

IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY

" A PRINCIPLE FOR APPROXIMATE SOLUTIONS OF QUEUEING SYSTEMS "

BY

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To Mary, for her infinite patience

in coping with a fifth member in our family: queues.

ABSTRACT

A principle for the approximate solution of queueing systems is established in the present thesis. The principle is based on the underlying idea of "similarity in the behaviour of queueing systems".

The notion of similarity and the problem of identifying the systems and the conditions under which these systems have a similar behaviour are discussed. "Similarity ratios" are determined and used to express measures of practical interest (basically the average queueing time of an arriving customer and the measures directly related to it) associated with a system A in terms of the same measures associated with a "similar" system B. Such expressions seem to be useful if the queueing system B is relatively easy to solve.

The principle is applied in deriving simple approximate formulae for the average queueing time of an arriving customer to a standard GI/G/1 system; the degree of approximation achieved under various conditions is investigated and found satisfactory for many practical purposes.

Possible application of the principle when dealing with more complicated systems is outlined and a specific example of non-preemptive multi-server systems is analyzed.

Finally, some of the potential improvements on the ways by which the principle can be applied are briefly discussed; it seems that fresh ideas and further research in the field are required.

A C K N O W L E D G E M E N T S

I wish to express my most sincere thanks to my supervisor Mr. S.C. Mathewson for his invaluable contribution in the whole course of the present work.

I am also indebted to my wife for her devotion, affection and many sacrifices and for her constant encouragement and persistent reminder that I should bring my task to an end in order to devote more time to her and to our two children.

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SYMBOLS AND NOMENCLATURE

This section contains definitions of symbols used frequently and consistently throughout the Text and the Appendices. Symbols which are used only occasionally in isolated sections of the text are not included here.

$A/B/r$	Notation for describing queueing systems where A indicates inter-arrival pattern and B indicates service pattern
$A, v_a = v_a^*/B, v_s = v_s^*/r$	Notation for describing queueing systems with special reference to the coefficients of variation of inter-arrival and service times
A.Q.T.	Average queueing time
$A(u)$	Distribution function of inter-arrival times
$A^*(s)$	Laplace transform of the distribution of inter-arrival times
$B(t)$	Distribution function of service times
$B^*(s)$	Laplace transform of the distribution of service times
D	Deterministic inter-arrival or service times
E	Erlang inter-arrival or service times
e	Relative percentage error incurred
Γ	Gamma inter-arrival or service times

G	General distribution for service times
GI	General independent distribution for inter-arrival times
k	Parameter of the erlang distribution when related to inter-arrival times
ℓ	Parameter of the erlang distribution when related to service times
λ	Average rate of arrival of customers
M	Exponential inter-arrival or service times (or equivalently Poisson arrival or service process)
μ	Average service rate
μ^{-1}	Average service time
r, r_i, r_j	Number of parallel servers
ρ	Traffic intensity ($= \lambda/\mu$ for single-server systems and $\lambda/r\mu$ for multi-server systems)
S_a, S_s	Similarity factors as defined in text
Standard system	System as defined in the first section of the relevant Chapter
v_a	Coefficient of variation of inter-arrival times
v_a^*, v_a'	Some specific value of v_a
v_s	Coefficient of variation of service times
v_s^*, v_s'	Some specific value of v_s

var()

variance

 $W(A/B/r)$ Average queueing time of an arriving customer to a system $A/B/r$ $W(A, v_a = v_a^*/B, v_s = v_s^*/r)$ Average queueing time of an arriving customer to a system $A/B/r$ with special reference to the coefficients of variation of inter-arrival and service times $\mu W()$

Average queueing time of an arriving customer measured in units of average service time

CHAPTER 1

INTRODUCTION

1. INTRODUCTION

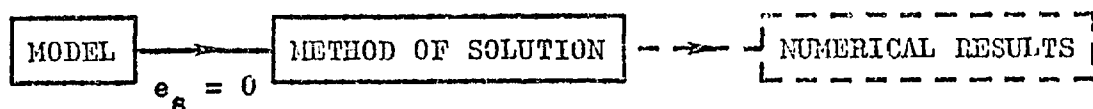
1.1 SEVENTY YEARS OF QUEUEING THEORY

Queueing theory has a history of approximately 70 years. In the early days of its development, starting back in the first decade of this century with the pioneering work of Erlang, research was generated by a desire to investigate and improve the behaviour of real life waiting line situations. Therefore, applications (mainly in the fields of telephone communication, machine servicing and maintenance and automobile traffic) ~~gave~~ ~~that~~ gave the inspiration for the foundation of queueing theory as it is known today. And as Morse³³ reveals, out of a total of 753 books and papers published on queueing theory before 1960, almost two thirds were devoted to theory in connection with some field of application.

The supplementary bibliography on queues prepared in 1966 by Saaty⁴¹ and the papers published thereafter seem, however, to indicate a gradual shift of interest: since 1960 more and more papers seem to have been motivated by a desire to solve models rather than problems; and my personal feelings on papers of this kind are, I think, quite accurately reflected in most of the views expressed fairly recently by Woolsey⁴⁹.

1.2 THE DISTINCTION BETWEEN MODEL AND PROBLEM SOLVING

Model solving is usually restricted to the transition:

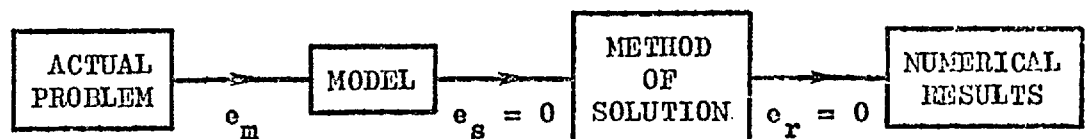


and most papers on model solving have three features in com-

mon, namely:

- a) that the error e_s incurred in the described transition is equal to zero;
- b) that the method of solution is rather complicated (due to inefficient mathematics?) even for conceptually simple models; and
- c) that numerical results are not given (and are indeed difficult to obtain).

Problem solving, on the other hand, starts with the actual problem and ends with numerical results. In simplified terms, three transitions are now required:



which implies that an error e_m is incurred, reflecting the fact that the model does not represent the actual problem accurately. Hence the questions:

- If interested in minimizing the cost associated with the total error incurred, why should one have the constraints $e_s = 0$ and $e_r = 0$, when e_m may be quite significant?
- If interested in keeping low the cost associated with the derivation of numerical results, what is the purpose of having both e_s and e_r equal to zero, since both constraints imply high mathematical skills, long learning and understanding processes, knowledge of sophisticated computing techniques and, in many cases, considerable amount of computer time?
- If interested in minimizing the total cost of solving the

problem, how can such an objective be reached?

1.3 ADVANCES IN PROBLEM SOLVING

In the context of the views expressed in section 1.2, advances in problem solving can be and are directed towards:

1.3.1 Publication of tables providing numerical results to well defined models

The consequences are:

- a) the cost associated with the derivation of numerical results is incurred only once per model;
- b) the two errors e_s and e_r remain equal to zero; and
- c) the error e_m is reduced in magnitude because access to more complicated models with known solutions becomes possible.

Such tables exist in the literature (see, for example, Kometani²⁵, Descloux¹⁰, Peck and Hazelwood³⁶, to name only a few of their authors). It is felt, however, that many more are kept in individual files without ever having been presented in published form.

1.3.2 Derivation of approximate methods of solution to specific models

The consequences are:

- a) the cost associated with the derivation of numerical results to those specific models is reduced;
- b) the errors e_s and/or e_r become positive; and

- c) the error e_m is reduced in magnitude due to the wider range of workable models.

Approximate methods of solution to specific models are not uncommon in the literature (see, for example, the related work of Wishart⁴⁸ on the GI/E_k/1, of Page³⁵ on the E₂/E₂/r system, of Everett¹¹ on the M/D/r system with possible extension to the GI/D/r system, etc.). Many of them are, however, not all that easy to apply.

1.3.3 Use of computer simulation

The consequences are:

- a) the error e_m ^{can be} reduced considerably;
- b) the errors e_s and e_r are non - zero; but
- c) the cost associated with the derivation of numerical results may be high
 - if e_s and e_r have to be kept low;
 - if the traffic intensity considered in the simulation model approaches unity; or
 - if the same basic simulation model has to be run several times for different values of some key variables.

Research in the field is quite active and is basically directed towards the increase of the method's efficiency. It is felt, however, that though many queueing models must have been solved by simulation, very few results are available in published form.

1.3.4 Derivation of principles for the approximate solution of queueing models

The consequences are:

- a) the cost associated with the derivation of numerical results is reduced;
- b) the errors e_s and/or e_r become positive; and
- c) the error e_m is reduced in magnitude because access to more complicated models becomes possible.

Examples of such principles are:

- a) the one established by Kingman²⁰ in 1961, whereby the queueing time distribution in a standard single - server of heavy traffic system operating under conditions does not depend on the distributions of inter - arrival and service times;
- b) the one established by Kingman²³ and Marshall²⁹, whereby the idea of deriving bounds on the average queueing time and/or other measures of practical interest yields information which is at once sufficiently simple to be of practical use and sufficiently general to apply to a wide class of queueing systems; and
- c) the one established by Page³⁵, whereby the average queueing time of an arriving customer to a standard $E_k/E_l/r$ system can be considered as a linear function of $1/k$ and $1/l$, if the traffic intensity is not too low.

1.4 THE PRESENT THESIS IN THE CONTEXT OF PROBLEM SOLVING

1.4.1 Its contribution

It is felt that the main contribution to problem solving of the present thesis lies in the establishment of one further principle for approximate solutions to queueing models. The principle is based on the underlying idea of "similarity" in the behaviour of queueing systems. "Similarity ratios" are

determined and used to express measures of practical interest (basically the average queueing time of an arriving customer and the measures directly related to it) associated with a system A (in most cases the standard GI/G/r system) in terms of the same measures associated with a "similar" system B. Such expressions seem to be useful if the queueing system B is easy to solve. (At this stage, a comparison between two "similar" queueing systems and two similar triangles may provide additional insight both into the principle itself and into its possible applications).

Minor contributions of the present thesis include:

- the possibility of detecting errors (both in the mathematical treatment of a model and in the computer program) by comparing analytical and approximate results;
- the various formulae summarized in Appendix 2;
- the analytical results presented in Appendix 3; and
- the simulation results which are given in Appendix 4.

1.4.2 The major problems encountered

- a) The identification of the queueing systems and of the conditions under which those systems have a similar behaviour

This problem seems to have no unique solution: three possible solutions are investigated and developed, some are only mentioned but many more can be envisaged.

- b) The derivation of a method for the determination of the similarity ratios

This problem seems to have no unique solution either: one method is applied, some are just discussed but many more can be envisaged.

- c) The almost complete lack of numerical results which were required for the justification of the principle and of the ways by which it is applied

A lot of effort and an incredible number of man - and CDC 6400 computer - hours were "spent" in trying to derive numerical results (either analytically or by simulation) for the average queueing time of an arriving customer to a standard GI/G/r system.

1.4.3 The results obtained

In general terms the degree of approximation achieved depends on the values of the various parameters involved: In some cases it is absolutely satisfactory for most practical purposes; in other cases it is just satisfactory; and in a few cases it may be considered as unsatisfactory. In all cases, however, it becomes better as the average queueing time to be estimated and, hence, the traffic intensity increase; in other words, the relative percentage errors incurred decrease in absolute value as the traffic intensity ρ increases, tending to 0 as ρ tends to 1.

1.4.4 Philosophy and organization of material

No attempt is made to present a theory of similarity in the behaviour of queueing systems or anything of this kind. Though such a theory may well be established in the future, my ideas and thoughts on the subject are at present immature and incomplete if not slightly confused. Moreover, instead of choosing an approach, whereby the most general case would be analyzed first, the presentation of ideas and results goes in line with

the progress of my research. However, at the expense of some repetition, each Chapter (with the exception of Chapters 9 and 10) is self-contained, so that the reader can follow and understand the text of Chapter n even if he possesses the Markovian property of memorylessness.

CHAPTER 2

APPROXIMATE FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME

IN THE STANDARD SYSTEM

$$\underline{GI / M / r}$$

An abridged version of this Chapter has been published in
Operational Research Quarterly (reference 5)

2. APPROXIMATE FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.Q.T.) IN THE STANDARD SYSTEM

$$\underline{GI / M / r}$$

2.1 INTRODUCTION

2.1.1 Brief description of the system

The system considered can be described as follows: there are r servers in parallel each having an exponential service time distribution, $1 - \exp(-\mu t)$. Intervals between successive arrivals are independently distributed with distribution function $A(u)$ such that there is zero probability that two or more customers arrive simultaneously. The average rate of arrival of customers is λ and the coefficient of variation of inter-arrival times $v_a = \sqrt{\text{var}(u)} \lambda$, such that $0 < v_a < 1$. Customers form into a single queue assigned to the first available server. The system is in statistical equilibrium, i.e. the traffic intensity $\rho = \lambda/r\mu < 1$.

2.1.2 On the derivation of analytical results for the A.Q.T.

The equilibrium theory of this type of queue is discussed by Kendall¹⁸, Winston⁴⁷ and Cox and Smith⁸. The theory was applied in order to obtain expressions for the A.Q.T.; these expressions for up to $r = 6$ are given in Appendix A2.1. It is interesting to note that the formulae for the A.Q.T. become increasingly complicated as r increases; and, also, that the evaluation of the formulae does not seem easy: in most cases numerical methods have to be used.

If the distribution of inter-arrival times is Erlang (see

Appendix A1.1) the alternative method of solution discussed by Morse³² is applicable, though the algebra required becomes quite tedious as r and the parameter k of the Erlang distribution increase. Appendix A2.2 gives the formulae for the A.Q.T. in the special case $k = 2$ for $r = 1, 2, 3, 4, 5, 6, 8$ and 10.

2.1.3 Alternative ways of obtaining results for the A.Q.T. (the practitioner's viewpoint)

In view of the major difficulties encountered when trying to apply the theory to practical problems, the practitioner may find adequate for the purpose of his study and therefore be content with:

- a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude with all other errors incurred due to the assumption that the queueing model $GI/M/r$ reflects the real life waiting line situation under study. An example of such a formula is the one suggested by Page³⁵ for use in the system $E_k/M/r$.
- b) The determination of bounds on the A.Q.T., provided that they are adequately tight for the purposes of the study. Kingman²², for example, studied the behaviour of the queue $GI/M/r$ under conditions of heavy traffic and obtained an extremely simple approximate formula for the average queueing time; a formula that gives an upper bound under unsaturated conditions. Brumelle¹, extending Marshall's²⁹ results, Kingman²³ and Köllerström²⁴ obtained simple bounds on the average queueing time for the more general system $GI/G/r$. Finally, Rolski³⁹ and Brumelle² obtained bounds

on the A.Q.T. for the system GI/M/r which are tighter than the ones derived from references 1,23 or 24 .

- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be emphasized, however, that simulation is not in itself a panacea: apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections 2.2, 2.3 and 2.4, approximate formulae for the A.Q.T. are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is based on the idea (introduced in Chapter 1) that two queueing systems of the same general form GI/M/r with:

- the same type of distribution of inter-arrival times;
- equal coefficients of variation of inter-arrival times;
- equal traffic intensities; but
- unequal number of servers,

have a similar behaviour. Due to the special structure of GI/M/r systems, the establishment of an approximate value for the A.Q.T. allows the establishment of approximate results for the queue-size and queueing-time distributions. These results are outlined in the last section of this chapter.

2.2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

2.2.1 Establishment of the basic structure of the formulae

Let $W(A(u)/M/r)$ denote the A.Q.T. of an arriving customer. One can reasonably expect that $\mu W(A(u)/M/r)$ depends on ρ , r and $A(u)$ only; furthermore, that for ρ , r and the type of $A(u)$ fixed, $\mu W(A(u)/M/r)$ is an increasing function of v_a with limiting values $\mu W(D/M/r)$, corresponding to the system $D/M/r$, if $v_a = 0$ and, say, $\mu W(A(u), v_a=1/M/r)$, corresponding to the system $A(u), v_a=1/M/r$, if $v_a = 1$.

Two such relationships between $\mu W(A(u)/M/r)$ and v_a are shown in Figure 2.1. They are both related to a queueing system of the general form $A(u)/M/r$ with the same type of distribution of inter-arrival times. Curve (1) represents a situation in which $\lambda = \lambda_i$, $\mu = \mu_i$ and $r = r_i$; curve (2) represents

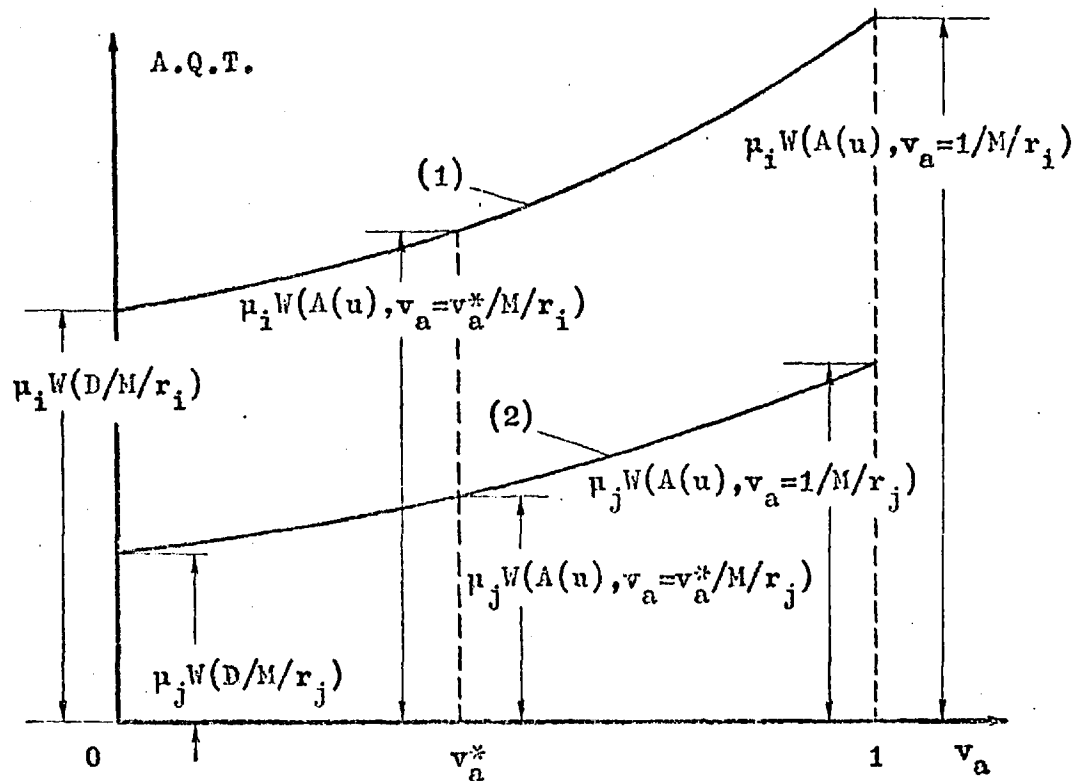


Fig. 2.1 $\mu W(A(u)/M/r)$ as a function of v_a for two queueing systems with the same type of distribution of inter-arrival times and equal traffic intensities.

a situation in which $\lambda = \lambda_j$, $\mu = \mu_j$ and $r = r_j > r_i$, such, however, that $\rho_i = \rho_j = \rho$.

We now proceed in writing a relationship of the general form:

$$\frac{p_j W(A(u), v_a = v_a^* / M / r_j)}{p_i W(A(u), v_a = v_a^* / M / r_i)} \approx S_a \frac{p_j W(A(u), v_a = 1 / M / r_j)}{p_i W(A(u), v_a = 1 / M / r_i)} + (1 - S_a) \frac{p_j W(D / M / r_j)}{p_i W(D / M / r_i)} \quad (2.1)$$

with v_a^* any value of v_a between 0 and 1 and S_a some "similarity factor", with limiting values 0 and 1, for $v_a^* = 0$ and 1 respectively, regardless of r_j , r_i , ρ or $A(u)$.

(2.1) is an expression giving $W(A(u), v_a = v_a^* / M / r_j)$ in terms of $W(A(u), v_a = v_a^* / M / r_i)$; consequently, if the relative percentage errors incurred are small,

a) the hypothesis that the systems $A(u), v_a = v_a^* / M / r_j$ and $A(u), v_a = v_a^* / M / r_i$ with $\rho_j = \rho_i$ have a similar behaviour would be established, the quantity on the right-hand side being regarded as the similarity ratio; and

b) (2.1) could be used as an approximate formula for the evaluation of $W(A(u), v_a = v_a^* / M / r_j)$ provided that $W(A(u), v_a = v_a^* / M / r_i)$ with $r_i = r_j - 1, r_j - 2, \dots$ or even 1 and the similarity ratio were both known.

2.2.2 A method for the determination of the similarity factor S_a

The method suggested here has three main advantages:

- a) it is simple in its application (provided that it can be applied);
- b) it provides a very simple expression for S_a ; and
- c) it is justified in terms of a) and b) of paragraph 2.2.1, particularly if the value of ρ is high, a situation likely to be encountered in practice and, also, one in which numerical results are usually difficult to obtain either analytically or by simulation.

It consists of two steps:

- a) the approximate formulae, for the average queueing times in the basic expression (2.1) under conditions of heavy traffic, are found.

For the system considered:

$$(pW(A(u), v_a = v_a^*/M/r))_{\rho \rightarrow 1} \approx \frac{(v_a^*)^2 + \rho^2}{2\rho(1-\rho)r} - \rho dW,$$

as suggested by Kingman²², valid for $0 < v_a^* < 1$;

- b) these formulae are then introduced into (2.1); simple algebra yields the desired expression for S_a .

For the system considered:

$$S_a = \frac{2(v_a^*)^2}{1 + (v_a^*)^2}$$

in which case (2.1) takes the form:

$$\begin{aligned} \frac{W(A(u), v_a = v_a^*/M/r_j)}{W(A(u), v_a = v_a^*/M/r_i)} &\approx \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(A(u), v_a = 1/M/r_j)}{W(A(u), v_a = 1/M/r_i)} + \\ &+ \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \end{aligned} \quad (2.2)$$

The method described is not the only one possible. An alternative method is presented in Chapter 10 and many more could be

envisaged.

2.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

2.3.1 The queuing system $\Gamma / M / r$

Consider first the case in which inter-arrival times have the gamma distribution (the gamma distribution is discussed in Appendix A1.1.1). For this case, $W(\Lambda(u), v_a = 1/M/r) = W(M/M/r)$, corresponding to the system with exponential inter-arrival times. Consequently (2.2) can be written:

$$\frac{W(\Gamma, v_a = v_a^*/M/r_j)}{W(\Gamma, v_a = v_a^*/M/r_i)} \approx \frac{2(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(M/M/r_j)}{W(M/M/r_i)} + \frac{1-(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \quad (2.3)$$

We now proceed in evaluating formula (2.3) for $2 \leq r_j \leq 6$, $1 \leq r_i < r_j - 1$, $0 < v_a^* < 1$ and $0 < \rho < 1$. Analytical values for $W(\Gamma/M/r)$ are given in Tables A3.3, A3.4, A3.5, A3.6, A3.7 and A3.8 of Appendix 3, the formulae used in the calculations being the ones shown in Appendix A2.1. Analytical values for $W(M/M/r)$ are given in Table A3.1. Finally, analytical results for $W(D/M/r)$ are presented in Table A3.2

a) Evaluation of formula (2.3) with $r_i = 1$

- For a given value of v_a^* , any value of the A.Q.T.

$W(\Gamma, v_a = v_a^*/M/r_j)$ can be approximated with a relative percentage error:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{th. val.}),$$

which:

is positive, thus implying that formula (2.3) provides an upper bound on the A.Q.T. to be estimated;

decreases as the A.Q.T. to be estimated and, hence, the traffic intensity increase, tending to 0 as $\rho \rightarrow 1$; and

is not very sensitive to the value of r_j .

Table 2.1 gives the relative percentage errors incurred if $r_i = 1$ is used in formula (2.3) for the evaluation

TABLE 2.1

Relative percentage errors incurred if (2.3) with $r_i=1$ is used for the evaluation of $W(\Gamma, v_a=0.5/M/r_j)$

r_j	2	3	4	5	6
$\mu W(\Gamma, v_a=0.5/M/r_j) = 0.1$	1.9	2.5	2.7	2.8	2.7
$\mu W(\Gamma, v_a=0.5/M/r_j) = 0.2$	1.2	1.5	1.6	1.6	1.6
$\mu W(\Gamma, v_a=0.5/M/r_j) = 0.5$	0.7	0.7	0.7	0.7	0.6

of $W(\Gamma, v_a=0.5/M/r_j)$ for $r_j = 2, 3, 4, 5$ and 6 and $\mu W(\Gamma, v_a=0.5/M/r_j) = 0.1, 0.2$ and 0.5 : an A.Q.T. of as small as $0.1/\mu$ can be approximated with a relative percentage error smaller than 3 per cent, at least for $r_j \leq 6$.

- For a given value of r_j , any value of $W(\Gamma, v_a=v_a^*/M/r_j)$ can be approximated with a relative percentage error which varies with v_a^* as originally expected: it tends to 0 as v_a^* approaches 0 or 1 and attains a maximum value somewhere in-between. Table 2.2 gives the relative percentage errors incurred if $r_i = 1$ is used in formula (2.3) for the evaluation of an A.Q.T. $\mu W(\Gamma, v_a=v_a^*/M/r_j)$ equal to 0.2 for $r_j = 2, 3, 4, 5$ and 6 and $v_a^* = 0 \dots 1$ in steps of 0.1.

b) Evaluation of formula (2.3) with $r_i = 2$

All conclusions stated in paragraph a) above hold: the relative percentage errors incurred are, however, smaller (see, f.e., Tables 2.3 and 2.4 which are equivalent to Tables 2.1 and 2.2 respectively).

TABLE 2.2

Relative percentage errors incurred if (2.3) with $r_i=1$ is used for the evaluation of an A.Q.T.

$\mu W(\Gamma, v_a = v_a^*/M/r_j)$ equal to 0.2

v_a^*	$r_j = 2$	$r_j = 3$	$r_j = 4$	$r_j = 5$	$r_j = 6$
0	0	0	0	0	0
0.1	0.1	0.2	0.2	0.2	0.1
0.2	0.3	0.4	0.4	0.4	0.4
0.3	0.5	0.6	0.7	0.7	0.7
0.4	0.8	1.1	1.2	1.2	1.1
0.5	1.2	1.5	1.6	1.6	1.6
0.6	1.5	1.8	1.9	1.9	1.9
0.7	1.6	1.9	2.0	1.9	1.9
0.8	1.4	1.6	1.7	1.7	1.7
0.9	0.9	1.1	1.2	1.1	1.1
1.0	0	0	0	0	0

TABLE 2.3

Relative percentage errors incurred if (2.3) with $r_i=2$ is used for the evaluation of $W(\Gamma, v_a=0.5/M/r_j)$

r_j	2	3	4	5	6
$\mu W(\Gamma, v_a=0.5/M/r_j) = 0.1$	0	0.6	1.1	1.4	1.5
$\mu W(\Gamma, v_a=0.5/M/r_j) = 0.2$	0	0.4	0.7	0.8	0.9
$\mu W(\Gamma, v_a=0.5/M/r_j) = 0.5$	0	0.2	0.3	0.4	0.4

TABLE 2.4

Relative percentage errors incurred if (2.3) with $r_i=2$ is used for the evaluation of an A.Q.T.

$\mu W(\Gamma, v_a = v_a^*/M/r_j)$ equal to 0.2

v_a^*	$r_j = 2$	$r_j = 3$	$r_j = 4$	$r_j = 5$	$r_j = 6$
0	0	0	0	0	0
0.1	0	0.0	0.1	0.1	0.1
0.2	0	0.1	0.2	0.2	0.2
0.3	0	0.2	0.3	0.4	0.4
0.4	0	0.3	0.5	0.6	0.7
0.5	0	0.4	0.7	0.8	0.9
0.6	0	0.5	0.8	0.9	1.0
0.7	0	0.5	0.9	1.0	1.1
0.8	0	0.4	0.7	0.8	0.9
0.9	0	0.2	0.5	0.6	0.6
1.0	0	0	0	0	0

c) Evaluation of formula (2.3) with $r_i = r_j - 1$

- For a given value of v_a^* , any value of $W(\Gamma, v_a = v_a^*/M/r_j)$ can be approximated with a relative percentage error which decreases very rapidly as r_j increases (see f.e. Table 2.5 and compare it with Tables 2.1 and 2.3).

- For a given value of v_a^* , any value of the total A.Q.T. per unit time $\lambda W(\Gamma, v_a = v_a^*/M/r_j)$ can be approximated with a relative percentage error which also decreases quite rapidly as r_j increases. Table 2.6 gives the relative percentage errors incurred if (2.3) with $r_i = r_j - 1$ is

TABLE 2.5

Relative percentage errors incurred if (2.3) with $r_i = r_j - 1$ is used for the evaluation of $W(\Gamma, v_a = 0.5/M/r_j)$

r_j	2	3	4	5	6
$\mu W(\Gamma, v_a = 0.5/M/r_j) \approx 0.1$	1.9	0.6	0.3	0.2	0.1
$\mu W(\Gamma, v_a = 0.5/M/r_j) \approx 0.2$	1.2	0.4	0.2	0.1	0.1
$\mu W(\Gamma, v_a = 0.5/M/r_j) \approx 0.5$	0.7	0.2	0.1	0.1	0.1

used for the evaluation of $W(\Gamma, v_a = 0.5/M/r_j)$ for $r_j = 2, 3, 4, 5$ and 6 and $\lambda W(\Gamma, v_a = 0.5/M/r_j) = 0.5$ and 1.5 . These two values correspond to $\rho = 0.66$ and 0.79 , if $r_j = 2$ and to $\rho = 0.74$ and 0.83 , if $r_j = 6$ respectively; or, in other words, to $\mu W(\Gamma, v_a = 0.5/M/r_j) = 0.38$ and 0.95 , if $r_j = 2$ and to $\mu W(\Gamma, v_a = 0.5/M/r_j) = 0.11$ and 0.30 , if $r_j = 6$ respectively.

TABLE 2.6

Relative percentage errors incurred if (2.3) with $r_i = r_j - 1$ is used for the evaluation of $W(\Gamma, v_a = 0.5/M/r_j)$

r_j	2	3	4	5	6
$\lambda W(\Gamma, v_a = 0.5/M/r_j) = 0.5$	0.9	0.4	0.2	0.1	0.1
$\lambda W(\Gamma, v_a = 0.5/M/r_j) = 1.5$	0.4	0.2	0.1	0.1	0.1

- For $r_i = r_j - 1$, (2.3) becomes a recursive formula. Consequently, if $W(\Gamma, v_a = v_a^*/M/r)$ with $r = r_i$ is known, one can obtain approximate results for the system $\Gamma, v_a = v_a^*/M/r$ with $r = r_j > r_i$ not only by using formula (2.3) directly but also by using formula (2.4):

$$W(I, v_a = v_a^*/M/r_j) \approx \prod_{k=r_i+1}^{r_j} \left[\frac{2(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(M/M/k)}{W(M/M/k-1)} + \frac{1-(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(D/M/k)}{W(D/M/k-1)} \right] W(I, v_a = v_a^*/M/r_i) \quad (2.4)$$

Formula (2.4) is more complicated than (2.3); the relative percentage errors incurred are, however, considerably smaller.

d) An example

Suppose that one is faced with the problem of determining the A.Q.T. of an arriving customer in a situation where:

- arrivals occur by an Erlang-2 distribution;
- service times are exponentially distributed;
- the number of servers is 10; and
- the traffic intensity is equal to 0.80.

Suppose, further, that, for one reason or another, no analytical model is available, which implies that the desired queueing time (given in Table A3.9 of Appendix 3) $\mu W(E_2/M/10) = 0.1276$ cannot be determined analytically.

One could, evidently, at this point approximate the desired value of 0.1276 by simulation. If, however, the A.Q.T. $\mu W(E_2/M/r)$ for $\rho = 0.80$ and $r = \text{any value } r_i < 10$ is known, the two formulae (2.3) and (2.4) give very good approximations indeed, provided that Tables for $\mu W(M/M/r)$ and $\mu W(D/M/r)$ are available (the values of $\mu W(D/M/r)$ for $r = 8$ and $r = 10$ which are required in the calculations were obtained from Page 35). Table 2.7 gives the relative percentage errors incurred as a function

of r_i . Even if only $\mu W(E_2/M/1)$ is known, the relative percentage error incurred in evaluating $\mu W(E_2/M/10)$ is as low as 2.5 or 1.3 per cent depending on whether formula (2.3) or (2.4) is used.

TABLE 2.7

Relative percentage errors incurred in evaluating

$\mu W(E_2/M/10)$ — a numerical example

r_j	Use of (2.3)	Use of (2.4)
1	2.5	1.3
2	1.8	0.9
3	1.3	0.7
4	0.9	0.5
5	0.7	0.4
6	0.5	0.3
8	0.2	0.1

2.3.2 The general case

The four distributions of inter-arrival times summarized in Appendices A1.1.3, A1.1.4, A1.1.5 and A1.1.6 are now considered. These distributions were selected in such a way so as to ensure that analytical results could be fairly easily obtained and if possible for $0 < v_a < 1$. Consequently they are most unlikely to represent any real system, except perhaps for a few values of x and y .

Since $v_a = 1$ is achieved for $x = 0$ and $y \rightarrow \infty$, instead of evaluating formula (2.2), we proceed in evaluating the slightly modified formula (2.5) :

$$\begin{aligned}
 \frac{W(\Lambda(u), v_a = v_a^*/M/r_j)}{W(\Lambda(u), v_a = v_a^*/M/r_i)} &\approx \frac{2(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(\Lambda(u), v_a = 1/\sqrt{1.02}/M/r_j)}{W(\Lambda(u), v_a = 1/\sqrt{1.02}/M/r_i)} + \\
 &+ \frac{1-(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \quad (2.5)
 \end{aligned}$$

Some of the analytical results for $pW(\Lambda(u), v_a = v_a^*/M/r_j)$ required in the calculations are tabulated in Appendix 3. In particular, Tables A3.10 to A3.15 give values of $pW(\sqcup/M/r)$ and $pW(\lrcorner/M/r)$ for $1 \leq r \leq 6$ and $0 < v_a^* < 1$; Tables A3.16 to A3.21 give values of $pW(\sqcup/M/r)$ and $pW(\lrcorner/M/r)$, again for $1 \leq r \leq 6$ and $0 < v_a^* < 1$; finally, Tables A3.22 and A3.23 are related to the queuing systems $\sqcup/M/r$ and $\lrcorner/M/r$ with $1 \leq r \leq 6$.

The approximations achieved in evaluating formula (2.5) - see f.e. Tables 2.8 and 2.9 - were all found to be as good as the ones stated in the special case of gamma arrivals already analysed in section 2.3.1: both formula (2.1) with $S_a = 2(v_a^*)^2/(1 + (v_a^*)^2)$ and the underlying idea that the models $\Lambda(u)/M/r_j$ and $\Lambda(u)/M/r_i$ have a similar behaviour seem justified.

TABLE 2.8

Relative percentage errors incurred if (2.5) with $r_i = 1$ is used for the evaluation of $pW(\lrcorner, v_a = \sqrt{1/6}/M/r_j)$

r_j	2	3	4	5	6
$pW(\lrcorner, v_a = \sqrt{1/6}/M/r_j) = 0.1$	1.9	2.1	2.2	2.1	2.0
$pW(\lrcorner, v_a = \sqrt{1/6}/M/r_j) = 0.2$	1.0	1.1	1.1	1.1	1.0
$pW(\lrcorner, v_a = \sqrt{1/6}/M/r_j) = 0.5$	0.3	0.4	0.4	0.4	0.3

TABLE 2.9

Relative percentage errors incurred if (2.5) with $r_i=1$ is used for the evaluation of $\mu W(\underline{A}, v_a = \sqrt{1/2}/M/r_j)$

r_j	2	3	4	5	6
$\mu W(\underline{A}, v_a = \sqrt{1/2}/M/r_j) = 0.1$	5.3	4.2	3.6	3.3	3.2
$\mu W(\underline{A}, v_a = \sqrt{1/2}/M/r_j) = 0.2$	2.1	1.8	1.8	1.7	1.6
$\mu W(\underline{A}, v_a = \sqrt{1/2}/M/r_j) = 0.5$	0.6	0.6	0.6	0.6	0.6

2.4 A MODIFIED SET OF FORMULAE FOR THE A.Q.T.

2.4.1 An assumption and the formulae

From the analysis so far, it seems justified to conclude that the systems $A(u), v_a = v_a^*/M/r_j$ and $A(u), v_a = v_a^*/M/r_i$ with $\rho_j = \rho_i$ do have a similar behaviour, the similarity ratio being the expression on the right hand side of formula (2.2). Such a conclusion, however, does not seem to be of any practical use if the distribution of inter-arrival times is other than the Erlang. In fact, given a distribution $A(u)$ with $v_a = v_a^*$, it may well be impossible even to define what $A(u)$ with $v_a = 1$ should be, let alone the calculation of $W(A(u), v_a = 1/M/r_j)$. Therefore the assumption (known to be valid under conditions of heavy traffic²²) is made that:

$$\left[C_a = \frac{W(A(u), v_a = 1/M/r_j)}{W(A(u), v_a = 1/M/r_i)} \right] = \left[C_m = \frac{W(M/M/r_j)}{W(M/M/r_i)} \right], \quad (2.6)$$

in which case formula (2.2) takes the much more interesting forms :

$$\frac{W(A(u), v_a = v_a^*/M/r_j)}{W(A(u), v_a = v_a^*/M/r_i)} \simeq \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(M/M/r_j)}{W(M/M/r_i)} +$$

$$+ \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \quad (2.7)$$

and

$$\frac{W(A(u), v_a = v_a^*/M/r_j)}{W(A(u), v_a = v_a^*/M/r_i)} = \prod_{k=r_i+1}^{r_j} \left[\frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(M/M/k)}{W(M/M/k-1)} + \right. \\ \left. + \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(D/M/k)}{W(D/M/k-1)} \right] \quad (2.8)$$

equivalent in all respects to the formulae (2.3) and (2.4).

2.4.2 On the evaluation of the formulae

For all four distributions of inter-arrival times considered in section 2.3.2 it so happens that $C_a > C_m$. In particular, the ratio C_a/C_m was found to be a decreasing function of all r_j , r_i and ρ , tending quite rapidly to the value of 1. In evaluating formula (2.7) instead of (2.2) one would therefore expect that the relative percentage errors incurred

- would not be affected for low values of v_a^* ;
- would be reduced in magnitude for medium values of v_a^* ;
- would become negative for values of v_a^* approaching unity; but
- would not be seriously affected for high values of ρ and r_j regardless of v_a^* .

In fact, it was found that even for $r_j = 2$ and $v_a^* \rightarrow 1$, the chance of underestimating $W(A(u), v_a = v_a^*/M/2)$ by more than 5 per cent occurs only for $\rho < 0.40$.

2.4.3 A numerical example

Consider the distributions of inter-arrival times presented in Appendices A1.1.3, A1.1.4 and A1.1.6 with $x = 0$, $y = 0$. The coefficients of variation of inter-arrival times are respectively equal to $1/\sqrt{6}$, $1/\sqrt{3}$ and $1/\sqrt{2}$. The evaluation of formulae (2.7) and (2.8) with $r_j = 6$, $r_i = 1$ and $\rho = 0.5, 0.6, 0.7, 0.8$ and 0.9 is summarized in Table 2.10.

TABLE 2.10

A numerical example ($r_j = 6$, $r_i = 1$)

Distributions of i-a times	A.Q.Ts and r.p.errors	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
 $v_a^* \approx 0.408$	Th. value	0.0055	0.0214	0.0664	0.1911	0.6401
	Form. (2.7)	0.0064	0.0227	0.0682	0.1934	0.6433
	e (%)	16.3	6.3	2.7	1.2	0.5
	Form. (2.8)	0.0055	0.0216	0.0671	0.1926	0.6426
	e (%)	0.5	1.1	1.1	0.8	0.4
 $v_a^* \approx 0.577$	Th. value	0.0107	0.0336	0.0913	0.2409	0.7613
	Form. (2.7)	0.0120	0.0355	0.0940	0.2448	0.7566
	e (%)	12.3	5.7	3.0	1.6	0.7
	Form. (2.8)	0.0104	0.0337	0.0924	0.2435	0.7660
	e (%)	-2.7	0.2	1.2	1.1	0.6
 $v_a^* \approx 0.707$	Th. value	0.0161	0.0456	0.1155	0.2894	0.8807
	Form. (2.7)	0.0175	0.0477	0.1185	0.2935	0.8869
	e (%)	9.0	4.6	2.6	1.4	0.7
	Form. (2.8)	0.0157	0.0457	0.1165	0.2921	0.8858
	e (%)	-2.4	0.2	1.0	1.0	0.6

Despite the fact that the average queueing time with $r_i = 1$

only is assumed to be known, the relative percentage errors incurred in evaluating $\mu W(A(u), v_a = v_a^*/M/6)$ are in all cases almost negligible; this is particularly true if formula (2.8) is used.

2.5 SOME FURTHER APPROXIMATE EQUILIBRIUM RESULTS FOR THE QUEUEING SYSTEM GI/M/r

It is known from Kendall's work on the system GI/M/r that, conditional on being non-zero, queueing time has an exponential distribution with mean $((1-z_0)r\mu)^{-1}$. z_0 is the only root within the unit circle of the equation:

$$z = \int_0^{\infty} \exp(-(1-z)r\mu u) dA(u) = \int_0^{\infty} \exp(-(1-z)\lambda u/\rho) dA(u) \quad (2.9)$$

and, therefore, it does not depend on the value of r but only on ρ and $A(u)$.

It is also known that the major difficulty encountered when dealing with the system GI/M/r lies in the calculation of the probability P that a customer has to queue; a difficulty which increases quite rapidly with r .

Having, however, determined an approximate value for $W(A(u)/M/r_j)$ and having assumed that $W(A(u)/M/r_i)$ is known, the obvious relation:

$$P = (1-z_0)r_j\mu_j W(A(u)/M/r_j) \quad (2.10)$$

provides an approximate value for P . In fact, if (2.9) can be solved, the value of z_0 is precisely known and, thus, P is determined with a relative percentage error equal to the one by which $W(A(u)/M/r_j)$ was determined. If, on the other hand, only $W(A(u)/M/r_i)$ is known, one can use formula (2.7) or preferably (2.8), set $r_i = 1$ and $r_j = r_i$, find an approximate value for $W(A(u)/M/1)$ and then an approximate value for

z_0 through the simple relation $\rho W(A(u)/M/1) = z_0/(1-z_0)$.

The relative percentage errors incurred in evaluating P are again low.

In a similar way one can derive approximate results for the equilibrium queue-size distribution:

$$p_k = \frac{1 - z_0}{(z_0)^r} \rho^k P \quad (k \geq r)$$

though the relative percentage errors incurred could sometimes become significant for high values of r if the precise value of z_0 is unknown.

It should be remembered that ρ has the same value throughout all these calculations.

If we finally adopt the more complete notation $P(A(u), v_a = v_a^*/M/r)$ for the probability that a customer has to queue in the system $A(u), v_a = v_a^*/M/r$, it is interesting to note that by introducing into (2.7) the expressions for the average queueing times as obtained from (2.10), we get the relationship :

$$\begin{aligned} \frac{P(A(u), v_a = v_a^*/M/r_j)}{P(A(u), v_a = v_a^*/M/r_i)} &= \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{P(M/M/r_j)}{P(M/M/r_i)} + \\ &+ \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{P(D/M/r_j)}{P(D/M/r_i)} \end{aligned} \quad (2.11)$$

(2.11) implies that, at least for the queueing system considered in this Chapter, the underlying idea of similarity in the behaviour of the systems $A(u), v_a = v_a^*/M/r_j$ and $A(u), v_a = v_a^*/M/r_i$ with $\rho_j = \rho_i$ need not be restricted to the average queueing time of an arriving customer and the measures directly related to it.

2.5 CONCLUSIONS

Formula (2.8) and the much simpler one (2.7) seem to give very good approximations for the average queueing time of an arriving customer in the system $GI/M/r$. In selecting the formula to be used and the value to be given to r_1 a compromise between desired accuracy and complexity of calculations has to be made. Both formulae can be simplified further, if the explicit formula for the approximate evaluation of $W(D/M/r)$, which will be presented in Chapter 7, is used. In case v_a^* is not close to unity, the explicit formula for the approximate evaluation of $W(A(u), v_a = v_a^*/M/1)$, which will be discussed in Chapter 8, simplifies the necessary calculations still further.

CHAPTER 3

APPROXIMATE FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME

IN THE STANDARD SYSTEM

$$\underline{M / G / r}$$

An abridged version of this Chapter has been accepted for publication in Operational Research Quarterly (reference 6)

3. APPROXIMATE FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.Q.T.) IN THE STANDARD SYSTEM $M / G / r$

3.1 INTRODUCTION

3.1.1 Brief description of the system

The system considered can be described as follows: there are r servers in parallel dealing with a stream of customers arriving at random at an average rate λ . Each server has an independently and identically distributed service time distribution $B(t)$. The average service time is μ^{-1} and the coefficient of variation of service times $v_s = \sqrt{\text{var}(t)} \mu$, such that $0 < v_s < 1$. Customers form into a single queue assigned to the first available server. The system is in statistical equilibrium, i.e. the traffic intensity $\rho = \lambda/r\mu < 1$.

3.1.2 On the derivation of analytical results for the A.Q.T.

The equilibrium theory of this type of queue does not seem to lead to usable results (see f.e. Syski⁴⁴ and Gross and Harris¹⁴); and the relatively simple numerical method of solution suggested by Descamps⁹ does not seem to give correct results for distributions of service times others than the exponential.

If the distribution of service times is Erlang, the method of solution discussed by Morse³² (together with the application of Rouché's theorem for the generation of additional equations) is applicable, though the algebra required becomes quite tedious as r and the parameter ℓ of the Erlang distribution increase. Appendix A2.3 gives the formulae for the A.Q.T. in

the special case $\ell = 2$, for $r = 1, 2, 3, 4$ and 5 ; and Appendix A2.4 gives the formula for the A.Q.T. in the system $M/E_3/2$. Shapiro⁴², Mayhugh and McCormick³⁰ and Heffer¹⁵ proposed numerical methods of obtaining fairly complete equilibrium results for the system $M/E_\ell/r$; from a practical viewpoint, however, the application of such methods is not always easy.

3.1.3 Alternative ways of obtaining results for the A.Q.T. (the practitioner's viewpoint)

In view of the major difficulties encountered when trying to apply the theory to practical problems, the practitioner may find adequate for the purpose of his study and therefore be content with:

- a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude with all other errors incurred due to the assumption that the queueing model $M/G/r$ reflects the real life waiting line situation under study. An example of such a formula is the one suggested by Page³⁵ for use in the system $M/E_\ell/r$.
- b) The determination of bounds on the A.Q.T., provided that they are adequately tight for the purposes of the study. Bounds on the A.Q.T. for the more general system $GI/G/r$ were obtained by Kingman²³, Brumelle¹ and Köllerström²⁴.
- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be empha-

sized, however, that simulation is not in itself a panacea; apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections 3.2, 3.3 and 3.4, approximate formulae for the A.Q.T. are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is based on the idea (introduced in Chapter 1) that two queueing systems of the same general form $M/G/r$ with:

- the same type of distribution of service times;
- equal coefficients of variation of service times;
- equal traffic intensities; but
- unequal number of servers,

have a similar behaviour.

3.2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

3.2.1 Establishment of the basic structure of the formulae

Let $W(M/B(t)/r)$ denote the A.Q.T. of an arriving customer. One can reasonably expect that $\rho W(M/B(t)/r)$ depends on ρ , r and $B(t)$ only; furthermore, that for ρ , r and the type of $B(t)$ fixed, $\rho W(M/B(t)/r)$ is an increasing function of v_s with limiting values $\rho W(M/D/r)$, corresponding to the system $M/D/r$, if $v_s = 0$ and, say, $\rho W(M/B(t), v_s=1/r)$, corresponding to the system $M/B(t), v_s=1/r$, if $v_s = 1$.

Two such relationships between $\rho W(M/B(t)/r)$ and v_s are shown in Figure 3.1. They are both related to a queueing system of the general form $M/B(t)/r$ with the same type of dis-

tribution of service times. Curve (1) represents a situation in which $\lambda = \lambda_i$, $\mu = \mu_i$ and $r = r_i$; curve (2) represents a situation in which $\lambda = \lambda_j$, $\mu = \mu_j$ and $r = r_j > r_i$, such, however, that $\rho_i = \rho_j = \rho$.

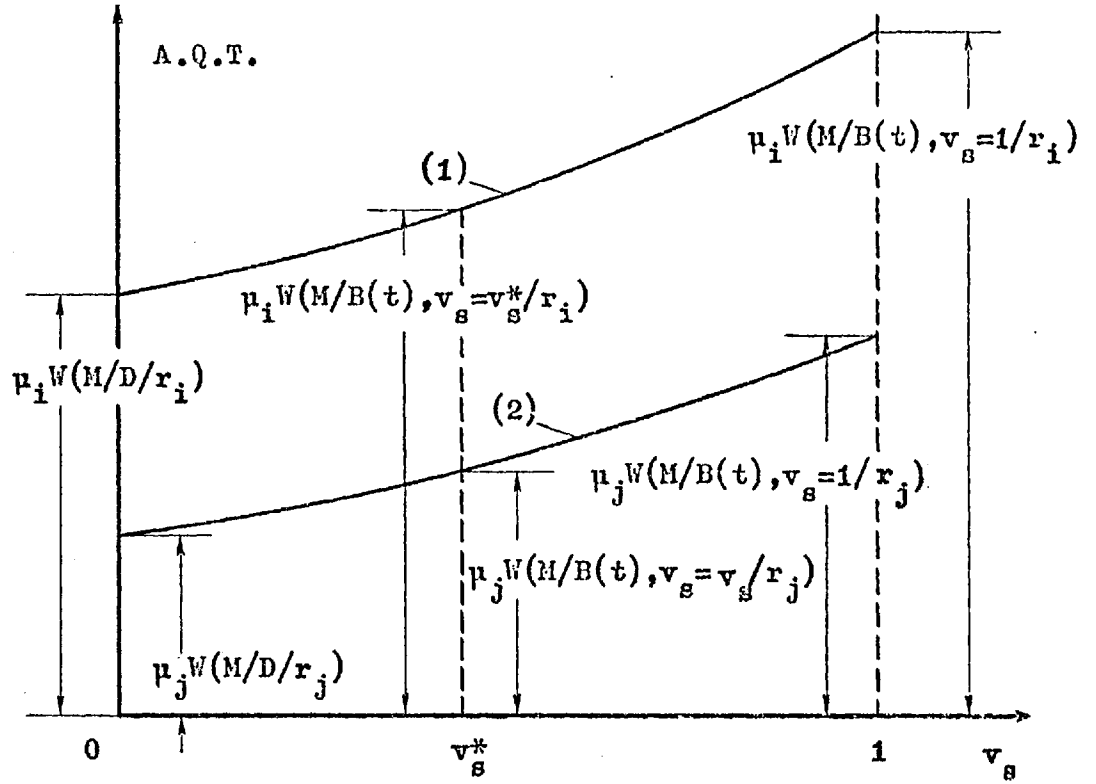


Fig. 3.1 $\mu W(M/B(t)/r)$ as a function of v_s for two queueing systems with the same type of distribution of service times and equal traffic intensities.

We now proceed in writing a relationship of the general form:

$$\frac{\mu_j W(M/B(t), v_s=v_s^*/r_j)}{\mu_i W(M/B(t), v_s=v_s^*/r_i)} \approx S_s \frac{\mu_j W(M/B(t), v_s=1/r_j)}{\mu_i W(M/B(t), v_s=1/r_i)} + (1 - S_s) \frac{\mu_j W(M/D/r_j)}{\mu_i W(M/D/r_i)} \quad (3.1)$$

with v_s^* any value of v_s between 0 and 1 and S_s some "similarity factor", with limiting values 0 and 1, for

$v_s^* = 0$ and 1 respectively, regardless of r_j , r_i , ρ or $B(t)$.

(3.1) is an expression giving $W(M/B(t), v_s = v_s^*/r_j)$ in terms of $W(M/B(t), v_s = v_s^*/r_i)$; consequently, if the relative percentage errors incurred are small,

- a) the hypothesis that the systems $M / B(t), v_s = v_s^* / r_i$ and $M / B(t), v_s = v_s^* / r_j$ with $\rho_i = \rho_j$ have a similar behaviour would be established, the quantity on the right-hand side being regarded as the similarity ratio; and
- b) (3.1) could be used as an approximate formula for the evaluation of $W(M / B(t), v_s = v_s^* / r_j)$ provided that $W(M / B(t), v_s = v_s^* / r_i)$ with $r_i = r_j - 1, r_j - 2, \dots$ or even 1 and the similarity ratio were both known.

3.2.2 A method for the determination of the similarity factor S_s

The method suggested here has three main advantages:

- a) it is simple in its evaluation (provided that it can be applied);
- b) it provides a very simple expression for S_s ; and
- c) it is justified in terms of a) and b) of paragraph 3.2.1, particularly if the value of ρ is high, a situation likely to be encountered in practice and, also, one in which numerical results are usually difficult to obtain either analytically or by simulation.

It consists of two steps:

- a) the approximate formulae, for the average queueing times in the basic expression (3.1) under conditions of heavy traffic, are found.

For the system considered:

$$(\mu W(M/B(t), v_s = v_s^*/r))_{\rho \rightarrow 1} = \frac{1 + \rho^2 (v_s^*)^2}{2\rho(1-\rho)r} - \mu dW,$$

as suggested by Kingman²⁵, valid for $0 < v_s^* < 1$;

b) these formulae are then introduced into (3.1); simple algebra yields the desired expression for S_s .

For the system considered:

$$S_s = \frac{2(v_s^*)^2}{1 + (v_s^*)^2}$$

in which case (3.1) takes the form:

$$\begin{aligned} \frac{W(M/B(t), v_s = v_s^*/r_j)}{W(M/B(t), v_s = v_s^*/r_i)} &\approx \frac{2(v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(M/B(t), v_s = 1/r_j)}{W(M/B(t), v_s = 1/r_i)} + \\ &+ \frac{1 - (v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} \end{aligned} \quad (3.2)$$

The method described is not the only one possible. An alternative method is presented in Chapter 10 and many more could be envisaged.

3.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

3.3.1 The queuing system $M / E_c / r$

Consider first the case in which service times have the Erlang distribution (the Erlang distribution is discussed in Appendix A1.2). For this case, $W(M/B(t), v_s = 1/r) = W(M/M/r)$, corresponding to the system with exponential service times. Consequently (3.2) can be written:

$$\frac{W(M/E, v_s = v_s^*/r_j)}{W(M/E, v_s = v_s^*/r_i)} \approx \frac{2(v_s^*)^2 W(M/M/r_j)}{1+(v_s^*)^2 W(M/M/r_i)} + \frac{1-(v_s^*)^2 W(M/D/r_j)}{1+(v_s^*)^2 W(M/D/r_i)} \quad (3.3)$$

We now proceed in evaluating formula (3.3) for $2 \leq r_j \leq 5$, $1 < r_i < r_j - 1$, $v_s^* = 1/\sqrt{2}$ and $0 < \rho < 1$; and, also, for $r_j = 2$, $r_i = 1$, $v_s^* = 1/\sqrt{3}$ and $0 < \rho < 1$. Analytical values for $W(M/E_2/r)$ and $W(M/E_3/2)$ are given in Table A3.25 of Appendix 3, the formulae used in the calculations being the ones shown in Appendices A2.3 and A2.4 respectively. Analytical values for $W(M/M/r)$ are given in Table A3.1. Finally, analytical results for $W(M/D/r)$ are presented in Table A3.24.

a) Evaluation of formula (3.3) with $r_i = 1$

If we set $r_i = 1$, formula (3.3) is greatly simplified:

$$W(M/E, v_s = v_s^*/r_j) \approx (v_s^*)^2 W(M/M/r_j) + (1-(v_s^*)^2) W(M/D/r_j) \quad (3.4)$$

Formula (3.4) coincides with the one suggested by Page³⁵.

- For $v_s^* = 1/\sqrt{2}$ or, in other words, for $\ell = 2$, any value of the A.Q.T. $W(M/E_2/r_j)$ can be approximated with a relative percentage error:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{th. val.}),$$

which:

is positive or negative depending on the values of the various parameters involved and tends to 0 as $\rho \rightarrow 1$;

is not sensitive to the value of r_j ; and

is very small in absolute value and cannot, therefore, be easily identified if analytical results are not available.

Table 3.1 gives the relative percentage errors incurred if formula (3.4) is used for the evaluation of $W(M/E_2/r_j)$ for $r_j = 2, 3, 4$ and 5 and $\mu W(M/E_2/r_j) = 0.05, 0.1, 0.2$ and 0.5: an A.Q.T. of as small as $0.05/\mu$

can be approximated with a relative percentage error not

TABLE 3.1

Relative percentage errors incurred if (3.4) is
used for the evaluation of $W(M/E_2/r_j)$

r_j	2	3	4	5
$\mu W(M/E_2/r_j) = 0.05$	0.1	-0.1	-0.2	-0.3
$\mu W(M/E_2/r_j) = 0.1$	-0.1	-0.3	-0.4	-0.5
$\mu W(M/E_2/r_j) = 0.2$	-0.2	-0.4	-0.4	-0.5
$\mu W(M/E_2/r_j) = 0.5$	-0.2	-0.3	-0.3	-0.4

exceeding 0.5 per cent in absolute value, at least for
 $r_j \leq 5$.

- For $r_j = 2$, any value of $W(M/E_\ell/2)$ can be approximated with a relative percentage error which seems to vary with v_g^* as originally expected: it tends to 0 as v_g^* approaches 0 or 1 and attains a maximum absolute value somewhere in-between. Table 3.2 gives the relative

TABLE 3.2

Relative percentage errors incurred if (3.4) is
used for the evaluation of an A.Q.T.

$\mu W(M/E_\ell/2)$ equal to 0.2

ℓ	v_g^*	e (%)
∞	0	0
3	$1/\sqrt{3}$	-0.3
2	$1/\sqrt{2}$	-0.2
1	1	0

percentage errors incurred if formula (3.4) is used for
the evaluation of an A.Q.T. $\mu W(M/E_\ell/2)$ equal to 0.2 ,

for $\ell = \infty, 3, 2$ and 1 ($v_g^* = 0, 1/\sqrt{3}, 1/\sqrt{2}$ and 1 respectively).

b) Evaluation of formula (3.3) with $r_i = 2$

All conclusions stated in paragraph a) above hold: the relative percentage errors incurred are, however, smaller (see, f.e., Table 3.3 which is equivalent to Table 3.1).

TABLE 3.3

Relative percentage errors incurred if (3.3) with $r_i = 2$ is used for the evaluation of $W(M/E_2/r_j)$

r_j	2	3	4	5
$\mu W(M/E_2/r_j) = 0.05$	0	-0.1	-0.1	-0.2
$\mu W(M/E_2/r_j) = 0.1$	0	-0.1	-0.2	-0.3
$\mu W(M/E_2/r_j) = 0.2$	0	-0.2	-0.3	-0.3
$\mu W(M/E_2/r_j) = 0.5$	0	-0.1	-0.2	-0.3

c) Evaluation of formula (3.3) with $r_i = r_j - 1$

- For $v_g^* = 1/\sqrt{2}$ or, in other words, for $\ell = 2$, any value of the A.Q.T. $W(M/E_2/r_j)$ can be approximated

TABLE 3.4

Relative percentage errors incurred if (3.3) with $r_i = r_j - 1$ is used for the evaluation of $W(M/E_2/r_j)$

r_j	2	3	4	5
$\mu W(M/E_2/r_j) = 0.05$	0.1	-0.1	-0.1	-0.0
$\mu W(M/E_2/r_j) = 0.1$	-0.1	-0.1	-0.1	-0.1
$\mu W(M/E_2/r_j) = 0.2$	-0.2	-0.1	-0.1	-0.1
$\mu W(M/E_2/r_j) = 0.5$	-0.2	-0.1	-0.1	-0.1

with a relative percentage error which decreases in absolute value as r_j increases (see, f.e., Table 3.4 and compare it with Tables 3.1 and 3.3).

- For $v_g^* = 1/\sqrt{2}$ or, in other words, for $\ell = 2$, any value of the total A.Q.T. per unit time $\lambda W(N/E_2/r_j)$ can be approximated with a relative percentage error which also decreases in absolute value as r_j increases.
- For $r_i = r_j - 1$, (3.3) becomes a recursive formula; this recursive formula is, however, much more complicated and, at least for the cases considered in this section, gives slightly worse approximations than the simple formula (3.4).

3.3.2 The general case

The four distributions of service times summarized in Appendix A1.2 are now considered. These distributions were selected in such a way for no specific reason, apart from a general desire to retain some degree of consistency throughout this presentation: the same distributions were selected as distributions of inter-arrival times when studying the standard system GI/M/r in Chapter 2; and the same distributions will be selected as distributions of both inter-arrival and service times in subsequent Chapters.

For the queueing system considered in this Chapter, no attempt was made to verify the approximate formula (3.2) or the slightly modified formula:

$$\begin{aligned}
\frac{W(M/B(t), v_s = v_s^*/r_j)}{W(M/B(t), v_s = v_s^*/r_i)} &\approx \frac{2(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(M/B(t), v_s = 1/\sqrt{1.02}/r_j)}{W(M/B(t), v_s = 1/\sqrt{1.02}/r_i)} + \\
&+ \frac{1-(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} \quad (3.5)
\end{aligned}$$

as was attempted in section 2.3.2 of Chapter 2 for the GI/M/r system, the reasons being:

- a) there seems to be no easy way of obtaining analytical results for either $W(M/B(t), v_s = v_s^*/r_j)$ and $W(M/B(t), v_s = 1/\sqrt{1.02}/r_j)$ or $W(M/B(t), v_s = v_s^*/r_i)$ and $W(M/B(t), v_s = 1/\sqrt{1.02}/r_i)$, if $r_i > 1$;
- b) possible evaluation of formula (3.5) by simulation seems meaningless: in fact, the error associated with simulation experiments may well prove significant for such a purpose; and
- c) the evaluation of a formula which, in its present form, is of no use to the practitioner, seems to provide no incentive for testing a hypothesis of questionable theoretical importance.

3.4 A MODIFIED SET OF FORMULAE FOR THE A.Q.T.

3.4.1 An assumption and the formulae

For a distribution of service times other than the Erlang, formula (3.2) is of no practical use. In fact, given a distribution $B(t)$ with $v_s = v_s^*$, it may well be impossible even to define what $B(t)$ with $v_s = 1$ should be, let alone the calculation of $W(M/B(t), v_s = 1/r_j)$. Therefore, the assumption (known to be valid under conditions of heavy traffic²³) is made that:

$$\left[c_b = \frac{W(M/B(t), v_s=1/r_j)}{W(M/B(t), v_s=1/r_i)} \right] = \left[c_m = \frac{W(M/M/r_j)}{W(M/M/r_i)} \right] ; \quad (3.6)$$

in which case formula (3.2) takes the much more interesting forms:

$$\begin{aligned} \frac{W(M/B(t), v_s=v_s^*/r_j)}{W(M/B(t), v_s=v_s^*/r_i)} &\approx \frac{2(v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(M/M/r_j)}{W(M/M/r_i)} + \\ &+ \frac{1 - (v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} \end{aligned} \quad (3.7)$$

and, for $r_i = 1$,

$$W(M/B(t), v_s=v_s^*/r_j) \approx (v_s^*)^2 W(M/M/r_j) + (1 - (v_s^*)^2) W(M/D/r_j) \quad (3.8)$$

equivalent in all respects to the two formulae (3.3) and (3.4).

It is worthwhile noting that formula (3.8) can be considered as a generalization of the well known Pollaczek - Khintchine formula:

$$\begin{aligned} W(M/B(t), v_s=v_s^*/1) &= \frac{\rho(1 + (v_s^*)^2)}{2(1 - \rho)\mu} = \\ &= (v_s^*)^2 W(M/M/1) + (1 - (v_s^*)^2) W(M/D/1) \end{aligned}$$

3.4.2 On the evaluation of the formulae

The effect of the assumption (3.6) on the relative percentage errors incurred in evaluating $W(M/B(t), v_s=v_s^*/r_j)$ by using formula (3.7) instead of (3.2) is difficult to assess. As, however, both formulae (3.7) and (3.8) were found to give very good approximations (indeed as good as the ones found in the case of Erlang service times already dealt with in section 3.3.1) it seems reasonable to conclude that the assumption in

(3.6) is justified.

Some of the results for $W(M/B(t), v_g = v_g^*/r_j)$ obtained by simulation are summed up in Table A4.13 of Appendix 4, the design of the simulation experiments being outlined in Appendix A2.6. In all cases studied, the approximate value for $W(M/B(t), v_g = v_g^*/r_j)$, as given in formula (3.8) and calculated by hand in a couple of minutes, lies within the 95% confidence limits which were determined by simulation after 12 or more minutes of CDC 6400 time.

3.4.3 A numerical example

Consider the distributions of service times corresponding to the distributions of inter-arrival times presented in Appendices A113, A114 and A116 with $x = 0$, $y = 0$. The coefficients of variation of service times are respectively equal to $1/\sqrt{6}$, $1/\sqrt{3}$ and $1/\sqrt{2}$. The evaluation of formula (3.8) with $r_j = 6$ and $\rho = 0.60, 0.70, 0.80$ and 0.90 is summarized in Table 3.5. The simulated values were obtained from Table A4.13 of Appendix 4 and the relative percentage error e was calculated as:

$$e = 100(\text{approximate value} - \text{simulated value})/(\text{simul. value})$$

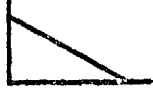
CONCLUSIONS

Formula (3.8) seems to give very good approximations for the A.Q.T. of an arriving customer in the system $M/G/r$. It can be simplified still further, if the explicit formula for the approximate evaluation of $W(M/D/r)$, which will be presented in Chapter 7, is used.

TABLE 3.5

A numerical example

(evaluation of formula 3.8 with $r_j = 6$)

Distributions of service times	Results	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
 $v_s^* = 0.408$	simulated value	0.0521	0.1154	0.2570	0.7235
	CDC 6400 time (min.)	12	16	16	32
	approximate value	0.0525	0.1158	0.2607	0.7307
	e (%)	0.8	0.3	1.4	1.0
 $v_s^* = 0.577$	simulated value	0.0589	0.1328	0.2956	0.8215
	CDC 6400 time (min.)	12	12	12	32
	approximate value	0.0584	0.1300	0.2949	0.8312
	e (%)	-0.8	-2.1	-0.2	1.2
 $v_s^* = 0.707$	simulated value	0.0637	0.1426	0.3329	0.9097
	CDC 6400 time (min.)	12	12	12	32
	approximate value	0.0643	0.1441	0.3290	0.9318
	e (%)	0.9	1.1	-1.2	2.4

CHAPTER 4

APPROXIMATE FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME

IN THE STANDARD SYSTEM

$$\underline{G_1 / G / r}$$

4. APPROXIMATE FORMULAS FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.Q.T.) IN THE STANDARD SYSTEM

$$\underline{GI / G / r}$$

4.1 INTRODUCTION

4.1.1 Brief description of the system

The system considered can be described as follows: there are r servers in parallel. Each server has an independently and identically distributed service time distribution $B(t)$. The average service time is μ^{-1} and the coefficient of variation of service times $v_s = \sqrt{\text{var}(t)} \mu$, such that $0 < v_s < 1$. Intervals between successive arrivals are independently distributed with distribution function $A(u)$ such that there is zero probability that two or more customers arrive simultaneously. The average rate of arrival of customers is λ and the coefficient of variation of inter-arrival times $v_a = \sqrt{\text{var}(u)} \lambda$, such that $0 < v_a < 1$. Customers form into a single queue assigned to the first available server. The system is in statistical equilibrium, i.e. the traffic intensity $\rho = \lambda / r\mu < 1$.

4.1.2 On the derivation of analytical results for the A.Q.T.

The existence of equilibrium results is discussed by Kiefer and Wolfowitz¹⁹. A general theory, due to Pollaczek⁴⁵, is also discussed by Le Gall²⁶, who in particular examines the case $r = 2$ in some detail. From a practical viewpoint, however, the results do not seem easy to handle.

In case the distributions of inter-arrival and service times are both Erlang, the alternative method of solution discussed

by Morse³² (together with the application of Rouché's theorem for the generation of additional equations) is applicable, though the algebra required becomes quite tedious as r and the parameters of the Erlang distributions increase; using this method, Poyntz and Jackson³⁷ determined the complete system of equations and suggested a way of calculating the steady state probabilities.

4.1.3 Alternative ways of obtaining results for the A.Q.T. (the practitioner's viewpoint)

In view of the major difficulties encountered when trying to apply the theory to practical problems, the practitioner may find adequate for the purpose of his study and therefore be content with:

- a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude with all other errors incurred due to the assumption that the queueing model $GI/G/r$ reflects the real life waiting line situation under study. An example of such a formula is the one suggested by Page³⁵ for use in the system $E_k/E_c/r$.
- b) The determination of bounds on the A.Q.T., provided that they are adequately tight for the purpose of the study. Bounds on the A.Q.T. were obtained by Kingman²³, Brumelle¹ and Köllerström²⁴.
- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be empha-

sized, however, that simulation is not in itself a panacea: apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections of this Chapter, approximate formulae for the A.Q.T. are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is based on the idea (introduced in Chapter 1) that two queueing systems of the same general form $GI/G/r$ with:

- the same type of distribution of inter-arrival times;
- equal coefficients of variation of inter-arrival times;
- the same type of distribution of service times;
- equal coefficients of variation of service times;
- equal traffic intensities; but
- unequal number of servers,

have a similar behaviour.

4.2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

4.2.1 Establishment of the basic structure of the formulae

Let $W(A(u)/B(t)/r)$ denote the A.Q.T. of an arriving customer. One can reasonably expect that $pW(A(u)/B(t)/r)$ depends on ρ , r , $A(u)$ and $B(t)$ only; furthermore that for ρ , r , $B(t)$ and the type of $A(u)$ fixed, $pW(A(u)/B(t)/r)$ is an increasing function of v_a with limiting values $pW(D/B(t)/r)$, corresponding to the system $D/B(t)/r$, if $v_a=0$ and, say, $pW(A(u), v_a=1/B(t)/r)$, corresponding to the system $A(u), v_a=1/B(t)/r$, if $v_a=1$.

Two such relationships between $\mu W(\Lambda(u)/B(t)/r)$ and v_a are shown in Figure 4.1. They are both related to a queueing

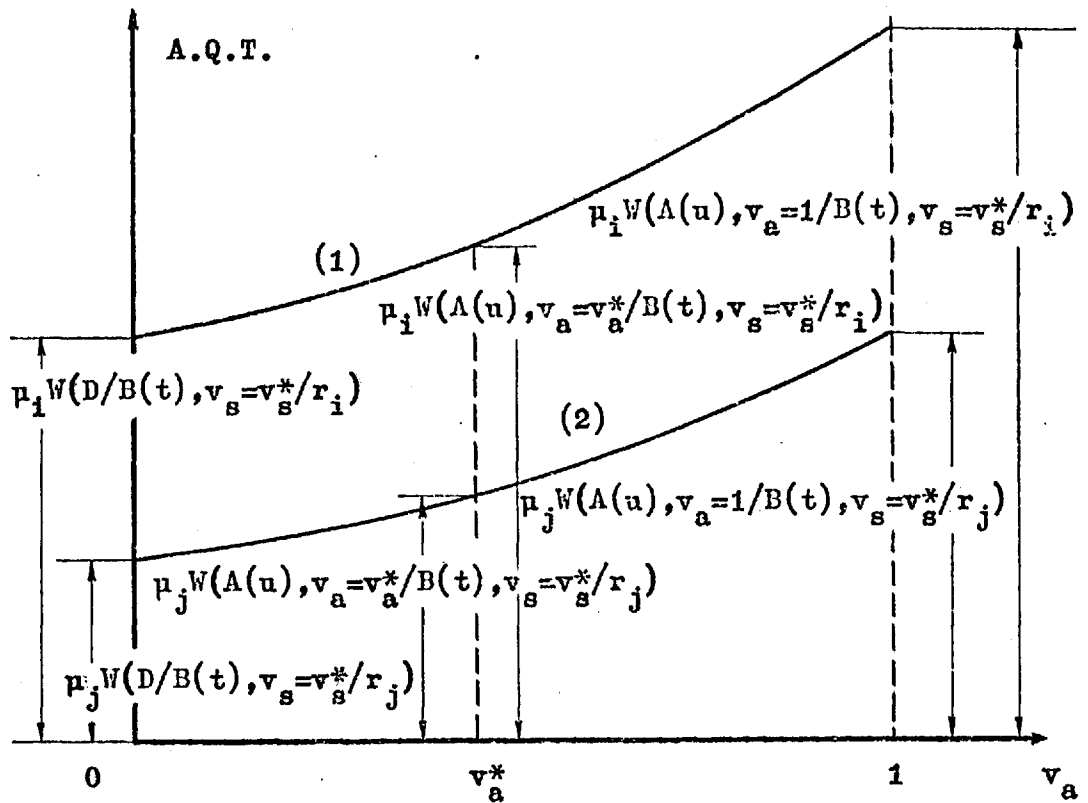


Fig. 4.1 $\mu W(\Lambda(u)/B(t)/r)$ as a function of v_a for two queueing systems with the same type of distribution of service times, equal coefficients of variation of service times, the same type of distribution of inter-arrival times and equal traffic intensities.

system of the general form $\Lambda(u)/B(t)/r$ with:

- the same type of distribution of service times;
- equal coefficients of variation of service times, say v_s^* ; and
- the same type of distribution of inter-arrival times.

Curve (1) represents a situation in which $\lambda = \lambda_i$, $\mu = \mu_i$ and $r = r_i$; curve (2) represents a situation in which $\lambda = \lambda_j$, $\mu = \mu_j$ and $r = r_j > r_i$, such, however, that $\rho_i = \rho_j = \rho$.

We now proceed in writing a relationship of the general form:

$$\frac{p_j W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{p_i W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \approx (1 - S_a) \frac{p_j W(D/B(t), v_s = v_s^*/r_j)}{p_i W(D/B(t), v_s = v_s^*/r_i)} + S_a \frac{p_j W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/r_j)}{p_i W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/r_i)} \quad (4.1)$$

with v_a^* any value of v_a between 0 and 1 and S_a some "similarity factor" with limiting values 0 and 1 for $v_a^* = 0$ and 1 respectively, regardless of r_j , r_i , ρ , $B(t)$ or $\Lambda(u)$.

One can also expect that for ρ , r , $\Lambda(u)$ and the type of $B(t)$ fixed, $pW(\Lambda(u)/B(t)/r)$ is an increasing function of v_s with limiting values $pW(\Lambda(u)/D/r)$, corresponding to the system $\Lambda(u)/D/r$, if $v_s = 0$ and, say, $pW(\Lambda(u)/B(t), v_s = 1/r)$, corresponding to the system $\Lambda(u)/B(t), v_s = 1/r$, if $v_s = 1$. A reasoning similar to the one associated with Figure 4.1 leads to the relationship:

$$\frac{p_j W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{p_i W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \approx (1 - S_s) \frac{p_j W(\Lambda(u), v_a = v_a^*/D/r_j)}{p_i W(\Lambda(u), v_a = v_a^*/D/r_i)} + S_s \frac{p_j W(\Lambda(u), v_a = v_a^*/B(t), v_s = 1/r_j)}{p_i W(\Lambda(u), v_a = v_a^*/B(t), v_s = 1/r_i)} \quad (4.2)$$

with v_s^* any value of v_s between 0 and 1 and S_s some "similarity factor" with limiting values 0 and 1 for $v_s^* = 0$ and 1 respectively, regardless of r_j , r_i , ρ , $\Lambda(u)$ or $B(t)$.

Both (4.1) and (4.2) are expressions giving $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ in terms of

$W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)$; consequently, if the relative percentage errors incurred are small,

- a) the hypothesis that the systems $\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j$ and $\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i$ with $\rho_j = \rho_i$ have a similar behaviour would be established, the quantities on the right hand side being regarded as similarity ratios;
- b) either (4.1) or (4.2) or indeed some convenient combination of them could be used as an approximate formula for the evaluation of $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ provided that $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)$ with $r_i = r_j - 1$, $r_j - 2$, ... or even 1 and the similarity ratios were both known.

4.2.2 A method for the determination of the similarity factors S_a and S_s

The method suggested here has three main advantages:

- a) it is simple in its evaluation (provided that it can be applied);
- b) it provides very simple expressions for S_a and S_s ; and
- c) it is justified in terms of a) and b) of paragraph 4.2.1, particularly if the value of ρ is high, a situation likely to be encountered in practice and, also, one in which numerical results are usually difficult to obtain either analytically or by simulation.

It consists of two steps:

- a) the approximate formulae, for the average queueing times in the basic expressions (4.1) and (4.2) under conditions of heavy traffic, are found.

For the system considered:

$$(pW(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r))_{\rho \rightarrow 1} = \frac{(v_a^*)^2 + \rho^2 (v_s^*)^2}{2\rho(1 - \rho)r} - p.dW$$

as suggested by Kingman²³, valid for $0 < v_a^* < 1$, $0 < v_s^* < 1$.

b) these formulae are then introduced into (4.1) and (4.2); simple algebra yields the desired expressions for S_a and S_s .

For the system considered:

$$S_a = \frac{(v_a^*)^2(1 + (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \quad \text{and} \quad S_s = \frac{(v_s^*)^2(1 + (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2},$$

in which case (4.1) and (4.2) take the forms:

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \approx \\ & \frac{(v_a^*)^2(1 + (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = 1/B(t), v_s = v_s^*/r_i)} + \\ & + \frac{(v_s^*)^2(1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(D/B(t), v_s = v_s^*/r_j)}{W(D/B(t), v_s = v_s^*/r_i)} \end{aligned} \quad (4.3)$$

and

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \approx \\ & \frac{(v_s^*)^2(1 + (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(A(u), v_a = v_a^*/B(t), v_s = 1/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = 1/r_i)} + \\ & + \frac{(v_a^*)^2(1 - (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(A(u), v_a = v_a^*/D/r_j)}{W(A(u), v_a = v_a^*/D/r_i)} \end{aligned} \quad (4.4)$$

respectively.

The method described is not the only one possible. An alternative method is presented in Chapter 10 and many more could be envisaged.

By setting $v_s^* = 0$ in (4.3) and $v_a^* = 0$ in (4.4), we get:

$$\frac{W(A(u), v_a = v_a^*/D/r_j)}{W(A(u), v_a = v_a^*/D/r_i)} \simeq \frac{W(A(u), v_a = 1/D/r_j)}{W(A(u), v_a = 1/D/r_i)} \quad (4.5)$$

and

$$\frac{W(D/B(t), v_s = v_s^*/r_j)}{W(D/B(t), v_s = v_s^*/r_i)} \simeq \frac{W(D/B(t), v_s = 1/r_j)}{W(D/B(t), v_s = 1/r_i)} \quad (4.6)$$

respectively.

Finally, by introducing (4.5) into (4.4), we obtain:

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \simeq \\ & \simeq \frac{(v_s^*)^2(1 + (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(A(u), v_a = v_a^*/B(t), v_s = 1/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = 1/r_i)} + \\ & + \frac{(v_a^*)^2(1 - (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(A(u), v_a = 1/D/r_j)}{W(A(u), v_a = 1/D/r_i)} ; \quad (4.7) \end{aligned}$$

and by introducing (4.6) into (4.3), we obtain:

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \simeq \\ & \simeq \frac{(v_a^*)^2(1 + (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = 1/B(t), v_s = v_s^*/r_i)} + \\ & + \frac{(v_s^*)^2(1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(D/B(t), v_s = 1/r_j)}{W(D/B(t), v_s = 1/r_i)} \quad (4.8) \end{aligned}$$

4.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

4.3.1 The queuing system $E_k / E_c / r$

Consider first the case in which both inter-arrival and service times have the Erlang distribution (the Erlang distribution is discussed in Appendix A1.1.2). For this case, $W(A(u), v_a = 1/D/r) = W(M/D/r)$ and $W(A(u), v_a = 1/B(t), v_s = v_s^*/r) = W(M/E, v_s = v_s^*/r)$, corresponding to the systems with exponential inter-arrival times; and $W(D/B(t), v_s = 1/r) = W(D/M/r)$ and $W(A(u), v_a = v_a^*/B(t), v_s = 1/r) = W(E, v_a = v_a^*/M/r)$, corresponding to the systems with exponential service times. Consequently, (4.7) and (4.8) can be written:

$$\frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r_j)}{W(E, v_a = v_a^*/E, v_s = v_s^*/r_i)} \approx \frac{(v_s^*)^2 (1 + (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(E, v_a = v_a^*/M/r_j)}{W(E, v_a = v_a^*/M/r_i)} + \frac{(v_a^*)^2 (1 - (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} \quad (4.9)$$

and

$$\frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r_j)}{W(E, v_a = v_a^*/E, v_s = v_s^*/r_i)} \approx \frac{(v_a^*)^2 (1 + (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(M/E, v_s = v_s^*/r_j)}{W(M/E, v_s = v_s^*/r_i)} + \frac{(v_s^*)^2 (1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \quad (4.10)$$

respectively.

Finally, by introducing either the approximate expression (2.3) established in Chapter 2 for

$$W(E, v_a = v_a^*/M/r_j) / W(E, v_a = v_a^*/M/r_i)$$

into (4.9) or the approximate expression (3.3) established in Chapter 3 for

$$W(M/E, v_s = v_s^*/r_j) / W(M/E, v_s = v_s^*/r_i)$$

into (4.10), we obtain:

$$\begin{aligned} \frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r_j)}{W(E, v_a = v_a^*/E, v_s = v_s^*/r_i)} &\simeq \frac{2(v_a^*)^2 (v_s^*)^2}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(M/M/r_j)}{W(M/M/r_i)} + \\ &+ \frac{(v_a^*)^2 (1 - (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} + \frac{(v_s^*)^2 (1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \end{aligned} \quad (4.11)$$

In case $(v_a^*)^2 = 1/2$, $(v_s^*)^2 = 1/2$ — the only one for which analytical results were available — the three expressions (4.9) , (4.10) and (4.11) can be written:

$$\frac{W(E_2/E_2/r_j)}{W(E_2/E_2/r_i)} \simeq \frac{3}{4} \frac{W(E_2/M/r_j)}{W(E_2/M/r_i)} + \frac{1}{4} \frac{W(M/D/r_j)}{W(M/D/r_i)} \quad (4.12)$$

$$\frac{W(E_2/E_2/r_j)}{W(E_2/E_2/r_i)} \simeq \frac{3}{4} \frac{W(M/E_2/r_j)}{W(M/E_2/r_i)} + \frac{1}{4} \frac{W(D/M/r_j)}{W(D/M/r_i)} \quad (4.13)$$

and

$$\begin{aligned} \frac{W(E_2/E_2/r_j)}{W(E_2/E_2/r_i)} &\simeq \frac{1}{2} \frac{W(M/M/r_j)}{W(M/M/r_i)} + \frac{1}{4} \frac{W(M/D/r_j)}{W(M/D/r_i)} + \\ &+ \frac{1}{4} \frac{W(D/M/r_j)}{W(D/M/r_i)} \end{aligned} \quad (4.14)$$

respectively.

An example (the queueing system $E_2/E_2/2$)

The evaluation of (4.12) , (4.13) and (4.14) is summarized in Table 4.1 : Analytical values for $W(E_2/E_2/2)$ were calculated by using the formula presented in Appendix A2.5. Analytical values for $W(E_2/E_2/1)$ were obtained by applying the

method discussed by Cox and Smith⁸; some of the values for $\rho = 0.30, \dots, 0.90$ in steps of 0.05 are included in Table A3.26 of Appendix 3. Analytical values for $W(E_2/M/1)$ and $W(E_2/M/2)$ are given in Tables A3.3 and A3.4 of Appendix 3, the formulae used in the calculations being the ones presented in Appendix A2.1. Analytical values for $W(M/E_2/2)$ are included in Table A3.25 of Appendix 3, the formula used in the calculations being the one presented in Appendix A2.3 for $r = 2$.

TABLE 4.1

An example: Evaluation of formulae (4.12), (4.13) and (4.14) in the special case $r_i = 1$, $r_j = 2$

ρ	A.Q.T. (analyt. results)	Use of form. (4.12)		Use of form. (4.13)		Use of form. (4.14)	
		A.Q.T.	e	A.Q.T.	e	A.Q.T.	e
0.40	0.057608	0.061673	7.1	0.062898	9.2	0.062805	9.0
0.45	0.083735	0.088268	5.4	0.089703	7.1	0.089545	6.9
0.50	0.118060	0.122982	4.2	0.124643	5.6	0.124408	5.4
0.55	0.163084	0.168319	3.2	0.170223	4.4	0.169901	4.2
0.60	0.222564	0.228044	2.5	0.230203	3.4	0.229784	3.2
0.65	0.302419	0.308084	1.9	0.310507	2.7	0.309984	2.5
0.70	0.412534	0.418332	1.4	0.421027	2.1	0.420393	1.9
0.75	0.570734	0.576618	1.0	0.579588	1.6	0.578839	1.4
0.80	0.812701	0.818633	0.7	0.821880	1.1	0.821011	1.0
0.85	1.221738	1.227686	0.5	1.231206	0.8	1.230217	0.7
0.90	2.047816	2.053748	0.3	2.057545	0.5	2.056429	0.4
0.95	4.540909	4.546842	0.1	4.550874	0.2	4.549639	0.2

Finally, analytical results for $W(M/M/r)$, $W(M/D/r)$ and $W(D/M/r)$ with $r = 1, 2$ are presented in Tables A3.1, A3.24 and A3.2 respectively. The relative percentage error e is defined as:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{th. value})$$

Of particular interest to the practitioner is the easily applicable formula (4.14). We therefore proceed in evaluating formula (4.14) only — though formula (4.12) gives better approximations than (4.14) — for $r_j \leq 6$, $1 < r_i < r_j - 1$, $[(v_a^*)^2 = (v_s^*)^2 = 1/2]$ and $0 < \rho < 1$. The analytical values for $W(E_2/E_2/r)$, $r > 2$, included in Table 4.2 were obtained from Page³⁵; and the analytical values for $W(E_2/E_2/1)$, included in the same Table, from Table A3.26 of Appendix 3. Finally, the

TABLE 4.2

Analytical values for $\mu W(E_2/E_2/r)$

ρ	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
0.30	0.130952	0.0235	0.0062			
0.40	0.235457	0.0576	0.0205	0.0085		
0.50	0.390388	0.1181	0.0512	0.0253	0.0142	0.0082
0.60	0.630647	0.2222	0.1103	0.0639	0.0400	0.0265
0.70	1.039055	0.4125	0.2275	0.1441	0.0988	0.0712

analytical values of $W(M/M/r)$, $W(M/D/r)$ and $W(D/M/r)$ required in the calculations were obtained from Tables A3.1, A3.24 and A3.2 of Appendix 3.

a) Evaluation of formula (4.14) with $r_i = 1$

Table 4.3 gives the relative percentage errors incurred if formula (4.14) with $r_i = 1$ is used for the approximate evaluation of $W(E_2/E_2/r_j)$. Table 4.4 gives the relative percentage errors incurred if formula (4.14) with $r_i = 1$ is used for the evaluation of an A.Q.T.

$\mu W(E_2/E_2/r_j)$ equal to 0.05 and 0.10.

TABLE 4.3

Relative percentage errors incurred if (4.14) with $r_i = 1$ is used for the evaluation of $W(E_2/E_2/r_j)$

ρ	$r_j=2$	$r_j=3$	$r_j=4$	$r_j=5$	$r_j=6$
0.30	16.0	41.5			
0.40	9.0	20.7	36.7		
0.50	5.4	11.3	20.9	26.9	37.4
0.60	3.2	6.9	9.9	13.7	17.5
0.70	1.9	3.7	5.5	7.3	9.2

TABLE 4.4

Relative percentage errors incurred if (4.14) with $r_i = 1$ is used for the evaluation of $W(E_2/E_2/r_j)$

r_j	2	3	4	5	6
$\mu W(E_2/E_2/r_j) = 0.05$	10	12	12	12	12
$\mu W(E_2/E_2/r_j) = 0.10$	6	8	7	7	7

b) Evaluation of formula (4.14) with $r_i = 2$

If formula (4.14) with $r_i = 2$ is used for the approximate evaluation of $W(E_2/E_2/r_j)$, the relative percentage errors incurred are smaller than the ones incurred if (4.14) with $r_i = 1$ is used instead (see, f.e., Tables 4.5 and 4.6 and compare them with Tables 4.3 and 4.4 respectively).

TABLE 4.5

Relative percentage errors incurred if (4.14) with $r_i = 2$ is used for the evaluation of $W(E_2/E_2/r_j)$

ρ	$r_j=2$	$r_j=3$	$r_j=4$	$r_j=5$	$r_j=6$
0.30	0	13.9			
0.40	0	7.4	19.1		
0.50	0	4.2	9.8	16.7	25.5
0.60	0	3.0	5.2	8.4	11.6
0.70	0	2.0	3.1	4.7	6.4

TABLE 4.6

Relative percentage errors incurred if (4.14) with $r_i = 2$ is used for the evaluation of $W(E_2/E_2/r_j)$

r_j	2	3	4	5	6
$\rho W(E_2/E_2/r_j) = 0.05$	0	4	6	7	8
$\rho W(E_2/E_2/r_j) = 0.10$	0	3	4	5	5

c) Evaluation of formula (4.14) with $r_i = r_j - 1$

- If formula (4.14) with $r_i = r_j - 1$ is used for the approximate evaluation of $W(E_2/E_2/r_j)$, the relative percentage errors incurred are smaller than the ones incurred if (4.14) with $r_i = 2$ is used instead (provided that $r_j > 3$); these relative percentage errors tend quite rapidly to 0 as r_j increases (see, f.e., Tables 4.7 and 4.8 and compare them with Tables 4.5 and 4.6 respectively; the noticeable discontinuity occurring at $\rho=0.50$, $r_j=4$ should be attributed to an uncorrect analytical value of the average queueing time).

TABLE 4.7

Relative percentage errors incurred if (4.14) with $r_i = r_j - 1$ is used for the evaluation of $W(E_2/E_2/r_j)$

ρ	$r_j=2$	$r_j=3$	$r_j=4$	$r_j=5$	$r_j=6$
0.30	16.0	13.9			
0.40	9.0	7.4	7.7		
0.50	5.4	4.2	6.3	1.9	5.0
0.60	3.2	3.0	1.9	2.2	1.9
0.70	1.9	2.0	1.4	1.4	1.3

TABLE 4.8

Relative percentage errors incurred if (4.14) with $r_i = r_j - 1$ is used for the evaluation of $W(E_2/E_2/r_j)$

r_j	2	3	4	5	6
$\mu W(E_2/E_2/r_j) = 0.05$	10	4	2	2	1
$\mu W(E_2/E_2/r_j) = 0.10$	6	3	2	1	1

- For $r_i = r_j - 1$, (4.14) becomes a recursive formula. Consequently, if $W(E_2/E_2/r)$ with $r = r_i$ is known, one can obtain approximate results for the system $E_2/E_2/r$ with $r = r_j > r_i$ not only by using formula (4.14) directly but also by using formula (4.15):

$$\frac{W(E_2/E_2/r_j)}{W(E_2/E_2/r_i)} \approx \prod_{k=r_i+1}^{r_j} \left[\frac{1}{2} \frac{W(M/M/k)}{W(M/M/k-1)} + \frac{1}{4} \frac{W(M/D/k)}{W(M/D/k-1)} + \frac{1}{4} \frac{W(D/M/k)}{W(D/M/k-1)} \right] \quad (4.15)$$

Formula (4.15) is more complicated than (4.14); the relative percentage errors incurred are, however, smaller.

(see, f.e., Tables 4.9 and 4.10 which summarize the evaluation of formula (4.15) with $r_i = 1$ and compare them with Tables 4.3 and 4.4 respectively).

TABLE 4.9

Relative percentage errors incurred if (4.15) with $r_i = 1$ is used for the evaluation of $W(E_2/E_2/r_j)$

ρ	$r_j=2$	$r_j=3$	$r_j=4$	$r_j=5$	$r_j=6$
0.30	16.0	32.1			
0.40	9.0	17.2	26.2		
0.50	5.4	9.8	16.8	19.1	24.6
0.60	3.2	6.4	8.3	10.8	12.8
0.70	1.9	3.4	4.9	6.3	7.6

TABLE 4.10

Relative percentage errors incurred if (4.15) with $r_i = 1$ is used for the evaluation of $W(E_2/E_2/r_j)$

r_j	2	3	4	5	6
$\mu W(E_2/E_2/r_j) = 0.05$	10	10	10	9	9
$\mu W(E_2/E_2/r_j) = 0.10$	6	7	6	6	6

4.3.2 The general case

The four distributions of inter-arrival and service times summarized in Appendix A1 are now considered. These distributions were selected in such a way for no specific reason, apart from a general desire of maintaining some degree of consistency throughout this presentation: the same distributions were selected in Chapters 2 and 3; and the same distributions will be selected as distributions of both inter-arrival and service times in subsequent Chapters.

For the queueing system considered in this Chapter, no attempt was made to verify the approximate formulae (4.7) and (4.8) or the slightly modified formulae which could be obtained by setting $v_a = 1/\sqrt{1.02}$ instead of $v_a = 1$ and $v_s = 1/\sqrt{1.02}$ instead of $v_s = 1$; there was no incentive for undertaking such a task.

4.4 A MODIFIED SET OF FORMULAE FOR THE A.Q.T.

4.4.1 An assumption and the formulae

For a distribution of either inter-arrival or service times other than the Erlang, formulae (4.7) and (4.8) are of no practical use. In fact, given a distribution of inter-arrival times $A(u)$ with $v_a = v_a^*$ or one of service times $B(t)$ with $v_s = v_s^*$, it may well be impossible even to define what $A(u)$ with $v_a = 1$ or $B(t)$ with $v_s = 1$ should be, let alone the calculation of the various average queueing times appearing on the right hand side of the two formulae. Therefore, the assumptions (known to be valid under conditions of heavy traffic²³) are made that:

$$a) \quad \frac{W(A(u), v_a=1/D/r_j)}{W(A(u), v_a=1/D/r_i)} \approx \frac{W(M/D/r_j)}{W(M/D/r_i)} \quad (4.16)$$

$$b) \quad \frac{W(A(u), v_a=v_a^*/B(t), v_s=1/r_j)}{W(A(u), v_a=v_a^*/B(t), v_s=1/r_i)} \approx \frac{W(A(u), v_a=v_a^*/M/r_j)}{W(A(u), v_a=v_a^*/M/r_i)} \quad (4.17)$$

$$c) \quad \frac{W(D/B(t), v_s=1/r_j)}{W(D/B(t), v_s=1/r_i)} \approx \frac{W(D/M/r_j)}{W(D/M/r_i)} \quad (4.18)$$

$$d) \frac{W(A(u), v_a=1/B(t), v_s=v_s^*/r_j)}{W(A(u), v_a=1/B(t), v_s=v_s^*/r_i)} \approx \frac{W(M/B(t), v_s=v_s^*/r_j)}{W(M/B(t), v_s=v_s^*/r_i)} \quad (4.19)$$

in which case formulae (4.7) and (4.8) take the more interesting forms:

$$\begin{aligned} \frac{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r_j)}{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r_i)} &\approx \frac{(v_a^*)^2(1-(v_s^*)^2)}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} + \\ &+ \frac{(v_s^*)^2(1+(v_a^*)^2)}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(A(u), v_a=v_a^*/M/r_j)}{W(A(u), v_a=v_a^*/M/r_i)} \end{aligned} \quad (4.20)$$

and

$$\begin{aligned} \frac{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r_j)}{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r_i)} &\approx \frac{(v_s^*)^2(1-(v_a^*)^2)}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} + \\ &+ \frac{(v_a^*)^2(1+(v_s^*)^2)}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(M/B(t), v_s=v_s^*/r_j)}{W(M/B(t), v_s=v_s^*/r_i)} \end{aligned} \quad (4.21)$$

respectively.

Finally by introducing either the approximate expression (2.7) established in Chapter 2 for $W(A(u), v_a=v_a^*/M/r_j) / W(A(u), v_a=v_a^*/M/r_i)$ into (4.20) or the approximate expression (3.7) established in Chapter 3 for $W(M/B(t), v_s=v_s^*/r_j) / W(M/B(t), v_s=v_s^*/r_i)$ into (4.21), we obtain:

$$\begin{aligned} \frac{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r_j)}{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r_i)} &\approx \frac{2(v_a^*)^2(v_s^*)^2}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(M/M/r_j)}{W(M/M/r_i)} + \\ &+ \frac{(v_a^*)^2(1-(v_s^*)^2)}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(M/D/r_j)}{W(M/D/r_i)} + \frac{(v_s^*)^2(1-(v_a^*)^2)}{(v_a^*)^2+(v_s^*)^2} \cdot \frac{W(D/M/r_j)}{W(D/M/r_i)} \end{aligned} \quad (4.22)$$

equivalent in all respects to formula (4.11).

4.4.2 Evaluation of the formulae

The effect of the assumptions (4.16) to (4.19) on the relative percentage errors incurred in evaluating $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ by using formulae (4.20) and (4.21) instead of (4.7) and (4.8) is difficult to assess. As, however, both formulae (4.20) and (4.21) were found to give good approximations in most of the queueing systems studied, it seems reasonable at this stage to accept the assumptions as valid and leave the investigation associated with their justification as a topic requiring further research.

We now proceed in evaluating formula (4.22) - the most promising from a practical viewpoint - for:

- $A(u) = D$, ,  and  with $v_a^2 = 0, 1/6, 1/3$ and $1/2$ respectively;
- $B(t) = D$, ,  and  with $v_s^2 = 0, 1/6, 1/3$ and $1/2$ respectively;
- $2 \leq r_j \leq 6$;
- $1 \leq r_i \leq r_j - 1$; and
- $0 < \rho < 1$.

Results for $pW(A(u)/B(t)/r_j)$ are summarized in Tables A4.5, A4.6, A4.7, A4.8, A4.9, A4.10, A4.11 and A4.12 of Appendix 4; results for $pW(A(u)/B(t)/1)$ are summarized in Tables A4.1, A4.2, A4.3 and A4.4 of Appendix 4, the design of the simulation experiments being outlined in Appendix A2.6. The analytical values of $W(M/M/r)$, $W(M/D/r)$ and $W(D/M/r)$ required in the calculations are given in Tables A3.1, A3.24 and A3.2 of Appendix 3.

The relative percentage error incurred if formula (4.22) is used for the approximate evaluation of $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ is defined as:

$$e = 100(\text{approximate value} - \text{simulated value})/(\text{simul. value}).$$

a) Evaluation of formula (4.22) with $r_i = 1$

- For given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, any value of the average queueing time $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ can be approximated with a relative percentage error which:

is positive or negative depending on the values of the various parameters involved;

decreases in absolute value as the A.Q.T. to be estimated and, hence, the traffic intensity increase, tending to 0 as ρ tends to 1; and

is not very sensitive to the value of r_j .

Table 4.11 gives the relative percentage errors incurred if $r_i = 1$ is used in formula (4.22) for the evaluation of $W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r_j)$ for $r_j = 2, 3, 4, 5$ and 6 and $pW(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r_j) = 0.2, 0.3, 0.4$ and 0.5 . The asterisks (*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated A.Q.T. (see Table A4.5 of Appendix 4).

- For a given value of r_j and a given distribution of inter-arrival times $A(u)$ with coefficient of variation v_a^* , any value of the A.Q.T. $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ can be approximated with a relative percentage error which seems to be a decreasing function of v_s^* (at least up to

$v_s^* = 0.7$). Table 4.12, for example, gives the relative

TABLE 4.11

Relative percentage errors incurred if (4.22) with $r_i = 1$ is used for the evaluation of

$$W(\triangleleft, v_a = \sqrt{1/2} / \sqcup, v_s = \sqrt{1/3} / r_j)$$

r_j	2	3	4	5	6
$pW(\triangleleft / \sqcup / r_j) = 0.2$	4.5*	5.4*	5.4*	5.2	5.0
$pW(\triangleleft / \sqcup / r_j) = 0.3$	3.4	3.9	3.8	3.4	3.2
$pW(\triangleleft / \sqcup / r_j) = 0.4$	2.8	3.0	2.8	2.5	2.2
$pW(\triangleleft / \sqcup / r_j) = 0.5$	2.4	2.5	2.1	1.9	1.6

percentage errors incurred if $r_i = 1$ is used in formula (4.22) for the evaluation of an average queueing time

$$pW(\triangleleft, v_a = \sqrt{1/6} / B(t), v_s = v_s^* / 6) \text{ equal to } 0.2. \text{ The}$$

TABLE 4.12

Relative percentage errors incurred if (4.22) with $r_i = 1$ is used for the evaluation of an A.Q.T.

$$pW(\triangleleft, v_a = \sqrt{1/6} / B(t), v_s = v_s^* / 6) \text{ equal to } 0.2$$

$B(t)$	$(v_s^*)^2$	e (%)
D	0	15*
$\triangleleft$	1/6	2.9
$\sqcup$	1/3	1.3
$\triangleleft$	1/2	0.8

construction of this Table was based on the relatively few results that are summarized in Table A4.10 of Appendix 4; therefore, the values of e in the Table should be viewed as rough estimates rather than as true values. The aster-

isk(*) indicates that the approximate value falls outside the corresponding 95 % confidence interval on the simulated A.Q.T.

- For a given value of r_j and a given distribution of service times $B(t)$ with coefficient of variation v_s^* , any value of the A.Q.T. $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)$ can be approximated with a relative percentage error which seems to be an increasing function of v_a^* (at least up to $v_a^* = 0.7$). Table 4.13, for example, gives the relative percentage errors incurred if $r_1 = 1$ is used in formula (4.22) for the evaluation of an average queueing time $\mu W(A(u), v_a = v_a^* / \underline{\quad}, v_s = \sqrt{1/3} / 6)$ equal to 0.2. The

TABLE 4.13

Relative percentage errors incurred if (4.22) with $r_1 = 1$ is used for the evaluation of an A.Q.T.

$\mu W(A(u), v_a = v_a^* / \underline{\quad}, v_s = \sqrt{1/3} / 6)$ equal to 0.2

$A(t)$	$(v_a^*)^2$	e (%)
D	0	-3.3*
	1/6	2.9
	1/3	3.4
	1/2	5.0

construction of this Table was based on the relatively few results that are summarized in Tables A4.9 , A4.10 , A4.11 and A4.12 of Appendix 4; therefore, the values of e in the Table should be viewed as estimates rather than as true values. The asterisk(*) indicates that the approximate value falls outside the corresponding 95 % confidence

interval on the simulated A.Q.T.

b) Evaluation of formula (4.22) with $r_i = 2$

All conclusions stated in paragraph a) above hold: the relative percentage errors incurred are, however, smaller in absolute value (see, for example, Table 4.14 and compare

TABLE 4.14

Relative percentage errors incurred if (4.22) with $r_i = 2$ is used for the evaluation of

$$W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r_j)$$

r_j	2	3	4	5	6
$pW(\triangle / \square / r_j) = 0.2$	0	2.3	2.8	3.1	3.3
$pW(\triangle / \square / r_j) = 0.3$	0	1.4	1.9	2.1	2.2
$pW(\triangle / \square / r_j) = 0.4$	0	1.0	1.4	-	-
$pW(\triangle / \square / r_j) = 0.5$	0	0.8	-	-	-

it with Table 4.11; though the Table is incomplete due to insufficient simulation results, the comparison is possible).

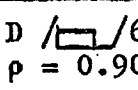
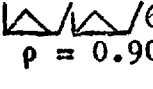
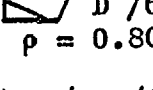
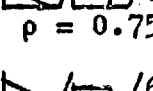
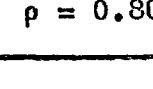
c) Evaluation of formula (4.22) with r_j fixed

For given values of ρ and r_j and for given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, the A.Q.T. of an arriving customer can be approximated with a relative percentage error which decreases in absolute value as r_i increases. Table 4.15, for example, gives the relative percentage errors incurred if (4.22) is used for the evaluation of the A.Q.T. in five 6-server

queueing systems. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated A.Q.T. The simulated values of the A.Q.T. used in the calculations are given in Tables A4.5 , A4.6 , A4.7 and A4.8 of Appendix 4.

TABLE 4.15

Relative percentage errors incurred if (4.22) is used for the evaluation of the A.Q.T. in five 6-server queueing systems

results system	simulated A.Q.T.	relative percentage errors				
		$r_i = 1$	$r_i = 2$	$r_i = 3$	$r_i = 4$	$r_i = 5$
 $\rho = 0.90$	0.1586	-3.9*	-1.7	-0.6	-0.1	-0.0
 $\rho = 0.90$	0.1771	3.3	2.6	2.1	1.6	1.0
 $\rho = 0.80$	0.0973	22.7*	15.2*	10.0*	5.8*	2.5
 $\rho = 0.75$	0.1012	8.8*	5.9*	4.1	2.6	1.4
 $\rho = 0.80$	0.2655	6.2*	4.1	2.9	1.9	1.0

fidence interval on the simulated A.Q.T. The simulated values of the A.Q.T. used in the calculations are given in Tables A4.5 , A4.6 , A4.7 and A4.8 of Appendix 4.

d) Evaluation of formula (4.22) with $r_i = r_j = 1$

- For given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, any value of the average queueing time $W(A(u), v_a^* = v_a^*/B(t), v_s^* = v_s^*/r_j)$ can be approximated with a relative percentage error which decreases very rapidly in absolute value as r_j increases (see, f.e., Table 4.16 and compare it with Tables 4.11 and 4.14; the asterisk(*) indicates that the approximate

value falls outside the corresponding 95 % confidence interval on the simulated A.Q.T.).

TABLE 4.16

Relative percentage errors incurred if (4.22) with $r_i = r_j - 1$ is used for the evaluation of

$$W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r_j)$$

r_j	2	3	4	5	6
$pW(\triangle / \square / r_j) = 0.2$	4.5*	2.3	1.2	0.9	0.8
$pW(\triangle / \square / r_j) = 0.3$	3.4	1.4	0.9	0.7	0.5
$pW(\triangle / \square / r_j) = 0.4$	2.8	1.0	0.6	0.5	0.3
$pW(\triangle / \square / r_j) = 0.5$	2.4	0.8	0.4	0.3	0.2

- For given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, any value of the total A.Q.T. per unit time $\lambda W(A(u), v_a = v_a^* / B(t), v_s = v_s^* / r_j)$ can be approximated with a relative percentage error which also decreases quite rapidly in absolute value as r_j increases.

TABLE 4.17

Relative percentage errors incurred if (4.22) with $r_i = r_j - 1$ is used for the evaluation of

$$W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r_j)$$

r_j	2	3	4	5	6
$\lambda W(\triangle / \square / r_j) = 0.5$	3.0	2.0	1.4	1.3	1.3
$\lambda W(\triangle / \square / r_j) = 1.5$	1.4	0.7	0.5	0.5	0.5

Table 4.17 gives the relative percentage errors incurred if formula (4.22) with $r_i = r_j - 1$ is used for the eva-

luation of $W(\underline{\Delta}, v_a = \sqrt{1/2} / \underline{\Delta}, v_s = \sqrt{1/3} / r_j)$ for $r_j = 2, 3, 4, 5$ and 6 and $\lambda W(\underline{\Delta}, v_a = \sqrt{1/2} / \underline{\Delta}, v_s = \sqrt{1/3} / r_j) = 0.5$ and 1.5 (These two values correspond to $\rho = 0.70$ and 0.84 if $r_j = 2$ and to $\rho = 0.76$ and 0.86 if $r_j = 6$ respectively; or, in other words, to $\mu W(\underline{\Delta}, v_a = \sqrt{1/2} / \underline{\Delta}, v_s = \sqrt{1/3} / r_j) = 0.36$ and 0.90 if $r_j = 2$ and to $\mu W(\underline{\Delta}, v_a = \sqrt{1/2} / \underline{\Delta}, v_s = \sqrt{1/3} / r_j) = 0.11$ and 0.29 if $r_j = 6$ respectively).

- For $r_i = r_j - 1$, (4.22) becomes a recursive formula. Consequently, if $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ with $r = r_i$ is known, one can obtain approximate results for the system $A(u), v_a = v_a^*/B(t), v_s = v_s^*/r$ with $r = r_j > r_i$ not only by using formula (4.22) directly but also by using formula (4.23):

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_j)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r_i)} \approx \\ & \approx \prod_{k=r_i+1}^{r_j} \left[\frac{2(v_a^*)^2(v_s^*)^2}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(M/M/k)}{W(M/M/k-1)} + \right. \\ & \quad + \frac{(v_a^*)^2(1-(v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(M/D/k)}{W(M/D/k-1)} + \\ & \quad \left. + \frac{(v_s^*)^2(1-(v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \cdot \frac{W(D/M/k)}{W(D/M/k-1)} \right] \quad (4.23) \end{aligned}$$

Formula (4.23) is more complicated than (4.22); the relative percentage errors are, however, somewhat smaller (see for example Table 4.18 and compare it with Table 4.11.

The Table is incomplete due to insufficient simulation re-

sults; the asterisks(*) indicate that the approximate val-

TABLE 4.18

Relative percentage errors incurred if (4.23) with $r_1 = 1$ is used for the evaluation of

$$W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r_j)$$

r_j	2	3	4	5	6
$pW(\triangle / \square / r_j) = 0.2$	4.5*	5.2*	5.1*	4.9	4.7
$pW(\triangle / \square / r_j) = 0.3$	3.4	3.8	3.6	3.3	3.0
$pW(\triangle / \square / r_j) = 0.4$	2.8	3.0	2.6	2.3	-
$pW(\triangle / \square / r_j) = 0.5$	2.4	2.4	2.0	-	-

ues fall outside the corresponding 95 % confidence interval on the simulated A.Q.T.).

- e) An example: Evaluation of formulae (4.22) and (4.23) with $r_1 = 1$ for some 6-server queueing systems

The values of ρ and the ones of the simulated A.Q.T., the approximate values of the A.Q.T. as obtained from formulae (4.22) and (4.23) and the corresponding relative percentage errors incurred are summarized in Table 4.19 for 33 in total 6-server queueing systems. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated average queueing time.

We now proceed in interpreting the results presented in Table 4.19.

- On average, the total A.Q.T. per unit time

$\lambda W(A(n)/B(t)/6)$ for the systems presented in Table 4.19 can be easily found equal to 1.02. In the familiar

TABLE 4.19

An example: Evaluation of formulae (4.22) and (4.23) with
 $r_1 = 1$ for 33 6-server queueing systems

Queueing system considered	Simulation results		Approximate results			
			Form. (4.22)		Form. (4.23)	
	ρ	A.Q.T.	A.Q.T.	e (%)	A.Q.T.	e (%)
D / Δ / 6	0.90	0.0773	0.0685*	-11.4	0.0685*	-11.4
"	0.95	0.2127	0.1982*	-6.8	0.1982*	-6.8
D / $\square$ / 6	0.90	0.1586	0.1524*	-3.9	0.1524*	-3.9
"	0.95	0.4259	0.4172	-2.1	0.4172	-2.1
D / ∇ / 6	0.85	0.1228	0.1183	-3.7	0.1183	-3.7
"	0.90	0.2505	0.2448	-2.3	0.2448	-2.3
Δ / D / 6	0.90	0.0716	0.0975*	36.2	0.0975*	36.2
"	0.95	0.2021	0.2323*	14.9	0.2323*	14.9
Δ / Δ / 6	0.85	0.0877	0.0946*	7.9	0.0943	7.6
"	0.90	0.1771	0.1829	3.3	0.1827	3.2
"	0.95	0.4531	0.4565	0.7	0.4564	0.7
Δ / $\square$ / 6	0.85	0.1356	0.1388	2.4	0.1384	2.1
"	0.90	0.2693	0.2710	0.6	0.2707	0.6
Δ / ∇ / 6	0.80	0.1023	0.1049	2.6	0.1043	2.0
"	0.85	0.1821	0.1839	1.0	0.1837	0.9
$\square$ / D / 6	0.85	0.0963	0.1248*	29.6	0.1248*	29.6
"	0.90	0.1819	0.2145*	17.9	0.2145*	17.9
$\square$ / Δ / 6	0.85	0.1538	0.1671*	8.6	0.1667*	8.4
"	0.90	0.2898	0.3026	4.4	0.3024	4.3

TABLE 4.19 (continued)

An example: Evaluation of formulae (4.22) and (4.23) with $r_1 = 1$ for 33 6-server queueing systems

Queueing system considered	Simulation results		Approximate results			
	ρ	A.Q.T.	Form. (4.22)		Form. (4.23)	
			A.Q.T.	e (%)	A.Q.T.	e (%)
$\square / \square / 6$	0.85	0.2052	0.2120	3.3	0.2115	3.1
"	0.90	0.3859	0.3919	1.6	0.3916	1.5
$\square / \triangle / 6$	0.80	0.1470	0.1532	4.2	0.1526	3.8
"	0.85	0.2506	0.2596	3.6	0.2593	3.5
$\triangle / D / 6$	0.80	0.0973	0.1194*	22.7	0.1194*	22.7
"	0.85	0.1598	0.1859*	16.3	0.1859*	16.3
"	0.90	0.2913	0.3211*	10.2	0.3211*	10.2
$\triangle / \triangle / 6$	0.80	0.1293	0.1439*	11.4	0.1432*	10.7
"	0.85	0.2134	0.2298*	7.8	0.2295*	7.5
$\triangle / \square / 6$	0.75	0.1012	0.1102*	8.8	0.1093*	8.0
"	0.80	0.1601	0.1700*	6.2	0.1694	5.8
"	0.85	0.2655	0.2754	3.7	0.2748	3.5
$\triangle / \triangle / 6$	0.75	0.1239	0.1293	4.3	0.1282	3.5
"	0.80	0.1970	0.2022	2.6	0.2013	2.2

queueing system M/M/1 the same value of the total average queueing time per unit time corresponds to $\rho = 0.62$.

- Unless v_a^* or $v_s^* = 0$ formula (4.23) gives better approximations than formula (4.22); this is particularly noticeable for relatively low values of ρ .

- In 17 out of the 33 queueing systems considered, the approximate value of the A.Q.T. as calculated by using formula (4.22) falls within the corresponding 95 % confidence interval on the simulated A.Q.T., the latter requiring approximately 8 minutes of CDC 6400 time to determine.
- In 20 out of the 33 queueing systems considered, both v_a^* and v_g^* are $\neq 0$. For those 20 systems, formulae (4.22) and (4.23) give approximate values for the A.Q.T. which fall 14 and 16 times respectively within the corresponding 95 % confidence interval on the simulated A.Q.T.

CONCLUSIONS

If v_g^* does not have a value too close to zero, formulae (4.22) and (4.23) with $r_i = 1$ seem to give reasonably good approximations for the average queueing time of an arriving customer in the system $GI/G/r$. (see, for example, Table 4.20 which gives the magnitude of the relative percentage errors incurred if formula (4.22) with $r_i = 1$ is used for the approximate evaluation of an A.Q.T. $pW(A(u)/B(t)/6)$ equal to 0.2).

Both formulae can be simplified further, if the explicit formulae for the approximate evaluation of $W(M/D/r)$ and $W(D/M/r)$, which will be presented in Chapter 7, are used. The formula for the approximate evaluation of $W(A(u), v_a = v_a^*/B(t), v_g = v_g^*/1)$ which will be derived in Chapter 5 simplifies the necessary calculations still further.

TABLE 4.20

Magnitude of the relative percentage errors incurred if formula (4.22) with $r_1=1$ is used for the approximate evaluation of an A.Q.T. $pW(A(u)/B(t)/6)$ equal to 0.2

$\begin{array}{c} A(u) \\ B(t) \end{array}$	$\begin{array}{c} D \\ (v_a^*)^2=0 \end{array}$	$\begin{array}{c} \triangle \\ (v_a^*)^2=1/6 \end{array}$	$\begin{array}{c} \square \\ (v_a^*)^2=1/3 \end{array}$	$\begin{array}{c} \nabla \\ (v_a^*)^2=1/2 \end{array}$
$\begin{array}{c} D \\ (v_s^*)^2=0 \end{array}$	-	10 ... 20	10 ... 20	10 ... 20
$\begin{array}{c} \triangle \\ (v_s^*)^2=1/6 \end{array}$	-10 .. -5	0 5	0 ... 10	5 ... 10
$\begin{array}{c} \square \\ (v_s^*)^2=1/3 \end{array}$	-5 ... 0	0 5	0 5	0 ... 10
$\begin{array}{c} \nabla \\ (v_s^*)^2=1/2 \end{array}$	-5 ... 0	0 5	0 5	0 5

C H A P T E R 5

APPROXIMATE FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME

IN THE STANDARD SYSTEM

$$\underline{G I / G / 1}$$

5. APPROXIMATE FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUING TIME (A.Q.T.) IN THE STANDARD SYSTEM

$$\underline{GI / G / 1}$$

5.1 INTRODUCTION

5.1.1 Brief description of the system

The system considered can be described as follows: there is one server. Intervals between successive arrivals are independently distributed with distribution function $A(u)$ such that there is zero probability that two or more customers arrive simultaneously. The average rate of arrival of customers is λ and the coefficient of variation of inter-arrival times $v_a = \sqrt{\text{var}(u)} \lambda$, such that $0 < v_a < 1$. Service times have a general distribution function $B(t)$. The average service time is μ^{-1} and the coefficient of variation of service times $v_s = \sqrt{\text{var}(t)} \mu$, such that $0 < v_s < 1$. Customers form into a single queue and get served in the order of their arrival. The system is in statistical equilibrium, i.e. the traffic intensity $\rho = \lambda / r\mu < 1$.

5.1.2 On the derivation of analytical results for the A.Q.T.

The equilibrium theory of this type of queue is discussed by Lindley²⁷. From a practical viewpoint, however, analytical results are not easy to obtain.

In case one of the two distributions involved is Erlang, Cox and Smith⁸ and Prabhu³⁸ obtained the Laplace transform of the queueing time distribution. If, however, the parameter of the

Erlang distribution has a high value and depending on the nature of the second distribution, the derivation of values for the A.Q.T. may present some problems.

5.1.3 Alternative ways of obtaining results for the A.Q.T. (the practitioner's viewpoint)

In view of the major difficulties encountered when trying to apply the theory to practical problems, the practitioner may find adequate for the purpose of his study and therefore be content with:

- a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude ^{errors} with all other incurred due to the assumption that the queueing model $GI/G/1$ reflects the real life waiting line situation under study. An example of such a formula is the one suggested by Page³⁵ for use in the system $E_k/E_l/1$.
- b) The determination of bounds on the A.Q.T., provided that they are adequately tight for the purpose of the study. Kingman²¹, for example, studied the behaviour of the standard $GI/G/1$ queue under conditions of heavy traffic and obtained an extremely simple formula for the A.Q.T.; a formula that gives an upper bound under unsaturated conditions. A lower bound on the A.Q.T. related to the nature of the distribution of inter-arrival times was obtained by Marshall²¹.
- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be empha-

sized, however, that simulation is not in itself a panacea: apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections of this Chapter, approximate formulae for the A.Q.T. are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is based on the idea (introduced in Chapter 1) that two queueing systems of the same general form GI/G/1 with:

- the same type of distribution of inter-arrival times;
- the same type of distribution of service times;
- equal traffic intensities; and
- either equal coefficients of variation of inter-arrival times and unequal coefficients of variation of service times or equal coefficients of variation of service times and unequal coefficients of variation of inter-arrival times,

have a similar behaviour.

5.2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

5.2.1 Establishment of the basic structure of the formulae

Let $W(A(u)/B(t)/1)$ denote the A.Q.T. of an arriving customer. One can reasonably expect that $pW(A(u)/B(t)/1)$ depends on $A(u)$ and $B(t)$ only; furthermore that for ρ , $B(t)$ and the type of $A(u)$ fixed, $pW(A(u)/B(t)/1)$ is an increasing function of v_a with limiting values $pW(D/B(t)/1)$, corresponding to the system $D/B(t)/1$, if $v_a=0$ and, say, $pW(A(u), v_a=1/B(t)/1)$, corresponding to the system $A(u), v_a=1/B(t)/1$ if $v_a = 1$.

Two such relationships between $\mu W(\Lambda(u)/B(t)/1)$ and v_a are shown in Figure 5.1. They are both related to a queuing

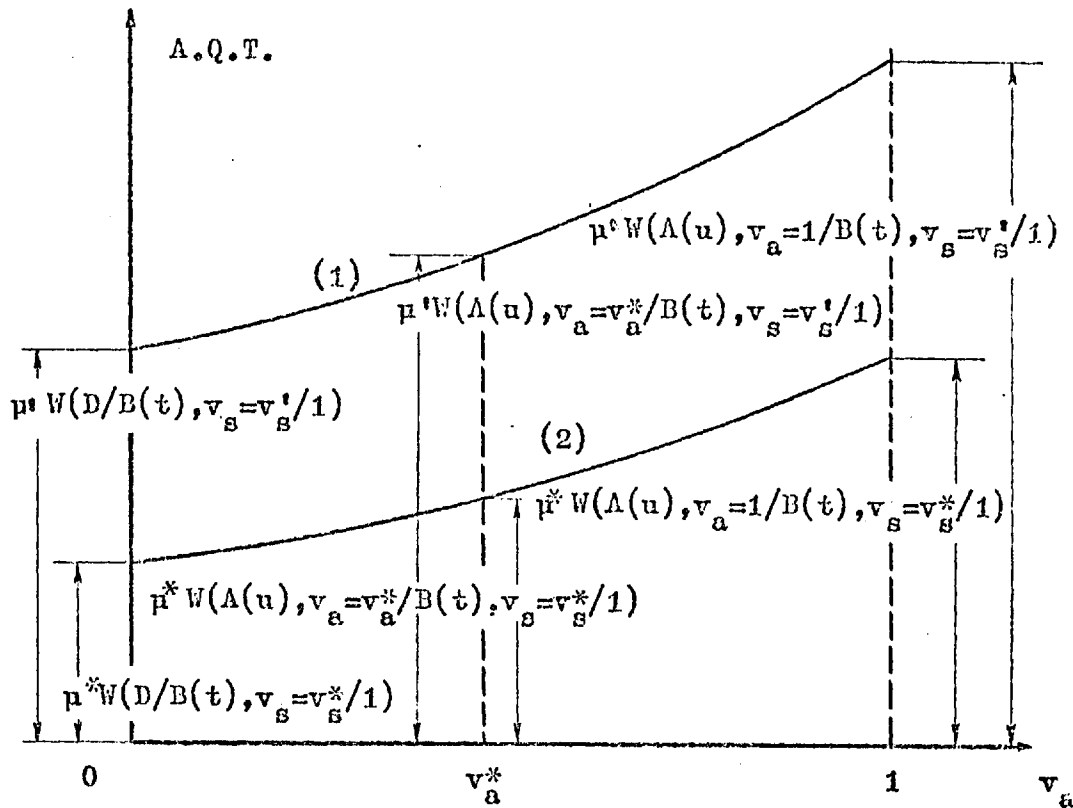


Fig. 5.1 $\mu W(\Lambda(u)/B(t)/1)$ as a function of v_a for two queuing systems with the same types of distributions of inter-arrival and service times and equal traffic intensities.

system of the general form $\Lambda(u)/B(t)/1$ with the same types of distributions of inter-arrival and service times. Curve (1) represents a situation in which $\lambda = \lambda'$, $\mu = \mu'$ and $v_s = v_s'$; curve (2) represents a situation in which $\lambda = \lambda^*$, $\mu = \mu^*$ such, however, that $\rho' = \rho^* = \rho$ and $v_s = v_s^* < v_s'$.

We now proceed in writing a relationship of the general form:

$$\begin{aligned}
& \frac{\mu^* W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{\mu' W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)} \approx (1 - S_a) \frac{\mu^* W(D/B(t), v_s = v_s^*/1)}{\mu' W(D/B(t), v_s = v_s^*/1)} \\
& + S_a \frac{\mu^* W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/1)}{\mu' W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/1)} \quad (5.1)
\end{aligned}$$

with v_a^* any value of v_a between 0 and 1 and S_a some "similarity factor" with limiting values 0 and 1 for $v_a^* = 0$ and 1 respectively, regardless of v_s^* , v_s^* , ρ , $\Lambda(u)$ or $B(t)$.

One can also expect that for ρ , $\Lambda(u)$ and the type of $B(t)$ fixed, $\mu W(\Lambda(u)/B(t)/1)$ is an increasing function of v_s with limiting values $\mu W(\Lambda(u)/D/1)$, corresponding to the system $\Lambda(u)/D/1$, if $v_s = 0$ and, say, $\mu W(\Lambda(u)/B(t), v_s = 1/1)$, corresponding to the system $\Lambda(u)/B(t), v_s = 1/1$, if $v_s = 1$. A reasoning similar to the one associated with Figure 5.1 leads to the relationship:

$$\begin{aligned}
& \frac{\mu^* W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{\mu' W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)} \approx (1 - S_s) \frac{\mu^* W(\Lambda(u), v_a = v_a^*/D/1)}{\mu' W(\Lambda(u), v_a = v_a^*/D/1)} \\
& + S_s \frac{\mu^* W(\Lambda(u), v_a = v_a^*/B(t), v_s = 1/1)}{\mu' W(\Lambda(u), v_a = v_a^*/B(t), v_s = 1/1)} \quad (5.2)
\end{aligned}$$

with v_s^* any value of v_s between 0 and 1 and S_s some "similarity factor" with limiting values 0 and 1 for $v_s^* = 0$ and 1 respectively, regardless of v_a^* , v_a^* , ρ , $\Lambda(u)$ or $B(t)$.

(5.1) is an expression giving $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$

in terms of $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ and (5.2) is an expression giving $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ in terms of $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$; consequently, if the relative percentage errors incurred are small,

- a) the hypothesis that the systems $\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1$ and $\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1$ with $\rho^* = \rho'$ and the systems $\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1$ and $\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1$ with $\rho^* = \rho'$ have a similar behaviour would be established, the quantities on the right hand sides being regarded as similarity ratios;
- b) either (5.1) or (5.2) or indeed some convenient combination of them could be used as an approximate formula for the evaluation of $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ provided that $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ and/or $W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ with $v_s^* = v_s^* + \Delta v_s^*$, $v_s^* + 2\Delta v_s^*$, ... or even 1, $v_a^* = v_a^* + \Delta v_a^*$, $v_a^* + 2\Delta v_a^*$, ... or even 1 and the two similarity ratios were known.

5.2.2 A method for the determination of the similarity factors S_a and S_s

The method suggested here has three main advantages:

- a) it is simple in its evaluation (provided that it can be applied);
- b) it provides very simple expressions for S_a and S_s ; and
- c) it is justified in terms of a) and b) of paragraph 5.2.1, particularly if the value of ρ is high, a situation likely to be encountered in practice and, also, one in which numerical results are usually difficult to obtain either analytically or by simulation.

It consists of two steps:

- a) the approximate formulae, for the average queuing times in the basic expressions (5.1) and (5.2) under conditions of heavy traffic, are found.

For the system considered:

$$(pW(A(u)/B(t)/1))_{\rho \rightarrow 1} = \frac{(v_a)^2 + \rho^2(v_s)^2}{2\rho(1 - \rho)}$$

as suggested by Kingman²¹, valid for $0 < v_a, v_s < 1$.

- b) these formulae are then introduced into (5.1) and (5.2); simple algebra yields the desired expressions for S_a and S_s .

For the system considered:

$$S_a = \frac{(v_a^*)^2(1 + (v_s')^2)}{(v_a^*)^2 + (v_s')^2} \quad \text{and} \quad S_s = \frac{(v_s^*)^2(1 + (v_a')^2)}{(v_a')^2 + (v_s^*)^2},$$

in which case (5.1) and (5.2) take the forms:

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)} \approx \\ & \frac{(v_a^*)^2(1 + (v_s')^2)}{(v_a^*)^2 + (v_s')^2} \cdot \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)}{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)} + \\ & + \frac{(v_s')^2(1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s')^2} \cdot \frac{W(D/B(t), v_s = v_s^*/1)}{W(D/B(t), v_s = v_s^*/1)} \end{aligned} \quad (5.3)$$

and

$$\frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)} \approx$$

$$\begin{aligned}
& \approx \frac{(v_s^*)^2(1 + (v_a')^2)}{(v_a')^2 + (v_s^*)^2} \cdot \frac{W(\Lambda(u), v_a = v_a^*/B(t), v_s = 1/1)}{W(\Lambda(u), v_a = v_a'/B(t), v_s = 1/1)} + \\
& + \frac{(v_a')^2(1 - (v_s^*)^2)}{(v_a')^2 + (v_s^*)^2} \cdot \frac{W(\Lambda(u), v_a = v_a^*/D/1)}{W(\Lambda(u), v_a = v_a'/D/1)} \quad (5.4)
\end{aligned}$$

respectively.

The method described is not the only one possible. An alternative method is presented in Chapter 10 and many more could be envisaged.

5.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

5.3.1 The queueing system $\Gamma / \Gamma / 1$

Consider first the case in which both inter-arrival and service times have a Gamma distribution (the gamma distribution is discussed in Appendix A1.1.1). For this case, $W(\Lambda(u), v_a = 1/B(t)/1) = W(M/\Gamma/1)$, corresponding to the system with exponential inter-arrival times and $W(\Lambda(u)/B(t), v_s = 1/1) = W(\Gamma/M/1)$, corresponding to the system with exponential service times. Consequently, (5.3) and (5.4) can be written:

$$\begin{aligned}
& \frac{W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1)}{W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s'/1)} \approx \frac{(v_s^*)^2(1 + (v_s^*)^2)}{(v_s^*)^2 + (v_s')^2} + \\
& + \frac{(v_s')^2(1 - (v_s^*)^2)}{(v_s^*)^2 + (v_s')^2} \cdot \frac{W(D/\Gamma, v_s = v_s^*/1)}{W(D/\Gamma, v_s = v_s'/1)} \quad (5.5)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1)}{W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s'/1)} \approx \frac{(v_s^*)^2(1 + (v_s')^2)}{(v_s')^2 + (v_s^*)^2} \cdot \frac{W(\Gamma, v_a = v_a^*/M/1)}{W(\Gamma, v_a = v_a^*/M/1)} + \\
& + \frac{(v_s')^2(1 - (v_s^*)^2)}{(v_s')^2 + (v_s^*)^2} \cdot \frac{W(\Gamma, v_a = v_a^*/M/1)}{W(\Gamma, v_a = v_a^*/M/1)}
\end{aligned}$$

$$+ \frac{(v_a')^2 (1 - (v_s^*)^2)}{(v_a')^2 + (v_s^*)^2} \cdot \frac{W(\Gamma, v_a = v_a^*/D/1)}{W(\Gamma, v_a = v_a'/D/1)} \quad (5.6)$$

respectively.

Both formulae can be greatly simplified if we set $v_a' = 1$ and $v_s' = 1$:

$$\begin{aligned} \frac{W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1)}{W(\Gamma, v_a = v_a^*/M/1)} &\simeq \frac{(v_s^*)^2 (1 + (v_s^*)^2)}{1 + (v_s^*)^2} + \\ &+ \frac{1 - (v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(D/\Gamma, v_s = v_s^*/1)}{W(D/M/1)} \end{aligned} \quad (5.7)$$

and

$$\begin{aligned} W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1) &\simeq (v_s^*)^2 W(\Gamma, v_a = v_a^*/M/1) + \\ &+ (1 - (v_s^*)^2) W(\Gamma, v_a = v_a^*/D/1) \end{aligned} \quad (5.8)$$

Note that the expression in (5.8) can be viewed as a generalization of the well known Pollaczek-Khintchine formula:

$$W(M/G/1) = \frac{\rho(1 + (v_s^*)^2)}{2(1 - \rho)\mu} = (v_s^*)^2 W(M/M/1) + (1 - (v_s^*)^2) W(M/D/1)$$

for a gamma instead of an exponential distribution of inter-arrival times.

If we now set $v_s^* = 0$ in (5.7) we obtain:

$$W(\Gamma, v_a = v_a^*/D/1) \simeq \frac{(v_a^*)^2}{1 + (v_a^*)^2} W(\Gamma, v_a = v_a^*/M/1) \quad (5.9)$$

and if we set $v_a^* = 0$ in (5.8) we obtain:

$$W(D/\Gamma, v_s = v_s^*/1) \simeq (v_s^*)^2 W(D/M/1) \quad (5.10)$$

Finally, by combining either (5.7) and (5.10) or (5.8) and (5.9) we obtain the simple expression:

$$W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1) \simeq \frac{(v_a^*)^2 + (v_s^*)^2}{1 + (v_a^*)^2} W(\Gamma, v_a = v_a^*/M/1) \quad (5.11)$$

We now proceed in evaluating formulae (5.5) , (5.6) , (5.7) , (5.8) and (5.11) for $0 < v_a^* , v_s^* , \rho < 1$, the emphasis being given on formula (5.11). Some analytical values for $pW(E/I/1)$ and $pW(I/E/1)$ are given in Table A3.26 of Appendix 3; the ones with 6 decimal digits were calculated by applying the method of solution outlined in Cox and Smith⁸ ; and the values with 4 decimal digits were obtained from the Table included in the book by Page³⁵ . Somewhat more detailed results for the systems $E_k/D/1$ and $D/E_k/1$ are given in Tables A3.27 and A3.28. Finally, the values of $W(I/M/1)$ required in the calculations were obtained from Table A3.3.

a) Evaluation of formula (5.11)

Any value of $W(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1)$ can be approximated with a relative percentage error:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{th. value})$$

which:

is positive for $0 < v_a^* , v_s^* , \rho < 1$, thus implying that the expression in (5.11) gives an upper bound on the A.Q.T. to be estimated;

decreases as the A.Q.T. to be estimated and, hence, the traffic intensity increase, tending to 0 as $\rho \rightarrow 1$; and

tends quite rapidly to 0 as either v_a^* or v_s^* tend to 1.

Table 5.1 gives an indication of the magnitude of e in evaluating an A.Q.T. $pW(\Gamma, v_a = v_a^*/\Gamma, v_s = v_s^*/1)$ equal to 0.5 , 1.0 and 1.5 (Note that these three values of the A.Q.T. of an arriving customer occur in the quite familiar system

M/M/1 for $\rho = 1/3$, 0.50 and 0.60 respectively).

TABLE 5.1

Maximum relative percentage errors incurred if formula (5.11) is used for the approximate evaluation of an A.Q.T. $\mu W(I, v_a = v_a^*/I, v_s = v_s^*/1)$ equal to 0.5, 1.0 and 1.5

$v_s^* =$	$\mu W \quad v_a^* =$	0 .. 0.4	0.4 .. 0.6	0.6 .. 0.8	0.8 .. 1.0
0 .. 0.4	0.5	23.0	16.7	11.4	5.3
	1.0	12.6	9.2	6.3	3.0
	1.5	8.4	6.4	4.4	2.1
0.4 .. 0.6	0.5	23.0	16.7	10.6	4.6
	1.0	12.6	9.2	5.9	2.8
	1.5	8.4	6.4	4.1	1.9
0.6 .. 0.8	0.5	20.4	14.2	8.6	3.7
	1.0	11.3	8.1	5.0	2.2
	1.5	7.9	5.7	3.5	1.6
0.8 .. 1.0	0.5	12.3	8.4	5.1	2.1
	1.0	6.8	4.8	3.0	1.3
	1.5	4.8	3.5	2.1	0.9

b) Evaluation of formulae (5.5), (5.6), (5.7) and (5.8)

Whereas (5.11) gives $W(I, v_a = v_a^*/I, v_s = v_s^*/1)$ in terms of one A.Q.T. only ($W(I, v_a = v_a^*/M/1)$, which is relatively easy to calculate) the four formulae considered in this paragraph are likely to present some problems in their application: in fact, average queueing times such as $W(I, v_a = v_a^*/D/1)$ or $W(D/I, v_s = v_s^*/1)$ are not always easy to evaluate analytically. As, however, these formulae give much better approximations than formula (5.11) it may be worthwhile to examine their application by means of an example.

An example

Suppose that one is faced with the problem of determining the A.Q.T. of an arriving customer in a situation where:

arrivals to a single server occur by an Erlang - 5 distribution;

service times have an Erlang - 3 distribution; and

the traffic intensity is equal to 0.80.

Suppose, further, that for one reason or another, no analytical model is available, which implies that the desired queuing time

$$\mu W(\Gamma, v_a = \sqrt{1/5} / \Gamma, v_s = \sqrt{1/3} / 1) = \mu W(E_5/E_3/1) = 0.8831$$

cannot be determined analytically.

Depending on how many of the average queuing times appearing in formulae (5.5) , (5.6) , (5.7) and (5.8) have a known value for $\rho = 0.80$, the approximations to the value 0.8831 summarized in Table 5.2 can be made.

TABLE 5.2

An example: Possible approximations to $\mu W(E_5/E_3/1)$

with $\rho = 0.80$

Formula used	Average queuing times having a known value for $\rho = 0.80$	Approximate value	e (%)
5.11	$\mu W(E_5/M/1) = 2.1523$	0.9566	8.3
5.8	$\mu W(E_5/M/1) = 2.1523$ $\mu W(E_5/D/1) = 0.2891$	0.9102	3.1
5.7	$\mu W(E_5/M/1) = 2.1523$ $\mu W(D/E_5/1) = 0.4604$ $\mu W(D/M/1) = 1.6927$	0.8686	-1.6
5.5	$\mu W(E_5/E_2/1) = 1.1946$ $\mu W(D/E_3/1) = 0.4604$ $\mu W(D/E_2/1) = 0.7568$	0.8704	-1.4

TABLE 5.2 (continued)

An example: Possible approximations to $\mu W(E_5/E_3/1)$ with $\rho = 0.80$

Formula used	Average queuing times having a known value for $\rho = 0.80$	Approximate value	e (%)
5.6	$\mu W(E_2/E_3/1) = 1.5419$ $\mu W(E_5/M/1) = 2.1523$ $\mu W(E_2/M/1) = 2.8440$ $\mu W(E_5/D/1) = 0.2891$ $\mu W(E_2/D/1) = 0.9033$	0.8972	1.6
5.6	$\mu W(E_3/E_3/1) = 1.1755$ $\mu W(E_5/M/1) = 2.1523$ $\mu W(E_3/M/1) = 2.4594$ $\mu W(E_5/D/1) = 0.2891$ $\mu W(E_3/D/1) = 0.5539$	0.8884	0.6
5.6	$\mu W(E_4/E_3/1) = 0.9914$ $\mu W(E_5/M/1) = 2.1523$ $\mu W(E_4/M/1) = 2.2674$ $\mu W(E_5/D/1) = 0.2891$ $\mu W(E_4/D/1) = 0.3861$	0.8840	0.1
5.6 (as a recursive formula)	$\mu W(M/E_3/1) = 2.6667$ $\mu W(M/M/1) = 4.0000$ $\mu W(E_2/M/1) = 2.8440$ $\mu W(E_3/M/1) = 2.4594$ $\mu W(E_4/M/1) = 2.2674$ $\mu W(E_5/M/1) = 2.1523$ $\mu W(M/D/1) = 2.0000$ $\mu W(E_2/D/1) = 0.9033$ $\mu W(E_3/D/1) = 0.5539$ $\mu W(E_4/D/1) = 0.3861$ $\mu W(E_5/D/1) = 0.2891$	0.8948	1.3

As it so often happens in practice, a compromise between desired accuracy and complexity of calculations has to be made.

5.3.2 The queueing system $E_k/G/1$

The four distributions of service times corresponding to the ones of inter-arrival times summarized in Appendices A1.1.3, A1.1.4, A1.1.5 and A1.1.6 are now considered. These distributions were selected in such a way so as to ensure that analytical results could be fairly easily obtained and if possible for $0 < v_s < 1$. Consequently, they are most unlikely to represent any real system, except perhaps for a few values of x and y .

For the queueing system $E_k/G/1$,

- a) No attempt is made to verify formula (5.3): there seems to be no easy way of obtaining analytical results for $W(D/B(t)/1)$.
- b) As $B(t)$ with $v_s = 1$ is achieved for $x = 0$ and $y \rightarrow \infty$, instead of evaluating formula (5.4) we proceed in evaluating the slightly modified formula (5.12):

$$\frac{W(E, v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(E, v_a = v_a'/B(t), v_s = v_s^*/1)} \approx S_s \frac{W(E, v_a = v_a^*/B(t), v_s = 1/\sqrt{1.02}/1)}{W(E, v_a = v_a'/B(t), v_s = 1/\sqrt{1.02}/1)} + (1 - S_s) \frac{W(E, v_a = v_a^*/D/1)}{W(E, v_a = v_a'/D/1)} \quad (5.12)$$

with

$$S_s = 1.02 \frac{(v_s^*)^2 (1/1.02 + (v_a')^2)}{(v_a')^2 + (v_s^*)^2} ;$$

the above expression for S_s can be easily obtained if the

method suggested in section 5.22 is applied to formula (5.12).

Formula (5.12) corresponds to formula (5.6) if $v_a' < 1$ and to formula (5.8) if $v_a' = 1$ established in section 5.3.1 for the case of gamma service times.

- c) By setting $v_s^* = 0$ and $v_s' = 1$ into (5.3) we find an approximate expression for $W(E, v_a = v_a^*/D/1)$. If this expression is introduced into (5.4) and v_a' is set equal to 1, we obtain the approximate formula:

$$\frac{W(E, v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(E, v_a = v_a^*/B(t), v_s = 1/1)} = \frac{(v_a^*)^2 + (v_s^*)^2}{1 + (v_a^*)^2} \quad (5.13)$$

which corresponds to formula (5.11) established in section 5.3.1 for the case of gamma service times.

As, however, $B(t)$ with $v_s = 1$ is achieved for $x = 0$ and $y \rightarrow \infty$, instead of evaluating (5.13) we proceed in evaluating the slightly modified formula:

$$\frac{W(E, v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(E, v_a = v_a^*/B(t), v_s = 1/\sqrt{1.02}/1)} \approx S_s \quad (5.14)$$

with

$$S_s = \frac{(v_a^*)^2 + (v_s^*)^2}{(v_a^*)^2 + 1/1.02},$$

the above expression for S_s being obtained if the method suggested in section 5.2.2 is applied to formula (5.14).

The approximations achieved in evaluating formulae (5.12) and (5.14) were all found to be as good as the ones stated in the special case of gamma service times already dealt with in section 5.3.1; see, for example, Table 5.3 which gives the relative ~~atime~~ percentage errors incurred if formulae (5.14) and (5.12) are used for the evaluation of

$\mu W(E, v_g=1/2 / \triangle, v_g=\sqrt{1/6} / 1) = \mu W(E_k / \triangle, v_g=\sqrt{1/6} / 1)$
 with $\rho = 0.80$ and compare it with Table 5.2 (cases of using
 formulae (5.11), (5.8) and (5.6)); the analytical results
 required in the calculations are included in Tables A5.29 and
 A5.27.

TABLE 5.3

An example: Possible approximations to

$$\mu W(E_k / \triangle, v_g=\sqrt{1/6} / 1) = 0.6747$$

$$(\rho = 0.80)$$

Formula used	Average queuing times having a known value for $\rho = 0.80$	Approximate value	ϵ (%)
5.14	$\mu W(E_k / \triangle, v_g=1/\sqrt{1.02}/1)=2.1560$	0.7301	8.2
5.12	$\mu W(M / \triangle, v_g=\sqrt{1/6} / 1) = 2.3333$ $\mu W(E_k / \triangle, v_g=1/\sqrt{1.02}/1)=2.1560$ $\mu W(M / \triangle, v_g=1/\sqrt{1.02}/1) = 3.9608$ $\mu W(E_k / D/1) = 0.3861$ $\mu W(M / D/1) = 2.0000$	0.6863	1.7
5.12	$\mu W(E_2 / \triangle, v_g=\sqrt{1/6} / 1) = 1.2176$ $\mu W(E_k / \triangle, v_g=1/\sqrt{1.02}/1)=2.1560$ $\mu W(E_2 / \triangle, v_g=1/\sqrt{1.02}/1)=2.7661$ $\mu W(E_k / D/1) = 0.3861$ $\mu W(E_2 / D/1) = 0.9033$	0.6811	0.9
5.12	$\mu W(E_3 / \triangle, v_g=\sqrt{1/6} / 1) = 0.8539$ $\mu W(E_k / \triangle, v_g=1/\sqrt{1.02}/1)=2.1560$ $\mu W(E_3 / \triangle, v_g=1/\sqrt{1.02}/1)=2.3606$ $\mu W(E_k / D/1) = 0.3861$ $\mu W(E_3 / D/1) = 0.5539$	0.6773	0.4

5.3.3 The queueing system GI/E₀/1

The four distributions of inter-arrival times summarized in Appendices A1.1.3, A1.1.4, A1.1.5 and A1.1.6 are now considered. These distributions were selected in such a way so as to ensure that analytical results could be fairly easily obtained and if possible for $0 < v_a < 1$. Consequently, they are most unlikely to represent any real system, except perhaps for a few values of x and y .

For the queueing system GI/E₀/1,

- a) No attempt is made to verify formula (5.4): there seems to be no easy way of obtaining analytical results for $W(\Lambda(u)/D/1)$.
- b) As $\Lambda(u)$ with $v_a = 1$ is achieved for $x = 0$ and $y \rightarrow \infty$, and as values of $\mu W(\Lambda(u), v_a = 1/\sqrt{1.02}/E/1)$ were found difficult to obtain analytically, instead of evaluating formula (5.3) we proceed in evaluating the modified formula (5.15):

$$\frac{W(\Lambda(u), v_a = v_a^*/E, v_s = v_s^*/1)}{W(\Lambda(u), v_a = v_a^*/E, v_s = v_s'/1)} \approx S_a \frac{W(\Lambda(u), v_a = 1/\sqrt{1.21}/E, v_s = v_s^*/1)}{W(\Lambda(u), v_a = 1/\sqrt{1.21}/E, v_s = v_s'/1)} + (1 - S_a) \frac{W(D/E, v_s = v_s^*/1)}{W(D/E, v_s = v_s'/1)} \quad (5.15)$$

with

$$S_a = 1.21 \frac{(v_a^*)^2 (1/1.21 + (v_s')^2)}{(v_a^*)^2 + (v_s')^2} ;$$

the above expression for S_a can be easily obtained if the method suggested in section 5.2.2 is applied to formula (5.15).

Formula (5.15) corresponds to formula (5.5) if $v'_a < 1$ and to formula (5.7) if $v'_a = 1$ established in section 5.3.1 for the case of gamma inter-arrival times.

- c) By setting $v_a^* = 0$ and $v'_a = 1$ into (5.4) we find an approximate expression for $W(D/E, v_a = v_a^*/1)$. If this expression is introduced into (5.3) and v'_a is set equal to 1, we obtain the approximate formula:

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/E, v_s = v_s^*/1)}{W(A(u), v_a = 1/E, v_s = v_s^*/1)} \approx \\ & \approx \frac{2((v_a^*)^2 + (v_s^*)^2)}{(1 + (v_a^*)^2)(1 + (v_s^*)^2)} \cdot \frac{W(A(u), v_a = v_a^*/M/1)}{W(A(u), v_a = 1/M/1)} \end{aligned} \quad (5.16)$$

which corresponds to formula (5.11) established in section 5.3.1 for the case of gamma inter-arrival times.

As, however, $A(u)$ with $v_a = 1$ is achieved for $x = 0$ and $y \rightarrow \infty$ and as values of $\mu W(A(u), v_a = 1/\sqrt{1.02}/E/1)$ were found difficult to obtain analytically, instead of evaluating (5.16) we proceed in evaluating the modified formula:

$$\frac{W(A(u), v_a = v_a^*/E, v_s = v_s^*/1)}{W(A(u), v_a = 1/\sqrt{1.21}/E, v_s = v_s^*/1)} \approx S_a \frac{W(A(u), v_a = v_a^*/M/1)}{W(A(u), v_a = 1/\sqrt{1.21}/M/1)} \quad (5.17)$$

with

$$S_a = \frac{(2.21/1.21)((v_a^*)^2 + (v_s^*)^2)}{(1 + (v_a^*)^2)(1/1.21 + (v_s^*)^2)},$$

the above expression for S_a being easily obtained if the method suggested in section 5.2.2 is applied to formula (5.17).

The approximations achieved in evaluating formulae (5.15) and (5.17) were all found to be as good as the ones stated in the special case of gamma inter-arrival times already dealt with in section 5.3.1 ; see, for example, Table 5.4 which gives the relative percentage errors incurred if formulae (5.17) and (5.15) are used for the evaluation of

$$\rho W(\triangle, v_a = \sqrt{1/6} / E, v_g = 1/2 / 1) = \rho W(\triangle, v_a = \sqrt{1/6} / E_4 / 1)$$

with $\rho = 0.80$ and compare it with Table 5.2 (cases of using formulae (5.11) , (5.7) and (5.5)); the analytical results required in the calculations are included in Tables A3.32 and A3.28 of Appendix 3.

TABLE 5.4

An example: Possible approximations to

$$\rho W(\triangle, v_a = \sqrt{1/6} / E_4 / 1) = 0.6934$$

$$(\rho = 0.80)$$

Formula used	Average queueing times having a known value for $\rho = 0.80$	Approximate value	c (%)
5.17	$\rho W(\triangle, v_a = 1/\sqrt{1.21}/E_4/1) = 2.5176$ $\rho W(\triangle, v_a = \sqrt{1/6} / M / 1) = 2.0929$ $\rho W(\triangle, v_a = 1/\sqrt{1.21}/M/1) = 3.8830$	0.8223	18.6
5.15	$\rho W(\triangle, v_a = \sqrt{1/6} / M / 1) = 2.0929$ $\rho W(\triangle, v_a = 1/\sqrt{1.21}/E_4/1) = 2.5176$ $\rho W(\triangle, v_a = 1/\sqrt{1.21}/M/1) = 3.8830$ $\rho W(D/E_4/1) = 0.3187$ $\rho W(D/M/1) = 1.6927$	0.6981	0.6
5.15	$\rho W(\triangle, v_a = \sqrt{1/6} / E_2 / 1) = 1.1461$ $\rho W(\triangle, v_a = 1/\sqrt{1.21}/E_4/1) = 2.5176$ $\rho W(\triangle, v_a = 1/\sqrt{1.21}/E_2/1) = 2.9580$ $\rho W(D/E_4/1) = 0.3187$ $\rho W(D/E_2/1) = 0.7568$	0.6803	-1.9

TABLE 5.4 (continued)

An example: Possible approximations to

$$\mu W(\triangle, v_a = \sqrt{1/6} / E_1 / 1) = 0.6934$$

$$(\rho = 0.80)$$

Formula used	Average queuing times having a known value for $\rho = 0.80$	Approximate value	e (%)
5.15	$\mu W(\triangle, v_a = \sqrt{1/6} / E_3 / 1) = 0.8415$	0.6803	-1.9
	$\mu W(\nabla, v_a = 1/\sqrt{1.21} / E_1 / 1) = 2.5176$		
	$\mu W(\nabla, v_a = 1/\sqrt{1.21} / E_2 / 1) = 2.6619$		
	$\mu W(D/E_1/1) = 0.3187$		
	$\mu W(D/E_3/1) = 0.4604$		

5.3.4 The general case

The four distributions of inter-arrival times summarized in Appendices A1.1.3, A1.1.4, A1.1.5 and A1.1.6 and the corresponding distributions of service times summarized in Appendix A1.2 are now considered. These distributions were selected in such a way for no specific reason apart from a general desire of maintaining some degree of consistency throughout this presentation: the same distributions were selected in Chapters 2, 3, 4 and also in sections 5.3.2 and 5.3.3 of the present Chapter; and the same distributions will be selected as distributions of both inter-arrival and service times in subsequent Chapters.

For the queueing system GI/G/1, no attempt was made to verify the approximate formulae (5.3) and (5.4); there was no real incentive for undertaking such a task.

5.4 A MODIFIED SET OF FORMULAE FOR THE A.Q.T.

5.4.1 An assumption and the formulae

For a distribution of either inter-arrival or service times other than the gamma, formulae (5.3) and (5.4) are of no practical use. In fact, given a distribution of inter-arrival times $A(u)$ with $v_a = v_a^*$ or one of service times $B(t)$ with $v_s = v_s^*$, it may well be impossible even to define what $A(u)$ with $v_a = 1$ or $B(t)$ with $v_s = 1$ should be, let alone the calculation of the various queueing times appearing on the right hand side of the two formulae. Therefore we restrict the analysis to $v_a' = 1$, $v_s' = 1$ and also make the assumptions (known to be valid under conditions of heavy traffic²²) that:

$$a) \left[C_b = \frac{W(A(u), v_a = v_a^*/B(t), v_s = 1/1)}{W(A(u), v_a = 1/B(t), v_s = 1/1)} \right] = \left[C_m = \frac{W(A(u), v_a = v_a^*/M/1)}{W(A(u), v_a = 1/M/1)} \right] \quad (5.18)$$

and

$$b) \left[C_a = \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)}{W(A(u), v_a = 1/B(t), v_s = 1/1)} \right] = \left[C_m = \frac{W(M/B(t), v_s = v_s^*/1)}{W(M/B(t), v_s = 1/1)} \right] \quad (5.19)$$

in which case formulae (5.13) and (5.16) take the very interesting forms:

$$W(E, v_a = v_a^*/B(t), v_s = v_s^*/1) \approx \frac{(v_a^*)^2 + (v_s^*)^2}{1 + (v_a^*)^2} W(E, v_a = v_a^*/M/1) \quad (5.20)$$

and

$$W(A(u), v_a = v_a^*/E, v_s = v_s^*/1) \approx \frac{(v_a^*)^2 + (v_s^*)^2}{1 + (v_a^*)^2} W(A(u), v_a = v_a^*/M/1) \quad (5.21)$$

respectively; whereas (5.3) and (5.4) can be written:

$$\begin{aligned}
\frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(A(u), v_a = v_a^*/B(t), v_s = 1/1)} &\approx \frac{2(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(M/B(t), v_s = v_s^*/1)}{W(M/B(t), v_s = 1/1)} + \\
&+ \frac{1-(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(D/B(t), v_s = v_s^*/1)}{W(D/B(t), v_s = 1/1)} \quad (5.22)
\end{aligned}$$

and

$$\begin{aligned}
\frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)} &\approx \frac{2(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(A(u), v_a = v_a^*/M/1)}{W(A(u), v_a = 1/M/1)} + \\
&+ \frac{1-(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(A(u), v_a = v_a^*/D/1)}{W(A(u), v_a = 1/D/1)} \quad (5.23)
\end{aligned}$$

respectively.

We now make the further assumptions (known to be valid under conditions of heavy traffic²²) that:

$$c) \quad \frac{W(A(u), v_a = v_a^*/B(t), v_s = 1/1)}{W(M/B(t), v_s = 1/1)} = \frac{W(A(u), v_a = v_a^*/M/1)}{W(M/M/1)} \quad (5.24)$$

$$d) \quad \frac{W(A(u), v_a = v_a^*/B(t), v_s = 1/1)}{W(D/B(t), v_s = 1/1)} = \frac{W(A(u), v_a = v_a^*/M/1)}{W(D/M/1)} \quad (5.25)$$

$$e) \quad \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)}{W(A(u), v_a = 1/M/1)} = \frac{W(M/B(t), v_s = v_s^*/1)}{W(M/M/1)} \quad (5.26)$$

$$f) \quad \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)}{W(A(u), v_a = 1/D/1)} = \frac{W(M/B(t), v_s = v_s^*/1)}{W(M/D/1)} \quad (5.27)$$

Formulae (5.22) and (5.23) can be greatly simplified:

$$\begin{aligned}
\frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(A(u), v_a = v_a^*/M/1)} &\approx \frac{(v_a^*)^2 (1 + (v_s^*)^2)}{1 + (v_a^*)^2} + \\
&+ \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(D/B(t), v_s = v_s^*/1)}{W(D/M/1)} \quad (5.28)
\end{aligned}$$

and

$$\begin{aligned}
W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1) &\approx (v_s^*)^2 W(A(u), v_a = v_a^*/M/1) + \\
&+ (1 - (v_s^*)^2) W(A(u), v_a = v_a^*/D/1) \quad (5.29)
\end{aligned}$$

Note that these two formulae can be considered as generalizations of the two formulae (5.7) and (5.8) established in section 5.3.1 for the case of gamma inter-arrival and service times.

If we now set $v_s^* = 0$ in (5.28) we obtain:

$$W(A(u), v_a = v_a^*/D/1) \approx \frac{(v_a^*)^2}{1 + (v_a^*)^2} W(A(u), v_a = v_a^*/M/1) \quad (5.30)$$

and if we set $v_a^* = 0$ in (5.29) we obtain:

$$W(D/B(t), v_s = v_s^*/1) \approx (v_s^*)^2 W(D/M/1) \quad (5.31)$$

Finally, by combining either (5.28) and (5.31) or (5.29) and (5.30) we obtain the simple expression:

$$\frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)}{W(A(u), v_a = v_a^*/M/1)} \approx \frac{(v_a^*)^2 + (v_s^*)^2}{1 + (v_a^*)^2} \quad (5.32)$$

It can be recognized that all three approximate formulae (5.11), (5.20) and (5.21) constitute special cases of for-

mula (5.32); the underlying assumptions, however, are:

- a) only, if dealing with the system $\Gamma / G / 1$;
- b) only, if dealing with the system $GI / \Gamma / 1$; and
- all six, if dealing with the system $GI / G / 1$.

5.4.2 On the evaluation of the formulae

As compared to the various intermediate formulae derived in section 5.4.1, formulae (5.20) , (5.21) and (5.32) are by far the simplest and undoubtedly the only ones that can prove useful to the practitioner. Therefore the investigation will be limited to the evaluation of the above-mentioned three formulae, though the evaluation of, say, formulae (5.28) and (5.29) may prove useful if interested in assessing the validity of the underlying assumptions (5.24) , (5.25) and (5.26) , (5.27) respectively; a topic requiring further research.

a) The queueing system $E_k/G/1$

For all four distributions of service times considered in section 5.3.2 it so happens that $C_b < C_m$ in assumption (5.18). In particular, the ratio C_b/C_m was found to be an increasing function of both ρ and v_a^* , tending to unity as ρ and v_a^* tend to 1. In evaluating formula (5.20) instead of (5.13) one would therefore expect that the relative percentage errors incurred

- would remain positive for all $0 < v_a^* , v_s^* , \rho < 1$;
- would be increased in magnitude mainly for low values of ρ and v_a^* and/or high values of v_s^* , the latter due to the relatively large value of the coefficient $((v_a^*)^2 + (v_s^*)^2) / (1 + (v_a^*)^2)$; but
- would not be seriously affected for high values of ρ ,

regardless of v_a^* or v_s^* .

Table 5.5 gives the maximum relative percentage errors incurred if formula (5.20) is used for the approximate evaluation of an A.Q.T. $pW(E, v_a = v_a^*/B(t), v_s = v_s^*/1)$ equal to 0.5, 1.0 and 1.5. (Note that these three values of the A.Q.T. of an arriving customer occur in the quite familiar system M/M/1 for $\rho = 1/3$, 0.50 and 0.60 respectively). Some of the analytical results used in the calculations are included in Tables A3.3, A3.27, A3.29, A3.30 and A3.31 of Appendix 3.

TABLE 5.5

Maximum relative percentage errors incurred if formula (5.20) is used for the approximate evaluation of an A.Q.T. $pW(E, v_a = v_a^*/B(t), v_s = v_s^*/1)$ equal to 0.5, 1.0 and 1.5

$v_s^* =$	$pW \backslash v_a^* =$	0 ... 0.4	0.4 ... 0.6	0.6 ... 0.8	0.8 ... 1.0
0 ... 0.4	0.5	35 ¹	19.6	11.4	5.3
	1.0	20	10.5	6.3	3.0
	1.5	15	7.4	4.5	2.1
0.4 ... 0.6	0.5	35	19.6	11.2	4.8
	1.0	20	10.6	6.4	2.8
	1.5	15	7.4	4.5	2.1
0.6 ... 0.8	0.5	35	19.6	11.0	4.2
	1.0	20	10.5	6.3	2.7
	1.5	15	7.2	4.4	2.0
0.8 ... 1.0	0.5	35	19.5	10.8	4.0
	1.0	20	10.0	6.0	2.4
	1.5	10	6.7	4.1	1.8

¹ The values in this column represent rough estimates rather than accurate values.

Irrespective of v_a^* and v_g^* formula (5.20) seems to give a useful upper bound on the A.Q.T. in the system $E_k/G/1$. If v_a^* and the A.Q.T. to be estimated are not too small, formula (5.20) gives quite good approximations for $W(E/B(t)/1)$.

b) The queueing system $GI/E_t/1$

For all four distributions of inter-arrival times considered in section 5.3.2 it so happens that $C_m < C_a$ in assumption (5.19). In particular, the ratio C_m/C_a was found to be an increasing function of both ρ and v_g^* , tending to unity as ρ and v_g^* tend to 1. In evaluating formula (5.21) instead of (5.16) one would therefore expect that the relative percentage errors incurred

- would be reduced in magnitude, becoming negative, for low values of ρ and v_g^* and/or high values of v_a^* , the latter due to the relatively high value of the coefficient $((v_a^*)^2 + (v_g^*)^2) / (1 + (v_a^*)^2)$;
- would be reduced in magnitude, remaining positive, for "medium" values of ρ , v_a^* and v_g^* ; but
- would not be seriously affected for high values of ρ , regardless of v_a^* or v_g^* .

Table 5.6 gives the maximum relative percentage errors (in absolute value) incurred if formula (5.21) is used for the evaluation of an A.Q.T. $\mu W(A(u), v_a = v_a^*/E, v_g = v_g^*/1)$ equal to 0.5, 1.0 and 1.5, the minus sign showing that the corresponding relative percentage error is negative in value. (Note that these three values of the A.Q.T. of an arriving customer occur in the quite familiar system $M/M/1$ for $\rho = 1/3$, 0.50 and 0.60 respectively). Some of the

analytical results used in the calculations are included in Tables A3.10 , A3.16 , A3.22 , A3.23 , A3.28 and A3.32 of Appendix 3.

TABLE 5.6

Maximum relative percentage errors (in abs.value) incurred if formula (5.21) is used for the approximate evaluation of an A.Q.T. $\mu W(A(u), v_a^* = v_a^*/E, v_g^* = v_g^*/1)$ equal to 0.5, 1.0 and 1.5 (the minus signs indicate that the corresponding relative percentage errors are negative in value)

$v_g^* =$	$\mu W \backslash v_a^* =$	0 ... 0.4	0.4 ... 0.6	0.6 ... 0.8	0.8 ... 1.0
0 ... 0.4	0.5 ¹	23.0	-20	-40	-55
	1.0 ¹	12.6	-15	-40	-50
	1.5 ¹	8.4	-10	-25	-35
0.4 ... 0.6	0.5	23.0	-11.0	-34.0	-46.5
	1.0	12.6	5.9	-20.5	-36.5
	1.5	8.4	4.1	-13.4	-25.0
0.6 ... 0.8	0.5	20.4	10.0	-25.4	-35.5
	1.0	11.3	6.0	-14.0	-28.0
	1.5	7.9	4.1	-8.9	-16.5
0.8 ... 1.0	0.5	12.3	6.1	-12.1	-20.8
	1.0	6.8	3.7	-7.2	-15.8
	1.5	4.8	2.6	-4.5	-9.7

¹ The values in this row for $v_a^* > 0.4$ represent rough estimates rather than accurate values.

Formula (5.21) does not give an upper bound on the A.Q.T. in the system $GI/E_g/1$. However, if v_a^* is not higher than, say, 0.7 and provided that the A.Q.T. to be estimated is not too small, formula (5.21) gives reasonably good approximations for $\mu W(A(u), v_a^* = v_a^*/E, v_g^* = v_g^*/1)$.

c) The queueing system GI/G/1

The effect of the assumptions (5.24) , (5.25) , (5.26) and (5.27) on the degree of approximation achieved if formula (5.32) is used for the evaluation of $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ is difficult to assess. As, however, formula (5.32) was found to give reasonably good approximations in many of the queueing systems studied, it seems sensible at this stage to accept the assumptions as valid and leave the investigation associated with their justification as a topic requiring further research.

We now proceed in evaluating formula (5.32) for:

- $A(u) = D$,  ,  and  with $v_a^2 = 0$, $1/6$, $1/3$ and $1/2$ respectively;
- $B(t) = D$,  ,  and  with $v_s^2 = 0$, $1/6$, $1/3$ and $1/2$ respectively; and
- $0 < \rho < 1$.

Results for $pW(A(u)/B(t)/1)$ are summarized in Tables A4.1 , A4.2 , A4.3 and A4.4 of Appendix 4, the design of the simulation experiments being outlined in Appendix A2.6. The analytical values of $pW(A(u), v_a = v_a^*/M/1)$ required in the calculations are given in Tables A3.2 , A3.10 , A3.22 and A3.23 of Appendix 3. The relative percentage error incurred if formula (5.32) is used for the approximate evaluation of $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ is defined as:

$$e = 100(\text{approximate value} - \text{simulated value})/(\text{sim. value})$$

Table 5.7 gives the relative percentage errors incurred if formula (5.32) is used for the evaluation of an A.Q.T. $pW(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$ equal to 0.5 , 1.0 and 1.5

(Note that these three values of the A.Q.T. of an arriving customer occur in the quite familiar system $M/M/1$ for $\rho = 1/3$, 0.50 and 0.60 respectively). The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated A.Q.T.

TABLE 5.7

Relative percentage errors incurred if formula (5.32) is used for the approximate evaluation of an A.Q.T. $\mu W(A(u), v_s = v_s^*/B(t), v_s = v_s^*/1)$ equal to 0.5, 1.0 and 1.5 (The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated A.Q.T.)

$B(t)$	$\Lambda(u)$ μW	D $v_s^* = 0$	$\triangle$ $v_s^* = 0.408$	$\sqcap$ $v_s^* = 0.577$	∇ $v_s^* = 0.707$
D $v_s^* = 0$	0.5		3.2*	-15.6*	-14.2*
	1.0	-	2.4	-7.9*	-8.2*
	1.5		2.2	-4.8*	-5.6*
$\triangle$ $v_s^* = 0.408$	0.5	29.4*	10.8*	-6.8*	-8.4*
	1.0	17.2*	6.6*	-2.8	-3.8*
	1.5	11.2*	3.6	-1.6	-1.8
$\sqcap$ $v_s^* = 0.577$	0.5	32.4*	15.4*	-1.8	-4.4*
	1.0	16.8*	9.1*	0.3	-1.0
	1.5	12.6*	5.9	1.0	0.4
∇ $v_s^* = 0.707$	0.5	24.6*	13.8*	1.6	-3.2
	1.0	13.6*	9.2*	1.7	-1.2
	1.5	10.2*	7.4*	2.4	-0.4

For the distributions of inter-arrival and service times considered in this section, it seems that for a distribution of inter-arrival times other than the deterministic, formula (5.32) gives reasonably good approximations if the A.Q.T. $\mu W(A(u)/B(t)/1)$ to be estimated has a relatively high value.

In fact, in 9 out of a total of 12 cases presented in Table 5.7 for which $v_a^* \neq 0$, the approximate value of an A.Q.T. equal to 1.5 falls within the corresponding 95 % confidence interval, the determination of which requires approximately 8 minutes of CDC 6400 computer time.

CONCLUSIONS

From a practical viewpoint it seems that the quite simple formula (5.32) may provide useful approximations for the A.Q.T. in the system GI/G/1 provided that this A.Q.T. has a relatively high value (see, f.e., Table 5.8 which combines some of the results presented in Tables 5.1 , 5.5 , 5.6 and 5.7). As, however, the degree of approximation achieved depends to a significant extent on the particular distributions of inter-arrival and service times involved, a reconsideration of the assumptions made in section 5.3.1 may well be justified.

TABLE 5.8

Magnitude of the relative percentage errors incurred if formula (5.32) is used for the approximate evaluation of an A.Q.T. $\mu W(A(u)/B(t)/1)$ equal to 1.5

$\begin{matrix} A(u) \\ B(t) \end{matrix}$	D $v_a^* = 0$	E_6 ,  $v_a^* = 0.408$	E_3 ,  $v_a^* = 0.577$	E_2 ,  $v_a^* = 0.707$
D $v_a^* = 0$	-	0 15	-5 15	-10 ... 10
E_6 ,  $v_a^* = 0.408$	10 25	0 10	-5 5	-10 5
E_3 ,  $v_a^* = 0.577$	10 25	0 10	-5 5	-10 5
E_2 ,  $v_a^* = 0.707$	10 20	0 10	-5 5	-5 5

Formula (5.32) can be simplified further if the explicit formula for the approximate evaluation of $W(\Lambda(u)/M/1)$ which will be discussed in Chapter 8 is used.

CHAPTER 6

APPROXIMATE FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME

IN THE STANDARD SYSTEM

$$\underline{GI / G / r}$$

(A GENERALIZATION OF THE ONES DERIVED IN CHAPTER 5)

6. APPROXIMATE FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.O.T.) IN THE STANDARD SYSTEM

$$\underline{GI / G / r}$$

(A GENERALIZATION OF THE ONES DERIVED IN CHAPTER 5)

6.1 INTRODUCTION

6.1.1 Brief description of the system

The system considered can be described as follows: there are r servers in parallel. Each server has an independently and identically distributed service time distribution $B(t)$. The average service time is μ^{-1} and the coefficient of variation of service times $v_s = \sqrt{\text{var}(t)} \mu$, such that $0 < v_s < 1$. Intervals between successive arrivals are independently distributed with distribution function $A(u)$ such that there is zero probability that two or more customers arrive simultaneously. The average rate of arrival of customers is λ and the coefficient of variation of inter-arrival times $v_a = \sqrt{\text{var}(u)} \lambda$, such that $0 < v_a < 1$. Customers form into a single queue assigned to the first available server. The system is in statistical equilibrium, i.e. the traffic intensity $\rho = \lambda/r\mu < 1$.

6.1.2 On the derivation of analytical results for the A.O.T.

The existence of equilibrium results is discussed by Kiefer and Wolfowitz¹⁹. A general theory, due to Pollaczek⁴⁵, is also discussed by Le Gall²⁶, who in particular examines the case $r = 2$ in some detail. From a practical viewpoint, however, the results do not seem easy to handle.

In case the distributions of inter-arrival and service times are both Erlang, the alternative method of solution discussed by Morse³² (together with the application of Rouché's theorem for the generation of additional equations) is applicable, though the algebra required becomes quite tedious as r and the parameters of the Erlang distributions increase; using this method, Poyntz and Jackson³⁷ determined the complete system of equations and suggested a way of calculating the steady state probabilities.

6.1.3 Alternative ways of obtaining results for the A.Q.T. (the practitioner's viewpoint)

In view of the major difficulties encountered when trying to apply the theory to practical problems, the practitioner may find adequate for the purpose of his study and therefore be content with:

- a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude with all other errors incurred due to the assumption that the queuing model $GI/G/r$ reflects the real life waiting line situation under study. An example of such a formula is the one suggested by Page³⁵ for use in the system $E_k/E_c/r$.
- b) The determination of bounds on the A.Q.T., provided that they are adequately tight for the purpose of the study. Bounds on the A.Q.T. were obtained by Kingman²³, Brumcl-¹lo and Köllerström²⁴.
- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds

on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be emphasized, however, that simulation is not in itself a panacea: apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections of this Chapter, approximate formulae for the A.Q.T. are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is based on the idea (introduced in Chapter 1) that two queueing systems of the same general form $GI/G/r$ with:

- the same type of distribution of inter-arrival times;
- the same type of distribution of service times;
- equal number of servers;
- equal traffic intensities; and
- either equal coefficients of variation of inter-arrival times and unequal coefficients of variation of service times or equal coefficients of variation of service times and unequal coefficients of variation of inter-arrival times,

have a similar behaviour.

6.2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

6.2.1 Establishment of the basic structure of the formulae

Let $W(\lambda(u)/B(t)/r)$ denote the A.Q.T. of an arriving customer. One can reasonably expect that $pW(\lambda(u)/B(t)/r)$ depends on r , $\lambda(u)$ and $B(t)$ only; furthermore, that for r ,

ρ , $B(t)$ and the type of $\Lambda(u)$ fixed, $\mu W(\Lambda(u)/B(t)/r)$ is an increasing function of v_a with limiting values $\mu W(D/B(t)/r)$, corresponding to the system $D/B(t)/r$, if $v_a = 0$ and, say, $\mu W(\Lambda(u), v_a=1/B(t)/r)$, corresponding to the system $\Lambda(u), v_a=1/B(t)/r$, if $v_a=1$.

Two such relationships between $\mu W(\Lambda(u)/B(t)/r)$ and v_a are shown in Figure 6.1. They are both related to a queuing system

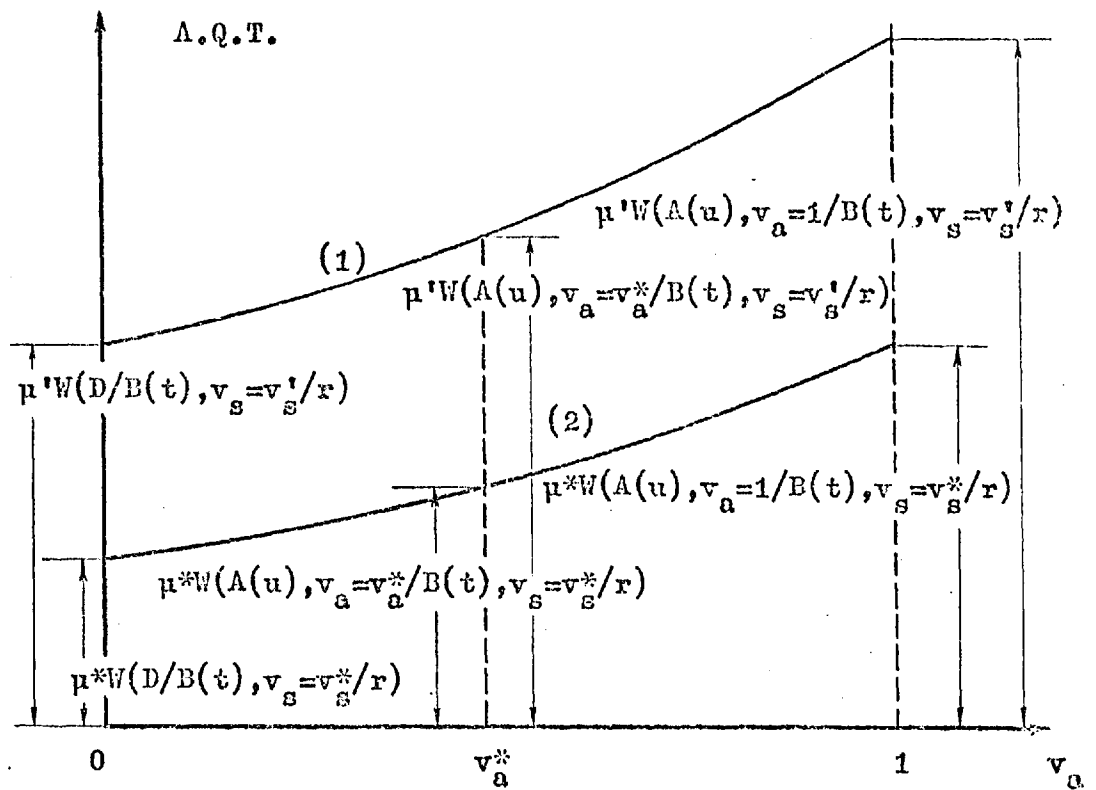


Fig. 6.1 $\mu W(\Lambda(u)/B(t)/r)$ as a function of v_a for two queuing systems with the same types of distributions of inter-arrival and service times, equal numbers of servers and equal traffic intensities.

system of the general form $\Lambda(u)/B(t)/r$ with the same types of distributions of inter-arrival and service times and equal numbers of servers. Curve (1) represents a situation in which $\lambda = \lambda'$, $\mu = \mu'$ and $v_s = v_s'$; curve (2) represents

a situation in which $\lambda = \lambda^*$, $\mu = \mu^*$, such, however, that $\rho' = \rho^* = \rho$ and $v_s = v_s^* < v_s'$.

We now proceed in writing a relationship of the general form:

$$\begin{aligned} \frac{\mu^* W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{\mu' W(A(u), v_a = v_a^*/B(t), v_s = v_s'/r)} &\approx (1 - S_a) \frac{\mu^* W(D/B(t), v_s = v_s^*/r)}{\mu' W(D/B(t), v_s = v_s'/r)} + \\ &+ S_a \frac{\mu^* W(A(u), v_a = 1/B(t), v_s = v_s^*/r)}{\mu' W(A(u), v_a = 1/B(t), v_s = v_s'/r)} \end{aligned} \quad (6.1)$$

with v_a^* any value of v_a between 0 and 1 and S_a some "similarity factor" with limiting values 0 and 1 for $v_a^* = 0$ and 1 respectively, regardless of v_s' , v_s^* , ρ , r , $A(u)$ or $B(t)$.

One can also expect that for r , ρ , $A(u)$ and the type of $B(t)$ fixed, $\mu W(A(u)/B(t)/r)$ is an increasing function of v_s with limiting values $\mu W(A(u)/D/r)$, corresponding to the system $A(u)/D/r$, if $v_s = 0$ and, say, $\mu W(A(u)/B(t), v_s = 1/r)$, corresponding to the system $A(u)/B(t), v_s = 1/r$, if $v_s = 1$. A reasoning similar to the one associated with Figure 6.1 leads to the relationship:

$$\begin{aligned} \frac{\mu^* W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{\mu' W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)} &\approx (1 - S_s) \frac{\mu^* W(A(u), v_a = v_a^*/D/r)}{\mu' W(A(u), v_a = v_a^*/D/r)} + \\ &+ S_s \frac{\mu^* W(A(u), v_a = v_a^*/B(t), v_s = 1/r)}{\mu' W(A(u), v_a = v_a^*/B(t), v_s = 1/r)} \end{aligned} \quad (6.2)$$

with v_s^* any value of v_s between 0 and 1 and S_s some "similarity factor" with limiting values 0 and 1 for $v_s^* = 0$ and 1 respectively, regardless of v_a' , v_a^* , ρ , r , $A(u)$ or $B(t)$.

(6.1) is an expression giving $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ in terms of $W(A(u), v_a = v_a^*/B(t), v_s = v_s'/r)$ and (6.2) is an expression giving $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ in terms of $W(A(u), v_a = v_a'/B(t), v_s = v_s^*/r)$; consequently, if the relative percentage errors incurred are small,

- a) the hypothesis that the systems $A(u), v_a = v_a^*/B(t), v_s = v_s^*/r$ and $A(u), v_a = v_a^*/B(t), v_s = v_s'/r$ with $\rho^* = \rho'$ and the systems $A(u), v_a = v_a^*/B(t), v_s = v_s^*/r$ and $A(u), v_a = v_a'/B(t), v_s = v_s^*/r$ with $\rho^* = \rho'$ have a similar behaviour would be established, the quantities on the right hand side being regarded as similarity ratios;
- b) either (6.1) or (6.2) or indeed some convenient combination of them could be used as an approximate formula for the evaluation of $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ provided that $W(A(u), v_a = v_a^*/B(t), v_s = v_s'/r)$ and/or $W(A(u), v_a = v_a'/B(t), v_s = v_s^*/r)$ with $v_s' = v_s^* + \Delta v_s^*$, $v_s^* + 2\Delta v_s^*$, ... or even 1, $v_a' = v_a^* + \Delta v_a^*$, $v_a^* + 2\Delta v_a^*$, ... or even 1 and the two similarity ratios were known.

6.2.2 A method for the determination of the similarity factors S_a and S_s

The method suggested here has three main advantages:

- a) it is simple in its evaluation (provided that it can be applied);

- b) it provides very simple expressions for S_a and S_s ; and
 c) it is justified in terms of a) and b) of paragraph 6.2.1, particularly if the value of ρ is high, a situation likely to be encountered in practice and, also, one in which numerical results are usually difficult to obtain either analytically or by simulation.

It consists of two steps:

- a) the approximate formulae, for the average queueing times in the basic expressions (6.1) and (6.2) under conditions of heavy traffic, are found.

For the system considered:

$$(pW(\Lambda(u)/B(t)/r))_{\rho \rightarrow 1} = \frac{(v_a)^2 + \rho^2(v_s)^2}{2\rho(1 - \rho)r}$$

as suggested by Kingman²³, valid for $0 < v_a, v_s < 1$.

- b) these formulae are then introduced into (6.1) and (6.2); simple algebra yields the desired expressions for S_a and S_s .

For the system considered:

$$S_a = \frac{(v_a^*)^2(1 + (v_s')^2)}{(v_a^*)^2 + (v_s')^2} \quad \text{and} \quad S_s = \frac{(v_s^*)^2(1 + (v_a')^2)}{(v_a')^2 + (v_s^*)^2},$$

in which case (6.1) and (6.2) take the forms:

$$\begin{aligned} & \frac{W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s'/r)} \approx \\ & \approx \frac{(v_a^*)^2(1 + (v_s')^2)}{(v_a^*)^2 + (v_s')^2} \cdot \frac{W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/r)}{W(\Lambda(u), v_a = 1/B(t), v_s = v_s'/r)} + \end{aligned}$$

$$+ \frac{(v_s')^2(1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s')^2} \cdot \frac{W(D/B(t), v_s = v_s^*/r)}{W(D/B(t), v_s = v_s'/r)} \quad (6.3)$$

and

$$\begin{aligned} & \frac{W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{W(\Lambda(u), v_a = v_a'/B(t), v_s = v_s^*/r)} \approx \\ & \approx \frac{(v_s^*)^2(1 + (v_a')^2)}{(v_a')^2 + (v_s^*)^2} \cdot \frac{W(\Lambda(u), v_a = v_a^*/B(t), v_s = 1/r)}{W(\Lambda(u), v_a = v_a'/B(t), v_s = 1/r)} + \\ & + \frac{(v_a')^2(1 - (v_s^*)^2)}{(v_a')^2 + (v_s^*)^2} \cdot \frac{W(\Lambda(u), v_a = v_a^*/D/r)}{W(\Lambda(u), v_a = v_a'/D/r)} \quad (6.4) \end{aligned}$$

respectively.

The method described is not the only one possible. An alternative method is presented in Chapter 10 and many more could be envisaged.

6.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

6.3.1 The queueing system $E_k/E_l/r$

Consider first the case in which both inter-arrival and service times have an Erlang distribution (the Erlang distribution is discussed in Appendix A1.1.2). For this case, $W(\Lambda(u), v_a = 1/B(t)/r) = W(M/E/r)$, corresponding to the system with exponential inter-arrival times and $W(\Lambda(u)/B(t), v_s = 1/r) = W(E/M/r)$, corresponding to the system with exponential service times. If, in addition, v_a' and v_s' are set equal to 1, (6.3) and (6.4) are greatly simplified:

$$\frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r)}{W(E, v_a = v_a^*/M/r)} \approx \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(M/E, v_s = v_s^*/r)}{W(M/M/r)} +$$

$$+ \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(D/E, v_s = v_s^*/r)}{W(D/M/r)} \quad (6.5)$$

and

$$\begin{aligned} \frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r)}{W(M/E, v_s = v_s^*/r)} &\approx \frac{2(v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(E, v_a = v_a^*/M/r)}{W(M/M/r)} + \\ &+ \frac{1 - (v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(E, v_a = v_a^*/D/r)}{W(M/D/r)} \end{aligned} \quad (6.6)$$

If we now set $v_s^* = 0$ in (6.5) we obtain:

$$\frac{W(E, v_a = v_a^*/D/r)}{W(E, v_a = v_a^*/M/r)} \approx \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(M/D/r)}{W(M/M/r)} \quad (6.7)$$

and if we set $v_a^* = 0$ in (6.6) we obtain:

$$\frac{W(D/E, v_s = v_s^*/r)}{W(M/E, v_s = v_s^*/r)} \approx \frac{2(v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W(D/M/r)}{W(M/M/r)} \quad (6.8)$$

Finally, by combining either (6.5) and (6.8) or (6.6) and (6.7) we obtain:

$$\frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r)}{W(E, v_a = v_a^*/M/r)} \approx \frac{2((v_a^*)^2 + (v_s^*)^2)}{(1 + (v_a^*)^2)(1 + (v_s^*)^2)} \cdot \frac{W(M/E, v_s = v_s^*/r)}{W(M/M/r)} \quad (6.9)$$

and by introducing into (6.9) formula (3.4) established in Chapter 3 for the approximate evaluation of $W(M/E, v_s = v_s^*/r)$ we get:

$$\begin{aligned}
 & \frac{W(E, v_a = v_a^*/E, v_s = v_s^*/r)}{W(E, v_a = v_a^*/M/r)} \approx \\
 & \approx \frac{2((v_a^*)^2 + (v_s^*)^2)}{(1+(v_a^*)^2)(1+(v_s^*)^2)} \cdot \frac{(v_s^*)^2 W(M/M/r) + (1-(v_s^*)^2) W(M/D/r)}{W(M/M/r)}
 \end{aligned} \tag{6.10}$$

Note that for $r = 1$ both formulae (6.9) and (6.10) coincide with formula (5.11) established in Chapter 5 for the approximate evaluation of $W(E, v_a = v_a^*/E, v_s = v_s^*/1)$.

In case $(v_a^*)^2 = 1/2$, $(v_s^*)^2 = 1/2$ (the only case for which analytical results were available) the two expressions (6.9) and (6.10) can be written:

$$W(E_2/E_2/r) \approx \frac{8}{9} \frac{W(M/E_2/r)}{W(M/M/r)} W(E_2/M/r) \tag{6.11}$$

and

$$W(E_2/E_2/r) \approx \frac{4}{9} \frac{W(M/M/r) + W(M/D/r)}{W(M/M/r)} W(E_2/M/r) \tag{6.12}$$

respectively.

An example (the queueing system $E_2/E_2/2$)

The evaluation of (6.11) and (6.12) is summarized in Table 6.1 : The analytical values for $W(E_2/E_2/2)$ were calculated by using the formula presented in Appendix A2.5 ; the values of $W(M/E_2/2)$, $W(E_2/M/2)$, $W(M/M/r)$ and $W(M/D/r)$ required in the calculations are included in Tables A3.25, A3.9, A3.1 and A3.24 of Appendix A3. The relative percentage error e is defined as:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{theor. value})$$

TABLE 6.1

An example: Evaluation of formulae (6.11) and (6.12)
in the special case $r = 2$

ρ	$\Lambda.Q.T.$ (analyt. results)	Use of form. (6.11)		Use of form. (6.12)	
		$\Lambda.Q.T.$	e	$\Lambda.Q.T.$	e
0.40	0.057608	0.064159	11.4	0.064045	11.2
0.45	0.083735	0.091642	9.4	0.091443	9.2
0.50	0.118060	0.127324	7.8	0.127024	7.6
0.55	0.163084	0.173692	6.5	0.173276	6.2
0.60	0.222564	0.234497	5.4	0.233952	5.1
0.65	0.302419	0.315651	4.4	0.314965	4.1
0.70	0.412534	0.427036	3.5	0.426202	3.3
0.75	0.570734	0.586475	2.8	0.585486	2.6
0.80	0.812701	0.829650	2.1	0.828500	1.9
0.85	1.221738	1.239862	1.5	1.238548	1.4
0.90	2.047816	2.067086	0.9	2.065604	0.9
0.95	4.540909	4.561293	0.4	4.559643	0.4

A comparison between Table 6.1 and Table 4.1 of Chapter 4 suggests that formulae (4.12), (4.13) and (4.14) give better approximations for $W(E_2/E_2/2)$ than formulae (6.11) and (6.12). Before, however, drawing general conclusions we proceed in evaluating the fairly easily applicable formula (6.12) for higher values of r . Table 6.3 gives the relative percentage errors incurred if formula (6.12) is used for the approximate evaluation of $pW(E_2/E_2/r)$ with $r = 1, 2, 3, 4, 5$ and 6. The analytical values for $pW(E_2/E_2/r)$, $r > 2$, included in Table 6.2 were obtained from Page³⁵; and the analytical values for $pW(E_2/E_2/1)$, included in the same Table, from Table A3.26 of Appendix 3. Finally, the values of $W(M/M/r)$,

TABLE 6.2

Analytical values for $\mu W(E_2/E_2/r)$

ρ	$r = 1$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$
0.30	0.130952	0.0235	0.0062			
0.40	0.235457	0.0576	0.0205	0.0085		
0.50	0.390388	0.1181	0.0512	0.0255	0.0142	0.0082
0.60	0.650647	0.2222	0.1103	0.0639	0.0400	0.0265
0.70	1.039055	0.4125	0.2275	0.1441	0.0988	0.0712

$W(M/D/r)$ and $W(E_2/M/r)$ required in the calculations were obtained from Tables A3.1, A3.24 and A3.9 of Appendix 3.

TABLE 6.3

Relative percentage errors incurred if (6.12) is used for the evaluation of $W(E_2/E_2/r)$

$\rho \backslash r$	1	2	3	4	5	6
0.30	10.3	16.7	24.2			
0.40	7.6	11.2	14.6	18.8		
0.50	5.5	7.6	9.4	13.8	14.1	17.1
0.60	4.0	5.1	6.7	7.2	8.2	9.1
0.70	2.6	3.3	3.9	4.6	5.2	5.8

A comparison between Table 6.3 and Tables 4.3 and 4.9 shows that as r increases, the relative percentage errors incurred in evaluating $W(E_2/E_2/r)$ become smaller if formula (6.12) instead of (4.14) or (4.15) is used. Furthermore, the relative percentage errors incurred, if formula (6.12) is used for the evaluation of an A.Q.T. $W(E_2/E_2/r)$ equal to a given value, decrease as the number of servers increases (see, for example,

TABLE 6.4

Relative percentage errors incurred if (6.12)
is used for the evaluation of $W(E_2/E_2/r)$

r	1	2	3	4	5	6
$\mu W(E_2/E_2/r) = 0.05$	15	12	9	8	7	7
$\mu W(E_2/E_2/r) = 0.10$	12	8	7	6	5	5

Table 6.4 and compare it with Tables 4.4 and 4.10).

6.3.2 The general case

The four distributions of inter-arrival times summarized in Appendices A1.1.3, A1.1.4, A1.1.5 and A1.1.6 and the corresponding distributions of service times summarized in Appendix A1.2 are now considered. These distributions were selected in such a way for no specific reason apart from a general desire of maintaining some degree of consistency throughout this presentation: the same distributions were selected in Chapters 2, 3, 4 and 5 and the same distributions will be selected as distributions of both inter-arrival and service times in subsequent Chapters.

For the queuing system considered in this Chapter, no attempt was made to verify the approximate formulae (6.3) and (6.4); there was no real incentive for undertaking such a task.

6.4 A MODIFIED SET OF FORMULAE FOR THE A.Q.T.

6.4.1 An assumption and the formulae

For a distribution of either inter-arrival or service times other than the Erlang, formulae (6.3) and (6.4) are of no prac-

tical use. In fact, given a distribution of inter-arrival times $A(u)$ with $v_a = v_a^*$ or one of service times with $v_s = v_s^*$, it may well be impossible even to define what $A(u)$ with $v_a = 1$ or $B(t)$ with $v_s = 1$ should be, let alone the calculation of the various average queueing times appearing on the right hand side of the two formulae. Therefore, we restrict the analysis to $v_a' = 1$, $v_s' = 1$ and also make the assumptions (known to be valid under conditions of heavy traffic²³) that:

$$a) \quad \frac{W(A(u), v_a=1/B(t), v_s=v_s^*/r)}{W(A(u), v_a=1/B(t), v_s=1/r)} \approx \frac{W(M/B(t), v_s=v_s^*/r)}{W(M/B(t), v_s=1/r)} \quad (6.15)$$

$$b) \quad \frac{W(A(u), v_a=v_a^*/B(t), v_s=1/r)}{W(M/B(t), v_s=1/r)} \approx \frac{W(A(u), v_a=v_a^*/M/r)}{W(M/M/r)} \quad (6.14)$$

$$c) \quad \frac{W(A(u), v_a=v_a^*/B(t), v_s=1/r)}{W(D/B(t), v_s=1/r)} \approx \frac{W(A(u), v_a=v_a^*/M/r)}{W(D/M/r)} \quad (6.15)$$

in which case formula (6.5) can be written:

$$\begin{aligned} \frac{W(A(u), v_a=v_a^*/B(t), v_s=v_s^*/r)}{W(A(u), v_a=v_a^*/M/r)} &\approx \frac{2(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(M/B(t), v_s=v_s^*/r)}{W(M/M/r)} + \\ &+ \frac{1-(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(D/B(t), v_s=v_s^*/r)}{W(D/M/r)} \end{aligned} \quad (6.16)$$

and

$$d) \quad \frac{W(A(u), v_a=v_a^*/B(t), v_s=1/r)}{W(A(u), v_a=1/B(t), v_s=1/r)} \approx \frac{W(A(u), v_a=v_a^*/M/r)}{W(A(u), v_a=1/M/r)} \quad (6.17)$$

$$e) \quad \frac{W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/r)}{W(\Lambda(u), v_a = 1/M/r)} \approx \frac{W(M/B(t), v_s = v_s^*/r)}{W(M/M/r)} \quad (6.18)$$

$$f) \quad \frac{W(\Lambda(u), v_a = 1/B(t), v_s = v_s^*/r)}{W(\Lambda(u), v_a = 1/D/r)} \approx \frac{W(M/B(t), v_s = v_s^*/r)}{W(M/D/r)} \quad (6.19)$$

in which case formula (6.4) can be written:

$$\begin{aligned} \frac{W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{W(M/B(t), v_s = v_s^*/r)} &\approx \frac{2(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(\Lambda(u), v_a = v_a^*/M/r)}{W(M/M/r)} + \\ &+ \frac{1-(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(\Lambda(u), v_a = v_a^*/D/r)}{W(M/D/r)} \end{aligned} \quad (6.20)$$

If we now set $v_s^* = 0$ in (6.16) we obtain:

$$\frac{W(\Lambda(u), v_a = v_a^*/D/r)}{W(\Lambda(u), v_a = v_a^*/M/r)} \approx \frac{2(v_a^*)^2}{1+(v_a^*)^2} \cdot \frac{W(M/D/r)}{W(M/M/r)} \quad (6.21)$$

and if we set $v_a^* = 0$ in (6.20) we obtain:

$$\frac{W(D/B(t), v_s = v_s^*/r)}{W(M/B(t), v_s = v_s^*/r)} \approx \frac{2(v_s^*)^2}{1+(v_s^*)^2} \cdot \frac{W(D/M/r)}{W(M/M/r)} \quad (6.22)$$

Finally, by combining either (6.16) and (6.22) or (6.20) and (6.21) we obtain:

$$\begin{aligned} \frac{W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{W(\Lambda(u), v_a = v_a^*/M/r)} &\approx \frac{2((v_a^*)^2 + (v_s^*)^2)}{(1+(v_a^*)^2)(1+(v_s^*)^2)} \cdot \\ &\cdot \frac{W(M/B(t), v_s = v_s^*/r)}{W(M/M/r)} \end{aligned} \quad (6.23)$$

and by introducing into (6.23) formula (3.7) established in Chapter 3 for the approximate evaluation of $W(M/B(t), v_s = v_s^*/r)$ we get:

$$\begin{aligned} & \frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)}{W(A(u), v_a = v_a^*/M/r)} \approx \\ & \approx \frac{2((v_a^*)^2 + (v_s^*)^2)}{(1+(v_a^*)^2)(1+(v_s^*)^2)} \cdot \frac{(v_s^*)^2 W(M/M/r) + (1-(v_s^*)^2) W(M/D/r)}{W(M/M/r)} \end{aligned} \quad (6.24)$$

Note that for $r = 1$ formulae (6.23) and (6.24) coincide with formula (5.32) established in Chapter 5 for the approximate evaluation of $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$.

6.4.2 On the evaluation of the formulae

As compared to the various intermediate formulae derived in section 6.4.1, formula (6.24) is by far the simplest and undoubtedly the only one that can prove useful to the practitioner. Therefore the investigation will be limited to the evaluation of formula (6.24), though the evaluation of, say, formulae (6.16) and (6.20) may prove useful in assessing the validity of the assumptions made.

In particular we proceed in evaluating formula (6.24) for:

- $A(u) = D$, $\triangleleft$, $\sqsubset$ and $\triangle$ with $v_a^2 = 0, 1/6, 1/3$ and $1/2$ respectively;
- $B(t) = D$, $\triangleleft$, $\sqsubset$ and $\triangle$ with $v_s^2 = 0, 1/6, 1/3$ and $1/2$ respectively;
- $r \leq 6$; and
- $0 < \rho < 1$.

Results for $\mu W(A(u)/B(t)/r)$ are summarized in Tables A4.5 , A4.6 , A4.7 , A4.8 , A4.9 , A4.10 , A4.11 and A4.12 of Appendix 4; and results for $\mu W(A(u)/B(t)/1)$ are summarized in Tables A4.1 , A4.2 , A4.3 and A4.4 of Appendix 4. Results for $\mu W(A(u)/M/r)$ are included in Tables A3.2 , A3.11 , A3.12 , A3.13 , A3.14 , A3.15 , A3.22 and A3.23 of Appendix 3; and the analytical values of $W(M/M/r)$ and $W(M/D/r)$ required in the calculations are given in Tables A3.1 and A3.24 of Appendix 3. The relative percentage error incurred if formula (6.24) is used for the approximate evaluation of $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ is defined as:

$$e = 100(\text{approximate value} - \text{simulated value})/(\text{simul. value})$$

a) Evaluation of formula (6.24) for $A(u)$ and $B(t)$ "fixed"

For given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, any value of the average queuing time $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ can be approximated with a relative percentage error which depends on ρ and r in the following way:

it is positive or negative depending on the values of ρ and r ;

it shows a trend of decreasing in absolute value as the A.Q.T. to be estimated and, hence, the traffic intensity increase, tending to 0 as ρ tends to 1; and

it does not seem to be sensitive to the value of r (at least for $r \leq 6$).

Table 6.5 gives the relative percentage errors incurred if formula (6.24) is used for the evaluation of an A.Q.T. $\mu W(\triangle, v_a = \sqrt{1/2} / \triangle, v_s = \sqrt{1/3} / r)$ equal to 0.2 , 0.3 , 0.4 and 0.5 for values of r ranging from 1 to 6.

The asterisks(*) indicate that the approximate values fall

outside the corresponding 95 % confidence interval on the simulated A.Q.T. (see Table A4.5 of Appendix 4).

TABLE 6.5

Relative percentage errors incurred if (6.24)

is used for the evaluation of

$$W(\triangle, v_a = \sqrt{1/2} / \triangle, v_s = \sqrt{1/3} / r)$$

r	1	2	3	4	5	6
$pW(\triangle / \triangle / r) = 0.2$	-10.8*	-2.0	0.9	1.8	2.4	2.7
$pW(\triangle / \triangle / r) = 0.3$	-7.9*	-0.8	1.3	2.0	2.2	2.0
$pW(\triangle / \triangle / r) = 0.4$	-5.9*	0.1	1.7	1.9	1.8	1.3
$pW(\triangle / \triangle / r) = 0.5$	-4.4*	0.7	1.8	1.7	1.2	0.5

A comparison between Table 6.5 and Tables 4.11 and 4.18 of Chapter 4 may be useful; but no general conclusions can be drawn.

b) Evaluation of formula (6.24) for r and A(u) "fixed"

For a given value of r and a given distribution of inter-arrival times A(u) with coefficient of variation v_a^* , any value of the A.Q.T. $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ can be approximated with a relative percentage error which seems to vary with v_s^* according to a U - shaped curve; at least for $v_s^* < 0.7$. Table 6.6, for example, gives the relative percentage errors incurred if formula (6.24) is used for the evaluation of an A.Q.T. $pW(\triangle, v_a = \sqrt{1/6} / B(t), v_s = v_s^*/6)$ equal to 0.2. The construction of this Table is based on the relatively few results that are summarized in Table A4.10 of Appendix 4; therefore, the values of e in the Table should be viewed as rough estimates rather than as accurate

values. The asterisk(*) indicates that the approximate value falls outside the corresponding 95 % confidence interval on the simulated A.Q.T. (A comparison between Table 6.6 and Table 4.12 of Chapter 4 may be useful).

TABLE 6.6

Relative percentage errors incurred if (6.24) is used for the evaluation of an A.Q.T.

$pW(\triangle, v_g = \sqrt{1/6} / B(t), v_g = v_g^* / 6)$ equal to 0.2

$B(t)$	$(v_g^*)^2$	e (%)
D	0	14*
	1/6	4.0
	1/3	4.5
	1/2	6.1

c) Evaluation of formula (6.24) for r and $B(t)$ "fixed"

For a given value of r and a given distribution of service times $B(t)$ with coefficient of variation v_g^* , any value of the A.Q.T. $W(\Lambda(u), v_a = v_a^* / B(t), v_g = v_g^* / r)$ can be approximated with a relative percentage error which seems to vary with v_a^* according to a U - shaped curve; at least for $v_a^* < 0.7$. Table 6.7, for example, gives the relative percentage errors incurred if formula (6.24) is used for the evaluation of an A.Q.T. $pW(\Lambda(u), v_a = v_a^* / \square, v_g = \sqrt{1/3} / 6)$ equal to 0.2. The construction of this Table is based on the relatively few results that are summarized in Tables A4.9, A4.10, A4.11 and A4.12 of Appendix 4; therefore, the values of e in the Table should be viewed as rough estimates rather than as accurate values. The asterisk(*)

indicates that the approximate value falls outside the corresponding 95 % confidence interval on the simulated A.Q.T. (A comparison between Table 6.7 and Table 4.13 of Chapter 4 may be useful).

TABLE 6.7

Relative percentage errors incurred if (6.24)

is used for the evaluation of an A.Q.T.

$\mu W(\Lambda(u), v_a = v_a^* / \sqrt{1/3} / 6)$ equal to 0.2

$\Lambda(u)$	$(v_a^*)^2$	e (%)
D	0	9*
	1/6	4.0
	1/3	1.3
	1/2	2.8

d) An example: Evaluation of formula (6.24) for some 6-server queuing systems

The values of ρ and the ones of the simulated A.Q.T., the approximate values of the A.Q.T. as obtained from formula (6.24) and the corresponding relative percentage errors incurred are summarized in Table 6.8 for 33 in total 6-server queuing systems. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated average queuing time.

We now proceed in interpreting the results presented in Table 6.8.

- On average, the total A.Q.T. per unit time

$\Lambda W(\Lambda(u)/B(t)/6)$ for the systems presented in Table 6.8

TABLE 6.8

An example: Evaluation of formula (6.24)
for 33 6-server queueing systems

Queueing system considered	Simulation results		Approximate results	
	ρ	A.Q.T.	A.Q.T.	e (%)
D / Δ / 6	0.90	0.0773	0.0886*	14.7
"	0.95	0.2127	0.2247*	5.6
D / $\square$ / 6	0.90	0.1586	0.1765*	11.3
"	0.95	0.4259	0.4484*	5.3
D / ∇ / 6	0.85	0.1228	0.1348*	9.8
"	0.90	0.2505	0.2655*	6.0
Δ / D / 6	0.90	0.0716	0.0934*	30.4
"	0.95	0.2021	0.2299*	13.8
Δ / Δ / 6	0.85	0.0877	0.0979*	11.7
"	0.90	0.1771	0.1857	4.8
"	0.95	0.4531	0.4587	1.2
Δ / $\square$ / 6	0.85	0.1356	0.1458	7.5
"	0.90	0.2693	0.2772	2.9
Δ / ∇ / 6	0.80	0.1023	0.1114*	9.0
"	0.85	0.1821	0.1940	6.5
$\square$ / D / 6	0.85	0.0963	0.1056*	9.7
"	0.90	0.1819	0.1944*	6.9
$\square$ / Δ / 6	0.85	0.1538	0.1569	2.0
"	0.90	0.2898	0.2899	0.0

TABLE 6.8 (continued)

An example: Evaluation of formula (6.24)
for 33 6-server queueing systems

Queueing system considered	Simulation results		Approximate results	
	ρ	A.Q.T.	A.Q.T.	e (%)
$\square / \square / 6$	0.85	0.2052	0.2077	1.2
"	0.90	0.3859	0.3848	-0.3
$\square / \triangle / 6$	0.80	0.1470	0.1534	4.3
"	0.85	0.2506	0.2593	3.5
$\triangle / D / 6$	0.80	0.0973	0.1012*	4.0
"	0.85	0.1598	0.1655*	3.6
"	0.90	0.2913	0.2990	2.6
$\triangle / \triangle / 6$	0.80	0.1293	0.1332	3.0
"	0.85	0.2134	0.2187	2.5
$\triangle / \square / 6$	0.75	0.1012	0.1041	2.9
"	0.80	0.1601	0.1645	2.7
"	0.85	0.2655	0.2714	2.2
$\triangle / \triangle / 6$	0.75	0.1239	0.1237	-0.2
"	0.80	0.1970	0.1961	-0.5

can be easily found equal to 1.02. In the familiar queueing system M/M/1 the same value of the total A.Q.T. per unit time corresponds to $\rho = 0.62$

- In 19 out of the 33 queueing systems considered, the approximate value of the A.Q.T. as calculated by using formula (6.24) falls within the corresponding 95 % con-

fidence interval on the simulated A.Q.T., the latter requiring approximately 8 minutes of CDC 6400 time to determine.

- In 20 out of the 33 queueing systems considered, both v_a^* and v_s^* are $\neq 0$. For those 20 systems formula (6.24) gives an approximate value for the A.Q.T., which falls within the corresponding 95 % confidence on the simulated A.Q.T., 18 times.
- A comparison between Table 6.8 and Table 4.19 of Chapter 4 shows that formula (6.24), though much simpler, does not give worse approximations than formulae (4.22) and (4.23).

CONCLUSIONS

Formula (6.24) seems to give reasonably good approximations for the A.Q.T. of an arriving customer in the system $GI/G/r$. (see, for example, Table 6.9 which gives the magnitude of the relative percentage errors incurred if formula (6.24) is used for the approximate evaluation of an average queueing time $\mu W(A(u)/B(t)/6)$ equal to 0.2). It can be simplified further if (2.7) or (2.8) established in Chapter 2 is used for the approximate evaluation of $W(A(u), v_a = v_a^*/M/r)$; and, also, if the explicit formulae for the approximate evaluation of $W(M/D/r)$ and $W(D/M/r)$ which will be presented in Chapter 7 are used. In case v_a^* is not close to unity, the explicit formula for the approximate evaluation of $W(A(u), v_a = v_a^*/M/1)$ which will be presented in Chapter 8 simplifies the necessary calculations still further.

TABLE 6.9

Magnitude of the relative percentage errors incurred
 if formula (6.24) is used for the approximate
 evaluation of an A.Q.T. $\mu W(\Lambda(u)/B(t)/6)$ equal to 0.2

$\Lambda(u) \backslash B(t)$	D $(v_a^*)^2 = 0$	$\triangle \backslash$ $(v_a^*)^2 = 1/6$	$\sqcap \backslash$ $(v_a^*)^2 = 1/3$	$\nabla \backslash$ $(v_a^*)^2 = 1/2$
D $(v_a^*)^2 = 0$	-	10 ... 20	0 10	0 5
$\triangle \backslash$ $(v_a^*)^2 = 1/6$	5 10	0 5	0 5	0 5
$\sqcap \backslash$ $(v_a^*)^2 = 1/3$	5 10	0 ... 10	0 5	0 5
$\nabla \backslash$ $(v_a^*)^2 = 1/2$	5 10	0 ... 10	0 5	-5 5

CHAPTER 7

APPROXIMATE EXPLICIT FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUING TIME

IN THE STANDARD SYSTEMS

M / D / r AND D / M / r

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7. APPROXIMATE EXPLICIT FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUING TIME (A.Q.T.) IN THE STANDARD SYSTEMS

$$\underline{M / D / r \quad \text{AND} \quad D / M / r}$$

7.1 INTRODUCTION

7.1.1 Brief description of the systems

The system $M/D/r$ is a special case of the more general queueing system $M/G/r$ already dealt with in Chapter 3. In the system $M/D/r$ the coefficient of variation of service times is equal to zero.

The system $D/M/r$ is a special case of the more general queueing system $GI/M/r$ already dealt with in Chapter 2. In the system $D/M/r$ the coefficient of variation of inter-arrival times is equal to zero.

7.1.2 On the derivation of analytical results for the A.Q.T.

The equilibrium theory of the queue $M/D/r$ can be found in almost all basic textbooks on queueing theory. Saaty⁴⁰, for example, discusses two methods of obtaining analytical results for the A.Q.T. The first method gives the results in terms of a probability generating function and requires the determination of $r - 1$ roots of an equation and, also, the differentiation of the function, a task that gets to be rather difficult as r gets large. The second method gives the A.Q.T. in explicit form; the explicit formula is, however, difficult to programme (underestimation of the A.Q.T. is likely for $\rho > 0.7$) and time consuming to evaluate. Both methods were

applied in order to obtain analytical results for the A.Q.T. in case $r \leq 10$; the explicit formula only was used in order to obtain analytical results for the A.Q.T. in case $r > 10$; (the results given by Shelton⁴⁵, though adequate for many practical purposes, do not provide the accuracy which was required in the present study).

The equilibrium theory of the queue D/M/r is discussed by Kendall¹⁸, Winsten⁴⁷ and Cox and Smith⁸. The theory was applied in order to obtain expressions for the A.Q.T.; these expressions for up to $r = 6$ are given in Appendix A2.1. It is interesting to note that the formulae for the A.Q.T. become increasingly complicated as r increases.

7.1.3 Alternative ways of obtaining results for the A.Q.T. (the practitioner's viewpoint)

In view of the major difficulties encountered when trying to apply the theory to practical problems (and indeed when trying to find results in published form) the practitioner may find adequate for the purpose of his study and therefore be content with:

a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude with all other errors incurred due to the assumption that the queueing model M/D/r or D/M/r reflects the real life waiting line situation under study. Examples of such formulae are the ones suggested by Molina and Crommelin (both are mentioned in the book by Kaufmann¹⁷) for use in the system M/D/r.

b) The determination of bounds on the A.Q.T., provided that

they are adequately tight for the purposes of the study. Bounds on the A.Q.T. for possible use in the system $M/D/r$ were obtained by Kingman²³, Brumelle¹ and Köllerström²⁴; bounds on the A.Q.T. for possible use in the system $D/M/r$ were obtained by Rolski³⁹ and Brumelle².

- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be emphasized, however, that simulation is not in itself a panacea: apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections 7.2 and 7.3 of this Chapter, simple explicit formulae for the approximate evaluation of the A.Q.T. in the systems $M/D/r$ and $D/M/r$ are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is the result of a desire aiming at simplifying the formulae established in Chapters 2, 3, 4 and 6 for the approximate evaluation of the A.Q.T. in the standard systems $GI/M/r$, $M/G/r$ and $GI/G/r$.

Due to the special structure of $D/M/r$ systems, the establishment of an approximate value for the A.Q.T. allows the establishment of approximate results for the queue-size and queueing-time distributions. These results are outlined in the last section of this Chapter.

7-2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

7.2.1 Establishment of the basic structure of the formulae

Let $W(M/D/r)$, $W(D/M/r)$ and $W(M/M/r)$ denote the A.Q.T. of an arriving customer in the systems $M/D/r$, $D/M/r$ and $M/M/r$ respectively. It is a well established fact that under conditions of heavy traffic²³ ($\rho \rightarrow 1$):

$$\frac{W(M/D/r)}{W(M/D/1)} = \frac{W(D/M/r)}{W(D/M/1)} = \frac{W(M/M/r)}{W(M/M/1)} = \frac{1}{r}.$$

We therefore proceed in writing two relationships of the general forms:

$$\frac{W(M/D/r)}{W(M/D/1)} \approx S_s \frac{W(M/M/r)}{W(M/M/1)} \quad (7.1)$$

and

$$\frac{W(D/M/r)}{W(D/M/1)} \approx S_a \frac{W(M/M/r)}{W(M/M/1)} \quad (7.2)$$

with S_s and S_a "similarity factors", functions of ρ and r , equal to 1 for $r = 1$ (regardless of ρ) and for $\rho \rightarrow 1$ (regardless of r).

7.2.2 Determination of the similarity factors S_a and S_s

In order to determine S_a and S_s the following heuristic approach was selected: Assuming $W(M/D/r)$, $W(D/M/r)$ and $W(M/M/r)$ to be known, values of S_s and S_a were calculated for $r = 2, \dots, 10$ and $\rho = 0.30, \dots, 0.95$ and simple analytical expressions for S_s and S_a were then sought and tested. The outcome of this investigation led to:

$$S_g = 1 + (1-\rho)(r-1) \frac{\sqrt{4+5r} - 2}{16\rho r},$$

$$\text{and } S_a = \begin{cases} 1 - (1-\rho)(r-1) \frac{\sqrt{4+5r} - 2}{4\rho r} & (= \Lambda) \quad \text{if } \Lambda \geq 0 \\ 0 & \text{if } \Lambda < 0 \end{cases}$$

Hence, formulae (7.1) and (7.2) can be written:

$$W(M/D/r) \approx \frac{1}{2} \left[1 + \frac{(1-\rho)(r-1)(\sqrt{4+5r} - 2)}{16\rho r} \right] W(M/M/r) \quad (7.3)$$

and

$$W(D/M/r) \approx \left[1 - \frac{(1-\rho)(r-1)(\sqrt{4+5r} - 2)}{4\rho r} \right] \frac{W(M/M/r)}{W(M/M/1)} W(D/M/1) \quad (7.4)$$

In case a quick evaluation of $W(D/M/1)$ is desired, the following simple formula is suggested:

$$W(D/M/1) \approx \left[\frac{\rho}{2(1-\rho)} - 0.325\rho(2-\rho) \right] \frac{1}{\mu} \quad (7.5)$$

with $1/\mu$ denoting the average service time.

We now proceed in evaluating formulae (7.3), (7.4) and (7.5) for $1 \leq r \leq 100$ and $0 < \rho < 1$.

7.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

7.3.1 The queuing system M/D/r

In evaluating formula (7.3) it is verified that any value of the A.Q.T. $W(M/D/r)$ can be approximated with a relative percentage error:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{theor. value}),$$

which:

is positive or negative depending on the values of r and ρ and tends to zero as $\rho \rightarrow 1$;

is not sensitive to the value of r ; and

is almost negligible for most practical purposes: in fact, an A.Q.T. as small as $0.02/\mu$ ($1/\mu$ denotes the service time) can be approximated with a relative percentage error not exceeding 1% in absolute value.

Table 7.1 gives the relative percentage errors incurred if (7.3) is used for the approximate evaluation of $W(M/D/r)$ for $r = 2, \dots, 10$ and $\rho = 0.30, \dots, 0.95$ in steps of 0.05. The analytical values of $W(M/D/r)$ are given in Table A3.24 of Appendix 3; the values of $W(M/M/r)$ required in the calculations are given in Table A3.1 of Appendix 3.

TABLE 7.1

Relative percentage errors incurred if (7.3) is used
for the approximate evaluation of $W(M/D/r)$

$$2 \leq r \leq 10$$

$\rho \backslash r$	2	3	4	5	6	7	8	9	10
0.30	.9	2.0							
0.35	.1	.7							
0.40	-.3	-.0	.2						
0.45	-.5	-.4	-.3						
0.50	-.6	-.6	-.5	-.5					
0.55	-.6	-.6	-.6	-.6	-.7	-.7			
0.60	-.5	-.6	-.6	-.6	-.6	-.7	-.7		
0.65	-.5	-.5	-.5	-.4	-.4	-.5	-.6	-.6	-.7
0.70	-.4	-.4	-.4	-.3	-.3	-.3	-.3	-.3	-.4
0.75	-.3	-.3	-.3	-.2	-.1	-.1	-.1	-.1	-.1
0.80	-.3	-.2	-.2	-.1	-.0	.0	.1	.1	.1
0.85	-.2	-.2	-.1	-.0	.0	.1	.2	.2	.3
0.90	-.1	-.1	-.0	.0	.1	.1	.2	.2	.3
0.95	-.1	-.0	-.0	.0	.0	.1	.1	.1	.2

Table 7.2 gives a more general view of the relative percentage errors incurred if (7.3) is used for the approximate evaluation of $W(M/D/r)$. The analytical values of $W(M/D/r)$ were calculated by using the explicit formula already referred to in section 7.1.2; the values of $W(M/M/r)$ required in the calculations were obtained from the Tables by Descloux¹⁰.

TABLE 7.2

Relative percentage errors incurred if (7.3) is used
for the approximate evaluation of $W(M/D/r)$

$$2 \leq r \leq 100$$

r	Av. Q. Times and rel. perc. errors	$\rho = 0.80$	$\rho = 0.85$	$\rho = 0.90$	$\rho = 0.95$
2	theor. value	0.903284	1.316698	2.146923	4.643948
	appr. value	0.900984	1.314305	2.144470	4.641463
	e (%)	-0.3	-0.2	-0.1	-0.1
10	theor. value	0.109828	0.185505	0.344582	0.856717
	appr. value	0.109989	0.185997	0.345543	0.858658
	e (%)	0.1	0.3	0.3	0.2
20	theor. value	0.055874	0.069359	0.144286	0.385506
	appr. value	0.055905	0.069688	0.145137	0.387376
	e (%)	0.1	0.5	0.6	0.5
50	theor. value	0.005356	0.014557	0.039526	0.130455
	appr. value	0.005276	0.014549	0.039838	0.131473
	e (%)	-1.5	-0.1	0.8	0.8
100	theor. value	0.000676	0.003110	0.012333	0.053693
	appr. value	0.000647	0.003052	0.012372	0.054018
	e (%)	-4.3	-1.9	0.3	0.8

7.3.2 The queuing system $D/M/r$

In evaluating formula (7.4) or (7.4) in combination with (7.5) it is verified that any value of the A.Q.T. $W(D/M/r)$

can be approximated with a relative percentage error:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{theor. value}),$$

which:

is positive or negative depending on the values of r and ρ and tends to zero as $\rho \rightarrow 1$;

is not sensitive to the value of r ; and

is almost negligible for most practical purposes.

The evaluation of formulae (7.4) and (7.4) in combination with (7.5) for $r = 1, \dots, 10$ and $\rho = 0.30, \dots, 0.95$ in steps of 0.05 is summarized in Tables 7.3 and 7.4 respectively. The analytical values of $W(D/M/r)$ for $r \leq 6$ are

TABLE 7.3

Relative percentage errors incurred if (7.4) is used
for the approximate evaluation of $W(D/M/r)$

$$2 \leq r \leq 10$$

$\rho \backslash r$	2	3	4	5	6	7	8	9	10
0.40	3.7								
0.45	3.1								
0.50	2.4	2.4							
0.55	1.8	2.3							
0.60	1.2	1.8	1.7	.5					
0.65	.8	1.2	1.4	1.1	.5				
0.70	.4	.7	.9	.9	.8	.3	-.4		
0.75	.2	.3	.4	.5	.5				
0.80	.0	.0	.0	.0	.0	.1	.1	.1	.0
0.85	-.0	-.1	-.2	-.2	-.2				
0.90	-.1	-.2	-.2	-.3	-.4	-.4	-.4	-.5	-.5
0.95	-.1	-.1	-.2	-.2	-.3				

Note: there is no entry in the table when $\rho W(D/M/r) < 0.02$

given in Table A3.2 of Appendix 3; the ones for $6 < r \leq 10$ were obtained from the book by Page³⁵. The values of $W(M/M/r)$ required in the calculations are given in Table A3.1 of Appendix 3.

TABLE 7.4

Relative percentage errors incurred if (7.4) in combination with (7.5) is used for the approximate evaluation of $W(D/M/r)$

$$1 \leq r \leq 10$$

$\rho \backslash r$	1	2	3	4	5	6	7	8	9	10
0.30	13.9									
0.35	8.0									
0.40	4.2	8.0								
0.45	1.9	5.1								
0.50	.5	2.9	2.9							
0.55	-.3	1.5	2.1							
0.60	-.6	.6	1.2	1.1	-.5					
0.65	-.7	.1	.6	.7	.5	-.2				
0.70	-.6	-.2	.1	.3	.3	.2	-.3	-1.0		
0.75	-.5	-.3	-.1	-.1	-.0	.0				
0.80	-.3	-.2	-.2	-.2	-.2	-.2	-.2	-.2	-.2	-.2
0.85	-.1	-.2	-.2	-.3	-.3	-.4				
0.90	-.0	-.1	-.2	-.2	-.3	-.4	-.4	-.4	-.5	-.5
0.95	.0	-.0	-.1	-.1	-.2	-.2				

Note: there is no entry in the table when $pW(D/M/r) < 0.02$

The evaluation of formula (7.4) in combination with (7.5) for $r = 20$, $\rho = 0.85$ / $r = 50$, $\rho = 0.90$ and $r = 100$, $\rho = 0.95$ is summarized in Table 7.5. For the cases considered in this Table no results for the A.Q.T. were available;

and the possibility of deriving them analytically was ruled out in favour of simulation, the design of the simulation experiments being outlined in Appendix A2.6. It is interesting

TABLE 7.5

Evaluation of (7.4) in combination with (7.5)
for some values of ρ and r

R e s u l t s	$r = 20$ $\rho = 0.85$	$r = 50$ $\rho = 0.90$	$r = 100$ $\rho = 0.95$
number of arrivals simulated	300,000	300,000	300,000
lower 95% confidence limit	0.0360	0.0198	0.0311
simulated value (s.v.)	0.0386	0.0213	0.0351
upper 95% confidence limit	0.0412	0.0227	0.0391
100(95% conf. interval)/(s.v.)	13.5	13.6	22.8
approximate value (a.v.)	0.0375	0.0210	0.0359
$e = 100(a.v. - s.v.)/(s.v.)$	-2.8	-1.4	2.5

to note that despite the variance reduction techniques used and the quite large number of arrivals simulated, the 95% confidence interval is wide as compared to the simulated A.Q.T. This could probably be attributed to the high values of ρ (see, for example, the work by Gebhard¹³).

7.4 SOME FURTHER APPROXIMATE EQUILIBRIUM RESULTS FOR THE QUEUEING SYSTEM D/M/r

It is known from Kendall's work on the system GI/M/r that, conditional on being non-zero, queueing time has an exponential distribution with mean $((1-z_0)rp)^{-1}$. In case of the system D/M/r, z_0 is the only root within the unit circle of the equation:

$$z = \exp(-(1 - z)/\rho) \quad (7.6)$$

It is also known that the major difficulty encountered when dealing with the system $D/M/r$ lies in the calculation of the probability P that a customer has to queue; a difficulty which increases quite rapidly with r .

Having, however, determined an approximate value for $W(D/M/r)$ the obvious relation:

$$P = (1 - z_0)r \mu W(D/M/r) \quad (7.7)$$

provides an approximate value for P . In fact, since the value of z_0 in (7.6) depends only on ρ , one can use the simple relation:

$$\mu W(D/M/1) = z_0 / (1 - z_0)$$

and express z_0 in terms of $\mu W(D/M/1)$. Thus (7.7) takes the form:

$$P = \frac{r \mu W(D/M/r)}{1 + \mu W(D/M/1)} \quad (7.8)$$

By introducing into (7.8) the approximate values of $\mu W(D/M/r)$ and $\mu W(D/M/1)$ as obtained from (7.4) and (7.5), (7.8) becomes an approximate explicit formula for P ; the relative percentage errors incurred in evaluating P are very small.

In a similar way one can derive approximate results for the equilibrium queue-size distribution:

$$p_k = \frac{1 - z_0}{z_0^r} \rho^k P \quad (k \geq r)$$

CONCLUSIONS

Formulae (7.3) , (7.4) and (7.5) give very good approximations for the average queueing time of an arriving customer in the systems $M/D/r$, $D/M/r$ and $D/M/1$ respectively. All three of them can be used in combination with the formulae established in previous Chapters for the approximate evaluation of the average queueing time of an arriving customer in more general multi-server systems: while simplifying them to a considerable extent, their effect on the relative percentage errors incurred is hardly noticeable.

C H A P T E R 8

APPROXIMATE FORMULAE

FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME

IN THE STANDARD SYSTEM

GI / M / 1

8. APPROXIMATE FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.Q.T.) IN THE STANDARD SYSTEM

$$\underline{GI / M / 1}$$

8.1 INTRODUCTION

8.1.1 Brief description of the system

The system considered is a special case of the more general queueing system $GI/M/r$ already dealt with in Chapter 2 and can be described as follows: there is one server having an exponential service time distribution, $1 - \exp(-\mu t)$. Intervals between successive arrivals are independently distributed with distribution function $A(u)$ such that there is zero probability that two or more customers arrive simultaneously. The average rate of arrival of customers is λ and the coefficient of variation of inter-arrival times $v_a = \sqrt{\text{var}(u)} \lambda$, such that $0 < v_a < 1$. Customers form into a single queue and get served in the order of their arrival. The system is in statistical equilibrium, i.e. the traffic intensity $\rho = \lambda/\mu < 1$.

8.1.2 On the derivation of analytical results for the A.Q.T.

The equilibrium theory of the queue $GI/M/1$ can be found in almost all basic textbooks on queueing theory. The application of the theory is relatively simple, though numerical methods have to be used. If, however, compared with the dual queue $M/G/1$, the evaluation of the A.Q.T. of an arriving customer in the system $GI/M/1$ can be considered as difficult.

8.1.3 Alternative ways of obtaining results for the A.Q.T.
(the practitioner's viewpoint)

In view of possible difficulties encountered when trying to apply the theory to practical problems, the practitioner may find adequate for the purpose of his study and therefore be content with:

- a) The use of some approximate formula, provided that the relative percentage error incurred is comparable in magnitude with all other errors incurred due to the assumption that the queueing model $GI/M/1$ reflects the real life waiting line situation under study. An example of such a formula is the one suggested by Page for use in the system $E_k/M/1$.
- b) The determination of bounds on the A.Q.T., provided that they are adequately tight for the purposes of the study. Kingman²¹, for example, studied the behaviour of the more general system $GI/G/1$ under conditions of heavy traffic and obtained an extremely simple formula for the A.Q.T.; a formula that gives an upper bound under unsaturated conditions. A lower bound on the A.Q.T. related to the nature of the distribution of inter-arrival times was obtained by Marshall²⁹.
- c) The analysis by simulation. Such an analysis has the advantage of providing both an approximate value and bounds on the A.Q.T. and, in addition, valuable information on other measures of practical interest. It should be emphasized, however, that simulation is not in itself a panacea: apart from the considerable number of possible pitfalls, it may well be considered as a rather expensive "way out".

In the following sections of this Chapter, approximate formulae for the A.Q.T. are derived and the degree of approximation achieved under various conditions is investigated. The derivation of the formulae is based on the idea (introduced in Chapter 1) that two queuing systems of the general forms $GI/M/1$ and $M/G/1$ respectively with:

- the coefficient of variation of inter-arrival times in the former equal to the coefficient of variation of service times in the latter; and
- equal traffic intensities,

have a similar behaviour.

8.2 DERIVATION OF APPROXIMATE FORMULAE FOR THE A.Q.T.

8.2.1 Establishment of the basic structure of the formulae

Let $W(A(u)/M/1)$ denote the A.Q.T. of an arriving customer in the system $GI/M/1$. One can reasonably expect that $pW(A(u)/M/1)$ depends on ρ and $A(u)$ only; furthermore, that for ρ and the type of $A(u)$ fixed, $pW(A(u)/M/1)$ is an increasing function of v_a with limiting values $pW(D/M/1)$, corresponding to the system $D/M/1$, if $v_a = 0$ and, say, $pW(A(u), v_a=1/M/1)$, corresponding to the system $A(u), v_a=1/M/1$, if $v_a = 1$. One such relationship between $pW(A(u)/M/1)$ and v_a is represented by curve (2) in Figure 8.1.

Let $W(M/B(t)/1)$ denote the A.Q.T. of an arriving customer in the system $M/G/1$. It is a well established fact that $pW(M/B(t)/1)$ depends on ρ and on the coefficient of variation of service times v_b only; consequently, for ρ fixed,

$\mu W(M/B(t)/1)$ is an increasing function of v_s with limiting values $\mu W(M/D/1)$, corresponding to the system $M/D/1$, if $v_s = 0$ and, say, $\mu W(M/B(t), v_s=1/1)$, corresponding to the system $M/B(t), v_s=1/1$, if $v_s = 1$. One such relationship between

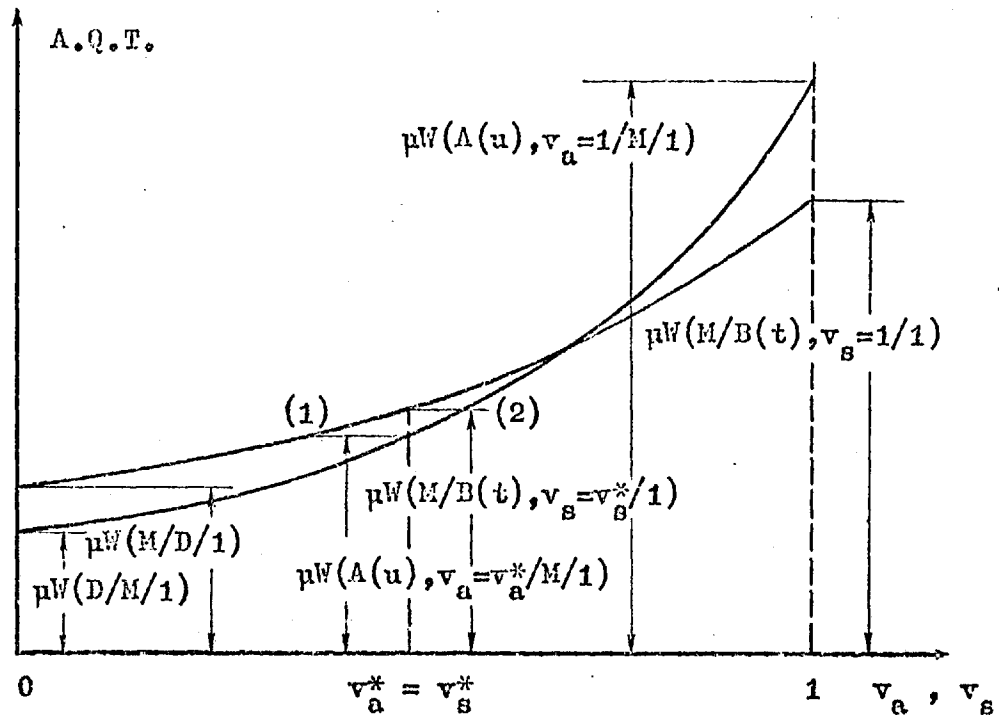


Fig. 8.1 $\mu W(A(u)/M/1)$ as a function of v_a and $\mu W(M/B(t)/1)$ as a function of v_s for two queueing systems with equal traffic intensities.

$\mu W(M/B(t)/1)$ and v_s is represented by curve (1) in Fig. 8.1. It is assumed that both curves in Fig. 8.1 are related to queueing systems with equal traffic intensities.

We now proceed in writing a relationship of the general form:

$$\frac{\mu W(A(u), v_a=v_s^*/M/1)}{\mu W(M/B(t), v_s=v_s^*/1)} \approx S \frac{\mu W(A(u), v_a=1/M/1)}{\mu W(M/B(t), v_s=1/1)} + (1-S) \frac{\mu W(D/M/1)}{\mu W(M/D/1)} \quad (8.1)$$

with v_s^* any value of v_s between 0 and 1 and S some "similarity factor" with limiting values 0 and 1 for

$v_a^* = 0$ and 1 respectively, regardless of ρ , $\Lambda(u)$ or $B(t)$.

(8.1) is an expression giving $\mu W(\Lambda(u), v_a = v_a^*/M/1)$ in terms of $\mu W(M/B(t), v_s = v_s^*/1)$; consequently, if the relative percentage errors incurred are small,

- a) the hypothesis that the systems $\Lambda(u), v_a = v_a^*/M/1$ and $M/B(t), v_s = v_s^*/1$ with equal traffic intensities have a similar behaviour would be established, the quantity on the right hand side being regarded as the similarity ratio; and
- b) (8.1) could be used as an approximate formula for the evaluation of $W(\Lambda(u), v_a = v_a^*/M/1)$ provided that $W(M/B(t), v_s = v_s^*/1)$ and the similarity ratio were both known.

It is known from the equilibrium theory of the queue $M/G/1$ that:

$$\mu W(M/B(t), v_s = v_s^*/1) = \frac{\rho (1 + (v_s^*)^2)}{2 (1 - \rho)},$$

a formula that can also be written under the equivalent forms:

$$W(M/B(t), v_s = v_s^*/1) = (v_s^*)^2 W(M/M/1) + (1 - (v_s^*)^2) W(M/D/1)$$

or

$$W(M/B(t), v_s = v_s^*/1) = (v_s^*)^2 W(M/B(t), v_s = 1/1) + (1 - (v_s^*)^2) W(M/D/1) \quad (8.2)$$

since

$$W(M/B(t), v_s = 1/1) = W(M/M/1) \quad (8.3)$$

regardless of $B(t)$.

Consequently, the expression in (8.1) can be simplified:

$$W(\Lambda(u), v_a = v_a^*/M/1) \approx S \frac{1 + (v_a^*)^2}{2} W(\Lambda(u), v_a = 1/M/1) +$$

$$+ (1 - S)(1 + (v_a^*)^2) W(D/M/1) \quad (8.4)$$

8.2.2 A method for the determination of the similarity factor

The method suggested here has three main advantages:

- a) it is simple in its application (provided that it can be applied);
- b) it provides a very simple expression for S ; and
- c) it is justified in terms of a) and b) of paragraph 8.2.1, particularly if the value of ρ is high, a situation likely to be encountered in practice and, also, one in which numerical results are usually difficult to obtain either analytically or by simulation.

It consists of two steps:

- a) the approximate formulae, for the average queueing times in the basic expression (8.4) under conditions of heavy traffic, are found.

For the system considered:

$$(\mu W(A(n), v_a = v_a^*/M/1))_{\rho \rightarrow 1} \simeq \frac{(v_a^*)^2 + \rho^2}{2\rho(1-\rho)},$$

- b) as suggested by Kingman²¹, valid for $0 < v_a^* < 1$; these formulae are then introduced into (8.4); simple algebra yields the desired expression for S .

For the system considered:

$$S = \frac{2(v_a^*)^2}{1 + (v_a^*)^2}$$

in which case (8.4) takes the form:

$$\begin{aligned}
 W(A(u), v_a = v_a^*/M/1) &\approx (v_a^*)^2 W(A(u), v_a = 1/M/1) + \\
 &+ (1 - (v_a^*)^2) W(D/M/1)
 \end{aligned} \quad (8.5)$$

Formula (8.5) can be viewed as a dual to the Pollaczek - Khintchine formula (8.2)

The method described is not the only one possible. An alternative method is presented in Chapter 10 and many more could be envisaged.

8.3 EVALUATION OF THE APPROXIMATE FORMULAE DERIVED

8.3.1 The queueing system $\Gamma / M / 1$

Consider first the case in which inter-arrival times have the gamma distribution (the gamma distribution is discussed in Appendix A111). For this case $W(A(u), v_a = 1/M/1) = W(M/M/1)$, corresponding to the system with exponential inter-arrival times. Consequently, (8.5) can be written:

$$W(\Gamma, v_a = v_a^*/M/1) \approx (v_a^*)^2 W(M/M/1) + (1 - (v_a^*)^2) W(D/M/1) \quad (8.6)$$

which coincides with the formula suggested by Page³⁵.

The expression in (8.6) satisfies the two results established by Fischer¹², namely:

- a) that the A.Q.T. of an arriving customer decreases monotonically in $1/(v_a^*)^2$; and
- b) that the A.Q.T. in $\Gamma/M/1$ is less than in the dual system $M/\Gamma/1$ since $W(D/M/1) < W(M/D/1)$ for all values of ρ .

Formula (8.6) gives very good approximations for $W(\Gamma, v_a = v_a^*/M/1)$. In fact, any value of the average queueing time $W(\Gamma, v_a = v_a^*/M/1)$ can be approximated with a relative per-

centage error:

$$e = 100(\text{approximate value} - \text{theoretical value})/(\text{theor. value})$$

which:

is positive, thus implying that the expression in (8.6) provides an upper bound on the A.Q.T. to be estimated;

decreases as the A.Q.T. to be estimated and, hence, the traffic intensity increase, tending quite rapidly to 0 as $\rho \rightarrow 1$; and

varies with v_a^* as originally expected: it tends to 0 as v_a^* approaches 0 or 1 and attains a maximum value somewhere in-between, namely for $v_a^* \approx 0.7$.

As an indication of the magnitude of e , Table 8.1 gives the relative percentage errors incurred if formula (8.6) is used for the evaluation of $W(\Gamma, v_a = v_a^*/M/1)$ for $\rho = 0.40, \dots, 0.90$ in steps of 0.10 and $(v_a^*)^2 = 1/1.21, 1/1.44, 1/2, 1/3, 1/4, 1/6, 1/9$ and $1/25$; and Table 8.2 gives the

TABLE 8.1

Relative percentage errors incurred if (8.6) is used
for the evaluation of $W(\Gamma, v_a = v_a^*/M/1)$

$\rho \backslash v_a^{*2}$	1/1.21	1/1.44	1/2	1/3	1/4	1/6	1/9	1/25
0.40	1.1	2.0	3.6	4.8	5.2	4.9	4.3	1.8
0.50	0.5	0.9	1.5	1.9	2.0	1.9	1.6	0.6
0.60	0.2	0.4	0.7	0.8	0.8	0.7	0.6	0.2
0.70	0.1	0.2	0.3	0.3	0.3	0.3	0.2	0.0
0.80	0.0	0.1	0.1	0.1	0.1	0.0	0.0	0.0
0.90	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

maximum relative percentage errors incurred if formula (8.6) is used for the evaluation of an A.Q.T. $\rho W(\Gamma, v_a = v_a^*/M/1)$ equal to 0.3, 0.5, 1.0 and 2.0. (The analytical values of $W(\Gamma/M/1)$ are given in Table A3.3 of Appendix 3).

TABLE 8.2

Maximum relative percentage errors incurred

if (8.6) is used for the evaluation

of $W(\Gamma, v_a = v_a^*/M/1)$

$\mu W(\Gamma, v_a = v_a^*/M/1)$	e (%)
0.3	5.5
0.5	2.2
1.0	0.7
2.0	0.2

8.3.2 The general case

The four distributions of inter-arrival times summarized in Appendices A1.13, A1.14, A1.15 and A1.16 are now considered. These distributions were selected in such a way so as to ensure that analytical results could be fairly easily obtained and if possible for $0 < v_a < 1$. Consequently they are most unlikely to represent any real system, except perhaps for a few values of x and y .

Since $v_a = 1$ is achieved for $x = 0$ and $y \rightarrow \infty$, instead of evaluating formula (8.5) we proceed in evaluating the slightly modified formula (8.7):

$$W(A(u), v_a = v_a^*/M/1) \approx S W(A(u), v_a = 1/\sqrt{1.02}/M/1) + (1 - S) W(D/M/1) \quad (8.7)$$

with $S = 1.02(v_a^*)^2$, the expression for S being easily obtained if the method suggested in section 8.2.2 is applied to formula (8.7).

The approximations achieved in evaluating formula (8.7) were found to be much worse than the ones stated in the special case

of gamma inter-arrival times already dealt with in section 8.3.1. Formula (8.7) does provide an upper bound on the A.Q.T. to be estimated; the relative percentage errors incurred are, however, quite large (see, for example, Table 8.3 which gives the relative percentage errors incurred if (8.7) is used for the evaluation of $W(\underline{L}, v_a = v_a^*/M/1)$, if $(v_a^*)^2 \geq 1/6$ and of $W(\underline{L}, v_a = v_a^*/M/1)$, if $(v_a^*)^2 < 1/6$ and compare it with Table 8.1).

TABLE 8.3

Relative percentage errors incurred if (8.7) is used for the evaluation of $W(\underline{L}, v_a = v_a^*/M/1)$ if $(v_a^*)^2 \geq 1/6$ and of $W(\underline{L}, v_a = v_a^*/M/1)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.21	1/1.44	1/2	1/3	1/4	1/6	1/9	1/25
0.40	8.4	17.0	31.0	40.5	41.8	38.1	34.2	17.7
0.50	4.8	9.4	16.4	20.4	20.5	18.2	15.4	7.5
0.60	2.8	5.3	8.9	10.8	10.6	9.2	7.6	3.6
0.70	1.6	2.9	4.8	5.7	5.5	4.8	3.8	1.8
0.80	0.8	1.5	2.4	2.8	2.7	2.3	1.8	0.8
0.90	0.3	0.6	0.9	1.0	1.0	0.9	0.7	0.3

8.4 A MODIFIED FORMULA FOR THE A.Q.T.

8.4.1 An assumption and the formula

For a distribution of inter-arrival times other than the gamma, formula (8.5) is of no practical use. In fact, given a distribution $A(u)$ with $v_a = v_a^*$, it may well be impossible even to define what $A(u)$ with $v_a = 1$ should be. Therefore, the assumption (known to be valid under conditions of heavy traffic²¹) is made that:

$$\left[C_a = W(\Lambda(u), v_a = 1/M/1) \right] = \left[C_m = W(M/M/1) \right] \quad (8.8)$$

in which case formula (8.5) takes the much more interesting form:

$$W(\Lambda(u), v_a = v_a^*/M/1) \approx (v_a^*)^2 W(M/M/1) + (1 - (v_a^*)^2) W(D/M/1) \quad (8.9)$$

8.4.2 On the evaluation of the formula

For all four distributions of inter-arrival times considered in section 8.3 it so happens that $C_a > C_m$. In particular, the ratio C_a/C_m was found to be a decreasing function of ρ tending to unity as ρ approaches unity but having a value significantly larger than 1 for low values of ρ . (Table 8.4, for example, gives the values of $\mu W(\Gamma, v_a = 1/\sqrt{1.02} / M/1)$, $\mu W(L_\Delta, v_a = 1/\sqrt{1.02} / M/1)$ and C'_a/C'_m).

TABLE 8.4

Comparison between

$\mu W(\Gamma, v_a = 1/\sqrt{1.02} / M/1)$ and $\mu W(L_\Delta, v_a = 1/\sqrt{1.02} / M/1)$

ρ	$\mu W(\Gamma, v_a = 1/\sqrt{1.02} / M/1)$ C'_m	$\mu W(L_\Delta, v_a = 1/\sqrt{1.02} / M/1)$ C'_a	C'_a/C'_m
0.20	0.2438	0.9638	3.95
0.30	0.4199	1.0477	2.49
0.40	0.6552	1.2072	1.84
0.50	0.9849	1.4771	1.50
0.60	1.4796	1.9242	1.30
0.70	2.3045	2.7103	1.18
0.80	3.9546	4.3283	1.09
0.90	8.9054	9.2521	1.04

Therefore:

- a) there seems to be no dual relation to (8.3) except for values of ρ approaching unity; and
- b) in evaluating formula (8.9) instead of (8.5) one would expect that the relative percentage errors incurred:

- would not be affected for " low " values of v_a^* ;
- would be reduced in magnitude, remaining positive, for " medium " values of v_a^* ;
- would be reduced in magnitude, becoming negative, for " high " values of v_a^* ; but
- would not be seriously affected for high values of ρ , regardless of v_a^* .

(See, for example, Table 8.5 which gives the relative percentage errors incurred if formula (8.9) is used for the eval-

TABLE 8.5

Relative percentage errors incurred if (8.9) is used for the evaluation of $W(\underline{u}, v_a = v_a^*/M/1)$ if $(v_a^*)^2 \geq 1/6$ and of $W(\underline{L}, v_a = v_a^*/M/1)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.21	1/1.44	1/2	1/3	1/4	1/6	1/9	1/25
0.40	-40.2	-34.3	-23.6	-13.2	-8.4	-4.3	1.5	1.6
0.50	-29.0	-24.6	-16.9	-9.6	-6.1	-3.2	-0.9	0.4
0.60	-19.9	-16.7	-11.4	-6.5	-4.1	-2.1	-0.8	0.1
0.70	-12.8	-10.6	-7.2	-4.1	-2.6	-1.3	-0.6	-0.0
0.80	-7.3	-6.0	-4.0	-2.3	-1.5	-0.7	-0.3	-0.0
0.90	-3.1	-2.6	-1.7	-1.0	-0.6	-0.3	-0.2	-0.0

uation of $W(\underline{u}, v_a = v_a^*/M/1)$, if $(v_a^*)^2 \geq 1/6$ and of $W(\underline{L}, v_a = v_a^*/M/1)$, if $(v_a^*)^2 < 1/6$ and compare it with Ta-

ble 8.3).

Tables 8.6 , 8.7 and 8.8 are similar to Table 8.5, though they are related to the other three of the four distributions

TABLE 8.6

Relative percentage errors incurred if (8.9) is used for the evaluation of $W(\underline{L}_1, v_a = v_a^*/M/1)$ if $(v_a^*)^2 \geq 1/3$ and of $W(\underline{L}_1, v_a = v_a^*/M/1)$ if $(v_a^*)^2 < 1/3$

$\rho \backslash (v_a^*)^2$	1/1.21	1/1.44	1/2	1/3	1/4	1/6	1/9	1/25
0.40	-40.3	-34.7	-24.9	-16.3	-9.1	-2.6	0.6	1.7
0.50	-29.0	-24.7	-17.4	-10.8	-6.4	-2.5	-0.6	0.5
0.60	-19.9	-16.8	-11.6	-7.0	-4.3	-1.9	-0.7	0.1
0.70	-12.8	-10.7	-7.3	-4.2	-2.6	-1.2	-0.5	-0.0
0.80	-7.3	-6.1	-4.1	-2.3	-1.5	-0.7	-0.3	-0.0
0.90	-3.1	-2.6	-1.7	-1.0	-0.6	-0.3	-0.1	-0.0

TABLE 8.7

Relative percentage errors incurred if (8.9) is used for the evaluation of $W(\underline{L}_1, v_a = v_a^*/M/1)$ if $(v_a^*)^2 \geq 1/2$ and of $W(\underline{L}_1, v_a = v_a^*/M/1)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.21	1/1.44	1/2	1/3	1/4	1/6	1/9	1/25
0.40	-40.3	-34.9	-25.6	-13.2	-6.8	-1.3	1.3	1.9
0.50	-29.1	-24.8	-17.7	-9.6	-5.5	-2.0	-0.3	0.5
0.60	-19.9	-16.8	-11.7	-6.5	-3.9	-1.7	-0.6	0.1
0.70	-12.8	-10.7	-7.3	-4.1	-2.5	-1.2	-0.5	-0.0
0.80	-7.3	-6.1	-4.1	-2.3	-1.4	-0.7	-0.3	-0.0
0.90	-3.1	-2.6	-1.7	-1.0	-0.6	-0.3	-0.1	-0.0

of inter-arrival times considered in section 8.3. Some of the analytical values for the average queueing times are given in

Tables A3.16 , A3.22 and A3.23 of Appendix 3.

TABLE 8.8

Relative percentage errors incurred if (8.9) is used
for the evaluation of $W(\underline{b}_n, v_a = v_a^*/M/1)$ if $(v_a^*)^2 \geq 1/2$
and of $W(\underline{b}_n, v_a = v_a^*/M/1)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.21	1/1.44	1/2	1/3	1/4	1/6	1/9	1/25
0.40	-13.3	-12.7	-11.8	-3.2	1.4	4.4	4.9	2.8
0.50	-10.8	-9.9	-9.7	-3.3	-0.6	1.2	1.7	1.0
0.60	-8.1	-7.1	-6.5	-2.6	-0.9	0.2	0.6	0.4
0.70	-5.5	-4.8	-4.2	-1.8	-0.8	-0.1	0.2	0.2
0.80	-3.3	-2.8	-2.4	-1.1	-0.5	-0.1	0.0	0.1
0.90	-1.5	-1.3	-1.0	-0.5	-0.2	-0.1	-0.0	0.0

A comparison between the five Tables 8.1 , 8.5 , 8.6 , 8.7 and 8.8 seems to suggest that from a practical viewpoint:

- if $v_a^* < 0.5$, formula (8.9) gives reliable approximations for $pW(A(u), v_a = v_a^*/M/1)$ regardless of $A(u)$;
- if $0.5 < v_a^* < 0.6$, formula (8.9) gives reliable approximations for $pW(A(u), v_a = v_a^*/M/1)$ regardless of $A(u)$ only if $\rho > 0.6$ or, in other words, only if $pW(A(u), v_a = v_a^*/M/1) > \sim 1$;
- if $0.6 < v_a^* < 0.7$, formula (8.9) gives reliable approximations for $pW(A(u), v_a = v_a^*/M/1)$ regardless of $A(u)$ only if $\rho > 0.7$ or, in other words, only if $pW(A(u), v_a = v_a^*/M/1) > \sim 2$; and
- if $v_a^* > 0.7$, formula (8.9) should better not be used because the degree of approximation achieved depends largely on the distribution function of inter-arrival times.

Formula (8.9) can be simplified further if the explicit formula (7.5) established in Chapter 7 is used for the approximate evaluation of $W(D/M/1)$; the relative percentage errors incurred are hardly affected.

CHAPTER 9

CONCLUSIONS

9. CONCLUSIONS

9.1 INTRODUCTION

9.1.1 Brief recapitulation

The idea of similarity in the behaviour of queueing systems introduced in Chapter 1 was applied in subsequent Chapters in order to express the average queueing time of an arriving customer in a system A as a product of a similarity ratio times the average queueing time of an arriving customer in a "similar" system B :

$$(A.Q.T.)_A \approx (S.R.)_{AB} (A.Q.T.)_B \quad (9.1)$$

In all cases studied, the system B is easier to handle than the system A ; and due to the assumptions made, the similarity ratio can be fairly easily calculated in each particular case. Therefore, the evaluation of the product $(S.R.)_{AB} (A.Q.T.)_B$ in (9.1) is easier than the direct evaluation of the A.Q.T. of an arriving customer in the system A. As, however, "easier" does not have the same meaning as "easy", the evaluation of $(S.R.)_{AB} (A.Q.T.)_B$ may in itself be difficult; and it is in fact difficult in some of the cases examined (see, for example, formula (4.22) established in Chapter 4). For this reason it might be reasonable to attempt:

- a) a simplification of the similarity ratio; but mainly
- b) an "extension" of the approximate formula (9.1) by expressing $(A.Q.T.)_B$ as:

$$(A.Q.T.)_B \approx (S.R.)_{BC} (A.Q.T.)_C \quad (9.2)$$

and, if necessary, $(A.Q.T.)_C$ as:

$$(A.Q.T.)_C \approx (S.R.)_{CD} (A.Q.T.)_D \quad (9.3)$$

until some system, say F , which can be easily handled.

9.1.2 On the simplification of the similarity ratio(s)

In all cases studied, the similarity ratios are expressed in terms of all or some of the following quantities:

- the coefficient of variation of inter-arrival times;
- the coefficient of variation of service times; and
- the average queueing times in the systems $M/M/r$, $M/D/r$ and $D/M/r$.

Consequently, the only possible difficulty in the evaluation of the similarity ratios lies in the calculation of $W(M/D/r)$ and/or $W(D/M/r)$; a difficulty, however, that is easily overcome if the explicit formulae (7.3), (7.4) and (7.5) established in Chapter 7 for the approximate evaluation of $W(M/D/r)$, $W(D/M/r)$ and $W(D/M/1)$ are used; as the corresponding relative percentage errors incurred are negligible, their use is absolutely justified.

9.1.3 On the "extension" of formula (9.1)

Let e_{AB} denote the relative percentage error incurred if formula (9.1) is used for the approximate evaluation of $(A.Q.T.)_A$; and e_{BC} the relative percentage error incurred if formula (9.2) is used for the evaluation of $(A.Q.T.)_B$. In case $(A.Q.T.)_A$ is evaluated as:

$$(A.Q.T.)_A \approx (S.R.)_{AB} \left[(S.R.)_{BC} (A.Q.T.)_C \right],$$

the relative percentage error incurred (c_{AC}) can be easily

found equal to:

$$e_{AC} = e_{AB} + e_{BC} + e_{AB} e_{BC} / 100 ;$$

and there is no difficulty in generalizing this relationship if, say, $(A.Q.T.)_C$ is further expressed as in (9.3).

In this Chapter, two simple explicit formulae for the average queueing time of an arriving customer in the standard system $GI/G/r$ are derived and evaluated. Their derivation is based on the idea of extending formula (9.1) as outlined in section 9.1.3 by using approximate formulae already established in previous Chapters.

9.2 DERIVATION OF TWO EXPLICIT FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.Q.T.) IN THE STANDARD SYSTEM $GI/G/r$

9.2.1 The first formula

If formulae:

- (4.22) with $r_j = r$ and $r_i = 1$;
- (7.3) ;
- (5.32) ;
- (7.4) ;
- (8.9) ; and
- (7.5)

are combined, formula (9.4) results.

$W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ is expressed in terms of r , ρ , v_a^* , v_s^* and $1/\mu$ only.

$$W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r) \approx (S.R.)_1 W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)$$

with:

$$(S.R.)_1 = \frac{2(v_a^*)^2 (v_s^*)^2}{(v_a^*)^2 + (v_s^*)^2} \frac{W(M/M/r)}{W(M/M/1)} + \frac{(v_a^*)^2 (1 - (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \frac{W(M/D/r)}{W(M/D/1)} +$$

$$+ \frac{(v_s^*)^2 (1 - (v_a^*)^2)}{(v_a^*)^2 + (v_s^*)^2} \frac{W(D/M/r)}{W(D/M/1)}$$

where:

$$\frac{W(M/D/r)}{W(M/D/1)} \approx \left[1 + \frac{(1-\rho)(r-1)(\sqrt{4+5r} - 2)}{16\rho r} \right] \frac{W(M/M/r)}{W(M/M/1)},$$

$$\frac{W(D/M/r)}{W(D/M/1)} \approx \left[1 - \frac{(1-\rho)(r-1)(\sqrt{4+5r} - 2)}{4\rho r} \right] \frac{W(M/M/r)}{W(M/M/1)}$$

and:

$$W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1) \approx (S.R.)_2 W(A(u), v_a = v_a^*/M/1)$$

where:

$$(S.R.)_2 = \frac{(v_a^*)^2 + (v_s^*)^2}{1 + (v_a^*)^2},$$

$$W(A(u), v_a = v_a^*/M/1) \approx (v_a^*)^2 W(M/M/1) + (1 - (v_a^*)^2) W(D/M/1) \quad \text{and}$$

$$W(D/M/1) \approx \left[\frac{\rho}{2(1-\rho)} - 0.325 \rho(2-\rho) \right] \frac{1}{\mu} \quad (9.4)$$

9.2.2 The second formula

If formulae:

$$- (6.24);$$

$$- (7.3);$$

- (2.7) with $r_j = r$ and $r_i = 1$;
- (7.4) ;
- (8.9) ; and
- (7.5)

are combined, formula (9.5) results.

$$W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r) \approx (S.R.)_3 W(A(u), v_a = v_a^*/M/r)$$

with:

$$(S.R.)_3 = \frac{2((v_a^*)^2 + (v_s^*)^2)}{(1 + (v_a^*)^2)(1 + (v_s^*)^2)} \frac{(v_s^*)^2 W(M/M/r) + (1 - (v_s^*)^2) W(M/D/r)}{W(M/M/r)}$$

where:

$$W(M/D/r) \approx 0.5 \left[1 + \frac{(1-\rho)(r-1)(\sqrt{4+5r} - 2)}{16\rho r} \right] W(M/M/r)$$

and

$$W(A(u), v_a = v_a^*/M/r) \approx (S.R.)_4 W(A(u), v_a = v_a^*/M/1)$$

where:

$$(S.R.)_4 = \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(M/M/r)}{W(M/M/1)} + \frac{1 - (v_a^*)^2}{1 + (v_a^*)^2} \cdot \frac{W(D/M/r)}{W(D/M/1)} \quad \text{and}$$

$$\frac{W(D/M/r)}{W(D/M/1)} \approx \left[1 - \frac{(1-\rho)(r-1)(\sqrt{4+5r} - 2)}{4\rho r} \right] \frac{W(M/M/r)}{W(M/M/1)}$$

and:

$$W(A(u), v_a = v_a^*/M/1) \approx (v_a^*)^2 W(M/M/1) + (1 - (v_a^*)^2) W(D/M/1)$$

where:

$$W(D/M/1) \approx \left[\frac{\rho}{2(1-\rho)} - 0.325 \rho(2-\rho) \right] \frac{1}{p} \quad (9.5)$$

$W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ is again expressed in terms of r , ρ , v_a^* , v_s^* and $1/p$ only.

9.3 EVALUATION OF THE FORMULAE

9.3.1 Evaluation of the formulae with respect to r

a) For given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, any value of the A.Q.T.

$\mu W(A(u), v_a^* = v_a^*/B(t), v_s^* = v_s^*/r)$ can be approximated with a relative percentage error which:

is positive or negative depending on the values of the various parameters involved;

decreases in absolute value as the A.Q.T. to be estimated and, hence, the traffic intensity increase, tending to 0 as ρ tends to 1; and

is not very sensitive to the value of r (at least for $1 < r \leq 6$) unless the A.Q.T. to be estimated is very small.

Tables 9.1 and 9.2 give the relative percentage errors incurred if formulae (9.4) and (9.5) respectively are used for the evaluation of $W(\triangle, v_a^* = \sqrt{1/2} / \square, v_s^* = \sqrt{1/3} / r)$ for $r = 1, 2, 3, 4, 5$ and 6 and

$\mu W(\triangle, v_a^* = \sqrt{1/2} / \square, v_s^* = \sqrt{1/3} / r)$ equal to $0.2, 0.3, 0.4, 0.5$ and 0.6 . The construction of the Tables is

based on the simulation results summarized in Table A4.5 of Appendix 4. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated A.Q.T.

b) For given distributions of inter-arrival and service times $A(u)$ and $B(t)$ with coefficients of variation v_a^* and v_s^* respectively, any value of the total A.Q.T. per unit time $\lambda W(A(u), v_a^* = v_a^*/B(t), v_s^* = v_s^*/r)$ can be approximated with a relative percentage error which:

is positive or negative depending on the values of the various parameters involved;

decreases in absolute value as ρ increases, tending to 0 as ρ tends to 1; and

is not very sensitive to the value of r (at least for $r \leq 6$) unless the total A.Q.T. per unit time to be estimated is very small.

Tables 9.3 and 9.4 give the relative percentage errors

TABLE 9.1

Relative percentage errors incurred if (9.4) is used for the evaluation of $W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r)$

r	1	2	3	4	5	6
$\mu W(\triangle / \square / r) = 0.2$	-27.0*	-6.9*	-0.5	1.8	3.0	3.7
$\mu W(\triangle / \square / r) = 0.3$	-19.3*	-4.0*	0.3	1.9	2.6	2.8
$\mu W(\triangle / \square / r) = 0.4$	-15.2*	-2.5	0.8	1.8	2.0	1.8
$\mu W(\triangle / \square / r) = 0.5$	-12.4*	-1.4	1.1	1.5	1.3	0.9
$\mu W(\triangle / \square / r) = 0.6$	-10.0*	-0.7	1.1	1.2	0.7	

TABLE 9.2

Relative percentage errors incurred if (9.5) is used for the evaluation of $W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r)$

r	1	2	3	4	5	6
$\mu W(\triangle / \square / r) = 0.2$	-27.0*	-8.0*	-2.4	-0.4	0.9	1.7
$\mu W(\triangle / \square / r) = 0.3$	-19.3*	-5.1*	-1.2	0.4	1.1	1.2
$\mu W(\triangle / \square / r) = 0.4$	-15.2*	-3.6	-0.4	0.6	0.7	0.5
$\mu W(\triangle / \square / r) = 0.5$	-12.4*	-2.2	0.1	0.5	0.3	-0.1
$\mu W(\triangle / \square / r) = 0.6$	-10.0*	-1.4	0.4	0.3	-0.2	

incurred if formulae (9.4) and (9.5) respectively are used for the evaluation of $W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r)$ for $r = 1, 2, 3, 4, 5$ and 6 and $\lambda W(\triangle, v_a = \sqrt{1/2} / \square, v_s = \sqrt{1/3} / r)$ equal to 0.5, 1.0 and 1.5. The construction of the Tables is based on the

simulation results summarized in Table A4.5 of Appendix 4. The asterisk(*) indicates that the approximate value falls outside the corresponding 95 % confidence interval on the simulated A.Q.T.

TABLE 9.3

Relative percentage errors incurred if (9.4) is used for the evaluation of $W(\triangle, v_a = \sqrt{1/2} / \square, v_g = \sqrt{1/3} / r)$

r	1	2	3	4	5	6
$\lambda W(\triangle / \square / r) = 0.5$	-7.3*	-3.0	-0.2	1.6	3.2	4.8
$\lambda W(\triangle / \square / r) = 1.0$	-3.1	-0.5	0.9	1.9	2.9	3.8
$\lambda W(\triangle / \square / r) = 1.5$	-1.4		1.1	1.7	2.3	2.9

TABLE 9.4

Relative percentage errors incurred if (9.5) is used for the evaluation of $W(\triangle, v_a = \sqrt{1/2} / \square, v_g = \sqrt{1/3} / r)$

r	1	2	3	4	5	6
$\lambda W(\triangle / \square / r) = 0.5$	-7.3*	-3.8	-2.0	-0.8	0.4	1.5
$\lambda W(\triangle / \square / r) = 1.0$	-3.1	-1.2	-0.3	0.4	1.1	1.7
$\lambda W(\triangle / \square / r) = 1.5$	-1.4		0.3	0.6	0.9	1.2

9.3.2 Evaluation of the formulae with respect to v_g^*

For a given value of $r > 1$ and a given distribution of inter-arrival times $A(u)$ with coefficient of variation v_a^* , any value of the A.Q.T. $W(A(u), v_a = v_a^* / B(t), v_g = v_g^* / r)$ can be approximated with a relative percentage error which seems to vary with v_g^* according to a U - shaped curve; at least for $v_g^* < 0.7$. Table 9.5, for example, gives the relative percentage errors if formulae (9.4) and (9.5) are used for the evalu-

ation of an A.Q.T. $\mu W(\triangle, v_a = \sqrt{1/6} / B(t), v_s = v_s^*/6)$ equal to 0.2. The construction of this Table is based on the relatively few results that are summarized in Table A4.10 of Appendix 4; therefore, the values of e in the Table should be viewed as rough estimates rather than as accurate values. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated A.Q.T.

TABLE 9.5

Relative percentage errors incurred if (9.4) and (9.5)

are used for the evaluation of an A.Q.T.

$\mu W(\triangle, v_a = \sqrt{1/6} / B(t), v_s = v_s^*/6)$ equal to 0.2

B(t)	$(v_s^*)^2$	e (%)	
		Use of (9.4)	Use of (9.5)
D	0	17.5*	14*
	1/6	6.0	4.1
	1/3	5.7	4.5
	1/2	6.8	6.2

9.3.3 Evaluation of the formulae with respect to v_a^*

For a given value of $r > 1$ and a given distribution of service times $B(t)$ with coefficient of variation v_s^* , any value of the A.Q.T. $W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/r)$ can be approximated with a relative percentage error which seems to vary with v_a^* according to a U - shaped curve; at least for $v_a^* < 0.7$. Table 9.6, for example, gives the relative percentage errors incurred if formulae (9.4) and (9.5) are used for the evaluation of an A.Q.T. $\mu W(A(u), v_a = v_a^* / \square, v_s = \sqrt{1/3} / r)$ equal to 0.2. The construction of this Table is based on the

relatively few results that are summarized in tables A4.9 , A4.10 , A4.11 and A4.12 of Appendix 4; therefore the values of e in the Table should be viewed as rough estimates rather than as accurate values. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence intervals on the simulated A.Q.T.

TABLE 9.6

Relative percentage errors incurred if (9.4) and (9.5) are used for the evaluation of an A.Q.T.

$\mu W(A(u), v_a = v_a^* / \square, v_s = \sqrt{1/3} / 6)$ equal to 0.2

A(u)	$(v_a^*)^2$	e (%)	
		Use of (9.4)	Use of (9.5)
D	0	7.9*	8.9*
	1/6	5.7	4.5
	1/3	2.7	0.6
	1/2	3.7	1.7

9.3.4 An example: Evaluation of formulae (9.4) and (9.5) for some 6 - server queueing systems

The values of ρ and the ones of the simulated A.Q.T. , the approximate values of the A.Q.T. as obtained from formulae (9.4) and (9.5) and the corresponding relative percentage errors incurred are summarized in Table 9.7 for 34 in total 6 - server queueing systems. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence interval on the simulated average queueing time.

TABLE 9.7

An example: Evaluation of formulae (9.4) and (9.5)
for 34 6 - server queueing systems

Queueing system considered	Simulation results		Approximate results			
	ρ	A.Q.T.	Form. (9.4)		Form. (9.5)	
			A.Q.T.	e (%)	A.Q.T.	e (%)
D / Δ / 6	0.90	0.0773	0.0870*	12.5	0.0884*	14.3
"	0.95	0.2127	0.2225*	4.6	0.2242*	5.4
D / $\square$ / 6	0.90	0.1586	0.1740*	9.7	0.1759*	10.9
"	0.95	0.4259	0.4454	4.6	0.4477*	5.1
D / ∇ / 6	0.85	0.1228	0.1327*	8.1	0.1343*	9.3
"	0.90	0.2505	0.2623*	4.7	0.2642*	5.5
Δ / D / 6	0.90	0.0716	0.0997*	39.2	0.0934*	30.4
"	0.95	0.2021	0.2369*	17.4	0.2298*	13.7
Δ / Δ / 6	0.85	0.0877	0.1016*	15.9	0.0980*	11.8
"	0.90	0.1771	0.1897	7.1	0.1856	4.8
"	0.95	0.4531	0.4630	2.2	0.4585	1.2
Δ / $\square$ / 6	0.85	0.1356	0.1482*	9.3	0.1459	7.6
"	0.90	0.2693	0.2796	3.8	0.2771	2.9
Δ / ∇ / 6	0.80	0.1023	0.1130*	10.5	0.1117*	9.3
"	0.85	0.1821	0.1954*	7.3	0.1941	6.6
$\square$ / D / 6	0.85	0.0963	0.1129*	17.3	0.1050*	9.1
"	0.90	0.1819	0.2027	11.4	0.1937*	6.5
$\square$ / Δ / 6	0.85	0.1538	0.1617	5.1	0.1559	1.4
"	0.90	0.2898	0.2952	1.8	0.2887	0.5

TABLE 9.7 (continued)

An example: Evaluation of formulae (9.4) and (9.5)
for $3\frac{1}{2}$ 6 - server queuing systems

Queuing system considered	Simulation results		Approximate results			
	ρ	A.Q.T.	Form. (9.4)		Form. (9.5)	
			A.Q.T.	e (%)	A.Q.T.	e (%)
$\square / \square / 6$	0.85	0.2052	0.2105	2.6	0.2064	0.6
"	0.90	0.3859	0.3913	0.5	0.3832	-0.7
$\square / \triangle / 6$	0.80	0.1470	0.1546	5.2	0.1521	3.5
"	0.85	0.2506	0.2602	3.8	0.2574	2.7
$\triangle / D / 6$	0.80	0.0973	0.1072*	10.2	0.1000	2.8
"	0.85	0.1598	0.1724*	7.9	0.1642	2.8
"	0.90	0.2913	0.3067*	5.3	0.2976	2.2
$\triangle / \triangle / 6$	0.80	0.1293	0.1370*	6.0	0.1316	1.8
"	0.85	0.2134	0.2230	4.5	0.2170	1.7
$\triangle / \square / 6$	0.75	0.1012	0.1062	4.9	0.1027	1.5
"	0.80	0.1601	0.1668	4.2	0.1628	1.7
"	0.85	0.2655	0.2737	3.1	0.2693	1.4
"	0.90	0.4919	0.4962	0.9	0.4912	-0.1
$\triangle / \triangle / 6$	0.75	0.1239	0.1244	0.4	0.1220	-1.5
"	0.80	0.1970	0.1968	-0.1	0.1940	-1.5

We now proceed in interpreting the results included in Table 9.7.

- a) On average the total A.Q.T. per unit time $\lambda W(A(u)/B(t)/6)$ for the systems presented in Table 9.7 can be easily found equal to 1.02. In the familiar queueing system $M/M/1$, the

same value of the total A.Q.T. per unit time corresponds to $\rho = 0.62$.

- b) In 17 out of the 34 queueing systems considered, the approximate value of the A.Q.T. as calculated by using formula (9.4) falls within the corresponding 95 % confidence interval on the simulated A.Q.T.
- c) In 22 out of the 34 queueing systems considered, the approximate value of the A.Q.T. as calculated by using formula (9.5) falls within the corresponding 95 % confidence interval on the simulated A.Q.T.
- d) In 21 out of the 34 queueing systems considered, both v_a^* and v_s^* are $\neq 0$. For those 21 systems:
 - formula (9.4) gives an approximate value for the A.Q.T. which falls within the corresponding 95 % confidence interval 16 times; and
 - formula (9.5) gives an approximate value for the A.Q.T. which falls within the corresponding 95 % confidence interval 19 times.
- e) A comparison between Tables 4.19 , 6.8 and 9.7 shows that formulae (9.4) and (9.5), though by far the simplest, do not seem to give worse approximations than formulae (4.22), (4.23) and (6.24).

FINAL CONCLUSIONS

1. Both formulae (9.4) and (9.5) are very simple in their evaluation: In fact they can be easily evaluated by hand; if, however, evaluated by computer, the time required for one evaluation is less than half a second (including compilation time) as compared to approximately eight minutes if the simulation method of solution is adopted (the times refer to

a CDC 6400 computer).

2. Both formulae seem to give reasonably good approximations for the A.Q.T. of an arriving customer in the system GI/G/r. Tables 9.8 and 9.9 give the magnitude of the relative percentage errors incurred if formulae (9.4) and (9.5) are used for the approximate evaluation of an A.Q.T. $\mu W(\Lambda(u)/B(t)/6)$ equal to 0.2.

TABLE 9.8

Magnitude of the relative percentage errors incurred if formula (9.4) is used for the approximate evaluation of an A.Q.T. $\mu W(\Lambda(u)/B(t)/6)$ equal to 0.2

$\Lambda(u) \backslash B(t)$	D $(v_a^*)^2=0$	 $(v_a^*)^2=1/6$	 $(v_a^*)^2=1/3$	 $(v_a^*)^2=1/2$
D $(v_s^*)^2 = 0$	-	15 ... 20	5 15	5 ... 10
 $(v_s^*)^2 = 1/6$	0 10	0 ... 10	0 5	0 ... 10
 $(v_s^*)^2 = 1/3$	0 10	0 10	0 5	0 5
 $(v_s^*)^2 = 1/2$	0 10	0 ... 10	0 10	-5 5

3. For v_a^* , v_s^* and r fixed, the relative percentage errors incurred decrease in absolute value as the average queueing time to be estimated and, hence, the traffic intensity increase.

4. For v_a^* , v_s^* and the average queueing time to be estimated

fixed, the relative percentage errors increase in magnitude with r .

TABLE 9.9

Magnitude of the relative percentage errors incurred if formula (9.5) is used for the approximate evaluation of an A.Q.T. $\mu W(\Lambda(u)/B(t)/6)$ equal to 0.2

$\Lambda(u) \backslash B(t)$	D $(v_a^*)^2 = 0$	 $(v_a^*)^2 = 1/6$	 $(v_a^*)^2 = 1/3$	 $(v_a^*)^2 = 1/2$
D $(v_s^*)^2 = 0$	-	10 ... 20	0 10	0 5
 $(v_s^*)^2 = 1/6$	5 10	0 5	0 5	0 5
 $(v_s^*)^2 = 1/3$	5 10	0 ... 10	0 5	0 5
 $(v_s^*)^2 = 1/2$	5 10	0 ... 10	0 5	-5 5

CHAPTER 10

FURTHER CONSIDERATIONS

AND

POSSIBLE EXTENTIONS

10. FURTHER CONSIDERATIONS AND POSSIBLE EXTENSIONS

10.1 INTRODUCTION

The purpose of this Chapter is to give rise rather than answers to questions. It is felt that the analysis so far is just a starting point towards the development of a theory based on the idea that queueing systems have many features in common all of which are reflected in their behaviour; and aimed at quantifying these common features in simple explicit form in order to provide usable formulae for complex systems which the existing theories have failed to provide.

Congestion in queueing systems, whether measured in terms of queue length, queueing time or any other way, is affected to a large extent by the value of ρ ; by the coefficients of variation of inter-arrival and service times; by the number of servers; by the priority discipline; and, "qualitatively", in the same way, regardless of the particular system considered or of the mathematical method of solution adopted for the model. Relatively little is, however, known about the quantitative effects on congestion of the above mentioned characteristics of queueing systems: these are well hidden by the usually complicated non-explicit expressions resulting from the mathematical treatment of queueing models.

The investigation carried out in previous Chapters gives rise to a hope that few of the parameters by which queueing systems are defined have significant effects on congestion and, furthermore, that these effects may not only qualitatively but

also quantitatively be similar.

The questions still to be answered are many. Some of them will be raised in the following sections of this Chapter. Most, however, will remain unanswered: it seems that fresh ideas are required.

10.2 ON THE DERIVATION OF BETTER APPROXIMATE FORMULAE FOR THE EQUILIBRIUM AVERAGE QUEUEING TIME (A.Q.T.) IN THE STANDARD SYSTEM $GI/G/r$

Looking back at the results obtained by using the various approximate formulae derived in previous Chapters, many questions arise:

- a) The same principle of similarity was applied in the same way for the determination of approximate formulae for the A.Q.T. in the systems $GI/M/r$, $M/G/r$ and $GI/G/r$ (see Chapters 2, 3 and 4). Why is the degree of approximation achieved so different? In particular, why is it so much worse in the general case? Is it because of the principle itself? Is it because of the way it was applied? Or is it because of the assumptions made?
- b) The same principle of similarity was applied in the same way for the determination of approximate formulae for the A.Q.T. in the systems $GI/G/1$ and $GI/G/6$ (see Chapters 5 and 6). Why is the degree of approximation achieved so different in case either of the two distributions involved has a coefficient of variation close to zero? Is it because of the principle itself? Is it because of the way it was applied? Or is it because of the assumptions made?

- c) In case of $M/G/r$ systems (see Chapter 3), the degree of approximation achieved does not seem to depend very much on the values of ρ and v_g . Why does not this happen in the case of other multi-server systems? Is it because of some inherent property of $M/G/r$ systems? Or is it because of the way by which the similarity factors were determined?
- d) In more general terms, could there be some other more efficient way of expressing the similarity in the behaviour of queueing systems? For example through a relationship of the form:

$$\left[\frac{W(GI/G/r_j)}{W(GI/G/r_i)} \right]_{\rho=\rho^*} \approx S_o \left[\frac{W(GI/G/r_j)}{W(GI/G/r_i)} \right]_{\rho=\rho_1} + (1 - S_o) \left[\frac{W(GI/G/r_j)}{W(GI/G/r_i)} \right]_{\rho=\rho_2}$$

with $\rho_1 < \rho^* < \rho_2$ and S_o some "similarity factor" with limiting values 0 and 1 for $\rho^* = \rho_2$ and ρ_1 respectively?

10.2.2 Alternative ways of determining the similarity factors

S_a and S_s

- a) Referring to some of the questions raised in the previous section, it may be worthwhile noticing that in Chapters 2, 3 and 4 all basic relationships (2.1), (3.1), (4.1) and (4.2) hold under conditions of heavy traffic regardless of the values given to the similarity factors S_a and S_s . In fact, as $\rho \rightarrow 1$, all four relationships take the degenerate form:

$$\frac{r_i}{r_j} = S \frac{r_i}{r_j} + (1 - S) \frac{r_i}{r_j}$$

Therefore the values of the similarity factors in sections 2.2.2 , 3.2.2 and 4.2.2 were determined under conditions slightly below the ones prevailing in heavy traffic by deducting ~~the~~ ^{an} quantity $\mu \Delta W$ from the expressions established by Kingman. But should this quantity be independent of v_a^* , v_s^* and r as it was assumed to be?

- b) Looking at Table 4.20 it becomes evident that formula (4.22) overestimates the A.Q.T. in systems of the general form $GI/D/r$ and underestimates it in systems of the general form $D/G/r$. Should one write formulae (4.5) and (4.6) as:

$$\frac{W(A(u), v_a = v_a^*/D/r_j)}{W(A(u), v_a = v_a^*/D/r_i)} \approx S'_a \frac{W(A(u), v_a = 1/D/r_j)}{W(A(u), v_a = 1/D/r_i)}$$

with $S'_a < 1$ and

$$\frac{W(D/B(t), v_s = v_s^*/r_j)}{W(D/B(t), v_s = v_s^*/r_i)} \approx S'_s \frac{W(D/B(t), v_s = 1/r_j)}{W(D/B(t), v_s = 1/r_i)}$$

with $S'_s > 1$ or should one first try to establish the validity of the assumptions in (4.16) and (4.18) ? In the former case how would one determine expressions for S'_a and S'_s and how should one modify section 4.3.1 in order to maintain consistency?

- c) In Chapters 5 and 6 the basic relationships (5.1) , (5.2) , (6.1) and (6.2) hold under conditions of heavy traffic only for the values of S_a and S_s determined in sections 5.2.2 and 6.2.2. (that is, in fact, the way

by which S_a and S_s were determined). Could this be the reason why the degree of approximation achieved worsens quite rapidly as ρ decreases?

Looking at (5.1) with $v_s^i = 1$, for example, it is reasonable to expect that the degree of similarity in the behaviour of the two systems $A(u), v_a = v_a^*/B(t), v_s = v_s^*/1$ and $A(u), v_a = v_a^*/B(t), v_s = 1/1$ is high when v_s^* is close to 1. If, however, v_s^* is close to 0, one would rather expect a high degree of similarity in the behaviour of the systems $A(u), v_a = v_a^*/B(t), v_s = v_s^*/1$ and $A(u), v_a = v_a^*/D/1$. One could therefore:

- split (5.1) into two relationships:

$$\frac{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)_1}{W(A(u), v_a = v_a^*/B(t), v_s = 1/1)} \approx (1 - S_{a1}) \frac{W(D/B(t), v_s = v_s^*/1)}{W(D/B(t), v_s = 1/1)} + S_{a1} \frac{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)}{W(A(u), v_a = 1/B(t), v_s = 1/1)}$$

and

$$\frac{W(A(u), v_a = v_a^*/D/1)}{W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)_2} \approx S_{a2} \cdot \frac{W(A(u), v_a = 1/D/1)}{W(A(u), v_a = 1/B(t), v_s = v_s^*/1)}$$

with

$$S_{a1} = \frac{2(v_a^*)^2}{1 + (v_a^*)^2} \quad \text{and} \quad S_{a2} = \frac{(v_a^*)^2(1 + (v_s^*)^2)}{(v_a^*)^2 + (v_s^*)^2}$$

as determined by the method suggested in section 5.2.2;

- write a relationship of the general form:

$$W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1) \approx S_a W(A(u), v_a = v_a^*/B(t), v_s = v_s^*/1)_1 +$$

$$+ (1 - S_a) W(\Lambda(u), v_a = v_a^*/B(t), v_s = v_s^*/1)_2$$

which now holds under conditions of heavy traffic regardless of S_a ; and

- try to determine an expression for S_a such that $S_a = 0$ and 1 for $v_s^* = 0$ and 1 respectively.

Would such a procedure lead to better results?

- d) The method for the determination of the similarity factors suggested in previous Chapters gives S_a and S_s in terms of v_a^* and/or v_s^* only. The same method can be easily modified to give S_a and S_s in terms of v_a^* and/or v_s^* and ρ . In fact, instead of using the expressions established by Kingman as approximate formulae for the A.Q.T. under conditions of heavy traffic, one can use the same expressions as upper bounds on the A.Q.T. under unsaturated conditions.

An example

Consider the relationship (5.2) with $v_a' = 1$ applied to a single-server queueing system where the distribution functions of inter-arrival and service times are both Erlang:

$$\begin{aligned} \frac{W(E, v_a = v_a^*/E, v_s = v_s^*/1)}{W(M/E, v_a = v_a^*/1)} &\approx S_s \frac{W(E, v_a = v_a^*/M/1)}{W(M/M/1)} + \\ &+ (1 - S_s) \frac{W(E, v_a = v_a^*/D/1)}{W(M/D/1)} \end{aligned} \quad (10.1)$$

By introducing into (10.1) the expressions for the upper bounds on the six average queueing times, we obtain after simple algebra:

$$S_s = \frac{(v_s^*)^2 (1+\rho^2)}{1+\rho^2 (v_s^*)^2}$$

instead of:

$$S_s = \frac{2(v_s^*)^2}{1+(v_s^*)^2}$$

which was obtained in section 5.2.2.

(10.1) now takes the form:

$$\begin{aligned} W(E, v_a = v_a^*/E, v_s = v_s^*/1) &\approx \\ &\approx \frac{(v_s^*)^2 (1+\rho^2) (1+(v_s^*)^2)}{2(1+\rho^2 (v_s^*)^2)} W(E, v_a = v_a^*/M/1) + \\ &+ \frac{(1-(v_s^*)^2) (1+(v_s^*)^2)}{1+\rho^2 (v_s^*)^2} W(E, v_a = v_a^*/D/1) \end{aligned}$$

and for $v_a^* = 0$

$$W(D/E, v_s = v_s^*/1) \approx \frac{(v_s^*)^2 (1+\rho^2) (1+(v_s^*)^2)}{2(1+\rho^2 (v_s^*)^2)} W(D/M/1)$$

or, if we set $(v_s^*)^2 = 1/\ell$,

$$W(D/E_\ell/1) \approx \frac{(1+\rho^2) (1+\ell)}{2\ell(\ell+\rho^2)} W(D/M/1) \quad (10.2)$$

instead of

$$W(D/E_\ell/1) \approx (1/\ell) W(D/M/1) \quad (10.3)$$

which is equivalent to (5.10) and can be derived from (10.2) by setting $\rho = 1$.

Table 10.1 gives the relative percentage errors incurred if formulae (10.2) and (10.3) are used for the approx-

imate evaluation of an average queuing time $\mu W(D/E_\ell/1)$ equal to 0.5, 1.0 and 1.5 for $\ell = 1, 2, 3, 5$ and 10.

TABLE 10.1

Relative percentage errors incurred if (10.2) and (10.3) are used for the approximate evaluation of $W(D/E_\ell/1)$

ℓ	Use of (10.2)			Use of (10.3)		
	$\mu W=0.5$	$\mu W=1.0$	$\mu W=1.5$	$\mu W=0.5$	$\mu W=1.0$	$\mu W=1.5$
1	0	0	0	0	0	0
2	6.4	3.2	2.1	16.6	9.2	6.4
3	9.6	4.7	3.3	21.1	11.6	8.1
5	12.1	6.3	4.4	22.9	12.6	8.5
10	14.1	7.7		23.0	12.5	

In this specific example the modified method for determining the similarity factor S_g is preferable to the one used in previous Chapters. It is not difficult, however, to find counter-examples.

10.2.3 Re-examination of the basic assumptions

In deriving approximate formulae for the A.Q.T. in systems where both distributions involved are gamma, no assumptions were required. When, however, generalizing the formulae to cover queueing systems where one or both distributions are arbitrary, a set of assumptions was made in order to make the formulae useful. Some of the assumptions seem justified; an example could be the assumption in (3.5). Most of them, however, need re-examination, possibly by taking into consideration moments of higher order. The formulae would become more complicated but at the same time more reliable.

10.3 ON THE DERIVATION OF APPROXIMATE FORMULAE FOR THE EQUILIBRIUM PROBABILITY THAT AN ARRIVING CUSTOMER JOINS THE QUEUE

When dealing with waiting line systems, apart from the average queuing time of an arriving customer and the measures directly related to it through Little's²⁸ formula, the practitioner may well be interested in other measures describing the behaviour of the system such as the probability that an arriving customer joins the queue. Hence the question: Could the idea of similarity in the behaviour of queuing systems provide approximate formulae for this probability?

Formula (2.11) established in section 2.5 of Chapter 2 suggests that such a possibility exists. It seems, however, that the similarity factors in (2.11) are identical to the ones used in (2.7) only by coincidence; some method has to be devised for the determination of the similarity factors in the general case.

10.4 ON THE EVALUATION OF APPROXIMATE SOLUTIONS TO MORE COMPLEX QUEUING SYSTEMS

10.4.1 Introduction

The investigation so far has been confined to standard GI/G/x systems; but there seems to be no fundamental reason, at least in principle, why the idea of similarity in the behaviour of queuing systems cannot be applied to more complex multi - server systems, the analytical treatment of which has failed to provide usable results. Examples of more complex systems could be:

a) multi - server systems with priorities;

- b) multi - server machine interference systems;
- c) multi - server systems with limited space; and
- d) multi - server systems with batch arrivals or service.

10.4.2 Basic limitations

When trying to apply the methodology developed in previous Chapters to complex systems, one is bound to encounter difficulties:

- a) the corresponding single - server systems may not have easy to handle analytical solutions; and
- b) the similarity ratios may not be easy to evaluate either because no method for the determination of the similarity factors can be devised or because the "boundary" multi - server systems have no analytical (not even approximate) solution.

10.4.3 An example: A multi - server system with exponential arrivals and two non - preemptive priorities with the same distribution of service times

Consider the following queueing system: There are r servers in parallel each having an independently and identically distributed service - time distribution $B(t)$. The average service time is μ^{-1} and the coefficient of variation of service times $v_s = \sqrt{\text{var}(t)} \mu$, such that $0 < v_s < 1$. There are two types of customers: 1 - customers and 2 - customers.

- 1 - customers arrive at random at an average rate λ_1 ;
- 2 - customers arrive at random at an average rate λ_2 . Two queues are formed: one with 1 - customers and the other with 2 - customers. Within each queue customers are ordered according to their arrival times but 1 - customers have a non -

preemptive priority over 2 - customers. The system is in statistical equilibrium, i.e. both traffic intensities $\rho_1 = \lambda_1/r\mu$ and $\rho_2 = \lambda_2/r\mu$ and also $\rho = \rho_1 + \rho_2$ are assumed less than 1.

Let $W_1(M_1/B(t)/r)$, $W_2(M_2/B(t)/r)$ and $W(M/B(t)/r)$ denote the average queuing times of a 1 - customer, of a 2 - customer and of a customer (whether 1 or 2) respectively. $W(M/B(t)/r)$ is equal to the A.Q.T. of an arriving customer in the standard $M/B(t)/r$ queuing system with $\lambda = \lambda_1 + \lambda_2$ and $\rho = \lambda/r\mu = \rho_1 + \rho_2$ and is related to $W_1(M_1/B(t)/r)$ and $W_2(M_2/B(t)/r)$ through:

$$W(M/B(t)/r) = (\rho_1/\rho) W_1(M_1/B(t)/r) + (\rho_2/\rho) W_2(M_2/B(t)/r) \quad (10.4)$$

One can reasonably expect that $\mu W_1(M_1/B(t)/r)$ depends on ρ_1 , ρ_2 , r and $B(t)$ only; furthermore, that for ρ_1 , ρ_2 , r and the type of $B(t)$ fixed, $\mu W_1(M_1/B(t)/r)$ is an increasing function of v_s with limiting values $\mu W_1(M_1/D/r)$, if $v_s = 0$ and, say, $\mu W_1(M_1/B(t), v_s=1/r)$, if $v_s = 1$.

Two such relationships between $\mu W_1(M_1/B(t)/r)$ and v_s are shown in Figure 10.1. They are both related to a queueing system of the same general form with the same type of distribution of service times. Curve (1) represents a situation in which $r = 1$; curve (2) represents a situation in which the number of servers is r . In both cases, however, the traffic intensities are assumed equal.

a) We now proceed in writing a relationship of the general form:

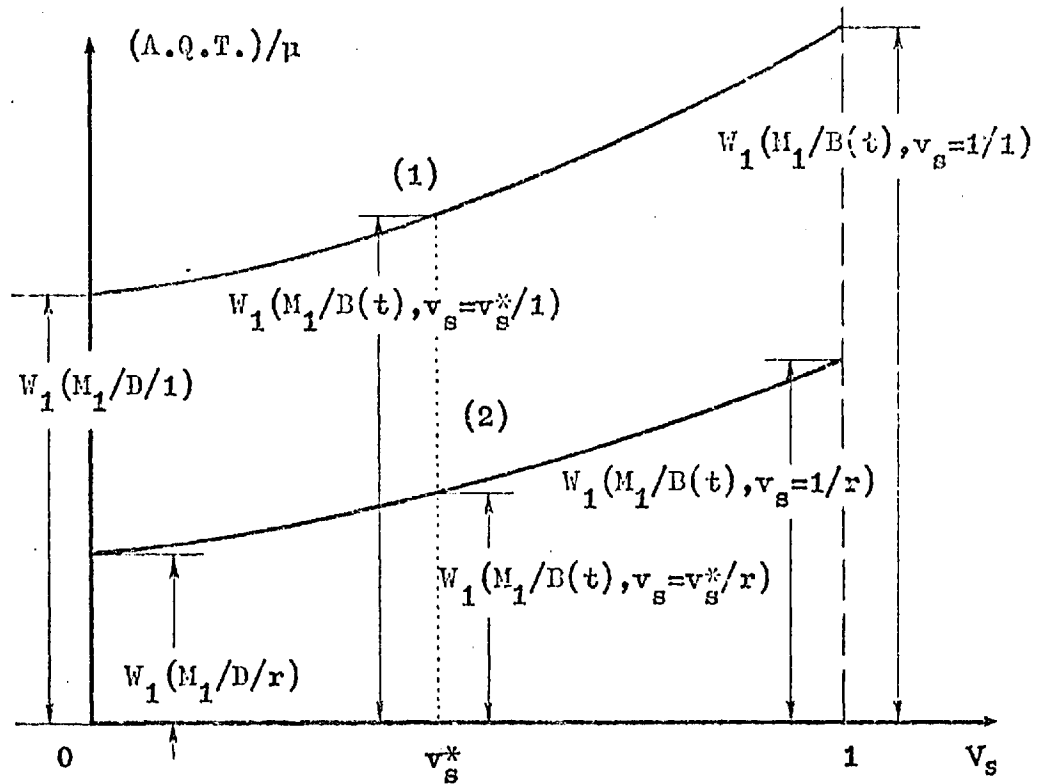


Fig. 10.1 $\rho W_1(M_1/B(t)/r)$ as a function of v_s for two non - preemptive priority queuing systems with the same type of distribution of service times and equal traffic intensities.

$$\frac{W_1(M_1/B(t), v_s=v_s^*/r)}{W_1(M_1/B(t), v_s=v_s^*/1)} = S_s \frac{W_1(M_1/B(t), v_s=1/r)}{W_1(M_1/B(t), v_s=1/1)} + (1 - S_s) \frac{W_1(M_1/D/r)}{W_1(M_1/D/1)} \quad (10.5)$$

with v_s^* any value of v_s between 0 and 1 and S_s some "similarity factor" with limiting values 0 and 1 for $v_s^* = 0$ and 1 respectively, regardless of r , ρ_1 , ρ_2 or $B(t)$.

b) We determine the expression:

$$S_s = \frac{2(v_s^*)^2}{1+(v_s^*)^2}$$

for the similarity factor S_g by applying the method suggested in section 3.2.2.

c) Finally we make the assumption that:

$$\frac{W_1(M_1/B(t), v_s=1/r)}{W_1(M_1/B(t), v_s=1/1)} = \frac{W_1(M_1/M/r)}{W_1(M_1/M/1)}.$$

(10.5) can be written:

$$\begin{aligned} \frac{W_1(M_1/B(t), v_s=v_s^*/r)}{W_1(M_1/B(t), v_s=v_s^*/1)} &\approx \frac{2(v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W_1(M_1/M/r)}{W_1(M_1/M/1)} + \\ &+ \frac{1 - (v_s^*)^2}{1 + (v_s^*)^2} \cdot \frac{W_1(M_1/D/r)}{W_1(M_1/D/1)} \end{aligned} \quad (10.6)$$

or, if the well known result

$$W_1(M_1/B(t), v_s=v_s^*/1) = \rho (1 + (v_s^*)^2) / (2(1 - \rho_1)\mu)$$

is used,

$$W_1(M_1/B(t), v_s=v_s^*/r) \approx (v_s^*)^2 W_1(M_1/M/r) + (1 - (v_s^*)^2) W_1(M_1/D/r). \quad (10.7)$$

If formula (3.8) established in Chapter 3 for the approximate evaluation of $W(M/B(t), v_s=v_s^*/r)$ and the expression for $W_1(M_1/B(t), v_s=v_s^*/r)$ in (10.7) are introduced in (10.4) we obtain:

$$W_2(M_2/B(t), v_s=v_s^*/r) \approx (v_s^*)^2 W_2(M_2/M/r) + (1 - (v_s^*)^2) W_2(M_2/D/r).$$

Cobham³ gives for $W_1(M_1/M/r)$ and $W_2(M_2/M/r)$ the very simple expressions:

$$W_1(M_1/M/r) = \frac{1 - \rho}{1 - \rho_1} W(M/M/r) \quad \text{and}$$

$$W_2(M_2/M/r) = \frac{1}{1 - \rho_1} W(M/M/r)$$

with $W(M/M/r)$ the average queueing time of an arriving customer to a standard $M/M/r$ system where $\rho = \rho_1 + \rho_2$.

Consequently, (10.7) and (10.8) can be written:

$$\begin{aligned} W_1(M_1/B(t), v_s=v_s^*/r) &\approx (v_s^*)^2 \frac{1 - \rho}{1 - \rho_1} W(M/M/r) + \\ &+ (1 - (v_s^*)^2) W_1(M_1/D/r) \end{aligned} \quad (10.9)$$

and

$$\begin{aligned} W_2(M_2/B(t), v_s=v_s^*/r) &\approx (v_s^*)^2 \frac{1}{1 - \rho_1} W(M/M/r) + \\ &+ (1 - (v_s^*)^2) W_2(M_2/D/r) \end{aligned} \quad (10.10)$$

with

$$W_2(M_2/D/r) = \frac{\rho}{\rho_2} W(M/D/r) - \frac{\rho_1}{\rho_2} W_1(M_1/D/r)$$

If the relative percentage errors are small, formulae (10.9) and (10.10) can be used for the approximate evaluation of $W_1(M_1/B(t), v_s=v_s^*/r)$ and $W_2(M_2/B(t), v_s=v_s^*/r)$. Though, it seems, $W_1(M_1/D/r)$ has to be evaluated by simulation, the use of (10.9) and (10.10) may prove useful:

- a) if the effect of $B(t)$ on the average queueing times is investigated;
- b) if the values of v_s^* and ρ_1 are relatively high in which case the second term on the right hand side of (10.9) and (10.10) is relatively small and, hence, $W_1(M_1/D/r)$ need not be evaluated accurately; and
- c) because $W_1(M_1/D/r)$ can be evaluated efficiently by simulation.

Table 10.2 summarizes the evaluation of formulae (10.9) and (10.10) for:

$$B(t) = \begin{array}{|} \hline \square \\ \hline \end{array} \text{ with } (v_s^*)^2 = 1/3 \text{ and } \begin{array}{|} \hline \triangle \\ \hline \end{array} \text{ with } (v_s^*)^2 = 1/2;$$

$$\rho_1 = 0.20 \text{ and } 0.40;$$

$$\rho = 0.80; \text{ and}$$

$$r = 2 \text{ and } 4.$$

The analytical values of $W(M/M/r)$ required in the calculations are given in Table A3.1 of Appendix 3; all other av-

TABLE 10.2

Evaluation of (10.9) and (10.10)

for some non-preemptive priority multi-server systems

($\rho = 0.8$)

B(t)	$(v_s^*)^2$	r	ρ_1	Use of (10.9)		Use of (10.10)	
				A.Q.T.	e(%)	A.Q.T.	e(%)
$\begin{array}{ } \hline \square \\ \hline \end{array}$	1/3	2	0.2	0.5211*	-2.8	1.4697	-0.1
$\begin{array}{ } \hline \triangle \\ \hline \end{array}$	1/2	"	"	0.3519*	-2.9	1.6578	2.2
$\begin{array}{ } \hline \square \\ \hline \end{array}$	1/3	4	"	0.1453*	-4.0	0.6175	-0.7
$\begin{array}{ } \hline \triangle \\ \hline \end{array}$	1/2	"	"	0.1556*	-3.8	0.6961	2.0
$\begin{array}{ } \hline \square \\ \hline \end{array}$	1/3	2	0.4	0.4137*	-1.5	1.9763	4.6
$\begin{array}{ } \hline \triangle \\ \hline \end{array}$	1/2	"	"	0.4585*	-3.1	2.2230	-1.6
$\begin{array}{ } \hline \square \\ \hline \end{array}$	1/3	4	"	0.1845*	-5.3	0.8261	5.1
$\begin{array}{ } \hline \triangle \\ \hline \end{array}$	1/2	"	"	0.2005*	-4.7	0.9302	-2.0

erage queueing times were obtained by simulation, the results of the simulation experiments being presented in Table A4.14 of Appendix 4. The asterisks(*) indicate that the approximate values fall outside the corresponding 95 % confidence

interval on the simulated A.Q.T. The relative percentage error incurred is defined as:

$$e = 100(\text{approximate value} - \text{simulated value})/(\text{simul. value})$$

With the idea of similarity in the behaviour of queueing systems always in mind, we observe that:

$$\frac{W_1(M_1/D/r)}{W_1(M_1/M/r)} \approx \left[\frac{W(M/D/r)}{W(M/M/r)} \right]_{\rho = \rho_1} \quad (10.11)$$

(see Table 10.3).

TABLE 10.3

Comparison between

$$W_1(M_1/D/r)/W_1(M_1/M/r) \quad \text{and} \quad W(M/D/r)/W(M/M/r)$$

the latter calculated for traffic intensity equal to ρ_1

r	ρ_1	$W_1(M_1/D/r)/W_1(M_1/M/r)$	$W(M/D/r)/W(M/M/r)$
2	0.2	0.584	0.581
4	"	0.669	0.687
2	0.4	0.547	0.542
4	"	0.613	0.601

Obviously, the results available are not enough to allow an assessment of the validity of (10.11) in connection with the values given to the various parameters involved. Assuming, however, that (10.11) holds (and it does hold for the cases presented in Table 10.2) formulae (10.9) and (10.10) are greatly simplified:

$$W_1(M_1/B(t), v_s = v_s^*/r) \approx \frac{1 - \rho}{1 - \rho_1} W(M/M/r) \left[(v_s^*)^2 + \right.$$

$$+ (1 - (v_s^*)^2) \left[\frac{W(M/D/r)}{W(M/M/r)} \right] \rho = \rho_1 \quad (10.12)$$

and

$$W_2(M_2/B(t), v_s = v_s^*/r) \approx (\rho/\rho_2) (1 - (v_s^*)^2) W(M/D/r) + \frac{W(M/M/r)}{1 - \rho_1} \left[(v_s^*)^2 - (1 - (v_s^*)^2) \frac{\rho_1(1-\rho)}{\rho_2} \left[\frac{W(M/D/r)}{W(M/M/r)} \right] \rho = \rho_1 \right] \quad (10.13)$$

with $W(M/M/r)$ and $W(M/D/r)$ calculated for traffic intensity $\rho (= \rho_1 + \rho_2)$.

The evaluation of (10.12) and (10.13) is extremely simple and can be completed in a couple of minutes by hand; the degree of approximation achieved is quite satisfactory for practical purposes (see Table 10.4).

TABLE 10.4

Evaluation of (10.12) and (10.13)
for some non-preemptive priority multi-server systems
($\rho = 0.8$)

B(t)	$(v_s^*)^2$	r	ρ_1	Use of (10.12)		Use of (10.13)	
				A.Q.T.	e(%)	A.Q.T.	e(%)
□	1/3	2	0.2	0.3204*	-3.0	1.4862	1.1
▤	1/2	"	"	0.3514*	-3.0	1.6702	3.0
□	1/3	4	"	0.1475*	-2.6	0.6254	0.5
▤	1/2	"	"	0.1573*	-2.7	0.7020	2.9
□	1/3	2	0.4	0.4118*	-2.0	1.9778	4.6
▤	1/2	"	"	0.4570*	-3.4	2.2241	-1.5
□	1/3	4	"	0.1823*	-4.4	0.8295	5.5
▤	1/2	"	"	0.1989*	-5.5	0.9327	-1.7

There seems to be no difficulty in extending the approximate results derived in this section to the case of more than two types of arriving customers.

CONCLUSIONS

The questions raised in this Chapter suggest that a lot of effort has still to be put into the investigation of the ways by which one can express and apply the idea of similarity in the behaviour of queueing systems.

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A P P E N D I C E S

A P P E N D I X 1

DISTRIBUTIONS

OF INTER-ARRIVAL AND SERVICE TIMES

CONSIDERED

A1.1 DISTRIBUTIONS OF INTER-ARRIVAL TIMES CONSIDEREDA1.1.1 GAMMA (Γ)

$$dA(u) = \frac{(k\lambda)^k u^{k-1} \exp(-k\lambda u)}{\Gamma(k)} du, \quad u \geq 0.$$

$$\text{mean}(u) = 1/\lambda$$

$$\text{var}(u) = 1/(k\lambda^2)$$

$$v_a^2 = 1/k$$

A1.1.2 ERLANG (E_k)

A special case of the Gamma distribution, where k is an integer and, hence, $\Gamma(k) = (k-1)!$

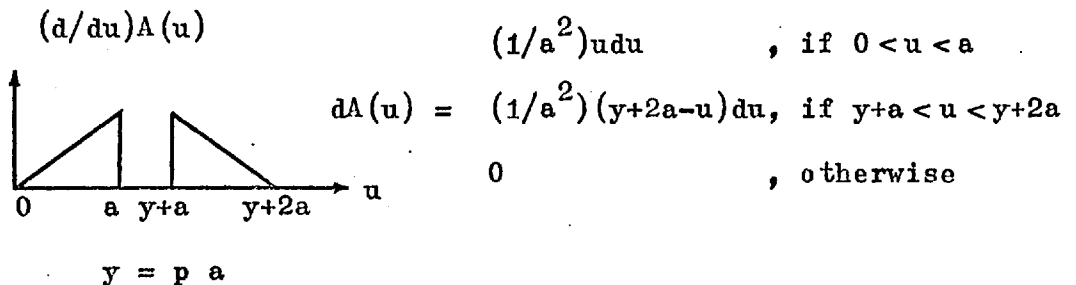
$$dA(u) = \frac{(k\lambda)^k u^{k-1} \exp(-k\lambda u)}{(k-1)!} du, \quad u \geq 0.$$

$$A^*(s) = (k\lambda)^k / (k\lambda + s)^k$$

$$\text{if } k = 1, v_a^2 = 1, dA(u) = \lambda \exp(-\lambda u) du \text{ and } A^*(s) = \lambda / (\lambda + s)$$

$$\text{if } k \rightarrow \infty, v_a^2 = 0, \text{ var}(u) = 0, u = 1/\lambda \text{ and } A^*(s) = \exp(-s)$$

A1.1.3 a.



$$\text{mean}(u) = (2 + p)a/2$$

$$\text{var}(u) = (2 + 4p + 3p^2)a^2/12$$

$$v_a^2 = (2 + 4p + 3p^2)/(12 + 12p + 3p^2)$$

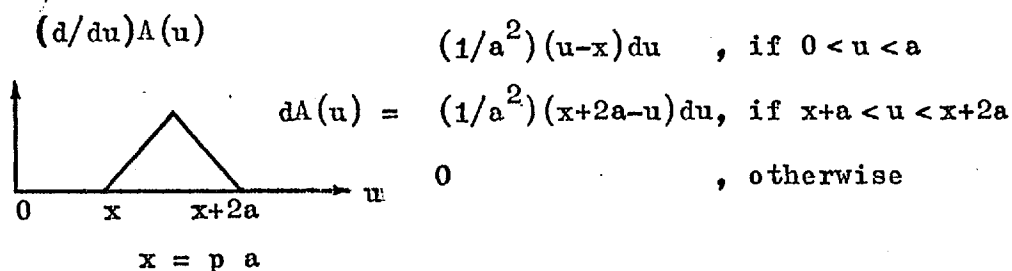
$$A^*(s) = (1 - (1+as)e^{-as} + (as-1)e^{-as})e^{-(1+p)as} / (as)^2$$

Correspondence between v_a^2 and p

v_a^2	1/1.02	1/1.21	1/1.44
p	$66 + \sqrt{4,522}$	$(358 + \sqrt{188,518})/63$	$(26 + \sqrt{1,512})/11$

v_a^2	1/2	1/3	1/4	1/6
p	$(2 + 2\sqrt{7})/3$	1	$(-2 + 2\sqrt{10})/9$	0

b.



$$\text{mean}(u) = (1 + p) a$$

$$\text{var}(u) = a^2/6$$

$$v_a^2 = 1/(6(1 + p)^2)$$

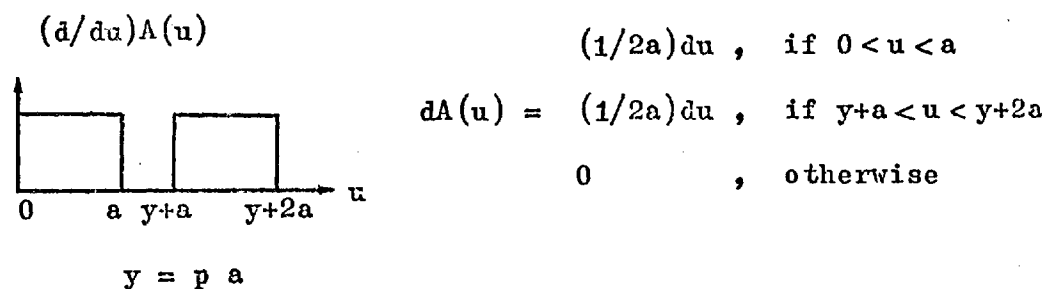
$$A^*(s) = e^{-pas}(1 - e^{-as})^2/(as)^2$$

Correspondence between v_a^2 and p

v_a^2	1/6	1/9	1/16	1/25
p	0	$(\sqrt{6} - 2)/2$	$(\sqrt{24} - 3)/3$	$(\sqrt{150} - 6)/6$

v_a^2	1/64	1/100	1/400
p	$(4\sqrt{6} - 3)/3$	$(\sqrt{150} - 3)/3$	$(\sqrt{600} - 3)/3$

A1.1.4 a.



$$\text{mean}(u) = (2 + p) a / 2$$

$$\text{var}(u) = (4 + 6p + 3p^2) a^2 / 12$$

$$v_a^2 = (4 + 6p + 3p^2) / (3(2 + p)^2)$$

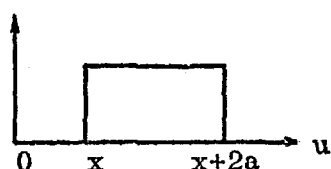
$$A^*(s) = (1 - e^{-as})(1 + e^{-(1+p)as}) / (2as)$$

Correspondence between v_a^2 and p

v_a^2	1/1.02	1/1.21	1/1.44	1/2	1/3
p	$49 + \sqrt{2,533}$	$(79 + \sqrt{11,253})/21$	$(14 + 16\sqrt{3})/11$	$2/\sqrt{3}$	0

b.

$$(d/du)A(u)$$



$$x = p a$$

$$dA(u) = \begin{cases} 1/2a, & \text{if } x < u < x+2a \\ 0, & \text{otherwise} \end{cases}$$

$$\text{mean}(u) = (1 + p) a$$

$$\text{var}(u) = a^2 / 3$$

$$v_a^2 = 1 / (3(1 + p)^2)$$

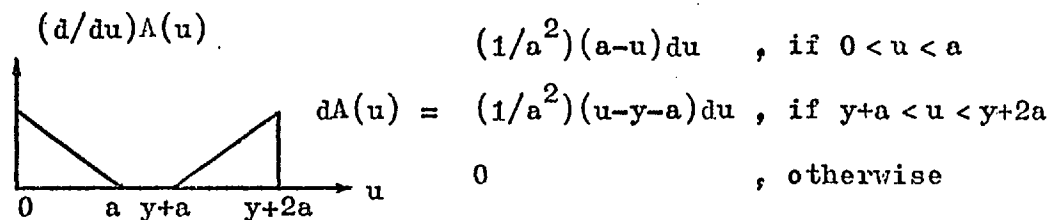
$$A^*(s) = e^{-pas}(1 - e^{-2as})$$

Correspondence between v_a^2 and p

v_a^2	1/3	1/4	1/6	1/9	1/16	1/25
p	0	$(\sqrt{12}-3)/3$	$\sqrt{2}-1$	$\sqrt{3}-1$	$(\sqrt{3}-0.75)/0.75$	$(\sqrt{75}-3)/3$

v_a^2	1/64	1/100	1/400
p	$(8\sqrt{3}-3)/3$	$(10\sqrt{3}-3)/3$	$(20\sqrt{3}-3)/3$

A1.1.5 a.



$$y = p a$$

$$\text{mean}(u) = (2 + p) a / 2$$

$$\text{var}(u) = (6 + 8p + 3p^2) a^2 / 12$$

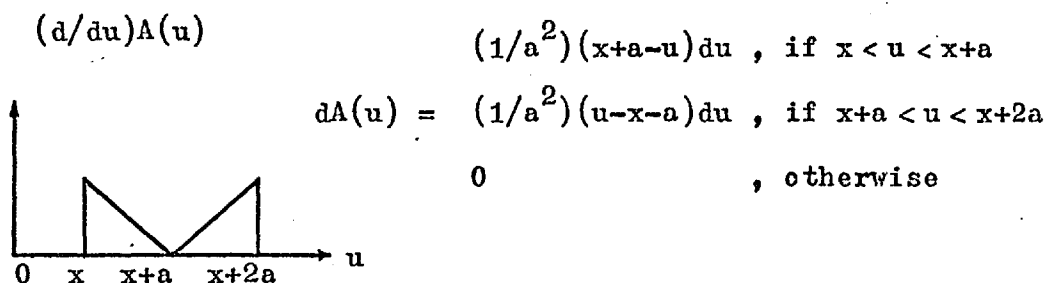
$$v_a^2 = (6 + 8p + 3p^2) / (3(2 + p)^2)$$

$$A^*(s) = ((as-1) + e^{-as} + (1-e^{-as}-ase^{-as})e^{-(1+p)as}) / (as)^2$$

Correspondence between v_a^2 and p

v_a^2	1/1.02	1/1.21	1/1.44	1/2
p	$32+\sqrt{1,122}$	$(116+\sqrt{43,318})/63$	$(2+\sqrt{312})/11$	0

b.



$$x = p a$$

$$\text{mean}(u) = (1 + p) a$$

$$\text{var}(u) = a^2 / 2$$

$$v_a^2 = 1 / (2(1 + p)^2)$$

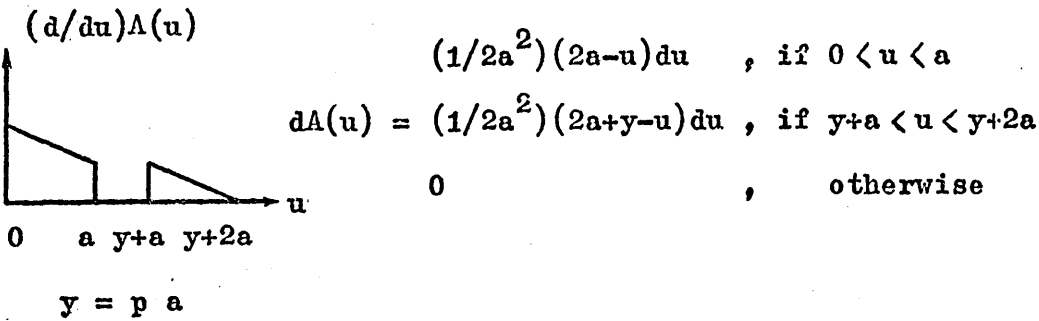
$$A^*(s) = ((as-1)e^{-pas} + 2e^{-(1+p)as} - (as+1)e^{-(2+p)as}) / (as)^2$$

Correspondence between v_a^2 and p

v_a^2	1/2	1/3	1/4	1/6	1/9
p	0	$(\sqrt{6}-2)/2$	$\sqrt{2}-1$	$\sqrt{3}-1$	$(3\sqrt{2}-2)/2$

v_a^2	1/16	1/25	1/64	1/100	1/400
p	$\sqrt{8}-1$	$(5\sqrt{2}-2)/2$	$4\sqrt{2}-1$	$5\sqrt{2}-1$	$10\sqrt{2}-1$

A1.1.6 a.



$$\begin{aligned} \text{mean}(u) &= (8 + 3p) a / 12 \\ \text{var}(u) &= (32 + 48p + 27p^2) a^2 / 144 \\ v_a^2 &= (32 + 48p + 27p^2) / ((8 + 3p)^2) \end{aligned}$$

$$A^*(s) = ((2as-1) + (1-as)e^{-as} + (as-1+e^{-as})e^{-(1+p)as}) / (2(as)^2)$$

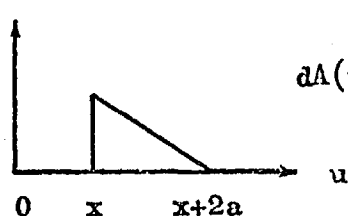
Correspondence between v_a^2 and p

v_a^2	1/1.02	1/1.21
p	$(-8+\sqrt{161,259})/309$	$(-168+4\sqrt{43,318})/789$

v_a^2	1/1.44	1/2	1
p	$(-88+8\sqrt{702})/249$	0	4/3

b.

$$(d/du)A(u)$$



$$x = p a$$

$$dA(u) = \begin{cases} (1/2a) (2a+x-u) du, & \text{if } x < u < x+2a \\ 0 & \text{, otherwise} \end{cases}$$

$$\text{mean}(u) = (2 + 3p) a / 3$$

$$\text{var}(u) = 2 a^2 / 9$$

$$v_a^2 = 2 / (2 + 3p)^2$$

$$A^*(s) = ((2as-1) + e^{-2as})e^{-pas} / (2(as)^2)$$

Correspondence between v_a^2 and p

v_a^2	1/2	1/3	1/4	1/6	1/9
p	0	$(\sqrt{6}-2)/3$	$(2\sqrt{2}-2)/3$	$(2\sqrt{3}-2)/3$	$(3\sqrt{2}-2)/3$

v_a^2	1/16	1/25	1/64	1/100	1/400
p	$(4\sqrt{2}-2)/3$	$(5\sqrt{2}-2)/3$	$(8\sqrt{2}-2)/3$	$(10\sqrt{2}-2)/3$	$(20\sqrt{2}-2)/3$

A1.2 DISTRIBUTIONS OF SERVICE TIMES CONSIDERED

The same as the ones analysed in A1.1 with:

t instead of u ;
 $B(t)$ " " $A(u)$;
 p " " λ ;
 ℓ " " k ; and
 $B^*(s)$ " " $A^*(s)$.

A P P E N D I X 2

THE EQUILIBRIUM AVERAGE QUEUEING TIME

OF AN ARRIVING CUSTOMER

IN SOME STANDARD SYSTEMS

A2.1 $\mu W(A(u)/M/r)$, $r = 1, 2, 3, 4, 5, 6$

METHOD USED : The one described in reference (8).

NOTATION : z is the only root within the unit circle of the equation $z = \int_0^\infty e^{-r(1-z)\mu u} dA(u)$.
 $A^*(k\mu) = \int_0^\infty e^{-k\mu u} dA(u)$, $k = 1, 2, 3, 4, 5$

$$\mu W(A(u)/M/r) = \frac{z}{r(1-z)} \frac{F_{r,1}}{(1-z)F_{r,2} + F_{r,3}}$$

A2.1.1 $r = 1$

$$F_{1,1} = 1$$

$$F_{1,2} = 0$$

$$F_{1,3} = 1$$

A2.1.2 $r = 2$

$$F_{2,1} = (1-2z)A^*(\mu)$$

$$F_{2,2} = 1$$

$$F_{2,3} = -A^*(\mu)$$

A2.1.3 $r = 3$

$$F_{3,1} = (1-3z)(2-3z)A^*(\mu)A^*(2\mu)$$

$$F_{3,2} = (2-3z)(1-A^*(\mu)) - (2+3z)A^*(2\mu)$$

$$F_{3,3} = 2A^*(\mu)A^*(2\mu)$$

A2.1.4 $r = 4$

$$F_{4,1} = (1-4z)(2-4z)(3-4z)A^*(\mu)A^*(2\mu)A^*(3\mu)$$

$$F_{4,2} = (2-4z)(3-4z)(1-A^*(\mu)-A^*(2\mu)+A^*(\mu)A^*(2\mu)) - \\ - (2+8z)(3-4z)A^*(3\mu)(1-A^*(\mu)) + (6+4z+16z^2)A^*(2\mu)A^*(3\mu)$$

$$F_{4,3} = -6A^*(\mu)A^*(2\mu)A^*(3\mu)$$

A2.1.5 r = 5

$$F_{5,1} = (1-5z)(2-5z)(3-5z)(4-5z)\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)$$

$$F_{5,2} = (2-5z)(3-5z)(4-5z)(1-\Lambda^*(\mu)-\Lambda^*(2\mu)-\Lambda^*(3\mu)+\Lambda^*(\mu)\Lambda^*(2\mu)+ \\ +\Lambda^*(\mu)\Lambda^*(3\mu)+\Lambda^*(2\mu)\Lambda^*(3\mu)-\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)) - \\ - (2+15z)(3-5z)(4-5z)\Lambda^*(4\mu)(1-\Lambda^*(\mu)-\Lambda^*(2\mu)+ \\ +\Lambda^*(\mu)\Lambda^*(2\mu)) + (6+5z+75z^2)(4-5z)(1-\Lambda^*(\mu))\Lambda^*(3\mu)\Lambda^*(4\mu)- \\ - (24+30z-25z^2+125z^3)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)$$

$$F_{5,3} = 24\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)$$

A2.1.6 r = 6

$$F_{6,1} = (1-6z)(2-6z)(3-6z)(4-6z)(5-6z)\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu) \\ \Lambda^*(5\mu)$$

$$F_{6,2} = (2-6z)(3-6z)(4-6z)(5-6z)(1-\Lambda^*(\mu)-\Lambda^*(2\mu)-\Lambda^*(3\mu)-\Lambda^*(4\mu)+ \\ +\Lambda^*(\mu)\Lambda^*(2\mu)+\Lambda^*(\mu)\Lambda^*(3\mu)+\Lambda^*(\mu)\Lambda^*(4\mu)+\Lambda^*(2\mu)\Lambda^*(3\mu)+ \\ +\Lambda^*(2\mu)\Lambda^*(4\mu)+\Lambda^*(3\mu)\Lambda^*(4\mu)-\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)- \\ -\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(4\mu)-\Lambda^*(\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)- \\ -\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)+\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)) - \\ - (2+24z)(3-6z)(4-6z)(5-6z)\Lambda^*(5\mu)(1-\Lambda^*(\mu)-\Lambda^*(2\mu)- \\ -\Lambda^*(3\mu)+\Lambda^*(\mu)\Lambda^*(2\mu)+\Lambda^*(\mu)\Lambda^*(3\mu)+\Lambda^*(2\mu)\Lambda^*(3\mu)- \\ -\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)) + \\ + (6+216z^2)(4-6z)(5-6z)\Lambda^*(4\mu)\Lambda^*(5\mu)(1-\Lambda^*(\mu)-\Lambda^*(2\mu)+ \\ +\Lambda^*(\mu)\Lambda^*(2\mu)) - (24+84z-216z^2+864z^3)(5-6z)\Lambda^*(3\mu) \\ \Lambda^*(4\mu)\Lambda^*(5\mu)(1-\Lambda^*(\mu)) + (120+96z+396z^2-864z^3+ \\ +1296z^4)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)\Lambda^*(5\mu)$$

$$F_{6,3} = -120\Lambda^*(\mu)\Lambda^*(2\mu)\Lambda^*(3\mu)\Lambda^*(4\mu)\Lambda^*(5\mu)$$

A2.2 $\mu W(E_2/M/r)$, $r = 1, 2, 3, 4, 5, 6, 8, 10$

METHOD USED : Algebraic solution of the equations of detailed balance for the equilibrium state, as given in reference (32).

NOTATION : $R = 2\rho r$
 $z = \sqrt{2\rho + 0.25} - 0.5$

$$\mu W(E_2/M/r) = \frac{(2\rho - z)R^{r-1}}{(1 + z - 2\rho)F_r}$$

A2.2.1 $r = 1$

$$F_1 = 1$$

A2.2.2 $r = 2$

$$F_2 = R + 3 + 2/R + 2z/R$$

A2.2.3 $r = 3$

$$F_3 = R^2 + R + 20 + 32/R + 12/R^2 + 12z/R^2 + 24z/R + 6z$$

A2.2.4 $r = 4$

$$F_4 = R^3 + 9R^2 + 54R + 216 + 504/R + 492/R^2 + 144/R^3 + 144z/R^3 + 432z/R^2 + 336z/R + 96z + 12zR$$

A2.2.5 $r = 5$

$$F_5 = R^4 + 12R^3 + 104R^2 + 672R + 3,096 + 9,264/R + 15,264/R^2 + 11,424/R^3 + 2,880/R^4 + 2,880z/R^4 + 10,560z/R^3 + 12,240z/R^2 + 6,000z/R + 1,560z + 240zR + 20zR^2$$

A2.2.6 $r = 6$

$$F_6 = R^5 + 15R^4 + 170R^3 + 1,520R^2 + 10,560R + 54,840 + 199,680/R + 462,720/R^2 + 602,400/R^3 + 380,160/R^4 +$$

$$\begin{aligned}
& + 86,400/R^5 + 86,400z/R^5 + 360,000z/R^4 + 521,280z/R^3 + \\
& + 351,360z/R^2 + 128,880z/R + 29,520z + 4,560zR + 480zR^2 + \\
& + 30zR^3
\end{aligned}$$

A2.2.8 r = 8

$$\begin{aligned}
F_8 = & R^7 + 21R^6 + 350R^5 + 4,872R^4 + 56,952R^3 + 554,820R^2 + \\
& + 4,430,160R + 28,271,040 + 139,214,880/R + \\
& + 504,887,040/R^2 + 1,271,410,560/R^3 + 2,075,068,800/R^4 + \\
& + 2,018,862,720/R^5 + 1,028,401,920/R^6 + 203,212,800/R^7 + \\
& + 203,212,800z/R^7 + 995,742,720z/R^6 + 1,862,461,440z/R^5 + \\
& + 1,792,143,360z/R^4 + 1,010,419,200z/R^3 + \\
& + 364,250,880z/R^2 + 90,397,440z/R + 16,396,800z + \\
& + 2,268,000zR + 245,280zR^2 + 20,832zR^3 + 1,344zR^4 + 56zR^5
\end{aligned}$$

A2.2.10 r = 10

$$\begin{aligned}
F_{10} = & R^9 + 27R^8 + 594R^7 + 11,232R^6 + 184,896R^5 + 2,654,064R^4 + \\
& + 33,070,464R^3 + 354,461,184R^2 + 3,225,216,960R + \\
& + 24,479,884,800 + 151,579,572,480/R + \\
& + 744,796,684,800/R^2 + 2,808,573,626,880/R^3 + \\
& + 7,816,780,661,760/R^4 + 15,353,847,613,440/R^5 + \\
& + 20,166,739,568,640/R^6 + 16,437,871,779,840/R^7 + \\
& + 7,318,796,083,200/R^8 + 1,316,818,944,000/R^9 + \\
& + 1,316,818,944,000z/R^9 + 7,157,851,545,600z/R^8 + \\
& + 15,577,654,579,200z/R^7 + 18,338,271,436,800z/R^6 + \\
& + 13,264,715,520,000z/R^5 + 6,357,410,841,600z/R^4 + \\
& + 2,134,576,281,600z/R^3 + 526,292,121,600z/R^2 + \\
& + 99,229,536,000z/R + 14,794,012,800z + \\
& + 1,790,328,960zR + 179,262,720zR^2 + 15,029,280zR^3 + \\
& + 1,058,400zR^4 + 61,920zR^5 + 2,880zR^6 + 90zR^7
\end{aligned}$$

A2.3 $\mu W(M/E_2/r)$, $r = 2, 3, 4, 5$

METHOD USED : Algebraic solution of the equations of detailed balance for the equilibrium state, as given in reference (32), together with application of Rouché's theorem for the generation of additional equations.

The server utilization equation :

$P(r-1) + \dots + jP(r-j) + \dots + P(0) = r(1-\rho)$, mentioned in reference (37) , was also used in order to test intermediate results.

NOTATION : $h = \rho/2$.

$P(n)$: The probability of having n units in the system.

$p(n; s_1, s_2)$: The probability of having n units in the system, s_1 servers with a unit in phase 1 and s_2 servers with a unit in phase 2.

$$\mu W(M/E_2/r) = \frac{N_r}{2hr(1-2h)D_r} - 1$$

A2.3.2 $r = 2$

$$N_2 = h(8+12h-6h^2-11h^3-4h^4)P(0)$$

$$D_2 = (2+3h)(1-2h)$$

with :

$$P(0) = (2+3h)(1-2h)/(2+7h+6h^2+2h^3)$$

$$P(1) = 4h(2+3h+h^2)P(0)/(2+3h)$$

$$= 2(1-2h) - 2P(0) .$$

A2.3.3 r = 3

$$N_3 = (72+200h-76h^2-408h^3-216h^4) - \\ - 2(36+184h+346h^2+312h^3+144h^4+27h^5)P(0)$$

$$D_3 = 4(11+28h+18h^2)$$

with :

$$P(0) = -2(1-2h)A_3/B_3$$

and

$$A_3 = b_7a_1 + b_8za_2 + z(b_9z+b_{10})a_3$$

$$B_3 = b_1a_1 + (b_2z+b_3)a_2 + (b_4z^2+b_5z+b_6)a_3$$

where :

$$b_1 = 2+8h+45h^2+84h^3+45h^4$$

$$b_6 = 22+56h+36h^2$$

$$b_2 = -4-28h-6h^2+96h^3+126h^4+54h^5$$

$$b_7 = -1$$

$$b_3 = h(44+112h+72h^2)$$

$$b_8 = 2+6h$$

$$b_4 = -12-92h-251h^2-324h^3-207h^4-54h^5$$

$$b_9 = 6+22h+18h^2$$

$$b_5 = -8+12h+64h^2+72h^3+36h^4$$

$$b_{10} = 4$$

$$a_1 = z^2(1-3x^2z)(3-z) - 2z(1-xz)(1+3xz) + 4xz^3$$

$$a_2 = (1-3x^2z+2xz)(z^2+3xz^2) - (1+2x-3x^2z)(z+3xz^2)$$

$$a_3 = (1-3x^2z)(1-3z) - 4xz - 2(x-1)(1+3x)z^2$$

$$x = 1 + h - hz \quad \text{and}$$

z is the only root within the unit circle of the equation :

$$9x^2z - 1 = 0 .$$

$$P(1) = (b_{11}P(0)-6P(0)+6(1-2h))/(b_7+b_8+b_9+b_{10})$$

$$\text{with } b_{11} = 9h^2+3h(1+h)(2+3h)(7+3h)-6$$

$$P(2) = (b_7+b_8+b_9)P(1)/2 + (b_{11}+6)P(0)/2$$

$$= 3(1-2h) - 2P(1) - 3P(0) .$$

A2.3.4 r = 4

$$N_4 = 3(4e_1 + 2e_2 + 3e_5 + e_6 + 2e_8 + e_{10} - e_7 - 2e_4)P(0) + 3(4e_{11} + 2e_{12} + 3e_{14} + e_{15} + 2e_{16} + e_{18})d_{23} + (56 - 52h)(1 - 2h) - 8(4 - 25h + 25h^2)$$

$$D_4 = 24$$

with :

$$P(0) = -d_{23}A_4/B_4$$

and

$$A_4 = z(e_{11}z^2 + e_{12}z + e_{13})a_1 + z(e_{14}z + e_{15})a_2 + (e_{16}z + e_{17})a_3 + e_{18}a_4$$

$$B_4 = (e_1z^3 + e_2z^2 + e_3z + e_4)a_1 + (e_5z^2 + e_6z + e_7)a_2 + (e_8z + e_9)a_3 + e_{10}a_4$$

where :

$$e_1 = d_{19}d_{24} + d_{20}$$

$$e_7 = 12h$$

$$e_{13} = 3$$

$$e_2 = 2(d_1d_{24} + d_2)$$

$$e_8 = d_{13}d_{24} + d_{14}$$

$$e_{14} = d_{11}$$

$$e_3 = 3(d_{24} - 4h)$$

$$e_9 = 2(d_3d_{24} + d_4)$$

$$e_{15} = 2d_5$$

$$e_4 = 4$$

$$e_{10} = d_9d_{24} + d_{10}$$

$$e_{16} = d_{13}$$

$$e_5 = d_{11}d_{24} + d_{12}$$

$$e_{11} = d_{19}$$

$$e_{17} = 2d_3$$

$$e_6 = 2(d_5d_{24} + d_6)$$

$$e_{12} = 2d_1$$

$$e_{18} = d_9$$

$$d_1 = (10h + (1+2h)(6+7h)(1+4h) + 2(1+h)(1+3h)(3+4h))c_1$$

$$d_2 = -8h(4h + h(6+7h) + 2h(1+2h) + (6+7h)(1+2h)^2 + 2(1+2h)(3+4h)(1+h)^2)c_1$$

$$d_3 = -0.5$$

$$d_7 = d_1 + d_3 + d_5$$

$$d_4 = 4h(1+2h)$$

$$d_8 = d_2 + d_4 + d_6$$

$$d_5 = 1+4h$$

$$d_9 = -2(1+3h)/3$$

$$d_6 = -8h(1+2h)$$

$$d_{10} = 16h(1+h)(1+2h)/3$$

$$d_{11} = (-20h(1+h) + 2(1+4h)(6+7h)(1+2h)^2 + 4(1+h)(1+2h)(1+3h)(3+4h))c_1$$

$$d_{12} = -8h(4h^2(6+7h) + 4h(1+2h)^2 + 2(6+7h)(1+2h)^3 +$$

$$+ 4(3+4h)(1+h)^2(1+2h)^2)c_1$$

$$d_{13} = -(10h + 2(1+h)(1+3h)(3+4h) - 4(1+4h)(1+2h)^2)c_1$$

$$d_{14} = 8h(2h(1+2h) - 8h^2 - 4(1+2h)^3 + 2(1+2h)(3+4h)(1+h)^2)c_1$$

$$d_{15} = (1+h)(30h - 12(1+4h)(1+2h)^2 - 2(1+3h)(3+4h)(7+12h))c_1/3$$

$$d_{16} = 8h(1+h)(2(1+h)(1+2h)(3+4h)(7+12h) + 24h^2 + 12(1+2h)^3 - 6h(1+2h))c_1/3$$

$$d_{17} = d_{15}/(4(1+h))$$

$$d_{18} = d_{16}/(4(1+h))$$

$$d_{19} = (12hd_1 + 5(1+h)((3+4h)d_{11} - 4hd_5) - 4hd_{13} + 12hd_{17} - 6(1+h)(2(1+h)d_{15} - 2hd_9))c_2$$

$$d_{20} = (12hd_2 + 5(1+h)((3+4h)d_{12} - 4hd_6) - 4hd_{14} + 12hd_{18} - 6(1+h)(2(1+h)d_{16} - 2hd_{10}))c_2$$

$$d_{21} = d_9 + d_{11} + d_{13} + d_{19}$$

$$d_{22} = d_{10} + d_{12} + d_{14} + d_{20}$$

$$d_{23} = 4(1-2h)/(3+2d_7+d_{21})$$

$$d_{24} = -(4+2d_8+d_{22})/(3+2d_7+d_{21})$$

$$c_1 = 1/(5(2+3h))$$

$$c_2 = 1/(3(8+9h))$$

$$a_1 = 3z^2(1-x)(1+4x) - 3(1-x)(1+4xz)$$

$$a_2 = -3xz(1+4xz) + 3xz^3(1+4x)$$

$$a_3 = -3xz(1+4xz) + 3xz^3(1+4x)$$

$$a_4 = 3z^3(1-xz)(1+4x) - 3z(1-xz)(1+4xz)$$

$$x = 1 + h - hz \quad \text{and}$$

z is the only root within the unit circle of the equation :

$$4x^2z - 1 = 0.$$

$$P(1) = d_{23} + d_{24}P(0)$$

$$P(2) = d_7P(1) + d_8P(0)$$

$$P(3) = d_{21}P(1) + d_{22}P(0)$$

$$= 4(1-2h) - 2P(2) - 3P(1) - 4P(0).$$

A2.3.5 $r = 5$

$$N_5 = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5 + 24(e_6 + e_7 + e_8 + e_9 + e_{10}) - f_6$$

$$D_5 = 120$$

with :

$$f_1 = 140.5 - 161h$$

$$f_2 = 128.5 - 161h$$

$$f_3 = 116.5 - 161h$$

$$f_4 = 104.5 - 161h$$

$$f_5 = 92.5 - 161h$$

$$f_6 = 462.5 - 2,210h + 2,210h^2$$

$$e_1 = p(4;4,0)$$

$$e_2 = p(4;3,1) + 2p(3;3,0)$$

$$e_3 = p(4;2,2) + 2p(3;2,1) + 3p(2;2,0)$$

$$e_4 = p(4;1,3) + 2p(3;1,2) + 3p(2;1,1) + 4p(1;1,0)$$

$$e_5 = p(4;0,4) + 2p(3;0,3) + 3p(2;0,2) + 4p(1;0,1) + 5P(0)$$

$$e_6 = 0$$

$$e_7 = p(4;3,1)$$

$$e_8 = 2p(4;2,2) + 2p(3;2,1)$$

$$e_9 = 3p(4;1,3) + 4p(3;1,2) + 3p(2;1,1)$$

$$e_{10} = 4p(4;0,4) + 6p(3;0,3) + 6p(2;0,2) + 4p(1;0,1)$$

$$p(1;1,0) = 5hP(0)$$

$$p(1;0,1) = -5hP(0) + P(1)$$

$$p(2;2,0) = 2.5h(2+5h)P(0) - 0.5P(1)$$

$$p(2;1,1) = -5h(2+5h)P(0) + (1+5h)P(1)$$

$$p(2;0,2) = w_7 + (2.5h(2+5h)+w_8)P(0) + (-(0.5+5h)+w_9)P(1)$$

$$p(3;3,0) = 5hs_1P(0) - t_1P(1)$$

$$p(3;2,1) = -w_7 + (-5hs_3-w_8)P(0) + (t_3-w_9)P(1)$$

$$p(3;1,2) = (2+5h)w_7 + (5hs_2+(2+5h)w_8)P(0) + (-t_2+(2+5h)w_9)P(1)$$

$$\begin{aligned}
p(3;0,3) &= u_2 w_7 + (5hs_7 + u_2 w_8)P(0) + (-t_7 + u_2 w_9)P(1) \\
p(4;4,0) &= 0.25w_7 + (5hs_4 + 0.25w_8)P(0) + (-t_4 + 0.25w_9)P(1) \\
p(4;3,1) &= -(7+15h)w_7/3 + (-5hs_5 - (7+15h)w_8/3)P(0) + \\
&\quad + (t_5 - (7+15h)w_9/3)P(1) \\
p(4;2,2) &= -u_1 w_7 + (-5hs_6 - u_1 w_8)P(0) + (t_6 - u_1 w_9)P(1) \\
p(4;1,3) &= u_3 w_7 + (5hs_8 + u_3 w_8)P(0) + (t_8 + u_3 w_9)P(1) \\
p(4;0,4) &= u_5 w_7 + (5hs_{11} + u_5 w_8)P(0) + (t_{11} + u_5 w_9)P(1)
\end{aligned}$$

and

$$\begin{aligned}
P(0) &= \frac{g_3(z_1) g_1(z_2) - g_1(z_1) g_3(z_2)}{g_2(z_1) g_3(z_2) - g_3(z_1) g_2(z_2)} \\
P(1) &= \frac{g_2(z_1) g_1(z_2) - g_1(z_1) g_2(z_2)}{g_3(z_1) g_2(z_2) - g_2(z_1) g_3(z_2)}
\end{aligned}$$

where :

$$\begin{aligned}
g_1(z_j) / w_7 &= 0.25d_{1j} - (7+15h)z_j d_{2j}/3 - (u_1 z_j^2 + 2z_j) d_{3j} + \\
&\quad + (u_3 z_j^3 + 2(2+5h)z_j^2) d_{4j} + \\
&\quad + (u_5 z_j^4 + 2u_2 z_j^3 + 3z_j^2) d_{5j} \quad (j=1,2) \\
g_2(z_j) &= (5hs_4 + 0.25w_8) d_{1j} + ((-5hs_5 - (7+15h)w_8/3)z_j + 10hs_1) d_{2j} + \\
&\quad + ((-5hs_6 - u_1 w_8)z_j^2 + 2(-5hs_3 - w_8)z_j + 7.5h(2+5h)) d_{3j} + \\
&\quad + ((5hs_8 + u_3 w_8)z_j^3 + 2(5hs_2 + (2+5h)w_8)z_j^2 - \\
&\quad - 15h(2+5h)z_j + 20h) d_{4j} + \\
&\quad + ((5hs_{11} + u_5 w_8)z_j^4 + 2(5hs_7 + u_2 w_8)z_j^3 + \\
&\quad + 3(2.5h(2+5h) + w_8)z_j^2 - 20hz_j + 5) d_{5j} \quad (j=1,2) \\
g_3(z_j) &= (-t_4 + 0.25w_9) d_{1j} + ((t_5 - (7+15h)w_9/3)z_j - 2t_1) d_{2j} + \\
&\quad + ((t_6 - u_1 w_9)z_j^2 + 2(t_3 - w_9)z_j - 1.5) d_{3j} + \\
&\quad + ((t_8 + u_3 w_9)z_j^3 + 2(-t_2 + (2+5h)w_9)z_j^2 + 3(1+5h)z_j) d_{4j} + \\
&\quad + ((t_{11} + u_5 w_9)z_j^4 + 2(-t_7 + u_2 w_9)z_j^3 + 3(-(0.5+5h) + w_9)z_j^2 + \\
&\quad + 4z_j) d_{5j} \quad (j=1,2)
\end{aligned}$$

$$w_1 = 5h(s_1 + s_2 - s_3 + s_7)$$

$$w_2 = -t_1 - t_2 + t_3 - t_7$$

$$w_3 = 1 + 5h + u_2$$

$$w_4 = 5h(s_4 - s_5 - s_6 + s_8 + s_{11})$$

$$w_5 = -t_4 + t_5 + t_6 + t_8 + t_{11}$$

$$w_6 = 0.25 - (7+15h)/3 - u_1 + u_3 + u_5$$

$$w_7 = 5(1-2h)/(3+2w_3+w_6)$$

$$w_8 = -(5+2w_1+w_4)/(3+2w_3+w_6)$$

$$w_9 = -(4+2w_2+w_5)/(3+2w_3+w_6)$$

$$u_1 = (4+4.5h)(7+15h)/3 + 0.5(1+h)(4+5h)$$

$$u_2 = (2+5h)(3+5h)/3 + 2u_1/3$$

$$u_3 = (3+5h)u_2 - 5h$$

$$u_4 = -h(320+535h) + 26(1+h)(4+5h)u_1 + 3(38+41h)u_3$$

$$u_5 = u_4/36 - (h-5(1+h)^3)(80+150h)/36$$

$$s_1 = (8+30h+25h^2)/6$$

$$s_2 = (4+30h+25h^2)/2$$

$$s_3 = (6+30h+25h^2)/2$$

$$s_4 = (42+220h+300h^2+125h^3)/24$$

$$s_5 = (26+190h+300h^2+125h^3)/6$$

$$s_6 = (4+4.5h)s_5 + 2.5hs_1 + 2(1+h)(4+5h)s_4$$

$$s_7 = (3+5h)s_2/3 + 2s_6/3 + 5h(2+5h)/3$$

$$s_8 = (3+5h)s_7 - 2.5h(2+5h)$$

$$s_9 = -10h(1+h)(12s_4+13s_3) - 5h(6s_5+9s_2) + 3(38+41h)s_8$$

$$s_{10} = s_9 - 12(h-5(1+h)^3)((8+10h)s_4+2s_5)$$

$$s_{11} = s_{10} + 26(1+h)(4+5h)s_6/36 - 35s_{10}/36$$

$$t_1 = (4+15h)/6$$

$$t_2 = (2+35h+50h^2)/2$$

$$t_3 = (3+25h+25h^2)/2$$

$$t_4 = (21+140h+150h^2)/24$$

$$t_5 = (13+165h+300h^2+125h^3)/6$$

$$t_6 = (4+4.5h)t_5 + 2.5ht_4 + 2(1+h)(4+5h)t_4$$

$$t_7 = (3+5h)t_2/3 + 2t_6/3 + 5h(1+5h)/3$$

$$t_8 = 5h(0.5+5h) - (3+5h)t_7$$

$$t_9 = 10h(1+h)(12t_4+13t_3) + 5h(6t_5+9t_2) + 3(38+41h)t_8$$

$$t_{10} = t_9 + 12(h-5(1+h)^3)((8+10h)t_4+2t_5)$$

$$t_{11} = t_{10}/36 - 26(1+h)(4+5h)t_6/36$$

$$\begin{aligned} d_{1j} = & z_j^2 y_{1j} (-20x_j z_j (1-x_j z_j) (3y_{1j} + x_j z_j y_{2j}) + \\ & + 24x_j (1-x_j z_j) (1+5x_j z_j) + 9z_j^2 (1-x_j z_j) (1+5x_j) y_{1j} - \\ & - z_j^2 (9y_{1j}^2 - x_j^2 z_j y_{2j}^2) - 5x_j^2 z_j^3 (1-x_j z_j) (1+5x_j) y_{2j}) \\ & (j=1,2) \end{aligned}$$

$$\begin{aligned} d_{2j} = & z_j^2 y_{1j} (6x_j (1+5x_j z_j) (4x_j + y_{1j}) + 9x_j z_j^2 (1+5x_j) y_{1j} - \\ & - 5x_j z_j (4x_j + y_{1j}) (x_j z_j y_{2j} + 3y_{1j}) + \\ & + x_j^2 z_j^3 (1+5x_j) (2-5x_j) y_{2j}) \\ & (j=1,2) \end{aligned}$$

$$\begin{aligned} d_{3j} = & z_j y_{1j} (15x_j z_j (1-z_j) y_{1j}^2 - 6x_j (1+5x_j z_j) y_{1j} + \\ & + 6x_j z_j^3 (1+5x_j) y_{1j} + 2x_j^2 z_j (1-z_j^2) y_{2j} + \\ & + 10x_j^3 z_j^2 (1-z_j) y_{2j}) \\ & (j=1,2) \end{aligned}$$

$$\begin{aligned} d_{4j} = & z_j y_{1j} (-9x_j (1+5x_j z_j) y_{1j} - 6x_j z_j^2 (1+5x_j) (4x_j z_j + y_{1j}) + \\ & + 5x_j z_j (4x_j z_j + y_{1j}) (x_j y_{2j} + 3y_{1j}) + \\ & + x_j^2 (1+5x_j z_j) (5x_j z_j - 2) y_{2j}) \\ & (j=1,2) \end{aligned}$$

$$\begin{aligned} d_{5j} = & y_{1j} (-9(1-x_j) (1+5x_j z_j) y_{1j} - 24x_j z_j^3 (1-x_j) (1+5x_j) + \\ & + 9y_{1j}^2 - x_j^2 z_j y_{2j}^2 + 20x_j z_j^2 (1-x_j) (x_j y_{2j} + 3y_{1j}) + \\ & + 5x_j^2 z_j (1-x_j) (1+5x_j z_j) y_{2j}) \\ & (j=1,2) \end{aligned}$$

$$y_{1j} = 1 - 5x_j^2 z_j$$

$$y_{2j} = 13 - 25x_j^2 z_j$$

$$x_j = 1 + h - h z_j$$

z_j ($j=1,2$) are the two roots within the unit circle of the equation : $(1 - 25x_j^2 z_j)(9 - 25x_j^2 z_j) = 0$.

$$P(2) = w_7 + w_8 P(0) + w_9 P(1)$$

$$P(3) = w_1 P(0) + w_2 P(1) + w_3 P(2)$$

$$P(4) = w_4 P(0) + w_5 P(1) + w_6 P(2)$$

$$= 5(1-2h) - 2P(3) - 3P(2) - 4P(1) - 5P(0) .$$

A2.4 $\mu W(M/E_3/2)$

METHOD USED : Algebraic solution of the equations of detailed balance for the equilibrium state, as given in reference (32), together with application of Rouché's theorem for the generation of one additional equation.

NOTATION : $h = \rho/3$

$P(n)$: The probability of having n units in the system.

$p(n; s_1, s_2, s_3)$: The probability of having n units in the system, s_1 servers with a unit in phase 1, s_2 servers with a unit in phase 2, and s_3 servers with a unit in phase 3.

$$\mu W(M/E_3/2) = \frac{N_2}{3hr(1-3h)D_2} - 1$$

with :

$$N_2 = 2(8-11\rho)P(0) + 11(1-\rho)p(1;1,0,0) + (14-11\rho)p(1;0,1,0) + (17-11\rho)p(1;0,0,1) - 2(8-31\rho+17\rho^2)$$

$$D_2 = 12$$

$$p(1;1,0,0) = 2hP(0)$$

$$p(1;0,1,0) = 2(1-3h)/(3(1+h)) + 2(2h^3+3h^2-1)P(0)/(3(1+h))$$

$$p(1;0,0,1) = 2(2+3h)(1-3h)/(3(1+h)) - 4(1+3h+3h^2+h^3)P(0)/(3(1+h))$$

$$P(0) = -2(1-\rho)A_2/B_2$$

and

$$\begin{aligned} A_2 &= b_3 + (2+3h)b_4 \\ B_2 &= 3(1+h)b_1 + 6h(1+h)b_2 + 2(-1+3h^2+2h^3)b_3 - \\ &\quad - 4(1+3h+3h^2+h^3)b_4 \end{aligned}$$

where :

$$b_1 = a_1 a_2 \quad b_2 = a_2 a_3 \quad b_3 = a_2 a_4 \quad b_4 = a_2 a_5$$

$$\begin{aligned} a_1 &= 6x(1-x) \\ a_2 &= 3x^2 z^2 (1+2x) + (1+2xz)(1+2x^3 z) - xz(1+2x)(1-4x^3 z) \\ a_3 &= 1 + x + 2x^3 z - 4x^4 z \\ a_4 &= 1 + 3x^2 z - 4x^3 z \\ a_5 &= 3xz(1-x) \end{aligned}$$

$$x = 1 + h - hz \quad \text{and}$$

z is the only root within the unit circle of the equation :

$$-1 - 7x^3 z + 8x^6 z^2 = 0 .$$

A2.5 $\mu W(E_2/E_2/2)$

METHOD USED : Algebraic solution of the equations of detailed balance for the equilibrium state, as given in reference (35), together with application of Rouché's theorem for the generation of one additional equation.

NOTATION : $P(n)$: The probability of having n units in the system.

$$\mu W(E_2/E_2/2) = \frac{N_2}{4\rho^2(1-\rho)D_2} - 1$$

with :

$$\begin{aligned} N_2 &= (2+14\rho+37\rho^2+25\rho^3-11\rho^4-21\rho^5-6\rho^6) - \\ &\quad - (2+18\rho+57\rho^2+73\rho^3+49\rho^4+15\rho^5+2\rho^6)P(0) \\ D_2 &= 4+15\rho+15\rho^2+6\rho^3 \end{aligned}$$

$$P(0) = A_2/B_2$$

and

$$\begin{aligned} A_2 &= d_1b_5 - d_2b_6 + b_1(d_4-d_3)/(2\rho) \\ B_2 &= d_1b_4 - d_2b_7 + (d_4b_3-d_3b_2)/(2\rho) \end{aligned}$$

where :

$$\begin{aligned} b_1 &= (1-\rho)(2+3\rho)^2 \\ b_2 &= 2(1+\rho)^2(2+2\rho-3\rho^2) \\ b_3 &= 2(2+10\rho+18\rho^2+11\rho^3+3\rho^4) \\ b_4 &= 2\rho(1+\rho)(5+12\rho+7\rho^2+2\rho^3) \\ b_5 &= \rho(1-\rho)(2+3\rho)(3+2\rho) \\ b_6 &= 2(1-\rho)(2+12\rho+17\rho^2+9\rho^3) \\ b_7 &= 2(2+14\rho+29\rho^2+29\rho^3+12\rho^4+2\rho^5) \end{aligned}$$

$$d_1 = \rho a_1 c_1 + \rho a_5 c_2$$

$$d_2 = \rho a_2 c_1 + \rho a_6 c_2$$

$$d_3 = \rho a_3 c_1 + \rho a_7 c_2$$

$$d_4 = \rho a_4 c_1 + \rho a_8 c_2$$

$$c_1 = (1+\rho+\rho z)(1+z) + 2x$$

$$c_2 = (1+2\rho)(1+z) + 2x$$

$$a_1 = (1+\rho)z^2(x-1) + \rho z^2(-1-\rho+\rho z)(1+x)$$

$$a_2 = (1+\rho)z^2(-1-\rho+\rho z)(x-1) + \rho z^3(1+x)$$

$$a_3 = 2(1+\rho)z(-2-2\rho+\rho z+x)(x-1) + 2\rho(1-\rho)z^2(1+x)$$

$$a_4 = 2(1+\rho)z(2-x)(x-1) + 2\rho z(-2-2\rho+2\rho z+x)(1+x)$$

$$a_5 = (1+\rho)z^2(-1-\rho+\rho z)(x-1) + \rho z^3(1+x)$$

$$a_6 = (1+\rho)z^3(x-1) + z^3(-\rho-\rho^2+\rho^2 z)(1+x)$$

$$a_7 = 2(1-\rho^2)z^2(x-1) + 2\rho z^2(-2-2\rho+\rho z+x)(1+x)$$

$$a_8 = 2(1+\rho)z(-2-2\rho+2\rho z+x)(x-1) + 2\rho z^2(2-x)(1+x)$$

$$x = (1+\rho)^2 z - \rho^2 z^2 \quad \text{and}$$

z is the only root within the unit circle of the equation :

$$(1+\rho)^2(1-x)^2 - \rho^2 z(1+x)^2 = 0 .$$

A2.6 DESIGN OF THE SIMULATION EXPERIMENTSA2.6.1 Generalities

Computer used : Imperial College CDC 6400

Simulation language : SIMON

Basic references : Conway⁴, Gross and Harris¹⁴, Hills and Mathewson¹⁶, Mitcell³¹, Naylor et al.³⁴ and Tocher⁴⁶.

Two typical programmes, one related to the standard system $\triangle / \square / 6$ with $\rho = 0.85$ and the other related to a four - server system with two classes of non - preemptive priority arrivals are given in Appendix 5.

A2.6.2 Determination of 95 % confidence limits on the simulated average queueing time

Of the two basic approaches discussed in the literature, namely:

- a) n independent simulation runs (replications) ; or
 - b) one long run, breaking up of the single run into n "blocks" and treatment of each block as a separate replication,
- a modified version of the former was selected.

The design of the simulation experiments is briefly described by the flow chart shown in Figure A2.1. Each of the n independent runs (replications) represents a sample of k runs, the $k - 1$ of them being antithetic equivalents to the k th. ρ_{ij} , P_{ij} and μW_{ij} denote respectively the traffic intensity, the probability that an arriving customer joins the queue and the A.Q.T. of an arriving customer as determined after the j th ($j = 1, \dots, k$) run of replication i ($i = 1, \dots, n$).

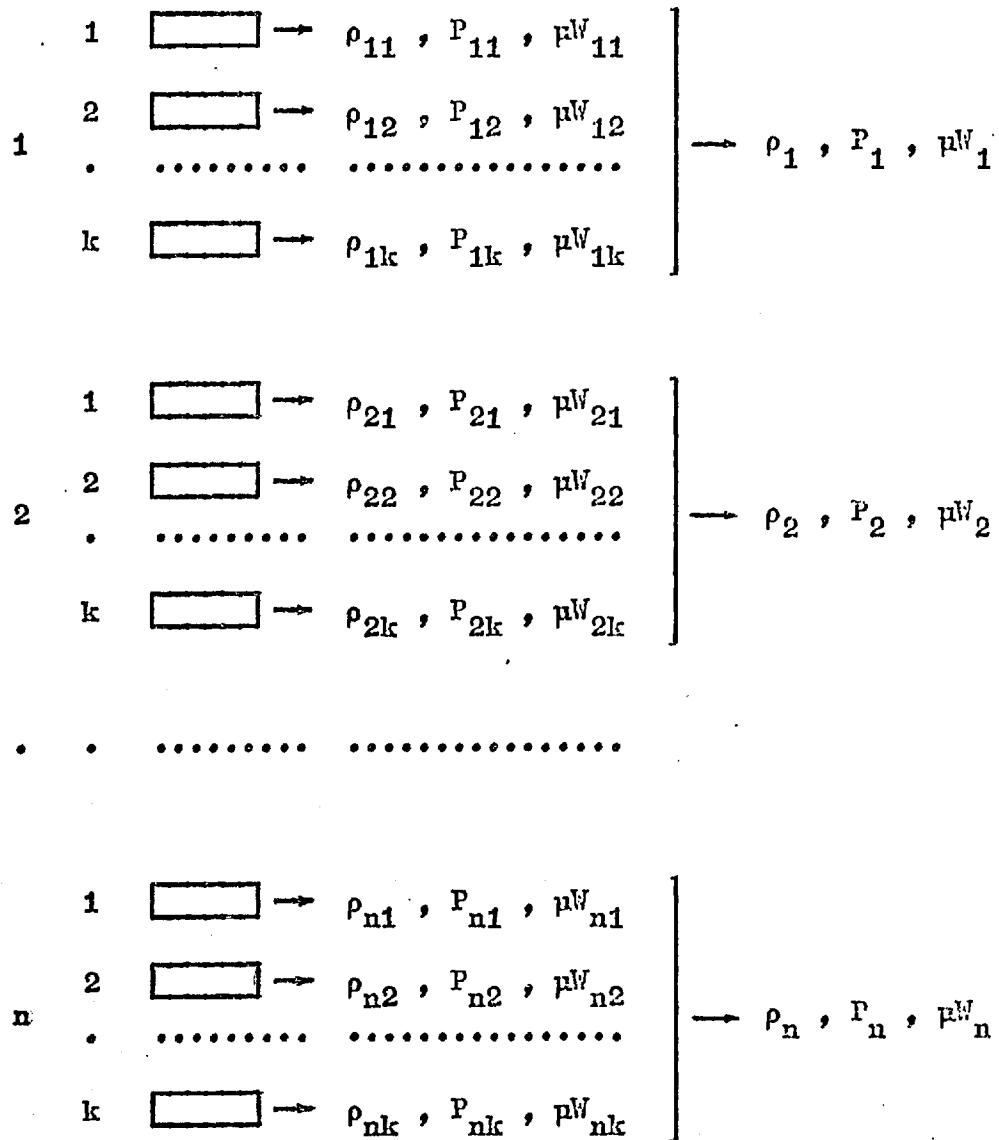


Fig. A2.1 Design of the simulation experiments

In all cases considered, each of the kn simulation runs had approximately the same length in terms of number of customers observed; therefore ρ_i , P_i and μW_i were calculated as $(\rho_{i1} + \rho_{i2} + \dots + \rho_{ik})/k$, $(P_{i1} + P_{i2} + \dots + P_{ik})/k$ and $(\mu W_{i1} + \mu W_{i2} + \dots + \mu W_{ik})/k$ and the simulated average queueing time μW as $(\mu W_1 + \mu W_2 + \dots + \mu W_n)/n$.

Three 95 % confidence intervals were determined:

- one by considering the n values of μW_i in isolation;
- one by considering linear regression of μW_i on ρ_i ; and

c) one by considering linear regression of μW_i on P_i .

The 95 % confidence interval referred to in the main body of the text and also in the Tables of Appendix 4 is the average of the three.

A2.6.3 Codification of the simulation experiments

Each of the queueing systems presented in the Tables of Appendix 4 is described by a "code" of the general form A.B.C.D.

A is related to the sequence of random numbers used in each of the k runs within a replication;

B is related to the sequence of random numbers used in each of the n replications;

C is related to the initial transient state which is ignored in the calculations. C is measured in terms of "number of customers ignored"; finally

D is related to the length of the simulation runs and is measured in terms of "number of customers observed in each of the kn runs".

With r_a , r_s , $r_{a'}$ and $r_{s'}$, denoting $\text{RANDY}(\text{ZZXZZ}(a))$, $\text{RANDY}(\text{ZZXZZ}(s))$, $\text{RANDY}(\text{ZZXZZ}(a'))$ and $\text{RANDY}(\text{ZZXZZ}(s'))$ respectively, where a , s , a' and s' are integers ranging from 1 to 10, the following tables give the values of A, B, C, D, presented in the Tables of Appendix 4, and their interpretation.

a). Values of A and their interpretation

A = 1		A = 2	
i - a times	service times	i - a times	service times
-	r_s	r_a	-
-	$1 - r_s$	$1 - r_a$	-
k = 2			

A = 3		A = 4		A = 5	
i - a times	service times	i - a times	service times	i - a times	service times
r_a	r_s	r_a	$1 - r_s$	r_a	r_s
$1 - r_a$	$1 - r_s$	$1 - r_a$	r_s	$1 - r_a$	r_s
r_s	r_a	$1 - r_s$	r_a	r_a	$1 - r_s$
$1 - r_s$	$1 - r_a$	r_s	$1 - r_a$	$1 - r_a$	$1 - r_s$
k = 4					

A = 6	
i - a times	service times
r_a	r_s
r_a	$1 - r_s$
$1 - r_a$	r_s
$1 - r_a$	$1 - r_s$
r_s	r_a
r_s	$1 - r_a$
$1 - r_s$	r_a
$1 - r_s$	$1 - r_a$
k = 8	

A = 7			
i - a times		service times	
1 - cus/ers	2 - cus/ers	1 - cus/ers	2 - cus/ers
r_a	$r_{a'}$	-	-
$1 - r_a$	$1 - r_{a'}$	-	-
$r_{a'}$	r_a	-	-
$1 - r_{a'}$	$1 - r_a$	-	-
k = 4			

$\Lambda = 8$			
i - a times		service times	
1 - cus/ers	2 - cus/ers	1 - cus/ers	2 - cus/ers
r_a	$1 - r_{a'}$	r_s	$r_{s'}$
$1 - r_a$	$r_{a'}$	$1 - r_s$	$1 - r_{s'}$
$r_{a'}$	$1 - r_a$	$r_{s'}$	r_s
$1 - r_{a'}$	r_a	$1 - r_{s'}$	$1 - r_s$
$k = 4$			

$\Lambda = 9$			
i - a times		service times	
1 - cus/ers	2 - cus/ers	1 - cus/ers	2 - cus/ers
r_a	$1 - r_{a'}$	r_s	$1 - r_{s'}$
$1 - r_a$	$r_{a'}$	$1 - r_s$	$r_{s'}$
$r_{a'}$	$1 - r_a$	$r_{s'}$	$1 - r_s$
$1 - r_{a'}$	r_a	$1 - r_{s'}$	r_s
$k = 4$			

B = 14				B = 15			
a	a'	s	s'	a	a'	s	s'
1	10	-	-	1	10	3	2
2	9	-	-	2	9	4	1
3	8	-	-	3	8	5	10
4	7	-	-	4	7	6	9
5	6	-	-	5	6	7	8
n = 5							

c) Values of C and their interpretation

C	No of customers "ignored"
1	200
2	50

d) Values of D and their interpretation

D	No of arrivals simulated in each of the kn runs
1	10,000
2	15,000
3	20,000

A P P E N D I X 3

TABLES OF VALUES

FOR THE EQUILIBRIUM AVERAGE QUEUING TIME

OF AN ARRIVING CUSTOMER

IN SOME STANDARD SYSTEMS

(OBTAINED ANALYTICALLY)

TABLE A3.1 $\mu W(M/M/r)$

$\rho \backslash r$	1	2	3	4	5
0.30	0.428571	0.098901	0.033347	0.013232	0.005754
0.35	0.538462	0.139601	0.052524	0.023194	0.011210
0.40	0.666667	0.190476	0.078431	0.037792	0.019900
0.45	0.818182	0.253918	0.112773	0.058424	0.033022
0.50	1.000000	0.333333	0.157895	0.086957	0.052149
0.55	1.222222	0.433692	0.217164	0.125999	0.079449
0.60	1.500000	0.562500	0.295620	0.179402	0.118076
0.65	1.857143	0.731602	0.401181	0.253158	0.172887
0.70	2.333333	0.960784	0.547049	0.357212	0.251892
0.75	3.000000	1.285714	0.757009	0.509434	0.369431
0.80	4.000000	1.777778	1.078652	0.745541	0.554113
0.85	5.666667	2.603604	1.623059	1.148860	0.872627
0.90	9.000000	4.263158	2.723537	1.969383	1.524986
0.95	19.000000	9.256410	6.046723	4.457095	3.511196

$\rho \backslash r$	6	7	8	9	10
0.30	0.002654	0.001275	0.000632	0.000320	0.000165
0.35	0.005743	0.003064	0.001685	0.000948	0.000543
0.40	0.011098	0.006443	0.003853	0.002358	0.001469
0.45	0.019746	0.012284	0.007869	0.005156	0.003440
0.50	0.033048	0.021771	0.014761	0.010233	0.007221
0.55	0.052870	0.036542	0.025977	0.018874	0.013953
0.60	0.081903	0.058948	0.043607	0.032951	0.025325
0.65	0.124254	0.092563	0.070819	0.055312	0.043918
0.70	0.186654	0.143209	0.112751	0.090563	0.073910
0.75	0.281101	0.221216	0.178491	0.146822	0.122644
0.80	0.431477	0.347098	0.286028	0.240123	0.204590
0.85	0.693389	0.568556	0.477141	0.407640	0.353241
0.90	1.233543	1.028510	0.876916	0.760595	0.668732
0.95	2.885296	2.441292	2.110422	1.854631	1.651171

TABLE A3.2 $\mu W(D/H/r)$

$\rho \backslash r$	1	2	3	4	5	6
0.30	0.042625	0.004799	0.000789	0.000153	0.000032	0.000007
0.35	0.075483	0.011266	0.002457	0.000632	0.000178	0.000053
0.40	0.120266	0.022320	0.006042	0.001926	0.000673	0.000249
0.45	0.179064	0.039544	0.012685	0.004784	0.001976	0.000866
0.50	0.255001	0.064914	0.023867	0.010290	0.004855	0.002427
0.55	0.352815	0.101101	0.041560	0.019970	0.010483	0.005827
0.60	0.479822	0.151977	0.068533	0.035994	0.020610	0.012480
0.65	0.647715	0.223535	0.108954	0.061602	0.037883	0.024600
0.70	0.876160	0.325706	0.169587	0.101990	0.066540	0.045763
0.75	1.200729	0.476335	0.262449	0.166285	0.113979	0.082202
0.80	1.692731	0.711115	0.411427	0.272504	0.194707	0.146088
0.85	2.518735	1.113345	0.672127	0.462438	0.342244	0.265435
0.90	4.178659	1.932989	1.211245	0.861178	0.656737	0.523775
0.95	9.172434	4.420007	2.862125	2.093790	1.638302	1.337956

TABLE A3.3 $\mu W(I/M/1)$

$\rho \frac{(\nabla^*)^2}{n}$	1	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.428571	0.419938	0.352882	0.296625	0.216612	0.151921
0.35	0.538462	0.528478	0.450717	0.385094	0.290857	0.213382
0.40	0.666667	0.655189	0.565600	0.489675	0.379891	0.288575
0.45	0.818182	0.805008	0.702026	0.614480	0.487253	0.380542
0.50	1.000000	0.984856	0.866334	0.765343	0.618034	0.493725
0.55	1.222222	1.204730	1.067713	0.950759	0.779692	0.634687
0.60	1.500000	1.479631	1.319971	1.183511	0.983499	0.813396
0.65	1.857143	1.833132	1.644831	1.483732	1.247242	1.045620
0.70	2.333333	2.304525	2.078514	1.885010	1.600623	1.357727
0.75	3.000000	2.964536	2.686237	2.447835	2.097168	1.797268
0.80	4.000000	3.954622	3.598449	3.293217	2.843980	2.459423
0.85	5.666667	5.604846	5.119547	4.703546	4.091027	3.566366
0.90	9.000000	8.905403	8.162747	7.526028	6.588284	5.784724
0.95	19.000000	18.807274	17.294158	15.996764	14.085725	12.447797

$\rho \frac{(\nabla^*)^2}{n}$	1/4	1/5	1/6	1/10	1/25	0
0.30	0.121493	0.104032	0.092780	0.071364	0.053561	0.042625
0.35	0.176269	0.154677	0.140613	0.113407	0.090175	0.075483
0.40	0.244285	0.218279	0.201219	0.167871	0.138909	0.120266
0.45	0.328332	0.297480	0.277142	0.237110	0.201951	0.179064
0.50	0.432523	0.396195	0.372167	0.324642	0.282586	0.255001
0.55	0.562972	0.520266	0.491953	0.435761	0.385772	0.352815
0.60	0.728989	0.678609	0.645150	0.578585	0.519149	0.479822
0.65	0.945330	0.885369	0.845497	0.766037	0.694902	0.647715
0.70	1.236694	1.164242	1.116021	1.019804	0.933507	0.876160
0.75	1.647639	1.557992	1.498289	1.379056	1.271974	1.200729
0.80	2.267384	2.152257	2.075552	1.922269	1.784483	1.692731
0.85	3.304203	3.146973	3.042185	2.832698	2.644277	2.518735
0.90	5.383050	5.142087	4.981465	4.660276	4.371279	4.178659
0.95	11.628882	11.137553	10.810010	10.154948	9.565427	9.172434

TABLE A3.4 $\mu W(\Gamma/M/2)$

$\rho \backslash (v^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0962	0.0762	0.0603	0.0396	0.0247
0.35	0.1362	0.1107	0.0902	0.0626	0.0419
0.40	0.1863	0.1547	0.1289	0.0934	0.0660
0.45	0.2489	0.2103	0.1784	0.1340	0.0988
0.50	0.3273	0.2805	0.2416	0.1868	0.1426
0.55	0.4264	0.3700	0.3228	0.2556	0.2007
0.60	0.5537	0.4856	0.4283	0.3461	0.2781
0.65	0.7210	0.6381	0.5682	0.4671	0.3829
0.70	0.9477	0.8456	0.7591	0.6335	0.5281
0.75	1.2693	1.1407	1.0313	0.8720	0.7375
0.80	1.7563	1.5884	1.4453	1.2361	1.0587
0.85	2.5739	2.3410	2.1422	1.8508	1.6028
0.90	4.2170	3.8551	3.5456	3.0910	2.7030
0.95	9.1612	8.4137	7.7735	6.8317	6.0260

$\rho \backslash (v^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0184	0.0129	0.0098	0.0074	0.0064
0.35	0.0328	0.0245	0.0196	0.0157	0.0140
0.40	0.0536	0.0420	0.0349	0.0292	0.0266
0.45	0.0824	0.0670	0.0573	0.0493	0.0457
0.50	0.1216	0.1016	0.0889	0.0781	0.0733
0.55	0.1743	0.1489	0.1324	0.1185	0.1121
0.60	0.2452	0.2132	0.1923	0.1744	0.1663
0.65	0.3418	0.3014	0.2750	0.2523	0.2418
0.70	0.4763	0.4252	0.3916	0.3626	0.3492
0.75	0.6711	0.6054	0.5620	0.5243	0.5070
0.80	0.9708	0.8835	0.8257	0.7754	0.7522
0.85	1.4795	1.3568	1.2753	1.2043	1.1715
0.90	2.5097	2.3168	2.1886	2.0766	2.0249
0.95	5.6238	5.2220	4.9544	4.7205	4.6123

TABLE A3.5 $nW(I/A/3)$

$\rho (v_A^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0322	0.0242	0.0181	0.0108	0.0060
0.35	0.0510	0.0397	0.0310	0.0199	0.0122
0.40	0.0764	0.0613	0.0493	0.0366	0.0222
0.45	0.1102	0.0906	0.0747	0.0535	0.0374
0.50	0.1546	0.1296	0.1092	0.0812	0.0595
0.55	0.2131	0.1817	0.1557	0.1196	0.0909
0.60	0.2905	0.2512	0.2185	0.1724	0.1350
0.65	0.3948	0.3456	0.3044	0.2456	0.1974
0.70	0.5391	0.4769	0.4245	0.3492	0.2867
0.75	0.7467	0.6667	0.5990	0.5010	0.4190
0.80	1.0650	0.9586	0.8682	0.7368	0.6260
0.85	1.6039	1.4540	1.3263	1.1398	0.9817
0.90	2.6934	2.4573	2.2557	1.9601	1.7084
0.95	5.9838	5.4905	5.0682	4.4476	3.9172

$\rho (v_A^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0041	0.0027	0.0019	0.0013	0.0011
0.35	0.0090	0.0063	0.0048	0.0037	0.0032
0.40	0.0173	0.0129	0.0103	0.0083	0.0075
0.45	0.0302	0.0237	0.0197	0.0164	0.0150
0.50	0.0495	0.0402	0.0344	0.0296	0.0275
0.55	0.0774	0.0647	0.0566	0.0498	0.0468
0.60	0.1173	0.1002	0.0893	0.0800	0.0758
0.65	0.1741	0.1516	0.1370	0.1245	0.1189
0.70	0.2564	0.2267	0.2073	0.1906	0.1830
0.75	0.3788	0.3393	0.3133	0.2909	0.2806
0.80	0.5714	0.5174	0.4817	0.4508	0.4366
0.85	0.9034	0.8257	0.7742	0.7294	0.7087
0.90	1.5833	1.4587	1.3759	1.3037	1.2704
0.95	3.6527	3.3886	3.2128	3.0593	2.9883

TABLE A3.6 $\mu W(I/M/4)$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0127	0.0091	0.0064	0.0035	0.0017
0.35	0.0224	0.0168	0.0125	0.0075	0.0042
0.40	0.0367	0.0285	0.0222	0.0142	0.0088
0.45	0.0569	0.0456	0.0367	0.0250	0.0166
0.50	0.0850	0.0698	0.0576	0.0413	0.0290
0.55	0.1234	0.1035	0.0873	0.0651	0.0480
0.60	0.1760	0.1503	0.1291	0.0996	0.0762
0.65	0.2489	0.2157	0.1881	0.1492	0.1177
0.70	0.3517	0.3087	0.2728	0.2215	0.1794
0.75	0.5022	0.4458	0.3983	0.3299	0.2731
0.80	0.7358	0.6595	0.5949	0.5014	0.4229
0.85	1.1349	1.0260	0.9333	0.7983	0.6843
0.90	1.9472	1.7735	1.6253	1.4084	1.2240
0.95	4.4103	4.0436	3.7298	3.2689	2.8754

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0011	0.0007	0.0004	0.0003	0.0002
0.35	0.0029	0.0019	0.0014	0.0010	0.0009
0.40	0.0066	0.0047	0.0036	0.0028	0.0025
0.45	0.0130	0.0098	0.0079	0.0065	0.0058
0.50	0.0236	0.0186	0.0156	0.0131	0.0121
0.55	0.0401	0.0328	0.0282	0.0245	0.0228
0.60	0.0652	0.0548	0.0482	0.0427	0.0403
0.65	0.1027	0.0884	0.0791	0.0713	0.0678
0.70	0.1591	0.1394	0.1266	0.1157	0.1107
0.75	0.2455	0.2184	0.2007	0.1855	0.1785
0.80	0.3844	0.3465	0.3215	0.2999	0.2900
0.85	0.6280	0.5722	0.5353	0.5033	0.4885
0.90	1.1325	1.0415	0.9811	0.9285	0.9042
0.95	2.6793	2.4836	2.3534	2.2397	2.1871

TABLE A3.7 $\mu W(\Gamma/M/5)$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0055	0.0037	0.0025	0.0012	0.0005
0.35	0.0108	0.0077	0.0056	0.0031	0.0016
0.40	0.0192	0.0145	0.0109	0.0066	0.0038
0.45	0.0321	0.0251	0.0197	0.0128	0.0081
0.50	0.0508	0.0409	0.0331	0.0229	0.0155
0.55	0.0777	0.0641	0.0533	0.0387	0.0276
0.60	0.1157	0.0976	0.0828	0.0625	0.0467
0.65	0.1698	0.1458	0.1259	0.0983	0.0762
0.70	0.2478	0.2160	0.1895	0.1520	0.1215
0.75	0.3639	0.3214	0.2856	0.2345	0.1923
0.80	0.5466	0.4881	0.4387	0.3674	0.3078
0.85	0.8618	0.7771	0.7052	0.6006	0.5126
0.90	1.5975	1.3710	1.2545	1.0844	0.9400
0.95	3.4741	3.1330	2.9340	2.5686	2.2568

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0003	0.0002	0.0001	0.0001	0.0001
0.35	0.0011	0.0007	0.0004	0.0003	0.0003
0.40	0.0027	0.0019	0.0014	0.0010	0.0009
0.45	0.0061	0.0045	0.0035	0.0028	0.0025
0.50	0.0123	0.0094	0.0077	0.0064	0.0058
0.55	0.0227	0.0182	0.0154	0.0131	0.0122
0.60	0.0395	0.0327	0.0284	0.0249	0.0233
0.65	0.0658	0.0560	0.0497	0.0444	0.0420
0.70	0.1069	0.0929	0.0838	0.0761	0.0726
0.75	0.1719	0.1520	0.1391	0.1279	0.1229
0.80	0.2787	0.2502	0.2314	0.2152	0.2078
0.85	0.4692	0.4263	0.3980	0.3735	0.3622
0.90	0.8685	0.7973	0.7502	0.7092	0.6903
0.95	2.1015	1.9466	1.8436	1.7537	1.7121

TABLE A3.8 $\mu W(I/M/6)$

$\rho \left(\frac{v^*}{a} \right)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0025	0.0016	0.0010	0.0005	0.0002
0.35	0.0055	0.0038	0.0026	0.0013	0.0006
0.40	0.0107	0.0078	0.0057	0.0033	0.0018
0.45	0.0191	0.0146	0.0112	0.0070	0.0042
0.50	0.0321	0.0254	0.0202	0.0135	0.0088
0.55	0.0516	0.0420	0.0343	0.0243	0.0168
0.60	0.0801	0.0668	0.0561	0.0415	0.0303
0.65	0.1219	0.1037	0.0888	0.0682	0.0520
0.70	0.1835	0.1588	0.1384	0.1098	0.0867
0.75	0.2768	0.2432	0.2151	0.1751	0.1423
0.80	0.4255	0.3786	0.3391	0.2823	0.2351
0.85	0.6846	0.6158	0.5576	0.4731	0.4021
0.90	1.2192	1.1072	1.0118	0.8726	0.7546
0.95	2.8546	2.6138	2.4079	2.1058	1.8483

$\rho \left(\frac{v^*}{a} \right)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0001	0.0001	0.0000	0.0000	0.0000
0.35	0.0004	0.0002	0.0002	0.0001	0.0001
0.40	0.0012	0.0008	0.0006	0.0004	0.0003
0.45	0.0031	0.0022	0.0016	0.0013	0.0011
0.50	0.0068	0.0051	0.0041	0.0033	0.0030
0.55	0.0136	0.0106	0.0089	0.0075	0.0069
0.60	0.0252	0.0206	0.0177	0.0153	0.0143
0.65	0.0445	0.0374	0.0329	0.0292	0.0275
0.70	0.0757	0.0652	0.0585	0.0528	0.0502
0.75	0.1266	0.1112	0.1013	0.0928	0.0890
0.80	0.2121	0.1896	0.1748	0.1621	0.1563
0.85	0.3672	0.3328	0.3101	0.2904	0.2814
0.90	0.6962	0.6382	0.5998	0.5664	0.5510
0.95	1.7200	1.5922	1.5072	1.4330	1.3988

TABLE A3.9 $\mu W(E_2/M/r)$

$\rho \backslash r$	1	2	3	4	5
0.30	0.216612	0.039564	0.010757	0.003462	0.001224
0.35	0.290857	0.062582	0.019876	0.007452	0.003066
0.40	0.379891	0.093427	0.033631	0.014247	0.006615
0.45	0.487253	0.133961	0.053453	0.025018	0.012811
0.50	0.618034	0.186773	0.081229	0.041289	0.022918
0.55	0.779692	0.255606	0.119584	0.065141	0.038668
0.60	0.983499	0.346093	0.172371	0.099569	0.062536
0.65	1.247242	0.467112	0.245585	0.149162	0.098272
0.70	1.600623	0.633493	0.349170	0.221453	0.151981
0.75	2.097168	0.871973	0.500991	0.329906	0.234502
0.80	2.843980	1.236078	0.736760	0.501356	0.367361
0.85	4.091027	1.850763	1.139799	0.798334	0.600633
0.90	6.588284	3.090990	1.960094	1.408359	1.084371
0.95	14.085725	6.831719	4.447623	3.268937	2.568612

$\rho \backslash r$	6		8		10
0.30	0.000460		0.000073		0.000013
0.35	0.001339		0.000286		0.000067
0.40	0.003257		0.000884		0.000264
0.45	0.006951		0.002287		0.000827
0.50	0.013466		0.005185		0.002192
0.55	0.024266		0.010636		0.005108
0.60	0.041462		0.020232		0.010796
0.65	0.068241		0.036418		0.021201
0.70	0.109755		0.063142		0.039514
0.75	0.175102		0.107326		0.071339
0.80	0.282281		0.182572		0.127640
0.85	0.473072		0.320314		0.233652
0.90	0.872565		0.614454		0.464588
0.95	2.105826		1.533870		1.195625

TABLE A3.10 $\mu W(\lambda\lambda/M/1)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(\lambda\lambda/M/1)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	1.0477	0.7716	0.5685	0.3379	0.2031
0.35	1.1161	0.8568	0.6594	0.4212	0.2699
0.40	1.2072	0.9566	0.7611	0.5149	0.3485
0.45	1.3254	1.0773	0.8803	0.6241	0.4425
0.50	1.4771	1.2262	1.0241	0.7547	0.5567
0.55	1.6718	1.4128	1.2018	0.9151	0.6979
0.60	1.9242	1.6512	1.4267	1.1169	0.8765
0.65	2.2573	1.9631	1.7192	1.3781	1.1082
0.70	2.7103	2.3848	2.1130	1.7288	1.4195
0.75	3.3535	2.9813	2.6686	2.2224	1.8582
0.80	4.3283	3.8830	3.5071	2.9663	2.5193
0.85	5.9645	5.3944	4.9109	4.2105	3.6252
0.90	9.2521	8.4285	7.7272	6.7049	5.8425
0.95	19.1420	17.5514	16.1920	14.1995	12.5045

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.1532	0.111602	0.0821	0.0621	0.0543
0.35	0.2106	0.160038	0.1257	0.1010	0.0911
0.40	0.2803	0.220869	0.1815	0.1521	0.1399
0.45	0.3654	0.296776	0.2521	0.2176	0.2030
0.50	0.4701	0.391631	0.3410	0.3009	0.2836
0.55	0.6007	0.511149	0.4535	0.4071	0.3868
0.60	0.7666	0.664019	0.5980	0.5440	0.5202
0.65	0.9826	0.864004	0.7876	0.7242	0.6960
0.70	1.2736	1.134148	1.0440	0.9685	0.9346
0.75	1.6840	1.516029	1.4070	1.3147	1.2731
0.80	2.3032	2.092903	1.9558	1.8387	1.7856
0.85	3.3394	3.059153	2.8754	2.7175	2.6453
0.90	5.4176	4.998058	4.7215	4.4822	4.3723
0.95	11.6628	10.826237	10.2717	9.7890	9.5665

TABLE A3.11 $\mu W(L\Delta/M/2)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(L\Delta/M/2)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.2874	0.1971	0.1344	0.0689	0.0352
0.35	0.3225	0.2342	0.1699	0.0972	0.0552
0.40	0.3683	0.2794	0.2125	0.1323	0.0818
0.45	0.4270	0.3354	0.2647	0.1761	0.1167
0.50	0.5017	0.4055	0.3297	0.2315	0.1622
0.55	0.5974	0.4945	0.4119	0.3023	0.2217
0.60	0.7214	0.6092	0.5180	0.3942	0.3003
0.65	0.8854	0.7605	0.6579	0.5162	0.4059
0.70	1.1090	0.9666	0.8486	0.6833	0.5517
0.75	1.4274	1.2601	1.1204	0.9222	0.7616
0.80	1.9114	1.7063	1.5337	1.2864	1.0832
0.85	2.7260	2.4572	2.2297	1.9011	1.6275
0.90	4.3661	3.9695	3.6322	3.1412	2.7279
0.95	9.3074	8.5263	7.8591	6.8817	6.0509

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0242	0.0159	0.0109	0.0077	0.0065
0.35	0.0402	0.0284	0.0211	0.0161	0.0142
0.40	0.0625	0.0465	0.0368	0.0297	0.0268
0.45	0.0927	0.0721	0.0595	0.0499	0.0459
0.50	0.1329	0.1072	0.0913	0.0788	0.0736
0.55	0.1865	0.1549	0.1350	0.1193	0.1125
0.60	0.2582	0.2195	0.1951	0.1753	0.1666
0.65	0.3553	0.3080	0.2779	0.2532	0.2422
0.70	0.4903	0.4320	0.3947	0.3636	0.3496
0.75	0.6854	0.6123	0.5651	0.5253	0.5074
0.80	0.9854	0.8905	0.8289	0.7764	0.7526
0.85	1.4942	1.3638	1.2786	1.2054	1.1719
0.90	2.5245	2.3239	2.1919	2.0777	2.0253
0.95	5.6387	5.2291	4.9577	4.7216	4.6127

TABLE A3.12 $\mu W(L\Delta/M/3)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(L\Delta/M/3)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.1134	0.0730	0.0464	0.0208	0.0092
0.35	0.1348	0.0931	0.0641	0.0332	0.0169
0.40	0.1626	0.1187	0.0866	0.0500	0.0284
0.45	0.1984	0.1512	0.1156	0.0725	0.0451
0.50	0.2445	0.1930	0.1531	0.1028	0.0685
0.55	0.3043	0.2472	0.2020	0.1432	0.1012
0.60	0.3827	0.3185	0.2669	0.1978	0.1465
0.65	0.4877	0.4142	0.3543	0.2725	0.2098
0.70	0.6322	0.5464	0.4757	0.3774	0.3000
0.75	0.8400	0.7369	0.6511	0.5302	0.4329
0.80	1.1581	1.0292	0.9210	0.7667	0.6405
0.85	1.6967	1.5248	1.3796	1.1704	0.9967
0.90	2.7856	2.5280	2.3092	1.9911	1.7238
0.95	6.0754	5.5610	5.1218	4.4789	3.9328

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0058	0.0034	0.0022	0.0014	0.0011
0.35	0.0115	0.0075	0.0053	0.0038	0.0033
0.40	0.0206	0.0145	0.0110	0.0085	0.0075
0.45	0.0344	0.0257	0.0205	0.0167	0.0151
0.50	0.0546	0.0426	0.0354	0.0299	0.0276
0.55	0.0833	0.0675	0.0578	0.0502	0.0469
0.60	0.1239	0.1034	0.0906	0.0804	0.0760
0.65	0.1814	0.1551	0.1385	0.1250	0.1191
0.70	0.2641	0.2304	0.2089	0.1911	0.1832
0.75	0.3870	0.3432	0.3151	0.2915	0.2808
0.80	0.5800	0.5214	0.4836	0.4514	0.4368
0.85	0.9123	0.8299	0.7761	0.7300	0.7090
0.90	1.5924	1.4630	1.3779	1.3044	1.2707
0.95	3.6620	3.3930	3.2149	3.0600	2.9885

TABLE A3.13 $\mu W(L\Delta/M/l)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(L\Delta/M/l)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0527	0.0319	0.0190	0.0075	0.0028
0.35	0.0663	0.0437	0.0285	0.0134	0.0061
0.40	0.0842	0.0592	0.0416	0.0222	0.0116
0.45	0.1079	0.0798	0.0592	0.0351	0.0205
0.50	0.1390	0.1072	0.0830	0.0534	0.0339
0.55	0.1801	0.1437	0.1153	0.0791	0.0539
0.60	0.2351	0.1929	0.1594	0.1152	0.0831
0.65	0.3098	0.2604	0.2204	0.1664	0.1255
0.70	0.4142	0.3551	0.3067	0.2400	0.1880
0.75	0.5659	0.4936	0.4336	0.3496	0.2825
0.80	0.8004	0.7084	0.6314	0.5220	0.4329
0.85	1.2003	1.0757	0.9707	0.8197	0.6948
0.90	2.0130	1.8239	1.6634	1.4304	1.2349
0.95	4.4763	4.0944	3.7684	3.2915	2.8866

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0016	0.0009	0.0005	0.0003	0.0002
0.35	0.0039	0.0023	0.0015	0.0011	0.0009
0.40	0.0080	0.0053	0.0039	0.0029	0.0025
0.45	0.0151	0.0108	0.0083	0.0066	0.0059
0.50	0.0263	0.0199	0.0161	0.0133	0.0121
0.55	0.0434	0.0343	0.0289	0.0247	0.0229
0.60	0.0691	0.0567	0.0490	0.0430	0.0404
0.65	0.1072	0.0905	0.0801	0.0716	0.0679
0.70	0.1641	0.1417	0.1277	0.1160	0.1108
0.75	0.2510	0.2210	0.2019	0.1859	0.1787
0.80	0.3903	0.3493	0.3228	0.3003	0.2902
0.85	0.6342	0.5751	0.5367	0.5037	0.4887
0.90	1.1390	1.0446	0.9825	0.9290	0.9044
0.95	2.6860	2.4868	2.3549	2.2402	2.1873

TABLE A3.14 $\mu W(L\Delta/M/5)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(L\Delta/M/5)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0269	0.0153	0.0085	0.0029	0.0010
0.35	0.0357	0.0224	0.0139	0.0059	0.0024
0.40	0.0477	0.0323	0.0218	0.0109	0.0052
0.45	0.0640	0.0460	0.0331	0.0186	0.0102
0.50	0.0861	0.0649	0.0492	0.0303	0.0184
0.55	0.1159	0.0909	0.0717	0.0476	0.0313
0.60	0.1567	0.1269	0.1035	0.0730	0.0513
0.65	0.2131	0.1773	0.1486	0.1102	0.0815
0.70	0.2932	0.2495	0.2139	0.1652	0.1276
0.75	0.4111	0.3566	0.3116	0.2489	0.1991
0.80	0.5952	0.5248	0.4660	0.3827	0.3152
0.85	0.9115	0.8149	0.7335	0.6168	0.5205
0.90	1.5581	1.4097	1.2838	1.1013	0.9483
0.95	3.5253	3.2224	2.9640	2.5861	2.2655

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0005	0.0002	0.0001	0.0001	0.0001
0.35	0.0014	0.0008	0.0005	0.0003	0.0003
0.40	0.0034	0.0022	0.0015	0.0011	0.0009
0.45	0.0072	0.0049	0.0037	0.0028	0.0025
0.50	0.0138	0.0101	0.0080	0.0065	0.0058
0.55	0.0247	0.0191	0.0158	0.0133	0.0122
0.60	0.0420	0.0338	0.0289	0.0250	0.0234
0.65	0.0689	0.0574	0.0503	0.0446	0.0421
0.70	0.1105	0.0945	0.0846	0.0764	0.0727
0.75	0.1759	0.1539	0.1399	0.1282	0.1230
0.80	0.2831	0.2522	0.2323	0.2155	0.2079
0.85	0.4739	0.4285	0.3991	0.3738	0.3623
0.90	0.8734	0.7997	0.7513	0.7095	0.6904
0.95	2.1067	1.9490	1.8447	1.7540	1.7122

TABLE A3.15 $\mu W(l\Delta/M/6)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(l\Delta/H/6)$ if $(v_a^*)^2 < 1/6$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0146	0.0078	0.0041	0.0012	0.0003
0.35	0.0204	0.0122	0.0072	0.0028	0.0010
0.40	0.0287	0.0187	0.0122	0.0056	0.0025
0.45	0.0402	0.0281	0.0197	0.0104	0.0054
0.50	0.0564	0.0416	0.0308	0.0182	0.0105
0.55	0.0788	0.0608	0.0471	0.0303	0.0193
0.60	0.1101	0.0881	0.0709	0.0488	0.0334
0.65	0.1544	0.1272	0.1056	0.0769	0.0559
0.70	0.2182	0.1844	0.1569	0.1197	0.0912
0.75	0.3135	0.2705	0.2352	0.1861	0.1475
0.80	0.4638	0.4075	0.3605	0.2943	0.2408
0.85	0.7243	0.6460	0.5802	0.4859	0.4083
0.90	1.2601	1.1385	1.0354	0.8862	0.7613
0.95	2.8963	2.6459	2.4323	2.1201	1.8553

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0002	0.0001	0.0000	0.0000	0.0000
0.35	0.0006	0.0003	0.0002	0.0001	0.0001
0.40	0.0016	0.0009	0.0006	0.0004	0.0003
0.45	0.0037	0.0024	0.0017	0.0013	0.0011
0.50	0.0077	0.0055	0.0042	0.0033	0.0030
0.55	0.0149	0.0112	0.0091	0.0075	0.0069
0.60	0.0270	0.0214	0.0180	0.0154	0.0143
0.65	0.0467	0.0384	0.0333	0.0293	0.0275
0.70	0.0783	0.0664	0.0590	0.0530	0.0503
0.75	0.1295	0.1126	0.1019	0.0930	0.0890
0.80	0.2154	0.1911	0.1755	0.1623	0.1564
0.85	0.3709	0.3345	0.3109	0.2907	0.2815
0.90	0.7002	0.6401	0.6007	0.5667	0.5512
0.95	1.7242	1.5942	1.5081	1.4333	1.3989

TABLE A3.16 $\mu W(\lambda/M/1)$ if $(v_a^*)^2 \geq 1/2$ and $\mu W(\lambda/M/1)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	1.0477	0.7758	0.5811	0.3631	0.2028
0.35	1.1161	0.8597	0.6681	0.4399	0.2698
0.40	1.2072	0.9586	0.7673	0.5288	0.3485
0.45	1.3254	1.0787	0.8847	0.6345	0.4425
0.50	1.4771	1.2272	1.0273	0.7626	0.5567
0.55	1.6718	1.4136	1.2042	0.9211	0.6980
0.60	1.9242	1.6518	1.4285	1.1214	0.8765
0.65	2.2573	1.9636	1.7206	1.3815	1.1082
0.70	2.7103	2.3851	2.1140	1.7313	1.4196
0.75	3.3535	2.9815	2.6693	2.2243	1.8582
0.80	4.3283	3.8832	3.5076	2.9677	2.5193
0.85	5.9645	5.3945	4.9112	4.2114	3.6252
0.90	9.2521	8.4285	7.7274	6.7054	5.8425
0.95	19.1420	17.5514	16.1921	14.1998	12.5045

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.1459	0.1017	0.0782	0.0610	0.0539
0.35	0.2048	0.1517	0.1223	0.1000	0.0907
0.40	0.2757	0.2140	0.1787	0.1512	0.1395
0.45	0.3617	0.2912	0.2497	0.2168	0.2027
0.50	0.4672	0.3871	0.3390	0.3002	0.2834
0.55	0.5984	0.5075	0.4519	0.4066	0.3866
0.60	0.7648	0.6611	0.5967	0.5436	0.5201
0.65	0.9812	0.8617	0.7865	0.7238	0.6958
0.70	1.2725	1.1323	1.0432	0.9682	0.9345
0.75	1.6832	1.5147	1.4064	1.3145	1.2730
0.80	2.3026	2.0919	1.9553	1.8386	1.7855
0.85	3.3390	3.0585	2.8751	2.7174	2.6453
0.90	5.4174	4.9976	4.7213	4.4821	4.3723
0.95	11.6627	10.8260	10.2716	9.7889	9.5665

TABLE A5.17 $nW(M/N/2)$ if $(v_g^*)^2 \geq 1/2$ and $nW(M/N/2)$ if $(v_g^*)^2 < 1/2$

$\rho \backslash (v_g^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.2874	0.1981	0.1372	0.0740	0.0352
0.35	0.3225	0.2349	0.1719	0.1013	0.0552
0.40	0.3683	0.2799	0.2140	0.1356	0.0818
0.45	0.4270	0.3358	0.2658	0.1788	0.1167
0.50	0.5017	0.4058	0.3305	0.2336	0.1622
0.55	0.5974	0.4947	0.4126	0.3039	0.2217
0.60	0.7214	0.6093	0.5185	0.3955	0.3003
0.65	0.8854	0.7606	0.6583	0.5172	0.4059
0.70	1.1090	0.9667	0.8489	0.6840	0.5518
0.75	1.4274	1.2602	1.1205	0.9227	0.7617
0.80	1.9114	1.7063	1.5338	1.2868	1.0832
0.85	2.7260	2.4572	2.2298	1.9014	1.6275
0.90	4.3661	3.9695	3.6322	3.1413	2.7279
0.95	9.3074	8.5263	7.8590	6.8817	6.0509
$\rho \backslash (v_g^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0230	0.0145	0.0103	0.0075	0.0064
0.35	0.0391	0.0269	0.0205	0.0160	0.0141
0.40	0.0615	0.0451	0.0362	0.0295	0.0267
0.45	0.0918	0.0708	0.0589	0.0497	0.0459
0.50	0.1322	0.1061	0.0907	0.0787	0.0735
0.55	0.1859	0.1538	0.1346	0.1191	0.1124
0.60	0.2576	0.2186	0.1947	0.1751	0.1666
0.65	0.3548	0.3073	0.2776	0.2531	0.2422
0.70	0.4899	0.4314	0.3944	0.3635	0.3496
0.75	0.6852	0.6118	0.5649	0.5253	0.5074
0.80	0.9852	0.8902	0.8287	0.7764	0.7526
0.85	1.4941	1.3636	1.2785	1.2053	1.1719
0.90	2.5245	2.3238	2.1918	2.0777	2.0253
0.95	5.6387	5.2291	4.9577	4.7216	4.6127

TABLE A3.18 $\mu W(M/M/3)$ if $(v_a^*)^2 \geq 1/2$ and $\mu W(M/M/3)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.1134	0.0733	0.0475	0.0225	0.0092
0.35	0.1348	0.0934	0.0649	0.0347	0.0169
0.40	0.1626	0.1189	0.0873	0.0513	0.0284
0.45	0.1984	0.1513	0.1161	0.0737	0.0451
0.50	0.2445	0.1931	0.1535	0.1037	0.0685
0.55	0.3043	0.2473	0.2024	0.1440	0.1012
0.60	0.3827	0.3186	0.2671	0.1984	0.1465
0.65	0.4877	0.4142	0.3545	0.2730	0.2098
0.70	0.6322	0.5464	0.4758	0.3777	0.3000
0.75	0.8400	0.7369	0.6512	0.5304	0.4329
0.80	1.1581	1.0292	0.9211	0.7669	0.6405
0.85	1.6967	1.5248	1.3796	1.1705	0.9967
0.90	2.7856	2.5280	2.3092	1.9911	1.7238
0.95	6.0754	5.5610	5.1218	4.4789	3.9328

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0055	0.0031	0.0020	0.0014	0.0011
0.35	0.0111	0.0071	0.0051	0.0038	0.0032
0.40	0.0203	0.0140	0.0108	0.0084	0.0075
0.45	0.0341	0.0252	0.0203	0.0166	0.0151
0.50	0.0542	0.0421	0.0352	0.0298	0.0276
0.55	0.0830	0.0670	0.0576	0.0501	0.0469
0.60	0.1236	0.1030	0.0905	0.0804	0.0760
0.65	0.1812	0.1547	0.1384	0.1250	0.1190
0.70	0.2639	0.2301	0.2088	0.1911	0.1832
0.75	0.3869	0.3430	0.3150	0.2914	0.2808
0.80	0.5799	0.5213	0.4835	0.4514	0.4368
0.85	0.9122	0.8298	0.7761	0.7300	0.7090
0.90	1.5924	1.4630	1.3779	1.3044	1.2707
0.95	3.6620	3.3930	3.2149	3.0600	2.9885

TABLE A3.19 $\mu W(LA/M/4)$ if $(v_a^*)^2 \geq 1/2$ and $\mu W(LM/M/4)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0527	0.0321	0.0194	0.0081	0.0028
0.35	0.0663	0.0438	0.0289	0.0141	0.0061
0.40	0.0842	0.0593	0.0419	0.0229	0.0116
0.45	0.1079	0.0799	0.0595	0.0357	0.0205
0.50	0.1390	0.1072	0.0833	0.0539	0.0359
0.55	0.1801	0.1438	0.1155	0.0795	0.0539
0.60	0.2351	0.1930	0.1596	0.1156	0.0831
0.65	0.3098	0.2604	0.2205	0.1667	0.1255
0.70	0.4142	0.3551	0.3068	0.2402	0.1880
0.75	0.5659	0.4936	0.4337	0.3497	0.2825
0.80	0.8004	0.7084	0.6315	0.5221	0.4329
0.85	1.2003	1.0757	0.9707	0.8198	0.6948
0.90	2.0130	1.8239	1.6634	1.4304	1.2349
0.95	4.4763	4.0944	3.7684	3.2914	2.8866

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0015	0.0008	0.0005	0.0003	0.0002
0.35	0.0037	0.0022	0.0015	0.0010	0.0009
0.40	0.0079	0.0051	0.0038	0.0028	0.0025
0.45	0.0149	0.0105	0.0082	0.0065	0.0059
0.50	0.0261	0.0196	0.0160	0.0133	0.0121
0.55	0.0433	0.0341	0.0288	0.0246	0.0229
0.60	0.0690	0.0564	0.0489	0.0429	0.0403
0.65	0.1071	0.0903	0.0800	0.0716	0.0679
0.70	0.1640	0.1416	0.1276	0.1160	0.1108
0.75	0.2509	0.2209	0.2018	0.1859	0.1787
0.80	0.3902	0.3492	0.3227	0.3003	0.2902
0.85	0.6342	0.5751	0.5366	0.5037	0.4887
0.90	1.1390	1.0445	0.9825	0.9290	0.9044
0.95	2.6860	2.4868	2.3549	2.2402	0.1875

TABLE A3.20 $pW(M/M/5)$ if $(v_a^*)^2 \geq 1/2$ and $pW(M/M/5)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/5
0.30	0.0269	0.0154	0.0087	0.0032	0.0010
0.35	0.0357	0.0225	0.0141	0.0065	0.0024
0.40	0.0477	0.0324	0.0220	0.0112	0.0052
0.45	0.0640	0.0461	0.0333	0.0189	0.0102
0.50	0.0861	0.0650	0.0493	0.0306	0.0184
0.55	0.1159	0.0910	0.0718	0.0479	0.0313
0.60	0.1567	0.1269	0.1036	0.0733	0.0513
0.65	0.2131	0.1774	0.1487	0.1104	0.0815
0.70	0.2932	0.2496	0.2140	0.1653	0.1276
0.75	0.4111	0.3566	0.3117	0.2490	0.1991
0.80	0.5952	0.5248	0.4660	0.3828	0.3152
0.85	0.9115	0.8149	0.7336	0.6169	0.5205
0.90	1.5581	1.4097	1.2838	1.1013	0.9484
0.95	3.5253	3.2224	2.9640	2.5861	2.2655

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0005	0.0002	0.0001	0.0001	0.0001
0.35	0.0014	0.0008	0.0005	0.0003	0.0003
0.40	0.0034	0.0021	0.0015	0.0011	0.0009
0.45	0.0071	0.0048	0.0036	0.0028	0.0025
0.50	0.0137	0.0100	0.0080	0.0065	0.0058
0.55	0.0246	0.0190	0.0157	0.0132	0.0122
0.60	0.0419	0.0337	0.0288	0.0250	0.0234
0.65	0.0688	0.0572	0.0502	0.0445	0.0421
0.70	0.1104	0.0944	0.0845	0.0763	0.0727
0.75	0.1758	0.1538	0.1398	0.1282	0.1230
0.80	0.2830	0.2521	0.2323	0.2155	0.2079
0.85	0.4739	0.4285	0.3990	0.3738	0.3623
0.90	0.8734	0.7997	0.7513	0.7095	0.6904
0.95	2.1067	1.9491	1.8447	1.7540	1.7122

TABLE A3.21 $\mu W(\Delta A/M/6)$ if $(v_a^*)^2 \geq 1/2$ and $\mu W(lm/M/6)$ if $(v_a^*)^2 < 1/2$

$\rho \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
0.30	0.0146	0.0078	0.0042	0.0014	0.0003
0.35	0.0204	0.0123	0.0073	0.0030	0.0010
0.40	0.0287	0.0188	0.0123	0.0058	0.0025
0.45	0.0402	0.0282	0.0198	0.0106	0.0054
0.50	0.0564	0.0416	0.0309	0.0184	0.0105
0.55	0.0788	0.0608	0.0472	0.0305	0.0193
0.60	0.1101	0.0881	0.0709	0.0490	0.0334
0.65	0.1544	0.1272	0.1056	0.0771	0.0559
0.70	0.2182	0.1844	0.1570	0.1198	0.0912
0.75	0.3135	0.2705	0.2352	0.1862	0.1475
0.80	0.4638	0.4075	0.3606	0.2944	0.2408
0.85	0.7243	0.6460	0.5802	0.4860	0.4083
0.90	1.2601	1.1385	1.0354	0.8862	0.7613
0.95	2.8963	2.6459	2.4323	2.1200	1.8553

$\rho \backslash (v_a^*)^2$	1/4	1/6	1/9	1/16	1/25
0.30	0.0002	0.0001	0.0000	0.0000	0.0000
0.35	0.0005	0.0003	0.0002	0.0001	0.0001
0.40	0.0015	0.0009	0.0006	0.0004	0.0003
0.45	0.0036	0.0023	0.0017	0.0013	0.0011
0.50	0.0077	0.0054	0.0042	0.0033	0.0030
0.55	0.0148	0.0111	0.0091	0.0075	0.0069
0.60	0.0269	0.0213	0.0180	0.0154	0.0143
0.65	0.0466	0.0383	0.0333	0.0293	0.0275
0.70	0.0783	0.0663	0.0590	0.0530	0.0503
0.75	0.1295	0.1126	0.1019	0.0930	0.0890
0.80	0.2154	0.1911	0.1755	0.1623	0.1564
0.85	0.3709	0.3345	0.3109	0.2907	0.2815
0.90	0.7002	0.6401	0.6007	0.5667	0.5518
0.95	1.7242	1.5942	1.5081	1.4333	1.3980

TABLE A3.22 $\mu W(\square, (v_a^*)^2 = \frac{1}{3} / M / r)$

$\rho \backslash r$	1	2	3	4	5	6
0.35	0.286021	0.0585	0.0179	0.0065	0.0026	0.0011
0.40	0.361124	0.0847	0.0294	0.0121	0.0055	0.0026
0.45	0.452346	0.1191	0.0460	0.0209	0.0104	0.0055
0.50	0.564377	0.1642	0.0694	0.0344	0.0186	0.0107
0.55	0.703945	0.2233	0.1020	0.0543	0.0316	0.0194
0.60	0.881125	0.3016	0.1471	0.0834	0.0515	0.0336
0.65	1.111756	0.4070	0.2103	0.1258	0.0817	0.0560
0.70	1.422252	0.5526	0.3004	0.1882	0.1278	0.0913
0.75	1.860193	0.7623	0.4333	0.2827	0.1992	0.1476
0.80	2.520778	1.0836	0.6407	0.4330	0.3153	0.2409
0.85	3.626194	1.6278	0.9968	0.6949	0.5205	0.4084
0.90	5.843075	2.7280	1.7238	1.2349	0.9484	0.7613
0.95	12.504723	6.0509	3.9328	2.8866	2.2655	1.8553

TABLE A3.23 $\mu W(\triangle, (v_a^*)^2 = \frac{1}{2} / M / r)$

$\rho \backslash r$	1	2	3	4	5	6
0.35	0.366260	0.0819	0.0271	0.0106	0.0046	0.0021
0.40	0.456460	0.1150	0.0426	0.0186	0.0089	0.0045
0.45	0.564256	0.1574	0.0640	0.0305	0.0160	0.0088
0.50	0.694945	0.2118	0.0932	0.0480	0.0270	0.0161
0.55	0.856137	0.2819	0.1329	0.0730	0.0437	0.0276
0.60	1.059213	0.3735	0.1868	0.1085	0.0685	0.0456
0.65	1.322041	0.4953	0.2611	0.1590	0.1051	0.0732
0.70	1.674383	0.6623	0.3655	0.2322	0.1596	0.1155
0.75	2.169807	0.9012	0.5180	0.3414	0.2429	0.1815
0.80	2.915449	1.2657	0.7544	0.5135	0.3764	0.2894
0.85	4.161299	1.8806	1.1579	0.8110	0.6102	0.4807
0.90	6.657348	3.1209	1.9786	1.4215	1.0945	0.8807
0.95	14.153585	6.8617	4.4664	3.2825	2.5791	2.1144

TABLE A3.24 $W(N/D/r)$

$\rho \backslash r$	1	2	3	4	5
0.30	0.214286	0.055259	0.020094	0.008456	0.003857
0.35	0.269231	0.076770	0.030843	0.014348	0.007244
0.40	0.333333	0.103511	0.045011	0.022699	0.012426
0.45	0.409091	0.136069	0.063416	0.034168	0.019977
0.50	0.500000	0.176741	0.087202	0.049652	0.030653
0.55	0.611111	0.227812	0.118032	0.070422	0.045503
0.60	0.750000	0.293036	0.158409	0.098376	0.066068
0.65	0.928571	0.378337	0.212283	0.136486	0.094747
0.70	1.166667	0.493610	0.286247	0.189708	0.135517
0.75	1.500000	0.656694	0.392188	0.266961	0.195526
0.80	2.000000	0.903284	0.553900	0.386095	0.289060
0.85	2.833333	1.316698	0.826924	0.588766	0.449451
0.90	4.500000	2.146923	1.377912	0.999966	0.776695
0.95	9.500000	4.643948	3.040186	2.244684	1.770788

$\rho \backslash r$	6	7	8	9	10
0.30	0.001851	0.000920	0.000469	0.000243	0.000128
0.35	0.003852	0.002123	0.001200	0.000692	0.000405
0.40	0.007168	0.004288	0.002634	0.001651	0.001051
0.45	0.012310	0.007868	0.005166	0.003462	0.002357
0.50	0.019938	0.013450	0.009321	0.006594	0.004742
0.55	0.030956	0.021834	0.015817	0.011697	0.008792
0.60	0.046674	0.034166	0.025678	0.019695	0.015352
0.65	0.069113	0.052198	0.040454	0.031985	0.025693
0.70	0.101603	0.078805	0.062680	0.050834	0.041872
0.75	0.150115	0.119117	0.096860	0.080263	0.067519
0.80	0.226565	0.183361	0.151954	0.128249	0.109828
0.85	0.358736	0.295360	0.248819	0.213342	0.185505
0.90	0.629966	0.526555	0.449973	0.391122	0.344582
0.95	1.456922				

TABLE A3.25 $\mu W(M/E_2/r)$

ρ^r	1	2	3	4	5
0.30	0.321429	0.077090	0.026659	0.010782	0.004761
0.35	0.403846	0.108308	0.041679	0.018726	0.009179
0.40	0.500000	0.147155	0.061813	0.030248	0.016133
0.45	0.613636	0.195417	0.088321	0.046387	0.026520
0.50	0.750000	0.255639	0.122946	0.068526	0.041511
0.55	0.916667	0.331546	0.168201	0.098606	0.062723
0.60	1.125000	0.428765	0.227852	0.139501	0.092505
0.65	1.392857	0.556177	0.307828	0.195690	0.134484
0.70	1.750000	0.728620	0.418025	0.274620	0.194654
0.75	2.250000	0.972845	0.576273	0.389681	0.283753
0.80	3.000000	1.342391	0.818262	0.567652	0.423226
0.85	4.250000	1.962231	1.227299	0.871021	0.663078
0.90	6.750000	3.207339	2.053361	1.487273	1.153309
0.95	14.250000	6.952694	4.546425	3.353894	2.643913

TABLE A3.25 $\mu W(M/E_2/2)$

ρ	A.Q.T.	ρ	A.Q.T.
0.30	0.069872	0.65	0.497661
0.35	0.097922	0.70	0.651177
0.40	0.132750	0.75	0.868486
0.45	0.175938	0.80	1.197130
0.50	0.229748	0.85	1.748347
0.55	0.297489	0.90	2.855297
0.60	0.384162	0.95	6.184679

TABLE A3.26 $\mu W(E/\Gamma/1)$ and $\mu W(\Gamma/E/1)$ $\rho = 0.30$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	0.419938	0.352882	0.296625	0.216612	0.151921
1/1.02	-	-	-	0.213127	0.148798
1/1.21	-	-	-	0.186082	0.124790
1/1.44	-	-	-	0.163388	0.105018
1/2	0.313421	0.251768	0.200974	0.130952	0.077544
1/3	0.277960	0.218520	0.170012	0.104293	0.055922
1/4				0.091438	0.045923
1/5				0.0839	
1/6				0.078949	0.036559
1/10				0.0693	
0	-	-	-	0.055259	0.020094

 $\rho = 0.30$ (continued)

$(v_s^*)^2 \backslash (v_a^*)^2$	1/4	1/5	1/6	1/10	0
1	0.121493	0.104032	0.092780	0.071364	0.042625
1/1.02	0.118590	0.101279			-
1/1.21	0.096450	0.080408			-
1/1.44	0.078494	0.063695			-
1/2	0.054146	0.041499	0.033753	0.0202	0.005781
1/3	0.035743	0.025325	0.019208		0.001120
1/4					0.000251
1/5		0.0148		0.0035	0.000061
1/6	0.020243	0.012495			
1/10		0.0085			0.000000
0	0.008456	0.003857		0.000128	0

All values with four decimal digits were obtained from the Tables included in reference (35).

TABLE A3.26 (continued)

$$\rho = 0.40$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	0.655189	0.565600	0.489675	0.379891	0.288575
1/1.02	-	-	-	0.374081	0.283127
1/1.21	-	-	-	0.328326	0.240942
1/1.44	-	-	-	0.290596	0.205726
1/2	0.489170	0.405173	0.334902	0.235457	0.155850
1/3	0.433882	0.352228	0.284393	0.189585	0.115527
1/4				0.167242	0.096433
1/5				0.1541	
1/6				0.145367	0.078207
1/10				0.1283	
0	-	-	-	0.103311	0.045011

$$\rho = 0.40 \text{ (continued)}$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/4	1/5	1/6	1/10	0
1	0.244285	0.218279	0.201219	0.167871	0.120266
1/1.02	0.239056	0.213198			-
1/1.21	0.198753	0.174171			-
1/1.44	0.165414	0.142121			-
1/2	0.118893	0.097937	0.084574	0.0596	0.027831
1/3	0.082218	0.063854	0.052432		0.009231
1/4					0.003562
1/5		0.0400		0.0149	0.001493
1/6	0.049594	0.034642			
1/10		0.0248		0.0059	0.000033
0	0.022699	0.012426		0.001051	0

All values with four decimal digits were obtained from the Tables included in reference (35).

TABLE A3.26 (continued)

$$\rho = 0.45$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/2	1/3	1/4	1/5
1/1.02	-	0.479955	0.373601	0.321606	0.290898
1/1.21	-	0.423033	0.319720	0.269578	0.240124
1/1.44	-	0.374825	0.274520	0.226245	0.198069
1/2	0.601116	0.305056	0.210059	0.165167	0.139347
1/3	0.533206	0.246738	0.157422	0.116282	0.093140
1/4		0.218220	0.132273		
1/6		0.190213	0.108088	0.071908	0.052414

$$\rho = 0.55$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/2	1/3	1/4	1/5
1/1.02	-	0.768420	0.623758	0.552247	0.509677
1/1.21	-	0.680319	0.538599	0.468858	0.427474
1/1.44	-	0.605414	0.466637	0.398704	0.358549
1/2	0.899832	0.496417	0.362922	0.298333	0.260493
1/3	0.798257	0.404620	0.276914	0.216151	0.181048
1/4		0.359438	0.235237		
1/6		0.314839	0.194683	0.139194	0.108017

$$\rho = 0.65$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/2	1/3	1/4	1/5
1/1.02	-	1.229728	1.028432	0.928335	0.868500
1/1.21	-	1.092625	0.894148	0.795723	0.737000
1/1.44	-	0.975725	0.780092	0.683392	0.625832
1/2	1.369500	0.804928	0.614464	0.521004	0.465675
1/3	1.215018	0.660260	0.475563	0.385893	0.333259
1/4		0.588699	0.407549		
1/6		0.517772	0.340774	0.256495	0.207883

TABLE A3.26 (continued)

 $\rho = 0.50$

$(V_s^*)^2 \backslash (V_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	0.984856	0.866334	0.765343	0.618034	0.493725
1/1.02	-	-	-	0.608948	0.484989
1/1.21	-	-	-	0.538000	0.417038
1/1.44	-	-	-	0.477787	0.359807
1/2	0.735512	0.622639	0.527340	0.390388	0.277718
1/3	0.652454	0.541992	0.449203	0.317038	0.210124
1/4				0.281045	0.177582
1/5				0.2597	
1/6				0.245600	0.146090
1/10				0.2177	
0	-	-	-	0.176741	0.087208

 $\rho = 0.50$ (continued)

$(V_s^*)^2 \backslash (V_a^*)^2$	1/4	1/5	1/6	1/10	0
1	0.432523	0.396195	0.372167	0.324642	0.255001
1/1.02	0.423997	0.387810			-
1/1.21	0.357861	0.322895			-
1/1.44	0.302472	0.268763			-
1/2	0.223759	0.192397	0.171996	0.1326	0.078415
1/3	0.159971	0.131330	0.112976		0.034185
1/4					0.017306
1/5		0.0867		0.0415	0.009526
1/6	0.101085	0.076259			
1/10		0.0566		0.0195	0.000837
0	0.049652	0.030653		0.004742	0

All values with four decimal digits were obtained
from the Tables included in reference (35).

TABLE A5.26 (continued)

$$\rho = 0.60$$

$(V_s^*)^2 \backslash (V_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	1.479631	1.519971	1.183511	0.983499	0.813396
1/1.02	-	-	-	0.969496	0.799728
1/1.21	-	-	-	0.859953	0.693066
1/1.44	-	-	-	0.766675	0.602684
1/2	1.105287	0.951260	0.820450	0.630647	0.471890
1/3	0.980566	0.828980	0.700708	0.515736	0.362769
1/4				0.459028	0.309596
1/5				0.4253	
1/6				0.402930	0.257608
1/10				0.3586	
0	-	-	-	0.293036	0.158409

$$\rho = 0.60 \text{ (continued)}$$

$(V_s^*)^2 \backslash (V_a^*)^2$	1/4	1/5	1/6	1/10	0
1	0.728989	0.678609	0.645150	0.578585	0.479822
1/1.02	0.715519	0.665269			-
1/1.21	0.610579	0.561479			-
1/1.44	0.521965	0.474062			-
1/2	0.394468	0.348844	0.318839	0.2600	0.175577
1/3	0.289165	0.246263	0.218303		0.089769
1/4					0.052967
1/5		0.1689		0.0951	0.033882
1/6	0.189356	0.150445			
1/10		0.1149		0.0501	0.006264
0	0.098376	0.066068		0.015352	0

All values with four decimal digits were obtained from the Tables included in reference (35).

TABLE A3.26 (continued)

 $\rho = 0.70$

$(v_s^*)^2 \backslash (v_\lambda^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	2.304525	2.078514	1.885010	1.600623	1.357727
1/1.02	-	-	-	1.578430	1.335853
1/1.21	-	-	-	1.404599	1.164776
1/1.44	-	-	-	1.256213	1.019177
1/2	1.721842	1.501407	1.313467	1.039055	0.807113
1/3	1.527677	1.309689	1.124286	0.854685	0.628469
1/4				0.763294	0.540624
1/5				0.7088	
1/6				0.672560	0.454063
1/10				0.6005	
0	-	-	-	0.493610	0.286247

 $\rho = 0.70$ (continued)

$(v_s^*)^2 \backslash (v_\lambda^*)^2$	1/4	1/5	1/6	1/10	0
1	1.236694	1.164242	1.116021	1.019804	0.876160
1/1.02	1.215005	1.142675			-
1/1.21	1.045552	0.974296			-
1/1.44	0.901635	0.831507			-
1/2	0.692747	0.624797	0.579827	0.4908	0.360305
1/3	0.517851	0.452559	0.409578		0.204136
1/4					0.132511
1/5		0.3198		0.2006	0.092855
1/6	0.348844	0.287578			
1/10		0.2247		0.1150	0.026338
0	0.189708	0.135517		0.041872	0

All values with four decimal digits were obtained
from the Tables included in reference (35).

TABLE A3.26 (continued)

 $\rho = 0.75$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/2	1/3	1/4	1/5
1/1.02	-	2.068429	1.768840	1.619391	1.529861
1/1.21	-	1.843204	1.546304	1.398431	1.309959
1/1.44	-	1.650753	1.356584	1.210348	1.122949
1/2	2.215183	1.368702	1.079546	0.936417	0.851141
1/3	1.965463	1.128739	0.845254	0.705820	0.623158
1/4		1.009570	0.729616		
1/6		0.891076	0.615300	0.481252	0.402596

 $\rho = 0.85$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/2	1/3	1/4	1/5
1/1.02	-	4.036124	3.511758	3.249764	3.092643
1/1.21	-	3.605475	3.083671	2.823161	2.667009
1/1.44	-	3.236885	2.717696	2.458736	2.303611
1/2	4.188817	2.695391	2.181039	1.925045	1.771930
1/3	3.716873	2.233088	1.724269	1.471846	1.321239
1/4		2.002774	1.497436		
1/6		1.773165	1.271980	1.024857	0.878175

 $\rho = 0.95$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/2	1/3	1/4	1/5
1/1.02	-	13.900083	12.262435	11.443679	10.952452
1/1.21	-	12.442963	10.807757	9.990399	9.500079
1/1.44	-	11.194217	9.561514	8.745623	8.256267
1/2	14.057903	9.356219	7.728172	6.915106	6.427650
1/3	12.666667	7.782647	6.159972	5.350340	4.865277
1/4		6.996716	5.377447		
1/6		6.211507	4.596332	3.791867	3.310608

TABLE A3.26 (continued)

$$\rho = 0.80$$

$(V_s^*)^2 \backslash (V_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	3.954622	3.598449	3.293217	2.843980	2.459423
1/1.02	-	-	-	2.805427	2.421173
1/1.21	-	-	-	2.503148	2.121519
1/1.44	-	-	-	2.244630	1.865673
1/2	2.955265	2.604583	2.304804	1.865268	1.491238
1/3	2.622210	2.273964	1.976703	1.541919	1.173494
1/4				1.381071	1.016156
1/5				1.2849	
1/6				1.220914	0.860170
1/10				1.0934	
0	-	-	-	0.903284	0.553900

$$\rho = 0.80 \text{ (continued)}$$

$(V_s^*)^2 \backslash (V_a^*)^2$	1/4	1/5	1/6	1/10	0
1	2.267384	2.152257	2.075552	1.922269	1.692731
1/1.02	2.229308	2.114294			-
1/1.21	1.931183	1.817172			-
1/1.44	1.676930	1.563979			-
1/2	1.305532	1.194648	1.120987	0.9744	0.756831
1/3	0.991425	0.883102	0.811348		0.460398
1/4					0.318699
1/5		0.6391		0.3916	0.237104
1/6	0.683450	0.579100			
1/10		0.4608		0.2644	0.087724
0	0.386095	0.289060		0.109828	0

All values with four decimal digits were obtained from the Tables included in reference (35).

TABLE A3.26 (continued)

$$\rho = 0.90$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/1.02	1/1.21	1/1.44	1/2	1/3
1	8.905403	8.162747	7.526028	6.588284	5.784724
1/1.02	-	-	-	6.500690	5.697419
1/1.21	-	-	-	5.813382	5.012612
1/1.44	-	-	-	5.224718	4.426506
1/2	6.656036	5.918772	5.287384	4.359048	3.565585
1/3	5.906313	5.171462	4.542560	3.618892	2.830881
1/4				3.249662	2.465097
1/5				3.0285	
1/6				2.881146	2.100707
1/10				2.5869	
0	-	-	-	2.146923	1.377912

$$\rho = 0.90 \text{ (continued)}$$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/4	1/5	1/6	1/10	0
1	5.383050	5.142087	4.981465	4.660276	4.178659
1/1.02	5.295908	5.055051			-
1/1.21	4.612541	4.372621			-
1/1.44	4.027942	3.789018			-
1/2	3.169914	2.932944	2.775171	2.4602	1.989492
1/3	2.438717	2.204197	2.048237		1.274838
1/4					0.923913
1/5		1.6265		1.1674	0.716760
1/6	1.713781	1.483130			
1/10		1.1981		0.7484	0.316452
0	0.999966	0.776695		0.344582	0

All values with four decimal digits were obtained from the Tables included in reference (35).

TABLE A3.27 $\mu W(E/D/1)$

ρ $(V_a^*)^2$	1	1/2	1/3	1/4	1/5	1/10
0.35	0.268231	0.076770	0.030843	0.014348	0.007244	0.000405
0.40	0.333333	0.103311	0.045011	0.022699	0.012426	0.001051
0.45	0.409091	0.136069	0.063416	0.034168	0.019977	0.002357
0.50	0.500000	0.176741	0.087202	0.049652	0.030653	0.004742
0.55	0.611111	0.227812	0.118032	0.070422	0.045503	0.008792
0.60	0.750000	0.293036	0.158409	0.098376	0.066068	0.015352
0.65	0.928571	0.378337	0.212283	0.136486	0.094747	0.025694
0.70	1.166667	0.493610	0.286247	0.189708	0.135517	0.041872
0.75	1.500000	0.656694	0.392188	0.266961	0.195526	0.067519
0.80	2.000000	0.903284	0.553900	0.386095	0.289060	0.109828
0.85	2.853333	1.316698	0.826924	0.588766	0.449451	0.185505
0.90	4.500000	2.146923	1.377912	0.999966	0.776695	0.344582
0.95	9.500000	4.643948	3.040186	2.244684	1.770788	0.837083

TABLE A3.28 $\mu W(D/E/1)$

ρ $(V_s^*)^2$	1	1/2	1/3	1/4	1/5	1/10
0.35	0.075483	0.014017	0.003735	0.001155	0.000387	0.000003
0.40	0.120266	0.027831	0.009231	0.003562	0.001493	0.000033
0.45	0.179064	0.048696	0.018895	0.008529	0.004187	0.000206
0.50	0.255001	0.078415	0.034185	0.017306	0.009526	0.000837
0.55	0.352815	0.119473	0.056946	0.031430	0.018840	0.002535
0.60	0.479822	0.175577	0.089769	0.052967	0.033882	0.006264
0.65	0.647715	0.252581	0.136636	0.084992	0.057197	0.013460
0.70	0.876160	0.360305	0.204136	0.132511	0.092855	0.026338
0.75	1.200729	0.516513	0.304137	0.204464	0.148054	0.048663
0.80	1.692731	0.756831	0.460398	0.318699	0.237104	0.087724
0.85	2.518735	1.164514	0.728427	0.516858	0.393343	0.160295
0.90	4.178659	1.989492	1.274838	0.923913	0.716760	0.316452
0.95	9.172434	4.481704	2.952918	2.164831	1.707339	0.806241

TABLE A3.29 $\mu W(E/\Delta/1)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(E/\Delta/1)$ if $(v_a^*)^2 < 1/6$ $\rho = 0.30$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	0.188278	0.111152	0.073579	0.051812	
1/1.21	0.167501	0.096236	0.062209	0.042859	0.017082
1/1.44	0.149608	0.083631	0.052814	0.035622	0.013412
1/2	0.123155	0.065359	0.039565	0.025712	0.008794
1/3	0.100429	0.049973	0.028756	0.017920	0.005559
1/4	0.089067	0.042377	0.023533		0.004172
1/6	0.077727	0.034856	0.018420	0.010716	0.002903
0	0.005781	0.001120	0.000251	0.000061	

 $\rho = 0.40$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	0.343184	0.233878	0.177320	0.142236	
1/1.21	0.305587	0.203065	0.150725	0.118633	0.068879
1/1.44	0.273237	0.176944	0.128577	0.099317	0.054941
1/2	0.225467	0.138973	0.097080	0.072488	0.036863
1/3	0.184495	0.106940	0.071182	0.051077	0.023729
1/4	0.164039	0.091130	0.058626		0.017949
1/6	0.143644	0.075496	0.046346	0.031117	0.012614
0	0.103311	0.045011	0.022699	0.012426	

 $\rho = 0.50$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	0.574013	0.428769	0.353058	0.305829	
1/1.21	0.511630	0.373412	0.301849	0.257381	0.186427
1/1.44	0.457981	0.326313	0.258814	0.217140	0.151150
1/2	0.378818	0.257602	0.197013	0.160305	0.103635
1/3	0.310984	0.199476	0.145727	0.114179	0.067723
1/4	0.277138	0.170762	0.120759		0.051558
1/6	0.243407	0.142377	0.096336	0.070790	0.036556
0	0.176741	0.087202	0.049652	0.030653	

TABLE A3.29 (continued)

 $\rho = 0.55$

$(V_S^*)^2 \backslash (V_A^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	0.732080	0.565473	0.479034	0.425495	
1/1.21	0.652861	0.493254	0.410800	0.359791	0.279357
1/1.44	0.584747	0.431697	0.353205	0.304820	0.228645
1/2	0.484263	0.341723	0.270072	0.226497	0.158874
1/3	0.398176	0.265475	0.200723	0.162325	0.104845
1/4	0.355227	0.227774	0.166864		0.080167
1/6	0.312425	0.190490	0.133709	0.101575	0.057152
0	0.227812	0.118032	0.070422	0.045503	

 $\rho = 0.60$

$(V_S^*)^2 \backslash (V_A^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	0.932082	0.740099	0.641134	0.580338	
1/1.21	0.831672	0.646614	0.551458	0.492981	0.402239
1/1.44	0.745353	0.566799	0.475466	0.419430	0.332307
1/2	0.618031	0.449925	0.365256	0.313781	0.234236
1/3	0.508960	0.350693	0.272834	0.226399	0.156413
1/4	0.454544	0.301568	0.227562		0.120269
1/6	0.400306	0.252952	0.183158	0.143056	0.086316
0	0.293036	0.158409	0.098376	0.066068	

 $\rho = 0.65$

$(V_S^*)^2 \backslash (V_A^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	1.191520	0.968026	0.853569	0.783768	
1/1.21	1.063746	0.847081	0.736278	0.668635	0.565132
1/1.44	0.953925	0.743676	0.636553	0.571192	0.470867
1/2	0.791957	0.592009	0.491317	0.430249	0.336635
1/3	0.653204	0.462991	0.368917	0.312651	0.227762
1/4	0.583970	0.399033	0.308752		0.176328
1/6	0.514949	0.335676	0.249615	0.199585	0.127562
0	0.378337	0.212283	0.136486	0.094747	

TABLE A3.29 (continued)

 $\rho = 0.70$

$(V_S^*)^2 \backslash (V_A^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	1.539662	1.275102	1.140407	1.058720	
1/1.21	1.375314	1.117481	0.986315	0.906681	0.786012
1/1.44	1.234087	0.982564	0.854951	0.777482	0.659754
1/2	1.025823	0.784401	0.662970	0.589547	0.477907
1/3	0.847398	0.615530	0.500459	0.431550	0.327770
1/4	0.758349	0.531694	0.420301		0.255721
1/6	0.669547	0.448546	0.341316	0.278417	0.186698
0	0.493610	0.286247	0.189708	0.135517	

 $\rho = 0.75$

$(V_S^*)^2 \backslash (V_A^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	2.029293	1.708069	1.545324	1.447008	
1/1.21	1.813671	1.499079	1.339773	1.243452	1.098317
1/1.44	1.628431	1.320037	1.164180	1.069947	0.927674
1/2	1.355293	1.056768	0.906852	0.816459	0.679900
1/3	1.121264	0.832058	0.688207	0.602030	0.472558
1/4	1.004434	0.720338	0.580011		0.371737
1/6	0.887883	0.609387	0.473113	0.392625	0.274143
0	0.656694	0.392188	0.266961	0.195526	

 $\rho = 0.80$

$(V_S^*)^2 \backslash (V_A^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	2.766081	2.360622	2.156009	2.032714	
1/1.21	2.473500	2.074589	1.873353	1.752032	1.569713
1/1.44	2.222220	1.829396	1.631528	1.512250	1.332821
1/2	1.851752	1.468551	1.276403	1.160804	0.986911
1/3	1.534295	1.160141	0.973749	0.862077	0.694635
1/4	1.375768	1.006592	0.823546		0.551045
1/6	1.217552	0.853885	0.674741	0.568375	0.410720
0	0.903284	0.553900	0.386095	0.289060	

TABLE A3.29 (continued)

 $\rho = 0.85$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	3.996698	3.451605	3.177383	3.012416	
1/1.21	3.575822	3.037203	2.766323	2.603329	2.358635
1/1.44	3.214479	2.681843	2.414269	2.253299	2.011579
1/2	2.681831	2.158567	1.896515	1.739101	1.502869
1/3	2.225351	1.710840	1.454238	1.300484	1.070198
1/4	1.997326	1.487641	1.234192		0.856030
1/6	1.769643	1.265346	1.015617	0.866755	0.645026
0	1.316698	0.826924	0.588766	0.449451	

 $\rho = 0.90$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	6.461289	5.637791	5.224536	4.976204	
1/1.21	5.783813	4.966728	4.556796	4.310451	3.940746
1/1.44	5.202390	4.391191	3.984520	3.740196	3.373570
1/2	4.345496	3.543422	3.142098	2.901237	2.540121
1/3	3.611073	2.817447	2.421278	2.183834	1.828265
1/4	3.244089	2.455120	2.061905		1.474089
1/6	2.877474	2.093748	1.704051	1.471072	1.122872
0	2.146923	1.377912	0.999966	0.776695	

 $\rho = 0.95$

$(v_s^*)^2 \backslash (v_a^*)^2$	1/2	1/3	1/4	1/5	1/8
1/1.02	13.860791	12.203422	11.373369	10.874987	
1/1.21	12.413551	10.762546	9.935807	9.439432	8.694638
1/1.44	11.172029	9.526820	8.703310	8.208963	7.467350
1/2	9.342720	7.706396	6.888078	6.397120	5.661068
1/3	7.774774	6.146590	5.333148	4.845389	4.114556
1/4	6.991036	5.367333	4.556646		3.342636
1/6	6.207694	4.589070	3.781685	3.297969	2.573400
0	4.643948	3.040186	2.244684	1.770788	

TABLE A3.30 $\mu W(E/\square, (v_s^*)^2 = \frac{1}{3}/1)$

$\rho \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5
0.35	0.137780	0.074602	0.046819	0.031929
0.40	0.183876	0.106439	0.071054	0.051363
0.45	0.240349	0.146942	0.103067	0.078005
0.50	0.310020	0.198357	0.144860	0.113747
0.55	0.397023	0.263975	0.199324	0.161262
0.60	0.507611	0.348773	0.270814	0.224547
0.65	0.651654	0.460624	0.366208	0.309885
0.70	0.845642	0.612697	0.497014	0.427786
0.75	1.119303	0.828749	0.684002	0.597225
0.80	1.532127	1.156353	0.968777	0.856217
0.85	2.222978	1.706577	1.448508	1.293588
0.90	3.608499	2.812718	2.414811	2.175940
0.95	7.772001	6.141408	5.325973	4.836549

TABLE A3.31 $\mu W(E/\triangle, (v_s^*)^2 = \frac{1}{2}/1)$

$\rho \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5
0.35	0.173148	0.104366	0.072717	0.054931
0.40	0.229800	0.146730	0.107483	0.084907
0.45	0.298868	0.199918	0.152313	0.124513
0.50	0.383734	0.266696	0.209681	0.176058
0.55	0.489354	0.351148	0.283236	0.242930
0.60	0.623228	0.459480	0.378533	0.330288
0.65	0.797199	0.601522	0.504387	0.446336
0.70	1.031058	0.793729	0.675581	0.604848
0.75	1.360475	1.065797	0.918818	0.830725
0.80	1.856844	1.477193	1.287595	1.173881
0.85	2.686799	2.166756	1.906851	1.750911
0.90	4.350314	3.551112	3.151532	2.911753
0.95	9.347367	7.713556	6.896591	6.406353

TABLE A3.32 $\mu W(l\Delta/E/1)$ if $(v_a^*)^2 \geq 1/6$ and $\mu W(l\Delta/E/1)$ if $(v_a^*)^2 < 1/6$ $\rho = 0.30$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	0.723243	0.711371	0.706594	0.704199	0.701976
1/1.44	0.499376	0.477004	0.465840	0.459175	0.451677
1/2	0.249017	0.214608	0.195095	0.182259	0.166200
1/3	0.123009	0.095515	0.081393	0.072851	0.063150
1/4	0.083344	0.062135	0.052041	0.046229	0.039899
1/6	0.053353	0.038336	0.031812	0.028221	0.024412
0	0.005781	0.001120	0.000251	0.000061	

 $\rho = 0.40$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	0.862051	0.837531	0.826977	0.821246	0.815261
1/1.44	0.654904	0.625880	0.613133	0.606218	0.599176
1/2	0.390118	0.350347	0.330185	0.317739	0.302903
1/3	0.221163	0.180087	0.159006	0.145885	0.136192
1/4	0.158464	0.121490	0.103544	0.092882	0.080778
1/6	0.107770	0.076824	0.063154	0.055611	0.047640
0	0.027831	0.009231	0.003562	0.001493	

 $\rho = 0.50$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	1.052928	1.004061	0.981826	0.969242	0.955563
1/1.44	0.843681	0.793066	0.770456	0.757944	0.744765
1/2	0.559852	0.502268	0.475470	0.460134	0.443381
1/3	0.355007	0.293269	0.263387	0.245572	0.224981
1/4	0.270007	0.209602	0.180809	0.163894	0.144710
1/6	0.196891	0.140547	0.114964	0.100649	0.085389
0	0.078415	0.034185	0.017306	0.009526	

TABLE A3.32 (continued)

 $\rho = 0.55$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	1.184767	1.118385	1.087645	1.070052	1.050745
1/1.44	0.967946	0.900539	0.869873	0.852644	0.834166
1/2	0.668433	0.595474	0.561801	0.542731	0.522193
1/3	0.443686	0.366253	0.329410	0.307878	0.283619
1/4	0.346679	0.269576	0.233031	0.211751	0.187885
1/6	0.261025	0.186911	0.152789	0.133531	0.112854
0	0.119473	0.056946	0.031430	0.018840	

 $\rho = 0.60$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	1.353851	1.265028	1.223324	1.199258	1.172665
1/1.44	1.124520	1.035142	0.993789	0.970260	0.944677
1/2	0.803262	0.709413	0.665932	0.641285	0.614740
1/3	0.554894	0.456552	0.410054	0.383171	0.353407
1/4	0.444261	0.345509	0.298732	0.271638	0.241548
1/6	0.344325	0.247585	0.202458	0.176765	0.148963
0	0.175577	0.089769	0.052967	0.033882	

 $\rho = 0.65$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	1.576182	1.458079	1.401998	1.369420	1.333225
1/1.44	1.328136	1.209780	1.154212	1.122267	1.087173
1/2	0.976807	0.854790	0.797728	0.765179	0.729895
1/3	0.698762	0.572371	0.512525	0.478024	0.440117
1/4	0.571725	0.444391	0.383867	0.348843	0.310156
1/6	0.454682	0.328516	0.268946	0.234742	0.197433
0	0.252581	0.136636	0.084992	0.057197	

TABLE A3.32 (continued)

 $\rho = 0.70$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	1.878168	1.720694	1.645190	1.601083	1.551849
1/1.44	1.602806	1.445264	1.370357	1.326928	1.278833
1/2	1.209295	1.048693	0.972737	0.929059	0.881301
1/3	0.891991	0.727189	0.648689	0.603326	0.553496
1/4	0.743987	0.577864	0.498416	0.452331	0.401455
1/6	0.605277	0.439615	0.360527	0.314755	0.264431
0	0.360305	0.204136	0.132511	0.092855	

 $\rho = 0.75$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	2.307210	2.094355	1.991421	1.930991	1.863252
1/1.44	1.991375	1.778557	1.676250	1.616507	1.549912
1/2	1.536703	1.321269	1.218218	1.158458	1.092542
1/3	1.164471	0.945023	0.839641	0.778421	0.710885
1/4	0.987860	0.766829	0.660320	0.598253	0.529521
1/6	0.819875	0.598722	0.492085	0.429908	0.361031
0	0.516513	0.304137	0.204464	0.148054	

 $\rho = 0.80$

$(v_a^*)^2 \backslash (v_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	2.958045	2.661922	2.517607	2.432491	2.356710
1/1.44	2.579269	2.283234	2.139539	2.055094	1.960436
1/2	2.030636	1.732310	1.588072	1.503751	1.409974
1/3	1.575816	1.273655	1.127265	1.041660	0.946602
1/4	1.356934	1.053009	0.905375	0.818845	0.722513
1/6	1.146054	0.841534	0.693376	0.606398	0.509357
0	0.756831	0.460398	0.318699	0.237104	

TABLE A5.52 (continued)

 $\rho = 0.85$

$(V_a^*)^2 \backslash (V_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	4.051721	3.616683	3.403155	3.276660	3.133782
1/1.44	3.565624	3.130689	2.917747	2.791887	2.650090
1/2	2.857879	2.420893	2.207531	2.081863	1.941044
1/3	2.264980	1.824323	1.608954	1.482136	1.340266
1/4	1.976221	1.533675	1.316987	1.189199	1.046015
1/6	1.694885	1.251365	1.033839	0.905336	0.761002
0	1.164514	0.728427	0.516858	0.393343	

 $\rho = 0.90$

$(V_a^*)^2 \backslash (V_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21	6.251424	5.538508	5.186269	4.976700	4.739101
1/1.44	5.547693	4.834870	4.483169	4.274185	4.037604
1/2	4.518513	3.803813	3.451759	3.242996	3.007393
1/3	3.648675	2.930451	2.576504	2.366678	2.130175
1/4	3.220724	2.500530	2.145217	1.934395	1.696557
1/6	2.799667	2.078194	1.721799	1.510055	1.270736
0	1.989492	1.274838	0.923913	0.716760	

 $\rho = 0.95$

$(V_a^*)^2 \backslash (V_s^*)^2$	1/2	1/3	1/4	1/5	1/7
1/1.21		11.326818			
1/1.44	11.511601	9.965256	9.196957		
1/2	9.512591	7.964495	7.195871		
1/3		6.258776			
1/4	6.965443				
1/6		4.572003	3.798911		
0	4.481704	2.932918	2.164831	1.707339	

A P P E N D I X A

TABLES OF RESULTS

FOR THE EQUILIBRIUM AVERAGE QUEUING TIME

OF AN ARRIVING CUSTOMER

IN SOME STANDARD SYSTEMS

(OBTAINED BY SIMULATION)

TABLE A4.1 STANDARD D/B(t)/1 QUEUEING SYSTEMS

Model description				Results of simulation runs					
B(t)	$(\frac{v}{s})^2$	ρ	code	ρ	Average	queuing time		95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
	1/6	0.85	1.1.1.1	0.8500	0.2741	0.2791	0.2841	0.0100	3.6
"	"	0.90	"	0.9000	0.5323	0.5461	0.5599	0.0276	5.1
"	"	0.95	"	0.9500	1.3194	1.3586	1.3977	0.0783	5.8
	1/3	0.80	1.1.2.1	0.7990	0.3942	0.4026	0.4110	0.0168	4.2
"	"	0.85	1.1.1.1	0.8490	0.6522	0.6691	0.6860	0.0338	5.0
"	"	0.90	1.1.2.1	0.8990	1.1737	1.2030	1.2323	0.0586	4.9
"	"	0.90	1.1.1.1	0.9000	1.1873	1.2162	1.2451	0.0578	4.8
"	"	0.95	"	0.9500	2.7447	2.8600	2.9752	0.2305	8.1
	1/2	0.75	1.1.2.1	0.7493	0.4675	0.4769	0.4864	0.0189	4.0
"	"	0.80	"	0.7993	0.6992	0.7164	0.7336	0.0345	4.3
"	"	0.85	1.1.1.1	0.8503	1.0878	1.1213	1.1548	0.0670	6.0
"	"	0.90	"	0.9003	1.8953	1.9510	2.0067	0.1114	5.7

TABLE A4.2 STANDARD $\Delta/B(t)/1$ QUEUING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(v_s^*)^2$	ρ	code	ρ	Average queuing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	0	0.85	2.2.1.1	0.8500	0.4113	0.4213	0.4313	0.0199	4.7
"	"	0.90	"	0.9000	0.6796	0.6968	0.7141	0.0344	4.9
"	"	0.95	"	0.9500	1.4651	1.5149	1.5647	0.0996	6.6
Δ	1/6	0.80	3.3.2.2	0.7999	0.5235	0.5419	0.5603	0.0369	6.8
"	"	0.85	3.3.1.1	0.8500	0.7607	0.8083	0.8559	0.0952	11.8
"	"	0.90	"	0.9000	1.2782	1.3708	1.4635	0.1854	13.5
"	"	0.90	3.3.2.2	0.9000	1.5109	1.3701	1.4293	0.1184	8.6
"	"	0.95	3.3.1.1	0.9500	2.7784	3.0435	3.3085	0.5301	17.4
$\square$	1/3	0.70	3.3.1.1	0.7000	0.3938	0.4118	0.4299	0.0361	8.8
"	"	0.75	"	0.7500	0.5412	0.5696	0.5981	0.0569	10.0
"	"	0.80	"	0.8000	0.7638	0.8093	0.8547	0.0909	11.2
"	"	0.85	"	0.8500	1.1437	1.2192	1.2947	0.1510	12.4
"	"	0.90	"	0.9000	1.9056	2.0656	2.2256	0.3200	15.5
∇	1/2	0.65	4.3.1.1	0.6502	0.4153	0.4290	0.4426	0.0273	6.4
"	"	0.70	"	0.7002	0.5589	0.5756	0.5924	0.0335	5.8
"	"	0.75	"	0.7502	0.7570	0.7841	0.8113	0.0544	6.9
"	"	0.80	"	0.8003	1.0559	1.1021	1.1483	0.0925	8.4
"	"	0.85	"	0.8503	1.5509	1.6362	1.7215	0.1706	10.4

TABLE A4.3 STANDARD $M/B(t)/1$ QUEUING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(v_s^*)^2$	ρ	code	ρ	Average queueing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	0	0.70	2.2.1.1	0.7000	0.4225	0.4290	0.4356	0.0131	3.1
"	"	0.85	"	0.8500	0.9617	0.9855	1.0093	0.0475	4.8
"	"	0.90	"	0.9000	1.4965	1.5324	1.5684	0.0718	4.7
$\triangle$	1/6	0.70	3.3.1.1	0.7000	0.5451	0.5680	0.5910	0.0458	8.1
"	"	0.80	"	0.8000	0.9254	0.9734	1.0214	0.0960	9.9
"	"	0.85	"	0.8500	1.3067	1.3814	1.4562	0.1496	10.8
"	"	0.90	"	0.9000	2.0817	2.2240	2.3662	0.2845	12.8
$\square$	1/3	0.60	3.3.1.1	0.6000	0.4338	0.4499	0.4660	0.0322	7.2
"	"	0.70	5.4.2.1	0.6986	0.6962	0.7106	0.7250	0.0288	4.1
"	"	0.80	"	0.7991	1.2110	1.2420	1.2730	0.0620	5.0
"	"	0.85	3.3.1.1	0.8500	1.6734	1.7947	1.9160	0.2426	13.5
"	"	0.90	"	0.9000	2.6735	2.9215	3.1695	0.4960	17.0
∇	1/2	0.60	4.3.1.1	0.6000	0.5225	0.5426	0.5627	0.0402	7.4
"	"	0.70	4.4.1.1	0.6989	0.8400	0.8700	0.9000	0.0600	6.9
"	"	0.80	"	0.7995	1.4639	1.5334	1.6030	0.1391	9.1
"	"	0.80	4.3.1.1	0.8003	1.4505	1.5236	1.5968	0.1463	9.6
"	"	0.85	"	0.8503	2.0510	2.1977	2.3443	0.2934	13.8

TABLE A4.4 STANDARD $\Delta/B(t)/1$ QUEUEING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(\tau_s^*)^2$	ρ	code	ρ	Average queueing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	0	0.60	2.2.1.1	0.5998	0.4160	0.4212	0.4264	0.0104	2.5
"	"	0.70	"	0.6998	0.6214	0.6330	0.6447	0.0234	3.7
"	"	0.80	"	0.7998	1.0325	1.0539	1.0754	0.0429	4.1
"	"	0.85	"	0.8498	1.4371	1.4688	1.5006	0.0635	4.3
"	"	0.90	"	0.8998	2.2370	2.2948	2.3527	0.1158	5.1
Δ	1/6	0.60	4.3.1.1	0.5999	0.4999	0.5130	0.5261	0.0262	5.1
"	"	0.70	4.4.2.3	0.6993	0.7776	0.7860	0.7944	0.0168	2.1
"	"	0.80	"	0.7993	1.3000	1.3223	1.3446	0.0446	3.4
"	"	0.80	4.3.1.1	0.7998	1.2758	1.3246	1.3733	0.0975	7.4
"	"	0.85	"	0.8498	1.7761	1.8700	1.9639	0.1878	10.0
$\square$	1/3	0.40	4.3.1.1	0.3999	0.2683	0.2772	0.2862	0.0179	6.5
"	"	0.50	"	0.4999	0.3966	0.4101	0.4235	0.0269	6.6
"	"	0.60	"	0.5999	0.5880	0.6092	0.6304	0.0424	7.0
"	"	0.70	"	0.6999	0.9084	0.9408	0.9731	0.0647	6.9
"	"	0.75	"	0.7499	1.1619	1.2078	1.2538	0.0919	7.6
"	"	0.80	"	0.7998	1.5351	1.6091	1.6830	0.1479	9.2
"	"	0.85	"	0.8498	2.0978	2.2828	2.4677	0.3698	16.2
∇	1/2	0.50	3.3.1.1	0.5000	0.4626	0.4789	0.4952	0.0326	6.8
"	"	0.60	"	0.6000	0.6919	0.7217	0.7515	0.0596	8.3
"	"	0.70	5.4.2.1	0.6985	1.0993	1.1219	1.1446	0.0453	4.0
"	"	0.75	3.3.1.1	0.7500	1.3762	1.4532	1.5301	0.1538	10.6
"	"	0.80	"	0.8000	1.8264	1.9470	2.0676	0.2413	12.4

TABLE A4.5 STANDARD Δ / $\square$ / r QUEUEING SYSTEMS ($r \leq 6$)

r	code	Results of simulation runs					
		p	Average queueing time			95% conf. interval (CI)	100 CI W
			lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
1	4.3.1.1	0.3999	0.2683	0.2772	0.2862	0.0179	6.5
"	"	0.4999	0.3966	0.4101	0.4235	0.0269	6.6
"	"	0.5999	0.5880	0.6092	0.6304	0.0424	7.0
"	"	0.6999	0.9084	0.9408	0.9731	0.0647	6.9
"	"	0.7499	1.1619	1.2078	1.2538	0.0919	7.6
"	"	0.7998	1.5351	1.6091	1.6830	0.1479	9.2
"	"	0.8498	2.0978	2.2828	2.4677	0.3698	16.2
2	"	0.5999	0.2061	0.2158	0.2255	0.0194	9.0
"	"	0.6999	0.3598	0.3732	0.3866	0.0268	7.2
"	"	0.7499	0.4828	0.5025	0.5222	0.0395	7.9
"	"	0.7999	0.6647	0.6990	0.7333	0.0686	9.8
3	"	0.6999	0.2009	0.2059	0.2110	0.0101	4.9
"	"	0.7499	0.2775	0.2893	0.3011	0.0236	8.2
"	"	0.7998	0.3959	0.4177	0.4395	0.0436	10.4
"	"	0.8498	0.5910	0.6381	0.6852	0.0942	14.8
4	"	0.7499	0.1823	0.1908	0.1994	0.0171	9.0
"	"	0.7998	0.2702	0.2846	0.2991	0.0289	10.1
"	"	0.8498	0.4126	0.4480	0.4834	0.0708	15.8
"	"	0.8998	0.6820	0.7911	0.9002	0.2182	27.6
5	"	0.7499	0.1294	0.1358	0.1423	0.0129	9.5
"	"	0.7998	0.1969	0.2086	0.2203	0.0234	11.2
"	"	0.8498	0.3088	0.3371	0.3655	0.0567	16.8
"	"	0.8998	0.5276	0.6104	0.6932	0.1656	27.1

TABLE A4.5 (continued)

r	code	Results of simulation runs					
		ρ	Average queueing time			95% conf. interval (CI)	100 CI W
			lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
6	4.3.1.1	0.7499	0.0959	0.1012	0.1066	0.0107	10.5
"	"	0.7998	0.1503	0.1601	0.1699	0.0195	12.2
"	"	0.8498	0.2422	0.2655	0.2888	0.0466	17.5
"	"	0.8998	0.4249	0.4919	0.5589	0.1340	27.2

TABLE A4.6 STANDARD D /  / r QUEUEING SYSTEMS ($\rho = 0.9$, $r \leq 6$)

r	code	Results of simulation runs					
		ρ	Average queueing time			95% conf. interval (CI)	100 CI W
			lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
1	1.1.1.1	0.9000	1.1873	1.2162	1.2451	0.0578	4.7
2	"	0.9000	0.5628	0.5755	0.5882	0.0254	4.4
3	"	0.9000	0.3559	0.3644	0.3728	0.0169	4.6
4	"	0.9000	0.2546	0.2605	0.2664	0.0118	4.5
5	"	0.9000	0.1936	0.1989	0.2042	0.0106	5.3
6	"	0.9000	0.1543	0.1586	0.1628	0.0085	5.4

TABLE A4.7 STANDARD $M / M / r$ QUEUEING SYSTEMS ($\rho = 0.9$, $r \leq 6$)

r	code	Results of simulation runs					
		ρ	Average queueing time			95% conf. interval (CI)	100 CI ----- W
			lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
1	3.3.1.1	0.9000	1.2782	1.3708	1.4635	0.1854	13.5
2	"	0.9000	0.5911	0.6413	0.6915	0.1004	15.7
3	"	0.9000	0.3715	0.4052	0.4390	0.0676	16.7
4	"	0.9000	0.2673	0.2900	0.3127	0.0455	15.7
5	"	0.9000	0.2041	0.2221	0.2401	0.0360	16.2
6	"	0.9000	0.1631	0.1771	0.1911	0.0280	15.8

TABLE A4.8 STANDARD $M / D / r$ QUEUEING SYSTEMS ($\rho = 0.8$, $r \leq 6$)

r	code	Results of simulation runs					
		ρ	Average queueing time			95% conf. interval (CI)	100 CI ----- W
			lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
1	2.2.1.1	0.7998	1.0325	1.0539	1.0754	0.0429	4.1
2	"	0.7998	0.4370	0.4471	0.4571	0.0201	4.5
3	"	0.7998	0.2555	0.2617	0.2678	0.0124	4.7
4	"	0.7998	0.1707	0.1755	0.1803	0.0095	5.4
5	"	0.7998	0.1235	0.1272	0.1310	0.0075	5.9
6	"	0.7998	0.0944	0.0973	0.1001	0.0057	5.9

TABLE A4.9 STANDARD D/B(t)/6 QUEUEING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(v_s^*)^2$	ρ	code	ρ	Average queueing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
	1/6	0.90	1.1.1.1	0.9000	0.0752	0.0773	0.0794	0.0042	5.4
"	"	0.95	"	0.9500	0.2063	0.2127	0.2190	0.0128	6.0
	1/3	0.90	1.1.1.1	0.9000	0.1543	0.1586	0.1628	0.0085	5.4
"	"	0.95	"	0.9500	0.4054	0.4259	0.4465	0.0410	9.6
	1/2	0.85	1.1.1.1	0.8503	0.1183	0.1228	0.1273	0.0091	7.4
"	"	0.90	"	0.9003	0.2409	0.2505	0.2601	0.0192	7.7

TABLE A4.10 STANDARD /B(t)/6 QUEUEING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(v_s^*)^2$	ρ	code	ρ	Average queueing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	0	0.90	2.2.1.1	0.9000	0.0690	0.0716	0.0743	0.0053	7.5
"	"	0.95	"	0.9500	0.1946	0.2021	0.2096	0.0150	7.4
	1/6	0.85	3.3.1.1	0.8500	0.0825	0.0877	0.0929	0.0103	11.8
"	"	0.90	"	0.9000	0.1631	0.1771	0.1911	0.0280	15.8
"	"	0.95	"	0.9500	0.4111	0.4531	0.4951	0.0840	18.5
	1/3	0.85	3.3.1.1	0.8500	0.1246	0.1356	0.1466	0.0220	16.2
"	"	0.90	"	0.9000	0.2452	0.2693	0.2935	0.0484	18.0
	1/2	0.80	4.3.1.1	0.8003	0.0959	0.1023	0.1086	0.0127	12.4
"	"	0.85	"	0.8503	0.1696	0.1821	0.1947	0.0251	13.8

TABLE A4.11 STANDARD $M/D(1)/6$ QUEUING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(v_s^*)^2$	ρ	code	ρ	Average queueing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	0	0.85	2.2.1.1	0.8500	0.0927	0.0963	0.0998	0.0071	7.4
"	"	0.90	"	0.9000	0.1762	0.1819	0.1877	0.0115	6.3
$\triangle$	1/6	0.85	3.3.1.1	0.8500	0.1447	0.1538	0.1629	0.0183	11.9
"	"	0.90	"	0.9000	0.2661	0.2898	0.3136	0.0476	16.4
$\square$	1/3	0.85	3.3.1.1	0.8500	0.1882	0.2052	0.2223	0.0340	16.6
"	"	0.90	"	0.9000	0.3467	0.3859	0.4250	0.0782	20.3
∇	1/2	0.80	4.3.1.1	0.8003	0.1376	0.1470	0.1564	0.0188	12.8
"	"	0.85	"	0.8503	0.2290	0.2506	0.2722	0.0432	17.3

TABLE A4.12 STANDARD $M/B(t)/6$ QUEUING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(v_B^*)^2$	ρ	code	ρ	Average queuing time			95% conf. interval (CI)	100 CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	0	0.80	2.2.1.1	0.7998	0.0944	0.0973	0.1001	0.0057	5.9
"	"	0.85	"	0.8498	0.1552	0.1598	0.1644	0.0092	5.8
"	"	0.90	"	0.8998	0.2821	0.2913	0.3006	0.0185	6.4
$\triangle$	1/6	0.80	4.3.1.1	0.7998	0.1230	0.1293	0.1355	0.0125	9.7
"	"	0.85	"	0.8498	0.1983	0.2134	0.2284	0.0301	14.1
$\square$	1/3	0.75	4.3.1.1	0.7499	0.0959	0.1012	0.1066	0.0107	10.5
"	"	0.80	"	0.7998	0.1503	0.1601	0.1699	0.0195	12.2
"	"	0.85	"	0.8498	0.2422	0.2655	0.2888	0.0466	17.5
"	"	0.90	"	0.8998	0.4249	0.4919	0.5589	0.1340	27.2
∇	1/2	0.75	3.3.1.1	0.7500	0.1147	0.1239	0.1331	0.0184	14.9
"	"	0.80	"	0.8000	0.1808	0.1970	0.2132	0.0324	16.4

TABLE A4.13 STANDARD $M/D(t)/r$ QUEUEING SYSTEMS

Model description				Results of simulation runs					
$B(t)$	$(\tau_g^*)^2$	r	code	ρ	Average queueing time			95% conf. interval (CI)	100CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
	1/6	6	5.5.2.2	0.5984	0.0507	0.0521	0.0536	0.0029	5.7
"	"	"	3.6.2.3	0.6981	0.1129	0.1154	0.1179	0.0050	4.2
"	"	"	3.7.2.3	0.7978	0.2496	0.2570	0.2644	0.0148	5.8
"	"	"	6.13.2.3	0.9002	0.7008	0.7235	0.7462	0.0454	6.5
	1/3	"	3.8.2.2	0.5993	0.0576	0.0589	0.0602	0.0026	4.6
"	"	"	4.9.2.2	0.6991	0.1267	0.1328	0.1390	0.0123	9.3
"	"	"	5.8.2.2	0.7986	0.2830	0.2956	0.3083	0.0253	8.6
"	"	"	6.9.2.3	0.8984	0.7695	0.8215	0.8734	0.1039	12.7
	1/2	"	3.3.2.2	0.5995	0.0595	0.0637	0.0679	0.0024	13.2
"	"	"	4.10.2.2	0.6995	0.1377	0.1426	0.1475	0.0098	6.9
"	"	"	4.3.2.2	0.7996	0.3191	0.3329	0.3467	0.0276	8.5
"	"	"	6.3.2.3	0.8990	0.8654	0.9097	0.9541	0.0887	9.8
	1/2	4	5.4.2.3	0.8000	0.5586	0.5730	0.5875	0.0289	5.0
"	"	10	5.11.2.3	0.8989	0.4757	0.4983	0.5209	0.0452	9.1
	1/3	3	3.12.2.3	0.5990	0.2058	0.2097	0.2137	0.0080	3.8
"	"	4	"	0.5989	0.1261	0.1293	0.1324	0.0063	4.9
"	"	"	4.12.2.3	0.7985	0.4943	0.5121	0.5299	0.0356	6.9

TABLE A4.14 $\rho W_1(M_1/B(t)/r)$ TO A MULTI-SERVER QUEUING SYSTEM WITH TWO TYPES OF ARRIVING CUSTOMERS, THE SAME DISTRIBUTION OF SERVICE TIMES FOR BOTH AND $\rho = 0.80$ (NON-PRE-EMPTIVE)

Model description				Results of simulation runs					
B(t)	r	ρ_1	code	ρ_1	Average queuing time			95% conf. interval (CI)	100CI W
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	2	0.2	7.14.1.1	0.1990	0.2570	0.2594	0.2618	0.0048	1.8
II	"	"	8.15.1.1	0.1991	0.3236	0.3302	0.3368	0.0132	4.0
II	"	"	9.15.1.1	0.1991	0.3560	0.3624	0.3688	0.0128	3.5
D	4	"	7.14.1.1	0.1990	0.1210	0.1248	0.1286	0.0076	6.1
II	"	"	8.15.1.1	0.1991	0.1494	0.1514	0.1534	0.0041	2.7
II	"	"	9.15.1.1	0.1991	0.1594	0.1617	0.1640	0.0046	2.9
D	2	0.4	7.14.1.1	0.3993	0.3195	0.3243	0.3291	0.0097	3.0
II	"	"	8.15.1.1	0.3992	0.4147	0.4201	0.4255	0.0108	2.6
II	"	"	9.15.1.1	0.3996	0.4662	0.4732	0.4801	0.0138	2.9
D	4	"	7.14.1.1	0.3993	0.1498	0.1525	0.1551	0.0054	3.5
II	"	"	8.15.1.1	0.3992	0.1876	0.1907	0.1938	0.0062	3.3
II	"	"	9.15.1.1	0.3996	0.2074	0.2104	0.2134	0.0060	2.8

TABLE A4.14 $\mu W_2(M_2/B(t)/r)$ TO A MULTI-SERVER QUEUING SYSTEM WITH
 (continued) TWO TYPES OF ARRIVING CUSTOMERS. THE SAME DISTRIBUTION
 OF SERVICE TIMES FOR BOTH AND $\rho = 0.80$ (NON-PREEMPTIVE)

Model description				Results of simulation runs					
B(t)	r	ρ_2	code	ρ_2	Average queueing time			95% conf. interval (CI)	100% -----
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	2	0.6	7.14.1.1	0.6000	1.0677	1.0935	1.1192	0.0515	4.7
П	"	"	8.15.1.1	0.6002	1.4317	1.4705	1.5093	0.0775	5.3
П	"	"	9.15.1.1	0.6001	1.5662	1.6219	1.6777	0.1115	6.9
D	4	"	7.14.1.1	0.6000	0.4496	0.4603	0.4709	0.0213	4.6
П	"	"	8.15.1.1	0.6007	0.6001	0.6221	0.6441	0.0440	7.1
П	"	"	9.15.1.1	0.6001	0.6563	0.6825	0.7088	0.0525	7.7
D	2	0.4	7.14.1.1	0.3993	1.4367	1.4829	1.5292	0.0925	6.2
П	"	"	8.15.1.1	0.3992	1.7431	1.8899	2.0366	0.2935	15.5
П	"	"	9.15.1.1	0.3996	2.1846	2.2588	2.3329	0.1483	6.6
D	4	"	7.14.1.1	0.3993	0.5967	0.6179	0.6392	0.0425	6.9
П	"	"	8.15.1.1	0.3992	0.7148	0.7860	0.8572	0.1424	18.1
П	"	"	9.15.1.1	0.3996	0.9164	0.9489	0.9814	0.0650	6.9

TABLE A4.14 $M(H/B(t)/r)$ TO A MULTI-SERVER QUEUING SYSTEM WITH
(continued) TWO TYPES OF ARRIVING CUSTOMERS, THE SAME DISTRIBUTION
OF SERVICE TIMES FOR BOTH AND $\rho = 0.80$ (NON-PREEMPTIVE)

Model description				Results of simulation runs					
B(t)	r	ρ	code	ρ	Average queuing time			95% conf. interval (CI)	100% -----
					lower 95% c.l.	sim/ed (W)	upper 95% c.l.		
D	2	0.8	7.14.1.1	0.7991	0.8686	0.8857	0.9028	0.0342	3.9
П	"	"	8.15.1.1	0.7993	1.1588	1.1848	1.2107	0.0519	4.4
П	"	"	9.15.1.1	0.7991	1.2645	1.3082	1.3519	0.0874	6.7
D	4	"	7.14.1.1	0.7990	0.3697	0.3767	0.3837	0.0140	3.7
П	"	"	8.15.1.1	0.7998	0.4884	0.5049	0.5214	0.0329	6.5
П	"	"	9.15.1.1	0.7991	0.5336	0.5528	0.5721	0.0395	7.0
D	2	"	7.14.1.1	0.7985	0.8809	0.9036	0.9262	0.0453	5.0
П	"	"	8.15.1.1	0.7985	1.0818	1.1549	1.2280	0.1462	12.7
П	"	"	9.15.1.1	0.7993	1.3263	1.3660	1.4056	0.0793	5.8
D	4	"	7.14.1.1	0.7985	0.3742	0.3852	0.3962	0.0220	5.7
П	"	"	8.15.1.1	0.7984	0.4521	0.4884	0.5247	0.0726	14.9
П	"	"	9.15.1.1	0.7993	0.5623	0.5796	0.5970	0.0347	6.0

A P P E N D I X 5

TWO TYPICAL SIMULATION PROGRAMMES

A5.1 A SIMULATION PROGRAMME RELATED TO THE / / 6 SYSTEM

```

PROGRAM GIGR(INPUT,OUTPUT,TAPE 2=INPUT,TAPE 3=OUTPUT)
INTEGER MAST1(500),MAST2(500),ZZXZZ(10),STATE,TIMES,CLOCK
INTEGER TAILM,COMES,SERVE,WORLD,QUEUE,IDLER,SIZEO,HEADO
INTEGER CUSTM(150),CLERK(6)
DIMENSION SINTM(19)
DIMENSION SERTM(19)
DIMENSION QNGTM(19)
DIMENSION QGTIM(19)
DIMENSION QGTME(19)
COMMON NIM,TAILM,MAST1,MAST2,ZZXZZ,STATE,NAME,TIMES,CLOCK,MEMBE

CALL SIMON
PRINT 1
1 FORMAT(*ENTER I*)
  READ 2,I
2 FORMAT(I4)
  CALL ENTIT(INPUT,1)
  CALL GROUP(CLERK,6,2)
  CALL SET(WORLD)
  CALL SET(QUEUE)
  CALL SET(IDLER)
  CALL HISTO(SINTM,5.,5.)
  CALL HISTO(SERTM,80.,80.)
  CALL HISTO(QNGTM,.001,25.)
  CALL HISTO(QGTIM,250.,40.)
  CALL HISTO(QGTME,650.,60.)

  CLOCK=0
  COMES=100
  SERVE=510

  DO 12 J=1,150
12 CALL ADDLA(J,WORLD)
    DO 11 J=1,6
11 CALL ADDLA(CLERK(J),IDLER)
    RN=RANDY(ZZXZZ(1))

```

```
      IF(RN-0.5)22,32,32
32  LATER=COMES*(2.-SQRT(2.*(1.-RN)))+0.5
      GO TO 95
22  LATER=COMES*SQRT(2.*RN)+0.5
95  CALL EVENT(INPUT,LATER,HEAD0(WORLD))
      CALL BEHEA(WORLD)

55  CALL FAZEA(K)
      IF(CLOCK-1020000)101,201,201
101 GO TO(140,160)K

140 CUSTM(STATE)=CLOCK
      CALL ADDLA(STATE,QUEUE)
      RNN=RANDY(ZZXZZ(1))
      IF(RNN-0.5)24,34,34
34  LATER=COMES*(2.-SQRT(2.*(1.-RNN)))+0.5
      GO TO 96
24  LATER=COMES*SQRT(2.*RNN)+0.5
96  CALL EVENT(INPUT,LATER,HEAD0(WORLD))
      CALL BEHEA(WORLD)
      ALATE=LATER
      IF(CLOCK-20000)151,151,161
161 CALL ADDTO(SINTM,ALATE)
151 GO TO 555

160 CALL ADDLA(STATE,WORLD)
      N=MEMNU(MEMBE)
      DELAY=CLERK(N)-CUSTM(STATE)
      IF(CLOCK-20000)152,152,162
162 CALL ADDTO(QNGTM,DELAY)
      CALL ADDTO(QGTIM,DELAY)
      CALL ADDTO(QGTME,DELAY)
152 CALL ADDLA(MEMBE,IDLER)

555 IF(SIZE0(QUEUE)*SIZE0(IDLER))55,55,65
65  NEXTT=2.*SERVE*(RANDY(ZZXZZ(10)))+0.5
      CALL EVENT(HEAD0(IDLER),NEXTT,HEAD0(QUEUE))
      ANEXT=NEXTT
      IF(CLOCK-20000)154,154,164
164 CALL ADDTO(SERTM,ANEXT)
```


154 N=HEAD0(IDLER)

N=MEMNU(N)

CLERK(N)=CLOCK

CALL BEHEA(IDLER)

CALL BEHEA(QUEUE)

GO TO 555

201 CALL WRITE(SINTI,1)

CALL WRITE(SENEM,2)

CALL WRITE(QNGTM,3)

CALL WRITE(QGTIM,4)

CALL WRITE(QGTME,5)

STOP

END

A5.2 A SIMULATION PROGRAMME RELATED TO A FOUR - SERVER SYSTEM
WITH TWO CLASSES OF NON - PREEMPTIVE PRIORITY ARRIVALS

```

PROGRAM PRIOR(INPUT,OUTPUT,TAPE 2=INPUT,TAPE 3=OUTPUT)
INTEGER MAST1(500),MAST2(500),ZZZZZ(10),STATE,TIMES,CLOCK
INTEGER TAILM,COME1,COME2,SERVE,WORL1,WORL2
INTEGER QUEU1,QUEU2,IDLER,SIZE0,HEAD0
INTEGER CUST1(50,2),CUST2(100,2),CLERK(4,2)
DIMENSION SINT1(19)
DIMENSION SINT2(19)
DIMENSION SERTM(19)
DIMENSION QGTM1(19)
DIMENSION QGTM2(19)
DIMENSION QGTME(19)
COMMON NIM,TAILM,MAST1,MAST2,ZZZZZ,STATE,NAME,TIMES,CLOCK,MEMBE

CALL SIMON
PRINT 1
1 FORMAT(*ENTER I*)
  READ 2,I
2 FORMAT(I4)
  CALL ENTIT(INPU1,1)
  CALL ENTIT(INPU2,2)
  CALL GROUP(CLERK,4,3)
  CALL SET(WORL1)
  CALL SET(WORL2)
  CALL SET(QUEU1)
  CALL SET(QUEU2)
  CALL SET(IDLER)
  CALL HIST0(SINT1,100.,100.)
  CALL HIST0(SINT2,25.,25.)
  CALL HIST0(SERTM,50.,50.)
  CALL HIST0(QGTM1,.001,50.)
  CALL HIST0(QGTM2,.001,50.)
  CALL HIST0(QGTME,.001,50.)

  CLOCK=0
  COME1=900
  COME2=300
  SERVE=720

```

```
DO 121 J=1,50
121 CALL ADDLA(J,WORL1)
DO 122 J=51,150
122 CALL ADDLA(J,WORL2)
DO 11 J=1,4
11 CALL ADDLA(CLERK(J,1),IDLER)

LATE1=-COME1*ALOG((RANDY(ZZXZZ(1))))+0.5
CALL EVENT(INSU1,LATE1,HEAD0(WORL1))
CALL BEHEA(WORL1)
LATE2=-COME2*ALOG(RANDY(ZZXZZ(10)))+0.5
CALL EVENT(INSU2,LATE2,HEAD0(WORL2))
CALL BEHEA(WORL2)

55 CALL FAZEA(K)
IF(CLOCK-2295000)101,201,201
101 GO TO(140,160,180)K

140 CUST1(STATE,1)=CLOCK
CALL ADDLA(STATE,QUEU1)
LATE1=-COME1*ALOG(RANDY(ZZXZZ(1))))+0.5
CALL EVENT(INSU1,LATE1,HEAD0(WORL1))
CALL BEHEA(WORL1)
ALAT1=LATE1
IF(CLOCK-45000)555,555,161
161 CALL ADDTO(SINT1,ALAT1)
GO TO 555

160 CUST2(STATE,1)=CLOCK
CALL ADDLA(STATE,QUEU2)
LATE2=-COME2*ALOG(RANDY(ZZXZZ(10)))+0.5
CALL EVENT(INSU2,LATE2,HEAD0(WORL2))
CALL BEHEA(WORL2)
ALAT2=LATE2
IF(CLOCK-45000)555,555,162
162 CALL ADDTO(SINT2,ALAT2)
GO TO 555
```

```
180 IF(STATE=50)25,25,26
  25 CALL ADDLA(STATE,WORL1)
    N=MEMNU(MEMBE)
    DELAY=CLERK(N,2)-CUST1(STATE,1)
    IF(CLOCK-45000)153,153,163
163 CALL ADDTO(QGTM1,DELAY)
    CALL ADDTO(QGTME,DELAY)
153 CALL ADDLA(MEMBE,IDLER)
    GO TO 555

  26 CALL ADDLA(STATE,WORL2)
    N=MEMNU(MEMBE)
    DELAI=CLERK(N,2)-CUST2(STATE,1)
    IF(CLOCK-45000)154,154,164
164 CALL ADDTO(QGTM2,DELA1)
    CALL ADDTO(QGTME,DELA1)
154 CALL ADDLA(MEMBE,IDLER)

555 IF(SIZE0(QUEUE1)*SIZE0(IDLER))556,556,65
  65 NEXTT=3.*SERVE*(1.-SQRT(RANDY(ZZXZZ(3))))+0.5
    CALL EVENT(HEAD0(IDLER),NEXTT,HEAD0(QUEUE1))
    ANEXT=NEXTT
    IF(CLOCK-45000)155,155,165
165 CALL ADDTO(SERTM,ANEXT)
155 NN=HEAD0(IDLER)
    N=MEMNU(NN)
    CLERK(N,2)=CLOCK
    CALL BEHEA(IDLER)
    CALL BEHEA(QUEUE1)
    GO TO 555

556 IF(SIZE0(QUEUE2)*SIZE0(IDLER))55,55,66
  66 NEXTT=3.*SERVE*(1.-SQRT(RANDY(ZZXZZ(2))))+0.5
    CALL EVENT(HEAD0(IDLER),NEXTT,HEAD0(QUEUE2))
    ANEXT=NEXTT
    IF(CLOCK-45000)156,156,166
166 CALL ADDTO(SERTM,ANEXT)
156 MM=HEAD0(IDLER)
    M=MEMNU(MM)
```

CLERK(M,2)=CLOCK
CALL DEMEA(IDLER)
CALL DEMEA(QUEUE2)
GO TO 556

201 CALL WRITE(SINT1,1)
CALL WRITE(SINT2,2)
CALL WRITE(SERTM,3)
CALL WRITE(QGTM1,4)
CALL WRITE(QGTM2,5)
CALL WRITE(QGTIME,6)

STOP
END