Errata and Addenda

Part A.

p. 34: The energy equation given here is applicable to a strictly incompressible fluid, not to gas flows at low Mach number.
p. 55: Remarks following equation (8) apply to the case $n = q \neq 1$
p. 65, line 16: For "equivalent" read "similar"
p. 68, 2nd Para: Note that it has been shown that the kinetic energy remains constant for the doubtful cases. This point is crucial to the applications which follow.
4th Para: An angular momentum theorem generally vaid for a viscous fluid can be demonstrated.

Additional References:
Preston, J. H. (1950), Aero. Quart., 1, p. 319.
Nanda, R. S. (1960), Appl. Sci. Res., A9, p. 85.
Also, Donaldson and Sullivan (1960) as in Part C.

Part B.

This part has been deleted at the request of the examiners.

Part C.

Additional References:
Hinze, J. O. (1959), "Turbulence," McGraw-Hill, New York.
Donaldson, C. duP. and Sullivan, R. D. (1960), Report AFROSR TN 60-1227, Aero. Res. Associates of Princeton and in 1960 Trans. Heat Transfer Institute, Stanford University Press.

STUDIES OF ROTATING FLUIDS:

a) Plane Axisymmetric Flow,

~~b) Forced Oscillations in a Rotating Fluid,~~

c) The Ranque-Hilsch Vortex Tube

by

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Volume I

Abstract

Three discrete studies of rotating fluids are presented.

In Part A the fundamentals of vortex flow are studied by restricting consideration to plane axisymmetric flows of an incompressible fluid. It is found to be possible to introduce the generalization of radial flow as from a constant line source along the axis without inhibiting the analytic development for either steady or non-steady cases. The solutions obtained are applicable to flow between porous rotating cylinders.

Part B consists of theoretical and experimental investigations of forced oscillations in a uniformly rotating liquid. Several new approaches are presented: a study of time-dependence of solutions, a similarity law generally applicable to rapid-forcing motions, and an analysis by the method of characteristics valid for slow-forcing motions. The similarity law leads to solutions for such cases as the oscillating and pulsating spheres. Next the problem of an oscillating disc is reconsidered and small corrections are made to previous work. Finally, an attempt is made to give some intuitive grasp of the general problem. The experiments demonstrate again the existence of "characteristic surfaces" for the slow-forcing regime.

Studies of vortex motion directed towards an understanding of the Ranque-Hilsch vortex tube comprise Part C.

Observations of flows of water through horizontal and vertical tubes are reported; numerous internal temperature and pressure measurements made in air are presented. Hypotheses of the mechanism by which energy separation is effected are reviewed; turbulent mixing is found to be the most plausible explanation. A set of approximate relations appropriate to high-swirl flows is derived from Reynolds' equations. They give insight into the dynamics of such flows and suggest that the Ranque effect is chiefly due to an outwards flow of work associated with the tangential shear stress and an outwards heat flux due to mixing of the compressible fluid across the radial pressure gradient. Finally, a class of axisymmetric solutions of the Navier-Stokes equations is derived and studied.

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General Introduction

The work presented here treats three widely differing aspects of rotating fluid flow. The first is an exact theoretical treatment of plane axisymmetric motions, the analysis concentrated on the case of a viscous, incompressible fluid. Another is the study of the linearized problem of small forced oscillations in a basically uniformly rotating liquid, together with the investigation of an experimental realization of the theoretical model. The last is a consideration of the complex swirling flow of a gas within a cylindrical chamber (the Ranque-Hilsch vortex tube) in which the thermodynamic aspects of the problem are of special interest and in which compressibility and turbulence play important parts.

The questions may well be asked: "Why are three so dissimilar studies grouped together; have they any real unity?" The equations of motion in terms of cylindrical co-ordinates hold the answer to these questions. They are obviously the most convenient starting point for the analysis of all axisymmetric flows as their use reduces the number of independent variables by one. And they also indicate the particular unity of swirling, axisymmetric flows. The special dynamics of such flows are indicated by the acceleration terms peculiar to this co-ordinate system: $\frac{v^2}{r}$ and $\frac{uv}{r}$, in the usual notation. The real unity of these problems lies not just in the rotary motion of the fluid, with its attendant

family of dynamic effects, but also in the axial symmetry of the flow patterns which dictates the use of the same theoretical tools in their analysis. This is not just a formal division based on the words axisymmetric and swirling, but a useful one based firmly on important physical and mathematical kinships.

It is not difficult to justify the study of this class of problems. Theoretically they are important because of the simplicity of the analysis in terms of cylindrical co-ordinates. Also, theoretical models with axial symmetry are among the easiest to realize experimentally. Physically, their importance is supported by such permanent factors as the rotation of the earth about its axis and the ease of manufacture of surfaces of revolution.

But the studies reported here are widely separated, although within the basically similar group of axisymmetric swirling flows. It will perhaps be as well to account for this mixed bag before reaching into it.

The present study of swirling fluids was begun as an investigation of the Ranque-Hilsch vortex tube. This problem presented a welter of the effects of compressibility, turbulence and heat transfer superimposed on the only partially understood basic dynamic phenomena. This complexity, which made much of the available literature all but incomprehensible because of its lack of precision, was the first major difficulty encountered.

It soon became apparent that a gradual approach would be necessary, beginning with the study of a simpler class of flows.

The second major problem encountered was that of dealing with the extensive literature pertinent to the study. The underlying unity of approach expedient for the wide range of axisymmetric swirling flow problems means that useful concepts and methods are very widely scattered throughout scientific literature. In addition to the literature of the vortex tube itself, there exist extensive bodies of material which might contain work pertinent to this investigation---the literature on vortex furnaces, cyclone separators of various kinds, drain vortices, and the rotary phenomena of meteorology. Further, the literature of applied mathematics is fruitful, the simplicity of axisymmetric problems having attracted mathematicians interested in testing their techniques. It was soon realized that a complete search of this vast range of material was quite impractical, let alone the mastery of its content. However, it seemed necessary to be at least on nodding terms with the various fields of study so that the chance of neglecting a really fruitful line of investigation might be reduced.

Thus the study began with an investigation of plane, axisymmetric flows, a general survey of previous work on rotating fluids being carried on at the same time. The search of the literature led to the discovery of many of

the results obtained independently by the analysis carried on in parallel. It was realized, of course, that such a simple situation must often have attracted attention in the past. But just where the accounts of such analyses were to be found was initially a mystery.

That any mystery whatsoever should exist about this essential building block in the understanding of swirling flows seemed undesirable. To remedy this situation it was decided to include a brief account of the theoretical work on plane axisymmetric flows, together with a somewhat more unified treatment of certain aspects of the problem. One of the chief aims of this investigation was to establish criteria for estimating whether certain three-dimensional problems were tractable or not.

Thus the first study reported herein is accounted for. The second arose from the general survey of the literature. An account of a theoretical and experimental investigation of forced oscillations in a rotating fluid was encountered. As apparatus existed in the laboratories of the Department of Aeronautics with which the experimental realization might be repeated, it was decided to attempt to do so. The attempt was successful and led to a consideration of the theoretical side of the problem. These investigations comprise the second study.

The basic investigations---those pertaining to the vortex tube---formed a matrix into which these two studies

were set, being carried on concurrently with them.

The reports on the three studies which follow are quite independent: the notation used in each is explained there; conclusions, references, and figures are to be found at the end of each part. A letter (A, B, or C) at the top of each page indicates to which part the page belongs.

Part A

Plane Axisymmetric Flow

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Notation

Only the frequently used symbols are listed here. Others are defined as introduced.

Independent Variables

r , θ , z cylindrical polar co-ordinates

t time

$\tau = \nu t$, $\tau' = \tau/\sigma$ modified measures of time

Fluid Properties

ρ density

μ viscosity

ν kinematic viscosity

h specific heat

k thermal conductivity

Flow Variables

u , v , w radial, tangential, and axial velocity components

p pressure

p_{rr} , $p_{\theta\theta}$ direct stress components

$\tau = p_{r\theta}$ shear stress

e internal energy

T temperature

Δ divergence

ζ vorticity

Q source strength per unit volume

Φ dissipation function

A

Non-dimensional Parameters

$C = \frac{ru}{\nu}$ source Reynolds number

$q = c/2$

$\sigma = \frac{\mu h}{K}$ Prandtl number

Generalized Cylinder Functions

$W_n(\alpha r)$, $W_n'(\alpha r)$ see equations (7.8) and (7.12)

Constants of Integration

A , B , a , b , etc.

1. Introduction

The theoretical study of rotating fluids is best begun by considering the simplest class of axisymmetric flows, those in which the flow pattern is the same in every plane perpendicular to the axis. For this case the flow variables are functions of a single spatial variable, the radial distance. This class of flows co tains the simplest two-dimensional flows, those due to a line source and a line vortex.

It is obvious that a mastery of this simple class of problems must precede the treatment of more complex problems in rotating fluids. After seeing how far the analysis can be extended in these simple cases we will be in a better position to judge the tractability of more complex problems. Further, some insight into the special phenomena associated with vortex flows should be derived from this study. Familiarity with the relevant mathematical techniques will be an additional benefit.

We shall interest ourselves in the exact treatment of this class of flows. In order that such solutions can be obtained it is necessary that highly idealized fluids be postulated. Only fluids with simple equations of state can be treated; for example, incompressible fluids, or those obeying the perfect gas law. Further, the transport properties of the fluid (viscosity and thermal conductivity) must either be neglected or specified in some

very simple way (say, as constant coefficients).

We begin the study by considering the work which has already been done on this topic, dealing not only with what has been done, but also giving some thought to why it has been done, and to what seemingly remains to be done.

2. Previous Work

Very many investigators have dealt with particular aspects of the general class of plane axisymmetric flows. They include not only scientists interested in such problems for their own sake or wishing to gain insight into some more complex problem, but also mathematicians desiring solely to test their techniques. Hence the contributions to this study are widely spread though the scientific literature. The preparation of a complete list and summary of the very numerous contributions is impractical and, in view of the repetitious and trivial nature of many of them, is probably not even desirable. But is has been observed that recent workers have often been unaware of their predecessors' treatments of similar problems. It does seem worthwhile then to prepare a brief guide to some of these previous studies.

The work dealt with here will be primarily that on laminar motions. The literature dealing with the various extensions to this subject---the stability study initiated by Taylor, the study of fully developed turbulent motions in a basically axisymmetric situation, the investigations of heat transfer in such situations, and so on---will not be considered. It appears that the relevant literature is well known to workers in these fields.

Recently a number of devices have aroused interest in basic vortex flows: the vortex tube, the vortex furnace, the swirl atomizer, the cyclone separator. In particular,

study of the vortex tube has resulted in consideration of the radial transfer of energy in swirling fluids. Although these studies usually start from a two-dimensional theoretical model, this is often an arbitrarily or empirically determined one. Thus such investigations usually do not fall within the bounds of the present consideration.

Rather than undertake the considerable task of listing and discussing all the known work, we adopt the policy of referring to readily available works containing discussions of relevant papers. While none of the bibliographies is complete in itself (if only because of age), combined they give fairly easy access to the previous work.

The discussion is facilitated by breaking up the general class into three groups of increasing complexity: radial flows, swirling flows, and spiral flows.

2.1. Radial Flows

Since radial flows are usually prescribed by requiring that continuity be preserved except at the axis, the mathematical treatment of this case is trivial for highly idealized fluids. Such flows usually emerge as special cases from more general analyses. However, Binnie and Hooker (1937) give separate consideration to the radial flow of a compressible, inviscid fluid. Wu (1956) has considered the steady radial flow of a viscous, compressible fluid, the introduction of this degree of complexity into the properties of the fluid making even this

geometrically simple flow worthy of special treatment. He refers to previous work by Ringleb, Hess, Sakurai, and Levey. Certain unsteady problems of radial flow will also be of interest. Fraenkel (1959) refers to treatments of an acoustic line source.

2.2. Swirling Flows

Steady flows of this type will only be of interest for a viscous fluid in which the different elements are able to influence one another. For the unsteady case only flows of an incompressible fluid can satisfy the condition of zero radial velocity. Thus the sort of fluid which has usually been considered under this category is viscous and incompressible.

Interest has concentrated on problems of two types: cylinder problems and vorticity diffusion problems. The cylinder problems involve the steady or unsteady flow conditioned by rotating or fixed cylindrical boundaries, the fluid being outside or within a cylinder, or perhaps occupying the annular region between two cylinders. The two simpler cases may be thought of as limiting cases of the last mentioned.

Many studies of cylinder problems are reported by Lamb (1932) and Bateman (1932). Lamb mentions Newton as being the first to attempt this sort of problem. Both he and Bateman give the analysis of the steady-state motion between concentric cylinders which is due to Stokes.

The elementary solutions for steady flow have also emerged as special cases from searches for exact solutions of the Navier-Stokes equations. Such analyses are given by Bateman and Rosenblatt (1935).

Bateman gives in addition the results of McLeod's analysis of the flow within a single cylinder for arbitrarily prescribed initial velocity distribution and velocity history at the wall. The solution was, of course, obtained as a series of normal solutions. Many other less general contributions to the study are mentioned by Bateman, including some in which solutions are fixed by specifying the torque to be applied to the fluid by the containing cylinders. Goldstein (1932) has worked out a number of unsteady problems of the cylinder variety, being primarily interested in demonstrating the applicability of operational methods to parabolic problems. Recently Yih (1959) has given a general treatment like that of McLeod to another cylinder problem, that of fluid motion within two concentric cylinders. Despite the considerable generality of these treatments it appears possible to apply boundary conditions in a somewhat more general manner. This matter will be taken up later.

The interest in vorticity diffusion problems arose because of their mathematical likeness to problems in the conduction of heat in solids. The mathematical apparatus for the solution of certain hydrodynamical problems was

to hand and the investigation often consisted simply of re-interpreting the results in terms of the new problem. Lamb and Bateman mention work by Rayleigh and Terazawa on problems of this kind. Goldstein deals with several such problems, the most general being those of any initial distribution of vorticity in an unbounded fluid and of any initial distribution of vorticity in a region bounded by a cylinder. Recently, de Neufville (1957) has reviewed the classical work of Helmholtz, Kelvin, and Oseen on this sort of problem and has generalized their results.

It will of course be realized that this approach is chiefly useful in dealing with certain initial value problems and suffers from the disadvantage of being relatively inflexible with respect to the prescription of physically meaningful boundary conditions. However, such problems often represent limiting cases of more realistic problems.

2.3. Spiral Flows

Here, as for purely radial flow, continuity is usually required except at the origin. This type of flow offers something of interest even for an inviscid fluid. Bickley (1953) gives an analysis of the steady spiral flow of an inviscid, perfect gas, based on the previous work of Taylor and Ringleb. Another contribution to this study is that of Binnie and Hooker in which an iterative method of obtaining solutions to compressible flow problems

(suggested by Taylor) is checked by treating compressible spiral flow. A more recent analysis using hodograph methods is that of Lay (1959). Unfortunately, the experimental realization of this interesting problem is not straightforward. However, there are three-dimensional counterparts to the problem in nature.

Another sort of spiral flow which has been given considerable attention is that of an incompressible, viscous fluid. The solution for this case is given by Bateman without comment as to its physical meaning as an example following a rather complicated analysis. He refers to Jeffrey and Watson as having previously produced the result. Recently Dornbrand (1950) and Einstein and Li (1955) have brought forward anew this solution and have attempted to apply it to more complex problems---the vortex tube and drain vortex, respectively. Pengelley (1957) has extended Dornbrand's work slightly. Solutions were fixed by Dornbrand using the method applied by Stokes in the sourceless case---considering the fluid to be contained within concentric cylinders at which the no-slip condition can be applied. Pengelley pointed out that suitable choice of one of the radii at which the no-slip condition is applied allows a no-shear condition to be applied at a predetermined radius. Apparently he has not realized that the prescription of zero shear can be made directly. Einstein and Li introduce a vertically moving

core along the axis and fix a solution for the two-dimensional outer flow using compatibility conditions on the core periphery. We see then that, from the point of view of possible applications, it is useful to be able to apply a variety of boundary conditions.

A case in which the unsteady spiral flow of an incompressible viscous fluid has been treated is given by Biesel (1955). The radial flow is due to a constant line source on the axis; the solution corresponding to an arbitrary initial swirl distribution is given in the form of a Fourier-Bessel integral. We shall later show that solutions can easily be given for various boundary conditions for this case of constant source strength. Fraenkel has also considered an unsteady problem in dealing with the propagation of a cylindrical sound pulse through a rotating gas.

A generalization which has not so far been made is that of radially moving cylindrical boundaries. If we allow porous-walled bounding cylinders (and these are necessary already for stationary bounding cylinders in radial and spiral flows) we can have such moving boundaries in all the cases discussed here without violating the restrictions implicit in the equations of motion. The ability to apply such boundary conditions would be especially useful in extending the results to three-dimensional flows. Further, there may be some advantage to be gained (with an eye to application to three-dimensional problems)

by allowing the requirement of continuity to be relaxed throughout the fluid, rather than just at the axis. These matters will be discussed subsequently.

3. Scope of the Present Work

We shall begin by deriving the equations governing plane, laminar, axisymmetric flow. These are the equations from which all the analyses mentioned above arise. Thereafter, attention will be confined primarily to the flow of an incompressible fluid of constant transport coefficients as governed by these equations. In order to make significant progress it will be found necessary to further restrict the treatment to cases of constant radial flow. The problem to be dealt with is a generalized cylinder problem---the determination of spiral flows subject to conditions applied on concentric cylinders.

Within these bounds it is possible to extend previous analyses considerably, particularly with regard to the way in which boundary data is supplied. This particular study has been undertaken as it involves a simple model with some of the essential features of the flows of practical interest. Also it is one study in the general class which has not yet been attacked in a systematic manner.

Parts of the analysis can be extended easily to deal with a fluid of variable density. But a simple analytic development still requires the assumption of constant viscosity. Since in a real fluid significant variations in density must be accompanied by temperature changes (with attendant alterations in viscosity) this restriction

seems somewhat unrealistic. Dornbrand and Pengelley have, however, used this approach to the compressible case.

4. Fundamental Considerations

The analysis of axisymmetric flows springs most naturally from a set of differential equations referred to cylindrical polar co-ordinates. Rather than take as starting point these equations in their accepted form, we develop them from the basic conservation principles, with the hope that this will render the analysis more meaningful. To this end we first consider the description of the velocity field in cylindrical co-ordinates, then the specification of stress components within the fluid in these co-ordinates, and finally the derivation of the equations themselves.

We are thus retracing the course followed in the general analysis of fluid flow using more general methods of describing flow phenomena---vector and tensor notations. It is in these general notations that the fundamental concepts are most directly and fully apprehended. But the specialization to axisymmetry imposed here takes us beyond the scope of these notations. They are, indeed, not specialized enough to extract the advantages of this restriction. To orient ourselves in this co-ordinate system we proceed systematically as in the more general development, familiarizing ourselves with the forms adopted by various entities in this special case and noting the features peculiar to this system of description.

4.1. Description of the Velocity Field

Consider a continuous velocity field in the neighbourhood of the point (r, θ). We shall describe this field by obtaining expressions for the total differentials of the velocity components in terms of derivatives at the point. To do this we consider the velocities at the two neighbouring points (r, θ) and $(r+\delta r, \theta+\delta\theta)$ as shown in Figure 1. Resolving $\bar{v}_2$ into components parallel to those of $\bar{v}_1$ we obtain

$$v_2' \doteq v_2 + u_2 \delta\theta$$
$$u_2' \doteq u_2 - v_2 \delta\theta$$

Then the velocity changes referred to axes radial and tangential at (r, θ) are given approximately by

$$\delta u = u_2' - u_1$$
$$\doteq u(r+\delta r, \theta+\delta\theta) - u(r, \theta) - v(r+\delta r, \theta+\delta\theta)\delta\theta$$
$$\delta v = v_2' - v_1$$
$$\doteq v(r+\delta r, \theta+\delta\theta) - v(r, \theta) + u(r+\delta r, \theta+\delta\theta)\delta\theta$$

In the limit as $\delta r, \delta\theta \to 0$ we have

$$du = \frac{\partial u}{\partial r} dr + r\left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right) d\theta$$
$$dv = \frac{\partial v}{\partial r} dr + r\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right) d\theta$$

The coefficients of these differential expressions are the components of a tensor specifying the variation of the velocity vector in the field. We split these coefficients up to display this tensor as the sum of a symmetric and an anti-symmetric tensor:

A

$$du = \frac{\partial u}{\partial r}dr + \frac{1}{2}\left[\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right]r\,d\theta - \frac{1}{2}\left[\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right]r\,d\theta$$

$$dv = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} + \frac{\partial v}{\partial r}\right]dr + \left[\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right]r\,d\theta + \frac{1}{2}\left[\frac{\partial v}{\partial r} + \frac{v}{r} - \frac{1}{r}\frac{\partial u}{\partial \theta}\right]dr$$

By considering the straining of two elemental lines of fluid, initially radially and tangentially directed, interpretations can be given to these coefficients. Those of the anti-symmetric tensor are seen to be the average angular velocity of the line elements, that is, the angular velocity of the fluid. The like coefficients of the symmetric tensor are seen to be the rate at which the lines are rotating towards one another, that is, the rate of shearing strain; and the unlike coefficients are seen to be simply the rates of direct strain in the radial and tangential directions. Hence the symmetric tensor is termed the rate-of-strain tensor.

The divergence of the fluid is the sum of the direct rates of strain. That is,

$$\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta}$$

On comparing the forms of these quantities with those referred to Cartesian co-ordinates we note that these are somewhat more complex. The development shows clearly how this complexity arises from the non-Cartesian reference system.

4.2. The Stress Components

The relationship between the stresses within a deforming viscous fluid and the rates of straining is taken to be (in the notation of Cartesian tensors):

$$p_{ij} = -(p + \tfrac{2}{3}\mu\Delta)\,\delta_{ij} + \mu e_{ij}$$

where p_{ij} are the components of the stress tensor,

e_{ij} are the components of the rate-of-strain tensor, $\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}$

δ_{ij} is the Kronecker delta.

In order to compute the stress components relative to our curvilinear frame of reference we consider the simultaneous description of a flow field by Cartesian and polar axes chosen so that the velocity components are locally parallel. The corresponding stress components in the two systems are then equal. That is,

$$p_{xx} = p_{rr}, \qquad p_{yy} = p_{\theta\theta}$$

$$p_{xy} = p_{yx} = p_{r\theta} = p_{\theta r} = \tau$$

Further, the relationships between the rates-of-strain can be obtained from

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$= \frac{\partial u}{\partial r}dr + \left(\frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r}\right)r\,d\theta$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$$

$$= \frac{\partial v}{\partial r}dr + \left(\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{u}{r}\right)r\,d\theta$$

by requiring that corresponding coefficients be equal.

Hence: $e_{xx} = e_{rr} = 2\frac{\partial u}{\partial r}$

$$e_{xy} = e_{r\theta} = e_{\theta r} = r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$e_{yy} = e_{\theta\theta} = 2\left(\frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r}\right)$$

Combining these results we obtain

$$p_{rr} = -p + \mu\left(2\frac{\partial u}{\partial r} - \frac{2}{3}\Delta\right)$$

$$p_{\theta\theta} = -p + \mu\left(2\frac{u}{r} + 2\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{2}{3}\Delta\right)$$

$$\tau = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)$$

The positive senses of the stresses acting on an elemental volume are shown in Figure 2a.

4.3. The Case of Axial Symmetry

Summarizing the results of the previous sections for the axisymmetric case which will subsequently be of interest:

Vorticity: $\zeta = \frac{\partial v}{\partial r} + \frac{v}{r} = \frac{1}{r}\frac{\partial}{\partial r}(rv)$

Divergence: $\Delta = \frac{\partial u}{\partial r} + \frac{u}{r} = \frac{1}{r}\frac{\partial}{\partial r}(ru)$

Direct Stresses: $p_{rr} = -p + 2\mu\left(\frac{\partial u}{\partial r} - \frac{\Delta}{3}\right)$

$$p_{\theta\theta} = -p + 2\mu\left(\frac{u}{r} - \frac{\Delta}{3}\right)$$

Shear Stress: $\tau = \mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right) = \mu r\frac{\partial}{\partial r}\left(\frac{v}{r}\right)$

The viscosity dependent terms of the direct stresses can be justified in terms of an underlying particle structure of the fluid. Consider the case of source-type flow ($\mu > 0$, $\frac{\partial u}{\partial r} < 0$). The effective pressure acting in the radial direction is augmented by the effect of viscosity,

while that acting tangentially is reduced. The radial effect results from the deceleration of the fluid as it moves outwards; particles from the outer fluid possess greater inwards momentum than that characterized by the pressure. The tangential effect results from the lateral spreading of the outwards moving fluid. As a result of this spreading (which is required for continuity), elements of fluid have a relative lateral velocity and particles impinging on a radial plane produce lesser reactions than would be the case in parallel flow. The effect of a positive divergence is to increase the effective pressure in every direction as particles of the fluid move farther apart. The coefficient of viscosity μ is thus seen to be generally representative of the ability of one portion of a continuous fluid to influence other parts dynamically.

The shearing stress relation given above can also be obtained from the basic ideas of viscosity derived by considering the shearing of fluid between parallel walls. It is necessary, however, to consider the relative shearing motion within the fluid by assuming a suitable rotating frame of reference.

4.4. Derivation of Equations governing Axisymmetric Flow

Differential equations will be derived directly by requiring the conservation of mass, momentum, and energy for suitable elemental control volumes---those shown in

Figure 2. The quantities entering into the conservation laws can be expressed in terms of differential changes in the flow variables. The limiting forms (as $dV \to 0$) of these differential expressions will be given. Then they will be combined to give the required equations.

4.4.1. Equation of Continuity

Consider a fixed annular volume as shown in Figure 2b. $dV = 2\pi r dr$. We require that in the incremental time δt the increase in mass within the volume be entirely due to the flow in through the cylindrical boundaries and to an axisymmetric source distribution given by $\rho_o Q$, where Q is the rate at which volume is produced per unit volume, referred to the standard density ρ_o.

a) Increase in mass within control volume
$$\frac{\partial \rho}{\partial t} dV \delta t$$
b) Net flow in through fixed boundaries
$$-\frac{1}{r}\frac{\partial}{\partial r}(\rho r u) dV \delta t$$
c) Production of mass within boundaries
$$\rho_o Q dV \delta t$$

Hence: $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u) = \rho_o Q$$

4.4.2. Radial Momentum Equation

Consider the segment of an annular region depicted in Figure 2a, moving with the fluid. $dV = r\,dr\,d\theta$. We require that the impulse in the radial direction given by forces at the boundaries be equal to the change in momentum in the radial direction, both effects considered in the

interval δt . We assume that any fluid created comes into existence with the local momentum.

a) Impulse due to surface forces

Due to symmetry the shear stresses do not contribute to this impulse. The impulse due to direct stresses is

$$\left(\frac{\partial p_{rr}}{\partial r} + \frac{p_{rr} - p_{\theta\theta}}{r}\right) dV \delta t$$

b) Momentum change due to change in radial velocity

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r}\right) dV \delta t$$

c) Momentum change due to rotation of the tangential velocity vector

$$-\rho \frac{v^2}{r} dV \delta t$$

The sketches of Figure 3 indicate how this result is derived.

Upon introducing the previously determined expressions for the stress components, we obtain

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r}\right)$$

$$= -\frac{\partial p}{\partial r} + \frac{4}{3}\mu\frac{\partial \Delta}{\partial r} + 2\left(\frac{\partial u}{\partial r} - \frac{\Delta}{3}\right)\frac{\partial u}{\partial r}$$

$$= -\frac{\partial p}{\partial r} + 2\frac{\partial u}{\partial r}\cdot\frac{\partial u}{\partial r} + \frac{2}{3}\left(2\mu\frac{\partial}{\partial r} - \frac{\partial u}{\partial r}\right)\Delta$$

4.4.3. Tangential Momentum Equation

Consider the tangentially directed momentum and impulse for the same elemental volume.

a) Impulse due to surface forces

In this case the direct stresses do not contribute. The impulse is

$$\left(\frac{\partial \tau}{\partial r} + 2\frac{\tau}{r}\right) dV \delta t$$

b) Momentum change due to change in tangential velocity

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r}\right) dV\delta t$$

c) Momentum change due to rotation of radial velocity vector

$$\rho\frac{uw}{r} dV\delta t$$

Once again Figure 3 may be referred to for clarification. Finally,

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{uw}{r}\right)$$
$$= \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} - \frac{w}{r^2}\right) + \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\frac{\partial \mu}{\partial r}$$

4.4.4. The Energy Equation

Consider the annular control volume of Figure 2b, moving with the fluid.

a) Increase in total energy (internal plus kinetic)

$$\rho\frac{D}{Dt}\left(e + \tfrac{1}{2}u^2 + \tfrac{1}{2}w^2\right) dV\delta t$$

b) Heat transfer into control volume

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) dV\delta t$$

c) Work done by direct stresses on fluid in control volume

$$\frac{1}{r}\frac{\partial}{\partial r}(ru p_{rr})\, dV\delta t$$

d) Work done by shearing stresses on the fluid

$$\frac{1}{r}\frac{\partial}{\partial r}(rw\tau)\, dV\delta t$$

Requiring that energy be conserved, we have

$$\rho\frac{D}{Dt}\left(e + \tfrac{1}{2}u^2 + \tfrac{1}{2}w^2\right)$$
$$= \frac{1}{r}\frac{\partial}{\partial r}\, r\left(k\frac{\partial T}{\partial r} + u p_{rr} + w\tau\right)$$

whence, using the momentum equations, we get

$$\rho \frac{De}{Dt} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right) - p\Delta$$
$$+\mu\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + 2\left(\frac{\partial u}{\partial r}\right)^2 + 2\left(\frac{u}{r}\right)^2 - \frac{2}{3}\Delta^2\right]$$

where the viscosity dependent term is the dissipation function, Φ .

5. Initiation of the Study

For an incompressible, sourceless fluid of constant viscosity, thermal conductivity, and specific heat the governing equations are

Continuity: $$\frac{\partial}{\partial r}(rv) = 0, \quad \Delta = 0 \tag{1}$$

Tangential Momentum:
$$\frac{\partial v}{\partial t} + u\left(\frac{\partial v}{\partial r} + \frac{v}{r}\right) = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right) \tag{2}$$

Radial Momentum:
$$\frac{\partial u}{\partial t} - \frac{u^2+v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{3}$$

Energy:
$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial r} = \frac{k}{\rho h}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\nu}{h}\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + \left(2\frac{u}{r}\right)^2\right] \tag{4}$$

where the coefficients (functions of ρ, μ, k, h) are constants.

The continuity equation has been used to simplify the last two equations. For a quiescent fluid the energy equation reduces to the simple equation of heat conduction in an idealized solid. The tangential velocity distribution contributes only a source distribution. But the radial velocity contributes in addition a convective effect.

The problem posed is that of determining the four dependent variables u, v, p, T as functions of the independent variables r, t. The equations are arranged above in order of complexity, the earlier having already been utilized to simplify the more complicated. Note that the first two equations completely specify the velocity field, independent of the latter pair. This is the result

of the rigorous idealizations that have been made.

It seems expedient to attack the system in the order in which the equations are listed. The integration of the first is straightforward. The tangential momentum equation is somewhat more difficult, but its solution will lead to the immediate integration of the other momentum equation and the consequent determination of the pressure distribution. The energy equation appears to be formidable even when the velocity components have been determined, but no simpler approach to it seems possible.

Beginning this line of attack, we integrate the continuity equation.

$$ru = f(t) \tag{5}$$

Once the arbitrary function has been specified the radial velocity component is known. There are many ways of specifying this function: prescribing the velocity at some fixed or moving point; prescribing a source strength; requiring that various physical conditions be satisfied, for example, a finite velocity at the axis or a solid bounding wall; and so on. But whatever the means of determining this arbitrary function, once it is specified the tangential momentum equation is reduced to an equation containing only one unknown function---the swirl velocity component.

Actually, this result can be obtained without the prior integration of the continuity equation. If we use the unintegrated continuity equation to eliminate the

radial velocity component from the tangential momentum equation we obtain an equation for the swirl component only. This equation can be written in terms of the specific angular momentum as

$$(a_r \frac{\partial}{\partial r} - a_{rr})\left[r a_t - \nu r^2 \frac{\partial}{\partial r}\left(\frac{a_r}{r}\right)\right] = 0$$

in which form it can be integrated immediately. An arbitrary function of time is introduced by this integration. comparison of the integrated form with the basic form (2) leads to the interpretation of the radial velocity in terms of the arbitrary function given in equation (5).

Despite this promising start we are at this point reduced to the study of a simple special case. No analytic method is known for integrating the tangential momentum equation containing an arbitrary function of time. Henceforth we restrict ourselves to the case of constant source (or sink) strength. We take

$$r u = \nu C \tag{6}$$

where $C = \frac{\rho r u}{\mu}$ is a dimensionless constant characteristic of the combination of radial flow rate and fluid properties. We shall refer to $|C|$ as the source Reynolds number.

With this restriction the tangential momentum equation can be written in several more convenient forms:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial r^2} + (1-C)\frac{1}{r}\frac{\partial v}{\partial r} - (1+C)\frac{v}{r^2} \tag{7}$$

where $\tau = \nu t$ is a modified measure of time,

$$\frac{\partial a}{\partial \tau} = r^{1+C}\frac{\partial}{\partial r}\left(\frac{1}{r^{1+C}}\frac{\partial a}{\partial r}\right) \tag{8}$$

where $a = rv$ is the specific angular momentum,

A

$$\frac{\partial \zeta}{\partial \tau} = r^{c-1} \frac{\partial}{\partial r}\left(r^{1-c} \frac{\partial \zeta}{\partial r}\right) \tag{9}$$

where ζ is the vorticity.

The equation for the pressure (3) becomes

$$\frac{\partial p}{\partial r} = \rho \frac{u^2 + w^2}{r} \tag{10}$$

and the pressure can be seen to be monotonically increasing outwards.

The energy equation can also be given in a considerably more compact form:

$$\frac{\partial T}{\partial \tau'} = r^{\sigma c - 1} \frac{\partial}{\partial r}\left[r^{1-\sigma c} \frac{\partial T}{\partial r}\right] + \frac{\mu}{K}\left[\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)^2 + \left(2\frac{w}{r}\right)^2\right] \tag{11}$$

where $\sigma = \frac{\mu h}{K}$ is the Prandtl number of the fluid,

$\tau' = \tau/\sigma$ is another modified measure of time.

This equation is formally very similar to the momentum equation expressed in terms of the vorticity (9). The analogy between heat and vorticity still holds in spite of the superposition of convection effects on those of conduction.

Having listed these results we proceed with the analysis arising from them, dealing first with the simple case of a stationary velocity field, that is, the case for which the swirl distribution as well as the radial flow is constant in time.

6. The Steady Velocity Field

The radial component of velocity is given by

$$ru = \nu C \tag{1}$$

6.1. Swirl Velocity Distribution

When the swirl distribution is stationary either equation (5.8) or (5.9) can be integrated directly. On integrating (5.8) twice we get

$$\begin{aligned} \nu &= A r^{C+1} + B/r && \text{for } C \neq -2 \\ \nu &= A \frac{\log(r/r_0)}{r} + B/r && \text{for } C = -2 \end{aligned} \tag{2}$$

where A , B , r_0 are constants.

The constants of integration can be specified in many ways; for example, by prescribing the velocity, angular velocity, shear stress, or vorticity on any two cylindrical surfaces, by giving two of these on one surface, by requiring that the solution remain finite in some region, by specifying the total amount of some quantity in some region---say, the total angular momentum, kinetic energy, or energy dissipation.

For the sourceless case we have

$$\nu = Ar + B/r$$

a superposition of "solid-body" rotation and a "potential" vortex. The potential vortex is also a solution for the general case, $C \neq 0$. The other solution for the radial flow case may be thought of as a generalization of the solid-body type rotation.

Those quantities which are determined solely by the

velocity distribution can now be determined; for example, the vorticity and the shear. The solid-body rotation and potential vortex are especially interesting in this connection. The former possesses constant vorticity but zero shear; the latter, zero vorticity associated with a shear distribution tending to infinity at the axis.

6.2. The Streamline Pattern

The differential equation of the streamlines is

$$r\frac{d\theta}{dr} = \frac{v}{u}$$

Integrating formally, we get $\theta = \int_{r_0}^{r} \frac{v}{ru}\,dr = \frac{1}{\nu C}\int_{r_0}^{r} v\,dr$

where we have taken $\theta = 0$ for $r = r_0$.

Introducing the velocity distributions just found we obtain the equations of the streamlines

$$\begin{aligned} \nu\theta &= \frac{1}{C}\left[\frac{A}{C+2}\left(r^{C+2} - r_0^{C+2}\right) + B\log\left(\frac{r}{r_0}\right)\right] && \text{for } C \neq -2 \\ \nu\theta &= -\frac{1}{2}\left[\frac{A}{2}\log\left(\frac{r}{r_0}\right) + B\right]\log\left(\frac{r}{r_0}\right) && \text{for } C = -2 \end{aligned} \qquad (3)$$

As $C \to 0$ the streamlines tend towards circles; as $|C| \to \infty$, towards straight lines.

6.3. Particle History

The differential equation for the radial position of a particle is $\frac{dr}{dt} = u$

Integrating: $t = \int_{r_0}^{r} \frac{dr}{u} = \frac{1}{\nu C}\int_{r_0}^{r} r\,dr = \frac{r^2 - r_0^2}{2\nu C}$

where we have taken $t = 0$ for $r = r_0$.

Then

$$r = r_0\left(1 + \frac{2C\tau}{r_0^2}\right)^{1/2} \qquad (4)$$

The θ co-ordinate of the particle can be determined as a function of time by introducing this value into the streamline equation (3). The particle paths are thus specified as functions of time.

The variations of particle velocity with time can be found by using (4) in conjunction with the velocity distributions (1) and (2). The acceleration history follows immediately.

6.4. Dynamics of the Flow

The pressure distribution can be found by integrating equation (5.10).

$$P/\rho = \int_{r_0}^{r} \frac{u^2}{r}\,dr - \frac{(\nu C)^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right) + p_0(t)/\rho$$

Introducing the steady-state velocity distribution we get

$$\frac{p - p_0}{\rho} = \frac{A^2}{2C+2}\left(r^{2C+2} - r_0^{2C+2}\right) + \frac{2AB}{C}\left(r^C - r_0^C\right) - \frac{B^2 + (\nu C)^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right) \quad \text{for } C \neq 0, -1, -2$$

$$\frac{p - p_0}{\rho} = \frac{A^2}{2}\left(r^2 - r_0^2\right) - 2AB \log\left(\frac{r}{r_0}\right) - \frac{B^2}{2}\left(\frac{1}{r^2} - \frac{1}{r_0^2}\right) \quad \text{for } C = 0$$

To specify the pressure distribution we must supply information sufficient to determine the functions $p_0(t)$ and $r_0(t)$. Once the pressure distribution is known the direct stress components can be found. The shear stress follows immediately from the shear distribution and is

not influenced by the pressure.

6.5. Thermodynamics of the Flow

We deal with the general case of non-steady temperature distribution by separating the problems associated with the steady source distribution due to dissipation and the possibly time-dependent boundary conditions.

Set $T = T_1(r) + T_2(r, \tau')$

the equations for T_1 and T_2 being

$$\frac{d}{dr}\left[r^{1-\sigma c}\frac{dT_1}{dr}\right] = -\frac{\mu}{K} r^{1-\sigma c}\left[\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)^2 + \left(2\frac{v}{r}\right)^2\right]$$

and

$$r^{1-\sigma c}\frac{\partial T_2}{\partial \tau'} = \frac{\partial}{\partial r}\left[r^{1-\sigma c}\frac{\partial T_2}{\partial r}\right]$$

The first equation can be integrated immediately, the processes of integration producing two arbitrary constants which may be set equal to zero as the boundary condition can be accounted for in determining T_2. The problem of determining a suitable solution of the partial differential equation for T_2 is similar to that of solving for the swirl velocity component in the time-dependent case to be dealt with later (compare with equation (5.9)). Hence this problem will not be dealt with in detail here. A solution can be obtained in the form of a series in terms of orthogonal cylinder functions of order $\left(\frac{\sigma c}{2} - 1\right)$. The usual boundary conditions of heat conduction problems can be satisfied by such solutions. The solution T_1 must, of course, be taken into account in obtaining T_2, so that the overall boundary and initial

conditions are satisfied.

For the case of steady temperature distribution, the introduction of the time-dependent term T_2 is not necessary.

6.6. General

We see that it is possible to obtain general solutions for the velocity, pressure, and temperature distributions when all these variables are independent of time. But even for this simplest case the number of ways in which the constants thrown up in the integrations can be specified is unreckonable. However, closed algebraic formulae can be obtained for most of the physically meaningful quantities in terms of which conditions on the solutions might be expressed. Thus the determination of the constants becomes simply the problem of solving six simultaneous algebraic equations.

6.7. Examples and Generalizations

In order to investigate the nature of the solutions just derived we shall examine several particular cases. The annular region between $r = 1$ and $r = 2$ is chosen for examination, conditions being specified at the boundaries to fix solutions within this region. It seems judicious to avoid regions including the axis so that the singularity there does not introduce an unreal complexity into the problem.

The following cases will be considered:

a) $\nu = 1$ for $r = 1$, $\nu = 1$ for $r = 2$

b) $\nu = 0$ for $r = 1$, $\nu = 1$ for $r = 2$

c) $\tau = 0$ for $r = 1$, $\nu = 1$ for $r = 2$

The first two are immediately interpretable in terms of rotating bounding cylinders. The last represents some features of the swirling flow with air core.

Representative velocity distributions for the three cases are given in Figures 4, 5 and 6. The most obvious common feature is the tendency towards the distribution corresponding to constant angular momentum for very large source Reynolds number (either due to large radial flow rate or to very small viscosity). For these limiting cases very abrupt adjustments are necessary near the exit wall. For those cases for which the inlet angular momentum is zero ((b) and (c), $C \to \infty$), the entire velocity change takes place at the exit wall.

These properties of the solution can be found from the general form, equation (2), on introducing values of the constants of integration determined by specifying ν_1 at r_1, ν_2 at r_2. We find that

$$\nu \longrightarrow \frac{r_1 \nu_1}{r} \qquad \text{for } r \neq r_2 \quad \text{as } C \to \infty$$

and a similar result as $C \to -\infty$

The shear distributions corresponding to these velocity distributions are given in Figures 7, 8 and 9. The most obvious feature common to all is that the shear is usually

highest on the exit side of the annulus. This generalization holds even for the case (c) in which zero shear is specified on the inner surface.

Both the effects discussed above indicate the tendency for a boundary layer to be formed at high source Reynolds number.

The distribution of dissipation or heat production within the annular region can be visualized easily from the shear distribution. (Considering the case of ν small, the dissipation function is proportional to the shear squared.) It is interesting to note that the effect of viscosity in those limiting cases where the dissipation is concentrated near the exit cylinder will be equivalent to a cylindrical heat source.

An index of the overall effect of viscosity is the dissipation per unit mass transferred through the system:

$$D = \frac{2\pi}{|2\pi r \rho u|} \int_{r_1}^{r_2} r \Phi \, dr = \frac{1}{|C|} \int_{r_1}^{r_2} r \left(\Phi / \mu \right) dr$$

This quantity can be quite simply computed for the limiting case of small radial flow. Then

$$\Phi = \mu \left(\frac{\partial v}{\partial r} - \frac{v}{r} \right)^2$$

Hence $$D = \frac{1}{|C|} \left[\frac{A^2 C^2 r^{2c+2}}{2c+2} - 4AB r^c - \frac{2B^2}{r^2} \right]_{r_1}^{r_2} \quad \text{for } C \neq 0, -1, -2$$

The variation of this index of dissipation with C is given in Figures 10, 11, 12 for the three cases considered. For the first two cases D is infinite for $C = 0$ since dissipation exists even for zero throughput. For the case (c) the shearless velocity distribution is found for zero

radial flow and the behaviour of D for small radial flows is quite different.

The complex forms adopted by the dissipation index for the various examples suggest, in conjunction with the principle of minimum dissipation of energy, that the two-dimensional flow patterns modelled here will not always be the means of transferring the required throughput which is actually adopted in real flows.

To illustrate this point we give an example relating to example (b). Consider a net radial flow characterized by $C = -10$. From Figure 11, $D = 1.24$ But this throughput could also be carried by $C = -18$ over half the transfer area and $C = -2$ over the other half. Then

$$D = \frac{1.56 + 0.55}{2} = 1.06$$

Of course, this crude calculation neglects the additional dissipation due to the more complex three-dimensional flow pattern. Note that for very high $|C|$ the dissipation is concentrated at the exit boundary and effective modifications of the flow pattern must introduce more significant three-dimensional effects. Nevertheless, the contention that the two-dimensional pattern is not always that of minimum dissipation seems plausible.

The most noticeable common feature of the various D vs. C plots is the existence of asymptotic values towards which the dissipation index tends for very large Reynolds numbers. These limiting values are of considerable

practical interest since the kinematic viscosities of the most common fluids are very small, so that $|C|$ may be large even for small radial flows.

An examination of a more general solution shows that

$$D \longrightarrow \frac{1}{2}\left(\frac{r_1 v_1}{r_2} - v_2\right)^2 \qquad \text{as } C \longrightarrow \infty$$

and a similar result as $C \longrightarrow -\infty$

where r_1 and r_2 are the inner and outer radii,

v_1 and v_2 are the velocities prescribed there.

The dissipation for the whole region is thus seen to be directly related to the velocity jump at exit. This is to be expected, since in the limiting case the whole of the dissipation does in fact take place just at the exit wall. We can see this by considering case (c) as $C \to -\infty$. Here the dissipation goes to zero although the velocity distribution tends to the same profile as for cases (a) and (b). The difference between these cases is that the shear remains bounded for case (c), while for the others it is infinite at the exit boundary.

The asymptotic values of D found on examining the special cases can be derived from the result given above. We have then a compact expression for the dissipation which covers those cases of practical interest for which $|C|$ is large.

The kinetic energy change at the exit wall is

$$E = \frac{1}{2}v_2^2 - \frac{1}{2}\left(\frac{r_1 v_1}{r_2}\right)^2$$

Then the total energy transferred to the fluid at the exit wall is

$$D + E = u_2\left(u_2 - \frac{r_1 u_1}{r_2}\right)$$

For case (b) as $C \to -\infty$ this quantity is zero, the dissipation being entirely due to loss of kinetic energy, no power being supplied by the stationary wall. For the case (c) as $C \to -\infty$ the energy added at the exit wall is again zero, both kinetic energy addition and shear work being negligible in this case.

The shear stress at the exit wall is given by

$$r_2 v_2 \tau = \rho r_2 u_2 (D+E)$$

or $$\tau = \rho u_2 \left(v_2 - \frac{r_1 v_1}{r_2}\right)$$

Note that $\tau \to \rho u_2 v_2$ as $r_2 \to \infty$

in accordance with the well-known asymptotic result for uniform boundary layer suction.

6.8. More General Steady Velocity Fields

If we relax the condition of continuity by allowing an axisymmetric distribution of steady sources within the fluid, the equation of continuity appropriate to this study becomes

$$\frac{1}{r}\frac{d}{dr}(ru) = Q$$

The equation governing tangential momentum remains unchanged (assuming the created fluid to come into existence with the local momentum):

$$u\left(\frac{dv}{dr} + \frac{v}{r}\right) = \nu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr} - \frac{v}{r^2}\right)$$

Rewriting in terms of the vorticity we get

$$u\,\zeta = \nu \frac{d\zeta}{dr}$$

Separating the variables and integrating twice:

$$w = \frac{A}{r}\int^r r \exp\left[\int^r \frac{u}{\nu}\,dr\right]dr + \frac{B}{r}$$

The continuity equation can, of course, be integrated.

$$u = \frac{1}{r}\int^r Q\,dr + \frac{C}{r}$$

Thus the whole velocity field can be determined, given Q and data sufficient to specify the constants of integration.

The real importance of this generalization is that it allows an immediate extension to three-dimensional flow. Note that the continuity equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) = -\frac{\partial w}{\partial z}$$

for three-dimensional axisymmetric flows. Thus there is a formal equivalence of two-dimensional flows with general source distribution and a class of three-dimensional flows. This class will be discussed further in a more appropriate place.

7. Non-steady Velocity Fields

We shall once again restrict ourselves to the simple class of flows for which the radial flow is stationary and equivalent to that due to a line source on the axis. The radial velocity component is then given as before by

$$ru = \nu C \tag{1}$$

The next problem which faces us, that which will occupy most of the present section, is that of determining the tangential velocity component. This second velocity component can be determined using one of the several forms of the equation governing tangential momentum: (5.7), (5.8), and (5.9). Note that the problem has been linearized by the integration of the continuity equation and very greatly simplified by the simple form we have adopted for the integral.

A general solution of one of these equations might prove a powerful tool in investigating the flows it governs. However, such classical methods leading to general solutions as those of Monge and Ampere fail for these equations. We are thus reduced to searching for particular solutions, hoping that they can be superposed to model physical situations of interest.

The solutions most useful in this respect are those independent of time which were previously determined (equations (6.2)) and those normal solutions involving orthogonal functions. Other normal solutions and a variety of singular solutions are known, but they are less

useful in building up solutions of general applicability, although in particular examples they may prove convenient tools.

The analysis arises most naturally from the equation written directly in terms of swirl velocity:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial r^2} + (1-c)\frac{1}{r}\frac{\partial v}{\partial r} - (1+c)\frac{v}{r^2} \qquad (2)$$

7.1. Elementary Normal Solutions

To find normal solutions we substitute

$$v = R(r)S(\tau)$$

into equation (2), obtaining

$$\frac{S'}{S} = \frac{R''}{R} + \frac{1-c}{r}\frac{R'}{R} - \frac{1+c}{r^2} = -\alpha^2 \qquad (3)$$

on noting that each of the members must be independent of r and τ and setting them equal to a negative definite constant. The negative constant gives rise to the most useful normal solutions. Those arising for other choices will be mentioned later.

The first member of (3) can be integrated immediately.

$$S = S_0 e^{-\alpha^2 \tau}$$

The second member can be written

$$R'' + \frac{1-c}{r}R' + \left(\alpha^2 - \frac{1+c}{r^2}\right)R = 0$$

Solutions in series form can be obtained using the standard methods. However, a simpler approach is to introduce the substitution

$$R = r^q P(r) \qquad \text{where} \quad q = \frac{c}{2} \qquad (4)$$

The equation for P is

$$P'' + \frac{1}{r}P' + \left[\alpha^2 - \frac{(q+1)^2}{r^2}\right]P = 0$$

This is Bessel's equation of order $(q+1)$, with solutions $J_{q+1}(\alpha r)$ and $Y_{q+1}(\alpha r)$, Bessel functions of the first and second kinds of order $(q+1)$.

Finally,

$$\begin{aligned} v &= r^q J_{q+1}(\alpha r) e^{-\alpha^2 \tau} \\ v &= r^q Y_{q+1}(\alpha r) e^{-\alpha^2 \tau} \end{aligned} \qquad (5)$$

are seen to be the elementary solutions of (2) arising from the choice of a negative constant in (3). The pair of solutions

$$\begin{aligned} v &= r^q I_{q+1}(\alpha r) e^{\alpha^2 \tau} \\ v &= r^q K_{q+1}(\alpha r) e^{\alpha^2 \tau} \end{aligned} \qquad (5')$$

arise from the choice of the constant in (3) to be $+\alpha^2$. The steady-state solutions obtained previously arise for $\alpha = 0$.

The solutions (5) are more useful than those (5') as they involve potentially orthogonal cylinder functions and also exhibit the property of remaining bounded for all positive time which is desirable if physical situations are to be represented.

The substitution (4) can be arrived at in several ways: by obtaining the solution in series and identifying its relationship to the Bessel functions, or by realizing that the solutions must involve Bessel functions and searching a table of equations whose solutions are of this type for the form (2). Jahnke and Emde give such a table.

A substitution equivalent to (4) might have been introduced earlier in the development,

$$\nu = r^{q} V(r, \tau)$$

leading to an equation equivalent to (2),

$$V_\tau = V_{rr} + \frac{1}{r} V_r - \left(\frac{q+1}{r}\right)^2 V$$

A somewhat more elegant treatment arises from this equation. However, some loss of physical significance is suffered, so we shall not investigate it further.

7.2. More Complex Solutions

The construction of other solutions from the elementary ones found above is made particularly simple by the linearity and homogeneity of the equation (2) and by the existence of potentially orthogonal solutions to this equation. A class of problems which can be dealt with in complete generality is that of motions of fluid within an annular region for which the solution is specified on the cylindrical boundaries for all times and throughout the region at some one instant. The same problems can be dealt with for a variety of other boundary conditions. Motion within a cylindrical region is a special case of this annular class, corresponding to the specification of zero swirl at the axis.

We shall consider at first somewhat simplified problems in order to introduce the notation gradually. Various extensions will be treated after the straightforward problem is discussed.

7.3. The Initial Value Problem

We first consider the problem of finding the flow within an annular region starting from an arbitrary swirl distribution and with constant values of swirl velocity prescribed on the boundaries:

$$v(r_1) = v_1 \qquad v(r_2) = v_2$$

The physical situation corresponding to this mathematical model is the flow of fluid contained within two concentric cylinders subsequent to some instant, the fluid being set in motion by prior rotation of the cylinders, the rotational speeds thenceforth being held constant. The walls of the cylinders will in general be porous to allow the radial flow to occur.

Consider a solution of form

$$v = a r^{2q+1} + b/r + \sum_{i=1}^{\infty} \left(a_i J_{q+1}(\alpha_i r) + b_i Y_{q+1}(\alpha_i r) \right) r^q e^{-\alpha_i^2 \tau} \quad (6)$$

We specify a and b by requiring that the steady part of the solution match the prescribed boundary conditions. That is,

$$v_1 = a r_1^{2q+1} + b/r_1$$

$$v_2 = a r_2^{2q+1} + b/r_2$$

give a and b.

The time-dependent part of the solution is thus zero at r_1 and r_2 for all time. Then each term must vanish at r_1 and r_2. To achieve this we require that

$$a_i J_{q+1}(\alpha_i r_1) + b_i Y_{q+1}(\alpha_i r_1) = 0$$

$$a_i J_{q+1}(\alpha_i r_2) + b_i Y_{q+1}(\alpha_i r_2) = 0$$

or, eliminating a_i and b_i,

$$J_{q+1}(\alpha_i r_1) Y_{q+1}(\alpha_i r_2) - J_{q+1}(\alpha_i r_2) Y_{q+1}(\alpha_i r_1) = 0 \quad (7)$$

This is an equation specifying suitable values of α_i. There are an infinite number of positive α_i which satisfy this condition.

The problem is simplified by introducing a new orthogonal function whose form is suggested by the result (7):

$$W_n(\alpha r) = J_{q+1}(\alpha r_1) Y_n(\alpha r) - Y_{q+1}(\alpha r_1) J_n(\alpha r) \quad (8)$$

This function is automatically zero at r_1 and vanishes for $r = r_2$, if the condition (7) is satisfied by α_i. These functions form a complete orthogonal set in the interval (r_1, r_2) for the choice of α_i dictated by (7).

A form of solution suitable for problems of this type is then

$$v = a r^{2q+1} + b/r + \sum_{i=1}^{\infty} A_i r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau} \quad (6')$$

To determine the coefficients A_i we require that the initial condition be satisfied. That is

$$f(r) = a r^{2q+1} + b/r + \sum_{i=1}^{\infty} A_i r^q W_{q+1}(\alpha_i r)$$

or $$\sum_{i=1}^{\infty} A_i W_{q+1}(\alpha_i r) = \frac{f(r)}{r^q} - a r^{q+1} - \frac{b}{r^{q+1}}$$

Hence, using the orthogonality relationships, the coefficients of the Fourier-Bessel expansion are defined by

A

$$A_i \int_{r_1}^{r_2} r W_{q+1}^2(\alpha_i r)\,dr = \int_{r_1}^{r_2} r^{1-q} f(r) W_{q+1}(\alpha_i r)\,dr$$

$$- a \int_{r_1}^{r_2} r^{2+q} W_{q+1}(\alpha_i r)\,dr - b \int_{r_1}^{r_2} \frac{W_{q+1}(\alpha_i r)}{r^q}\,dr$$

The fully specified integrals can be evaluated as follows:

$$\int_{r_1}^{r_2} r W_{q+1}^2\,dr = \frac{r_2^2 W_q^2(\alpha_i r_2) - r_1^2 W_q^2(\alpha_i r_1)}{2}$$

$$\int_{r_1}^{r_2} r^{2+q} W_{q+1}\,dr = \frac{r_1^{q+2} W_q(\alpha_i r_1) - r_2^{q+2} W_q(\alpha_i r_2)}{\alpha_i} \qquad (9)$$

$$\int_{r_1}^{r_2} \frac{W_{q+1}}{r^q}\,dr = \frac{1}{\alpha_i}\left[\frac{W_q(\alpha_i r_1)}{r_1^q} - \frac{W_q(\alpha_i r_2)}{r_2^q}\right]$$

For later notational convenience we introduce

$$a_i \int_{r_1}^{r_2} r W_{q+1}^2(\alpha_i r)\,dr = \int_{r_1}^{r_2} r^{2+q} W_{q+1}(\alpha_i r)\,dr$$

$$B_i \int_{r_1}^{r_2} r W_{q+1}^2(\alpha_i r)\,dr = \int_{r_1}^{r_2} \frac{W_{q+1}(\alpha_i r)}{r^q}\,dr \qquad (9')$$

The problem of determining the solution is thus reduced to a purely computational one. In practice, the computation may prove to be far from trivial, as suitable tables of functions may not be available. The evaluation of the integral involving $f(r)$ may also prove to be a

major difficulty as very few indefinite integrals involving cylinder functions of general order are known. That the evaluations given above can be made seems almost fortuitous since only for these particular relationships between powers of r in the integrand and order of the cylinder function are indefinite integrals known. One sort of problem that can be dealt with, however, is that in which the initial distribution of velocity corresponds to some stationary distribution with the same radial flow.

7.4. The Boundary Value Problem

Next consider the problem of finding the motion from rest specified by giving the solution at two fixed points as functions of time,

$$v(r_1) = v_1(\tau) \qquad v(r_2) = v_2(\tau)$$

The corresponding physical situation is the motion of the fluid between concentric cylinders excited by arbitrary rotations of the bounding cylinders. We restrict ourselves to consideration of continuous velocity distributions since we are interested in dealing with physically meaningful solutions. In particular, this implies that the bounding cylinders are initially at rest.

Given these data we can determine $a(\tau)$ and $b(\tau)$ from

$$v_1(\tau) = a r_1^{2q+1} + b/r_1$$

$$v_2(\tau) = a r_2^{2q+1} + b/r_2$$

We shall deal with the problem in the first instance by approximating the functions a and b by discontinuous functions, jumps in a and b occurring at intervals $\Delta\tau$. During each interval (say the jth, from $j\Delta\tau$ to $(j+1)\Delta\tau$) a and b will be constant (a_j and b_j) and the problem is essentially an initial value problem like that just dealt with.

The simplest approach is to form a solution

$$w(\tau) = \sum_{j=0}^{n} w_j$$

where $w_j = 0$ for $\tau < \tau_j$

$$w_j = \Delta a_j r^{2q+1} + \Delta b_j / r + \sum_{i=1}^{\infty} \Delta B_{ij} r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau} \quad \text{for } \tau > \tau_j$$

with $\Delta a_j = a_j - a_{j-1}$, $\Delta b_j = b_j - b_{j-1}$ the jumps from the previous interval.

We determine the coefficients ΔB_{ij} from the condition of continuity of the solution at $\tau_j = j\Delta\tau$, that is, by requiring that each increment be initially zero:

$$w_j(j\Delta\tau) = 0$$

Hence, by the treatment of the previous section,

$$\Delta B_{ij} = -e^{j\alpha_i^2 \Delta\tau}\left[\Delta a_j A_i + \Delta b_j B_i\right]$$

Then

$$v(n\Delta\tau) = a_n r^{2q+1} + b_n/r$$
$$-\sum_{i=1}^{\infty} A_i \left\{ \sum_{0}^{n} \Delta a_j e^{j\alpha_i^2 \Delta\tau} \right\} r^q W_{q+1}(\alpha_i r) e^{-n\alpha_i^2 \Delta\tau}$$
$$-\sum_{i=1}^{\infty} B_i \left\{ \sum_{0}^{n} \Delta b_j e^{j\alpha_i^2 \Delta\tau} \right\} r^q W_{q+1}(\alpha_i r) e^{-n\alpha_i^2 \Delta\tau}$$

We next consider the form taken in the limit as $n \to \infty$, $\Delta\tau \to 0$, setting $j\Delta\tau = \lambda$. Since the values Δa_j and Δb_j have been chosen to approximate the continuous functions

$$\Delta a_j \longrightarrow \frac{\partial a}{\partial \lambda} d\lambda, \quad \Delta b_j \longrightarrow \frac{\partial b}{\partial \lambda} d\lambda$$

Hence $v(\tau) = a(\tau) r^{2q+1} + b(\tau)/r$

$$-\sum_{i=1}^{\infty} \left\{ \int_0^{\tau} \left(A_i \frac{\partial a}{\partial \lambda} + B_i \frac{\partial b}{\partial \lambda} \right) e^{\alpha_i^2 \lambda} d\lambda \right\} r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau}$$

On noting that

$$\int_0^{\tau} \frac{\partial a}{\partial \lambda} e^{\alpha_i^2 \lambda} d\lambda = a(\tau) e^{\alpha_i^2 \tau} - \alpha_i^2 \int_0^{\tau} a(\lambda) e^{\alpha_i^2 \lambda} d\lambda$$

we obtain the alternative form:

$$v(\tau) = \sum_{i=1}^{\infty} \alpha_i^2 \left\{ \int_0^{\tau} (A_i a + B_i b) e^{\alpha_i^2 \lambda} d\lambda \right\} r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau}$$

Although both solutions converge to the same values in the interior of the region, their behaviour on the boundaries is not the same. The first satisfies the boundary conditions by virtue of the method of defining a and b. The second form does not converge to the prescribed values at r_1 and r_2.

7.5. The Initial Value, Boundary Value Problem

We now deal with the problem combining the features of those just considered, that of finding a solution with specified values on two cylindrical surfaces,

$$u(r_1) = u_1(\tau) \qquad u(r_2) = u_2(\tau)$$

and with a specified initial value,

$$u(0) = f(r)$$

It is possible to obtain the solution by superposing the results obtained above, but we shall start fresh and use slightly more sophisticated methods to evolve the solution. The method to be used relies upon Duhamel's theorem relating to solutions of a homogeneous equation in the following form:

If $g(r,\tau)$ is a solution of a homogeneous differential equation such that $g(r,0) = 0$ and $g(r_1,\tau)=1$, $g(r_2,\tau)=0$,

then $$u(r,\tau) = \int_0^\tau f(\lambda) \frac{\partial g}{\partial \tau}(r, \tau-\lambda)\, d\lambda$$

is a solution of the equation such that $u(r_1,\tau) = f(\tau)$

$$u(r_2,\tau) = 0$$

$$u(r, 0) = 0$$

To demonstrate these results we integrate by parts:

$$u(r,\tau) = \int_0^\tau f(\lambda) \frac{\partial g}{\partial \lambda}(r, \tau-\lambda)\, d\lambda$$

$$= -f(\tau)g(r,0) + f(0)g(r,\tau) + \int_0^\tau f'(\lambda) g(r, \tau-\lambda)\, d\lambda$$

From this form the results given above are seen to be the consequences of the assumptions of the theorem.

Using this result we can obtain a formal solution for our problem:

$$v = v_i + \int_0^\tau v_1(\lambda) \frac{\partial g_1}{\partial \tau}(r, \tau-\lambda)\, d\lambda + \int_0^\tau v_2(\lambda) \frac{\partial g_2}{\partial \tau}(r, \tau-\lambda)\, d\lambda$$

where v_i , g_1 , g_2 are solutions of equation (2) for the swirl component chosen so that:

v_i	g_1	g_2	
0	1	0	for $r = r_1$
0	0	1	for $r = r_2$
$f(r)$	0	0	for $\tau = 0$

The problem is thus reduced to that of determining solutions with the required properties. Note that all are initial value problems of the type considered previously. Following the method of Section 7.3. we obtain

$$g_1 = \frac{r_1}{r}\left[\frac{r^{2q+2} - r_2^{2q+2}}{r_1^{2q+2} - r_2^{2q+2}}\right] - \frac{r_1}{r_1^{2q+2} - r_2^{2q+2}} \sum_{i=1}^{\infty} (a_i - r_2^{2q+2} B_i) r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau}$$

The solution g_2 is obtained by interchanging r_1 and r_2 . The third solution v_i is just a special case of that obtained in Section 7.3.

Using these results we can write

$$v_1(\lambda) \frac{\partial g_1}{\partial \tau}(r, \tau-\lambda) + v_2(\lambda) \frac{\partial g_2}{\partial \tau}(r, \tau-\lambda) = \sum_{i=1}^{\infty} \alpha_i^2 \{ a_i a(\lambda) + B_i b(\lambda) \} r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau}$$

Finally, the required solution is

$$v(r,\tau) = \sum_{i=1}^{\infty} \left\{ \frac{\int_{r_1}^{r_2} r^{1-q} f(r) W_{q+1}(\alpha_i r)\, dr}{\int_{r_1}^{r_2} r\, W_{q+1}^2(\alpha_i r)\, dr} + \left[\alpha_i^2 \int_0^{\tau} (A_i\, a(\lambda) + B_i\, b(\lambda)) e^{\alpha_i^2 \lambda}\, d\lambda \right] \right\} r^q W_{q+1}(\alpha_i r) e^{-\alpha_i^2 \tau}$$

Note that this form of the solution does not converge to the specified values at the boundaries r_1 and r_2. An alternative form can be obtained as in the previous section which converges suitably at the boundaries.

7.6. More General Conditions on the Solution

Consider the problem of determining a solution such that

$$v(r, 0) = f(r)$$

and

$$\begin{aligned} \alpha_1 v(r_1) + \beta_1 \frac{\partial v}{\partial r}(r_1) &= D_1 \\ \alpha_2 v(r_2) + \beta_2 \frac{\partial v}{\partial r}(r_2) &= D_2 \end{aligned} \tag{10}$$

or, more compactly,

$$\begin{aligned} v(r_1) + \lambda_1 \frac{\partial v}{\partial r}(r_1) &= C_1 \\ v(r_2) + \lambda_2 \frac{\partial v}{\partial r}(r_2) &= C_2 \end{aligned} \tag{10'}$$

where α_1, α_2, β_1, β_2, λ_1, λ_2, D_1, D_2, C_1, C_2 are prescribed constants and $f(r)$ is an arbitrary initial swirl distribution. The boundary conditions (10) are of the generalized Sturm-Liouville type. The alternative forms (10') are introduced solely to contract the algebra.

We choose a solution of form (6). The equations to determine a and b are here

$$\begin{aligned} a r_1^{2q+1} + b/r_1 + \lambda_1[a(2q+1)r_1^{2q} - b/r_1^2] &= c_1 \\ a r_2^{2q+1} + b/r_2 + \lambda_2[a(2q+1)r_2^{2q} - b/r_2^2] &= c_2 \end{aligned} \qquad (10'')$$

for $q \neq -1$

(The analysis for the case $q = -1$ will not be given. It is very similar to that which will be given for the more general case.)

We require that the time-dependent part of the solution vanish suitably at r_1 and r_2. Hence

$$r_1^q[a_i J_{q+1}(\alpha_i r_1) + b_i Y_{q+1}(\alpha_i r_1)] + \lambda_1 \frac{d}{dr}\left[a_i J_{q+1}(\alpha_i r) r^q + b_i Y_{q+1}(\alpha_i r) r^q\right]_{r=r_1} = 0$$

$$r_2^q[a_i J_{q+1}(\alpha_i r_2) + b_i Y_{q+1}(\alpha_i r_2)] + \lambda_2 \frac{d}{dr}\left[a_i J_{q+1}(\alpha_i r) r^q + b_i Y_{q+1}(\alpha_i r) r^q\right]_{r=r_2} = 0$$

whence we obtain, on eliminating a_i and b_i, the equation prescribing the eigenvalues α_i.

$$\begin{aligned} &\left\{r_1^q J_{q+1}(\alpha_i r_1) + \lambda_1 \frac{d}{dr}[r_1^q J_{q+1}(\alpha_i r_1)]\right\} \times \\ &\quad \left\{r_2^q Y_{q+1}(\alpha_i r_2) + \lambda_2 \frac{d}{dr}[r_2^q Y_{q+1}(\alpha_i r_2)]\right\} \\ &= \left\{r_1^q Y_{q+1}(\alpha_i r_1) + \lambda_1 \frac{d}{dr}[r_1^q Y_{q+1}(\alpha_i r_1)]\right\} \times \\ &\quad \left\{r_2^q J_{q+1}(\alpha_i r_2) + \lambda_2 \frac{d}{dr}[r_2^q J_{q+1}(\alpha_i r_2)]\right\} \end{aligned} \qquad (11)$$

Here too there are an infinite number of positive solutions.

Again we adopt an auxiliary orthogonal function whose form is suggested by the equation defining the eigenvalues.

$$W_n'(\alpha r) = \left\{ r_1^q J_{q+1}(\alpha r_1) + \lambda_1 \frac{d}{dr}\left[r_1^q J_{q+1}(\alpha r_1) \right] \right\} Y_n(\alpha r) - \left\{ r_1^q Y_{q+1}(\alpha r_1) + \lambda_1 \frac{d}{dr}\left[r_1^q Y_{q+1}(\alpha r_1) \right] \right\} J_n(\alpha r) \qquad (12)$$

Now $r_1^q W_{q+1}'(\alpha r_1) + \lambda_1 \frac{d}{dr}\left[r_1^q W_{q+1}'(\alpha r_1) \right] \equiv 0$

and $r_2^q W_{q+1}'(\alpha r_2) + \lambda_2 \frac{d}{dr}\left[r_2^q W_{q+1}'(\alpha r_2) \right] = 0$

is the condition for the α_i given above

But we can write these results in the form:

$$\left(1 + q\frac{\lambda_1}{r_1}\right) W_{q+1}'(\alpha_i r_1) + \lambda_1 \frac{d}{dr} W_{q+1}'(\alpha_i r_1) = 0$$
$$\left(1 + q\frac{\lambda_2}{r_2}\right) W_{q+1}'(\alpha_i r_2) + \lambda_2 \frac{d}{dr} W_{q+1}'(\alpha_i r_2) = 0 \qquad (13)$$

where the conditions imposed are seen to imply the orthogonality of the set $W_{q+1}'(\alpha_i r)$ specified by (11) in the region (r_1, r_2). It is necessary to demonstrate that these functions as well as the functions $r^q W_{q+1}'(\alpha_i r)$ form an orthogonal set in order that the integrals arising from the steady-state solution can be evaluated as in the simpler cases treated previously.

Note that the results just given reduce to those of Section 7.3. (equations (7) and (8)) for $\lambda_1 = \lambda_2 = 0$.

The solution of form (6) can now be written as

$$u(r, \tau) = a r^{2q+1} + b/r + \sum_{i=1}^{\infty} A_i' r^q W_{q+1}'(\alpha_i r) e^{-\alpha_i^2 \tau} \qquad (14)$$

This result is formally identical to (6'). (We shall deal only with those simple cases of Dini series that are formally similar to Fourier-Bessel series.) The method

of choosing the A_L' to satisfy the initial condition is identical with that of Section 7.3. The formulae for the integrals analogous to equations (9) will be somewhat more complex evaluations from the same well-known indefinite integrals from which the previous results were obtained.

We have now shown how the initial value problem can be treated subject to the more general conditions (10). But the method of Section 7.5. reduces the combined initial value, boundary value problem to initial value problems. Thus we can deal with the combined problem for the more general conditions:

$$v(r, 0) = f(r)$$

$$\alpha_1 v(r_1) + \beta_1 \frac{\partial v}{\partial r}(r_1) = D_1(\tau)$$

$$\alpha_2 v(r_2) + \beta_2 \frac{\partial v}{\partial r}(r_2) = D_2(\tau)$$

where α_1 , α_2 , β_1 , β_2 are still constants.

The solution will be formally equivalent to that given at the end of Section 7.5. with the slightly altered notation of the present section.

7.7. Conditions with Physical Meaning

Consider the equation

$$\alpha v + \beta \frac{\partial v}{\partial r} = D$$

where $v = v(r, \tau)$, $D = D(r, \tau)$

It can by physically meaningful only if β/α has the dimensions of length and D those of αv , unless α or β is zero. The condition will only be physically significant

if we prescribe some meaningful property at the point of application. It then appears that D must be some point function (or perhaps a point function integrated over a cylindrical surface) and that β/α must be a length associated with a point function.

Some interpretations which satisfy these requirements are:

	α	β	D
a)	1	0	v
b)	0	1	$\frac{\partial v}{\partial r}$
c)	$-1/r$	1	$\frac{\partial v}{\partial r} - \frac{v}{r}$
d)	$1/r$	1	$\frac{\partial v}{\partial r} + \frac{v}{r}$
e)	r	0	rv
f)	$-\mu/r$	μ	$\mu\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$
g)	$-2\pi\mu r$	$2\pi\mu r^2$	$2\pi\mu r^2\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)$

Note that the conditions (13) on the generalized cylinder functions adopt particularly simple forms for these cases.

The use of most of these forms is obvious. The condition on the derivative of the velocity can be used to specify the position of a maximum or minimum of velocity. The condition on the shear can be used to simulate an air core. The last case gives the shear moment on a cylinder.

One condition that cannot be applied in this way is the non-linear one that a certain power be exerted on a

cylindrical surface by shearing stresses. Nor is it possible to apply two conditions on the same surface as an orthogonal set of eigenfunctions cannot then be defined. Nevertheless, we are able to specify directly a wide variety of boundary conditions using this approach.

7.8. Steady-state Solutions

Consider the solution (14) to the initial value problem with fixed boundary conditions. The part of the solution arising from the initial condition dies away with time in all cases, leaving only the steady-state terms. The coefficients of these steady-state terms can in general be determined directly from (10"). But some slight difficulty arises for the case $C_1 = C_2 = 0$. For this case it is obvious that $a = b = 0$ is always a solution of the system (10"). But for certain combinations of λ_1 and λ_2 other values of a and b can satisfy the equations.

We shall discuss two such cases which can be interpreted physically; the others all involve boundary conditions which lack immediate physical significance.

a) Zero source flow, zero shear specified at both boundaries.

$q = 0$, $\lambda_1 = -r_1$, $\lambda_2 = -r_2$, $b = 0$, a unspecified

b) Any source strength, zero vorticity specified at both boundaries.

$\lambda_1 = r_1$, $\lambda_2 = r_2$, $a = 0$, b unspecified

It is our task to decide whether the motion will tend towards these states or towards quiescence ($a = b = 0$) and

to determine the unspecified parameters. To this end we shall develop a theorem relating to the angular momentum in an annular region.

The doubtful cases can also be found by seeking those series solutions which give rise to constant kinetic energy within the annular region between the cylinders on which data is supplied. The use of Parseval's theorem to simplify the condition of constant kinetic energy leads quickly to the critical boundary conditions. Thus the consideration of kinetic energy, although at first seemingly promising additional insight, in fact gives nothing new.

7.8.1. An Angular Momentum Theorem

A fairly general angular momentum principle may be stated as:

The variation of angular momentum in a region fixed in space is due only to moments of forces acting at the boundaries, the moments of body forces acting on the fluid within the boundaries, and the net convection of angular momentum into the region through the boundaries.

Using the linear momentum principle this statement can be shown to hold for an arbitrary region provided that the fluid is inviscid. It is not, in general, true for a viscous fluid. But we can develop such a result in a restricted form, suitable for the present problem.

Consider a fixed annular region through which there is an axisymmetric flow of an incompressible fluid of constant viscosity, the source strength not necessarily constant.

For this region the angular momentum principle is

$$A.M.(t) = A.M.(0) + \int_0^t (M_1 + M_2)\,dt + \int_0^t F(r_1 v_1 - r_2 v_2)\,dt$$

$$A.M.(t) = \int_{r_1}^{r_2} 2\pi\rho r^2 v(t)\,dr$$

$$A.M.(0) = \int_{r_1}^{r_2} 2\pi\rho r^2 v(0)\,dr$$

$$M_1 + M_2 = -2\pi\mu r_1^2\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)_{r_1} + 2\pi\mu r_2^2\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)_{r_2}$$

$$F = 2\pi r\rho\mu = 2\pi\mu C$$

Introducing these definitions we obtain

$$\rho\int_{r_1}^{r_2} r^2\left[v(t) - v(0)\right]dr$$

$$= \mu\int_0^t \left[r^2\left(\frac{\partial v}{\partial r} - \frac{v}{r}\right)\right]_{r_1}^{r_2} dt - \mu\int_0^t C\left[rv\right]_{r_1}^{r_2} dt$$

$$= \mu\int_0^t \left[r^2\frac{\partial v}{\partial r} - (1+C)rv\right]_{r_1}^{r_2} dt$$

But

$$\int_0^t r^2 \frac{\partial w}{\partial t} dt = r^2 \left[w(t) - w(0) \right]$$

$$\int_{r_1}^{r_2} \left[r^2 \frac{\partial^2 w}{\partial r^2} + (1-c) r \frac{\partial w}{\partial r} - (1+c) w \right] dr$$

$$= \left[r^2 \frac{\partial w}{\partial r} - (1+c) r w \right]_{r_1}^{r_2}$$

So that

$$\int_{r_1}^{r_2} \int_0^t r^2 \left[\frac{\partial w}{\partial \tau} - \frac{\partial^2 w}{\partial r^2} - \frac{1-c}{r} \frac{\partial w}{\partial r} + \frac{1+c}{r^2} w \right] dr\, d\tau = 0$$

The truth of the angular momentum principle is thus seen to depend on that of the corresponding equation governing linear momentum in the tangential direction, equation (5.7).

7.8.2. Application to Doubtful Cases

The case (a) given above can be settled immediately using this result. Here

$$F = M_1 = M_2 = 0$$

so that $$A.M.(t) = A.M.(0)$$

whence $$a = \frac{4}{r_2^4 - r_1^4} \int_{r_1}^{r_2} r^2 w(0) dr$$

is the previously unspecified value. We are thus able not only to state that the final condition will not be quiescent, but also to determine it precisely.

The case (b) is not so easily dealt with. Here

$$\tau = -2\mu \frac{v}{r}$$

and the angular momentum relationship is

$$A.M.(t) - A.M.(0) = 2\pi\mu \int_0^t (2+c)(r_1 v_1 - r_2 v_2)\,dt$$

Although we cannot say anything definite about the general case, two special cases can be discussed using our result.

We note first that an initial swirl distribution of form b/r will not be altered by shear or convection.

Also, for the particular case $C = -2$ the angular momentum in the annular region remains constant. Then

$$b = \frac{2}{r_2^2 - r_1^2} \int_{r_1}^{r_2} r^2 v(0)\,dr$$

gives the undetermined parameter for this case. (The conditions (10") from which the doubtful cases were derived do not hold for the case $C = -2$. However, this doubtful case is also thrown up by the special conditions for that case. That this is to be expected may be seen immediately by noting that the parameter b plays the same role in both the solutions (6.2).)

Finally: there are certain cases for which we have not been able to determine the state towards which the system tends with increasing time using the methods set forth. The one last treated is the only one in this category which can be interpreted simply in physical terms. And even this case is highly artificial.

7.9. Pressure and Temperature Distributions

The pressure field corresponding to a particular velocity distribution can be determined as in Section 6.4. on introducing the more complex function giving the swirl and on supplying sufficient data to fix the pressure level.

Methods similar to those of Section 6.5. seem advisable in determining the temperature distribution.

Take $T = T_1 + T_2$

such that

$$\frac{\partial T_1}{\partial \tau'} = r^{\sigma c - 1}\frac{\partial}{\partial r}\left[r^{1-\sigma c}\frac{\partial T_1}{\partial r}\right] + \frac{\mu}{K}\left[\left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)^2 + \left(2\frac{w}{r}\right)^2\right]$$

determines T_1, and

$$\frac{\partial T_2}{\partial \tau'} = r^{\sigma c - 1}\frac{\partial}{\partial r}\left[r^{1-\sigma c}\frac{\partial T_2}{\partial r}\right]$$

determines T_2.

To find T_1 we must deal with a non-homogeneous equation, the dissipation term being of extremely complex form. In order to indicate how this problem might be dealt with we shall consider a simple case---the flow conditioned by fixed cylindrical boundaries (a special case of the initial value problem of Section 7.3.). The source term in the equation for T_1 is then of form:

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \Phi_{ij}(r)\, e^{-(\alpha_i^2 + \alpha_j^2)\tau}$$

where the Φ_{ij} are known functions.

Let us assume a representation for T_1 of the same form:

$$T_1 = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} F_{ij}(r) e^{-(\alpha_i^2 + \alpha_j^2)\tau}$$

Introducing these expressions into the equation and equating the coefficients of the time-dependent factors to zero, we obtain

$$-\sigma(\alpha_i^2 + \alpha_j^2) F_{ij} = r^{\sigma c - 1} \frac{d}{dr}\left[r^{1-\sigma c} \frac{dF_{ij}}{dr}\right] + \Phi_{ij}$$

---a series of ordinary differential equations for the functions F_{ij}.

Even in this uncomplicated case the method proposed is far from simple. Its extension to more complex cases will make analytic solution of the problem even more unlikely.

Assuming that the function T_1 has been determined in the manner suggested, let us turn to the evaluation of T_2. The method of dealing with the homogeneous equation for T_2 has been discussed in Section 6.5. Once again, account must be taken of the values of the solution T_1 at the positions where data are supplied.

7.10. Further Extensions

There are several more general models which are governed by the equations (1) to (4) or Section 5: those for which boundary conditions are specified along arbitrary curves in r,τ-space, and those for which the radial flow is variable.

It is possible that the present analysis can be extended to deal with the former class of problems. The normal solutions from which the analysis springs are still valid. The problem can perhaps be treated as an inverse one---that of determining the conditions to be prescribed at some pair of fixed points in order that the solution adopt certain values at a pair of moving points.

But the methods used here seem ill-adapted to dealing with the case of variable source strength. The previously immutable bases of the analysis---the elementary solution---lose their applicability. The treatment of Section 6.8. may give some indication of the degree of complexity to be expected in dealing with more general radial flows.

It is worth noting that alternative representations for the solutions given above can often be obtained by using the integral transform approach. These representations may be valuable if numerical evaluation of the solutions is required. The less general classical approach through normal solutions has been used here because it is simpler and keeps closer to the physical problems which are of real interest. This technique seemed to be a more powerful constructive tool, although this may be due in part to its familiarity.

8. Conclusions

The previous analyses of swirling flows of an incompressible fluid within cylindrical boundaries can easily be extended to include the case of constant radial flow for both the steady and non-steady swirl cases. The class of solutions obtained in this manner is capable of giving considerably more insight into the complex flows encountered in practical situations since the effects of radial convection can be studied. Further, some of the essential features of boundary layers can be studied using these models.

Using generalized Sturm-Liouville conditions to specify solutions it is possible to determine time-dependent solutions satisfying a variety of physically meaningful boundary conditions. These solutions may represent some features of steady, three-dimensional flows. However, the eddy loops (or reverse flows) commonly found in such three-dimensional patterns introduce complications not found in the plane model. A limited extension to three-dimensional flow is possible starting from a stationary two-dimensional model which includes a radially varying source distribution.

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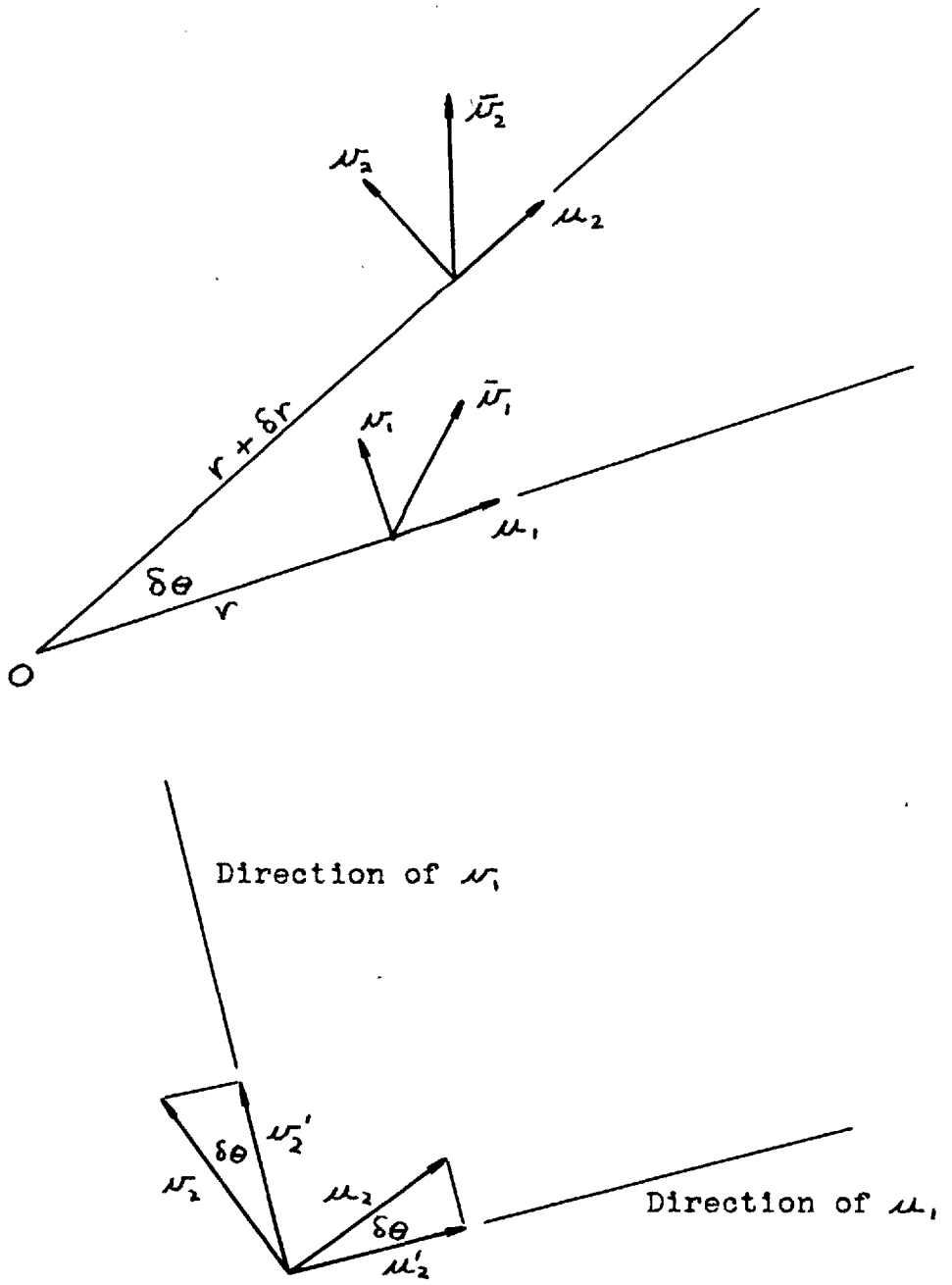


Figure 1. Description of the Velocity Field.

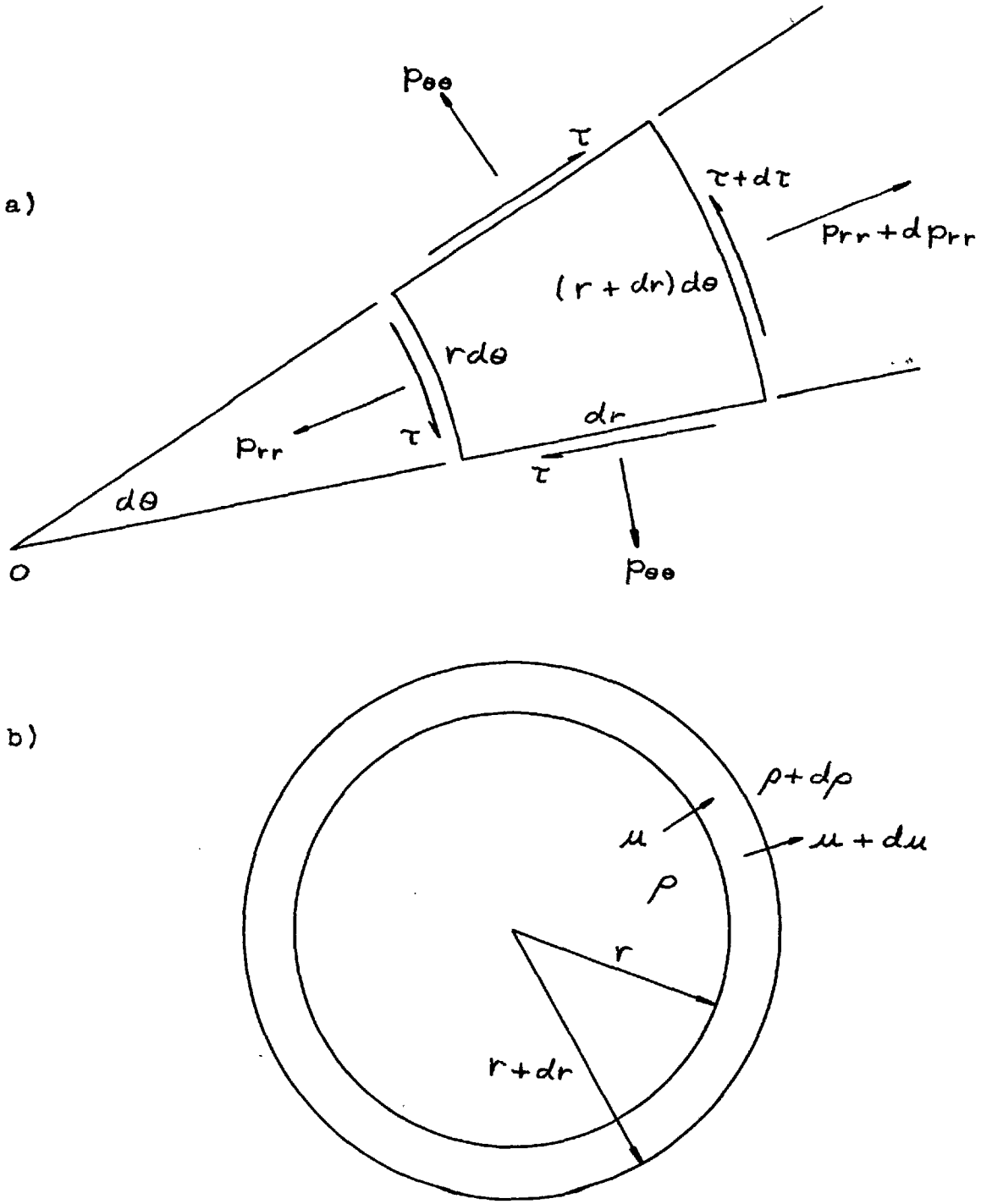


Figure 2. Fundamental Control Volumes

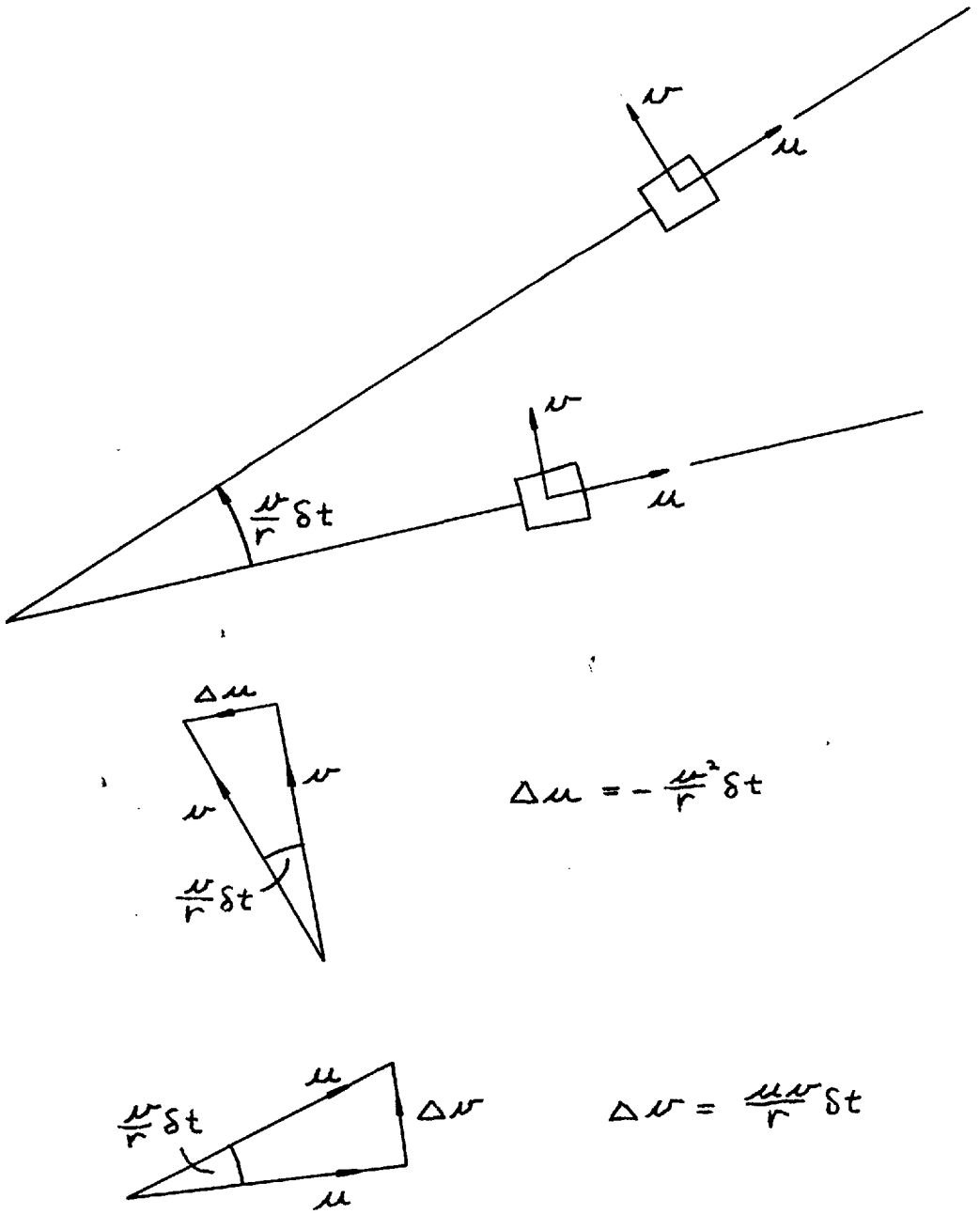


Figure 3. Velocity Changes due to Rotation of Velocity Vectors

Figure 4.

Velocity Distribution for Example (a)

$\nu = 1$ for $r = 1$

$\nu = 1$ for $r = 2$

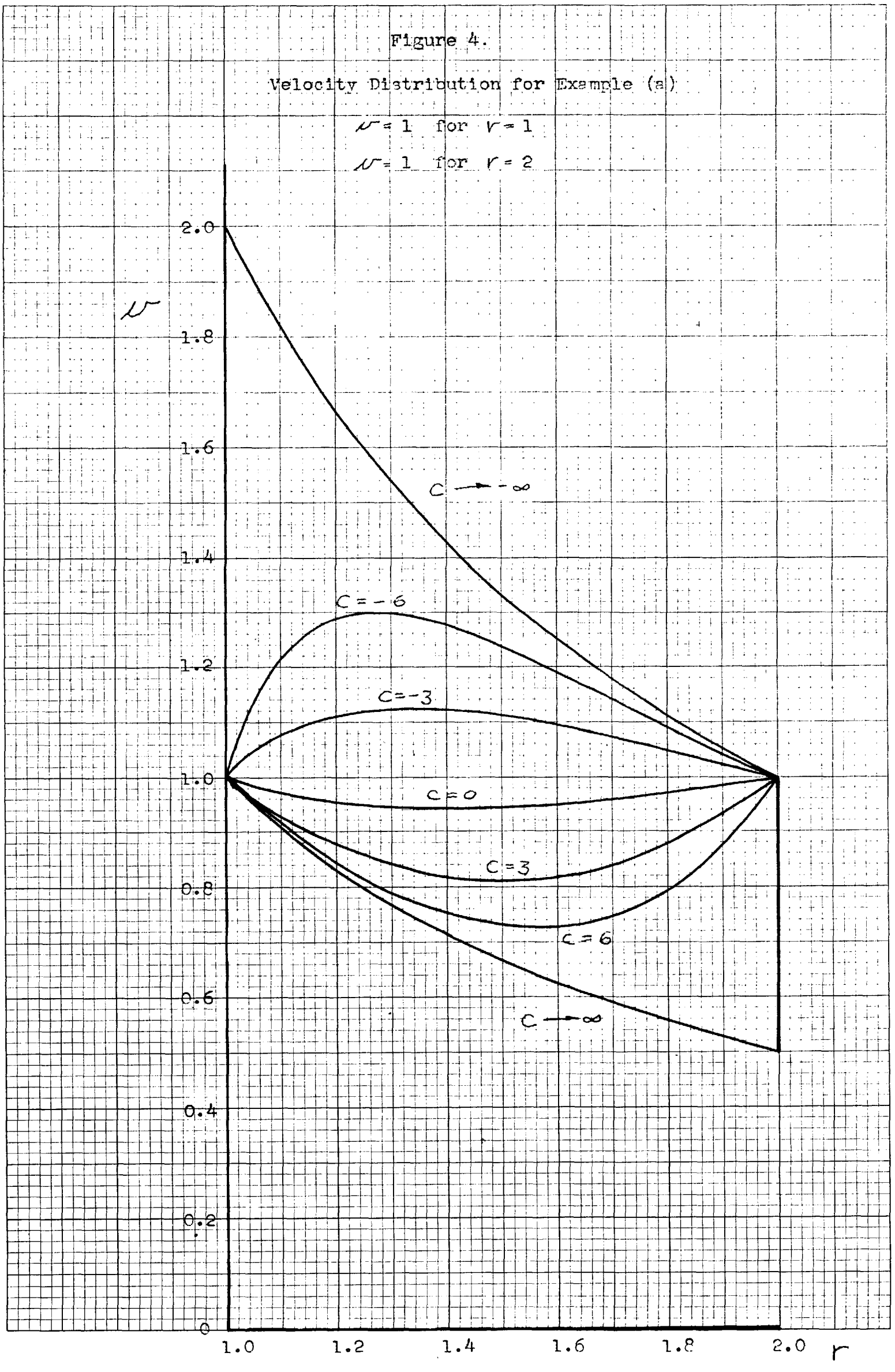


Figure 5.

Velocity Distribution for Example (b)

$v = 0$ for $r = 1$

$v = 1$ for $r = 2$

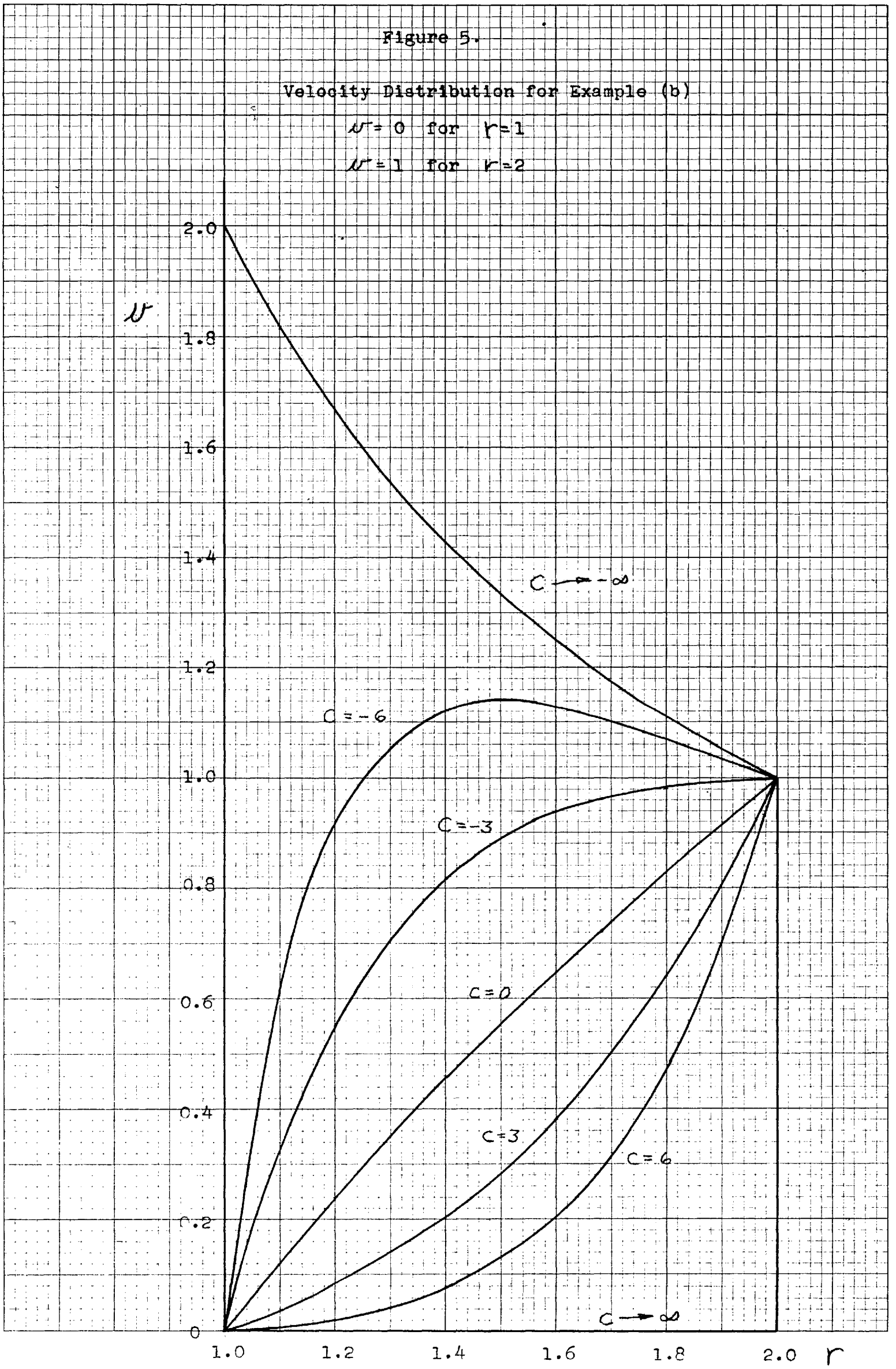


Figure 6.

Velocity Distribution for Example (c)

$\tau = 0$ for $r = 1$

$v = 1$ for $r = 2$

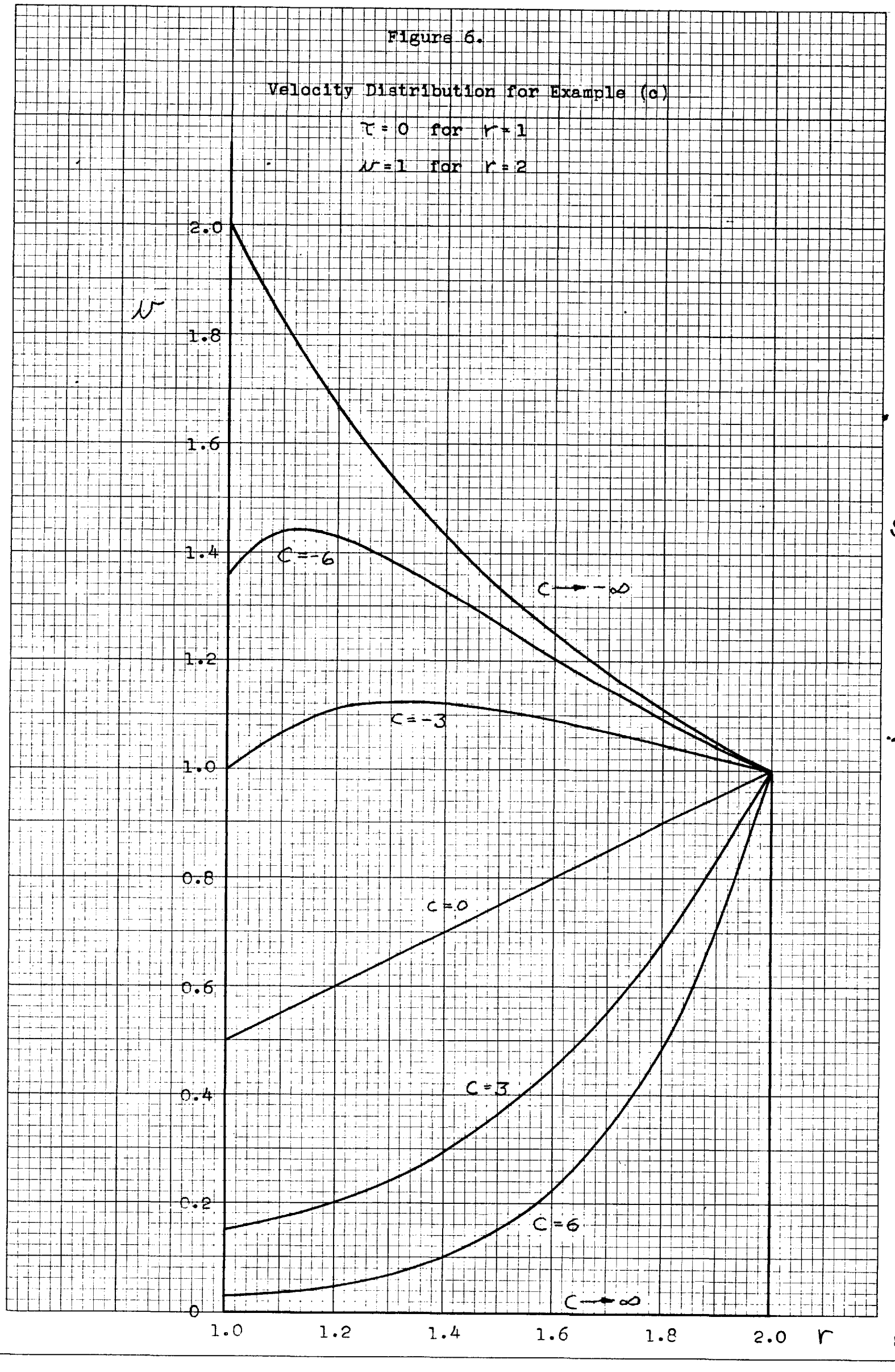


Figure 7.

Shear Distribution

for Example (a)

$\nu = 1$ for $r = 1$

$\nu = 1$ for $r = 2$

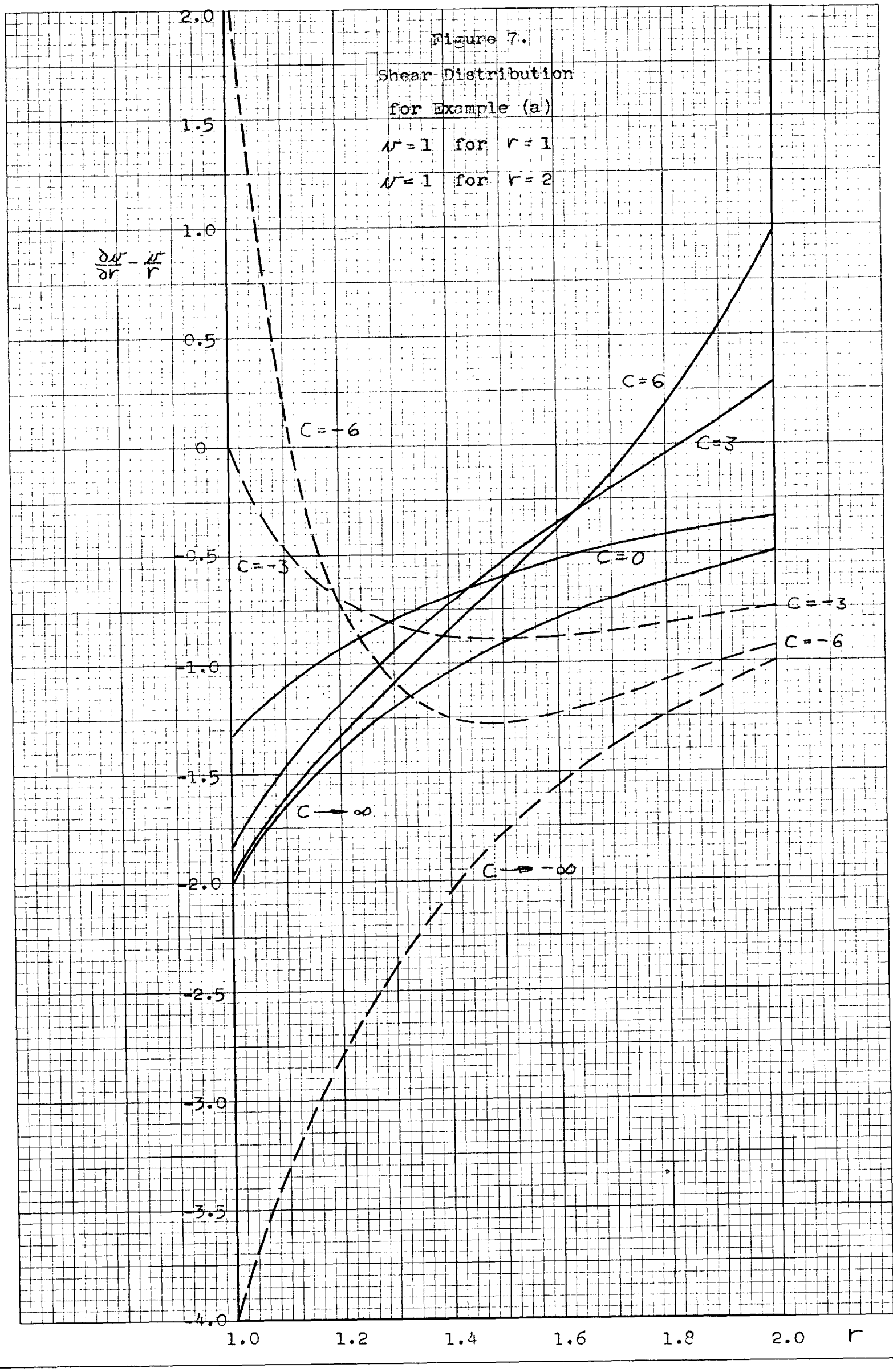


Figure 8.

Shear Distribution

for Example (b)

$\nu = 0$ for $r = 1$

$\nu = 1$ for $r = 1$

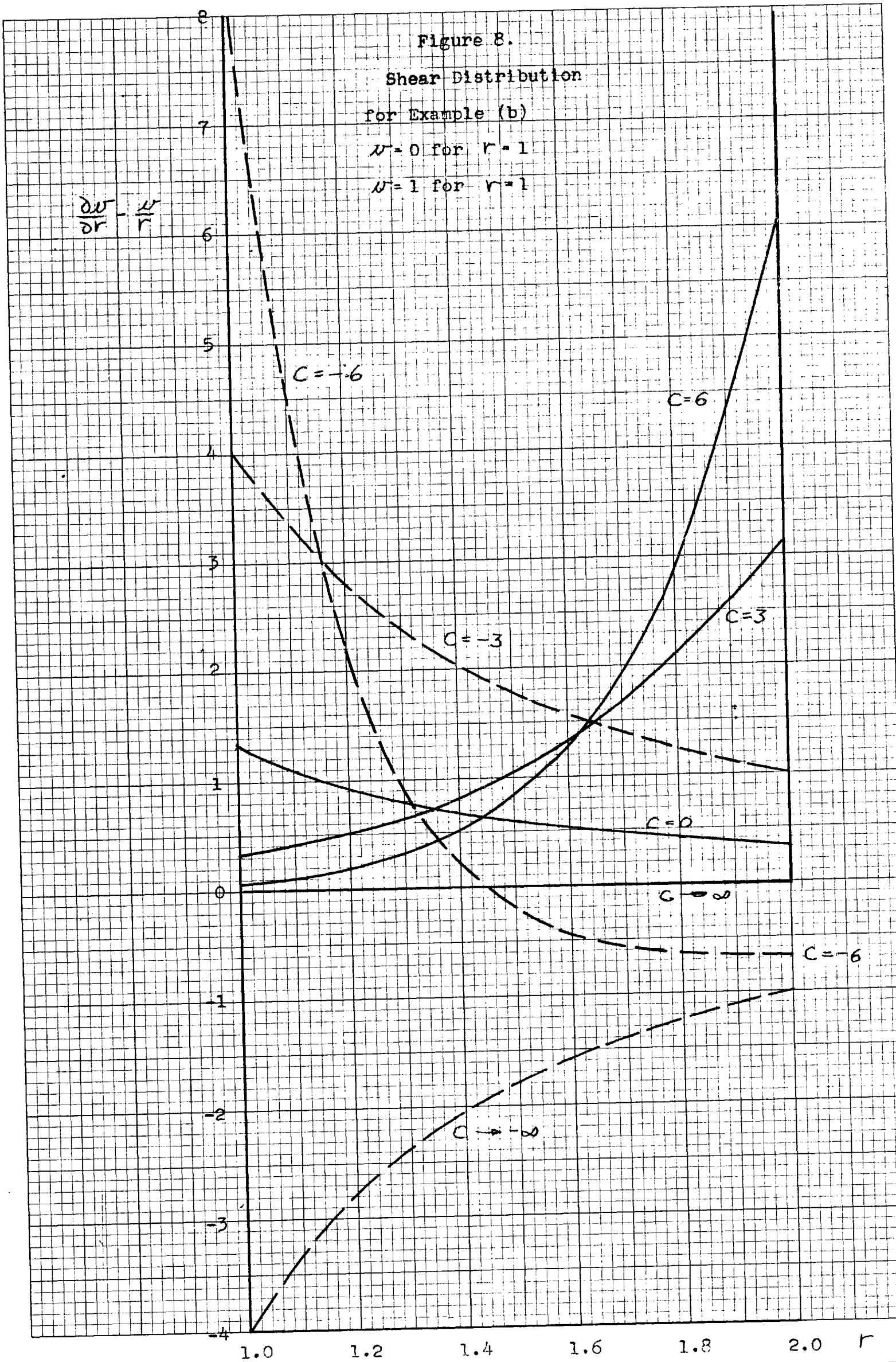


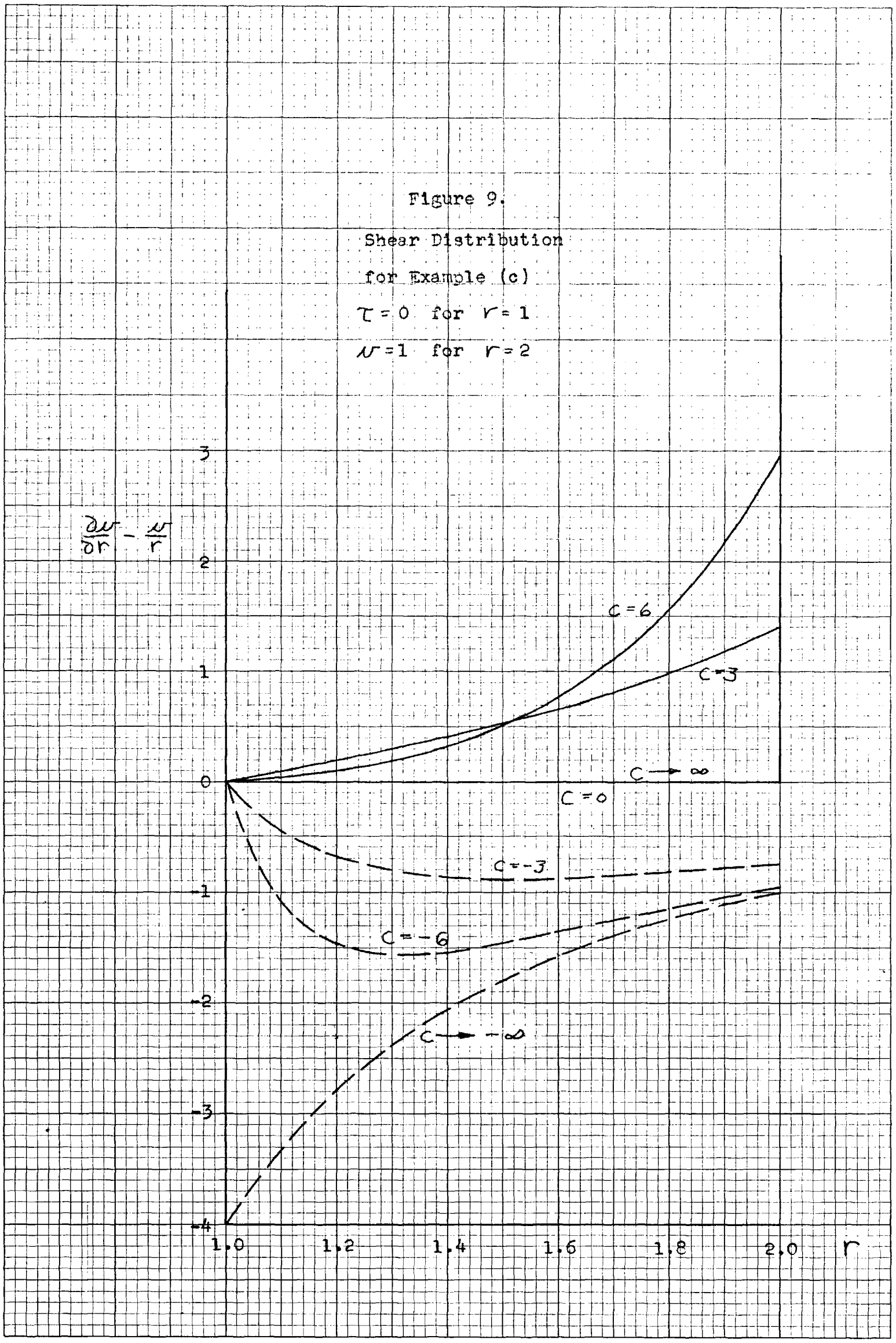
Figure 9.

Shear Distribution

for Example (c)

$\tau = 0$ for $r = 1$

$\nu = 1$ for $r = 2$



A

Figure 10.

Dissipation for Example (a)

$\nu = 1$ for $r = 1$

$\upsilon = 1$ for $r = 2$

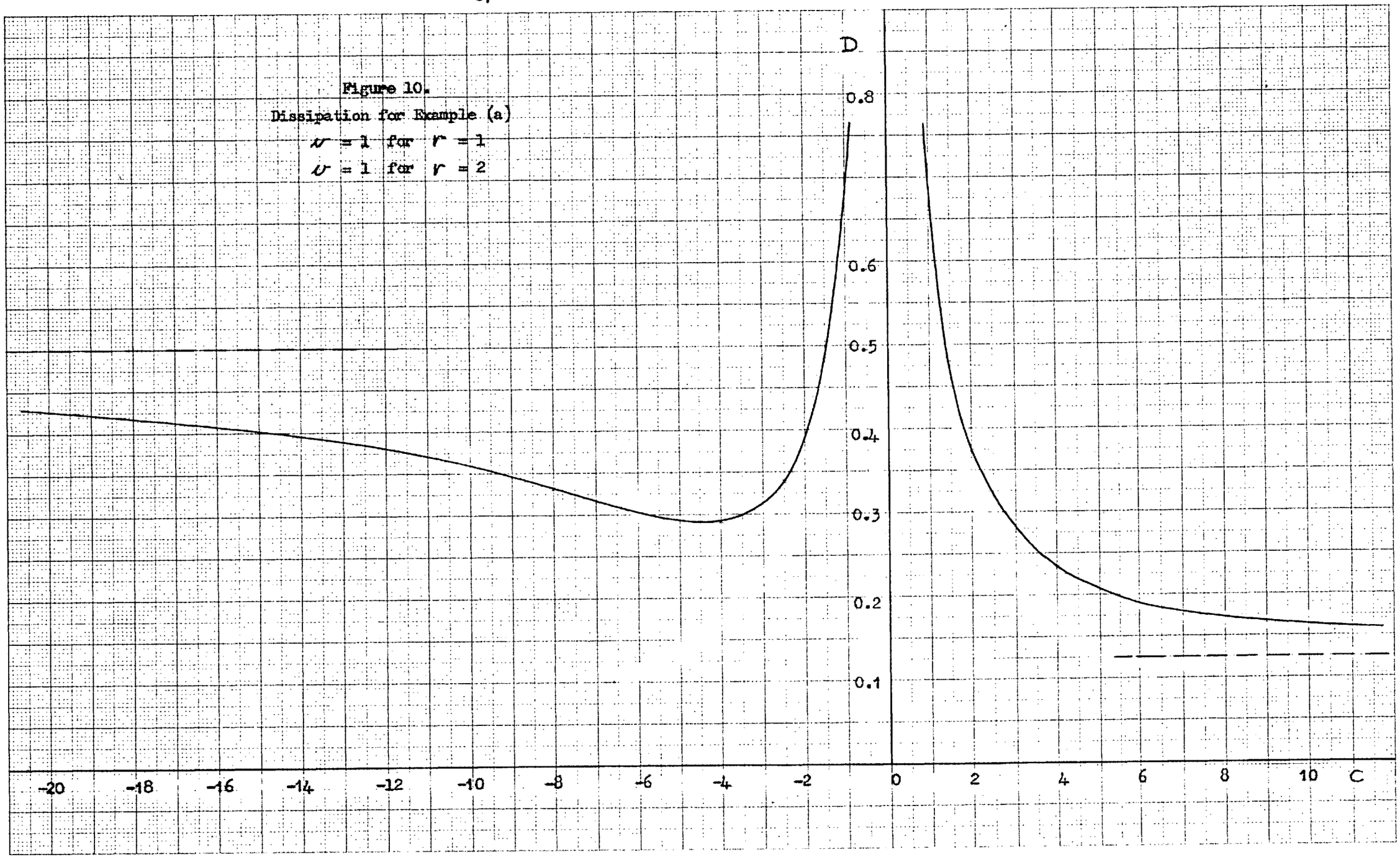
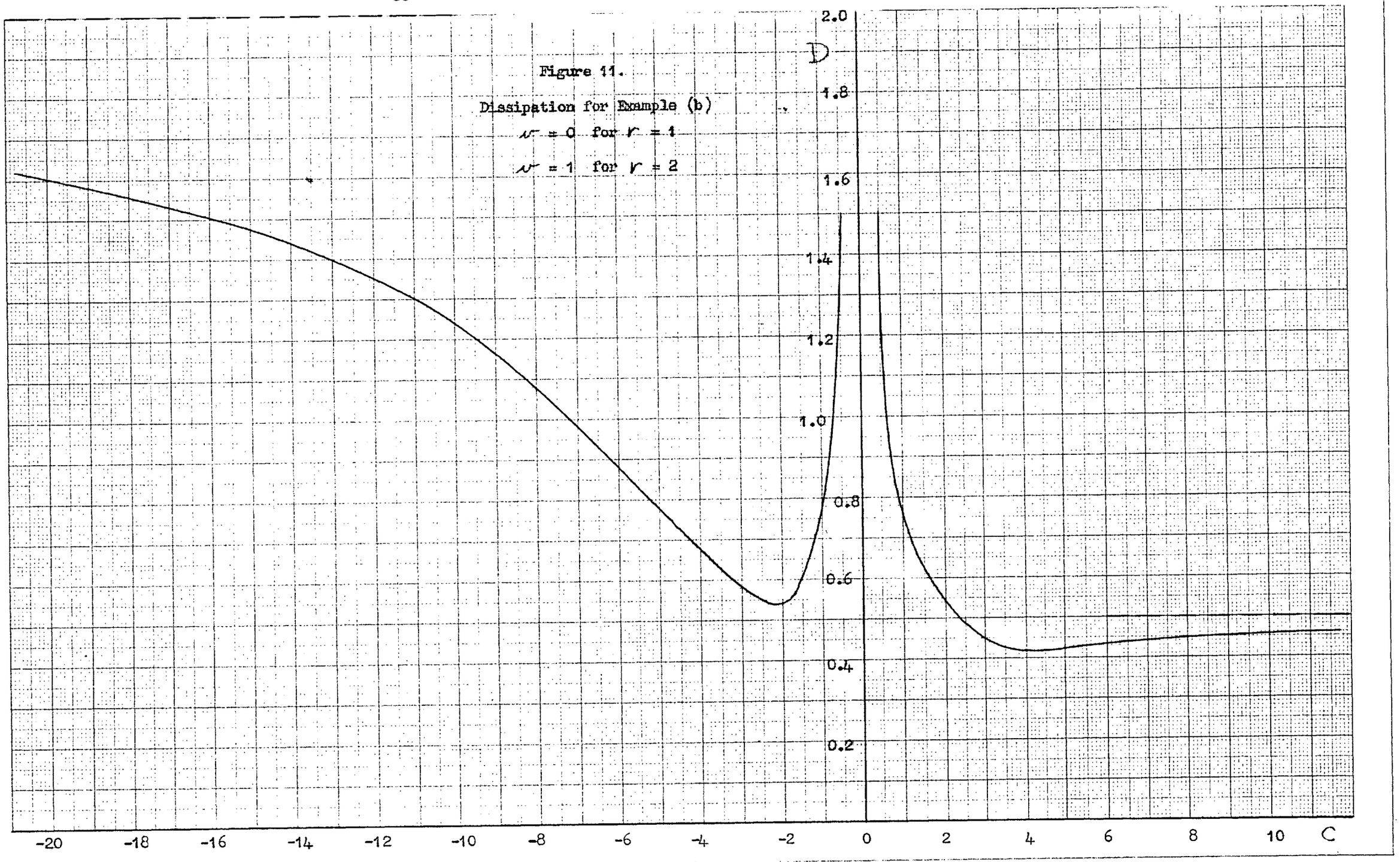


Figure 11.

Dissipation for Example (b)

$\nu = 0$ for $r = 1$

$\nu = 1$ for $r = 2$



A

Figure 12.

Dissipation for Example (c)

$\tau = 0$ for $r = 1$

$\nu = 1$ for $r = 2$

