

CONTINUOUS UNWRAPPED PHASE

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ABSTRACT

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A standard Fourier transform algorithm computes the phase modulo- 2π , the principal values of the phase at sampled frequencies. Conventional methods to obtain the unwrapped or continuous phase curve are based on those principal values. Instead of unwrapping the principal values of the phase, this paper presents an alternative method based on the integration of phase derivatives, which precisely gives a continuous unwrapped phase.

KEY WORDS: continuous phase, phase spectrum, phase unwrapping, unwrapped phase.

INTRODUCTION

Phase unwrapping is an aspect of digital data processing which is of much interest in various disciplines such as optics, electromagnetism, digital image processing, synthetic aperture radar interferometry, and seismic data processing. Methods exist to solve this problem though each has disadvantages. The methods and the problems in seismic phase unwrapping have been summarised by Shatilo (1992). I here describe a new method of phase calculation based on adaptive integration of the phase derivatives, which in theory would provide a continuous phase precisely.

UNWRAPPING OF THE PHASE MODULO- 2π

The Fourier transform of a time series $x(t)$ may be expressed in polar form

$$X(\omega) = |X(\omega)| \exp\{j \arg[X(\omega)]\} , \quad (1)$$

where $|X(\omega)|$ and $\arg[X(\omega)]$ are commonly referred to as the amplitude spectrum and phase spectrum respectively, and ω is the angular frequency. A standard Fourier transform algorithm computes the phase modulo- 2π . Interpretation of the modulo- 2π phase spectrum is difficult and is unsuitable for further processing in many cases (Ulrych, 1971; Cambois and Stoffa, 1993). The representation of the phase spectrum by an unwrapped curve is therefore important.

A conceptually simple algorithm for computing $\arg[X(\omega)]$ is based on the phase modulo- 2π , i.e.,

$$-\pi < \text{ARG}[X(k)] \leq \pi , \quad (2)$$

where $\text{ARG}[X(k)]$ is the principal value of the phase sampled at frequency $\omega_k = 2\pi k/N$, and is achieved using standard inverse tangent routines.

Unwrapping the principal-value phase is based on the relation

$$\arg[X(k)] = \text{ARG}[X(k)] + 2\pi r(k) , \quad (3)$$

where $r(k)$ denotes an integer that determines the appropriate multiple of 2π to be added to the principal value at frequency ω_k . A practical implementation for computing $r(k)$ from $\text{ARG}[X(k)]$, starting with $r(0) = 0$, is summarised as follows (Oppenheim and Schaffer, 1989):

1. If $\text{ARG}[X(k)] - \text{ARG}[X(k-1)] > 2\pi - \epsilon_1$, then $r(k) = r(k-1) - 1$;
2. If $\text{ARG}[X(k)] - \text{ARG}[X(k-1)] < -(2\pi - \epsilon_1)$, then $r(k) = r(k-1) + 1$;
3. Otherwise $r(k) = r(k-1)$;
4. Repeat steps 1-3 for $1 \leq k \leq N/2$.

After $r(k)$ is determined, equation (3) can be used to compute $\arg[X(k)]$ for $0 \leq k \leq N/2$. The values of $\arg[X(k)]$ for $N/2 < k \leq N-1$ can be obtained by symmetry. Please note that we here assume $x(t)$ is real.

The above algorithm works well if the samples of $\text{ARG}[X(k)]$ are close

enough together to detect the discontinuities. The parameter ϵ_1 is a tolerance, recognising that the magnitude of the difference between adjacent samples of the principal-value phase will always be less than 2π . If ϵ_1 is too large, a discontinuity will be indicated where there is none. If ϵ_1 is too small, the algorithm will miss a discontinuity falling between two adjacent samples of a rapidly varying unwrapped phase function $\arg[X(\omega)]$. Some authors, such as Tribolet (1977), proposed methods with the aim of overcoming that problem.

Tribolet's (1977) algorithm determines $r(k)$ using both the principal values of the phase and the phase derivatives. It is assumed that we know the phase derivative

$$\arg'[X(k)] = \{d_\omega \arg[X(\omega)]\}_{(\omega=2\pi k/N)} \quad (4)$$

at all values of k . Applying trapezoidal numerical integration to the samples of the phase derivative, we then could estimate $\arg[X(k)]$ from $\arg[X(k-1)]$,

$$\widetilde{\arg}[X(k)] = \arg[X(k-1)] + (\Delta\omega/2)\{\arg'[X(k-1)] + \arg'[X(k)]\} \quad (5)$$

This estimate is said to be *consistent* if for some $\epsilon_2 < \pi$ ($\epsilon_2 = 0.5\pi$ is often used) an integer $r(k)$ exists such that

$$|\widetilde{\arg}[X(k)] - \text{ARG}[X(k)] - 2\pi r(k)| < \epsilon_2 \quad (6)$$

Obviously, the estimate improves with decreasing step size $\Delta\omega$ in equation (5). Initially, $\Delta\omega = 2\pi/N$ as provided by the DFT. If equation (6) cannot be satisfied by an integer $r(k)$, then $\Delta\omega$ is halved and a new estimate of $\widetilde{\arg}[X(k)]$ is computed with the new step size. Increasingly accurate estimates of $\widetilde{\arg}[X(k)]$ are computed until equation (6) can be satisfied by an integer $r(k)$. That resulting $r(k)$ is then used in equation (3) to finally compute $\arg[X(k)]$.

Gelius (1996) used the trigonometric smoothing technique for the phase unwrapping. Given a set of permissible values at frequency ω_k for equation (3), this method seeks to determine $r(k)$ by minimising the second derivative of the prediction, $\Phi(k)$, of the unwrapped phase curve,

$$S = \int_0^\pi [\Phi''(\theta)]^2 d\theta \rightarrow \text{minimum} \quad , \quad (7)$$

where S is called the smoothness operator. The prediction $\Phi(\theta)$ in equation (7) is represented by the sine series expansion,

$$\Phi(\theta) = \sum_{n=1}^{N-1} A_n \sin(n\theta) \quad , \quad (8)$$

where the coefficient is defined by

$$A_n = (2/N) \sum_{k=1}^{N-1} \{ \text{ARG}[X(k)] + 2\pi r(k) \} \sin(kn\pi/N) , \quad (9)$$

in terms of the sampled function in equation (3).

Both algorithms described above need first to detect an integer $r(k)$ and then to remove the discontinuity in the curve of principal-value phase. Instead of unwrapping the principal values of the phase, I present here an implementation based on an adaptive integration of the phase derivatives, which directly produces a continuous unwrapped phase.

CONTINUOUS UNWRAPPED PHASE

The continuous unwrapped phase may be obtained directly by integrating the phase derivatives. The phase derivative $\arg'[X(\omega)]$ can be derived straightforwardly from equation (1). Performing operations of $d_\omega \log$ on both sides of equation (1), we obtain

$$\arg'[X(\omega)] = \text{Im}[X'(\omega)/X(\omega)] , \quad (10)$$

where $X'(\omega)$ is the derivative of $X(\omega)$ with respect to the frequency ω , and provided $X(\omega) \neq 0$. If $x(t)$ is a stable sequence [so that the region of convergence of $X(\omega)$ includes the unit circle], then both $X(\omega)$ and $X'(\omega)$ exist and are continuous functions of ω . To avoid problems where $|X(\omega)|$ is close to zero, equation (10) is stabilised by adding a factor σ^2 to the denominator,

$$\arg'[X(\omega)] = \text{Im}[X^*(\omega)X'(\omega)/\{X^*(\omega)X(\omega) + \sigma^2\}] , \quad (11)$$

where the asterisk denotes the conjugate, and the small value of σ^2 should be normalised to the maximum magnitude of $X(\omega)$. The unwrapped phase may then be obtained *precisely* from the integration,

$$\arg[X(\omega)] = \int_0^\omega \arg'[X(\theta)]d\theta , \quad (12)$$

with the initial condition given by

$$\arg[X(\omega)]_{(\omega=0)} = 0 . \quad (13)$$

Integration in equation (12) is performed adaptively. Supposing that we have the unwrapped phase at frequency ω_{k-1} , the unwrapped phase at frequency ω_k can be expressed as

$$\arg[X(\omega_k)] = \arg[X(\omega_{k-1})] + \int_{\omega_{k-1}}^{\omega_k} \arg'[X(\theta)]d\theta . \quad (14)$$

The phase change between ω_{k-1} and ω_k is computed by the trapezoidal rule,

$$\Delta\phi[\omega_{k-1}, \omega_k] = [\Delta\omega/(2L)] \sum_{l=1}^L \{ \arg'[X(l-1)] + \arg'[X(l)] \} , \quad (15)$$

where $\Delta\omega = \omega_k - \omega_{k-1}$ is discretized into L segments with $[\omega]_{(l=0)} = \omega_{k-1}$ and $[\omega]_{(l=L)} = \omega_k$. Initially, we have $L = 1$. We then increase L by a factor 2, i.e., halving each integration segment. A *continuous* criterion will be satisfied by gradually increasing the number L until there is no significant change in $\Delta\phi[\omega_{k-1}, \omega_k]$ between two iterations. For the n -th iteration, we set $L = 2^n$ and

$$\Delta\phi^{(n)} = (1/2)\Delta\phi^{(n-1)} + (\Delta\omega/L) \sum_{i=1}^{L/2} \arg'[X(2i-1)] , \quad (16)$$

where $\Delta\phi^{(n-1)}$ is the estimate of the phase change at the previous iteration. Equation (16) improves the accuracy of the calculation.

IMPLEMENTATION

Recall that the derivatives of the Fourier transform in equation (10) can be calculated by

$$X'(\omega) = -j \text{FT}[tx(t)] . \quad (17)$$

Therefore, the phase derivative $\arg'[X(k)]$, at frequency ω_k , is specifically the imaginary part of the ratio of the DFT of $-jtx(t)$ to the DFT of $x(t)$. The DFT's at $k = 0, \dots, N-1$ points are computed using an FFT routine. The DFT at a required frequency within (ω_{k-1}, ω_k) is calculated using a Z-transform. For a sequence $x(t) = \{x_0, x_1, \dots, x_{N-2}, x_{N-1}\}$, its Fourier transform is written as

$$X(\omega) = x_0 + Z\{x_1 + \dots + Z[x_{N-2} + Z(x_{N-1})]\} , \quad (18)$$

where $Z = \exp(-j\omega)$. In this manner, the calculation of DFT at a specified frequency is five times faster than a standard DFT computation.

We can see from equation (10) that, when $X(k) \rightarrow 0$, the estimate of $\arg'[X(k)]$ and then the estimate of the phase $\arg[X(k)]$ may have considerable errors. Therefore, a threshold, THLD, is required in the program. If at frequency ω_k , the relative amplitude $20[\log_{10}|X(k)| - \log_{10}|X|_{\max}] < \text{THLD}$ (in dB), I simply replace the phase by the (unwrapped) principal-value [obtained by the conventional method i.e. equation (3)]. In the examples shown in Fig. 1 and Fig. 2, THLD = 80 db is applied.

Adaptively reducing the step length of the integration in the frequency direction improves the accuracy of the resultant phase. In practice, however, the seismic trace $x(t)$ is often not stable because it includes a number of notches. At each notch point, the phase curve will have a $\pm \pi$ discontinuity (Poggiagliolmi et al., 1982). Therefore, the iteration of equation (16) should stop if the continuous criterion has not been satisfied after a sufficient number of iterations.

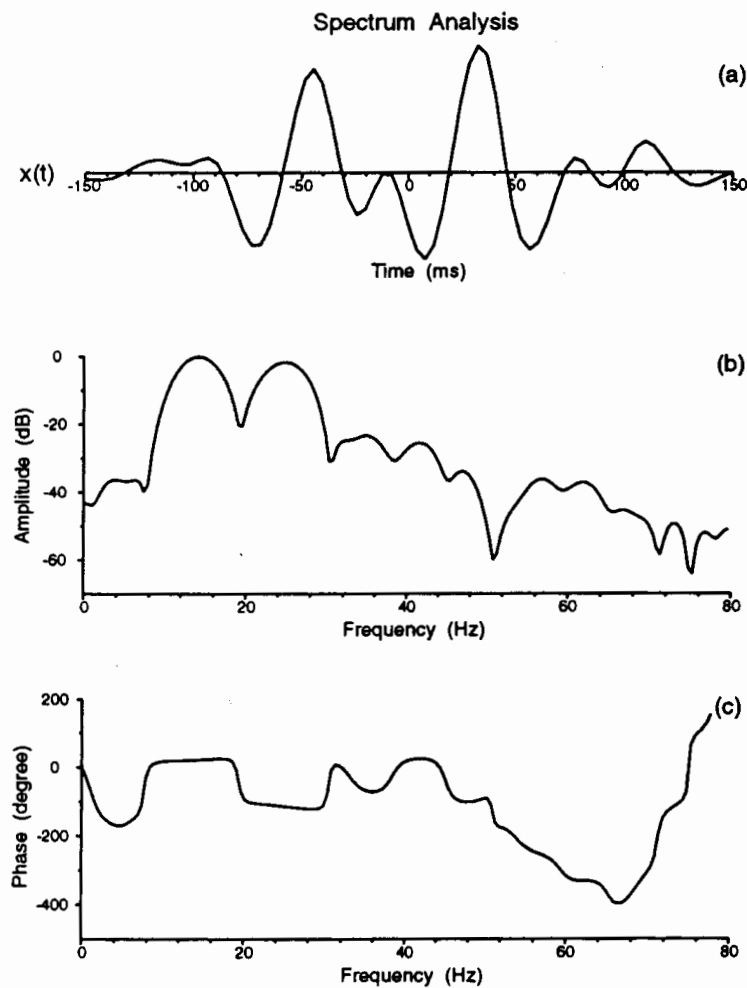


Fig. 1. Phase unwrapping: (a) input sequence $x(t)$, where t is time in msec; (b) the amplitude spectrum; and (c) the unwrapped continuous phase spectrum.

In the test problems (shown in Figs. 1 and 2), I set the maximum number of iterations to 4. After we have done the phase unwrapping, we need to check the sign of the phase spectrum. If the sign is negative, it needs to be reversed. In theory, the sign can be easily determined from the sign of $X(\omega)$ at $\omega = 0$ (Oppenheim and Schaffer, 1989). This method was used by Tribolet (1977). In practice, however, the amplitude of $X[0]$ is often very small and its sign is not reliable because of noise contamination in the seismic trace. In my implementation, the sign is determined by using the correlation between two imaginary parts of input and output spectra.

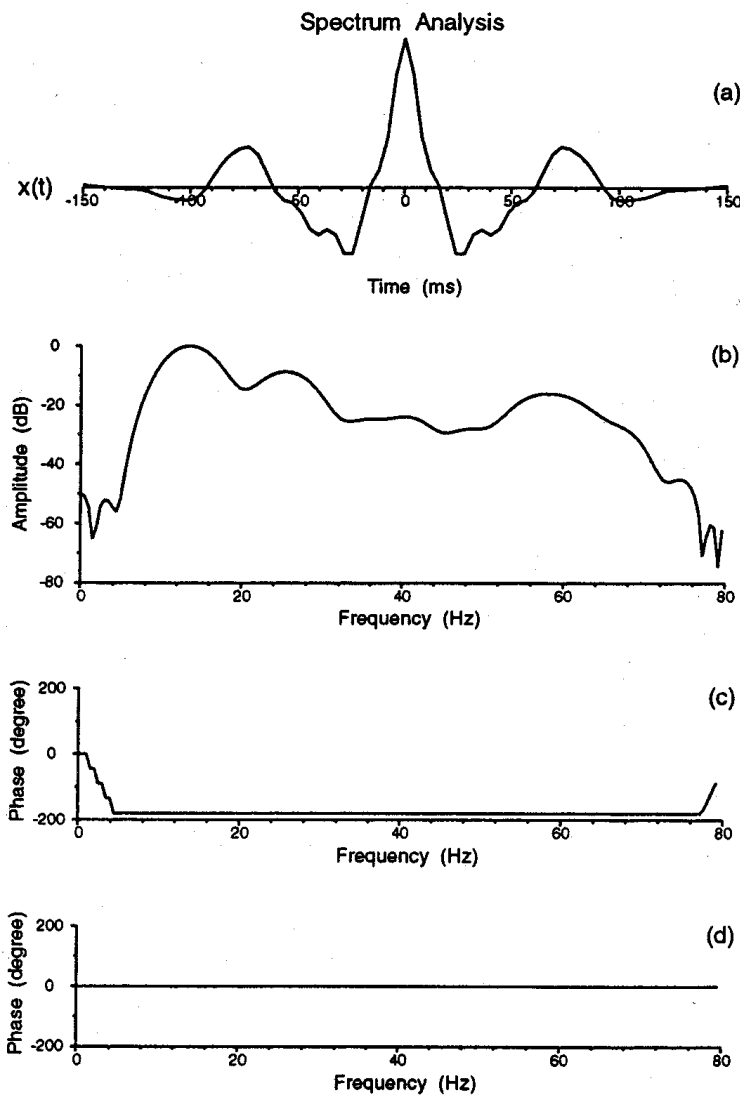


Fig. 2. Phase unwrapping: (a) input sequence $x(t)$; (b) the amplitude spectrum; (c) unwrapped phase spectrum, using Tribolet's (1977) method; and (d) continuous phase spectrum obtained using the adaptive integration method described in this paper.

Finally, a linear component is removed from the phase curve. It is evaluated by

$$\tau = \left[\int_0^T t x^2(t) dt \right] / \left[\int_0^T x^2(t) dt \right], \quad (19)$$

where T is the duration of $x(t)$. The time τ indicates the centre point of the signal, i.e., the point of equilibrium of the linearly weighted signal energy.

DISCUSSION

A new implementation of a continuous phase calculation has been presented. The implementation is based on the adaptive integration of phase derivatives, which defines precisely the continuous unwrapped phase. The integration was also used by Tribolet (1977) but only for checking the consistency between this estimate and the principal value of the phase [see equation (6)]. When the ratio of signal to noise is poor, the principal value of the phase is not reliable as is often the case in seismic. For example, the signal in Fig. 2a is the auto-correlation of a seismic trace and is expected to show a zero phase spectrum (Fig. 2d). However, Tribolet's method fails to find the value of $r(k)$ in equation (6) at those frequency components with low amplitude, as seen from Fig. 2c. The result obtained by using the method presented in this paper is shown in Fig. 2d. The comparison between Fig. 2c and Fig. 2d demonstrates the value of the adaptive integration method.

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REFERENCES

- Cambois, G. and Stoffa, P., 1993. Surface-consistent phase decomposition in the log/Fourier domain. *Geophysics*, 58: 1099-1111.
- Gelius, L.-J., 1996. An introduction to trigonometric smoothing. *J. of Seismic Explor.*, 5: 291-301.
- Oppenheim, A.V. and Schaffer, R.W., 1989. *Discrete-Time Signal Processing*. Prentice-Hall International, Inc., New Jersey.
- Poggiagliolmi, E., Berkhout, A.J. and Boone, M.M., 1982. Phase unwrapping, possibilities and limitations. *Geophys. Prosp.*, 30: 281-291.
- Shatilo, A.P., 1992. Seismic phase unwrapping: methods, results, problems. *Geophys. Prosp.*, 40: 211-225.
- Tribolet, J.M., 1977. A new phase unwrapping algorithm. *IEEE Transact. Acoust., Speech and Signal Process.*, ASSP-25: 170-177.
- Ulrych, T.J., 1971. Application of homomorphic deconvolution to seismology. *Geophysics*, 36: 650-660.