghost coupling in supergravity). In this chapter, we shall introduce the relevant techniques of the modern phase space approach.

We start by reviewing the basic concepts of symplectic geometry, including the graded case. We then go on to look at how this can be used to describe the dynamics of a system described by a Lagrangian with constraints. The Dirac-Bergmann algorithm is outlined, together with a globalisation due to Gotay, Nester, and Hinds. The additional structure available in the case of constraints arising from a group action on phase space is summarised. We

finish with a description of various characterisations of the observables of the system.

1.2 Symplectic Geometry

The arena of classical mechanics is the phase space associated with the dynamical system. Here we introduce the standard methods of symplectic geometry used to describe the phase space mathematically (1).

The most important structure on a symplectic manifold is the symplectic form, which is defined in the following way. Given a two-form σ on a manifold $\mathcal{J}$, we can define a linear mapping b_s for each $s \in \mathcal{J}$, by

$$\begin{aligned} b_s : T_s \mathcal{J} &\longrightarrow T_s^* \mathcal{J} \\ X_s &\longrightarrow l_{X_s} \sigma_s \end{aligned} \tag{1.2.1}$$

where $l_{X_s} \sigma_s : T_s \mathcal{J} \longrightarrow \mathbb{R}$ by

$$(l_{X_s} \sigma_s) Y_s = \sigma_s (X_s, Y_s) \tag{1.2.2}$$

Such a two-form σ is said to be nondegenerate if the mapping b_s is an isomorphism for each $s \in \mathcal{J}$. (In finite dimensions, it is clear that such a mapping is an isomorphism iff it is injective iff it is surjective. In

infinite dimensions more care must be taken to phrase this condition correctly.) (2)

A manifold $\mathcal{J}$ is said to be symplectic if there is a two-form σ on $\mathcal{J}$, with the properties that it is closed, i.e. $d\sigma = 0$, and nondegenerate. Such a σ is called a symplectic form.

Since σ is a two-form, it is easily seen that $\mathcal{J}$ must be even dimensional since otherwise σ would be degenerate. Given the manifold $\mathbb{R}^{2n}$ with coordinates $(q^1, q^2, \dots, q^n, p_1, p_2, \dots, p_n)$, we can define a one-form

$$\Theta = p_i dq^i \quad (1.2.3)$$

Then the two-form $\sigma = -d\Theta$, is a symplectic form on $\mathbb{R}^{2n}$. Explicitly

$$\sigma = dq^i \wedge dp_i \quad (1.2.4)$$

That σ is nondegenerate is clear from (1.2.4). This construction can be generalised to define Θ and σ on the cotangent bundle of any manifold Q , so that (1.2.3) and (1.2.4) hold locally. The one-form Θ is called the Liouville form on $\mathcal{J} = T^*Q$ and is defined by

$$\langle \Theta_r, X \rangle = \langle r, \pi_* X \rangle_{\pi(r)} \quad \forall X \in T_r(T^*Q) \quad (1.2.5)$$

where $r \in T^*Q$, $\pi: T^*Q \rightarrow Q$ is the projection, and $\langle, \rangle$ denotes the natural pairing between tangent and cotangent spaces. (If Q is finite dimensional, and we take coordinates $(q^1, q^2, \dots, q^n)$ on Q and the natural coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ on T^*Q , then locally Θ is given by (1.2.3).) In classical dynamics, this is the case which arises most often, when Q is the configuration space, and T^*Q is (momentum) phase space. On the other hand, not every symplectic manifold is a cotangent bundle, not even all the symplectic manifolds in classical mechanics. This is especially true in field theory, where the conjugate momenta of the canonical fields are typically a dense subspace of the dual of the tangent space to the configuration space. However, it is true that locally one can find coordinates such that (1.2.4) is true in some neighbourhood about any given point in any finite dimensional symplectic manifold. (In order to extend this result to infinite dimensions, care must be taken to interpret the nondegeneracy of σ in the correct manner, but since we are only interested in the finite dimensional case here, we will not pursue this matter.) A coordinate neighbourhood in which (1.2.4) holds is called a symplectic chart, or a canonical coordinate system. The existence of symplectic charts about any point of a finite dimensional

symplectic manifold is guaranteed by Darboux's theorem.

The symplectic form allows us to classify certain submanifolds of $\mathcal{S}$. At any point $s \in \mathcal{S}$, the two-form σ_s is a skew-symmetric and nondegenerate bilinear operator on the vector space $T_s \mathcal{S}$. For any subspace $V \subset T_s \mathcal{S}$ we can define the symplectic complement $V^\perp$ of V by

$$V^\perp = \{ w \in T_s \mathcal{S} : \sigma_s(w, v) = 0 \ \forall v \in V \} \quad (1.2.6)$$

It is clearly seen that

$$\dim V^\perp + \dim V = \dim T_s \mathcal{S} \quad (1.2.7)$$

Now, if we have a submanifold $K \subset \mathcal{S}$, we can consider the subspace $T_s K \subset T_s \mathcal{S}$ and its symplectic complement $(T_s K)^\perp$ at each point $s \in K$. Then we say

- a) K is isotropic at s iff $T_s K \subset (T_s K)^\perp$
- b) K is coisotropic at s iff $T_s K \supset (T_s K)^\perp$
- c) K is lagrangian at s iff it is both isotropic and coisotropic i.e. $T_s K = (T_s K)^\perp$
- d) K is symplectic at s iff $T_s K \cap (T_s K)^\perp = \{0\}$

A submanifold is said to be isotropic, coisotropic, lagrangian, or symplectic if it is this at each point.

Note that if K is symplectic as a submanifold of $\mathcal{S}$

then it is a symplectic manifold in its own right with symplectic form $\sigma|_K$, since this form is still nondegenerate.

If K is isotropic then the restriction of σ to K is identically zero i.e. $\sigma|_K = 0$.

The coisotropic submanifolds are the ones in which we will be interested, but we note in passing the importance of lagrangian submanifolds in many fields such as geometric quantisation (3), and dynamics (4).

We now consider how to use these ideas to describe the dynamics on a phase space. This makes use of the symplectic form in an essential way. Consider Hamilton's equations with Hamiltonian H :

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \quad ; \quad \dot{p}_i = -\frac{\partial H}{\partial q^i} \quad (1.2.8)$$

Then, if ζ_H is the tangent vector to the dynamical trajectories we have

$$\zeta_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \quad (1.2.9)$$

If we now assume $(q^1, \dots, q^n, p_1, \dots, p_n)$ is a canonical coordinate system, we see that (1.2.9) satisfies

$$l_{\xi_H} \sigma = dH \quad (1.2.10)$$

We call ξ_H the Hamiltonian vector field generated by H. Thus to each $f \in C^\infty(\mathcal{G})$ we can associate a local one parameter group of transformations (the integral curves of the corresponding Hamiltonian vector field). These transformations are symplectic i.e. $\psi^* \sigma = \sigma$. Diffeomorphisms from $\mathcal{G}$ to $\mathcal{G}$ which preserve the symplectic form can be characterised infinitesimally by

$$L_X \sigma = 0 \quad (1.2.11)$$

or equivalently

$$d(l_X \sigma) = 0 \quad (1.2.12)$$

Those infinitesimal transformations which preserve σ are called locally Hamiltonian vector fields. From (1.2.12), it is clear that all locally Hamiltonian vector fields are Hamiltonian if and only if all closed one-forms are exact.

It is natural to ask whether the space of Hamiltonian vector fields is closed under the commutator bracket. A simple calculation shows that for $f, g \in C^\infty(\mathcal{G})$,

$$[\xi_f, \xi_g] = \xi_h \quad (1.2.13)$$

where $h = \mathcal{J}_f(g)$.

With this in mind we define the Poisson bracket of two functions f and g by

$$\{f, g\} = -\mathcal{J}_f(g) \quad (1.2.14)$$

The mapping $(f, g) \rightarrow \{f, g\}$ is skew-symmetric and satisfies the Jacobi identities and so makes the set of all smooth functions into a real Lie algebra with the Poisson bracket as the Lie bracket. From the equations (1.2.13) and (1.2.14) we have

$$[\mathcal{J}_f, \mathcal{J}_g] = -\mathcal{J}_{\{f, g\}} \quad (1.2.15)$$

In classical mechanics it is traditional to call the smooth functions on phase space, observables. We see that this construction has made the observables into a Lie algebra.

With (1.2.15) in mind, we see that the mapping

$$f \longrightarrow -\mathcal{J}_f$$

from $C^\infty(\mathcal{J})$ to $\text{Ham}(\mathcal{J})$, the Hamiltonian vector fields on $\mathcal{J}$, is a Lie algebra homomorphism, whose kernel is the real numbers (i.e. the constant functions). This gives a short exact

sequence

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(\mathcal{S}) \rightarrow \text{Ham}(\mathcal{S}) \rightarrow 0$$

This description of dynamical systems is adequate to deal with many classical systems but in the light of quantum theory, it is interesting to see if it could accomodate anticommuting objects. There are a number of motivations for this, but primarily it is hoped that modelling the classical version of such theories on a graded phase space may bring insight into the behaviour of such quantum systems (5). This is especially true in supersymmetric theories where the supersymmetry demands that we use a formalism which classically accomodates the different statistics of the variables. Various ways of doing this extension have been suggested (6).

These methods allow us to define a closed nondegenerate two-form on a graded phase space $\overline{\mathcal{S}}$, which is antisymmetric on even vectors and symmetric on odd vectors. We can also define a super Poisson Bracket $\{, \}$, which in canonical coordinates is given by

$$\{F, G\} = \frac{\delta^r F}{\delta q^A} \frac{\delta^l G}{\delta p_A} - (-1)^{\eta_G \eta_F} \frac{\delta^r G}{\delta q^A} \frac{\delta^l F}{\delta p_A} \quad (1.2.16)$$

where r and l denote right and left derivatives and η_F is

equal to 0 if F is even, and equal to 1 if F is odd. Here

$q^A = (q^i, \theta^\alpha)$, $p_A = (p_i, \epsilon_\alpha)$. This bracket has the properties

$$(1) \quad \{q^i, q^j\} = \{p_i, p_j\} = \{\theta^\alpha, \theta^\beta\} = \{\epsilon_\alpha, \epsilon_\beta\} = 0$$

$$\{q^i, p_j\} = \delta^i_j = -\{p_j, q^i\}$$

(1.2.17)

$$\{\theta^\alpha, \epsilon_\beta\} = \delta^\alpha_\beta = \{\epsilon_\beta, \theta^\alpha\}$$

(2) If F, G, H are homogeneous superfunctions (i.e. either even or odd):

$$a) \quad \{F, G\} = -(-1)^{n_F n_G} \{G, F\}$$

$$b) \quad \{F, G+H\} = \{F, G\} + \{F, H\}$$

$$c) \quad \{F, GH\} = (-1)^{n_F n_G} G\{F, H\} + \{F, G\}H$$

$$d) \quad (-1)^{n_F n_G} \{F, \{H, G\}\} + (-1)^{n_F n_H} \{H, \{G, F\}\} + (-1)^{n_G n_H} \{G, \{F, H\}\} = 0$$

This last equation is the superJacobi equation.

With this framework of symplectic geometry behind us, we will now move on to considering the description of dynamical systems in which constraints play a role, and how such constraints arise. For now we observe that a supercanonical transformation generated by an odd function would be nilpotent by virtue of the superJacobi identities, i.e.

$$\delta^2 F = \{Q, \{Q, F\}\} = \frac{1}{2}(-1)^{n_F} \{F, \{Q, Q\}\} = 0 \quad (1.2.18)$$

provided that

$$\{Q, Q\} = 0 \quad (1.2.19)$$

Equation (1.2.19) is a nontrivial condition if Q is odd.

This property has been used to describe B.R.S.T. symmetry in the path integral formalism (7), an approach closely related to the one outlined in this thesis.

1.3 Constrained Dynamics And The Dirac-Bergmann Algorithm

One of the common methods used to describe a dynamical system involves the use of a Lagrangian. Here, one is given a smooth function on velocity phase space i.e. on the tangent bundle TQ , of configuration space. The action associated with a particular path $q'(t)$ is defined by

$$S = \int_{t_1}^{t_2} L(q'(t), \dot{q}(t)) dt \quad (1.3.1)$$

The dynamical trajectory between two configurations q_1 and q_2 of the system is the path which extremises the action S . However, the tangent bundle of a manifold is not automatically a symplectic manifold, and so we cannot automatically use the language of Section 1.2. This can be corrected in certain circumstances by using the Lagrangian to pull back the canonical structures on T^*Q . (8) This is done in the following way.

We associate with the Lagrangian $L: TQ \rightarrow \mathbb{R}$ the fibre derivative $FL: TQ \rightarrow T^*Q$, defined by

$$FL(V)W = \left. \frac{d}{dt} L(V + tW) \right|_{t=0} \quad (1.3.2)$$

where $V, W \in T_q Q$

In other words, $FL(V)W$ is the derivative of L at $(q, V) \in TQ$,

in the direction W in the fibre $T_q Q$. Locally (1.3.2) is

$$FL(q', \dot{q}') = \left(q^i, \frac{\partial L}{\partial \dot{q}^i} \right) \quad (1.3.3)$$

However, for arbitrary L , this is not necessarily either injective or surjective. The Lagrangian L is called nondegenerate if FL is a local diffeomorphism. In this case,

$$\det \left| \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0 \quad (1.3.4)$$

If this is true, we can pull back the canonical structures on T^*Q to TQ . This means that $\sigma_L^* (FL)^* \sigma$ is a symplectic form on TQ . This approach has been used to discuss dynamical systems in a geometric formulation using special symplectic manifolds of which the above is an example. These can be constructed even if L is degenerate (9).

The symplectic structure on TQ can also be used to derive the Euler-Lagrange equations associated with L . (8,10)

In the most common examples of Lagrangian systems, the Lagrangian is expressed as the integral of a local Lagrangian density. This occurs in classical field theory, a case we will return to later.

If the condition (1.3.4) is not satisfied, then this

is a sign that the physically allowed classical states, and their evolution, is limited to a submanifold $\mathcal{K}$ of the full (momentum) phase space $\mathcal{J}$. In this case, care must be taken in going over to the phase space description from the Lagrangian one. This procedure is called the Dirac-Bergmann algorithm (11). In outline this is as follows.

We are given a Lagrangian with

$$\det \left| \frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} \right| = 0 \quad (1.3.5)$$

Locally, we wish to express everything in terms of momenta p_i rather than velocities $\dot{q}^i$ where

$$p_i = \frac{\partial L}{\partial \dot{q}^i} \quad (1.3.6)$$

However, this change of variables can only be effected if equation (1.3.5) is not true. To get around this, the equations (1.3.6) are separated into two classes :

- 1) a set (say $i=1,2,\dots,R$) in which the velocities can be expressed as a function of the momenta,
- 2) a set in which the velocities do not appear

$$p_r = q_r(q, p_i) \quad r=R+1, \dots, N \quad (1.3.7)$$

Here $N = \dim Q$.

The equations (1.3.7) are called primary constraints.
We can write these as

$$G_r(q,p) = p_r - g_r(q,p_i) = 0 \quad (1.3.8)$$

The evolution and physical states of the system, is restricted to a submanifold Γ_p of the full phase space $\mathcal{S}$. If we call a function weakly zero $f \approx 0$, if it vanishes on Γ_p , then the equations of motion can be written in the form

$$\begin{aligned} \dot{q}^i &\approx \{q^i, H_c + \dot{q}^r G_r\} \\ \dot{p}_i &\approx \{p_i, H_c + \dot{q}^r G_r\} \end{aligned} \quad (1.3.9)$$

where

$$H_c = p_i \dot{q}^i - L(q, \dot{q}) \quad (1.3.10)$$

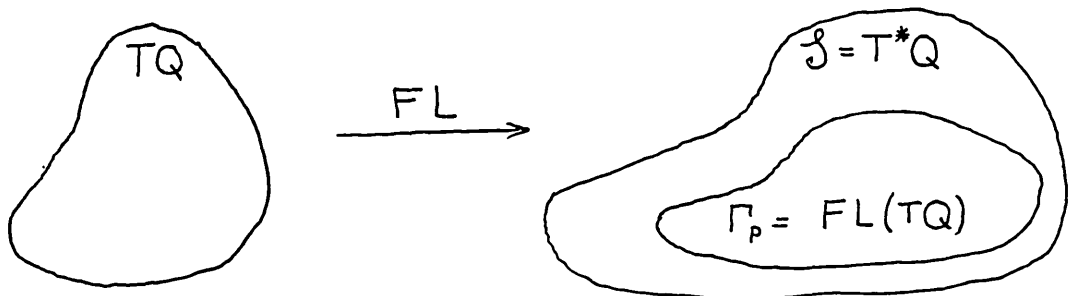
and H_c is expressed as a function of q , the R p_i , and the $N-R$ velocities $\dot{q}^r$.

Note that (1.3.9) contains undetermined functions $\dot{q}^r$. We will return to this problem later.

At this stage, it must be checked that the primary constraints G_r are preserved in time. This may generate new constraints (termed secondary constraints). These also must

be preserved in time, leading to tertiary constraints, etc. If the Lagrangian is admissible (i.e. the action does have an extremum) the procedure will end with a set of constraints defining a submanifold of $\mathcal{S}$. Before describing this manifold, we mention a "globalisation" of the Dirac-Bergmann algorithm, developed by Gotay, Nestor, and Hinds.(12)

When the Lagrangian is degenerate, the fibre derivative FL is only into T^*Q . The full phase space is not accessible to the dynamics, only the primary constraint manifold $\Gamma_p = FL(TQ)$



In general, Γ_p is not a symplectic manifold, i.e. $j^*\sigma$ is presymplectic (closed but degenerate). The Hamiltonian H_c is defined on Γ_p by $H_c = E^\sigma(FL)^{-1}$ where

$$E = \frac{\partial L}{\partial \dot{q}^i} \dot{q}^i - L \quad (1.3.11)$$

However, the equations involving the dynamical trajectories (1.2.10)

$$L_{\mathcal{H}} \sigma|_{\Gamma_p} = dH_c|_{\Gamma_p} \quad (1.3.12)$$

may not have any solutions for $\mathcal{H}$, and in general will not have a unique solution. The idea of Gotay, Nestor and Hinds is to develop an iterative scheme to allow for this. Call $\Gamma_p = M_1$. Using the mapping b introduced earlier, we consider two cases.

- a) dH_c outside the range of b for all $s \in M_1$
- b) there exist points $M_2 \subset M_1$ where $dH_c|_{M_2}$ is in the range of $b|_{M_2}$
(We assume M_1 is a submanifold of M_1)

In case (a), (1.3.12) has no solutions, and we cannot continue. This case is the case of an inadmissible Lagrangian. In case (b), we must ensure that the Hamiltonian vector field $\mathcal{H}_c$ is tangent to M_2 . This means that

$$\mathcal{H}_c = j_2 * \hat{X} \quad (1.3.13)$$

where $j_2: M_2 \rightarrow M_1$ is the inclusion and $\hat{X} \in TM_2$, i.e. $\mathcal{H}_c$ must be the push forward of a vector field on M_2 . This will not be true, in general, and we must look for a manifold $M_3 \subset M_2$ defined by

$$M_3 = \{m \in M_2; dH_c(m) \in b(j_2 * TM_2)\} \quad (1.3.14)$$

Continuing in this way, we generate a sequence of

submanifolds

$$\dots \xrightarrow{j_4} M_3 \xrightarrow{j_3} M_2 \xrightarrow{j_1} M_1 = \Gamma_p \xrightarrow{j} T^*Q = \mathcal{J} \quad (1.3.15)$$

with

$$M_{j+1} = \{ m \in M_j ; dH_c(m) \in b(j_{j*} TM_j) \} \quad (1.3.16)$$

For finite dimensional systems, this sequence terminates in one of the following ways. (We assume this is true in infinite dimensional cases of physical interest). Either

- (i) $M_k = \{ \}$
- (ii) $\dim M_k = 0$
- (iii) $M_k = M_{k+1}, \dim M_k \neq 0$

Case (i) is only true for inadmissible systems. Case (ii) is also not of interest, since this means that M_k consists of isolated points, allowing no room for dynamics. Case (iii), however, is of interest. Note that the vector field defined by

$$(l_{\mathcal{J}_H} \sigma - dH_c) \Big|_{M_k} = 0 \quad (1.3.17)$$

is not unique, since M_k will be presymplectic, and so $\sigma|_{M_k}$ will be degenerate.

It can be shown that the above approach is a globalisation of the Dirac-Bergmann algorithm, in the sense that the steps from M_2 to M_3 to M_4 etc, correspond to checking the consistency of the constraints in time. It should be noted however, that the arbitrariness in the Dirac-Bergmann algorithm lies in the Hamiltonian, while here it lies in the degeneracy of the presymplectic form. However the two approaches can be shown to be equivalent (13).

The end result of either of these approaches is the fact that the whole phase space $\mathcal{S}$ is not accessible to the dynamical system. In other words, the physically relevant states make up a submanifold $\mathcal{K}$ of $\mathcal{S}$, and the evolution from an initial state remains in $\mathcal{K}$ for all time. The types of theory which arise can be classified according to the classification for submanifolds given in section 1.2. (14). Thus if $\mathcal{K}$ is coisotropic, we have a first class system, while if $\mathcal{K}$ is symplectic, we have a second class system. A system is said to be completely integrable if $\mathcal{K}$ is lagrangian. If $\mathcal{K}$ is not any of the above then the system is said to have mixed constraints.

We now consider the first and second class systems separately. So assume we have a constraint hypersurface $\mathcal{K} \subset \mathcal{S}$ and a Hamiltonian H .

If $\mathcal{K}$ is symplectic, then the restriction of σ to $\mathcal{K}$ is still symplectic. This means that the solution to (1.3.17) is unique. Thus given initial data (contained in $\mathcal{K}$), the dynamical path is uniquely determined. Thus we have a standard phase space construction, with the addition of an embedding of $\mathcal{K}$ in some larger extended phase space. In order to take this embedding into account it is customary to work with Dirac brackets on $\mathcal{S}$, which are defined by the symplectic form on $\mathcal{K}$. (15)

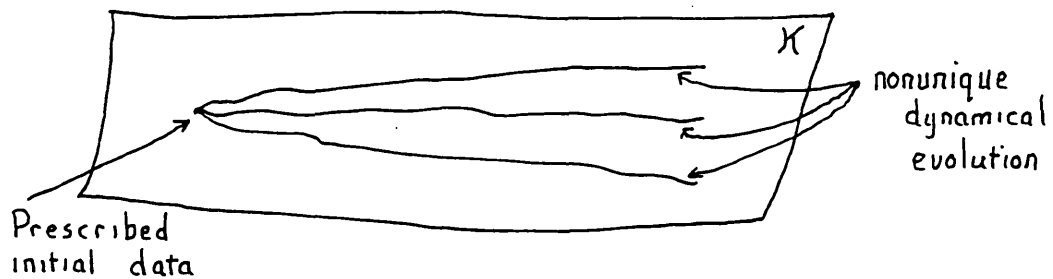
It can also be shown that if $\mathcal{K}$ is of mixed class, then one can find a second class submanifold $\bar{\mathcal{K}}$ of $\mathcal{S}$ with $\mathcal{K}$ a first class submanifold of $\bar{\mathcal{K}}$ (15). Thus we need only discuss the case where $\mathcal{K}$ is a first class submanifold of $\mathcal{S}$. (Note that this reduction must be done after the completion of the Dirac-Bergmann. This caveat is necessary since it seems that second class constraints can generate first class constraints during the algorithm, while first class constraints only generate first class constraints. For a discussion of this matter see (16).)

Thus it is sufficient to consider the case of $\mathcal{K}$ a first class submanifold of $\mathcal{S}$. In this case, we see that the restriction of σ to $\mathcal{K}$ is degenerate. Thus the solution for ξ_μ in equation (1.3.17) is not unique. We are free to add to this solution any vector field in $\ker \sigma$, where

$$\ker \sigma = \{Z \in TK ; i_Z \sigma = 0\}$$

(1.3.18)

When we then solve for the integral curves of $\tilde{X}_H$ we have a whole family of trajectories, corresponding to different choices of $\tilde{X}_H$.



It is highly desirable to resolve this redundancy in the choice of Hamiltonian vector fields and also to remove the redundancy in the description of the space of physical states, and reduce to a phase space $\hat{K}$ which is in one-to-one correspondence with the space of physical states ($\hat{K}$ is called the true degrees of freedom associated with K). In order that the dynamical evolution is unambiguous after initial data is prescribed, $\hat{K}$ must be symplectic.

In order to proceed, we note that since K is coisotropic, $T_s K = (T_s K)^\perp$ for each $s \in K$. This allows us to form the quotient space $T_s K / (T_s K)^\perp$ which has a symplectic structure inherited from $T_s K$. Moreover, $(TK)^\perp$ is an involutive subbundle of TK and so will define a foliation of K . This

foliation, which we call $\mathcal{K}^\perp$, can be used to form locally the quotient $\mathcal{K}/\mathcal{K}^\perp$, which we identify with $\hat{\mathcal{K}}$, the true degrees of freedom. The tangent space to $\mathcal{K}/\mathcal{K}^\perp$ at one of its points may be identified with $T_{\mathcal{K}/(\mathcal{T},\mathcal{K})}^\perp$, and so is symplectic. (For this construction to function globally, the foliation must satisfy certain technical conditions, but we do not discuss this here.) The Hamiltonian H on $\mathcal{K}$ can be reduced to $\hat{H}$ on $\hat{\mathcal{K}}$ and so we have an unambiguous phase space formalism for the true dynamics of the system associated with H and $\mathcal{K}$. Note that since $\hat{\mathcal{K}}$ is a quotient manifold, it is not, in general, realizable as a submanifold of $\mathcal{S}$.

While this discussion of the dynamics on first class constraint submanifolds shows why different classes of constraints should be dealt with in different ways it is usually impractical to do the calculations explicitly in examples of interest. We now turn our attention to more practical approaches to this problem.

First we note that in section 1.2 we used the symplectic form to express statements about vector fields as statements about forms or functions. With this in mind, we make the following definitions.

We define $\mathcal{C}$, the set of constraint functions on $\mathcal{S}$ by

(14)

$$C = \{f \in C^\infty(\mathcal{S}); f|_K = \text{const.}\} \quad (1.3.19)$$

Now, if X is a well-behaved manifold with $\dim K = 2n - k$, we expect to be able to choose k members of C , which we denote by $\phi_\alpha: \alpha = 1, \dots, k$, such that K is actually $\{\phi_\alpha = 0; \alpha = 1, \dots, k\}$. This can certainly be done locally, and we assume it can be done globally without restricting the properties of K too much. We will then show that the type of theory we have is reflected in the properties of the constraints ϕ_α .

We define other subsets of $C^\infty(\mathcal{S})$ by

$$B = \{f \in C^\infty(\mathcal{S}); \{f, g\}|_K = 0; \forall g \in C\} \quad (1.3.20)$$

$$F = C \cap B$$

Elements in B are called first class functions, while those in F are called first class constraints. With these definitions, it can be shown that

- 1) $f \in C \Leftrightarrow (\zeta_f)_s \in (T_s K)^\perp; s \in K$
- 2) $f \in B \Leftrightarrow (\zeta_f)_s \in T_s K; s \in K$

It is easily seen that if the constraints satisfy $\det\{\phi_\alpha, \phi_\beta\} \neq 0$ then $\phi_\alpha \notin F$, and moreover, K is symplectic. The constraints ϕ_α are called second class constraints. Note that we must exercise some care over the choice of the actual constraints used and over the use of this terminology. If we had chosen ϕ_i^2 , say, instead of ϕ_i , then $\phi_i^2 \in C$, but $\phi_i^2 \in F$. We need to

impose some nondegeneracy requirement on the choice of constraints ϕ_α . This problem is normally dealt with when discussing observables (e.g. (10)), a subject to which we will return. We require the constraints to be nondegenerate in the sense that $\{d\phi_\alpha; \alpha=1, \dots, k\}$ are all linearly independent. Alternatively, this means that if a function is zero on the zero set of one of the ϕ_α , then $f=h\phi_\alpha$ with $h \in C^\infty(\mathcal{S})$, together with the requirement that the constraints are independent in the usual sense. We assume henceforth that the constraints are chosen to satisfy this criterion. We note that this means that the Hamiltonian vector fields associated with the constraints are not zero when restricted to $\mathcal{K}$. We will use the other characterisation of nondegenerate constraints later.

With these ideas in mind, it is clear that if $\mathcal{K}$ is characterised by first class constraints then $\mathcal{K}$ is coisotropic.

We now consider how to describe the dynamics of the true degrees of freedom $\hat{\mathcal{K}}$, from the framework of a coisotropic submanifold $\mathcal{K}$ in an extended phase space $\mathcal{S}$. This is usually forced upon us, since working with $\hat{\mathcal{K}}$ is either difficult or impossible.

We consider the case where we have a first class constrained submanifold $\mathcal{K}$ described by k constraints ϕ_α .

(Since $\mathcal{K}$ is coisotropic, $k < n = \frac{1}{2} \dim \mathcal{S}$). The physically allowed states are represented in $\mathcal{K}$, but this representation is not unique, i.e. different points of $\mathcal{K}$ represent the same physical state. These points are connected by canonical transformations generated by the constraints. Also, the true observables are smooth functions on the true degrees of freedom $\hat{\mathcal{K}}$. We wish to relate these true observables to some functions on $\mathcal{S}$. Clearly the functions on $\mathcal{S}$ which we wish to consider should be those which preserve the dynamical flows. However there are two ways of characterising this. Either the condition can be imposed that an observable f is first class (or weakly invariant) or that it is invariant i.e.

$$\{f, \phi_\alpha\} = 0 \quad \alpha = 1, 2, \dots, k \quad (1.3.22)$$

Note that if f is weakly invariant then the constraints are preserved when restricted to $\mathcal{K}$ (i.e. the constraint hypersurface is preserved), a property which we would wish the observables to have.

In what follows, we will take the invariant functions as the more fundamental objects. However, we also note that we have a natural equivalence on the set of functions $f \in C^\infty(\mathcal{S})$:

$$f \sim g \Leftrightarrow f|_K = g|_K \quad (1.3.23)$$

We call this relation weak equivalence (or weak equality)

We will show in Chapter Three that a weakly invariant function is, in fact, weakly equal to an invariant function. Why then worry about the choice of criterion? This is, in part, motivated by the path integral approach to quantum theory. We will return to this later, but in the path integral approach, the measure used is not zero on $\mathcal{J}-K$, and so there will be contributions from off the constraint surface. This suggests that we should take care with the behaviour we allow to occur off K .

We now define the physical observables as weak equivalence classes of invariant functions. From the definition of $\hat{K}$, the true degrees of freedom, we see that such physical observables can be associated in a one-to-one manner with the true observables. We note however, that we will need to develop an algebra of physical observables, which reduces to the Poisson bracket on the true observables.

Before going on to develop these ideas, we consider the properties of the constraints a little more closely. We have k first class constraints ϕ_α , which satisfy $\{\phi_\alpha, \phi_\beta\}|_K = 0$. This means

$$\{\phi_\alpha, \phi_\beta\} = U_{\alpha\beta}^\gamma \phi_\gamma \quad (1.3.24)$$

where $U_{\alpha\beta}^\gamma$ are the (first-order) structure functions. These structure functions are not defined uniquely -- we can add a term of the form $W_{[\alpha\beta]}^{[\gamma\delta]} \phi_\delta$ to them without violating (1.3.24). (We will discuss this matter in greater detail in Chapter Three.) However, for any given theory we can usually choose the simplest form for the $U_{\alpha\beta}^\gamma$ that is possible. It is clear that, if possible, we should take the $U_{\alpha\beta}^\gamma$ to be zero. This is a very special case and would correspond to an abelian group acting on phase space. A slightly more general case is when the $U_{\alpha\beta}^\gamma$ are constant: this would correspond to having a general Lie group acting on the phase space. This example is worth a closer look since this case is of relevance to Yang-Mills theory and is also the case which was used to motivate this line of work as outlined in Chapter Two. We outline the structures which are of interest when a Lie group $\mathcal{D}$ is acting on a phase space. The point to note is that the constraints can be identified with the generators of the gauge group. This occurs in many of the theories of importance, like Yang-Mills. The analysis which we will outline, with finite dimensional $\mathfrak{g}$, and Lie group $\mathcal{D}$, is a model of these types of theories, which still contains the important structures or the canonical approach to these

theories.

We can define a mapping γ from the Lie algebra of $\mathcal{D}$, $\mathcal{L}(\mathcal{D})$ to the set of vector fields of $\mathcal{S}$, by allowing the one parameter subgroups of $\mathcal{D}$ to act on $\mathcal{S}$. Thus $\lambda \in \mathcal{L}(\mathcal{D})$ is mapped to the vector field whose value at $s \in \mathcal{S}$ is the tangent vector to the curve

$$s \longrightarrow [\exp(-t\lambda)] s \quad (1.3.25)$$

at the origin. This mapping γ is a Lie algebra homomorphism. The action of $\mathcal{D}$ is assumed to be symplectic, so the Lie derivative of σ with respect to $\gamma(\lambda) = \zeta_\lambda$ is zero. Thus ζ_λ is a locally Hamiltonian vector field. We assume that $\mathcal{S}$ is topologically trivial (17). Thus we must have a smooth function ϕ_λ such that

$$L_{\zeta_\lambda} \sigma = d\phi_\lambda \quad (1.3.26)$$

It is natural to ask if this mapping from $\mathcal{L}(\mathcal{D})$ to $\mathcal{C}^0(\mathcal{S})$ is a Lie algebra homomorphism for some appropriate choice of ϕ_λ .

(Remember, ϕ_λ is only defined up to the addition of a constant) In fact, while this is not true in general, for almost all the groups used in physical theories such a mapping does exist and is unique (8,17). (For this to be so, it is sufficient that $\mathcal{D}$ be semisimple, since then the

relevant cohomology groups of $\mathcal{L}(\mathcal{D})$ are trivial (18)) There is a technique for expressing this homomorphism $P: \mathcal{L}(\mathcal{D}) \rightarrow \mathcal{C}^\infty(\mathcal{S})$, which has many important properties (e.g. (19)). This uses the momentum map (8).

A mapping

$$\mathcal{J} : \mathcal{S} \longrightarrow \mathcal{L}(\mathcal{D})^*$$

is a momentum mapping for the action if, for every $\lambda \in \mathcal{L}(\mathcal{D})$,

$$i(\xi_\lambda)\sigma = dP(\lambda) \quad (1.3.27)$$

where

$$P(\lambda)(s) = \langle \mathcal{J}(s), \lambda \rangle \quad (1.3.28)$$

where $\langle, \rangle$ is the natural pairing between dual spaces. It can be shown that if the Hamiltonian is also invariant under the action of $\mathcal{D}$, then $\mathcal{J}$ is an integral for the dynamical flows (8). This, in fact, was the original motivation for developing momentum mapping techniques. However, in our case we wish to identify the constraint hypersurface $\mathcal{K}$ with $\mathcal{J}^{-1}(0)$. (Whether $\mathcal{J}^{-1}(0)$ is a manifold i.e. whether $0 \in \mathcal{L}(\mathcal{D})^*$ is a regular value of $\mathcal{J}$ is discussed with great care in (19). This reference also notes that the space of solutions for classical relativistic field theory is very closely related to the zero set of a momentum mapping, which is, of course,

our own motivation for identifying $\mathcal{K}$ with $\bar{J}^{-1}(0)$. With this identification, we can write

$$\bar{J}(s) = \phi_\alpha(s) \eta^\alpha \quad (1.3.29)$$

where $\{\eta^\alpha\}$ is a basis of $\mathcal{L}(\mathcal{D})^*$. Then we can choose as our k constraints the functions ϕ_α . Moreover, since P is a Lie algebra homomorphism,

$$\{\phi_\alpha, \phi_\beta\} = U_{\alpha\beta}^\gamma \phi_\gamma \quad (1.3.30)$$

where the $U_{\alpha\beta}^\gamma$ are the structure constants of $\mathcal{D}$. Thus $\mathcal{K} \equiv \bar{J}^{-1}(0)$ is coisotropic.

This construction is very powerful and allows a global description of the true degrees of freedom whenever a Lie group acts properly and freely on $\mathcal{S}$. (20)

This characterisation of the constraints by the use of the momentum map is crucial to certain aspects of the analysis of Chapter Two. However, we will show that a fairly complete discussion of constrained theories along the lines of Chapter Two can be developed in the absence of a straightforward symplectic group action on $\mathcal{S}$. This is important, since several theories such as gravity, supergravity (21), and the n -dimensional membrane (22), have

constraints satisfying (1.3.24) with the $U_{\alpha\beta}^{\gamma}$ functions on $\mathcal{S}$.

We recap on the framework which we wish to describe. We have a coisotropic submanifold $\mathcal{K}$ of dimension $2n-k$, which is contained in a $2n$ dimensional phase space $\mathcal{S}$. The submanifold $\mathcal{K}$ is defined by k constraints $\phi_{\alpha}=0$, which satisfy

$$\{\phi_{\alpha}, \phi_{\beta}\} = U_{\alpha\beta}^{\gamma} \phi_{\gamma} \quad (1.3.31)$$

The ϕ_{α} also satisfy certain nondegeneracy requirements. The canonical transformations generated by the constraints, which we will refer to as gauge transformations, connect different points in $\mathcal{K}$ which describe the same physical state. The physical observables are the invariant functions, modulo the weak equivalence relation.

C H A P T E R T W O

GROUP ACTIONS AND B.R.S.T. SYMMETRY

2.1 Introduction

The previous chapter introduced the language of symplectic geometry as used in dynamics. We have also seen that much more structure is available when we have a group action and momentum mapping on $\mathcal{S}$. In the next two chapters we characterise the physical observables using the language of homological algebra. A summary of the relevant definitions and concepts can be found in the Appendix.

In this chapter we recap on the techniques, developed by McMullan (22) in the case of a symplectic group action on $\mathcal{S}$. We will characterise the invariant functions in terms of the ghost complex, using the momentum map, and the techniques of homological algebra. We indicate how this could be generalised.

By introducing the antighost complex, we develop a homological expression of the equivalence structure on functions. We will also consider the resolution properties of this complex.

These two sequences will be combined and the B.R.S.T. operator constructed. The physical observables will be related to the cohomology of the ghost number complex. The analysis of constraints is summarised, as is the definition of the algebra of physical observables.

We finish the chapter with a discussion of the reasons for needing to formulate an extension of these methods to deal with cases where the coisotropic submanifold $\mathcal{K}$ is not necessarily describable as the zero set of a momentum map.

2.2 The Ghost Sequence

In this section we develop a homological characterisation of invariant functions.

The invariant functions are those which are constant on the orbits of $\mathcal{D}$, i.e. those which satisfy

$$\gamma(\lambda) f = \{ \lambda, f \} = 0 \quad (2.2.1)$$

for all $\lambda \in \mathcal{L}(\mathcal{D})$. If we write $\mathcal{J} = \phi_\alpha \eta^\alpha$, where $\{\eta^\alpha\}$ is a basis of $\mathcal{L}(\mathcal{D})^*$ then it is clear that an invariant function is characterised by

$$\delta_1 f = \{ \mathcal{J}, f \} = \{ \phi_\alpha, f \} \eta^\alpha = 0 \quad (2.2.2)$$

where the Poisson bracket is extended to $C^\infty(\mathcal{S}) \otimes \mathcal{L}(\mathcal{D})^*$ by acting on the $C^\infty(\mathcal{S})$ part. In order to seek a complex connected with δ_1 we define $\Omega^{r,0}$ to be the space of $C^\infty(\mathcal{S})$ -valued r -forms on $\mathcal{L}(\mathcal{D})^*$, thus

$$\Omega^{r,0} = C^\infty(\mathcal{S}) \otimes \wedge^r \mathcal{L}(\mathcal{D})^* \quad (2.2.3)$$

Then an element of $\Omega^{r,0}$ is of the form

$$\alpha = f_{\alpha_1, \alpha_2, \dots, \alpha_r} \eta^{\alpha_1} \eta^{\alpha_2} \dots \eta^{\alpha_r} \quad (2.2.4)$$

where $\eta^{\alpha_1}\eta^{\alpha_2} = -\eta^{\alpha_2}\eta^{\alpha_1}$, $f_{\alpha_1, \dots, \alpha_r}$ is a totally antisymmetric set of smooth functions on $\mathcal{S}$. We have defined δ_i on $\Omega^{0,0}$. It can be shown (22) that there is a unique extension of δ_i to $\Omega^{r,0}$, such that it is linear, an antiderivation, satisfies $\delta_i^2 = 0$ and restricts to (2.2.2) when acting on $\Omega^{0,0}$. This extension is also local : i.e. if U is an open set in $\mathcal{S}$ then

$$(\delta_i \alpha)|_U = \delta_i(\alpha|_U) \quad \alpha \in \Omega^{r,0} \quad (2.2.5)$$

The action of δ_i on an element of the form (2.2.4) is given by

$$\delta_i f = \{ \phi_{\alpha}, f \} \eta^{\alpha} \quad (2.2.6)$$

$$\delta_i \eta^{\alpha} = -\frac{1}{2} U_{\beta\gamma}^{\alpha} \eta^{\beta} \eta^{\gamma} \quad (2.2.7)$$

and is extended by using its properties as an antiderivation. In (2.2.7) the $U_{\beta\gamma}^{\alpha}$ are the structure constants of the group $\mathcal{D}$, as before.

These definitions allow us to introduce the ghost complex

$$C^{\infty}(\mathcal{S}) \xrightarrow{\delta_i} \Omega^{1,0} \xrightarrow{\delta_i} \Omega^{2,0} \xrightarrow{\delta_i} \dots \xrightarrow{\delta_i} \Omega^{k,0} \xrightarrow{\delta_i} 0 \quad (2.2.8)$$

Then the invariant functions are the elements of $H^0(\delta)$ where

$$H^0(\delta) = \ker \delta \cap C^\infty(\mathcal{G}) \quad (2.2.9)$$

However this characterisation of the invariant functions leaves no room for weakly invariant functions, nor have we introduced the weak equivalence relation. We will resolve this in the next section.

We also note two apparent difficulties with (2.3.8)

- a) The construction appears to be basis dependant
- b) It is not clear what significance the higher cohomology groups $H^i(\delta)$, $i > 0$ have.

It is clear that if we hope to extend these methods to field theory, we must produce a basis independant construction (23). Such a development has been described in (22), where it is shown that the ghost complex is exactly the complex needed to calculate the cohomology of $\mathcal{L}(\mathcal{D})$ with coefficients in $C^\infty(\mathcal{G})$, which is defined by

$$H^*(\mathcal{L}(\mathcal{D}), C^\infty(\mathcal{G})) := \text{Ext}_{\mathcal{L}(\mathcal{D})^p}^* (\mathbb{R}, C^\infty(\mathcal{G})) \quad (2.2.10)$$

where $\mathcal{L}(\mathcal{D})^p$ is the universal enveloping algebra of $\mathcal{L}(\mathcal{D})$, and Ext is a functor which can be characterised in terms of exact sequences modulo a particular equivalence relation (24).

We note, however that this approach relies heavily on the properties of the group $\mathcal{D}$, which we will lose in the more general cases we wish to discuss. In this case, we will try to model the ghost complex and the definitions (2.2.6) and (2.2.7) rather than take the more abstract approach. We note here (and will show later) that in general the ghost complex will fail to be a complex, that is, δ_i as defined above, will not satisfy $\delta_i^2 = 0$.

2.3 The Antighost Sequence

We now consider how to characterise the equivalence class structure on the functions. In the case we are considering here, we can use the momentum mapping to do this. Introduce

$$\Omega^{\circ,1} = C^\infty(\mathcal{G}) \otimes \mathcal{L}(\mathcal{D}) \quad (2.3.1)$$

and a basis $\{\rho_\alpha\}$ of $\mathcal{L}(\mathcal{D})$ which is dual to $\{\eta^\alpha\}$. Then $\omega \in \Omega^{\circ,1}$ can be written as

$$\omega = f^\alpha \rho_\alpha \quad (2.3.2)$$

with $f^\alpha \in C^\infty(\mathcal{G})$. Introduce $\delta_\circ: \Omega^{\circ,1} \rightarrow \Omega^{\circ,0}$ by

$$\begin{aligned} \delta_\circ \omega &= \langle J, \omega \rangle \\ &= \phi_\alpha f^\alpha \end{aligned} \quad (2.3.3)$$

Then if we take $\delta_\circ \Omega^{\circ,0} = 0$, we find that

$$H_\circ(\delta_\circ) = \frac{\ker \delta_\circ \cap \Omega^{\circ,0}}{\text{im } \delta_\circ \cap \Omega^{\circ,0}} = \frac{C^\infty(\mathcal{G})}{C_K^\infty(\mathcal{G})} \quad (2.3.4)$$

Since, in this expression, $C_K^\infty(\mathcal{G})$ is the set of all functions which are zero on K , we have imposed exactly the correct

structure on $C^\bullet(\mathcal{S})$.

We now wish to extend this to find a way of intertwining the ghost complex and the mapping δ_0 above. As is shown in (23), the correct observation to make is that the space $C^\bullet(\mathcal{S}) \otimes \wedge \mathcal{L}(\mathcal{D})$ is an associative algebra with identity, which is also an $\mathcal{L}(\mathcal{D})$ -algebra, and which is positively graded by the exterior algebra of $\mathcal{L}(\mathcal{D})^*$. (25)

An $\mathcal{L}(\mathcal{D})$ -algebra is a pair $(\mathcal{S}, \sigma)$ where $\mathcal{S}$ is a positively graded associative algebra and $\sigma: \mathcal{L}(\mathcal{D}) \rightarrow \mathcal{S}$ is a linear map satisfying

$$\sigma(\lambda)s = s\sigma(\lambda) \quad ; \quad \lambda \in \mathcal{L}(\mathcal{D}), s \in \mathcal{S} \quad (2.3.5)$$

Such an algebra is automatically a $\mathcal{L}(\mathcal{D})$ -space.

Associated with such an $\mathcal{L}(\mathcal{D})$ -space, one can define a Koszul complex (25), which defines a linear operator δ_0 on the tensor product $\mathcal{S} \otimes \wedge \mathcal{L}(\mathcal{D})$ with $\delta_0^2 = 0$. For us this complex has two important properties

- a) the zero homology is the type of object we are looking for i.e. it is the weak equivalence relation on $C^\bullet(\mathcal{S})$;
- b) the properties of a Koszul complex in general allow us to deduce that

$$H_i(\mathcal{S} \otimes \wedge \mathcal{L}(\mathcal{D})) = 0 \quad (i \geq 1) \Leftrightarrow H_1(\mathcal{S} \otimes \wedge \mathcal{L}(\mathcal{D})) = 0 \quad (2.3.6)$$

See (22,25) for details.

However, the object of this thesis is to consider theories where the constraints do not arise from a straightforward group action. We will show that while we lose the language of Koszul complexes, we can still introduce a complex with the same effective properties (a) and (b) provided the constraints we use are nondegenerate in the sense discussed in Chapter One. Let us introduce Ω^{or} analogously to $C^\infty(\mathcal{S}) \otimes \wedge^r \mathcal{L}(\mathcal{O})$. Thus if $\alpha \in \Omega^{or}$, we have

$$\alpha = f^{\alpha_1, \alpha_2, \dots, \alpha_r} \rho_{\alpha_1} \rho_{\alpha_2} \dots \rho_{\alpha_r} \quad (2.3.7)$$

where the $\{\rho_\alpha\}$ are a set of anticommuting elements. Define δ_o on $\bigoplus_{r=0}^k \Omega^{or}$ by demanding that it acts linearly and

$$\delta_o \alpha = r \phi_\alpha f^{\alpha_1, \dots, \alpha_r} \rho_{\alpha_1} \dots \rho_{\alpha_r} \quad (2.3.8)$$

Then $\delta_o : \Omega^{or} \rightarrow \Omega^{or-1}$. Moreover, $\delta_o^2 = 0$. Thus we have a complex (the antighost complex)

$$0 \xrightarrow{\delta_o} \Omega^{ok} \xrightarrow{\delta_o} \Omega^{o, k-1} \xrightarrow{\delta_o} \dots \xrightarrow{\delta_o} \Omega^{o1} \xrightarrow{\delta_o} \Omega^{oo} \xrightarrow{\delta_o} 0 \quad (2.3.9)$$

It is clear that this complex satisfies (a) above. While (b) is not applicable, we can show that its conclusion (that (2.3.9) is exact at $\Omega^{oi} \ i > 0$) is, in fact, true.

We prove exactness at $\Omega^{\sigma'}$. The other cases are similar. We use a proof due to McMullan (22).

We note that the nondegeneracy requirement on the constraints allows us to deduce the following :

(*) If $f, g \in C^\infty(\mathcal{G})$ such that $f\phi_1 = g\phi_2$ then $f = h\phi_2$ and $g = h\phi_1$, with $h \in C^\infty(\mathcal{G})$

Let $\omega \in \Omega^{\sigma'}$ with $\delta_0 \omega = 0$. Then we can write $\omega = f^\alpha \phi_\alpha$, and we know that

$$(2.3.10) \quad \phi_\alpha f^\alpha = 0$$

We denote the zero set of $(\phi_1, \dots, \phi_k)$ by Z_r . Thus the constraint hypersurface $\mathcal{K}$ is Z_0 .

On Z_1 , (2.3.10) becomes $\phi_1 f^1 = 0$. Hence we have

$$f^1 = 0 \text{ on } Z_1, \quad (2.3.11)$$

On Z_2 , (2.3.10) becomes

$$(2.3.12) \quad \phi_1 f^1 + \phi_2 f^2 = 0$$

By (*) above, this means

$$f^1 = h\phi_2 \text{ and } f^2 = -h\phi_1$$

$$\text{i.e. on } Z_2, \quad (2.3.13) \quad f^i = \phi_j \epsilon^{ji} h$$

where i and j run over $1, 2$.

We assume that the construction has been carried out to Z_s . That is, if we have a set of functions $f^1, f^2, \dots, f^s$ with $\phi_i f^i = 0$ (i running over $1, 2, \dots, s$) then $f^i = \phi_j F^{ji}$ for some $F^{ji} \in C^\infty(\mathcal{G})$ which are antisymmetric in i and j .

Then consider (2.3.10) on Z_{s+1} . (In what follows, i and j run over $1, 2, \dots, s$ only)

$$\phi_{s+1} f^{s+1} + \phi_i f^i = 0 \quad (2.3.14)$$

$\phi_i f^i = 0$ on Z_s , so by the induction hypothesis

$$f^i = \phi_j F^{ji} \quad \text{on } Z_s \quad (2.3.15)$$

Thus on Z_{s+1}

$$f^i = \phi_j F^{ji} + \phi_{s+1} h^i \quad (2.3.16)$$

(Here we have used the nondegeneracy of the constraints).

$h^i \in C^\infty(\mathcal{G})$ for each $i = 1, 2, \dots, s$

Substituting (2.3.16) into (2.3.14), gives

$$\phi_{s+1} f^{s+1} + \phi_i \phi_{s+1} h^i = 0 \quad \text{on } Z_{s+1}$$

i.e.,
$$f^{s+1} = -\phi_i h^i \quad \text{on } Z_{s+1}$$

Thus $f = \phi_b^a F^{ba}$ (a, b running over $1, 2, \dots, s+1$) on Z_{s+1} where

$$F^{ba} = \left(\begin{array}{ccc|c} & & F^{ji} & -h^i \\ \hline & & & \\ \hline & & h^i & 0 \end{array} \right)$$

Thus by induction, if ω is such that (2.3.10) holds, then

$$f^\alpha = \phi_\beta F^{\beta\alpha} \quad (2.3.17)$$

Defining $r \in \Omega^{02}$ by $r = \frac{1}{2} F^{\beta\alpha}_{\beta\alpha}$ then we see that $\omega = \delta_0 r$.

Hence the complex (2.3.9) is exact at Ω^{01} .

We see then, that although we can no longer use the Koszul complex construction, the result is still true : we can define a complex which is exact except at Ω^{00} , which introduces the correct equivalence class structure on $C^\infty(\mathcal{G})$.

In fact, we can generalise this even more. We can define

$$\Omega^{rs} = C^\infty(\mathcal{G}) \otimes \wedge^r \mathcal{L}(\mathcal{D})^* \otimes \wedge^s \mathcal{L}(\mathcal{D}) \quad (2.3.18)$$

with $\delta_0 : \Omega^{rs} \rightarrow \Omega^{r+s-1}$ by

$$\delta_0(V_{\beta_1, \dots, \beta_r}^{\alpha_1, \dots, \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_r} \rho_{\alpha_1} \dots \rho_{\alpha_s}) = (-1)^r \phi_\alpha V_{\beta_1, \dots, \beta_r}^{\alpha \alpha_1 \dots \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_r} \rho_{\alpha_1} \dots \rho_{\alpha_s} \quad (2.3.19)$$

Then for each $r \geq 0$, the complex

$$0 \rightarrow \Omega^{rk} \rightarrow \Omega^{rk-1} \rightarrow \dots \rightarrow \Omega^{r2} \rightarrow \Omega^{r1} \rightarrow \Omega^{r0} \rightarrow 0 \quad (2.3.20)$$

is exact except at Ω^{r0} . It is clear that if we lack a group action, but introduce two set of anticommuting variables $\{\eta^\alpha\}$ and $\{\rho_\beta\}$, and define a complex as above,

then the result is still true.

We now go on to show how to intertwine the ghost and antighost complexes and so introduce the B.R.S.T. operator in order to describe the physical observables.

2.4 The B.R.S.T. Operator

First we recap on what we have shown in the previous sections.

We have defined various spaces

$$\Omega^{rs} = C^\infty(\mathcal{G}) \otimes \wedge^r \mathcal{L}(\mathcal{D})^* \otimes \wedge^s \mathcal{L}(\mathcal{D}) \quad (2.4.1)$$

whose elements are of the form

$$\omega \in \Omega^{rs} \Leftrightarrow \omega = f_{\beta_1, \dots, \beta_r}^{\alpha_1, \dots, \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_r} \rho_{\alpha_1} \dots \rho_{\alpha_s} \quad (2.4.2)$$

where $\{\eta^\beta\}$ is a basis for $\mathcal{L}(\mathcal{D})^*$ and $\{\rho_\alpha\}$ is a basis for $\mathcal{L}(\mathcal{D})$. Together, the space

$$\Omega = \sum_{r=0}^k \sum_{s=0}^k \Omega^{rs} \quad (2.4.3)$$

is a bigraded, associative algebra. (The two gradings are those associated with the exterior algebra of $\mathcal{L}(\mathcal{D})$ and $\mathcal{L}(\mathcal{D})^*$) The multiplication of two elements of Ω , is defined by allowing the functional part in (2.4.2) to commute with everything, and requiring the ghosts and antighosts to satisfy

$$\eta^\alpha \eta^\beta = -\eta^\beta \eta^\alpha; \quad \rho_\alpha \rho_\beta = -\rho_\beta \rho_\alpha; \quad \rho_\alpha \eta^\beta = -\eta^\beta \rho_\alpha \quad (2.4.4)$$

We have two operators defined on (parts of) Ω , δ_0 which is defined on all Ω^{rs} , and δ_1 which is defined on Ω^{r0} ($r \geq 0$)

It can be shown that δ_1 can be extended to all of Ω by requiring that it be an antiderivation, defined on functions and ghosts by (2.2.6) and (2.2.7) and acting on antighosts by

$$\delta_1 \rho_\alpha = - U_{\alpha\beta}^{\gamma} \eta^\beta \rho_\gamma \quad (2.4.5)$$

Then $\delta_1^2 = 0$ as before, and moreover

$$\delta_1 \delta_0 + \delta_0 \delta_1 = 0 \quad (2.4.6)$$

We have thus defined the ghost-antighost double complex

$$\begin{array}{ccccccc}
 \Omega^{00} & \rightarrow & \Omega^{10} & \rightarrow & \Omega^{20} & \rightarrow & \dots \rightarrow \Omega^{k0} \rightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \Omega^{01} & \rightarrow & \Omega^{11} & \rightarrow & \Omega^{21} & \rightarrow & \dots \rightarrow \Omega^{k1} \rightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \Omega^{02} & \rightarrow & \Omega^{12} & \rightarrow & \Omega^{22} & \rightarrow & \dots \rightarrow \Omega^{k2} \rightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \Omega^{0k} & \rightarrow & \Omega^{1k} & \rightarrow & \Omega^{2k} & \rightarrow & \dots \rightarrow \Omega^{kh} \rightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & & 0 & & 0 & & 0
 \end{array} \quad (2.4.7)$$

where the vertical arrows represent δ_0 and the horizontal

arrows are δ_1 .

In (2.4.7) each column is exact, while each row is a complex.

The definition (2.4.5) can be shown to be unique provided that we demand that δ_1 satisfies

$$\delta_1 \delta_0 + \delta_0 \delta_1 = 0$$

$$\delta_1^2 = 0$$

and that the $U_{\alpha\beta}^\gamma$ are constant (22). We will return to this point later.

From this bigraded complex, we can define a singly graded complex in a natural way (26). Define

$$\Omega^q = \sum_{r+s=q} \Omega^{rs} \quad \delta = \delta_0 + \delta_1 \quad (2.4.8)$$

Then $\delta: \Omega^q \rightarrow \Omega^{q+1}$, and $\delta^2 = 0$. Hence we have a complex, the ghost number complex :

$$0 \rightarrow \Omega^{-k} \rightarrow \dots \rightarrow \Omega^{-1} \rightarrow \Omega^0 \rightarrow \Omega^1 \rightarrow \dots \rightarrow \Omega^k \rightarrow 0 \quad (2.4.9)$$

This complex will allow us to discuss the physical observables.

The operator δ is called the B.R.S.T. operator. It is an antiderivation. If $\omega \in \Omega^q$ satisfies $\delta\omega = 0$ we shall call

it B.R.S.T. closed, while if $\omega = \delta r$ for some $r \in \Omega^{q+1}$, we will call it B.R.S.T. exact.

Before considering the homology of the ghost number complex, we should note that the ghost-antighost double complex can, in fact, be developed without reference to bases in $\mathcal{L}(\mathcal{D})$ and $\mathcal{L}(\mathcal{D})^*$. We refer to (22) for details.

The B.R.S.T. cohomology is defined as

$$H^q(\delta) = \frac{\ker \delta: \Omega^q \rightarrow \Omega^{q+1}}{\text{im } \delta: \Omega^{q-1} \rightarrow \Omega^q} \quad (2.4.9)$$

It can be then shown that $H^q(\delta) = 0$ if $q < 0$, and that $H^0(\delta)$ corresponds precisely with the physical observables. It can also be shown that a weakly invariant function on $\mathcal{G}$ determines a unique equivalence class in $H^0(\delta)$, showing that the analysis can deal with the weakly invariant functions. Since we will extend these results in Chapter Three, we will not discuss the details here.

One can also develop an algebra of observables using the super Poisson bracket acting on elements of Ω . This has fundamental brackets:

- 1) On $C^\infty(\mathcal{G})$ it is the Poisson bracket
- 2) $\{\eta^\alpha, \eta^\beta\} = \{\rho_\alpha, \rho_\beta\} = 0$
- 3) $\{\eta^\alpha, \rho_\beta\} = \{\rho_\beta, \eta^\alpha\} = \delta^\alpha_\beta$
- 4) $\{f, \eta^\alpha\} = \{f, \rho_\alpha\} = 0 \quad \forall f \in C^\infty(\mathcal{G})$

Moreover one can show that if $Q \in \Omega'$ is defined by

$$Q = \phi_\alpha \eta^\alpha + \frac{1}{2} \rho_\gamma U_{\alpha\beta}^\gamma \eta^\beta \eta^\alpha \quad (2.4.10)$$

then

$$\{Q, Q\} = 0 \quad (2.4.11)$$

and

$$\delta\omega = \{Q, \omega\} \quad (2.4.12)$$

We now consider what happens when we try to apply this construction to more general class of theories where the structure constants become functions on $\mathcal{F}$.

2.5 Difficulties

We consider what happens when we lose the action of the group $\mathcal{D}$. Thus we have a coisotropic submanifold $\mathcal{K} \subset \mathcal{S}$, defined by constraints $\phi_\alpha = 0$ $\alpha=1, \dots, k$, which satisfy

$$\{\phi_\alpha, \phi_\beta\} = U_{\alpha\beta}^\gamma \phi_\gamma \quad (2.5.1)$$

One might hope that the formalism would deal with this, with the loss of some of the geometric flavour. After all, the antighost complex could still be constructed, and is still exact, provided we have chosen the constraints correctly. Unfortunately there is a larger problem which obstructs the definition of the ghost complex. This is the fact that δ_1^2 is no longer zero if the $U_{\alpha\beta}^\gamma$ are functions on $\mathcal{S}$. To see this consider

$$\begin{aligned} \delta_1^2 &= \delta_1(\{\phi_\alpha, f\} \eta^\alpha) \\ &= \left(\left(\frac{1}{2} U_{\beta\alpha}^\gamma + H_{\beta\alpha}^\gamma \right) \{\phi_\gamma, f\} + \frac{1}{2} \phi_\gamma \{U_{\beta\alpha}^\gamma, f\} \right) \eta^\beta \eta^\alpha \end{aligned} \quad (2.5.2)$$

where we assume

$$\delta_1 \eta^\gamma = H_{\beta\alpha}^\gamma \eta^\beta \eta^\alpha \quad (2.5.3)$$

Thus if $U_{\alpha\beta}^\gamma \in C^\infty(\mathcal{S})$ then we cannot make $\delta_1^2 = 0$ for any choice of $H_{\beta\alpha}^\gamma$.

That this is an important question can be seen from the fact that there are several theories which have structure functions. For example, the superHamiltonian constraint $\mathcal{H}_\perp$ in gravity satisfies

$$\{ \mathcal{H}_\perp(x), \mathcal{H}_\perp(y) \} = \{ g^u \mathcal{H}_i(x) + g^u \mathcal{H}_i(y) \} \partial_j \delta(x, y) \quad (2.5.4)$$

where $\mathcal{H}_i$ is the supermomentum constraint and the g^i_j are the components of the three metric, which are the configuration variables (27). Other examples include supergravity (30) and the n dimensional relativistic membrane ((28) and Chapter Three). In Chapter Three we develop the necessary modifications to cope with these difficulties.

C H A P T E R T H R E E

B.R.S.T. SYMMETRY IN HIGHER ORDER THEORIES

3.1 Introduction

In this chapter we develop the extension necessary to generalise the homological description of B.R.S.T. symmetry to the case of a general coisotropic constraint submanifold. We have seen that if the constraints can be described by a momentum mapping, a number of techniques from homological algebra can be used to describe the physical observables. However, we also know that one essential part of this construction, the ghost complex, is no longer available to us. On the other hand, the antighost sequence seems to have all the actual properties which we used before, it has merely lost some of its geometrical flavour. It is clear that we will no longer be able to identify the ghosts and antighosts as elements in the Lie algebra of the symmetry group, but it may be that we can still extend the phase space $\mathcal{S}$ by $2k$ odd degrees of freedom, and so describe the physical observables.

In section 3.2 we describe the higher order structure functions, and develop an iterative formula for the n th

order functions. We then give an example of a theory which has nonzero structure functions of all possible orders, the n dimensional membrane.

We then go on to show that by considering antiderivations associated with these structure functions, we can define a generalised B.R.S.T. operator and a ghost number complex associated with it. However we note that this solution is not unique, and we find a form for the most general form for the B.R.S.T. operator.

By applying the techniques of homological algebra, we show that the physical observables, and all the physically relevant quantities are independent of the particular form of the B.R.S.T. operator used, once its action on functions is defined.

We then consider the relationship between the homology and the observables, and weakly invariant functions. We also consider the construction of an algebra of observables.

3.2 Structure Functions Associated with Constraints

We have shown that the development of a homological approach to B.R.S.T. symmetry along the lines of Chapter Two has problems when the structure constants become functions on $\mathcal{S}$. However, there will be relationships between the structure functions and the constraints which arise from the Jacobi identity for the Poisson bracket. In this section we give a systematic development of these relationships.

We are given a set of k functions ϕ_α , satisfying

$$\{\phi_\alpha, \phi_\beta\} = U_{\alpha\beta}^{\gamma} \phi_\gamma \quad (3.2.1)$$

The ϕ_α also satisfy the nondegeneracy condition described earlier. If we take the Poisson bracket of (3.2.1) with we get

$$\{\{\phi_\alpha, \phi_\beta\}, \phi_\delta\} = \{U_{\alpha\beta}^{\gamma}, \phi_\delta\} \phi_\gamma + U_{\alpha\beta}^{\epsilon} U_{\epsilon\delta}^{\gamma} \phi_\gamma \quad (3.2.2)$$

If we antisymmetrise on $\alpha\beta\delta$, the left hand side is zero by Jacobi's identity and we get

$$\left(\{U_{[\alpha\beta}^{\gamma}, \phi_{\delta]}\} - U_{[\alpha\beta}^{\epsilon} U_{\delta]\epsilon}^{\gamma} \right) \phi_\gamma = 0 \quad (3.2.3)$$

However this is an equation of the form

$$\delta_0 \omega = 0 \quad (3.2.4)$$

By the exactness of the antighost complex, which we showed is true even in the most general cases, we can deduce

$$\{U_{[\alpha\beta]}^{\gamma}, \phi_\delta\} - U_{[\alpha\beta]}^{\epsilon} U_{\delta] \epsilon}^{\gamma} = \phi_\tau V_{\alpha\beta\delta}^{\tau\gamma} \quad (3.2.5)$$

where $V_{\alpha\beta\delta}^{\tau\gamma}$ is a collection of $C^\infty(\mathcal{G})$ functions which are antisymmetric in $\tau\gamma$ and in $\alpha\beta\delta$. We can continue and take the Poisson bracket of (3.2.5) with a constraint and then antisymmetrise, giving

$$\begin{aligned} \phi_\tau \left[\frac{1}{2} \{U_{[\alpha\beta]}^{\tau}, U_{\gamma\delta]}^{\epsilon}\} - \{V_{[\alpha\beta\gamma]}^{\tau\epsilon}, \phi_\delta\} + U_{\kappa[\alpha}^{\epsilon} V_{\beta\gamma\delta]}^{\tau\kappa} \right. \\ \left. + \frac{3}{2} U_{[\alpha\beta]}^{\kappa} V_{\gamma\delta]\kappa}^{\tau\epsilon} - U_{\kappa[\alpha}^{\tau} V_{\beta\gamma\delta]}^{\epsilon\kappa} \right] = 0 \end{aligned} \quad (3.2.6)$$

Again it can be deduced that the term in brackets in (3.2.6) is of the form $\phi_\mu W_{\alpha\beta\gamma\delta}^{\mu\tau\epsilon}$ where $W_{\alpha\beta\gamma\delta}^{\mu\tau\epsilon}$ is a collection of functions which are antisymmetric in $\mu\tau\epsilon$ and in $\alpha\beta\gamma\delta$.

We now proceed to show that we can continue like this indefinitely, or rather, until we run out of indices. That is, the algorithm continues until we try and produce a structure function with $k+1$ suffixes, which, of course, is zero by the antisymmetrisation property.

Theorem

The $(n+1)$ th structure functions $Q_{\beta_1, \dots, \beta_{n+2}}^{(n+1)} \alpha_1, \dots, \alpha_{n+1}$ are defined in terms of the structure functions of order p ($0 \leq p \leq n$), by

$$(n+1) Q_{\beta_1, \dots, \beta_{n+2}}^{(n+1)} \alpha_1, \dots, \alpha_{n+1} = B_{[\beta_1, \dots, \beta_{n+2}]}^{(n)} [\alpha_1, \dots, \alpha_n] \quad (3.2.7)$$

where

$$\begin{aligned} B_{\beta_1, \dots, \beta_{n+2}}^{(n)} \alpha_1, \dots, \alpha_n &= \frac{1}{2} \sum_{p=0}^n (-1)^{np+1} \left\{ Q_{\beta_1, \dots, \beta_{p+1}}^{(p)} \alpha_1, \dots, \alpha_p, Q_{\beta_{p+2}, \dots, \beta_{n+2}}^{(n-p)} \alpha_{p+1}, \dots, \alpha_n \right\} \\ &\quad - \sum_{p=0}^{n-1} (-1)^{n(p+1)} (p+1)(n-p+1) Q_{\beta_1, \dots, \beta_{p+2}}^{(p+1)} \alpha_1, \dots, \alpha_{p+1} \gamma, Q_{\beta_{p+3}, \dots, \beta_{n+2}}^{(n-p)} \alpha_{p+2}, \dots, \alpha_n \gamma \end{aligned} \quad (3.2.8)$$

In this expression, the zeroth order structure functions are the constraints

$$Q_{\alpha}^{(0)} = \phi_{\alpha} \quad (3.2.9)$$

Also note

$$Q_{\alpha\beta}^{(1)} \gamma = -\frac{1}{2} U_{\alpha\beta}^{\gamma} \quad (3.2.10)$$

The proof parallels one in (29), where a similar calculation is done for the case of mixed (i.e. fermionic and bosonic) constraints.

Proof

The proof is by induction. If $n=0$, then

$$\overset{(0)}{B}_{[\beta_1, \beta_2]} = -\frac{1}{2} \{Q_{\beta_1}, Q_{\beta_2}\} = -\frac{1}{2} U_{\beta_1 \beta_2}^\gamma \phi_\gamma \quad (3.2.11)$$

This is in agreement with (3.2.10).

Assume that the structure functions of order n ($0 \leq n \leq m$) have been defined as in the theorem. We now show that it is possible to define the $(m+1)$ th structure functions according to (3.2.7).

We take the Poisson bracket of $\phi_{\beta_{m+2}}$ with

$$m \overset{(m)}{Q}_{\beta_1, \dots, \beta_{m+1}}^{\alpha_1, \dots, \alpha_m} \phi_{\alpha_m} - \overset{(m-1)}{B}_{[\beta_1, \dots, \beta_{m+1}]}^{\alpha_1, \dots, \alpha_{m-1}} = 0 \quad (3.2.12)$$

and antisymmetrise with respect to $\beta_1, \beta_2, \dots, \beta_{m+2}$. This gives

$$\begin{aligned} 0 = m \left\{ \overset{(m)}{Q}_{[\beta_1, \dots, \beta_{m+1}]}^{\alpha_1, \dots, \alpha_{m-1}, \alpha_m}, \phi_{\beta_{m+2}} \right\} \phi_{\alpha_m} + 2m \overset{(m)}{Q}_{[\beta_1, \dots, \beta_{m+1}]}^{\alpha_1, \dots, \alpha_m} \overset{(1)}{Q}_{\beta_{m+2}] \alpha_m}^\gamma \phi_\gamma \\ - \left\{ \overset{(m-1)}{B}_{[\beta_1, \dots, \beta_{m+1}]}^{\alpha_1, \dots, \alpha_{m-1}}, \phi_{\beta_{m+2}} \right\} \end{aligned} \quad (3.2.13)$$

We need to simplify the last term (3.2.13). To do this, we note that for $0 \leq n \leq m$ we can substitute (3.2.7) into (3.2.8) and solve for $\{\overset{(n)}{Q}, \phi\}$ to get :

$$\begin{aligned} \left\{ \overset{(n)}{Q}_{[\beta_1, \dots, \beta_{n+1}]}^{\alpha_1, \dots, \alpha_n}, \phi_{\beta_{n+2}} \right\} = (-1)^{n+1} \overset{(n+1)}{Q}_{\beta_1, \dots, \beta_{n+2}}^{\alpha_1, \dots, \alpha_{n+1}} \phi_{\alpha_{n+1}} \\ - \frac{1}{2} \sum_{p=1}^{n-1} (-1)^{n(p+1)} \left\{ \overset{(p)}{Q}_{[\beta_1, \dots, \beta_{p+1}]}^{\alpha_1, \dots, \alpha_p}, \overset{(n-p)}{Q}_{\beta_{p+2}, \dots, \beta_{n+2}}^{\alpha_{p+1}, \dots, \alpha_n} \right\} \\ - \sum_{p=0}^{n-1} (-1)^{np} (p+1)(n-p+1) \overset{(p+1)}{Q}_{[\beta_1, \dots, \beta_{p+2}]}^{\alpha_1, \dots, \alpha_p, \gamma} \overset{(n-p)}{Q}_{\beta_{p+3}, \dots, \beta_{n+2}}^{\alpha_{p+1}, \dots, \alpha_n} \phi_\gamma \end{aligned} \quad (3.2.14)$$

Thus we can use Jacobi's identity on $\{\overset{(m-1)}{B}, \phi\}$ and substitute (3.2.14) into the resulting expression.

When this is done we are left with

$$\overset{(m)}{B}_{[\alpha_1, \dots, \alpha_m]}^{\phi_{\alpha_m}} = 0 \quad (3.2.15)$$

(In order to be left with (3.2.15) we must note that the sum of terms of the form

$$\left\{ \overset{(p)}{Q}, \left\{ \overset{(q)}{Q}, \overset{(m-p-q)}{Q} \right\} \right\}$$

which appears, is zero by Jacobi's identity.)

Hence by the exactness of the δ_0 mapping we can define $\overset{(m+1)}{Q}$ by (3.2.7).

Note that the structure functions are not uniquely defined. They are unique only up to the addition of a term of the form $\delta_\lambda(-)$, i.e. the addition of an antisymmetric combination of constraints. Note also that if we change the choice of $\overset{(p)}{Q}$, say, to $\overset{(p)}{Q} + \phi.W$ for some W , then this will change the possible choices for all $\overset{(p)}{Q}$ with $l \geq p$. We will have to take this behaviour into account, when using these structure functions to define antiderivations on Ω .

3.3 Some Examples

In this section, we take a brief look at situations where the analysis in 3.2 is valid. As mentioned earlier, gravity has structure functions present in the Poisson bracket algebra for the superHamiltonian constraint. In fact, it can be shown (7,28) that despite the appearance of the canonical coordinates $g_{(i)}^{ij}$ in this expression, the higher order structure functions, $Q^{(i)} (i \geq 2)$ can be taken to be zero. In the classification of Fradkin et al (7), a theory in which $Q^{(m)} \neq 0$ while $Q^{(n)} = 0 (n > m)$ is called a theory of rank m. Thus gravity is of rank one. Note that rank is not an intrinsic characterisation of a theory. By using the ambiguity in the structure functions, that is by adding an antisymmetric combination of constraints, we can change (in general, increase) the rank of a theory. Clearly gravity cannot be of lower rank than one, so this is the minimal rank possible for the theory.

There are examples of theories of rank greater than one. It can be shown that supergravity is an example of a theory of rank two (30).

The discussion of structure functions in the previous section showed that the rank of a theory is, at most, equal to the number of constraints less one. It is reasonable to

ask if there is any theory which must have this maximal rank. There is one known example, the n dimensional relativistic membrane (28), embedded in d dimensional minkowski space. The action for this theory is

$$S[X^A(x^\mu)] = - \int [-^{(n+1)}g]^{1/2} d^{n+1}x \quad (3.3.1)$$

In this expression, X^A are coordinates in the d dimensional flat spacetime ($A=1, \dots, d$), $X^A = X^A(x^\mu)$ are the parametric equations of the membrane trajectory, and $^{(n+1)}g$ is the determinant of the metric induced on the trajectory. Here the indices $\mu=0, 1, \dots, n$, while latin indices run over $1, 2, \dots, n$ and refer to the membrane at a given time.

Since the action in (3.3.1) is invariant under reparameterisations of the trajectory, the Legendre transformation is singular, and so the techniques of the Dirac-Bergmann algorithm must be used. This leads to an action in the canonical form:

$$S_H[X^A, P_A, N, N^k] = \int dx^0 \int dx (\dot{X}^A P_A - N^L \mathcal{H}_L - N^k \mathcal{H}_k) \quad (3.3.2)$$

In this expression N^L and N^k are arbitrary Lagrangian multipliers, and the superHamiltonian and supermomentum constraints are given by

$$\mathcal{H}_1 = \frac{1}{2} (P_A P^A + g) \quad ; \quad \mathcal{H}_k = P_A X^A_{,k} \quad (3.3.3)$$

respectively, where g is the determinant of the riemannian metric

$$g_{rs}(x) = \eta_{AB} X^A_{,r}(x) X^B_{,s}(x) \quad (3.3.4)$$

induced on the membrane at a given instant x^0 . These constraints generate canonical transformations which leave the action invariant (once appropriate variations of the lapse and shift functions are prescribed). These canonical transformations correspond to the reparameterisation transformations.

This theory has rank n . The explicit expressions for the structure functions are simplest if written in terms of

$$\Omega^{(m)}_{\alpha_1, \alpha_2, \dots, \alpha_m} = Q^{(m)}_{\beta_1, \dots, \beta_{m+1}} \eta^{\beta_1} \dots \eta^{\beta_{m+1}} \quad (3.3.5)$$

where the $\{\eta^\beta\}$ are a set of $n+1$ anticommuting elements. In this case we have (28),

$$\begin{aligned} \Omega^{(1)}{}^\perp &= \eta^\perp \eta^k_{,k} - \eta^\perp_{,k} \eta^k \\ \Omega^{(1)}{}^k &= g^{kj} g \eta^\perp \eta^\perp_{,j} + \eta^j \eta^k_{,j} \\ \Omega^{(u)}{}^{k_1, \dots, k_u}(x^1, \dots, x^u) &= \frac{2^{u-1}}{u!} \frac{\delta^u g}{\delta g_{k_u j_u} \dots \delta g_{k_1 j_1}} \eta^\perp \eta^\perp_{,j_1} \dots \eta^\perp_{,j_u} \delta(x^1, x^1) \dots \delta(x^{u-1}, x^u) \end{aligned} \quad (3.3.6)$$

The reason that the membrane differs in this respect from gravity can be traced back to the fact that the metric of the membrane involves the derivatives of the canonical variables (i.e. the $\chi^A_{,k}$). This in itself is a consequence of the embedding of the membrane in a larger geometric structure.

Thus we see that the higher order structure functions do play a role in theories of physical interest, and we even have a plausible model in which structure functions of the highest order are nonzero. We will return to this example when we consider B.R.S.T. symmetry in quantisation in Chapter Four.

3.4 B.R.S.T. Symmetry in Higher Order Theories

We have already seen that much of the structure of the cohomological approach is lost when we do not have a momentum map. However, we have also seen that some of the structure does remain valid, notably the antighost complex. The main difficulty is the fact that the ghost complex is no longer a complex since $\delta_1^2 \neq 0$. We now consider what extensions are necessary to allow this approach to be applied to higher order theories.

We are given a symplectic manifold $\mathcal{S}$ and a coisotropic submanifold $\mathcal{K}$ described by k constraints

$$\phi_\alpha = 0 \quad \alpha = 1, 2, \dots, k \quad (3.4.1)$$

These constraints, being first class, satisfy

$$\{\phi_\alpha, \phi_\beta\} = U_{\alpha\beta}^\gamma \phi_\gamma \quad (3.4.2)$$

Thus we can use the theorem in section 3.2 to define higher order structure functions. We extend the phase space $\mathcal{S}$ by the addition of $2k$ anticommuting coordinates $(\eta^\alpha, \rho_\beta)$. Then we denote by Ω^{rs} the set of all expressions of the form

$$V_{\beta_1 \dots \beta_r}^{\alpha_1 \dots \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_r} \rho_{\alpha_1} \rho_{\alpha_2} \dots \rho_{\alpha_s} \quad (3.4.3)$$

where $V_{\beta_1, \dots, \beta_r}^{\alpha_1, \dots, \alpha_s}$ is a set of $C^\infty(\mathcal{G})$ functions, totally antisymmetric in $\alpha_1, \alpha_2, \dots, \alpha_s$, and $\beta_1, \beta_2, \dots, \beta_r$. We refer to the η^α as ghosts, and to the ρ_α as antighosts. (This mimics the introduction of the Lie algebra of the group $\mathcal{D}$ in Chapter Two. We clearly cannot seek an identification of η^α and ρ_β with bases of $\mathcal{L}(\mathcal{D})$ or its dual, but we will show that enough structure remains to allow an essentially unique extension of the methods of Chapter Two).

We define

$$\Omega = \bigoplus_{r,s} \Omega^{r,s} \quad (3.4.4)$$

Then Ω is a positively bigraded space, with the gradings being given by the number of ghosts, and the number of antighosts. We define the ghost number of an element in $\Omega^{r,s}$ to be $r-s$. Then Ω can be singly graded by ghost number.

$$\Omega^q = \bigoplus_{r-s=q} \Omega^{r,s} \quad (3.4.5)$$

We wish to define an antiderivation which maps Ω^q to Ω^{q+1} , and which reduces to the analysis of Chapter Two when we have a group action and momentum map available.

With this in mind, we investigate the difficulties

which arise in the formalism of Chapter Two.

As we saw, we could still define the antiderivation δ_0 , which defined a complex

$$0 \rightarrow \Omega^{rk} \rightarrow \Omega^{rk-1} \rightarrow \dots \rightarrow \Omega^{r2} \rightarrow \Omega^{r1} \rightarrow \Omega^{r0} \rightarrow 0 \quad (3.4.6)$$

for each r . This complex is exact except at Ω^{r0} . The antiderivation δ_0 is defined by

$$\delta_0(V_{\beta_1, \dots, \beta_r}^{\alpha_1, \dots, \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_r} \rho_{\alpha_1} \dots \rho_{\alpha_s}) = (-1)^r \phi_{\gamma} V_{\beta_1, \dots, \beta_r}^{\gamma \alpha_1 \dots \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_r} \rho_{\alpha_1} \dots \rho_{\alpha_s} \quad (3.4.7)$$

and is extended to Ω by linearity.

The difficulties which arise, appear in the ghost complex. If we seek a homological characterisation of invariant functions, we clearly wish to retain the antiderivation δ_1 , with its action on $f \in C^\infty(\mathfrak{g})$ given by

$$\delta_1 f = \{ \phi_\alpha, f \} \eta^\alpha \quad (3.4.8)$$

However, in this case,

$$\delta_1^2 f = \left((H_{\beta\alpha}^\gamma + \frac{1}{2} U_{\beta\alpha}^\gamma) \{ \phi_\gamma, f \} + \frac{1}{2} \phi_\gamma [U_{\beta\alpha}^\gamma, f] \right) \eta^\beta \eta^\alpha \quad (3.4.9)$$

where we have written

$$\delta_1 \eta^\alpha = H_{\beta\gamma}^\alpha \eta^\beta \eta^\gamma \quad (3.4.10)$$

and assumed that δ_1 is an antiderivation. It is clear that $\delta_1^2 f \neq 0$ for any choice of $H_{\alpha\beta}^\gamma$, provided that the $U_{\alpha\beta}^\gamma$ are not constant. This stops the development of the ghost complex as a complex. On the other hand, if as before, we define

$$\delta_1 \eta^\alpha = -\frac{1}{2} U_{\beta\gamma}^\alpha \eta^\beta \eta^\gamma \quad (3.4.11)$$

then

$$\delta_1^2 f = \delta_0 \left(\frac{1}{2} \{ U_{\alpha\beta}^\gamma, f \} \eta^\alpha \eta^\beta \rho_\gamma \right) \quad (3.4.12)$$

Thus, if we define

$$\delta_2 f = -\frac{1}{2} \{ U_{\alpha\beta}^\gamma, f \} \eta^\alpha \eta^\beta \rho_\gamma \quad (3.4.13)$$

we have

$$(\delta_1^2 + \delta_0 \delta_2) f = 0 \quad (3.4.14)$$

Thus we have the situation

$$\begin{array}{ccccc} f \in \Omega^{\infty} & \xrightarrow{\delta_1} & \Omega^{10} & \xrightarrow{\delta_1} & \Omega^{20} \\ & \searrow \delta_2 & & & \downarrow \\ & & & & \Omega^{21} \ni \delta_2 f \end{array} \quad (3.4.15)$$

This suggests that we should introduce an antiderivation $\delta: \Omega^r \rightarrow \Omega^{r+2s+1}$, and then define the B.R.S.T. operator δ as $\delta_0 + \delta_1 + \delta_2$, where the properties of $\delta_0, \delta_1, \delta_2$ are chosen in order to satisfy $\delta^2 = 0$. (Note that we have yet to define $\delta_1 \eta$ or $\delta_2 \rho$). Consider an arbitrary element $\omega \in \Omega^r$. Then we can decompose its image under δ^2 by

$$\begin{array}{rclcl}
 & \delta_0^2 \omega & & & \\
 & \uparrow & & & \\
 \delta_0^2 \omega & \rightarrow & (\delta_1 \delta_0 + \delta_0 \delta_1) \omega & & \\
 \uparrow & & \uparrow & & \\
 \omega & \rightarrow & \delta_1 \omega & \rightarrow & (\delta_1 \delta_0 + \delta_0^2 + \delta_0 \delta_1) \omega \\
 & \searrow & & \uparrow & \\
 & & \delta_2 \omega & \rightarrow & (\delta_2 \delta_1 + \delta_1 \delta_2) \omega \\
 & & & & \searrow \\
 & & & & \delta_2^2 \omega
 \end{array} \tag{3.4.16}$$

We see that we must define δ_2 with the properties

$$\begin{aligned}
 \delta_0^2 &= 0 & (a) \\
 \delta_0 \delta_1 + \delta_1 \delta_0 &= 0 & (b) \\
 \delta_0 \delta_1 + \delta_1^2 + \delta_2 \delta_0 &= 0 & (c) \\
 \delta_1 \delta_2 + \delta_2 \delta_1 &= 0 & (d) \\
 \delta_2^2 &= 0 & (e)
 \end{aligned} \tag{3.4.17}$$

We already know that (3.4.17)(a) is true. If we define δ_1 by (2.2.6/7) and (2.3.20), then (3.4.17)(b) is true as before. We need only consider the properties of δ_2 .

With the definition of $\delta_1 f$ by (3.4.13) we see that (3.4.17)(c) is true when acting on functions.

Consider $\delta_1^2 \eta$:

$$\delta_1 \eta^\alpha = -\frac{1}{2} \left[\{ \phi_\beta, u_{\gamma\delta}^\alpha \} - u_{\epsilon\beta}^\alpha u_{\gamma\delta}^\epsilon \right] \eta^\beta \eta^\gamma \eta^\delta \quad (3.4.18)$$

Then from the definition of the structure functions (equations 3.2.5) we can express this as

$$\delta_1 \eta^\alpha = -\frac{1}{2} \phi_\epsilon V_{\beta\gamma\delta}^{\epsilon\alpha} \eta^\beta \eta^\gamma \eta^\delta \quad (3.4.19)$$

If we define

$$\delta_2 \eta^\alpha = \frac{1}{2} V_{\beta\gamma\delta}^{\alpha\epsilon} \eta^\beta \eta^\gamma \eta^\delta \rho_\epsilon \quad (3.4.20)$$

then equation (3.4.17)(c) is true when acting on ghosts.

Similarly by simplifying $(\delta_1 \delta_0 + \delta_1^2) \rho_\alpha$, one can show that if $\delta_2 \rho_\alpha$ is defined by

$$\delta_2 \rho_\alpha = -\frac{2}{3} V_{\alpha\beta\gamma}^{\delta\epsilon} \eta^\beta \eta^\gamma \rho_\delta \rho_\epsilon \quad (3.4.21)$$

then (3.4.17)(c) is true when acting on antighosts. Thus if we define δ_2 by (3.4.13), (3.4.20) and (3.4.21), and extend to the whole of Ω by using its properties as a antiderivation, then (3.4.17)(c) is true.

Rather than try and impose the remainder of (3.4.17) on f , η and ρ , which would impose conditions on the structure functions, we proceed to the general case, using the above calculation as a guide.

We have defined the bigraded space Ω , with the antiderivations δ_0 and δ_1 , defined on it.

$$\begin{array}{ccccccc}
 \Omega^{00} & \longrightarrow & \Omega^{10} & \longrightarrow & \Omega^{20} & \longrightarrow & \dots \longrightarrow \Omega^{k0} \longrightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \Omega^{01} & \longrightarrow & \Omega^{11} & \longrightarrow & \Omega^{21} & \longrightarrow & \dots \longrightarrow \Omega^{k1} \longrightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \Omega^{02} & \longrightarrow & \Omega^{12} & \longrightarrow & \Omega^{22} & \longrightarrow & \dots \longrightarrow \Omega^{k2} \longrightarrow 0 \\
 \vdots & & \vdots & & \vdots & & \vdots \\
 \Omega^{0k} & \longrightarrow & \Omega^{1k} & \longrightarrow & \Omega^{2k} & \longrightarrow & \dots \longrightarrow \Omega^{kk} \longrightarrow 0 \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & & 0 & & 0 & & 0
 \end{array} \quad (3.4.22)$$

Here we make the convention that the vertical arrows represent δ_0 , while the horizontal ones represent δ_1 .

We postulate the existence of antiderivations $\delta_1, \delta_2, \dots, \delta_k$, where

$$\delta_i : \Omega^{rs} \longrightarrow \Omega^{r+i, s+i-1} \quad (3.4.23)$$

which are such that $\delta = \delta_0 + \delta_1 + \dots + \delta_k$ satisfies

$$\delta^2 = 0 \quad (3.4.24)$$

From this, we can derive conditions on the δ_i :

$$\begin{aligned}
 \delta_0^2 &= 0 \\
 \delta_0 \delta_1 + \delta_1 \delta_0 &= 0 \\
 \delta_0 \delta_2 + \delta_1^2 + \delta_2 \delta_0 &= 0 \\
 \delta_0 \delta_3 + \delta_1 \delta_2 + \delta_2 \delta_1 + \delta_3 \delta_0 &= 0 \\
 &\vdots
 \end{aligned}$$

$$\begin{aligned}
& \delta_{k-2} \delta_k + \delta_{k-1}^2 + \delta_k \delta_{k-2} = 0 \\
& \delta_{k-1} \delta_k + \delta_k \delta_{k-1} = 0 \\
& \delta_k^2 = 0
\end{aligned} \tag{3.4.25}$$

From the point of view of proving the existence of these antiderivations, it is easier to look for antiderivations δ_i ($i \geq 0$) satisfying (3.4.23), and all equations

$$\sum_{i=0}^m \delta_i \delta_{m-i} = 0 \tag{3.4.26}$$

for all $m \geq 0$. Then the conditions (3.4.23) imply $\delta_i = 0$ for $i > k$.

We can now show that antiderivations $\delta_0, \delta_1, \dots, \delta_k$ with the properties (3.4.25) do exist.

Given constraints ϕ_α satisfying (3.4.2), we can use the results of section 3.2 to choose a set of structure functions of all orders $\{Q_{\alpha}^{(0)}, Q_{\alpha, \alpha_2}^{(1)}, \dots, Q_{\alpha, \alpha_2, \dots, \alpha_m}^{(k-1)}\}$. Then if we define

$$\delta_m f = \left\{ Q_{\beta_1, \dots, \beta_m}^{(m-1), \alpha_1, \dots, \alpha_{m-1}}, f \right\} \eta^{\beta_1} \eta^{\beta_2} \dots \eta^{\beta_m} \rho_{\alpha_1} \rho_{\alpha_2} \dots \rho_{\alpha_{m-1}} \tag{3.4.27}$$

$$\delta_m \eta^\alpha = (-1)^{m+1} Q_{\beta_1, \dots, \beta_{m+1}}^{(m), \alpha, \alpha_2, \dots, \alpha_m} \eta^{\beta_1} \dots \eta^{\beta_{m+1}} \rho_{\alpha_2} \dots \rho_{\alpha_m} \tag{3.4.28}$$

$$\delta_m \rho_\alpha = (m+1) Q_{\alpha, \beta_2, \dots, \beta_{m+1}}^{(m), \alpha_1, \dots, \alpha_m} \eta^{\beta_2} \dots \eta^{\beta_{m+1}} \rho_{\alpha_1} \dots \rho_{\alpha_m} \tag{3.4.29}$$

for $0 \leq m \leq k$, extend δ_m to $\Omega^{\leq k}$ as an antiderivation and to Ω by linearity, then the resulting antiderivations satisfy (3.4.25).

We prove this by assuming these forms for δ_m ($0 \leq m \leq s$), and proving that to satisfy

$$\delta_{s+1}\delta_0 + \delta_s\delta_1 + \dots + \delta_1\delta_s + \delta_0\delta_{s+1} = 0 \quad (3.4.30)$$

we can take δ_{s+1} defined by the above equations. The result will then follow from induction.

It is clear from section 3.3 that for $m=0$ and 1 the above equations are

$$\begin{aligned} \delta_0 \eta &= \delta_0 f = 0 & \delta_0 \rho_\alpha &= \phi_\alpha \\ \delta_1 \eta^\alpha &= -\frac{1}{2} U_{\beta\gamma}^\alpha \eta^\beta \eta^\gamma & \delta_1 \rho_\alpha &= -U_{\alpha\beta}^\gamma \eta^\beta \rho_\gamma \\ \delta_1 f &= \{ \phi_\alpha, f \} \eta^\alpha \end{aligned} \quad (3.4.31)$$

These antiderivations satisfy

$$\delta_0^2 = 0 \quad ; \quad \delta_0 \delta_1 + \delta_1 \delta_0 = 0$$

Assume we take $\delta_0, \delta_1, \dots, \delta_s$ as above. We show that to satisfy (3.4.30), we can take δ_{s+1} in line with the above expressions.

In order to do this consider the action of

$$\delta_{s+1}\delta_0 + \delta_s\delta_1 + \dots + \delta_1\delta_s$$

on each of functions, ghosts and antighosts. Note that for functions and ghosts, the term $\delta_{s+1}\delta_0$ drops out, so that we

have no difficulty

$$\begin{aligned}
 \sum_{i=1}^s \delta_i \delta_{s+1-i} f &= \sum_{i=1}^s \delta_i \left(\left\{ Q_{\beta_1, \dots, \beta_{s+1-i}}^{(s-i)} \right\} \eta^{\beta_1} \dots \eta^{\beta_{s+1-i}} \rho_{\alpha_1} \dots \rho_{\alpha_{s-i}} \right) \\
 &= \sum_{i=1}^s \delta_i \left(\left\{ Q_{\beta_1, \dots, \beta_{s+1-i}}^{(s-i)} \right\} \eta^{\beta_1} \dots \eta^{\beta_{s+1-i}} \rho_{\alpha_1} \dots \rho_{\alpha_{s-i}} \right) \\
 &\quad + \sum_{i=1}^s \left\{ Q_{\beta_1, \dots, \beta_{s+1-i}}^{(s-i)} \right\} (s-i+1) (\delta_i \eta^{\beta_1}) \eta^{\beta_2} \dots \eta^{\beta_{s+1-i}} \rho_{\alpha_1} \dots \rho_{\alpha_{s-i}} \\
 &\quad + \sum_{i=1}^s \left\{ Q_{\beta_1, \dots, \beta_{s+1-i}}^{(s-i)} \right\} \eta^{\beta_1} \dots \eta^{\beta_{s+1-i}} (s-i) (-1) (\delta_i \rho_{\alpha_1}) \rho_{\alpha_2} \dots \rho_{\alpha_{s-i}}
 \end{aligned}$$

where we have used the antiderivation property of the δ_i and the antisymmetry in $\alpha_1, \dots, \alpha_{s-i}$ and $\beta_1, \dots, \beta_{s+1-i}$. If we substitute from (3.4.27-29) and simplify we get

$$\sum_{i=1}^s \delta_i \delta_{s+1-i} f = \left\{ -B_{\beta_1, \dots, \beta_{s+1}}^{(s-1)} \right\} f + \left\{ \phi_2, f \right\} s(-1)^{s+1} Q_{\beta_1, \dots, \beta_{s+1}}^{(s)} \eta^{\beta_1} \dots \eta^{\beta_{s+1}} \rho_{\alpha_1} \dots \rho_{\alpha_{s-1}} \quad (3.4.32)$$

where $B^{(s-1)}$ is given by (3.2.8). Then, by the definition of $Q^{(s)}$ in (3.2.7), we have

$$\sum_{i=1}^s \delta_i \delta_{s+1-i} f = \delta_0 \left(- \left\{ Q_{\beta_1, \dots, \beta_{s+1}}^{(s)} \right\} \eta^{\beta_1} \dots \eta^{\beta_{s+1}} \rho_{\alpha_1} \dots \rho_{\alpha_s} \right) \quad (3.4.33)$$

Thus if we take

$$\delta_{s+1} f = \left\{ Q_{\beta_1, \dots, \beta_{s+1}}^{(s)} \right\} \eta^{\beta_1} \dots \eta^{\beta_{s+1}} \rho_{\alpha_1} \dots \rho_{\alpha_s} \quad (3.4.34)$$

then (3.4.30) is true on functions. This choice for $\delta_{s+1} f$ is

in line with (3.4.27).

Similarly we can show

$$\begin{aligned} \sum_{i=1}^s \delta_i \delta_{s+1-i} \eta^\alpha &= (-1)^s \tilde{B}_{\beta_1, \dots, \beta_{s+2}}^{(s) \alpha_1, \dots, \alpha_{s-1}} \eta^{\beta_1} \dots \eta^{\beta_{s+2}} \rho_{\alpha_1, \dots, \rho_{\alpha_{s-1}}} \\ &= + \delta_o (-1)^{s+1} (s+1) Q_{\beta_1, \dots, \beta_{s+2}}^{(s+1) \alpha_1, \dots, \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_{s+2}} \rho_{\alpha_1, \dots, \rho_{\alpha_s}} \end{aligned} \quad (3.4.35)$$

Thus we can take

$$\delta_{s+1} \eta^\alpha = (-1)^s (s+1) Q_{\beta_1, \dots, \beta_{s+2}}^{(s+1) \alpha_1, \dots, \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_{s+2}} \rho_{\alpha_1, \dots, \rho_{\alpha_s}} \quad (3.4.36)$$

again, in line with (3.4.28).

When considering antighosts, we must note that the $\delta_{s+1} \delta_o$ term is not zero. On the other hand, since $\delta_{s+1} \delta_o \rho_\alpha = \delta_{s+1} \phi_\alpha$, and we already know the expression for this, this causes no difficulties.

$$\begin{aligned} \sum_{i=1}^{s+1} \delta_i \delta_{s+1-i} \rho_\alpha &= (s+2) \tilde{B}_{\alpha \beta_1, \dots, \beta_{s+1}}^{(s) \alpha_1, \dots, \alpha_s} \eta^{\beta_1} \dots \eta^{\beta_{s+1}} \rho_{\alpha_1, \dots, \rho_{\alpha_s}} \\ &= - \delta_o ((s+2) Q_{\alpha \beta_1, \dots, \beta_{s+1}}^{(s+1) \alpha_1, \dots, \alpha_{s+1}} \eta^{\beta_1} \dots \eta^{\beta_{s+1}} \rho_{\alpha_1, \dots, \rho_{\alpha_{s+1}}}) \end{aligned}$$

Thus we can choose

$$\delta_{s+1} \rho_\alpha = (s+2) Q_{\alpha \beta_1, \dots, \beta_{s+1}}^{(s+1) \alpha_1, \dots, \alpha_{s+1}} \eta^{\beta_1} \dots \eta^{\beta_{s+1}} \rho_{\alpha_1, \dots, \rho_{\alpha_{s+1}}} \quad (3.4.37)$$

This is again in line with (3.4.29)

Thus, by induction, we can find antiderivations $\delta_o, \delta_1, \dots, \delta_k$,

with the property that $(\delta_0 + \delta_1 + \dots + \delta_k)^2 = 0$ and which reduces to the form of Chapter Two when we have structure constants.

However, the question arises as to the uniqueness of this extension. There are two sources of nonuniqueness:

- 1) As discussed in section 3.2, there is an ambiguity in the definitions of the higher order structure functions;
- 2) In the calculation above, we substituted an expression of the form $\phi \cdot Q^{(s)}$ for a $B^{(s-1)}$ at three separate points. We could have chosen a different $Q^{(s)}$ on each occasion.

We now look for the most general form for $\{\delta_i\}$, in terms of a particular choice $\{\hat{\delta}_i\}$.

First, we make a choice of structure functions $Q^{(1)}, Q^{(2)}, Q^{(3)}, \dots, Q^{(k-1)}$. From this we can find expressions for $\delta_0, \delta_1, \dots, \delta_k$ with the aid of (3.4.27-29). We denote this particular choice by $\hat{\delta}_0, \hat{\delta}_1, \dots, \hat{\delta}_k$. We now look for the most general form for $\{\delta_i\}$.

We wish to keep δ_0 as before, as its resolution property is crucial to the approach. We also want to keep

$$\delta_i f = \{\phi_i, f\} \eta^\alpha = \hat{\delta}_i f \quad (3.4.38)$$

We now consider the most general form for $\delta_i \eta$ and $\delta_i \rho$. From (3.4.9) we have

$$\delta_1 f = \left((H_{\alpha\beta}^{\gamma} - \frac{1}{2} U_{\beta\alpha}^{\gamma}) \{\phi_r, f\} + \frac{1}{2} \phi_r \{U_{\alpha\beta}^{\gamma}, f\} \right) \eta^{\alpha} \eta^{\beta} \quad (3.4.39)$$

where

$$\delta_1 \eta^{\alpha} = H_{\beta\gamma}^{\alpha} \eta^{\beta} \eta^{\gamma} \quad (3.4.40)$$

The only possibility for $H_{\beta\gamma}^{\alpha}$ if this expression is to be of the form $\delta_0(-)$ is

$$H_{\alpha\beta}^{\gamma} = Q_{\alpha\beta}^{\gamma} + \phi_{\delta} F_{[\alpha\beta]}^{[\delta\gamma]} \quad (3.4.41)$$

for some undetermined $F_{\alpha\beta}^{\delta\gamma}$. Thus we can write

$$\delta_1 \eta^{\alpha} = \hat{\delta}_1 \eta^{\alpha} - \delta_0 \chi_1 \eta^{\alpha} \quad (3.4.42)$$

where $\chi_1: \Omega^2 \rightarrow \Omega^2$ is a derivation with properties

$$\begin{aligned} \chi_1 f &= 0 \quad \forall f \in C^{\infty}(\mathcal{S}) \\ \chi_1 \eta^{\alpha} &= F_{[\beta\gamma]}^{[\alpha\delta]} \eta^{\beta} \eta^{\gamma} \rho_{\delta} \end{aligned} \quad (3.4.43)$$

If we consider

$$\delta_1 \delta_0 \rho_{\alpha} = \delta_1 \phi_{\alpha} = \{\phi_{\beta}, \phi_{\alpha}\} \eta^{\beta} = \phi_{\gamma} U_{\beta\alpha}^{\gamma} \eta^{\beta} \quad (3.4.44)$$

Then $\delta_1 \rho_{\alpha}$ must be of the form

$$\delta_1 \rho_\alpha = 2 Q_{\alpha\beta}^r \eta^\beta \rho_r + \Phi_z G_{[\alpha\beta]}^{[z\delta]} \eta^\beta \rho_\delta \quad (3.4.45)$$

for some arbitrary $G_{\alpha\beta}^{rs}$. Again we can write this as

$$\delta_1 \rho_\alpha = \hat{\delta}_1 \rho_\alpha - \delta_0 \kappa_1 \rho_\alpha \quad (3.4.46)$$

where we extend the derivation $\kappa_1: \Omega^{rs} \rightarrow \Omega^{(r+1)s+1}$ to act on the antighosts by

$$\kappa_1 \rho_\alpha = \frac{1}{2} G_{[\alpha\beta]}^{rs} \eta^\beta \rho_s \rho_\delta \quad (3.4.47)$$

This suggests we write

$$\delta_1 = \hat{\delta}_1 - \delta_0 \kappa_1 \quad (3.4.48)$$

However the right hand side is not an antiderivation. To get around this note that if we took

$$\delta_1 = \hat{\delta}_1 + \kappa_1 \delta_0 - \delta_0 \kappa_1 \quad (3.4.49)$$

then in the case of interest above, (3.4.49) reduces to (3.4.48), and the right hand side is an antiderivation (This can be shown by noting that $\kappa_1 \delta_0 - \delta_0 \kappa_1$ is the commutator of an antiderivation and a derivation, and so is an antiderivation.) Thus the most general solution for δ_1 ,

consistant with $\delta_0^2 = \delta_1 \delta_0 + \delta_0 \delta_1 = 0$ is (3.4.49), with χ_1 an arbitrary derivation.

We now look for the most general expression for δ_2 , satisfying $\delta_2 \delta_0 + \delta_1^2 + \delta_0 \delta_2 = 0$ where δ_1 is given by (3.4.45). We wish to write δ_2 in terms of $\hat{\delta}_1$, $\hat{\delta}_2$ and χ_1 . By close examination of the relevant equations, we see that we pick up another arbitrary derivation

$$\chi_2 : \Omega^{rs} \rightarrow \Omega^{r+2, s+2} \quad (3.4.50)$$

The expression for δ_2 is

$$\delta_2 = \hat{\delta}_2 + \chi_2 \delta_0 - \delta_0 \chi_2 + \chi_1 \delta_1 - \delta_1 \chi_1 + \frac{1}{2} \chi_1 (\chi_1 \delta_0 - \delta_0 \chi_1) - \frac{1}{2} (\chi_1 \delta_0 - \delta_0 \chi_1) \chi_1, \quad (3.4.51)$$

This is clearly a antiderivation.

We can write these expressions in a way which makes the form of the general δ_i apparent. first we define $[\alpha, \beta]$ as $\alpha\beta - \beta\alpha$ where β is an antiderivation and α is a derivation. Then $[\alpha, \beta]$ is antiderivation. We can write the general form of δ_1 and δ_2 as

$$\delta_1 = \hat{\delta}_1 + [\chi_1, \delta_0] \quad (3.4.52)$$

$$\delta_2 = \hat{\delta}_2 + [\kappa_1, \delta_0] + [\kappa_1, \hat{\delta}_1] + \frac{1}{2} [\kappa_1, [\kappa_1, \delta_0]] \quad (3.4.53)$$

By the same approach as above, we can show that

$$\begin{aligned} \delta_3 = & \hat{\delta}_3 + [\kappa_3, \delta_0] + [\kappa_2, \hat{\delta}_1] + [\kappa_1, \hat{\delta}_2] \\ & + \frac{1}{2} [\kappa_1, [\kappa_1, \hat{\delta}_1]] + \frac{1}{2} [\kappa_2, [\kappa_1, \delta_0]] \\ & + \frac{1}{2} [\kappa_1, [\kappa_2, \delta_0]] + \frac{1}{3!} [\kappa_1, [\kappa_1, [\kappa_1, \delta_0]]] \end{aligned} \quad (3.4.54)$$

where

$$\kappa_3 : \Omega^{rs} \longrightarrow \Omega^{r+3 \ s+3} \quad (3.4.55)$$

is an arbitrary derivation.

It is clear what the general form for δ_j ($3 \leq j \leq k$) will be:- it will be a sum of $\hat{\delta}_j$ with all multiple, distinct commutators of the form

$$\frac{1}{r!} [\kappa_{l_1}, [\kappa_{l_2}, \dots [\kappa_{l_r}, \hat{\delta}_a]] \dots] \quad (3.4.56)$$

with $\sum_{s=1}^r l_s + a = j$ where $\kappa_i : \Omega^{rs} \rightarrow \Omega^{r+is+i}$ are arbitrary derivations.

(In the next section we will write it in a form more amenable to calculations)

Thus we have found the most general choice of antiderivations $\{\delta_j\}$ which have the desired properties. However, unlike the case in Chapter Two, these

antiderivations are not unique. The uniqueness in Chapter Two can be traced back to two conditions.

- 1) The choice of structure constants. (Requiring them to be constant automatically sets the higher order functions to be zero)
- 2) Requiring that $\delta_1^2 = 0$ stopped the introduction of any arbitrary derivations.

This ambiguity in the definitions of the $\{\delta_i\}$ has no physical content. This is because the quantities of physical interest (e.g. the observables) are calculated from the ghost number complex. This is the singly graded complex

$$\Omega^q = \bigoplus_{r+s=q} \Omega^{rs} \quad (3.4.57)$$

with differential operator

$$\delta = \delta_0 + \delta_1 + \dots + \delta_k \quad (3.4.58)$$

The operator δ is called the B.R.S.T. operator and

$$\delta : \Omega^q \longrightarrow \Omega^{q+1} \quad (3.4.59)$$

In the next section we shall prove that despite the arbitrariness entering into δ via the undetermined derivations $\{\kappa_i\}$, that the cohomology associated with the

ghost number complex is unique.

3.5 Uniqueness of Ghost Number Complex

We have seen that an extension of the methods of Chapter Two is possible, and that such an extension is not unique. We now show that this is not a real problem. The quantities of physical interest e.g. the observables, are to be related to the cohomology of the ghost number complex. If we can show that for any two choices of $\delta_0, \delta_1, \dots, \delta_k$, the cohomology of this complex is the same, then this will suffice, since no arbitrariness will be attached to the observables.

In order to show that the cohomology is the same, we construct a chain equivalence between the ghost number complex defined using $\delta = \sum \delta_i$ and $\hat{\delta} = \sum \hat{\delta}_i$. In order to do this we must put the expressions (3.4.52) etc. into a form amenable to calculation.

It can be shown that the general form for δ_m ($0 \leq m \leq k$) is

$$\delta_m = \hat{\delta}_m + \sum_{s \geq 0; t \geq 0} \frac{(-1)^t}{s! t!} \chi_{i_1} \chi_{i_2} \dots \chi_{i_s} \hat{\delta}_a \chi_{i_{s+1}} \dots \chi_{i_{s+t}} \quad (3.5.1)$$

where the sum is over all s and t , and the indices satisfy $\sum_{j=1}^{s+t} i_j + a = m$. In this expression the χ_i are arbitrary derivations with $\chi_i: \Omega^{rs} \rightarrow \Omega^{r+i, s+i}$.

To see that this is the same as (3.4.52), consider a term in (3.5.1) with all χ_i different. This term has contributions from all the commutators with $\chi_{i_1}, \dots, \chi_{i_s}$ present in that order and $\chi_{i_{s+1}}, \dots, \chi_{i_{s+t}}$ present in that order. There are

$$\frac{(s+t)!}{t! s!} \quad (3.5.2)$$

ways of doing this. Since each commutator is multiplied by

$$\frac{1}{(s+t)!} \quad (3.5.3)$$

and would gain a minus sign for every derivation on the right of $\hat{\delta}_a$, we reach (3.5.1). The case with some of the the same is similar.

Thus the expression for δ_m is equivalent to that developed in section 3.4. This means that it is an antiderivation as required. (The form for δ_m in section 3.4 is explicitly an antiderivation.) We show that (3.5.1) satisfy (3.4.26) directly.

The desired identity (3.4.26) is

$$\sum \delta_a \delta_{b-a} = 0 \quad (3.5.4)$$

where b is fixed, and a sums over all the integers. (We take the convention $\delta_a = 0; a > k$ and $\delta_a = 0; a < 0$). Substituting (3.5.1) into (3.5.4) gives a sum of terms of the form

$$C x_{i_1} \dots x_{i_s} \hat{\delta}_c x_{i_{s+1}} \dots x_{i_t} \hat{\delta}_d x_{i_{t+1}} \dots x_{i_n} \quad (3.5.5)$$

where $C \in \mathbb{R}$ and $\sum_{j=1}^n i_j + c + d = b$. We show this is zero.

Case (i) $t=s$

In this case, we alter the order of summation (the sum is finite) and sum over c and d subject to

$$\sum_{j=1}^n i_j + c + d = b \quad (3.5.6)$$

holding $\{i_j\}$ constant, and then over $\{i_j\}$. However, this is zero since we know the $\{\hat{\delta}_a\}$ satisfy

$$\sum \hat{\delta}_b \hat{\delta}_{a-b} = 0 \quad (3.5.7)$$

already.

Case (ii) $t > s$

We need to show $C=0$ for each distinct term of the form (3.5.5). We write it as

$$\underbrace{x_{i_1} \dots x_{i_s} \delta_c x_{i_{s+1}} \dots x_{i_{s+v}}}_{\text{From first factor}} \underbrace{x_{i_{s+v+1}} \dots x_{i_t} \delta_d x_{i_{t+1}} \dots x_{i_n}}_{\text{From second factor}}$$

Thus

$$\begin{aligned}
 C &= \sum_{v=0}^{t-s} \frac{(-1)^v}{s! v!} \frac{(-1)^{n-t}}{(t-s-v)!(n-t)!} \\
 &= \frac{(-1)^{n-t}}{s!(n-t)!(t-s)!} \sum_{v=0}^{t-s} \binom{t-s}{v} (-1)^v \\
 &= 0
 \end{aligned}$$

by properties of the binomial coefficients.

This is an explicit check that the δ_m defined by (3.5.1) satisfy (3.5.2) and acts as a check on the previous calculation.

We can thus define two different ghost number complexes, one with $\delta = \bar{\sum} \delta_i$, the other with $\hat{\delta} = \bar{\sum} \hat{\delta}_i$. We need to construct a chain equivalence K from one to the other.

$$\begin{array}{ccccccccccccccc}
 0 & \xrightarrow{\hat{\delta}} & \Omega^{-k} & \xrightarrow{\hat{\delta}} & \Omega^{-k+1} & \xrightarrow{\hat{\delta}} & \dots & \xrightarrow{\hat{\delta}} & \Omega^{-1} & \xrightarrow{\hat{\delta}} & \Omega^0 & \xrightarrow{\hat{\delta}} & \Omega^1 & \xrightarrow{\hat{\delta}} & \dots & \xrightarrow{\hat{\delta}} & \Omega^k & \xrightarrow{\hat{\delta}} & 0 \\
 & & \downarrow K & & \downarrow K & & & & \downarrow K & & \downarrow K & & \downarrow K & & & & \downarrow K & & \\
 0 & \xrightarrow{\delta} & \Omega^{-k} & \xrightarrow{\delta} & \Omega^{-k+1} & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \Omega^{-1} & \xrightarrow{\delta} & \Omega^0 & \xrightarrow{\delta} & \Omega^1 & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \Omega^k & \xrightarrow{\delta} & 0
 \end{array} \quad (3.5.8)$$

K must satisfy

$$K \hat{\delta} = \delta K \quad (3.5.9)$$

i.e. K is a chain map, and

$$K \tilde{K} = \tilde{K} K = 1 \quad (3.5.10)$$

for some other chain map $\tilde{K}$. (See Appendix)

We prove that

$$K = 1 + \sum_{r \geq 0} \frac{1}{r!} \kappa_{i_1, \dots, i_r} \quad (3.5.11)$$

is the required mapping, where the sum is over all $r \geq 0$, and all values of $i_1, i_2, \dots, i_r$.

It is clear that $K: \Omega^q \rightarrow \Omega^q$. We prove that K is a chain map. The left hand side of (3.5.9) is a sum of terms, of the form

$$\sum_{r \geq 0} \frac{1}{r!} \kappa_{i_1, \dots, i_r} \hat{\delta}_a \quad (3.5.12)$$

where the sum is over all $r \geq 0$ and over all values of the $i_1, i_2, \dots, i_r$. The typical term of the right hand side of (3.5.9) is

$$\sum_{s \geq 0; t \geq 0} C(s, t) \kappa_{i_1, \dots, i_s} \hat{\delta}_a \kappa_{i_{s+1}, \dots, i_{s+t}} \quad (3.5.13)$$

where $C(s, t) \in \mathbb{R}$.

To prove the equality of these expressions we separate the cases: (i) $t \neq 0$, (ii) $t = 0$

We fix values of $l_1, l_2, \dots, l_{s+t}$ and α .

Case (i) $t \neq 0$

$$C(s, t) = \sum_{u=0}^t \frac{(-1)^u}{s! u!} \frac{1}{(t-u)!} = \frac{1}{s! t!} \sum_{u=0}^t \binom{t}{u} (-1)^u = 0$$

To see this, look at the possible contributions to a particular term in (3.5.13).

$$\begin{aligned} & \left(\frac{1}{s!} \kappa_{i_1} \dots \kappa_{i_s} \hat{\delta}_\alpha \right) \left(\frac{1}{t!} \kappa_{i_{s+1}} \dots \kappa_{i_{s+t}} \right) + \left(\frac{1}{s!} \kappa_{i_1} \dots \kappa_{i_s} \hat{\delta}_\alpha \kappa_{i_{s+1}} \right) \left(\frac{1}{(t-1)!} \kappa_{i_{s+2}} \dots \kappa_{i_{s+t}} \right) \\ & + \text{etc.} \end{aligned} \quad (3.5.15)$$

Case (ii) $t = 0$

In this case, the term must arise from 1δ and so

$$C(s, 0) = \frac{1}{s!} \quad (3.5.16)$$

This completes the proof that K is a chain mapping between the two ghost number complexes.

We can define another chain mapping by defining $\tilde{K}$ using the derivations $\{-\kappa_i\}$ rather than $\{\kappa_i\}$ i.e.

$$\tilde{K} = 1 + \sum_{r \geq 0} \frac{(-1)^r}{r!} \kappa_{i_1} \dots \kappa_{i_r} \quad (3.5.16)$$

Then we can show that $\tilde{K}$ is a chain mapping in the opposite direction to K ,

$$\hat{\delta} \tilde{K} = \tilde{K} \delta \quad (3.5.17)$$

and also

$$\tilde{K}K = 1 = K\tilde{K} \quad (3.5.18)$$

This completes the construction of a chain equivalence between the two complexes.

In summary, we have shown that the extension of the methods of Chapter Two to the general case of constraints satisfying $\{\phi_\alpha, \phi_\beta\} = \mathcal{U}_{\alpha\beta}^\gamma \phi_\gamma$ with the $\mathcal{U}_{\alpha\beta}^\gamma$ functions on $\mathcal{S}$, is possible but not unique. However, the physically relevant quantities (such as observables), which are all derived from the ghost number complex, are unique up to isomorphism.

We now go on to analyse the relationship between the physical observables and the homology of the ghost number complex.

3.6 Properties of Ghost Number Complex

Before considering the homology of the ghost number complex, we define the order of a theory. A theory of order n is one in which the family of antiderivations $\{\delta_i\}$ are taken to have $\delta_n \neq 0; \delta_i = 0 (i > n)$. Note that this is not an invariant characterisation of a theory. Even if we choose a set of structure functions and used them to define a family $\{\delta_i\}$ (similar to the construction of $\hat{\delta}_i$ in section 3.5), the ambiguity in the structure functions would introduce an ambiguity in the number of nonzero antiderivations. (In fact, there is a greater freedom than this in the $\{\delta_i\}$ since not every choice of derivations corresponds to a choice of structure functions. We will return to this point later.) On the other hand, we can hope to find a choice of antiderivations which would minimise the order of a particular theory.

In the classification above, we see that the case discussed in (22) and Chapter Two is a theory of order one. It can be shown that gravity is a theory of order two, and supergravity is a theory of order three. In the case of the n dimensional membrane, the n th order structure function is independent of phase space variables, so it is a theory of order n .

Note that gravity is a theory of order two and Yang-Mills a theory of order one, unlike the classification of Fradkin et al, where both are of rank one.

For a theory of order n , the antiderivations must satisfy

$$\begin{aligned}
 \delta_0^2 &= 0 \\
 \delta_1 \delta_0 + \delta_0 \delta_1 &= 0 \\
 \delta_2 \delta_0 + \delta_1^2 + \delta_0 \delta_2 &= 0 \\
 &\vdots \\
 \delta_n \delta_{n-1} + \delta_{n-1} \delta_n &= 0 \\
 \delta_n^2 &= 0
 \end{aligned} \tag{3.6.1}$$

We saw that in constrained theories, we could consider the physical observables to be the functions on $\mathcal{S}$ which are invariant, modulo the weak equality relation. Following the case in Chapter Two, we would like to relate $H^0(\delta)$ (the zero homology of the ghost complex) with these physical observables. In order to do this, we need to show two things:-

- 1) That we can always choose a representative from any equivalence class in $H^0(\delta)$ with no ghosts or antighosts;
- 2) That we can relate this function on $\mathcal{S}$ to the invariant functions.

The definition of $H^0(\delta)$ is

$$H^0(\delta) = \frac{\ker \delta \cap \Omega^0}{\delta \Omega^{-1}} \tag{3.6.2}$$

Now, an element in Ω^0 can be written uniquely as

$$\omega_0 + \omega_1 + \dots + \omega_k \quad (3.6.3)$$

where $\omega_s \in \Omega^{ss} \quad (0 \leq s \leq k)$

Consider a theory of order n . Then we have derivations $\delta_0, \delta_1, \dots, \delta_n$ satisfying (3.6.1). Pick an element $\omega \in \Omega^0$ closed under $\delta = \sum \delta_i$. Then this element is of the form (3.6.3). Since ω is closed under δ

$$\delta \omega = 0 \quad (3.6.4)$$

Then we project out the image of ω under δ into the various Ω^{rs} e.g. for a second order theory

$$(3.6.5)$$

This would give rise to equations (for a second order theory)

$$\begin{aligned} \delta_0 \omega_0 &= 0 \\ \delta_1 \omega_0 + \delta_0 \omega_1 &= 0 \\ \delta_2 \omega_0 + \delta_1 \omega_1 + \delta_0 \omega_2 &= 0 \\ \delta_2 \omega_1 + \delta_1 \omega_2 + \delta_0 \omega_3 &= 0 \\ &\vdots \end{aligned} \quad (3.6.6)$$

We now return to the case of a order n theory.

We separate out two cases (i) $\omega_0 = 0$ (ii) $\omega_0 \neq 0$

Case (i) $\omega_0 = 0$

Let r be the smallest integer such that $\omega_r \neq 0$. Then writing out the equations gives us

$$\begin{aligned} \delta_0 \omega_r &= 0 \\ \delta_1 \omega_r + \delta_0 \omega_{r+1} &= 0 \\ \delta_2 \omega_r + \delta_1 \omega_{r+1} + \delta_0 \omega_{r+2} &= 0 \\ \delta_2 \omega_{r+1} + \delta_1 \omega_{r+2} + \delta_0 \omega_{r+3} &= 0 \\ &\vdots \end{aligned} \tag{3.6.7}$$

These equations are of the form

$$\sum_a \delta_a \omega_{b-a} = 0 \tag{3.6.8}$$

where the sum is over all integers a , with the convention $\delta_a = 0$ if $a \notin \{0, 1, \dots, n\}$, and $\omega_b = 0$ if $b \notin \{r, r+1, \dots, k\}$.

Now the first equation in (3.6.7), together with the exactness of the antighost complex, implies that there exists $\gamma_r \in \Omega^{r+1}$ such that

$$\omega_r = \delta_0 \gamma_r \tag{3.6.8}$$

$$\begin{aligned} \text{Consider } \delta_0(\omega_{r+1} - \delta_1 \gamma_r) &= \delta_0 \omega_{r+1} + \delta_1 \delta_0 \gamma_r && (\text{since } \delta_0 \delta_1 = -\delta_1 \delta_0) \\ &= \delta_0 \omega_{r+1} + \delta_1 \omega_r \\ &= 0 \end{aligned}$$

by the second equation in (3.6.7). Exactness of δ implies

$$\omega_{r+1} = \delta_1 \gamma_r + \delta_0 \gamma_{r+1} \quad (3.6.9)$$

This suggests that

$$\omega_a = \sum_b \delta_b \gamma_{a-b} \quad (r \leq a \leq k) \quad (3.6.10)$$

for some γ_a with $\gamma_a \in \Omega^{a+1}$ ($r \leq a \leq k-1$). Again the convention is that δ_b is zero outside the range $0 \leq b \leq n$, and $\gamma_a = 0$ if $a < r$.

We prove this by induction.

Equation (3.6.8-9) show that (3.6.10) is true for $a=r$ and $a=r+1$. We assume it to be true for $r \leq a \leq m < k$, and show that it must then be true for $a=m+1$. Consider

$$\begin{aligned} \delta_0 \left(\omega_{m+1} - \sum_{a=1}^n \delta_a \gamma_{m+1-a} \right) &= \delta_0 \omega_{m+1} + \sum_{a=1}^n \sum_{b=1}^a \delta_b \delta_{a-b} \gamma_{m+1-a} \\ &= \sum_{a=0}^{m+1} \delta_a \omega_{m+1-a} - \sum_{a=n}^{2n} \left(\sum_{b=0}^n \delta_b \delta_{a-b} \right) \gamma_{m+1-a} \end{aligned} \quad (3.6.11)$$

where we have used equations (3.6.1) to substitute for $\delta_0 \delta_a$ and also the induction hypothesis to express the ω_a in terms of the γ_a . But the last term in (3.6.11) is zero by (3.6.1) and the first term is zero by the assumption that $\omega = \sum \omega_a$ is closed under δ . Thus there exists a $\gamma_{m+1} \in \Omega^{m+1, m+2}$ such that

$$\omega_{m+1} - \delta_0 \gamma_{m+1} + \sum_{a=1}^n \delta_a \gamma_{m+1-a} = \sum_{a=0}^n \delta_a \gamma_{m+1-a} \quad (3.6.12)$$

Thus (3.6.10) is true for $a=m+1$. By induction it is true for $r \leq a \leq k$.

Thus if we have a B.R.S.T. closed element with $\omega = \omega_r + \omega_{r+1} + \dots + \omega_k$ with $\omega_a \in \Omega^{aa}$ then the ω_a satisfy (3.6.10)

Define

$$\chi = \chi_r + \chi_{r+1} + \dots + \chi_k \quad (3.6.13)$$

Then $\chi \in \Omega^{-1}$ and moreover, (by 3.6.10)

$$\omega = \delta \chi \quad (3.6.14)$$

Thus a B.R.S.T. closed element of Ω° , which has no contribution from $\Omega^{\circ\circ} = C^\infty(\mathcal{G})$ is, in fact, exact. Such an element makes no contribution to $H^0(\delta)$.

For a second order theory, the proof above corresponds to constructing the elements χ_r in (3.6.14).

$$(3.6.15)$$

It is clear that the proof above can be extended to show

that $H^q(\delta) = 0$ if $q < 0$.

We now consider nonzero contributions to $H^0(\delta)$. These must have $\omega \neq 0$. The above proof is clearly inapplicable since the nonexactness of the antighost complex at Ω^{∞} does not allow us to deduce $\omega_0 = \delta_0 \gamma_0$ for some $\gamma_0 \in \Omega^1$.

Case (ii) ($\omega \neq 0$)

In this case we write

$$\omega = f + \omega_1 + \dots + \omega_k \quad (3.6.16)$$

with $f \in C^\infty(\mathcal{G})$ and $\omega_a \in \Omega^{\infty}$ ($1 \leq a \leq k$). We will show that there exists $\psi \in \Omega^1$ such that $\delta(\delta_0 \psi + \omega_1 + \dots + \omega_k) = 0$. This will enable us to relate $H^0(\delta)$ and the observables.

We give the proof for a theory of order two. The general case is similar.

In this case, ψ must satisfy

$$\begin{cases} \delta_1 \delta_0 \psi + \delta_0 \omega_1 = 0 \\ \delta_2 \delta_0 \psi + \delta_1 \omega_1 + \delta_0 \omega_2 = 0 \end{cases} \quad (3.6.17)$$

Then if we write $\psi = Z^\alpha \rho_\alpha$ and $\omega_i = V_i^\beta \eta^\beta \rho_\beta$, then these equations are

$$\begin{aligned} \{ \phi_\alpha, Z^\beta \phi_\beta \} &= V_\alpha^\beta \phi_\beta \\ \{ U_{\alpha\beta}^\gamma, Z^\delta \phi_\delta \} &= \{ \phi_\alpha, V_\beta^\gamma \} - \{ \phi_\beta, V_\alpha^\gamma \} + V_\alpha^\delta U_{\beta\delta}^\gamma - V_\beta^\delta U_{\alpha\delta}^\gamma \\ &\quad - V_\delta^\gamma U_{\alpha\beta}^\delta + \delta_0(\omega_1) \end{aligned} \quad (3.6.18)$$

In order for $\overline{Z}^\alpha$ to exist, we must show that the right hand side satisfies certain integrability conditions. These conditions arise because of the form of the left hand side and, (we will show) are satisfied due to the closure of ω under δ . Then the techniques of Pfaffian forms allows us to conclude that a solution exists (31). Rather than proceed with this on (3.6.18) which leads to a long and tricky calculation, we will show that the integrability conditions are satisfied by using the properties of the antiderivations. This is much simpler and can obviously be generalised to a theory of arbitrary order.

Equation (3.6.17) is equivalent to

$$\begin{cases} \delta_1 \psi = \omega_1 + \delta_0 \tau_1 \\ \delta_2 \psi = \omega_2 + \delta_1 \tau_1 + \delta_0 \tau_2 \end{cases} \quad (3.6.19)$$

where $\tau_1 \in \Omega^{12}$ and $\tau_2 \in \Omega^{13}$ are as yet entirely arbitrary. The integrability conditions of (3.6.19) are

$$\left. \begin{aligned} \delta_2 (\omega_1 + \delta_0 \tau_1) + \delta_1 (\omega_2 + \delta_1 \tau_1 + \delta_0 \tau_2) &= 0 \\ \delta_2 (\omega_2 + \delta_1 \tau_1 + \delta_0 \tau_2) &= 0 \end{aligned} \right\} \quad (3.6.20)$$

$$\left. \begin{aligned} \delta_0 (\delta_1 \omega_2 + \delta_2 \omega_1) &= 0 \\ \delta_2 (\delta_0 \omega_2 + \delta_1 \omega_1) &= 0 \end{aligned} \right\} \quad (3.6.21)$$

Since ω is closed we have conditions relating ω_1 , ω_2 and f . These are

$$\left. \begin{aligned} \delta_1 \omega_2 + \delta_2 \omega_1 &= -\delta_0 \omega_3 \\ \delta_0 \omega_2 + \delta_1 \omega_1 &= -\delta_2 f \end{aligned} \right\} \quad (3.6.22)$$

This ensures (3.6.21) is satisfied, but not necessarily (3.6.20). In order to correct this, we substitute (3.6.22) into (3.6.20) and use the resulting equations to solve for τ_1 and τ_2 . This gives

$$\left. \begin{aligned} \omega_3 + \delta_0 \tau_3 + \delta_1 \tau_2 + \delta_2 \tau_1 &= 0 \\ \omega_4 + \delta_2 \tau_2 + \delta_1 \tau_3 + \delta_0 \tau_4 &= 0 \end{aligned} \right\} \quad (3.6.23)$$

where τ_3 and τ_4 are arbitrary. (We have used the closure of $f + \omega_1 + \dots$ to substitute for $\delta_1 \omega_2$ in terms of ω_3 and ω_4 .) We view (3.6.23) as equations for τ_1, τ_2 and τ_4 ^{use τ_3, τ_4} to ensure that the integrability conditions are met. These conditions are

$$\left. \begin{aligned} \delta_2 (\omega_3 + \delta_0 \tau_3) + \delta_1 (\omega_4 + \delta_1 \tau_3 + \delta_0 \tau_4) &= 0 \\ \delta_2 (\omega_4 + \delta_1 \tau_3 + \delta_0 \tau_4) &= 0 \end{aligned} \right\} \quad (3.6.24)$$

$$\left. \begin{aligned} \delta_0 (\delta_1 \omega_4 + \delta_2 \omega_3) &= 0 \\ \delta_2 (\delta_0 \omega_4 + \delta_1 \omega_3) &= 0 \end{aligned} \right\} \quad (3.6.25)$$

Using the closure of ω , (3.6.25) are satisfied and (3.6.24) become

$$\left. \begin{aligned} \omega_5 + \delta_1 \tau_3 + \delta_1 \tau_4 + \delta_0 \tau_5 &= 0 \\ \omega_6 + \delta_2 \tau_4 + \delta_1 \tau_5 + \delta_0 \tau_6 &= 0 \end{aligned} \right\} \quad (3.6.26)$$

These equations are similar to (3.6.23) but we have moved down two rows. Continuing in this way, we will eventually reach the bottom of the complex and the integrability conditions will be satisfied. This shows that given an element $\omega \in \Omega^0$ which has a nonzero contribution from Ω^{∞} , we can find $\psi \in \Omega^0$ such that

$$\gamma = \delta_0 \psi + \omega_1 + \omega_2 + \dots + \omega_k \quad (3.6.27)$$

is closed. However, the proof in case (i) shows that $\gamma = \delta \mu$ for some $\mu \in \Omega^{-1}$. Thus $F = f - \delta_0 \psi$ is closed. Moreover, F is a function on $\mathcal{S}$.

This shows that every equivalence class in contains a representative which is a function on the original phase space. Such a representative is, of course, closed and so satisfies

$$\delta_i F = 0 \quad (1 \leq i \leq n) \quad (3.6.28)$$

(It is clear that the above proof can be extended to the case of order n theories). Note that (3.6.28) means that F is invariant and commutes with the structure functions.

Then, since

$$\delta\Omega^T \cap \Omega^\infty = \{f \in C^\infty(\mathcal{G}) ; f|_X = 0\} \quad (3.6.29)$$

we have

$$H^0(\delta) = \{f \in C^\infty(\mathcal{G}) ; \delta f = 0\} / \sim \quad (3.6.30)$$

where the relation $\sim$ is given by

$$f_1 \sim f_2 \Leftrightarrow f_1|_X = f_2|_X$$

i.e. weak equality.

Thus we have seen that $H^0(\delta)$ correctly describes the observables. At first sight, however, it may seem that we are only picking up a restricted class of observables. This is not so, as we will now show by describing how a weakly invariant function can be dealt with in this formalism.

We will show that a weakly invariant function can be extended to a B.R.S.T. closed object by the addition of terms involving ghosts and antighosts. This extension will define a unique equivalence class in $H^0(\delta)$. Again we only give the proof for a theory of order two. The proof is easily generalised.

Given a weakly gauge invariant function $f \in C^\infty(\mathcal{G})$, we must find $\omega_1, \omega_2, \dots, \omega_k$ with $\omega_r \in \Omega^r$ such that

$$\delta (f + \omega_1 + \omega_2 + \dots + \omega_k) = 0 \quad (3.6.31)$$

We saw that weakly invariant functions satisfy

$$\{\phi_\alpha, f\} = V_\alpha^\beta \phi_\beta \quad (3.6.32)$$

for some V_α^β . We define

$$\omega_1 = -V_\alpha^\beta \eta^\alpha \rho_\beta \in \Omega'' \quad (3.6.33)$$

Then

$$\delta_1 f = \delta_0 \omega_1 \quad (3.6.34)$$

Consider the following diagram

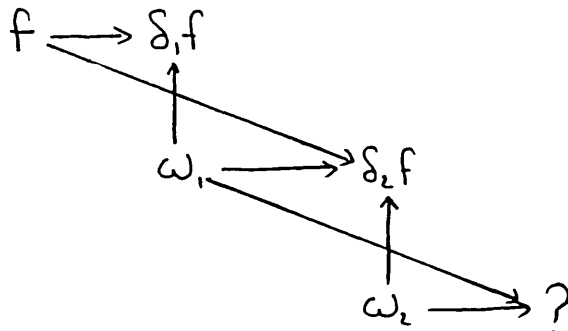
$$\begin{array}{ccc} f & \xrightarrow{\delta_1} & \delta_1 f \\ & \searrow & \uparrow \\ & & \omega_1 \xrightarrow{\quad} \delta_2 f \end{array} \quad (3.6.35)$$

Then

$$\begin{aligned} \delta_0(\delta_2 f - \delta_1 \omega_1) &= \delta_0 \delta_2 f - \delta_0 \delta_1 \omega_1 \\ &= -\delta_1^2 f + \delta_1 \delta_0 \omega_1 \\ &= -\delta_1(\delta_1 f - \delta_0 \omega_1) \\ &= 0 \end{aligned}$$

By exactness of δ_0 , there exists an $\omega_1 \in \Omega^2$ such that

$$\delta_2 f = \delta_1 \omega_1 + \delta_0 \omega_2 \quad (3.6.36)$$



Again we look at the image of the elements at "?" above, under δ_0 ,

$$\begin{aligned}
 \delta_0(\delta_2 \omega_1 + \delta_1 \omega_2) &= -\delta_2 \delta_0 \omega_1 - \delta_1^2 \omega_1 - \delta_1 \delta_0 \omega_2 \\
 &= -\delta_2 \delta_1 f - \delta_1(\delta_1 \omega_1 + \delta_0 \omega_2) \\
 &= -\delta_2 \delta_1 f - \delta_1 \delta_2 f \\
 &= 0
 \end{aligned}$$

Again, by exactness we find $\omega_3 \in \Omega^{33}$ such that

$$\delta_2 \omega_1 + \delta_1 \omega_2 + \delta_0 \omega_3 = 0 \quad (3.6.37)$$

We can continue in this way until we reach Ω^{kk} . Thus we can see that

$$\delta(f - \omega_1 - \omega_2 - \dots - \omega_k) = 0$$

Thus to each weakly invariant function, we can associate a well-defined equivalence class in $H^p(\delta)$.

This shows that $H^0(\delta)$ is correctly describing the observables. We are used to the case in theories of order one, whereby a weakly invariant function is weakly equivalent to an invariant function. These results show that in theories of order greater than one, that a weakly invariant function $f \in C^{\infty}(\mathcal{P})$ is not only weakly equal to an invariant function $F \in C^{\infty}(\mathcal{P})$, but that F actually commutes with the structure functions associated with the constraints. Note also, that it is important to use all of Ω^0 in describing the observables, in order to accomodate weakly invariant functions. Note that the proof in case (ii) of the homology can be modified to show that in a theory of order n , we need only add n terms to a weakly invariant function in order to have

$$\delta (f + \omega_1 + \dots + \omega_n) = 0$$

3.7 Algebra of Observables

As was mentioned in Chapter Two, we must consider the definition of a bracket on the observables. Such a bracket should have certain properties (22), in order that it be compatible with the observables. Some of these properties are

- 1) $\omega_1, \omega_2 \in \Omega^0 \Rightarrow \{\omega_1, \omega_2\} \in \Omega^0$ and linearity
- 2) $\delta\omega_1 = \delta\omega_2 = 0 \Rightarrow \delta\{\omega_1, \omega_2\} = 0$
- 3) $\delta\omega_1 = 0 \Rightarrow \{\omega_1, \delta\psi_1\} = \delta\psi_2 \quad : \quad \psi_1, \psi_2 \in \Omega^{-1}$
- 4) On physical observables $\{, \}$ is a Poisson bracket

For the case of a theory of order one, it was shown that the superPoisson bracket $\{, \}$ on Ω , (introduced in Chapter One) satisfied these properties, where

$$\delta\omega = \{Q, \omega\} \quad (3.7.1)$$

and

$$\{Q, Q\} = 0 \quad (3.7.2)$$

Q is odd (i.e. $Q \in \Omega^1$), and could be taken as

$$Q = \phi_\alpha \eta^\alpha + \frac{1}{2} \rho_r \mathcal{U}_{\alpha\beta}^r \eta^\beta \eta^\alpha \quad (3.7.3)$$

It is natural to ask if this property can be extended to theories of higher order.

The fundamental superPoisson brackets are

- 1) If $f, g \in C^\infty(\mathcal{G})$ then it reduces to the Poisson bracket
- 2) $\{\eta^\alpha, \eta^\beta\} = \{\rho_\alpha, \rho_\beta\} = 0$
- 3) $\{\eta^\alpha, \rho_\beta\} = \{\rho_\beta, \eta^\alpha\} = \delta^\alpha_\beta$
- 4) $\{f, \eta^\alpha\} = \{f, \rho_\alpha\} = 0 \quad \forall f \in C^\infty(\mathcal{G})$

Can we find a form for Q which satisfies (3.7.1) and (3.7.2) with these definitions?

If we use the $\{\hat{\delta}_\alpha\}$ of section 3.6 then we can take

$$Q = \phi_\alpha \eta^\alpha + \sum_{p>0} Q^{(p)}_{\beta_1 \dots \beta_p, \alpha_1 \dots \alpha_p} \eta^{\beta_1} \dots \eta^{\beta_p} \rho_{\alpha_1} \dots \rho_{\alpha_p} \quad (3.7.4)$$

then it can be shown that

$$\hat{\delta}_\omega = \{Q, \omega\} \quad (3.7.5)$$

$$\text{and} \quad \{Q, Q\} = 0 \quad (3.7.6)$$

However, what about the more general form $\{\delta_\alpha\}$? It is clear that for some choices of $\{\chi_i\}$ a choice of Q is not possible. This is because if we satisfy (3.7.5), then, for example,

$$\begin{aligned} \hat{\delta}_1 \eta^\alpha &= Q^{(1)}_{\beta\gamma} \eta^\beta \eta^\gamma \\ \hat{\delta}_1 \rho_\alpha &= 2 Q^{(1)}_{\alpha\beta} \eta^\beta \rho_\gamma \end{aligned} \quad (3.7.7)$$

If we take $\chi_1 \eta^\alpha = H^\alpha_\gamma \eta^\gamma \rho_\beta$; $\chi_1 \rho_\alpha = G^\gamma_\alpha \eta^\beta \rho_\gamma$ then the new choice for

$\delta_1 = \hat{\delta}_1 + [\chi_1, \delta_0]$ would be of the same form if and only if $H_{\gamma\delta}^{\alpha\beta} = G_{\gamma\delta}^{\alpha\beta}$. On the other hand if this is true, it corresponds to a different choice of structure functions. If we call this alternative choice $\overset{(P)}{R}$ and construct $R \in \Omega^{-1}$ in the same way we constructed Q then

$$\delta\omega = \{R, \omega\} \quad (3.7.8)$$

We would like to relate KQ and R . It can be shown that

$$R = KQ \quad (3.7.9)$$

Moreover since K is a chain map

$$K\{Q, \omega\} = \{R, K\omega\} \quad (3.7.10)$$

for all $\omega \in \Omega^0$.

It is natural to ask if K preserves the Poisson bracket, provided it is constructed out of derivations which correspond to a change in choice of structure functions. This is very difficult to prove in general, but is true in the lower order cases. Note that K also commutes with multiplication

$$K(\omega_1) K(\omega_2) = K(\omega_1 \omega_2) \quad (3.7.11)$$

The detailed properties of K are still under investigation.

We have shown in this chapter that the methods of (22) can be extended in an essentially unique way to higher order theories, preserving many of the properties, if not the geometric flavour. While the particular choice of antiderivations is open to some degree of freedom, the homology of the ghost number complex is uniquely defined by the requirement that the action of δ_i on functions on $\mathcal{S}$ is given by

$$\delta_i f = \{ \phi_\alpha, f \} \eta^\alpha \quad (f \in C^\infty(\mathcal{S}))$$

Moreover, an algebra of observables can be developed for a particular class of choices of $\{\delta_i\}$, in which the B.R.S.T. operator is given by the supercanonical transformation associated with an odd element $Q \in \Omega^1$. The fact that this approach deals differently with classical theories according to their rank, while the standard analysis of constrained canonical theories deals with all first class theories in exactly the same way, can be motivated by pointing out that the quantisation (at least in S matrix theory) of such theories is sensitive to the order of a theory. In the quantum theory, it is known that the application of this kind of analysis gives the correct form of the S matrix for the theory.(30).

The above approach, using the B.R.S.T. operator has allowed us to develop some very elegant results in the theory of constrained dynamical systems, but apart from the association of the observables with the zero homology of δ , none of the results is new. Given a particular B.R.S.T. closed object, trying to find the function on $\mathcal{S}$ in the same equivalence class is the same problem as choosing a gauge, and just as difficult. On the other hand, we could show such a choice is possible in principle, while to prove this in the Dirac analysis requires complicated calculations at the Poisson bracket level. We are also able to keep a phase space description at all times. While these computational advantages are all to the good, the question must arise as to whether this is enough to justify the unusual language used. The main justification for this formalism must depend on its relationship to quantisation, and in the next chapter we will argue that the methods extended here are exceptionally well suited to quantisation.

C H A P T E R F O U R

QUANTISATION

4.1 Introduction

As we have discussed earlier, the gauge invariance of a theory is a source of difficulty when it comes to quantisation. The gauge invariance of the field tells us that some of the modes of that field are nondynamical. This suggests that we should reduce the size of the phase space. However, if we wish to have a Poincare covariant quantisation scheme, we would require to enlarge the phase space to include the constraints (32). In simpler theories, like electrodynamics, constraints can be imposed to define the subspace of physical states, and it can be shown that this can be preserved in time. However, even in Yang-Mills (a first order theory) the auxiliary field introduced does not satisfy a free field equation and so this technique will not work.

In this chapter, we relate the work in this thesis to the B.R.S.T. symmetry used in covariant path integrals, using the example of the n dimensional membrane. We then consider the possible relevance of the B.R.S.T. operator

outside perturbation theory and the possibility of a quantisation scheme based directly on the homological approach.

We end with some general comments on the content of this thesis.

4.2 B.R.S.T. Symmetry in Quantisation

Since Faddeev and Popov included ghost terms in the action in the functional integral approach, in order to account for the closed fictitious loops Feynmann had introduced to preserve unitarity in Yang-Mills (36) and the subsequent observation by Becchi, Rouet and Stora (37) and Tyutin (38) that the Lagrangian was invariant under a global transformation, there have been many attempts to give a natural interpretation of this symmetry and the ghosts and antighosts. The symmetry is nilpotent when acting on the gauge field and ghosts (but not on antighosts). These various approaches are based on finding some mathematical structure which mimics the transformation properties. (Especially the properties of the ghost transformation which is very reminiscent of the way Maurer-Cartan forms of a Lie group transform under the exterior derivative.) There are two main approaches, the differential form approach (39,40) and the superspace approach (41). While they are quite different in outlook both share several problems.

They both use the nilpotency of the B.R.S.T. transformations in a crucial way (as does this thesis). However the transformation is not nilpotent on antighosts. In order to rectify this difficulty, an auxiliary field is introduced. This can be done by using an appropriate gauge

fixing term, and we are left with an action invariant under B.R.S.T. transformations, which are nilpotent on all fields. However, the antighosts do not admit a geometrical interpretation so a new transformation, the anti-B.R.S.T. transformation is introduced. Essentially this transformation is the same as the B.R.S.T. transformation but with the role of ghosts and antighosts interchanged (48). This allows a geometric interpretation in the spirit of this phenomenological approach.

However, one of the major properties of the B.R.S.T. transformation is that a B.R.S.T. invariant Lagrangian can be shown to lead to a unitary S matrix. It also allows an easy derivation of Ward-Takahashi identities. None of this requires anti-B.R.S.T. invariance. Also (22) shows a way of interpreting the ghosts and antighosts using the approach of Chapter Two and (7) which does not require anti-B.R.S.T. symmetry. We consider the generalisation of this approach to the higher order theories (Note however, that in the 'canonical operator theory' of Kugo and Ojima (34), we are left with the same physical subspace of the quantum Hilbert space if we use both B.R.S.T. symmetry and anti-B.R.S.T. symmetry, as if we had just used B.R.S.T. symmetry (41)).

The essential property of gauge fixing terms is to introduce the primary constraints as dynamical variables.

The action can be written in the following form:

$$S = \int dt (\dot{q}^i p_i - h - \lambda^\alpha \phi_\alpha - \gamma_\beta \phi^\beta) \quad (4.2.1)$$

where h is weakly invariant, ϕ_α are first class constraints of any rank, λ^α and γ_β are multipliers and ϕ^β is a gauge fixing term. Poincare covariance requires that ϕ^β contains terms with $\dot{\lambda}$ present. We take ϕ^β to be of the form

$$\phi^\beta = -\dot{\lambda} + \gamma^\beta \quad (4.2.2)$$

where the γ^β can depend on q^i, p_i, λ^α and γ_β . This gives an action of the form

$$S = \int dt (\dot{q}^A p_A - h - \lambda^\alpha \phi_\alpha - \gamma_\beta \gamma^\beta) \quad (4.2.3)$$

where $q^A = (q^i, \lambda^\alpha)$; $p_A = (p_i, \gamma_\beta)$ which is a theory with $n+k$ degrees of freedom. Here $2n$ is the dimension of the original phase space $\mathcal{S}$ with coordinates (q^i, p_i) . To ensure that this is the original theory which we started with, we wish to impose the $2k$ constraints $\Phi_a = (\phi_\alpha, \gamma_\beta) = 0$. On the $2(n+k)$ dimensional phase space above, these constraints have Poisson bracket

$$\{\Phi_a, \Phi_b\} = U_{ab}^c \Phi_c \quad (4.2.4)$$

where

$$U_{ab}^c = \begin{pmatrix} U_{\alpha\beta}^{\gamma} & 0 \\ 0 & 0 \end{pmatrix} \quad (4.2.5)$$

The main theorem of Fradkin et al (7) states that if we add $4k$ anticommuting dimensions (η^a, ρ_a) and construct the B.R.S.T. operator of Chapter Three (making a choice such that we can define a B.R.S.T. charge Q), then the functional integral Z_ψ is independent of the odd function ψ . Here Z_ψ is

$$Z_\psi = \int dq dp d\eta d\rho \exp i \int dt (p_A \dot{q}^A + p_a \dot{\eta}^a - H_\psi) \quad (4.2.6)$$

where $H_\psi = H + \{\psi, Q\}$
and H is the B.R.S.T. closed object related to h .

In (22), it was shown that for first order theories we could derive the configuration space path integral by choosing a polarisation (3), and integrating out some of the anticommuting variables. Because of this, the antighost is no longer conjugate to the ghost and we would no longer expect the same geometric interpretation. We can go on to identify the ghosts as a basis of $\Lambda^1 \mathcal{L}(\mathcal{D})^*$, and the antighosts as a basis of $\Lambda^1 \mathcal{L}(\mathcal{R}^*)$. This gives a natural geometric interpretation of ghosts and antighosts, and can be related to the differential form approach mentioned earlier. However in the case of higher order theories, while

we can still relate Chapter Three to (4.2.6), we have already lost the geometric flavour. This is unfortunate but seems to be inevitable unless we can find some way of introducing a Lie group. This seems unlikely since, a priori, a first class constraint manifold need not be associated with the zero set of some momentum map of some Lie group. On the other hand, despite the loss of this interpretation, the calculation does still work and a connection can be made between the homological classical description and the quantum theory.

In (28), it is shown that for the n dimensional membrane, the full extent of this formalism must be used in constructing the B.R.S.T. invariant effective action. This shows that in the effective action coupling terms of order Q^n in ghost fields appear, which persist in the Lagrangian version, providing we can integrate over the momenta. (This requires a suitable choice of gauge fixing.)

In the case of supergravity we must also use the generalisation developed here (or rather its field theory version), to show correctly that four ghost interactions occur(42). Here another problem occurs. This is due to having a gauge algebra which only closes on shell. In (42) it is shown that a form for Q which satisfies

$$\{Q, Q\} = 0$$

due to the field equations is possible. However this makes the homology arguments of Chapter Three rather artificial. This point needs further study.

The form of (4.2.6) suggests that any terms of the form $\{\psi, Q\}$ do not affect physical amplitudes. In the paper by Kugo and Ojima (34), a canonical operator formalism is developed which is manifestly covariant. They define a self adjoint, nilpotent B.R.S.T. operator $\hat{Q}$ for Yang-Mills. This procedure is justified, formally, by its equivalence to the path integral techniques. They show that any unphysical particles are confined by a 'quartet mechanism'. The main requirement is that physical states are characterised by

$$\hat{Q} |phys\rangle = 0 \quad (4.2.7)$$

and any states of the form

$$\hat{Q} |anything\rangle \quad (4.2.8)$$

are physically trivial. The true physical states are then defined as physical states modulo trivial states. (In (41) it is shown that if an anti-B.R.S.T. operator is defined, $\hat{\bar{Q}}$ and physical states are required to be annihilated by $\hat{\bar{Q}}$ as well as by (4.2.7) then the resulting physical states are the same.)

Physical observables must take physical states to physical states e.g.

$$[\hat{F}, \hat{Q}] = 0 \quad (4.2.9)$$

but operators of the form

$$\hat{G} = [\hat{F}, Q] \text{ for any } \hat{F} \quad (4.2.10)$$

are to be trivial. (They take physical states to states of zero norm which are trivial, i.e. make no contribution to matrix elements.)

In (43) the formal equivalence of this approach at the operator level to the noncovariant canonical quantisation where physical states are given by

$$\hat{\phi}_\alpha |phys\rangle = 0 \quad \alpha = 1, 2, \dots, k \quad (4.2.11)$$

is shown, where $\hat{\phi}_\alpha$ is a quantised version of the constraints.

However, all this work (e.g.(34)) is essentially for first order theories, and for Yang-Mills in particular. If we are to consider a quantisation scheme based on (22) and Chapter Three, we should perhaps note that there appears to be uses for B.R.S.T. symmetry outside perturbation theory

and the path integral formalism. For the first order theories, it has been conjectured that to deal with relationships like (4.2.11) we could introduce a "quantised ghost complex"

$$L^2(N, \mathbb{C}) \xrightarrow{\bar{\delta}} L^2(N, \mathbb{C}) \otimes \wedge \mathcal{L}(\mathcal{D})^* \xrightarrow{\bar{\delta}} \dots \dots \quad (4.2.12)$$

where N is the configuration space associated with the extended phase space, $\mathcal{S} = T^*N$. This approach is quite promising since we can formally define $\bar{\delta}$ by

$$\bar{\delta} = \eta^\alpha \hat{\phi}_\alpha + \frac{1}{2} U_{\alpha\beta}^\gamma \eta^\alpha \eta^\beta \frac{\partial}{\partial \eta^\gamma} \quad (4.2.13)$$

which is self adjoint in an appropriate superHilbert space. We could then relate $\bar{\delta}$ to $\hat{Q}$. However, in higher order theories, the ghost complex does not exist:- this is why we required to define higher order theories the way we did. This approach requires investigation. In particular the relationship of the work of Kugo and Ojima with the idea of (4.2.12) must be clarified. For example, in order to make precise the definition of physical states as closed states modulo exact states, we must prove that the quantised ghost complex is a resolution of the physical states $H^0(\mathcal{D}, L^2(N, \mathbb{C}))$. This problem is under investigation. It is likely that a good understanding of the first order theories is required

before identifying which structures will allow an extension to the higher order theories.

4.3 Conclusions

In this thesis, we have shown that the analysis of first order constrained theories presented in (22) can be extended to any first class theory. This analysis showed many interesting properties, mainly

- 1) That the extension was not entirely unique but the physically relevant quantities were well defined and essentially unique;
- 2) The use of the language and concepts of homology which allowed us to deal with a number of complicated calculations in a simple way;
- 3) The natural way in which the superphase space interpretation arises;
- 4) That unlike the standard classical approach, this approach seems to mirror the sensitivity of the quantum theory on the order of the theory.

However, this work has been on finite dimensional models of field theory. To be truly relevant these results must be extended to the infinite dimensional case. There is a well-developed formalism for such cases in symplectic geometry, and the algorithm of Gotay, Nestor and Hinds is also relevant in the infinite dimensional case. There is also a large literature (24 and references therein) on momentum mappings. This suggests that there is plenty of

structure with which to extend the first order case to field theory e.g. Yang-Mills. This problem is being actively investigated.

A P P E N D I X

In this appendix some of the terms used in this thesis are defined and explained. Useful references are (24,25,26).

Given a ring R , a module over R , is an abelian group K , with an operation $(r,a) \rightarrow ra$ for $r \in R$ and $a \in K$, such that

$$r(a_1 + a_2) = ra_1 + ra_2 \quad ; \quad (r_1 + r_2)a = r_1a + r_2a$$

$$(r_1 r_2)a = r_1(r_2 a) \quad ; \quad 1a = a$$

Given a sequence of R modules K_i $i=1,2,\dots,n$, and a sequence of homomorphisms $\delta_i: K_i \rightarrow K_{i+1}$ then we call

$$\xrightarrow{\delta_{i-1}} K_i \xrightarrow{\delta_i} K_{i+1} \xrightarrow{\delta_{i+1}} K_{i+2} \rightarrow \dots$$

a complex if $\delta_{i+1}\delta_i = 0$. A double complex is a double sequence of R modules K_{mn} , together with two homomorphisms δ, d such that $\delta^2 = d^2 = \delta d + d\delta = 0$.

$$\begin{array}{ccccc} & & \uparrow & & \uparrow \\ \delta & \rightarrow & K_{mn} & \xrightarrow{\delta} & K_{m+1,n} & \xrightarrow{\delta} \\ & & \uparrow d & & \uparrow d \\ \delta & \rightarrow & K_{m,n+1} & \xrightarrow{\delta} & K_{m+1,n+1} & \xrightarrow{\delta} \\ & & | & & | \end{array}$$

A complex is said to be exact at K_i if

$$\ker \delta_i = \operatorname{im} \delta_{i-1}$$

An element k , of K_i is said to be closed if $\delta_i k = 0$ and exact if $k = \delta_{i-1}c$ where $c \in K_{i-1}$. Given a complex, the i th homology group is defined as

$$H_i(K) = \frac{\ker \delta_i}{\operatorname{im} \delta_{i-1}}$$

Given two complexes K, K' a chain transformation $f: K \rightarrow K'$ is a family of homomorphisms $f_i: K_i \rightarrow K'_i$ such that $\delta'_i f_i = f_{i+1} \delta_i$. A chain homotopy s between two chain transformations $f, g: K \rightarrow K'$ is a family of homomorphisms $s_i: K_i \rightarrow K'_{i-1}$ such that

$$\delta'_{i-1} s_i + s_{i+1} \delta_i = f_i - g_i$$

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & K_{i-1} & \xrightarrow{\delta_{i-1}} & K_i & \xrightarrow{\delta_i} & K_{i+1} \longrightarrow \cdots \\
 & & \downarrow f_{i-1} & \swarrow s_i & \downarrow f_i & \swarrow s_{i+1} & \downarrow f_{i+1} \\
 \cdots & \longrightarrow & K'_{i-1} & \xrightarrow{\delta'_{i-1}} & K'_i & \xrightarrow{\delta'_i} & K'_{i+1} \longrightarrow \cdots
 \end{array}$$

We write $s: f \simeq g$.

A chain transformation $f: K \rightarrow K'$ is said to be a chain equivalence if there is another chain transformation $g: K' \rightarrow K$ and homotopies $s: gf \simeq 1_K$, $t: fg \simeq 1_{K'}$. In this case, the homology groups are isomorphic, $H_i(K) \cong H_i(K')$.

A complex

$$\cdots \longrightarrow K_n \longrightarrow K_{n-1} \longrightarrow \cdots \longrightarrow K_1 \longrightarrow K_0 \longrightarrow 0$$

is a resolution of the module C if $H_i(K) = 0$ for $i > 0$, and $H_0(K) \cong C$.

A linear operator T on Ω^{ij} is defined to be of degree p if

$$T: \Omega^{ij} \longrightarrow \bigoplus_p \Omega^{i+p, j+p}$$

i.e. $T: \Omega^k \rightarrow \Omega^{k+p}$. T is a derivation if its degree is even and it obeys the Liebnitz rule

$$T(\omega_1 \omega_2) = (T\omega_1)\omega_2 + \omega_1(T\omega_2)$$

T is an antiderivation if its degree is odd and it obeys

$$T(\omega_1 \omega_2) = (T\omega_1)\omega_2 + (-1)^{\text{degree } \omega_1} \omega_1(T\omega_2)$$

All the derivations χ_i are of degree zero.

These definitions come with appropriate modifications from (31), Chapter IV A where they are stated for linear operators on the de Rham complex.

R E F E R E N C E S

- 1) A short account of the relevant symplectic geometry appears in: A. Weinstein Lectures on Symplectic Manifolds, (Regional Conference Series in Maths; No. 29)
See also the appropriate parts of (8) and (17)
- 2) P.R. Chernoff and J. E. Marsden; Properties of Infinite Dimensional Hamiltonian Systems; Lecture Notes in Maths: No. 425 (Springer Verlag 1974)
- 3) N. Woodhouse; Geometric Quantisation (Clarendon Press, Oxford 1980)
- 4) J. Kijowski and W.M. Tulczyjew: A Symplectic Framework for Field Theories : Lecture Notes In Physics No. 107 (Springer Verlag 1979)
- 5) R. Casalbuoni : On the Quantisation of Systems with Anticommuting Variables : Nuovo Cim. 33A No. 1 (1976)
A. Barducci and D. Dominici : Nuovo Cim. 37A 385 (1977)
- 6) A. Rogers : A Global Theory of Supermanifolds J. Math. Phys 21 No.6 (1980)
B. Kostant : Differential Geometric Methods in Mathematical Physics, Lecture Notes in Maths 570 (Springer Verlag 1977). See also relevant parts of (22)

- 7) E.S. Fradkin and G.A. Vilkovisky : Quantisation of
Relativistic Systems with Constraints : CERN Preprint
TH2332 (1977)

I. A. Batalin and G.A. Vilkovisky : Relativistic S Matrix
of Dynamical Systems with Boson and Fermion Constraints
Phys Lett 69b No. 3 (1977)

E.S. Fradkin and T.E. Fradkina : Quantisation of
Relativistic Systems with Boson and Fermion First and
Second Class Constraints : Phys Lett 72b 343 (1978)
- 8) R. Abraham and J.E. Marsden Foundations of Mechanics
(Second Edition : Benjamin/Cummings)
- 9) J. Sniatycki and W.M. Tulczyjew Generating Forms of
Lagrangian Submanifolds : Indiana Univ. Maths. Journal
Vol. 22 No. 3 (1972)

W.M. Tulczyjew : The Legendre Transformation Annales de
l'Institut de Henri Poincare A XXVII No. 1 (1977)
- 10) K. Sundermeyer : Constrained Dynamics (Lecture Notes in
Physics 169 Springer Verlag 1982)
- 11) P.A.M. Dirac : Lectures on Quantum Mechanics Belfer
School of Science (Yeshiva Univ. NY 1964) See also (10)
and references therein
- 12) M.J. Gotay J.M. Nester G. Hinds J. Math Phys 19 2388
(1978)
- 13) This is discussed in both (12) and (10)

- 14) M.R. Menzio and W.M. Tulczyjew : Infinitesimal
Symplectic Relations and Generalised Hamiltonian Dynamics
: Annales de l'Institut de Henri Poincare A Vol XXVII No.
4 (1978)
- 15) J. Sniatycki : Dirac Brackets In Geometric Dynamics :
Annales de l'Institut de Henri Poincare A Vol XX No.4
(1974)
- 16) M.J. Gotay and J.M. Nester : Apartheid in the Dirac
Theory of Constraints J Phys A 17 3063 (1984)
- 17) The difference between topological triviality and
triviality in a global sense is pointed out in
C.J. Isham : Lectures in ; Relativity, Groups and Topology
: 1983 les Houches Summer School.
- 18) N. Jacobson Lie Algebras (Interscience Publishers London
(1966)
- 19) J.Arms, J. Marsden, V. Moncrief. Commun Math Phys 78 455
(1981)
- 20) J. Marsden and A. Weinstein: Rep Math Phys 5 121 (1974)
- 21) P. van Nieuwenhuizen: Supergravity Phys Rep 68 No.4
(1981)
- 22) D McMullan : PhD Thesis, Imperial College, London (1984)
D McMullan : Constraints and BRS Symmetry, Imperial
College preprint TP/83-84/21
- 23) D McMullan : To Appear
- 24) S. MacLane : Homology (Springer Verlag 1963)

- 25) W. Greub, S. Halperin, K. Vanstone : Connections, Curvature and Cohomology Vol III (Academic Press 1976)
- 26) R. Bott and L. W. Tu : Differential Forms in Algebraic Topology, Graduate Texts In Maths No. 60 (Springer Verlag 1982)
- 27) For canonical quantisation, especially of gravity see
 C.J. Isham : An Introduction to Quantum Gravity in Quantum Gravity: An Oxford Symposium, eds C.J. Isham, R. Penrose, D.W. Sciama (Clarendon Press, Oxford 1975)
 K. Kuchar : Canonical Methods of Quantisation in Quantum Gravity 2, eds C.J. Isham, R. Penrose, D.W. Sciama (Clarendon Press, Oxford 1981)
- 28) M. Henneaux : Transition Amplitude in the Quantum Theory of the Relativistic Membrane, Phys Lett 120B 179 (1983)
- 29) M. Henneaux : Forme Hamiltonienne de l'Integral de Chemin pour les Theories a Invariance de Jauge, University Libre de Bruxelles (1983)
- 30) E.S. Fradkin and M. A. Vasiliev : Hamiltonian Formalism, Quantisation, And S Matrix for Supergravity, Phys Lett 72B no. 1 (1977)
- 31) Y. Choquet-Bruhat, C deWitt-Morette, M. Dillard-Bleick : Analysis, Manifolds, and Physics, (Revised Edition, North Holland 1982)
- 32) C. Itzykson and J. B. Zuber : Quantum Field Theory (Mcgraw Hill 1980)

- 33) See relevant parts of (22) and (34)
- 34) T. Kugo and I. Ojima : Suppl. Prog. Theor. Phys No. 66
(1979)
- 35) L. D. Faddeev and V. N. Popov : Phys Lett 25B 29 (1967)
- 36) R. P. Feynmann : Acta Physica Polenca 24 697 (1963)
- 37) C. Becchi, A. Rouet, R. Stora Ann. Phys (NY) 98 287
(1976)
- 38) I. V. Tyutin : Int Report FIAN 39 (1975)
- 39) J. Thierry-Mieg : J. Math Phys 21 (1980) See also (40)
- 40) J.M. Leinaas and K. Olaussen : Phys Letts 108B no. 3
L. Bonora and P Cotta-Ramusino : Commun. Math. Phys. 87
589 (1983)
- 41) L. Bonora and M. Tonin : Phys Lett 98B no. 12 (1981)
- 42) R.E. Kallosh : Nucl Phys B141 141 (1978)
B. deWitt and J. W. van Holten : Phys Letts 79B 389 (1978)
- 43) R. Marnelius : Phys Lett 99B 467 (1981)
- 44) J. Sniatycki : Constraints and Quantisation in Lecture
Notes in Maths No. 1037 (Springer Verlag 1983)
- 45) B. S. DeWitt : Phys Rev 162 1195 (1967)
- 46) See (27) and references therein
- 47) C. Becchi, A. Rouet, R. Stora in Renormalisation Theory
eds G. Velo and A.S. Wightmann, NATO Advanced Study
Institute Series C23 (D. Reidal Publishing Co. 1976)