

NULL CONTROLLABILITY OF A CLASS
OF NONLINEAR SYSTEMS

by

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ABSTRACT

The concept of controllability and the development of the topic for linear and nonlinear systems, following the initial work in the 1960's, is discussed and some of the relevant work in stability theory is introduced in Chapter One.

Mathematical tools, including some stability theory techniques, are given in the next chapter and this is followed by a discussion of a set of possible bounds upon the nonlinear function $f(y, u_y, t)$ which occurs in the nonlinear state-space equation $\dot{y} = A(t)y(t) + B(t)u_y(t) + f(y, u_y, t)$.

Chapter Three introduces the first of the new results concerning the null controllability of these nonlinear systems when the bound upon the norm of the nonlinearity involves only state variables. Two theorems are presented and discussed.

The extension of these results to encompass nonlinearities bounded by state and control is given in Chapter Four where three more norm bounds are considered and corresponding null controllability results presented.

Weaker restrictions can be placed upon the nonlinearity in the form of inner product bounds and in Chapter Five it is shown how null controllability results can be derived when the nonlinearity is bounded in this way. An inner product condition for the uniqueness of the nonlinear system solution is given.

Chapter Six contains a result showing the asymptotic equivalence of a pair of linear and nonlinear state equation provided the nonlinearity is weakly linear. This is followed by some examples of the preceding results. On the basis of the possible implementation which is given next, the results are discussed and compared. Some areas of possible extensions are indicated.

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CHAPTER 1

- 1.1 Representation and structure of dynamical systems.
- 1.2 Definitions of controllability for linear dynamical systems.
- 1.3 Some nonlinear controllability definitions.
- 1.4 Controllability of nonlinear systems
- 1.5 Stability of nonlinear differential equations.
- 1.6 Outline of the new results.

1.1 THE REPRESENTATION AND STRUCTURE OF DYNAMICAL SYSTEMS

The relationship between the signals entering and leaving a plant may be mathematically described in more than one way. It is often useful to postulate the existence of additional variables whose behaviour governs the interaction between the incoming and outgoing signals. As long as the number of these variables in the mathematical model is finite we have a finite dimensional system, and these additional variables are known as the "states" of the plant. In order to discuss more easily the concept of controllability we shall briefly develop the general structure of dynamical systems and then consider various special types of system for which controllability results are known. Following Kalman, Falb and Arbib^{K1} we first define the states by

DEFINITION 1.1

The set X , with elements $x \in X$, where $x: T \rightarrow R^n$ is called the state set if knowledge of the state x at any time $t \in T$ is enough to fully describe the system at time t .

The state represents the internal dynamics of the plant and hence relates the two other fundamental concepts: the inputs to the system and the outputs from the system. In general the outputs from the system depend upon the present inputs to the system and the present state. The states may be regarded as some form of information storage or accumulation of past events. In common with the state, the inputs and outputs also relate to the independent parameter set T and may be seen as a mapping from T into an appropriate space, e.g.

$u : T \rightarrow U \subset R^m$ the input map and

$y : T \rightarrow Y \subset R^p$ the output map.

These concepts are formalised in the following definition

DEFINITION 1.2

S is called a dynamical system and is represented by $\{r, X, \Psi, \phi, g\}$ such that

(1) $r = \{u | u: T \rightarrow U\}$

(2) X is the set of states

(3) $\Psi = \{y | y: T \rightarrow Y\}$

(4) $\phi(t_1; x_0, t_0, u)$ is defined by

$$x(t_1) = \phi(t_1; x_0, t_0, u) \quad \forall t_1 \geq t_0$$

(5) $\exists$ a function g s.t. $g: T \times X \times U \rightarrow R^p$

$$\text{and } y(t) = g(x(t), t, u(t)) \quad \forall x \in X, u(t) \in r$$

$$t \in T$$

This definition gives the system dynamics in terms of the function ϕ . We usually require that systems be causal and this in turn requires a special form of system dynamics. The independent time set T is an order set and hence time has a preferred direction and thus the past behaviour of the system may influence the future but not vice versa. In addition the function ϕ is required to satisfy the important composition property where the state at any time t_2 , given the state at time t_0 , can also be obtained from the state at any intermediate time t_1

$$\phi(t_2; x_0, t_0, u) = \phi(t_2; \phi(t_1; x_0, t_0, u), t_1, u)$$

$$\forall t_0 \leq t_1 \leq t_2$$

Such a definition of a dynamical system is too general to enable us to develop much mathematical theory, or to provide a basis for practical deduction; we require the imposition of more structure upon the system.

A special class of dynamical systems is one where the dynamics of the states may be represented by the set of

first-order differential equations

$$\frac{dx(t)}{dt} = f(x(t), u(t), t) \quad 1.1$$

where $x(t)$ is a $n \times 1$ vector and

$u(t)$ is a $m \times 1$ vector

and $f: R^n \times U \times T \rightarrow R^n$

The output $y(t)$ is also considered to be a finite dimensional vector given by

$$y(t) = g(x(t), t) \quad 1.2$$

where $g: R^n \times T \rightarrow Y$

In general it will not be possible to determine the states exactly by examining the output $y(t)$; in fact there may well be more states than outputs. The problem of reconstructing the states given the values of the outputs is a property of the function g which Kalman called "observability". In a similar way it may not be possible to force the state to take any given value by a particular choice of input $u(t)$; this is the "controllability" problem. These problems were initially founded in the framework of linear systems when the general functions $f(\cdot, \cdot, t)$ and $g(\cdot, t)$ in equations 1.1 and 1.2 are replaced by linear transformations represented by matrices.

DEFINITION 1.3

A linear time-varying dynamical system with linear output is given by the equation pair

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad 1.3$$

$$y(t) = C(t)x(t) \quad 1.4$$

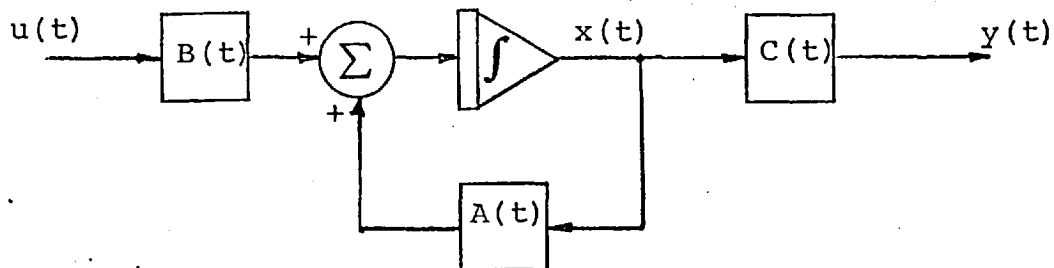
where $x(t) \in R^n$, $u(t) \in U \subset R^m$

$y(t) \in Y \subset R^p$, $t \in T$

and the matrices $A(t)$, $B(t)$ and $C(t)$ have appropriate dimensions.

This type of system is frequently represented by the block diagram in Fig. 1.1.

FIG.1.1



BLOCK DIAGRAM OF LINEAR TIME-VARYING
DYNAMIC SYSTEM

In treating linear dynamical systems it was recognised at an early stage that certain regularity assumptions on the models were important for the validity of the techniques which were being employed. It was originally thought that these were mathematical devices; however it was soon realised that these qualitative properties were of importance in their own right. This was first emphasised in Kalman, Ho and Narendra^{K2}. The authors show how controllability and observability are particularly important in the design of linear optimal controllers and linear filters. For example, in the case of the optimal controller, if the linear system is controllable then the linear feedback system, obtained by using the time optimal control with quadratic cost, is an asymptotically stable system. The concepts of controllability and observability are also important in the context of mathematical model building, where one may start with an

input/output model obtained experimentally and require a statespace model in order to use the analytic design techniques available. In this case it is usually desirable that the model has the minimum possible dimension whilst still satisfying the input/output description. It can be shown that this minimality is closely related to the principles of controllability and observability.

1.2 DEFINITIONS OF CONTROLLABILITY FOR LINEAR DYNAMICAL SYSTEMS

Consider the following question, originally posed in Kalman, Ho and Narendra^{K2} :

"Can any initial state of a given dynamic system be transferred to any desired final state in a finite length of time by some control action?"

It is from this basic question that the property of controllability was developed. If the answer is in the affirmative then the given system is said to be controllable. A current standard definition of controllability is

DEFINITION 1.4

A dynamical system is said to be "completely controllable" on the interval (t_0, t_1) if for any initial state x_0 and final state x_1 , there exists a control $u(t) \in U$ such that the condition $x(t_0) = x_0$ implies $x(t_1) = x_1$.

Kalman et al^{K2} produced a criterion to test this property in linear dynamical systems which involves considering the rank of the controllability matrix, $W(t)$, given by

$$W(t) \triangleq \int_{t_0}^t \phi(t_0, \tau) B(\tau) B^T(\tau) \phi^T(t_0, \tau) d\tau$$

where $\phi(\cdot, \cdot)$ is the state transition matrix for the linear system 1.3.

In practice, as the transition matrix may not be known analytically, it is not generally possible to use this criterion. The authors also developed a simple test by which the controllability of linear time invariant dynamical systems may be examined.

THEOREM 1.1 (KALMAN, HO & NARENDRA 1963)

The linear time invariant system given by

$$\dot{x} = Ax + Bu \tag{1.5}$$

$$y = Cx \tag{1.6}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and

A, B and C are matrices of appropriate dimensions

is completely controllable if and only if the n by $m \cdot n$ matrix

$$(B \ AB \ \dots \ A^{n-1}B) \tag{1.7}$$

is of full rank.

We note that there exists a similar analytic criterion for observability of linear time invariant systems. The extension of these simple tests to time-varying systems is not straightforward and to fully describe the controllability of time-varying systems we need to introduce two more definitions:

A special property of linear time invariant systems is that if they are controllable, then the state transfer may be made in an arbitrarily short interval of time, by some control action. Thus if a time-varying system were to be controllable, over some small time interval in the time period of interest, it would be possible to effect the state transfer in this short

interval. This would not imply the controllability of the system over the whole time period. In order to distinguish between these two cases Silverman and Meadows^{S2} define a more restrictive property - total controllability:

DEFINITION 1.5

A dynamical system is said to be "totally controllable" on the interval (t_0, t_1) if it is completely controllable on any open subinterval in (t_0, t_1) .

Any time invariant system that is completely controllable is also totally controllable: the system matrices are constant, so if they satisfy the criterion in Theorem 1.1 at any time in (t_0, t_1) , this property must hold on any open interval in (t_0, t_1) .

It is important for many applications to have a somewhat stronger controllability definition than these basic ones. In the above case we note that it is possible to transfer between any two states in an arbitrarily short time; however this may imply the use of controls which are arbitrarily large. The following definition from Silverman and Meadows^{S2} avoids this practical difficulty.

DEFINITION 1.6

A dynamical system is said to be "uniformly controllable" on (t_0, t_1) if there exists a $t_f \in (t_0, t_1)$ and continuous function $\alpha: \mathbb{R} \rightarrow \mathbb{R}$ such that for any initial state x_0 there exists a control $u(t) \in U$ such that $x(t_0) = x_0$ and $\phi(t_f; x_0, t_0, u) = 0$ and

$$\int_{t_0}^{t_f} ||u(s)||^2 ds \leq \alpha(||x_0||)$$

Note that all uniformly controllable systems are

controllable, but the converse is not true: in general it is not possible to transfer a linear system exactly to zero in a finite time using bounded control energy.

Silverman and Meadows^{S2} derive a similar criterion to that given in Theorem 1.1 which can be used with time-varying linear systems:

THEOREM 1.2 (SILVERMAN & MEADOWS 1967)

The linear time-varying system given by equation 1.3 is:

- a) completely controllable on (t_0, t_1) if the matrix

$$(P_0(t), P_1(t), \dots, P_{n-1}(t))$$

has full rank for some $t \in (t_0, t_1)$

- b) totally controllable on (t_0, t_1) if the matrix

$$(P_0(t), P_1(t), \dots, P_{n-1}(t))$$

has full rank on any subinterval of (t_0, t_1)

where $P_0(t) = B(t)$

$$\text{and } P_{j+1}(t) = -A(t)P_j(t) + P_j(t) \quad j = 0, 1, \dots, n-2$$

There exists a parallel theorem for the observability of time-varying systems with output map given by equation 1.4. Notice that the criterion for linear time invariant systems is a necessary and sufficient condition that the system be completely controllable. In the case of time-varying systems, however, it is only possible to give a sufficient condition for controllability. It is possible that there exist systems which are completely controllable and yet do not satisfy the condition given in Theorem 1.2(a).

1.3 SOME NONLINEAR CONTROLLABILITY DEFINITIONS

The controllability of linear systems is a qualitative property which characterises the extent to which the system

states may be influenced by control actions. In a control sense it may be seen as a measure of the "wellposedness" of the system equations and hence controllability has a considerable bearing throughout linear systems theory. When attempting to extend this property of nonlinear systems it must be recognised that, in order to put some additional structure on the problem, it has been necessary to use a series of somewhat restricted definitions. Because of this, the rather special properties of a linear controllable system do not carry over to nonlinear systems, with the result that the controllability of nonlinear systems has been reduced to something similar to a stability problem.

One assumption that is generally made, in order to simplify the controllability problem for nonlinear systems, is to take the desired final state always to be the origin. We shall generally call this problem the null controllability problem and assume a transformation of the state space has been made such that $x(t_f) = 0$.

In an early attempt at tackling the null controllability problem Markus^{M1} considered the nonlinear autonomous equation

$$\frac{dx}{dt} = f(x, u) \quad 1.8$$

By taking a local linearisation around the origin, he produced a local null controllability result analogous to the preceding condition given for linear systems in equation 1.7.

Consideration of the rank of the matrix

$$\begin{pmatrix} B, AB, \dots, A^{n-1}B \end{pmatrix} \text{ where } \quad 1.9$$

$$A \triangleq \frac{\partial f}{\partial x}(0,0) \text{ and } B \triangleq \frac{\partial f}{\partial u}(0,0)$$

enables Markus to derive the result that the system 1.8 will be null controllable within some neighbourhood of the origin,

provided that matrix 1.9 has full rank and that the space of admissible controls, U , contains zero and $u(t)$ can be arbitrarily small whilst still spanning R^m . In order that the null solution be a trivial solution to equation 1.8 he also required that $f(0,0) = 0$. 1.10

No further developments followed this approach and later workers in the field have concentrated upon controllability definitions and derivations which do not lean heavily upon linear system results. The criterion of Markus may, however, be useful when using a "global" theorem which cannot show controllability exactly to the origin, but only to within a neighbourhood of the origin. The final check upon the controllability of the system may then be made using the result given above.

More recent work in the field of nonlinear controllability has led to the introduction of many new definitions and the structure of these definitions depends upon the techniques used in the derivation of the results. Some of these definitions are:

DEFINITION 1.7 (HERMES^{H2})

The dynamical system given by 1.1 is said to be "locally controllable" along a solution $\phi(\cdot; x_0, t_0, u)$ corresponding to control $u(t)$ if for some $t_1 > t_0$ all points in some neighbourhood of $\phi(t_1; x_0, t_0, u)$ are obtainable at time t_1 using some $u(t) \in U$.

DEFINITION 1.8 (HAYNES & HERMES^{H2})

The dynamical system given by 1.2 is said to be "locally-locally controllable" if for any $\epsilon > 0$ there exists a $\delta > 0$; $\delta < \epsilon$ such that all points in a " δ neighbourhood" of x_0 can be

attained by a trajectory not leaving an " ε neighbourhood" of x_0 .

DEFINITION 1.9 (GERSHWIN & JACOBSON^{G1})

The dynamical system given by 1.1 is said to be "controllable from (x_0, t_0) to $(0, t_f)$ " if for some $u(t) \in U$ the solution of 1.1 with $x(t_0) = x_0$ is such that $x(t_f) = 0$ where t_f is a preassigned terminal time. If a system is controllable for all x_0 at $t = t_0$ it will be called "completely controllable from t_0 to $(0, t_f)$ ".

DEFINITION 1.10 (MIRZRA & WOMACK^{M2})

An initial state of the dynamical system given by 1.1 is said to be controllable at t_0 if for some $t_1 > t_0$ there exists a piecewise continuous control function $u(t)$ defined on $[t_0, t_1]$ such that if $\phi(t; x_0, t_0, u)$ exists then $\phi(t_1; x_0, t_0, u) = 0$.

DEFINITION 1.11 (TOKUMARU, ADACHI & INOUE^{T1})

If for an arbitrary given $\tau > 0$ there exists a $u(t) \in U$ which transfers the initial state x_0 given at time $t = t_0$ to the state x_1 at time t_1 : $(t_1 - t_0) < \tau$ then the state x_0 is said to be "well controllable to x_1 ".

DEFINITION 1.12 (BEN LASHIHER & STOREY^{B2})

The dynamical system given by 1.1 is $(\alpha, \beta, t_1 \in (t_0, T))$ controllable if there exists a control $u(t) \in U$ such that for any trajectory $x(t) = \phi(t; t_0, x_0, u)$ the condition $x_0 \in \alpha$ implies $x(t_1) \in \beta$.

These definitions form a cross-section of typical definitions used. Every contributor to the field of controllability is consistent and this implies that each writer must introduce a series of definitions ; for instance, definition 1.11 is one of a series of ten definitions given by

Tokumaru, Adachi and Inoue. From the above set of definitions we can see that, although there is a general consensus of opinion which says that controllability involves the transformation of one state to another in the state space, the particular definitions used to study nonlinear system vary in detail on a number of points. For instance only definition 1.12 requires that the transformation be from a set of points in the state space to another set of points. This is due to the fact that the definition used by Ben Lashiher & Storey is an extension of the definition of final stability introduced by Wiess as a more practical type of stability. Some of the above definitions define a final time t_1 'a priori' whereas others do not; instead they are satisfied with the existence of a finite time interval over which the phenomenon holds true. The definitions used in this work will be similar to definition 1.12; however, they will use some of the ideas introduced by other authors.

1.4 CONTROLLABILITY OF NON LINEAR SYSTEMS

In this section we shall look at the results of two or three papers in the field and analyse their progress in working towards some general theory of controllability for nonlinear systems. At this stage it is useful to point out that many of the approaches used to obtain criteria are not obviously related. It would seem that it is sometimes possible for an author to be working in a similar field and almost accidentally derive a controllability criterion which is applicable to some class of nonlinear systems.

Davison and Kunze (1970)^{D2} give sufficient conditions

for a nonlinear system, with controls appearing linearly, to be controllable in either a global or local sense. They consider the system given by

$$\dot{x} = A(t,x)x + B(t,x)u \quad 1.11$$

In order to infer something about the behaviour of this system they show necessary and sufficient conditions that the simpler system given by

$$\dot{x} = A(t,z)x + B(t,z)u \quad 1.12$$

be completely (or totally) controllable. Here z is some continuous function on (t_0, t_f) taking values in R^n . Then for each choice of z , 1.12 is a linear system with solutions parameterised by z . Thus the controllability Gramian (see Silverman & Meadows^{S2}) is given by

$$G(t_0, t_f; z) = \int_{t_0}^{t_f} \phi(t_0, \tau; z) B(\tau, z) B^T(\tau, z) \phi^T(t_0, \tau; z) d\tau \quad 1.13$$

Using a Schauder Fixed Point Theorem the authors are able to show that if

$$a) \quad \inf_{z \in C^n(t_0, t_f)} G(t_0, t_f; z) \geq C > 0 \quad 1.14$$

b) The elements of $A(t, x)$, $B(t, x)$ are bounded, continuous in x and piecewise continuous in t

then there must exist at least one function z^* which is also a solution to 1.12 with any required initial and final states x_0 and x_1 . Hence $x^* = z^*$ will be a solution to 1.11 on the interval (t_0, t_f) with end point conditions $x(t_0) = x_0$ and $x(t_f) = x_1$. Using the controllability matrix introduced by Silverman and Meadows they establish a simple computational criterion for the controllability of single input nonlinear systems.

The controllability studies of Gershwin and Jacobson^{G1} (1971) centre around the joint ideas of Lyapunov functions and optimal controllers. Their main result is very similar to a stability theorem; however, by application of this in particular cases, some well known results are rederived and new results for nonlinear systems presented. They demand the existence of a Lyapunov function: a scalar function $V(x,t)$, with first derivatives existing and such that

$$\lim_{t \rightarrow t_f} c(t) \neq 0 \Rightarrow \lim_{t \rightarrow t_f} V(c(t), t) = \infty \quad 1.13$$

Then, if a solution to equation 1.1 with initial condition $x(t_0) = x_0$ exists and is unique and if the total time derivative of $V(x,t)$ over the interval (t_0, t_f) is uniformly bounded above, the system 1.1 is controllable from (x_0, t_0) to $(0, t_f)$ in the sense of definition 1.9.

By applying this technique to a linear time-varying system, as in equation 1.3, and by choosing a Lyapunov function of the quadratic form, Gershwin & Jacobson are able to rederive Kalman's original criterion, with the restriction of the feedback control law to the form

$$u(t) = -\frac{1}{2} B^T(t) S(t) x(t) \quad 1.14$$

where the Lyapunov function is given by

$$V(x,t) = x^T(t) S(t) x(t) \quad 1.15$$

Other examples given by the authors show the use of this method for various forms of nonlinear system where equation 1.3 is perturbed in some way on the right hand side. For instance example 2 yields conditions that the following equation be controllable (Definition 1.9).

The nonlinear system given by

$$\dot{x} = A(t)x(t) + B(t)u(t) + h(x,t) \quad 1.16$$

is controllable from (x_0, t_0) to $(0, t_f)$ using the control law

$$u(t) = -\frac{1}{2} B^T(t)S(t)x(t) + q(x,t) \quad 1.17$$

if the linear system 1.3 is controllable from (x_0, t_0) to $(0, t_f)$ and if there exists some function $q(x,t)$ such that $\dot{V}$ given by

$$\dot{V} = 2x^T(t)S(t) \left(B(t)q(x,t) + h(x,t) \right). \quad 1.18$$

is uniformly bounded above. We also require that solutions due to the control given by 1.17 exist and are unique. We can see that, in common with other forms of control law of this type, if we wish to control the systems given by 1.3 and 1.16 exactly to zero in finite time then the control used approaches infinity as the solution approaches zero. This is in order that the quadratic Lyapunov function, 1.15, should satisfy the limiting condition 1.13 at $t=t_f$. That is to say we require

$$\lim_{t \rightarrow t_f} (S^{-1}(t)) = 0 \quad 1.19$$

Despite this drawback this is an important paper which adds to the structure of the problem and uses some of the mathematical tools from stability theory to good effect.

More sophisticated stability theory techniques are used by Ben Lashiher and Story ^{B2} when tackling the controllability problem from definition 1.12. Instead of using a total time derivative of the Lyapunov function to provide information about the solutions of the system the authors use a scalar comparison function to ensure that the system solutions are sufficiently stable under appropriate controls.

Comparison functions, which will be discussed more fully in the next chapter, are a more general way of putting some measure upon the Lyapunov functions used with the system equations. The usual Lyapunov function is a scalar function $V(x,t)$ which is positive definite but has a negative total time derivative $\dot{V}(x,t)$, which determines the stability of the system considered. Comparison functions, however, enable the bounds upon the system solution to be implied from a scalar function without demanding either of these restrictions. By suitably choosing a comparison function it is usually possible to obtain the normal Lyapunov-type results as a special case of this method.

In common with other forms of stability, the final stability concept, from which definition 1.12 stems, considers the possible value of the solution norm at the initial and final time. In the case of final stability we are concerned that a solution which is in some zone α at time $t=t_0$ (where $\alpha \triangleq \{x \mid \|x\| \leq a\}$) should then necessarily lie in β at time $t=t_1$ (where $\beta \triangleq \{x \mid \|x\| \leq \beta\}$). It is for this reason that definition 1.12 defines controllability from some zone α into a zone β . Considering that we are concerned with non-linear systems this would seem to be more realistic than demanding, as Gershwins & Jacobson do, that the final solution be exactly equal to zero. By use of examples Ben Lashihir and Storey show how this method, of using a scalar comparison function, may be applied to certain specific types of control system and a suitable control law determined, such that the system is controllable in the sense of definition 1.12.

1.5 STABILITY OF NONLINEAR DIFFERENTIAL EQUATIONS

From the above two papers it would seem that there exist stability techniques that can usefully be applied to the controllability problem. To illustrate this we next consider two papers which give us some information about the asymptotic behaviour of differential systems.

Lovelady^{L2} considers two nonlinear systems, without any control functions, and gives conditions which will imply that they are asymptotically equivalent. That is to say

$$\lim_{t \rightarrow \infty} ||w(t) - z(t)|| = 0 \quad 1.20$$

where $w(t)$ and $z(t)$ are solutions to the two nonlinear differential equations

$$\dot{w}(t) = F(t, w(t)) \quad 1.21$$

$$\dot{z}(t) = F(t, z(t)) + G(t, z(t)) \quad 1.22$$

where $w(t), z(t) \in \mathbb{R}^n$ and F and G take values in $\mathbb{R}^n$

He assumes that the n vector valued function F is such that solutions to 1.21 exist, are unique and can be continued indefinitely to the right. He then proceeds to show the existence of solutions to 1.22, despite the fact that G need not be differentiable, by considering a mapping between the two solutions when the function G is "driven" by the solution to 1.21. The verification of a fixed point in this map confirms the existence of solutions to 1.22 and the asymptotic equivalence follows under much the same assumptions. As G is not differentiable it is necessary for Lovelady to use a more generalised Dini derivative in order to examine the behaviour of terms like $||w(t) - z(t)||$ as time evolves. Scalar comparison functions are also used throughout as a way to obtain a bound upon the solution to the differential inequalities

arising in the proof.

One of the five conditions Lovelady requires is that for any $x, y \in \mathbb{R}^n$ and positive c there exists a function $\alpha(t)$ such that

$$|| (x-y) - c (F(t,x) - F(t,y)) || \geq (1-c\alpha(t)) ||x-y|| \quad 1.23$$

If $F(t, w(t))$ is a linear function it is possible to say more about the function $\alpha(t)$ which satisfies this inequality and we shall introduce, in Chapter Two, a scalar function of the matrix $H(t)$ (say) which will satisfy this relationship when $F(t, y(t)) = H(t)y(t)$. Obviously in this case $\alpha(t)$ will, in some way, be related to the eigen-values of $H(t)$.

Another recent paper presenting stability results which are of relevance to the controllability problem is by Pao^{P1}, an author who more usually works with infinite dimensional systems. This is evident in the approach Pao uses in this paper when he develops a stability theory for finite dimensional systems using as the basic tools the ideas of inner products and equivalence of inner products. This enables him to give Lyapunov-like results without resorting to the actual construction of a Lyapunov function.

The precise form of the equations examined by Pao is

$$\dot{x} = A(t)x + f(x, t) \quad 1.24$$

where $x(t) \in \mathbb{R}^n$

f takes values in $\mathbb{R}^n$ and $A(t)$ is a continuous matrix of appropriate dimension

f is a continuous function of x at each t but is not necessarily differentiable. Many theorems concerning the stability of the equation 1.24 are given but here we examine only Theorem 3.2 as being typical of the approach used by the

author. Pao shows that if the matrix $A(t)$ "dominates" the nonlinearity at every time and if $A(t)$ is a stability matrix then the equation 1.24 is asymptotically stable to the null solution. In order to characterise this dominance Pao considers the nonlinearity to be bounded in the following way

$$\langle f(x,t), x \rangle \leq k(t) \|x\|^2 \quad 1.25$$

and shows that if $\beta(t)$ is the smallest eigen-value of $-\frac{1}{2}(A(t)+A^T(t))$ and if $\beta(t) \geq k(t)$ then the system given by 1.24 is stable to the null solution.

Pao arrives at this result by considering an estimate of $\frac{d}{dt} \|x\|^2$ arising from the differential equation 1.24.

Integration of this inequality using a comparison function yields the estimate of $\|x(t)\|$ given by

$$\|x(t)\| \leq \exp \left\{ \int_{t_0}^t -\beta(s) + k(s) ds \right\} \|x_0\| \quad 1.26$$

Now if $-\beta(s)+k(s) \leq 0 \forall t \in (t_0, \infty)$ then $\|x(t)\| \rightarrow 0$ as $t \rightarrow \infty$ and the solution is asymptotically stable.

Pao extends this type of result to cover various situations. For example when the inner product bound upon f given by 1.25 is of the form

$$\langle f(x,t), Vx \rangle \leq k(t) \|x\|^2 \quad 1.27$$

where V is a positive definite symmetric matrix and if $A(t)$ is a stability matrix which exists as $t \rightarrow \infty$, i.e.

$\lim_{t \rightarrow \infty} A(t) = A$ exists, then a similar result concerning dominance is shown.

These papers represent two particular ways of studying the behaviour of differential equations. The approaches used will be reflected in the results presented in the following chapters.

Much of the classical stability theory for differential equations is developed in Hahn^{H1}, where most of the traditional stability techniques are outlined. For discussion upon differential and integral inequalities and their relevance to the stability problem see either Lakshmikantham and Leela^{L1} or Walter^{W1}.

Most of the essential theory in the area can be found well presented in the short monograph by Coppel^{C1}.

1.6 OUTLINE OF THE NEW RESULTS

We consider now the results presented in this thesis; they concern the controllability of the equation

$$\dot{x} = A(t)x(t) + B(t)u(t) + f(x,u,t) \quad 1.28$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$

and f is a continuous function taking values on $\mathbb{R}^n$.

The new results which follow consider the controllability of equation 1.28 by extending the ideas which have been presented in sections 1.5 and 1.6. Both Lovelady and Pao are concerned with the behaviour of differential equations which have no control term, but here we are interested in determining whether or not a control exists which ensures that the above system has an adequately stable behaviour. In order to structure the nonlinearity in the equation, various bounds upon $f(x,u,t)$ are introduced in Chapter Two and these are conditions not previously considered in discussing the controllability of equation 1.28. The approach used by Lovelady when describing the asymptotic behaviour of systems is to consider the norm of the system solutions. Pao, on the

other hand, describes the stability of differential equations through inner products. Correspondingly the bounds used upon $f(x,u,t)$ take two basic forms: those where norms are used to restrict $f(x,u,t)$ and those where inner products are used and the equivalent results are given in Chapters Three and Four and Chapters Five and Six. Throughout the following work a greater emphasis than usual is placed upon the measure of a matrix given by Dahlquist^{D1} and it is through this scalar function that we characterise the behaviour of the linear components in equation 1.28.

Chapter Three presents the first new controllability theorem for equation 1.28 and in this case we consider the nonlinearity to be bounded in norm by some scalar multiple of the norm of the solution. If the linear section of the equation is totally controllable then, by relating the solution of the nonlinear equation 1.28 to the solution of its linear counterpart 1.3, we show that this implies the controllability of the nonlinear equation using linear feedback control laws. As we do not assume that the nonlinearity $f(x,u,t)$ satisfies a Lipschitz condition a major part of the proof is taken up in showing that a controllable solution to equation 1.28 exists.

In Chapter Four we show how these results can be extended to give more general results for the case where the nonlinearity $f(x,u,t)$ is more easily bounded in norm by some combination of the norm of the solution and the norm of the control. In this case we consider a specific linear feedback control law which is closely related to that used by Gershwin & Jacobson and hence to the optimal control law for a linear system with

a quadratic cost function. Using this control law we develop two more theorems concerning the controllability of equation 1.28 when the nonlinearity is bounded in norm: in one case the bound consists of the sum of scalar multiples of the norm of the solution and the norm of the control and in the second case it consists of a scalar multiple of the product of these two norms. In this chapter we also show how the uniqueness of the controllable solution may be shown if the nonlinearity satisfies a Lipschitz condition.

These bounds upon $f(x,u,t)$ involving norms may in some cases be unnecessarily pessimistic and in Chapter Five the nonlinearity is considered to be bounded, not in norm but instead in an inner product with the state $x(t)$. Using some of the techniques adopted by Pao for his stability results, together with comparison functions and the measure given by Dahlquist, we derive some new controllability results concerning equation 1.28; here the inner product constraints on the nonlinearity have inner product terms on the right hand side involving either just the state or else both the state and the control. These results are extended to include the case where the inner products considered have structure characterised by the inclusion of a real positive definite symmetric matrix. This yields a result which is related to the more usual Lyapunov-type conditions.

A new theorem which gives a simple equivalence result for equation 1.28 when the nonlinearity can be shown to be weakly linear in $x(t)$ is presented in Chapter Six. A heuristic argument is given and this is backed up by a rigorous proof which follows the results of earlier chapters.

In the final chapter the main results are considered in the light of their development and we indicate ways in which they might be used to establish the controllability of a given nonlinear plant. In conclusion the relationships of the results obtained are discussed with respect to those given by previous authors and in the light of this discussion possible extensions of value are proposed.

CHAPTER 2

- 2.1 Inner products and norms.
- 2.2 The measure $\mu\{\cdot\}$ and its properties.
- 2.3 Derivatives, Lipschitz conditions and comparison functions.
- 2.4 An application of these techniques in stability theory.
- 2.5 Norm bounds.
- 2.6 Weak linearity bounds.
- 2.7 Inner product bounds.

2.1 INNER PRODUCTS AND NORMS

This chapter is almost exclusively concerned with the mathematical preliminaries for use in subsequent chapters. Both the stability problem and the controllability problem involve the estimation of vectors and matrices in order to predict the behaviour of nonlinear differential equations as time evolves. The solutions of these nonlinear equations take values in the linear vector space R^n and, in order to put some structure on R^n , we define the usual inner product of x and y ; with $x, y \in R^n$, and components x_i, y_i by:

$$\langle x, y \rangle \triangleq \sum_{i=1}^n x_i y_i \quad 2.1$$

Using this we can define a distance measure on R^n . First, however, it is convenient to define the measure from a more general point of view: any real valued scalar function, $|x|$, of $x \in R^n$ which satisfies the following three axioms is said to be a norm of vector x .*

- (1) $|x| \geq 0$ with $|x| = 0$ iff $x = 0$)
 - (2) $|\alpha x| = |\alpha| \cdot |x|$ for any scalar α)
 - (3) $|x + y| \leq |x| + |y| \quad \forall x, y \in R^n$)
- 2.2

The inner product defined by 2.1 may be used to construct a norm on R^n by considering $\langle x, x \rangle$. The resulting measure is called the Euclidean norm and is given by:

$$||x|| \triangleq \langle x, x \rangle^{\frac{1}{2}} = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}} \quad 2.3$$

It is clear that $||x||$ satisfies the three norm axioms. The norm of a square matrix D is any real valued function $|D|$

*NOTE: We use the symbol $|\cdot|$ to represent any general norm in R^n . However, when the absolute value of a scalar quantity is required we shall use the same symbol. The correct usage may be inferred from the context.

which satisfies the above norm axioms and also satisfies:

$$|D_1 \cdot D_2| \leq |D_1| \cdot |D_2| \quad 2.4$$

The norm chosen for the vectors in R^n may be used in forming a norm for matrices and this matrix norm is said to be subordinate to the vector norm, i.e.:

$$|D| \triangleq \sup_{|x| \neq 0} \frac{|Dx|}{|x|} = \sup_{|x|=1} |Dx| \quad 2.5$$

It is known that all norms on a finite dimensional space, such as R^n , are equivalent, i.e. if we choose any two possible norms, $|\cdot|_1$ and $|\cdot|_2$, then there exist two positive constants α and β such that:

$$\alpha |y|_1 \leq |y|_2 \leq \beta |y|_1 \quad \forall y \in R^n \quad 2.6$$

2.2 THE MEASURE $\mu(\cdot)$ AND ITS PROPERTIES

In subsequent chapters we shall make use of another measure for matrices which does not satisfy the three norm axioms, but is subordinate to the norm chosen for vectors. Consider the scalar function defined by:

$$\mu(D) \triangleq \lim_{h \rightarrow 0^+} \frac{|I+hD| - 1}{h} \quad 2.7$$

This function, sometimes known as the "logarithmic norm", was introduced by Dahlquist^{D1} and has been used to some extent in the stability literature (see Lakshmikantham and Leela^{L1}).

We make note of some properties of this scalar measure $\mu(\cdot)$: the function is convex and continuous for continuous D , its convergence is locally uniform and it can take negative values.

From the definition of $\mu(\cdot)$ we can see that

$$\begin{aligned} \mu(\alpha D) &= \alpha \mu(D) & \alpha > 0 \\ \mu(D+E) &\leq \mu(D) + \mu(E) \end{aligned} \quad \begin{array}{l}) \\) \\) \end{array} \quad 2.8$$

$$\text{and also } |\mu(D)| \leq |D| \quad 2.9$$

From equation 2.8 notice that, although $\mu(\cdot)$ can take negative values, it does satisfy the remaining two norm axioms: scalar multiplication and triangle inequality.

NOTE 2.1: Functions with these properties are called Kamke functions or K-norms in the stability literature and Walter^{W1} uses K-norms to derive stability and existence theorems for many classes of differential inequality.

Using 2.9 and 2.8 we can also see that:

$$|\mu(A) - \mu(B)| \leq |A-B| \quad 2.10$$

If we choose the Euclidean norm for vectors then two interesting characterisations of the matrix norm and $\mu(\cdot)$ are available.

$$\text{If } ||x|| = \left(\sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}$$

$$\begin{aligned} \text{then } ||D|| &= \text{square root of largest eigenvalue of } D^T D \\ \text{and } \mu(D) &= \text{largest eigenvalue of } \frac{1}{2}(D^T + D) \end{aligned} \quad \begin{array}{l}) \\) \end{array} \quad 2.11$$

Thus, when the norm chosen for vectors is Euclidean, then $\mu(\cdot)$ is some measure of the symmetric part of a matrix.

These two characterisations of the matrix norm and $\mu(\cdot)$ are related by an inequality which simplifies to an equality when D is a normal matrix.

Consider first the case for the equality:

We define the symmetric part of D by

$$D_S \triangleq \frac{1}{2}(D + D^T) \text{ and the unsymmetric part of } D \text{ by}$$

$$D_U \triangleq \frac{1}{2}(D - D^T); \text{ and notice that } D_U, \text{ being skew-symmetric}$$

has the following property:

$$D_U^T = -D_U$$

Now D_U and D_S commute if and only if D is a normal matrix (i.e. iff $DD^T = D^TD$).

$$\begin{aligned} \text{So if we consider } ||e^D||^2 &= \lambda_{\max} \left(e^{D_U} e^{D_S} e^{D_S^T} e^{D_U^T} \right) \\ &= \lambda_{\max} (e^{2D_S}) \quad \text{iff } D \text{ is normal} \end{aligned}$$

$$\text{Then } ||e^D||^2 = \exp \{ \lambda_{\max} (2D_S) \} = \exp \{ \lambda_{\max} (D+D^T) \}$$

$$\text{Hence } ||e^D|| = \exp \{ \mu(D) \} \quad \text{iff } D \text{ is normal} \quad 2.12$$

The following proof of the inequality relationship for these characterisations, when D is not a normal matrix, follows Wiess and Lee^{W3}.

The Rayleigh quotient representation of an eigenvalue (see ~~2-6~~⁸³) yields:

$$\begin{aligned} 2\mu(D) &= \max_{||x|| \neq 0} \frac{\langle x, (D^T+D)x \rangle}{||x||^2} \\ &= \max_{x(\cdot)} \int_0^1 \frac{\langle x(t), (D^T+D)x(t) \rangle}{||x(t)||^2} dt \end{aligned}$$

And if we make the limitation $\dot{x} = Dx(t)$

$$\begin{aligned} &> \max_{\substack{x(\cdot) \\ \dot{x}=Dx}} \int_0^1 \frac{\dot{x}^T(t)x(t) + x^T(t)\dot{x}(t)}{||x(t)||^2} dt \\ &> \max_{x_0} \int_{x_0}^{x(1)} \frac{d(||x(t)||^2)}{||x(t)||^2} \geq \max_{x_0} \ln \frac{||x(1)||^2}{||x_0||^2} \\ &= \ln \max_{x_0} \frac{||e^D x_0||^2}{||x_0||^2} \geq \ln(||e^D||)^2 \end{aligned}$$

$$\text{So we have } 2\mu(D) \geq \ln (||e^D||)^2$$

$$\mu(D) \geq \ln ||e^D|| \quad 2.13$$

with equality if $D^TD = DD^T$, i.e. if D is normal.

It is due to this relationship that the measure $\mu(\cdot)$ may be called a logarithmic norm.

From the computational viewpoint $\mu(\cdot)$ is a very useful estimate of $||e^D||$ and has the advantage that it only requires

exponentiation of a scalar rather than exponentiation of a matrix. The application of either measure requires eigenvalue evaluation.

2.3 DERIVATIVES, LIPSCHITZ CONDITIONS & COMPARISON FUNCTIONS

The nonlinear equations that we consider are not necessarily differentiable and in order to overcome this difficulty we consider one of the four Dini derivatives that is given by:

$$D^+M(t) \triangleq \limsup_{h \rightarrow 0^+} \frac{1}{h} (M(t+h) - M(t)) \quad 2.14$$

This is an upper righthand derivative of $M(t)$, some vector valued function; there also exist lower and left hand derivatives and these are denoted by D_+, D^-, D_- . We shall only consider the derivative given by 2.14⁴ in this thesis. We note that if a derivative exists for some function $M(t)$ then it also exists for $|M(t)|$. (See Coppel^{C1}).

When considering the stability of

$$\dot{x} = f(x, t) \quad 2.15$$

a Lipschitz condition upon f in x is often demanded, this will guarantee the existence and uniqueness of solutions to 2.15. If f satisfies the following inequality then it is said to be Lipschitz in x :

$$||f(x_1, t) - f(x_2, t)|| \leq L ||x_1 - x_2|| \quad 2.16$$

The Lipschitz constant L is a bound upon the absolute value of $f_x(x, t)$ and hence contains no information as to whether the solutions are increasing or decreasing. To avoid this we could consider a one-sided Lipschitz condition of the form (for x scalar)

$$f(x_1, t) - f(x_2, t) \leq L(x_1 - x_2) \quad \text{for } x_1 > x_2 \quad 2.17$$

where we now allow L to be negative.

For some common nonlinear ordinary differential equations the right hand side of the equation does not satisfy a Lipschitz condition. The equation

$$\dot{x} = x^2 \quad \text{with } x \in \mathbb{R} \quad 2.18$$

is a simple example. In this case

$$f(x_1, t) - f(x_2, t) = x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2)$$

which is not $\leq L(x_1 - x_2)$ for $x_1 > x_2$.

A common procedure to extend the Lipschitz condition to cover equations of this type is to consider the function $f(x, t)$ only on some set for which x is bounded. Call $D \subset \mathbb{R}^n$ the set of all x with norm less than $\bar{x}$ say, i.e.

$$D \triangleq \{x \mid x \in \mathbb{R}^n, \|x\| < \bar{x}\} \quad 2.19$$

Then in the above example we can see that the Lipschitz condition is satisfied for $x_1, x_2 \in D$ with $L \geq 2\bar{x}$. The bounds that we introduce upon $f(x, u, t)$ at the end of this chapter will be treated in a similar way, so that they may be extended to fit functions which do not contain only linear terms.

The estimate of the growth rate of the solution to the nonlinear equation, arrived at by using some of the above techniques, generally takes the form of a differential inequality. An important method of extending this to yield an estimate of the solution to the inequality involves looking at the extremal solutions of the corresponding differential equation. This theory hinges around comparison theorems which are extensively discussed in Lakshmikantham & Leela^{L1}.
Theorem 2.1 (LAKSHMIKANTHAM & LEELE)

Let D be some open set in $\mathbb{R}$ and let $g(x, t)$ be a continuous scalar function defined on $(t_0, t_0 + a) \times D$ over which the maximal

solution $r(t)$ of

$$\dot{r} = g(r, t) \quad r(t_0) = r_0 \quad \text{exists everywhere.}$$

Let $m(t)$ be a scalar function defined on (t_0, t_0+a) with initial condition $m(t_0) \leq r_0$. Then for some fixed Dini derivative (say D^+)

$$D^+m(t) \leq g(m(t), t) \quad \forall t \in (t_0, t_0+a)$$

implies

$$m(t) \leq r(t) \quad \forall t \in (t_0, t_0+a)$$

This comparison theorem is one of the simplest as it involves only scalar functions. For the proof of this theorem and others in a similar vein see (L1).

Comparison theorems of this type say that if some unknown function, $m(t)$, has an initial condition less than some known function, $r(t)$, and grows less quickly than $r(t)$, then over the whole range of definition of $r(t)$, the unknown function, $m(t)$, is always less than $r(t)$. In many cases we use this theorem to avoid outright integration of a differential inequality and then the unknown function is the derivative on the lefthand side of the inequality and the known function is the assumed solution of the differential equality that is the upper bound. Here the initial conditions are equal.

This type of theorem makes it possible to avoid some of the usual problems associated with Lyapunov functions. For instance, if the function $m(t)$ were replaced by a Lyapunov function $V(x, t)$, then the theorem will yield an estimate of $V(x, t)$ for any given solution $x(t)$ and by choice of end point conditions upon $V(x, t)$ it is possible to avoid the requirement that the Lyapunov function be positive definite with non-positive derivative.

2.4 AN APPLICATION OF THESE TECHNIQUES IN STABILITY THEORY

In order to illustrate the means by which a stability result may be derived using some of the above techniques let us consider the stability of the differential equation

$$\dot{x}(t) = A(t)x(t) \quad 2.20$$

with initial condition $x(t_0) = x_0$

From the stability viewpoint we are concerned with the rate of change of $||x(t)||$ so let us consider

$$D^+ ||x(t)|| = \limsup_{h \rightarrow 0^+} \frac{1}{h} (||x(t+h)|| - ||x(t)||) \quad 2.21$$

Or from equation 2.20

$$D^+ ||x(t)|| = \limsup_{h \rightarrow 0^+} \frac{1}{h} (||x(t) + h A(t)x(t)|| - ||x(t)||) \quad 2.22$$

$$\leq \limsup_{h \rightarrow 0^+} \frac{1}{h} (||I + hA(t)|| ||x(t)|| - ||x(t)||) \quad 2.23$$

From the definition of the logarithmic norm $\mu(\cdot)$ we can see that:

$$D^+ ||x(t)|| \leq \mu(A(t)) ||x(t)|| \quad 2.24$$

We postulate now the existence of a differential equation

$$\dot{r} = g(r, t) = \mu(A(t))r(t) \quad 2.25$$

with initial condition $r(t_0) = ||x_0||$

As $\mu(A(t))$ is continuous if $A(t)$ consists of all continuous elements, let us assume that equation 2.25 has a unique (maximal) solution. This will certainly be true if equation 2.20 has a unique solution. Now using the comparison theorem we can say that

$$||x(t)|| \leq r(t) \quad \forall t \in (t_0, \infty) \quad 2.26$$

But equation 2.25 has a well known solution given by:

$$r(t) = r(t_0) \exp \left\{ \int_{t_0}^t \mu(A(s)) ds \right\} \quad 2.27$$

Substitution of 2.27 and the initial condition into 2.26 gives

$$||x(t)|| \leq ||x_0|| \exp \left\{ \int_{t_0}^t \mu(A(s)) ds \right\} \quad 2.28$$

NOTE 2.2: If we make the change of variable $t=-t$ we can derive a lower bound for $||x(t)||$ given by

$$||x_0|| \exp \left\{ - \int_{t_0}^t \mu(-A(s)) ds \right\} \leq ||x(t)|| \quad 2.29$$

From the above development, notice that the stability behaviour of the vector equation 2.20 is governed by the stability behaviour of scalar equation 2.25. This, in turn, depends upon the estimate of $||x(t)||$ given by 2.28.

Hence we can say that equation 2.25 is stable, that is to say bounded as $t \rightarrow \infty$, if:

$$\limsup_{t \rightarrow \infty} \int_{t_0}^t \mu(A(s)) ds < \infty \quad 2.30$$

and that it is asymptotically stable, that is to say eventually the solution will become arbitrarily close to the zero solution, if

$$\lim_{t \rightarrow \infty} \int_{t_0}^t \mu(A(s)) ds = -\infty \quad 2.31$$

If we wish to imply that the solution is always decreasing in norm as time evolves, that is uniformly asymptotically stable, then we need to examine equation 2.28 at every time, i.e.

$$\mu(A(s)) \leq -\alpha < 0 \quad \forall t > t_0 \quad 2.32$$

The above results are well known in stability theory and their derivation is given here in some detail only to

illustrate the use of the mathematical tools we have introduced. Notice that, although the results have been given using the Euclidean norm for vectors and hence $\mu(\cdot)$ subordinate to this norm, the above analysis will hold equally well for any consistent norm in R^n .

2.5 NORM BOUNDS

This section introduces some restrictions which may be imposed upon the vector valued function $f(x,u,t)$ so that similar procedures may be developed to examine the behaviour of the control differential equation:

$$\dot{x} = A(t)x + B(t)u + f(x,u,t) \quad 2.33$$

Some qualitative properties of this equation, such as controllability and asymptotic behaviour, may be implied from the linear counterpart if the nonlinear function f is bounded in some way. We shall always consider $f(x,u,t)$ to be a continuous function taking values in R^n :

more rigorously $f \in C(R^n \times R^m \times (t_0, \infty), R^n)$

When these bounds are used to derive controllability results we shall mainly be concerned with linear feedback control policies of the form $u(t) = K(t)x(t)$, where $K(t)$ is a feedback matrix of appropriate dimension.

Let Ω represent those $u(t)$ which are constructed in this way:

$$\Omega \triangleq \{ u(t) \mid u(t) = K(t)x(t); u \in R^m, x \in R^n, t \in (t_0, \infty) \} \quad 2.34$$

Also let Ω_k represent a more restricted set of admissible controls parameterised by the maximum allowable norm of the feedback matrix:

$$\Omega_k \triangleq \{ u(t) \mid u(t) \in \Omega, \|K(t)\| \leq k \} \quad 2.35$$

Consider first the Lipschitz condition, given in equation 2.16 applied to the function $f(x,u,t)$:

$$\begin{aligned} ||f(x_1,u,t)-f(x_2,u,t)|| &\leq L ||x_1-x_2|| & 2.36 \\ \forall x_1,x_2 &\in \mathbb{R}^n \\ \forall u &\in \Omega \\ \forall t &\in (t_0,\infty) \end{aligned}$$

Here f satisfies a Lipschitz condition in x for every $u \in \Omega$, and it is known (see Bellman ⁸³~~2.6~~) that this is enough to guarantee the existence and uniqueness of a solution to the control equation 2.33 using continuous control policies. If we consider bounds which do not imply that f is Lipschitz in x , then not only will we be concerned with showing that a particular initial state has a controllable solution, but also in showing that such a solution exists.

We often require that the solution $x(t) \equiv 0$ be a trivial solution of the equation 2.33 in order that if at any time, $t = \bar{t}$ say, the solution should have value $x(\bar{t}) = 0$ then this implies $x(t) = 0 \forall t > \bar{t}$.

This property requires that we demand

$$f(0,0,t) = 0 \quad \forall t \in (t_0,\infty) \quad 2.37$$

A weaker bound upon f than the Lipschitz condition and one which still makes some restriction upon the growth rate of f but does not imply differentiability is given by

$$\begin{aligned} ||f(x,u,t)|| &\leq p(t) ||x(t)|| & 2.38 \\ \forall x &\in D \subset \mathbb{R}^n \\ u &\in \Omega_k, \quad t \in (t_0,\infty) \end{aligned}$$

Notice first that this inequality can be minimised at each time using $p(t)$ rather than minimised over the whole interval using a constant, say, L . It is of course possible

to replace $p(t)$ by $L = \sup_{t \in (t_0, \infty)} p(t)$ and the inequality will still be valid. Note, also, that we have restricted $u(t)$ to be contained in Ω_k rather than the more general class Ω and thus $p(t)$ is implicitly a function of k . However, by restricting both $u(t)$ and $x(t)$ to lie in closed domains in their respective spaces we can always find a scalar $p(t)$ which will satisfy equation 2.38. This restriction, of $x \in D$ and $u(t) \in \Omega_k$, is analogous to the restrictions made in order to extend the Lipschitz condition to cover functions which are not necessarily less than linear. When using this bound upon f to show null controllability of nonlinear systems we will ensure that $x(t)$ is contained in D throughout and the choice of the restricted class of feedback controls Ω_k will ensure some bounded region of control.

This inequality bound upon f can be made more general by explicitly including the control term on the right hand side. In this case we need not restrict the feedback matrix norm 'a priori' and can admit controls in Ω .

We may thus consider

$$\|f(x, u, t)\| \leq p(t) \|x(t)\| + q(t) \|u(t)\| \quad 2.39$$

$$\forall x \in D \subset \mathbb{R}^n$$

$$u \in \Omega, t \in (t_0, \infty)$$

If some knowledge of the overall performance of the non-linearity exists then it is possible to choose between different inequality bounds such as 2.38 and 2.39 in order to reduce the scalar multipliers $p(t)$ and $q(t)$ for any given nonlinearity.

With this in mind we can also look at the inequality

bound given by

$$\|f(x, u, t)\| \leq p(t) \left(\|x(t)\| \cdot \|u(t)\| \right)^{\frac{1}{2}} \quad 2.40$$

$\forall x \in D \subset \mathbb{R}^n$

$u \in \Omega, t \in (t_0, \infty)$

For nonlinearities with dominant terms which are multiplicative in nature this inequality may well yield a smaller value of $p(t)$ to be used in the subsequent analysis. To this extent the theory is adaptable to fit the nonlinearity in the most appropriate way and is consistent with the body of opinion claiming that nonlinearities should be treated individually without any attempts at general theories.

It is interesting to note that the control term $\|u(t)\|$ appears explicitly on the right hand side of the inequalities 2.39 and 2.40 and this provides some motivation for considering the minimum energy control which will also satisfy the controllability boundary value conditions. We are interested in keeping the term $\|u(t)\|$ as small as possible commensurate with $u(t)$ being an acceptable control for the system. In this context we can relate the control policy which would be chosen to the controllability matrix given by Kalman and discussed in Chapter One:

$$W(t) = \int_{t_0}^t \phi(t, \tau) B(\tau) B^T(\tau) \phi^T(t, \tau) d\tau \quad 2.41$$

and we can also see a relationship with $\det\{W^{-1}(t)\}$ which has been used to characterise the degree of controllability of linear systems (see Johnson ^{J1}).

The three norm bounds upon $f(x, u, t)$ given above do not allow, at any time instant, negative terms to occur on the right hand side. These bounds are similar to the Lipschitz

condition 2.16, in that they are restrictions upon the absolute size of the nonlinearity. They ignore any cases where the nonlinearity may be stabilizing. It will later be shown that the solution to the equation 2.33 is related in norm to the solution of the corresponding linear equation 1.3 by the time integral of the right hand side of the bound used to restrict the nonlinearity. As in the stability results developed in section 2.4 this will be via the monotonic exponential operator. Thus it would be useful to be able to accept negative values of $p(t)$ which would be reflected in the estimate of $D^+||x(t)||$, via the exponential.

Rather than just measuring the size of the nonlinearity we wish to also incorporate some knowledge of its direction into the inequalities. To this end we shall now consider some inner product bounds upon $f(x,u,t)$.

2.6 WEAK LINEARITY BOUNDS

As an introduction to bounds involving inner products let us first consider the weak linearity condition. Although this bound includes inner products it does not exhibit the property we desire, namely negative values of $p(t)$. It is, however, an interesting bound to consider as it incorporates some information about the direction of f . The nonlinearity $f(x,u,t)$ is said to be weakly linear if there exists some $p(t) > 0$ such that:

$$\langle f(x,u,t), \tilde{z} \rangle \leq p(t) | \langle x, z \rangle | \quad 2.42$$

$$\forall x \in DCR^n$$

$$u \in \Omega_k$$

$$t \in (t_0, \infty)$$

$$\forall z \in R^n$$

We are obviously concerned with the way this condition relates to those given previously. First it is interesting to note that weak linearity does not imply that a function is Lipschitz, nor is the converse true.

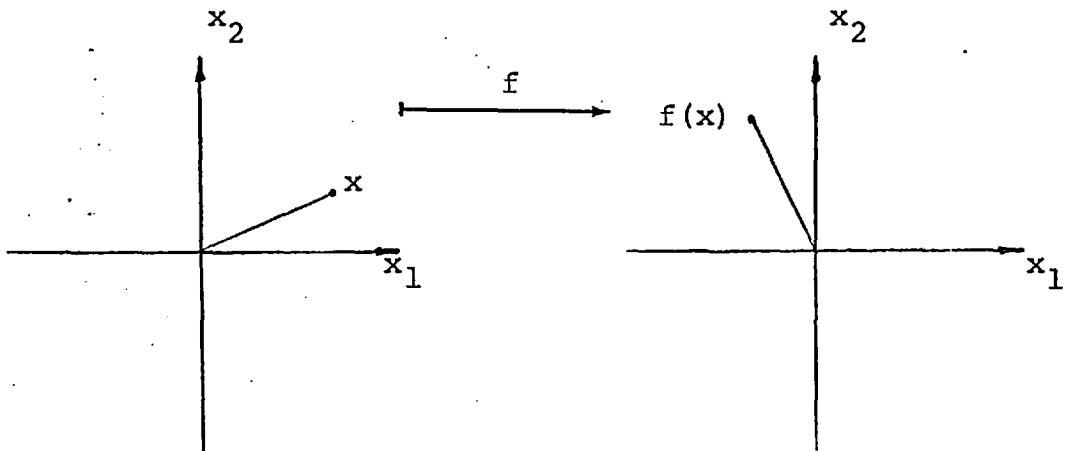
To clarify this point consider the following two examples.

Example 2.1 A function $f(x,u,t)$ that is Lipschitz in x but is not weakly linear:

Consider the rotation in R^2 given by

$$f(x,u,t) = f \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad 2.43$$

f here is independent of u and t and takes vectors x in R^2 and rotates them anticlockwise through $\pi/2$



a) To show that $f(x,u,t)$ defined in equation 2.43 is Lipschitz:

Choose any two vectors in R^2 , say: $(x_1, x_2)^T$, $(y_1, y_2)^T$ and then the left hand side of the Lipschitz inequality 2.16 becomes:

$$||f(x,u,t) - f(y,u,t)|| = (-x_2 + y_2)^2 + (x_1 - y_1)^2$$

Now the right hand side of the inequality is:

$$L ||x(t) - y(t)|| = L \left((x_1 - y_1)^2 + (-x_2 + y_2)^2 \right)^{\frac{1}{2}}$$

So we can see that any Lipschitz constant $L \geq 1$ will satisfy inequality 2.16 for the particular case when $f(x,u,t)$ is defined by 2.43.

b) To show that $f(x,u,t)$ defined in equation 2.43 is not weakly linear we need only consider any vector in R^2 , say $(x_1, x_2)^T$. Now the left hand side of the weak linearity condition is

$$\langle f(x,u,t), z \rangle = (-x_2, x_1) \cdot (z_1, z_2)^T \quad \forall z \in R^2$$

If we make the particular choice of $z = (-x_2, x_1)^T$ then:

$$\langle f(x,u,t), z \rangle = ||(-x_2, x_1)||^2$$

However for this choice of z the right hand side of the weak linearity condition becomes:

$$|\langle x, z \rangle| = |(x_1, x_2) \cdot (-x_2, x_1)^T| = 0$$

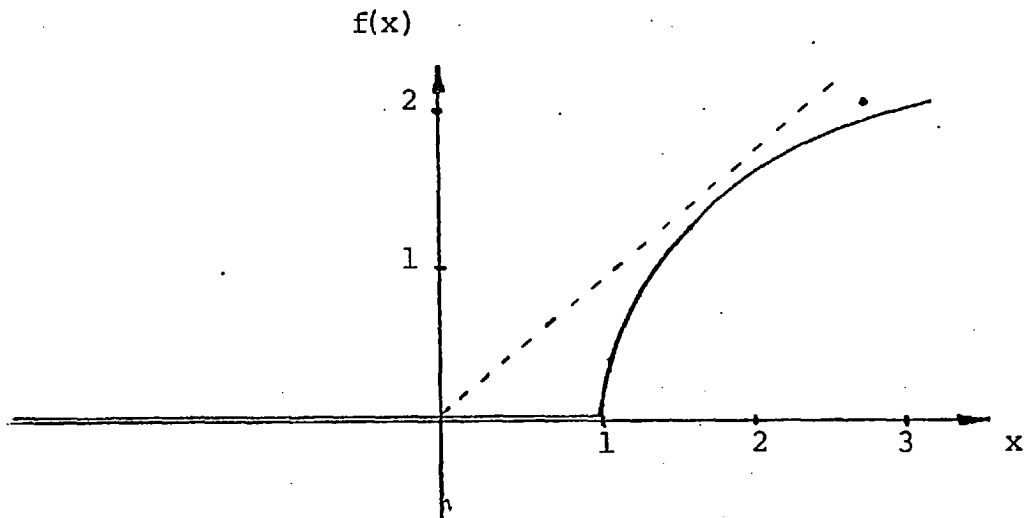
As there is no choice of $p(t)$ ($\neq \infty$) which will satisfy inequality 2.42 for this case, we can see that this particular $f(x,u,t)$ is not weakly linear.

Example 2.2 A function of $f(x,u,t)$ that is weakly linear but is not Lipschitz in x .

Consider the function in R given by

$$f(x,u,t) = \begin{cases} 0 & \forall x < 1 \\ \frac{3}{2} (x-1)^{1/3} & \forall x \geq 1 \end{cases} \quad 2.44$$

In this example f is again independent of u and t and has the following graph:



a) To show that $f(x,u,t)$ defined in 2.44 is weakly linear:

Consider this function when $x < 0$. Then in this case the weak linearity condition

$\langle f(x,u,t), z \rangle \leq p(t) |\langle x, z \rangle|$ is trivially satisfied for any positive $p(t)$. When $x \geq 1$ the weak linearity condition becomes:

$$\frac{3}{2} (x-1)^{1/3} \cdot z \leq p(t) \cdot |x \cdot z|$$

Due to the positivity of both x and $f(x,u,t)$ we can cancel z without loss of generality and then:

$$\frac{3}{2} (x-1)^{1/3} \leq p(t) \cdot x.$$

By consideration of the graph we can see that there will be a $p(t)$ which will satisfy this inequality for $x > 0$.

b) To show that $f(x,u,t)$ as defined in 2.44 is not Lipschitz in x .

Consider any two points $x_1, x_2 \in \mathbb{R}$. Then substitution in the Lipschitz condition will yield:

$$\left| \frac{3}{2} (x_1-1)^{1/3} - \frac{3}{2} (x_2-1)^{1/3} \right| \leq p(t) |x_1 - x_2|$$

if we make the change of variable $y_1 = (x_1-1)$; $y_2 = (x_2-1)$

this becomes:

$$\left| \frac{3}{2} y_1^{1/3} - \frac{3}{2} y_2^{1/3} \right| \leq p(t) |y_1 - y_2|$$

Now, without loss of generality, choose $y_1 \geq y_2 \geq 0$ and we can then write:

$$\frac{3}{2} (y_1^{1/3} - y_2^{1/3}) \leq p(t) (y_1 - y_2)$$

We need a $p(t)$ such that:

$$\begin{aligned} \frac{2}{3} p(t) &\geq \left(\frac{y_1^{1/3} - y_2^{1/3}}{y_1 - y_2} \right) = \left(\frac{y_1 - y_2}{y_1^{1/3} - y_2^{1/3}} \right)^{-1} \\ &= (y_1^{2/3} + y_1^{1/3} y_2^{1/3} + y_2^{2/3})^{-1} \end{aligned}$$

which is unbounded as $y_1, y_2 \rightarrow 0$.

Thus the Lipschitz constant for 2.44 needs to be unbounded at $x=1$ and hence the function does not satisfy a Lipschitz condition.

We may extend the weakly linear condition 2.42 for $f(x, u, t)$, in the same way as with the norm inequality 2.38, to explicitly involve the control term on the right hand side. In order to match the dimensions of the general n vector z and the control $u(t)$, which has dimension m , the scaling factor $q(t)$ is generalised in this inner product term, in the form of the $m \times n$ matrix $Q(t)$:

$$\langle f(x, u, t), z \rangle \leq p(t) |\langle x, z \rangle| + |\langle Q(t)u(t), z \rangle| \quad 2.45$$

$$\forall z \in \mathbb{R}^n, x \in D \subset \mathbb{R}^n$$

$$\forall u \in \Omega$$

$$t \in (t_0, \infty)$$

These weak linearity conditions, although incorporating some information about the direction of f , are still restricted to positive definite terms on the right hand side. In fact the norm inequality 2.38 is a less restrictive condition upon $f(x, u, t)$ than the weak linearity condition 2.42, for any function f which satisfies 2.38 will also

satisfy 2.42.

Consider the weak linearity condition:

$$\langle f(x,u,t), z \rangle \leq p(t) |\langle x, z \rangle|$$

For the particular choice $z = f(x,u,t)$ this becomes:

$$||f(x,u,t)||^2 \leq p(t) |\langle x(t), f(x,u,t) \rangle|$$

or equivalently

$$||f(x,u,t)||^2 \leq p(t) ||x(t)|| \cdot ||f(x,u,t)||$$

Division by $||f(x,u,t)||$ gives the norm bound

$$||f(x,u,t)|| \leq p(t) ||x(t)||.$$

The following two diagrams indicate the properties of these two inequalities and the possible range of values of $f(x,u,t)$ they admit for various $p(t)$:

Figure 2.1 Circles parameterised by $p(t)$ for the norm bound 2.38 and typical vector $x(t) \in \mathbb{R}^2$

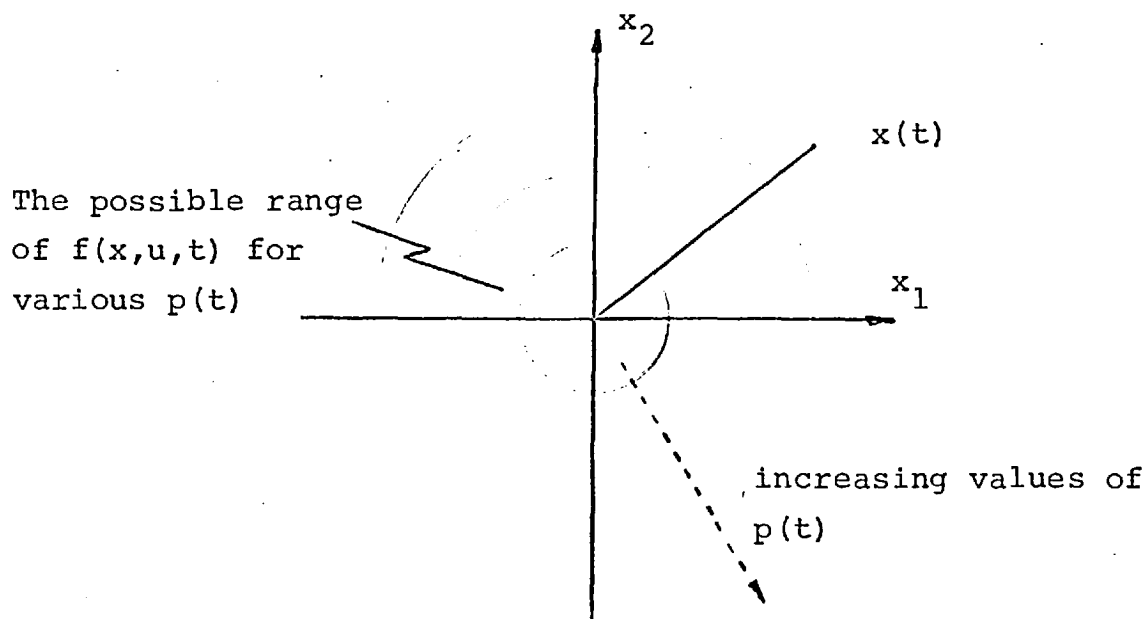
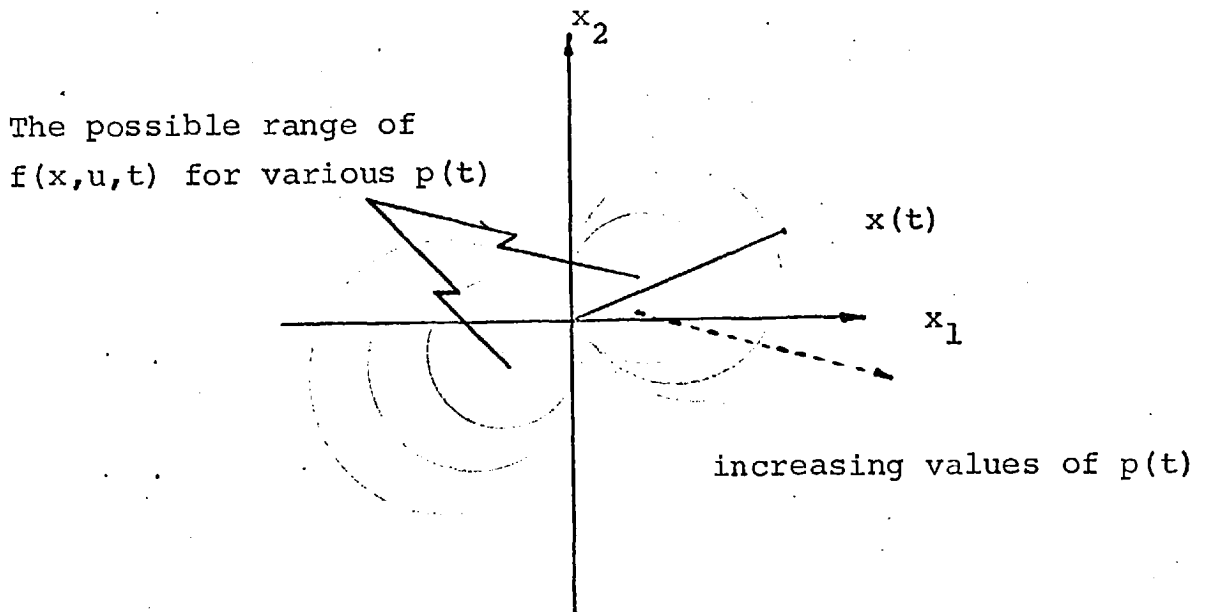


Figure 2.2 Circles parameterised by $p(t)$ for the weak linearity bound 2.44 and typical vector $x(t) \in \mathbb{R}^2$



The relationship shown above for the norm inequality 2.38 and the weak linearity condition 2.44 can be readily extended to cover the two equivalent conditions which have $u(t)$ explicitly on the right hand side: inequality 2.39 and inequality 2.45.

2.7 INNER PRODUCT BOUNDS

The specific choice of $z = f(x,u,t)$ in the weak linearity condition yields the norm bound 2.39. An alternative choice for z , in order to generalise the weak linearity condition, is $z = x(t)$. This substitution yields the following bound:

$$\langle f(x,u,t), x(t) \rangle \leq p(t) \|x(t)\|^2 \quad 2.46$$

$$\forall x \in \mathbb{D} \subset \mathbb{R}^n$$

$$u \in \Omega_k$$

$$t \in (t_0, \infty)$$

In this case we are relating the projection of $f(x,u,t)$ in the direction of $x(t)$ to the size of the vector $x(t)$ itself. So this bound incorporates the information about the size of the nonlinearity f and its direction - both relative to the state vector $x(t)$. Obviously, as we have made a specific choice for the arbitrary vector z , this bound is implied by the weak linearity bound. It is also implied by the norm bound 2.38:

The left hand side of the inner product bound 2.46 can always be written as:

$$\langle f(x,u,t), x(t) \rangle \leq \|f(x,u,t)\| \cdot \|x(t)\|$$

and if we assume f satisfies 2.38 then substitution on the right hand side yields:

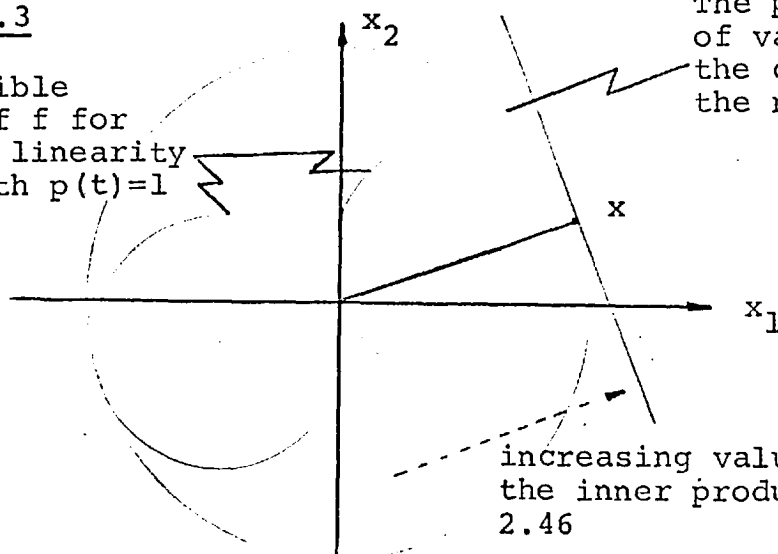
$$\langle f(x,u,t), x(t) \rangle \leq p(t) \|x(t)\|^2$$

In inequality 2.46 negative values of the scalar $p(t)$ are permitted and this gives some additional information about the direction of $f(x,u,t)$; a negative $p(t)$ will imply that f is a stable function.

The following diagram illustrates the possible range of values of $f(x,u,t)$ when parameterised by $p(t)$ for the case $f(x,u,t) \in \mathbb{R}^2$:

Figure 2.3

The possible values of f for the weak linearity bound with $p(t)=1$



The possible range of values of f for the case $p(t)=1$ in the norm bound

increasing values of $p(t)$ in the inner product inequality 2.46

This diagram also shows the relationship between inequalities that we have just shown theoretically. Any $f(x,u,t)$ which satisfies the weak linearity condition will also satisfy the norm inequality and hence will also satisfy the inner product inequality.

Notice that we can also introduce yet another inequality bound upon $f(x,u,t)$ which is the extension of the inner product condition 2.46 to include a term $\langle Q(t)u(t), x(t) \rangle$ on the right hand side where $Q(t)$ has appropriate size.

Thus:

$$\langle f(x,u,t), x(t) \rangle \leq p(t) \|x(t)\|^2 + q(t) |\langle Q(t)u(t), x(t) \rangle| \quad 2.47$$

$$\forall x \in DCR^n$$

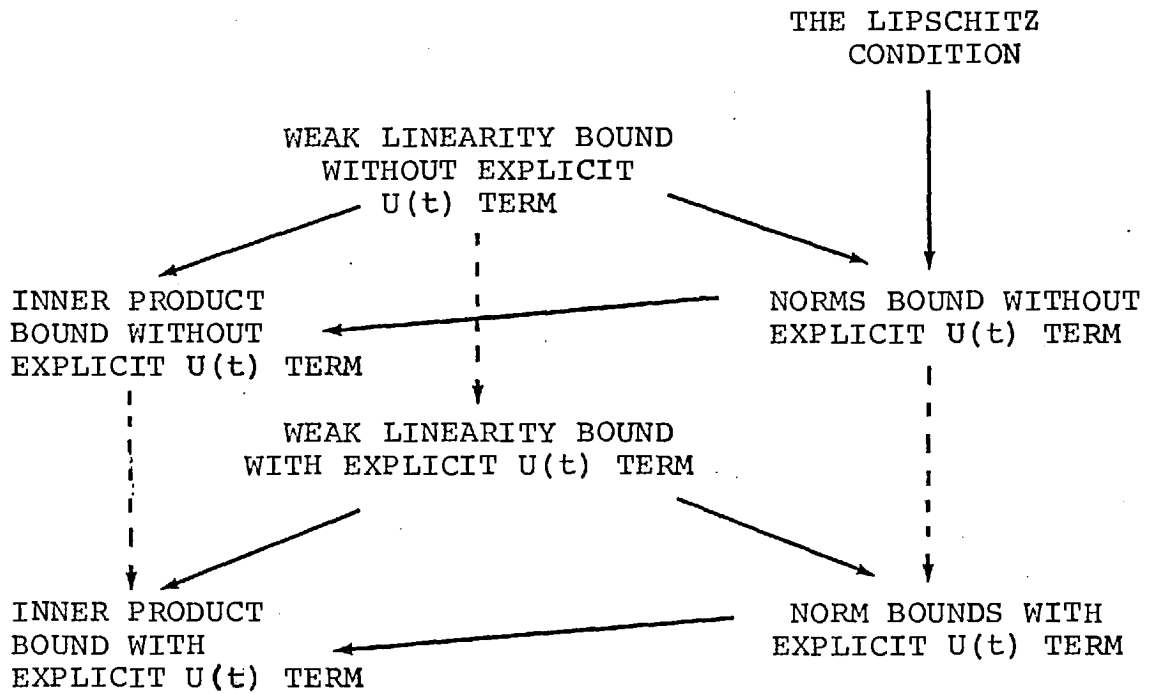
$$u \in \Omega$$

$$t \in (t_0, \infty)$$

Again both $p(t)$ and $q(t)$ can take negative values and the matrix $Q(t)$ ensures that the inner product can be formed. Some special choices for $Q(t)$ can be made when using 2.47.

The relationships between the above bounds upon $f(x,u,t)$ are illustrated in the following diagram. A full line ($\longrightarrow$) indicates that any function $f(x,u,t)$ which satisfies the bound at the starting point will also satisfy the bound at the end of the line. A broken line ($---\rightarrow$) indicates that the bound at the end of the line is more general than the bound at the start of the line, due to the introduction of control terms specifically on the right hand side.

Figure 2.4



In conclusion we add that when considering controllability in finite time we may be satisfied with these bounds upon $f(x,u,t)$ holding over the interval (t_0, T) . This will not restrict the functions in any additional way and may admit a function not previously covered by these inequalities.

CHAPTER 3

- 3.1 Introduction.
- 3.2 $\mu\{\cdot\}$ and controllable linear systems.
- 3.3 Statement of a null controllability theorem.
- 3.4 Proof of the null controllability theorem.
 - 3.4.1 Some initial definitions.
 - 3.4.2 The continuity of map L .
 - 3.4.3 $L\{D(2\sigma)\}$ is an equicontinuous family of functions.
 - 3.4.4 Comparison of the solutions to equations 3.1 and 3.2.
 - 3.4.5 Extension of this property to cover (t_0, ∞) .
- 3.5 Discussion of the null controllability result.
- 3.6 A stronger null controllability result.

3.1 INTRODUCTION

This chapter introduces the definitions of controllability used to describe the control behaviour of the nonlinear differential equations with which we are concerned. A basic lemma is presented relating the logarithmic norm of $A(t)+B(t)K(t)$ for a linear system to the controllability properties of this system and, using this result, the first null controllability theorem for nonlinear systems is established.

Since we need to distinguish between the solutions and controls of the nonlinear and linear systems the following notation will be used throughout: we shall use $y(t)$ to represent the solution to the nonlinear system which is given by

$$\dot{y}(t) = A(t)y(t) + B(t)u_y(t) + f(y, u_y, t) \quad 3.1$$

where the state $y(t) \in \mathbb{R}^n$ and the control $u_y(t) \in U_y \subset \mathbb{R}^m$ with $t \in (t_0, \infty)$

The nonlinearity $f(y, u_y, t)$ is a vector valued function taking values in $\mathbb{R}^n$ and $A(t)$ and $B(t)$ are matrices of continuous, *banded*, possibly time-varying, elements which have appropriate dimensions. Usually $f(y, u_y, t)$ will be considered to be continuous, but not necessarily differentiable, in y for every u_y and t .

The linear system, whose solution $x(t)$ will be used to establish the controllability of equation 3.1, is given by

$$\dot{x}(t) = A(t)x(t) + B(t)u_x(t) \quad 3.2$$

where the state $x(t) \in \mathbb{R}^n$ and the control $u_x(t) \in U_x \subset \mathbb{R}^n$ with $t \in (t_0, \infty)$

The matrices $A(t)$ and $B(t)$ are those defined in

equation 3.1 and hence this equation describes the linear dynamics in the nonlinear equation 3.1.

The theorem will show the existence of a linear state variable feedback control for equation 3.1 with initial condition $y(t_0) = y_0$ such that a solution to the nonlinear equation exists and that given an arbitrary $\epsilon > 0$ then there is some time t_f when $\|y(t_f)\| < \epsilon$. In order to formalise these ideas we now introduce the relevant definitions of controllability. These definitions embody the ideas of those given in Chapter One, but have the advantages of suitability and simplicity. None of the concepts put forward here are new and in fact, by small modifications to the theorems and their proofs, many of the results may be stated using controllability definitions given by other authors. In addition to the following definitions we shall also require the definitions of complete, total and uniform controllability stated in Chapter One.

When considering the nonlinear system 3.1 we are interested in controllability to within some neighbourhood of the trivial solution $y(t) = 0$. This restricted form of controllability can be defined as

DEFINITION 3.1

The nonlinear system 3.1 is said to be "null controllable" on (t_0, T) from y_0 , if for any $\epsilon > 0$ there exists an admissible control $u_y(t)$ and a time $t_f \in (t_0, T)$ such that $y(t_0) = y_0$ implies $\|y(t_f)\| < \epsilon$.

This definition is similar to that given by Ben Lashier and Storey^{B2}; in their case, however, the initial condition y_0 is not defined specifically but restricted to be contained

in some domain α , where α is usually defined by the sphere round the origin given by $||y(t_0)|| < a$.

The other slight difference between the definitions is that whereas Ben Lashiher and Storey fix the final time in the definition statement, in our case the time at which the ϵ bound on the solution is satisfied is unfixed though bounded above by T .

Although this definition has been developed primarily for use with nonlinear systems, where we have made the assumption that the desired final state is the origin, it also has meaning when applied to linear systems. Obviously any completely controllable linear system is also null controllable, but if the admissible control set is bounded in some way the converse may not hold.

By analogy with the complete and total controllability definitions for linear systems we may also define total null controllability.

DEFINITION 3.2

A system is said to be "totally null controllable" on (t_0, T) from y_0 if it is null controllable from y_0 on every open subinterval in (t_0, T) .

The total controllability concepts defined here and in Definition 1.5 require that a controllability condition be satisfied upon an arbitrarily short time interval; for this reason we shall only consider these definitions with systems where the available controls are unrestricted in magnitude.

3.2 $\mu(\cdot)$ AND CONTROLLABLE LINEAR SYSTEMS

Throughout this work we shall be concerned with bounding above the scalar matrix function

$\mu(A(t) + B(t)K(t))$ in order to show that various linear bounds upon the nonlinear system 3.1 are stable to the null solution. Now the measure $\mu(\cdot)$ is related to the eigenvalues of the matrix considered and in Chapter Two it was stated that in the specific case of the Euclidean norm upon vectors $\mu(D(t))$ took the value of $\frac{1}{2} \lambda_{\text{Max}}(D(t) + D^T(t))$.

It is well known **B1** that, if the linear system 3.2 is completely controllable, then by choice of the feedback matrix $K(t)$ it is possible to arbitrarily assign the eigenvalues of the matrix $A(t) + B(t)K(t)$. In particular, if the linear system is null controllable, then there exists some $K(t)$ such that $A(t) + B(t)K(t)$ is a stability matrix. Unfortunately the controllability of the linear system 3.2 is not a sufficient condition to enable the arbitrary assignment of $\mu(A(t) + B(t)K(t))$ by choice of some feedback matrix and some stronger condition upon the linear system is required.

We shall frequently demand

$$\mu(A(t) + B(t)K(t)) \leq -\alpha(t) ; \quad \alpha(t) > 0 \quad \forall t \in (t_0, t_f) \quad 3.3$$

given that the system 3.2 is either totally controllable or totally null controllable on (t_0, t_f) and the following lemma points out some of the sufficient conditions for this to be satisfied.

We shall require the following two definitions in addition to some well known matrix properties.

DEFINITION 3.2

The square $n \times n$ matrix D with elements d_{ij} is said to be "diagonally dominant" if

$$\sum_{\substack{i=1 \\ i \neq j}}^n |d_{ij}| \leq |d_{jj}|$$

$$\sum_{\substack{j=1 \\ j \neq i}}^n |d_{ij}| \leq |d_{ii}|$$

DEFINITION 3.3

The square $n \times n$ matrix D with elements d_{ij} is said to be "diagonally dominant to degree ϵ " if

$$\sum_{\substack{i=1 \\ i \neq j}}^n |d_{ij}| \leq |d_{jj}| - \epsilon$$

$$\sum_{\substack{j=1 \\ j \neq i}}^n |d_{ij}| \leq |d_{ii}| - \epsilon$$

LEMMA 3.1

If the system 3.2 is totally controllable on (t_0, t_f) using a control $u_x(t) \in \Omega$ then any of the following are sufficient conditions that there exists a $K(t)$ so that 3.3 is satisfied.

- 1) $B(t)$ has rank n
- 2) There exists $K(t)$ such that

$\lambda_{\max} (A(t) + B(t)K(t)) \leq -\alpha(t)$ and $A(t) + B(t)K(t)$ is a symmetric matrix.

- 3) There exists $K(t)$ such that

$\lambda_{\max} (A(t) + B(t)K(t)) \leq -\alpha(t)$ and $A(t) + B(t)K(t)$ is a normal matrix.

- 4) There exists a $K(t)$ such that $A(t)+B(t)K(t)$ is a stability matrix, diagonally dominant to degree $\alpha(t)$.
- 5) There exists a $K(t)$ such that $A(t)+B(t)K(t)$ is a stability matrix with positive off-diagonal elements and there exist positive r_i , $i=1 \dots n$ such that

$$\sum_{i=1}^n r_i [A(t)+B(t)K(t)]_{ij} \leq -\beta(t)$$

$$\sum_{i=1}^n r_i [A(t)+B(t)K(t)]_{ji} \leq -\beta(t) \quad j=1, \dots, n$$

with $\beta(t) \geq \max_i (r_i) \cdot \alpha(t) > 0$.

PROOF

Although the conditions 1) - 5) overlap to some extent it is simpler and more enlightening to show them individually and the subsequent proof is divided into the appropriate sections.

- 1) If the matrix $B(t)$ has rank n then there exist $K(t)$ such that each element of $A(t)+B(t)K(t)$ can be individually determined and clearly this is a sufficient condition for this lemma.
- 2) If the feedback matrix can be chosen so that $A(t)+B(t)K(t)$ is symmetric then the matrix is unchanged as $\frac{1}{2} (A(t)+B(t)K(t) + A(t)+B(t)K(t)^T)$ and hence the eigenvalues are unchanged; so

$$\mu(A(t)+B(t)K(t)) = \lambda_{\max} (A(t)+B(t)K(t))$$

and Section 2) of the lemma is proved.

- 3) The field of values of a matrix is defined by

$$F(D) \triangleq \{x^T D x \mid x \text{ is a complex } n \text{ vector with } x^T x = 1\}$$

In the case of a normal matrix $F(D)$ is the compact closure of the set of eigenvalues of D .

From Wiedlandt^{W2}, $F(D) = F(D^T)$

$$\text{and } F(D+E) \leq F(D) + F(E)$$

So

$$F \left[(A(t)+B(t)K(t)) + (A(t)+B(t)K(t))^T \right] \\ \leq 2 F[A(t)+B(t)K(t)]$$

Then if $A(t)+B(t)K(t)$ is normal the value of $\mu(A(t)+B(t)K(t))$ can be bounded above by $\lambda_{\max}(A(t)+B(t)K(t))$ and Section 3) of the lemma is proved. Notice that, as all symmetric matrices are normal, this also proves Section 2) of the lemma.

4) If E is a diagonally dominant matrix then consider the square, symmetric matrix $D \triangleq \frac{1}{2}(E+E^T)$.

$$\sum_{\substack{i=1 \\ i \neq j}}^n |d_{ij}| = \sum_{\substack{i=1 \\ i \neq j}}^n \frac{1}{2} [|e_{ij}| + |e_{ji}|] \\ \leq \sum_{\substack{i=1 \\ i \neq j}}^n \frac{1}{2} [|e_{ij}| + |e_{ji}|]$$

From the diagonal dominance of E

$$\sum_{\substack{i=1 \\ i \neq j}}^n |d_{ij}| \leq \frac{1}{2} [|e_{jj}| + |e_{jj}|]$$

But $d_{jj} = e_{jj}$ and so

$$\sum_{\substack{i=1 \\ i \neq j}}^n |d_{ij}| \leq |d_{jj}| \text{ and } D \text{ is also diagonally dominant.}$$

In a similar way, if the matrix E is diagonally dominant to

degree ϵ then the matrix D is also diagonally dominant to degree ϵ .

For notational convenience we write

$$D(t) \triangleq \frac{1}{2} (A(t) + B(t)K(t) + A(t) + B(t)K(t)^T)$$

then if $A(t) + B(t)K(t)$ is diagonally dominant to degree ϵ so is $D(t)$.

Consider the eigenvector $z(t)$ corresponding to

$\lambda_{\text{Max}}(D(t))$. Then

$$\lambda_{\text{Max}}(D(t)) z_i(t) = \sum_{j=1}^n d_{ij}(t) z_j(t) \quad \forall t \in (t_0, t_f)$$

Now if $|z_m(t)| = \max_i |z_i(t)|$

$$\lambda_{\text{Max}}(D(t)) z_m(t) = d_{mm}(t) z_m(t) + \sum_{\substack{j=1 \\ j \neq m}}^n d_{mj}(t) z_j(t)$$

$$\lambda_{\text{Max}}(D(t)) - d_{mm}(t) \leq \frac{\sum_{\substack{j=1 \\ j \neq m}}^n d_{mj}(t) z_j(t)}{z_m(t)}$$

Or

$$|\lambda_{\text{Max}}(D(t)) - d_{mm}(t)| \leq \sum_{\substack{j=1 \\ j \neq m}}^n d_{mj}(t)$$

From the diagonal dominance of $A(t) + B(t)K(t)$ and hence $D(t)$

$$|\lambda_{\text{Max}}(D(t)) - d_{mm}(t)| \leq |d_{mm}(t)| - \alpha(t)$$

Now $D(t)$ has negative diagonal elements equal to those of $A(t) + B(t)K(t)$ and so the maximum eigenvalue of $D(t)$ is contained in an interval on the real line bounded away from zero by $-\alpha(t)$ and section 4) of the lemma is proved.

5) If the feedback matrix can be chosen so

$$\sum_{i=1}^n r_i [A(t) + B(t)K(t)]_{ij} \leq -\beta(t)$$

$$\sum_{i=1}^n r_i [A(t) + B(t)K(t)]_{ji} \leq -\beta(t)$$

Then if $d_{ij}(t)$ are the elements of $D(t)$ defined above

$$\begin{aligned} \sum_{i=1}^n r_i d_{ij}(t) &\leq \frac{1}{2} \{-\beta(t) - \beta(t)\} \\ &\leq -\beta(t) \end{aligned}$$

Consider the eigenvector $z(t)$ corresponding to $\lambda_{\text{Max}}(D(t))$ and define $w_i(t) \triangleq z_i(t)/r_i$.

Then

$$\lambda_{\text{Max}}(D(t)) \cdot r_i w_i(t) = \sum_{j=1}^n r_j d_{ij}(t) w_j(t) \quad \forall t \in (t_0, t_f)$$

Now if $|w_m(t)| = \max_i |w_i(t)|$

$$\lambda_{\text{Max}}(D(t)) r_m w_m(t) = r_m w_m(t) d_{mm}(t) + \sum_{\substack{j=1 \\ j \neq m}}^n d_{mj}(t) \cdot r_j w_j(t)$$

$$\lambda_{\text{Max}}(D(t)) r_m \leq r_m d_{mm}(t) + \frac{1}{w_m(t)} \sum_{\substack{j=1 \\ j \neq m}}^n d_{mj}(t) r_j w_j(t)$$

So

$$|\lambda_{\text{Max}}(D(t)) r_m - r_m d_{mm}(t)| \leq \sum_{j=1}^n |d_{mj} \cdot r_j| \left| \frac{w_j(t)}{w_m(t)} \right|$$

and simplifying the right hand side this becomes

$$|\lambda_{\text{Max}}(D(t)) r_m - r_m d_{mm}(t)| \leq -r_m d_{mm}(t) - \beta(t)$$

Or

$$|\lambda_{\text{Max}}(D(t)) - d_{mm}(t)| \leq -d_{mm}(t) - \beta(t)/r_m$$

Then if $\beta(t) \geq \max_i (r_i) \alpha(t)$

$$|\lambda_{\text{Max}}(D(t)) - d_{mm}(t)| \leq -d_{mm}(t) - \alpha(t)$$

and, similarly to section 4) the maximum eigenvalue of $D(t)$ is bounded away from zero by $-\alpha(t)$ and section 5) of the lemma is proved.

In addition to the sufficient conditions for the existence

of $K(t)$ such that 3.3 is satisfied given in Lemma 3.1 there also exists a stronger correspondence between the controllability of the linear equation and the logarithmic norm $\mu(A(t)+B(t)K(t))$.

The scalar function $\mu(\cdot)$ can be defined in terms of any matrix norm subordinate to a vector norm on R^n and in those cases where the norm for vectors takes the form $\|z\|_V \triangleq \langle z, Vz \rangle$ (where V is some real, positive definite, symmetric matrix), then complete controllability of the linear equation 3.2 is a sufficient condition that, given any $\alpha(t)$, there exists some feedback matrix $K(t)$ and some V such that

$$\mu_V(A(t)+B(t)K(t)) \leq -\alpha(t) \quad \text{for some } t \in (t_0, t_f)$$

If the linear equation is totally controllable then this can be replaced by

$$\mu_V(A(t)+B(t)K(t)) \leq -\alpha(t) \quad \forall t \in (t_0, t_f)$$

This relationship is discussed in greater detail in Chapter Five where the work is formulated in this framework. It should be noted at this stage that the results in this and the following chapter do not depend upon the fact that the Euclidean norm is used and that the absence of this assumption does not substantially alter the form of the results or their derivation.

Throughout these two chapters it is assumed that there exists some $K(t)$ such that inequality 3.3 is satisfied at least with $\alpha(t) = 0$ and, depending upon the form of the bound upon the nonlinearity $f(\cdot, \cdot, t)$, the results enable some value to be placed upon the upper bound on $\mu(A(t)+B(t)K(t))$.

We can now develop the main theorem of this chapter. Here we use the inequality bound, 2.38, upon the nonlinear function $f(y, u_y, t)$ and recall that for this bound to be useful it is essential to restrict the state $y(t)$ to be contained in some bounded region of the state space D , say.

As there is no $||u_y(t)||$ term appearing explicitly on the right hand side of the bound it is also necessary to restrict the controls to be contained in Ω_k .

3.3 STATEMENT OF A NULL CONTROLLABILITY THEOREM

THEOREM 3.1

The nonlinear differential equation given by 3.1 with initial condition $y(t_0) = y_0$ is null controllable on (t_0, ∞) using control $u_y(t) \in \Omega_k$ provided that the linear system given by 3.2 with initial condition $x(t_0) = y_0$ is totally null controllable on (t_0, ∞) using $u_x(t) \in \Omega$, and the following conditions are satisfied:

- 1) $f(0,0,t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) $\exists p(t)$, scalar, $p(t) > 0$ such that:

$$||f(y, u_y, t)|| \leq p(t) ||y(t)|| \quad y \in D \subset \mathbb{R}^n, u_y(t) \in \Omega_k$$

$$t \in (t_0, \infty)$$
- 3) For scalars c_1, c_2 and λ with
 $0 < c_1 < \frac{1}{2}$ and $c_1/(1-c_1) \leq \lambda < 1$

Then there exist $\{t_i\}$, with $t_i < t_{i+1}$ in (t_0, ∞) with $i = 0, 1, \dots, N$ such that over each interval (t_i, t_{i+1}) the following are satisfied:

- a) $\int_{t_i}^{t_{i+1}} p(r) dr \leq c_1$
- b) $|t_{i+1} - t_i| \leq c_2 < \infty$
- c) $\exists u_x(t) \in \Omega_k$ such that

$$\int_{t_i}^{t_{i+1}} \mu(A(s) + B(s)K(s)) ds \leq \ln \left(\frac{\lambda - (1+\lambda)c_1}{1-c_1} \right)$$

NOTE 3.1

This theorem statement provides a means of looking at the null controllability of 3.1 by considering the behaviour of the linear system 3.2 and the "size" of the nonlinearity. In this case, the admissible controls Ω_k are given 'a priori' and a value for $p(t)$, which will be valid for any possible control in Ω_k , calculated. This value would then be used to determine the intervals over which condition 3a) is satisfied. It is necessary that these intervals be bounded above so that any number of such intervals (t_i, t_{i+1}) will exist in (t_0, ∞) ; condition 3b) ensures that this is so. If there exists a control in the admissible controls Ω_k which will satisfy condition 3c) over each interval (t_i, t_{i+1}) then the nonlinear system will be null controllable from y_0 on (t_0, ∞) using controls in Ω_k .

Notice that the term on the right hand side of the inequality in condition 3^c) is negative. Thus the smaller the time interval (t_i, t_{i+1}) needed to satisfy the condition 3a) then the more negative it will be necessary for $\mu\{A(s) + B(s)K(s)\}$ to be in order to satisfy condition 3b). It is via this mechanism that the linear system is compared with the nonlinear system to determine if the linear section of 3.1 "dominates" the nonlinearity.

3.4 PROOF OF THE NULL CONTROLLABILITY THEOREM

The procedure shown in the proof is given for the interval (t_0, t_1) but can be applied to any of the open intervals (t_i, t_{i+1}) in (t_0, ∞) . The null controllability definition demands that for any $\epsilon > 0$ there exists a time t_f

in (t_0, ∞) such that $\|y(t_f)\| < \epsilon$. The following analysis shows that over the typical interval (t_0, t_1) the solution of the nonlinear system can be made decreasing in norm.

A map L is established, similar to that used in Lovelady^{L2}, which takes solutions of system 3.2 into solutions of 3.1 when the nonlinearity is driven by the states and controls appropriate to the linear system. This map, L , is shown to be a continuous map from a domain of solutions $D\mathbb{R}^n$ back into D ; when mapped through L this domain of functions forms an equicontinuous set and a Schauder Fixed Point Theorem is then used to show the existence of a solution to the given system 3.1. The lemma and condition 3) are then used to show the existence of a linear state variable feedback control policy forcing the solution of 3.1 to some fraction of its initial value. Finally it is shown that this transfer may be accomplished using a continuous feedback matrix $K(t)$.

3.4.1 SOME INITIAL DEFINITIONS

Choose $\sigma \in \mathbb{R}$, $\sigma > 0$ such that for a solution $x(t)$ to system 3.2 with control $u_x(t) \in \Omega_k$ and initial condition $x(t_0) = y_0$

$$\|x(t)\| \leq \sigma \quad \forall t \in (t_0, t_f)$$

Consider $t_1 > t_0$ such that

$$\int_{t_0}^{t_1} p(r) \, dr \leq c_1 < \frac{1}{2}$$

and define B as

$$B \triangleq \left\{ x(t) \mid x(t) \text{ a solution to 3.2 over } (t_0, t_1) \text{ with initial condition } x(t_0), u_x(t) \in \Omega_k \right\} \quad 3.5$$

and also

$$N(x(s)) \triangleq \left\{ d \mid \sup_{s \in (t_0, t_1)} \|x(s)\| = d, x(s) \in B \right\} \quad 3.6$$

We can now define the domain of solutions which will be considered; the domain of solutions are given by

$$D(d) \triangleq \left\{ x(s) \in B \mid N(x(s)) \leq d, s \in (t_0, t_1), d > 0 \right\} \quad 3.7$$

The map L , which relates solutions of equation 3.1 to solutions of equations 3.2 when they have the same feedback matrix and initial conditions, is defined by considering any $z(t) \in B$.

$L(z(t)) \triangleq w(t)$ where $w(t)$ is the solution to the following differential equation

$$\begin{aligned} \dot{w}(t) &= A(t)w(t) + B(t)u_w(t) + f(z, u_z, t) \\ w(t_0) &= x(t_0) \end{aligned} \quad \begin{array}{l}) \\) \\) \\) \\) \end{array} \quad \begin{array}{l} \\ \\ \\ t \in (t_0, t_1) \end{array} \quad 3.8$$

Now consider the range of map L when $z(t) \in D(\sigma/c_1)$ and define $y(t) = L(z(t))$.

Under this mapping $y(t)$ represents the solution to the differential equation

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + B(t)u_y(t) + f(z, u_z, t) \\ y(t_0) &= x(t_0) \end{aligned} \quad \begin{array}{l}) \\) \\) \end{array} \quad \begin{array}{l} \\ t \in (t_0, t_1) \end{array} \quad 3.9$$

This equation describes the behaviour of a solution to equation 3.1 when the nonlinearity is driven by the state and control appropriate to ^{any admissible pair (z, u_z)} solution of the linear system 3.2.

Now if $R(t) \triangleq \|y(t) - x(t)\|$ the norm of the difference between the solution of equations 3.9 and 3.2 then $R(t_0) = 0$.

Using the upper right Dini derivative we can see that if the feedback matrices $K_x(t)$, ^{$K_z(t)$} and $K_y(t)$ are restricted to be equal for each system then

$$D^+R(t) = \limsup_{h \rightarrow 0^+} \frac{1}{h} (\|y(t) - x(t) + h(A(t) + B(t)K(t))y(t) - h(A(t) + B(t)K(t))x(t) + hf(z, u_z, t)\| - \|y(t) - x(t)\|)$$

So

$$D^+R(t) \leq \limsup_{h \rightarrow 0^+} \frac{1}{h} (\|y(t) - x(t) + h(A(t) + B(t)K(t))(y(t) - x(t))\| + h\|f(z, u_z, t)\| - \|y(t) - x(t)\|)$$

By definition of $R(t)$ this can be written as

$$D^+R(t) \leq \limsup_{h \rightarrow 0^+} \frac{1}{h} (\|I + h(A(t) + B(t)K(t))\| R(t) - R(t) + h\|f(z, u_z, t)\|)$$

Using the definition of the logarithmic norm $\mu(\cdot)$

$$D^+R(t) \leq \mu(A(t) + B(t)K(t))R(t) + \|f(z, u_z, t)\| \quad 3.10$$

Using the total controllability of the linear system together with the conditions in Lemma 3.1 we restrict the choice of the feedback matrix to a $K(t)$ such that

$$D^+R(t) \leq -\alpha(t) R(t) + \|f(z, u_z, t)\| \quad \alpha(t) > 0 \\ t \in (t_0, t_1)$$

As in the stability example given earlier it is necessary to use a comparison function to give an estimate of $R(t)$.

We postulate the existence of a solution $r(t)$ to the scalar differential equation

$$D^+r(t) = -\alpha(t)r(t) + \|f(z, u_z, t)\| \quad 3.11$$

over (t_0, t_1) with initial condition $r(t_0) = R(t_0) = 0$

It is well known that the solution to 3.11 is given by

$$r(t) = \int_{t_0}^t \|f(z, u_z, s)\| \exp\left\{\int_s^t -\alpha(r) dr\right\} ds$$

then from the comparison function an estimate of $R(t)$ is

given by

$$R(t) \leq \int_{t_0}^t \|f(z, u_z, s)\| \exp\left\{\int_s^t -\alpha(r) dr\right\} ds \quad 3.12$$

But as the feedback matrix $K(t)$, for both $u_x(t)$ and $u_y(t)$, is chosen such that $\alpha(r) \geq 0$, $r \in (t_0, t_1)$, then

$$R(t) \leq \int_{t_0}^t ||f(z, u_z, s)|| ds$$

Now from the inequality bound placed upon f in the theorem condition 2)

$$\begin{aligned} R(t) &\leq \sup_{s \in (t_0, t)} ||z(s)|| \int_{t_0}^{t_1} p(r) dr \\ &\leq \sup_{s \in (t_0, t_1)} ||z(s)|| \int_{t_0}^{t_1} p(r) dr \end{aligned} \quad 3.13$$

However $z(s) \in D(\sigma/c_1)$, and hence

$$R(t) \leq (\sigma/c_1) \cdot c_1 = \sigma$$

Thus for $z(t) \in B$

$$\begin{aligned} ||L(z(t))|| &= ||y(t)|| \leq ||y(t) - x(t)|| + ||x(t)|| \\ &\leq \sigma + \sigma = 2\sigma \end{aligned}$$

So for the worst choice of $c_1 = \frac{1}{2}$ we have $L: D(2\sigma) \rightarrow D(2\sigma)$, and for any $c_1 \leq \frac{1}{2}$ L is an injective map from $D(\sigma/c_1)$ into $D(2\sigma)$.

3.4.2 THE CONTINUITY OF MAP L

The nonlinear function f is a continuous function over $R^n \times R^m \times (t_0, \infty)$ and so the restriction of f to (t_0, t_1) is uniformly continuous.

NOTE: If f is uniformly continuous then, given $\epsilon > 0$

$\exists \delta, \delta^1 > 0$ such that $y_1, y_2 \in D$ $||y_1(t) - y_2(t)|| < \delta$,

$||u_{y_1}(t) - u_{y_2}(t)|| < \delta^1$ implies that

$$||f(y_1, u_{y_1}, t) - f(y_2, u_{y_2}, t)|| < \epsilon \quad \forall t \in (t_0, t_1)$$

To show the continuity of map L , consider two solutions $x_1(t), x_2(t) \in B$ such that $\|x_1(t) - x_2(t)\| < \delta$ and $\|u_{x_1}(t) - u_{x_2}(t)\| < \delta^1$.

The two corresponding solutions of the equation 3.9 are given by

$$y_i(t) = L(x_i(t)) \quad i = 1, 2 \quad t \in (t_0, t_1)$$

Now define $P(t) \triangleq \|y_1(t) - y_2(t)\|$, $P(t_0) = 0$ and use a similar development to that which yielded bounds upon $R(t)$ to make an estimate of $P(t)$ -

$$\begin{aligned} D^+P(t) &= \limsup_{h \rightarrow 0^+} \frac{1}{h} (\|x_1(t) - x_2(t) + h(A(t) + B(t)K(t))x_1(t) \\ &\quad - L(A(t) + B(t)K(t))x_2(t) + h(f(x_1, u_{x_1}, s) - f(x_2, u_{x_2}, s))\| \\ &\quad + \|x_1(t) - x_2(t)\|) \end{aligned}$$

Then, following the previous procedure

$$D^+P(t) \leq \mu(A(t) + B(t)K(t))P(t) + \|f(x_1, u_{x_1}, s) - f(x_2, u_{x_2}, s)\| \quad 3.14$$

The lemma 3.1 and a comparison function yield the estimate

$$P(t) \leq \int_{t_0}^t \|f(x_1, u_{x_1}, s) - f(x_2, u_{x_2}, s)\| \exp \left\{ \int_s^t -\alpha(r) dr \right\} ds$$

So if the linear system 3.2 is totally null controllable and the nonlinear function f is continuous, then

$$P(t) \leq (t - t_0)\varepsilon = \varepsilon^1 \quad t \in (t_0, t_1) \quad 3.15$$

and hence the restriction of L to $D(2\sigma)$ is continuous.

3.4.3 $L(D(2\sigma))$ IS AN EQUICONTINUOUS FAMILY OF FUNCTIONS

The definition of equicontinuity given by Coppel^{C1} is:

"A family of functions F defined on an interval I is said to be equicontinuous at a point $\bar{t} \in I$ if for every $\varepsilon > 0$ $\exists \delta > 0$ such that for all $f \in F$

$$||f(\bar{t}^1) - f(\bar{t})|| < \varepsilon \quad \text{provided} \quad ||\bar{t}^1 - \bar{t}|| < \delta, \quad \bar{t}^1 \in I "$$

Consider a solution to equation 3.2 given by $x(t) \in B$ and consider also the solution to equation 3.9 given by $z(t) = L(x(t))$.

At the times s_1, s_2 ; $s_1 < s_2$ in (t_0, t_1)

$$\begin{aligned} ||z(s_2) - z(s_1)|| &\leq \int_{s_1}^{s_2} ||\dot{z}(r)|| dr \\ &\leq \int_{s_1}^{s_2} ||A(r)z(r) + B(r)K(r)|| dr + \int_{s_1}^{s_2} ||f(x, u_x, r)|| dr \end{aligned} \quad 3.16$$

Now there exists some ϕ such that

$$\phi \geq ||A(r)z(r) + B(r)K(r)|| \quad \forall r \in (t_0, t_1)$$

and some ψ such that

$$\psi \geq ||f(x, u_x, r)|| \quad \forall r \in (t_0, t_1)$$

So, from inequality 3.16,

$$||z(s_2) - z(s_1)|| \leq (\phi + \psi)(s_2 - s_1)$$

and this holds for any possible solution in $D(2\sigma)$; hence

from the definition it can be seen that the family of functions $L(D(2\sigma))$ is equicontinuous at every point in (t_0, t_1) .

We now know enough about the map L to enable us to use a

Schauder Fixed Point Theorem and establish the existence of

a solution $y(t) \in D(2\sigma)$ such that

$$y(t) = L(y(t)) \quad y(t_0) = x(t_0) = y_0 \quad 3.17$$

Here $y(t)$ is a fixed point under the map L which takes solutions, in $D(2\sigma)$, of the equation 3.2 into solutions of the equation 3.9. The substitution of this fixed point $y(t)$ into equation 3.9 yields the original nonlinear differential equation 3.1: we have thus shown the existence of a solution to system 3.1 over (t_0, t_1) with initial condition $y(t_0) = y_0$.

Note that at this stage, although the existence of a solution to equation 3.1 has been shown, such a solution is not necessarily unique; not enough is known about the map L to invoke a Schauder Fixed Point Theorem which guarantees the uniqueness of the fixed point.

3.4.4 COMPARISON OF THE SOLUTIONS TO EQUATIONS 3.1 AND 3.2

As before consider $x(t) \in B$, $x(t_0) = y_0$, and now let $y(t)$ be a solution to equation 3.1 with $y(t_0) = y_0$.

Consider next the difference between these two solutions given by $Q(t) \triangleq ||y(t) - x(t)||$; $Q(t_0) = 0$. We wish to be able to bound $Q(t)$ above and hence say something about the behaviour of the nonlinear system solution given knowledge of the linear system solution. The estimate for $Q(t)$ is again developed using a Dini derivative -

$$D^+Q(t) \leq \limsup_{h \rightarrow 0^+} \frac{1}{h} (||y(t) - x(t) + h(A(t) + B(t)K(t))(y(t) - x(t))|| + h||f(y, u_y, t)|| + ||y(t) - x(t)||) \quad 3.18$$

From the definition of $\mu(\cdot)$

$$\begin{aligned} D^+Q(t) &\leq \mu(A(t) + B(t)K(t))Q(t) + ||f(y, u_y, t)|| \\ D^+Q(t) &\leq -\alpha(t)Q(t) + ||f(y, u_y, t)|| \quad t \in (t_0, t_1) \end{aligned} \quad 3.19$$

Now from condition 2) in the theorem statement we can bound the nonlinearity, and

$$\begin{aligned} D^+Q(t) &\leq -\alpha(t)Q(t) + p(t)||y(t)|| \\ &\leq (-\alpha(t) + p(t))Q(t) + p(t)||x(t)|| \end{aligned} \quad 3.20$$

From this differential inequality an estimate of $Q(t)$ can be found using a scalar comparison function. Inequality 3.20 has the same form as equation 3.11 and the

same procedure is valid; however by writing the estimate of $Q(t)$ in a slightly expanded form, we have

$$Q(t) \leq \exp\left\{\int_{t_0}^t [-\alpha(s) + p(s)] ds\right\} \cdot \int_{t_0}^t \exp\left\{\int_{t_0}^s [\alpha(r) - p(r)] dr\right\} \cdot p(s) ||x(s)|| ds \quad 3.21$$

If we define

$$I \triangleq \int_{t_0}^t \exp\left\{\int_{t_0}^s [\alpha(r) - p(r)] dr\right\} p(s) ds$$

we can bound $Q(t)$ above by

$$Q(t) \leq \exp\left\{\int_{t_0}^t [-\alpha(s) + p(s)] ds\right\} \cdot \max_{s \in (t_0, t_1)} ||x(s)|| \cdot I \quad 3.22$$

To arrive at an estimate for $Q(t)$ from inequality 3.22 some bound is required for the integral I . Integration by parts yields

$$I = \left[\int_{t_0}^s p(r) dr \cdot \exp\left\{\int_{t_0}^s [\alpha(r) - p(r)] dr\right\} \right]_{s=t_0}^{s=t} \\ - \int_{t_0}^t \int_{t_0}^s p(r) dr \cdot (\alpha(s) - p(s)) \cdot \exp\left\{\int_{t_0}^s [\alpha(r) - p(r)] dr\right\} ds$$

Substitution of the boundary conditions and expanding the second term gives

$$I = \int_{t_0}^t p(r) dr \cdot \exp\left\{\int_{t_0}^t [\alpha(r) - p(r)] dr\right\} - \int_{t_0}^t \int_{t_0}^s p(r) dr \cdot \alpha(s) \cdot \exp\left\{\int_{t_0}^s [\alpha(r) - p(r)] dr\right\} ds \\ + \int_{t_0}^t \int_{t_0}^s p(r) dr \cdot p(s) \cdot \exp\left\{\int_{t_0}^s [\alpha(r) - p(r)] dr\right\} ds.$$

So, as the second term is positive and as $\int_{t_0}^s p(r)dr$ is maximised at $s=t$

$$I \leq \int_{t_0}^t p(r)dr \cdot \exp \left\{ \int_{t_0}^t [\alpha(r) - p(r)]dr \right\} + \int_{t_0}^t p(r)dr \cdot I. \quad 3.23$$

Now from condition three in the theorem statement it is known that $\int_{t_0}^t p(r)dr \leq c_1 < \frac{1}{2}$ and so, after manipulation 3.23 becomes

$$I \leq \exp \left\{ \int_{t_0}^t [\alpha(r) - p(r)]dr \right\} \cdot \frac{\int_{t_0}^t p(r)dr}{\left(1 - \int_{t_0}^t p(r)dr\right)}$$

Substituting this estimate for I into inequality 3.22 the bound upon $Q(t)$ is given by

$$Q(t) \leq \max_{s \in (t_0, t)} ||x(s)|| \cdot \exp \left\{ \int_{t_0}^t [-\alpha(r) + p(r)]dr \right\} \cdot \exp \left\{ \int_{t_0}^t [\alpha(r) - p(r)]dr \right\} \cdot \frac{\int_{t_0}^t p(r)dr}{\left(1 - \int_{t_0}^t p(r)dr\right)}$$

$$Q(t) \leq \max_{s \in (t_0, t)} ||x(s)|| \cdot \frac{c_1}{1 - c_1} \quad 3.24$$

But we have chosen $K(t)$ such that $u_x(t)$ and $u_y(t)$ belong to Ω_k and $\mu\{A(s) + B(s)K(s)\} < 0$. In this case $||x(s)||$, $s \in (t_0, t_1)$ takes its maximum value at $s = t_0$ and $\max_{s \in (t_0, t_1)} ||x(s)|| = ||x_0||$.

So, from 3.24, at $t = t_1$

$$||y(t_1) - x(t_1)|| \leq ||x_0|| \cdot \left(\frac{c_1}{1 - c_1} \right)$$

We require an estimate of $||y(t_1)||$ and from the above

$$\begin{aligned} ||y(t_1)|| &\leq ||y(t_1) - x(t_1)|| + ||x(t_1)|| \\ &\leq ||x_0|| \left(\frac{c_1}{1 - c_1} \right) + ||x(t_1)|| \end{aligned} \quad 3.25$$

However

$$||x(t_1)|| \leq ||x_0|| \exp \left\{ \int_{t_0}^{t_1} \mu(A(s) + B(s)K(s)) ds \right\}$$

Now if condition 3) in the theorem statement is satisfied then substitution into inequality 3.25 gives

$$||y(t_1)|| \leq ||x_0|| \lambda = ||y_0|| \lambda \quad 3.26$$

where $\lambda < 1$

So over the interval (t_0, t_1) we have shown that $||y(t_1)||$ is some fraction of the initial value $||y_0||$.

3.4.5 EXTENSION OF THIS PROPERTY TO COVER (t_0, ∞)

The above procedure can be applied to each consecutive interval on (t_0, ∞) over which condition three holds. Each of these intervals is of bounded length and hence for any $\epsilon > 0$ there exists some N such that $||y_0|| \lambda^N < \epsilon$ and $t_N < \infty$. By the above method we have shown that there exists some time $T \leq t_N < \infty$ such that for any $\epsilon > 0$, $||y(T)|| < \epsilon$. This shows the existence of a control $u_y(t) \in \Omega_k$ which ensures that the nonlinear system 3.1 is null controllable from y_0 on (t_0, ∞) . It remains to show the existence of a continuous feedback matrix $K(t)$.

From the theorem statement we know that over each interval (t_i, t_{i+1})

$$\int_{t_i}^{t_{i+1}} \mu(A(s) + B(s)K(s)) ds < \text{some constant } \gamma \text{ say} \quad 3.27$$

and as $u_Y(t) \in \Omega_k$ then at every $t \in (t_i, t_{i+1})$, $||K(t)|| \leq k$.

Consider two adjacent intervals (t_{i-1}, t_i) and (t_i, t_{i+1}) . Over each interval there exists a control feedback matrix $K_1(t)$, $t \in (t_{i-1}, t_i)$ and $K_2(t)$, $t \in (t_i, t_{i+1})$, each of which satisfies 3.27. These two feedback matrices lead to boundary values for $\mu(A(s) + B(s)K(s))$ of, say,

$$\lim_{t \rightarrow t_{i-1}^+} \mu(A(t) + B(t)K_1(t)) = a_1$$

$$\lim_{t \rightarrow t_i^-} \mu(A(t) + B(t)K_1(t)) = a_2$$

and

$$\lim_{t \rightarrow t_i^+} \mu(A(t) + B(t)K_2(t)) = b_1$$

$$\lim_{t \rightarrow t_{i+1}^-} \mu(A(t) + B(t)K_2(t)) = b_2$$

Now if $a_2 = b_1$ we can choose $K_1(t) = K_2(t)$ and then $K(t)$ is continuous over (t_{i-1}, t_{i+1}) . In general, however, $a_2 \neq b_1$ and in this case we note that either

$$\mu(A(t_i) + B(t_i)K_1(t_i)) > \mu(A(t_i) + B(t_i)K_2(t_i)) \quad 3.28$$

or vice versa.

Without loss of generality assume that the condition given by 3.28 exists; it is necessary that $K_1(t)$ and $K_2(t)$ both satisfy 3.27 and $||K(t)|| \leq k$. To show that $K(t)$ can be continuous assume that the condition given by 3.28 exists. Then let $K_1(t_i) = K_2(t_i)$ and choose $K_1(t)$ to be continuous over (t_{i-1}, t_i) .

Then as $||K_2(t_i)|| \leq k$ we know that $||K_1(t_i)|| \leq k$ and also, from 3.28

$$\int_{t_{i-1}}^{t_i} \mu(A(s)+B(s)K_1(s))ds \leq \bar{\gamma} \leq \gamma$$

and condition 3) of the theorem is still satisfied. By this method we can see that $K(t)$ can be chosen to be continuous over (t_0, ∞) such that the nonlinear system is null controllable from y_0 on (t_0, ∞) and the theorem is proved.

3.5 DISCUSSION OF THE NULL CONTROLLABILITY RESULT

This theorem provides a means of comparing the solution to the nonlinear equation 3.1 with the solution to the linear equation 3.2. The nonlinearity is bounded in norm and thus it is necessary always to consider the "worst case" situation when the nonlinearity $f(y, u_y, t)$ is regarded as unstable. The theorem compares the possible values of $\mu(A(t)+B(t)K(t))$ (when $u_y(t) \in \Omega_k$) at every time with the largest possible value of $\|f(y, u_y, t)\|$ at every time. The fact that a norm is used to bound f means that no information about the direction of f is assumed in the analysis. The bound upon $f(y, u_y, t)$ involves no control terms explicitly and hence the value of $\|f(y, u_y, t)\|$ is taken to be the maximum possible at each time $t \in (t_0, t_f)$. The size of the nonlinearity is characterised by the inequality bound in condition 2) together with the length of the intervals (t_i, t_{i+1}) . If a control exists which makes the linear equation sufficiently stable over each of these intervals then the nonlinear system is null controllable. This restriction upon the linear system is given by condition 3).

The above proof does not show the uniqueness of the solution to 3.1 when the nonlinearity is required to satisfy

the bound upon the norm given in condition 2). The fixed point for the map L may not be unique and hence more than one solution may exist. This can be avoided by constructing the proof in such a way that the mapping $L : D(2\sigma) \rightarrow D(2\sigma)$ is a contraction. The additional restriction upon f in order that this be satisfied is a Lipschitz condition and we consider this in detail in the next chapter (see Coppel^{C1} or for the original text, Schauder^{S1}).

In general the lack of a uniqueness condition does not affect the controllability condition. Any solution which is shown to exist, via the Schauder Fixed Point theorem will satisfy those conditions given in 3.4.4 and hence will be null controllable.

In the theorem statement, condition 3), we give the bound

$$\int_t^t \mu(A(s) + B(s)K(s)) ds \leq \ln \left(\frac{\lambda - (1+\lambda)c_1}{1 - c_1} \right)$$

We require this in order that the bound upon $\|y(t)\|$ at the end of each interval will be a uniform one over (t_0, t_f) . In practise if $\lambda = 1$ this bound could be replaced by

$$\int_t^t \mu(A(s) + B(s)K(s)) ds < \ln \left(\frac{1 - 2c_1}{1 - c_1} \right)$$

and the strict inequality here should ensure that, in any nonpathological cases, the solution of 3.1 will satisfy $\|y(t_f)\| < \epsilon$ for any $\epsilon > 0$ at some time t_f .

3.6 A STRONGER NULL CONTROLLABILITY RESULT

The above result may be modified to yield a stronger controllability theorem: in certain cases it may be useful to be able to say more about the solution of equation 3.1 than is implied by Theorem 3.1 and so an upper bound upon the solution $y(t)$ is imposed in the following definition of strong null controllability.

DEFINITION 3.3

The system given by equation 3.1 is said to be "strongly null controllable" from y_0 on (t_0, T) if there exists an admissible control $u_y(t)$ which implies that given any $\epsilon > 0$ there exists a time $t_f \in (t_0, T)$ such that for all $t > t_f, t \in (t_0, T)$

$$\|y(t)\| < \epsilon.$$

Theorem 3.1 may be modified to yield a strong null controllability result for system 3.1 when the nonlinearity is bounded by inequality 2.38. In this case, rather than requiring the property given by condition 3) to hold upon consecutive time intervals (t_i, t_{i+1}) we require that this condition be satisfied for every t in (t_0, ∞) so that there exists a $\bar{t}$ such that

$$\int_t^{\bar{t}} \mu(A(s) + B(s)K(s)) ds \leq \ln \frac{\lambda - (1+\lambda)c_1}{1 - c_1} \quad 3.28$$

In this way the solution behaviour over any interval $(t, \bar{t})$ upon which this condition is satisfied is reflected in the solution behaviour over subsequent intervals. It is then sufficient to ask that there exists an admissible control which ensures that the solution is decreasing over $(t_0, \bar{t}_0)$, an interval upon which 3.28 holds. This will ensure that the

solution to 3.1 is bounded above by some decreasing function of time and hence the system 3.1 will be strongly null controllable. This is more rigorously stated in the following corollary.

COROLLARY 3.1

The system given by equation 3.1 is strongly null controllable from y_0 on (t_0, ∞) if the linear system given by 3.2 is totally null controllable on (t_0, ∞) and if:

- 1) $f(0,0,t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) $||f(y, u_y, t)|| \leq p(t) ||y(t)|| \quad \forall t \in (t_0, \infty)$
 $y(t) \in DCR^n, u_y(t) \in \Omega_k$

where $p(t)$ is a continuous, non zero, positive function of t .

- 3) For scalars c_1, c_2 and λ with $0 < c_1 < \frac{1}{2}$ and $c_1/(1-c_1) \leq \lambda < 1$.
Then $\forall t \in (t_0, \infty)$ there exists $\bar{t}$ such that:

- a) $\int_t^{\bar{t}} p(r) dr = c_1$
- b) $|\bar{t} - t| \leq c_2 < \infty$
- c) $\exists u_x(t) \in \Omega_k$ such that

$$\int_t^{\bar{t}} \mu(A(s) + B(s)K(s)) ds \leq \ln \frac{\lambda - (1+\lambda)c_1}{1 - c_1}$$

- 4) The control $u_y(t) = K(t)y(t)$, with $K(t)$ as defined above is such that the nonlinear system 3.1 has solution $||y(t)||$ bounded above by a monotonic decreasing scalar function $r(t)$; $r(t_0) = ||\dot{y}(t_0)||$ over the interval $(t_0, \bar{t}_0)$.

PROOF:

The existence of a solution to equation 3.1 may be inferred using the method given in sections 3.4.2 - 3.4.4. It remains to show that such a solution is strongly null controllable.

Consider the condition

$$\int_{\bar{t}}^{\bar{t}} p(r) dr = c_1 \quad 3.29$$

If $p(r) > 0 \forall r \in (t_0, \infty)$ then for each $t \in (t_0, \infty)$ there exists a unique $\bar{t}$ such that 3.29 is satisfied, and also, given any $\varepsilon > 0$ there exists a unique $\bar{\varepsilon} > 0$ such that

$$\int_{t+\varepsilon}^{\bar{t}+\bar{\varepsilon}} p(r) dr = c_1$$

Note that this relationship does not hold if, over any time interval, $p(r)$ has zero value. However, this is a special limiting case: $p(r)=0$ implies a zero value for the non-linearity $f(y, u_y, t)$, however, this may still be bounded by some positive value for $p(r)$.

Now from the monotonicity of the bound upon solution over (t_0, t_0^-) there exists $0 \leq \delta < 1$ such that

$$||y(t_0+\varepsilon)|| = \delta ||y(t_0)||$$

But, as the corollary demands that condition 3) be satisfied $\forall t \in (t_0, \infty)$ we can apply the proof of Theorem 3.1 to $(t_0+\varepsilon, \bar{t}_0+\bar{\varepsilon})$ and (t_0, t_0^-) :

$$||y(t_0^-)|| \leq \lambda ||y(t_0)||$$

$$\text{and } ||y(t_0^-+\bar{\varepsilon})|| \leq \lambda ||y(t_0+\varepsilon)||$$

In order to show that $||y(t)||$ is bounded above by a decreasing function we need to show some upper bound for $||y(t_0^-+\bar{\varepsilon})||$.

From the above inequalities

$$\begin{aligned} ||y(t_0^-+\bar{\varepsilon})|| &\leq \lambda \cdot \delta \cdot ||y(t_0)|| \\ &\leq \delta^1 r(t_0) \text{ where } \delta^1 < 1 \end{aligned}$$

By extending this procedure forward in time we can show the existence of some monotonic decreasing, positive scalar

function $r(t)$ such that

$$||y(t)|| \leq r(t) \quad \forall t \in (t_0, \infty)$$

This in turn shows that the solution is strongly null controllable according to definition 3.3 and the corollary is proved.

It is not possible by this means to show that the solution to equation 3.1 with control $u_y(t)$ is itself monotonic decreasing in norm because the theorem gives no information about a lower bound for $||y(t)||$.

NOTE 3.2. The stability results quoted in chapter two may be used directly to give bounds upon $||y(t)||$: consider the stability of the nonlinear differential equation

$$\dot{y}(t) = (A(t) + B(t)K(t)) y(t) + f(y, K(t)y, t) \quad 3.30$$

If there exists a $K(t)$ such that $u_y(t) \in \Omega_k$ and

$$\mu(A(s) + B(s)K(s)) + p(s) < 0 \quad \forall s \in (t_0, \infty) \quad 3.31$$

then the differential equation 3.30 is uniformly stable.

This shows the existence of a control $u_y(t)$ such that for any $\epsilon > 0$ there exists a time $t_f \in (t_0, \infty)$ so that for all $t > t_f$ $||y(t)|| < \epsilon$ and hence the system given by 3.1 is strongly null controllable.

CHAPTER 4

4.1 Introduction.

4.2 Examination of solution to matrix Riccati equation.

4.3 A null controllability theorem for nonlinearity
bound 2.40.

4.4 Proof of theorem 4.1.

4.4.1 The existence of a solution to equation 4.4.

4.4.2 Comparison of the solutions to equations 4.2
and 4.4.

4.5 Null controllability of equation 4.4 with uniformly
controllable linear equation 4.2.

4.6 For a nonlinearity bound involving only terms in control.

4.7 For a nonlinearity bound involving terms in state and
control.

4.8 Conclusions.

4.1 INTRODUCTION

The bound used to restrict the nonlinearity when developing the results in the previous chapter is independent of control. The extension of these results to the case where the bound involves terms in control is considered in this chapter. Of the bounds introduced in Chapter Two the norm bounds involving control terms are considered here. A similar approach to that given in Chapter Three is used, but the norm of possible feedback control matrices is not restricted beforehand. Instead the feedback matrix is of the form $K(t) = -\frac{1}{2} B^T(t)S(t)$ where $S(t)$ is a solution to the matrix Riccati equation

$$\frac{d}{dt} S(t) = -A^T(t)S(t) - S(t)A(t) + S(t)B(t)B^T(t)S(t) \quad 4.1$$

The matrix solution $S(t)$ to this equation, when considered in norm, can be related to the growth of the solution of the linear system equation

$$\dot{x}(t) = A(t)x(t) + B(t)u_x(t) \quad 4.2$$

with the control $u_x(t) = -\frac{1}{2} B^T(t)S(t)x(t)$

and $x(t) \in D\mathbb{R}^n$, $t \in (t_0, \infty)$

The introduction of this control into the previous analysis leads to restrictions not only upon the size of the scalar multipliers but also upon the integral of $\|B^T(t)\|$ over the time intervals considered.

Consider the bound given by

$$\|f(y, u_y, t)\|^2 \leq p(t) \|y(t)\| \|u_y(t)\| \quad 4.3$$

(2.40)

The use of a linear feedback control policy, together with this inequality bound leads to a restriction similar to

that used in Chapter Three. Say $u_y(t) = K(t) y(t)$, then 4.3 implies

$$||f(y, u_y, t)|| \leq \left(p(t) ||K(t)|| \right)^{\frac{1}{2}} ||y(t)||$$

and in this case the scalar function of time $\left(p(t) ||K(t)|| \right)^{\frac{1}{2}}$ depends explicitly upon the feedback matrix. The use of a control where the matrix $S(t)$ is a solution to the Riccati equation 4.1 enables that part of the function $\left(p(t) ||K(t)|| \right)^{\frac{1}{2}}$ which is dependent upon $S(t)$ to be characterised by a function involving the logarithmic norm of the matrix $\left[A(t) - \frac{1}{2} B(t) B^T(t) S(t) \right]$. As the solution to the linear equation 4.2 and the solution to the nonlinear equation

$$\dot{y}(t) = A(t)y(t) + B(t)u_y(t) + f(y, u_y, t) \quad 4.4$$

$$\text{when } u_y(t) = \frac{1}{2} B^T(t) S(t) y(t)$$

$$\text{and } y(t) \in DCR^n, t \in (t_0, \infty)$$

are also characterised by a term involving the logarithmic norm of $\left[A(t) - \frac{1}{2} B(t) B^T(t) S(t) \right]$ this enables these relationships to be combined using a similar technique to that used in Chapter Three.

4.2 EXAMINATION OF SOLUTION TO MATRIX RICCATI EQUATION

The control which minimises the cost function

$$J = \int_{t_0}^{t_f} ||u_x(t)||^2 dt$$

for the linear system 4.2 is known to be of the form

$$u_x(t) = -\frac{1}{2} B^T(t) S(t) x(t)$$

where $S(t)$ is a real positive definite symmetric matrix which satisfies the Riccati equation 4.1 with boundary condition given by $S(t_f) = S_f$, say.

If the matrix $S(t)$ satisfies the Riccati equation then the term $\frac{d}{dt} ||S(t)||$ can be bounded and this in turn implies a bound upon the value of $||S(t)||$ at any time.

LEMMA 4.1

The matrix solution $S(t)$ to the Riccati equation 4.1 with end point condition $S_f = S(t_f)$ satisfies the following bound:

$$||S(t)|| \leq ||S_f|| \exp\left\{ \int_t^{t_f} \mu\left(A(s) - \frac{B(s)B^T(s)S(s)}{2}\right) ds \right\} \quad 4.5$$

PROOF

From the usual definition of a derivative

$$\frac{d||S(t)||}{dt} = \lim_{h \rightarrow 0} \left[\frac{||S(t+h)|| - ||S(t)||}{h} \right]$$

Then for $\frac{d||S(t)||^2}{dt}$

$$\begin{aligned} \frac{d||S(t)||^2}{dt} &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\max_{||z_1||=1} \langle z_1 S(t+h), S(t+h) z_1 \rangle \right. \\ &\quad \left. - \max_{||z_2||=1} \langle z_2 S(t), S(t) z_2 \rangle \right] \quad 4.6 \end{aligned}$$

$$\begin{aligned} &\leq \lim_{h \rightarrow 0} \frac{1}{h} \left[\max_{||z_1||, ||z_2||=1} \langle z_1 S(t+h) - z_2 S(t), S(t+h) z_1 \rangle \right. \\ &\quad \left. + \max_{||z_1||, ||z_2||=1} \langle z_2 S(t), S(t+h) z_1 - S(t) z_2 \rangle \right] \end{aligned}$$

Now as $h \rightarrow 0$ $z_1 = z_2 = z$ and

$$\begin{aligned} \frac{d||S(t)||^2}{dt} &\leq \left[\max_{||z||=1} \langle z dS(t)/dt, S(t) z \rangle \right. \\ &\quad \left. + \max_{||z||=1} \langle z S(t), dS(t)/dt z \rangle \right] \quad 4.7 \end{aligned}$$

$$\frac{d||S(t)||^2}{dt} \leq 2 \cdot \max_{||z||=1} \langle z dS(t)/dt, S(t) z \rangle$$

However $S(t)$ is a solution to the Riccati equation 4.1 and hence

$$\frac{d||S(t)||^2}{dt} \leq 2 \cdot \max_{||z||=1} z \left(-A^T(t)S(t) - S(t)A(t) + S(t)B(t)B^T(t)S(t) \right), S(t)z > 0 \quad 4.8$$

From the definition of the logarithmic norm $\mu(\cdot)$

$$\frac{d||S(t)||^2}{dt} \leq 2\mu \left(-A^T(t)S(t) - S(t)A(t) + S(t)B(t)B^T(t)S(t) \right) \cdot ||S(t)||$$

which leads to

$$\frac{d||S(t)||}{dt} \leq \mu \left(-A^T(t)S(t) - S(t)A(t) + S(t)B(t)B^T(t)S(t) \right) \quad 4.9$$

Now from the properties of $\mu(\cdot)$

$$\begin{aligned} \frac{d||S(t)||}{dt} &\leq \mu \left(-A^T(t)S(t) + \frac{S(t)B(t)B^T(t)S(t)}{2} \right) \\ &\quad + \mu \left(-S(t)A(t) + \frac{S(t)B(t)B^T(t)S(t)}{2} \right) \end{aligned}$$

Hence from the symmetry of the function $\mu(\cdot)$

$$\frac{d||S(t)||}{dt} \leq 2\mu \left(-S(t)A(t) + \frac{S(t)B(t)B^T(t)S(t)}{2} \right) \quad 4.10$$

Now $S(t)$ is a symmetric, positive definite matrix and so, from the structure of the logarithmic norm subordinate to the Euclidean norm for vectors used here and from the eigenvalue properties given by Wielandt^{W2}, the right hand side of equation 4.10 can be written as

$$\frac{d||S(t)||}{dt} \leq 2||S(t)|| \cdot \mu \left(-A(t) + \frac{B(t)B^T(t)S(t)}{2} \right) \quad 4.11$$

This scalar differential inequality bears a strong resemblance to the inequality bound upon $||x(t)||$, the solution to 4.2, which arises when using the techniques shown in the previous chapter. In a similar way a scalar

comparison function for equation 4.11 yields the estimate

$$\int_{s=t}^{s=t_f} \frac{1}{||S(s)||} d||S(s)|| \leq \int_t^{t_f} \mu \left(-A(s) + \frac{B(s)B^T(s)S(s)}{2} \right) ds \quad 4.12$$

By change of variable $s = -s$ inequality 4.11 also yields the estimate

$$\begin{aligned} \int_t^{t_f} -\mu \left(+A(s) - \frac{B(s)B^T(s)S(s)}{2} \right) ds \\ \leq \int_{s=t}^{s=t_f} \frac{1}{||S(s)||} d||S(s)|| \end{aligned}$$

This leads to an estimate for $S(t)$ given by

$$||S(t)|| \leq ||S_f|| \exp \left\{ \int_t^{t_f} \mu \left(A(r) - \frac{B(r)B^T(r)S(r)}{2} \right) dr \right\} \quad 4.13$$

and the lemma is proved.

NOTE 4.1

From the inequality 4.11 a number of possible bounds upon $||S(t)||$ are available using the usual techniques for differential inequalities. We wish to be able to bound $||S(t)||$ by a term directly related to the exponential function used to bound the estimates of the solutions to equations 4.2 and 4.4. Notice also that the sign of the matrix $A(t) - \frac{B(t)B^T(t)S(t)}{2}$ which would be the resultant state matrix for the linear equation 4.2 using the above control has been changed in equation 4.11 describing the evolution of $||S(t)||$.

Using this lemma and the bound upon the nonlinearity given by inequality 4.3 a technique similar to that used in Chapter Three yields a null controllability condition for the nonlinear equation 4.4. Here the control is bounded at the end of each interval (t_i, t_{i+1}) and the Riccati equation

solved in reverse time, as in optimal control, to yield a value of $S(t)$ which then satisfies the bound upon $||S(t)||$. It is through this bound upon $||S(t_{i+1})||$ that the value of $||u_Y(s)||$ over the intervals (t_i, t_{i+1}) is restricted. As the feedback matrix has the form $-\frac{1}{2} B^T(t)S(t)$ it is also necessary to impose some integral bound on the value of $\frac{||B^T(t)||}{2}$.

4.3 A NULL CONTROLLABILITY THEOREM FOR NONLINEARITY

BOUND 2.40

THEOREM 4.1

The nonlinear differential equation given by 4.4 with initial condition $y(t_0) = y_0$ is null controllable on (t_0, ∞) using control $u_Y(t) \in \Omega$ provided that the linear system given by equation 4.2 with initial condition $x(t_0) = y_0$ is totally null controllable on (t_0, ∞) using $u_X(t) \in \Omega$ and that the following conditions are satisfied:

1) $f(0,0,t) = 0 \quad \forall t \in (t_0, \infty)$

2) $\exists p(t)$, scalar, $p(t) > 0$ such that

$$||f(y, u_Y, t)||^2 \leq p(t) ||y(t)|| \cdot ||u_Y(t)||$$

$$y(t) \in DCR^n, u_Y(t) \in \Omega, t \in (t_0, \infty)$$

3) There exist times $\{t_i\}$, $i=0,1, \dots \in (t_0, \infty)$

and a feedback matrix $K(t) = -\frac{1}{2} B^T(t)S(t)$ where $S(t)$ is the solution to the matrix Riccati equation 4.1 with end point condition $S(t_{i+1}) = S_{i+1}$ such that

$$a) \int_{t_i}^{t_{i+1}} p(s)^{\frac{1}{2}} \left(\frac{||B^T(s)||}{2} \right)^{\frac{1}{2}} ||S_{i+1}||^{\frac{1}{2}} ds \leq c_1 < \frac{1}{2}$$

$$b) \quad |t_{i+1} - t_i| \leq c_2 < \infty$$

c) $\exists S(t)$, a solution to the Riccati equation, with S_{i+1} satisfying a) and

$$\int_{t_i}^{t_{i+1}} \mu(A(t) - \frac{B(t)B^T(t)S(t)}{2}) dt < \ln(1-c_1)$$

NOTE 4.2

This theorem enables the null controllability of equation 4.4 to be determined from the behaviour of the equivalent linear system 4.2 under the particular control law given by $-\frac{1}{2} B^T(t)S(t)x(t)$. In this case a value for $p(t)$, which holds for $u_y(t) \in \Omega \subset \mathbb{R}^m$ and $y(t) \in D \subset \mathbb{R}^n$, is determined and this is used, together with the value $\|B^T(t)\|$, to give the relationship between the length of (t_i, t_{i+1}) and the final value of $\|S_{i+1}\|$. As before, it is necessary to bound the time intervals above, in order that any finite number of such intervals will exist in (t_0, ∞) . The existence of a matrix $S(t)$ which satisfies the above condition implies that the system given by 4.4 is null controllable using the above control law.

4.4 PROOF OF THEOREM 4.1

The proof has a similar structure to that used for Theorem 3.1; however there are differences in detail due to the use of Lemma 4.1 together with a knowledge of the particular structure of the control law. As with Theorem 3.1 the proof is applied to time intervals (t_i, t_{i+1}) and is illustrated here for (t_0, t_1) . The map L is used to compare solutions to equations 4.2 and 4.4 and to show that the solutions are suitably bounded at the end of each interval.

4.4.1 THE EXISTENCE OF A SOLUTION TO EQUATION 4.4

Choose $t_1 \in (t_0, \infty)$ and $S(t_1) = S_1$ such that

$$\int_{t_0}^{t_1} \left(p(s) \frac{||B^T(s)||}{2} \right)^{\frac{1}{2}} ds ||S_1||^{\frac{1}{2}} \leq C_1 < \frac{1}{2}$$

Now consider $\sigma \in \mathbb{R}$, $\sigma > 0$ such that, for a solution $x(t)$ to equation 4.2 with initial condition $x(t_0) = y_0$ and control $u_x(t) = -\frac{1}{2} B^T(t)S(t)x(t)$ and $S(t)$ a solution to 4.1 with $S(t_1) = S_1$,

$$||x(t)|| < \sigma \quad \forall t \in (t_0, t_1)$$

Define, as in Chapter Three,

$$B \triangleq \{x(t) | x(t) \text{ a solution to 4.2 over } (t_0, t_1),$$

$$x(t_0) = y_0, u_x(t) = -\frac{1}{2} B^T(t)S(t)x(t)\} \quad 4.14$$

$$N(x(s)) \triangleq \{d | \sup_{(t_0, t_1)} ||x(s)|| = d, x(s) \in B\} \quad 4.15$$

$$D(d) \triangleq \{x(s) \in B | N(x(s)) \leq d, s \in (t_0, t_1), d > 0\} \quad 4.16$$

The map $L(z(t)) \triangleq w(t)$ is defined exactly as in Chapter Three and hence $w(t)$ is the solution to equation 3.8.

Now consider the range of map L when $z(t) \in D(\sigma/c_1)$ and define $y(t) = L(z(t))$

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + B(t)u_y(t) + f(z, u_z, t) \\ y(t_0) &= x(t_0) \end{aligned} \quad \begin{matrix}) \\) \\ t \in (t_0, t_1) \end{matrix} \quad 4.17$$

Now if $R(t) = ||y(t) - x(t)||$ then, as in Chapter Three, an estimate for $D^+R(t)$ is given by

$$D^+R(t) \leq \mu \left(A(t) + \frac{B(t)B^T(t)S(t)}{2} \right) R(t) + ||f(z, -\frac{1}{2}B^T S z, t)|| \quad 4.18$$

As the system given by equation 4.2 is totally null controllable on (t_0, t_f) , by using Lemma 3.1 equation 4.18 can be written as

$$D^+R(t) \leq -\alpha(t)R(t) + ||f(z, -\frac{1}{2}B^T S z, t)|| \quad 4.19$$

and a comparison function of the usual form gives

$$R(t_1) \leq \int_{t_0}^{t_1} ||f(z, -\frac{1}{2} B^T S z, s)|| \cdot \exp\{ \int_s^{t_1} -\alpha(r) dr \} ds \quad 4.20$$

The nonlinearity $f(\cdot, \cdot, t)$ can be bounded as in inequality 4.3 and this, together with the defined control policy, leads to

$$R(t_1) \leq \int_{t_0}^{t_1} \left(p(s) \cdot \left| \left| \frac{B^T(s) S(s)}{2} \right| \right| \right)^{\frac{1}{2}} ||z(s)|| \cdot \exp\{ \int_s^{t_1} -\alpha(r) dr \} ds \quad 4.21$$

Then from Lemma 4.1

$$R(t_1) \leq \int_{t_0}^{t_1} \left(p(s) \left| \left| \frac{B^T(s)}{2} \right| \right| \right)^{\frac{1}{2}} ||S_1||^{\frac{1}{2}} ||z(s)|| \cdot \exp\{ \int_s^{t_1} -2\alpha(r) dr \} ds \quad 4.22$$

Then

$$R(t_1) \leq \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \left| \frac{B^T(s)}{2} \right| \right| \right)^{\frac{1}{2}} ds \cdot ||S_1||^{\frac{1}{2}} \cdot \sup_{s \in (t_0, t_1)} \left(||z(s)|| \cdot \exp\{ \int_s^{t_1} -2\alpha(r) dr \} \right)$$

and since $\sup_{s \in (t_0, t_1)} \exp\{ \int_s^{t_1} -2\alpha(r) dr \} \leq 1$

the estimate for $R(t_1)$ becomes

$$R(t_1) \leq \sup_{s \in (t_0, t_1)} ||z(s)|| \cdot \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \left| \frac{B^T(s)}{2} \right| \right| \right)^{\frac{1}{2}} ds \cdot ||S_1||^{\frac{1}{2}} \quad 4.23$$

Noticing that $z(s) \in D(\sigma/c_1)$ then the condition 3a) in the Theorem yields

$$R(t_1) \leq (\sigma/c_1) \cdot c_1 = \sigma$$

and thus for any $z(t) \in B$

$$||L(z(t))|| = ||y(t)|| \leq ||y(t) - x(t)|| + ||x(t)|| \leq 2\sigma$$

and then L is an onto map from $D(2\sigma) \rightarrow D(2\sigma)$.

The continuity of map L and the equicontinuity of the family $L(D(2\sigma))$ follow in exactly the same way as sections 3.4.2 and 3.4.3 respectively. Again a Schauder Fixed Point Theorem may be used to show the existence of a solution $y(t)$ such that $y(t) = L(y(t))$; $y(t_0) = x(t_0) = y_0$ and this $y(t)$ is a solution to the nonlinear equation 4.4.

4.4.2 COMPARISON OF THE SOLUTIONS TO EQUATIONS 4.2 AND 4.4

As before let $x(t) \in B$, $x(t_0) = y_0$ and $y(t)$ be a solution to equation 4.4 with $y(t_0) = y_0$.

$$Q(t) \triangleq ||y(t) - x(t)||; Q(t_0) = 0$$

Using the Dini derivative an upper bound upon $Q(t_1)$ is developed in a similar way to section 3.4.4; however, because of the specific form of the control policy and also because of the different bound upon the nonlinearity, the resultant condition differs slightly from that given in Chapter Three:

$$D^+Q(t) \leq \limsup_{h \rightarrow 0^+} \frac{1}{h} (||y(t) - x(t) + h(A(t) + B(t)K(t))(y(t) - x(t))|| + h||f(y, u_y, t)|| + ||y(t) - x(t)||)$$

Then we know that

$$D^+Q(t) \leq -\alpha(t)Q(t) + ||f(y, -\frac{1}{2}B^T S y, t)|| \quad 4.24$$

using the control policy defined by $K(t) = -\frac{1}{2}B^T(t)S(t)$.

Bounding $||f(\cdot, \cdot, t)||$ by the inequality 4.3, the equation 4.24 becomes

$$D^+Q(t) \leq -\alpha(t)Q(t) + p(t)^{\frac{1}{2}} ||-\frac{1}{2}B^T(t)S(t)||^{\frac{1}{2}} ||y(t)||$$

Thus

$$D^+Q(t) \leq \left(-\alpha(t) + p(t)^{\frac{1}{2}} \left| \left| -\frac{1}{2}B^T(t)S(t) \right| \right|^{\frac{1}{2}} \right) Q(t) \\ + p(t)^{\frac{1}{2}} \left| \left| -\frac{1}{2}B^T(t)S(t) \right| \right|^{\frac{1}{2}} \|x(t)\|$$

This yields the estimate for $Q(t_1)$

$$Q(t_1) \leq \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left| \left| -\frac{1}{2}B^T(s) \right| \right|^{\frac{1}{2}} \|S(s)\|^{\frac{1}{2}} \|x(s)\| \cdot \\ \exp \left\{ \int_s^{t_1} -\alpha(r) + p(r)^{\frac{1}{2}} \left| \left| \frac{B^T(r)}{2} \right| \right|^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} ds \quad 4.25$$

From the Lemma 4.1

$$\|S(s)\|^{\frac{1}{2}} \leq \|S_1\|^{\frac{1}{2}} \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\} \quad 4.26 \\ \text{where } S_1 = S(t_1)$$

and also, from the stability theory in Chapter Two

$$\|x(s)\| \leq \|x_0\| \exp \left\{ \int_{t_0}^s -\alpha(r) dr \right\} \quad 4.27$$

Combining 4.26 and 4.27

$$\|S(s)\|^{\frac{1}{2}} \|x(s)\| \leq \|x_0\| \|S_1\|^{\frac{1}{2}} \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\}$$

Introducing this into inequality 4.25 and expanding the integral

$$Q(t_1) \leq \|x_0\| \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \cdot \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \left| \frac{B^T(s)}{2} \right| \right|^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} \cdot \right. \\ \left. \exp \left\{ \int_{t_0}^s \alpha(r) - p(r)^{\frac{1}{2}} \left(\left| \left| \frac{B^T(r)}{2} \right| \right|^{\frac{1}{2}} \cdot \|S(r)\|^{\frac{1}{2}} dr \right\} ds \cdot \right. \\ \left. \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) + p(r)^{\frac{1}{2}} \left(\left| \left| \frac{B^T(r)}{2} \right| \right|^{\frac{1}{2}} \|S(r)\| dr \right\} \right) \quad 4.28$$

This can be written as

$$Q(t_1) \leq \|x_0\| \exp \left\{ \int_{t_0}^{t_1} -2\alpha(r) dr + p(r)^{\frac{1}{2}} \left(\left| \left| \frac{B^T(r)}{2} \right| \right|^{\frac{1}{2}} \right. \right. \\ \left. \left. \|S(r)\| dr \right\} \cdot I \quad 4.29$$

where

$$I \triangleq \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left\| \frac{B^T(s)}{2} \right\| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} \exp\left\{ \int_{t_0}^s \alpha(r) - p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} ds \quad 4.30$$

As in Chapter Three an estimate for I is needed. This follows much the same procedure; however, it is necessary to make some additional estimates due to the inclusion of the term $\|S(\cdot)\|^{\frac{1}{2}}$.

Integration of equation 4.30 by parts yields

$$\begin{aligned} I &= \int_{t_0}^s p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} dr \\ &\quad \exp\left\{ \int_{t_0}^s \alpha(r) - p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} \Bigg|_{s=t_0}^{s=t_1} \\ &\quad - \int_{t_0}^{t_1} \int_{t_0}^s p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} dr \\ &\quad \left[\alpha(s) - p(s)^{\frac{1}{2}} \left(\left\| \frac{B^T(s)}{2} \right\| \right)^{\frac{1}{2}} \|S(s)\|^{\frac{1}{2}} \right] \\ &\quad \cdot \exp\left\{ \int_{t_0}^s \alpha(r) - p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} ds \quad 4.31 \end{aligned}$$

So, substituting limits in the first term and neglecting the negative semidefinite term

$$\begin{aligned} &- \int_{t_0}^{t_1} \int_{t_0}^s p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} ds \cdot \alpha(s) \cdot \\ &\quad \exp\left\{ \int_{t_0}^s \alpha(r) - p(r)^{\frac{1}{2}} \left(\left\| \frac{B^T(r)}{2} \right\| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} ds \end{aligned}$$

the estimate for I becomes

$$\begin{aligned}
I &\leq \int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} dr. \\
&\quad \exp \left\{ \int_{t_0}^{t_1} \alpha(r) - p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} \\
&\quad + \int_{t_0}^{t_1} \int_{t_0}^s p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} dr. \\
&\quad p(s)^{\frac{1}{2}} \left(\left| \frac{B^T(s)}{2} \right| \right)^{\frac{1}{2}} \|S(s)\|^{\frac{1}{2}}. \\
&\quad \exp \left\{ \int_{t_0}^s \alpha(r) - p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} ds \quad 4.32
\end{aligned}$$

The second term in this estimate is less than

$$\begin{aligned}
&\int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} dr. \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \frac{B^T(s)}{2} \right| \right)^{\frac{1}{2}} \|S(s)\|^{\frac{1}{2}} \\
&\quad \exp \left\{ \int_{t_0}^s \text{----} dr \right\} ds
\end{aligned}$$

Then, as $\|S(s)\|^{\frac{1}{2}} \leq \|S_1\|^{\frac{1}{2}} \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\}$, the bound upon I becomes

$$\begin{aligned}
I &\leq \int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S_1\| dr. \\
&\quad \exp \left\{ \int_{t_0}^{t_1} \alpha(r) - p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S(r)\|^{\frac{1}{2}} dr \right\} \\
&\quad + \int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} \|S_1\|^{\frac{1}{2}} dr. \quad I
\end{aligned}$$

Substitution of the resulting estimate for I into 4.29 yields

$$Q(t_1) \leq \|x_0\| \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \cdot \frac{\int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} dr \cdot \|S_1\|^{\frac{1}{2}}}{\left(1 - \int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(\left| \frac{B^T(r)}{2} \right| \right)^{\frac{1}{2}} dr \cdot \|S_1\|^{\frac{1}{2}} \right)}$$

This yields an estimate for $||y(t_1)||$ given by

$$||y(t_1)|| \leq ||x_0|| \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \cdot \left(\frac{\int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(||\frac{B^T(r)}{2}|| \right)^{\frac{1}{2}} dr ||S_1||^{\frac{1}{2}}}{1 - \int_{t_0}^{t_1} p(r)^{\frac{1}{2}} \left(||\frac{B^T(r)}{2}|| \right)^{\frac{1}{2}} dr ||S_1||^{\frac{1}{2}}} + 1 \right) \quad 4.33$$

Now, as $||x_0|| = ||y_0||$ and the integral is bounded as in condition 3a) of the theorem statement, then

$$||y(t_1)|| \leq ||y_0|| \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \cdot \frac{1}{1 - c_1} \quad 4.34$$

The existence of a control policy such that condition 3c) is satisfied ensures that

$$||y(t_1)|| < \lambda ||y_0|| \quad 0 < \lambda < 1$$

This procedure, which has been applied above to the interval (t_0, t_1) , may be shown for each consecutive interval (t_i, t_{i+1}) . Just as for Theorem 3.1 this shows that the nonlinear system 4.4 with nonlinearity bounded by 4.3 is null controllable provided that the conditions of the theorem can be satisfied.

NOTE 4.3

As in Theorem 3.1 the null controllability of the nonlinear system 4.4 hinges around the controllability of the linear system 4.2. It is necessary that the linear system be sufficiently responsive to control in order to dominate the nonlinearity $f(\cdot, \cdot, t)$. This domination is characterised by condition 3). However, the specific form of the control is now given and this leads to a set of

conditions which determine the control feedback gain throughout the interval. In this case the length of the interval (t_i, t_{i+1}) is not bounded above purely by condition 3a) but instead an extra degree of flexibility is introduced by the term $||S_{i+1}||^{\frac{1}{2}}$. The circular nature of the conditions is completed via condition 3c) which specifies a bound over (t_i, t_{i+1}) relating the linear system matrices to the solution of the Riccati equation with $S(t_{i+1}) = S_{i+1}$.

This circularity of condition 3) is similar to that given in Theorem 3.1. However, because the control has additional structure imposed upon it, the condition 3a) yielding the necessary bound upon $R(t)$ is, in this case, too stringent. In fact, in this theorem the condition 3a) may be replaced by the weaker condition

$$\int_{t_i}^{t_{i+1}} p(s)^{\frac{1}{2}} \left(\left| \frac{B(s)}{2} \right| \right)^{\frac{1}{2}} ds ||S_{i+1}||^{\frac{1}{2}} \leq c_1 < 1 \quad 4.34.1$$

This, together with condition 3c), is enough to ensure that the value of $R(t_{i+1})$ (in equation 4.22 $R(t_1)$) satisfies

$$R(t_{i+1}) \leq \left(\sigma / c_1 \right) c_1$$

Consider the case for the interval (t_0, t_1) and notice that, from equation 4.22 we can write

$$R(t_1) \leq ||x_0|| \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \frac{B(s)}{2} \right| \right)^{\frac{1}{2}} ds ||S_1||^{\frac{1}{2}}.$$

$$\exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \quad 4.35$$

owing to the fact that

$$||x(s)|| \leq ||x_0|| \exp \left\{ \int_{t_0}^s -\alpha(r) dr \right\}$$

$$\text{and} \quad \sup_{s \in (t_0, t_1)} \exp\left\{\int_s^{t_1} -\alpha(r) dr\right\} \leq 1$$

This is analogous to 4.23 and for this inequality to yield a bound upon $R(t_1)$ we need

$$\int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \frac{B(s)}{2} \right| \right)^{\frac{1}{2}} ds \cdot \|S_1\|^{\frac{1}{2}} \cdot \exp\left\{\int_{t_0}^{t_1} -\alpha(r) dr\right\} \leq c_1 < \frac{1}{2} \quad 4.36$$

This bound could replace condition 3a) in the theorem. However the condition 3c), used to show that $Q(t_1)$ is sufficiently small, remains the same. This condition results from examining inequality 4.34; an equivalent, but more cumbersome, bound resulting from 4.33 is

$$\exp\left\{\int_{t_0}^{t_1} -\alpha(r) dr\right\} \leq \left[1 - \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \frac{B(s)}{2} \right| \right)^{\frac{1}{2}} ds \cdot \|S_1\|^{\frac{1}{2}} \right] \quad 4.37$$

Now if $\mu\left(A(s) - \frac{B(s)B^T(s)}{2} S(s)\right)$ satisfies inequality 4.37 then the bound 4.36, which is analogous to condition 3a) is also satisfied and hence bound 4.36 is redundant. However, in order to derive 4.37 it was assumed that

$$0 < 1 - \int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \frac{B(s)}{2} \right| \right)^{\frac{1}{2}} ds \cdot \|S_1\|^{\frac{1}{2}}$$

and so it is still necessary that this be satisfied.

To summarise this note: the modified condition 3a) given by 4.36 is implied by the condition 3c) or 4.37 but in order that 4.37 may be derived it is necessary that

$$\int_{t_0}^{t_1} p(s)^{\frac{1}{2}} \left(\left| \frac{B(s)}{2} \right| \right)^{\frac{1}{2}} ds \cdot \|S_1\|^{\frac{1}{2}} \leq c_1 < 1 \quad 4.38$$

Hence the Theorem 4.1 may be restated with condition 3a) replaced by the inequality 4.38.

4.5 NULL CONTROLLABILITY OF EQUATION 4.4 WITH UNIFORMLY CONTROLLABLE LINEAR EQUATION 4.2

The conditions upon the linear system in Theorem 4.1 and the control law used have a close resemblance to the conditions demanded of a uniformly controllable linear system. In the case of uniform controllability bounds can be placed upon the system which are independent of time: another definition of uniform controllability, which is equivalent to definition 1.6, bounds the controllability matrix $W(t, t+\sigma)$ above and below by scalars which are dependent upon the interval length σ but independent of t . Notice that a controllable system satisfies this boundedness condition upon the controllability matrix except that the bound will not be independent of time - hence the name uniform.

Kalman^{K2} has shown that if a linear system is uniformly controllable, there exists a continuous feedback matrix, of the form $-\frac{1}{2}B^T(t)S(t)$, such that the linear system under this control is exponentially stable.

The system is given by

$$\dot{x}(t) = \left(A(t) - \frac{B(t)B^T(t)}{2} S(t) \right) x(t) \quad 4.39$$

and there exist positive scalars M and β_1 , independent of time, such that

$$||x(s)|| \leq ||x_0|| M \exp(-\beta_1(s-t_0))$$

A consequence of this condition is that the transition matrix corresponding to equation 4.39 can be bounded by

$$||\phi(s, t_0)|| \leq N \exp(-\beta_2(s-t_0))$$

where N and β_2 are positive scalars independent of t_0 .

These necessary and sufficient conditions for the exponential

stability of 4.39 are determined by the uniform controllability of 4.2 and are very similar to the bounds implied by condition 3c) in Theorem 4.1. If condition 3c) is modified to become a uniform bound upon $\mu\left(A(s) + \frac{B(s)B^T(s)}{2} S(s)\right)$ then this can imply that the differential equation 4.39 is exponentially stable.

Consider the following result in which condition 3c) is replaced by a bound upon $\mu(\cdot)$ which is independent of time. Then the condition 3d) need only be checked over one interval whilst still ensuring the null controllability of the nonlinear system over (t_0, ∞) .

THEOREM 4.2

The nonlinear differential equation given by 4.4 with initial condition $y(t_0) = y_0$ is null controllable on (t_0, ∞) using control $u_y(t) \in \Omega$ provided that the linear system given by equation 4.2 with initial condition $x(t_0) = y_0$ is totally controllable on (t_0, ∞) and the following conditions are satisfied

$$1) \quad f(0,0,t) \equiv 0 \quad \forall t \in (t_0, \infty)$$

$$2) \quad \exists p(t) > 0, \text{ scalar such that}$$

$$\|f(y, u_y, t)\|^2 \leq p(t) \|y(t)\| \|u_y(t)\|$$

$$y(t) \in \mathbb{R}^n, u_y(t) \in \Omega \subset \mathbb{R}^m$$

$$t \in (t_0, \infty)$$

3) There exist $\{t_i\} \ i=0,1,2 \dots \in (t_0, \infty)$ and a feedback matrix $K(t)$ defined on (t_i, t_{i+1}) with $K(t) = -\frac{1}{2}B^T(t)S(t)$, where $S(t)$ is a solution to the Riccati equation 4.1 with end condition $S(t_{i+1}) = S_f$ such that

$$a) \int_{t_i}^{t_{i+1}} p^{\frac{1}{2}}(s) \left(\left\| \frac{B^T(s)}{2} \right\| \right)^{\frac{1}{2}} \|S_f\|^{\frac{1}{2}} ds \leq c_1 < \frac{1}{2}$$

$$b) |t_{i+1} - t_i| \leq c_2 < \infty$$

c) For all $\bar{t} \in (t_0, \infty)$ the above feedback with end condition $S(\bar{t}) = S_f$ is such that

$$\mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) \leq -\beta < 0 \quad s \in (\bar{t} - c_2, \bar{t})$$

$$d) -\beta \cdot (t_{j+1} - t_j) \leq \ln(1 - c_1)$$

where (t_j, t_{j+1}) is the smallest of the intervals (t_i, t_{i+1}) .

PROOF

The proof follows from the proof of Theorem 4.1, the only change required is to notice that the combination of conditions 3c) and 3d) in the theorem comprise a sufficient condition that 3c) of Theorem 4.1 be satisfied: if condition 3c) is satisfied then certainly, at times $\bar{t} = t_{i+1}$, we can write

$$\exp \left\{ \int_{t_i}^{t_{i+1}} \mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) ds \right\} \leq \exp \{-\beta(t_{i+1} - t_i)\}$$

and if $\beta(t_{i+1} - t_i)$ satisfies condition 3d) on the smallest interval (t_j, t_{j+1}) it also satisfies 3d) on the remaining intervals.

It is interesting to notice how 3c) and 3d) are related to uniform controllability via exponential stability of equation 4.39. The transition matrix $\phi(\cdot, \cdot)$ of equation 4.39 can be related to $\mu(\cdot)$ via

$$\|\phi(s, t_0)\| \leq \exp \left\{ \int_{t_0}^s \mu \left(A(r) - \frac{B(r)B^T(r)}{2} S(r) \right) dr \right\}$$

and so, if $\mu(\cdot)$ is uniformly bounded as in 3c), the linear

equation 4.39 has transition matrix such that.

$$||\phi(s, t_0)|| \leq \exp(-\beta(s-t_0))$$

and hence is exponentially stable. Equation 4.39 is also exponentially stable if 4.2 is uniformly controllable.

4.6 FOR A NONLINEARITY BOUND INVOLVING ONLY TERMS IN CONTROL

As an introduction to the situation where the nonlinearity is bounded in norm by inequality 2.39, it is instructive to consider the special case when the bound upon the term $||f(\cdot, \cdot, t)||$ depends only upon control.

$$||f(y, u_y, t)|| \leq q(t) ||u_y(t)||$$

$$\forall y \in D \subset \mathbb{R}^n, u_y \in \Omega \subset \mathbb{R}^m \quad 4.40$$

$$t \in (t_0, \infty)$$

The choice of this bound upon $f(\cdot, \cdot, t)$ is motivated by the fact that, when the control is of the form specified in Theorem 4.1, a weaker restriction than that used in Theorem 3.1 can be demanded in condition 3c). In the proof of Theorem 4.1 the bound upon $Q(t_1)$ given by 4.29 contains an additional term of the form $\exp\{\int_s^{t_1} -\alpha(r)dr\}$. For the controls considered $-\alpha(r) \leq 0$ and hence this term is less than unity. This is used to advantage to weaken the conditions demanded of the linear system 4.2. The ability to bound the nonlinearity by 4.40 would enable us to introduce a term $\exp\{\int_s^{t_1} -2\alpha(r)dr\}$ at an equivalent position in the bound upon $Q(t_1)$ and this would allow a still weaker restriction upon the linear components in the nonlinear equation.

We shall show later that the analysis which leads to a result when bound 4.40 is in force may be generalised and

applied to the problem when the nonlinearity is bounded by a linear function of $||y(t)||$ and $||u_y(t)||$.

THEOREM 4.3

The nonlinear equation given by 4.4, with initial condition $y(t_0) = y_0$ is null controllable on (t_0, ∞) using control $u_y(t) \in \Omega$ provided that the linear system given by equation 4.2, with initial condition $x(t_0) = y_0$ is totally null controllable and the following conditions are satisfied:

- 1) $f(0,0,t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) There exists $q(t)$, scalar $q(t) > 0$ such that

$$||f(y, u_y, t)|| \leq q(t) ||u_y(t)||$$

$$y(t) \in DCR^n, u_y(t) \in \Omega CR^m$$

$$t \in (t_0, \infty)$$
- 3) There exists $\{t_i\}$, $i=0,1,2 \dots \in (t_0, \infty)$ and a feedback matrix $K(t)$ defined upon (t_i, t_{i+1}) by $K(t) = -\frac{1}{2}B^T(t)S(t)$ where $S(t)$ is a solution to the Riccati equation 4.1 with end condition $S(t_{i+1}) = S_{i+1}$ such that
 - a) $\int_{t_i}^{t_{i+1}} q(s) \left(||\frac{B^T(s)}{2}|| \right) ds ||S_{i+1}|| \leq c_1 < \frac{1}{2}$
 - b) $|t_{i+1} - t_i| \leq c_2 < \infty$
 - c) There exists $S(t)$, a solution to the Riccati equation such that

$$\int_{t_i}^{t_{i+1}} \mu \left\{ A(s) - \frac{B(s)B^T(s)S(s)}{2} \right. \\ \left. + q(s) \left(||\frac{B^T(s)}{2}|| \right) ||S_{i+1}|| \right\} ds < 0 \quad 4.41$$

PROOF

The proof follows the usual procedures given in detail in the previous theorems. As in Theorem 4.1 the restrictions given in condition 3) are those bounds necessary to impose the desired upper limits upon $R(t_1)$ and $Q(t_1)$. As in the previous cases the bound upon $R(t_1)$, in order to use the Schauder Fixed Point Theorem upon $L(\cdot)$, follows without difficulty. The development of the bound for $Q(t_1)$ follows that for Theorem 4.1 but the introduction of the extra exponential term $\exp\{\int_s^{t_1} -\alpha(r)dr\}$ leads to the simpler bound in condition 3c).

With $R(t)$ defined as before the expression equivalent to 4.21 is

$$R(t_1) \leq \int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| \|S(s)\| \|z(s)\| \exp\left\{\int_s^{t_1} -\alpha(r)dr\right\} ds \quad 4.42$$

From the lemma bounding $\|S(s)\|$ this may be written

$$R(t_1) \leq \int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| \|S_1\| \|z(s)\| \exp\left\{\int_s^{t_1} -3\alpha(r)dr\right\} ds \quad 4.43$$

As $\alpha(r)$ is positive a sufficient condition for $R(t_1) < \sigma$ is

$$\int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ds \|S_1\| \leq c_1 < \frac{1}{2} \quad 4.44$$

and condition 3a) ensures that this is satisfied.

The Schauder Fixed Point Theorem again shows the existence of a solution to equation 4.2 over (t_0, t_1) . If the difference between the solutions to 4.2 and 4.4 is

given by $Q(t) \triangleq ||y(t) - x(t)||$ then, equivalent to 4.25, we have

$$Q(t_1) \leq \int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ||S(s)|| \cdot ||x(s)|| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) + q(r) \left\| \frac{B^T(r)}{2} \right\| ||S(r)|| dr \right\} ds \quad 4.45$$

This may be written as

$$Q(t_1) \leq ||x_0|| \exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \cdot \int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ||S_1|| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\} \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) + q(r) \left\| \frac{B^T(r)}{2} \right\| ||S(r)|| dr \right\} ds \quad 4.46$$

Or equivalently

$$Q(t_1) \leq ||x_0|| \cdot \exp \left\{ \int_{t_0}^{t_1} -2\alpha(r) + q(r) \left\| \frac{B^T(r)}{2} \right\| ||S(r)|| dr \right\} \cdot I$$

where

$$I \triangleq \int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ||S_1|| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\} \cdot \exp \left\{ \int_{t_0}^s \alpha(r) - q(r) \left\| \frac{B^T(r)}{2} \right\| ||S(r)|| dr \right\} ds$$

However, as in Theorem 4.1, if

$$\int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ||S_1|| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\} \leq 1$$

then a bound for I is

$$I \leq \frac{\int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ||S_1|| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\} ds}{1 - \int_{t_0}^{t_1} q(s) \left\| \frac{B^T(s)}{2} \right\| ||S_1|| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) dr \right\} ds \cdot \exp \left\{ \int_{t_0}^{t_1} \alpha(r) - q(r) \left\| \frac{B^T(r)}{2} \right\| ||S(r)|| dr \right\}} \quad 4.47$$

So the bound for $Q(t_1)$ given by 4.46 becomes .

$$Q(t_1) \leq ||x_0|| \exp\{\int_{t_0}^{t_1} -\alpha(r)dr\}.$$

$$\frac{\int_{t_0}^{t_1} q(s) ||\frac{B^T(s)}{2}|| ||S_1|| \cdot \exp\{\int_s^{t_1} -\alpha(r)dr\}ds}{1 - \int_{t_0}^{t_1} q(s) ||\frac{B^T(s)}{2}|| ||S_1|| \cdot \exp\{\int_s^{t_1} -\alpha(r)dr\}ds} \quad 4.48$$

Note that, apart from the extra terms, $\exp\{\int_s^{t_1} -\alpha(r)dr\}$, this expression has the same form as that given for Theorem 4.2 in inequality 4.33.

This leads to a bound upon $y(t_1)$ given by

$$||y(t_1)|| \leq ||x_0|| \exp\{\int_{t_0}^{t_1} -\alpha(r)dr\}.$$

$$\frac{1}{1 - \int_{t_0}^{t_1} q(s) ||\frac{B^T(s)}{2}|| ||S_1|| \cdot \exp\{\int_s^{t_1} -\alpha(r)dr\}ds}$$

Now in order that $||y(t_1)|| \leq \lambda ||y_0||$, $0 < \lambda < 1$

we need

$$\exp\{\int_{t_0}^{t_1} -\alpha(r)dr\} < 1 - \int_{t_0}^{t_1} q(s) \cdot ||\frac{B^T(s)}{2}|| ||S_1|| \cdot$$

$$\exp\{\int_s^{t_1} -\alpha(r)dr\}ds \quad 4.49$$

$$\begin{aligned} \text{Now } 1 - \exp\{\int_{t_0}^{t_1} -\alpha(r)dr\} &= \int_{s=t_0}^{s=t_1} \frac{d}{ds} \exp\{\int_s^{t_1} -\alpha(r)dr\}ds \\ &= \int_{t_0}^{t_1} \alpha(s) \exp\{\int_s^{t_1} -\alpha(r)dr\}ds. \end{aligned}$$

So 4.49 is satisfied if

$$0 < \int_{t_0}^{t_1} (\alpha(s) - q(s) ||\frac{B^T(s)}{2}|| ||S_1||) \cdot \exp\{\int_s^{t_1} -\alpha(r)dr\}ds$$

This requires that

$$\int_{t_0}^{t_1} (\alpha(s) - q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| S_1 \right\|) ds > 0 \quad 4.50$$

which is exactly condition 3c) of Theorem 4.3 and the proof of the theorem is completed in the same way as Theorem 4.1.

The implications of this result are clear from bound 4.50: if there exists a control law such that the linear section of the nonlinear equation "dominates" the maximum possible value of $\|f(\cdot, \cdot, t)\|$ over the intervals (t_i, t_{i+1}) then such a control will drive the nonlinear system towards the null solution. An obvious sufficient condition which illustrates more clearly this "dominance" is

$$-\mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) > q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| S_{i+1} \right\| \quad \forall s \in (t_i, t_{i+1})$$

The scalar term $-\mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right)$ represents the dynamic performance of the linear section of the nonlinear equation and if this is greater than the scalar term $q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| S_{i+1} \right\|$ which is known to bound the value of $\|f(y, u_y, s)\|$ over the interval (t_i, t_{i+1}) then the solution of the nonlinear equation has norm which decreases over the interval.

4.7 FOR A NONLINEARITY BOUND INVOLVING TERMS IN STATE AND CONTROL

We are now in a position to consider the case where the nonlinearity is bounded by

$$\begin{aligned} \|f(y, u_y, t)\| &\leq p(t) \|y(t)\| + q(t) \|u_y(t)\| & 4.51 \\ &\forall y(t) \in D \subset \mathbb{R}^n, u_y(t) \in \Omega \subset \mathbb{R}^m & (2.39) \\ &t \in (t_0, \infty) \end{aligned}$$

It can be seen from the previous section that the form of the bound on the nonlinearity implies the form of the restriction upon the linear section of the nonlinear equation given in condition 3c) of the theorems. Condition 3c) derives from the need to put an upper bound upon $Q(t_1)$. If the nonlinearity is bounded by 4.51 then it is possible to rewrite this bound so that a similar approach to Theorems 3.1 or 4.1 or 4.3 may be used.

For instance, following 3.1, when 4.51 is the bound used, $Q(t_1)$ is given by

$$Q(t_1) \leq \int_{t_0}^{t_1} \left(p(s) + q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| S(s) \right\| \right) \cdot \left\| x(s) \right\| \cdot \exp \left\{ \int_s^{t_1} -\alpha(r) + p(r) + q(r) \left\| \frac{B^T(r)}{2} \right\| \left\| S(r) \right\| dr \right\} ds \quad 4.52$$

then defining a new scalar $\bar{p}(t)$

$$\bar{p}(t) \triangleq \left(p(t) + q(t) \left\| \frac{B^T(t)}{2} \right\| \left\| S(t) \right\| \right) \quad t \in (t_0, \infty)$$

gives a condition 3c) of the form,

$$\exp \left\{ \int_{t_0}^{t_1} -\alpha(r) dr \right\} \leq \frac{1 - 2 \int_{t_0}^{t_1} \bar{p}(s) ds}{1 - \int_{t_0}^{t_1} \bar{p}(s) ds} \quad 4.53$$

The most useful result can be obtained when the approach used in the proof is similar to that adopted for Theorem 4.3. In this case the proof takes advantage of additional terms of the form $\exp \left\{ \int_s^{t_1} -2\alpha(r) dr \right\}$ and this results in an interesting condition imposed upon the linear system in condition 3c) of the theorem.

In order to extend Theorem 4.3 it is necessary to define another $\bar{p}(t)$ by

$$\bar{p}(t) \triangleq \exp\left\{\int_t^{t_1} 2\alpha(r)dr\right\} p(t) \quad \forall t \in (t_0, t_1) \quad 4.54$$

Now the expression 4.52 bounding $Q(t_1)$ can be written

$$Q(t_1) \leq \int_{t_0}^{t_1} \left(\bar{p}(s) + q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| s_1 \right\| \right) \cdot \exp\left\{\int_s^{t_1} -2\alpha(r)dr\right\} \cdot \left\| x(s) \right\| \cdot \exp\left\{\int_s^{t_1} -\alpha(r) + p(r) + q(r) \left\| \frac{B^T(r)}{2} \right\| \left\| s(r) \right\| dr\right\} ds$$

This bound for $Q(t_1)$ has the same form as 4.46 and the analysis proceeds exactly as before. The condition that results in order that $\left\| y(t_1) \right\| < \lambda \left\| y_0 \right\|$, $0 < \lambda < 1$, equivalent to 4.50 is

$$\int_{t_0}^{t_1} \left(\alpha(s) - q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| s_1 \right\| - \bar{p}(s) \right) ds > 0 \quad 4.55$$

or, substituting from 4.54

$$\int_{t_0}^{t_1} \alpha(s) - q(s) \left\| \frac{B^T(s)}{2} \right\| \left\| s_1 \right\| - p(s) \cdot \exp\left\{\int_s^{t_1} 2\alpha(r)dr\right\} ds > 0 \quad 4.56$$

The following theorem states more rigorously the ideas put forward above. Although the equation 4.54 is introduced only to enable the analysis of Theorem 4.3 to be applied to the bound 4.51, it can also be seen in a slightly different light: the term $\exp\left\{\int_s^{t_1} -\alpha(r)dr\right\}$ is related to the transition matrix $\phi(s, t_1)$ of the linear equation

$$\dot{x}(t) = \left(A(t) - \frac{B(t)B^T(t)S(t)}{2} \right) x(t)$$

In fact, from Chapter Two, we know

$$\left\| \phi(s, t_1) \right\| \leq \exp\left\{\int_s^{t_1} -\alpha(r)dr\right\}$$

The term $p(s) \cdot \exp \left\{ \int_s^{t_1} -2\alpha(r) dr \right\}$ is certainly greater than

$||\phi(s, t_1)||^2 p(s)$ which can be regarded as the net effect of $p(t)$ upon the linear system trajectory carried forward in time to be compounded at t_1 .

THEOREM 4.4

The nonlinear equation given by 4.4, with initial condition $y(t_0) = y_0$, is null controllable on (t_0, ∞) using control $u_y(t) \in \Omega$ provided that the linear system given by equation 4.2, with initial condition $x(t_0) = y_0$, is totally null controllable and the following conditions are satisfied.

- 1) $f(0, 0, t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) There exist scalars $p(t), q(t) > 0$ such that

$$||f(y, u_y, t)|| \leq p(t) ||y(t)|| + q(t) ||u_y(t)||$$

$$y(t) \in D \subset \mathbb{R}^n, u_y(t) \in \Omega \subset \mathbb{R}^m$$

$$t \in (t_0, \infty)$$
- 3) There exist $\{t_i\}$, $i=0, 1, \dots \in (t_0, \infty)$ and a feedback matrix $K(t)$ defined upon (t_i, t_{i+1}) with

$$K(t) = -\frac{1}{2} B^T(t) S(t)$$
 and end condition $S(t_{i+1}) = S_{i+1}$ such that
 - a) $\int_{t_i}^{t_{i+1}} p(s) + q(s) ||\frac{B^T(s)}{2}|| ||S_{i+1}|| ds \leq c_1$
 - b) $|t_{i+1} - t_i| \leq c_2 < \infty$
 - c) There exists $S(t)$ a solution to the Riccati equation such that

$$\int_{t_i}^{t_{i+1}} \mu \left(A(s) - \frac{B(s) B^T(s) S(s)}{2} \right) + q(s) ||\frac{B^T(s)}{2}|| ||S_{i+1}||$$

$$+ p(s) \exp \left\{ \int_s^{t_{i+1}} 2\mu \left(A(r) - \frac{B(r) B^T(r) S(r)}{2} \right) dr \right\} ds < 0$$

PROOF

The proof follows exactly that given for Theorem 4.3 with the substitution of

$$\bar{p}(s) \exp\left\{\int_s^{t_{i+1}} -2\alpha(r)dr\right\} = p(s) \quad s \in (t_i, t_{i+1})$$

in order that the bounds are of the required form.

It is interesting to note here two points regarding the restriction 3c):

- An obvious sufficient condition which can replace bound 3c) and which may be simpler to evaluate is

$$\begin{aligned} \int_{t_i}^{t_{i+1}} p(s) ds \cdot \exp\left\{\int_{t_i}^{t_{i+1}} 2\alpha(r)dr\right\} &\leq \int_{t_i}^{t_{i+1}} \alpha(s) ds \\ &- \int_{t_i}^{t_{i+1}} q(s) \left\| \frac{B^T(s)}{2} \right\| ds \|S_{i+1}\| \end{aligned} \quad 4.57$$

or equivalently

$$\begin{aligned} \int_{t_i}^{t_{i+1}} p(s) ds \cdot \frac{\|x_0\|^2}{\|x(t_1)\|^2} &\leq \int_{t_i}^{t_{i+1}} -\mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) ds \\ &- \int_{t_i}^{t_{i+1}} q(s) \left\| \frac{B^T(s)}{2} \right\| ds \|S_{i+1}\| \end{aligned} \quad 4.58$$

- If the nonlinearity is bounded by 4.51 and $q(t) = 0$ over (t_0, ∞) then how does this result relate to that of Theorem 3.1?

The bound upon $Q(t_{i+1})$ in Theorem 3.1 requires that

$$\exp\left\{\int_{t_i}^{t_{i+1}} -\alpha(r)dr\right\} \leq \frac{1 - 2 \int_{t_i}^{t_{i+1}} p(s) ds}{1 - \int_{t_i}^{t_{i+1}} p(s) ds} \quad 4.59$$

whereas the equivalent bound upon $Q(t_{i+1})$ in Theorem 4.4 with $q(t) = 0$ requires that

$$\int_{t_i}^{t_{i+1}} \alpha(s) ds - \int_{t_i}^{t_{i+1}} p(s) \cdot \exp \left\{ \int_s^{t_{i+1}} 2\alpha(r) dr \right\} ds \geq 0$$

and a sufficient condition for this to be satisfied is

$$\int_{t_i}^{t_{i+1}} \alpha(s) \cdot \exp \left\{ \int_s^{t_{i+1}} -2\alpha(r) dr \right\} ds \geq \int_{t_i}^{t_{i+1}} p(s) ds$$

or

$$\frac{1}{2} \left[\int_{t_i}^{t_{i+1}} \frac{d}{ds} \exp \left\{ \int_s^{t_{i+1}} -2\alpha(r) dr \right\} ds \right] \geq \int_{t_i}^{t_{i+1}} p(s) ds \quad 4.60$$

So one condition that ensures that the bound on $Q(t_{i+1})$ in Theorem 4.4 is sufficiently small is

$$\exp \left\{ \int_{t_i}^{t_{i+1}} -2\alpha(r) dr \right\} \leq 1 - 2 \int_{t_i}^{t_{i+1}} p(s) ds$$

But any $\alpha(r)$ which satisfies this will also satisfy

$$\exp \left\{ \int_{t_i}^{t_{i+1}} -2\alpha(r) dr \right\} \leq \left(1 - \int_{t_i}^{t_{i+1}} p(s) ds \right)^2$$

and hence will satisfy

$$\exp \left\{ \int_{t_i}^{t_{i+1}} -\alpha(r) dr \right\} \leq 1 - \int_{t_i}^{t_{i+1}} p(s) ds \quad 4.61$$

A comparison of inequalities 4.59 and 4.61 shows that any $\alpha(r)$ which satisfies the conditions of Theorem 3.1 will also satisfy inequality 4.61 which is a sufficient condition for Theorem 4.4 in the special case when $q(t) = 0$. So, when the nonlinearity is bounded by 2.38, Theorem 4.4 with $q(t) = 0$ will allow bigger values of $p(s)$ and hence a larger class of nonlinearities for any particular $\alpha(r)$ on (t_i, t_{i+1}) than Theorem 3.1. In Theorem 3.1, however, the form of the feedback is not specified and the conditions are somewhat simpler. This may make the implementation of

the earlier theorem more attractive when the nonlinear bounds involve only $||x(t)||$.

4.8 CONCLUSIONS

The results given in this chapter illustrate several methods of approaching the null controllability problem for the case where the nonlinear function can be bounded by a first order term involving state and/or control. The control law used has a specific form related to the minimum energy controller for the linear system 4.2.

Gershwin and Jacobson^{G1} use a similar control law in their paper for the cases where the nonlinearity can be written

$$f(y, u_y, t) = - m(y, t) \cdot y(t) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \\ \\ m(-, t) \text{ scalar} \end{array} \quad 4.62a$$

$$\text{or } f(y, u_y, t) = - m(y, u_y, t) \cdot y(t) \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \\ \\ m(-, t) \text{ scalar} \end{array} \quad 4.62b$$

In the cases where $m(-, t)$ is positive semidefinite for this specific choice of control law Gershwin & Jacobson conclude that this linear control policy will force the system to be null controllable. They are unable to handle those nonlinearities where the scalar $m(-, t)$ can take negative values corresponding to an unstable function. In their Example Two, Gershwin and Jacobson show that the system 4.4 is null controllable using a linear feedback control of the form

$$u_y(t) = -\frac{1}{2}B^T(t)S(t)y(t)$$

if the nonlinearity is a stable function which can be written as in 4.62. This situation is a special case of the Theorems discussed in this chapter: if f can be written to satisfy 4.62

then it can certainly satisfy at least one of the norm bounds considered here. The bounds used in the last two chapters do not differentiate between stable and unstable functions and hence are applicable to those cases where $m(\cdot, t)$ has either positive or negative values.

In the light of Gershwin and Jacobson's results and their statement that stable functions of the form 4.62 (positive values of $m(\cdot, t)$) may be ignored when designing the controller, it is possible to restate some of the above results for the particular case where the nonlinearities satisfy either 4.62a or 4.62b. For instance, Theorem 3.1 can be applied to nonlinearities satisfying 4.62a when $m(y, t)$ is negative, but Gershwin and Jacobson state that the nonlinearity can be ignored for $m(y, t)$ positive. So if the nonlinearity f is written as

$$f(y, u_y, t) = -m(y, t) \cdot y(t)$$

$$m(y, t) \text{ scalar}$$

$$y(t) \in D \subset \mathbb{R}^n, u_y(t) \in \Omega_k \subset \mathbb{R}^m$$

$$t \in (t_0, \infty)$$

and then if $p(t)$ in bound 2.38 is defined by

$$p(t) \triangleq \sup_{y \in D} |m(y, t)| : m(y, t) < 0$$

$$p(t) \triangleq 0 : m(y, t) \geq 0$$

the setting of $p(t) = 0$ when the nonlinearity is stable means that when the intervals (t_i, t_{i+1}) are calculated the effect of the nonlinearity is ignored and the control chosen as if for a linear system. During the periods when $p(t) \neq 0$ the analysis from Theorem 3.1 applies as before.

Although the fact that the previous theory does not

distinguish between stable and unstable nonlinearities enables the consideration of a wider class of functions $f(\cdot, \cdot, t)$ it also leads to the demand for unnecessarily large values of control in some situations and this is due to the fact that we have chosen to bound the nonlinearities by norm functions. In an effort to use information about the direction of the vector $f(\cdot, \cdot, t)$ inner product bounds are considered in the following chapter.

CHAPTER 5

- 5.1 Introduction.
- 5.2 An upper bound for solution norms using inner products.
- 5.3 A null controllability theorem for inner product bounded nonlinearities.
- 5.4 Uniqueness of the solution to equation 5.1.
- 5.5 Extension of these results to equivalent inner products.
- 5.6 For the case where the nonlinearity is bounded by inner products involving control terms.
- 5.7 Conclusions.

5.1 INTRODUCTION

The inner product bounds upon nonlinear functions discussed in Chapter Two are of two types: those where the function is weakly linear and hence the inner products involve an arbitrary vector $z \in \mathbb{R}^n$ and those bounds where this arbitrary vector is specifically chosen to be $y(t) \in \mathbb{R}^n$, the state vector. It is these latter bounds that are considered here. These bounds are the logical extension of the norm bounds, used in Chapters Three and Four, to allow negative terms to occur on the righthand side and thus take into account the occasions when the nonlinear function is a stable one.

In this chapter we shall consider the null controllability properties of the nonlinear system

$$\begin{aligned} \dot{y}(t) &= A(t)y(t) + B(t)u_y(t) + f(y, u_y, t) \\ &\text{where } y(t) \in \mathbb{R}^n, u_y(t) \in \mathbb{R}^m \\ &\quad f(y, u_y, t) \in \mathbb{R}^n \end{aligned} \tag{5.1}$$

and where $A(t)$, $B(t)$ are matrices of continuous, possibly timevarying, elements which have appropriate dimensions.

The definition of null controllability given in Chapter Three will be considered, but in addition it will be useful to look at another definition of null controllability where the norm of the solution at the final time is specified in the definition.

DEFINITION 5.1

The system given by 5.1 is said to be "null controllable from (y_0, t_0) to (ϵ, T) " if there exists an admissible control $u_y(t)$ such that $y(t_0) = y_0$ implies that $\|y(T)\| < \epsilon$.

This definition is exactly equivalent to that given by Ben Lashiher and Storey^{B2}; Definition 4.5 with their $\alpha = y_0$ and $\beta = \varepsilon$ and their time interval J being (t_0, T) . The results of the previous two chapters could have been cast in the framework of this definition; however, it is then necessary to state a relationship between the length of the interval (t_0, T) , the length of the intervals (t_i, t_{i+1}) $i = 0, 1, 2, \dots$ and the size of the multiplier $\lambda < 1$ such that $\|y(t_{i+1})\| \leq \lambda \|y(t_i)\|$. This is not required for definition 3.1.

The above definition is closely related to the concept of final stability, introduced initially by Weiss (see for instance **W3**), and this serves to illustrate the parallel between controllability and stability when the controllability problem is set up with the desired final state contained in a sphere around the origin. This parallel relationship can lead to other definitions of controllability each corresponding to an equivalent definition in stability theory.

The results given previously used the scalar comparison functions as a fundamental tool in the proof to bound above the solution to a differential inequality of the form $D^+m(t) \leq -\alpha(t)m(t) + g(m(t), t)$. In order to use a comparison theorem for a differential inequality it is necessary to ensure that the function $g(m(t), t)$ is positive semidefinite, as stated in Lakshmikantham and Leela^{L1}. This is not necessarily so in the case where inner product bounds are used. For this reason the approach given previously cannot be applied to yield a null controllability result for equation 5.1 when the nonlinearity $f(\cdot, \cdot, t)$ is

bounded by

$$\langle f(y, u_y, t), y(t) \rangle \leq p(t) \|y(t)\|^2 \quad 5.2$$

$$\text{when } f(y, u_y, t) \in R^n, y(t) \in DCR^n, u_y(t) \in \Omega C R^m \quad (2.46) \\ t \in (t_0, \infty)$$

or by

$$\langle f(y, u_y, t), y(t) \rangle \leq p(t) \|y(t)\|^2 \\ + q(t) |\langle Q(t) u_y(t), y(t) \rangle| \quad 5.3$$

$$\text{when } f(y, u_y, t) \in R^n, y(t) \in DCR^n, u_y(t) \in \Omega C R^m \quad (2.47) \\ t \in (t_0, \infty)$$

Instead a more direct method is appropriate where the solution norm to the nonlinear equation 5.1 is directly bounded above using estimates derived from the inner product.

The next section considers, as an example, this technique applied to a differential equation involving no control terms.

5.2 AN UPPER BOUND FOR SOLUTION NORMS USING INNER PRODUCTS

Consider the differential equation

$$\dot{x}(t) = A(t)x(t) + g(x, t) \quad 5.4$$

$$\text{where } g(\cdot, t), x(t) \in R^n$$

Now, by definition, we can write

$$D^+ |x(t)|^2 = \limsup_{h \rightarrow 0^+} \frac{1}{h} |x(t+h)|^2 - |x(t)|^2 \quad 5.5$$

Here, as in Chapter Two, $|\cdot|$ represents a scalar function of R^n satisfying the norm axioms. Now if $|\cdot|$ is the norm subordinate to the inner product $\langle \cdot, \cdot \rangle$ following Pao^{P1}, equation 5.5 can be written

$$D^+ |x(t)|^2 = \limsup_{h \rightarrow 0^+} \frac{1}{h} \langle x(t+h), x(t+h) \rangle - \langle x(t), x(t) \rangle$$

From the linearity of the inner product this is equivalent to

$$D^+ |x(t)|^2 = \limsup_{h \rightarrow 0^+} \frac{1}{h} \left[\langle x(t+h) - x(t), x(t+h) \rangle - \langle x(t), x(t+h) - x(t) \rangle \right] \quad 5.6$$

As $h \rightarrow 0^+$ this becomes

$$\begin{aligned} D^+ |x(t)|^2 &= 2 \langle D^+ x(t), x(t) \rangle \\ &= 2 \langle (A(t)x(t) + g(x, t)), x(t) \rangle \end{aligned}$$

Now in the special case $g(x, t) = 0$

$$D^+ |x(t)|^2 = 2 \langle A(t)x(t), x(t) \rangle \quad 5.7$$

If $\langle a, b \rangle$ is defined as $a^T b$ then

$$D^+ ||x(t)||^2 = x^T(t) [A(t) + A^T(t)] x(t)$$

$$D^+ ||x(t)||^2 \leq 2\mu(A(t)) ||x(t)||^2$$

$$\text{or } D^+ ||x(t)|| \leq \mu(A(t)) ||x(t)||$$

which is equivalent to the bound upon $||x(t)||$ given in Chapter Two, section 2.5, as a demonstration of the use of the logarithmic norm in stability theory and a scalar comparison function yields an estimate of $||x(t_1)||$

$$||x(t_1)|| \leq ||x(t_0)|| \cdot \exp\left\{\int_{t_0}^{t_1} \mu(A(s)) ds\right\} \quad 5.8$$

For the above example, if the perturbing function $g(x, t)$ is non zero this leads to an addition term in equation 5.7 of the form $\langle g(x, t), x(t) \rangle$ and if this term can be bounded by some scalar multiple, $p(t)$ say, of $||x(t)||^2$ then an upper bound for $D^+ ||x(t)||^2$ is

$$D^+ ||x(t)||^2 \leq 2[\mu(A(t)) + p(t)] ||x(t)||^2$$

In a similar fashion this can give the estimate for $||x(t_1)||$

$$||x(t_1)|| \leq ||x(t_0)|| \cdot \exp\left\{\int_{t_0}^{t_1} \mu(A(s)) + p(s) ds\right\}$$

Inequality 5.8 results from the choice of the inner

product $\langle a, b \rangle = a^T b$ and the norm, the Euclidean norm, subordinate to that inner product.

The choice of other equivalent inner products yield similar inequalities: consider those equivalent inner products which can be characterised by a real symmetric positive definite matrix V .

$$\text{Say } \langle a, b \rangle_V = a^T V b$$

This is an inner product in R^n and it satisfies the criterion for equivalence of inner products $\langle, \rangle$ and $\langle, \rangle_V$ with the two constants α_1, α_2 set to be the largest and smallest eigenvalues of V , i.e.

$$\lambda_{\min}(V) \|a\|^2 \leq \|a\|_V^2 \leq \lambda_{\max}(V) \|a\|^2 \quad 5.9$$

Under this inner product equation 5.7 becomes

$$D^+ \|x(t)\|_V^2 = 2 \langle A(t)x(t), Vx(t) \rangle + \langle g(x, t), Vx(t) \rangle$$

or in particular, if $g(x, t) = 0$

$$D^+ \|x(t)\|_V^2 = x^T(t) \left[A(t)V + VA^T(t) \right] x(t) \quad 5.10$$

Now as $\mu(A(t))$ is defined by

$$\mu(A(t)) = \lim_{h \rightarrow 0^+} \frac{1}{h} \left[\|I - hA(t)\| - 1 \right]$$

we can equivalently define

$$\mu_V(A(t)) = \lim_{h \rightarrow 0^+} \frac{1}{h} \left[\|I + hA(t)\|_V - 1 \right]$$

and then

$$\begin{aligned} \mu_V(A(t)) &\leq \lambda_{\max}(V) \mu(A(t)) \\ \text{and } \mu_V(A(t)) &= \frac{1}{2} \lambda_{\max} \left[A(t)V + VA^T(t) \right] \end{aligned}$$

So 5.10 becomes

$$D^+ \|x(t)\|_V \leq \mu_V(A(t)) \|x(t)\| \quad 5.11$$

and the stability behaviour of equation 5.4 can be deduced by considering the inequality

$$||x(t_1)||_V \leq ||x_0|| \exp\left\{\int_{t_0}^{t_1} \mu_V(A(t)) dt\right\} \quad 5.12$$

The above methods enable the direct consideration of the controllability of equation 5.1 by examining the stability of the equation when controls are applied.

If the nonlinearity is bounded in inner product by a function of $||y(t)||^2$ then the above method may be used to give an upper bound for $D^+||y(t)||$. This differential inequality can yield a bound for $||y(t)||$ via a scalar comparison function because there will be no additive term upon the right hand side. Considerations on these lines yield the following controllability theorem.

5.3 A NULL CONTROLLABILITY THEOREM FOR INNER PRODUCT BOUNDED NONLINEARITIES

This first theorem results from considering the simplest inner product bound upon the nonlinearity and as such it is equivalent to Theorem 3.1; however, in this case the scalar $p(t)$ is allowed to take negative values. As the bound is independent of $u_Y(t)$ upon the right hand side it is necessary to assume that the control $u_Y(t)$ is contained in the bounded control set Ω_k and the value of $p(t)$ is such that the bound is valid for all $u_Y(t) \in \Omega_k$.

THEOREM 5.1

The nonlinear system given by 5.1 is null controllable from (y, t_0) to (ϵ, T) provided that

- 1) $f(0, 0, t) = 0 \quad \forall t \in [t_0, \infty)$
- 2) There exists $p(t)$, scalar, such that

$$\langle f(y, u_Y, t), y(t) \rangle \leq p(t) ||y(t)||^2$$

$$\forall y(t) \in D \subset \mathbb{R}^n, u_Y(t) \in \Omega_k \subset \mathbb{R}^m$$

$$t \in [t_0, \infty)$$

- 3) There exists an admissible control $u_y(t)$
 $= K(t)y(t)$, $u_y(t) \in \Omega_k$ such that

$$\int_{t_0}^T \mu(A(s)+B(s)K(s)) + p(s) ds \leq \ln \frac{\epsilon}{||y_0||}$$

PROOF

As the condition 2) upon the nonlinearity is not a Lipschitz condition, it is first necessary to show the existence of a solution to 5.1. This can be done using a similar technique to that used in Chapters Three and Four. Notice that if $f(\cdot, \cdot, t)$ satisfies condition 2) it also satisfies

$$||f(y, u_y, t)|| \leq |p(t)| ||y(t)||$$

Then a fixed point for the map L may be shown in the usual way and this will imply the existence of a solution to 5.1. Alternatively a solution may be shown to exist locally using the well known techniques outlined in Coppel^{C1}. The assumption that $A(t)$ and $B(t)$ are continuous and that $f(y, K(t)y, t)$ is continuous for $y(t) \in D$ ensures that the local solution to equation 5.1 may be extended arbitrarily to the right.

Alternatively the existence problem may be circumvented by assuming, as Ben Lashihier and Storey^{B2} do, that the nonlinearity is sufficiently smooth so that a solution to 5.1 always exists for continuous controls.

The remainder of the proof is concerned with showing the solution of the nonlinear system, with a control $u_y(t) \in \Omega_k$ to be stable to the trivial solution $y(t) \equiv 0$.

Consider the expression for the derivative $D^+ ||y(t)||^2$ given by

$$D^+ ||y(t)||^2 = \limsup_{h \rightarrow 0^+} \frac{1}{h} \left[||y(t+h)||^2 - ||y(t)||^2 \right] \quad 5.13$$

This will lead to an upper bound for $D^+ ||y(t)||^2$ given by

$$D^+ ||y(t)||^2 \leq 2 \langle \{A(t) + B(t)K(t)\} y(t), y(t) \rangle + f(y, K_y, t), y(t) > \quad 5.14$$

From the linearity of the inner product this can be split into two terms:

$$D^+ ||y(t)||^2 \leq 2 \langle \{A(t) + B(t)K(t)\} y(t), y(t) \rangle + 2 \langle f(y, K_y, t), y(t) \rangle \quad 5.15$$

Now the nonlinearity $f(\cdot, \cdot, t)$ satisfies the bound given by 5.3 and so this yields

$$D^+ ||y(t)|| \leq \mu \{A(t) + B(t)K(t)\} ||y(t)|| + p(t) ||y(t)|| \quad 5.16$$

where the scalar $p(t)$ is not necessarily positive.

The scalar comparison function which is needed to derive a bound for $||y(t)||$ from 5.16 is the simple function which satisfies

$$\dot{m}(t) = \left\{ \mu \{A(t) + B(t)K(t)\} + p(t) \right\} m(t) \quad 5.17$$

This has the well known solution

$$m(t) = m(t_0) \exp \left\{ \int_{t_0}^t (\mu \{A(s) + B(s)K(s)\} + p(s)) ds \right\}$$

so the corresponding inequality bound resulting from 5.16, after the use of a scalar comparison function is

$$||y(t)|| \leq ||y_0|| \exp \left\{ \int_{t_0}^t (\mu \{A(s) + B(s)K(s)\} + p(s)) ds \right\} \quad 5.18$$

By hypothesis there exists a control $u_y(t) = K(t)y(t)$ such that

$$\exp \left\{ \int_{t_0}^T (\mu \{A(s) + B(s)K(s)\} + p(s)) ds \right\} \leq \frac{\varepsilon}{||y_0||}$$

Then by substitution in 5.18

$$||y(T)|| \leq \varepsilon \text{ and the theorem is proved.}$$

The corresponding result for the case where definition 3.1

is considered and the final time, T , is not stated 'a priori' is given below.

COROLLARY 5.1

The nonlinear system given by 5.1 is null controllable from y_0 on (t_0, ∞) provided that conditions 1) and 2) of Theorem 5.1 are satisfied and there exists a control

$u_y(t) \in \Omega_k$ such that

$$3) \quad \lim_{t \rightarrow \infty} \int_{t_0}^t \mu(A(s) + B(s)K(s)) + p(s) \, ds = -\infty$$

PROOF

This result follows from Theorem 5.1 noticing that the controllability behaviour of equation 5.1 is determined by the stability behaviour of the scalar equation 5.17. Condition 3) of the corollary implies that 5.17 is asymptotically stable and this in turn implies that the nonlinear system is null controllable from y_0 on (t_0, ∞) and so the corollary is proved.

For the case when $p(t)$ in 5.2 is a positive scalar these two results can be related to those of the previous Chapters. Consider Corollary 5.1; if $p(t)$ is positive then the inner product bound 5.2 is equivalent to the norm bound 2.38 and hence the main restriction upon the linear section of the equation occurs in condition 3) of both theorems. We have shown in the previous chapter that the restriction of Theorem 4.4 to the case where $q(t) = 0$ ^{is implied by} ~~is implied by~~ Theorem 3.1. The condition 3) for this special case of Theorem 4.4 is certainly satisfied if

$$\int_{t_i}^{t_{i+1}} \mu(A(s) + B(s)K(s)) + p(s) \, ds < 0 \quad i=0,1,2 \dots \quad 5.19$$

$\{t_i\} \in (t_0, \infty)$

and if this is satisfied then condition 3) of Corollary 5.1 is also satisfied. So if there exists a control which satisfies 5.19 then this control will certainly satisfy the appropriate condition in either Corollary 5.1 or the special case of Theorem 4.4 with $q(t) = 0$.

NOTE 5.1

A simple condition which is sufficient to imply 5.19 is

$$\mu(A(s) + B(s)K(s)) + p(s) < 0 \quad \forall s \in (t_0, \infty) \quad 5.20$$

For the case when the nonlinearity can be bounded independently of control, or when the nonlinearity itself is independent of control, condition 5.20 reflects the fact that the nonlinear system is always null controllable if the linear system is totally controllable. This is equivalent to Gershwins and Jacobson's ^{G1} result in Example 5.

The relationship between Theorems 5.1 and 4.4 is somewhat concealed because the two theorems are stated for different controllability definitions. Theorem 4.4 can be modified to apply to null controllability from (y_0, t_0) to (ϵ, T) if the number of intervals and the size of λ (in

$\|y(t_{i+1})\| \leq \lambda \|y(t_i)\|$) are related to the ratio $\frac{\epsilon}{\|y_0\|}$. In this case inequality 4.55 can be replaced by (notice

$q(s) = 0$ and $p'(s) \leq p(s) \quad \forall s \in (t_i, t_{i+1})$)

$$\int_{t_i}^{t_{i+1}} -\alpha(s) + p(s) ds \leq \left(\frac{\lambda-1}{\lambda} \right) \quad \begin{array}{l} \lambda < 1 \\ \{t_i\} \in (t_0, T) \\ i=0, 1, 2, \dots \end{array} \quad 5.21$$

in order that the bound upon $\|y(t_{i+1})\|$ is $\lambda \|y(t_i)\|$.

For N of these intervals this modification to Theorem 4.4 would require $\lambda^N \|y_0\| < \epsilon$ and this, together with 5.21 implies

$$\int_{t_0}^T -\alpha(s) + p(s) ds \leq N \left(\frac{\lambda-1}{\lambda} \right) \quad 5.22$$

The equivalent condition for Theorem 5.1 is

$$\int_{t_0}^T -\alpha(s) + p(s) ds \leq \ln \frac{\varepsilon}{||y_0||} \quad 5.23$$

But by definition of λ we know that

$$N \ln \lambda \leq \ln \varepsilon / ||y_0||$$

then, as $\frac{\lambda-1}{\lambda} < \ln \lambda$, if there exists an admissible control which satisfies 5.22 this control certainly satisfies 5.23.

From the above we can see that the special version of Theorem 4.4 modified explicitly to cover the case where $q(t) = 0$ and the null controllability is from (y_0, t_0) to (ε, T) will always be more restrictive than the result from Theorem 5.1 which also admits a wider class of nonlinearities. Notice, however, that Theorem 5.1 says less about the solution to the nonlinear equation over the interval (t_0, T) than does the restricted Theorem 4.4, in the sense that Theorem 4.4 places an upper bound upon the solution at each of the times t_i whereas no intermediate restriction is placed on the solution in Theorem 5.1.

5.4 UNIQUENESS OF THE SOLUTION TO EQUATION 5.1

It has been argued by Ben Lashiher and Storey^{B2} that in the controllability problems where the target and starting points are characterised by sets in the state space it is no longer necessary to show uniqueness of the solution to the system equation when the control is applied. If all the solutions originating in a certain zone under the action of a particular control can be shown to be contained in the desired zone at the final time, then this is enough for the

problem to be solved and to say any more about the solution is unnecessary and beyond the scope of the controllability problem. In the case where the nonlinearity $f(\cdot, \cdot, t)$ is bounded by 5.2 or 5.3 there exists a particularly attractive way of showing uniqueness of the solution asking slightly less than the conventional Lipschitz condition.

The normal Lipschitz condition for $f(\cdot, \cdot, t)$, with the linear state variable feedback control $u_y(t) \in \Omega_k$ is

$$\begin{aligned} ||f(y_1, u_{y_1}, t) - f(y_2, u_{y_2}, t)|| &\leq L ||y_2(t) - y_1(t)|| \quad 5.24 \\ y_1(t), y_2(t) &\in D \subset \mathbb{R}^n \\ t &\in (t_0, \infty) \end{aligned}$$

Pao ^{P1} has shown the uniqueness of solutions to linear differential equations with nonlinear disturbing functions by asking a onesided condition which when modified to include control terms becomes

$$\begin{aligned} < f(y_1, u_{y_1}, t) - f(y_2, u_{y_2}, t), y_1(t) - y_2(t) > \\ &\leq k(t) ||y_1(t) - y_2(t)||^2 \quad 5.25 \\ y_1(t), y_2(t) &\in D \subset \mathbb{R}^n \\ t &\in (t_0, \infty), u_{y_1}(t), u_{y_2}(t) \in \Omega_k \end{aligned}$$

Following Pao's technique we can show the uniqueness of solutions to the nonlinear system 5.1 when $f(\cdot, \cdot, t)$ satisfies the condition 5.25 instead of inequality 5.2.

Consider any two solutions $y_1(t)$ and $y_2(t)$ to equation 5.1 with control $u_{y_i}(t) = K(t)y_i(t)$ $i=1,2$ and initial conditions $y_1(t_0) = y_2(t_0) = y_0$. Then if $R(t) \triangleq ||y_1(t) - y_2(t)||^2$, $R(t_0) = 0$ we have, from 5.14

$$\begin{aligned} D^+ R(t) &\leq 2 < \{ A(t) + B(t)K(t) \} R(t) \\ &\quad + f(y_1, u_{y_1}, t) - f(y_2, u_{y_2}, t), y_1(t) - y_2(t) > \end{aligned}$$

Hence

$$D^+R(t) \leq 2\mu(A(t)+B(t)K(t)) \cdot R(t) + \langle f(y_1, u_{y_1}, t) - f(y_2, u_{y_2}, t), y_1(t) - y_2(t) \rangle$$

Now if $f(\cdot, \cdot, t)$ satisfies 5.25 this becomes

$$D^+R(t) \leq \left[2\mu(A(t)+B(t)K(t)) + k(t) \right] R(t)$$

with initial condition $R(t_0) = 0$.

Hence for bounded values of $\mu(A(t)+B(t)K(t))$ and $k(t)$

$$R(t) = ||y_1(t) - y_2(t)|| = 0$$

and the solutions to 5.1 with initial condition $y(t_0) = y_0$ are thus unique.

Notice also that if $f(\cdot, \cdot, t)$ satisfies 5.25 then it also satisfies 5.2 if $f(0, 0, t) = 0$ by substitution of $y_2(t) = 0$ in 5.25.

5.5 EXTENSION OF THESE RESULTS TO EQUIVALENT INNER PRODUCTS

The above results have been confined to the case where the Euclidean norm in R^n has been used for vectors and matrices. Interesting parallels can be seen when the same results are developed using the norm $|\cdot|_V$.

THEOREM 5.2

The nonlinear system given by 5.1 is null controllable from (y_0, t_0) to (ϵ, T) provided there exists some positive definite real symmetric matrix V defining the inner product

$\langle \cdot, \cdot \rangle_V$ and

$$1) \quad f(0, 0, t) = 0 \quad \forall t \in (t_0, \infty)$$

2) There exists $p_V(t)$, scalar, such that

$$\langle f(y, u_y, t), y(t) \rangle_V \leq p_V(t) ||y(t)||^2$$

$$\forall y(t) \in D \subset R^n, u_y(t) \in \Omega_k \subset R^m$$

$$t \in (t_0, \infty)$$

- 3) There exists an admissible control $u_Y(t) = K(t)y(t)$ such that

$$\frac{1}{\lambda_{\min}(V)} \int_{t_0}^T \mu_V(A(s)+B(s)K(s)) + p_V(s) ds \leq \ln \frac{\varepsilon}{\|y_0\|}$$

PROOF

The proof is essentially that given for Theorem 5.1 with the substitution of $\|\cdot\|_V$ for the Euclidean norm.

Assuming the existence of a solution has been shown in one way or another the upper bound for $D^+ \|y(t)\|_V^2$, equivalent to 5.15 is

$$D^+ \|y(t)\|_V^2 \leq 2 \langle \{A(t)+B(t)K(t)\}y(t), Vy(t) \rangle + 2 \langle f(y, u_Y, t), Vy(t) \rangle \quad 5.26$$

From the bound demanded in condition 2) the second term may be bounded above by $p_V(t) \|y(t)\|^2$ whilst the first term can be bounded by a term in $\mu_V(\cdot)$. This leads to

$$D^+ \|y(t)\|_V^2 \leq 2 \mu_V(A(t)+B(t)K(t)) \|y(t)\|^2 + 2p_V(t) \|y(t)\|^2 \leq 2 \left[\mu_V(A(t)+B(t)K(t)) + p_V(t) \right] \|y(t)\|^2$$

Equivalently

$$D^+ \|y(t)\|^2 \leq \frac{2}{\lambda_{\min}(V)} \left[\mu_V(A(t)+B(t)K(t)) + p_V(t) \right] \|y(t)\|^2 \quad 5.27$$

A comparison function etc. lead to $\|y(t)\|$ satisfying the inequality

$$\|y(t)\| \leq \|y_0\| \exp \left\{ \int_{t_0}^t \frac{1}{\lambda_{\min}(V)} \mu_V(A(s)+B(s)K(s)) + p_V(s) ds \right\}$$

and the theorem follows.

It is possible to consider this problem from a stability point of view and then, for the nonlinear system with feedback gain $K(t)$ to be sufficiently stable so that the null controllability is from (y_0, t_0) to (ε, T) , it is necessary that the value of $\mu_V(\cdot)$ dominates the value of $p_V(t)$ so that

the sum $\mu_V(A(s)+B(s)K(s)) + p_V(s)$ has a negative integral over (t_0, T) . The upper bound upon this integral is determined by $\|y_0\|$, ε and V in the form of its smallest eigenvalue. The symmetric positive definite matrix V bears a strong resemblance to the choice of a quadratic Lyapunov function. In this case the value of $\mu_V(A(s)+B(s)K(s))$ is serving an equivalent purpose to the negative definiteness of the matrix $-R(t)$ in the Lyapunov matrix equation which would in this case be

$$\{A(t)+B(t)K(t)\}V + V\{A(t)+B(t)K(t)\}^T = -R(t) \quad 5.28$$

The similarity with quadratic Lyapunov functions is more clear in the equivalent inner product formulation of Corollary 5.1.

COROLLARY 5.2

The nonlinear system given by 5.1 is null controllable from y_0 on (t_0, ∞) provided that conditions 1) and 2) of Theorem 5.2 are satisfied and there exists a control

$u_Y(t) \in \Omega_k$ such that

$$3) \quad \lim_{t \rightarrow \infty} \int_{t_0}^t \mu_V(A(s)+B(s)K(s)) + p(s) ds = -\infty$$

PROOF

The proof follows that of Corollary 5.1 and in this case the asymptotic stability of the scalar comparison function used to imply a bound upon $\|y(t)\|$ from 5.28 ensures the null controllability of equation 5.1 from (y_0, t_0) . The more stringent condition of uniform stability of the scalar equation will also guarantee the null controllability of 5.1 and a sufficient condition for uniform stability would be

$$\mu_V(A(s)+B(s)K(s)) + p_V(s) < 0 \quad \forall s \in (t_0, \infty) \quad 5.29$$

The resemblance to the Lyapunov matrix equation is now clearer: inequality 5.29 is the equivalent to the demand that $\dot{V}(y,s)$ be negative.

In the case of a quadratic Lyapunov function $y^T(s)Vy(s)$ the value of $\dot{V}(y,s)$ for equation 5.1 is given by

$$\begin{aligned} \dot{V}(y,s) &= 2 y^T(s)V\dot{y}(s) \\ &= 2 \{ y^T(s) \cdot V \cdot (A(s)+B(s)K(s)y(s) + f(y,Ky,s)) \} \\ &\leq 2 \{ y^T(s)V[A(s)+B(s)K(s)]y(s) \} + 2p_V(s) ||y(s)||^2 \end{aligned}$$

Then $\dot{V}(y,s) < 0$ if inequality 5.29 is satisfied and hence the system 5.1 with feedback matrix $K(t)$ is shown to be stable under the conditions implied by inequality 5.29.

This particular formulation of the Lyapunov function when applied to the controllability problem is similar to the work of Gershwin and Jacobson where the matrix V is consistently chosen to be the matrix $S(t)$ used in the feedback law $K(t) = -\frac{1}{2} B^T(t)S(t)$. Ben Lashiher and Storey also produce results of a similar form in their example 6.3 where final stability is applied, in a controllability sense, to the simple equation

$$\dot{x}(t) = A(t)x(t) + u(t) \quad 5.30$$

Ben Lashiher and Storey consider this linear case and use only the Euclidean norm in their example, however the framework exists for this technique to be applied at least to problems of linear systems in the form of

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad 5.31$$

using the equivalent norm $|\cdot|_V$.

This is not explicitly stated by the authors.

5.6 FOR THE CASE WHERE THE NONLINEARITY IS BOUNDED BY INNER PRODUCTS INVOLVING CONTROL TERMS

In Chapter Four results were presented which extended those of Chapter Three to the case where control terms are introduced on the right hand side of the norm bound upon $f(\cdot, \cdot, t)$ and where the controls are contained in Ω rather than Ω_k . In order to develop the results it was necessary to put an implicit restriction upon $K(t)$ in the form of $S(t_{i+1})$. The extension of the above results to the case where $f(\cdot, \cdot, t)$ is bounded by inner product terms involving control is considered in this section.

The above inner product bound upon $f(\cdot, \cdot, t)$ involves no terms in control explicitly on the right hand side, however, the extension to the case where $u_y(t)$ is involved can be made using similar techniques. As an introduction to this form of bound consider first the case where $f(\cdot, \cdot, t)$ is bounded by

$$\begin{aligned} \langle f(y, u_y, t), y(t) \rangle_V \leq q_V(t) \langle Q(t)u_y(t), y(t) \rangle \quad 5.32 \\ y(t) \in DCR^n, u_y(t) \in \Omega CR^m \\ t \in (t_0, \infty) \end{aligned}$$

where $Q(t)$ is some arbitrary $n \times m$ matrix of possibly time varying elements taking the control vector into R^n . Notice that the general V inner product occurs on the left hand side of this inequality but it is bounded above by a multiple of a Euclidean inner product. For the case where feedback controls are considered the development is similar to that of the previous theorems; however, in this case the scalar multiplier depends explicitly upon $K(t)$. Special case results are available for some choices of $q_V(t)$ and $Q(t)$.

THEOREM 5.3

The nonlinear system given by 5.1 is null controllable from (y_0, t_0) to (ϵ, T) provided there exists some positive definite real symmetric matrix V defining the inner product $|\cdot|_V$ and

- 1) $f(0,0,t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) There exists some $q_V(t)$, scalar, and $Q(t)$ a $n \times m$ matrix such that

$$\begin{aligned} \langle f(y, u_Y, t), y(t) \rangle_V &\leq q_V(t) \langle Q(t) u_Y(t), y(t) \rangle \\ y(t) &\in D \subset \mathbb{R}^n, u_Y(t) \in \Omega \subset \mathbb{R}^m \\ t &\in (t_0, \infty) \end{aligned}$$

- 3) There exists an admissible control $u_Y(t) = K(t)y(t)$ such that

$$\begin{aligned} \frac{1}{\lambda_{\min}(V)} \int_{t_0}^T \mu_V(A(s) + B(s)K(s)) + q_V(s) \mu_V(Q(s)K(s)) ds \\ \leq \ln \frac{\epsilon}{\|y_0\|} \end{aligned}$$

PROOF

Again, as we assume $f(\cdot, \cdot, t)$ to be continuous in y for every u_Y , the existence of a solution to equation 5.1 may be shown in a variety of ways. The bounding of such a solution to ensure the null controllability of 5.1 to (ϵ, T) requires that we consider

$$D^+ |y(t)|_V^2 \leq 2 \langle \{A(t) + B(t)K(t)\} y(t) + f(y, u_Y, t), Vy(t) \rangle$$

From the bound in condition 2) this may be written

$$\begin{aligned} D^+ |y(t)|_V^2 &\leq 2 \langle \{A(t) + B(t)K(t)\} y(t), Vy(t) \rangle \\ &\quad + 2q_V(t) \langle Q(t)K(t)y(t), y(t) \rangle \end{aligned}$$

or equivalently

$$D^+ |y(t)|_V^2 \leq 2 \left[\mu_V(A(t) + B(t)K(t)) + q_V(t) \mu(Q(t)K(t)) \right] ||y(t)||^2 \quad 5.33$$

This may be rewritten involving only Euclidean norms and use of a scalar comparison function results in $||y(t)||$ being bounded by

$$||y(t)|| \leq ||y_0|| \exp \left\{ \int_{t_0}^t \frac{1}{\lambda_{\min}(V)} \left[\mu_V(A(s) + B(s)K(s)) + q_V(s) \mu(Q(s)K(s)) \right] ds \right\}$$

and the theorem follows.

This inner product bound upon $f(\cdot, \cdot, t)$ exhibits an extra degree of freedom over the previous bounds and this is characterised by the choice of the matrix $Q(t)$. One choice of the matrix which yields interesting results is

$$Q(t) = B(t) \quad \forall t \in (t_0, \infty)$$

In this case the bound given by 5.33 becomes

$$D^+ |y(t)|_V^2 \leq 2 \mu_V(A(t) + B(t)K(t)) + q_V(t) \mu(B(t)K(t)) ||y(t)||^2 \quad 5.34$$

The specific choice of the feedback matrix of $K(t) = -\frac{1}{2}B^T(t)S(t)$, where $S(t)$ is the solution to the matrix Riccati equation, leads to

$$D^+ |y(t)|_V^2 \leq 2 \left[\mu_V(A(t) - \frac{B(t)B^T(t)}{2} S(t)) + q_V(t) \mu\left(-\frac{B(t)B^T(t)}{2} S(t)\right) \right] ||y(t)||^2$$

and, as the product $B(t)B^T(t)S(t)$ is a real symmetric matrix, the bound for $D^+ ||y(t)||$ is given by

$$D^+ ||y(t)|| \leq \left[\mu_V(A(t) - \frac{B(t)B^T(t)}{2} S(t)) - q_V(t) \mu\left(\frac{B(t)B^T(t)}{2} S(t)\right) \right] \frac{1}{\lambda_{\min}(V)} ||y(t)||$$

This leads to the following bound upon $||y(T)||$;

$$\begin{aligned}
||y(T)|| \leq ||y_0|| \exp \left\{ \int_{t_0}^T \frac{1}{\lambda_{\min}(V)} \mu_V \left(A(s) - \frac{B(s)B^T(s)S(s)}{2} \right) ds \right\} \\
\cdot \exp \left\{ \int_{t_0}^T \frac{1}{\lambda_{\min}(V)} \cdot -q_V(s) \mu \left(\frac{B(s)B^T(s)S(s)}{2} \right) ds \right\} \quad 5.35
\end{aligned}$$

Then, from previous results, the (ϵ, T) null controllability of 5.1 is guaranteed by the existence of an $S(t)$ such that the linear part of the equation is null controllable and

$$-\frac{1}{\lambda_{\min}(V)} \int_{t_0}^T q_V(s) \mu \left(\frac{B(s)B^T(s)S(s)}{2} \right) ds < 0 \quad 5.36$$

COROLLARY 5.3

The nonlinear system given by 5.1 is null controllable from (y_0, t_0) to (ϵ, T) provided that conditions 1) and 2) of Theorem 5.3 are satisfied with $Q(t) \triangleq B(t)$, $\int_{t_0}^T q_V(s) \geq 0$ and

3) There exists a control of the form

$u_Y(t) = -\frac{1}{2}B(t)S(t)x(t)$, with $S(t)$ a solution to the Riccati equation, such that

$$\int_{t_0}^T \frac{1}{\lambda_{\min}(V)} \mu_V \left(A(s) + \frac{B(s)B^T(s)S(s)}{2} \right) ds \leq \ln \frac{\epsilon}{||y_0||}$$

PROOF

The proof of Corollary 5.3 follows from the above discussion and inequality 5.36, noticing that

$\mu \left(\frac{B(s)B^T(s)S(s)}{2} \right)$ is positive semi definite.

Then $\int_{t_0}^T q_V(s) ds \geq 0$ is enough to ensure that 5.36 is satisfied.

This corollary is of value for nonlinearities depending only upon control. If the function f has an inner product upon $y(t)$ of the same sign as inner product $B(t)u_Y(t)$ upon $y(t)$, and this is characterised by $\int_{t_0}^T q_V(s) ds > 0$, then

from the point of view of null controllability of 5.1 the nonlinearity may be neglected.

The class of functions $f(\cdot, \cdot, t)$ which satisfies the above inequality bound is somewhat limited due to the fact that the right hand side of the bound can take both positive and negative values, depending upon the choice of $K(t)$. Consider the scalar example of

$$f(y, u_y, t) = u_y [\sin(y) + 1]$$

For this function the bound becomes

$$u_y \sin(y) + 1 \cdot y \leq q_V \cdot Q \cdot u_y \cdot y. \quad \begin{matrix} \forall u_y \in \Omega_k \\ y \in D \end{matrix}$$

Without loss of generality in this scalar case assume $Q = 1$, then as the product $u_y \cdot y$ can take both positive and negative values according to the choice of feedback, it is not possible to find a q_V such that the above inequality is satisfied. The bound 5.32 can be modified to admit a wider class of functions, including the example given above, but then some of the advantages of bound 5.32 are lost.

Consider

$$\begin{aligned} \langle f(y, u_y, t), y(t) \rangle_V \leq q_V(t) | \langle Q(t) u_y(y), y(t) \rangle | \quad 5.37 \\ \forall y(t) \in D \subset \mathbb{R}^n, u_y(t) \in \Omega \subset \mathbb{R}^m \\ t \in (t_0, \infty) \end{aligned}$$

In this case the sign of $\langle Q(t) u_y(t), y(t) \rangle$ is rendered irrelevant by the absolute value function whilst $q_V(t)$ can still take negative values if the left hand side is always less than zero.

In the case of this bound Theorem 5.3 follows almost as before, however in condition 3) it is necessary to replace $\mu(Q(s)K(s))$ by $|\mu(Q(s)K(s))|$ to account for the absolute value

introduced in the inner product.

The most general inner product bound considered here, 5.3, which is a combination of 5.2 and 5.37 can be used in a similar formulation to the previous theorems to yield another null controllability theorem.

In this case $u_Y(t) \in \Omega$ and for generality the absolute value inner product is used for the control term on the right hand side. The resulting theorem is the inner product equivalent to Theorem 4.4 which is the most general theorem involving norm bounds.

THEOREM 5.4

The nonlinear system given by 5.1 is null controllable from (y_0, t_0) to (ϵ, T) provided there exists some positive definite real symmetric matrix V defining the inner product $\langle \cdot, \cdot \rangle_V$ and

$$1) \quad f(0, 0, t) = 0 \quad \forall t \in (t_0, \infty)$$

2) There exist $p_V(t), q_V(t)$ scalars and $Q(t)$ a $n \times m$ matrix such that

$$\begin{aligned} \langle f(y, u_Y, t), y(t) \rangle_V &\leq p_V(t) \|y(t)\|^2 \\ &\quad + q_V(t) |\langle Q(t) u_Y(t), y(t) \rangle| \\ &\quad y(t) \in D \subset \mathbb{R}^n, u_Y(t) \in \Omega \subset \mathbb{R}^m \\ &\quad t \in (t_0, \infty) \end{aligned}$$

3) There exists an admissible control $u_Y(t)$ such that

$$\begin{aligned} \frac{1}{\lambda_{\min}(V)} \int_{t_0}^T \mu_V(A(s) + B(s)K(s)) + p_V(s) \\ + q_V(s) |\mu(Q(s)K(s))| ds \leq \ln \frac{\epsilon}{\|y_0\|} \end{aligned}$$

PROOF

The structure of the proof is essentially similar to those given previously: a solution having been shown to exist, an upper bound upon $||y(t)||$ is deduced from

$$D^+|y(t)|^2_V \leq 2 < [A(t)+B(t)K(t)] y(t) + f(y, u_y, t), Vy(t) >$$

In this case the bound upon $f(\cdot, \cdot, t)$ leads to

$$D^+|y(t)|^2_V \leq 2 < [A(t)+B(t)K(t)] y(t), Vy(t) > + 2p_V(t) ||y(t)||^2 + 2q_V(t) | < Q(t)K(t)y(t), y(t) > |$$

5.38

which becomes

$$D^+|y(t)|^2_V \leq 2 \{ \mu_V(A(t)+B(t)K(t)) + p_V(t) + q_V(t) | \mu(Q(t)K(t)) | \} ||y(t)||^2$$

A comparison function leads to the bound

$$||y(t)|| \leq ||y_0|| \exp \{ \int_{t_0}^t \frac{1}{\lambda_{\min}(V)} \{ \mu_V(A(s)+B(s)K(s)) + p_V(s) + q_V(s) | \mu(Q(s)K(s)) | \} ds \}$$

and the theorem follows.

Although Theorem 5.4 is the equivalent result to Theorem 4.4 for the case where the nonlinearity is bounded by an inner product, the two results are not easily shown to be explicitly related. However, Theorem 5.1 has already been shown to be connected with Theorem 4.4 when $q(s) = 0$ and a similar connection can be shown between Theorem 5.4 and Theorem 4.4.

Theorem 4.4 concerns null controllability from y_0 on (t_0, ∞) whereas Theorem 5.4 considers null controllability from (y_0, t_0) to (ϵ, T) , but Theorem 4.4 can be rederived to give a sufficient condition for null controllability from

(y_0, t_0) to (ε, T) if the interval (t_0, T) is divided into N subintervals (t_i, t_{i+1}) over which the solution norm satisfies

$$||y(t_{i+1})|| \leq \lambda ||y(t_i)||.$$

This is guaranteed if the crucial condition 3c) is replaced by

$$\int_{t_i}^{t_{i+1}} \mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) + q(s) \left\| \frac{B^T(s)}{2} \right\| ||S_{i+1}|| + p'(s) ds \leq \left\{ \frac{\lambda-1}{\lambda} \right\} \quad 5.39$$

Then if $||\bar{S}|| \triangleq \max_i ||S_i||$ a sufficient condition for 5.39 to be satisfied is

$$\int_{t_0}^T \mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) + q(s) \left\| \frac{B^T(s)}{2} \right\| \cdot ||\bar{S}|| + p(s) ds \leq N \left\{ \frac{\lambda-1}{\lambda} \right\} \quad 5.40$$

because $p(s) \geq p'(s) \quad \forall s \in (t_0, T)$.

Theorem 5.4, for the special case where V is the identity matrix and the control is chosen so that

$K(t) = -\frac{1}{2}B^T(t)S(t)$, has condition 3) given by

$$\int_{t_0}^T \mu \left(A(s) - \frac{B(s)B^T(s)}{2} S(s) \right) + p_I(s) + q_I \left\| \frac{B^T(s)}{2} \right\| \cdot ||S(s)|| ds \leq \ln \frac{\varepsilon}{||y_0||} \quad 5.41$$

and it is then seen that 5.40 is also a sufficient condition for 5.41 to be satisfied if $q(s) \geq q_I(s) ||Q(s)||$.

So although Theorems 5.4 and 4.4 have not been shown to be interdependent they can be shown to be interrelated by showing that inequality 5.41 is a sufficient condition for special cases of both theorems.

5.7 CONCLUSIONS

The results presented in this chapter, concerning null controllability when the nonlinearity is bounded in inner

product, are related to stability results of a similar nature. If the nonlinearity is sufficiently well understood to be bounded by some inner product this may enable a null controllability result to be developed using the criteria presented in this chapter, which generally involve weaker conditions that are easier to satisfy than those of Chapters Three and Four. In cases where an inner product bound can be found then, as shown in Chapter Two, a norm bound can also be found. The converse, however, is not always true and so, although the results of this chapter are generally similar to those of Chapter Four, cases may exist where the nonlinearity cannot be bounded by inner products and a norm bound must be used.

These results have been shown within the more flexible framework of equivalent inner products and this facility can be useful when bounding the nonlinear vector function. It has been indicated in the first part of this chapter how in general both types of null controllability result (either from (y_0, t_0) to (ϵ, T) or from y_0 on (t_0, T)) can be derived using these techniques.

CHAPTER 6

6.1 Introduction

6.2 Asymptotic equivalence of linear and nonlinear systems
when the nonlinearity is weakly linear.

6.3 Examples of some of the preceding results.

6.4 A conceptual procedure for a basic implementation.

6.5 Conclusions and suggested areas of further work.

6.1 INTRODUCTION

This final chapter can conveniently be divided into three major parts: a result showing the asymptotic equivalence of the linear and nonlinear systems which have been the basis of the previous results is followed by a series of examples illustrating the form which our results would take when applied to nonlinear plants. These examples of the various results have certain points in common and this has enabled a procedure describing the form of a computational implementation to be developed. The third part of the chapter uses this procedure as a means of collectively discussing the results, summing up the work and putting forward areas of further research. However, before this concluding stage the one further result concerning asymptotic equivalence is presented.

6.2 ASYMPTOTIC EQUIVALENCE OF LINEAR AND NONLINEAR SYSTEMS WHEN THE NONLINEARITY IS WEAKLY LINEAR

It has been shown in Chapter Two that, of the considered bounds upon the nonlinearity, weak linearity is the most restrictive: any function which satisfies the weak linearity condition

$$\begin{aligned} \langle f(y, u_y, t), z \rangle &\leq p(t) |\langle y(t), z \rangle| & 6.1 \\ \forall z \in \mathbb{R}^n, y(t) \in D \subset \mathbb{R}^n \\ u_y(t) \in \Omega_k, t \in (t_0, \infty) \end{aligned}$$

will also satisfy a corresponding norm bound. Obviously then, if the nonlinear system given by

$$\dot{y} = A(t)y(t) + B(t)u_y(t) + f(y, u_y, t) \quad 6.2$$

has a perturbing function which satisfies 6.1 it may be

possible to derive a null controllability result for this system using the criteria given in Chapters Three and Four. An additional result is available when $f(\cdot, \cdot, t)$ is weakly linear. In Chapters Three and Four the solution to equation 6.2 is compared with the solution to the equation

$$\dot{x} = A(t)x(t) + B(t)u_x(t) \quad 6.3$$

when the feedback control law is chosen to be the same in each case. In these results we were concerned with the behaviour of $z(t) \triangleq y(t) - x(t)$ in order to show the null controllability of equation 6.2.

Now following Lovelady^{L2} we can define equations 6.2 and 6.3 to be "asymptotically equivalent" if $\|z(t)\| \rightarrow 0$ as $t \rightarrow \infty$ under some choice of feedback matrix $K(t)$. In the earlier chapters the nonlinear system 6.2 was shown to be null controllable with a control which also drives the linear system to the origin. In this case $\|z(t)\| \rightarrow 0$ and the systems are asymptotically equivalent. In this section the systems are explicitly shown to be asymptotically equivalent and an upper bound is given for $\|z(t)\|$ over (t_0, ∞) .

THEOREM 6.1

The nonlinear system given by equation 6.2 and the linear system given by equation 6.3 under the control laws defined by $u_y(t) = K(t)y(t)$ and $u_x(t) = K(t)x(t)$ and with initial conditions $x(t_0) = y(t_0) = x_0$ are asymptotically equivalent if

- 1) $f(0, 0, t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) $\langle f(y, u_y, t), z \rangle \leq p(t) |\langle y(t), z \rangle|$
 $\forall z \in \mathbb{R}^n, y(t) \in DCR^n$
 $u_y(t) \in \Omega_K, t \in (t_0, \infty)$

$$3) \quad \int_{t_0}^{t_1} \mu(A(s)+B(s)K(s)) + p(s) \, ds \rightarrow -\infty \text{ as } t_1 \rightarrow \infty$$

PROOF

The existence of a solution to 6.2 can be shown using one of the methods discussed earlier. If the solution is required to be unique it will be necessary to also ask that a Lipschitz condition is satisfied because although the nonlinearity is weakly linear it does not necessarily satisfy a Lipschitz condition.

Given that a solution to equation 6.2 exists then to show that the two solutions are asymptotically equivalent consider the function $z(t) \triangleq y(t) - x(t)$; $z(t_0) = 0$. Now from Chapter Five we know that an upper bound for $||z(t)||$ can be deduced from the differential inequality

$$D^+ ||z(t)||^2 \leq 2\mu(A(t)+B(t)K(t)) ||z(t)||^2 + 2p(t) | \langle y(t), z(t) \rangle | \quad 6.5$$

and so, to show that equations 6.2 and 6.3 are asymptotically equivalent under the feedback controls defined by $K(t)$, it is enough to show that the above inequality is asymptotically stable to the null solution.

Now from 6.5

$$D^+ ||z(t)||^2 \leq 2\{\mu(A(t)+B(t)K(t)) + p(t)\} ||z(t)||^2 + 2p(t) | \langle x(t), z(t) \rangle | \quad 6.6$$

or equivalently

$$D^+ ||z(t)||^2 \leq 2\{\mu(A(t)+B(t)K(t)) + p(t)\} ||z(t)||^2 + 2p(t) | \langle x(t), y(t) \rangle - ||x(t)||^2 | \quad 6.7$$

The term $| \langle x(t), y(t) \rangle - ||x(t)||^2 |$ in 6.7 involves implicitly $z(t)$ and so a comparison function, similar to

those used in previous chapters, cannot be applied with the equation in this form. However, if an upper bound for this term can be found which is independent of $z(t)$ then the analysis can proceed in the usual way.

$$\begin{aligned} \text{A bound for } \langle x(t), y(t) \rangle \text{ can be derived from} \\ D^+ \langle x(t), y(t) \rangle \leq 2\mu \{A(t) + B(t)K(t)\} \langle x(t), y(t) \rangle \\ + \langle f(y, u_y, t), x(t) \rangle \end{aligned} \quad 6.8$$

Using condition 2) from the theorem this becomes

$$\begin{aligned} D^+ \langle x(t), y(t) \rangle \leq 2\mu \{A(t) + B(t)K(t)\} \langle x(t), y(t) \rangle \\ + p(t) |\langle x(t), y(t) \rangle| \end{aligned}$$

Hence

$$\begin{aligned} D^+ \langle x(t), y(t) \rangle \leq \{2\mu \{A(t) + B(t)K(t)\} \\ + p(t) \operatorname{sgn}[\langle x(t), y(t) \rangle]\} \langle x(t), y(t) \rangle \end{aligned} \quad 6.9$$

$$\text{Now } x(t_0) = y(t_0) = x_0 \text{ and so } \langle x(t_0), y(t_0) \rangle = ||x_0||^2$$

A scalar comparison function for inequality 6.9 is

$$D^+ w(t) = \{2\mu \{A(t) + B(t)K(t)\} + p(t) \operatorname{sgn} [w(t)]\} w(t)$$

which has solution

$$w(t) = w(t_0) \exp \left\{ \int_{t_0}^t 2\mu \{A(s) + B(s)K(s)\} + p(s) \operatorname{sgn} [w(s)] \, ds \right\}$$

Now, as $p(t) > 0 \quad \forall t \in (t_0, \infty)$

$$w(t) \leq w(t_0) \exp \left\{ \int_{t_0}^t 2\mu \{A(s) + B(s)K(s)\} + p(s) \, ds \right\}$$

and hence an upper bound for $\langle x(t), y(t) \rangle$ from inequality 6.9 is

$$\langle x(t), y(t) \rangle \leq ||x_0||^2 \exp \left\{ \int_{t_0}^t 2\mu \{A(s) + B(s)K(s)\} + p(s) \, ds \right\} \quad 6.10$$

This upper bound for $\langle x(t), y(t) \rangle$ together with the bound for $||x(t)||^2$ given by

$$||x(t)||^2 \leq ||x_0||^2 \exp \left\{ \int_{t_0}^t 2\mu \{A(s) + B(s)K(s)\} \, ds \right\}$$

enables the term $|\langle x(t), y(t) \rangle - ||x(t)||^2|$ in 6.7 to be

bounded by a function independent of $z(t)$. Substitution of this bound into inequality 6.7 gives

$$D^+ ||z(t)||^2 \leq 2\{\mu(A(t)+B(t)K(t)) + p(t)\} ||z(t)||^2 + 2p(t) ||x_0||^2 \exp\{\int_{t_0}^t 2\mu(A(s)+B(s)K(s))ds\} \cdot |\exp\{\int_{t_0}^t p(s)ds\} - 1| \quad 6.11$$

This inequality is now quite suitable for the application of a scalar comparison function and this yields

$$||z(t)||^2 \leq \int_{t_0}^t 2p(s) ||x_0||^2 \cdot \exp\{\int_{t_0}^s -2\alpha(r)dr\} \cdot |\exp\{\int_{t_0}^s p(r)dr\} - 1| \cdot \exp\{\int_s^t 2(-\alpha(r)+p(r))dr\}ds \quad 6.12$$

where we have made the substitution $-\alpha(r) = \mu(A(r)+B(r)K(r))$.

So simplifying this expression

$$||z(t)||^2 \leq ||x_0||^2 \exp\{\int_{t_0}^t -2\alpha(r)dr\} \cdot \int_{t_0}^t 2p(s) |\exp\{\int_{t_0}^s p(r)dr\} - 1| \cdot \exp\{\int_s^t 2p(r)dr\}ds \quad 6.13$$

Now the second half of the righthand side of this inequality satisfies

$$\begin{aligned} & \int_{t_0}^t 2p(s) |\exp\{\int_{t_0}^s p(r)dr\} - 1| \cdot \exp\{\int_s^t 2p(r)dr\}ds \\ & \leq \exp\{\int_{t_0}^t 2p(r)dr\} \cdot \int_{t_0}^t 2p(s) \{\exp\{\int_{t_0}^s -p(r)dr\} \\ & \quad - \exp\{\int_{t_0}^s -2p(r)dr\}\}ds \quad 6.14 \end{aligned}$$

But

$$\begin{aligned}
 & \int_{t_0}^t 2p(s) \left\{ \exp\left\{\int_{t_0}^s -p(r) dr\right\} - \exp\left\{\int_{t_0}^s -2p(r) dr\right\} \right\} ds \\
 &= -2 \exp\left\{\int_{t_0}^s -p(r) dr\right\} \Big|_{s=t_0}^{s=t} + \exp\left\{\int_{t_0}^s -2p(r) dr\right\} \Big|_{s=t_0}^{s=t} \\
 &= \left[\exp\left\{\int_{t_0}^t -p(r) dr\right\} - 1 \right]^2 \quad 6.15
 \end{aligned}$$

Substitution of equation 6.15 and inequality 6.14 into 6.13 yields

$$\|z(t)\|^2 \leq \|x_0\|^2 \cdot \exp\left\{\int_{t_0}^t 2(-\alpha(r) + p(r)) dr\right\} \cdot \left[\exp\left\{\int_{t_0}^t -p(r) dr\right\} - 1 \right]^2$$

and from condition 3) the theorem follows.

6.3 EXAMPLES OF SOME OF THE PRECEDING RESULTS

By the use of some simple examples this section illustrates how our results may be used to examine the null controllability of nonlinear systems. The examples consist of straightforward applications of some of the typical results to systems with nonlinearities of the appropriate type and serve to indicate the form which the results would take when used upon a plant with known nonlinear dynamics.

Example 1

As an example of the application of Theorem 3.1 consider the system $\dot{y} = Ay + Bu_y + f(y)$

$$\text{where}^* \quad A = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad f(y) = \begin{bmatrix} 0 \\ -(\text{sat}(y_2) + 1)y_2 \end{bmatrix} \quad \text{E1.}$$

Examining the conditions of Theorem 3.1 notice that the

* $\text{sat}(y_2) \triangleq y_2; |y_2| < 1$
 $= 1; \text{ otherwise}$

matrix pair A, B is completely controllable and that the following conditions of the theorem are satisfied:

- 1) $f(0,0,t) = 0, \quad \forall t \in (t_0, \infty)$
- 2) There exists a $p(t)$ so that $\|f(y)\| \leq p(t) \|y\|$

$$\|f(y)\| = \sqrt{(\text{sat}(y_2)+1)^2 y_2^2} \leq 2\sqrt{y_2^2} \leq 2\|y\|$$

and the choice $p(t) = 2 \quad \forall t \in (t_0, \infty)$ satisfies this condition.

$$3) a) \quad \int_{t_i}^{t_{i+1}} p(r) dr \leq c_1 < \frac{1}{2}$$

The choice $|t_{i+1} - t_i| = 1/10$ will ensure that this condition is satisfied with $c_1 = 1/5$.

$$b) \quad |t_{i+1} - t_i| \leq c_2 < \infty$$

This is certainly satisfied from condition 3a).

$$c) \quad \int_{t_i}^{t_{i+1}} \mu(A(s) + B(s)K(s)) ds \leq \ln \left(\frac{\lambda - (1+\lambda)c_1}{1-c_1} \right)$$

For the choice of $c_1 = 1/5$ and interval $|t_{i+1} - t_i| < 1/10$ this condition is equivalent to

$$1/10 \mu(A+BK) < \ln 3/4$$

or

$$\mu(A+BK) < -2.8$$

The choice of the feedback gain matrix $[k_1 \ k_2]$ such that the above condition is satisfied is equivalent to the choice of k_1 and k_2 so that

$$\mu \begin{bmatrix} k_1 & k_2 \\ 1 & -3 \end{bmatrix} < -2.8$$

The choice $k_1 = -2.9, k_2 = -1$ satisfies the theorem conditions and hence $E1$ is null controllable on (t_0, ∞) . This choice of K makes $(A+BK) + (A+BK)^T$ diagonal and so $\mu(A+BK)$ is easily determined. This feedback matrix does not lead to $A+BK$

satisfying any of the conditions given in Lemma 3.1 and this emphasises that the Lemma gives just sufficient conditions for $\mu(A+BK)$ to be bounded above. In this particular example if $a_{22} = -4$ then the choice $k_1 = -4$, $k_2 = 1$ makes $A+BK$ symmetric with $\mu(A+BK) = -3$ and the conditions of Theorem 3.1 are again satisfied.

The control law is linear and to some extent independent of the nonlinearity: it will drive the system to the origin whenever the nonlinearity satisfies condition 2) with $p(t) = 2$. This flexibility in the structure of the nonlinearity in the system which can be controlled by a particular feedback law is due to the somewhat stronger demands made upon the linear part of the equations in the form of a bound upon $\mu(A+BK)$. As our conditions have been derived using norm bounds upon the system equations and because we have shown that the norm of the solution tends to zero this has led to no restrictions upon the form of the nonlinearity satisfying the norm bound. This can be illustrated by considering Example 5 in Gershwin and Jacobson. They consider the case where

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad f(y) = \begin{bmatrix} 0 \\ -(\text{sat}(y_2)+1)y_2 \end{bmatrix}$$

and show controllability to the origin with control defined by $u = -\frac{1}{2} Ky - (\text{sat}(y_2)+1)y_1$

It is not possible to show the above system to be null controllable using Theorem 3.1 because of these stronger demands upon the linear part of the equation - there is no choice of K which makes $\mu(A+BK)$ sufficiently negative to dominate the nonlinearity. The example has shown how, if the linear section of the equation is sufficiently controllable,

our results, unlike those of Gershwin and Jacobson, are not concerned with the exact form of the nonlinearity.

Example 2

An example with time-varying parameters is necessary to illustrate the use of Corollary 3.1 because the corollary is an obvious consequence of Theorem 3.1 if these parameters are time invariant. Let us consider a simple alteration to Example 1 where

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -3 \end{bmatrix} ; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; f(y, t) = \begin{bmatrix} 0 \\ \frac{1}{(1+t)} (\text{sat}(y_2) + 1)y_2 \end{bmatrix} \quad E2$$

Then, examining individually the conditions of the Corollary on the interval $(0, \infty)$

- 1) $f(0, t) = 0 \quad \forall t \in (0, \infty)$
- 2) There exists some $p(t)$ such that $||f(y, t)|| \leq p(t) ||y||$

Following Example 1 let us choose $p(t) = \frac{2}{1+t}$

- 3) a) There exists $\bar{t}$ such that $\int_t^{\bar{t}} p(r) dr = c_1 < \frac{1}{2}$

Such a time $\bar{t}$ is unique and in this case is given by

$$\bar{t} = \exp\{\frac{1}{2}c_1\} \cdot (1+t) - 1$$

$$\text{and so } \bar{t}_0 = \exp\{\frac{1}{2}c_1\} - 1$$

- b) $|\bar{t} - t| \leq c_2 < \infty$

On the interval bounded away from infinity this condition is satisfied although the lengths of consecutive intervals are increasing.

- c) $\int_t^{\bar{t}} \mu(A+BK) ds \leq \ln \frac{\lambda - (1+\lambda)c_1}{1-c_1}$

Say we choose again $c_1 = 1/5$ then we need to choose

k_1, k_2 such that

$$(\exp\{1/10\} - 1)(1+t) \cdot \mu(A+BK) \leq -0.29$$

But in this case the intervals (t_i, t_{i+1}) are increasing and so if this condition is satisfied at $t=0$, $\bar{t}=t_0$ it is satisfied $\forall t \in (0, \infty)$.

So we need to satisfy

$$\mu(A+BK) \leq -0.29 \frac{1}{(\exp\{1/10\}-1)} = -2.75$$

and the choice of $k_1=-2.8$, $k_2=-1$ ensures that this is satisfied.

- 4) In this example, because the linear section of the equations are not time-varying, the condition which requires that $\|y(t)\|$ be bounded above by a monotonic decreasing function $r(t)$ is satisfied by the solution to

$$\dot{r}(t) = \mu(A+BK) r(t) ; \quad r(t_0) = \|y_0\|$$

In situations where $\mu(A+BK)$ is not constant and hence not necessarily negative $\forall t \in (t_0, \bar{t}_0)$ this definition of the function $r(t)$ would not suffice and some other function must be determined.

Hence the system given by equation E2 is shown to be strongly null controllable using the conditions given in Corollary 3.1.

Example 3

The introduction of control terms into the bounds upon the nonlinearities enables systems with nonlinearities which involve control terms to be examined. This form of nonlinearity is not inappropriate for consideration using the results of Chapter Three provided that the state is contained within some subset of the state-space, but the introduction of control explicitly into the bounds enables

this form of nonlinearity to be considered with possibly smaller values for the scalar multipliers. The results presented in Chapter Four require the solution of a Riccati equation and, in order that these examples can be worked by hand, only scalar systems are considered. The choice of scalar examples still enables the application of the theorems to be illustrated, however it is important to note that in the scalar case the controllability of the linear equation is guaranteed.

As an example of Theorem 4.1 consider the nonlinear control system given by

$$\dot{y}(t) = a(t)y(t) + b(t)u_y(t) + f(y, u_y, t) \quad E3$$

where $a(t) = 1$; $b(t) = 2$

and $f(y, u_y, t) = (\text{sat}(y))^{\frac{1}{2}} u_y^{\frac{1}{2}} (1 - \exp(-t))$

Examining the conditions of the theorem we can see that the linear system is controllable and

- 1) $f(0, 0, t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) There exists $p(t)$ so that

$$||f(y, u_y, t)||^2 \leq p(t) ||y(t)|| \cdot ||u_y(t)||$$

In this example

$$\begin{aligned} ||f(y, u_y, t)||^2 &\leq | \text{sat}(y) \cdot u_y (1 - \exp(-t)) |^2 \\ &\leq |y| \cdot |u_y| \end{aligned}$$

and this condition can be satisfied with

$$p(t) = 1 \quad \forall t \in (t_0, \infty)$$

- 3) For times $\{t_i\}$ and feedback matrix $K(t) = -\frac{1}{2}B^T(t)S(t)$ the following conditions must be satisfied:

$$a) \int_{t_i}^{t_{i+1}} p^{\frac{1}{2}}(s) ||\frac{B(s)}{2}||^{\frac{1}{2}} ||S_{i+1}||^{\frac{1}{2}} ds \leq c_1 < \frac{1}{2}$$

If we choose $|t_{i+1} - t_i| = 1/6$ and $c_1 = 1/3$ then this requires

$$\frac{1}{6} \left[1 \cdot \left(\frac{2}{3} \right)^{\frac{1}{2}} ||s_{i+1}||^{\frac{1}{2}} \right] \leq 1/3$$

$$||s_{i+1}|| \leq 4$$

This condition defines the end condition for the solution to the Riccati equation which describes the evolution of $S(t)$. In this scalar example the Riccati equation simplifies to

$$\dot{s}(t) = -2as(t) + b^2 s^2(t)$$

and it is well known that (see for example Barnett^{B1}) this equation has solution given by

$$s(t) = v_2(t) / v_1(t) \text{ where}$$

$$\dot{v}_1(t) = av_1(t) - b^2 v_2(t)$$

$$\dot{v}_2(t) = -av_2(t)$$

and so solving these equations and substituting

$$s(t) = \frac{s(t_{i+1}) \exp(2a(t_{i+1}-t))}{1 + b^2 / 2a \cdot s(t_{i+1}) (\exp(2a(t_{i+1}-t_i)) - 1)} \quad t \in (t_i, t_{i+1})$$

The values of a, b and $|t_{i+1} - t_i|$ given in this example lead to

$$s(t_i) = \frac{4 \cdot e^{1/3}}{1 + 2.4 \cdot (e^{1/3} - 1)} = 1.33$$

b) $|t_{i+1} - t_i| \leq c_2 < \infty$

This is certainly satisfied by the above choice of

$$|t_{i+1} - t_i| = 1/6.$$

c) There exists $S(t)$, a solution to the Riccati equation with $||s_{i+1}||$ satisfying condition a) such that

$$\int_{t_i}^{t_{i+1}} \mu \left(A(t) - \frac{B(t)B^T(t)S(t)}{2} \right) dt \leq \ln(1-c_1)$$

This condition reduces to

$$\int_{t_i}^{t_{i+1}} \left(1 - \frac{4}{2} s(t) \right) dt \leq \ln(2/3) = -0.4$$

Substituting for $s(\frac{t}{\#})$ from above this becomes

$$\begin{aligned}
 & \int_{t_i}^{t_{i+1}} \left(1 - \frac{2 \cdot 4 \cdot e^{2(t_{i+1}-t)}}{1 + 2 \cdot 4 \cdot (e^{2(t_{i+1}-t)} - 1)} \right) dt \\
 &= \int_{t_i}^{t_{i+1}} \left(\frac{-1}{8/7 e^{2(t_{i+1}-t)} - 1} \right) dt \\
 &= \frac{1}{2} \log \left[\frac{d-1}{d} \right] \Big|_{d=8/7 e^{1/3}}^{d=8/7} \\
 &= \frac{1}{2} \log \frac{1.4}{4.2} = -0.55 < \ln(2/3) = -0.4
 \end{aligned}$$

and hence the conditions of the theorem are satisfied
and the system given by E3 is shown to be null controllable
on (t_0, ∞) .

Example 4

Theorem 4.3 considers the case where the nonlinearity is bounded by a term involving $||u_y(t)||$ and this example again requires the solution of a Riccati equation. As an illustration of this theorem consider the equation

$$\dot{y}(t) = ay(t) + bu_y(t) + f(y, u_y, t)$$

where $a = 2$, $b = 3$ and $f(y, u_y, t) = e^{-2t} \cdot \frac{(y(t) \cdot \sin y(t))}{10} \cdot u_y(t)$

then, if the solutions are restricted to evolve in the sphere where $||y(t)|| \leq d$, $f(y, u_y, t)$ can be bounded by a term $q(t) ||u_y(t)||$.

Examining individually the conditions of the theorem gives

- 1) $f(0, 0, t) = 0 \quad \forall t \in (t_0, \infty)$
- 2) There exists $q(t)$ such that $||f(y, u_y, t)|| \leq q(t) ||u_y(t)||$

For $y(t) \in D \subset \mathbb{R}^n$ where D is those solutions with $||y(t)|| \leq 10$ then we can choose either $q(t) = 1$ or more exactly,
 $q(t) = e^{-2t}$ over $(0, \infty)$.

3) For times $\{t_i\}$ and feedback matrix $-\frac{1}{2}B^T(t)S(t)$ the following are to be satisfied:

$$a) \int_{t_i}^{t_{i+1}} q(s) \left[\left| \frac{B^T(s)}{2} \right| \right] ds \cdot ||S_{i+1}|| \leq c_1 < \frac{1}{2}$$

Say we choose $q(s) = e^{-2s}$ then we need

$$\frac{3}{2} \int_{t_i}^{t_{i+1}} e^{-2s} ds ||S_{i+1}|| < \frac{1}{2}$$

$$\text{or equivalently } ||S_{i+1}|| < \frac{2}{3} \frac{e^{2t_i}}{(1 - e^{-2(t_{i+1} - t_i)})}$$

$$b) |t_{i+1} - t_i| \leq c_2 < \infty$$

The choice $c_2 = \frac{1}{2}$ gives values of

$$||S_1|| = \frac{2}{3} \frac{1}{(1 - 1/e)} ; ||S_2|| = \frac{2}{3} \frac{e}{(1 - 1/e)} ; \text{ etc}$$

c) There exists $S(t)$, a solution to the Riccati equation with end condition $||S_{i+1}||$ satisfying condition a), such that

$$\int_{t_i}^{t_{i+1}} \mu \left(A(t) - \frac{B(s)B^T(s)S(s)}{2} + q(s) \left| \frac{B^T(s)}{2} \right| ||S_{i+1}|| \right) ds < 0$$

For the choice of $c_2 = \frac{1}{2}$ then $||S_1|| < 1/.995$ so if we put $||S_1|| = 1$ in the Riccati solution given in the previous example then $s(t)$ is given as

$$s(t) = \frac{e^{2(t_1 - t)}}{1 + 9/4 (e^{2(t_1 - t)} - 1)} \quad t \in (0, \frac{1}{2})$$

$$s(t) = \frac{4 e^{2(\frac{1}{2} - t)}}{-5 + 9e^{2(\frac{1}{2} - t)}} \quad t \in (0, \frac{1}{2})$$

Now using this value of $s(t)$ the above condition becomes

$$\int_{t_0=0}^{t_1=\frac{1}{2}} \left(2 - \frac{9}{2} \cdot \frac{4e^{2(\frac{1}{2}-s)}}{-5+9e^{2(\frac{1}{2}-s)}} \right) + e^{-2s} \left(\frac{3}{2} \right) \cdot 1.05 ds \leq 0$$

Or equivalently

$$\int_{t_0=0}^{t_1=\frac{1}{2}} \frac{-20}{-5+9e^{2(\frac{1}{2}-s)}} ds + 3.15 \int_{t_0=0}^{t_1=\frac{1}{2}} e^{-2s} ds \leq 0$$

which after integration becomes

$$4.1 \frac{1}{2} \ln \left[\frac{d-1}{d} \right]_{d=9/5}^{d=9/5 \cdot e} + 3.15 \left[\frac{1}{-2} e^{-2s} \right]_{s=t_0=0}^{s=t_1=\frac{1}{2}}$$

$$= 2 \ln \frac{4/5 \cdot e}{(9/5 e - 1)} - 1.57(e^{-1}-1)$$

$$= 2 \ln (0.58) + 1.57 \cdot 0.63$$

$$= 2 \cdot -0.55 + .99 < 0 \quad - \text{the conditions of the theorem}$$

are satisfied over the interval $(0, \frac{1}{2})$ and it only remains to extend this to the intervals (t_1, t_2) etc. In this case we have chosen $q(t) = e^{-2t}$ and so, if the choice $|t_2 - t_1| = \frac{1}{2}$ is made then condition 3a) will permit greater values of $||S_2||$ than those used for $||S_1||$ above and hence the appropriate condition 3c) will be satisfied. This process can be carried forward to the remaining intervals. Alternatively it is quite possible to choose a larger bound for $|t_2 - t_1|$ with a consequentially smaller bound for $||S_2||$. It is over the interval (t_0, t_1) that a nonlinearity, bounded above by e^{-2t} , has greatest effect upon the linear equation and the behaviour of the solution over remaining intervals can be inferred from the behaviour over (t_0, t_1) . The nonlinear system given by equation E4 has been shown to be null controllable from y_0 , $||y_0|| \leq 5$ on $(0, \infty)$.

Example 5

The results presented in Chapter Five are somewhat different from those presented in previous chapters and have the advantage that only one integral need be evaluated to check the null controllability of the nonlinear system.

The following example of an application of Theorem 5.1 :

serves to illustrate how these results, which all have essentially the same form, might be implemented.

Consider the null controllability of the following system from (y_0, t_0) to (ϵ, T) where $\|y_0\| = 3$, $\epsilon = 1/10$, $t_0 = 0$ and $T = 3$.

$$\dot{y} = A y(t) + B u_y(t) + f(y, u_y, t) \quad E5$$

$$\text{where } A = \begin{bmatrix} 3 & 6 \\ 2 & -4 \end{bmatrix}; \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix};$$

$$\text{and } f(y, u_y, t) = \begin{bmatrix} (y_1 - y_2) (1 - \exp\{-t\}) \\ (y_1 + y_2) (1 - \exp\{-t\}) \end{bmatrix}$$

with initial condition $y(0) = y_0$, $t_0 = 0$.

The conditions required in Theorem 5.1 are that the linear system with matrices defined by $A(t)$, $B(t)$ be controllable and

- 1) $f(0, 0, t) = 0 \quad \forall t \in (0, \infty)$
- 2) There exists $p(t)$ such that

$$\langle f(y, u_y, t), y(t) \rangle \leq p(t) \|y(t)\|^2$$

For this particular example $f(y, u_y, t)$ satisfies the above bound with $p(t) = (1 - \exp\{-t\})$. The influence of the rotation increases as $t \rightarrow \infty$ and $(1 - \exp\{-t\}) \rightarrow 1$.

- 3) There exists an admissible feedback control such that

$$\int_{t_0}^T \mu(A(s) + B(s)K(s)) + p(s) \, ds \leq \ln \frac{\epsilon}{\|y_0\|}$$

In this application the above condition becomes

$$\int_0^3 \{\mu(A+BK) + (1 - e^{-s})\} ds \leq \ln \frac{1}{30} = -3.4$$

The choice of $K = \begin{bmatrix} -4 & -1 \end{bmatrix}$ gives

$$A+BK = \begin{bmatrix} -5 & 4 \\ 2 & -5 \end{bmatrix}$$

and hence $\frac{1}{2} \{(A+BK) + (A+BK)^T\}$ is given by

$$\begin{bmatrix} -5 & 3 \\ 3 & -5 \end{bmatrix} \text{ which has eigenvalues } -2, -8 \text{ and so}$$

$$\mu(A+BK) = -2$$

Condition 3) requires that $\int_0^3 (-2 + 1 - e^{-s}) ds \leq -3.4$

which is easily seen to be satisfied.

In this example the system, given by E5, has been shown to be null controllable from $(3,0)$ to $(1/10,3)$ using controls defined by the feedback matrix $[-4 \ -1]$. In fact, the system has been shown to be null controllable from $(y_0: ||y_0|| \leq 3, 0)$ to $(\epsilon = 1/10, 3)$.

The other results given in Chapter Five take a similar form and examples of these results have a similar structure.

6.4 A CONCEPTUAL PROCEDURE FOR A BASIC IMPLEMENTATION

The preceding examples illustrate how some of the controllability results can be used to ascertain whether some simple systems are null controllable. Examination of these examples enables a procedure which would form the basis of a computer implementation to be established. In this procedure the basic steps and decisions made in a typical implementation are formalised and some possible areas of computational difficulty indicated.

STAGE 1:

Each of the theorems requires that certain 'a priori' information about the system is available. This includes:

- a) a full knowledge of the linear part of the dynamics and hence the elements of the matrices $A(t)$ and $B(t)$;
- b) a knowledge of the nonlinearity $f(\cdot, \cdot, t)$ such that it can be bounded by an appropriate inequality;
- c) the range of possible initial conditions, or a specific value for $y(t_0)$ must be given;
- d) some of the theorems require that an upper

limit be placed upon $||K(t)||$, and hence the possible control feedback matrices are restricted.

STAGE 2:

It is necessary that the linear section of the equation be completely or totally controllable and this can be verified using well known criteria given in the constant case by Kalman^{K2}, or in the time-varying case by Silverman and Meadows^{S2}.

STAGE 3:

Determine, from the knowledge of the nonlinearity, permissible values for the scalars used in the inequality bound. If the state $y(t)$ needs to be bounded, $y(t) \in D \subset R^n$, in order to yield a value for these scalars then D is that sphere in the state space given by $||y(t)|| \leq 2||y_0||$. Likewise, if the control $u_y(t)$ needs to be bounded in terms of $y(t)$ a value for a bound upon $||K(t_1)||$ has been specified in Stage 1.

STAGE 4:

Choose a time t_{i+1} and if necessary $||S_{i+1}||$ such that condition 3a) is satisfied. The integral that forms the basis of condition 3a) in the appropriate theorem requires that the length of the interval being considered (initially (t_0, t_1) and subsequently (t_i, t_{i+1})) and the maximum value of $||S_{i+1}||$ are balanced against each other so that the integral can be bounded by $c_1 < \frac{1}{2}$.

STAGE 5:

[In theorems where the form of the feedback matrix is left undefined this stage is omitted.]

Using the value of $||S_{i+1}||$ specify a feedback control matrix over the interval being considered. A value for

$S(t_{i+1}) = S_{i+1}$ which satisfies the bound upon $||S_{i+1}||$ is used via a reverse time solution of the Riccati equation to define $S(t)$ over (t_i, t_{i+1}) .

STAGE 6:

Check condition 3c) in the theorem statement using the interval (t_i, t_{i+1}) , feedback matrix and scalar multipliers specified earlier.

If the condition is satisfied over (t_i, t_{i+1}) proceed to the next time interval and return to Stage 3.

If the condition is not satisfied either return to Stage 5 and choose another feedback matrix satisfying the previous conditions or, if this is not possible, return to Stage 4 and choose another time t_{i+1} and $||S_{i+1}||$ which satisfy the inequality bound.

The results of Chapters Three and Four consider the nonlinearity to be bounded by norms and take a similar form whereas those of Chapter Five consider the nonlinearity to be bounded by inner products and take a slightly different form. The inner product results consider only the interval (t_0, T) in the integral inequalities and these inequalities need only be satisfied once but the earlier theorems require the interval to be split into (t_i, t_{i+1}) over which the necessary inequalities are satisfied. Thus an application of the results in Chapter Five would only require one run through the above procedure.

In Stage 3 the choice of scalars which will bound the nonlinearity can sometimes simplify the remainder of the criteria. In many cases the nonlinearity can be crudely bounded by a constant or more accurately by some time-varying

function and it may be worth making a series of tests using the simple constant bound before entering into the more complex situation of a timevarying function $p(t)$ (or $q(t)$).

The choice of the constant c_1 in Stage 4 appears to be straightforward in applications of those results presented in Chapter Four: a reduction in the value of c_1 does lead to a less negative value on the righthand side of the bound in condition 3c), however this is compensated for by either a smaller value for $||S_{i+1}||$ or a smaller time interval over which to evaluate condition 3c). An application of the results in Chapter Three seems to require a slightly different choice for c_1 because a value close to $\frac{1}{2}$ leads to almost impossible conditions in 3c). However, because in this case the feedback matrix is not implicitly restricted by a bound upon $||S_{i+1}||$ as it is in Chapter Four this may not prove prohibitive.

There will be a number of choices of elements of the matrix S_{i+1} which will satisfy the constraint upon $||S_{i+1}||$ and when propagated via the Riccati equation these will lead to different possible feedback control matrices. Having made a choice of S_{i+1} the Riccati equation solution can be obtained using established computer algorithms and the feedback is defined by $K(t) = \frac{1}{2} B^T(t)S(t)$ over the time interval considered.

The checking of condition 3c) undertaken in Stage 6 will require that, having formed the matrix $A(t)+B(t)K(t)$, its logarithmic norm subordinate to the vector norm considered be calculated: in the case of the Euclidean norm this will involve eigenvalue calculation but for some other norms this

may only involve the summation of row or column elements. The numerical integration of the scalar resulting from the formation of the lefthand side of the condition 3c) would use conventional algorithms.

6.5 CONCLUSIONS AND SUGGESTED AREAS OF FURTHER WORK

The conceptual procedure given in the previous section provides a means of collectively describing the results. Generally these fall into a framework where the linear dynamics are compared with the additive nonlinearity in some integral inequality. In some cases a control law is derived and this is used in the comparison of the linear and nonlinear sections. Some well known computational problems are immediately evident in the procedure but these can be tackled using conventional methods. The reverse time solution of the matrix Riccati equation, given an end point condition for $S(t_{i+1})$, has been considered in some detail and suitable computer programs exist. The calculation of the logarithmic norm of a matrix may involve evaluation of the largest eigenvalue and again this is relatively simple. Note that the use of the logarithmic norm does mean that some of the computational load associated with a more direct method involving evaluation of

$$||\exp\{[A(s)+B(s)K(s)ds]\}|| \text{ is avoided.}$$

The procedure also serves to illustrate that in the algorithm it is necessary to choose some values for the parameter c_1 , the interval (t_i, t_{i+1}) and, if called for, $||S(t_{i+1})||$. These values are interrelated and for some simple systems the choice is straightforward; more care needs

to be exercised when choosing these values for larger systems. Given this upper bound on $||S(t_{i+1})||$ it is necessary to choose the elements of $S(t_{i+1})$ and one possible way of doing this would be to choose $S(t_{i+1})$ diagonal and change the elements of the matrix if this is not satisfactory.

The procedure follows the form of the theorems closely and does not attempt to ensure that if a possible feedback matrix exists, it is always generated in the algorithm. This convergence property would be a function of the choice of elements of $S(t_{i+1})$ which is made given a bound upon $||S(t_{i+1})||$. It may be possible to improve this aspect of the procedure by drawing upon the work in the area of optimal control where the Riccati equation is used in a similar way.

The computational complexity of the algorithm is much reduced when the nonlinearity can be bounded by a constant or monotonic decreasing value for $p(t)$, or alternatively when the linear section of the equation is not timevarying. In some applications the computational load may be rather high but because the property of controllability is a qualitative one it should only be necessary to test the plant once for some given initial conditions.

The results require that the logarithmic norm, a symmetric scalar function, of the system matrices can be bounded above by some scalar dependent upon the nonlinearities. But, it has been shown in Chapter Three that it is not always possible to assign the value of $\mu(A(t)+B(t)K(t))$ by choice of $K(t)$ even when the matrices $A(t)$, and $B(t)$ form a controllable pair. From this point of view the work presented here places

slightly stronger conditions than usual upon the linear equation. On the other hand our approach has enabled weak conditions to be placed upon the nonlinearity. This means that the nonlinearity need not be fully described by some functional relationship but instead need only be sufficiently understood to be bounded in either norm or inner product.

The development of a practical algorithm to test the null controllability of nonlinear systems requires that some aspects of the procedure be examined in greater detail and some of these have been mentioned above. The problems associated with choosing c_1 , $|t_{i+1} - t_1|$ and $||S(t_{i+1})||$ would form the basis of any additional work and these areas are worthy of consideration in some depth if the results are to be fully implemented. It may be possible, via analytic work, to show the length of the interval (t_i, t_{i+1}) to be immaterial in some applications. Due to the interdependent nature of the conditions the length of this interval can be shown to be irrelevant if in Theorem 3.1 both the linear section of the equation and the scalar bound $p(t)$ are constant. Further work may enable this property to be extended to more general situations.

In the field of linear systems theory a deeper study of the relationship between the controllable pair $A(t), B(t)$ and the possibility of assigning $\mu(A(t) + B(t)K(t))$ by choice of feedback matrix $K(t)$ could lead to an improvement in Lemma 3.1 and consequentially strengthen the subsequent results. In Chapter Four it was indicated that, by choice of a quadratic norm for vectors, it was always possible to bound $\mu(A(t) + B(t)K(t))$ to be less than any arbitrary value.

This is one way of weakening the restrictions upon the linear part of the equations and instead putting further requirements upon the nonlinearity. The choice of the norm under which to structure an application of the results is similar to the choice of a quadratic Lyapunov function but again this area of work requires further investigation.

Alternatively work could be centred around the possibility of finding a better scalar bound to the solution of the nonlinear and linear systems. In some cases the function used here can be a rather coarse measure of stability of the trajectories and further results could lead to weaker condition upon the nonlinear system.

Throughout this thesis the controllability results have required that the final state is the origin. This is a common simplification of the problem when considering nonlinear systems and stems from the analogy with stability theory, where, without loss of generality, one can assume that the stability properties are being considered with regard to the null solution. For linear systems the concept of controllability is considerably more general than this. Our controllability definitions, and those of Ben Lashiher and Storey^{B2} are suitable for situations where the final zone is not necessarily centred around the origin and it may be possible to show null controllability from (α, t_0) to (β, T) where $\max(\|y\| \leq \beta) \leq \min(\|y\| \leq \alpha)$ using similar arguments to those used here. When the time interval is finite the requirement that

$\exp \left\{ \int_{t_0}^T \mu(A(s) + B(s)K(s)) ds \right\}$ is negative does not restrict

the final state to be zero but only to have norm less than the initial state. Using this property it may be possible to show the linear and nonlinear systems to have solutions sufficiently close together at time T to ensure that the nonlinear system satisfies the controllability condition.

Interconnected systems have dynamic behaviour which can be bounded by functions of each basic system. An application of some of the techniques used here would result in a system of smaller dimension whose stability properties imply bounds upon the larger system. Siljak ^{S3} has considered interconnected systems and following his work should enable null controllability criteria to be developed for these systems. Difficulty may be encountered in characterising the non-dynamic interconnections between the systems when it is necessary to distinguish between connections which add to, or detract from, the controllability of the total system.

The analysis of a linear system with a nonlinear feedback function may be simplified by using a technique similar to the above to show asymptotic equivalence with the same linear dynamics under a linear feedback law. This might be extended to show the null controllability of linear, or possibly nonlinear, systems with nonlinear state variable feedback.

In Chapter Four the bounds upon the nonlinearity can be regarded as a special case of the quadratic cost function used in optimal control and the form of the feedback matrix is the same as that which would result from an application of optimal control theory. This also applies to the work of Gershwins and Jacobson (see also Feljd ^{F1}). This similarity

might be exploited by using, in the feedback calculation the solution to the optimal control problem with cost function based upon the righthand side of the inequality bound on the nonlinearity. The null controllability problem would then be reduced to examining the behaviour of the nonlinear system when controls are applied which minimise a positive function which is an upper bound for the norm of the nonlinearity.

Comparisons between the results presented here and those of other researchers in the area of null controllability fall into two categories: those where the analysis has some common ground and those where the main similarity is in the problem considered. In the initial stages of their paper Gershwin and Jacobson discuss some problems using a similar control policy and although their results are of a slightly different form it has been possible to make direct comparisons. The approach of Ben Lashiher and Storey bears some similarity to the earlier chapters of this work and sometimes their results can be case in the framework used here. However it should be noted that although the same mathematical tools are sometimes used, they are often used in different situations: whereas the authors use scalar comparison functions to bound directly the systems considered, comparison functions are used in this work only as a method of circumventing an integral inequality. Other null controllability results cannot be compared so directly with our results but again some superficial resemblances can be seen in the methods used. The Schauder Fixed Point Theorem is used by some authors ^{M2} , ^{V1} , to show the existence of a map between solutions of linear and

nonlinear systems and hence show the existence of a controllable solution.

The results presented in this thesis yield some sufficient conditions for the null controllability of a class of nonlinear systems, based upon techniques derived from stability theory, where nonlinear differential equations have been examined in some detail. A number of different ways of characterising the nonlinearities are considered and a series of null controllability results derived. Some comparisons have been drawn between these results and previous work in the area. It is considered by some researchers that nonlinear systems are so complex that they need to be considered individually and that it is very difficult to develop general results. By considering linear systems with an added nonlinearity this work provides a way of classifying certain nonlinearities and examining the null controllability of some nonlinear systems.

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