

Studies on the Deuteron

by

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Abstract

We extend the technique of Cutkosky for calculating anomalous threshold discontinuities to deal with particles with spin. The dominance of these discontinuities, their relative simplicity and their connection with potential theory are then exploited to give a relativistic treatment of various deuteron processes.

For the np-d vertex we are able to reproduce the results of Blankenbecler and Cook with considerably less labour and a corresponding clarification of the approximations made.

For the deuteron form factor the method yields relativistic corrections to the elastic e-d scattering formula, giving corrections to the neutron form factor deduced from this reaction.

Finally for deuteron electrodisintegration we obtain a 'covariant Born approximation' for the final state. For the large momentum transfers for which this approximation is valid we show that the final state interaction may be neglected.

Contents.

Abstract	2
Contents	3
Preface	5
Introduction	6
A Brief Discussion of Cutkosky's Method.	9

Part I. - The np-d Vertex.

1. Scalar Vertex.	13
(a) Connection with potential theory.	13
(b) Analytic structure from potential theory.	15
(c) Anomalous threshold.	17
(d) Comparison with Blankenbecler and Cook.	18
2. Spinor Vertex.	21
(a) Form and normalisation.	21
(b) Anomalous threshold.	25

Part II - e-d Elastic Scattering.

3. Kinematics.	28
4. Nucleon Form Factors.	35
5. The Deuteron Form Factor.	37
(a) Scalar Vertex.	38
(i) Potential theory.	38
(ii) Anomalous threshold.	39
(b) Spinor Vertex.	40
(i) Form and normalisation.	40
(ii) Anomalous threshold.	44

6.	Dispersion Integrals.	48
(a)	Bare form factors.	48
(b)	3π cut.	52
(c)	S-wave contribution.	56
(d)	D-wave contribution.	56
7.	Relativistic Corrections to the Jankus Formula.	60
8.	Neutron Form Factors.	63
9.	Recent Developments.	66

Part III - Deuteron Electrodissintegration

10.	Introduction.	70
11.	Kinematics.	72
12.	Physical and Spectral Regions.	75
13.	Partial Waves.	79
14.	Single Dispersion Relations.	83
(a)	Pole Terms.	83
(b)	Anomalous Threshold.	88
15.	Double Dispersion Relations.	91
(a)	Scalar Case.	92
(b)	Spinor Case.	96
16.	Final-State Interaction and Intersecting Ellipses.	99
Appendix A.	Deuteron parameters.	102
Appendix B.	The geometry of Fig. 6.	103
Appendix C.	The geometry of Fig. 17.	105
Appendix D.	The geometry of Fig. 32.	108
	References.	110

Preface

The work described in this thesis was carried out under the supervision of Professor A. Salam between October 1961 and October 1963 in the Department of Physics, Imperial College, University of London.

Except where otherwise stated, the material contained herein is original and has not been previously presented for a degree in this or any other university.

The author is extremely grateful to Professor Salam for his interest and encouragement, to Dr. T.W.B. Kibble for very many helpful discussions, and to Mr. C. Lovelace for providing the original suggestion out of which this thesis has grown.

He is also indebted to the Department of Scientific and Industrial Research for financial support.

Introduction

Apart from the normal thresholds of dispersion theory, which occur at the squared masses of possible intermediate states, there appear in problems involving weakly bound structures such as the deuteron* the so-called anomalous thresholds, whose position depends on all the masses involved. They were first discovered in perturbation theory by Karplus, Sommerfeld and Wichmann⁽¹⁾ and the fact that they were associated with compound structure effects was recognised by several authors. The explicit connection of these thresholds with the analytic structure of potential theory as developed by Martin ⁽²⁾ was shown for the np-d vertex by Blankenbecler and Cook ⁽³⁾ and for the scalar deuteron form factor by Blankenbecler and Nambu ⁽⁴⁾, while the position of the partial-wave cuts for deuteron photodisintegration was shown to be the same in both treatments by De Alfaro and Rossetti ⁽⁵⁾.

One of the main features of the anomalous threshold discontinuities as discussed by these authors turned out to be their relative simplicity compared with the normal threshold discontinuities. A very elegant method for calculating these discontinuities was subsequently developed by Cutkosky ⁽⁶⁾. His discussion was, however, mainly of pedagogic value, since it was confined to the case of spinless particles. We have been able to extend the method to the case of particles with spin and are therefore in a

*

A weakly bound state C of two particles A and B is one whose mass satisfies $m_C^2 > m_a^2 + m_b^2$

position to exploit the other features of anomalous thresholds, viz. their dominance in deuteron processes and their close connection with potential theory.

The np-d vertex has already been studied in the spinor case by Blankenbecler and Cook. These authors calculate the absorptive parts coming from the triangle diagram of Fig. 5 and find the anomalous threshold discontinuity by analytic continuation. This discontinuity is the discontinuity then used for the calculation of the asymptotic D-S ratio ρ . In Part I we apply Cutkosky's method to this vertex and are able to reproduce these results with a considerable saving of labour.

Part II is essentially the calculation of Blankenbecler and Nambu with the inclusion of spin. In this part we particularly exploit the correspondence with potential theory. With the use of the single photon exchange formula of Glaser and Jakšić⁽⁷⁾ and the dispersion theory of the deuteron form factor we are able to find relativistic corrections to the elastic e-d scattering formula obtained in potential theory by Jankus⁽⁸⁾ and modified to include nucleon structure by Glendenning and Kramer⁽⁹⁾. This results in changes in the neutron form factors deduced from this reaction, and may ultimately be important in deciding the sign of F_{1n} .

In Part III we are concerned with deuteron electrodisintegration. The dominance of pole terms and anomalous thresholds in the single dispersion relations has already been exploited by Durand⁽¹⁰⁾, whose work we review. The effect of the final state interaction was investigated by Durand in

potential theory using partial waves. This treatment, suitable for small momentum transfers, should be supplemented by another, suitable for larger momentum transfers. We exploit the dominance of anomalous thresholds in the double dispersion relations to obtain such a treatment. It turns out that for larger momentum transfers the final state interaction is negligible.

We confine ourselves to processes involving the deuteron because of its very small binding energy, or large radius. This makes it a reasonable assumption that anomalous thresholds should be dominant. For other possible vertices, e.g. $\Sigma - \bar{K}N$, the dominance is not nearly so marked.

The notation we use is that of ref. ⁽¹⁵⁾.

A Brief Discussion of Cutkosky's Method

Consider a scalar Feynman amplitude

$$(1) \quad F = \int \frac{\Pi d^4 k}{A_1 \dots A_n}, \quad A_i = m_i^2 - q_i^2$$

Feynman parametrisation gives

$$(2) \quad F = (n-1)! \int \frac{\Pi d\alpha}{D^n} \Pi d^4 k \delta(1-\Sigma \alpha) \quad \text{where } D = \Sigma \alpha_i A_i$$

Landau⁽¹¹⁾ shows that the singularities of F are then given by

$$(3) \quad \begin{cases} \alpha = 0 & \text{(line removed)} \\ \text{or } \frac{\partial D}{\partial \alpha} = 0 & \text{(line on mass shell)} \end{cases}$$

for each line of the graph.

The discontinuity across the cut arising from such a singularity is obtained as follows (^{6a}):

We consider the case of $\alpha_i = 0$ for $i > m$, and suppose that $q_1 \dots q_m$ depend only on the loop momenta $k_1 \dots k_\nu$. Then we transform the $k_1 \dots k_\nu$ integrations to $q_1^2 \dots q_m^2$, together with angular variables ξ_i .

Then

$$(4) \quad F = \int_{a_1}^{b_1} dq_1^2 \dots \int_{a_m}^{b_m} dq_m^2 \Pi \int d\xi_i \Pi \int_{j>\nu} d^4 k_j \quad \frac{1}{JA_1 \dots A_n}$$

where J is the Jacobian of the transformation.

If we suppose all the integrations have been performed except that over q_1^2 ,

$$F = \int_{a_1}^{b_1} \frac{dq_1^2}{m_1^2 - q_1^2} F_{(1)}(q_1^2)$$

The singularity of this integral arises from a pinching of the q_1^2 contour between the denominator and a singularity of $F_{(1)}(q_1^2)$. We can replace the q_1^2 contour by a small circle surrounding $q_1^2 = m_1^2$ and a contour which is not pinched.

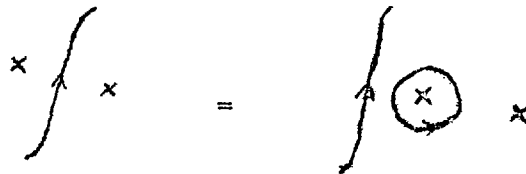


FIG. 1

The singularity then arises only from the small circle, or in other words we may replace $(m_1^2 - q_1^2)^{-1}$ by $\pm 2\pi i \delta(q_1^2 - m_1^2)$

A similar argument applies to end-point singularities, and the net result is that the discontinuity, F_m , of the graph in question is

$$F_m = (2\pi i)^m \int \prod d^4 k \frac{\delta_p(q_1^2 - m_1^2) \dots \delta_p(q_m^2 - m_m^2)}{A_{m+1} \dots A_n}$$

Here we have transformed back to the loop variables; however in many cases it will be convenient to keep to q_i^2 + angular variables as variables of integration.

If we now sum over all Feynman graphs with the same skeleton the vertices involved become renormalised coupling constants, exact (physical)

scattering amplitudes, production amplitudes etc. All the lines remaining in the reduced graph are on the mass-shell.

The preceding considerations lend themselves to an elegant geometrical construction, viz. the dual diagram ⁽¹¹⁾.

Since momentum is conserved at every vertex of the reduced diagram we may draw a dual diagram, in which vectors meeting at a point of the reduced diagram become the sides of a closed polygon. The Landau conditions $q_i^2 = m_i^2$ imply that the lengths of the sides are given by the corresponding masses. The Landau loop equation $\sum \alpha_i q_i = 0$ for the position of the singularity now becomes a condition of linear dependence for vectors meeting in a point of the dual diagram.

Thus, if we consider a vertex function, Fig. 2, we see that the singularity occurs when the dual diagram, Fig. 3, can be drawn in a plane. Further, since all the α 's are positive, P must lie inside the $\triangle ABC$.

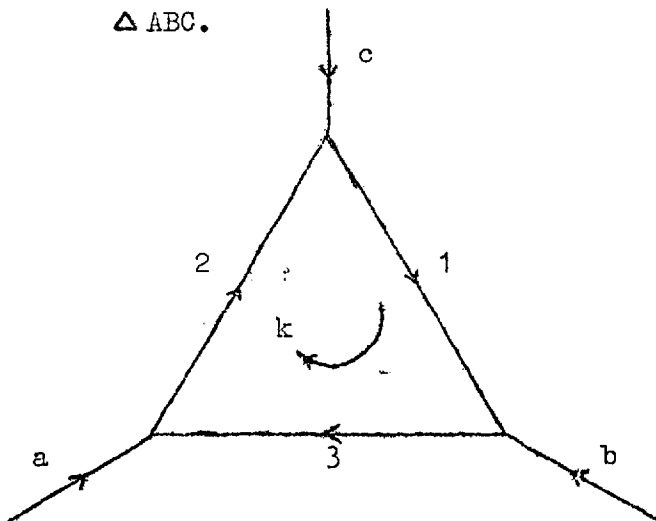


Fig. 2

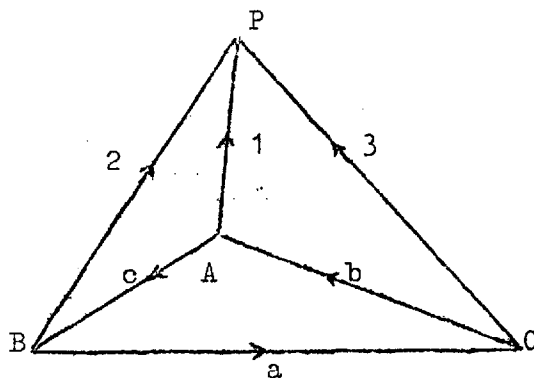


Fig. 3

This gives a geometrical construction for the anomalous threshold of the vertex function, considered, e.g., as a function of p_a^2 . Only for certain values of $m_b^2, m_c^2, m_1^2, m_2^2, m_3^2$ will the construction be possible.

Furthermore Cutkosky's discontinuity formula can be expressed in terms of the geometry of Fig. 3

$$(6) \quad F_3 = \frac{1}{2\pi} \int d^4k \, \delta_p(q_1^2 - m_1^2) \delta_p(q_2^2 - m_2^2) \delta_p(q_3^2 - m_3^2)$$

The constraints $q_i^2 = m_i^2$ force P to lie on a circle in [4], radius κ , on which we take the angular variable to be ψ^* .

Then, transforming to variables q_i^2, ψ ,

$$d^4k = \kappa d\psi d^3\underline{k} = \kappa d\psi \Pi(dq_i^2) J_3^{-1}$$

where

$$J_3 = 8 \det q_{i\alpha} = 2^4 \kappa A, \text{ where } A \text{ is the area of } \triangle ABC.$$

Hence

$$(7) \quad F_3 = \kappa J_3^{-1} = \frac{1}{2^4 A}$$

Hence both the position of the anomalous threshold and the discontinuity are obtaining in a simple manner from the geometry of Fig. 3.

* A helpful analogy is obtained by reducing the dimensionality by one. The locus in [3] of a point P at fixed distances from two points A,B is a circle in a plane perpendicular to AB. The circle cuts any plane containing A,B in two points P,P', mirror images in the line AB, and the radius of the circle is the distance of either of these points from AB.

Part I - The np-d Vertex.

This basic vertex enters all impulse-type calculations of deuteron processes, and will be shown below to be intimately related to the deuteron wave-function ⁽¹²⁾.

1. Scalar Vertex.

(a) Connection with Potential Theory.

We consider the vertex

$$(8) \quad \Gamma(t) = \langle n | f_p(0) | d \rangle, \text{ with } t = p'^2$$

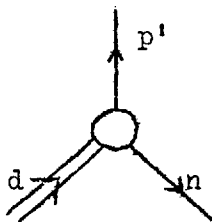


FIG.4

Supplying Γ with a momentum-conserving delta-function, we obtain

$$(9) \quad \begin{aligned} K(t) &= (2\pi)^4 \delta^4(p' + n - d) \langle n | f_p(0) | d \rangle, \\ &= i \int e^{inx + ip'y} K_x K_y \langle 0 | T(\psi(y) \psi(x)) | d \rangle d^4x d^4y \end{aligned}$$

by contraction on the neutron.

Changing to relative and centre-of mass coordinate gives

$$K(t) = (2\pi)^4 \delta^4(p' + n - d) i \int e^{ikr} K_x K_y \mathcal{T}(r, d) d^4r$$

where

$\Phi(r, d) = \langle 0 | T(\psi(r/2)\psi(-r/2)) | d \rangle$ is the Bethe-Salpeter amplitude for the deuteron, and $k = \frac{1}{2}(p' - n)$.

In terms of the Fourier transform

$$\Phi(k, d) = \int d^4r e^{ikr} \Phi(r, d)$$

$K(t)$ becomes

$$(10) \quad K(t) = (2\pi)^4 \delta^4(p' + n - d) i K_p K_n \Phi(k, d)$$

$\Phi(k, d)$ satisfies the Bethe-Salpeter equation

$$(11) \quad K_p K_n \Phi(k, d) = i \int \frac{d^4k'}{(2\pi)^4} V(k, k') \Phi(k', d),$$

while the Schrodinger wave-function $\varphi(\underline{k}, d)$ is given ⁽¹³⁾ by

$$(12) \quad \varphi(\underline{k}, d) = \int \frac{dk_0}{2\pi} \Phi(k, d)$$

For an instantaneous potential $V(k, k') = V(\underline{k} - \underline{k}')$ the Bethe-Salpeter equation reduces to

$$\begin{aligned} K_p K_n \Phi(k, d) &= i \int \frac{d^3\underline{k}'}{(2\pi)^3} V(\underline{k} - \underline{k}') \varphi(\underline{k}', d) \\ &= -i\Gamma(\underline{k}) \end{aligned}$$

$$\therefore \Phi(k, d) = -i K_n^{-1} K_p^{-1} \Gamma(\underline{k})$$

We can now perform the integral in (12); since Γ is now a function of $\underline{k}$ only, it comes outside the integral, and we find

$$(13) \quad \varphi(\underline{k}, d) = \frac{\Gamma(\underline{k})}{4m(\alpha^2 + \underline{k}^2)}$$

The normalisation of φ has yet to be fixed; this can be done by a comparison of the potential and dispersion theory treatments of n-p or of the deuteron form factor

(b) Analytic Structure from Potential Theory.

Having found the relation between Γ and φ let us examine the analytic properties of φ (c.f. (2))

With a potential of the form

$$V(r) = \int_{\mu}^{\infty} C(\gamma) e^{-\gamma r} d\gamma$$

we try to solve the configuration-space Schrödinger equation,

$$(14) \quad \left[\frac{d^2}{dr^2} - \alpha^2 - V(r) \right] (r\varphi) = 0,$$

with a wave-function of the form

$$(15) \quad \varphi(r) = \frac{N}{\sqrt{4\pi}} \frac{e^{-\alpha r}}{r} \left(1 + \int_{\mu}^{\infty} \rho(\beta) e^{-\beta r} d\beta \right)$$

Substitution of these Ansätze into (14) gives a recurrence relation for ρ :

$$(16) \quad \beta(\beta - 2\alpha)\rho(\beta) = C(\beta) + \int_{\mu}^{\beta-\mu} C(\beta-\gamma)\rho(\gamma) d\gamma,$$

yielding

$$(17) \quad \left\{ \begin{array}{ll} \rho(\beta) = 0 & \beta < \mu \\ \rho(\beta) = \frac{C(\beta)}{\beta(\beta+2\alpha)} & \mu < \beta < 2\mu \\ \rho(\beta) = \frac{1}{\beta(\beta+2\alpha)} \left\{ C(\beta) + \int_{\mu}^{\beta-\mu} \frac{C(\beta-\gamma)C(\gamma)d\gamma}{\gamma(\gamma+2\alpha)} \right\} & 2\mu < \beta < 3\mu \end{array} \right.$$

etc.

The momentum-space wave-function is

$$\begin{aligned}
 \varphi(\underline{k}) &= \int e^{i\underline{k} \cdot \underline{r}} \varphi(\underline{r}) d^3 \underline{r} \\
 (18) \quad &= N\sqrt{4\pi} \left\{ \frac{1}{\alpha^2 + \underline{k}^2} + \int_{\mu+\alpha}^{\infty} \frac{\rho(\beta-\alpha)}{\beta^2 + \underline{k}^2} d\beta \right\} \\
 &= N\sqrt{4\pi} \left\{ \frac{1}{\alpha^2 - y} + \int_{(\mu+\alpha)^2}^{\infty} \frac{\rho(\sqrt{y'}-\alpha) dy'}{2\sqrt{y'} (y'-y)} \right\}
 \end{aligned}$$

with $y = -\underline{k}^2$, showing that $\varphi(y)$ has a pole at $y = \alpha^2$ and a cut from $(\mu+\alpha)^2$, with absorptive part $\pi N\sqrt{4\pi} \frac{\rho(\sqrt{y}-\alpha)}{2\sqrt{y}}$

From the second of eqns. (17) we see that

$$\rho(\sqrt{y}-\alpha) = \frac{C(\sqrt{y}-\alpha)}{y-\alpha^2} \quad \text{for } (\alpha+\mu)^2 < y < (\alpha+2\mu)^2$$

For a pure Yukawa potential $C(\gamma) = \lambda$, so that the absorptive part is

$$\text{Im}\varphi(y) = \frac{\pi N\lambda\sqrt{4\pi}}{2(y-\alpha^2)\sqrt{y}}, \quad (\alpha+\mu)^2 < y < (\alpha+2\mu)^2$$

From (13) we deduce that Γ has a cut from $(\alpha+\mu)^2$, with absorptive part :

$$\text{Im}\Gamma(y) = - \frac{\pi\lambda\Gamma_0}{2\sqrt{y}} \quad \text{in the region } (\alpha+\mu)^2 < y < (\alpha+2\mu)^2$$

Here we have written Γ_0 for $\Gamma(\alpha^2)$, corresponding to $p'^2 = m^2$.

We will now show that the above result, with relativistic corrections, is reproduced by the anomalous threshold.

(b) Anomalous Threshold.

In the dispersion theory for $\Gamma(t)$ we find that the lowest cut comes from the anomalous threshold graph, Fig. 5, whose dual diagram is Fig. 6.

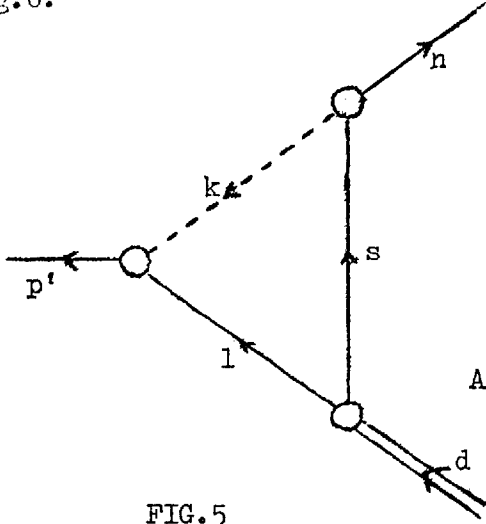


FIG. 5

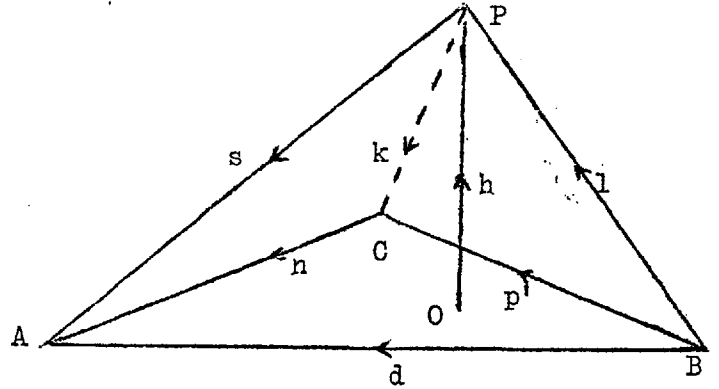


FIG. 6

The anomalous threshold is that value of t for which Fig. 6 becomes plane, Fig. 7.

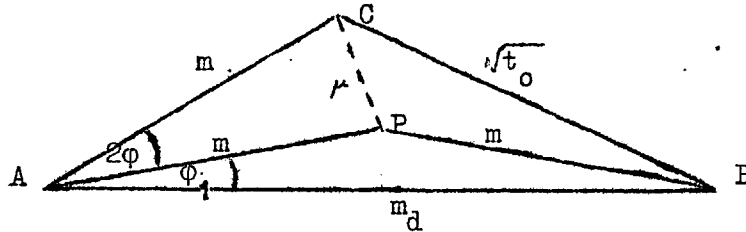


FIG. 7

Thus

$$t_0 = m^2 + m_d^2 - 2mm_d \cos(\varphi_1 + 2\varphi)$$

$$= m^2 + m_d^2 - 2mm_d \left\{ \frac{m_d}{2m} \left(1 - \frac{\mu^2}{2m^2} \right) - \frac{\alpha\mu}{m^2} \left(1 - \frac{\mu^2}{4m^2} \right)^{\frac{1}{2}} \right\}$$

$$= m^2 + 2\mu^2 + 4\alpha\mu$$

It is convenient to work with the variable y introduced in the previous section :

$$t - m^2 = 2(y - a^2)$$

The anomalous threshold in y is

$$(20) \quad y_0 = (a + \mu)^2, \text{ in agreement with the previous section.}$$

From (7), the discontinuity across the cut is $\frac{g^2 \Gamma_0}{16A}$.

For small y this reduces to $\frac{g^2 \Gamma_0}{16m \sqrt{y}}$, which, with a suitable identification of λ , will be recognized as the discontinuity from potential theory, eqn. (19).

More accurately,

$$A = \frac{1}{2} m_d n_2,$$

where n_2 is the height of C above AB:

$$(21) \quad n_2^2 = y - \frac{(y-a^2)^2}{m_d^2}$$

(d) Comparison with Blankenbecler and Cook.

Before considering the spinor vertex it may be instructive to compare this treatment with that of Blankenbecler and Cook ⁽³⁾ and to show how the anomalous threshold arises in the latter treatment.

After transforming to relative and centre-of-mass variables the total contribution from Fig. 5 (not just the anomalous part) is

$$(22) \quad \text{Im } \Gamma(t) = \frac{\lambda |\underline{Q}|}{\sqrt{t}} \frac{1}{2\pi} \int \frac{d\Omega}{A - 2\underline{N} \cdot \underline{Q}},$$

where A , $|\underline{Q}|$ and $|\underline{N}|$ are fairly complicated functions of t .

$$\text{i.e. } \text{Im } \Gamma(t) = \frac{\lambda |\underline{Q}|}{\sqrt{t}} \frac{1}{4|\underline{N}||\underline{Q}|} \log \frac{A + 2|\underline{N}||\underline{Q}|}{A - 2|\underline{N}||\underline{Q}|}$$

Below the physical threshold $t = (m_d + m)^2$, $|\underline{N}|$ is imaginary, and writing $|\underline{N}| = -in$ we get

$$\text{Im } \Gamma(t) = \frac{\lambda}{\sqrt{t}} \frac{1}{2n} \tan^{-1} \left(\frac{2n |\underline{Q}|}{A} \right) \quad t < (m_d + m)^2$$

The logarithm has a singular point at $A^2 = 4|\underline{N}|^2 |\underline{Q}|^2$, which gives the anomalous threshold t_0 , though the approximations made are not nearly so clear as in our treatment.

Also $A=0$ for $t=3m^2$. So the arctangent passes on to another branch, and to get back on to the correct branch (vanishing at $t = (m+\mu)^2$) we have to take the path of integration round the branch-point t_0 .

The path of integration is then as shown in Fig. 8, and the effective absorptive part in the anomalous region $t_0 < t < (m+\mu)^2$ is

$$\text{Im}_a \Gamma(t) = \frac{\lambda}{\sqrt{t}} \frac{\pi}{2n}$$

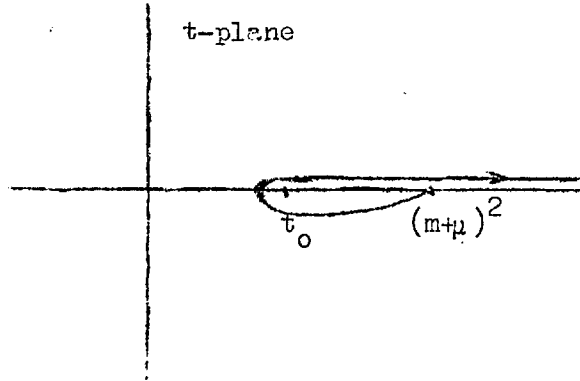


FIG.7

Alternatively we can start with the dispersion relation for small m_d , when there is no anomalous threshold, and analytically continue it in m_d . At $m_d^2 = 2m^2 + m\mu$ the branch point t_0 comes out of the second sheet at $t = (m+\mu)^2$ and then moves to its final position. The path of integration has to be deformed to avoid it, giving the path of integration of Fig.7.

Finally it is instructive to leave (22) in integral form. We have

$$\text{Im}I(t) = \frac{\lambda}{\sqrt{t}} \frac{i}{2n} \int \frac{dz}{B-z} \quad \text{with } B = -\frac{A}{2nq}$$

The singularity arises when $B^2=1$, and is then an end-point singularity in the z -integration.

B as a function of t follows the path shown in Fig. 8.

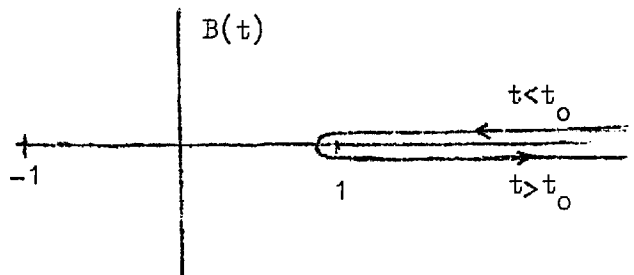


FIG.8

Thus for $t > t_0$ we have to distort the contour of integration to avoid passing through the singularity. We therefore get the contour shown in Fig.9, which is equivalent to that of Fig. 10.

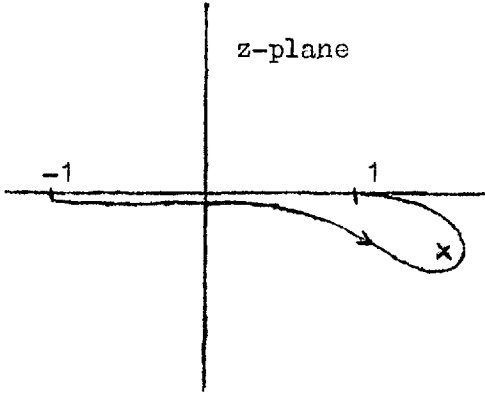


FIG.9

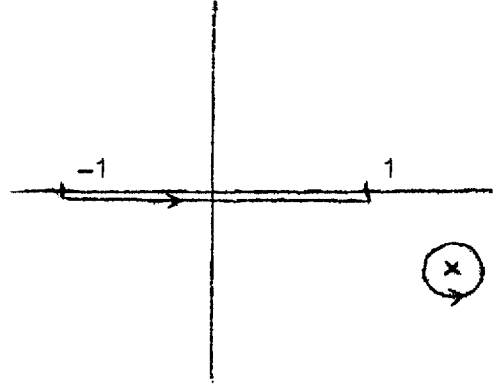


FIG.10

Hence the singular part of the integral comes from a small circle around $z = B$. The reader will notice a strong similarity between this discussion and the derivation of the Cutkosky formula.

2. Spinor Vertex.

(a) Form & Normalisation.

Following Blankenbecler and Cook we write the spinor vertex as

$$\Gamma(t) v(n) = \langle n | f_p(0) | d \rangle$$

We have to form the scalar matrix Γ out of momentum vectors, γ -matrices and the deuteron polarisation pseudovector ξ ($\xi \cdot d = 0$).

Moreover the vertex must be linear in ξ .

Thus the possible forms are $\xi \cdot n \not{L}, \xi \cdot n \not{p}', \not{L} \not{p}'$ and we write

$$(23) \quad \Gamma(t) = - (F(t) \not{L} + G(t) \xi \cdot n) + (\not{p}' - m)(H(t) \not{L} + I(t) \xi \cdot n)$$

In forming (23) we have already used covariance and parity. Charge conjugation does not further restrict the number of possible invariants, but tells us that F, G etc. are real functions for real t not on the cut.

To find the connection with the wave-function we take the non-relativistic form of $\bar{u}(p)\Gamma(t)v(n)$. To do this it is convenient to change to the other set of γ -matrices, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. Then we reduce the Dirac spinors to Pauli spinors by

$$u(p) = (p_0 + m)^{\frac{1}{2}} \left(1 - \frac{\gamma_5 \underline{\sigma} \cdot \underline{p}}{p_0 + m} \right) \chi_p \text{ etc.}$$

and work in the rest frame of the deuteron, so that $\xi = (0, \underline{\xi})$

The result is

$$\begin{aligned} \bar{u}(p)\Gamma(t)v(n) &= i\chi_p^+ \underline{\sigma} \cdot \underline{\xi} \chi_n^* \left[((p_0 + m) + \frac{1}{3}(p_0 - m))F(t) + \frac{2}{3}(p_0^2 - m^2)G(t) \right] \\ &\quad - \frac{2i}{3} \chi_p^+ (\underline{\sigma} \cdot \hat{p} \hat{\xi} \cdot \hat{p} - \underline{\sigma} \cdot \underline{\xi}) \chi_n^* [(p_0 - m)F(t) - (p_0^2 - m^2)G(t)] \end{aligned}$$

In potential theory the nucleon spins are represented by Pauli spinors α, β , combining to form a triplet state.

The three triplet states are $\alpha_1\alpha_2, \frac{1}{\sqrt{2}}(\alpha_1\beta_2+\alpha_2\beta_1), \beta_1\beta_2$.

We can represent these by writing α_1, β_1 as column vectors and α_2, β_2 as row vectors and forming the direct product of the two sets of spinors.

In this representation any triplet state must be of the form.

$$\frac{i}{\sqrt{2}} \underline{\sigma} \cdot \underline{\xi} \underline{\sigma}_y \text{ where } \underline{\xi} \text{ is some unit vector.}$$

$$(25) \quad \text{Thus } S_o = \frac{i}{\sqrt{2}} \underline{\sigma} \cdot \underline{\xi} \underline{\sigma}_y \text{ is the representation of the deuteron S-state.}$$

It is well-known ⁽¹⁴⁾ that the D-state wave-functions can be introduced by acting on the S-state wave-functions with the tensor operator

$$S_{12} = 3 \underline{\sigma}_1 \cdot \underline{\hat{p}} \underline{\sigma}_2 \cdot \underline{\hat{p}} - \underline{\sigma}_1 \cdot \underline{\sigma}_2$$

Hence near the pole $\underline{p}^2 = -a^2$ the total spin wave-function is

$$(26) \quad S = \left(1 + \frac{\rho}{\sqrt{8}} S_{12} \right) S_o$$

where ρ is the asymptotic D-S ratio.

In the present representation $\underline{\sigma}_1$ is represented by premultiplying by $\underline{\sigma}$, and $\underline{\sigma}_2$ by postmultiplying by $\underline{\sigma}^T$.

Then, performing the algebra, we find that

$$(27) \quad \frac{1}{\sqrt{8}} S_{12} S_o = -\frac{i}{2} (3 \underline{\xi} \cdot \underline{\hat{p}} \underline{\sigma} \underline{\hat{p}} \cdot \underline{\sigma} \underline{\xi}) \underline{\sigma}_y$$

We are now in a position to evaluate the spin factors in (24). Using (13) and evaluating the factors at the pole we find ⁽¹²⁾

$$(28) \quad \begin{cases} \left(\frac{8\pi}{m} \right)^{\frac{1}{2}} N = F_0 - \frac{1}{3} \frac{\varepsilon}{m} (mG_0) \\ \left(\frac{8\pi}{m} \right)^{\frac{1}{2}} \rho N = -\frac{\sqrt{2}}{3} \frac{\varepsilon}{m} \left(\frac{1}{2} F_0 - mG_0 \right), \end{cases}$$

where the overall normalisation is obtained from neutron-deuteron scattering and we have neglected terms in ρ^2 .

Solving (27) for F_0 and G_0 we finally obtain

$$(29) \quad \begin{cases} F_0 = \left(\frac{8\pi}{m} \right)^{\frac{1}{2}} N (1 + \rho/\sqrt{2}) \\ G_0 = F_0 \left(\frac{1}{m_d} + \frac{3\rho m}{\sqrt{2}\alpha^2} \right) \end{cases}$$

N is the asymptotic normalisation constant already introduced in (15). It is related to the triplet effective range r_e by

$$(30) \quad N^2 = \frac{2\alpha}{1 - \alpha r_e}$$

Our present approach is slightly different from, and perhaps clearer than that of ref. (12). We have taken the covariant vertex and reduced it to its non-relativistic form; in ref. (12) a suitable covariant form is synthesised from the known non-relativistic form (26).

Away from the pole it is clear that the combinations $F(t) + (t - m^2)H(t)/2m$ and $G(t) + (t - m^2)I(t)/2m$ essentially play the role of S- and D-state wave-

functions respectively.

Let us now examine the analytic structure of the invariant functions F, G .

(b) Anomalous Threshold.

The analytic structure of F and G is the same as that of Γ in the scalar case, and the anomalous threshold arises from the same diagram, Fig. 5. The position of the threshold is unchanged, but the evaluation of the discontinuities is now rather more complex.

We have*

$$(31) \quad \text{Im}\Gamma(t')v(n) = -\frac{1}{32A'} \frac{1}{2\pi} \oint d\psi \ 3g^2 \gamma_5 (\not{1}+m)(F\not{\xi}+G\not{\xi}\cdot s)(\not{s}-m)\gamma_5 v(n)$$

In their treatment of the vertex, Blankenbecler and Cook ⁽³⁾ supply $\text{Im}\Gamma$ with a free-particle spinor $\bar{u}(p)$ on the left, with the justification that they will ultimately only be concerned with F_0, G_0 . This is not sufficient justification, as we shall see, and we will not use this approximation.

Eqn. (31) can be reduced to

$$(32) \quad \text{Im}\Gamma(t')v(n) = -\frac{3g^2}{32A'} \frac{1}{2\pi} \oint d\psi [(\not{p}'-m)(F\not{\xi}+G\not{\xi}\cdot 1)\not{k} - (2F\not{\xi}\not{k} - F\not{u}^2\not{\xi} + G\not{u}^2\not{\xi}\cdot 1)]v(n)$$

The main problem now is to perform the circular integral $\frac{1}{2\pi} \oint d\psi$, which in the scalar case simply reduces to unity.

* The factor 3 arises from isotopic spin considerations, c.f. ref. ⁽³⁾

We do this by writing every vector q as $\hat{q}+h$, where h is the vector $\overrightarrow{OP}$ and $\hat{q}$ is the projection of q on to the plane ABC, Fig.6.

$$\text{Now} \quad h = \kappa(0,0, \cos\psi, \sin\psi),$$

so that

$$(33) \quad \left\{ \begin{array}{l} \frac{1}{2\pi} \int d\psi h_{\mu} = \frac{1}{2\pi} \int d\psi h_{\mu} h_{\nu} h_{\kappa} = 0 \text{ etc.} \\ \frac{1}{2\pi} \int d\psi = 1, \quad \frac{1}{2\pi} \int d\psi h_{\mu} h_{\nu} = \frac{1}{2} \kappa^2 (g_{\mu\nu} - \hat{g}_{\mu\nu}) \end{array} \right.$$

The projections of momentum vectors are expressible by elementary geometry in terms of d, n . $\hat{\xi}$ is also expressible in such terms by use of the restriction $\xi \cdot d = 0$.

In the present case, the circular integrals are

$$(34) \quad \left\{ \begin{array}{l} \frac{1}{2\pi} \int d\psi \not{k} = \hat{\not{k}}, \quad \frac{1}{2\pi} \int d\psi \xi \cdot \not{1} = \xi \cdot \hat{\not{1}}, \\ \frac{1}{2\pi} \int d\psi \xi \cdot \not{k} = \xi \cdot \hat{\not{k}} + \frac{1}{2} \kappa^2 (\not{k} - \hat{\not{k}}) \end{array} \right.$$

The geometry of Fig.6 is considered in detail in Appendix B. Performing the circular integrals and collecting terms together in the form

$$\text{Im } \Gamma v(n) = [- (\text{Im} F \not{k} + \text{Im} G \xi \cdot n) + (\not{p}' - m) (\text{Im} H \not{k} + \text{Im} I \xi \cdot n)] v(n),$$

we find the following expressions for the various imaginary parts :

$$(35) \quad \left\{ \begin{aligned}
 \text{Im F} &= \frac{3g^2}{32A'} F_0 \left[\mu^2 + \frac{(y' - \mu^2)^2}{4y'} - (t' - m^2) \frac{y' + \mu^2}{4y'} \right] \\
 \text{Im G} &= \frac{3g^2}{32A'} \left[(4mF_0 + (t' - m^2)G_0) \left(\frac{3(y' + \mu^2)^2}{16y'^2} - \frac{\mu^2}{4y'} \right) \right. \\
 &\quad \left. + \mu^2 G_0 \frac{y' - \mu^2}{2y'} - (t' - m^2) \frac{y' + \mu^2}{4y'} \right] \\
 \text{Im H} &= - \frac{3g^2}{32A'} \left[mF_0 \frac{y' + \mu^2}{2y'} + G_0 \frac{(y' - \mu^2)^2}{8y'} \right] \\
 \text{Im I} &= - \frac{3g^2}{32A'} F_0 \left[\left(\frac{3(y' + \mu^2)^2}{8y'^2} - \frac{\mu^2}{2y'} \right) - \frac{y' + \mu^2}{2y'} \right]
 \end{aligned} \right.$$

Because of the approximation already mentioned of multiplying by a free-particle spinor $\bar{u}(p)$, Blankenbocler and Cook did not obtain expressions for ImH, ImI. Their expressions for ImF, ImG are obtained from ours by omitting the terms in $(t' - m^2)$. The expression for ImG was used to calculate G_0 in terms of F_0 and G_0 , thus giving G_0/F_0 and hence ρ . They obtained a not unreasonable value of $\rho = 0.03$.

The extra terms in $(t' - m^2)$ unfortunately make the dispersion integral using the non-relativistic form of the absorptive part divergent. The neglect of these terms corresponds to a particular choice of subtraction constant, namely zero. This may seem plausible, but it is nevertheless an assumption, and the problem is typical of the ambiguities which arise with spinor numerators.

Part II -- e-d Elastic Scattering

3. Kinematics.

Denote the initial and final electron momenta by k, k' . We treat the electron as an extreme relativistic particle, i.e. neglect m_e^2 .

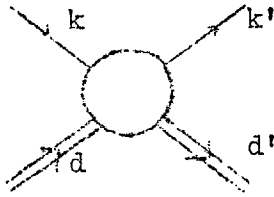


FIG. 11

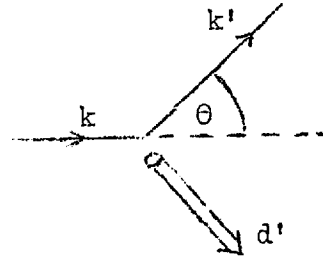


FIG. 12

$$(36) \quad k + d = k' + d' \quad \text{with } k^2 = k'^2 = 0, \quad d^2 = d'^2 = m_d^2.$$

Use of (36) shows that all scalar products can be expressed in terms of the two invariants

$$(37) \quad \begin{cases} \alpha = k \cdot d = k' \cdot d' \\ \text{and} \quad \beta = k' \cdot d = k \cdot d' \end{cases}$$

The two remaining scalar products are

$$(38) \quad k \cdot k' = \alpha - \beta \quad \text{and} \quad d \cdot d' = m_d^2 + \alpha - \beta,$$

and α and β are related to q^2 by

$$(39) \quad \alpha - \beta = -\frac{1}{2}q^2$$

Specialising to the lab. frame, where $d = (m_d, 0)$, we have

$$\alpha = m_d E, \quad \beta = m_d E'$$

where E and E' are the initial and final electron energies.

Then from (38) and (39):

$$(40) \quad q^2 = 2m_d(E' - E) = -2EE'(1 - \cos\theta)$$

where θ is the lab. scattering angle, Fig. 12.

This relation gives the elastic scattering energy,

$$(41) \quad E' = \frac{E}{1 + \frac{2E}{m_d} \sin^2 \frac{1}{2}\theta}$$

and the momentum transfer in terms of energy and angle :

$$(42) \quad q^2 = \frac{-4E^2 \sin^2 \frac{1}{2}\theta}{1 + \frac{2E}{m_d} \sin^2 \frac{1}{2}\theta}$$

Also of interest is the expression for the three-momentum transfer in the lab. frame,

$$(43) \quad \underline{q}_L^2 = -q^2 \left(1 - \frac{q^2}{4m_d^2} \right)$$

Finally, the differential cross-section is :

$$(44) \quad \frac{d\sigma}{d\Omega} = \left(\frac{e}{4\pi} \right)^2 \left(\frac{1}{2E} \right)^2 \frac{1}{16m_d^2} \frac{1}{E^2 \sin^4 \frac{1}{2}\theta} \sum \sum |\overline{M}|^2$$

where the summations are over initial and final spins, and in calculating M we limit ourselves to single-photon exchange, Fig. 13.

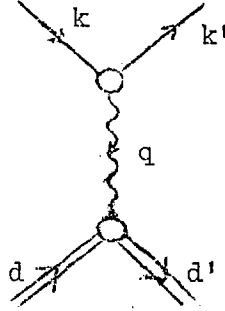


FIG. 13

Thus $M = \bar{e}u(k')\gamma^\mu u(k)F_{\mu d}$,

the last factor being the deuteron form-factor discussed in § 5(b).

Hence $\sum \bar{\sum} |M|^2 = 2 \left(\frac{1}{2} q^2 g_{\mu\nu} + k'_\mu k_\nu + k'_\nu k_\mu \right) T^{\mu\nu}$

where

(45) $T_{\mu\nu} = \frac{1}{3} \sum \sum F_{\mu d} F_{\nu d}$

In $T_{\mu\nu}$ we may drop terms in q_μ because of current conservation at the electron vertex. Hence we may write

$T_{\mu\nu} = J_0 g_{\mu\nu} + J_1 D_\mu D_\nu$, with $D = d + d'$

Then

(46)
$$\begin{aligned} \sum \bar{\sum} |M|^2 &= 2 [J_1 (2m_d^2 q^2 + 8\alpha\beta) + J_0 q^2] \\ &= \frac{16m_d^2 E^2 \cos^2 \frac{21}{2}\theta}{1 + \frac{2E}{m_d} \sin^2 \frac{21}{2}\theta} [J_1 - J_0 / 2m_d^2 \tan^2 \frac{21}{2}\theta] \\ &= -4 m_d^2 q^2 [J_1 \cot^2 \frac{21}{2}\theta - J_0 / 2m_d^2] \end{aligned}$$

So, substituting in (44) we get

$$(47) \left\{ \begin{aligned} \frac{d\sigma}{d\Omega} &= \left(\frac{e^2}{4\pi} \right)^2 \left(\frac{1}{2E} \right)^2 \frac{\cos^2 \frac{21}{2}\theta}{\sin^4 \frac{41}{2}\theta} \frac{1}{1 + \frac{2E}{m_d} \sin^2 \frac{21}{2}\theta} [\mathfrak{J}_1 - \mathfrak{J}_0 / 2m_d^2 \tan^2 \frac{21}{2}\theta] \\ &= \sigma_{NS} [\mathfrak{J}_1 - \mathfrak{J}_0 / 2m_d^2 \tan^2 \frac{21}{2}\theta] \\ &= \sigma_{NS} G^2, \text{ say} \end{aligned} \right.$$

We must now evaluate (45), using the form (72) for $F_{\mu d}$. The spin sums are performed according to $\sum_{\xi} \xi_{\mu} \xi_{\nu}^* = \frac{\delta_{\mu\nu}}{m_d^2} - \epsilon_{\mu\nu}$

After a certain amount of algebra we obtain (c.f. (7))

$$(48) \left\{ \begin{aligned} 3\mathfrak{J}_1 &= (F_{1d})^2 \left[2 + \left(1 - \frac{q^2}{2m_d^2} \right)^2 \right] - 2F_{1d}F'_{3d} \frac{q^2}{m_d^2} \left(1 - \frac{q^2}{2m_d^2} \right) + (F'_{3d})^2 \frac{q^4}{m_d^4} \\ &\quad - 2(F_{2d})^2 \frac{q^2}{4m_d^2} \left(1 - \frac{q^2}{2m_d^2} \right) + F_{2d}F_{1d} \frac{q^2}{m_d^2} \left(1 - \frac{q^2}{2m_d^2} \right) - F_{2d}F'_{3d} \frac{q^4}{m_d^4} \\ \mathfrak{J}_0 &= \frac{2}{3} (F_{2d})^2 \frac{q^2}{4m_d^2} \left(1 - \frac{q^2}{2m_d^2} \right) \end{aligned} \right.$$

where, as in (15), we have written $F'_{3d} = F_{3d} \left(1 - \frac{q^2}{4m_d^2} \right)$.

$\mathfrak{J}_1$ can be manipulated into a sum of squares :

$$(49) \quad 3\mathfrak{J}_1 = 2(F_{1d})^2 + \left[F_{1d} - \frac{q^2}{2m_d^2} (F_{1d} - F_{2d} + 2F'_{3d}) \right]^2 - 2 \frac{q^2}{4m_d^2} (F_{2d})^2$$

This expression is now easily transformed in terms of the other deuteron form factors defined by (77), giving*

$$3J_1 = 2 \left(F_{ch} + \frac{q^2}{6m_d^2} F_Q \right)^2 + \left(F_{ch} - \frac{q^2}{3m_d^2} F_Q \right)^2 - 2 \frac{q^2}{4m_d^2} (F_{2d})^2$$

$$(50) \quad \text{i.e. } J_1 = F_{ch}^2 - \frac{2}{3} \frac{q^2}{4m_d^2} F_{mag}^2 + \frac{q^4}{18m_d^4} F_Q^2$$

Substituting this last form into G^2 we finally obtain

$$(51) \quad G^2 = F_{ch}^2 - \frac{2}{3} \frac{q^2}{4m_d^2} (F_{mag})^2 (1 + 2(1 - q^2/4m_d^2) \tan^2 \frac{1}{2}\theta) + \frac{q^4}{18m_d^4} F_Q^2$$

This form, first written down by Gourdin ⁽¹⁶⁾, is a very satisfactory one. The cross-section is diagonal in F_{ch} , F_{mag} and F_Q , and the three terms will later be shown to correspond very closely to the three terms of the Jankus cross-section ⁽⁸⁾. It is actually a particular case of a general formula due to Durand, De Celles and Marx ⁽¹⁷⁾, who define relativistic multipole form-factors for the electromagnetic vertex functions and show that the cross-section is diagonal in terms of them.

Of course, since we do not restrict ourselves to orthogonal transformations among the form factors, there are many other combinations which

* In ⁽¹⁵⁾ we essentially wrote down the first two terms of (50), but did not at the time realise that the $\frac{q^4}{18m_d^4}$ term had such a simple form.

diagonalise the cross-section.* Eqn. (49) is a case in point; the particular combinations appearing here are those used by Cabibbo and Gatto (18) in their consideration of electron-positron annihilation into pairs of vector bosons.

We may mention here the form of the annihilation cross-section ($e\bar{e} \rightarrow d\bar{d}$) in terms of $\mathcal{J}_0$ and $\mathcal{J}_1$. Barnes (19) has shown that if we consider the scattering cross-section $d\sigma/dt$ rather than $d\sigma/d(\cos\theta)$ we can obtain the annihilation cross-section $d\sigma/ds$ by simply interchanging s and t in the flux and phase-space factors on the right-hand side. $\Sigma \bar{\Sigma} |M|^2$ should be written in terms of invariants and then reinterpreted in terms of the annihilation energy and angle E_c, θ_c .

Taking the last form of eqn. (46) we can express the factor $\cot^2 \frac{1}{2}\theta$ in terms of the invariant

$$\Phi = \frac{(s-u)^2}{(t-4m_e^2)(t-4m_d^2)}$$

For the scattering process

$$\Phi = 1 + \frac{\cot^2 \frac{1}{2}\theta}{1 - q^2/4m_d^2},$$

while for the annihilation process

$$\Phi = \cos^2 \theta_c$$

Hence

$$\Sigma \bar{\Sigma} |M|^2 = 4m_d^2 q^2 [\mathcal{J}_1 (1 - q^2/4m_d^2) \sin^2 \theta_c + \mathcal{J}_0/2m_d^2]$$

* Excluding θ -dependent transformation coefficients there is a singly infinite set of such combinations.

$$= -q^4 \left[\mathcal{J}_1 v^2 \sin^2 \vartheta_c - \frac{2\mathcal{J}_0}{q^2} \right]$$

where $v^2 = 1 - \frac{4m_d^2}{q^2}$

So, transforming the other factors, we finally obtain for the annihilation cross-section

$$(52) \quad \frac{d\sigma}{d\Omega_c} = \left(\frac{e^2}{4\pi} \right)^2 \frac{v}{32E_c^2} \left[\mathcal{J}_1 v^2 \sin^2 \vartheta_c - \frac{2\mathcal{J}_0}{q^2} \right]$$

On substituting the forms (50) and (48) for $\mathcal{J}_1$ and $\mathcal{J}_0$ this becomes

$$(53) \quad \frac{d\sigma}{d\Omega_c} = \left(\frac{e^2}{4\pi} \right)^2 \frac{v^3}{32E_c^2} \left[(|F_{ch}|^2 + \frac{q^4}{18m_d^4} |F_Q|^2) \sin^2 \vartheta_c + \frac{q^2}{6m_d^2} |F_{mag}|^2 (1 + \cos^2 \vartheta_c) \right]$$

Equally we could write the cross-section in the form used by Cabibbo and Gatto by using the form (49) for $\mathcal{J}_1$.

The $A + B \tan^2 \frac{1}{2} \vartheta$ form of G^2 is simply a consequence of the assumption of single-photon exchange, the photon carrying of course spin 1. This can be seen most clearly in the annihilation channel, where, c.f. eqn.(53), the ϑ_c dependence is $C + D \cos 2\vartheta_c$. The formula of Blankenbecler ⁽²⁰⁾ cannot be correct, as it does not have this feature.

Our next task is to discuss and evaluate the various deuteron form factors. To do this we need to know the nucleon form factors, which we now review briefly.

4. Nucleon Form Factors.

The nucleon form factors are introduced by

$$(54) \quad \langle p' | j_\mu(0) | p \rangle = \bar{u}(p') (F_1(q^2) \gamma_\mu + i \sigma_{\mu\nu} q_\nu F_2(q^2) / 2m) u(p)$$

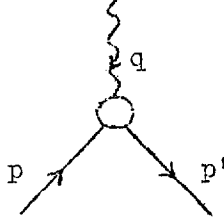


FIG.14

The form of the right-hand side is dictated by covariance, parity and current conservation. Charge conjugation does not further restrict the number of form factors, but tells us that they are real for real q^2 not on the cut.

To see the significance of the F_i we take the non-relativistic form of (54). We work in the Breit frame, in which

$$p' = (E, \frac{1}{2}\underline{q}) , \quad p = (E, -\frac{1}{2}\underline{q})$$

$$\text{i.e.} \quad P = (2E, 0) , \quad q = (0, \underline{q})$$

and reduce the Dirac spinors to Pauli spinors as in § 2(a)

The reduction of the spatial components of (54) is straightforward, and shows that

$$(55) \quad F_m = F_1 + F_2$$

is the magnetic form factor, with normalisation :

$$(56) \quad F_m(0) = \mu, \text{ the total magnetic moment in units } e/2m$$

The reduction of the temporal component leads to

$$\langle p' | j_0(0) | p \rangle = 2m \chi'^* F_c \chi,$$

showing that

$$(57) \quad F_c = F_1 + \frac{q^2}{4m^2} F_2$$

is the charge form-factor ⁽²¹⁾, with normalisation :

$$(58) \quad F_c(0) = 1$$

The inverse relations to (55) and (57) are

$$(59) \quad F_1 = \frac{F_c - q^2/4m^2 F_m}{1 - q^2/4m^2}, \quad F_2 = \frac{F_m - F_c}{1 - q^2/4m^2}$$

with corresponding static limits

$$(60) \quad F_1(0) = 1$$

$$F_2(0) = \kappa, \text{ the anomalous magnetic moment in units } e/2m$$

The Rosenbluth cross-section ⁽²²⁾ for elastic e-N scattering is

$$(61) \quad \frac{d\sigma}{d\Omega} = \sigma_{NS} \left[F_1^2 - \frac{q^2}{4m^2} (2(F_1 + F_2)^2 \tan^2 \frac{1}{2}\theta + F_2^2) \right],$$

which diagonalises in terms of F_c and F_m

$$(62) \quad \frac{d\sigma}{d\Omega} = \frac{\sigma_{NS}}{1 - q^2/4m^2} \left[F_c^2 - \frac{q^2}{4m^2} F_m^2 (1 + 2(1 - q^2/4m^2) \tan^2 \frac{1}{2}\theta) \right]$$

$$\text{Here } \sigma_{\text{NS}} = \left(\frac{e^2}{4\pi}\right)^2 \left(\frac{1}{2E}\right)^2 \frac{\cos^2 \frac{21}{2}\theta}{\sin^4 \frac{41}{2}\theta} \frac{1}{1 + 2E/m \sin^2 \frac{21}{2}\theta} = \sigma_{\text{mott}}$$

is the Mott cross-section for nucleons.

The reader will notice the strong similarity between the forms of eqns. (62) and (51).

By means of spinor identities eqn. (54) can be written in a number of ways*. The way we find most convenient in subsequent work is

(63)

$$\langle p' | j_\mu(0) | p \rangle = e \bar{u}(p') (F_m \gamma_\mu - F_2 \frac{p_\mu}{2m}) u(p).$$

Since the deuteron is isoscalar we shall be concerned with the isoscalar form factors[†]

$$(64) \quad F_i^S = F_i^n + F_i^p$$

5. The Deuteron Form Factor.

- the vertex which essentially determines the elastic scattering. In the scalar case we will show once again the close

* The separation into charge and magnetic-moment terms is perhaps most clearly shown by the form

$$\langle p' | \bar{j}_\mu(0) | p \rangle = e \bar{u}(p) (F_c \frac{p_\mu}{2m} + F_m \frac{r_\mu}{4m^2}) u(p) \times (1 - q^2/4m^2)^{-1}$$

The temporal component of $r_\mu = -\frac{1}{6} \epsilon_{\mu\nu\lambda\kappa} p_\nu q_\lambda \epsilon_{\kappa\alpha\beta\gamma} \gamma_\alpha \gamma_\beta \gamma_\gamma$ vanishes identically in the Breit frame.

† We strongly recommend this notation for the iso-scalar and -vector combinations, and also our choice of normalisation, eqn.(60). In the literature one often finds the combination F_m^S written out in full as $F_1^p + F_1^n + \kappa F_2^p + \kappa F_2^n$, which adds nothing to the clarity and a great deal to the algebraic clutter.

connection between the anomalous threshold and wave-function calculations and will use this in the spinor case to obtain relativistic corrections to the wave-function calculations (Jankus formula).

(a) Scalar Vertex.

(i) Potential Theory

In potential theory the deuteron form factor (neglecting nucleon electromagnetic structure) appears as ⁽²⁾

$$(65) \quad f_d = \int \varphi^*(\underline{r}) e^{\frac{1}{2} i \underline{q} \cdot \underline{r}} \varphi(\underline{r}) d^3 r$$

We will work in the Breit frame, where $\underline{q} = (0, \underline{q})$ and write $s = q^2$.

Using the form (15) for φ ,

$$f_d = N^2 [I(2\alpha) + 2 \int_{\mu}^{\infty} I(2\alpha + \beta) \rho(\beta) d\beta + \int_{\mu}^{\infty} \int_{\mu}^{\infty} \rho(\beta) \rho(\gamma) I(2\alpha + \beta + \gamma) d\beta d\gamma]$$

where

$$\begin{aligned} I(x) &= \frac{1}{4\pi} \int e^{\frac{1}{2} i \underline{q} \cdot \underline{r}} \frac{e^{-x r}}{r^2} d^3 r \\ &= \int_x^{\infty} \frac{d\gamma}{\gamma^2 + (\frac{1}{2} \underline{q})^2} \end{aligned}$$

Thus I has a cut in the $\underline{q}^2$ plane from $\underline{q}^2 = -4x^2$, with absorptive part

$$\text{Im } I = \frac{\pi}{\sqrt{s}}$$

Hence f_d has three cuts, the first beginning at $s = 16\alpha^2$, with absorptive part

$$(66) \quad \text{Im}^T f_d = \frac{\pi N^2}{\sqrt{s}}$$

We will now show how this result, with relativistic corrections, is reproduced by the lowest anomalous threshold in dispersion theory

(ii) Anomalous Threshold.

The form factor is

$$(67) \quad eF_d = \langle d' | j(0) | d \rangle$$

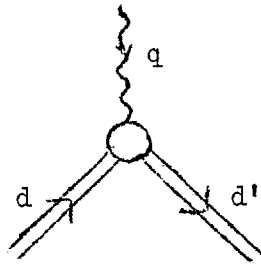


FIG. 15

Dispersing in $s=q^2$, we find that the lowest threshold is an anomalous threshold arising from the reduced graph of Fig. 16, whose dual diagram we draw in Fig. 17.

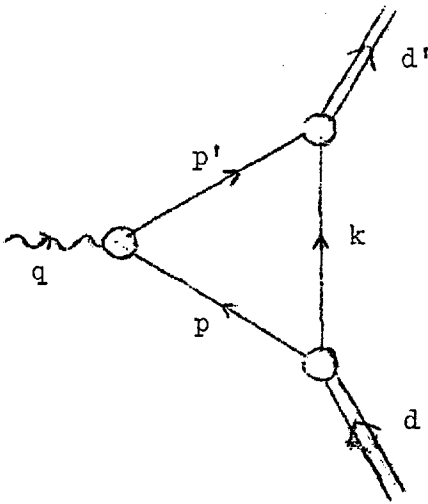


FIG. 16

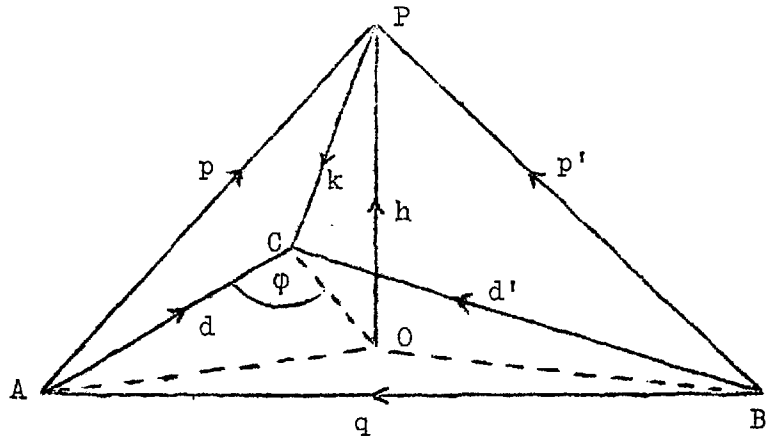


FIG. 17

The anomalous threshold is that value of s for which Fig. 17 becomes plane, Fig. 18.

$$(68) \quad \sqrt{s_0} = 2m_d \sin \phi_0$$

$$\div 2m_d \cdot \frac{2a}{m_d}$$

$$\text{i.e. } \sqrt{s_0} \div 4a, \text{ in agreement}$$

with the previous section.

The absorptive part on the cut is

$$\text{Im } T_{F_d} = \frac{\Gamma_o^2}{32A}$$

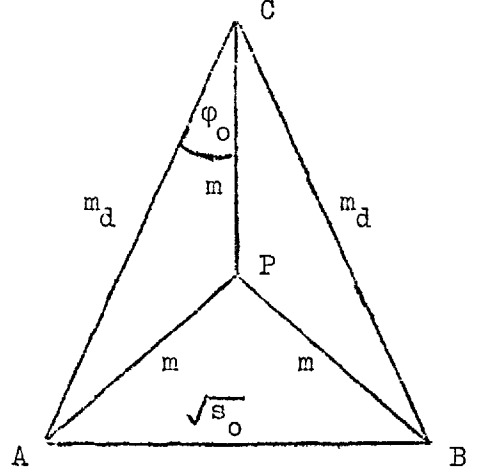


FIG. 18

$$\text{and } A = \frac{1}{2}m_d \sqrt{s(1 - s/4m_d^2)}$$

Thus with the identification $\Gamma_o^2 = 16\pi m_d N^2$ we have

$$(69) \quad \text{Im } T_{F_d} = \frac{\pi N^2}{\sqrt{s(1 - s/4m_d^2)}}$$

We will return to this equation later, when we are considering the value of F_d itself.

This section has been concerned with the 'bare' deuteron form factor, i.e. the form factor for structureless nucleons. To obtain the actual form factor we have to multiply by the isoscalar nucleon form factor.

(b) Spinor Vertex.

(i) Form and Normalisation*

We write the spinor vertex as

$$(70) \quad eF_{\mu d} = \langle d' | j_\mu(0) | d \rangle$$

*

The author is indebted to Loyal Durand III for help on this section.

The vector $F_{\mu d}$ is to be formed out of momentum vectors and the two polarisation pseudovectors ξ, ξ' . It is a bilinear form in ξ, ξ' , and must satisfy the current conservation condition $q \cdot F_d = 0$.

The only scalar at our disposal is q^2 , while $\xi \cdot d' = \xi \cdot q$ etc. It is then easily seen that $F_{\mu d}$ must have the form

$$(71) \quad F_{\mu d} = (A \xi \cdot q \xi'^* \cdot q + B \xi \cdot \xi'^*) D_\mu + C (\xi_\mu \xi'^* \cdot q - \xi'^* \xi_\mu \cdot q) \\ + E (\xi \cdot q \xi'^* \cdot q q_\mu - \frac{1}{2} q^2 (\xi_\mu \xi'^* \cdot q + \xi'^* \xi_\mu \cdot q))$$

We still have one invariance principle to apply, namely invariance under charge conjugation.

This gives

$$F_{\mu d}(\xi', \xi; d; d) = - F_{\mu d}(\xi, \xi'; -d, -d'),$$

which implies that E must vanish.

Thus we may finally write

$$(72) \quad F_{\mu d} = - F_{1d}(q^2) \xi'^* \xi D_\mu - F_{2d}(q^2) (\xi_\mu \xi'^* \cdot q - \xi'^* \xi_\mu \cdot q) + F_{3d}(q^2) \frac{\xi'^* \cdot q \xi \cdot q}{m_d^2} D_\mu$$

To see the significance of the various F_{id} , let us take the non-relativistic form of (72) in the Breit frame, in which

$$d' = (E, \frac{1}{2}q), \quad d = (E, -\frac{1}{2}q)$$

$$\text{i.e. } D = (2E, 0), \quad q = (0, q)$$

Taking first of all the spatial components of $F_{\mu d}$ we have

$$\langle d' | j_i(0) | d \rangle = e F_{2d}(q^2) (\xi_i \xi'^* \cdot q - \xi'^* \xi_i \cdot q)$$

Hence

$$(73) \quad \langle \underline{j} \cdot \underline{A} \rangle = ieF_{2d}(q^2) (\underline{\xi}'^* \times \underline{\xi}) \cdot \text{curl } \underline{A}$$

With the identification of $\underline{\xi}'^* \times \underline{\xi}$ as the expectation value of the deuteron spin operator (2^3) , eqn.(73) shows that F_{2d} is the magnetic form factor,* with normalisation

$$(74) \quad F_{2d}(0) = \mu_d \text{ in units } e/2m_d$$

The separation of the charge and quadrupole-moment form factors is somewhat more difficult; we take the non-relativistic form of the temporal component of $F_{\mu d}$:

$$(75) \quad F_{0d} = -2EF_{1d} \xi'^* \cdot \xi - F_{2d} (\xi_o \xi'^* \cdot q - \xi'_o \xi \cdot q) + \frac{2E}{m_d^2} F_{3d} \xi'^* \cdot q \xi \cdot q$$

We take into account the supplementary condition $\xi \cdot d = 0$ and the normalisation condition $\xi^2 = -1$ by writing

$$\xi = (\xi_1, \xi_2, \frac{E}{m_d} \xi_3, \frac{-|q|}{2m_d} \xi_3)$$

$$\xi' = (\xi'_1, \xi'_2, \frac{E}{m_d} \xi'_3, \frac{|q|}{2m_d} \xi'_3)$$

where the third axis is in the direction of $\underline{q}$, and $\underline{\xi}$, $\underline{\xi}'$ are now unit 3-vectors.

* The separation of the magnetic-moment term is most clearly effected by writing the coefficient of F_{2d} as

$$r_\mu = \epsilon_{\mu\nu\lambda\alpha} D_{\nu\lambda} \epsilon_{\kappa\alpha\beta\gamma} \xi'_\alpha \xi_\beta D_{\gamma} = 2(\xi_\mu \xi'^* \cdot q - \xi'_\mu \xi \cdot q) + 2D_\mu \xi \cdot q \xi'^* \cdot q$$

Once again the temporal component of r_μ vanishes identically in the Breit frame,

Eqn.(75) then reduces to

$$\begin{aligned}
 F_{od} &= 2EF_{1d} \left(\frac{\xi'^* \cdot \xi + \xi' \cdot q \xi \cdot q}{2m_d^2} \right) - 2EF_{2d} \frac{\xi \cdot q \xi'^* \cdot q}{2m_d^2} + 2EF_{3d} \frac{\xi \cdot q \xi' \cdot q}{m_d^2} \\
 \text{i.e. } \langle d' | j_o(0) | d \rangle &= 2Ee \left\{ F_{1d} \xi'^* \cdot \xi + F_Q \frac{\xi \cdot q \xi' \cdot q}{2m_d^2} \right\} \\
 (76) \qquad &= 2Ee \left\{ F_{ch} \xi'^* \cdot \xi + \frac{F_Q}{2m_d^2} \left(\xi \cdot q \xi'^* \cdot q - \frac{1}{3} q^2 \xi' \cdot \xi \right) \right\}
 \end{aligned}$$

where we have written

$$(77) \quad \begin{cases} F_Q &= 2F'_{3d} + F_{1d} - F_{2d} \\ F_{ch} &= F_{1d} - \frac{q^2}{6m_d^2} F_Q \end{cases}$$

The form of (76) shows that F_{ch} and F_Q are the charge and quadrupole-moment form factors with normalisation

$$(78) \quad \begin{cases} F_{ch}(0) &= 1 \\ F_Q(0) &= Q, \text{ the quadrupole moment in units } e/m_d^2 \end{cases}$$

To complete (77) we will also write

$$(79) \quad F_{mag} = F_{2d}$$

The static limits for the F_{id} are easily obtained from (77) and (78); they are wrongly quoted in ref. (7).

(ii) Anomalous Threshold.

We are now in a position to evaluate the various deuteron form factors. Expressions for these have essentially been given by Jankus ⁽⁸⁾ in potential theory. We will obtain the contribution to them from the lowest anomalous threshold. These will be found to be the same as the form factors calculated in potential theory using only the asymptotic part of the deuteron wave-function, together with relativistic corrections. We will apply these relativistic corrections to the complete form factors (i.e. those calculated with the complete wave-function).

The position of the cuts for the F_{id} is the same as for F_d ; in particular the lowest anomalous threshold is the same, arising once again from Fig. 16, with dual diagram Fig. 17. The evaluation of the discontinuities is now, however, rather more difficult.

We have *

$$(80) \quad \text{Im}^T F_{\mu d}(s) = \frac{1}{32A} \frac{1}{2\pi} \oint d\psi \text{Tr} \left[(F_m^S \gamma_\mu - F_2^S \not{p}/2m)(\not{p}+m)(F_o \not{\epsilon} + G_o \xi \cdot k)(\not{k}-m) \times \right. \\ \left. \times (F_o \not{\epsilon}' + G_o \xi' \cdot k)(\not{p}'+m) \right]$$

To calculate the circular integrals we use the same method as described for the np-d vertex, and in particular eqns. (33). We have here the additional complication of a fairly long trace to calculate. However the calculation is greatly eased by a high degree of symmetry, absent in

* The superscript T on Im^T means the contribution from the lowest anomalous threshold, corresponding to the tail of the deuteron wave-function. We have also written $s = q^2$.

the case of the np-d vertex.

The radius of the [4]circle is now given by

$$(81) \quad \kappa^2 = m^2 - \frac{m_d^2}{4\cos^2\varphi} \approx (\alpha^2 - 1/16^s) \sec^2\varphi$$

and the projections of the deuteron polarisation pseudovectors are

$$(82) \quad \begin{cases} \hat{\xi} = \xi \cdot q \left(\frac{q}{s} + \frac{D}{4m_d^2 \cos^2\varphi} \right) \\ \hat{\xi}'^* = \xi' \cdot q \left(\frac{q}{s} - \frac{D}{4m_d^2 \cos^2\varphi} \right) \end{cases}$$

These and other projections are obtained in Appendix C, where the geometry of Fig. 17 is discussed in detail.

The angle φ , given by $\sin\varphi = \sqrt{s}/2m_d$, is an extremely convenient variable, and we will express most of our results in terms of it.

For convenience we consider the terms in F_o^2 , $F_o G_o$, G_o^2 in turn. The traces involving F_o^2 are found to be

$$(83) \quad \frac{1}{2\pi} \int d\psi \frac{F_o^2}{\cos^2\varphi} \left\{ F_2^S [\xi' \cdot \xi D_\mu (m_d^2 - \frac{1}{2}s) \cos 2\varphi + \xi \cdot q \xi' \cdot q D_\mu \cos 2\varphi + 8\xi \cdot k \xi' \cdot k P_\mu \cos^2\varphi] \right. \\ \left. - F_m^S [\xi' \cdot \xi D_\mu (m_d^2 \cos 2\varphi + \frac{1}{2}s) + 2m_d^2 (\xi_\mu \xi' \cdot q - \xi' \cdot q \xi_\mu) + 8\xi \cdot k \xi' \cdot k P_\mu \cos^2\varphi] \right\}$$

in which we have already performed the integrals 1,h. The only ψ integration remaining is

$$(84) \quad \frac{1}{2\pi} \int d\psi \xi \cdot k \xi' \cdot k P_\mu = \frac{\kappa^2 \cos 2\varphi}{4 \cos^2\varphi} \xi' \cdot \xi D_\mu + \frac{\kappa^2}{4 \cos^2\varphi} (\xi_\mu \xi' \cdot q - \xi' \cdot q \xi_\mu) \\ - D_\mu \frac{\xi \cdot q \xi' \cdot q}{8 \cos^2\varphi} \left[\frac{\cos^2\varphi}{(2 \cos^2\varphi)^2} + \frac{2\kappa^2}{m_d^2} \left(\frac{\cos^2 2\varphi}{\sin^2 2\varphi} + \frac{1}{2 \cos^2\varphi} \right) \right]$$

Expressing (83) in terms of F_1^S and F_m^S and sorting out the contributions to the various F_{id} , we get

$$(85) \left\{ \begin{aligned} \left. \text{Im}^T F_{1d} \right|_{F_o^2} &= \frac{F_o^2}{32A \cos^2 \varphi} \left\{ F_1^S (m_d^2 - \frac{1}{2}s + 2\kappa^2) \cos 2\varphi + F_m^S \cos^2 \varphi \right\} \\ \left. \text{Im}^T F_{2d} \right|_{F_o^2} &= \frac{F_o^2}{32A \cos^2 \varphi} \left\{ 2\kappa^2 F_1^S + 2m_d^2 F_m^S \right\} \\ \left. \frac{1}{m_d^2} \text{Im}^T F_{3d} \right|_{F_o^2} &= \frac{F_o^2}{32A \cos^2 \varphi} \left\{ -F_1^S \left[\cos 2\varphi - \frac{\cos 2\varphi}{(2 \cos^2 \varphi)^2} - \frac{2\kappa^2}{m_d^2} \left(\frac{\cos^2 2\varphi}{\sin^2 2\varphi} + \frac{1}{2 \cos^2 \varphi} \right) \right] \right. \\ &\quad \left. + F_m^S \cos 2\varphi \right\} \end{aligned} \right.$$

The traces involving $F_o G_o$ are found to be

$$(86) \quad \frac{1}{2\pi} \int d\psi F_o G_o \left\{ -\frac{2}{m} F_2^S \left[\frac{1}{2} m_d^2 (\xi \cdot k \xi'^* \cdot q - \xi'^* \cdot k \xi \cdot q) P_\mu + (8m^2 - s) \xi \cdot k \xi'^* \cdot k P_\mu \right] \right. \\ \left. + \frac{m F_m^S}{\cos^2 \varphi} \left[\frac{1}{2} s (\xi_\mu \xi'^* \cdot q - \xi'^* \cdot q) + D_\mu \xi \cdot q \xi'^* \cdot q + 16 \xi \cdot k \xi'^* \cdot k P_\mu \cos^2 \varphi \right] \right\}$$

The ψ integrations still to be performed are (84) and

$$(87) \quad \frac{1}{2\pi} \int d\psi (\xi \cdot k \xi'^* \cdot q - \xi'^* \cdot k \xi \cdot q) P_\mu = -\kappa^2 (\xi_\mu \xi'^* \cdot q - \xi'^* \cdot q \xi_\mu) \\ + \frac{D_\mu \xi \cdot q \xi'^* \cdot q}{4 \cos^2 \varphi} \left(\frac{\cos 2\varphi}{\cos^2 \varphi} + \frac{2\kappa^2}{m_d^2} \right)$$

From (86), (87) and (84) we get the contributions to the F_{id} . We will be content with $\text{Im } F_{1d}$, $\text{Im } F_{2d}$ to first order (in s and α^2) and $\text{Im } F_{3d}$ to zeroth order. In this approximation

$$(88) \quad \left\{ \begin{array}{l} \text{Im } F_{1d} \Big|_{F_o G_o} = - \frac{F_o G_o}{32 A \cos^2 \varphi} \cdot 4 m \kappa^2 \cos 2 \varphi F_1^S \\ \text{Im } F_{2d} \Big|_{F_o G_o} = - \frac{F_o G_o}{32 A \cos^2 \varphi} \cdot m (4 \kappa^2 + \frac{1}{2} s) F_m^S \\ \text{and } \text{Im } F_{3d} \Big|_{F_o G_o} = \frac{F_o G_o}{32 A \cos^2 \varphi} \cdot 2 m (\frac{1}{4} - \frac{2 \kappa^2}{m_d^2} \frac{\cos^2 2 \varphi}{\sin^2 2 \varphi}) F_1^S \end{array} \right.$$

We will also quote the result of the traces involving G_o^2 :

$$(89) \quad \frac{1}{2\pi} \int d\psi. 2 G_o^2 \xi_{\cdot k} \xi_{\cdot k}^* [(4 \alpha^2 - \frac{1}{2} s) P_\mu F_2^S - (4 \alpha^2 P_\mu - s k_\mu) F_m^S] ,$$

but will neglect these, as their contributions to the absorptive parts are all one order smaller than those of (88).

Finally we collect together the contributions to the absorptive parts from (85) and (88), using the values of F_o, G_o given in (29)

$$(90) \quad \left\{ \begin{array}{l} \text{Im } F_{1d} = C \left\{ F_1^S m_d^2 \cos^2 2 \varphi + F_m^S s \cos^2 \varphi - \sqrt{2} \rho F_1^S (6 m^2 \frac{\kappa^2}{\alpha^2} - m_d^2) \right\} \\ \text{Im } F_{2d} = C \left\{ 2 \kappa^2 F_1^S + F_m^S (2 m_d^2 - \frac{1}{4} s - 2 \kappa^2) - \sqrt{2} \rho F_m^S (3 m^2 \cdot \frac{2 \kappa^2 + \frac{1}{4} s}{\alpha^2} - 2 m_d^2) \right\} \\ \frac{1}{m_d^2} \text{Im } F_{3d} = C \left\{ -\frac{1}{2} F_1^S + F_m^S + 3 \sqrt{2} F_1^S \frac{m^2}{\alpha^2} (\frac{1}{4} - \frac{2 \kappa^2}{s}) \right\} \end{array} \right.$$

where

$$(91) \quad C = \frac{\pi N^2 \sec^2 \varphi}{2m_d \sqrt{s(1-s/4m_d^2)}}$$

Let us transform the absorptive parts of F_{1d} , F_{3d} into absorptive parts for F_{ch} , F_Q . We get

$$(92) \quad \begin{cases} \frac{1}{m_d^2} \operatorname{Im}^T F_Q &= C.3\rho/\delta F_1^S \frac{m^2}{\alpha^2} \left(\frac{1}{4} - \frac{2\kappa^2}{s} \right) \quad \text{to zeroth order} \\ \operatorname{Im}^T F_{ch} &= C \cdot m_d^2 F_c^S \quad \text{to first order} \end{cases}$$

These expressions exemplify once again the greater simplicity of the problem in terms of F_{ch} and F_Q . We note also that the combination of the nucleon form factors appearing in $\operatorname{Im}^T F_{ch}$ turns out to be F_c^S , while the absorptive part of F_Q^T is proportional to ρ .

6. Dispersion Integrals.

(a) Bare Form Factor.

Let us return for a moment to eqn. (69)

$$(69) \quad \operatorname{Im}^T F_d = \frac{\pi N^2}{\sqrt{s(1-s/4m_d^2)}}$$

The dispersion integral is

$$(93) \quad F_d^T(s) = \frac{1}{\pi} \int \frac{ds'}{s'-s} \operatorname{Im}^T F_d(s')$$

Writing $s = -4m_d^2 x^2$ and transforming the integration variable to φ'

$$(93') \quad F_d^T(s) = \frac{N^2}{m_d} \int \frac{d\varphi'}{x^2 + \sin^2 \varphi'}$$

$$(94) \quad = \frac{N^2}{m_d} \left[\frac{-1}{x\sqrt{(1+x^2)}} \tan^{-1} \frac{x \cot \varphi'}{\sqrt{(1+x^2)}} \right]_{\alpha/m}^{\bar{\varphi}}$$

The upper limit $\bar{\varphi}$ is $\pi/6$ if we take the integration up to the normal two-nucleon threshold $s = 4m^2$; we may then ignore the contribution from the upper limit compared to that from the lower limit.

Hence

$$(95) \quad F_d^T(s) = \frac{N^2}{m_d} \cdot \frac{1}{x\sqrt{(1+x^2)}} \tan^{-1} \frac{m_d x}{2\alpha\sqrt{(1+x^2)}}$$

We want to compare this with the corresponding result in potential theory.

Recalling that

$$(66) \quad \text{Im}^T f_d(s) = \frac{\pi N^2}{\sqrt{s}}$$

we find that

$$\begin{aligned} f_d^T(s) &= N^2 \int_{16\alpha^2}^{\infty} \frac{ds'}{s' - s} \frac{1}{\sqrt{s'}} \\ &= \frac{N^2}{m_d} \int_{2\alpha/m_d}^{\infty} \frac{dy'}{y'^2 + x^2}, \end{aligned}$$

where we have written $s' = 4m_d^2 y'^2$

Finally we get

$$\begin{aligned}
 (96) \quad f_d^T(s) &= \frac{N^2}{m_d} \cdot \frac{1}{x} \tan^{-1} \frac{m_d x}{2a} \\
 &= \frac{2N^2}{\sqrt{-s}} \tan^{-1} \frac{\sqrt{-s}}{4a},
 \end{aligned}$$

which can also be written as

$$\begin{aligned}
 f_d^T(s) &= \frac{N^2}{4\pi} \int e^{-2ar} e^{\frac{1}{2}iq \cdot r} d^3r \quad (\text{c.f. eqn. (65)}) \\
 (97) \quad &= N^2 \int e^{-2ar} j_0\left(\frac{1}{2}|q|r\right) dr
 \end{aligned}$$

It is the purpose of this section to find the relativistic correction transforming f_d into F_d .

Comparing the forms of (95) and (96) we expand the inverse tangents in powers of α/mx and x for $\alpha/m \ll x$, and obtain

$$\begin{aligned}
 (98) \quad R = \frac{F_d^T(s)}{f_d^T(s)} &= \frac{1}{\sqrt{(1+x^2)}} \frac{\frac{\pi}{2} - (\alpha/mx)(1+x^2)^{\frac{1}{2}} + \frac{1}{3}(\alpha/mx)^3(1+x^2)^{3/2} - \dots}{\frac{\pi}{2} - \alpha/mx + \frac{1}{3}(\alpha/mx)^3 - \dots} \\
 &= 1 - x^2 \left[\frac{1}{2} + \frac{1}{\pi} \left(\frac{\alpha}{mx} \right) \left\{ 1 - \left(\frac{\alpha}{mx} \right)^2 \right\} + \dots \right]
 \end{aligned}$$

$$(99) \quad \sim \frac{1}{\sqrt{(1+x^2)}} \quad \text{for } \alpha/m \ll x$$

For smaller values of x there is no such simple form of R^* , and we have to use the form (98).

An alternative and possibly simpler way of regarding the relativistic correction is to remark that eqn. (69) can be written from eqn.(43) as

$$\text{Im}^T F_d = \frac{\pi N^2}{\sqrt{-q_L^2}},$$

which suggests strongly that we should write

$$(100) \quad F_d^T(s) = N^2 \int e^{-2\alpha r} j_0\left(\frac{1}{2}|q|_L r\right) dr$$

This is in fact a reversion to the original form of the Jankus formula, where the argument of the spherical Bessel function was $\frac{1}{2}|q|_L r$. It was then argued (^{25,26}) that in the relativistic form $|q|_L$ should be replaced by $\sqrt{-q^2}$. However Gourdin (¹⁶) showed that to get the correct non-relativistic formula one should work in the Breit frame, giving forms like (97), so the relativistic corrections are now seen to be the expression of the Lorentz transformation from the Breit frame back to the lab. frame, in which the experiment is actually carried out. We would like to argue, therefore, that the modification of the Jankus formula by replacement of lab. three-momenta by four-momenta should not be made.

* In (¹⁵) the inverse tangents were expanded in powers of mx/a , giving

$$(99a) \quad R \sim \frac{1}{1+x^2}$$

However this is only true for very small values of x . The error was later corrected in (²⁴).

Force is lent to this argument by an examination of eqn. (51), where it can be seen that the coefficient of $2 \tan^2 \frac{1}{2} \theta$ is not $\frac{-q^2}{6m_d^2} (F_{\text{mag}})^2$, but rather $\frac{|q|_L^2}{6m_d^2} (F_{\text{mag}})^2$. [The other term in $(F_{\text{mag}})^2$ and that in F_Q^2 are always so small that the change from $-q^2$ to $|q|_L^2$ makes a negligible difference].

(b) 3π cut

Comparison of (91) with (69) shows that $\text{Im}^T F_d$ enters all the expressions for $\text{Im}^T F_{id}$ as a factor :

$$(101) \quad \left\{ \begin{array}{l} \text{Im}^T F_{ch} = \text{Im}^T F_d \cdot \frac{F_c^S}{\cos^2 \varphi} \\ \text{Im}^T F_{2d} = \frac{\text{Im}^T F_d}{m_d^2 \cos^2 \varphi} \left\{ 2\kappa^2 F_1^S + F_m^S (2m_d^2 - \frac{1}{4}s - 2\kappa^2) \right. \\ \qquad \qquad \qquad \left. - \sqrt{2} \rho F_m^S (3m^2 \frac{2\kappa^2 + \frac{1}{4}s}{a^2} - 2m_d^2) \right\} \\ \text{Im}^T F_Q = \text{Im}^T F_d \cdot 3\rho\sqrt{8} F_1^S \frac{m^2}{a^2} (\frac{1}{4} - \frac{2\kappa^2}{s}) \end{array} \right.$$

In fact the absorptive parts have the form

$$(102) \quad \text{Im}^T F_{id} = \text{Im}^T F_d \Lambda (F_1^S, F_m^S),$$

where Λ is a linear function of F_1^S and F_m^S .

Let us consider a simplified case of (102), namely that which

would obtain for scalar particles*

$$(103) \quad \text{Im}^T F_{ch} = \text{Im}^T F_d \cdot F_c^S$$

In addition to the anomalous threshold cut from Fig. 16, F_{ch} has a 3π cut, Fig. 19.

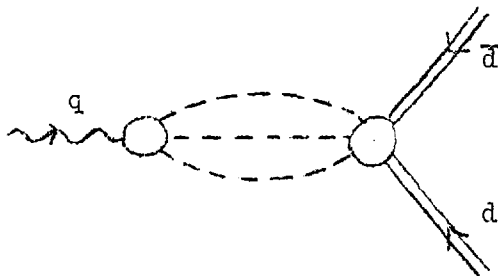


FIG. 19

The nucleon form factor F_c^S of course also has a 3π cut, so that Fig. 16 has a 3π cut arising from Fig. 20.

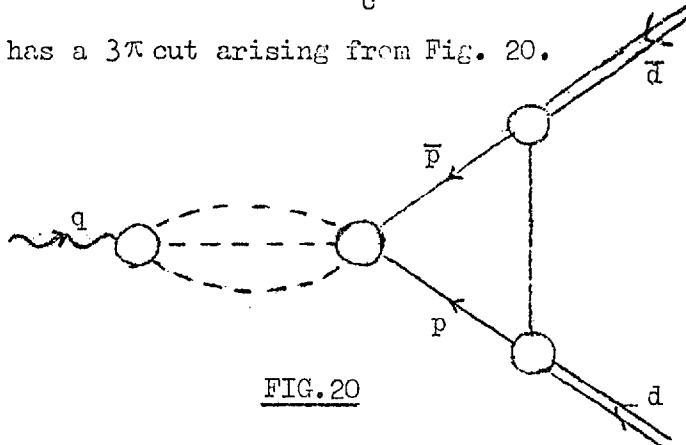


FIG. 20

These considerations suggest that, taking into account the anomalous threshold and the 3π cut,

$$(104) \quad F_{ch}^T = F_d^T F_c^S,$$

and we will now show that this is so.

*

The argument here is due to Cutkosky (^{6b}).

Let us write

$$F_{ch}^{T'} = F_d^T F_c^S$$

Then certainly F_{ch}^T and $F_{ch}^{T'}$ have the same discontinuity across the anomalous threshold cut :

$$(105) \quad [F_{ch}^T]_0 = [F_{ch}^{T'}]_0$$

Writing the discontinuity across the 3π cut as $[]_s$ we have

$$[F_{ch}^{T'}]_s = F_d^T [F_c^S]_s$$

$$\text{and} \quad [[F_{ch}^{T'}]_s]_0 = [F_d^T]_0 [F_c^S]_s$$

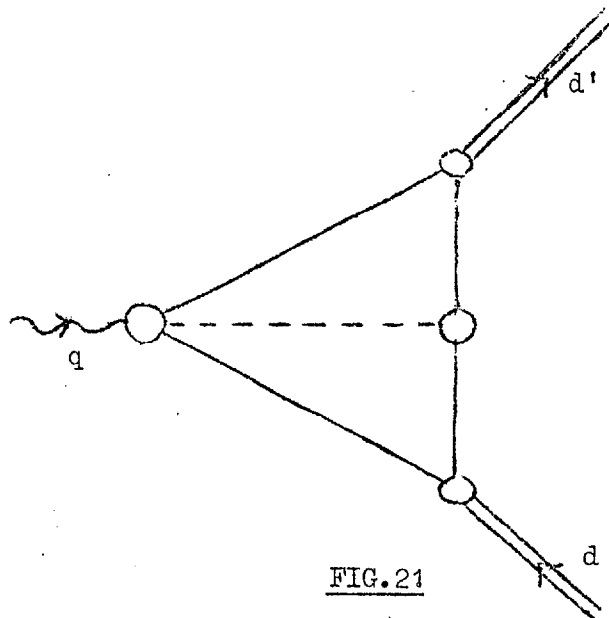
But from Fig. 20 we see that

$$(106) \quad \begin{aligned} [[F_{ch}^T]_s]_0 &= [F_d^T]_0 [F_c^S]_s \\ &= [[F_{ch}^{T'}]_s]_0 \end{aligned}$$

Thus as far as the anomalous threshold cut and the 3π cut are concerned F_{ch}^T and $F_{ch}^{T'}$ have the same analytic structure. They are obviously equal at the static point $s = 0$. Hence

$$F_{ch}^T = F_{ch}^{T'}, \text{ giving eqn. (104)}$$

The argument can be extended to include the next anomalous threshold arising from Fig. 21.



If the left-hand vertex is approximated by nucleon poles, the absorptive part of F_{ch} again contains $\text{Im } F_d$ as a factor, and the same argument applies. The contribution arising from the non-pole-term part of the left-hand vertex is known as the "non-additive" part (6b).

We can also easily extend the argument to eqn.(102), and may write

$$(107) \quad F_{id}^T = F_d^T \Lambda(F_1^S, F_m^S)$$

The transition from (102) to (107) is of great utility. On the right-hand side we now have form factors in the physical region, whereas an attempt to perform the dispersion integrals explicitly would involve integrations over nucleon form factors in the unphysical region, where they are unknown. The form of (107) also makes possible a close comparison between the dispersion theory and potential theory results.

The ρ -contribution to the absorptive parts must belong to the 'bare' form factor, involving, as they do, parameters belonging to the deuteron wave-function. Hence we shall have to perform the dispersion integrals for these contributions to obtain the various bare form factors appearing in the spinor case.

(c) S-wave contribution

Applying the results of the previous section, we find for the S-wave parts ($\rho = 0$)

$$(108) \quad \left\{ \begin{array}{l} F_{ch}^T = \frac{F_d^T}{\cos^2 \varphi} F_c^S \\ F_{2d}^T|_S = \frac{F_d^T}{\cos^2 \varphi} \left\{ \left(2 - \frac{s}{4m_d^2} - \frac{2\kappa^2}{m_d^2} \right) F_m^S + \frac{2\kappa^2}{m_d^2} F_1^S \right\} \\ F_Q^T|_S = 0 \end{array} \right.$$

which reproduces the correct static limits (74), (78). (Q would be zero for $\rho = 0$).

(d) D-wave contribution

(i) Let us first perform the dispersion integral for the bare form factor associated with F_Q , eqn. (101).

The integral we have to perform is

$$I = 3\rho \sqrt{8} \frac{m^2}{\alpha^2} \cdot \frac{1}{\pi} \int_{s'-s} \frac{ds'}{s'-s} \left(\frac{1}{4} - \frac{2\kappa^2}{s'} \right) \text{Im}^T F_d(s')$$

Recalling eqn.(81) we may write this as

$$(109) \quad I = 3\rho\sqrt{8} \frac{m^2}{\alpha^2} \frac{1}{\pi} \int \frac{ds'}{s'-s} \left(\frac{3}{8} - \frac{2\alpha^2}{s'} \right) \text{Im}^T F_d(s')$$

The integration of the first term simply yields $F_d^T(s)$ as in (93) while the integration of the second is like a subtraction in (93) and yields

$$(110) \quad \frac{1}{\pi} \int \frac{ds'}{s'-s} \frac{\alpha^2}{s'} \text{Im}^T F_d(s') = \frac{\alpha^2}{4m_d^2 x^2} \frac{N^2}{m_d} \left[\frac{1}{x\sqrt{(1+x^2)}} \tan^{-1} \frac{x \cot \varphi'}{\sqrt{(1+x^2)}} - \cot \varphi' \right]_{\alpha/m}^{\Phi}$$

$$\approx \frac{\alpha^2}{4m_d^2 x^2} \frac{N^2}{m_d} \left[-\frac{1}{3} x^2 \cot^3 \varphi' \right]_{\alpha/m}^{\Phi} \quad (x \text{ small})$$

$$\approx \frac{1}{48} F_d^T(s),$$

neglecting the contribution from the upper limit.

Hence

$$I = 3\rho\sqrt{8} \frac{m^2}{\alpha^2} F_d^T(s) \left(\frac{3}{8} - \frac{1}{24} \right)$$

$$= \rho\sqrt{8} \frac{m^2}{\alpha^2} F_d^T(s),$$

giving

$$(111) \quad F_Q^T = F_1^S \rho\sqrt{8} \frac{m^2}{\alpha^2} F_d^T$$

Taking the static limit of this equation we find

$$\begin{aligned} F_Q^T(0) &= \rho \sqrt{8} \frac{m^2}{\alpha^2} \frac{N^2}{2\alpha} \simeq m_d^2 \left(\frac{\rho N^2}{\sqrt{8} \alpha^3} \right) \\ (112) \quad &\simeq Q^T m_d^2, \end{aligned}$$

since the contribution to Q from the tail of the deuteron wave function is ⁽¹⁴⁾

$$(113) \quad Q^T = \frac{\rho N}{\sqrt{8} \alpha^3}$$

Eqn.(112) now completes the static limits (78) (The ρ term in F_{2d} does not alter the static limit of F_{2d} , as we shall see)

(ii) Next let us complete the evaluation of the bare form factor associated with F_{mag} , eqn. (101).

We have to perform the integral

$$(114) \quad J = \sqrt{2} \rho \cdot \frac{1}{\pi} \int \frac{ds'}{s'-s} \left(3m^2 \cdot \frac{2s'^2 + \frac{1}{4}s'}{\alpha^2} - 2m_d^2 \right) \text{Im}^T F_d(s')$$

The difficulty here is the factor s'/α^2 in (). We cannot perform this integral directly as we did with α^2/s' , since the contribution from the upper limit Φ would be the more important. We do it by remarking that

$$\text{Im}^T F_{ch} \Big|_{\rho} = 0, \quad \text{c.f. eqn. (101).}$$

We therefore require that

$$F_{ch}^T \Big|_{\rho} = 0$$

i.e.

$$(115) \quad F_{1d}^T - \frac{1}{6} \frac{s}{m_d^2} F_Q^T \Big|_\rho = 0$$

where F_Q^T is determined from (111)

This equation enables us to find to what an imaginary part of s in the bare form factor corresponds.

Writing

$$\frac{1}{\pi} \int \frac{ds'}{s'-s} f(s') \operatorname{Im} F_d^T(s') = \langle f(s) \rangle F_d^T(s)$$

and using $\operatorname{Im} F_{1d}^T \Big|_\rho$ as given in eqn. (90), we find that

$$(116) \quad \begin{cases} \text{i.e.} & \langle x^2 \rangle = \frac{2}{3} a^2 - \frac{1}{18} s \\ \text{or} & \frac{1}{16} \langle s \rangle = \frac{1}{3} a^2 + \frac{1}{18} s \end{cases}$$

Applying these results to the evaluation of J we find

$$\begin{aligned} J &= \sqrt{2} \rho F_d^T(s) \left(-2m^2 + \frac{3}{8} \frac{m^2}{a^2} \langle s \rangle \right) \\ &= \sqrt{2} \rho F_d^T(s) \cdot \frac{m^2}{a^2} \cdot \frac{s}{3} \end{aligned}$$

Hence

$$\begin{aligned} (117) \quad F_{2d}^T \Big|_\rho &= - \frac{J F_m^S}{m_d^2} \\ &= - \frac{F_d^T}{m_d^2} F_m^S \rho \sqrt{2} \frac{m^2}{a^2} \cdot \frac{s}{3} \\ &\approx - \frac{1}{6} (\mu_p + \mu_n) Q^T s \end{aligned}$$

from (113) and (95).

Now with neglect of w^2 the magnetic scattering term in the Jankus formula, eqn (121) , is

$$(118) \quad - \frac{2}{3} \frac{q^2}{4m_d^2} (2 \tan^2 \frac{1}{2} \theta + 1) (\mu_p + \mu_n)^2 (2 \int u^2 j_0(\frac{1}{2} q r) dr + \sqrt{2} \int u w j_2(\frac{1}{2} q r) dr)^2$$

The second integral here is

$$(119) \quad \sqrt{2} \int u w j_2(\frac{1}{2} q r) dr \approx - \frac{1}{6} Q q^2 ,$$

which corresponds very closely to (117).

7. Relativistic Corrections to the Jankus Formula.

Let us now write down the full Jankus formula ⁽⁸⁾

$$(120) \quad G^2 = \left(\int (u^2 + w^2) j_0(\frac{1}{2} q r) dr \right)^2 \\ - \frac{2}{3} \frac{q^2}{4m^2} (2 \tan^2 \frac{1}{2} \theta + 1) \left(\int \left\{ [(\mu_p + \mu_n)(u^2 + w^2) - \frac{3}{2}(\mu_p + \mu_n - \frac{1}{2})w^2] j_0(\frac{1}{2} q r) \right. \right. \\ \left. \left. + 2^{-\frac{1}{2}} w [(\mu_p + \mu_n)(u + 2^{-\frac{1}{2}} w) + 3 \cdot 8^{-\frac{1}{2}} w] j_2(\frac{1}{2} q r) \right\} dr \right)^2 \\ + \left(\int 2 w (u - 8^{-\frac{1}{2}} w) j_2(\frac{1}{2} q r) dr \right)^2$$

If we neglect w^2 this simplifies considerably to

$$(121) \quad G^2 = \left(\int u^2 j_0(\frac{1}{2} q r) dr \right)^2 \\ - \frac{2}{3} \frac{q^2}{4m^2} (2 \tan^2 \frac{1}{2} \theta + 1) (\mu_p + \mu_n)^2 \left(\int u^2 j_0(\frac{1}{2} q r) dr + 2^{-\frac{1}{2}} \int u w j_2(\frac{1}{2} q r) dr \right)^2 \\ + \left(\int 2 u w j_2(\frac{1}{2} q r) dr \right)^2$$

The usual procedure to generalise this to include relativistic effects and nucleon structure ^(25,26) has been to take the 'q' in the spherical Bessel functions as $\sqrt{-q^2}$ and to make the modifications $1 \rightarrow F_1^S$, $\mu_p + \mu_n \rightarrow F_m^S$. This is in fact only an inspired guess, based on the idea that F_1^S and F_m^S describe the charge and magnetic moment structure, and an analogy with the Rosenbluth formula, where the formula for point nucleons

$$\frac{d\sigma}{d\Omega} = \sigma_{NS} \left\{ 1 - \frac{q^2}{4m^2} (2\mu^2 \tan^2 \frac{1}{2}\theta + \kappa^2) \right\},$$

generalises to the form (61).

However the long-held belief that F_1^S describes the charge structure is now recognised to be erroneous, and we may remind the reader of the alternative form of the Rosenbluth formula in terms of F_c, F_m , eqn.(62), which shows that an analogy with the Rosenbluth formula is rather ambiguous.

Let us write (121) as

$$(122) \quad G^2 = f_c^2 - \frac{2}{3} \frac{q^2}{4m^2} (2 \tan^2 \frac{1}{2}\theta + 1) (\mu_p + \mu_n)^2 f_m^2 + f_Q^2$$

and let us compare this with (51)

$$(51) \quad G^2 = F_{ch}^2 - \frac{2}{3} \frac{q^2}{4m_d^2} (2 \tan^2 \frac{1}{2}\theta (1 - q^2/4m_d^2) + 1) (F_{mag})^2 + \frac{q^4}{18m_d^4} F_Q^2$$

Recalling the principle of this Part, we will regard results proved with a superscript ^T as holding without that superscript.

Then from (108) we have

$$(123) \quad F_{ch} = F_d \frac{F_c^S}{1 - s/4m_d^2} \quad \text{to first order}$$

and, recognising f_c as our old friend f_d , we may write F_d alternately as

$$(124) \quad F_d(s) = f_c \left(\left| \underline{q} \right|_L \right)$$

or as

$$(125) \quad F_d(s) = \frac{f_c(s)}{\sqrt{(1-s/4m_d^2)}} \quad \text{for } -s > 16a^2,$$

from eqns. (100) and (99) respectively.

From (108), (117) and (119) we have

$$(126) \quad F_{mag} = - \frac{2f_m(s)}{\sqrt{(1-s/4m_d^2)}} \frac{F_m^S}{1 - s/4m_d^2} \quad \text{to zeroth order,}$$

while eqns. (112) and (119) yield

$$(127) \quad f_Q = -\frac{\sqrt{2}}{6} Q q^2 = -\frac{1}{\sqrt{18}} F_Q \frac{q^2}{m_d^2} \quad \text{to first order,}$$

In view of the fact that (126) only holds to zeroth order, the first order terms involving F_c^S , we will restrict our remarks to small angle scattering.

In this case we can write (51) as

$$(128) \quad G^2 = \frac{f_c^2 (F_c^S)^2}{(1-s/4m_d^2)^3} + \frac{2}{3} \frac{q_L^2}{4m^2} (2 \tan^2 \frac{1}{2} \theta + 1) f_m^2 (F_m^S)^2 + f_Q^2 (F_c^S)^2$$

The first term may alternatively be written from (124) as

$$(129) \quad F_c^2 (|q|_L) = \left(\frac{F_c^S}{1 - s/4m_d^2} \right)^2$$

This is the form we prefer; the first form has been written for convenience in finding the relativistic corrections to the results of Glendenning and Kramer (⁹).

8. Neutron Form Factors.

The whole object of electron-deuteron scattering is of course to get at the neutron form factors. We cannot find them directly because of the absence of free neutron targets, and the deuteron, being a source of quasi-free neutrons, offers the next best opportunity.

The impulse-approximation formula for deuteron electrodisintegration (^{10,27}) is of the form

$$(130) \quad \frac{d^2\sigma}{dE'_e d\Omega} = \frac{\sigma_{\text{mott}}}{4\pi} \frac{m^2}{pE} N^2 \left(\frac{1}{\alpha^2} - \frac{2}{\beta^2 - \alpha^2} \log \frac{\beta^2}{\alpha^2} + \frac{1}{\beta^2} \right) (G_p + G_n),$$

where G_p and G_n are the proton and neutron Rosenbluth form factors multiplying G_{NS} in eqns. (61) and (62).

From the form of (62) it is clear that inelastic scattering can not give us any information about the sign of F_c^n . There is of course no ambiguity in the signs of the other nucleon form factors, since they have non-zero static limits. In terms of F_1^n , there is also an ambiguity of sign, corresponding to slightly different values of F_2^n . The choice of the solution has been largely a matter of fashion.

Elastic scattering, however, offers the possibility of determining the sign of F_1^n (or F_c^n), since the neutron form factors enter the formula in the isoscalar combination $F_1^S = F_1^D + F_1^n$. The sign of F_1^S must be positive and F_1^n is determined as

$$(131) \quad F_1^n = F_1^S - F_1^D$$

Since $F_1^n(0) = 0$, relativistic corrections may well be of decisive importance in determining the sign of F_1^n away from the static point. An unfortunate feature in practice is that F_1^n appears as the difference of two nearly equal numbers, each with a certain error, so that F_1^n is not very well determined. However, recent experiments at Orsay ⁽²⁸⁾ and elsewhere have greatly reduced the experimental uncertainty in F_1^D and F_1^S , and the relativistic corrections and the experimental uncertainties are now about the same size.

The elastic scattering formula which has previously been used is that quoted by Glendenning and Kramer ⁽⁹⁾.

$$(132) \quad G^2 = f_c^2 (F_1^S)^2 - \frac{2}{3} \frac{q^2}{4m^2} (2 \tan^2 \frac{1}{2} \theta + 1) f_m^2 (F_m^S)^2 + f_Q^2 (F_1^S)^2$$

From this formula they obtained F_1^S and hence F_1^n from (131).

The relativistic correction from (128) is that F_1^S is replaced by $F_c^S (1 - s/4m_d^2)^{-3/2}$.

Hence their value for F_1^n is not really F_1^n , but rather

$$(133) \quad z = F_c^S (1 - s/4m_d^2)^{-3/2} - F_1^D$$

Thus to order η , with $\eta = -q^2/4m_d^2$,

$$\begin{aligned} z &= F_1^n - 3/2 \eta F_1^p - 4\eta F_2^s \\ &= F_1^n - 4\eta F_1^p (\mu_p + \mu_n - 5/8) \end{aligned}$$

$$(134) \quad \therefore \quad F_1^n = z + 1.02\eta F_1^p$$

We now quote the results of Glendenning and Kramer, together with the correction term of (134), in Table I.

Table I.

q (f ⁻¹)	F ₁ ^p exp.	z		1.02ηF ₁ ^p
		Lower	Upper	
0.99	0.901	- 0.120	- 0.024	0.003
1.07	0.889	- 0.100	0.001	0.003
1.36	0.839	- 0.060	0.058	0.005
1.51	0.809	- 0.093	0.015	0.006
1.79	0.748	- 0.110	0.003	0.007
1.99	0.706	- 0.132	-0.021	0.0085
2.22	0.658	- 0.048	-0.086	0.010
2.41	0.619	- 0.130	-0.019	0.011
2.61	0.578	- 0.111	0.020	0.012
2.82	0.538	- 0.125	-0.001	0.013

The effect of the correction is thus seen to be a slight increase of F₁ⁿ, though it still retains a small negative bias.

We recall here the result of the scattering of slow neutrons off atomic electrons (²⁹) :

$$(135) \quad F_1^{n'}(0) = 0$$

[This is not the same as saying that the neutron charge-radius is zero; the r.m.s. radius is given by

$$1/6 \langle r^2 \rangle_c = F_c^{n'}(0) = \kappa_n/4m^2]$$

Hence the effect of the relativistic correction is to bring F_1^n more in line with the neutron-electron interaction, which is quite satisfactory.

Grossetête and Lehmann (²⁸) have also applied our correction to their very accurate experiment, and find the same result. Their results are written in terms of F_c^n , which is of course a trivial modification.

On the other hand an experiment at Stanford (³⁰) seems to give the result $F_c^n = 0$, which is in disagreement with both the Orsay results and the neutron-electron interaction. Since both sets of experimenters use the same theory we can offer no explanation of this discrepancy.

9. Recent Developments.

Since the appearance of (¹⁵) several other papers have appeared on the subject of relativistic correction to elastic e-d scattering, and experimenters seem to have at least agreed to replace F_1^S by F_c^S , which is certainly an improvement, though we may remark that, owing to the quasi-cancellation of the nucleon anomalous magnetic moments $\kappa_p + \kappa_n \approx 0$, corrections of the type $(1 - s/4m_d^2)^n$ will be more important than the transition from F_1^S to F_c^S .

Neuer and Urban (³¹) approached the problem by the method of Jankus, but including in the non-relativistic Hamiltonian terms which Jankus had

neglected. (Darwin term and spin-orbit term).

Their result was that $(F_1^S)^2$ was replaced by

$$\begin{aligned} (F_1^S)^2 &\longrightarrow F_1^S \left[F_1^S + \frac{q^2}{4m^2} (F_1^S + 2F_2^S) \right] \\ &= \frac{(F_c^S - xF_m^S)(F_c^S + xF_m^S)}{1 - x}, \text{ where } x = \frac{q^2}{4m^2} \\ &= \frac{(F_c^S)^2}{1 - x} \quad \text{to order } x \end{aligned}$$

Hence F_1^S is replaced by $F_c^S(1 - s/4m_d^2)^{-2}$. Referring back to eqn.(99a) we see that this is just the form we would obtain for very small s ($-s < 16a^2$).

Gourdin ⁽¹⁶⁾ uses the form (51) for G^2 , and evaluates the deuteron form factors in the impulse approximation, stating the assumptions of this approximation very clearly and making quite an extensive use of the general properties of angular momentum operators (e.g. the Wigner-Eckart theorem) in deriving the bare form factors. He finds that F_1^S is replaced by F_c^S , but his formula, calculated in potential theory, is of course still subject to relativistic correction.

An attempt at making the calculation of Gourdin relativistic has been made by Tanh Van ⁽³²⁾ using the Bethe-Salpeter equation. However he makes the basic assumption that the interaction kernel $W(p, p')$ can be given a separable form $W(p, p') = g(p)g(p')$, where $g(p) = (p^2 + \beta^2)^{-1}$. His justification for this is that it corresponds to the Hulthén wave-function in the non-relativistic limit, but since the interaction kernel can not in

fact be separable, and since many different forms of $W(p,p')$ can give the same non-relativistic limit, we find it hard to see why we should prefer this form of W to any other.*

The method we have adopted for dealing with the non-asymptotic part of the wave-function is to apply the relativistic correction obtained for the asymptotic wave-function to the complete wave-function, and we might say a few words about this.

The next highest threshold (^{6b}) arises from Fig.21, the position of the threshold being given by the dual diagram as $t_1 = 4(2\alpha + \mu)^2$. The main contribution to the absorptive parts comes from nucleon pole-terms in the left-hand vertex of Fig. 21, i.e. from Figs. 22 and 23, where the lines marked $\neq$ are off the mass-shell

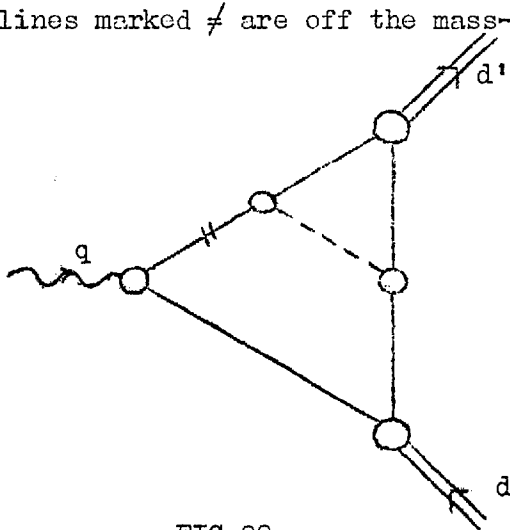


FIG.22

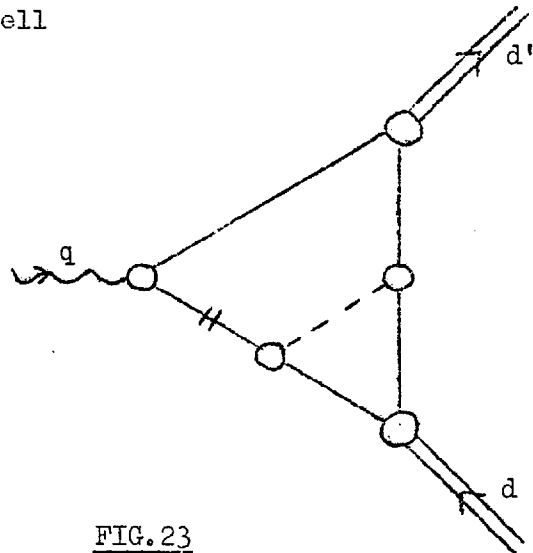


FIG.23

It is clear that these diagrams represent structure (off-mass-shell) corrections to the np-d vertex $\Gamma(t)$, as a detailed examination of the

* The author is indebted to C. Lovelace for this remark.

absorptive parts will confirm. An argument similar to that of Durand ⁽³³⁾ shows that off-mass-shell effects for the nucleon form factors are likely to be small compared with the structure effects for $\Gamma(t)$. Hence the inclusion of the second anomalous threshold is equivalent to replacing Γ_0 by $\Gamma(t)$ in (80). The relativistic factors arising from the trace and the circular integrals will remain the same. Hence the overall relativistic corrections we have found will apply to the complete wave-function. This method of including the short-range wave-function has the great advantage over ⁽³²⁾ that we are able to make use of the detailed and extensive investigations of the deuteron wave-function of, e.g., Glendenning and Kramer ⁽³⁴⁾.

Part III - Deuteron Electrodissintegration

10. Introduction

Electrodissintegration is more complicated kinematically (since there are more degrees of freedom in the final state), but simpler dynamically than elastic scattering. The greater dynamic simplicity lies in the fact that the asymptotic deuteron wave-function has its counterpart in dispersion theory as a pole term, which is much easier to calculate than the absorptive parts of § 5. The lowest anomalous threshold in the single dispersion relations now corresponds to the rest of the deuteron wave-function. The single dispersion relations have been investigated by Durand (¹⁰), who thus obtained a 'covariant Hulthén model' of electrodissintegration. He confined his analysis to outgoing electron energies and angles corresponding to the quasi-elastic peak in the cross-section, where the electron virtually scatters off a single nucleon, the other nucleon remaining as a 'spectator'.

Since there are now two nucleons in the final state, an important feature of the scattering may well be a strong final-state interaction between these two particles. Durand investigated this in potential theory, using the partial-wave expansion and finding the difference between the cross-sections calculated (a) with the free-particle final-state wave-functions and (b) with the actual ones, which must have the appropriate phase of nucleon-nucleon scattering. These results cannot be regarded as complete, as the final-state effect was only calculated for two momentum-

transfers, and also the wave-functions were approximated by those derived from an 'equivalent square-well potential'. Both these shortcomings should be eliminated in a work projected by Durand^{*}, in which the calculation will be programmed for a computer and the wave-functions calculated from a phenomenological n-p potential.

We might enquire whether anything can be learned of the final-state interaction by a more modest programme. Jankus⁽⁸⁾ estimated it by using the Born approximation in the framework of potential theory. This is hardly reliable, however, as the Born approximation is only valid for large q , when the problem is no longer non-relativistic. However for large q we can use an interaction in dispersion theory corresponding to the Born approximation; namely we consider the lowest anomalous threshold in the double dispersion relations, corresponding to the graph of Fig. 33. This can be expected to give us a reliable estimate of the final-state interaction for large q .

The kinematics have been considered by Gourdin⁽³⁵⁾, and we will quote his results for completeness[†]. We will also consider the single dispersion relations again by the Cutkosky method and see how Durand's results arise from this approach.

* private communication.

† The formalism used for elastic scattering, eqns.(44)-(46) is based on the approach of this paper.

11. Kinematics.

As in the case of elastic scattering, we consider only single-photon exchange, so that we are primarily concerned with the virtual photo-disintegration matrix element.

$$M_{\mu} = \langle n p | j_{\mu} | d \rangle = \bar{u}(p) A_{\mu} v(n)$$

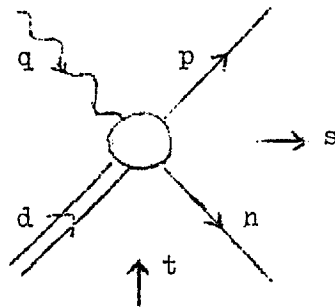


FIG.24.

The scalar invariants are defined as

$$s = (p+n)^2$$

$$t = (d-n)^2$$

$$u = (d-p)^2$$

In terms of CM variables

$$(136) \quad \begin{cases} s = 4E^2 = 4(m^2 + \underline{p}^2) \\ t - m^2 = -2(\alpha^2 + \underline{p}^2 - \frac{1}{4}q^2 - \underline{p} \cdot \underline{q}) \end{cases}$$

Further useful relations are

$$(137) \quad \begin{cases} q_0 = \frac{4(\underline{p}^2 + \alpha^2) + q^2}{2\sqrt{s}} \\ |\underline{q}|^2 = |\underline{d}|^2 = \frac{(q^2 - (\sqrt{s} - m_d)^2)(q^2 - (\sqrt{s} + m_d)^2)}{4s} \end{cases}$$

The connection with the lab. scattering angle and electron energies is

$$(138) \quad \left\{ \begin{array}{l} q^2 = -4E_e E'_e \sin^2 \frac{1}{2}\vartheta \\ \text{and } E'_e = \frac{E_e - 1/m (p^2 + a^2)}{1 + E_e/m \sin^2 \frac{1}{2}\vartheta} \end{array} \right.$$

The quasi-elastic peak in the cross-section (c.f. ⁽³⁶⁾ and § 14 (a)) occurs at $p^2 = -\frac{1}{4}q^2$, giving

$$(139) \quad E'_e = \frac{E_e - \varepsilon}{1 + 2E_e/m \sin^2 \frac{1}{2}\vartheta}$$

We will write $\hat{p} \cdot \hat{q} = \cos \varphi$, $q_-^2 = -q^2$, and p for $|p|$ where this can cause no confusion.

Following Gourdin ⁽³⁵⁾ we write

$$(140) \quad \left\{ \begin{array}{l} T_{\mu\nu} = \frac{1}{3} \sum_{\text{spins}} \langle np | j_\mu | d \rangle \langle d | j_\mu^+ | np \rangle \\ = g_{\mu\nu} J_0 + d_\mu d_\nu J_1 + \frac{1}{2} (n_\mu d_\nu + n_\nu d_\mu) J_3 + p_\mu p_\nu J_2, \end{array} \right.$$

since we may once again drop contributions proportional to q_μ .

Integration over an unobserved angle then gives the cross-section as

$$(141) \quad \frac{d^2\sigma}{dE'_e d(\cos\vartheta)} = \frac{e^2}{4\pi} \left(\frac{1}{8\pi} \right)^2 \frac{m_p}{E} \left(\frac{\cos^2 \frac{1}{2}\vartheta}{4E_e^2 \sin^4 \frac{1}{2}\vartheta} \right) (m_d^2 V_1 - 2V_0 \tan^2 \frac{1}{2}\vartheta),$$

$$\text{where } V_{0,1} = \int_{-1}^1 U_{0,1} dz$$

$$(142) \quad \left\{ \begin{array}{l} U_0 = J_0 + \frac{1}{2}(1-z^2)K_p^2 J_2 \\ U_1 = J_1 + \frac{1}{W^2} \left[\left(p_0 + \frac{K_p}{K_d} z q_0 \right)^2 - \frac{1}{2} q^2 \frac{K_p^2}{K_d^2} (1-z^2) \right] J_2 + \frac{1}{W} \left(p_0 + \frac{K_p}{K_d} z q_0 \right) J_3 \end{array} \right.$$

(The notation here is $W=2E$, $K_p = p$, $K_d = |\underline{d}|$, $z = -\cos\phi$)

Just as in the case of elastic scattering, essentially the same form factor serves to describe the annihilation process $e\bar{e} \rightarrow \bar{d}np$. Taking the cross-section as a function of E_p, E_n and $\cos\theta$, where θ is the angle between the line of collision and the normal to the production plane, we find, after integration over unobserved angles,

$$\frac{d\sigma}{dE_p dE_n d(\cos\theta)} = \frac{1}{4\pi} \left(\frac{e^2}{4\pi} \right)^2 \frac{1}{4q^4} \left[-2J_0 + \frac{1}{2}(1+\cos^2\theta) (J_1 K_p^2 + J_3 K_p K_n \cos\alpha + J_2 K_n^2) \right]$$

where α is the angle between $\underline{p}$ and $\underline{n}$. This formula, reinterpreted appropriately, gives the differential cross-section for annihilation into any three particles. In particular the form $H_{\mu rst} \epsilon_{\nu lmn} p^r n^s q^t p^l n^m q^n$ for $T_{\mu\nu}$ reproduces the cross-section for annihilation into three pions given by Cabibbo and Gatto (³⁷). Note once again the $C+D\cos 2\theta$ form of the cross-section.

Eqns. (142) simplify somewhat for special kinematical situations. At the peak we may neglect q_0 , while near the proton pole, which occurs at $\cos\phi = 1 + \frac{q^2}{2p^2}$, we may neglect $1-z^2$.

The equations then reduce to

$$(142') \quad \begin{cases} U_0 &= J_0 \\ U_1 &= J_1 + \frac{1}{4} J_2 + \frac{1}{8} J_3 \end{cases}$$

and $T_{\mu\nu}$ to

$$(140') \quad T_{\mu\nu} = g_{\mu\nu} U_0 + P_\mu P_\nu U_1,$$

where $P = n+p$ (Near the pole we may effectively put $d=2n$, c.f. §14(a))

12. The Physical and Spectral Regions.

We consider the Mandelstam diagram for Γ_μ . It has the rather unusual feature of an 'imaginary mass' iq_- , but this causes no difficulty, although the relation

$$s+t+u = 2m^2 + m_d^2 - q_-^2$$

causes the triangle of reference to shrink to a point for $q_-^2 = 2m^2 + m_d^2$ and thereafter to become inverted.

The boundary of the physical region is ⁽³⁸⁾

$$(143) \quad stu = m^2(m_d^2 + q_-^2)^2 - s(m_d^2 - m^2)(m^2 + q_-^2)$$

(Aids in drawing this curve are that it touches $u, t = (m_d + m)^2$, $s=0, 4m^2$ and crosses $u, t=0$ on the line $s = \frac{m^2(m_d^2 + q_-^2)^2}{(m_d - m)^2(m^2 + q_-^2)}$)

The nucleon poles are at $u, t = m^2$ and the anomalous thresholds in the single dispersion relations are at $u, t = t_0 = m^2 + 2\mu(\mu + 2\alpha)$.

A very useful variable is y , defined by

$$(144) \quad t - m^2 = 2(y - \alpha^2)$$

In terms of y the proton pole is at $y = \alpha^2$ and the lowest anomalous threshold at

$$(145) \quad y_0 = (\mu + \alpha)^2$$

The boundary of the double spectral function is given by the square diagram, Fig. 33, as ^(6a, 38)

$$(146) \quad J^2 = 16 \begin{vmatrix} 2m^2 & 2m^2 - m_d^2 & \mu^2 & 2m^2 - s \\ 2m^2 - m_d^2 & 2m^2 & m^2 + \mu^2 - t & 2m^2 + q^2 \\ \mu^2 & m^2 + \mu^2 - t & 2\mu^2 & \mu^2 \\ 2m^2 - s & 2m^2 + q^2 & \mu^2 & 2m^2 \end{vmatrix} = 0$$

in which we take the larger root in t .

The anomalous threshold t_0 is obtained by setting the minor

$$(147) \quad \begin{vmatrix} 2m^2 & 2m^2 - m_d^2 & \mu^2 \\ 2m^2 - m_d^2 & 2m^2 & m^2 + \mu^2 - t \\ \mu^2 & m^2 + \mu^2 - t & 2\mu^2 \end{vmatrix} = 0,$$

yielding (145)

Of interest later on will be the roots y_1, y_2 ($y_1 > y_2$) of (146) for $s = 4m^2 + q^2$. Setting $y = \alpha^2 + \mu^2 + 2\alpha\mu x$ we obtain after a certain amount of algebra:

$$(148) \quad x = \frac{\alpha\mu}{q^2} \pm \left[\left(1 + \frac{\mu^2}{q^2} \right) \left(1 + \frac{\alpha^2}{q^2} - \frac{q^2}{4m^2 + q^2} \right) \right]^{\frac{1}{2}}$$

For $q_-^2 \gg \mu^2$ we can expand the square-root, obtaining

$$(149) \quad \begin{cases} y_1 & \doteq (\alpha + \mu)^2 \left(1 + \frac{\alpha \mu}{q_-^2} \right) \\ y_2 & \doteq (\alpha - \mu)^2 \end{cases}$$

Also of interest is

$$(150) \quad s(t_0) = 4m^2 + q_-^2 + 4\alpha(\mu + \alpha) + \frac{q_-^2}{m^2} \alpha(\mu + 2\alpha)$$

And finally, we quote the value of s at which the boundary of the physical region touches $u = (m_d - m)^2$:

$$(151) \quad s = (m_d^2 + q_-^2) \frac{m}{m_d - m}$$

We show the general features of the physical and spectral regions in Fig. 25, for $q_-^2 = m^2$. Of particular note is the closeness of the two near $s = 4m^2 + q_-^2$, as already indicated in eqns. (150) and (151).

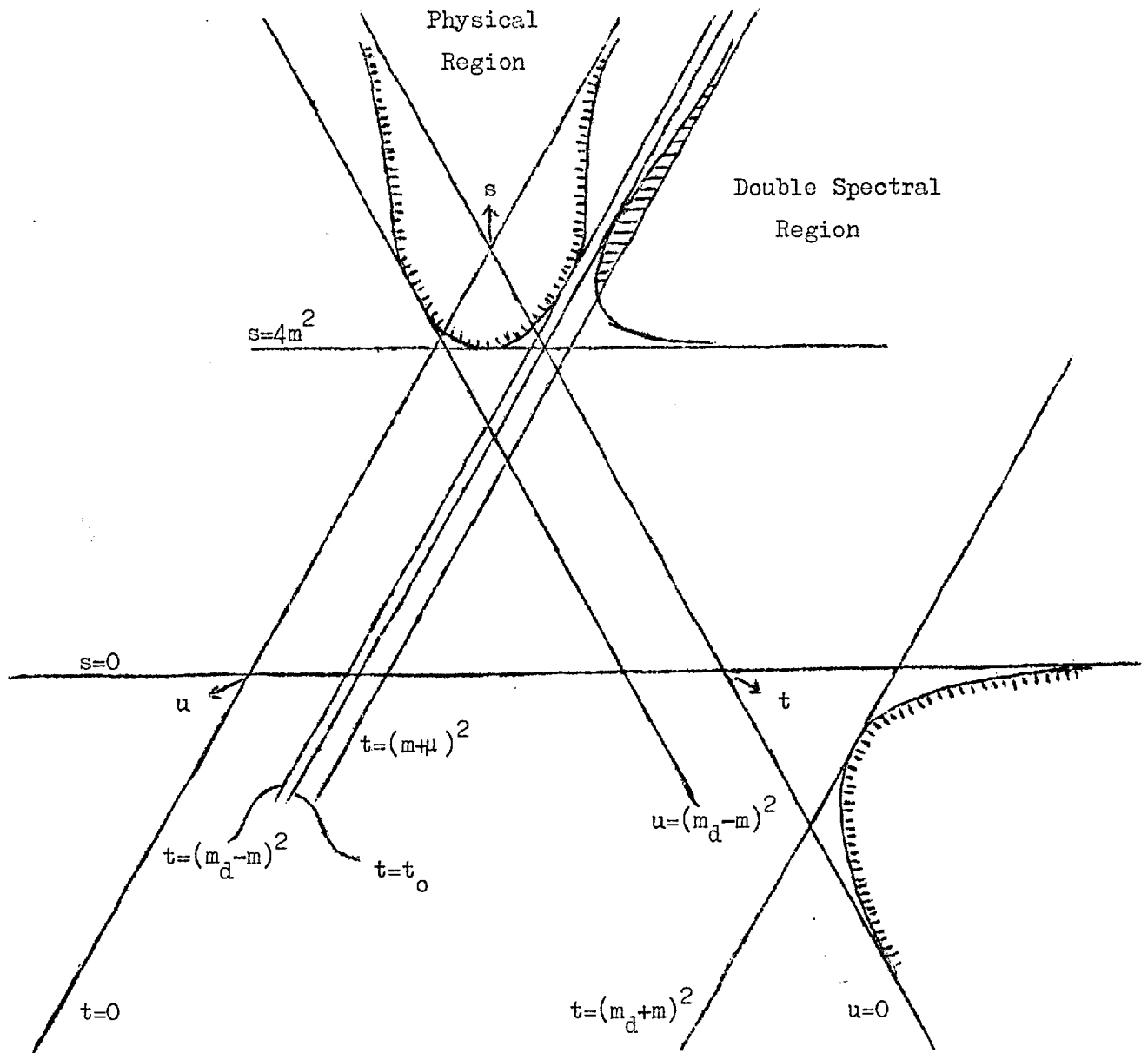


Fig.25

13. Partial Waves.

It does not seem to be generally realised that the cuts for the partial wave amplitudes can also be obtained very easily from the Mandelstam diagram, and we think it worth-while at this stage to demonstrate the procedure.

The partial-wave amplitude $A_1(s)$ is defined as

$$A_1(s) = \frac{1}{2} \int_{-1}^1 d(\cos\varphi) P_1(\cos\varphi) A(s, t)$$

Considering only the dispersion integral in t , we have

$$A_1(s) = \frac{1}{2} \int_{-1}^1 d(\cos\varphi) P_1(\cos\varphi) \int_{t_0} \frac{dt' \rho(s, t')}{t' - t(s, \cos\varphi)}$$

This last integral is singular when we have an end-point singularity i.e. when

$$(152) \quad t' = t(s, \pm 1)$$

The roots s_+ , s_- of this equation are clearly the s -values of the points in which the line $t=t'$ cuts the boundary of the physical region.

Thus to find the cuts we take search lines $t=t'$ for t' greater than the threshold of the spectral region. The roots s_+ , s_- then trace out the cuts in the s -plane. The cuts are actually lines of branch-points.

A similar procedure applies for the cuts due to the u -channel. In the present case the Mandelstam diagram is symmetric in u and t , so that this gives no extra information.

So for electrodisintegration we first consider the pole term $t'=m^2$.

This gives two complex conjugate roots very near to the value of s given in (151). Then for $t' \geq t_0$ we have two complex conjugate roots, whose real part decreases until the roots come together for $t' = (m_d + m)^2$ at $s = -(m_d^2 + q^2) \times \frac{m}{m_d + m}$. Thereafter the roots are real, one tending to zero, the other to $-\infty$. We thus get the cuts illustrated in Fig. 26.

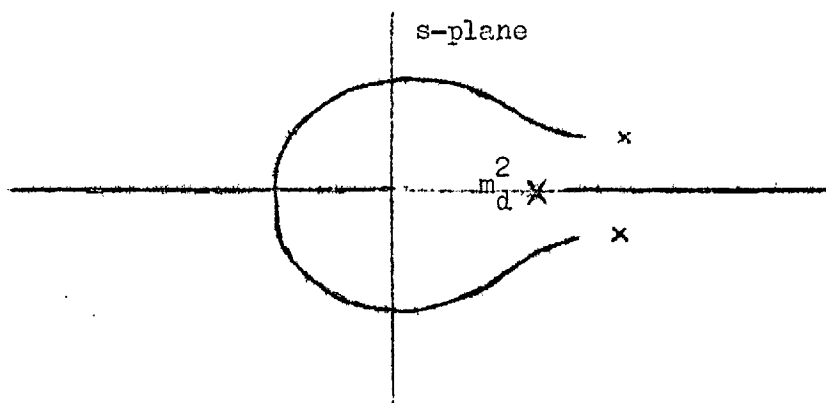


FIG.26

If we take q small and transform into the p -plane, the cuts in the neighbourhood of $s = 4m^2$ become those shown in Fig. 27, which are those quoted by Bosco ⁽³⁹⁾ and De Alfaro and Rossetti ⁽⁵⁾, and obtained from potential theory by a similar method to § 5(a)

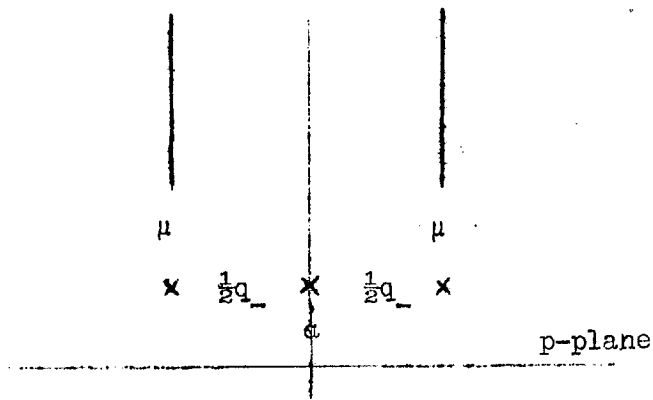


FIG.27

Bosco (⁴⁰) has attempted to use the dispersion relations associated with Fig. 27 to calculate the final-state interaction. However, this does not seem a very fruitful approach, as he fits the parameters from electrodisintegration near the break-up threshold and then applies the formula to electrodisintegration at the quasi-elastic peak, where the kinematics are vastly different (c.f. Loyal Durand's remarks at the recent Stanford Conference (⁴¹)).

We shall not attempt to use the partial-wave dispersion relations, as it seems to the author that the partial-wave approach is applicable for small q , where the potential theory formalism of Durand might be expected to give a good approximation for the final-state interaction. Durand's approach becomes cumbersome for high momentum-transfers, since we have to take into account many partial-waves.* This is where dispersion theory can be useful; it is then better not to split the amplitude into partial-waves.

Before leaving the subject of partial-waves we might mention that we have applied the present method to the examples considered by Kennedy and Spearman in their paper on partial-wave cuts for a general process (⁴²), and that it shows up clearly a slight mistake in the position of the cuts for photoproduction. The Mandelstam diagram is shown in Fig. 28.

* The conversion from q^2 to the corresponding lab. kinetic energy for np scattering is approximately $T = 20q^2$ where T is in MeV and q in f^{-1} .

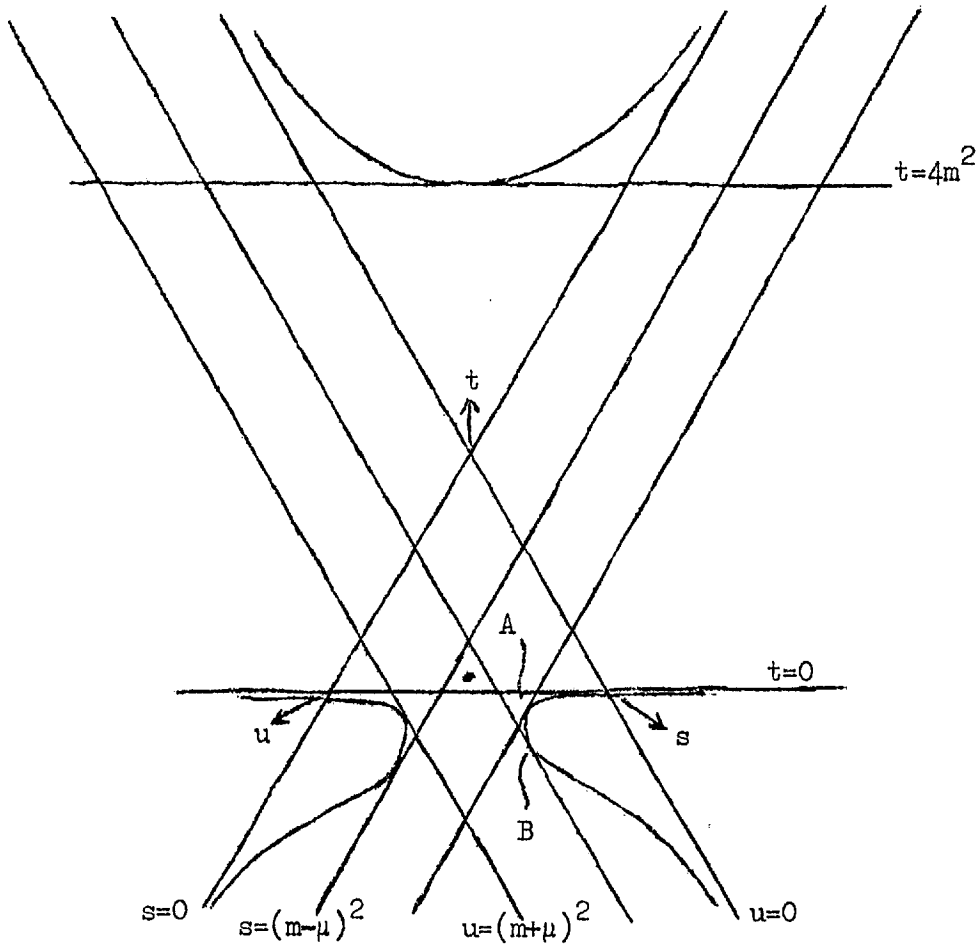


FIG. 28

We are concerned here with partial-waves in the u -channel due to the s -channel. Kennedy and Spearman give $u = \frac{m}{m+\mu} (m^2 - m\mu - \mu^2)$ as the upper end of the left-hand cut, corresponding to the point A where the two roots of u coincide. It is however clear from Fig. 28 that the upper end of the cut is in fact $u = (m - \mu)^2$, corresponding to the point B. This is because the larger root does not fall monotonically to zero, but

first increases to $(m-\mu)^2$, and then falls to zero. A similar mistake occurs in Fayazuddin's work on photoproduction of strange particles ⁽⁴³⁾.

We hope that this brief discussion of the graphical method of finding partial-wave cuts has been sufficient to show its utility, especially in getting the 'feel' of the process under discussion.

14. Single Dispersion Relations.

In this section we discuss the single dispersion relations, which lead to the Durand-Goldberg formula, and in particular the term in G_0 , which Durand neglects.

(a) Pole Terms

The contribution of the proton pole, Fig. 29, to M_μ is

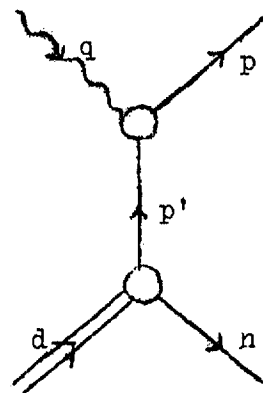


FIG. 29

$$(153) \quad M_\mu^\pi = -e\bar{u}(p)(F_m\gamma_\mu - F^2/mP_\mu)(\not{p}'+m)(F_0\not{\xi}+G_0\xi.n)v(n) \times \frac{1}{t-m^2}$$

We can write the denominator variously as

$$(154) \quad t-m^2 = 2(y-a^2) = -2p|\underline{q}|(x-\cos\phi), \quad x = \frac{a^2+p^2-\frac{1}{4}q^2}{p|\underline{q}|}$$

The position of the quasi-elastic peak ⁽¹⁰⁾ is obtained by

maximising $\int (t-m^2)^{-2} dz$. With the help of eqn. (137) and some algebra this gives

$$(155) \quad p^2 = \frac{1}{4}q^2 + \alpha^2 \left(1 + \frac{2p^2}{m^2} \right),$$

or, to a good approximation, $p = \frac{1}{2}q$.

With these kinematics we find from (154) that

$$(156) \quad y = -2p^2 (1 - \cos\phi)$$

Durand also exploited the weighting of the denominator in (153) towards small values of y by putting $d=2n$ in the numerator, which gets rid of the term in $\xi.n$, since $\xi.d=0$.

In the lab. frame we find yet another expression for $t-m^2$:

$$(157) \quad t-m^2 = -2(\alpha^2 + n^2) + O(n^4),$$

so that the above approximation is equivalent to putting $y=0$, or $z=-1$.

The numerator of (153) then simplifies considerably, giving

$$(153') \quad M_{\mu}^T = - \frac{2mF_0}{t-m^2} \bar{e}u(p) (F_m \gamma_{\mu} - F_2/m p_{\mu}) \not{z} v(n)$$

To get the cross-section Durand then essentially forms the $T_{\mu\nu}$ of (140') with this form of M_{μ}^T and finds that

$$(158) \quad \left\{ \begin{array}{l} V_0^T = 2\pi \frac{e^2}{m^2} \frac{N^2}{p^2 \alpha^2} q^2 F_m^2 \\ V_1^T = 2\pi \frac{e^2}{m^2} \frac{N^2}{p^2 \alpha^2} (F_1^2 - q^2/4m^2 F_2^2), \end{array} \right.$$

giving

$$(159) \quad \frac{d^2\sigma}{dE'_e d\Omega} = \frac{\sigma_{\text{mott}}}{4\pi} \frac{m^2}{pE} \frac{N^2}{a^2} G_p ,$$

where F_0 has been expressed in terms of N from (29) with neglect of ρ .

Eqn. (159) will be recognised as the proton tail contribution to the Durand-Goldberg formula (130).

However, great care must be exercised with the term in G_0 . Here we have neglected it on the grounds that $\xi.n = 0$. However, in view of the relative sizes of F_0 and mG_0 , eqn. (29), we see that $\xi.n$ has only to be of the order a^2/m for the G_0 term to be non-negligible in comparison with that in F_0 .

Let us therefore examine the form of $T_{\mu\nu}$ without making the above approximation.

After some algebra we find that

$$(160) \quad \int T_{\mu\nu}^T dz = \int \frac{dz}{(t-m^2)^2} \frac{e^2}{12m^2} \text{Tr}[(\not{p}+m)(F_m\gamma_\mu - F^2/m^2 p_\mu)N^T(F_m\gamma_\nu - F^2/m^2 p_\nu)]$$

where

$$(161) \quad N^T = 2(F_0 - mG_0) \left\{ 2(F_0 - mG_0)n_2^2 (\not{p}'+m) \right. \\ \left. + 2(y-a^2) \left[F_0 \left(\frac{y-a^2}{m_d^2} \not{p} - \frac{\not{k}}{2} \right) - G_0 n_2^2 \right] \right\} \\ + (3F_0^2 - G_0^2 n_2^2) \left\{ (\not{p}'+m)(m_d^2 + m^2 - t) - (\not{p}-m)(t-m^2) \right\}$$

and we have written $k=d - 2n$. n_2^2 is given in Eqn. (21)

The leading terms in (161) are

$$(162) \quad N^T \doteq (3F_o^2 m_d^2 - 8m_y F_o G_o) (y+m)$$

The combination of F_o and G_o in this equation will be recognised as essentially the S-wave part of eqn. (24). Using the values of F_o, G_o given in (29) we find

$$(163) \quad N^T = 3m_d^2 N^2 \frac{8\pi}{m} (y+m) [1 + \rho\sqrt{2} (1 - \frac{y}{\alpha^2})]$$

From this last equation we see that the approximation which in fact leads to (159) is not $y = 0$, but $y = \alpha^2$ or $t = m^2$. As we have seen, the approximation $y = 0$ must be supplemented by a neglect of ρ to give the correct form of (159).

If we do not make the approximation $y = \alpha^2$, but simply evaluate (160) with the form (163) for N^T , we find, using the relations

$$(164) \quad - \int_0^{q^2} \frac{dy}{(y-\alpha^2)^2} \doteq \frac{1}{\alpha^2}, \quad - \int_0^{q^2} \frac{dy}{y-\alpha^2} = \log \frac{\alpha^2}{q^2},$$

that the factor α^{-2} in (159) is replaced by $\lambda \alpha^{-2}$, where

$$(165) \quad \lambda = 1 + \rho\sqrt{2} \log \frac{q^2}{\alpha^2},$$

which represents quite an appreciable correction to (159).

This is another example of a difficulty of the type found in § 2(a). We have put in the mass-shell forms of the vertices in Fig. (29), and yet the spinor factors and traces have produced terms in $(t-m^2)$ in the numerator. Any terms in which a $(t-m^2)$ appears should clearly be considered in conjunction with the single dispersion-relations, which

in dispersion theory represent off-mass-shell corrections to the pole terms. Since in the contribution from the pole term we have F_0, G_0 etc., it seems in the spirit of this approach to put $t = m^2$ in the remaining terms.

The ambiguity in the present case is emphasized by two factors, firstly that the coefficient of these extra terms is large, and secondly that in the integration over the angle ϕ we are integrating over values of t far away from the pole. The latter difficulty can be avoided by not integrating over ϕ , ie. by making coincidence experiments of the type suggested by Durand in his first paper (³⁶) and recently carried out at Cornell (⁴⁴). These experiments pick out the kinematical situations in which first the proton and then the neutron is a spectator particle, corresponding to $z = -1, +1$. In this way the experiments are performed close to the poles $t, u = m^2$. It is perhaps significant that the form factors obtained from these experiments differ from earlier determinations using the Durand formula.

Returning to the present problem, we regard the term in $(y-a^2)$ in eqn. (163) as a contribution from the non-asymptotic wave-function, which we will now discuss.

First we must mention that the treatment of the neutron pole term is exactly similar, and will give the neutron tail contribution to (130). The n-p interference term is known to be entirely negligible ^{at} the peak (¹⁰) and will be dropped.

(b) Anomalous Threshold Cuts

The next most important contribution to the single dispersion integrals comes from the anomalous threshold diagram, Fig. 31, which is a special case of the general reduced graph, Fig. 30.

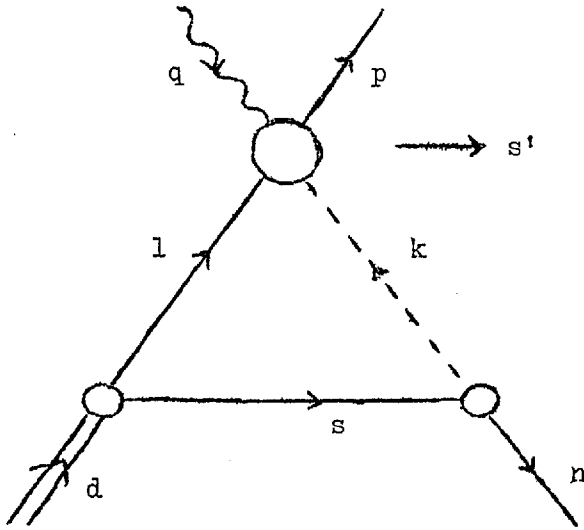


FIG. 30

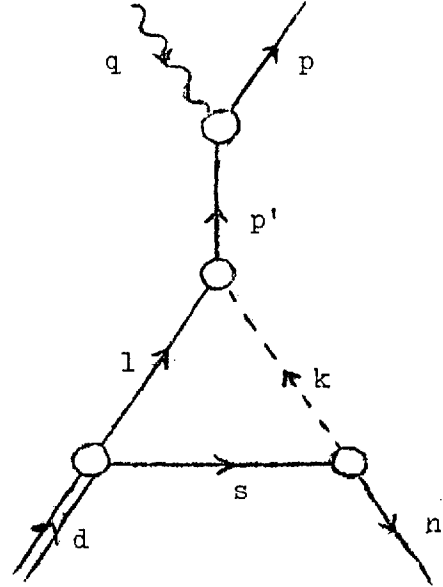


FIG. 31

The contribution of this diagram is

$$\begin{aligned}
 \text{Im}^A M_\mu &= -\frac{3g^2}{32A'} \cdot \frac{1}{2\pi} \oint \frac{d\psi}{t'-m^2} e\bar{u}(p)(F_m \gamma_\mu - F_{2/m} p_\mu)(\not{p}'+m)\gamma_5(\not{\chi}+m) \times \\
 &\quad \times (F_0 \not{\xi} + G_0 \xi \cdot s)(\not{s}-m)\gamma_5 v(n) \\
 (166) \quad &= -e\bar{u}(p)(F_m \gamma_\mu - F_{2/m} p_\mu) \frac{\not{p}'+m}{t'-m^2} \times \\
 &\quad \times \frac{3g^2}{32A'} \cdot \frac{1}{2\pi} \oint d\psi (\not{\chi}-m)(F_0 \not{\xi} + G_0 \xi \cdot 1)\not{v}(n)
 \end{aligned}$$

Eqn. (166) clearly factorises, the second factor being the imaginary part of the np-d vertex with the proton off the mass-shell (§ 2(b)). i.e.

$$(167) \quad \text{Im} M_{\mu}^A = \bar{e}u(p) (F_m \gamma_{\mu} - F_2/m p_{\mu}) \frac{\not{p}+m}{t'-m^2} \text{Im} \Gamma(y') v(n)$$

A discussion of ordinary singularities similar to that of § 6(b) then enables us to write

$$(168) \quad M_{\mu}^A = \bar{e}u(p) (F_m \gamma_{\mu} - F_2/m p_{\mu}) \frac{\not{p}+m}{t-m^2} \Gamma(y) v(n),$$

in which

$$-(\not{p}'+m)\Gamma(y) = [(\not{p}'+m)F(t) - (t-m^2)H(t)] \not{\epsilon} + [(\not{p}'+m)G(t) - (t-m^2)I(t)] \not{\xi}_n$$

Durand's result then follows by neglecting the term in $\not{\xi}_n$, putting $\not{p}'=m$ and writing

$$(169) \quad D(t) = \frac{2mF - (t-m^2)H}{2mF_0(t-m^2)} = \frac{1}{t-m^2} - \frac{1}{t-v^2},$$

where $v^2 = m^2 + 2(\beta^2 - \alpha^2)$.

This form of $D(t)$ is chosen since it corresponds to the Hulthén wave-function ⁽⁴⁵⁾

$$(170) \quad \varphi = \frac{N}{\sqrt{4\pi}} \frac{e^{-\alpha r} - e^{-\beta r}}{r}$$

The whole of the cut for Γ has been replaced by a suitably-placed pole. It appears that the cross-section is fairly insensitive to the form of wave-function used (unlike elastic scattering), so that the Hulthén wave-function offers an adequate description of the scattering.

The combined contributions of (153') and (168) then give

$$(171) \quad M_{\mu}^I = -2mF_o D(t) \bar{u}(p)(F_m \gamma_{\mu} - F_2/mP_{\mu})v(n),$$

or equivalently

$$(172) \quad N^I = 3F_o^2 m_d^2 \left(1 - \frac{y-\alpha^2}{y-\beta^2}\right) (\not{y}+m),$$

which, neglecting ρ , gives the proton term in (130).

The essential simplifications lie in the form of $D(t)$, eqn. (169), which seems to be justified by the wave-function approach, and the transition, whose accuracy is unfortunately not easy to estimate, from (167) to (168). Any attempt to perform the dispersion integrals for the invariant amplitudes of M_{μ} would involve considerable difficulties, notably unknown subtraction constants.

The treatment of the neutron terms, or the dispersion-relations in u , proceeds on exactly similar lines. Once again we neglect the n - p interference term. The final result is eqn. (130).

15. Double Dispersion Relations.

We will confine our attention to the square diagram, Fig. 33, which is another special case of Fig. 30.

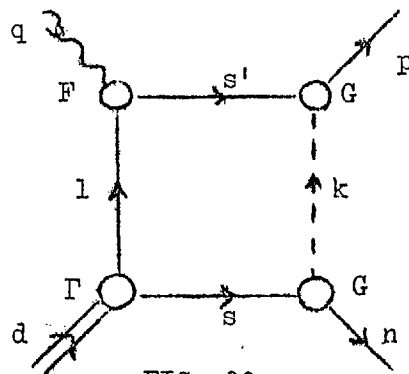


FIG. 33

Let us first ignore the complications due to spin.

(a) Scalar Case.

Writing

$$(173) \quad M^{II}(s, t) = \frac{1}{\pi} \oint_{t_0} \frac{dt'}{t' - t} A_2(s, t')$$

the contribution to A_2 from Fig. 33 is :

$$(174) \quad A_2(s, t) = \frac{FG^2}{32A'} \frac{1}{2\pi} \oint \frac{d\psi}{s' - m^2 - i\epsilon}$$

We find from the dual diagram that $(s' - m^2)$ is of the form

$$s' - m^2 = a + b \cos \psi ,$$

so that the circular integral $I_0 = \frac{1}{2\pi} \oint \frac{d\psi}{s' - m^2 - i\epsilon}$ is

$$(175) \quad I_0 = \begin{cases} (a^2 - b^2)^{-\frac{1}{2}} , & a^2 > b^2 \\ i(b^2 - a^2)^{-\frac{1}{2}} , & b^2 > a^2 \end{cases}$$

Fig.34 shows the part of the double spectral region of present interest.

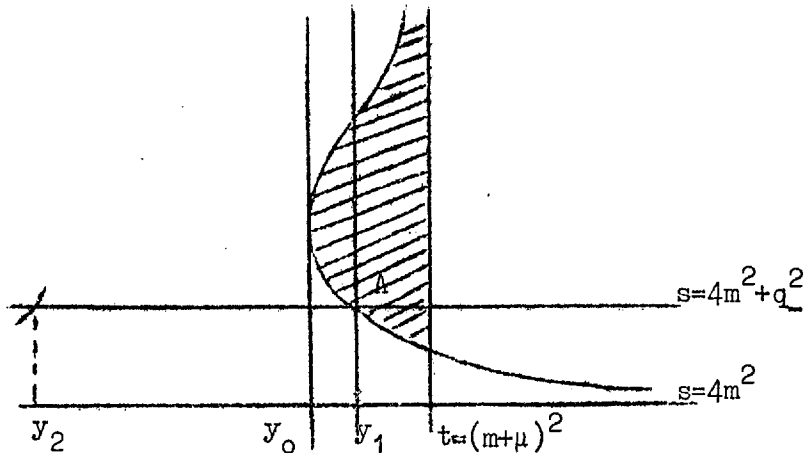


FIG. 34

The singular point $b^2 = a^2$ where I_0 becomes imaginary is obviously A, the point where the line of constant s , $s = 4m^2 + q_-^2$, meets the boundary of the double spectral region. The important feature of eqn. (175) is that the principal part of $A_2(s, t')$ vanishes inside the double spectral region.

This enables us to express $A_2(s, t')$ in terms of $a(s, t')$.

$$\text{For } A_2(s, t') = \frac{1}{\pi} \int_{s_-}^{s_+} \frac{ds'}{s' - s - i\epsilon} a(s', t'),$$

from which it is clear that

$$\text{Im} A_2(s, t') = a(s, t') \quad \text{for } s_- < s < s_+$$

(s_-, s_+ are the s -values of the points in which the line of constant t cuts the double spectral region).

Eqn. (175) thus shows that

$A_2(s, t') = ia(s, t')$ inside the double spectral region, and further that outside the double spectral region $A_2(s, t')$ is the analytic continuation of $ia(s, t')$.

This enables us to express $A_2(s, t')$ in terms of the simple form $(6a)$ for $a(s, t')$:

$$\begin{aligned} a(s, t') &= \frac{1}{4J}, \text{ where } J \text{ is given by (146)} \\ (176) \quad \therefore A_2(s, t') &= \frac{F \Gamma G^2 i}{4J} \end{aligned}$$

From (146) and specialising to $s = 4m^2 + q_-^2$,

$$(177) \quad J^2 = 64q_-^2 (4m^2 + q_-^2)(y - y_1)(y - y_2)$$

so that finally

$$(178) \quad \text{II} \quad \text{M}(s, t) = \Lambda \int_{y_0}^{y_1} \frac{dy'}{y' - y} \frac{1}{\sqrt{(y_1 - y')(y' - y_2)}} ,$$

where

$$(179) \quad \Lambda = \frac{\pi \Gamma G^2}{32 \pi \sqrt{q^2(4m^2 + q^2)}}$$

There is a fundamental difference between (178) and (167), in that the factor Λ' no longer remains; it has been absorbed in $\sqrt{(b^2 - a^2)}$ to give J . Hence we can no longer avoid actually performing the dispersion integrals by an argument of the kind leading from (167) to (168). (This does not seem to have been realised by Cutkosky, who does apply such an argument to the similar case of π -d scattering ^(6b)).

Let us therefore proceed with the evaluation of eqn. (178). Since we will only require the interference term of (178) with (171), we need only ReM^{II} ; so that the integration on y' now runs up to y_1 :

$$(180) \quad \text{Re M}^{\text{II}}(s, t) = \Lambda \int_{y_0}^{y_1} \frac{dy'}{y' - y} \frac{1}{\sqrt{(y_1 - y')(y' - y_2)}}$$

Changing the variable of integration to u , given by $u^2 = \frac{y_1 - y'}{y' - y_2}$, yields

$$(181) \quad \text{ReM}^{\text{II}}(s, t) = \Lambda \int_0^{u_0} \frac{2du}{L_1 + u^2 L_2} , \text{ where } L(y') = y' - y$$

Hence

$$\text{ReM}^{\text{II}}(s, t) = \frac{2\Lambda}{\sqrt{(L_1 L_2)}} \tan^{-1} u_0 \sqrt{\frac{L_2}{L_1}}$$

From (148) we find that

$$(183) \quad u_o^2 \doteq \frac{1}{4} \left[\frac{(\alpha+\mu)^2}{q^2} + \frac{(2\alpha+\mu)^2}{4m^2} - \frac{\alpha\mu}{4m^2+q^2} \right] ,$$

which is small for momentum-transfers for which the Born approximation in the final state can be expected to be valid, viz. $p \gg \mu$.

Eqn. (182) then reduces to

$$(184) \quad \text{Re} M^{\text{II}}(s, t) = \frac{2\Lambda u_o}{L_1} ,$$

which is negligible compared to the contribution from the single dispersion integral :

$$M^A(s, t) = \frac{F\Gamma^2}{2L(\alpha^2)} \cdot \frac{1}{32m_d} \cdot \frac{2}{\sqrt{-y}} \tan^{-1} \sqrt{\frac{-y}{y_o}} ,$$

the determining factor being the very close proximity of y_o and y_1 as displayed in (149).

If we do not wish to make the approximation leading to (184), the interference term of M^{II} and M^{I} is best evaluated by using the form (181) and performing the y -integration first. The u -integration can then by a change of scale be transformed into the form

$$I = \frac{\lambda}{\sqrt{(x_1 x_2)}} \int_0^{v_o} \frac{dv}{1+v^2} \log \frac{a^2+b^2v^2}{c^2+d^2v^2} ,$$

where we have written $x = y - \alpha^2$. This last integral can be expressed as a sum of imaginary parts of Spence functions of complex arguments (Clausen functions).

(b) Spinor Case.

Eqn. (174) is now replaced by

$$\begin{aligned}
 \bar{u}(p)A_2(s, t')v(n) &= \frac{-3g^2}{32A'} \frac{1}{2\pi} \oint \frac{d\psi}{s' - m^2 - i\epsilon} \bar{u}(p) \not{F}_m \gamma_\mu + \not{F} 2/m^1 \mu (\not{\chi} - m) (\not{F}_0 \not{\xi} - G_0 \xi \cdot s) \not{v}(n) \\
 (185) \qquad &= \frac{-3g^2}{32A'} \frac{1}{2\pi} \oint \frac{d\psi}{s' - m^2 - i\epsilon} \bar{u}(p) \not{F}_m \gamma_\mu + \not{F} 2/m^1 \mu P v(n)
 \end{aligned}$$

where

$$\begin{aligned}
 (186) \qquad P &= [2(F_0 - mG_0) \not{F} + (2k \cdot l + \mu^2) G_0 - G_0 \not{k}] \xi \cdot l \\
 &\quad - F_0 [(2k \cdot l + \mu^2) + \not{k} \not{l}] \not{\xi} + 2F_0 \xi \cdot k \not{l}
 \end{aligned}$$

The circular integrations in (185) can now in principle be performed, yielding the imaginary parts of the invariant amplitudes of M_μ . However, since the integrations involve up to four powers of h , they will involve considerable algebraic difficulties. Moreover it is not at all clear which are the amplitudes free from kinematic singularities (c.f. (46)).

These difficulties can be obviated by using (185) in conjunction with (153) to obtain the contribution of the interference term of the pole diagram and the square graph to the imaginary parts of V_0 and V_1 . We take advantage of the fact that the operations of circular integration, summation over ξ and taking the trace commute, and perform the last two operations first. In the circular integrations then remaining we have only two powers of h .

Neglecting the term in $\xi \cdot n$, the numerator of the pole term is

$$-2emF_0 \bar{u}(p) (\not{F}_m \gamma_\nu - \not{F} 2/m^1 \nu) \not{\xi} v(n).$$

We therefore have to perform the trace :

$$(187) \quad S_{\mu\nu} = -2emF_0 \sum_{\xi} \text{Tr} [(\not{p} + m) \not{\epsilon} (F_m \gamma_{\mu} + \frac{F^2}{m} 1_{\mu}) P (\not{p} - m) \not{\epsilon} (F_m \gamma_{\nu} - \frac{F^2}{m} P_{\nu})]$$

The summation over ξ yields

$$(188) \quad \sum_{\xi} P (\not{p} - m) \not{\epsilon} = - (A - B \not{\epsilon}) (\not{p} + m),$$

where

$$(189) \quad \left\{ \begin{array}{l} A = 2F_0 [2(y' - a^2) + \mu^2] - \frac{2}{m_d} (y' - a^2)^2 G_0 \\ \text{and } B = 2mF_0 - 2(y' - a^2) G_0 \end{array} \right.$$

and then, performing the trace, we find after some algebraic endeavour :

$$(190) \quad S_{\mu\nu} = 8emF_0 \left\{ A [m^2 F_m^2 q \cdot k_{\mu} \epsilon_{\mu\nu} \frac{1}{2} \frac{F^2}{2m} P_{\mu} P_{\nu} (F_1 P \cdot k + F_m q \cdot k) \right. \\ \left. - \frac{1}{2} q^2 F_m^2 \frac{F^2}{2m} P_{\mu} k_{\nu} + m^2 F_m^2 \frac{F^2}{2m} P_{\mu} P_{\nu}] \right. \\ \left. - (2F_m B - \frac{F^2}{m^2} A) (\frac{1}{2} F_1 k \cdot P P_{\nu} + \frac{1}{2} F_m q^2 k_{\nu}) k_{\mu} \right. \\ \left. + B \mu^2 [(F_1^2 - \frac{q^2}{4m^2} F_2^2) P_{\mu} P_{\nu} + q^2 F_m^2 g_{\mu\nu} + F_2^2 F_m^2 k_{\mu} P_{\nu}] \right\}$$

Then

$$(191) \quad \text{Im} T_{\mu\nu} = \frac{-3g^2}{32A'} \cdot \frac{1}{12m^2} \cdot \frac{1}{t-m^2} \cdot \frac{1}{2\pi} \oint d\psi S_{\mu\nu}$$

The circular integrals remaining are of the form

$$I_{mn} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi (\cos\psi)^m (\sin\psi)^n}{a + b \cos\psi - i\epsilon}$$

I_{mn} evidently vanishes for n odd. Writing $I_m = I_{m0}$ we can express $I_{m,2n}$ in terms of the I_r by expansion of $(\sin\psi)^{2n}$, giving

$$I_{m,2n} = \sum_{r=0}^n (-1)^r \binom{n}{r} I_{m+2r}.$$

The I_m are evaluated by writing $(\cos\psi)^m$ as $[(a+b\cos\psi) - a]^m b^{-m}$ and expanding the binomial. I_0 is given in eqn. (175), while

$$(192) \quad \begin{cases} I_1 &= \frac{1}{b} - \frac{a}{b} I_0 \\ I_2 &= -\frac{a}{b} I_1, \quad I_{02} = I_0 - I_2 \end{cases}$$

With the aid of these equations and the geometry of Fig. 32 we finally obtain

$$(193) \quad \begin{cases} \text{Im } U_0 &= \frac{e^2 F_0 g^2}{32m^2 \sqrt{y'}(y-a^2)} F_m^2 \frac{q^2}{4\lambda^2} [2Am(1 + \frac{q^2}{4m^2}) - aB] \\ \text{Im } U_1 &= \frac{e^2 F_0 g^2}{32m^2 \sqrt{y'}(y-a^2)} \left[\frac{A}{2\lambda^2} \frac{q^2}{4m^2} (F_1 F_c - F_m (F_m F_c)) + \frac{8\mu^2}{2\lambda^2} F_2 F_c \frac{q^2}{4m^2} \right. \\ &\quad \left. + \frac{F_m B}{1+q^2/4m^2} \left\{ F_c \left(\frac{k_1}{2m} + \frac{a}{4\lambda^2} \frac{q^2}{4m^2} + \frac{k_2 k_1}{2m_d n_2} \right) + \frac{q^2}{4m^2} F_m \left(\frac{a}{4\lambda^2} - \frac{k_2 k_1}{2m_d n_2} \right) \right\} \right] \end{cases}$$

where, in view of the considerations of the previous section, we have neglected contributions proportional to I_0 .

A and B have already been given in (189), while approximate expressions for k_1, k_2 a and λ are obtained from Fig. 32 (c.f. Appendix D) as

$$(194) \left\{ \begin{array}{lcl} k_1 & = & \frac{y' - \alpha^2}{m_d} \\ k_2 & = & - \frac{y' + \mu^2 - \alpha^2}{2n_2} \\ a & = & \frac{1}{n_2^2} \left[y'(\alpha^2 + \mu^2 - y') - \frac{q_-^2}{m_d^2} (y' - \alpha^2)(y' - \mu^2 + \alpha^2) \right] \\ \text{and } \lambda^2 & = & - q_-^2 \left(1 + \frac{q_-^2}{4m^2} \right) \end{array} \right.$$

A comparison of (193) with a typical contribution to $\text{Im } U_0$ from (167):

$$\text{Im } U_0 = \frac{e^2 F_0^2 g^2}{32m \sqrt{y'}(y' - \alpha^2)} \frac{q_-^2}{y' - \alpha^2} F_m^2 \left[\mu^2 + \frac{(y' - \mu^2)^2}{4y'} \right]$$

shows that the ratio of the two contributions is of the order μ^2/q_-^2 , which must be small for the use of the square diagram as a final-state correction to be meaningful. Similarly, the ratio of the respective contributions to $\text{Im } U_1$ is of order μ^2/m^2 , which is small.

Thus we see that the final-state correction is small for two main reasons. Firstly, the range of integration for I_0 is very small, and secondly the terms not involving I_0 do not have the factor $(y' - \alpha^2)^{-1}$ which appears in the single dispersion integrals and which makes this contribution appreciable.

16. Final-State Interaction and Intersecting Ellipses.

We would like to close this Part with a brief discussion of the effect of final-state interaction on the method of intersecting ellipses, which is widely used to obtain form factors from the Rosenbluth formula (61)

and (62) and the deuteron break-up formula (130).

Taking (62) as an example we see that this is the equation of an ellipse in (F_c, F_m) space whose characteristics are functions of E and θ .

$$\text{i.e. } C(E, \theta) = A(E, \theta)(F_c(q^2))^2 + B(E, \theta)(F_m(q^2))^2$$

Choosing several values of E and θ which give the same value of q^2 gives us several ellipses, which should have the common intersections $(\pm F_c, \pm F_m)$. It appears ⁽⁴⁷⁾ that with the present analysis using eqn. (130) the ellipses fail to intersect in a common point, and one may enquire if a final-state correction can rectify this. To answer this question it is easier to talk in terms of $\tan^2 \frac{1}{2} \theta$ plots, whose relation to intersecting ellipses we will explain.

The other way of obtaining F_c and F_m from (62) is to plot the ratio $R = \frac{\frac{d\sigma}{d\Omega}}{\sigma_{NS}}$ against $\tan^2 \frac{1}{2} \theta$. Clearly this should give a straight line with intercept R_0 on the R-axis given by

$$(195) \quad R_0 = \frac{\frac{F_c^2}{c} - q^2/4m^2 F_m^2}{1 - q^2/4m^2},$$

and slope m given by

$$(196) \quad m = 2F_m^2 \frac{q^2}{4m^2}$$

If the experimental points do not lie on a line then clearly the cross-section is not of the form $A(q^2) + B(q^2)\tan^2 \frac{1}{2} \theta$, and this corresponds to the ellipses having no common point of intersection. The final-state interaction cannot, whatever its nature, affect the $A+B \tan^2 \frac{1}{2} \theta$ form,

which is simply a consequence of single-photon exchange.

Even if the experimental points^{do} lie on a line, this is still not necessarily compatible with the Rosenbluth form ⁽⁴⁸⁾. For real F_c and F_m we must of course have R_0 and m positive, and further

$$(197) \quad m \lesssim 2(1 - q^2/4m^2) R_0.$$

A slope greater than this value corresponds to imaginary intersections of the ellipses. The final-state interaction can reduce the slope of the graph to a permissible value, i.e. it can make imaginary points of intersection real.

Since the non-intersection of ellipses is not affected by the final-state interaction it seems that we must look elsewhere for the reason. A possible reason lies in enhanced two-photon effects, which are currently under investigation in polarisation experiments at Orsay ⁽⁴⁹⁾ and in e^+ , e^- -p experiments at Stanford ⁽⁵⁰⁾. The size of these effects is not yet certain.

Appendix A

Writing $\alpha^2 = m\varepsilon$, where ε is the deuteron binding-energy, 2.225 MeV, approximate values for the various masses are

$$m_d = 9.5 \text{ f}^{-1}$$

$$m = 4.75 \text{ f}^{-1}$$

$$\mu = 0.71 \text{ f}^{-1}$$

$$\alpha = 0.23 \text{ f}^{-1}$$

These can be converted to MeV by $1\text{f}^{-1} = 197.2 \text{ MeV}$. A useful rough conversion into pion masses is obtained by noting that $\mu^2 \doteq \frac{1}{2} \text{ f}^{-2}$.

The deuteron static properties are

$$\mu_d = 0.857 \text{ }^e/2m$$

$$Q = 24.6 \text{ }^e/m_d^2$$

The asymptotic D-S ratio ρ is usually taken as

$$\rho = 0.03$$

Finally the triplet range

$$r_e = 1.7\text{f}$$

determines the asymptotic normalisation constant N^2 , which in turn determines the parameter β in the Hulthén wave-function (170) as

$$\beta = 1.43\text{f}^{-1}$$

Appendix B.

In Fig. B1 we show the plane ABC, in which we take axes as indicated.

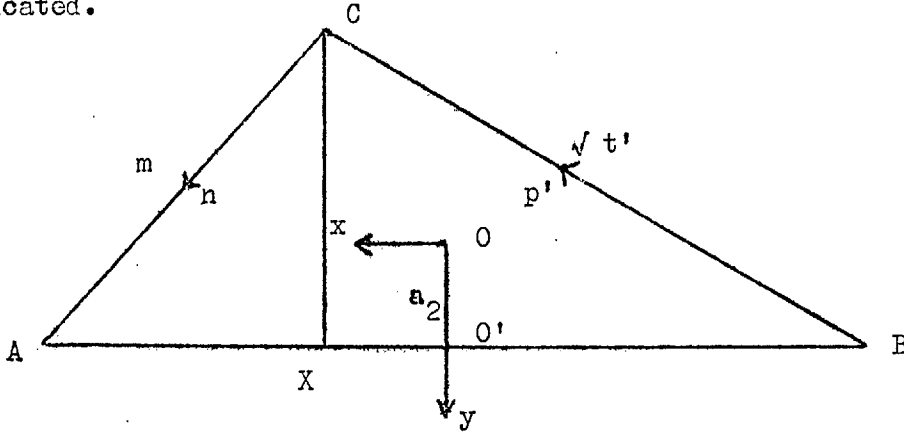


FIG. B1

From the plane geometry alone, knowing that O is equidistant from A and B, we find

$$(B1) \quad c_1 = \frac{y' - a^2}{m_d}$$

$$(B2) \quad n_2 \cong \sqrt{y'}$$

a_2 and κ are now determined by

$$(B3) \quad \begin{cases} m^2 &= a_2^2 + \frac{1}{4} m_d^2 + \kappa^2 \\ \mu^2 &= (a_2 - \sqrt{y'})^2 + \frac{(y' - a^2)^2}{m_d^2} + \kappa^2, \end{cases}$$

whence

$$(B4) \quad a_2 = \frac{y' - \mu^2 + a^2}{2\sqrt{y'}}$$

$$(B5) \quad \kappa^2 = \frac{((a+\mu)^2 - y')(y' - (a-\mu)^2)}{4y'}$$

From (B2) and (B4) we find

$$(B6) \quad c_2 = - \frac{y' + \mu^2 - a^2}{2\sqrt{y'}}$$

The figure is now completely determined, and we are in a position to find the projections of the various vectors on to ABC.

$$(B7) \quad \hat{k} = (c_1, c_2) \\ \cong \frac{y' + \mu^2 - a^2}{4y'} (p' - n)$$

We only need $\hat{l}$ in the combination $\xi \cdot \hat{l}$; this is easy to calculate, since

$$(B8) \quad \xi \cdot \hat{l} = \xi \cdot (d - \hat{s}) \\ = - (\xi \cdot n + \xi \cdot \hat{k})$$

Finally we need the projection of ξ . This is obtained from the supplementary condition $\xi \cdot d = 0$, giving

$$(B9) \quad \hat{\xi} = (0, \frac{\xi \cdot n}{n_2}) \\ = \frac{\xi \cdot n}{2y'} (n - p')$$

The equations developed now enable us to evaluate the circular integrals in eqn.(34).

while the projections of the polarisation pseudovectors, obtained from $\xi \cdot d$
 $= \xi'^* \cdot d' = 0$ are

$$(C6) \quad \begin{cases} \hat{\xi} &= \xi \cdot q \left(\frac{1}{2m_d \cos \varphi}, \frac{1}{\sqrt{s}} \right) = \xi \cdot q \left(\frac{D}{4m_d^2 \cos^2 \varphi} + \frac{q}{s} \right) \\ \hat{\xi}'^* &= \xi'^* \cdot q \left(-\frac{1}{2m_d \cos \varphi}, \frac{1}{\sqrt{s}} \right) = \xi'^* \cdot q \left(\frac{-D}{4m_d^2 \cos^2 \varphi} + \frac{q}{s} \right) \end{cases}$$

Then, forming the various scalar products, we have

$$(C7) \quad \begin{cases} \hat{\xi} \cdot \hat{k} &= \frac{\xi \cdot q}{4 \cos^2 \varphi} \\ \hat{\xi}'^* \cdot \hat{k} &= \frac{-\xi'^* \cdot q}{4 \cos^2 \varphi} \end{cases}$$

$$(C8) \quad \hat{\xi}'^* \cdot \hat{\xi} = \frac{\xi \cdot q \xi'^* \cdot q \cos 2\varphi}{4m_d^2 \cos^2 \varphi \sin^2 \varphi}$$

We are now able to evaluate the circular integrals of (84) and (87).

Using eqns. (33) we find that

$$\frac{1}{2\pi} \int d\psi \xi \cdot k \xi'^* \cdot k P_\mu = \hat{P}_\mu (\xi \cdot k \hat{\xi}'^* \cdot k + \frac{1}{2} \chi^2 \check{\xi}'^* \cdot \check{\xi}) - \chi^2 (\check{\xi}'^* \cdot \hat{\xi} \cdot k + \check{\xi}_\mu \xi'^* \cdot \hat{k}),$$

where we have written $\check{q} = q - \hat{q}$ for any vector q :

Thus

$$\begin{aligned} \frac{1}{2\pi} \int d\psi \xi \cdot k \xi'^* \cdot k P_\mu &= \frac{1}{2} \chi^2 \hat{P}_\mu \xi'^* \cdot \xi - \chi^2 (\xi'^* \cdot \hat{\xi} \cdot k + \xi_\mu \xi'^* \cdot \hat{k}) \\ &\quad + \hat{P}_\mu \xi \cdot k \xi'^* \cdot k - \frac{1}{2} \chi^2 \hat{P}_\mu \hat{\xi}'^* \cdot \hat{\xi} + \chi^2 (\hat{\xi}'^* \cdot \hat{\xi} \cdot k + \hat{\xi}_\mu \hat{\xi}'^* \cdot \hat{k}) \end{aligned}$$

The use of eqns. (C5), (C7) and (C8) then yields the result of eqn. (84).

Similarly

$$\begin{aligned}
 & \frac{1}{2\pi} \int d\psi (\xi \cdot k \xi'^* \cdot q - \xi'^* \cdot k \xi \cdot q) \\
 &= \hat{P}_\mu (\xi \cdot \hat{k} \xi'^* \cdot q - \xi'^* \cdot \hat{k} \xi \cdot q) - \kappa^2 (\check{\xi}_\mu \xi'^* \cdot q - \check{\xi}'^*_\mu \xi \cdot q) \\
 &= \hat{P}_\mu (\xi \cdot \hat{k} \xi'^* \cdot q - \xi'^* \cdot \hat{k} \xi \cdot q) - \kappa^2 (\xi_\mu \xi'^* \cdot q - \xi'^*_\mu \xi \cdot q) + \kappa^2 (\hat{\xi}_\mu \xi'^* \cdot q - \hat{\xi}'^*_\mu \xi \cdot q)
 \end{aligned}$$

which gives (87) on using eqns. (C5), (C6) and (C7).

Finally, we determine κ by the geometry of Fig. 17, giving

$$(C9) \quad \kappa^2 = m^2 - \frac{m_d^2}{4\cos^2\varphi}$$

Appendix D.

The geometry of PABC has already been dealt with in Appendix B.

This gives the values of k_1 and k_2 quoted in (194).

The position of D is determined from

$$(D1) \quad \begin{cases} s &= (d_1 - a_1)^2 + (d_2 - a_2)^2 + \lambda^2 \\ q^2 &= (d_1 + a_1)^2 + (d_2 - a_2)^2 + \lambda^2 \\ m^2 &= (d_1 - c_1)^2 + (d_2 - c_2)^2 + \lambda^2 \end{cases}$$

where we have written $c_3 = \lambda$.

After some algebra we find

$$d_1 = \frac{2m^2 + q^2}{m_d}$$

$$(D2) \quad 2d_2 n_2 = \mu^2 + 2(y' - a^2) \frac{2m^2 + q^2}{m_d^2}$$

$$(D3) \quad \lambda^2 = q^2 \left(1 + \frac{q^2}{4m^2} \right)$$

We are now in a position to find s' , and thus a and b .

$$\text{Thus } s' = d_1^2 + d_2^2 + (\kappa \cos \psi - d_3)^2 + \kappa^2 \sin^2 \psi$$

$$= d_1^2 + d_2^2 + \lambda^2 + \kappa^2 - 2d_3 \kappa \cos \psi$$

$$(D4) \quad \therefore b = -2\lambda\kappa$$

and

$$\begin{aligned}
 a &= d_1^2 + d_2^2 - \kappa^2 + \kappa^2 - m^2 \\
 &= \kappa^2 + m^2 + 2d_2 a_2 - a_1^2 - a_2^2 \quad \text{from (D1)} \\
 &= 2(\kappa^2 + a_2 d_2) \quad \text{from (B3)}
 \end{aligned}$$

Hence finally

$$(D5) \quad a = \frac{1}{n_2} \left[y' (\alpha^2 + \mu^2 - y') - \frac{q_d^2}{m_d^2} (y' - \alpha^2)(y' - \mu^2 + \alpha^2) \right]$$

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Dispersion Theory of the Deuteron Form Factor and Elastic e-d Scattering (*).

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Summary. — The anomalous threshold discontinuities of the deuteron form factors are calculated by an extension of Cutkosky's method to the spinor case. The form factors are then evaluated in a « tail » approximation and substituted into the single photon exchange term for elastic e-d scattering. The resulting expression is very similar to the Jankus formula, but shows that the usual modification $e \rightarrow eF_1^S$ should more correctly be $e \rightarrow eF_{ch}^S(1 - s/4m_d^2)^{-1}$. The bare form factors should also be multiplied by $(1 - s/4m_d^2)^{-1}$. Small angle scattering is re-analysed with the new formula; the changes are largely masked by the experimental uncertainties, but result in a small increase in the neutron form factor F_1^n .

Introduction.

Up to now the neutron form factors have been obtained from elastic e-d scattering by means of the Jankus formula ⁽¹⁾, modified to allow for nucleons with electromagnetic structure ⁽²⁻⁴⁾. This « modification », however, is not unique, as all we have to go on is the analogy with the Rosenbluth formula, which can be written in a number of different ways ⁽⁵⁾, and the static limits, which are the same, *e.g.*, for F_1 and F_{ch} ⁽⁶⁾.

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For a further discussion of F_{ch} , F_{mag} see K. J. BARNES: *Phys. Lett.*, **1**, 166 (1962).

The first step in a completely relativistic approach has been made by GLASER and JAKŠIĆ⁽⁷⁾, who derive the differential cross-section for the elastic scattering in the single photon exchange (SPE) approximation; this involves three deuteron form factors F_0 , F_2 , G_0 (or equivalently F_{id} , $i=1 \dots 3$). To complete this work we need to know the covariant expressions for these form factors in terms of those of the nucleons.

In the present paper we are able to go some way towards this; we calculate the contributions to the absorptive parts of the deuteron form factors from the lowest anomalous threshold, using the method of Cutkosky⁽⁸⁾, extended slightly to include spin (Section 2). The calculation is the relativistic counterpart of simply using the tail of the deuteron wave function.

Then we evaluate the form factors to the order required by the SPE term and substitute them therein (Section 3). In this way we obtain a modified Jankus formula, which differs from the usual one, and shows the more fundamental significance of F_{ch} .

In Section 4 we apply the formula to the results of GLENDENNING and KRAMER⁽⁴⁾ and find a small increase in F_1^n , which lends further weight to these authors' speculation that F_1^n could be zero up to about $q = 3F^{-1}$ (remembering that $F_1^{n'}(0)$ is very small⁽⁹⁾). The disagreement with inelastic scattering still of course remains.

Notation. — As in⁽¹⁰⁾, except that we use covariant normalization

$$\langle k_i | k'_j \rangle 2\pi \delta^{(+)}(k_i^2 - m_i^2) 2\pi \delta^{(+)}(k'_j{}^2 - m_j^2) = (2\pi)^4 \delta^4(k_i - k'_j) 2\pi \delta^{(+)}(k_i^2 - m_i^2) \delta_{ij}$$

and normalize F_2 by $F_2(0) = \kappa$ (in units $e/2m$).

1. — The SPE formula.

With $D = d' + d$, $q = d' - d$ we write the deuteron form factor as⁽¹¹⁾

$$(1) \quad F_{\mu d} = \langle d' | j_\mu(0) | d \rangle = e \{ -F_{1d}(q^2) \xi'^* \cdot \xi D_\mu - F_{2d}(q^2) (\xi_\mu \xi'^* \cdot q - \xi_\mu'^* \xi \cdot q) + \\ + F_{3d}(q^2) D_\mu \xi'^* \cdot q \xi \cdot q / m_d^2 \}.$$

(7) V. GLASER and B. JAKŠIĆ: *Nuovo Cimento*, **5**, 1197 (1957).

(8) R. E. CUTKOSKY: (a) *Journ. Math. Phys.*, **1**, 429 (1960); (b) *Proc. of the 1961 Ann. Int. Conf. on High-Energy Physics at Rochester*, p. 236.

(9) See e.g. ref. (10), p. 22.

(10) S. D. DRELL and F. ZACHARIASEN: *Electromagnetic Structure of Nucleons* (London, 1962)

(11) L. DURAND III: *Phys. Rev.*, **123**, 1393 (1961). The different appearance of (1) is of course due to the different metric.

The static limits are ⁽¹²⁾

$$(2) \quad \begin{cases} F_{1d}(0) = 1, \\ F_{2d}(0) = \mu_d \quad \text{in units } e/2m_d^2, \\ F_{3d}(0) = \frac{1}{2}Q + \frac{1}{2}(\mu_d - 1), \quad Q \text{ in units } e/m_d^2 \end{cases}$$

Then the single photon exchange approximation gives (cf. (7))

$$(3) \quad dQ_1 = dq_0 \left\{ \frac{1}{3} (F_{1d})^2 \left[2 + \left(1 - \frac{q^2}{2m_d^2} \right)^2 \right] - \frac{2}{3} F_{1d} F'_{3d} \frac{q^2}{m_d^2} \left(1 - \frac{q^2}{2m_d^2} \right) + \right. \\ \left. + \frac{1}{3} F_{2d} F_{1d} \frac{q^2}{m_d^2} \left(1 - \frac{q^2}{2m_d^2} \right) - \frac{2}{3} (F_{2d})^2 \frac{q^2}{4m_d^2} \left(1 + 2 \operatorname{tg}^2 \frac{1}{2} \vartheta - \frac{q^2}{2m_d^2} \sec^2 \frac{1}{2} \vartheta \right) - \right. \\ \left. - \frac{1}{3} F_{2d} F'_{3d} \frac{q^1}{m_d^4} + \frac{1}{3} (F'_{3d})^2 \frac{q^1}{m_d^4} \right\},$$

where we have written $F'_{3d} = F_{3d}(1 - q^2/4m_d^2)$. In the notation of Glaser and Jakšić $\eta = -q^2/4m_d^2$.

That this formula must have the form

$$dQ_1 = dq_0 (A(q^2) + B(q^2) \operatorname{tg}^2 \frac{1}{2} \vartheta)$$

is probably well known; it can easily be shown in a similar fashion to a proof of GOURDIN ⁽¹³⁾ for the inelastic scattering. The formula of BLANKENBECLER ⁽¹⁴⁾ does not appear to have this feature.

We shall only require (3) to order q^2 ; to this order it reads

$$(4) \quad dQ_1 = dq_0 \left\{ \left[F_{1d} - \frac{2}{3} \frac{q^2}{4m_d^2} (2F_{3d} + F_{1d} - F_{2d}) \right]^2 - \frac{2}{3} \frac{q^2}{4m_d^2} (F_{2d})^2 (1 + 2 \operatorname{tg}^2 \frac{1}{2} \vartheta) \right\}.$$

It is already clear that unless it is contained in F_{1d} there is to this order no interference term between the anomalous magnetic moment- and charge-scattering, in contrast to ⁽¹⁴⁾, since the static limit of $F_{3d} + F_{1d} - F_{2d}$ is simply Q .

⁽¹²⁾ In ref. (7) the contribution to $F_{3d}(0)$ from the deuteron's anomalous magnetic moment is omitted, while in ref. ⁽¹¹⁾ the units should involve the deuteron mass rather than the nucleon mass and the Q refers to a quadrupole operator $r^2 P_2(\cos \vartheta)$ rather than the more usual $2r^2 P_2(\cos \vartheta)$. The static values are obtained by evaluating the 0 and i components of (1) in the brick-wall frame $d = (E, -\frac{1}{2}\mathbf{q})$, ($d = E, \frac{1}{2}\mathbf{q}$). The author is very grateful to Loyal Durand for help on this point.

⁽¹³⁾ M. GOURDIN: *Nuovo Cimento*, **21**, 1094 (1961).

⁽¹⁴⁾ R. BLANKENBECLER: (unpublished) discussed in ref. ⁽³⁾ and by R. HOFSTADTER: *Ann. Rev. Nucl. Sci.*, **7**, 259 (1957).

2. - Absorptive parts of F'_{id} from lowest anomalous threshold.

2'1. - The Cutkosky graph ^(8b) for the lowest anomalous threshold is shown in Fig. 1 and the corresponding dual diagram in Fig. 2.

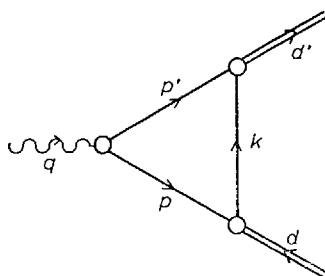


Fig. 1. - The reduced diagram for the lowest anomalous threshold of the deuteron form factor.

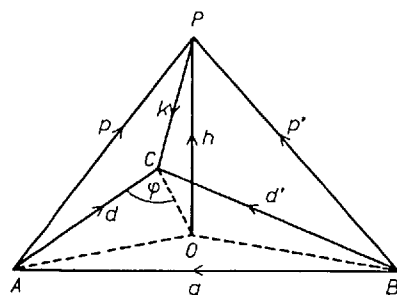


Fig. 2. - The dual diagram corresponding to Fig. 1.

The vertices occurring in Fig. 1 are as follows:

$$(5) \quad \begin{cases} \langle p' | j_{\mu}^S(0) | p \rangle = e \bar{u}(p') (F_1^S \gamma_{\mu} + i \sigma_{\mu\nu} q_{\nu} F_2^S / 2m) u(p), \\ \qquad \qquad \qquad = e \bar{u}(p') (F_{\max}^S \gamma_{\mu} - P_{\mu} F_2^S / 2m) u(p), \end{cases} \quad P = p + p',$$

with static values

$$(6) \quad F_1^S(0) = 1, \quad F_2^S(0) = \kappa_p + \kappa_n \quad \text{in units } e/2m$$

(the superscript S means « isotopic scalar »).

The relation between $F_{1,2}$ and $F_{ch.mag}$ is

$$(7) \quad F_{ch} = F_1 + q^2/4m^2 \cdot F_2, \quad F_{mag} = F_1 + F_2,$$

$$(8) \quad \langle k | f(0) | d \rangle = - (F_0 \xi \cdot \gamma + G_0 \xi \cdot k) v(k) \quad \text{on the mass shell,}$$

with ⁽¹⁵⁾

$$(9) \quad \begin{cases} F_0 = \left(\frac{8\pi}{m} \right)^{\frac{1}{2}} \frac{(1 + q/\sqrt{2}) N}{(1 + q^2)^{\frac{1}{2}}}, \\ G_0 = F_0 \cdot \frac{1}{m_d} + \frac{3qm}{\sqrt{2}\alpha^2} (1 + q/\sqrt{2})^{-1}, \end{cases}$$

where $N = (2\alpha/(1 - \alpha r_e))^{\frac{1}{2}}$.

⁽¹⁵⁾ R. BLANKENBECLER, M. L. GOLDBERGER and F. R. HALPERN: *Nucl Phys.*, **12**, 629 (1959).

The absorptive part may be found by the standard contraction procedures, and proceeds very similarly to ⁽¹⁶⁾.

We will not go through this, however, as the answer can be written down almost immediately by inspection of Fig. 1; it reads (cf. ^(8a)):

$$(10) \quad \text{Im}^T F_{\mu d}(s) = \frac{1}{2} \cdot \frac{1}{16A} \cdot \frac{1}{2\pi} \int_0^{2\pi} d\psi \cdot \\ \cdot \text{Tr} [(F_{\text{mag}}^S \gamma_\mu - P_\mu F_2^S/2m)(\gamma \cdot p + m)(F_0 \gamma \cdot \xi + G_0 \xi \cdot k)(\gamma \cdot k - m) \cdot \\ \cdot (F_0 \gamma \cdot \xi'^* + G_0 \xi'^* \cdot k)(\gamma \cdot p' + m)],$$

where A is the area of ΔABC , and the superscript T on Im^T means the contribution from Fig. 1, corresponding to the tail of the deuteron wave function. With our metric $s = q^2$.

2'2. The circular integration. — In the scalar case the integration over ψ is trivial: in the present case we have to take it seriously.

We write every vector q as $\hat{q} + h$, where h is the vector **OP** and q is the projection of $\hat{q}$ on to the plane ABC , and then make use of the formulae

$$(11) \quad \left\{ \begin{array}{l} \frac{1}{2\pi} \int d\psi h_\mu = \frac{1}{2\pi} \int d\psi h_\mu h_r h_\lambda = 0 \text{ etc. ,} \\ \frac{1}{2\pi} \int d\psi = 1, \quad \frac{1}{2\pi} \int d\psi h_\mu h_\nu = \frac{1}{2} K^2 (g_{\mu\nu} - \hat{g}_{\mu\nu}), \end{array} \right.$$

where K is the radius of the [4] circle on which P moves.

From the geometry we readily see that

$$(12) \quad K^2 = m^2 - m_d^2/(4 \cos^2 \varphi) \simeq (\alpha^2 - s/16) \sec^2 \varphi.$$

Having performed the integrations (11), we will have expressions involving 4-vectors q and also their projections $\hat{q}$. These latter can be expressed in terms of 4-vectors—for momenta it is easy, while for ξ, ξ' we have

$$(13) \quad \left\{ \begin{array}{l} \hat{\xi} = \xi \cdot q \left(\frac{q}{s} + \frac{D}{4m_d^2 \cos^2 \varphi} \right), \\ \hat{\xi}'^* = \xi'^* \cdot q \left(\frac{q}{s} - \frac{D}{4m_d^2 \cos^2 \varphi} \right), \end{array} \right.$$

⁽¹⁶⁾ R. BLANKENBECLER and Y. NAMBU: *Nuovo Cimento*, **18**, 595 (1960).

obtained from

$$(14) \quad \xi \cdot d = \xi'^* \cdot d' = 0.$$

We find φ an extremely convenient variable and will express most of our results in terms of it ($\sin \varphi = \sqrt{s}/2m_d$).

A very similar method can be applied to the anomalous threshold for the np-d vertex, eq. (8), reproducing the results of ⁽¹⁷⁾ for $\text{Im } F_2$ and $\text{Im } G_2$ very simply.

2'3. *The absorptive parts.* — The traces involving F_0^2 are found to be

$$(15) \quad \frac{1}{2\pi} \int d\varphi \frac{F_0^2}{\cos^2 \varphi} \left\{ F_2^S [\xi'^* \cdot \xi D_\mu (m_d^2 - \frac{1}{2}s) \cos 2\varphi + \xi \cdot q \xi'^* \cdot q D_\mu \cos 2\varphi + \right. \\ \left. + 8\xi \cdot k \xi'^* \cdot k P_\mu \cos^2 \varphi] - \right. \\ \left. - F_{\text{mag}}^S [\xi'^* \cdot \xi D_\mu (m_d^2 \cos 2\varphi + \frac{1}{2}s) + 2m_d^2 (\xi_\mu \xi'^* \cdot q - \xi_\mu'^* \xi \cdot q) + 8\xi \cdot k \xi'^* \cdot k P_\mu \cos^2 \varphi] \right\},$$

in which we have already performed the integrals involving 1, h . The only φ integration remaining is

$$(16) \quad \frac{1}{2\pi} \int d\varphi \xi \cdot k \xi'^* \cdot k P_\mu = \frac{K^2 \cos 2\varphi}{4 \cos^2 \varphi} \xi'^* \cdot \xi D_\mu + \frac{K^2}{4 \cos^2 \varphi} (\xi_\mu \xi'^* \cdot q - \xi_\mu'^* \xi \cdot q) - \\ - \frac{D_\mu \xi \cdot q \xi'^* \cdot q}{8 \cos^2 \varphi} \left[\frac{\cos 2\varphi}{(2 \cos^2 \varphi)^2} + \frac{2K^2}{m_d^2} \left(\frac{\cos^2 2\varphi}{\sin^2 2\varphi} + \frac{1}{2 \cos^2 \varphi} \right) \right].$$

Expressing (15) in terms of F_1^S and F_{mag}^S and sorting out the contributions to the various F_{id} we get:

$$(17) \quad \left\{ \begin{array}{l} \text{Im}^T F_{1d} \Big|_{F_0^2} = \frac{F_0^2}{32A \cos^2 \varphi} \{ F_1^S (m_d^2 - \frac{1}{2}s + 2K^2) \cos 2\varphi + F_{\text{mag}}^S s \cos^2 \varphi \}, \\ \text{Im}^T F_{2d} \Big|_{F_0^2} = \frac{F_0^2}{32A \cos^2 \varphi} \{ 2K^2 F_1^S + 2m_d^2 F_{\text{mag}}^S \}, \\ \frac{1}{m_d^2} \text{Im}^T F_{3d} \Big|_{F_0^2} = \frac{F_0^2}{32A \cos^2 \varphi} \cdot \\ \cdot \left\{ -F_1^S \left[\cos 2\varphi - \frac{\cos 2\varphi}{(2 \cos^2 \varphi)^2} - \frac{2K^2}{m_d^2} \left(\frac{\cos^2 2\varphi}{\sin^2 2\varphi} + \frac{1}{2 \cos^2 \varphi} \right) \right] + F_{\text{mag}}^S \cos 2\varphi \right\}. \end{array} \right.$$

⁽¹⁷⁾ R. BLANKENBECLER and L. F. COOK jr.: *Phys. Rev.*, **119**, 1745 (1960).

The traces involving $F_0 G_0$ are found to be

$$(18) \quad \frac{1}{2\pi} \int d\psi F_0 G_0 \left\{ -\frac{2}{m} F_2^S \left[\frac{1}{2} m_a^2 (\xi \cdot k \xi'^* \cdot q - \xi'^* \cdot k \xi \cdot q) P_\mu + (8m^2 - s) \xi \cdot k \xi'^* \cdot k P_\mu \right] + \right. \\ \left. + \frac{m F_{\text{mag}}^S}{\cos^2 \varphi} \left[\frac{1}{2} s (\xi_\mu \xi'^* \cdot q - \xi_\mu'^* \xi \cdot q) + D_\mu \xi \cdot q \xi'^* \cdot q + 16 \xi \cdot k \xi'^* \cdot k P_\mu \cos^2 \varphi \right] \right\}.$$

The ψ integrations still to be performed are (16) and

$$(19) \quad \frac{1}{2\pi} \int d\psi (\xi \cdot k \xi'^* \cdot q - \xi'^* \cdot k \xi \cdot q) P_\mu = -K^2 (\xi_\mu \xi'^* \cdot q - \xi_\mu'^* \xi \cdot q) + \\ + \frac{D_\mu \xi \cdot q \xi'^* \cdot q}{4 \cos^2 \varphi} \left(\frac{\cos 2\varphi}{\cos^2 \varphi} + \frac{2K^2}{m_a^2} \right).$$

From (18), (19) and (16) we get the contributions to the F_{ia} . We will be content with $\text{Im } F_{1a}$, $\text{Im } F_{2a}$ to 1st order (in q^2 and α^2) and $\text{Im } F_{3a}$ to zeroth order. In this approximation

$$(20) \quad \left\{ \begin{array}{l} \text{Im}^T F_{1a} \Big|_{F_0 G_0} = -\frac{F_0 G_0}{32A \cos^2 \varphi} 4mK^2 \cos 2\varphi F_1^S, \\ \text{Im}^T F_{2a} \Big|_{F_0 G_0} = -\frac{F_0 G_0}{32A \cos^2 \varphi} \cdot m(4K^2 + \frac{1}{2}s) \cdot F_{\text{mag}}^S, \\ \text{and} \\ \text{Im}^T F_{3a} \Big|_{F_0 G_0} = \frac{F_0 G_0}{32A \cos^2 \varphi} \cdot 2m \left(\frac{1}{4} - \frac{2K^2}{m_a^2} \frac{\cos^2 2\varphi}{\sin^2 2\varphi} \right) F_1^S. \end{array} \right.$$

We will also quote the result of the traces involving G_0^2 :

$$(21) \quad \frac{1}{2\pi} \int d\psi \cdot 2G_0^2 \xi \cdot k \xi'^* \cdot k [(4\alpha^2 - \frac{1}{2}s) P_\mu F_2^S - (4\alpha^2 P_\mu - s k_\mu) F_{\text{mag}}^S],$$

but will neglect these, as their contributions to the absorptive parts are all one order smaller than those of (20).

Finally we collect together the contributions to the absorptive parts from

(17) and (20), using the values of F_0 , G_0 given in (9). The result is

$$(22) \quad \left\{ \begin{aligned} \operatorname{Im}^T F_{1d} &= \frac{\pi N^2 \sec^2 \varphi}{2mm_d \sqrt{s(1-s/4m_d^2)}} \left\{ F_1^S m_d^2 \cos^2 2\varphi + F_{\max}^S s \cos^2 \varphi - \right. \\ &\quad \left. - \sqrt{2} \rho F_1^S \left(\frac{6m^2 K^2}{\alpha^2} - m_d^2 \right) \right\}, \\ \operatorname{Im}^T F_{2d} &= \frac{\pi N^2 \sec^2 \varphi}{2mm_d \sqrt{s(1-s/4m_d^2)}} \left\{ 2K^2 F_1^S + F_{\max}^S (2m_d^2 - \frac{1}{4}s - 2K^2) - \right. \\ &\quad \left. - \sqrt{2} \rho F_{\max}^S \left(3m^2 \cdot \frac{2K^2 + \frac{1}{4}s}{\alpha^2} - 2m_d^2 \right) \right\}, \\ \frac{1}{m_d^2} \operatorname{Im}^T F_{3d} &= \frac{\pi N^2 \sec^2 \varphi}{2mm_d \sqrt{s(1-s/4m_d^2)}} \left\{ -\frac{1}{2} F_1^S + F_{\max}^S + 3\sqrt{2} \rho F_1^S \frac{m^2}{\alpha^2} \left(\frac{1}{4} - \frac{2K^2}{s} \right) \right\}. \end{aligned} \right.$$

3. - Dispersion integrals.

3'1. *The «bare» form factor and the S-wave contribution.* - If all the spinor factors are omitted and the nucleon assumed to have no electromagnetic structure, the result corresponding to eq. (22) is simply ^(8,16)

$$(23) \quad \operatorname{Im}^T F_d = \frac{1}{32A} F_0^2 = \frac{\pi N^2}{\sqrt{s(1-s/4m_d^2)}},$$

which corresponds to

$$(24) \quad F_d^T(x) = \frac{N^2}{m_d} \left[\frac{-1}{x\sqrt{(1+x^2)}} \operatorname{tg}^{-1} \frac{x \cot \varphi}{\sqrt{(1+x^2)}} \right]_{\alpha/m}^{\varphi}, \quad s = -4m_d^2 x^2,$$

on doing the dispersion integral.

In what follows we will ignore the contribution from the upper limit φ , which is $\pi/6$ if we take the integration up to the normal two-nucleon threshold $s = 4m^2$.

The corresponding results in potential theory are ⁽¹⁸⁾

$$(25) \quad \operatorname{Im}^T f_d = \frac{N^2 \pi}{\sqrt{s}},$$

⁽¹⁸⁾ See ref. ⁽¹⁶⁾ and also: A. MARTIN: Lecture notes from 1961 summer school in Yugoslavia (CERN 1775 Th/195).

giving

$$(26) \quad f_d^T(x) = \frac{2N^2}{\sqrt{-s}} \operatorname{tg}^{-1} \frac{\sqrt{-s}}{4\alpha} = \int_0^\infty \exp[-2\alpha r] j_0\left(\frac{1}{2}qr\right) dr.$$

With neglect of ϱ the absorptive parts in (22) are of the form

$$\operatorname{Im}^T F_{\text{id}} = \operatorname{Im}^T F_d A(F_1^S, F_{\text{mag}}^S)$$

and we will adopt the point of view that the same relation should hold without the superscript T . This is what we call the « tail » approximation.

Then, with neglect of « nonadditive parts »^(8b), we may write

$$F_{\text{id}} = F_d A(F_1^S, F_{\text{mag}}^S).$$

This procedure gives⁽¹⁹⁾

$$(27) \quad \begin{cases} F_{1d} = \frac{F_d}{\cos^2 \varphi} F_{\text{ch}}^S, \\ F_{2d} = \frac{F_d}{\cos^2 \varphi} \left[\left(2 - \frac{s}{4m_d^2} - \frac{2K^2}{m_d^2} \right) F_{\text{mag}}^S + \frac{2K^2}{m_d^2} F_1^S \right], \\ F_{3d} = \frac{F_d}{\cos^2 \varphi} \left[F_{\text{mag}}^S - \frac{1}{2} F_1^S \right], \end{cases}$$

which reproduces the correct static limits (2) (Q would be zero for $\varrho = 0$).

3'2. The D-wave contribution. — We regard the extra factors accompanying ϱ as belonging to the bare form factor and integrate accordingly.

i) Thus for F_{3d} we have to integrate $\operatorname{Im}^T F_d(s) \cdot 3\varrho \sqrt{2} \cdot \frac{m^2}{\alpha^2} \left(\frac{1}{4} - \frac{2K^2}{s} \right)$.

In view of (12) this means that we have to integrate $(\alpha^2/s) \operatorname{Im}^T F_d(s)$. This is quite simple as it is like a subtraction in (24), and yields

$$(28) \quad \begin{aligned} \frac{1}{\pi} \int \frac{ds'}{s' - s} \frac{\alpha^2}{s'} \operatorname{Im}^T F_d(s') &= \frac{\alpha^2}{4m_d^2 x^2} \cdot \frac{N^2}{m_d} \left[\frac{1}{x\sqrt{(1+x^2)}} \operatorname{tg}^{-1} \frac{x \operatorname{ctg} \varphi}{\sqrt{(1+x^2)}} - \operatorname{ctg} \varphi \right]_{x/m}^\varphi, \\ &\simeq \frac{\alpha^2}{4m_d^2 x^2} \cdot \frac{N^2}{m_d} \left[-\frac{1}{3} x^2 \operatorname{ctg}^3 \varphi \right]_{x/m}^\varphi, \quad (x \text{ small}) \\ &\simeq \frac{1}{48} F_d^T(x), \end{aligned}$$

neglecting the contribution from the upper limit.

⁽¹⁹⁾ We do not regard the factor $2m/m_d$ as significant here, as it is neglected in several places in the derivation of eq. (9) (ref. (15)).

Thus the ϱ contribution to F_{3d}^T is

$$(29) \quad \begin{aligned} F_{3d}^T \Big|_{\varrho} &= F_d^T \cdot 3\varrho\sqrt{2} \cdot \frac{m^2}{\alpha^2} \left(\frac{3}{8} - \frac{1}{24} \right) F_1^S, \\ &= F_1^S \varrho\sqrt{2} \cdot \frac{m^2}{\alpha^2} \cdot F_d^T. \end{aligned}$$

Taking static limits we get

$$(30) \quad \begin{aligned} F_{3d}^T(0) \Big|_{\varrho} &= \varrho\sqrt{2} \cdot \frac{m^2}{\alpha^2} \cdot \frac{N^2}{2\alpha} \simeq m_d^2 \cdot \left(\frac{1}{2} \frac{\varrho N^2}{\sqrt{8}\alpha^3} \right), \\ &\simeq \frac{1}{2} Q^T m_d^2, \end{aligned}$$

since the contribution to Q from the tail of the deuteron wave function is ⁽²⁰⁾

$$(31) \quad Q^T = \frac{\varrho N^2}{\sqrt{8}\alpha^3}.$$

Thus (30) completes the correct static limit for F_{3d} .

ii) Now let us consider the ϱ contribution to the imaginary part of $F_{1d} - \frac{1}{3}(s/m_d^2)F_{3d}$, which we see we require from (4). It is

$$\begin{aligned} \text{Im } F_{1d} - \frac{1}{3} \frac{s}{m_d^2} \text{Im } F_{3d} \Big|_{\varrho} &= -\text{Im } F_d \cdot \sqrt{2} \varrho \frac{F_1^S}{m_d^2} \left[6m^2 \frac{K^2}{\alpha^2} - m_d^2 + \frac{m^2}{\alpha^2} \left(\frac{s}{4} - 2K^2 \right) \right] \\ &= 0 \text{ to this order.} \end{aligned}$$

This result enables us to determine to what an imaginary part of s in the bare form factor corresponds. (This can not be done as in (28), (29) for α^2/s , as in this case the contribution from the upper limit is the more important.) We do it by requiring that

$$(32) \quad F_{1d} - \frac{1}{3} \frac{s}{m_d^2} F_{3d} \Big|_{\varrho} = 0,$$

where $F_{3d}|_{\varrho}$ is determined from (29).

Let us write

$$(33) \quad \frac{1}{\pi} \int \frac{ds'}{s' - s} f(s') \text{Im}^T F_d(s') = \langle f(s) \rangle F_d^T(x).$$

⁽²⁰⁾ See, e.g. H. A. BETHE and P. MORRISON: *Elementary Nuclear Theory* (New York, 1956).

Then (32) gives

$$\varrho\sqrt{2}\left(\frac{3}{2}\frac{\langle K^2 \rangle}{\alpha^2} - 1\right) + \frac{1}{3}s\frac{\varrho}{2\sqrt{2}\alpha^2} = 0,$$

i.e.

$$(34) \quad \langle K^2 \rangle = \frac{2}{3}\alpha^2 - \frac{1}{18}s, \quad \text{or} \quad \frac{1}{16}\langle s \rangle = \frac{1}{3}\alpha^2 + \frac{1}{18}s,$$

iii) We can now consider the ϱ contribution to F_{2d} .

From (22)

$$\text{Im}^T F_{2d} \Big|_{\varrho} = -\frac{\text{Im}^T F_d}{m_d^2} \cdot \sqrt{2}\varrho \left(-2m^2 + \frac{3}{8}\frac{m^2}{\alpha^2}s\right) F_{\text{mag}}^S.$$

So, applying (34),

$$(35) \quad \begin{aligned} F_{2d}^T \Big|_{\varrho} &= -\frac{F_d^T}{m_d^2} \cdot F_{\text{mag}}^S \cdot \sqrt{2}\varrho \frac{m^2}{\alpha^2} \cdot \frac{s}{3}, \\ &\simeq -\frac{1}{6}(\mu_v + \mu_n)Q^T s, \end{aligned}$$

from (24) and (31).

Now with neglect of w^2 the magnetic scattering term in the Jankus formula is, cf. (1),

$$(36) \quad -\frac{2}{3}\frac{q^2}{4m_d^2}\left(2\text{tg}^2\frac{1}{2}\vartheta + 1\right)(\mu_p + \mu_n)^2\left(2\int u^2 j_0\left(\frac{1}{2}qr\right)dr + \sqrt{2}\int uwj_2\left(\frac{1}{2}qr\right)dr\right)^2.$$

The second integral here is

$$\sqrt{2}\int uwj_2\left(\frac{1}{2}qr\right)dr \simeq -\frac{1}{6}Qq^2,$$

which corresponds very closely to (35).

3'3. - Finally, combining (27), (32), (35) with eq. (4) we get our main result:

$$(37) \quad dQ_1 = dq_0 \left\{ F_d^2 \left(\frac{F_{\text{ch}}^S}{1 - s/4m_d^2} \right)^2 - \frac{2}{3}\frac{q^2}{4m_d^2} G_d^2 \left(1 + 2\text{tg}^2\frac{1}{2}\vartheta \right) (F_{\text{mag}}^S)^2 \right\},$$

to order q^2 . F_d and G_d are the relativistic generalizations of the bare charge and magnetic moment form factors respectively occurring in the Jankus formula.

Comparison of (24) and (26) suggests that

(38)
$$F_d \simeq \frac{f_d}{1 - s/4m_d^2},$$

and similarly for G_d .

The occurrence of $F_{ch}^s/(1-s/4m_d^2)$ in place of the more usual F_1^s in eq. (37) is not really surprising in view of the results of ⁽⁶⁾, where it is shown that it is F_{ch} and not F_1 which measures the interaction with the electrostatic field. Also it is well known that the Rosenbluth formula takes on a simpler form when expressed in terms of F_{ch} and F_{mag} ⁽⁵⁾.

The factor $1-s/4m_d^2$ is, among other things, the ratio $-q^2/q^2$ where q is the 3-momentum transfer in the laboratory system.

One would perhaps expect the magnetic scattering term, which to zeroth order contains F_{mag}^s , to contain $F_{mag}^s/(1-s/4m_d^2)$ to first order. From (27) this does not appear to be the case, and in fact to first order we are unable to give any simple expression.

4. - In view of the remarks at the end of the last section we can offer no new interpretation of the results of ref. ⁽³⁾; the corrections will in any case be small.

For small angle scattering, however, we take eq. (37) seriously and use it to reinterpret the results of ref. ⁽⁴⁾. In principle the difference between F_{ch} and F_1 should be very important for the neutron, since $F_1^n(0) = 0$, but in fact this tends to be swamped by the experimental uncertainties.

Equation (37) implies that the last column of Table I of ref. ⁽⁴⁾ is not really

TABLE I. - *The relevant parts of Table I of ref. ⁽⁴⁾ together with the correction term of (40).*

$q \text{ (F}^{-1}\text{)}$	$F_1^p \text{ exp}$	z		$1.52 \eta F_1^p$
		lower	upper	
0.99	0.901	- 0.120	- 0.024	0.004
1.07	0.889	- 0.100	0.001	0.004
1.36	0.839	- 0.060	0.058	0.006
1.51	0.809	- 0.093	0.015	0.008
1.79	0.748	- 0.110	0.003	0.010
1.99	0.706	- 0.132	- 0.021	0.012
2.22	0.658	- 0.048	0.086	0.014
2.41	0.619	- 0.130	- 0.019	0.015
2.61	0.578	- 0.111	0.020	0.016
2.82	0.538	- 0.125	- 0.001	0.018

F_1^n , but

$$(39) \quad z = \frac{F_{ch}^S}{(1 - s/4m_d^2)^2} - F_1^n.$$

(The extra factor of $(1 - 2/4m_d^2)^{-1}$ comes from (38).)

Thus to order η , with $\eta = -q^2/4m_d^2$,

$$(40) \quad \left\{ \begin{array}{l} z = F_1^n - 2\eta F_1^p - 4\eta F_2^S, \\ \quad = F_1^n - 4\eta F_1^p(\mu_p + \mu_n - \frac{1}{2}), \\ \therefore F_1^n = z + 1.52\eta F_1^p. \end{array} \right.$$

From this table we see that the effect of (40) is to increase F_1^n slightly, though it still retains a small negative bias.

5. - Discussion.

We would like here to comment on the « tail » approximation of Section 3'1. This seems to us the most reasonable (and simplest!) way of including the effect of the higher thresholds. An attempt to calculate the higher anomalous thresholds directly, besides involving one in a prohibitive amount of spinology, also involves one in the rather unsatisfactory business of taking pole terms ⁽⁸⁾. The 3π cut is largely taken into account via the nucleon form factors; the rest is the « nonadditive » part of ⁽⁸⁾.

* * *

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RIASSUNTO (*)

Con una estensione del metodo di Cutkosky al caso spinoriale, si calcolano le discontinuità di soglia anomale dei fattori di forma del deutone. Si valutano poi i fattori di forma in una approssimazione « a coda » e si sostituiscono nel termine di scambio di un solo fotone dello scattering elastico e.d. L'espressione risultante è molto simile alla formula di Jankus, ma mostra che la sostituzione usuale $e \rightarrow e F_1^S$ dovrebbe essere più correttamente $e \rightarrow e F_{ch}^S (1 - s/4m_d^2)^{-1}$. I nudi fattori di forma dovrebbero anche essere moltiplicati per $(1 - s/4m_d^2)^{-1}$. Si riesamina con la nuova formula lo scattering a piccolo angolo; i cambiamenti sono abbondantemente nascosti dalle incertezze sperimentali, ma producono un piccolo aumento del fattore di forma F_1^n del neutrone.

(*) Traduzione a cura della Redazione.

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Dispersion Theory of the Deuteron Form Factor and Elastic e-d Scattering.

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The relation between F_d and f_d is not as simple as stated in this paper. Equation (38) is true only for $x \ll \alpha/m$, or $q \ll 0.5 F^{-1}$. For $x \gg \alpha/m$ the relation is

$$F_d \approx \frac{f_d}{\sqrt{(1 - s/4m_d^2)}},$$

and eq. (40) should be replaced by

$$F_1^{\pi} = z + 1.02 \eta F_1^{\pi}.$$

The corrected values for the last column of Table I are

$$3, 3, 5, 6, 7, 8.5, 10, 11, 12, 13 (\cdot 10^{-3}).$$

We would like to take this opportunity to remark that Gourdin's diagonalization ⁽¹⁾ of (3) as

$$dQ_1 = dq_0 \left\{ F_{ch}^2 - \frac{2}{3} \frac{q^2}{4m_d^2} (F_{2d})^2 \left(1 + 2 \operatorname{tg}^2 \frac{1}{2} \vartheta \right) + \frac{1}{18} \frac{q^4}{m_d^4} F_0^2 \right\},$$

with

$$F_{ch} = F_{1d} - \frac{q^2}{6m_d^2} F_0, \quad F_0 = 2F'_{3d} + F_{1d} - F_{2d},$$

now enables us to extend (37) to include the quadrupole term.

⁽¹⁾ M. GOURDIN: Orsay preprint Th 21/7, submitted to *Nuovo Cimento*.

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