

Vortices in Bose-Einstein Condensates

by

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Declaration of Originality

I hereby declare that, aside from all references identified, the work in this thesis is my own.

Signed:

A handwritten signature in black ink, appearing to read "E. J. Allen". The signature is written in a cursive style with a large initial "E".

Date: 14th of October, 2019.

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I am grateful to my supervisor, Professor Henrik Jensen, and to Professor Serafim Kalliadasis for their advice, guidance and continued support. I will also like to acknowledge the conversations I had with the late Professor John Elgin, as well as with Dr Ryan Barnett.

Dedication

I dedicate this work to my late father,
Distinguished Emeritus Professor Vincent Olusegun Sowemimo Olunloyo,
FAS, FNSE, FAEng, NNOM.

Abstract

This thesis surveys aspects of the very broad topic of vortex dynamics in Bose-Einstein condensates. The Gross-Pitaevskii equation (GPE) is an important and popular tool in this regard, and we discuss a number of its properties which are relevant to the study of vortex dynamics. Using the GPE, we explore the dynamics of point vortices in Bose-Einstein condensates, writing hydrodynamic relations we have not encountered elsewhere in the literature. For incompressible condensates, we pay particular attention to discussions of the structure of the Hamiltonian system, and we revisit the topic of linear stability analysis for a steadily rotating, regular polygon of vortices. Furthermore, geometric mechanics is a powerful tool in the study of fluid dynamics, and we make use of the language of geometric mechanics, in particular the Lie derivative, to gain insights into some of the consequences of describing superfluid dynamics using the GPE. We have not encountered the relations we have derived as a result of this effort elsewhere in the literature. The topics of vortex formation, general approaches to describing vortex dynamics and phonon emission are also discussed.

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1 Introduction

This thesis is concerned with the study of vortices in Bose-Einstein condensates (BECs). We pay particular attention to the dynamics of point vortices in two-dimensional BECs (i.e. films which have a thickness of the order of the superfluid correlation length [1]). Such two-dimensional systems are less complicated than their three-dimensional counterparts and may be realized for example by slicing a three-dimensional BEC into ‘pancake’ pieces using one-dimensional optical lattices [2]. In particular we are concerned with the study of single-component, ultra-cold, atomic BECs in the limit as temperature approaches absolute zero. In this temperature limit, the BEC behaves as a superfluid and is often modelled using the Nonlinear Schrödinger Equation (NLSE). The particular NLSE often employed in this case is the Gross-Pitaevskii Equation (GPE). Superfluidity leads to interesting behaviour such as frictionless flow, quenching of the moment of inertia and, in particular, formation of quantized vortices [3].

Vortices have been described as ‘persistent circulating flow patterns’ [4]. They occur in widely varying scientific and mathematical contexts, including hydrodynamics, superfluids, and nonlinear optics [4]. Vortex dynamics are relevant in the description of natural phenomena such as sunspots and dust devils [4]. Vortices of inviscid superflow are quantized vortices [5]. Such vortices sustain any rotational motion of a superfluid [5]. Quantized vortices are stable and well defined topological defects, and differ from ordinary vortices found in conventional fluids in a number of ways [5]. First, the circulation of a classical vortex can take on any value, unlike that for a quantized vortex which is quantized as the name suggests [5]. Second, a quantized vortex cannot decay by viscous diffusion as would occur in a classical fluid, given that a quantized vortex is a vortex of inviscid superfluid [5]. Third, a quantized vortex has a very thin core, of the order of the superfluid coherence length [5]. Given that the vortex core does not decay by viscous diffusion and is very thin, the position of a quantized vortex present in the superfluid can always be identified [5]. A quantized vortex is therefore of greater stability and definiteness than a classical vortex [5]. The phenomenon of Bose-Einstein condensation has generated an environment where many features of the nonlinear dynamics of vortices of single and multiple charge, and of vortex crystals and lattices can be both theoretically studied and experimentally observed [4]. It is often thought that Bose-Einstein condensation is a macroscopic quantum phenomenon [5], given that bosons form a macroscopic wave function (also known as an order parameter) which extends over the whole system [5] below a critical temperature in which bosons are present in a single-particle ground state [5]. Hydrodynamics dominated

by quantized vortices is known as quantum hydrodynamics, while turbulence consisting of quantized vortices is called quantum turbulence (QT) [5]. Turbulence has been one of the great mysteries of nature for a long time, with discussions dating back to the time of Leonardo da Vinci [5]. Turbulence has been intensely studied in multiple fields, but it is not yet completely understood [5], as turbulence is a complex dynamical phenomenon featuring strong nonlinearity [5]. The potential for drawing useful comparisons between QT and classical turbulence demonstrates the importance of studying QT [5]. To buttress this point, we note that it was pointed out by Leonardo da Vinci that ‘turbulence in a classical viscous fluid appears to be comprised of vortices’ [5]. These vortices are unstable, appearing and disappearing repeatedly [5]. Furthermore, circulation is neither conserved nor identical for each vortex [5]. QT is made up of a tangle of quantized vortices that have the same circulation [5]. Therefore, it may be easier to study a QT system than classical turbulence [5].

This study consists of five main sections besides this introduction, which is the first. Section two includes a literature review of vortex formation, approaches to modelling vortex dynamics and phonon emission. Here, we survey the literature and examine a variety of means by which vortices are formed in superfluid BECs. A simple picture of vortex formation is that it can be understood as a local suppression (evacuation) of the BEC which may occur if the BEC is rotating [6]. Given that vortex formation involves a local suppression of the BEC, any other mechanism of BEC suppression may disrupt vortex formation [6]. For example, BEC depletion by a strong interaction between the bosonic particles is one such process [6]. The topic of condensate surface modes, given its relation to vortex formation is also discussed in this thesis. Regarding the mechanistic details of the dynamics of vortex formation under rotation, an important part of the discussion is a break in circular symmetry of the trapping potential, which is able to excite condensate surface modes and potentially form vortices. In the absence of trap anisotropies, the condensate surface modes themselves may generate the necessary break in symmetry to form vortices. We further examine in this section two broad approaches to how, in principle, the equations of motion for vortices in BECs may be written in a two-dimensional setting. That is, the microscopic approach, and the effective action theory approach. The equation of motion, even for a single vortex in a two-dimensional setting, has been a controversial topic for a long time [7]. In our study of effective action theories, we pay particular attention to the quantum electrodynamical (QED) approach of Arovas and Freire [8], given its potential to provide a means of making progress in the development of an analytically tractable theory to simultaneously describe vortex motion and phonon emission. The topic of vortex mass, another controversial topic in the literature, is also surveyed in this section. Furthermore, we examine the topic

of phonon radiation by accelerating vortices in two, as well as three dimensional settings. Superfluid vortices are known to be unstable to acceleration, with the emission of phonons being a possible decay mechanism, similar to the emission of Larmor radiation by accelerating charges [9]. It is known also that phonon emission occurs during vortex reconnection, itself caused by the close approach of vortex lines [10]. Studying phonon emission is useful in order to study the relaxational kinetics of superfluid turbulence [11]. It has also been shown that phonon radiation may contribute to the total vorticity through ‘twisted phonon modes’ [12]. Studies of phonon emission by superfluid vortices abound in the literature. Of interest is the examination of the phonon radiation pattern caused by vortex acceleration, especially where more than one vortex is concerned, given the need for further development in this area [13]. In particular we examine a pair of key papers ([9] & [13]) which demonstrate this phenomenon via numerical solution of the GPE. We also discuss some of the complications which arise when considering the problem of vortex motion in a three-dimensional setting as opposed to a two-dimensional setting. Both vortex motion, and phonon radiation are discussed.

Section three contains a discussion of aspects of Gross-Pitaevskii (GP) theory, which has been useful in the description of superfluid BEC dynamics (in particular, dilute BECs are well described by the GP mean-field model) ([14], [15]). On the other hand, high density and strong repulsive interactions hinder the use of GP mean-field theory in theoretical treatment of superfluid ^4He [16]. Alkali-metal atomic BEC’s are dilute gases with weak interactions, and the GP mean-field model is more applicable in their case [16]. The vortex solution of the GPE is discussed.

In section four, we specialize our study of vortex dynamics using the GPE to the more tractable point vortex model, and further restrict our study to two-dimensional motion. We examine how a point vortex distribution is written using the GPE wave function, and then specialize even further to the case of an incompressible condensate. We study a number of properties of the resulting Hamiltonian system, notably the integrability of systems of four or fewer vortices. Besides these, we provide in this thesis our independent solution to the problem of linear stability analysis of the motion of a steadily rotating, regular polygon of point vortices following a literature survey of this topic.

Section five is where methods of geometric mechanics are applied to our study of the GPE. However, unlike in section three the studies carried out in this section are applicable to BECs considered in two, as well as three spatial dimensions. The most important mathematical tool employed in this section is the Lie derivative. It allows us to find the time derivatives of integrals with moving boundaries in a convenient manner, which is of interest in the study of superfluid vortex dynamics.

Section six of this thesis details the conclusions of our study and ideas for future work.

2 Review of Vortex Formation, Approaches to describing Vortex Dynamics and Phonon Emission

Given that we are studying vortex dynamics, it will be of interest to discuss in this section various methods by which vortices are generated in the first place. This will be followed by a closer look at the non-equilibrium process of vortex formation in rotating condensates (noting that rotating condensates will be discussed at several points in this thesis). Following our discussions of vortex formation we then restrict our studies to two-dimensional systems and consider other ways of modelling the motion of vortices in a BEC besides assuming the presence of point vortices in an incompressible condensate. This is done in the form of a discussion of two broad approaches to this task, an effective action theory approach, and a microscopic approach. The topic of vortex mass is also discussed. This discussion of vortex mass is largely carried out in the context of effective action theories however.

In writing vortex-only effective action theories to describe the motion of multiple vortices in a BEC, it is difficult to extract information about sound wave (phonon) emission, which is a characteristic of accelerating vortices. We discuss phonon emission by accelerating vortices in a two dimensional, as well as a three dimensional setting.

2.1 Concerning Vortex Formation

The first part of this section will deal with the broad topic of vortex formation. A number of vortex generation techniques are discussed. Manipulation of the BEC through changes to the trapping potential feature prominently in this discussion, though a variety of techniques are discussed.

2.1.1 Methods of Vortex Generation

Vortex formation in BECs of alkali atom gases has been studied intensively, experimentally and theoretically [17]. The GPE has played a prominent role in this effort and is more readily applicable to alkali atom gases than say, superfluid ^4He [16]. High density and strong repulsive interactions complicate theoretical treatments of superfluid ^4He [16]. Furthermore, two-dimensional studies are more tractable theoretically than three-dimensional studies and are well represented in the literature. We see in [18] that two-dimensional (and reportedly one-dimensional as well) condensates can be made by reducing

the number of atoms of a condensate held in a strongly anisotropic trap. If we consider a condensate held in an axially symmetric trap, the length scales which characterize the condensate are the radius of the condensate $R_{\perp}$, the axial half length R_z (z is chosen axis in this case), the scattering length a which is the parameter for the strength of interaction between BEC atoms and the condensate healing length ε . When $R_z > \varepsilon > R_{\perp}$ the condensate exhibits one-dimensional behaviour. When $R_{\perp} > \varepsilon > R_z$, we have a condensate exhibiting two-dimensional behaviour. For a condensate, two-dimensional behaviour could be achieved for instance by adjusting the trapping frequency ω_z in the anisotropic trap to be much greater than roughly equal ω_x and ω_y frequencies, along with reducing the number of atoms to form a disk-shaped condensate. A notable approach to studying vortex formation in BECs is the study of the stability of the axially symmetric state in which a vortex is absent with respect to perturbations of the order parameter [17]. Another notable approach involves the use of a London model in which vortices are modelled as independent point-like objects, along with calculating the frequency required to destroy the surface barrier preventing vortex penetration through the Thomas-Fermi boundary [17]. Prominent among existing approaches to this problem is the use of numerical techniques to solve the GPE.

An important method of vortex formation is breaking the trap symmetry of the BEC trap (see for example [19] and [20]). Commonly, this is done by stirring a rotating condensate trapped in a potential. For example, superposing a rotating laser beam on a rotating harmonic trap containing an atomic BEC results in a rotating harmonic potential with an adjustable anisotropy, leading to the nucleation of one or possibly several vortices depending on the harmonic potential's rotation frequency [21]. The BEC is stirred by the anisotropy in the harmonic potential. Other studies indicating that vortices are generated in a trapped rotating Bose-Einstein condensate where the trapping potential has an anisotropy are found in references [22], [23], [24], and [25]. However, it has been shown theoretically that a BEC rotating in a potential may result in vortex nucleation even without an anisotropy in the potential [26]. This is seen for example in [26] in a harmonic trap, and in a quadratic-plus-Gaussian trap. It has been asserted [26] that the formation of the vortex is due to the spontaneous shape deformation of the system caused by condensate surface modes. We note also that it is possible to generate vortices in a BEC without the application of any external rotation [5]. A quench of an alkali system through the BEC transition temperature can lead to spontaneous vortex formation without the application of any external rotation (as noted in [5] and [27]).

We have encountered a variety of geometries of potentials applied in the study of vortex formation in rotating condensates, such as harmonic [21] and toroidal [23]. We pay some attention now to the

double-well potential. The double-well potential is a significant model potential [28] which is simple and yet rich in physics [28]. In [28], it served as a testing ground for Feynman’s rule which in a more complicated geometric confinement than the more commonly studied harmonic potential. Feynman’s rule is an important rule which links the total number of vortices N_v , with the angular frequency of rotation [28]. It was argued by Feynman that a superfluid rotating with angular frequency Ω which has reached the steady state should be regarded as a classical fluid [28]. This results in a relation between the number of vortices N_v , located in an area A and Ω given by [28]

$$2\pi\hbar\frac{N_v}{m} = 2\Omega A.$$

This rule was initially formulated for uniform superfluid helium and m above refers to the mass of a helium atom [28]. Feynman’s rule has been studied extensively for BECs trapped in single harmonic potentials [28]. It is worth noting however, that in a two-dimensional, numerical study [28] of vortex formation in a rotating double well potential, the presence of so-called ‘hidden vortices’ is required to recover the Feynman rule. A hidden vortex has no visible core but has a phase singularity [28]. External potentials have a strong influence on the core size of a hidden vortex [28]. The authors [28] point out that these hidden vortices are reminiscent of ‘ghost vortices’ which have also appeared in numerical studies. For simply connected traps, ghost vortices can be found at the outskirts of the condensate [28]. For multiply connected traps, ghost vortices may be found in the condensate cloud’s outer region, where the condensate density is very small [28]. Like hidden vortices, ghost vortices have invisible cores as well as phase singularities. However, while a hidden vortex carries angular momentum, a ghost vortex does not [28].

Besides rotating potentials and quenching through the BEC transition temperature, vortices in atomic BECs may also be generated by dynamical phase imprinting and topological phase imprinting [29]. It is known that it is possible to form a multiply charged vortex by topological phase imprinting [29] (see [30] for further details). Topological phase imprinting leads to vortex generation by acquisition of a Berry’s phase. A Berry’s phase in this case is a condensate wave function phase difference which is imparted to the system by causing the system to adiabatically traverse the parameter space of the system’s Hamiltonian [31]. This could mean for example adiabatically changing the confining magnetic field as is done in [31]. However, given that the energy of the vortex is proportional to the square of its winding number, the multiply charged vortex is energetically unstable [29]. Resultantly, the multiply charged vortex is expected to split into a number of vortices with smaller winding numbers [29]. It is known from theoretical studies

that the splitting process, as well as the vortex stability are influenced by atomic interaction in the BEC [29]. Changing the symmetry of the trap potential distortion allows for the exertion of a degree of control over the splitting pattern [29]. In BECs of sodium atoms, doubly, as well as quadruply charged vortices have been experimentally realized [29].

Vortices may also be generated by the expansion of a previously confined condensate, for instance as demonstrated theoretically in [32] for BECs with repulsive interactions. The method discussed entails confining the BEC in a magnetic trap, and then further confining the BEC in an impenetrable hard wall potential. Upon being released from the magnetic trap, the repulsive energy of the condensate is converted into kinetic energy and the BEC expands rapidly towards the hard wall [32]. If the BEC is expanding in a circular box, vortices may be generated in a particular spatial location by introducing a static potential dip, and letting the BEC flow across the potential dip [32]. Still on the topic of vortex formation during condensate growth, it has been shown experimentally [33] that if there is a confining harmonic potential, and the single harmonic potential well is partitioned into three sections by a barrier, enabling the simultaneous formation of three independent uncorellated BECs, the BECs could either automatically merge together during their growth or if (in the case of high energy barriers) the barriers are removed after their formation. The merging process may cause vortex formation in the resulting BEC, depending on the initial relative phases of the condensates and the merging rate [33]. Another theoretical proposal to produce vortices in a BEC (determined via a numerical study of the GPE) is through the decay of dark solitons [34].

The use of vibrating structures in the superfluid is another way to generate vortices [5]. For instance, using a thin superconducting wire vibrating in a superfluid in resonance with a Lorentz force (due to an alternating electric current flowing through the wire in the presence of a static magnetic field) [5]. For a low driving force, the wire vibrates smoothly. However, beyond a certain value of the driving force, there is a sudden drop in the velocity of the wire, and the energy injected into driving the wire escapes and is used to excite quantum turbulence [5].

An important reference for the study of vortex formation techniques is [35]. The article discusses differences between two-dimensional turbulence and three-dimensional turbulence. For instance the vortices are now rectilinear in two-dimensional systems, and Kelvin waves can be ignored. In two-dimensional incompressible superfluids, there is the conserved quantity of enstrophy (total squared vorticity) which is not conserved in three dimensional superfluids [35]. Furthermore, as we intuitively expect, the particular method used to generate turbulence has a strong influence on the vortex state which forms [35]. Stirring

methods tend to impart linear and/or angular momentum and excite collective modes. It is possible to make disordered distributions of vortices with predominantly one sign of circulation through the formation of transient states in rotating condensates and the decay of giant vortices [35]. Inter-component hydrodynamical instabilities provide a means to generate quantum turbulence. Should miscible two-component condensates be made to counter-propagate, then vortex nucleation may occur above a critical relative speed [35]. If the condensates are immiscible, the condensates may mix due to a Rayleigh-Taylor instability at the interfacial boundary, and then a Kelvin-Helmholtz instability following this mixing can produce vortex bundles [35]. We further learn from this paper that vortices in a BEC may be generated by dragging an obstacle through the BEC in the form of a localised potential ([36], [37] and [38]). It is implied in [35] that quantum turbulence seems to capture the essential properties of turbulence and is therefore a potential tool for studying turbulence, buttressing a similar point made in the introduction of this thesis.

Lastly, we note that there are considerable differences in physics between trapped ultracold gases of atoms with attractive s -wave interactions and those with repulsive interactions, which are more common [39]. For example, quantized vortex formation under rotation is not necessarily an energetically favourable transition in attracting BECs, unlike for repulsive BEC gases [39].

2.1.2 A Look at Vortex Formation away from Equilibrium and under Rotation

We shall have a closer look at the mechanistic details of vortex formation, for the particular case of a rotating condensate. We shall examine reference [14] which is focused on nonlinear dynamics of vortex lattice formation in a rotating BEC. Energy dissipation is an important consideration in this paper given that vortex formation dynamics away from equilibrium is an energy-losing process [14]. This paper studies the mechanism of vortex lattice formation in a rotating BEC with repulsive interactions in a two-dimensional setting. In two-dimensional analysis however, effects such as vortex bending are ignored [14]. Imaginary time propagation of the GPE is a very useful scheme to find equilibrium states of the condensate, but the dynamical process towards such states cannot be elucidated by this method [14]. This paper is concerned with such dynamical processes and the GPE is numerically analyzed in order to understand these.

A key part of the study is the presence of a rotating, symmetry-breaking potential [14]. As the potential breaks rotational symmetry, angular momentum is transferred to the condensate through the excitation of surface modes or vortex formation [14]. In some scenarios, defined by the parameters of the

GPE chosen, the rotating anisotropic potential does not result in vortex nucleation but excites mainly (along with other modes) a quadrupole mode. The anisotropy of the rotating potential and the nonlinear atomic interaction cause different surface modes to couple to one other, resulting in complex nonlinear dynamics [14]. Mode coupling is clarified in this paper using the quasiparticle projection method [14]. For some rotation frequencies, a simple oscillation of the condensate is observed. However, a change in the frequency with which the condensate rotates may cause resonant excitation of the quadrupole mode resulting in chaotic behaviour due to dynamic instability of the rotating condensate (Impure resonance, in practice. Single resonance with the quadrupole mode does not take place due to the coupling of the quadrupole mode with higher energy modes through the nonlinear interaction [14]). Resonant behaviour (resonant excitation of the quadrupole modes) causes violent density and phase fluctuations of the condensate surface with surface ripples generated but these do not develop into quantized vortices in the energy-conserving scenario [14]. Increasing the trap anisotropy expands the range of the rotation frequencies which may result in resonance excitation [14].

In a different scenario, a phenomenological dissipative term is added to the GPE, and energy is no longer conserved. Vortex nucleation may now occur due to the Landau instability. Mean field interaction strength is kept the same as before dissipation was added, and also a condensate rotation frequency near the (impure) resonant value for the quadrupole mode resonant excitation which was explored in the absence of dissipation and did not lead to vortex nucleation is now explored with the dissipative term added in. Other parameters for simulation, aside from the newly added dissipation term, remain the same. The condensate initially undergoes a quadrupole oscillation, which is damped due to dissipation [14]. The boundary surface of the condensate then becomes unstable and ripples that move along the condensate surface are generated [14]. These ripples on the surface then develop into vortex cores. These vortices move towards the rotation axis due to the rotating drive, but at the same time they are pushed apart by repulsive interactions between one another. The result is a vortex lattice of which the vortex density is determined by the rotation frequency [14].

This paper is a good demonstration of how the topic of surface modes is a part of the broader discussion of vortex formation, along with how a modified GPE, the dissipative GPE, may be needed as a non-energy-conserving model in order to study the dynamics of vortex formation. In another study involving a dissipative GPE [40] it has been found that the stability of vortices is weakened by an increase in attractive dipolar strength of interaction between atoms. It was also found that increasing the repulsive contact interactions between atoms can increase the number of vortices formed in a trapped rotating BEC

[40]. The dissipative GPE is a finite-temperature model [22].

2.2 Concerning Approaches to Describing Vortex Dynamics

Broadly, this subsection reviews effective action (of which GPE theory is one) and microscopic approaches to describing the dynamics of vortices.

2.2.1 Effective Action Theories in Two Dimensions

As far as finding the equations of motion of superfluid vortices is concerned, significant work has been done in the area of effective action theories. Such theories have the well known benefit of replacing complex microscopic theories with a simpler theory written in terms of variables deemed relevant [41]. Furthermore, effective action theories are useful to extract the qualitative features of vortex dynamics, such as the Magnus force, which are more difficult to extract from numerical evolution of the time-dependent NLSE [42]. We examine a number of such theories in this subsection, beginning with, and paying particular attention to, the work of Arovas and Freire [8] in which the link between QED and superfluidity as expressed via an NLSE formulation is explored. The starting point for our discussion of their work (which is similar to our discussion in [43]) is the Ginzburg-Landau Lagrangian density [8]

$$L_{GL}(\psi^*, \psi) = \psi^* (i\hbar\partial_t + eZ^0)\psi - \frac{1}{2m} \left| \left(\frac{\hbar}{i}\nabla + \frac{e}{c}\mathbf{Z} \right) \psi \right|^2 - \lambda \left(|\psi|^2 - \frac{\bar{\sigma}}{m} \right)^2 \quad (2.1)$$

which we recognise is an NLSE Lagrangian density. We shall reinterpret the electrodynamical terms in this Lagrangian density in order to apply to a BEC. The mass of a boson is represented by m , while c , the speed of sound and, e , the ‘charge’ are arbitrary parameters. Z^μ is an external gauge field, and together with e describes an externally imposed ‘current’ in the form $\frac{e}{m}\epsilon^{\mu\nu\lambda}\partial_\mu Z_\lambda$. This external current could represent a global rotation of the superfluid [8]. ψ is the order parameter and is written as $\psi = \left(\frac{\tilde{\sigma}}{m}\right)^{\frac{1}{2}} e^{i\alpha}$ where $\tilde{\sigma}$ is the mass density of the condensate. Lastly, $\bar{\sigma} = \langle \tilde{\sigma} \rangle$ represents the average mass density and $\lambda = \frac{m^2 c^2}{2\bar{\sigma}}$. We have introduced the phase $\alpha = \theta + \chi$. In our discussion, $\theta = \theta(\mathbf{r}, t)$ is a smooth function while $\chi = \chi(\mathbf{r}, t)$ is a singular function. Both the density and current densities of the point vortex distribution can be written in terms of θ and χ [43]. Defining a current density tensor [43],

$$J^\mu = \frac{\hbar c}{m} \epsilon^{\mu\nu\lambda} \partial_\nu \partial_\lambda \chi \quad (2.2)$$

where $\partial_\mu = (\frac{1}{c}\partial_t, \partial_x, \partial_y)$. We can then write

$$J^0 = \frac{\hbar c}{m} \sum_i^N k_i \delta(x - x_i) \delta(y - y_i), \quad (2.3a)$$

$$J^1 = \frac{\hbar c}{m} \sum_i^N k_i \frac{dx_i}{dt} \delta(x - x_i) \delta(y - y_i) \quad (2.3b)$$

and

$$J^2 = \frac{\hbar c}{m} \sum_i^N k_i \frac{dy_i}{dt} \delta(x - x_i) \delta(y - y_i), \quad (2.3c)$$

which are not written in [8], and allow us to write an effective vortex action explicitly in terms of the point vortex coordinates [43]. The position of the i th vortex is given by (x_i, y_i) . An effective action written explicitly in terms of the vortex coordinates also does not appear in [8] and will be provided in this thesis. Substituting $\psi = \left(\frac{\tilde{\sigma}}{m}\right)^{\frac{1}{2}} e^{i\alpha}$ into equation (2.1), we can then write

$$L_{GL} = -\frac{\hbar\tilde{\sigma}}{m} \left(\partial_t \theta + \partial_t \chi - \frac{e}{\hbar} Z^0 \right) - \frac{\hbar^2 \tilde{\sigma}}{2m^2} \left| \nabla \theta + \nabla \chi + \frac{e}{\hbar c} \mathbf{Z} \right|^2 - \frac{\hbar^2}{8m^2 \tilde{\sigma}} |\nabla \tilde{\sigma}|^2 - \frac{\lambda}{m^2} (\tilde{\sigma} - \bar{\sigma})^2.$$

Next, the effective substitution

$$\tilde{\mathbf{Q}} = \frac{\hbar\tilde{\sigma}}{m} \left(\nabla \theta + \nabla \chi + \frac{e}{\hbar c} \mathbf{Z} \right)$$

is then made [8]. We have described this substitution as effective given that the above expression for $\tilde{\mathbf{Q}}$ is not given in [8]. We can then write

$$L_{GL} = -\frac{\hbar\tilde{\mathbf{Q}}}{m} \cdot \left(\nabla \theta + \nabla \chi - \frac{e}{\hbar c} \mathbf{Z} \right) - \frac{\hbar\tilde{\sigma}}{m} \left(\partial_t \theta + \partial_t \chi - \frac{e}{\hbar} Z^0 \right) + \frac{|\tilde{\mathbf{Q}}|^2}{2\tilde{\sigma}} - \frac{\hbar^2}{8m^2 \tilde{\sigma}} |\nabla \tilde{\sigma}|^2 - \frac{\lambda}{m^2} (\tilde{\sigma} - \bar{\sigma})^2.$$

The guage field $A^\mu = (A^0, \mathbf{A})$ is then introduced. A^μ is such that

$$(\tilde{\sigma} - \bar{\sigma}, \tilde{\mathbf{Q}}) = -\frac{\bar{\sigma}}{c} \left(\hat{\mathbf{k}} \cdot \nabla \times \mathbf{A}, c\hat{\mathbf{k}} \times \nabla A^0 + \hat{\mathbf{k}} \times \partial_t \mathbf{A} \right). \quad (2.4)$$

It is a guage field to be integrated out at a later stage. The phonons are integrated out as a result of this integration. c is considered finite in compressible condensates and infinite in incompressible condensates. It is expected that equations of motion which retain c will describe vortex motion in a compressible condensate [43]. It can be shown with some work that the dynamics described by L_{GL} are equivalent to the dynamics described by [8]

$$L_{eff} = -\frac{\bar{\sigma}}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\bar{\sigma}}{c} J^\mu (A_\mu + a_\mu) + \frac{e\bar{\sigma}}{mc} \epsilon^{\mu\nu\lambda} Z_\mu \partial_\nu (A_\lambda + a_\lambda) + \frac{\bar{\sigma}}{2c} \frac{B|\mathbf{E}|^2}{(1 - \frac{B}{c})} - \frac{\bar{\sigma}}{8} \frac{|\nabla B|^2}{(1 - \frac{B}{c})}$$

where $a^\mu = (0, \frac{cy}{2}, -\frac{cx}{2})$ and ε is the superfluid coherence length, defined by $\varepsilon = \frac{\hbar}{mc}$. $F_{\mu\nu}$ is the field

strength tensor and is written as [8]

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y \\ -E_x & 0 & -B \\ -E_y & B & 0 \end{pmatrix}.$$

By neglecting terms in L_{eff} which are greater than second order in $F_{\mu\nu}$ or involve higher derivatives acting on A^μ [8], a (2+1)-dimensional QED Lagrangian density is recovered,

$$L_{QED} = -\frac{\bar{\sigma}}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\bar{\sigma}}{c} J^\mu (A_\mu + a_\mu) + \frac{e\bar{\sigma}}{mc} \epsilon^{\mu\nu\lambda} Z_\mu \partial_\nu (A_\lambda + a_\lambda). \quad (2.5)$$

It is known that superfluid dynamics for a homogeneous superfluid in two spatial dimensions is equivalent to (2+1) QED [44]. Setting Z_μ to zero, we seek the extrema of the action $S = \int \int L_{QED} d\mathbf{r} dt$ with respect to A^μ . This results in the EL equations

$$\nabla^2 A^\mu - \frac{1}{c^2} \partial_t^2 A^\mu = -\frac{J^\mu}{c} \quad (2.6)$$

where $\mu = 0, 1, 2$. The equations represented by (2.6) are in fact Maxwell's equations [8], and take the form of the inhomogenous wave equation. We then seek a Green's function, G such that

$$\nabla^2 G - \frac{1}{c^2} \partial_t^2 G = -\delta(x - x_1) \delta(y - y_1) \delta(t).$$

It can be shown that [43]

$$G = \frac{1}{2\pi} \frac{\Theta\left(t - \frac{r}{c}\right)}{\left(t^2 - \frac{r^2}{c^2}\right)^{\frac{1}{2}}}$$

where $r = \{(x - x_1)^2 + (y - y_1)^2\}^{\frac{1}{2}}$ and Θ represents the Heaviside step function. Compressibility of the condensate results in retardation effects given that the retarded time $t - \frac{r}{c}$ appears as the argument of Θ . It follows that [43]

$$A^\mu = \frac{\hbar}{m} \int_{-\infty}^{\infty} \sum_i^N k_i P^\mu \frac{\Theta\left[t - t_1 - \frac{\langle\{x-x_i(t_1)\}^2 + \{y-y_i(t_1)\}^2\rangle^{\frac{1}{2}}}{c}\right]}{\left[(t - t_1)^2 - \frac{\langle\{x-x_i(t_1)\}^2 + \{y-y_i(t_1)\}^2\rangle}{c^2}\right]^{\frac{1}{2}}} dt_1. \quad (2.7)$$

$P^\mu = 1$ for $\mu = 0$, $\frac{dx_i(t_1)}{cdt_1}$ for $\mu = 1$, and $\frac{dy_i(t_1)}{cdt_1}$ for $\mu = 2$. It is briefly mentioned in [8] that the above expression for A^μ gives rise to the effective vortex action. We give a full derivation ([43] - see Appendix A.1) for the Lagrangian density, L_{min} of the extremized action S_{min} (extremized with respect to the gauge

field). It turns out that

$$L_{min} = -\frac{\bar{\sigma} J^\mu (A_\mu + 2a_\mu)}{2c}. \quad (2.8)$$

This derivation is analogous to writing down equation (3.20). The effective vortex-only action, S_{min} , then takes the form

$$S_{min} = S_1 + S_2 \quad (2.9)$$

where

$$S_1 = \frac{\bar{\sigma} h}{2m} \int_{-\infty}^{\infty} \sum_i^N \left\{ y_i(t) \frac{dx_i(t)}{dt} - x_i(t) \frac{dy_i(t)}{dt} \right\} dt$$

and

$$S_2 = -\frac{\bar{\sigma} h^2}{4\pi m^2} \times \sum_{\substack{i,j \\ i \neq j}}^N \sum_{\substack{i,j \\ i \neq j}}^N k_i k_j \int_{-\infty}^{\infty} \left[\int_{-\infty}^{s_{R_{i,j}}} \left[\frac{1}{\left\{ (t-t_1)^2 - \frac{R_{i,j}^2(t,t_1)}{c^2} \right\}^{\frac{1}{2}}} \left\langle 1 - \frac{1}{c^2} \left\{ \frac{dx_i(t)}{dt} \frac{dx_j(t_1)}{dt_1} + \frac{dy_i(t)}{dt} \frac{dy_j(t_1)}{dt_1} \right\} \right\rangle \right] dt_1 \right] dt.$$

We note that $R_{i,j}(t, t_1) = \left\langle \{(x_i(t) - x_j(t_1))^2 + \{y_i(t) - y_j(t_1)\}^2\}^{\frac{1}{2}} \right\rangle$ and $s_{R_{i,j}} = t - \frac{R_{i,j}(t, s_{R_{i,j}})}{c}$. Our calculated expression for S_{min} in which the QED vortex action is written explicitly in terms of the point vortex coordinates (using equations (2.3a) - (2.3c)) is not written in [8], and we have not encountered S_{min} elsewhere in the literature. Charge neutrality of the superfluid may be implemented by the condition $\sum_i^N k_i = 0$. It is also noteworthy that the above effective action suggests that the motion of a system of vortices depends on the previous motions of the system, suggesting the existence of ‘memory’ forces which we shall encounter in our discussion of microscopic considerations of vortex dynamics. Arovas & Freire note [8] that the Magnus force on a vortex may be extracted from the above theory by consideration of what will happen to the vortex in the presence of a background superflow, and they consider a Galilean-transformed Lagrangian density in order to achieve this. It is important to note that the above action, though it is a vortex-only action does take into account how density variations of the condensate will affect the motion of the vortices.

Another paper in which a vortex-only action is calculated for a two-dimensional BEC, and in which density fluctuations of the condensate are accounted for in describing the vortex motion, is the work of Lucas and Surówka in reference [7]. Their approach is not focused on QED, as that of Arovas and Freire is.

We find it convenient to employ some of their notation in discussing their work. They present a systematic calculation of the effective action of multiple superfluid vortices under the assumption that the underlying continuum action is the GP action [7]. Their analysis is correct in the dilute limit of vortices where the vortex core size is small compared to the intervortex spacing, and where fluctuations in the numbers of atoms in vortex cores are negligible [7]. Their calculation is perturbative, with the perturbative parameter being the ratio of the vortex core size, ε to the intervortex spacing, $\bar{r}$. Furthermore their calculation is valid at $\left\{O\left(\frac{\varepsilon^2}{\bar{r}^2}\right)\right\}$. The starting point for the analysis in this case is the GP action, which we will write as

$$S = \iint \left\{ \psi^* \left(i\partial_t + \frac{\nabla^2}{2m} - V \right) \psi - \frac{g}{2} (\psi^* \psi)^2 \right\} d\mathbf{r} dt$$

(in units such that $\hbar = 1$). The external potential V is taken as a negative constant in order that the theory's lowest energy state corresponds to a superfluid state [7]. g is the rescaled interparticle interaction constant and m is the boson mass. We can write the Euler-Lagrange (EL) equation

$$i\partial_t \psi = -\frac{1}{2m} \nabla^2 \psi + V\psi + g|\psi|^2 \psi$$

which will be discussed in more detail at a later stage in this thesis. The wave function ψ is treated in their analysis as

$$\psi = \rho_0^{\frac{1}{2}} e^{\chi + i\theta}, \psi^* = \rho_0^{\frac{1}{2}} e^{\chi - i\theta}$$

where

$$\rho_0 = -\frac{V}{g}$$

and both χ and θ are real. Essentially, their analysis is based on the key substitutions

$$\chi = \chi_{PV} + \delta\chi \tag{2.10a}$$

and

$$\theta = \theta_{PV} + \delta\theta. \tag{2.10b}$$

where χ_{PV} and θ_{PV} are the values that χ and θ will take assuming a point vortex ansatz while $\delta\chi$ and

$\delta\theta$ are fluctuations in χ and θ respectively. We point out that θ_{PV} takes the form [7]

$$\theta_{PV}(\mathbf{r}, t) = \sum_{n=1}^N k_n \arctan \frac{y - Y_n(t)}{x - X_n(t)} = \sum_{n=1}^N k_n \theta_n.$$

$X_n(t)$ in the above represents the horizontal or x -component of the trajectory of an individual vortex labelled n . $Y_n(t)$ represents the vertical or y -component on the other hand while k_n represents the vortex winding number or charge. The action S is then expanded as

$$S(\chi, \theta) = S_{PV} + \int \int \left\{ \begin{pmatrix} \frac{\delta S}{\delta \chi} & \frac{\delta S}{\delta \theta} \end{pmatrix} \begin{pmatrix} \delta \chi \\ \delta \theta \end{pmatrix} + \begin{pmatrix} \delta \chi \\ \delta \theta \end{pmatrix}^T \begin{pmatrix} \frac{\delta^2 S}{\delta \chi \delta \chi} & \frac{\delta^2 S}{\delta \chi \delta \theta} \\ \frac{\delta^2 S}{\delta \theta \delta \chi} & \frac{\delta^2 S}{\delta \theta \delta \theta} \end{pmatrix} \begin{pmatrix} \delta \chi \\ \delta \theta \end{pmatrix} \right\} d\mathbf{r} dt \quad (2.11)$$

where S_{PV} is a point vortex action. We note our independent derivation of a point vortex Lagrangian (to be discussed in section 4) and mention that their approach to writing S_{PV} differs from ours. A path integral formalism is employed in order to find the effective vortex action. That is, they consider [7]

$$Z = \int D\chi D\theta e^{iS(\chi, \theta)} \quad (2.12)$$

and integrate Z over the $\delta\chi$ and $\delta\theta$ modes, in order to obtain

$$Z = \int e^{\{iS_{eff}(X_n)\}} d\mathbf{X}_n.$$

While the resultant action will retain information on the influence of BEC density fluctuations on vortex motion, it will only be able to describe the vortex motion, as opposed to simultaneous description of vortex motion and BEC density variations. It will turn out that [7]

$$S_{eff} = S_{PV} - \frac{1}{2} \int \int \int \int \frac{\delta S}{\delta R}(\mathbf{r}, t) G_{R\tilde{S}}(\mathbf{r}, \mathbf{r}', t, t') \frac{\delta S}{\delta \tilde{S}}(\mathbf{r}', t') d\mathbf{r} dt d\mathbf{r}' dt' + \frac{i}{2} \text{tr} \log G$$

where the indices R , $\tilde{S}$ and T (see the equation below) refer to the fields χ and θ . G can be obtained by solving the equations

$$\frac{\delta S}{\delta R \delta \tilde{S}} G_{\tilde{S}T}(\mathbf{r}, \mathbf{r}', t, t') = -\delta_{RT} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t').$$

Analysis of the above action in the case of a vortex-antivortex pair (see [7] for details) shows that for two-dimensional, zero-temperature, superfluid turbulence, sound-mediated interactions are not capable of preventing an inverse energy cascade [7]. We note at this stage that experimental confirmation of the inverse cascade has been reported [45]. Put simply, the inverse energy cascade refers to a situation where energy flows from small length scales to large length scales [46]. An important theory underpinning this

experimental work was formulated by Onsager [47] in which he provided a statistics-based explanation for the formation of large, isolated vortices in unsteady flow in two dimensions. Deducible from Onsager's work is that the steady state behaviour of the two-dimensional turbulent system of quantized vortices may be described using a Boltzmann distribution with negative temperature [45]. Johnstone et al's paper [45] is the first experimental confirmation of Onsager's negative temperature states in a two-dimensional flow in a BEC. In this work, vortices are generated (in a two-dimensional setting) by moving a grid barrier (made up of an array of laser beams) through an oblate atomic cloud. It is yet another case in the literature of lasers being used to generate vortices. For an isolated system [45], the steady state of the turbulent system (two-dimensional vortex motion in this case) can be caused by evaporative heating, whereby the 'coldest' vortices are removed by annihilation of vortex-antivortex pairs which drives the inverse energy cascade. Another group which observed experimentally these Onsager clusters of vortices in a two-dimensional superfluid flow is that of Gauthier et al [48]. The inverse energy cascade is interesting given that large scale flow structures [46] are a result of two-dimensional turbulence which is in effect the production of order from disorder (turbulence). Interestingly, three-dimensional quantum turbulence in the steady state however exhibits a direct energy cascade whereby vortices break up over time and the system appears chaotic [45].

In the case of Hatsuda et al. [42], an adiabatic action for the Bose superfluid (referred to in their paper as a neutral superfluid, as opposed to a superconductor which is a charged superfluid) is calculated using topological Ginzburg-Landau theory which is described in [49]. They show that in the Bose superfluid, the centre of vorticities is the sole degree of freedom. It is mentioned also that the vortex energy is finite only when the total vorticity of the system is zero. Similarly to Arovas and Freire in [8], it is noted here that Galilean invariance is used to derive the Magnus force.

The topological Ginzburg-Landau theory which is built upon in this paper is a field theoretic approach which has the conventional NLSE as one of its field equations [42]. Helmholtz and Kelvin's theorems are guiding principles for the introduction of the topological terms and collective coordinates for the vortex centre [42]. In their approach, both the topological and source terms give rise to the Magnus force. The starting point for their action takes, for N vortices, the form (making use of some of the notation found in [42])

$$S = \iint \left\{ \psi^* (i\hbar\partial_0 + \hbar a_0) \psi - \frac{1}{2m} |(i\hbar\partial_i + \hbar a_i) \psi|^2 - g \left(|\psi|^2 - \rho_0 \right)^2 + \frac{\hbar}{2m} \epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} f_{\rho\sigma} + b_{\mu\nu} J^{\mu\nu} \right\} d\mathbf{r} dt - U_{pin}(X)$$

where

$$J^{\mu\nu}(x) = \sum_a^N \gamma_a \iint \left\langle \frac{\partial X_a^{[\mu}}{\partial \tau} \frac{\partial X_a^{\nu]}}{\partial \sigma} \delta^{(4)} \{x - X_a(\sigma, \tau)\} \right\rangle d\tau d\sigma.$$

As usual, ψ represents the wave function or order parameter, while m is the mass of a boson. a_μ is a vector potential and represents vortex singularities. g is a constant, while the term $g \left(|\psi|^2 - \rho_0 \right)^2$ takes the form of a potential which discourages the deviation of the number density $|\psi|^2$ from the value ρ_0 [49]. g is a constant, $\frac{mc_s^2}{2\rho_0}$ where c_s is the zero sound velocity. Compared to the standard Ginzburg-Landau Lagrangian density which we encountered earlier, the ‘topological term’ $\epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} f_{\rho\sigma}$ is new. $\epsilon^{\mu\nu\rho\sigma}$ is described as the complete antisymmetric tensor [49]. $b_{\mu\nu}$ is a rank-two antisymmetric tensor potential, and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$. $J^{\mu\nu}$ represents the vorticity current and $U_{pin}(X)$ is the pinning potential (which has no corresponding term in the standard Ginzburg-Landau Lagrangian density) with X representing the vortex coordinate. Regarding describing the vorticity current, $\gamma_a = \frac{2\pi k_a \hbar}{m}$ with integer k_a and $\frac{2\pi\hbar}{m}$ denotes vorticity unit. σ and τ represent parameters for the trajectory of the vortex.

In writing the above, the authors have incorporated into their theory the fact that in superfluids there are situations where the vortices become trapped by pinning potentials instead of moving with the background superfluid flow [42]. If the background flow becomes strong enough, depinning occurs and vortices are freed from the trap and can move [42]. Candidates for depinning mentioned in this paper are the Magnus force and quantum tunnelling.

To write down the effective action, the substitution $\psi = \rho^{\frac{1}{2}} e^{i\theta}$ is made and the resultant action is [42]

$$S = \iint \left\{ \hbar \rho (-\partial_0 \theta + a_0) - \frac{\hbar^2 \rho}{2m} |\nabla \theta + \mathbf{a}|^2 - \frac{\hbar^2}{8m\rho} |\nabla \rho|^2 - g(\rho - \rho_0)^2 + \frac{\hbar}{2m} \epsilon^{\mu\nu\rho\sigma} b_{\mu\nu} f_{\rho\sigma} + b_{\mu\nu} J^{\mu\nu} \right\} d\mathbf{r} dt - U_{pin}(X).$$

In the presence of a uniform background flow ($\mathbf{v}_{bg} = \text{constant}$) and in the absence of vortex excitations, the superfluid has a stationary state characterized by [42]

$$a^\mu = 0, \quad \rho = \rho_0, \quad \theta = \theta_{bg} \equiv \frac{m}{\hbar} \mathbf{v}_{bg} \cdot \mathbf{r} - \frac{m}{2\hbar} |\mathbf{v}_{bg}|^2 t.$$

The effective action is then written using the same path integral method as encountered in the work of Lucas and Surówka [7]. Phonon fluctuation is accounted for by expanding the amplitude and phase variables around the stationary solution,

$$\rho = \rho_0 + \delta\rho, \quad \theta = \theta_{bg} + \delta\theta.$$

The effective Lagrangian is then found by Gaussian integration with respect to $\delta\rho$ and $\delta\theta$. With some work (see [42] for details), this is found to be [42]

$$Z = \int D(\delta\rho, \delta\theta, a_\mu, b_{\mu\nu}) e^{iS(\rho, \theta, a_\mu, b_{\mu\nu}; X)} = e^{iS_{eff}(X)}$$

where (in a two-dimensional setting)

$$S_{eff}(X) = \int (L_{berry} + L_{ve} + L_{kin} + L_{int}) dt - U_{pin}(X),$$

with L_{Berry} , the Berry phase term (which is responsible for the Magnus force [42]) taking the form,

$$L_{Berry} = \frac{m\rho_0}{2} \sum_a^N \gamma_a \left\{ \hat{\mathbf{k}} \cdot (\dot{\mathbf{X}}_a - \mathbf{v}_{bg}) \times (\mathbf{X}_a - \mathbf{v}_{bg}t) \right\}.$$

L_{ve} is the vortex energy term,

$$L_{ve} = -\frac{m\rho_0}{4\pi} \left(\sum_a^N \gamma_a \right)^2 \ln \left(\frac{R}{d} \right),$$

where d and R are both ultraviolet and infrared cutoffs respectively. L_{kin} is the vortex kinetic term,

$$L_{kin} = \frac{1}{2} m_{eff} (\dot{\mathbf{X}}_{CV})^2$$

where

$$\mathbf{X}_{CV} = \sum_a^N \frac{\gamma_a}{\gamma_0} (\mathbf{X}_a - \mathbf{v}_{bg}t), \quad \gamma_0 \equiv \frac{2\pi\hbar}{m} \quad \& \quad m_{eff} = \left(\frac{m\rho_0}{4\pi c_s^2} \right) \gamma_0^2 \ln \left(\frac{R}{d} \right).$$

$\mathbf{X}_{CV}$ is the centre of vorticity of the system with m_{eff} being the inertial mass of the kinetic term. Lastly, we have the vortex interaction term

$$L_{int} = \sum_a^N \sum_{\substack{b \\ a \neq b}}^N \left(\frac{m\rho_0}{4\pi} \right) \gamma_a \gamma_b \ln \left(\frac{|\mathbf{X}_{ab}|}{d} \right).$$

In the above action, the effect of the phonons on the vortex motion is taken into account, however, as before, phonon radiation patterns cannot be calculated from this action.

Integrating out fluctuations in the wave function describing the superfluid using a path integral formulation as a means to writing down an effective vortex action is a recurring theme in the literature. Another paper using this technique is that of Wang et al. [50] in which they derive a two-dimensional effective vortex-position-only action for the superfluid by integrating out environmental (i.e. non-vortex) condensate phase and density fluctuations. To provide a simple mathematical summary of their work,

consider the action as written by Lucas and Surówka [7], i.e.

$$S = \iint \left\{ \psi^* \left(i\partial_t + \frac{\nabla^2}{2m} - V \right) \psi - \frac{g}{2} (\psi^* \psi)^2 \right\} d\mathbf{r} dt$$

and unlike what was done in reference [7], consider an expansion of the wave function in the form

$$\psi(\mathbf{r}, t) = \psi_V \{ \mathbf{r} - \mathbf{r}_V(t) \} + \delta\psi(\mathbf{r}, t)$$

where $\psi_V(\mathbf{r} - \mathbf{r}_V)$, a function of $\mathbf{r} - \mathbf{r}_V$, is a local minimum of $\varepsilon\{\psi(\mathbf{r})\}$, which is equal to

$$\int \left(\frac{1}{2m} |\nabla\psi|^2 + V|\psi|^2 + \frac{g}{2} |\psi|^4 \right) d\mathbf{r},$$

an energy functional for a given value of the vortex position $\mathbf{r}_V$. The fluctuations $\delta\psi$ and $\delta\psi^*$ are then integrated out [7]. In their paper, they also confirm the presence of the Magnus force.

It is worth noting that Popov's paper [51] also employs the path integral formulation to write down an effective action. As noted by Popov, the main idea in his work is the integration over 'fast' (high momentum [52]) variables and then over 'slow' (low momentum [52]) variables using different perturbation theory schemes at each stage. In this paper, he obtains the important result that there is equivalence between a two-dimensional phonon-vortex system and relativistic two-dimensional electrodynamics if the phonon-phonon interaction and the dispersion of the phonon spectrum are neglected [51].

2.2.2 Vortex Mass

An important quantity in vortex dynamics which is often sought from an effective action theory is the vortex mass. In the case of the QED effective action theory derived in [8], we shall perform a detailed calculation based on their work, along with some discussions. Prior to this however, we shall have a more broad discussion of the calculation of vortex mass in the literature. The three papers which form the basis of this discussion illustrate how the estimation of the vortex mass has varied over the years, from negligible to potentially infinite.

It will be useful in our discussions of vortex mass to understand the concepts of dynamic, and inertial mass. Dynamic mass [53], m_{dyn} may be extracted from Newton's second law as the ratio of the total force acting on an object to its acceleration. For a vortex line, this force usually accounts for the Magnus force, interactions due to other vortex lines and viscous force [53]. Conventionally, the dynamic mass is assumed to be equal to the inertial mass m_{inert} which is the coefficient of $\frac{1}{2}|\mathbf{v}|^2$ in the expression for the

kinetic energy of an object with velocity $\mathbf{v}$ [53].

We begin our discussion of the literature with a paper on superfluid helium [54] in which vortices are described as massless fluid configurations. According to [54], the motion of a vortex relative to the local superfluid induces a small additional irrotational flow which gives kinetic energy to the system and causes the vortex to have an effective mass. The effective vortex mass is negligible for practical purposes [54], but the form of the hydrodynamic equations describing the flow in the presence of the vortex is made more lucid by taking into account an effective vortex mass [54]. Considering the classical flow around a hard core vortex which has velocity $\mathbf{v}_L$ in a fluid which, at infinity, is motionless, the vortex mass may be estimated. For this calculation, the authors assume that the fluid is located within a large cylindrical bucket of radius R where both the cylinder and rotation axes are along the z direction. In cylindrical coordinates (r, z, Φ) , and further considering the vortex to be instantaneously at the origin, this flow takes the form [54]

$$\mathbf{v} = \frac{\varepsilon^2}{r^2} \mathbf{v}_L + \frac{\hbar}{mr} \hat{\Phi}$$

where ε is the coherence length and m is the mass of an atom of the element making up the superfluid. Associated with the above flow is a kinetic energy of the form [54]

$$E_{kin} = \int \left\{ \frac{1}{2} |\mathbf{v}_L|^2 \rho \pi \varepsilon^2 + \rho \pi \left(\frac{\hbar}{m} \right)^2 \ln \frac{R}{\varepsilon} \right\} dz$$

where ρ is the superfluid mass density. The coefficient of $\frac{1}{2} |\mathbf{v}_L|^2$ can be interpreted as the effective mass per unit length of the vortex line m_{core} , which is the mass per unit length of superfluid displaced by the core of the vortex [54]. To be clear,

$$m_{core} = \rho \pi \varepsilon^2.$$

The second paper [53] involves a discussion of Baym and Chandler's paper [54] regarding the calculation of vortex mass in a superfluid in the zero-temperature limit. In Duan's calculation, it is shown that the condensate compressibility contributes to a vortex mass which is significantly bigger than the vortex normal core mass calculated above. Whereas Baym and Chandler's work is based on a classical fluid model, Duan's work is based on Bose liquid theory, itself based on the central assumption that macroscopic occupation of a single quantum state, the zero momentum state is a characteristic of homogeneous Bose liquids at rest [55]. This contribution is due to gauge-symmetry breaking and vortex topology [53]. A vortex is considered by Duan as a core region of the size of the coherence length ε together with

an outside circulating supercurrent. The inertial and dynamical mass of the vortex according to Duan's calculation turn out to be equivalent and divergent with the system size.

The inertial mass of the vortex is calculated by equating the energy change of the system due to the global density change induced by the moving vortex line to the kinetic energy of the vortex line. First, the global density change of the system (outside the normal core) caused by the moving vortex (given an instant velocity with a magnitude of v by the author) is calculated. Next, the energy change for a unit length of the vortex line is computed in terms of this density change. The energy change will take the form of a typical kinetic energy expression involving the square of the initial given speed of the vortex. Area integration will result in an inertial mass given by [53]

$$m_{inert} = m_{core} \ln \frac{L}{\varepsilon}$$

where L is the sample size and it is clear that m_{inert} is logarithmically divergent with the system size. To calculate m_{dyn} , the total momentum of the system outside the vortex core is considered. Integrating the product of the change in density outside the core and the superfluid velocity (which may be calculated from the gradient of the phase of the order parameter in the theory - also outside the vortex core), we end up with the product of m_{dyn} and the given vortex velocity. It will turn out that m_{dyn} equals m_{inert} . Baym and Chandler, in [54] considered the frequency of the 'inertial wave' mode, which is a type of collective mode of a vortex lattice, whose existence is entirely as a result of a non-zero vortex mass [53]. Baym and Chandler showed that the frequency of the mode is inversely proportional to the vortex mass, and since in their case they relied only on the core mass and not external contributions to the vortex mass, they obtain a frequency so high they question the validity of their work [53]. However, with m_{inert} and m_{dyn} calculated above, the frequency calculated for this inertial wave mode may be reduced by an order of magnitude [53].

We turn to the work [56] of Thouless and Anglin who study vortex mass in superfluids at low frequencies. They apply their method to the Gross-Pitaevskii model and derive a formula for the vortex mass. In this analysis, the mass is shown to be dependent on the form of the pinning force which sustains the motion of the vortex while the Magnus force is present. The vortex mass is found to diverge as the radius of the pinning potential tends to zero [56]. This suggests that an unambiguous vortex mass may not exist and could depend on the measurement process [56]. A notable feature of their work is their careful consideration of how velocity is imparted to the vortex. They have pointed out that it is an omission for Duan [53] to give the vortex an instantaneous velocity without considering how the velocity is acquired.

They consider an external pinning potential which attracts the low density vortex core by repelling the fluid particles. They drag the vortex along a circular path by moving the pinning potential itself around a circular trajectory. The vortex therefore moves under the influence of a combination of Magnus and centripetal forces, which will be equal to the product of the vortex mass and acceleration. Furthermore, they consider slow enough orbits of the vortex that the effect of phonon wave radiation by an orbiting vortex may be neglected. They calculate a mass of the form [56]

$$M_V \approx 4\pi \int_{r_c}^R a_1 \ln r_c J_0(r) \left\{ \frac{d\psi(r)}{dr} + \frac{|\psi(r)|}{r} \right\} dr$$

where R is a length of the order of unity, a_1 is an integration constant, J is a Bessel function of the first kind, r_c is the radius of the pinning potential, and $|\psi(r)|$ represents the magnitude of the GPE order parameter ψ at r . This result is in agreement with Duan in the sense that compressibility at large r gives rise to logarithmically divergent mass in the low frequency limit [56]. It is worth noting that Popov [51] has also argued that compressibility at large r gives rise to a divergent vortex mass [56]. In Popov's work, the energy E required to generate a vortex at rest is determined, and the vortex mass is found by the Einstein formula, i.e $M_V = \frac{E}{c_s^2}$ where c_s is the speed of sound in the superfluid [56].

Having examined the literature, we return to focus on the QED approach of Arovas and Freire, which is of particular interest. We revert also to our previous notations for this theory employed in section 2.2.1. We begin by providing a detailed derivation (based on our previous work in [43]) of the vortex mass which they have obtained for the case of an isolated vortex. Only a summary of this calculation appears in [8].

From equation (2.9), the self-interaction part of the action be written for an isolated vortex as [43]

$$S_{self} = -\frac{k^2 h^2 \bar{\sigma}}{4\pi m^2 c} \int_{-\infty}^{\infty} \left\langle \int_0^{\infty} \left(1 - \frac{d\mathbf{X}(u)}{du} \cdot \frac{d\mathbf{X}(u+\sigma)}{du} \right) \left\{ \frac{\Theta(\sigma - |\mathbf{X}(u+\sigma) - \mathbf{X}(u)|)}{\sqrt{\sigma^2 - |\mathbf{X}(u+\sigma) - \mathbf{X}(u)|^2}} \right\} d\sigma \right\rangle du.$$

The distance variables σ and u are obtained by multiplying time variables t and t_1 by the speed of sound in the superfluid, c . $\mathbf{X}$ is the vortex location vector. The winding number of a single vortex is represented by k . Performing a Taylor expansion of the denominator, and approximating the distance $|\mathbf{X}(u+\sigma) - \mathbf{X}(u)|$ as being very small, we can write

$$S_{self} = -\frac{k^2 h^2 \bar{\sigma}}{4\pi m^2 c} \int_{-\infty}^{\infty} \left\langle \int_0^{\infty} \left\{ 1 - \frac{d\mathbf{X}(u)}{du} \cdot \frac{d\mathbf{X}(u+\sigma)}{d(u+\sigma)} \right\} \left\{ \frac{1}{\sigma} + \frac{|\mathbf{X}(u+\sigma) - \mathbf{X}(u)|^2}{2\sigma^3} + \dots \right\} d\sigma \right\rangle du.$$

From S_{self} , consider the terms

$$S_{self, M'(w)} = -\frac{k^2 h^2 \bar{\sigma}}{4\pi m^2 c} \int_{-\infty}^{\infty} \left\langle \int_0^{\infty} \left\{ -\frac{1}{\sigma} \frac{d\mathbf{X}(u)}{du} \cdot \frac{d\mathbf{X}(u+\sigma)}{d(u+\sigma)} + \frac{|\mathbf{X}(u+\sigma) - \mathbf{X}(u)|^2}{2\sigma^3} \right\} d\sigma \right\rangle du,$$

in order to identify the inertial term and find $M'(w)$. With some work (see Appendix A.2) it can be shown that

$$S_{self, M'(w)} = k^2 \mu c \int_{-\infty}^{\infty} \frac{dw}{2\pi} w^2 \left| \hat{\mathbf{X}}(w) \right|^2 \int_0^{\infty} \left\{ \frac{\cos(w\sigma)}{\sigma} - \frac{1 - \cos(w\sigma)}{w^2 \sigma^3} \right\} d\sigma.$$

The variable w has been introduced through fourier transformation of the vortex location vector $\mathbf{X}$. The dimensionality of w at this stage of the calculation is length^{-1} . Making the substitutions $\tilde{w} = cw$ and $\tilde{\sigma} = \frac{\sigma}{c}$, and then following this by switching the dummy variable $\tilde{w}$ with w and likewise switching the dummy variable $\tilde{\sigma}$ with σ , we can write

$$S_{self, M'(w)} = \frac{k^2}{4\pi} \int_{-\infty}^{\infty} M'(w) w^2 \left| \hat{\mathbf{X}}(w) \right|^2 dw. \quad (2.13)$$

In equation (2.13), the dummy variable w is now a frequency variable. It turns out that

$$M'(w) = 2\mu \int_{\delta}^{\infty} \left\{ \frac{\cos(w\sigma)}{\sigma} - \frac{1 - \cos(w\sigma)}{w^2 \sigma^3} \right\} d\sigma$$

where, according to [8], $\mu = \pi \varepsilon^2 \bar{\sigma}$ represents the vortex ‘core mass’, given that ε is the coherence length and $\bar{\sigma}$ is the average density. The problem with describing μ as the core mass however, is that given the expression for μ , the vortex would effectively be taken as circle of radius ε , contradicting the point vortex description of the model. $\left| \hat{\mathbf{X}}(w) \right|$ in equation (5.13) has a dimensionality of $\text{length} \times \text{time}$ which is a result of the equivalence of the dimensionality of $S_{self, M'(w)}$ and S_{self} . The integrand in equation (5.13) resembles the typical kinetic energy expression given by $\frac{1}{2} m_{inert} |\mathbf{v}|^2$ where $w \left| \hat{\mathbf{X}}(w) \right|$ plays the role of $|\mathbf{v}|$ in frequency space. Additionally, $M'(w)$ has a dimensionality of mass, and it will therefore be interpreted as an inertial mass term. Using integration by parts, it can be shown that

$$M'(w) = -\mu \left\langle \text{Ci}(|w|\delta) + \frac{\{1 - \cos(w\delta)\}}{w^2 \delta^2} + \frac{\sin(w\delta)}{w\delta} \right\rangle.$$

The temporal cut-off δ is introduced to regularize the integral required to calculate the mass [8]. The function $\text{Ci}(x)$ is the cosine integral such that $\text{Ci}(x) = -\int_x^{\infty} \frac{\cos(t)}{t} dt$. Accounting for the vortex core structure is beyond the QED theory [8] under examination. A ln expression for the mass may be obtained

by considering w to tend to zero, in which case

$$M'(w) \approx -\mu \ln(|w| \delta).$$

We have deduced that the model assumes that the inverse fourier transform of the frequency-dependent mass of the vortex is equal to zero for every value of the ‘time’ argument of the inverse fourier transform less than zero. We deduce this as the frequency-dependent mass has been assumed (in reference [8]) to have both real and imaginary parts with a Kramers-Kronig relation between them. Exploiting this relationship, and using [57], we calculate this imaginary part, $M''(w)$ (see Appendix A.3) as

$$M''(w) = \mu \left\{ \frac{\cos(w\delta) - 1}{w} - \text{sgn}(w) \text{si}(|w| \delta) + \frac{\sin(w\delta)}{w^2 \delta^2} - \frac{1}{w\delta} \right\} \quad (2.14)$$

where sgn is the sign function. $M''(w)$ can be used to calculate power dissipation per unit frequency by a moving vortex [8]. We note that the above functional expression for $M''(w)$ does not appear in their paper. Given that $M''(w)$ is an odd function, then

$$S_{self, M'(w)} = \frac{k^2}{4\pi} \int_{-\infty}^{\infty} M'(w) w^2 \left| \hat{\mathbf{X}}(w) \right|^2 dw = \frac{k^2}{4\pi} \int_{-\infty}^{\infty} \{M'(w) + iM''(w)\} w^2 \left| \hat{\mathbf{X}}(w) \right|^2 dw.$$

Verification of the Kramers-Kronig relations was carried out by successfully calculating $M'(w)$ using $M''(w)$.

It will be important in the future however, to carefully examine the effective action, in equation (2.9) in order to obtain the functional form of the vortex mass in a case where more than one vortex is present. We are also interested in the explicit vortex equations of motion when more than one vortex is present. Even a simple case, such as a vortex pair will be a useful case study. These may then be combined with a study of any resultant phonon radiation patterns, which are the subject of the next subsection.

2.2.3 Microscopic Considerations in describing Vortex Dynamics

We briefly discuss microscopic considerations in describing superfluid vortex dynamics. The work of Thompson [52] serves as a useful template to discuss this topic, in this case in a two-dimensional setting. We employ the notation found in this work. A useful summary of the forces at play in an isolated superfluid vortex line is provided by the following equation [52],

$$M_V \ddot{\mathbf{r}}_V(t) - \mathbf{F}_M - \mathbf{F}_I - \mathbf{F}_{\parallel} - \mathbf{F}_{boundary} - \mathbf{F}_{pinning} = \mathbf{F}_{fluct}(t),$$

where M_V is the mass of the vortex and $\mathbf{r}_V(t)$ is the position of the vortex core. Moving on to describe the forces, $\mathbf{F}_M$ represents the superfluid Magnus force, taking the form [52]

$$\mathbf{F}_M = \rho_s \kappa_V \times \{\dot{\mathbf{r}}_V(t) - \mathbf{v}_s\}$$

where ρ_s is the superfluid density, $\mathbf{v}_s$ is the superfluid velocity and κ_V represents the circulation along the vortex line. $\mathbf{F}_I$ is the Iordanskii force and is similar to the Magnus force but different in that it results from the motion of the vortex relative to normal fluid as opposed to the superfluid [52]. There have been a number of controversies surrounding the description of this force [52]. $\mathbf{F}_I$ can be written as [52]

$$\mathbf{F}_I = \rho_n \kappa_V \times \{\dot{\mathbf{r}}_V(t) - \mathbf{v}_n\}$$

where ρ_n is the normal fluid density and $\mathbf{v}_n$ is the normal fluid velocity. $\mathbf{F}_{||}$ is a longitudinal damping force which, like the Iordanskii force also arises due to the presence of normal fluid. It is usually written as [52]

$$\mathbf{F}_{||} = -\gamma \{\dot{\mathbf{r}}_V(t) - \mathbf{v}_n\}$$

where γ is a temperature-dependent damping coefficient. $\mathbf{F}_{boundary}$, as noted by Thompson is exerted on the vortex as a result of boundary conditions. $\mathbf{F}_{pinning}$ accounts for all forces due to imperfections of the container walls as well as forces due to impurities in the fluid. $\mathbf{F}_{fluct}$ is a fluctuating force, with correlations related to the effective damping forces. Further details of this force may be found in Thompson's work [52].

Interestingly, it is shown in Thompson's work [52] that the equation of vortex motion depends on the past motion of the vortex if a quantum description of the vortex motion is applied. For instance, both the longitudinal damping force and Iordanskii forces will depend on past vortex motion, i.e. 'memory' forces are a part of their description [52]. Such a 'memory-based' description of vortex motion also appears to apply to the motion of a collection of superfluid vortices, as we encountered in our examination of effective action theories describing superfluid vortices.

2.3 Concerning Phonon Emission

Phonon emission in two spatial dimensions will be studied by examination of key papers in this area. Some aspects of the complexity of describing phonon emission by accelerating vortices in three dimensions relative to two dimensions are discussed.

2.3.1 Examination of Phonon Radiation by Accelerating Vortices in Two Spatial Dimensions

Accelerating vortices radiate phonons. The numerical study of Parker et al. [9] provides a clear illustration of this concept in a two-dimensional setting. In their work, the vortex executes relatively simple, precessional motion, and the phonon radiation pattern is clearly elucidated as being a spiral pattern. Such spiral patterns are amenable to mathematical description. It is challenging however, even in this relatively simple scenario, to provide an analytically tractable model (as opposed to numerical simulation of the GPE) which simultaneously describes both the vortex motion and the phonon radiation pattern in a self-consistent manner while properly accounting for how motion is imparted to the vortex. This paper is useful for study in order to make progress in the effort to provide such a model, and we discuss aspects of their study in this part of the thesis.

Parker et al. study phonon emission by a vortex precessing in a dimple trap, embedded in a weaker harmonic trap which results in sound emission in a dipolar radiation pattern [9]. The observed phonon radiation pattern can be changed by alteration of the dimple depth [9]. The precession of the vortex is due to the inhomogeneous background density which can be interpreted as the result of a Magnus force due to the density gradient [9]. Acceleration of the vortex produces sound emission, which can be examined in the simulation. The simulation is such that the vortex-line is effectively devoid of curvature and therefore Kelvin waves can be ignored [9]. The quasi-two-dimensional ‘pancake’ BECs which this simulation is applicable to are susceptible to strong phase fluctuations, but these are suppressed in the very low temperature limit [9].

Analysis is based on the GPE, written in their paper as

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V_{ext}\psi + \bar{g}|\psi|^2\psi - \mu\psi \quad (2.15)$$

where ψ is the wave function, m is the atomic mass, $\mu = n\bar{g}$ is the chemical potential with n being the atomic density and $\bar{g} = \left(\frac{4\pi\hbar^2 a}{m}\right)$ where a is the s -wave scattering length which is considered to be positive. V_{ext} is the external confining potential given by

$$V_{ext} = V_0 \left(1 - e^{-\frac{2r^2}{w_0^2}}\right) + \frac{m(\omega_r^2 r^2 + \omega_z^2 z^2)}{2}$$

with w_0 being the waist of the Gaussian dimple, ω_r being the harmonic trap frequency in the x and y directions, while ω_z is the harmonic trap frequency in the z direction. The trap parameters were chosen in a way that it would be possible to observe phonon emission by the accelerating vortex before the thermodynamic lifetime of the vortex expired [9]. If the conditions $\omega_z \gg \omega_r$ and $l_z \gg a$ are met, where l_z is the transverse harmonic oscillator length, given by $\left(\frac{\hbar}{m\omega_z}\right)^{\frac{1}{2}}$, then the GPE can be reduced to a two-dimensional form which has a modified coefficient $g_{2D} = \frac{\bar{g}}{(2\pi l_z^2)^{\frac{1}{2}}}$ [9]. Measurement of the vortex energy is done by integration of the system energy,

$$E = \int \left\{ \frac{\hbar^2}{2m} |\nabla \psi|^2 + (V_{ext} - \mu) |\psi|^2 + \frac{\bar{g}}{2} |\psi|^4 \right\} d\mathbf{r}$$

across a ‘vortex region’ [9] which is a circle of radius $\frac{5\hbar}{(m\bar{g})^{\frac{1}{2}}}$ (five times the healing length) centred on the vortex core. In their paper [9] the use of the two-dimensional GPE is justified by the good agreement obtained by comparing the energy of the three-dimensional and two-dimensional form of the GPE over time. When $V_0 \gg \mu$ indicating a deep dimple, sound reabsorption occurs, and the vortex energy shows a sinusoidal pattern [9]. For shallow dimples (i.e. dimples such that $V_0 < \mu$), the vortex energy shows a monotonic decrease, indicating energy loss due to phonon emission [9].

The trajectory of the vortex for both shallow and deep dimples are measured. For the deep dimple, the vortex is unable to escape the harmonic region of the dimple but its trajectory is influenced by collective excitations of the superfluid [9]. For the shallow dimple, the vortex escapes the harmonic region of the dimple and moves in a spiral pattern towards lower density regions [9]. In this case, sound wave emission can be observed in the density distribution surrounding the vortex as it moves [9]. The sound waves are emitted in a direction perpendicular to the instantaneous direction of the motion of the vortex in a dipolar pattern, and the spiralling motion of the vortex modifies the dipolar pattern into a swirling distribution [9]. This is illustrated in figure 1.

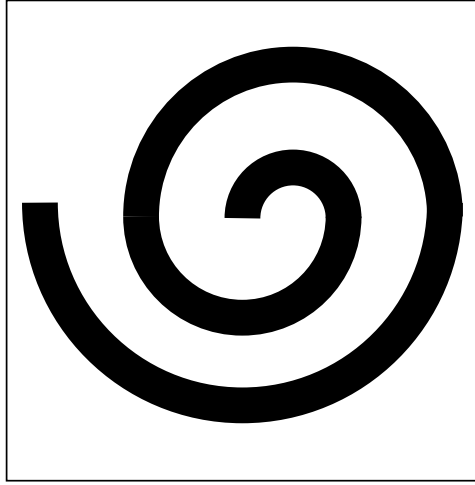


Figure 1: Simplified illustration of the swirling radiation distribution pattern produced by the spiralling vortex motion (adapted from [13]) in two spatial dimensions. The white and black colourings of the illustration represent the polarities of the sound wave emission. The rectangular box in the image represents the boundary of the observation area in this illustration.

As part of their analysis, they also calculate that for the shallow dimple, the power emitted from the accelerating vortex is proportional to the ratio of the square of the vortex acceleration to the vortex precession frequency. In calculating this power law, they report a dimensionless proportionality coefficient, $\bar{\beta}$, which is noted to be in close agreement with the calculation for this same dimensionless coefficient found by mapping a vortex in a homogeneous superfluid to a QED system [9]. We note for the sake of clarity that an inhomogeneous condensate system is not equivalent to a QED system. A homogeneous condensate, i.e. one with no distortion of the condensate distribution in space which is described by a nonlinear Schrödinger-type energy functional is however, equivalent to a QED system [44]. In the QED approach, the vortex is the charge while sound corresponds to electromagnetic radiation [44]. Both the electric and magnetic fields may be explicitly calculated if the vortex velocity is known in the QED approach, and thus this approach provides an avenue to analyze the phonons through such means as calculating the Poynting vector [44]. The QED approach however, will be less applicable in cases where there are severe distortions in the condensate distribution by the potential. Furthermore, in the paper employing the QED approach which they refer to (i.e. reference [44]), the vortex is given a velocity indicating circular motion without specifying how such a velocity would arise. Such a specification is required to improve the theory. In spite of these shortcomings however, a good agreement of calculated values of $\bar{\beta}$, as mentioned earlier, is obtained. The theory will need improvement in order to describe

the vortex motion seen in reference [9] (which is clearly not circular as seen during sound emission in the simulation), and at the same time predict the phonon radiation pattern. Regarding spiral patterns of the form we have encountered in this section, the mathematical form of a differential equation which could produce such a spiral pattern is given in [58], and may be worth looking at as a guide for algebraic manipulations which may be required for an appropriate theory.

When more than one vortex is present in the superfluid, there is an energetic interaction between these vortices. In the simplified point vortex model (which we study in section 4.3), a logarithmic interaction is present. The emission of sound waves by an accelerating vortex is an energetic process. The emission of sound waves where more than one vortex is present in the superfluid condensate is fundamentally different from the emission of sound waves from a single vortex due in part to the energetic interaction between the vortices. One such paper studying this scenario in a two-dimensional setting is that of Barenghi et al in [13]. In their paper, they consider the rotational motion of a vortex-vortex pair in a homogeneous system. The vortices radiate quadrupolar sound waves as they rotate about their central point. A simple illustration of the double-armed spiral wave pattern which they observed is illustrated as follows.

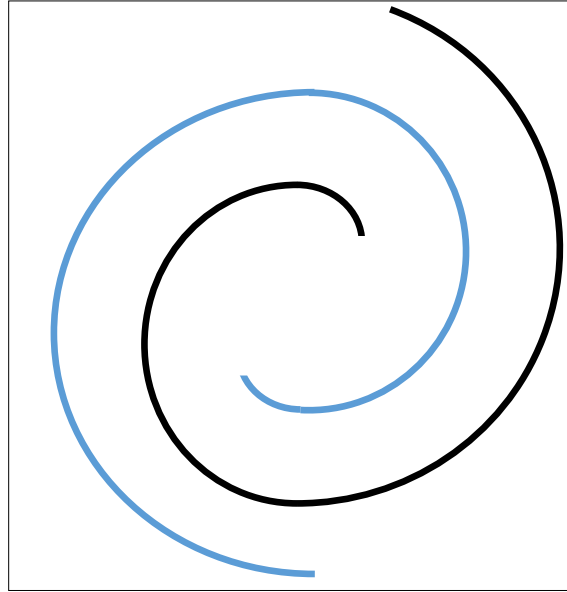


Figure 2: In the above simplified illustration (adapted from [13]), each colour in the phonon radiation pattern corresponds to an arm of the spiral pattern, with each vortex contributing an arm. Both blue and black represent the same polarity of phonon wave emission, while white represents a different polarity. Again, the rectangular box in the image represents the boundary of the observation area in this illustration.

They consider also a three-vortex interaction, in particular the close approach of a vortex-antivortex pair towards a single vortex. In their simulation the polarity of the single vortex and the nearest neighbour in the pair is the same. Acceleration of these interacting vortices produces a sound burst [13]. Furthermore, the energy loss caused by the sound burst results in a reduction in the separation of the vortex pair [13]. They also find that the magnitude of the energy loss is reduced by moving the initial position of the pair further away from the single vortex. It is reported in their study that if the single vortex and its nearest neighbour in the pair have opposite charges, it is likely a reconnection, i.e. pair formation, occurs between these vortices while the other vortex is left behind.

2.3.2 Vortex Motion and Phonon Radiation in Three Spatial Dimensions

We have mentioned that accelerating vortices produce phonons. Therefore, it is useful to have an understanding of vortex motion in three dimensions in order to understand phonon emission in three dimensional systems. We shall discuss briefly some of the complications which arise when describing vortice motion in three dimensions as opposed to two dimensions. Consider a simple illustration of point vortex motion in two dimensions.

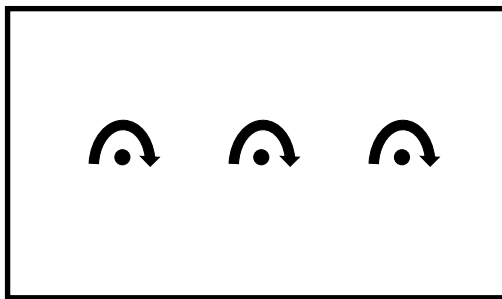


Figure 3: A simple illustration of three point vortices in a plane. The arrows indicate the direction of superfluid circulation.

A transition to a three-dimensional picture based on the above illustration can be made by consideration of vortex filaments, which are illustrated in figure 4.

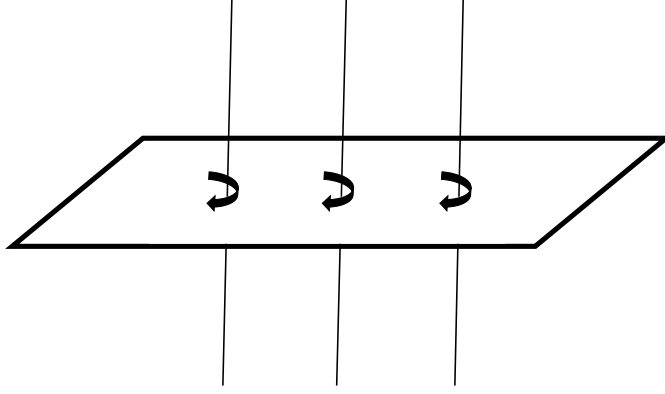


Figure 4: An illustration of the point vortices in figure 3 from a three-dimensional perspective. The point vortices now appear as vortex filaments.

We note that according to reference [5], the idea of a vortex filament was introduced in 1858 by Helmholtz. The vortex filament essentially extends the idea of a point vortex to three dimensions. It is clearly an approximation, but it is one that has been useful in studying vortex motion in three dimensions. Figure 4 still approximates a simple scenario in which the motion of the vortices is effectively two-dimensional. However, consideration of bending will complicate the description of the vortices beyond the two-dimensional picture. An example of how vortex bending may arise is found in the consideration of vortex generation in disk-shaped condensates [59]. If a disk-shaped condensate rotating faster than the critical angular velocity required for the generation of a metastable vortex [59], it is possible that the equilibrium vortex shape is curved [59]. The description of processes such as vortex reconnection (breaking and rejoining of vortex lines as a result of the touching of vortex filaments of equal and opposite circulation [60]) is more complicated in a three-dimensional setting due to such phenomena as vortex bending (see illustration in figure 8) as opposed to their description in a two-dimensional setting (as seen for example in reference [13]).

Besides bending, superfluid vortices may also form rings. As a result of reconnection processes, vortex loops continually break up and join together, forming a vortex tangle, which is a self-maintaining structure [61].

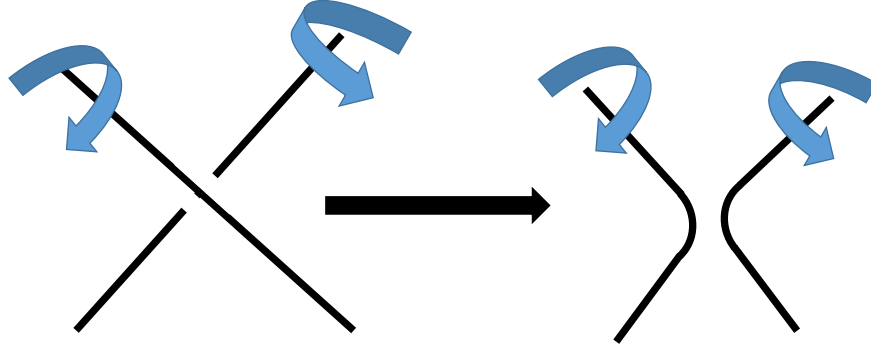


Figure 5: Illustration of the process of vortex reconnection (adapted from [62]).

Vortex tangle formation will further complicate the description of vortex dynamics in three, as opposed to two spatial dimensions.

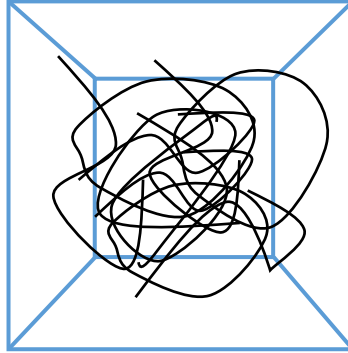


Figure 6: Illustration of a vortex tangle (adapted from [63]), viewed from one face of a cuboid, with the cuboid representing the boundary of the observation area.

Following our discussion of some of these complications, we now discuss practical ways in which the dynamics of the vortices may be studied in a three-dimensional setting. We begin by noting that three-dimensional vortex motion may be studied through numerical simulation of the GPE. Besides this however, another popular technique used to study vortex dynamics in superfluids is the Biot-Savart model. According to [10], quantized vortices are modelled as filaments in the Biot-Savart model, and move in accordance with the principles of classical fluid mechanics. When normal fluid is absent, the vortex

elements move with the local superfluid velocity [10], which is composed of any imposed superflow as well as flows caused by other vortices or by other parts of the same vortex [10]. The superfluid velocity at point $\mathbf{r}_0$ due to other vortices, or parts of the same vortex may be calculated using the Biot-Savart law [10]

$$\mathbf{v}_s(\mathbf{r}_0) = \frac{\kappa}{4\pi} \int \frac{(\mathbf{r} - \mathbf{r}_0) \times d\mathbf{r}}{|\mathbf{r} - \mathbf{r}_0|^3}$$

where κ is the quantum of circulation and the integration is carried out over all vortex filaments. When the local superfluid velocity is due in large part to the local curvature of the vortex line close to $\mathbf{r}_0$, then the local induction approximation (LIA) is sufficient to calculate the superfluid velocity at $\mathbf{r}_0$ [10]. In this case, the superfluid velocity at $\mathbf{r}_0$ is given by

$$\mathbf{v}^{(\text{loc})}_s(\mathbf{r}_0) = \frac{\kappa}{4\pi} \ln \left(\frac{1}{|\mathbf{r}''| \varepsilon} \right) (\mathbf{r}' \times \mathbf{r}'')_{\mathbf{r}=\mathbf{r}_0}$$

where ε is the superfluid coherence length and the primes denote differentiation with respect to the arc length measured along the vortex filament [10]. The above Biot-Savart model differs from a classical model only in the sense that the quantization of the vortex has been taken into account [10]. When quantum effects become important, such as on scales on the order of the superfluid coherence length, the model will fail [10]. Furthermore, as noted in [10], the LIA is not useful when long-range vortex interactions are important. The Biot-Savart model has also been used to describe vortex dynamics in the presence of normal fluid, see [64]. In [64], the presence of normal fluid and hence, the existence of mutual friction, is considered through the Schwarz model. The Schwarz model is, possibly the most important tool to understand superfluid turbulence in the presence of normal fluid [65] and is discussed in [65]. Hamiltonian equations have also been written for a vortex filament in a rotating BEC, with soliton solutions discussed [66]. In reference [66], the LIA in the dynamics of the filament is applied. For the LIA equation under consideration, the work is based on developing a variational principle for the equation. The Thomas-Fermi approximation is also in place.

Regarding phonon radiation, the fact that any oscillatory motion of the vortex results in phonon radiation [10] means that this is a more complicated problem in three-dimensions as opposed to two dimensions, given the wider variety of possible oscillations which may be accounted for in three dimensions as opposed to two. Kelvin waves (displacements of the vortex core which are helical in nature [13], such waves may be formed through vortex reconnections) which cannot be accounted for in a two-dimensional description of vortex dynamics play an important role in describing phonon radiation in three dimensions, as do vortex

reconnections. Consider for instance, the case of superfluid ^4He at temperatures low enough that the normal fluid fraction is insignificant. The frequencies of oscillations associated with vortex motion on the scale of the mean spacing between vortex lines [10] is often too small to produce significant phonon radiation [10]. Much higher oscillation frequencies are needed, and these are associated with smaller length scales [10]. Turbulent motion tends to transfer energy to smaller length scales [10] (although this is not the case in two-dimensional studies, see [45] for instance), increasing the likelihood of significant phonon radiation occurring [10]. Looking at a pair of vortex lines approaching one another within a vortex tangle, a reconnection, may occur, which in itself results in phonon radiation [10]. Kelvin waves produced by the reconnection may produce phonon radiation given their oscillatory nature. However, in ^4He , phonon radiation during reconnections is insignificant compared with phonon radiation from Kelvin waves [10]. Vortex reconnections produce Kelvin waves with a range of frequencies, of which only those that are sufficiently high may lead to efficient phonon radiation [10]. Most of the frequencies produced by the reconnection however are too low to achieve efficient phonon radiation [10]. Successive reconnections however, result in Kelvin waves with large wave amplitudes, and this results in nonlinear interactions [10]. These nonlinear interactions result in an energy transfer to the higher frequencies required for phonon emission [10]. This process, by which such high frequency Kelvin waves are produced, is known as a Kelvin wave cascade.

An important advance in the study of vortex dynamics is the ability to image in real-time the reconnection and interaction of two vortex lines, described in [67]. The focus of this work is on elongated BECs of ultracold atoms confined by magnetic harmonic potentials. An innovative imaging method is used in this work which combines numerical simulations of the NLSE with experimental technique. The experimental technique involves repeatedly extracting a small amount of the atoms from the BEC (i.e. outcoupling these atoms). The outcoupled atoms are then imaged through the absorption by these atoms of a probe laser beam after the outcoupled atoms have been spatially separated from the trapped condensate. The NLSE is used to simulate the outcoupling process to determine the effect a vortex (or vortices) of a given position and orientation in the BEC should have on the image of the outcoupled atoms. To put it simply, we want to know what the NLSE simulation tells us we should see in the image if we know where the vortex (or vortices) is.

3 The Gross-Pitaevskii Equation

Following our review section, we turn to a more detailed examination of the GPE. We begin this section with a summary of the derivation of the time-independent GPE given by Pethick and Smith [68], and similar to the summary found in [69]. In a Bose-Einstein condensate at low energy, and in the coordinate representation, the effective interaction between two particles takes the form $U_0\delta(\mathbf{r} - \mathbf{r}')$. U_0 is a constant, δ is the Dirac-delta function while $\mathbf{r}$ and $\mathbf{r}'$ are the positions of the two particles. The mean-field approach is employed, and it is assumed that the wave function is a symmetrized product of single-particle wave functions. In a fully condensed condensate, all bosons are in the same single-particle state $\phi(\mathbf{r})$ (normalised so that $\int |\phi(\mathbf{r})|^2 d\mathbf{r} = 1$), and the N -particle wave function can be written as $\prod_{i=1}^N \phi(\mathbf{r}_i)$. This wave function however does not take into consideration correlations generated by inter-atomic interactions.

These correlation effects are considered using the effective interaction $U_0\delta(\mathbf{r} - \mathbf{r}')$. The effective Hamiltonian is of the form,

$$H = \sum_{i=1}^N \left\{ \frac{p_i^2}{2m} + V(\mathbf{r}_i) \right\} + U_0 \sum_{i=1}^N \sum_{\substack{j=1 \\ i \neq j}}^N \delta(\mathbf{r}_i - \mathbf{r}_j). \quad (3.1)$$

The boson's mass is represented by m while the momentum of the boson is represented by p . $V(\mathbf{r})$ represents the external potential. The expectation value of the above Hamiltonian which corresponds to the N -particle wave function $\prod_{i=1}^N \phi(\mathbf{r}_i)$ is found to be

$$E = N \int \left\{ \frac{\hbar^2}{2m} |\nabla \phi(\mathbf{r})|^2 + V(\mathbf{r}) |\phi(\mathbf{r})|^2 + \frac{(N-1)}{2} U_0 |\phi(\mathbf{r})|^4 \right\} d\mathbf{r}, \quad (3.2)$$

which consists of a kinetic term, a potential term and an interaction term respectively from left to right. The prefactor of $\frac{N(N-1)}{2}$ in the interaction term represents the number of ways of making pairs of bosons. We then introduce the condensate wave function $\eta(\mathbf{r}) = N^{\frac{1}{2}}\phi(\mathbf{r})$, and assume a large value of N so we may neglect terms of order $\frac{1}{N}$, reducing equation (3.2) to

$$E = \int \left(\frac{\hbar^2}{2m} |\nabla \eta(\mathbf{r})|^2 + V(\mathbf{r}) |\eta(\mathbf{r})|^2 + \frac{1}{2} U_0 |\eta(\mathbf{r})|^4 \right) d\mathbf{r}. \quad (3.3)$$

The sign of U_0 defines either an attractive or repulsive interaction. U_0 is positive for repulsive interactions,

and negative for attractive interactions. The GPE is recovered by minimising the energy, E in equation (3.3), with respect to independent variations of η and η^* . This minimization is subject to the normalization constraint $N = \int |\eta(\mathbf{r})|^2 d\mathbf{r}$ which states that the number of condensate particles is a constant. Minimization of the energy in this way leads to the equation

$$-\frac{\hbar^2}{2m}\nabla^2\eta(\mathbf{r}) + V(\mathbf{r})\eta(\mathbf{r}) + U_0 |\eta(\mathbf{r})|^2 \eta(\mathbf{r}) = \mu\eta(\mathbf{r}), \quad (3.4)$$

which is a time-independent NLSE, the time-independent GPE. μ is the chemical potential, and is a Lagrange multiplier employed in the derivation of equation (3.4) to ensure that N remains a constant. Equation (3.4) has the form of a linear Schrödinger equation where the potential acting on the particles is the sum of a nonlinear term and the external potential. The chemical potential, μ serves as the eigenvalue. A time-dependent generalization of equation (3.4) is used to study dynamical problems of condensates. The time-dependent generalization of equation (3.4) takes the form [68] ,

$$-\frac{\hbar^2}{2m}\nabla^2\eta(\mathbf{r}, t) + V(\mathbf{r})\eta(\mathbf{r}, t) + U_0 |\eta(\mathbf{r}, t)|^2 \eta(\mathbf{r}, t) = i\hbar\partial_t\eta(\mathbf{r}, t). \quad (3.5)$$

We can rewrite equation (3.5) in ‘natural’ units which effectively imply that $m = \hbar = 1$ ([70], [43]). This is essentially a change of variables in order to more conveniently capture the ‘mathematical essence’ [71] of the GPE. Reconsider equation (3.5) and let

$$x' = \frac{x}{\varepsilon}, y' = \frac{y}{\varepsilon} \text{ and } z' = \frac{z}{\varepsilon}$$

where ε is the condensate healing length. Let

$$t' = \frac{t}{\tau}$$

where τ is such that

$$\eta(\mathbf{r}, t) = \bar{\eta}(\mathbf{r}) e^{-i\frac{t}{\tau}}$$

noting that $\bar{\eta}(\mathbf{r})$ represents the solution of the time-independent NLSE. Furthermore let

$$\eta' = \frac{\eta}{\eta_0}$$

where η_0 is the wave function of a uniform condensate at rest and is equal to $\left(\frac{\mu}{U_0}\right)^{\frac{1}{2}}$. Making the above

substitutions we have a dimensionless NLSE

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi, \quad (3.6)$$

which can be said to be written in ‘natural’ or ‘healing length’ units. ψ is the rescaled wave function, V is the rescaled external potential (and is time-independent in all of our considerations), and g is the rescaled interparticle interaction constant. The structural properties of equation (3.6) shall be examined more closely in the following subsection.

3.1 Structural Properties of the Gross-Pitaevskii Equation

The primary reference for this subsection of the thesis is Sulem and Sulem’s book [72]. Our discussions in this subsection follow closely our discussions in reference [69]. We discuss the Hamiltonian and Lagrangian structures of the GPE, as well as the application of Noether’s theorem to obtain conserved quantities.

3.1.1 Lagrangian Structure

We consider a more general form of the NLSE encountered in equation (3.6), that is,

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + f(|\psi|^2)\psi. \quad (3.7)$$

We impose that the above equation is defined in $\mathbf{R}^d$, with ψ denoting a smooth solution that vanishes at infinity along with its derivatives. f is a smooth function of $|\psi|^2$ and we define, following [72], the relationship $F(\lambda) = \int_0^\lambda f(\lambda)d\lambda$. Consider the Lagrangian density

$$L_d = \frac{i}{2}(\psi^*\partial_t\psi - \psi\partial_t\psi^*) - \frac{1}{2}|\nabla\psi|^2 - F(|\psi|^2), \quad (3.8)$$

which has the corresponding action

$$S(\psi, \psi^*) = \int_{t_0}^{t_1} \int_{\mathbf{R}^d} L_d d\mathbf{r} dt, \quad (3.9)$$

which is a functional of admissible regular functions for which $\psi(\mathbf{r}, t_0) = \psi_0(\mathbf{r})$ and $\psi(\mathbf{r}, t_1) = \psi_1(\mathbf{r})$. If the variation $\delta S = S(\psi + \delta\psi, \psi^* + \delta\psi^*) - S(\psi, \psi^*)$ vanishes at the time end points t_0 and t_1 , it can be shown through integration by parts that the NLSE, equation (3.7), is the corresponding EL equation.

3.1.2 Hamiltonian Structure

The Hamiltonian density, H_d , follows from the Lagrangian density, L_d in equation (3.8),

$$H_d = \frac{i}{2} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) - L_d. \quad (3.10)$$

From the properties of ψ outlined in section 3.1.1, it can be shown that the Hamiltonian $H = \int H_d d\mathbf{r}$ is a conserved quantity of the NLSE, equation (3.7). Using the EL equations associated with L_d , the variation of the Hamiltonian H , can be shown to take the form

$$\delta H = \int (i\delta\psi^* \partial_t \psi - i\partial_t \psi^* \delta\psi) d\mathbf{r}, \quad (3.11)$$

and the NLSE can be written as

$$i\partial_t \psi = \frac{\delta H}{\delta \psi^*}. \quad (3.12)$$

With an appropriate Poisson bracket, equation (3.12) can be rewritten as

$$i\partial_t \psi = \{\psi, H\}.$$

3.1.3 Noether's Theorem

We rewrite the action, equation (3.9), as

$$S(\psi, \psi^*) = \int_{t_o}^{t_1} \int_{\mathbf{R}^d} L_d(\psi, \nabla\psi, \partial_t\psi, \psi^*, \nabla\psi^*, \partial_t\psi^*) d\mathbf{r}dt, \quad (3.13)$$

where, following [72], $\xi = (t, \mathbf{r}) = (\xi_0, \xi_1, \dots, \xi_d)$, $\partial_0 = \partial_t, \partial = (\partial_0, \partial_1, \dots, \partial_d)$ and $\Psi = (\Psi_1, \Psi_2) = (\psi, \psi^*)$. By Noether's theorem, 'if the action (equation (3.9)) is invariant by the infinitesimal transformation of the dependent and independent variables $\psi \mapsto \psi + \delta\psi$ and $\xi \mapsto \xi + \delta\xi$ respectively then the conservation law

$$\partial_{\xi_\mu} (L_d \delta\xi_\mu + \partial_{\partial_\mu \Psi_i} \delta\Psi_i) = 0, \quad (3.14)$$

holds' [72]. Integrating over space and time variables, it follows from equation (3.14) that

$$\int \{(\partial_{\partial_t \psi} L_d) (\partial_t \psi \delta t + \nabla\psi \cdot \delta\mathbf{r} - \delta\psi) + (\partial_{\partial_t \psi^*} L_d) (\partial_t \psi^* \delta t + \nabla\psi^* \cdot \delta\mathbf{r} - \delta\psi^*) - L_d \delta t\} d\mathbf{r} \quad (3.15)$$

is a conserved quantity. For example, if the action is invariant under the phase shift transformation

$\tilde{\psi} = e^{is}\psi$ (which, in the case of infinitesimal s gives $\delta\psi = is\psi$), and at the same time $\delta t = \delta\mathbf{r} = 0$, then the particle number, i.e. $\int |\psi|^2 d\mathbf{r}$ is a conserved quantity. Should the action be invariant under the infinitesimal transformation $t \rightarrow t + \delta t$ with $\delta\psi = \delta\psi^* = \delta\mathbf{r} = 0$, then the Hamiltonian is conserved. In the case where the action is invariant under the infinitesimal space translation $\mathbf{r} \rightarrow \mathbf{r} + \delta\mathbf{r}$ and at the same time $\delta\psi = \delta\psi^* = \delta t = 0$ then the linear momentum $i \int (\psi \nabla \psi^* - \psi^* \nabla \psi) d\mathbf{r}$ is conserved. In a similar statement, invariance of the action under rotation through angle $\delta\theta$ around an axis $\mathbf{n}$, with $\delta\mathbf{r} = \delta\theta \mathbf{n} \times \mathbf{r}$ and $\delta\psi = \delta\psi^* = \delta t = 0$ results in the angular momentum $i \int \mathbf{r} \times (\psi^* \nabla \psi - \psi \nabla \psi^*) d\mathbf{r}$ being conserved.

Conservation of angular momentum in two, as well as three spatial dimensions was verified by manipulation of the above expression for angular momentum [69]. Employing that ψ is a smooth solution of the NLSE (written as $i\partial_t\psi = -\nabla^2\psi + g|\psi|^2\psi$ in our calculation) with the aforementioned properties, the expression $\frac{d}{dt} \{ \mathbf{J} = i \int \mathbf{r} \times (\psi^* \nabla \psi - \psi \nabla \psi^*) d\mathbf{r} \}$ was found to be zero. Using a Hamiltonian formulation, with a Poisson Bracket written by Zakharov and Manakov [73] in a cartesian coordinate system, conservation of angular momentum was again verified.

We also found [69] that $\hat{\mathbf{i}} \cdot \mathbf{J} = \hat{\mathbf{j}} \cdot \mathbf{J} = 0$, as well as the quantization $\hat{\mathbf{k}} \cdot \mathbf{J} = n$, where n is an integer. The normalization condition for ψ for our calculations was that $\int |\psi|^2 d\mathbf{r} = 1$.

3.1.4 Wave Function Collapse

Part of our studies on the structural properties of the GPE included the topic of wave function collapse. Berkshire and Gibbon [74] note that wave function collapse in the NLSE occurs when the amplitude of the wave function becomes infinite due to the collapse of the initial data to the centre of mass in a finite time. Wave function collapse and vortex formation are related in the sense that both involve singularities of the NLSE wave function.

In one spatial dimension, the cubic NLSE of the form

$$i\psi_t = -\partial_x^2\psi + g|\psi|^2\psi$$

with $g < 0$ is integrable by means of the Inverse Scattering transform (see ([75] & [76]) for a description of the method) as shown by Zakharov and Shabat [77]. In higher dimensions where $\partial_x^2 \rightarrow \nabla^2$ then it is possible for wave function collapse to occur [78].

We examined a number of references (such as ([79], [80] & [81])) on this topic, focusing on the time-rate of wave function collapse close to the time the singularity appears, with different authors obtaining

different time rates, even within the same number of spatial dimensions. An important idea in this area is that mechanical collapse, i.e. the decrease of the moment of inertia of the system, $I = \int |\psi|^2 |\mathbf{r}|^2 d\mathbf{r}$ to zero is synonymous with wave function collapse [79]. It has been pointed out in numerical studies that collapse actually occurs before I reaches zero however [79]. As we examined the work of Salahuddin [79], we considered what form the virial relation would take in the case of a harmonic potential, i.e. $V = \frac{1}{2}k' |\mathbf{r}|^2$, with k' being a constant. With some work, we found that for the NLSE given by equation (3.6), the virial relation takes the form

$$\frac{d^2 I}{dt^2} = 4H - 4k' I$$

where $H = \int \left(\frac{1}{2} |\nabla \psi|^2 + V |\psi|^2 + g |\psi|^4 \right) d\mathbf{r}$, the Hamiltonian, incorporating the external potential. The above equation has a solution

$$I(t) = \frac{2H}{k'} \sin^2 \left\{ (k')^{\frac{1}{2}} t \right\} + I^{(0)} \cos \left\{ 2(k')^{\frac{1}{2}} t \right\} + \left(\frac{dI}{dt} \Big|_{t=0} \right) \frac{1}{2(k')^{\frac{1}{2}}} \sin \left\{ 2(k')^{\frac{1}{2}} t \right\}.$$

We have not encountered the above virial relation or its solution in the literature.

3.2 Ground-State Solutions of the Gross-Pitaevskii Equation

In the language of physics, the ground-state of a system typically refers to the lowest energy state of the system. It is useful to recall at this stage that the GPE has been written by minimization of the condensate energy, and therefore any solution of the GPE represents a ground state of the BEC. The Thomas-Fermi solution is one such ground state, albeit an approximate one. The Thomas-Fermi solution is commonly discussed in the literature and is of significant practical importance. We shall discuss this regime in some detail in this thesis, following [68].

For a BEC with a large number of atoms, and having repulsive interactions, the Thomas-Fermi approximation whereby the kinetic energy term $\left\{ -\frac{1}{2} \nabla^2 \psi(\mathbf{r}) \right\}$ is small compared with other terms of the time-independent GPE and can be neglected is applied, and the GPE in natural units takes the form

$$\left\{ V(\mathbf{r}) + g |\psi(\mathbf{r})|^2 \right\} \psi(\mathbf{r}) = \mu \psi(\mathbf{r}).$$

Therefore we have the solution

$$V(\mathbf{r}) + g |\psi(\mathbf{r})|^2 = \mu$$

and we can write that

$$|\psi(\mathbf{r})|^2 = \frac{\mu - V(\mathbf{r})}{g}$$

(which is the Thomas-Fermi solution of the GPE) for $\mu > V(\mathbf{r})$ and

$$|\psi(\mathbf{r})|^2 = 0$$

if $\mu < V(\mathbf{r})$. This is because $|\psi(\mathbf{r})|^2$ cannot be negative. We note that we can write that

$$\psi(\mathbf{r}) = \left\{ \frac{\mu - V(\mathbf{r})}{g} \right\}^{\frac{1}{2}} e^{i\bar{\theta}}$$

as a Thomas-Fermi solution. The choice $\bar{\theta} = 0$ is often made in the literature. The condensate cloud therefore has a boundary of $V(\mathbf{r}) = \mu$. To examine the cloud in a little more detail, consider an isotropic potential of the form

$$V(\mathbf{r}) = \frac{1}{2}\omega^2 (x^2 + y^2 + z^2)$$

where ω represents the trap frequency. At the edge of the cloud,

$$V(\mathbf{r}) = \mu.$$

The resultant equation

$$\frac{1}{2}\omega^2 (x^2 + y^2 + z^2) = \mu$$

gives us the value of the Thomas-Fermi radius R , as

$$R = \left(\frac{2\mu}{\omega^2} \right)^{\frac{1}{2}}.$$

For an anisotropic potential, we consider at the edge of the cloud that

$$\frac{1}{2} (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2) = \mu, \tag{3.16}$$

where ω_x , ω_y and ω_z are the trap frequencies in the x , y and z directions respectively. The extension of the cloud in the x direction, R_x , is given by solving equation (3.16) when $\omega_y = \omega_z = 0$. Therefore

$$R_x = \left(\frac{2\mu}{\omega_x^2} \right)^{\frac{1}{2}}.$$

Similarly,

$$R_y = \left(\frac{2\mu}{\omega_y^2} \right)^{\frac{1}{2}}$$

and

$$R_z = \left(\frac{2\mu}{\omega_z^2} \right)^{\frac{1}{2}}.$$

Let us re-examine the central mathematical effect of the Thomas-Fermi approximation on the GPE, that is, the removal of the kinetic energy term from the time-independent GPE. Through order analysis [68], we shall examine whether neglecting the kinetic energy term is consistent with the properties of a cloud with a large number of atoms. We saw that

$$|\psi(\mathbf{r})|^2 = \frac{\mu - V}{g}. \quad (3.17)$$

We can write that

$$V(\mathbf{r}) = V(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla V(\mathbf{r}_0) + \dots$$

where $\mathbf{r}_0$ refers to the cloud surface. Near the edge of the cloud,

$$V(\mathbf{r}) \approx V(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla V(\mathbf{r}_0).$$

Recalling that

$$V(\mathbf{r}_0) = \mu,$$

we can write that

$$V(\mathbf{r}) \approx \mu + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla V(\mathbf{r}_0).$$

Therefore

$$\mu - V(\mathbf{r}) \approx (-1) (\mathbf{r} - \mathbf{r}_0) \cdot \nabla V(\mathbf{r}_0) = (\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{F}$$

where

$$\mathbf{F} = -\nabla V(\mathbf{r}_0).$$

We recognise $\mathbf{F}$ as the force exerted by the potential on a condensate particle at the edge of the cloud

[68]. We can now write a solution to equation (3.17) in the form

$$\psi(\mathbf{r}) \approx \left\{ \frac{(\mathbf{r} - \mathbf{r}_0) \cdot \mathbf{F}}{g} \right\}^{\frac{1}{2}}.$$

Following [68], the co-ordinate in the $\nabla V(\mathbf{r}_0)$ direction will be labelled x , and the position of the surface will be labelled as x_0 . Then for $x \leq x_0$ i.e. inside the cloud,

$$\psi(\mathbf{r}) \propto (x_0 - x)^{\frac{1}{2}}.$$

and it follows that

$$\frac{d}{dx}\psi(\mathbf{r}) \propto (x_0 - x)^{-\frac{1}{2}}.$$

Therefore the energy functional kinetic energy term $\frac{1}{2}|\nabla\psi|^2$ behaves as a term proportional to

$$\left\{ (x_0 - x)^{-\frac{1}{2}} \right\}^2 = (x_0 - x)^{-1}.$$

We note [68] that the kinetic energy per unit area behaves as

$$\int \frac{1}{x_0 - x} dx = -\ln(x_0 - x)$$

which will diverge as x tends to x_0^- .

We return to the issue of examining the size of the kinetic term compared to other terms in the energy functional. We recall that the energy functional to be minimized takes the form

$$E(\psi) - \mu N = \int \left(\frac{1}{2} |\nabla\psi|^2 + (V - \mu) |\psi|^2 + \frac{1}{2} g |\psi|^4 \right) d\mathbf{r}.$$

It will be convenient to perform order analysis on a per-atom basis following [68], and we consider

$$\frac{E(\psi) - \mu N}{N} = \frac{\int \left(\frac{1}{2} |\nabla\psi|^2 + (V - \mu) |\psi|^2 + \frac{1}{2} g |\psi|^4 \right) d\mathbf{r}}{\int |\psi|^2 d\mathbf{r}}.$$

Now

$$|\psi|^2 \propto (x - x_0).$$

Therefore

$$\frac{|\nabla\psi(\mathbf{r})|^2}{2|\psi|^2} \propto \frac{1}{2(x - x_0)^2}.$$

We recall that

$$V \approx \mu + (\mathbf{r}_0 - \mathbf{r}) \cdot \mathbf{F} = \mu - (x - x_0) \tilde{F}$$

and therefore we will write that

$$\frac{(V - \mu) |\psi|^2}{|\psi|^2} \propto \tilde{F} (x_0 - x).$$

We note also that the $\frac{g|\psi|^4}{|\psi|^2}$ term in Thomas-Fermi Regime is such that

$$\frac{g |\psi|^4}{|\psi|^2} \propto \frac{\tilde{F} (x - x_0)}{2}.$$

The $\frac{(V - \mu) |\psi|^2}{|\psi|^2}$ term is of greater magnitude than the $\frac{g|\psi|^4}{|\psi|^2}$ term. For consideration of the dominance of the kinetic energy (i.e. consideration of whether or not the kinetic energy term has the greatest magnitude of all three terms looked at individually), it will be sufficient to compare it with the $\frac{(V - \mu) |\psi|^2}{|\psi|^2}$ term. We look at the inequality

$$\frac{|\nabla \psi(\mathbf{r})|^2}{2} > (V - \mu) |\psi|^2$$

and determine the value of $x_0 - x$ for which

$$\frac{1}{2(x_0 - x)} > \tilde{F} (x_0 - x)^2.$$

The critical value of $x_0 - x$, δ for which the kinetic energy term is dominant is such that

$$\delta = \left(\frac{1}{2\tilde{F}} \right)^{\frac{1}{3}}.$$

Therefore kinetic energy term is dominant compared to the $V - \mu$ term for

$$x_0 - x < \delta.$$

The Thomas-Fermi solution is valid when the kinetic energy term is not dominant. As we have performed our analysis in one spatial dimension, the fraction of the volume for which the Thomas-Fermi solution is not valid is estimated to be

$$\propto \frac{\delta}{R}$$

where R is the Thomas-Fermi radius. It turns out that $\frac{\delta}{R}$ is a quantity which is known to be small when

a large number of particles are present [68].

3.3 Hydrodynamic Relations of the Gross-Pitaevskii Equation

It is well known that mathematical manipulation of the GPE can reveal the hydrodynamic relations the GPE implies. We perform this activity in this subsection, in the absence of vorticity and condensate rotation.

3.3.1 Continuity Equation

We consider the NLSE written in natural units, taking the form

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi. \quad (3.18)$$

Multiplying the NLSE by ψ^* , then

$$i\psi^*\partial_t\psi = -\frac{1}{2}\psi^*\nabla^2\psi + V|\psi|^2 + g|\psi|^4. \quad (3.19)$$

Taking the complex conjugate of the NLSE, then

$$-i\partial_t\psi^* = -\frac{1}{2}\nabla^2\psi^* + V\psi^* + g|\psi|^2\psi^*.$$

Multiplying the complex conjugate of the NLSE by ψ , we can write that

$$-i\psi\partial_t\psi^* = -\frac{1}{2}\psi\nabla^2\psi^* + V|\psi|^2 + g|\psi|^4. \quad (3.20)$$

Subtracting equation (3.20) from equation (3.19), we can write that

$$i\psi^*\partial_t\psi + i\psi\partial_t\psi^* = -\frac{1}{2}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*)$$

and it follows that

$$i\partial_t(|\psi|^2) = -\frac{1}{2}\nabla \cdot (\psi^*\nabla\psi - \psi\nabla\psi^*). \quad (3.21)$$

We recognise that equation (3.21) is a continuity equation. Making the substitution

$$\psi = \sigma e^{i\alpha}, \quad (3.22)$$

then it follows that

$$|\psi|^2 = \sigma^2$$

and σ^2 can be interpreted as an expression for density. Furthermore, from equation (3.22), we can make the following statements:

$$\nabla\psi = e^{i\alpha} (\nabla\sigma + i\sigma\nabla\alpha),$$

$$\nabla\psi^* = e^{-i\alpha} (\nabla\sigma - i\sigma\nabla\alpha),$$

$$\psi^*\nabla\psi = \sigma (\nabla\sigma + i\sigma\nabla\alpha),$$

$$\psi\nabla\psi^* = \sigma (\nabla\sigma - i\sigma\nabla\alpha)$$

and it follows that

$$\psi^*\nabla\psi - \psi\nabla\psi^* = \sigma\nabla\sigma + i\sigma^2\nabla\alpha - \sigma\nabla\sigma + i\sigma^2\nabla\alpha = 2i\sigma^2\nabla\alpha.$$

Returning to equation (3.21),

$$i\partial_t\sigma^2 = -\frac{1}{2}\nabla \cdot (2i\sigma^2\nabla\alpha),$$

and therefore the continuity equation can be written as

$$\partial_t\sigma^2 + \nabla \cdot (\sigma^2\nabla\alpha) = 0. \tag{3.23}$$

It follows that $\nabla\alpha$ can be interpreted as a superfluid velocity term. We can write

$$\nabla\alpha = \mathbf{v}$$

to emphasize this point. We note that equation (3.23) holds in both two and three spatial dimensions.

3.3.2 Analogue of the Euler Equation

Having obtained the continuity equation in terms of the variables σ and α , we turn to writing an analogue of the Euler equation for the GPE using these same variables. We recall the GPE, equation

(3.18),

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi.$$

As before, let

$$\psi = \sigma e^{i\alpha}.$$

We can make the following statements:

$$\partial_t\psi = e^{i\alpha}(\partial_t\sigma + i\sigma\partial_t\alpha)$$

and

$$i\partial_t\psi = e^{i\alpha}(-\sigma\partial_t\alpha + i\partial_t\sigma).$$

We saw that

$$\nabla\psi = e^{i\alpha}(\nabla\sigma + i\sigma\nabla\alpha).$$

Therefore, it follows that

$$\begin{aligned}\nabla \cdot \nabla\psi &= \nabla \cdot (e^{i\alpha}\nabla\sigma) + i\nabla \cdot \{(\sigma\nabla\alpha)e^{i\alpha}\} \\ &= e^{i\alpha}\nabla^2\sigma + \{\nabla\sigma \cdot (ie^{i\alpha}\nabla\alpha)\} + i\nabla \cdot (\sigma e^{i\alpha}\nabla\alpha) \\ &= e^{i\alpha}(\nabla^2\sigma + i\nabla\sigma \cdot \nabla\alpha) + i\{\nabla(\sigma e^{i\alpha}) \cdot \nabla\alpha + \sigma e^{i\alpha}\nabla^2\alpha\} \\ &= e^{i\alpha}(\nabla^2\sigma + i\nabla\sigma \cdot \nabla\alpha) + e^{i\alpha}(i\nabla\sigma \cdot \nabla\alpha - \sigma|\nabla\alpha|^2 + i\sigma\nabla^2\alpha).\end{aligned}$$

We note that

$$V\psi = V\sigma e^{i\alpha}$$

and

$$g|\psi|^2\psi = g\sigma^3 e^{i\alpha}.$$

Upon factoring out $e^{i\alpha}$, the GPE becomes

$$-\sigma\partial_t\alpha + i\partial_t\sigma = -\frac{1}{2}\nabla^2\sigma - \frac{i}{2}\nabla\sigma \cdot \nabla\alpha - \frac{i}{2}\nabla\sigma \cdot \nabla\alpha + \frac{\sigma}{2}|\nabla\alpha|^2 - \frac{i}{2}\sigma\nabla^2\alpha + V\sigma + g\sigma^3. \quad (3.24)$$

From the real part of equation (3.24), we can write that

$$-\sigma \partial_t \alpha = -\frac{1}{2} \nabla^2 \sigma + \frac{1}{2} \sigma |\nabla \alpha|^2 + V \sigma + g \sigma^3,$$

from which it follows that

$$-\partial_t \alpha = -\frac{1}{2\sigma} \nabla^2 \sigma + \frac{1}{2} |\nabla \alpha|^2 + V + g \sigma^2.$$

Taking ∇ of the above,

$$-\partial_t \nabla \alpha = \frac{1}{2} \nabla (|\nabla \alpha|^2) - \nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g \sigma^2 \right).$$

We shall write

$$\nabla \alpha = \mathbf{v},$$

recalling the appearance of $\nabla \alpha$ in the continuity equation. Therefore we can write

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g \sigma^2 \right) - (\mathbf{v} \times \text{curl} \mathbf{v}). \quad (3.25)$$

which is the analogue of the Euler equation we have sought. $(\partial_t + \mathbf{v} \cdot \nabla)$ is recognizable as a material derivative. The term $\left(\frac{\nabla^2 \sigma}{2\sigma} - V - g \sigma^2 \right)$ is known as the quantum pressure term [68], it serves the role of describing forces due to spatial variation of the condensate density [68]. In the absence of vortices, $\text{curl} \mathbf{v} = 0$ and equation (3.25) can be rewritten as

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = \nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g \sigma^2 \right).$$

Equation (3.25) is applicable in two and three spatial dimensions.

3.4 Condensate Surface Modes

An important topic in the study of superfluid vortices is the topic of condensate surface modes or condensate collective oscillations. These play an important role in the discussion of how superfluid vortices are formed. The GPE provides a useful framework for discussing these modes. We revisit the GPE as written in equation (3.18), given by

$$i \partial_t \psi = -\frac{1}{2} \nabla^2 \psi + V \psi + g |\psi|^2 \psi.$$

In this subsection, $V = V(\mathbf{r})$. Small amplitude oscillations are an important class of time-dependent solutions of the GPE [82]. For such oscillations, the changes in space and time of the order parameter with respect to the equilibrium configuration are small [82]. These oscillations emphasize the collective behaviour exhibited by interacting Bose gases in many cases [82]. Small amplitude oscillations may also admit a quantum description, which we shall discuss here [82]. Large amplitude oscillations may also be studied, such as the important problem of expansion of gas after releasing a trap [82]. However, we do not focus on studying such large amplitude oscillations in this thesis. To study the small oscillations, we consider the order parameter written in the form [82]

$$\psi(\mathbf{r}, t) = \psi'(\mathbf{r}, t)e^{-i\mu t} = \{\psi_0(\mathbf{r}) + \nu(\mathbf{r}, t)\} e^{-i\mu t}$$

where μ is the rescaled chemical potential. In line with how equation (3.6) was written, μ has been rescaled to units such that effectively, $m = \hbar = 1$. $\psi = \psi_0(\mathbf{r})e^{-i\mu t}$ represents a solution of the time-dependent GPE written in equation (3.6). ν is a small quantity for which we look for solutions of the form [82]

$$\nu(\mathbf{r}, t) = \sum_i \{u_i(\mathbf{r})e^{-i\omega_i t} + v_i^*(\mathbf{r})e^{i\omega_i t}\}$$

where ω_i is the oscillation frequency. The functions $u_i(\mathbf{r})$ and $v_i(\mathbf{r})$ are determined by solving the GPE in the linear limit. By collecting all the terms evolving in time like $e^{-i\omega_i t}$ and $e^{i\omega_i t}$, one obtains the following pair of differential equations, the Bogoliubov equations [82];

$$\omega_i u_i(\mathbf{r}) = \left\{ \hat{H}_0 - \mu + 2g\sigma^2(\mathbf{r}) \right\} u_i(\mathbf{r}) + g\sigma^2(\mathbf{r})v_i(\mathbf{r}) \quad (3.26a)$$

and

$$-\omega_i v_i(\mathbf{r}) = \left\{ \hat{H}_0 - \mu + 2g\sigma^2(\mathbf{r}) \right\} v_i(\mathbf{r}) + g\sigma^2(\mathbf{r})u_i(\mathbf{r}) \quad (3.27b)$$

where $\sigma^2(\mathbf{r}) = |\psi_0(\mathbf{r})|^2$ similarly to written previously and $\hat{H}_0 = -\frac{1}{2}\nabla^2 + V(\mathbf{r})$ with the index i being a label for the i th solution. The formalism of the above equations was developed by Pitaevskii to describe oscillations of a vortex line [82]. These equations provide the eigenfrequencies as well as the amplitudes u and v of the small amplitude oscillations of the system [82]. The solutions of equations (3.26a) and (3.27b) exhibit interesting properties which, as seen in [82], we shall discuss here.

One such property is that two solutions with frequencies ω_i and ω_j where $\omega_i \neq \omega_j$, satisfy the

orthogonality relation

$$\int (u_i^* u_j - v_i^* v_j) d\mathbf{r} = 0. \quad (3.28)$$

We note also that for each solution with amplitudes u_i and v_i and corresponding frequency ω_i , one can find another solution with amplitudes v_i^* and u_i^* and corresponding frequency $-\omega_i$ [82]. The physical oscillation represented by one of these solutions is the same as the physical oscillation represented by the other [82].

Equations (3.26a) and (3.27b) provide a platform to discuss the concepts of dynamical instability and energetic instability with respect to the small amplitude oscillations. It is easily shown from equations (3.26a) and (3.27b) that [82]

$$(\omega_i - \omega_i^*) \int (|u_i|^2 - |v_i|^2) d\mathbf{r} = 0, \quad (3.29)$$

which shows that, unless $\int (|u_i|^2 - |v_i|^2) d\mathbf{r} = 0$, the Bogoliubov equations (equations (2.26a) and (2.27b)) only allow solutions for which $\omega_i - \omega_i^* = 0$, that is, real frequency solutions. Dynamical instability of the system is associated with the occurrence of a complex frequency. Energetic instability on the other hand, can only destabilise the system in the presence of dissipative terms in the GPE which drive the system towards configurations with lower energy. The application of the concept of energetic stability to the topic of these small oscillations can be understood through the consideration of what is known as Landau instability. To describe the Landau instability, we consider the change in energy associated with the small oscillations around equilibrium (details of a sample calculation are seen in [82]). The Landau instability occurs when there exists a negative addition to the equilibrium value of the system energy due to one of the oscillating modes which results in an energetic instability [14].

Regarding a quantum description of the small oscillations, these can be illustrated [68] by specialization to a case where the condensate is subjected to a spherical trapping, and furthermore such that the Thomas-Fermi regime is in place. Under these conditions, it is possible to describe the total angular momentum of the oscillation with a quantum number l , while the projection of the total angular momentum on the polar axis may be described with a quantum number, m [68]. We note that $l = 2$ modes are called quadrupole modes [82]. Quadrupole modes are important in discussions of vortex formation as we have seen. For our discussion of the quantization of the oscillations, it will be convenient to revert back to the hydrodynamic description of the GPE, as was done in [68] in order to discuss this same topic. We base our presentation of this discussion on that found in [68].

We revisit the hydrodynamic formulation of the GPE (equation (3.18)) focusing on the hydrodynamic relations we recently derived [68], written in the form

$$\partial_t \sigma^2 + \nabla \cdot (\sigma^2 \mathbf{v}) = 0 \quad (3.30a)$$

and

$$\partial_t \mathbf{v} = -\nabla \left(\tilde{\mu} + \frac{1}{2} |\mathbf{v}|^2 \right), \quad (3.30b)$$

where

$$\tilde{\mu} = V + \sigma^2 g - \frac{\nabla^2 \sigma}{2\sigma}. \quad (3.30c)$$

We then write the density as $\sigma^2 = \sigma_{eq}^2 + \delta\sigma^2$, where σ_{eq}^2 is the equilibrium density (thus satisfying the time-dependent GPE) and $\delta\sigma^2$ is the deviation of the density from its equilibrium value [68]. Linearization of equations (3.30a) - (3.30c) by treating $\mathbf{v}$ and $\delta\sigma^2$ as small results in [68]

$$\partial_t \delta\sigma^2 = -\nabla \cdot (\sigma_{eq}^2 \mathbf{v}) \quad (3.31a)$$

and

$$\partial_t \mathbf{v} = -\nabla \delta\tilde{\mu} \quad (3.31b)$$

where $\delta\tilde{\mu}$ is obtained [68] by linearization of equation (3.30c). Taking the time derivative of equation (3.31a), along with elimination of the velocity using equation (3.31b) leads to the equation [68]

$$\partial_{t^2}^2 \delta\sigma^2 = \nabla \cdot (\sigma_{eq}^2 \nabla \delta\tilde{\mu}), \quad (3.32)$$

which describes the small oscillations of the condensate in an arbitrary potential V [68]. Neglecting the quantum pressure term (due to the implementation of the Thomas-Fermi assumption), $\tilde{\mu}$ reduces to $\sigma^2 g + V$, and we can write

$$\delta\tilde{\mu} = g\delta\sigma^2. \quad (3.33)$$

Inserting this result into equation (3.32), the density disturbance can be shown to satisfy the equation [68]

$$\partial_{t^2}^2 \delta\sigma^2 = g \nabla \cdot (\sigma_{eq}^2 \nabla \delta\sigma^2). \quad (3.34)$$

Considering oscillations of the form $\delta\sigma^2 \propto e^{-i\omega t}$ with respect to time dependence, equation (3.34) simplifies to [68]

$$-\omega^2 \delta\sigma^2 = g \left(\nabla \sigma_{eq}^2 \cdot \nabla \delta\sigma^2 + \sigma_{eq}^2 \nabla^2 \delta\sigma^2 \right). \quad (3.35)$$

Now $\sigma_{eq}^2 = \frac{\mu-V}{g}$ and therefore equation (3.35) becomes

$$\omega^2 \delta\sigma^2 = \left\{ \nabla V \cdot \nabla \delta\sigma^2 - (\mu - V) \nabla^2 \delta\sigma^2 \right\}. \quad (3.36)$$

Now for a spherical trap with $V(\mathbf{r}) = \frac{\omega_0^2 r^2}{2}$ with ω_0 being the trap frequency, we can find a solution of the form

$$\delta\sigma^2 = \bar{C} r^l Y_{lm}(\theta, \varphi) e^{-i\omega t}$$

where r , θ , and φ are spherical polar coordinates. Y_{lm} is a spherical harmonic and the previously mentioned quantum numbers are now manifest. We further note that $\omega^2 = l\omega_0^2$ and that $\bar{C}$ is an arbitrary constant. Solutions with $l = 2$ are called quadrupole modes [83]. Consider an anisotropic trap, with [68]

$$V = \frac{\omega_0^2 (r^2 + \lambda^2 z^2)}{2}$$

where $\lambda = \frac{\omega_z}{\omega_0}$ and $\omega_z \neq \omega_0$. ω_0 is the trap frequency in both the x and y directions, while ω_z represents the trap frequency in the z direction. For Ioffe-Pritchard type traps, it is possible to continuously adjust λ by varying currents in the coils [68]. For the above trap, the equilibrium density in the Thomas-Fermi approximation is equal to [68]

$$\sigma_{eq}^2 = \frac{\mu}{g} \left(\frac{R^2 - r^2 - \lambda^2 z^2}{R^2} \right)$$

with $R^2 = \frac{2\mu}{\omega_0^2}$. We can find for example solutions of the form [68]

$$\delta\sigma^2 \propto r^l Y_{l,\pm l}(\theta, \varphi)$$

with $\omega^2 = l\omega_0^2$ and $l = 2$ again corresponding to quadrupole modes [83].

3.5 Vortices

The only vortices admissible in a BEC are quantized vortices, unlike for classical fluids. The special properties of superfluids (compared to classical fluids for instance) are due to the fact that their motions are restricted by the fact that the superfluid velocity is proportional to the gradient of the BEC wave function phase [68], as we saw earlier. The GPE allows us to describe the presence of vortices in the condensate, which shall now be examined more closely for the case of a single vortex solution.

3.5.1 The Vortex Solution

Consider again, the GPE, written in natural units as

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi.$$

We shall examine the form of a single vortex solution in an infinite medium (with a uniform potential, $V = 0$) in which case we write the GPE as

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + g|\psi|^2\psi.$$

Let us assume a solution of the GPE of the form

$$\psi = Ae^{-i\mu t}.$$

where μ is the chemical potential. Assuming that A has no time dependence, then

$$0 = -\frac{1}{2}\nabla^2 A - \mu A + g|A|^2 A. \quad (3.37)$$

A vortex line corresponds to a solution of equation (3.37) possessing a symmetry relative to some axis [84]. This can be achieved by writing equation (3.37) in cylindrical coordinates r , θ and z and setting

$$A = e^{il\theta} F(r)$$

where $F(r)$ is real and l is the vortex winding number. We shall assume $l = \pm 1$ for mathematical convenience. Therefore

$$0 = -\frac{F'}{2r}e^{il\theta} - \frac{F''}{2}e^{il\theta} + \frac{F}{2r^2}e^{il\theta} - \mu F e^{il\theta} + gF^3 e^{il\theta}.$$

Factor out $e^{il\theta}$, and we can write that

$$0 = -\frac{F'}{2r} - \frac{F''}{2} + \frac{F}{2r^2} - \mu F + gF^3. \quad (3.38)$$

For mathematical convenience in examination of the vortex solution, following [68], we will rescale r and F in order to write equation (3.38) in terms of just two variables, a rescaled r variable and a rescaled F variable. We begin the search for this variable by examination of equation (3.38) at large distances from the axis of symmetry, where it is assumed that

$$-\frac{F'}{2r} - \frac{F''}{2} + \frac{F}{2r^2} \rightarrow 0$$

as $r \rightarrow \infty$. Therefore, at an infinite distance from the axis of symmetry (where $F = F_0$)

$$0 = -\mu F_0 + g F_0^3,$$

and it follows that

$$F_0 = \left(\frac{\mu}{g} \right)^{\frac{1}{2}}.$$

However, at small distances from the axis of symmetry, then the dominant expression of the right hand side of equation (3.38) is $-\frac{F'}{2r} - \frac{F''}{2} + \frac{F}{2r^2}$, in which case equation (3.38) now takes the form

$$-\frac{F'}{2r} - \frac{F''}{2} + \frac{F}{2r^2} = 0.$$

We shall now split the expression on the right hand side of equation (3.38) into two parts, one part which we expect to be dominant close to the axis of symmetry (A1), and another part which we expect to be dominant far from the axis of symmetry (A2). For both parts, we shall set $r = r_c$ which is the crossover variable and perform scale analysis of the individual terms. Consider

$$A1 = -\frac{F'}{2r} - \frac{F''}{2} + \frac{F}{2r^2}.$$

Performing scale analysis:

$$-\frac{F'}{2r} \sim -\frac{F}{2r_c^2},$$

$$-\frac{F''}{2} \sim -\frac{F}{2r_c^2}$$

and

$$\frac{F}{2r^2} \sim \frac{F}{2r_c^2}.$$

Therefore, in the crossover region

$$A1 = O\left(\frac{F}{2r_c^2}\right).$$

Now

$$A2 = -\mu F + gF^3.$$

At $r = r_c$,

$$O(A2) = O(\mu F) = O(gF_0^2 F).$$

Equating $O\left(\frac{F}{2r_c^2}\right)$ and $O(\mu F)$, then

$$\frac{F}{2r_c^2} = \mu F$$

which means that

$$r_c = \frac{1}{(2\mu)^{\frac{1}{2}}}.$$

We are now ready to introduce the rescaled r and F variables, and we write that

$$x = \frac{r}{r_c}$$

and

$$\chi = \frac{F}{F_0}.$$

We have scaled the distance r to the crossover length r_c and we have scaled the condensate wave function amplitude to its value F_0 at an infinite distance from the vortex. By this change of variables, we can rewrite equation (3.38) as

$$0 = -\frac{1}{x} \frac{d}{dx} \left(x \frac{d\chi}{dx} \right) + \frac{\chi}{x^2} - \chi + \chi^3.$$

A plot of the numerical solution of the above takes the form (from [68])

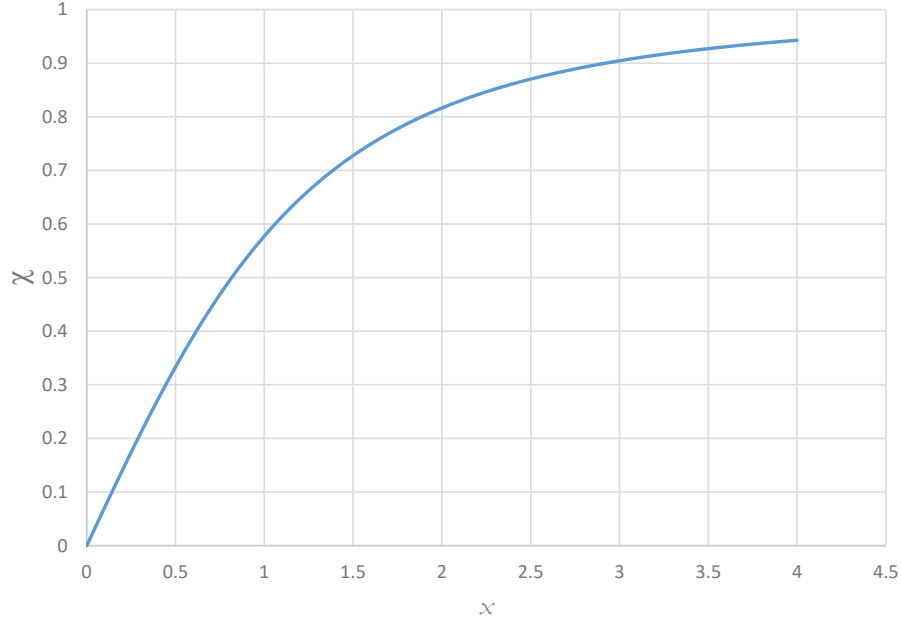


Figure 7: Plot of χ against x . A notable feature of this plot is the decrease of the condensate density to zero at the centre of the vortex, which is expected given how the condensate kinetic energy diverges on the vortex axis (see [68] and discussion in section 4.1 of this thesis).

3.5.2 Energetic Favourability of Vortex Formation

Having examined the vortex solution, let us next examine a condition under which vortex formation is energetically favourable. Our aim will be to calculate the difference between the energy,

$$E = \int \left(\frac{1}{2} |\nabla \psi|^2 + V |\psi|^2 + \frac{g}{2} |\psi|^4 \right) d\mathbf{r} \quad (3.39)$$

(to be calculated per unit length) of a uniform condensate system and a condensate system in which a vortex is present, under the condition that both systems have the same number of particles. We consider the vortex solution wave function of the time independent NLSE written as $f(r)e^{i\theta}$ and insert this solution into the energy expression of equation (3.39), with V set to zero, resulting in

$$\int \left\{ \frac{1}{2} \left(\frac{\partial f}{\partial r} \right)^2 + \frac{1}{2} \frac{f^2}{r^2} + \frac{g}{2} f^4 \right\} dr.$$

In cylindrical coordinates, $d\mathbf{r} = r d\theta dr dz$. Therefore

$$\frac{d\mathbf{r}}{dz} = r d\theta dr.$$

It will be of interest to examine

$$\epsilon = \int \left\{ \frac{1}{2} \left(\frac{\partial f}{\partial r} \right)^2 + \frac{1}{2} \frac{f^2}{r^2} + \frac{g}{2} f^4 \right\} \frac{d\mathbf{r}}{dz} \quad (3.40)$$

which represents the energy per unit length along the vortex axis (z axis). Furthermore, we restrict our consideration of the above integral only to atoms at a distance r_c (roughly the size of the vortex core radius [70]) from the vortex and up to the finite distance D from the vortex to obtain a well-defined integral (a slight modification of the calculation in [68]). The condition $D \gg r_c$ is assumed. Performing the θ integral right away, we obtain that

$$\epsilon = \int_{r_c}^D \left\{ \frac{1}{2} \left(\frac{\partial f}{\partial r} \right)^2 + \frac{1}{2} \frac{f^2}{r^2} + \frac{g}{2} f^4 \right\} 2\pi r dr.$$

We turn our attention to the uniform system. We shall now calculate the total energy per unit length along the z axis of a uniform condensate (with the same number of particles per unit length along the z axis as the vortex system) enclosed in a cylinder of radius D . In general, the system energy per unit volume at a given value of r can be written as

$$\frac{1}{2} |\nabla \psi|^2 + V |\psi|^2 + \frac{g}{2} |\psi|^4$$

where ψ is the condensate wave function. For uniform condensate where

$$|\nabla \psi| = V = 0$$

then the energy per unit volume is equal to

$$\frac{g}{2} |\psi|^4 = \frac{g}{2} \bar{n}^2$$

where

$$\bar{n} = |\psi|^2$$

with $\bar{n}$ being the average density for a uniform system. We recall that we are examining a cylinder of radius D with circular face area πD^2 . Therefore the average density of the uniform system, which is equal

to the ratio of the number of particles per unit length (ν) to the circular face area of the cylinder, is equal to $\frac{\nu}{\pi D^2}$. To be clear,

$$\bar{n} = \frac{\nu}{\pi D^2}.$$

as noted in [68]. The number of particles per unit length is (for both the vortex and the uniform system)

$$\nu = \int f^2 \frac{d\mathbf{r}}{dz} = \int \int f^2 r dr d\theta.$$

We perform the θ integral and recall r ranges from r_c to D , resulting in

$$\nu = \int_{r_c}^D 2\pi f^2 r dr = \int_{r_c}^D 2\pi r (f_0^2 - f_0^2 + f^2) dr = \int_{r_c}^D (2\pi r) (f_0^2) dr - \int_{r_c}^D (2\pi r) (f_0^2 - f^2) dr. \quad (3.41)$$

Now f_0 is the density of the vortex system far from the vortex. Therefore

$$\nu \approx \pi D^2 f_0^2 - \int_{r_c}^D (2\pi r) (f_0^2 - f^2) dr.$$

Energy per unit volume of the uniform system, as we have seen already is equal to $\frac{\bar{n}^2 g}{2}$. Therefore, energy per unit length of the uniform system, the product of energy per unit volume and circular surface area, can be written as

$$\epsilon_0 = \frac{\bar{n}^2 g}{2} \times \pi D^2 = \frac{\bar{n}^2 g \pi D^2}{2}.$$

Now $\bar{n} = \frac{\nu}{\pi D^2}$. Therefore

$$\epsilon_0 = \frac{\nu^2}{\pi^2 D^4} \times \frac{g \pi D^2}{2} = \frac{g}{2\pi D^2} \times \nu^2.$$

Now

$$\nu^2 \approx \pi^2 D^4 f_0^4 - 2\pi D^2 f_0^2 \int_{r_c}^D (f_0^2 - f^2) 2\pi r dr = \pi^2 D^4 f_0^4 - 4\pi^2 D^2 f_0^2 \int_{r_c}^D (f_0^2 - f^2) r dr.$$

Therefore

$$\epsilon_0 \approx \frac{\pi}{2} g D^2 f_0^4 - f_0^2 g \int_{r_c}^D (f_0^2 - f^2) 2\pi r dr. \quad (3.42)$$

In writing equation (3.42) we neglected (see [68]) terms of order $\left\{ \int_{r_c}^D (f_0^2 - f^2) 2\pi r dr \right\}^2$ in ν^2 . This is due

to the assumption that $D \gg r_c$. The difference in energy per unit length between the vortex and uniform systems is given by

$$\epsilon - \epsilon_0 \approx \int_{r_c}^D \left\{ \frac{1}{2} \left(\frac{df}{dr} \right)^2 + \frac{1}{2} \frac{f^2}{r^2} + \frac{g}{2} f^4 + g f_0^2 (f_0^2 - f^2) - \frac{g f_0^4}{2} \right\} 2\pi r dr.$$

Now

$$\frac{g}{2} f^4 + g f_0^2 (f_0^2 - f^2) - \frac{g f_0^4}{2} = \frac{g}{2} (f^2 - f_0^2)^2.$$

Therefore

$$\epsilon - \epsilon_0 \approx \int_{r_c}^D \left\{ \frac{1}{2} \left(\frac{df}{dr} \right)^2 + \frac{1}{2} \frac{f^2}{r^2} + \frac{g}{2} (f_0^2 - f^2)^2 \right\} 2\pi r dr = \epsilon_v. \quad (3.43)$$

We recall our rescaled variables, $x = \frac{r}{r_c}$ and $\chi = \frac{f}{f_0}$ and rewrite equation (3.43) in terms of the rescaled variables, resulting in

$$\epsilon_v = \pi f_0^2 \int_1^{\frac{D}{r_c}} \left\{ \left(\frac{d\chi}{dx} \right)^2 + \frac{\chi^2}{x^2} + \frac{1}{2} (1 - \chi^2)^2 \right\} x dx. \quad (3.44)$$

Observing the plot of the vortex solution, it is reasonable to estimate that $\left(\frac{d\chi}{dx} \right)^2$ and $\frac{1}{2} (1 - \chi^2)^2$ are negligible for $r \gg r_c$. To logarithmic accuracy we may estimate ϵ_v by neglecting these terms [68] in the integrand of the integral in equation (3.44). In doing so we find that

$$\epsilon_v = \pi f_0^2 \int_1^{\frac{D}{r_c}} \frac{\chi^2}{x^2} x dx \approx \pi f_0^2 \ln \left(\frac{D}{r_c} \right)$$

noting that for $r \gg r_c$, $\chi \sim 1$. r_c may be considered a lower distance cut-off for the vortex core radius [70]. Should the numerical solution of the NLSE be used in evaluating the vortex energy, the missing numerical factor in the argument of the logarithm will turn out to be 1.464 [68], that is

$$\epsilon_v = \pi f_0^2 \ln \left(1.464 \frac{D}{r_c} \right).$$

We are now in a position to estimate the critical angular velocity for vortex formation. To carry out this task it will be useful to calculate the energy of a given state viewed from the perspective of a rotating frame. We shall initially consider two states, E_1 which possesses a non-zero angular momentum (and therefore is a state in which at least one vortex is present) and E_0 , a state with no angular momentum.

Both E_1 and E_0 are considered to represent energy values in an inertial, non-rotating reference frame.

We have that

$$\Delta E^{rot} = E_1^{rot} - E_0^{rot}$$

is the energy difference between E_1 in a frame rotating with angular velocity $\mathbf{\Omega}$ which is E_1^{rot} and E_0 in the same rotating reference frame, referred to as E_0^{rot} . Now

$$E_1^{rot} = E_1 - \mathbf{\Omega} \cdot \mathbf{L} \quad (3.45)$$

where $\mathbf{\Omega}$ is the angular velocity and $\mathbf{L}$ is the angular momentum of the condensate. We shall rewrite equation (3.45) as

$$E_1^{rot} = E_1 - |\mathbf{\Omega}| |\mathbf{L}| \cos \Xi$$

and we shall choose, following references ([68] & [70]) that $\mathbf{\Omega}$ and $\mathbf{L}$ are in the same direction so that $\mathbf{\Omega} \cdot \mathbf{L} = |\mathbf{\Omega}| |\mathbf{L}| = \Omega L$. Therefore we can write that

$$E_1^{rot} = E_1 - \Omega L.$$

We note that

$$E_0^{rot} = E_0.$$

Therefore

$$\Delta E^{rot} = E_1 - E_0 - \Omega L.$$

Vortex formation is favoured under the condition that

$$\Delta E^{rot} < 0.$$

Therefore the condition for energetic favourability of vortex formation is that

$$E_1 - E_0 - \Omega L < 0,$$

equivalent to

$$\Omega > \frac{E_1 - E_0}{L}.$$

The value of $E_1 - E_0$ per unit length is known (ϵ_v). The form of L will be clarified. In general

$$L = |\mathbf{L}| = |\mathbf{r} \times \mathbf{p}| = |\mathbf{r}| |m_{total} \mathbf{v}| |\sin \varphi|$$

where $\mathbf{r}$ is the displacement vector, $\mathbf{v}$ is the velocity vector and m_{total} represents total mass. For the vortex system, the angle between $\mathbf{r}$ and $\mathbf{v}$, φ , is equal to $\frac{\pi}{2}$. Therefore

$$|\mathbf{L}| = m_{total} |\mathbf{v}| |\mathbf{r}|.$$

For the condensate, m_{total} is equivalent to the product of the mass of a boson, m and the particle number density n , integrated over the condensate volume. As the condensate is treated as a continuum, we can write (seen also in [70])

$$L = \int m n v r d\mathbf{r} = \int \int \int m n v r^2 dr d\theta dz$$

where v is the condensate speed. Now, we shall calculate L for the vortex state with height H (similar to a calculation seen in [70]) and as mentioned previously, with the same number of particles per unit length as the uniform state. The integral over z will be done first, resulting in

$$L = H \int \int m n v r^2 dr d\theta.$$

Reverting to natural units,

$$L = H \int \int n v r^2 dr d\theta.$$

We know that

$$v = |\nabla \theta| = \frac{1}{r},$$

therefore

$$L = H \int \int n r dr d\theta.$$

We make the approximation, since $D \gg r_c$, that n is effectively a constant and can be pulled out of the integral. We recall that integration over r is carried out from $r = r_c$ to $r = D$. It follows that

$$L = nH \left(\frac{D^2}{2} - \frac{r_c^2}{2} \right) (2\pi) \approx nH\pi D^2.$$

We recall that

$$\epsilon_v = \pi n \ln \left(1.464 \frac{D}{r_c} \right).$$

Therefore we will write that

$$E_1 - E_0 = H \pi n \ln \left(1.464 \frac{D}{r_c} \right).$$

It follows that we can estimate the critical velocity as

$$\Omega_c = \frac{E_1 - E_0}{L} = \frac{1}{D^2} \ln \left(1.464 \frac{D}{r_c} \right).$$

This investigation shows that if the condensate is rotating at a speed greater than Ω_c then the vortex state is favourable compared to the uniform state. Though the above investigation is for a specific case, determination of energetic favourability of vortex formation will in general involve the examination of expression (3.39) in order to compare the energies of different states.

4 Point Vortex Motion in Two Spatial Dimensions

We have looked at a single vortex solution of the GPE, and it is naturally of interest to consider the dynamics of multiple vortices. This will be explored using the point vortex model, which is a simplification. We shall further restrict our study to two-dimensional, incompressible condensates in the latter part of this section.

4.1 The GPE Wave Function: Point Vortex Distributions in Two Dimensions

It is useful to begin this discussion by consideration of a scenario in which the point vortex solution is applicable, using [7]. We recall that in natural units, we can write that

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi.$$

Consider a uniform medium where $V = 0$, resulting in

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + g|\psi|^2\psi.$$

Furthermore, let $\psi = \sigma e^{i\alpha} e^{-i\mu t}$, resulting in

$$\partial_t\psi = \partial_t(\sigma e^{i\alpha}) e^{-i\mu t} + (\sigma e^{i\alpha}) e^{-i\mu t}(-\mu i).$$

Labelling $\sigma e^{i\alpha} = A$, then the GPE becomes

$$i\{(\partial_t A) e^{-i\mu t} - i\mu A e^{-i\mu t}\} = -\frac{1}{2}\nabla^2 A e^{-i\mu t} + g|A|^2 e^{-i\mu t} A.$$

Therefore

$$i\partial_t A = -\frac{1}{2}\nabla^2 A - \mu A + g|A|^2 A$$

where μ is the chemical potential. We consider a uniform medium where $\mu > 0$. Following our discussion in [43], the wave function phase α is split into a smooth part θ , and a singular part, χ , in the form $\alpha = \theta + \chi$. If we assume a dilute limit where the density of vortices is small (further details in [7]), then

the point vortex ansatz whereby

$$\chi = \sum_i^N k_i \tan^{-1} \left(\frac{y - y_i}{x - x_i} \right),$$

may be applied. k_i is the charge of the i th vortex while (x_i, y_i) is the location of the i th vortex in the group of N vortices. Furthermore, the point vortex ansatz assumes that the condensate wave function amplitude may be written in the form e^{iB} where

$$B = \sum_i^N B_0(x - x_i, y - y_i)$$

and $B_0(r)$ is the solution to

$$B_0'' + (B_0')^2 + \frac{B_0'}{r} - \frac{1}{r^2} = \frac{e^{2B_0} - 1}{\zeta^2}$$

where $\zeta = \frac{1}{2\mu}$ with boundary conditions that $B_0(\infty) = 0$ and $B_0(0) = -\infty$. We note that $\sigma = 0$ at the positions given by (x_i, y_i) and that χ is singular at the location of a point vortex. Employing Green's theorem in the plane, it can be shown that [43]

$$\iint_{\bar{A}} (\partial_x \partial_y \alpha - \partial_y \partial_x \alpha) dx dy = \iint_{\bar{A}} (\partial_x \partial_y \chi - \partial_y \partial_x \chi) dx dy = \oint_C (\partial_x \chi dx + \partial_y \chi dy)$$

where the vortex distribution is enclosed by area $\bar{A}$. The contour C bounds the area $\bar{A}$. Defining a vortex as 'a point defect around which the change in phase for a complete journey around a closed contour enclosing the point defect is a multiple' [43], k_i of 2π , we can write

$$\oint_C (\partial_x \chi dx + \partial_y \chi dy) = 2\pi \sum_i^N k_i.$$

The above definition of a vortex is consistent with the requirement that the wave function is single-valued.

From the above, it follows that we can write

$$(\partial_x \partial_y - \partial_y \partial_x) \chi = 2\pi \sum_i^N k_i \delta(x - x_i) \delta(y - y_i) = \rho. \quad (4.1a)$$

Similarly, it can be written that

$$(\partial_y \partial_t - \partial_t \partial_y) \chi = 2\pi \sum_i^N k_i \frac{dx_i(t)}{dt} \delta(x - x_i) \delta(y - y_i) = J_x \quad (4.1b)$$

and

$$(\partial_t \partial_x - \partial_x \partial_t) \chi = 2\pi \sum_i^N k_i \frac{dy_i(t)}{dt} \delta(x - x_i) \delta(y - y_i) = J_y. \quad (4.1c)$$

It is noteworthy that equation (4.1a) is consistent with the well known relation (see [68] and [82] for instance)

$$\nabla \times \mathbf{v} = 2\pi k \delta(\mathbf{r}) \hat{\mathbf{k}} \quad (4.2)$$

for a vortex lying along the z axis. k represents the vortex winding number. We also recognize the similarity between equations (4.1a) - (4.1c) and equations (2.3a) - (2.3c). As was done in [68], consider a purely azimuthal flow in a trap i.e. external potential, which is rotationally invariant about the z axis. Taking r as the distance to the axis of the trap it follows that the flow speed at r is $\frac{k\hbar}{mr}$ [68] (reverting back to equation (3.5)) which will diverge on the axis of the trap [68]. To deal with this problem, the wave function is conventionally taken to vanish on the vortex line [68]. As we have seen, the point vortex ansatz is such that the wave function vanishes on the vortex line. This is accompanied in the model by a singularity in the wave function phase.

4.2 Hydrodynamic Relations of Rotating Condensates in the presence of Point Vortex Distributions

In the presence of a point vortex distribution, and incorporating condensate rotation in the $x-y$ plane at a constant rate Ω (which is easily removed by setting Ω to zero), we shall revisit the hydrodynamic relations of the GPE.

4.2.1 Continuity Equation

Following our derivation in [43], we consider (see Appendix B.1)

$$i\partial_t \psi = -\frac{1}{2} \nabla^2 \psi + V\psi + g|\psi|^2 \psi + i\Omega (y\partial_x \psi - x\partial_y \psi). \quad (4.3)$$

It can be readily shown (by considering $i\psi^* \partial_t \psi + i\psi \partial_t \psi^*$) that

$$i\partial_t \sigma = \frac{1}{2} \nabla \cdot \left\{ (\psi \nabla \psi^* - \psi^* \nabla \psi) + i\Omega (y\sigma^2 \hat{\mathbf{i}} - x\sigma^2 \hat{\mathbf{j}}) \right\}. \quad (4.4)$$

As before we define $\mathbf{v} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} = \nabla \alpha$. We also define $\mathbf{\Omega} = \Omega \hat{\mathbf{k}}$ and $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. We can then rewrite

equation (4.4) as

$$\partial_t \sigma^2 + \nabla \cdot \langle \sigma^2 \{ \mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r}) \} \rangle = 0. \quad (4.5)$$

which we recognise as a continuity equation. The expression $\mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r})$ is the condensate velocity as observed from an inertial reference frame where the rotation of the condensate at frequency Ω may be observed.

4.2.2 Analogue of the Euler Equation

To discuss the analogue of the Euler equation, we follow closely our derivation seen in [43]. To derive the Euler equation analogue, we substitute $\psi = \sigma e^{i\alpha}$ into equation (3.3) directly and then split into real and imaginary parts. This leads to the pair of equations:

$$\partial_t \sigma = - \left\{ (\nabla \sigma) \cdot (\nabla \alpha) + \frac{\sigma}{2} \nabla^2 \alpha \right\} + \Omega \{ y \partial_x \sigma - x \partial_y \sigma \} \quad (4.6a)$$

and

$$-(\partial_t \alpha)(\sigma) = -\frac{1}{2} (\nabla^2 \sigma - \sigma |\nabla \alpha|^2) + V \sigma + g \sigma^3 - \Omega \sigma (y \partial_x \alpha - x \partial_y \alpha). \quad (4.6b)$$

To be clear, the equation obtained from the imaginary part (equation 4.6a), is equivalent to the continuity equation seen in equation (4.5). From the real part, equation (4.6b), it can be shown that

$$\partial_t \alpha + \frac{1}{2} |\nabla \alpha|^2 + \Omega (x \partial_y \alpha - y \partial_x \alpha) = - \left(g \sigma^2 - \frac{\nabla^2 \sigma}{2\sigma} + V \right). \quad (4.7)$$

Differentiating equation (4.7) with respect to x , we find

$$\partial_t v_x + (\mathbf{v} \cdot \nabla) v_x - (J_y - \rho v_y) + \Omega \partial_x (x \partial_y \alpha - y \partial_x \alpha) = -\partial_x \left(g \sigma^2 - \frac{\nabla^2 \sigma}{2\sigma} + V \right). \quad (4.8a)$$

We can similarly differentiate equation (4.7) with respect to y to get

$$\partial_t v_y + (\mathbf{v} \cdot \nabla) v_y - (J_x - \rho v_x) + \Omega \partial_y (x \partial_y \alpha - y \partial_x \alpha) = -\partial_y \left(g \sigma^2 - \frac{\nabla^2 \sigma}{2\sigma} + V \right). \quad (4.8b)$$

Consider the expression,

$$\Omega \nabla (x \partial_y \alpha - y \partial_x \alpha) = \Omega (\partial_y \alpha + x \partial_x \partial_y \alpha - y \partial_x^2 \alpha) \hat{\mathbf{i}} + \Omega (-\partial_x \alpha + x \partial_y^2 \alpha - y \partial_y \partial_x \alpha) \hat{\mathbf{j}}.$$

We recall that $\boldsymbol{\Omega} = \Omega \hat{\mathbf{k}}$, and it follows that $\Omega(\partial_y \alpha \hat{\mathbf{i}} - \partial_x \alpha \hat{\mathbf{j}}) = -(\boldsymbol{\Omega} \times \mathbf{v})$. This allows us to write

$$\Omega \nabla (x \partial_y \alpha - y \partial_x \alpha) = -(\boldsymbol{\Omega} \times \mathbf{v}) + \Omega \rho \mathbf{r} + \Omega (x \partial_y v_x - y \partial_x v_x) \hat{\mathbf{i}} + \Omega (x \partial_y v_y - y \partial_x v_y) \hat{\mathbf{j}}.$$

We note that $(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla = -\Omega y \partial_x + \Omega x \partial_y$ which leads to $\{(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\} \mathbf{v} = \Omega (x \partial_y v_x - y \partial_x v_x) \hat{\mathbf{i}} + \Omega (x \partial_y v_y - y \partial_x v_y) \hat{\mathbf{j}}$. Therefore

$$\Omega \nabla (x \partial_y \alpha - y \partial_x \alpha) = -(\boldsymbol{\Omega} \times \mathbf{v}) + \Omega \rho \mathbf{r} + \{(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\} \mathbf{v}.$$

It follows that equations (4.8a) and equation (4.8b) may be combined, leading to

$$\partial_t \mathbf{v} + \langle \{\mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r})\} \cdot \nabla \rangle \mathbf{v} - \left\{ (\mathbf{J} - \rho \mathbf{v}) \times \hat{\mathbf{k}} \right\} - (\boldsymbol{\Omega} \times \mathbf{v}) + \Omega \rho \mathbf{r} = -\nabla \left(g \sigma^2 - \frac{\nabla^2 \sigma}{2\sigma} + V \right) \quad (4.9)$$

where $\mathbf{J} = J_x \hat{\mathbf{i}} + J_y \hat{\mathbf{j}}$. For convenience we define $\mathbf{w} = \rho \hat{\mathbf{k}}$ and it follows that $\mathbf{w} \times (\mathbf{r} \times \boldsymbol{\Omega}) = \rho \boldsymbol{\Omega} \mathbf{r}$. Equation (4.9) becomes

$$\partial_t \mathbf{v} + \langle \{\mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r})\} \cdot \nabla \rangle \mathbf{v} - \left\{ (\mathbf{J} - \rho \mathbf{v}) \times \hat{\mathbf{k}} \right\} - (\boldsymbol{\Omega} \times \mathbf{v}) - \{\mathbf{w} \times (\boldsymbol{\Omega} \times \mathbf{r})\} = -\nabla \left(g \sigma^2 - \frac{\nabla^2 \sigma}{2\sigma} + V \right)$$

from which it can be deduced that

$$\frac{D\mathbf{v}}{Dt} + \left\langle \{\rho(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) - \mathbf{J}\} \times \hat{\mathbf{k}} \right\rangle = -\nabla \left(g \sigma^2 - \frac{\nabla^2 \sigma}{2\sigma} + V \right) + (\boldsymbol{\Omega} \times \mathbf{v}) \quad (4.10)$$

where

$$\frac{D\mathbf{v}}{Dt} = \partial_t \mathbf{v} + \langle \{\mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r})\} \cdot \nabla \rangle \mathbf{v}$$

is the expression for the material derivative of the superfluid velocity $\mathbf{v}$. If both $\{\rho(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) - \mathbf{J}\}$ and Ω are set to zero, then equation (4.10) is analogous to the Euler equation in fluid dynamics. We have not encountered equation (4.10), which is a force equation, anywhere in the literature.

4.2.3 Convection of the Vorticity Distribution with the Superfluid Flow

Having derived equation (4.10), we are now in a position to calculate (again following our work in [43]) the material derivative of the point vortex distribution in the superfluid via $\frac{D\mathbf{w}}{Dt}$. We derive the relationships:

$$\nabla \times \{(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\} \mathbf{v} = \Omega (\nabla \cdot \mathbf{v}) \hat{\mathbf{k}} + \{(\boldsymbol{\Omega} \times \mathbf{r}) \cdot \nabla\} \mathbf{w}, \quad (4.11a)$$

$$\nabla \times (\mathbf{v} \cdot \nabla) \mathbf{v} = (\mathbf{v} \cdot \nabla) \mathbf{w} + (\nabla \cdot \mathbf{v}) \mathbf{w} \quad (4.11b)$$

and

$$\nabla \times (\boldsymbol{\Omega} \times \mathbf{v}) = (\boldsymbol{\Omega} \nabla \cdot \mathbf{v}) \hat{\mathbf{k}}. \quad (4.11c)$$

The equations (4.11a) - (4.11c) are very useful in the evaluation of the curl of equation (4.10), which with some work is found to be

$$\frac{D\mathbf{w}}{Dt} + (\nabla \cdot \mathbf{v}) \mathbf{w} + [\nabla \times \langle \rho \{ \mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r}) \} - \mathbf{J} \rangle] = 0. \quad (4.12)$$

It is interesting to consider a scenario [43] where $\rho \{ \mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r}) \} - \mathbf{J} = 0$. The mathematical statement $\rho \{ \mathbf{v} + (\boldsymbol{\Omega} \times \mathbf{r}) \} - \mathbf{J} = 0$ means that the superfluid velocity in the inertial reference frame (where the superfluid may be observed to rotate at angular frequency Ω) is equal to the sum of the velocities of the vortices in the distribution. If at the same time, we impose the condition that the condensate flow is incompressible, i.e. $\nabla \cdot \mathbf{v} = 0$, then it follows that

$$\frac{D\mathbf{w}}{Dt} = 0 \quad (4.13)$$

which indicates that the point vortex distribution will be convected with the flow [43].

4.3 Point Vortex Motion in Incompressible Condensates

Having examined the point vortex model in a compressible BEC, we now demonstrate how equations of motion for multiple point vortices in an incompressible BEC may be written. We have written the derivation below in an independent manner and we are yet to encounter the derivation below elsewhere in the literature. For a uniform condensate, the Lagrangian density corresponding to an incompressible condensate effectively takes the form

$$L_\alpha = -\partial_t \alpha - \frac{1}{2} |\nabla \alpha|^2. \quad (4.14)$$

Making the substitution

$$\mathbf{Q} = \nabla \alpha, \quad (4.15)$$

where $\mathbf{Q}$ is a Hubbard-Stratonovich field, allows us to write

$$L_\alpha := -\partial_t \alpha - \mathbf{Q} \cdot \nabla \alpha + \frac{1}{2} |\mathbf{Q}|^2. \quad (4.16)$$

Looking back to section 2, the mass current $\tilde{\mathbf{Q}}$ is analogous to the Hubbard-Stratonovich field $\mathbf{Q}$. The EL equation for α takes the form $\nabla \cdot \mathbf{Q} = 0$ and motivates the introduction of an ‘electromagnetic’ potential in the form

$$\mathbf{Q} = \partial_y \Phi \hat{\mathbf{i}} - \partial_x \Phi \hat{\mathbf{j}}.$$

It is also convenient to introduce a vector potential of a uniform ‘magnetic’ field in the form

$$\mathbf{P} = \frac{1}{2} (y \hat{\mathbf{i}} - x \hat{\mathbf{j}}).$$

These introductions allow us to write (see Appendix C.1)

$$L_\alpha := (\mathbf{P} \cdot \mathbf{J}) - \rho \Phi + \frac{1}{2} |\nabla \Phi|^2. \quad (4.17)$$

where $\mathbf{J} = J_x \hat{\mathbf{i}} + J_y \hat{\mathbf{j}}$ as before. The EL equation for Φ is

$$-\nabla^2 \Phi = \rho \quad (4.18)$$

and has the solution $\Phi = -\sum_i^N k_i \ln \left(\frac{1}{|\mathbf{r} - \mathbf{r}_i|} \right)$. Φ performs a similar role to the gauge field A^μ encountered in equation 2.4 in the sense that they are both integrated out in the course of writing the Lagrangian.

We note the relation

$$|\nabla \Phi|^2 = \nabla \cdot (\Phi \nabla \Phi) - \Phi \nabla^2 \Phi := -\Phi \nabla^2 \Phi = \rho \Phi. \quad (4.19)$$

Employing equations (4.18) and (4.19), and ignoring self-interactions, we can write

$$\int L_\alpha d\mathbf{r} = \pi \sum_i^N k_i \left(-x_i \frac{dy_i}{dt} + y_i \frac{dx_i}{dt} \right) - \pi \sum_i^N \sum_{\substack{j \\ i \neq j}}^N k_i k_j \ln \left(\frac{1}{|\mathbf{r}_i - \mathbf{r}_j|} \right). \quad (4.20)$$

which is the desired Lagrangian describing point vortex motion in an incompressible, homogeneous condensate. Consideration of the EL equations for (x_i, y_i) leads to the equations of motion for the vortices:

$$\dot{z}_k = \sum_{j \neq k}^N i k_j \frac{z_k - z_j}{|z_k - z_j|^2}, \quad (4.21)$$

which is a simpler system to deal with compared to vortex motion in a compressible condensate. For

instance, for a pair of identical vortices, i.e vortices of identical winding number k , we immediately recognize a solution of the form

$$z_1 = \bar{r}e^{i\Lambda t}, z_2 = -\bar{r}e^{i\Lambda t}$$

where $\bar{r}$ is a constant and the frequency with which the identical pair corotate about one another is given by $\Lambda = \frac{k}{2\bar{r}^2}$. Equation (4.21) is well known as the form for two-dimensional, hydrodynamic point vortex motion. By equation (4.21), a point vortex makes no contribution to its own velocity [85]. Equation (4.21) may also be derived from Helmholtz's vorticity theorems [85].

Having clarified how the link between our study of BECs and such a hydrodynamic system can be made, we shall examine some properties of the motion of vortices in an incompressible condensate in the next subsection. A two-dimensional theory is applicable for instance to a rotating superfluid not rotating too fast and where the flow is restricted everywhere to a plane perpendicular to the axis of rotation [86]. In this case, straight line vortices parallel to the rotation axis are generated [86], and the arrangement is equal to a collection of point vortices with identical circulation or charge [86]. Studies of point vortex motion in incompressible condensates, though a simplified model, will be relevant for application to understanding the behaviour of a condensate in this scenario.

4.4 Some General Properties of the Motion of Point Vortices in a Homogeneous, Incompressible Condensate

We discuss in this subsection some of the properties of the motion of superfluid point vortices whose motion is described by equation (3.21). It is noteworthy that the only difference between the motion described by equation (4.21) and the motion of vortices in an inviscid incompressible fluid in conventional fluid mechanics is that the circulation around the superfluid vortex is quantized, whereas the circulation around a vortex in conventional fluid mechanics is not quantized [86]. The well known Hamiltonian and Poisson bracket formalism for these equations is discussed along with verification of known conserved quantities of the system. We move on to discuss the integrability of the superfluid point vortex motion, using these conserved quantities. Aref and Pomphrey in their paper [85] state the known, important result that the motion of three point vortices described by equation (4.21) is completely integrable, regardless of their charge, and they provide a summary of the proof. We provide in this thesis a more detailed proof

of this statement. Aref and Pomphrey's paper [85] is the primary reference for our discussions of the integrability of the point vortex motion in this section, and we employ much of their notation.

On the other hand, a system of four or more vortices is not, in general, an integrable system. This is clear as it has been shown that a system of four identical vortices is not integrable [85]. Aref and Pomphrey demonstrate this by employing a method to reduce the degrees of freedom of a Hamiltonian describing identical vortices by two, and apply this method to a system of three, and four identical vortices respectively. In particular, they show that the system of four identical vortices may be reduced to a Hamiltonian of the form found in coupled nonlinear oscillators, with resonant interaction terms suggesting nonintegrable behaviour. They verify this nonintegrable behaviour by numerical experiments. We discuss their method in this thesis, and its application to systems of three, and four identical vortices. In particular we provide details, not provided in their research paper, of the lengthy derivation process for the Hamiltonian of the form seen in coupled nonlinear oscillators.

Lastly, and still on the topic of motion of identical vortices, we reconsider the interesting problem of the linear stability analysis of the motion of a steadily rotating vortex polygon. We conduct a literature survey, and present our independent solution to the problem.

4.4.1 Hamiltonian Formalism for Two-Dimensional Point Vortex Dynamics

We recall that

$$\dot{z}_k = \frac{1}{2} \sum_{j \neq k}^N i k_j \frac{z_k - z_j}{|z_k - z_j|^2},$$

which we obtained from the GPE, by consideration of natural units. However, the above equation more commonly takes the form [85]

$$\dot{z}_\alpha^* = \frac{1}{2\pi i} \sum_{\beta \neq \alpha}^N \frac{\kappa_\beta}{z_\alpha - z_\beta}, \quad (4.22)$$

in conventional fluid dynamics. In this subsection, α and β are indices used to count vortices. Equation (4.22) is obtained from equation (4.21) by the simple transformation $k_j \rightarrow \frac{\kappa_j}{2\pi}$. For the sake of convenience, we make use of equation (4.22), as seen in [85], for the rest of this section of the thesis. Equation (4.22) may also be written as the following pair of equations [85]

$$\kappa_\alpha \dot{x}_\alpha = \partial_{y_\alpha} H_{pv}, \quad (4.23a)$$

$$\kappa_\alpha \dot{y}_\alpha = -\partial_{x_\alpha} H_{pv} \quad (4.23b)$$

which can also be written as [85]

$$\kappa_\alpha^{\frac{1}{2}} \dot{x}_\alpha = \frac{\partial_{y_\alpha} H_{pv}}{\kappa_\alpha^{\frac{1}{2}}}, \quad (4.24a)$$

$$\kappa_\alpha^{\frac{1}{2}} \dot{y}_\alpha = -\frac{\partial_{x_\alpha} H_{pv}}{\kappa_\alpha^{\frac{1}{2}}} \quad (4.24b)$$

where

$$H_{pv} = -\frac{1}{4\pi} \sum_{\alpha}^N \sum_{\substack{\beta \\ \alpha \neq \beta}}^N \kappa_\alpha \kappa_\beta \ln |z_\alpha - z_\beta|. \quad (4.25)$$

Defining for instance that $q_\alpha = \kappa_\alpha^{\frac{1}{2}} x_\alpha$, $p_\alpha = \kappa_\alpha^{\frac{1}{2}} y_\alpha$ for $\kappa_\alpha > 0$ and $q_\alpha = \left| \kappa_\alpha^{\frac{1}{2}} \right| y_\alpha$, $p_\alpha = \left| \kappa_\alpha^{\frac{1}{2}} \right| x_\alpha$ for $\kappa_\alpha < 0$, then equations (4.24a) and (4.24b) take the form of Hamilton's canonical equations,

$$\dot{q}_\alpha = \partial_{p_\alpha} H_{pv}, \quad (4.26a)$$

$$\dot{p}_\alpha = -\partial_{q_\alpha} H_{pv} \quad (4.26b)$$

with q_α, p_α being canonically conjugate variables for $\alpha = 1, \dots, N$. H_{pv} is the point vortex Hamiltonian. The number of degrees of freedom of the system is equal to the number of vortices [85]. Having described a Hamiltonian formalism for the system, we next define [85] the Poisson bracket $[f, g]$ of the quantities f and g by

$$[f, g] = \sum_{\alpha}^N \partial_{q_\alpha} f \partial_{p_\alpha} g - \partial_{p_\alpha} f \partial_{q_\alpha} g = \sum_{\alpha}^N \frac{(\partial_{x_\alpha} f \partial_{y_\alpha} g - \partial_{y_\alpha} f \partial_{x_\alpha} g)}{\kappa_\alpha}$$

We note that the fundamental Poisson brackets are [85]

$$[z_\alpha, z_\beta] = 0, [z_\alpha, z_\beta^*] = -2i \frac{\delta_{\alpha, \beta}}{\kappa_\alpha}.$$

4.4.2 Conserved Quantities, and Integrability of a System of Three Vortices

We prove that a system of three vortices is completely integrable. The starting point for our proof is verification that the quantities described by Q , P and L^2 are conserved quantities of the motion. Q , P ,

and L^2 are described in equations (4.27a) and (4.27b) that follow, that is

$$\sum_{\alpha}^N \kappa_{\alpha} z_{\alpha} = Q + iP \quad (4.27a)$$

and

$$\sum_{\alpha}^N \kappa_{\alpha} |z_{\alpha}|^2 = L^2. \quad (4.27b)$$

Consider $[H, P]$. Now

$$\begin{aligned} \left[\sum_{\substack{\beta \ c \\ \beta \neq c}}^N \sum_{\substack{\beta \ c \\ \beta \neq c}}^N \kappa_c \kappa_{\beta} \ln |z_c - z_{\beta}|, \sum_{\beta} \kappa_{\beta} y_{\beta} \right] &= \sum_{\alpha}^N \frac{1}{\kappa_{\alpha}} \left(\frac{\partial}{\partial x_{\alpha}} \sum_{\substack{\beta \ c \\ \beta \neq c}}^N \sum_{\substack{\beta \ c \\ \beta \neq c}}^N \kappa_c \kappa_{\beta} \ln |z_c - z_{\beta}| \right) \times \kappa_{\alpha} \\ &= \sum_{\alpha}^N \sum_{\beta}^N \sum_{\substack{\beta \ c \\ \beta \neq c}}^N \kappa_c \kappa_{\beta} \frac{\partial_{x_{\alpha}} |z_c - z_{\beta}|}{|z_c - z_{\beta}|}. \end{aligned}$$

We note that

$$|z_c - z_{\beta}| = \{(x_c - x_{\beta})^2 + (y_c - y_{\beta})^2\}^{\frac{1}{2}}$$

and therefore

$$\partial_{x_{\alpha}} |z_c - z_{\beta}| = \frac{(x_c - x_{\beta})(\delta_{c,\alpha} - \delta_{\beta,\alpha})}{|z_c - z_{\beta}|}.$$

It follows that

$$\sum_{\alpha}^N \sum_{\beta}^N \sum_{\substack{\beta \ c \\ \beta \neq c}}^N \kappa_c \kappa_{\beta} \frac{\partial_{x_{\alpha}} |z_c - z_{\beta}|}{|z_c - z_{\beta}|} = \sum_{\alpha}^N \sum_{\substack{\beta \\ \alpha \neq \beta}}^N \kappa_{\alpha} \kappa_{\beta} \frac{(x_{\alpha} - x_{\beta})}{|z_{\alpha} - z_{\beta}|^2} - \sum_{\alpha}^N \sum_{\substack{\beta \\ \alpha \neq c}}^N \kappa_c \kappa_{\alpha} \frac{(x_c - x_{\alpha})}{|z_c - z_{\alpha}|^2} = 0.$$

Therefore

$$[H, P] = 0.$$

By a very similar calculation,

$$[H, Q] = 0.$$

We have therefore shown that P and Q are conserved quantities of the motion. We evaluate next, $[H, L^2]$.

Consider

$$\left[\sum_{\substack{\beta \\ \beta \neq c}}^N \sum_c^N \kappa_c \kappa_\beta \ln |z_c - z_\beta|, \sum_{\alpha}^N \kappa_{\alpha} (x_{\alpha}^2 + y_{\alpha}^2) \right].$$

Now,

$$\partial_{x_{\alpha}} \sum_{\substack{\beta \\ \beta \neq c}}^N \sum_c^N \kappa_c \kappa_\beta \ln |z_c - z_\beta| = 2 \sum_{\beta}^N \kappa_{\alpha} \kappa_{\beta} \frac{(x_{\alpha} - x_{\beta})}{|z_{\alpha} - z_{\beta}|^2}$$

and

$$\partial_{y_{\alpha}} \sum_{\substack{\beta \\ \beta \neq c}}^N \sum_c^N \kappa_c \kappa_\beta \ln |z_c - z_\beta| = 2 \sum_{\beta}^N \kappa_{\alpha} \kappa_{\beta} \frac{(y_{\alpha} - y_{\beta})}{|z_{\alpha} - z_{\beta}|^2}.$$

Also, it is easily shown that $\partial_{x_{\alpha}} \sum_{\beta}^N \kappa_{\beta} (x_{\beta}^2 + y_{\beta}^2) = 2\kappa_{\alpha} x_{\alpha}$ and $\partial_{y_{\alpha}} \sum_{\beta}^N \kappa_{\beta} (x_{\beta}^2 + y_{\beta}^2) = 2\kappa_{\alpha} y_{\alpha}$. From these, it can be shown that

$$\left[\sum_{\substack{\beta \\ \beta \neq c}}^N \sum_c^N \kappa_c \kappa_\beta \ln |z_c - z_\beta|, \sum_{\alpha}^N \kappa_{\alpha} (x_{\alpha}^2 + y_{\alpha}^2) \right] = 4 \sum_{\alpha}^N \sum_{\substack{\beta \\ \alpha \neq \beta}}^N \kappa_{\alpha} \kappa_{\beta} \frac{(x_{\alpha} y_{\beta} - x_{\beta} y_{\alpha})}{|z_{\alpha} - z_{\beta}|^2} = 0.$$

It follows therefore that

$$[H, L^2] = 0.$$

Following this, we consider $[P^2 + Q^2, L^2]$. Now

$$[P^2 + Q^2, L^2] = [P^2, L^2] + [Q^2, L^2] = [P, L^2] P + P [P, L^2] + [Q, L^2] Q + Q [Q, L^2].$$

Also,

$$[P, L^2] = \left[\sum_{\beta}^N \kappa_{\beta} y_{\beta}, \sum_{\alpha}^N \kappa_{\alpha} (x_{\alpha}^2 + y_{\alpha}^2) \right] = - \sum_{\alpha=1}^N 2\kappa_{\alpha} x_{\alpha} = -2Q$$

and

$$[Q, L^2] = \left[\sum_{\beta}^N \kappa_{\beta} x_{\beta}, \sum_{\alpha}^N \kappa_{\alpha} (x_{\alpha}^2 + y_{\alpha}^2) \right] = \sum_{\alpha=1}^N 2\kappa_{\alpha} y_{\alpha} = 2P.$$

Therefore,

$$\begin{aligned} [P^2, L^2] + [Q^2, L^2] &= [P, L^2] P + P [P, L^2] + [Q, L^2] Q + Q [Q, L^2] \\ &= (-2Q) P + P (-2Q) + (2P) Q + Q (2P) = 0. \end{aligned}$$

That is, we have shown that

$$[P^2 + Q^2, L^2] = 0.$$

Lastly,

$$[P^2 + Q^2, H] = [P^2, H] + [Q^2, H] = [P, H] P + P [P, H] + [Q, H] Q + Q [Q, H] = 0.$$

Therefore

$$[P^2 + Q^2, H] = 0.$$

It follows that H , L^2 and $P^2 + Q^2$ are three integrals in involution for the system which has three degrees of freedom. Two-dimensional three vortex motion is therefore integrable for arbitrary vortex strengths.

4.4.3 Reducing the Degrees of Freedom of a Hamiltonian for a System of Identical Vortices

This process is achieved by a series of canonical transformations [85]. The first of these is

$$Q_n + iP_n = N^{-\frac{1}{2}} \sum_{\alpha} e^{i\frac{2\pi n}{N}(\alpha-1)} z_{\alpha}$$

where $n = 0, 1, \dots, N - 1$. This is a canonical transformation from the conjugate variables x_{α} and y_{α} to the conjugate variables Q_{α} and P_{α} . A key result of this canonical transformation is that Q_0 and P_0 are both constants of the motion and may be set to zero without loss of generality [85]. They will not appear in the final, transformed Hamiltonian [85]. This corresponds to the removal of a degree of freedom. The second canonical transformation is given by [85]

$$(2J_n)^{\frac{1}{2}} e^{i\theta_n} = Q_n + iP_n$$

in which we transform the conjugate variables Q_n and P_n to the new conjugate variables J_n and θ_n . No

degrees of freedom are lost at this stage. However, the action variable J_n , is such that [85]

$$\sum_{n=1}^{N-1} J_n = \frac{1}{2} L^2.$$

We recognise L^2 as a constant of the motion and this motivates the final canonical transformation from the conjugate variables J_n and θ_n to the conjugate variables I_n and ϕ_n . This is done using a type F_3 generating function,

$$F_3(J_1, \dots, J_{N-1}, \phi_1, \dots, \phi_{N-1}) = \sum_{k=1}^{N-1} \sum_{l=1}^{N-1} C_{kl} J_k \phi_l.$$

C_{kl} is a matrix of constant coefficients to be defined when the cases $N = 3$ and $N = 4$ are studied in detail. The final transformation is written so that

$$I_{N-1} = \sum_{n=1}^{N-1} J_n$$

and therefore I_{N-1} is a constant of the motion and its conjugate variable will be absent from the final Hamiltonian. Therefore, it is at this canonical transformation that a second degree of freedom is removed. The final canonical transformation is given by

$$I_k = \sum_{l=1}^{N-1} C_{lk} J_l, \tag{4.28a}$$

$$\theta_k = \sum_{l=1}^{N-1} C_{kl} \phi_l. \tag{4.28b}$$

We apply the above reduction procedure to a system of three vortices, as was done in [85], providing the details of the algebra involved in writing down the reduced Hamiltonian. These details are not given in [85]. For a system of three vortices we have $Q_0 + iP_0 = \frac{(z_1 + z_2 + z_3)}{3^{\frac{1}{2}}}$, $Q_1 + iP_1 = \frac{(z_1 + e^{\frac{2\pi i}{3}} z_2 + e^{-\frac{2\pi i}{3}} z_3)}{3^{\frac{1}{2}}}$ and $Q_2 + iP_2 = \frac{(z_1 + e^{-\frac{2\pi i}{3}} z_2 + e^{\frac{2\pi i}{3}} z_3)}{3^{\frac{1}{2}}}$. We then invoke the inverse of the first transformation i.e.

$$z_\alpha = N^{-\frac{1}{2}} \sum_{n=0}^{N-1} e^{-i\frac{2\pi n}{N}(\alpha-1)} (Q_n + iP_n) \tag{4.29}$$

where $n = 0, 1, \dots, N-1$. Setting $Q_0 = P_0 = 0$ and inserting the second canonical transformation then it can be shown that

$$|(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)|^2 = J_1^3 + J_2^3 - 2(J_1 J_2)^{\frac{3}{2}} \cos(3\theta_1 - 3\theta_2).$$

The final canonical transformation is such that

$$\theta_2 - \theta_1 = \phi_1, J_1 = I_2 - I_1 \quad \& \quad J_2 = I_1 + I_2,$$

from which it follows that

$$|(z_1 - z_2)(z_2 - z_3)(z_1 - z_3)|^2 = 16 \left\{ I_2 (I_2^2 + 3I_1^2) - (I_2^2 - I_1^2)^{\frac{3}{2}} \cos(3\theta_1 - 3\theta_2) \right\}.$$

Therefore

$$H_{pv} = -\frac{1}{4\pi} \ln \left\langle 16 \left\{ I_2 (I_2^2 + 3I_1^2) - (I_2^2 - I_1^2)^{\frac{3}{2}} \cos(3\theta_1 - 3\theta_2) \right\} \right\rangle. \quad (4.30)$$

We note that $I_2 = \frac{J_1 + J_2}{2}$ is a constant of the motion. Therefore the new Hamiltonian has just one degree of freedom. As noted in [85], I_1 has the simple geometrical interpretation,

$$I_1 = \frac{\sigma_{123} A_{123}}{3^{\frac{1}{2}}}$$

where σ_{123} is the orientation (+1 for a clockwise arrangement of vortices 1, 2 and 3) and A_{123} is the area of the triangle which the three vortices span. It has been shown [85] that this process of reducing the degrees of freedom of the Hamiltonian allows for the description of the evolution of the system in terms of $(I_1)^2$. We elucidate this process in this thesis. Now

$$\dot{I}_1 = -\frac{\partial H_{pv}}{\partial \phi_1} = \frac{-3 (I_2^2 - I_1^2)^{\frac{3}{2}} \sin(3\phi_1)}{4\pi \left\{ I_2 (I_2^2 + 3I_1^2) - (I_2^2 - I_1^2)^{\frac{3}{2}} \cos(3\theta_1 - 3\theta_2) \right\}}.$$

Using equation (4.30) we can write

$$\dot{I}_1 = -\frac{\partial H_{pv}}{\partial \phi_1} = \frac{12e^{4\pi H_{pv}} (I_2^2 - I_1^2)^{\frac{3}{2}} \sin(3\phi_1)}{\pi}.$$

We seek to remove ϕ_1 from the picture, to this end we square the above equation and write

$$(\dot{I}_1)^2 = \frac{144e^{8\pi H_{pv}} (I_2^2 - I_1^2)^3 \sin^2(3\phi_1)}{\pi^2}.$$

We return to equation (4.30), and from equation (4.30) we can show that

$$\sin^2(3\phi_1) = \frac{(I_2^2 - I_1^2)^3 - \left\{ I_2 (I_2^2 + 3I_1^2) - \frac{e^{-4\pi H_{pv}}}{16} \right\}^2}{(I_2^2 - I_1^2)^3}.$$

We can then write that

$$\left(\dot{I}_1\right)^2 = \frac{144e^{8\pi H_{pv}} \left\langle (I_2^2 - I_1^2)^3 - \left\{ I_2 (I_2^2 + 3I_1^2) - \frac{e^{-4\pi H_{pv}}}{16} \right\}^2 \right\rangle}{\pi^2}.$$

The above can be solved exactly, as seen in [85]. It is useful to define $\tilde{I} = \left(\frac{I_1}{I_2}\right)^2$ and also define the scaled time variable

$$\hat{\tau} = \frac{3t}{2\pi\varpi^2 L^2},$$

where ϖ is given by $\left(\frac{\theta_0^{(N)}}{\theta}\right)^{\frac{1}{2}}$. $\theta_0^{(N)}$ is the minimum value of θ for a given value of N . Now, $\theta = L^6 e^{4\pi H_{pv}}$ (assigning $P = Q = 0$ and $\kappa_\alpha = 1$). ϖ is such that $0 < \varpi \leq 1$. Furthermore, defining that $D_0 = \tilde{I}^{(0)}$, $D_1 = \max\{0, I^{(2)}\}$, $D_2 = \min\{0, I^{(2)}\}$, $D_3 = I^{(1)}$, $\gamma^2 = \frac{D_1 - D_0}{D_1 - D_3}$, and $\bar{\omega} = \frac{\{(D_0 - D_2)(D_1 - D_3)\}^{\frac{1}{2}}}{2}$, then we have the solution [85]

$$\tilde{I} = \frac{D_0 - D_3 \gamma^2 \text{sn}(\bar{\omega} \hat{\tau})}{I - \gamma^2 \text{sn}^2(\bar{\omega} \hat{\tau})}.$$

The area variable $\tilde{I}$ pulses with period given by $T = \frac{8\pi\varpi^2 L^2 K(m)}{3\bar{\omega}}$ for $0 < \varpi < 2^{-\frac{1}{2}}$, and $T = \frac{4\pi\varpi^2 L^2 K(m)}{3\bar{\omega}}$ for $2^{-\frac{1}{2}} < \varpi < 1$, where $m^2 = \frac{(D_0 - D_1)(D_2 - D_3)}{(D_0 - D_2)(D_1 - D_3)}$ in the above definition of T . K is a complete elliptic integral of the first kind while sn represents the elliptic sine function. For $\varpi = 2^{-\frac{1}{2}}$ a collapse of the vortex triangle into a collinear configuration occurs and the motion is aperiodic [85]. Discussion of the regimes $0 < \varpi < 2^{-\frac{1}{2}}$, $\varpi = 2^{-\frac{1}{2}}$ and $2^{-\frac{1}{2}} < \varpi \leq 1$, as noted in [85] can be found in [87] and [88].

The reduction process for the four-vortex system is more algebraically involved however. The first canonical transformation reads, in this case as

$$z_1 = \frac{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)}{2}, z_2 = \frac{(-i\varepsilon_1 - \varepsilon_2 + \varepsilon_3)}{2}, z_3 = \frac{(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3)}{2} \text{ and } z_4 = \frac{(i\varepsilon_1 - \varepsilon_2 - i\varepsilon_3)}{2}.$$

where for ease of notation, following [85], we have written $Q_n + iP_n = \varepsilon_n$. We recall that Q_0 and P_0 are constants of the motion, and we set $Q_0 = P_0 = 0$ without loss of generality. It will be useful to note that [85]

$$(z_1 - z_3)(z_2 - z_4) = i(\varepsilon_3^2 - \varepsilon_1^2)$$

and

$$(z_1 - z_2)(z_2 - z_3)(z_3 - z_4)(z_1 - z_4) = 2\varepsilon_1\varepsilon_2^2\varepsilon_3 - \varepsilon_2^4 - \frac{1}{4}(\varepsilon_1^2 + \varepsilon_3^2)^2,$$

given that the Hamiltonian takes the form

$$H_{pv} = -\frac{1}{2\pi} \{ \ln |(z_1 - z_3)(z_2 - z_4)(z_1 - z_2)(z_1 - z_4)(z_2 - z_3)(z_3 - z_4)| \}.$$

The second canonical transformation may be applied through the substitution $\varepsilon_n = (2J_n)^{\frac{1}{2}} e^{i\theta_n}$. If while making the second canonical transformation, we at the same time make a partial substitution of the third canonical transformation, i.e. we make the substitutions

$$\theta_1 = \phi_1 + \phi_2 + \phi_3, \theta_2 = \phi_3 \text{ and } \theta_3 = -\phi_1 + \phi_2 + \phi_3,$$

we can then write that [85]

$$\begin{aligned} & \prod_{1 \leq k < l < 4} |z_k - z_l| = \\ & 2 \left| \{ J_1 J_3 (J_1 - J_3) \cos(2\phi_1) + (J_1^3 - J_3^3) \cos(6\phi_1) + 4J_1 J_2^2 \cos(2\phi_1 - 4\phi_2) - 4J_2^2 J_3 \cos(2\phi_1 + 4\phi_2) \right. \\ & \quad + 8J_1^{\frac{1}{2}} J_2 J_3^{\frac{3}{2}} \cos(2\phi_1 + 2\phi_2) + i (J_1^3 + J_3^3) \sin(6\phi_1) - 8J_1^{\frac{3}{2}} J_2 J_3^{\frac{1}{2}} \cos(2\phi_1 - 2\phi_2) \\ & \quad + i \{ J_1 J_3 (J_1 + J_3) \sin(2\phi_1) + 4J_1 J_2^2 \sin(2\phi_1 - 4\phi_2) \} \\ & \quad \left. + i \left\langle +4J_2^2 J_3 \sin(2\phi_1 + 4\phi_2) - 8J_1^{\frac{3}{2}} J_2 J_3^{\frac{1}{2}} \sin(2\phi_1 - 2\phi_2) - 8J_1^{\frac{1}{2}} J_2 J_3^{\frac{3}{2}} \sin(2\phi_1 + 2\phi_2) \right\rangle \right|. \end{aligned}$$

We now invoke the third canonical transformation in full. The details of the lengthy algebraic process from this stage, up to writing down the Hamiltonian which has the form of a Hamiltonian for a coupled nonlinear oscillator problem does not appear in Aref and Pomphrey's paper [85]. Moving on, we can write

$$J_1 = \frac{(I_1 + I_2)}{2}, J_2 = I_3 - I_2 \text{ and } J_3 = \frac{(I_2 - I_1)}{2}.$$

Therefore we can now write

$$\prod_{1 \leq k < l < 4} (|z_k - z_l|)^2 = F(I_1, I_2, \phi_1, \phi_2)$$

where $F(I_1, I_2, \phi_1, \phi_2)$ is defined in Appendix C.2. With some work (see Appendix C.3), the transformed Hamiltonian can be written as [85]

$$H_{pv}(I_1, I_2, \phi_1, \phi_2) = -\frac{\ln \{h(I_1, I_2, \phi_1, \phi_2)\}}{4\pi},$$

where h is of the form

$$h(I_1, I_2, \phi_1, \phi_2) = h_1(I_1, I_2, \phi_1) - (I_3 - I_2) \{h_2(I_1, I_2, \phi_2) + h_{12}(I_1, I_2, \phi_1, \phi_2)\},$$

where

$$h_1(I_1, I_2, \phi_1) = 4 \{ I_1^2 \cos^2(2\phi_1) + I_2^2 \sin^2(2\phi_1) \} \\ \times \left\langle 16(I_3 - I_2)^2 \{ (I_3 - I_2)^2 + I_2^2 - I_1^2 \} + \{ I_1^2 \sin^2(2\phi_1) + I_2^2 \cos^2(2\phi_1) \}^2 \right\rangle,$$

$$h_2(I_1, I_2, \phi_2) = 4 (I_1^2 + I_2^2) (I_2^2 - I_1^2)^{\frac{1}{2}} \\ \times \left\langle \{ 16(I_3 - I_2)^2 + I_2^2 - I_1^2 \} \cos(2\phi_2) - (I_3 - I_2) (I_2^2 - I_1^2)^{\frac{1}{2}} \cos(4\phi_2) \right\rangle$$

and

$$h_{12}(I_1, I_2, \phi_1, \phi_2) = g_{12}(I_1, I_2, \phi_1, \phi_2) + g_{12}(-I_1, I_2, \phi_1, -\phi_2)$$

where

$$g_{12}(I_1, I_2, \phi_1, \phi_2) = 8 (I_3 - I_2) (I_2 - I_1)^2 I_1 I_2 \cos(4\phi_1 - 4\phi_2) \\ - 8 (I_2 - I_1) (I_2^2 - I_1^2)^{\frac{1}{2}} \{ I_1 I_2 (I_2 - I_1) + 4(I_1 + I_2)(I_2 - I_3)^2 \} \\ \times \cos(4\phi_1 - 2\phi_2) - 2 (I_2^2 - I_1^2)^{\frac{3}{2}} (I_2 - I_1)^2 \cos(8\phi_1 - 2\phi_2) \\ + 2 (I_2^2 - I_1^2) (I_3 - I_2) (I_2 - I_1)^2 \cos(8\phi_1 - 4\phi_2).$$

Aref and Pomphrey note that this is a tedious process but provide no derivation in their paper. We have provided an independent derivation of the above Hamiltonian in Appendix C.3. It can be shown that h serves as the effective Hamiltonian of the system with h_1 and h_2 corresponding to individual ‘oscillators’ [85]. Resonance overlap terms are embodied in h_{12} . The presence of chaotic solutions for this system was confirmed numerically by generating suitable Poincaré sections with a chaotic appearance, details of which may be found in their paper [85].

4.4.4 Linear Stability Analysis of the Motion of a Steadily Rotating, Regular Polygon of Point Vortices

In this subsection (which is based on our work in [89]), we consider a regular polygon of identical vortices, rotating at constant angular velocity. We calculate, using the cartesian coordinate system and from the perspective of a frame corotating with the vortex polygon, the equations of motion governing small displacements of the individual vortices from their equilibrium positions for any vortex polygon. Following this, we are able to calculate the smallest number of vortices in the polygon for which an

exponential instability of the deviation from equilibrium motion will occur. It turns out such an instability will first be manifest in a vortex octagon. We further confirm in our calculation the existence of linear instability for polygons with larger numbers of vortices than an octagon. This problem originates from Lord Kelvin’s work on the stability of stationary rotation of a regular vortex polygon [90].

Thomson, in his Adam’s prize essay [91] carried out the first linear stability analysis calculation of this kind ([90], [92]). His approach involved consideration of a stream function due to the vortex polygon with each vortex slightly displaced from steady, equilibrium rotation [89]. In Thomson’s calculation, he carries out linear stability analysis on successive vortex polygons in turn, beginning with a vortex triangle. Thomson comes to the conclusion that linear instability is first observed for a vortex heptagon. Furthermore, his calculation does not prove that linear instability exists for polygons with a greater number of vortices than seven, though it may be expected intuitively. It was pointed out however (see [90] and [93] for details) that Thomson made an error in his calculation regarding the vortex heptagon and therefore linear instability does not manifest in the vortex heptagon. It does in fact manifest in the vortex octagon and beyond [92]. Dritschel extended Thomson’s work by considering vortices of finite size for equilibrium configurations of two to eight vortices [94]. We note that Dritschel’s work is the only mention of finite-size vortices in this subsection. To quote [95] ‘the main result for the finite-area vortices is that instability occurs at any N when the area of the vortices becomes sufficiently large’. All other mentions of vortices in this subsection, unless stated otherwise, refer to point vortices.

Another prominent linear stability analysis calculation of this type was carried out by Havelock [92]. Working in polar coordinates, Havelock also calculates the equations of motion governing small displacements of the vortices from their equilibrium motion for a polygon of arbitrary size, proving also that linear instability manifests in the vortex octagon and above. However, he does not observe these displacements from the corotating frame as we do, where a different picture, one excluding the rotation of the entire polygon, may be obtained. Aref [96] performs essentially the same calculation as Havelock in his paper, with the main difference being the use of a cartesian coordinate system in Aref’s calculation. Aref however in his calculation does not employ a rotating frame as we have done.

The question of the nonlinear stability of the vortex polygon in the plane is fully addressed in [90]. For seven or fewer vortices, the vortex polygon motion is nonlinearly stable, while for eight or more vortices the vortex polygon motion is nonlinearly unstable. Consideration of how the nonlinear stability of the motion of a vortex polygon depends on the curvature of the surface where the motion takes place may be found in [97].

To summarise, our approach is approach similar to Havelock's but employs a corotating frame, and a cartesian coordinate system. We recover the same results as were obtained previously for linear stability analysis. However, we have not seen the calculation of the equations of motion governing the displacements while employing a rotating frame for the case of the planar vortex N -gon, anywhere in the literature. We are aware of the discussion of transforming a Hamiltonian describing the system to a corotating frame in the geometrical approach of Boatto [97] to this problem. In his work however, he does not calculate the equations governing the displacements. The following presentation of our calculation is based on our work in [89].

We return to the equations of motion for point vortices in an incompressible, homogeneous condensate in the form of equation (4.22),

$$\dot{z}_k = \frac{i\kappa}{2\pi} \sum_{j \neq k}^N \frac{z_k - z_j}{|z_k - z_j|^2}$$

where k and j are indices used to label vortices. We employ a frame rotating at an angular frequency of Ω_f in our analysis. Employing the Transport Theorem in analytical dynamics, we can write that

$$\dot{z}_k = \frac{i\kappa}{2\pi} \sum_{j \neq k}^N \frac{z_k - z_j}{|z_k - z_j|^2} - i\Omega_f z_k \quad (4.31)$$

in which case we have now written the equations of vortex motion from the perspective of the rotating frame. To proceed with our linear stability analysis, we make the ansatz

$$z_k = r_p e^{i\theta_k} \quad (4.32)$$

where

$$\theta_k = \theta_1 + \frac{2\pi(k-1)}{N} \quad (4.33)$$

and r_p is the radius of the polygon. For equation (4.32) to represent the steadily rotating vortex polygon, it is necessary that $\dot{r}_p = 0$. To check if this is true, we insert equation (4.32) into equation (4.31). For convenience, we set $k = 1$ in equation (4.31). Now

$$\begin{aligned} z_k - z_j &= r_p (e^{i\theta_k} - e^{i\theta_j}) = r_p \left(e^{i\frac{\theta_k}{2}} e^{i\frac{\theta_k}{2}} - e^{i\frac{\theta_j}{2}} e^{i\frac{\theta_j}{2}} \right) = r_p \left(e^{i\frac{\theta_k}{2}} e^{i\frac{\theta_k}{2}} e^{i\frac{\theta_j}{2}} e^{-i\frac{\theta_j}{2}} - e^{i\frac{\theta_j}{2}} e^{i\frac{\theta_j}{2}} e^{i\frac{\theta_k}{2}} e^{-i\frac{\theta_k}{2}} \right) \\ &= r_p e^{i\frac{\theta_k+\theta_j}{2}} \left(e^{i\frac{\theta_k-\theta_j}{2}} - e^{-i\frac{\theta_k-\theta_j}{2}} \right) = 2ir_p e^{i\frac{\theta_k+\theta_j}{2}} \sin \left(\frac{\theta_k-\theta_j}{2} \right). \end{aligned}$$

Therefore,

$$\frac{z_k - z_j}{|z_k - z_j|^2} = \frac{2ir_p e^{i\frac{\theta_k + \theta_j}{2}} \sin\left(\frac{\theta_k - \theta_j}{2}\right)}{4r_p^2 \sin^2\left(\frac{\theta_k - \theta_j}{2}\right)} = \frac{ie^{i\frac{\theta_k + \theta_j}{2}}}{2r_p \sin\left(\frac{\theta_k - \theta_j}{2}\right)}.$$

It follows that equation (4.31) with $k = 1$ can be written as

$$\dot{r}_p + ir_p \dot{\theta}_1 = -\frac{\kappa}{4\pi r_p} \sum_{j \neq 1}^N \frac{e^{i\frac{\theta_j - \theta_1}{2}}}{\sin\left(\frac{\theta_1 - \theta_j}{2}\right)} - i\Omega_f r_p$$

which is equal to

$$\dot{r}_p + ir_p \dot{\theta}_1 = -\frac{\kappa}{4\pi r_p} \sum_{j \neq 1}^N \left\{ \cot\left(\frac{\theta_j - \theta_1}{2}\right) - i \right\} - i\Omega_f r_p.$$

Compare real and imaginary parts of the above equation, resulting in

$$\dot{r}_p = -\frac{\kappa}{4\pi r_p} \sum_{j \neq 1}^N \cot\left(\frac{\theta_j - \theta_1}{2}\right) \quad (4.34)$$

and

$$\dot{\theta}_1 = \frac{\kappa(N-1)}{4\pi r_p^2} - \Omega_f. \quad (4.35)$$

We recall that $\theta_k = \theta_1 + \frac{2\pi(k-1)}{N}$. Therefore we can write that

$$\dot{r}_p = \frac{\kappa}{4\pi r_p} \sum_{j=2}^N \cot\left\{\frac{\pi(j-1)}{N}\right\}. \quad (4.36)$$

It can be shown with some work (see Appendix C.4) that

$$\dot{r}_p = 0$$

in which case the ansatz, equation (4.32) has satisfied a necessary condition to represent a steadily rotating regular vortex polygon. The other condition is that $\dot{\theta}_1$ in equation (4.35) evaluates to a constant. We will choose that this constant is zero, by making the convenient and important choice

$$\Omega_f = \frac{\kappa(N-1)}{4\pi r_p^2} \quad (4.37)$$

in order to remove the observation of the steady rotation of the entire polygon in the rotating frame. The linear stability analysis of the system is then carried out by consideration of the ansatz

$$z_k = z_{k0} + \delta z_k \quad (4.38)$$

in equation (4.31) where $z_{k0} = \rho e^{i\frac{2\pi(k-1)}{N}}$ is the equilibrium position for vortex k and δz_k represents a small displacement. Terms of order $(\delta z_k)^2$ will be neglected in this analysis. Now

$$z_k - z_j = r_p \left(e^{i\frac{2\pi(k-1)}{N}} - e^{i\frac{2\pi(j-1)}{N}} \right) + \delta z_k - \delta z_j.$$

We re-introduce $\theta_k = \frac{2\pi(k-1)}{N}$ and $\theta_j = \frac{2\pi(j-1)}{N}$. It follows that

$$z_k - z_j = 2ir_p e^{i\frac{\theta_k + \theta_j}{2}} \left\{ \sin \left(\frac{\theta_k - \theta_j}{2} \right) \right\} + \delta z_k - \delta z_j.$$

Therefore

$$(z_k - z_j) (z_k^* - z_j^*) = |z_k - z_j|^2 = 4r_p^2 \sin^2 \left(\frac{\theta_k - \theta_j}{2} \right) \left\{ 1 + \frac{ie^{i\frac{\theta_k + \theta_j}{2}}}{2r_p \sin \left(\frac{\theta_k - \theta_j}{2} \right)} (\delta z_k^* - \delta z_j^*) - \frac{ie^{-i\frac{\theta_k + \theta_j}{2}}}{2r_p \sin \left(\frac{\theta_k - \theta_j}{2} \right)} (\delta z_k - \delta z_j) \right\} + O \left\{ (\delta z)^2 \right\}.$$

We next evaluate $\frac{z_k - z_j}{|z_k - z_j|^2}$. Now

$$\left(|z_k - z_j|^2 \right)^{-1} = \frac{1}{4r_p^2 \sin^2 \left(\frac{\theta_k - \theta_j}{2} \right)} \left\{ 1 - \frac{ie^{i\frac{\theta_k + \theta_j}{2}}}{2r_p \sin \left(\frac{\theta_k - \theta_j}{2} \right)} (\delta z_k^* - \delta z_j^*) + \frac{ie^{-i\frac{\theta_k + \theta_j}{2}}}{2r_p \sin \left(\frac{\theta_k - \theta_j}{2} \right)} (\delta z_k - \delta z_j) \right\} + O \left\{ (\delta z)^2 \right\}.$$

Therefore, it follows that

$$\frac{z_k - z_j}{|z_k - z_j|^2} = \frac{ie^{i\frac{\theta_k + \theta_j}{2}}}{2r_p \sin \left(\frac{\theta_k - \theta_j}{2} \right)} + \frac{e^{i(\theta_j + \theta_k)}}{4r_p^2 \sin^2 \left(\frac{\theta_k - \theta_j}{2} \right)} (\delta z_k^* - \delta z_j^*) + O \left\{ (\delta z)^2 \right\}.$$

We consider the sum $\sum_{j \neq k}^N \frac{e^{i\frac{\theta_j - \theta_k}{2}}}{\sin \left(\frac{\theta_k - \theta_j}{2} \right)}$ which is equivalent to

$$\sum_{j \neq k}^N \frac{\cos \left(\frac{\theta_k - \theta_j}{2} \right) + i \sin \left(\frac{\theta_j - \theta_k}{2} \right)}{\sin \left(\frac{\theta_k - \theta_j}{2} \right)} = \left\{ \sum_{j \neq k}^N \cot \left(\frac{\theta_k - \theta_j}{2} \right) \right\} - i(N-1).$$

We recall at this stage our manipulation of equation (4.36), and can write that

$$\sum_{j \neq k}^N \cot \left(\frac{\theta_1 - \theta_j}{2} \right) = 0.$$

We note also that

$$\dot{z}_k = \dot{z}_{k0} + \dot{\delta z}_k = \dot{\delta z}_k.$$

Based on the calculations we have done so far, we can write that

$$\dot{\delta z}_k = \frac{i\kappa(N-1)}{4\pi r_p} e^{i\theta_k} + \frac{i\kappa}{8\pi r_p^2} \sum_{j \neq k}^N \frac{e^{i(\theta_j + \theta_k)}}{\sin^2\left(\frac{\theta_k - \theta_j}{2}\right)} (\delta z_k^* - \delta z_j^*) - i\Omega_f z_k.$$

We recall that

$$\Omega_f = \frac{\kappa(N-1)}{4\pi\rho^2}.$$

Therefore

$$\frac{i\kappa(N-1)}{4\pi r_p} e^{i\theta_k} - i\Omega_f z_k = -i\Omega_f \delta z_k.$$

It follows that (as seen in [89])

$$\dot{\delta z}_k = \frac{i\kappa}{8\pi r_p^2} \left\{ \sum_{j \neq k}^N \frac{e^{i(\theta_j + \theta_k)}}{\sin^2\left(\frac{\theta_k - \theta_j}{2}\right)} (\delta z_k^* - \delta z_j^*) - 2(N-1)\delta z_k \right\}. \quad (4.39)$$

We have therefore found the complete set of equations which govern the small displacements of the vortices from equilibrium motion, when viewed from the perspective of a frame corotating with an angular velocity which is equal to the equilibrium angular velocity of the regular polygon. In what follows, we shall turn equation (4.39) into an eigenvalue problem in order to determine the smallest number of vortices for which linear instability manifests.

We begin by taking the complex conjugate of equation (4.39), resulting in

$$\dot{\delta z}_k^* = \frac{i\kappa}{8\pi r_p^2} \left\{ \sum_{j \neq k}^N \frac{e^{-i(\theta_j + \theta_k)}}{\sin^2\left(\frac{\theta_k - \theta_j}{2}\right)} (\delta z_k - \delta z_j) - 2(N-1)\delta z_k^* \right\}. \quad (4.40)$$

We define

$$\lambda_j^{(k)} = \frac{e^{i(\theta_j + \theta_k)}}{\sin^2\left(\frac{\theta_k - \theta_j}{2}\right)}.$$

We further define that

$$\lambda^{(p)} = \sum_{j \neq p}^N \lambda_j^{(p)} = \lambda_p^{(p)}.$$

Next, we define the column matrix ξ , such that

$$\xi = \begin{pmatrix} \delta z_1 \\ \delta z_2 \\ \vdots \\ \delta z_N \\ \delta z_1^* \\ \delta z_2^* \\ \vdots \\ \delta z_N^* \end{pmatrix} = \begin{pmatrix} u \\ u^* \end{pmatrix}.$$

Therefore equations (4.39) and (4.40) may be written as a matrix equation in the form

$$\dot{\xi} = \frac{i\kappa}{8\pi r_p^2} M \xi$$

where

$$M = \begin{pmatrix} -2(N-1)I & \Gamma \\ -\Gamma^+ & 2(N-1)I \end{pmatrix},$$

noting that I is an $N \times N$ identity matrix and the matrix Γ is such that $\Gamma_{ab} = (-1)^{(1-\delta_{a,b})} \lambda_b^{(a)}$. Our exercise in linear stability analysis of the system is concerned with the examination of the nature of the eigenvalues of the square matrix M . We consider

$$M\xi = \zeta\xi. \tag{4.41}$$

Equation (4.41) can be rewritten as

$$\begin{pmatrix} -2(N-1)I & \Gamma \\ -\Gamma^+ & 2(N-1)I \end{pmatrix} \begin{pmatrix} u \\ u^* \end{pmatrix} = \zeta \begin{pmatrix} u \\ u^* \end{pmatrix},$$

from which it follows that

$$-2(N-1)u + \Gamma u^* = \zeta u \tag{4.42a}$$

and

$$2(N-1)u^* - \Gamma^+ u = \zeta u^*. \tag{4.42b}$$

From equations (4.42a) and (4.42b), it can easily be shown that

$$\Gamma^+ \Gamma u^* = \left\{ (2N-2)^2 - \zeta^2 \right\} u^*. \quad (4.43)$$

We shall now focus on equation (4.43), and examine Γ . It can be shown (see Appendix C.5) that

$$\Gamma = \begin{pmatrix} 1 & & & & \\ & e^{i\theta_3} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & e^{i\theta_{2N+1}} \end{pmatrix} \begin{pmatrix} \lambda^{(1)} & -\lambda_2^{(1)} & -\lambda_3^{(1)} & \dots & -\lambda_N^{(1)} \\ -\lambda_N^{(1)} & \lambda^{(1)} & -\lambda_2^{(1)} & \dots & -\lambda_{N-1}^{(1)} \\ \vdots & & & & \\ \vdots & & & & \\ -\lambda_2^{(1)} & \dots & \dots & \dots & \lambda^{(1)} \end{pmatrix}, \quad (4.44)$$

that is,

$$\Gamma = \Theta C$$

where C is the illustrated circulant matrix. Now

$$\Gamma^+ = C^+ \Theta^+.$$

Therefore

$$\Gamma^+ \Gamma = C^+ \Theta^+ \Theta C = C^+ C.$$

We can now revisit equation (4.43) and rewrite this equation as

$$\Gamma^+ \Gamma u^* = C^+ C u^* = \left\{ (2N-2)^2 - \zeta^2 \right\} u^*. \quad (4.45)$$

The eigenvalues of C are complex conjugates of the eigenvalues of C^+ (see Appendix C.6). This allows us to write that

$$\zeta(N, j)^2 = (2N-2)^2 - |\tilde{\rho}_j|^2 \quad (4.46)$$

where

$$\tilde{\rho}_j = -4 \sum_{k=1}^{N-1} \frac{(\omega^k)^2 \{1 - (\omega^j)^k\}}{(1 - \omega^k)^2}, \quad (4.47)$$

noting that $\omega^k = e^{i\frac{2\pi k}{N}}$ and $j = 0, 1, \dots, N-1$. To evaluate $\tilde{\rho}_j$ it will be convenient to consider the contour integral $I_1 = \frac{1}{2\pi i} \int_{C_1} \left(\frac{z^{N-1}}{z^N - 1} \right) \frac{z^2(1-z^j)}{(1-z)^2} dz$ where C_1 is illustrated as follows.

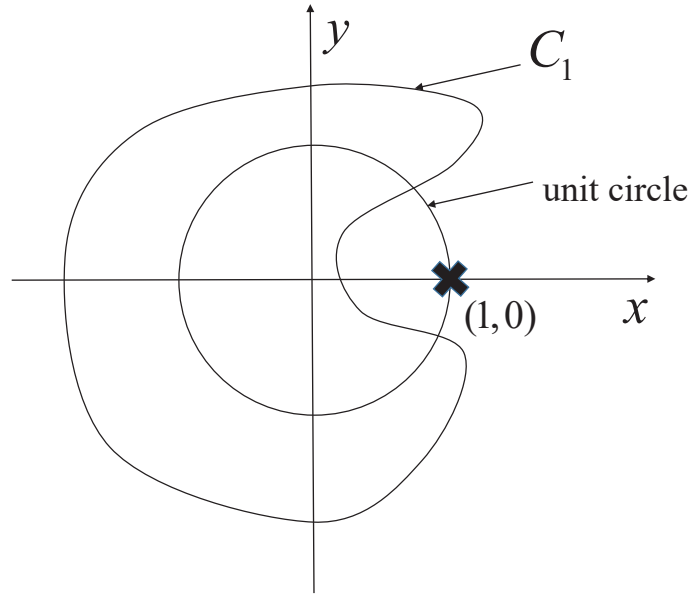


Figure 8: Illustration of contour C_1 (adapted from [89]). The contour is drawn so as to as to enclose all the roots of $z^N = 1$ except the root $z = 1$.

Employing the residue theorem (see Appendix C.7),

$$I_1 = \frac{1}{N} \sum_{k=1}^{N-1} \frac{\{1 - (\omega^k)^j\} (\omega^k)^2}{(1 - \omega^k)^2 (N)}. \quad (4.48)$$

Now $I_1 = \frac{1}{2\pi i} \int_{C_1} f(z) dz$ is equivalent to the contour integral given by

$$I_2 = \frac{1}{2\pi i} \int_{C_2} \left(\frac{z^{N-1}}{z^N - 1} \right) \frac{z^2(1 - z^j)}{(1 - z)^2} dz$$

with C_2 illustrated as follows.

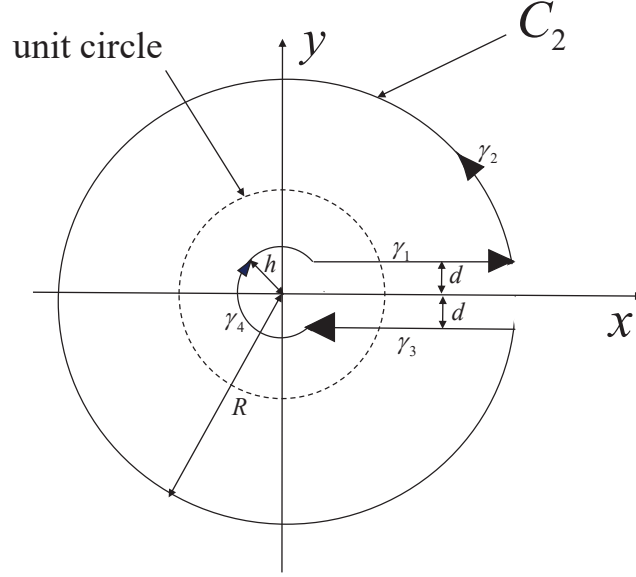


Figure 9: Illustration of contour C_2 (adapted from [89]). In the course of evaluating I_2 , we take the limits, $d \rightarrow 0$, $h \rightarrow 0$ and $R \rightarrow \infty$.

We note that $I_2 = I_1$ because the singularities of the expression $\left(\frac{z^{N-1}}{z^N-1}\right) \frac{z^2(1-z^j)}{(1-z)^2}$ trapped inside C_1 are the same as the singularities of this same expression trapped inside C_2 . We find that (see Appendix C.7)

$$I_2 = \frac{j(j+N+2)}{2N} - j.$$

Now, as noted previously, $I_2 = I_1$, and therefore

$$\frac{1}{N} \sum_{k=1}^{N-1} \frac{\left\{1 - (\omega^k)^j\right\} (\omega^k)^2}{(1 - \omega^k)^2} = \frac{j(j+N+2)}{2N} - j = \frac{j^2 - Nj + 2j}{2}.$$

We can now write that

$$\tilde{\rho}_j = -4 \sum_{k=1}^{N-1} \frac{\left\{1 - (\omega^k)^j\right\} (\omega^k)^2}{(1 - \omega^k)^2} = -4 \left(\frac{j^2 - Nj + 2j}{2} \right) = (2j)(N - j - 2).$$

We can now calculate ζ_j^2 explicitly. We recall that $\zeta_j^2 = (2N-2)^2 - |\tilde{\rho}_j|^2$ which given the above calculation

can be shown to be

$$\zeta(N, j)^2 = 4(j+1)(N-j-1) \{ (N-1)(1-j) + j(j+1) \}$$

where $j = 0, 1, \dots, N-1$. We can now prove that linear instability i. e. $\zeta^2(N, j) < 0$ occurs for $N \geq 8$. Calculating $\zeta(N, j)^2$ exhaustively for $2 \leq N \leq 7$ reveals that there are no negative values for $\zeta(N, j)^2$ in this range. To complete the proof, we can focus on $j = 2$. Now

$$\zeta(N, 2)^2 = 12(N-3)(7-N).$$

We know that for $N \geq 7$, $12(N-3) > 0$. We now focus on $7-N$ and we see this is clearly negative for $N \geq 8$. Therefore we are guaranteed a negative value of $\zeta(N, j)^2$ for $N \geq 8$. Linear instability thus sets in at $N = 8$ and occurs for every $N \geq 8$. Regarding the degeneracy of eigenvalues, it is easily shown that

$$\zeta(N, j) = \zeta \{N, N - (j+2)\}.$$

5 Writing down Hydrodynamic Relations arising from the Gross-Pitaevskii Equation using the Language of Geometric Mechanics

We have mentioned the fact that the only vortices permitted in superfluids are quantized vortices. From a modelling perspective, the quantization of the superfluid circulation integral,

$$\oint_c \mathbf{v} \cdot d\mathbf{r}$$

(evaluated along a closed contour) is a consequence of both the fact that the superfluid velocity $\mathbf{v}$ is proportional to the gradient of the wave function phase and an insistence that the condensate wave function is single-valued. If the contour c is not moving, it is a simple matter to evaluate the time derivative of the above integral, which will be zero given that the integral evaluates to a constant.

In our exploration of vortex dynamics in a BEC, we shall introduce some complexity to this situation, and geometric mechanics, in particular the Lie derivative provides a powerful tool for this exploration. The Lie derivative of an object with respect to a vector field measures the rate of change of the object as it is transported by the flow of the vector field [98]. We shall consider (in two and three dimensions) instead a moving contour, one moving with the same velocity as the superfluid. Furthermore we shall consider other integrals (in three dimensions) relevant to the study of superfluid vortex dynamics. The one-form basis set dx^j , $j = 1, 2, 3$ is written as $dx^1 = dx$, $dx^2 = dy$ and $dx^3 = dz$ in this section. Unit vectors will be represented by the familiar $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ vectors.

This section is divided into two broad subsections, one for the study of two dimensional BECs, and one for the study of BECs in three dimensions. In both scenarios, we perform analyses in the absence and presence of condensate rotation, as well as in the absence and presence of vortices.

Typically, for each integral we consider, we initially perform our analysis in the absence of vortices and in the absence of condensate rotation. Complexity is then built up by adding the vortices and then condensate rotation in turn (where applicable). We have not encountered the geometric equations produced in this section elsewhere in the literature.

5.1 Consideration of the GPE in Two Spatial Dimensions

In two spatial dimensions, we concern ourselves primarily with the time derivative of the circulation integral.

5.1.1 Time Derivative of the Circulation Integral (Vortices and Condensate Rotation absent)

We begin our exploration with the time derivative of the circulation integral, written as

$$\frac{d}{dt} \oint_{c(\mathbf{v})} \mathbf{v} \cdot d\mathbf{r}, \quad (5.1)$$

where $c(\mathbf{v})$ is a contour moving with the superfluid velocity $\mathbf{v}$. We can write that [99]

$$\frac{d}{dt} \oint_{c(\mathbf{v})} \mathbf{v} \cdot d\mathbf{r} = \oint_{c(\mathbf{v})} \left(\frac{\partial}{\partial t} + L_{\mathbf{v}} \right) (\mathbf{v} \cdot d\mathbf{r}) \quad (5.2)$$

noting that the Lie derivative symbol L has made an appearance. We shall now evaluate the terms appearing on the right hand side of equation (5.1). Now,

$$\partial_t(\mathbf{v} \cdot d\mathbf{r}) = \partial_t \mathbf{v} \cdot d\mathbf{r}.$$

$L_{\mathbf{v}} \mathbf{v} \cdot d\mathbf{r}$ is to be considered in greater detail. By Cartan's formula

$$L_{\mathbf{v}} (\mathbf{v} \cdot d\mathbf{r}) = v \rfloor d(\mathbf{v} \cdot d\mathbf{r}) + d(v \rfloor \mathbf{v} \cdot d\mathbf{r}).$$

$d(\mathbf{v} \cdot d\mathbf{r})$ will be considered. In two dimensions,

$$\begin{aligned} d(v_1 dx + v_2 dy) &= dv_1 \wedge dx + dv_2 \wedge dy \\ &= (\partial_x v_1 dx + \partial_y v_1 dy) \wedge dx + (\partial_x v_2 dx + \partial_y v_2 dy) \wedge dy \\ &= \partial_y v_1 (dy \wedge dx) + \partial_x v_2 (dx \wedge dy) \\ &= (dx \wedge dy) (\partial_x v_2 - \partial_y v_1). \end{aligned}$$

where $\wedge$ is the wedge product. Next, we will consider $v \rfloor d(\mathbf{v} \cdot d\mathbf{r})$.

$$\begin{aligned} v \rfloor d(\mathbf{v} \cdot d\mathbf{r}) &= (v_1 \partial_x + v_2 \partial_y) \rfloor (dx \wedge dy) (\partial_x v_2 - \partial_y v_1) \\ &= v_1 \partial_x \rfloor (dx \wedge dy) (\partial_x v_2 - \partial_y v_1) + v_2 \partial_y \rfloor (dx \wedge dy) (\partial_x v_2 - \partial_y v_1) \\ &= v_1 dy (\partial_x v_2 - \partial_y v_1) + v_2 \partial_y \rfloor (dy \wedge dx) (\partial_y v_1 - \partial_x v_2) \\ &= v_1 dy (\partial_x v_2 - \partial_y v_1) + v_2 dx (\partial_y v_1 - \partial_x v_2) \\ &= v_2 (\partial_y v_1 - \partial_x v_2) dx + v_1 (\partial_x v_2 - \partial_y v_1) dy. \end{aligned}$$

In two dimensions,

$$\mathbf{v} \times \text{curl} \mathbf{v} = \{-v_2 (\partial_y v_1 - \partial_x v_2)\} \hat{\mathbf{i}} + \{-v_1 (\partial_x v_2 - \partial_y v_1)\} \hat{\mathbf{j}}.$$

We have therefore shown that

$$v \rfloor d(\mathbf{v} \cdot d\mathbf{r}) = -(\mathbf{v} \times \text{curl} \mathbf{v}) \cdot d\mathbf{r}.$$

Next consider $v \rfloor \mathbf{v} \cdot d\mathbf{r}$.

$$\begin{aligned} v \rfloor \mathbf{v} \cdot d\mathbf{r} &= (v_1 \partial_x + v_2 \partial_y) \rfloor (v_1 dx + v_2 dy) \\ &= v_1 \partial_x \rfloor v_1 dx + v_2 \partial_y \rfloor v_2 dy \\ &= v_1^2 + v_2^2. \end{aligned}$$

Therefore

$$d(v \rfloor \mathbf{v} \cdot d\mathbf{r}) = d(|\mathbf{v}|^2) = \nabla(|\mathbf{v}|^2) \cdot d\mathbf{r}.$$

We can write that

$$L_{\mathbf{v}}(\mathbf{v} \cdot d\mathbf{r}) = \left\{ -(\mathbf{v} \times \text{curl} \mathbf{v}) + \nabla(|\mathbf{v}|^2) \right\} \cdot d\mathbf{r}.$$

It follows that

$$(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r}) = \left\{ \partial_t \mathbf{v} - (\mathbf{v} \times \text{curl} \mathbf{v}) + \nabla(|\mathbf{v}|^2) \right\} \cdot d\mathbf{r}. \quad (5.3)$$

Recalling equation (3.25),

$$\partial_t \mathbf{v} = -\frac{1}{2} \nabla(|\mathbf{v}|^2) + \nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right).$$

Therefore, in two dimensions, we can write the geometric form of the Euler equation analogue as

$$(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r}) = \left\{ \nabla \left(\frac{1}{2} |\mathbf{v}|^2 + \frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right) + (\text{curl} \mathbf{v} \times \mathbf{v}) \right\} \cdot d\mathbf{r}. \quad (5.4)$$

In the absence of vortices, this is equivalent to

$$(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r}) = \nabla \left(\frac{1}{2} |\mathbf{v}|^2 + \frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right) \cdot d\mathbf{r} = d\bar{\Phi} \quad (5.5)$$

and we obtain the result

$$\frac{d}{dt} \oint_{c(\mathbf{v})} \mathbf{v} \cdot d\mathbf{r} = 0. \quad (5.6)$$

This is because $\text{curl} \mathbf{v} = 0$ in the absence of vortices. Therefore, even if the contour is moving with the superfluid, the time derivative of the circulation integral remains the same as the case when the contour is not moving.

5.1.2 Time Derivative of the Circulation Integral (Vortices present, Condensate Rotation absent)

We can build on our work in 5.1.1 and now consider the effect of the presence of vortices on the time derivative of the circulation integral, expression (5.1). In this case, we can re-examine the geometric equation (5.4) and note that $\text{curl} \mathbf{v}$ is no longer necessarily equal to zero.

Therefore, in two spatial dimensions, the circulation integral will not necessarily vanish in the presence of vortices. This is due to the fact that $(\partial_t + L_V) \mathbf{v} \cdot d\mathbf{r}$ does not necessarily take the form of an exact differential given that the $\mathbf{v} \times \text{curl} \mathbf{v}$ term does not necessarily vanish.

5.1.3 Time Derivative of the Circulation Integral (Vortices and Condensate Rotation present)

We can further build on the complexity encountered in section 5.1.2, by incorporating a rotating condensate (rotating at a constant angular speed Ω in the $x - y$ plane) into the GPE. We recall at this stage our work in section 4.2, and we can write that

$$i\partial_t \psi = -\frac{1}{2} \nabla^2 \psi + V\psi + g|\psi|^2 \psi - i\Omega(x\partial_y - y\partial_x)\psi.$$

We further recall that the continuity equation takes the form

$$\partial_t \sigma^2 = -\nabla \cdot \{ \sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \}.$$

It can be shown that the Euler equation analogue takes the form

$$\partial_t \mathbf{v} = \nabla \left(-V - g\sigma^2 + \frac{1}{2\sigma} \nabla^2 \sigma + \frac{1}{2} |\mathbf{v}|^2 \right) - \nabla \{ (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v} \}.$$

In examining the time derivative of the circulation integral, we consider

$$L_{\mathbf{v}+\mathbf{\Omega}\times\mathbf{r}}(\mathbf{v}\cdot d\mathbf{r}),$$

the implication being that we are examining a contour moving with the velocity of the superfluid as viewed from an inertial reference frame not moving with the fluid. Now

$$(v + \mathbf{\Omega} \times \mathbf{r}) \rfloor d(\mathbf{v} \cdot d\mathbf{r}) = -\{(\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \times \text{curl}\mathbf{v}\} \cdot d\mathbf{r}.$$

$d\{(v + \mathbf{\Omega} \times \mathbf{r}) \rfloor \mathbf{v} \cdot d\mathbf{r}\}$ is to be considered next. We find that

$$(v + \mathbf{\Omega} \times \mathbf{r}) \rfloor \mathbf{v} \cdot d\mathbf{r} = (\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \cdot \mathbf{v}.$$

Therefore

$$\begin{aligned} L_{\mathbf{v}+\mathbf{\Omega}\times\mathbf{r}}(\mathbf{v}\cdot d\mathbf{r}) &= -\{(\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \times \text{curl}\mathbf{v}\} \cdot d\mathbf{r} + d\{(\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \cdot \mathbf{v}\} \\ &= -\{(\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \times \text{curl}\mathbf{v}\} \cdot d\mathbf{r} + \nabla\{(\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \cdot \mathbf{v}\} \cdot d\mathbf{r}. \end{aligned}$$

For a rotating condensate, the Euler equation analogue may therefore be recast in the geometric form

$$(\partial_t + L_{\mathbf{v}+\mathbf{\Omega}\times\mathbf{r}})(\mathbf{v} \cdot d\mathbf{r}) = \left\{ \nabla \left(\frac{1}{2} |\mathbf{v}|^2 + \frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right) - (\mathbf{v} + \mathbf{\Omega} \times \mathbf{r}) \times \text{curl}\mathbf{v} \right\} \cdot d\mathbf{r}. \quad (5.7)$$

We note that

$$\frac{d}{dt} \oint_{c(\mathbf{v}+\mathbf{\Omega}\times\mathbf{r})} \mathbf{v} \cdot d\mathbf{r} = \oint_{c(\mathbf{v}+\mathbf{\Omega}\times\mathbf{r})} (\partial_t + L_{\mathbf{v}+\mathbf{\Omega}\times\mathbf{r}})(\mathbf{v} \cdot d\mathbf{r}). \quad (5.8)$$

Setting the external potential in the GPE to zero and recalling the form of $\text{curl}\mathbf{v}$ for the point vortex ansatz, then we have found the form of the time derivative for the circulation integral in the presence of point vortices in a uniform medium from the perspective of a rotating reference frame. In the presence of vortices $(\partial_t + L_{\mathbf{v}+\mathbf{\Omega}\times\mathbf{r}})(\mathbf{v} \cdot d\mathbf{r})$ is no longer necessarily equal to an exact differential and therefore the right hand side of equation (5.8) is not necessarily equal to zero. Rotation of the condensate can be removed by setting $\mathbf{\Omega}$ to zero.

5.2 Consideration of the GPE in three spatial dimensions

In two dimensions, we considered the time derivative of the circulation integral. In three dimensions, among the new integrals we are able to consider are the time derivative of the mass of a moving volume of superfluid and the time derivative of the vorticity flux from a moving surface.

5.2.1 Time Derivative of the Circulation Integral (Vortices and Condensate Rotation absent)

We return to consideration of

$$(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r}).$$

As expected, the details of this calculation will be different in the three dimensional case compared to the two dimensional case. In three dimensions,

$$\mathbf{v} \cdot d\mathbf{r} = v_1 dx + v_2 dy + v_3 dz$$

and therefore

$$\partial_t(\mathbf{v} \cdot d\mathbf{r}) = \partial_t v_1 dx + \partial_t v_2 dy + \partial_t v_3 dz.$$

We shall now evaluate $L_{\mathbf{v}}(\mathbf{v} \cdot d\mathbf{r})$. As before,

$$L_{\mathbf{v}}(\mathbf{v} \cdot d\mathbf{r}) = v \rfloor d(\mathbf{v} \cdot d\mathbf{r}) + d(v \rfloor \mathbf{v} \cdot d\mathbf{r}).$$

In three dimensions,

$$d(\mathbf{v} \cdot d\mathbf{r}) = (dx \wedge dy)(\partial_x v_2 - \partial_y v_1) + (dy \wedge dz)(\partial_y v_3 - \partial_z v_2) + (dz \wedge dx)(\partial_z v_1 - \partial_x v_3).$$

Therefore

$$\begin{aligned} & v \rfloor d(\mathbf{v} \cdot d\mathbf{r}) \\ &= \\ & dx \{v_2(\partial_y v_1 - \partial_x v_2) + v_3(\partial_z v_1 - \partial_x v_3)\} + dy \{v_1(\partial_x v_2 - \partial_y v_1) + v_3(\partial_z v_2 - \partial_y v_3)\} \\ & \quad + dz \{v_2(\partial_y v_3 - \partial_z v_2) + v_1(\partial_x v_3 - \partial_z v_1)\}. \end{aligned}$$

In three dimensions,

$$\text{curl} \mathbf{v} = (\partial_y v_3 - \partial_z v_2) \hat{\mathbf{i}} + (\partial_z v_1 - \partial_x v_3) \hat{\mathbf{j}} + (\partial_x v_2 - \partial_y v_1) \hat{\mathbf{k}}$$

and it follows that

$$\begin{aligned}
& \mathbf{v} \times \text{curl} \mathbf{v} \\
& = \\
& \{v_2 (\partial_x v_2 - \partial_y v_1) + v_3 (\partial_x v_3 - \partial_z v_1)\} \hat{\mathbf{i}} + \{v_3 (\partial_y v_3 - \partial_z v_2) + v_1 (\partial_y v_1 - \partial_x v_2)\} \hat{\mathbf{j}} \\
& \quad + \{v_1 (\partial_z v_1 - \partial_x v_3) + v_2 (\partial_z v_2 - \partial_y v_3)\} \hat{\mathbf{k}}.
\end{aligned}$$

Therefore,

$$v \rfloor d(\mathbf{v} \cdot d\mathbf{r}) = -(\mathbf{v} \times \text{curl} \mathbf{v}) \cdot d\mathbf{r}.$$

Next, consider that

$$\begin{aligned}
& v \rfloor \mathbf{v} \cdot d\mathbf{r} \\
& = \\
& (v_1 \partial_x + v_2 \partial_y + v_3 \partial_z) \rfloor (v_1 dx + v_2 dy + v_3 dz) \\
& = \\
& v_1^2 + v_2^2 + v_3^2.
\end{aligned}$$

It follows that

$$d\{v \rfloor d(\mathbf{v} \cdot d\mathbf{r})\} = d(\mathbf{v} \cdot \mathbf{v}) = \nabla(\mathbf{v} \cdot \mathbf{v}) \cdot d\mathbf{r}.$$

Therefore we have found that

$$L_{\mathbf{v}}(\mathbf{v} \cdot d\mathbf{r}) = \left\{ -(\mathbf{v} \times \text{curl} \mathbf{v}) + \nabla(|\mathbf{v}|^2) \right\} \cdot d\mathbf{r}.$$

Recalling equation (3.25), we can write the geometric equation

$$(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r}) = \left\{ \nabla \left(\frac{1}{2} |\mathbf{v}|^2 + \frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right) + (\text{curl} \mathbf{v} \times \mathbf{v}) \right\} \cdot d\mathbf{r},$$

which is identical in appearance to the geometric form of the Euler equation analogue we encountered in our study of a two-dimensional condensate. In the absence of vortices and in the absence of a condensate rotation, the quantity $(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r})$ is once again found to be an exact differential with the result that

$$\frac{d}{dt} \oint_{c(\mathbf{v})} \mathbf{v} \cdot d\mathbf{r} = 0.$$

5.2.2 Time Derivative of the Circulation Integral (Vortices present, Condensate Rotation absent)

In section 5.2.1 we found the geometric equation

$$(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r}) = \left\{ \nabla \left(\frac{1}{2} |\mathbf{v}|^2 + \frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right) + (\text{curl} \mathbf{v} \times \mathbf{v}) \right\} \cdot d\mathbf{r},$$

where $(\partial_t + L_{\mathbf{v}})(\mathbf{v} \cdot d\mathbf{r})$ is no longer necessarily equal to an exact differential as $\text{curl} \mathbf{v}$ is no longer taken to vanish as is done for the case where vortices are absent. As a result, the time derivative of the circulation integral does not necessarily vanish.

5.2.3 Time Derivative of the Circulation Integral (Vortices Present, Condensate Rotation present)

We proceed to write the GPE in three dimensions in a manner that takes into account condensate rotation, making the choice that the condensate rotation is restricted to the $x - y$ plane. To this end, we revisit the GPE,

$$i\partial_t \psi = -\frac{1}{2} \nabla^2 \psi + V\psi + g|\psi|^2 \psi,$$

and make the change of variables,

$$\begin{pmatrix} x' \\ y' \\ z' \\ t' \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}.$$

It will turn out that for the condensate undergoing rotation in the $x - y$ plane that we can write the GPE in the form

$$i\partial_t \psi = -\frac{1}{2} \nabla^2 \psi + V\psi + g|\psi|^2 \psi + i\Omega (y\partial_x - x\partial_y) \psi,$$

where

$$\nabla^2 = \partial_{x^2}^2 + \partial_{y^2}^2 + \partial_{z^2}^2$$

and Ω is a constant (equivalent to $\frac{d\theta}{dt}$) which is the angular rotation rate. In the usual manner, let

$$\psi = \sigma e^{i\alpha},$$

and the continuity equation takes the familiar form

$$\partial_t \sigma^2 = -\nabla \cdot \{\sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})\}$$

where $\mathbf{v} = \nabla \alpha$. It will turn out that, in three dimensions, the Euler equation analogue takes the familiar form

$$\partial_t \mathbf{v} = \nabla \left(-V - g\sigma^2 + \frac{1}{2\sigma} \nabla^2 \sigma + \frac{1}{2} |\mathbf{v}|^2 \right) - \nabla \{(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v}\}.$$

To consider the circulation integral, we again look at

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}) (\mathbf{v} \cdot d\mathbf{r}),$$

and we obtain the familiar geometric equation,

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}) (\mathbf{v} \cdot d\mathbf{r}) = \left\{ \nabla \left(\frac{1}{2} |\mathbf{v}|^2 + \frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right) - (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \times \text{curl} \mathbf{v} \right\} \cdot d\mathbf{r}.$$

As vortices are present, the time derivative of the circulation integral, i.e.

$$\frac{d}{dt} \oint_{c(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})} \mathbf{v} \cdot d\mathbf{r}$$

does not necessarily vanish.

5.2.4 Time Derivative of the Mass of a moving Volume (Condensate Rotation absent)

In three spatial dimensions, we are in a position to consider the time derivative of a mass of moving volume of fluid. We shall assume that the volume of fluid involved, $Volume(\mathbf{v})$ is bounded by a surface moving with the fluid. We shall therefore evaluate

$$\frac{d}{dt} \int_{Volume(\mathbf{v})} \sigma^2 d^3x = \int_{Volume(\mathbf{v})} (\partial_t + L_{\mathbf{v}}) \sigma^2 d^3x.$$

Now

$$(\partial_t + L_{\mathbf{v}}) (\sigma^2 d^3x) = \partial_t (\sigma^2 d^3x) + L_{\mathbf{v}} \sigma^2 d^3x = (\partial_t \sigma^2) d^3x + L_{\mathbf{v}} \sigma^2 d^3x.$$

We shall evaluate

$$L_{\mathbf{v}}(\sigma^2 d^3 x) = v \rfloor d(\sigma^2 d^3 x) + d(v \rfloor \sigma^2 d^3 x).$$

We note that

$$d(\sigma^2 d^3 x) = d\sigma^2 \wedge dx \wedge dy \wedge dz = (\nabla \sigma^2 \cdot d\mathbf{r}) \wedge dx \wedge dy \wedge dz = 0.$$

It follows that

$$v \rfloor d(\sigma^2 d^3 x) = 0.$$

Now,

$$\begin{aligned} & v \rfloor \sigma^2 d^3 x \\ &= \\ & (v_1 \partial_x + v_2 \partial_y + v_3 \partial_z) \rfloor (\sigma^2 dx \wedge dy \wedge dz) \\ &= \\ & v_1 \sigma^2 dy \wedge dz + v_2 \sigma^2 dz \wedge dx + v_3 \sigma^2 dx \wedge dy. \end{aligned}$$

It can be shown that

$$d(v \rfloor \sigma^2 d^3 x) = \{\text{div}(\sigma^2 \mathbf{v})\} d^3 x.$$

We can therefore recast the continuity equation in the geometric form

$$(\partial_t + L_v)(\sigma^2 d^3 x) = (\partial_t \sigma^2) d^3 x + \{\text{div}(\sigma^2 \mathbf{v})\} d^3 x = \{\partial_t \sigma^2 + \text{div}(\sigma^2 \mathbf{v})\} d^3 x = 0. \quad (5.9)$$

Therefore we have found that

$$\frac{d}{dt} \int_{\text{Volume}(\mathbf{v})} \sigma^2 d^3 x = 0.$$

The above equation holds whether vortices are present or absent.

5.2.5 Time Derivative of the Mass of a moving Volume (Rotating Frame present)

Let us consider a moving volume for a condensate rotating about the z axis (as we encountered in

section 5.2.3). Consider

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\sigma^2 d^3 x) = (\partial_t \sigma^2) d^3 x + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}(\sigma^2 d^3 x).$$

Now

$$L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}(\sigma^2 d^3 x) = (v + \boldsymbol{\Omega} \times \mathbf{r}) \lrcorner d(\sigma^2 d^3 x) + d\{(v + \boldsymbol{\Omega} \times \mathbf{r}) \lrcorner \sigma^2 d^3 x\}.$$

We note that

$$d(\sigma^2 d^3 x) = d\sigma^2 \wedge d^3 x = 0.$$

Now

$$\begin{aligned} & (v + \boldsymbol{\Omega} \times \mathbf{r}) \lrcorner (\sigma^2 dx \wedge dy \wedge dz) \\ &= \\ & \sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})_1 dy \wedge dz + \sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})_2 dz \wedge dx + \sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})_3 dx \wedge dy \\ &= \\ & \sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot d\mathbf{S}. \end{aligned}$$

Furthermore,

$$d\{(v + \boldsymbol{\Omega} \times \mathbf{r}) \lrcorner \sigma^2 d^3 x\} = \boldsymbol{\nabla} \cdot \{\sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})\} d^3 x.$$

Therefore

$$L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}(\sigma^2 d^3 x) = \boldsymbol{\nabla} \cdot \{\sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})\} d^3 x.$$

It follows that the geometric form of the continuity equation for a rotating, three dimensional condensate in this case is

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\sigma^2 d^3 x) = \langle \partial_t \sigma^2 + \boldsymbol{\nabla} \cdot \{\sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})\} \rangle d^3 x = 0 \quad (d^3 x) = 0 \quad (5.10)$$

as

$$\partial_t \sigma^2 = -\boldsymbol{\nabla} \cdot \{\sigma^2 (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})\}$$

from the continuity equation. Therefore it has been established that

$$\frac{d}{dt} \int_{Volume(\mathbf{v}+\mathbf{\Omega}\times\mathbf{r})} \sigma^2 d^3x = 0$$

whether vortices are present or not.

5.2.6 Time derivative of the Vorticity Flux through a moving Surface (Condensate Rotation absent)

For a surface $S(\mathbf{v})$ bound by an arbitrary circuit moving with the superfluid, the vorticity flux through the surface can be written as

$$\int_{S(\mathbf{v})} \text{curl} \mathbf{v} \cdot d\mathbf{S}.$$

In the absence of vortices $\text{curl} \mathbf{v} = 0$ and the vorticity flux integral will evaluate to zero as expected. To evaluate the time derivative of the above integral in the presence of vortices, we shall look at

$$(\partial_t + L_{\mathbf{v}}) (\text{curl} \mathbf{v} \cdot d\mathbf{S}).$$

Going forward we shall write $\text{curl} \mathbf{v} = \boldsymbol{\omega}$. Now

$$(\partial_t + L_{\mathbf{v}}) (\boldsymbol{\omega} \cdot d\mathbf{S}) = (\partial_t \boldsymbol{\omega}) \cdot d\mathbf{S} + L_{\mathbf{v}} (\boldsymbol{\omega} \cdot d\mathbf{S}).$$

We can write

$$L_{\mathbf{v}} (\boldsymbol{\omega} \cdot d\mathbf{S}) = v \rfloor d(\boldsymbol{\omega} \cdot d\mathbf{S}) + d(v \rfloor \boldsymbol{\omega} \cdot d\mathbf{S}).$$

Now,

$$d(\boldsymbol{\omega} \cdot d\mathbf{S}) = (\nabla \cdot \boldsymbol{\omega}) d^3x.$$

We can then show that

$$v \rfloor d(\boldsymbol{\omega} \cdot d\mathbf{S}) = (\nabla \cdot \boldsymbol{\omega}) \mathbf{v} \cdot d\mathbf{S}.$$

With some work (see Appendix D.1), we can show that

$$d(v \rfloor \boldsymbol{\omega} \cdot d\mathbf{S}) = \text{curl}(\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{S}.$$

Therefore

$$(\partial_t + L_{\mathbf{v}})(\boldsymbol{\omega} \cdot d\mathbf{S}) = \{\partial_t \boldsymbol{\omega} + (\nabla \cdot \boldsymbol{\omega}) \mathbf{v} + \text{curl}(\boldsymbol{\omega} \times \mathbf{v})\} \cdot d\mathbf{S}$$

We recall from equation (3.25) that

$$\partial_t \mathbf{v} = -\frac{1}{2} \nabla (|\mathbf{v}|^2) + \nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right).$$

Taking the curl of both sides, the curl of

$$\nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right)$$

will be assumed to vanish. Therefore

$$\partial_t \boldsymbol{\omega} = \frac{1}{2} \left(-\nabla \times \nabla |\mathbf{v}|^2 \right).$$

Making use of the identity

$$-\frac{1}{2} \nabla (|\mathbf{v}|^2) = -(\mathbf{v} \cdot \nabla) \mathbf{v} - (\mathbf{v} \times \boldsymbol{\omega})$$

we obtain the geometric description of the vorticity flux as

$$(\partial_t + L_{\mathbf{v}})(\boldsymbol{\omega} \cdot d\mathbf{S}) = \left\langle \text{curl}(\boldsymbol{\omega} \times \mathbf{v}) - \text{curl} \left\{ \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) \right\} + \mathbf{v}(\nabla \cdot \boldsymbol{\omega}) \right\rangle \cdot d\mathbf{S}. \quad (5.11)$$

Under the condition that

$$\mathbf{v}(\nabla \cdot \boldsymbol{\omega}) = \text{curl} \left\{ \nabla \left(\frac{|\mathbf{v}|^2}{2} \right) \right\} - \text{curl}(\boldsymbol{\omega} \times \mathbf{v}),$$

then

$$(\partial_t + L_{\mathbf{v}})(\boldsymbol{\omega} \cdot d\mathbf{S}) = 0$$

and vorticity flux from the moving surface is zero. In writing the geometric equation (5.11), We note that we obtained an expression for

$$\frac{d}{dt} \int_{S(\mathbf{v})} \boldsymbol{\omega} \cdot d\mathbf{S} = \int_{S(\mathbf{v})} (\partial_t + L_{\mathbf{v}}) \boldsymbol{\omega} \cdot d\mathbf{S}.$$

5.2.7 Time Derivative of the Vorticity Flux through a moving Surface (Condensate Rotation present)

We shall in this instance evaluate the geometric expression

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\boldsymbol{\omega} \cdot d\mathbf{S}) = (\partial_t \boldsymbol{\omega}) \cdot d\mathbf{S} + (L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\boldsymbol{\omega} \cdot d\mathbf{S})$$

where the condensate is performing rotation about the z axis, as encountered in section 5.2.3. With some work, we can show that

$$(L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\boldsymbol{\omega} \cdot d\mathbf{S}) = d\langle \{\boldsymbol{\omega} \times (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})\} \cdot d\mathbf{r} \rangle + (\nabla \cdot \boldsymbol{\omega})(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot d\mathbf{S}.$$

Therefore we can write that

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\boldsymbol{\omega} \cdot d\mathbf{S}) = \langle \partial_t \boldsymbol{\omega} + \text{curl} \{ \boldsymbol{\omega} \times (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \} \rangle \cdot d\mathbf{S} + (\nabla \cdot \boldsymbol{\omega})(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot d\mathbf{S}.$$

We recall that for a rotating condensate, the Euler equation analogue takes the form

$$\partial_t \mathbf{v} = \nabla \left(-V - g\sigma^2 + \frac{1}{2\sigma} \nabla^2 \sigma + \frac{1}{2} |\mathbf{v}|^2 \right) - \nabla \{ (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v} \}.$$

Taking the curl of both sides, and again assuming that the curl of $\nabla \left(\frac{\nabla^2 \sigma}{2\sigma} - V - g\sigma^2 \right)$ will vanish, then

$$\partial_t \boldsymbol{\omega} = \text{curl} \left\langle -\frac{1}{2} \nabla (|\mathbf{v}|^2) - \nabla \{ (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v} \} \right\rangle.$$

We recall that

$$-\frac{1}{2} \nabla (|\mathbf{v}|^2) = -(\mathbf{v} \cdot \nabla) \mathbf{v} - (\mathbf{v} \times \boldsymbol{\omega}).$$

Therefore

$$\partial_t \boldsymbol{\omega} = \text{curl} \langle \boldsymbol{\omega} \times \mathbf{v} - (\mathbf{v} \cdot \nabla) \mathbf{v} - \nabla \{ (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v} \} \rangle.$$

It follows that with some work, we can write the rotational analogue of the geometric equation (5.11) as

$$\begin{aligned} & (\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}})(\boldsymbol{\omega} \cdot d\mathbf{S}) \\ &= \\ & \left[\text{curl} \left\langle (\boldsymbol{\omega} \times \mathbf{v}) - \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \{ (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v} \} + \boldsymbol{\omega} \times \boldsymbol{\Omega} \times \mathbf{r} \right\rangle + (\nabla \cdot \boldsymbol{\omega})(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}) \right] \cdot d\mathbf{S}. \end{aligned} \tag{5.12}$$

Furthermore, under the condition that

$$\operatorname{curl} \left\langle (\boldsymbol{\omega} \times \mathbf{v}) - \left(\frac{|\mathbf{v}|^2}{2} \right) - \nabla \{ (\boldsymbol{\Omega} \times \mathbf{r}) \cdot \mathbf{v} \} + \boldsymbol{\omega} \times \boldsymbol{\Omega} \times \mathbf{r} \right\rangle = - (\nabla \cdot \boldsymbol{\omega}) (\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})$$

then

$$(\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}) (\boldsymbol{\omega} \cdot d\mathbf{S}) = 0.$$

In writing the geometric equation (5.12), we note that we have obtained an expression for

$$\frac{d}{dt} \int_{S(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})} \boldsymbol{\omega} \cdot d\mathbf{S} = \int_{S(\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r})} (\partial_t + L_{\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{r}}) (\boldsymbol{\omega} \cdot d\mathbf{S}).$$

6 Conclusions and future work

Given that a considerable portion of this thesis studies vortex dynamics, it is of interest to examine how vortices are formed. This topic is reviewed in section two. Focusing on two-dimensional vortex motion, reviews are also provided concerning the effective action and microscopic approaches to describing vortex dynamics. Vortex mass is discussed. We also survey the topic of vortex motion and phonon radiation in two and three dimensions in this section. In our discussion of three dimensional vortex motion, we discuss some of the practical approaches taken to describe the motion of three dimensional vortices. This complements the work on two dimensional vortex motion which has been the primary focus of this thesis.

Concerning the description of phonon emission from accelerating vortices, particular attention has been paid to the work of Arovas and Freire, in which the relationship between the dynamics of a two-dimensional, homogeneous superfluid and QED is exploited to write down an effective point vortex action. We have written down, based on their work, the full functional form of the frequency-dependent inertial mass of an isolated vortex. We have not encountered this functional form in their paper, or anywhere else in the literature. Another new result, based on their work, is an expression for the effective vortex action which accounts explicitly for the positions of the vortices. Aside from these, algebraic details of key steps of their work, which are not present in their paper are also elucidated in this thesis. We have also identified a framework within their theory to calculate the phonon radiation patterns which may result from the motion of a system of two or more vortices. We recall that the effective action takes the form as seen in equation (5.9). In principle, the velocities of the vortices can be extracted from this action. We further recall the form of the extremized guage field A^μ of L_{QED} ,

$$A^\mu = \frac{\hbar}{m} \int_{-\infty}^{\infty} \sum_i^N k_i P^\mu \frac{\Theta \left[t - t_1 - \frac{\langle \{x-x_i(t_1)\}^2 + \{y-y_i(t_1)\}^2 \rangle^{\frac{1}{2}}}{c} \right]}{\left[(t - t_1)^2 - \frac{\langle \{x-x_i(t_1)\}^2 + \{y-y_i(t_1)\}^2 \rangle}{c^2} \right]^{\frac{1}{2}}} dt_1$$

where P^μ is given by 1 for $\mu = 0$, $\frac{dx_i(t_1)}{cdt_1}$ for $\mu = 1$, and $\frac{dy_i(t_1)}{cdt_1}$ for $\mu = 2$. If the velocities calculated from the effective action are known, the form of A^μ may be calculated. Calculation of the phonon radiation

pattern involves recalling the transformation given by

$$\left(\tilde{\sigma} - \bar{\sigma}, \tilde{\mathbf{Q}}\right) = -\frac{\bar{\sigma}}{c} \left(\hat{\mathbf{k}} \cdot \nabla \times \mathbf{A}, c\hat{\mathbf{k}} \times \nabla A^0 + \hat{\mathbf{k}} \times \partial_t \mathbf{A}\right)$$

from which the condensate density can be calculated as

$$\tilde{\sigma} = \bar{\sigma} \left(1 + \frac{\hat{\mathbf{k}} \cdot \nabla \times \mathbf{A}}{c}\right).$$

It is worth noting that the calculation of the difference $\tilde{\sigma} - \bar{\sigma}$ resembles a calculation of a magnetic field, while the calculation of $\tilde{\mathbf{Q}}$ resembles an electric field calculation. The obstacle to implementing this method is the successful calculation of the vortex velocities from the effective action. This makes their theory particularly interesting for study as a possible means to provide some insights into simultaneously describing phonon motion and radiation where systems of multiple vortices are concerned (besides numerical simulation of the GPE). It may be useful to study geometric equations related to the GPE to make progress in studying this problem.

Following section two, we look more closely at describing vortex dynamics using the GPE, beginning with a derivation of the GPE itself. Following this derivation of the GPE, we discuss a number of its structural properties. Regarding the topic of GPE wave function collapse in two spatial dimensions, we have derived and solved the virial relation for the case of a harmonic potential, a result we have not encountered elsewhere in the literature. The GPE is a nonlinear partial differential equation, and therefore it is not surprising that analytic solutions are not always readily accessible. However, if the Thomas-Fermi regime is applicable, then analytic solutions of the GPE are more readily obtained. The Thomas-Fermi regime is commonly encountered in the literature and is discussed in this thesis. We then illustrate the hydrodynamic nature of the GPE through its hydrodynamic relations, given that the study of vortex dynamics has long been understood (even prior to the discovery of BECs) as a topic in hydrodynamics. We take care to illustrate how the notion of the gradient of the condensate wave function phase being proportional to the superfluid velocity arises. Our study of the hydrodynamic relations of the GPE facilitates our discussion of condensate surface modes, which themselves are an important topic in the field of BEC vortex dynamics given the important role they play in the description of vortex formation. Vortices are mathematically introduced in this work by discussion of a single-vortex solution of the GPE, followed by a discussion of conditions whereby vortex formation is energetically favourable.

In section four, we describe the point-vortex model, and derive hydrodynamic relations for the two dimensional GPE in the presence of point vortices while incorporating a constant rate of condensate rota-

tion. These are novel equations. The point vortex model in a compressible condensate is then simplified to that of a point vortex model in an incompressible condensate. We present our independent derivation of the Lagrangian describing the motion of multiple point vortices in an incompressible (and non-rotating) condensate. Maintaining this focus on point vortices in two dimensional, incompressible BECs, we explore properties of the resulting Hamiltonian system. Notably, regarding study of the integrability of the Hamiltonian system, we have independently provided the full algebraic details of a transformation which reduces the degrees of freedom of the Hamiltonian representing four identical vortices from four to two. The reduced Hamiltonian resembles the form found when coupled nonlinear oscillators are involved in a problem [85]. In our study of the Hamiltonian system, we have derived in the context of linear stability analysis, the equations of motion governing the displacements of the vortices from equilibrium motion in a steadily rotating regular vortex polygon (when viewed from the perspective of a frame corotating with the regular vortex polygon). We have not encountered this calculation elsewhere in the literature. In particular, from our calculation, we recover the known result that the smallest number of vortices for which linear instability manifests is eight. This thesis includes a literature survey of the problem of linear stability analysis of rotating vortex polygons.

Regarding exploration of the GPE, the thesis concludes with the examination of a number of hydrodynamic integrals using the language of geometric mechanics. We derive novel geometric equations for superfluid BEC dynamics as a result of this effort. Our work in section four is applicable to a broader range of scenarios in superfluid BEC dynamics than encountered in section three. For example, a three dimensional BEC system rotating at a constant angular speed (and in which incompressibility has not been imposed) is examined. The presence or absence of vortices, as well as the presence or absence of condensate rotation, are recurring themes in our examination of the BEC superfluid dynamics using the GPE.

Concerning future work, It is of interest to re-examine the action, equation (2.9), and deduce an analytical expression for the vortex equations of motion, starting with a simple case, for example a system with just two vortices. It will be of use to retain the effect of condensate compressibility in any approximate or fully analytical schemes. It will also be of interest to determine how the presence of more than vortex affects the expression for the vortex mass in this theory. We note that progress continues to be made concerning mass calculations for an isolated vortex, noting the work of reference [100]. Returning to the results of Barenghi et al. [13], the motion of a corotating vortex pair, along with phonon emission, has been simulated. We could, upon successful application of the framework we have

considered based on work in [8], compare the calculated phonon radiation pattern with the results of this simulation. Furthermore, if a suitable approximation scheme for the action, equation (2.9), is found such that equations of vortex motion similar to equation (4.31) can be written, it might be of interest to calculate the effect of condensate compressibility on the linear stability analysis calculation we have performed in this thesis.

On the topic of vortex formation, which we have surveyed in this thesis, we are interested in studying the NLSE to gain further insights into how defects are formed in a two dimensional setting [43]. It will be interesting to study the time evolution of contours of constant value, with the contours representing the real and imaginary parts of the GPE wave function [43]. The intersection of real and imaginary contours of zero value will indicate points where the condensate density becomes zero, and may indicate vortex formation [43].

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APPENDICES

APPENDIX A

A.1 Derivation of L_{min}

We recall that (for $Z_\mu = 0$)

$$L_{QED} = \frac{\bar{\sigma}}{2}(E^2 - B^2) - \frac{\bar{\sigma}}{c}J^\mu(A_\mu + a_\mu).$$

Now $E_x = \frac{1}{c}\partial_t A_1 - \partial_x A_0$ and $E_y = \frac{1}{c}\partial_t A_2 - \partial_y A_0$. Therefore

$$\begin{aligned} E^2 &= E_x^2 + E_y^2 \\ &= (\partial_x A_0)(\partial_x A_0) + (\partial_y A_0)(\partial_y A_0) + \frac{1}{c^2}(\partial_t A_1)(\partial_t A_1) + \frac{1}{c^2}(\partial_t A_2)(\partial_t A_2) \\ &\quad - \frac{2}{c}(\partial_x A_0)(\partial_t A_1) - \frac{2}{c}(\partial_y A_0)(\partial_t A_2). \end{aligned}$$

Now $(\partial_x A_0)(\partial_x A_0) + (\partial_y A_0)(\partial_y A_0) := -A_0 \nabla^2 A_0$ and $\frac{1}{c^2}(\partial_t A_1)(\partial_t A_1) + \frac{1}{c^2}(\partial_t A_2)(\partial_t A_2) := -\frac{1}{c^2}A_1 \partial_t^2 A_1 - \frac{1}{c^2}A_2 \partial_t^2 A_2$. It follows that

$$E^2 := -A_0 \nabla^2 A_0 - \frac{1}{c^2}A_1 \partial_t^2 A_1 - \frac{1}{c^2}A_2 \partial_t^2 A_2 - \frac{2}{c}(\partial_x A_0)(\partial_t A_1) - \frac{2}{c}(\partial_y A_0)(\partial_t A_2).$$

We note that $-\frac{1}{c}(\partial_x A_0)(\partial_t A_1) - \frac{1}{c}(\partial_y A_0)(\partial_t A_2) := \frac{1}{c}(A_1 \partial_t \partial_x A_0 + A_2 \partial_t \partial_y A_0)$ and $-\frac{1}{c}(\partial_x A_0)(\partial_t A_1) - \frac{1}{c}(\partial_y A_0)(\partial_t A_2) := \frac{1}{c}A_0(\partial_x \partial_t A_1 + \partial_y \partial_t A_2)$. We can write

$$\begin{aligned} E^2 &:= -\left\{A_0 \nabla^2 A_0 - \frac{1}{c}A_0(\partial_x \partial_t A_1 + \partial_y \partial_t A_2)\right\} \\ &\quad - \frac{1}{c^2}(A_1 \partial_t^2 A_1 + A_2 \partial_t^2 A_2 - cA_1 \partial_t \partial_x A_0 - cA_2 \partial_t \partial_y A_0). \end{aligned}$$

At this stage of the calculation we recall the EL equations i.e. equation (2.6) while noting that $A^0 = A_0$, $A^1 = -A_1$ and $A^2 = -A_2$. Therefore

$$\begin{aligned} E^2 &:= -\frac{1}{c}\left\{A_0\left(\frac{1}{c}\partial_t^2 A_0 - \partial_x \partial_t A_1 - \partial_y \partial_t A_2\right)\right\} \\ &\quad - \frac{1}{c^2}\left\{A_1\left(c^2 \nabla^2 A_1\right) + A_2\left(c^2 \nabla^2 A_2\right) - cA_1 \partial_t \partial_x A_0 - cA_2 \partial_t \partial_y A_0\right\} \\ &\quad + \frac{1}{c}A_0 J^0 + \frac{1}{c}A_1 J^1 + \frac{1}{c}A_2 J^2. \end{aligned}$$

Now $\frac{1}{c}\partial_t^2 A_0 - (\partial_x \partial_t A_1 + \partial_y \partial_t A_2) = 0$ by the Lorentz gauge condition. Therefore

$$E^2 := -\frac{1}{c^2} \{A_1 (c^2 \nabla^2 A_1) + A_2 (c^2 \nabla^2 A_2) - cA_1 \partial_t \partial_x A_0 - cA_2 \partial_t \partial_y A_0\} \\ + \frac{1}{c} A_0 J^0 + \frac{1}{c} A_1 J^1 + \frac{1}{c} A_2 J^2.$$

We note that $\partial_t \nabla A_0 = \nabla \partial_t A_0$ and we recall the Lorentz gauge condition. Therefore $-\nabla \partial_t A_0 = -c \nabla (\partial_x A_1 + \partial_y A_2)$, from which it can be shown that

$$E^2 := (A_1 \partial_x \partial_y A_2 - A_1 \partial_y^2 A_1) + A_2 \partial_y \partial_x A_1 - A_2 \partial_x^2 A_2 + \frac{1}{c} A_0 J^0 + \frac{1}{c} A_1 J^1 + \frac{1}{c} A_2 J^2.$$

Now

$$B^2 = (\partial_x A_2)^2 + (\partial_y A_1)^2 - 2(\partial_x A_2)(\partial_y A_1).$$

Noting the relations $A_1 \partial_x (\partial_y A_2) := -(\partial_y A_1)(\partial_x A_2)$, $A_2 \partial_x (\partial_y A_1) := -(\partial_y A_1)(\partial_x A_2)$, $A_1 \partial_y (\partial_y A_1) := -(\partial_y A_1)^2$ and $A_2 \partial_x (\partial_x A_2) := -(\partial_x A_1)^2$, then it can be shown that

$$E^2 - B^2 := \frac{1}{c} J^\mu A_\mu.$$

It follows that L_{QED} simplifies to

$$L_{min} = -\frac{\bar{\sigma} J^\mu (A_\mu + 2a_\mu)}{2c}.$$

A.2 Extraction of $M'(w)$ from $S_{self, M'(w)}$

We recall that

$$S_{self, M'(w)} = -\frac{k^2 h^2 \bar{\sigma}}{4\pi m^2 c} \int_{-\infty}^{\infty} \left\langle \int_0^{\infty} \left\{ -\frac{1}{\sigma} \frac{d\mathbf{X}(u)}{du} \cdot \frac{d\mathbf{X}(u+\sigma)}{d(u+\sigma)} + \frac{|\mathbf{X}(u+\sigma) - \mathbf{X}(u)|^2}{2\sigma^3} \right\} d\sigma \right\rangle du,$$

We begin by noting that, as was done in [8], we can write that $\beta(u) = \frac{d\mathbf{X}(u)}{du}$. Now

$$\int_{-\infty}^{\infty} \beta(u) \cdot \beta(u+\sigma) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \beta(u) \cdot \beta(v) e^{iw\{v-(u+\sigma)\}} dw dv du \\ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\langle \left\{ \int_{-\infty}^{\infty} \beta(u) e^{-iwu} du \cdot \int_{-\infty}^{\infty} \beta(v) e^{-i w v} dv \right\} e^{i w \sigma} \right\rangle dw.$$

Employing integration by parts,

$$\int_{-\infty}^{\infty} \beta(u) e^{-i w u} du = i w \int_{-\infty}^{\infty} \mathbf{X}(u) e^{-i w u} du = i w \left\{ \hat{\mathbf{X}}(w) \right\}^*.$$

Therefore

$$\int_{-\infty}^{\infty} \beta(u) e^{-i w u} du \cdot \int_{-\infty}^{\infty} \beta(v) e^{-i w v} dv = w^2 \left| \hat{\mathbf{X}}(w) \right|^2.$$

It follows that

$$\int_{-\infty}^{\infty} \beta(u) \cdot \beta(u + \sigma) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ w^2 \left| \hat{\mathbf{X}}(w) \right|^2 e^{i w \sigma} \right\} dw.$$

Assuming $\left| \hat{\mathbf{X}}(-w) \right|^2 = \left| \hat{\mathbf{X}}(w) \right|^2$, then

$$\int_{-\infty}^{\infty} \beta(u) \cdot \beta(u + \sigma) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ w^2 \left| \hat{\mathbf{X}}(w) \right|^2 \cos(w\sigma) \right\} dw.$$

We consider next that

$$\int_{-\infty}^{\infty} \left| \mathbf{X}(u + \sigma) - \mathbf{X}(u) \right|^2 du = \int_{-\infty}^{\infty} \left\{ \mathbf{X}(u + \sigma) \cdot \mathbf{X}(u + \sigma) + \mathbf{X}(u) \cdot \mathbf{X}(u) - 2\mathbf{X}(u + \sigma) \cdot \mathbf{X}(u) \right\} du.$$

Now

$$\begin{aligned} \int_{-\infty}^{\infty} \mathbf{X}(u + \sigma) \cdot \mathbf{X}(u) du &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{X}(v) \cdot \mathbf{X}(u) e^{i w \{v - (u + \sigma)\}} du dv dw \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\mathbf{X}}(w) \right|^2 \cos(w\sigma) dw. \end{aligned}$$

Note also that,

$$\int_{-\infty}^{\infty} \mathbf{X}(u) \cdot \mathbf{X}(u) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{X}(v) \cdot \mathbf{X}(u) e^{i w (v - u)} du dv dw = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\mathbf{X}}(w) \right|^2 dw.$$

Similarly,

$$\int_{-\infty}^{\infty} \mathbf{X}(u + \sigma) \cdot \mathbf{X}(u + \sigma) du = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \hat{\mathbf{X}}(w) \right|^2 dw.$$

Therefore

$$\begin{aligned} \int_{-\infty}^{\infty} \left\{ |\mathbf{X}(u + \sigma)|^2 + |\mathbf{X}(u)|^2 - 2\mathbf{X}(u + \sigma) \cdot \mathbf{X}(u) \right\} du &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \left| \hat{\mathbf{X}}(w) \right|^2 - \left| \hat{\mathbf{X}}(w) \right|^2 \cos(w\sigma) \right\} dw \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \left\{ \frac{|\hat{\mathbf{X}}(w)|^2 w^2}{w^2} - \frac{|\hat{\mathbf{X}}(w)|^2 w^2 \cos(w\sigma)}{w^2} \right\} dw. \end{aligned}$$

Having evaluated $\int_{-\infty}^{\infty} \frac{d\mathbf{X}(u)}{du} \cdot \frac{d\mathbf{X}(u+\sigma)}{d(u+\sigma)} du$ and $\int_{-\infty}^{\infty} |\mathbf{X}(u + \sigma) - \mathbf{X}(u)|^2 du$ then it is easily shown that

$$S_{self, M'(w)} = \frac{k^2}{4\pi} \int_{-\infty}^{\infty} M'(w) w^2 \left| \hat{\mathbf{X}}(w) \right|^2 dw,$$

where

$$M'(w) = 2\mu \int_{\delta}^{\infty} \left\{ \frac{\cos(w\sigma)}{\sigma} - \frac{1 - \cos(w\sigma)}{w^2 \sigma^3} \right\} d\sigma.$$

A.3 Calculation of $M''(w)$.

Exploiting the Kramers-Kronig relations, then

$$M''(w) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{M''(w')}{w' - w} dw'.$$

Noting that $\delta > 0$, from [57],

$$-\frac{\mu}{\pi} P \int_{-\infty}^{\infty} \frac{\text{Ci}(|w'| \delta)}{w' - w} dw' = \mu \text{sgn}(w) \text{si}(|w| \delta),$$

$$\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{1 - \cos(w' \delta)}{(w' \delta)^2 (w' - w)} dw' = \frac{\sin(w \delta)}{w^2 \delta^2} - \frac{1}{w \delta}$$

and

$$\frac{\mu}{\pi} P \int_{-\infty}^{\infty} \frac{\sin(w' \delta)}{(w' \delta)(w' - w)} dw' = \frac{\mu}{8} \frac{\cos(w \delta) - 1}{w}.$$

Therefore

$$M''(w) = \mu \left\{ \frac{\cos(w \delta) - 1}{w} - \text{sgn}(w) \text{si}(|w| \delta) + \frac{\sin(w \delta)}{w^2 \delta^2} - \frac{1}{w \delta} \right\},$$

where $\text{si}(x) = -\int_x^\infty \frac{\sin(t)}{t} dt$.

APPENDIX B

B.1 Derivation of $i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi + i\Omega(y\partial_x\psi - x\partial_y\psi)$

We recall the GPE,

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + V\psi + g|\psi|^2\psi.$$

We now consider a change of variables corresponding to a change to a rotating reference frame. Consider the substitution

$$\begin{pmatrix} x' \\ y' \\ t' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix}$$

noting that $\theta = \theta(t)$. Now

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x} \frac{\partial}{\partial y'} + \frac{\partial t'}{\partial x} \frac{\partial}{\partial t'}.$$

Given that

$$\frac{\partial x'}{\partial x} = \cos\theta, \frac{\partial y'}{\partial x} = \sin\theta \text{ and } \frac{\partial t'}{\partial x} = 0,$$

then

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial x'} + \sin\theta \frac{\partial}{\partial y'}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\cos\theta \frac{\partial}{\partial x'} + \sin\theta \frac{\partial}{\partial y'} \right) \left(\cos\theta \frac{\partial}{\partial x'} + \sin\theta \frac{\partial}{\partial y'} \right) \\ &= \cos^2\theta \frac{\partial^2}{\partial x'^2} + \cos\theta \sin\theta \frac{\partial}{\partial x'} \frac{\partial}{\partial y'} + \sin\theta \cos\theta \frac{\partial}{\partial y'} \frac{\partial}{\partial x'} + \sin^2\theta \frac{\partial^2}{\partial y'^2}. \end{aligned}$$

Similarly,

$$\frac{\partial}{\partial y} = \frac{\partial y'}{\partial y} \frac{\partial}{\partial y'} + \frac{\partial x'}{\partial y} \frac{\partial}{\partial x'} + \frac{\partial t'}{\partial y} \frac{\partial}{\partial t'},$$

where

$$\frac{\partial y'}{\partial y} = \cos \theta, \frac{\partial x'}{\partial y} = -\sin \theta \quad \text{and} \quad \frac{\partial t'}{\partial y} = 0.$$

Therefore

$$\frac{\partial}{\partial y} = \cos \theta \frac{\partial}{\partial y'} - \sin \theta \frac{\partial}{\partial x'}$$

and we can write that

$$\begin{aligned} \frac{\partial^2}{\partial x^2} &= \left(\cos \theta \frac{\partial}{\partial y'} - \sin \theta \frac{\partial}{\partial x'} \right) \left(\cos \theta \frac{\partial}{\partial y'} - \sin \theta \frac{\partial}{\partial x'} \right) \\ &= \cos^2 \theta \frac{\partial^2}{\partial y'^2} - \cos \theta \sin \theta \frac{\partial}{\partial y'} \frac{\partial}{\partial x'} - \sin \theta \cos \theta \frac{\partial}{\partial x'} \frac{\partial}{\partial y'} + \sin^2 \theta \frac{\partial^2}{\partial x'^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x^2} &= (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2}{\partial y'^2} + (\cos^2 \theta + \sin^2 \theta) \frac{\partial^2}{\partial x'^2} \\ &= \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial x'^2}. \end{aligned}$$

We turn next to consider the time derivative, beginning with

$$\frac{\partial}{\partial t} = \frac{\partial t'}{\partial t} \frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} + \frac{\partial y'}{\partial t} \frac{\partial}{\partial y'}.$$

However

$$\frac{\partial t'}{\partial t} = 1$$

and

$$\frac{\partial x'}{\partial t} = (-x \sin \theta - y \cos \theta) \dot{\theta}.$$

Therefore

$$\frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} = (-x \sin \theta - y \cos \theta) \dot{\theta} \frac{\partial}{\partial x'}.$$

Similarly,

$$\frac{\partial y'}{\partial t} = (x \cos \theta - y \sin \theta) \dot{\theta}$$

and therefore

$$\frac{\partial y'}{\partial t} \frac{\partial}{\partial y'} = (x \cos \theta - y \sin \theta) \dot{\theta} \frac{\partial}{\partial y'}.$$

However,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

Therefore,

$$\frac{\partial x'}{\partial t} \frac{\partial}{\partial x'} = -y' \dot{\theta} \frac{\partial}{\partial x'}$$

and

$$\frac{\partial y'}{\partial t} \frac{\partial}{\partial y'} = x' \dot{\theta} \frac{\partial}{\partial y'}.$$

It follows that

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \dot{\theta} \left(x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} \right).$$

Let $\dot{\theta} = \Omega$, a constant. We can write

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} + \Omega \left(x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} \right).$$

Therefore in the new coordinates

$$i \left\{ \frac{\partial}{\partial t'} + \Omega \left(x' \frac{\partial}{\partial y'} - y' \frac{\partial}{\partial x'} \right) \right\} \psi = -\frac{1}{2} \nabla'^2 \psi + V \psi + g |\psi|^2 \psi,$$

and we can write that

$$i \partial_{t'} \psi = -\frac{1}{2} \nabla'^2 \psi + V \psi + g |\psi|^2 \psi + i \Omega (y' \partial_{x'} - x' \partial_{y'}) \psi.$$

Considering notation, replace x' with x , y' with y and t' with t , resulting in

$$i \partial_t \psi = -\frac{1}{2} \nabla^2 \psi + V \psi + g |\psi|^2 \psi + i \Omega (y \partial_x - x \partial_y) \psi.$$

APPENDIX C

C.1 Derivation of $L_\alpha := \mathbf{P} \cdot \mathbf{J} - \Phi\rho + \frac{1}{2} |\nabla\Phi|^2$

The incompressible condensate Lagrangian density takes the form

$$L_\alpha = -\partial_t\alpha + \frac{1}{2} |\nabla\alpha|^2.$$

The EL equation for α is

$$\nabla \cdot (\nabla\alpha) = \nabla^2\alpha = 0.$$

We are therefore inspired to write that

$$\nabla\alpha = \partial_y\Phi\hat{\mathbf{i}} - \partial_x\Phi\hat{\mathbf{j}}$$

and it follows that

$$\frac{1}{2} |\nabla\alpha|^2 = \frac{1}{2} |\nabla\Phi|^2.$$

Let

$$\mathbf{P} = \frac{y}{2}\hat{\mathbf{i}} - \frac{x}{2}\hat{\mathbf{j}} = P_1\hat{\mathbf{i}} + P_2\hat{\mathbf{j}}.$$

Therefore

$$\partial_y P_1 - \partial_x P_2 = 1$$

and we can write that

$$-\partial_t\alpha = -\partial_t\alpha (\partial_y P_1 - \partial_x P_2) = -\partial_t\alpha\partial_y P_1 + \partial_t\alpha\partial_x P_2 := P_1 (\partial_y\partial_t\alpha) - P_2 (\partial_x\partial_t\alpha).$$

Now

$$\alpha (\nabla^2\alpha) = 0$$

and

$$\alpha (\nabla^2\alpha) := -\Phi\rho.$$

We note that

$$(\partial_t \mathbf{P}) \cdot (\partial_y \alpha \hat{\mathbf{i}} - \partial_x \alpha \hat{\mathbf{j}}) = 0$$

and that

$$(\partial_t \mathbf{P}) \cdot (\partial_y \alpha \hat{\mathbf{i}} - \partial_x \alpha \hat{\mathbf{j}}) := -P_1 (\partial_t \partial_y \alpha) + P_2 (\partial_t \partial_x \alpha).$$

Therefore we can write that

$$L_\alpha := P_1 (\partial_y \partial_t \alpha - \partial_t \partial_y \alpha) + P_2 (\partial_t \partial_x \alpha - \partial_x \partial_t \alpha) - \Phi \rho + \frac{1}{2} |\nabla \Phi|^2 = P_1 J_x + P_2 J_y - \Phi \rho + \frac{1}{2} |\nabla \Phi|^2,$$

which is equivalent to writing that

$$L_\alpha := \mathbf{P} \cdot \mathbf{J} - \Phi \rho + \frac{1}{2} |\nabla \Phi|^2.$$

C.2 Definition of $F(\phi_1, \phi_2, I_1, I_2)$

$$F(\phi_1, \phi_2, I_1, I_2) = F_1 + F_2 + F_3$$

where

$$\begin{aligned} F_1 = & \frac{1}{4} \left\langle \left\{ I_1^2 (I_1^2 + 3I_2^2)^2 \cos^2(6\phi_1) + I_2^2 (I_2^2 + 3I_1^2)^2 \sin^2(6\phi_1) \right\} \right\rangle \\ & + \frac{1}{4} \left\langle (I_2^2 - I_1^2)^2 \{ I_1^2 \cos^2(2\phi_1) + I_2^2 \sin^2(2\phi_1) \} + 4(I_1 + I_2)^2 (I_3 - I_2)^4 \right\rangle \\ & + 16(I_2 - I_1)^2 (I_3 - I_2)^4 + 4(I_1 + I_2)^3 (I_3 - I_2)^2 (I_2 - I_1) + 4(I_1 + I_2) (I_3 - I_2)^2 (I_2 - I_1)^3 \\ & + \frac{1}{2} (I_2^2 - I_1^2) \{ (I_1^2) (I_1^2 + 3I_2^2) \cos(2\phi_1) \cos(6\phi_1) + (I_2^2) (I_2^2 + 3I_1^2) \sin(2\phi_1) \sin(6\phi_1) \} \\ & + 4(I_2^2 - I_1^2) (I_1 + I_2) (I_3 - I_2)^2 \{ I_1 \cos(2\phi_1) \cos(2\phi_1 - 4\phi_2) + I_2 \sin(2\phi_1) \sin(2\phi_1 - 4\phi_2) \} \\ & + 4(I_2^2 - I_1^2) (I_2 - I_1) (I_3 - I_2)^2 \{ -I_1 \cos(2\phi_1) \cos(2\phi_1 + 4\phi_2) + I_2 \sin(2\phi_1) \sin(2\phi_1 + 4\phi_2) \}, \end{aligned}$$

$$F_2 =$$

$$\begin{aligned} & 4(I_3 - I_2)(-1)(I_1 + I_2)^{\frac{3}{2}}(I_2 - I_1)^{\frac{1}{2}}(I_2^2 - I_1^2) \{I_1 \cos(2\phi_1) \cos(2\phi_1 - 2\phi_2) + I_2 \sin(2\phi_1) \sin(2\phi_1 - 2\phi_2)\} \\ & + 4(I_3 - I_2)(I_2 - I_1)^{\frac{3}{2}}(I_1 + I_2)^{\frac{1}{2}}(I_2^2 - I_1^2) \{I_1 \cos(2\phi_1) \cos(2\phi_1 + 2\phi_2) - I_2 \sin(2\phi_1) \sin(2\phi_1 + 2\phi_2)\} \\ & + 4(I_1 + I_2)(I_3 - I_2)^2 \{I_1(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) + I_2(I_2^2 + 3I_1^2) \sin(6\phi_1) \sin(2\phi_1 - 4\phi_2)\} \\ & + 4(I_2 - I_1)(I_3 - I_2)^2 \{-I_1(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 + 4\phi_2) + I_2(I_2^2 + 3I_1^2) \sin(6\phi_1) \sin(2\phi_1 + 4\phi_2)\} \end{aligned}$$

and

$$F_3 =$$

$$\begin{aligned} & 4(I_1 + I_2)^{\frac{3}{2}}(I_3 - I_2)(I_2 - I_1)^{\frac{1}{2}} \{-I_1(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 - 2\phi_2)\} \\ & + 4(I_1 + I_2)^{\frac{3}{2}}(I_3 - I_2)(I_2 - I_1)^{\frac{1}{2}} \{-I_2(I_2^2 + 3I_1^2) \sin(6\phi_1) \sin(2\phi_1 - 2\phi_2)\} \\ & + 4(I_2 - I_1)^{\frac{3}{2}}(I_3 - I_2)(I_1 + I_2)^{\frac{1}{2}} \{I_1(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 + 2\phi_2)\} \\ & - 4(I_2 - I_1)^{\frac{3}{2}}(I_3 - I_2)(I_1 + I_2)^{\frac{1}{2}} \{I_2(I_2^2 + 3I_1^2) \sin(6\phi_1) \sin(2\phi_1 + 2\phi_2)\} \\ & - 32(I_2^2 - I_1^2)(I_3 - I_2)^4 \cos(4\phi_1) + (I_1 + I_2)(I_3 - I_2)^2 - 32(I_1 + I_2)^{\frac{3}{2}}(I_3 - I_2)(I_2 - I_1)^{\frac{1}{2}} \cos(2\phi_2) \\ & + (I_2 - I_1)(I_3 - I_2)^3 (-32)(I_1 + I_2)^{\frac{1}{2}}(I_2 - I_1)^{\frac{3}{2}} \cos(2\phi_2) - 32(I_1 + I_2)^2(I_3 - I_2)^2(I_2 - I_1)^2 \cos(4\phi_1) \\ & + 32(I_1 + I_2)(I_1 + I_2)^{\frac{1}{2}}(I_3 - I_2)^3(I_2 - I_1)^{\frac{3}{2}} \cos(4\phi_1 - 2\phi_2) \\ & + 32(I_2 - I_1)(I_1 + I_2)^{\frac{3}{2}}(I_3 - I_2)^3(I_2 - I_1)^{\frac{1}{2}} \cos(4\phi_1 + 2\phi_2). \end{aligned}$$

C.3 Algebra involved in the derivation of the Hamiltonian which takes the form of a Coupled Nonlinear Oscillator Problem

We begin with the simplification of $\prod_{1 \leq k < l < 4} (|z_k - z_l|)^2$, and we consider

$$\begin{aligned} & \frac{1}{4} \left(\left\{ I_1^2(I_1^2 + 3I_2^2)^2 \cos^2(6\phi_1) + I_2^2(I_2^2 + 3I_1^2)^2 \sin^2(6\phi_1) \right\} + (I_2^2 - I_1^2)^2 \{ I_1^2 \cos^2(2\phi_1) + I_2^2 \sin^2(2\phi_1) \} \right) \\ & + \frac{1}{8} (I_2^2 - I_1^2) \{ (I_1^2)(I_1^2 + 3I_2^2) \cos(2\phi_1) \cos(6\phi_1) + (I_2^2)(I_2^2 + 3I_1^2) \sin(2\phi_1) \sin(6\phi_1) \} \\ & - 32 \left\{ (I_2^2 - I_1^2)(I_3 - I_2)^4 + (I_1 + I_2)^2(I_3 - I_2)^2(I_2 - I_1)^2 \right\} \cos(4\phi_1). \end{aligned}$$

We call the above collection of terms A_1 . It is useful to note that

$$\cos(6\phi_1) = 4 \cos^3(2\phi_1) - 3 \cos(2\phi_1),$$

$$\cos^2(6\phi_1) = 16 \cos^6(2\phi_1) + 9 \cos^2(2\phi_1) - 24 \cos^4(2\phi_1),$$

$$\cos(2\phi_1) \cos(6\phi_1) = 4 \cos^4(2\phi_1) - 3 \cos^2(2\phi_1),$$

$$\sin(6\phi_1) = 3 \sin(2\phi_1) - 4 \sin^3(2\phi_1),$$

$$\sin(2\phi_1) \sin(6\phi_1) = 3 \sin^2(2\phi_1) - 4 \sin^4(2\phi_1)$$

and

$$\sin^2(6\phi_1) = 9 \sin^2(2\phi_1) + 16 \sin^6(2\phi_1) - 24 \sin^4(2\phi_1).$$

It can be shown that

$$\begin{aligned} A_1 = & \{64I_1^2 \cos^2(2\phi_1)\} (I_3 - I_2)^2 \{(I_3 - I_2)^2 + I_2^2 - I_1^2\} + \{64I_2^2 \sin^2(2\phi_1)\} (I_3 - I_2)^2 \{(I_3 - I_2)^2 + I_2^2 - I_1^2\} \\ & + 4 \{I_1^6 \cos^2(2\phi_1) \sin^4(2\phi_1) + 2I_1^4 I_2^2 \sin^2(2\phi_1) \cos^4(2\phi_1) + I_1^2 I_2^4 \cos^6(2\phi_1)\} \\ & + 4 \{I_2^6 \sin^2(2\phi_1) \cos^4(2\phi_1) + 2I_1^2 I_2^4 \sin^4(2\phi_1) \cos^2(2\phi_1) + I_1^4 I_2^2 \sin^6(2\phi_1)\} \\ & - 32(I_1^2 + I_2^2)(I_3 - I_2)^2 \{(I_3 - I_2)^2 + (I_2^2 - I_1^2)\} \\ = & h_1(I_1, I_2, \phi_1) - 32(I_1^2 + I_2^2)(I_3 - I_2)^2 \{(I_3 - I_2)^2 + (I_2^2 - I_1^2)\}. \end{aligned}$$

Next, we look at the calculated expression for $\prod_{1 \leq k < l < 4} (|z_k - z_l|)^2$, and we pick out

$$\begin{aligned} & 16(I_1 + I_2)^2(I_3 - I_2)^4 + 16(I_2 - I_1)^2(I_3 - I_2)^4 \\ & + 16(I_1 + I_2)^3(I_2 - I_1)(I_3 - I_2)^2 + 16(I_2 - I_1)^3(I_1 + I_2)(I_3 - I_2)^2 \\ = & 32(I_1^2 + I_2^2)(I_3 - I_2)^2 \{(I_3 - I_2)^2 + (I_2^2 - I_1^2)\}. \end{aligned}$$

Therefore,

$$\begin{aligned} & A_1 + 16(I_1 + I_2)^2(I_3 - I_2)^4 + 16(I_2 - I_1)^2(I_3 - I_2)^4 \\ & + 16(I_1 + I_2)^3(I_2 - I_1)(I_3 - I_2)^2 + 16(I_2 - I_1)^3(I_1 + I_2)(I_3 - I_2)^2 \\ = & h_1(I_1, I_2, \phi_1). \end{aligned}$$

Let us now, from within $\prod_{1 \leq k < l < 4} (|z_k - z_l|)^2$, examine $B_1 + B_2$, where

$$\begin{aligned} B_1 &= 4(I_3 - I_2)^2(I_1 + I_2) \times \\ &\left\{ (I_1)(I_2^2 - I_1^2) \cos(2\phi_1) \cos(2\phi_1 - 4\phi_2) + (I_2)(I_2^2 - I_1^2) \sin(2\phi_1) \sin(2\phi_1 - 4\phi_2) \right. \\ &\left. + (I_1)(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) + (I_2)(I_2^2 + 3I_1^2) \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) \right\}, \end{aligned}$$

noting that $B_1 = B_1(I_1, I_2, \phi_1, \phi_2)$ and $B_2 = B_2(I_1, I_2, \phi_1, \phi_2) = B_1(-I_1, I_2, \phi_1, -\phi_2)$. We begin by examination of the coefficient of $(I_3 - I_2)^2 I_1^4$ which is equal to

$$\begin{aligned} &4 \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) + 4 \cos(6\phi_1) \cos(2\phi_1 + 4\phi_2) \\ &- 4 \cos(2\phi_1) \cos(2\phi_1 - 4\phi_2) - 4 \cos(2\phi_1) \cos(2\phi_1 + 4\phi_2) \\ &= -4 \cos(4\phi_2) + 2 \cos(8\phi_1 - 4\phi_2) + 2 \cos(8\phi_1 + 4\phi_2). \end{aligned}$$

Therefore the coefficient of $(I_3 - I_2)^2 I_1^4$ in $B_1 + B_2$ is

$$-4 \cos(4\phi_2) + 2 \cos(8\phi_1 - 4\phi_2) + 2 \cos(8\phi_1 + 4\phi_2).$$

Next, we look at the coefficient of $(I_3 - I_2)^2 I_2^4$ in $B_1 + B_2$, which is equal to

$$\begin{aligned} &4 \sin(2\phi_1) \sin(2\phi_1 - 4\phi_2) + 4 \sin(6\phi_1) \sin(2\phi_1 - 4\phi_2) \\ &+ 4 \sin(2\phi_1) \sin(2\phi_1 + 4\phi_2) - 4 \sin(6\phi_1) \cos(2\phi_1 + 4\phi_2) \\ &= 4 \cos(4\phi_2) - (2) \{ \cos(8\phi_1 - 4\phi_2) + \cos(8\phi_1 + 4\phi_2) \}. \end{aligned}$$

Next, we look at the coefficient of $(I_3 - I_2)^2 I_1^2 I_2^2$ in $B_1 + B_2$, which is equal to

$$\begin{aligned} &4 \cos(2\phi_1) \cos(2\phi_1 - 4\phi_2) - 4 \sin(2\phi_1) \sin(2\phi_1 - 4\phi_2) \\ &+ 12 \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) + 12 \sin(6\phi_1) \sin(2\phi_1 - 4\phi_2) \\ &+ 4 \cos(2\phi_1) \cos(2\phi_1 + 4\phi_2) - 4 \sin(2\phi_1) \sin(2\phi_1 + 4\phi_2) \\ &+ 12 \cos(6\phi_1) \cos(2\phi_1 + 4\phi_2) + 12 \sin(6\phi_1) \sin(2\phi_1 + 4\phi_2) \\ &= 16 \langle \cos \{4(\phi_1 - \phi_2)\} + \cos \{4(\phi_1 + \phi_2)\} \rangle. \end{aligned}$$

Next, we look at the coefficient of $(I_3 - I_2)^2 I_1^3 I_2$ in $B_1 + B_2$, which is equal to

$$\begin{aligned}
& -4 \cos(2\phi_1) \cos(2\phi_1 - 4\phi_2) + 4 \cos(2\phi_1) \cos(2\phi_1 + 4\phi_2) \\
& + 12 \sin(6\phi_1) \sin(2\phi_1 - 4\phi_2) - 12 \sin(6\phi_1) \sin(2\phi_1 + 4\phi_2) \\
& - 4 \sin(2\phi_1) \sin(2\phi_1 - 4\phi_2) + 4 \sin(2\phi_1) \sin(2\phi_1 + 4\phi_2) \\
& + 4 \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) - 4 \cos(6\phi_1) \cos(2\phi_1 + 4\phi_2) \\
& = -8 \{ \cos 4(\phi_1 - \phi_2) - \cos 4(\phi_1 + \phi_2) \} - 4 \{ \cos(8\phi_1 - 4\phi_2) - \cos(8\phi_1 + 4\phi_2) \}.
\end{aligned}$$

Next, we look at the coefficient of $(I_3 - I_2)^2 I_1 I_2^3$ in $B_1 + B_2$, which is equal to

$$\begin{aligned}
& 4 \cos(2\phi_1) \cos(2\phi_1 - 4\phi_2) - 4 \cos(2\phi_1) \cos(2\phi_1 + 4\phi_2) \\
& + 4 \sin(2\phi_1) \sin(2\phi_1 - 4\phi_2) - 4 \sin(2\phi_1) \sin(2\phi_1 + 4\phi_2) \\
& + 12 \cos(6\phi_1) \cos(2\phi_1 - 4\phi_2) - 12 \cos(6\phi_1) \cos(2\phi_1 + 4\phi_2) \\
& + 4 \sin(6\phi_1) \sin(2\phi_1 - 4\phi_2) - 4 \sin(6\phi_1) \sin(2\phi_1 + 4\phi_2) \\
& = 8 \{ \cos 4(\phi_1 + \phi_2) - \cos 4(\phi_1 - \phi_2) \} - 4 \{ \cos(8\phi_1 + 4\phi_2) - \cos(8\phi_1 - 4\phi_2) \}.
\end{aligned}$$

To finish off, we look at $\prod_{1 \leq k < l < 4} (|z_k - z_l|)^2$, and extract

$$\begin{aligned}
C_1 &= 4(I_3 - I_2)(I_1 + I_2) \times \\
& \{ (I_1^2 - I_2^2)(I_1 \cos(2\phi_1) \cos(2\phi_1 - 2\phi_2) + I_2 \sin(2\phi_1) \sin(2\phi_1 - 2\phi_2) \\
& - I_1(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 - 2\phi_2) - I_2(I_2^2 + 3I_1^2) \sin(6\phi_1) \sin(2\phi_1 - 2\phi_2) \} \\
& + 4(I_3 - I_2)(I_2 - I_1)^{\frac{3}{2}}(I_1 + I_2)^{\frac{1}{2}} \times \\
& \{ (I_1^2 - I_2^2)(-I_1 \cos(2\phi_1) \cos(2\phi_1 + 2\phi_2) + I_2 \sin(2\phi_1) \sin(2\phi_1 + 2\phi_2) \\
& + I_1(I_1^2 + 3I_2^2) \cos(6\phi_1) \cos(2\phi_1 + 2\phi_2) - I_2(I_2^2 + 3I_1^2) \sin(6\phi_1) \sin(2\phi_1 + 2\phi_2) \}.
\end{aligned}$$

Within C_1 , we consider first the coefficient of $(I_3 - I_2)(I_2^2 - I_1^2)^{\frac{1}{2}} I_1^4$ which is equal to

$$\begin{aligned}
& 4 \cos(2\phi_1) \cos(2\phi_1 - 2\phi_2) + 4 \cos(2\phi_1) \cos(2\phi_1 + 2\phi_2) \\
& - 4 \cos(6\phi_1) \cos(2\phi_1 - 2\phi_2) - 4 \cos(6\phi_1) \cos(2\phi_1 + 2\phi_2) \\
& = 4 \cos(2\phi_2) - 2 \cos(8\phi_1 - 2\phi_2) - 2 \cos(8\phi_1 + 2\phi_2).
\end{aligned}$$

Therefore the coefficient of $(I_3 - I_2)(I_2^2 - I_1^2)^{\frac{1}{2}}I_1^4$ is equal to

$$4 \cos(2\phi_2) - 2 \cos(8\phi_1 - 2\phi_2) - 2 \cos(8\phi_1 + 2\phi_2).$$

Looking at C_1 , we move on to consider the coefficient of $(I_3 - I_2)(I_2^2 - I_1^2)^{\frac{1}{2}}I_2^4$ which is equal to

$$\begin{aligned} & -4 \sin(2\phi_1) \sin(2\phi_1 - 2\phi_2) - 4 \sin(2\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & -4 \sin(6\phi_1) \sin(2\phi_1 - 2\phi_2) - 4 \sin(6\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & = -4 \cos(2\phi_2) + 2 \{ \cos(8\phi_1 - 2\phi_2) + \cos(8\phi_1 + 2\phi_2) \}. \end{aligned}$$

We return to looking at C_1 again, we consider the coefficient of $(I_3 - I_2)(I_2^2 - I_1^2)^{\frac{1}{2}}I_1^2I_2^2$ which is equal to

$$\begin{aligned} & -4 \cos(2\phi_1) \cos(2\phi_1 - 2\phi_2) + 4 \sin(2\phi_1) \sin(2\phi_1 - 2\phi_2) \\ & +4 \sin(2\phi_1) \sin(2\phi_1 + 2\phi_2) - 4 \cos(2\phi_1) \cos(2\phi_1 + 2\phi_2) \\ & -12 \cos(6\phi_1) \cos(2\phi_1 - 2\phi_2) - 12 \sin(6\phi_1) \sin(2\phi_1 - 2\phi_2) \\ & -12 \cos(6\phi_1) \cos(2\phi_1 + 2\phi_2) - 12 \sin(6\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & = -16 \{ \cos(4\phi_1 - 2\phi_2) + \cos(4\phi_1 + 2\phi_2) \}. \end{aligned}$$

Next, we look at C_1 again, and examine the coefficient of $(I_3 - I_2)(I_2^2 - I_1^2)^{\frac{1}{2}}I_1^3I_2$ which is equal to

$$\begin{aligned} & 4 \cos(2\phi_1) \cos(2\phi_1 - 2\phi_2) + 4 \sin(2\phi_1) \sin(2\phi_1 - 2\phi_2) \\ & -4 \cos(2\phi_1) \cos(2\phi_1 + 2\phi_2) - 4 \sin(2\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & -4 \cos(6\phi_1) \cos(2\phi_1 - 2\phi_2) - 12 \sin(6\phi_1) \sin(2\phi_1 - 2\phi_2) \\ & +4 \cos(6\phi_1) \cos(2\phi_1 + 2\phi_2) + 12 \sin(6\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & = -4 \{ \cos(8\phi_1 + 2\phi_2) - \cos(8\phi_1 - 2\phi_2) \} + 8 \{ \cos(4\phi_1 - 2\phi_2) - \cos(4\phi_1 + 2\phi_2) \}. \end{aligned}$$

Lastly from C_1 we consider the coefficient of $(I_3 - I_2)(I_2^2 - I_1^2)^{\frac{1}{2}}I_1I_2^3$, which is equal to

$$\begin{aligned} & -4 \sin(2\phi_1) \sin(2\phi_1 - 2\phi_2) - 4 \cos(2\phi_1) \cos(2\phi_1 - 2\phi_2) \\ & +4 \cos(2\phi_1) \cos(2\phi_1 + 2\phi_2) + 4 \sin(2\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & -12 \cos(6\phi_1) \cos(2\phi_1 - 2\phi_2) + 12 \cos(6\phi_1) \cos(2\phi_1 + 2\phi_2) \\ & -4 \sin(6\phi_1) \sin(2\phi_1 - 2\phi_2) + 4 \sin(6\phi_1) \sin(2\phi_1 + 2\phi_2) \\ & = 8 \{ \cos(4\phi_1 - 2\phi_2) - \cos(4\phi_1 + 2\phi_2) \} + 4 \{ \cos(8\phi_1 + 2\phi_2) - \cos(8\phi_1 - 2\phi_2) \}. \end{aligned}$$

Therefore we can write that

$$\prod_{1 \leq k < l < 4} (|z_k - z_l|)^2 = F_4 + F_5 + F_6 = h$$

where

$$\begin{aligned} F_4 = & h_1(I_1, I_2, \phi_1) + \cos(4\phi_2)(I_3 - I_2)^2(4)(I_2^4 - I_1^4) \\ & + \cos(2\phi_2) \left\{ (I_3 - I_2)^3(I_2^2 - I_1^2)^{\frac{1}{2}}(-64)(I_1^2 + I_2^2) + (I_2^2 - I_1^2)^{\frac{1}{2}}(I_3 - I_2)(4I_1^4 - 4I_2^4) \right\} \\ & + \{\cos(4\phi_1 - 4\phi_2)\} \left\{ 16(I_3 - I_2)^2 I_1^2 I_2^2 - 8(I_1^3 I_2)(I_3 - I_2)^2 - 8(I_1 I_2^3)(I_3 - I_2)^2 \right\} \\ & + \{\cos(4\phi_1 + 4\phi_2)\} \left\{ 16(I_3 - I_2)^2 I_1^2 I_2^2 + 8(I_1^3 I_2)(I_3 - I_2)^2 + 8(I_1 I_2^3)(I_3 - I_2)^2 \right\}, \end{aligned}$$

$$\begin{aligned} F_5 = & \{\cos(8\phi_1 - 4\phi_2)\} (I_3 - I_2)^2(-4I_1^3 I_2 - 2I_2^4 + 2I_1^4 + 4I_2^3 I_1) \\ & + \{\cos(8\phi_1 + 4\phi_2)\} (I_3 - I_2)^2(4I_1^3 I_2 - 2I_2^4 + 2I_1^4 - 4I_2^3 I_1) \\ & \{\cos(8\phi_1 - 2\phi_2)\} (I_3 - I_2) \left\{ -4(I_1 I_2^3)(I_2^2 - I_1^2)^{\frac{1}{2}} + 4(I_1^3 I_2)(I_2^2 - I_1^2)^{\frac{1}{2}} + (2I_2^4)(I_2^2 - I_1^2)^{\frac{1}{2}} - (2I_1^4)(I_2^2 - I_1^2)^{\frac{1}{2}} \right\} \\ & \{\cos(8\phi_1 + 2\phi_2)\} (I_3 - I_2) \left\{ 4(I_1 I_2^3)(I_2^2 - I_1^2)^{\frac{1}{2}} - 4(I_1^3 I_2)(I_2^2 - I_1^2)^{\frac{1}{2}} + (2I_2^4)(I_2^2 - I_1^2)^{\frac{1}{2}} - (2I_1^4)(I_2^2 - I_1^2)^{\frac{1}{2}} \right\} \end{aligned}$$

and

$$\begin{aligned} F_6 = & \cos(4\phi_1 - 2\phi_2) \left\{ (I_3 - I_2)^3(32)(I_2^2 - I_1^2)^{\frac{1}{2}}(I_2^2 - I_1^2)(I_3 - I_2)(8)(I_2^2 - I_1^2)^{\frac{1}{2}}(I_2^3 I_1 + I_1^3 I_2 - 2I_1^2 I_2^2) \right\} \\ & + \cos(4\phi_1 + 2\phi_2) \left\{ (I_3 - I_2)^3(32)(I_2^2 - I_1^2)^{\frac{1}{2}}(I_2^2 - I_1^2)(I_3 - I_2)(8)(I_2^2 - I_1^2)^{\frac{1}{2}}(-I_2^3 I_1 - I_1^3 I_2 - 2I_1^2 I_2^2) \right\}. \end{aligned}$$

C.4 Evaluation of the Time-rate of Change of the Magnitude of the Polygon Radius in the Linear Stability Ansatz

Consider equation (4.36) when N is even. For the middle term in the sum, $j = \frac{N}{2} + 1$. We note that

$$\lim_{j \rightarrow \frac{N}{2} + 1} \cot \left\{ \frac{\pi(j-1)}{N} \right\} = 0.$$

Next we consider terms to the left and right of the middle term, that is, we consider $j = \frac{N}{2} + 1 - s$

and $j = \frac{N}{2} + 1 + s$ terms respectively. s is an integer such that $1 \leq s \leq \frac{N}{2} - 1$. For $j = \frac{N}{2} + 1 - s$, $\cot \left\{ \frac{\pi(j-1)}{N} \right\} = \cot \left(\frac{\pi s}{N} + \frac{\pi}{2} \right)$, and for $j = \frac{N}{2} + 1 + s$, $\cot \left\{ \frac{\pi(j-1)}{N} \right\} = \cot \left(-\frac{\pi s}{N} + \frac{\pi}{2} \right)$. Consider that

$$\cot \left(A + \frac{\pi}{2} \right) = \frac{\frac{1}{\tan\left(\frac{\pi}{2}\right)} - \tan(A)}{\frac{\tan(A)}{\tan\left(\frac{\pi}{2}\right)} + 1}.$$

Let $A = \frac{s\pi}{N}$, and note that

$$\lim_{\tilde{s} \rightarrow s, \hat{s} \rightarrow 1} \frac{\tan\left(\frac{\tilde{s}\pi}{N}\right)}{\tan\left(\frac{\hat{s}\pi}{2}\right)} = 0.$$

We note also that

$$\lim_{\hat{s} \rightarrow 1} \frac{1}{\tan\left(\frac{\hat{s}\pi}{2}\right)} = 0.$$

Therefore

$$\lim_{\tilde{s} \rightarrow s, \hat{s} \rightarrow 1} \cot \left(-\frac{\pi \tilde{s}}{N} + \frac{\pi \hat{s}}{2} \right) + \cot \left(\frac{\pi \tilde{s}}{N} + \frac{\pi \hat{s}}{2} \right) = 0.$$

Employing the above limits, $\dot{r}_p$ will be set to zero for even N .

We return to equation (4.36) and consider the case when N is odd. In this case we need only consider two types of terms, i.e. terms for which $j = \frac{N}{2} + 1 + (q + \frac{1}{2})$ and $j = \frac{N}{2} + 1 - (q + \frac{1}{2})$. q is an integer such that $0 \leq q \leq \frac{N-3}{2}$. For $j = \frac{N}{2} + 1 + (q + \frac{1}{2})$, $\cot \left\{ \frac{\pi(j-1)}{N} \right\} = \cot \left\{ \frac{\pi(q+\frac{1}{2})}{N} + \frac{\pi}{2} \right\}$ and for $j = \frac{N}{2} + 1 - (q + \frac{1}{2})$, $\cot \left\{ \frac{\pi(j-1)}{N} \right\} = \cot \left\{ -\frac{\pi(q+\frac{1}{2})}{N} + \frac{\pi}{2} \right\}$. Now

$$\cot \left\{ \frac{\pi(q+\frac{1}{2})}{N} + \frac{\pi}{2} \right\} = \frac{\frac{1}{\tan\left(\frac{\pi}{2}\right)} - \tan \left\{ \frac{\pi(q+\frac{1}{2})}{N} \right\}}{\frac{\tan \left\{ \frac{\pi(q+\frac{1}{2})}{N} \right\}}{\tan\left(\frac{\pi}{2}\right)} + 1}$$

and

$$\cot \left\{ -\frac{\pi(q+\frac{1}{2})}{N} + \frac{\pi}{2} \right\} = \frac{\frac{1}{\tan\left(\frac{\pi}{2}\right)} + \tan \left\{ \frac{\pi(q+\frac{1}{2})}{N} \right\}}{-\frac{\tan \left\{ \frac{\pi(q+\frac{1}{2})}{N} \right\}}{\tan\left(\frac{\pi}{2}\right)} + 1}.$$

Now

$$\lim_{\tilde{q} \rightarrow q, \hat{q} \rightarrow 1} \frac{\tan \left\{ \frac{\pi(\tilde{q}+\frac{1}{2})}{N} \right\}}{\tan \left(\frac{\hat{q}\pi}{2} \right)} = 0$$

and

$$\lim_{\hat{q} \rightarrow 1} \frac{1}{\tan\left(\frac{\hat{q}\pi}{2}\right)} = 0.$$

Therefore

$$\lim_{\tilde{q} \rightarrow q, \hat{q} \rightarrow 1} \cot\left\{\frac{\pi(\tilde{q} + \frac{1}{2})}{N} + \frac{\pi\hat{q}}{2}\right\} + \cot\left\{-\frac{\pi(\tilde{q} + \frac{1}{2})}{N} + \frac{\pi\hat{q}}{2}\right\} = 0.$$

Employing the above limits, $\dot{r}_p$ will also be set to zero for odd N . Therefore, for equation (4.36), we write

$$\dot{r}_p = 0.$$

C.5 Proof of Equation (4.44)

We shall now focus on equation (4.44), and examine Γ . Consider the proposition that

$$\lambda_k^{(1+j)} = e^{i\theta_{1+2j}} \lambda_{k-j}^{(1)}$$

where $1+j \neq k$, $j \neq k$ and $k > j$. We shall prove this relationship. Now,

$$\lambda_k^{(1+j)} = \frac{e^{i(\theta_{1+j} + \theta_k)}}{\sin^2\left(\frac{\theta_{1+j} - \theta_k}{2}\right)}.$$

We note that

$$\theta_{1+j} + \theta_k = \frac{2\pi(j+k-1)}{N} = \theta_{j+k},$$

and

$$\theta_{1+j} - \theta_k = \frac{2\pi(j-k+1)}{N} = \theta_{j-k+2}.$$

Therefore

$$\lambda_k^{(1+j)} = \frac{e^{i(\theta_{j+k})}}{\sin^2\left(\frac{\theta_{j-k+2}}{2}\right)}.$$

Consider next

$$e^{i\theta_{1+2j}} \lambda_{k-j}^{(1)} = \frac{e^{i(\theta_{1+2j} + \theta_{1+\theta_{k-j}})}}{\sin^2\left(\frac{\theta_{1+\theta_{k-j}}}{2}\right)}.$$

Now

$$\theta_{1+2j} + \theta_1 + \theta_{k-j} = \frac{2\pi(j+k-1)}{N} = \theta_{j+k}.$$

Also,

$$\theta_1 - \theta_{k-j} = \frac{2\pi(j-k+1)}{N} = \theta_{j-k+2}.$$

Therefore

$$e^{i\theta_{1+2j}} \lambda_{k-j}^{(1)} = \frac{e^{i(\theta_{j+k})}}{\sin^2\left(\frac{\theta_{j-k+2}}{2}\right)}.$$

It follows that $\lambda_k^{(1+j)} = e^{i\theta_{1+2j}} \lambda_{k-j}^{(1)}$ where $1+j \neq k$, $j \neq k$ and $k > j$. Consider next, the proof of the statement

$$\lambda_k^{(1+j)} = e^{i\theta_{1+2j}} \lambda_{N-(j-k)}^{(1)}$$

under the condition that $j \geq k$. Consider the LHS,

$$\lambda_k^{(1+j)} = \frac{e^{i(\theta_{1+j}+\theta_k)}}{\sin^2\left(\frac{\theta_{1+j}-\theta_k}{2}\right)}$$

which as we have already seen is equal to $\frac{e^{i(\theta_{j+k})}}{\sin^2\left(\frac{\theta_{j-k+2}}{2}\right)}$. Consider the RHS,

$$e^{i\theta_{1+2j}} \lambda_{N-(j-k)}^{(1)} = \frac{e^{i\{\theta_{1+2j}+\theta_{N-(j-k)}\}}}{\sin^2\left\{\frac{\theta_{N-(j-k)}}{2}\right\}}.$$

Now

$$\frac{\theta_{N-(j-k)}}{2} = \pi - \frac{\pi(j-k+1)}{N}.$$

Therefore,

$$\sin\left\{\frac{\theta_{N-(j-k)}}{2}\right\} = \sin\left(\frac{\theta_{j-k+2}}{2}\right)$$

and it readily follows that

$$\sin^2\left(\frac{\theta_{N-(j-k)}}{2}\right) = \sin^2\left(\frac{\theta_{j-k+2}}{2}\right).$$

Consider next

$$e^{i\theta_{1+2j}} e^{i\theta_{N-(j-k)}} = e^{i\{\theta_{1+2j} + \theta_{N-(j-k)}\}}.$$

Now,

$$\theta_{1+2j} = \frac{\pi(4j)}{N}$$

and

$$\theta_{N-j+k} = \frac{2\pi(N-j+k-1)}{N}.$$

It follows that

$$\theta_{1+2j} + \theta_{N-j+k} = 2\pi + \frac{2\pi(j+k-1)}{N}.$$

Therefore

$$e^{i\theta_{1+2j}} e^{i\theta_{N-(j-k)}} = e^{i(\theta_{j+k})}.$$

We have therefore proven that $\lambda_k^{(1+j)} = e^{i\theta_{1+2j}} \lambda_{k-j}^{(1)}$ in the situation where $1+j \neq k$, $j \neq k$ and $k > j$. It has also been proven that $\lambda_k^{(1+j)} = e^{i\theta_{1+2j}} \lambda_{N-(k-j)}^{(1)}$ for the situation where $1+j \neq k$ and $j \geq k$. It follows from the above that

$$\lambda^{(1+j)} = e^{i(\theta_{1+2j})} \lambda^{(1)}.$$

From the above relations it can be shown that

$$\Gamma = \begin{pmatrix} 1 & & & & \\ & e^{i\theta_3} & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & e^{i\theta_{2N+1}} \end{pmatrix} \begin{pmatrix} \lambda^{(1)} & -\lambda_2^{(1)} & -\lambda_3^{(1)} & \cdots & -\lambda_N^{(1)} \\ -\lambda_N^{(1)} & \lambda^{(1)} & -\lambda_2^{(1)} & \cdots & -\lambda_{N-1}^{(1)} \\ \vdots & & & & \\ \vdots & & & & \\ -\lambda_2^{(1)} & \cdots & \cdots & \cdots & \lambda^{(1)} \end{pmatrix}.$$

C.6 Proof that the Eigenvalues of C are Complex Conjugates of the Eigenvalues of C^+ and derivation of Equation (4.47)

Beginning with the circulant matrix,

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ c_{N-2} & c_{N-1} & c_0 & \cdots & c_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix},$$

and therefore

$$C^T = \begin{pmatrix} c_0 & c_{N-1} & c_{N-2} & \cdots & c_1 \\ c_1 & c_0 & c_{N-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{N-1} & c_{N-2} & c_{N-3} & \cdots & c_0 \end{pmatrix}.$$

It follows that

$$(C^T)^* = C^+ = \begin{pmatrix} c_0^* & c_{N-1}^* & c_{N-2}^* & \cdots & c_1^* \\ c_1^* & c_0^* & c_{N-1}^* & \cdots & c_2^* \\ c_2^* & c_1^* & c_0^* & \cdots & c_3^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{N-1}^* & c_{N-2}^* & c_{N-3}^* & \cdots & c_0^* \end{pmatrix}.$$

We now set $C^+ = D$ where

$$D = \begin{pmatrix} d_0 & d_1 & d_2 & \cdots & d_{N-1} \\ d_{N-1} & d_0 & d_1 & \cdots & d_{N-2} \\ d_{N-2} & d_{N-1} & d_0 & \cdots & d_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & d_3 & \cdots & d_0 \end{pmatrix}.$$

We see that

$$d_k = c_{N-k}^*.$$

Let us now write down the eigenvalues of D . The eigenvalues of D are given by

$$\Upsilon_j^D = d_0 + d_1\omega_j + d_2(\omega_j)^2 + d_3(\omega_j)^3 + \dots + d_{N-1}(\omega_j)^{N-1}$$

where $j = 0, 1, \dots, N-1$. But $\omega_j = e^{i\frac{2\pi j}{N}}$ and so we can write

$$\Upsilon_j^D = d_0 + d_1e^{i\frac{2\pi j}{N}} + d_2(e^{i\frac{2\pi j}{N}})^2 + d_3(e^{i\frac{2\pi j}{N}})^3 + \dots + d_{N-1}(e^{i\frac{2\pi j}{N}})^{N-1}.$$

Writing d 's in terms of c 's.

$$\Upsilon_j^D = c_0^* + c_{N-1}^*e^{i\frac{2\pi j}{N}} + c_{N-2}^*(e^{i\frac{2\pi j}{N}})^2 + c_{N-3}^*(e^{i\frac{2\pi j}{N}})^3 + \dots + c_1^*(e^{i\frac{2\pi j}{N}})^{N-1}.$$

For the matrix C we have

$$\Upsilon_j^C = c_0 + c_1e^{i\frac{2\pi j}{N}} + c_2(e^{i\frac{2\pi j}{N}})^2 + c_3(e^{i\frac{2\pi j}{N}})^3 + \dots + c_{N-1}(e^{i\frac{2\pi j}{N}})^{N-1}.$$

Consider the illustration

$$e^{i\frac{2\pi j}{N}} = e^{i(\frac{2\pi j}{N} - 2\pi j)} = e^{-i(\frac{2\pi j}{N})(N-1)}.$$

With the above illustration in mind, we can write

$$\Upsilon_j^D = c_0^* + \sum_{k=1}^{N-1} c_{N-k}^* e^{i(\frac{2\pi j}{N})k}$$

and

$$\Upsilon_j^C = c_0 + \sum_{k=1}^{N-1} c_{N-k} e^{i(\frac{2\pi j}{N})(N-k)} = c_0 + \sum_{k=1}^{N-1} c_{N-k} e^{i(\frac{2\pi j}{N})(N-k)} e^{-i(\frac{2\pi Nj}{N})} = c_0 + \sum_{k=1}^{N-1} c_{N-k} e^{-i(\frac{2\pi j}{N})k}.$$

Therefore

$$(\Upsilon_j^C)^* = \Upsilon_j^D.$$

Indeed the eigenvalues of C are complex conjugates of the eigenvalues of C^+ . We move on to find a simplified expression for the eigenvalues of C , which we refer to now as $\tilde{\rho}_j$. We can then write

$$\tilde{\rho}_j = \lambda^{(1)} + \left\{ -\lambda_2^{(1)} \right\} \omega_j + \left\{ -\lambda_3^{(1)} \right\} (\omega_j)^2 + \left\{ -\lambda_4^{(1)} \right\} (\omega_j)^3 + \dots + \left\{ -\lambda_N^{(1)} \right\} (\omega_j)^{N-1}$$

where, as before, $j = 0, 1, \dots, N-1$. Now

$$\lambda^{(1)} = \lambda_2^{(1)} + \lambda_3^{(1)} + \lambda_4^{(1)} + \dots + \lambda_N^{(1)} = \frac{e^{i\theta_2}}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{e^{i\theta_3}}{\sin^2\left(\frac{\theta_3}{2}\right)} + \frac{e^{i\theta_4}}{\sin^2\left(\frac{\theta_4}{2}\right)} + \dots + \frac{e^{i\theta_N}}{\sin^2\left(\frac{\theta_N}{2}\right)}.$$

We recall that $\theta_k = \frac{2\pi(k-1)}{N}$. Therefore

$$\lambda^{(1)} = \frac{e^{i(\frac{2\pi}{N})}}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{e^{i(\frac{2\pi}{N})(2)}}{\sin^2\left(\frac{\theta_3}{2}\right)} + \frac{e^{i(\frac{2\pi}{N})(3)}}{\sin^2\left(\frac{\theta_4}{2}\right)} + \dots + \frac{e^{i(\frac{2\pi}{N})(N-k)}}{\sin^2\left(\frac{\theta_N}{2}\right)}.$$

Now $\omega_j = \omega^j = e^{i\frac{2\pi j}{N}}$. We can write therefore that

$$\lambda^{(1)} = \frac{\omega_1}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_2}{\sin^2\left(\frac{\theta_3}{2}\right)} + \frac{\omega_3}{\sin^2\left(\frac{\theta_4}{2}\right)} + \dots + \frac{\omega_{N-1}}{\sin^2\left(\frac{\theta_N}{2}\right)}.$$

Clearly, $\lambda_3^{(1)} = \frac{\omega_1}{\sin^2\left(\frac{\theta_3}{2}\right)}$, $\lambda_4^{(1)} = \frac{\omega_3}{\sin^2\left(\frac{\theta_4}{2}\right)}$, $\lambda_5^{(1)} = \frac{\omega_4}{\sin^2\left(\frac{\theta_5}{2}\right)}$ and so on. We note that $\lambda_N^{(1)} = \frac{\omega^{N-1}}{\sin^2\left(\frac{\theta_N}{2}\right)}$. We can then write

$$\begin{aligned} \tilde{\rho}_j = & \frac{\omega_1}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_2}{\sin^2\left(\frac{\theta_3}{2}\right)} + \frac{\omega_3}{\sin^2\left(\frac{\theta_4}{2}\right)} + \dots + \frac{\omega_{N-1}}{\sin^2\left(\frac{\theta_N}{2}\right)} + \frac{-(\omega_1)(\omega_j)}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{-(\omega_2)(\omega_j)^2}{\sin^2\left(\frac{\theta_3}{2}\right)} + \frac{-(\omega_3)(\omega_j)^3}{\sin^2\left(\frac{\theta_4}{2}\right)} \\ & + \dots + \frac{-(\omega_{N-1})(\omega_j)^{N-1}}{\sin^2\left(\frac{\theta_N}{2}\right)}. \end{aligned}$$

Therefore

$$\tilde{\rho}_j = \frac{\omega_1(1 - \omega_j)}{\sin^2\left(\frac{\theta_2}{2}\right)} + \frac{\omega_2\{1 - (\omega_j)^2\}}{\sin^2\left(\frac{\theta_3}{2}\right)} + \frac{\omega_3\{1 - (\omega_j)^3\}}{\sin^2\left(\frac{\theta_4}{2}\right)} + \dots + \frac{\omega_{N-1}\{1 - (\omega_j)^{N-1}\}}{\sin^2\left(\frac{\theta_N}{2}\right)}.$$

Therefore

$$\tilde{\rho}_j = \sum_{k=1}^{N-1} \frac{\omega^k \{1 - (\omega^j)^k\}}{\sin^2\left(\frac{\theta_{k+1}}{2}\right)}.$$

Now

$$\sin^2\left(\frac{\theta_{k+1}}{2}\right) = -\left(\frac{e^{-i\frac{\theta_{k+1}}{2}}}{2}\right)^2 (e^{i\theta_{k+1}} - 1)^2,$$

but $\theta_{k+1} = \frac{2\pi k}{N}$ and therefore $e^{i\theta_{k+1}} = \omega^k$. It follows that

$$\sin^2\left(\frac{\theta_{k+1}}{2}\right) = -\frac{(\omega^k - 1)^2}{4\omega^k}.$$

Therefore

$$\tilde{\rho}_j = -4 \sum_{k=1}^{N-1} \frac{(\omega^k)^2 \{1 - (\omega^j)^k\}}{(1 - \omega^k)^2}.$$

C.7 Evaluation of the Integrals I_1 and I_2

We begin with $I_1 = \frac{1}{2\pi i} \int_{C_1} \frac{z^{N-1}}{z^N - 1} f(z) dz$. Now

$$I_1 = \frac{1}{2\pi i} \int_{C_1} \left(\frac{z^{N-1}}{z^N - 1} \right) \frac{z^2(1 - z^j)}{(1 - z)^2} dz.$$

Let

$$f(z) = \left(\frac{z^{N-1}}{z^N - 1} \right) \frac{z^2(1 - z^j)}{(1 - z)^2} = \frac{A(z)}{B(z)},$$

where $A(z) = \frac{z^{N+1}(1 - z^j)}{(1 - z)^2}$. We shall calculate the sum of the residues of $f(z)$ inside C_1 . Consider

$$A(\omega^k + \delta z) = \frac{(\omega^k)^{N+1} \left(1 + \frac{\delta z}{\omega^k}\right)^{N+1} \left\{1 - (\omega^k)^j \left(1 + \frac{\delta z}{\omega^k}\right)^j\right\}}{\left\{1 - (\omega^k) \left(1 + \frac{\delta z}{\omega^k}\right)\right\}^2}$$

Examining the numerator, and performing an expansion to $O(\delta z)$, the result is

$$\begin{aligned} & (\omega^k)^{N+1} \{1 - (\omega^k)^j\} + (\delta z)(\omega^k)^{N+1} \left\{ \frac{N+1}{\omega^k} - \frac{(\omega^k)^{j-1}}{(N+1)^{-1}} - j(\omega^k)^{j-1} \right\} \\ & = (\omega^k)^{N+1} \{1 - (\omega^k)^j\} + \bar{F}_1 \delta z. \end{aligned}$$

We now examine the denominator of $A(z)$, and perform an expansion to $O(\delta z)$, resulting in

$$\frac{1}{(1 - \omega^k)^2} + \frac{\delta z}{(1 - \omega^k)^3}.$$

It follows that, to $O(\delta z)$,

$$\begin{aligned} A(z) &= \frac{(\omega^k)^{N+1} \{1 - (\omega^k)^j\}}{(1 - \omega^k)^2} + \frac{(\omega^k)^{N+1} \{1 - (\omega^k)^j\}}{(1 - \omega^k)^3} \delta z + \frac{\bar{F}_1}{1 - \omega^k} \delta z \\ &= \frac{(\omega^k)^{N+1} \{1 - (\omega^k)^j\}}{(1 - \omega^k)^2} + \bar{F}_2 \delta z \end{aligned}$$

Consider next, $B(z) = z^N - 1$. As with $A(z)$, consider $B(\omega^k + \delta z)$, which to $O(\delta z)$, is equal to $N\delta z(\omega^k)^{N-1}$.

It follows therefore, following the expansion of both $A(z)$ and $B(z)$ in terms of δz , that the coefficient of

$\frac{1}{\delta z}$ in the expansion of $f(z)$ is

$$\frac{\{1 - (\omega^k)^j\} (\omega^k)^2}{(1 - \omega^k)^2(N)}.$$

Therefore, employing the residue theorem,

$$I_1 = \frac{1}{2\pi i} \int_{C_1} \left(\frac{z^{N-1}}{z^N - 1} \right) \frac{z^2(1 - z^j)}{(1 - z)^2} dz = \frac{1}{N} \sum_{k=1}^{N-1} \frac{\{1 - (\omega^k)^j\} (\omega^k)^2}{(1 - \omega^k)^2(N)}.$$

Next we look at $I_2 = \frac{1}{2\pi i} \int_{C_2} \frac{z^{N-1}}{z^N - 1} f(z) dz$. We begin by noting that the integrals along γ_1 and γ_3 cancel out one another. We move to consider the integral along γ_2 . Along γ_2 , $z = Re^{i\theta}$ and we note that R is large. We note that

$$\frac{z^{N-1}}{z^N - 1} = \frac{1}{z \left(1 - \frac{1}{z^N}\right)} = \frac{1}{z} \left(1 + \frac{1}{z^N}\right) + \dots$$

and also that

$$\frac{z^2(1 - z^j)}{(z - 1)^2} = \frac{1 - z^j}{\left(1 - \frac{1}{z}\right)^2} = (1 - z^j) \left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right).$$

Along γ_2 , $dz = izd\theta$. Therefore,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\gamma_2} f(z) dz &= \frac{1}{2\pi i} \int_{\theta=0}^{2\pi} \left\{ \frac{1}{z} \left(1 + \frac{1}{z^N}\right) + \dots \right\} (1 - z^j) \left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right) iz d\theta \\ &= \frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left(1 + \frac{1}{z^N} + \dots\right) (1 - z^j) \left(1 + \frac{2}{z} + \frac{3}{z^2} + \dots\right) d\theta. \end{aligned}$$

We recall that R is very large, and write that the above is approximately equal to

$$\frac{1}{2\pi} \int_{\theta=0}^{2\pi} \left(1 - z^j \sum_{l=0}^{\infty} \frac{l+1}{z^l}\right) d\theta = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} (1 - j - 1) d\theta = -j.$$

Importantly we note that $\int_{\theta=0}^{2\pi} z^{j-l} d\theta = 0$, for $j \neq l$, $z = Re^{i\theta}$. Therefore

$$\lim_{R \rightarrow \infty} \frac{1}{2\pi i} \int_{\gamma_2} f(z) dz = -j.$$

Next we consider the integral along γ_4 . Now

$$I_2 = \frac{1}{2\pi i} \int_{C_2} f(z) dz = \frac{1}{2\pi i} \int_{\gamma_4} f(z) dz - j = -R_{f(z)} - j,$$

where $R_{f(z)}$ is the residue of $f(z)$ at $z = 1$. We recall that

$$f(z) = \left(\frac{z^{N-1}}{z^N - 1} \right) \frac{z^2(1 - z^j)}{(1 - z)^2}.$$

At $z = 1$, we have a second order pole, and therefore

$$R_{f(z)} = \lim_{z \rightarrow 1} \left\langle \frac{d}{dz} \{ (z - 1)^2 f(z) \} \right\rangle.$$

Now

$$(z - 1)^2 f(z) = \frac{z^{N+1} - z^{N+1+j}}{z^N - 1}$$

and

$$\frac{d}{dz} \{ (z - 1)^2 f(z) \} = \frac{z^{2N} - (1 + j)z^{2N+j} - (N + 1)z^N + (N + 1 + j)z^{N+j}}{(z^N - 1)^2} = \frac{p(z)}{q(z)}$$

where $p(z) = z^{2N} - (1 + j)z^{2N+j} - (N + 1)z^N + (N + 1 + j)z^{N+j}$ and $q(z) = (z^N - 1)^2$. We will use L'Hopital's rule to evaluate the required limit. Now

$$\left. \frac{d^2 q}{dz^2} \right|_{z=1} = 2N^2$$

and

$$\left. \frac{d^2 p}{dz^2} \right|_{z=1} = -jN(j + N + 2).$$

Therefore

$$R_{f(z)} = \frac{-j(j + N + 2)}{2N}.$$

It follows that

$$I_2 = \frac{j(j + N + 2)}{2N} - j.$$

APPENDIX D

D.1 Derivation of $d(v \rfloor \boldsymbol{\omega} \cdot d\mathbf{S}) = \mathbf{curl}(\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{S}$

Consider the $v \rfloor \boldsymbol{\omega} \cdot d\mathbf{S}$ term. We write

$$\begin{aligned}
 & v_1 \partial_x + v_2 \partial_y + v_3 \partial_z \\
 & \rfloor \\
 & \omega_1 (dy \wedge dz) + \omega_2 (dz \wedge dx) + \omega_3 (dx \wedge dy) \\
 & = \\
 & -v_1 \omega_2 dz + v_1 \omega_3 dy + v_2 \omega_1 dz - v_2 \omega_3 dx - v_3 \omega_1 dy + v_3 \omega_2 dx \\
 & = \\
 & (v_3 \omega_2 - v_2 \omega_3) dx + (v_1 \omega_3 - v_3 \omega_1) dy + (v_2 \omega_1 - v_1 \omega_2) dz.
 \end{aligned}$$

Noting that

$$\begin{aligned}
 & \mathbf{v} \times \boldsymbol{\omega} \\
 & = \\
 & \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ v_1 & v_2 & v_3 \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix} \\
 & = \\
 & (v_2 \omega_3 - v_3 \omega_2) \hat{\mathbf{i}} - (v_1 \omega_3 - v_3 \omega_1) \hat{\mathbf{j}} + (v_1 \omega_2 - v_2 \omega_1) \hat{\mathbf{k}} \\
 & = \\
 & (v_2 \omega_3 - v_3 \omega_2) \hat{\mathbf{i}} + (v_3 \omega_1 - v_1 \omega_3) \hat{\mathbf{j}} + (v_1 \omega_2 - v_2 \omega_1) \hat{\mathbf{k}},
 \end{aligned}$$

then

$$v \rfloor \boldsymbol{\omega} \cdot d\mathbf{S} = -(\mathbf{v} \times \boldsymbol{\omega}) \cdot d\mathbf{r}.$$

We can then write that

$$\begin{aligned}
& d(v] \boldsymbol{\omega} \cdot d\mathbf{S}) \\
& = \\
& d\{(\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{r}\} \\
& = \\
& \partial_y (\boldsymbol{\omega} \times \mathbf{v})_1 dy \wedge dx + \partial_z (\boldsymbol{\omega} \times \mathbf{v})_1 dz \wedge dx \\
& + \partial_x (\boldsymbol{\omega} \times \mathbf{v})_2 dx \wedge dy + \partial_z (\boldsymbol{\omega} \times \mathbf{v})_2 dz \wedge dy \\
& + \partial_x (\boldsymbol{\omega} \times \mathbf{v})_3 dx \wedge dz + \partial_y (\boldsymbol{\omega} \times \mathbf{v})_3 dy \wedge dz \\
& = \\
& (dx \wedge dy) \{\partial_x (\boldsymbol{\omega} \times \mathbf{v})_2 - \partial_y (\boldsymbol{\omega} \times \mathbf{v})_1\} \\
& + (dy \wedge dz) \{\partial_y (\boldsymbol{\omega} \times \mathbf{v})_3 - \partial_z (\boldsymbol{\omega} \times \mathbf{v})_2\} \\
& + (dz \wedge dx) \{\partial_z (\boldsymbol{\omega} \times \mathbf{v})_1 - \partial_x (\boldsymbol{\omega} \times \mathbf{v})_3\}.
\end{aligned}$$

Noting that

$$\begin{aligned}
& \text{curl} \mathbf{A} \\
& = \\
& \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ A_1 & A_2 & A_3 \end{vmatrix} \\
& = \\
& (\partial_y A_3 - \partial_z A_2) \hat{\mathbf{i}} + (\partial_z A_1 - \partial_x A_3) \hat{\mathbf{j}} + (\partial_x A_2 - \partial_y A_1) \hat{\mathbf{k}},
\end{aligned}$$

we can then write

$$d(v] \boldsymbol{\omega} \cdot d\mathbf{S}) = \text{curl}(\boldsymbol{\omega} \times \mathbf{v}) \cdot d\mathbf{S}.$$