

APPLICATIONS OF HIGHER SYMMETRY GROUPS

TO PARTICLE PHYSICS

BY

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PREFACE

The work presented in this thesis was carried out in the Department of Theoretical Physics, Imperial College, London, between October 1963 and October 1966 under the supervision of Professors P.T. Mathews and Abdus Salam. The author wishes to thank them, and Dr. R. Delbourgo and Dr. J. Strathee for their continued guidance and encouragement.

Except where stated in the text, the work described is original and has not been submitted in this or any other University for any other degree. The thesis is based principally upon two papers by the author and two others with the author as a collaborator. He is grateful to his collaborators, Dr. R. Delbourgo, Mr. C.O. Nwachuku and Mr. P. Mahanta, for their permission to include the joint work in his thesis. He is also grateful to the members of the Theoretical Physics Department, especially Messrs. C. L. Cook and R. J. Rivers for several stimulating discussions. The award of a scholarship to the author by the University of Ghana is gratefully acknowledged.

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ABSTRACT

The predictions of $U(6,6)$ for the baryon-baryon scattering and the crossed processes are compared in Chapter II with the available experimental data, with particular reference to forward scattering and the threshold limit. The results in the annihilation channel are fairly satisfactory, whereas those in the direct channel are decidedly not so at the low energies where the comparison is effected. In the course of this investigation we derive an important relation between the non-relativistic (Wolfenstein-Ashkin) and the relativistic (GGMW) amplitudes of the baryon-baryon scattering, from which it is now possible to translate the analysis of scattering and polarisation experiments in terms of one set of amplitudes into the other.

In Chapter III, we derive the mass formula for $U(6) \otimes U(6)$, the little group of $U(6,6)$, and show that it gives the same mass relations as the $SU(6)$ formula, while in Chapter IV, we speculate on the possibility of using the infinite-dimensional unitary representations of non-compact groups to classify the strongly interacting particles, and display some of the techniques which will make non-compact groups more amenable to computation. The last Chapter deals with the recent conjecture

that the low-lying baryon states may not belong to any pure representation but may in fact be in a mixed representation. This is studied in the context of $U(6,6)$ and various predictions following from such an assumption are made and compared with experiment where the agreement is found to be very encouraging.

INTRODUCTION

Attempts to understand the structure of the strong interactions led Gell-Mann and Ne'eman to propose in 1961 the 'eight-fold' way theory of elementary particles.¹ The successes of the theory in classifying and predicting masses of the elementary particles stirred a number of authors to a consideration of some other higher symmetries. As a result the SU(6) symmetry was proposed in the summer of 1964 by Gursey and Radicati² and independently by Sakita³, in analogy to the Wigner supermultiplet SU(4) symmetry of nuclear forces⁴. The internal symmetry group SU(3) and the non-relativistic spin group SU(2) were here combined in a non-trivial way, with the octet and decuplet baryons placed in a 56-dimensional representation of the group ($8 \times 2 + 10 \times 4 = 56$), and the octet of pseudoscalar mesons and the nonet of vector mesons in a 35-dimensional representation ($8 \times 1 + 9 \times 3 = 35$). The SU(6) symmetry represented a completely new departure in particle physics. Previously there existed only purely internal symmetries such as SU(3) or purely geometrical ones such as the Poincare' group.

Although SU(6) predicted additional results (apart from the earlier SU(3) ones), perhaps, the most successful one being

the magnetic moment ratio of the proton and the neutron, $\mu_p/\mu_n = -3/2$, it is good as a symmetry in the static state only since it is non-relativistic. A relativistic generalisation of this symmetry was therefore, sought for and achieved by a number of authors⁵ who found that the simplest but non-trivial relativistic extension is the group $SL(6, C)$.

Meanwhile, Dashen and Gell-Mann also proposed the group $U(6) \otimes U(6)$ as an approximate symmetry group of strong interactions⁶. This is also a static group, which when boosted to a finite momentum $\underline{p} \neq 0$, gives us the non-compact $U(6, 6)$ symmetry of Salam, Delbourgo, Strathdee and other authors⁷. While $SL(6, C)$ results can all be obtained in the $U(6, 6)$ symmetry, the latter has the added advantage of providing a definite value for the proton magnetic moment which is in fair agreement with experiment. Also $U(6, 6)$ explains the hitherto mysterious equality of the electric and magnetic form factors.

Both these schemes, however, were not without their difficulties. Soon after they were proposed, it was realised that the symmetries were not compatible in general, with unitarity, one of the basic notions of S-matrix theory.⁸ Concentrating henceforth on the $U(6, 6)$ group, we consider this aspect of the theory further in Chapter I as well as some other difficulties of the symmetry and the various attempts made to

remedy them. Chapter I is not new but is essential to our understanding of the rest of this work. In Chapter II, we assume that the unitarity terms mentioned earlier are small and test the consequences of such an assumption for the octet baryon scattering processes while we consider in Chapter III the maximal compact $U(6) \otimes U(6)$ group and derive the mass formula for this group which, since it is the little group of $U(6,6)$, should also hold for the latter symmetry.

In spite of the unitarity difficulties, however, it did not take long for Dothan et al to see the usefulness of using non-compact groups in other context⁹. What had so far been used in the $SL(6,C)$ and $U(6,6)$ groups are their finite-dimensional non-unitary representation properties. However, it was noted by Gell-Mann and his collaborators that in analogy with the well-known fact¹⁰ that the entire spectrum of the hydrogen atom forms an infinite-dimensional representation of the $O(4,1)$ group, with its maximal compact subgroup $O(4)$, a similar idea might be used to study the classification of elementary particles using the infinite-dimensional representations of some non-compact groups. This is the subject considered in Chapter IV where we speculate on the possible use of the non-compact $U(6,6)$ as a 'rung' symmetry.

10.

Following the success of the Adler-Weisberger relation¹¹, it has been proposed that no pure representation exists (at least for the baryons) in the chiral (or collinear) $U(3) \otimes U(3)$ but that a great deal of mixing of the stable baryons and the low-lying baryon states takes place¹². We conjecture that a similar situation occurs in $U(6,6)$ and study the consequences of this in the final Chapter for the weak and electromagnetic processes.

CHAPTER I

REVIEW OF U(6,6) SYMMETRY

1. The Algebra of U(6,6)

U(6,6) is the group of all linear transformations on the complex 12 x 12 matrices of non-zero determinant which leaves invariant the Hermitian quadratic form¹³

$$\begin{aligned}
 H &= \bar{Z}_1 Z_1 + \bar{Z}_2 Z_2 + \dots + \bar{Z}_6 Z_6 - \bar{Z}_7 Z_7 - \dots - \bar{Z}_{12} Z_{12} \\
 &= \sum_{i=1}^6 \bar{Z}_i Z_i - \sum_{i=7}^{12} \bar{Z}_i Z_i
 \end{aligned} \tag{1.1}$$

The algebra of U(6,6) is defined in terms of the sixteen 12x12 Dirac matrices

$$\begin{aligned}
 \gamma_R = 1, \gamma_\mu, \sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu], i\gamma_\mu\gamma_5, \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3, \\
 \text{where } R = 1, 2, \dots, 16; \mu, \nu = 0, 1, 2, 3, \\
 \text{with } \left\{ \gamma_\mu, \gamma_\nu \right\} = 2g_{\mu\nu} \text{ and } g_{\mu\nu} = [1, -1, -1, -1].
 \end{aligned}$$

Taking γ_0 hermitian and $\underline{\gamma}$ anti-hermitian, the set of γ -matrices obeys the defining property

$$\gamma_0 \gamma_R \gamma_0 = \gamma_R^+ \tag{1.2}$$

of the U(6,6) algebra. With this property we can choose our matrices as

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad \text{where}$$

1 and σ_i are the 6x6 unit and Pauli matrices respectively.

Thus of the sixteen Dirac matrices, there are 8 hermitian ones:

$1, \sigma_-, \gamma_0, \gamma_0 \sigma_- = i\gamma\gamma_5$ and 8 anti-hermitian ones: $\gamma_5, \gamma_5 \sigma_- = -i\sigma_-, i\gamma_0\gamma_5, i\gamma_0\gamma_5 \sigma_- = \underline{1}$ where $\underline{\sigma} = (\sigma_{23}, \sigma_{31}, \sigma_{12})$ $\sigma_0 = (\sigma_{01}, \sigma_{02}, \sigma_{03})$. Note that all hermitian matrices are block diagonal and all anti-hermitian ones are block off-diagonal.

The generators J_A^B ($A, B = 1, 2, \dots, 12$) of the $U(6, 6)$ algebra will obey the commutation relations

$$[J_B^A, J_D^C] = \delta_B^C J_D^A - \delta_D^A J_B^C \quad (1.3)$$

with

$$(J_B^A)^+ = (\gamma_a J_{\gamma_0}^B)_A$$

We pass to a 'hermitian' basis by making use of the γ_R defined above together with the unitary spin matrices T^i ($i = 0, 1, \dots, 8$) which define the fundamental $U(3)$ representation and where we have chosen $T^i = \frac{1}{2} \lambda^i$ ($i=1, \dots, 8$) with λ^i defined by Gell-Mann¹ and $T^0 = \frac{1}{\sqrt{6}}$. Thus $T^i T^j = \frac{1}{2} (d^{ijk} + if^{ijk}) T^k$, $i, j, k = 0, 1, \dots, 8$, where d^{ijk}, f^{ijk} ($i, j, k=1, \dots, 8$) have been given by Gell-Mann¹ and $d^{0jk} = \sqrt{\frac{2}{3}} \delta^{jk}$, $f^{0jk} = 0$. It follows that

$$\text{Tr} (T^i T^j) = \frac{1}{2} \delta^{ij} \quad (1.4)$$

The generators J_R^i can now be constructed. These are defined as

$$J_R^i = (\gamma_R)_\alpha^\beta (T^i)_r^s J_{\beta s}^{\alpha r} \quad (1.5)$$

where

$$A = \alpha r; B = \beta s; \alpha, \beta = 1, 2, 3, 4; r, s = 1, 2, 3.$$

The transformed commutation relations now read

$$\begin{aligned}
 [J^i, J^j] &= if^{ijk} J^k, [J^i, J_5^j] = if^{ijk} J_5^k \\
 [J^i, J_{\mu\nu}^j] &= if^{ijk} J_{\mu\nu}^k, [J^i, J_\mu^j] = if^{ijk} J_\mu^k \\
 [J^i, J_{\mu 5}^j] &= if^{ijk} J_{\mu 5}^k \\
 [J_5^i, J_5^j] &= -if^{ijk} J^k, [J_5^i, J_{\mu\nu}^j] = \frac{1}{2} if^{ijk} \epsilon_{\mu\nu\kappa\lambda} J_{\kappa\lambda}^k \quad (1.6) \\
 [J_5^i, J_\mu^j] &= id^{ijk} J_{\mu 5}^k, [J_5^i, J_{\mu 5}^j] = id^{ijk} J_\mu^k \\
 [J_\mu^i, J_\nu^j] &= if^{ijk} g_{\mu\nu} J^k - id^{ijk} J_{\mu\nu}^k \\
 [J_\mu^i, J_{\nu 5}^j] &= id^{ijk} g_{\mu\nu} J_5^k + \frac{1}{2} if^{ijk} \epsilon_{\mu\nu\kappa\lambda} J_\kappa^k \\
 [J_\lambda^i, J_{\mu\nu}^j] &= id^{ijk} (g_{\lambda\mu} J_\nu^k - g_{\lambda\nu} J_\mu^k) - if^{ijk} \epsilon_{\lambda\mu\nu\kappa} J_\kappa^k \\
 [J_{\mu 5}^i, J_{\nu 5}^j] &= id^{ijk} J_{\mu\nu}^k - if^{ijk} g_{\mu\nu} J^k \\
 [J_{\lambda 5}^i, J_{\mu\nu}^j] &= id^{ijk} (g_{\lambda\mu} J_{\nu 5}^k - g_{\lambda\nu} J_{\mu 5}^k) - if^{ijk} \epsilon_{\lambda\mu\nu\kappa} J_\kappa^k \\
 [J_{\lambda\mu}^i, J_{\nu\kappa}^j] &= id^{ijk} (g_{\lambda\nu} J_{\mu\kappa}^k + g_{\lambda\kappa} J_{\mu\nu}^k - g_{\lambda\mu} J_{\nu\kappa}^k - g_{\lambda\nu} J_{\mu\kappa}^k) \\
 &\quad + if^{ijk} \left\{ (g_{\lambda\mu} g_{\nu\kappa} - g_{\lambda\nu} g_{\mu\kappa}) J^k - \epsilon_{\lambda\mu\nu\kappa} J_5^k \right\}
 \end{aligned}$$

These commutation relations are obviously cumbersome. They can, however, be put into a convenient form if we adopt a new notation (J^i, J_{mn}^i) in place of the J_R^i where $m, n = 0, 1, 2, 3, 5, 6$ and we use the identification

$$\gamma_{\mu\nu} = \sigma_{\mu\nu}, \quad \gamma_{\mu 5} = i\gamma_\mu \gamma_5, \quad \gamma_{\mu 6} = \gamma_\mu, \quad \gamma_{56} = \gamma_5$$

Then all the above commutation relations collapse into the following forms¹⁴

$$\begin{aligned}
[J^i, J^j] &= if^{ijk} J^k \\
[J^i, J^j_{mn}] &= if^{ijk} J^k_{mn} \\
[J^i_{k\bar{l}}, J^j_{mn}] &= if^{ijk} [(g_{km} g_{ln} - g_{kn} g_{ml}) J^{kl} + \frac{1}{2} \epsilon_{klmni} J^{kl}] \\
&\quad + id^{ijk} [g_{kn} J^k_{lm} + g_{lm} J^k_{kn} - g_{km} J^k_{ln} - g_{ln} J^k_{km}]
\end{aligned} \tag{1.7}$$

where ϵ_{klmn} is the totally antisymmetric term, with $\epsilon_{012356} = 1$

Using Eqn.(1.5) the eleven Casimir operators of SU(6,6) can easily be built up and calculated. These are

$$C_1 = J^A_{B A}, \quad C_2 = J^A_{B C} J^B_{C A}, \quad \dots \quad C_{11} = J^A_B \dots J^B_A$$

They are in general very long and unwieldy except for the quadratic Casimir operator C_1 , which is simply

$$C_1 = 2(J^i J^i - J^i_5 J^i_5 - J^i_{\lambda 5} J^i_{\lambda 5} + J^i_{\lambda \lambda + \frac{1}{2}} J^i_{\lambda \lambda + \frac{1}{2}}).$$

2. The multispinor representations of U(6,6)

The U(6,6) theory assumes that the fundamental entity for strong interactions is a 12-component Dirac quark. Under a U(6,6) transformation the quark spinor undergoes the change

$$\Psi_A \longrightarrow S^B_A \Psi_B$$

$$\text{with} \quad S = \exp(i \epsilon^i_{R R} J^i_R) = \exp(i \epsilon^A_{B A} J^B_A) \tag{1.8}$$

where $A, B = 1, 2, \dots, 12$ and all the ϵ 's are real. The antiquark spinor also defined by

$$\begin{aligned}
\bar{\Psi}^A &= \bar{\Psi}^{\alpha r} = (\Psi_{\beta r})^+ (\gamma_0)_{\beta}^{\alpha}, \quad \alpha, \beta = 1, 2, 3, 4 \\
r &= 1, 2, 3
\end{aligned} \tag{1.9}$$

transforms as

$$\bar{\Psi}^A \longrightarrow \bar{\Psi}^B (S^{-1})^A_B \quad \text{and}$$

leaves $\bar{\Psi}^A \Psi_A$ invariant. The kinetic energy term $\bar{\Psi}^A \partial_\mu \hat{\Psi}_A$, however,

is not left invariant under $U(6,6)$. Thus the free

Lagrangian breaks the symmetry. Finite-dimensional non-

unitary representations of $U(6,6)$ may be built up from products

of quarks and antiquarks. These multispinors transform

according to the general rule:

$$\bar{\Psi}^{CD\dots}_{AB\dots} \longrightarrow S^{A'}_A S^{B'}_B \dots (S^{-1})^C_{C'} (S^{-1})^D_{D'} \dots \bar{\Psi}^{C'D'\dots}_{A'B'\dots} \quad (1.10)$$

where S is given by Eqn.(1.8).

Following the usual procedure¹⁵ we assign to the 12-component quark the baryon number $B=\frac{1}{3}$. The baryons are then constructed from three quark states and the mesons from quark-antiquark states.

Under $U(6,6)$, these states decompose as follows

$$12 \otimes 12 = 220 + 36_4 + 572 + 572$$

$$12 \otimes \bar{12} = 1 + 14_3,$$

where in terms of the Young Tableau¹⁶, the 220 is completely antisymmetric $[1^3]$, the 36₄ completely symmetric, $[3]$, and the 572 is of the mixed symmetry type $[2,1]$. The decomposition of these $U(6,6)$ multiplets with respect to the subgroup

$U(3) \otimes U(2,2)$ is given by the following

$$14_3 = (8,15) + (8,1) + (1,15)$$

$$220 = (10, \bar{4}) + (8, 20') + (1, 20)$$

$$36_4 = (10, 20) + (8, 20') + (1, \bar{4})$$

$$572 = (10, 20') + (8, 20') + (8, 20) + (8, \bar{4}) + (1, 20'),$$

where the first number in each bracket denotes the $U(3)$ representation and the second the $U(2,2)$ representation. The two $U(2,2)$ representations denoted by $\underline{20}$ and $\underline{20}'$ are of the symmetry types $[3]$ and $[2,1]$ respectively.

As in the $SU(6)$ symmetry we assign the spin $\frac{1}{2}$ and spin $3/2$ baryons to the $\underline{36}_4$ representation of $U(6,6)$ and the spin 0 and 1 mesons to the $\underline{14}_3$ representation. Then the following reductions of the products are relevant:

$$\underline{14}_3 \otimes \underline{14}_3 = 1 + \underline{14}_3 + \underline{14}_3 + \underline{4} + \underline{21}_2 + \underline{500}_5 + \underline{500}_5^* + \underline{59}_4 + \underline{0} \quad (1.11a)$$

$$\underline{36}_4 \otimes \underline{36}_4^* = 1 + \underline{14}_3 + \underline{59}_4 + \underline{0} + \underline{126}_4 + \underline{12} \quad (1.11b)$$

$$\underline{14}_3 \otimes \underline{36}_4 = \underline{36}_4 + \underline{572} + \underline{160}_{16} + \underline{35}_{100} \quad (1.11c)$$

The single occurrence of $\underline{14}_3$ in (1.11b) makes the baryon-meson trilinear coupling unique. Hence, the F/D ratio is determined by $U(6,6)$ theory.

3. Identification with physical states

The regular representation $\underline{14}_3$ decomposes under $SU(3) \otimes U(2,2)$ into

$$\phi_B^A = [\phi_5^i + \gamma_5^i \phi_5^i + i\gamma_\mu \gamma_5^i \phi_{\mu 5}^i + \gamma_\mu \phi_{\mu 5}^i + \frac{1}{2} \gamma_\mu \phi_{\mu 5}^i]_{\beta}^{\alpha} (T^i)_{\alpha}^{\beta} \quad (1.12)$$

where Greek letters indicate $U(2,2)$ indices, running from 1 to 4 and Latin indices stand for the $SU(3)$ part and run from one to three. Eqn.(1.12), as it stands can lead to no physical

consequences unless it is made to represent physical particle states. This is accomplished by requiring that ϕ_B^Λ satisfies some equations of motion, called the Bargmann-Wigner¹⁷ equations.

Applying the Bargmann-Wigner equations

$$\begin{aligned} (\not{p})^\gamma_\alpha \phi^\beta_{\gamma q} &= \mu \phi^\beta_{\alpha q} \\ (\not{p})^\beta_\gamma \phi^\gamma_{\alpha q} &= -\mu \phi^\beta_{\alpha q} \end{aligned} \quad (1.13)$$

(where μ is the meson mass),

to each $U(2,2)$ index results in the elimination of the scalar part of the 14_3 and the relation of the axial-vector to the pseudoscalar and the tensor to the vector part. Eqn.(1.12) now reduces to

$$\phi_B^\Lambda = [(1+\not{q}/\mu) \left\{ \gamma_5 P_q^\mu + \gamma_\mu V_q^\mu \right\}]^\alpha_\beta \quad (1.14)$$

where P and V are the $SU(3)$ pseudoscalar and vector nonets respectively, and μ is the mean mass of the 14_3 representation.

Similarly the baryons are described by the fully symmetric tensor $\Psi_{\{ABC\}}$ of the 36_4 dimensional representation. Under $SU(3) \otimes U(2,2)$, this decomposes to give

$$\begin{aligned} \Psi_{\{ABC\}} &= \frac{1}{2} \sqrt{\frac{3}{2}} D_{\alpha\beta\gamma, pqr} + \epsilon_{pqr} V_{[\alpha\beta\gamma]} \\ &\quad \frac{1}{2} \sqrt{\frac{1}{6}} [\epsilon_{pqs} N_{[\alpha\beta]\gamma, r}^s + \epsilon_{qrs} N_{[\beta\gamma]\alpha, p}^s + \epsilon_{rps} N_{[\gamma\alpha]\beta, q}^s] \end{aligned} \quad (1.15)$$

where D is completely symmetric in both its spinor and unitary spin indices, $V_{[\alpha\beta\gamma]}$ is completely antisymmetric and $N_{[\alpha\beta]\gamma, r}^s$ has mixed symmetry in spinor indices and is traceless

in unitary spin indices. The Bargmann-Wigner equations ensure that D describes a particle of spin $3/2^+$, N a particle of spin $\frac{1}{2}^+$ and V vanishes identically on account of its complete antisymmetry. The relative parities of the decuplet and the octet are the same. As a result of the equations of motion, we can now write D and N in the forms

$$D_{\alpha\beta\gamma, pqr}^{(p)} = (\gamma_\mu C)_{\alpha\beta} D_{\gamma\mu, pqr}^{(p)} + \frac{1}{2m} (\sigma_{\mu\nu} C)_{\alpha\beta} [p_\mu D_{\gamma\nu, pqr}^{(p)} - p_\nu D_{\gamma\mu, pqr}^{(p)}]$$

with the Rarita-Schwinger conditions¹⁸

$$(\not{p}-m) D_{\mu, pqr}^{(p)} = 0, \quad \gamma_\mu D_{\mu, pqr}^{(p)} = 0$$

i.e.

$$D_{\alpha\beta\gamma, pqr}^{(p)} = \frac{1}{m} [(\not{p}+m)\gamma_\mu C]_{\alpha\beta} D_{\mu\gamma, pqr}^{(p)} \quad (1.16)$$

and

$$N_{[\alpha\beta]\gamma, r}^S = \frac{1}{m} [(\not{p}+m)\gamma_5 C]_{\alpha\beta} N_{\gamma, r}^S(p) \quad (1.17)$$

$$(\not{p}-m) N_r^S(p) = 0;$$

where C is the charge conjugation matrix satisfying the properties

$$(C^{-1})^{\alpha\beta} = -(C^{-1})^{\beta\alpha}$$

$$C_{\alpha\beta} (C^{-1})^{\beta\gamma} = \delta_\alpha^\gamma \quad (1.18)$$

$$(\gamma_\mu C)_{\alpha\beta} = (\gamma_\mu C)_{\beta\alpha},$$

and m is the mean mass of the $\underline{36}_L$ representation. It can be

shown from the above properties then that of the 16 Dirac

matrices $(\gamma^R C)_{\alpha\beta}$; $(\gamma_\mu C)_{\alpha\beta}$ and $(\sigma_{\mu\nu} C)_{\alpha\beta}$ are symmetric whilst

$C_{\alpha\beta}$, $(\gamma_5 C)_{\alpha\beta}$ and $(i\gamma_\mu \gamma_5 C)_{\alpha\beta}$ are antisymmetric. D_{pqr} , the decuplet

wave function of SU(3) is defined as

$$\begin{aligned}
D_{111} &= N^{*++} & D_{112} &= N^{*+} & D_{122} &= N^{*0} & D_{222} &= N^{*-} \\
D_{113} &= \frac{Y^{*+}}{\sqrt{3}} & D_{123} &= \frac{Y^{*0}}{\sqrt{6}} & D_{223} &= \frac{Y^{*-}}{\sqrt{3}} \\
D_{133} &= \frac{E^{*0}}{\sqrt{3}} & D_{233} &= \frac{E^{*-}}{\sqrt{3}} \\
D_{333} &= \Omega^{-}.
\end{aligned}$$

The normalisation is fixed by demanding that $\bar{D}^{pqr} D_{pqr} = \frac{10}{\sum_{A=1}^4 \bar{\Psi}(A) \Psi(A)}$ where $\Psi(A)$ represents the decuplet state A . Owing to its importance in our later work we defer to a later Chapter, the explicit particle correspondence of the N_s^r . It should be emphasized that the $U(6,6)$ multiplets after the application of the Bargmann-Wigner equations no longer possess the full symmetry. They are now reduced to a number of subgroup multiplets, fully discussed in Section 5.

With the definitions (1.14) (1.15) and (1.17) we can now write down the various currents

$$J^{Rj} = \bar{\Psi}^{ABC} (\gamma^R \gamma^j)_A^{A'} \Psi_{A'BC}$$

and obtain all the successful results of $SU(6)$ and more. All these can be found in the review article of Delbourgo et al¹⁴.

4. Troubles with $U(6,6)$ symmetry and attempts to remedy them.

Because $U(6,6)$ is non-compact, its representations can be either (1) finite-dimensional and therefore non-unitary or (2) infinite-dimensional but unitary, corresponding to the existence of an infinite number of particles all with the same

mass, a physically unrealistic situation¹⁹. We shall in Chapter II study the implications of using the finite-dimensional non-unitary scheme and consider in Chapter IV some speculations concerning the use of infinite-dimensional unitary representations.

Shortly after the advent of U(6,6) explicit calculations were carried out to study the direct implications of the scheme for a variety of experimentally accessible reactions. In particular, several authors²⁰ studied strict U(6,6) predictions for pseudo-scalar meson-baryon octet scattering and found that in addition to reproducing the SU(6) results like the Johnson-Treiman relations²¹, viz.

$$\frac{1}{2}[\sigma(K^+p) - \sigma(K^-p)] = \sigma(K^0p) - \sigma(\bar{K}^0p) = \sigma(\pi^+p) - \sigma(\pi^-p) \quad (1.19)$$

where $\sigma(MB)$ means elastic $\sigma(MB \rightarrow MB)$, the theory predicted other results concerned with polarisation effects. It was explicitly shown that the scheme predicts zero polarisation for various reactions, in particular the reaction

$$K^- + p \longrightarrow \Xi^- + K^+ \quad (1.20)$$

These predictions are found to be in violent disagreement with experiment. Thus, in the case of Eqn.(1.20), for example at K^- lab momentum $P_{lab} = 1.95 \text{ BeV/c}$ the average Ξ^- polarisation is²² 0.77 ± 0.24 .

It has also been pointed out that U(6,6) scheme comes into conflict with the principle of unitarity of the physical S-matrix⁸.

This can be demonstrated as follows. The unitarity condition is defined as

$$\text{Im}\langle f|T|i\rangle = (2\pi)^4 \sum_n \langle f|T^\dagger|n\rangle \langle n|T|i\rangle \delta^4(p_n - p_i),$$

where, in the summation over the intermediate states n , we must sum only over positive energy states. The conflict of $U(6,6)$ with unitarity comes about, by noting that the projection operator $\sum_n |n\rangle \langle n| = \mathbb{P} + m$ is not $U(6,6)$ invariant. It must, however, be remarked that even $SU(3)$ is compatible with unitarity only if we ignore the physical mass differences produced by the symmetry-breaking in S-matrix elements. A corresponding solution to the $U(6,6)$ theory is (apart from requiring mass degeneracy) the use of inhomogeneous $U(6,6)$ under which the kinetic energy term is invariant under the group. We consider the inhomogeneous $U(6,6)$ further in Section 6. It is therefore not surprising that $SU(3)$ predictions for scattering cross-sections are not in general in agreement with experiment²³.

5. The derivative and non-derivative couplings

In order to circumvent some of the difficulties of the theory, some authors introduced the notion of derivative coupling (kinetic spurion) as a symmetry breaking scheme²⁴. The use of these momentum-dependent spurions still reproduced most of the good results of $U(6,6)$, such as the Johnson-Treiman

relations and did not predict any more zero polarisation for the processes considered earlier²⁵. However, in some cases, the number of amplitudes appearing in the calculations is so many that this greatly reduces the predictive power of the theory. An ingenious method of finding the subgroup which remains after the application of some special forms of the derivative coupling was given by Harari and Lipkin²⁶. These special forms of the derivative couplings also reduce the number of amplitudes considerably. The amplitudes involved are now dictated by the structure of the subgroup.

The derivative coupling is of the form $\not{p} = p'_\mu \gamma_\mu$. With all the γ_μ present, the $U(6,6)$ symmetry is broken and reduced to $SU(3)$ and hence no new predictions are possible. However, it can happen that for a particular process, one or more of the component of the momentum is zero for all the particles present in a particular Lorentz frame. In this case, we obtain a subgroup of the symmetry which remains unperturbed to all orders in the symmetry breaking. Thus

(1) if in a particular state all the particles are at rest, then we have $p_x = p_y = p_z = 0$ and thus the symmetry is broken by $\gamma_0 p_0$ spurions only. The subgroup of $U(6,6)$ which commutes with γ_0 remains a good symmetry for these states and can be used to classify particles at rest only. This is the non-chiral

$U(6) \otimes U(6)$ group of Dashen and Gell-Mann, defined by the generators

$$\frac{1}{2}(1+\gamma_0) \sigma_i T^j \quad i=0,1,2,3; \quad j=0,1,\dots,8$$

with $\sigma_0 = 1$

(2) Suppose that in a given set of processes the particles involved are moving in the z -direction, say, in a particular frame. Then $p_x = p_y = 0$ for all the particles and the symmetry is broken by a $\not{p} = \gamma_0 p_0 - \gamma_z p_z$ spurion. The subgroup of $U(6,6)$ which commutes with γ_0 and γ_z becomes a good symmetry for these processes. This is the $SU(6)_H$ group [hybrid $SU(6)$] with the generators

$$T^i, \quad \sigma_{12} T^i, \quad \gamma_0 \sigma_{23} T^i, \quad \gamma_0 \sigma_{31} T^i,$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$.

It is to be noted that this group can only be used for collinear processes such as vertex functions, form factors, forward and backward scattering and two-body decays and it is compatible with unitarity if all intermediate states are also collinear.

$SU(6)_H$ has another peculiar property when it comes to the classification of particles. Thus, for example, the 35 of the non-relativistic $SU(6)$, called $SU(6)_S$ decomposes into the following $SU(3) \otimes SU(2)$ multiplets: $(8,3), (8,1)$ and $(1,3)$. Under W - S flip, the $S_z = 0$ parts of the two representations will be interchanged. This interchange occurs within the 35 $SU(6)_H$ and $SU(6)_S$. But the corresponding flip between the $(1,3)$ and $(1,1)$ will

bring a term from the $\underline{35}$ of $SU(6)_W$ into $\underline{1}$ of $SU(6)_S$ and vice-versa. Thus if $(1,3)$ of $SU(6)_S$ is ω^0 and the $(1,1)$ X^0 , the $(1,3)$ of $SU(6)_W$ will be composed of ω_1^0 , X^0 , ω_{-1}^0 and its singlet will be ω_0^0 . The baryons, however, are unaffected by this W-S transition. It is precisely this peculiarity which allows the $\rho \rightarrow 2\pi$ and $N^* \rightarrow N + \pi$ decays, processes forbidden by $SU(6)_S$, contrary to experiment.

(3) For the particular case where, say, p_x is zero for all particles, $\not{p}$ reduces to the form

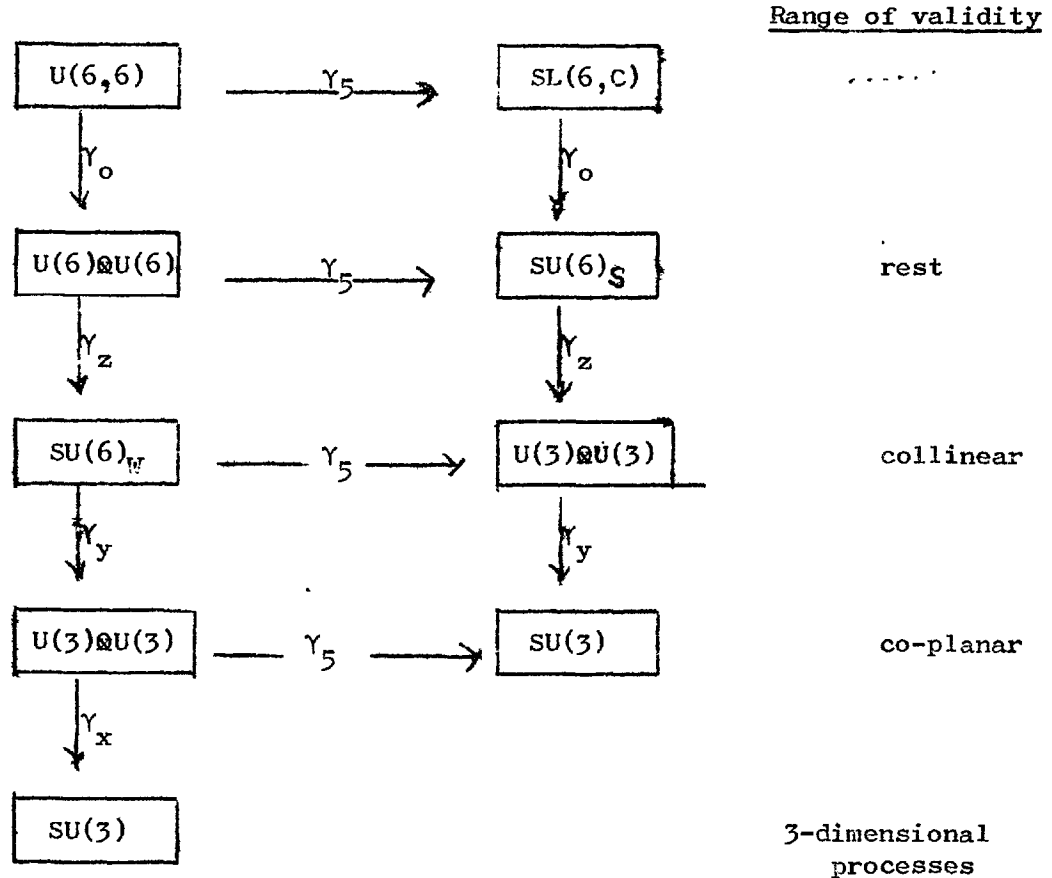
$$\not{p} = \gamma_0 p_0 - \gamma_y p_y - \gamma_z p_z$$

The subgroup which commutes with γ_0 , γ_y , γ_z as well as with all Lorentz transformations in the y-z plane is the coplanar $U(3) \otimes U(3)$ group generated by $\frac{1}{2}(1 + \gamma_0 \sigma_{23}) T^i$.

All these subgroups of $U(6,6)$ were also obtained independently by Dashen and Gell-Mann, using the algebra of current⁶.

A further set of $U(6,6)$ subgroups can be obtained by considering the momentum independent spurions²⁴ that arise from insertions of $\gamma_5 \otimes \gamma_5$ matrices at the allowed places in a $U(6,6)$ invariant amplitude. The symmetry is then reduced in such a way that the generators of the resulting subgroups commute with γ_5 . This procedure automatically splits the pseudoscalar singlet X^0 from the 143 meson states of $U(6,6)$.

All the allowed hierarchy of subgroups can be shown diagrammatically as follows²⁷:

FIG. I - $U(6,6)$ and its subgroups

$A \xrightarrow{\gamma} B$ means the generators of A that commute with γ generate B.

The generators of $SL(6, C)$ are :

$$T^i, \gamma_5 T^i, \underline{\sigma} T^i, \gamma_5 \underline{\sigma} T^i$$

By performing the Weyl unitary trick²⁸ on the $SL(6, C)$ generators, we obtain the chiral $U(6) \otimes U(6)$, with generators $\frac{1}{2}(1 \pm i\gamma_5) \sigma_i T^S$

This has found useful applications in the algebra of current

densities²⁹. The $SU(6)_S$ is the Gursey-Radicati-Sakita group with

the well-known generators T^j , $\underline{\sigma} T^j$. We can similarly obtain the generators of the collinear $U(3) \otimes U(3)$ to be $\frac{1}{2}(1 \pm \sigma_2) T^j$.

6. The inhomogeneous $U(6,6)$

One of the difficulties of the symmetry was also the fact that the free Lagrangian, as we have already pointed out, and the Bargmann-Wigner equations are not invariant under the full group. It was in order to circumvent this difficulty that some authors introduced the idea of 143 momenta into the theory^{14,30}. This automatically leaves the free Lagrangian and the Bargmann-Wigner equations invariant under the full group, called the inhomogeneous $U(6,6)$, which is obtained by adjoining to $U(6,6)$ a translation group of 143 dimensions. The representations of this new group are now unitary. The physical implications of these extra 139 momenta are clearly difficult to interpret.

If we denote the 143 generators of the translation group T_{143} , by the notation P_B^A with $P_A^A = 0$ ($A, B = 1, 1 \dots, 12$), then by requiring that every physical state can be brought to rest, i.e.

$$P_B^A = m(\gamma_O)_B^A,$$

we see that the group of transformations which leaves P invariant is the little group $U(6) \otimes U(6)$. It should be noted, however, that the unitarity problem is still unaffected by this formulation as soon as we make contact with physical reality, i.e. as soon as we put the extra 139 momenta equal to zero.

7. U(6,6) S-matrix calculations

As already stated, some of the calculations based on complete U(6,6) invariance disagree rather badly with experiment. This might be due to the absence of the unitarity corrections which must be incorporated in the theory. To see this clearly, we proceed as follows. Let us decompose the full $SU(3) \otimes 1_{\frac{1}{2}}$ - the group remaining after the application of the Bargmann-Wigner equations - invariant matrix elements into three parts, i.e.

$$M = M_0 + M_1 + M_2$$

M_0 has the full homogeneous U(6,6) symmetry and therefore possesses no derivative terms, M_1 contains the derivative terms and M_2 represents what we call the unitarity corrections. For certain reactions, such as forward and backward scattering²⁵, M_1 collapses into M_0 and we are then left with

$$M = M_0 + M_2$$

In Chapter II, we shall be testing the assumption that the M_0 terms dominate in certain situations. Thus, we consider in detail, the octet baryon scattering and because of experimental comparisons, we shall mostly deal with the forward scattering where enough data exists.

CHAPTER IIOCTET BARYON SCATTERING IN $U(6,6)$

The successful applications of $U(6,6)$ to three-point functions prompted similar calculations for four-point functions such as meson-baryon scattering in the direct and crossed channels and led to the belief that the predictions would be equally good. Apart from certain limiting situations such as forward scattering and annihilations at rest, this hope has been largely disappointing. However, $U(6,6)$ is only expected to provide the starting approximation to any calculation, perturbation-theoretic or otherwise.

It is this first approximation for the scattering of two baryon octets (N) that we shall investigate here³¹. As already explained in Chapter I, in addition to the regular four-point interaction, there are the derivative couplings which follow directly from the ideas of the inhomogeneous $U(6,6)$ group. We shall only consider the regular interactions with the modification of replacing the coupling constants by unknown form factors. Already at this level the calculations involved are complicated enough to make the task of including all the derivative interaction terms seem truly formidable. However, the neglect of these amplitudes is not serious for it so happens that in the special kinematical situation of forward scattering they reduce to regular form^{24,25}. At the same time we disregard the baryon-decuplet scattering

processes which together with NN scattering form an integral part of the $U(6,6)$ programme. This we have been forced to do owing to the absence of enough experimental data on such processes.

There are some important observations concerning the validity of our results which deserve special emphasis. These stem from the fact that the pole approximation preserves the inhomogeneous $U(6,6)$ structure³², and even with the derivative couplings such quantities as total cross sections remain easy to compute. This situation applies with force to all peripheral mechanisms where only the exchange (Born) amplitude is considered, e.g. $p\bar{p} \rightarrow Y\bar{Y}$ with meson exchange. (An investigation in this direction has been carried out using the absorptive model and the results obtained are in fair agreement with experiment³³.) However, any improvement of the scheme which comes to terms with unitarity, such as using the Born term as the potential input of an S-matrix N/D calculation, will necessarily destroy this last vestige of $U(6,6)$ symmetry. Part of these unitary corrections again fall into $U(6,6)$ form; the remainder we have disregarded in this work as a starting approximation. The extent to which this neglect is justified can only be judged by comparing the consequences with experiment and these are none too striking.

2. The baryon-baryon $U(6,6)$ amplitudes

The inhomogeneous $U(6,6)$ theory classifies particles through the compact little group $U(6) \otimes U(6)$. The 56 baryon

states belong to the $(56,1)$ multiplet and when boosted to finite momentum $\underline{p} \neq \underline{0}$ will fall into the 36_4 $U(6,6)$ representation. From equation (1.16) and (1.17) the explicit decomposition of the 36_4 is

$$2m \Psi_{ABC}^{(p)} = \left(\frac{3}{2}\right)^{\frac{1}{2}} [(p+m)\gamma_\mu C] D_{\mu\gamma, pqr} + \left(\frac{1}{6}\right)^{\frac{1}{2}} \left\{ [(p+m)\gamma_5 C]_{\alpha\beta} \epsilon_{pqrs} N_{\gamma r}^s + \text{cyclic in } ABC \right\} \quad (2.1)$$

where $A, B, C = 1, 2, \dots, 12$, $\alpha, \beta, \gamma = 1, 2, 3, 4$; $p, q, r, s = 1, 2, 3$.

The modification to take account of mass differences between the multiplet members is described later.

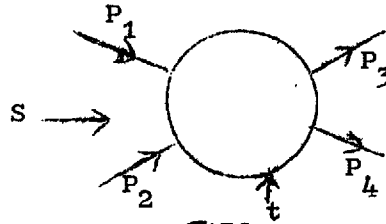
From the product

$$36_4 \times \overline{36}_4 = 1 + 14_3 + 59_4 + 126_4 + 12,$$

we see that the amplitudes which describe the scattering

$N_1 + N_2 \rightarrow N_3 + N_4$ (and by crossing $N_1 + \bar{N}_3 \rightarrow N_4 + \bar{N}_2$) are given by

FIG. II



$$\begin{aligned} M(s, t, u) = & A(s, t, u) \bar{\Psi}^{ABC}(p_3) \Psi_{ABC}(p_1) \bar{\Psi}^{DEF}(p_4) \Psi_{DEF}(p_2) \\ & - \tilde{A}(s, t, u) \bar{\Psi}^{ABC}(p_3) \Psi_{ABC}(p_2) \bar{\Psi}^{DEF}(p_4) \Psi_{DEF}(p_1) \\ & + B(s, t, u) \bar{\Psi}^{ABC}(p_3) \Psi_{DBC}(p_1) \bar{\Psi}^{DEF}(p_4) \Psi_{AEF}(p_2) \\ & - \tilde{B}(s, t, u) \bar{\Psi}^{ABC}(p_3) \Psi_{DBC}(p_2) \bar{\Psi}^{DEF}(p_4) \Psi_{AEF}(p_1) \end{aligned} \quad (2.2)$$

where s, t, u are the usual Mandelstam variables defined as

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad \text{and } u = (p_1 - p_4)^2.$$

Applying the generalised Pauli principle to the amplitude (2.2) we obtain the symmetry relations:

$$A(s, t, u) = \tilde{A}(s, u, t), \quad B(s, t, u) = \tilde{B}(s, u, t).$$

It proves more convenient to cast (2.2) in the form

$$\begin{aligned} M(s, t, u) = & A J_A^A(p_3, p_1) J_B^B(p_4, p_2) - \tilde{A} J_A^A(p_3, p_2) J_B^B(p_4, p_1) \\ & + B J_B^A(p_3, p_1) J_A^B(p_4, p_2) - \tilde{B} J_B^A(p_3, p_2) J_A^B(p_4, p_1) \end{aligned} \quad (2.3)$$

where $J_B^A(p', p) = \tilde{\Psi}^{ACD}(p') \Psi_{BCD}(p)$. The expressions for

$J_p^{RQ} = (\gamma^R)_\beta^\alpha J_{\alpha p}^{\beta Q}$ have been worked out by Salam et al.,⁷ and

facilitate calculation of

$$J_B^A J_A^B = \frac{1}{4} \text{Tr}(J^R J^R),$$

where we define

$$J^R J^R = J_0 J_0 - J_5 J_5 + J_\mu J_\mu - J_{\mu 5} J_{\mu 5} + \frac{1}{2} J_{\mu\nu} J_{\mu\nu}$$

As they are crucial to our work we list the baryon-octet contributions to the J^R :

$$\begin{aligned} J_p^Q(p', p) &= \frac{P^2}{4m^2} [(\bar{N}N)_{p,F}^Q + \delta_p^Q \text{Tr}(\bar{N}N)] + \dots \\ J_{5,p}^Q(p', p) &= \frac{P^2}{4m^2} [(\bar{N}\gamma_5 N)_{p,D+2/3F}^Q - 1/3 \delta_p^Q \text{Tr}(\bar{N}\gamma_5 N)] + \dots \quad (2.4) \\ J_{\mu 5,p}^Q(p', p) &= \frac{P^2}{4m^2} [i(\bar{N}\gamma_\mu \gamma_5 N)_{p,D+2/3F}^Q - 1/3 i \delta_p^Q \text{Tr}(\bar{N}\gamma_\mu \gamma_5 N)] + \dots \\ J_{\mu,p}^Q(p', p) &= \frac{P_\mu}{2m} [(\bar{N}N)_{p,1/3F-D}^Q + \frac{4}{3} \delta_p^Q \text{Tr}(\bar{N}N)] + \frac{P^2}{4m^2} [(\bar{N}\gamma_\mu N)_{p,D+2/3F}^Q \\ &\quad - 1/3 \delta_p^Q \text{Tr}(\bar{N}\gamma_\mu N)] + \dots \end{aligned}$$

$$J_{\mu\nu,p}^q(\mathbf{p}', \mathbf{p}) = \frac{i}{4m^2} (P_\mu q_\nu - P_\nu q_\mu) [(\bar{N}N)_{p, 1/3F-D+4/3}^q \delta_p^q \text{Tr}(\bar{N}N)] \\ + \frac{P_\mu^2}{4m^2} [(\bar{N} \sigma_{\mu\nu} N)_{p, D+2/3F} \frac{1}{3} \delta_p^q \text{Tr}(\bar{N} \sigma_{\mu\nu} N)] + \dots$$

Here $\mathbf{q} = \mathbf{p} - \mathbf{p}'$, $P = \mathbf{p} + \mathbf{p}'$, $N = N(\mathbf{p})$, $\bar{N} = \bar{N}(\mathbf{p}')$, and

$$\text{Tr}(\bar{N}N) = \bar{N}_{\mathbf{q} \mathbf{p}}^{\mathbf{p} \mathbf{q}} \quad (2.5)$$

$$(\bar{N}N)_{p,F}^q = \bar{N}_{\mathbf{r} \mathbf{p}}^{\mathbf{q} \mathbf{r}} - \bar{N}_{\mathbf{p} \mathbf{r}}^{\mathbf{r} \mathbf{q}}$$

$$(\bar{N}N)_{p,D}^q = \bar{N}_{\mathbf{r} \mathbf{p}}^{\mathbf{q} \mathbf{r}} + \bar{N}_{\mathbf{p} \mathbf{r}}^{\mathbf{r} \mathbf{q}}$$

and $+$... means that we are dropping the decuplet contributions.

In reducing J^R from the form given in Ref.7 to the above form, we have used the relation

$$\bar{N}(\mathbf{p}') r_\mu N(\mathbf{p}) = \bar{N}(\mathbf{p}') [P_\mu^2 \gamma_\mu - 2mP_\mu] N(\mathbf{p}) \\ = -\bar{N}(\mathbf{p}') [2im \sigma_{\mu\nu} q_\nu + q_\mu^2 \gamma_\mu] N(\mathbf{p}),$$

with $r_\mu = \epsilon_{\mu\nu\kappa\lambda} P_\nu q_\kappa \gamma_\lambda \gamma_5$.

Further, we shall make use of the SU(3) identifications

$$N_1^1 = 1/\sqrt{2} (\Sigma^0 + 1/\sqrt{3} \Lambda), \quad N_1^2 = \Sigma^+, \quad N_1^3 = \mathbf{p} \\ N_2^1 = \Sigma^-, \quad N_2^2 = 1/\sqrt{2} (-\Sigma^0 + 1/\sqrt{3} \Lambda), \quad N_2^3 = \mathbf{n} \\ N_3^1 = \Xi^-, \quad N_3^2 = -\Xi^0, \quad N_3^3 = -\sqrt{2/3} \Lambda \quad (2.6)$$

At this stage, we deem it appropriate to make some remarks about derivative and other amplitudes before closing this section. We know that the amplitude as it stands blatantly violates unitarity; in addition to (2.2) we should include spurion terms, first derivative terms constructed by $P_B^A =$

$(\gamma)_{\alpha}^{\beta} \delta_p^q$ insertions that are necessarily present in the framework of inhomogeneous $U(6,6)$ and secondly, higher spurions of the type $\sum_R C_R (\gamma_R \otimes \gamma_R)$ that make up the totality of $SU(3) \otimes$ Poincaré'-group amplitudes. In this Chapter we shall be testing the assumption that the regular amplitudes of the initial $U(6,6)$ approximation by and large dominate the processes in question. Even if it turns out that we need the spurion terms at the very minimum to explain certain experimental results, our conclusions will still apply to forward-scattering situations, when these kinematic terms reduce to the homogeneous $U(6,6)$ form.

3. Elastic scattering of protons by the baryon octet

Relations 2.4 in conjunction with the amplitude 2.3 make it an easy matter to pick out any desired process. We shall naturally focus our attention on processes which can be physically realised, i.e. those in which the proton plays an active role. It is sufficient to consider the elastic scatterings pp , pn , $p\Lambda$, $p\Sigma^+$, $p\Xi^-$ as all other processes can be deduced from these. Except for $p\Xi^-$ scattering, the other cross-sections are all observable as the interactions of hyperons with protons can be studied in the same pictures which record their production³⁴.

We now express our results for each case as linear combinations of the basic entities :

$$\begin{aligned}
S &= \bar{N}(p_3)N(p_1)\bar{N}(p_4)N(p_2) \\
P &= \bar{N}(p_3)\gamma_5 N(p_1)\bar{N}(p_4)\gamma_5 N(p_2) \\
A &= \bar{N}(p_3)i\gamma_\mu \gamma_5 N(p_1)\bar{N}(p_4)i\gamma_\mu \gamma_5 N(p_2) \\
V &= \bar{N}(p_3)\gamma_\mu N(p_1)\bar{N}(p_4)\gamma_\mu N(p_2) \\
T &= \frac{1}{2}\bar{N}(p_3)\sigma_{\mu\nu} N(p_1)\bar{N}(p_4)\sigma_{\mu\nu} N(p_2)
\end{aligned} \tag{2.7}$$

as well as the interchanged ($1 \leftrightarrow 2$) quantities $\tilde{S}, \tilde{P}, \tilde{A}, \tilde{V}, \tilde{T}$. In writing down the amplitude we would have the occasion to make use of the following very useful identities³⁵:

$$\begin{aligned}
1/2m\bar{N}(p_4)N(p_2)\bar{N}(p_3)[\not{p}_2+\not{p}_4]N(p_1) \\
= \frac{1}{2}\left[\frac{(s-u)}{4m^2}(S-P)+V-\frac{t}{4m^2}T\right], \\
\frac{1}{4m^2}\bar{N}(p_4)N(p_2)[(p_2+p_4)_\mu(p_2-p_4)_\nu - (p_2+p_4)_\nu(p_2-p_4)_\mu]\bar{N}(p_3)\sigma_{\mu\nu}N(p_1) \\
= \left[\frac{(s-u)}{4m^2}(-S-P)+V-\frac{t}{4m^2}T\right]
\end{aligned} \tag{2.8}$$

Also we introduce the abbreviations

$$x=s/4m^2, \quad y=t/4m^2, \quad z=u/4m^2; \quad x+y+z=1.$$

and note that

$$J_A^A(p', p) = \text{Tr} J(p', p) = 3(1-q^2/4m^2)\text{Tr}(\bar{N}(p')N(p)).$$

Compiling the results, we have :

$$\begin{aligned}
M_{pp} &= 9\bar{A}(1-y)^2 S - 9\tilde{A}(1-z)^2 \tilde{S} + \frac{1}{4}B(1-y)^2 [5S+17/9(V+T-A-P)] + (x-z) \\
&\quad (1+y)20/9S+8/9(1-y)[V-yT-(x-z)P] + \frac{1}{4}\tilde{B}(1-z)^2 [5\tilde{S}+17/9\tilde{V} \\
&\quad (\tilde{V}+\tilde{T}-\tilde{A}-\tilde{P})] + 20/9(x-y)(1+z)\tilde{S}+8/9(1-z)[\tilde{V}-\tilde{A}-\tilde{T}-(x-y)\tilde{P}] \} \tag{2.9}
\end{aligned}$$

$$\begin{aligned}
M_{pn} = & 9A(1-y)^2 S + \frac{1}{4}B \left\{ (1-y)^2 [4S - 8/9(V+T-A-P)] + 16/9(x-z)(1+y)S \right. \\
& + 28/9(1-y)[V-yT-(x-z)P] \left. \right\} + \frac{1}{4}\tilde{B} \left\{ (1-z)^2 [\tilde{S} + 25/9(\tilde{V}+\tilde{T}-\tilde{A}-\tilde{P})] \right. \\
& + 4/9(x-y)(1+z)\tilde{S} + 20/9(1-z)[\tilde{V}-\tilde{T}-(x-y)\tilde{P}] \left. \right\} \quad (2.10)
\end{aligned}$$

$$\begin{aligned}
M_{p\Lambda} = & 9A(1-y)^2 S + \frac{1}{4}B [3(1-y)^2 S + 2(x-z)(1+y)S + (1-y)[V-yT-(x-z)P] \\
& - \frac{1}{4}\tilde{B} 3/2(1-z)^2 [\tilde{S} + \tilde{V} + \tilde{T} - A - P] \quad (2.11)
\end{aligned}$$

$$\begin{aligned}
M_{p\Sigma^+} = & 9A(1-y)^2 S + \frac{1}{4}B \left\{ (1-y)^2 [4S + 16/9(V+T-A-P)] + 4/9(x-z)(1+y)S \right. \\
& + 16/9(1-y)[V-yT-(x-z)P] \left. \right\} + \frac{1}{4}\tilde{B} \left\{ (1-z)^2 [\tilde{S} + 1/9(\tilde{V}+\tilde{T}-\tilde{A}-\tilde{P})] \right. \\
& + 16/9(x-y)(1+z)\tilde{S} - 8/9(1-z)[\tilde{V}-\tilde{T}-(x-y)\tilde{P}] \left. \right\} \quad (2.12)
\end{aligned}$$

$$\begin{aligned}
M_{p\Sigma^-} = & 9A(1-y)^2 S + \frac{1}{4}B \left\{ (1-y)^2 [2S - 4/9(V+T-A-P)] + 8/9(x-z)(1+y)S \right. \\
& + 14/9(1-y)[V-yT-(x-z)P] \left. \right\} \quad (2.13)
\end{aligned}$$

$$\begin{aligned}
M_{p\Xi^-} = & 9A(1-y)^2 S + \frac{1}{4}B \left\{ (1-y)^2 [S + 1/9(V+T-A-P)] + 16/9(x-z)(1+y)S \right. \\
& - 8/9(1-y)[V-yT-(x-z)P] \left. \right\}. \quad (2.14)
\end{aligned}$$

4. Conventional relativistic forms for NN scattering

Before casting our results in the standard GGMW form³⁶, we first extend the considerations given in their paper from SU(2)- (nucleon-nucleon scattering) to SU(3)-space. Contrary to what has been the general belief³⁷, there are in general two additional amplitudes 8_{FD} , 8_{DF} , in addition to the usual six amplitudes corresponding to 27 , 10 , $\bar{10}$, 8_D , 8_F and 1 . However, in the above relations which we are going to consider, time reversal invariance allows us to eliminate these two transition amplitudes 8_{DF} and 8_{FD} ³⁸. Thus in place of the $I=0$, $I=1$ amplitudes of SU(2), we now construct the SU(3)

amplitudes corresponding to 27, $\bar{10}$, 10, 8_D , 8_F and 1 intermediate states for the direct channel. Making use of the Clebsch-Gordan tables³⁹ for SU(3) these are linearly related to the six processes considered in Section 3 as follows:

$$\begin{bmatrix} M^{(27)} \\ M^{(\bar{10})} \\ M^{(10)} \\ M^{(8D)} \\ M^{(8F)} \\ M^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & -5/2 & -5/4 & 15/4 & 0 \\ -1 & -2 & 9/2 & 1/4 & -3/4 & 0 \\ 1 & 4 & -8 & 0 & -4 & 8 \end{bmatrix} \begin{bmatrix} M_{pp} \\ M_{pn} \\ M_{p\Lambda} \\ M_{p\Sigma^+} \\ M_{p\Sigma^-} \\ M_{pE^-} \end{bmatrix} \quad (2.15)$$

Notice that $M^{(27)}$ and $M^{(\bar{10})}$ correspond to the isotriplet and isosinglet parts of nucleon-nucleon scattering. We are now able to write the above NN scattering amplitudes under the assumption of SU(3) invariance in terms of the GGMW form

$$M^{(I)} = F_1^{(I)}(S-\tilde{S}) + F_2^{(I)}(T+\tilde{T}) - F_3^{(I)}(A-\tilde{A}) + F_4^{(I)}(V+\tilde{V}) - F_5^{(I)}(P-\tilde{P}) \quad (2.16)$$

We infer from the Pauli principle that

$$\begin{aligned} F_i^{(27, 8_D, 1)}(s, t, u) &= (-1)^{i+1} F_i^{(27, 8_D, 1)}(s, u, t) \\ F_i^{(\bar{10}, 10, 8_F)}(s, t, u) &= (-1)^i F_i^{(\bar{10}, 10, 8_F)}(s, u, t) \end{aligned} \quad (2.17)$$

which conditions provide a useful check on our work. To cast our amplitudes in the form (2.16) from (2.15) we make use of the Fierz crossing matrix

$$\begin{bmatrix} \tilde{S} \\ \tilde{V} \\ \tilde{T} \\ \tilde{A} \\ \tilde{P} \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & -1 & -1 \\ 4 & -2 & 0 & -2 & 4 \\ 6 & 0 & -2 & 0 & -6 \\ -4 & -2 & 0 & -2 & -4 \\ -1 & 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} S \\ V \\ T \\ A \\ P \end{bmatrix} \quad (2.18)$$

from which we obtain the useful matrix relation

$$\begin{bmatrix} S+\tilde{S} \\ V+\tilde{V} \\ T+\tilde{T} \\ A+\tilde{A} \\ P+\tilde{P} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ -2 & 0 & 0 & 1 & -2 \\ -3 & 0 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} S-\tilde{S} \\ V-\tilde{V} \\ T-\tilde{T} \\ A-\tilde{A} \\ P-\tilde{P} \end{bmatrix} \quad (2.18a)$$

The final expressions are collected in Tables I and II.

In addition to those pN amplitudes already given in equations (2.9) to (2.14), we may now evaluate some inelastic amplitudes. Since the process $p+\Sigma^0 \rightarrow p+\Lambda$ is particularly interesting from the point of view of crossing to $p\bar{p} \rightarrow \Lambda \bar{\Sigma}^0$ we present the result in detail,

$$\sqrt{3}M(p\Sigma|p\Lambda) = \frac{1}{4}B \left\{ \frac{5}{3}(1-y)^2 (V+T -A-P) + \frac{2}{3}(x-z)(1+y)S - \frac{7}{3}(1-y)[V-yT - (x-z)P] \right\} - \frac{1}{4}B \left\{ \frac{1}{2}(1-z)^2 (3\tilde{S}-\tilde{V}-\tilde{T}+\tilde{A}+\tilde{P}) + (1-z)[\tilde{V}-z\tilde{T}-(x-y)\tilde{P}] \right\}. \quad (2.19)$$

5. Reduction to non-relativistic forms for NN scattering.

While it is certainly possible to express physically interesting quantities such as cross sections $d\sigma$, polarisations $\langle P \rangle$, etc. directly in terms of the F_i (for instance through the helicity amplitudes listed in GGMW), we prefer first to pass into the conventional non-relativistic parametrisation where the connections with $d\sigma$ and $\langle P \rangle$ are widely known and have been extensively tabulated⁴⁰. For convenience we reproduce these in Table III.

This non-relativistic reduction is straightforwardly, if tediously, carried out as follows. From equations (2.16) and (2.18), we have the expansion

$$\begin{aligned} 4M = & (3F_1 + 6F_2 - 4F_3 + 4F_4 - F_5)S + (-F_1 - 2F_3 + 2F_4 + F_5)V + (-F_1 + 2F_2 - F_5)T \\ & + (F_1 - 6F_3 - 2F_4 - F_5)A + (F_1 - 6F_2 - 4F_3 + 4F_4 - 3F_5)P \end{aligned} \quad (2.20)$$

We work in the centre-of-mass frame with $\underline{p}$ and $\underline{p}'$ denoting the initial and final relative momenta and, following standard practice, define the unit vectors

$$\hat{\underline{q}} = \frac{\underline{p} - \underline{p}'}{|\underline{p} - \underline{p}'|}, \quad \hat{\underline{p}} = \frac{\underline{p} + \underline{p}'}{|\underline{p} + \underline{p}'|}, \quad \hat{\underline{n}} = \frac{\underline{p} \times \underline{p}'}{|\underline{p} \times \underline{p}'|}$$

with $|\underline{p}'| = |\underline{p}|$ and the projections

$$\sigma_q, \sigma_p, \sigma_n = \hat{\underline{e}} \cdot \hat{\underline{q}}, \hat{\underline{e}} \cdot \hat{\underline{p}}, \hat{\underline{e}} \cdot \hat{\underline{n}} \text{ respectively.}$$

As stated earlier, we choose our γ matrices such that

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \underline{\gamma} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}, \quad i\gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with $U(p) = 1/\sqrt{E+m} \begin{pmatrix} E+m \\ \underline{\sigma} \cdot \underline{p} \end{pmatrix} \chi$; where χ is a two-component spinor.

We thus obtain:

$$\bar{U}(p')U(p) = [(E+m)-(E-m)\cos Q] \chi'^{\dagger} \chi + i(E-m)\sin Q \chi'^{\dagger} \underline{\sigma} \cdot \hat{n} \chi$$

$$\bar{U}(p')\gamma_0 U(p) = [(E+m)+(E-m)\cos Q] \chi'^{\dagger} \chi - i(E-m)\sin Q \chi'^{\dagger} \underline{\sigma} \cdot \hat{n} \chi$$

$$\bar{U}(p')\underline{\gamma} U(p) = \underline{p} \chi'^{\dagger} \chi - i \chi'^{\dagger} \underline{\sigma} \cdot \hat{q} \chi$$

$$\bar{U}(p')i\gamma_5 U(p) = i\underline{q} \chi'^{\dagger} \chi + \chi'^{\dagger} \underline{\sigma} \cdot \hat{p} \chi$$

$$\begin{aligned} \bar{U}(p')\underline{\sigma} U(p) &= [(E+m)+(E-m)\cos Q] \chi'^{\dagger} \underline{\sigma} \chi - i(E-m)\sin Q \chi'^{\dagger} \underline{\sigma} \cdot \hat{n} \chi \\ &\quad - \frac{1}{2}\underline{p}/E+m \chi'^{\dagger} \underline{\sigma} \cdot \hat{p} \chi + \frac{1}{2}\underline{q}/E+m \chi'^{\dagger} \underline{\sigma} \cdot \hat{q} \chi \end{aligned}$$

$$\bar{U}(p')i\gamma_5 \gamma_5 U(p) = \chi'^{\dagger} \underline{\sigma} \cdot \underline{p} \chi$$

$$\begin{aligned} \bar{U}(p')i\underline{\gamma} \gamma_5 U(p) &= [(E+m)-(E-m)\cos Q] \chi'^{\dagger} \underline{\sigma} \chi + i(E-m)\sin Q \chi'^{\dagger} \underline{\sigma} \cdot \hat{n} \chi \\ &\quad - \frac{1}{2}\underline{q}/E+m \chi'^{\dagger} \underline{\sigma} \cdot \hat{q} \chi + \frac{1}{2}\underline{p}/E+m \chi'^{\dagger} \underline{\sigma} \cdot \hat{p} \chi \end{aligned}$$

$$\bar{U}(p')\gamma_5 U(p) = -i \chi'^{\dagger} \underline{\sigma} \cdot \hat{q} \chi$$

We can now calculate the various quantities S, V, T, A and P

by proceeding to the c.o.f m. frame where

$$p_1 = (E, \underline{p}) \quad p_3 = (E, -\underline{p}')$$

$$p_2 = (E, -\underline{p}) \quad p_4 = (E, \underline{p}')$$

The conventional non-relativistic expansion then reads⁴¹

$$M = \alpha \underline{1} \underline{1} + \beta \underline{\sigma}_n \cdot \underline{\sigma}_n + \gamma (\underline{\sigma}_n \cdot \underline{1} + \underline{1} \cdot \underline{\sigma}_n) + \delta \underline{\sigma}_q \cdot \underline{\sigma}_q + \epsilon \underline{\sigma}_p \cdot \underline{\sigma}_p \quad (2.21)$$

M being understood to act between the two-component spinors χ

viz $Q \otimes Q' \rightarrow X_3^* X_1 X_4^* X_2$. $\alpha, \beta, \gamma, \delta$ and ϵ are, of course, complex functions of the centre-of-mass energy $E = (\mathbf{p}^2 + m^2)^{\frac{1}{2}} = (\mathbf{s}^2)^{\frac{1}{2}}$ and the scattering angle $\theta [\cos \theta = \mathbf{p} \cdot \mathbf{p}' / p^2 = (t-u)/(t+u)]$.

In terms of the GGMW amplitudes, the $\alpha, \beta, \gamma, \delta, \epsilon$ are of the following forms⁴²

$$\alpha = [E^2 + m^2 - 3(E^2 - m^2) \cos \theta] F_1 + 2[m(2E + m) - (E^2 - m^2) \cos \theta + (E - m)^2 \cos^2 \theta] F_2 \quad (2.22)$$

$$- 2(2E^2 + m^2 F_3 + 2[E(E + 2m) + (E - m)^2 \cos^2 \theta] F_4 + (E^2 - m^2)(1 + \cos \theta) F_5$$

$$\beta = -[E^2 + m^2 + (E^2 - m^2) \cos \theta] F_1 + 2[m(2E - m) + (E^2 - m^2) \cos \theta + (E - m)^2 \cos^2 \theta] F_2$$

$$+ 2[2E^2 + m^2 - 2(E^2 - m^2) \cos \theta] F_3 + 2[E(2m - E) + (E - m)^2 \cos^2 \theta] F_4$$

$$- (E^2 - m^2)(1 + \cos \theta) F_5 \quad (2.23)$$

$$\gamma = (E^2 - m^2) \sin \theta (F_1 + F_3) - (E - m)^2 \sin^2 \theta (F_2 + F_4) \quad (2.24)$$

$$\delta = -[E^2 + m^2 + (E^2 - m^2) \cos \theta] F_1 + 2[m^2 + 2(E^2 - m^2) \cos \theta] F_2 + 2[-E^2 + 4m^2 + (E^2 - m^2) \cos \theta] F_3$$

$$+ 2[E^2 - (E^2 - m^2) \cos \theta] F_4 + (E^2 - m^2)(-3 + \cos \theta) F_5 \quad (2.25)$$

$$\epsilon = -[E^2 + m^2 + (E^2 - m^2) \cos \theta] F_1 + 2m^2 F_2 + 2[5E^2 - 2m^2 + (E^2 - m^2) \cos \theta] F_3$$

$$+ 2[E^2 + (E^2 - m^2) \cos \theta] F_4 + (E^2 - m^2)(1 + \cos \theta) F_5 \quad (2.26)$$

These connections between the F_i and the $\alpha, \beta, \gamma, \delta$ and ϵ may prove valuable in their own right, without reference to U(6,6).

6. Physical observables and non-relativistic parameters

At this stage, we find it convenient to state the physical observables in terms of the non-relativistic parameters:

- (a) The differential cross section for an initially unpolarised beam is

$$d\sigma/d\Omega = 1/128\pi^2 s^2 [|\alpha|^2 + |\beta|^2 + 2|\gamma|^2 + |\delta|^2 + |\epsilon|^2] \quad (2.27)$$

(b) The polarisation $\langle P \rangle$ of the final baryon (perpendicular to the scattering plane) is proportional to $\text{Im}[\gamma^*(\alpha+\beta)]$.

From (22), (23) and (24) we get

$$\begin{aligned} \langle P \rangle &= -8(E^2 - m^2)[2mE - (E - m)^2 \cos^2 \theta] \\ &\quad \times \sin \theta \text{Im}[(F_1 + F_3)^*(F_2 + F_4)] \\ &\sim p^2 \sin \theta \text{Im}[(F_1 + F_3)^*(F_2 + F_4)] \end{aligned} \quad (2.28)$$

near threshold.

(c) The spin-singlet and spin-triplet scattering lengths, a_s and a_T (obtained at threshold), are given by

$$\alpha(0) = m^2(3a_T + a_s), \quad \beta(0) = m^2(a_T - a_s).$$

Hence,

$$a_T = 2(F_2 + F_4)/s \rightarrow l m^2, \quad a_s = 2(F_1 - 3F_3)/s \rightarrow l m^2 \quad (2.29)$$

(d) The optical theorem relates the total cross section to forward scattering as follows :

$$\sigma^{\text{tot}}(s) = \text{Im}\alpha(s, 0)/l m^2 [s(s - l m^2)]^{\frac{1}{2}}, \quad (2.30)$$

where

$$\alpha(s, 0) = \frac{1}{2}s(2F_4 + F_5 - F_1 - 2F_3) + 2m^2(2F_1 + 3F_2 - F_3 + F_4 - F_5) \quad (2.31)$$

(e) Connections with other experimental observables may be obtained from standard tables. For example, the correlation parameter $C_{q\bar{p}}$ is proportional to $\text{Im}[\gamma^*(\delta - \epsilon)]$ and so on.

7. Incorporation of mass-differences

We wish now to make some observations on the question of mass differences. Throughout the whole of the previous work we have assumed the baryon octet degenerate at a common mass m . At high energies this is probably a good approximation, but at low energies where we shall be comparing scattering lengths, polarisations, etc., this will certainly not do. There is to date no well-defined prescription for taking exact account of mass corrections within any symmetry scheme, but a plausible method is to make comparisons of amplitudes for each channel at the same Q values after correcting for phase space and flux factors⁴³. Q is defined as $Q = E^* - m_3 - m_4$ where E^* = total energy in the centre of mass system, and m_3, m_4 are the masses of the two outgoing particles. This choice has the virtue that the thresholds for each section are superimposed at $Q = 0$ and that the opening up of new channels which, through unitarity, will affect the reactions under consideration is also superimposed. Secondly, at the same Q -value, each cross section σ is weighted by a relativistically invariant phase factor F , such that

$$|M|^2 = E^{*2} p_{in}/p_{out} \sigma = F\sigma.$$

$|M|^2$ represents the square of the scattering amplitudes computed from $U(6,6)$ and p_{in} and p_{out} are the momenta of the incident and outgoing particles in the C.M.system. The kinematic factors that relate cross sections to matrix

elements are, of course, different for each channel. We shall follow this approach and thereby need modify the external momentum factors. Consider typically the forward-scattering case where we meet the factor $s(s-4m^2)$ in the degenerate situation

$m_1 = m_2 = m_3 = m_4 = m$. If $m_1 = m_2$ and $m_3 = m_4$ we make the modification $s(s-4m^2) \rightarrow s[(s-4m_1^2)(s-4m_2^2)]^{\frac{1}{2}}$, whereas if $m_1 = m_3$ and $m_2 = m_4$ we make the modification $s(s-4m^2) \rightarrow [s-(m_1-m_2)^2][s-(m_1+m_2)^2]$.

Similar replacements may be envisaged for the various factors occurring in equations (2.9)-(2.14) and (2.19), and will be used particularly for $p\bar{p} \rightarrow N\bar{N}$ reactions. Although they correctly incorporate threshold effects, these prescriptions at best can only be a rough way of handling mass corrections. However, any more sophisticated treatment must make the theoretical analysis many orders of magnitude more difficult and really quite unnecessary, bearing in mind that $U(6,6)$ is, by its very nature, an approximate theory.

8. Comparisons with pN scattering experiments

In the direct channel, apart from the extremely well-studied nucleon-nucleon problem, there is a scarcity of data all of which is confined to Y_p scattering as Ξ processes are entirely out of reach at the present time. Nevertheless, we shall concentrate on the seven processes $pp \rightarrow pp$, $pn \rightarrow pn$, $p\Lambda \rightarrow p\Lambda$, $\bar{p}\Sigma^+ \rightarrow p\Sigma^+$, $p\Lambda \rightarrow p\Sigma^0$ and

$p \Xi^- \rightarrow p \Xi^-$ since the last is physically attainable in the crossed channel $p\bar{p} \rightarrow N\bar{N}$. The analysis will be confined to comparisons of scattering lengths, polarisations, and total cross sections obtained from forward scattering.

The F_i for each of these processes is collected in Tables IV and V, whilst in Table VI we have for convenience listed the combinations $(F_2 + F_4)_{s=4m^2}$, $(F_1 - 3F_3)_{s=4m^2}$ and

$$[\frac{1}{2}s(-F_1 - 2F_3 + 2F_4 + F_5) + 2m^2(2F_1 + 3F_2 - F_3 + F_4 - F_5)]_{t=0},$$

which have physical relevance. Theoretical predictions are then easily read off. Notice that in the threshold limit

$$a(s, t, u) \rightarrow \tilde{a}(s, t, u) \rightarrow a(l_m^2, 0, 0) \text{ but that otherwise } a,$$

$\tilde{a}$, B , $\tilde{B}$ remain distinct amplitudes in general. However, if we neglect all but S waves (i.e. neglect the t, u dependence of these amplitudes) $a = \tilde{a}$ and $B = \tilde{B}$ and it is this special situation that has been studied for $SU(6)$ by Barger and Rubin⁴⁴.

We shall make no such restriction unless we have no other option, such as when we consider low-energy $p\bar{p} \rightarrow N\bar{N}$ cross sections.

(a) Firstly, regarding the question of polarisation which is given by the interference of $(F_1 + F_3)$ and $(F_2 + F_4)$, we immediately conclude from Tables IV and V that the outgoing proton may well be polarised for the reactions considered. To illustrate, let us take proton-proton scattering where

$$(F_1 + F_3)^{(pp)} = [9/2 A - 8/9 B] (1-y)^2 + 5/18 B (1+y) (x-z) + (y \leftrightarrow z),$$

$$(F_2 + F_4)^{(pp)} = [9/2 A + 13/9 B] (1-y)^2 + 5/18 B (1+y) (x-z) - (y \leftrightarrow z)$$

The interference takes place between amplitudes of opposite symmetry types $A \pm \tilde{A}$, $B \pm \tilde{B}$ and is obviously nonzero. Similarly for each of the reactions in Tables IV and V there is never a single amplitude and polarisation of the final proton is quite feasible within $U(6,6)$ symmetry. Evidently there exist processes which we have not considered such as $p \Xi^- \rightarrow n \Xi^0$ which are predicted to give vanishing polarisations on the basis of $U(6,6)$ because they involve just one amplitude. However, once derivative couplings are admitted into the theory, such conclusions will no longer apply.

(b) The six independent scattering lengths of $SU(3)$ are reduced to just two by $U(6,6)$. Thus in addition to $SU(3)$ predictions such as $a_s^{\Sigma^+p} = a_s^{np}$, we have the relations (see Table VI)

$$\begin{aligned} a_s^{np} &= a_T^{np}, & a_s^{\Lambda p} &= a_T^{\Lambda p} \\ 8a_s^{\Lambda p} &= a_T^{\Sigma^+p} + 7a_T^{np} & 6a_s^{\Lambda p} &= a_s^{\Sigma^-p} + 5a_s^{np} \end{aligned} \quad (2.32)$$

The only experimental evidence regarding these scattering lengths after eliminating possible Coulomb effects is as follows^{40,45}

$$\begin{aligned} a_s^{np} &= -23.680 \pm 0.028F, & a_T^{np} &= 5.399 \pm 0.011F \\ a_s^{\Lambda p} &= -2.46 F & a_T^{\Lambda p} &= -2.07F \\ -3 \lesssim a_T^{\Sigma^+p} &\lesssim 1.7F, & 4F &\gtrsim a_s^{\Sigma^+p} \gtrsim 30F \end{aligned} \quad (2.33)$$

The very first theoretical conclusion $a_s^{np} - a_T^{np}$ is so badly violated that comparison with $a_T^{\Sigma^+p} - 8a_s^{\Lambda p}$ in equation (2.32) is rendered meaningless. Most likely the discrepancy with $U(6,6)$ lies in the great sensitivity of the results to the position of the deuteron poles (real and virtual) near threshold. Such excuses have already been invoked to account for deviations from $SU(3)$ symmetry²³ and therefore the lack of equality of singlet and triplet nucleon-nucleon scattering lengths could well represent only small $U(6,6)$ symmetry breakdown. We must mention that kineton terms from inhomogeneous $U(6,6)$ reduce to regular couplings and cannot affect this result. However, it might be appropriate to point out that the equality of the singlet and triplet Λp scattering lengths is in good agreement with experiment. This equality is interesting since no resonance or bound state seems to exist in Λp S-wave scattering giving support to the excuse already invoked to explain the equality of the np scattering lengths.

(c) In view of these remarks it is desirable to make comparisons with experiment at somewhat higher energies. Unfortunately except for nucleon-nucleon scattering, this is not readily available. The present available $\bar{Y}p$ scattering data are summarised in Table VII. On the theoretical side, making use of equations (2.30) and (2.31), we obtain, apart

from the standard $SU(3)$ relations such as

$$\sigma^{\text{tot}}(\Xi^0 p) = \sigma^{\text{tot}}(\Sigma^- p),$$

the extra predictions (unchanged even with derivative couplings),

$$\begin{aligned} \sigma^{\text{tot}}(\Sigma^+ p) - 3\sigma^{\text{tot}}(\Sigma^- p) &= 4\sigma^{\text{tot}}(np) - 6\sigma^{\text{tot}}(\Lambda p), \\ 3\sigma^{\text{tot}}(\Sigma^+ p) &= 8\sigma^{\text{tot}}(\Sigma^- p) + 4\sigma^{\text{tot}}(\Xi^- p) \quad (2.34) \\ &= 3\sigma^{\text{tot}}(np) - 4\sigma^{\text{tot}}(\bar{K}^+ p). \end{aligned}$$

The fact that the first relation is violated at these low energies cannot again be regarded as a serious objection; $\sigma(np)$ shows very large variations in this region and, quite apart from that, Coulomb forces will muddy the strong interaction amplitudes. Until data is available at considerably higher momenta, equation (2.34) will not prove a useful test of $U(6,6)$ theory.

(d) It has been shown^{1.6} by Dashen and Gell-Mann that for coplanar nucleon-nucleon scattering the invariance under $U(3) \otimes U(3)$, which is the invariance group left with insertion of the irregular couplings of inhomogeneous $U(6,6)$, of the form $\not{p} = p_0 \gamma_0 - p_1 \gamma_x - p_2 \gamma_y$, leads to the prediction that the correlation parameter, $C_{pq} = 0$. We easily reproduce this result and extend it to all NN scattering processes using Tables IV and V, by noting that $\not{p} = \not{p} \equiv 0$. This is the

single constraint on the F_i imposed by $U(6,6)$ invariance.

This condition also implies $R' = -A$, $R = A'$,

$$1 - C_{NN}(90^\circ) = 2[1 - D(90^\circ)] \quad (2.35)$$

(For the definitions of these parameters, see Table III.)
on nucleon-nucleon scattering.

The available experimental data for all these correlation parameters are collected in Tables VIII - XI. Apart from $R' = -A$ prediction, the agreement with experiment of the others is not very encouraging.

9. Comparison with $p\bar{p} \rightarrow N\bar{N}$ experiments

By applying crossing symmetry to the direct channel, $p(1) + N(2) \rightarrow p(3) + N(4)$ we obtain the amplitude for $p(1) + \bar{p}(2) \rightarrow \bar{N}(3) + N(4)$. This is expressed by the substitution law^{L7} $p_2 \rightarrow -p_3$, $N(p_2) \rightarrow N(-p_3)$ in equations (2.2) and (2.3). At the same time $A(s,t,u) \rightarrow \bar{A}(s,t,u) = A(t,s,u)$, etc. We could also apply crossing to obtain the processes $p\bar{N} \rightarrow p\bar{N}$ but, apart from the elastic nucleon-anti-nucleon scattering, these reactions do not hold much physical interest, and we shall therefore address ourselves only to $p\bar{p} \rightarrow N\bar{N}$ for which there is considerable data, especially in the form of total cross sections.

One immediate consequence of $U(6,6)$ invariance is the vanishing of all inelastic (absorptive) cross sections $p\bar{p} \rightarrow$

$N\bar{N}$ at threshold. The only surviving amplitude is then $\tilde{a}_{J_B}^{B(p_4, p_1) J_C^C(p_3, p_2)}$. It is interesting that this symmetry scheme automatically leads to the dominance of $p\bar{p}$ elastic scattering without recourse to physical arguments. On the experimental side it is known that $\sigma(p\bar{p})$ and $\sigma(n\bar{n})$ have values of about 100 mb and 20 mb very near threshold, while the data⁴⁸ on $Y\bar{Y}$ production at 3 GeV/c $\bar{p}$ laboratory momenta show $\sigma(Y\bar{Y}) \approx 50 \mu\text{b}$. At this energy $\sigma(p\bar{p}) \approx 20 \text{ mb}$ and $\sigma(n\bar{n}) \lesssim 1 \text{ mb}$. Thus the $U(6,6)$ prediction is in fair agreement with experiment.

To make more detailed comparisons with $U(6,6)$ we must calculate the $\bar{F}_i$ above threshold; for $p\bar{p} \rightarrow Y\bar{Y}$ we cannot use the Johnson-Treiman device²¹ to circumvent equation (2.27). However, though differential cross sections are known for these processes we will limit ourselves to comparing total cross sections and for that purpose we make the S-wave approximation (supposedly we are near threshold) of discarding all angular dependence (t-u) and retaining those terms to first order in $(s-l_{\mu}^2)$. $d\sigma/d\Omega$ often shows a strong peaking in the forward direction indicating a peripheral mechanism with K^* exchange and invalidating this approach, but as far as rough comparisons are concerned our approximation is permissible. In this limit the crossed non-relativistic amplitudes (2.22) - (2.26) become

$$\begin{aligned}
(\bar{\alpha} - \bar{\beta})/4m^2 &= \frac{1}{2}(x-1)[-2\bar{F}_1 - 5\bar{F}_2 + 5\bar{F}_3 - \bar{F}_4 + \bar{F}_5], \\
(\bar{\alpha} + \bar{\beta})/4m^2 &= [1 + 3/2(x-1)][-\bar{F}_1 + \bar{F}_2 - \bar{F}_3 + \bar{F}_4], \\
\bar{\gamma}/4m^2 &= \frac{1}{2}i(x-1)^{\frac{1}{2}}[\bar{F}_1 + 2\bar{F}_2 + \bar{F}_3 + s\bar{F}_4] - 2(x-1)(\bar{F}_2 + \bar{F}_4), \\
(\bar{\sigma} + \bar{\epsilon})/4m^2 &= (-\bar{F}_1 + 2\bar{F}_2 + 2\bar{F}_3 + \bar{F}_5) + \frac{1}{2}(x-1)[- \bar{F}_1 + 3\bar{F}_2 + \bar{F}_3 - \bar{F}_4 + 2\bar{F}_5], \\
(\bar{\sigma} - \bar{\epsilon}) &= 0,
\end{aligned}$$

with $x-1 = (s/4m^2 - 1)$ as the expansion parameter. We then obtain the following theoretical leading behaviours of the cross sections from equation (2.27) valid for $(x-1) \ll 1$:

$$\begin{aligned}
\sigma(pp) &\longrightarrow 81|\tilde{A}|^2/x, \\
\sigma(nn) &\longrightarrow 25(x-1)|\tilde{B}|^2/4x, \\
\sigma(\Lambda\bar{\Lambda}) &\longrightarrow 81(x-1)|\tilde{B}|^2/16x, \\
\sigma(\Sigma\bar{\Lambda}) &\longrightarrow 3(x-1)|\tilde{B}|^2/16x, \\
\sigma(\Sigma^+\bar{\Sigma}^-) &\longrightarrow (x-1)|\tilde{B}|^2/4x, \\
\sigma(\Sigma^-\bar{\Sigma}^+) &\longrightarrow 19(x-1)^4|\tilde{B}|^2/648x, \\
\sigma(\Xi^-\bar{\Xi}^+) &\longrightarrow 2(x-1)^4|\tilde{B}|^2/81x.
\end{aligned} \tag{2.36}$$

Experimentally $p\bar{p}$ annihilations at 3 GeV/c laboratory momentum have yielded

$$\begin{aligned}
\sigma(pp) &= 21 \text{ mb} \\
\sigma(nn) &= 0 \pm 1.3 \text{ mb} \\
\sigma(\Lambda\bar{\Lambda}) &= 117 \pm 18 \text{ } \mu\text{b}
\end{aligned} \tag{2.37}$$

$$\begin{aligned}
\sigma(\Sigma^0 \bar{\Lambda}) &= 51 \pm 8 \text{ } \mu\text{b} \\
\sigma(\Sigma^+ \bar{\Sigma}^+) &= 36 \pm 16 \text{ } \mu\text{b} \\
\sigma(\Sigma^- \bar{\Sigma}^-) &= 10 \pm 4 \text{ } \mu\text{b} \\
\sigma(\Xi^- \bar{\Xi}^-) &= 2 \pm 1 \text{ } \mu\text{b}
\end{aligned} \tag{2.37}$$

A first crude comparison of equations (2.36) and (2.37) is achieved by taking a common mass $\approx 1.1\text{GeV}$ for the baryon octet (when $x-1 \approx 0.6$). This gives the rough theoretical estimates,

$$\sigma(n\bar{n}) : \sigma(\Lambda\bar{\Lambda}) : \sigma(\Sigma^0 \bar{\Lambda}) : \sigma(\Sigma^+ \bar{\Sigma}^+) \approx 125:100:4:5,$$

$$\sigma(\Sigma^- \bar{\Sigma}^-) : \sigma(\Xi^- \bar{\Xi}^-) \approx 3:1;$$

and in the event that the S dependence of the amplitudes is neglected so that $B = \tilde{B}$, we have

$$\sigma(\Sigma^- \bar{\Sigma}^-) : \sigma(\Sigma^+ \bar{\Sigma}^+) \approx 1:6.$$

A more 'correct' computation of the theoretical ratios (not entirely in accordance with the standard rules because the Q values of the processes are slightly different) must take account of the phase-space factors in equation (2.35) and the different masses involved. The prescriptions of Section 7 give the corrected U(6,6) amplitudes to be

$$\sigma(n\bar{n}) : \sigma(\Lambda\bar{\Lambda}) : \sigma(\Sigma^0 \bar{\Lambda}) : \sigma(\Sigma^+ \bar{\Sigma}^+) \approx 130:80:3:4$$

$$\sigma(\Sigma^- \bar{\Sigma}^-) : \sigma(\Xi^- \bar{\Xi}^-) \approx 25:1$$

and

$$\sigma(\Sigma^- \bar{\Sigma}^-) : \sigma(\Sigma^+ \bar{\Sigma}^+) \approx 1:6,$$

as before, if $B \approx \tilde{B}$.

It should be emphasized that in view of the approximations made to derive equation (2.36) the quantitative results cannot be wholly reliable, particularly as $(x-1)_{nn}^{-} \approx 1.1$ and $(x-1)_{\Lambda\bar{\Lambda}}^{-} \approx 0.75$ are hardly small parameters. Moreover, the introduction of derivative couplings will in general radically change the answers although not in the approximation which gave equation (2.36) when we extrapolated from the forward direction. Therefore the most we can hope for is for qualitative successes of $U(6,6)$; these we do obtain except for the over estimate of $\sigma(\Lambda\bar{\Lambda})$. It is a curious feature that in contrast to $SU(6)$, $U(6,6)$ is fairly satisfactory in the annihilation channels because it somehow incorporates absorption effects. In the direct scattering channel it seems to fail at low energies just as with $SU(6)$; it is conceivable that this blame is carried by bound-state effects.

Finally we remark that our comments about polarisations in Section 5(a) carry over unmodified to the crossed channel, viz, for the processes considered there will in general be polarisations.

CHAPTER IIITHE $U(6) \otimes U(6)$ MASS FORMULA1. Introduction

Once the classification of particles in any symmetry scheme is decided, the next stage is the choice of a suitable mass-breaking operator since it is known that in real life the masses of the particles are not degenerate. Whilst this is fairly straightforward in the various symmetries studied so far, the situation in $U(6,6)$ is a bit complicated by the fact that because it is a non-compact group, the usual techniques used for compact groups may not apply in this case. The situation is, however, not as gloomy as all that, for we are in general interested in the rest mass of a particle so that all that our interest lies in, is the little group, which is the maximal compact $U(6) \otimes U(6)$; in other words the mass formula is essentially a static problem. We shall show below how the little group $U(6) \otimes U(6)$ arises and thence consider its appropriate mass operator.

2. The little group of $U(6,6)$

As has been emphasized in the previous Chapter, because the homogeneous $U(6,6)$ is non-compact, its finite-dimensional

representations are non-unitary and so cannot be associated directly with physical particles. In order to do this, let us, for example, consider a multispinor $\phi_{A_1 A_2 \dots}^{B_1 \dots}$ belonging to some irreducible representation of $U(6,6)$. For the invariant scalar product we must take

$$\phi_{A_1 A_2}^{B_1} \phi_{B_1}^{A_1 A_2} = \phi_{A_1 A_2}^{B_1} (\gamma_o)_{B_1}^{B'_1} (\phi_{A'_1 A'_2}^{B'_1})^* (\gamma_o)_{A'_1}^{A_1} (\gamma_o)_{A'_2}^{A_2}, \quad (3.1)$$

where the asterisk denotes complex conjugation and

$$(\gamma_o)_A^B = (\gamma_o)_\alpha^\beta \delta_r^s \quad (3.2)$$

Equation (3.1) is evidently not positive definite. We can, however, project out a positive definite subspace by restricting the set $\phi_{A_1 A_2 \dots}^{B_1 \dots}$ to those on which γ_o acting on this multispinor, takes one value, for example

$$(\gamma_o)_{A'_1}^{A_1} \phi_{A'_1 A'_2 \dots}^{B_1 \dots} = \phi_{A_1 A_2 \dots}^{B_1 \dots} \quad (3.3)$$

$$(\gamma_o)_{A_2}^{A'_2} \phi_{A_1 A'_2 \dots}^{B_1 \dots} = \phi_{A_1 A_2 \dots}^{B_1 \dots}$$

for lower indices and

$$(\gamma_o)_{B'_1}^{B_1} \phi_{A_1 A_2 \dots}^{B'_1 \dots} = -\phi_{A_1 A_2 \dots}^{B_2 \dots} \quad (3.4)$$

for upper indices.

Equations (3.3) and (3.4) are, in fact, the Bargmann-Wigner¹⁷ equations for $p = 0$. Thus the set of multispinors satisfying

(3.3) and (3.4) is now not invariant under the full group.

Instead, the $U(6,6)$ representation has now been reduced under the subgroup consisting of those matrices J_A^B for which

$$J \gamma_0 J^{-1} = \gamma_0 \quad (3.5)$$

i.e.

$$J^{-1} = J^+.$$

This is the maximal compact subgroup $U(6) \otimes U(6)$, composed of the generators $\frac{1}{2}(1 \pm \gamma_0) \sigma_\mu T^i$ as already pointed out in Chapter I. Thus the indefinite space of the original representation breaks up in this way into a collection of subspaces, invariant under $U(6) \otimes U(6)$, each of which is positive definite. The prescriptions (3.3) and (3.4) single out a particular one of these.

Now, under $U(6) \otimes U(6)$, the basic spinors Ψ_A and $\bar{\Psi}^A$ reduce in the following way:

$$\Psi_A = (6, 1)_+ \oplus (1, 6)_- \quad (3.6)$$

$$\bar{\Psi}^A = (\bar{6}, 1)_+ \oplus (1, \bar{6})_-$$

where the appended sign indicates the value taken by γ_0 in the respective subspaces. Charge conjugation invariance⁴⁹

$$\Psi_{\alpha r} \longrightarrow C_{\alpha\beta} \bar{\Psi}^{\beta r} \quad (3.7)$$

with $C^{-1} \gamma_0 C = -\gamma_0$,

implies that under this operation

$$(6,1) \longrightarrow (1,\bar{6})$$

(3.8)

$$\text{and } (1,6) \longrightarrow (\bar{6},1) .$$

Thus the relative signs chosen in equations (3.3) and (3.4) ensure the invariance of these equations under charge conjugation. The spinors Ψ_A and $\bar{\Psi}^A$ are now associated, in particular, with the $(6,1)$ and $(1,\bar{6})$ or quark and antiquark respectively. The equations (3.3) and (3.4) exclude the other set, the pseudo-quark and antipseudo-quark $(1,6)$ and $(\bar{6},1)$. Hence under the $U(6) \otimes U(6)$ symmetry and the first type of quark and antiquarks¹⁶, the 36 meson states with $J^P=0^-$ and $J^P=1^-$ will transform like $(6,\bar{6})$ and the 56 baryon states with $J^P=\frac{1}{2}^+$ and $J^P=\frac{3}{2}^+$ will transform like $(56,1)$.

3. The symmetry breaking operator in $U(6) \otimes U(6)$

We are now in a position to study how the symmetry breaking term in the Hamiltonian transforms⁵⁰. As has been shown in the derivation of the $SU(6)$ mass formula⁵¹, the analogous $SU(3)$ assumption that the mass operator transforms under $SU(6)$ like the $I=J=Y=0$ component of the $\underline{35}$ representation leads to a mass formula which leaves the ρ and π , Ξ^* and Ξ , etc. degenerate. In other words, such an assumption does not succeed in spin-splitting the pseudoscalar and vector mesons, for example. It does seem obvious *then* that for higher symmetries, one cannot assume that the symmetry-breaking term belongs to the adjoint

representation without landing into serious trouble.

The simplest way of determining the symmetry-breaking terms is to calculate the reduced matrix elements of all possible tensor operators for a given multiplet of particles and then to neglect those contributions which are either zero or are extremely small compared to the other terms. Such a method has been applied to the $SU(3)$ mass formula⁵² and we shall follow this approach, but taking a hint from the work of Bég and Singh⁵³, we shall restrict ourselves to the assumption that under $SU(3)$ we consider only the tensors that are either singlet or transform like the $I=J=Y=0$ member of an octet. They will, needless to say, be singlets under $SU(2)$.

4. The baryon mass formula

On the way from the full $U(6) \otimes U(6)$ down to $SU(3)$, we could pass through $SU(6)$ or through $SU(3) \otimes SU(3)$ symmetry as an intermediate stage. We choose mainly for convenience to follow the first chain, i.e. $U(6) \otimes U(6) \rightarrow SU(6) \rightarrow SU(3) \otimes SU(2)$. Thus, for the $U(6) \otimes U(6)$ baryon states $(56,1)$, we have the relation

$$(56,1) \otimes (\bar{56},1) = (1,1) \oplus (35,1) \oplus (405,1) \oplus (2695,1) \quad (3.9)$$

The $SU(3) \otimes SU(2)$ contents of 35, 405, and 2695 are of course well-known. From equation (3.9) we can write the baryon mass operator as

$$M = M^{1(1,1)+M^8(35,1)+M^1(405,1)+M^8(405,1)+M^8(2695,1)} \quad (3.10)$$

where the superscript refers to the SU(3) contents of ^{the}irreducible representations of the subscripts. Applying now the Wigner-Eckart theorem to equation (3.10), making use of the ~~56~~⁵⁶ Clebsch-Gordan coefficients⁵⁴ of SU(6), we obtain five equations involving the physical masses N, Σ, Λ, N^* and Y^* , and the reduced matrix elements $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 corresponding to the tensors $(1,1), (35^8,1), (405^1,1), (405^8,1)$ and $(2695^8,1)$ respectively. We write these in the matrix form

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} 0 & 1/7 & 1/7 & 0 & 5/7 \\ 1/6\sqrt{6/7} & -1/24\sqrt{6/7} & -1/8\sqrt{6/7} & 5/6\sqrt{6/7} & -5/6\sqrt{6/7} \\ 0 & -\sqrt{10}/14 & \sqrt{10}/14 & 0 & \sqrt{10}/7 \\ -1/\sqrt{14} & 3/(4\sqrt{14}) & 1/(4\sqrt{14}) & 1/\sqrt{14} & -1/\sqrt{14} \\ -1/3\sqrt{3/7} & -1/6\sqrt{3/7} & 1/2\sqrt{3/7} & 1/3\sqrt{3/7} & -1/3\sqrt{3/7} \end{bmatrix} \begin{bmatrix} N \\ \Sigma \\ \Lambda \\ N^* \\ Y^* \end{bmatrix} \quad (3.11)$$

Putting in experimental values of the baryons⁵⁵, we obtain the values

$$\begin{aligned} \alpha_1 &= 1319 & \alpha_3 &= 104 & \alpha_5 &= -2 \\ \alpha_2 &= 144 & \alpha_4 &= 24 \end{aligned} \quad (3.12)$$

from which we note that α_5 is small compared to the other reduced matrix elements and can therefore be dropped. Doing this gives us the mass relation

$$2N^* + 3\Lambda = 2N + 2Y^* + \Sigma \quad (3.13)$$

which is very well satisfied experimentally and justifies a posteriori the Beg-Singh baryon mass formula. No further contribution can be neglected without obtaining disagreement with experiment. For example, if we discard the $(405^8, 1)$ contribution (because it is fairly small), then we have

$$4N^* + \Lambda + 3\Sigma = 4Y^* + 4N. \quad (3.14)$$

so that from equations (3.13) and (3.14) we have

$$\Lambda = \Sigma$$

which is physically out by about 10%. Thus in order to lift this degeneracy we have to retain the $(405^8, 1)$ contribution.

The mass operator for the baryons can now be written as

$$M = M_{(1,1)}^1 + M_{(35,1)}^8 + M_{(405,1)}^1 + M_{(405,1)}^8 \quad (3.15)$$

which happens to be exactly the same as the $SU(6)$ one⁵³.

5. The meson mass formula

The $U(6) \otimes U(6)$ mass formula for the mesons is definitely of more interest than that for the baryons owing to the presence of the pseudoscalar meson X^0 (960 MeV). We shall proceed in much the same way as in the above and consider the reducible tensor $(6, \bar{6}) \otimes (\bar{6}, 6)$ in terms of its irreducible representations. Then we have

$$(6, \bar{6}) \otimes (\bar{6}, 6) = (1, 1) \oplus (35, 1) \oplus (1, 35) \oplus (35, 35) \quad (3.16)$$

The SU(6) contents of these irreducible representations are given by the relations

$$\begin{aligned} (1, 1) &= 1 \\ (35, 1) &= 35 \\ (1, 35) &= 35 \\ (35, 35) &= 1 + 35_D + 35_F + 189 + 280 + 2\bar{80} + 405 \end{aligned} \quad (3.17)$$

Charge conjugation invariance (or equivalently, the equality of masses of say, the K and $\bar{K}$) eliminates the antisymmetric tensors from (35, 35).

Also it is easily seen that the application of this same invariance leads to the equality of the reduced matrix elements corresponding to (35, 1) and (1, 35). Thus if, following the Bég-Singh type of assumption, we drop the (189; 8), the (189; 27) and the (405; 27) contributions, the mass operator we are left with may be written as

$$\begin{aligned} M^2 &= M_{(1,1)}^{1;1} + M_{(35,1)+(1,35)}^{35;8} + M_{(35,35)}^{1;1} + M_{(35,35)}^{35;8} \\ &+ M_{(35,35)}^{189;1} + M_{(35,35)}^{405;1} + M_{(35,35)}^{405;8} \quad (3.18) \end{aligned}$$

where the superscript notation $(\alpha; \beta)$ represents the SU(6) and SU(3) contents of the irreducible representation respectively. Equation (3.18) is the U(6) SU(6) analogue of the Bég-Singh mass

formula and we are following the usual practice of using the squared mass for the mesons. It would be interesting to see whether this predicts any new mass relations.

From (3.18) it is now a simple matter to write down the equations for the masses in terms of the reduced matrix elements corresponding to the $(1,1)^{1;1}$, $[(35,1)+(1,35)]^{35,8}$, $(35,35)^{1;1}$, $(35,35)^{35;8}$, etc. by using the 35 $\otimes$ 35 Clebsch-Gordan coefficients⁵⁴ for SU(6). We collect these equations into the matrix equation (3.19), where the particle label represents the particle mass squared, and we have normalised the reduced matrix elements in such a way that if we use all the ten parameters of the mass operator, the $(1,1)^{1;1}$ contribution will be equal to the mean mass of the meson multiplet.

$$\begin{bmatrix} \pi \\ K \\ \rho \\ K^* \\ \omega_1 \\ X_1 \\ 2\eta_8 \leftrightarrow X_1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 2 & -9 & -9 & -18 \\ 1 & -1 & -1 & -1 & -9 & -9 & 9 \\ 1 & 2 & -1 & 2 & 1 & 5 & 14 \\ 1 & -1 & -1 & -1 & 1 & 5 & -7 \\ 1 & 0 & -1 & 0 & 16 & -16 & 0 \\ 1 & 0 & 35 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 32 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ \frac{1}{2}b \\ (\frac{1}{35})^{\frac{1}{2}}c \\ \frac{1}{4}(\frac{1}{2})^{\frac{1}{2}}d \\ \frac{1}{2}(1/10)^{\frac{1}{2}}e \\ \frac{1}{2}(1/14)^{\frac{1}{2}}f \\ \frac{1}{4}(1/10)^{\frac{1}{2}}g \end{bmatrix} \quad (3.19)$$

with

$$\begin{aligned} \omega_8 &= a - 2b' - c' - 2d' + 2e' + 5f' - 14g' \\ \eta_8 &= a - 2b' - c' - 2d' - 9e' - 9f' + 18g' \\ \sqrt{2}(\omega_1 \leftrightarrow \omega_8) &= 4b' + 4d' - 20g' \end{aligned} \quad (3.20)$$

where b' , c' , etc. are equal to the normalised reduced matrix elements of the column matrix on the right-hand side of equation (3.19). Note that the unphysical masses, $\omega_1, \omega_8, \eta_8, X_1$, $\eta_8 \leftrightarrow X_1, \omega_1 \leftrightarrow \omega_8$ are related to the physical masses (with the usual mixing angles of $\sin \alpha = \frac{1}{\sqrt{3}}$ and $\tan \beta = 0.19$ for the ω and ϕ, η and X^0 respectively) by the equations

$$3\omega_8 = 4K^* - \rho \quad (3.21a)$$

$$3\eta_8 = 4K - \pi \quad (3.21b)$$

$$\omega_1 + \omega_8 = \omega + \phi \quad (3.21c)$$

$$X_1 + \eta_8 = X + \eta \quad (3.21d)$$

$$(\omega_1 \leftrightarrow \omega_8)^2 = \omega_1 \omega_8 - \omega \phi \quad (3.21e)$$

$$(X_1 \leftrightarrow \eta_8)^2 = \eta_8 X_1 - \eta X \quad (3.21f)$$

Solving for equation (3.19) in terms of the masses, we obtain the following results

$$\begin{aligned} \sqrt{a} &= 778 & \sqrt{d} &= 325 \\ \sqrt{b} &= -429 & \sqrt{e} &= 432 & \sqrt{g} &= -35. \\ \sqrt{c} &= 225 & \sqrt{f} &= 365 \end{aligned} \quad (3.22)$$

It is thus seen that the $(35, 35)^{405;8}$ reduced matrix element is very small compared to the others and can therefore be safely neglected. Doing this gives us the very well-known mass relation

$$K^* - K = \rho - \pi, \quad (3.23)$$

a result which was first obtained in SU(6) by Pais⁵⁶ and is in impressive agreement with experiment. Also from equation

(3.21e), we obtain the Schwinger mass relation⁵⁷.

$$(\omega - \rho)(\phi - \rho) = 4/3(K^* - \rho)(\omega + \phi - 2K^*) \quad (3.24)$$

also an SU(6) result. We see therefore from the above that U(6)@U(6) does not give us any new mass relations. The above results are not incompatible with the general Beg-Singh formula, but are only obtainable from it by imposing special conditions which are not dictated by the tensor analysis alone.

Of particular interest now are the reduced matrix elements α_1 and a . In the absence of any symmetry breaking, these are the multiplet masses for the baryons and the mesons respectively. From equation (3.12), α_1 is about 1320 MeV and from equation (3.22), the meson central mass is about 780 MeV. Thus the presence of the X^0 particle raises the central mass fairly from the SU(6) value of about 750 MeV to 780 MeV. Feeding these into the relation

$$g_{ps}^2/4\pi = 25/9(2\alpha_1/a)^2 g_v^2/4\pi, \quad (3.25)$$

where g_v is related to the rate for $\rho \rightarrow 2\pi$, we obtain using the Gell-Mann, Sharp and Wagner⁵⁸ result for $\rho \rightarrow 2\pi$, the gratifying result that

$$g_{ps}^2/4\pi \sim 15.9 \quad (3.26)$$

to be compared with the experimental value of 14.7. It is rather puzzling that in spite of the rather unsatisfactory

$U(6) \otimes U(6)$ result for g_A/g_V , the agreement with experiment should be so good in this strict $U(6) \otimes U(6)$ limit.

In conclusion, then, we propose that the meson mass operator for $U(6) \otimes U(6)$ should be

$$M^2 = M_{(1,1)}^{1;1} + M_{(35,1)+(1,35)}^{35;8} + M_{(35,35)}^{1;1} + M_{(35,35)}^{35;8} + M_{(35,35)}^{189;1} + M_{(35,35)}^{405;1} \quad (3.27)$$

showing that the spin-splitting is produced by the singlet-contributing terms of 189 and 405. Knowing now how the symmetry-breaking term transforms, we can write down equations (3.15) and (3.27) in the usual form by making use of the results obtained by Beg-Singh⁵³ for the various tensor operators which we reproduce here for completeness

$$\begin{aligned} M_{35}^8 &= a_1 + b_1 Y + c_1 [2S(S+1) - C_2^{(4)} + \frac{1}{4}Y^2] \\ M_{189}^1 &= a_2 + b_2 [2J(J+1) - C_2^{(3)}] \\ M_{405}^1 &= a_3 + b_3 [2J(J+1) + C_2^{(3)}] \\ M_{405}^8 &= a_4 + b_4 [2J(J+1) + C_2^{(3)}] + 21/8 [2S(S+1) - C_2^{(4)} + \frac{1}{4}Y^2] \\ &\quad + 3 [2I(I+1) - \frac{1}{2}Y^2 + 2N(N+1) - 2S(S+1)] \end{aligned} \quad (3.28)$$

where J , Y , I , S and N refer to the total spin, hypercharge, isotopic spin and the spins of the strange and non-strange quark respectively; $C_2^{(3)}$ and $C_2^{(4)}$ are the quadratic

Casimir operators of $SU(3)$ and $SU(4)$, and a_i, b_i and c_i depend only on the Casimir operators of $SU(6)$.

Substituting these in equations (3.15) and (3.27) and making use of the following identities which hold only for the baryons

$$\begin{aligned} 2J(J+1) - c_2^{(3)} &= -9/2 \\ 2S(S+1) - c_2^{(4)} + \frac{1}{4}Y^2 &= -8Y - 15/2 \\ I(I+1) - \frac{1}{4}Y^2 - N(N+1) + S(S+1) &= -Y + \frac{3}{4} \end{aligned} \quad (3.29)$$

we obtain for the baryons the formula

$$M = a + bJ(J+1) + cY + d[I(I+1) - \frac{1}{4}Y^2] \quad (3.30)$$

while for the mesons, we obtain the result

$$M^2 = a + bJ(J+1) + cC_2^{(3)} + d[2S(S+1) - C_2^{(4)} + \frac{1}{4}Y^2] + ef \quad (3.31)$$

where e in (3.31) is zero for all the particles except X_1 when it is 36.

In Table XII we collect together the quantum numbers of all the baryons and mesons with their corresponding Casimir operators.

Equation (3.30) now furnishes us with an explanation for the empirically known, but hitherto unexplained, fact that the mass formula for the baryon octet can be used with the same coefficients for the decuplet in broken $SU(3)$.

CHAPTER IV

NON-COMPACT GROUPS AND THE CLASSIFICATION OF HADRONS

1. Introduction

Until a year or two ago, only compact groups were used in classifying energy levels of baryons and mesons. However, several authors^{9,59} have lately proposed that the infinite-dimensional representations of some non-compact groups might possibly be used in classifying strongly interacting particles. It is then hoped that in analogy to the hydrogen atom¹⁰, all hadrons could be accommodated in infinite-dimensional representations, thereby forming a ladder in which all the hadrons lie on the rungs. So far the groups $U(6,6)$, $GL(6, \mathbb{C})$ and $U(6) \otimes U(6) \otimes O(3,1)$ have been considered. $U(6,6)$ and $GL(6, \mathbb{C})$ start with the maximal compact subgroups $U(6) \otimes U(6)$ and $U(6)$ respectively and quark-antiquarks (s-wave mesons) are added to these in a symmetrical manner. In this way, the maximum spin increases by one unit and also new $SU(3)$ representations are introduced. On the other hand $U(6) \otimes U(6) \otimes O(3,1)$ begins with a given representation of the maximal compact subgroup $U(6) \otimes U(6) \otimes O(3)$ and keeps exciting its constituent quarks into higher orbital states. Once again the maximum spin increases by one unit each time such an excitation takes place but no

higher $SU(3)$ representations are introduced.

Besides the well-known fact⁶⁰ that in some of these suggested schemes the mesons and baryons do not couple, it has also been recently stated that existing experimental evidence seems not to support all three possibilities⁶¹. In spite of these, we wish to speculate⁶² on the possibility of using yet another non-compact group $U(6,6) \supset U(6,6)$. This is the group which results when we relativistically boost the unitary representations⁶³ of the rest-symmetry $U(6,6)$ and has $U(6,6)_p$ as the little group and a chain of subgroups $GL(6, \mathbb{C})$ for 2-momentum processes; $U(3,3)$ for 3-momentum processes and $GL(3, \mathbb{C})$ for 4-momentum processes.

Apart from the little group $U(6,6)_p$ (appropriate to a particle with 4-momentum p_μ), there is another $U(6,6)$ subgroup called $U(6,6)_D$ by Delbourgo et al, which contains the Lorentz transformations. These two groups also have the distinguishing feature that $U(6,6)_p$ can be defined as the set of matrices which commute with $\not{p} \times \underline{\tau}_3$, where $\underline{\tau}$ are the Pauli matrices, while $U(6,6)_D$ is generated by $\gamma_C T^i$ and $\gamma_{nc} T^j$, with

$$\begin{aligned} \gamma_C &= 1, \quad \underline{\tau}, \gamma_0, \gamma_0 \underline{\sigma} \\ \gamma_{nc} &= \gamma_5, \gamma_5 \underline{\sigma}, i\gamma_0 \gamma_5, i\gamma_0 \gamma_5 \underline{\sigma} \\ \underline{\sigma} &= (\sigma_{23}, \sigma_{31}, \sigma_{12}), \quad \sigma_{\mu\nu} = i\gamma_\mu \gamma_\nu \quad \mu \neq \nu \end{aligned} \tag{4.1}$$

and T^j are the usual $SU(3)$ matrices. As is well-known, the Bargmann-Wigner equations (which intrinsically break the symmetry) will ensure a unitary norm even though the representations of $U(6,6)_D$ we start with are non-unitary. We shall therefore consider the infinite-dimensional representations of $U(6,6) \otimes U(6,6)$ as the tower (or ladder) symmetry and take as the "rung" symmetry the finite-dimensional (non-unitary) $U(6,6)_D$ and note that it is here acting as the maximal "compact" subgroup in analogy to the $GL(6, \mathbb{C})$ reduction to $U(6)$, say. It should be stressed that, although the group we are studying may or may not be the correct one, one of our main purposes here is to display some of the techniques that will make non-compact group representations more amenable to computation.

2. The $U(\nu, \nu) \otimes U(\nu, \nu)$ representations

In studying the representations of $U(6,6) \otimes U(6,6) \subset U(12,12)$, we shall, as already stated, carry out the reduction relative to the subgroup $U(6,6)_D$. This group has been chosen since it contains the compact rest-symmetry $U(6) \otimes U(6)$, which has been shown to be a good strong interaction symmetry, as a subgroup. In carrying out the reduction, therefore, we shall follow the procedure proposed by Dothan et al⁹ and used extensively by Delhourgo et al⁵⁹. The idea of the formalism is to construct an infinite ladder which creates an invariant

space in which the irreducible representation of the parent group acts, and on each rung of the ladder, to put the irreducible finite-dimensional representations of the maximal compact subgroup.

Although we are mainly interested in the $U(6,6) \otimes U(6,6)$ group, owing to the simplicity involved in working with the general group, we shall consider below some of the representations of $U(\nu, \nu) \otimes U(\nu, \nu)$. Following reference 9, we define creation and annihilation operators $\bar{a}_\alpha$ and a^α with $\alpha = 1, 2, \dots, 2\nu$ for fictitious boson quarks and another set of creation and annihilation operators $\bar{b}^\alpha$ and b_α for equally fictitious boson antiquarks. These obey the commutation relations

$$\begin{aligned} [\bar{a}_\alpha, a^\beta] &= \delta_\alpha^\beta, \\ [b_\alpha, \bar{b}^\beta] &= \delta_\alpha^\beta, \\ [b_\alpha, a^\beta] &= [b_\alpha, \bar{a}_\beta] = \dots = 0 \end{aligned} \quad (4.2)$$

where

$$a^\beta = (\bar{a}_\alpha)^\dagger (\gamma_0)^\beta_\alpha, \quad \bar{b}^\alpha = (b_\beta)^\dagger (\gamma_0)^\alpha_\beta \quad \text{with } \gamma_0 \text{ as defined}$$

in Chapter I. The generators of $U(\nu, \nu) \otimes U(\nu, \nu)$ are now given by $\frac{1}{2}[M_\alpha^\beta + N_\alpha^\beta]$ where

$$M_\alpha^\beta = \bar{b}^\beta c_\alpha + a^\beta \bar{a}_\alpha, \quad N_\alpha^\beta = \bar{b}^\beta \bar{a}_\alpha + a^\beta b_\alpha$$

and these satisfy the commutation relations

$$\begin{aligned}
[M_\alpha^\beta, M_\gamma^\delta] &= \delta_\alpha^\delta M_\gamma^\beta - \delta_\gamma^\beta M_\alpha^\delta \\
[M_\alpha^\beta, N_\gamma^\delta] &= \delta_\alpha^\delta N_\gamma^\beta - \delta_\gamma^\beta N_\alpha^\delta \\
[N_\alpha^\beta, N_\gamma^\delta] &= \delta_\alpha^\delta M_\gamma^\beta - \delta_\gamma^\beta M_\alpha^\delta
\end{aligned} \tag{4.3}$$

$$\text{with } (M_\alpha^\beta)^+ = (\gamma_o M \gamma_o)^\beta_\alpha, \quad (N_\alpha^\beta)^+ = (\gamma_o N \gamma_o)^\alpha_\beta$$

It is clear from the above relations that M_α^β are the generators of $U(6,6)_D$. To construct the irreducible representations of $U(\nu, \nu) \otimes U(\nu, \nu)$, it is feasible to fix on a lowest or "ground" level and apply the generators M_α^β and N_α^β to it repeatedly. Whilst N_α^β will lead to new levels, M_α^β will produce a mixing of states at each $U(\nu, \nu)_D$ rung. The representation then appears as a tower of finite-dimensional non-unitary representations of $U(\nu, \nu)_D$. With the vacuum state $|0\rangle$ defined as

$$a^\alpha |0\rangle = b_\alpha |0\rangle = 0, \tag{4.4}$$

it is seen that the operator N_α^α is projected out of the vacuum state into $(\bar{a}_\alpha \bar{b}^\alpha) |0\rangle$. Hence we choose the lowest state as

$$|f\rangle = \sum_n f_n (\bar{a} \cdot \bar{b})^n |0\rangle \tag{4.5}$$

$$\text{with } \langle 0 | (b \cdot a)^n (\bar{b} \cdot \bar{a})^n | 0 \rangle = (-1)^n n! \frac{(2\nu+n-1)!}{(2\nu-1)!}.$$

The constructions of the representations of $U(\nu, \nu)$ and $GL(\nu, C)$ have been discussed elsewhere and the generalisation to $U(\nu, \nu) \otimes U(\nu, \nu)$ follows closely on those lines.

We construct an irreducible representation from an infinite sequence of symmetrised, traceless tensors

$$\left\{ \left\{ \Psi(0) \quad \Psi(1), \dots, \Psi(1), \dots \right\} \right\}$$

where

$$\begin{aligned} \Psi &= \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} \\ &= \sum_n f_n(1) \left\{ \bar{b}^{\beta_1} \dots \bar{b}^{\beta_1} \bar{a}_{\alpha_1} \dots \bar{a}_{\alpha_{r+1}} - \text{traces} \right\} (\bar{b}^\gamma \bar{a}_\gamma)^n |0\rangle, \end{aligned} \quad (4.6)$$

$l = 0, 1, 2, \dots, 2\nu$ and r is the difference between the number of lower and upper indices. $M_\alpha^\beta, N_\alpha^\beta$ are then defined by the relations

$$\begin{aligned} M_\alpha^\beta \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} &= \sum_{j=1}^l \delta_{\alpha_j}^{\beta_j} \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots (j) \dots \beta_1} \\ &\quad - \sum_{i=1}^{r+1} \delta_{\alpha_i}^{\beta_i} \Psi_{\alpha \alpha_1 \dots (i) \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} + \frac{1}{2\nu} \delta_{\alpha}^{\beta} \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1}, \\ N_\alpha^\beta \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} &= A_1 \Psi_{\alpha \alpha_1 \dots \alpha_{r+1}}^{\beta \beta_1 \dots \beta_1} \\ &\quad + B_1 \left\{ \sum_j \delta_{\alpha_j}^{\beta_j} \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta \beta_1 \dots (j) \dots \beta_1} + \sum_i \delta_{\alpha_i}^{\beta_i} \Psi_{\alpha \alpha_1 \dots (i) \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} \right. \\ &\quad \left. - \frac{2}{\rho} \sum_{ij} \delta_{\alpha_i}^{\beta_j} \Psi_{\alpha \alpha_1 \dots (i) \dots \alpha_{r+1}}^{\beta \beta_1 \dots (j) \dots \beta_1} - \frac{2l+r}{\nu} \delta_{\alpha}^{\beta} \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} \right. \\ &\quad \left. - A_{l+1} \left\{ \sum_{ij} (\delta_{\alpha_j}^{\beta_j} \delta_{\alpha_i}^{\beta_i} - \frac{1}{\rho} \delta_{\alpha_j}^{\beta_i} \delta_{\alpha_i}^{\beta_j}) \Psi_{\alpha_1 \dots (i) \dots \alpha_{r+1}}^{\beta_1 \dots (j) \dots \beta_1} \right. \right. \\ &\quad \left. \left. - \frac{1}{\rho} \sum_{i \neq i'} (\delta_{\alpha_i}^{\beta_j} \delta_{\alpha_{i'}}^{\beta_i} + \delta_{\alpha_i}^{\beta_i} \delta_{\alpha_{i'}}^{\beta_j}) \Psi_{\alpha \alpha_1 \dots (i) \dots (i') \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} \right\} \right. \end{aligned} \quad (4.8)$$

$$\begin{aligned}
& \frac{1}{\rho_\ell} \sum_{i \neq j'} \left(\delta_{\alpha_i}^{\beta_j} \delta_{\alpha}^{\beta_{j'}} + \delta_{\alpha_i}^{\beta_{j'}} \delta_{\alpha}^{\beta_j} \right) \Psi_{\alpha_1 \dots (i) \dots \alpha_{r+1}}^{\beta_1 \dots (j) \dots (j') \dots \beta_1} \\
& + \frac{2}{\rho_\ell(\rho_\ell - 1)} \sum_{\substack{i' > i \\ j' > j}} \left(\delta_{\alpha_i}^{\beta_j} \delta_{\alpha_{i'}}^{\beta_{j'}} + \delta_{\alpha_i}^{\beta_{j'}} \delta_{\alpha_{i'}}^{\beta_j} \right) \Psi_{\alpha_1 \dots (i) \dots (i') \dots \alpha_{r+1}}^{\beta_1 \dots (j) \dots (j') \dots \beta_1} \Bigg\}
\end{aligned}$$

where the notation $\beta_1 \dots (j) \dots \beta_1$ indicates that β_j is missing from the sequence $\beta_1 \beta_2 \dots \beta_1$ and

$$A_1^2 = \frac{(2l+r+\nu)^2 - \lambda^2}{(2l+r+\nu)(2l+r+\nu+1)},$$

$$B_1 = \frac{\lambda}{2l+r+\nu}, \quad (4.9)$$

$$\rho_\ell = 2l+r+\nu-2$$

and λ is a real number defined below in the next section.

We demand also that A_1, B_1 be non-zero. Thus we have obtained a two-parameter set of representations of $U(\nu, \nu) \otimes U(\nu, \nu)$.

That they are unitary follows from noticing the invariance of the positive bilinear forms

$$\sum_1 \sum_{\alpha\beta} \frac{1}{1!(1+r)!} \left| \Psi_{\alpha_1 \dots \alpha_{r+1}}^{\beta_1 \dots \beta_1} \right|^2 \quad (4.10)$$

This invariance can be verified on using the generators as defined in equations (4.7) and (4.18).

3. The Eigenvalue problem

The reduction under $U(\nu, \nu)_D$ of each level in the $U(\nu, \nu) \otimes U(\nu, \nu)$ ladder is unique. In the $U(\nu, \nu) \otimes U(\nu, \nu)$ ladder, however, there exists an infinite number of distinct irreducible representations of particle states, repeated throughout the tower, the only way of distinguishing among the same particle state being in the assignment of different values of the λ , the Casimir operator. Thus the $U(6,6)_D$ rungs of say, the meson tower will be made up of the series

$$1 \oplus 143 \oplus 5940 \oplus \dots$$

Under $U(6) \otimes U(6)$, for example, the following decompositions occur:

$$\begin{aligned} 1 &= (1,1) \\ 143 &= (1,1) + (6, \bar{6}) + (\bar{6}, 6) + (35,1) + (1,35) \\ 5940 &= (1,1) + (6, \bar{6}) + (\bar{6}, 6) + (35,1) + (1,35) + (21, \bar{21}) \\ &\quad + (\bar{21}, 21) + (35,35) + (120, \bar{6}) + (6, \bar{120}) + (\bar{120}, 6) \\ &\quad + (\bar{6}, 120) + (405,1) + (1,405) \end{aligned}$$

The various $(1,1)$ singlet states, for instance, coming from the $U(6,6)_D$ irreducible representations 1, 143 and 5940 are distinct particle states and are only distinguished from each other by the different values of λ they take. We shall now determine the spectrum of λ , where for convenience we consider the meson tower. Let us construct the $U(\nu, \nu)_D$ irreducible

particle state of the $U(\nu, \nu) \otimes U(\nu, \nu)$ by demanding that M_α^β , N_α^α satisfy the equations

$$\begin{aligned} (M_\alpha^\beta - \frac{1}{2\nu} \delta_\alpha^\beta M_Y^\gamma) \Psi_{\alpha_1 \dots \alpha_1}^{\beta_1 \dots \beta_1} &= 0 \\ N_\alpha^\alpha \Psi_{\alpha_1 \dots \alpha_1}^{\beta_1 \dots \beta_1} &= \lambda \Psi_{\alpha_1 \dots \alpha_1}^{\beta_1 \dots \beta_1} \end{aligned} \quad (4.11)$$

Notice that for the special case where we consider the singlet of $U(\nu, \nu)_D$, $\beta_i = \alpha_i$. We are especially interested in knowing under what conditions, if any, the spectrum of λ is discrete. Applying equation (4.6) to equation (4.11), we obtain the recurrence relation

$$f_n(1) - \lambda f_{n+1}(1) - (n+2)(n+\nu+1)f_{n+2}(1) = 0 \quad (4.12)$$

By means of the function

$$y = \sum_n f_n(1) x^n,$$

Equation (4.12) is replaced by the equivalent differential equation

$$xy''(x) + \nu y'(x) - (x - \lambda)y(x) = 0 \quad (4.13)$$

To solve this, we make the substitution

$$y(x) = e^{-x} Z, \quad x = \frac{1}{2} \xi$$

Then equation (4.13) reduces to the confluent hypergeometric equation

$$\xi \frac{d^2 Z}{d\xi^2} + (\nu - \xi) \frac{dZ}{d\xi} - \frac{1}{2}(\nu - \lambda)Z = 0 \quad (4.14)$$

which has the solution⁶⁴

$$Z = {}_1F_1\left\{\frac{1}{2}(\nu - \lambda); \nu; \xi\right\} \quad \text{and hence}$$

$$y(x) = e^{-x} {}_1F_1\left\{\frac{1}{2}(\nu - \lambda); \nu; 2x\right\} \quad (4.15)$$

Note that only the above solution is allowed since ν is necessarily an integer and we are imposing the boundary condition that the solution be finite at $x = 0$. It now remains to see under what conditions the orthogonality relation

$$\int_0^{\infty} (\lambda' - \lambda) y_{\lambda'}^*(x) y_{\lambda}(x) x^{\nu-1} dx$$

$$= [y_{\lambda'}^* x^{\nu} y'_{\lambda}(x) - x^{\nu} y_{\lambda'}^*(x) y'_{\lambda}(x)]_0^{\infty}$$

$$= 0 \quad (4.16)$$

holds.

From the properties⁶⁴ of ${}_1F_1\left\{\frac{1}{2}(\nu - \lambda); \nu; 2x\right\}$, we see that this function increases with x for positive $\frac{1}{2}(\nu - \lambda)$. However, if $\frac{1}{2}(\nu - \lambda)$ is zero or negative, then it reduces to a finite polynomial which tends to zero for large x . Hence for condition (4.16) to be fulfilled, it is sufficient that $\frac{1}{2}(\nu - \lambda)$ be zero or negative, i.e.

$$\frac{1}{2}(\nu - \lambda) = -n \quad \text{for } n=0, 1, 2, \dots \quad (4.17)$$

Thus the discrete spectrum of λ is given by the relation

$$\lambda = \nu + 2n. \quad \text{Now for } \frac{1}{2}(\nu - \lambda) = -n$$

$$Z = {}_1F_1(-n; \nu; 2x)$$

and then ${}_1F_1(-n; \nu; 2x)$ is related to the generalised Laguerre polynomial by the equation

$$L_n^{\nu-1}(2x) = \binom{n+\nu-1}{n} {}_1F_1(-n; \nu; 2x) \quad (4.18)$$

i.e.

$${}_1F_1(-n; \nu; 2x) = \sum_{m=0}^n \binom{n}{m} \left\{ m! \binom{m+\nu-1}{\nu-1} \right\}^{-1} (-2)^m x^m$$

The orthogonality condition now becomes

$$\int_0^\infty x^{\nu-1} y_{\lambda'}^*(x) y_\lambda(x) dx = N^2 \delta_{\lambda\lambda'} \quad (4.19)$$

where

$$N^2 = 2^{-\nu} \frac{1}{(\Gamma(\nu))^2} \left(\frac{\Gamma(\nu+n)}{\Gamma(1+n)} \right)^{3/2} \left(\frac{\Gamma(\nu+n')}{\Gamma(1+n')} \right)^{3/2} \quad (4.19a)$$

and $n' = \frac{1}{2}(\lambda' - \nu)$.

Hence the required solution is

$$\begin{aligned} y(x) &= e^{-x} {}_1F_1(-n; \nu; 2x) \\ &= \sum_{p=0}^{\infty} \sum_{m=0}^n (-1)^{m+p} 2^m \binom{n}{m} \frac{1}{p!} [m! \binom{m+\nu-1}{\nu-1}]^{-1} x^{m+p} \end{aligned} \quad (4.20)$$

Thus with the condition $0 \leq x < \infty$, we obtain a complete set of solutions by taking $\lambda = \nu + 2n$, implying that the spectrum of λ is determined. The coefficients f_m can now, in principle, be evaluated. These are, however, complicated as can be seen above and so for purposes of illustration, we put $n=0$ (corresponding to $\lambda = \nu$) in equation (4.20). In this very special case, we have

$$f_m = (-1)^m 1/m!$$

and hence from equation (4.5) the lowest state is now defined as

$$|if\rangle = \sum_m (-1)^m 1/m! (\bar{a}, \bar{b})^m |0\rangle$$

which clearly betrays the well-known fact⁶⁵ that $U(6,6)$ symmetry does not act on physical particle states but rather on the purely formal indices. The natural way to employ the finite-dimensional (and therefore non-unitary) representations of $U(6,6)_D$ is to make use of the Bargmann-Wigner equations which serve to restrict the physical states to positive definite sectors of the Hilbert space. Under this projection procedure the above state will become normalisable with a discrete spectrum λ , a pleasant result physically.

Equation (4.11) can now be used for the construction of higher meson states. For example, with this as the lowest state $\phi(\lambda)$, the relation

$$N_Y^\gamma \phi_\alpha^\beta(\lambda) = (N_\alpha^\beta - \frac{\lambda}{2\nu} \phi_\alpha^\beta) \phi(\lambda) \quad (4.21)$$

will take us to the next state, etc.

4. The trilinear coupling

The general prescription for coupling of towers has been discussed in reference⁶⁰ and we shall in this section employ this technique to determine the first few coupling coefficients

of a $U(\nu, \nu) \otimes U(\nu, \nu)$ tower, assuming that coupling exists in general. We shall also consider only the coupling of two baryon towers to a meson one. The general form of such a trilinear invariant will be

$$I = \sum_{1_1 1_2 1_3} \sum_{\xi} [1_1 1_2 1_3]_{\xi} (\bar{\Psi}(1_1) \bar{\Psi}(1_2) \phi(1_3))_{\xi} \quad (4.22)$$

where $(\bar{\Psi}(1_1) \bar{\Psi}(1_2) \phi(1_3))$ denotes a $U(\nu, \nu)_D$ invariant term, ξ labels the distinct invariants which may be obtained with the same three tensors and the coefficients $[1_1 1_2 1_3]$ must be determined so as to make the sum a $U(\nu, \nu) \otimes U(\nu, \nu)$ invariant. To achieve this, is to impose the condition

$$N_{\alpha}^{\beta} I = 0 \quad (4.23)$$

and from this we obtain a number of recursion relations for the various coupling coefficients. The meson-baryon invariant vertex function for any determination of the recoupling coefficients follows from equation (4.22). Applying equation (4.23) to equation (4.22) and equating appropriate coefficients leads to a number of equations. Since in actual practice, only the first few terms are of interest, we characteristically equate the coefficients of

$$\bar{\Psi}_1 \gamma_1 \gamma_2^{\beta} \phi, \quad \bar{\Psi}_1 \gamma_1^{\beta} \gamma_2^{\beta} \phi, \quad \bar{\Psi}_1 \gamma_1^{\beta} \gamma_2^{\beta} \phi^{\alpha}_1 \quad \text{and} \quad \bar{\Psi}_1 \gamma_1 \gamma_2 \gamma_3 \phi^{\beta}_{\alpha}$$

respectively to zero and obtain the following equations

$$\frac{3(\lambda_1 + \lambda_2)}{3 + \nu} [000] - \left\{ \frac{\nu^2 - \lambda_3^2}{\nu(\nu+1)} \right\}^{\frac{1}{2}} [001] = 0, \quad (4.24)$$

$$\begin{aligned} \frac{3}{3 + \nu} \left\{ \frac{(3 + \nu)^2 - \lambda_1^2}{(3 + \nu)(4 + \nu)} \right\}^{\frac{1}{2}} [101] + \left\{ \frac{(3 + \nu)^2 - \lambda_2^2}{(3 + \nu)(4 + \nu)} \right\}^{\frac{1}{2}} [011] \Big] + \frac{(\lambda_1 + \lambda_2)}{3 + \nu} [001] \\ = 2 \left\{ \frac{(2 + \nu)^2 - \lambda_3^2}{(2 + \nu)(3 + \nu)} \right\}^{\frac{1}{2}} [002], \end{aligned} \quad (4.25)$$

$$\begin{aligned} \left\{ \frac{(3 + \nu)^2 - \lambda_1^2}{(3 + \nu)(4 + \nu)} \right\}^{\frac{1}{2}} [101] + \left\{ \frac{(3 + \nu)^2 - \lambda_2^2}{(3 + \nu)(4 + \nu)} \right\}^{\frac{1}{2}} [011] + \frac{2\lambda_3}{\nu(2 + \nu)} [001] \\ + \frac{4}{(1 + \nu)(2 + \nu)} \left\{ \frac{(2 + \nu)^2 - \lambda_3^2}{(2 + \nu)(3 + \nu)} \right\}^{\frac{1}{2}} [002] - \left\{ \frac{\nu^2 - \lambda_3^2}{\nu(\nu+1)} \right\}^{\frac{1}{2}} [000] \end{aligned} \quad (4.26)$$

Following Bisiacchi and Fronsdal⁶⁶, we choose $[001]$ arbitrarily and normalise it to unity. We then have

$$[000] = \frac{3 + \nu}{3(\lambda_1 + \lambda_2)} \left\{ \frac{\nu^2 - \lambda_3^2}{\nu(\nu+1)} \right\}^{\frac{1}{2}}, \quad (4.27)$$

$$\text{and } \left\{ \frac{(2 + \nu)^2 - \lambda_3^2}{(2 + \nu)(3 + \nu)} \right\}^{\frac{1}{2}} [002] = \frac{1}{2\nu(\nu+4)(\nu^2 + 2\nu+3)(\lambda_1 + \lambda_2)} \quad (4.28)$$

$$\times [(\lambda_1 + \lambda_2)^2 \nu(1 + \nu)(2 + \nu) - 6(\lambda_1 + \lambda_2)\lambda_3(1 + \nu) + (2 + \nu)(3 + \nu)(\nu^2 - \lambda_3^2)]$$

A similar method can, of course, be used to derive the recoupling coefficients for the meson-meson vertex tower.

CHAPTER V

REPRESENTATION MIXING IN U(6,6)

1. Introduction

In this final Chapter we shall report on some investigations we have carried out, following the suggestion of Dashen and Gell-Mann¹² that, probably the low-lying baryon states do not belong to any pure representation. In fact, the recent applications of the algebra of current components to weak and electromagnetic interactions indicate that the assumption that the expectation values of the commutators of the time components of the vector and axial vector currents between stable octet baryons are saturated by the octet and decuplet baryons is rather unsatisfactory since it leads to some predictions, especially the G_A/G_V ratio, which are not in good agreement with experiment. It was in an effort to remedy the situation that it was suggested by Dashen and Gell-Mann that there is presumably the possibility of some mixing of the stable baryons with the other low-lying baryon states. This is motivated physically by the observation that the spin $3/2^+$ decuplet, spin $\frac{1}{2}^+$ singlet and octet and the spin $3/2^-$ octet make dominant contributions to the Adler-Weisberger sum rule

$$1 - (G_V/G_A)^2 = \frac{2}{\pi} \left(\frac{M}{g_{\pi NN}} \right)^2 \int_0^\infty \frac{q d\nu}{\nu} [\sigma_\nu^{\text{tot}}(\pi^+ p) - \sigma_\nu^{\text{tot}}(\pi^- p)], \quad (5.1)$$

where $\sigma^{\text{tot}}(\pi+p)$ is the total cross-section for elastic $\pi+p$ scattering, ν is the pion energy in the lab.frame, q is the magnitude of the three-momentum of the pion in the lab.system and M and μ are the nucleon and pion masses respectively.

Since the above sum rule converges fairly rapidly, much of the mixing will be with well-defined resonances. We shall, in the following, study the consequences of this representation mixing conjecture⁶⁷ in $U(6,6)$, first by assuming that the stable octet baryons are in the $(220, L=0)$ and $(364, L=0)$ mixed representation and secondly (because the first suggestion does not give encouraging results, as we shall demonstrate presently), that they are in the $(220, L=1)$ and $(364, L=0)$ state. In both cases, we shall, following previous authors⁶⁸ who considered the problem in $U(3) \otimes U(3)$, assume that the decuplet is in the pure $(364, L=0)$ multiplet. Again, owing to the fact that the results of the 364 representation are well-known, we shall mainly be interested in the calculations involving the 220.

2. The $(220, L=0)$ and $(364, L=0)$ representation mixing

The $(220, L=0)$ contains an octet of spin $\frac{1}{2}^+$ so that, apart from the stable octets belonging to the 364, there should exist another set of positive parity spin $\frac{1}{2}$ octet. We believe therefore that the recent discoveries of the resonances⁶⁹ like $N^*(1450, \frac{1}{2}^+)$ and $Y_o^*(1663, \frac{1}{2}^+)$ serve as an incentive to

the consideration of the $(220, L=0)$ representation. Suppose that the helicity $\frac{1}{2}$ state of the octet baryons of 220 and 364 mix in such a way that the baryon wave function can be written as

$$|B\rangle = \sin\alpha |220\rangle + \cos\alpha |364\rangle \quad (5.2)$$

where α is the mixing angle. It is now a simple matter to calculate the necessary theoretical predictions resulting from such an assumption.

The 220 fully anti-symmetric tensor $\Psi[A_1 A_2 A_3]$, leaving out the singlet contribution, may be written as

$$\begin{aligned} \Psi_{[A_1 A_2 A_3]}^{(p)} = \frac{1}{6m} \sqrt{\frac{1}{6}} [& (\epsilon_{r_3 r_1}^s N_{[\alpha_1 \alpha_2] \alpha_3, r_2}^s - \epsilon_{r_2 r_3}^s N_{[\alpha_1 \alpha_2] \alpha_3, r_1}^s) \\ & + \text{cyclic in } A_1 A_2 A_3], \end{aligned} \quad (5.3)$$

where

$$N_{[\alpha_1 \alpha_2] \alpha_3, r_2}^s(p) = \left(\frac{(\not{p} + m) \gamma_5 C}{2m} \right)_{\alpha_1 \alpha_2} N_{\alpha_3 r_2}^s(p) \quad (5.3a)$$

Making use of the mixed symmetry properties

$$\begin{aligned} N_{[\alpha_1 \alpha_2] \alpha_3}^{+N} N_{[\alpha_2 \alpha_3] \alpha_1}^{+N} N_{[\alpha_3 \alpha_1] \alpha_2}^{+N} &= 0 \\ \epsilon_{r_1 r_2}^s N_{r_3}^{+N} + \epsilon_{r_2 r_3}^s N_{r_1}^{+N} + \epsilon_{r_3 r_1}^s N_{r_2}^{+N} &= 0, \end{aligned} \quad (5.4)$$

equation (5.3) reduces to the convenient form

$$\begin{aligned} \Psi_{[A_1 A_2 A_3]}^{(p)} = \frac{1}{2m} \sqrt{\frac{1}{6}} [& [(\not{p} + m) \gamma_5 C]_{\alpha_2 \alpha_3} \epsilon_{r_1 r_2}^s N_{\alpha_1 r_2}^s(p) \\ & - [(\not{p} + m) \gamma_5 C]_{\alpha_1 \alpha_2} \epsilon_{r_2 r_3}^s N_{\alpha_3 r_1}^s(p)] \end{aligned} \quad (5.5)$$

From this we can now write down the currents which interest us.

With the definition

$$J_D^C = \bar{\Psi}^{AA_2C} \Psi_{AA_2D} = (\gamma_R^{Tj})_D^C J_R^j,$$

where

$$J_R^j = \bar{\Psi}_{(p')}^{AA_2A_3} (\gamma_R^{Tj})_{A_1}^{A_2} \Psi_{A_1A_2A_3}^{(p)}, \quad (5.6)$$

and $\gamma_R = 1, \gamma_\mu, \gamma_5, i\gamma_\mu\gamma_5, \sigma_{\mu\nu}$,

we obtain, on using the relation $P_\mu^2 \gamma_\mu = r_\mu + 2mP_\mu$

(notations are as defined in Chapter II, Section 2.)

$$J_\mu^j = \frac{1}{12m^2} [\text{Tr}(T_j) \left\{ (\bar{N}r_\mu N)_S + 2mP_\mu (\bar{N}N)_S \right\} \\ + 2mP_\mu (\bar{N}N)_F - (\bar{N}r_\mu N)_D], \quad (5.7)$$

$$J_{\mu 5}^j = \frac{P^2}{12m^2} [\text{Tr}(T_j) (\bar{N}i\gamma_\mu\gamma_5 N)_S - (\bar{N}i\gamma_\mu\gamma_5 N)_D]. \quad (5.8)$$

From equations (5.2), (5.7) and (5.8) and the U(6,6) results

for the 364, we get the following relations

$$\frac{D}{F} = \frac{1}{2}(3 - \tan^2 \alpha) \quad (5.9)$$

$$\frac{G_A}{G_V} = \frac{5 - \tan^2 \alpha}{3 + \tan^2 \alpha} \quad (5.10)$$

From equations (5.9) and (5.10), we obtain the result

$$G_A/G_V = \frac{D + F}{3F - D} \quad (5.11)$$

in agreement with the U(3)@U(3) result obtained by Cabibbo and

Ruegg⁶⁸. If we take the experimental G_A/G_V result of

$G_A/G_V \simeq 1.2$ and substitute it in Eq.(5.11), we obtain the result $\frac{D}{F} \simeq 1.18$, which is bad compared with the experimental result of $D/F \simeq 1.75$.

3. The (220, L=1) and (364, L=0) representation mixing

The bad prediction in Section 2 (Eq.5.11) now prompts us to consider other possible states. We shall now assume that the mixing occurs between the octets of the (364, L=0) and the (220, L=1) states, thereby introducing automatically the negative parity resonance states such as $Y^*(1405, \frac{1}{2}^-)$, $N^*(1512, \frac{3}{2}^-)$ and $Y^*(1520, \frac{3}{2}^-)$. We shall further assume that the positive helicity state of the stable octet baryons is given by

$$|B, J=\frac{1}{2}\rangle = \cos\theta |364, L=0\rangle + \sin\theta |220, L=1\rangle, \quad (5.12)$$

while the decuplet, as already stated, is in the pure (364, L=0) multiplet. At first sight assumption (5.12) seems to be inconsistent, since we know that the proton has a definite parity. However, we usually refer to the intrinsic parity of the proton, i.e. in its rest frame. A proton moving with any definite momentum, finite or infinite, does not have a definite parity since the momentum changes sign under space inversion⁷⁰.

The baryon (220, L=1) kinetic supermultiplet theory of Gatto et al⁷¹ described by the tensor product of the kinetic tensor and of the irreducible representation 220 will have the

$SU(3) \otimes SU(2)_W$ content of

$$(8, \frac{1}{2}) + (8, 3/2) + (1, \frac{1}{2}) + (1, 3/2) + (1, 5/2).$$

We are only interested in the $(8, \frac{1}{2})$ part. Defining the wave-function for the $(220, L=1)$ to be

$$\begin{aligned} \Psi_{ABCD}^{(p)E} = \frac{g}{6m^3} \sqrt{\frac{1}{6}} \Big\{ & [(\not{p} + m)\gamma_5 C]_{\alpha\beta} [(\not{p}_\mu + m\gamma_\mu)\gamma_5]^\rho (\epsilon_{cas} N_{\rho b}^S + \epsilon_{cbs} N_{\rho a}^S) (\gamma_\mu)_\delta^\epsilon \delta_d^e \\ & + [(\not{p} + m)\gamma_5 C]_{\beta\gamma} [(\not{p}_\mu + m\gamma_\mu)\gamma_5]^\rho (\epsilon_{abs} N_{\rho c}^S + \epsilon_{acs} N_{\rho b}^S) (\gamma_\mu)_\delta^\epsilon \delta_d^e \\ & + [(\not{p} + m)\gamma_5 C]_{\gamma\alpha} [(\not{p}_\mu + m\gamma_\mu)\gamma_5]^\rho (\epsilon_{bcs} N_{\rho a}^S + \epsilon_{bas} N_{\rho c}^S) (\gamma_\mu)_\delta^\epsilon \delta_d^e \Big\} \quad (5.13) \end{aligned}$$

where $A = (\alpha, a)$, $B = (\beta, b)$, ... with $\alpha = 1, 2, 3, 4$, $a = 1, 2, 3$,

and g is a normalisation constant.

Using Eq.(5.4), Eq.(5.13) reduces to the form

$$\begin{aligned} \Psi_{ABCD}^E = \frac{g}{2m^3} \sqrt{\frac{1}{6}} \Big\{ & [(\not{p} + m)\gamma_5 C]_{\beta\gamma} [(\not{p}_\mu + m\gamma_\mu)\gamma_5]^\rho \epsilon_{abs} N_{\rho c}^S(p) (\gamma_\mu)_\delta^\epsilon \delta_d^e \\ & - [(\not{p} + m)\gamma_5 C]_{\alpha\beta} [(\not{p}_\mu + m\gamma_\mu)\gamma_5]^\rho \epsilon_{bcs} N_{\rho a}^S (\gamma_\mu)_\delta^\epsilon \delta_d^e \Big\} \quad (5.14) \end{aligned}$$

Hence, as in Section 2, we can now write down the various

currents from which the D/F and G_A/G_V results are obtained.

Notice, however, that here we have, in general, 16 amplitudes:

$$\begin{aligned} \bar{\Psi}_{ABCD}^E (\gamma_{Tj}^R)_A^F \Psi_{FBCD}^E, \quad \bar{\Psi}_{ABCD}^E (\gamma_{Tj}^R)_D^F \Psi_{ABCF}^E \\ \bar{\Psi}_{ABCD}^E (\gamma_{Tj}^R)_F^E \Psi_{ABCD}^F, \quad \bar{\Psi}_{ABCD}^E (\gamma_{Tj}^R)_A^F \Psi_{EBCD}^E \end{aligned}$$

$$\begin{aligned}
\bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^E{}_F \Psi^{F}{}_{EBCD} &, \quad \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_B \Psi^{E}{}_{EFCD} \\
\bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_D \Psi^{E}{}_{EBCF} &, \quad \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_A \Psi^{E}{}_{EBDC} \\
\bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_B \Psi^{E}{}_{EFDC} &, \quad \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_C \Psi^{E}{}_{EBDF} \\
\bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_D \Psi^{E}{}_{EBFC} &, \quad \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^E{}_F \Psi^{F}{}_{EBDC} \\
\bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_A \Psi^{E}{}_{FBDC} &, \quad \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_C \Psi^{E}{}_{ABDF} \\
\bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^F{}_D \Psi^{E}{}_{ABFC} &, \quad \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^E{}_F \Psi^{F}{}_{ABDC}, \quad (5.15)
\end{aligned}$$

Some of which are related by charge conjugation invariance.

It can be shown, however, that all these produce the same D/F and G_A/G_V ratios. This is not surprising since we know that at infinite momentum for which all the Adler-Weisberger type of results are obtained, chiral $U(3) \otimes U(3)$ reduces to the collinear $U(3) \otimes U(3)$, a subgroup of $U(6,6)^{12,72}$. Picking out one particular amplitude, say

$$J_{\mu}^{Rj} = \bar{\Psi}^{ABCD}(\gamma_{Tj}^R)^{A'}{}_A \Psi^{E}{}_{A'BCD}(p)$$

we obtain the relevant currents as follows :

$$\begin{aligned}
J_{\mu}^{jj} = \frac{g^2}{2m^6} \Bigg\{ & 2\text{Tr}(T_j) \left[\frac{1}{4}(P^2 - q^2)(\bar{N}r_{\mu}N)_S - \frac{1}{2}mP_{\mu}(3P^2 + q^2)(\bar{N}N)_S \right] \\
& - \frac{1}{2}(P^2 - q^2)(\bar{N}r_{\mu}N)_D - \frac{1}{2} \cdot 2mP_{\mu}(3P^2 + q^2)(\bar{N}N)_F \Bigg\},
\end{aligned}$$

$$\begin{aligned}
J_{\mu 5}^j &= \frac{g^2}{2m^6} \left\{ \text{Tr}(T_j) P^2 \left[\frac{1}{2} (P^2 - q^2) (\bar{N} i \gamma_\mu \gamma_5 N)_S - 4mi q_\mu (\bar{N} \gamma_5 N)_S \right] \right. \\
&\quad \left. + 4mi P^2 q_\mu (\bar{N} \gamma_5 N)_D - \frac{1}{2} P^2 (P^2 - q^2) (\bar{N} i \gamma_\mu \gamma_5 N)_D \right\} \\
J_{\mu \nu}^j &= \frac{g^2}{2m^6} \left\{ 2P^2 \text{Tr}(T_j) \left[\frac{1}{4} (P^2 + 3q^2) (\bar{N} \sigma_{\mu\nu} N)_S - 2mi (\bar{N} (q_\nu \gamma_\mu - q_\mu \gamma_\nu) N)_S \right] \right. \\
&\quad + 4mi P^2 (\bar{N} (q_\nu \gamma_\mu - q_\mu \gamma_\nu) N)_D - \frac{P^2}{2} (P^2 + 3q^2) (\bar{N} \sigma_{\mu\nu} N)_D \\
&\quad \left. - i(3P^2 + q^2) (P_\mu q_\nu - q_\mu P_\nu) (\bar{N} N)_F \right\} \quad (5.16)
\end{aligned}$$

From these and Eq.(5.12), we obtain the relations

$$\frac{D}{F} = \frac{2 \cos^2 \theta + 1}{2 \cos^2 \theta}, \quad \frac{G_A}{G_V} = \frac{1}{3} (4 \cos^2 \theta + 1), \quad (5.17)$$

and since the decuplet is pure,

$$G^* = \frac{4}{3} \cos \theta,$$

where G^* is the strength of the axial vector transition between the nucleon and the $N^*(1238)$. From Eq.(5.17), follows the relation

$$\frac{G_A}{G_V} = \frac{1}{3} \frac{D+F}{D-F} \quad (5.18)$$

first obtained by Gatto et al⁶⁸ using a different approach .

It is found that if we fix the mixing angle⁷³ at $\theta = 37^\circ$,

we have the results

$$G_A/G_V = 1.18, \quad D/F = 1.78, \quad \text{and} \quad G^* = 1.05, \quad (5.19)$$

to be compared with the experimental values of

$$G_A/G_V = 1.18 \pm 0.02, \quad D/F = 1.75 \pm 0.03 \quad \text{and} \quad G^* \sim 1.$$

4. The magnetic moment ratio

The consistent impressive agreement of the results in Eq.(5.19) with experiment encourages us to take this particular mixing a step further. It is well-known that the pure 364 representation predicts a magnetic moment ratio for the proton and neutron which is in excellent agreement with experiment. It may probably be that we have therefore to pay a price in this case and it will be interesting to see how the representation mixing now affects this result.

The total effective vector current (on the mass shell) for the (220, L=1) is defined as

$$J_{\mu, \text{tot}}^j = j_{\mu}^j + \frac{i}{\mu} q_{\nu} j_{\nu\mu}^j, \quad (5.20)$$

so that we have

$$\begin{aligned} J_{\mu, \text{tot}}^j &= 2(3P^2 + q^2) \frac{P_{\mu}}{2m} (1 + \frac{q^2}{2m_{\mu}^2}) (\bar{N}N)_F \\ &\quad + \frac{1}{2}(P^2 - q^2) (1 + \frac{2m}{\mu}) (\bar{N}r_{\mu} N)_D \end{aligned} \quad (5.21)$$

Assuming, as in the 364 case, that the photon couples with

the baryon through an effective gauge-invariant "interaction"

$\bar{\psi}(q^2) F_{\mu\nu} M_{\mu\nu}^u$, where $M_{\mu\nu}^u$ is the U-scalar meson combination ($M^u = M_3 + \frac{1}{\sqrt{3}} M_8$), we obtain for the electromagnetic form factors, the expressions

$$J_{\mu}^C = \frac{P_{\mu}}{2m} (1 + \frac{q^2}{2m_{\mu}^2}) (\bar{N}N)_F F(q^2) \quad (5.22)$$

$$J_{\mu}^M = \frac{1}{m^2} \left(1 + \frac{2m}{\mu}\right) (\bar{N} r_{\mu} N)_D F(q^2), \quad (5.23)$$

where $F(q^2)$ is proportional to $f(q^2)/q^2 - \mu^2$.

From Eq.(5.23) and the 364 result, the ratio of the proton and neutron magnetic moments is predicted to be

$$\frac{\mu_P}{\mu_n} = - \frac{1 + (\cancel{3} - 1) \cos^2 \theta}{2 + 2(\kappa - 1) \cos^2 \theta}, \quad (5.24)$$

where κ is the ratio of the reduced matrix elements corresponding to the 220 and 364. We thus have one extra free adjustable parameter κ , which we will fit below.

Denoting now, by μ^* the N^*-N transition magnetic moment, we have the result

$$\frac{\mu^*}{\mu_P} = \frac{2\sqrt{2}\kappa \cos \theta}{(3\kappa - 1) \cos^2 \theta + 1} \quad (5.25)$$

Comparing Eq.(5.25) with the latest analysis of photo-production data, using the mixing angle of $\theta = 37^\circ$, it is seen that the recent data of ⁷⁴

$$\mu^* = (1.25 \pm 0.02) (\mu^*)_{U(6,6)} \quad (5.26)$$

where

$$(\mu^*)_{U(6,6)} = \frac{2\sqrt{2}}{3} \mu_P,$$

enables us to obtain a very close result if $\kappa = 3$.

In this case, Eq.(5.25) becomes

$$\mu^* = (1.18) (\mu^*)_{U(6,6)}$$

and Eq. (5.24) reduces to

$$\frac{\mu_p}{\mu_n} = - \frac{8 \cos^2 \theta + 1}{4 \cos^2 \theta + 2}, \quad (5.27)$$

in agreement with the result reported in reference ⁷⁵ obtained by assuming the hypothesis of partially conserved tensor current. With the known mixing angle, Eq.(5.27) then becomes

$$\frac{\mu_p}{\mu_n} = -1.34,$$

to be compared with the present available experimental result of -1.46. Thus it is noticed that the representation mixing of (220, L=1) and (364, L=0) does not reduce the ratio of the magnetic moments in any significant way.

TABLE B: $M^{(27)}$ and $M^{(8_D)}$ amplitudes deduced from $U(6,6)$;

$$x = s/l_m^2, \quad y = t/l_m^2, \quad z = u/l_m^2$$

$\begin{smallmatrix} I \\ F \end{smallmatrix}$	27	8_D
F_1	$(\frac{9}{2}A - \frac{8}{9}B)(1-y)^2$ $+ \frac{1}{18}B[5(1+y)(x-z)+2(1-y)]$ $+ (y \leftrightarrow z)$	$(\frac{9}{2}A + \frac{1}{2}B)(1-y)^2$ $+ \frac{1}{4}B(1-y) + (y \leftrightarrow z)$
F_2	$(\frac{9}{4}A + \frac{7}{9}B)(1-y)^2$ $+ \frac{1}{36}B[(7+3y)(x-z)-4(1-y)]$ $- (y \leftrightarrow z)$	$(\frac{9}{4}A - \frac{1}{8}B)(1-y)^2$ $+ \frac{1}{8}B(x-z-2)(1-y) - (y \leftrightarrow z)$
F_3	$-\frac{1}{9}B(1-y) + (y \leftrightarrow z)$	$-\frac{1}{4}B(1-y) + (y \leftrightarrow z)$
F_4	$(\frac{9}{4}A + \frac{2}{3}B)(1-y)^2$ $+ \frac{1}{36}B[(3+7y)(x-z)+$ $4(1-y)] - (y \leftrightarrow z)$	$(\frac{9}{4}A - \frac{1}{8}B)(1-y)^2 +$ $+ \frac{1}{8}B(2-x+z)(1-y) - (y \leftrightarrow z)$
F_5	$\frac{1}{9}B(1-y)(3+2y-2z)$ $+ (y \leftrightarrow z)$	$\frac{1}{4}B(1-y)(3+2y-2z)$ $+ (y \leftrightarrow z)$

TABLE Ib : $M^{(1)}$ amplitudes deduced from $U(6,6)$;

$$x = s/l_m^2, \quad y = t/l_m^2, \quad z = u/l_m^2$$

$\begin{smallmatrix} I \\ R \end{smallmatrix}$	1	
F_1	$\begin{aligned} & \left(\frac{2}{2}A + 2B\right)(1-y)^2 \\ & + \frac{1}{2}B[(1+y)(x-z) - 2(1-y)] \\ & + (y \longleftrightarrow z) \end{aligned}$	
F_2	$\begin{aligned} & \left(\frac{2}{4}A - B\right)(1-y)^2 \\ & + \frac{1}{4}B[(-1+3y)(x-z) + 4(1-y)] \\ & - (y \longleftrightarrow z) \end{aligned}$	
F_3	$B(1-y) + (y \longleftrightarrow z)$	
F_4	$\begin{aligned} & \frac{2}{4}A(1-y)^2 \\ & + \frac{1}{4}B[(3-y)(x-z) - 4(1-y)] \\ & - (y \longleftrightarrow z) \end{aligned}$	
F_5	$\begin{aligned} & -B(1-y)(3+2y-2z) \\ & + (y \longleftrightarrow z) \end{aligned}$	

TABLE IIa : $M^{(\bar{1}\bar{0})}$ and $M^{(10)}$ amplitudes deduced from $U(6,6)$;

$$x=s/lm^2, y=t/lm^2, z=u/lm^2$$

F_i $\begin{smallmatrix} I \\ \hline \end{smallmatrix}$	$\bar{1}\bar{0}$	10
F_1	$(\frac{9}{2}A + \frac{2}{3}B)(1-y)^2$ $+ \frac{1}{6}B[(1+y)(x-z) + 4(1-y)]$ $- (y \leftrightarrow z)$	$(\frac{9}{2}A - \frac{5}{3}B)(1-y)^2$ $+ \frac{1}{6}B[-(1+y)(x-z) - 10(1-y)]$ $- (y \leftrightarrow z)$
F_2	$(\frac{9}{4}A + \frac{1}{6}B)(1-y)^2$ $+ \frac{1}{12}B[(5-3y)(x-z) - 8(1-y)]$ $+ (y \leftrightarrow z)$	$(\frac{9}{4}A + \frac{5}{6}B)(1-y)^2$ $+ \frac{1}{12}B[(1-3y)(x-z) - 4(1-y)]$ $+ (y \leftrightarrow z)$
F_3	$-\frac{2}{3}B(1-y) - (y \leftrightarrow z)$	$-\frac{1}{3}B(1-y) - (y \leftrightarrow z)$
F_4	$(\frac{9}{4}A - \frac{1}{2}B)(1-y)^2$ $+ \frac{1}{12}B[(-3+5y)(x-z) + 8(1-y)]$ $+ (y \leftrightarrow z)$	$(\frac{9}{4}A + \frac{1}{2}B)(1-y)^2$ $+ \frac{1}{12}B[(-3+y)(x-z) + 4(1-y)]$ $+ (y \leftrightarrow z)$
F_5	$\frac{2}{3}B(1-y)(3+2y-2z)$ $- (y \leftrightarrow z)$	$\frac{1}{3}B(1-y)(3+2y-2z)$ $- (y \leftrightarrow z)$

(8_F)
TABLE IIb : M amplitudes deduced from U(6,6);

$$x=s/l_m^2, \quad y=t/l_m^2, \quad z=u/l_m^2$$

I F _i	8 _F	
F ₁	$\left(\frac{9}{2}A + \frac{5}{6}B\right)(1-y)^2$ $+ \frac{1}{12}B[4(1+y)(x-z) - 5(1-y)]$ $- (y \longleftrightarrow z)$	
F ₂	$\left(\frac{9}{4}A - \frac{7}{24}B\right)(1-y)^2$ $+ \frac{1}{24}B[(-1+9y)(x-z) + 10(1-y)]$ $+ (y \longleftrightarrow z)$	
F ₃	$\frac{5}{12}B(1-y) - (y \longleftrightarrow z)$	
F ₄	$\left(\frac{9}{4}A + \frac{1}{8}B\right)(1-y)^2$ $+ \frac{B}{24}[(9-y)(x-z) - 10(1-y)]$ $+ (y \longleftrightarrow z)$	
F ₅	$- \frac{5}{12}B(1-y)(3+2y-2z)$ $- (y \longleftrightarrow z)$	

TABLE III : Measured quantities in terms of the Wolfenstein
Parameters

$d\sigma$	$= \alpha ^2 + \beta ^2 + 2 \gamma ^2 + \delta ^2 + \epsilon ^2$
$d\sigma\langle P \rangle$	$= 2\text{Re } \gamma^*(\alpha + \beta)$
$d\sigma(1-D)$	$= 2\delta^2 + 2\epsilon^2 = \mu^2 + \nu^2$ where $\mu = (\delta + \epsilon)$, $\nu = (\delta - \epsilon)$
$d\sigma R$	$= [\alpha^2 - \beta^2 + \text{Re } \nu \mu^*] \cos \theta / 2 - 2\text{Im} \gamma (\alpha^* - \beta^*) \sin \theta / 2$
$d\sigma A$	$= -[\alpha^2 - \beta^2 + \text{Re } \nu \mu^*] \sin \theta / 2 - 2\text{Im} \gamma (\alpha^* - \beta^*) \cos \theta / 2$
$d\sigma R'$	$= (\alpha^2 - \beta^2 - \text{Re} \mu \nu^*) \sin \theta / 2 + 2\text{Im} \gamma (\alpha^* - \beta^*) \cos \theta / 2$
$d\sigma A'$	$= (\alpha^2 - \beta^2 - \text{Re} \mu \nu^*) \cos \theta / 2 - 2\text{Im} \gamma (\alpha^* - \beta^*) \sin \theta / 2$
$d\sigma C_{pq}$	$= 2\text{Im} \gamma (\delta^* - \epsilon^*)$
$d\sigma(1-C_{NN})$	$= \alpha - \beta ^2 + \mu^2$
$d\sigma C_{nnn}$	$= \text{Re} \gamma (\alpha^* + \beta^*)$
$d\sigma C_{npq}$	$= \text{Im} [(\alpha - \beta) \mu^* - (\alpha + \beta) \nu^*]$
$d\sigma C_{(rxq)np}$	$= \cos \theta / 2 \text{Im} [(\alpha + \beta) \mu^* - (\alpha - \beta) \nu^*] + 2 \sin \theta / 2 \text{Re} \gamma \mu^*$

TABLE IVa: The $F_i^{pp,np}$ as deduced from $U(6,6)$;

$$x=s/l_m^2, \quad y=t/l_m^2, \quad z=u/l_m^2$$

Process F_i	$pp \rightarrow pp$	$pn \rightarrow pn$
F_1	$\left[\frac{2}{2}A - \frac{8}{9}B\right](1-y)^2$ $+ \frac{1}{18}B[5(1+y)(x-z)+2(1-y)]$ $+ (y \leftrightarrow z)$	$\left[\frac{2}{2}A - \frac{1}{9}B\right](1-y)^2$ $+ \frac{B}{18}[1_4(1+y)(x-z)+7(1-y)]$ $- \frac{1}{18}B[1_4(1-z)^2+5(1-z)]$ $- (x-y)(1+z)]$
F_2	$\left[\frac{2}{4}A + \frac{7}{9}B\right](1-y)^2$ $+ \frac{B}{36}[(7+3y)(x-z)-1_4(1-y)]$ $- (y \leftrightarrow z)$	$\left[\frac{2}{4}A + \frac{17}{36}B\right](1-y)^2$ $+ \frac{1}{36}B[(11-3y)(x-z)-1_4(1-y)]$ $- \frac{1}{36}B[11(1-z)^2+10(1-z)]$ $- (x-y)(1_4-6z)]$
F_3	$- \frac{1}{9}B(1-y) + (y \leftrightarrow z)$	$- \frac{7}{18}B(1-y) + \frac{5}{18}B(1-z)$
F_4	$\left[\frac{2}{4}A + \frac{2}{3}B\right](1-y)^2$ $+ \frac{1}{36}B[(3+7y)(x-z)+1_4(1-y)]$ $- (y \leftrightarrow z)$	$\left[\frac{2}{4}A + \frac{1}{12}B\right](1-y)^2$ $+ \frac{1}{36}B[(-3+11y)(x-z)+1_4(1-y)]$ $- \frac{1}{36}B[21(1-z)^2-10(1-z)]$ $+ (x-y)(6-1_4z)]$
F_5	$\frac{1}{9}B(1-y)(3+2y-2z)$ $+ (y \leftrightarrow z)$	$\frac{7}{18}B(1-y)(3+2y-2z)$ $- \frac{5}{18}B(1-z)(3+2z-2y)$

TABLE IVb: The $F_i \equiv P$ as deduced from $U(6,6)$;
 $x=s/l_m^2$, $y=t/l_m^2$, $z=u/l_m^2$

Process F_i	$p \Xi^- \rightarrow p \Xi^-$
F_1	$\left[\frac{9}{2}A + \frac{7}{18}B\right](1-y)^2$ $+ \frac{1}{18}B[4(1+y)(x-z) - 2(1-y)]$
F_2	$\left[\frac{9}{4}A - \frac{1}{36}B\right](1-y)^2$ $+ \frac{1}{36}B[(2+6y)(x-z) + 4(1-y)]$
F_3	$\frac{1}{9}B(1-y)$
F_4	$\left[\frac{9}{4}A + \frac{1}{12}B\right](1-y)^2$ $+ \frac{1}{36}B[(6+2y)(x-z) - 4(1-y)]$
F_5	$- \frac{1}{9}B(1-y)(3+2y-2z)$

TABLE Va : The $F_i^{p\Lambda, p\Lambda/p_{\Lambda\Sigma}}$ deduced from $U(6,6)$;

$$x=s/l_m^2, \quad y=t/l_m^2, \quad z=u/l_m^2$$

F_i Process	$p\Lambda \rightarrow p\Lambda$	$\sqrt{3}(p\Lambda \rightarrow p\Sigma^0)$
F_1	$\frac{9}{2}A(1-y)^2$ $+ \frac{1}{8}B[2(1+y)(x-z)+(1-y)]$ $- \frac{3}{4}\tilde{B}(1-z)^2$	$\frac{1}{24}B[2(1+y)(x-z)-7(1-y)$ $-4(1-y)^2] + \frac{1}{4}\tilde{B}[(1-z)-(1-z)^2]$
F_2	$[\frac{9}{4}A + \frac{5}{16}B](1-y)^2$ $+ \frac{1}{16}B[(3+y)(x-z)-2(1-y)]$ $- \frac{3}{8}\tilde{B}(1-z)^2$	$\frac{1}{48}B[2(1+y)(x-z)+14(1-y)$ $+(1-y)^2] - \frac{1}{8}\tilde{B}[(1-z)(x-y)$ $-2(1-z)+2(1-z)^2]$
F_3	$- \frac{1}{8}B(1-y)$	$\frac{7}{24}(1-y) - \frac{1}{4}\tilde{B}(1-z)$
F_4	$[\frac{9}{4}A + \frac{3}{16}B](1-y)^2$ $+ \frac{1}{16}B[(1+3y)(x-z)+2(1-y)]$ $- \frac{3}{8}B(1-z)^2$	$\frac{1}{48}B[(9-5y)(x-z)-14(1-y)$ $+ 15(1-y)^2] + \frac{1}{8}\tilde{B}[(1-z)(x-y)$ $-2(1-z)]$
F_5	$\frac{1}{8}B(1-y)(3+2y-2z)$	$- \frac{7}{24}B(1-y)(3+2y-2z)$ $+ \frac{1}{4}\tilde{B}(1-z)(3+2z-2y)$

TABLE Vb : The $F_i^p \Sigma^{\pm}$ as deduced from $U(6,6)$;

$$x=s/4m^2, \quad y=t/4m^2, \quad z=u/4m^2$$

F_i Process	$p \Sigma^+ \rightarrow p \Sigma^+$	$p \Sigma^- \rightarrow p \Sigma^-$
F_1	$\begin{aligned} & [\frac{9}{2}A - \frac{23}{18}B](1-y)^2 \\ & + \frac{1}{18}B[(1+y)(x-z)+4(1-y)] \\ & + \frac{\tilde{B}}{18}[4(1+z)(x-y)-2(1-z) \\ & + 7(1-z)^2] \end{aligned}$	$\begin{aligned} & [\frac{9}{2}A - \frac{1}{18}B](1-y)^2 \\ & + \frac{1}{36}B[4(1+y)(x-z)+7(1-y)] \end{aligned}$
F_2	$\begin{aligned} & [\frac{9}{4}A + \frac{29}{36}B](1-y)^2 \\ & + \frac{1}{36}B[(5-3y)(x-z)-8(1-y)] \\ & - \frac{\tilde{B}}{36}[(2+6z)(x-z)+4(1-z) \\ & - (1-z)^2] \end{aligned}$	$\begin{aligned} & [\frac{9}{4}A + \frac{17}{72}B](1-y)^2 \\ & + \frac{1}{72}B[(11-3y)(x-z) \\ & - 14(1-y)] \end{aligned}$
F_3	$-\frac{2}{9}B(1-y) + \frac{\tilde{B}}{9}(1-z)$	$-\frac{7}{36}B(1-y)$
F_4	$\begin{aligned} & [\frac{9}{4}A + \frac{7}{12}B](1-y)^2 \\ & + \frac{1}{36}B[(-3+5y)(x-z) \\ & + 8(1-y)] - \frac{\tilde{B}}{36}[(6+2z)(x-y) \\ & - 4(1-z)+3(1-z)^2] \end{aligned}$	$\begin{aligned} & [\frac{9}{4}A + \frac{1}{24}B](1-y)^2 \\ & + \frac{1}{72}B[(-3+11y)(x-z) \\ & + 14(1-y)] \end{aligned}$
F_5	$\begin{aligned} & \frac{2}{9}B(1-y)(3+2y-2z) \\ & - \frac{\tilde{B}}{9}(1-z)(3+2z-2y) \end{aligned}$	$\frac{7}{36}B(1-y)(3+2y-2z)$

TABLE VI : Scattering lengths and forward scattering

amplitudes for various $pN \rightarrow pN$ processes; $x=s/4m^2$

Reaction	$a_s = 2(F_1 - 3F_3)_{x=1}$	$a_T = 2(F_2 + F_4)_{x=1}$	$\alpha(s,0)/2m^2 = x(-F_1 - 2F_3 + 2F_4 + F_5) + (2F_1 + 3F_2 - F_3 + F_4 - F_5)$
$pp \rightarrow pp$	$18a - \frac{2}{3}B$	0	$36a + 10xB - \frac{32}{3}x^2\tilde{B} - 18x^3\tilde{a}$
$pn \rightarrow pn$	$9a - \frac{1}{3}B$	$9a - \frac{1}{3}B$	$36a + 8xB - \frac{28}{3}x^2\tilde{B}$
$p\Lambda \rightarrow p\Lambda$	$9a$	$9a$	$36a + 6xB - 6x^2\tilde{B}$
$p\Sigma^+ \rightarrow p\Sigma^+$	$9a - \frac{1}{3}B$	$9a + \frac{7}{3}B$	$36a + 8xB - \frac{4}{3}x^2\tilde{B}$
$p\Sigma^- \rightarrow p\Sigma^-$	$9a + \frac{5}{3}B$	$9a + \frac{7}{9}B$	$36a + 4xB$
$p\Sigma^0 \rightarrow p\Sigma^0$	$9a + \frac{1}{3}B$	$9a + \frac{5}{9}B$	$36a + 2xB$
$p\Lambda \rightarrow p\Sigma^0$	$-B/\sqrt{3}$	$B/\sqrt{3}$	...

TABLE VII : Low-momentum elastic baryon-baryon scattering

Process	Effective mean momentum (MeV/C)	(mb)
$p\Sigma^- \rightarrow p\Sigma^-$	155	232 ± 47^a
$p\Sigma^+ \rightarrow p\Sigma^+$	155	185 ± 55^a
$p\Lambda \rightarrow p\Lambda$	136	214 ± 46^b
$p\Lambda \rightarrow p\Lambda$	165	156 ± 27^b
$pn \rightarrow pn$	163	689 ± 5^c
$pp \rightarrow pp$	150	500 ± 50^d

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c. R.Wilson, The Nucleon-Nucleon Interaction (Interscience
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TABLE VIII : Experimental comparison of the prediction $C_{pq}=0$

T_{LAB} (MeV)	$\theta_{\text{c.m.}}$ (deg.)	Value ^a
52.3	90°	0.10 ± 0.14
380	30°	0.12 ± 0.10
400	90°	0.32 ± 0.09
450	90°	0.37 ± 0.14
660	90°	0.22 ± 0.18

a Data taken from R. Wilson, Ref. 40

TABLE IX : Experimental comparison of the prediction $R' = -A$

T_{LAB} (MeV)	$\theta_{\text{c.m.}}$	R'	T_{LAB} (MeV)	$\theta_{\text{c.m.}}$	$-A$
140^a	31.4°	0.625 ± 0.062	139	31.1°	0.368 ± 0.032
	41.7°	0.548 ± 0.062		41.4°	0.344 ± 0.031
	52.0°	0.470 ± 0.069		51.7°	0.311 ± 0.035
	61.8°	0.343 ± 0.058		61.9°	0.231 ± 0.046
	72.1°	0.466 ± 0.095		72.0°	0.189 ± 0.056
	82.2°	0.190 ± 0.177		82.1°	0.099 ± 0.079
213^b	30°	0.491 ± 0.025	213	30°	0.400 ± 0.019
	40°	0.390 ± 0.024		40°	0.317 ± 0.019
	50°	0.177 ± 0.022		50°	0.205 ± 0.021
	60°	0.120 ± 0.025		60°	0.102 ± 0.025
	70°	-0.277 ± 0.045		70°	0.012 ± 0.036
	80°	-0.208 ± 0.068		80°	0.090 ± 0.046
	90°	-0.340 ± 0.104		90°	0.180 ± 0.077

a O.N.Jarvis et al, Nucl.Phys., 50,529 (1964)

b K.Gotow and F.Lobkowicz, Phys.Rev., 136, B.1345 (1964)

TABLE X : Experimental comparison of the prediction $R = A'$

T_{LAB} (MeV)	$\theta_{\text{c.m.}}$	R	A'
430 ^a	30°	0.06 $\pm$ 0.11	0.47 $\pm$ 0.20
	45°	0.10 $\pm$ 0.11	0.06 $\pm$ 0.11
	60	0.13 $\pm$ 0.08	0.06 $\pm$ 0.09
	75	0.47 $\pm$ 0.07	0.22 $\pm$ 0.08
	90°	0.47 $\pm$ 0.05	0.36 $\pm$ 0.07
	105°	0.35 $\pm$ 0.11	0.01 $\pm$ 0.11
	120°	0.34 $\pm$ 0.18	0.08 $\pm$ 0.04

a R.F.Roth, Princeton Technical Report No.33, 1964 (unpublished

Report quoted by P.B.Kantor et al, Phys.Rev., 140, B.1008 (1965)

TABLE XI : Experimental comparison of the prediction

$$2[1-D_{NN}(90^\circ)] = [1-C_{NN}(90^\circ)]$$

T_{LAB} (MeV)	$2[1-D_{NN}(90^\circ)]$	T_{LAB} (MeV)	$1-C_{NN}(90^\circ)$
310	1.16 ± 0.12^d	310	$0.16^{+0.22}_{-0.10}^b$
430 ^c	0.66 ± 0.20	425 ^a	0.34 ± 0.15
660	0.14 ± 0.34	650	0.07 ± 0.20

a Data averaged over $T=400$ and 450 MeV

b I.M.Vasilevski et al., Zh.Eksperim i Teor.Fiz.,39, 889 (1960)

[English translation - JETP 12, 616, (1961)]

c R.F.Roth (see footnote in Table X)

d Extrapolation of data in R.Wilson , Ref.40

TABLE XII : Quantum number of mesons and baryons

(centre dots mean "not an eigenstate".

Particle	I	N	S	J	$c_2^{(3)}$	$c_2^{(4)}$
π	1	0	0	0	6	8
ρ	1	1	0	1	6	8
K	1/2	1/2	1/2	0	6	15/4
K^*	1/2	1/2	1/2	1	6	15/4
ω_1	0	0	1	1	...	0
ω_8	0	1	0	1	...	8
η_8	0	0	0	0	...	0
χ_1	0	0	0	0	...	0
N	1/2	1/2	0	1/2	6	63/4
Σ	1	1	1/2	1/2	6	9
Λ	0	0	1/2	1/2	6	9
Ξ	1/2	1/2	1	1/2	6	15/4
N^*	3/2	3/2	0	3/2	12	63/4
Y^*	1	1	1/2	3/2	12	9
Ξ^*	1/2	1/2	1	3/2	12	15/4
Ω^-	0	0	3/2	3/2	12	0

In general, if we denote the dimensionality of a particular representation of $SU(3)$ and $SU(4)$ by $D(p,q)$ and $D(p,q,r)$ respectively then $c_2^{(3)} = 2/3(p^2 + q^2 + pq + 3p + 3q)$

$$c_2^{(4)} = 3/4(p^2 + 4p + q^2 + 2q + r^2)$$

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