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Résumé

Aspects globaux de la réductibilité des cocycles quasi-périodiques à valeurs dans des groupes de Lie compacts semi-simples

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Résumé

Ce mémoire porte sur l'étude des cocycles quasi-périodiques à valeurs dans des groupes de Lie compacts semi-simples. Nous nous restreindrons au cas des cocycles à une fréquence. Nous démontrons que, pour un ensemble de fréquences de mesure de Lebesgue pleine, l'ensemble des cocycles C^∞ qui sont C^∞-réductibles sont C^∞-denses. De plus, sous la même condition arithmétique, nous démontrerons que tout cocycle (quitte à l'itérer afin de simplifier suffisamment l'homotopie du lacet dans le groupe), est presque tore-réductible (c'est-à-dire qu'il peut être conjugué arbitrairement proche à des cocycles prenant valeurs dans un sous-groupe abelien donné de G).

Le premier pas de la démonstration sera l'obtention de deux invariants de la dynamique, qu'on appellera *énergie* et *degré*, qui distinguent en particulier les cocycles (presque-)réductibles des cocycles non-réductibles. On entamera ensuite la démonstration du théorème principal. Nous démontrons dans un second temps qu'un algorithme dit de *renormalisation* permet de ramener l'étude de tout cocycle à celle des perturbations de modèles simples indexés par le degré. Nous analysons ensuite ces perturbations par des méthodes inspirées de la *théorie K.A.M.*.

Mots-clefs

Théorie K.A.M. , Renormalisation , Cocycles Quasi-périodiques , Groupes de Lie Compacts Semi-simples

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Global aspects of the reducibility of quasiperiodic cocycles in semisimple compact Lie groups

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Abstract

In this mémoire we study quasiperiodic cocycles in semi-simple compact Lie groups. For the greatest part of our study, we will focus ourselves to one-frequency cocyles. We will prove that C^∞-reducible cocycles are dense in the C^∞ topology, for a full measure set of frequencies. Moreover, we will show that every cocycle (or an appropriate iterate of it, if homotopy appears as an obstruction) is almost torus-reducible (i.e. can be conjugated arbitrarily close to cocycles taking values in an abelian subgroup of G). In the course of the proof we will firstly define two invariants of the dynamics, which we will call *energy* and *degree* and which give a preliminary distinction between (almost-)reducible and non-reducible cocycles. We will then take up the proof of the density theorem. We will show that an algorithm of *renormalization* converges to perturbations of simple models, indexed by the degree. Finally, we will analyze these perturbations using methods inspired by *K.A.M.* theory.

Keywords

K.A.M. theory , Renormalization , Quasiperiodic Cocycles , Compact Semisimple Lie Groups

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Chapter 1

Introduction

This mémoire is concerned with the study of quasiperiodic cocycles in semisimple compact Lie groups. Cocycles are discrete dynamical systems whose phase space is a fibered space $X \times E \to X$. Fibered dynamics is given by the iteration of a mapping of the type

$$\begin{array}{rccc} (T,f): & X\times E & \to & X\times E \\ & (x,e) & \mapsto & (Tx, f(x,e)) \end{array}$$

where T is a mapping of X into itself, and $f : X \times E \to E$. Consequently, the fiber $\{x\} \times E$ is mapped into the fiber $\{Tx\} \times E$ following $e \mapsto f(x, e)$. The notation $SW(X, E)$ for the set of such dynamical systems is classical. If E is a group or a space on which a group acts, this kind of fibered dynamics is called a cocycle. We can then note a cocycle by

$$\begin{array}{rccc} (T,f): & X\times E & \to & X\times E \\ & (x,e) & \mapsto & (Tx, f(x).e) \end{array}$$

with $f : X \to E$ and the dot . stands for the group multiplication or action.

The n-th iterate of the cocycle (T, f), $n \geq 1$, is of the form

$$(T,f)^n.(x,e) = (T^n x, f(T^{n-1}x) \circ \cdots \circ f(x).e)$$

We say that two cocycles over the same transformation $\psi_i = (T, f_i) \in SW(X, E)$, $i = 1, 2$, are (semi-)*conjugate* iff there exists $g : X \to E$ such that

$$\psi_1 \circ (id, g) = (id, g) \circ \psi_2$$

and we remark that it is a notion stronger than that of dynamical (semi-)conjugation by a mapping $h : X \times E \circlearrowleft$ satisfying $\psi_1 \circ h = h \circ \psi_2$, since conjugation of cocycles preserves the fibered-space structure of $X \times E$.

In general contexts we suppose that (X, μ), the *basis* of the dynamics, is a measured space and that T is ergodic with respect to μ. A particular case, which brings us to the subject of our study, occurs when $X = \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, a d-dimensionan torus, and $T = R_\alpha : x \mapsto x + \alpha$, is a minimal translation, and therefore uniquely ergodic with respect to the Haar measure on the torus. These cocycles are called *quasiperiodic* and α is called the *frequency* of the cocycle.

We remark here that, depending on the structure of E, the *fibers*, we can define measurable cocycles, or C^k-differentiable cocycles, with $k \in \mathbb{N} \cup \{\infty, \omega\}$ (ω stands for real analytic), according to the regularity of the mapping f.

Cocycles in linear groups come up naturally in dynamical systems. For example, if φ is a diffeomorphism of the torus $\mathbb{T}^d$, its differential defines a cocycle on $T\mathbb{T}^d \approx \mathbb{T}^d \times \mathbb{R}^d$ in a natural way by

$$(x, y) \mapsto (\varphi(x), D\varphi(x).y)$$

An other class of examples, closer to our subject, is that of fibered linear flows. Such a flow is defined as the fundamental solution of the system of ODEs

$$\begin{array}{rcl} X' & = & F(\theta).X \\ \theta' & = & \omega = (\alpha, 1) \in \mathbb{T}^{d+1} \end{array}$$

where $F : \mathbb{T}^{d+1} \to g$, and g is a matrix algebra in $M_N(\mathbb{R})$. The map of first return in the vertical circle $\mathbb{T} \hookrightarrow \mathbb{T}^d \times \mathbb{T}$ is a cocycle on $\{0 \ \times \mathbb{T}^d \times \mathbb{R}^N\}$. It is quasiperiodic if $\{\mathbb{R}\omega \mod \mathbb{Z}^{d+1}\}$ is dense in $\mathbb{T}^{d+1}$.

When α is rational, such flows are perfectly understood thanks to the Floquet representation of solutions: The solutions of the system of ODEs

$$X' = U(\theta).X \tag{1.1}$$
$$\theta' = k \in \mathbb{Q}^d \tag{1.2}$$

are of the form $X(t) = B(kt+\theta_0).e^{tU_0(\theta_0)}.X_0$ where $B(\cdot)$ is a $2\mathbb{Z}^d$-periodic map in the matrix group of algebra g. For cocycles, the Floquet representaiton of a solution corresponds to conjugation of a cocycle to a constant one: a cocycle (R_α, f) is called constant if $f : \mathbb{T}^d \to G$ is a constant mapping. Such cocycles are called *reducible*.

This normal form theorem breaks down when α is irrational, and the goal of the theory is to examine the density properties of Floquet-type solutions, and the possibility of approximation of any given vector field with a field admitting Floquet-type solutions. We now restrict ourselves to the case of cocycles in $\mathbb{T}^d \times G$, where G is a semisimple compact Lie group, such as $SU(N)$ or $SO(N)$.

The first step is the study of vector fields close to constants, or cocycles close to constants. In this setting the breakdown of Floquet theory is attributed to *small divisor* phenomena. The use of K.A.M. machinery allows, however, to give an affirmative answer to both questions, for a full measure set of frequencies (called Diophantine) (see [Kri99a] and the references therein). Moreover, Floquet representations occur in full measure sets for generic one-parameter families of cocycles. On the other hand, we know that the Floquet representation does break down for generic cocycles (cf [Eli02]).

The second step was carried out in a first time by R. Krikorian in his article [Kri01] on one-frequency cocycles in $SU(2)$. There, under a full measure condition on the frequency, the density of reducible cocycles is shown to be true when $G = SU(2)$. However, not all cocycles can be conjugated arbitrarily close to a reducible cocycle (which amounts to the approximation by Floquet-type solutions in the formulation above). It is however shown that a weaker form of approximation can be obtained : every cocycle can be conjugated arbitrarily close to a cocycle taking values in the subgroup of diagonal matrices. Cocycles conjugated to diagonal ones are called *torus-reducible* and those who can be conjugated arbitrarily close to diagonal cocycles are caled *almost torus-reducible*.

The goal of this mémoire is to generalize these results to more general contexts. Here, some complications due to homotopy have to be faced.

Theorem 1.1. *For a.e. frequency in $\mathbb{T}$, and for any given semisimple compact Lie group G, reducible cocycles are dense in $SW^{\infty,1}(\mathbb{T}, G)$, the subspace of $SW^{\infty}(\mathbb{T}, G)$ formed by cocycles homotopic to $Id : \mathbb{T} \times G \circlearrowleft$.*

For cocycles non-homotopic to the identity, we obtain

Theorem 1.2. *Under the same hypotheses, if a cocycle $(\alpha, A(\cdot)) \in SW^{\infty,m}(\mathbb{T}, G)$, the subspace of cocycles whose m-th iterate is homotopic to the Id, $(\alpha, A(\cdot))^m \in SW^{\infty,1}(\mathbb{T}, G)$ and it is accumulated by reducible cocycles in $SW^{\infty,1}(m\mathbb{T}, G)$.*

where we remark that the approximation holds for cocycles of longer periods. The second part of R. Krikorian's theorem also admits a generalization.

Theorem 1.3. *Under the same hypotheses on α and G, every cocycle in $SW^{\infty,1}(\mathbb{T}, G)$ is almost torus-reducible.*

again, for non-homotopic cocycles we have to consider an iterate:

Theorem 1.4. *Under the same hypotheses on α and G, for every cocycle $(\alpha, A(\cdot)) \in SW^{\infty,m}(\mathbb{T}, G)$, there exists χ dividing m such that $(\alpha, A(\cdot))^\chi \in SW^{\infty,m/\chi}(\mathbb{T}, G)$ is almost torus-reducible.*

We remark that almost torus-reducibility holds without passing to longer periods, but only by iterating.

The proof follows the line of argument of R. Krikorian's article cited above, but we also use some techniques appearing in [Fra04], and adapt them to the more general algebraic setting. A brief presentation of the argument is as follows.

A theorem by R. Krikorian stating that a cocycle is C^∞-reducible iff the sequence of its iterates is precompact in C^∞. This fact allows us to construct examples of cocycles that are not reducible, and in fact not even almost reducible. These are the periodic geodesics of the group over any irrational rotation. Such examples demostrate the fact that almost reducibility is not a global phenomenon, and explains the use of almost torus-reducibility in its place.

Subsequently, the renormalization scheme allows us to identify such cocycles as the only models of cocycles for which we can rule out almost reducibility. Disregarding homotopy issues, if there is no way of conjugating a given cocycle close to a constant one, then it can be conjugated close to a periodic geodesic.

Finally, we take up the local study of the different types of models of dynamics, where a diophantine condition on the rotation is needed so that K.A.M. techniques can be applicable. The local study of periodic geodesics over such rotations shows that almost reducibility to the model holds in a small enough neighbourhood, under the condition that the scheme is initiated on a cocycle obtained as a limit of renormalization. Moreover, we show that cocycles who are driven by renormalization towards shorter geodesics (or constant cocycles) are dense in a neighbourhood of these models. Finally, we prove the almost reducibility theorem for cocycles close to constants, and reprove the theorem of local density of reducible cocycles.

The global almost torus-reducibility theorem has now been proved case-by-case, and the global density theorem follows from a finite induction argument. If the cocycle can be conjugated to a neighbourhood of a constant cocycle, it is accumulated by reducible ones. If not, it is almost torus-reducible, and the corresponding periodic geodesic is accumulated by cocycles conjugate close to shorter periodic geodesics. Since the periodic geodesics form a discrete set, after a finite number of similar steps and arbitrarily small perturbations we obtain a cocycle that can be conjugated close to constant ones, and the theorem has been proved.

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Chapter 2

Basic definitions and tools from Algebra and Analysis

2.1 Algebra

2.1.1 Compact Lie groups

For this section, we refer mainly to [Die75], whose notations we have adapted, and [Hel62], as well as [Bum04], [DK00] and [Kri99a]. By G we will denote a real connected compact semi-simple Lie group, and sometimes refer to it simply as a compact group, even though some of the results presented below are true in a more general context.

Generalities

Let G be a real connected compact Lie group, furnished with a Haar measure, both left- and right-invariant, inducing an invariant Riemannian metric $d(\cdot,\cdot)$. The following theorem by Gleason-Montgomery-Zipplin supplies a sufficient condition of algebraic and topological nature under which a topological group is a Lie group.

Theorem 2.1. *Let G' be a locally compact topological group satisfying the additional hypothesis of non-existence of small subgroups. Then G' is a Lie group.*

The hypothesis of non-existence of small subgroups is the following: there exists a neighborhood of the Id which contains only the trivial topological subgroup Id. The study of real compact groups is capital in the theory of Lie groups, as explained in [Die75] and their study is to a large extent reduced to that of semisimple ones [1]. Under these hypotheses, G admits a representation in a unitary group. For this reason, with the exception of this first chapter, we will abuse the notation A^* for the inverse of $A \in G$.

The tangent space to the identity of G, furnished with the linear antisymmetric commutator $[\cdot,\cdot] : g \times g \to g$, is by definition its Lie algebra g. The endomorphisms of g preserving its structure as a Lie algebra are denoted by $Aut(g)$, while the linear transformations of the vector space g are denoted by $GL(g)$.

The exponential mapping

$$\begin{array}{rccc} \exp: & g & \to & G \\ & h & \mapsto & \exp(h) = e^h \end{array}$$

is defined as the time one of the flow generated by the right-invariant vector field associated to h. As a result of the compactness of the group, the exponential mapping is onto.

We can define the adjoint representation of the group into its algebra as the differential at the Id of the mapping $B \mapsto A.B.A^{-1}$, for $A \in G$ fixed. We thus obtain a linear automorphism

$$\begin{array}{rccc} Ad_A: & h & \mapsto & \frac{d}{dt} A.B(\cdot).A^{-1}|_{t=0} \end{array}$$

1. Semisimplicity will be defined later on by means of a geometric criterion.

where $B(\cdot)$ is defined locally around 0 and such that $\frac{d}{dt}B(\cdot)|_{t=0} = h_{Id}$ and $B(0) = Id$. The image of G in $Aut(g)$ under its adjoint representation is noted by $Inn(g)$ and called the *inner automorphisms* of g. It is a subgroup of $Aut(g)$ since $[Ad_A.h, Ad_A.h'] = Ad_A.[h, h']$ holds for all h and h' in g. We also have that $A \in Z_G$, if, and only if, $Ad(A) = Id_g$. Therefore, $Inn(g)$ is naturally isomorphic to G/Z_G, which is the Lie group that has the same Lie algebra as G, but whose center is reduced to the Id.

We can also define the adjoint representation of g into itself via the commutator: for h fixed, let $ad_h : h' \longmapsto [h, h']$ and the two representations are connected by $Ad(e^h).h' = e^{ad(h)}.h'$, for all $h, h' \in g$. This representation induces a bilinear symmetric form on g, called the *Cartan-Killing form*, by

$$\langle h, h' \rangle = -tr(ad_h \circ ad_{h'})$$

The group and the algebra are *semi-simple* if the Cartan-Killing form is non-degenerate, and it can be proved that a semi-simple group is compact if it is *positive definite*, in which case the algebra is said to be compact. The Riemannian structure induced by a non-degenerate Cartan-Killing form is compatible with the one induced by the Haar measure.

The center of G, denoted by Z_G is finite and its cardinal will be denoted by c_G.

The adjoint action of G is an isometry of g with respect to the Cartan-Killing form and the latter satisfies identically

$$\langle [h, h'], h'' \rangle = \langle h, [h', h''] \rangle$$

We will use the following theorem, from [Die75] (*21.6.9*).

Theorem 2.2. *Let G be a connected real Lie group (not necessarily compact or semisimple), Z_G its center and g its Lie algebra. Then, the following conditions are equivalent:*

1. *The group G/Z_G is compact*
2. *The group G is isomorphic to a product $\mathbb{R}^m \times G_1$, where G_1 is compact*
3. *The group $\tilde{G}$, the universal covering of G, is isomorphic to a product $\mathbb{R}^n \times K$, where K is a compact semi-simple simply connected Lie group.*
4. *The Lie algebra g is decomposed in $\mathfrak{c} \oplus \mathfrak{D}(g)$, where $\mathfrak{c}$ is its center and $\mathfrak{D}(g)$ is the derived algebra $[g, g]$. The restriction of the Cartan-Killing form to $\mathfrak{D}(g)$ is positive non-degenerate. Under these assumptions, $\mathfrak{D}(g)$ is isomorphic to the Lie algebra of K; its center Z_K is finite; G is isomorphic to $\tilde{G}/D$, where D is a discrete subroup of $\mathbb{R}^n \times Z_K$, Z_G is isomorphic to $(\mathbb{R}^n \times Z_K)/D$ and if we call $\check{G} = G/Z_G$, we have $Z_{\check{G}} = \{Id\}$. The groups $Ad(\tilde{G})$, $Ad(G)$ and $Ad(K)$, viewed as subroups of $Aut(g)$ are all equal and isomorphic to $G/Z_G = Inn(G)$.*

Tori and maximal tori

For this section, see [Die75], *21.7*. From this section on, the group is supposed compact semisimple, even though some of the results hold if we drop the semisimpleness assumption.

Let us consider a connected abelian subgroup $\mathcal{H} \subset G$. Since G is compact, $\mathcal{H}$ is in fact homomorphic to a torus $\mathbb{R}^d/\mathbb{Z}^d$ and for this reason we will call it a torus of G. Using standard arguments, we can find $\mathcal{T}$, a maximal abelian subgroup of G (called a *maximal torus*[2]), containing $\mathcal{H}$. Maximality is understood in the sense of set-theoretical inclusion. We have then

Proposition 2.3.
1. *Let $\mathcal{T}$ be a maximal torus of G. Then, for any $A \in G$, there exists $S \in G$ such that $S.A.S^{-1} \in \mathcal{T}$. In particular, G agrees set-theoretically with the union of its maximal tori.*
2. *If $\mathcal{T}$ and $\mathcal{T}'$ are maximal tori of G, there exists $S \in G$ such that $S.\mathcal{T}'.S^{-1} = \mathcal{T}$.*
3. *The centralizer $Z_G(\mathcal{H})$ of a torus $\mathcal{H} \subset G$ is connected and, if $S \in Z_G(\mathcal{H})$, then $\mathcal{H} \cup \{S\}$ is contained in a maximal torus.*
4. *The neutral component $Z_G^0(S)$ of $Z_G(S)$ (the centralizer of $S \in G$) is equal to the union of the maximal tori containing S.*

2. Since the maximal tori of matrix groups such as $SU(w+1)$ are formed by diagonal matrices, we will sometimes abuse the word diagonal and its derivatives.

By *2*, the dimension of maximal tori of G is a constant depending only on the group. It is called the *rank* of G and we will denote it by w.

We now give the following definition, concerning the number of maximal tori containing an element of G.

Definition 2.4. An element in $S \in G$ is called *regular* if it is contained in a unique maximal torus, and *generic* if it generates it topologically, i.e. if the closure of $\{S^k\}_{k\in\mathbb{Z}}$ is the torus itself. It will be called *singular* if it is not regular.

Genericity is a generic condition, just as minimality is generic in the space of translations in $\mathbb{T}^w$. Therefore, S is regular iff the neutral component of $Z_G(S)$ is a maximal torus and a generic element is a fortiori regular. An example of a regular but not generic element can be constructed by using any root of 1 in $\mathbb{S}^1$ other than -1. We refer the reader to section *1.1.2* for the canonical embedding $\mathbb{S}^1 \hookrightarrow SU(2)$ which gives an example of such an element in a Lie group. More concretely, we consider elements of $SU(2)$ of the type

$$\begin{bmatrix} \exp(2i\pi p/q) & 0 \\ 0 & \exp(-2i\pi p/q) \end{bmatrix}$$

with $p, q \in \mathbb{Z}$, $q \neq 0$ and $\frac{p}{q} \neq 1, \frac{1}{2}$. It can be verified that if S is of this form, it commutes only with diagonal matrices (which are the maximal tori in $SU(2)$, as we will see in subsection *1.1.2*), but $S^q = Id$, so that S is not generic.

The normalizer of a maximal torus $N_G(\mathcal{T}) = \{S \in G,\ S.\mathcal{T}.S^{-1} \subset \mathcal{T}\}$ in general strictly contains the torus. The factor group $W(\mathcal{T}) = N_G(\mathcal{T})/\mathcal{T}$ is a finite group, called the *Weyl group* with respect to $\mathcal{T}$. Since any two tori are obtained by conjugation, all the groups $W(\mathcal{T})$ are all isomorphic one to another and therefore isomorphic to the Weyl group of G, noted W. For any torus $\mathcal{H}$, the group $W_{\mathcal{H}} = N_G(\mathcal{H})/Z_G(\mathcal{H})$ is a subgroup of W and we have equality if $\mathcal{H}$ is maximal. In the particular case where $S \in G$ is given, the intersection of all maximal tori containing S is a torus $\mathcal{H}$ of G, and the group $W_{\mathcal{H}}$ is isomorphic to W if, and only if, $\mathcal{H}$ is a maximal torus, i.e. if s is regular.

We also reprove the following lemma, a proof of which using techniques from Algebraic Geometry can be found in [Kri99a] (lemma 2.3.14).

Lemma 2.5. *Let G be a real compact semi-simple Lie group. Then, there exists a minimal positive integer χ_G, depending only on the group G and smaller than $\#W!$ with the following property. If K, a subset of G, is abelian, then*

$$K^{\chi_G} = \{k^{\chi_G}, k \in K\}$$

is a subset of a maximal torus of G. The integer χ_G is equal to 1 if $G = SU(w+1)$.

Proof. Let $A, B \in G$ such that $ABA^{-1}B^{-1} = Id$ and suppose that there does not exist a maximal torus passing by both A and B.

We can fix, therefore, a maximal torus $\mathcal{T}_A$ containing A. Since $B \notin \mathcal{T}_A$, there exists $\tilde{A} \in \mathcal{T}_A$ such that

$$B\tilde{A}B^{-1}\tilde{A}^{-1} = C' \neq Id$$

where $C' \in Z_G^0(A)$. For this, one only need consider a path $\Phi(\cdot) : [0,1] \to \mathcal{T}_A$ connecting $\tilde{A}$ to the Id and to verify that then $\Phi(\cdot)B\Phi(\cdot)^{-1}B^{-1}$ commutes with A at all times, while it connects C' with the Id. In particular, each generic element $\tilde{A}$ is such.

We also remark that, by the commutation relation $ABA^{-1}B^{-1} = Id$, we find directly that such a B defines a whole class $B.Z_G^0(A)$ of elements commuting with A but not on the same torus as A.

Since $C = C'.\tilde{A} \in Z_G^0(A)$ and generic (it is conjugate to $\tilde{A}$ by B), it defines a unique maximal torus $\mathcal{T}_C$. By point *2* of Proposition 2.2 applied to the group $Z_G^0(A)$ gives the existence of an $S \in Z_G^0(A)$ such that $S^{-1}.\mathcal{T}_C.S = \mathcal{T}_A$, which implies that

$$S.B.\mathcal{T}_A.B^{-1}.S^{-1} = \mathcal{T}_A$$

Therefore, $S.B$ is in $W(\mathcal{T}_A)$, and does not depend on the maximal torus $\mathcal{T}_A$. Therefore, $Ad(S.B)|_{\mathcal{T}_A}$ is of finite order, so that $(S.B)^n \in \mathcal{T}_A$ for some n not bigger than $\#W$. Thus, $B^n \in Z_G^0(A)$,

which implies that $Ad(B)$ preserves some maximal torus passing by A, and therefore B^n is in this torus. □

For any given group G, the constant χ_G of this lemma is to be chosen optimal. We also prove

Lemma 2.6. *If $\tilde{G}$ is a semi-simple compact group and $G = \tilde{G}/K$ where $K \subset Z_{\tilde{G}}$ is a non-trivial subgroup, then $\chi_{\tilde{G}} \leq \chi_G$, and $\chi_{\tilde{G}}$ divides χ_G.*

Proof. For the first part, we use the fact that the commutator group of a semisimple compact Lie group is the group itself, and we let $\tilde{A}, \tilde{B} \in \tilde{G}$ such that $\tilde{A}.\tilde{B}.\tilde{A}^{-1}.\tilde{B}^{-1} = S \in K$, $S \neq Id$. From the commutation relation it follows that, if we call π the projection $\tilde{G} \to G$, then $\pi(\tilde{A}) \neq \pi(\tilde{B})$. On the other hand, $\pi(\tilde{A})$ and $\pi(\tilde{B})$ commute in G.

Consider now $\mathcal{T}_{\tilde{A}} \subset \tilde{G}$, a maximal torus containing $\tilde{A}$. Then, $\mathcal{T}_{\tilde{A}}.\tilde{B} \neq \mathcal{T}_{\tilde{A}}$, $(\mathcal{T}_{\tilde{A}}.\tilde{B})/K \neq \mathcal{T}_{\tilde{A}}/K$, $\pi(\mathcal{T}_{\tilde{A}}.\tilde{B})$ and $\pi(\tilde{A})$ commute.

Therefore, there exist elements in G who commute, but whose preimages in $\tilde{G}$ do not.

The second statement follows from the fact that if $\tilde{A}, \tilde{B} \in \tilde{G}$ do not commute, but $A = \pi(\tilde{A})$ and $B = \pi(\tilde{B})$ do commute in G, then $\tilde{A}.\tilde{B}.\tilde{A}^{-1}.\tilde{B}^{-1} = S \in K$. Consequently, $\tilde{A}^{\#K}$ and $\tilde{B}^{\#K}$ commute. □

This second lemma clarifies the relation of the constant χ_G with the homotopy of a group, since any non-simply connected group is obtained by factorization of a simply connected one as in the statement of the lemma. Finally, we will need the following last lemma in this series. In order to keep the proof simple, and since the lemma itself will be used in such a context, we use the language of toral algebras. We therefore postpone it to the appendix 11. The reader is however encouraged to read the proof, since it actually comes up in chapter 6.

Lemma 2.7. *Let G be as before and suppose that it is simply connected. Then, $\chi_G = 1$. In other words, if G is simply connected, commuting elements can be simultaneously diagonalized.*

Toral algebras and root-spaces

Let us consider an abelian subalgebra $\mathfrak{h}$ of g, i.e. a subalgebra for which the restriction of the Lie bracket to $\mathfrak{h} \times \mathfrak{h}$ vanishes identically. It is the Lie algebra of a torus $\mathcal{H}$ of G, and for this reason it will be called a *toral algebra*. If $\mathcal{H}$ is contained in $\mathcal{T}$, a maximal torus whose Lie algebra is $\mathfrak{t}$, then $\mathfrak{h} \subset \mathfrak{t}$. The Lie algebra $\mathfrak{t}$ is a maximal toral algebra and its dimension is w. The relations between different (maximal) toral algebras are analogous to those between (maximal) tori. We also give the definition of regular and singular elements of g.

Definition 2.8. An element $s \in g$ will be called *regular* if it is contained in a unique maximal toral algebra, and *generic* if $\exp(\mathbb{R}s)$ is dense in the maximal torus containing it. If it is not regular, it will be called *singular*.

If $s \in g$ is regular, then $Z_G(\exp(\mathbb{R}s))$ is a maximal torus of G.

A special case of Lie group is the n-dimensional torus $\mathbf{U}^n = \{(z_i) \in (\mathbb{C}^*)^n,\ |z_i| = 1\}$. Its Lie algebra is $i\mathbb{R}^n \subset \mathbb{C}^n$, where $\mathbb{C}^n$ is the Lie algebra of the ambient group $(\mathbb{C}^*)^n$. The kernel of the group homomorphism $\exp : i\mathbb{R}^n \to \mathbf{U}^n$ is $2i\pi\mathbb{Z}^n$ and every character χ of $\mathbf{U}^n$ (i.e. a group homomorphism $\mathbf{U}^n \to S^1$) can be written as

$$\chi(u) = e^{\gamma(i\xi_1,\ldots,i\xi_n)}$$

if $u = (e^{i\xi_1}, ..., e^{i\xi_n})$, so that $\gamma(2i\pi m_1, ..., 2i\pi m_n) \in 2i\pi\mathbb{Z}$ if all $m_i \in \mathbb{Z}$. For the same reasons, if $\mathcal{T}$ is a maximal torus and $\mathfrak{t}$ its Lie algebra, we find that the restriction of exp in $\mathfrak{t}$ defines a group homomorphism, whose kernel is a lattice $\Gamma_{\mathfrak{t}} \subset \mathfrak{t}$. If χ is a character of $\mathcal{T}$ and $h \in \mathfrak{t}$, then

$$\chi(\exp(h)) = e^{2i\pi\rho(h)}$$

for some $\rho \in \mathfrak{t}^*$, such that $\rho(h) \in \mathbb{Z}$ for all $h \in \Gamma_{\mathcal{T}}$. The set of such ρ is $\Gamma^*_{\mathcal{T}}$, the dual lattice of $\Gamma_{\mathcal{T}}$. The elements of $2i\pi\Gamma^*_{\mathcal{T}} \subset \mathfrak{t}^*_{\mathbb{C}}$ (the dual space of the complexified Lie algebra $\mathfrak{t}_{\mathbb{C}}$) are the weights of $\mathfrak{t}$. They are $\mathbb{R}$-linear mappings $\mathfrak{t} \to i\mathbb{R} \subset \mathbb{C}$. The lattice $2i\pi\Gamma^*_{\mathfrak{t}}$ will be denoted by $P(G, \mathfrak{t})$ or

simply $P(G)$, since the lattices corresponding to different tori are obtained by the action of an inner endomorphism.

Let us now fix a maximal toral algebra $\mathfrak{t}$ of g and the corresponding torus. We will denote by $\mathfrak{t}_{\mathbb{C}}$ and $g_{\mathbb{C}}$ the corresponding complexified spaces and we introduce the involution $c(x+iy) = x - iy$, where $x, y \in g$. The Cartan-Killing form extends to a complex symmetric (and therefore not Hermitian) bilinear form.

The properties of the representation of $\mathcal{T} = \exp(\mathfrak{t})$ in $g_{\mathbb{C}}$ imply the existence of a finite subset Δ of $2i\pi\Gamma^*_{\mathcal{T}}$ such that $g_{\mathbb{C}}$ decomposes into the sum of $\mathfrak{t}_{\mathbb{C}}$ and subspaces E_ρ such that, for all $e_\rho \in E_\rho$ and $h \in \mathfrak{t}$,

$$ad_h.e_\rho = 2i\pi\rho(h)e_\rho$$

Such weights are called the roots of G with respect to $\mathcal{T}$. We note that if $\rho \in \Delta$, then $-\rho \in \Delta$ and $E_{-\rho} = c(E_\rho)$. If $\rho+\rho'$, $\rho' \neq -\rho$, is a root, we have $[E_\rho, E_{\rho'}] \subset E_{\rho+\rho'}$ and the bracket is 0 otherwise. We will denote by $Q(g)$ the lattice generated by the roots over $\mathbb{Z}$. We have clearly $Q(g) \subset P(g)$.

In the more familiar context of $SU(w+1)$, one retrieves these concepts by fixing a basis for $\mathbb{C}^{w+1}$, thus fixing the torus of diagonal matrices and the corresponding lattice of preimages of the Id in $i\mathbb{R}^w$. Each non-diaconal entry at position (i,j) in $su(w+1)$, the vector space of traceless skew-symmetric $(w+1)\times(w+1)$ matrices, corresponds to a space E_ρ, and the root functional simply to $\rho_{ij} : diag(2i\pi a_k) \mapsto (a_i - a_j)$. In particular, the number of roots is greater than w, and there are relations of linear dependence between them.

A vector $h \in \mathfrak{t}$ is regular if, and only if, $\rho(h) \neq 0$ for all roots ρ, or equivalently, if $\exp(h)$ does not belong to any of the subgroups $\chi_\rho^{-1}(1) \subset \mathfrak{t}$, where $\chi_\rho(\exp(h)) = e^{2i\pi\rho(h)}$ is the character whose weight is ρ.

For any given ρ, there exists a subalgebra $\mathbb{R}h_\rho \oplus \mathbb{C}j_\rho \approx su(2)$ so that the following multiplication table and orthogonality relations hold true.

$$\begin{aligned}
[th_\rho, zj_\rho] &= 2iztj_\rho, \\
[zj_\rho, wj_\rho] &= 2\frac{Re(iz\bar{w})}{|h_\rho|}h_\rho, \ \forall z, w \in \mathbb{C} \\
\langle j_\rho, h_\rho\rangle &= \langle ij_\rho, h_\rho\rangle = \langle j_\rho, ij_\rho\rangle = 0 \\
|zj_\rho| &= |z|, \text{ and } |h_\rho| > 0 \\
[j_\rho, j_{\rho'}] &\in E_{\rho+\rho'}
\end{aligned}$$

Here, $h_\rho \in \mathfrak{t}$, the complex plane $\mathbb{C}j_\rho$ stands for the two real-dimensional plane underlyign $E_\rho \oplus E_{-\rho}$ and we remind that $E_{\rho+\rho'} = \{0\}$ if $\rho+\rho'$ is not a root. We stress that the complex directions $\mathbb{C}j_\rho$ complexify to 2-dimensional complex planes in $g_{\mathbb{C}}$. The subalgebras thus constructed are called *root spaces*.

Basis of the root system

Using the standard properties of semi-simple compact Lie algebras, we can obtain a basis $\tilde{\Delta}$ of Δ such that if

$$\rho' = \sum_{\rho\in\tilde{\Delta}} m_{\rho'\rho}\rho \tag{2.1}$$

then the $m_{\rho'\rho}$ are all integers of the same sign. This results in the existence of a partition $\Delta = \Delta_+ \cup (-\Delta_+)$ such that $\rho \in \Delta_+$ (resp. $\in -\Delta_+$) if, and only if, all the integers in the above sum are positive (resp. negative).

We then have the following decomposition:

$$\begin{aligned}
g &= \mathfrak{t} \oplus \bigoplus_{\rho\in\Delta_+} \mathbb{C}j_\rho \\
&= \bigoplus_{\rho\in\tilde{\Delta}} \mathbb{R}h_\rho \oplus \bigoplus_{\rho\in\Delta_+} \mathbb{C}j_\rho
\end{aligned} \tag{2.2}$$

with

$$\begin{aligned} \langle j_\rho, j_{\rho'} \rangle &= 0 \\ |j_\rho| &= 1 \\ \langle h_\rho, h_{\rho'} \rangle &\lessgtr 0 \\ [h, j_\rho] &= 2i\pi\rho(h) j_\rho, \forall h \in \mathfrak{t} \end{aligned}$$

The basis $\tilde{\Delta}$ will be call a Weyl's basis for the root system and the decomposition of g as above the root system decomposition with respect to the maximal abelian algebra $\mathfrak{t}$. The choice of such a basis is a simultaneous diagonalization of all operators $ad_h : s \mapsto [h, s], s \in g$, for $h \in \mathfrak{t}$. These operators therefore commute.

A special case of mappings $\mathfrak{t} \to G$ which will be of great importance in our study is that of *periodic geodesics* of the group, which are one-parameter subgroups isomorphic to $\mathbb{S}^1$. Since such mappings can be viewed as mappings $\mathfrak{t} \to \mathcal{T} \subset G$, where $\mathcal{T}$ is a maximal torus of G, this geodesic admits a lift in $\mathfrak{t}$ (the Lie algebra of $\mathcal{T}$). Since, additionally, any basis of $\mathfrak{t}$ can be obtained by the action of $W(G)$ on a single one, we can consider a fixed basis and introduce the notations

$$E_r(\cdot) = \exp(\sum\nolimits_\rho (2\pi r_\rho h_\rho \cdot)) \text{ and} \tag{2.3}$$

$$E_{r,a}(\cdot) = \exp(\sum\nolimits_\rho (2\pi (r_\rho \cdot + a_\rho) h_\rho) \tag{2.4}$$

where $r \in \mathbb{Z}^w$ and $a \in \mathbb{R}^w$ characterize the geodesic, modulo the action of $W(G)$.

Finally, by duality, there exists a basis $(H_\rho)_{\rho \in \tilde{\Delta}}$ of $\mathfrak{t}$, such that $\rho'(H_\rho) = \delta_{\rho,\rho'}$. We then have

$$H_\rho = \sum_{\rho \in \tilde{\Delta}} k_{\rho'\rho} H_{\rho'}$$

with $k_{\rho'\rho}$ rational numbers of the same sign. We can suppose that they are of the form

$$\frac{l_{\rho'\rho}}{D}$$

where $D \in \mathbb{N}^*$ and $|l_{\rho'\rho}| \leq e$, with $e \in \mathbb{N}^*$. Clearly, $\#\tilde{\Delta} = w$, the rank of the group, while $q = \#\Delta_+ = \frac{1}{2}(f - w)$, where f is the real dimension of g.

2.1.2 Notation and algebra in $SU(2)$ and $SO(3)$

Since a certain variety of the phenomena that we will describe are already present when G is $SU(2)$ or $SO(3)$ we will construct some concrete examples in these cases and we will compare the results obtained in this mémoire with the ones already existing in the literature. In order to do this, we need to describe explicitly the notions defined in the previous section in this more concrete setting.

We bring to the attention of the reader the fact that in these concrete examples we introduce a normalization of the Cartan-Killing form which is not compatible with the one used in the abstract case.

The group $SU(2)$

Let us denote the matrix $S \in SU(2)$, $S = \begin{bmatrix} t & z \\ -\bar{z} & \bar{t} \end{bmatrix}$, where $(t, z) \in \mathbb{C}^2$ and $|t|^2 + |z|^2 = 1$, by $\{t, z\}_{SU(2)}$. The subscript will be suppressed from the notation, unless necessary. The manifold $SU(2)$ is naturally identified with $\mathbb{S}^3 \subset \mathbb{C}^2$ through $G \ni \{t, z\}_{SU(2)} \mapsto (t, z) \in \mathbb{S}^3$. The law of multiplication is pushed forward to the mapping of $\mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$ given by $\{t_1, z_1\}.\{t_2, z_2\} = \{t_1 t_2 - z_1 \bar{z}_2, t_1 z_2 + \bar{t}_2 z_1\}$. In particular, $SU(2)$ is simply connected. Inversion is pushed forward to the involution of S^3 $\{t, z\} \mapsto \{\bar{t}, -z\}$. The circle $\mathbb{S}^1$ is naturally embedded in G as the group of diagonal matrices, which is a maximal torus of $SU(2)$. In particular $w = 1$, the rank of $SU(2)$.

The Lie algebra $su(2)$ is naturally isomorphic to $\mathbb{R}^3 \approx \mathbb{R}\times\mathbb{C}$ equipped with its vector and scalar product. The element $s = \begin{bmatrix} it & u \\ -\bar{u} & -it \end{bmatrix}$ will be denoted by $\{t,u\}_{su(2)} \in \mathbb{R}\times\mathbb{C}$, or $\{t,\mathcal{R}u,\mathcal{I}u\}_{\mathfrak{g}} \in \mathbb{R}^3$. The scalar product will sometimes be denoted by

$$t_1t_2 + \mathcal{R}(u_1\bar{u}_2) = t_1t_2 + \mathcal{R}u_1.\mathcal{R}u_2 + \mathcal{I}u_1.\mathcal{I}u_2$$

Mappings with values in $su(2)$ will be denoted by

$$U(\cdot) = \{U_t(\cdot), U_z(\cdot)\}_{su(2)}$$

in these coordinates, where $U_t(\cdot)$ is a real-valued and $U_z(\cdot)$ is a complex-valued function.

The adjoint action of $h \in su(2)$ on itself is pushed-forward to twice the vector product:

$$ad_{\{1,0\}}.\{0,1,0\} = 2\{0,0,1\}$$

plus cyclic permutations and the Cartan-Killing form, normalized by $\langle h,h'\rangle = -\frac{1}{8}tr(ad(h)\circ ad(h'))$ is pushed-forward to the scalar product of $\mathbb{R}^3$. The periodic geodesics of the group for the induced Riemannian structure are of the form $S.\exp(\{2\pi r\cdot,0\}_{su(2)}).S^*$, $S \in SU(2)$. Under this normalization, the minimal length for a geodesic leaving from the Id and arriving at $-Id$ is π, and the minimal length of a periodic geodesic is twice as much, i.e. 2π. We also find directly that the preimages of the Id in the maximal toral algebra of diagonal matrices are points of coordinates in the lattice $2\pi\mathbb{Z}$. The first critical point of the exponential mapping is $\{\pi,0\}$, and the critical value is $-Id$.

The adjoint action of the group on its algebra is pushed-forward to the action of $SO(3)$ on $\mathbb{R}\times\mathbb{C}$. In particular, the diagonal matrices, of the form $\exp(\{t,0\}_{su(2)})$, fix the real direction and act by multiplication by e^{2it} in the complex direction.

Finally, the Weyl group of $SU(2)$ is naturally isomorphic to the multiplicative group $\{1,-1\}$ and two representatives are the matrices

$$Id \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the action of the second one being a reflection with respect to the complex plane:

$$(-1)_W.\{t,0\}_{su(2)} = \{-t,0\}_{su(2)}$$

The group $SO(3)$

Since $SO(3) = SU(2)/\{\pm Id\}$, we will identify the Lie algebras of the two groups and keep the same normalization for the Cartan-Killing form. In particular, all non-zero vectors in $so(3)$ are regular. From the normalization of the Cartan-Killing form, it follows that the minimal length of a periodic geodesic, typically

$$\begin{pmatrix} \cos(2\pi\cdot) & \sin(2\pi\cdot) & 0 \\ -\sin(2\pi\cdot) & \cos(2\pi\cdot) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.5}$$

is π. This geodesic admits a lift in $SU(2)$ to a geodesic connecting the Id with $-Id$, $\exp(\{\pi\cdot,0\}_{su(2)})$, and therefore is is not homotopic to a constant in $SO(3)$. On the other hand

$$\begin{pmatrix} \cos(4\pi\cdot) & \sin(4\pi\cdot) & 0 \\ -\sin(4\pi\cdot) & \cos(4\pi\cdot) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

admits a lift to $E_1(\cdot) = \exp(\{2\pi\cdot,0\}_{su(2)})$, and therefore it is a closed geodesic, homotopic to a constant and with minimal length. For this reason we will use the notation

$$E_{1/2}(\cdot) = R_{2\pi\cdot} = \begin{pmatrix} \cos(2\pi\cdot) & \sin(2\pi\cdot) & 0 \\ -\sin(2\pi\cdot) & \cos(2\pi\cdot) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for geodesics as in eq. 2.5. From the above, it follows that the preimages of the Id in the maximal toral algebra of standard rotations are points of coordinates in the lattice $\pi\mathbb{Z} \subset \mathbb{R}$.

We also remark that $\chi_{SO(3)} = 2$, and we investigate the relation of this fact with the fact that $SO(3) = SU(2)/\{\pm Id\}$. It is a common fact that if two matrices A and B in $SU(2)$ commute, they can be simultaneously diagonalized, or in our terminology, they belong to the same maximal torus. Since the torus of reference in this group is that of diagonal matrices, we can rephrase this into the existence of $S \in SU(2)$ such that $S.A.S^*$ and $S.B.S^*$ belong to $\mathcal{T}$. On the other hand, not any two commuting matrices in $SO(3)$ can be simultaneously conjugated to the torus of standard rotations. In order to see this, we consider the torus $\mathcal{T}/\{\pm Id\}$ of standard rotations and an element $\pi(\{e^{i\theta}, 0\}) \neq Id_{SO(3)}$ in it (π is the canonical projection $SU(2) \to SO(3)$). It commutes with all the matrices in $\mathcal{T}/\{\pm Id\}$, and conversely, if (a preimage of) it commutes with a matrix in $SU(2)$, this matrix is in $\mathcal{T}$. A simple calculation shows that

$$\{e^{i\theta}, 0\}.\{a, b\}.\{e^{-i\theta}, 0\}.\{\bar{a}, -b\} = \{|a| + e^{2i\theta}|b|, ab(1 + e^{2i\theta})\}$$

We see, therefore, that the equation

$$A.B.A^*.B^* = -Id$$

admits the solutions

$$\begin{aligned} A &= \{\pm i, 0\} \\ B &= \{0, b\} \end{aligned}$$

with $b \in S^1$. This family of solutions obviously does not intersect the standard torus, and its projection to $SO(3)$ gives the rotation

$$\pi(A) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which commutes with standard rotations, but also with the one parameter family

$$\pi(B(\theta)) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

for $\theta \in [0, 2\pi]$, which does not intersect the standard torus.

Finally, we introduce the standard basis of $so(3)$,

$$j_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; j_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; j_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

In the identification $su(2) \approx so(3)$, the vector j_1 corresponds to $\{1, 0\}_{su(2)}$, and the plane generated by j_2 and j_3 to the complex plane $\mathbb{C} \hookrightarrow su(2)$.

2.2 Calculus

2.2.1 Calculus in Lie groups

Let us now define the differential operator $L : C^\infty(\mathbb{T}, G) \to C^\infty(\mathbb{T}, g)$, as

$$L(A)(\cdot) = \partial A(\cdot).A^*(\cdot) \tag{2.6}$$

This operator uses the structure of the group in order to simplify the expressions of derivatives. Since a Lie group is a manifold, the derivative of a G-valued path is a section of its tangent bundle. The fact that the tangent bundle is trivial (i.e. trivializable in a canonical way) allows us to define the derivative of a path as a curve in a single vector space, naturally isomorphic to the

fibers, instead of a section of the tangent bundle. This is done by using the group structure of the manifold and mapping canonically, for each $x \in \mathbb{T}$, the tangent space $T_{A(x)}G \subset TG$ at $A(x) \in G$ to $g = T_{Id}G$ via the differential of left translation, $S \mapsto SA(x)^*$. [3] If the underlying group is simply $\mathbb{R}_+^*$, this is just a way of normalizing the derivative of the exponential as

$$\phi'(x) = \left(\frac{d}{dx}e^{\phi(x)}\right)e^{-\phi(x)}$$

instead of letting it grow along with $e^{\phi(x)}$. If the underlying group is $\mathbb{S}^1$, the tangent vector deviates from the vertical line oriented upwards when the base point is not $1 \in \mathbb{S}^1$, and right multiplication rectifies it to the standard one.

The basic properties of this operator are as follows.

1. $L(A(\cdot).B(\cdot)) = a(\cdot) + Ad(A(\cdot)).b(\cdot)$, where we introduce a notational convention: $L(A(\cdot)) = a(\cdot)$, $L(B(\cdot)) = b(\cdot)$ and similarly for any mapping $\mathbb{T} \to G$ (eventually $\mathbb{R} \to G$).
2. Derivation of $A^*(\cdot).A(\cdot) = Id$ gives $a^*(\cdot) = -Ad(A^*(\cdot)).a(\cdot)$ (notice the abuse of notation)
3. If $A(\cdot) = \exp(\phi(\cdot))$, where $\phi : C^\infty(\mathbb{R}, \mathfrak{t})$, with $\mathfrak{t} \subset g$ a toral algebra, then $a(\cdot) = \phi'(\cdot)$
4. $\partial L(A(\cdot).B(\cdot)) = \partial a(\cdot) + Ad(A(\cdot)).\partial b(\cdot) + [a(\cdot), Ad(A(\cdot)).b(\cdot)]$
5. If $A(\cdot) = \exp(U(\cdot))$ is C^0-close to the identity, then

$$a(\cdot) = \frac{\exp(ad(U(\cdot))) - Id}{ad(U(\cdot))}\partial U(\cdot)$$

where $\frac{\exp(ad(U(\cdot)))-Id}{ad(U(\cdot))}$ represents the operator $\sum_0^\infty \frac{1}{(n+1)!}ad(U(\cdot))^n$.

Property 1 implies the following fact. Let K be a subgroup of Z_G, $\tilde{G} = G/K$ and $\pi : G \to \tilde{G}$ the canonical projection. We remind that the Lie algebras of G and $\tilde{G}$ are naturally isomorphic. Then, if $A(\cdot) : \mathbb{T} \to G$ is C^1, $\tilde{A}(\cdot) = \pi(A(\cdot)) : \mathbb{T} \to \tilde{G}$ is also C^1-smooth (possibly of period smaller than 1) and property 1 implies that $\tilde{a}(\cdot) = a(\cdot)$, with a slight abuse of notation. If, keeping the same notations, we suppose that $\tilde{A}(\cdot)$ is given and we choose a lift $A(\cdot)$ of $\tilde{A}(\cdot)$ to G (i.e. $\pi(A(\cdot)) = \tilde{A}(\cdot)$), then $A(\cdot)$ will be at most $\#K$ periodic and $A(\cdot + 1) = S.A(\cdot)$, where $S \in K$. Then we find again that the derivatives of $A(\cdot)$ and $\tilde{A}(\cdot)$ in g are equal. Or, equivalently, the derivative in g of a curve in G depends on the isomorphism class of the Lie algebra, and not the group itself, a quite natural fact, since different groups with the same Lie algebra are locally indistinguishable.

Finally, property 4 implies the following bounds for derivatives of products, which will be useful in the estimates of iterates of cocycles.

Proposition 2.9. *We have the following estimates concerning the derivatives.*

1. *Let $A_i : \mathbb{T} \to G$, $i = 1, \cdots, n$, be smooth enough, and let $m_0 = \max_i \|a_i\|_\infty$ and $m_s = \max_{i,0\le\sigma\le s} \|\partial^s a_i\|_\infty$. Then we have*

$$\|\partial^s(a_1...a_n)\|_\infty \le C_s m_s(1+m_0)^s n^{s+1}$$

2. *Let $U \in C^\infty(\mathbb{T}, g)$ be small enough in $C^0(\mathbb{T}, g)$ and $A : \mathbb{T} \to G$. Then*

$$C_s^{-1}(1 - \|U(\cdot)\|_0)^s \|U(\cdot)\|_{s+1} \le \|\partial^s L e^{U(\cdot)}\|_s \le C_s(1 + \|U(\cdot)\|_0)^s \|U(\cdot)\|_{s+1}$$

and

$$\begin{aligned} \|Ad(A(\cdot)).U(\cdot)\|_0 &= \|U(\cdot)\|_0 \\ \|L(\exp(Ad(A(\cdot)).U(\cdot))\|_s &\le C_s(1 + \|U(\cdot)\|_0)^s(1 + \|a(\cdot)\|_0)^s(1 + \|a(\cdot)\|_s)\|U(\cdot)\|_{s+1} \end{aligned}$$

If we admit a uniform bound on the term $\|U(\cdot)\|_0$,

$$\|L(\exp(Ad(A(\cdot)).U(\cdot))\|_s \le C_s(1 + \|a(\cdot)\|_0)^s(1 + \|a(\cdot)\|_s)\|U(\cdot)\|_{s+1}$$

3. In fact, since the action of G in the following chapter will be chosen to be left multiplication, the choice of right multiplication in the definition of the operator L slightly complicates the calculations in chapter 5. However, it becomes more natural in the local theory of models, where traditionally perturbations are chosen to act on the left.

Proof. 1. Since $\partial(Ad(A(\cdot)).b(\cdot))) = Ad(A(\cdot)).\partial b(\cdot) + [a(\cdot), Ad(A(\cdot)).b(\cdot)]$, induction in n, convexity inequalities and counting of the terms gives the result.

2. If we let $h \in g$, we calculate

$$\partial Ad(B(\cdot)).h = [b(\cdot), Ad(B(\cdot)).h]$$

so that the derivative in the Lie algebra of $Inn(G) \approx G/Z_G$ (which is a subalgebra of $gl(g)$ naturally isomorphic to g) reads

$$(\partial Ad(B(\cdot)).Ad(B^*(\cdot)) = ad_{b(\cdot)}$$

and

$$C^{-1} \|B\|_s \leq \|Ad(B(\cdot))\|_s \leq C \|B\|_s$$

where the constant is universal since it depends only on the adjoint representation of g into itself[4].

If we call $\tilde{U}(\cdot) = Ad(A(\cdot)).U(\cdot)$, we have

$$Le^{\tilde{U}(\cdot)} = Ad(A(\cdot)).Le^{U(\cdot)} + (Id - Ad(e^{\tilde{U}(\cdot)}))a(\cdot)$$

Therefore,

$$\begin{aligned} \|Le^{\tilde{U}(\cdot)}\|_s &\leq \|Ad(A(\cdot)).Le^{U(\cdot)}\|_s + \|(Id - Ad(e^{\tilde{U}(\cdot)}))a(\cdot)\|_s \\ &\lesssim \|Le^{U(\cdot)}\|_s + \|Ad(A(\cdot))\|_s \|Le^{U(\cdot)}\|_0 + \|U(\cdot)\|_0 \|a(\cdot)\|_s + \|Ad(e^{\tilde{U}(\cdot)})\|_s \|a(\cdot)\|_0 \\ &\lesssim \|e^{U(\cdot)}\|_s + (1 + \|a(\cdot)\|_{s-1}) \|e^{U(\cdot)}\|_1 + \|U(\cdot)\|_0 \|a(\cdot)\|_s + \|e^{\tilde{U}(\cdot)}\|_s \|a(\cdot)\|_0 \end{aligned}$$

In particular, for $s = 1$,

$$\begin{aligned} \|Le^{\tilde{U}(\cdot)}\|_0 &\leq \|e^{U(\cdot)}\|_1 + \|A(\cdot)\|_0 \|e^{U(\cdot)}\|_1 + \|U(\cdot)\|_0 \|A(\cdot)\|_1 \\ &\leq \|e^{U(\cdot)}\|_1 + \|U(\cdot)\|_0 (1 + \|a(\cdot)\|_0) \end{aligned}$$

and the announced estimates follow from these expressions.

□

For $X \in g$ small enough, let $K(X).\Delta X = (D(e^X).\Delta X).e^{-X}$. Then, $K(X)$ is in $GL(g)$, since the exponential is a local diffeomorphism. This operator can be calculated by

$$\begin{aligned} K(X).\Delta X &= \frac{e^{ad(X)} - Id}{ad(X)}.\Delta X \\ &= \Delta X + \frac{1}{2}[X, \Delta X] + \frac{1}{3!}[X, [X, \Delta X]] + ... \end{aligned}$$

In particular, $L \exp(U(\cdot)) = K(U(\cdot)).\partial U(\cdot)$, and

$$K(-X).\Delta X = Ad(e^{-X}).K(X).\Delta X$$

The transposed operator with respect to the Cartan-Killing form is found by

$$K(X)^T = K(-X)$$

as shows the expansion of the operator. Since $K(X)$ is invertible for small enough X, we will sometimes use the notational convention

$$\widetilde{\Delta X} = K(X).\Delta X$$

In fact, for X small enough, $K(X)$ is close to being an isometry, since

$$K(-X).K(X) = Id + \frac{1}{12} ad_X^2 + O(|X|^4)$$

as shows a direct calculation.

4. This expression is in fact not totally accurate, since in the C^0 norm we should replace the Riemannian distance from $B(\cdot)$ to the Id with the $\min(d(B(\cdot), S), S \in Z_G)$. We keep this fact implicit in order to keep notation simpler.

2.2.2 Functional spaces

We will consider the space $C^\infty(\mathbb{T}^d, g)$ equipped with the standard maximum norms

$$\begin{aligned} \|U\|_s^0 &= \max_{\mathbb{T}} |\partial^s U(\cdot)| \\ \|U\|_s &= \max_{0\le\sigma\le s} \|U\|_\sigma^0 \end{aligned}$$

for $s \geq 0$, and the Sobolev norms

$$\|U\|_{H^s}^2 = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s |\hat{U}(k)|^2$$

where $\hat{U}(k) = \int U(\cdot) e^{-2i\pi kx}$ are the Fourier coefficients of $U(\cdot)$. The fact that the injections $H^{s+d/2}(\mathbb{T}^d, g) \hookrightarrow C^s(\mathbb{T}^d, g)$ and $C^s(\mathbb{T}^d, g) \hookrightarrow H^s(\mathbb{T}^d, g)$ for all $s \geq 0$ are continuous is classical.

For mappings $\mathbb{R} \supset I \to g$ we will use the norms

$$\|U(\cdot)\|_{s,I} = \max_{0\le\sigma\le s} \|\partial^\sigma U(\cdot)\|_{L^\infty(I)}$$

where $I \subset \mathbb{R}$ is an interval. If $I = [0, T]$, we will replace I by T in these notations.

The space $C^\infty(\mathbb{T}^d, G)$ is equipped with the following norms (we remind the convention $LA(\cdot) = a(\cdot)$)

$$\begin{aligned} \|A\|_0 &= \max_{\mathbb{T}} d(A(\cdot), Id) \\ \|A\|_s^0 &= \|a(\cdot)\|_{s-1}^0 \\ \|A\|_s &= \|a\|_{s-1} \\ \|A(\cdot)\|_{s,I} &= \|a\|_{s,I} \end{aligned}$$

for $s \geq 1$. We note that in the $\|\cdot\|$ norms of G-valued maps we omit the C^0 norm, irrelevant for the majority of the arguments since G is supposed to be compact.

We will also use the convexity or Hadamard-Kolmogorov inequalities (see [Kol49]) ($U \in C^\infty(\mathbb{T}^d, g)$):

$$\|U(\cdot)\|_\sigma \leq C_{s,\sigma} \|U\|_0^{1-\sigma/s} \|U\|_s^{\sigma/s} \tag{2.7}$$

for $0 \leq \sigma \leq s$, and the inequalities concerning the composition (see [Kri99a]) of functions:

$$\|\phi \circ (f+u) - \phi \circ f\|_s \leq C_s \|\phi\|_{s+1} (1 + \|f\|_0)^s (1 + \|f\|_s) \|u\|_s \tag{2.8}$$

which hold for C^s norms.

We will use the truncation operators

$$\begin{aligned} T_N f(\cdot) &= \sum_{|k|\le N} \hat{f}(k) e^{2i\pi k\cdot} \\ \dot{T}_N f(\cdot) &= T_N f(\cdot) - \hat{f}(0) \\ R_N f(\cdot) &= \sum_{|k|>N} \hat{f}(k) e^{2i\pi k\cdot} \text{ and} \\ \dot{T} f(\cdot) &= T f(\cdot) - \hat{f}(0) \end{aligned}$$

These operators satisfy the estimates

$$\begin{aligned} \|T_N f(\cdot)\|_{C^s} &\leq C_{s,s'} N^{(d+1)/2} \|f(\cdot)\|_{C^s} \\ \|R_N f(\cdot)\|_{C^s} &\leq C_{s,s'} N^{s-s'+d+1} \|f(\cdot)\|_{C^{s'}} \end{aligned}$$

The Fourier spectrum of a function will be denoted by

$$\hat{\sigma}(f) = \{k \in \mathbb{Z}^d,\ \hat{f}(k) \neq 0\}$$

Finally, we will need the space of real analytic mappins $C^\omega_\delta(\mathbb{T}, SU(2))$, defined as the space of analytic mappings $\mathbb{T} \to SU(2)$ (the reader can think of mappings whose matrix coefficients for the adjoint action are real analytic), and admit an analytic extension to $\mathbb{T} \times [-\delta i, \delta i]$ for $\delta > 0$. The norm of this space is the analytic norm $\|A\|_\delta = \sup_{x\in\mathbb{T},|\epsilon|<\delta} \|A(x + i\epsilon)\|$. The norm on the righthand side of the definition here is the operator norm associated to the hermitian product on $\mathbb{C}^2$, which is legitimate since $SU(2)$ complexifies to $SL(2, \mathbb{C})$, so that the values of the mapping for complex values of the variable in the basis are in fact in $SL(2, \mathbb{C})$.

2.2.3 The Nash-Moser inverse function theorem

This theorem, stated and proved in [Ham82], will be used in the proof of a normal form theorem in chapter 8. The theorem is stated in the context of tame mappings between Fréchet spaces, for which the model is, say, $\mathcal{E} = C^\infty(X, \mathbb{R})$ spaces with X a compact manifold. Since the Nash-Moser theorem will be used in exactly such a context, we will restrict the generality and slightly simplify the statement. In our discussion, we give an overview of *section 4.2* of [Kri99a], rather than referring directly to the R. Hamilton's article.

In what follows, $\mathcal{E}$ denotes a Fréchet space and $\|\cdot\|_s$, $s \in \mathbb{N}$ the family of its seminorms defining the usual topology of $\mathcal{E}$ by the system of neighborhoods of $0 \in \mathcal{E}$

$$U_{i,j} = x \in \mathcal{E}, \|x\|_s < (j + 1)^{-1}, 0 \leq s \leq i$$

where $i, j \in \mathbb{N}^*$

The absence of a norm defining the topology of the space $\mathcal{E}$ makes the definition of a differential more difficult than in Banach spaces, but directional derivation (Gâteaux derivative) is nonetheless well defined. Let $U \subset E$ be an open subset and E and F be topological vector spaces. A continuous mapping $f : U \to F$ is called *Gâteaux-differentiable* if there exists a mapping

$$\begin{array}{rccc} Df: & U \times E & \to & F \\ & (x, h) & \mapsto & Df(x).h \end{array}$$

continuous jointly in (x, h) and linear in h, and such that the following limit exists and verifies

$$\lim_{t\to 0} \frac{f(x + th) - f(x)}{t} = Df(x).h$$

Higher order differentiability is defined inductively.

A Fréchet space is called *graded* if the semi-norms defining its topology are increasing in strength. In the context of $C^\infty(\mathbb{T}, g)$, this amounts to

$$\|\cdot\|_s \leq \|\cdot\|_{s+1}$$

which is verified by the definition of the norms. In fact, any Fréchet space can be made into a graded one, by a simple change in the definition of its seminorms. A graded Fréchet space is called *tame* if there exists a family of continuous operators T_N, indexed by $N \in \mathbb{N}^*$, satisfying the properties of the truncation operators introduced just above:

$$\begin{array}{rcl} \|T_N e\|_{s'} & \leq & C_{s.s'} N^{s-s'+r} \|e\|_s \\ \|R_N e\|_s & \leq & C_{s,s'} N^{s'-s-r} \|e\|_{s'} \end{array}$$

for all $0 \leq s \leq s'$, for a uniform choice of $r \in \mathbb{N}$ and positive constants $C_{s.s'}$, and where $R_N = Id - T_N$. In general contexts, such operators T_N are called *smoothing operators*, following the properties of truncation in the usual concrete examples.

A linear application T between graded Fréchet spaces $\mathcal{E}$ and $\mathcal{F}$ will be called *tame* if there exists $l \in \mathbb{N}$ and a family of positive constants C_s such that

$$\|Te\|_s \leq C_s \|e\|_{s+l}, \ \forall s \in \mathbb{N}, \ \forall e \in \mathcal{E}$$

where we have used the same notation for seminorms in both spaces.

If $\mathcal{E}$ and $\mathcal{F}$ are graded Fréchet spaces, $U \subset E$ is a neighborhood of 0 and $P : (U, 0) \to (\mathcal{F}, 0)$ is a mapping, we will say that it is tame if there exists $l \in \mathbb{N}$, "the loss of derivatives", and a family of positive constants C_s such that

$$\|Pf\|_s \leq C_s(1 + \|f\|_{s+l}),\ \forall s \in \mathbb{N}$$

The following proposition of [Ham82] shows that the basic properties of differential calculus in Banach spaces survive in this context.

Proposition 2.10. *Let X be a compact manifold, E and F two finite-dimensional vector spaces. Then*

1. *The spaces $C^\infty(X, E)$ and $C^\infty(E, F)$, furnished with the C^s norms are tame Fréchet spaces*
2. *Composition*
$$\begin{array}{ccc} C^\infty(X,E) \times C^\infty(E,F) & \to & C^\infty(X,F) \\ (f,g) & \mapsto & g \circ f \end{array}$$
is a tame mapping.
3. *If $g \in C^\infty(E, F)$ is fixed, then*
$$\begin{array}{rccc} \beta_g : & C^\infty(X,E) & \to & C^\infty(X,F) \\ & f & \mapsto & g \circ f \end{array}$$
is C^∞-tame, and its differential is given by
$$\begin{array}{rccc} D\beta_g : & C^\infty(X,E) & \to & C^\infty(X,F) \\ & \Delta f & \mapsto & D\beta_g(f).\Delta f \end{array}$$
where $D\beta_g(f).\Delta f$ represents the mapping $x \mapsto Dg(f(x)).\Delta f(x)$.
4. *If $g \in C^\infty(E, F)$ is fixed, then*
$$\begin{array}{rccc} \beta_g : & C^\infty(X,F) & \to & C^\infty(X,E) \\ & f & \mapsto & f \circ g \end{array}$$
is C^∞-tame, and its differential is given by
$$\begin{array}{rccc} D\beta_g : & C^\infty(X,E) & \to & C^\infty(X,F) \\ & \Delta f & \mapsto & (x \mapsto \Delta f(g(x)).g'(x)) \end{array}$$
5. *if $f \in C^\infty(E, E)$ is invertible, then there exists U, a neighborhood of f, such that*
$$\begin{array}{ccc} U & \to & C^\infty(E,E) \\ g & \mapsto & g^{-1} \end{array}$$
is C^∞-tame, where g^{-1} is the inverse function of g in the usual sense.

Finally, we introduce some notation before stating the version of Hamilton's theorem, proved in [Kri99a], that we will use.

Let $\mathcal{E}$ and $\mathcal{F}$ be two tame Fréchet spaces, $U \subset \mathcal{E}$ a neigborhood of 0, $r \in \mathbb{N}^*$ and $(C_s)_s$ a family of positive constants indexed by $\mathbb{N}$. We denote by $\mathbf{A}(r, (C_s)_s, U)$ the set of tame mappings $f : U \to \mathcal{F}$ such that $f(0) = 0$ and

$$\|f(x)\|_s \leq C_s(1 + \|x\|_{s+r}),\ \forall s \in \mathbb{N}$$

The set $\mathbf{A}^k(r, (C_s)_s, U)$, $k \in \mathbb{N} \cup \{\infty\}$, is formed by functions f such that $D^l f \in \mathbf{A}(r, (C_s)_s, U)$, for $0 \leq l \leq k$. If there exists a tame continuous mapping $L : U \times F \to E$, linear in the second variable, such that for all $x \in U$, $Df(x)$ is invertible and of inverse $L(x)$, we will write, by abuse of notation, $f \in \mathbf{A}^{-1}(r, (C_s)_s, U)$. If f is both in $\mathbf{A}^k(r, (C_s)_s, U)$ and $\mathbf{A}^{-1}(r, (C_s)_s, U)$, we will write $f \in \mathbf{A}^{k,-1}(r, (C_s)_s, U)$.

With this notation, we have

Theorem 2.11. *Let $f \in \mathbf{A}^{2,-1}(r,(C_s)_s,U)$. Then, there exist r_1, ε_1, depending only on r, U and the family C_s, such that f^{-1}, the inverse of f, exists and is well defined in V_1, the neighborhood of $0 \in \mathcal{F}$ defined by $\{\|y\|_{r_1} \leq \varepsilon_1\}$. Moreover, for $y \in V_1$, $\|f^{-1}(y)\|_s \leq c_s(1+\|y\|_{s+r_1})$.*

The following version is a uniform one, and allows estimates on the size of the neighborhood where inversion takes place.

Theorem 2.12. *Let $f \in \mathbf{A}^{2,-1}(r,(C_s)_s,U)$, whose derivative Df admits a tame inverse in U, and $M > 0$ such that $(Df)^{-1} \in \mathbf{A}(r,(MC_s)_s,U)$. Then, there exists $\nu > 0$ (one can chose $\nu = 2$), r_1 and ε_1, depending only on r, U and the family C_s, such that f^{-1} exists and is well defined in a neighborhood of $0 \in \mathcal{F}$ of the form $\{\|y\|_{r_1} \leq M^{-\nu}\varepsilon_1\}$. In this neighborhood, $f^{-1} \in \mathbf{A}^2(r_1,(M^{-\nu}C_s)_s)$*

2.3 Arithmetics, continued fraction expansion

A deep introduction into arithmetics and continued fraction expansion can be found in [Khi63]. A geometric interpretation of the algorithm, followed by an introduction to K.A.M. theory can be found in [Arn83].

Let us introduce some notation:
- $|||\alpha||| = |\alpha|_{\mathbb{Z}} = dist(\alpha,\mathbb{Z}) = \min_{\mathbb{Z}} |\alpha - l|$
- $[\alpha]$ the integer part of α
- $\{\alpha\}$ the fractional part of α
- $G(\alpha) = \{\alpha^{-1}\}$, the Gauss map.

where α is a real number.

Consider $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ fixed, and let $p_{-2} = q_{-1} = 0$ and $p_{-1} = q_{-2} = 1$. Then $(p_n/q_n)_{n\geq 0}$ is the sequence of best rational approximations of α defined recursively as follows. Let $\alpha_n = G^n(\alpha) = G(\alpha_{n-1})$, $a_n = [\alpha_{n-1}^{-1}]$,[5] and $\beta_n = \prod_0^n \alpha_k$. Then the Euclidean division of β_{n-2} by β_{n-1} reads

$$\beta_{n-2} = a_n\beta_{n-1} + \beta_n$$

and

$$\begin{aligned} q_n &= a_n q_{n-1} + q_{n-2} \\ p_n &= a_n p_{n-1} + p_{n-2} \end{aligned}$$

p_n and q_n are strictly positive for $n \geq 1$ and $\beta_n = (-1)^n(q_n\alpha - p_n)$. We have

$$\begin{aligned} \frac{1}{q_n + q_{n+1}} &< \beta_n < \frac{1}{q_{n+1}} \\ |||q_{n-1}\alpha||| &< |||k\alpha|||, \ \forall\ 0 < k < q_n \end{aligned}$$

Moreover, for all n,

$$\alpha_n\alpha_{n-1} < \frac{1}{2}$$

and consequently

$$\beta_n < 2^{-n/2}$$

Two basic facts which distinguish continued fractions in one-dimension from the higher dimensional analogues are the following:

$$q_{n-1}\beta_n + q_n\beta_{n-1} = 1$$

and

$$\begin{vmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{vmatrix} = (-1)^n$$

The following notion is essential in K.A.M. theory. It is related with the quantification of the closeness of rational numbers to certain classes of irrational numbers.

5. We hope that the reader will not confuse the natural numbers a_n introduced in the continued fractions expansion with the mappings $a_n(\cdot)$ defined in chapter 5.

Definition 2.13. We will denote by $DC(\gamma,\tau)$ the set of numbers α in $\mathbb{T}\setminus\mathbb{Q}$ such that for any $k\neq 0$, $|\alpha k|_{\mathbb{Z}}\geq\frac{\gamma^{-1}}{|k|^{\tau}}$. Such numbers are called *Diophantine*. The set $DC(\gamma,\tau)$, for $\tau>1$ fixed and $\gamma\in\mathbb{R}_{+}^{*}$ is of positive Haar measure in $\mathbb{T}$. If we fix τ and let γ run through the positive real numbers, we obtain $\cup_{\gamma>0}DC(\gamma,\tau)$ which is of full Haar measure.

The numbers that do not satisfy any Diophantine condition are called *Liouvillean*. They form a residual set of 0 Lebesgue measure.

This last following definition concerns the relation of the approximation of an irrational number with its continued fractions representation.

Definition 2.14. Denote by $RDC(\gamma,\tau)$ the set of *recurrent Diophantine* numbers. It is the set of all $\alpha\in\mathbb{T}\setminus\mathbb{Q}$ such that $G^{n}(\alpha)\in DC(\gamma,\tau)$ for infinitely many n. It is a set of full Lebesgue measure since the Gauss map is ergodic with respect to a smooth measure.

In contexts where the parameters γ and τ are not significant, they will be omitted in the notation of both sets.

Chapter 3

Cocycles in $\mathbb{T}^d \times G$

3.1 Definitions and general properties

Let $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{T}^d$, $d \in \mathbb{N}^*$, be a topologically minimal translation. This property is, by Kronecker's theorem, equivalent to $(\alpha_1, ..., \alpha_d, 1) \in \mathbb{R}^{d+1}$ being linearly independent over $\mathbb{Z}$. The translation $\mathbb{T}^d \circlearrowleft$ will sometimes be denoted by $R_\alpha : x \mapsto x + \alpha \mod (\mathbb{Z}^d)$. In the greatest part of our study we will limit ourselves to the case $d = 1$, where minimality is equivalent to α being an irrational number.

If $A(\cdot) \in C^s(\mathbb{T}^d, G)$, $s \in \mathbb{N} \cup \infty$, then the couple $(\alpha, A(\cdot))$ acts on the fibered space $\mathbb{T}^d \times G \to \mathbb{T}^d$ defining a diffeomorphism by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S)$$

for any $(x, S) \in \mathbb{T}^d \times G$. We will call such an action a *quasiperiodic cocycle over* R_α (or simply a cocycle). The space of such actions is denoted by $SW^s_\alpha(\mathbb{T}^d, G) \subset Diff^s(\mathbb{T}^d \times G)$ and d is the number of *frequencies* of the cocycle. When d and G are clearly defined by the context, we will abbreviate the notation to SW^s_α. Cocycles are a class of fibered diffeomorphisms, since fibers of $\mathbb{T}^d \times G$ are mapped into fibers, and the mapping from one fiber to another in general depends on the base point. A closed subspace of $SW^s_\alpha(\mathbb{T}^d, G)$ which will be of use will be the space $SW^{s,1}_\alpha(\mathbb{T}^d, G)$ of cocycles *homotopic to constants*. In this context, this last condition amounts to $A(\cdot)$ being homotopic to constants in G.

The space $\bigcup_\alpha SW^s_\alpha(\mathbb{T}^d, G)$, where the union is over minimal translations, will be denoted by $SW^s(\mathbb{T}^d, G)$. The space $SW^s_\alpha(\mathbb{T}^d, G)$ inherits the topology of $C^s(\mathbb{T}^d, G)$, and $SW^s(\mathbb{T}^d, G)$ has the standard product topology of $\mathbb{T}^d \times C^s(\mathbb{T}^d, G)$. The space SW^s is consequently not a complete metric space, something which makes reasoning more delicate when one wishes to vary the frequency. In this work, the frequency is to be considered fixed. We note that cocycles are defined over more general maps and in more general contexts of regularity and structure of the basis and fibers, including Lie groups such as $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$ and their classical subgroups.

If we consider a representation of G on $\mathbb{C}^{w'}$, the action of the cocycle can be also defined on $\mathbb{T}^d \times \mathbb{C}^{w'}$, simply by replacing S by a complex vector. A particular case of representation that we will strongly use is the adjoint representation of G in g.

We can define two projections

$$\begin{array}{rccc} \pi_1 : & SW^s(\mathbb{T}^d, G) & \to & \mathbb{T}^d \\ & (\alpha, A(\cdot)) & \mapsto & \alpha \end{array} \tag{3.1}$$

and

$$\begin{array}{rccc} \pi_2 : & SW^s(\mathbb{T}^d, G) & \to & C^s(\mathbb{T}^d, G) \\ & (\alpha, A(\cdot)) & \mapsto & A(\cdot) \end{array}$$

and we can identify $C^s(\mathbb{T}^d, G)$ with $\pi_1^{-1}(0)$. This identification set aside, α is to be considered minimal.

The n-th iterate of the action is given by

$$(\alpha, A(\cdot))^n.(x, S) = (n\alpha, A_n(\cdot)).(x, S) = (x + n\alpha, A_n(x).S)$$

where $A_n(\cdot)$ represents the *quasiperiodic product* of matrices equal to

$$A_n(\cdot) = A(\cdot + (n-1)\alpha)...A(\cdot)$$

for positive iterates. Negative iterates are found as inverses of positive ones:

$$\begin{aligned}(\alpha, A(\cdot))^{-n} &= ((\alpha, A(\cdot))^n)^{-1} \\ &= (-n\alpha, A^*(\cdot - n\alpha)...A^*(\cdot - \alpha))\end{aligned}$$

The regularity needed for the main results is in fact C^∞, but some of the intermediate results are proved in very low regularity.

We will also need to consider the case where $G_j \hookrightarrow G$, $j = 1, 2$, and $G_1 \cap G_2 = Id$. If $(\alpha, A_j(\cdot))$ are cocycles in $\mathbb{T}^d \times G_j$ over the same rotation in the basis, then we can define a cocycle in $\mathbb{T}^d \times G$ by their product and we have $SW^s_\alpha(\mathbb{T}^d, G_1) \times SW^s_\alpha(\mathbb{T}^d, G_2) \hookrightarrow SW^s_\alpha(\mathbb{T}, G)$. We will denote this cocycle by $(\alpha, A_1 \times A_2(\cdot))$. Clearly, the subgroup $G_1 \times G_2 \subset G$, as well as G_1 and G_2, are invariant under the action of the product cocycle.

Cocycles appear as Poincaré mappings of *quasiperiodic skew-systems*, the continuous time equivalent of cocycles. Such systems are defined by the ODE in $\mathbb{T}^d \times G$

$$\begin{cases} \frac{d}{dx}X = F(x).X \\ \dot{x} = \omega = (\alpha, 1) \end{cases} \tag{3.2}$$

Here, $F(\cdot) : \mathbb{T}^d \to g$, and $\omega \in \mathbb{R}^{d+1}$. The flow of this ODE satisfies

$$\Phi_t(x, S) = (x + t\omega, X(x, t).S) \tag{3.3}$$

where the first factor is a dense geodesic in $\mathbb{T}^{d+1}$ and the second factor satisfies the ODE

$$\frac{d}{dt}X(x, t) = F(x + t\omega).X(x, t) \tag{3.4}$$

and we consider the fundamental solution, i.e. the one with initial values $X(x, 0) = Id$. The first return map on the vertical circle in $\mathbb{T}^{d+1} = \mathbb{T}^d \times \mathbb{T}$ defines a cocycle in $\mathbb{T}^d \times G$. The study of such ODEs is an attempt to extend Floquet theory in ODEs with quasi-periodic (instead of periodic) time-dependent coefficients, and reducibility amounts to the existence of a time-independent conjugation to a constant vector field.

3.2 Classes of cocycles with simple dynamics, conjugation

The cocycle $(\alpha, A(\cdot))$ is called a constant cocycle if $A(\cdot) = A \in G$ is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices

$$(\alpha, A)^n = (n\alpha, A^n)$$

and the dynamics become easy to describe.

Another, more general, distinct class of cocycles having relatively simple dynamics is given by the mappings $A(\cdot)$ taking values in a maximal torus of G. Such a cocycle will be called *abelian*. The interest of abelian cocycles lies in the the existence of a discrete invariant that distinguishes qualitatively different types of dynamics (c.f. chapter 4), as well as in the fact that they represent the most general cocycles whose iteration can be made explicit. More precisely, let $(\alpha, A(\cdot))$ be an abelian cocycle, and call $\mathcal{T}$ a maximal torus in which $A(\cdot)$ takes its values. Let us also fix a basis (h_ρ) of $\mathfrak{t}$, the Lie algebra of $\mathcal{T}$. With these choices, there exists a vector $r = (r_\rho) \in \mathbb{Z}^w$, periodic functions of zero mean value $\phi_\rho(\cdot)$ and $a = \sum_\rho a_\rho h_\rho \in \mathfrak{t}$, such that

$$A(\cdot) = \exp(\sum\nolimits_\rho (2\pi r_\rho \cdot + \phi_\rho(\cdot) + a_\rho)h_\rho) \tag{3.5}$$

The iterates of such a cocycle can be calculated explicitly, namely

$$(\alpha, A(\cdot))^n = (n\alpha, \exp(\sum\nolimits_\rho (2\pi r_\rho n(\cdot + \frac{(n-1)}{2}\alpha) + S^\alpha_n \phi_\rho(\cdot) + na_\rho)h_\rho))$$

where $S_n^\alpha\phi(\cdot)$ stands for the Birkhoff sum of the function $\phi(\cdot)$ over the translation by α

$$S_n^\alpha\phi(\cdot) = \sum_{k=0}^{n-1} \phi(\cdot + k\alpha)$$

Clearly, if the values of the mapping $A(\cdot)$ over some fibers x and $x+\alpha$ do not commute, the calculation is no longer true.

The group $C^s(\mathbb{T}^d, G) \hookrightarrow SW^s(\mathbb{T}^d, G)$ acts by *dynamical conjugation*: Let $B(\cdot) \in C^s(\mathbb{T}^d, G)$ and $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}^d, G)$. Then we define

$$\begin{aligned} Conj_{B(\cdot)}.(\alpha, A(\cdot)) &= (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1} \\ &= (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot)) \end{aligned}$$

which is in fact a change of variables within each fiber of the product $\mathbb{T}^d \times G$. The dynamics of $Conj_{B(\cdot)}.(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))$ are essentially the same, since

$$(Conj_{B(\cdot)}.(\alpha, A(\cdot)))^n = (n\alpha, B(\cdot + n\alpha).A_n(\cdot).B^{-1}(\cdot))$$

Definition 3.1. Two cocycles $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$ in $SW_\alpha^s(\mathbb{T}^d, G)$ are called C^s-*conjugate* modulo $m \in \mathbb{N}^*$ iff there exists $B(\cdot) \in C^s(m\mathbb{T}, G)$ such that $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}.(\alpha, A(\cdot))$. We will use the notation

$$(\alpha, A(\cdot)) \sim (\alpha, \tilde{A}(\cdot))$$

to state that the two cocycles are conjugate to each other.

We remark that it is indeed possible for cocycles in $SW_\alpha^s(\mathbb{T}^d, G)$ to be conjugate modulo m with $m \geq 2$. We will encounter such examples later on.

Since the qualitative features of the dynamics are determined up to a conjugation, we are interested in the density properties of the orbits of the classes described above and this motivates the following definition:

Definition 3.2. A cocycle will be called C^s-*reducible* (modulo m) iff it is C^s-conjugate (modulo m) to a constant, and C^s-*torus-reducible* (modulo m) iff it is C^s-conjugate (modulo m) to an abelian cocycle.

In these definitions the conjugacy is implicitly supposed to take its values in G. If we fix a unitary representation of $G \hookrightarrow U(w')$, we can define a somewhat weaker notion of (torus-)reducibility, namely reducibility in the ambient group $U(w')$, which is referred to as $U(w')$ (torus-)reducibility.

Finally, since it is known that not all cocycles are reducible (e.g. generic abelian cocycles over Liouvillean rotations, but also cocycles over Diophantine rotations, even though this result is hard to obtain, see [Eli02]) we also need the following concept.

Definition 3.3. A cocycle $(\alpha, A(.))$ is said to be *almost reducible* mod m, $m \in \mathbb{N}^*$, if there exists sequence of conjugations $B_n(\cdot) \in C^\infty(m\mathbb{T}^d, G)$, such that $Conj_{B_n(\cdot)}.(\alpha, A(\cdot))$ becomes arbitrarily close to constants in the C^∞ topology, i.e. iff there exists (A_n), a sequence in G, such that

$$A_n^* \left(B_n(\cdot + \alpha)A(\cdot)B_n^*(\cdot)\right) \to Id$$

in $C^\infty(m\mathbb{T}^d, G)$. Almost torus-reducibility is defined in an analogous way.

The last and weakest notion of reducibility is that of *quasi-reducibility*. It is a version of almost reducibility where we allow $B_n(\cdot) \in C^\infty(P_n\mathbb{T}^d, G)$ and $P_n \to \infty$ and we demand that

$$A_n^* B_n(P_n \cdot + \alpha)A(P_n\cdot)B_n^*(P_n\cdot) \to Id$$

in $C^\infty(\mathbb{T}^d, G)$. This notion is better adapted to cases where construction of a conjugation is not possible without a loss of periodicity. This phenomenon was present in [Kri99a], but we have been able to overcome the difficulties related to it in chapter 9.

3.3 Some considerations and results on reducibility of cocycles

Let us now study the construction of conjugacies in some cases which have come to be considered simple. For this reason, we will allow G to be a more general Lie group than in the rest of the work.

The question of conjugation within the class of constant cocycles has been settled in [Kri99a] (see section 4.3 for the statement of the theorem). As we have already pointed out the significance of abelian cocycles, we will proceed to the study of conjugation within this class in the aforementioned section. Let us, however, study right-away a particular case, which lies in the core of K.A.M. theory.

The simplest case of cocycles occurs when the group G is abelian. Cocycles in $\mathbb{T} \times \mathbb{R}$ where studied for example in [Yoc95] in the resolution of the linearized equation satisfied by commuting diffeomorphisms of $\mathbb{S}^1$. We also refer to [Arn83] and [KH96]. A cocycle in this setting has the form $(\alpha, \phi(\cdot))$ and the dynamics are given by

$$(x, y) \mapsto (x + \alpha, \phi(x) + y)$$

Suppose now that two cocycles over the same rotation α are given, $(\alpha, \phi_1(\cdot))$ and $(\alpha, \phi_2(\cdot))$. These cocycles are conjugate to each other iff there exists $\psi(\cdot) : \mathbb{T} \to \mathbb{R}$ such that

$$\psi(\cdot + \alpha) + \phi_1(\cdot) - \psi(\cdot) = \phi_2(\cdot)$$

or

$$\psi(\cdot + \alpha) - \psi(\cdot) = \phi_2(\cdot) - \phi_1(\cdot)$$

Since $\mathbb{R}$ is abelian, conjugating $(\alpha, \phi_1(\cdot))$ to $(\alpha, \phi_2(\cdot))$ is equivalent to conjugating $(\alpha, \phi(\cdot))$ to $(\alpha, 0)$, where $\phi(\cdot) = \phi_1(\cdot) - \phi_2(\cdot)$, so that this last equation reads

$$\psi(\cdot + \alpha) - \psi(\cdot) = \phi(\cdot) \tag{3.6}$$

This equation, whose unknown is the conjugant ψ is called a *linear cohomological equation.*

Periodicity of the functions implies that $\hat{\phi}(0) = 0$ (or equivalently $\hat{\phi}_1(0) = \hat{\phi}_2(0)$) is a necessary condition for the existence of such a ψ. For this reason $\hat{\phi}(0)$ is called the *obstruction* of such an equation. If $\hat{\phi}(0) \neq 0$, we can only solve the equation

$$\begin{aligned} \psi(\cdot + \alpha) - \psi(\cdot) &= \phi(\cdot) - \hat{\phi}(0) \\ &= \dot{T}\phi(\cdot) \end{aligned}$$

Application of the Fourier transform shows that such a ψ must satisfy, for $k \neq 0$,

$$\hat{\psi}(k) = \frac{1}{e^{2i\pi k\alpha} - 1}\hat{\phi}(k)$$

Since α is irrational (the equivalent to minimality of R_α for one-frequency cocycles), $e^{2i\pi k\alpha} - 1$ is never 0, and therefore eq. 3.6 admits as a solution at least a "formal Fourier series", so long as $\phi(\cdot)$ is, say, in L^2. The convergence of the Fourier series of such a solution is related to *small denominator* phenomena, since $e^{2i\pi k\alpha}$ is arbitrarily close to 1 for an infinite number of k (e.g. for k equal to some q_n, the denominators of continued fractions approximations).

The convergence of $\sum \hat{\psi}(k)e^{2i\pi k\cdot}$ is guaranteed under some smoothness conditions on ϕ and/or some arithmetic conditions on α. A relevant combination of hypotheses in our context gives the following classical lemma.

Lemma 3.4. *Let $\alpha \in DC(\gamma, \tau)$ and $\phi(\cdot) \in C^\infty(\mathbb{T}, \mathbb{R})$. Then, there exists $\psi \in C^\infty(\mathbb{T}, \mathbb{R})$ such that*

$$\psi(x + \alpha) - \psi(x) = \phi(x) - \hat{\phi}(0)$$

The solution ψ satisfies the estimate $\|\psi\|_s \lesssim \gamma \|\phi\|_{s+\tau+1/2}$, where the constant is universal and depends only on s. The solution is unique modulo additive constants.

We remind the reader that Diophantine numbers are defined in definition 2.13, and that such a condition is necessary. For the convenience of the reader, we include the proof of the lemma.

Proof. Using the diophantine condition on α, we find that for $k \neq 0$

$$|\hat{\psi}(k)| \leq \gamma |k|^{\tau} |\hat{\phi}(k)|$$

Therefore,

$$\begin{aligned} \|\psi\|_s^2 &\leq \sum_{k\in\mathbb{Z}} (1+|k|)^{2s+1} |\hat{\psi}(k)|^2 \\ &\lesssim \gamma^2 \sum_{k\in\mathbb{Z}} (1+|k|)^{2s+1} |k|^{2\tau} |\hat{\phi}(k)|^2 \\ &\lesssim \gamma^2 \sum_{k\in\mathbb{Z}} (1+|k|)^{2s+2\tau+1} |\hat{\phi}(k)|^2 \\ &\lesssim \gamma^2 \|\phi\|_{s+\tau+1/2}^2 \end{aligned}$$

where the constant may change from one line to another. □

We also remark that in the general case where $d \geq 1$, the same result holds if we replace the estimate by

$$\|\psi\|_s \lesssim \gamma \, \|\phi\|_{s+\tau+d/2}$$

On the other hand, if α is Liouvillean, no solution exists in any function space for generic C^∞ (or even C^ω) functions. This kind of arithmetic obstructions to reducibility is therefore already present in the simplest of algebraic settings, and we will encounter in the guise of *resonances* in the dynamics in the fibers in chapter 9.

A second kind of obstructions to reducibility is the one related to homotopy. Cocycles conjugate to each other are also homotopic to each other, independently of the arithmetic properties of the rotation in the basis. In order to be precise, let us suppose that $(\alpha, A_i(\cdot))$. $i = 1, 2$, are continuous and conjugate to each other, and that $B(\cdot)$ is continuous. The path in $C^0(\mathbb{T}^d, G)$

$$B(\cdot + (1-t)\alpha) A_2(\cdot) B^{-1}(\cdot)$$

deforms continuously $A_1(\cdot)$ to $B(\cdot)A_2(\cdot)B^{-1}(\cdot)$. Since the homotopy group of a compact Lie group G is that of $\mathcal{T} \hookrightarrow G$, where $\mathcal{T}$ is a maximal torus, we can deform continuously $A_2(\cdot)$ to $\tilde{A}(\cdot)$ and $B(\cdot)$ to $\tilde{B}(\cdot)$ which they take values in $\mathcal{T}$. As a consequence, both $A_1(\cdot)$ and $A_2(\cdot)$ are homotopic to $\tilde{A}(\cdot)$. In particular, a reducible cocycle is homotopic to constants, and so are almost reducible cocycles.

An example of a cocycle which is not reducible and not almost reducible for this exact reason is

$$(\alpha, R_{2\pi \cdot}) \in SW^\infty(\mathbb{T}^d, SO(3))$$

for any minimal α. This cocycle, as well as C^0-small perurbations of it, are not reducible. Its second iterate,

$$(2\alpha, R_{4\pi(\cdot + \alpha/2)})$$

admits a lift in $SW^\infty(\mathbb{T}^d, SU(2))$, which is $(2\alpha, E_1(\cdot + \alpha/2))$ in the notation that we have adopted. Since $SU(2)$ is simply connected, there is no topological obstruction to the reducibility of this cocycle, but it is in fact not reducible. The reason is a non-trivial theorem by R. Krikorian (cf. theorem 3.8) which is still of topological nature, but this time the topological space is $C^\infty(\mathbb{T}^d, SU(2))$ and not the group $SU(2)$ itself. Since the reason for the non-reducibility of $(2\alpha, E_1(\cdot + \alpha/2))$ is not a discrete invariant depending continuously on the mapping in the fibers, one could expect (and it is in fact true, cf. theorem 3.11) that arbitrarily small perturbations of this cocycle become almost reducible.

One could call such obstructions geometric ones, since one of the results that we will prove is that, at least for one-frequency coycles, cocycles which are not conjugate in any way close to constant ones are those whose dynamics are similar to those of some periodic geodesic of G.

The third type of obstructions is of purely dynamical nature and they were constructed in [Eli02]. The obstruction in this case is unique ergodicity of the dynamics in $\mathbb{T}^d \times SO(3)$, where the preserved measure is the product of the Haar measures on the basis and the fibers. Since a constant cocycle (α, R_ϑ) admits many invariant measures, unique ergodicity is incompatible with reducibility. The construction of these cocycles by a K.A.M. scheme shows in fact that non-reducible cocycles are not exotic objects that live far from the well understood constant cocycles, since they are in fact almost reducible.

We close this chapter by stating some results in the existing litterature concerning the reducibility of quasiperiofic skew-systems in compact Lie groups. Some of these theorems are stated in the continuous-time case, but their proofs work equally well in the discrete-time case.

Firstly, let us give two positive results in the local setting, by R. Krikorian, the first one being on the density of reducible cocycles in the neighborhood of constants, proved in [Kri99a].

Theorem 3.5. *Let G be a compact semisimple Lie group $\omega \in DC_d(\gamma, \sigma)$, i.e. such that*

$$|(k, \omega)| \geq \frac{\gamma^{-1}}{|k|^\sigma}, \ k \in \mathbb{Z}^d \backslash \{0\}$$

Then, there exist $\varepsilon_0, s_0 > 0$ and a natural number χ_G, depending only on the group G, such that, for all $A \in g$ and $F \in C^\infty(\mathbb{T}^d, g)$ satisfying $\|F\|_{s_0} \leq \varepsilon_0$ and every $\varepsilon, s > 0$, there exists $F' \in C^\infty(\chi_G \mathbb{T}^d, g)$ such that $(\omega/2\pi, A + F'(\cdot))$ is reducible $\mod \chi_G$ and satisfies $\|F - F'\|_s \leq \varepsilon$. If $G = SO(3)$, or $SU(w+1)$, one can take $\chi_G = 1$. In other words, reducible systems are dense in

$$\{A + F(\cdot), \ A \in g, \ F \in C^\infty(\mathbb{T}^d, g), \ \|F\|_{s_0} \leq \varepsilon_0\}, \ g = so(3), \ su(w+1)$$

A step of the proof of the theorem is the proof of quasi-reducibility of systems satisfying the same smallness condition.

The density properties are quite delicate and (as shows the proof) reducibility of any given system cannot be concluded, at least in the K.A.M. constructive sense. For this reason, such theorems are in general coupled with theorems where one embeds an individual system or cocycle in a family of such systems depending on parameters, and then studies the measure-theoretic abundance of reducible systems with respect to the parameters. The first such theorems, proofs of which are in the scope of classical K.A.M. theory, asserted that for generic one-parameter families there is reducibility for a set of positive measure in the parameter space. Moreover, the measure tends to be full as the size of the perturbation goes to 0 (the "K.A.M. tongues"). For this part of the theory, one can consult [Eli88] and [Kri99a]. These theorems have been improved to prevalence theorems, i.e. theorems asserting reducibility in full measure for generic one-parameter families. This improvement has been possible only for analytic cocycles, in contrast with positive measure theorems who have also been proved in the C^∞ case ([Kri99a]).

The prevalence theorem that we cite next was proved in [Kri99b].

Theorem 3.6. *Let G and ω as above, $h > 0$, $\Lambda \subset \mathbb{R}$ an interval and $A \in g$, generic. Then, there exists ϵ_0, depending on Λ, A, ω, h, such that, if $F(\cdot) \in C^\omega_h(\mathbb{T}^d, G)$ satisfies $|F|_h < \epsilon_0$, then for a.e. $\lambda \in \Lambda$, $(\omega/2\pi, \lambda A + F(\cdot))$ is reducible $\mod \chi_G$. If $G = SO(3)$, or $SU(w+1)$, one can take $\chi_G = 1$*

The local picture as we know it is completed by H. Eliasson's theorem, which we have already discussed.

Theorem 3.7. *Let ω as above and let $h > 0$. Then, there exists a constant $C = C(h, \sigma)$ such that the set of $F(\cdot) \in C^\omega_h(\mathbb{T}^d, so(3))$ for which the system $(\omega/2\pi, F(\cdot))$ has a unique invariant measure is a G_δ-dense set.*

We also cite the following theorem of qualitative nature, proved in [Kri99a] (theorem *2.2.3*)

Theorem 3.8. *Let G be a compact connected semisimple Lie group, $\alpha \in \mathbb{T}^d$ a minimal translation and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$, such that $\{(\alpha, A(\cdot))^n\}_{n \in \mathbb{Z}}$ be precompact in $SW^\infty(\mathbb{T}^d, G)$ for the C^∞ topology. Then, there exists $A_0 \in G$ and $B(\cdot) \in C^\infty(\mathbb{R}^d / \tilde{\chi}_G \mathbb{Z}^d, G)$ such that*

$$A(\cdot) = B(\cdot + \alpha) A_0 B^*(\cdot)$$

The integer $\tilde{\chi}_G$ can be chosen equal to $c_G \chi_G$, and if $\chi_G = 1$ it can be chosen equal to 1.

We remind that χ_G is a constant depending only on G. We also remark that the inverse is trivially true, and we give the immediate corollary

Corollary 3.9. *A cocycle $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$ is reducible, possibly modulo $\tilde{\chi}_G$, if, and only if, there exists $q \in \mathbb{Z}^*$ such that $(\alpha, A(\cdot))^q$ is reducible.*

As for the problem of reducibility in an ambient group and intrinsic reducibility, C. Chavaudret has shown in [Cha11] the following theorem.

Theorem 3.10. *Consider a skew-system in $\mathbb{T}^d \times G$, where G is $Sp(n, \mathbb{R})$ (the real symplectic group of dimension n), $SL(n, \mathbb{R})$ (the group of real $n \times n$ matrices of determinant 1), $O(n)$ (the n-dimensional orthogonal group), or $U(n)$. Suppose, moreover, that the system is $GL(n, \mathbb{C})$-reducible. Then, it is G-reducible* mod 2. *It is reducible* mod 1 *when $G = U(n)$.*

The equivalence of $U(w')$- and G-reducibility is unknown.

Finally, we conclude with the results obtained more recently in the global setting, i.e. where there is no assumption of closeness to constants for the studied system. Although it was preceded by some preliminary results by K. Fraczek ([Fra00]), the first general result of global density is due to R. Krikorian ([Kri01]):

Theorem 3.11. *There exists $\Sigma \subset \mathbb{T}$, of full measure such that, for $\alpha \in \Sigma$, reducible cocycles are dense in $SW^\infty_\alpha(\mathbb{T}, SU(2))$ in the C^∞ topology.*

The set Σ as it is defined is slightly thinner than RDC (see definition 2.14). The proof of the theorem relies in the renormalization of dynamics, which brakes down the proof of the theorem to the local study of a collection of models (periodic geodesics of $SU(2)$), followed by the actual study of the models using K.A.M. techniques.

A corollary of the proof of the theorem is the quantization of the length of rescaled iterates of the cocycle, for rotations in Σ and in regularity C^2. This result was generalized by K. Fraczek in [Fra04]:

Theorem 3.12. *For every $(\alpha, A(\cdot)) \in SW^1(\mathbb{T}, SU(2))$, there exists a measurable and bounded function $\psi : \mathbb{T} \to su(2)$ such that*

$$\frac{1}{n} LA_n(\cdot) = \frac{1}{n} a_n(\cdot) \to \psi(\cdot)$$

Lebesgue a.e.. Moreover, ψ is invariant by the dynamics of the cocycle,

$$Ad(A(\cdot))\psi(\cdot) = \psi(\cdot + \alpha)$$

and therefore $|\psi(\cdot)|$ is a.e. a constant.

This constant was defined to be the *degree* of the cocycle and in the same paper it was shown that it is *quantized*.

Theorem 3.13. *If $(\alpha, A(\cdot)) \in SW^2(\mathbb{T}, SU(2))$, then its degree is in $2\pi\mathbb{N}$.*

We remark that no arithmetic conditions, other than irrationality, are assumed for α, and that from the proof (which uses renormalization) it follows that the degree as defined by K. Fraczek generalizes the length of "renormalization representatives" as used by R. Krikorian in his proof.

Subsequently, K.Fraczek showed that

Theorem 3.14. *If $(\alpha, A_i(\cdot)) \in SW^2(\mathbb{T}, SU(2))$, $i = 1, 2$, are measurably conjugate to each other then their degrees are equal.*

Finally, K.Fraczek refined the local study of obstructions and showed that if the rotation in the basis is RDC and the cocycle is of degree r, it is conjugate to the obstruction, thus obtaining the following global density theorem.

Theorem 3.15. *Reducible cocycles are dense in $SW^\infty_\alpha(\mathbb{T}, SU(2))$ in the C^∞ topology if $\alpha \in RDC$.*

3.4 Statement of the main theorem and plan of the proof

After having established the basic vocabulary, we can state the main theorem of this mémoire.

Theorem 3.16. *Let $\alpha \in RDC$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, G)$. If $1 \leq m \leq \chi_G$ is the minimal natural number such that $(\alpha, A(\cdot))^m$ is homotopic to constants, then $(\alpha, A(\cdot))^m$ is accumulated by reducible cocycles in $SW^\infty_{m\alpha}(m\mathbb{T}, G)$.*

In other words, for such α, any cocycle $(\alpha, A(\cdot))$ in $\mathbb{T} \times G$ has an iterate which is accumulated by reducible cocycles and the maximal number of iterations needed in order to satisfy this property depends only on the group G. We remind that the definition of the constant χ_G is given by lemma 2.5.

We can also give a version of the theorem concerning the cocycle itself, and not perturbations of it. Moreover, its statement does not involve any loss of periodicity, only iteration for reasons of homotopy.

Theorem 3.17. *Let $\alpha \in RDC$, and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, G)$ and m as above. Then, there exists $\chi \in \mathbb{N}^*$, $\chi | m$, such that $(\alpha, A(\cdot))^\chi$ is almost torus-reducible in $SW^\infty_{\chi\alpha}(\mathbb{T}, G)$.*

The proof of these theorems, based on intermediate results some of which have an interest on their own, occupies the remaining of this mémoire. It combines techniques used in the proof of theorems 3.11, 3.15 and 3.5, which nonetheless have had to be adapted in order to be applicable in this more general context. Let us give a general plan of the proof.

Abelian cocycles Firstly, in chapter 4, we study briefly the basic examples of non-reducible cocycles obtained easily from theorem 3.8. These are cocycles C^∞-conjugate to ones of the form $(\alpha, E_r(\cdot))$, where $E_r(\cdot)$ is a 1-periodic geodesic of G (for the notation, see subsection 2.1.1). If $(\alpha, A(\cdot))$ is such a cocycle, we find that by

$$\lim \frac{1}{q_n} LA_{q_n}(\cdot) \tag{3.7}$$

with q_n as in section 2.3, we can define a smooth curve $\mathbf{a}(\cdot) : \mathbb{T} \to g \setminus \{0\}$ which is invariant under the dynamics:

$$Ad(A(\cdot)).\mathbf{a}(\cdot) = \mathbf{a}(\cdot + \alpha)$$

The invariance of the curve and the minimality of α imply that the class of the vectors $\mathbf{a}(\cdot)$ in g mod $Inn(g)$ is well defined, and that the curve is in fact traced in a sphere in g (since the adjoint action of G is an isometry). For such cocycles, in fact, this class is essentially the degree $r \in \mathbb{Z}^w$ of the geodesic $E_r(\cdot)$.

Conversely, the existence of such an invariant curve $\mathbf{a}(\cdot)$ implies that, given any vector $s \in g$ in the class of $\mathbf{a}(\cdot)$, we can conjugate the given cocycle to $(\alpha, \tilde{A}(\cdot))$ commuting with s. Two cases are possible : either the vector s is *regular* (in the algebraic sense of the first chapter), or it is *singular*. In the fist case, since only one maximal torus $\mathcal{T}$ passes by s, $Ad(\tilde{A}(\cdot)).s = s$ implies that $\tilde{A}(\cdot)$ takes values in $\mathcal{T}$, i.e. $(\alpha, A(\cdot))$ is torus-reducible. However, if s is singular, torus-reducibility cannot be concluded by an algebraic reasoning. Such an argument can only produce what we will call a *splitting* of the dynamics in proposition 4.5. The splitting consists of two subgroups, G_+ and G_0, with G_+ abelian and given by the intersection of all maximal tori passing by s, and with G_0 being the largest subgroup of G commuting with G_+ and intersecting with G_+ only at the Id. If the vector s is singular, the cocycle $(\alpha, A(\cdot))$ can be conjugated to a cocycle taking values in $G_0 \times G_+ \hookrightarrow G$, which is not abelian. If the invariant curve is obtained as in eq. 3.7, we can additionally conclude that the part of the dynamics in G_+ carries all the linear growth of the derivatives of the cocycle, and the derivatives in the G_0-part grow sublinearly.

The energy of a cocycle In chapter 5 we begin the general study of cocycles. We drop the assumption that the cocycle $(\alpha, A(\cdot))$ is torus-reducible, but we define, following K. Fraczek, an invariant curve of regularity only $L^2(\mathbb{T}, g)$. Due to the curve's low regularity, we cannot derive conclusions finer than the fact that if it is *non-degenerate* (i.e. if it is not equal to 0 for $a.e.\, x \in \mathbb{T}$),

the cocycle is not reducible. On the other hand, since the invariance relation holds *a.e.*, we can still define the type of *a.e.* vector in the image of $\mathbf{a}(\cdot)$ as the degree of the cocycle. The length of these vectors is an invariant of the dynamics (just as the degree), which we will call energy. There is a shift in terminology with respect to K. Fraczek's work. There, the degree of a cocycle in $\mathbb{T} \times SU(2)$ is defined as a numerical quantity (which corresponds to our energy), since classes of vectors in $su(2) \approx \mathbb{R}^3$ modulo $Inn(su(2)) \approx SO(3)$ are spheres centered in the origin, thus completely determined by their radius. In more general groups, however, the length of vectors in the image of $\mathbf{a}(\cdot)$ is an insufficient invariant, and therefore we need to keep the term *degree* for their type. Subsequently, we show that higher-order derivatives (if they exist) carry no additional information, since $\frac{1}{n^{\sigma+1}} \partial^\sigma L A_n(\cdot) \to 0$ in L^2, provided that $\partial^\sigma L A(\cdot) \in L^2$.

The most important properties of the energy are those proven in Proposition 5.3 : the energy of the n-th iterate is nothing else but n times the energy of the cocycle, and the energy of the cocycle $(\alpha, A(\cdot))$ is not bigger than the energy of the path $A(\cdot) : \mathbb{T} \to G$ (i.e. the L^2 norm of its derivative), or its length (i.e. the L^1 norm of its derivative). These L^2 and L^1 norms decrease as n increases due to two causes. One is the averaging, already present in abelian groups : if $\phi : \mathbb{T} \to \mathbb{R}$, then its Birkhoff sums converge to its mean value, $\hat{\phi}(0)$, which is certainly smaller than $\|\phi\|_{L^i}, i = 1, 2$. The other is the non-alignment of $Ad(A(\cdot)).a_{n-1}(\cdot + \alpha)$ with $a(\cdot)$ (cf. the invariance relation satisfied by $\mathbf{a}(\cdot)$), which results in an at most linear growth of lengths, due to the triangle inequality.

Renormalization The low regularity of the *limit object of the dynamics*, the curve $\mathbf{a}(\cdot)$, calls for a more sophisticated means of study of the dynamics. This means is known as the renormalization scheme, and we present and apply it in chapter 6, following R. Krikorian. In this context, renormalization could be motivated by the observation that, thanks to the invariance of the curve,

$$Ad(A_{(-1)^n q_n}(\cdot)).\mathbf{a}(\cdot) = \mathbf{a}(\cdot + \beta_n) \tag{3.8}$$

where the quantities q_n and β_n are given by the continued fractions algorithm (see section 2.3). This fact, together with the fact that the passage in the limit in eq. 3.8 is still justified in L^2, shows that for large n, and around every point ν of Lebesgue continuity of $\mathbf{a}(\cdot)$, $A_{(-1)^n q_n}(\cdot)$ looks like a piece of geodesic of speed $(-1)^n q_n \mathbf{a}(\nu)$. Let us fix 0 as such a point for definiteness.

However, the fact that $A_{(-1)^n q_n}(\cdot) \approx \exp((-1)^n q_n \mathbf{a}(0)\cdot)$ does not imply, at least not directly, that $(\alpha, A(\cdot)) \sim (\alpha, \exp(\mathbf{a}(0)\cdot))$. In order to establish such a fact, we need to consider $A_{(-1)^n q_n}(\cdot)$ coupled with the preceding iterate in the continued fractions algorithm, $A_{-(-1)^n q_{n-1}}(\cdot)$. For n large enough, $A_{-(-1)^n q_{n-1}}(\cdot)$ obviously also approaches the limit object. The couple of cocycles

$$(\beta_{n-1}, A_{-(-1)^n q_{n-1}}(\cdot)) \text{ and } (\beta_n, A_{(-1)^n q_n}(\cdot))$$

is to be considered as a linearly independent couple of commuting cocycles which completely describes the dynamics of the original cocycle. Indeed, it is obtained by lifting the continued fractions algorithm and interpreting division at each step as iteration of the commuting pair

$$(1, Id) \text{ and } (\alpha, A(\cdot))$$

Here, 1 codes the initial scale of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, α the dynamics in the basis, $A(\cdot)$ the dynamics in the fibers. Finally, Id codes the 1-periodicity of the mapping $A(\cdot)$, as well as the fact that the space in which the dynamics takes place is $\mathbb{T} \times G$, i.e. that fibers $\{x\} \times G$ and $\{x + 1\} \times G$ are isomorphic via the identity mapping. Both couples $((1, Id), (\alpha, A(\cdot)))$ and $((\beta_{n-1}, A_{-(-1)^n q_{n-1}}(\cdot)), (\beta_n, A_{(-1)^n q_n}(\cdot)))$ act naturally on $\mathbb{R} \times G$. They generate the same $\mathbb{Z}^2$ action, modulo a change of basis of generators, coded by a matrix in $GL(2, \mathbb{Z})$.

After n steps of the algorithm of continued fractions applied on the initial couple $((1, Id), (\alpha, A(\cdot)))$, we obtain $((\beta_{n-1}, A_{-(-1)^n q_{n-1}}(\cdot)), (\beta_n, A_{(-1)^n q_n}(\cdot)))$. Let us interpret the quantities defining the second one. In a scale reduced to β_{n-1}, the dynamics of the cocycle $(\alpha, A(\cdot))$ are represented by the cocycle $(\beta_n, A_{(-1)^n q_n}(\cdot))$, but acting on a space whose fiber $\{x + \beta_{n-1}\} \times G$ is isomorphic to $\{x\} \times G$ by $S \to A_{-(-1)^n q_{n-1}}(x).S$ (and not by the Id). We remark that the distortion of the fibers accumulates as x increases, so that x in this context is more naturally considered as a real-valued variable.

We can, however, zoom in, in order to change the scale of the dynamics back to 1. The zoom, which makes sense since the cocycles act on $\mathbb{R} \times G$, gives the commuting couple

$$(1, \tilde{A}_{-(-1)^n q_{n-1}}(\cdot)) \text{ and } (\alpha_n, \tilde{A}_{(-1)^n q_n}(\cdot))$$

Finally, we can restore the fibers by a conjugation $(0, B(\cdot)) : \mathbb{R} \to G$ such that

$$B(\cdot) = \tilde{A}_{-(-1)^n q_{n-1}}(\cdot + 1).\tilde{A}_{-(-1)^n q_{n-1}}(\cdot))^*$$

We then obtain a commuting couple of the type

$$(1, Id) \text{ and } (\alpha_n, \check{A}_n(\cdot))$$

which we call a *renormalization representative* of $(\alpha, A(\cdot))$. A renormalization representative is a cocycle over a different rotation (α_n instead of α), and of shorter period (β_{n-1} instead of 1) than the original one. It represents, nonetheless, completely the original cocycle in the following sense. Conjugation of the representative, or arbitrarily small perturbations of it translate back to conjugations of the original cocycle. Consequently, all reducibility and accumulation by reducible cocycles properties are shared by the two objects.

The important point comes from our first observation in this paragraph (cf. the formula in eq. 3.8). In our vocabulary, it means that around a typical point $\nu \in \mathbb{T}$, the two cocycles

$$(1, \tilde{A}_{-(-1)^n q_{n-1}}(\cdot)) \text{ and } (\alpha_n, \tilde{A}_{(-1)^n q_n}(\cdot))$$

defining the commuting pair at the n-th step of the algorithm converge to the same limit object of the dynamics, $\mathbf{a}(\nu)$. This allows us to obtain a renormalization representative as an expression of the limit object:

$$\check{A}_n(\cdot) \approx \exp(\mathbf{a}(\nu)\cdot)$$

Since the representative is a 1-periodic cocycle, we obtain the quantization of the degree (theorem 6.1):

$$e^{\mathbf{a}(\nu)} = Id, \; a.e. \; \nu \in \mathbb{T}$$

We have hidden some problems concerning homotopy and the algebraic type of the degree (regular or singular), but roughly the convergence of the renormalization scheme shows the following thing. If we are willing to allow the arithmetic properties of the rotation to deteriorate, we can simplify the expression of our cocycle to a perturbation of its degree. Then, any conjugation or perturbation that we construct for the representative (therefore in a distorted phase-space and locally in x) is translated into a conjugation or perturbation of the original cocycle, well defined for all $x \in \mathbb{T}$ and 1-periodic.

The homotopy class of a cocycle manifests itself, when significant, in the context of lemma 2.5. Therefore, it is significant only if the degree is not a regular vector and only if the cocycle is non-homotopic to constants within G_0. The way to cope with this issue is to consider a sublattice of the action, and thus commuting pairs of the type

$$(\chi, Id) \text{ and } (\chi\alpha_n, \check{A}_n(\cdot))$$

as renormalization representatives. The number $\chi \in \mathbb{N}^*$ is such that "$A(\cdot)$ is homotopic to constants in the part of the group that matters".

The preceding arguments sum up essentially to saying that the ergodicity of the dynamics in the basis results in the cancellation of non-commutativity in the fibers. Moreover, they interpret the energy $(\alpha, A(\cdot))$ as the length of the path coding the dynamics of the q_n-th iterate of $(\alpha, A(\cdot))$ in the scale β_{n-1}. Moreover, they show how commutativity appears robustly when the lengths in the proper scale are in accordance with the asymptotic characteristics of the dynamics: initially, non-commutativity makes lengths decrease, until the ergodic theorem cancels the non-commutativity. Finally, let us remark that this derivation shows that the fact that the group G is not commutative does not affect the asymptotic properties of the dynamics and only marginally (appearing through homotopy) the models of the dynamics: the quantization of the degree is the same as for cocyles in $\mathbb{T} \times \mathbb{T}^w$ and the models of dynamics are (non-abelian) perturbations of geodesics in $\mathbb{T}^w \hookrightarrow G$.

Finally, the observation that conjugation between cocycles acts on the limit curve by algebraic conjugation (see the proof of proposition 5.3), can serve as a motivation for K. Fraczek's theorem on the measurable invariance of the degree, which we generalize and strengthen in terms of its regularity assumptions (theorem 6.7). Since the invariant curve defining the degree of a C^1 cocycle is measurable, it is reasonable to expect that measurable conjugation of C^1 cocycles, which preserves the properties of their limit curves, should preserve the degree. We do not know, however, a proof of this fact not using renormalization, although this property is not related to it.

A priori estimates for perturbations of regular cocycles The effect of non-commutativity is exploited in chapter 7. At this point, we suppose that renormalization of a given cocycle is not bound to simplify its dynamics, i.e. that the cocycle $(\alpha, A(\cdot))$ is given as a perturbation of a geodesic $(\alpha, E_{r,a}(\cdot))$ whose derivative is equal to the degree of $(\alpha, A(\cdot))$. Under this assumption, we obtain some necessary conditions satisfied by the perturbation, summed up in lemma 7.1.

In order to motivate these estimates, let us restrain ourselves to the case where $G = SU(2)$ and consider a cocycle whose dynamics in the fibers are given by a perturbation of $E_r(\cdot) = \exp(\{2\pi r\cdot, 0\}_{su(2)})$, with $r \in \mathbb{N}^*$ (for the notation, see subsection 2.1.2). We remark that a perturbation in the diagonal direction (i.e. in the direction commuting with the privileged direction of the dynamics) dies out thanks to the additive ergodic theorem. We suppose, therefore, that the perturbation is non-zero only in the complex direction of $su(2) \approx \mathbb{R}\times\mathbb{C}$, and write the perturbation in the form $U(\cdot) : x \mapsto \{0, U_z(\cdot)\}_{su(2)}$, with $U_z(\cdot) : \mathbb{T} \to \mathbb{C}$ small enough. Therefore, we consider a cocycle of the form $(\alpha, E_{r,a}(\cdot)e^{U(\cdot)})$, with such a mapping $U(\cdot)$, restrained in the plain orthogonal to e_r.

Then again, in this simplified context, two possibilities can be distinguished. Either $0 \in \mathbb{C}$ is in the bounded component of the complement of the curve $U_z(\cdot)$, or in the unbounded one, depending on the significance of the constant part of the perturbation over its non-constant part.

For the first case, let us suppose moreover that the perturbation has only one frequency, and thus of the form $U_z(\cdot) = \varepsilon e^{2i\pi k\cdot}$ with $k \neq 0$, so that $U(\cdot) = \{0, \varepsilon e^{2i\pi k\cdot}\}$. [1] In this case, $\partial U(\cdot) = \{0, 2i\pi\varepsilon k e^{2i\pi k\cdot}\}$ is orthogonal to $U(\cdot)$, and thus does not commute with it. Transfer of $\partial e^{U(\cdot)}$ to $g = T_{Id}$ gives the first order term $\partial U(\cdot)$ and the second order term $\frac{1}{2}[U(\cdot), \partial U(\cdot)]$. This second term is, by elementary vector algebra in $\mathbb{R}^3$, parallel to $e_r = LE_{r,a}(\cdot)$, i.e. in the real line of $\mathbb{R}\times\mathbb{C}$, and its value is $\{2\pi\varepsilon^2 k, 0\}$.

Since $[\cdot,\cdot] \equiv 2\cdot\times\cdot$, we obtain that if $k > 0$ (i.e. if the circle of radius ε in $\mathbb{C}$ is traversed in the direction of e_r) all terms contribute positively in the calculation of the L^2 norm of the derivative. If, however, $k < 0$, the bracket $\{2\pi\varepsilon^2 k, 0\}$ is in the opposite direction than e_r : the path $A(\cdot)$ is traversed with a speed equal to $2\pi(r - \varepsilon^2|k|)$, constantly smaller than $2\pi r$ at least in the diagonal direction. On the other hand, the term in the complex direction, $\partial U(\cdot)$, contributes to the L^2 norm by $(2\pi k\varepsilon)^2$. Therefore, if $k \leq -2r$, this drift in the real direction is compensated by the movement within the complex one. On the other hand, for $-2r + 1 \leq k < 0$, the contribution remains negative. The same calculation when all Fourier modes are authorized in the perturbation shows that, at least in the order of ε^2, the frequencies are not mixed, so that the conclusion remains valid: if the perturbation rotates clockwise in $\{0\}\times\mathbb{C}$ around $\mathbb{R}$ sufficiently slowly, the counter-clockwise rotation of the geodesic does not compensate it, and the cocycle cannot be of degree r.

Up to now there is no dynamics in the estimates, but it comes into effect when the perturbation is of the second type, i.e. if 0 is in the unbounded component of the complement of $U_z(\cdot)$. In this case, the constant part is the principal part of the perturbation, so let us suppose that the perturbation is actually constant :

$$A(\cdot) = E_r(\cdot)e^{\{0,z\}}$$

where $z \in \mathbb{C}^*$ is small. The path is merely a (right) translate of the geodesic $E_r(\cdot)$, so that is length is that of the geodesic. Surprisingly enough, in the second iterate of this cocycle the

1. The coefficient ε should in fact be chosen in $\mathbb{C}^*$, but its absolute value is the significant part, as the imaginary part only introduces a phase in the perturbation, which by the geometric argument given below should se easily seen to be irrelevant. The parameter ε remains, nonetheless a 2-dimensional real parameter for considerations concening the dimension of obstructions.

previous phenomenon appears:

$$\begin{aligned} A(\cdot+\alpha).A(\cdot) &= E_r(\cdot+\alpha)e^{\{0,z\}}.E_r(\cdot)e^{\{0,z\}} \\ &= E_{2r}(\cdot)E_r(\alpha)e^{\{0,z\ e^{-4i\pi r\cdot}\}}.e^{\{0,z\}} \end{aligned}$$

The constants $E_r(\alpha)$ and $e^{\{0,z\}}$ are insignificant, and the previous reasoning applies with r replaced by $2r$. In both cases, the fact that estimation of the length is continuous in the C^1 topology completes the picture when $G = SU(2)$. For definiteness and comparability with the general case, we remind that not r, but $2r$ corresponds to r_ρ in the notation for general Lie groups.

Finally, since $su(2)$ is embedded in any semi-simple compact Lie algebra, and since the interaction of different such subalgebras is of higher order, the result remains true for any compact algebra g.

These estimates allow a preliminary description of the configuration in $SW^\infty_\alpha(\mathbb{T}, G)$ of the conjugacy classes of periodic geodesics (section 7.7), for any $\alpha \in \mathbb{T} \setminus \mathbb{Q}$. From the estimates, we obtain directly that every model $(\alpha, E_r(\cdot))$ is in the boundary of an open set (in the C^1 topology) of cocycles of energy strictly smaller than $|e_r|$. Since any cocycle $(\alpha, A(\cdot))$ has renormalization representatives $\mod \chi$ arbitrarily close to such a geodesic, we can perturb such a renormalization representative and obtain a cocycle of smaller energy. This perturbed cocycle is the renormalization representative of a cocycle $(\chi\alpha, A'(\cdot)) \in SW^\infty(\mathbb{T}, G)$. However, due to the distortion-restoration of the fibers and the rescaling, we cannot relate the smallness of the perturbation of the renormalization representatives to the distance of the cocycles $(\alpha, A(\cdot))$ and $(\alpha, A'(\cdot))$. We can say, nonetheless, that $A(\cdot)$ can be continuously deformed to $A'(\cdot)$, and that, if we go deep enough into renormalization, we can make the perturbation of the representative arbitrarily small.

As a conclusion, we can say that the different conjugacy classes are not too far from each other, unlike their discrete repartition in $SW^\infty(\mathbb{T}, \mathbb{T}^w)$. Additionally, it is precisely the non-abelian character of G which destroys this structure. On the other hand, this argument cannot be pushed any further, and a more precise conclusion demands a study of the renormalization representatives as cocycles. If some renormalization representative of $(\alpha, A(\cdot))$ can be conjugated arbitrarily close to the limit object, since the distortion of the fibers and the rescaling factor have been kept constant, the same follows for the cocycle $(\alpha, A(\cdot))$ by inverting the renormalization scheme. Likewise, if the renormalization representative is accumulated by cocycles reducible to the limit object, we can conclude that the property of accumulation by lower energy cocycles holds equally for the cocycle $(\alpha, A(\cdot))$ itself.

Perturbations of regular geodesics The local study of the limit objects starts in chapter 8 with the local study of regular geodesics. From this chapter on, unlike with the previous ones, we suppose that the rotation α satisfies some recurrent Diophantine condition (see definition 2.14), so that K.A.M. and K.A.M.-like methods are applicable.

In this chapter we suppose that the cocycle $(\alpha_n, E_{r,a_n}(\cdot).e^{U_n(\cdot)})$ is given, where $\alpha_n \in DC$ with *a priori* fixed constants, $U_n(\cdot) : \mathbb{T} \to g$ satisfies some smallness condition in C^∞ and $E_{r,a_n}(\cdot)$ is a regular geodesic. The hypothesis that $\alpha_n \in DC$ is satisfied by the definition of RDC if we consider a subsequence in the renormalization scheme. Moreover, we suppose that the degree of the renormalization representative of some cocycle is r, and therefore equal to the degree of $(\alpha_n, E_{r,a_n}(\cdot))$.

These assumptions pose the problem as that of the study of perturbations of regular geodesics over Diophantine rotations, the perturbations being of the same degree as the geodesics. In order to simplify the notation and since we have established the connection of this setting with the global problem, we suppose that the cocycle is given in the form $(\alpha, E_{r,a}(\cdot).e^{U(\cdot)})$ is given, where $\alpha \in DC$.

The regularity of $E_{r,a}(\cdot)$ implies that for every embedding $su(2) \hookrightarrow g$ corresponding to a root ρ, there exists an integer $r_\rho \neq 0$ such that $Ad(E_{r,a}(\cdot)).j_\rho = 2i\pi r_\rho j_\rho$. So, we suppose, with no serious loss of generality [2] that $G = SU(2)$, and that $E_{r,a}(\cdot) = \{e^{2i\pi r(\cdot+a)}, 0\}_{SU(2)}$.

The a priori estimates described just above indicate that only a part of the spectrum of the perturbation is significant. It corresponds to the part of the perturbation of the the limit object which cannot be evened out by the additive ergodic theorem.

2. The only loss is that we are forced to consider only $r_\rho \in 2\mathbb{Z}^*$ rather than in $\mathbb{Z}^*$, but this does not affect our discussion. The discussion remains true of one formally substitutes $r = 1/2$.

Before connecting this fact with the local theory of regular geodesics, let us make a small digression and state a more general remark on the local theory of models of cocycles. Let us suppose that the cocycle

$$(\alpha, A(\cdot))$$

is a model of dynamics in which we are interested, and let us consider a perturbation of it in the form

$$(\alpha, A(\cdot)e^{U(\cdot)})$$

where $U(\cdot) : \mathbb{T} \to g$ is small. What one would wish in the context of the local theory of $(\alpha, A(\cdot))$ would be to find a conjugation, hopefully of the order of $U(\cdot)$, conjugating the perturbed cocycle to the exact model. The conjugant is supposed to be of the form $\exp(Y(\cdot))$, where again $Y(\cdot)$ is supposed to be small. The equation satisfied by such a $Y(\cdot)$ is a non-linear one, namely

$$e^{Y(\cdot+\alpha)}.A(\cdot).e^{U(\cdot)}.e^{-Y(\cdot)} = A(\cdot)$$

or

$$e^{Ad(A^*(\cdot)).Y(\cdot+\alpha)}.e^{U(\cdot)}.e^{-Y(\cdot)} = Id$$

The exact model acts on the conjugant by algebraic conjugation along the translation by α. Since this equation is non-linear, we opt for the solution of the linearized one and therefore renounce in single-handedly conjugating the perturbed cocycle to the exact model. The linearized equation reads

$$Ad(A^*(\cdot)).Y(\cdot + \alpha) - Y(\cdot) = -U(\cdot) \tag{3.9}$$

By solving this equation, we can only reduce $(\alpha, A(\cdot))$ to

$$e^{Y(\cdot+\alpha)}.A(\cdot).e^{U(\cdot)}.e^{-Y(\cdot)} = A(\cdot)e^{U'(\cdot)}$$

with $U'(\cdot)$ much smaller than the initial pertrubation $U(\cdot)$ (we say that it is "of second order"). This will hopefully (i.e. under some relevant assumptions) give rise to an iterative scheme which will settle the question. Unfortunately, even this is not possible, at least not in general, as, for a given $A(\cdot)$, the image of the operator

$$Y(\cdot) \mapsto Ad(A^*(\cdot)).Y(\cdot + \alpha) - Y(\cdot) \tag{3.10}$$

may not be closed in $C^\infty(\mathbb{T}, g)$, or its closure may not be equal to $C^\infty(\mathbb{T}, g)$. In the first case, some first order phenomena can not be reduced with good estimates by a conjugation of the same order (we will call such phenomena *resonances* in chapter 9) [3] . In the second case, the perturbation in the complementary space cannot be reduced by a conjugation of the same order. We will see in the case of regular cocycles that such perturbations cannot be reduced by conjugations of any order. The space complementary to the image of the operator constitutes the *dynamical obstructions* to reducibility to the exact model. Perturbations supported in the obstructions are the exit door to the conjugacy class of the model. This is quite straightforward in the case of regular geodesics, and quite trickier in the case of constant cocycles.

Let us now end our digression and return to the special case under consideration. The conjugation equation, studied in section 8.3, then reads

$$Ad(E^*_{r,a}(x)).Y(x + \alpha) - Y(x) = -U(x)$$

which, if we note $Y(\cdot) = \{Y_t(\cdot), Y_z(\cdot)\}_{su(2)}$, decomposes into

$$\begin{aligned} Y_t(\cdot + \alpha) - Y_t(\cdot) &= -U_t(\cdot) \\ e^{-4i\pi r(\cdot+a)}.Y_z(\cdot + \alpha) - Y_z(\cdot) &= -U_z(\cdot) \end{aligned}$$

In the first equation, small divisor phenomena appear, but they are related only to α, so that the assumption that $\alpha \in DC$ settles this part of the perturbation (see lemma 3.4). We remind

3. Exact resonances belong actually to the second case of obstructions.

that the only topological obstruction to the solution of the equation concerns the mean value of the rhs function, which is an abelian constant, so that its dynamics are of the same type as those of $(\alpha, E_{r,a}(\cdot))$, only with a different a.

The second equation reads, in the space of frequencies,

$$e^{2i\pi((k+2r)\alpha-2a)}\hat{Y}_z(k+2r) - \hat{Y}_z(k) = \hat{U}_z(k) \tag{3.11}$$

for all $k \in \mathbb{Z}$. It is a difference equation of step $2r$ in the space of rapidly decaying sequences, so that its free parameters are $2r$ terms of the sequence. Since these parameters, which are in fact the obstructions, are freely chosen, we can place them in the interval $\{-2r+1, \cdots, 0\}$. We thus establish that $(\alpha, E_{r,a}(\cdot))$ perturbed by obstructions will have different dynamics if the obstructions are strong enough: in that case, the a priori estimates show that the cocycle cannot be of degree r. This observation gives rise to the K.A.M. scheme of [Kri01], a variant of which we use in chapter 10. This method, sufficient for the proof of the density theorem, shows that if the cocycle $(\alpha, A(\cdot))$ is close enough to $(\alpha, E_{r,a}(\cdot))$, the smallness of the perturbation depending on $\alpha \in DC$, and if $(\alpha, A(\cdot))$ is of degree r, then it is smoothly conjugate to $(\alpha, E_{r,a'}(\cdot))$. A fundamental reason for the convergence of the scheme (apart from the legitimate assumption on the degree) is the absence of small divisor phenomena in the dynamics in the fibers: there are no such phenomena in eq. 3.11 and as a consequence the conjugation not only always exists, but it also is of the order of the perturbation.

This last observation triggers a more general question. Since the core of the difficulties of K.A.M. theory is absent in this context, can we say more? This is done by dropping the assumption on the degree of $(\alpha, A(\cdot))$, thus authorizing non-zero obstructions.

The answer could be seen to be affirmative in the following way. Let us assume that $(\alpha, E_{r,a}(\cdot)e^{U(\cdot)})$ is given, and let us write $U(\cdot) = U_{cob}(\cdot) + U_{ob}(\cdot)$, where $U_{cob}(\cdot)$ is in the image [4] of the operator introduced in eq. 3.10, and $U_{ob}(\cdot)$ is in a complementary space, and it is spectrally supported in $\{-2r+1, \cdot s, 0\}$. Then, there exists $Y(\cdot) \in C^\infty(\mathbb{T}, g)$ such that

$$Ad(E_{-r,-a}(x)).Y(x+\alpha) - Y(x) = -U_{cob}(x)$$

Moreover, as we will see in section 8.2, $Y(\cdot)$ satisfies good (tame) estimates with respect to $U(\cdot)$, where the constants and the loss of derivatives depend only on α, and therefore are constant throughout the scheme. Consequently, we can write

$$\begin{aligned} E_{r,a}(\cdot)e^{U(\cdot)} &= E_{r,a}(\cdot)e^{U_{cob}(\cdot)+U'(\cdot)}e^{U_{ob}(\cdot)} \\ &= e^{-Y(\cdot+\alpha)}E_{r,a'}(\cdot)e^{U'(\cdot)}e^{U_{ob}(\cdot)}e^{Y(\cdot)} \end{aligned}$$

In these equations, $U'(\cdot)$ may change from one line to another, but it is quadratic with respect to $U(\cdot)$ and $Y(\cdot)$, and $a' = a + \hat{U}_t(0)$. The next step, however, has to be different, since, in general, $U_{ob}(\cdot)$ will be of the order of $U(\cdot)$. This takes us to the study of perturbations of models of the type

$$(\alpha, E_{r,a}(\cdot).e^{P(\cdot)})$$

where $P(\cdot)$ takes values in the space of obstructions [5] and satisfies some fixed a priori bound, and $U(\cdot)$ is at least quadratic with respect to $P(\cdot)$. Then, conjugation in the neighborhood of this model reads

$$e^{Y(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)}.e^{P(\cdot)}.e^{-Y(\cdot)} = E_{r,a'}(\cdot).e^{U'(\cdot)}.e^{P'(\cdot)}$$

where $Y(\cdot)$ is of the order of $U(\cdot)$, $U'(\cdot)$ is quadratic with respect to $U(\cdot)$, and $P'(\cdot)$ is close to $P(\cdot)$ up to a correction of the order of $U(\cdot)$.

4. *Cob* stands for coboundary and *Ob* for obstructions

5. We keep implicit the dependence of the obstructions with respect to the abelian constant a, since for a' close to a the space of obstructions of $(\alpha, E_{r,a'}(\cdot))$ is a finite-dimensional vector space, canonically isomorphic to the obstructions of $(\alpha, E_{r,a}(\cdot))$. In order to see this, is suffices to check that, in the case where $G = SU(2)$ we can normalize a to 0 by introducing a translation $x \mapsto x + \lambda$, and then verify that, in proposition 8.6, the obstructions Γ behave well under such translations for λ seen now as a parameter close to 0. Finally, the dimension depends only on r.

Given the a priori bounds on $P(\cdot)$, and anticipating that $Y(\cdot)$ be of the order of $U(\cdot)$, we find that, up to quadratic terms, the preceeding equation is equivalent to

$$e^{Y(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)}.e^{-Y(\cdot)}.e^{P(\cdot)} = E_{r,a'}(\cdot).e^{U'(\cdot)}.e^{P'(\cdot)}$$

where we have used the fact that the operator $Id - Ad(\exp(Y(\cdot)))$ is of the order of $Y(\cdot)$ (by the estimates on the Hausdorff-Campbell formula of section 11.3), and thus $Ad(\exp(Y(\cdot))).P(\cdot) = P(\cdot)$, plus terms of the order of $U(\cdot)$ which we have incorporated in $U(\cdot)$. [6] Rewriting this last equation in a more familiar form gives

$$e^{Y(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)}.e^{-Y(\cdot)} = E_{r,a'}(\cdot).e^{U'(\cdot)}.e^{P'(\cdot)-P(\cdot)}$$

which is exactly the case treated previously:

$$e^{Y(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)}.e^{-Y(\cdot)} = E_{r,a'}(\cdot).e^{U'(\cdot)}.e^{U_{Ob}(\cdot)}$$

and the step is concluded if we call $P'(\cdot) = P(\cdot) + U'_{Ob}(\cdot)$, $a' = a + \hat{U}_t(0)$, and $U'(\cdot)$ the quadratic term. Iteration is possible, since if the a priori bounds are satisfied with some margin by $P(\cdot)$, then they are equally satisfied with some margin by $P'(\cdot)$, and the diagonal constant a' is not too far from a. Therefore, the scheme whose iterative step we just described should converge and produce a conjugation $\tilde{Y}(\cdot)$ satisfying

$$e^{\tilde{Y}(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)}.e^{-\tilde{Y}(\cdot)}.e^{P(\cdot)} = E_{r,a'}(\cdot).e^{\tilde{P}(\cdot)}$$

where $\tilde{P}(\cdot)$ takes its values in the space of obstructions.

The scheme in the way we presented it looks more than a classical fixed-point argument than a K.A.M. scheme, and this is true in a way, since the loss of derivatives and the constants entering the estimates depend only on r and α, which are kept constant throughout the scheme, and small divisors are only a secondary phenomenon. The actual proof, however, is a convergent classical K.A.M. scheme, or its equivalent, the Hamilton inverse function theorem (theorem 2.11). The exactness of the statements of theorems 8.1 and 8.4 is precisely due to the secondary effect of small divisor phenomena in this context.

Perturbations of singular geodesics Since the natural place of chapter 10, at least from the dynamical point of view, is actually before chapter 9, let us try to motivate the need for a more accurate local theory of constant cocycles, which made chapter 9 necessary. We also hope that it will serve the reader as an introduction for the difficulties of the K.A.M. scheme finally applied in chapter 10.

Singular cocycles stand in the middle between regular ones and cocycles of 0 energy, in the sense that their obstruction to reducibility, the invariant curve $\mathbf{a}(\cdot)$ is non-degenerate, but in the same time weaker than the obstruction of a regular cocycle. The first manifestation of this fact is already present in renormalization, since a singular cocycle can be renormalized to a commuting pair of the type

$$(1, \tilde{C}_0^{(n)}(\cdot)) \text{ and } (\alpha_n, E_r(\cdot)\tilde{A}_0^{(n)}(\cdot))$$

where $\tilde{A}_0^{(n)}(\cdot)$ and $\tilde{C}_0^{(n)}(\cdot)$ are close to constants commuting with $e_r = LE_r = \mathbf{a}(0)$. However, unlike for regular cocycles, in general we cannot bring this pair to the canonical form

$$(1, Id) \text{ and } (\alpha_n, E_r(\cdot)\tilde{A}_0^{(n)}(\cdot))$$

unless we iterate χ_0 times first, where χ_0 is the constant χ_{G_0} related to G_0, the largest semisimple subgroup of G not commuting with $\mathbf{a}(0)$. We can, nonetheless, always obtain a commuting pair of the type

$$(\chi_0, Id) \text{ and } (\chi_0\alpha_n, E_{\chi_0 r}\tilde{A}_0^{(n)}(\cdot))$$

6. There is a hidden classical fixed point argument here. In the final proof that we give it does not appear in this way, but at this point the reader can think of this trick as a trial-and-error procedure, where the terms coming from $[Id - Ad(\exp(Y(\cdot)))].P(\cdot)$ are gradually incorporated in $U(\cdot)$ until equilibrium is reached. The absence of small divisors guarantees convergence.

with $\tilde{A}_0^{(n)}(\cdot)$ close to a constant $\tilde{A}_0^{(n)}$ belonging to the same maximal torus as $E_r(\cdot)$. By rescaling by a factor of χ_0 and introducing some obvious notation, we can bring ourselves to the case where the 1-periodic coycle $(\alpha_n, E_r(\cdot)\tilde{A}_0^{(n)}e^{U_n(\cdot)})$ is given. The *RDC* condition on α allows us to assume that $\alpha_n \in DC$, so by still simplifying the notation we obtain the cocycle $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ with $U(\cdot) \in C^\infty(\mathbb{T}, g)$ arbitrarily small and $\alpha \in DC$.

A second manifestation of the intermediate character of singular cocycles is the fact that they can be obtained as renormalization representatives of positive energy perturbations of regular cocycles, something that should become clear to the reader in the body of the mémoire.

This intermediate character makes itself clearer in their local theory, i.e. in the reduction of cocycles of the form $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ of degree r, with $e_r \in g$ a singular vector. In this case, the exact model that we consider is the normal form $(\alpha, E_r(\cdot)A)$, and the conjugation equation 3.9 then reads

$$Ad(A^*E_r^*(\cdot)).Y(\cdot + \alpha) - Y(\cdot) = -U(\cdot)$$

We can directly partition roots into two sets. The roots $\rho \in \Delta_+$ such that $[e_r, j_\rho] \neq 0$ called $I^{(+)}$, and the roots satisfying $[e_r, j_\rho] = 0$, called $I^{(0)}$. The fact that the geodesic is singular implies that both sets are non-empty: The emptiness of $I^{(+)}$ corresponds to a degenerate curve $\mathbf{a}(\cdot)$, while the emptiness of $I^{(0)}$ corresponds to a regular geodesic. The fact that $E_r(\cdot)$ and A belong to the same maximal torus allows us to write the equations separately in each eigenspace. The equation for the coordinates in the torus is the same as the one for $Y_t(\cdot)$ in 3.11, while the one in the direction $\rho \in I^{(+)}$ is of the same type as that for $Y_z(\cdot)$. If, however, $\rho \in I^{(0)}$, then the direction j_ρ commutes with e_r, so that only the constant part of the normal form acts on the conjugation via the operator associated to the cocycle $(\alpha, E_r(\cdot)A)$ as in eq. 3.10. This is the characteristic of the perturbation theory of constant cocycles and the source of small divisor phenomena in the fibers.

The same observation holds in fact for the effect of perturbations of the normal form in the directions j_ρ. If $\rho \in I^{(+)}$, the positive energy part of the dynamics, any perturbation of the normal form $(\alpha, E_r(\cdot)A)$ in the direction j_ρ with suitably chosen frequencies has energy strictly smaller than $|e_r|$. On the other hand, any small enough perturbation in j_ρ with $\rho \in I^{(0)}$, the 0 energy part of the dynamics, does not affect the energy, since it is evened out by the Ergodic theorem. Continuity of such estimates with respect to the C^1 norm allows us to conclude that if $U(\cdot)$ is ε-small in C^1 with ε small enough, then a perturbation of the order of ε is sufficient in order to obtain a cocycle of smaller energy. One only needs to chose

$$(\alpha, E_r(\cdot)Ae^{U(\cdot)}e^{zj_\rho})$$

with $|z|$ big enough, but of the order of ε, and $\rho \in I^{(+)}$. In other words, a theorem that can allow the proof of the global density theorem is the almost reducibility to normal forms, but with a controllable growth of the conjugations with respect to the decay of perturbations. Let us state what we need more clearly. Suppose that for $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ of degree r and with $U(\cdot)$ small enough, there exists a sequence of conjugations $B_n(\cdot) : \mathbb{T} \to G$ such that

$$Conj_{B_n}(\alpha, E_r(\cdot)Ae^{U(\cdot)}) = (\alpha, E_r(\cdot)A_ne^{U_n(\cdot)})$$

with A_n some sequence of constants commuting with $E_r(\cdot)$, and $U_n(\cdot) \to 0$ in C^∞. Then, conjugation is equivalent to

$$\begin{aligned} E_r(\cdot)Ae^{U(\cdot)} &= B_n^*(\cdot + \alpha)E_r(\cdot)A_ne^{U_n(\cdot)}B_n(\cdot) \\ &= B_n^*(\cdot + \alpha)E_r(\cdot)A_nB_n(\cdot)e^{Ad(B_n^*(\cdot)).U_n(\cdot)} \end{aligned}$$

This is an instance of a more general fact. If $(\alpha, A(\cdot))$ is a model of dynamics and $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$ is another expression of the same model, then

$$(\alpha, \tilde{A}(\cdot)e^{Ad(B(\cdot)).U(\cdot)}) = Conj_{B(\cdot)}(\alpha, A(\cdot)e^{U(\cdot)})$$

We can say that *dynamical conjugation acts on perturbations by algebraic conjugation.*

Therefore we can perturb $(\alpha, E_r(\cdot)A_ne^{U_n(\cdot)})$ and obtain a cocycle of smaller energy. If $Ad(B_n^*(\cdot)).U_n(\cdot) \to 0$ in C^∞, then the perturbation of the initial cocyle $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ is also arbitrarily small. This is a necessary condition for the density of reducible cocycles.

The independence of positive and 0-energy phenomena, at least in the first order, motivates a scheme where the initial cocycle $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ is supposed to be of degree r, and the perturbation is small enough. Then, at each step the conjugation in the positive and 0-energy parts are applied separately for the linearized problem, so that $(\alpha, E_r(\cdot)A_ne^{U_n(\cdot)})$ is conjugated to

$$(\alpha, E_r(\cdot)A_ne^{ObU_n(\cdot)}e^{U_{n+1}(\cdot)})$$

where $ObU_n(\cdot)$ are the positive energy obstructions of $U_n(\cdot)$, as defined in the study of regular geodesics. Then, since $U_{n+1}(\cdot)$ is quadratic with respect to $U_n(\cdot)$, $ObU_n(\cdot)$ must also be quadratic, otherwise the cocycle $(\alpha, E_r(\cdot)A_ne^{ObU_n(\cdot)}e^{U_{n+1}(\cdot)})$, which is conjugate to $(\alpha, E_r(\cdot)A_ne^{U_n(\cdot)})$ would not be of degree r, a contradiction. The scheme is therefore ready to be iterated.

Such a strategy, however, would demand the almost reducibility theorem for perturbations of constants, which was not available. Its weaker equivalent, the almost quasi-reducibility theorem, is sufficient for the proof of the local density theorem by the following trick, used by R. Krikorian [7]. The 1-periodic cocycle $(\alpha, Ae^{U(\cdot)}))$, sufficiently close to the constant (α, A), after n steps of the K.A.M. scheme is conjugated by $B_n(\cdot) \in C^\infty(cD^n\mathbb{T}, G)$ to a D^n-periodic cocycle $(\alpha, A_ne^{U_n(\cdot)}))$, where $U_n(D^n\cdot) \to 0$ in C^∞ exponentially fast. Then, the mapping $U_n(\cdot)$ is proved to be increasingly well approximated by 1-periodic mappings $U'_n(\cdot)$, something with is used in order to obtain a new sequence of conjugations $B'_n(\cdot) \in C^\infty(\chi_G\mathbb{T}, G)$, reducing the cocycle $(\alpha, Ae^{U(\cdot)}))$ to $(\alpha, A_ne^{\tilde{U}_n(\cdot)}))$, where still $\tilde{U}_n(\cdot)$ goes to $0 \in C^\infty(\chi_G\mathbb{T}, g)$ exponentially fast, while conjugations grow polynomially. Then, embedding of the cocycle $(\alpha, A_ne^{\tilde{U}_n(\cdot)}))$ in a suitably chosen one-parameter family and use of the reducibility in positive measure guarantees that there exists a parturbation of $(\alpha, A_ne^{\tilde{U}_n(\cdot)}))$ of the order of $\tilde{U}_n(\cdot)$ which is reducible. The conjugation reducing this last cocycle is of the order of the perturbation.

A relevant observation is that in general there exist $\rho \in I^{(+)}$ and $\rho' \in I^{(0)}$ such that $[j_\rho, j_{\rho'}] \neq 0$, for example the vectors j_1 and j_2 in the example of a singular cocycle in the next section. As a result, the independence of the reduction in the positive ($I^{(+)}$) and 0 ($I^{(0)}$) energy parts of the dynamics holds only in the first order, and the loss of periodicity in conjugation within the G_0 propagates to the rest of the perturbation. This makes the procedure of regaining periodicity more delicate. Suppose that we have obtained the D^n-periodic cocycle $(\alpha, E_r(\cdot)A_ne^{U_n(\cdot)})$, with $U_n(\cdot)$ very small. Imitation of R. Krikorian's trick for regaining periodicity would amount to writing the mapping in the form

$$E_r(\cdot)A_ne^{U_n(\cdot)} = E_r(\cdot)A_ne^{U_n^{(0)}(\cdot)}e^{U_n^{(+)}(\cdot)}$$

where $U_n^{(0)}(\cdot) \approx \pi_{g_0}U_n(\cdot)$ and $U_n^{(+)}(\cdot)$ has a 0 projection in g_0. By the local theory, there exists a perturbation of the cocycle $(\alpha, A_ne^{U_n^{(0)}(\cdot)})$ of the order of $U_n(\cdot)$ such that the new cocycle, $(\alpha, A_ne^{\tilde{U}_n^{(0)}(\cdot)})$ is reducible and the conjugation $\tilde{Y}_n^{(0)}(\cdot)$ is of the order of $\tilde{U}_n^{(0)}(\cdot)$. The cocyle

$$(\alpha, E_r(\cdot)A_ne^{\tilde{U}_n^{(0)}(\cdot)}e^{U_n^{(+)}(\cdot)})$$

can then be conjugated to the one defined by the mapping

$$E_r(\cdot)A'_ne^{Ad(\tilde{Y}_n^{(0)}(\cdot)).U_n^{(+)}(\cdot)})$$

The non-commutativity of the positive and 0-energy parts, however, results in the perturbation being quadratic, but non-zero, in the 0-energy part. Therefore, only one step of reduction can be made without having to deal with small divisor phenomena. It should be possible to iterate the procedure via a fixed point argument, but it is obviously a complication in the proof.

We think that problems arising from the loss of periodicity in the local theory of constant cocycles are not impossible to overcome. Neverthelesss, the complications in the proof of the global density theorem, such us the need for a priori estimates on perturbations of singular geodesics, where the perturbations have arbitrarily long periods, or the adaptation of the argument using the reducibility in positive measure would make the argument very complicated. Finally, the existent local theory does not fit into, and we were not able to make it fit into, the point of view that we have taken up concerning the role of obstructions in the dynamics. These reasons motivated the content of chapter 9, which we will resume now.

7. We present a simplified version of it, for the sake of comprehension of the exposition.

Revisiting the local theory We now treat the case which, with respect to the dynamics of the initial cocycle, corresponds to the curve $\mathbf{a}(\cdot)$ being degenerate to $0 \in g$, i.e. the case of 0 energy. Under this assumption, the initial cocycle $(\alpha, A(\cdot))$ can be renormalized to commuting pairs of the form

$$(\beta_{n-1}, \tilde{C}^{(n)}) \text{ and } (\beta_n, \tilde{A}^{(n)})$$

up to an arbitrarily small error, and therefore has renormalization representatives modulo χ_G arbitrarily close $(\chi_G\alpha_n, \tilde{A}_0^{(n)})$, where $\tilde{A}_0^{(n)} \in G$ is a constant. These cocycles are χ_G-periodic, but we can rescale the torus $\mathbb{T}$ and see them as 1-periodic cocycles over the rotation α_n. Again, the *RDC* condition on α implies that for a subsequence of such renormalization representatives, α_n is in *DC*, which we will suppose from now on. Finally, we simplify the notation and suppose that we are given the cocycle $(\alpha, Ae^{U(\cdot)})$, with $\alpha \in DC$, $A \in G$, and $U(\cdot) : \mathbb{T} \to g$ satisfying some smallness condition.

The reason for the differences between the theory of perturbations of periodic geodesics and that of constants is the importance of small divisor phenomena. These phenomena are in the core of the local theory, and the complications that they cause are not a failure of the methods. The proof of theorem 3.5 relies on a generalized K.A.M. scheme, following the results of H. Eliasson (see theorem 3.7). In this context a cocycle of the form $(\alpha, Ae^{U(\cdot)})$ is given, with $U(\cdot) : \mathbb{T} \to g$ small enough. Then, the local conjugation equation 3.9 reads

$$Ad(A^*).Y(\cdot + \alpha) - Y(\cdot) = -U(\cdot)$$

The operator associated the constant cocycle (α, A) always has a positive codimension image in $C^\infty(\mathbb{T}, g)$. This codimension is bounded by f, the real dimension of g, equal to $w + 2q$, where w is the rank and q is the number of positive roots. The first w constraints, always present, come from the toral coordinates, where the reduction equation is as for $Y_t(\cdot)$ in eq. 3.11. These obstructions, just as in the local study of regular geodesics, are not important, since they are always constants, and thus of the same type as the exact model. The rest $2q$ constraints come from the complex directions in g. For the direction j_ρ the equation reads

$$e^{-2i\pi a_\rho}.Y_\rho(\cdot + \alpha) - Y_\rho(\cdot) = -U_\rho(\cdot)$$

or, in the frequency space,

$$(e^{-2i\pi(a_\rho - k\alpha)} - 1).\hat{Y}_\rho(k) = -U_\rho(k) \tag{3.12}$$

Clearly, this equation is not solvable for the mode $k_\rho \in \mathbb{Z}$ if it satisfies

$$a_\rho - k_\rho\alpha \in \mathbb{Z} \tag{3.13}$$

These obstructions are not always are present, since such a k_ρ not always exists. If the k_ρ satisfying eq. 3.13 (which is automatically unique, since $\alpha \notin \mathbb{Q}$)) is equal to 0, the obstruction is of the same type as the constant model around which we have linearized. Therefore, in this case the obstruction is not important. On the other hand, if $k_\rho \neq 0$, then the corresponding obstruction is a non-constant perturbation and thus of a different type than (α, A). For this reason, we distinguish this case and call such a root ρ a *resonant root*. A constant in G is resonant if it has a resonant root.

The first results in the direction of theorem 3.6 were obtained by systematically ignoring resonant constants, and this approach allowed the proof of reducibility for a positive measure set in the parameter space. It is based on classical K.A.M. theory which in this context amounts to supposing that up to a rapidly growing truncation order N_n, the roots a_ρ are K_n-away from resonances, with K_n comparable to N_n. Then, for frequencies up to N_n the equation 3.12 can be solved with good estimates, while if some C^{s_0} norm (s_0 is fixed throughout the scheme) is small enough with respect to N_n, the frequencies not eliminated form a quadratic term. In this procedure, one needs at each step, by imposing transversality conditions to the one-parameter family, to assure that the non-resonance conditions are satisfied for a sufficiently large set of parameters.

If the cocycle is not embedded in such a family, in order to make reduction possible, one has to be able to cope with resonant constants. In this part, we follow and adapt the proof of theorem 3.5, a corollary of the proof of which is local quasi-reducibility (see definition 3.3). The proof of this theorem was directly based on the approach taken up by H. Eliasson. In section 9.4 we reinterpret

H. Eliasson's reduction of resonances as a reduction of obstructions. More precisely, let us suppose that at the n-th step of the algorithm we have obtained a cocycle $(\alpha, A_n e^{U_n(\cdot)})$ by successive conjugations acting on $(\alpha, A.e^{U(\cdot)})$. Then, either the constant A_n is sufficiently away (i.e. K_n-away) from resonances up to the order of truncation N_n, or not. In the first case, all obstructions are constant, so the preceding procedure works. In the second case, we partition the roots into three classes. The ones for which the obstruction is constant, called I_0 in lemma 9.7, those for which the obstruction is non-constant, I_r, and those for which there is no obstruction, I_d. In the corollary (cor. 9.9) to this lemma, we prove that if the perturbation $U_n(\cdot)$ of the constant cocycle (α, A_n) were spectrally supported in the obstructions, then the cocycle $(\alpha, A_n.e^{U_n(\cdot)})$ would be reducible, but by a conjugation far from the identity. Our interpretation of the role of the obstructions calls us to rewrite the cocycle in the form

$$(\alpha, A_n.e^{U_n(\cdot)}) = (\alpha, A_n.e^{Ob_n U_n(\cdot)}.e^{U'_n(\cdot)})$$

where $Ob_n U_n(\cdot)$ stands for the obstructions of $U_n(\cdot)$ with respect to conjugation in the neighborhood of (α, A_n) (i.e. to the operator 3.10), and $U'_n(\cdot)$ has only second order terms in the space of obstructions. Corollary 9.9 suggests that $(\alpha, A_n.e^{Ob_n U_n(\cdot)})$ is a good candidate for the model around which we linearize, and the technical part of the scheme confirms it: if we center the truncation on the obstructions by means of the operator $T^{(\mathbf{k})}_{N_n}$ defined in section 9.5 (see eq. 9.10 and 9.5) conjugation works like in the presence of only constant obstructions. In this way, we can conjugate $(\alpha, A_n.e^{Ob_n U_n(\cdot)}.e^{U'_n(\cdot)})$ to $(\alpha, A_n.e^{Ob_n U_n(\cdot)}.e^{\tilde{U}_{n+1}(\cdot)})$ with a close-to-the-identity conjugation, where $\tilde{U}_{n+1}(\cdot)$ is quadratic with respect to $U_n(\cdot)$. The estimates show that reduction of the principal part of the new perturbation, $Ob_n U_n(\cdot)$, does not change the quadratic character of $\tilde{U}_{n+1}(\cdot)$, so that, by conjugating with the reduction of obstructions, we obtain the cocycle $(\alpha, A_n.e^{U_{n+1}(\cdot)})$ with $U_{n+1}(\cdot)$ much smaller than $U_n(\cdot)$, and iteration is possible. The iteration of this K.A.M. scheme proves directly the almost reducibility of any cocycle sufficiently close to a constant one, and the comparison of the (polynomial) rate of growth of conjugants versus the (exponential) decay of the norms of perturbations proves the local density of reducible cocycles. This is the content of corollary 9.14, which is the conjugation lemma used in the iteration. Convergence of the scheme (which immediately proves local almost reducibility, i.e. theorem 9.2) is proved in section 9.6, and the section is concluded with the proof of the (already known) local density theorem (theorem 9.1).

In fact, we could skip the reduction of the obstructions at each step and construct a K.A.M. scheme for the reduction of any given cocycle (satisfying the smallness conditions) arbitrarily close to reducible ones by a product of converging conjugations. This is another natural concept of almost reducibility. Then, comparison of the norm of the conjugant reducing the limit cocycles to constants with the decay of the perturbation would yield the same results. This would be in fact more faithful to our interpretation of the role of obstructions, but the construction of such a scheme is needlessly complicated.

3.5 A concrete example

In order to make clear the proof of global density theorem, we will take up the study of an example in $\mathbb{T} \times SU(3)$. It is a simply connected group, for which $\chi_{SU(3)} = 1$, so we avoid problems related to homotopy, which will be coped with in section 7.7. For $SU(3)$ we have $w = 2$, $q = 3$ and $f = 8$, for the rank, the number of roots and the real dimension of $su(3)$.

Let us introduce the notations

$$\begin{aligned} j_1 &= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; j_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \; j_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \\ h_1 &= \begin{bmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix} \end{aligned}$$

We also note that $[j_l, ij_l] = 2h_l$, for $l = 1, 2, 3$, and normalize the Cartan-Killing norm so that the

norms of all vectors are equal to 1. Finally, we introduce the corresponding numbering of roots,

$$\begin{aligned} \rho_1 &: h \mapsto (t_1 - t_2) \\ \rho_2 &: h \mapsto (t_2 - t_3) \\ \rho_3 &: h \mapsto (t_1 - t_3) \end{aligned}$$

for

$$h = 2i\pi \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}$$

with $t_1 + t_1 + t_3 = 0$. Then, $\{\rho_1, \rho_2\}$ form the Weyl's basis, $\rho_3 = \rho_1 + \rho_2$ and $\{\rho_1, \rho_2, \rho_3\}$ are the positive roots. For a mapping $U(\cdot) : \mathbb{T} \to su(3)$ we will use the coordinates

$$U(\cdot) = \sum\nolimits_{i=1,2} U_i^{\mathfrak{t}}(\cdot) h_i + \sum\nolimits_{i=1,2,3} U_i(\cdot) j_i$$

Let, now, $A(\cdot) : \mathbb{T} \to SU(3)$ be such that $\|LA(\cdot)\|_{L^1} \geq |e_{(1,3)}|$, where

$$e_{(1,3)} = \begin{bmatrix} 2i\pi & 0 & 0 \\ 0 & 4i\pi & 0 \\ 0 & 0 & -6i\pi \end{bmatrix} = 2\pi h_1 + 6\pi h_2$$

For this vector, $\rho_1(e_{(1,3)}) = -1$, $\rho_2(e_{(1,3)}) = 5$, $\rho_3(e_{(1,3)}) = 4$. Since all these integers are non-zero, the vector is regular.

By proposition 5.3, the condition on the length of the path $A(\cdot) : \mathbb{T} \to SU(3)$ is necessary for the cocycle $(\alpha, A(\cdot))$ to be of degree $e_{(1,3)}$, and it is satisfied if, say, $\|LA(\cdot)\|_{L^1} \geq 200\pi$.

The energy of the cocycle Let us suppose that the sequence $\mathbf{a}_n(\cdot) = LA_n(\cdot)$ converges to a non-degenerate curve $\mathbf{a}(\cdot) \in L^2(\mathbb{T}, g)$, and suppose moreover that *a.e.* vector in the image of $\mathbf{a}(\cdot)$ is a regular vector in $su(3)$. At this point, the results proved in chapter 5, namely corollary 5.7 and lemma 5.10 allow us to make some preliminary conclusions. The corollary shows that the cocycle can be measurably diagonalized. Finally, the lemma shows that $n^{-(s+1)}\partial^s LA_n(\cdot)$ goes to 0 in L^2. Estimation of the decay of norms in L^2 uses the fact that $\|\partial^s LA_n(\cdot)\|_\infty$ grow no faster than n^{s+1}, by lemma 5.9.

Renormalization of a regular cocycle The next step in the study of the dynamics of the cocycle is renormalization of the dynamics, which we will follow under the assumption that $(\alpha, A(\cdot))$ is a regular cocycle. The convergence of the scheme (theorem 6.3) shows that, for almost every $\nu \in \mathbb{T}$, the cocycle $(\alpha, A(\cdot))$ can be renormalized to commuting pairs

$$\begin{aligned} &(1, \tilde{C}_\nu^{(n)}(\cdot)) \\ &(\alpha_n, \tilde{A}_\nu^{(n)}(\cdot)) \end{aligned}$$

(notice that we have already introduced the rescaling factor) satisfying the following property. Both mappings $\tilde{C}_\nu^{(n)}(\cdot)$ (which codes the distortion of the fibers) and $\tilde{A}_\nu^{(n)}(\cdot)$ (which codes the rescaled dynamics in the distorted phase space) approach $\mathbf{a}(\nu)$:

$$\begin{aligned} \tilde{A}_\nu^{(n)}(x) &= \tilde{A}_\nu^{(n)}(0).\exp((-1)^n q_n \beta_{n-1} \mathbf{a}(\nu) x + O(\varepsilon_n(\nu))) \\ \tilde{C}_\nu^{(n)}(x) &= \tilde{C}_\nu^{(n)}(0).\exp(-(-1)^n q_{n-1} \beta_{n-1} \mathbf{a}(\nu) x + O(\varepsilon_n(\nu))) \end{aligned}$$

where both constants $\tilde{C}_\nu^{(n)}(0)$ and $\tilde{A}_\nu^{(n)}(0)$ commute with $\exp(\mathbb{R}\mathbf{a}(\nu))$. Since we assume that $\mathbf{a}(\nu)$ is a regular vector, we have that $\rho_i(\mathbf{a}(\nu)) \neq 0$, for $i = 1, 2, 3$, which is clearly equivalent to the three diagonal entries being distinct. Elementary linear algebra the shows that both constants $\tilde{C}_\nu^{(n)}(0)$ and $\tilde{A}_\nu^{(n)}(0)$ have to be diagonal.

Commutation of the couple implies directly that

$$\tilde{C}_\nu^{(n)}(0).\tilde{A}_\nu^{(n)}(0) \exp(\mathbf{a}(\nu)x) = \tilde{A}_\nu^{(n)}(0) \exp(\mathbf{a}(\nu)(x+1).\tilde{C}_\nu^{(n)}(0)$$

Using the commutation of the constants with $\mathbf{a}(\nu)$, we find that

$$\tilde{C}^{(n)}_{\nu}(0).\tilde{A}^{(n)}_{\nu}(0)\exp(\mathbf{a}(\nu)) = \tilde{A}^{(n)}_{\nu}(0).\tilde{C}^{(n)}_{\nu}(0)$$

up to an arbitrarily small error. Since both constants are diagonal, the above equation implies that $\exp(\mathbf{a}(\nu)) = Id$, in accordance with theorem 6.1.

If we normalize the commuting couple, we obtain a representative which is arbitrarily close to

$$(\alpha_n, \tilde{A}^{(n)}_{\nu}(0)e^{\mathbf{a}(\nu)x})$$

This is the content of theorem 6.3 in the case $G = SU(3)$ and with the additional hypothesis that $\mathbf{a}(\nu)$ is regular.

Perturbations of regular geodesics In order to be able to illustrate the local study of regular geodesics more clearly, we reintroduce the assumption that the degree of $(\alpha, A(\cdot))$ is $e_{(1,3)}$, so that the renormalization representatives have the form

$$(\alpha_n, E_{(1,3),a_n}(\cdot)e^{U(\cdot)}))$$

where

$$E_{(1,3),a_n}(\cdot) = \begin{pmatrix} e^{2i\pi\cdot} & 0 & 0 \\ 0 & e^{4i\pi\cdot} & 0 \\ 0 & 0 & e^{-6i\pi\cdot} \end{pmatrix} . \begin{pmatrix} e^{2i\pi a_{1,n}} & 0 & 0 \\ 0 & e^{2i\pi(a_{2,n}-a_{1,n})} & 0 \\ 0 & 0 & e^{-2i\pi a_{2,n}} \end{pmatrix}$$

And we also suppose that n is such that $\alpha_n \in DC$.

There are exactly three embeddings of $su(2)$ in $su(3)$ sending the positive root of $su(2)$ to a positive one in $su(3)$, corresponding to the three positive roots of $su(3)$. Since $[e_{(1,3)}, j_1] = -2i\pi j_1$, $[e_{(1,3)}, j_2] = 10i\pi j_2$ and $[e_{(1,3)}, j_3] = 8i\pi j_3$, theorem 8.1 implies that any small enough perturbation of this periodic geodesic can be conjugated to a cocycle of the form

$$(\alpha_n E_{(1,3),a_n+a'_n}(\cdot).e^{P(\cdot)})$$

with $P(\cdot)$ of the form

$$\exp\left(\hat{P}_1(0)j_1 + \sum\nolimits_{-9}^{0} \hat{P}_2(k)e^{2i\pi k\cdot} j_2 + \sum\nolimits_{-7}^{0} \hat{P}_k(k)e^{2i\pi k\cdot} j_3\right)$$

where the $\hat{P}_i(k)$ are small complex numbers, and a'_n is close to 0. Then, the a priori estimates of the chapter 7 show that the cocycle is of degree $(1,3)$ if, and only if, all the $\hat{P}_i(k)$ in this previous expression are 0.

Moreover, they show that if these coefficients are non-zero, the cocycle is of energy strictly less than $|e_{(1,3)}|$. Even though it is possible to perturb this cocycle to a regular one (if we perturb it by $\hat{P}_1(0) \neq 0$), we will examine the perturbation by $\hat{P}_2(0) \neq 0$. This perturbation is in the subalgebra associated to the root ρ_2, so that is will lead renormalization a cocycle of degree $e_{(1,l)}$, with l equal to $0, 1$ or 2. Let us suppose that the perturbed cocycle is in fact of degree $e_{(1,2)}$, where

$$e_{(1,2)} = \begin{bmatrix} 2i\pi & 0 & 0 \\ 0 & 2i\pi & 0 \\ 0 & 0 & -4i\pi \end{bmatrix} = 2\pi h_1 + 4\pi h_2$$

We remark that the complex direction $\mathbb{C}j_1$ commutes with this vector, unlike the other two complex directions.

Renormalization of a singular cocycle The hypothesis that the degree of the new perturbed cocycle $(\alpha_n, A_1(\cdot))$ is $e_{(1,2)}$ implies that it can be renormalized to commuting pairs of the form $(m > n)$

$$\begin{aligned} \tilde{A}^{(m)}_{1,\nu}(x) &= \tilde{A}^{(m)}_{1,\nu}(0).\exp((-1)^m q_n \beta_{m-1}\mathbf{a}_1(\nu)x + O(\varepsilon_m(\nu))) \\ \tilde{C}^{(m)}_{1\nu}(x) &= \tilde{C}^{(m)}_{1,\nu}(0).\exp(-(-1)^m q_{n-1}\beta_{m-1}\mathbf{a}_1(\nu)x + O(\varepsilon_m(\nu))) \end{aligned}$$

where both constants $\tilde{C}_\nu^{(m)}(0)$ and $\tilde{A}_\nu^{(m)}(0)$ commute with $\exp(\mathbb{R}\mathbf{a}_1(\nu))$ and we have called $\mathbf{a}_1(\cdot)$ the invariant curve of the cocycle $(\alpha, A_1(\cdot))$.

Clearly, commutation of the two constants with $\mathbf{a}_1(\nu) \approx e_{(1,2)}$ no longer implies that the constants are diagonal. They are constrained, however, in the subgroup

$$\begin{pmatrix} SU(2) & 0 \\ 0 & 1 \end{pmatrix} \times \exp(\mathbb{R}e_{(1,2)}) = G_0 \times G_+$$

which is the splitting (cf. definition 4.7 and theorem 4.5) associated to $e_{(1,2)}$.

Then, as in the case of a regular cocycle, we can conjugate the pair to

$$\begin{array}{c} (1, \tilde{C}_\nu^{(m)}(0)) \\ (\alpha_m, \tilde{A}_\nu^{(m)}(0).\exp(\mathbf{a}_1(\nu)x)) \end{array}$$

plus error terms. Since this is still a commuting pair, we have, as in the regular case,

$$\tilde{C}_\nu^{(m)}(0).\tilde{A}_\nu^{(m)}(0)\exp(\mathbf{a}_1(\nu)) = \tilde{A}_\nu^{(m)}(0).\tilde{C}_\nu^{(m)}(0)$$

Since both constants take values in the splitting $G_0 \times G_+$ and the factor G_+ is abelian, the commutator of the constants is in $SU(2)$. Therefore we obtain that $\mathbf{a}_1(\nu)$ has to be a preimage of the Id. Then, the preceding equation simplifies to

$$\tilde{C}_\nu^{(m)}(0).\tilde{A}_\nu^{(m)}(0) = \tilde{A}_\nu^{(m)}(0).\tilde{C}_\nu^{(m)}(0)$$

up to an arbitrarily small error. Since, now, the semisimple factor G_0 of the splitting is in fact $SU(2)$, we obtain once again that the constants are both diagonal. For general groups, though, this is not true and in order to gain this property we may need to consider powers of the constants, a complication explained in the proof of theorem 6.3.

Following the procedure already described, we obtain the renormalization representative

$$(\alpha_m, \tilde{A}_{1,\nu}^{(n)}(0)e^{\mathbf{a}_1(\nu)x})$$

only with a different constant $\tilde{A}_{1,\nu}^{(m)}(0)$, which is on the same torus as $\mathbf{a}_1(\nu)$. This is the content of theorem 6.3 in the case $G = SU(3)$ and for a singular $\mathbf{a}_1(\nu)$.

Perturbations of singular geodesics This leads us to the local study of singular geodesics. Here, we suppose that the given cocycle is of the form

$$(\alpha_m, E_{(1,2)}(\cdot)Ae^{U(\cdot)})$$

with $A \in G$ on the same torus as $E_{(1,2)}(\cdot)$, and $U(\cdot) \in C^\infty(\mathbb{T}, g)$ satisfying some smallness condition depending on $\alpha_m \in DC$. We also suppose that the cocycle is of degree $(1, 2)$. Then, the iterative procedure described in chapter 10 allows us to conjugate this cocycle arbitrarily close to cocycles in normal form:

$$(\alpha_m, E_{(1,2)}(\cdot)A_k e^{U_k(\cdot)})$$

with $A_k \in G$ on the same torus as $E_{(1,2)}(\cdot)$, and $U_k(\cdot)$ converging exponentially fast to $0 \in C^\infty(\mathbb{T}, g)$, while the conjugations grow only polynomially. Then, we need only to perturb the cocycle like

$$(\alpha_n, E_{(1,2)}(\cdot)A_k e^{U_k(\cdot)} e^{z_k j_i})$$

where $z_k \in \mathbb{C}^*$ is of the order of $U_k(\cdot)$ and $i = 2, 3$, in order to obtain a cocycle of smaller enery.

We remark that, since in general we cannot hope to conjugate the cocycle to the normal form $(\alpha_n, E_{(1,2)}(\cdot)A_k)$, we cannot control the way in which the energy will decrease : all models $E_{(j,k)}$ with $j = 0, 1$ and $k = 0, 1, 2$ are possible, with the exception of $(1, 2)$.

Renormalization of 0 energy cocycles After a finite number of perturbations (the maximal number of perturbations is bounded by a number depending only on the degree of the cocyle) $(\alpha, A(\cdot))$, all of which can be made arbitrarily small, and renormalization of the perturbed cocycles and conjugation of renormalization representatives, we will obtain a cocycle of 0 energy. This cocyle can be renormalized to commuting pairs of the type $(l > m)$

$$(1, \tilde{C}_\nu^{(l)}(0))$$
$$(\alpha_l, \tilde{A}_\nu^{(l)}(0))$$

plus error terms. Then, commutation of the pair shows that the two constants commute, up to an arbitrarily small error. In $SU(3)$ this implies that they are both diagonal, so that we can obtain the renormalization representative of the form

$$(\alpha_l, Ae^{U(\cdot)})$$

In general groups, however, commutativity is gained after iteration.

Perturbations of constant cocycles This is the last step in our study, at least from the dynamical point of view, and it is described in chapter 9. Here, we suppose that $\alpha_l \in DC$, and theorem 9.1 allows us to perturb the renormalization representative $(\alpha_l, Ae^{U(\cdot)})$ to $(\alpha_l, Ae^{U'(\cdot)})$, which is reducible.

Since the induction was finite and the perturbations at each step could be made arbitrarily small, the global density theorem follows.

Chapter 4

Dynamics of abelian cocycles

In this chapter we attempt a survey of the models of cocycles whose dynamics can be characterized as simple. We try to point out what remains true when we embed these "toy cocycles" in the larger class that we seek to study in the rest of this mémoire.

In order to simplify the presentation, we suppose that the regularity of the mappings is C^∞, but most of the proofs use very low regularity. The results will be extended in weaker regularity in the following chapters, where the precise statements will be given, but in the proofs we will use the notions, techniques and observations that arise naturally in this phenomenological part of the study. We note nonetheless that the regularity typically needed in the proofs is no more than C^2 for the cocycles and for conjugations.

Since, in particular, minimal translations on the torus are also (uniquely) ergodic, some of the proofs work also in the measurable case, with the necessary modifications of the proofs and of the statements of the results.

Finally, since not all proofs work when $d \geq 2$ (e.g. those using the Denjoy-Koksma inequality, see below), and since the conclusions are significant precisely in the case $d = 1$, we suppose, as for the greatest part of the remaining of this thesis, that the cocycle has only one frequency.

4.1 Cocycles in $SW(\mathbb{T}, \mathbb{S})$

The simplest case of a cocycle in a compact (but not semisimple) Lie group is that of a cocycle in $\mathbb{T} \times \mathbb{S} = \mathbb{T} \times \mathbb{S}^1$. Any given $(\alpha, A(\cdot)) \in SW(\mathbb{T}, \mathbb{S})$ can be written in the form

$$(\alpha, E_r(\cdot)e^{2i\pi(\phi(\cdot)+\theta)})$$

with $\phi : \mathbb{T} \to \mathbb{S}$ such that $\int \phi = 0$ and $\theta \in \mathbb{R}$, and where, for $r \in \mathbb{Z}$, we have noted by $E_r(\cdot) : \mathbb{S} \circlearrowleft$ the map $e^{2i\pi\theta} \mapsto e^{2i\pi r\theta}$.

Conjugation in $SW(\mathbb{T}, \mathbb{S})$ is related with the solution of a linear cohomological equation (see lemma 3.4), and conjugation between constant cocycles is realized by conjugations of the type $E_r(\cdot)$.

Such cocycles admit a first classification by homotopy, for if $r' \neq r$, then the cocycles $(\alpha, E_r(\cdot))$ and $(\alpha, E_{r'}(\cdot))$ are not conjugate, and the choice of non-trivial ϕ and θ clearly cannot change this fact. Even more, if α satisfies some diophantine condition, r and $\theta + \alpha\mathbb{Z}$ completely determine the class of conjugacy of the cocycle.

However, since the source of interest in $SW(\mathbb{T}, \mathbb{S})$ comes from the canonical embedding $\mathbb{S} \hookrightarrow SU(2)$, classification by homotopy is not the good starting point. In fact, even in view of the embedding $\mathbb{S} \mod \{\pm 1\} \hookrightarrow SO(3)$, where there are homotopy issues, this proves to be insufficient, since the second iterate of a cocycle non-homotopic to constants in $SW(\mathbb{T}, SO(3))$ becomes homotopic to constants, whereas this is not true in $SW(\mathbb{T}, \mathbb{S})$. The concept apt for generalization proves to be the fact that non-trivial homotopy in $SW(\mathbb{T}, \mathbb{S})$ is equivalent to linear growth of the derivatives of its iterates. This can be done by observing that the rate of growth of derivatives equals the degree of the mapping in absolute value.

In order to obtain it, let us consider a particular case for the differential operator L introduced in 2.2.1. We identify $\mathbb{R}$ with $T_1\mathbb{S}$ and trivialize $T\mathbb{S}$ by $T\mathbb{S} \ni (e^{2i\pi\theta}, s) \mapsto (e^{2i\pi i\theta}, e^{-2i\pi\theta}s) \in \mathbb{S} \times \mathbb{R}$. Application of the operator L amounts to projecting the differential to the second coordinate in the trivial tangent bundle. A simple calculation then shows that $\frac{1}{n}LA_n(\cdot) \to 2\pi r$, where $A(\cdot)$ is as in the beginning of the paragraph.

If we consider $SW(\mathbb{T}, \mathbb{S}) \hookrightarrow SW(\mathbb{T}, SU(2))$ as the subgroup of diagonal cocycles, this cocycle assumes the form $(\alpha, E_r(\cdot))$, where the notation is now as in paragraph 2.1.2. The relics of homotopy is this type of growth of the derivative, which is in the direction of the vector $\{1, 0\}_{su(2)}$. Factorization of $SU(2)$ by $\pm Id$ and consideration of $SW(\mathbb{T}, \mathbb{S}) \hookrightarrow SW(\mathbb{T}, SO(3))$ shows that in this case the allowed rates of growth of derivatives are in $\frac{1}{2}2\pi\mathbb{Z}$ where the factor $\frac{1}{2}$ is precisely the cardinal of $\{\pm Id\}$.

In a first attempt to exit from the abelian world, we can conjugate the cocycle $(\alpha, A(\cdot)) \in SW(\mathbb{T}, \mathbb{S}) \hookrightarrow SW(\mathbb{T}, SU(2))$ by an arbitrary mapping $B(\cdot) : \mathbb{T} \to SU(2)$, and call the cocycle thus obtained $(\alpha, \tilde{A}(\cdot))$. Since $(\alpha, \tilde{A}_n(\cdot)) = (\alpha, B(\cdot + n\alpha)A_n(\cdot)B^*(\cdot))$ and using the properties of the operator L, we discover that

$$\begin{aligned} \tilde{a}^*_n(\cdot) = L(\tilde{A}^*_n(\cdot)) &= b(\cdot) + Ad(B(\cdot)).a^*_n(\cdot) + Ad(B(\cdot)A^*_n(\cdot)).b^*(\cdot + n\alpha) \\ &= -n.Ad(B(\cdot)).2\pi r\{1, 0\}_{su(2)} + o(n) \end{aligned}$$

(we remind the notational convention following which $a_n(\cdot) = LA_n(\ .)$ and $b(\cdot) = LB(\cdot)$).

The study in the following section is just a higher-dimensional analogue of what we just saw.

4.2 Dynamics of torus-reducible cocycles

4.2.1 A first definition of the degree

Let $(\alpha, A(\cdot))$ be a C^∞-torus-reducible cocycle in $SW^\infty(\mathbb{T}, G)$, $\mathcal{T}$ a maximal torus and $(h_\rho)_{\rho\in\tilde{\Delta}}$ a basis of $\mathfrak{t}$. Therefore, $A(\cdot)$ is of the type

$$A(x) = B(x + \alpha). \exp(\sum\nolimits_{\rho\in\tilde{\Delta}} 2\pi(r_\rho x + \phi_\rho(x) + a_\rho)h_\rho).B^{-1}(x) \tag{4.1}$$

with $\phi_\rho(\cdot) : \mathbb{T} \to \mathbb{R}$ of mean value 0, $a_\rho \in \mathbb{T}$, $r \in \mathbb{Z}^w$ and $B(\cdot) : \mathbb{T} \to G$.

After reminding the notation $a_n(\cdot) = LA_n(\cdot)$, where the operator L is defined in section 2.2.1, we can readily prove

Proposition 4.1. *Let $(\alpha, A(\cdot)) \in SW^1$ be C^1 torus-reducible. Then the length of the n-th iterate normalized by n is quantized:*

$$\frac{1}{n}|a_n(\cdot)| \to |\sum\nolimits_\rho 2\pi r_\rho h_\rho|$$

uniformly and $r_\rho \in \mathbb{Z}, \forall\rho \in \tilde{\Delta}$. The limit does not depent on the conjugation $B(\cdot)$ and is therefore an invariant under C^1 conjugation.

The definition of this quantity will be extended to all C^1 cocycles and called "energy" in the next chapter. The important point will be that the quantization does not change when we consider non-abelian cocycles.

Proof. Taking the derivative of the formula for the n-th iterate gives

$$a_n(x) = b(x + n\alpha) + Ad(B(x + n\alpha)).\sum\nolimits_\rho 2\pi r_\rho(n + S^\alpha_n\phi'_\rho(x))h_\rho - Ad(A_n(x)).b(x)$$

(we remind the notational convention following which $a_n(\cdot) = LA_n(\cdot)$ and $b(\cdot) = LB(\cdot)$).

If $r = 0$, we find that

$$a_n(x) = b(x + n\alpha) + Ad(B(x + n\alpha)).\sum\nolimits_\rho S^\alpha_n\phi'_\rho(x)h_\rho - Ad(A_n(x)).b(x)$$

so that $\frac{1}{n}a_n(\cdot)$ tends uniformly to 0 by unique ergodicity of R_α with respect to the Haar measure on $\mathbb{T}$.

If, now, $r \in \mathbb{Z}^d \setminus \{0\}$, we find that

$$a_n(x) = Ad(B(x+n\alpha)).\sum\nolimits_\rho 2\pi r_\rho(n + S^\alpha_n \phi'_\rho(x))h_\rho + c_n(x) \tag{4.2}$$

where$\|c_n(\cdot)\|_{L^\infty} \leq 2\,\|b(\cdot)\|_{L^\infty}$, so that

$$\|a_n(x) - Ad(B(x+n\alpha)).\sum\nolimits_\rho 2\pi r_\rho n h_\rho\|_{L^\infty} = o(n)$$

which implies the statement since the adjoint action is an isometry of g. □

A special class of (torus-)reducible cocycles in the case $r = 0$ is the one consisting of those reducible to a constant in Z_G. If we let $(\alpha, A(\cdot)) = (\alpha, B(\cdot+\alpha).C.B^*(\cdot))$, with $C \in Z_G$ be such a cocycle, we have

$$a_n(\cdot) = b(\cdot + n\alpha) - Ad(B(\cdot + n\alpha).B^*(\cdot)).b(\cdot)$$

and therefore

$$|\partial^s a_{q_n}(\cdot)| = O(\beta_n) \tag{4.3}$$

where the constant depends on s and $B(\cdot)$.

Calculation of higher order derivatives (still in the case $r = 0$) and a similar calculation show that higher derivatives share the same property:

$$\frac{1}{n}\partial^s a_n(x) \quad \to \quad 0, \text{ and} \tag{4.4}$$

$$|\partial^s a_{q_n}(\cdot)| \quad \leq \quad C_s,\ \forall\ s \geq 0 \tag{4.5}$$

In the special case where $n = q_k$, a denominator of a best rational approximation, the Denjoy-Koksma inequality (see [Her79]), gives estimates on the speed of convergence. According to this inequality, for any function $\varphi(\cdot) : \mathbb{T} \to \mathbb{R}$ of bounded variation,

$$|S^\alpha_{q_k}\varphi(\cdot) - q_k \int_{\mathbb{T}} \varphi(\cdot)| \leq Var(\varphi)$$

where $Var(\varphi)$ stands for the total variation of $\varphi(\cdot)$. Application of this inequality in our case and use of the fact that $\int_{\mathbb{T}} \phi'_\rho = 0$ gives

$$|a_{q_n}(\cdot)| \leq C \max \|\phi_\rho\|_1$$

Similar estimations in the case where $r \neq 0$ give only

$$\frac{1}{n^2}\partial^s a_n(\cdot) \to 0$$

uniformly.

Examination of the proof of the previous proposition shows that C^1 torus-reducibility implies in fact more than the existence of a discrete invariant when $r \neq 0$: for large n,

$$\mathbf{a}_n(\cdot) = \frac{1}{n} a_n(\cdot) \tag{4.6}$$

is uniformly bounded away from 0. It follows from eq. (4.2) that if $n_k\alpha \to 0$ in $\mathbb{T}$, then

$$\mathbf{a}_{n_k}(\cdot) \to \mathbf{a}(\cdot) = Ad(B(\cdot)).\sum\nolimits_\rho 2\pi r_\rho h_\rho$$

uniformly. Moreover, whenever $n_k\alpha \to \gamma$ in $\mathbb{T}$, $\mathbf{a}_{n_k}(\cdot) \to Ad(B(\cdot+\gamma)).\sum_\rho 2\pi r_\rho h_\rho$, that is to $\mathbf{a}(\cdot+\gamma)$.

Therefore, $B(\cdot)$ satisfies

$$Ad(B^*(\cdot)).\mathbf{a}(\cdot) = \sum\nolimits_\rho 2\pi r_\rho h_\rho \tag{4.7}$$

but this equation in general does not determine $B(\cdot)$, nor the dynamics of the cocycle, as we will see later on.

As for $\mathbf{a}(\cdot)$, we can prove, in fact, something more:

Lemma 4.2. *Let $(\alpha, A(\cdot)) \in SW^1$ be C^1 torus-reducible, and let $r \neq 0$. Then, $\frac{1}{q_n} a_{q_n}(\cdot)$ converges uniformly (in C^0) to a C^1 continuous curve $\mathbf{a}(\cdot) : \mathbb{T} \to g$.*

The curve $\mathbf{a}(\cdot)$ satisfies the invariance relation

$$Ad(A(\cdot)).\mathbf{a}(\cdot) = \mathbf{a}(\cdot + \alpha) \tag{4.8}$$

Proof. We need only to prove the invariance relation. Let $n \in \mathbb{N}$. The cocycles $(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))^{q_n}$ commute, since they are iterates of the same diffeomorphism. We have, therefore, $A(\cdot + q_n\alpha).A_{q_n}(\cdot) = A_{q_n}(\cdot + \alpha).A(\cdot)$. Taking the derivative of this formula gives

$$Ad(A(x + q_n\alpha)).a_{q_n}(x) - a_{q_n}(x + \alpha) = Ad(A_{q_n}(x + \alpha)).a(x) - a(x + q_n\alpha)$$

If we divide both sides by q_n and let n go to infinity, we obtain the announced formula by uniform convergence. □

We can immediately prove

Corollary 4.3. *The modulus of $\mathbf{a}(\cdot)$ is constant. Moreover, the image of $\mathbf{a}(\theta)$ under $Inn(g)$ is a constant subset of g, not depending on θ.*

Proof. Using the invariance relation (4.8) we find

$$|\mathbf{a}(\cdot + \alpha)| = |Ad(A(\cdot)).\mathbf{a}(\cdot)| = |\mathbf{a}(\cdot)|$$

Since R_α is minimal on the torus and $|\mathbf{a}(\cdot)|$ is invariant under R_α, it is constant. The relation (4.8) itself shows that, for a fixed $h \in g$, the set

$$\{\theta \in \mathbb{T},\ \exists\ B \in G\ s.t.\ Ad(B).\mathbf{a}(\theta) = h\}$$

is invariant under R_α. If for h, h' the corresponding sets intersect, the two vectors can be conjugated in $Inn(g)$, so that minimality of R_α allows us to conclude. □

We remark that the fist statement of the corollary is exactly the definition of the energy of the cocycle given in 4.1, but its proof in this context gives an additional reason (the invariance of the curve under the dynamics) which is the property that we will use when we examine the inverse problem of determining the dynamics under the assumption of the existence of $\mathbf{a}(\cdot)$.

Finally, let us suppose that $(\alpha, A_i(\cdot)) \in SW^1$, $i = 1, 2$, are conjugate by $\tilde{B}(\cdot) \in C^1$ and that $(\alpha, A_1(\cdot))$ satisfying the hypotheses of the previous lemma. Then, $(\alpha, A_2(\cdot))$ also satisfies these hypotheses, and the curves $\mathbf{a}_i(\cdot)$ associated to $(\alpha, A_i(\cdot))$ as in the lemma satisfy

$$\mathbf{a}_1(\cdot) = Ad(\tilde{B}(\cdot))\mathbf{a}_2(\cdot)$$

as shows a direct calculation. Therefore, the class of $\mathbf{a}(\cdot)$ in g modulo $Inn(g)$ is in fact a second quantized invariant under C^1 conjugation. This invariant takes values in the lattice of preimages of the Id in g, modulo the action of $Inn(g)$. The definition of this invariant will be extended to what we will call the *degree* in the next chapter.

The conclusion of chapter 6 is, in fact, that the two quantized invariants defined in this section keep the same quantization in their values when aptly extended to all C^1 cocycles. Chapter 6 already implies that, unlike for cocycles in $\mathbb{T} \times \mathbb{T}^w$, the conjugacy classes of abelian models for cocycles in $\mathbb{T} \times G$ should not be expected to form a space of discrete classes in SW^1.

4.2.2 The failure of the converse derivation

We have seen that C^1 torus reducibility implies the existence of some nice objects, which are invariant under conjugation of the dynamics.

Let us now suppose that a cocycle $(\alpha, A(\cdot))$ is given such that the normalized sequence $\mathbf{a}_{q_n}(\cdot) = \frac{1}{q_n} a_{q_n}(\cdot)$ tends uniformly to a C^σ curve $\mathbf{a}(\cdot)$, with $\sigma \geq 1$. Inspection of the proof of lemma 4.2 shows that in fact the curve $\mathbf{a}(\cdot)$ satisfies automatically the conclusions of the lemma, and in particular $|\mathbf{a}_{q_n}(\cdot)|$ tends to a positive constant C. Let us suppose that $C > 0$, in which case we say that $\mathbf{a}(\cdot)$ is *non-degenerate*.

Lemma 4.2 and its corollary allow us to transform algebraic conjugations acting on the invariant curve into dynamic conjugations acting on the cocycle in the following way.

Lemma 4.4. *If there exists a (non-degenerate) curve* $\mathbf{a}(\cdot)$ *in* g *satisfying eq. 4.8, there exists* $h \in g \setminus \{0\}$ *a vector in the Lie algebra and a* c_G*-periodic mapping* $B(\cdot)$ *such that* $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$ *satisfies*

$$Ad(\tilde{A}(\cdot)).h = h$$

Proof. Let us consider $I_{\mathbf{a}} = \{\mathbf{a}(\theta), \theta \in \mathbb{T}\}$ and $h \in g$ such that $Ad(B)\mathbf{a}(\theta) = h$ for some $\theta \in \mathbb{T}$ and $B \in G$. The set of vectors of g that can be conjugated to h is given by $Ad(\check{G}).h$, where $\check{G} = G/Z_G$ and the Lie algebra of $\check{G}$ is naturally isomorphic to g. Moreover, if $h' = Ad(B).h$ for some $B \in \check{G}$, then for any $B' \in \mathcal{Z}_h = \{S \in G, Ad(S).h = h\}$,

$$h' = Ad(B.B').h$$

Finally, if B_1 and B_2 are such that $h' = Ad(B_i).h$, $i = 1, 2$, then $B = B_1 B_2^*$ satisfies

$$Ad(B).h = h$$

so that $B \in \mathcal{Z}_h$. Therefore, $Ad(\check{G}).h$ is naturally isomorphic to $G/\mathcal{Z}_h$.

The set $\mathcal{Z}_h = Z_G(\exp(\mathbb{R}h))$ is a subgroup of $\check{G}$. Let us call $\tilde{g}$ the Lie algebra of $\mathcal{Z}_h$. We remark that the manifold (in fact symmetric space) $G/\mathcal{Z}_h$ has, at each point, a tangent space canonically isomorphic to $g/\tilde{g}$ (the quotient is taken in the category of vector spaces): this is true at the image of Id under the projection $\pi : \check{G} \to G/\mathcal{Z}_h$, and any other coset is obtained by left multiplication in $\check{G}$, as $S.\mathcal{Z}_h = S.(Id.\mathcal{Z}_h)$. The projection is locally a submersion in the neighborhood of the Id, as for all $s \in g$,

$$d\pi_{Id}(s) = s - \pi_{\tilde{g}} s$$

where $\pi_{\tilde{g}}$ is the orthogonal projection on $\tilde{g}$ with respect to the Cartan-Killing form. Therefore, the kernel of $d\pi_{Id}$ is $\tilde{g}$. It follows that π is globally a sumbersion and the kernel of the differential at $S \in \check{G}$ is $(R_S)_* \tilde{g}$, where $R_S : H \mapsto SH$ is the right multiplication in $\check{G}$.

Consequenlty, since the curve $\mathbf{a}(\cdot) : \mathbb{T} \to g$ takes values in $Ad(\check{G}).h \approx G/\mathcal{Z}_h$, it admits a lift $\check{B} : \mathbb{R} \to \check{G}$ of regularity C^σ

$$\mathbf{a}(\cdot) = Ad(\check{B}(\cdot)).h$$

Since $\mathbf{a}(\cdot)$ is periodic,

$$Ad(\check{B}^*(\cdot + 1)\check{B}(\cdot)).h = h$$

and therefore $D(\cdot) = \check{B}^*(\cdot + 1)\check{B}(\cdot)$ takes values in $\mathcal{Z}_h$.

Anticipating lemma 6.10 (for the proof of which the semisimplicity of G is irrelevant), we write $D(\cdot) = \check{D}(\cdot + 1)\check{D}^*(\cdot)$, with $\check{D}(\cdot) \in C^\sigma(\mathbb{R}, \mathcal{Z}_h)$. The mapping

$$\tilde{B}(\cdot) = \check{B}(\cdot)\check{D}(\cdot)$$

is in $C^\sigma(\mathbb{T}, \check{G})$, since

$$\begin{aligned} \tilde{B}(\cdot + 1) &= \check{B}(\cdot + 1)\check{D}(\cdot + 1) \\ &= \check{B}(\cdot + 1)D(\cdot)\check{D}(\cdot) \\ &= \check{B}(\cdot)\check{D}(\cdot) \end{aligned}$$

Since $Ad(\check{D}(\cdot)).h = h$, we have

$$\mathbf{a}(\cdot) = Ad(\tilde{B}(\cdot)).h$$

and, $\tilde{B}(\cdot)$ admits a c_G-periodic lift $B(\cdot) \in C^\sigma(c_G\mathbb{T}, G)$ satisfying the same equation owing to the fact that

$$B^*(\cdot + 1)B(\cdot) \in Z_G$$

Using, now the invariance relation satisfied by $\mathbf{a}(\cdot)$, we see that

$$\begin{aligned} Ad(A(\cdot))\mathbf{a}(\cdot) &= \mathbf{a}(\cdot + \alpha) \\ Ad(A(\cdot)B(\cdot)).h &= Ad(B(\cdot + \alpha)).h \\ Ad(B^*(\cdot + \alpha)A(\cdot)B(\cdot)).h &= h \end{aligned}$$

Therefore, the lemma is proved. □

The description of the dynamics of the cocycle $(\alpha, \tilde{A}(\cdot))$ is given by the following proposition.

Proposition 4.5. *Let $(\alpha, A(\cdot))$ be a cocycle in $SW(\mathbb{T}, G)$ satisfying the hypothesis of lemma 4.4. Then, there exists an abelian subgroup $G_+ = \mathcal{T}_h$, and a semisimple subgroup G_0 (G_0 may be trivial) commuting with G_+ and such that $G_0 \cap G_+ = Id$ satisfying the following properties. The cocycle $(\alpha, A(\cdot))$ can be conjugated $\mod c_G$ to a cocycle of the form*

$$(\alpha, \tilde{A}(\cdot)) = (\alpha, \tilde{A}_0(\cdot) \times \tilde{A}_+(\cdot))$$

The cocycle $(\alpha, \tilde{A}_+(\cdot)) \in SW(\mathbb{T}, G_+)$ is abelian and

$$\frac{1}{n}\tilde{a}_{+,n}(\cdot) \to h$$

and the cocycle $(\alpha, \tilde{A}_0(\cdot)) \in SW(\mathbb{T}, G_0)$ is such that

$$\frac{1}{n}\tilde{a}_{0,n}(\cdot) \to 0$$

The conjugation can be chosen of the same regularity as the invariant curve.

Corollary 4.6. *If $G = SU(2)$, any cocycle satisfying the hypothesis of lemma 4.4 can be conjugated to an abelian one by a 2-periodic transfer function.*

Proof of the Corollary. Since every torus in $SU(2)$ is a maximal torus, and since the hypothesis of lemma implies that G_+ is non-trivial, we obtain that $G_0 = \{Id\}$. Therefore, in the notation of the theorem, $(\alpha, \tilde{A}(\cdot))$ is abelian. □

This proposition motivates the following definition.

Definition 4.7. Let $(\alpha, A(\cdot)) \in SW(\mathbb{T}, G)$ and suppose that there exist G_0 and G_+ as in theorem 4.5 and $B(\cdot)$ such that $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$ takes values in $G_0 \times G_+ \hookrightarrow G$. We will then say that $(\alpha, A(\cdot))$ admits a *splitting* of the dynamics

Proof of Proposition 4.5. Let h and $B(\cdot)$ as in the proof of the previous lemma.

If h is a regular element of the algebra, the condition $Ad(\tilde{A}(\cdot)).h = h$ is equivalent to $\tilde{A}(\cdot)$ taking values on the unique maximal torus passing by h. Therefore, in this case G_+ is a maximal torus and $G_0 = \{Id\}$.

Let us now suppose that h is singular. Since $Ad(\tilde{A}(\cdot))$ fixes h, $\tilde{A}(\cdot)$ takes values in $\mathcal{Z}_h = Z_G(\exp(\mathbb{R}h))$. As g is the Lie algebra of a compact group, it follows from Theorem 2.2 that $\mathcal{Z}_h$ is a Lie group whose Lie algebra $\tilde{g} = \ker(ad_h)$ splits in the direct sum of g_+, the intersection of all toral algebras containing h, and $g_0 = [\tilde{g}, \tilde{g}]$, the Lie algebra of $\mathcal{Z}_h / \exp(g_+)$. By construction, g_+ and g_0 commute.

A concrete description of the decomposition can be obtained via a root space decomposition of g associated to a torus containing g_+. Then, g_0 is generated by the vectors h_ρ and j_ρ corresponding to roots ρ such that $\rho(h) = 0$, and therefore by constuction g_0 is semi-simple. Then, g_+ is $g_0^\perp \cap \tilde{g}$.

Finally, $G_0 \cap G_+ = Id$ where $G_0 = \exp(g_0)$ and $G_+ = \exp(g_+)$. If w_0 and w_+ are the respective ranks of G_0 and G_+, we have $w_0 + w_+ = w$ and G_+ is a torus of dimension $0 < w_+ \leq w$ and equality holds iff h is regular.

Since $(\alpha, \tilde{A}(\cdot)) \in SW^s(\mathbb{T}, \tilde{G})$, and $\tilde{g} = g_0 \oplus g_+$, we can easily verify that the normalized derivatives of $(\alpha, \tilde{A}_+(\cdot))$ converge to h, and those of $(\alpha, \tilde{A}_0(\cdot))$ converge to 0:

$$\begin{aligned} L\tilde{A}_{q_n}(\cdot) &= b^*(\cdot + q_n\alpha) + Ad(B^*(\cdot + q_n\alpha)).a_{q_n}(\cdot) + Ad(B^*(\cdot + q_n\alpha)A_{q_n}(\cdot))b(\cdot) \\ &= q_n Ad(B^*(\cdot))\mathbf{a}(\cdot) + o(q_n) \\ &= q_n h + o(q_n) \end{aligned}$$

Since $L\tilde{A}(\cdot) = a_+(\cdot) + a_0(\cdot)$, and the two components are iterated separately thanks to the commutativity of G_+ and G_0, we obtain that

$$\frac{1}{q_n}(a_+)_{q_n}(\cdot) \to h$$
$$\frac{1}{q_n}(a_0)_{q_n}(\cdot) \to 0$$

□

Heuristically, the existence of an invariant curve obtained as the limit of normalized derivatives allows us to abelianize only a part of the dynamics. The fact that the part of the dynamics controlled by the invariant curve is exactly the part where there is linear growth of the derivatives allows us to conclude that in the remaining part derivatives grow sublinearly. Sublinear growth of the derivatives is true for almost reducible cocycles, and, as we will see in chapter 6, the converse is true, in the sense that sublinear growth characterizes dynamics of close-to-constant cocycles. On the other hand, it is already clear that linear growth of the derivatives is the characteristic of non-local dynamics.

A concrete example of a curve $\mathbf{a}(\cdot)$ for which we cannot conclude torus-reducibility can be seen in $SU(4)$. If the points of the curve belong to $Inn(g).h$, where

$$h = \begin{bmatrix} \lambda i & 0 & 0 & 0 \\ 0 & -\lambda i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then commutation with h, which is the only exploitable information, does not provide anything on the dynamics within the subgroup

$$\begin{bmatrix} Id_2 & 0 \\ 0 & SU(2) \end{bmatrix}$$

which is the G_0 factor of the corresponding decomposition. The factor G_+ is given by the subgroup image of

$$\begin{bmatrix} (\lambda_1 + 2\lambda_2)i & 0 & 0 & 0 \\ 0 & -\lambda_1 i & 0 & 0 \\ 0 & 0 & -\lambda_2 i & 0 \\ 0 & 0 & 0 & -\lambda_2 i \end{bmatrix}$$

under the exponential mapping, where $\lambda_1, \lambda_2 \in \mathbb{R}$. We remark that the only diagonal vector in g_0 of the Weyl basis of $su(4)$ is

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

while the vectors

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

are neither in g_0 nor in g_+. The only vector of the Weyl's basis which is in g_+ is the remaining one,

$$\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which accounts for the fact that the corresponding complex direction

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is the only one in the root-space decomposition commuting with g_0.

Concrete examples exist, in fact, in any group of dimension higher than $SU(2)$, namely in $SU(3)$, where vectors of the type

$$\begin{bmatrix} 2\lambda i & 0 & 0 \\ 0 & -\lambda i & 0 \\ 0 & 0 & -\lambda i \end{bmatrix}$$

commute with the subgroup

$$\begin{bmatrix} 1 & 0 \\ 0 & SU(2) \end{bmatrix}$$

and the corresponding abelian algebra g_+ is generated by the vector

$$\begin{bmatrix} 2\lambda i & 0 & 0 \\ 0 & -\lambda i & 0 \\ 0 & 0 & -\lambda i \end{bmatrix}$$

itself.

If we anticipate proposition 4.9, we find directly that $(\alpha, A(\cdot))$ is torus-reducible if, and only if, $(\alpha, \tilde{A}_0(\cdot))$ is torus-reducible in G_0.

Since $\tilde{A}_+(\cdot)$ takes its values in a torus and it is periodic, it is of the form

$$\tilde{A}_+(\cdot) = \sum\nolimits_\rho 2\pi h_\rho(r_\rho \cdot + \phi_\rho(\cdot))$$

where the $\phi_\rho(\cdot)$ are periodic. By resolving a linear cohomological equation in G_+, we find that if α is Diophantine, $(\alpha, \tilde{A}(\cdot))$ can be conjugated to a cocycle of the form

$$\big(\alpha, \tilde{A}_0(\cdot) \times (E_r(\cdot).A)\big)$$

where $E_r(\cdot) = \exp(\sum_\rho 2\pi r_\rho h_\rho \cdot)$ and $A \in G_+$. In the particular case where $\tilde{A}_0(\cdot) = A_0 \in G_0$, such a splitting will be called a *normal form*.

The construction of the splitting of the dynamics shows that in general we cannot expect to avoid phenomena of coexistence of close-to constants cocycles with far-from-constants cocycles. The two cases where this does not occur is either when iteration and normalization converge to constants, or when the vector obtained by the construction is regular, and it should be expected that this dichotomy is generic under some reasonable Diophantine conditions. Since every non-zero vector in $su(2)$ is regular, the result in $SU(2)$ assumes a more elegant form, namely that the existence of an invariant curve implies the existence of a 2-periodic conjugation to an abelian cocycle. This form, however, cannot be obtained in full generality for any given group G.

4.3 Conjugation of abelian cocycles

The action of conjugacies on the class of constant cocycles has been exhaustively described in [Kri99a] (Prop. *2.5.9*):

Proposition 4.8. *Let (α, A_i), $i = 1, 2$ be two constant cocycles, conjugate by $B(\cdot) \in C^\infty(m\mathbb{T}^d, G)$, where $B(0) = Id$ and $m \in \mathbb{N}^*$. Then, for any character χ_ρ, $\chi_\rho(A_1).\chi_\rho(A_2^{-1}) \in \exp(2i\pi/m\mathbb{Z})$, and $B(\cdot)$ is a group homomorphism $m\mathbb{T}^d \to G$.*

Since conjugation between abelian cocycles is less restrictive, the result is less precise. On the other hand it is somewhat more elegant, since it is related to the action of Weyl's group, which is a finite group.

Proposition 4.9. *Let $(\alpha, \tilde{A}(\cdot))$ and $(\alpha, A(\cdot))$ in $SW(\mathbb{T}, \mathcal{T}) \hookrightarrow SW(\mathbb{T}, G)$ such that*

$$(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$$

and consider the corresponding splittings $\tilde{G}_0 \times \tilde{G}_+ \hookrightarrow G$ and $G_0 \times G_+ \hookrightarrow G$. Then

$$\begin{aligned} (\alpha, \tilde{A}_+(\cdot)) &= Conj_{B(\cdot)}(\alpha, A_+(\cdot)) \\ (\alpha, \tilde{A}_0(\cdot)) &= Conj_{B(\cdot)}(\alpha, A_0(\cdot)) \end{aligned}$$

The action of $B(\cdot)$ on $G_+ \to \tilde{G}_+$ projects naturally to the action of a representative of $W(G_+) \subset W(G)$.

The following corollary is immediate.

Corollary 4.10. *The integer vector r which distinguishes qualitatively different models of dynamics of abelian cocycles is determined up to the action of $W(G)$.*

Before passing on to the proof of the proposition, we remark that the hypothesis that $\tilde{A}(\cdot)$ and $A(\cdot)$ take values on the same torus is not restrictive, since any two tori in G are obtained by a (constant algebraic) conjugation acting on a standard one.

Proof of Proposition 4.9. Let us suppose that

$$(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$$

holds, for a certain $B(\cdot) \in C^0(\mathbb{T}, G)$.

The mappings $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$ take values in the same maximal torus $\mathcal{T}$, whose toral algebra is $\mathfrak{t}$. In this case, $B(\cdot)$ is determined up to left and right multiplication with a constant in $\mathcal{T}$, as shows the formula

$$\tilde{A}(\cdot) = B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot)$$

Let, now, $h, h' \in \mathfrak{t}$ be any two vectors. Then, the same formula, together with the fact that the restriction of the adjoint action of both $A(\cdot)$ and $\tilde{A}(\cdot)$ in $\mathfrak{t}$ is the identity, gives directly

$$\langle Ad(B(\cdot + \alpha)).h, h' \rangle = \langle Ad(B(\cdot)).h, h' \rangle$$

In particular, if we let these two vectors run through a basis of $\mathfrak{t}$, we see by the minimality of R_α that the restriction of $Ad(B(\cdot))$ in $\mathfrak{t}$ is in fact a constant.

If we denote by $B_{\rho,h}(\cdot)j_\rho$ the projection of the image of $Ad(B(\cdot)).h$ on the plane $\mathbb{C}j_\rho$, and also use the notation $Ad(\tilde{A}(\cdot)).j_\rho = \exp(\tilde{a}_\rho(\cdot)).j_\rho$, we find that

$$B_{\rho,h}(\cdot + \alpha) = \exp(-\tilde{a}_\rho(\cdot))B_{\rho,h}(\cdot)$$

In particular, for all ρ and h, we have $|B_{\rho,h}(\cdot)| = C$, a constant. Therefore, if $C \neq 0$, $B_{\rho,h}(\cdot) : \mathbb{T} \to C.\mathbb{S}^1$ admits a lift $C\exp(b_{\rho,h}(\cdot))$ which satisfies

$$b_{\rho,h}(\cdot + \alpha) - b_{\rho,h}(\cdot) = \tilde{a}_\rho(\cdot) \mod 2i\pi\mathbb{Z}$$

We now disinguish two cases. If $\tilde{a}_\rho(\cdot) : \mathbb{R} \to i\mathbb{R}$ is of the form $2i\pi(\tilde{r}_\rho \cdot + \varphi_\rho(\cdot))$ with $r_\rho \in \mathbb{Z}^*$ (i.e. if $h_\rho \notin g_0$) and $\varphi_\rho(\cdot)$ periodic, then $b_{\rho,h}(\cdot)$ cannot be a lift of a periodic function, unless it is constant and therefore $B_{\rho,h}(\cdot)$ must be equal to 0. On the other hand, if $\tilde{r}_\rho = 0$, the equation may or not admit smooth or even continuous solutions depending on the arithmetic properties of α, but in the particular case where $\hat{\varphi}_\rho(0) \in \alpha\mathbb{Z}^*$ (such constants are called resonant) it may also admit solutions of the form

$$b_{\rho,h}(\cdot) = 2i\pi r\cdot$$

which correspond to mappings $c_G\mathbb{T} \to G$ If we admit that the conjugant be m-periodic, the resonant constants can belong to $\alpha\frac{1}{m}\mathbb{Z}^*$, and then the equation admits non-local solutions of the form

$$b_{\rho,h}(\cdot) = 2i\pi\frac{r}{m}\cdot$$

a phenomenon already observed in [Kri99a]. We remark that there is no loss of periodicity apart from the phenomena observed in conjugation between constant cocycles.

Therefore, $\langle Ad(B(\cdot))h_{\rho'}, j_\rho \rangle$ can be non-zero only if $Ad(\tilde{A}(\cdot))j_\rho = C\exp(2i\pi(\varphi_{\rho'}(\cdot)))$ with $C > 0$. Since the roles of $\tilde{A}(\cdot)$ and $A(\cdot)$ are symmetric, the same holds for $A(\cdot)$.

If we now turn to the study of $B_{\rho,\rho'}(\cdot) = \langle Ad(B(\cdot))j_\rho, j_{\rho'} \rangle$, we find that

$$B_{\rho,\rho'}(\cdot + \alpha) = \exp(\tilde{a}_{\rho'}(\cdot) - a_\rho(\cdot))B_{\rho,\rho'}(\cdot)$$

Consequently, $B_{\rho,\rho'}(\cdot)$ can be non-zero only if $\tilde{a}_{\rho'}(\cdot) - a_\rho(\cdot)$ is a periodic function, which amounts to $\tilde{r}_{\rho'} = r_\rho$.

We conclude that the class of conjugant $B(\cdot)$ is well defined in $W(G_+) \approx W(\tilde{G}_+) \subset W(G)$, where $G_0 \times G_+ \hookrightarrow G$ and $\tilde{G}_0 \times \tilde{G}_+ \hookrightarrow G$ are the splittings corresponding to $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$. Since any two different maximal tori are obtained by an algebraic conjugation in G and all the groups $W(\mathcal{T})$ are naturally isomorphic to W, independently of such conjugations, we have proved the proposition □

Therefore, the additional symmetries of the torus, owing to the fact that it is embedded in a Lie group G, result in the existence of conjugations between models which are not conjugate otherwise. The action of Weyl's group reduces the number of abelian models of dynamics, by taking into account the symmetries of $\mathcal{T}$ induced by the action of G. For example, the cocycles $(\alpha, E_r(\cdot))$ and $(\alpha, E_{-r}(\cdot))$, $r \in \mathbb{N}^*$ are not conjugate in $SW(\mathbb{T}, \mathbb{S})$, but they are conjugate in $SW(\mathbb{T}, SU(2))$.

If the rotation in the basis is Diophantine, the vectors r and $(a_\rho)_{\rho \in \tilde{\Delta}}$ completely describe the dynamics of an abelian cocycle up to conjugation (cf. eq. (3.5) for the notation), as shows the resolution of a classical linear cohomological equation. If the rotation is Liouvillean, we cannot conclude anything more than the density in C^∞ topology in the space of abelian cocycles of the orbits of models of the type

$$(\alpha, \exp(\sum\nolimits_\rho (2\pi(r_\rho \cdot + a_\rho)h_\rho)$$

Since any basis of a maximal torus of G can be obtained by the action of $W(G)$ on a single one, we can consider a fixed basis and use the notations

$$E_r(\cdot) = \exp(\sum\nolimits_\rho (2\pi r_\rho h_\rho \cdot)) \text{ and} \tag{4.9}$$

$$E_{r,a}(\cdot) = \exp(\sum\nolimits_\rho (2\pi(r_\rho \cdot + a_\rho)h_\rho) \tag{4.10}$$

In the particular case where $G = SU(2)$, the possible abelian models are exhausted by

$$(\alpha, \exp(\{2\pi r \cdot + \phi(\cdot) + a, 0\})$$

where $r \in \mathbb{Z}$. The action of Weyl's group identifies r with $-r$ (and $\phi(\cdot)$ and a with $-\phi(\cdot)$ and $-a$) in the space of abelian cocycles, so that we need only consider $r \in \mathbb{N}$ in the description of the abelian models.

Chapter 5

The energy of a cocycle in $\mathbb{T} \times G$

5.1 Introduction

In the previous chapter (proposition 4.1), we observed the phenomenon of quantization of the asymptotic normalized length of torus-reducible cocycles in $\mathbb{T} \times G$ over any irrational rotation. This phenomenon was first observed in [Kri01] in his study of cocycles in $\mathbb{T} \times SU(2)$ (cf. also [Fra00] for a similar result for cocycles over the golden ratio). In this more restricted algebraic context, R. Krikorian proved the quantification of this same quantity for cocycles over rotations in the set σ of theorem 3.11 as an intermediate result of the proof of the global density of reducible cocycles in the non-local case. The proof uses the local density theorem, theorem 3.5, and the renormalization scheme that we will be presented and implemented in chapter 6. Under some arithmetic conditions, somewhat stricter than a simple recurrent Diophantine one, renormalization was proved to converge towards either constant cocycles, or the normal forms $(\alpha_n, E_r(\cdot + \theta))$, where $E_r(\cdot) = \exp(2i\pi rh\cdot)$ and $\theta \in \mathbb{T}$ is a constant and $r \in \mathbb{N}^*$, which are geodesics of $SU(2)$ and thus a special case of abelian cocycles. These normal forms, called obstructions, are not reducible by virtue of theorem 3.8. They are proved to be C^∞-accumulated by reducible cocycles by establishing a dichotomy: if a cocycle in the neighborhood of a normal form is not reducible to it, renormalization converges to a normal form of smaller length, and cocycles of the latter type form a dense set in the neighborhood of normal forms. Induction and use of the local version of the theorem concludes the proof, and the quantization of the asymptotic length of C^2 cocycles has been proved under the arithmetic condition imposed.

This result was generalized in [Fra04] (cf. theorems 3.13 and 3.15). There, it was proved that the conjugation to the obstructions under the condition defining Σ implies conjugation under a simple recurrent Diophantine condition [1]. Moreover, it was proved by K. Fraczek with the use of ergodic theorems and renormalization that the quantization phenomenon is observed for C^2 cocycles independently of any arithmetic conditions, and the asymptotic normalized length, or energy, of the cocycle was defined to be its degree. Additionally, it was proved, using a technique similar to the one used in order to obtain proposition 4.5, that every cocycle in $\mathbb{T} \times SU(2)$ of positive degree is measurably torus-reducible. This result cannot be generalized in more general groups, as it is based essentially on the fact that the tori of $SU(2)$ are of dimension 1, or, in other terms, the fact that phenomena of positive degree cannot be observed in coupling with local phenomena.

Here, we extend these results to cocycles in compact groups, where the complexity of the geometry forces a change in terminology. The fact that a maximal toral algebra in $su(2)$ is the real line in $\mathbb{R} \times \mathbb{C}$, results in the classification of the different abelian models by an invariant in $2\pi\mathbb{N}$, as already discussed, equal to the mean speed of a reparametrized periodic geodesic. However, this invariant is insufficient for more general groups. For example, the cocycles obtained by the

1. This result can be strengthened by establishing a normal form theorem in the neighbourhood of $(\alpha, E_r(.))$ with a Diophantine α, which asserts that reducibility to $(\alpha, E_r(.))$ is of codimension $2r$, and that the dichotomy proved by R. Krikorian persists under the weaker artithmetic condition. This will be the object of chapter 8

geodesics $E_{r_i}(\cdot)$ for $r_1 = (3, 0, 4)$ and $r_2 = (5, 0, 0)$, i.e.

$$\begin{bmatrix} e^{6i\pi\cdot} & 0 & 0 & 0 \\ 0 & e^{-6i\pi\cdot} & 0 & 0 \\ 0 & 0 & e^{8i\pi\cdot} & 0 \\ 0 & 0 & 0 & e^{-8i\pi\cdot} \end{bmatrix} \text{ and } \begin{bmatrix} e^{10i\pi\cdot} & 0 & 0 & 0 \\ 0 & e^{-10i\pi\cdot} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in $SU(4)$, who have the same speed (or degree, according to K. Fraczek's terminology), cannot be conjugated over any irrational rotation as shows Proposition 4.9. Therefore, we keep the term degree for the vector in $2\pi\mathbb{Z}^w$, defined modulo the action of Weyl's group, i.e. modulo permutations of its coordinates and changes of sign, and call its length energy.

5.2 Definition and basic properties of the energy

Let $(\alpha, A(\cdot)) \in SW^s$, with $\alpha \in \mathbb{T}\backslash\mathbb{Q}$, $s \geq 1$, and

$$a(\cdot) = LA(\cdot) = \partial A(\cdot).A^*(\cdot) \in C^{s-1}(\mathbb{T}, g)$$

(see section 2.2.1). Since the n-th iterate of the cocycle for $n \geq 1$ is given by $(n\alpha, A_n(\cdot)) = (n\alpha, A(\cdot + (n-1)\alpha)...A(\cdot))$, the derivative of its inverse reads

$$a_n^*(\cdot) = LA_n^*(\cdot) = a^*(\cdot) + Ad(A^*(\cdot)).a_{n-1}^*(\cdot + \alpha)$$

as imply the properties of the differential operator L (see section 2.2.1). Therefore, if we define the unitary operator

$$\begin{array}{rccc} \mathcal{U}: & L^2(\mathbb{T}, g) & \to & L^2(\mathbb{T}, g) \\ & b(\cdot) & \mapsto & Ad(A^*(\cdot)).b(\cdot + \alpha) \end{array}$$

we have $a_n^*(\cdot) = \sum_0^{n-1} \mathcal{U}^k a^*(\cdot)$. We note that this operator is in fact associated to the cocycle $(\alpha, A(\cdot))$ and should be noted by $\mathcal{U}_{(\alpha, A(\cdot))}$, but the simplified notation will be used, since the cocycle is to be considered fixed.

By the von Neumann ergodic theorem, there exists $\mathbf{a}_+^*(\cdot) \in L^2(\mathbb{T}, g)$, $\mathcal{U}$-invariant, such that

$$\mathbf{a}_+^*(\cdot) = \lim_{n\to\infty} \frac{1}{n} a_n^*(\cdot)$$

in L^2 and Lebesgue a.e. U-invariance implies that $|\mathbf{a}_+^*(\cdot)|$ is constant a.e., since

$$|\mathbf{a}_+^*(\cdot)| = |Ad(A^*(\cdot)).\mathbf{a}_+^*(\cdot + \alpha)| = |\mathbf{a}_+^*(\cdot + \alpha)|$$

and α is an ergodic translation on the torus.

In the same way we define $\mathbf{a}_-^*(\cdot)$ as

$$\mathbf{a}_-^*(\cdot) = -\lim_{n\to\infty} \frac{1}{n} a_{-n}^*(\cdot)$$

Since $A_{-n}(\cdot + n\alpha)A_n(\cdot) = Id$, we have

$$\begin{array}{rcl} a_n^*(\cdot) & = & -Ad(A_n^*(\cdot)).a_{-n}^*(\cdot + n\alpha) \\ & = & -\mathcal{U}^n.a_{-n}^*(\cdot) \end{array}$$

and consequently we find that

$$\|\frac{1}{n} a_n^*(\cdot) - \mathbf{a}_+^*(\cdot)\|_{L^2} = \|-\frac{1}{n} a_{-n}^*(\cdot) - \mathbf{a}_+^*(\cdot)\|_{L^2}$$

which implies that

$$\mathbf{a}^*(\cdot) = \mathbf{a}_+^*(\cdot) = \mathbf{a}_-^*(\cdot)$$

is unambiguously defined.

In a similar way, the fact that $A_n^*(\cdot)A_n(\cdot) = Id$, and therefore $a_n^*(\cdot) = -\mathcal{U}^n.a_n(\cdot - n\alpha)$, implies that

$$\frac{1}{n}a_n(\cdot) \to -\mathbf{a}^*(\cdot)$$

in L^2 when $n \to \pm\infty$, and we are justified to define

$$\mathbf{a}(\cdot) = \lim_{n\to\pm\infty} \frac{1}{n}a_n(\cdot) = -\mathbf{a}^*(\cdot)$$

As before, $\mathbf{a}(\cdot)$ satisfies the invariance equation 4.8

$$Ad(A(\cdot)).\mathbf{a}(\cdot) = \mathbf{a}(\cdot + \alpha)$$

and $|\mathbf{a}(\cdot)|$ is constant *a.e.*. The invariance equation implies that the measurable families of sub-algebras $g_+(\cdot)$ and $g_0(\cdot)$, associated to $\mathbf{a}(\cdot)$ as in section 4.2, satisfy the same invariant relations *a.e.*. The proofs are obtained as in the afore-mentioned section just by replacing minimality and continuity by ergodicity and measurability, respectively.

Definition 5.1. Let $(\alpha, A(\cdot))$ be a C^1 cocycle. The number

$$\|\mathbf{a}(\cdot)\|_{L^2} = \lim_{n\to\pm\infty} \frac{1}{n} \|a_n(\cdot)\|_{L^2} = |\mathbf{a}(\cdot)| \ a.e.$$

will be called the *energy* of $(\alpha, A(\cdot))$ and denoted by $en(\alpha, A(\cdot))$. The set where $|\mathbf{a}(\cdot)| = en(\alpha, A(\cdot))$ is invariant by R_α and thus of full measure.

We remark that the derivative of $A(\cdot)$ in g and the inner endomorphism $Ad(A^*(\cdot))$ (and therefore the operator $\mathcal{U}$) depend only on the Lie algebra g and not on the topology of the group G having g as its Lie algebra. Theorem 3.8 implies directly that a cocycle of positive energy is not reducible. There exist however cocycles of 0 energy which are not reducible, as we will see in section 7.7.

Since the invariance of the curve gives directly that $Ad(A_{q_n}(\cdot)).\mathbf{a}(\cdot) = \mathbf{a}(\cdot + (-1)^n\beta_n)$, the following corollary is immediate

Corollary 5.2. $\mathbf{a}(\cdot)$ *and* $A_{q_n}(\cdot)$ *asymptotically commute:*

$$Ad(A_{q_n}(\cdot)).\mathbf{a}(\cdot) \to \mathbf{a}(\cdot) \ \textit{in} \ L^2$$

We observe that the mapping $\mathbf{a}(\cdot)$ generalizes the one defined for C^1-torus-reducible cocycles, with uniform convergence replaced by convergence in L^2, and the invariance of the degree in the latter case persists in this more general setting:

Proposition 5.3. *The energy of a cocycle is invariant by conjugation in* $C^1(\mathbb{T}, G)$. *Moreover,*

$$\begin{aligned} \text{en}\,(\alpha, A(\cdot))^n &= |n|.\text{en}\,(\alpha, A(\cdot)), \forall n \in \mathbb{Z} \\ \text{en}\,(\alpha, A(\cdot)) &\leq \|a(\cdot)\|_{L^i}, \ i = 1, 2, \infty \end{aligned}$$

Proof. Let $B(\cdot) \in C^1(\mathbb{T}, G)$. Then, calling $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}(\alpha, A(\cdot))$,

$$\tilde{a}_n(\cdot) - Ad(B(\cdot + n\alpha)).a_n(\cdot) = b(\cdot + n\alpha) + Ad(B(. + n\alpha).A_n(\cdot)).b^*(\cdot)$$

so that

$$||\tilde{a}_n(\cdot)| - |a_n(\cdot)|| \leq Cst$$

and $en(\alpha, \tilde{A}(\cdot)) = en(\alpha, A(\cdot))$.

The second relation follows directly from the fact that

$$(\alpha, A_n(\cdot))^k = (\alpha, A(\cdot))^{nk}$$

and the third from the triangle inequality. □

Corollary 5.4. *The ergodic cocycles in* $\mathbb{T} \times SO(3)$ *constructed in [Eli02] are of zero energy.*

In fact, we can prove that all cocycles in the regime of the local theory as in [Kri95] and [Kri99a] are of zero energy, despite the eventual loss of periodicity.

We also note the following simple fact.

Lemma 5.5. *Let* $(\alpha, A(\cdot)) = (\alpha, A_1(\cdot) \times A_2(\cdot))$. *Then,*

$$\mathrm{en}((\alpha, A(\cdot))) = \mathrm{en}((\alpha, A_1(\cdot))) + \mathrm{en}((\alpha, A_2(\cdot)))$$

In view of these properties of the invariant curve and the discussion in section 4.2.2, we are motivated to give the following definition

Definition 5.6. A cocycle (of positive energy) will be called *regular* if, and only if, the curve $\mathbf{a}(\cdot)$ consists a.e. of regular vectors, and *singular* otherwise. This property is invariant by C^1-conjugation.

We remind that a vector in g is called regular if, and only if, it is contained in a single maximal torus, so that a cocycle being regular or singular is an algebraic property of its dynamics, and should not be confused with the regularity of the cocycle, i.e. the regularity of the mapping $\mathbb{T} \to G$ defining the dynamics in the fibers.

The following corollary of Proposition 4.5 is immediate:

Corollary 5.7. *If the cocycle* $(\alpha, A(\cdot))$ *is regular and of positive energy, then it can be measurably conjugated to an abelian cocycle. In particular, all cocycles in* $\mathbb{T} \times SU(2)$ *of positive energy have this property.*

The hypothesis of the corollary is optimal as shows the following example. Let $\alpha \in DC$, $E_r(\cdot)$ be a periodic geodesic in $SU(2)$ and $(\alpha, A\exp(F(\cdot))$ be a non-reducible cocycle in $SU(2)$. It is well known that such cocycles exist, and they are even abundant in the neighborhoods of constants (see, e.g., [Eli02]). The hypothesis that $\alpha \in DC$ implies that such a cocycle is not torus-reducible, since for a Diophantine rotations reducibility and torus-reducibility coincide. Finally, we consider $SU(2) \times SU(2) \hookrightarrow SU(4)$ and define $(\alpha, A(\cdot)) = (\alpha, E_r(\cdot) \times A\exp(F(\cdot))$, which is a cocycle in $\mathbb{T} \times SU(4)$. This cocycle is of positive energy, but not regular, and clearly not C^∞-torus-reducible, by virtue of theorem 4.9.

This is, in fact the best possible generalization of K. Fraczek's theorem which states that all cocycles of positive energy in $\mathbb{T} \times SU(2)$ can be measurably conjugated to abelian ones, obtained as a corollary of theorem 3.12 of the same paper.

If we try to adapt the arguments in section 4.2 in the measurable context, which is, in fact, imitating K. Fraczek's proof of the afore-mentioned result, but for a singular invariant curve, we obtain a void statement. We cannot conclude anything more than the existence of a measurable conjugacy and the corresponding (measurable) splitting

$$\begin{array}{rcl} \tilde{G}_0 \times \tilde{G}_+ & \hookrightarrow & G \\ (\alpha, \tilde{A}(\cdot)) & = & (\alpha, \tilde{A}_0(\cdot) \times \tilde{A}_+(\cdot)) \end{array}$$

where $(\alpha, \tilde{A}_+(\cdot))$ is "of positive energy" and $(\alpha, \tilde{A}_0(\cdot))$ is in general non-abelian and "of zero energy".

The definition of the energy can be extended in a rather awkward way to some measurable abelian cocycles by using Birkhoff's ergodic theorem. More precisely, let us consider the case where $G = SU(2)$ and a measurable abelian cocycle $(\alpha, A(\cdot))$. If the mapping $A(\cdot)$ admits a lift in g of the form $\exp(2i\pi(r \cdot + \phi(\cdot))h)$ with $r \in \mathbb{Z}^*$ and $\phi(\cdot) : \mathbb{T} \to \mathbb{R}$ an integrable function such that $\int \phi(\cdot) = 0$, [2] then the n-th iterate is calculated as in the abelian case:

$$A_n(\cdot) = \exp((2\pi r(n \cdot + \frac{n(n-1)}{2}\alpha + S_n^\alpha \phi(\cdot))h))$$

Clearly, normalization by n of the lift of $A_n(\cdot)$ in g and passage to the limit gives $2\pi r$, which is what we would want to define as the degree of $(\alpha, A(\cdot))$. In fact this generalization is cyclic in the

2. This is in fact not any more restrictive than the existence of such a lift, since, if we consider $A(\cdot + \gamma)$ (such a choice of γ is possible since $r \neq 0$) instead of $A(\cdot)$, we can eliminate $\int \phi(\cdot)$ without affecting the dynamics.

sense that it supposes that the degree can be defined, as the existence of a lift of this form is not guaranteed in the measurable context.

This generalization cannot in general be applied to $(\alpha, \tilde{A}_+(\cdot))$, which is abelian by construction, and thus we do not know if the cocycle $(\alpha, \tilde{A}_+(\cdot))$ remains a cocycle of positive energy under this extended definition. On the other hand, there is no known extension to (non-abelian) cocycles of regularity lower than H^1, and it is highly probable that there does not exist any, so that the statement that $en(\alpha, \tilde{A}_0(\cdot)) = 0$ is devoid of any meaning, unless $(\alpha, \tilde{A}_0(\cdot))$ is found to be at least measurably torus-reducible. However, since in $SU(2)$ exactly one of the factors of the splitting is non-trivial (which is not always the case, even if it should be so "generically", in any given compact group), and the conclusion that all positive energy cocycles are measurable torus-reducible is important, even though it does not survive in more general contexts.

Finally, let us revisit the expression $a_n^*(\cdot) = \sum_0^{n-1} \mathcal{U}^k a^*(\cdot)$. Using the triangle inequality, as well as the fact that $\mathcal{U}$ is a unitary operator, we find that

$$\|a_n^*(\cdot)\|_{L^2} \leq n \|a^*(\cdot)\|_{L^2}$$

It is a classical fact that equality holds if, and only if, all the vectors composing the sum are colinear. In this particular case, this amounts to the existence of a positive constant λ such that

$$\mathcal{U}a^*(\cdot) = \lambda a^*(\cdot)$$

Since $|\lambda| = \|\mathcal{U}a^*(\cdot)\|_{L^2} / \|a^*(\cdot)\|_{L^2}$ (we assume that $\|a^*(\cdot)\|_{L^2} \neq 0$), we find that $\lambda = 1$, and the ergodic sum reduces to

$$a_n^*(\cdot) = na^*(\cdot)$$

Therefore, in the case of maximal growth of the derivatives of the iterates of $(\alpha, A(\cdot))$, $\mathbf{a}^*(\cdot) = a^*(\cdot)$, and $en(\alpha, A(\cdot)) = \|a^*(\cdot)\|_{L^2}$. Reinserting the expression for the operator $\mathcal{U}$ we obtain the relation

$$a^*(\cdot) = Ad(A^*(\cdot)).a^*(\cdot + \alpha)$$

or

$$a^*(\cdot + \alpha) = -a(\cdot)$$

Thus,

$$\begin{aligned} a_2(\cdot) &= a(\cdot + \alpha) + Ad(A(\cdot + \alpha)).a(\cdot) \\ &= 2a(\cdot + \alpha) \end{aligned}$$

and inductively

$$a_n(\cdot) = na(\cdot + n\alpha)$$

Since

$$\frac{1}{n} a_n(\cdot) = a(\cdot + n\alpha) \to \mathbf{a}(\cdot)$$

and α is minimal, this can be true only if $a(\cdot)$ is constant and equal to, say, h. The assumption that $A(\cdot)$ is H^1 implies, therefore, that it is of the form

$$\exp(h\cdot).A_0$$

The condition that $\mathcal{U}a(\cdot) = a(\cdot)$ and the periodicity of $A(\cdot)$ imply respectively that

$$\begin{aligned} Ad(A_0).h &= h \\ \exp(h) &= Id \end{aligned}$$

and we have proved

Theorem 5.8. *The operator $\mathcal{U}$ associated to a cocycle $(\alpha, A(\cdot))$ of regularity H^1 admits $a(\cdot)$ as an eigenvector with positive eigenvalue if, and only if, the cocycle is a periodic geodesic, and therefore C^∞. The eigenvalue is equal to 1.*

It seems reasonable to speculate that the attractors of the dynamics in $SW^1(\mathbb{T}, G)$ should be the periodic geodesics of the group, which is in fact proved to be true in the next chapter.

Another distinct case of the action of the operator $\mathcal{U}_{(\alpha,A(\cdot))}$ is the one associated to abelian cocycles, which is slightly more general than that of the periodic geodesics. For such a cocycle, we have

$$Ad(A(\cdot)).a(\cdot + \alpha) = a(\cdot + \alpha)$$

Supposing that this relation is verified, we find that the sum defining $\mathbf{a}(\cdot)$ is reduced to a simple ergodic sum over R_α, so that $\mathbf{a}(\cdot) = \hat{a}(0)$, a constant. This constant is fixed by $Ad(A(\cdot))$, as we have seen in the previous chapter.

5.3 Higher-order derivatives

Let us start by stating and proving a lemma on the growth of C^s norms of the iterates of a smooth cocycle:

Lemma 5.9. *The n-th iterate of the cocycle $(\alpha, A(.)) \in SW^s$ satisfies the estimates*

$$\begin{aligned} \|\partial^\sigma a_n(\cdot)\|_0 &\leq C_\sigma n^{\sigma+1}(1 + \|a(\cdot)\|_\sigma) \|a(\cdot)\|_0^{\sigma+1}, \text{ or} \\ \|A_n(\cdot)\|_s &\leq C_s n^s(1 + \|a(\cdot)\|_s) \|a(\cdot)\|_0^s \end{aligned}$$

where the constant C_σ depends only on σ, and $0 \leq \sigma \leq s-1$. Therefore, the normalized derivatives $\frac{1}{n^{\sigma+1}}\partial^\sigma a_n(\cdot)$ are bounded uniformly for n.

Proof. The proof is obtained by applying Proposition 2.9 and using the translation invariance of the norms. □

The result given above can be considerably refined:

Lemma 5.10. *Let $1 \leq \sigma \leq s-2$. Then $\frac{1}{n^{\sigma+1}}\partial^\sigma a_n(\cdot) \to 0$ in L^2, as $n \to \pm\infty$.*

Proof. The estimates of Lemma 5.9 and convexity inequalities imply that it is sufficient to prove the lemma for $\sigma = 1$, and the fact that $\mathbf{a}(\cdot) = -^*(\cdot)$ implies that it is sufficient to prove the result for $\frac{1}{n^2}\partial a_n^*(\cdot)$.

The properties of the differential operator L imply that

$$\partial a_n^*(\cdot) = \sum_0^{n-1} \mathcal{U}^{k-1}\partial a^*(\cdot) + \sum_{k=0}^{n-2}\sum_{l=k+1}^{n-1} [\mathcal{U}^k a^*(\cdot), \mathcal{U}^l a^*(\cdot)]$$

As has already been proved, $\frac{1}{n}\sum_0^{n-1} \mathcal{U}^{k-1}\partial a^*(\cdot)$ converges in L^2, and therefore

$$\frac{1}{n^2}\sum_0^{n-1} \mathcal{U}^{k-1}\partial a^*(\cdot)$$

tends to 0 as $n \to \infty$, and, since $[\cdot, \cdot]$ is antisymmetric, we have to prove that

$$\frac{1}{n^2}\sum_{k=0}^{n}\sum_{l=0}^{k} [\mathcal{U}^k a^*(\cdot), \mathcal{U}^l a^*(\cdot)] \to 0 \text{ in } L^1(\mathbb{T}, g)$$

which is the object of lemma 11.1 in Appendix 11.

Since the sequence $(\frac{1}{n^2}\partial a_n^*(\cdot))_n$ is uniformly bounded in L^∞, we obtain the convergence in L^2, and thus we can prove the same result for higher derivatives. The case $n \to -\infty$ is treated analogously. □

The two lemmas of this section imply that the essential information on the asymptotic behaviour of the dynamics of a cocycle should be contained in at most the first derivative, since the mere existence in L^2 of a derivative of a higher order implies that, after rescaling, it converges to 0 in the same space. This is the object of the next chapter.

Chapter 6

$\mathbb{Z}^2$ actions on $\mathbb{R} \times G$ and the renormalization scheme

In this chapter, we present and apply the main tool used in the study of non-local phenomena of the dynamics of cocycles in $\mathbb{T} \times G$, where G is typically $SL(2, \mathbb{R})$ or $SU(2)$. It can be described essentially as a division algorithm, since it is the outcome of the lift of the euclidean division algorithm for the continued fractions expansion to the dynamics of a cocycle over an irrational rotation. The lift of this algorithm divides the limit object of the dynamics obtained in the previous chapter, i.e. the invariant curve $\mathbf{a}(\cdot)$, by the group of iterates of the cocycle. Since, however, the limit object lives in a low-regularity space (L^2), the algorithm should be expected to diverge (at least a priori) in C^∞.

Another interpretation of the function of renormalization in the study of quasiperiodic cocycles could be the reduction of the study of general global objects to the local study of perturbations of normal forms, obtained as the basins of attraction of renormalization. The property used in the reduction of the global study to the different local ones is the preservation of the quality of dynamics under renormalization, and it is precicely due to the fact that renormalization divides the limit object of the dynamics by the group of iterates of the cocycle.

An application of a weaker, yet renormalization-like, algorithm to the study of cocycles in $\mathbb{T} \times SU(2)$ dates back to [Ryc92], with some interesting results and questions for research, but the application is restricted to the golden ratio, and it is based on the very fact that, for the golden ratio φ, iteration of R_φ coincides with the continued fractions algorithm. This scheme was used also in [Fra00], where it was used in the proof of the quantization of the degree of a cocycle, as it is defined in the same paper.

Renormalization was introduced in the study of cocycles in the same product space in [Kri01], under a form which is applicable to any irrational rotation, and the convergence results where proved for a full measure set of frequencies. The algorithm was proved to converge for C^2 cocycles over such rotations, and then the local study of the normal forms towards which the renormalization converges gave the first global density result for C^∞ cocycles in $\mathbb{T} \times SU(2)$. In the study of the convergence of renormalization, two functionals where introduced. The one based on L^2 norms (called $J^{(2)}$) was used to measure the convergence and the second, the length functional (called $J^{(1)}$), was proved to take discrete values in a way that generalizes the result of K. Fraczek, obtained for cocycles over the golden ratio.

These results where generalized in manifold directions in [Fra04]. In a first time, the algorithm was proved to converge independently of any arithmetic conditions on the (irrational) rotation, for C^1 regularity. The two functionals introduced by R. Krikorian where proved to be alternative definitions of the degree, a quantity that can be defined without the use of renormalization and for C^1 cocycles over any irrational rotation. In particular, the regularity needed for the equivalence was only C^1 for $J^{(2)}$, but C^2 for $J^{(1)}$. Since it is $J^{(1)}$ that is used in the identification of the normal forms towards which renormalization converges, the quantization of the degree was proved in regularity C^2. Re-examination and improvement of the local study made in [Kri01] loosened the

arithmetic condition for the global density of reducible cocycles to a *RDC*. Finally, it was proved that C^2 cocycles that are measurably conjugate have the same degree.

In the presentation of the algorithm, we have adapted the one in [FK09], used for the study of cocycles in $\mathbb{T} \times SL(2, \mathbb{R})$. The proof of the convergence of renormalization is based on K. Fraczek's approach as it is given in [Fra04], but the notation is adapted to that used in [Kri01].

The generalizations of the results are essentially two-fold, the first direction being of algebraic nature and the second one concerning regularity. By its definition, the algorithm does not depend on the fibers G of the product space, but only on the fact that the cocycle has only one frequency, since continued fractions algorithms are less efficient when more frequencies are involved. The tool is therefore ready to be applied to the study of cocycles in any compact Lie group, where the complexity of the observable phenomena outside the case studied so far is revealed (see the discussion on the splitting of the dynamics, cf. section 4.2.2). The second direction is based on a rather simple observation, which establishes the equivalence of the three different possible definitions of the degree for C^1 cocycles, and this allows us to prove the quantization of the degree for cocycles of such regularity. In fact, observation of the proof shows that the real regularity needed is H^1 and this regularity represents the threshold under which the degree of a cocycle cannot be defined. For this, see [Kat01] for the study of the "parabolic map" of $\mathbb{T}^2$, given by $(x, y) \to (x + \alpha, x + y)$. It is the additive version of the dynamics of periodic geodesics in $SU(2)$. Therein it is established that all invariant distributions of this mapping, other than the Haar measure on the torus, are in $H^{-1}(\mathbb{T}^2)$, the space of distributions in $\mathbb{T}^2$ that can be continuously extended to functionals on $H^1(\mathbb{T}^2)$.

Summing up, in this chapter we prove that the degree of a cocycle in $SW^1(\mathbb{T}, G)$, for any compact Lie group G, admits a definition in a way that naturally extends the degree as defined for cocycles in $SW^1(\mathbb{T}, SU(2))$. This degree is the topological degree of a periodic geodesic, viewed as a mapping of $\mathbb{T}$ into a maximal torus of G, and therefore takes discrete values. The periodic geodesics of the group are the normal forms which arise as topological obstructions to reducibility, thus marking the passage from local to global. Finally, the measurable invariance of the degree is proved to be true in regularity C^1 and persists in this more general algebraic setting.

6.1 Synopsis of the chapter

After having introduced the notion of $\mathbb{Z}^2$ actions and defined the renormalization algorithm for such actions, we generalize the definition of the energy of a cocycle to actions and subsequently study the convergence of the algorithm in order to obtain the following key theorems to the proof of the global density theorem.

The first one concerns the *quantization* of the length of the vectors belonging to the invariant curve $\mathbf{a}(\cdot)$ (see section 5.2 for the notation), which is the same as for abelian cocycles:

Theorem 6.1. *Let $(\alpha, A(\cdot)) \in SW^1(\mathbb{T}, G)$, with G a semisimple compact Lie group and w the dimension of its maximal tori. Then, the energy of the cocycle satisfies*

$$\mathrm{en}(\alpha, A(\cdot)) = 2\pi \| \sum\nolimits_{\rho \in \tilde{\Delta}} r_\rho h_\rho \|, \ r \in \mathbb{N}^w$$

where the vector $h = 2\pi \sum_{\rho \in \tilde{\Delta}} r_\rho h_\rho \in g$ is a preimage of the Id and $\mathbf{a}(\nu) \in Inn(g).h$ for a.e. $\nu \in \mathbb{T}$.

This theorem, whose proof is in p. 79, motivates the following definition

Definition 6.2. The vector $r \in \mathbb{N}^w$, defined modulo the action of Weyl's group and invariant under C^1 conjugation, is the degree of $(\alpha, A(\cdot))$. It is denoted by $\deg((\alpha, A(\cdot)))$ and it satisfies

$$\exp(2\pi \sum\nolimits_{\rho \in \tilde{\Delta}} r_\rho . h_\rho) = Id$$

The degree of an abelian cocycle coincides with the topological degree of the corresponding mapping $E_r(\cdot) : \mathbb{T} \to \mathcal{T}$, which justifies the terminology.

The following theorems concern the convergence of renormalization *per se*. For the definition of the unknown quantities in their statements (mostly the operators $(\Lambda_{\chi_0})_*$ and $\tilde{\mathcal{R}}^n_\nu$ acting on Φ), we address the reader to section 6.3.

The first theorem asserts that renormalization of a positive energy cocycle $(\alpha, A(\cdot))$, homotopic to constants, converges towards geodesics of speed equal to the degree of $(\alpha, A(\cdot))$:

Theorem 6.3. *Let $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $2 \leq s \leq \infty$, homotopic to the Id, and $\mathrm{en}(\alpha, A(\cdot)) > 0$. Then, for a.e. $\nu \in \mathbb{T}$, there exists a sequence of conjugations $D_{n,\nu}(\cdot) \in H^{s-1}(\mathbb{R}, G)$ satisfying the following properties: $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n_\nu\Phi$ is normalized and $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n\Phi(e_2)(\cdot)$ is arbitrarily close to*

$$A^{(n)}_\nu . \exp(\mathbf{a}(\nu)(\cdot))$$

in $H^{s-1}(\mathbb{T}, G)$, with $A^{(n)}_\nu \in \mathcal{Z}_\nu$. The set of such $\nu \in \mathbb{T}$ is invariant under R_α.

We have called $\mathcal{Z}_\nu = Z_G(\exp(\mathbb{R}\mathbf{a}(\nu)))$. For cocycles non-homotopic to constants, we have the following version of the theorem.

Theorem 6.4. *Let $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $2 \leq s \leq \infty$ and $\mathrm{en}(\alpha, A(\cdot)) > 0$. Then, for a.e. $\nu \in \mathbb{T}$, there exists a sequence of conjugations $D_{n,\nu}(\cdot) \in H^{s-1}(\mathbb{R}, G)$ satisfying the following properties: $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n_\nu\Phi$ is normalized, and $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n\Phi(e_2)(\cdot)$ is arbitrarily close to*

$$A^{(n)}_\nu . \exp(\mathbf{a}(\nu)\cdot) . A^{(n)}_{\nu,0}(\cdot)$$

in $H^{s-1}(\mathbb{T}, G)$. Here $A^{(n)}_{\nu,0}(\cdot) : \mathbb{T} \to G_0$ is a geodesic of minimal length in its homotopy class, of order χ in the homotopy group of G_0, and $A^{(n)}_\nu \in \mathcal{Z}_\nu$ anti-commutes with $A^{(n)}_{\nu,0}(\cdot)$: $A^{(n)}_\nu . A^{(n)}_{\nu,0}(\cdot) . (A^{(n)}_\nu)^* = A^{(n)}_{\nu,0}(\cdot)^*$. The renormalized action $Conj_{D_{n,\nu}(\cdot)}\Lambda_\chi . \tilde{\mathcal{R}}^n_\nu\Phi$, which corresponds to the renormalization representative of the χ-th iterate of $(\alpha, A(\cdot))$, satisfies the conclusions of the previous theorem. The constant χ satisfies $0 \leq \chi \leq \chi_0$, with χ_0 the constant of lemma 2.5 corresponding to G_0, the 0-energy component of the splitting associated to $(\alpha, A(\cdot))$. This constant is taken equal to 1 if $G_0 = \{Id\}$.

Thus, homotopy appears in the part of the dynamics which is of 0 energy, if it is not trivial.

The second couple of theorems concerns the cocycles of 0, and asserts that renormalization of 0-energy cocycles converges toward constant cocycles, again modulo iteration by χ times:

Theorem 6.5. *Let $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $2 \leq s \leq \infty$, homotopic to the Id, and $\mathrm{en}(\alpha, A(\cdot)) = 0$. Then, for every $\nu \in \mathbb{T}$, there exists a sequence of conjugations $D_{n,\nu}(\cdot) \in H^{s-1}(\mathbb{R}, G)$ such that $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n\Phi$ is normalized and $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n\Phi(e_2)(\cdot)$ is arbitrarily close to a constant.*

The version for cocycles non-homotopic to constants reads

Theorem 6.6. *Let $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $2 \leq s \leq \infty$ such that $(\alpha, A(\cdot))^m$ is homotopic to the Id. and $\mathrm{en}(\alpha, A(\cdot)) = 0$. Then, for every $\nu \in \mathbb{T}$, there exists a sequence of conjugations $D_{n,\nu}(\cdot) \in H^{s-1}(\mathbb{R}, G)$ such that $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n\Phi$ is normalized and $Conj_{D_{n,\nu}(\cdot)}\tilde{\mathcal{R}}^n\Phi(e_2)(\cdot)$ is arbitrarily close to $A_0.\tilde{A}(\cdot)$. Here, $\tilde{A}(\cdot)$ a geodesic of minimal length in the homotopy class of $A(\cdot)$ and $A_0 \in G$ anti-commutes with $\tilde{A}(\cdot)$. The action $Conj_{D_{n,\nu}(\cdot)}\Lambda_m.\tilde{\mathcal{R}}^n\Phi$ is arbitrarily close to a normalized constant action.*

We then proceed to a brief discussion on the preservation of the homotopy under renormalization before extending a result known since [Fra04], namely the dependence of the degree of the cocycle (now defined as a vector and not as a numerical quantity) on the rotation in the basis to this more general context.

The degree of a C^1-cocycle can be well defined, we have seen that it results immediately from the definitions that C^1 conjugations leave the degree invariant. However, using the convergence of renormalization, we can prove a stronger proposition:

Proposition 6.7. *Let $(\alpha, A_i(\cdot)) \in SW^1(\mathbb{T}, G)$, $i = 1, 2$, be conjugate by $B(\cdot) : \mathbb{T} \to G$ and let us suppose that $B(\cdot)$ is measurable. Then*

$$\deg(\alpha, A_1(\cdot)) = \deg(\alpha, A_2(\cdot))$$

The proof of this proposition occupies section *5.8*.

6.2 $\mathbb{Z}^2$ actions

In order to cope with the expected divergence in C^∞ of the approximation of the limit object $\mathbf{a}(\cdot)$, we will have to examine the dynamics in finer scales, which sums up to considering tori $\gamma^{-1}\mathbb{T} = \mathbb{R}/\gamma^{-1}\mathbb{Z}$, where γ goes to infinity in a controlled way, related to the arithmetic properties of α. In order to avoid such non-uniform objects, we consider cocycles in $SW^s(\mathbb{R}, G)$ and express periodicity in a more abstract way, which nonetheless proves to be practical for the study of the dynamics.

Let, therefore, $(\alpha, A(\cdot))$ and $(\beta, B(\cdot))$ be commuting cocycles in $SW^s(\mathbb{R}, G)$, with $\alpha, \beta \in (0, 1]$, not necessarily irrational. Commutation of these cocycles sums up to the following relation satisfied by the mappings $A(\cdot)$ and $B(\cdot)$

$$B(\cdot + \alpha).A(\cdot) = A(\cdot + \beta).B(\cdot)$$

We can then define an *abelian action* of $\mathbb{Z}^2$ on $\mathbb{R} \times G$ [1] by

$$\Phi\,(k, l)\,.\,(x, S) = (\alpha, A(\cdot))^k \circ (\beta, B(\cdot))^l.\,(x, S)$$

The space of such actions will be denoted by $\Lambda^s(\mathbb{R}, G)$ and we will say that the action Φ is generated by $(\alpha, A(\cdot))$ and $(\beta, B(\cdot))$. We will denote by $e_1 = (1, 0)$ and $e_2 = (0, 1)$ the standard basis of the module $\mathbb{Z}^2$. We will also use the notation

$$\begin{pmatrix} (\alpha, A(\cdot)) \\ (\beta, B(\cdot)) \end{pmatrix}$$

for such an action Φ.

Definition 6.8. An action will be called *constant* if $\Phi\,(k, l)$ is a constant cocycle for every $(k, l) \in \mathbb{Z}^2$. It will be called *normalized* if $\Phi(e_1) = (1, Id)$, in which case $\pi_2(\Phi(e_2))$ is automatically 1-periodic. A cocycle $(\alpha, A(\cdot))$ with $A(\cdot)$ 1-periodic is thus naturally identified with the action generated by $(1, Id)$ and $(\alpha, A(\cdot))$. We will say that this action is *associated* to the cocycle $(\alpha, A(\cdot))$.

An action is constant iff its generators are constant, and we remind this fact since we will examine actions through their generators, and not abstractly.

The space $C^s(\mathbb{R}, G)$ acts on Λ^s by the generalization of conjugation in $SW^s(\mathbb{T}, G)$: if $B(\cdot) \in C^s(\mathbb{R}, G)$, then $\Phi' = Conj_{B(\cdot)}\Phi$ is given by

$$\Phi'(k, l) = Conj_{B(\cdot)}(\Phi(k, l)),\ \forall\ (k, l) \in \mathbb{Z}^2$$

and this gives rise to the following natural definitions.

Definition 6.9. Two actions will be called *conjugate* if they belong to the same orbit under the action of conjugacies.

An action will be called *reducible* if it is conjugate to a normalized constant action, and *torus-reducible* if it is conjugate to a normalized action such that $\pi_2(\Phi(e_2))$ takes its values on a torus of G, which with no loss of generality can be assumed to be a fixed torus $\mathcal{T}$.

Naturally, two actions are conjugate iff their generators are, but with transfer functions in $\mathbb{R} \to G$.

If $d(\cdot)$ is a (semi)metric on a space of applications taking values in G, then it induces a (semi)metric on Λ^s defined by

$$\max_{i=1,2} d(\pi_2(\Phi_1(e_i)), \pi_2(\Phi_2(e_i)))$$

i.e. the max of the distances of their generators. In the special case of the $C^s(I)$ topology, where $I \subset \mathbb{R}$ or $\mathbb{T}$ and $s \geq 0$, we will use the notations

$$\begin{aligned} \|\Phi\|^0_{s,I} &= \max_{i=1,2} \|\pi_2(\Phi(j_i))\|_{s,I} \\ d_{s,I}(\Phi_1, \Phi_2) &= \max_{i=1,2} \|\pi_2(\Phi_1(j_i)) - \pi_2(\Phi_2(j_i))\|_{s,I} \end{aligned}$$

1. and likewise on the product of $\mathbb{R}$ with any space on which G acts.

and

$$\begin{aligned} \|\Phi\|_{s,I} &= \max_{1\le\sigma\le s} \|\Phi\|_{\sigma,I} \\ d_{s,I}(\Phi_1,\Phi_2) &= \max_{1\le\sigma\le s} d_{\sigma,I}(\Phi_1,\Phi_2) \end{aligned}$$

We can now prove

Lemma 6.10. *Let $\Phi \in \Lambda^s(\mathbb{R},G)$ such that $\pi_1(\Phi(1,0)) = 1$. Then there exists $B(\cdot) \in C^s(\mathbb{R},G)$ such that $Conj_{B(\cdot)}\Phi$ is normalized. $B(\cdot)$ satisfies, for $1 \le \sigma \le s-1$ and $T \in \mathbb{N}^*$, the estimates*

$$\|B(\cdot)\|_{\sigma,T} \le C_\sigma T^\sigma (1 + \|\pi_2(\Phi(e_1)\|_{\sigma,T}) \, \|\pi_2(\Phi(e_1))\|_{0,T}$$

and the normalized action satisfies

$$\|Conj_{B(\cdot)}\Phi(e_2)\|_{s,1} \le C_s(1 + \|\pi_2(\Phi(e_1))\|_{1,2}) \, \|\pi_2(\Phi(e_2)\|_{s,1}^{\max} + \|\pi_2(\Phi(e_1)\|_{\sigma,2}^{\max} \, \|\pi_2(\Phi(e_1))\|_{1,1}$$

Proof. In order to prove the first part, we need to solve the equation

$$B(\cdot+1) = B(\cdot).C^*(\cdot) \tag{6.1}$$

where $C(\cdot) = \pi_2(\Phi(1,0))$. Let us fix $B_0(\cdot) \in C^s([0,1],G)$ such that

$$\begin{aligned} B_0(0) &= Id \text{ and } \partial^\sigma B_0(0) = 0,\ 1 \le \sigma \le s \\ \partial^\sigma B_0(\cdot)|_{t=1} &= \partial^\sigma (B_0(\cdot).C^*(\cdot))|_{t=0},\ 0 \le \sigma \le s \\ \|B_0(\cdot)\|_{\sigma,1} &\le C_\sigma \, \|C(\cdot)\|_{\sigma,1}\,,\ 1 \le \sigma \le s-1 \end{aligned}$$

Then, define for all $j \in \mathbb{Z}$

$$B(\cdot+j) = B_0(\cdot).C_j^*(\cdot)$$

It can be verified inductively that $B(\cdot)$ is $C^s(\mathbb{R},G)$ and by its very definition it solves the equation (6.1). Moreover, if $\sigma, T \in \mathbb{N}$, it satisfies

$$\|B(\cdot)\|_{\sigma,[T,T+1]} = \|B_0(\cdot).C_T(\cdot)\|_{\sigma,1}$$

and the estimate can be proved by imitating the proof of lemma 5.9. □

Since $B(\cdot) \in C^s(\mathbb{R},G)$ is 1-periodic if $Conj_{B(\cdot)}(1,Id) = (1,Id)$, the following lemma is an immediate consequence of the definitions and the previous lemma.

Lemma 6.11. *A cocycle is (torus-)reducible if, and only if, the $\mathbb{Z}^2$ action associated to it is (torus-)reducible. Likewise, it is accumulated by (torus-)reducible cocycles if, and only if, the corresponding action is.*

We remark that, in view of the proof of the previous lemma, an action is torus-reducible if it is conjugate to an action Φ' such that $\pi_2(\Phi'(e_1))$ and $\pi_2(\Phi'(e_2))$ take their values on the same torus, and $\pi_1(\Phi'(e_1)) = 1$.

However, unlike with actions associated to cocycles in, say $\mathbb{T} \times SU(w+1)$, we cannot always normalize a constant action to another constant one for the following reason. If

$$\Phi = \begin{pmatrix} (1,C) \\ (\alpha,A) \end{pmatrix}$$

is a constant action, then C and A commute. However, if G is not simply connected, then they need not belong to the same maximal torus. A natural conjugation normalizing the cocycle is $B(\cdot) = \exp(-H\cdot)$, where $\exp(H) = C$. But, unless $Ad(A).H = H$,

$$\Phi' = \begin{pmatrix} (1,Id) \\ (\alpha, \exp(-H(\cdot+\alpha)).A.\exp(H\cdot)) \end{pmatrix}$$

is not constant, but with a good choice of the conjugant it is a geodesic of minimal length in its class, as shows the proof of lemma 2.7. Normalization can be achieved by considering the action

given by a sub-lattice of $\mathbb{Z}^2$. If $(\alpha, \exp(-H(\cdot + \alpha)).A.\exp(H\cdot))^m$ is homotopic to constants, then there exists a preimage $H' \in g$ of C and $0 \leq \chi \leq \chi_G$ such that $Ad(A^\chi).H' = H'$. Therefore, if we consider the action $\tilde{\Phi}$ generated by $(1, C)$ and $(\alpha, A)^\chi$, and conjugate by $B'(\cdot) = \exp(-H'\cdot)$, we obtain the constant action

$$Conj_{B'(.\ )}\tilde{\Phi} = \begin{pmatrix} (1, Id) \\ (\chi\alpha, \exp(-\chi H'\alpha).A^\chi) \end{pmatrix}$$

Therefore, we have the following lemma.

Lemma 6.12. *Let Φ be a constant action with generators $(1, C)$ and (α, A). This action can be normalized to a constant action iff A and C belong to a common maximal torus. If not, it can be normalized to an action Φ' associated to a cocycle $(\alpha, A(\cdot))$ non-homotopic to constants, and more precisely to a geodesic of minimal length in its class. The minimal $\chi \in \mathbb{N}$ such that C and A^χ belong to a common maximal torus and the order of $A(\cdot)$ in the homotopy group of G are the same number.*

The action generated by $(1, C)$ and $(\alpha, A)^\chi$, corresponding to the sub-action $\mathbb{Z} \oplus \chi\mathbb{Z} \hookrightarrow \mathbb{Z}^2$ can be normalized to a constant action.

This last action is $\Lambda_\chi.\Phi$ in the notation of the next section. Since normalized actions are associated to cocycles conjugate to each other (cf. proof of lemma 6.20, and since homotopy is preserved by fibered conjugation, a different choice of conjugant will not solve the problem. Either we do not normalize the action to a constant one, or we consider an iterate of the cocycle, the action associated to which can be normalized to a constant one.

6.3 Renormalization of actions

We now present the basic tool for the study of the dynamics of cocycles via their identification with the corresponding $\mathbb{Z}^2$-action. Given the irrational number α, we generate an infinity of base-changes of the module $\mathbb{Z}^2$, and these base-changes are pulled back to changes of generators of the $\mathbb{Z}^2$- action corresponding to $(\alpha, A(\cdot))$. These successive base-changes, given by the continued fractions algorithm applied to α, provide us with a subsequence of iterates of $(\alpha, A(\cdot))$, which, being adapted to α, reflects its arithmetic properties, and whose derivatives converge to $\mathbf{a}(\cdot)$.

We now define the actions that are used in the renormalization:

- $GL(2, \mathbb{Z})$ acts on Λ^s by change of basis of the module $\mathbb{Z}^2$. If $\Phi \in \Lambda^s$ and $P \in GL(2, \mathbb{Z})$, then $P_*\Phi(k, l) = \Phi(P^{-1}.(k, l)),\ \forall\ (k, l) \in \mathbb{Z}^2$, i.e. $P_*\Phi$ is generated by $P^{-1}.e_1$ and $P^{-1}.e_2$
- $\mathbb{R}$ acts by translations in the basis: for $\nu \in \mathbb{R}$:

  $$\pi_2(T_\nu\Phi(k, l)) = \pi_2(\Phi((k, l))(. + \nu),\ \forall\ (k, l) \in \mathbb{Z}^2$$

  and we remind that π_2 is the projection on the mapping defining the cocycle, defined right after π_1, the projection on the frequency (cf. eq. 3.1).
- $\mathbb{R}_+^*$ acts by dilatations of the basis: if $\lambda \in \mathbb{R}_+^*$, then

  $$\begin{aligned} \pi_1(M_\lambda\Phi(k, l)) &= \lambda^{-1}\pi_1(\Phi((k, l))(\cdot) \\ \pi_2(M_\lambda\Phi(k, l)) &= \pi_2(\Phi((k, l))(\lambda\cdot),\ \forall\ (k, l) \in \mathbb{Z}^2 \end{aligned}$$

- We also define the action of the semigroup $\mathbb{N}^*$ by the restriction to the a sub-lattice. If $m \in \mathbb{N}^*$, the action $\Lambda_m.\Phi$ is generated by $\Phi(e_1)$ and $\Phi(me_2)$.

We note that the first three actions commute with each other, since dilatations and translations commute with iteration. Conjugation commutes with base changes, while it satisfies the following invariance properties

$$\begin{aligned} T_\nu \circ Conj_{B(\cdot)} &= Conj_{B(\cdot+\nu)} \circ T_\nu \\ M_\lambda \circ Conj_{B(\lambda\cdot)} &= Conj_{B(\cdot)} \circ M_\lambda \end{aligned}$$

and he equalities

$$\|M_\lambda\Phi\|_{s,T} = \lambda^s \|M_\lambda\Phi\|_{s,\lambda^{-1}T}$$

follow directly from the definitions.

Since the loss of information described by lemma 6.12 is inherent, we give the following definition

Definition 6.13. Given a $\mathbb{Z}^2$ action Φ and $m \in \mathbb{N}^*$, the action $\Lambda_m.\Phi$ will be called the m-lattice of Φ. An action Φ' such that $\Lambda_m.\Phi$ is normalized will be called normalizable modulo m.

It can be seen directly that the action Φ is (torus-)reducible (resp. accumulated by reducible actions) iff $M_\lambda P_* \Phi$ is (torus-)reducible (resp. accumulated by reducible actions) for some (and thus for all) $P \in GL(2,\mathbb{Z})$, and $\lambda = \pi_1(P_*\Phi(1,0))$, and the same holds for T_ν. We can therefore generalize the different notions of reducibility of $\mathbb{Z}^2$-actions as defined and used above to

Definition 6.14. An action Φ will be called reducible if it is conjugate to a constant action Φ' such that $\pi_2(\Phi'(e_1)) = Id$, and torus-reducible if it is conjugate to an action Φ' such that $\pi_2(\Phi'(e_1))$ and $\pi_2(\Phi'(e_2))$ take their values on the same maximal torus $\mathcal{T} \subset G$.

It will be called (torus-)reducible modulo $m \in \mathbb{N}^*$ iff $\Lambda_m.\Phi$ is (torus-)reducible.

Given an action Φ such that $\pi_1(\Phi(e_1)) = 1$, the n-th renormalized action can now be defined as

$$\mathcal{R}^n\Phi = M_{\beta_{n-1}} \circ (Q_n)_*\Phi$$

where, following the notations introduced in section *1.3*,

$$Q_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$$

is in $GL(2,\mathbb{Z})$ and the Q_n satisfy the recursive relation

$$Q_n = Q_{n-1}\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = Q_{n-1}A_n$$

so that

$$\tilde{\mathcal{R}}^n\Phi = M_{\alpha_{n-1}} \circ (A_n)_*\mathcal{R}^{n-1}\Phi$$

For the action Φ associated to the cocycle $(\alpha, A(\cdot))$,

$$\begin{aligned}
\tilde{\mathcal{R}}^n\Phi(e_2) &= (\alpha_n, A_{(-1)^n q_n}(\beta_{n-1}\cdot)) = (\alpha_n, A^{(n)}(\beta_{n-1}\cdot)) = (\alpha_n, \tilde{A}^{(n)}(\cdot)) \\
\tilde{\mathcal{R}}^n\Phi(e_1) &= (1, A_{(-1)^n q_{n-1}}(\beta_{n-1}\cdot)) = (1, A^{(n-1)}(\beta_{n-1}\cdot)) = (1, \tilde{C}^{(n)}(\cdot)) \\
(\beta_n, A^{(n)}(\cdot)) &= (\beta_{n-1}, A^{(n-1)}(\cdot))^{-a_n} \circ (\beta_{n-2}, A^{(n-2)}(\cdot))
\end{aligned}$$

We will also use the notations

$$\begin{aligned}
\tilde{\mathcal{R}}^n_\nu\Phi &= T_{-\nu}\tilde{\mathcal{R}}^n(T_\nu\Phi) \\
\mathcal{R}^n_\nu\Phi &= T_{-\nu}(Q_n)_*(T_\nu\Phi)
\end{aligned}$$

and omit the subscript when $\nu = 0$.

The actions $\tilde{\mathcal{R}}^n_\nu\Phi$ can be normalized in a non canonical way: there exists $B(\cdot) = B_{n,\nu}(\cdot) \in C^\infty(\mathbb{R}, G)$ such that $Conj_{B(\cdot)}\tilde{\mathcal{R}}^n_\nu\Phi$ is normalized.

Definition 6.15. For an action Φ associated to the cocycle $(\alpha, A(\cdot))$, we will call such a $Conj_{B(\cdot)}\tilde{\mathcal{R}}^n_\nu\Phi(e_2)$ a *renormalization representative* of $(\alpha, A(\cdot))$.

The choice of a renormalization representative is non-canonical, since it depends on the choice of $B(\cdot)$, but since different renormalization representatives are cocycles conjugate to each other (cf. proof of lemma 6.20), the indeterminacy is not important for the results that we seek to obtain. However, we will see that when renormalization goes deep enough, so that $\mathbf{a}(\cdot)$ is approximated well enough, it indicates a natural way of normalizing the actions.

Similarly, we call a renormalization representative $\mod m$ the cocycle $Conj_{B(\cdot)}\Lambda_m.\tilde{\mathcal{R}}^n_\nu\Phi(e_2)$, when this last action is normalized $\mod m$.

From the above, it follows that

Proposition 6.16. *The cocycle $(\alpha, A(\cdot))$ is (torus-)reducible (resp. accumulated by (torus-)reducible cocycles) if there exist n and ν such that $\mathcal{R}^n_\nu\Phi$ is (torus-)reducible (resp. accumulated by (torus-)reducible cocycles), where Φ is the action associated to $(\alpha, A(\cdot))$.*

We remark that the action associated to a cocycle is (torus-)reducible mod m, iff the m-th iterate of the cocycle is (torus-)reducible. We also need to make clear that renormalization does not commute with the action of $\mathbb{N}^*$. This is because the continued fractions of $m\gamma$ and those of γ are not related. They become related if we place $m\gamma$ in $m\mathbb{T}$, which we will do in the proof of our main theorem.

Using lemmas 5.9 and 5.10, we can prove the a priori estimates

Proposition 6.17. *For the action Φ associated to the cocycle $(\alpha, A(\cdot))$, we have*

$$\begin{aligned} \left\|\tilde{\mathcal{R}}_\nu^n\Phi\right\|_{\sigma,\mathbb{R}}^{\max} &\lesssim K_\sigma \\ \left\|\partial^\sigma\tilde{c}^{(n)}(\cdot)\right\|_{L^2(I)} &\to 0 \\ \left\|\partial^\sigma\tilde{a}^{(n)}(\cdot)\right\|_{L^2(I)} &\to 0 \end{aligned}$$

for all $1\le\sigma\le s-1$ and any compact interval I and for any $\nu\in\mathbb{T}$.

Proof. As we have seen, for any $n\in\mathbb{Z}^*$, the derivative of $A_n(\cdot)$ satisfies

$$\|a_n(\cdot)\|_0 \le n\,\|a(\cdot)\|_0$$

In particular, $\|a^{(n)}(\cdot)\|_0 = \|LA^{(n)}(\cdot)\|_0 \le q_n\,\|a(\cdot)\|_0$, and since

$$\tilde{a}^{(n)}(\cdot) = \beta_{n-1}a^{(n)}(\beta_{n-1}\cdot)$$

and $q_n\beta_{n-1}<1$, the result is proved for the C^0 norms of the first derivatives. Similarly,

$$\|\partial a_n(\cdot)\|_0 \lesssim n\,\|\partial a(\cdot)\|_0 + n^2\,\|a(\cdot)\|_0^2$$

Since, now, $\partial\tilde{a}^{(n)}(\cdot) = \beta_{n-1}^2\partial a^{(n)}(\beta_{n-1}\cdot)$, the estimate follows as before.

Finally, the convergence to 0 in $L^2(I)$ for $\partial^\sigma\tilde{a}^{(n)}(\cdot)$, $\sigma\ge 1$, follows from the fact that

$$\begin{aligned} \partial\tilde{a}^{(n)}(\cdot) &= \beta_{n-1}^2\partial a^{(n)}(\beta_{n-1}\cdot) \\ &= q_n^2\beta_{n-1}^2\partial\mathbf{a}^{(n)}(\beta_{n-1}\cdot) \end{aligned}$$

and $\partial\mathbf{a}^{(n)}(\cdot)\to 0$ in $L^2(\mathbb{T})$, so that $\partial\mathbf{a}^{(n)}(\beta_{n-1}\cdot)$ in $L^2(I)$, for any compact interval I.

The estimates on $\tilde{c}^{(n)}(\cdot)$ and its derivatives are obtained in exactly the same way, and we just remark that the numerical factor that gains convergence for $\tilde{c}^{(n)}(\cdot)$ (as does $q_n\beta_{n-1}$ for $\tilde{a}^{(n)}(\cdot)$) is now $q_{n-1}\beta_{n-1} = \alpha_{n-1}q_{n-1}\beta_{n-2}$, which is in fact more than what is needed in order to establish convergence. The factor α_{n-1} will be interpreted as a scaling factor in the proof of the convergence of the scheme. □

A remark

A. Avila's theory of $SL(2,\mathbb{C})$ cocycles (cf. [Avi09]), gives immediately the following quantization theorem for analytic cocycles in $\mathbb{T}\times SU(2)$. Before stating it, we remind that $SU(2)$ complexifies to $SL(2,\mathbb{C})$ and remark that, if $A(\cdot)\in C_\delta^\omega(\mathbb{T},SU(2))$, i.e. if $A(\cdot)$ is analytic and admits a holomorphic extension to the band $|\Re(z)|<\delta$, then

$$\frac{d}{d\varepsilon}\,\|A(x_0+\varepsilon i)\|\,|_{\varepsilon=0^+} = |\pi_{\mathfrak{t}}(a(x_0))|$$

where $\mathfrak{t}$ is a maximal torus [2] passing by $A(x_0)$ and $\|\cdot\|$ stands for the operator norm associated to the Hermitian norm on $\mathbb{C}^2$. This relation holds for the following simple reason. If we assume that

$$A(0) = \{A_{\mathfrak{t}}(0), 0\}$$

is diagonal and $A_{\mathfrak{t}}(x) = exp(is_0 + is_1x + O(x^2))$, then the increment of the norm as we pass to the complex plane is given by the increment of the real part of $A_{\mathfrak{t}}(0+i\varepsilon)$ which is precisely s_1. The general case follows.

2. In fact in the case where $A(x_0)=\pm Id$ one should consider the maximum of such projections on all maximal tori, which is equal to the norm of the derivative

Theorem 6.18. *The energy of analytic cocycles in $\mathbb{T} \times SU(2)$ is quantized in $2\pi\mathbb{N}$.*

Proof. Suppose that the curve $\mathbf{a}(\cdot)$ is non-degenerate, so that the cocycle is measurably diagonalizable. Moreover, for *a.e.* point x_0,

$$\frac{1}{q_n}\frac{d}{d\varepsilon}\|A_{q_n}(x_0+\varepsilon i)\|\,|_{\varepsilon=0^+} \underset{n\to\infty}{\to} |\mathbf{a}(x_0))|$$

This fact implies that the acceleration of a cocycle, as defined in [Avi09], is equal to its energy as has been defined herein. In the same paper by A. Avila, it is shown that this quantity is quantized in $2\pi\mathbb{N}$. □

As we will see later on, this fact persists in more general contexts and in lower regularity.

6.4 Energy of actions

Using lemma 6.10, we can define the energy of a $\mathbb{Z}^2$ action, in a way that generalizes the definition given for cocycles:

Definition 6.19. The energy of a normalized action Φ equals the energy of $\Phi(e_2)$, i.e.

$$en(\Phi) = en(\Phi(e_2))$$

If $M_\lambda\Phi$ is normalized, then

$$en(\Phi) = \lambda^{-1}.en(M_\lambda\Phi(e_2))$$

The pertinence of the definition is established by the following lemma.

Lemma 6.20. *The definition of the energy does not depend on the choice of the conjugacy that normalizes the action, provided that it is C^1.*

Proof. Let $B_i(\cdot) \in C^1(\mathbb{R}, G)$, $i = 1, 2$ be such that $\Phi_i' = Conj_{B_i(\cdot)}\Phi$ be normalized. Then, if we let $B(\cdot) = B_1(\cdot).B_2^*(\cdot)$, it satisfies $Conj_{B(\cdot)}(1, Id) = (1, Id)$, and therefore it is 1-periodic. Since $\Phi_1' = Conj_{B(\cdot)}\Phi_2'$, which implies that $\Phi_1'(e_2) = Conj_{B(\cdot)}\Phi_2'(e_2)$, and the energy of a cocycle is invariant under the action of C^1 periodic conjugations (by proposition 5.3), the result has been proved. □

The following corollary is immediate.

Corollary 6.21. *The energy of an action is invariant under C^1 conjugation of actions.*

The energy of an action is related with the renormalization scheme in the following way (notice that we need only consider renormalization of normalized actions):

Lemma 6.22. *The sequences d_n^i given by*

$$d_n^i = \beta_n\|a^{(n-1)}(\cdot)\|_{L^i(\mathbb{T})} + \beta_{n-1}\|a^{(n)}(\cdot)\|_{L^i(\mathbb{T})}$$

for $i = 1, 2$ are decreasing and converge to $\mathrm{en}(\Phi)$. As a consequence, the energy of an action is invariant under renormalization:

$$\mathrm{en}(\Phi) = \mathrm{en}(\tilde{\mathcal{R}}_\nu^n\Phi)$$

for all $n \in \mathbb{N}^*$ and $\nu \in \mathbb{T}$. Moreover, the energy of a cocycle is equal to the energy of any of its renormalization representatives.

Proof. By virtue of the recurrence relation satisfied by $\mathcal{R}^n\Phi$ and the triangle inequality, we find that

$$\|a^{(n)}(\cdot)\|_{L^i} \le \|a^{(n-2)}(\cdot)\|_{L^i} + a_n\|a^{(n-1)}(\cdot)\|_{L^i}$$

Since $\beta_{n-2} = a_n\beta_{n-1} + \beta_n$, we find

$$\beta_{n-1}\|a^{(n)}(\cdot)\|_{L^i} + \beta_n\|a^{(n-1)}(\cdot)\|_{L^i} \le \beta_{n-1}\|a^{(n-2)}(\cdot)\|_{L^i} + \beta_{n-2}\|a^{(n-1)}(\cdot)\|_{L^i}$$

and the sequences d_n^i are decreasing. Since $\frac{1}{q_n}\|a^{(n)}(\cdot)\|_{L^i} \to en(\Phi)$ for $i = 1, 2$, and since $\beta_{n-1}q_n + \beta_n q_{n-1} = 1$, we immediately obtain the proof of the first part of the lemma.

The second part follows from the first and the invariance of the energy under smooth enough conjugation. □

This quantity was introduced in [Kri01] and was used in the proof of the convergence of the renormalization scheme. The observation that this quantity equals the degree is due to K. Fraczek. Another important quantity used in the calculation of the energy of an action is its original definition by R. Krikorian as the length functional J^1 defined as follows. Let us consider the sequence

$$\begin{aligned} J_n^1(\nu) &= \int_\nu^{\nu+\beta_{n-1}} |a^{(n)}(\cdot)| + \int_\nu^{\nu+\beta_n} |a^{(n-1)}(\cdot)| \\ &= \int_0^1 |\tilde{a}_\nu^{(n)}(\cdot)| + \int_0^{\alpha_n} |\tilde{a}_\nu^{(n-1)}(\cdot)| \end{aligned}$$

We can prove immediately

Lemma 6.23. *For $\nu \in \mathbb{T}$ fixed, $(J_n^1(\nu))_n$ is decreasing, and $(J_n^1(\nu))_n$ converges uniformly to the energy of the action.*

Proof. We calculate

$$\begin{aligned} J_n^1(\nu) &= \int_\nu^{\nu+\beta_{n-1}} |a^{(n)}(\cdot)| + \int_\nu^{\nu+\beta_n} |a^{(n-1)}(\cdot)| \\ &\leq \int_\nu^{\nu+\beta_{n-1}} \sum_1^{a_k} |a^{(n-1)}(\cdot + \beta_{n-2} - j\beta_{n-1})| + \int_\nu^{\nu+\beta_{n-1}} |a^{(n-2)}(\cdot)| + \int_\nu^{\nu+\beta_n} \left|a^{(n-1)}(\cdot)\right| \\ &\leq \int_\nu^{\nu+\beta_{n-2}} |a^{(n-1)}(\cdot)| + \int_\nu^{\nu+\beta_{n-1}} |a^{(n-2)}(\cdot)| = J_{n-1}^1(\nu) \end{aligned}$$

which proves the first part.

We call

$$J^1(\nu) = \lim_{n\to\infty} J_n^1(\nu)$$

which is defined for all $\nu \in \mathbb{T}$.

Let us now observe that $A^*(\cdot + (-1)^n q_n \alpha).A^{(n)}(\cdot) = A^{(n)}(\cdot - \alpha).A^*(\cdot)$, so that

$$\left||a^{(n)}(\cdot)| - |a^{(n)}(\cdot - \alpha)|\right| \leq |a^*(\cdot)| + |a^*(\cdot + (-1)^n q_n \alpha)| \leq 2\,\|a^*(\cdot)\|_{L^\infty}$$

and we find

$$\begin{aligned} |J_n^1(\nu) - J_n^1(\nu+\alpha)| &\leq \left|\int_\nu^{\nu+\beta_{n-1}} |a^{(n)}(\cdot)| - |a^{(n)}(\cdot - \alpha)|\right| + \left|\int_\nu^{\nu+\beta_n} |a^{(n-1)}(\cdot)| - |a^{(n-1)}(\cdot - \alpha)|\right| \\ &\leq 2\,\|a^*(\cdot)\|_{L^\infty}\,(\beta_{n-1} + \beta_n) \end{aligned}$$

and, by the definition of $J^1(\nu)$, $J^1(\nu + \alpha) = J^1(\nu)$, for all $\nu \in \mathbb{T}$. Since $J^1(\cdot)$ is defined as a pointwise limit of a decreasing sequence of continuous functions, and the invariance relation holds everywhere, $J^1(\cdot) = J^1$, a constant. We now find that

$$\begin{aligned} \|J_n^1(\cdot)\|_{L^1} &= \int_{\mathbb{T}} J_n^1(\nu) d\nu \\ &= \beta_{n-1}\|a^{(n)}(\cdot)\|_{L^1} + \beta_n \|a^{(n-1)}(\cdot)\|_{L^1} \end{aligned}$$

by Fubini's theorem and the invariance of the Haar-Lebesgue measure on the torus. The Lebesgue monotone convergence theorem and the previous lemma conclude the proof of convergence of $J_n^1(\cdot)$ in L^1. Since $(J_n^1(\cdot))$ is a sequence of continuous functions on a compact space converging pointwise to a continuous function, the convergence is in fact uniform. □

The proof of this lemma is essentially as in [Fra04], except for this last observation, which is the reason for the improvement of our results in terms of regularity. The functional $J^1(\cdot)$ is the quantity used in the proof of the convergence of the renormalization scheme, where it appears as the length of the path of $\mathcal{R}_\nu^n \Phi$ after the natural normalization, and by this observation we relate it *a priori* to the energy of the cocycle, whereas in K. Fraczek's article they were related *a posteriori*.

The factors in the definition of $J_n^1(\cdot)$ in this last lemma, are exactly the respective lengths of the curves $\tilde{R}_\nu^n.e_1(\cdot)$ and $\tilde{R}_\nu^n.e_2(\cdot)$ for times in $[0,\alpha_n]$ and $[0,1]$ respectively. The factor α_n appears as the rescaling factor which we encountered in the proof of 6.17. For these reasons, the argument ν is to be considered as the base point of renormalization, i.e. the point around which we can rescale and obtain a good image of the global dynamics. The fact that the choice of the point is irrelevant is justified by the integration in the proof, which connects the pointwise information of the functional $J_n^1(\cdot)$ with the global (in $\mathbb{T}$) information of d_n^1.

6.5 Convergence of the scheme

As indicates the conclusion of the previous section, the sequence

$$u_n(\nu) = (-1)^n \int_\nu^{\nu+\beta_{n-1}} a^{(n)}(\cdot) - (-1)^n \int_\nu^{\nu+\beta_n} a^{(n-1)}(\cdot)$$

should be expected to converge to $\mathbf{a}(\nu)$. The study of this fact, which in other words is the invariance of the limit object of the dynamics under renormalization, is the object of the section. We begin by the following proposition.

Proposition 6.24. *The sequence $(u_n(\cdot))$ converges to $\mathbf{a}(\cdot)$ in L^i, $i = 1, 2$.*

Proof. Using the notation $\frac{1}{q_n} a^{(n)}(\cdot) = \mathbf{a}^{(n)}(\cdot)$, we have

$$\begin{aligned} \left| \int_\nu^{\nu+\beta_{n-1}} a^{(n)}(\cdot) \right| &\leq q_n \int_\nu^{\nu+\beta_{n-1}} |\mathbf{a}^{(n)}(\cdot)| \\ &\leq q_n \beta_{n-1} \sup_n \|\mathbf{a}^{(n)}(\cdot)\|_{L^\infty} \end{aligned}$$

which is finite, as has already been proved. Thus, $(u_n(\cdot))$ is uniformly bounded in L^∞.

Using the fact that $q_n\beta_{n-1} + q_{n-1}\beta_n = 1$, we find

$$u_n(\nu) - \mathbf{a}(\nu) = (-1)^n \int_\nu^{\nu+\beta_{n-1}} a^{(n)}(\cdot) - q_n\beta_{n-1}\mathbf{a}(\nu) - (-1)^n \int_\nu^{\nu+\beta_n} a^{(n-1)}(\cdot) - q_{n-1}\beta_n \mathbf{a}(\nu)$$

Let us assume for simplicity that n is even. Then

$$\begin{aligned} \left| \int_\nu^{\nu+\beta_{n-1}} a^{(n)}(y)dy - q_n\beta_{n-1}\mathbf{a}(\nu) \right| &\leq q_n \int_\nu^{\nu+\beta_{n-1}} |\mathbf{a}^{(n)}(y) - \mathbf{a}(\nu)| dy \\ &\leq q_n \int_\nu^{\nu+\beta_{n-1}} |\mathbf{a}^{(n)}(y) - \mathbf{a}^{(n)}(\nu)| + |\mathbf{a}^{(n)}(\nu) - \mathbf{a}(\nu)| dy \\ &\leq \frac{1}{\beta_{n-1}} \int_\nu^{\nu+\beta_{n-1}} |\mathbf{a}^{(n)}(y) - \mathbf{a}^{(n)}(\nu)| + |\mathbf{a}^{(n)}(\nu) - \mathbf{a}(\nu)| \end{aligned}$$

and, since $\mathbf{a}^{(n)}(\cdot) \to \mathbf{a}(\cdot)$ a.e., we need only to prove the same thing for the first term. This done in lemma 11.2 in Appendix 11.

The other term can be estimated in the same way in both arguments, and likewise the case of an odd n in the second one and therefore, $(u_n(\cdot))$ converges almost everywhere to $\mathbf{a}(\cdot)$. The bounded convergence theorem concludes the proof of convergence in L^1, and the uniform boundedness of the sequence grants automatically convergence in L^2. □

Corollary 6.25. *For a.e. $\nu \in \mathbb{T}$,*

$$\int_\nu^{\nu+\beta_{n-1}} a^{(n)}(\cdot) - (-1)^n q_n \beta_{n-1} \mathbf{a}(\nu) \to 0$$

and the set where there is convergence, denoted by $\mathcal{E}$, is invariant by R_α.

Let us point out that for all $\nu \in \mathbb{T}$,

$$\begin{aligned} |u_n(\nu)| &= |\int_\nu^{\nu+\beta_{n-1}} a^{(n)}(\cdot) - \int_\nu^{\nu+\beta_n} a^{(n-1)}(\cdot)| \\ &\leq \int_\nu^{\nu+\beta_{n-1}} |a^{(n)}(\cdot)| + \int_\nu^{\nu+\beta_n} |a^{(n-1)}(\cdot)| \\ &= J_n^1(\nu) \end{aligned}$$

This inequality implies directly that

$$\limsup |u_n(\nu)| \leq en(\alpha, A(\cdot)),\ \forall\ \nu \in \mathbb{T}$$

and, if the energy of the cocycle is 0, $u_n(\nu)$ converges to 0 in C^0 topology.

We now use the commutation of the generators and their asymptotic commutation with $\mathbf{a}(\cdot)$. We begin by remarking that in the general case where $\gamma(\cdot)$ is a curve defined locally around 0 and taking values in g, the integral of the ODE

$$\begin{aligned} L\Gamma(\cdot) &= \gamma(\cdot) \\ \Gamma(0) &= \Gamma_0 \end{aligned}$$

does not satisfy

$$\Gamma(\nu) = \exp(\int_0^\nu \gamma(\cdot)).\Gamma_0$$

unless $\int_0^x \gamma(\cdot)$, or $\Gamma(x)$, and $\gamma(x)$ commute for all $x \in [0, \nu]$. Such a relation is verified up to an error in L^1 by $\mathbf{a}^{(n)}(\cdot)$ and $A^{(n)}(\cdot)$:

$$Ad(A^{(n)}(\cdot)).\mathbf{a}^{(n)}(\cdot) = \mathbf{a}^{(n)}(\cdot) + \varepsilon_{n,\cdot} \tag{6.2}$$

where $\varepsilon_{n,\cdot} \in L^1$ is of the order of $\|\mathbf{a}^{(n)}(\cdot) - (-1)^n \mathbf{a}(\cdot)\|_{L^1} + \|\mathbf{a}(\cdot + \beta_n) - \mathbf{a}(\cdot)\|_{L^1}$. This fact, along with the continuous dependence of solutions of ODE's and the fact that $a_n(\cdot) = LA_n(\cdot)$, implies that

$$\begin{aligned} \tilde{A}^{(n)}(\nu + x) &= \tilde{A}^{(n)}(\nu).\exp(\int_\nu^{\nu+\beta_{n-1}x} a^{(n)}(\cdot) + O(\varepsilon_{n,\nu}(x))) \\ \tilde{C}^{(n)}(\nu + x) &= \tilde{C}^{(n)}(\nu).\exp(\int_\nu^{\nu+\beta_{n-1}x} a^{(n-1)}(\cdot) + O(\varepsilon_{n,\nu}(x))) \end{aligned}$$

for $x \in [0, 1]$. In particular,

$$\tilde{A}^{(n)}(\nu + 1) = \tilde{A}^{(n)}(\nu).\exp(\int_\nu^{\nu+\beta_{n-1}} a^{(n)}(\cdot) + O(\varepsilon_{n,\nu}(1))) \tag{6.3a}$$

$$\tilde{C}^{(n)}(\nu + \alpha_n) = \tilde{C}^{(n)}(\nu).\exp(\int_\nu^{\nu+\beta_n} a^{(n-1)}(\cdot) + O(\varepsilon_{n,\nu}(\alpha_n))) \tag{6.3b}$$

and, using the preceding calculations (proposition 6.24 and its corollary), we find

$$\begin{aligned} \tilde{A}^{(n)}(\nu + 1) &= \tilde{A}^{(n)}(\nu).\exp((-1)^n q_n \beta_{n-1} \mathbf{a}(\nu) + O(\varepsilon_{n,\nu}(1))) \\ \tilde{C}^{(n)}(\nu + \alpha_n) &= \tilde{C}^{(n)}(\nu).\exp(-(-1)^n q_{n-1} \beta_n \mathbf{a}(\nu) + O(\varepsilon_{n,\nu}(\alpha_n))) \end{aligned}$$

These formulas can be written as

$$\begin{aligned} \tilde{A}^{(n)}(\nu + 1) &= \tilde{A}^{(n)}(\nu).\exp(K_n(\nu) + O(\varepsilon_n(\nu))) \\ \tilde{C}^{(n)}(\nu + \alpha_n) &= \tilde{C}^{(n)}(\nu).\exp(L_n(\nu) + O(\varepsilon_n(\nu))) \end{aligned}$$

where we call

$$\begin{aligned} K_n(\nu) &= (-1)^n q_n \beta_{n-1} \mathbf{a}(\nu) \\ L_n(\nu) &= -(-1)^n q_{n-1} \beta_n \mathbf{a}(\nu) \end{aligned}$$

If the cocycle is smoother, we can sharpen the convergence result using the following lemma

Lemma 6.26. *Let $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $2 \leq s \leq \infty$. If $\mathrm{en}(\alpha, A(\cdot)) > 0$, there exists a full measure set $\mathcal{E}_s \subset \mathbb{T}$ such that for each $\nu \in \mathcal{E}_s$,*

$$\begin{aligned} \tilde{a}^{(n)}(\cdot) - (-1)^n q_{n-1}\beta_n \mathbf{a}(\nu) &\to 0 \text{ uniformly in } [\nu-1, \nu+1] \\ |\mathbf{a}(\nu)| &= \mathrm{en}(\alpha, A(\cdot)) \\ \partial^\sigma \tilde{a}^{(n)}(\cdot) &\to 0 \text{ uniformly in } [\nu-1, \nu+1],\ 1 \leq \sigma \leq s-2 \\ \frac{1}{\beta_{n-1}} \int_{\nu-1}^{\nu+1} |\partial^{s-1} \tilde{a}^{(n)}(\cdot)| &\to 0 \end{aligned}$$

We stress that there is no uniformity in ν or σ, but only in each interval $[\nu-1, \nu+1]$. The sets $\mathcal{E}_s$ are invariant under R_α.

If $\mathrm{en}(\alpha, A(\cdot)) = 0$, we have

$$\partial^\sigma \tilde{a}^{(n)}(\cdot) \to 0 \text{ uniformly in } [\nu-1, \nu+1],\ 0 \leq \sigma \leq s-2$$

for all $\nu \in \mathbb{T}$.

Proof. We can apply lemma 11.2 of appendix 11 to $\frac{1}{q_n^s}\partial^{s-1} a^{(n)}(\cdot)$, which has been proved to converge to 0 in L^2 and obtain a full measure set of in $\mathbb{T}$ for which

$$\frac{1}{\beta_{n-1} q_n^s} \int_{\nu-\beta_{n-1}}^{\nu+\beta_{n-1}} |\partial^{s-1} a^{(n)}(\cdot)| \to 0$$

holds. If $\mathcal{E}_s$ is the intersection of such sets for $0 \leq \sigma \leq s$ with the set where $|\mathbf{a}(\cdot)| = en(\alpha, A(\cdot))$, it is of full measure and the rest of the properties follow from the convergence results already proved. The invariance of the sets $\mathcal{E}_s$ follows from the $\mathcal{U}$-invariance of the limit curve $\mathbf{a}(\cdot)$. □

Under these assumptions, the mismatch function $\varepsilon_{n,\nu}(\cdot)$ in eq. (6.3) is in fact C^{s-1} in the dynamic variable x for every fixed $\nu \in \mathbb{T}$, and tends to 0 in H^{s-1}, with the same local convergence properties as in the previous lemma.

We can now prove the quantization theorem:

Proof of theorem 6.1. Let us suppose for the moment that $0 \in \mathcal{E}_s$, for simplicity in notation. For n even and big enough,

$$\tilde{C}^{(n)}(x) = \tilde{C}^{(n)}(0).\exp(L_n(0).(x) + O(\varepsilon_{n,0}(x)))$$

uniformly around 0, with $\tilde{C}^{(n)}(0) \in \mathcal{Z}_0 = Z_G(\exp(\mathbb{R}\mathbf{a}(0)))$ up to a small error, i.e. $Ad(\tilde{C}_0^{(n)}).\mathbf{a}(0) = \mathbf{a}(0) + O(\varepsilon_n)$ and the same holds for $\tilde{A}^{(n)}(x)$. The fact that the generators of the renormalized action commute implies that

$$\exp(\mathbf{a}(0) + O(\varepsilon_n))) = \tilde{C}^{(n)}(0)\tilde{A}^{(n)}(0)(\tilde{C}^{(n)}(0))^*(\tilde{A}^{(n)}(0))^*$$

where we have used that $q_n\beta_{n-1} + q_{n-1}\beta_n = 1$. Since the commutator of two constants in $\mathcal{Z}_0 = G_0 \times G_+$ is in G_0, and since $\exp(\mathbf{a}(0)) \in G_+$, we immediately obtain that

$$\exp(\mathbf{a}(0)) \in G_0 \cap G_+ = \{Id\}$$

Since the choice of the base-point of renormalization is not significant, we have proved theorem 6.1. □

Let us now proceed to the proof of theorems 6.3 and 6.5, and their analogues for cocycles non-homotopic to constants.

Proof of Theorems 6.3 and 6.4. The conjugation

$$\tilde{D}_n(x) = \exp(-L_n(0)\frac{x(x-1)}{2})$$

taking values in $\exp(\mathbb{R}\mathbf{a}(0))$ reduces the action $\tilde{\mathcal{R}}^n\Phi$ to the one generated by

$$(1, \tilde{C}^{(n)}(0).\exp(O(\varepsilon_n)))$$
$$(\alpha_n, \tilde{A}^{(n)}(0).\exp(\mathbf{a}(0)(\cdot) + O(\varepsilon_n)))$$

where again $\tilde{A}^{(n)}(0) \in \mathcal{Z}_0$ is a different constant and we have used the fact that $L_n(0)$ and $K_n(0)$ are collinear with $\mathbf{a}(0)$, as well as the commutation relation 6.2. Let us recall the splitting of $\mathcal{Z}_0 = G_0 \times G_+ \hookrightarrow G$ (cf. section 4.2) which corresponds to $\mathbf{a}(0)$. We can decompose, up to a small error, $\tilde{C}^{(n)}(0)$ into $C_+.C_0$, where $C_0 \in G_0$ and $C_+ \in G_+$. Then, if $H_+ \in g_+$ is a preimage of C_+, the conjugant $\exp(-H_+\cdot)$ reduces the action to

$$(1, C_0.\exp(O(\varepsilon_n))) \tag{6.4}$$
$$(\alpha_n, \tilde{A}^{(n)}(0).\exp(\mathbf{a}(0)(\cdot) + O(\varepsilon_n))) \tag{6.5}$$

with $\tilde{A}^{(n)}(0)$ replaced by $\tilde{A}^{(n)}(0)\exp(-H_+\alpha_n)$. We remark that the first generator is a perturbation of a constant cocycle in $\mathbb{T} \times G_0$. Let us also split $\tilde{A}^{(n)}(0)$ into $A_+.A_0$, as we have done with $\tilde{C}_0^{(n)}$.

In order to normalize the action, we have to reduce the constant C_0 to the Id, and the natural conjugant is $\exp(-H_0\cdot)$, where $H_0 \in g_0$ is a preimage of C_0. If A_0 and C_0 are on a common maximal torus in G_0, this conjugation conjugates the action to the one generated by

$$(1, \exp(O(\varepsilon_n))) \tag{6.6}$$
$$(\alpha_n, \exp(-H_0\alpha_n)\tilde{A}^{(n)}(0).\exp(\mathbf{a}(0)(\cdot) + O(\varepsilon_n))) \tag{6.7}$$

and a close-to-the Id can normalize this action to an action associated to a cocycle of the same form.

If, however, C_0 and A_0, the proof of lemma 2.7 shows that we can choose $H_0 \in g_0$, a preimage of C_0, satisfying the following condition. The vector H_0 decomposes into H_1+H_2, where $Ad(A_0).H_1 = H_1$ and $Ad(A_0).H_2 = -H_2$, and then $\exp(2H_2) = Id$. We choose H_2 of minimal length, and the minimal χ, $1 \leq \chi \leq \chi_0 \leq \chi_G$, such that $Ad(A^\chi).H_2 = H_2$. Then conjugation by $\exp(-H_0\cdot)$ reduces the action to the one generated by

$$(1, \exp(O(\varepsilon_n))) \tag{6.8}$$
$$(\alpha_n, \exp(-H_0\alpha_n)A_+.\exp(\mathbf{a}(0)(\cdot)).\exp(-H_0\cdot).A_0.\exp(H_0\cdot)\exp(O(\varepsilon_n))) \tag{6.9}$$

As shows the proof of lemma 2.7, $\exp(-H_0\cdot).A_0.\exp(H_0\cdot)$ is a periodic geodesic in G_0, it is non homotopic to constants, and its order in the homotopy group of G_0 is precisely χ. Consequently, if, as previously normalize this action by a close-to-the Id conjugation reducing the first generator to $(1, Id)$, we obtain the first part of theorem 6.4. The second part follows from the choice of χ made above.

Translating this construction by ν, composing with the algebraic conjugation $D(\nu)$ that diagonalizes $\mathbf{a}(\nu)$, and modifying accordingly the notation, we can obtain the same result for any $\nu \in \mathcal{E}_s$. This proves theorems 6.3 and 6.4 in their full generality. □

Proof of Theorems 6.5 and 6.6. Repeating the arguments in the 0 energy case, and remembering that under this assumption $J_n^1(\cdot) \to 0$ uniformly proves theorem 6.5.

□

We close this section with the following remark.

Remark 6.27. We stress that if the cocycle is regular, the constant χ_G is irrelevant, as follows from the proof. In particular, if $G = SO(3)$, this constant equals 2, but all non-zero vectors in $g = so(3) \approx su(2)$ are regular. We will examine the consequences of this phenomenon in section 7.7.

6.6 Preservation of homotopy under renormalization

From the above arguments we can draw the rather surprising conclusion that, in the global as in the local theory, the greatest part of the analysis of the dynamics of a cocycle is made in the isomorphism class of the Lie algebra of the group. In particular, for a given cocycle $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, the definition of the energy and the invariant curve do not depend directly on the group, but only on $a(\cdot)$ (which however should be the derivative of a closed path $\mathbb{T} \to G$) and $Ad(A(\cdot)) \in C^s(\mathbb{T}, Inn(g))$, which both remain the same if we replace G by G/K, where $K \subset Z_G$ and $(\alpha, A(\cdot))$ by $(\alpha, \pi(A(\cdot))) \in SW^s(\mathbb{T}, G/K)$, where $\pi : G \to G/K$ is the canonical projection. In particular, since the functionals measuring the convergence of the renormalization are calculated in the same way, they depend only in the same indirect way on the group. These properties are not necessarily conserved when we pass to a covering of G, and we will examine how this affects the study of the dynamics.

The difference between the groups becomes important in the solution of the equation

$$\exp(h) = Id$$

When we renormalize cocycles in $\mathbb{T} \times (G/K)$, a finer lattice in $\mathbb{Z}^w$ is admissible than the one admissible for cocycles in $\mathbb{T} \times G$, and this is the case even for regular cocycles. In the case of close-to-a-constant and singular cocycles, the constant χ_G becomes important. Since the constants χ_G and $\chi_{G/K}$ are different, we have to iterate a different number of times in order to obtain the necessary commutativity for the normalization of the renormalized action.

A manifestation of the first phenomenon is the quantization of $E_r(\cdot)$ in $SU(2)$, where the levels of energy allowed are in $2\pi\mathbb{N}$, whereas the quantization in $SO(3)$ is in $\pi\mathbb{N}$. The factor $1/2$ is clearly due to the cardinal of the center of $SU(2)$, equal to $\{\pm Id\}$, which in this case is also the reason why $\chi_{SU(2)} = 1$, while $\chi_{SO(3)} = 2$.

It is nonetheless possible to renormalize a cocycle without losing track of its homotopy class, but this seems to be obtained in a more natural, but still not canonical, way by passing on to a group with simpler homotopy [3].

Let, therefore, $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$ and $\tilde{G}$ be the universal cover of G, so that $G = \tilde{G}/K$, where $K \subset Z_{\tilde{G}}$. Then, $A(\cdot)$ admits a lift $\tilde{A}(\cdot)$ in $\tilde{G}$ which is at most $\#K$-periodic. There exists, however, $S \in K$ such that $\tilde{A}(\cdot + 1) = S.\tilde{A}(\cdot)$. The choice of S depends on the chosen lift, but the smallest integer m such that $S^m = Id$ is in fact a characteristic of the homotopy class of $A(\cdot)$. Therefore, even though the couple $((1, Id), (\alpha, \tilde{A}(\cdot)))$ does not define a $\mathbb{Z}^2$- action on $\mathbb{T} \times \tilde{G}$, it satisfies the commutativity relation

$$(1, Id) \circ (\alpha, \tilde{A}(\cdot)) = (0, S) \circ (\alpha, \tilde{A}(\cdot)) \circ (1, Id) \tag{6.10}$$

Let us firstly study cocycles satisfying this weaker commutation relation, and let two cocycles $U, V \in SW^s(\mathbb{T}, \tilde{G})$ commute modulo $S \in Z_{\tilde{G}}$:

$$U \circ V = (0, S) \circ V \circ U$$

These cocycles do not define a $\mathbb{Z}^2$ action, but we can always define a function

$$\begin{array}{rccc} \Phi_l : & \mathbb{Z}^2 & \to & SW^s(\mathbb{T}, \tilde{G}) \\ & (m, n) & \longmapsto & U^m \circ V^n \end{array}$$

or

$$\begin{array}{rccc} \Phi_r : & \mathbb{Z}^2 & \to & SW^s(\mathbb{T}, \tilde{G}) \\ & (m, n) & \longmapsto & V^n \circ U^m \end{array}$$

3. A similar way to the one we have chosen is to associate a symbol to the action associated to the cocycle. The symbol codes the homotopy class of the path in G. It is invariant by the all the actions used in renormalization, except for the action of $\mathbb{N}^*$, which accounts for the fact that composition of a path with itself simplifies its homotopy. However, we find this presentation needlessly abstract, and we prefer the one given in the text.

which is not a group homomorphism. This property can be regained if we factorize $\tilde{G}$ modulo K, or if we restrict it to the submodule $(\#K.\mathbb{Z})^2$. It is equally regained if we consider the adjoint action of $\tilde{G}$ on g and the induced $\mathbb{Z}^2$ action.

Let us examine Φ_l. A matrix in $GL(2,\mathbb{Z})$ does not act on it in a canonical way. It can be nonetheless verified that the change of basis of $\mathbb{Z}^2$ by a matrix in $GL(2,\mathbb{Z})$ preserves the relation modulo an inversion of S which is not significant, as it corresponds to the change of orientation of the path induced by renormalization. If

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z})$$

and $\tilde{U} = U^a \circ V^c$ and $\tilde{V} = U^d \circ V^d$, then

$$\tilde{U} \circ \tilde{V} = (0, S^{\pm 1}) \circ \tilde{U} \circ \tilde{V}$$

according to the sign of the determinant. Therefore, if $P \in GL(2,\mathbb{Z})$, the action $\tilde{\Phi} = P_* \Phi_l$ is well defined and

$$\tilde{\Phi}_l(e_1) \circ \tilde{\Phi}_l(e_2) = (0, S^{\pm 1}) \circ \tilde{\Phi}_l(e_2) \circ \tilde{\Phi}_l(e_1)$$

so that the commutation relation is preserved, and the only reason for which it is non-canonical is the original choice of Φ_l instead of Φ_r. Additionally, it can be verified directly that the renormalization and normalization of the action $\tilde{\Phi}$ associated to the couple $((1, Id), (\alpha, \tilde{A}(\cdot)))$ as if it were a $\mathbb{Z}^2$ action preserve this relation, in view of the fact that $S \in Z_{\tilde{G}}$:

$$\tilde{R}_\nu^n \tilde{\Phi}(e_1) \circ \tilde{R}_\nu^n \tilde{\Phi}(e_2) = (0, S^{(-1)^n}) \circ \tilde{R}_\nu^n \tilde{\Phi}(e_2) \circ \tilde{R}_\nu^n \tilde{\Phi}(e_1)$$

and this property persists after conjugation of actions. For example, if $(1, C(\cdot))$ and $(\alpha, A(\cdot))$ satisfy

$$C(\cdot + \alpha).A(\cdot) = S.A(\cdot + 1).C(\cdot)$$

and $B : \mathbb{R} \to G$ normalizes $(1, C(\cdot))$ to $(1, Id)$, then $\tilde{A}(\cdot) = B(\cdot + \alpha)A(\cdot)B^*(\cdot)$ satisfies

$$\tilde{A}(\cdot + 1) = S.\tilde{A}(\cdot)$$

Since S codes the homotopy class of the cocycle, we find that the renormalization in fact preserves homotopy, in contrast with simple iteration of the cocycle.

Finally, since all the estimates concerning the measurement of the convergence of the scheme depend only on the isomorphism class of g, this renormalization of non-abelian actions has the same characteristics as the renormalization of the action associated to $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$.

6.7 Dependence of the degree on the frequency

In this section we simplify and generalize an argument given in [Fra04]. It was used in the construction of two cocycles $(\alpha_i, A(\cdot)) \in SW^\infty(\mathbb{T}, SU(2))$, $i = 1, 2$, given by the same mapping $A(\cdot) : \mathbb{T} \to SU(2)$, but different rotations in the basis satisfying $\alpha_1 - \alpha_2 \neq 1/2$, and such that $en(\alpha_1, A(\cdot)) \neq en(\alpha_2, A(\cdot))$. Since any compact group contains subgroups locally isomoprhic to $SU(2)$, we present the argument in the context of $SU(2)$, and use the fact that it is of perturbative nature in order to obtain the same result in any compact group.

From the results so far, it is clear that if $\|a(\cdot)\|_{L^1} < 2\pi$ then $en(\alpha, A(\cdot)) = 0$, independently of α. Moreover, a simple calculation shows that, for $E_r(\cdot) = \{e^{2i\pi r \cdot}, 0\}_{SU(2)}$, $r \in \mathbb{N}^*$, then $en(\alpha, E_r(\cdot + \theta)) = 2\pi r$, again independently of α, and of $\theta \in \mathbb{R}$. However, if $A(\cdot) = E_r(\cdot)A_0$, where A_0 is a non-diagonal constant, then $\|a_2(\cdot)\|_{L^1} < 4\pi r$, so that $en(\alpha_1, A(\cdot)) < 2\pi r$ (see [Kri01] or the following chapter). Thus, our goal should be the construction of a mapping $A(\cdot) : \mathbb{T} \to SU(2)$ such that $(\alpha_1, A(\cdot))$ is conjugate to $(\alpha_1, E_r(\cdot))$ whereas $(\alpha_2, A(\cdot))$ is conjugate to $(\alpha_2, E_r(\cdot)A_0)$, with A_0 as before.

Let us therefore consider $A(\cdot) = \Phi(\cdot + \alpha_1).E_r(\cdot).\Phi(-\cdot)$, where $\Phi(\cdot) : \mathbb{T} \to \mathcal{T}'$ is a (non-trivial) homomorphism ($\mathcal{T}'$ is a maximal torus, not necessarily the standard one). By construction, $(\alpha_1, A(\cdot)) \sim (\alpha_1, E_r(\cdot))$, and therefore $en(\alpha_1, A(\cdot)) = 2\pi r$. On the other hand,

$$Conj_{\Phi(\cdot)}(\alpha_2, A(\cdot)) = \Phi(\cdot + \alpha_2).A_0.E_r(\cdot).\Phi(-\cdot)$$

where $A_0 = \Phi(\alpha_1 - \alpha_2)$, so that $(\alpha_2, A(\cdot)) \sim (\alpha_2, A_0 E_r(\cdot))$, and $en(\alpha_2, A(\cdot)) < 2\pi r$, provided that A_0 is non-diagonal. This last property is satisfied, say, if $\Phi(\alpha_1 - \alpha_2) \notin G_+ \times G_0$.

In particular, we see that the property that the cocycle $(\alpha, A(\cdot))$ be (torus-)reducible should be expected to depend both on the rotation and the mapping defining the cocycle.

Following the remark in the beginning of the section, similar constructions can be carried out with $SU(2)$ replaced by any compact Lie group giving

Theorem 6.28. *Let $\alpha_1, \alpha_2 \in \mathbb{T}\backslash\mathbb{Q}$ be such for some torus homomorphism $\Phi : c_G\mathbb{T} \to G$, $\Phi(\alpha_1 - \alpha_2) \notin Z_G$. Then, there exists a mapping $A(\cdot) : \mathbb{T} \to G$ such that* $\mathrm{en}(\alpha_1, A(\cdot)) \neq \mathrm{en}(\alpha_2, A(\cdot))$.

The condition on the α_i is satisfied if $\alpha_1 - \alpha_2 \notin \frac{1}{c_G \psi_G}\mathbb{Z}$ where c_G is the cardinal of Z_G and ψ_G is a constant in $\mathbb{N}^*$ depending only on the group G.

6.8 Proof of measurable invariance of the degree (prop. 6.7)

In this section, we prove proposition 6.7. There is a similar proof in [Fra04], but for cocycles of regularity C^2. The improvement of the result in terms of regularity is granted by the improved results for the convergence of renormalization. There is also the slight complication due to the fact that for general compact groups the degree is a vector and not a number as in $SU(2)$.

Let us call Φ_i, $i = 1, 2$ the actions associated to the cocycles $(\alpha, A_i(\cdot))$. If we use the fact that $\tilde{\mathcal{R}}_\nu^n \Phi_2 = Conj_{\tilde{B}_\nu(\cdot)} \tilde{\mathcal{R}}_\nu^n \Phi_1$, we find that

$$\tilde{B}_\nu(\cdot + \alpha_n)\tilde{A}_{\nu,1}^{(n)}(\cdot + 1)\tilde{C}_{\nu,1}^{(n)}(\cdot)^* \tilde{B}_\nu^*(\cdot) = \tilde{A}_{\nu,2}^{(n)}(\cdot + 1)\tilde{C}_{\nu,2}^{(n)}(\cdot)^*, \ a.e.$$

We have seen (cf. formula 6.3 and those that follow it) that for almost every ν we have

$$A_{\nu,1}^{(n)}(\cdot + 1)C_{\nu,1}^{(n)}(\cdot)^* \to A_{\nu,1}^{(n)} \exp((-1)^n \mathbf{a}_1(\nu)\cdot)(C_{\nu,1}^{(n)})^*$$

uniformly in $[-1, 1]$, where $A_{\nu,1}^{(n)}$ and $C_{\nu,1}^{(n)}$ are constants, and similarly for $\tilde{A}_{\nu,2}^{(n)}(\cdot)\tilde{C}_{\nu,2}^{(n)}(\cdot)^*$.

The set of ν such that both $\tilde{B}_\nu(x + \alpha_n)$ and $\tilde{B}_\nu(x + 1)$ tend to $\tilde{B}_\nu(0)$ for *a.e.* $x \in [-1, 1]$ is of full measure. Therefore, $\mathbf{a}_1(\nu)$ and $\mathbf{a}_2(\nu)$ are *a.e.* algebraically conjugate and thus

$$\deg((\alpha, A_1(\cdot))) = \deg((\alpha, A_2(\cdot)))$$

Chapter 7

A priori estimates on perturbations of regular geodesics

As has been shown in the previous chapter, the renormalization scheme converges, independently of any arithmetical conditions, to a model of dynamics given by $(\alpha, E_r(\cdot).A)$, where $r \in \mathbb{Z}^w$ and $A \in G$ commutes with $E_r(\cdot)$. However, convergence is gained by the introduction of the scaling factors β_{n-1}^{-1}, since otherwise convergence is possible only in L^2, where lives $\mathbf{a}(\cdot)$, the limit object of the dynamics. This is actually the way lemma 6.11 becomes useful in the study of the dynamics, since it shows the limitations of renormalization. We cannot conclude the desired density properties, unless we stop renormalization after a finite number of steps, and then take up the local study of the limit cocycles, but viewed as cocycles and not as actions any more. Therefore, the local theory of geodesics is the next natural step in our study.

The non-triviality of G_+ implies the existence of an obstruction to the reducibility of the cocycle and is the core of the passage from local to global phenomena. In this chapter we begin the study of the case where this obstruction is maximal. By maximal we mean that $e_r = L(E_r(\cdot))$ is a regular vector in g, i.e. it belongs to a unique maximal torus which we will call $\mathfrak{t}$. Since the property of a cocycle being regular is invariant under conjugation and renormalization, fact that follows directly from the invariance of the degree under renormalization (cf. Proposition 6.24), we can study the dynamics of perturbations of periodic geodesics whose derivative is a regular vector in g, without loss of generality. Another characterization of regular cocycles is the triviality of the zero-energy component G_0 for the splitting associated to a positive energy cocycle.

The convergence of the renormalization and the invariance of the degree under renormalization allow us to pose the problem in the following perturbative setting. We can suppose that the cocycle $(\alpha, A(\cdot))$ is of degree r, with $|r| > 0$, and a C^∞ perturbation of a regular periodic geodesic of degree r. In other words, we suppose that $A(\cdot)$ can be written in the form

$$E_r(\cdot).A.\exp(U(\cdot))$$

where $e_r = L(E_r(\cdot))$ and $A \in G$ commute (i.e. $Ad(A).e_r = e_r$), and $U(\cdot)$ is small. A smallness condition of the type $\|U(\cdot)\|_{H^1} \leq C$, where $0 < C < 1$ is some fixed constant will in fact be sufficient. The fact that e_r is regular implies that $E_r(\cdot)$ and A take values on the same maximal torus $\mathbb{T}$. Then, the choice of $\tilde{\Delta} \subset \Delta_+$, a basis for the roots of g, with respect to the maximal abelian algebra $\mathfrak{t}$, the Lie algebra of $\mathcal{T}$ allows us to write $E_r(\cdot)$ in the form $\exp(\sum 2\pi r_\rho h_\rho \cdot)$ and A in the form $\exp(\sum 2\pi a_\rho h_\rho)$ and use the notation

$$E_{r,a}(\cdot) = E_r(\cdot).A$$

as in eq. 2.3. Therefore, the models of cocycles which we are interested in are written in the form

$$(\alpha, E_{r,a}(\cdot)\exp(U(\cdot)))$$

with $U(\cdot) = \sum_{\rho \in \tilde{\Delta}} U_\rho^{\mathfrak{t}}(\cdot) h_\rho + \sum_{\rho \in \Delta_+} U_\rho(\cdot) j_\rho$, small enough in $C^\infty(\mathbb{T}, g)$.

It turns out that some estimates on the perturbation can be deduced, under the assumption that $(\alpha, E_{r,a}(\cdot)\exp(U(\cdot)))$ is of degree r. It has already been proved (cf. Proposition 5.3) that $\|\mathbf{a}_n(\cdot)\|_{L^i} \geq en(\alpha, A(\cdot))$, for $i = 1, 2$ and for all $n \in \mathbb{N}^*$. In particular, $\|a(\cdot)\|_{L^1} \geq en(\alpha, A(\cdot))$, and this gives a first estimate, which is however inadequate for the following reason. Let $(a, E_r(\cdot).e^{\{0,z\}})$ be a constant perturbation of a periodic geodesic in $SU(2)$. We find directly that $\|L(E_r(\cdot).e^{\{0,z\}})\|_{L^i} = 2\pi r$. However, it is known since [Kri01] that such a cocycle, and even small enough perturbations in C^1 (the smallness depending on $|z|$ and α^{-1}) can be of energy strictly smaller than $2\pi r$. The reason for this is the non-commutativity of $e^{\{0,z\}}$ and $E_r(\cdot)$, which results in the second iterate being of energy strictly smaller than $4\pi r$. We point out that, since the invariant curve $\mathbf{a}(\cdot)$ has been defined by simple iteration, and not renormalization, we can obtain the needed estimates using the derivative of $(2\alpha, A_2(\cdot))$, and not by estimating the functional $J^{(1)}$ on $R^1\Phi$, where again Φ is the $\mathbb{Z}^2$ action associated to the cocycle $(\alpha, A(\cdot))$. This is one of the drawbacks of the techniques used in [Kri01], since it results in an implicit dependence of the smallness condition on the frequency. The lack of uniformity results in the set Σ of theorem 3.11 being a strict subset of RDC.

Further iteration gives expressions that are too complicated for the estimates, so we will restrict the estimations in those of the first two iterates. They prove to be sufficient, since the influence of the first order terms is already important after two iterations, and higher-order phenomena are outside the scope of reduction schemes.

7.1 Notation

As mentioned before, the perturbation

$$U(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} U^{\mathfrak{t}}_\rho(\cdot)h_\rho + \sum\nolimits_{\rho\in\Delta_+} U_\rho(\cdot)j_\rho$$

is supposed to be small in a topology to be made precise. We will mostly use is expression in Fourier coefficients

$$U(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} \sum\nolimits_{k\in\mathbb{Z}} \hat{U}^{\mathfrak{t}}_\rho(k)e^{2i\pi k\cdot}h_\rho + \sum\nolimits_{\rho\in\Delta_+} \sum\nolimits_{k\in\mathbb{Z}} \hat{U}_\rho(k)e^{2i\pi k\cdot}j_\rho$$

so that

$$\partial U(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} \sum\nolimits_{k\in\mathbb{Z}} 2i\pi k\hat{U}^{\mathfrak{t}}_\rho(k)e^{2i\pi k\cdot}h_\rho + \sum\nolimits_{\rho\in\Delta_+} \sum\nolimits_{k\in\mathbb{Z}} 2i\pi k\hat{U}_\rho(k)e^{2i\pi k\cdot}j_\rho$$

Since $U(\cdot)$ is to be seen as a perturbation of $E_{r,a}(\cdot)$, it is reasonable to impose that $\sum_{\rho\in\tilde{\Delta}} \hat{U}^{\mathfrak{t}}_\rho(0)h_\rho = 0$, i.e. that there is no constant part in the perturbation in the torus $\mathfrak{t}$. This can be obtained by using the following fact. We can write

$$\exp(U(\cdot)) = \exp(\sum\nolimits_{\rho\in\tilde{\Delta}} \hat{U}^{\mathfrak{t}}_\rho(0)h_\rho).\exp(\tilde{U}(\cdot) - \sum\nolimits_{\rho\in\tilde{\Delta}} \hat{U}^{\mathfrak{t}}_\rho(0)h_\rho)$$

where $\tilde{U}(\cdot) - \dot{T}U(\cdot)$ is of second order. The mapping

$$h \longmapsto \int \pi_{\mathfrak{t}}(\exp^{-1}(e^h.e^{\tilde{U}(\cdot)}))$$

is a local diffeomorphism for $\tilde{U}(\cdot)$ taking values in a neighborhood of 0 in $\mathfrak{t}$, which gives the result.

Finally if $E_{r,a}$ is a regular geodesic (see eq. 2.3) and $e_r = LE_{r,a}(\cdot)$, for any $\rho \in \Delta_+$ there exist $\tilde{r}_\rho \in \mathbb{Z}^*$ such that

$$[e_r, j_\rho] = 2i\pi\tilde{r}_\rho.j_\rho$$

and $\|\tilde{r}\| \leq C.en(\alpha, A(\cdot))$, where the constant depends only on the group G and we call $\tilde{r} = (r_\rho)_{\rho\in\Delta_+} \in \mathbb{Z}^q$.

7.2 Synopsis of the chapter

In this chapter, we obtain an estimate similar to the one used in [Kri01] in the control of the low frequencies of a perturbation of the normal form of degree r, under the assumption that the perturbed cocycle is still of the same degree r.

In the context of this mémoire, the estimates are valid if the cocycle is a perturbation of a regular geodesic. As can be seen if one follows the calculations, the result remains true if the normal form is a singular geodesic coupled with a constant in G_0 and the perturbation is 0 in g_0. Let us, however, state the main result of the chapter in the first, more transparent context.

Lemma 7.1. *There exists a positive constant C, depending only on r, such that if $(\alpha, E_{r,a}(\cdot).e^{U(\cdot)})$ is of degree r, where $U(\cdot)$ is small enough in H^1, then it satisfies*

$$\|\Lambda^0_{\tilde{r}} U(\cdot)\|_{L^2} \leq C \|(Id - \Lambda_{\tilde{r}})\partial U(\cdot)\|_{L^2}$$

The truncation operator $\Lambda^0_{\tilde{r}}$ is defined in def. 7.3. It is a truncation in low frequencies, corresponding to the phenomenon that we described in the plan of the proof (p. 37), only in a more complicated geometry. Firstly, we use the fact that interactions between the different subalgebras isomporphic to $su(2)$ are of higher order, in order to break the estimates into a superposition of estimates in simpler geometry. In this context, we can say that if in $(su(2))_\rho$, the subalgebra corresponding to the root ρ, the perturbation is too eccentric (i.e. the non-abelian constant part is dominant), or if the perturbation spins around h_ρ too fast in the negative direction, then the perturbation in $(su(2))_\rho$ contributes negatively to the energy of the cocycle. And this is exactly the description of the part of $\Lambda^0_{\tilde{r}}$ concerning $(su(2))_\rho$. If, now, the perturbed cocycle is assumed to be of degree r, then the contributions of the different parts of the perturbation must sum up to a non-negative contribution, and this grants the a priori estimate of the lemma.

Using these estimates, we can obtain a first theorem describing (with an inadequate precision) the configuration of the conjugacy classes of periodic geodesics. Let us introduce some notation before stating the theorem proved above. Let $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $s \geq 1$, and let $1 \leq \chi \leq \chi_G$ be the smallest positive integer such that $A_\chi(\cdot)$ is homotopic to constants. Let also Φ be the $\mathbb{Z}^2$ action associated to $(\alpha, A(\cdot))$. Then, we have

Theorem 7.2. *Let G be a compact semi-simple Lie group and $(\alpha, A(\cdot)) \in SW^s(\mathbb{T}, G)$, $s \geq 1$, and let χ as above. Then, if $\mathrm{en}(\alpha, A(\cdot)) > 0$, $(\alpha, A(\cdot))^\chi$ has renomalization representatives $\mod \chi$ which are arbitrarily close to cocycles of smaller energy. All cocycles of zero energy in the homotopy class of $(\alpha, A(\cdot))$ satisfy the following property: renormalization $\mod \chi$ of the χ-lattice associated to such cocycles converges to constant actions.*

The proof of this theorem occupies section 7.7. Since the proof relies heavily on the phenomenon of non-commutativity exploited in section 7.4, it is clear that this property is due to the fact that G is non-commutative.

This theorem is a density theorem for 0-energy actions, and a density theorem for close-to constant normalized actions, modulo a finite covering of the torus. However, it does not assert that reducible and almost-reducible cocycles are dense in $SW^\infty(\mathbb{T}, G)$ modulo χ_G, for the following reason. Let $(\alpha, A(\cdot))$ be a cocycle as in the theorem and Φ be the associated action. The theorem only asserts that for *a.e.* $\nu \in \mathbb{T}$, $\tilde{\mathcal{R}}^{(n)}_\nu \Phi$ converges to models accumulated by lower energy cocycles. If we wish to obtain a density theorem, we have to be able to normalize $\tilde{\mathcal{R}}^{(n)}_\nu \Phi$ to Φ' and to prove that $\Phi'(e_2)$ is accumulated by lower energy cocycles, and then apply lemma 6.11. This, however, is a stronger assertion.

The theorem, which holds for all irrational rotations, only allows us to construct perturbations of $\tilde{\mathcal{R}}^{(n)}_\nu \Phi$ which are of smaller energy, but cannot be made arbitrarily small unless we allow n to grow. It allows us, nonetheless, to conclude a fundamental difference between cocycles in $SW^\infty(\mathbb{T}, \mathbb{T}^w)$, which is abelian and not semi-simple, and $SW^\infty(\mathbb{T}, G)$, with G compact semisimple. We saw in the introductory study of abelian cocycles (chapter 3) that these cocycles are classified by homotopy, and therefore form discrete classes, and iteration of a path $\mathbb{T} \to \mathbb{T}^w$ non-homotopic to constants will not make it homotopic to constants. On the other hand, when we study abelian cocycles in G, and therefore consider $SW^\infty(\mathbb{T}, \mathbb{T}^w) \hookrightarrow SW^\infty(\mathbb{T}, G)$, with w the rank of G, we find that the

limit object of the dynamics is quantized in the same lattice as in $SW^\infty(\mathbb{T}, \mathbb{T}^w)$, but the classes are no longer discrete (eventually modulo iteration).

The situation is intermediate in compact groups which are not semi-simple. Let us consider, for example, the cocycle in $\mathbb{T} \times U(2)$ defined by

$$A(\cdot) = \begin{bmatrix} e^{2i\pi k\cdot} & 0 \\ 0 & e^{2i\pi k\cdot} \end{bmatrix}$$

over any irrational, with $k \in \mathbb{Z}^*$. This cocycle admits a non-trivial limit curve in $u(2)$ consisting of the sole vector

$$\begin{bmatrix} 2i\pi k & 0 \\ 0 & 2i\pi k \end{bmatrix}$$

We remark that $A(\cdot)$ takes values in the center of $U(2)$, which is naturally isomorphic to $\mathbb{S}^1$. Since, now, $U(2)$ locally around the Id is isomorphic to a neigborhood of $(1, Id)$ in $\mathbb{S}^1 \times SU(2)$, we can readily see that all small enough perturbations of $(\alpha, A(\cdot))$ have the same limit curve. Clearly, the reason for this is that the limit object of the dynamics of $(\alpha, A(\cdot))$ lives in the direction in which the semisimplicity criterion fails for $u(2)$.

7.3 Estimation of the energy of the path $A(\cdot)$

A direct calculation shows that

$$a(\cdot) = L(A(\cdot)) = e_r + Ad(E_{r,a}(\cdot)).u(\cdot)$$

where $u(\cdot) = L\exp(U(\cdot))$, so that the estimates of the energy of the path $A(\cdot)$ in G are significant only when $U(\cdot)$ is non-constant. The remaining case will be studied later in this section. Using the Ad-invariance of the Cartan-Killing form we find

$$\|a(\cdot)\|^2_{L^2} = en^2 + 2\int_{\mathbb{T}} \langle e_r, u(\cdot)\rangle + \|u(\cdot)\|^2_{L^2}$$

and the assumption that the cocycle is of degree r implies

$$2\int_{\mathbb{T}} \langle e_r, u(\cdot)\rangle + \|u(\cdot)\|^2_{L^2} \geq 0$$

Using the expansion of the operator L and the fact that $U(\cdot)$ is to be considered small, we have the estimate

$$\begin{aligned} u(\cdot) = Le^{U(\cdot)} &= \partial U(\cdot) + \frac{1}{2}[U(\cdot), \partial U(\cdot)] + O(|U(\cdot)|^2 |\partial U(\cdot)|) \\ &= \tilde{u}(\cdot) + O(|U(\cdot)|^2 |\partial U(\cdot)|) \end{aligned}$$

We now find that

$$\begin{aligned} 2\langle e_r, \tilde{u}(\cdot)\rangle &= 2\langle e_r, \partial U(\cdot)\rangle + \langle e_r, [U(\cdot), \partial U(\cdot)]\rangle \\ &= 2\langle e_r, \pi_{\mathfrak{t}}\partial U(\cdot)\rangle + \langle e_r, [U(\cdot), \partial U(\cdot)]\rangle \\ &= 2\langle e_r, \pi_{\mathfrak{t}}\partial U(\cdot)\rangle + \langle [e_r, U(\cdot)], \partial U(\cdot)\rangle \\ &= 2\langle e_r, \pi_{\mathfrak{t}}\partial U(\cdot)\rangle + \sum_{\rho\in\Delta_+} Re(2i\pi\tilde{r}_\rho U_\rho(\cdot)\partial\bar{U}_\rho(\cdot)) \end{aligned}$$

and therefore

$$2\int \langle e_r, \tilde{u}(\cdot)\rangle = \int Re(\sum\nolimits_{\rho\in\Delta_+} 4\pi^2\tilde{r}_\rho \sum\nolimits_{k\in\mathbb{Z}} k|\hat{U}_\rho(k)|^2)$$

The error term can be bounded by

$$\begin{aligned}\int |U(\cdot)|^2 \, |\partial U(\cdot)| &\leq \|U\|_{L^\infty} \int |U(\cdot)| \, |\partial U(\cdot)| \\ &\leq \|U\|_{L^\infty} \|U\|_{L^2} \|\partial U\|_{L^2} \\ &\lesssim \|U\|_{H^1}^3\end{aligned}$$

since H^1 is continuously embedded in L^∞. Consequently,

$$2\int \langle e_r, u(\cdot)\rangle = \sum_{\rho\in\Delta_+} 4\pi^2 \tilde{r}_\rho \sum\nolimits_{k\in\mathbb{Z}} k|\hat{U}_\rho(k)|^2 + O(\|U\|_{H^1}^3)$$

We also have

$$\begin{aligned}\|\tilde{u}\|_{L^2}^2 &= \|\partial U(\cdot)\|_{L^2}^2 + \|[U(\cdot), \partial U(\cdot)]\|_{L^2}^2 \\ &= \|\partial \pi_{\mathfrak{t}} U(\cdot)\|_{L^2}^2 + \sum_{\rho\in\Delta_+} \|\partial U_\rho(\cdot)\|_{L^2}^2 + O(\|U\|_{H^1}^4)\end{aligned}$$

Therefore,

$$2\int \langle u(\cdot), e_r\rangle + \|u\|_{L^2}^2 \leq \sum_{\rho\in\Delta_+} 4\pi^2 \tilde{r}_\rho \sum_{k\in\mathbb{Z}} k|\hat{U}_\rho(k)|^2 + \|\partial \pi_{\mathfrak{t}} U(\cdot)\|_{L^2}^2 + \sum_{\rho\in\Delta_+} \|\partial U_\rho(\cdot)\|_{L^2}^2 + O(\|U\|_{H^1}^3)$$

Since

$$\sum_{\rho\in\Delta_+} \|\partial U_\rho(\cdot)\|_{L^2}^2 = \sum_{\rho\in\Delta_+} \sum_{k\in\mathbb{Z}} 4\pi^2 k^2 |\hat{U}_\rho(k)|^2$$

and

$$\sum_{\rho\in\Delta} 4\pi^2 \tilde{r}_\rho \sum_{k\in\mathbb{Z}} k|\hat{U}_\rho(k)|^2 + \sum_{\rho\in\Delta_+} \|\partial U_\rho(\cdot)\|_{L^2}^2 = 4\pi^2 \sum_{\rho\in\Delta_+} \sum_{k\in\mathbb{Z}} (\tilde{r}_\rho + k)k|\hat{U}_\rho(k)|^2$$

we find that

$$2\int \langle u(\cdot), e_r\rangle + \|u\|_{L^2}^2 \leq 4\pi^2 \sum_{\rho\in\Delta_+} \sum_{k\in\mathbb{Z}} (\tilde{r}_\rho + k)k|\hat{U}_\rho(k)|^2 + \|\partial \pi_{\mathfrak{t}} U(\cdot)\|_{L^2}^2 + O(\|U\|_{H^1}^3)$$

The fact that $(\tilde{r}_\rho + k)k < 0$ for k between 0 and $-\tilde{r}_\rho$ motivates the following definition.

Definition 7.3. Let $\tilde{r}_\rho \in \mathbb{Z}^*$, $\rho \in \Delta_+$. We will call $I_\rho = \{1, 2, ... - \tilde{r}_\rho - 1\}$ if $\tilde{r}_\rho < 0$, and $I_\rho = \{-1, -2, .., -\tilde{r}_\rho + 1\}$ if $\tilde{r}_\rho > 0$. We also let $\Lambda_{\tilde{r}}$ and Λ_ρ be the truncation operators

$$\begin{aligned}\Lambda_{\tilde{r}} U(\cdot) &= \sum\nolimits_{\rho\in\Delta_+} \Lambda_\rho U_\rho(\cdot) j_\rho \\ &= \sum\nolimits_{\rho\in\Delta_+} \sum\nolimits_{k\in I_\rho} \hat{U}_\rho(k) e^{2i\pi k\cdot} j_\rho\end{aligned}$$

We will also use the non-homogeneous truncation operator

$$\begin{aligned}\Lambda_{\tilde{r}}^0 U(\cdot) &= \sum\nolimits_{\rho\in\Delta_+} \Lambda_\rho^0 U_\rho(\cdot) j_\rho \\ &= \Lambda_{\tilde{r}} U(\cdot) + \sum\nolimits_{\rho\in\Delta_+} \hat{U}_\rho(0) j_\rho\end{aligned}$$

The image of the operator $\Lambda_{\tilde{r}}^0$ in L^2 will be denoted by $\mathcal{P}_{\tilde{r}}$.

Using these notations, we have

$$\begin{aligned}2\int \langle u(\cdot), e_r\rangle + \|u\|_{L^2}^2 &\leq -4\pi^2 \|\Lambda_{\tilde{r}} U(\cdot)\|_{L^2}^2 + \sum\nolimits_{\rho\in\Delta} (\tilde{r}_\rho + 1)^2 \|(Id - \Lambda_\rho)\partial U_\rho(\cdot)\|_{L^2}^2 \\ &\quad + \|\partial \pi_{\mathfrak{t}} U(\cdot)\|_{L^2}^2 + O(\|U\|_{H^1}^3)\end{aligned}$$

The condition that the cocycle is of degree r implies that

$$4\pi^2 \left\|\Lambda_r U(\cdot)\right\|_{L^2}^2 \leq \sum\nolimits_{\rho\in\Delta} (\tilde{r}_\rho + 1)^2 \left\|(Id - \Lambda_\rho)\partial U_\rho(\cdot)\right\|_{L^2}^2 + \left\|\partial \pi_{\mathfrak{t}} U(\cdot)\right\|_{L^2}^2 + O(\|U\|_{H^1}^3) \tag{7.1}$$

or, in a more compact form

Lemma 7.4. *Let $(\alpha, E_{r,a}(\cdot)\exp(U(\cdot)))$ be of degree r. Then, there exists a positive constant such that if $\|U\|_{H^1}$ is small enough*

$$\left\|\Lambda_{\tilde{r}} U(\cdot)\right\|_{L^2} \lesssim \left\|(Id - \Lambda_{\tilde{r}})\partial U(\cdot)\right\|_{L^2}$$

The constant depends only on the degree.

We remark that this inequality does not give any information on $\hat{U}(0)$.

7.4 Estimation of the energy of the second iterate

For the second iterate of $(\alpha, A(\cdot)) = (\alpha, E_{r,a}(\cdot).e^{U(\cdot)})$, we have

$$(2\alpha, A_2(\cdot)) = (2\alpha, E_{r,a}(\cdot + \alpha).e^{U(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)})$$

and therefore

$$a_2(\cdot) = a(\cdot + \alpha) + Ad(A(\cdot + \alpha)).a(\cdot)$$

so that

$$\left\|a_2(\cdot)\right\|_{L^2}^2 = 2\left\|a(\cdot)\right\|_{L^2}^2 - 2\int \langle a^*(\cdot + \alpha), a(\cdot)\rangle$$

We have already seen that

$$a(\cdot) = e_r + Ad(E_{r,a}(\cdot)).(\partial U(\cdot) + \frac{1}{2}[U(\cdot), \partial U(\cdot)]) + O(|U(\cdot)|^2.|\partial U(\cdot)|)$$

and in the same way we find

$$-a^*(\cdot) = \partial U(\cdot) - \frac{1}{2}[U(\cdot), \partial U(\cdot)] + e_r + [e_r, U(\cdot)] - \frac{1}{2}[U(\cdot), [e_r, U(\cdot)]] + O(|U(\cdot)|^3 + |U(\cdot)|^2.|\partial U(\cdot)|)$$

Since we have already estimates on $\|\dot{T}U\|_{L^2}$, we will split $U(\cdot)$ in $\hat{U}(0) + \dot{T}U(\cdot) = U_0 + \dot{U}(\cdot)$ where $U_0 = \hat{U}(0)$ is orthogonal to e_r. With these notations, we find that

$$\begin{aligned}\left\|a_2(\cdot)\right\|_{L^2}^2 \quad \leq \quad & 2\left\|a(\cdot)\right\|_{L^2}^2 + 2en^2 - \left\|[e_r, U(\cdot)]\right\|_{L^2}^2 + 2\int \left\langle [e_r, U_0], Ad(E_{r,a}(\cdot)).\partial\dot{U}(\cdot)\right\rangle + \\ & 2\int \left\langle \partial\dot{U}(\cdot + \alpha) + [e_r, \dot{U}(\cdot + \alpha)], Ad(E_{r,a}(\cdot)).\partial\dot{U}(\cdot)\right\rangle + O(\|U\|_{H^1}^3)\end{aligned}$$

so that

$$\begin{aligned}\left\|a_2(\cdot)\right\|_{L^2}^2 \quad \leq \quad & 4en^2 + 2\int \left\langle \partial\dot{U}(\cdot) + [e_r, \dot{U}(\cdot)], \partial\dot{U}(\cdot)\right\rangle - \left\|[e_r, U(\cdot)]\right\|_{L^2}^2 + \\ & 2\int \left\langle \partial\dot{U}(\cdot + \alpha) + [e_r, U(\cdot + \alpha)], Ad(E_{r,a}(\cdot)).\partial\dot{U}(\cdot)\right\rangle + O(\|U\|_{H^1}^3) \\ \leq \quad & 4en^2 + C_1\|\partial\dot{U}(\cdot)\|_{L^2}^2 - |[e_r, U_0]|\,(|[e_r, U_0]| - \|\partial\dot{U}(\cdot)\|_{L^2}) + O(\|U\|_{H^1}^3) \\ \leq \quad & 4en^2 + C_1\|\partial\dot{U}(\cdot)\|_{L^2}^2 - C_2\,|[e_r, U_0]|\,(|[e_r, U_0]| - \|\partial\dot{U}(\cdot)\|_{L^2})\end{aligned}$$

if $\|U\|_{H^1}$ is small enough. We find, therefore, that

Lemma 7.5. *Let $(\alpha, E_{r,a}(\cdot).e^{U(\cdot)+U_0})$ be a perturbation (small enough in H^1) of the cocycle $(\alpha, E_{r,a}(\cdot))$. Let also, without loss of generality, $\hat{U}(0) = 0$ and $U_0 \neq 0$. Then, there exists a constant C_0 depending only on r such that if*

$$\|\partial U\|_{L^2} \leq C_0|U_0|$$

then $(\alpha, E_{r,a}(\cdot).e^{U_0}e^{U(\cdot)})$ is of energy strictly smaller than that of $(\alpha, E_{r,a}(\cdot))$.

This was first proved in [Kri01], but the constant implicitly depended on α.

7.5 End of the proof of lemma 7.1

Using the estimates obtained so far, we can prove, lemma 7.1:

Proof of lemma 7.1. The estimate on the L^2 norm of the derivative of the first iterate implies that

$$\|\Lambda_r U(\cdot)\|_{L^2} \leq C' \|(Id - \Lambda_r)\partial U(\cdot)\|_{L^2}$$

for some constant C'. Since

$$\|\Lambda_r^0 U(\cdot)\|_{L^2}^2 = \|\Lambda_r U(\cdot)\|_{L^2}^2 + |\hat{U}(0)|^2$$

$\left\|\Lambda_r^0 U(\cdot)\right\|_{L^2} \geq M \left\|(Id - \Lambda_r^0)\partial U(\cdot)\right\|_{L^2}$ for M big enough would imply that $\|\partial U\|_{H^1} \leq C_0|U_0|$, which contradicts the hypothesis. □

7.6 Further estimates

Use of the invariance of the degree under C^1 conjugation allows us in fact to control some of the low frequencies in the spectrum of the perturbation. It follows immediately from the definitions that, for $B(\cdot) = \exp(U(\cdot))$,

$$\begin{aligned} Conj_{B(\cdot)}(\alpha, E_{r,a}(\cdot).e^{U(\cdot)}) &= (\alpha, e^{U(\cdot+\alpha)}.E_{r,a}(\cdot)) \\ &= (\alpha, E_{r,a}(\cdot).e^{V(\cdot)}) \end{aligned}$$

with $V(\cdot) = Ad(E_{r,a}^*(\cdot)).e^{U(\cdot+\alpha)}$, so that

$$\hat{V}_\rho(k) = e^{2i\pi k\alpha}\hat{U}_\rho(k + 2r_\rho)$$

Inserting this relation in the estimates obtained in the previous sections gives the estimate

$$\|\Lambda_{-\tilde{r}}^0 U\|_{L^2} \lesssim \|(Id - \Lambda_{-\tilde{r}}^0)\partial U\|_{L^2}$$

More generally, since conjugation by a periodic geodesic $E_{r'}$ acts by translation on the frequencies of $U(\cdot)$ [1], we can see conjugation as an action on the truncation operator. If, now, Λ is a translate of $\Lambda_{\tilde{r}}^0$ under this action, then we have the same type of estimate, with Λ in the place of $\Lambda_{-\tilde{r}}^0$.

Finally, the conjugation $D(\cdot) = \exp(U(\cdot)/2)$ imitates the effects of two iterations and normalization of the derivative, since

$$\begin{aligned} Conj_{D(\cdot)}(\alpha, E_{r,a}(\cdot).e^{U(\cdot)}) &= (\alpha, e^{U(\cdot+\alpha)/2}.E_{r,a}(\cdot).e^{U(\cdot)/2}) \\ &= (\alpha, E_{r,a}(\cdot).e^{V(\cdot)/2}.e^{U(\cdot)/2}) \end{aligned}$$

with $V(\cdot)$ as previously, since

$$\begin{aligned} (\alpha, E_{r,a}(\cdot).e^{U(\cdot)})^2 &= (2\alpha, E_{r,a}(\cdot + \alpha).e^{U(\cdot+\alpha)}.E_{r,a}(\cdot).e^{U(\cdot)}) \\ &= (2\alpha, E_{r,a}(2\cdot + \alpha).e^{V(\cdot)}e^{U(\cdot)}) \end{aligned}$$

A calculation shows, however, that the effect of non-commutativity of e_r with the constant part of the perturbation is not seen in $L(E_{r,a}(\cdot).e^{V(\cdot)/2}.e^{U(\cdot)/2})$. As a result, estimation of the energy of the path of $E_{r,a}(\cdot).e^{V(\cdot)/2}.e^{U(\cdot)/2}$ gives only the trivial estimate

$$\|u(\cdot)\|_{L^2}^2 \geq |\int \langle u^*(\cdot + \alpha), Ad(E_{r,a}(\cdot)).u(\cdot)\rangle|$$

which shows, nonetheless, that the use of Cauchy-Schwartz inequality in the estimation of the energy of the path of the second iterate does not result into too rough estimates.

1. It also changes the constant A, by it is not important since the constant does not enter in the estimates.

7.7 Proof of theorem 7.2

Let $E_{r,a}(\cdot)$ be a periodic geodesic in G, not necessarily regular, and the corresponding splitting $G_0 \times G_+ \hookrightarrow G$. If the zero energy component is not trivial, and $A_0 e^{U_0(\cdot)} \in C^\infty(\mathbb{T}, G_0)$ is a perturbation of a constant $A_0 \in G_0$, it is clear that $(\alpha, E_{r,a}(\cdot)A_0 e^{U_0(\cdot)})$ is of the same degree, for an irrational α, since renormalization of the perturbed cocycle will converge to some $E_{r,a'}(\cdot)$. On the other hand, the preceding estimates show that a small perturbation spectrally supported in some I_ρ with $\rho(e_r) \neq 0$, and therefore not commuting with $E_{r,a}(\cdot)$, decreases the energy of the cocycle. The same thing holds for renormalization representatives of cocycles non-homotopic to constants. If $A \in G_0$ anti-commutes with $\exp(H\cdot) : 2\mathbb{T} \to G_0$, then the G_0-valued cocycle is of energy 0. Therefore, no G_0-valued perturbation can change the energy of the cocycle. We also point out that the perturbed cocycle is in the same homotopy class as $E_{r,a}(\cdot)$.

Therefore, the picture is the following. If we renormalize a cocycle in a given class of homotopy, the scheme converges towards an action within the same class, which can be of positive or zero energy. Let us begin the study with the two simplest non-abelian groups, $SU(2)$ and $SO(3)$, where all geodesics are regular, but $SO(3)$ is not simply connected and $\chi_{SO(3)} = 2$, while $SU(2)$ is simply connected and $\chi_{SU(2)} = 1$.

Cocycles in $SU(2)$ and $SO(3)$

The positive energy abelian models in $SU(2)$, all of them regular, are given by the periodic geodesics $E_r(\cdot)$, $r \in \mathbb{N}^*$

$$\begin{pmatrix} e^{2i\pi r\cdot} & 0 \\ 0 & e^{-2i\pi r\cdot} \end{pmatrix}$$

These models, after projection to $SO(3)$, become the positive energy models which are homotopic to the identity:

$$\begin{pmatrix} \cos(4\pi r\cdot) & \sin(4\pi r\cdot) & 0 \\ -\sin(4\pi r\cdot) & \cos(4\pi r\cdot) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Perturbation of such a model, say by a small constant in non-standard configuration (which we will note by A throughout this paragraph) reduces the energy. In the special case of $E_1(\cdot)$, such a perturbation will yield a cocycle which, after renormalization, will converge to close-to-constant dynamics. Consideration of its homotopy class (or, equivalently, the fact that it admits a 1-periodic lift to $SU(2)$), implies that in $SO(3)$ as in $SU(2)$, it can be normalized modulo 1, and passage to a 2-lattice is not necessary. Therefore, in both groups, such models are accumulated by cocycles of 0 energy modulo 1.

In $SO(3)$, however, we also have the class of geodesics non-homotopic to constants. given by the periodic geodesics $E_{r+1/2}(\cdot)$, $r \in \mathbb{N}$:

$$\begin{pmatrix} \cos(2\pi(2r+1)\cdot) & \sin(2\pi(2r+1)\cdot) & 0 \\ -\sin(2\pi(2r+1)\cdot) & \cos(2\pi(2r+1)\cdot) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Such geodesics, seen in $SU(2)$, emanate from the Id, and after having passed r times by the Id end in $-Id$. Clearly, the second iterate of such a cocycle is homotopic to constants. The same calculations as before show that a perturbation of $E_{r+1/2}(\cdot)$ with $r > 0$ by A drives the renormalization towards some $E_{r'+1/2}(\cdot)$, with $r' < r$. A remarkable phenomenon is encountered when we reach the normal form $E_{1/2}(\cdot)$, which is the minimal energy model of dynamics simultaneously non-reducible and non-homotopic to constants. Once again, the estimations show that renormalization of $E_{1/2}(\cdot).A$ will converge towards constant actions, or equivalently that for n big enough $\tilde{\mathcal{R}}^n\Phi$ (the action associated to a $(\alpha, E_{1/2}(\cdot).A)$) will be close to a constant action. The consideration of its homotopy class shows that in this case we will have to iterate $\tilde{\mathcal{R}}^n\Phi(e_2)$ before normalizing the action. Therefore, $(\Lambda_2)_*\tilde{\mathcal{R}}^n\Phi$, and consequently the second iterate of $E_{1/2}(\cdot).A$, is in the class of local dynamics $\mod 2$, whereas $E_{1/2}(\cdot).A$ itself cannot be. In fact, Φ is expected to converge

towards constants like, say

$$\begin{aligned}\pi_2(\tilde{\mathcal{R}}^n(e_1)) &= \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ \pi_2(\tilde{\mathcal{R}}^n(e_2)) &= \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\end{aligned}$$

who commute, but are not on the same maximal torus. Normalization of the action itself drives it back to the neighborhood of translate of $E_{1/2}(\cdot)$, since the natural choice is the conjugant $E_{1/2}(\frac{1}{2}\cdot)$, and

$$E_{1/2}(\frac{1}{2}(\cdot+\alpha)).\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}.E_{1/2}(\frac{1}{2}\cdot)^T = E_{1/2}(\frac{1}{2}\alpha)\begin{pmatrix}\cos(2\pi\cdot) & \sin(2\pi\cdot) & 0\\ \sin(2\pi\cdot) & -\cos(2\pi\cdot) & 0\\ 0 & 0 & -1\end{pmatrix}$$

On the other hand, the square of both constants is the Id, so that $(\Lambda_2)_*\tilde{\mathcal{R}}^n\Phi$ can be normalized without any problem. The factor 2 in Λ_2 in this case is due to the fact that $\chi_{SO(3)} = 2$. In fact, this very example shows that there exist cocycles in $\mathbb{T}\times SO(3)$ which become reducible after iteration, since, if we call

$$\begin{aligned}A(\cdot) &= \begin{pmatrix}\cos(2\pi\cdot) & \sin(2\pi\cdot) & 0\\ \sin(2\pi\cdot) & -\cos(2\pi\cdot) & 0\\ 0 & 0 & -1\end{pmatrix}\\ &= \begin{pmatrix}\cos(2\pi\cdot) & -\sin(2\pi\cdot) & 0\\ \sin(2\pi\cdot) & \cos(2\pi\cdot) & 0\\ 0 & 0 & 1\end{pmatrix}.\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}\end{aligned}$$

then, for any irrational α, $\pi_2(2\alpha, A_2(\cdot))$ is equal to

$$\begin{pmatrix}\cos(2\pi\alpha) & -\sin(2\pi\alpha) & 0\\ \sin(2\pi\alpha) & \cos(2\pi\alpha) & 0\\ 0 & 0 & 1\end{pmatrix}$$

However, $(\alpha, A(\cdot))$ itself is not reducible for any α, since $A(\cdot)$ is non-homotopic to the Id.

Schematically, we can say that renormalization of $E_{1/2}(\cdot).A$ converges towards constant actions, but normalization without iteration drives us back away from constant cocycles. Iteration and normalization allow us to conjugate the dynamics to a model close to a constant (which in fact is in the conjugacy class of (α, Id)). We can say that, if we do not authorize iteration, the geodesic $E_{1/2}(\cdot)$ takes up the role of constants in its homotopy class.

Cocycles of smaller energy in the neighborhood of regular geodesics

In this case, the value of χ_G is irrelevant with the normalization of the action, since we can conjugate the action to a normalized one and arbitrarily close to some $(\alpha_n, E_{r,a}(\cdot))$, without iterating. Any small enough constant non-diagonal perturbation of such a model reduces the energy, but in general it may lead renormalization to singular dynamics and not directly to constants. For example, renormalization of

$$\begin{pmatrix}e^{2i\pi\cdot} & 0 & 0\\ 0 & e^{4i\pi\cdot} & 0\\ 0 & 0 & e^{-6i\pi\cdot}\end{pmatrix}.\exp\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & z\\ 0 & -\bar{z} & 0\end{pmatrix}$$

in $SU(3)$, with $z\in\mathbb{C}$ small, may converge to $\begin{pmatrix}e^{2i\pi\cdot} & 0 & 0\\ 0 & e^{2i\pi\cdot} & 0\\ 0 & 0 & e^{-4i\pi\cdot}\end{pmatrix}$, which is singular. In the neighborhood of a singular geodesic χ_{G_0} becomes relevant, since normalization without iteration

may, as in $SO(3)$, drive the algorithm away from constant actions. Therefore, in general the local picture in the neighborhood of regular geodesics is that such models are accumulated by cocycles of smaller energy, which however may be conjugate (close) to the corresponding geodesics or (close) to constant actions only modulo $1 \leq m \leq \chi_G$.

The picture is simpler in groups like $SU(w+1)$ where the constant equals one, and therefore all cocycles of positive energy are accumulated by cocycles of smaller energy, which can be conjugated (close) to the corresponding geodesics or (close) to constant actions modulo 1.

Cocycles of smaller energy in the neighborhood of singular geodesics

In the case of a singular cocycle of positive energy, renormalization and conjugation of actions is not affected until it reaches the form of eq. 6.7, which is (after some simplification of the notation)

$$\begin{pmatrix} (1, C_0.\exp(O(\varepsilon_n))) \\ (\alpha_n, A_0.\exp(e_r \cdot + O(\varepsilon_n))) \end{pmatrix}$$

with $C_0 \in G_0$. We also recall the corresponding splitting $G_0 \times G_+ \hookrightarrow G$, and suppose that there are no error terms, i.e. that

$$\tilde{\mathcal{R}}^{(n)}\Phi = \begin{pmatrix} (1, C_0) \\ (\alpha_n, A_0.\exp(e_r \cdot)) \end{pmatrix}$$

Since $\mathbf{a}(0)$ is singular, depending on the homotopy of the cocycle in G_0, we may not be able to conjugate C_0 to the Id with a conjugant commuting with A_0, but this can be assured by iterating at most $\chi_0 = \chi_{G_0}$ times. This fact is a more general case of the phenomenon observed in perturbations of $E_{1/2}(\cdot)$-like models in $SO(3)$. An action like the one we are studying may be associated to a cocycle non-homotopic to the Id, and renormalization may converge to constants for a part of the non-triviality of its homotopy class while the rest is still close to a geodesic in the same homotopy class. At this step, we can iterate χ_0 times and obtain a cocycle of singular dynamics, but whose homotopy within the 0 energy component allows conjugation of the action to the actual geodesic. This action, $(\Lambda_{\chi_0})_*\tilde{\mathcal{R}}^{(n)}\Phi$, can thus be conjugated to

$$\begin{pmatrix} (1, Id) \\ (\chi_0\alpha_n, A_0.\exp(\chi_0 e_r \cdot)) \end{pmatrix}$$

with a different A_0, but still commuting with $\mathbf{a}(0)$. A constant non-abelian perturbation of $A_0.\exp(\chi_0\mathbf{a}(0)\cdot)$ in a complex direction j_ρ such that $[e_r, j_\rho] \neq 0$ will drive renormalization to a lower energy model.

We can, nonetheless, perturb $\tilde{\mathcal{R}}^{(n)}\Phi$ to a lower energy cocycle without iterating. Let $H_0 \in g_0$ be a preimage of C_0. Then, conjugation by $\exp(-H_0\cdot)$ gives the action

$$\begin{pmatrix} (1, Id) \\ (\alpha_n, \exp(e_r \cdot).A_0(\cdot)) \end{pmatrix}$$

where the zero-energy component $A_0(\cdot) = \exp(-H_0(\cdot + \alpha_n)).A_0.\exp(H_0\cdot)$ may not be constant. If we chose a root ρ such that $h_\rho \in g_+$, then j_ρ commutes with all vectors in g_0, so that the action

$$\begin{pmatrix} (1, Id) \\ (\alpha_n, \exp(e_r \cdot).A_0(\cdot).\exp(z j_\rho)) \end{pmatrix}$$

will be of smaller energy. Iteration of this procedure, and renormalization and normalization of the resulting actions will eventually give a cocycle of minimal energy in the homotopy class of the initial cocycle and the associated action

$$\begin{pmatrix} (1, Id) \\ (\alpha_n, \exp(e_{r'} \cdot).A_0) \end{pmatrix}$$

where A_0 is a different constant and $e_{r'}$ will be 0 iff the cocycle were homotopic to constants. If we renormalize a m-lattice of such an action, with $1 \leq m \leq \chi_G$ depending only on the homotopy class of the cocycle, renormalization will converge to constants.

Chapter 8

Perturbations of regular geodesics

Based on the a priori estimates obtained in the previous chapter and on techniques developed in [Kri99a] (chapter 4) and in [Kri01], we begin the local study of regular obstructions by proving that under a Diophantine condition on the rotation in the basis and a smallness condition (related to the Diophantine condition) on the perturbation, the cocycle $(\alpha, E_{r,a}(\cdot).e^{U(\cdot)})$ can be reduced to $(\alpha, E_{r,a'}(\cdot).e^{P(\cdot)})$, where a' is a constant in the torus, and $P(\cdot)$ is a Fourier polynomial spectrally supported in $I_{\tilde{r}}$. Already, the estimates and the discussion of the previous chapter show that, unless $P(\cdot) = 0$, such a $(\alpha, E_{r,a'}(\cdot).e^{P(\cdot)})$ is not of degree r, which gives a good description of the local structure of the conjugacy class of $(\alpha, E_{r,a}(\cdot))$, where in fact the diagonal constant a is not very important.

The motivation of the result becomes easier after the study of the linearized equation, at the end of section 8.2, but let us briefly describe the procedure. In the previous chapter we saw that some low frequencies in the spectrum of the perturbation of a regular geodesic do not allow, if they are dominant, the perturbed cocycle to be of energy $|e_r|$. It is reasonable, therefore, to think of them as obstructions to reducibility to $E_{r,a}(\cdot)$. The first reaction to this observation was that of R. Krikorian in [Kri01], where he supposed that a given cocycle in $\mathbb{T} \times SU(2)$, perturbation of a periodic geodesic of the group (automatically regular), is actually of the same degree as the geodesic. Then, he showed that if the rotation is Diophantine, a convergent K.A.M. scheme can be defined which constructs a conjugation reducing the cocycle to the periodic geodesic. At each step, the obstructions to reducibility were shown to die out along with the perturbation thanks to the assumption on the degree.

A careful study of the procedure shows that a K.A.M. scheme (and not a fixed-point argument) is needed because there is a loss of derivatives in the estimates, which, however, is uniform throughout the scheme. Therefore, it is possible that a more accurate result can be obtained, and this is done by dropping the assumption that the given cocycle is of the same degree as the geodesic, but retaining only the assumption that the perturbation is small. Then, the Nash-Moser inverse function theorem is used in order to show the following. We can split any given perturbation, at least in the level of linear terms, into the obstructions plus a coboundary. The obstructions cannot be eliminated, but we can consider them as part of the model around which linearize. Then, the coboundary can still be eliminated, thanks to the absence of small divisors. The fact that this procedure can be carried out with uniform estimates provided that the obstructions are small enough, allows the use of the inverse function theorem, which guarantees that the exact problem can be solved. In other words, we can split any perturbation into two parts. The first one, $P(\cdot)$, is the one that we cannot hope to eliminate. But if we consider the model $(\alpha, E_{r,a}(\cdot)e^{P(\cdot)})$ as the model of the dynamics, then the remaining part of the perturbation can be eliminated by conjugation. Finally, the a priori estimates show that the cocycle is of degree r if, and only if, this principal part $P(\cdot) = 0$.

8.1 Synopsis of the chapter

In this section we apply Hamilton's inverse function theorem in order to obtain a normal form theorem for cocycles in the neighborhood of a regular geodesic. We remind that $\mathcal{P}_r$ was introduced in the previous chapter (def. 7.3). We denote by $\mathcal{E}_0$ the space of mappings in $f \in C^\infty(\mathbb{T}, g)$ such that $\int \pi_{\mathfrak{t}}(f) = 0$. This tame Fréchet space inherits its topology from $\mathcal{E} = C^\infty(\mathbb{T}, g)$, which in turn is the topology induced by the family of seminorms

$$\|U\|_s = \max_{0 \leq \sigma \leq s} (\|\partial_\sigma U\|_\infty)$$

for $U \in \mathcal{E}$. We bring to the attention of the reader that these norms correspond to the norms $\|U\|_s^{\max}$ in the notation of the first chapter. Finally, $\mathfrak{t}$ denotes the unique maximal toral algebra of g containing e_r.

With these notations, the normal form theorem reads

Theorem 8.1. *Let $\alpha \in DC(\gamma, \tau)$, $r \in \mathbb{Z}^w$ such that $e_r \in \mathfrak{t}$ be regular, and $a \in \mathfrak{t}$. Then there exists $\mathcal{W} = \mathcal{W}(\alpha, r)$, a neighborhood of 0 in $\mathcal{E}$, such that for any $U(\cdot) \in \mathcal{W}$, there exists a unique $(\lambda, B, P) \in \mathfrak{t} \times \mathcal{E}_0 \times \mathcal{P}_r$ such that*

$$E_{r,a}(\cdot).e^{U(\cdot)} = e^{B(\cdot+\alpha)}.E_{r,a+\lambda}(\cdot).e^{P(\cdot)}.e^{-B(\cdot)}$$

The triple (λ, B, P) satisfies tame estimates with respect to U and λ depends continuously on it.

We remark that the important characteristic of $\mathcal{P}_r$ is not its support in the space of frequencies, but rather the fact that $\mathcal{P}_r$ and $Ad(E_{r,a}(\cdot)).\mathcal{P}_r$ have disjoint supports, for all $a \in \mathfrak{t}$, as we have seen in the previous chapter.

We also have the following corollary.

Corollary 8.2. *The orbits of $(\alpha, E_{r,a}(\cdot))$, with $a \in \mathfrak{t}$ and $\alpha \in DC(\gamma, \tau)$, under the adjoint action of $\mathcal{E}_0$ form locally a tame Fréchet manifold of codimension $2\sum_{\rho \in \Delta_+} |\tilde{r}_\rho| + w$ in $SW^\infty(\mathbb{T}, G)$. Or, more importantly, reducibility to a normal form is a property with codimension $2\sum_{\rho \in \Delta_+} |\tilde{r}_\rho|$ in the neighborhood of $(\alpha, E_{r,a}(\cdot))$ in $SW^\infty_\alpha(\mathbb{T}, G)$.*

A direct application in this particular case of the a priori estimates on perturbations of regular geodesics shows that a such cocycle in normal form is of energy smaller than that of $(\alpha, E_{r,a}(\cdot))$, unless $P(\cdot) = 0$. We thus obtain

Theorem 8.3. *Reducibility to a regular periodic geodesic over a Diophantine rotation $(\alpha, E_{r,a}(\cdot))$ is locally of codimension $2\sum_{\rho \in \Delta} |\tilde{r}_\rho|$. All cocycles of degree r in this neighborhood of $E_{r,a}(\cdot)$ are reducible to a normal form.*

Combining this theorem with the renormalization scheme, we obtain the following result.

Theorem 8.4. *If $\alpha \in RDC$ and the cocycle $(\alpha, A(\cdot))$ is regular of degree r, then $(\alpha, A(\cdot))$ is C^∞-conjugate to $(\alpha, E_{r,a}(\cdot))$ for some $a \in \mathfrak{t}$.*

Proof. If the cocycle $(\alpha, A(\cdot))$ is regular, then, by theorem 6.3, renormalization converges towards models of the type $(\alpha_n, E_{r,a}(\cdot))$. We remind that under the assumption that the cocycle is regular, the constant χ_G becomes irrelevant, and therefore there exist renormalization representatives $(\alpha_n, \tilde{A}_n(\cdot))$ arbitrary close in the C^∞ topology to cocycles of the type $(\alpha_n, E_{r,a}(\cdot))$.

Since $\alpha \in RDC(\gamma, \tau)$, $\alpha_n \in DC(\gamma, \tau)$ infinitely many times. Therefore, the α_n can be chosen uniformly Diophantine and the normal form theorem can be applied to some $(\alpha_n, \tilde{A}_n(\cdot))$. The hypothesis that $(\alpha_n, \tilde{A}_n(\cdot))$ is of degree r allows us to conclude. □

Finally, for cocycles in $\mathbb{T} \times SU(2)$ we have the following result.

Theorem 8.5. *If $\alpha \in RDC$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, SU(2))$ is of non-zero degree, then it is C^∞-conjugate to $(\alpha, E_r(\cdot + \theta))$ for some $\theta \in \mathbb{T}$. Reducible cocycles in $\mathbb{T} \times SU(2)$ are dense in the total space, if the rotation satisfies an RDC.*

Proof. Every non-zero vector in $su(2)$ is regular, so that all positive energy cocycles are regular too. Therefore, as in the proof of theorem 8.4, we find that all cocycles of positive energy are conjugate to the corresponding periodic geodesic.

Since every such cocycle is accumulated by cocycles of lower energy, after a finite number of arbitrarily small perturbations we can obtain a cocycle of 0 energy. Then, renormalization of such a cocycle converges to constants (we remind that $\chi_{SU(2)} = 1$), and the local density theorem 3.5 allows us to conclude. □

This density result was first obtained in [Kri01] under a somewhat stricter (and easily recognizable as an artefact of the method) arithmetic condition of full measure in $\mathbb{T}$. The calculations in the proofs already indicate the existence of a local Fréchet manifold of cocycles reducible to the normal form $(\alpha, E_r(\cdot + \theta))$ and show that cocycles of smaller energy form an open dense set in the neighborhood of normal forms. The arithmetic condition imposed was loosened to a simple *RDC* in [Fra04], but without the stronger result on the codimension and the local description of the conjugacy classes.

8.2 Local reduction lemmas

Let us firstly write and solve the linearized cohomological equation of reduction of the perturbation to a second-order one. We suppose that conjugation by a small $B(\cdot) \in C^\infty(\mathbb{T}, g)$ reduces the perturbation $U(\cdot)$ to a smaller one, noted by $V(\cdot)$

$$\exp(B(x+\alpha)).E_{r,a}(x).\exp(U(x)).\exp(-B(x)) = E_{r,a}(x).\exp(V(x))$$

Linearization of this equation, under the assumption that $U(\cdot)$, $D(\cdot)$ and $V(\cdot)$ are small in C^0 and $V(\cdot) \ll U(\cdot)$, i.e. is of second order with respect to $U(\cdot)$, gives

$$Ad(E_{-r,-a}(x)).B(x+\alpha) - B(x) = -U(x)$$

Introducing the coordinates $B(\cdot) = \sum_{\rho\in\tilde{\Delta}} B^{\mathrm{t}}_\rho(\cdot).h_\rho + \sum_{\rho\in\Delta_+} B_\rho(\cdot).j_\rho$, as for $U(\cdot)$, we have the equations

$$B^{\mathrm{t}}_\rho(\cdot+\alpha) - B^{\mathrm{t}}_\rho(\cdot) = -U^{\mathrm{t}}_\rho(\cdot),\ \rho \in \tilde{\Delta} \tag{8.1}$$

$$e^{-2i\pi(\tilde{r}_\rho\cdot + a_\rho)}.B_\rho(\cdot+\alpha) - B_\rho(\cdot) = -U_\rho(\cdot),\ \rho \in \Delta_+ \tag{8.2}$$

where $2\pi\tilde{r}_\rho = \langle e_r, h_\rho\rangle \neq 0$ and $r_\rho \in \mathbb{N}^*$ for all $\rho \in \tilde{\Delta}$ (but in general, $r_\rho \in \mathbb{Z}^*$, for $\rho \in \Delta_+$).

The set of equations involving the coordinates on the torus (eq. (8.1)) are classical linear cohomological equations for which we refer to lemma 3.4. We remind that the only topological obstruction to the solution of the equation concerns the mean value of the rhs function, which must be 0.

Solving the equation (8.2) is less classical, so let us consider its general form,

$$e^{2i\pi(mx+c)}f(x+\alpha) - f(x) = g(x)$$

with $m \in \mathbb{Z}^*$ and $c \in \mathbb{T}$. The constant c can easily be seen to be irrelevant, since composition by a translation $x \mapsto x - c/m$ gives the equation the form

$$e^{2i\pi mx}f(x+\alpha) - f(x) = g(x)$$

where we have kept the same notation for $f(\cdot)$ and $f(\cdot - c/m)$, similarly for $g(\cdot)$.

Application of the Fourier transform to this last form of the equation gives

$$e^{2i\pi(k-m)\alpha}\hat{f}(k-m) - \hat{f}(k) = \hat{g}(k) \tag{8.3}$$

for all $k \in \mathbb{Z}$.

Let us suppose for simplicity that m is positive, the remaining case being treated similarly. The forward solution of the equation is then found by forward iteration of the following formula:

$$\hat{f}(k) = e^{-2i\pi k\alpha}(\hat{g}(k+m) + \hat{f}(k+m))$$

This gives

$$\hat{f}_+(k) = \sum_{j=1}^{\infty} e^{-2i\pi j(k+\frac{j-1}{2}m)\alpha}\hat{g}(k+jm) \tag{8.4}$$

Backward iteration of the formula

$$\hat{f}(k) = \hat{g}(k) + e^{2i\pi(k-m)\alpha}\hat{f}(k-m)$$

gives the backward solution

$$\hat{f}_-(k) = \sum_{j=0}^{\infty} e^{2i\pi j(k-\frac{(j+1)}{2}m)\alpha}\hat{g}((k-j)m) \tag{8.5}$$

The forward and the backward solution are constructed independently, and $f_+(\cdot) = \sum_k \hat{f}_+(k)e^{2i\pi k\cdot}$ (resp. $f_- = (\cdot)\sum_k \hat{f}_-(k)e^{2i\pi k\cdot}$) is in $C^\infty(\mathbb{T}, g)$ only if the summation is for $k > l$ (rep. $k \leq l'$) where $l \in \mathbb{Z}$ (resp. l') is to be fixed. Clearly, determination of all the Fourier coefficients of the solution imposes that $l' = l - 1$. If, now we define $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(k) = \begin{cases} \hat{f}_+(k), & k > l \\ \hat{f}_-(k), & k \leq l \end{cases}$$

Then, $\hat{f}(\cdot)$ is the Fourier series of a function $f : \mathbb{T} \to \mathbb{C}$ which solves the eq. 8.3 for $k \notin \{l+1, ..., l+m\}$, as shows the form of the equation. For the remaining values of k, we find that

$$e^{2i\pi(k-m)\alpha}\hat{f}(k-m) - \hat{f}(k) = e^{2i\pi(k-m)\alpha}\hat{f}_-(k-m) - \hat{f}_+(k)$$

The a priori estimates obtained in the previous chapter suggest that we chose $l = -m$, since, with this choice, the set of frequencies in $\{l+1, ..., l+m\}$ is equal to the set I_ρ of def. 7.3 if $m = r_\rho$. In other words, the frequencies in $\{l+1, ..., l+m\}$ are, for a given choice of l, the obstruction to the solution of eq. 8.3 and we use the arbitrariness of the choice of l in order to place the obstructions in the frequencies controlled by the a priori estimates.

For this reason, we will abuse the notation of I_ρ and denote the set $\{-m+1, ..., 0\}$ by I_m.

For the given choice of l, depending on m, we define $\hat{f} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{f}(k) = \begin{cases} \hat{f}_+(k), & k > m \\ \hat{f}_-(k), & k \leq m \end{cases}$$

and $f : \mathbb{T} \to \mathbb{C}$ is the inverse Fourier transform of $\hat{f}(\cdot)$.

Since, for any $p \in \mathbb{N}$, $|k|^p|\hat{g}(k)| \to 0$ as $|k| \to \infty$, we obtain the estimate

$$\|f\|_s \leq C_s \|g\|_{s+3}$$

and for truncations up to an arbitrary order N, we find that

$$\begin{aligned} \|T_N f\|_s &\leq C_s N^2 \|g\|_s \\ \|R_N f\|_s &\leq C_{s,s'} N^{s-s'+3} \|g\|_{s'} \end{aligned}$$

with $s \geq s' \geq 0$. Clearly, if for any given m we choose $l = -m$, the constants in the estimates depend only on m.

We have therefore proved

Proposition 8.6. *The function $f = \mathcal{F}^{-1}(\hat{f})$ as defined above solves the equation*

$$f(x+\alpha) - e^{2i\pi mx} f(x) = g(x) - \Gamma_m g(x)$$

where $\mathcal{F}\{\Gamma_m g\}(k) = \hat{g}(k) - (e^{2i\pi(k-m)\alpha}\hat{f}_-(k-m) - \hat{f}_+(k))$ for $k \in \{-m+1, .., 0\}$ and 0 otherwise, is a trigonometric polynomial of order at most $m-1$. The following estimates hold for all $s \geq 0$:

$$\begin{aligned} \|f\|_s &\lesssim \|g\|_{s+3} \\ \|\Gamma_m g\|_s &\lesssim \|g\|_s \end{aligned}$$

where the constants depend only on m and s.

The mapping $\Phi : g \mapsto (f, \Gamma_m g)$ is an invertible continuous linear application from $H^{s+3}(\mathbb{T}, \mathbb{R})$ to $H^s(\mathbb{T}, \mathbb{R}) \times \{\mathcal{F}^{-1}(\sum_{I_m} a_k e^{2i\pi k\cdot}), a_k \in \mathbb{C}\}$.

Summing up what we have proved, we can state the following proposition.

Proposition 8.7. *Let $E_{r,a}(\cdot) = \exp(\sum_{\rho\in\tilde{\Delta}} 2i\pi h_\rho(r_\rho \cdot + a_\rho))$ be a regular periodic geodesic, $U(\cdot) = \sum_{\rho\in\tilde{\Delta}} U^{\mathfrak{t}}_\rho(\cdot).h_\rho + \sum_{\rho\in\Delta_+} U_\rho(\cdot).j_\rho$ be small enough in $C^{\tau+3}(\mathbb{T}, g)$ and $\alpha \in DC(\gamma, \tau)$.*

Then, there exist $B(\cdot)$ and $\Gamma_{\tilde{r}}U(\cdot)$ (notice the abuse in the notation) in $C^\infty(\mathbb{T}, g)$, where

$$\begin{aligned} B(\cdot) &= \sum_{\rho\in\tilde{\Delta}} B^{\mathfrak{t}}_\rho(\cdot).h_\rho + \sum_{\rho\in\Delta_+} B_\rho(\cdot).j_\rho \\ \Gamma_{\tilde{r}}U(\cdot) &= \sum_{\rho\in\Delta_+} \Gamma_\rho U_\rho(\cdot).j_\rho = \sum_{\rho\in\Delta_+}\sum_{k\in I_\rho} p_\rho(k) e^{2i\pi k\cdot}.j_\rho \end{aligned}$$

solving the equation

$$Ad(E_{r,a}(\cdot)^*).B(\cdot+\alpha) - B(\cdot) = -U(\cdot) + \Gamma_{\tilde{r}}U(\cdot) + \sum_{\rho\in\tilde{\Delta}} \hat{U}^{\mathfrak{t}}_\rho(0).h_\rho$$

with the following estimates:

$$\begin{aligned} \|B(\cdot)\|_s &\leq \gamma C_s \|U(\cdot)\|_{s+\tau+2} \\ \|\Gamma_{\tilde{r}}U(\cdot)\|_s &\leq C_s \|U(\cdot)\|_s \end{aligned}$$

where the constants depend only on r, but not on a or α.

The $B^{\mathfrak{t}}_\rho(\cdot)$ are unique up to an additive constant in $\mathbb{R}$ (thus unique if we impose that their constant Fourier coefficient be 0). The rest $\Gamma_r U(\cdot)$ is uniquely determined by $U(\cdot)$ and the choice of its spectral support.

Let us also define the spaces in which the obstructions take values

Definition 8.8. We will denote by $\mathcal{P}_{\tilde{r}}$ the image of the mapping $\Gamma_{\tilde{r}}$ in $C^\infty(\mathbb{T}, g)$ and by $\mathcal{P}^0_{\tilde{r}}$ the space $\mathfrak{t} \oplus \mathcal{P}_{\tilde{r}}$.

This proposition indicates that a K.A.M.-like scheme can be used, just as in [Kri01]. The scheme is defined by replacing the perturbation $U(\cdot)$ by a Fourier-truncation of a sufficiently high order N, and solving the linearized equation for the low frequencies. This reduces the perturbation to the sum of a term of order $C_s N^a \|U\|_0 \|U\|_s$ and the rest $\Gamma_h U(\cdot)$. The hypothesis that the cocycle is of degree r gives the estimate

$$\|\Lambda^0_{\tilde{r}}U(\cdot)\|_s \lesssim \|(Id - \Lambda^0_{\tilde{r}})U(\cdot)\|_s$$

This estimate implies that the rest is in fact of second order with respect to the original perturbation, and the reduction can be iterated. The congergence of the scheme shows that all small enough perturbations of $(\alpha, E_{r,a}(\cdot))$ of degree r are C^∞ conjugate to a $(\alpha, E_{r,a'}(\cdot))$. The reason why this is a purely local theorem, i.e. if the conjugation exists it can be chosen small, is the fact that the arithmetic properties of the diagonal constant a in $E_{r,a}(\cdot)$ are irrelevant and the only small denominator phenomena are those appearing in the reduction of the toral coordinates. Since,

however, they are related only to α, and not to a, they are uniform throughout the K.A.M. scheme and do not pose any problems to the convergence of the scheme.

This fact enables us to choose a Nash-Moser inverse function theorem approach (see [Ham82]). The proof of such a theorem consist essentially in the construction of a convergent classical K.A.M. scheme which replaces the fixed point argument in the standard inverse function theorem for Banach spaces. Heuristically, the theorem states that if the equation of reduction to the normal form has can be uniquely solved in all the tangent spaces in a neighborhood of 0 and with uniform estimates and uniform loss of derivatives, then the K.A.M. scheme converges. In other words, if we can make one step of the scheme (i.e. solve the linearized equation) in each tangent space, but with uniform estimates and loss of derivatives, then the inversion of the mapping by the Nash-Moser theorem does the rest of the convergent scheme.

Nonetheless, K.A.M. theory is not made obsolete by the Nash-Moser theorem, since the latter is, as all such theorems, of purely local nature. As a consequence, an inverse function approach can only construct conjugations of the same order as the perturbation of the normal form. On the other hand, one can construct reducible cocycles for which the hypothesis (apart from regularity assumptions) of the theorem (i.e. the resolution of the linearized equation with good estimates) fails. Consider, eg, $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ and $k \in \mathbb{Z}^*$. Then, the cocycles in $\mathbb{T} \times SU(2)$ of the form

$$(\alpha, Ae^{U(\cdot)}) \tag{8.6}$$

with $A = \{e^{i\pi k\alpha}, 0\}_{SU(2)}$ and $U(\cdot) = \{0, \varepsilon e^{2i\pi k\cdot}\}_{su(2)}$ ($\varepsilon \in \mathbb{R}^*$ is small) can be reduced by the 2-periodic conjugation $E_{-k/2}(\cdot) = \{e^{-i\pi k\cdot}, 0\}_{SU(2)}$ to the constant cocycle

$$(\alpha, E_{-k/2}(\alpha)A \exp(\{0, \varepsilon\}_{su(2)}))$$

as shows a direct calculation. The conjugation, however is of order one, while the perturbation is of order ε. We refer the reader who is unfamiliar with this kind of problems to Chapter 9 for the proof that the linearized equation cannot be solved for this cocycle due to the fact that the constant A is *resonant*. As a consequence, there is no good candidate for a conjugation of the same order as the perturbation, and the inverse function theorem approach fails, even though the cocycle is reducible. On the other hand, conjugation by $E_{-k/2}(\cdot)$ can be integrated in a K.A.M. scheme as a process of *reduction of resonances* (as in, say, [Eli01]), if we renounce in constructing a convergent sequence of conjugations.

After this digression, let us return to the subject of this chapter and point out that the phenomena that obstruct the existence of a conjugation of the order of the perturbation are absent, as indicates proposition 8.7. This fact results in the exactness of theorem 8.3.

In order to establish the contrast between theorem 8.3 and the phenomena observed in the local theory, we present, somewhat prematurely, the equivalent to our theorem in the local case, as obtained by R. Krikorian in [Kri99a], after having introduced some notation. Let $u \in G$, then call Γ_u a space complementary to $Im(Id - Ad_u)$ in g and

$$l_u : g \to \ker(Id - Ad_u)$$

be the projection with respect to the Cartan-Killing form. The Fréchet space of smooth mappings $\mathbb{T}^d \to g$ such that $l_u(B(0)) = 0$ will be denoted by $\mathcal{E}_u$. An element u of G is called Diophantine with respect to $\alpha \in \mathbb{T}$ if the eigenvalues of its adjoint action have this property, i.e. if

$$|u_\rho - k\alpha|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^\tau}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$. We can now state

Theorem 8.9 (R. Krikorian)**.** *Let $u_0 \in G$ be in $DC_\alpha(\gamma, \tau)$ and let $\Gamma = \Gamma_{u_0}$ and $\mathcal{E} = \mathcal{E}_{u_0}$. Let also α be Diophantine. Then, there exist an $\varepsilon > 0$ and $s > 0$ such that if $u \in C^\infty(\mathbb{T}^d, G)$ and $\|u - u_0\|_s < \varepsilon$, there exists a unique couple $(B, C) \in \mathcal{E} \times \Gamma$ such that*

$$u(\cdot) = e^C . e^{B(\cdot+\alpha)} . u_0 e^{-B(\cdot)}$$

The smallness condition ε is given by $c.\gamma^{c'}$, where the constants c and c' do not depend on γ.

The contrast between the two theorems resides in the factor e^C of eq. 8.9, which measures the failure of the inverse function theorem to obtain reducibility. As will be made clear in chapter 9 (and as has been made clear in [Kri99a]) the factor e^C is precisely due to phenomena resembling to those observed in the discussion following eq. 8.6.

The local normal form theorem can be used in the proof of a theorem for the reducibility in positive measure in the parameter space for generic one-parameter families. However, the stronger theorem of reducibility in full measure is true, but cannot be obtained by the inverse function theorem, since it is based on the reduction of resonances.

8.3 Preliminary transformations and notation

Let us denote by $\exp^{-1}$ the inverse of the exponential mapping, defined in a neighborhood of the Id, and taking values in a neighborhood of $0 \in g$. The space orthogonal to $\mathfrak{t}$ for the Cartan-Killing form, $\mathfrak{t}^\perp$, is the complex space spanned by the vectors j_ρ, $\rho \in \Delta_+$. The orthogonal projection on $\mathfrak{t}$ will be denoted by $\pi_\mathfrak{t}$. Finally, we will denote by $\mathcal{V} = C^\infty(\mathbb{T}, g)$ and call $\mathcal{W}$ a neighborhood of $0 \in \mathcal{V}$.

We can easily eliminate the obstruction to the solution of eq. (8.1) using the following fact. The mapping

$$h \mapsto \int \pi_\mathfrak{t}(\exp^{-1}(e^h.e^{U(\cdot)}))$$

is a local diffeomorphism in a neighborhood of 0 in $\mathfrak{t}$, called $\mathcal{N}$ (cf. also *Corollary 8.1* of [Kri01]). We can therefore introduce a fibration of $\mathcal{W}$ into $\mathcal{N} \times \mathcal{W}_0$, where $\mathcal{W}_0$=$\{U \in \mathcal{W}, \pi_\mathfrak{t}(\hat{U}(0)) = 0\}$.

Therefore, if $\|U(\cdot)\|_0 \in \mathcal{W}$ is small enough, there exists a unique couple $(h, U_0(\cdot)) \in \mathcal{N} \times \mathcal{W}_0$ such that

$$\begin{aligned} E_{r,a}(\cdot).A.e^{U(\cdot)} &= E_{r,a}(\cdot).e^h.e^{U_0(\cdot)} \\ &= E_{r,a+h}(\cdot)e^{U_0(\cdot)} \end{aligned}$$

where $U_0(\cdot) \in \mathcal{W}_0$ and the mapping $U(\cdot) \mapsto U_0(\cdot)$ is continuous from C^s into itself, for all s. The smallness condition on the perturbation can be weakened to a smallness condition on $\|U(\cdot) - \pi_\mathfrak{t}(\hat{U}(0))\|_0$. Finally, we can chose $\mathcal{W}_0$ so that $Ad(\mathcal{T}).\mathcal{W}_0 = \mathcal{W}_0$.

Since the rest $C \in \Gamma$ in the local normal form theorem is needed in order to absorb the topological obstruction to the resolution of the linearized cohomological equation, we are lead to replace it by a more general mapping, since the topological obstruction for the solution of (8.2) is no longer a constant.

Essentially, the normal form theorem should measure the failure to solve the equation

$$Conj_{\exp(B(\cdot))}.(\alpha, E_{r,a}(\cdot).e^{U(\cdot)}) = (\alpha, E_{r,a}(\cdot))$$

This failure will be represented by the additional terms e^λ and $e^{D(\cdot)}$ in the following formula:

$$E_{r,a}(\cdot).e^{U(\cdot)} = e^{-B(\cdot+\alpha)}.E_{r,a+\lambda}(\cdot).e^{B(\cdot)}.e^{D(\cdot)}$$

This equation, solved for the perturbation term, gives

$$e^{U(\cdot)} = E_{r,a}(-\cdot).e^{-B(\cdot+\alpha)}.E_{r,a+\lambda}(\cdot).e^{B(\cdot)}.e^{D(\cdot)} \tag{8.7}$$

The formula that we seek to obtain is

$$E_{r,a}(\cdot).e^{U(\cdot)} = e^{-B(\cdot+\alpha)}.E_{r,a'}(\cdot).e^{P(\cdot)}.e^{B(\cdot)}$$

with $P(\cdot)$ a Fourier polynomial taking values in $\mathfrak{t}^\perp$.

We remind that for $X \in g$ small enough and ΔX in the tangent space at X, we call

$$\widetilde{\Delta X} = K(X).\Delta X = (D(e^X).\Delta X).e^{-X}$$

We shall write $\Delta B = \sum \Delta B(k).e^{2i\pi kx}$ for a variation of $B(\cdot) \in \mathcal{V}$, where $\Delta B(k) = \sum_{\rho \in \tilde{\Delta}} \Delta B^\mathfrak{t}_\rho(k).h_\rho + \sum_{\rho \in \Delta_+} \Delta B_\rho(k).j_\rho$ are the Fourier coefficients. Mappings will not be denoted by $B(\cdot)$, but simply by B, in order to avoid confusion with the notation $\Delta B(k)$.

We also recall the notation $R_\alpha : x \mapsto x + \alpha$ for the translation of $\mathbb{T}$ into itself.

8.4 Proof of the normal form theorem

Let us therefore consider the smooth mapping $\tilde{\Psi} : ((\mathfrak{t},0)\times(g,0)^3\ ;\mathbb{T},\mathfrak{t}) \to ((g,0)\ ;\mathbb{T},\mathfrak{t})$

$$(\lambda, X, Y, Z;\ x, a) \mapsto \exp^{-1}(E_{r,a}(-x).e^{-X}.E_{r,a+\lambda}(x).e^{Y}.e^{Z})$$

where x and a are presently seen as parameters.

We remark that (8.7) can be solved by inverting the mapping

$$\Psi(\lambda, B, D) = \tilde{\Psi}(\lambda, B\circ R_a, B, D; a)$$

and, therefore, the next step is to prove the differentiability properties required by Hamilton's theorem and choose the functional spaces so that the mapping will be invertible.

The rest of this section is thus occupied by the proof of

Proposition 8.10. *Suppose that $\alpha \in DC(\gamma,\tau)$, $r \in \mathbb{Z}^w$ is such that $e_r \in g$ is regular and let $a \in \mathfrak{t}$, the unique maximal torus passing by e_r. Then there exists $\tilde{\mathcal{W}}$, a neighborhood of 0 in the local Fréchet manifold*

$$\mathcal{V}_r = \{(\lambda, B, D) \in \mathfrak{t}\times\mathcal{E}_0\times\mathcal{E},\ D \in Ad(e^{-B}).\mathcal{P}_{\tilde{r}}\}$$

such that the mapping

$$\begin{array}{rccc} \Psi: & \tilde{\mathcal{W}} & \to & \mathcal{E} \\ & (\lambda,B,D) & \mapsto & \exp^{-1}(E^*_{r,a}(\cdot).e^{B\circ R_\alpha}.E_{r,a+\lambda}(\cdot).e^{-B}.e^{-D)}) \end{array}$$

satisfies the conditions of Hamilton's theorem (theorem 2.11) for all $(\lambda,B,D)\in\tilde{\mathcal{W}}$. Moreover, $\tilde{\mathcal{W}}$ can be chosen so that $(\lambda,B,D)\in\tilde{\mathcal{W}}$ if and only if $(\lambda, B\circ R_\theta, D\circ R_\theta)\in\tilde{\mathcal{W}}$, for all $\theta\in\mathbb{T}$. The size of the neighborhood in $\mathcal{E}$ is bounded by

$$\|u\|_{s_0} \le Cst.\max(\gamma,1)^{-2}$$

where $s_0\in\mathbb{N}$ and the constants do not depend on γ, but only on r.

Regularity of the mapping

Let us begin by proving

Proposition 8.11. *Ψ is a C^∞-tame mapping $\mathcal{N}\times\mathcal{W}^2\to\mathcal{E}$ defined in a neighborhood of $0\in\mathfrak{t}\times\mathcal{E}^2$.*

Proof. Let us call $U=\tilde{\Psi}(\lambda,X,Y,Z)$. A direct calculation shows that

$$K(U).\Delta U = Ad(E_{r,a}(-\cdot)e^{-X})(-\widetilde{\Delta X}+\Delta\lambda+Ad(E_{r,a+\lambda}(\cdot)).\widetilde{\Delta Y}+Ad(E_{r,a+\lambda}(\cdot).e^{Y}).\widetilde{\Delta Z}$$

We can now define $\Psi:\mathcal{N}\times\mathcal{W}^2\to\mathcal{E}$ by

$$\Psi(\lambda,B,D)=\tilde{\Psi}(\lambda,B\circ R_\alpha,B,D;A)$$

If we pose now $U=\Psi(\lambda,B,D)\in\mathcal{E}$ and $\Delta U=(\Delta\lambda,\Delta B,\Delta D)$ (we remind that $\widetilde{\Delta U}=K(U).\Delta U$) we see that

$$Ad(e^{B\circ R_\alpha}E_{r,a}(\cdot)).\widetilde{\Delta U} = -\widetilde{\Delta B}\circ R_\alpha+\Delta\lambda+Ad(E_{r,a+\lambda}(\cdot)).\widetilde{\Delta B}+Ad(E_{r,a+\lambda}(\cdot).e^{B}).\widetilde{\Delta D}$$

which proves the proposition. □

Construction of the inverse

We now try to restrict the space on which Ψ is defined, in order to obtain inversibility for the restricted mapping. More precisely, we will define $\mathcal{V}_r$, the local Fréchet manifold of proposition 8.10 so that, if we still denote by Ψ the mapping $\mathcal{V}_r \to \mathcal{V}$, $D\Psi$ is bijective at each point of $\mathcal{V}_r$, and the inverse satisfies tame estimates.

We thus solve the equation

$$\Delta U' = -\widetilde{\Delta B} \circ R_\alpha + Ad(E_{r,a+\lambda}(\cdot)).\widetilde{\Delta B} + \Delta\lambda + Ad(E_{r,a+\lambda}(\cdot).e^B).\widetilde{\Delta D}$$

for any $\Delta U' \in \mathcal{E}$, thus establishing the surjectivity of $D\Psi$ in the tangent space at $(\lambda, B, D) \in \mathcal{N} \times \mathcal{W}^2$. We will solve the equation in the following equivalent form

$$\Delta U'' = Ad(E^*_{r,a+\lambda}(\cdot)).\widetilde{\Delta B} \circ R_\alpha - \widetilde{\Delta B} - \Delta\lambda - Ad(e^B).\widetilde{\Delta D} \tag{8.8}$$

where

$$\|\Delta U''\|_s \leq C_s(1 + \|B\|_s)\|\widetilde{\Delta U}\|_s$$

It can be seen directly that $\pi_{\mathfrak{t}}(\widetilde{\Delta B}(0))$ is not determined by the equation, so, in order to gain uniqueness for $\Delta B(0)) \in \mathcal{W}_0$, we will use $\pi_{\mathfrak{t}}(\widetilde{\Delta B}(0))$ as free parameters.

Direct application of Proposition 8.7 grants the existence of $(\Delta b, \Delta p) \in \mathcal{V}_0 \times \mathcal{P}^0_{\tilde{r}}$ such that

$$\Delta U'' = Ad(E^*_{r,a+\lambda}(\cdot)).\Delta b \circ R_\alpha - \Delta b + \Delta p$$

and $(\Delta b, \Delta p)$ satisfy tame estimates with respect to ΔU, uniformly for a and λ:

$$\begin{aligned} \|\Delta b\|_s &\leq C'_s \|\Delta U''\|_{s+\tau+3} \leq C'_{s+\tau+3}(1 + \|B\|_{s+\tau+3})\|\widetilde{\Delta U}\|_{s+\tau+3} \\ \|\Delta p\|_s &\leq C'_s \|\Delta U''\|_s \leq C'_s(1 + \|B\|_s)\|\widetilde{\Delta U}\|_s \end{aligned}$$

where $C'_s = \max(1, \gamma)C''_s(r)$. We will omit the prime in the notation of this constant.

We can therefore rewrite eq. 8.8 in the form

$$Ad(E^*_{r,a+\lambda}(\cdot)).\Delta b \circ R_\alpha - \Delta b + \Delta p = Ad(E^*_{r,a+\lambda}(\cdot)).\widetilde{\Delta B} \circ R_\alpha - \widetilde{\Delta B} - \Delta\lambda - \widetilde{\Delta P} \tag{8.9}$$

We have called $\widetilde{\Delta P} = Ad(e^B).\widetilde{\Delta D}$, which by convexity inequalities satisfies

$$\|\widetilde{\Delta P}\|_s \leq C_s(1 + \|B\|_s)\|\widetilde{\Delta D}\|_s$$

The situation for the inversion of $\Delta D \to \widetilde{\Delta P}$ and the decomposition of $\widetilde{\Delta P}$ into a coboundary plus an obstruction, however, is slightly more complicated. Observation of the formula that we seek to prove indicates that we should seek D in $Ad(e^{-B}).\mathcal{P}_{\tilde{r}}$. For such a D, we have $\widetilde{\Delta D} = Ad(e^{-B}).\widetilde{\Delta P}$, where $\widetilde{\Delta P} = K(P).\Delta P$ for $P, \Delta P \in \mathcal{P}_{\tilde{r}}$, which justifies the notation $\widetilde{\Delta P}$. For a given such P and ΔP, Proposition 8.7 gives the existence of ΔD_{cob} and a Fourier polynomial $\Gamma_{\tilde{r}}.\widetilde{\Delta P}$ such that

$$\widetilde{\Delta P} = Ad(E^*_{r,a+\lambda}(\cdot)).\Delta D_{cob} \circ R_\alpha - \Delta D_{cob} + \Gamma_r.\widetilde{\Delta P}$$

The family of mappings $\tilde{\mathcal{P}}_r$ $\circlearrowleft$

$$\Delta P \mapsto \tilde{\Gamma}_{\tilde{r}}(P).\Delta P = \Gamma_{\tilde{r}} \circ K(P).\Delta P$$

is a family of linear mappings of between vector spaces of the same finite dimension, depending on the parameter $P \in \mathcal{P}_r$. For $P = 0$, $\tilde{\Gamma}_{\tilde{r}}(0) = Id$, and the estimates in Proposition 8.7 imply that the family depends continuously on P, so that ΔD_{cob} and $\tilde{\Gamma}_r \Delta P$ satisfy uniform estimates with respect to ΔP, for all P sufficiently small in C^0:

$$\begin{aligned} \|\Delta D_{cob}\|_s &\leq C_{s+\tau+3}(1 + \|P\|_{s+\tau+3})\|\Delta P\|_{s+\tau+3} \\ \|\tilde{\Gamma}_{\tilde{r}}(P).\Delta P\|_s &\leq C_s(1 + \|P\|_s)\|\Delta P\|_s \end{aligned}$$

Similar estimates are also satisfied with respect to the original variable ΔD

$$\begin{aligned} \|\Delta D_{cob}\|_s &\leq C_{s+\tau+3}(1+\|B\|_{s+\tau+3}+\|P\|_{s+\tau+3})\|\Delta D\|_{s+\tau+3} \\ \|\tilde{\Gamma}_{\tilde{r}}(P).\Delta P\|_s &\leq C_s(1+\|B\|_s+\|P\|_s)\|\Delta D\|_s \end{aligned}$$

so long as $\|B\|_{s+\tau+3}, \|P\|_{s+\tau+3} \leq 1$.

We can therefore solve the equation

$$\tilde{\Gamma}_r.\Delta P = \pi_{\mathfrak{t}^\perp}\Delta p$$

with good estimates and write the equation for the remaining terms in the form

$$Ad(E^*_{r,a+\lambda}(\cdot)).\Delta b' \circ R_\alpha - \Delta b' + \pi_{\mathfrak{t}}\Delta p = Ad(E^*_{r,a+\lambda}(\cdot)).\Delta\tilde{B} \circ R_\alpha - \widetilde{\Delta B} - \Delta\lambda$$

where $\Delta b' = \Delta b - \Delta D_{cob}$. Projection on $\mathfrak{t}$ gives $\Delta\lambda = -\pi_{\mathfrak{t}}\Delta p$, and finally we find $\widetilde{\Delta B} = \Delta b' + \Delta c$, where $\Delta c \in \mathfrak{t}$ is to be determined.

Let us now study the inversion of the change of variables $\Delta B \mapsto \widetilde{\Delta B}$.

Lemma 8.12. *Let $\widetilde{\Delta B} \in \mathcal{V}_0$ satisfy 8.9 (it is then automatically unique, by virtue of Proposition 8.7). Then there exists a unique $\Delta c \in \mathfrak{t}$ such that $\Delta B = K^{-1}(B).(\widetilde{\Delta B} + \Delta c) \in \mathcal{V}_0$. Moreover, the mapping $\widetilde{\Delta B} \to \Delta B$ is tame.*

Proof. Let us call $\Delta B' = K(B)^{-1}.\widetilde{\Delta B}$. Since B is to be considered small, $\int K^{-1}(B)$ is invertible as for $\|B\|_0$ small enough

$$\| \int K^{-1}(B) - Id\| \leq C\|B\|_0$$

in operator norm. Therefore, we can find a unique $\Delta c \in \mathfrak{t}$ such that $\pi_{\mathfrak{t}} \int K^{-1}(B).\Delta c = \pi_{\mathfrak{t}}\Delta B'_{\mathfrak{t}}(0)$. Let, now,

$$\Delta B = \Delta B' - K^{-1}(B).b$$

Then, $\int \pi_{\mathfrak{t}}(\Delta B) = 0$ and thus $\Delta B \in \mathcal{E}_0$. Additionally, $K(B).\Delta B'$ and $K(B).\Delta B$ differ only by a constant diagonal matrix, so that eq. 8.9 is satisfied with $K(B).\Delta B$ in the place of $\widetilde{\Delta B}$. Finally,

$$\begin{aligned} \|\Delta B\|_s &\lesssim \|\Delta B'\|_s + (1+\|B\|_s)|\Delta c| \\ &\lesssim (1+\|B\|_s)\left(\|\widetilde{\Delta B}\|_s + (1+\|B\|_0)\|\widetilde{\Delta B}\|_s\right) \end{aligned}$$

Admitting a uniform bound, say $\|B\|_0 < 1$, we obtain the desired estimate

$$\|\Delta B\|_s \leq C_s(1+\|B\|_s)\|\widetilde{\Delta B}\|_s$$

□

We remark that, thanks to the local nature of the setting, the changes of variables are continuous mappings from $C^s(\mathbb{T},\mathbb{R})$ into itself, with norms bounded by constants depending only on $\mathcal{N}$ and $\mathcal{W}$, for each s. Therefore, the loss of derivatives is only due to the inversion of the operator involved in the linearized equation studied in the previous section.

Chapter 9

Revisiting the local theory

The next natural step in our study, since we follow the path "global-to-local", should be to treat the case where the obstruction to reducibility is intermediate, i.e. the perturbations of singular geodesics. As we will see in the next chapter, the study of singular geodesics is reduced to a parallel application of the local reduction lemmas of the local theory as we will develop it in this chapter with the K.A.M. scheme used in [Kri01].

For this reason, we will present the local theory, for which the main reference is [Kri99a]. We will adapt and simplify the proof of theorem 3.5 proved in the reference. Additionally, we improve the local almost quasi-reducibility theorem obtained as a corollary of the proof of theorem 3.5 to a local almost reducibility theorem (see chapter 2 for the definitions of these notions).

The proof of the local density theorem by R. Krikorian works in the context of a unitary representation of G. If one reads carefully the proofs, they can see that the adjoint representation of G on g works just as well. It is not a faithful representation, but since its kernel Z_G is discrete, it does not interfere significantly in the theory. In order to be able to use the Fourier transform, the unitary group in which G was embedded, was in turn embedded in a space of matrices with complex coefficients. In our context, the role of the ambient matrix vector space is taken up by the linear transformations of the real vector space g, an object which is again defined intrinsically.

Summing up, we re-obtain in a slightly more general context the well known theorem of the density of reducible cocycles in the neighborhood of constant cocycles. We also prove the local almost-reducibility theorem, which strengthens the almost quasi-reducibility theorem of R. Krikorian. In addition, the proof of the density theorem is significantly simpler, since it does not use the reducibility in positive measure for 1-parameter families. This last element becomes important in the study of perturbations of singular geodesics that we will take up in the next chapter, and allows an elegant proof of a slightly sharper and more natural global density theorem.

9.1 Notation

Throughout this chapter, we suppose that a 1-periodic perturbation of a constant cocycle is given

$$(\alpha, A.e^{U(\cdot)})$$

We also suppose that $\alpha \in DC(\gamma, \tau)$, i.e.

$$|k\alpha|_{\mathbb{Z}} = dist(k\alpha, \mathbb{Z}) \geq \frac{\gamma^{-1}}{|k|^{\tau}},\ k \in \mathbb{Z}^*$$

We remind that we denote by w the rank of G, by q the number of positive roots of G, and by f the dimension of the real vector space g and by $\mathfrak{t} = \mathfrak{t}_a$ a maximal toral algebra passing by a, a preimage of A in g by exp. If $\tilde{\Delta}$ are the roots in a basis of Weyl with respect to $\mathfrak{t}$, we denote by $(H_\rho)_{\rho\in\tilde{\Delta}}$ the dual basis of $\mathfrak{t}$. The choice of a Weyl's basis induces a decomposition of g into a direct sum of subspaces

$$g = (\oplus_{\rho\in\tilde{\Delta}}\mathbb{R}H_\rho) \oplus (\oplus_{\rho\in\Delta_+}\mathbb{C}j_\rho)$$

where the (H_ρ) span $\mathfrak{t}$ and the $(j_\rho) \cup (ij_\rho)$, where $i = \sqrt{-1}$, form an orthonormal basis of the space othogonal to $\mathfrak{t}$ with respect to the Cartan-Killing form. We remind the reader that if $s \in \mathfrak{t}$, then

$$[s, j_\rho] = 2i\pi\rho(s)j_\rho = 2i\pi s_\rho j_\rho$$

in the notations that we have adopted. For mappings $U(\cdot) : \mathbb{T} \to g$ we use the notation

$$\begin{aligned} U(\cdot) &= \sum\nolimits_{\rho\in\tilde{\Delta}} U^{\mathfrak{t}}_\rho(\cdot)H_\rho + \sum\nolimits_{\rho\in\Delta_+} U_\rho(\cdot)j_\rho \\ &= U^{\mathfrak{t}}(\cdot) + \sum\nolimits_{\rho\in\Delta_+} U_\rho(\cdot)j_\rho \end{aligned}$$

Since for every $\rho' \in \Delta_+$, there exist positive integers $m_{\rho',\rho}$ such that

$$\rho' = \sum_{\tilde{\Delta}} m_{\rho',\rho}\rho$$

there exist rational numbers $p_{\rho',\rho} = l_{\rho',\rho}/D$ with $|l_{\rho',\rho}| \leq b$, such that

$$H_{\rho'} = \sum_{\tilde{\Delta}} p_{\rho',\rho}H_\rho$$

where $D, b \in \mathbb{N}^*$ are considered fixed.

We also call $c = c_G$, the cardinal of Z_G.

Finally, a real number β will be called *Diophantine with respect to* α if it satisfies

$$|\beta - k\alpha|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^\tau}$$

The set of such numbers will be denoted by $DC_\alpha(\gamma, \tau)$.

9.2 Synopsis of the chapter

In this chapter, we revisit, simplify and sharpen the local theory, as known since [Kri99a], namely theorem 3.5, and obtain

Theorem 9.1. *Let $\alpha \in DC(\gamma, \tau)$. Then, there exist $\epsilon > 0$ and $s_0 \in \mathbb{N}^*$ such that, if $U(\cdot) \in C^\infty(\mathbb{T}, g)$ satisfies $\|U(\cdot)\|_{s_0} < 1$, $\|U(\cdot)\|_0 < \epsilon$ and $A \in G$, then the cocycle $(\alpha, Ae^{U(\cdot)})$ is accumulated in the C^∞ topology by reducible cocycles.*

This theorem was in fact already known in a slightly weaker form, as a comparison with theorem 3.5 can show. Its proof was based on a local quasi-reducibility theorem (see definition 3.3 and the discussion below), and obtained from it in a slightly more complicated way than we derive it from the following local almost reducibility theorem, which in turn was not known.

Theorem 9.2. *Let $\alpha \in DC(\gamma, \tau)$. Then, there exist $\epsilon > 0$ and $s_0 \in \mathbb{N}^*$ such that, if $\|U(\cdot)\|_{s_0} < 1$, $\|U(\cdot)\|_0 < \epsilon$ and $A \in G$, then the cocycle $(\alpha, Ae^{U(\cdot)})$ is almost reducible.*

The proof of this theorem is grants control of the growth of the conjugants, which is polynomial, versus an exponential decay of the perturbations. This fact grants the preceding density theorem. It is with the same argument that R. Krikorian obtained his local density theorem from his almost quasi-reducibility, but with a complication. The problem of longer periods (the difference between almost reducibility and quasi-reducibility) made it necessary to use the classical K.A.M. theory and reducibility in positive measure in the parameter space. We managed to overcome this difficulty and rid the argument of this complication by reinterpreting H. Eliasson's generalized K.A.M. scheme. We followed the adapted version of it for compact Lie groups as in [Kri99a], but interpreted the reduction of resonances (see section 9.4), as the reduction of the good linear model in the presence of resonances. This fact allowed us a better control of the procedure and the recovery of the loss of periodicity (inherent in the theory in general compact groups) at each step of the K.A.M scheme, which gives almost reducibility, instead of quasi-reducibility.

These last two theorems admit global analogues. Since we use the convergence of renormalization, we strengthen the arithmetic condition to a recurrent Diophantine one and obtain the following statements.

Theorem 9.3. *Let $\alpha \in RDC$ and $(\alpha, A(\cdot)) \in SW^\infty_\alpha(\mathbb{T}, G)$ be of degree 0, and let $m \in \mathbb{N}^*$ be the minimal natural number such that $(\alpha, A(\cdot))^m$ is homotopic to the Id. Then, the cocycle $(\alpha, A(\cdot))^m$ is almost reducible.*

The global analogue of the local density theorem is as follows

Theorem 9.4. *Let $\alpha \in RDC$. Then, reducible cocycles are dense in the class of 0-degree cocycles in $SW^\infty_\alpha(\mathbb{T}, G)$ that are homotopic to constants.*

Since we have already seen that positive energy cocycles in $SW^\infty(\mathbb{T}, SU(2))$ are accumulated by cocycles of smaller energy, by a simple induction argument we can obtain the following theorem.

Theorem 9.5. *Let $\alpha \in RDC$. Then, reducible cocycles are dense in $SW^\infty_\alpha(\mathbb{T}, SU(2))$. The same holds for cocycles in $SW^\infty_\alpha(\mathbb{T}, SO(3))$ which are homotopic to constants. If $(\alpha, A(\cdot)) \in SW^\infty_\alpha(\mathbb{T}, SO(3))$ is not homotopic to constants, there exist arbitrarily small perturbations of $(\alpha, A(\cdot))^2 \in SW^\infty_{2\alpha}(\mathbb{T}, SO(3))$ (thus one-periodic perturbations), such that $(\alpha, A'(\cdot))^2 \in SW^\infty_{2\alpha}(2\mathbb{T}, SO(3))$ is accumulated by reducible cocycles (the perturbed cocycle is considered as a 2-periodic cocycle).*

Proof. The only part which is not clear is the one on cocycles in $\mathbb{T} \times SO(3)$ which are not homotopic to constants.

Let $(\alpha, A(\cdot))$ be such a cocycle of energy 0. Then it has renormalization representatives $\mod 2$ which are arbitrarily close to constants, and therefore it is accumulated (in $C^\infty(2\mathbb{T}, SO(3))$) by reducible cocycles.

If $(\alpha, A(\cdot))$ has positive energy, then its second iterate can be conjugated to a normal form $(2\alpha, R_{4r\pi(\cdot+\theta)})$ by a 1-periodic conjugation. Therefore, we can perturb $(\alpha, A(\cdot))^2$ to a cocycle of 0 energy. However, renormalization of the second iterate has to be done in $SW^\infty(2\mathbb{T}, SO(3))$ if we want to draw conclusions for $(\alpha, A(\cdot))$. Renormalization of the action generated by $(2, Id)$ and $(2\alpha, A'(\cdot))$, where $A'(\cdot) = A_2(\cdot)e^{U(\cdot)}$ (after a change of scale by a factor 2) drives us to the case treated first in the proof. □

This complication will have to be dealt with in the proof of the main theorem of this mémoire in the case of multiply connected groups.

The theorems in the local case are of course valid for cocycles with an arbitrary number of frequencies, but their global analogues are not known to hold, since their proof requires renormalization of the dynamics.

9.3 Strategy of the proof of theorem 9.2

The proof of the local almost reducibility theorem is reduced at a first time to writing and solving with good estimates the reduction of a perturbation of a constant cocycle to a perturbation of order two, and then iterating the procedure by means of a K.A.M. scheme.

The convergence of the scheme needed for the second part has been taken from [FK09], so for the time being we focus on the first part, the *linear problem*.

Let us assume that a $Y(\cdot) : \mathbb{T} \to g$, small enough, can reduce the perturbation $U(\cdot)$ to $V(\cdot)$, with $V(\cdot)$ of order two with respect to $U(\cdot)$. The equation satisfied by such a conjugant is

$$e^{Y(\cdot+\alpha)}.A.e^{U(\cdot)}.e^{-Y(\cdot)} = A'.e^{V(\cdot)}$$

or

$$e^{Ad(A^*).Y(\cdot+\alpha)}.e^{U(\cdot)}.e^{-Y(\cdot)} = A^*A'.e^{V(\cdot)}$$

Linearization of this equation under the smallness assumptions implies that such a $Y(\cdot)$ must satisfy the equation

$$Ad(A^*).Y(\cdot+\alpha) - Y(\cdot) = -U(\cdot) \tag{9.1}$$

We introduce the coordinates

$$U(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} U^{\mathrm{t}}_\rho(\cdot)H_\rho + \sum\nolimits_{\rho\in\Delta_+} U_\rho(\cdot)j_\rho$$

where $U_\rho^{\mathfrak{t}}(\cdot)$ are real and $U_\rho(\cdot)$ complex 1-periodic functions. We will write $Y(\cdot)$ in an analogous way. The linearized equation then decomposes into two different types of equations as follows.

The simpler case is that of the abelian part, which reads

$$Y_\rho^{\mathfrak{t}}(\cdot+\alpha) - Y_\rho^{\mathfrak{t}}(\cdot) = -U_\rho^{\mathfrak{t}}(\cdot)$$

for which we refer the reader to lemma 3.4.

The second type of equation is the one that characterizes the reduction of perturbations of constants. It reads

$$e^{-2i\pi\rho(a)}Y_\rho(\cdot+\alpha) - Y_\rho(\cdot) = -U_\rho(\cdot) \tag{9.2}$$

for $\rho \in \Delta_+$. Application of the Fourier transform gives

$$(e^{2i\pi(k\alpha-\rho(a))} - 1)\hat{Y}_\rho(k) = -\hat{U}_\rho(k),\ k \in \mathbb{Z} \tag{9.3}$$

Therefore, the Fourier coefficient $\hat{Y}_\rho(k)$, $k \in \mathbb{Z}$, is not determined by the equation if

$$k\alpha - \rho(a) \in \mathbb{Z}$$

or equivalently if

$$a \in P(G) + \alpha P(G)$$

For the definition of $P(G)$, see section 2.1.1. If $a \notin P(G) + \alpha P(G)$, but is close to it, reduction of the corresponding Fourier coefficients deteriorates the estimates.

In order to avoid the loss of derivatives, a problem already encountered in Lemma 3.4, we solve the eq. 9.3 for only a finite number of coefficients, which we will determine depending on the arithmetic properties of a.

For this equation, we can distinguish three different possibilities.

The first one, which gives rise to a classical K.A.M. scheme as in [Kri99a], occurs when $\rho(a)$ is far from $k\alpha$,

$$|\rho(a) - k\alpha|_{\mathbb{Z}} \geq K^{-1} \tag{9.4}$$

for all $0 < |k| \leq N$ and some $K > 0$, comparable with N. Unlike with classical K.A.M., we suppose explicitly that the inequality 9.4 is violated for $k = 0$. Let us denote by I_0 the roots $\rho \in \Delta_+$ for which a_ρ satisfies this property. Then, we can solve eq. 9.2 for $0 < |k| \leq N$, i.e. we solve eq. 9.1 with $U_\rho(\cdot)$ replaced by the homogeneous truncaction $\dot{T}_N U_\rho(\cdot)$, and obtain a solution to the equation

$$e^{-2i\pi\rho(a)}Y_\rho(\cdot+\alpha) - Y_\rho(\cdot) = -T_N U_\rho(\cdot) + \hat{U}_\rho(0)$$

with $Y_\rho(\cdot)$ of the order of $U_\rho(\cdot)$.

The second one, the most gentle in fact, occurs when $\rho(a)$ is far from $k\alpha$,

$$|\rho(a) - k\alpha|_{\mathbb{Z}} \geq K^{-1}$$

for all $0 \leq |k| \leq N$, in which case we will say that it is Diophantine with respect to α. Let us denote by I_{nr} the roots $\rho \in \Delta_+$ for which a_ρ satisfies this property. Then, we can solve eq. 9.3 for $0 \leq |k| \leq N$, i.e. we solve eq. 9.2 with $U_\rho(\cdot)$ replaced by the truncaction $T_N U_\rho(\cdot)$, and obtain a solution $Y_\rho(\cdot)$ of the order of $U_\rho(\cdot)$.

The third and last case occurs when the inequality 9.4 is violated by some k_ρ with $0 < |k_\rho| \leq N$. We will show that with no loss of generality we can restrict our study to the case where $|k_\rho| \leq N'$, with $N' < N$, but with $N - N'$ of the order of N. Moreover, with a good choice of the parameters K and N, we can ensure that if such a k_ρ exists, then it is unique, and, additionally, that the inequality 9.4 is verified for $0 < |k - k_\rho| \leq N$. Finally, we will ensure that, if I_r is the set of such roots, then $I_0 \cup I_d \cup I_r = \Delta_+$. For the roots in I_r we can only solve the equation

$$e^{-2i\pi\rho(a)}Y_\rho(\cdot+\alpha) - Y_\rho(\cdot) = -\dot{T}_N^{(k_\rho)}$$

where

$$\dot{T}_N^{(k_\rho)} = \sum_{0<|k-k_\rho|\leq N} \hat{U}_\rho(k)e^{2i\pi k\cdot} \tag{9.5}$$

when again $Y_\rho(\cdot)$ is of the order of $U_\rho(\cdot)$.

Therefore, we have solved the equation 9.1, but with $U(\cdot)$ on the rhs replaced by

$$\dot{T}_N U^{\mathfrak{t}}(\cdot) + \sum_{\rho\in I_d} T_N U_\rho(\cdot) j_\rho + \sum_{\rho\in I_r} \dot{T}_N^{(k_\rho)} U_\rho(\cdot) j_\rho + \sum_{\rho\in I_0} \dot{T}_N U_\rho(\cdot) j_\rho$$

If we call

$$ObU(\cdot) = \hat{U}^{\mathfrak{t}}(0) + \sum_{\rho\in I_r} U_\rho(k_\rho) e^{2i\pi k_\rho \cdot} j_\rho + \sum_{\rho\in I_0} \hat{U}_\rho(0) j_\rho$$

the equation that we have solved reaches the form

$$Ad(A^*).Y(\cdot+\alpha) + U(\cdot) - Y(\cdot) = ObU(\cdot) + \text{Rest}$$

where the term "Rest" represents the rests of the truncation operators $\dot{T}_N^{(k_\rho)}$, $\dot{T}_N$ and T_N, accordingly with the subspace of g. The corresponding result for the non-linear problem is the conjugaction of $(\alpha, Ae^{U(\cdot)})$ to $(\alpha, Ae^{ObU(\cdot)}e^{U'(\cdot)})$, with $U'(\cdot)$ of second order with respect to $U(\cdot)$.

We remark that, since we have solved a linearized equation, where second order terms are neglected, linearization of the equation

$$Conj_{\exp(Y(\cdot))}(\alpha, Ae^{ObU(\cdot)}e^{\tilde{U}(\cdot)}) = (\alpha, Ae^{ObU(\cdot)}e^{U'(\cdot)})$$

with $\tilde{U}(\cdot) = U(\cdot) - ObU(\cdot) + O(|U|^2)$ and $U'(\cdot)$ of second order, gives exactly the same equation, solution and estimates. We can therefore interpret the procedure as linearization in the neighborhood of a the non-constant cocycle which incorporates the obstructions to the solution of the linearized equation.

Iteration of such a conjugation lemma is thus not possible, since the term $ObU(\cdot)$ is of the order of $U(\cdot)$. Moreover, since $ObU(\cdot)$ has non-zero Fourier coefficients exactly where the linear theory fails, a different approach has to be taken.

This approach resembles to, without being identical with, the reduction of resonances of [Eli92] and [Kri99a], and in this context it consists in constructing a conjugation of a controlled norm that reduces the exact model around which we linearize, $(\alpha, Ae^{ObU(\cdot)})$, to a constant. A good candidate for such a conjugation is a periodic geodesic $e^{-H\cdot} : \mathbb{T} \to \mathcal{T}_A$, with $\mathcal{T}_A$ a maximal torus passing by A. We have, then,

$$Conj_{\exp(-H\cdot)}(\alpha, Ae^{ObU(\cdot)}) = (\alpha, e^{-H\alpha} A \exp(Ad(e^{-H\cdot}).ObU(\cdot)))$$

so that the only possibly non-constant term is $Ad(e^{-H\cdot}).ObU(\cdot)$. Given the form of $ObU(\cdot) : \mathbb{T} \to g$, we calculate

$$\begin{aligned} Ad(e^{-H\cdot}).ObU(\cdot) &= \hat{U}^{\mathfrak{t}}(0) + Ad(e^{-H\cdot})(\sum_{\rho\in I_r} U_\rho(k_\rho) e^{2i\pi k_\rho \cdot} j_\rho + \sum_{\rho\in I_0} \hat{U}_\rho(0) j_\rho) \\ &= \hat{U}^{\mathfrak{t}}(0) + (\sum_{\rho\in I_r} U_\rho(k_\rho) e^{2i\pi (k_\rho - \rho(H))\cdot} j_\rho + \sum_{\rho\in I_0} \hat{U}_\rho(0) e^{-2i\pi\rho(H)\cdot} j_\rho \end{aligned}$$

Consequently, $ObU(\cdot)$ can be made constant after such a conjugation if we can solve the equations $\rho(H) = k_\rho$ for $\rho \in I_r$, and $\rho(H) = 0$ for $\rho \in I_0$. This is done in section 9.4, where it is shown also that such an H can be constructed so as to satisfy $|H| \lesssim N'$, where we remind that $|k_\rho| \leq N'$.

With such an H, we have

$$Conj_{\exp(-H\cdot)}(\alpha, Ae^{ObU(\cdot)}e^{\tilde{U}(\cdot)}) = (\alpha, \tilde{A} \exp(Ad(e^{-H\cdot}).U'(\cdot)))$$

where $U'(\cdot)$ is quadratic with respect to $U(\cdot)$, and therefore we can iterate the procedure.

The second part of the reduction, the construction of the conjugation $\exp(-H\cdot)$, shows that the obstructions to the solution of the linear equation are reducible cocycles. Therefore, the linearization of the problem in the beginning was linearization in the neighborhood of reducible, instead of constant, cocycles.

There is in fact a slight complication due to the fact that $e^{-H\cdot}$ is not in general 1-periodic, but this is resolved by post-conjugating with another periodic geodesic, as our approach allows us to treat this problem of longer periods at each step of the algorithm, instead of the accumulated loss of periodicity after having applied the algorithm a sufficient number of times.

9.4 Resonances and reduction of resonant modes

The preceding section has shown the importance of the relation of $\rho(a) \mod \mathbb{Z}$, for $\rho \in \Delta_+$, with $k\alpha \mod \mathbb{Z}$. More precisely, if $\rho(a) = k_\rho \alpha \mod \mathbb{Z}$ for some $k_\rho \in \mathbb{Z}$, then the corresponding Fourier coefficient $\hat{U}_\rho(k_\rho)$ of $U_\rho(\cdot)$ cannot be eliminated in the linear equation. In the perturbative setting, if this equality is approximate and $\rho(a) - k_\rho \mod \mathbb{Z}$ is small with respect to N^{-1}, then reduction of the corresponding Fourier coefficient deteriorates the estimates. This motivates the following definition.

Definition 9.6. A constant $\beta \in \mathbb{R}$ such that

$$k\alpha - \beta \in \mathbb{Z}$$

for some $k \in \mathbb{Z}^*$ is called (α-)*resonant*. The set of such constants is denoted by $Res(\alpha)$ and its complement, the set of *non-resonant* roots, $NR(\alpha)$.

Let $I \subset \mathbb{Z}^*$. If there exists $k \in I$ such that

$$|\beta - k\alpha|_{\mathbb{Z}} = dist(\beta - k\alpha, \mathbb{Z}) \leq K^{-1}$$

we will say that β is in $RS(\alpha, K)$ in I. In the case where $I = \{k \in \mathbb{Z}, 0 < |k| \leq N\}$, the sets of such constants will be denoted by $RS(\alpha, N, K)$ and $DS(\alpha, N, K)$ respectively. We will then say that β is K-resonant to the order N.

A constant $a \in g$ is in $RS(\alpha, K)$ in I (resp. in $RS(\alpha, N, K)$) if there exists a root in $\rho \in \Delta_+$ such that $\rho(a)$ is in $RS(\alpha, K)$ in I (resp. in $RS(\alpha, N, K)$). If no such root exists, $a \in DS(\alpha, K)$ in I (resp. in $DS(\alpha, N, K)$).

We stress that we consider 0 as non-resonant, i.e. we do not authorise $k = 0$ in the definition of Res and that of RS.

With this notation, for any given $a \in g$ and $\mathfrak{t} = \mathfrak{t}_a$, a maximal toral algebra passing by a, we can introduce a partition of the roots in Δ_+ in

$$\begin{aligned} I_r &= \{\rho \in \Delta_+,\ \rho(a) = k_\rho \alpha + l_\rho \in \alpha\mathbb{Z}^* + \mathbb{Z}\} \\ I_0 &= \{\rho \in \Delta_+,\ \rho(a) = l_\rho \in \mathbb{Z}\} \\ I_d &= \{\rho \in \Delta_+,\ \rho(a) \notin \alpha\mathbb{Z} + \mathbb{Z}\} = \Delta_+ \backslash (I_r \cup I_0) \end{aligned} \tag{9.6}$$

Let us also associate to each $\rho \in I_r$ the (unique) k_ρ satisfying $\rho(a) = k_\rho \alpha + l_\rho$. This notation is to be fixed throughout the chapter. We can now state

Lemma 9.7. *Let $a \in \mathfrak{t} \subset g$, call $(a_\rho)_{\rho \in \Delta_+} = (\rho(a))_{\rho \in \Delta_+} \in \mathbb{C}^q$ and define the corresponding partition of the roots in Δ_+, as in eq. 9.6. Then, there exists $H \in \mathfrak{t}$ such that, for all $\rho \in I_r$, we have $\rho(H) = k_\rho$, and for all $\rho \in I_0$, $\rho(H) = 0$. The vector H is of the form $\sum \frac{k'_\rho}{D} H_\rho$, where the summation is over a subset of $(I_r \cup I_d) \cap \tilde{\Delta}$, and $k'_\rho \in \mathbb{Z}$.*

We remind that $q = \#\Delta_+$ denotes the number of positive roots, which are not linearly independent. We also recall that any root ρ' in $\Delta_+ \backslash \tilde{\Delta}$ can be written in the form $\sum_{\rho \in \tilde{\Delta}} m_{\rho',\rho} \rho$ where the $m_{\rho',\rho}$ are positive integers. The entries of inverses of invertible submatrices of $(m_{\rho',\rho})$ are rational numbers of the form $l_{\rho,\rho'}/D$, where $|l_{\rho,\rho'}| \leq e$ is an integer and $D \in \mathbb{N}^*$ and $l_{\rho,\rho'}$ and D are not necessarily coprime.

Proof. Let us now call

$$\begin{aligned} \tilde{I}_r &= I_r \cap \tilde{\Delta} \\ \tilde{I}_0 &= I_0 \cap \tilde{\Delta} \\ \tilde{I}_d &= I_d \cap \tilde{\Delta} \end{aligned}$$

It is then clear that

$$I_r \subset vect(\tilde{I}_r, \tilde{I}_d) \cap \Delta_+$$

so that we can chose I_r^l a linearly independent subset of I_r, and complete it with I_0^l, a maximal set of linearly independent roots in $I_0 \setminus \text{vect}(\tilde{I}_0)$ in order to obtain a maximal linearly independent subset of roots in $\tilde{I}_r \cup \tilde{I}_d$. Then, we can solve the linear system

$$\begin{aligned}\sum_{\rho\in\tilde{I}_r\cup\tilde{I}_d} m_{\rho',\rho}u_\rho &= k_{\rho'},\ \rho' \in I_r^l \\ \sum_{\rho\in\tilde{I}_r\cup\tilde{I}_d} m_{\rho',\rho}u_\rho &= 0,\ \text{if } \rho' \in I_0^l\end{aligned}$$

and the u_ρ are rational numbers of the form k'_ρ/D, and $k'_\rho = Dk_\rho$ if $\rho \in \tilde{I}_r$. The vector

$$\sum_{\rho\in\tilde{I}_r\cup\tilde{I}_d} \frac{k'_\rho}{D} H_\rho$$

by construction satisfies the conclusions of the lemma, since the resonance of a linear combination of resonant roots is the linear combination of the resonances: if $\rho' = \sum_{\rho\in\tilde{I}_r} m_{\rho',\rho}\rho$ is resonant, $k_{\rho'} = \sum_{\rho\in\tilde{I}_r} m_{\rho',\rho}, k_\rho$. □

Remark 9.8. If we suppose that $|k_\rho| \leq N$, for all $\rho \in I_r$, then there exists a constant b depending only on G such that $|k'_\rho| \leq bDN$, for all $\rho \in \tilde{I}_r \cup \tilde{I}_d$. This follows from the fact that the resonances k_ρ for the roots in I_r and the integers k'_ρ defined for roots in $\tilde{I}_r \cup \tilde{I}_d$ are related by submatrices of the Cartan matrix of the group.

The notation for the integers k'_ρ as solutions to the linear system of the proof of lemma 9.7 is also to be considered fixed throughout the chapter.

Before stating a direct corollary, we remind that for a function $f : \mathbb{T} \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we call $\sigma(f) = \{k \in \mathbb{Z}, \hat{f}(k) \neq 0\}$.

Corollary 9.9. *Let $A \in G$, $U(\cdot) \in C^\infty(\mathbb{T}, g)$ and H constructed as in lemma 9.7. Then, if $U(\cdot)$ is spectrally supported in the obstructions and the resonances,*

$$\begin{aligned}\sigma(U_\rho^{\mathfrak{t}}(\cdot)) &\subset \{0\} \\ \sigma(U_\rho(\cdot)) &\subset \{0\} \text{ if } \rho \in I_0 \\ \sigma(U_\rho(\cdot)) &\subset \{k_\rho\} \text{ if } \rho \in I_r \\ \sigma(U_\rho(\cdot)) &= \emptyset \text{ otherwise}\end{aligned}$$

the cocycle $(\alpha, Ae^{U(\cdot)})$ is reducible mod 1.

Proof. If $U(\cdot)$ satisfies the hypothesis of the corollary, then it has the form

$$U(\cdot) = \hat{U}^{\mathfrak{t}}(0) + \sum_{\rho\in I_r} U_\rho(k_\rho)e^{2i\pi k_\rho \cdot} j_\rho + \sum_{\rho\in I_0} \hat{U}_\rho(0)j_\rho \tag{9.7}$$

If, now, the vector H satisfies the conclusions of lemma 9.7, we find directly that

$$\begin{aligned}e^{-H(\cdot+\alpha)}Ae^{U(\cdot)}e^{H\cdot} &= e^{-H\alpha}A\exp(Ad(e^{-H\cdot}).U(\cdot)) &(9.8)\\ &= e^{-H\alpha}A\exp(Ad(e^{-H\cdot}).(U^{\mathfrak{t}}(0) + \sum U_\rho(\cdot)j_\rho) &(9.9)\end{aligned}$$

and we calculate

$$\begin{aligned}Ad(e^{-H\cdot}).(U^{\mathfrak{t}}(0) + \sum U_\rho(\cdot)j_\rho) &= U^{\mathfrak{t}}(0) + Ad(e^{-H\cdot}).(\sum_{\rho\in I_r} e^{2i\pi k_\rho\cdot}\hat{U}_\rho(k_\rho)j_\rho + \sum_{\rho\in I_0}\hat{U}_\rho(0)j_\rho) \\ &= U^{\mathfrak{t}}(0) + \sum_{\rho\in I_r} e^{2i\pi(k_\rho-\rho(H))\cdot}\hat{U}_\rho(k_\rho)j_\rho + \sum_{\rho\in I_0}\hat{U}_\rho(0)j_\rho\end{aligned}$$

Since, by construction of the vector H, $k_\rho = \rho(H)$, we find that

$$Ad(e^{-H\cdot}).U(\cdot) = s \in g$$

This shows that

$$Conj_{\exp(-H\cdot)}(\alpha, Ae^{U(\cdot)}) = (\alpha, \tilde{A})$$

i.e. that $(\alpha, Ae^{U(\cdot)})$ is reducible mod cD via the torus morphism $e^{-H\cdot}$.

Since $e^{-H(\cdot+1)}e^{H\cdot}$ is constant, we can imitate the last part of the proof of proposition *2.2.4* in [Kri99a] and regain periodicity.

Firstly, we show that there exists a maximal torus passing both by e^{-H} and $\tilde{A}$, where

$$\tilde{A} = e^{-H\alpha}A\exp(s)$$

We initially show that $s \in g$ commutes with e^H:

$$\begin{aligned} Ad(e^{-H}).s &= U^{\mathfrak{t}}(0) + Ad(e^{-H}).(\sum_{\rho\in I_r}\hat{U}_\rho(k_\rho)j_\rho + \sum_{\rho\in I_0}\hat{U}_\rho(0)j_\rho) \\ &= U^{\mathfrak{t}}(0) + (\sum_{\rho\in I_r}\hat{U}_\rho(k_\rho)e^{-2i\pi\rho(H)}j_\rho + \sum_{\rho\in I_0}\hat{U}_\rho(0)e^{-2i\pi\rho(H)}j_\rho) \\ &= s \end{aligned}$$

since for $\rho \in I_r \cup I_0$, $\rho(H) \in \mathbb{Z}$. Therefore, e^{-H} and $e^{-H\alpha}A\exp(\lambda s)$ commute for $\lambda \in [0,1]$. Since $e^{-H\alpha}A$ and $e^{-H\cdot}$ are on the same maximal torus $\mathcal{T}_A \subset Z_G^0(e^{-H})$ (the neutral component of $Z_G(e^{-H})$), this implies that $\tilde{A} \in Z_G^0(e^{-H})$, and consequently there exists $\mathcal{T}_{\tilde{A}}$, a maximal torus passing both by e^{-H} and $\tilde{A}$.

Therefore, there exists a group morphism $C(\cdot) : \mathbb{R} \to \mathcal{T}_{\tilde{A}}$ such that $C(1) = e^{-H}$, so that

$$\begin{aligned} Ae^{U(\cdot)} &= e^{-H(\cdot+\alpha)}\tilde{A}e^{H\cdot} \\ &= e^{-H(\cdot+\alpha)}C(\cdot+\alpha).C(-\alpha)\tilde{A}C^*(\cdot)e^{H\cdot} \\ &= e^{-H(\cdot+\alpha)}C(\cdot+\alpha).A'.C^*(\cdot)e^{H\cdot} \end{aligned}$$

Therefore, $Ae^{U(\cdot)}$ is reducible by $C^*(\cdot)e^{-H\cdot}$ which is 1-periodic since

$$\begin{aligned} C^*(\cdot+1)e^{-H(\cdot+1)} &= C^*(\cdot)C^*(1)e^{-H}e^{-H\cdot} \\ &= C^*(\cdot)e^{-H\cdot} \end{aligned}$$

□

This corollary concludes what we need to prove in the linear algebraic context before passing on to the regime of perturbation theory. The connection of these results with K.A.M. theory is made with the following lemma. It is a generalization (already found in [Kri99a]) of a rather simple observation, which constitutes the basis of L. H. Eliasson's results for quasiperiodic skew-systems in $\mathbb{T}^d \times SO(3)$ (see [Eli88]) and those based upon them. It asserts that for a suitably chosen K, big enough with respect to N, there can be at most one resonant mode.

Lemma 9.10. *Let $\alpha \in DC(\gamma,\tau)$, $N', N, \in \mathbb{N}^*$, and $K > 0$ such that $K \geq 2^{\tau+1}\gamma N^\tau$. Then,*

1. *If $\beta \in RS(\alpha, N, K)$, there exists a unique k_0, for which*

$$|\beta - k\alpha|_{\mathbb{Z}} < K^{-1},\ 0 < |k| \leq N$$

Moreover,

$$|\beta - k\alpha|_{\mathbb{Z}} \geq K^{-1},\ 0 < |k - k_0| \leq N'$$

for any N' such that $K \geq 2\gamma {N'}^{\tau}$ and

$$|\beta - k\alpha|_{\mathbb{Z}} \geq K^{-1},\ 0 < |k - k_0| \leq N'$$

for any N' such that $K \geq 2\gamma {N'}^{\tau}$. In particular, $|\beta|_{\mathbb{Z}} \geq K^{-1}$.

2. *If $\beta \in RS(\alpha, N, K)$, then β is non-resonant (in the linear-algebraic sense, i.e $\beta - k\alpha \notin \mathbb{Z}$) to the order $N_{nr} < (\gamma^{-1}K)^{1/\tau}$ for $k \in \mathbb{Z}^*$ with $0 < |k - k_0| \leq N_{nr}$.*
3. *If $\beta \in RS(\alpha, N, K)$, then $|\beta|_Z \geq K^{-1}$*

Proof. 1. Let $k_1, k_2 \in \mathbb{Z}$ with $0 < |k_i| \leq N$ and $k_1 \neq k_2$, such that $|\beta - k_i\alpha|_{\mathbb{Z}} \leq K^{-1}$, for $i = 1, 2$, i.e.

$$|\beta - k_i\alpha - l_i| \leq K^{-1}$$

for some $l_i \in \mathbb{Z}$. We have therefore, with $k = k_1 - k_2$, so that $0 < |k| \leq 2N$, and $l = l_1 - l_2$

$$|k\alpha - l| \leq 2K^{-1}$$

Since $\alpha \in DC(\gamma, \sigma)$, $k \neq 0$ implies

$$|k\alpha|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^\tau} \geq \frac{\gamma^{-1}}{(2N)^\tau}$$

This is possible only if

$$\frac{\gamma^{-1}}{(2N)^\tau} < 2K^{-1}$$

which contradicts the choice of K and N.

2. For the second part, we need only remark that

$$\begin{aligned} |\beta - k\alpha|_{\mathbb{Z}} &= |\beta - k_0\alpha + k - k_0\alpha|_{\mathbb{Z}} \\ &\geq |k - k_0\alpha|_{\mathbb{Z}} - |\beta - k_0\alpha|_{\mathbb{Z}} \\ &\geq \frac{\gamma^{-1}}{|k - k_0|^\tau} - K^{-1} \\ &\geq K^{-1} \end{aligned}$$

provided that

$$K \geq 2\gamma|k - k_0|^\tau$$

3. Finally, since

$$|\beta - k_0\alpha - l_0| \leq K^{-1}$$

we have

$$\begin{aligned} |\beta|_{\mathbb{Z}} &\geq |k_0\alpha|_{\mathbb{Z}} - K^{-1} \\ &\geq \frac{\gamma^{-1}}{|k_0|^\tau} - K^{-1} \\ &\geq 2^{\tau+1}K^{-1} - K^{-1} \\ &\geq K^{-1} \end{aligned}$$

□

If we fix K and N' satisfying $K \geq 2\gamma {N'}^\tau$ and we assume that resonances occur only to the order $N < N'$, we have the following corollary.

Corollary 9.11. *1. If $\beta \in RS(\alpha, N, K)$, then $|\beta|_{\mathbb{Z}} \geq (2(\frac{N'}{N})^\tau - 1)K^{-1}$.*

2. If $|\beta|_{\mathbb{Z}} < K^{-1}$, then $\beta \in DS(\alpha, N', K)$.

This lemma and its corollary are necessary in order to apply the linear-algebraic reduction of resonances to a K.A.M. scheme, since in the latter case constants are only approximately Diophantine, 0 or resonant. A resonant constant can have only one resonant mode, but, if the constants are not chosen suitably, they may be more than one when we allow closeness to resonances up to K^{-1}. On the other hand, we need to be able to solve exactly the linear system that reduces the resonant modes to constants.

9.5 The local conjugation lemma

Before stating and proving the proposition which will be used in the K.A.M. scheme, we will fix some notation. We let b be the maximum of the operator norms of (inverses of) square submatrices of $(m_{\rho',\rho})$, D as before and $\tilde{b} = 2b$. For a given $N \in \mathbb{N}^*$ we let $N_0 = N$, $N_i = \tilde{b}N_{i-1}$, $i = 1, ..., q+1$. To every order of truncation we associate the sets of roots

$$\begin{aligned} I_r^{(i)} &= Res(K, N_i) \\ I_0^{(i)} &= \{\rho \in \Delta_+,\ |a_\rho| < K^{-1}\} \\ DS^{(i)} &= DS(K, N_i) \\ I_d^{(i)} &= DS^{(i)} \backslash I_0^{(i)} \end{aligned}$$

We also define the $k_\rho \in \mathbb{Z}$ equal to 0 for all roots in $I_0^{(i)}$, and k_ρ if $\rho \in I_r^{(i)}$, and we will sometimes omit the superscript (i) for simplicity in notation. We remark that if we fix a constant A in G and a maximal torus passing by it, the root-space decomposition of g results in the canonical definition of the Fourier modes of mappings in $C^\infty(\mathbb{T}, g)$ in

$$\mathfrak{Z} = (\bigsqcup_{\rho \in \tilde{\Delta}} \mathbb{Z}_\rho^{\mathfrak{t}}) \bigsqcup (\bigsqcup_{\rho \in \Delta_+} \mathbb{Z}_\rho)$$

where the first factor accounts for the coordinates in the maximal torus (and therefore for real functions) and the second for $\mathbb{C}$-valued functions in the directions j_ρ. The spectrum σ of a mapping $U(\cdot) \in C^\infty(\mathbb{T}, g)$ is viewed in a canonical way as a subset of $\mathfrak{Z}$ as

$$\sigma(U) = (\bigsqcup_{\rho \in \tilde{\Delta}} \sigma(U_\rho^{\mathfrak{t}})) \bigsqcup (\bigsqcup_{\rho \in \Delta_+} \sigma(U_\rho))$$

In this notation, we can define $\mathbf{k} = \mathbf{k}_a \subset \mathfrak{Z}$ such that $(\mathbf{k})_\rho = \{k_\rho\}$ if $\rho \in I_0 \cup I_r$ and $(\mathbf{k})_\rho = \emptyset$ otherwise, so that the hypothesis of lemma 9.9 amounts to $\sigma(ObU(\cdot)) \subset \mathbf{k}_A$. We will need the notation $|\mathbf{k}| = \max_{\rho \in \Delta^+} |(\mathbf{k})_\rho|$ where by convention $|\emptyset| = 0$.

Clearly, the frequencies of mappings in $C^\infty(D\mathbb{T}, g)$ belong to the finer set

$$\mathfrak{Z}_D = (\bigsqcup_{\rho \in \tilde{\Delta}} \frac{1}{D}\mathbb{Z}_\rho^{\mathfrak{t}}) \bigsqcup (\bigsqcup_{\rho \in \Delta_+} \frac{1}{D}\mathbb{Z}_\rho)$$

and $\mathfrak{Z} \hookrightarrow \mathfrak{Z}_D$ in a canonical way. Even though $\mathfrak{Z}$ is not a lattice, the structure of additive group within each components remains of interest, since the adjoint action of a cD-periodic geodesic translates the frequencies within each component of $\mathfrak{Z}$ without mixing them:

$$\sigma(Ad(e^{H\cdot}).U(\cdot)) = (\bigsqcup_{\rho \in \tilde{\Delta}} \sigma(U(\cdot)_\rho^{\mathfrak{t}})_\rho) \bigsqcup (\bigsqcup_{\rho \in \Delta_+} (\sigma(U_\rho(\cdot)) - \frac{k'_\rho}{D})_\rho$$

if H is of the form $\sum \frac{k'_\rho}{D} H_\rho$.

To such a partition of the roots, we associate the truncation operators

$$T_{N_i}^{(\mathbf{k})} U(\cdot) = \dot{T}_{N_i} U^{\mathfrak{t}}(\cdot) + \sum_{\rho \in I_d} T_{N_i} U_\rho(\cdot) j_\rho + \sum_{\rho \in I_r} \dot{T}_{N_i}^{(k_\rho)} U_\rho(\cdot) j_\rho + \sum_{\rho \in I_0} \dot{T}_{N_i} U_\rho(\cdot) j_\rho \quad (9.10)$$

$$ObU(\cdot) = \hat{U}^{\mathfrak{t}}(0) + \sum_{\rho \in I_r} U_\rho(k_\rho) e^{2i\pi k_\rho \cdot} j_\rho + \sum_{\rho \in I_0} \hat{U}_\rho(0) j_\rho \quad (9.11)$$

$$R_{N_i}^{(\mathbf{k})} U(\cdot) = U(\cdot) - T_{N_i}^{(k)} U(\cdot) - ObU(\cdot) \quad (9.12)$$

and we refer the reader to eq. 9.5 for the definition of the less familiar quantities. Finally, we define the pierced box

$$\dot{\mathcal{B}}(\mathbf{k}, N_i) \subset (\bigsqcup_{\rho \in \tilde{\Delta}} \mathbb{Z}_\rho) \bigsqcup (\bigsqcup_{\rho \in \Delta_+} \mathbb{Z}_\rho)$$

to be the spectral support of mappings in the image of $T^{(\mathbf{k})}_{N_i}$, and the box $\mathcal{B}(\mathbf{k}, N_i)$ as the spectral support of mappings in the image of $T^{(\mathbf{k})}_{N_i} + Ob$.

Proposition 9.12. *Let $\alpha \in DC(\gamma, \tau)$, $N \in \mathbb{N}^*$, and $K > 0$ such that $K \geq 2^{\tau+1}\gamma\tilde{b}^{q+1}N^\tau$. Then for any $A \in G$, there exists $0 \leq i \leq q$ and $\mathbf{k} = \mathbf{k}_A \in \mathfrak{Z}$ with $|\mathbf{k}_A| \leq N_i$, such that*

1. *$Ad(A) \in DS(\alpha, K)$ in $\dot{\mathcal{B}}(\mathbf{k}_A, N_{i+1})$ and is non-resonant in $\dot{\mathcal{B}}(\mathbf{k}_A, 2N_{i+1}) \subset \mathfrak{Z}$.*
2. *If $\sigma(U(\cdot)) \subset \mathbf{k}$, then there exists a cD -periodic torus morphism $B(\cdot) = \exp(-H_A\cdot) : cD\mathbb{T} \to \mathcal{T}_A$ with $\sigma(Ad(B(\cdot)))$ included in $\mathcal{B}(\emptyset, bDN_i) \times \{\emptyset\} \subset \mathfrak{Z}_D$, reducing $(\alpha, Ae^{U(\cdot)})$ to a constant (α, A').*

We remind that in fact $(\alpha, Ae^{U(\cdot)})$ is reducible modulo 1. In order to establish this stronger fact, one need only follow the proof of corollary 9.9 in this perturbed setting. The vectors H and s remain the same ("resonant modes are exact"), but A in the perturbed setting is at most K^{-1}-away from the A considered in the exact setting ("resonances are approximate"). This indefiniteness of A only tilts the torus in which we regain periodicity (the mapping $C(\cdot)$ of the proof), but the argument proving the existence of the torus remains valid since K^{-1} is small, and $Ad(A).H = H$ and $Ad(e^H).s = s$ in both cases.

Proof. Let $A \in G$, $\mathcal{T}_A$ a maximal torus passing by A, and $(e^{2i\pi a_\rho})_\rho$ the eigenvalues of the adjoint action of A. For $0 \leq i \leq q$ we observe that, since $I^{(i)}_r \subset I^{(i+1)}_r$, $DS^{(i+1)} \subset DS^{(i)}$ and $DS^{(i)} \cup I^{(i)}_r = \Delta_+$, there exists such an i for which $I^{(i)}_r = I^{(i+1)}_r$, so that

$$\Delta_+ = DS^{(i+1)} \cup I^{(i)}_r$$

is a partition of the root system. Immitating the proof of the previous lemma, we define $\tilde{I}_r = I^{(i)}_r \cap \tilde{\Delta}$, $\tilde{I}_d = I^{(i)}_d \cap \tilde{\Delta}$ and $\tilde{I}_0 = I^{(i)}_0 \cap \tilde{\Delta}$.

It follows directly from the definitions that if $\rho \in span(\tilde{I}_0)$, then $|a_\rho| \leq bK^{-1}$. On the other hand, if $\rho \in I^{(i)}_r$, then $|a_\rho| \geq (2\tilde{b} - 1)K^{-1} > bK^{-1}$, so that $I^{(i)}_r \not\subset \text{span}(\tilde{I}_0)$. As a consequence, it remains true in the perturbative setting that the resonance of a linear combination of resonant roots is the linear combination of the resonances: if $\rho' = \sum_{\rho \in \tilde{I}_r} m_{\rho',\rho}\rho$ and $\rho' \in I^{(i)}_r$, then $k_{\rho'} = \sum_{\rho \in \tilde{I}_r} m_{\rho',\rho}k_\rho$.

We can therefore adapt the proof of the lemma 9.7 to this setting and obtain a vector H of the form $\sum_{\rho \in \tilde{\Delta}} \frac{k'_\rho}{D} H_\rho$ satisfying

$$\begin{aligned} \rho(H) &= k_\rho, \text{ if } \rho \in I^{(i)}_r \\ \rho(H) &= 0, \text{ if } \rho \in I_0 \\ |k'_\rho| &\leq bDN_i,\ k'_\rho \in \mathbb{Z} \end{aligned}$$

Therefore, if $\sigma(U(\cdot))$ is contained in the resonant modes of the directions in $I^{(i)}_r$ and the constant terms in I_0, the cocycle $(\alpha, Ae^{U(\cdot)})$ is reducible by $\exp(-H\cdot)$. □

Since the set of frequencies described in the second point of the previous proposition is exactly the set of obstructions to resolving the linear cohomological equation with good estimates, we can now prove the following proposition, after having introduced the notation

$$\begin{aligned} N^{(u)}_i &= N_i + N_{i+1} \\ N^{(l)}_i &= N_i - N_{i-1} \end{aligned}$$

for $1 \leq i \leq q$. Both $N^{(u)}_i$ and $N^{(l)}_i$ are of the order of N.

Proposition 9.13. *Let $\alpha \in DC(\gamma, \tau)$ and $K \geq 2^{\tau+1}\gamma\tilde{b}^{q+1}N^\tau$. Let, also, $(\alpha, Ae^{U(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ with*

$$c_{1,0}K(N^{(u)}_q)^2\varepsilon_0 < 1$$

where $\varepsilon_s = \|U\|_s$. Then, there exists $0 \leq i \leq q$ and a conjugation $Y(\cdot) \in C^\infty(\mathbb{T}, g)$ such that

$$e^{Y(\cdot+\alpha)}.A.e^{U(\cdot)}.e^{-Y(\cdot)} = Ae^{ObU(\cdot)}e^{U'(\cdot)}$$

The cocycle $(\alpha, Ae^{ObU(\cdot)})$ is reducible by a cD-periodic torus morphism $B(\cdot) : cD\mathbb{T} \to \mathcal{T}_A$ with $\sigma(Ad(B(\cdot)))$ included in $\mathcal{B}(\emptyset, bDN_i) \times \{0\} \in \mathfrak{Z}_D$. The mapping $B(\cdot)$ is $exp(-H\cdot)$ as constructed in lemma 9.7 .

Finally, the following estimates hold:

$$\begin{aligned} \|Y(\cdot)\|_s &\leq c_{1,s}K(N_i^{(u)})^{s+1/2}\varepsilon_0 \\ \|Ob(U(\cdot))\|_s &\leq c_{1,s}N_i^{s+1/2}\varepsilon_0 \end{aligned}$$

and $U'(\cdot) \in C^\infty(\mathbb{T}, g)$ is such that

$$\varepsilon'_s \leq c_{2,s}K^2(N_i^{(u)})^2((N_i^{(u)})^s\varepsilon_0 + \varepsilon_s)\varepsilon_0 + C_{s,s'}(N_{i+1}^{(l)})^{s-s'+2}\varepsilon_{s'}$$

Proof. Let us write the cohomological equation in the neighborhood of (α, A):

$$e^{Y(\cdot+\alpha)}Ae^{U(\cdot)}e^{-Y(\cdot)} = A'(\cdot)e^{U'(\cdot)}$$

where $A'(\cdot)$ is not too far from A and $U'(\cdot)$ is supposed to be much smaller than $U(\cdot)$. We remark that since we want to obtain a purely local conjugation proposition we allow $A'(\cdot)$ to be non-constant and to incorporate the obstructions to the solution of the linear equation. The following form of the equation is therefore more adapted to the problem

$$e^{Ad(A^*)Y(\cdot+\alpha)}e^{U(\cdot)}e^{-Y(\cdot)} = A^*A'(\cdot)e^{U'(\cdot)}$$

Since $A^*A'(\cdot) = exp(V(\cdot))$ is of the same order as $U(\cdot)$, we can linearize the preceeding equation to

$$Ad(A^*)Y(\cdot + \alpha) + U(\cdot) - Y(\cdot) = V(\cdot)$$

For the constant $A \in G$, we fix the the index i of the previous proposition and use the partition $\Delta_+ = I_r \cup I_d \cup I_0$ and $\mathbf{k} = \mathbf{k}_A \in \mathfrak{Z}$ associated to i. We have the following estimates

$$\begin{aligned} \|T_{N_{i+1}}^{(\mathbf{k}_A)}U(\cdot)\|_s &\leq C_s(N_i^u)^{s+1/2}\varepsilon_0 \\ \|R_{N_{i+1}}^{(\mathbf{k}_A)}U(\cdot)\|_s &\leq C_{s,s'}(N_{i+1}^l)^{s-s'+2}\varepsilon_{s'} \\ \|ObU(\cdot)\|_s &\leq C_s(N_i)^{s+1/2}\varepsilon_0 \end{aligned}$$

concerning the corresponding truncation operators.

The definition of the partition $I_r \cup I_d \cup I_0$ implies that the equation

$$Ad(A^*)Y(\cdot + \alpha) + U(\cdot) - Y(\cdot) = ObU(\cdot) + R_{N_{i+1}}^{(\mathbf{k}_A)}U(\cdot)$$

can be solved, with the estimate

$$\|Y(\cdot)\|_s \leq C_sK(N_i^u)^{s+1/2}\varepsilon_0$$

Coming back to the non-linear problem (see also section 11.3 of the appendix), we find that

$$e^{Y(\cdot+\alpha)}.A.e^{U(\cdot)}.e^{-Y(\cdot)} = Ae^{ObU(\cdot)}e^{U'(\cdot)}$$

where $U'(\cdot)$ is of second order with respect to $Y(\cdot)$ and $U(\cdot)$ (and $ObU(\cdot)$), so that

$$\varepsilon'_s \leq c_{2,s}K^2(N_i^{(u)})^2((N_i^{(u)})^s\varepsilon_0 + \varepsilon_s)\varepsilon_0 + C_{s,s'}(N_{i+1}^{(l)})^{s-s'+2}\varepsilon_{s'}$$

Finally, by the construction of the mapping $ObU(\cdot)$ and the vector k_A, it follows that the cocycle $(\alpha, Ae^{ObU(\cdot)})$ is reducible by the torus morphism constructed in the previous proposition, and therefore reducible modulo cD. □

Corollary 9.14. *Let $\alpha \in DC(\gamma, \tau)$ and $K \geq 2^{\tau+1}\gamma\tilde{b}^{q+1}N^\tau$. Let, also, $(\alpha, Ae^{U(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ with*

$$c_{1,0}K(N_q^{(u)})^2\varepsilon_0 < 1$$

where $\varepsilon_s = \|U\|_s$. *Then, there exists* $0 \leq i \leq q$ *and a conjugation* $G(\cdot) \in C^\infty(\mathbb{T}, G)$ *such that*

$$G(\cdot + \alpha).A.e^{U(\cdot)}.G^*(\cdot) = A'e^{U'(\cdot)}$$

The mapping $U'(\cdot)$ *is* 1*-periodic, and* $G(\cdot)$ *and* $U(\cdot)$ *satisfy the following estimates*

$$\begin{aligned} \|G(\cdot)\|_s &\leq c_{1,s}N_i^s + c_{1,s}K(N_i^{(u)})^{s+1/2}\varepsilon_0 \\ \varepsilon'_s &\leq c_{2,s}K^2(N_i^{(u)})^2((N_i^{(u)})^s\varepsilon_0 + \varepsilon_s)\varepsilon_0 + C_{s,s'}(N_{i+1}^{(l)})^{s-s'+2}\varepsilon_{s'} \end{aligned}$$

Proof. We only need to apply successively the conjugations constructed in the proof of corollary 9.9. By the corollary we have

$$Conj_{\exp(Y(\cdot))}(\alpha, Ae^{U(\cdot)}) = (\alpha, Ae^{ObU(\cdot)}e^{\tilde{U}(\cdot)})$$

where $\tilde{U}(\cdot)$ is quadratic with respect to $U(\cdot)$ and $Y(\cdot)$. We observe that

$$\sigma(Ad(e^{-H\cdot})) \subset \mathcal{B}(\emptyset, bDN_i)$$

so that

$$\sigma(Ad(e^{-H\cdot})R_N^{\mathbf{k}}U(\cdot) \subset \mathcal{B}(\emptyset, bDN^l)^c$$

which implies

$$\|(Ad(e^{-H\cdot})R_N^{\mathbf{k}}U(\cdot)\|_s \leq C_{s,s'}(N_{i+1}^l)^{s-s'+2}\varepsilon_{s'}$$

By construction, the conjugation by $e^{-H\cdot}$ satisfies

$$Conj_{\exp(-H\cdot)}(\alpha, Ae^{ObU(\cdot)}e^{\tilde{U}(\cdot)}) = (\alpha, Ae^s e^{U'(\cdot)})$$

where $s \in g$ and $U'(\cdot) = Ad(e^{-H\cdot})\tilde{U}(\cdot)$.

Finally, we can choose the torus morphism $C(\cdot)$ constructed in the corollary for the reduction of the period of the conjugation to be of minimal C^1 norm, so that the adjoint action of $C(\cdot)$ affects the norms by constants depending only on G. Since the mapping $G(\cdot) = C(\cdot)e^{-H\cdot}e^{Y(\cdot)}$ is 1-periodic, it satisfies the conclusions of the corollary. □

The proof of this corollary concludes the preparation for the K.A.M. scheme.

9.6 Iteration and conclusion

In this section, we will follow [FK09] since the scheme needed in order to conclude is only a special case of *Theorem 10*, proved in the appendix C of the paper. For the sake of completeness, we present the context and the statement of the theorem.

Let $a > 0$, $0 < \sigma_0 \leq 1$, $M, m, \mu, \overline{\mu} > 0$ such that

$$\frac{M}{m} < 1 + \sigma_0,\ \frac{m}{\mu} < 1,\ 0 \leq 2\overline{\mu} < \mu$$

We point out that $\overline{\mu}$ is taken positive in the paper, but inspection of the proof shows that $\overline{\mu} = 0$ is admissible.

For such choices of parameters, there exists $g > 0$

$$\frac{1}{\mu - \overline{\mu}} < g < \min(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\overline{\mu}})$$

and $0 < \sigma < \sigma_0$ such that

$$1 + \sigma < \mu - \overline{\mu}g$$

that is

$$\frac{1+\sigma}{\mu - \overline{\mu}} < g < \min(\frac{1+\sigma_0}{M}, \frac{1}{m}, \frac{1}{\overline{\mu}})$$

For $s, \overline{s}$, let $C_{s,\overline{s}} : [0, \infty) \to [1, \infty)$ be a family of continuous functions on $[0, 2]$ such that $C_{s,\overline{s}}(t) = 1$ if $t > 2$, increasing with respect to $s, \overline{s} \geq 0$ and let $\overline{C}_s = \max_{0 \leq t \leq 2} C_{s,s}(t)$.

Under these assumptions we get the following theorem

Theorem 9.15. *There exists $s_0 > 0$ such that if $\varepsilon_{p,s}$ is a double sequence satisfying*

$$\varepsilon_{p+1,s} \leq C_{s,\overline{s}}(1+\lambda_p^a\varepsilon_{p,0})(\lambda_p^{a+Ms}\varepsilon_{p,0}^{1+\sigma_0} + \lambda_p^{a+ms}\varepsilon_{p,0}\varepsilon_{p,s} + \lambda_p^{a-(\overline{s}-s)\mu}(\varepsilon_{p,\overline{s}} + \lambda_p^{\overline{s}\,\overline{\mu}}\varepsilon_{p,0}))$$

for any $s, \overline{s}, p \in \mathbb{N}$, where $\lambda_p = L^{(1+\sigma)^p}$, $L = \overline{C}_{s_0}$ and if

$$\varepsilon_{0,0} \leq (\overline{C}_{s_0})^{-\frac{s_0}{g}},\ \varepsilon_{0,s_0} \leq 1$$

then, for any $s \in \mathbb{N}$ $\varepsilon_{p,s} = O(\lambda_p^{-\infty})$. Moreover, s_0 does not depend on the sequence $(C_s)_s$ and can be taken of the form $(a+1)\xi(\sigma_0, M, m, \mu, \overline{\mu})$.

Remark 9.16. A small comment on the theorem and its proof is due. In its statement, the theorem asserts that, provided that the smallness conditions are satisfied and the inductive estimates hold for all positive integers p, then the K.A.M. scheme converges exponentially fast. In practice, however, the inductive estimates hold under certain smallness conditions which have to be verified at each step. In our case, the smallness conditions, which are related to the applicability of the Hausdorff-Campbell formula, are of the type $CL_p^a\varepsilon_{p,0} < 1$ in the notation of the theorem above.

This problem is resolved by the very first lemma in the proof of the theorem 9.15 which asserts that if γ_0 is big enough (the largeness depending on the parameters of the theorem), $s_0 = g\gamma_0$ and $b = \kappa\gamma_0$ where $\kappa > 0$ is such that

$$\frac{1+\sigma+\kappa}{\mu-\overline{\mu}} < g$$

and if the smallness assumptions of the theorem for $p = 0$ are satisfied, then we have inductively

$$\begin{aligned}\varepsilon_{p,0} &< \lambda_p^{-\gamma_0}\\ \varepsilon_{p,s_0} &< \lambda_p^b\end{aligned}$$

which justifies the use of the inductive estimates. We have omitted the constants in the expressions for the sake of simplicity.

9.6.1 Choice of constants

In order to apply theorem 9.15, we begin by fixing some notation. We call

$$\varepsilon_{n,s} = \|U_n(\cdot)\|_s$$

We also define the sequence of orders of truncation N_n and the diophantine constants K_n and we need to have

$$\begin{aligned}K_n &\geq 2^{\tau+1}\gamma\tilde{b}^{q+1}N_n^\tau\\ N_{n,i} &= \tilde{b}^i N_n\\ N_{n,i}^l &\geq N_n\\ N_{n,i}^u &\leq 2\tilde{b}^{q+1}N_n\end{aligned}$$

If we also take into account the customary definition of $N_n = N_{n-1}^{1+\sigma} = N_1^{(1+\sigma)^{n-1}}$, where $N = N_1$ and σ are to be fixed, we may also chose K_n to grow as a power of N_n, so that we have

$$\begin{aligned}N_n &= N^{(1+\sigma)^{n-1}}\\ K_n &= N_n^{\tau+\nu} \geq 2^{\tau+1}\gamma\tilde{b}^{q+1}N_n^\tau\end{aligned}$$

where for $\nu > 0$ fixed, the last inequality holds if N is big enough.

9.6.2 Estimates

In view of corollary 9.14, we have the following situation. A cocycle $(\alpha, Ae^{U(\cdot)}) = (\alpha, A_1 e^{U_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ is given, and we want to construct inductively a sequence of 1-periodic conjugations

$$\begin{aligned} L_n(\cdot) &= G_n(\cdot)...G_1(\cdot) \\ &= C_n(\cdot)e^{-H_n\cdot}e^{Y_n(\cdot)}...C_1(\cdot)e^{-H_1\cdot}e^{Y_1(\cdot)} \end{aligned}$$

such that

$$(\alpha, A_1 e^{U_1(\cdot)}) = Conj_{L_n(\cdot)}(\alpha, A_n e^{U_n(\cdot)})$$

At each step, $Y_n(\cdot)$ is to be given by the local conjugation lemma 9.13, the cD-periodic torus morphism $e^{-H_n\cdot}$ reduces the obstructions of $U_n(\cdot)$ seen as a perturbation of the constant cocycle (α, A_n), and $C_n(\cdot)$ commutes with

$$A_{n+1} = e^{-H_n\alpha}A_n \exp(Ad(e^{-H\cdot}).Ob_n U_n(\cdot))$$

and $C_n(1) = e^{-H_n}$, so that $G_n(\cdot)$ and consequently $Ad(G_n(\cdot))$ are 1-periodic.

Let us now express the inductive estimates at the n-th step as a function of N_n. The corollary 9.14 is applicable at the step n if

$$c_{1,0}K_n(N_{n,q}^u)^2\varepsilon_{n,0} < 1$$

or, with a different constant $c_{1,0}$,

$$c_{1,0}N_n^{\tau+\nu+2}\varepsilon_{n,0} \leq 1 \tag{9.13}$$

We find that

$$\begin{aligned} \varepsilon_{n+1,s} &\leq c_{2,s}N_n^{2\tau+2\nu+2}(N_n^s\varepsilon_{n,0} + \varepsilon_{n,s})\varepsilon_{n,0} + C_{s,s'}N_n^{s-s'+2}\varepsilon_{n,s'} \\ \|G_n(\cdot)\|_s &\leq c_{1,s}N_n^{\tau+\nu+s+1/2}\varepsilon_{n,0} + c_{3,s}N_n^{s+1/2} \end{aligned}$$

or, in the form of the estimates as they appear in theorem 9.15

$$\varepsilon_{n+1,s} \leq C_{s,\overline{s}}(N_n^{2\tau+2\nu+2+s}\varepsilon_{n,0}^2 + N_n^{2\tau+2\nu+2}\varepsilon_{n,0}\varepsilon_{n,s} + N_n^{2-(\overline{s}-s)}\varepsilon_{n,\overline{s}}) \tag{9.14}$$

Therefore, the estimates satisfy the hypotheses of theorem 9.15 with

$$\begin{aligned} a &= 2\tau + 2\nu + 2 \\ M &= 1 \\ \sigma_0 &= 1 \\ m &= 0 \\ \mu &= 1 \\ \overline{\mu} &= 0 \end{aligned}$$

and with these choices, we can determine σ as in theorem 9.15.

9.6.3 Iteration and convergence

A choice of a big enough $N \geq \overline{C}_{s_0}$ implies that the smallness condition 9.13 is satisfied provided that $\varepsilon_{0,0} \leq N^{-\frac{s_0}{g}}$. Therefore, the scheme can be initiated, and the remark following theorem 9.15 justifies the application of corollary 9.14 at each step, and consequently the inductive estimates in eq. 10.5 are valid.

Therefore, 9.15 is applicable and gives that, for all $s \in \mathbb{N}$,

$$\varepsilon_{n,s} = O(N_n^{-\infty})$$

i.e. that for any $s \in \mathbb{N}$ and for any $\sigma \in N$, there exists a positive constant $C = C(s, \sigma)$ such that

$$\varepsilon_{n,s} \leq CN_n^{-\sigma}$$

For shortness, we will write $\|U_n(\cdot)\| = O_{C^\infty}(N_n^{-\infty})$.

9.6.4 Proof of theorem 9.1

A direct consequence of the application of the generalized K.A.M. scheme of theorem 9.15 is the proof of theorem 9.2, since the convergence of every $\varepsilon_{n,s}$ with s fixed and $n \to \infty$ is precisely the definition of almost reducibility.

The fact that $\varepsilon_{n,s}$ converges exponentially fast to 0 becomes important in the proof of the local density theorem:

Proof of theorem 9.1. In the previous section, we obtained that, if $U_1(\cdot)$ satisfies the smallness conditions of theorem 9.15, we can construct a sequence of conjugations $L_n(\cdot) \in C^\infty(c\mathbb{T}, G)$ such that

$$\begin{aligned} \|L_n(\cdot)\|_s &\lesssim nN_n^s \\ (\alpha, A_1 e^{U_1(\cdot)}) &= Conj_{L_n(\cdot)}(\alpha, A_n e^{U_n(\cdot)}) \\ \|U_n(\cdot)\|_s &= O(N_n^{-\infty}) \end{aligned}$$

We have constructed the sequence of conjugations $L_n(\cdot)$ satisfying

$$\begin{aligned} A_1 e^{U_1(\cdot)} &= L_n(\cdot + \alpha) A_n e^{U_n(\cdot)} L_n^*(\cdot) \\ &= L_n(\cdot + \alpha) A_n L_n^*(\cdot) e^{\tilde{U}_n(\cdot)} \end{aligned}$$

where $\tilde{U}_n(\cdot) = Ad(L_n(\cdot))U_n(\cdot)$. Since

$$\left\|\tilde{U}_n(\cdot)\right\|_s \lesssim \varepsilon_{n,s} + nN_n^s \varepsilon_{n,0}$$

we obtain directly that $\tilde{U}_n(\cdot) \to 0$ in the C^∞ topology. Therefore, $L_n(\cdot + \alpha) A_n L_n^*(\cdot) \to A_1 e^{U_1(\cdot)}$, and by construction the cocycles

$$(\alpha, L_n(\cdot + \alpha) A_n L_n^*(\cdot))$$

are reducible. □

The proof of the global analogues of the theorems are obtained in the same way as theorem 8.4 from theorem 8.3, and we only point out that the local theorems are true for cocycles in $SW(\mathbb{T}^d, G)$ for any $d \in \mathbb{N}^*$, the only difference being that the loss of derivatives in the local conjugation lemmas is $s + d/2$ in place of $s + 1/2$ and the resulting non-significant changes in the choice of constants in the K.A.M. scheme.

Chapter 10

Perturbations of singular geodesics

We now return to the study of the non-local problem. More specifically, we take up the local study the remaining case of singular geodesics, which is the final step for the proof of the theorem on the global density of reducible cocycles and its consequences.

In this setting, we suppose that e_r is a singular vector in g, so that neither of the factors in the splitting $G_0 \times G_+$ is trivial. The models toward which renormalization converges, modulo an iteration at most χ_{G_0} times, are of the form $E_r(\cdot)A$, where $E_r(\cdot)$ and A commute.

We fix $\mathfrak{t}$, a maximal toral algebra in g and $\mathcal{T}$ the corresponding torus passing by A. We can therefore, write such a cocycle in the form

$$(\alpha, E_r(\cdot).A.e^{U(\cdot)})$$

where $A \in \mathcal{T}$ and $U(\cdot)$ is small in the C^∞ topology.

The particularity of the local study of singular geodesics is the coexistence of local and global phenomena, which demands a posteriori estimates on the rest, as in the study of regular geodesics, together with a reduction scheme as in the local case. The independence of the two parts of the dynamics in the first order allows us, however, to apply separately the reduction within G_0 and within the directions that do not commute with G_+.

10.1 Notation

Throughout this chapter, we suppose given a 1-periodic perturbation of a 1-periodic geodesic of the form

$$(\chi\tilde{\alpha}, E_{\chi\tilde{r}}(\cdot)A\exp(U(\cdot)))$$

where $E_r(\cdot)$ and A commute, and $1 \leq \chi \leq \chi_0$. We call

$$(\alpha, E_r(\cdot)A\exp(\tilde{U}(\cdot))) = (\chi\tilde{\alpha}, E_{\chi\tilde{r}}(\cdot)A\exp(U(\cdot)))$$

We also suppose that $\alpha \in DC(\gamma, \tau)$, i.e.

$$|k\alpha|_{\mathbb{Z}} = dist(k\alpha, \mathbb{Z}) \geq \frac{\gamma^{-1}}{|k|^\tau}$$

This is implied by the hypothesis that $\tilde{\alpha} \in DC(\tilde{\gamma}, \tau)$, only with a different constant γ. We also suppose that e_r is singular, so that the corresponding splitting $G_0 \times G_+$ has no trivial factors.

We remind that we denote by w (w_0) the rank of G (resp. G_0), by q (resp. q_0) the number of positive roots of G (resp. G_0), and by f (resp. f_0) the dimension of the real vector space g (resp. g_0). If $\tilde{\Delta}$ are the roots in a basis of Weyl, we denote by $\tilde{\Delta}_0$ the roots in the corresponding basis of the root system of $\mathfrak{t}_0 = \mathfrak{t} \cap g_0$, by $I^{(0)}$ the positive roots of G_0 and $I^{(+)} = \Delta_+ \setminus I^{(0)}$. In other terms,

$$\begin{aligned} I^{(+)} &= \{\rho \in \Delta_+, \rho(e_r) \neq 0\} \\ I^{(0)} &= \{\rho \in \Delta_+, \rho(e_r) = 0\} \end{aligned}$$

We also remind that for $\rho' \in I^{(0)}$, there exist positive integers $(m_{\rho',\rho})_\rho, \rho' \in I^{(0)}$ such that

$$\rho' = \sum_{\tilde{\Delta}_0} m_{\rho',\rho}\rho$$

there exist rational numbers $p_{\rho',\rho} = q_{\rho',\rho}/D_0$ with $|q_{\rho',\rho}| \leq b_0$, such that

$$H_{\rho'} = \sum_{\tilde{\Delta}_0} p_{\rho',\rho}H_\rho$$

where $D_0, b_0 \in \mathbb{N}^*$ are considered fixed. The matrices in these relations are submatrices of the matrices satisfying the same relations in g, and consequently $D_0 \leq D$ and $b_0 \leq b$.

We also define the orthogonal projections $\pi_0 : g \to g_0$ and $\pi_+ = id - \pi_0$. [1]

As we have already seen in the concrete examples in chapter 4.2, the vectors H_ρ corresponding to roots in $\tilde{\Delta} \setminus \tilde{\Delta}_0$ do not form a basis of g_+. This fact is nonetheless not significant, so we complete the basis $(H_\rho)_{\rho\in\tilde{\Delta}_0}$ to $(H_\rho)_{\rho\in\tilde{\Delta}}$, a basis of g, with a set of orthonormal vectors in g_+, indexed by roots in $\Delta_+ \setminus \tilde{\Delta}_0$, even though the completion of the basis is non-canonical.

On the other hand, wherever the properties of the vector e_r are important, the reader should rather think in terms of the dual of the Weyl's basis of $\mathfrak{t}$ extending that of $\mathfrak{t}_0$, even though this basis is not used explicitly. The complex directions $j_\rho \in g$, $\rho \in \tilde{\Delta}$ are those associated to such a Weyl's basis of $\mathfrak{t}$.

We also use the notation $c_0 = c_{G_0}$ and $\mathcal{Z} = Z^0_G(\exp(\mathbb{R}e_r))$.

10.2 Synopsis of the chapter

In a first time, as in the previous chapters, we prove the local almost reducibility to normal forms in the case of singular geodesics:

Theorem 10.1. *Let $\alpha \in DC(\gamma,\tau)$ and $r \in \mathbb{N}^w$ such that $E_r(\cdot)$ be a 1-periodic singular geodesic and $A \in Z^0_G(\exp(\mathbb{R}e_r)) = \mathcal{Z}$. Then, there exist $\epsilon > 0$ and $s_0 \in \mathbb{N}^*$ such that if*

$$\begin{aligned} \|U(\cdot)\|_0 = \varepsilon_0 &< \epsilon \\ \|U(\cdot)\|_{s_0} = \varepsilon_{s_0} &< 1 \end{aligned}$$

and the cocycle $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ is of degree r, then it is almost reducible to $(\alpha, E_r(\cdot)A')$ where also $A' \in \mathcal{Z}$: There exists a sequence of conjugations $B_n(\cdot) \in C^\infty(\mathbb{T}, G)$ and a sequence of constants $A_n \in \mathcal{Z}$, such that

$$((E_r(\cdot)A_n)^* \left(B_n(\cdot+\alpha).E_r(\cdot)Ae^{U(\cdot)}.B_n^*(\cdot)\right) \to Id$$

in $C^\infty(\mathbb{T}, G)$.

We point out that the failure of exact reducibility to the normal form is due to small divisor phenomena in G_0. All the complications in this part of the theory, in comparison with the theory of regular geodesics, are due to the presence of the non-trivial factor G_0. Consequently, all the proofs and the statements of the theorems stand between the simplicity and the sharpness of those in Chapter 8 and the complications of the local theory of the previous chapter.

Besides almost reducibility to periodic geodesics, we can also obtain the local density of cocycles reducible to periodic geodesics in the class of cocycles of degree r:

Theorem 10.2. *Let $\alpha \in DC(\gamma,\tau)$, $r \in \mathbb{N}^w$ such that $E_r(\cdot)$ be a 1-periodic singular geodesic and $A \in \mathcal{Z}$. Then, cocycles reducible to $(\alpha, E_r(\cdot)A')$ where $A' \in \mathcal{Z}$ are dense in the class of cocycles of the form $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ which are of degree r and $U(\cdot)$ satisfies the smallness conditions of the previous theorem. More formally, The closure of*

$$\{Conj_{B(\cdot)}(\alpha, E_r(\cdot)A'), B(\cdot) \in C^\infty(\mathbb{T}, G), A' \in \mathcal{Z}\}$$

in SW^∞_α contains all cocycles in

$$\{(\alpha, E_r(\cdot)A).e^{U(\cdot)}), A \in \mathcal{Z}, U(\cdot) \in C^\infty(\mathbb{T}, G), \varepsilon_0 < \epsilon, \varepsilon_{s_0} < 1, \deg(\alpha, E_r(\cdot)A).e^{U(\cdot)}) = r\}$$

1. Note that π_+ is not a projection on g_+ but on $(g_0)^\perp$.

The proof of this theorem is obtained by the comparison of the rate of convergence of the scheme with the explosion of the norms of the reducing conjugations, exactly as in the local theory.

As previously, if we relax the arithmetic condition on the frequency to an RDC, we obtain the global analogues of these theorems, namely

Theorem 10.3. *Let $\alpha \in RDC$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, G)$ be of degree $r \in \mathbb{N}^w$ such that $E_r(\cdot)$ be a 1-periodic singular geodesic. Moreover, let $m \in \mathbb{N}^*$ be the minimal natural number such that $(\alpha, A(\cdot))^m$ is homotopic to the Id. Then, there exists $\chi \in \mathbb{N}$, with $1 \leq \chi \leq m$ and $\chi | m$, such that $(\alpha, A(\cdot))^\chi$ can be conjugated arbitrarily close in $C^\infty(\mathbb{T}, G)$ to cocycles of the form $(\chi\alpha, E_{\chi r}(\cdot)A)$, where $A \in \mathcal{Z}$.*

We also prove the following theorem.

Theorem 10.4. *Let α and $(\alpha, A(\cdot))$ as in the previous theorem. Then, for the same $\chi \in \mathbb{N}^*$, $(\alpha, A(\cdot))^\chi$ is accumulated in $SW^\infty_{\chi\alpha}(\mathbb{T}, G)$ by cocycles conjugate to $(\chi\alpha, E_{\chi r}(\cdot)A)$ in $SW^\infty_{\chi\alpha}(\mathbb{T}, G)$, where $A \in \mathcal{Z}$.*

Finally, we state the following corollary

Corollary 10.5. *Let $\alpha \in RDC$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, G)$ be of positive energy, and m as above. Then, there exists $\chi \in \mathbb{N}$, with $1 \leq \chi \leq m$ such that $(\alpha, A(\cdot))^\chi$ is accumulated in $C^\infty(\mathbb{T}, G)$ by cocycles of smaller energy.*

Proof. Since $(\alpha, A(\cdot))^\chi$ is almost reducible to a cocycle of the form $(\alpha, E_r(\cdot)A)$ where $Ad(A).e_r = e_r$, and for all $a \in g$ small enough and such that $[a, e_r] \neq 0$, $(\chi\alpha, E_{\chi r}(\cdot)Ae^a)$ is of smaller energy than $(\alpha, A(\cdot))^\chi$, the result follows. □

We can now obtain the main theorem of this thesis, the theorem of global density of reducible cocycles, assuming the proofs of the theorems previously stated in this section.

10.3 Proof of the global density theorem

The proof of theorem 3.16 is in fact a finite induction argument and uses the yet unproven theorems of this chapter, as well as the theorems proved in chapters 8 and 9.

Proof. Let $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}, G)$. If the cocycle is of energy 0, the conclusion of the theorem is true by the theorems proved in the study of the local case, see the synopsis of chapter 9, and more specifically theorem 9.3.

If the cocycle is of degree $r \neq 0$, then, by corollary 10.5, there exists $1 \leq \chi \leq \min(m, \chi_0)$ and $\chi | m$ such that $(\alpha, A(\cdot))^\chi$ is accumulated by cocycles of smaller energy. Let us consider $(\chi\alpha, A'(\cdot))$, a perturbation of $(\alpha, A(\cdot))^\chi$, and assume that $en(\chi\alpha, A'(\cdot)) > 0$, since if $en(\alpha, A'(\cdot)) = 0$ the proof is concluded. If $en(\chi\alpha, A'(\cdot)) > 0$, we need to renormalize $(\chi\alpha, A'(\cdot))$. Since we want to relate the result of the renormalization scheme with the dynamics of the cocycle $(\alpha, A(\cdot))$, we cannot apply the continued fractions algorithm to $\chi\alpha \in \mathbb{T}$, even though $(\chi\alpha, A'(\cdot)) \in SW^\infty_{\chi\alpha}(\mathbb{T}, G)$. We can, nonetheless, consider $\chi\alpha \in \chi\mathbb{T}$, which places $(\chi\alpha, A'(\cdot)) \in SW^\infty_{\chi\alpha}(\chi\mathbb{T}, G)$. Rescaling by χ we obtain the cocycle $(\alpha, A'(\chi^{-1}\cdot)) \in SW^\infty_\alpha(\mathbb{T}, G)$, which is of smaller energy than $(\alpha, A(\cdot))$, and whose order in the homotopy group is smaller.

If to $(\alpha, A(\cdot))$ we associate the splitting of the dynamics $G_0 \times G_+ \hookrightarrow G$, and to $(\alpha, A'(\chi^{-1}\cdot))$ the splitting $G'_0 \times G'_+ \hookrightarrow G$, we find that $G_0 \hookrightarrow G'_0$ in a canonical way. This is due to the fact that the splitting is preserved by renormalization, since the degree is preserved. Therefore, if r' is the degree of $(\alpha, A'(\chi^{-1}\cdot))$, G'_0 (or rather g'_0) is characterized by the equation $[e_{r'}, s] = 0$ for $s \in g$. This equation is satisfied for all $s \in g$ such that $[e_r, s] = 0$, since the perturbations that decrease the energy of the cocycle $(\alpha, A(\cdot))$ do not commute with e_r. More informally, such perturbations shrink G_+ and and consequently make G_0 grow.

If we denote by χ' the natural number corresponding to $(\alpha, A'(\chi^{-1}\cdot))$, χ divides χ' and thus

$$(\alpha, A(\cdot))^{\chi'} = (\chi\alpha, A_\chi(\cdot))^{\chi'/\chi}$$

is accumulated in $SW^{\infty}_{\chi'\alpha}(\chi\mathbb{T}, G)$ by cocycles of energy smaller than $en(\alpha, A'(\chi^{-1}\cdot))$, just by virtue of the corresponding theorem of accumulation by cocycles of smaller energy applied to $(\alpha, A'(\chi^{-1}\cdot))$.

Since the perturbations can be chosen arbitrarily small, we can repeat this step a finite number of times, until we obtain a cocycle for which $en = 0$, or, equivalently, $G'_+ = \{Id\}$ and $G'_0 = G$. □

10.4 Strategy of the proof of theorem 10.1

As announced in the previous chapter, the proof local almost reducibility to singular geodesics consists essentially in a parallel reduction of the positive and 0-energy parts of the perturbation. The linear equation corresponding to the first part is the one encountered in chapter 8, while the one corresponding to the latter is that of the local theory.

This coexistence of close-to-constants and far-from-constants dynamics does not allow the use of an inverse function theorem as in the case of regular geodesics, since the reduction of resonant modes in G_0 has to be treated with non-converging conjugations. As we will see, this forces a non-uniform choice for the spectral support of the obstructions, since the reduction of resonant modes does not commute with all $j_\rho, \rho \in I^{(+)}$. This is in fact a sufficient reason for the inapplicability of the inverse function theorem, as we know that reduction of resonant modes cannot be avoided for generic cocycles. We also need to adapt the a priori estimates obtained for perturbations of regular geodesics (see chapter 7) in order to control the obstructions in the positive-energy regime, as perturbations within G_0 will increase the energy of the path.

In order to cope with these problems, we use the K.A.M. scheme of the previous chapter combined with the one described in [Kri01], after having proved that the local reduction lemmas for local and non-local dynamics can actually be combined.

Let us, therefore, consider $(\alpha, A(\cdot)) = (\alpha, E_r(\cdot)A\exp(U(\cdot)))$, a 1-periodic perturbation of a singular geodesic, and assume that a $Y(\cdot) : \mathbb{T} \to g$, small enough, can reduce the perturbation $U(\cdot)$ to $V(\cdot)$, with $V(\cdot)$ of second order with respect to $U(\cdot)$. Let us call $(\alpha, A'(\cdot))$ the cocycle thus obtained. The equation satisfied by such a conjugant is

$$e^{Y(\cdot+\alpha)}.E_r(\cdot)A.e^{U(\cdot)}.e^{-Y(\cdot)} = E_r(\cdot)A'.e^{V(\cdot)}$$

Linearization of this equation under the smallness assumptions implies that such a $Y(\cdot)$ must satisfy the equation

$$Ad(E^*_r(\cdot)A^*).Y(\cdot+\alpha) - Y(\cdot) = -U(\cdot) \tag{10.1}$$

The linearized equation then decomposes into two different types of equations (and a third for the coordinates in the torus) as follows.

Projection of the equation on g_0 and introduction of some obvious notation gives

$$Ad(A^*)Y^{(0)}(\cdot+\alpha) + U^{(0)}(\cdot) - Y^{(0)}(\cdot) = Ob_0U(\cdot) + R^{0,(\mathbf{k}_A)}_{N_{i+1}}U(\cdot)$$

where $Y(\cdot)$ satisfies good estimates and the cocycle $(\alpha, A.e^{Ob_0U(\cdot)})$ is reducible by $B(\cdot) : c_0D_0\mathbb{T} \to G_0 \hookrightarrow G$.

Projection of the equation on the positive energy component gives

$$\begin{aligned} Ad(E_r(\cdot)^*.A^*).Y^{(+)}(\cdot+\alpha) + U^{(+)}(\cdot) - Y^{(+)}(\cdot) &= \Gamma_{\tilde{r}}U^{(+)}(\cdot) + \sum_{\Delta_{++}} \hat{U}^{\mathrm{t}}_\rho(0).H_\rho + \sum_{\Delta_{++}} R_{N_{i+1}}U_\rho(\cdot)j_\rho \\ &= Ob_+U(\cdot) + \sum_{\Delta_{++}} R_{N_{i+1}}U_\rho(\cdot)j_\rho \end{aligned}$$

If, now, we call $ObU(\cdot) = Ob_0U(\cdot) + Ob_+U(\cdot)$, we find that

$$Conj_{\exp(Y(\cdot))}(\alpha, E_r(\cdot)Ae^{U(\cdot)}) = (\alpha, E_r(\cdot)A'e^{ObU(\cdot)}e^{V(\cdot)})$$

where $ObU(\cdot)$ is of the order of $U(\cdot)$, and $V(\cdot)$ is of second order. Since, however, $\pi_0ObU(\cdot) = Ob_0U(\cdot)$ may not be constant, the length of the paths $A'(\cdot)$ and $A'_2(\cdot)$ will be greater than $|e_r|$ and $2|e_r|$ respectively, and therefore the a priori estimates cannot give any additional information.

This fact shows that, before applying the a priori estimates, we have to reduce the 0-energy part of the obstruction. This reduction may translate the positive energy part of the obstruction, since $\rho(H)$ may be non-zero also for roots in $I^{(+)}$. This fact forces a modification in the definition of the operator $\Gamma_{\tilde{r}}$, so that, the frequencies of $Ad(e^{-H\cdot}).\Gamma_{\tilde{r}}$ belong to the standard intervals $[0, -r_\rho - 1] \cap \mathbb{Z}$ or $[-r_\rho + 1, 0] \cap \mathbb{Z}$, depending on the sign of r_ρ (see def. 7.3). Moreover, since the conjugation (called $C(\cdot)$ in lemma 9.9) takes values in a different torus, it may mix the frequencies of the positive energy obstructions, so we need to apply the estimates for a D-periodic perturbation of the geodesic, obtain their smallness, and then regain periodicity.

The conjugation lemma that we obtain this way gives rise to a slightly non-standard K.A.M. scheme for the following reason. We have no direct estimates for the C^0 norm of the perturbation after conjugation, and we cannot have any, since the only way to control it is through the C^1 norm. Therefore, the Kolmogorov-Hadamard interpolation inequalities have to be applied between the C^1 and C^{s_0} norms, instead of C^0 and C^{s_0}, and we can then use the fact that the C^0 norm of the first derivative controls the C^0 norm of the function in order to gain smallness of all norms and continue the procedure.

10.5 A priori estimates

In this section we adapt the apriori estimates obtained for perturbations of regular geodesics in the case of perturbations of singular geodesics, and we prove a similar lemma. The differences are in fact due to the different definition of the truncation operator, taking into account the fact that e_r is singular, and to the fact that perturbations of longer periods need to be considered. Therefore, in what follows we let $U(\cdot) : P\mathbb{T} \to g$, where $P \in \mathbb{N}^*$.

Estimates on the energy of the path

If $U(\cdot)$ is small and P-periodic, then the path $A(\cdot) = E_{Pr}(\cdot).A.e^{\tilde{U}(\cdot)}$, where $\tilde{U}(\cdot) = U(P\cdot)$ is 1-periodic. We remind that, since

$$U(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} \sum\nolimits_{k\in\mathbb{Z}} \hat{U}^{\mathfrak{t}}_\rho(k) e^{2i\pi P^{-1}k\cdot} H_\rho + \sum\nolimits_{\rho\in\Delta_+} \sum\nolimits_{k\in\mathbb{Z}} \hat{U}_\rho(k) e^{2i\pi P^{-1}k\cdot} j_\rho$$

where

$$\hat{U}^{(\mathfrak{t})}_\rho(k) = \int_0^1 U^{(\mathfrak{t})}_\rho(P\cdot) e^{-2i\pi k\cdot}$$

we find immediately that

$$\tilde{U}(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} \sum\nolimits_{k\in\mathbb{Z}} \hat{U}^{\mathfrak{t}}_\rho(k) e^{2i\pi k\cdot} h_\rho + \sum\nolimits_{\rho\in\Delta_+} \sum\nolimits_{k\in\mathbb{Z}} \hat{U}_\rho(k) e^{2i\pi k\cdot} j_\rho$$

Estimation of its energy gives

$$\|a(\cdot)\|^2_{L^2} = P^2 en^2 + 2\int \langle e_{Pr}, \tilde{u}(\cdot)\rangle + \|\tilde{u}(\cdot)\|^2_{L^2}$$

This last equation implies that, in the first order, the projection of $u(\cdot)$ on g_0 contributes only (positively) in the term $\|u(\cdot)\|^2_{L^2}$, and not in $\int\langle e_r, u(\cdot)\rangle$, which justifies the notation

$$\begin{aligned} I^{(+)} &= \{\rho \in \Delta_+, \rho(e_r) \neq 0\} \\ I^{(0)} &= \{\rho \in \Delta_+, \rho(e_r) = 0\} \end{aligned}$$

We also modify the definition of the truncation operators:

Definition 10.6. Let $\tilde{r}_\rho \in \mathbb{Z}^*$ for $\rho \in I^{(+)}$, and $P \in \mathbb{N}^*$. We will call $I_{P,\rho} = \{1, 2, ... - \tilde{r}_\rho P + 1\}$ if $\tilde{r}_\rho < 0$, and $I_{P,\rho} = \{-1, -2, .., -\tilde{r}_\rho P - 1\}$ if $\tilde{r}_\rho > 0$. We also let $\Lambda_{P\tilde{r}}$ and $\Lambda_{P,\rho}$ be the truncation operators

$$\begin{aligned} \Lambda_{P\tilde{r}} U(\cdot) &= \sum\nolimits_{\rho\in I^{(+)}} \Lambda_{P,\rho} U_\rho(\cdot) j_\rho \\ &= \sum\nolimits_{\rho\in I^{(+)}} \sum\nolimits_{k\in I_{P,\rho}} \hat{U}_\rho(k) e^{2i\pi k\cdot} j_\rho \end{aligned}$$

We will also use the non-homogeneous truncation operator [2]

$$\begin{aligned}\Lambda^0_{P\tilde{r}}U(\cdot) &= \sum\nolimits_{\rho\in I^{(+)}} \Lambda^0_{P,\rho}U_\rho(\cdot)j_\rho \\ &= \Lambda_{P\tilde{r}}U(\cdot) + \sum\nolimits_{\rho\in I^{(+)}} \hat{U}_\rho(0)j_\rho\end{aligned}$$

A calculation similar to the one made in the regular case gives

$$\|a(\cdot)\|^2_{L^2} = P^2 en^2 + \sum_{\rho\in I^{(+)}} 4\pi^2 P\tilde{r}_\rho \sum\nolimits_{k\in\mathbb{Z}} k|\hat{U}_\rho(k)|^2) + \sum_{\rho\in\Delta_+}\sum_{k\in\mathbb{Z}} 4\pi^2k^2|\hat{U}_\rho(k)|^2 + \|\partial\pi_{\mathfrak{t}}\tilde{U}(\cdot)\|^2_{L^2} + O(\|\tilde{U}\|^3_{H^1})$$

and the resulting bound

$$\|a(\cdot)\|^2_{L^2} \leq P^2 en^2 - \sum_{\rho\in I^{(+)}} 4\pi^2\|\Lambda_{Pr}\tilde{U}(\cdot)\|^2_{L^2} + \|\partial(Id - \Lambda_{P\tilde{r}})\tilde{U}(\cdot)\|^2_{L^2} + O(\|\tilde{U}\|^3_{H^1})$$

so that the following lemma is true

Lemma 10.7. *Let $(\alpha, E_r(\cdot)A\exp(U(\cdot)))$ be of degree r and $U(\cdot)\in C^\infty(P\mathbb{T}, g)$ be such that $U(P\cdot)$ is small enough in H^1. Then, there exists a positive constant such that*

$$\|\Lambda_{P\tilde{r}}U(P\cdot)\|_{L^2} \lesssim \|\partial(Id - \Lambda_{P\tilde{r}})U(P\cdot)\|_{L^2}$$

The dependence of the constant on P is not important for us, since in fact $P = D$, the loss of periodicity in the reduction of resonant modes.

Estimation of the energy of the second iterate

We use the bound on the L^2 norm of the derivative, obtained in the regular case

$$\|a_2(\cdot)\|^2_{L^2} \leq 4P^2 en^2 + C_1\|\partial\tilde{U}(\cdot)\|^2_{L^2} - C_2\|[e_r, \tilde{U}\hat{}\,(0)]\|(\big\|[e_r, \tilde{U}\hat{}\,(0)]\big\| - \big\|\partial\tilde{U}(\cdot)\big\|_{L^2})$$

Since $\big\|[e_r, \tilde{U}\hat{}\,(0)]\big\|^2 = \sum_{\rho\in I^{(+)}} 4\pi^2\tilde{r}^2_\rho|\hat{U}_\rho(0)|^2$, the following lemma holds.

Lemma 10.8. *Let $(\alpha, E_r(\cdot)Ae^{\tilde{U}(\cdot)})$ be a P-periodic perturbation (small enough in H^1) of the cocycle $(\alpha, E_r(\cdot)A)$. Let also, without loss of generality, $\hat{U}(0)$ be orthogonal to e_r and $\sum_{\rho\in I^{(+)}}|\hat{U}_\rho(0)|^2 \neq 0$. Then, there exists a constant C_0 depending only on r such that if*

$$\|\partial\tilde{U}\|^2_{L^2} \leq C_0 \sum_{\rho\in I^{(+)}} |\hat{U}_\rho(0)|^2$$

then $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ is of energy strictly smaller than that of $(\alpha, E_r(\cdot)A)$.

Conclusion

In the same way, we conclude that the following a priori estimate holds for perturbations of singular geodesics, which is to be compared with lemma 7.1.

Lemma 10.9. *There exists a positive constant C, depending on r but not on A and $U(\cdot)$, such that if $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ is of degree r, where $U(\cdot)$ is P-periodic and small enough in H^1, then it satisfies*

$$\big\|\Lambda^0_{P\tilde{r}}U(P\cdot)\big\|_{L^2} \leq C\,\|(Id - \Lambda_{P\tilde{r}})\partial U(P\cdot)\|_{L^2}$$

We remark that the only difference with the estimates obtained previously is in the fact that the perturbation within g_0 contributes only in increasing the energy of the path. This is expressed in the estimates by the fact that functions in g_0 are in the kernel of the truncation operator $\Lambda_{P\tilde{r}}$. Since these estimates give no information on the pertrubation if the order of the perturbation in g_0 is bigger than the one not commuting with g_+, we need to reduce both parts.

2. The superscript 0 does not refer to g_0.

10.6 Local conjugation lemma

Let us assume that a $Y(\cdot) : \mathbb{T} \to g$, small enough, can reduce the perturbation $U(\cdot)$ to $V(\cdot)$ with $V(\cdot) \ll U(\cdot)$. The equation satisfied by such a conjugant is

$$e^{Y(\cdot+\alpha)}.E_r(\cdot).A.e^{U(\cdot)}.e^{-Y(\cdot)} = E_r(\cdot).A'.e^{V(\cdot)}$$

or

$$(Ad(E_r^*(\cdot)A^*)e^{Y(\cdot+\alpha)}).e^{U(\cdot)}.e^{-Y(\cdot)} = A^*A'.e^{V(\cdot)}$$

Linearization of this equation under smallness assumtions implies that such a $Y(\cdot)$ must satisfy the equation

$$Ad(E_r^*(\cdot)A^*).Y(\cdot+\alpha) - Y(\cdot) = -U(\cdot)$$

Let us introduce the coordinates

$$U(\cdot) = \sum\nolimits_{\rho\in\tilde{\Delta}} U_\rho^{\mathfrak{t}}(\cdot)H_\rho + \sum\nolimits_{\rho\in\Delta_+} U_\rho(\cdot)j_\rho$$

where as usual $U_\rho^{\mathfrak{t}}(\cdot)$ are real and $U_\rho(\cdot)$ complex periodic functions. We will write $Y(\cdot)$ in an analogous way. The linearized equation then decomposes in three different types of equations as follows.

The simpler case is that of the diagonal part, which reads

$$Y_\rho^{\mathfrak{t}}(\cdot+\alpha) - Y_\rho^{\mathfrak{t}}(\cdot) = -U_\rho^{\mathfrak{t}}(\cdot)$$

for which we refer the reader to lemma 3.4. In a similar way, if $\rho \in I^{(+)}$, i.e. if $\rho(e_r) \neq 0$, then the equation in the direction j_ρ is of the type of the lemma 8.6. Finally, for roots in $I^{(0)}$, we refer to proposition 9.13.

Putting all the corresponding estimates together gives the following lemma concerning the solution of the linearized equation.

Lemma 10.10. *Let $\alpha \in DC(\gamma,\tau)$ and $(\alpha, E_r(\cdot).A.e^{U(\cdot)}) \in SW^\infty(\mathbb{T},G)$ and $K \geq 2^{\tau+1}\gamma\tilde{b}_0^{q_0+1}N^\tau$. We remind that we call $\varepsilon_s = \|U\|_s$. Then, there exists $0 \leq i \leq q_0$ and mapping $Y(\cdot) \in C^\infty(\mathbb{T},g)$ and $U'(\cdot) \in C^\infty(\mathbb{T},g)$ such that*

$$Ad(E_r^*(\cdot).A^*)Y(\cdot+\alpha) + U(\cdot) - Y(\cdot) = ObU(\cdot) + \tilde{R}_nU(\cdot)$$

and $Conj_{\exp(Y(\cdot))}(\alpha, E_r(\cdot).A.e^{U(\cdot)}) = (\alpha, E_r(\cdot).A.e^{ObU(\cdot)}.e^{U'(\cdot)})$. The conjugant $Y(\cdot)$ satisfies the estimates

$$\|Y(\cdot)\|_s \leq c_{1,s}K(N_i^u)^{s+1/2}\varepsilon_0$$

The mapping $ObU(\cdot) = Ob_{(\alpha,E_r(\cdot).A)}U(\cdot)$ satisfies the property that there exists a torus morphism $B(\cdot) : c_0D_0\mathbb{T} \to G_0 \hookrightarrow G$ such that

$$\pi_{g_0}(Ad(B(\cdot))ObU(\cdot)) = U_0 \in g_0 \oplus g_+ \tag{10.2}$$

$$\Lambda^0_{D_0\tilde{r}}(Ad(B(\cdot))ObU(\cdot)) = Ad(B(\cdot))ObU(\cdot) - U_0 \tag{10.3}$$

We have the following estimates

$$\begin{array}{rcl} \|ObU(\cdot)\|_s & \leq & C_sN_i^{s+1/2}\varepsilon_0 \\ \|\tilde{R}_NU(\cdot)\|_s & \leq & C_{s,s'}(N_{i+1}^l)^{s-s'+2}\varepsilon_{s'} \\ \varepsilon_s' & \leq & c_{2,s}K^2(N_i^u)^2((N_i^u)^s\varepsilon_0+\varepsilon_s)\varepsilon_0 + C_{s,s'}(N_{i+1}^l)^{s-s'+2}\varepsilon_{s'} \end{array}$$

We remind that $N_{i+1} = \tilde{b}_0N_i$, $N_i^u = N_i + N_{i+1}$, and $N_i^l = N_i - N_{i-1}$.

Proof. The coordinates $Y_\rho^{\mathfrak{t}}(\cdot)$, $\rho \in \tilde{\Delta}$, as has been shown, satisfy the equation

$$Y_\rho^{\mathfrak{t}}(\cdot+\alpha) + U_\rho^{\mathfrak{t}}(\cdot) - Y_\rho^{\mathfrak{t}}(\cdot) = \hat{U}_\rho^{\mathfrak{t}}(0) + R_{N_i}U_\rho^{\mathfrak{t}}(\cdot)$$

for which the estimates follow directly.

We now focus on the roots in $I^{(0)}$. Following the solution of the linear equation in the close-to-constants case, we chose a suitable i between 0 and q_0 and partition the roots in $I^{(0)}$ into $I^{(0)}_{res} \cup I^{(0)}_{nr}$, the resonant and the non-resonant ones. We also define Ob_0, the projection operator on $C^\infty(\mathbb{T}, g_0)$ which projects to the resonant modes $\mathbf{k} = (k_\rho)_{\rho \in I^{(0)}}$ and the constant ones for roots close to 0. Finally, we use the truncation operator $R^{(\mathbf{k}),0}_{N_{i+1}}$ on $C^\infty(\mathbb{T}, g_0)$. With these notations, we have

$$\sum_{\rho \in I^{(0)}} [Ad(A^*)Y_\rho(\cdot + \alpha) + U_\rho(\cdot) - Y_\rho(\cdot)]j_\rho = Ob_0 U(\cdot) + R^{(\mathbf{k}),0}_{N_{i+1}} U(\cdot)$$

As in local theory (see proposition 9.13), there exists a torus morphism $B(\cdot) : c_0 D_0 \mathbb{T} \to G_0$ satisfying

$$|LB(\cdot)| \leq b_0 D_0 N_i$$

and such that

$$Ad(B(\cdot))Ob_0 U(\cdot) = U_0$$

a constant in g_0. The announced estimates on ε'_s, $Ob_0 U(\cdot) = \pi_0 ObU(\cdot)$ and the conjugation are obtained in the same way as in the proposition treating the purely local case.

Finally, we consider the roots in $I^{(+)}$ and define the integers k'_ρ for such roots by the relation

$$[LB(\cdot), j_\rho] = 2i\pi \frac{k'_\rho}{D_0} j_\rho$$

Clearly, if k'_ρ is non-zero, the reduction of resonant modes in G_0 translates the Fourier coefficients of the rest of the linear cohomological equation for roots in $I^{(+)}$. The a priori estimates of lemma 10.9, however, give us information only on a part of the spectrum of the perturbation determined by r. In order to take this effect into account, we need to translate the rest of the equation by $-\frac{k'_\rho}{D_0}$ in each direction, which amounts in fact to resolving the auxiliary equation

$$\sum\nolimits_{\rho \in I^{(+)}} [Ad(A^* E^*_r(\cdot))\tilde{Y}_\rho(\cdot + \alpha) + U_\rho(\cdot) - \tilde{Y}_\rho(\cdot)]j_\rho = Ad(B(\cdot)) \sum\nolimits_{\rho \in I^{(+)}} U_\rho(\cdot) j_\rho + \Gamma_{D_0 r}[Ad(B(\cdot)) \sum\nolimits_{\rho \in I^{(+)}} U_\rho(\cdot) j_\rho]$$

for which the given function function $\tilde{U}(\cdot) = Ad(B(\cdot)) \sum_{\rho \in I^{(+)}} U_\rho(\cdot) j_\rho$ satisfies the estimate $\tilde{\varepsilon}_s \leq C_s N_i^{s+1/2} \varepsilon_s$, since its Fourier coefficients are translated. By solving this equation, we obtain a rest corresponding to $\tilde{U}$ in standard configuration. If we conjugate the equation back by $Ad(B^*(\cdot))$, we obtain $Y_\rho(\cdot)$

$$\sum\nolimits_{\rho \in I^{(+)}} Y_\rho(\cdot) j_\rho = Ad(B^*(\cdot)) \sum\nolimits_{\rho \in I^{(+)}} \tilde{Y}_\rho(\cdot) j_\rho$$

which solves the cohomological equation for $U(\cdot)$ but with a translated rest. The rest $Ad(B^*(\cdot)).\Gamma_{D_0 r}[\tilde{U}(\cdot)]$ has the appropriate spectral support for eq. 10.3 to be satisfied. The periodicity is regained in the same way as in the local case, since the mapping $\tilde{Y}_\rho(\cdot)$ is obtained by simple operations on the Fourier coefficients of

$$Ad(B(\cdot)) \sum\nolimits_{\rho \in I^{(+)}} U_\rho(\cdot) j_\rho$$

which commute with algebraic conjugation by $Ad(B(\cdot))$.

This translation deteriorates the estimates for the rest obtained in proposition 8.6 for C^s norms by a factor $(b_0 N_i)^{s+1/2}$, while the rest of the estimates remain as in the proposition.

This last part is proved in more detail in lemma 11.3 in the appendix, for the sake of completeness. □

We remark that, by using smallness assumptions on the C^1 norm of the perturbation, we can show that the significant part of the obstruction is its projection on g_0, i.e. that $ObU(\cdot) - Ob_0 U(\cdot)$ is of the order of $U'(\cdot)$. Therefore, the non-constant part of the rest in the complement of g_0 can be incorporated to $U'(\cdot)$, thus giving the cocycle $(\alpha, E_r(\cdot).A.e^{Ob_0 U(\cdot)} e^{U'(\cdot)})$ where

$$(\alpha, E_r(\cdot).A.e^{Ob_0 U(\cdot)}) = Conj_{B(\cdot)}(\alpha, E_r(\cdot).B(\alpha)A. \exp(\frac{1}{D_0} \int_0^{D_0} Ad(B(\cdot))Ob_0 U(\cdot)))$$

so that the following corollary is true.

Corollary 10.11. *Let $\alpha \in DC(\gamma,\tau)$, $A \in \mathcal{Z} = Z_G(\exp(\mathbb{R}e_r)$, $(\alpha, E_r(\cdot).A.e^{U(\cdot)}) \in SW^\infty(\mathbb{T},G)$ be of degree r, and $K \geq 2^{\tau+1}\gamma\tilde{b}_0^{q_0+1}N^\tau$. Let, also,*

$$c_{1,0}K(N_{q_0}^u)^3\varepsilon_1 < 1$$

where $\varepsilon_s = \|U\|_s$. Then, there exists $0 \leq i \leq q_0$ and a conjugation $G(\cdot) = C(\cdot)B(\cdot)e^{Y(\cdot)} \in C^\infty(\mathbb{T},g)$ such that

$$G(\cdot+\alpha)E_r(\cdot)Ae^{U(\cdot)}G^*(\cdot) = E_r(\cdot)A'e^{U'(\cdot)}$$

where also $A' \in \mathcal{Z}$. The conjugant $G(\cdot)$ satisfies the estimates

$$\|G(\cdot)\|_s \leq c_{1,s}(K(N_i^u)^{s+1/2}\varepsilon_0 + N_i^{s+1/2})$$

Finally, if we call

$$\begin{aligned}\tilde{\varepsilon}_s &= \|\partial U(\cdot)\|_s\\ \tilde{\varepsilon}'_s &= \|\partial U'(\cdot)\|_s\\ \varepsilon'_0 &= \|U'(\cdot)\|_0\end{aligned}$$

we have the following estimates:

$$\begin{aligned}\varepsilon'_0 &\leq C_2\tilde{\varepsilon}'_0\\ \tilde{\varepsilon}'_s &\leq c_{2,s}K^2(N_i^u)^2((N_i^u)^s\tilde{\varepsilon}_0 + \tilde{\varepsilon}_s)\tilde{\varepsilon}_0 + C_{s,s'}(N_{i+1}^l)^{s-s'+2}\tilde{\varepsilon}_{s'}\end{aligned}$$

We note that the smallness condition is now imposed on ε_1, so that the a priori estimates can be applied. If we admit an a priori bound on ε_0, the ε_1 in the smallness condition can be substituted by $\tilde{\varepsilon}_0$. We also note that the quantities with tildes are the norms of the *derivative* of the perturbation.

Proof. We need only to show that the positive energy part of the obstruction $\pi_+ObU(\cdot)$ is of second order with respect to $U(\cdot)$. By construction of the mappings,

$$\begin{aligned}B(\cdot+\alpha)e^{Y(\cdot+\alpha)}E_r(\cdot)Ae^{U(\cdot)}e^{-Y(\cdot)}B^*(\cdot) &= E_r(\cdot)Ae^{Ad(B(\cdot))ObU(\cdot)}e^{\tilde{U}'(\cdot)}\\ &= E_r(\cdot)A'e^{\Lambda^0_{D_0\tilde{r}}Ad(B(\cdot)))ObU(\cdot)}e^{\tilde{U}'(\cdot)}\end{aligned}$$

where $A' = A\exp(Ad(B(\cdot)).Ob_0U(\cdot)) \in \mathcal{Z}$ is a constant, and $\delta'_s = \|\tilde{U}'(\cdot)\|_s$ satisfies

$$\delta'_s \leq c_{2,s}K^2(N_i^u)^2((N_i^u)^s\varepsilon_0 + \varepsilon_s)\varepsilon_0 + C_{s,s'}(N_{i+1}^l)^{s-s'+2}\varepsilon_{s'}$$

Under the assumption that $\delta'_1 \leq C$, where $C \ll 1$ is a fixed positive number, condition which is implied by the smallness assumption of the statement, $c_{1,0}K(N_{q_0}^u)^3\varepsilon_1 < 1$, the a priori estimates (lemma 10.9) are applicable and give

$$\|\Lambda^0_{D_0r}Ad(B(\cdot))Ob_+U(\cdot)\|_0 \leq C_1\|\partial\tilde{U}'(\cdot)\|_0$$

Therefore, if we call

$$U'(\cdot) = \exp^{-1}(e^{Ad(B(\cdot))ObU(\cdot)}e^{\tilde{U}'(\cdot)})$$

and

$$\tilde{\varepsilon}_s = \|\partial U(\cdot)\|_s \text{ and } \tilde{\varepsilon}'_s = \|\partial U'(\cdot)\|_s$$

we have

$$\begin{aligned}\varepsilon'_0 &\leq C_2\tilde{\varepsilon}'_0\\ \tilde{\varepsilon}'_s &\leq c_{2,s}K^2(N_i^u)^2((N_i^u)^s\tilde{\varepsilon}_0 + \tilde{\varepsilon}_s)\tilde{\varepsilon}_0 + C_{s,s'}(N_{i+1}^l)^{s-s'+2}\tilde{\varepsilon}_{s'}\end{aligned}$$

where we have used the a priori bound for ε_0.

Finally, conjugation by $C(\cdot)$ deteriorates the estimations by multiplicative constants and regains periodicity. □

10.7 Iteration and conclusion

10.7.1 Choice of constants

We now fix some notation similar to the one in the previous chapter. We call

$$\begin{aligned}\tilde{\varepsilon}_{n,s} &= \|\partial U_n(\cdot)\|_s \\ \varepsilon_{n,0} &= \|U_n(\cdot)\|_0\end{aligned}$$

We also define the sequence of orders of truncation N_n and the diophantine constants K_n as before

$$\begin{aligned}K_n &= N_n^{\tau+\nu} \\ N_n &= N^{(1+\sigma)^{n-1}} \\ N_{n,i}^l &\geq N_n \\ N_{n,i}^u &\leq 2\tilde{b}^{q+1}N_n\end{aligned}$$

where for $\nu > 0$ fixed, $K_n \geq 2^{\tau+1}\gamma\tilde{b}_0^{q_0+1}N_n^\tau$ provided that N is big enough.

10.7.2 Estimates

We now wish to apply the corollary 10.11 to the cocycle $(\alpha, E_r(\cdot).A.e^{U(\cdot)}) = (\alpha, E_r(\cdot).A_1.e^{U_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ where $A \in Z_G(\exp(\mathbb{R}e_r))$ in order to construct inductively a sequence of 1-periodic conjugations

$$\begin{aligned}L_n(\cdot) &= G_n(\cdot)...G_1(\cdot) \\ &= C_n(\cdot)e^{-H_n\cdot}e^{Y_n(\cdot)}...C_1(\cdot)e^{-H_1\cdot}e^{Y_1(\cdot)}\end{aligned}$$

such that

$$(\alpha, E_r(\cdot).A_1.e^{U_1(\cdot)}) = Conj_{L_n(\cdot)}(\alpha E_r(\cdot).A_n.e^{U_n(\cdot)})$$

with $A_n \in Z_G(\exp(\mathbb{R}e_r))$ and $U_n(\cdot) \to 0$ in C^∞ faster than the rate of growth of L_n. At each step, $Y_n(\cdot)$ is to be given by the local conjugation lemma 10.10, $B(\cdot)$ reduces the obstructions in g_0, and $C(\cdot)$ regains periodicity of the conjugants.

Let us therefore express the inductive estimates and smallness conditions at the n-th step as a function of N_n. Firstly, we admit a bound $\|U_1\|_0 \leq C_0 \ll 1$, which also has to be proven uniform for all n. If this is the case, the lemma of local reduction is applicable at the step n if

$$c_{1,1}K_n(N_{n,q}^u)^3\tilde{\varepsilon}_{n,0} < 1$$

or, with a different constant $c_{1,1}$,

$$c_{1,1}N_n^{\tau+\nu+3}\tilde{\varepsilon}_{n,0} < 1 \tag{10.4}$$

We find that

$$\begin{aligned}\tilde{\varepsilon}_{n+1,s} &\leq c_{2,s}K_n^2(N_{n,q_0}^u)^2((N_{n,q_0}^u)^s\tilde{\varepsilon}_{n,0} + \tilde{\varepsilon}_{n,s})\tilde{\varepsilon}_{n,0} + C_{s,s'}(N_{n,q_0+1}^l)^{s-s'+2}\tilde{\varepsilon}_{n,s'} \\ \varepsilon_{n+1,0} &\leq C_2\tilde{\varepsilon}_{n+1,0}\end{aligned}$$

or, since N_{n,q_0}^u and N_{n,q_0}^l are of the order of N_n,

$$\begin{aligned}\tilde{\varepsilon}_{n+1,s} &\leq c_{2,s}K_n^2(N_n)^2((N_n)^s\tilde{\varepsilon}_{n,0} + \tilde{\varepsilon}_{n,s})\tilde{\varepsilon}_{n,0} + C_{s,s'}(N_n)^{s-s'+2}\tilde{\varepsilon}_{n,s'} \\ \varepsilon_{n+1,0} &\leq C_2\tilde{\varepsilon}_{n+1,0}\end{aligned}$$

or, in the form of the estimates as they appear in theorem 9.15

$$\tilde{\varepsilon}_{n+1,s} \leq C_{s,\overline{s}}(N_n^{2\tau+2\nu+2+s}\tilde{\varepsilon}_{n,0}^2 + N_n^{2\tau+2\nu+2}\tilde{\varepsilon}_{n,0}\tilde{\varepsilon}_{n,s} + N_n^{2-(\overline{s}-s)}\tilde{\varepsilon}_{n,\overline{s}}) \tag{10.5}$$

Therefore, the estimates satisfy the hypotheses of theorem 9.15 with

$$\begin{aligned} a &= 2\tau + 2\nu + 3 \\ M &= 1 \\ \sigma_0 &= 1 \\ m &= 0 \\ \mu &= 1 \\ \overline{\mu} &= 0 \end{aligned}$$

and with these choices, we can determine σ as in theorem 9.15.

10.7.3 Iteration and convergence

A choice of a big enough $N \geq \overline{C}_{s_0}$ implies that the smallness condition 10.4 is satisfied provided that $\tilde{\varepsilon}_{0,0} \leq N^{-\frac{s_0}{g}}$. Therefore, the scheme can be initiated, and the remark following theorem 9.15 justifies the application of corollary 10.11 at each step, and consequently the inductive estimates in eq. 10.5 are valid. It is crucial that at each step the estimate $\varepsilon_{n,0} \leq C_2 \tilde{\varepsilon}_{n,0}$ implies that the C^0 norm, which we do not control directly dies out along with the C^0 norm of the first derivative, so that the use of the a priori estimates is justified.

Therefore, theorem 9.15 is applicable and gives that, for all $s \in \mathbb{N}$,

$$\tilde{\varepsilon}_{n,s} = O(N_n^{-\infty})$$

which implies that $\varepsilon_{n,s} = O(N_n^{-\infty})$ as well. The density theorem and the accumulation by cocycles of smaller energy are concluded as before, since conjugations grow only polynomially with N_n.

10.8 Proof of theorem 10.1

Finally, a word on the proof of the local almost reducibility and local density of cocycles reducible to the normal forms.

These theorems are obtained in the same way as in their counterparts for perturbations of constant cocycles. The K.A.M. scheme whose convergence we proved in the preceding section constructs a sequence of conjugations that conjugate a given cocycle $(\alpha, E_r(\cdot)Ae^{U(\cdot)})$ ever closer to one of the form $(\alpha, E_r(\cdot)A')$, on the assumption of a smallness condition on $U(\cdot)$ and provided that

$$deg(\alpha, E_r(\cdot)Ae^{U(\cdot)}) = r$$

Therefore, local almost reducibility of all such cocycles to normal forms is an immediate corollary of the convergence of the scheme. Finally, the given cocycle is conjugated exponentially close to the normal form, whereas the norms of conjugations grow only polynomially. This proves the local density theorem.

Chapter 11

Appendix

11.1 Missing proofs of some lemmas

We begin by the lemma concerning the abelian subsets of simply connected Lie groups.

Proof of Lemma 2.7. Let $A, B \in G$ be commuting elements and suppose that $\mathcal{T} \ni A$ is a maximal torus, such that $Ad(B).\mathcal{T} = \mathcal{T}$. Such a torus exists, by the proof of lemma 2.5. If $Ad(B)|_{\mathcal{T}} = Id$, then $\mathcal{T}$ contains both A and B, so the proof is finished. If not, let us consider $\Phi(\cdot) : [0,1] \to \mathcal{T}$ be a geodesic of minimal length connecting the Id with A and such that $\tilde{\Phi} = Ad(B).\Phi : [0,1] \to \mathcal{T}$ intersects Φ only at its extremities. This amounts to considering $a \in \mathfrak{t}$ ($\mathfrak{t}$ is the Lie algebra of $\mathcal{T}$), a preimage of A, such that $Ad(B).a \neq a$. This preimage satisfies, however, $Ad(B).a \in \mathfrak{t}$. This implies that a decomposes in $a_1 + a_2$, both in $\mathfrak{t}$, such that $Ad(B).a_1 = a_1$ (and a_1 is possibly 0), and $Ad(B).a_2 \neq a_2$, but $a_2 \neq 0$. Since A and B commute,

$$e = a - Ad(B).a = (a_1 + a_2) - Ad(B).(a_1 + a_2) = a_2 - Ad(B).a_2 \neq 0$$

is a preimage of the Id. Since $e + Ad(B).e = a_2 - Ad(B^2).a_2$, and since $Ad(B)|_{\mathcal{T}}$ is of finite order m, $\sum_0^m Ad(B^i).e = 0$. We call χ the minimal positive integer for which this sum is 0. We also remark that $\langle e, a_2 \rangle = -\langle e, Ad(B).a_2 \rangle$, so that if we call a' the projection of a_2 parallel to e, we have $\exp(2a') = Id$.

If we assume, now, that G is simply connected, the paths Φ and $\tilde{\Phi}$ are homotopic with fixed endpoints. In particular, there exists such a homotopy $\Psi_t(\cdot)$ such that for $t \in [0,1]$ fixed, $\Psi_t(\cdot)$ is a minimal geodesic connecting the Id with A, $\Psi_0(\cdot) = \Phi(\cdot)$, and $\Psi_1(\cdot) = \tilde{\Phi}(\cdot)$. It remains to show that we can find a homotopy Ψ such that for some fixed parameter value t_0, $Ad(B).L\Psi_{t_0}(\cdot) = L\Psi_{t_0}(\cdot)$.

The fact that the homotopy $\Psi_t(\cdot)$ consists of geodesics implies that there exists a path $B(\cdot) : [0,1] \to G$ such that $\Psi_t(\cdot) = Ad(B(t)).\Psi_0(\cdot)$. This path can be continuously deformed to a path still denoted by $B(\cdot)$, such that $Ad(B(\cdot)).(a - a') = a - a'$. The corresponding homotopy is still denoted by Ψ. This choice is possible, since $Ad(B(0)).a = a$. Therefore, for some parameter value t_0, $Ad(B.B(\cdot)).a' = a'$, and there exists a maximal torus passing both by A and B, since $\Psi_{t_0}(1) = A$ and $Ad(B).L\Psi_{t_0}(\cdot) = L\Psi_{t_0}(\cdot)$.

If the group is not simply connected, the argument remains valid if we replace B by B^χ, with $\chi \leq m$.

□

We now give the proof of the lemma on Lie brackets of Birkhoff sums used in the proof of lemma 5.10:

Lemma 11.1. *Let $u \in L^2(\mathbb{T}, \mathbb{R}^d)$, $\mathcal{U}$ a unitary operator in $L^2(\mathbb{T}, \mathbb{R}^d)$, and $[\cdot,\cdot]$ a bilinear antisymmetric application on $\mathbb{R}^d \times \mathbb{R}^d$, invariant by $\mathcal{U}$: $[\mathcal{U}\cdot, \mathcal{U}\cdot] = \mathcal{U}.[\cdot,\cdot]$. Then,*

$$\frac{1}{n^2} \sum_{k=0}^{n} \sum_{l=0}^{k} [\mathcal{U}^k u, \mathcal{U}^l u] \to 0 \text{ in } L^1(\mathbb{T}, \mathbb{R}^d)$$

Proof. Let

$$\phi = \lim \frac{1}{n} \sum_{k=0}^{n} \mathcal{U}^k u$$

which is the projection of u on the subspace of U-invariant functions, and call $\psi = u - \phi$. We have immediately

$$\lim \frac{1}{n} \sum_{k=0}^{n} \mathcal{U}^k \psi = 0$$

in L^2. By the antisymmetry of the bracket,

$$\sum_{k=0}^{n} \sum_{l=0}^{k} [\mathcal{U}^k u, \mathcal{U}^l u] = \sum_{k=0}^{n} k[\mathcal{U}^k \psi, \phi] + \sum_{k=0}^{n} (n-k)[\phi, \mathcal{U}^k \psi] + \sum_{k=0}^{n} \sum_{l=0}^{k} [\mathcal{U}^k \psi, \mathcal{U}^l \psi] \tag{11.1}$$

If we call $s_n = \| \sum_{k=0}^{n} \mathcal{U}^k \psi \|_{L^2}$, we can rearrange the terms in $\sum_{k=0}^{n} (n-k) \mathcal{U}^k \psi$ and obtain

$$\| \sum_{k=0}^{n} (n-k) \mathcal{U}^k \psi \|_{L^2} \leq \sum_{k=0}^{n} s_k$$

If, for $\varepsilon > 0$ we let $N > 1$ such that for $n \geq N$, $s_n \leq n\varepsilon$, then we have

$$\begin{aligned} \sum_{k=0}^{n} s_k &= \sum_{k=0}^{N} s_k + \sum_{k=N}^{n} s_k \\ &\leq \frac{1}{2} N(N+1) \|\psi\|_{L^2} + \frac{\varepsilon}{2} n(n+1) \end{aligned}$$

Choosing n big enough ($n \gg N$), we can make $\frac{1}{n^2} \sum_{k=0}^{n} s_k$ smaller than ε.

Since $\sum_{k=0}^{n} k \mathcal{U}^k \psi + \sum_{k=0}^{n} (n-k) \mathcal{U}^k \psi = n \sum_{k=0}^{n} \mathcal{U}^k \psi$, we find that $\frac{1}{n^2} \sum_{k=0}^{n} k \mathcal{U}^k \psi$ also converges to 0. Consequently, the first two sums in eq. (11.1) converge to 0 in L^1.

Now, we estimate, with a similar choice of ε and N, $\frac{1}{(nN)^2} \sum_{k=0}^{nN} \sum_{l=0}^{k} [\mathcal{U}^k \psi, \mathcal{U}^l \psi]$ in L^1. The summation can be decomposed into summation over $\frac{1}{2} n(n-1)$ squares of the type

$$\sum_{k=jN}^{(j+1)N-1} \sum_{l=iN}^{(i+1)N-1} [\mathcal{U}^k \psi, \mathcal{U}^l \psi] = [\sum_{k} \mathcal{U}^k \psi, \sum_{l} U^l \psi], \ 1 \leq i, j \leq n-1$$

which, by invariance of the norms by U, we can bound by

$$\| \sum_{k=jN}^{(j+1)N-1} \sum_{l=iN}^{(i+1)N-1} [\mathcal{U}^k \psi, \mathcal{U}^l \psi] \|_{L^1} \lesssim N^2 \varepsilon$$

and n triangles of the type

$$\mathcal{U}^{jN} \sum_{k=0}^{N} \sum_{l=0}^{k} [\mathcal{U}^k \psi, \mathcal{U}^l \psi], \ 1 \leq j \leq n-1$$

which can be bounded by

$$\| \sum_{k=0}^{N} \sum_{l=0}^{k} [\mathcal{U}^k \psi, \mathcal{U}^l \psi] \|_{L^1} \lesssim \frac{1}{2} N(N-1) \|\psi\|_{L^2}$$

Thus,

$$\frac{1}{(nN)^2} \| \sum_{k=0}^{nN} \sum_{l=0}^{k} [\mathcal{U}^k \psi, \mathcal{U}^l \psi] \|_{L^1} \lesssim \frac{\|\psi\|_{L^2}}{2n} + \frac{\varepsilon}{2}$$

and an appropriate choice of n gives the desired estimate. □

The following, used in the proof of proposition 6.24, can be also found in [Fra04] (lemma 2.4).

Lemma 11.2. *Let $(f_n(\cdot))$ be a sequence of non-negative, uniformly bounded measurable functions, such that $f_n(\cdot) \to 0$ almost surely with respect to the lebesgue measure μ. Then, if (c_n) is a sequence of positive numbers converging to 0, we have*

$$\frac{1}{c_n}\int_x^{x+c_n} f_n(\cdot) \to 0,\ a.s.(x)$$

Proof. By the Egorov theorem, for any $k \in \mathbb{N}^*$, there exists a closed set B_k of Lebesgue measure at least $1 - \frac{1}{k}$ such that

$$\limsup_{x\in B_k} f_n(x) = 0$$

We denote by A_k the Lebesgue density points of B_k, i.e. the points $x \in B_k$ for which

$$\lim_{\varepsilon\to 0^+} \frac{meas([x-\varepsilon, x+\varepsilon])}{2\varepsilon} = 1$$

This property is true for *a.e.* point of B_k. Then, we find that for $x \in A_k$,

$$\begin{aligned}\frac{1}{c_n}\int_x^{x+c_n} f_n(\cdot) &= \frac{1}{c_n}\int_{[x,x+c_n]\cap A_k} f_n(\cdot) + \frac{1}{c_n}\int_{[x,x+c_n]\setminus A_k} f_n(\cdot)\\ &\leq \limsup_{x\in A_k} f_n(x) + M\frac{\mu([x,x+c_n]\setminus A_k)}{c_n}\end{aligned}$$

where M is a uniform bound for $f_n(\cdot)$. Since the right hand side converges to 0, the lemma is proved. □

11.2 Lemma on conjugation close to singular geodesics

Finally, we state and prove the lemma needed in lemma 10.10 in order to quantify the influence of the translation of Γ_m in the space of frequencies on the solution of 8.3. More precisely,

Lemma 11.3. *Let $m \in \mathbb{Z}^*$ and $g(\cdot) \in C^\infty(\mathbb{T}, \mathbb{C})$. Then, there exist positive constants C_s such that, for any $N, N' \in \mathbb{N}$ with $0 \leq N' < N$, there exists $f(\cdot)$, a solution to eq.*

$$T_N f(\cdot + \alpha) - e^{2i\pi m x\cdot} T_N f(\cdot) = g(\cdot) - \Gamma_{N',m} g(\cdot) + r_N(\cdot)$$

with $\sigma(\Gamma_{N',m} g(\cdot)) \subset \{-m+1-N', .., -N'\}$ satisfying the estimates

$$\begin{aligned}\|T_N f\|_s &\leq C_s (N^{(u)})^{s+3} \|g\|_{s+3}\\ \|r_N\|_s &\leq C_s (N^{(l)})^{s'-s+3} \|g\|_s\\ \|\Gamma_{N',m} g\|_s &\leq C_s (N')^{s+1/2} \|g\|_0\end{aligned}$$

where $N^{(u)} = N + N'$ and $N^{(l)} = N - N'$. The constants do not depend on N and N'.

Proof. Let us revisit the formulas (8.4) and (8.5) for the forward and backward solution, respectively. The solution of the auxiliary equation

$$\phi(\cdot + \alpha) - e^{2i\pi m\cdot}\phi(\cdot) = e^{-2iN'\pi\cdot} g(\cdot) - \Gamma_m(e^{-2iN'\pi\cdot} g(\cdot))$$

gives the formulas

$$\hat{\phi}_+(k) = \sum_{j=1}^{\infty} e^{-2i\pi j(k+N'+\frac{j-1}{2}m)\alpha} \hat{g}(k + jm + N')$$

$$\hat{\phi}_-(k) = \sum_{j=0}^{\infty} e^{2i\pi j(k+N'-\frac{j+1}{2}m)\alpha} \hat{g}((k-j)m + N')$$

respectively for the Fourier coefficients of the forward and backward solution. If we call $f = e^{2iN'\pi\cdot}\phi$, then it satisfies

$$f(\cdot+\alpha) - e^{2i\pi m\cdot} f(\cdot) = g(\cdot) - \Gamma_{N',m} g(\cdot)$$

with

$$\Gamma_{N',m} g(\cdot) = e^{2iN'\pi\cdot}\Gamma_m(e^{-2iN'\pi\cdot} g(\cdot))$$

The forward and backward sums for $f(\cdot)$ become

$$\hat{f}_+(k) = \sum_{j=1}^{\infty} e^{-2i\pi j(k+N'+\frac{j-1}{2}m)\alpha}\hat{g}(k+jm)$$

$$\hat{f}_-(k) = \sum_{j=0}^{\infty} e^{2i\pi j(k+N'-\frac{(j+1)}{2}m)\alpha}\hat{g}((k-j)m)$$

This implies the estimates of the lemma exactly as in the proof of 8.6, since N' appears only in the factors $e^{-2i\pi j(k+N'+\frac{j-1}{2}m)\alpha}$ and $e^{2i\pi j(k+N'-\frac{(j+1)}{2}m)\alpha}$ which do not affect the estimates, and in the translation of the truncation operators. Moreover, inspection of the proof shows that if N' is in $\frac{1}{D}\mathbb{Z}$, but $g(\cdot)$ is 1-periodic, the solution $f(\cdot)$ is in fact 1-periodic. □

In the proof of lemma 10.10 we actually use a vector-valued version of the lemma above, with the different m uniformly bounded by the size of resonances in G_0.

11.3 Estimate on the Hausdorff-Campbell formula without linear terms

The following estimate is useful in K.A.M. theory, since the latter passes by the solution of liearized equations coming from non-linear problems. When we use these approximate solutions to attack the initial problem, we have to estimate the series of the Hausdorff-Campbell formula (see [Hel62]) without constant and linear terms. This is done in the appendix of [Kri99a], or in [FK09] (eq. 6.1 and 6.2) as follows.

If Q is quadratic in (f,g), i.e. if Q is C^2, $Q(0,0)=0$ and $DQ(0,0)=0$, we have

$$\|Q(f,g)\|_s \leq C_s(1+\|f\|_0+\|g\|_0)^{s+1}(\|f\|_0+\|g\|_0)(\|f\|_s+\|g\|_s)$$

which simplifes to

$$\|Q(f,g)\|_s \leq C_s(\|f\|_0+\|g\|_0)(\|f\|_s+\|g\|_s)$$

if we admit a priori bounds for $\|f\|_0$ and $\|g\|_0$.

Bibliography

[AK70] D. V. Anosov & A. B. Katok – "New examples in smooth ergodic theory. Ergodic diffeomorphisms", *Trudy Moskov. Mat. Obšč.* **23** (1970), p. 3–36.

[AK06a] A. Avila & R. Krikorian – "Monotonic cocycles, lyapunov exponents and rigidity", *preprint* (2006).

[AK06b] — , "Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles", *Ann. of Math. (2)* **164** (2006), no. 3, p. 911–940.

[Arn83] V. I. Arnol'd – *Geometrical methods in the theory of ordinary differential equations*, Springer, 1983.

[Avi09] A. Avila – "Global theory of one-frequency Schrodinger operators I: stratified analyticity of the Lyapunov exponent and the boundary of nonuniform hyperbolicity", (2009).

[Bum04] D. Bump – *Lie groups*, Graduate Texts in Mathematics, vol. 225, Springer-Verlag, New York, 2004.

[Cha11] C. Chavaudret – "Reducibility of quasiperiodic cocycles in linear Lie groups", *Ergodic Theory Dynam. Systems* **31** (2011), no. 3, p. 741–769.

[Die75] J. Dieudonné – *Eléments d'analyse, 5*, Gauthier-Villars, 1975.

[DK00] J. J. Duistermaat & J. A. C. Kolk – *Lie groups*, Universitext, Springer-Verlag, Berlin, 2000.

[DK07] D. Dolgopyat & R. Krikorian – "On simultaneous linearization of diffeomorphisms of the sphere", *Duke Math. J.* **136** (2007), no. 3, p. 475–505.

[Eli88] L. H. Eliasson – "Perturbations of stable invariant tori for Hamiltonian systems", *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **15** (1988), no. 1, p. 115–147 (1989).

[Eli92] — , "Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation", *Comm. Math. Phys.* **146** (1992), no. 3, p. 447–482.

[Eli01] L. H. Eliasson – "Almost reducibility of linear quasiperiodic systems", *Proceedings of Symposia in Pure Mathematics* **69** (2001).

[Eli02] — , "Ergodic skew-systems on $T^d \times SO(3, R)$", *Ergodic Theory and Dynamical Systems* **22** (2002), p. 1429–1449.

[FK09] B. Fayad & R. Krikorian – "Rigidity results for quasiperiodic $\mathrm{SL}(2, \mathbb{R})$-cocycles", *J. Mod. Dyn.* **3** (2009), no. 4, p. 497–510.

[Fra00] K. Fraczek – "On cocycles with values in the group $SU(2)$", *Monaschefte für Mathematik* (2000), p. 279–307.

[Fra04] — , "On the degree of cocycles with values in the group $SU(2)$", *Israel Journal of Mathematics* **139** (2004), p. 293–317.

[Gev18] M. Gevrey – "Sur la nature analytique des solutions des équations aux dérivées partielles. Premier mémoire", *Ann. Sci. École Norm. Sup. (3)* **35** (1918), p. 129–190.

[Hal60] P. R. Halmos – *Lectures on ergodic theory*, Chelsea Publishing Co., New York, 1960.

[Ham82] R. S. Hamilton – "The inverse function theorem of nash and moser", *Bulletin of the American Mathematical Society* **7** (1982), no. 1, p. 65–222.

[Hel62] S. Helgason – *Differential geometry and symmetric spaces*, Academic Press, 1962.

[Her79] M. R. HERMAN – "Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations", *Inst. Hautes Études Sci. Publ. Math.* (1979), no. 49, p. 5–233.

[Kat01] A. KATOK – "Cocycles, cohomology and combinatorial constructions in ergodic theory", in *Smooth ergodic theory and its applications (Seattle, WA, 1999)*, Proc. Sympos. Pure Math., vol. 69, Amer. Math. Soc., Providence, RI, 2001, In collaboration with E. A. Robinson, Jr., p. 107–173.

[KH96] A. KATOK & B. HASSELBLATT – *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1996.

[Khi63] A. Y. KHINTCHINE – *Continued fractions*, P. Noordhoff Ltd, 1963.

[Kol49] A. KOLMOGOROFF – "On inequalities between the upper bounds of the successive derivatives of an arbitrary function on an infinite interval", *Amer. Math. Soc. Translation* **1949** (1949), no. 4, p. 19.

[Kri95] R. KRIKORIAN – "Réductibilité presque partout des systèmes quasi périodiques analytiques dans le cas SO(3)", *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), no. 8, p. 1039–1044.

[Kri99a] R. KRIKORIAN – *Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts, astérisque 259*, Société Mathématique de France, 1999.

[Kri99b] R. KRIKORIAN – "Réductibilité presque partout des flots fibrés quasi-périodiques à valeurs dans des groupes compacts", *Ann. Sci. École Norm. Sup. (4)* **32** (1999), no. 2, p. 187–240.

[Kri01] R. KRIKORIAN – "Global density of reducible quasi-periodic cocycles on $T^1 \times SU(2)$", *Annals of Mathematics* **154** (2001), p. 269–326.

[KS87] K. M. KHANIN & Y. G. SINAĬ – "A new proof of M. Herman's theorem", *Comm. Math. Phys.* **112** (1987), no. 1, p. 89–101.

[Mil69] J. MILNOR – *Morse theory*, Princeton University Press, 1969.

[Rud66] W. RUDIN – *Real and complec analysis*, McGraw-Hill, 1966.

[Ryc92] M. RYCHLIK – "Renormalization of cocycles and linear ODE with almost-periodic coefficients", *Invent. Math.* **110** (1992), no. 1, p. 173–206.

[Sha97] I. R. SHAFAREVICH – *Basic notions of algebra*, Springer, 1997.

[SM95] C. L. SIEGEL & J. K. MOSER – *Lectures on celestial mechanics*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Translated from the German by C. I. Kalme, Reprint of the 1971 translation.

[Yoc95] J.-C. YOCCOZ – *Petits diviseurs en dimension 1, astérisque 231*, Société Mathématique de France, 1995.