COEFFICIENT DIFFERENCES AND THE DOMAIN OF VALUES OF SOME UNIVALENT FUNCTIONS.

by

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Abstract

It is our purpose to examine aspects of the theory of univalent functions. The first half of the thesis is concerned with the successive coefficients of functions $\Sigma\, a_n z^n$ of the class S. For instance we prove that $\overline{\lim}\ \big||a_{n+1}| - |a_n|\big| \leq 1$. The second part of the thesis investigates the range of values for univalent functions mapping into a surface D.

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0. Introduction

Suppose that $f(z)$ is analytic on a complex manifold D. Then $f(z)$ is univalent in D if

$$f(z_1) \neq f(z_2)$$

whenever z_1, z_2 are distinct points in D. The study of univalent functions was initiated by Riemann and others in the secondhalf of the nineteenth century. The fundamental problem was to prove that every simply connected complex manifold may be mapped by some univalent function into the complex sphere. This theorem was stated by Riemann and proved by Poincaré and Koebe.

0.1 The class S.

Koebe [1907] introduced the class S of univalent functions $f(z)$ defined on the unit disk $E = \{z : |z| < 1\}$ with normalization conditions

$$(0.1)\quad f(0) = 0,\ f'(0) = 1.$$

These conditions ensure that S is a compact subset of the space of analytic functions on E. Consequently the maximum value of linear functionals on S is attained. In 1916 the cycle of elementary distortion theorems, e.g.

$$(0.2)\quad \frac{r}{(1+r)^2} \leq |f(r)| \leq \frac{r}{(1-r)^2},\quad f \in S,$$

was completed by Bieberbach. He also made the famous conjecture

$$(0.3)\quad |a_n| \leq n\ ,$$

for $\sum_1^\infty a_n z^n \in S$. The so called Koebe function $z/(1 - z)^2$ is thought to be extremal in (0.3). However the asymptotic result of Hayman [1955]

$$(0.4)\quad |a_n| \leq n, \qquad n > N(f)$$

is still the only sharp result of a universal nature.

When $\sum_{n=1}^{\infty} a_{2n-1}z^{2n-1} \in S$ Littlewood and Paley [1932] showed that

$$(0.5)\quad |a_{2n-1}| \leq A \quad \dagger$$

Fekete and Szegö [1933] together with Schaeffer and Spencer [1943] proved that 1 is not the best possible constant in (0.5). Nevertheless Hayman [1955] obtained

$$(0.6)\quad |a_{2n-1}| \leq 1, \quad n > N(f).$$

The coefficient problem for S is related to the question of finding bounds for the difference of successive coefficients, i.e. $d_n = ||a_{n+1}| - |a_n||$, for $\sum_{1}^{\infty} a_n z^n \in S$. Goluzin [1946] showed that

$$d_n \leq A\, n^{\frac{1}{4}} \log n, \quad n \geq 2,$$

which was improved by Biernacki [1956] to

$$d_n \leq A(\log n)^{3/2}.$$

Finally Hayman [1963] proved that

$$(0.7)\quad d_n \leq A.$$

The next problem was to obtain estimates of the absolute constant in (0.7). Our remarks about odd functions show that $A > 1$. Milin's method of quadratic inequalities has proved useful in this regard. The value of A has been pushed down to 9 [Milin 1968], 4.18 [I'lina 1968] and finally 3.61 [Grinspân 1976].

It is a natural question to ask for results like

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† Henceforth A denotes absolute constants, which are usually not the same value.

(0.4) and (0.6) for d_n. We prove

Theorem 0.1 Suppose that $\sum_1^\infty a_n z^n \in S$ then

$$(0.8)\quad \overline{\lim}\ \big||a_{n+1}| - |a_n|\big| \leq 1.$$

Several partial versions of (0.8) have already been obtained. Let $S_1(\alpha)$ be the subclass of S where

$$(0.9)\quad \lim_{r\to 1} (1-r)^2 M(r,f) = \alpha > 0.^{\dagger}$$

Hayman [1955] showed that $\alpha < 1$ except for $f(z) = z/(1 - e^{i\theta}z)^2$. Hayman [1963] obtained

Theorem 1 : Suppose that $\sum_1^\infty a_n z^n \in S_1(\alpha)$ then

$$(0.10)\quad |a_{n+1}| - |a_n| \to \alpha.$$

Pommerenke [1963] studied the class $S_t \subset S$ such that $f(E)$ is starlike with respect to 0. [§] He showed that for $\sum_1^\infty a_n z^n \in S_t$ there exists θ and $\delta > 0$ such that

$$(0.11)\quad |a_{n+1} - e^{i\theta} a_n| = O(n^{-\delta}),$$

except for $z/(1 - ze^{i\theta})(1 - ze^{i\phi})$. Consequently Theorem 0.1 holds for starlike functions. Eke [1967 b] extended the Hayman asymptotic theory to more general classes. Eke defined functions $f(z) \in S$ to have maximal growth on two distinct rays $\arg z = \theta_j$, $(j = 1,2)$, if there is a positive constant C and a sequence of points $r_n e^{i\theta_{j,n}}$ $(j = 1,2)$, such that

$$(1 - r_n)\ |f(r_n e^{i\theta_{j,n}})| > C$$

and

$$\theta_{j,n} \to \theta_j,$$

as $r_n \to 1$. This subclass will be denoted by $S_2(\theta_1,\theta_2)$ or just S_2.

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† $M(r,f) = \max_\theta |f(re^{i\theta})|$

§ S_t was introduced by Alexander [1915].

Functions satisfying (0.9) are said to have maximum growth on one ray and one writes $S_1 = \bigcup_{0<\alpha\leq 1} S_1(\alpha)$. Then Eke [1967b] proved

Theorem 2: Suppose that $\sum_1^\infty a_n z^n \in S - (S_1 \cup S_2)$, then

$$(0.12) \quad d_n \to 0.$$

We shall show in Theorem 1.1 that $\overline{\lim}\, d_n \leq 1$ in S_2. This, together with (0.10) and (0.12), implies Theorem 0.1. In Chapter 1 we also examine the case of equality in (0.8). One finds that

$$\overline{\lim}\, d_n = 1$$

if $f(z) = z/(1 - ze^{i\theta})(1 - ze^{i\phi})$. We conjecture that

$$\overline{\lim}\, d_n < 1, \quad n > N(f),$$

except for these functions.

The main result of Chapter 2 is the extension of (0.11) to the subclass C of close-to-convex functions, i.e. $\mathbb{C} - f(E)$ is the closure of a union of disjoint open half lines.††

The next Chapter contains sharp inequalities of the form (0.7) for various subclasses of S. For instance if $\sum_1^\infty a_n z^n \in C$ then

$$(0.13) \quad |n|a_n| - m|a_m|| \leq |n^2 - m^2|,$$

which solves a conjecture of M.S. Robertson [1966, 1970]. Unfortunately this stronger form of the Bieberbach conjecture does not extend to S [Jenkins, 1968].

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††Kaplan [1955], Lewandowski [1958].

0.2 Univalent functions mapping into Riemann surfaces.

Suppose that $a \in D$ is a fixed point and let $S_a(D)$ be the class of functions $f(z)$ which are univalent on E, take values in D and satisfy

$$f(0) = a.$$

If D is simply connected and hyperbolic then the theory of uniformization gives

Theorem 3 : We have that

$$(0.14)\quad \{f(r) : f \in S_a(D)\} = \{F(z) : |z| \leq r\},$$

where F is the univalent function mapping E onto D with $F(0) = a$.

The left hand side of (0.14) defines the domain of values of $S_a(D)$. For surfaces of higher connectivity we find that the domain of values may be very naturally considered to be a subset of a covering space of D. We then investigate the domain of values on general surfaces.

In Chapter 5 we examine the special case $D = \mathbb{C} - \{0\}$ and $a = 1$. It is found that the range of values is given by an elliptic modular function.

Chapter 1.

The asymptotic bounds for adjacent coefficients

1.1. Introduction

We shall prove

Theorem 1.1. For $\sum_1^\infty a_n z^n \in S_2$,

$$(1.1)\quad \overline{\lim_{n\to\infty}} \big||a_{n+1}| - |a_n|\big| \leq 1.$$

Our proof depends on certain approximation results for S_2 due to Eke [1967b] combined with a special version of the Grunsky Inequalities [Golusin 1946]. The second part of the proof is basically the method Hayman [1963] used to establish (0.10). This method was used by Eke to obtain (0.12) and also by Lucas [1969] to generalise (0.10) to more general classes. However we shall need the method to treat special subsequences of n.

Firstly we give the S_2 theory of Eke and briefly describe the proof of his fundamental approximation theorem. (These results of Eke are not widely known). Then the proof of (1.1) is given. Next we show that there exists a univalent function $z + \sum_2^\infty a_n z^n$ with real coefficients such that

$$\overline{\lim}\, |a_{n+1} - e^{i\phi} a_n| = \infty$$

for every ϕ. Consequently the analogue of (0.11) fails for this class. Also we examine the case when $\overline{\lim}\, \big||a_{n+1}| - |a_n|\big| = 1$ and find that, in contrast to previous asymptotic results, the extremals consist of more than just one function (and its rotations).

1.2. The theory of Eke

Eke [Theorem 8, p. 150, 1967 b] proved

Theorem 4. Suppose that $f(z) = z + \sum_2^\infty a_n z^n \in S_2(\theta_1,\theta_2)$. Now define

factors $\alpha_j(n)$ $(j=1,2)$ $(n=1,2,3,\ldots)$ by

$$(1.2)\quad \alpha_j(n) = f\left[\left(1-\frac{1}{n}\right)e^{i\theta_j}\right]\, n^{-1}.$$

Then if

$$f_n(z) = \sum_{j=1}^{2} \frac{\alpha_j(n)}{(1-ze^{i\theta_j})} = \sum_{m=0}^{\infty} c_m z^m,$$

we have

$$(1.3)\quad a_n = c_n + o\,(\max(\alpha_1(n), \alpha_2(n))).$$

This result is based on the classical technique of Hardy and Littlewood used previously in this context by Hayman [1955]. We begin with the formula

$$(1.4)\quad n\,a_n = \frac{\rho^{1-n}}{2\pi}\int_{-\pi}^{\pi} f'(\rho e^{i\theta})\, e^{-i(n-1)\theta}\,d\theta,$$

where ρ will be chosen as a function of r later. Choose ε between 0 and 1 and let $\Delta_{n,j}$ be the domain

$$\{z : \frac{\varepsilon}{n} < |1-ze^{-i\theta_j}| < \frac{1}{\varepsilon n},\ |\arg(1-ze^{-i\theta_j})| < \frac{\pi}{2}-\varepsilon\}.$$

The next step is to show that

$$(1.5)\quad \frac{f(z)}{f_n(z)} \to 1 \quad\text{and}\quad \frac{f'(z)}{f_n'(z)} \to 1, \quad \text{as } n\to\infty \text{ uniformly for}$$

$z \in \Delta_{n,1} \cup \Delta_{n,2}$. Actually the proof also uses estimates for $f(z)$ along the ray $\arg z = \theta_j$ derived from theorems similar to the Ahlfors' distortion theorem [Eke, Lemma 12, 196.7 a] We call $\Gamma_j = \{re^{i\theta} : |\theta-\theta_j| < B(1-r)\}$ † a major arc. The complementary minor arcs $\{re^{i\theta} \notin \Gamma_1 \cup \Gamma_2\}$ are denoted by $\tilde{\Gamma}$. Now define $r_n = (1-\frac{1}{n})$. Eke establishes (after 10 pages!) that if $\eta > 0$ we may choose ε so that the major arcs $\Gamma_1 \cup \Gamma_2 \subset \Delta_{n,1}(\varepsilon) \cup \Delta_{n,2}$ and for all large n

$$(1.6)\quad \int_{re^{i\theta}\in\tilde{\Gamma}} \{|f'(r_n e^{i\theta})| + |f_n'(r_n e^{i\theta})|\}\,d\theta < \eta\alpha_n n,$$

where $\alpha_n = \max(\alpha_1(n), \alpha_2(n))$. To establish Theorem 4 (1.5),(1.6) are

† B is a constant independent of r.

applied to formula (1.4). [Eke p. 184-201, 1967b]. Actually if n/m is bounded above and below by positive constants independent of n,m then the Δ_{nj} and Δ_{mj} overlap. Thus the same approximating function for Δ_{nj} may be used for Δ_{mj}. In particular

$$(1.7)\quad \frac{f_n(z)}{f_m(z)} \to 1$$

as $n,m \to \infty$. Formula (1.7) shows that we may use the same function f_n to estimate a_n and a_{n+1}. Thus combining these results of Eke and calculating the c_n in (1.3) we obtain

Lemma 1.1 With the above notation

$$(1.8)\quad a_n = \alpha_1(n)\, e^{-in\theta_1} + \alpha_2(n)\, e^{-in\theta_2} + o(\alpha_n)$$

$$(1.9)\quad a_{n+1} = \alpha_1(n)\, e^{-i(n+1)\theta_1} + \alpha_2(n)\, e^{-i(n+1)\theta_2} + o(\alpha_n),$$

where

$$\alpha_n = \max\,(\alpha_1(n),\, \alpha_2(n)).$$

1.3 The proof of theorem 1.1.

We shall consider subsequences n_k according to whether or not

$$(1.10)\quad \alpha_{n_k} = O(1).$$

For a particular subsequence n_k such that α_{n_k} is bounded we show (in section 1.5) that

$$(1.11)\quad \overline{\lim}\, \big||a_{n_k+1}| - |a_{n_k}|\big| \leq 1.$$

The next few sections are concerned with the case $\alpha(n_k) \to \infty$. We show that

$$(1.12)\quad |a_{n_k+1}| - |a_{n_k}| \to 0$$

in this case. The proof of (1.1) is immediate, for if

$$\overline{\lim}\, \big||a_{n+1}| - |a_n|\big| = \beta > 1$$

there would be a subsequence n_k such that

$$|a_{n_k+1}| - |a_{n_k}| \to \beta\ .$$

However (1.11) and (1.12) tell us that α_{n_k} does not contain subsequences which are bounded or tend to ∞. This is impossible. It remains to prove our two assertions.

1.4 The Golusin Inequality.

Suppose that $f(z) \in S$ and define $g(\xi) = 1/f(1/\xi)$ $(|\xi| > 1)$. The Grunsky coefficients are defined by

$$(1.13) \quad \log \frac{g(\xi_1) - g(\xi_2)}{(\xi_1 - \xi_2)} = - \sum_{k=1}^{\infty} \sum_{\ell=1}^{\infty} b_{k\ell} \xi_1^{-k} \xi_2^{-\ell}$$

for $|\xi_j| > 1$. [Grunsky 1939]. The bilinear form of the Grunsky inequality [Pommerenke 1975, p.64] states that

$$(1.14) \quad \Big| \sum_k \sum_\ell b_{k\ell} \mu_\ell \lambda_k \Big|^2 \leq \sum_k \frac{1}{k} |\mu_k|^2 \sum_k \frac{1}{k} |\lambda_k|^2,$$

for arbitrary complex μ_ℓ, λ_k. If we set $|\xi_1| = |\xi_2| = \frac{1}{r} > 1$, $\mu_\ell = \xi_1^{-2}$, $\lambda_k = \xi_2^{-2}$ we deduce the chord distortion theorem of Goluzin (1946).

Theorem 5: Suppose that $f(z) \in S$ and $g(\xi) = 1/f(1/\xi)$.

Then

$$(1.15) \quad (1 - r^2) \leq \left| \frac{g(z_1) - g(z_2)}{z_1 - z_2} \right| ,$$

whenever $|z_1| = |z_2| = \frac{1}{r} > 1$.

1.5. The proof of (1.1) for bounded α_{n_k}.

We assume the notation of the previous sections, supposing $f(z) = z + \sum_2^{\infty} a_n z^n \in S_2(\theta_1, \theta_2)$ and $\alpha(n_k) = O(1)$.

We prove

Lemma 1.2 Suppose that $|\alpha_2(n_k)| \leq |\alpha_1(n_k)|$ for some subsequence n_k then

$$|\alpha_2| \leq - \frac{1}{|e^{i\theta_1} - e^{i\theta_2}|} + o(1).$$

We substitute $z_j = e^{-i\theta_j}/r$ into Theorem 5 to obtain

$$(1-r)|f(re^{i\theta_2}| \leq \left| 1 - \frac{f(re^{i\theta_2})}{f(re^{i\theta_1})} \right| \frac{r}{(1+r)} .$$

Then put $r = 1 - \frac{1}{n}$ and note the definition (in theorem 4) of α_1 and α_2, i.e.

$$|\alpha_2| \leq \frac{2(n-1)}{|e^{i\theta_1} - e^{i\theta_2}|(2n-1)} , \quad n = n_k .$$

This proves the lemma.

From lemma 1.1. we find that

$$|a_{n_k+1} - a_{n_k} e^{-i\theta_1}| = |\alpha_2| \, |e^{-i\theta_2} - e^{-i\theta_1}| + o(\alpha),$$

$$\leq 1 + o(1)$$

from lemma 1.2 and as $\alpha = O(1)$, thus proving equation (1.11).

1.6 The Hayman bounds for coefficient differences.

Biernacki [1956] introduced the following formula for coefficient differences for $f \in S$. †

$$(n-1)a_{n-1} - n\, z_1 a_n = \frac{1}{2\pi i} \int_{|z|=\rho} \frac{(z-z_1)f'(z)}{z^n} dz .$$

We put $r_1 = 1 - \frac{1}{n_k}$, $z_1 = r_1 e^{i\phi_1}$ so that with this choice of $\arg z_1$.

$$|f(z_1)| = M(r_1, f).$$

Following Hayman [1963] we put $M_1 = M(r_1, f)$ and for $\nu = 1,2,3...$ define

$$M_\nu = 2^{1-\nu} M_1, \quad \text{and} \quad G_\nu = \frac{M_\nu^2 R^2}{M_\nu^2 + R^2} .$$

Hayman makes use of Green's formula to establish

Theorem 6: With the above notation and $R = |f(\rho e^{i\theta})|$, we have

$$(1.16) \quad \big||a_n| - |a_{n-1}|\big| \leq A \sum_{\nu=1}^{\infty} \Big\{ \int_{1-3/n}^{1-2/n} \rho d\rho \int_0^{2\pi} |\rho e^{i\theta} - z_1|^2 \, G_\nu(R) \, d\theta \Big\}^{\frac{1}{2}} .$$

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† Actually we prove $|a_{n_k-1}| - |a_{n_k}| \to 0$. In fact by (1.7) $|a_{n_k+m-1}| - |a_{n_k+m}| \to 0$ for any fixed any m since $\alpha_{n_k} \to \infty$ implies $\alpha_{n_k+m} \to \infty$.

Further, if $1 - 3/n < \rho < 1 - 2/n$ we have

$$(1.17)\quad \int_0^{2\pi} |\rho e^{i\theta} - z_1|^2 \, G_\nu(R)\, d\theta \leq I_1(\nu) + I_2(\nu),$$

where

$$(1.18)\quad I_1(\nu) = A\, 2^{-\nu/2} M_1^{\frac{1}{2}} + A\, 2^{-(1/4)\nu} n^{1/2} M_1^{1/4}$$

and

$$(1.19)\quad I_2(\nu) \leq 4 \iint_{|z|\leq\rho} |z - z_1|^2 \, |f'(z)|^2 \, r \log\left(\frac{\rho}{r}\right) dr d\theta$$

the integral being taken over all $z = re^{i\theta}$, such that $|f(z)| < M_\nu$.

To estimate $I_1(\nu)$ we note that $f(z)$ does not have maximal growth on one ray, i.e.

$$M_1 = M(f, 1 - \frac{1}{n}) = o(n^2),$$

and hence

$$(1.20)\quad I_1(\nu) = 2^{-(1/4)\nu} o(n).$$

The integral in equation (1.19) is estimated by the following result of Hayman [1963] .

Lemma 1. Suppose that $\tau_\mu = \{|z| < \rho : |f(z)| < M_\mu\}$, and $1 - \frac{3}{n} < \rho < 1 - \frac{2}{n}$ then

$$\iint_{\tau_\mu} |z - z_1|^2 \, |f'(z)|^2 \, r \log\left(\frac{\rho}{r}\right) dr d\theta \leq A\, n\, 2^{-\mu/2} \quad .$$

We also need a further result of Hayman [1963, p.234] .

Lemma 2. Suppose that $z_j = r_j e^{i\phi_j} (j = 1,2)$, where $0 < r_j < 1$ and $|f(z_1)| \geq |f(z_2)|$, then

$$|f(z_1)|^{1/4} \, |f(z_2)|^{3/4} \leq \frac{A}{(1-r_1)^{\frac{1}{2}}(1-r_2)^{1/2}|z_1 - z_2|} \quad .$$

1.7 The proof of (1.12)

For a small positive constant c define

$$(1.21)\quad a(r_1,c) = \{ z : |z| < r_1,\ |f(z)| > cM_1 \} .$$

We choose ν_o by the condition

$$(1.22)\quad 2^{-\nu_o - 1} \leq c < 2^{-\nu_o} .$$

Then from Lemma 1 and (1.19), (1.22) we have

$$(1.23)\quad I_2(\nu) \leq A\, n\, 2^{-\nu/2} < A\, n (2c)^{1/4}\, 2^{-\nu/4}$$

for $\nu \geq \nu_o$.

Now if we can show that

$$(1.24)\quad M_1(1 - r)^{\frac{1}{2}}\,(1 - r_1)^{\frac{1}{2}}\,|z - z_1| = o(1),$$

for $z = re^{i\theta} \in a(r_1, c)$ as $n_k = \frac{1}{(1-r_1)} \to \infty$ then the following argument establishes (1.12). From Lemma 1

$$(1.25)\quad I_2(\nu) \leq A\, n\, 2^{-\nu_o/2} + \iint |z - z_1|^2\, |f'(z)|^2\, r \log \left(\frac{\rho}{r}\right) dr d\theta,$$

for $\nu < \nu_o$, and where the integral is taken over $\{|z| < \rho : M_{\nu_o} \leq |f(z)| < M_\nu\}$. As $f(z) \in S_2$ $\lim M(r,f) = \infty$, hence there exists r_o such that

$$M_{\nu_o} \geq M(\tfrac{1}{2}, f)$$

for all $r_1 > r_o$ and therefore for all points z, which satisfy $|f(z)| > M_{\nu_o}$. Thus, we have $\frac{1}{2} < |z| < 1$,

$$\log \frac{\rho}{r} = \frac{1}{r}\left(r \log \frac{\rho}{r}\right) < 2(\rho - r) < 2(1 - r)$$

and from (1.25) we obtain

$$I_2(\nu) < A\, n\, 2^{-\nu_o/2} + 2 \iint_{\substack{M_{\nu_o} < |f(z)| < M_\nu \\ |z| < \rho}} |z - z_1|^2 |f'(z)|^2 (1-r)\, r d\theta dr,$$

$$< A\, n\, (2c)^{\frac{1}{2}} + o\left(\iint_{\substack{M_{\nu_o} < |f(z)| < M_\nu \\ |z| < \rho}} \frac{|f'(z)|^2}{M_1^2 (1-r_1)}\, r\, dr d\theta \right),$$

for $\nu < \nu_o$, $M_1 > 2^{\nu_o} M(\frac{1}{2}, f)$, where we have used equations (1.22) and (1.24).

Now as $f(z)$ is univalent,

$$\iint_{|f(z)|<M_\nu} |f'(z)|^2 \, r dr d\theta \leq \pi M_1^2 .$$

Therefore

$$I_2(\nu) < A\, n\, (2c)^{\frac{1}{2}} + o(n),$$

for $\nu < \nu_o$, and from (1.23)

$$I_2(\nu) < A\, n\, (2c)^{1/4}\, 2^{-\nu/4}$$

for $\nu \geq \nu_o$, as $n = n_k \to \infty$. Thus, as $n_k \to \infty$,

$$(1.26)\quad \sum_{\nu=1}^{\infty} \left\{ \int_{1-3/n}^{1-2/n} I_2(\nu)\, \rho d\rho \right\}^{\frac{1}{2}} < \nu_o [A\,(2c)^{\frac{1}{2}} + o(1)]^{\frac{1}{2}} + A\,(2c)^{\frac{1}{8}} \sum_{\nu=\nu_o}^{\infty} 2^{-\nu/8}.$$

Also from (1.20),

$$(1.27)\quad \sum_{\nu=1}^{\infty} \left\{ \int_{1-3/n}^{1-2/n} I_1(\nu)\, \rho d\rho \right\}^{\frac{1}{2}} = o(1) \left(\sum_{\nu=1}^{\infty} 2^{-\nu/4} \right)^{1/2},$$

as $n_k \to \infty$. Select $n = n_o(c)$ so large that the $o(1)$ terms in (1.26) and (1.27) are both less than $c^{\frac{1}{2}}$ for $n_k > n_o$.

Then for all $n = n_k > n_o$,

$$\left| |a_{n_k}| - |a_{n_k - 1}| \right| \leq A \sum_{\nu=1}^{\infty} \left\{ \int_{1-3/n}^{1-2/n} [I_1(\nu) + I_2(\nu)]^{\frac{1}{2}}\, \rho d\rho \right\}$$

$$\leq A \sum_{\nu=1}^{\infty} \left\{ \int_{1-3/n}^{1-2/n} I_1(\nu)\, \rho\, d\rho \right\}^{\frac{1}{2}} + A \sum_{\nu=1}^{\infty} \left\{ \int_{1-3/n}^{1-2/n} I_2(\nu)\, \rho d\rho \right\}^{\frac{1}{2}}$$

$$\leq \nu_o\, A\, c^{1/4} + A\, c^{\frac{1}{8}} + A\, c^{\frac{1}{2}}$$

$$\leq A\,(c^{\frac{1}{8}} - c^{1/4} \log c),$$

from (1.26), (1.27) and Theorem 6. this completes the proof of the second case as c is an arbitrary positive number. It remains to prove equation (1.24).

1.8 The proof of (1.24)

In the notation of the previous sections, we have

Lemma 3 . [Eke 1967b.p.210]. Suppose that $r_1 e^{i\phi_1}$ satisfies $|f(r_1 e^{i\phi_1})| = M(r_1,f)$ and $z_2 = r_2 e^{i\phi_2} \in a(r_1,c)$.

Then

$$(1.28)\quad c\, M_1 (1 - r_1)^{\frac{1}{2}} (1 - r_2)^{\frac{1}{2}} \leq A\, M(1 - \delta, f)\, \delta,$$

where

$$\delta = \frac{1}{2\pi} |\phi_1 - \phi_2| \geq (1 - r_2).$$

From this we prove

Lemma 1.3 Suppose that $\alpha(n_k) \to \infty$ and $z_2 = r_2 e^{i\phi_2}$ belongs to $a(r_1,c)$ as $n_k \to \infty$. Then, as $n_k \to \infty$

$$|\phi_1 - \phi_2| \to 0.$$

Since $z_2 \in a(r_1,c)$ substitution into Lemma 2 gives

$$(1.29)\quad c^{3/4} M_1 \leq \frac{A}{(1-r_1)|z_1-z_2|},$$

as $|f(z_1)| = M_1$, $|f(z_2)| > c\, M_1$ and $|z_2| \leq |z_1| = r_1$.

Now assuming $\alpha = \alpha_1$, say, we have

$$M_1 \geq |f(r_1 e^{i\theta_1})|$$

and by hypothesis we have

$$(1.30)\quad \alpha(n_k) = (1 - r_1)\, |f(r_1 e^{i\theta_1})| \to \infty$$

with $r_1 = 1 - \frac{1}{n_k}$, as $n_k \to \infty$. From (1.29) we obtain

$$\frac{A}{|z_1-z_2|} \to \infty,$$

as $n_k \to \infty$. Hence $|z_1 - z_2| \to 0$ as $n_k \to \infty$. This proves the lemma.

Now define $\delta = \frac{1}{2\pi}\,|\phi_1 - \phi_2|$ and compare δ with $(1 - r_2)$. If $\delta < (1 - r_2)$, then for $r_1 > r_o(\varepsilon)$ say,

$$(1.31)\quad M_1 = |f(z_1)| < \varepsilon\,(1 - r_1)^{-2},$$

as $f(z)$ does not have maximal growth, see (0.9), on one ray. Also

$$(1.32)\quad c\,M_1 \leq |f(z_2)| < \varepsilon\,(1 - r_2)^{-2}.$$

Hence from (1.31) and (1.32)

$$(1.33)\quad c^{3/4}\,M_1 \leq \varepsilon\,(1 - r_2)^{-\frac{1}{2}}(1 - r_1)^{-\frac{1}{2}}(1 - r_2)^{-1}$$

and

$$(1.34)\quad |z_1 - z_2| \leq 2\pi\delta + (1 - r_2) \leq (2\pi + 1)\,(1 - r_2).$$

Inequalities (1.33) and (1.34) imply (1.24).

If $\delta \geq (1 - r_2)$, we have

$$|z_1 - z_2| \leq (2\pi + 1)\delta,$$

and $\delta \to 0$ as $n_k = \frac{1}{1-r_1} \to \infty$ in view of Lemma 1.3.

Now Lemma 3 gives

$$c\,M_1\,(1 - r_1)^{\frac{1}{2}}\,(1 - r_2)^{\frac{1}{2}}\,|z_1 - z_2| \leq A\,M\,(1 - \delta, f)\,\delta^2.$$

The right hand side tends to zero as $n_k = (1 - r_1)^{-1} \to \infty$. For by Lemma 1.3 $\delta = \frac{1}{2\pi}\,|\phi_1 - \phi_2| \to 0$, and as $f(z)$ does not have maximal growth on one ray, we have $\delta^2\,M\,(1 - \delta, f) \to 0$.

This completes the proof of (1.24) and so of Theorem 1.1.

1.9. Some Counterexamples.

In this section we examine how the inequality $\overline{\lim}\;\big||a_{n+1}| - |a_n|\big| \leq 1$ may be improved. Suppose that

$$(1.35)\quad \sum_1^\infty a_k z^k = \frac{z}{(1 - ze^{-i\theta_1})(1 - ze^{-i\theta_2})} = \sum_{k=1}^\infty \frac{\sin k\phi}{\sin \phi}\, e^{\frac{1}{2}ik\psi} z^k,$$

where $\psi = -(\theta_1 + \theta_2)$, $\phi = (-\theta_1 + \theta_2)/2$. Then

$$|a_{n+1}| - |a_n| = \frac{1}{\sin \phi} (|\sin (n+1)\phi| - |\sin n\phi|).$$

If ϕ is a rational multiple of π, then $|a_n|$ is periodic and assumes only a finite number of values and so does $|a_{n+1}| - |a_n|$. If $\sin n\phi = 0$, then $a_n = 0$ and $|a_{n+1}| = 1$, so that $|a_{n+1}| - |a_n| = 1$ for infinitely many n. On the other hand if ϕ/π is irrational then

$$\left| \frac{\sin (n \mp 1)\phi}{\sin \phi} \right| < \left| \frac{\sin n\phi}{\sin \phi} \right| + 1$$

so that $||a_{n+1}| - |a_n|| < 1$ for all n. This suggests that for $f(z) \in S$ we have $||a_{n+1}| - |a_n|| < 1$ for $n > N(f)$ unless $f(z)$ takes the form (1.35) with rational ϕ/π or $f(z)$ is a Koebe function. We shall prove this conjecture for close-to-convex functions in the next chapter. Notice that equality is attained in (1.1) for functions of form (1.35). If ϕ/π is irrational then for any λ, $-1 \leq \lambda \leq 1$, there is a subsequence n_k such that $|a_{n_k+1}| - |a_{n_k}| \to \lambda$. Consequently the asymptotic coefficients difference theorem is unlike the previous results of Hayman in which there is only one extremal function (and its rotations).

Hayman actually proved a somewhat stronger version of (0.10). He showed that

$$(1.36) \quad |a_{n+1} - e^{-i\theta} a_n| \to \alpha(f),$$

where θ is the direction of maximal growth. In the next chapter we shall show that for each close-to-convex function $f(z)$,

$$(1.37) \quad |a_{n+1} - e^{i\theta} a_n| \to \lambda$$

for some θ and $\lambda \leq 1$. We now note that this is not true in S.

Theorem 1.2 : There exists a function $\sum_1^\infty a_k z^k \in S$ with real coefficients such that for every ϕ,

$$\overline{\lim} \, |a_{n+1} - e^{i\phi} a_n| = \infty.$$

The basic idea depends on a result of Eke (1967b, p. 179). Let $\mu(r)$ be a positive decreasing function on $(0,1)$ such that $\lim_{r\to 0} \mu(r) = 0$. Eke constructed a function $f(z)$ in $S_2(o,\pi)$ with real coefficients such that

$$(1.38) \quad \overline{\lim} \, \frac{\log (1-r) \, f(re^{i\theta_j})}{\mu(r) \, (\log \frac{1}{(1-r)})^{\frac{1}{2}}} = \infty,$$

for $j = 1,2$. Now let r_n be a sequence, $r_n \nearrow 1$, such that (1.38) occurs for $r = r_n$ and $j = 1$. Let ν_n be a subsequence of integers such that

$$(1.39) \quad 1 - \frac{1}{\nu_n} \leq r_n \leq 1 - \frac{1}{(\nu_n+1)} .$$

We now define α_1, α_2 as in section 1.2. Then from (1.39) and Lemma 20 (Eke 1967 b, p 204) we may choose $\mu(r)$ so that

$$(1.40) \quad \alpha_1(r_n) \to \infty .$$

Lemma 2 and (1.40) implies $\alpha_2(r_n) \to 0$. Hence from Lemma 1.1

$$|a_{n_\nu+1} - e^{i\phi} a_{n_\nu}| = |\alpha_1| \, |e^{-i\theta_1} - e^{-i\phi}| + o(\alpha_1) + O(\alpha_2).$$

Thus (1.40) implies

$$|a_{n_\nu+1} - e^{i\phi} a_{n_\nu}| \to \infty,$$

for $\phi \neq \theta_1 \pmod{2\pi}$. A similar result follows from $\phi \neq \theta_2 \pmod{2\pi}$ and we deduce Theorem 1.2.

1.20 Applications

Hayman (1963) applied (0.7) to univalent functions with gaps to obtain

Theorem 7: Suppose that $\sum_1^\infty a_k z^k \in S$ and that $a_{n_k} = 0$ for a subsequence satisfying $n_{k+1} - n_k \leq K$ then

$$|a_k| \leq A\, K.$$

Our result enables us to state a crisper asymptotic theorem.

Theorem 1.3: Suppose that $\sum_1^\infty a_k z^k \in S$ such that $a_{n_k} = 0$ for a subsequence satisfying $\overline{\lim}\ (n_{k+1} - n_k)$

$$\overline{\lim}\ |a_k| \leq \frac{K}{2}\ .$$

Although we are unable to obtain the result that for $f \in S$, $||a_{n+1}| - |a_n|| \leq 1$ for $n > N(f)$ we do have

Theorem 1.4: Suppose that $\sum_1^\infty a_k z^k \in S$ and $d_f = \text{dist}\,(o, \mathbb{C} - f(E))$ then

$$||a_{n+1}| - |a_n|| \leq 4d_f$$

for $n > N(f)$.

This theorem is related to an old problem of Littlewood, i.e.

$$|a_n| \leq 4\, d_f\, n.$$

The Koebe $\frac{1}{4}$ theorem (Pommerenke [1975], p.22) shows that

$$d_f > \frac{1}{4}$$

unless $f(z) = z/(1-ze^{i\theta})^2$. From our results

$$||a_{n+1}| - |a_n|| \leq 4\, d_f$$

for $n > N(f)$, unless $f(z) = z/(1-ze^{i\theta})^2$. This proves our statement.

Chapter 2.

Asymptotic bounds for close-to-convex functions.

2.1 Introduction: It has long been known that by the addition of some geometric property, interesting subclasses of S may be obtained. For instance Alexander [1915] introduced the class S_t of functions such that $f(E)$ is starlike with respect to 0. In recent years Brickman, Sheil-Small et al. have developed an elegant theory of linear problems for many of these classes (see Schober [1975], chapters 1,2,3). In particular the Bieberbach conjecture is true for these classes.

For nonlinear problems, like the question of bounds for $||a_{n+1}| - |a_n||$, much less is known. In this chapter we improve Theorem 0.1 and extend (0.11) (Pommerenke [1963]) to the class of close-to-convex functions.

Theorem 2.1: *Suppose that* $z + \sum_2^\infty a_k z^k$ *is close-to-convex in* E. *Then there exist numbers* λ, θ *such that*

$$(2.1)\quad |a_{n+1} - e^{i\theta} a_n| \to \lambda, \quad \text{as} \quad n \to \infty.$$

Furthermore $0 \le \lambda \le 1$ *with* $\lambda = 1$ *only for the functions* (1.35).

In particular note that $||a_{n+1}| - |a_n|| \le 1$ for $n \ge N(f)$ with strict inequality except for the functions given by $z/(1-ze^{i\theta_1})(1-ze^{i\theta_2})$ with rational $(\theta_1 - \theta_2)/\pi$. We conjecture that $N(f) = 1$. This is known to be true for odd close to convex functions (Pommerenke [1975], p. 55) and starlike functions (Leung [1978]).

The final example of chapter 1 showed that (2.1) fails for univalent functions with real coefficients. We shall prove that this example is essentially the exceptional case.

Theorem 2.2: *Suppose that* $f(z) = \sum_1^\infty a_k z^k \in S$ *and that* a_k *is real* $(k = 2,3,\ldots)$, $f(z)$ *has maximal growth on two rays* $\arg z = \mp\theta$, *where* $0 < \theta < \pi$. *Then* (2.1) *holds. Equality occurs only for the functions* $z/(1-2z\cos\theta + z^2)$, $0 < \theta < \pi$.

The class S_R (Dieudonné [1931]) consists of the functions in S with real coefficients. From Theorem 2.2 we deduce that for $\sum_1^\infty a_k z^k \in S_R - S_2(0,\pi)$ $||a_{n+1}| - |a_n|| \leq 1$ for $n > N(f)$.

2.2. Preliminary Results

Suppose that $g(z) \in S_t$, i.e. $g(E)$ is starlike with respect to 0. Thus we have $t\,w \in f(E)$ $(0 < t < 1)$ for all $w \in g(E)$. Consequently $\arg g(re^{i\theta})$ is increasing as θ increases. Hence

$$(2.2) \quad \mathrm{Re}\left(\frac{zg'}{g}\right) > 0,$$

in E. We solve (2.2) to find that

$$g(z) = z \exp\left[\int_0^z \frac{p(z) - 1}{z}\right] dz$$

where $p(z) = 1 + p_1 z + \ldots$ is regular in E and has positive real part in E. (Pommerenke [1975], p.40). By use of the Poisson integral we find the Herglotz representation formula for $p(z)$ (Hayman [1964], p. 180 - 181). That is

$$(2.3) \quad p(z) = \int_T \left(\frac{1+z\bar{x}}{1-z\bar{x}}\right) d\mu(x)$$

for some probability measure μ on the unit circle $T = \{|x| = 1\}$. The class P of functions having form (2.3) is often used in connection with moment problems such as in Chapter 3. (Pommerenke [1975], p.40). Substituting for $p(z)$ we find

$$(2.4) \quad g(z) = z \exp -2 \int_T \log(1-z\bar{x})\, d\mu.$$

The idea of close-to-convexity (Kaplan [1953]) is a significant generalisation of starlike functions. For $f \in S$ is close-to-convex if $\mathbb{C} - f(E)$ is the closure of the union of disjoint open half lines (Lewandowski [1958]). The symbol C will denote the set of such functions. For our purposes the analytic criterion of Kaplan [1953], is more useful. For $f(z) \in C$ if there is a $g(z) \in S_t$ and a constant

$e^{i\gamma}$ such that

$$(2.5) \quad \mathrm{Re}\left\{e^{i\gamma}\left(\frac{zf'}{g}\right)\right\} > 0$$

in E.

From the Herglotz formula (2.3) it is easy to deduce a representation for univalent functions with real coefficients. For $\mathrm{Im}\, f(z) > 0$ when $\mathrm{Im}\, z > 0$ and hence

$$\mathrm{Re}(1 - z^2)\ \frac{f(z)}{z} > 0.$$

Therefore

$$(2.6) \quad f(z) = \frac{z}{(1-z^2)} \int_T \frac{1 + \bar{x}z}{1 - \bar{x}z}\, d\rho(\bar{x})$$

for some probability measure ρ on T. However we may rewrite (2.6) as

$$(2.7) \quad f(z) = \int_{-1}^{1} \frac{z\, d\rho_1}{(1-2tz + z^2)}$$

where $\rho_1(\cos t) = \rho(e^{it})$ is a probability measure on $[-1,1]$.

Given a probability measure μ on T we define the trigonometric moments as

$$\mu_k = \int_T \bar{x}^k d\mu.$$

The following fundamental result of measure theory plays a major part in the proof of Theorem 2.1.

Theorem 8: (N. Wiener, 1924). Under the above conditions

$$(2.8) \quad \lim_{n\to\infty} \frac{1}{n}\left\{\sum_{k=0}^{n} \mu_k\, e^{ik\theta}\right\} = \mu(\{e^{-i\theta}\}).$$

Hayman [1961] used Theorem 8 to prove that for functions p(z) defined by (2.3)

$$(2.9) \quad \lim_{r\to 1} (1-r)\, p(re^{i\theta}) = 2\mu(\{e^{-i\theta}\}).$$

Now this equation, with (2.5) and theorem 4 of chapter 1 could be used to prove a somewhat weaker version of Theorem 2.1; however we prefer to obtain a direct proof.

Finally we shall need a theorem of Paley and Littlewood (Pommerenke [1975], p. 131).

Theorem 9: Suppose that $f(z) = \sum_{k=1}^{\infty} a_k z^k \in S$ and

$$M(r,f) = O((1-r)^{-\alpha}),$$

then

$$|a_n| = O(n^{\alpha-1}) \qquad (\alpha > \tfrac{1}{2}).$$

2.2 Proof of Theorem 2.1.

Suppose that $f(z) = z + \sum_{2}^{\infty} a_k z^k \in C$. Then (2.5) implies the existence of $g(z) \in S_t$, $p(z) \in P$ and a constant c, $c = \frac{1+e^{-2i\gamma}}{2}$, such that

$$(2.10) \quad zf'(z) = \left(\frac{1+c}{2} + \frac{1-c}{2} p(z) \right) g(z).$$

We now represent $p(z)$ and $g(z)$ byprobability measures μ and ν as in (2.3) and (2.4). For any finite positive measure ρ on the unit circle T we define

$$\alpha = \alpha(\rho) = \sup \{\rho\{x\} : x \in T\},$$

i.e. the mass of the largest atom of ρ. Our proof will then depend on the representing measure ν for the starlike function g. There are 4 cases to examine.

(i) The representing measure ν is supported by a single atom x_o (i.e. $\nu = \delta_{x_o}$ where δ is the usual Dirac function).

(ii) The measure ν is supported by two atoms of equal weight, i.e. $\nu = \frac{1}{2}(\delta_{x_1} + \delta_{x_2})$.

Otherwise the proof depends on the value of $\alpha(\nu)$.

(iii) $1 > \alpha(\nu) \geq \frac{1}{2}$

(iv) $\frac{1}{2} > \alpha(\nu) \geq 0$.

2.3 Proof for case (i)

As $\nu = \delta_{x_o}$ (and without loss of generality $x_o = 1$) we deduce that

$$g(z) = z/(1 - z)^2.$$

Now suppose that $\lambda_0 = \mu(\{1\})$ then from (2.10) and (2.3)

$$(2.11)\quad f(z) = \frac{(1+c)z}{2(1-z)} + \frac{(1-c)\lambda_0 z}{2(1-z)^2} + \frac{(1-c)(1-\lambda_0)}{2} f_1(z),$$

where

$$(2.12)\quad f_1'(z) = \frac{1}{(1-z)^2} \int_T \left(\frac{1+z\bar{x}}{1-z\bar{x}}\right) \frac{d(\mu-\lambda_0\delta_1)}{(1-\lambda_0)} \quad ,$$

unless $\lambda_0 = 1$, in which case $f_1 = 0$.

Lemma 2.1 Suppose that $f_1(z) = \sum_1^\infty A_n z^n$. Then

$$(2.13)\quad |(n+1)\, A_{n+1} - n\, A_n| = o(n).$$

From (2.12) we write $(1-z)\, f_1'(z) = \sum_0^\infty H_n z^n$ where

$$H_n = 2 \sum_{k=0}^{n-1} q_{n-k} + 1, \quad q_k = \int_T \bar{x}^k \, d\, \frac{\mu-\lambda_0\delta_1}{1-\lambda_0} \quad .$$

Hence Theorem 8 implies

$$|(n+1)\, A_{n+1} - n\, A_n| = |H_n| = o(n).$$

Summing (2.13) we obtain from the lemma

$$|A_{n+1} - A_n| = o(1).$$

The coefficients of the other terms in (2.11) are easily estimated. Thus

$$|a_{n+1} - a_n| \to \frac{|1-c|}{2} \lambda_0.$$

The next few sections show that maximal growth only occurs in case (i).

2.4 Proof of case (ii)

Now we have $g(z) = z/(1-\bar{x}_1 z)(1-\bar{x}_2 z)$, and without loss of generality

$$1 = x_1 \mp \bar{x}_2 = e^{i\phi}$$

and

$$\mu(\{\bar{x}_1\}) = \lambda_1 \geq \lambda_2 = \mu(\{\bar{x}_2\}) \quad .$$

From (2.10) we obtain

$$(2.14)\quad f'(z) = \frac{(1+c)}{2(1-z)(1-e^{i\phi}z)} + \frac{1-c}{2}\left\{(1-\lambda_1-\lambda_2)\, f_2'(z) + \frac{\lambda_1(1+z)}{(1-z)^2(1-e^{i\phi}z)} + \frac{\lambda_2(1+e^{i\phi}z)}{(1-z)(1-e^{i\phi}z)^2}\right\}$$

where

$$f_2'(z) = \int_T \frac{1+z\bar{x}}{1-z\bar{x}}\, d\mu_1(x)/(1-z)(1-e^{i\phi}z),$$

and

$$\mu_1 = (\mu - \lambda_1\delta_1 - \lambda_2\delta_{\bar{x}_2})/(1-\lambda_1-\lambda_2),$$

provided that $\lambda_1 + \lambda_2 \neq 1$: otherwise $f_2 = 0$. The coefficients in the last terms are estimated by

Lemma 2.2. Suppose that $\lambda_1 \geq \lambda_2$ and

$$\sum_{k=1}^{\infty} kA_k z^{k-1} = \frac{\lambda_1(1+z)}{(1-z)^2(1-e^{i\phi}z)} + \frac{\lambda_2(1+ze^{i\phi})}{(1-z)(1-e^{i\phi}z)^2}$$

then

$$|A_{n+1} - A_n| = 2\lambda_2 + O(n^{-1}) .$$

We suppose that

$$\sum_{k=1}^{\infty} B_n z^n = \int_0^z \frac{(1+z)}{(1-z)^2(1-e^{i\phi}z)}\, dz, \quad \phi \neq 0,$$

$$\sum_{k=1}^{\infty} C_n z^n = \int_0^z \frac{(1+ze^{i\phi})}{(1-z)(1-e^{i\phi}z)^2}\, dz.$$

Thus by computation

$$B_n = \frac{2}{(1-e^{i\phi})} + O(n^{-1}) ,$$

and thus

$$|B_{n+1} - B_n| = O(n^{-1}).$$

Also

$$|(n+1)\, C_{n+1} - n\, C_n| = (2n+1),$$

and

$$C_n = O(1).$$

Thus

$$|A_{n+1} - A_n| \leq \lambda_1 |B_{n+1} - B_n| + \lambda_2 |C_{n+1} - C_n| = 2\lambda_2 + O(n^{-1}),$$

which proves the lemma.

Lemma 2.3 *Suppose that* $f_2(z) = \sum_1^\infty D_n z^n$ *then*

$$|D_{n+1} - D_n| = o(1).$$

The proof is similar to Lemma 2.1 and is omitted.

Finally the first term of (2.14) ,

$$\sum_{n=1}^{\infty} n\, L_n z^{n-1} = \frac{1}{(1-z)(1-ze^{i\phi})}$$

yields

$$|L_{n+1} - L_n| = o(1).$$

Combining these results shows that

$$|a_{n+1} - a_n| \to |1 - c|\lambda_2$$

which completes the proof of (ii) as $\lambda_2 \leq \frac{1}{2}$.

2.5 Proof for (iii) and (iv).

In case (iii) we may assume that

$$\alpha(\nu) = \nu(\{1\}) \geq \tfrac{1}{2}.$$

Then if we define $G(z) = (1-z)g(z)$, (2.4) shows that

$$G(z) = z \exp -2 \int_T \log(1-z\bar{x})\, d(\nu - \frac{\delta_1}{2}).$$

A calculation shows

$$\operatorname{Re} \frac{zG'}{G} > \tfrac{1}{2}, \quad z \in E,$$

so that $G(z)$ is starlike. We may apply Theorem 9 to estimate the coefficients of $G(z)$ via

Lemma 2.4: Suppose that ν is a finite positive measure on T. Define a function $h(z)$ via (2.4) and let $\alpha = \alpha(\{\nu\})$ such that $0 < \alpha < \frac{1}{2}$. Then for every $\varepsilon > 0$,

$$M(r,h) = O((1-r)^{-2(\alpha+\varepsilon)}).$$

Choose $J = \{e^{i\theta} \in T : |\theta - \theta_o| < \delta\}$ such that $\nu(J) < \alpha + \varepsilon$. Then from equation (2.4)

$$\log \left| \frac{h(re^{i\theta_o})}{re^{i\theta_o}} \right| \leq -2 \int_J \log |1 - re^{i\theta_o} \bar{x}|\, d\nu$$

$$-2 \int_{T-J} \log |1 - re^{i\theta_o} \bar{x}|\, d\nu .$$

The first integral is bounded by $2(\alpha + \varepsilon) \log \frac{1}{(1-r)}$ and the second by $\nu(T) \log \frac{1}{|\sin \delta|}$. We prove the Lemma noting that there exists $\delta > 0$ such that $\nu(J) < \alpha + \varepsilon$ for all θ_o.

Thus by Theorem 9 the coefficients of $G(z)$ are $O(n^{\beta})$, for some $\beta < 0$. From (2.10)

$$z\,(1-z)\,f'(z) = \left(\frac{1+c}{2} + \frac{1-c}{2}\,p(z)\right) G(z),$$

and thus

$$(2.15)\quad |(n+1)\,a_{n+1} - n\,a_n| = O(n^{\beta+1}),$$

where we noted

Lemma 2.5. (Pommerenke [1975], p 40). Suppose that $\sum_0^{\infty} p_k z^k$ is defined by (2.3) then $|p_k| \leq 2$.

We now apply Lemma 2.5 and Theorem 9 to deduce that if $g(z) = \sum_1^{\infty} b_n z^n$ then

$$|b_n| = O(n^{\gamma})$$

for $\gamma < 1$. Hence by (2.11) $|a_n| = O(n^{\gamma})$. This, together with (2.15) implies

$$|a_{n+1} - a_n| \leq \frac{|(n+1)a_{n+1} - n\,a_n|}{(n+1)} + \frac{|a_n|}{(n+1)}$$

$$= O(n^{\delta}), \quad \delta < 0.$$

This completes the proof of case (iii).

In case (iv) the above arguments show that the coefficients b_n of $g(z)$ satisfy

$$|b_n| = O(n^{-\delta})$$

where $0 < \delta < 1$, and hence we have

$$|a_n| = O(n^{-\delta})$$

in view of (2.10). Thus the proof of Theorem 2.1 is complete.

2.6 The proof of theorem 2.2

From chapter 1 the only open case is S_2. From representation (2.6) and (2.7) if $f(z)$ has maximal growth on $\arg z = \theta$ ρ_1 must have an atom at $\cos\theta$. Also ρ_1 could not have any other atoms on $[-1,1]$, for as $f(z)$ is univalent it can have growth like $\frac{1}{(1-r)}$ on at most 2 rays (Hayman [1958], p 34). Now we use (2.6) and Theorem 7 (similarly to previous sections) to conclude that

$$(1-r)\ f(re^{\pm i\theta}) \to \frac{\lambda e^{\pm i\theta}}{(1-e^{\pm 2i\theta})}, \text{ as } r \to 1,$$

where $\lambda = \rho(\{\cos\theta\})$. The method of Eke mentioned in chapter 1 gives

$$\alpha_1 \sim \frac{\lambda e^{i\theta}}{(1-e^{2i\theta})}$$

$$\alpha_2 \sim \frac{\lambda e^{-i\theta}}{(1-e^{-2i\theta})} \quad .$$

The order of approximation is $o(1)$ as $\max\{|\alpha_1|,|\alpha_2|\} = O(1)$.
Substitution into Lemma 1.1 shows that

$$|a_{n+1} - e^{i\theta}a_n| \to \lambda.$$

Chapter 3.

Exact bounds for the difference of adjacent coefficients.

3.1 Introduction: Suppose that $A \subset S$. In this chapter sharp bounds for $||a_{n+1}| - |a_n||$, $\sum_1^\infty a_k z^k \in A$, will be obtained. Milin and Lebedev introduced a method of exponentiating the Grunsky inequality to obtain good bounds for coefficient problems in S. The following inequality is the basis for bounding $||a_{n+1}| - |a_n||$ in S and in its subclasses.

Theorem 10: Suppose that $(\alpha_k)_1^\infty$, $(\beta_k)_0^\infty$ are sequences of complex numbers connected by the formal exponential relation

$$(3.1)\quad \sum_{k=0}^{\infty} \beta_k z^k = \exp\Big(\sum_{k=1}^{\infty} \alpha_k z^k\Big).$$

Then we define

$$(3.2)\quad \sigma_n = \frac{\sum_{k=0}^{n} |\beta_k|^2}{(n+1)} \exp \sum_{k=1}^{n} \Big(\frac{1}{(k+1)} - k\Big(1-\frac{k}{n+1}\Big)\Big) |\alpha_k|^2),$$

for $n = 0,1,2,\ldots$ and find that

$$(3.3)\quad \sigma_n \leq \sigma_{n-1} \leq \ldots \leq \sigma_0 = 1.$$

For $\alpha_k = c^{k/k}$, $\beta_k = c^k$ $(|c| = 1)$ we obtain inequality

This theorem is found in Milin's book [1977, p. 32]. It was used recently by Grinspân [1976] to show that

$$||a_{n+1}| - |a_n|| < 3.61,$$

in S.

For subclasses of S one expects sharp bounds. Pommerenke [1963] showed that

$$||a_{n+1}| - |a_n|| \leq \frac{3}{4} e^2$$

for starlike functions and conjectured that 1 is the correct bound. Sheil-Small [1970] reduced the bound to 2. Finally Leung [1978] used the Milin-Lebedev inequality, combined with an inequality of MacGregor, to

deduce that Pommerenke's conjecture is correct.

A similar inequality for the class C of close-to-convex functions was advanced by M.S. Robertson ([1966], [1970]). He conjectured that

$$(3.4)\quad \big||n\,a_n| - |m\,a_m|\big| \leq |n^2 - m^2|,$$

for $\sum_1^\infty a_k z^k \in C$. In 1978 the author obtained the proof of this conjecture and a variety of other bounds for other subclasses (to appear in the Journal of the London Mathematical Society). The method relied on repeated use of Theorem 10, together with properties of probability measures. Simultaneously Leung [1979] proved (3.4) by an inequality similar to (3.3). In a letter to the author Leung wrote that he had conjectured the existence of generalised Milin inequalities. It was easy to 'extract' just such an inequality from the author's work on generalisations of (3.4). We shall present this formulation.

Now for each $f(z) = \sum_{k=1}^\infty a_k z^k \in C$, from section 2.2, there exists $g(z)$ starlike in E and $q(z)$, satisfying $\operatorname{Re} q(z) > 0$ in E, such that

$$(3.5)\quad z f'(z) = q(z)\, g(z).$$

Thus the class C is obtained from S_t via certain multipliers $q(z)$. Schober [1976, p 22] conjectured that for each $\sum_{k=1}^\infty b_k z^k \in S_t$ there exists θ such that

$$(3.6)\quad |b_{k+1} - e^{i\theta} b_k| \leq 1$$

for $k = 1,2,3,\ldots$ Suppose now that $q(z) = \sum_0^\infty q_k z^k$ is an arbitrary multiplying function in (3.4). Then (3.6) implies that

$$\begin{aligned}(3.7)\quad \big|(n+1)\,|a_{n+1}| - n|a_n|\big| &\leq |(n+1)\,a_{n+1} - e^{i\theta} n\, a_n| \\ &\leq \sum_{k=0}^{n} |q_{n-k}|\,|b_{k+1} - e^{i\theta} b_k| \\ &\leq \sum_{k=0}^{n} |q_{n-k}|.\end{aligned}$$

For $q(z)$ that satisfy $\operatorname{Re} q(z) > 0$, we have (by Lemma 2.5) $q_0 = 1$ and $|q_k| \leq 2$. Thus Robertson's conjecture follows from Schober's if $m = n+1$ and hence generally.

Unfortunately the author cannot prove Schober's inequality butwe can prove (3.7), and for more general functions then $g(z) \in S_t$. We shall consider $g(z)$ defined by a representation of the form

$$(3.8)\quad g(z) = z \exp\left(\delta \int_T \log(1 - zx)\, d\mu\right)$$

where μ is a probability measure on the unit circle T.

For instance if the parameter $\delta = -(1+e^{-2i\alpha})$ we obtain

$$(3.9)\quad \operatorname{Re} e^{i\alpha} \frac{zg'}{g} > 0$$

in E. Equation (3.9) describes exactly those univalent functions which have a range $f(E)$ which is spiralike with respect to 0, i.e. $w \in f(E)$ implies $w \exp(-e^{-i\alpha}t) \in f(E)$ for $t > 0$. This subclass, S_p, was introduced by Špaček [1933].

Our first result is:

Theorem 3.1. Suppose that $\sum_{n=1}^{\infty} a_n z^n$ is equal to

$$f(z) = z\, q(z) \exp\left\{-(1+e^{-2i\alpha}) \int_T \log(1-zx)\, d\mu(x)\right\}$$

for some probability measure μ on T and function $q(z)$ which has coefficients q_n ($q_0 = 1$) dominated by (positive) coefficients Q_n.

Then, provided that $Q_1 \geq Q_2 \geq \ldots \geq Q_n \geq \ldots$, we have

$$(3.10)\quad \int_T \log|a_{n+1} - x\, a_n|\, d\mu(x) \leq \log(1 + Q_1 + \ldots + Q_n).$$

The inequality is sharp and equality is achieved for $\mu = \delta_a$, $\alpha = 0$.

We obtain immediately

Theorem 3.2. Suppose that $\sum_{1}^{\infty} a_n z^n = q(z)\, g(z)$ for some $g(z)$ spiralike in E and $q(z)$ as in Theorem 3.1.

Then

$$(3.11)\quad \big|\,|a_{n+1}| - |a_n|\,\big| \le 1 + Q_1 + \ldots + Q_n.$$

The functions $q(z) = 1 + \sum_1^\infty Q_n z^n$ and $g(z) = z/(1-z)^2$ show that the inequality is sharp.

The condition that $Q_1 \ge \ldots \ge Q_n$ may not be best possible and so we are unable to prove the most general possible version of (3.7). However (3.10) cannot be true for general sequences Q_n as we show by a counterexample. Theorem 3.2 contains Leung's result (extended to S_p) and Robertson's conjecture as special cases.

Corollary 3.1. Suppose that $z + \sum_2^\infty a_n z^n \in S_p$ then $\big|\,|a_{n+1}| - |a_n|\,\big| \le 1$.

Corollary 3.2. Suppose that $z + \sum_2^\infty a_n z^n \in C$ then

$$\big|\,n|a_n| - m|a_m|\,\big| \le |n^2 - m^2|.$$

We now return to the representation (3.8). Suppose that $\delta = -2(1-\lambda)$, $(\lambda \le 0)$. Then

$$\operatorname{Re} \frac{zg'}{g} > \lambda,$$

and $g(z)$ is said to be starlike of order λ in E. Although such functions are generally not univalent, Sheil-Small and others (Keogh [1976]) have developed an elegant linear theory of convolution and extreme points for the functions starlike of order λ. Our techniques give

Theorem 3.3 Suppose that $\sum_1^\infty a_n z^n$ is starlike of order $\lambda \le 0$.

Then

$$\big|\,|a_{n+1}| - |a_n|\,\big| \le \frac{\Gamma(1-2\lambda+n)}{\Gamma(1-2\lambda)\,\Gamma(n+1)}.$$

Equality holds for $z(1-z)^{2(\lambda-1)}$.

We begin by presenting a generalised Milin-Lebedev inequality. The above results are then obtained.

3.2 A generalised Milin-Lebedev inequality

To prove Theorem 3.1 we require

Theorem 3.4 Suppose that α_n, β_n are related by (3.1) as in Theorem 10.

Then if $Q_1 \geq Q_2 \geq \ldots > 0$, and $Q_o = 1$, we have

$$(3.12)\quad \sum_{k=0}^{n} Q_{n-k}|\beta_k|^2 \leq (1 + Q_1, + \ldots + Q_n)\ \mathcal{E}$$

where

$$\mathcal{E} = \exp\{\sum_{k=1}^{n} (A_{k,n}|\alpha_k|^2 - B_{k,n})\}$$

and $A_{k,n}$, $B_{k,n}$ are functions of $(Q_1,\ldots,Q_n)$ and k,n. Equality holds for $\beta_n=1$ and $\alpha_n = \frac{1}{n}$ $(N = 1,2,\ldots)$.

We define sums

$$\Gamma_k = |\beta_{n-1}|^2 + \ldots + |\beta_{n-k}|^2,$$

for $1 \leq k \leq n$. Then using partial summation we write

$$\sum_{o}^{n} Q_{n-k}\ |\beta_k|^2 = |\beta_n|^2 + \lambda_1\ \Gamma_1 + \ldots + \lambda_n\ \Gamma_n\ ,$$

where the numbers λ_k are defined by

$$\lambda_k = Q_k - Q_{k+1}\ (1 \leq k \leq n-1) \quad \text{and} \quad \lambda_n = Q_n.$$

The monotone property ensures that $\lambda_k \geq 0$. Thus

$$(3.13)\quad \log\ (|\beta_n|^2 + Q_1|\beta_{n-1}|^2 + \ldots + Q_n)$$

$$= \log \Gamma_n + \log\ (\frac{|\beta_n|^2}{\Gamma_n} + \lambda_1 \frac{\Gamma_1}{\Gamma_n} + \ldots + \lambda_{n-1} \frac{\Gamma_{n-1}}{\Gamma_n} + \lambda_n\ ).$$

Now conclusion (3.3) of Theorem 10 may be written as

$$(3.14)\quad \exp\ [\sum_{j=1}^{n-1} \{\frac{1}{(j+1)} - j(1-\frac{j}{n})|\alpha_j|^2\} - \sum_{j=1}^{n-k-1} \{\frac{1}{(j+1)} - j(1-\frac{j}{n-k})|\alpha_j|^2\}\ ]$$

$$\leq \frac{n}{(n-k)} \quad \frac{|\beta_o|^2+\ldots+|\beta_{n-k-1}|^2}{|\beta_o|^2+\ldots+|\beta_{n-1}|^2}\ .$$

Let us define the left hand side of (3.14) to be $\exp\ S_k$. Thus, using $e^x \geq x + 1$, we have

$$(3.15)\quad \frac{\Gamma_k}{\Gamma_n} = 1 - \frac{(|\beta_o|^2 + \ldots + |\beta_{n-k-1}|^2)}{(|\beta_o|^2 + \ldots + |\beta_{n-1}|^2)}$$

$$\leq 1 - \frac{(n-k)}{n} \quad (1 + S_k)$$

$$= \frac{k}{n} - \frac{(n-k)}{n} \quad S_k \ .$$

Now the β_n are defined by

$$\beta_n = \frac{1}{n} \sum_{k=1}^{n} k \, \alpha_k \beta_{n-k} \ ,$$

from which we deduce

$$(3.16) \quad \frac{|\beta_n|^2}{\Gamma_n} \leq \frac{1}{n^2} \sum_{k=1}^{n} k^2 |\alpha_k|^2 \ ,$$

via the Cauchy inequality. Also from (3.3)

$$(3.17) \quad \log \Gamma_n < \log n + \sum_{k=1}^{n-1} \{k(1 - \frac{k}{n}) |\alpha_k|^2 - \frac{1}{(k+1)}\}.$$

Substitution of (3.15), (3.16), (3.17) into (3.13) yields

$$\log (|\beta_n|^2 + Q_1 |\beta_{n-1}|^2 + \ldots + Q_n |\beta_o|^2)$$

$$\leq \sum_{k=1}^{n-1} \{k(1 - \frac{k}{n}) \ |\alpha_k|^2 - \frac{1}{(k+1)} \} + \log \sum_{k=1}^{n} (\frac{k^2 |\alpha_k|^2}{n} + k \ \lambda_k - (n-k) \lambda_k S_k)$$

$$\leq \log (1 + Q_1 + \ldots Q_n) + \sum_{k=1}^{n-1} \{k(1 - \frac{k}{n}) |\alpha_k|^2 - \frac{1}{k+1} \}$$

$$+ \frac{\sum_{k=1}^{n} \{\frac{k^2 |\alpha_k|^2 - 1}{n} - (n-k) \ \lambda_k S_k \}}{(1 + Q_1 + \ldots + Q_n)} \ ;$$

we use $\log (1 + x) \leq x$ and since $\sum_{1}^{n} k \ \lambda_k = Q_1 + \ldots + Q_n$,

this proves the theorem. It is easy to check that inequality holds for $\alpha_k = \frac{1}{k}$.

3.3 Proof of Theorem 3.1

We have $f(z) = \sum_{k=1}^{\infty} a_k z^k = (1 + \sum_{1}^{\infty} q_k z^k) \ g(z)$, where $g(z)$ is defined by equations (3.8) and (3.9).

We define

$$(3.18)\quad (1 - zx)\, g(z)/z = \sum_{k=0}^{\infty} \beta_k z^k .$$

The integral representation (3.8), with $\delta = -(1 + e^{-2i\alpha})$, implies that

$$\sum_{0}^{\infty} \beta_k z^k = \exp\left(\sum_{k=1}^{\infty} \alpha_k z^k\right),$$

the coefficient α_k is given by

$$(3.19)\quad \alpha_k = \frac{1}{k}\{(1 + e^{-2i\alpha})\, \mu_k - x^k\}$$

and μ_k are the moments $\int_T y^k d\mu(y)$ of the probability measure μ.

Now Theorem 3.4 implies that

$$(3.20)\quad \log\left(\sum_{k=0}^{n} Q_{n-k}|\beta_k|^2\right) \leq \log(1 + Q_1 + \ldots + Q_n) + \sum_{k=1}^{n} (A_{k,n}|\alpha_k|^2 - B_{k,n}).$$

Also

$$\sum_{k=1}^{\infty} (a_{k+1} - xa_k)\, z^k = (1 - zx)\, f(z) = (1 + \sum_{1}^{\infty} q_k z^k)(1 - zx)\, g(z) ,$$

so that, setting $q_o = 1$, we have

$$|a_{n+1} - x\, a_n| \leq \left|\sum_{k=0}^{n} q_{n-k}\, \beta_k\right|$$

$$\leq \sum_{k=0}^{n} Q_{n-k}^{\frac{1}{2}}\, Q_{n-k}^{\frac{1}{2}}\, |\beta_k|$$

$$\leq \left(\sum_{k=0}^{n} Q_{n-k}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{n} Q_{n-k}\, |\beta_k|^2\right)^{\frac{1}{2}}$$

by Cauchy's inequality. Hence using (3.20) we deduce that

$$(3.21)\quad \log|a_{n+1} - x\, a_n| \leq \log(1 + Q_1 + \ldots + Q_n) + \tfrac{1}{2} \sum_{k=1}^{n} (A_{k,n}|\alpha_k|^2 - B_{k,n}).$$

We now integrate both sides with respect to the measure μ noting

Lemma 3.1. $\int_T |\alpha_k|^2 \, d\mu(x) = \frac{1}{k^2}$.

The lemma follows from (3.19) as

$$(3.22) \quad k^2|\alpha_k|^2 = |1 + e^{-2i\alpha}|^2 \, |\mu_k|^2 - 2 \operatorname{Re} (1 + e^{2i\alpha}) \, \overline{\mu}_k \, x^k + 1$$

and

$$\mu_k = \int_T \overline{x}^k \, d\mu(x) = \int_T x^k \, d\mu(x) ,$$

as μ is a real measure. Consequently integrating (3.22) we obtain the lemma from

$$|1 + e^{-2i\alpha}|^2 - 2 \operatorname{Re} (1+ e^{-2i\alpha}) = 0.$$

Thus

$$\int_T \log |a_{n+1} - x \, a_n| \, d_\mu(x) \leq \log (1 + Q_1 + \ldots + Q_n) + \tfrac{1}{2} \sum_{k=1}^{n} (\frac{A_{k,n}}{k^2} - B_{k,n}).$$

The case of equality in Theorem 3.4 shows that the final sum is zero. This proves Theorem 3.1.

Notice that equality is achieved for $f(z) \in S_p$, i.e. for $\alpha = 0$ and $q_k = Q_k$ $(k = 0,1,\ldots,n)$ equality holds.

3.4 A counterexample

We show that Theorem 3.1 is not true for general Q_k. Consider the function

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \equiv (1 + \frac{1}{\varepsilon} z^3)(\sum_{k=1}^{\infty} b_k z^k)$$

where $g(z) = \sum_{k=1}^{\infty} b_k z^k$ is starlike and $\varepsilon > 0$. This has the form of Theorem 3.1 with $Q_3 = \frac{1}{\varepsilon}$, $Q_k = 0$ $(k \neq 3)$, and $\alpha = 0$.

Then

$$a_4 - x\, a_3 = (b_4 - x\, b_3) + \frac{1}{\varepsilon}\,.$$

If Theorem 3.1 were true for this set of Q_k we should deduce

$$\int_T \log |a_4 - x\, a_3|\, d\mu(x) \leq \log\,(1 + \tfrac{1}{\varepsilon}),$$

i.e.

$$\int_T \log |1 + \varepsilon(b_4 - \bar{x}\, b_3)|\, d\mu \leq \log\,(1 + \varepsilon).$$

We let ε tend to zero and note that

$$\log |1 + \varepsilon w| = \operatorname{Re} \varepsilon w + O(\varepsilon^2).$$

Thus

$$\operatorname{Re}\{b_4 - b_3 \int_T x\, d\mu\} \leq 1.$$

However, as $g(z)$ is starlike, we recall from (2.4) that

$$g(z) = z \exp\,(-2 \int_T \log\,(1 - xz)\, d\mu)$$

and thus

$$b_2 = 2 \int_T x\, d\mu.$$

Therefore

$$(3.23)\quad \operatorname{Re}\,(b_4 - \tfrac{1}{2}b_2b_3) \leq 1.$$

Since $e^{-i\theta}g(e^{i\theta}z) \in S_t$ for all θ, (3.23) becomes

$$\operatorname{Re}\, e^{3i\theta}(b_4 - \tfrac{1}{2}b_2b_3) \leq 1,$$

for all θ. Hence

$$(3.24)\quad |b_4 - \tfrac{1}{2}b_2b_3| \leq 1.$$

Schober([1976], p. 22) writes that R. Barnard has noted that (3.24) is false in S_t. As Schober neglects to include Barnard's argument we shall now show that (3.24) is false in S_t. Consider the starlike function $z/(1 + z)^{\varepsilon}(1-z)^{2-\varepsilon}$, $\varepsilon \geq 0$. Its power series expansion is $z + b_2z^2 + b_3z^3 + b_4z^4 + \ldots$ where

$$b_2 = 2 - 2\varepsilon,$$

$$b_3 = 3 - 4\varepsilon + 2\varepsilon^2,$$

$$b_4 = 4 - \frac{20}{3}\,\varepsilon + O(\varepsilon^2),$$

so that

$$b_4 - \tfrac{1}{2}b_2b_3 = (4 - \frac{20}{3}\varepsilon + O(\varepsilon^2)) - (1-\varepsilon)(3-4\varepsilon + O(\varepsilon^2))$$

$$= 1 + \frac{\varepsilon}{20} + O(\varepsilon^2),$$

which, for small positive ε, contradicts (3.24). Thus also Theorem 3.1 cannot be true with the values of Q_k given above.

3.5 The proof of Theorem 3.2 and of Corollaries 3.1, 3.2.

Theorem 3.2 is an immediate consequence of Theorem 3.1. For equation (3.10) implies the existence of $x \in T$ such that

$$Q_n + Q_{n-1} + \ldots + Q_1 + 1 \geq |a_{n+1} - x\, a_n|$$

$$\geq ||a_{n+1}| - |a_n||,$$

proving the theorem. Corollary 3.1 is just the special case $Q_1 = \ldots = Q_n = 0$. We intend to prove a more general version of corollary 3.2.

Theorem 3.5 *Suppose that* $f(z) = \sum_1^\infty a_n z^n$ *satisfies*

$$(3.25)\quad \mathrm{Re}\{e^{i\alpha}(\frac{zf'}{g} - \lambda)\} \geq 0$$

for some $g \in S_p$, *real* α, $\lambda \leq 1$. *Then*

$$(3.26)\quad |(n+1)|a_{n+1}| - n|a_n|| \leq 1 + 2(1-\lambda)n.$$

The function $\int_0^z \frac{1+(1-2\lambda)z}{(1-z)^3}\, dz$ *shows that* (3.26) *is sharp.*

Remark: The class of functions satisfying (3.26) is the set of close-to-convex functions of type λ. (Hamilton and Tuan [1978]).

We may write

$$zf'(z) = q(z)g(z)$$

where $\mathrm{Re}\, e^{i\alpha}(q - \lambda) > 0$. Then we apply Lemma 2.6, noting that

$$q(z) = \lambda + (1-\lambda)\{\frac{(1-e^{-2i\alpha}}{2} + \frac{(1+e^{-2i\alpha})}{2} p(z)\},$$

for some function $p(z)$ in the class P (of section 2.3). Consequently the coefficients of $q(z)$ are bounded by $Q_n = 2(1-\lambda)$.

Thus we apply Theorem 3.2 and deduce our result.

If $f(z)$ is close-to-convex, then $\lambda = 0$, and we have

$$|k|a_k| - (k-1)\,|a_{k-1}|| \leq (2k-1).$$

Therefore, using the triangle inequality we deduce for $n \geq m$

$$|n|a_n| - m|a_m|| \leq \sum_{k=m+1}^{n} ||ka_k| - |(k-1)\,a_{k-1}||$$

$$\leq \sum_{k=m+1}^{n} (2k-1)$$

$$= (n^2 - m^2).$$

3.6 Proof of Theorem 3.3

The Milin-Lebedev inequality, Theorem 10, is a special case of Theorem 11. (Milin [1970] pp 32-37). Suppose that α_n, β_n are related by (3.1) as in Theorem 10. We define

$$(3.27)\quad \Delta_n(s) = \sum_{k=1}^{n} \left(\frac{k|\alpha_k|^2}{s^2} - \frac{1}{k} \right), \quad s > 0,$$

and coefficients $d_n(s)$ by

$$(3.28)\quad \frac{1}{(1-z)^s} = \sum_{n=0}^{\infty} d_n(s) z^n.$$

Then for $s \geq 1$

$$(3.29)\quad |\beta_n| \leq d_n(s) \exp\left(\frac{s}{2d_n(s)} \sum_{k=1}^{n} d_{n-k}(s-1)\,\Delta_k(s) \right).$$

For functions $g(z) = \sum_{k=1}^{\infty} a_k z^k$ that are starlike of order λ we have representation (3.8) with $\delta = -2(1-\lambda)$. Thus

$$(3.30)\quad \log \frac{g(z)}{z} = -2(1-\lambda) \int_T \log(1-xz)\, d\mu$$

for some probability measure μ on the unit circle T. As before we define for $y \in T$,

$$(1-zy)\frac{g(z)}{z} = \sum_{k=0}^{\infty} \beta_k z^k = \exp\left(\sum_{k=1}^{\infty} \alpha_k z^k \right),$$

α_k, β_k are functions of y defined by

$$(3.31)\quad \alpha_k = \frac{1}{k}\,(2(1-\lambda)\mu_k - y^k),$$

and

$$\beta_k = a_{k+1} - y\,a_k,$$

where $\mu_k = \int_T x^k d\mu$ are the moments of μ. We apply Theorem 11 with $s = 1 - 2\lambda$ and obtain

$$\int_T \log|\beta_n|\,d\mu \le \log d_n(1-2\lambda) + \frac{(1-2\lambda)}{2d_n(1-2\lambda)} \sum_{k=1}^{n} d_{n-k}(-2\lambda) \int_T \Delta_k(1-2\lambda)\,d\mu.$$

It suffices to show $\int_T \Delta_k(1-2\lambda)\,d\mu \le 0$. By Theorem 11 this is equivalent to

Lemma 3.2 $\quad \int_T \left(\frac{k|\alpha_k|^2}{(1-2\lambda)^2} - \frac{1}{k}\right) d\mu \le 0.$

From (3.31) $\quad |\alpha_k|^2 = \frac{1}{k^2}\{4(1-\lambda)^2\,|\mu_k|^2 - 4(1-\lambda)\,\mathrm{Re}\,\bar{\mu}_k y^k + 1\}.$

We now integrate with respect to μ. Thus

$$\int_T |\alpha_k|^2\,d\mu = \frac{1}{k^2}(4(1-\lambda)^2|\mu_k|^2 - 4(1-\lambda)|\mu_k|^2 + 1),$$

in view of $\mu_k = \int y^k d\mu$. Consequently

$$\int_T \left(\frac{k|\alpha_k|^2}{(1-2\lambda)^2} - \frac{1}{k}\right) d\mu = \frac{\{4(1-\lambda)|\mu_k|^2(-\lambda) + 1 - (1-2\lambda)^2\}}{k(1-2\lambda)^2}$$

$$= \frac{4(1-\lambda)\lambda(1-|\mu_k|^2)}{k(1-2\lambda)^2}$$

which is non positive as $|\mu_k|^2 \le 1$ and $\lambda \le 0$. This completes the proof of the Lemma.

Thus $\int_T \log|a_{n+1} - x\,a_n|\,d\mu \le \log\, d_n(1-2\lambda)$ from which we deduce the existence of $x_o \in T$ such that

$$|a_{n+1} - x_o a_n| \le d_n(1-2\lambda).$$

Hence we have

$$||a_{n+1}| - |a_n|| \leq |a_{n+1} - x_o a_n| \leq \frac{\Gamma(1-2\lambda+n)}{\Gamma(1-2\lambda)\Gamma(n+1)}$$

using the formulae for d_n. This completes the proof of Theorem 3.3.

3.8 A result for odd spiral-like functions

Robertson has conjectured the following inequality for odd univalent functions (Pammerenke [1975], p. 39),

$$1 + |a_3|^2 + \ldots + |a_{2n-1}|^2 \leq n.$$

This is known to be true for odd starlike functions since all $|a_n| \leq 1$. For spiral-like functions we prove the stronger

Theorem 3.7 *Suppose that* $g(z) = \sum_0^\infty a_m z^m \in S_p$, *and* $g(z)$ *is odd. Then*

$$(3.42)\quad \frac{(m-k)}{m} \leq \frac{|\beta_o|^2 + \ldots + |\beta_{m-k-1}|}{|\beta_o|^2 + \ldots + |\beta_m|^2}$$

where

$$|\beta_{2k}| = |\beta_{2k+1}| = |a_{2k+1}|, \quad k = 0, 1, 2\ldots$$

Let us define as in (3.18)

$$\sum_0^\infty \beta_m z^m = (1-zx)\, g(z)/z$$

$$= \exp\left(\sum_1^\infty \alpha_n z^n\right).$$

Thus

$$\beta_m = a_{m+1} - x\, a_m,$$

so that $|\beta_m|$ has the value given in Theorem 3.7, since g is odd. Then we use equation (3.14), i.e.

$$\sum_{j=1}^{m-1} \left\{ \frac{1}{j+1} - j\left(1 - \frac{j}{n}\right) |\alpha_j|^2 \right\} - \sum_{j=1}^{m-k-1} \left\{ \frac{1}{j+1} - j\left(1 - \frac{j}{m-k}\right) |\alpha_j|^2 \right\}$$

$$\leq \log \left\{ \frac{m}{(m-k)} \, \frac{|\beta_o|^2 + \ldots + |\beta_{m-k-1}|^2}{|\beta_o|^2 + \ldots + |\beta_{m-1}|^2} \right\}.$$

Next we integrate this inequality with respect to the measure μ in the representation (3.8), with $\delta = -(1 + e^{-i2\alpha})$. Then Lemma 3.1 implies

that

$$\int_T \log \left(\frac{|\beta_o|^2 + \ldots + |\beta_{m-k-1}|^2}{|\beta_o|^2 + \ldots + |\beta_{m-1}|^2} \right) d\mu \geq \log \frac{(m-k)}{m} .$$

Since β_n is independent of x we deduce Theorem 3.7.

Remarks.

(i) The function $z/(1-z^2)$ shows that the inequality is sharp.

(ii) The case that $m-k=1$, $m=2n$ gives Robertson's inequality for S_p.

CHAPTER 4

Covering theorems for univalent functions.

4.1. Introduction

The theory of Riemann surfaces (Ahlfors and Sario [1960]) has been extensively studied. Suppose that D is an open Riemann surface and $a \in D$. A great amount of work has been concerned with functions f that are analytic on E and take values in D. In this chapter we investigate the range of values of the class of such functions $f(z)$ that are univalent on E and normalised by the condition $f(0) = a$. We denote this class by $S_a(D)$.

If D is simply connected then $S_a(D)$ is normal provided that D is hyperbolic. In that case the problem of determining the range of values, i.e.

$$(4.1)\quad \{f(r) : f \in S_a(D)\},$$

is just an application of the uniformisation theorem and the principle of subordination. For domains of connectivity 2 or greater these questions are still open. At a recent conference at Durham and in a paper of Duren and Schober [1980] the question of finding the range of values of $S_1\{0 < |w| < \infty\}$ was raised. The author [1980] was able to announce preliminary results on this problem.

Theorem 4.1. *Suppose that* $w = \lambda(\tau)$ *is the elliptic modular function* † *mapping* τ-*points* $\infty, 0, 1$ *to* w-*points* $1, \infty, 0$. *Then*

$$\{f(r) : f \in S_1(0 < |w| < \infty)\} = \{\lambda(\tau) : \operatorname{Im} \tau \geq 2\nu(r)\},$$

where

$$\nu(r) = 4\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}} \Big/ \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-(1-r^2)t^2)}}$$

is the modulus of $E - [0, r]$.

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† See Ahlfors [1966 a], p.269 or section 5.2.

In this chapter we give explicit results for doubly connected domains D and partial results for other classes. Hence we assume that D is conformally equivalent to a sphere with m holes and g handles. Thus we are assuming that D has analytic curves or punctured holes as boundaries. †

Problem (4.1) may be regarded as a generalisation of

$$(4.2) \quad \max \{|f'(0)| : f \in S_a(D)\},$$

i.e. the radius of mapping problem. This problem, for the case that $a = \infty$ and $D \subset \mathbb{C}$, is the extremal problem of Grötzsch-Pólya. (Schiffer and Spencer [1954] †† p.382). If $D = \mathbb{C} - \{w_1, \ldots, w_n\}$ then Grötzsch [1930] and Lavrent'ev [1934] showed that the extremal function $f(z)$ for (4.2) satisfies the quadratic differential equation

$$\left(\frac{zf'}{f - a} \right)^2 \frac{b_{m-1} f(z)^{m-1} + \ldots + b_o}{(f(z) - w_1)\ldots(f(z) - w_m)} = 1 .$$

However as the $b_{m-1}, \ldots, b_o$ are unknown, this equation cannot be solved.

The other reason to study (4.1) is the point functionals on $S_a(D)$, i.e. we wish to maximise

$$(4.3) \quad \operatorname{Re} \Phi(f(z))$$

over $S_a(D)$, where Φ is analytic on some suitable cover of D. In the following chapter a generalisation of Theorem 1 is used to study problems of the form (4.3). In fact (4.1) does not allow one to consider the most general point functionals, e.g. $\Phi(w) = i \log w$ and $D = \{0 < |w| < \infty\}$. For this reason, in section 4.2, we introduce the range of values as a covering surface X of D. In section 4.3 we characterise the range of values by a certain extremal problem. This is solved by means of the Jenkins-Strebel differential. In section 4.4 the explicit answers are given for doubly connected regions.

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† In section 4.1 our results are for arbitrary surfaces.

†† Their book "Functionals on finite Riemann surfaces" will be referred to as FRS.

Next we apply the theory for $S_a(0 < |w| < 1)$ to solve a problem of Huckemann, raised at a conference at Durham in 1976. The following section gives the complete theory of the range of values of S_1. This proves to be useful for the following chapter. We conclude with a discussion of the possibility of generalising our results to domains of higher connectivity.

4.2 The Covering Space X.

Let Y be the universal covering surface of $D - \{a\}$. We assume that Y is hyperbolic, i.e. there is a univalent mapping $y: \{\tau : \mathrm{Im}\,\tau > 0\} \to Y$. The space X is obtained by "adding" the point a to Y. We now briefly describe this process (see Chapter 1 of Nevanlinna's book [1953] for a complete account).

First we normalise the map $y(\tau)$. Without loss of generality we assume that the winding operation T about a (considered as an element of the fundamental group of $D - \{a\}$) corresponds to a translation $T : \tau \to \tau + 2$ of the τ-plane. The latter can always be achieved by means of a bilinear transformation. The space X is then defined to be $(Y/T) \cup \{a\}$. The normalisation of y ensures that the map

$$z = \exp(\pi i \tau(y))$$

maps X univalently onto $|z| < 1$, i.e. this map is the uniformiser of X. Hence X is a simply connected cover of D which covers the point a once and every other point infinitely often.

Example: If $D = \{0 < |w| < \infty\}$, $a = 1$ then the map y is the elliptic modular function $\lambda(\tau)$ of Theorem 4.1. Here we do not distinguish between the mappings λ which map onto $D - \{1\}$, Y or X.

The space X arose long ago (Littlewood [1944], p. 196) in another context. It is the range of analytic functions $f: E \to D$ such that $f(0) = a$ and $f(z) \neq a$ for $z \neq 0$. From this point of view $f(z)$ has a position on X defined by the (endpoint) homotopy equivalence

class of the arc

$$\gamma(t) = f(tz), \quad (0 \leq t \leq 1).$$

The reader may recall that (homotopy) equivalence classes of arcs are used to define the universal covering surface (Ahlfors and Sario [1960]).

Thus it is clear that each $f \in S_a(D)$ may be considered to map into X : the homotopy class of $f[0,z]$ defines $f(z) \in X$. The mapping is analytic, and note that $f(E)$ is a proper subset of X which has the property that distinct points do not lie over the same point of D. Now the previous paragraph shows that each point of X is essentially a homotopy equivalence class of arcs $\gamma : [0,1] \to D$ of the following type §

(4.4) Suppose that $b \in D$ and $a \neq b$ then

(i) $\gamma(0) = a, \quad \gamma(1) = b$

(ii) $\gamma(t) \neq a, \quad 0 \leq t \leq 1.$

However, for each $f \in S_a(D)$, $f[0,z]$ is a simple arc. Our first lemma shows that X is the correct carrier for the range of values.

Lemma 4.1: Suppose that γ is an arc which satisfies (4.4). Then γ is homotopic to a simple arc γ_1.

This exercise in topology is relegated to Appendix 4.1.Finally we note that the Riemann mapping theorem ensures that each simple arc γ_1 is homotopic to $f[0,r]$ for some $f \in S_a(D)$ and $r < 1$. Thus we define the range of values of $S_a(D)$ to be

(4.5) $\{f(r) \in X : f \in S_a(D)\}$.

The extremal problem which defines (4.5) is

Problem 1. Let us call a simply connected domain $\Delta \subset D$ admissible if Δ contains an arc homotopic to γ_1. Then $f(E) = \Delta$ for some $f \in S_a(D)$

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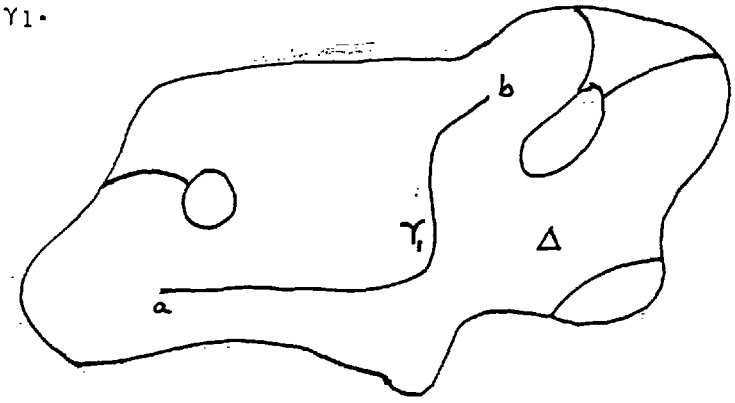
§ Except for the trivial map $\gamma : [0,1] \to \{a\}$.

and the problem is to determine

$$(4.6)\quad \min\{|z| : f(z) = b\},$$

over the class of admissible Δ.

The class $S_a(D) \cup \{a\}$ is compact and hence there is a function $f_o \in S_a(D)$ that is extremal for (4.6). The solution depends on b and γ_1.



4.3 The extremal problem.

The theory of extremal length combined with the variational technique of Schiffer and Spencer (FRS,pp 273 - 396) provides the solution to (4.6). Recall that D is a finite surface minus a finite number of points $w_1,\ldots, w_n$.

Consider a family F of closed curves C in D such that F is closed under homotopies. A measurable metric ρ is admissible if

$$(4.7)\quad \int_C \rho|dw| \geq 1$$

for all $C \in F$. The module m of F is the quantity

$$(4.8)\quad \inf_{\rho} \int\int_D \rho^2\, dA,$$

where ρ is admissible.

Jenkins [1957] used the technique of Schiffer and Spencer to obtain

Theorem 12 The extremal metric ρ is equal to $|Q|^{\frac{1}{2}}$ for a quadratic differential Q which satisfies

(i) $Q\,dw^2$ is regular on $\overline{D} - \{w_1,\ldots,w_n\}$.

(ii) $Q\,dw^2$ has, at worst, simple poles at $w_1, \ldots, w_n$,

(iii) Noncritical trajectories, i.e. those curves $Q\,dw^2 \geq 0$ which do not originate at poles or zeros of Q, are members of F.

(iv) The boundary of D is a subset of the critical trajectories.

From this we prove

Theorem 13. Suppose that f_o is extremal for the range of value problem (4.6). Then there is a quadratic differential Q on D with the following properties:

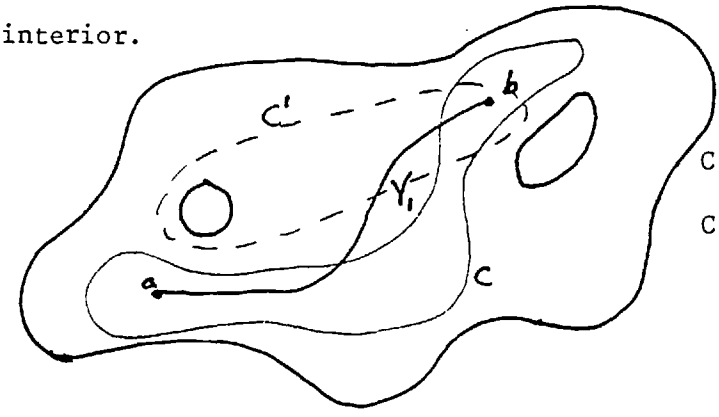
(i) Q is regular on $\overline{D}$ except at isolated points and a,b.

(ii) Q has simple poles at a and b and, at worst, at isolated points of the boundary of D

(iii) The critical trajectory $\Gamma : Q\,dw^2 \geq 0$ bounds $f_o(E)-f_o([0,r_o])$, where r_o is the minimum in (4.6).

Consider problem (4.8) for the surface $D_1 = D - \{a,b\}$ and F the class of closed curves C of the following type.

A curve C is admissible if it is homotopic (in D_1) to a curve C_1 which has a simply connected subset of D which contains γ_1 in its interior.



C is admissible ————

C' is not admissible. — — —

Then with Theorem 12 we obtain a quadratic differential Q defined on $\overline{D}$. Furthermore the module of F is

$$m = \iint_D |Q|\,dA.$$

Now consider any doubly connected domain $B \subset D$ such that the separating

curves of B are members of F. The modulus of B is given by $\frac{1}{2\pi}\log(\frac{1}{s})$ if B is conformally equivalent to $\{s < |w| < 1\}$. However the module of the family F_B of separating curves in B is equal to the modulus (Ohtsuka [1970], p.148). Thus as $F_B \subset F$ (Ohtsuka [1970], p.77)

$$(4.9)\quad \frac{1}{2\pi}\log\frac{1}{s} \leq m.$$

In particular for $f \in S_a(D)$, admissible for problem (4.6), we see that $B = f(E) - f[0,z]$, $(f(z) = b)$, is an admissible doubly connected subset of D. Now, (Ohtsuka [1970] p.179), $E - [0,r]$ has modulus

$$(4.10)\quad \nu(r) = 4\,\frac{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}}{\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-(1-r^2)t^2)}}}$$

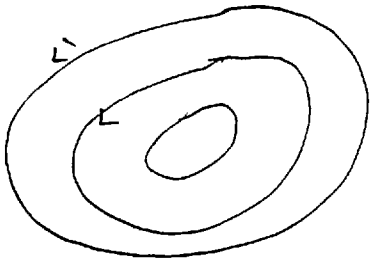
Thus B has modulus $\nu(r)$ and by (4.9)

$$(4.11)\quad \nu(r) \leq m.$$

The remainder of the proof is concerned with reversing (4.11).

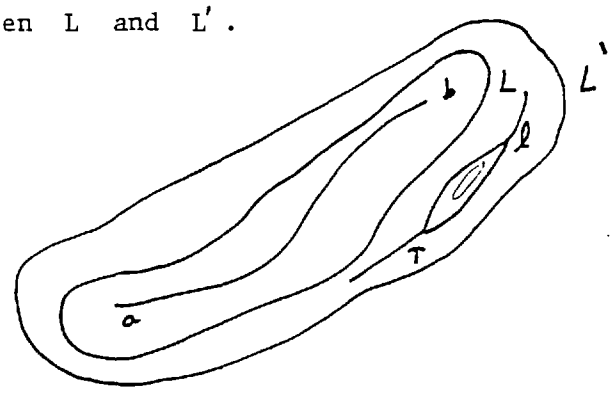
First we show that the critical trajectories of Q bound a doubly connected domain B_o.

Let L be a noncritical trajectory of Q. From Theorem (12),(iii), the interior of L is a simply connected subset of D which contains a,b. Since trajectories only meet at zeros or poles then for any other noncritical trajectory L′ either L is interior to L′ or vice versa, for the region between L and L′ is an annular region (from (iii) and the definition of F)



The only poles inside L′ are also inside L. There can be no zeros of Q between L and L′, otherwise there would be a critical trajectory T

between L and L'. Such a critical trajectory is not a simple curve but branches at its zeros. However these branches cut off subregions R which lie between L and L'.



Suppose that ℓ bounds such a subregion. Then the trajectories interior to ℓ do not contain a,b in their interiors. Consequently these trajectories do not belong to F, which contradicts Theorem 12, part (iii). Therefore the region between L and L' contains no critical trajectories.

Now let B_o be the union of the regions between noncritical trajectories. The above remarks show that this union may be taken to be the union of increasing doubly connected sets. Hence B_o is doubly connected and one component, Γ_1 say, of $\overline{D} - B_o$ lies in D. This is the component which contains a,b. Also the boundary of B_o (in $\overline{D}$) is the same as the set of critical trajectories of Q.

Next we show that the modulus of B_o is equal to the module m of F. The differential

$$(4.12) \qquad dH = \sqrt{Q}\ dw$$

is analytic on B_o (as Q has no poles or roots in B_o). dH has real period w_1 along the cycle J_1 separating Γ_1 from the other component Γ_2 of $\overline{D} - B_o$ since $Q\ dw^2 \geq 0$ on Γ_1. Hence for any separating curve C of B_o

$$\int_C |Q|^{\frac{1}{2}}\ |dw| \geq \left| \int_C Q^{\frac{1}{2}}\ dw \right| = |w_1|.$$

Also on trajectories C,

$$|Q|^{\frac{1}{2}}\ |dw| = Q^{\frac{1}{2}} dw$$

and

$$|w_1| = \left| \int_C Q^{\frac{1}{2}}\ dw \right| = \int_C |Q|^{\frac{1}{2}}\ |dw|.$$

Hence by (4.7) $|w_1| \geq 1$ and by (4.8) and the extremal property of $|Q|^{\frac{1}{2}}$ we have

$$|w_1| = 1.$$

Thus, as $dH = \sqrt{Q}\, dw$ is real on the boundary of B_o, the multivalued function

$$(4.12) \quad H = \int \sqrt{Q}\, dw$$

maps B_o onto an infinite strip of the form

$$(4.13) \quad \Sigma = \{0 < \operatorname{Im} H < \frac{u_2}{2}\}.$$

Remark 4.1 Note that we may have $u_2 > 0$ by choosing the sign of $\sqrt{Q}$ correctly.

As H has period $w_1 = 1$ the function $v = \exp(-2\pi i H)$ maps B_o univalently onto $\{1 < |v| < \exp(\pi u_2)\}$. Thus B_o has modulus $\frac{u_2}{2}$. However

$$(4.14) \quad \frac{u_2}{2} = \int_D \int |Q|\, dA = m$$

as H maps B_o, cut from Γ_1 to Γ_2, onto a region of area $\frac{u_2}{2}$.

Now let $f_o \in S_a(D)$ map E onto $B_o \cup \Gamma_1$, and $f_o(r_o) = b$. Then it is well known (Ohtsuka [1970], p. 172) that $E - f_o^{-1}(\Gamma_1)$ has modulus less than or equal to the modulus of $E - [0, r_o]$. Thus B_o has modulus $m \leq \nu(r_o)$. From (4.11) we deduce

$$m = \nu(r_o).$$

Consequently $f_o^{-1}(\Gamma_1) = [0, r_o]$ and thus

$$(4.15) \quad \Gamma_1 = f_o[0, r_o].$$

To complete the proof of the theorem we have only to note that Q does have simple poles at a and b. However (4.15) shows that the critical trajectory Γ_1 has endpoints at a and b. The local behaviour of quadratic differentials (Jenkins [1965], Chapter 2) shows that $Q\, dw^2$ has simple poles at a and b.

Remark 4.2. As Γ_1 does not branch (by (4.15)) $Q\,dw^2$ has no zeros on Γ_1. We showed that $Q\,dw^2$ has no zeros in B_o. The only poles are at a,b and isolated boundary points. Since $Q\,dw^2 \geq 0$ on the analytic boundary curves of D then we may continue $Q\,dw^2$ to the double of D, defining

$$Q(\tilde{w})\,d(\tilde{w})^2 = \overline{(Q(w)dw^2)}$$

where $\tilde{w}$ is dual to w on the double. If α is a zero of Q on the boundary of D such that k trajectories in D begin at α, then $2k+2$ trajectories originate at α in the double. Thus α is a zero of even order. The only other zeros occur at the branch points of $D \cap \Gamma_2$.

Remark 4.3. We have not shown the uniqueness of the extremal metric Q and hence the uniqueness of f_o. Actually we have shown that for each extremal Q there is an extremal f_o.

4.4 Results for doubly connected domains

Suppose that, without loss of generality, D is given by

$$(4.16)\quad \{s < |w| < 1\},$$

where $0 \leq s < 1$ and $0 < a < 1$, or

$$(4.17)\quad \{0 < |w| < \infty\},$$

and $a = 1$.

In this section we provide unique representations for the quadratic differential Q of section 4.3. We characterise $\sqrt{Q}\,dw$ as an analytic differential on a 4 sheeted cover of D. However first

we prove

Lemma 4.2 (i) $Q\,dw^2$ has no zeros in D.

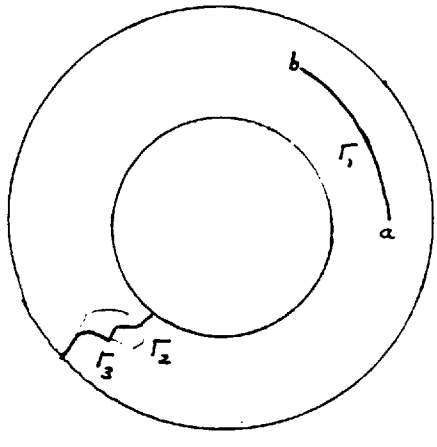
(ii) In case (4.16) $Q\,dw^2$ has one double zero on $|w| = 1$, and if $s > 0$ another on $|w| = s$.

(iii) In case (4.17) and in case (4.16), $s = 0$, $Q\,dw^2$ has a simple pole at 0.

(iv) In case (4.17) $Q\,dw^2$ has a simple pole at ∞.

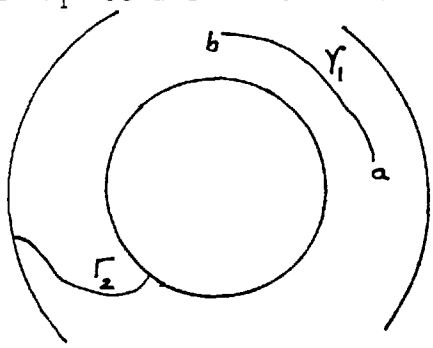
As $Q\,dw^2 \geq 0$ on the boundaries then Q may be continued into $0 < |w| < \infty$ as a meromorphic function. Theorem 13, (ii) shows that in case (4.17) and case (4.16), $s = 0$, $Q\,dw^2$ has, at worst, a simple pole at 0. Also $Q\,dw^2$ has, at worst, a simple pole at ∞, for (4.17). This together with Remark 4.2, reduces the proof to showing that $\Gamma_2 \cap D$ is a single unbranched analytic curve running between the components of $\mathbb{C} - D$. Now in case (4.16), $s > 0$, $\Gamma_2 \cap D$ meets the boundary at zeros of order 2. In the other case, 0 is the starting point of $\Gamma_2 \cap D$. The local property of trajectories shows that 0 is a simple pole. Similarly in (iv).

However the maximal property of $B_0 = D - (\Gamma_1 \cup \Gamma_2)$ shows that if Γ_3 is any subset Γ_2 which connects the components of D then $D - (\Gamma_1 \cup \Gamma_3)$ has modulus equal to $D - (\Gamma_1 \cup \Gamma_2)$. Hence $\Gamma_3 = \Gamma_2$ and Γ_2 cannot branch.



We now restrict our attention to case (4.16), $s > 0$. The construction of the covering surface $\mathcal{D}$ is as follows.

(i) Let D_1 be a two-sheeted cover of D with no branch points.



The curve γ_1, which defines our extremal problem, now appears as a pair γ_+, γ_- on D_1.

(ii) We cut D_1 along γ_+ and γ_- and join one side of the upper curve γ_+ to the opposite side of γ_-, and vice versa. Thus we obtain a two-sheeted cover D_2 of D with branch points at a, b and with γ_1 as a branch line.

(iii) Finally we add the Schottky double $\tilde{D}_2$ of D_2 (FRS pp 29 - 33), joining D_2 to $\tilde{D}_2$ at $|w| = 1$, $|w| = s$.

In appendix 4.2 it is shown that $\mathcal{D}$ is a closed orientable surface of genus 3.

Lemma 4.3 *dH is an analytic differential on* $\mathcal{D}$

Certainly dH is an analytic differential on B_o (and its three images which we constructabove). Let Γ_2 be the component of Γ which joins the inner and outer boundary of D. Then by Lemma 4.2, as we move across Γ_2, dH remains analytic and changes sign. Also as we move across Γ_1, which is homotopic to γ_1, dH changes sign. At the branch point a the local uniformiser p is defined by

$$w - a = p^2.$$

Now

$$dH = \sqrt{Q}\, dw \sim \frac{c}{\sqrt{w-a}}\, dw = 2c\, dp,$$

thus dH is analytic at a. Similarly dH is analytic near b. Thus dH is analytic on D_2. Also Lemma 4.2 shows that Q has zeros of order 2 at exactly two points, $e^{i\phi}$ and $s e^{i\theta}$ say, where Γ_2 meets the boundary of D.

Also $Q\,dw^2 \geq 0$ on the boundary of D. Thus dH is defined on $\mathcal{D}$ by reflection across the boundary, i.e.

$$(4.18)\quad dH(\tilde{p}) = \overline{dH(p)},$$

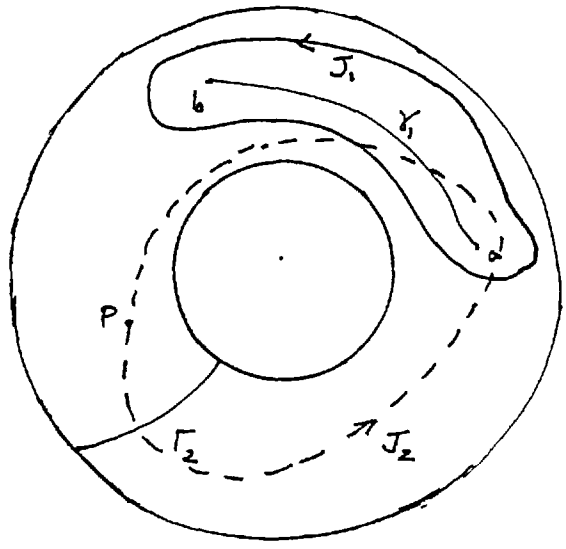
where $\tilde{p}$ is a point 'dual' to $p \in D_2$. This shows that dH is analytic on $\mathcal{D}$ with zeros at $re^{i\theta}$ and $e^{i\phi}$, and their two images on D_1.

We now wish to apply the following theorem of Riemann (FRS pp.64-70).

Theorem 14 Suppose that $\mathcal{D}$ is a closed orientable surface of genus h, with $2h$ base cycles $J_1,\ldots,J_{2h}$. Then for any set of $2h$ real numbers u_k there exists a unique analytic differential dH on $\mathcal{D}$ such that

$$u_k = \mathrm{Im} \int_{J_k} dH\,,$$

$k = 1,\ldots, 2h.$



The diagram shows two base cycles for $\mathcal{D}$. Cycle J_1 was introduced in section 4.3. Cycle J_2 begins on the upper sheet of D_2, at P say, crossing Γ_2 onto the lower sheet. Then J_2 crosses back to the upper sheet by passing through Γ_1. The cycles J_3 and J_4 are dual to J_1 and J_2 on the double. Also there are cycles J_5 and J_6 which are generated by the doubling of D_2: For instance J_5 consists of Γ_2 and its double.

We now calculate the periods $w_k = \int_{J_k} dH$. In section 4.3 we established that

$$\text{Im } w_1 = 0$$

$$\text{Im } w_2 = u_2 > 0$$

The other integrals, from (4.18), are

$$w_3 = \int_{J_1} \overline{dH} = \overline{w_1}\,, \quad w_4 = \int_{J_2} \overline{dH} = \overline{w}_2$$

and since dH is real on Γ,

$$\text{Im } w_5 = \text{Im } w_6 = 0.$$

Hence dH is uniquely determined by the 6 numbers $(0,u_2, 0,-u_2, 0,0)$. We first obtain a solution dH_o for the Riemann problem for numbers $(0,1,-1,0,0,0)$ Then u_2 is chosen by $\int_{J_1} dH_o = 1/u_2$, and $dH = u_2\, dH_o$. Thus we have

Theorem 4.2 *Given* D *and* γ_1 *we construct* $\mathcal{D}$ *and* dH_o *to solve the* $(0,1,0,-1,0,0)$ *Riemann problem on* $\mathcal{D}$. *Then the least* r *such that* $f[0,r]$ *is homotopic to* γ_1, *for some* $f \in S_a(D)$, *is given by*

$$(4.19)\quad \nu(r) = \frac{1}{2\int_{J_1} dH_o}\,.$$

4.5 The solution of a problem of F. Huckemann

In this section we use the theory for $S_a(0 < |w| < 1)$ to answer a question of F. Huckemann. Lemma 4.2 shows that $Q\,dw^2$ has simple poles at a,b. Also $Q\,dw^2$ has a simple pole at 0 and double zero $e^{i\phi}$ on $|w| = 1$. By analytic continuation, Q has poles at $1/\bar{a}$ and $1/\bar{b}$, hence

$$(4.20)\quad Q(w) = \frac{c\, e^{i\theta}(w - e^{i\phi})^2}{(w-a)(w-b)(\bar{a}w-1)(\bar{b}w-1)}\,,$$

for some $c > 0$.

The parameters θ, ϕ may be determined by the condition that $Q\,dw^2 \geq 0$ on $|w| = 1$ and $\operatorname{Im} \int_{J_1} \sqrt{Q}\,dw = 0$. In any case problem (4.5) is solved by $\nu(r) = \frac{1}{2}\int_{J_1} \sqrt{Q}\,dw$.

The problem of Huckemann is as follows. Let us consider the class G of continua C in $E - \{0\}$ which join a and b, such that $E - C$ is connected. He called C a natural continuum if there exists a quadratic differential σ on $E - \{0\}$ such that

(i) $\sigma\,dw^2 \geq 0$ on C and $|w| = 1$

(ii) $0,a,b$ are the only poles of σ in $|w| \leq 1$.

Huckemann asked if each homotopy class of G contains a natural continuum.

We pose problem (4.6), as $\gamma_1 \in G$, for the family $S_a(0 < |w| < 1)$. Then the quadratic differential $Q\,dw^2$ of (4.15) has the required properties, as the trajectory running from a to b is homotopic to γ_1. In fact we obtain the much more general

Theorem 4.3 *Suppose that D is a finite Riemann surface minus a number of points, $w_1,\ldots,w_n$. Then for each arc C_1 which satisfies (4.4) there exists a quadratic differential Q on $\overline{D}$ such that*

(i) *$Q\,dw^2 \geq 0$ on ∂D*

(ii) *$Q\,dw^2$ has simple poles at a,b and perhaps at $w_1,\ldots,w_n$ (and nowhere else). At least one w_k is a pole.*

(iii) *The trajectory $\Gamma_1 : Q\,dw^2 \geq 0$ running from a to b is homotopic to C_1.*

Remark 4.4 The above questions arose in the context of solving a problem of D. Gaier (see Huckemann [1977]).

4.6 The class S_1

From Lemma 4.2 we find that $Q\,dw^2$ has no zeros in D but has simple poles at 0 and ∞, i.e.

$$(4.21)\quad Q(w) = \frac{k}{w(w-1)(w-b)}$$

for some constant k. The value of $\mathrm{Arg}(k)$ is determined from

$$\mathrm{Im} \int_{J_1} \sqrt{Q}\, dw = 0.$$

For the class S_1 our results are quite explicit. It is well known (Ahlfors [1966 a], p. 269-274) that if $w_1 = \int_{J_1} \sqrt{Q}\, dw$ and $w_2 = \int_{J_2} \sqrt{Q}\, dw$, then

$$(4.22)\quad b = \lambda\left(\tfrac{1}{2}\frac{w_1}{w_2}\right), \qquad \mathrm{Im}\,\frac{w_1}{w_2} = 2\nu(r).$$

However, the elliptic modular function $\lambda(\tau)$ uniformises the universal covering space X of $\mathbb{C} - \{0,1\}$ and hence, following 4.2, uniformises the covering space X for S_1. This, together with (4.17) implies the following generalisation of Theorem 4.1.

Theorem 4.4 The range of values of S_1 is

$$\{\lambda(\tau) \in X : \mathrm{Im}\,\tau > 2\nu(r)\}.$$

Remark We proved that the above theorem is equivalent to a certain extremal length problem associated with the quadruple $0,1,\infty,b$. This problem has attracted considerable interest. Teichmüller [1938] gave applications to the Ahlfors' distortion theorem. Schiffer [1946] solved the equivalent modulus problem. A simple proof may be found in Ahlfors' "Lectures on Quasiconformal Mappings" ([1966 b] chapter 2). Krzyz [1970] applied it to the question of Koebe sets of univalent functions.

We conclude this section by giving a description of the extremal functions $f_o(z) \in S_1$ for point functional problems. The omitted set

$$\Gamma_2 = C - f_o(E)$$

is often studied in this context. The geometry of Γ_2 will now be described.

Equation (4.21) defines Γ_2 as the solution of a quadratic differential equation $kdw^2/w(w-1)(w-b) > 0$. The trajectories of the quadratic differential $dw^2/w(w-1)$ are hyperbolae with foci at 0 and 1. Thus Γ_2 must meet each of these hyperbolae at w with angle

$$(4.23) \quad \tfrac{1}{2} \arg \frac{1}{k} (w-b).$$

In particular, at each point of the half line,

$$w = b + kt, \quad t \geq 0,$$

Γ_2 is orthogonal to ellipses with foci at 0 and 1. From (4.21) one sees that Γ_2 makes angle $\arg (-\frac{b}{k})$ at 0.

The curves Γ_1, Γ_2 display some symmetry. The module problem (of section 4.3) shows that the domain $B = \mathbb{C} - (\Gamma_1 \cup \Gamma_2)$ is invariant under linear maps

$$T_1 w = w$$

$$T_2 w = \frac{b}{w}$$

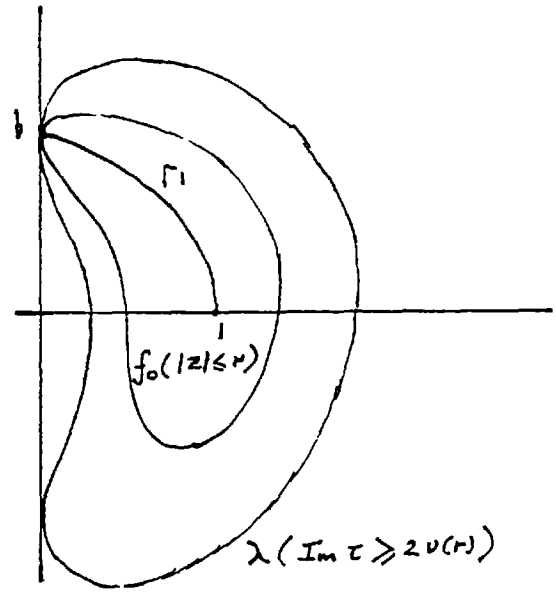
$$T_3 w = \frac{b(w-1)}{(w-b)}$$

$$T_4 w = \frac{w-b}{w-1}$$

and no others. Here we use the uniqueness of f_o.

For instance $T_2(\Gamma_2) = \Gamma_2$, and thus there is $w \in \Gamma_2$ such that $T_2(w) = w$. Then, choosing the sign correctly, $\sqrt{b} \in \Gamma_2$. Similarly $-\sqrt{b} \in \Gamma_1$.

Duren and Schober [1980] found that the extremal functions for the problem $\min \{\text{Re } f(r) : f \in S_1\}$ exhibit many similar properties. At the point on $\text{Re } b = m$, with minimum r such that $f_o(r) = b$, Γ_1 is orthogonal to $\text{Re } b = m$, for Γ_1 is $f_o[0,r]$ which is orthogonal to $f_o(|z| \leq r)$, which is tangent to $\text{Re } b = m$.



The earlier results of this section show that Γ_1 has angle $\arg(b-1) + \arg(-\frac{b}{k})$ at b. Thus k may be taken to be $b(b-1)$. In general k has argument $-\operatorname{Arg}\int_{\Gamma_1} \frac{dw}{\sqrt{w(w-1)(w-b)}}$. Thus the curve Γ_o of points b where $\operatorname{Re} b = \min\{\operatorname{Re} f(r) : f \in S_1\}$ may be found from

$$0 = \operatorname{Arg}_{b(b-1)} \frac{1}{\Gamma_1} \int \frac{dw}{\sqrt{w(w-1)(w-b)}} .$$

It is of interest to know when f is elementary, i.e.

$$(4.24)\quad f(z) = \left(\frac{1 + c_1 z}{1 + c_2 z}\right)^2, \qquad |c_1| = 1,$$

or equivalently Γ_2 is a ray and Γ_1 an arc of a circle. If $|b| = 1$ or $b > 0$, then uniqueness shows that f(z) has form (4.24). For if $|b| = 1$, then f_o and $1/\overline{f}_o(\overline{z})$ is extremal. Thus $w \in \Gamma_1$ implies $1/\overline{w} \in \Gamma_1$, consequently Γ_1 is the subarc of a circle. In this case uniqueness shows that $f(z) = \left(\frac{1 + e^{i\theta} z}{1 + e^{-i\theta} z}\right)^2$ where $\theta = \pi - \arccos(\frac{1}{4} \arg b)$. This also follows from results of Mori [1956] and Tiechmüller [1938].

If $b > 0$ then $f(z) = \left(\frac{1 + z}{1 - z} \right)^2$ $(b \geq 1)$, $f(z) = \left(\frac{1-z}{1+z} \right)^2$ $(b \leq 1)$, once again by uniqueness. This proves one side of

Theorem 4.5 The extremal functions have form (4.24) if and only if γ_1 is homotopic to $[1,b]$, where $b > 0$ or $|b| = 1$.

The differential inequality $\frac{k\, dw^2}{w(w-1)(w-b)} \geq 0$ for Γ_2 which has the form

$$w = At, \qquad t \geq 0$$

for some A gives

$$(4.25) \quad \operatorname{Im} \frac{\bar{k}}{A} (At-1)(At-b) = 0.$$

Expanding (4.25) in powers of t and setting the coefficients equal to 0 implies.

$$(4.26) \quad \operatorname{Im} A\bar{k} = 0,$$

$$(4.27) \quad \operatorname{Im} \bar{k} (b + 1) = 0$$

and

$$(4.28) \quad \operatorname{Im} (b\bar{k}/A) = 0.$$

Now if $A = \alpha_1 e^{i\alpha_2}$, $b = \beta_1 e^{i\beta_2}$, $\bar{k} = \delta_1 e^{i\delta_2}$ then (4.26) and (4.28) imply

$$\beta_2 + 2\delta_2 = 0 \quad [\pi], \text{ respectively,}$$

Substituting $\beta_2 = -2\delta_2$ in (4.27) we obtain

$$(\beta_1 - 1) \sin \frac{\beta_2}{2} = 0,$$

and hence $\beta_1 = 1$ or $\beta_2 = 0$. The other possibility, $\beta_2 + 2\delta_2 = \pi$, leads to $(\beta_1 + 1) \cos \frac{\beta_2}{2} = 0$, i.e. $\beta_2 = \pi$. Then $\delta_2 = 0$ which, from (4.26), implies $\alpha_2 = 0$. Thus $\Gamma_2 = \{ w > 0 \}$ and $1 \in \Gamma_2$ which is impossible. This completes the proof of Theorem 4.5.

4.7 Conclusions

It is interesting to consider if canonical solutions of (4.5) can be obtained for surfaces of higher connectivity. Given an extremal metric $|Q|^{\frac{1}{2}}$ one may construct a four sheeted cover of $\mathcal{D}$ upon which dH is single valued. Unfortunately $\mathcal{D}$ will have branch points at the zeros of $Q\,dw^2$ and branch lines on the critical trajectories.

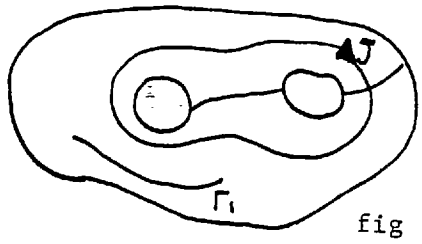


fig (i)

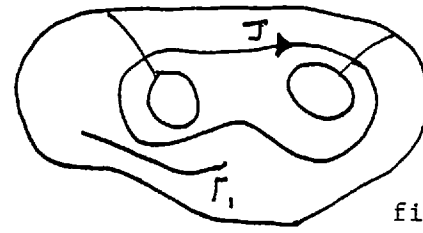


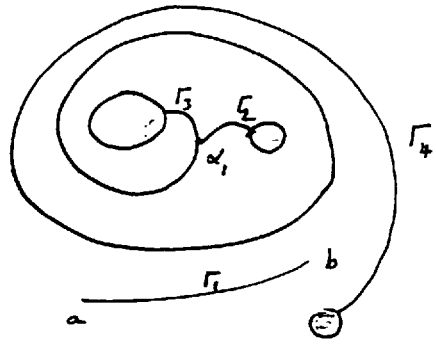
fig (ii)

The above diagrams illustrate just what sort of problems can arise. In fig (i) dH has opposite signs as we pass around J, while in fig (ii) dH has the same sign. Thus even when $Q\,dw^2$ has no zeros in D the locating of branch lines causes problems.

Actually given points $\alpha_1,\ldots,\alpha_p \in D$ and a system of cuts $\Gamma_1,\ldots,\Gamma_n$ such that $D - \{a,b,\alpha_1,\ldots,\alpha_p,\Gamma_1,\ldots,\Gamma_n\}$ is doubly connected, then one may pose problems which give rise to quadratic differentials $Q\,dw^2$ with roots at $\alpha_1,\ldots,\alpha_p$ and branch cuts homotopic to $\Gamma_1,\ldots,\Gamma_n$. The idea is to consider families F as in Theorem 12 and relate them to subclasses of $S_a(D)$. Thus for each geometric condition "$\alpha_1,\ldots,\alpha_p$ and $\Gamma_1,\ldots,\Gamma_n$" which is well posed, i.e. the construction of the covering surface is well defined, there is a unique extremal Q and f. In this case one constructs dH via Theorem 14.

This leaves the original problem as a question of maximising $\iint |Q|\,dA$ for families of admissible surfaces associated with $\alpha_1,\ldots,\alpha_p$ and $\Gamma_1,\ldots,\Gamma_n$. Topological considerations similar to the proof of Lemma 4.2 showthat if, say, D is a finite surface of genus 0

bounded by m analytic curves then one may assume that $p \leq (m-2)$. It is not so easy to cut down the way in which the cuts Γ_k may vary, e.g.



$m=3$.

In any case (4.6) is reduced to a difficult problem in calculus of a large number of complex variables.

Appendix 4.1 Proof of Lemma 4.1

We may suppose that D is a covering of the plane. Furthermore it is enough to consider piecewise analytic arcs γ. Let T be the set of $s \in (0,1]$ such that the sub-arc γ_s of γ running from a to $\alpha(s)$ is homotopic to a simple piecewise analytic curve β_s. We prove the lemma by showing that T is an open/closed nonempty set.

For any disk Δ, centred at a, inside D, there is a positive δ such that the subarc γ_s is contained in Δ for $s < \delta$. Thus as γ_s is homotopic to an interval $[a,\gamma(s)]$ we have $(0,\delta) \subset T$.

Suppose that $s_o \in T$. As β_{s_o} is a piecewise analytic simple arc we may find a small disk Δ centred at $\gamma(s_o)$ such that $\alpha = \Delta \cap \beta_{s_o}$ is a simple path. Let us choose a positive δ such that, for any s, $|s - s_o| < \delta$, the subarc χ of γ between $\gamma(s_o)$

and $\gamma(s)$ is contained inside Δ. Then $\alpha v \chi$ † is homotopic to the interval $[\alpha(0),\gamma(s)]$. Consequently as γ_s is homotopic to $\beta_{s_o} v \chi$ we have that γ_s is homotopic to $(\beta_{s_o} - \alpha) v [\alpha(0),\gamma(s)]$. Thus $(s_o - \delta, s_o + \delta) \subset T$.

Finally we suppose that $s_n \in T$ and $s_n \to s_o$. Let χ_n be the subarc of γ running from $\gamma(s_o)$ to $\gamma(s_n)$. Then we choose a disk Δ of D, centred at $\gamma(s_n)$ and with radius ε, such that for large fixed n, the subarc χ_n is contained in a disk of radius $\frac{\varepsilon}{2}$ centred at $\gamma(s_n)$. Now let α be the component of $\Delta \cap \beta_{s_n}$ which ends at $\gamma(s_n)$ and $d = \min (\frac{\varepsilon}{3}, \text{dist}(\gamma(s_n), \beta_{s_n} - \alpha))$. ††

Next we define a homeomorphism h of D by

(i) $h(w) = w$, $w \notin \Delta$,

(ii) $h(\gamma(s_n) + re^{i\theta}) = \gamma(s_n) + (\frac{\varepsilon(\varepsilon-d)}{2\varepsilon-d} + r\frac{\varepsilon}{2\varepsilon-d}) e^{i\theta}$, $\frac{d}{2} \leq r \leq \varepsilon$,

(iii) $h(\gamma(s_n) + re^{i\theta}) = \gamma(s_n) + \frac{\varepsilon}{d} re^{i\theta}$, $r \leq \frac{d}{2}$.

The function h is chosen to be piecewise analytic (in its real variables) and to map the disk of radius $\frac{d}{2}$ onto a disk of radius $\frac{\varepsilon}{2}$ (both centred at $\gamma(s_n)$). Thus β_{s_n} is homotopic to the piecewise analytic simple arc $h(\beta_{s_n})$. Furthermore $h(\beta_{s_n})$ has a single component in the disk of radius $\frac{\varepsilon}{2}$. We now use the later argument of "T is open" to show that γ_{s_o} is homotopic to a piecewise analytic simple arc. This completes the proof of the Lemma.

† α v χ denotes α followed by χ

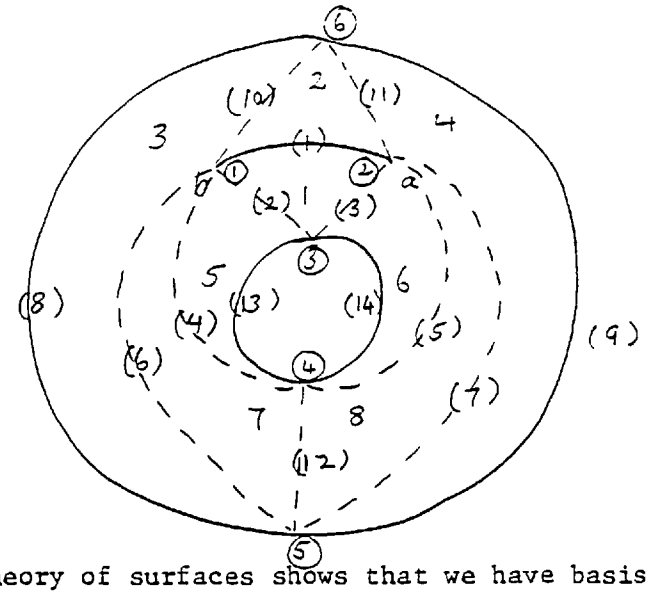
†† Also we require that d is chosen so that β_{s_n} has a single component in $|\gamma(s_n) - w| < \frac{d}{2}$.

Appendix 4.2 The genus of ∞

We triangulate $D - \gamma'$ by a system of 6 vertices, 14 edges and 8 triangles. Each triangle may be orientated in a positive (anti clockwise) direction. Then we see that D_2 is orientable, and that it is triangulated by 10 vertices, 28 edges and 16 triangles. Hence $\mathcal{D}$ is orientable and is triangulated by 16 vertices, 52 edges and 32 triangles. The Euler formulae shows that $\mathcal{D}$ has genus g given by

$$g = 1 + \frac{1}{2}(52-16-32) = 3.$$

Thus $\mathcal{D}$ is equivalent to a sphere with 3 handles.



○ vertices
( ) edges
triangles

The theory of surfaces shows that we have basis of 6 closed curves. We now show that these can be taken as $J_1, \ldots, J_6$. However examination of each of these cycles shows that each cycle is not equivalent to some combination of the others. Thus the cycles are independent.

Chapter 5

Nonzero univalent functions.

5.1 Introduction

In this chapter we use the techniques of the previous chapter to investigate some point functional problems for the class S_1 of functions $f(z) = 1 + a_1 z + a_2 z^2 + \ldots$ univalent and nonvanishing in the unit disk. The sharp bound

$$\left(\frac{1-r}{1+r}\right)^2 \leq |f(z)| \leq \left(\frac{1+r}{1-r}\right)^2 ,$$

$|z| = r$, $f \in S_1$, is well known. (Hayman [1958] p.95). Duren and Schober [1980] have given some results for the minimum real part problem in S_1, e.g. if $m(r) = \min \{\text{Re } f(r) : f \in S_1\}$ then $m(r) = \left(\frac{1-r}{1+r}\right)^2$ for $r \leq 3 - \sqrt{8}$. The function $\left(\frac{1+z}{1-z}\right)^2$ is extremal in these problems. For $r \geq \frac{1}{4}$ implicit results were obtained.

In the last chapter a complete answer to the range of value problem was obtained in terms of the elliptic modular function. This immediately gives implicit answers to all point functional problems. The problem is to determine explicit bounds. The first question considered is the argument problem in S_1, i.e. to find

$$\phi(r) = \sup \{\text{Arg } f(r) : f \in S_1\}.$$

For this the theory of the covering space X is essential. We show that

$$\phi(r) = 4 \arcsin r,$$

for $0 \leq r \leq R$ where R is constant, $R > \frac{1}{\sqrt{2}}$. As $r \to 1$ we find that

$$\phi(r) = \log\frac{1}{(1-r)} + \log 8 + o(1).$$

In section 5.10 we give applications to conformal mapping. In the final section we show that

$$m(r) = \frac{-4}{(1-r)^2}\ (1 + o(1)), \quad \text{as } r \to 1.$$

5.2 Elliptic modular functions.

We now state some properties of the function $\lambda(\tau)$ which was introduced in 4.1 and 4.2. Let Δ_o be

$$\{\tau : 0 < \mathrm{Re}\ \tau < 1, \quad |\tau - \tfrac{1}{2}| > \tfrac{1}{2}\}.$$

Then $w = \lambda(\tau)$ is to map Δ_o univalently onto the half plane $\{w : \mathrm{Im}\ w > 0\}$ such that the τ points $0,1,\infty$ map to the w points $\infty,0,1$. Analytic reflection in circles and lines gives a continuation of λ throughout $\{\mathrm{Im}\ \tau > 0\}$ and defines λ as a mapping onto the covering space X.

The process of partitioning $\{\mathrm{Im}\ \tau > 0\}$ by the images of Δ_o under reflection, for example, gives the fundamental triangles Δ_n which are bounded by arcs of the circles $C_{n,1} = \{|\tau - \frac{1}{2n}| = \frac{1}{2n}\}$ and $C_{n+1,1} = \{|\tau - \frac{1}{2(n+1)}| = \frac{1}{2(n+1)}\}$, $C_{n+1,2} = \{|\tau - \frac{(2n+1)}{2n(n+1)}| = \frac{1}{2n(n+1)}\}$.

Notice that $\mathrm{Im}\ (\log w)$ is well defined on X, or equivalently $\mathrm{Arg}\ \lambda(\tau)$ has period 2. Actually the symmetry properties of $\lambda(\tau)$ show that λ maps $\{\mathrm{Re}\ \tau = \frac{1}{2}\}$ onto $\{|w| = 1\}$, and $\{0 < \mathrm{Re}\ \tau < \frac{1}{2}\}$ onto $\{|w| > 1\}$. Symmetry also shows that

$$\lambda(\tfrac{1}{2} + s + it) = 1 \,/\, \overline{\lambda(\tfrac{1}{2} - s + it)}$$

and thus

$$\mathrm{Arg}\ \lambda\ (\tfrac{1}{2} + s + it) = \mathrm{Arg}\ \lambda\ (\tfrac{1}{2} - s + it).$$

Consequently to find

$$\max\ \{\mathrm{Arg}\ \lambda(\tau) : \mathrm{Im}\ \tau = t\}$$

we may restrict ourselves to $\{0 < \mathrm{Re}\ \tau < \frac{1}{2}\}$. In particular note that

$$\lambda(\tfrac{1+i}{2}) = -1,$$

which shows that -1 corresponds to modulus $\frac{1}{4}$.

Finally there are the product representations

$$(5.1)\quad \lambda(\tau) = \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n-1}}{1 - q^{2n-1}}\right)^8, \quad q = e^{\pi i \tau},$$

and if $\tau' = -\frac{1}{\tau}$, $\lambda_1(\tau') = \lambda(-\frac{1}{\tau})$,

$$(5.2)\quad \lambda_1(\tau') = \frac{1}{16p} \prod_{n=1}^{\infty} \left(\frac{1 + p^{2n-1}}{1 - p^{2n}}\right)^8,$$

where $p = e^{i\pi\tau'}$. These results may be found in Copson [1935,p 431].

5.3 The argument of nonzero univalent functions

Now we consider the problem

$$\max \{\text{Arg } f(r) : f \in S_1\},$$

denoting this quantity by $\phi(r)$. Our first result is

Theorem 5.1 *When* $r \le R$, *where* R *is a constant in the range* $(.901,.913)$,

$$\phi(r) = 4 \arcsin (r) .$$

The extremal functions are $\left(\frac{1 + e^{i\theta}z}{1 + e^{-i\theta}z}\right)^2$, *where*

$$\theta = \pi - \arccos r.$$

Our proof depends on Theorems 4.1 and 4.4, for we consider the function $\phi(w) = -i\log w$ defined on the covering space X. Thus the solution to the problem

$$\max \{ \text{Arg } f(r) : f \in S_1 \} = \max_s \{\text{Arg } \lambda(s + it) : \tfrac{1}{2}t = \nu(r)\},$$

and the extremal functions are described in section 4.6. If $\phi \le \pi$ the simpler Theorem 4.1 is sufficient, however the full generality of Theorem 4.4 is required for the case $\phi > \pi$.

Remarks 5.1 As special cases we deduce that for $f \in S_1$,

$$\text{Re } f(z) > 0$$

if $|z| \le \frac{1}{2}\sqrt{2 - \sqrt{2}} = 0.382$. This result was given as Theorem 7 in Duren and Schober [1980].

Remarks 5.2 Also we obtain $|\text{Arg } f(z)| \le \pi$ for $|z| \le \frac{1}{\sqrt{2}}$. In fact this is generalised to give

Theorem 5.2 *If* n *is a positive integer and* r_n *is defined by the equation* $\nu(r_n) = \frac{1}{4n}$ *then*

$$\phi(r_n) = n\pi .$$

From this we deduce

Corollary 5.1 $\phi(r) = \log\frac{1}{(1-r)} + O(1)$.

However this result may be refined to

Theorem 5.3 We have as $r \to 1$,

$$\phi(r) = \log\frac{1}{(1-r)} + \log 8 + O(1-r).$$

Remarks 5.3. The above results show a remarkable similarity to those for the argument of the derivative of univalent functions. Suppose that $g(z) \in S$. Then

$$|\arg g'(z)| \leq 4 \arcsin |z|, \qquad |z| \leq \frac{1}{\sqrt{2}}$$

and

$$|\arg g'(z)| \leq \pi + \log\frac{|z|^2}{(1-|z|^2)}, \quad |z| > \frac{1}{\sqrt{2}}$$

(Golusin [1936]). However since $\frac{1}{\sqrt{2}} < R$ and $\log 8 < \pi - \log 2$ the two bounds, though identical for $r < \frac{1}{\sqrt{2}}$ are different for r close to 1.

5.4. Proof of Theorem 5.1

Our proof follows from the discussion of section 5.3 together with

Lemma 5.1 The function $\psi(s,t) = \operatorname{Arg} \lambda(s+it)$ for fixed t, $t \geq \frac{1}{2}$, has an absolute maximium at $s = \frac{1}{2}$.

The symmetry results of section 5.2 show that we may restrict our attention to the region

$$\Omega = \{s + it : 0 \leq s \leq \tfrac{1}{2}, \ |s + it - \tfrac{1}{2}| \geq \tfrac{1}{2}\},$$

for λ maps Ω univalently onto $\{\operatorname{Im} w \geq 0, \ |w| \geq 1\}$.

Consequently it is enough to prove that

$$(5.3) \quad \frac{\partial \psi}{\partial s} > 0$$

in Ω. The minimum principle reduces this to showing that

$$\frac{\partial \operatorname{Arg} \lambda}{\partial s} \geq 0$$

as $s + it$ tends to the boundary of Ω.

Now λ maps the line $\{it : t > 0\}$ into the line $\{u : u > 1\}$, also $\psi > 0$ in the interior of Ω. Thus $\frac{\partial\psi}{\partial s} \geq 0$ on $\{it : t > 0\}$. Similarly $\frac{\partial\psi}{\partial s} \geq 0$ on $\{\frac{1}{2} + it : t \geq \frac{1}{2}\}$ and on the quarter circle $\{\frac{1}{2} + \frac{1}{2}e^{iw} : \frac{\pi}{2} < w < \pi\}$. Finally we deal with the behaviour of λ near the singularities at 0 and ∞.

Lemma 5.2 As $\tau \to \infty$ (0) in Ω $\frac{\partial\psi}{\partial s} \to 0$ $(+\infty)$.

First we deal with uniform convergence in Ω to ∞. The product representation (5.1) yields

$$\frac{\partial\psi}{\partial s} = 16\,\pi \sum_{n=1}^{\infty} \operatorname{Re} \frac{(2n-1)q^{2n-1}}{(1-q^{2(2n-1)})} \quad ,$$

$q = e^{i\pi(s+it)}$. If we write $\rho = e^{-\pi t}$ then

$$\frac{\left|\frac{\partial\psi}{\partial s}\right|}{} \leq 16\pi \sum_{n=1}^{\infty} \frac{(2n-1)\rho^{2n-1}}{(1-\rho^{2(2n-1)})} \quad ,$$

$$\leq 64\pi \sum_{n=1}^{\infty} n\,\rho^{2n-1}, \qquad \rho < \tfrac{1}{2}.$$

Thus

$$\left|\frac{d\psi}{ds}\right| \leq \frac{64\rho\pi}{(1-\rho^2)^2} \to 0, \quad \text{as} \quad \rho \to 0.$$

This proves the first half of the lemma.

The behaviour of λ in the cusp of Ω near 0 is best developed by introducing a transformation

$$\tau' = s' + it' = -1/\tau,$$

which transforms Ω onto a region

$$\Omega' = \{\tau' : -1 \leq \operatorname{Re}\ \tau' \leq 0,\ |\tau' + 1| \geq 1\}.$$

However $\lambda(\tau(\tau'))$ is a modular function which maps the τ' points $0,1,\infty$ to w points $1,0,\infty$. From the product representation (5.2) one finds

$$\log\,(\lambda_1(\tau')) = -i\pi\tau' - \log 16 + O(e^{-t'})$$

and hence

$$\frac{d}{d\tau'} \log \lambda = (-i\pi + O(e^{-t'}))$$

as $\tau' \to \infty$ in Ω'. Hence

$$\begin{aligned}\frac{\partial \psi}{\partial s} &= \operatorname{Im}\left(\frac{d \log \lambda_1}{d\tau'} \quad \frac{d\tau'}{ds}\right) \\ &= \operatorname{Im}(-\pi i + O(e^{-t})) \times \frac{1}{\tau^2} \\ &= -\pi(s'^2 - t'^2) + |\tau'|^2 \cdot O(e^{-t}) \\ &\to +\infty\end{aligned}$$

as $s' + it' \to \infty$ uniformly inside Ω'. This completes the proof of lemma 5.2.

Thus we have shown (5.3) inside Ω. By symmetry one sees that

$$\frac{\partial \psi}{\partial s} < 0$$

in $\{s + it : \frac{1}{2} \leq s \leq 1, \ |s + it - \frac{1}{2}| \geq \frac{1}{2}\}$. This completes the proof of Lemma 5.1.

Now Theorem 5.1 , in the case $r \leq \frac{1}{\sqrt{2}}$, is proved as follows. Firstly we note that λ maps $\frac{1}{2} + \frac{1}{2}i$ to -1. Consequently Lemma 5.1 applies to the case $\phi \leq \pi$. In fact λ maps $\{\frac{1}{2} + it : t \geq \frac{1}{2}\}$ to the semicircle $\{e^{i\omega}: 0 \leq \omega \leq \pi\}$. Thus the extremal functions f_o are those associated with $|w| = 1$. Theorem 4.5 shows that f_o has the form

$$f_o(z) = \left(\frac{1 + c_1 z}{1 + c_2 z}\right)^2$$

for some c_j, $|c_j| = 1$, $j = 1,2$. A simple computation shows that

$$\max \{\operatorname{Arg}(1 + c_1 z) : |z| = r, \ |c_1| = 1\} = \arcsin r$$

with equality for $c_1 = \pi - \arccos r$, $z = r$. Also

$$\min \{\operatorname{Arg}(1 + c_2 z) : |z| = r, \ |c_2| = 1\} = -\arcsin r,$$

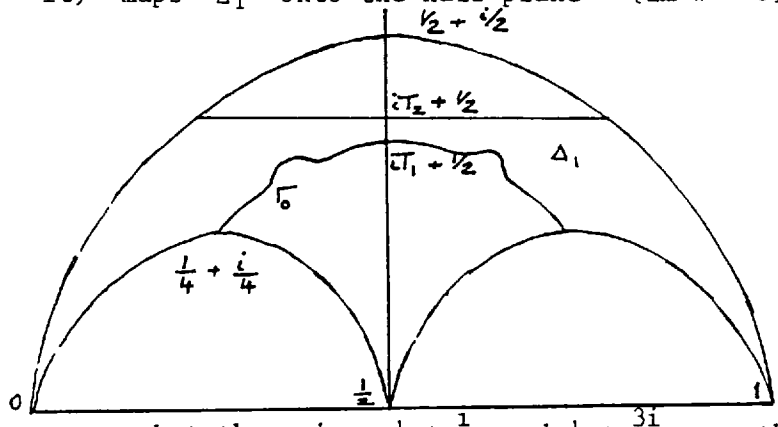
with equality for $c_2 = -\pi + \arccos r$, $z = r$. This concludes the proof of Theorem 5.1. for $r \leq 1/\sqrt{2}$.

5.5 Completion of the proof of Theorem 5.1.

Suppose that Δ_1 is the curvilinear triangle

$$\{\tau : |\tau - \tfrac{1}{2}| < \tfrac{1}{2},\ |\tau - \tfrac{1}{4}| > \tfrac{1}{4},\ |\tau - \tfrac{3}{4}| > \tfrac{1}{4}\}.$$

Now $\lambda(s + it)$ maps Δ_1 onto the half plane $\{\text{Im } w < 0\}$.



In 5.6 we prove that the points $\frac{1}{4} + \frac{1}{4}$ and $\frac{1}{4} + \frac{3i}{4}$ are the points τ, satisfying $\text{Arg } \lambda(\tau) = 2\pi$, and $0 \le \text{Re } \tau \le 1$, with greatest imaginary part. Consequently, with r_2 defined by $\nu(r_2) = \frac{1}{8}$, there are two extremal functions in S_1 for the argument problem. Thus somewhere between $r_1 = \frac{1}{\sqrt{2}}$ and r_2 Theorem 5.1 breaks down.

We must examine the level set $\Gamma : \frac{\partial \psi}{\partial s} = 0$ in Δ_1. By symmetry $\{\frac{1}{2} + it : 0 < t < \frac{1}{2}\}$ is part of Γ. Geometric considerations, similar to those in Lemma 5.2, show that $\frac{\partial \psi}{\partial s} \ge 0$ for $(s-\frac{1}{2})^2 + t^2 = \frac{1}{4}$, $s \le \frac{1}{2}$ and $(s - \frac{1}{4})^2 + t^2 = \frac{1}{16}$, $s \le \frac{1}{4}$. This is true for $(s - \frac{3}{4})^2 + t^2 = \frac{1}{16}$, $s \le \frac{3}{4}$. Otherwise on the boundary $\frac{\partial \psi}{\partial s} \le 0$. Finally, $\frac{\partial \psi}{\partial s}$ has the required properties near the cusps. Hence the maximum principle and symmetry shows that Γ divides Δ_1 into 4 regions. Also by geometric considerations $\frac{\partial \psi}{\partial s} = 0$ at $\frac{1}{4} + \frac{i}{4}$ and $\frac{1}{4} + \frac{3i}{4}$. Consequently there is an analytic level curve Γ_0 of Γ which joins these two points to some point $\frac{1}{2} + iT_1$. For $t < T_1$ the maximum of ψ cannot occur on $s = \frac{1}{2}$. In fact for $t < T_1$, $\frac{\partial^2 \psi}{\partial s^2} > 0$, while if $t > T_1$ we have $\frac{\partial^2 \psi}{\partial s^2} < 0$, for $s = \frac{1}{2}$. Thus T_1 is the unique

zero of $\frac{\partial^2\psi}{\partial s^2}$ on $s = \frac{1}{2}$. In Appendix 5.1 we calculate T_1 to a high degree of accuracy. Using (5.1) we also calculate the corresponding value of $\phi(R_1)$ and hence R_1 itself via

$$R_1 = \sin \frac{\phi(R_1)}{4} \quad .$$

Unfortunately we cannot show that Γ_o increases as it passes from $\frac{1}{4} + \frac{i}{4}$ to $\frac{1}{2} + iT_1$. However in Appendix 5.1 we show that $\frac{\partial\psi}{\partial s} > 0$ for $t > T_2$ and $s + it \in \Delta_1$, $s \leq \frac{1}{2}$. This is done by showing that $\frac{\partial^2\psi}{\partial s^2} < 0$ for $\frac{1}{2} - \sqrt{\frac{1}{4} + T_2^2} \leq s \leq \frac{1}{2}$. Since Lemma 5.1 shows that

$$\frac{\partial\psi}{\partial s} > 0$$

for $(s - \frac{1}{2})^2 + t^2 = \frac{1}{4}$, $s \leq \frac{1}{2}$, and also we know that $\frac{\partial\psi}{\partial s} = 0$, $s = \frac{1}{2}$, there cannot be a root of $\frac{\partial\psi}{\partial s}$ on $t = T_2$, $\frac{1}{2} - \sqrt{\frac{1}{4} + T_2^2} < s < \frac{1}{2}$. Consequently Γ_o lies below the line $t = T_2$, once again by the minimum principle

5.6 Proof of Theorem 5.2

Suppose that $L_n = \{\tau : \text{Arg } \lambda(\tau) = n\pi\}$, $n = 1,2,3\ldots$, then the previous results show that to find

$$r_n = \inf \{r : \text{Arg } f(r) = n\pi, \quad f \in S_o\}$$

is equivalent to finding

$$t_n = \sup \{t : s + it \in L_n\}.$$

The connection between the two expressions is

$$\nu(r_n) = \frac{t_n}{2} \quad .$$

Hence to obtain Theorem 5.2 we prove

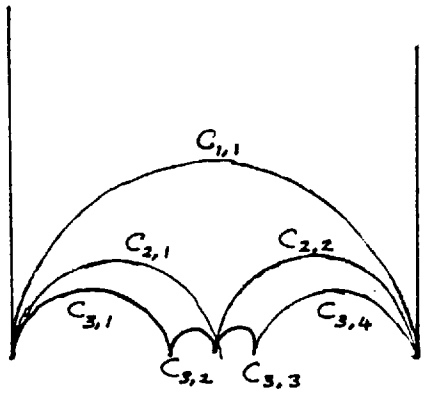
Lemma 5.3 $\quad \sup \{\text{Im } \tau : \tau \in L_n\} = \frac{1}{2n}$.

Consider the curvilinear triangles which arise in the definition of $\lambda(\tau)$. Each such triangle is mapped univalently onto an upper or lower half plane. Consequently $\frac{1}{\pi}$ Arg $\lambda(\tau)$ is an integer only on the boundaries of these triangles. As before we restrict our attention to $\{\tau : 0 < \text{Re } \tau < 1\}$. Let $C_{1,1}$ be the circle $\{|\tau - \frac{1}{2}| = \frac{1}{2}\}$.

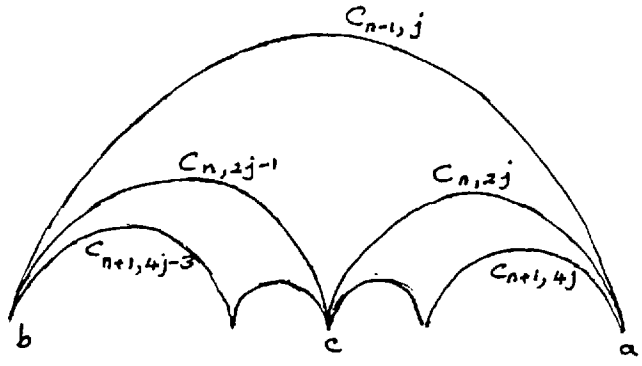
This is the first step of an inductive labelling of the boundary arcs. At the second step we have circles

$$C_{2,1} = \{|\tau - \tfrac{1}{4}| = \tfrac{1}{4}\}$$

$$C_{2,2} = \{|\tau - \tfrac{3}{4}| = \tfrac{1}{4}\}$$



The circles $C_{n,j}(j = 1,\ldots,2^{n-1})$ determine the circles $C_{n,j}$ of the $(n+1)^{st}$ step as follows. Suppose that circles $C_{n,2j-1}$ and $C_{n,2j}$ together with $C_{n-1,j}$ bound a fundamental triangle a,b,c.



Then we reflect $C_{n-1,j}$ and $C_{n,2j}$ in $C_{n,2j-1}$ to obtain circles $C_{n+1,4j-3}$ and $C_{n+1,4j-2}$. Similarly we obtain $C_{n+1,4j-1}$ and $C_{n+1,4j}$. Now let the circle $C_{n,j}$ have radius $X_{n,j}$. Then we find that

$$(5.4)\quad X_{n+1,4j-3} = \frac{X_{n,2j-1}}{2} + \frac{X_{n,2j-1}^2}{4X_{n,2j} + 2X_{n,2j-1}}$$

$$(5.5)\quad X_{n+1,4j-2} = \frac{X_{n,2j-1}}{2} - \frac{X_{n,2j-1}^2}{4X_{n,2j} + 2X_{n,2j-1}}$$

with similar formulae for $X_{n+1,4j-1}$ and $X_{n+1,4j}$. Now at the second step $X_{2,1} = X_{2,2} = \frac{1}{4}$ and $\frac{X_{2,1}}{X_{2,2}} = 1$. Equations (5.4) and (5.5) give

$$(5.6) \quad \frac{X_{n+1,4j-3}}{X_{n+1,4j-2}} = 1 + \frac{X_{n,2j-1}}{X_{n,2j}} .$$

We reflect circles about an axis through c which is normal to the line a,b and interchanging indices we obtain

$$(5.7) \quad \frac{X_{n+1,4j}}{X_{n+1,4j-1}} = 1 + \frac{X_{n,2j}}{X_{n,2j-1}} .$$

Consequently we prove by induction that

$$(5.8) \quad \frac{1}{(k-1)} \leq \frac{X_{k,2j}}{X_{k,2j-1}} \leq (k-1) .$$

The inequality (5.8) clearly holds for $k = 2$, $j = 1$, and (5.6), (5.7) show that if they hold for $k \geq 2$, then they hold for $k+1$. We deduce

Lemma 5.4 We have with the above notation

$$\max_m X_{k,m} = \frac{1}{2k}, \quad k = 2,3,\ldots$$

The result is clearly true for k = 2. Next (5.4),(5.5) show by induction that

$$X_{n,1} = \frac{1}{2n} , \quad X_{n,2} = \frac{1}{2n(n-1)} , \quad n = 2,3\ldots$$

which yields the upper bound in Lemma 5. 4. Next assume that Lemma 5.4 holds for $k = n$. Then (5.4) and (5.5) show that

$$X_{n+1,4j-2} \leq X_{n+1,4j-3} = X_{n,2j-1} \left\{ \frac{X_{n,2j-1} + X_{n,2j}}{X_{n,2j-1} + 2X_{n,2j}} \right\}$$

$$\leq \frac{1}{2n} \left(\frac{1 + \frac{1}{n-1}}{1 + \frac{2}{n-1}} \right) ,$$

in view of (5.8). Thus Lemma 5.4 holds for n + 1 and the proof of that Lemma is complete, and so that of Theorem 5.2.

5.7 Proof of Corollary 5.1

First we state

Lemma 5.5 $\frac{1}{\nu(r)} = \frac{4}{\pi} \log 16 \left(\frac{r^2}{1-r^2} + O(1) \right).$

Suppose that $\Lambda(R)$ is the modulus of $\mathbb{C} - [-1,0] - [R,\infty)$. Then the following relations are known (Ahlfors [1973, pp 74-76])

$$16\, R \leq e^{2\pi\Lambda(R)} \leq 16(R+1)$$

$$\tfrac{1}{4} = \Lambda(R)\, \Lambda\left(\frac{1}{R}\right)$$

and

$$\nu(r) = \tfrac{1}{2}\Lambda\left(\frac{1}{r^2} - 1\right).$$

Hence

$$\frac{1}{\nu(r)} = \frac{2}{\Lambda\left(\frac{1}{r^2}-1\right)} = 8\,\Lambda\left(\frac{r^2}{1-r^2}\right)$$

i.e.

$$\frac{1}{\nu(r)} = \frac{8}{2\pi} \log 16 \left(\frac{r^2}{1-r^2} + O(1)\right).$$

This proves the lemma.

Suppose that r_n is defined by

$$\nu(r_n) = \frac{1}{4n}$$

and $r_n < r < r_{n+1}$. Then Theorem 5.2 implies that

$$\phi(r) = n\pi + O(1).$$

From Lemma 5.5

$$n = \frac{1}{4\nu(r_n)} = \frac{1}{\pi} \log 16 \left(\frac{r_n^2}{1-r_n^2} + O(1) \right)$$

$$= \frac{1}{\pi} \log \frac{1}{(1-r)} + O(1).$$

Hence

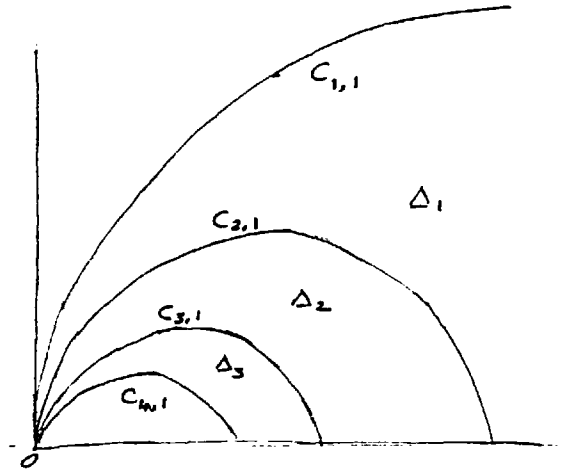
$$\phi(r) = \log \frac{1}{(1-r)} + O(1).$$

5.8 Preliminary results for Theorem 5.3

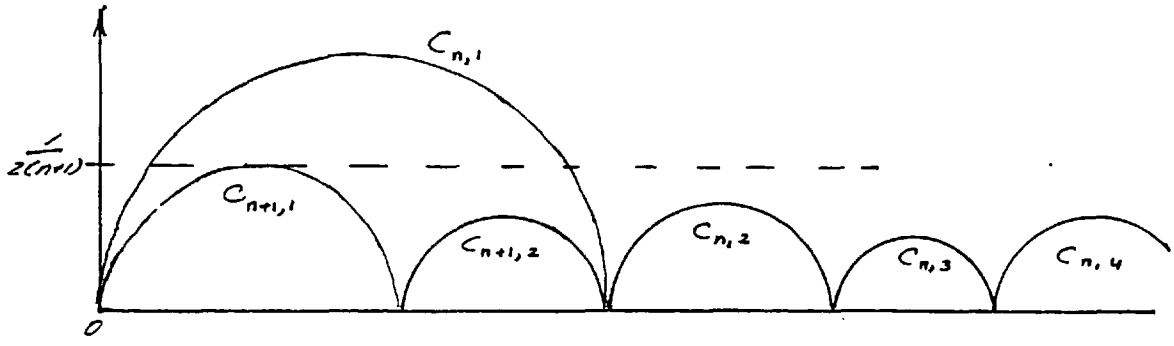
The problem in Theorem 5.1 for $\phi > \pi$, is to show that the curve $\Gamma : \frac{\partial \psi}{\partial s} = 0$ behaves in a nice manner. We now prove that if ϕ is a value with

$$n\pi \leq \phi \leq (n+1)\pi ,$$

$n = 2,3...$ then if $\text{Arg } \lambda(\tau) = \phi$ we may restrict our attention to τ inside the curvilinear triangle Δ_n bounded by $C_{n,1}$ and $C_{n+1,1}$ and $C_{n+1,2}$.



It will be shown that if τ^* is any value outside Δ_n such that $\text{Arg } \lambda(\tau^*) = \phi$ then $\text{Im } \tau^* < \frac{1}{2(n+1)}$. Now since $\phi \geq n\pi$ then τ^* lies in a region bounded above by the circles of the n^{th} step. If τ^* is bounded above by $C_{n,1}$ or $C_{n+1,1}$, there is nothing to show.

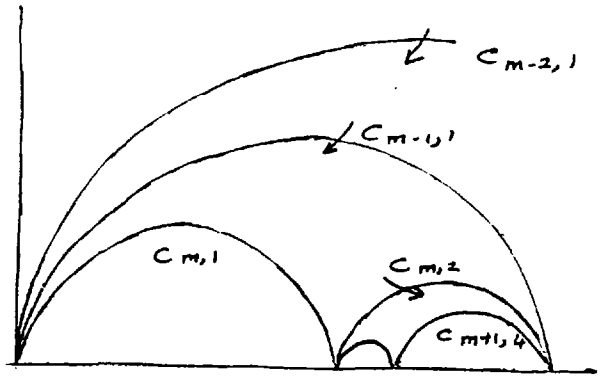


Consequently (in the notation of section 5.7) it suffices to show

Lemma 5.6 $\max \{X_{n,j} : 1 < j < 2^{n-1}\} < \frac{1}{2(n+1)}$.

Let us define $\rho_n = \max \{X_{n,j} : 1 < j < 2^{n-1}\}$. The production of circles $C_{n,j}$ $(n \geq 2)$ must be identical to the process of producing

$C_{2,1}$, $C_{3,1}$,...,$C_{m-1,1}$ for some m, $2 \leq m \leq n$, but differs at the mth step.



Then $X_{m,1} = \frac{1}{2m}$ and $X_{m,2} = \frac{1}{2m(m-1)}$. Thus

$$\frac{X_{m,2}}{X_{m,1}} = \frac{1}{(m-1)} .$$

Thus, from equation (5.7), $\frac{X_{m+1,4}}{X_{m+1,3}} = 1 + \frac{1}{(m-1)}$.

Consequently, for circles $C_{n,2j-1}$ and $C_{n,2j}$, $n \geq m+1$, we have by equation (5.6),(5.7) and induction that

$$\frac{1}{n-m+\frac{1}{m-1}} \leq \frac{X_{n,2j-1}}{X_{n,2j}} \leq n-m+\frac{1}{m-1} .$$

Next we use equation (5.4) and the definition of ρ_n to deduce

$$\rho_n \leq \frac{\rho_{n-1}}{2} \left(1 + \frac{1}{1 + \frac{2}{(n-m-1+\frac{1}{(m-1)})}} \right)$$

$$= \rho_{n-1} \frac{(n - m + \frac{1}{m-1})}{(n-m+1 + \frac{1}{m-1})} .$$

Hence the induction hypothesis $\rho_{n-1} < \frac{1}{2n}$ implies that

$$\rho_n \leq \frac{1}{2n} \frac{(n-m+\frac{1}{m-1})}{(n-m+1+\frac{1}{m-1})} < \frac{1}{2(n+1)} ,$$

for $2 \leq m \leq (n-1)$ [†]. This completes the proof of Lemma 5.6.

5.9 Proof of Theorem 5.3

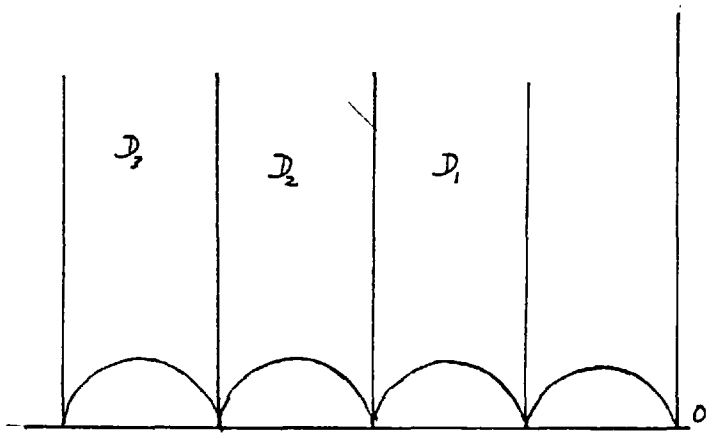
The above results show that we may restrict ourselves to the region

$$(5.10) \qquad R_n = \{\tau \in \Delta_n : \operatorname{Im} \tau \geq \frac{1}{2(n+1)}\}.$$

Consider the transformation

$$\tau' = s' + it' = -\frac{1}{\tau}$$

which maps each Δ_n onto a curvilinear triangle D_n with vertex at ∞.



From equation (5.2) we deduce that

$$\operatorname{Arg} \lambda_1(\tau') = -\pi s' + O(e^{-t'\pi}).$$

However the level curves of $\operatorname{Arg} \lambda_1(\tau')$ correspond to the level curves of $\operatorname{Arg} \lambda(\tau)$. That is if $L\phi = \{\tau \in \Delta_n : \operatorname{Arg} \lambda(\tau) = \phi\}$ then τ' maps

---

† As $\dfrac{n+1}{n} \leq \dfrac{n-m+\frac{1}{m-1}+1}{n-m+\frac{1}{m-1}}$,

i.e. $\dfrac{1}{n} \leq \dfrac{1}{(n-m+\frac{1}{m-1})}$.

$L\phi$ to a curve $L\phi'$ which has equation

$$\tau'(t) = -\frac{\phi}{\pi} + it + O(e^{-n\pi}),$$

where t is a real parameter and $\pi n \leq \phi \leq \pi(n+1)$. The problem of maximising $\mathrm{Im}\ \tau$ on $L\phi$ is equivalent to finding the smallest positive a such that the circle

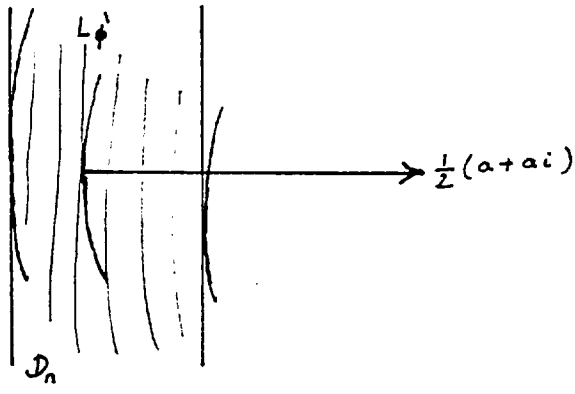
$$(5.11)\quad \left|\tau' - \frac{ia}{2}\right| = \frac{a}{2}$$

touches the level curve $L\phi'$. For

$$\max\{\mathrm{Im}\ \tau : \tau \in L\phi\} = \frac{1}{a}\ .$$

Notice that the line $\mathrm{Im}\ \tau = \frac{1}{2(n+1)}$, maps to a circle $|\tau' - (n+1)i| = (n+1)$ in the τ' plane. Hence the region R_n corresponds to a subregion R_n' of D_n contained in

$$(n+1) - \sqrt{(2n+1)} < \mathrm{Im}\ \tau' < (n+1) + \sqrt{2n+1}\ .$$



Consequently the smallest circle, of form (5.11), which touches $L\phi'$ is found to have radius

$$\frac{a_\phi}{2} = \frac{\phi}{\pi} + O(e^{-n\pi})$$

Hence

$$\max \{\operatorname{Im} \tau : \tau \in L\phi \cap D_n\} = \frac{1}{\frac{2\phi}{\pi} + O(e^{-n\pi})} .$$

The extremal point τ_o corresponds to r in the unit disk related by

$$\nu(r) = \tfrac{1}{2} \operatorname{Im} \tau_o .$$

Thus from Lemma 5.6 we conclude that

$$\frac{2\phi}{\pi} + O(e^{-n}) = \frac{1}{2\nu(r)} = \frac{2}{\pi} \log 16 \left(\frac{r^2}{1-r^2} + O(1) \right).$$

Therefore, noting $n = 0(\phi/\pi) + 0(1)$ and thus $0(e^{-n}) = 0(e^{-\phi})$, we derive

$$\frac{2\phi}{\pi} + O(e^{-\phi}) = \frac{2}{\pi} \log 16 \left(\frac{r^2}{1-r^2} + O(1) \right).$$

From this we obtain

$$\phi(r) = \log \frac{1}{(1-r)} + \log 8 + O(1-r),$$

which completes the proof of Theorem 5.3.

5.10 Applications to conformal mapping

In section 4.3 we noted that the range of value problem for S_1 is equivalent to an extremal length problem. We say that a doubly connected set $B \subset D$ is admissible if one component of $D - B$ contains a continuum homotopic to γ_1. The class of separating curves in B is admissible for the extremal length problem. Conversely the solution of the extremal length problem gives a doubly connected domain B_o. Consequently the extremal length problem is equivalent to the problem of maximising the modulus of admissible B. The following theorem is equivalent to the argument problem for S_1.

Theorem 5.4 Suppose that Γ_1, Γ_2 are disjoint continua in $\mathbb{C}$ such that Γ_1 joins the ray Arg w = 0 to the ray Arg w = ϕ ($0 < \phi \leq \pi$) and Γ_2 joins 0 to ∞. Then for these domains,

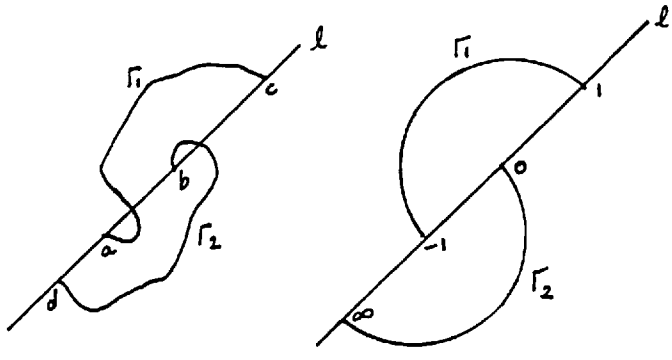
$$B_o = \mathbb{C} - \{te^{i\phi/2} : t < 0\} - \{e^{iw} : 0 \leq w \leq \phi\}$$

has maximum modulus.

Without loss of generality assume $1 \in \Gamma_1$. The above comments show that the extremal domain is $f_o(E) - f_o[0,r]$, where f_o is extremal in Theorem 5.1 with $r = \sin \frac{\phi}{4}$.

Remark 5.4 Mori [1956] obtained the above result with the further restriction that $1, e^{i\phi} \in \Gamma_1$. This result has important applications in the theory of quasiconformal mappings. [Ahlfors 1966, b, p.47]

Remark 5.5 In particular for $\phi = \pi$ we infer the following result. Suppose that a,b,c,d are ordered points on a line ℓ. Let Γ_1 and Γ_2 be disjoint continua such that $a,c \in \Gamma_1$ and $b,d \in \Gamma_2$. Then $\mathbb{C} - (\Gamma_1 \cup \Gamma_2)$ has maximum modulus if a,b,c,d are harmonic conjugates and if the Γ_i are semicircles orthogonal to ℓ. As λ maps $\frac{1}{2} + \frac{i}{2}$ to -1 the maximum modulus $\frac{1}{4}$.



Teichmüller [1938] and Schiffer [1946] gave the extremal domain when $a,b \in \Gamma_1$ and $c,d \in \Gamma_2$. In this case the extremal Γ_j are intervals.

Remark 5.6 Suppose that D is a connected domain in the u + iv plane. Further suppose that each vertical line u = constant intersects D in

a set of length at most 2π. Let d be the hyperbolic metric for D (see Ahlfors [1973], chapter 1). Various authors (see Hayman [1958], p.85) have considered the problem of minimising the hyperbolic distance between the lines $u = u_1$ and $u = u_2$. It is found that the strip, $|v| \leq \pi$, is extremal. A similar problem is to minimise the hyperbolic distance between the horizontal lines $v_1 = 0$, say, and $v_2 = \theta$. For the special case where $\exp(D)$ is contained in a simply connected set which omits 0 we obtain from the above

$$\min d(s_1, s_2 + i\theta) = \log\left(\frac{1 + \sin\frac{\theta}{4}}{1 - \sin\frac{\theta}{4}}\right)$$

for $\theta \leq \theta_o$, where $\theta_o = 4 \arcsin R$. Notice that for large θ the strip is not extremal.

Remark 5.7 Suppose that $g(z) \in S$, and $b \notin g(E)$. A number of results exist on the rotation of $g(|z| < r)$ about such points. Since $f(z) = 1 - \frac{g(z)}{b} \in S$ we are interested in

Lemma 5.7 *Suppose that* $f(z) \in S_1$ *then*

$$(5.12)\quad \operatorname{Arg}\frac{f(z_1)}{f(z_2)} \leq \phi\left(\frac{2r}{(1+r)^2}\right),$$

for $|z_1| = |z_2| \leq r$ *and* ϕ *given in section* 5.3.

For $f(z) \in S_1$ define $h(z)$ in S_1 by

$$h(z) = f\left(\frac{z + z_2}{1 + \overline{z}_2 z}\right) / f(z_2).$$

Then we define z_o by

$$z_o = \frac{z_1 - z_2}{1 - \overline{z}_2 z_1},$$

so that

$$z_1 = \frac{z_o + z_2}{1 + \overline{z}_2 z_o}.$$

Now if $|z_1| = |z_2| \leq r$ then $|z_o| \leq \frac{2r}{(1 + r^2)}$. Thus

$$\text{Arg } \frac{f(z_1)}{f(z_2)} = \text{Arg } h(z_o)$$

$$\leq \phi\left(\frac{2r}{1+r^2}\right).$$

The inequality is sharp. If $h_o(z)$ is extremal for the argument problem $\max \{\text{Arg } h\left(\frac{-2r}{1+r^2}.\right) : h \in S_1\}$ then

$$f_o(z) = h_o\left(\frac{z-r}{1-rz}\right)/h_o(-r)$$

is extremal in (5.12) for $z_2 = -z_1 = r$ as

$$\text{Arg } \frac{f_o(z_1)}{f_o(z_2)} = \text{Arg } \frac{h_o\left(\frac{-2r}{1+r^2}\right)}{h_o(o)} = \phi\left(\frac{2r}{1 + r^2}\right).$$

Consequently we deduce from Theorems 5.1 and 5.3 and Lemma 5.7

Theorem 5.5 Suppose that $g(z) \in S$, and $b \notin g(E)$ then

$$(5.13)\quad \sup\left\{\text{Arg } \frac{b - g(z_1)}{b - g(z_2)} : |z_1| = |z_2| = r\right\} = 8 \arctan(r)$$

for $\frac{2r}{1 + r^2} \leq R$, for some $R. \in (.901, .91)$, and

$$(5.14)\quad \sup\left\{\text{Arg } \frac{b - g(z_1)}{b - g(z_2)} : |z_1| = |z_2| = r\right\} = \log\frac{1}{(1-r)} + \log 16 + o(1)$$

as $r \to 1$. The function $z/(1-z)^2$ is extremal in (5.13).

Suppose that $A \subset \mathbb{C}$, let co A denote the convex hull of A. Clunie has asked for the largest r such that

$$(5.15)\quad \text{co } g(|z| < r) \subset g(E),$$

for all $g \in S$. From Theorem 5.5 we deduce

Theorem 5.6 For $r < \sqrt{2} - 1$ we have (5.15). The function $z/(1-z)^2$ shows that this result is sharp.

For if $|z_1| = |z_2| \leq r$ satisfy

$$t\, g(z_1) + (1-t)g(z_2) = b \notin g(E)$$

for some $0 < t < 1$ then

$$\left| \operatorname{Arg} \frac{b - g(z_1)}{b - g(z_2)} \right| = \pi$$

and by (5.13)

$$\frac{2r}{(1+r^2)} \geq \frac{1}{\sqrt{2}},$$

i.e.

$$r \geq \sqrt{2} - 1.$$

Remark 5.8 The same argument applied to Remark 5.6 shows that if D is a doubly connected set with modulus $m > \frac{1}{4}$ then

$$(\text{co } \Gamma_1) \cap \Gamma_2 = \phi,$$

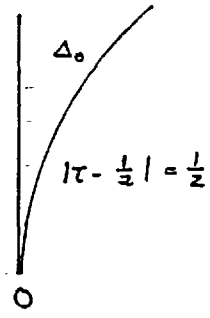
where Γ_1, Γ_2 are the bounded and unbounded components respectively of $\mathbb{C} - D$. Ahlfors [1973, p.74] shows that if $m > \frac{1}{2}$ then Γ_1, Γ_2 may be separated by a circle in D.

5.11. The asymptotic formulae for $m(r)$

We now turn to the minimum real part problem. Suppose that f_o is the extremal function. Without loss of generality we assume that $\operatorname{Im} f_o(r) > 0$. Then Duren and Schober [1980, pp. 208 - 211] show that $\Gamma_2 = \mathbb{C} - f_o(E)$ is confined to the lower half plane and $\Gamma_1 = f_o[0,r]$ is confined to the upper half plane $\{\operatorname{Im} w \geq 0\}$. Thus Γ_1 does not wind about 0 or 1. The results of section 4.2 show

that $b = f_o(r)$ correspond to τ points in the triangle

$$\Delta_o = \{ \tau : 0 < \mathrm{Re}\ \tau < 1, \ |\tau - \tfrac{1}{2}| > \tfrac{1}{2} \} .$$



The previous technique reduces the minimum real part problem to finding

$$(5.15) \quad \min \ \mathrm{Re}\ \lambda(s + it),$$

where s is real, $s + it \in \Delta_o$ and t fixed, $t = 2\nu(r)$.

Consequently

$$(5.16) \quad 0 \le s \le \tfrac{1}{2} - \sqrt{(\tfrac{1}{4} - t^2)} = t^2 + 0(t^4), \quad t \to 0.$$

We use the transformation $\tau' = -\frac{1}{\tau}$ and representation (5.2) to obtain

$$\mathrm{Re}\ \lambda(s + it) = \frac{1}{16} \mathrm{Re}\ e^{\frac{\pi i}{\tau}} (1 + o(1)),$$

as $0 \le s \le t^2 + 0(t^3), \quad t \to 0.$

Thus

$$(5.17) \quad \mathrm{Re}\ \lambda(s + it) = \frac{(1 + o(1))}{16} (\cos\frac{\pi s}{(s^2 + t^2)}) \ e^{\frac{\pi t}{(s^2+t^2)}}$$

From (5.16)

$$e^{\frac{\pi t}{(s^2 + t^2)}} = e^{\frac{1}{\pi t}(1 + 0(t^2))}$$

$$= (1 + o(1)) \ e^{\frac{1}{\pi t}} , \ t \to 0.$$

Also

$$-1 = \min \{\cos\frac{\pi s}{(s^2 + t^2)} : 0 \le s \le \tfrac{1}{2} - \sqrt{(\tfrac{1}{4} - t^2)}\},$$

with equality for $s = \frac{1}{2} - \sqrt{(\frac{1}{4} - t^2)}$. Hence from (5.15) and

(5.17) we obtain

$$(5.18)\quad m(r) \sim -\frac{1}{16}\, e^{\frac{\pi}{t}}\,(1 + o(1)),$$

where $t = 2\nu(r)$. Now from Lemma 5.6

$$\frac{1}{t} = \frac{1}{2\nu(r)} = \frac{2}{\pi}\left(\log\frac{1}{(1-r)} + \log 8 + o(1)\right).$$

Therefore by (5.18) we obtain

$$m(r) = -\frac{1}{16}\; e^{2\left(\log\frac{1}{(1-r)} + \log 8 + o(1)\right)}$$

$$= \frac{-4(1 + o(1))}{(1-r)^2} \quad .$$

Remark 5.9 The maximum real part problem is of exactly the same growth. In fact a similar analysis shows that for fixed δ,

$$\max\{\mathrm{Re}\; e^{i\delta} f(r) : f \in S_1\} = \frac{4(1 + o(1))}{(1-r)^2}\quad , \qquad r \to 1.$$

Appendix 5.1

This appendix is divided into 2 parts. First we calculate the root T_1 of $\frac{\partial^2\psi}{\partial s^2}$ on $s = \frac{1}{2}$ to a high degree of accuracy. Secondly we show that $\frac{\partial^2\psi}{\partial s^2} < 0$ for $t = T_2$, $s + it \in \Delta_1$. The values T_j are found to be

$$T_1 = .354731 \pm 10^{-6},$$

and $$T_2 = .3617 \quad .$$

5.1.1 The calculation of T_1

From equation (5.1) we have

(1) $$\psi(s + it) = 8 \operatorname{Im} \log \prod_{n=1}^{\infty} \left(\frac{1 + q^{2n-1}}{1 - q^{2n-1}}\right), \quad q = e^{i\pi(s + it)},$$

and hence expanding $\log(1 \pm q^{2n-1})$,

$$-\frac{1}{\pi^2}\frac{\partial^2\psi}{\partial s^2} = 16 \operatorname{Im} \sum_{n=1}^{\infty} (2n-1)^2 \sum_{m=1}^{\infty} (2m-1)\, q^{(2m-1)(2n-1)},$$

for $s = \frac{1}{2}$, $q = i\rho$ where $\rho = e^{-\pi t}$ and thus

(2) $$-\frac{1}{\pi^2}\frac{\partial^2\psi}{\partial s^2} = 16 \sum_{n=1}^{\infty} (2n-1)^2 \sum_{m=1}^{\infty} (2m-1)(-1)^{2mn-n-m} \rho^{(2m-1)(2n-1)}.$$

In general we write $\zeta = \rho e^{i\theta}$, $\theta = \pi s - \frac{\pi}{2}$ and find

$$-\frac{1}{\pi^2}\frac{\partial^2\psi}{\partial s^2} = \operatorname{Im}\, 16\, i \sum_{n=1}^{\infty} (2n-1)^2 \sum_{m=1}^{\infty} (2m-1)(-1)^{2mn-n-m} \zeta^{(2m-1)(2n-1)}$$

i.e.

(3) $$-\frac{1}{\pi^2}\frac{\partial^2\psi}{\partial s^2} = 16 \sum_{n=1}^{\infty} (2n-1)^2 \sum_{m=1}^{\infty} (2m-1)(-1)^{2mn-n-m} \rho^{(2m-1)(2n-1)} \cos(2m-1)(2n-1)\theta.$$

We truncate the power series in ρ after 15 terms (see overleaf) to obtain

(4) $$A = \zeta - 12\zeta^3 + 30\zeta^5 - 56\zeta^7 + 117\zeta^9 - 132\zeta^{11} + 182\zeta^{13} - 360\zeta^{15}.$$

Now for $s = \frac{1}{2}$ we use (2) only, and consider

$$A_1 = 1 - 12\rho^2 + 30\rho^4 - 56\rho^6 + 117\rho^8 - 132\rho^{10} + 182\rho^{12} - 360\rho^{14}.$$

The coefficients of $\rho^{(2m-1)(2n-1)}$ in (3)

n \ m	2m-1	1	2	3	4	5	6	7	8
	2n-1	1	3	5	7	9	11	13	15
1	1	1	-3	5	-7	9	-11	13	15
2	3	-9	27	-45					
3	5	25	-75						
4	7	-49							
5	9	81							
6	11	-121							
7	13	169							
8	15	-225							

The term in n^{th} row, m^{th} column is

$$(2n-1)^2(2m-1)\ (-1)^{(2mn-m-n)}.$$

The error term E_1 due to truncation is now calculated. We shall assume (and it will be justified later) that $\rho < .33$. Let $E_{1,n}$ be the truncation error in the n^{th} row.

$$E_{1,1} \leq \left|\sum_{m=9}^{\infty} (2m-1)\ \rho^{(2m-2)}(-1)^m\right|$$

$$\leq \frac{17\rho^{16}}{(1+\rho^2)} + \frac{2\rho^{16}}{(1+\rho^2)^2} < 19\rho^{16} .$$

$$E_{1,2} < \left| 9 \sum_{m=4}^{\infty} (2m-1)(-1)^m\ \rho^{3(2m-1)-1}\right|$$

$$< 81\rho^{20}$$

$$E_{1,3} < 175\rho^{24} .$$

The error in rows $n = 4 \ldots 8$ is

$$E_{1,n} = (2n-1)^2 \left| \sum_{m=2}^{\infty} (2m-1)(-1)^m\ \rho^{(2n-1)(2m-1)-1}\right|$$

$$< (2n-1)^2\ \ 3\rho^{3(2n-1)-1},$$

and for $n \geq 9$

$$E_{1,n} = (2n-1)^2 \left| \sum_{1}^{\infty} (2m-1)^2(-1)^m \rho^{(2n-1)(2m-1)-1} \right|$$

$$< (2n-1)^2 \ 3\rho^{(2n-2)} .$$

Consequently the total error is

$$E_1 < 19\rho^{16} + 81\rho^{20} + 175\rho^{24} + 3\sum_{4} (2n-1)^2\rho^{3(2n-1)-1}$$

$$+ 3 \sum_{9}^{\infty} (2n-1)^2 \ \rho^{2n-2}.$$

Also

$$\sum_{4}^{\infty} (2n-1)^2 \ \rho^{3(2n-1)} < \sum_{4}^{\infty} (2n-1)^2 \ \rho^{3(2n-1)-1}$$

$$< 49\rho^{20} \sum_{0}^{\infty}(2n+1)^2 \ \rho^{6n}$$

$$< 200\rho^{20}/(1-\rho^6)^3$$

and

$$\sum_{9}^{\infty} (2n-1)^2 \ \rho^{2n-2} < 17^2 \ \rho^{16} \sum_{0}^{\infty} (2n+1)^2 \ \rho^{2n}$$

$$< 1100 \ \rho^{16}/(1-\rho^2)^3 .$$

We add these results together and obtain

$$E_1 < 1600 \ \rho^{16},$$

recalling that $\rho \leq .33$.

We also need to estimate the derivative, i.e. bound

$$\sum_{n=1}^{\infty} (2n-1)^2 \sum_{m=1}^{\infty} (2m-1)((2m-1)(2n-1)-1) \ \rho^{(2m-1)(2n-1)-2}$$

for ρ in the interval $e^{-\pi/2} < \rho < e^{-\pi/4}$. For if α is a lower bound for the derivative near the root ρ_1 of the polynomial (4)

then the true root is given by $\rho_1 \pm E_1/\alpha$. Hence T_1 may be calculated to known accuracy.

We truncate the derivative after 7 terms i.e. we use

(5) $\rho(-24 + 120\rho^2 - 336\rho^4 + 928\rho^6)$.

The truncation error E_2 is calculated as above and one finds that

$$E_2 < 1000\rho^9, \quad \rho < .33 .$$

5.1.2 Calculating T_1.

(i) A few guesses show that the root of

$$1 - 12\delta + 30\delta^2 - 56\delta^3 + 117\delta^4 - 132\delta^5 + 182\delta^6 - 360\delta^7.$$

is given by

$$\delta_1 = .1076532 \pm 10^{-7}.$$

Thus

$$\rho_2 = \sqrt{\delta_1} = .32810547 \pm 10^{-7}.$$

(ii) Using the truncated series for the derivative and the error E_2 we find that

$$|\alpha| > 10.$$

Thus the true value of ρ_1 is given by

$$.328105 \pm 10^{-6}.$$

(iii) Then

$$T_1 = \frac{1}{n} \log \frac{1}{\rho_1} \pm 10^{-6} = .35473 \pm 10^{-6}.$$

(iv) For $s = \frac{1}{2}$, expression (1) becomes

$$\psi = 16 \sum_{n=1}^{\infty} (-1)^{2n-1} \arctan \rho^{2n-1}.$$

Thus for ρ_1 we have

$$\psi_1 = (16 \sum_{1}^{15} (-1)^{n-1} \arctan \rho_1^{2n-1}) + E_5$$

where

$$E_5 < 16.57 \left(\sum_{9}^{\infty} \rho_1^{2n-1} + \sum_{1}^{\infty} (2n-1)(10^{-6})^{2n-1} \right)$$

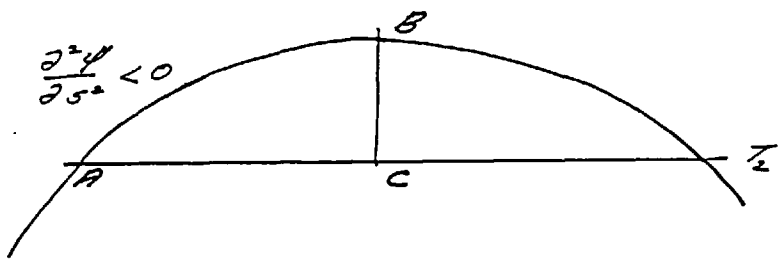
$$< 1000 \left(\frac{\rho_1^{17}}{(1-\rho_1^2)} + \frac{2.10^{-6}}{(1-10^{-12})^2} \right)$$

$$< 10^{-2}.$$

From Theorem 5.1 we have $\phi = \psi_1 = 4\arcsin R$, and thus $R_1 = \sin \frac{\psi_1}{4} = .913 \pm 10^{-3}$.

5.1.3 Calculating T_2

We have to show that $\frac{\partial^2 \psi}{\partial s^2} < 0$ for $t = T_2$, or $\rho = \rho_2$.



$\frac{\partial^2 \psi}{\partial s^2} \geq 0$ on ABC

We truncate (3) after 3 terms and consider

$$A = \rho \cos \theta - 12 \rho^3 \cos 3\theta + 30 \rho^5 \cos 5\theta.$$

We divide the truncation error into 2 parts:

$$E_3 = 56 \rho^7 + 117 \rho^9 + 132 \rho^{11} + 182 \rho^{13} + 360 \rho^{15}$$

and let E_4 be the remaining terms. As in 5.1.1 we calculate that the error in the n^{th} row is

$$E_{4,1} \leq \left| \sum_{m=9}^{\infty} (2m-1) \rho^{2m-1} \right| \qquad (n=1)$$

$$< 18 \rho^{15} \left| \sum_{m=1}^{\infty} m \rho^{2m} \right|$$

$$< \frac{18 \rho^{17}}{(1-\rho^2)^2} < 23 \rho^{17}$$

as $\rho < .32$.

$$E_{4,2} < \left| 9 \sum_{m=4}^{\infty} (2m-1)\, \rho^{3(2m-1)} \right| \qquad (n=2)$$

$$< 18\, \rho^{15} \left| \sum_{m=1}^{\infty} m\, \rho^{6m} \right| < \frac{18\, \rho^{21}}{(1-\rho^6)^2}$$

$$< 19\, \rho^{21}.$$

For $n = 3,\ldots,8$

$$E_{4,j} < (2n-1)^2 \left| \sum_{m=2}^{\infty} (2m-1)\, \rho^{(2n-1)(2m-1)} \right|$$

$$< 4(2n-1)^2\, \rho^{3(2n-1)}$$

and if $n \geq 9$

$$E_{4,j} < 3(2n-1)^2\, \rho^{(2n-1)}.$$

The total error is

$$E_4 < 23\rho^{17} + 19\rho^{21} + \sum_{4}^{\infty} 4(2n-1)^2\rho^{3(2n-1)} + \sum_{4}^{\infty} 3(2n-1)^2\, \rho^{2n-1}$$

$$< 24\rho^{17} + 64\rho^{21} \sum_{1}^{\infty} n^2\rho^{6(2n-1)} + 243\rho^{17} \sum_{1}^{\infty} n^2\rho 2(n-1)$$

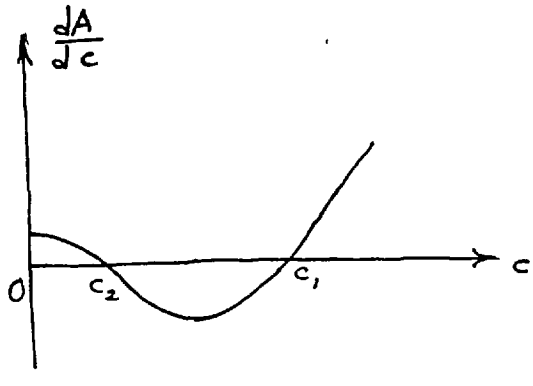
$$< 24\rho^{17} + 64\rho^{21} \frac{(1+\rho^6)}{(1-\rho^6)^3} + \frac{243\rho^{17}(1+\rho^2)}{(1-\rho^2)^3}$$

$$< 450\rho^{17} < .000003 \quad \text{if} \quad \rho < .33 \ .$$

Now when $\rho < .323$, $E_3 < .025635$. Thus

$$E_3 + E_4 < .025638$$

for $\rho < .323$.



We wish to consider $A_\theta = (\rho \cos\theta - 12\rho^3 \cos 3\theta + 30\rho^5 \cos 5\theta)$ where θ is restricted by $s + it \in \Delta_1$, i.e. $(s^2 - \frac{1}{2})^2 + t^2 < .25$.

Since $t = \frac{1}{\pi} \log \frac{1}{\rho}$ this implies that

(5) $$\theta \max = \pi(.25 - (\frac{1}{\pi} \log \frac{1}{\rho})^2)^{\frac{1}{2}} \text{ rads.}$$

Also without loss of generality $\theta > 0$. Thus we wish to find an optimum ρ such that

(6) $$.026 \leq \min \{A : 0 \leq \theta \leq \theta \max\}.$$

Writing $\cos\theta = c$, we have

$$A = \rho c - 12\rho^3(4c^3 - 3c) + 30\rho^5(16c^5 - 20c^3 + 5c)$$

and we examine

$$\frac{dA}{dc} = c^4(2400\rho^5) - c^2(1800\rho^5 + 144\rho^3) + (150\rho^5 + \rho)$$

to find the extrema.

The extreme values occur for

(7) $$c^2_{1,2} = \frac{(1800\rho^5 + 114\rho^3) \pm [(1800\rho^5 + 114\rho^3)^2 - 9600\rho^5(150\rho^5 + \rho)]^{\frac{1}{2}}}{4800\rho^5} .$$

The condition that the larger value of $c \geq 1$ is equivalent to

$$9000\rho^4 - 1363\rho^2 + 12 \leq 0$$

i.e.

(8) $$.297 \leq \rho \leq .4667.$$

We found that $\rho_1 = .3218$ and suspect that ρ_2 is equal to or slightly less than ρ_1. Now in range (8), $c = 1$, i.e. $\theta = 0$, is the local minimium. From (6) we find the largest ρ such that

$$.026 < \rho - 12\rho^3 + 30\rho^5.$$

We compute that $\rho_2 = .321$ is the optimum value. Now there are other values θ_o such that A_θ is a stationary point. Equation (7) shows that for $\rho = \rho_2$

$$\theta_o > 1.29 \text{ (rads)},$$

while on $\rho = \rho_2$ from (5) we find that

$$\theta \text{ max} = 1.09955 \text{ (rads)}.$$

Thus at ρ_2 the absolute minimum occurs at $\theta = 0$. As in 5.1.2 we have

$$T_2 = \frac{1}{\pi} \log \frac{1}{\rho_2} = .3617$$

and calculate

$$R_2 = .901.$$

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